SOLUTION OF THE 33RD PALIS-PUGH PROBLEM FOR GRADIENT-LIKE DIFFEOMORPHISMS OF A TWO-DIMENSIONAL SPHERE

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Abstract. In the present paper, a solution to the 33rd Palis-Pugh problem for gradient-like diffeomorphisms of a two-dimensional sphere is obtained. It is precisely shown that with respect to the stable isotopic connectedness relation there exists countable many of equivalence classes of such systems. 43 words.

1. Introduction. The problem of the existence of an arc with no more than a countable (finite) number of bifurcations connecting structurally stable systems (Morse-Smale systems) on manifolds is on the list of fifty Palis-Pugh problems [28] under number 33. In this paper, this problem is solved for gradient-like diffeomorphisms of a two-dimensional sphere.

First the notion of rough (or structural stable) system was introduced in the classical paper by A. Andronov and L. Pontryagin [3]. In 1976, S. Newhouse, J. Palis, F. Takens [24] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with a little perturbation. In the same year, S. Newhouse and M. Peixoto [25] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse-Smale flows. It follows from the result of G. Fleitas [9] that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one. For Morse-Smale diffeomorphisms given on manifolds of any dimension, examples of systems that cannot be connected by a stable arc are known. In this direction, the question naturally arises of finding an invariant that uniquely determines the equivalence class of the Morse-Smale diffeomorphism with respect to the relation of the connection by a stable arc.

According to [23], for diffeomorphisms of a closed manifold $M^n$ with a finite limit set, the stability of the arc $\{f_t \in Diff(M^n), t \in [0,1]\}$ is characterized by a finite number of bifurcation values $0 < b_1 < \cdots < b_q < 1$, while the bifurcation diffeomorphism $\varphi_{b_i}, i \in \{1, \ldots, q\}$ has the following properties:

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1) diffeomorphism \( \varphi_{b_i} \) has exactly one non-hyperbolic periodic orbit, namely a flip or a non-critical saddle-node, while the arc unfolds generically through a bifurcation value;

2) all invariant manifolds of the periodic points of the diffeomorphism \( \varphi_{b_i} \) intersect transversally and it has no cycles.

We say that diffeomorphisms \( f_0, f_1 : M^n \to M^n \) belong to the same \textit{component of stable connectedness} if in the space of diffeomorphisms they can be connected by an arc with the properties described above.

Classification from the point of view of the introduced equivalence relation is already non-trivial for orientation-preserving diffeomorphisms of the circle \( S^1 \). Here a countable set of such classes appears, each of which is uniquely determined by the rotation number of the rough transformation of the circle [26], which is \( \frac{k}{m} \), where \( k \in (\mathbb{N} \cup 0), m \in \mathbb{N}, k < m, (k, m) = 1 \).

In the present paper we also prove that here is a countable set of the components of the stable connectedness for orientation-preserving gradient-like diffeomorphisms of 2-sphere.

Namely, consider \( S^1 \) as the equator of the sphere \( S^2 \). Then the structurally stable diffeomorphism of the circle with exactly two periodic orbits of the period \( m \in \mathbb{N} \) and the rotation number \( \frac{k}{m} \) can be extended to the diffeomorphism \( \phi_{k,m} : S^2 \to S^2 \), which has two fixed sources at the north and south poles. Denote by \( C_{k,m} \) the component of stable connectedness of the diffeomorphism \( \phi_{k,m} \) and by \( C_{-k,m} \) the component of stable connectedness of the diffeomorphism \( \phi_{-k,m} \). Also denote by \( C_0 \) the component of stable connectedness of the source-sink diffeomorphism \( \phi_0 \).

**Theorem 1.1.** Every orientation-preserving gradient-like diffeomorphism of 2-sphere belongs to one of the components \( C_0, C_{k,m}, C_{-k,m}, k, m \in \mathbb{N}, k < m/2, (k, m) = 1 \). Herewith:

- the components \( C_0, C_{k,m}, C_{-k,m}, k, m \in \mathbb{N}, k < m/2, (k, m) = 1 \) are pairwise disjoint;
- \( C_{k,m} = C_{m-k,m} \), \( C_{-k,m} = C_{m-k,m} \), \( C_{1,2} = C_{1,2}^{-} = C_{0,1} = C_{0,1}^{-} = C_{0} \).

Notice that it follows from a result by P. Blanchard [6] that \( \phi_{k,m}, \phi_{k',m'} : S^2 \to S^2 \) belong to different components for \( m = 2^r \cdot q, m' = 2^{r'} \cdot q' \), where integers \( r, r' \geq 0 \) and positive integers \( q \neq q' \). He obtained some necessary conditions for the connection of Morse-Smale diffeomorphisms on the surface by a stable arc. However, the question on sufficient conditions was not considered in [6].

The obtained result is closely related with the classification of periodic homeomorphisms of a two-dimensional sphere obtained by B. von Kerekjarto [18]. The topological conjugacy class of the periodic transformation of the period \( m \) on a 2-sphere is also completely determined by the rotation number \( \frac{k}{m} \) around the north pole-south pole axis. Since any orientation-preserving gradient-like diffeomorphism is topologically conjugate to a composition of a periodic homeomorphism with a one-time shift of a gradient-like flow [5], [14], the natural question is about an interrelation between these rotation numbers.

The proof of the theorem 1.1 shows that they are not coincide in general. In any case the construction of a stable arc between diffeomorphisms is an independent problem that does not directly follow from the topological classification of diffeomorphisms. In support of this, it suffices to note that all orientation-preserving source-sink systems on the \( n \)-sphere are pairwise topologically conjugate for a fixed \( n \). However, they are not connected by a stable arc in general, for example, for
2. Backgrounds.

2.1. Morse-Smale diffeomorphisms. Let a diffeomorphism \( f : M^n \to M^n \) be given on a smooth closed (compact without boundary) \( n \)-manifold \((n \geq 1)\) \( M^n \) with a metric \( d \).

A point \( x \in M^n \) is called \textit{wandering} for \( f \) if there is an open neighborhood \( U_x \) of the point \( x \), such that \( f^n(U_x) \cap U_x = \emptyset \) for all \( n \in \mathbb{N} \). Otherwise, the point \( x \) is called \textit{non-wandering}. The set of non-wandering points \( f \) is called \textit{non-wandering set} and is denoted by \( \Omega_f \).

For example, all the limit points of a diffeomorphism are non-wandering. Recall that a point \( y \in M^n \) is called a \( \omega \)-\textit{limit point} for a point \( x \in M^n \) if there exists a sequence \( t_k \to +\infty \), \( t_k \in \mathbb{Z} \) such that \( \lim_{t_k \to +\infty} d(f^{t_k}(x), y) = 0 \). The set \( \omega(x) \) of all \( \omega \)-limit points for the point \( x \) is called \( \omega \)-\textit{limit set}. Replacing \( +\infty \) with \( -\infty \) determines the \( \alpha \)-\textit{limit set} \( \alpha(x) \) for the point \( x \). The set \( L_f = cl ( \bigcup_{x \in M^n} \omega(x) \cup \alpha(x) ) \) is called the \textit{limit set} of the diffeomorphism \( f \).

If the set \( \Omega_f \) is finite, then every point \( p \in \Omega_f \) is periodic, we denote by \( m_p \in \mathbb{N} \) the period of the periodic point \( p \). Any periodic point \( p \) is associated with \textit{stable} and \textit{unstable} manifolds defined as follows

\[
W^s_p = \{ x \in M^n : \lim_{k \to +\infty} d(f^{km_p}(x), p) = 0 \},
\]

\[
W^u_p = \{ x \in M^n : \lim_{k \to +\infty} d(f^{-km_p}(x), p) = 0 \}.
\]

Stable and unstable manifolds are called \textit{invariant manifolds}. It is said that the periodic orbits \( O_1, \ldots, O_k \) form a \textit{cycle} if \( W^s_{O_i} \cap W^u_{O_{i+1}} \neq \emptyset \) for \( i \in \{1, \ldots, k\} \) and \( O_{k+1} = O_1 \).

A periodic point \( p \in \Omega_f \) is called \textit{hyperbolic} if the absolute values of the eigenvalues of the Jacobi matrix \( \left( \frac{\partial f^{km_p}}{\partial x} \right)_p \) are not equal one. If all of them are less (greater) than one, then \( p \) is called the \textit{sink} (\textit{source}) point. Sink or source points are called \textit{nodal}. If a hyperbolic periodic point is not \textit{nodal}, then it is called \textit{saddle point}.

The stable \( W^s_p \) and the unstable \( W^u_p \) manifolds of the periodic point \( p \) are injective immersions of the spaces \( \mathbb{R}^{q_p} \) and \( \mathbb{R}^{n-q_p} \), where \( q_p \) is the number of eigenvalues of the Jacobi matrix, modulo large ones (see, for example, [31]). The number \( \nu_p \), equal to \( +1 (-1) \) if the map \( f^{m_p}|_{W^s_p} \) preserves (changes) the orientation of \( W^u_p \), is called an \textit{orientation type} of \( p \). The path-connected component of the set \( W^u_p \setminus p \) (\( W^s_p \setminus p \)) is called an \textit{unstable} (\textit{stable}) \textit{separatrix} of the point \( p \).

A closed \( f \)-invariant set \( A \subset M^n \) is called an \textit{attractor} of a discrete dynamical system generated by \( f \) if it has a compact neighborhood \( U_A \) such that \( f(U_A) \subset int U_A \) and \( A = \bigcap_{k \geq 0} f^k(U_A) \). The neighborhood \( U_A \) is called \textit{trapping}. A repeller is defined as an attractor for \( f^{-1} \). If the trapping neighborhood of an attractor \( A \) is the complement of a trapping neighborhood of a repeller \( R \) then pair \( A, R \) is called \textit{dual}.

A diffeomorphism \( f : M^n \to M^n \) is called \textit{Morse-Smale}, if

1) the non-wandering set \( \Omega_f \) consists of a finite number of hyperbolic orbits;
2) manifolds \( W^s_p, W^u_q \) intersect transversely for any non-wandering points \( p, q \).
A Morse-Smale diffeomorphism is called gradient-like if the condition $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different $\sigma_1, \sigma_2 \in \Omega_f$ implies that $\dim W^s_{\sigma_1} < \dim W^u_{\sigma_2}$. The Morse-Smale flow is similar defined and is called gradient-like in the absence of periodic trajectories.

Notice that the gradient-like diffeomorphism on a surface has no heteroclinic points – intersection points of the invariant manifolds of different saddle points.

**Proposition 1.** [14, Lemma 3.3] All saddle separatrices of an orientation-preserving gradient-like diffeomorphism $f$ on a surface has the same period $\mu_f \in \mathbb{N}$.

A homeomorphism $\phi : M^2 \to M^2$ is called periodic of order $\mu \in \mathbb{N}$ if $\phi^\mu = \text{id}$ and $\phi^j \neq \text{id}$ for any positive integer $j < \mu$.

**Proposition 2.** [14, Theorems 3.1, 3.3] Every orientation-preserving gradient-like diffeomorphism $f$ on a surface is topologically conjugate to a composition of a periodic homeomorphism $\phi_f$ of the period $\mu_f$ with the one-time shift of a gradient-like flow. Moreover, $f|_{\Omega_f} = \phi_f|_{\Omega_f}$.

We denote by $G$ the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere $S^2$. According to the classification given by B. Kerekjarto [18], an orientation-preserving periodic homeomorphism of the period $\mu$ on 2-sphere has periodic points of only two periods $1$ and $\mu$, while the set of its fixed points is not empty. This gives us the following corollary of the propositions 1 and 2 for the class $G$.

**Proposition 3.** Any $f \in G$ has periodic points of only two periods $1$ and $\mu_f$ (possibly $\mu_f = 1$). Moreover,

1) if $f$ has saddle points with the negative orientation type then all such points are fixed and $\mu_f = 2$;

2) any saddle point with a positive orientation type has the period $\mu_f$.

### 2.2. Stable arcs in the space of diffeomorphisms.

Consider a one-parameter family of diffeomorphisms (arc) $\varphi_t : M^n \to M^n, t \in [0,1]$. Denote by $Q$ the set of arcs $\{ \varphi_t \}$, that begin and end in Morse-Smale diffeomorphisms and every diffeomorphism $\varphi_t$ has a finite limit set.

According to [23], an arc $\{ \varphi_t \} \in Q$ is called stable if it is an internal point of the equivalence class with respect to the following relation: arcs $\{ \varphi_t \}, \{ \tilde{\varphi}_t \} \in Q$ are called conjugate, if there are homeomorphisms $h : [0,1] \to [0,1], H_t : M^n \to M^n$ such that $h(b)$ is a bifurcation value for every bifurcation value $b, H_t \varphi_t = \tilde{\varphi}_{h(t)}, H_t, t \in [0,1]$ and $H_t$ continuously depends on $t$ (see figure 2.2).

In [23] also established that the arc $\{ \varphi_t \} \in Q$ is stable if and only if all its arcs are structurally stable diffeomorphisms with the exception of a finite number of bifurcation points $\varphi_{b_i}, i = 1, \ldots, g$ such that $\varphi_{b_i}$:

1) has a unique non-hyperbolic periodic orbit, which is a non-critical saddle-node or flip;

2) has no cycles;

3) the invariant manifolds of all periodic points intersect transversally;

4) the arc $\varphi_t$ unfolds generically through $\varphi_{b_i}$.

Recall the definition of unfolding generically arc $\varphi_t$ through the saddle-node or flip. We give the definition for a fixed non-hyperbolic point, in the case when it has a period $k > 1$, a similar definition is given for the arc $\varphi^{k}_t$.

One says that an arc $\{ \varphi_t \} \in Q$ unfolds generically through a saddle-node bifurcation $\varphi_{b_i}$ (see figure 2), if in some neighborhood of the non-hyperbolic point $(p, b_i)$
the arc $\varphi_{1}$ is conjugate to the arc

$\tilde{\varphi}_{1}(x_1, x_2, \ldots, x_{1+n_u}, x_{2+n_u}, \ldots, x_n) = \left( x_1 + 2x_1^2 + \tilde{t}, \pm 2x_2, \ldots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \ldots, \frac{\pm x_n}{2} \right),$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

In the local coordinates $(x_1, \ldots, x_n, \tilde{t})$ the bifurcation occurs at time $\tilde{t} = 0$ and the origin $O \in \mathbb{R}^n$ is a saddle-node point. The axis $Ox_1$ is called a central manifold, the half-space \{
$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, x_{2+n_u} = \cdots = x_n = 0$\} is the unstable manifold, half-space \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0, x_2 = \cdots = x_{1+n_u} = 0\} is the stable manifold of the point $O$.

If $p$ is a saddle-nodal point of the diffeomorphism $\varphi_{b_i}$, then there exists a unique $\varphi_{b_i}$-invariant foliation $F_{p}^{ss}$ with smooth leaves such that $W_{p}^{s}$ is a leaf of this foliation [17]. $F_{p}^{ss}$ is called a strongly stable foliation (see figure 3). A similar strongly unstable foliation is denoted by $F_{p}^{uu}$. A point $p$ is called s-critical, if there exists some hyperbolic periodic point $q$ such that $W_{q}^{u}$ non-transversally intersect some leaf of the foliation $F_{p}^{ss}$; u-criticality is defined similarly. Point $p$ is called
- semi-critical if it is either s- or u-critical;
- bi-critical if it is s- and u-critical;
- non-critical if it is not semi-critical\(^1\).

**Remark 1.** For surface diffeomorphisms, the non-criticality of the saddle-node point $p$ means the absence of intersection of the central manifold of the point $p$ with the separatrices of saddle points.

\(^1\)For the first time, the effect of arc instability in a neighborhood of a non-critical saddle-node was discovered in 1974 by V. Afraimovich and L. Shilnikov [1], [2]. The existence of invariant foliations $F_{p}^{ss}$, $F_{p}^{uu}$ was also proved earlier in the works of V. Lukyanov and L. Shilnikov [19].
One says that an arc \( \{ \varphi_t \} \in Q \) unfolds generically through a flip bifurcation \( \varphi_{b_i} \) (see figure 4), if in some neighborhood of the non-hyperbolic point \((p, b_i)\) the arc \( \varphi_t \) is conjugate to the arc

\[
\hat{\varphi}_t(x_1, x_2, \ldots, x_{1+n_u}, x_{2+n_u}, \ldots, x_n) = \\
\left(-x_1(1 \pm \hat{t}) + x_1^3, \pm 2x_2, \ldots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \ldots, \frac{\pm x_n}{2}\right),
\]

where \((x_1, \ldots, x_n) \in \mathbb{R}^n, |x_i| < 1/2, |\hat{t}| < 1/10.\n
We say that Morse-Smale diffeomorphisms \( f_0, f_1 : M^n \to M^n \) belong to the same component of stable connectedness, if in the space of diffeomorphisms they can be connected by a stable arc.

A diffeomorphism source-sink on a closed manifold is a gradient-like diffeomorphism having exactly two fixed points: a source and a sink. The ambient manifold for such diffeomorphisms is always a n-sphere.

**Proposition 4.** [27, Theorem 1] Every source-sink diffeomorphisms \( f_0, f_1 : S^2 \to S^2 \) are connected by an arc without bifurcations.

### 2.3. Reduction of confluence objects to a canonical form.

To construct an arc that unfolds generically through a saddle-node or flip bifurcation, it is necessary to reduce the confluence objects to a canonical form. In this section, we give necessary facts that make it possible, without loss of generality, to consider any diffeomorphism by linear in a neighborhood of a hyperbolic point, and the closure of any saddle separatrix of a 2-diffeomorphism lying on a smooth arc.
Theorem 5.8 (Tom’s theorem on the continuation of isotopy)
Let \( Y \) be a smooth manifold without boundary, \( X \) be a smooth compact submanifold of \( Y \) and \( \{ f_t : X \to Y, t \in [0, 1] \} \) is a smooth isotopy such that \( f_0 \) is the inclusion of \( X \) in \( Y \). Then for any compact set \( A \subset Y \), containing the isotopy support \( \text{supp}\{ f_t \} \) there exists a smooth isotopy \( \{ g_t \in \text{Diff}(Y), t \in [0, 1] \} \) such that \( g_0 = \text{id} \), \( g_t|_X = f_t|_X \) for any \( t \in [0, 1] \) and \( \text{supp}\{ g_t \} \) belongs to \( A \).

Proposition 6. [11, Theorem 1 (A “pathwise” Franks’ lemma)]
Let a diffeomorphism \( \varphi_0 : M^n \to M^n \) has an isolated hyperbolic point \( r_0 \) of the period \( m_0 \) and let \( (U_0, h) \) is a local map of the manifold \( M^n \) such that \( r_0 \in U_0, h(r_0) = O \). Then for any hyperbolic automorphism \( Q \), having the same index as the automorphism \( (D\varphi_0^{m_0})_{r_0} \), there exist neighborhoods \( U_1, U_2 \) of the point \( r_0, U_2 \subset U_1 \subset U_0 \), and the arc \( \varphi_t : M^n \to M^n, t \in [0, 1] \) without bifurcations such that:

1) the diffeomorphism \( \varphi_1, t \in [0, 1] \), coincides with the diffeomorphism \( \varphi_0 \) outside the set \( \bigcup_{k=0}^{m-1} \varphi_k(U_1) \);

2) \( O_{r_0} = \bigcup_{k=0}^{m-1} \varphi_k(r_0) \) is an isolated hyperbolic orbit of period \( m_0 \) of the same index as the automorphism \( (D\varphi_0^{m_0})_{r_0} \), for every \( \varphi_t \);

3) \( W^s_{\varphi_0}(\varphi_t) = W^s_{\varphi_0}(\varphi_0) \) and \( W^u_{\varphi_0}(\varphi_t) = W^u_{\varphi_0}(\varphi_0) \) outside the set \( \bigcup_{k=0}^{m-1} \varphi_k(U_1) \);

4) the diffeomorphism \( h\varphi_t^{m_0}h^{-1} \) coincides with the diffeomorphism \( Q \) on the set \( h(U_2) \).

Proposition 7. [21, Lemma 2] Let a diffeomorphism \( \varphi_0 : M^2 \to M^2 \) has an isolated hyperbolic sink \( \omega_0 \) and an isolated hyperbolic saddle \( \sigma_0 \) such that the unstable separatrix \( \gamma_{\omega_0} \) of the saddle \( \sigma_0 \) lies in the sink basin \( W^s_{\omega_0} \) and has the same period \( m \), as the sink \( \omega_0 \). Let \( (V_0, \psi_0) \) be a local chart in \( \omega_0 \) such that the diffeomorphism \( \psi_0\varphi_0^m \psi_0^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linear contraction \( g(x_1, x_2) = (x_1/2, x_2/2) \). Then there exist neighborhoods \( V_1, V_2 \) of the point \( \omega_0, V_2 \subset V_1 \subset V_0 \) and the arc \( \varphi_t : M^2 \to M^2, t \in [0, 1] \) without bifurcations with the following properties:

1) the diffeomorphism \( \varphi_1, t \in [0, 1] \) coincides with the diffeomorphism \( \varphi_0 \) outside the set \( \bigcup_{k=0}^{m-1} \varphi_k(V_1) \) and \( \bigcup_{k=0}^{m-1} \varphi_k(\omega_0) \) is the hyperbolic sink orbit of the period \( m \) for all \( \varphi_t \);

2) the diffeomorphism \( \varphi_1 \) coincides with the diffeomorphism \( \varphi_0 \) on the set \( \bigcup_{k=0}^{m-1} \varphi_k \)

\( (V_2) \) and \( \psi_0(\gamma_{\varphi_1} \cap V_2) \subset OX_1 \), where \( \gamma_{\varphi_1} \) is an unstable separatrix of the saddle \( \sigma_0 \) with respect to the diffeomorphism \( \varphi_1 \).

2.4. Necessary information from the graph theory. Recall some definitions from the graph theory (see, for example, [16]).

Graph \( \Gamma \) is a pair \((V_\Gamma, E_\Gamma)\), where \( V_\Gamma \) is a set of vertices, and \( E_\Gamma \) is a set of pairs of vertices, called edges.

Two vertices are called adjacent, if they are connected by an edge (that is, they form an edge), and the edge in this case is called incidental to each of the vertices. The number of edges incident to a vertex is called the degree of the vertex.

\[ \text{supp}\{ f_t \} \] the closure of the set \{ \( x \in X : f_t(x) \neq f_0(x) \) for some \( t \in [0, 1] \) \}. 

The support
The set of vertices \( \{v_1, (v_1, v_2), v_2, \ldots, v_{k-1}, (v_{k-1}, v_k), v_k\} \) is called path of length \( k \). A path is called a cycle, if \( v_1 = v_k \). A graph without cycles is called acyclic. A graph is called connected, if every two of its vertices are connected by a path. Tree is a connected acyclic graph, that is, any two of its vertices are connected in exactly one way.

Everywhere below \( \Gamma \) is a tree.

Every tree with at least 2 vertices has at least two hanging vertices, that is, vertices of degree 1. Then every such tree \( \Gamma \) is uniquely associated with the sequence \( \Gamma_0, \Gamma_1, \ldots, \Gamma_s \) trees such that \( \Gamma_0 = \Gamma \), \( \Gamma_i \) contain one or two vertices and for any \( i \in \{1, \ldots, s\} \), the tree \( \Gamma_i \), is obtained from \( \Gamma_{i-1} \) y removing all its hanging vertices. All vertices of the tree \( \Gamma_s \) are called central vertices of the tree \( \Gamma \) and if \( \Gamma_s \) has an edge, then it is called central edge of the tree \( \Gamma \).

A tree \( \Gamma \) is called central if it has exactly one central vertex, and bi-central, otherwise.

Vertex rank \( x \in \Gamma \), denoted by \( \text{rank}(x) \), is defined by the formula

\[
\text{rank}(x) = \max\{i : x \in \Gamma_i\}.
\]

It follows from the definition that if the vertices \( v, w \) are incident to an off-center edge, then \( |\text{rank}(v) - \text{rank}(w)| = 1 \), and the central vertices of the bi-central tree have the same rank.

Automorphism \( P_{\Gamma} \) of the tree \( \Gamma \) is a bijective map of the set \( \Gamma \) onto itself, preserving the adjacency, i.e.

\[
(u, v) \in E_{\Gamma} \Leftrightarrow (P_{\Gamma}(u), P_{\Gamma}(v)) \in E_{\Gamma}.
\]

The automorphism \( P_{\Gamma} \) can be represented as a superposition of cyclic permutations. Then the set \( \Gamma \) can be decomposed into \( P_{\Gamma} \)-orbits – subsets invariant under the permutations. It is clear that every \( P_{\Gamma} \)-orbit consists of vertices of the same rank and if the tree is central (bi-central), then its central vertex (central edge) remains fixed for any automorphism.

The automorphism \( P_{\Gamma} \) naturally induces a map of the set of edges \( E_{\Gamma} \), which we will also denote by \( P_{\Gamma} \).

**Proposition 8.** [12, Corollary 2.2] Let \( (v, w) \in E_{\Gamma} \) be an off-center edge and \( \text{rank}(v) < \text{rank}(w) \). Then the period of the edge \( (v, w) \) is equal to the period of the vertex \( v \).

3. **Dynamics of gradient-like surface diffeomorphisms.**

3.1. **Dynamics on an arbitrary surface.** Consider an orientation-preserving gradient-like diffeomorphism \( f \), defined on a smooth orientable closed surface \( M^2 \). In this section, we describe the general dynamic properties of such diffeomorphisms.

Denote by \( \Omega^s_f, \Omega^s_1, \Omega^s_r \) the set of sinks, saddles and sources of \( f \). For any (possibly empty) \( f \)-invariant set \( \Sigma \subset \Omega^s_1 \) we set

\[
A_{\Sigma} = \Omega^s_1 \cup W^u_{\Sigma}, \ R_{\Sigma} = \Omega^s_r \cup W^s_{\Omega^s_1 \setminus \Sigma}.
\]

It follows from [13] that \( A_{\Sigma}, R_{\Sigma} \) are dual attractor and repeller. The set

\[
V_{\Sigma} = M^2 \setminus (A_{\Sigma} \cup R_{\Sigma})
\]
Figure 5. Illustration to the proof of lemma 3.1

is called characteristic space. We denote by \( \hat{V} \Sigma \) the orbit space of the action of the diffeomorphism \( f \) on \( V \Sigma \) and by \( p_\Sigma : V \Sigma \to \hat{V} \Sigma \) the natural projection. According to [15], each connected component of the manifold \( \hat{V} \Sigma \) is diffeomorphic to a two-dimensional torus.

**Lemma 3.1.** For every orientation-preserving gradient-like diffeomorphism \( f : M^2 \to M^2 \) there exists a set \( \Sigma \), such that the orbit space \( \hat{V} \Sigma \) is connected.

**Proof.** Let \( \Sigma_0 = \emptyset \) and consider the corresponding dual attractor and repeller \( A_{\Sigma_0} = \Omega_1^0 \) and \( R_{\Sigma_0} = \Omega_2^0 \cup W_{\Omega_1^0} \). If the orbit space of \( \hat{V} \Sigma_0 \) is connected, then \( \Sigma = \Sigma_0 \).

Otherwise, denote by \( \hat{V}_1, \ldots, \hat{V}_l \) the connected components of the space \( \hat{V} \Sigma_0 \). For any saddle point \( \sigma \in \Omega_1^1 \) we set \( \hat{L}_\sigma^u = p_{\Sigma_0}(W^u_{\sigma} \setminus \sigma) \). Due to [14], the set \( \hat{L}_\sigma^u \) consists of two closed curves if \( \nu_\sigma = +1 \) and one closed curve if \( \nu_\sigma = -1 \).

Consider the dual attractor and repeller for the set \( \Sigma_i \), that is, \( A_{\Sigma_i} = \Omega_1^0 \cup W_{\Omega_1^0} \) and \( R_{\Sigma_i} = \Omega_2^0 \). In this case, the repeller has dimension zero and, by [13], the attractor \( A_{\Sigma_i} \) is connected. Then, up to the renumbering of the components \( \hat{V}_1, \ldots, \hat{V}_l \), there is a sequence of saddle points \( \sigma_1, \ldots, \sigma_{l-1} \) such that the set \( \hat{L}_{\sigma_i}^u \) consists of two closed curves \( \hat{L}_{\sigma_i}^{1u} \subset \hat{V}_i, \hat{L}_{\sigma_i}^{2u} \subset \hat{V}_{i+1} \) (see figure 3.1). Let \( O_{\sigma_i} \) denote the orbit of the point \( \sigma_i \). Let

\[
\Sigma_i = \Sigma_0 \cup \bigcup_{j=1}^{i} W^u_{O_{\sigma_j}}, \quad i \in \{1, \ldots, l - 1\}.
\]

According to [15], the orbit space of \( \hat{V} \Sigma_i \) consists of \( l - i \) connected components. Thus, the space \( \hat{V} \Sigma_{i-1} \) is connected, and \( \Sigma = \Sigma_{l-1} \) is the desired set.

For any gradient-like diffeomorphism \( f : M^2 \to M^2 \) and a set \( \Sigma \), satisfying the conditions of the lemma 3.1, let

\[
A_f = A_{\Sigma}, \quad R_f = R_{\Sigma}, \quad V_f = V_{\Sigma}.
\]
Figure 6 shows a phase portrait of a gradient-like diffeomorphism $f$ with a pair $A_f, R_f$.

![Figure 6](image_url)

**Figure 6.** An example of an attractor $A_f$ and repeller $R_f$ for a gradient-like diffeomorphism $f$

Note that the set $\Sigma$, satisfying the conditions of Lemma 3.1, is not unique. Thus in Figure 7 depicted diffeomorphism $f \in G$ with two choices of the pair $A_f, R_f$.

![Figure 7](image_url)

**Figure 7.** For the shown diffeomorphism, there are two ways to choose the pair $A_f, R_f$: 1) $A_f = \partial W^u_\sigma$, $R_f = \alpha$ and 2) $A_f = \omega \cup f(\omega)$, $R_f = \partial W^s_\sigma$

For any chosen pair $A_f, R_f$ let

$$\hat{V}_f = \hat{V}_\Sigma, \hat{p}_f = \hat{p}_\Sigma.$$ 

Then the set $\hat{V}_f$ is connected and homeomorphic to the torus, while the set $V_f$ is not connected in general. Denote by $m_f$ the number of connected components of the set $V_f$. Note that the number $m_f$ depends on the choice of the pair $A_f, R_f$. For example, for a diffeomorphism in the figure 7 in case 1) $m_f = 1$, and in case 2) $m_f = 2$. 
In any case the set $V_f$ is diffeomorphic to $(\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$ and the restriction of the diffeomorphism $f$ to $V_f$ is topologically conjugate by a homeomorphism $h_f : V_f \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$ to periodic contraction $a_{m_f} : (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f} \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$, given by the formula

$$a_{m_f}(x, y, i) = \begin{cases} (x, y, i + 1), & i = 0, \ldots, m_f - 2, \\ (x/2, y/2, 0), & i = m_f - 1. \end{cases}$$

Let $c_i = h_f^{-1} (S^1 \times \{i\}), i = 0, \ldots, m_f - 1, c = h_f^{-1} (S^1 \times \mathbb{Z}_{m_f})$ and $\hat{c} = p_f (c)$ (see figure 6, where the curve $c$ is shown). Curve $\hat{c}$ is called equator, it is a simple closed curve on the torus $\hat{V}_f$ and its homotopy type is uniquely defined by a diffeomorphism $f$, that is, does not depend on the choice of the conjugating homeomorphism $h_f$.

### 3.2. Dynamics on the two-dimensional sphere.

Let us recall that $G$ is the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere $S^2$. Consider $f \in G$. In this case, the attractor and repeller $A_f, R_f$ can be described in more detail. To do this, note that $\bigcup_{i=0}^{m_f-1} f^i(c)$ divides the sphere $S^2$ into two disjoint parts $U$ and $V$ such that

$$f(U) \subset U, A_f = \bigcap_{j \in \mathbb{N}} f^j(U), \quad f^{-1}(V) \subset V, R_f = \bigcap_{j \in \mathbb{N}} f^{-j}(V).$$

**Figure 8.** Illustration to the lemma 3.2

**Lemma 3.2.** For any diffeomorphism $f \in G$ (up to a consideration of the diffeomorphism $f^{-1}$) the following is true (see figure 8):

1) the set $U$ consists of $m_f$ pairwise disjoint disks $D_f, f(D_f), \ldots, f^{m_f-1}(D_f)$ such that $f^{m_f}(\text{cl} D_f) \subset \text{int} D_f$;

2) the attractor $A_f$ consists of $m_f$ connected components $A, f(A), \ldots, f^{m_f-1}(A)$ such that $A = \bigcap_{j \in \mathbb{N}} f^{m_f}(D_f)$ and $f^{m_f}(A) = A$;

3) repeller $R_f$ is connected.
Proof. It follows from the definition of the equator \( \hat{c} \) that the set \( c \) consists of \( m_f \) simple closed curves \( c_0, \ldots, c_{m_f-1} \) on the sphere \( S^2 \) such that \( c_{i+1} = f(c_i), i = 0, \ldots, m_f - 2 \). Since there are a finite number of such curves, among them there exists a curve \( f^k(c), k \in \{0, \ldots, m_f-1\} \), bounding the disk \( D_f : \text{int } D_f \cap \bigcup_{i=0}^{m_f-1} f^i(c) = \emptyset \). For definiteness, we assume that the disk \( D_f \) is a connected component of the set \( U \) (otherwise, this holds for the diffeomorphism \( f^{-1} \)).

Since the restriction of the diffeomorphism \( f^{m_f} \) to \( D_f \cap V_f \) is associated with linear contraction, \( f^{m_f}(\text{cl } D_f) \subset \text{int } D_f \). Thus, the set \( A = \bigcap_{j \in \mathbb{N}} f^{jm_f}(D_f) \) is connected. Since \( A_f = \bigcap_{j \in \mathbb{N}} f^{jm_f}(U) \), \( A \) is a connected component of the attractor \( A_f \) and \( D_f = (D_f \cap V_f) \cup A \). Further, we consider separately two cases: (1) \( m_f = 1 \), (2) \( m_f > 1 \).

(1) If \( m_f = 1 \), then \( A_f = A, R_f = \bigcap_{j \in \mathbb{N}} f^{-j}(S^2 \setminus D_f) \) are connected attractor and repeller and the lemma is proved.

(2) If \( m_f > 1 \), then \( f(c) \cap (D_f \cap V_f) = \emptyset \) due to conjugation to periodic contraction, also \( f(c) \cap A = \emptyset \), since \( f(c) \subset V_f \). So \( f(D_f) \cap D_f = \emptyset \) since \( f(c) \cap D_f = \emptyset \). Therefore, the disk \( f(D_f) \) contains the connected component \( f(A) \) of the attractor \( A_f \), which does not intersect with \( A \). Reasoning in the same way, we get \( m_f \) of pairwise disjoint connected components \( A, f(A), \ldots, f^{m_f-1}(A) \) of the attractor \( A_f \), this means that the attractor \( A_f \) consists of one orbits of the period \( m_f \). Thus, the set \( U \) is the union of pairwise disjoint disks \( D_f, f(D_f), \ldots, f^{m_f-1}(D_f) \). This implies that the set \( V = S^2 \setminus U \) is connected, that implies the connectedness of the repeller \( R_f \).

4. Proof of the theorem 1.1. In this section, we give a scheme of the proof of theorem 1.1 with links to statements that will be proven in the following sections.

Recall that by \( G \) we denote the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere \( S^2 \) and in the section 5 we constructed a family of model diffeomorphisms \( \phi_{k,m}, \phi_0 \in G \). We denote by \( C_{k,m} \), the component of stable connectedness of the diffeomorphism \( \phi_{k,m} \) and we denote by \( C^-_{k,m} \) the component of stable connectedness of the diffeomorphism \( \phi_{k,m}^{-1} \). We denote by \( C_0 \) the component of the stable isotopic connection of the source-drain diffeomorphism \( \phi_0 \).

We show that any diffeomorphism \( f \in G \) belongs to one of the components \( C_0, C_{k,m}, C^-_{k,m}, \forall k, m \in \mathbb{N}, k < m/2, (k, m) = 1 \).

Proof. Let \( f \in G \). By lemma 3.2, diffeomorphism \( f \) (with respect to \( f^{-1} \)) has a (not unique) dual pair \( A_f, R_f \), in which the repeller \( R_f \) is connected and the attractor consists of the \( m_f \) connected components of the period \( m_f \). Let us show that \( f \) is connected by a stable arc either with the diffeomorphism \( \phi_0 \), or with the diffeomorphism \( \phi_{k,m}, (k, m) = 1 \).

It follows from the results of sections 6, 7 that if the non-wandering set of \( f \) contains a fixed sink or a saddle of negative orientation type, then there exists a fixed pair \( A_f, R_f \) (\( m_f = 1 \)). Otherwise, \( m_f = \mu_f \) for any pairs \( A_f, R_f \), where \( \mu_f \) is a period of non-wandering points of the diffeomorphism \( f \) that is different from 1.

Denote by \( G_1 \) the subset of \( G \) consisting of diffeomorphisms of \( f \) for which there exists a fixed pair \( A_f, R_f \) (\( m_f = 1 \)) and by \( G_{m}, m > 1 \) the subset of \( G \), consisting of \( f \) diffeomorphisms for which \( m_f = m \) for any pair \( A_f, R_f \).
By the lemmas 6.3, 6.4, any diffeomorphism \( f \in G_1 \) connected by a stable arc with the diffeomorphism \( \phi_0 \). By the lemmas 8.1, 8.2, 8.3, 8.5, any diffeomorphism \( f \in G_m \) is connected by a stable arc with some diffeomorphism \( \phi_{k,m} \).

Finitely, in section 9 we give a complete classification of the model diffeomorphisms \( \phi_{k,m} \) with respect to the stable connectedness. \( \square \)

5. Model diffeomorphisms. In this section, we give an exact description of the model diffeomorphisms \( \phi_{k,m}, \phi_0 : S^2 \to S^2 \).

For \( m \in \mathbb{N} \) we define a vector field on the plane \( \mathbb{R}^2 \) using the following system of differential equations in polar coordinates \((r, \varphi)\)

\[
\begin{aligned}
\dot{r} &= -r(r - 1), \\
\dot{\varphi} &= -\varphi \left( \varphi - \frac{\pi}{m} \right) \ldots \left( \varphi - \frac{(2m-1)\pi}{m} \right).
\end{aligned}
\]

Denote by \( \chi^t_m \) the flow induced by this vector field and denote by \( \chi_m \) its one-time shift. The resulting diffeomorphism has a hyperbolic source at the origin \( O \), hyperbolic saddles at \( A_{2i} \left( 1, \frac{2\pi i}{m} \right) \) and hyperbolic sinks at \( A_{2i+1} \left( 1, \frac{2\pi i+1}{m} \right), i \in \{0, \ldots, m-1\} \) (see figure 9).

For any integer \( k \geq 0 \) such that \( k < m \), \( (k, m) = 1 \) we define a diffeomorphism \( \theta_{k,m} : \mathbb{R}^2 \to \mathbb{R}^2 \) as follows \( \theta_{k,m}(r, \varphi) = (r, \varphi + \frac{2\pi k}{m}) \). We define the diffeomorphism \( \bar{\phi}_{k,m} : \mathbb{R}^2 \to \mathbb{R}^2 \) by the formula

\[
\bar{\phi}_{k,m} = \theta_{k,m} \circ \chi.
\]

By construction, the non-wandering set of the diffeomorphism \( \bar{\phi}_{k,m} \) coincides with the non-wandering set of the diffeomorphism \( \chi_m \), and all sink points form a unique orbit of the period \( m \) and all saddle points form a unique orbit of the period \( m \) of the diffeomorphism \( \phi_{k,m} \).
Consider the standard two-dimensional sphere
\[ S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} . \]
Denote by \( S(0, 0, -1) \) south pole and define a stereographic projection \( \vartheta : S^2 \setminus \{S\} \rightarrow \mathbb{R}^2 \) by the formula
\[ \vartheta(x_1, x_2, x_3) = \left( \frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3} \right) . \]
Then the inverse map \( \vartheta^{-1} : \mathbb{R}^2 \rightarrow S^2 \setminus \{S\} \) is given by the formula
\[ \vartheta^{-1}(x_1, x_2) = \left( \frac{2x_1}{x_1^2 + x_2^2 + 1}, \frac{2x_2}{x_1^2 + x_2^2 + 1}, \frac{1 - (x_1^2 + x_2^2)}{x_1^2 + x_2^2 + 1} \right) . \]
Define a diffeomorphism \( \phi_{k,m} : S^2 \rightarrow S^2 \) by the formula
\[ \phi_{k,m}(s) = \begin{cases} \vartheta^{-1} \circ \tilde{\phi}_{k,m} \circ \vartheta(s), & s \in S^2 \setminus \{S\}, \\ S, & s = S. \end{cases} \]
By construction, the diffeomorphism \( \phi_{k,m} \) is a gradient-like diffeomorphism of a 2-sphere with the following non-wandering set (see figure 10):

- fixed source points:
  - at the north pole \( \alpha_1 = \vartheta^{-1}(O) \), at the south pole \( \alpha_2 = S \);
- saddle and sink orbits of the period \( m \) at the equator:
  - saddle orbit \( \mathcal{O}_s = \{\vartheta^{-1}(A_0), \vartheta^{-1}(A_2), \ldots, \vartheta^{-1}(A_{2m-2})\} \),
  - sink orbit \( \mathcal{O}_S = \{\vartheta^{-1}(A_1), \vartheta^{-1}(A_3), \ldots, \vartheta^{-1}(A_{2m-1})\} \).

Let us define \( \chi_0 \) as one-time shift of the flow \( \chi^t_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by the vector field \( \dot{r} = -r \). Define a diffeomorphism \( \phi_0 : S^2 \rightarrow S^2 \) by the formula
\[ \phi_0(s) = \begin{cases} \vartheta^{-1} \circ \chi_0 \circ \vartheta(s), & s \in S^2 \setminus \{S\}, \\ S, & s = S. \end{cases} \]
By construction, the diffeomorphism \( \phi_0 \) is a source-sink diffeomorphism with the source \( \alpha = \vartheta^{-1}(O) \) and the sink \( \omega = S \).
6. Diffeomorphisms of class $G_1$. Recall that by $G_1$ we denote the subset $G$, consisting of diffeomorphisms $f$, for which there exists a fixed pair $A_f, R_f$ ($m_f = 1$).

6.1. Attractor structure. Let $f \in G_1$. We associate the attractor $A_f$ with the graph $\Gamma_f$ so that its vertices $V_f$ are in one-to-one correspondence with periodic points, and the edges $E_f$ — with saddle separatrices of the diffeomorphism $f$, belonging to the attractor $A_f$. Moreover, the diffeomorphism $f$ naturally induces the automorphism $P_f$ of the graph $\Gamma_f$.

Lemma 6.1. If $f \in G_1$, then the graph $\Gamma_f$ is a tree.

Proof. We show that if the attractor $A_f$ is different from the sink, then it does not contain cycles.

Suppose the opposite: $A_f$ contains a cycle formed by the closures of the unstable manifolds of saddle points $\sigma_1, \ldots, \sigma_r$. Then the closed curve $\bigcup_{i=1}^r clW^u_{\sigma_i}$ bounds a disk $d \subset D_f$. It implies that one of the stable separatrices of every saddle $\sigma_i$ lies in the disk $d$. Consequently, the closure of this separatrix lies in the disk $d$. Thus, $R_f \cap D_f \neq \emptyset$, which contradicts lemma 3.2. □

The following lemma follows directly from the proposition 8.

Lemma 6.2. If $f \in G_1$ and the attractor $A_f$ of the diffeomorphism $f$ is different from the sink, then exactly one of the following statements is true:

1) $A_f = clW^u_{\sigma}$, where $\sigma$ is a saddle point with a negative orientation type;

2) there is a saddle point $\sigma \in A_f$ with a positive orientation type and a sink point $\omega \in A_f$ such that $m_\sigma = m_\omega$, $\omega \in clW^s_\sigma$ and the intersection $W^s_\sigma \cap A_f$ consists of exactly one unstable separatrix of the saddle $\sigma$ and the sink $\omega$.

6.2. Trivialization of the attractor for $f \in G_1$. Denote by $H_1$ a subset of $G_1$, consisting of diffeomorphisms for which the attractor $A_f$ consists of one sink orbit.

Lemma 6.3. Any diffeomorphism $f \in G_1$ is connected by a stable arc with some diffeomorphism $g \in H_1$, coinciding with $f$ on $S^2 \setminus D_f$.

Proof. We divide the set $G_1$ into subsets $G_1 = G_{1,1} \cup G_{1,2} \cup \cdots \cup G_{1,\lambda} \cup \ldots$, where $\lambda \in \mathbb{N}$ is the number of sink orbits in the attractor $A_{f_\lambda}$ for a diffeomorphism $f_\lambda \in G_{1,\lambda}$. Note that $G_{1,1} = H_1$, then to prove the statement it is enough to construct a stable arc connecting a diffeomorphism $f_\lambda \in G_{1,\lambda}$, $\lambda > 1$ with a diffeomorphism $f_{\lambda-1} \in G_{1,\lambda-1}$.

Let $f = f_\lambda$. By lemma 6.2 there exist points $\sigma, \omega \in A_f$ such that $q_\omega = 0, q_\sigma = 1, \omega \in clW^u_\sigma$ and the intersection $W^u_\sigma \cap A_f$ consists of exactly one unstable separatrix $\gamma$ of the saddle $\sigma$ and the sink $\omega$, while the period of $\omega$, we denote it by $m$. By lemma 7, without loss of generality, we can assume that there exists a local map $(U, \psi)$ of $S^2$ such that $\omega \in U$, $\psi(\omega) = O, f^m(U) \subset U \subset D_f$ and $\psi(\gamma \cup U) \subset OX_1$. According to lemma 6.2 for the diffeomorphism $f$ two cases are possible: 1) $\nu_\sigma = -1$; 2) $\nu_\sigma = 1$. We construct the desired arc separately for each case.

1) In this case $A_f = W^u_\sigma \cup \omega \cup f(\omega)$ and $m = 2$. Let $l = W^u_\sigma \cup \psi^{-1}(OX_1) \cup f(\psi^{-1}(OX_1))$. Then $l$ is a smooth curve containing $A_f$, the points $\omega, f(\omega)$ are internal and $f(l) \subset l$. Let $\Pi_1 = \{ (x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2} \}$. Define the diffeomorphism $\hat{\phi} : \Pi_1 \rightarrow \mathbb{R}^2$ by the formula

$$\hat{\phi}(x_1, x_2) = \left( -\frac{11}{10} x_1 + x_3, -\frac{x_2}{2} \right).$$
By construction $\tilde{\varphi}(\Pi_1) \subset \text{int} \; \Pi_1$, the diffeomorphism $\tilde{\varphi}$ has a saddle point $O$ and a sink periodic orbit $\{P_0, \tilde{\varphi}(P_0)\}$, where $P_0(-x_0,0), \tilde{\varphi}(P_0) = (x_0,0), x_0 \in (0,1/2)$. Let $\Pi_2 = \varphi(\Pi_1)$. We choose a closed neighborhood $V$ of the attractor $A_f$ and a diffeomorphism $\beta : V \to \Pi_1$ so that $f(V) \subset \text{int} \; V$, $\beta(V \cap \Pi_1) = O \times 1 \cap \Pi_1$, $\beta(f(V)) = \Pi_2$, $\beta(0) = P_0$ and $\beta(f(0)) = \tilde{\varphi}(P_0)$ (see figure 11). Let $\tilde{f} = \beta \beta^{-1} : \Pi_1 \to \Pi_2$. Then on the set $\Pi_2$ the family of maps $\chi_t : \Pi_2 \to \mathbb{R}^2$ is correctly defined by the formula

$$\chi_t = (1-t)\tilde{f} + t\tilde{\varphi}.$$

By construction $\chi_t(\Pi_2) \subset \text{int} \; \Pi_2$ for all $t \in [0,1]$. Note that the origin is a fixed saddle point for the diffeomorphism $\chi_t$ and the points $P_0, \tilde{\varphi}(P_0)$ form a sink orbit. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t\vert_{\Pi_2}$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$ and $\xi_t(\Pi_2) \subset \text{int} \; \Pi_1$. By proposition 5, there exists an isotopy $\Xi_t : \Pi_1 \to \Pi_1$, coinciding with $\xi_t$ on $\Pi_2$ and identical on $\partial \Pi_1$. Let

$$f_t = \tilde{f} \Xi_t.$$

Note that $f_t = \tilde{\varphi}$ on $\Pi_2$. Let $\Pi_3 = \tilde{\varphi}(\Pi_2)$. Define the arc $\eta_t : \Pi_3 \to \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(-x_1 \left(1 + \frac{1}{10}(1-2t)\right) + x_2^3, -\frac{x_2}{2}\right).$$

By construction, $\eta_t(\Pi_3) \subset \text{int} \; \Pi_3$ for all $t \in [0,1]$, in addition, the isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ connects the identity map with the diffeomorphism $\tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_3) \subset \text{int} \; \Pi_2$. By the proposition 5, there exists an isotopy $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$, which coincides with $\zeta_t$ on $\Pi_3$ and is identical outside $\Pi_2$. Let

$$\Theta_t = \tilde{\varphi}\theta_t.$$

Then the desired arc is the product of the arcs $f_t, \Theta_t : S^2 \to S^2$, where $f_t$ coincides with $\tilde{f}$ outside $V$, $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in V$ and $\Theta_t$ coincides with $f_1$ outside $f_1(V)$, $\Theta_t(z) = \beta^{-1}(\Theta_1(\beta(z)))$ for $z \in f_1(V)$.

2) In this case the saddle $\sigma$ and the sink $\omega$ have the same period $m$. Let $l = W_\sigma^u \cup \tilde{\varphi}^{-1}(OX)$. Then $l$ is a smooth curve containing $\gamma$ and for which the points $\omega, \sigma$ are internal. Let $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < \frac{\gamma}{m}\}$, $\Pi_2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \frac{\gamma}{m}\}$,
\[ \tilde{U}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \frac{2}{3}\}. \]

Define the diffeomorphism \( \tilde{\varphi} : \tilde{U}_1 \to \mathbb{R}^2 \) by the formula
\[
\tilde{\varphi}(x_1, x_2) = \left( x_1 + 2x_1^2 - \frac{1}{10}x_2, \frac{x_2}{2} \right).
\]

By construction, the diffeomorphism \( \tilde{\varphi} \) has a saddle point \( P_1(0, x_1), x_1 \in (0, 1/2) \) and a sink point \( P_2(-x_2, 0), x_2 \in (0, 1/2) \). Let \( \Pi_2 = \tilde{\varphi}(\Pi_1) \).

We choose a closed neighborhood \( V \) of the arc \( \gamma \), an open neighborhood of \( U_1 \supset V \) of the arc \( \gamma \) and a diffeomorphism \( \tilde{\beta} : U \rightarrow \tilde{U}_1 \) so that \( \tilde{\beta}(\sigma) = P_1, \tilde{\beta}(\omega) = P_2, \tilde{\beta}(U_1) = O_{1} \cap \tilde{U}_1, \tilde{\beta}(V) = \Pi_1, \tilde{\beta}(f^m(V)) = \Pi_2 \) (see figure 12). Let \( \tilde{f} = \tilde{\beta}f^m\tilde{\beta}^{-1} : U \rightarrow \tilde{\varphi}(\tilde{U}_1) \). Then on the set \( \Pi_1 \) the family of maps \( \chi_t : \Pi_1 \rightarrow \Pi_2 \) is correctly defined by the formula
\[
\chi_t = (1 - t)\tilde{f} + t\tilde{\varphi}.
\]

![Figure 12. Illustration to the lemma 6.3, case 2)](image)

Note that the point \( P_1 \) is a fixed saddle point and the point \( P_2 \) is a fixed sink point for the diffeomorphism \( \chi_t \). In addition, the isotopy \( \xi_t = \tilde{\beta}^{-1}\chi_t|_{\Pi_1} : \Pi_1 \rightarrow \Pi_1 \) connects the identity map with the diffeomorphism \( \tilde{f}^{-1}\tilde{\varphi} \). By proposition 5, there exists an isotopy \( \Xi_t : \tilde{U}_1 \rightarrow \tilde{U}_1 \), coinciding with \( \xi_t \) on \( \Pi_1 \) and identical on \( \partial \tilde{U}_1 \). Let \( \tilde{f}_t = \tilde{f}\Xi_t \).

Note that \( \tilde{f}_1 = \tilde{\varphi} \) on \( \Pi_1 \). Define the arc \( \eta_t : \Pi_1 \rightarrow \mathbb{R}^2 \) by the formula
\[
\eta_t(x_1, x_2) = \left( x_1 + 2x_1^2 + \frac{1}{10}(2t - 1)x_2, \frac{x_2}{2} \right).
\]

By construction \( \eta_t(\Pi_1) \subset \Pi_2 \) for all \( t \in [0, 1] \), in addition, the isotopy \( \zeta_t = \tilde{\varphi}^{-1}\eta_t \) connects the identity map with the diffeomorphism \( \tilde{\varphi}^{-1}\eta_t \) and \( \zeta_t(\Pi_2) \subset \Pi_1 \). By proposition 5, there exists an isotopy \( \theta_t : \tilde{U}_2 \rightarrow \tilde{U}_2 \), coinciding with \( \zeta_t \) on \( \Pi_1 \) and identical on \( \partial \tilde{U}_2 \). Let \( \tilde{\Theta}_t = \tilde{\varphi}\theta_t \).

Let \( U_2 = \beta^{-1}(\tilde{U}_2) \). Then the desired arc is the product of the arcs \( f_t, \Theta_t : S^2 \rightarrow S^2 \), where \( f_t \) coincides with \( f \) outside \( \bigcup_{k=0}^{m-1} f^k(U_1) \), \( f_t(z) = f(z) \) for \( z \in f^k(U_1), k \in \{0, \ldots, m - 2\} \) and \( f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(f^{1-m}(z)))) \) for \( z \in f^{m-1}(U_1); \Theta_t \) coincides
with \( f_1 \) outside \( \bigcup_{k=0}^{m-1} f^k(U_2) \), \( \Theta_t(z) = f_1(z) \) for \( z \in f_1^t(U_2) \), \( k \in \{0, \ldots, m-2\} \) and \( \Theta_t(z) = \beta^{-1}(\tilde{\Theta}_t(\beta(f_1^{-m}(z)))) \) for \( z \in f^{m-1}(U_2) \).

### 6.3. Trivialization of the repeller for \( f \in H_1 \)

**Lemma 6.4.** Any diffeomorphism \( f \in H_1 \) is connected by a stable arc with diffeomorphism \( \phi_0 \).

**Proof.** Let \( f \in H_1 \). Then the diffeomorphism \( f^{-1} \) belongs to the class \( G_1 \) and has a connected attractor \( A_{f^{-1}} = R_f \) in the disk \( D_{f^{-1}} = S^2 \setminus \text{int} D_f \).

By lemma 6.3 there exists a stable arc \( \Gamma_{f^{-1}, h, t} \) connecting the diffeomorphism \( f^{-1} \) with some diffeomorphism \( h \in H_1 \) and such that \( \Gamma_{f^{-1}, h, t} = f^{-1} \) on \( D_f \). By construction, the diffeomorphism \( h \) is a source-sink diffeomorphism, as well as \( h^{-1} \). Thus, the arc \( \Gamma_{f^{-1}, h, t} \) connects the diffeomorphism \( f^{-1} \) with the source-sink diffeomorphism.

Then the arc \( \Gamma_{f^{-1}, h, t} \) connects the diffeomorphism \( f \) with a source-sink diffeomorphism. By the proposition 4 it can be connected by an arc without bifurcations with the diffeomorphism \( \phi_0 \).

### 7. Properties of the number \( m_f \)

Denote by \( G^+ \) the subset of \( G \), consisting of diffeomorphisms all of whose saddle points have a positive orientation type. Let \( G^- = G \setminus G^+ \).

#### 7.1. Diffeomorphisms \( f \in G^- \)

**Lemma 7.1.** \( G^- \subset G_1 \).

**Proof.** Let \( f \in G^- \). Choose a pair \( A_f, R_f \), satisfying lemma 3.1. By proposition 3, \( \mu_f = 2 \) and, therefore, \( m_f \leq 2 \).

If \( m_f = 1 \), then the lemma is proved. Consider the case \( m_f = 2 \). Let \( \sigma \) be a saddle with a negative orientation type. By the lemma 3.2, all periodic points belonging to the attractor \( A_f \) have a period at least two. According to proposition 3, \( \sigma \) is a fixed point, therefore \( \sigma \) does not belong to the attractor \( A_f \). Adding \( W^u_\sigma \) to \( A_f \), we get a new attractor \( \hat{A}_f \) and a dual repeller \( \hat{R}_f \) to it. By construction, \( \hat{A}_f \) is connected and lies in the disk, just like the \( \hat{R}_f \) repeller. Thus \( \hat{m}_f = 1 \) for the pair \( \hat{A}_f, \hat{R}_f \).

#### 7.2. Diffeomorphisms \( f \in G^+ \)

Recall that by proposition 3, for any diffeomorphism \( f \in G^+ \) there exists a natural number \( \mu_f \) such that all periodic (non-fixed) points of the diffeomorphism \( f \) have period \( \mu_f \).

**Lemma 7.2.** For any diffeomorphism \( f \in G^+ \) the number \( m_f \) is uniquely determined, that is, it does not depend on the choice of the pair \( A_f, R_f \). Moreover, \( m_f = 1 \), if the diffeomorphism \( f \) has at least one fixed sink and \( m_f = \mu_f \) otherwise.

**Proof.** It follows from lemma 3.2 that all periodic points of the attractor \( A_f \) of the diffeomorphism \( f \in G^+ \) have a period at least \( m_f \) and there is at least one sink point of the period \( m_f \). If \( m_f > 1 \), then, by proposition 3, all periodic points of the attractor \( A_f \), and therefore all sinks of \( f \) have period \( \mu_f = m_f \). Thus, the number \( m_f \) is uniquely determined, that is, it does not depend on the choice of the pair \( A_f, R_f \). Moreover, \( m_f = 1 \), if \( f \) has at least one fixed sink, and \( m_f = \mu_f \) otherwise.
Remark 2. If $m_f = 1$ for some diffeomorphism $f \in G^+$, then $\mu_f$ may be different from 1 (see the picture 13).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13}
\caption{A diffeomorphism $f \in G^+$, for which $m_f = 1$, $\mu_f = 3$}
\end{figure}

Thus, the set $G^+ \setminus G_1$ is represented as pairwise disjoint subsets of

$$G^+ \setminus G_1 = G_2 \cup \ldots \cup G_m \cup \ldots$$

such that $m_f = \mu_f = m$ for any diffeomorphism $f \in G_m$, $m > 1$.

8. Diffeomorphisms of the class $G_m, m > 1$.

8.1. Trivialization of the attractor for $f \in G_m$. For $m > 1$ denote by $H_m$ the subset of $G_m$, consisting of diffeomorphisms for which the attractor $A_f$ consists of one sink orbit $\mathcal{O}_\omega$ (of the period $m$ by lemma 3.2).

**Lemma 8.1.** Any diffeomorphism $f \in G_m$ is connected by a stable arc with some diffeomorphism $g \in H_m$, coinciding with $f$ on $S^2 \setminus (D_f \cup \ldots \cup f^{m-1}(D_f))$ (see figure 14).

**Proof.** By the lemma 3.2, the attractor $A_f$ of $f \in G_m$, $m > 1$ lies on the disjoint union of disks $D_f, \ldots, f^{m-1}(D_f)$ and the diffeomorphism $f^m|_{D_f}$ is conjugate to linear contraction. Let $g_0$ be a 2-sphere diffeomorphism coinciding with $f^m$ on $D_f$ and having a unique hyperbolic source in $S^2 \setminus D_f$. By construction, $g_0 \in G_1$. By lemma 6.3 there exists a stable arc $g_t : S^2 \rightarrow S^2$, $t \in [0, 1]$, connecting the diffeomorphism $g_0$ with the diffeomorphism $g_1 \in H_1$ and such that $g_t = g_0$ on $S^2 \setminus D_f$. Then the desired arc $f_t$ has the form

$$f_t(x) = \begin{cases} f(x), & x \in f^i(D_f), \ i = 0, \ldots, m-2, \\ g_t(f^{1-m}(x)), & x \in f^{m-1}(D_f), \\ f(x), & x \in S^2 \setminus (D_f \cup f(D_f) \cup \ldots \cup f^{m-1}(D_f)). \end{cases}$$

□
Figure 14. Transition from the diffeomorphism \( f \in G_m \) to the diffeomorphism \( g \in H_m \)

Figure 15. Curve \( C_\sigma \)

**Lemma 8.2.** For any diffeomorphism \( f \in H_m \) there exists a saddle orbit \( O_\sigma \) of period \( m \) such that \( cl W^u_{O_\sigma} \) is a \( f \)-invariant closed curve \( C_\sigma \) (see figure 15).

**Proof.** Let \( f \in H_m \). Then \( A_f \) consists of \( m \) sink points \( \omega, \ldots, f^{m-1}(\omega) \). Since the set \( cl W^u_{\Sigma_j} \) is connected, there exists a saddle point \( \sigma \in \Sigma_j \) such that the connected components of the set \( W^u_{\sigma} \setminus \sigma \) are in the different sink basins. Since the period of the point \( \sigma \) coincides with the period of its separatrices and is \( m \), the set \( C_\sigma = cl \bigcup_{i=0}^{m-1} W^u_{f^i(\sigma)} \) is a closed curve. \( \square \)

**Lemma 8.3.** For any saddle orbit \( O_\sigma \), satisfying the conclusion of lemma 8.2, there exists a unique number \( k < m/2 \), \((k, m) = 1\) such that the map \( f_{|C_\sigma} \) is topologically conjugate to a rough transformation of the circle with the rotation number \( \frac{k}{m} \).
Proof. Let \( C_\sigma \) be an \( f \)-invariant closed curve constructed in lemma 8.3. Then the homeomorphism \( f|_{C_\sigma} : C_\sigma \to C_\sigma \) is topologically conjugate to an orientation-preserving rough circle transformation. By [20] there is a unique number \( k < m/2 \), \((k,m) = 1\) such that \( k/m \) is the rotation number of this transformation. Suppose that there exists another closed curve \( \tilde{C}_\sigma \), that satisfies lemma 8.3. We show that the homeomorphism \( f|_{\tilde{C}_\sigma} : \tilde{C}_\sigma \to \tilde{C}_\sigma \) is topologically conjugate to a rough transformation of a circle with the same rotation number \( k/m \).

For this we note that the set \( \tilde{A}_f = C_\sigma \) is a connected attractor of the diffeomorphism \( f \) and divides the sphere \( S^2 \) into two disks \( D_1 \) and \( D_2 \). The dual repeller \( \tilde{R}_f = R_f \setminus W^u_\sigma \) consists of two \( f \)-invariant connected components \( R_1 \subset D_1, R_2 \subset D_2 \). Since the curve \( \tilde{C}_\sigma \) is \( f \)-invariant, it lies in the closure of one of the disks, suppose \( D_1 \), for definiteness (see figure 16).

Similarly to the lemma 6.1 it can be shown that the repeller \( R_1 \) is a tree, we denote it by \( \Gamma_1 \). Moreover, the diffeomorphism \( f \) induces an automorphism \( R_{f_1} \), for which all edges have a period \( m \). This means (see section 2.4), that the graph is central, that is, the repeller \( R_1 \) contains a single fixed point, which is the source, we denote it by \( \alpha_1 \). Denote by \( l_1 \) the connected component \( W^u_\sigma \setminus \sigma \), belonging to the disk \( D_1 \). Let \( a_1 = cl(l_1) \setminus (l_1 \cup \sigma) \) (see the picture 16). Then in the tree \( R_1 \) there is a unique path \( L_{a_1,\alpha_1} \), connecting the source \( a_1 \) with the source \( \alpha_1 \). It follows from the properties of the tree that the path \( L_{a_1,\alpha_1} \) consists of vertices of pairwise different ranks, that is, \( L_{a_1,\alpha_1} \cap f(L_{a_1,\alpha_1}) = \alpha_1 \).

Let \( L_1 = l_1 \cup L_{a_1,\alpha_1} \). Then the set \( L_1 = L_1 \cup f(L_1) \cup \cdots \cup f^{m-1}(L_1) \) divides the disk \( D_1 \) into \( m \) of pairwise disjoint parts \( D_1, \omega_1, \ldots, D_1, f^{m-1}(\omega) \), each of which contains a single sink \( \omega, \ldots, f^{m-1}(\omega) \), accordingly, in its closure. Moreover, by continuity, the diffeomorphism \( f \) induces on the components \( D_1, \omega_1, \ldots, D_1, f^{m-1}(\omega) \) a rotation with the same rotation number as on the circle \( C_\sigma \). Since any saddle point lying inside such a part has unstable separatrices going to the same sink, \( \tilde{\sigma} \in L_1 \). Thus, the homeomorphism \( f|_{\tilde{C}_\sigma} \) is topologically conjugate to the rough transformation of the circle with the rotation number \( k/m \). □
Lemma 8.4. For any diffeomorphism $f \in G_m$, $m > 1$ the following properties hold:

- there exists a simple closed $f$-invariant curve (may be not unique) $C_f$ composed by the closures of the unstable manifolds of saddle points;
- for all such curves $C_f$ the homeomorphisms $f|_{C_f}$ have the same rotation number $\frac{k}{m}$, $k < m/2$, $(k, m) = 1$.

Proof. According lemmas 6.1 and 8.1, every connected component of $A_f$ is a tree for $f \in G_m$, $m > 1$. Moreover, by lemma 8.1, there is a diffeomorphism $g \in H_m$, coinciding with $f$ on $S^2 \setminus (D_f \cup \ldots \cup f^{m-1}(D_f))$. Let $A$ be a connected component of $A_f$ belonging to $D_f$. Then for the saddle orbit $O$, satisfying the conclusion of lemma 8.2, denote by $A_\sigma$ the intersection $cl W^u_{O_\sigma} \cap A$. If $A_\sigma$ consists of a unique point then $C_f = cl W^u_{O_\sigma}$. In the opposite case $A_\sigma$ consists of two vertex of the tree $A_f$. Let $c_\sigma$ be a simple path connected them. Then $C_f = cl W^u_{O_\sigma} \cup c_\sigma$. By lemma 8.3, for all such curves $C_f$ the homeomorphisms $f|_{C_f}$ have the same rotation number $\frac{k}{m}$, $k < m/2$, $(k, m) = 1$.

For $k \in (\mathbb{N} \cup 0)$, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$ we denote by $G_{k,m}$ the subset of $G_m$ such that $f|_{C_f}$ is topologically conjugate to a rough transformation of the circle with the rotation number $\frac{k}{m}$ for any $f \in G_{k,m}$. We denote by $H_{k,m}$ the subset of $G_{k,m}$ consisting from diffeomorphisms with unique sink orbit.

8.2. Trivialization of the repeller for $f \in H_{k,m}$. Denote by $F_{k,m}$ the subset of $H_{k,m}$, consisting of diffeomorphisms having a repeller $R_f$, containing a unique saddle orbit (of the period $m$ by lemma 8.2).

Lemma 8.5. Any diffeomorphism $f \in H_{k,m}$ is connected by a stable arc with some diffeomorphism $g \in F_{k,m}$.

Proof. The set $A_f = cl W^u_{O_\sigma}$ is a connected attractor, homeomorphic to a circle. Then there exists a neighborhood $K$ of this attractor diffeomorphic to an annulus and such that $f(K) \subset int K$. Then the set $S^2 \setminus K$ consists of two disjoint disks $D_1, D_2$. Denote by $g_t$, a 2-sphere diffeomorphism coinciding with $f$ on $D_t$ and having a unique hyperbolic sink in $S^2 \setminus D_t$. By construction, $g_t \in G_t$. By lemma 6.4 there exists a stable arc $g_{i,t} : S^2 \to S^2$, $t \in [0, 1]$, connecting the diffeomorphism $g_t$ with the source-sink diffeomorphism, while $g_{i,t} = g_t$ on $S^2 \setminus f^{-1}(D_t)$. Define the arc $f_i$ by the formula

$$f_i(x) = \begin{cases} f(x), & x \in f^{-1}(K), \\ g_{i,t}(x), & x \in f^{-1}(D_t). \end{cases}$$

Lemma 8.6. Any diffeomorphism $f \in F_{k,m}$ is connected by an arc without bifurcations with diffeomorphism $\phi_{k,m}$.

Proof. Let $f \in F_{k,m}$. Then the non-wandering set of diffeomorphism $f$ consists of one saddle orbit $O_\sigma = \{\sigma, f(\sigma), \ldots, f^{m-1}(\sigma)\}$, one sink orbits $O_\omega = \{\omega, f(\omega), \ldots, f^{m-1}(\omega)\}$ and fixed sources $\alpha_1, \alpha_2$. Moreover, the closures of unstable saddle separatrices form a circle

$$C_\sigma = W^u_{O_\sigma} \cup O_\sigma.$$

Also a similar non-wandering set has a diffeomorphism $\phi_{k,m}$, which we will denote by $\phi$ for brevity (see figure 20). By proposition 7 the circle $C_\sigma$ can be considered smooth. Since all orientation-preserving diffeomorphisms of a 2-sphere are smoothly
isotopic to the identity map (see, for example, [30]), the circle $C_\sigma$ can be considered to coincide with the similar circle of the diffeomorphism $\phi$, we can also consider the same non-wandering sets of two diffeomorphisms. By proposition 6, we can assume that the diffeomorphisms $f$ and $\phi$ coincide in some neighborhoods of the periodic points.

Since the circle $C_\sigma$ is an attractor of both diffeomorphisms, there exist smooth annulus $K_f, K_\phi$, containing $C_\sigma$ and such that $f(K_f) \subset \text{int} K_f, \phi(K_\phi) \subset \text{int} K_\phi$. We choose a diffeomorphism $\beta : K_f \to K_\phi$ so that $\beta|_{C_\sigma} = \text{id}$ and $\beta(f(K_f)) = \phi(K_\phi)$. Let $\tilde{f} = \beta f \beta^{-1} : K_\phi \to \phi(K_\phi)$. Then on the set $K_\phi$ the family of maps $\chi_t : \phi(K_\phi) \to S^2$ is correctly defined by the formula\[ \chi_t = (1 - t)\tilde{f} + t\phi. \]

By construction, $\chi_t(\phi(K_\phi)) \subset \text{int} \phi(K_\phi)$ for all $t \in [0, 1]$. Note that the circle $C_\sigma$ is $\chi_t$-invariant and $C_\sigma = W^{u}_{\sigma} \cup O_\omega$ for any diffeomorphism $\chi_t$. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\phi(K_\phi)}$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\phi$ and $\xi_t(\phi(K_\phi)) \subset \text{int} K_\phi$. By proposition 5, there is an isotopy $\Xi_t : K_\phi \to K_\phi$, }
coinciding with $\xi_t$ on $\phi(K_{\phi})$ and identical on $\partial K_{\phi}$. Let

$$\tilde{f}_t = \tilde{f}_{\Xi t}.$$  

Note that $\tilde{f}_1 = \phi$ on $\phi(K_{\phi})$. Define the arc $f_t : S^2 \to S^2$ so that $f_t$ coincides with $f$ outside $K_f$, $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in K_f$.

Let $\gamma = f_1$ and $D_i, i = 1, 2$ denotes the connected component of the set $S^2 \setminus C_\sigma$, containing $\alpha_i$. Let $\gamma_i = \gamma|_{D_i}$. By construction, there is a neighborhood $V_{\alpha_i}$ of the point $\alpha_i$, where $\gamma_i|_{V_{\alpha_i}} = \phi|_{V_{\alpha_i}}$. Define the diffeomorphism $\psi_{\gamma_i} : D_i \to D_i$ by the formula

$$\psi_{\gamma_i}(w) = \phi^k(\gamma^{-k}_i(w)),$$

where $k \in \mathbb{Z}$ such that $\gamma_i^{-k}(w) \in \phi(K_{\phi})$ for $w \in D_i$. Then $\gamma_i = \psi_{\gamma_i}^{-1}\phi\psi_{\gamma_i}$. If $\psi_{\gamma_i}$ can be smoothly extended to the point $\alpha_i$ by the condition $\psi_{\gamma_i}(\alpha_i) = \alpha_i$, then, by [30], there exists a smooth isotopy $\rho_{i,t} : S^2 \to S^2$ such that $\rho_{i,0} = \psi_{\gamma_i}$, $\rho_{i,1} = id$. Let $\delta_{i,t} = \rho_{i,t}^{-1}\phi\rho_{i,t}$. Denote by $\delta_t : S^2 \to S^2$ the arc coinciding $\delta_{i,t}$ on $D_i$ and from $\phi$ to $\phi(K_{\phi})$. Then the product of the arcs $f_t$ and $\delta_t$ is the desired arc.
In the case when at least one of the diffeomorphisms $\psi_{\gamma, i}, i \in \{1, 2\}$ can not be smoothly continued to $\alpha$, we show that there is an arc $\zeta_{i,t} : D_i \to D_i$, connecting the diffeomorphism $\zeta_{i,0} = \gamma_i|_{D_i}$ with some diffeomorphism $\zeta_{i,1}$ such so that $\psi_{\gamma, 1}$ can be smoothly extended to $\alpha$ by the condition $\psi_{\gamma, 1}(\alpha_i) = \alpha_i$.

For definiteness, let $i = 1$. $\mathbb{B}_r(O) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $K_r = \mathbb{B}_r(O) \setminus \phi(int \mathbb{B}_r(O))$ for $r > 0$. Let $\gamma_i = \vartheta\gamma_i\vartheta^{-1}$ and $\gamma_i = \vartheta\gamma_i\vartheta^{-1}$, where $\vartheta : S^2 \setminus \{\alpha_2\} \to \mathbb{R}^2$ is the stereographic projection and $\vartheta(\alpha_1) = O$. Then there exists $r_0 \in (0, 1)$ such that $\gamma_i = \vartheta = \vartheta\vartheta^{-1}$ on $\mathbb{B}_{r_0}$ and the ring $K_{r_0}$ is the fundamental domain of the diffeomorphism of $\vartheta$ (and $\gamma_i$) and $int \mathbb{B}_1(O) \setminus \{O\}$. Represent $T^2$ as the orbit space of $(\vartheta \mathbb{B}_1(O) \setminus \{O\})/\hat{\vartheta}$. Let $p : \mathbb{B}_{r_0} \setminus \{O\} \to T^2$ denote the natural projection. Then the curves $a = p(Ox_1), b = p(\partial\mathbb{B}_{r_0})$ are the generators of the fundamental group $\pi_1(T^2)$. Since $\psi_{\gamma_1}$ translates the orbits $\hat{\vartheta}$ into the orbits $\gamma_1$ and $K_{r_0}$ is a common fundamental domain for $\vartheta, \gamma_1$ on $int \mathbb{B}_1(O) \setminus \{O\}$, then $\gamma_i$ is projected onto $T^2$ by the formula $\vartheta = p\vartheta p^{-1}$. Then the induced isomorphism $\psi_{\gamma_i} : \pi_1(T^2) \to \pi_1(T^2)$ preserves the homotopy class of the generator $a$ and, therefore, is given matrix

$$
\begin{pmatrix}
1 & n_0 \\
0 & 1
\end{pmatrix}
$$

for some integer $n_0$.

The arc $\zeta_{i,t}$ will be the smooth product of the arcs $\nu_t$ and $\mu_t$, where

1) $\nu_t$ is a smooth arc without bifurcations connecting the diffeomorphism $\nu_0 = \gamma_1$ with the diffeomorphism $\nu_1$ such that $\vartheta_{\nu_1}$ induces an identical isomorphism in $\pi_1(T^2)$;

2) $\mu_t$ is a smooth arc without bifurcations connecting the diffeomorphism $\mu_0 = \nu_1$ with the diffeomorphism $\mu_1$ such that $\vartheta_{\nu_1} = id$, which means $\psi_{\nu_1} = \vartheta^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, that is, the diffeomorphism $\psi_{\nu_1}$ is smoothly continued to $\alpha$.

1) We introduce the polar coordinates $r, \varphi$ on $\mathbb{R}^2$. Define the diffeomorphism $\hat{\vartheta}_t$ by the formula $\hat{\vartheta}_t(re^{i\varphi}) = \begin{cases} re^{i\varphi}, r > r_0; \\
re^{i\varphi + 4\pi r^2 (1 - \frac{r_0}{r})}, \frac{r_0}{2} \leq r \leq r_0; \\
re^{i\varphi + 2\pi r^2}, r < \frac{r_0}{2}. \end{cases}$

Let $\theta_t = \vartheta^{-1}\hat{\theta}_t\vartheta : D_1 \to D_1 \setminus \{\alpha_1\} \to D_1 \setminus \{\alpha_1\}$, then $\theta_t$ can be smoothly continued to $\alpha$ by the condition $\theta_t(\alpha_1) = \alpha_1$. Moreover, by construction, $\psi_{\theta, \gamma_1}$ induces an identical isomorphism on $\pi_1(T^2)$. Thus, $\nu_t = \theta_t \gamma_1 : D_1 \to D_1$ is the desired arc.

2) Here we are dealing with a diffeomorphism $\nu_1 : D_1 \to D_1$ such that a diffeomorphism $\psi_{\nu_1}$ induces an identical isomorphism in $\pi_1(T^2)$. Then, by [29, 7], the diffeomorphism $\psi_{\nu_1}$ is smoothly isotopic to the identity map. We choose a cover $U = \{U_1, \ldots, U_q\}$ of the torus $T^2$ consisting of disks such that a connected component of the set $p^{-1}(U_i)$ is a subset of $K_{r_i}$ for some $r_i$ such that $B_r(O) \subset \phi(B_{r_1}(O))$. By [4, Lemma de fragmentation] there exist smoothly isotopic to the identity diffeomorphisms $\hat{w}_1, \ldots, \hat{w}_q : T^2 \to T^2$ such that

i) for each $i = \Gamma, q$ there exists $U_{\hat{w}(i)} \subset U$ such that for each $t \in [0, 1]$ the map $\hat{w}_t$ is identical outside $U_{\hat{w}(i)}$, where $\{\hat{w}_i, t\}$ is the smooth isotopy between the identity map and $\hat{w}_i$;

ii) $\hat{w}_t = \hat{w}_1 \ldots \hat{w}_q$. 

Let \( \bar{w}_{t,i} : D_1 \to D_1 \) be a diffeomorphism that coincides with \((p|_{K_i})^{-1}\bar{w}_{t,i}p\) on \(K_i\) and is identical outside \(K_i\). Let \( \bar{\mu}_t = \bar{v}_1\bar{w}_{1,t} \ldots \bar{w}_{q,t} : D_1 \setminus \{\alpha_1\} \to D_1 \setminus \{\alpha_1\}\). By construction, \( \bar{\mu}_0 = \bar{v}_1 \) and \( \bar{\mu}_1 \) has the property: \( \bar{\psi}_{\mu_1} = \bar{w}_{q}^{-1} \ldots \bar{w}_{1}^{-1}\bar{\psi}_{\mu_1} = \bar{w}_{q}^{-1} \ldots \bar{w}_{1}^{-1} \bar{w}_1 \ldots \bar{w}_q = id. \)

\[\]

9. Classification of the model diffeomorphisms with respect to the stable connectedness. The classification directly follows from two lemmas below.

**Lemma 9.1.** There is a stable arc connecting the diffeomorphism \( \phi_{0,1}(\phi_{1,2}, \phi_{1,2}^{-1}, \phi_{0,1}^{-1}) \) with diffeomorphism \( \phi_0 \).

**Proof.** We show how to construct a stable arc connecting:
1) \( \phi_{1,2}^{-1} \) with \( \phi_0 \); 2) \( \phi_{0,1}^{-1} \) with \( \phi_0 \).

For diffeomorphisms \( \phi_{1,2}, \phi_{0,1} \) the constructions are similar.

1) Let \( f = \phi_{1,2}^{-1} \) (see picture 21). Consider a smooth curve \( l = cl W^u \setminus \{\omega_2\} \), for which points \( \sigma, f(\sigma) \) are internal, while \( f(l) \subset l \).

Let \( \Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2}\} \). Define a diffeomorphism \( \bar{\varphi} : \Pi_1 \to \mathbb{R}^2 \) formula along the axis \( Ox_1 \)

\[ \bar{\varphi}(x_1, x_2) = \left( -\frac{9}{10}x_1 - x_1^3, -\frac{x_2}{2} \right). \]

By construction \( \bar{\varphi}(\Pi_1) \subset int \Pi_1 \), diffeomorphism \( \bar{\varphi} \) has a sink point \( O \) and saddle periodic orbit \( \{P_0, \bar{\varphi}(P_0)\} \), where \( \bar{\varphi}(P_0) = (x_0, 0), P_0 = (x_0, 0) \), \( x_0 \in (0, 1/2) \). Let \( \Pi_2 = \bar{\varphi}(\Pi_1) \). We choose a closed neighborhood \( V \) of arc \( l \) and diffeomorphism \( \beta : V \to \Pi_1 \) in the following way \( f(V) \subset int V \), \( \beta(l \cap V) = Ox_1 \cap \Pi_1 \), \( \beta(f(V)) = \Pi_2 \), \( \beta(\omega) = P_0 \) and \( \beta(f(\omega)) = \bar{\varphi}(P_0) \) (see picture 21). Set \( \bar{f} = \beta f \beta^{-1} : \Pi_1 \to \Pi_2 \).

Then on the set \( \Pi_2 \) correctly defined family of maps \( \chi_t : \Pi_2 \to \mathbb{R}^2 \) by the formula

\[ \chi_t = (1-t)\bar{f} + t\bar{\varphi}. \]

\[\]

**Figure 21.** Illustration to the lemma 9.1, the case 1)

By construction \( \chi_t(\Pi_2) \subset int \Pi_2 \) for all \( t \in [0, 1] \). Note that the origin is a fixed sink point for the diffeomorphism \( \chi_t \) and the points \( P_0, \bar{\varphi}(P_0) \) form a saddle orbit. In addition, the isotopy \( \xi_t = \bar{f}^{-1}\chi_t|_{\Pi_2} \) connects the identity map with the
diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$ and $\xi_t(\Pi_2) \subset \text{int} \Pi_1$. By proposition 5, there exists an isotopy $\Xi_t : \Pi_1 \rightarrow \Pi_1$, coinciding with $\xi_t$ on $\Pi_2$ and identical on $\partial \Pi_1$. Let $\tilde{f}_t = \tilde{f}\Xi_t$.

Note that $\tilde{f}_1 = \tilde{\varphi}$ on $\Pi_2$. Let $\Pi_3 = \tilde{\varphi}(\Pi_2)$. Define the arc $\eta_t : \Pi_3 \rightarrow \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(-x_1 \left(1 + \frac{1}{10} (2t - 1)\right), -\frac{x_2}{2}\right).$$

By construction $\eta_t(\Pi_3) \subset \text{int} \Pi_3$ for all $t \in [0, 1]$, in addition, isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ connects the identity map with a diffeomorphism $\tilde{\varphi}^{-1}\eta_t$ and $\zeta_t(\Pi_3) \subset \text{int} \Pi_2$. By proposition 5, there exists isotopy $\Theta_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, coinciding with $\zeta_t$ on $\Pi_3$ and identical outside $\Pi_2$. Let $\tilde{\Theta}_t = \tilde{\varphi}\Theta_t$.

Let $\delta_t = f_t \ast \Theta_t : S^2 \rightarrow S^2$, where $f_t$ coincides with $f$ outside $V$, $f_t(z) = \beta^{-1}(f_t(\beta(z)))$ for $z \in V$ and $\Theta_t$ coincides with $f_1$ outside $f_1(V)$, $\Theta_t(z) = \beta^{-1}(\Theta_t(\beta(z)))$ for $z \in f_1(V)$. Phase portrait of diffeomorphism $\delta_1$ depicted on the picture 22.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure22.png}
\caption{Illustration to the lemma 9.1, case 1)
}
\end{figure}

Having done similar constructions in a neighborhood of the arc $\partial W^s_{\sigma}$, we connect the diffeomorphism $\delta_1$ with the source-sink diffeomorphism by a stable arc with one flip bifurcation. By the proposition 4, any source-sink diffeomorphism is connected by an arc without bifurcations with the diffeomorphism $\phi_0$.

2) Let $f = \phi_{0,1}^{-1}$ (see figure 23). For the diffeomorphism $f$ the saddle $\sigma$ and the drain $\omega_1$ are fixed. Let $l = \partial x_1 x_2 \cap S^2$ and denote by $\gamma \subset l$ an arc bounded by the points $\omega_1$, $\sigma$ and not containing $\alpha$. Set $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2}\}$, $\tilde{U}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{3}{4}\}$, $\tilde{U}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{3}{4}\}$. Define the diffeomorphism $\tilde{\varphi} : \tilde{U}_1 \rightarrow \mathbb{R}^2$ by the formula

$$\tilde{\varphi}(x_1, x_2) = \left(x_1 + 2x_1^2 - \frac{1}{10} \frac{x_2}{2}\right).$$
By construction, a diffeomorphism $\tilde{\varphi}$ has a saddle point $P_1(0, x_1), x_1 \in (0, 1/2)$ and sink point $P_2(-x_2, 0), x_2 \in (0, 1/2)$. Let $\Pi_2 = \varphi(\Pi_1)$.

We choose a closed neighborhood $V$ of arc $\gamma$, open neighborhood $U_1 \supset V$ of arc $\gamma$ and diffeomorphism $\beta : U_1 \rightarrow \tilde{U}_1$ so that $\beta(\sigma) = P_1, \beta(\omega_1) = P_2, \beta(U \cap U_1) = Ox_1 \cap \tilde{U}_1, \beta(V) = \Pi_1, \beta(f(V)) = \Pi_2$ (see picture 23). Let $f = \beta f^{-1} : \tilde{U}_1 \rightarrow \tilde{\varphi}(\tilde{U}_1)$. Then on the set $\Pi_1$ correctly defined family of mappings $\chi_t : \Pi_1 \rightarrow \Pi_2$ by the formula

$$\chi_t = (1 - t)f + t\tilde{\varphi}.$$ 

Note that the point $P_1$ is a fixed saddle point and the point $P_2$ is a fixed sink for diffeomorphism $\chi_t$. In addition, isotopy $\xi_t = f^{-1}\chi_t|_{\Pi_1} : \Pi_1 \rightarrow \Pi_1$ connects the identity map with a diffeomorphism $f^{-1}\tilde{\varphi}$. By virtue of the proposition 5, there exists isotopy $\Xi_t : \tilde{U}_1 \rightarrow \tilde{U}_1$, coinciding with $\xi_t$ on $\Pi_1$ and identical on $\partial\tilde{U}_1$. Let

$$\tilde{f}_t = f\Xi_t.$$ 

Notice, that $\tilde{f}_t = \tilde{\varphi}$ on $\Pi_1$. Define an arc $\eta_t : \Pi_1 \rightarrow \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(x_1 + 2x_1^2 + \frac{1}{10}(2t - 1), \frac{x_2}{2}\right).$$ 

By construction $\eta_t(\Pi_1) \subset \Pi_2$ for all $t \in [0, 1]$, in addition, isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t \tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_1) \subset \Pi_1$. By virtue of the proposition 5, there exists isotopy $\theta_t : \tilde{U}_2 \rightarrow \tilde{U}_2$, coinciding with $\zeta_t$ on $\Pi_1$ and identical on $\partial\tilde{U}_2$. Let

$$\tilde{\Theta}_t = \tilde{\varphi}\theta_t.$$ 

Let $U_2 = \beta^{-1}(\tilde{U}_2)$ and $\delta_t = f_t \ast \Theta_t : S^2 \rightarrow S^2, f_t$ coincides with $f$ out of $U_1$ and $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in U_1$; $\Theta_t$ coincides with $f_t$ out of $U_2$ and $\Theta_t(z) = \beta^{-1}(\tilde{\Theta}_t(\beta(z)))$ for $z \in U_2$. By construction, a diffeomorphism $\delta_1$ is a source-sink diffeomorphism. By virtue of the proposition 4, any source-sink diffeomorphism is connected by an arc without bifurcations with a diffeomorphism $\phi_0$. □

**Lemma 9.2.** Diffeomorphism $\phi_{k, m}, k < m/2, m > 2$ is connected by a stable arc with a diffeomorphism $\phi_{k', m'}$ if and only if $m' = m, k' = m - k$; and is not connected with any diffeomorphism $\phi_{k', m'}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure23.png}
\caption{Illustration to the lemma 9.1, case 2)
\end{figure}
Proof. Assume that diffeomorphism $\phi_{k,m}$, $k < m/2$, $m > 2$ is connected by a stable arc $\varphi_t$ with some diffeomorphism $\phi_{k',m'}$ such that $k/m \neq k'/m'$. By remark 1, all diffeomorphisms on $\varphi_t$, except bifurcations, belong to class $G$. Firstly, let us show that all these $\varphi_t$ belongs to the same subclass $G_m$.

Indeed, in the opposite case, as $\phi_{k,m} \in G_m$, there is a stable arc $f_t$ with unique bifurcation value $b$ such that $f_0 \in G_m$ and $f_1 \in G_m$, $\tilde{m} \neq m$. By proposition 3, $f_0$ has the periodic points of exactly two periods $1$ and $m$, moreover, all saddle points have the positive type of the orientation and the period $m$. As a flip bifurcation is connected with an appearing or disappearing of points of two different periods $k$ and $2k$, then for $f_b$ is impossible to be disappearing because $m > 2$. If $f_b$ is an appearing, then $k = 1$ or $k = m$ and, hence, $f_1$ necessary has periodic points of three different periods $1, 2, m$ or $1, m, 2m$, that is impossible according to proposition 3. Thus, $f_b$ is a saddle-node bifurcation.

A saddle-node bifurcation is connected with an appearing or disappearing of saddle of the positive type of orientation and node points of the same period. If $f_b$ is an appearing then, by proposition 3, this period equals $m$ and, hence, by lemma 7.2, $\tilde{m} = m$, that is a contradiction. If $f_b$ is a disappearing then there are two possibilities: 1) $f_b$ has no saddle points; 2) $f_b$ has a saddle point. In the case 1) $f_b$ has a saddle-node cycle, that contradicts to definition of the stable arc. In the case 2), by lemma 7.2, $f_1 \in G_m$. Thus, $m = \tilde{m}$.

Let us assume now that $k \neq k'$. There are two possibilities: 1) $\varphi_t$ has no bifurcation at all; 2) $\varphi_t$ contains bifurcations. In the case 1) $\varphi_0$ is topologically conjugated with $\varphi_1$. Hence, they are conjugated on the equator, where $\varphi_0$ is a rough transformation of the circle with the rotation number $\frac{k}{m}$, $k < \frac{m}{2}$ and $\varphi_1$ — with $\frac{k'}{m}$. By [20], it implies $k' = m - k$. Let us show that there indeed exists a stable arc $\varphi_t$, connecting $\phi_{k,m}$ with $\phi_{m-k,m}$. To do this, denote by $\Theta_t : S^2 \rightarrow S^2$ a rotation $S^2$ on the angle $2\pi t$ around an axis passing through points $(1, 0, 0)$ and $(-1, 0, 0)$. Then $\varphi_t = \Theta_t \phi_{k,m} \Theta_t^{-1}$.

In the case 2) let us show that all these $\varphi_t$ belongs to the same subclass $G_{k,m}$.

Indeed, in the opposite case, as $\phi_{k,m} \in G_{k,m}$, there is a stable arc $f_t$ with unique bifurcation value $b$ such that $f_0 \in G_{k,m}$ and $f_1 \in G_{k,m}$, $\tilde{k} \neq k$. Similar to the arguments above it is possible to show that for $f_b$ is impossible to be a flip bifurcation. Thus, $f_b$ is a saddle-node bifurcation connected with an appearing or disappearing of saddle of the positive type of orientation and node points of the same period $m$. By proposition 8.4, $f_b$ possesses a simple closed $f$-invariant curve $C_f$ composed by the closures of the unstable manifolds of saddle points such that the homeomorphism $f|_{C_f}$ has the rotation number $\frac{k}{m}$. If $f_b$ is an appearing then the curve is preserved for $f_1$ and, hence, $\tilde{k} = k$, that is a contradiction. If $f_b$ is a disappearing then there are two possibilities: 1) the disappearing points do not belong to $C_f$; 2) the disappearing points belong to $C_f$. In the case 1) the curve $C_f$ is preserved for $f_1$ and, hence, $\tilde{k} = k$, that is a contradiction. In the case 2) $f_b$ has periodic points different from saddle-node on $C_f$ (in the opposite case we have a saddle-node cycle) then, by [20], $f_b|_{C_f}$ and $f_1|_{C_f}$ has the rotation number $\frac{k}{m}$. Thus, $k = \tilde{k}$.

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