Research Article

Bounds of a Unified Integral Operator via Exponentially $(s, m)$-Convexity and Their Consequences

Yi Hu,1,2 Ghulam Farid,3 Zijiang,1,2 and Kahkashan Mahreen3

1School of Information Science and Technology, South China Business College of Guangdong University of Foreign Studies, 510545 Guangzhou, China
2Institute for Intelligent Information Processing, South China Business College of Guangdong University of Foreign Studies, 510545 Guangzhou, China
3Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan

Correspondence should be addressed to Ghulam Farid; faridphdsms@hotmail.com

Received 20 November 2019; Accepted 23 April 2020; Published 22 May 2020

1. Introduction

Fractional integral inequalities are useful in the field of fractional calculus. Many mathematicians have introduced fractional differential, fractional integral, and fractional conformable integral operators in this field (see [1–14]). Recently, several mathematical inequalities have been introduced via $(s, m)$-convexity (see [15, 16]). The goal of this paper is to obtain the bounds of all integral operators explained in Remarks 5 and 6 in a unified form for exponentially $(s, m)$-convex functions.

Definition 1 (see [6]). Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let $g$ be an increasing and positive function on $(a, b]$, having a continuous derivative $g'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[a, b]$ of order $\mu$ where $\mathcal{R}(\mu) > 0$ are defined by

$$
\mu^f_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t) - g(x))^{\mu-1} g'(t)f(t)dt, \quad x > a,
$$

(1)

$$
\mu^f_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(t) - g(x))^{\mu-1} g'(t)f(t)dt, \quad x < b,
$$

(2)

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [17]). Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let $g$ be an increasing and positive function on $(a, b]$, having a continuous derivative $g'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[a, b]$ of order $\mu$ where $\mathcal{R}(\mu) > 0, k > 0$, are defined by

$$
\mu^k_{a^+} f(x) = \frac{1}{k\Gamma(\mu)} \int_x^b (g(t) - g(x))^{(\mu/k)-1} g'(t)f(t)dt, \quad x > a,
$$

(3)

$$
\mu^k_{b^-} f(x) = \frac{1}{k\Gamma(\mu)} \int_a^x (g(t) - g(x))^{(\mu/k)-1} g'(t)f(t)dt, \quad x < b,
$$

(4)
where $\Gamma_k(\cdot)$ is the gamma function given as follows [18]:

$$\Gamma_k(x) = \frac{\Gamma(x)}{\Gamma(x+k)}, \quad R(x) > 0. \quad (5)$$

A fractional integral operator containing an extended generalized Mittag-Leffler function in its kernel is defined as follows:

**Definition 3 (see [19]).** Let $\omega, \mu, a, l, y, c \in \mathbb{C}, R(\mu), R(a), R(l) > 0, R(c) > R(y) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + R(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $I_{\mu,n,a,a}^{\gamma,k,c}\ f$ and $I_{\mu,n,a,b}^{\gamma,k,c} \ f$ are defined by

$$\left(I_{\mu,n,a,a}^{\gamma,k,c}\ f\right)(x; p) = \sum_{n=0}^{\infty} \beta_p(y + nk, c - y) \frac{(c)_n}{\Gamma(n + \alpha)} \frac{t^n}{t^\mu}, \quad x \geq a,$$

$$\left(I_{\mu,n,a,b}^{\gamma,k,c} \ f\right)(x; p) = \int_{a}^{x} (x - t)^{a-1} E_{\mu,n,(a)}^{\gamma,k,c} (\omega(x - t)\mu; p) f(t) dt,$$

where

$$E_{\mu,n,(a)}^{\gamma,k,c}(t; \nu) = \sum_{n=0}^{\infty} \beta_p(y + nk, c - y) \frac{(c)_n}{\Gamma(n + \alpha)} t^n.$$

Remark 5.

(i) Let $\phi(x) = x^\beta \Gamma(\beta)/k \Gamma_k(\beta), \ k > 0, \ \beta > k, \ \text{and} \ \rho = 0$, in unified integral operators (9) and (10). Then, generalized Riemann-Liouville fractional integral operators (3) and (4) are obtained.

(ii) For $k = 1$, (3) and (4) fractional integrals coincide with (1) and (2) fractional integrals, which further produce the following fractional and conformable integrals:

(1) By taking $g$ as identity function, (3) and (4) fractional integrals coincide with Riemann-Liouville fractional integrals [6].

(2) For $k = 1$, along with $g$ as the identity function, (3) and (4) fractional integrals coincide with Riemann-Liouville fractional integrals [6].

(3) For $k = 1$ and $g(x) = x^\rho/p, \ p > 0$, (3) and (4) produce fractional integrals defined by Chen and Katugampola in [1].

(4) For $k = 1$ and $g(x) = x^{t+s}/(t+s)$, (3) and (4) produce generalized conformable fractional integrals defined by Khan and Khan in [5].

(5) If we take $g(x) = (x-a)^s/s, \ s > 0$, in (3) and $g(x) = - (b-x)^s/s, \ s > 0$, in (4), then conformable $(k, s)$-fractional integrals are achieved as defined by Habib et al. in [3].

(6) If we take $g(x) = x^{t+s}/(1+s)$, then conformable fractional integrals are achieved as defined by Sarikaya et al. in [11].

(7) If we take $g(x) = (x-a)^s/s, \ s > 0$, in (3) and $g(x) = - (b-x)^s/s, \ s > 0$, in (4) with $k = 1$, then conformable fractional integrals are achieved as defined by Jarad et al. in [4].

(8) If we take $\omega = p = 0$ and $\phi(t) = \Gamma(\mu) x^\mu f(x) \Gamma_1(x, \omega, \mu, \phi)$ in (9) and (10), then generalized $k$-fractional integral operators are achieved as defined by Tunc et al. in [13].

(9) If we take $k = 1$ and $\phi(t) = \Gamma(\mu) \exp(-At)/\mu, \ A = (1 - \mu)/\mu, \ \mu > 0$, in (3) and (4), then generalized fractional integral operators with an exponential kernel are obtained [2].

Remark 6. Let $\phi(x) = x^\beta$ and $g(x) = x, \ \beta > 0$, in unified integral operators (9) and (10). Then, fractional integral operators (6) and (7) are obtained, which along with different settings of $k, \delta, l, c, y$ in generalized Mittag-Leffler function give the following integral operators:

(1) By setting $p = 0$, fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Salim and Faraj in [10].
(2) By setting \( l = \delta = 1 \), fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Rahman et al. in [9].

(3) By setting \( p = 0 \) and \( l = \delta = 1 \), fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Srivastava and Tomovski in [12].

(4) By setting \( p = 0 \) and \( l = \delta = k = 1 \), fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Prabhakar in [8].

(5) By setting \( p = \omega = 0 \), fractional integral operators (6) and (7) reduce to the left-sided and right-sided Riemann-Liouville fractional integrals.

(6) By setting \( p = \omega = 0 \) in fractional integral operators (9) and (10) we get \((1/\Gamma(\mu))(\Phi^{\mu,\delta,k,c}_{a,b}\, f)(x) = (\varphi^{\mu,\omega,k,c}_{a,b}\, f)(x,0;0)\) and \((1/\Gamma(\mu))(\Phi^{\mu,\delta,k,c}_{a,b}\, f)(x,0;0)\) where \((\varphi^{\mu,\omega,k,c}_{a,b}\, f)(x)\) and \((\Phi^{\mu,\delta,k,c}_{a,b}\, f)(x)\) are defined in [21].

Convex functions play a very vital role in the theory of inequalities. Motivated by its analytical interpretation, many other extended and generalized notions have been defined in literature. These notions are used to extend and generalize a lot of classical results (see [15, 16, 22, 23] and references therein). A generalized notion called exponentially \((s,m)\)-convexity is defined in [24].

Definition 7. Let \( s \in [0, 1] \) and \( I \subseteq [0, \infty) \) be an interval. A function \( f : I \to \mathbb{R} \) is said to be exponentially \((s,m)\)-convex in the second sense, if

\[
 f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y), \quad (11)
\]

holds for all \( x, y \in I, m \in [0, 1] \) and \( t \in \mathbb{R} \).

One can note the deductible definitions in the following remark:

Remark 8.

(i) For \( m = 1 \), (11) produces the definition of exponentially \( s \)-convex function

(ii) For \( \alpha = 0 \), (11) produces the definition of \((s,m)\)-convex function

(iii) For \( \alpha = 0 \) and \( m = 1 \), (11) produces the definition of \( s \)-convex function in the second sense

(iv) For \( \alpha = 0 \) and \( m = 1 \), (11) produces the definition of the convex function

(v) For \( \alpha = 0 \) and \( s = 1 \), (11) produces the definition of the \( m \)-convex function

In the upcoming section, bounds of unified integral operators for exponentially \((s,m)\)-convex functions are given in different forms. Bounds of associated fractional and conformable integral operators which are known in literature are also deduced. The Hadamard inequality is derived for exponentially \((s,m)\)-convex functions. Its diverse conformable and fractional versions are presented. A modulus inequality is established by using exponentially \((s,m)\)-convexity of \(|f'|\). In Section 3, boundedness and continuity of these operators are given.

2. Main Results

Bounds of unified integral operators (9) and (10), by using exponentially \((s,m)\)-convexity, are established in the following theorem.

**Theorem 9.** Let \( f : [a, b] \to \mathbb{R}, 0 \leq a < b \), be a positive exponentially \((s,m)\)-convex function with \( m \in (0, 1] \) and \( g : [a, b] \to \mathbb{R} \) be a differentiable and strictly increasing function. Also let \( \phi/x \) be an increasing function on \([a, b] \). If \( a, f, y, c \in \mathbb{C}, \mu \geq 0, \delta \geq 0, \) and \( 0 < k \leq \delta + \mu \), then for \( x \in (a, b) \), we have

\[
 \left( f^{\delta, k}_{a,b} \right)(x, \omega; p) + \left( f^{\omega,\delta,k}_{a,b} \right)(x, \omega; p)
 \leq K_{x}^{1} E_{\mu,a,l}^{\delta,k} g_{p}(a) \cdot \phi \left( \frac{f(x/m \omega(x/m)) - f(a)}{\omega(x/m)} - \Gamma(s + 1) \frac{b}{a} \right) + K_{x}^{1} E_{\mu,a,l}^{\delta,k} \left( g_{p}(a) - \Gamma(s + 1) \frac{b}{a} \right) \frac{f(b)}{\omega(b)} \left( \frac{f(x/m \omega(x/m)) - f(a)}{\omega(x/m)} \right)
 \leq K_{x}^{1} E_{\mu,a,l}^{\delta,k} \left( g_{p}(a) - \Gamma(s + 1) \frac{b}{a} \right) \frac{f(b)}{\omega(b)} \left( \frac{f(x/m \omega(x/m)) - f(a)}{\omega(x/m)} \right).
\]

**Proof.** Under the suppositions of \( \phi \) and \( g \), we can obtain

\[
 \frac{\phi(g(x) - g(t))}{g(x) - g(t)} \leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)} \quad \text{for } t \in [a, x), x \in (a, b). \]

Multiplying with \( E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(t))^\mu; p) g'(t) \), we can obtain

\[
 K_{x}^{1} E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(t))^\mu; p) g'(t) \leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)} \cdot E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(t))^\mu; p) g'(t). \]

By using \( E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(t))^\mu; p) \leq E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(a))^\mu; p) \), the following inequality is obtained:

\[
 K_{x}^{1} E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(t))^\mu; p) g'(t) \leq K_{x}^{1} E_{\mu,a,l}^{\delta,k} (\omega(g(x) - g(a))^\mu; p) g'(t). \]

Using exponentially \((s,m)\)-convexity of \( f \), we have

\[
 f(t) \leq \left( \frac{x - t}{x - a} \right)^{s} \frac{f(x)}{\omega(x)} + m \left( \frac{t - a}{x - a} \right)^{s} \frac{f(x/m \omega(x/m))}{\omega(x/m)}. \]
Multiplying (15) and (16) and integrating over \([a, x]\), we can obtain
\[
\int_a^x K_x^{\mu} \left( E_{\mu, a}^{\gamma, c} , g ; \phi \right) f(t) \, dt \, g(t(1)) \leq \int_a^x K_x^{\mu} \left( E_{\mu, a}^{\gamma, c} , g ; \phi \right) f(t) \, dt \, g(t(1)).
\]

By using Definition 4 and integrating by parts, the following inequality is obtained:
\[
\left( g^{E_{\mu, a}^{\gamma, c}} f \right)(x, \omega; p) \leq K_x^{\mu} \left( E_{\mu, a}^{\gamma, c} , g ; \phi \right) g'(t) \leq K_x^{\mu} \left( E_{\mu, a}^{\gamma, c} , g ; \phi \right) g'(t).
\]

Using exponentially \((s, m)\)-convexity of \(f\), we have
\[
f(t) \leq \frac{t-x}{b-x} f(b) + m \frac{b-t}{b-x} \frac{f(x)}{e^{\alpha b}} + \left( \frac{b-t}{b-x} \right)^{s} \frac{f(x)}{e^{\alpha b}}.
\]

Adopting the same pattern as we did for (15) and (16), we obtained the following inequality from (19) and (20):
\[
\left( g^{E_{\mu, a}^{\gamma, c}} f \right)(x, \omega; p) \leq K_x^{\mu} \left( E_{\mu, a}^{\gamma, c} , g ; \phi \right) f(b) + m \frac{b-t}{b-x} \frac{f(x)}{e^{\alpha b}} - \frac{f(x)}{e^{\alpha b}} I_x \left( \frac{f(x)}{e^{\alpha b}} I_x \right).
\]

By adding (18) and (21), (12) can be obtained.

**Remark 10.**

1. If we put \(p = \omega = 0\) in (12), then the result for exponentially convex functions can be obtained for the integral operators defined in [23].
2. If we put \(p = \alpha = 0\) in (12), then Theorem 1 in [23] can be obtained.
3. If we put \((s, m) = (1, 1)\) and \(\alpha = 0\) in (12), Theorem 8 in [20] can be obtained.
4. If we put \((s, m) = (1, 1)\) and \(p = \omega = 0\) in (12), then the result for exponentially convex functions can be obtained for the integral operators defined in [23].
5. If we put \((s, m) = (1, 1)\), \(p = \omega = 0\), and \(\alpha = 0\) in (12), then the result for convex functions can be obtained for the integral operators defined in [23].
6. If we put \(\phi(t) = t^\mu, \mu > 0, \alpha = 0, p = \omega = 0\), and \((s, m) = (1, 1)\) in (12), then Proposition 10 in [20] can be obtained.
7. If we put \(\phi(t) = t^\mu, \mu > 0, g(x) = x, \alpha = 0, p = \omega = 0\), and \((s, m) = (1, 1)\) in (12), then Corollary 1 in [26] can be obtained.

**Proposition 11.** Let \(\phi(t) = t^\mu, \mu \geq 1\). Then, (12) gives the following bound for fractional integral operators defined in [6] for \(\mu \geq 1\):
\[
\left( I \phi f \right)(x) + \left( I \phi f \right)(x) \leq \frac{(g(x) - g(a))^{\mu - 1} - \mu}{\Gamma(\mu)} \left( \frac{f(x)}{e^{\alpha b}} g(b) - \frac{f(a)}{e^{\alpha b}} I_x \left( \frac{f(x)}{e^{\alpha b}} I_x \right) \right).
\]

**Proposition 12.** Let \(g(x) = I(x) = x\) and \(p = \omega = 0\). Then, (12) gives the following bound for integral operators defined in [27]:
\[
\left( I \phi f \right)(x) + \left( I \phi f \right)(x) \leq \frac{\phi(x-a)}{x-a} \left( \frac{f(x)}{e^{\alpha b}} g(b) - \frac{f(a)}{e^{\alpha b}} I_x \left( \frac{f(x)}{e^{\alpha b}} I_x \right) \right).
\]

**Corollary 13.** If we take \(\phi(t) = \Gamma(\mu) t^{\mu/k} \Gamma_{\phi}(\mu)\) and \(p = \omega = 0\), then (12) gives the following bound for fractional integral
operators defined in [17] for $\mu \geq k$:

\[
\left(\frac{\mu^k f}{\Gamma_{a\varphi}}\right)(x) + \left(\frac{\mu^k f}{\Gamma_{b\psi}}\right)(x) \leq \frac{(g(x) - g(a))^{(\mu k) - 1}}{k!} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) - \frac{\Gamma(s + 1)}{(x - a)^s} \right)
\]

\[
\cdot \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right) + \frac{(g(b) - g(x))^{(\mu k) - 1}}{k!} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right)
\]

\[
\cdot g(x) - \frac{\Gamma(s + 1)}{(x - a)^s} \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right).
\]

(24)

Corollary 14. If we take $\phi(t) = t^\varphi, p = \omega = 0$, and $g(x) = I(x) = x$, then (12) gives the following bound for Riemann-Liouville fractional integrals defined in [6] for $\mu \geq 1$:

\[
\left(\frac{\mu^k f}{\Gamma_{a\varphi}}\right)(x) + \left(\frac{\mu^k f}{\Gamma_{b\psi}}\right)(x) \leq \frac{(x - a)^{\mu k - 1}}{\Gamma(\mu)}
\]

\[
\cdot \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) - \frac{\Gamma(s + 1)}{(x - a)^s} \right)
\]

\[
\cdot \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right) + \frac{(b - x)^{\mu k - 1}}{\Gamma(\mu)} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) \right)
\]

\[
\cdot g(x) - \frac{\Gamma(s + 1)}{(b - x)^s} \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right).
\]

(25)

Corollary 15. If we take $\phi(t) = \Gamma(\mu) t^{\varphi k}/k! \Gamma_{a\varphi}(\mu)$, $p = \omega = 0$, and $g(x) = I(x) = x$, then (12) gives the following bound for fractional integral operators defined in [7] for $\mu \geq k$:

\[
\left(\frac{\mu^k f}{\Gamma_{a\varphi}}\right)(x) + \left(\frac{\mu^k f}{\Gamma_{b\psi}}\right)(x) \leq \frac{(x - a)^{(\mu k) - 1}}{k!} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) - \frac{\Gamma(s + 1)}{(x - a)^s} \right)
\]

\[
\cdot \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right) + \frac{(b - x)^{(\mu k) - 1}}{k!} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) \right)
\]

\[
\cdot g(x) - \frac{\Gamma(s + 1)}{(b - x)^s} \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right).
\]

(26)

Corollary 16. If we take $\phi(t) = t^\varphi, p = \omega = 0$, and $g(x) = x^\varphi/\rho$, $\rho > 0$, then (12) gives the following bound for fractional integral operators defined in [1]:

\[
\left(\frac{\mu^k f}{\Gamma_{a\varphi}}\right)(x) + \left(\frac{\mu^k f}{\Gamma_{b\psi}}\right)(x) \leq \frac{(x^\varphi - a^\varphi)^{\mu k - 1}}{\Gamma(\mu)}
\]

\[
\cdot \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) - \frac{\Gamma(s + 1)}{(x - a)^s} \right)
\]

\[
\cdot \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right) + \frac{(b - x)^{\mu k - 1}}{\Gamma(\mu)} \left( \frac{m f(x/m)}{e^{\alpha(x/m)}} g(x) - \frac{f(a)}{e^{\alpha a}} g(a) \right)
\]

\[
\cdot g(x) - \frac{\Gamma(s + 1)}{(b - x)^s} \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right).
\]

(27)

The following lemma is essential to prove the next result.

Lemma 17. Let $f : [0,\infty) \rightarrow \mathbb{R}$ be an exponentially $(s, m)$-convex function with $m \in (0, 1]$. If $0 \leq a < b$ and $f(x)e^{ax} = f((a + b - x)/m)e^{a((a+b-x)/m)}$, then the following inequality holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2^s} \left(1 + m\right) f(x), \quad x \in [a, b].
\]

(28)

Proof. Since $f$ is an exponentially $(s, m)$-convex, therefore, the following inequality is valid:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2^s} f(x) + \frac{mf((a + b - x)/m)}{2^s e^{a((a+b-x)/m)}}.
\]

(29)

By using $f(x)e^{ax} = f((a + b - x)/m)e^{a((a+b-x)/m)}$ in the above inequality, we get (28).

The following result provides the Hadamard inequality for unified integral operators defined in (9) and (10).

Theorem 18. Under the assumptions of Theorem 9, in addition if $f(x)e^{ax} = f((a + b - x)/m)e^{a((a+b-x)/m)}$, then we have

\[
\frac{h(\alpha) a^{2f((a + b)/2)}(1 + m)}{(1 + m)} \left( \frac{g_{\text{Hadamard}}(a, \omega; p)}{g_{\text{Hadamard}}(a, \omega; p)} \right)
\]

\[
+ \left( g_{\text{Hadamard}}(a, \omega; p) \right) \left( b, \omega; p \right) \leq \left( g_{\text{Hadamard}}(a, \omega; p) \right) \left( b, \omega; p \right)
\]

\[
+ \left( g_{\text{Hadamard}}(a, \omega; p) \right) \left( b, \omega; p \right) \leq 2k^2 \left( b, \omega; p \right) \left( E_{\text{Hadamard}}(b, \omega; p) \right)
\]

\[
\cdot f(b) \left( \frac{f(b)}{e^{\alpha b}} g(b) - \frac{f(a)}{e^{\alpha a}} \right) \left( \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right)
\]

\[
\cdot g(x) - \frac{\Gamma(s + 1)}{(b - x)^s} \left( \frac{m f(x/m)^s}{e^{\alpha(x/m)}} I_x g(x) - \frac{f(a)}{e^{\alpha a}} I_\alpha g(x) \right).
\]

(30)
Proof. Under the assumptions of $\phi$ and $g$, we can obtain:

$$
\frac{\phi(g(x) - g(a))}{g(x) - g(a)} \leq \frac{\phi(b) - g(a))}{g(b) - g(a)}.
$$

(31)

Multiplying with $E^{\gamma,k,c}_{\mu,a,l} (\omega(g(x) - g(a))^\mu ; p)g'(x)$, we can obtain:

$$
K^a_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right) g'(x) \leq \frac{\phi(b) - g(a))}{g(b) - g(a)}
\cdot E^{\gamma,k,c}_{\mu,a,l} (\omega(g(x) - g(a))^\mu ; p)g'(x).
$$

(32)

By using $E^{\gamma,k,c}_{\mu,a,l} (\omega(g(x) - g(a))^\mu ; p) \leq E^{\gamma,k,c}_{\mu,a,l} (g(b) - g(a))^\mu ; p)$, the following inequality is obtained:

$$
K^a_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right) g'(x) \leq K^a_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right) g'(x).
$$

(33)

Using exponentially $(s, m)$-convexity of $f$, we have

$$
f(x) \leq \left( \frac{x - a}{b - a} \right)^s f(b) + m \left( \frac{b - x}{b - a} \right)^s f(a/m).
$$

(34)

Multiplying (33) and (34) and integrating the resulting inequality over $[a, b]$, we can obtain

$$
\int_a^b \left[ \frac{f(b)}{b - a} \right]^s d(g(x)) \leq m \frac{f(a/m)}{e^a/m} \int_a^b \left( \frac{b - x}{b - a} \right)^s f(a/m) d(g(x)).
$$

By using Definition 4 and integrating by parts, the following inequality is obtained:

$$
\left( g^{\phi,f}_{\mu,a,l} \right) (b, \omega ; p) \leq K^\phi_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right)
\cdot \frac{f(b)}{e^b} g(b) - m \frac{f(a/m)}{e^a/m} g(a) - \frac{\Gamma(s + 1)}{(b - a)^s}
\cdot \frac{f(b)^s}{e^b} \int_0^\omega (a - m) \cdot \frac{f(a/m)^s}{e^a/m} I_{a'} g(b).
$$

(36)

Omitting (34) and (37):

$$
\left( g^{\phi,f}_{\mu,a,l} \right) (a, \omega ; p) \leq K^\phi_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right)
\cdot \frac{f(b)}{e^b} g(b) - m \frac{f(a/m)}{e^a/m} g(a) - \frac{\Gamma(s + 1)}{(b - a)^s}
\cdot \frac{f(b)^s}{e^b} \int_0^\omega (a - m) \cdot \frac{f(a/m)^s}{e^a/m} I_{a'} g(b).
$$

(38)

By adding (36) and (38), following inequality can be obtained:

$$
\left( g^{\phi,f}_{\mu,a,l} \right) (b, \omega ; p) + \left( g^{\phi,f}_{\mu,a,l} \right) (a, \omega ; p)
\leq K^\phi_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right)
\cdot \frac{f(b)}{e^b} g(b) - m \frac{f(a/m)}{e^a/m} g(a) - \frac{\Gamma(s + 1)}{(b - a)^s}
\cdot \frac{f(b)^s}{e^b} \int_0^\omega (a - m) \cdot \frac{f(a/m)^s}{e^a/m} I_{a'} g(b).
$$

(39)

Multiplying both sides of (28) by $K^a_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right) g'(x)$, and integrating over $[a, b]$ we have

$$
f \left( \frac{a + b}{2} \right) \int_a^b \frac{f(b)}{e^b} \int_0^\omega (a - m) \cdot \frac{f(a/m)^s}{e^a/m} I_{a'} g(b).
$$

(40)

From Definition 4, the following inequality is obtained:

$$
h(a)f \left( \frac{a + b}{2} \right) \frac{2^s}{(1 + m)} \left( g^{\phi,f}_{\mu,a,l} \right) (a, \omega ; p)
\leq \left( g^{\phi,f}_{\mu,a,l} \right) (b, \omega ; p).
$$

(41)

Similarly multiplying both sides of (28) by $K^a_x \left( E^{\gamma,k,c}_{\mu,a,l} \cdot g ; \phi \right) g'(x)$ and integrating over $[a, b]$, we have

$$
h(a)f \left( \frac{a + b}{2} \right) \frac{2^s}{(1 + m)} \left( g^{\phi,f}_{\mu,a,l} \right) (a, \omega ; p)
\leq \left( g^{\phi,f}_{\mu,a,l} \right) (b, \omega ; p).
$$

(42)

By adding (41) and (42) following inequality is obtained:

$$
h(a)f \left( \frac{a + b}{2} \right) \frac{2^s}{(1 + m)} \left( g^{\phi,f}_{\mu,a,l} \right) (a, \omega ; p)
\cdot \left( g^{\phi,f}_{\mu,a,l} \right) (b, \omega ; p).
$$

(43)

Using (39) and (43), inequality (30) can be achieved.

Remark 19.
If we put $p = \omega = 0$ in (30), then result for exponentially $(s, m)$-convex functions can be obtained for the integral operators defined in [23].

If we put $(s, m) = (1, 1)$ and $\alpha = 0$ in (30), then Theorem 22 in [20] can be obtained.

If we put $(s, m) = (1, 1)$ and $p = \omega = 0$ in (30), then the result for exponentially convex functions can be obtained for the integral operators defined in [23].

If we put $p = \omega = \alpha = 0$ in (30), then Theorem 3 in [23] can be obtained.

If we put $(s, m) = (1, 1)$ and $p = \omega = \alpha = 0$ in (30), then the result for convex functions can be obtained for the integral operators defined in [23].

**Corollary 20.** If we put $\phi(t) = \Gamma(\mu) t^{\mu k}/\Gamma_k(\mu)$ and $p = \omega = 0$, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [17]:

$$\frac{h(a)2^{\mu k} f((a + b)/2)}{\Gamma_k(\mu)(m + 1)} (g(b) - g(a))^\mu \leq \left( \frac{\mu k}{\mu} I_b f(a) + \mu I_a f(b) \right).$$

**Corollary 21.** If we put $\phi(t) = t^\alpha$ and $p = \omega = 0$, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [6]:

$$\frac{h(a)2^{\mu k} f((a + b)/2)}{\mu \Gamma_k(\mu)(m + 1)} (g(b) - g(a))^\mu \leq \left( \frac{\mu k}{\mu} b I_b f(a) + \mu I_a f(b) \right).$$

**Corollary 22.** If we put $\phi(t) = \Gamma(\mu) t^{\mu k}/\Gamma_k(\mu)$, $p = \omega = 0$ and take $g$ as identity function, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [7]:

$$\frac{h(a)2^{\mu k} f((a + b)/2)}{\mu \Gamma_k(\mu)(m + 1)} (b - a)^\mu \leq \left( \frac{\mu k}{\mu} f(a) + \mu I_a f(b) \right).$$

**Corollary 23.** If we put $\phi(t) = t^\mu$, $p = \omega = 0$ and take $g$ as the identity function, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [6]:

$$\frac{h(a)2^{\mu k} f((a + b)/2)}{\mu \Gamma_k(\mu)(m + 1)} (g(b) - g(a))^\mu \leq \left( \frac{\mu k}{\mu} I_b f(a) + \mu I_a f(b) \right).$$

**Theorem 24.** Let $f : I \rightarrow \mathbb{R}$ be a differentiable function. If $| f' |$ is exponentially $(s, m)$-convex with $m \in (0, 1]$ and $g : I \rightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\phi(x)$ be an increasing function on $I$ then for $a, b \in I$, $a < b$. If $a, b, c \in \mathbb{C}$, $p, \mu \geq 0$, $\delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, we have

$$\left( g^\phi_{\mu a, l} f \ast g \right)(x, \omega; p) + \left( g^\phi_{\mu b, l} f \ast g \right)(x, \omega; p) \leq K^a \left( g^\phi_{\mu a, l} f \ast g \right)(x, \omega; p) + K^b \left( g^\phi_{\mu b, l} f \ast g \right)(x, \omega; p).$$

where

$$\left( g^\phi_{\mu a, l} f \ast g \right)(x, \omega; p) = \int_a^x K^a \left( g^\phi_{\mu a, l} f \ast g \right)(t) dt(g(t)),$n

$$\left( g^\phi_{\mu b, l} f \ast g \right)(x, \omega; p) = \int_a^x K^b \left( g^\phi_{\mu b, l} f \ast g \right)(t) dt(g(t)).$$

**Proof.** Using exponentially $(s, m)$-convexity of $| f' |$, we have

$$| f' (t) | \leq \left| \frac{x - t}{x - a} \right|^s | f'(a) | \frac{c^m}{c^{\xi(m)}} + m \left( \frac{t - a}{x - a} \right)^s | f'(x/m) | \frac{c^m}{c^{\xi(m)}}.$$
The inequality (50) can be written as follows:

\[-\left(\frac{x-t}{x-a}\right)^{s} |f'(a)| + m \left(\frac{t-a}{x-a}\right)^{s} \left|\frac{f'(x/m)}{a^{n(x/m)}}\right| \leq f'(t) \leq \left(\frac{x-t}{x-a}\right)^{s} |f'(a)| + m \left(\frac{t-a}{x-a}\right)^{s} \left|\frac{f'(x/m)}{a^{n(x/m)}}\right|.\]  

(51)

Let us consider the second inequality of (51)

\[f'(t) \leq \left(\frac{x-t}{b-x}\right)^{s} |f'(a)| + m \left(\frac{t-a}{b-x}\right)^{s} \left|\frac{f'(x/m)}{a^{n(x/m)}}\right|.\]  

(52)

Multiplying (15) and (52) and integrating over \([a, x] \), we can obtain:

\[\int_{a}^{x} K_{s}^{\rho,\delta,k,c}(g, \phi) f(t) d(g(t)) \leq \left(\frac{f'(a)}{a^{n(a)}} - K_{s}^{\rho,\delta,k,c}(g, \phi) \right) \int_{a}^{x} \left(\frac{x-t}{x-a}\right)^{s} d(g(t)) + m \left(\frac{t-a}{x-a}\right)^{s} \left|\frac{f'(x/m)}{a^{n(x/m)}}\right| \int_{a}^{x} \left(\frac{t-a}{x-a}\right)^{s} d(g(t)).\]  

(53)

By using (9) of Definition 4 and integrating by parts, the following inequality is obtained:

\[\left(\int g_{\rho,\mu,l,a}^{\psi,k,c}(f, x, \omega ; p) \right) \leq K_{s}^{\rho,\delta,k,c}(g, \phi) \times \left(\frac{f'(b)}{a^{n(b)}} g(b) - m \left(\frac{f'(x/m)}{a^{n(x/m)}}\right) g(x) - \frac{f'(a)}{a^{n(a)}} I_{x} g(x) - \frac{f'(b)}{a^{n(b)}} I_{x} g(b)\right).\]  

(54)

Now using exponentially \((s, m)\)-convexity of \(|f'|\), we have

\[\left|\int g_{\rho,\mu,l,b}^{\psi,k,c}(f, x, \omega ; p) \right| \leq K_{s}^{\rho,\delta,k,c}(g, \phi) \times \left(\frac{f'(b)}{a^{n(b)}} g(b) - m \left(\frac{f'(x/m)}{a^{n(x/m)}}\right) g(x) - \frac{f'(a)}{a^{n(a)}} I_{x} g(x) - \frac{f'(b)}{a^{n(b)}} I_{x} g(b)\right).\]  

(55)

From (54) and (55), the following inequality is observed:

\[\left|\int g_{\rho,\mu,l,a}^{\psi,k,c}(f, x, \omega ; p) \right| \leq K_{s}^{\rho,\delta,k,c}(g, \phi) \times \left(\frac{f'(b)}{a^{n(b)}} g(b) - m \left(\frac{f'(x/m)}{a^{n(x/m)}}\right) g(x) - \frac{f'(a)}{a^{n(a)}} I_{x} g(x) - \frac{f'(b)}{a^{n(b)}} I_{x} g(b)\right).\]  

(56)

Now using exponentially \((s, m)\)-convexity of \(|f'|\), we have

\[|f'(t)| \leq \left(\frac{x-t}{b-x}\right)^{s} |f'(a)| + m \left(\frac{t-a}{b-x}\right)^{s} \left|\frac{f'(x/m)}{a^{n(x/m)}}\right|.\]  

(57)

On the same pattern as we did for (15) and (50), one can obtain the following inequality from (19) and (57):

\[\left|\int g_{\rho,\mu,l,b}^{\psi,k,c}(f, x, \omega ; p) \right| \leq K_{s}^{\rho,\delta,k,c}(g, \phi) \times \left(\frac{f'(b)}{a^{n(b)}} g(b) - m \left(\frac{f'(x/m)}{a^{n(x/m)}}\right) g(x) - \frac{f'(a)}{a^{n(a)}} I_{x} g(x) - \frac{f'(b)}{a^{n(b)}} I_{x} g(b)\right).\]  

(58)

By adding (56) and (58), inequality (48) can be achieved.

Remark 25.

(1) If we put \(\alpha = 0\) in (48), then Theorem 2 in [23] can be obtained

(2) If we put \((s, m) = (1, 1)\) in (48), then the result for exponentially convex functions can be obtained for the integral operators defined in [23]

(3) If we put \(\rho = \omega = 0\) in (48), then the result for exponentially \((s, m)\)-convex functions can be obtained for the integral operators defined in [23]

(4) If we put \((s, m) = (1, 1)\) and \(\alpha = 0\) in (48), then Theorem 25 in [20] can be obtained

3. Boundedness and Continuity

Theorem 26. Under the assumptions of Theorem 9, the following inequality holds for \(m\)-convex functions:

\[\left(\int g_{\rho,\mu,l,a}^{\psi,k,c}(f, x, \omega ; p) + \int g_{\rho,\mu,l,b}^{\psi,k,c}(f, x, \omega ; p) \right) \leq K_{s}^{\rho,\delta,k,c}(g, \phi) \left(\frac{f'(b)}{a^{n(b)}} g(b) - m \left(\frac{f'(x/m)}{a^{n(x/m)}}\right) g(x) - \frac{f'(a)}{a^{n(a)}} I_{x} g(x) - \frac{f'(b)}{a^{n(b)}} I_{x} g(b)\right).\]  

(59)
Proof. If we put $s = 1$ and $\alpha = 0$ in (17), we have
\[
\int_a^b K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) f(t) d(g(t)) \leq f(a) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \int_a^b \left( \frac{x-t}{x-a} \right) d(g(t)) + m\left( \frac{x}{m} \right) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \int_a^x \left( \frac{t-a}{x-a} \right) d(g(t)).
\]
(60)

By using Definition 4 and integrating by parts, the following inequality is obtained:
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) (m\left( \frac{x}{m} \right) x - g(a))
\]
\[
\times \left( mf \left( \frac{x}{m} \right) + f(a) \right).
\]
(61)

which can be written as
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) (m\left( \frac{x}{m} \right) x - g(a))
\]
\[
\times \left( mf \left( \frac{x}{m} \right) + f(a) \right).
\]
(62)

Similarly from (21), the following inequality holds:
\[
\int_a^b K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) f(t) d(g(t)) \leq f(b) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \int_a^b \left( \frac{t-x}{b-x} \right) d(g(t)) + m\left( \frac{x}{m} \right) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \int_a^b \left( \frac{b-t}{b-x} \right) d(g(t)).
\]
(63)

which further simplifies as follows:
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \left( g(b) - g(x) \right) \left( mf \left( \frac{x}{m} \right) + f(b) \right).
\]
(64)

From (62) and (64), (59) can be obtained.

Corollary 27. If we take $m = 1$ in Theorem 26, then the following inequality holds for convex functions:
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) + \left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p)
\]
\[
\leq K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) (g(x) - g(a)) (f(x) + f(a))
\]
\[
+ K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right) (g(b) - g(x)) (f(x) + f(b)).
\]
(65)

Theorem 28. With assumptions of Theorem 26, if $f \in L_\infty[a, b]$, then unified integral operators for $m$-convex functions are bounded and continuous.

Proof. From (62), we have
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)
\]
\[
\times \left( g(b) - g(a) \right) (m+1) f\|_{\infty},
\]
(66)

which further gives
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K f\|_{\infty},
\]
(67)

where $K = (g(b) - g(a)) (m+1) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)$.

Similarly, from (64), the following inequality holds:
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K f\|_{\infty},
\]
(68)

The boundedness is established; also, they are linear therefore the continuity is obtained.

Corollary 29. If we take $m = 1$ in Theorem 28, then unified integral operators for convex functions are bounded and continuous and the following inequalities hold:
\[
\left( g \phi \right)^{E_\gamma^{\delta,k_c} f} (x, \omega ; p) \leq K f\|_{\infty},
\]
(69)

where $K = 2(g(b) - g(a)) K_x^s \left( E_\gamma^{\delta,k_c} \mu_{\lambda,d} \cdot g ; \phi \right)$.

4. Concluding Remarks

The class of exponentially $(s, m)$-convex functions contains $(s, m)$-convex, exponentially convex, $s$-convex, $m$-convex, and convex functions. The unified integral operators defined in (9) and (10) produce almost all fractional and conformable integral operators which have been defined independently in recent decades. We have established the bounds of sum of unified integral operators of exponentially $(s, m)$-convex functions. Also a Hadamard inequality for these operators is established. In conclusion, the presented results reproduce plenty of results for operators composed in Remarks 5 and 6 for functions deduced in Remark 8.

Data Availability

There is no additional data required for the finding of results of this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
References

[1] H. Chen and U. N. Katugampola, "Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for generalized fractional integrals," Journal of Mathematical Analysis and Applications, vol. 446, no. 2, pp. 1274–1291, 2017.

[2] S. S. Dragomir, "Inequalities of Jensen’s type for generalized $g$-fractional integrals of functions for which the composite $f \circ g$ is convex," Fractional Differential Calculus, vol. 8, no. 1, pp. 127–150, 2018.

[3] S. Habib, S. Mubeen, and M. N. Naeem, "Chebyshev type integral inequalities for generalized $k$-fractional conformable integrals," Journal of Inequalities and Special Functions, vol. 9, no. 4, pp. 53–65, 2018.

[4] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," Advances in Difference Equations, vol. 2017, no. 1, 2017.

[5] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," Journal of Computational and Applied Mathematics, vol. 346, pp. 378–389, 2019.

[6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Elsevier, New York-London, 2006.

[7] S. Mubeen and G. M. Habibullah, "$k$-fractional integrals and applications," International Journal of Contemporary Mathematical, vol. 7, no. 2, pp. 89–94, 2012.

[8] T. R. Parbhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," Yokohama Mathematical Journal, vol. 19, pp. 7–15, 1971.

[9] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen, and M. Arshad, "The extended Mittag-Leffler function via fractional calculus," Journal of Nonlinear Sciences and Applications, vol. 10, pp. 4244–4253, 2013.

[10] T. O. Salim and A. W. Faraj, "A generalization of Mittag-Leffler function and integral operator associated with integral calculus," Journal of Fractional Calculus and Applied Analysis, vol. 3, no. 5, pp. 1–13, 2012.

[11] M. Z. Sarikaya, Z. Dahmani, and M. E. Kiris, "$(k,s)$-Riemann-Liouville fractional integral and applications," Hacetettepe Journal of Mathematics and Statistics, vol. 1, no. 45, pp. 77–89, 2016.

[12] H. M. Srivastava and Z. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel," Applied Mathematics and Computation, vol. 211, no. 1, pp. 198–210, 2009.

[13] T. Tunc, H. Budak, F. Usta, and M. Z. Sarikaya, "On new generalized fractional integral operators and related fractional inequalities," https://www.researchgate.net/publication/313650587.

[14] D. Uçar, V. F. Hatipoğlu, and A. Akinçali, "Fractional integral inequalities on time scales," Open Journal of Mathematical Sciences, vol. 2, no. 1, pp. 361–370, 2018.

[15] I. A. Bloch and I. İşcan, "Integral inequalities for differentiable harmonically $(s,m)$-preinvex functions," Open Journal of Mathematical Analysis, vol. 1, no. 1, pp. 25–33, 2017.

[16] S. Kermausuor, "Simpson’s type inequalities for strongly $(s,m)$-convex functions in the second sense and applications," Open Journal of Mathematical Analysis, vol. 3, no. 1, pp. 74–83, 2019.

[17] Y. C. Kwon, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville $k$-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," IEEE Access, vol. 6, pp. 64946–64953, 2018.

[18] S. Mubeen and A. Rehman, "A note on k-Gamma function and Pochhammer k-symbol," Journal of Mathematical Sciences, vol. 6, no. 2, pp. 93–107, 2014.

[19] M. Andrić, G. Farid, and J. Pečarić, "A further extension of Mittag-Leffler function," Fractional Calculus and Applied Analysis, vol. 21, no. 5, pp. 1377–1395, 2018.

[20] Y. C. Kwon, G. Farid, S. Ullah, W. Nazeer, K. Mahreem, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," IEEE Access, vol. 7, pp. 126283–126292, 2019.

[21] G. Farid, "Existence of an integral operator and its consequences in fractional and conformable integrals," Open Journal of Mathematical Sciences, vol. 3, no. 1, pp. 210–216, 2019.

[22] G. Farid, A. U. Rehman, and Q. U. Ain, "$k$-fractional integral inequalities of Hadamard type for $(h - m)$-convex functions," Computational Methods for Differential Equations, vol. 8, no. 1, pp. 119–140, 2020.

[23] Y. C. Kwon, G. Farid, S. M. Kang, B. K. Bangash, and S. Ullah, "Derivation of bounds of several kinds of operators via $(s,m)$-convexity," Advances in Difference Equations, vol. 2020, no. 1, 2020.

[24] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, "Generalized fractional integral inequalities for exponentially $(s,m)$-convex functions," Journal of Inequalities and Applications, vol. 2020, no. 1, 2020.

[25] G. Farid, W. Nazeer, M. Saleem, S. Mehmood, and S. Kang, "Bounds of Riemann–Liouville fractional integrals in general form via convex functions and their applications," Mathematics, vol. 6, no. 11, p. 248, 2018.

[26] G. Farid, "Some Riemann – Liouville fractional integral inequalities for convex functions," The Journal of Analysis, vol. 27, no. 4, pp. 1095–1102, 2019.

[27] M. Z. Sarikaya and F. Ertuğrul, "On the generalized Hermite-Hadamard inequalities," https://www.researchgate.net/publication/321760443.