GENERALIZED REDUCED-RANK DECOMPOSITIONS USING SWITCHING AND ADAPTIVE ALGORITHMS FOR SPACE-TIME ADAPTIVE PROCESSING

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ABSTRACT

This work presents generalized low-rank signal decompositions with the aid of switching techniques and adaptive algorithms, which do not require eigen-decompositions, for space-time adaptive processing. A generalized scheme is proposed to compute low-rank signal decompositions by imposing suitable constraints on the filtering and by performing iterations between the computed subspace and the low-rank filter. An alternating optimization strategy based on recursive least squares algorithms is presented along with switching and iterations to cost-effectively compute the bases of the decomposition and the low-rank filter. An application to space-time interference suppression in DS-CDMA systems is considered. Simulations show that the proposed scheme and algorithms obtain significant gains in performance over previously reported low-rank schemes.

Index Terms— Low-rank modelling, adaptive algorithms, alternating optimization, switched systems, interference suppression.

1. INTRODUCTION

Low-rank signal processing is an area that is central for dealing with high-dimensional data, low-sample support situations and large optimization problems that has gained considerable attention in the last decades [1][2]. The origins of low-rank modelling and signal processing lie in the problem of feature selection encountered in statistical signal processing, which refers to a dimensionality reduction process whereby a data space is transformed into a feature space [2]. The fundamental idea is to devise a decomposition that performs dimensionality reduction so that the data vector can be represented by a reduced number of effective features and yet retain most of the intrinsic information content of the input data [2]. The goal is to find an appropriate trade-off between model bias and variance in a cost-effective way, yielding a reconstruction error as small as desired.

Prior work has shown that low-rank adaptive filters [3][4] are cost-effective techniques for modelling a number of practical problems in acoustics, communications, radar and sonar, and for dealing with large filters and situations of short data records. Several low-rank adaptive filtering methods have been proposed in the last decade or so [3][4]. Among these techniques are eigen-decomposition techniques [4], the multistage Wiener filter (MSWF) [4], the auxiliary vector filtering (AVF) algorithm [5], the interpolated reduced-rank filters [4], the reduced-rank filters based on joint and iterative optimization (JIO) [8] and joint iterative interpolation, decimation, and filtering (JIDF) [9]. Key problems with previously reported low-rank adaptive schemes are the modelling of certain low-rank signals and the design of multichannel processing schemes [3][4]. Low-rank signals that exhibit highly correlated statistical features and operate in the presence of high-power signals (eg. jamming signals) constitute a challenge for existing methods. Moreover, most available methods require either separate structures for multichannel processing [9] or exhibit high complexity [5][4].

In this work, a generalized scheme is devised to compute low-rank signal decompositions with switching techniques and adaptive algorithms, without the need for eigen-decompositions. The proposed generalized low-rank decomposition with switching (GLRDS) scheme computes the subspace and the low-rank filter that best match the problem of interest with very fast convergence speed and low complexity. By imposing constraints on the decomposition and performing iterations between the computed subspace and the low-rank filter, the proposed GLRDS scheme obtains smaller reconstruction errors than existing methods. In order to compute the parameters required in the signal decomposition and the low-rank filter, an alternating optimization strategy based on recursive least squares (RLS) algorithms is presented along with switching and iterations to cost-effectively compute them. Unlike existing schemes, the GLRDS efficiently lends itself to multichannel processing. An application to space-time interference suppression in DS-CDMA systems is considered. Simulations show that the GLRDS scheme and algorithms obtain significant gains in performance over existing schemes.

The paper is organized as follows. Section 2 formulates the problem. Section 3 presents the proposed GLRDS scheme and least squares (LS) design. Section 4 presents the alternating optimization strategy along with recursive algorithms. Section 5 presents and discusses the simulation results and Section 6 draws the conclusions.

2. PROBLEM STATEMENT

In this section, the fundamental ideas of low-rank signal processing are presented. The main design problems for a decomposition that performs dimensionality reduction are discussed. An approach based on linear algebra and a linear signal model, which is sufficiently general to account for numerous applications and topics, is adopted. Consider the following linear signal model at time instant $i$ with $M$ samples organized in a vector as given by

$$ r[i] = H s[i] + n[i], \quad i = 1, 2, \ldots, P $$

(1)

where $r[i]$ is the $M \times 1$ observed signal vector with the $M$ samples to be processed, $H$ is the $M \times M$ matrix that describes the mixing nature of the model, $s[i]$ is the $M \times 1$ signal vector that is generated by a given source, $n[i]$ is an $M \times 1$ vector of noise samples, and $P$ is the number of observed signal vectors or the data record size.

Fig. 1. Block diagram with the stages of low-rank signal processing.
In low-rank signal processing, the main idea is to process the observed signal $r[i]$ in two stages, as illustrated in Fig. 1. The first stage corresponds to the dimensionality reduction, whereas the second is responsible for the signal processing in a lower-dimensional subspace. The dimensionality reduction is performed by a mapping represented by a decomposition matrix $S_D = [s_1 \ldots s_d \ldots s_D]$ with dimensions $M \times D$, where $D$ is the rank ($D < M$) that projects $r[i]$ onto a $D \times 1$ reduced-dimension data vector $r_D[i]$ and $s_d$ is the $d$th column of $S_D$. This relationship is expressed by

$$r_D[i] = S_D^H r[i] = \sum_{d=1}^D s_d^H r[i] q_d,$$

where $q_d$ is a $D \times 1$ vector with a one in the $d$th position and zeros elsewhere. $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian transpose, respectively. Key design criteria for the matrix $S_D$ and the dimensionality reduction are the reconstruction error, the computational complexity and the compression ratio $CR = M/D$. These parameters usually depend on the application and the design requirements.

After the dimensionality reduction, an algorithm is used to perform the signal processing task on the reduced-dimension observed vector $r_D[i]$ according to the designer’s aims. The resulting scheme with $D$ elements shall benefit from a reduced number of parameters, which may lead to lower complexity, smaller requirements for storage, faster convergence and superior tracking capability. In the case of filtering by a $D \times K$ matrix $W = [w_{D,1} w_{D,2} \ldots w_{D,1}]$, we have the following output $K \times 1$ vector estimate

$$\hat{x}[i] = W_D^H S_D^H r[i] = W_D^H \sum_{d=1}^D s_d^H r[i] q_d$$

$$= \sum_{k=1}^K w_{D,k}^H \left( \sum_{d=1}^D s_d^H r[i] q_d \right) q_k.$$  

We consider low-rank algorithms with the aid of linear design techniques. In order to process $r[i]$ with low-rank techniques, we need to solve the mean-square error (MSE)-based optimization problem

$$[S_D, W_{D,\text{opt}}] = \arg\min_{S_D, W} E[\|x[i] - W_D S_D^H r[i]\|^2],$$

where $x[i]$ is the desired signal and $E[\cdot]$ stands for the expected value operator. The optimal solution $W_{D,\text{opt}}$ of the problem in (3) is obtained by fixing $S_D$, taking the gradient terms of the argument with respect to $W_D$ and equating them to a zero matrix which yields

$$W_{D,\text{opt}} = R^{-1} P = (S_D^H R S_D)^{-1} S_D^H P,$$

where $R = E[r[i]r^H[i]] = S_D^H R S_D$ is the $D \times D$ low-rank correlation matrix, $R = E[r[i]r^H[i]]$ is the $M \times M$ full-rank correlation matrix, $P = E[x[i]x^H[i]] = S_D^H P S_D$ is the $D \times K$ cross-correlation matrix of the low-rank model. The associated MMSE for a rank-$D$ matrix filter is expressed by

$$\text{MMSE} = \sigma^2_N - \text{tr} \left[ P^H S_D (S_D^H R S_D)^{-1} S_D^H P \right],$$

where $\sigma^2_N = E[|x[i]|^2|x[i]|^2] + \text{tr}[\cdot]$ stands for trace. The optimal solution $S_{D,\text{opt}}$ of the problem in (6) is obtained by fixing $w_{D}[i]$, taking the gradient terms of the associated MMSE in (6) with respect to $S_D$, and equating them to a zero matrix. Considering the eigen-decomposition of $R = \Phi \Lambda \Phi^H$, where $\Phi$ is an $M \times M$ unitary matrix with the eigenvectors of $R$ and $\Lambda$ is an $M \times M$ diagonal matrix with the eigenvalues of $R$ in decreasing order, we have

$$S_{D,\text{opt}} = \Phi_{1:M,1:D},$$

where $\Phi_{1:M,1:D}$ is a $M \times D$ unitary matrix that corresponds to the signal subspace and contains the $D$ eigenvectors associated with the $D$ largest eigenvalues of the unitary matrix $\Phi$. In our notation, the subscript represents the number of components in each dimension. For example, the $M \times D$ matrix $\Phi_{1:M,1:D}$ contains the $D$ first columns of $\Phi$, where each column has $M$ elements.

The previous development suggests that the central element for constructing low-rank techniques is the design of $S_D$ since the MMSE in (3) depends on $P$, $R$ and $S_D$. The quantities $P$ and $R$ are common to both low-rank and full-rank designs, however, the matrix $S_D$ plays a key role in the dimensionality reduction and in the performance. In what follows, a cost-effective scheme for computing $S_D, W_{D,\text{opt}}$ and the remaining statistical quantities is presented.

### 3. PROPOSED GLR DESIGN AND LS DESIGN

The idea of the proposed GLRDS scheme, shown in Fig. 2, is to introduce constraints in the decomposition matrix $S_D$ to substantially reduce the number of parameters for filtering. Since this usually affects the reconstruction error of the algorithm, a switching mechanism is incorporated to provide the $S_D$ matrix with alternative bases. Similar ideas have been reported in the literature of automatic control and, more specifically, in the area of switched control techniques [11][12]. Consider the following vector estimate:

$$\hat{x}_b[i] = W_{D,b}^H S_{D,b}^H [i] r[i]$$

$$= W_{D,b}^H [i] \left( \frac{\sum_{d=1}^D q_d d_{d,b}^H C_{d,b}[i]}{\sum_{d=1}^D q_d d_{d,b}^H C_{d,b}[i]} \right) r[i]$$

$$= W_{D,b}^H [i] \left( \frac{\sum_{d=1}^D q_d d_{d,b}^H C_{d,b}[i]}{\sum_{d=1}^D q_d d_{d,b}^H C_{d,b}[i]} \right) s_{d,b}[i],$$

where $d_{d,b}[i]$ is the $M \times 1$ shaping vector employed to mould the $d$th column of the matrix $S_{D,b}[i]$ which is expressed as

$$d_{d,b}[i] = [0 \ldots 0 1 0 \ldots 0]^{T},$$

and $b = 1, \ldots, B$ is the index of parallel switching branches. The quantity $\gamma$ is the number of zeros chosen according to a given design criterion. In this work, we use patterns created by $\gamma = \gamma = (j - 1)(M/D) + (b - 1)$ due to their simplicity and satisfactory performance. These patterns lead to $I_M \times 1$ basis vectors $s_{d,b}[i]$. The $M \times I_d$ matrices $C_{s,b}[i]$ and $C_r[i]$ are Hankel matrices with
shifted versions of \( s_{d,b}[i] \) and \( r[i] \) described by

\[
C_r[i] = \begin{bmatrix}
  r_0[i] & r_1[i] & \ldots & r_i[i] \\
  r_1[i] & r_2[i] & \ldots & r_i[i] \\
  \vdots & \vdots & \ddots & \vdots \\
  r_M[i] & r_{M-1}[i] & \ldots & 0 \\
  r_{M-1}[i] & 0 & \ldots & 0
\end{bmatrix}.
\] (10)

A similar structure to the above can be obtained for \( s_{d,b}[i] \). In order to design the GLRDS scheme, the parameters \( I_d, D \) and \( B \) must be chosen, and the filters \( s_{d,b}[i] \) and \( W_D[i] \) need to be computed by solving the optimization problem

\[
\begin{align}
\arg & \min_{s_{d,b}[i],W_D[i]} \sum_{i=1}^{t} \lambda^{-t-i} ||x[i] - \hat{x}[i]||^2, \\
& d = 1, \ldots, D, \quad b = 1, \ldots, B.
\end{align}
\] (11)

Fixing \( W_D[i] \) and solving the problem for \( s_{d,b}[i] \), we obtain

\[
s_{d,b}[i] = R_{d,b}^{-1}[i] (p_{d,b}[i] s_{d,b}[i]),
\] (12)

where

\[
R_{d,b} = \sum_{i=1}^{t} \lambda^{-t-i} q_d[i] W_D[i] W_d^{-H}[i] q_d[i] C_d^{-T}[i] d_{d,b} d_{d,b}^H C_d^{-1}[i]
\]

and

\[
p_{d,b}[i] = \sum_{i=1}^{t} \lambda^{-t-i} q_d[i] W_D[i] W_d^{-H}[i] q_d[i] C_d^{-T}[i] x[i] d_{d,b} d_{d,b}^H C_d^{-1}[i].
\]

Once the \( s_{d,b}[i] \) are computed, we can build the corresponding decomposition matrix \( S_{D,b}[i] \). According to the schematic in Fig. 2, a selection is performed among the \( B \) available matrices as follows

\[
S_{D,b}[i] = S_{D,b}[i] \quad \text{when } b_a = \arg \min_{1 \leq b \leq B} \| x[i] - \hat{x}[i] \|^2,
\]

Now fixing \( s_{d,b}[i] \) and solving the problem for \( W_D[i] \), we have

\[
W_D[i+1] = R^{-1}[i] P_D[i].
\] (14)

5. SIMULATIONS

The performance of the GLRDS scheme and RALS algorithms is assessed via simulations for space-time interference suppression. We consider the uplink of a DS-CDMA system with symbol interval \( T_c \), chip period \( T_c \), QPSK modulation, spreading gain \( N = T/T_c \), \( K \) users, and equipped with a uniform antenna array with \( J \) elements. The antenna elements are \( d = \lambda_c/2 \), where \( \lambda_c \) is the carrier wavelength. Assuming that the channel is constant during each symbol and the receiver is synchronized with the main path, the received signal after filtering by a chip-pulse matched filter and sampled at chip rate yields the \( M \times 1 \) received vector

\[
r[i] = \sum_{k=1}^{K} A_k x_k[i] p_{d,k}[i] + \eta[i] + j[i] + n[i],
\] (24)

where \( M = J(N + L_p - 1) \), the complex Gaussian noise vector is \( n[i] = [n_1[i], \ldots, n_M[i]]^T \) with \( E[n[i]] n_H[i] = \sigma_n^2 I_L \). The \( J L_p \times 1 \) channel vector \( h_k[i] \) contains the complex gains of the channel from user \( k \) to each antenna element. The \( M \times 1 \) spatial signature for user \( k \) is \( p_{d,k}[i] = F_k h_k[i] \), where \( F_k \) is an \( M \times J L_p \) matrix with shifted versions of the signature sequence \( s_k = [a_k(1) \ldots a_k(N)]^T \) of user \( k \) that performs convolution of the channel \( h_k[i] \) with \( s_k \). The signatures are randomly generated with \( N = 16 \). For the simulations, we use the initial values \( W_D[1] = [I, 0_{K \times (K - 1)}]^T \) and \( S_D[0] = [I_0, 0_{D \times (M-D)}]^T \), assume \( L = 9 \) as an upper bound, use 3-path channels with relative powers given by 0, −3 and −6 dB,
where in each run the spacing between paths is obtained from a discrete uniform random variable between 1 and 2 chips and average the experiments over 200 runs. The power and the phase of each path is time-varying and follows Clarke’s model [13]. The system has a power distribution among the users for each run that follows a log-normal distribution with associated standard deviation equal to 1.5 dB and there is a sinusoidal jamming signal $f[i]$ with a power level 20 dB above the average signal-to-noise ratio (SNR) of the users, which are jointly demodulated.

We compare the proposed GLRDS scheme and RALS with the Full-rank RLS [1], the eigen-decomposition (EIG) [3], the MSWF [4], the AVF [5], the JIO [8] and JIDF [9] techniques. We consider the MSE performance versus the rank $D$ space-time receivers process $r[i]$ with $M = 75$ samples per symbol. The results in Fig. 3 show that the best rank for the GLRDS scheme is $D = 4$ (used in the next experiments) and that it is very close to the optimal MMSE. The results in Fig. 4 show that the best rank for the GLRDS scheme is $D = 4$ and it is very close to the optimal MMSE. The number of elements $I_d$ required to construct the decompositions is often small (a uniform $I_d = 3$ for $d = 1, \ldots, D$ is used here throughout) and a designer can employ non-uniform lengths according the low-rank modelling needs. The number of iterations $T$ is also typically small and allows the RALS to converge faster. Our studies with systems with different sizes suggest that $D$ is relatively invariant to the system size, which brings considerable computational savings to the GLRDS scheme and allows a suitable low-rank modelling and a very fast convergence performance. In practice, the rank $D$ can be adapted in order to account for time-varying scenarios and models, ensuring good performance and tracking after convergence.

In the next experiment we evaluate the average BER performance against the number of received symbols for the GLRDS and RALS, and the existing schemes and algorithms, as depicted in Fig. 4. The packet size is $P = 1500$ symbols and the adaptive filters are trained with 200 symbols and then are switched to decision-directed mode to continue the adaptation. The results show that the proposed GLRDS scheme has a much better performance than the existing approaches and is able to adequately track the desired signal. A stability and convergence analysis of the proposed scheme based on control-theoretic arguments [11][12], including tracking and steady-state performance, conditions and proofs are not included here due to lack of space and are intended for a future paper.

![Fig. 3. MSE performance versus rank (D). Parameters: $\lambda = 0.999$, $P_D[0] = 0.011$, $P_d[0] = 0.011$.](image)

![Fig. 4. BER performance versus number of received symbols. Parameters: $\lambda = 0.999$, $P_D[0] = 0.011$, $P_d[0] = 0.011$.](image)

6. CONCLUSIONS

This work has proposed the GLRDS scheme and the RALS algorithms for performing low-rank adaptive filtering, and has considered their application to space-time interference suppression in DS-CDMA systems. The GLRDS scheme provides a way for computing generalized low-rank signal decompositions with switching techniques and adaptive algorithms, which does not require eigen-decompositions. The results of simulations show that the proposed GLRDS scheme and the RALS algorithms obtain quite significant gains in performance over previously reported low-rank schemes.

7. REFERENCES

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