Exponential bounds for inhomogeneous random graphs in a Gaussian case

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Abstract

Rank-1 inhomogeneous random graphs are a natural generalization of Erdős-Rényi random graphs. In this generalization each node is given a weight. Then the probability that an edge is present depends on the product of the weights of the nodes it is connecting. In this paper, we give precise and uniform exponential bounds on the size, weight and surplus of rank-1 inhomogeneous random graphs where the weight of the nodes behave like a random variable with finite third moments. We focus on the case where the mean degree of a random node is equal to 1 (critical regime), or slightly larger than 1 (barely supercritical regime). These bounds will be used in follow up papers to study a general class of random minimum spanning trees. They are also of independent interest since they show that these inhomogeneous random graphs behave like Erdős-Rényi random graphs even in a barely supercritical regime. The proof relies on novel concentration bounds for sampling without replacement and a careful study of the exploration process.

Index terms — Random, Graphs, Inhomogeneous, Networks
1 Introduction

1.1 The model

Consider \( n \in \mathbb{N} \) vertices labeled 1, 2, ..., \( n \). For a vector of weights \( \mathbf{W} = (w_1, w_2, ..., w_n) \), where \( 0 < w_n \leq w_{n-1} \leq ... \leq w_1 \), we create the inhomogeneous random graph associated to \( \mathbf{W} \) and to \( p \leq +\infty \) in the following way:

Each potential edge \( \{i, j\} \) is in the graph with probability \( 1 - e^{-w_i w_j p} \) independently from everything else. This gives a random graph that we call the rank-1 inhomogeneous random graph associated to \( \mathbf{W} \) and \( p \leq +\infty \).

One can couple the graphs for the different values of \( p \) as follow: Let \( K_n \) be the complete graph of size \( n \). To every potential edge \( \{i, j\} \), associate independently the random capacity \( E_{\{i, j\}} \) which is an exponential random variable of rate \( w_i w_j \). The weights are then used to create a sequence of graphs. For each \( p \in [0, +\infty] \) let \( G(\mathbf{W}, p) \) be the graph on \( \{1, 2, ..., n\} \) containing the edges of weight at most \( p \). So the edge set of \( G(\mathbf{W}, p) \) is:

\[
\{ \{i, j\} | E_{\{i, j\}} \leq p \}.
\]

Then \( (G(\mathbf{W}, p))_{p \in [0, +\infty]} \) is an increasing sequence of graphs for inclusion, and for each fixed value of \( p \), this constructions matches the first one. We will use both construction interchangeably in this paper.

Figure 1: An inhomogeneous random graph of size \( n = 20000 \). The node weights are i.i.d with Pareto distribution of parameters \( \frac{2}{3}, 4 \), and \( p = \frac{5}{n} \). These parameters correspond to typical graphs that will be studied in this chapter.
1.2 Definition of the exploration process

Before stating our main theorems, we define the exploration process of $G(W,p)$ seen as a graph from the sequence $(G(W,p))_{p \in [0, +\infty)}$ for a fixed $p$. All the results of this paper are proven by a careful study of this process. It is based on an "horizontal" exploration of the graph, called the breadth-first walk (BFW). The BFW constructs the spanning forest of $G(W,p)$, called the exploration forest. This is a forest consisting of spanning trees of all the connected components of $G(W,p)$, constructed in a particular way.

For each potential edge $\{i, j\}$ recall the definition of $E_{\{i,j\}}$ from the model presentation. The BFW operates by steps, define the following sets of vertices.

- $(U(i))_{n \geq 1}$ is the sequence of sets of undiscovered vertices at each step.
- $(D(i))_{n \geq 1}$ is the sequence of sets of discovered but not yet explored vertices at each step.
- $(F(i))_{n \geq 1}$ is the sequence of sets of explored vertices at each step.

First, choose a vertex $i$ with probability:

$$P(v(1) = i) = \frac{w_i}{\ell_n},$$

and call it $v(1)$. Let $V'$ be the set of all vertices labels, and $U(1) = V' \setminus \{v(1)\}$, $D(1) = \{v(1)\}$. At step 2, $v(1)$ is explored. It is thus not present in $D(2)$ and moved to $F(2)$. We call children of $v(1)$ the vertices $j$ that are unexplored at step 1 and such that $E_{\{j,v(1)\}} \leq p$. Those children are moved to $D(2)$ and become discovered but not yet explored. Let $c(1)$ be the number of children of $v(1)$. Call them $(v(2), v(3), ..., v(c(1) + 1))$ in increasing order of their $E_{\{j,v(1)\}}$'s.

For $i \geq 1$, denote the set $\{v(1), v(2), ..., v(i)\}$ by $V_i$. Hence, at step 2 we have:

- $U(2) = V \setminus V_{c(1)+1}$.
- $D(2) = V_{c(1)+1} \setminus V_1$.
- $F(2) = V_1$.

Now, at step 3, $v(2)$ becomes explored and its children $\{v(c(1) + 2), v(c(1) + 3), ..., v(c(1) + c(2) + 3)\}$ become discovered but not yet explored. The BFW continues like this, node $v(i)$ becomes explored at step $i + 1$, and its children are discovered at the same step. If the set of discovered nodes becomes empty at some step $i$, this means that the exploration of a connected component is finished. In that case, move on to the next step by choosing a vertex $j$ with probability proportional to its weight $w_j$ among the unexplored vertices and calling it $v(i)$ (like we did for $v(1)$) and exploring it. This construction ensures that a child has exactly one parent, since a child is always discovered while the process is exploring its parent. This ensures that we are constructing a forest. It is the exploration forest. We call the trees in that forest the exploration trees. By construction, exploration trees are spanning trees of the connected components of $G(W,p)$. We say that a connected component is discovered at step $i$ if its first node discovered by the BFW is $v(i)$. Similarly, we say that a connected component is explored at step $i$ if its last node discovered by the
BFW is \( v(i - 1) \). Generally, let \( c(i) \) be the number of children of the node labeled \( v(i) \). The exploration process associated to the BFW above is defined as follow for \( n - 1 \geq i \geq 0 \):

\[
L'_0 = 1, \\
L'_{i+1} = L'_i + c(i + 1) - 1.
\]

The reflected exploration process is defined by

\[
L_0 = 1, \\
L_{i+1} = \max(L_i + c(i + 1) - 1, 1).
\]

Figure 2: Example of a graph with ordered nodes. The integers correspond to the order in the exploration process. The edges in red correspond to the edges of the exploration trees. The labels of the nodes are not represented.

Figure 3: The exploration process of the graph in Figure 2.

The increment of the process \( L' \) at step \( i \) is the number of nodes added to the set of discovered nodes in the BFW after exploring node \( i \). This number is at least \(-1\) if the node being explored has no children. The process \( L' \) contains
a lot of information about $G(W, p)$. For example, each time a connected component is explored $L’$ attains a new minimum. Using $L’$ transforms geometrical questions about the graph, such as "Is there a connected component of size proportional to $n$?" into questions regarding random walks such as "Is there an excursion of $L’$ above its past minimum of size proportional to $n$?".

Moreover, the order of appearance of the nodes in the exploration process corresponds to a size-biased sampling. Formally, we have for $i \in \{1, 2, ..., n-1\}$ and $j \in \{1, 2, ..., n\}$,

$$\Pr(v(1) = j) = \frac{w_j}{\ell_n}.$$  

$$\Pr(v(i + 1) = j \mid V_i) = \frac{w_j I(j \notin V_i)}{\ell_n - \sum_{k=1}^{i} w_{v(k)}}.$$  

The proof of this fact uses only elementary results on exponential random variables. It is a widely known and used result (Aldous [1997], Bhamidi et al. [2010], Broutin et al. [2020] ...). We sketch the proof here.

**Proof.** By construction:

$$\Pr(v(1) = j) = \frac{w_j}{\ell_n}.$$  

Then for $v(2)$, if $c(1)$, the number of children of $v(1)$, is 0 then, by definition, for any $j \geq 1$:

$$\Pr(v(2) = j \mid V_1, c(1) = 0) = \frac{w_j I(j \notin V_1)}{\ell_n - w_{v(1)}}.$$  

Moreover if $c(1) \geq 1$, this means that there exists at least one $j \geq 1$ such that $j \neq v(1)$ and $E_{v(1)}(v) \leq p$. By the absence of memory property of exponential random variables, for any $j \geq 1$:

$$\Pr(v(2) = j, c(1) \geq 1 \mid V_1) = \Pr(v(2) = j \mid V_1) - \Pr(v(2) = j, c(1) = 0 \mid V_1)$$  

$$= \Pr(\arg\min_{k \neq v(1)} (E_{k,v(1)}) = j \mid V_1) - \Pr(\arg\min_{k \neq v(1)} (E_{k,v(1)}) = j \mid V_1) \Pr(c(1) = 0 \mid V_1)$$  

$$= \Pr(\arg\min_{k \neq v(1)} (E_{k,v(1)}) = j \mid V_1) \Pr(c(1) \geq 1 \mid V_1).$$  

By well known properties of exponential random variables, since conditionally on $V_1$ the $(E_{k,v(1)})_{k \neq v(1)}$’s are independent, we have:

$$\Pr(\arg\min_{k \neq v(1)} (E_{k,v(1)}) = j \mid V_1) = \frac{w_j I(j \notin V_1)}{\ell_n - w_{v(1)}}.$$  

This shows the statement for $v(2)$, and we can move to subsequent nodes by induction.  

**1.3 Conditions and main theorem**

The weights in $W$ depend implicitly on $n$. We will assume the following conditions on $W$ in the entire paper.

**Conditions 1.** There exists some positive random variable $W$ such that:
(i) The distribution of a uniformly chosen weight $w_X$ converges weakly to $W$.

(ii) $E[W^3] < \infty$.

(iii) $E[W^2] = E[W]$.

(iv) $\ell_n = E[W]n + o(n^{2/3})$.

(v) $\sum_{k=1}^{n} w_k^2 = E[W^2]n + o(n^{2/3})$.

(vi) $\sum_{k=1}^{n} w_k^3 = E[W^3]n + o(1)$.

(vii) $\max_{1 \leq n} w_i = o(n^{1/3})$.

Conditions (i), (ii) and (iii) ensure that the weak limit of $w_{v(1)}$ has a finite variance and mean 1. Condition (iii) can be ensured by changing the value of $p$.

Conditions (iv), (v) and (vi) ensure that asymptotically the sum of the weights behaves like the sum of independent identically distributed (i.i.d.) copies of $W$. Moreover, to further ease notations, as $n^{1/3} \geq w_1$, we will always use $n^{1/3}$ in our inequalities, even when $w_1$ would be sufficient. An important case to keep in mind is when $(w_1, w_2, ..., w_n)$ are realizations of random variables $(W_1, W_2, ..., W_n)$ which are i.i.d. with distribution $W$. In that case Conditions (iv), (v) and (vi) are consequences of Conditions (ii) and (iii) (see Bhamidi et al. [2010] for a proof).

We define the size of a connected component $C$, with vertices set $V(C)$, of $G(W, p)$ as the number of vertices in $C$. The distance between two vertices of $C$ is the number of edges in the smallest (in number of edges) path between them. We also define the weight of $C$ as:

$$\sum_{i \in V(C)} w_i.$$ 

We call surplus (or excess) of $C$ the number of edges that have to be removed from it in order to make it a tree. For instance, the surplus of a tree is 0, and the surplus of a cycle is 1.

Write $C = \frac{E[W^3]}{E[W^2]}$, and $p_{fn} = \frac{\ell_n^{3/2} + f_n}{\ell_n^{3/2}}$. We can now state the main theorems of this paper. Of course, these theorems hold only under Conditions II.

**Theorem 1 (Size and weight of the giant component).** Let $1 \geq \epsilon' > 0$. Then for $f_n = o(n^{1/3})$ large enough. Consider the following event:

The largest connected component of $G(n, W)$ has its size in the interval

$$\left[ \frac{2(1 - \epsilon'/2)f_n \ell_n^{2/3}}{C}, \frac{2(1 + \epsilon'/2)f_n \ell_n^{2/3}}{C} \right],$$

and its weight in the interval

$$\left[ \frac{2(1 - \epsilon')f_n \ell_n^{2/3}}{C}, \frac{2(1 + \epsilon')f_n \ell_n^{2/3}}{C} \right].$$

Bhamidi et al. [2010] shows that in that case the probability that the conditions hold tend to 1 when $n$ tend to infinity. However, since we need concentration bounds, our weights need to verify these conditions deterministically.
Then if Conditions 1 hold, there exists a positive constant $A > 0$ that only depend on the distribution of $W$, and such that the probability of this event not happening is at most:

$$A \exp \left( \frac{-f_n}{A} \right).$$

**Theorem 2 (The excess of the giant component).** Let $Exc$ be the excess of the largest connected component of $G(n, W)$. Then if Conditions 2 hold, there exists a positive constant $A > 0$ that only depends on the distribution of $W$ such that:

$$\mathbb{P}(Exc \geq Af_n^{3/4}) \leq A \exp \left( \frac{-f_n}{A} \right).$$

**Theorem 3 (The sizes and weights of the small components).** Let $1 > \epsilon' > 0$ then for $f_n = o(n^{1/4})$ large enough, for any $1 \geq \epsilon > 0$ Consider the following events:

- All the connected components discovered before the largest connected component in the exploration process of $G(n, W)$ have size smaller than
  $$\ell_{n/3}^{2/3} / f_n^{1-\epsilon},$$
  and weight smaller than
  $$\frac{(1 + \epsilon') \ell_{n/3}^{2/3}}{f_n^{1-\epsilon}}.$$

- All the connected components discovered after the largest connected component in the exploration process of $G(n, W)$ have size smaller than
  $$\ell_{n/3}^{2/3} / f_n^{1},$$
  and weight smaller than
  $$\frac{(1 + \epsilon') \ell_{n/3}^{2/3}}{f_n^{1}}.$$

Then if Conditions 3 hold, there exists a positive constant $A > 0$ that only depends on the distribution of $W$ such that the probability of one of those events not happening is at most:

$$A \left( \exp \left( \frac{-f_n}{A} \right) + \exp \left( \frac{-\sqrt{f_n}}{A} \right) + \exp \left( \frac{-\ell_{n}^{1/8}}{A} \right) \right).$$

**Theorem 4 (The excess of the small components).** Let $Exc_0$ be the the sum of the excesses of the connected components discovered before the largest connected component in the exploration process of $G(n, W)$. And let $Exc_1$ be the maximal excess of the connected component discovered after the largest connected component.

Then if Conditions 4 hold, there exists a positive constant $A > 0$ that only depends on the distribution of $W$ such that, for any $1 \geq \epsilon > 0$:

$$\mathbb{P}(Exc_0 \geq Af_n^{2}) \leq A \exp \left( \frac{-f_n^{2/3}}{A} \right).$$
\[ P(\text{Exc}_1 \geq A f_n^*) \leq A \left( \exp \left( \frac{-f_n \ln(\sqrt{f_n})}{A} \right) + \exp \left( -\sqrt{f_n} A \right) + \exp \left( -n^{1/8} A \right) \right). \]

These theorems give precise bounds on the size, weight and excess of not
only the largest connected component but also the other small connected com-
ponents of the graph \( G(n, W) \) in the barely supercritical regime, and in the
critical regime when \( f_n \) is a large enough constant. As a direct corollary of
those theorems, we also obtain convergence results when \( f_n \to +\infty \) (see Corol-
ary 38.2). Statements concerning the largest connected component and the
connected components discovered before it are proven in Section 4. While state-
ments concerning the connected components discovered after the largest one are
proven in Section 5. Moreover, at the cost of heavier notations, Theorem 38
provides a more precise statement than the one we presented in Theorem 3.

**Notation:** In the remainder of the paper we drop the \( n \) from \( f_n \). \( f \) will
always be the critical parameter. Moreover we will always assume \( f = o(n^{1/3}) \)
and \( f \geq F \), where \( F > 0 \) is a constant independent of \( n \) which is large enough
for all our theorems to hold. Similarly the variables \( m = m_n \), \( l = l_n \), \( h = h_n \)
and \( y = y_n \) will always depend on \( n \). The letters \( A, A', A'' \ldots \) will be used for
large positive constants that may only depend on the distribution of \( W \).

### 1.4 Motivation and previous work

If \( w_i = 1 \) for all \( i \), then the edge capacities \( (E_{(i,j)}) \) are i.i.d.. In that case
\( G(W, p) \) is an Erdős-Rényi random graph. This is why the rank-1 inhomoge-
neous random graph model is a natural generalization of Erdős-Rényi random
graphs. There are several variations of inhomogeneous random graphs. The
original inhomogeneous graph model was introduced by Aldous in his pioneer
work on the multiplicative coalescent (Aldous 1997), in this article he proved
convergence of the component weights to a suitable limit. Then this model
was further studied in Aldous and Limic 1998. The model we study here is
closely related to the so called Norros-Reittu model (Norros and Reittu 2006).
The difference between their model and ours being that their model allows for
multi-edges. This, however, has no incidence on our proofs. And everything we
show here still holds for their model. Other models of inhomogeneous random
graphs include the Britton-Deijfen-Martin-Löf (Section 3 in Addario-Berry et al.
2006) model, where edge \( \{i,j\} \) is present with probability:

\[
\frac{w_i w_j}{n + w_i w_j}.
\]

And the Chung-Lu model (chapter 5, Section 3 in Chung and Lu 2006), where
edge \( \{i,j\} \) is present with probability:

\[
\frac{w_i w_j}{\ell_n}.
\]

This definition supposes that \( \max_{i,j} (w_i w_j) \leq \ell_n \). we could have chosen some
other representation of the edge probabilities. However, under our conditions
and regime, all the results that we will prove are also true for those models.
Generally, it is easy to see that all the theorems we prove here under Conditions
will still hold for any of the models above. The choice of
\[ p_f = \frac{\ell^{1/3}+f}{\ell_n}, \]
with \( f = o(n^{1/3}) \) is motivated by the phase transition that appears in the following theorem (proved in Bollobás et al. [2007]).

**Theorem 5.** Take \( G(W, c \ell_n) \) and suppose that Conditions 1 are verified, then the following results hold with high probability:

- **Subcritical regime** If \( c < 1 \) then the largest connected component is of size \( o(n) \).
- **Supercritical regime** If \( c > 1 \) then the largest connected component is of size \( \Theta(n) \) and for any \( i > 1 \) the \( i \)-th largest connected component is of size \( o(n) \).
- **Critical regime** If \( c = 1 \) then for any \( i \geq 1 \) the \( i \)-th largest connected component is of size \( \Theta(n^{2/3}) \).

From this theorem it appears that there is a phase transition at \( c = 1 \). Just as in the Erdős-Rényi model, the right scale to look at the phase transition is for \( c_n = 1 + \frac{\lambda}{n^{1/3}} \), with \( \lambda > 0 \) a constant. Which explains our choice of \( p_f \). This is the so called critical window. In Theorems 1, 2, 3, and 4 we look at \( c \sim 1 \) and \( f \) that is either a large constant, or that goes to infinity but stays \( o(n^{1/3}) \). The latter is what we call the barely supercritical regime.

Plenty of work was done on \( G(W, \lambda) \) with \( \lambda \) constant. The most recent and comprehensive one being in Broutin, Duquesne, and Wang [2018] and Broutin et al. [2020]. Aldous was the first to study the closely related multiplicative coalescent in Aldous [1997]. In Bhamidi et al. [2010] it is shown, under Conditions 1, that the sequence of sizes of the connected components, properly rescaled, converges to a random vector. In Bhamidi et al. [2017] this result is further extended, under stronger conditions than Conditions 1, by showing that the sequence of connected components of the whole graph, seen as metric spaces, when properly rescaled, converge to a limit sequence of compact metric spaces. Moreover, under Conditions 1 up to a multiplicative constant, this limit object has the distribution of the scaling limit of Erdős-Rényi random graphs (presented in Addario-Berry et al. [2012]). This shows that there is an invariance principle, although we have a generalization of Erdős-Rényi random graphs the limit objects are just rescaled versions of one another.

However, unlike the Erdős-Rényi case (see Addario-Berry, Broutin, and Reed [2009]), there is no uniform study when \( f \) moves through the critical window. For instance, there are no known concentration results that depend on \( f \) for the size of the largest component of rank-1 inhomogeneous random graphs. Moreover, there are no known concentration results for the barely supercritical regime. These are the cases that we treat in this paper.

This study has other implications for another object. For \( n \in \mathbb{N} \), assign i.i.d., uniform random variables on \((0,1)\), that we call weights, to the edges of a complete graph of size \( n \). Then the random minimum spanning tree (random MST) is the (almost surely unique) connected subgraph with \( n \) vertices that minimizes the sum of the weights. It is a tree. In the Article by Addario-Berry et al. [2017] it is proven that when rescaling the distances by \( n^{-1/3} \), the random

\[ 1 \] We say that a sequence of events \( E_n \) holds with high probability if \( \lim_{n \to \infty} P(E_n) = 1 \)
MST converges to a compact tree-like metric space. The proof in [Addario-Berry et al., 2017] relies heavily on a uniform study of the critical Erdős-Rényi graph through the critical window and in the barely supercritical regime (done before in Article Addario-Berry et al. [2009]).

In order to do the same for the rank-1 inhomogeneous random graphs, instead of putting i.i.d. weights on a complete graph, put capacity \( E_{\{i,j\}} \) on edge \( \{i,j\} \) and construct the minimum spanning tree for those capacities. Call such a tree the inhomogeneous random MST. Clearly, this tree can be coupled with rank-1 inhomogeneous random graphs in the same fashion as in [Addario-Berry et al., 2017]. One can ask whether that tree, when properly rescaled, also converges to a continuous random tree-like metric space. And if the answer is yes, will this metric space be a rescaled version of the scaling limit of the random MST in [Addario-Berry et al., 2017]? A positive answer would show that there is still an invariance principle for those trees.

We intend on answering these questions in follow up papers, and the bounds we prove in this paper will be crucial in our future proofs.

The biggest difficulty in proving our theorems is that the weight discovered at step \( i \) of the exploration process depend on the weights discovered before it. Those weights appear in a size-biased fashion. This is why we show new concentration inequalities for size-biased sampling without replacement. We also make use of the note [Ben-Hamou et al., 2018] in order to estimate the deviations of the sum of variables sampled without replacement. Another difficulty is that we cannot rely on known results (for example results in Article Luczak [1990]) that were proved for Erdős-Rényi graphs. Everything has to be done separately for inhomogeneous random graphs.

There are other interesting problems that require more work. For instance there is the case of power law distributions for the node weights. Conditions ensure that a uniform node weight behaves like a random variable with finite third moment. One can change those conditions, and allow the variable to follow a power law distribution of parameter \( \tau > 3 \). If \( \tau > 4 \), then we are in the case of finite third moments treated here. However, when \( \tau \leq 4 \), we expect the results to be vastly different. Informal arguments show that in that case the scaling limit of the minimum spanning tree should be mutually singular with the scaling limit of random MST. This intuition is due to the appearance of Levy trees when studying those graphs (see [van der Hofstad, Kliem, and van Leeuwaarden, 2018] for further discussion of this model).

Finally another totally different set of questions regard biased sampling without replacement. Let \( n \geq 1 \) be an integer and \((a_1, a_2, ..., a_n)\) be decreasing real number. Moreover let \((p_1, p_2, ..., p_n)\) be positive real numbers such that:

\[
\sum_{k=1}^{n} p_k = 1.
\]

Let \((V(1), V(2), ..., V(n))\) be a vector random variables that correspond to indices sampled without replacement in the following way, for any \( i \in \{1, 2, ..., n-1\} \)
$\{1, 2, ..., n-1\}$:
\[
P(V(1) = j) = p_j,
\]
\[
P(V(i + 1) = j | (V(1), V(2), ..., V(i))) = \frac{p_j \mathbb{I}(V(j) \not\in (V(1), ..., V(i)))}{\sum_{k=1}^{n} p_k - \sum_{k=1}^{i} p_{V(k)}}.
\]

Consider also $(J(1), J(2), ..., J(n))$ that is a vector of independent random variables with the same distribution as $V(1)$. The $J(i)$'s correspond to indices sampled with replacement. Remark that size-biased sampling is a special case of biased sampling. While working on this paper two questions arose regarding these two samplings. First, under which set of conditions do we have the following inequality for any $n \geq m \geq l$ and real number $x \geq 0$:
\[
P\left(\left|\sum_{k=l}^{m} a_{V(i)} - \mathbb{E}[a_{V(i)}]\right| \geq x\right) \leq P\left(\left|\sum_{k=l}^{m} a_{J(i)} - \mathbb{E}[a_{J(i)}]\right| \geq x\right).
\]

This inequality means that biased sampling without replacement is more concentrated around its mean than biased sampling with replacement. The main idea behind this conjecture is that sampling without replacement tends to auto-concentrate itself around its mean. For instance, if for some $i \geq 1$, $V(i) = j$ and $a_j$ is very large, then we will not draw the same index $j$ in subsequent rounds. But in biased sampling with replacement, the same "bad" event can keep happening.

We were not able to find any trivial counter example to this inequality, so it could be true that it holds without any further assumptions. If not, then under which set of assumptions does it hold? With such an inequality it would be easy to answer the question regarding inhomogeneous random graphs with power law distribution presented in the paragraph above.

Another question is for the ordered case. Suppose now that $p_1 \geq p_2 \geq ... \geq p_n$. This means that larger $a_i$'s have larger probabilities of being drawn first. This is again a general case of size-biased sampling. Is it true then that for any $n - 1 \geq m \geq 1$, and real numbers $(x_1, x_2, x_3, ..., x_n)$
\[
P\left(a_{V(1)} \geq x_1, a_{V(2)} \geq x_2, ..., a_{V(m)} \geq x_m\right) \geq P\left(a_{V(2)} \geq x_1, a_{V(3)} \geq x_2, ..., a_{V(m+1)} \geq x_m\right),
\]
and also
\[
P\left(a_{J(1)} \geq x_1, a_{J(2)} \geq x_2, ..., a_{J(m)} \geq x_m\right) \geq P\left(a_{V(1)} \geq x_1, a_{V(2)} \geq x_2, ..., a_{V(m)} \geq x_m\right).
\]

In Lemma 35, we prove those inequalities for $m = 1$. With some more work, we can prove them for $m = 2$ also. We conjecture that they are in fact true for all $1 \leq m \leq n - 1$.

## 2 Bounding the weights

A well known fact is that the sum of weights sampled uniformly without replacement verifies slightly better Chernoff concentration inequalities as the sum of weights sampled uniformly with replacement (See [Serfling 1974](#)). No such general result is available for size-biased sampling.

In this section we will always assume that Conditions [4](#) are verified. We will prove concentration bounds for the weights sampled in size-biased order and without replacement under some conditions.
2.1 First concentration result and the mean

The following theorem, from Article Ben-Hamou et al. [2018], is a first important step in comparing the sum of the \((w_{v(i)})_i\)'s with the sum of i.i.d. copies of a random variable.

**Theorem 6.** Let \(0 < l \leq m \leq n\) be two integers, and \(J(1), J(2), \ldots, J(n)\) be i.i.d. random variables with the distribution of \(v(l)\), then for any convex function \(g:\)

\[
E \left[ g \left( \sum_{i=l}^{m} w_{v(i)} \right) \right] \leq E \left[ g \left( \sum_{i=l}^{m} w_{J(i)} \right) \right].
\]

Generally, concentration bounds that use Chernoff’s inequality are based on the fact that:

\[
E \left[ \exp \left( \sum_{i=l}^{m} w_{J(i)} \right) \right] = E \left[ \exp (w_{J(1)}) \right]^m.
\]

Hence, taking \(g\) to be the exponential function in Theorem 6 shows a Chernoff type inequality. This means that upper bounds that use Chernoff’s inequality (first used in Bernstein [1924]) and which hold for size-biased sampling with replacement are still true for size-biased sampling without replacement. This fact will be used later in the proofs. This is true in particular for Bernstein’s inequality [Bernstein 1924] which stems from Chernoff’s bound.

The following lemmas give an estimation of the mean of \(w_{v(i)}\). This first Lemma is already shown in one of the proofs that appear in Bhamidi et al. [2010], we prove it here again for clarity.

**Lemma 7.** Suppose that Conditions 1 hold. Then for any \(0 < l = o(n)\), and \(i \in \{1, 2, 3\}\) we have:

\[
\sum_{k=1}^{l} w^i_k = o(n).
\]

**Proof.** We do the proof for \(i = 3\), the other cases can be proved similarly or deduced easily from this case. Recall that the weights \((w_1, w_2, \ldots, w_n)\) are taken in decreasing order. For any \(K > 0:\)

\[
\sum_{k=1}^{l} w^3_k \leq \sum_{k=1}^{l} w^3_k \mathbb{1}(w_k \leq K) + \sum_{k=1}^{n} w^3_k \mathbb{1}(w_k > K) \leq \frac{LK^3}{\ell_n} + \sum_{k=1}^{n} \frac{w^3_k \mathbb{1}(w_k > K)}{\ell_n}.
\]

By the weak convergence in Conditions 1:

\[
\lim_{n \to \infty} \left( \frac{\sum_{k=1}^{n} w^3_k \mathbb{1}(w_k \leq K)}{n} \right) = E[W^3 \mathbb{1}(W \leq K)],
\]

and by the fact that:

\[
\sum_{k=1}^{n} w^3_k = E[W^3]n + o(n),
\]
it follows that:

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{w_k^3 \mathbb{1}(w_k > K)}{\ell_n} \right) = \frac{1}{\mathbb{E}[W]} \left( \mathbb{E}[W^3] - \mathbb{E}[W^3 \mathbb{1}(W \leq K)] \right)
\]

\[
= \frac{\mathbb{E}[W^3 \mathbb{1}(W > K)]}{\mathbb{E}[W]}. 
\]

Since \(\mathbb{E}[W^3] < \infty\):

\[
\lim_{K \to \infty} \left( \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{w_k^3 \mathbb{1}(w_k > K)}{\ell_n} \right) \right) = 0.
\]

Together with the fact that and \(l = o(n)\), letting \(n\) go to infinity then \(K\) go to infinity in Equation (1) yields:

\[
\sum_{k=1}^{l} \frac{w_k^3}{\ell_n} = o(1). \tag{2}
\]

**Lemma 8.** Suppose that Conditions 1 hold. Recall that \(C = \frac{\mathbb{E}[W^3]}{\mathbb{E}[W]}\). For any \(l = o(n)\):

\[
\mathbb{E}[w_{v(l)}^2] = C + o(1).
\]

**Proof.** We have using Lemma 7:

\[
\sum_{k \in V_{l-1}} \frac{w_k^3}{\ell_n} \leq \sum_{k=1}^{l-1} \frac{w_k}{\ell_n} = o(1).
\]

Hence:

\[
\mathbb{E}[w_{v(l)}^2] = \mathbb{E} \left[ \sum_{k \notin V_{l-1}} \frac{w_k^2}{\ell_n} - \sum_{k \in V_{l-1}} \frac{w_k^3}{\ell_n} \right]
\]

\[
= \mathbb{E} \left[ \sum_{k \notin V_{l-1}} \frac{w_k^2}{\ell_n} \right] (1 + o(1)) \tag{3}
\]

\[
= C(1 + o(1)) - \mathbb{E} \left[ \sum_{k \in V_{l-1}} \frac{w_k^3}{\ell_n} \right] (1 + o(1)) + o(1).
\]

In order to finish the proof we use Lemma 7 again:

\[
\mathbb{E} \left[ \sum_{k \in V_{l-1}} \frac{w_k^3}{\ell_n} \right] \leq \sum_{k=1}^{l-1} \frac{w_k^3}{\ell_n} = o(1). \tag{4}
\]

From Equations (3) and (4) we obtain:

\[
\mathbb{E}[w_{v(l)}^2] = C + o(1), \tag{5}
\]

which finishes the proof. \(\square\)

12
Lemma 9. Suppose that Conditions \[\square\] hold. Let \( l = o(n) \), we have:
\[
E[w_{v(l)}] = 1 + o(1).
\]

Proof. As in the proof of Lemma \[\square\] we have:
\[
E(w_{v(l)}) = \frac{E[W^2]}{E[W]} (1 + o(1)) - \frac{\sum_{k \in \mathcal{V}_{l-1}} w_k^2}{\ell_n} (1 + o(1))
\]
\[
= \frac{E[W^2]}{E[W]} (1 + o(1)).
\]
Recalling that \( \frac{E[W^2]}{E[W]} = 1 \) ends the proof. \( \square \)

By the same argument we also have:

Lemma 10. Suppose that Conditions \[\square\] hold. Let \( l = o(n) \). For any \( 0 < i < l \) we have:
\[
E(w_{v(i)}w_{v(l)}) = 1 + o(1).
\]

Proof. We have using Lemma \[\square\]
\[
E(w_{v(i)}w_{v(l)}) = \mathbb{E} \left[ \sum_{i \notin \mathcal{V}_{l-1}} \frac{w_i^2}{\ell_n} - \sum_{i' \in \mathcal{V}_{l-1}} w_{i'}^2 \right]
\]
\[
= \mathbb{E} \left[ \sum_{i \notin \mathcal{V}_{l-1}} \frac{w_i^2}{\ell_n} (1 + o(1)) \right]
\]
\[
= 1 + o(1),
\]
which ends the proof. \( \square \)

Thanks to these lemmas, we obtain a more precise estimation of the mean of \( w_{v(l)} \).

Lemma 11. Suppose that Conditions \[\square\] hold. For any \( l = o(n) \), we have:
\[
E[w_{v(l)}] = 1 + \frac{l}{\ell_n} (1 - C) + o \left( \frac{l + n^{2/3}}{n} \right).
\]

Proof. By definition:
\[
E[w_{v(l)}] = \mathbb{E} \left[ \sum_{i \notin \mathcal{V}_{l-1}} \frac{w_i^2}{\ell_n} - \sum_{i' \in \mathcal{V}_{l-1}} w_{i'}^2 \right].
\]
Moreover, by Lemma \[\square\]
\[
E \left[ w_{v(l)} \right] = \mathbb{E} \left[ \sum_{i \notin \mathcal{V}_{l-1}} \frac{w_i^2}{\ell_n} \left( 1 + \frac{\sum_{i' \in \mathcal{V}_{l-1}} w_{i'}}{\ell_n} \right) \right]
\]
\[
= \mathbb{E} \left[ \sum_{i \notin \mathcal{V}_{l-1}} \frac{w_i^2}{\ell_n} \left( 1 + \frac{\sum_{i' \in \mathcal{V}_{l-1}} w_{i'}}{\ell_n} \right) \right] + o \left( \frac{l}{n} \right).
\]
By Lemmas 7, 8 and 9 it follows that:

\[
E[w_{v(i)}] = \mathbb{E} \left[ \sum_{i \in V_{l-1}} \frac{w_i^2}{\ell_n} \left( 1 + \frac{\sum_{i' \in V_{l-1}} w_{i'}}{\ell_n} \right) \right] + o \left( \frac{1}{n} \right) \\
= \sum_{i=1}^n \frac{w_i^2}{\ell_n} + \mathbb{E} \left[ \frac{\sum_{i' \in V_{l-1}} w_{i'}}{\ell_n^2} \left( \sum_{i=1}^n w_i^2 \right) \right] - \mathbb{E} \left[ \frac{\sum_{i \in V_{l-1}} w_i^2}{\ell_n} \right] \\
= \sum_{i=1}^n \frac{w_i^2}{\ell_n} + \mathbb{E} \left[ \frac{\sum_{i' \in V_{l-1}} w_{i'}}{\ell_n^2} \left( \sum_{i=1}^n w_i^2 \right) \right] - \mathbb{E} \left[ \frac{\sum_{i \in V_{l-1}} w_i^2}{\ell_n} \right] \\
= 1 + \mathbb{E} \left[ \frac{\sum_{i' \in V_{l-1}} w_{i'}}{\ell_n^2} \left( \sum_{i=1}^n w_i^2 \right) \right] - \mathbb{E} \left[ \frac{\sum_{i \in V_{l-1}} w_i^2}{\ell_n} \right] + o \left( \frac{1}{n^2/3} \right) \\
= 1 + \frac{1}{\ell_n} (1 - C) + o \left( \frac{1 + \frac{n^{2/3}}{n}}{n} \right).
\]

\(\Box\)

Observe that with the assumption that \(\mathbb{E}[W^2] = \mathbb{E}[W]\), the Cauchy-Schwarz inequality implies that:

\[1 - C = \left( 1 - \frac{\mathbb{E}[W^3]}{\mathbb{E}[W]} \right) \leq 0,\]

so asymptotically \(E(w_{v(i)})\) decreases with \(i\). Lemma 35 shows that in fact, it decreases all the time.

### 2.2 A more precise concentration inequality

In order to obtain concentration inequalities for size-biased sampling without replacement, we will use a randomization trick. The main idea here is that taking weights without replacement is the same as putting exponential "clocks" on each weight and taking a weight when its clock rings.

More precisely let \((T_i)_{i \leq n}\) be a sequence of independent exponential random variables with respective rates \((w_i/\ell_n)_{i \leq n}\). Define the following quantities for \(x \geq 0\):

\[
N(x) = \sum_{k=1}^n \mathbb{1}(T_k \leq x), \\
X(x) = \sum_{k=1}^n w_k \mathbb{1}(T_k \leq x).
\]

By basic properties of exponential random variables, \((v'(1), v'(2), ..., v'(n))\), the distinct random indices of the \(T_i\)’s taken in increasing order, i.e:

\[T_{v'(1)} \leq T_{v'(2)} \leq \cdots \leq T_{v'(n)},\]

14
are distributed as a size-biased sample taken without replacement.

Moreover the following equality holds:

\[ X(x) = \sum_{k=1}^{n} w_{v(k)} \mathbb{1}(N(x) \geq k). \]

Since \( N(x) \) and \( X(x) \) are sums of independent random variables, we can apply Bernstein’s inequality \( \text{Bernstein [1924]} \) in order to obtain the following lemma. We let \( w_{v(0)} = 0 \).

**Lemma 12.** Suppose that Conditions 1 hold. For any \( x \geq 0 \) and \( t \geq 0 \), the following holds:

\[
P(|X(x) - \mathbb{E}[X(x)]| \geq t) \leq 2 \exp \left( -\frac{t^2}{2 \left( n^{1/3} + x \right)} \right),
\]

and

\[
P(|N(x) - \mathbb{E}[N(x)]| \geq t) \leq 2 \exp \left( -\frac{t^2}{2(t+x)} \right).
\]

The following conditions will always be verified in this section. They give a regime where our concentration bounds hold.

**Conditions 2.** We say that \((a(n), b(n))\) verifies Conditions 3 if for all \( n \) large enough:

\[
\exp \left( -\frac{b(n)^2}{A(b(n)n^{1/3} + a(n))} \right) < 1/4,
\]

\[
\lim_{n \to \infty} a(n) = \lim_{n \to \infty} b(n) = +\infty.
\]
\[ a(n) = o(n), \]
\[ b(n) = O(a(n)), \]
\[ a(n) = O \left( b(n) \ell_{n}^{1/3} \right), \]

and:
\[ (a(n))^{2} = O \left( b(n) \ell_{n} \right), \]

where \( \bar{A} > 0 \) is independent of \( n \) and larger than all the other constants \( A, A', A'' \ldots \) that appear in this paper.

The condition \( b(n) = O(a(n)) \) is not necessary, but it makes some computations easier and will be true in all the practical cases in this paper. Moreover, notice that if \( (a(n), b(n)) \) verifies Conditions \( 2 \) then for any \( A > 0 \) the couple \( (a(n), b(n)) \) will also verify those conditions. We want to prove that there exists an \( A > 0 \) such that if \( (m, y) \) verify Conditions \( 2 \) then:
\[
P \left[ \sup_{i \leq m} \left| \sum_{k=1}^{i} w_{v(k)} - \mathbb{E} \left[ \sum_{k=1}^{i} w_{v(k)} \right] \right| \geq y \right] \leq A \exp \left( \frac{-y^{2}}{A(yn^{1/3} + m)} \right).
\]

In order to do so, we will use the fact that if \( N(u_{n}) \geq m \) for some \( u_{n} > 0 \) then:
\[
\sup_{i \leq m} \left| \sum_{k=1}^{i} w_{v'(i)} - \mathbb{E} \left[ \sum_{k=1}^{i} w_{v'(i)} \right] \right| \leq \sup_{x \leq u_{n}} \left| X(x) - \sum_{k=1}^{N(x)} \mathbb{E} \left[ w_{v'(i)} \right] \right|.
\]

Then we will show concentration of the right-hand side of the above inequality. The following fact will be used through this whole section. For any \( x \geq 0 \):
\[
x \geq 1 - e^{-x} \geq x - \frac{x^{2}}{2}. \tag{6}
\]

We start by showing the following lemma:

**Lemma 13.** Suppose that Conditions \( 1 \) hold. Let \( (a(n), b(n)) \) verify Conditions \( 2 \) then there exists a constant \( A > 0 \) such that for all \( n \) large enough:
\[
P \left[ \sup_{x \leq a(n)} \mathbb{E} [X(x)] - \sum_{k=1}^{N(x)} \mathbb{E} [w_{v(i)}] \geq b(n) \right] \leq P \left[ \inf_{x \leq a(n)} N(x) - \mathbb{E} [N(x)] \leq \frac{-b(n)}{A} + 1 \right],
\]

and:
\[
P \left[ \inf_{x \leq a(n)} \mathbb{E} [X(x)] - \sum_{k=1}^{N(x)} \mathbb{E} [w_{v(i)}] \leq -b(n) \right] \leq P \left[ \sup_{x \leq a(n)} N(x) - \mathbb{E} [N(x)] \geq \frac{b(n)}{A} - 1 \right],
\]

and the same inequalities hold without the sup and inf.
Proof. Let $x \leq a(n)$. By Equation (6) and Conditions 1:

$$
\mathbb{E}[X(x)] = \sum_{k=1}^{n} w_k \mathbb{P}(T_k \leq x)
= \sum_{k=1}^{n} w_k \left(1 - \exp\left(-\frac{w_k x}{\ell_n}\right)\right)
\leq \sum_{k=1}^{n} w_k^2 \frac{x}{\ell_n}
= x\left(1 + o(n^{-1/3})\right).
$$

(7)

For any $b'(n)$ such that $(a(n), b'(n))$ verify Conditions 2, there exists $A' > 0$ such that:

$$
x^2 \leq a(n)^2 \leq A' b'(n) \ell_n.
$$

Denote $\lceil \mathbb{E}[N(x)] - b'(n) \rceil$ by $u$. By Conditions 1 and Equation (6) we obtain:

$$
u \geq x - b'(n) - \frac{\sum_{k=1}^{n} w_k^2 x^2}{2\ell_n^2}
\geq x - b'(n) - \frac{x^2}{2\ell_n} + o\left(\frac{x^2}{\ell_n}\right)
\geq x - b'(n) - \frac{A' b'(n)}{2} + o\left(\frac{x^2}{\ell_n}\right).
$$

(8)

Moreover by Condition 2

$$
u^2 \leq (x + b'(n))^2
\leq 2x^2 + 2b'(n)^2
\leq 2A' \ell_n b'(n) + 2b'(n)^2
\leq A'' \ell_n b'(n),
$$

where $A'' > 0$ is some large constant. By Equations (8), (9), Conditions 2 and Lemma 11 we have:

$$
\sum_{k=1}^{n} \mathbb{E} [w_{v(i)}] = \sum_{k=1}^{n} \left(1 + \frac{k}{\ell_n} (1 - C) + o\left(\frac{u^2 + un^{1/3}}{n}\right)\right)
\geq u + \frac{u^2}{2\ell_n} (1 - C) + o\left(\frac{u^2 + un^{1/3}}{n}\right)
\geq x - A''' b'(n),
$$

(10)

where $A''' > 0$ is a large constant. Inequalities (7) and (10) and Conditions 2 yield:

$$
\mathbb{E}[X(x)] - \sum_{k=1}^{n} \mathbb{E} [w_{v(i)}] \leq A''' b'(n) + o(xn^{-1/3}).
$$

And of course, since $\mathbb{E} [w_{v(i)}]$ is positive for all $i \leq n$, the same inequality holds if we replace $u$ by $u' \geq u$. This show that:

$$
\left(\mathbb{E}[X(x)] - \sum_{k=1}^{N(x)} \mathbb{E} [w_{v(i)}] \geq 2A''' b'(n)\right) \Rightarrow (N(x) \leq \mathbb{E}[N(x)] - b'(n) + 1)
$$
Taking \( b(n) = 2A''b'(n) \) proves the first inequality of the lemma, the second inequality is proved similarly.

Similarly we have the following lemma for which we omit the proof

**Lemma 14.** Suppose that Conditions 3 hold. Let \((a(n), b(n))\) verify Conditions 2. Then there exists a constant \( A > 0 \) such that for all \( n \) large enough:

\[
\begin{align*}
\Pr \left[ \sup_{x \leq a(n)} \mathbb{E} [X(a(n)) - X(x)] - \sum_{k=N(x)}^{N(a(n))} \mathbb{E} [w_{v(i)}] \geq b(n) \right] \\
\leq \Pr \left[ \inf_{x \leq a(n)} N(a(n)) - N(x) - \mathbb{E}[N(a(n)) - N(x)] \leq \frac{-b(n)}{A} + 1 \right],
\end{align*}
\]

and:

\[
\begin{align*}
\Pr \left[ \inf_{x \leq a(n)} \mathbb{E} [X(a(n)) - X(x)] - \sum_{k=N(x)}^{N(a(n))} \mathbb{E} [w_{v(i)}] \leq -b(n) \right] \\
\leq \Pr \left[ \sup_{x \leq a(n)} N(a(n)) - N(x) - \mathbb{E}[N(a(n)) - N(x)] \geq \frac{b(n)}{A} - 1 \right],
\end{align*}
\]

and the same inequalities hold without the sup and inf.

These lemmas will allow us to prove the following concentration inequality. Recall that \( m = m(n) \) and \( y = y(n) \) depend implicitly on \( n \).

**Lemma 15.** Suppose that Conditions 4 hold. There exist a constant \( A > 0 \) such that if \((x_n, y)\) verifies Conditions 3 then:

\[
\Pr \left( \left| X(x_n) - \sum_{k=1}^{N(x_n)} \mathbb{E}[w_{v(i)}] \right| \geq y \right) \leq A \exp \left( \frac{-y^2}{A(yn^{1/3} + x_n)} \right),
\]

**Proof.** By the union bound:

\[
\begin{align*}
\Pr \left[ \left| X(x_n) - \sum_{k=1}^{N(x_n)} \mathbb{E}[w_{v(i)}] \right| \geq y \right] \\
\leq \Pr \left[ \left| X(x_n) - \mathbb{E}[X(x_n)] \right| \geq \frac{y}{2} \right] + \Pr \left[ \left| \mathbb{E}[X(x_n)] - \sum_{k=1}^{N(x_n)} \mathbb{E}[w_{v(i)}] \right| \geq \frac{y}{2} \right].
\end{align*}
\]  

(11)

We bound separately each term of the right-hand side of Equation (11). Lemma 12 states that:

\[
\Pr \left[ \left| X(x_n) - \mathbb{E}[X(x_n)] \right| \geq \frac{y}{2} \right] \leq 2 \exp \left( \frac{-y^2}{8(yn^{1/3} + x_n)} \right).  
\]  

(12)

Using Equations (12), Lemma 13 on \((x_n, y/2)\) and Lemma 12 to bound the second expression in the right-hand side of Equation (11) expression in Equation (11) shows that:

\[
\Pr \left( \left| X(x_n) - \sum_{k=1}^{N(x_n)} \mathbb{E}[w_{v(i)}] \right| \geq y \right) \leq A' \exp \left( \frac{-y^2}{A'(yn^{1/3} + x_n)} \right),
\]

18
where \( A' > 0 \) is a large constant.

In order to prove concentration inequalities on the \( N(t) \) and \( X(t) \) for all \( t \) in some interval, we use the chaining method. This method consists of crafty discretizations of the "time" parameter space in order to derive general bounds for all "times". The method is explained in chapter 13 of Boucheron et al. [2013]. It is attributed to Kolmogorov, and it has been vastly used and improved by Dudley (Dudley [1973]) and Talagrand (for instance Talagrand [2005]).

**Lemma 16.** Suppose that Conditions 7 hold. There exist a constant \( A > 0 \) such that, for any \((m, y)\) that verify Conditions 2:

\[
P \left( \sup_{0 \leq t \leq m} (X(t) - E[X(t)]) \geq y \right) \leq A \exp \left( \frac{-y^2}{A(g_n/t + m)} \right).
\]

**Proof.** Recall that \( w_1 \geq w_2 \geq w_3 \geq ... \) and for \( i \leq n \) write:

\[
X_i(t) = \sum_{k=1}^{n} w_k \mathbb{1}(T_k \leq t).
\]

By Bernstein’s inequality and basic computations, for any \( u > 0 \) and \( s < t \):

\[
P \left( \left| X_i(t) - X_i(s) - E[X_i(t) - X_i(s)] \right| \geq \sqrt{2(t - s) \sum_{k=i+1}^{n} \frac{w_k^3}{\ell_n} u + uw_{i+1}} \right) \leq 2 \exp(-u).
\]

For \( i \geq 0 \) let:

\[
\Gamma_i = \left\{ m \frac{k}{2^i}, 0 \leq k < 2^i \right\} \cup \left\{ T_k, 1 \leq k < 2^i \right\}.
\]

Let \( f_i : [0, m] \rightarrow \max\{z \in \Gamma_i, t > z\} \). We have, by definition of \( f_i \) and \( \Gamma_i \), for any \( t \leq m \):

\[
X(t) - X(f_i(t)) = \sum_{k=1}^{n} w_k \mathbb{1}(f_i(t) < T_k \leq t)
\]

\[
= \sum_{k=2^i}^{n} w_k \mathbb{1}(f_i(t) < T_k \leq t)
\]

\[
= X_{2^i-1}(t) - X_{2^i-1}(f_i(t)).
\]

Since \( f_i(t) \) is measurable with respect to the \( (T_k)_{k \leq 2^i} \)'s. And conditionally on \( f_i(t) \), \( X(t) - X(f_i(t)) \) is a sum of independent random variables. We can apply Bernstein’s inequality to obtain similarly to Equation (13):

\[
P \left( \left| X(t) - X(f_i(t)) - E[X(t)] - X(f_i(t)) \right| \geq \sqrt{2(t - f_i(t)) \sum_{k=2^i}^{n} \frac{w_k^3}{\ell_n} u + uw_{2^i}} \right)
\]

\[
\leq 2 \exp(-u).
\]

Let:

\[
\rho_i = \sqrt{3 \frac{m}{2^i} C(u(i + 1)) + u(i + 1)w_{2^i}}.
\]
Since \((t - f_i(t)) \leq \frac{m}{2}\) and \(\sum_{k=1}^{n} \frac{w_k^3}{\tau_n} = C(1 + o(1))\). Inequality [14] with \(u' = u(i + 1)\) yields:

\[
P(|X_i(f_i(t)) - X_i(t) - \mathbb{E}[X_i(f_i(t)) - X_i(s)]| \geq \rho_i) \leq 2 \exp(-u(i + 1)).
\]

The classical chaining argument is that any \(0 \leq t \leq m\) can be written as:

\[
t = \sum_{i=0}^{\infty} (f_{i+1}(t) - f_i(t)),
\]

This gives us by union bound, if we suppose that \(u > \ell_n(4)\):

\[
\begin{align*}
\mathbb{P} \left( \sup_{0 \leq t \leq m} |X(t) - \mathbb{E}[X(t)]| \geq \sum_{i=0}^{\infty} \rho_i \right) &\leq \sum_{i=0}^{\infty} \sum_{t \in \Gamma_{i+1}} \mathbb{P} \left( |X_i(t) - X_i(f_i(t)) - \mathbb{E}[X_i(t) - X_i(f_i(t))]| \geq \rho_i \right) \\
&\leq \sum_{i=0}^{\infty} \sum_{t \in \Gamma_{i+1}} 2 \exp(-u(i + 1)) \\
&\leq \sum_{i=0}^{\infty} 2^{i+3} \exp(-u(i + 1)) \\
&\leq \frac{8e^{-u}}{1 - e^{-(u-\ell_n(2))}} \\
&\leq Ae^{-u},
\end{align*}
\]

where \(A > 0\) is some large constant and with the convention that \(w_k = 0\) if \(k \geq n\). Now notice that as \(\sum_{k=1}^{n} w_k^3 \leq An\) for some constant \(A\), we have for any \(i \geq 0\), \(w_{2i} \leq \frac{A^{1/3}n^{1/3}}{2^{i/3}}\). Hence:

\[
\sum_{i=1}^{\log(n)} (i + 1)w_{2i} \leq \sum_{i=1}^{+\infty} A^{1/3}(i + 1)n^{1/3} \frac{2^{i/3}}{2^{i/3}} \\
\leq A'n^{1/3},
\]

where \(A' > 0\) is some large constant. With Equation [16], a simple computation shows that there exists \(A > 0\) such that:

\[
\sum_{i=0}^{\infty} \rho_i = A' \left( \sqrt{ma} + un^{1/3} \right),
\]

Replacing in Equation [15] give just another way of writing Bernstein’s inequality, we finish by taking for instance:

\[
u = \frac{y^2}{2A'^2(n^{1/3}y + m)},
\]

which also ensures that \(u > \ln(4)\) by Conditions 2.

The following three lemmas have similar proofs, and their proofs are thus omitted.
Lemma 17. Suppose that Conditions 1 hold. There exists $A > 0$ such that, for any $(m, y)$ that verifies Conditions 2,

$$P \left( \sup_{0 \leq t \leq m} |N(m) - N(m - t) - E[N(m) - N(m - t)]| \geq y \right) \leq A \exp \left( \frac{-y^2}{A(y + m)} \right).$$

Lemma 18. Suppose that Conditions 1 hold. There exists $A > 0$ such that, for any $(m, y)$ that verify Conditions 2,

$$P \left( \sup_{0 \leq t \leq m} |N(t) - E[N(t)]| \geq y \right) \leq A \exp \left( \frac{-y^2}{A(y + m)} \right).$$

Lemma 19. Suppose that Conditions 1 hold. There exists $A > 0$ such that, for any $(m, y)$ that verifies Conditions 2,

$$P \left( \sup_{0 \leq t \leq m} |X(m) - X(m - t) - E[X(m) - X(m - t)]| \geq y \right) \leq A \exp \left( \frac{-y^2}{A(y n^{1/3} + m)} \right).$$

Now we can prove the concentration of the size-biased sum of weights sampled without replacement.

Theorem 20. Suppose that Conditions 1 hold. There exists a constant $A > 0$ that satisfies the following, for $(m, y)$ that verifies Conditions 2, we have:

$$P \left[ \sup_{0 \leq i \leq j \leq m} \left| \sum_{k=i}^{j} w^*(k) - E \left[ \sum_{k=i}^{j} w^*(k) \right] \right| \geq y \right] \leq A \exp \left( \frac{-y^2}{A(y n^{1/3} + m)} \right).$$

Proof. Let $l(m)$ be such that $E[N(l(m))] = m$. If $E = \{N(l(m) + y)) \geq m\}$ holds, then:

$$\sup_{0 \leq i \leq j \leq m} \left| \sum_{k=i}^{j} w^*(k) - E \left[ \sum_{k=i}^{j} w^*(k) \right] \right| \leq \sup_{0 \leq x \leq y \leq 3(l(m) + y)} \left| X(z) - X(x) - E \left[ w^*(k) \right] \right|.$$

We only bound:

$$P \left[ \inf_{i \leq j \leq m} \sum_{k=i}^{j} w^*(k) - E \left[ \sum_{k=i}^{j} w^*(k) \right] \leq -y \right],$$

as the argument for bounding the other part is the same. By union bound with the event $E$:

$$P \left( \inf_{i \leq j \leq m} \sum_{k=i}^{j} w^*(k) - E \left[ \sum_{k=i}^{j} w^*(k) \right] \leq -y \right) \leq P \left( E, \inf_{i \leq j \leq m} \sum_{k=i}^{j} w^*(k) - E \left[ \sum_{k=i}^{j} w^*(k) \right] \leq -y \right) + P(\bar{E}) \quad (17)$$

$$\leq P \left( \inf_{0 \leq x \leq z \leq 3(l(m) + y)} X(z) - X(x) - E \left[ w^*(k) \right] \leq -y \right) + P(\bar{E}).$$
Note that by Conditions 1, for $n$ large enough:

\[
E \left[ N \left( \frac{\ell_n}{9} \right) \right] \geq \sum_{k=1}^{n} \left( \frac{w_i}{9} - \frac{w_i^2}{162} \right) \\
\geq \frac{\ell_n}{11} (1 + o(1)) \\
\geq \frac{\ell_n}{12}.
\] (18)

Since $(E[N(x)])_{x \geq 0}$ is an increasing function, by Equation (18), $l(m) \leq \ell_n/9$.

Hence, by Equation (6):

\[
E \left[ N(l(m)) \right] = m \\
\geq l(m) - \sum_{k=1}^{N(z)} E \left[ w_v(i) \right] \leq -y \\
\geq 8l(m)/9.
\] (19)

By Lemma 12 and Equation (19):

\[
P(\bar{E}) \leq A \exp \left( \frac{-y^2}{A(y + m)} \right),
\] (20)

for some large constant $A > 0$. Now we need to prove that:

\[
P \left( \inf_{0 \leq x \leq z \leq l(m) + y} X(z) - X(x) - \sum_{k=N(z)}^{N(x)} E \left[ w_v(i) \right] \leq -y \right) \leq A \exp \left( \frac{-y^2}{A(yn^{1/3} + x)} \right).
\] (21)

By equation (19):

\[
3(l(m) + y) \leq 4m + 3y.
\] (22)

Let:

\[
C = \left\{ \inf_{0 \leq x \leq z \leq 4m + 3y} X(z) - X(x) - \sum_{k=N(z)}^{N(x)} E \left[ w_v(i) \right] \leq -y \right\},
\]

and:

\[
B = \left\{ X(4m + y) - \sum_{k=0}^{N(4m + 3y)} E \left[ w_v(i) \right] \leq -y/2 \right\}.
\]

Also, write

\[
(x^*, z^*) = \inf \left\{ 0 \leq x \leq z \leq 4m + 3y : X(z) - X(x) - \sum_{k=N(z)}^{N(x)} E \left[ w_v(i) \right] \leq -y \right\},
\]

where the infimum is taken in lexicographical order. And, by convention, $\inf(\emptyset) = (0, 4m + 3y)$. Let:

\[
D := \left\{ X(x^*) - \sum_{k=1}^{N(x^*)} E \left[ w_v(k) \right] \geq y/4 \right\} \text{ or } \left\{ X(4m + y) - X(z^*) - \sum_{k=N(z^*)}^{N(4m + 3y)} E \left[ w_v(k) \right] \geq y/4 \right\}.
\]
If $C$ happens then one of the events $B$ or $D$ happens. By Lemma 15

$$P(B) \leq A \exp\left(\frac{-y^2}{A(yn^{1/3} + m)}\right).$$

(23)

By Lemma 13 and union bound:

$$P\left(X(x^*) - \sum_{k=1}^{N(x^*)} E\left[w_{v(k)}\right] \geq y/4\right) \leq P\left(\sup_{t \leq 4m + 3y} X(t) - \sum_{k=1}^{N(t)} E\left[w_{v(k)}\right] \geq y/4\right)$$

$$\leq P\left(\sup_{t \leq 4m + 3y} X(t) - E[X(t)] \geq y/8\right) + P\left(\sup_{t \leq 4m + 3y} N(t) - E[N(t)] \geq \frac{y}{A} + 1\right).$$

(24)

where $A > 0$ is the positive constant that appears in Lemma 13. And by the same arguments, using Lemma 14 gives:

$$P\left(X(4m + 3y) - X(z^*) - \sum_{k=N(z^*)}^{N(4m + 3y)} E\left[w_{v(k)}\right] \geq y/4\right)$$

$$\leq P\left(\sup_{t \leq 4m + 3y} X(4m + 3y) - X(t) - \sum_{k=N(t)}^{N(4m + 3y)} E\left[w_{v(k)}\right] \geq y/4\right)$$

(25)

$$\leq P\left(\sup_{t \leq 4m + 3y} X(4m + 3y) - X(t) - E[X(4m + 3y) - X(t)] \geq y/8\right)$$

$$+ P\left(\sup_{t \leq 4m + 3y} N(4m + 3y) - N(t) - E[N(4m + 3y) - N(t)] \geq \frac{y}{A} + 1\right).$$

The union bound using Inequality (24) and (25) alongside Lemmas 16, 17, 18 and 19 yields:

$$P(D) \leq A'' \exp\left(\frac{-y^2}{A''(yn^{1/3} + m)}\right)$$

(26)

Hence, from Equations (23) and (26) we obtain:

$$P(C) \leq P(B) + P(D)$$

$$\leq A'' \exp\left(\frac{-y^2}{A''(yn^{1/3} + m)}\right).$$

(27)

This proves Equation (21). We can then bound Equation (17) by using Equation (26) and Equation (27) which finishes the proof.

In the above theorems we started the sums from one for the sake of clarity. The following general theorem has a similar proof.

**Theorem 21.** Suppose that Conditions 4 hold. There exists a constant $A > 0$ such that, if $1 \leq l \leq m$ is such that $(\sqrt{m(m-1)}, y)$ verify Conditions 2 $m - l \to \infty$ and $y = O(m - l)$ then:

$$P\left[\sup_{t \leq l \leq m} \left|\sum_{k=1}^{j} w_{v(k)} - E\left[\sum_{k=1}^{j} w_{v(k)}\right]\right| \geq y\right] \leq A \exp\left(\frac{-y^2}{A(yn^{1/3} + (m - l))}\right).$$

23
3 Bounds on the exploration process

In this section we prove concentration inequalities for the exploration process and related processes. These various inequalities will be used in the following sections. Recall that \( f = o(n) \) is the critical parameter and \( p_f = \frac{1}{\ell_0} + \frac{1}{\ell_f} \). In the rest of this section we consider the BFW of \( G(W, p_f) \).

For \( 0 \leq i \leq n \) and \( 0 \leq j \leq n \) define:

\[
Y(i, j) = \mathbb{1}(\text{There is an edge between nodes } i \text{ and } j).
\]

Then by definition of the BFW we have:

\[
L_0 = 1, \\
X_{i+1} = \sum_{j \notin V(i+L_i)} Y(v(i+1), j) - 1, \\
L_{i+1} = \max(L_i + X_{i+1}, 1).
\]

Recall also that:

\[
L'_0 = 1, \\
L'_{i+1} = L'_i + X_{i+1}.
\]

When seen as processes of \( i \), \( L' \) is equal to \( L \) until we finish discovering the first connected component. After that \( L' = L - 1 \) until the second connected component is discovered, then \( L' = L - 2 \) and so on. Generally \( L' \) is equal to \( L \) minus the number of connected components fully discovered. We say that the process \( L \) visits 0 in \( i \) if \( L'_i = \min_{j \leq i} L'_j \).

One of the difficulties in studying this process lies in the fact that \( X_{i+1} \) depends on \( L_i \). In the case of simple Erdős–Rényi random graphs, Addario-Berry et al. [2009] use a different exploration process where the children of a node being explored are taken uniformly. This allows them to use a simpler and close enough process in order to circumvent this problem. If we want to do like them, in our case the naive way to define such a process would be as follows, for \( h \geq 0 \):

\[
L'^0_0 = 1, \\
X'^0_{i+1} = \sum_{j \notin V(i+L_i+1)} Y(v(i+1), j) - 1, \\
L'^{h}_{i+1} = L'^{h}_i + X'^{h}_{i+1}.
\]

In that case \( L^0 \) is always above \( L' \) and in general \( L^h_i \leq L'_i \) as long as \( L_i < h + 1 \). \( L^0 \) is used to bound \( L' \) (and thus \( L \)) from above while \( L^h \) for \( h \) large enough would be used to bound it from below. However, in our case we sort the discovered children of a node by the weights of their edges. Hence, it is very likely that the indicator functions present in \( L'_i \) but not in \( L^h_i \) for \( h > L_i \) will be equal to 1 and hence \( L^h_i \) would be too far away from \( L'_i \). This is why we will use a martingale technique that we present now.

Note that for \( i \geq 1 \), \( L_i \) is \( \sigma(X_1, X_2, ..., X_i) \) measurable. Let \( (F_i)_{i \geq 1} \) be the increasing sequence of \( \sigma \)-fields such that \( F_i \) is the \( \sigma \)-field generated by \( V(i + L_i) \) and the \( (X_k)_{k \leq i} \)'s, with the convention that \( V(k) = V \) when \( k \geq n \). Then for any \( i \geq 1 \), \( X_i \) is measurable with respect to \( F_i \) and moreover we have:

\[
\mathbb{E}[X_i | F_{i-1}] + 1 = \sum_{k > i + L_{i-1} - 1} (1 - e^{-w_{v(i)}w_{v(k)}p_f}).
\]
Figure 5: The reflected exploration process of the graph in Figure 1 with time rescaled by $20000^{2/3}$ and space is rescaled by $20000^{1/3}$.

And we have the following fact:

Fact 22. Let

$$\tilde{L}_i = \sum_{k=0}^{i} \mathbb{E}[X_k | \mathcal{F}_{k-1}],$$
with the convention that $X_0 = 1$. Then for any $l \geq 0$, the process $(L'(i) - L'(l) - (\tilde{L}_i - \tilde{L}_l))_{i \geq l}$ is a martingale with respect to $({\mathcal{F}_i})_{i \geq l}$.

This fact allows us to use Bernstein’s inequality for martingales [Freedman 1975]. Then in order to bound $L'_i$ from below, we will use the fact that $(L'_i - \tilde{L}_i)_{i \geq l}$ is a martingale, and for $i \geq 1$ as long as $L_i \leq h$ we have:

$$\mathbb{E}[X_i | {\mathcal{F}_{i-1}}] + 1 \geq \sum_{k > i + h} (1 - e^{-w_{v(i)}w_{v(k)}p_f}).$$

This is why we define the following process, for $i \geq 1$ and $h \geq 0$:

$$\tilde{L}^h_m = m - 1 = \sum_{i=1}^{m} \sum_{k > i + h} (1 - e^{-w_{v(i)}w_{v(k)}p_f}),$$

then $\tilde{L}^h_m$ will be close to, and greater than $\tilde{L}_m$ as long as $h \geq L_i$ and $h$ is not too large. A second important fact is that while constructing the exploration process, we never inspect the potential surplus edges, namely the $Y(v(i), v(j))$’s where $i \geq 1$ and $i + 1 \leq j \leq i + L_i - 1$. This means that:

**Fact 23.** Conditionally on $\mathcal{F}_n$, the $\sigma$-field generated by $\mathcal{V}$ and the $(X_k)_{k \leq n}$’s, the random variables

$$Y(v(i), v(j))_{1 \leq i, j \leq i + L_i - 1},$$

are independent Bernoulli random variables of parameters

$$(1 - e^{-w_{v(i)}w_{v(j)}p_f})_{1 \leq i, i + 1 \leq j \leq i + L_i - 1}.$$ 

Moreover, for $h \geq 0$ and $i \geq 0$ define

$$\hat{L}'(k) = \sum_{i=0}^{k} X_i \mathbb{1}(X_i \leq 2n^{1/3}),$$

and if we write $d(i)$ for the degree of node $i$, then $d(i)$ is a sum of independent Bernoulli variables. Hence, when Conditions 1 hold, by the classical Bernstein inequality [Bernstein 1924] we have:

$$\mathbb{P}(d(i) \geq w_i + n^{1/3}) \leq \exp \left( -\frac{(n^{1/3})^2}{2(n^{1/3} + w_i)} \right).$$

By using Conditions 1 we have for $n$ large enough:

$$\mathbb{P}(\exists (k, h), \hat{L}'(k) \neq L'(k)) \leq \sum_{i=1}^{n} \mathbb{P}(d(i) \geq w_i + n^{1/3})$$

$$\leq \sum_{i=1}^{n} \exp \left( -\frac{(n^{1/3})^2}{2(n^{1/3} + w_i)} \right)$$

$$\leq A \exp \left( -\frac{n^{1/3}}{A} \right),$$

where $A$ is some large constant, this probability is smaller than the ones we will get in this section and the one following it. It is also clear that Fact 22 also
holds if we replace \( L'(k) \) by \( L'(k) \) and \( X_i \) by \( X_i \). Hence, we will assume that the increments of \( L' \) are smaller than \( 2n^{1/3} \). And we will assume the same of \( L^0 \). This will make computations lighter, as Bernstein’s inequality requires a bound on the maximal increment of the process. We will not have to do a union bound in each calculation and consider the case where \( L'(k) \neq L'(k) \).

This convention will be used up to Section 5, after that we will use the fact that the increments of \( L'(k) \) are even smaller when \( k \) is large enough.

A direct corollary of Lemma 11 is the following:

**Corollary 23.1.** For all \( m \geq l \geq 1 \) such that \( m = o(n) \), and \( h = o(n) \):

\[
E(\tilde{L}^h - \tilde{L}^h_{l-1}) = (m-l) \left( f \ell_n^{-1/3} - C(m + l) + 2h \right) + O \left( n^2 - l^2 + (m-l)(h + n^{2/3}) \right).
\]

**Proof.** For any \( l - 1 \leq i \leq m \), let:

\[
\tilde{X}^h_{i+1} = \sum_{j \not \in V(i + 1 + h)} (1 - e^{-w_{v(i)}w_{v(j)}p_f}) - 1,
\]

then:

\[
\tilde{L}^h_{i+1} = \tilde{L}^h_i + \tilde{X}^h_{i+1},
\]

and:

\[
E[\tilde{X}^h_i] + 1 = E \left[ \sum_{j \geq i+1 + h} 1 - \exp(-w_{v(i)}w_{v(j)}p_f) \right].
\]

By Conditions 1, \( w_{v(i)}w_{v(j)}p_f = o(1) \) deterministically for any \((i, j)\). The bounds giving \( O \) and \( o \) in the following expectations can thus be chosen to be deterministic. By Equation (6) we have:

\[
E[\tilde{X}^h_i] + 1 = E \left[ \sum_{j \geq i+1 + h} w_{v(i)}w_{v(j)}p_f(1 + O(w_{v(i)}w_{v(j)}p_f)) \right]
\]

\[
= E \left[ w_{v(i)} \left( 1 + f \ell_n^{-1/3} + O \left( \sum_{j=1}^{m} w_{v(i)}w_{v(j)}^2p_f^2 \right) \right) - \sum_{j < i+1 + h} w_{v(i)}w_{v(j)}p_f(1 + o(1)) \right]
\]

\[
= E \left[ w_{v(i)} \left( 1 + f \ell_n^{-1/3} + o \left( n^{-2/3} \right) \right) - \sum_{j < i+1 + h} w_{v(i)}w_{v(j)}p_f(1 + o(1)) \right].
\]

We use Lemmas 8 and 11 to do the proper replacements in Equation (32):

\[
E[\tilde{X}^h_i] = -1 + \left( 1 - \frac{i(1-C)}{\ell_n} + o \left( \frac{i + n^{2/3}}{n} \right) \right) \left( 1 + \frac{f}{\ell_n^{1/3}} \right)(1 + o(1))
\]

\[
- E \left[ \sum_{j < i+1 + h} w_{v(i)}w_{v(j)}p_f(1 + o(1)) \right].
\]

Finally, Lemma 10 yields:

\[
E[\tilde{X}^h_i] = -1 + \left( 1 + \frac{i(1-C)}{\ell_n} + o \left( \frac{i + n^{2/3}}{n} \right) \right) \left( 1 + \frac{f}{\ell_n^{1/3}} \right)(1 + o(1)) - \frac{i + h}{\ell_n}(1 + o(1)).
\]

Summing over \( i \) ends the proof. \( \square \)
We will first show concentration results for \( \hat{L}^h \) before moving to \( L \). We start by stating a set of conditions that will ensure the theorems hold.

**Conditions 3.** We say that \((a(n), b(n), c(n), d(n))\) verifies Conditions if:

\[
a(n) + c(n) = o(n),
\]

and:

\[
limit_{n} (a(n) - b(n)) = +\infty,
\]

and

\[
d(n) = O(a(n) - b(n)),
\]

and

\[
\left( \sqrt{(a(n) - b(n))(a(n) + c(n))}, d(n) \right)
\]

verify Conditions.

We start with the following technical lemma. Concentration follows here from the concentration of the ordered weights proved in the previous section.

**Lemma 24.** Suppose that Conditions hold. There exists a constant \( A > 0 \) such that, if \((m, l, h, y)\) verifies Conditions then the following is true:

\[
\mathbb{P} \left( \sup_{t \leq i \leq j \leq m} \left( \left| \hat{L}^h_i - \tilde{L}^h_i - \mathbb{E} \left[ \tilde{L}^h_j - \tilde{L}^h_i \right] \right| \right) \geq y \right) \leq A \exp \left( \frac{-y^2}{A(n^{1/3} + m - l)} \right),
\]

Proof. Let:

\[
D = \mathbb{P} \left( \sup_{t \leq i \leq j \leq m} \left( \left| \hat{L}^h_i - \tilde{L}^h_i - \mathbb{E} \left[ \tilde{L}^h_j - \tilde{L}^h_i \right] \right| \right) \geq y \right)
\]

Since \( p_f \geq 1/n \) and \( m - l = o(n) \). Conditions and Equation yield:

\[
\sum_{k=i+1}^{j} \sum_{k' > k + h} (1 - e^{-w(v(k))w(v(k'))p_f}) \leq \mathbb{E} \left[ 1 - e^{-w(v(k))w(v(k'))p_f} \right]
\]

\[
= \sum_{k=i+1}^{j} \left( \sum_{k' > k + h} w(v(k))w(v(k'))p_f - \mathbb{E}[w(v(k))w(v(k'))p_f] \right) + O(1).
\]

Moreover, recall, by our conditions, that \( y = y(n) \) and \( \lim_{n \to \infty} y(n) = +\infty \).

Since

\[
\sum_{k' > k + h} w(v(k))w(v(k'))p_f = \sum_{k'=1}^{n} \sum_{k' > k + h} w(v(k))w(v(k'))p_f - \sum_{k' \leq k + h} w(v(k))w(v(k'))p_f,
\]

we obtain by the union bound for \( n \) large enough:

\[
D \leq \mathbb{P} \left( \sup_{t \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} \left( \sum_{k' > k + h} w(v(k))w(v(k'))p_f - \mathbb{E} \left[ \sum_{k'=i+1}^{j} \left( \sum_{k' > k + h} w(v(k))w(v(k'))p_f \right) \right] \right) \right| \geq y/2 \right)
\]

\[
\leq \mathbb{P} \left( \sup_{t \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} \left( \sum_{k' \leq k + h} w(v(k))w(v(k'))p_f - \mathbb{E} \left[ \sum_{k'=i+1}^{j} \left( \sum_{k' \leq k + h} w(v(k))w(v(k'))p_f \right) \right] \right) \right| \geq y/4 \right)
\]

\[
+ \mathbb{P} \left( \sup_{t \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} w(v(k)) - \mathbb{E} \left[ \sum_{k=i+1}^{j} w(v(k)) \right] \geq \frac{y}{4l_{np}p_f} \right) \right).
\]

(33)
Since $\ell_n pf \leq 2$, by Conditions 3 we can apply Theorem 21 with $(m, l, y)$ to obtain:

$$P \left( \sup_{1 \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} w_{v(k)} - E \left[ \sum_{k=i+1}^{j} w_{v(k)} \right] \right| \geq \frac{y}{4\ell_n pf} \right) \leq A \exp \left( \frac{-y^2}{A(yn^{1/3} + m - l)} \right).$$

(34)

By injecting Inequality (34) in Inequality (33), bounding $D$ amounts to bounding:

$$P \left( \sup_{1 \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} \left( \sum_{k' \leq k+h} w_{v(k)} w_{v(k')} pf \right) - E \left[ \sum_{k=i+1}^{j} \left( \sum_{k' \leq k+h} w_{v(k)} w_{v(k')} pf \right) \right] \right| \geq y/4 \right).$$

We focus on proving a one-sided version of this inequality, the other half of the inequality is proven similarly:

$$P \left( \sup_{1 \leq i \leq j \leq m} \left| \sum_{k=i+1}^{j} \left( \sum_{k' \leq k+h} w_{v(k)} w_{v(k')} pf \right) - E \left[ \sum_{k=i+1}^{j} \left( \sum_{k' \leq k+h} w_{v(k)} w_{v(k')} pf \right) \right] \right| \geq y/4 \right) .$$

By Lemmas 8 and 10 for any $l \leq i \leq j \leq m$:

$$E \left[ \sum_{k=i+1}^{j} \left( \sum_{k' \leq k+h} w_{v(k)} w_{v(k')} pf \right) \right] = \frac{j^2 - i^2 + 2(j - i)h}{2\ell_n} (1 + o(1)).$$

(35)

By a simple computation, Conditions 3 imply that $(m + h, \frac{y(m+h)}{16(m-l)})$ verify Conditions 2. Using this with Theorem 21 yields for $n$ large enough:

$$P \left( \sup_{1 \leq k \leq m} \left| \sum_{k' \leq k+h} w_{v(j)} - E \left[ \sum_{k' \leq k+h} w_{v(j)} \right] \right| \geq \frac{y}{16 pf (m-l)} \right) \leq P \left( \sup_{1 \leq k \leq m+h} \left| \sum_{k=1}^{k} w_{v(j)} - E \left[ \sum_{k=1}^{k} w_{v(j)} \right] \right| \geq \frac{y(m+h)}{16(m-l)} \right) \leq A \exp \left( -\frac{y^2(m+h)^2}{A(y(m+h)(m-l)n^{1/3} + (m+h)(m-l))^2} \right) \leq A \exp \left( -\frac{y^2}{A(yn^{1/3} + (m-l))} \right).$$
Hence, by the above inequality and Equation (35) the union bound yields:

\[
P \left( \sup_{l \leq k \leq m} \left( \sum_{k' \leq k + h} w_{v(k')} \left( \sum_{k' \leq k + h} w_{v(k')} p_f \right) \right) - E \left[ \sum_{k=1}^j \left( \sum_{k' \leq k + h} w_{v(k')} w_{v(k')} p_f \right) \right] \right) \geq y/4 \]

\[
\leq P \left( \sup_{l \leq k \leq m} \left| \sum_{k' \leq k + h} w_{v(k')} - E \sum_{k' \leq k + h} w_{v(k')} \right| \geq \frac{y}{16p_f(m-l)} \right)
\]

\[
+ P \left( \sup_{l \leq k \leq m} \left( \sum_{k=1}^j w_{v(k)} \left( \frac{y}{16(m-l)} \right) \right) \geq y/8 \right)
\]

\[
\leq A \exp \left( \frac{-y^2}{8(1+m)} \right) + P \left( \sup_{l \leq k \leq m} \left( \sum_{k=1}^j w_{v(k)} \left( \frac{y}{16(m-l)} \right) \right) \geq y/8 \right)
\]

By Lemma \[9\] for any \( k \leq m \):

\[
E \left[ \sum_{k' \leq k + h} w_{v(k')} p_f \right] = \frac{(k + h)(1 + o(1))}{\ell_n}. \tag{37}
\]

Moreover, notice that for any \( l \leq i \leq j \leq m \):

\[
\sum_{k=i+1}^j \frac{w_{v(k)} k + h}{\ell_n} = \frac{i}{\ell_n} \sum_{k=i+1}^j w_{v(k)} + \frac{h}{\ell_n} \sum_{k=i+1}^j w_{v(k)} + \left( \frac{1}{\ell_n} \right) \sum_{k=i+1}^j \sum_{k' = k}^j w_{v(k')}.
\]

By Conditions \[3\] we have for any \( l \leq i \leq j \leq m \):

\[
\frac{j^2 - i^2 + 2(j - i)h}{2\ell_n} = O(y). \tag{38}
\]

Moreover, by Conditions \[3\] \( y \leq A(m-l) \), for some large constant \( A > 0 \). Hence,
by the union bound, Equation (36) becomes:
\[
\mathbb{P}\left( \sup_{i \leq j \leq m} \left( \frac{i}{\ell_n} \sum_{k=i+1}^{j} (w_{v(k)} - E[w_{v(k)}])(1 + o(1)) \right) \geq \frac{y}{48} \right)
\]
\[
+ \mathbb{P}\left( \sup_{i \leq j \leq m} \left( \frac{1}{\ell_n} \sum_{k=i+1}^{j} (w_{v(k)} - E[w_{v(k)}])(1 + o(1)) \right) \geq \frac{y}{48} \right)
\]
\[
+ A \exp\left( -\frac{y^2}{A(y^{1/3} + (m - l))} \right) + \mathbb{P}\left( \sup_{i \leq j \leq m} \left( \sum_{k=i+1}^{j} w_{v(k)} \right) \geq 2\frac{y}{\Lambda} \right) .
\]
Notice that we implicitly use Equation (38) in the above Inequality in order to make the \(o\) factors match and at the cost of taking \(y/48\). This is why we are able to write:
\[
\mathbb{P}\left( \sup_{i \leq j \leq m} \left( \frac{i}{\ell_n} \sum_{k=i+1}^{j} (w_{v(k)} - E[w_{v(k)}])(1 + o(1)) \right) \geq \frac{y}{48} \right),
\]
instead of:
\[
\mathbb{P}\left( \sup_{i \leq j \leq m} \left( \frac{i}{\ell_n} \sum_{k=i+1}^{j} (w_{v(k)}(1 + o(1)) - E[w_{v(k)}](1 + o(1)) \right) \geq \frac{y}{24} \right) .
\]
By Conditions 3, we can apply Theorem 21 with \((m - l, y/48)\) to obtain:
\[
\mathbb{P}\left( \sup_{i \leq j \leq m} \left( \sum_{k=i+1}^{j} w_{v(k)} - E \left[ \sum_{k=i+1}^{j} w_{v(k)} \right] \right) \geq \frac{y}{48} \right) \leq A \exp\left( -\frac{y^2}{A(y^{1/3} + m - l)} \right) .
\]
We finish by noticing that the first three probabilities in the right-hand side of Inequality (39) are all smaller than the left hand-side of (40).

**Theorem 25.** Suppose that Conditions 4 hold. There exists a constant \(A > 0\) such that, if \((m, l, 0, y)\) verifies Conditions 5 then the following holds:
\[
\mathbb{P}\left( \sup_{i \leq u \leq w \leq m} \left| L_{w}^{0} - L_{u}^{0} - E[L_{w}^{0} - L_{u}^{0}] \right| \geq y \right) \leq A \exp\left( -\frac{y^2}{A(y^{1/3} + m - l)} \right) .
\]

**Proof.** For \(i \geq 0\) let \(\mathcal{F}_{i}^{0}\) be the sigma-field generated by \(V_{i+1}\) and the random variables \((X_{k}^{0})_{k \leq i}\). Write:
\[
D_{1} = \mathbb{P}\left( \sup_{i \leq u \leq w \leq m} \left| L_{w}^{0} - L_{u}^{0} - \sum_{i=u+1}^{w} E \left[ X_{i}^{0} | \mathcal{F}_{i}^{0} \right] \right| \geq y/2 \right) ,
\]


and
\[
D_2 = \mathbb{P} \left( \sup_{t \leq u \leq w \leq m} \left| \sum_{i=u+1}^{w} \mathbb{E} \left[ X_i^0 | \mathcal{F}_{i-1}^0 \right] - \mathbb{E}[L_w^0 - L_u^0] \right| \geq y/2 \right).
\]

Then, by the union bound:
\[
\mathbb{P} \left( \sup_{t \leq u \leq w \leq m} \left| L_w^0 - L_u^0 - \mathbb{E}[L_w^0 - L_u^0] \right| \geq y \right) \leq D_1 + D_2.
\]

We start by bounding \( D_1 \). We have by the union bound:
\[
D_1 \leq \mathbb{P} \left( \sup_{t \leq u \leq m} \left| L_w^0 - L_u^0 - \sum_{i=t+1}^{w} \mathbb{E} \left[ X_i^0 | \mathcal{F}_{i-1}^0 \right] \right| \geq y/4 \right) + \mathbb{P} \left( \sup_{t \leq u \leq m} \left| L_w^0 - L_u^0 - \sum_{i=t+1}^{w} \mathbb{E} \left[ X_i^0 | \mathcal{F}_{i-1}^0 \right] \right| \geq y/4 \right).
\]

Notice that
\[
\left( L_w^0 - L_u^0 - \sum_{i=t+1}^{w} \mathbb{E} \left[ X_i^0 | \mathcal{F}_{i-1}^0 \right] \right)_{w \geq t}
\]
is a martingale with respect to the \( (\mathcal{F}_i^0)_{i \geq t} \). Moreover:
\[
\mathbb{E}[X_i^0 | \mathcal{F}_{i-1}^0] = \sum_{k \in \mathcal{V}, k' \notin \mathcal{V}} \mathbb{E}[Y(v(i), k)Y(v(i), k') | \mathcal{F}_{i-1}^0] \leq w_{v(i)} + w_{v(i)}^2.
\]

Applying Theorem 6 to the \( (w_{v(i)}^2) \)'s, let \( J(1), J(2), \ldots \) be i.i.d copies of \( v(l + 1) \). We have by Lemma 8 the following Bernstein inequality:
\[
\mathbb{P} \left( \sum_{i=l+1}^{m} w_{v(i)}^2 \geq \sum_{i=l+1}^{m} 2\mathbb{E}[w_{v(i)}^2] + 2yn^{1/3} \right)
\]
\[
\leq \mathbb{E} \left[ \exp \left( \frac{\sum_{i=l+1}^{m} w_{v(i)}^2}{\exp \left( \frac{\sum_{i=l+1}^{m} 2C\mathbb{E}[w_{v(i)}^2] + 2yn^{1/3}}{\exp \left( \frac{\sum_{i=l+1}^{m} 2C\mathbb{E}[w_{v(i)}^2] + 2yn^{1/3}}{2C(m - l)(1 + o(1)) + 2yn^{1/3}} \right) \right)} \right) \right]
\]
\[
\leq \mathbb{E} \left[ \exp \left( \frac{-2C(m - l)(1 + o(1)) + 2yn^{1/3} - \sum_{i=l+1}^{m} \mathbb{E}[w_{v(i)}^2]}{(Ayn + A(m - l)n^{2/3} + \sum_{i=l+1}^{m} \mathbb{E}[w_{v(i)}^2])} \right) \right]
\]
\[
\leq \mathbb{E} \left[ \exp \left( \frac{-2C(m - l)(1 + o(1)) + 2yn^{1/3} - \sum_{i=l+1}^{m} \mathbb{E}[w_{v(i)}^2]}{An^{2/3} \left( yn^{1/3} + (m - l) + \sum_{i=l+1}^{m} \mathbb{E}[w_{v(i)}^2] \right)} \right) \right],
\]

where line 3 of the equation is a Chernoff bound which yields Bernstein’s inequality in line 4 (as in the original proof of Bernstein [1924]), and we used the
fact that, by Conditions 1, we have $E[w^4_{J(i)}] \leq n^{2/3}E[w^3_{J(i)}]$. Now notice that by definition, and by Lemma 8, for any $i \geq l + 1$:

$$E[w^2_{J(i)}] = C(1 + o(1)). \quad (42)$$

Since $(m, l, 0, y)$ verifies Conditions 3, we can apply Theorem 21 on $(m, l, 2y)$ to obtain:

$$P \left( \sum_{i=l+1}^{m} w_{v(i)} \geq \sum_{i=l+1}^{m} E[w_{v(i)}] + 2y \right) \leq A \exp \left( -\frac{y^2}{A (y n^{1/3} + (m - l))} \right), \quad (43)$$

where $A > 0$ is a large enough constant. By Equations (41), (42) and (43) and by Bernstein's inequality for martingales (Theorem 2.1 in Freedman [1975]) we obtain:

$$D_1 \leq A' \exp \left( -\frac{y^2}{A' (y n^{1/3} + m - l)} \right). \quad (44)$$

In order to bound $D_2$ notice that the sum inside $D_2$ is equal to the one in Lemma 24 when $h = 0$ by definition. This finishes the proof. $\square$

Since $L^0$ is always greater than $L'$ deterministically, Theorem 25 gives us the following theorem.

**Theorem 26.** Suppose that Conditions 1 hold. Let $4f \ell_n^{1/3} \geq m \geq \frac{f \ell_n^{1/3}}{C}$, then there exists $A > 0$ and $A' > 0$ such that for any $\epsilon > 0$:

$$P \left( \max_{1 \leq i \leq m} (L_i) \geq \frac{10 f \ell_n^{1/3}}{C} \right) \leq A \exp \left( -\frac{f}{A} \right).$$

**Proof.** By definition:

$$\max_{1 \leq i \leq m} (L_i) \leq \max_{1 \leq u \leq v \leq m} (L_v^0 - L_u^0).$$

Hence

$$P \left( \max_{1 \leq i \leq m} (L_i) \geq \frac{10 f \ell_n^{1/3}}{C} \right) \leq P \left( \max_{1 \leq u \leq v \leq m} (L_v^0 - L_u^0) \geq \frac{10 f \ell_n^{1/3}}{C} \right). \quad (45)$$

From Corollary 23.1

$$\min_{1 \leq u \leq v \leq m} (E[L_v^0 - L_u^0]) = \min_{1 \leq u \leq v \leq m} \left( (v - u) \left( f \ell_n^{1/3} - \frac{C(v + u)}{2 \ell_n} \right) (1 + o(1)) + 1 \right)$$

$$\geq \min_{1 \leq u \leq v \leq m} \left( -\frac{C(v + u)(v - u)}{2 \ell_n} (1 + o(1)) + 1 \right)$$

$$\geq -9 f \ell_n^{1/3}. \quad (46)$$

We finish by injecting Equation (46) in (45) and using Theorem 25 with $(m, 1, 0, \ell_n^{1/3})$. $\square$

The same method that we used to bound the term $D_1$ in the proof of Theorem 25 directly yields

33
Theorem 27. Suppose that Conditions 1 hold. There exists a constant $A > 0$ such that, if $(m, l, 0, y)$ verifies Conditions 3, then the following holds:

$$
\mathbb{P} \left( \sup_{l \leq u \leq v \leq m} \left| L'_v - L'_u - E[\tilde{L}_v - \tilde{L}_u] \right| \geq y \right) \leq A \exp \left( -\frac{y^2}{A(yn^{1/3} + m - l)} \right).
$$

4 The structure of the giant component

The bounds in the previous section will allow us to determine the structure of the giant component of $G(W, p_f)$. We write $H^*_f$ for the component of $G(W, p_f)$ being explored at time $\frac{f^2}{3}$. We will prove that this component is the largest one with high enough probability. Informally, the BFW has a random unbiased part plus a drift (its expectation). Corollary 23.1 shows that the drift of $L^0$ is a parabola that has its maximum at $\frac{f^2}{3}$. Given the concentration of $L^0$, and if we also assume that it behaves like $L$, it follows that $L$ also has its maximum around $\frac{f^2}{3}$. Now recall that $L$ corresponds to the number of nodes discovered but not yet explored. It is then naturally maximal when the exploration process is in a large connected component. Hence $H^*_f$ should be the largest component. In this section we will prove this rigorously. Then we will prove in the following section that the other connected components are small enough.

![Figure 7: The largest connected component of the graph in Figure 1. Its size is 2654.](image)

4.1 The size of the giant component

Theorem 28. Suppose that Conditions 1 hold. Let $1 > \epsilon' > 0$. For $f$ large enough and for any $1 \geq \epsilon > 0$ consider the following event:

The exploration of $H^*_f$ starts before time $\frac{f^{2/3}}{\epsilon'}$ and ends between times $\frac{1}{f^{2/3}}$.

\[3\]In the rest of the proof, and in order to ease notations we do not use integer part notations for the indices and instead abuse notation by using real indices in our sums sometimes.
and $2(1+\epsilon')f^2\ell_n^{2/3}$.

Then there exists a positive constant $A > 0$ such that the probability of this event not happening is at most

$$A \exp \left( \frac{-f^\epsilon}{A} \right).$$

**Proof.** Let $t_1 = \frac{\ell_n^{2/3}}{f^{1-\epsilon}C}$, $t_2 = \frac{2(1-\epsilon')f^2\ell_n^{2/3}}{C}$ and $t_3 = \frac{2(1+\epsilon')f^2\ell_n^{2/3}}{C}$.

In order to prove this theorem we need to bound the probability that $L$ visits zero between times $t_1$ and $t_2$ and also the probability that $L$ does not visit 0 between times $t_2$ and $t_3$. Recall that for any $i$:

$$\tilde{L}_i = \sum_{k=1}^i \mathbb{E}[X_k | F_{k-1}].$$

We start by the probability of the first event. Recall that by definition $L \geq L'$.

We will thus focus on $L'$. For any $h > 0$, $\tilde{L}$ is at least $\tilde{L}_h$ until the first time $i$ when $L_i \geq h$.

Let $h = \frac{10f^2\ell_n^{1/3}}{C}$. Then by Theorem 26 and Conditions 1:

$$P \left( \sup_{1 \leq j \leq t_2} L_j \geq h \right) \leq A \exp \left( \frac{-f^\epsilon}{A} \right). \tag{47}$$

Now divide the interval $[t_1, t_2]$ by introducing intervals of the form $[t_i', t_{i+1}']$ with

$$t_i' = t_1 + \frac{2i+1}{f^{1-\epsilon}C}.$$ 

This subdivision is necessary in order to respect Conditions when we apply our concentration theorems. We stop at $t_i' = t_2$ by truncating the last interval. By Corollary 23 and a straightforward calculation, for $i < i - 1$:

$$\min_{t_i' \leq j \leq t_{i+1}'} \mathbb{E}(\tilde{L}^h_j) \geq \frac{2^{1+1} f^\epsilon \ell_n^{1/3}}{2C}, \tag{48}$$

and:

$$\min_{t_{i-1}' \leq j \leq t_i'} \mathbb{E}(\tilde{L}^h_j) \geq \frac{2^{i+1} f^\epsilon \ell_n^{1/3}}{2C}. \tag{49}$$

A simple computation shows that we can apply Theorem 24 to $\tilde{L}^h$ between 1 and $t_{i+1}$ in order to obtain the following inequalities for $i < i - 1$ and for $i$:

$$P \left( \inf_{t_i' \leq j \leq t_{i+1}'} (\tilde{L}^h_j - \mathbb{E}(\tilde{L}^h_j)) \leq -\frac{2^{i-1} f^\epsilon \ell_n^{1/3}}{2C} \right) \leq A \exp \left( \frac{-f^\epsilon}{A} \right),$$

$$P \left( \inf_{t_{i-1}' \leq j \leq t_i'} (\tilde{L}^h_j - \mathbb{E}(\tilde{L}^h_j)) \leq -\frac{f^\epsilon \ell_n^{1/3}}{4C} \right) \leq A \exp \left( \frac{-f^\epsilon}{A} \right). \tag{50}$$
By the union bound using Equations (48), (49) and (50), we get:

\[
P \left( \inf_{t_1 \leq t \leq t_2} L_j \leq 0 \right) \\
\leq \sum_{i=0}^{\infty} \left( \inf_{t' \leq t_i} (L_{i'} - \bar{L}_j) - \frac{2^{i-1} e f 2^{i} t_{n}^{1/3}}{2C} \right) + \sum_{i=0}^{\infty} \left( \inf_{t' \leq t_i} (L_{i'} - \bar{L}_j) - \frac{2^{i-1} e f \ell_{n}^{1/3}}{2C} \right) \\
\leq \sum_{i=0}^{\infty} \left( \inf_{t' \leq t_i} (L_{i'} - \bar{L}_j) - \frac{2^{i-1} e f 2^{i} t_{n}^{1/3}}{2C} \right) + \sum_{i=0}^{\infty} \left( \inf_{t' \leq t_i} (L_{i'} - \bar{L}_j) - \frac{2^{i-1} e f \ell_{n}^{1/3}}{2C} \right) \\
\leq \sum_{i=0}^{\infty} A \exp \left( \frac{-2^{i-1} f t_{n}}{A} \right) + \sum_{i=0}^{\infty} A \exp \left( \frac{-f t_{n}}{A} \right) \\
\leq A' \exp \left( \frac{-f t_{n}}{A'} \right),
\]

here the constant $A' > 0$ is large enough and of course these inequalities only hold for $n$ large enough.

We now show that $L$ visits 0 between times $t_2$ and $t_3$. Recall that $(Z(i))_{i \leq n}$ is defined by $Z(i) = L_i - L_{i'}$. Then if $L_{t_2} \leq -Z(t_2)$, it means that $L'$ attained a new minimum between $t_2$ and $t_3$, i.e., $L$ visited 0 between $t_2$ and $t_3$. Also, by construction, $Z(i) = -\min_{j \leq i} (L_{i'}) + 1$. Since $L'$ is deterministically smaller than $L^0$, if $L_{t_3}^0 \geq -Z(t_2)$ then $L_{t_3}^0 \geq -Z(t_2)$. Therefore, it is sufficient to bound $P(L_{t_3}^0 \geq -Z(t_2)).$ We do so by introducing an intermediate term:

\[
P(L_{t_3}^0 \geq -Z(t_2)) \leq P \left( L_{t_3}^0 \geq \frac{\epsilon' f 2^{i} t_{n}^{1/3}}{C} \right) + P \left( Z(t_2) \geq \frac{\epsilon' f 2^{i} t_{n}^{1/3}}{C} \right) \\
\leq P \left( L_{t_3}^0 \geq \frac{\epsilon' f 2^{i} t_{n}^{1/3}}{C} \right) + P \left( Z(t_2) \geq \frac{\epsilon' f 2^{i} t_{n}^{1/3}}{C} \right),
\]

we bound each one of the two terms of the right-hand side of (52) separately.
First:
\[
P\left(Z(t_2) \geq \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right) \leq P\left(Z(t_1) \geq \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right) + P\left(Z(t_2) > Z(t_1)\right).
\]

Since \(Z(t_2) > Z(t_1)\) occurs precisely if \(L\) visits 0 between \(t_1\) and \(t_2\) we already know by Equation (51) that:
\[
P(Z(t_2) > Z(t_1)) \leq A' \exp \left(\frac{-f^c}{A'}\right).
\]  

By definition \(Z(t_1) \geq r\) precisely if \(L_i < 1 - r\) for some \(i \leq t_1\). By Corollary 23.1, for any \(i \leq t_1\):
\[
E(L_i) \geq 0.
\]

Using this inequality alongside Inequality (47) and Theorems 24 and 27 yields:
\[
P\left(Z(t_1) \geq \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right)
\]
\[
= P\left(\inf_{1 \leq i \leq t_1} (L_i') \leq 1 - \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right)
\]
\[
\leq P\left(\inf_{1 \leq j \leq t_1} (\tilde{L}_j) \leq - \frac{\epsilon f^2 \ell_n^{1/3}}{4C}\right) + P\left(\inf_{1 \leq j \leq t_1} (L_j' - \tilde{L}_j) \leq - \frac{\epsilon f^2 \ell_n^{1/3}}{4C}\right) + P\left(\sup_{1 \leq j \leq t_1} L_j \geq h\right)
\]
\[
\leq A \exp \left(\frac{-f^c}{A}\right).
\]  

By the union bound between Equations (53) and (54) we get:
\[
P\left(Z(t_2) \geq \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right) \leq A \exp \left(\frac{-f^c}{A}\right).
\]  

Furthermore, by Corollary 23.1
\[
E[L_{t_3}^0] \leq - \frac{2\epsilon f^2 \ell_n^{1/3}}{C}.
\]

By this fact and Theorem 25 we obtain:
\[
P\left(L_{t_3}^0 \geq - \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right) \leq P\left(L_{t_3}^0 - E[L_{t_3}^0] \geq \frac{\epsilon f^2 \ell_n^{1/3}}{C}\right)
\]
\[
\leq A' \exp \left(\frac{-f^c}{A'}\right).
\]

Injecting Inequalities (55) and (56) in Inequality (52) yields:
\[
P(L_{t_3}^0 \geq -Z(t_2)) \leq A \exp \left(\frac{-f^c}{A}\right),
\]

and this finishes the proof. \(\square\)
The following theorem gives a lower and upper bound on the total weight of $H^*_f$.

**Theorem 29.** Suppose that Conditions 1 hold. Let $1 > \epsilon' > 0$. For $f$ large enough and for any $1 \geq \epsilon > 0$, let $t_1 = \frac{\ell^{2/3}}{f\epsilon\ell'}$, $t_2 = \frac{2(1-\epsilon')f^2/3}{C}$ and $t_3 = \frac{2(1+\epsilon')f^2/3}{C}$.

There exists a constant $A > 0$ such that the probability that the total weight of $H^*_f$ is less than $t_2 - t_1 - \epsilon'(t_2 - t_1)$ or more than $t_3 + \epsilon't_3$ is at most $A \exp\left(-\frac{f\epsilon'}{A}\right)$.

**Proof.** Let $E$ be the event that $L_i$ visits 0 for an $t_1 \leq i \leq t_2$ or $L_i$ does not visit 0 for any $t_2 \leq i \leq t_3$. For $n$ large enough, Theorem 28 states that there exists $A > 0$ such that:

$$\mathbb{P}(E) \leq A \exp\left(-\frac{f\epsilon'}{A}\right).$$

If $E$ does not hold, the total weight of $H^*_f$ is larger than:

$$T = \sum_{i=t_1}^{t_2} w_{v(i)}.$$

By Lemma 11

$$\mathbb{E}[T] = (t_2 - t_1) + o(t_2 - t_1).$$

By Theorem 21 there exist positive constants $A'', A'''$ such that:

$$\mathbb{P}[T \leq \mathbb{E}(T) - \epsilon'(t_2 - t_1)] \leq A'' \exp\left(-\frac{\epsilon' f^{2/3}}{A''}\right),$$

hence by the union bound the total weight of $H^*_f$ is less than $t_2 - t_1 - \epsilon'(t_2 - t_1)$ with probability at most:

$$\mathbb{P}[T \leq (t_2 - t_1) - \epsilon'(t_2 - t_1)] + \mathbb{P}(E) \leq A' \exp\left(-\frac{f^2}{A'}\right),$$

where $A > 0$ is a large constant. Moreover when $E$ does not hold the total weight of $H^*_f$ is less than:

$$T' = \sum_{i=0}^{t_3} w_{v(i)}.$$

By the same arguments $H^*_f$ is more than $t_3 + \epsilon't_3$ with probability at most:

$$\mathbb{P}[T' \geq t_3 + \epsilon't_3] + \mathbb{P}(E) \leq A' \exp\left(-\frac{f'}{A'}\right).$$

\[\square\]
4.2 The excess of the giant component.

The previous theorems give us information about the size of $H_f^*$. We now turn to its surplus. Recall that the surplus (or excess) is the number of edges we need to remove from a connected graph in order to make it a tree. The excess of a general graph is the sum of excesses of its connected components.

**Theorem 30.** Suppose that Conditions hold. Let $\text{Exc}$ be the excess of $H_f^*$, there exists a positive constant $A > 0$ such that:

$$\mathbb{P}(\text{Exc} \geq Af^3) \leq A \exp \left( -\frac{f}{A} \right).$$

**Proof.** By construction, if a component is discovered between times $t_1$ and $t_2$ of the process, then its excess is precisely

$$\sum_{i=t_1}^{t_2} \sum_{j=i+1}^{L_i-1} Y(v(i), v(j)).$$

Let $m = \frac{3t^2}{2c}$. By Theorem 26

$$\mathbb{P} \left( \sup_{1 \leq i \leq m} (L_i) \geq \frac{10f^2t^{1/3}}{C} \right) \leq A'' \exp \left( -\frac{f}{A''} \right). \quad (57)$$

By Theorem 28, there exists a constant $A' > 0$ such that the probability that $H_f^*$ has size more than $m$ is at most:

$$A' \exp \left( -\frac{f}{A} \right). \quad (58)$$

Let $E$ be the event that $H_f^*$ has size less than $m$ and $L_i \leq \frac{10f^2t^{1/3}}{C}$ for all $1 \leq i \leq m$. By the union bound between Inequalities (57) and (58) we get:

$$\mathbb{P}(\bar{E}) \leq A'' \exp \left( -\frac{f}{A''} \right), \quad (59)$$

for some large constants $A'' > 0$. Let $R = \frac{10f^2t^{1/3}}{C}$ and:

$$U(R, i) = \sum_{j=i+1}^{L_{i-1}+1-1} Y(v(i), v(j)) + \sum_{j=L_{i-1}+1}^{R+i} Y'(v(i), v(j)),$$

with $Y'(i, j)$ being a Bernoulli random variable independent of everything else and having the same distribution as $Y(i, j)$ for $i \neq j$. We have thus by the union bound for any $l \geq 0$:

$$\mathbb{P}(\text{Exc} \geq l) \leq \mathbb{P} \left( \left\lfloor H_f^* \right\rfloor \sum_{i=1}^{L_{i-1}+1-1} Y(v(i), v(j)) \geq l, E \right) + \mathbb{P}(\bar{E})$$

$$\leq \mathbb{P} \left( \sum_{i=1}^{m} U(R, i) \geq l \right) + \mathbb{P}(\bar{E}) \quad (60)$$

39
Conditionally on \( \mathcal{F}_n \) the \( U(R,i) \)'s are sums of independent Bernoulli random variables. This is true because the first sum in the definition of \( U(R,i) \) consists on independent Bernoulli random variables as stated in Fact 23. Moreover, for any \((i,j)_{1 \leq i, 1+i \leq j \leq L, i+j-1}\) by Equation 6:

\[
E[Y(v(i),v(j))|\mathcal{F}_n] \leq w_{v(i)}w_{v(j)}p_f,
\]

and

\[
E[Y(v(i),v(j))^2|\mathcal{F}_n] \leq w_{v(i)}w_{v(j)}p_f.
\]

The first inequality yields:

\[
E\left[\sum_{i=1}^{m} U(R,i) \middle| \mathcal{F}_n\right] \leq \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)}w_{v(j)}p_f.
\]

Hence, by Bernstein’s inequality:

\[
P\left(\sum_{i=1}^{m} U(R,i) \geq l + \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)}w_{v(j)}p_f \middle| \mathcal{F}_n\right) \leq \exp\left(-\frac{l^2}{2l + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)}w_{v(j)}p_f}\right).
\]

(61)

Denote by \( J_1, J_2, \ldots, J_n \) i.i.d. copies of \( v(1) \). From Lemma 8 there exists a constant \( A' > 0 \) such that:

\[
E\left[p_f \sum_{k=0}^{\left\lceil \frac{m}{2}\right\rceil} (k+2)R \sum_{j=kR+1}^{j=kR+1} w_{v(i)}^2\right] \leq A'mRp_f.
\]

(62)

Moreover, by Cauchy-Schwarz’s inequality:

\[
P\left(\sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)}w_{v(j)}p_f \geq (A' + 1)mRp_f\right) \leq P\left(p_f \sum_{k=0}^{\left\lceil \frac{m}{2}\right\rceil} (k+2)R \sum_{j=kR+1}^{j=kR+1} w_{v(i)}^2 \geq (A' + 1)mRp_f\right)
\]

\[
\leq P\left(p_f \sum_{k=0}^{\left\lceil \frac{m}{2}\right\rceil} 2R \sum_{j=kR+1}^{j=kR+1} w_{v(i)}^2 \geq (A' + 1)mRp_f\right).
\]

Hence, by Theorem 6 applied on the \((w_{v(i)}^2)_{1 \leq i \leq m} \)'s and Inequality (62) we have the following Chernoff bound which yields a Bernstein’s inequality 40.
Here the penultimate inequality uses the fact that \( \mathbb{E}[w_v^4(1)] \leq n^2/3 \mathbb{E}[w_v^2(1)] \) and Lemma 8. We have that \( m R p_f = A f^3 \) for some \( A > 0 \). By Equations 59, 60, 61 and 63, the union bound yields:

\[
\mathbb{P}( \text{Exc} \geq (A' + 1)m R p_f ) \leq \mathbb{P} \left( \sum_{i=1}^{m} U(R, i) \geq (A' + 1)m R p_f \right) + \mathbb{P}[\bar{E}]
\]

\[
\leq \exp \left( \frac{-(m R p_f)^2}{2m R p_f + 2(A' + 1)m R p_f} \right) + A'' \exp \left( -f A' \right)
\]

\[
+ \mathbb{P} \left( \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_v(i) w_v(j) p_f \geq (A' + 1)m R p_f \right)
\]

\[
\leq A'' \exp \left( \frac{-(m R p_f)^2}{A''(m R p_f)} \right) + A'' \exp \left( -f A' \right)
\]

\[
\leq A \exp \left( -\frac{f}{A} \right),
\]

where \( A > 0 \) is a large enough constant.

**4.3 The excess of the components discovered before the largest connected component.**

**Theorem 31.** Suppose that Conditions 1 hold. Let \( \text{Exc}_0 \) be the total excess of the components discovered before the largest component. There exists \( A > 0 \) such that for any \( 0 < \epsilon \leq 1 \):

\[
\mathbb{P}( \text{Exc}_0 \geq A f^\epsilon ) \leq A \exp \left( -\frac{\epsilon}{A} \right).
\]

**Proof.** We know from Theorem 28 that for any \( 0 < \bar{\epsilon} \leq 1 \) the exploration of the largest component starts before time \( m = \frac{f_1}{f_1 - \bar{c}} \) with probability at least:

\[
1 - A \exp \left( -\frac{\bar{\epsilon} f}{A} \right).
\]
In that case the total excess of components discovered before the largest one is at most:

\[
\sum_{i=0}^{m} \sum_{j=i+1}^{L_{i-1+i-1}} Y(v(i), v(j)).
\]

By Corollary 23.1 and Conditions 1 for any \(0 \leq i \leq j \leq m\):

\[
\mathbb{E}(L^0(j) - L^0(i)) \leq \frac{f \ell_n^{1/3}}{C}.
\]

By this fact and Theorem 25 applied on \((m, 0, 0, y)\), there exists an \(A > 0\) such that:

\[
P \left( \sup_{0 \leq i \leq j \leq m} (L^0(j) - L^0(i)) \geq \frac{2f \ell_n^{1/3}}{C} \right) \leq A \exp \left( \frac{-f^2}{A} \right).
\]

Remark that, deterministically,

\[
\sup_{0 \leq k \leq m} L(k) \leq \sup_{0 \leq i \leq j \leq m} (L'(j) - L'(i)) \leq \sup_{0 \leq i \leq j \leq m} (L^0(j) - L^0(i)),
\]

hence:

\[
P \left( \sup_{0 \leq i \leq j \leq m} L_i \geq \frac{2f \ell_n^{1/3}}{C} \right) \leq P \left( \sup_{0 \leq i \leq j \leq m} (L^0(j) - L^0(i)) \geq \frac{2f \ell_n^{1/3}}{C} \right),
\]

\[
\leq A \exp \left( \frac{-f^2}{A} \right).
\]

Let \(R = \frac{2f \ell_n^{1/3}}{C}\). Let \(E\) be the event \(\{\max_{0 \leq i \leq m} L_i \leq R\}\) and the exploration of the largest component starts before time \(m\). Recall the definition of \(U(R, i)\) from Theorem 30. We have for any \(l \geq 0\) by the union bound:

\[
P(\text{Exc}_0 \geq l) \leq P \left( \sum_{i=0}^{m} U(R, i) \geq l \right) + P[E].
\]

We use the same idea as in Theorem 30. By Bernstein’s inequality (Bernstein [1924]):

\[
P \left( \sum_{i=1}^{m} U(R, i) \geq l + \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)} w_{v(j) \mathcal{P}f} \mathcal{F}_n \right) \leq \exp \left( \frac{-l^2}{2l + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{v(i)} w_{v(j) \mathcal{P}f}} \right).
\]

Denote by \(J_1, J_2, ..., J_n\) i.i.d. copies of \(v(1)\). Similarly to Equation [32], there exists a constant \(A' > 0\) such that:

\[
\mathbb{E} \left[ \mathcal{P}f \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2R \left( \sum_{j=kR+1}^{(k+2)R} w_{J_i}^2 \right) \right] \leq A' m R \mathcal{P}f.
\]
And similarly to Equation (63) we have for any \( \lambda \geq 0 \):
\[
P \left( \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{e(i)} w_{e(j)} p_f \geq (A' + 1)m R f^{\lambda \bar{\epsilon}} p_f \right) \leq \mathbb{E} \left[ \exp \left( \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} 2 \left( \sum_{i=kR+1}^{(k+2)R} w_{v_i}^2 \right) \exp \left( -(A' + 1)m f^{\lambda \bar{\epsilon}} \right) \right) \right] \leq A \exp \left( -\frac{m^2 f^{2\lambda \bar{\epsilon}}}{A(m^{\frac{2}{3}} f^{\lambda \bar{\epsilon}} + m^{\frac{2}{3}} \bar{\epsilon})} \right) \leq A'' \exp \left( -\frac{f^{(\lambda+1)\bar{\epsilon}-1}}{A''} \right). \tag{69} \]

And also, Equations (64) and (65) yield:
\[
P(\bar{E}) \leq A' \exp \left( -\frac{f^{\bar{\epsilon}}}{A'} \right). \tag{70} \]

By Equations 66, 67, 69, and 70 the union bound yields for \( A'' > 0 \) large enough:
\[
P \left( Ex_0 \geq (A' + 2)m R f^{\lambda \bar{\epsilon}} p_f \right) \leq \mathbb{P} \left( \sum_{i=1}^{m} U(R, i) \geq (A' + 2)m R f^{\lambda \bar{\epsilon}} p_f \right) + \mathbb{P}[\bar{E}] \leq \exp \left( -\frac{(m R f^{\lambda \bar{\epsilon}} p_f)^2}{A''^2 (m R f^{\lambda \bar{\epsilon}} p_f)} \right) + A'' \exp \left( -\frac{f^{\bar{\epsilon}}}{A''} \right) + \mathbb{P} \left( \sum_{i=1}^{m} \sum_{j=i+1}^{(R+i)} w_{e(i)} w_{e(j)} p_f \geq (A' + 1)m R f^{\lambda \bar{\epsilon}} p_f \right) \leq A''' \exp \left( -\frac{(m R f^{\lambda \bar{\epsilon}} p_f)^2}{A''''^2 (m R f^{\lambda \bar{\epsilon}} p_f)} \right) + A'' \exp \left( -\frac{f^{\bar{\epsilon}}}{A''} \right) + A'''' \exp \left( -\frac{f^{(\lambda+1)\bar{\epsilon}-1}}{A''''} \right),
\]

where \( A > 0 \) is a large enough constant. Moreover, we have for \( n \) large enough:
\[
m R f^{\lambda \bar{\epsilon}} p_f \geq \frac{1}{C^2} f^{2(2+\lambda)\bar{\epsilon}-1}
\]

for some large constant \( A > 0 \). Hence, if we take:
\[
\lambda = \frac{2}{\epsilon},
\]
and:
\[
\epsilon = (2 + \lambda)\bar{\epsilon} - 1.
\]

We obtain \( \bar{\epsilon} = \epsilon/2 \) and:
\[
(1 + \lambda)\bar{\epsilon} - 1 = \frac{\epsilon}{2}.
\]

This proves the inequality of the theorem. \( \square \)
5 The structure of the tail’s components

5.1 Preliminaries

We call tail of the exploration process the part of it that starts after $H^*_f$ is fully explored and ends at $n$. In order to get bounds on the size, weight and excess of the tail, we will use two main ideas. Firstly we use an appropriate division of the interval that start after the exploration of $H^*_f$, and ends in $n$. Secondly we make use of the fact that the further we go in the exploration the smaller the weights we discover. These two ideas are formalized below. The rest of the proofs uses similar techniques to the ones presented in Section 4, but with the added complexity of incorporating these two ideas.

For $i \geq 1$, write:

$$\bar{k}_i = i^2 f((i + 1)^2 - i^2).$$

For $\bar{k}_i > k \geq 0$, and as long as $t_{ik} < \ell_n^{11/12}$, write:

$$t_{ik} = t + \frac{(i^2 - 1)f\ell_n^{2/3}}{C} + \frac{k\ell_n^{2/3}}{C^i f},$$

with $t = \frac{2(1 - \epsilon') f\ell_n^{2/3}}{C}$ and where $1/2 > \epsilon' > 0$ is fixed from here on. Moreover, let $(\tilde{i}, \tilde{k})$ be the first time when $\tilde{t}_{ik} \geq \ell_n^{11/12}$. For any $k > \tilde{k}$ let:

$$\tilde{t}_{ik} = t + \frac{(\tilde{i}^2 - 1)f\ell_n^{2/3}}{C} + \frac{k\ell_n^{2/3}}{C^i f},$$

$(\tilde{i}, \tilde{k})$ depends implicitly on $\epsilon'$. Moreover, by construction $\tilde{i}^2 f = o(n^{1/3})$. We are only interested in $t_{ik} \leq n$, and for simplicity, since there is no real difficulty in dealing with the boundaries, we assume everything is well truncated.

This construction gives a division of the interval between $t$ and $n$ in the following way: Take intervals of the form $[t_{i0}, t_{i+1}]$. Such intervals get larger and larger. Divide each one of them into small intervals of the form $[t_{ik}, t_{i(k+1)}]$ that get smaller with $i$. The main idea here is that the large intervals, those where $i$ changes, represent phases of the exploration where we will find connected components that are of size at most the size of small intervals $[t_{ik}, t_{i(k+1)}]$. Moreover Conditions 3 will be verified inside the small intervals for good enough deviation values, which will allow us to use all our concentration theorems. We start by showing that the maximum weight gets smaller the further we explore the tail.

Lemma 32. Suppose that Conditions 2 hold. There exists a constant $A > 0$ such that:

For any $1 \leq i \leq \tilde{i}$, the probability of discovering a weight larger than $\frac{\ell_n^{1/3}}{i^{2/3} f}$ in the BFW after time $t_0^i$ is less than:

$$A \exp \left( -\frac{i\sqrt{f}}{A} \right).$$

Proof. Recall that $(T_i)_{i \leq n}$ is a sequence of independent exponential variables with rates $(w_i/\ell_n)_{i \leq n}$. And that for any $x > 0$:

$$N(x) = \sum_{k=1}^{n} \mathbb{1}(T_k \leq x),$$
Moreover, recall that by the properties of exponential random variables, the order statistic indices \((\tilde{v}(1), \tilde{v}(2), ... \tilde{v}(n))\) of the \((T_k)_{k \leq n}\) have the same distribution as \((v(1), v(2), ... v(n))\).

Let \(x = t_0^i/2\), then by Lemma \ref{lem:order_statistic} Conditions \ref{cond:order_statistic} and obvious bounds:

\[
P(N(x) \geq t_0^i) \leq A \exp\left(\frac{-t_0^i}{A}\right). \tag{71}
\]

This equation shows that at time \(x\), the weights with indices \((\tilde{v}(t_0^i), \tilde{v}(t_0^i + 1), ... \tilde{v}(n))\) will not be picked yet with high probability. Denote the event \(\{N(x) \geq t_0^i\}\) by \(E\). For any \(k\) such that \(w_k \geq \frac{\ell^{1/3}}{i \sqrt{f}}\), we have:

\[
P(T_k \geq x, \bar{E}) \leq \exp\left(\frac{-i \sqrt{f}}{A}\right),
\]

this equation shows that a large weight has a large probability of being picked before time \(x\).

Recall that by Conditions \ref{cond:order_statistic}

\[
\sum_{k=1}^{n} w_k^3 = (\mathbb{E}[W^3] + o(1))n.
\]

Hence, the total number of weights larger than \(\frac{\ell^{1/3}}{i \sqrt{f}}\) is less than \(A' \ell^3 f^{3/2}\), where \(A' > 0\) is a large enough constant.

This yields:

\[
P\left(\sup_{k \geq t_0^i} (w_{v(k)}) \geq \frac{\ell^{1/3}}{i \sqrt{f}}\right) \leq P(E) + \sum_{k=1}^{n} P(T_k \geq x, \bar{E}) \mathbb{I}\left(w_k \geq \frac{\ell^{1/3}}{i \sqrt{f}}\right)
\]

\[
\leq \exp\left(\frac{-t_0^i}{A}\right) + A' \mathbb{E}[W^3] f^{3/2} \exp\left(\frac{-i \sqrt{f}}{A}\right)
\]

\[
\leq A'' \exp\left(\frac{-i \sqrt{f}}{A''}\right), \tag{72}
\]

whith \(A'' > 0\) a large constant and \(f\) large enough.

We now use the same notations as in the proof above. For \(0 \leq i \leq \tilde{i}\). Let \(B\) be the event that no weight larger than \(\frac{\ell^{1/3}}{i \sqrt{f}}\) is present after time \(t_0^i\). Then for any \(t_0^i \leq x\), with the notation of Section 2 when \(B\) holds we have:

\[
X(x) - X(u) = \sum_{k=1}^{n} w_k \mathbb{I}(u \leq T_k \leq x) \mathbb{I}\left(w_k \leq \frac{\ell^{1/3}}{i \sqrt{f}}\right).
\]

And:

\[
N(x) - N(u) = \sum_{k=1}^{n} \mathbb{I}(u \leq T_k \leq x) \mathbb{I}\left(w_k \leq \frac{\ell^{1/3}}{i \sqrt{f}}\right).
\]
Moreover, clearly:
\[
E \left[ \sum_{k=1}^{n} w_k \mathbb{1}(u \leq T_k \leq x) \right] \leq E[X(x) - X(u)],
\]
and
\[
E \left[ \sum_{k=1}^{n} \mathbb{1}(u \leq T_k \leq x) \right] \leq E[N(x) - N(u)].
\]
By those remarks, when \( B \) holds one can redo the proofs of Theorems 20 by only taking nodes with weights smaller than \( \frac{\ell_n^{1/3}}{\sqrt{f}} \). Then use the union bound with Lemma 32 to obtain the following theorem which is in the spirit of Theorem 20.

**Theorem 33.** Suppose that Conditions 4 hold. There exists a constant \( A > 0 \) such that the following holds: If \((m, l, 0, y)\) verify Conditions 3 and there exists \( i \leq \tilde{i} \) such that \( l \geq \tilde{t}_0 \), and \( m \leq \tilde{t}_0 \) then:

\[
P \left( \sup_{0 \leq u \leq w \leq m} \sum_{k=u}^{w} w_v(k) - E \left[ \sum_{k=u}^{w} w_v(k) \right] \geq y \right) \leq A \exp \left( \frac{-y^2}{A \left( y \frac{\ell_n^{1/3}}{\sqrt{f}} + m - l \right)} \right) + A \exp \left( \frac{-iy \sqrt{f}}{A} \right).
\]

Moreover we have by Bernstein’s inequality:

\[
P \left( \exists (h, j), j \geq \tilde{t}_0, X^0 \geq \frac{2\ell_n^{1/3}}{\sqrt{f}} \right) \leq \sum_{k=\tilde{t}_0}^{n} P \left( d_{\tilde{v}(k)} \geq \frac{2\ell_n^{1/3}}{\sqrt{f}} \right)
\leq P(\tilde{B}) + \sum_{k=0}^{n} \mathbb{1} \left( w_k \leq \frac{\ell_n^{1/3}}{\sqrt{f}} \right) P \left( d_k \geq \frac{2\ell_n^{1/3}}{\sqrt{f}} \right)
\leq P(\tilde{B}) + A' \exp \left( \frac{-\ell_n^{1/3}}{A' \sqrt{f}} \right)
\leq A \exp \left( \frac{-iy \sqrt{f}}{A} \right)
\]

(73)

where \( A \) is a large constant. This shows that, similarly to what we did in Section 2, one can assume that \( L^0 \) and \( L \) have increments of size at most \( \frac{2\ell_n^{1/3}}{\sqrt{f}} \) after time \( \tilde{t}_0 \). Using this fact, one can redo the proofs of Theorems 25 after time \( \tilde{t}_0 \). Then use the union bound with Lemma 32 and Equation (73) to obtain the following theorem which is in the spirit of Theorem 25.

**Theorem 34.** Suppose that Conditions 4 hold. There exists a constant \( A > 0 \) such that the following holds: Let \((m, l, y)\) be such that \((m, l, 0, y)\) verifies Conditions 3 and there exists \( i \leq \tilde{i} \) such that \( l \geq \tilde{t}_0 \), and \( m \leq \tilde{t}_0 \). We have:

\[
P \left( \sup_{0 \leq u \leq w \leq m} L^0_u - L^0_u - E[L^0_u - L^0_u] \geq y \right) \leq A \exp \left( \frac{-y^2}{A \left( y \frac{\ell_n^{1/3}}{\sqrt{f}} + m - l \right)} \right) + A \exp \left( \frac{-iy \sqrt{f}}{A} \right).
\]

46
We will also need the following lemma. It states that the weights get smaller in probability the further we go in the exploration. For $1 \leq k \leq n$ let $w_k = w_k^i$ if $w_k \leq \ell_1/n$ and $w_k^i = \ell_1/n$ otherwise.

**Lemma 35.** Let $1 \leq u \leq w \leq n$, then for any $x \geq 0$ and $1 \leq i \leq \tilde{i}$:

$$\mathbb{P}(w^i_{v(u)} \geq x) \leq \mathbb{P}(w^i_{v(u)} \geq x).$$

**Proof.** Recall that $V_u = (v(1), v(2), \ldots, v(u))$ for any $n \geq i \geq 1$. It is sufficient to prove the lemma for $w = u + 1$. In that case we have:

$$\mathbb{P}(w^i_{v(u)} \geq x|V_{u-1}) = \sum_{k \notin V_{u-1}} w_k \mathbb{I}(w_k^i \geq x) \sum_{k' \notin V_{u-1}} w_{k'}^i.$$

Let:

$$U = \sum_{k \notin V_{u-1}} w_k \mathbb{I}(w_k^i \geq x),$$

and

$$V = \sum_{k \notin V_{u-1}} w_k.$$

Since $V \geq U$ we have:

$$\mathbb{P}(w^i_{v(u+1)} \geq x|V_{u-1}) = \sum_{k \notin V_{u-1}} \mathbb{P}(v(u) = k|V_{u-1}) \mathbb{P}(w^i_{v(u+1)} \geq x|V_{u-1}, v(i) = k)$$

$$= \sum_{k \notin V_{u-1}} w_k \left( \frac{U - w_k \mathbb{I}(w_k^i \geq x)}{V - w_k} \right)$$

$$= \sum_{k \notin V_{u-1}, w_k^i \geq x} w_k \left( \frac{U - w_k}{V - w_k} \right) + \sum_{k \notin V_{u-1}, w_k^i < x} w_k \left( \frac{U}{V - w_k} \right)$$

$$\leq \sum_{k \notin V_{u-1}, w_k^i \geq x} w_k \left( \frac{U - x}{V - x} \right) + \sum_{k \notin V_{u-1}, w_k^i < x} w_k \left( \frac{U}{V - x} \right)$$

$$= \frac{U}{V} \left( \frac{U - x}{V - x} \right) + \left( \frac{V - U}{V - x} \right) \left( \frac{U}{V - x} \right)$$

$$= \frac{U}{V}$$

$$= \mathbb{P}(w^i_{v(u)} \geq x|V_{u-1}).$$

With this lemma in hand we can deal with the case when $m > t_0^i$.

**Theorem 36.** Suppose that Conditions 3 hold. There exists a constant $A > 0$ such that the following holds:

For $t_0^i < u \leq w$ and for any $y \geq 0$:

$$\mathbb{P} \left[ \sum_{k=u}^{w} (w(v(k)) - 1) \geq y \right] \leq A \exp \left( -\frac{y^2}{A n \sqrt{n} + w - u} \right) + A \exp \left(-\frac{-y \sqrt{F}}{A} \right)$$
Proof. Let $\mathcal{A}$ be the event that no weight discovered after time $t_0^{\frac{2}{3}}$. Let $(J(i))_{i \geq u}$ be i.i.d with the distribution of $v(u)$. Theorem 1 from Ben-Hamou et al. [2018] still applies for the $(w_i^{\gamma})_{v \geq 1}$’s and we get similarly to Theorem 6:

\[
P\left(\sum_{k=1}^{w}(w_v(k) - 1) \geq y, \mathcal{A}\right) \leq \mathbb{E}\left[\exp\left(\sum_{k=1}^{w}(w_v(k) - 1)\right)\right] \leq \mathbb{E}\left[\exp\left(\sum_{k=1}^{w}(w_v(k) - 1)\right)\right].
\]

By Lemma 35 we can apply an ordered coupling argument (Theorem 7.1 of den Hollander [2012]) in order to obtain:

\[
\mathbb{E}\left[\exp\left(\sum_{k=u}^{w}(w_v(j(k)) - 1)\right)\exp(-y)\right] \leq \mathbb{E}\left[\exp\left(\sum_{k=u}^{w}(w_v(j'(k)) - 1)\right)\exp(-y)\right].
\]

where the $J'(k)$’s are i.i.d random variables with the distribution of $v(t_0 + 1)$. Moreover by Lemma 11 we have for any $k \geq u$:

\[
\mathbb{E}[w_{j'(k)}] \leq 1
\]

and by Lemma 8:

\[
\mathbb{E}[w_{j'(k)}] \leq C(1 + o(1))
\]

Hence, by Equation 74 and the Chernoff bound in Equation 75 we obtain the following Bernstein’s inequality:

\[
P\left(\sum_{k=u}^{w}(w_v(k) - 1) \geq y, \mathcal{A}\right) \leq 2 \exp\left(-\frac{-y^2}{A\left(y^{2/3} + \sum_{k=u}^{w}\mathbb{E}\left[(w_v(j(k))^2\right]\right)}\right) \leq A \exp\left(-\frac{-y^2}{A\left(y^{2/3} + w - u\right)}\right).
\]

Moreover, by Theorem 32

\[
P(\mathcal{A}) \leq A \exp\left(-\frac{-y^2}{A}\right).
\]

We finish the proof by union bound between these last two inequalities.

By the same method, we obtain the following theorem which deals with the $w_{v(i)}$’s.

48
Theorem 37. Suppose that Conditions [7] hold. There exists a constant \( A > 0 \) such that the following holds:
For \( i < \tilde{i} \) and \( t_i^0 \leq u \leq t_i \) and for any \( y \geq 0 \):
\[
P \left( \sum_{k=u}^{w} (w_{v(k)}^2 - \mathbb{E}[w_{v(t_i^0)}^2]) \geq y \right) \leq A \exp \left( \frac{-y^2}{A \frac{\ell^2}{3} (y + w - u)} \right) + A \exp \left( \frac{-y \sqrt{f}}{A} \right).
\]
And for \( t_i^0 < u \leq w \) and for any \( y \geq 0 \):
\[
P \left( \sum_{k=u}^{w} (w_{v(k)}^2 - \mathbb{E}[w_{v(t_i^0)}^2]) \geq y \right) \leq A \exp \left( \frac{-y^2}{A \frac{\ell^2}{3} (y + w - u)} \right) + A \exp \left( \frac{-y \sqrt{f}}{A} \right).
\]

5.2 The size of connected components discovered after the largest connected component

We can now prove the main theorem on the concentration of the sizes of the components discovered after \( H_i^* \). In order to do that we will once again study the event that \( L \) visits 0 in some intervals.

Theorem 38. Suppose that Conditions [7] are verified. Let \( i^* \in \mathbb{N} \) be the time at which the exploration of \( H_i^* \) ends. There exists a constant \( A > 0 \) such that the following is true:
There exists an \( \tilde{i} \geq i \geq 1 \) and \( \bar{k}i > k \geq 0 \), such that \( L \) does not visit 0 between times \( t_i^k - t + i^* \) and \( t_i^{k+1} - t + i^* \), or times \( t_i^k - t + i^* \) and \( t_i^{k+1} - t + i^* \) is at most:
\[
A \exp \left( \frac{-y \sqrt{f}}{A} \right) + A \exp \left( \frac{-y^{1/8}}{A} \right).
\]

Proof. By Theorem 28
\[
P \left( \frac{2(1 + \epsilon') f \ell^{2/3}}{C} \geq i^* \geq \frac{2(1 - \epsilon') f \ell^{2/3}}{C} \right) \geq 1 - A \exp \left( \frac{-y \sqrt{f}}{A} \right). \tag{76}
\]

Define \( E_k^i \) as the event that \( L \) does not visit 0 between times \( t_i^k - t + i^* \) and time \( t_{k+1}^i - t + i^* \), or \( t_i^k - t + i^* \) and \( t_i^{k+1} - t + i^* \) if \( k = k_i \).

Deterministically, for any \( 0 \leq u \leq w \leq n \):
\[
P \left( L_w' - L_u' \geq 0 \right) \leq P \left( L_w^0 - L_u^0 \geq 0 \right), \tag{77}
\]
so it is sufficient to focus on \( L^0 \).

We start by dealing with \( (i, k) = (1, 0) \), then the rest of the proof consists in repeating the arguments we will give for \( (i, k) = (1, 0) \) with an induction.
In order to show that \( L \) visits 0 between \( i^* \) and \( i^* + \frac{\ell^{2/3}}{C} \), recall that \( t =
\[ \frac{2(1 - \epsilon^2) f 2/3}{C} \text{ and let } E \text{ be the event } t + \frac{2\epsilon' f 2/3}{C} \geq i^* \geq t. \] Then:

\[ P \left( L^0_{i^* + \frac{2\epsilon' f 2/3}{C}} - L^0_{i^*} \geq 0 \right) = P \left( E, \left\{ L^0_{i^* + \frac{2\epsilon' f 2/3}{C}} - L^0_{i^*} \geq 0 \right\} \right) + P(\bar{E}) \]

\[ \leq P \left( \sup_{t \leq u \leq t + \frac{2\epsilon' f 2/3}{C}} L^0_{u + \frac{2\epsilon' f 2/3}{C}} - L^0_{u} \geq 0 \right) + P(\bar{E}). \] (78)

Divide the interval between \( t \) and \( t + \frac{2\epsilon' f 2/3}{C} \) by introducing intermediate terms of the form: \( t_j' = t + \frac{2\epsilon' f 2/3}{C} \). Let \( \bar{j} \) be the largest integer such that \( t_{\bar{j}}' \leq t + \frac{2\epsilon' f 2/3}{C} \), and suppose everything is well truncated i.e \( t_{\bar{j}}' = t + \frac{2\epsilon' f 2/3}{C} \). Equation (78) then yields:

\[ P \left( \sup_{t \leq u \leq t + \frac{2\epsilon' f 2/3}{C}} L^0_{u + \frac{2\epsilon' f 2/3}{C}} - L^0_{u} \geq 0 \right) \leq \sum_{j=1}^{\bar{j}} P \left( \sup_{t_{j-1}' \leq u \leq t_j'} L^0_{u + \frac{2\epsilon' f 2/3}{C}} - L^0_{u} \geq 0 \right). \] (79)

For \( j \geq j \geq 1 \):

\[ y_j = \frac{\ell_{n}^{1/3}}{2C} (1 - 2\epsilon') + \frac{\ell_{n}^{1/3} (j - 1)}{2f^2C}. \]

By Corollary 23.1 and straightforward calculations:

\[ \sup_{t_{j-1}' \leq u \leq t_j'} E \left[ L^0_{k + \frac{2\epsilon' f 2/3}{C}} - L^0_{k} \right] \leq \frac{3}{4} E \left[ L^0_{t_j'} - L^0_{t_{j-1}'} \right] \]

\[ \leq \frac{-3y_j}{2}. \]

Moreover, for any \( j \geq j \geq 1 \), \((t_j', t_{j-1}', 0, y_j)\) verify Conditions 3. Hence, by Theorem 34 and the fact that, by definition, \( \bar{j} \leq 2f^2 \):

\[ \sum_{j=1}^{\bar{j}} P \left( \sup_{t_{j-1}' \leq u \leq t_j'} L^0_{u + \frac{2\epsilon' f 2/3}{C}} - L^0_{k} \geq 0 \right) \leq \sum_{j=1}^{\bar{j}} A \exp \left( \frac{-y_j^2}{A \left( y_j \ell_{n}^{1/3} + f^{-1} \ell_{n}^{2/3} \right)} \right) + A \exp \left( -\frac{-\sqrt{J}}{A'} \right). \]

\[ \leq A' f^2 \exp \left( -\frac{-\sqrt{J}}{A'} \right) \]

\[ \leq A'' \exp \left( -\frac{-\sqrt{J}}{A''} \right), \] (80)

we finish the initialization by injecting Inequalities (76) and (80) in (78).

We now move to the heredity property. Write

\[ \mathcal{E}_{i,k} := \cup_{(u,v) \leq (i,k)} E^u \cup \bar{E}. \]

Suppose that the following inequality holds for \((i,k)\):

\[ P (\mathcal{E}_{i,k}) \leq A \exp \left( \frac{-\sqrt{J}}{A} \right) + A \sum_{j=0}^{i} (i + 1)^2 \exp \left( \frac{-i\sqrt{J}}{A} \right) + Ak \exp \left( \frac{-i\sqrt{J}}{A} \right), \] (81)
where \( A > 0 \) is a large enough constant that does not depend on \((i, k)\).

For now suppose that \((i, k) \leq (\tilde{i}, \tilde{k})\). we want to prove a similar inequality for \((i, k + 1)\) if \(k + 1 < \tilde{k}_i\), or \((i + 1, 0)\) if not. Suppose we are in the case \(k + 1 < \tilde{k}_i\), the other case is similar. Write \( t_0 = t'_k, t_1 = t'_{k+1} + \frac{2r'f\ell_n^{2/3}}{C}\). By definition of \( \mathcal{E}_{(i,k)} \):

\[
P(\mathcal{E}_{(i,k+1)}) \leq P \left( \sup_{t_0 \leq u \leq t_1} \left( L^0_{u + \frac{\ell_n^{2/3}}{C} t} - L^0_u \right) \geq 0 \right) + P(\mathcal{E}_{(i,k)}) .
\]

By using a similar division to the one used in Inequality (80) we get again:

\[
P \left( \sup_{t_0 \leq u \leq t_1} \left( L^0_{u + \frac{\ell_n^{2/3}}{C} t} - L^0_u \right) \geq 0 \right) \leq A \exp \left( -\frac{i \sqrt{r}}{A} \right) .
\]

This finishes the induction in the case where \((i, k) \leq (\tilde{i}, \tilde{k})\).

Now suppose that \((i, k) > (\tilde{i}, \tilde{k})\), we cannot directly use Theorem 34 because \(t'_k\) might be of order \(n\). Thus, we use will use the coupling argument of Theorem 36. Similarly to Equation (79) we need to bound:

\[
\sum_{j=1}^{\tilde{j}} P \left( \sup_{t'_{j-1} \leq u \leq t'_j} \left( L^0_{u + \frac{\ell_n^{2/3}}{C} t} - L^0_u \right) \geq 0 \right) ,
\]

with \(t'_j = t'_k + \frac{2\ell_n^{2/3}}{C} j\) and \(\tilde{j}\) the largest integer such that \(t'_j \leq t'_k + \frac{2\ell_n^{2/3}}{C}\). Let

\[
y = \frac{\ell_n^{1/3} (\tilde{i}^2 - 1)}{8i^2C^2} .
\]

We have for any \(u > t'_k\):

\[
L^0_{u + \frac{\ell_n^{2/3}}{C} t} - L^0_u = \sum_{r = u + 1}^{u + \frac{\ell_n^{2/3}}{C} t} \sum_{r' > u} \left( 1 - \exp \left( -w_{v(r)}w_{v(r')}\rho_f \right) \right) = \frac{\ell_n^{2/3}}{C^2 f} \left( \sum_{r = u}^{u + \frac{\ell_n^{2/3}}{C} t} w_{v(r)} \right) \left( 1 + \frac{f}{\ell_n^{1/3}} - \sum_{r' = 1}^{u} w_{v(r')}\rho_f \right) - \frac{\ell_n^{2/3}}{C^2 f} (83)
\]

for \(u \geq 0\) let \(A_1(u)\) be the event:

\[
A_1(u) = \left\{ \sum_{r = u + 1}^{u + \frac{\ell_n^{2/3}}{C} t} w_{v(r)} < \frac{\ell_n^{2/3}}{C^2 f} y + y/2 \right\} \cap \left\{ \left( \sum_{r' = 1}^{\tilde{i}} w_{v(r')} / \ell_n \right) > \frac{\tilde{i}^2}{2} \right\} .
\]

51
Then, if $A_1(u)$ holds, then Equation (83) yields:

$$L^0_{u + \ell_n^{2/3} A} - \tilde{L}^0_u \leq -\frac{y}{2}$$

Let also $A_2(u)$ be the event:

$$\left\{ \sum_{v = u}^{u + \ell_n^{2/3} A} (w^2_{v(r)}) \leq 8y n^{1/3} + 2 \ell_n^{2/3} f \right\}.$$  

Then by Bernstein’s inequality for martingales (Freedman [1975]):

$$\mathbb{P}\left( \sup_{v_{j-1} \leq u \leq v_j} \left( L^0_{u + \ell_n^{2/3} A} - L^0_u - (\tilde{L}^0_{u + \ell_n^{2/3} A} - \tilde{L}^0_u) \right) \geq \frac{y}{2}, \cap_{v_{j-1} \leq u \leq v_j} A_2(u) \right)$$

$$\leq \mathbb{P}\left( \sup_{v_{j-1} \leq u \leq v_j} \left( L^0_{u + \ell_n^{2/3} A} - L^0_{v_{j-1}} - (\tilde{L}^0_{u + \ell_n^{2/3} A} - \tilde{L}^0_{v_{j-1}}) \right) \geq \frac{y}{4}, \cap_{v_{j-1} \leq u \leq v_j} A_2(u) \right)$$

$$+ \mathbb{P}\left( \sup_{v_{j-1} \leq u \leq v_j} (L^0_u - L^0_{v_{j-1}} - (\tilde{L}^0_u - \tilde{L}^0_{v_{j-1}}) \leq -\frac{y}{4}, \cap_{v_{j-1} \leq u \leq v_j} A_2(u) \right)$$

$$\leq \exp\left( -\frac{-\frac{y}{4}}{A'} \right),$$  \hspace{1cm} (84)

where the last inequality uses the fact that $y^2 = \Theta(\ell_n^{2/3})$.

By Theorem 36, for any $u > t_k^i$:

$$\mathbb{P}\left( \sum_{v = u}^{u + \ell_n^{2/3} A} (w^2_{v(r)} - 1) \geq y/2 \right) \leq A \exp\left( -\frac{y^2}{A r^{1/3} + \ell_n^{2/3} f} \right) + A \exp\left( -\frac{-\frac{y}{2}}{A'} \right)$$

$$\leq A' \exp\left( -\frac{-\frac{y}{2}}{A'} \right),$$  \hspace{1cm} (85)

with $A' > 0$ a large constant. By Theorem 37 we also get:

$$\mathbb{P}(A_2(u)) \leq A' \exp\left( -\frac{-\frac{y}{2}}{A'} \right).$$  \hspace{1cm} (86)

By Theorem 21 and straightforward computations we obtain:

$$\mathbb{P}\left( \sum_{v' = 1}^{t_k^i} w_{v'(r')} \leq \frac{\ell_n^{2/3}}{2} \right) \leq A' \exp\left( -\frac{-\frac{y}{2}}{A'} \right).$$  \hspace{1cm} (87)

52
By the union bound between inequalities \((84), (85), (86)\) and \((87)\) we obtain
\[
\Pr \left( \sup_{t'_{j-1} \leq u \leq t'_j} \left( L^0_{u + t'^{2/3}_j} - L^0_u \right) \geq 0 \right) \leq \exp \left( \frac{-\tilde{\gamma} \sqrt{\tilde{t}^*}}{A'} \right) + \sum_{u=t'_{j-1}}^{t'_j} \Pr(\tilde{A}_1) + \sum_{u=t'_j}^{t'_j} \Pr(\tilde{A}_2)
\]
\[
\leq A'' (t'_j - t'_{j-1}) \exp \left( \frac{-\tilde{\gamma} \sqrt{\tilde{t}^*}}{A''} \right),
\]
where \(A'' > 0\) is a large constant. Since \(t'_{k} > t'^{11/12}_n\):
\[
\ell^{11/12}_n \leq t + \left( \frac{\ell^2}{t} \right) f \ell^{2/3}_n + k \ell^{2/3}_n + C^2 f
\]
equation \((88)\) yields for \(n\) large enough:
\[
\sum_{j=1}^{\tilde{j}} \Pr \left( \sup_{t'_{j-1} \leq u \leq t'_j} \left( L^0_{u + t'^{2/3}_j} - L^0_u \right) \geq 0 \right) \leq A'' (t'_j - t'_0) \exp \left( \frac{-\tilde{\gamma} \sqrt{\tilde{t}^*}}{A''} \right),
\]
\[
\leq A \exp \left( \frac{-\tilde{\gamma} \sqrt{\tilde{t}^*}}{A} \right),
\]
where \(A > 0\) is a large constant. This finishes the proof of the induction of Equation \((81)\). By that same equation we obtain for \(n\) and \(f\) large enough:
\[
\Pr \left( \bigcup_{(u,v) \leq (\tilde{t}^*_n)} E^u_v \cup E \right) \leq A \exp \left( -\frac{\sqrt{\tilde{t}^*}}{A} \right) + A \sum_{i=0}^{\tilde{j}} (i + 1)^2 \exp \left( -\frac{\tilde{\gamma} \sqrt{\tilde{t}^*}}{A} \right) + A' \exp \left( -\frac{n^{1/8}}{A'} \right)
\]
\[
\leq A' \exp \left( -\frac{\sqrt{\tilde{t}^*}}{A''} \right) + A'' \exp \left( -\frac{n^{1/8}}{A''} \right).
\]

This theorem shows that, after exploring the largest connected component, we discover small connected components that become smaller and smaller the further the exploration process goes. From that, one can get multiple corollaries. A first one is that the total weights of the components also gets smaller and smaller. The proof is the same as that of Theorem \([29]\) and is omitted.

**Corollary 38.1.** Suppose that Conditions \([7]\) hold. There exists a constant \(A > 0\) such that the following holds:

For any \(\epsilon > 0\), the probability that there exists an \(i \geq 0\) and \(k_i \geq k \geq 0\), such that a connected component discovered between times \(t^*_k - t + i^*\) and \(t^*_k - t + i^*\) (or times \(t^*_k - t + i^*\) and \(t'^*_k - t + i^*\)) in the exploration process has total weight larger than \((1 + \epsilon)(t^*_k - t^*_k)\) (or \((1 + \epsilon)(t'^*_k - t'^*_k)\)), where \(i^* \in \mathbb{N}\) is the time when the exploration of \(H^*_k\) ends, is at most:
\[
A \exp \left( -\frac{\sqrt{\tilde{t}^*}}{A} \right) + A \exp \left( -\frac{n^{1/8}}{A} \right).
\]
Another fact we can deduce from Theorem 38 is the following convergence in probability. Its proof is straightforward from Theorems 28 and 38.

**Corollary 38.2.** Recall that \( f = f(n) \) is such that \( f(n) = o(n^{1/3}) \). Suppose that \( \lim_{n \to \infty} f(n) = +\infty \). Let \( (|C_1|, |C_2|, |C_3|, ...) \) denote the sequence of sizes of the connected components of \( G(n, W, p f(n)) \) taken in decreasing order, with the convention \( |C_i| = 0 \) if there is no \( i \)-th largest component. We have the following convergence in probability for any \( p > 7/3 \) as \( n \to \infty \):

\[
\left( \frac{|C_1|}{2f(n)\ell_2^{1/3}}, \frac{|C_2|}{\ell_2^{1/3}}, \frac{|C_3|}{\ell_2^{1/3}}, \ldots \right) \to_d (C, 0, 0, \ldots),
\]

in \( \ell^p \), the usual \( p \) norm.

**Proof.** By Theorem 28, for any \( 1 > \epsilon' > 0 \) there exists a constant \( A > 0 \) such that for \( n \) large enough:

\[
P\left( \left| \frac{|C_1|}{2f(n)\ell_2^{1/3}} - C \right| \geq (3\epsilon')^p \right) \leq A \exp\left( -f(n)^{1/2}/A \right). \tag{89}
\]

We showed in the end of the proof of Theorem 38 that \( \ell_2^{1/4} = O(f^{2/3}) \), this yields \( \lim_n \epsilon(f(n)) = 0 \). Using those remarks alongside Theorems 38 and 28, there exists a constant \( A > 0 \) such that:

\[
P\left( \sum_{k \geq 2} \left| \frac{|C_k|}{\ell_2^{1/3}} \right|^p \geq A \epsilon(f(n)) \right) \leq A \exp\left( -f(n)^{1/2}/A \right) + A \exp\left( -n^{1/8}/A \right). \tag{90}
\]

The corollary follows by the union bound Inequalities (89) and (90).

**Note:** If we change \( \ell_2^{1/12} \) to \( \ell_2^{1-\epsilon''} \) for \( 1/3 > \epsilon'' > 0 \) arbitrarily small in the definition of \( t_k^i \) then Theorem 38 will hold with the term \( n^{1/8} \) being replaced by \( n^{-\epsilon''} \). And this shows that Corollary 38.2 holds in fact for any \( p > 2 \). Moreover, with the same technique one can also obtain the same convergence for the sequence of weights of the connected components of \( G(W, p f(n)) \). It is also easy to show that if \( f(n) \) is of order \( n^\epsilon \) for some \( \epsilon > 0 \) then this convergence will hold in expectation for any moment larger than 1.
5.3 The excess of the tail

We showed that after discovering the giant component all the other components have size less than $t_n^{1/3}/f$ with high probability. We call excess of a discrete interval between 1 and $n$, the number of excess edges discovered in that interval of time during the exploration process, regardless of which connected component they belong to. In the following theorem we will first focus on getting bounds on the excess of small intervals, then getting bounds on the excess of the tail will be straightforward by using Theorem 38.

**Theorem 39.** Suppose that Conditions 1 hold. There exists a constant $A > 0$ such that the following is true:

For $i \geq i \geq 1$, for $k_i \geq k \geq 0$ let $\text{Exc}_i^k$ be the excess of the interval $[t_i^k, t_{i+1}^k)$.

For any $\epsilon > 0$:  

$$
\mathbb{P}\left(\sup_{k_i \leq k \geq 0} (\text{Exc}_i^k) \geq f^\epsilon\right) \leq A \exp\left(-\frac{f^\epsilon \ln(i/\sqrt{f})}{A}\right) + A \exp\left(-\frac{\sqrt{f}}{A}\right) + A \exp\left(-\frac{n^{1/8}}{A}\right).
$$

**Proof.** Let $k < k_i$. If $t_i^k \leq t_n^{11/12}$, by Theorem 34:  

$$
\mathbb{P}\left(\sup_{t_i^k-1 \leq u \leq w \leq t_i^k+1} (L_u^0 - L_w^0 - E[L_u^0 - L_w^0]) \geq t_n^{1/3}\right) \leq A \exp\left(-\frac{i \sqrt{f}}{A'}\right).
$$

By Corollary 23.1 for any $t_i^k-1 \leq u \leq w \leq t_i^k+1$:  

$$
E[L_u^0 - L_w^0] \leq 0.
$$

With the above inequality, Equation 91 yields:

$$
\mathbb{P}\left(\sup_{t_i^k-1 \leq u \leq w \leq t_i^k+1} (L_u^0 - L_w^0) \geq t_n^{1/3}\right) \leq A \exp\left(-\frac{i \sqrt{f}}{A'}\right).
$$

And in fact, notice that this inequality also holds for $t_i^k > t_n^{11/12}$ by the method used to obtain Inequality 88. Denote the event "no connected component discovered after time $t_i^k$ has size larger $t_n^{1/3}$" by $\mathbb{G}$. When $\mathbb{G}$ holds, $L$ visits 0 in any interval of size $t_n^{1/3}$ after $t_i^k$. In that case:

$$
\sup_{t_i^k \leq r \leq t_i^k+1} L(r) \leq \sup_{t_i^k-1 \leq u \leq w \leq t_i^k+1} (L_u^0 - L_w^0).
$$

This fact and Equation 92 yield:

$$
\mathbb{P}\left(\sup_{t_i^k \leq r \leq t_i^k+1} L_r \geq t_n^{1/3}\right) \leq A' \exp\left(-\frac{i \sqrt{f}}{A'}\right) + \mathbb{P}(\mathbb{G}).
$$

Let $\mathcal{M} = \left\{\sup_{t_i^k \leq r \leq t_i^k+1} L_r \leq t_n^{1/3}\right\}$. By Equation 93 and Theorem 38 we obtain:

$$
\mathbb{P}(\mathcal{M}) \leq A \exp\left(-\frac{\sqrt{f}}{A}\right) + A \exp\left(-\frac{n^{1/8}}{A}\right) + A' \exp\left(-\frac{i \sqrt{f}}{A'}\right).
$$
By the union bound:

\[ \mathbb{P}(\text{Exc}_k^i \geq l + \mathbb{E}[\text{Exc}_k^i]) \leq \mathbb{P}(\text{Exc}_k^i \geq l + \mathbb{E}[\text{Exc}_k^i], \mathcal{M}) + \mathbb{P}(\mathcal{M}). \]  

(95)

Now we use the same method we used in Lemma 30. Let \( R = t_{n/3} \) and define \( \tilde{t} = t_{k+1} - t_k \). By Lemma 8,

\[
\mathbb{E} \left[ p_f \sum_{r=t_{k-1}}^{t_k} 2R \left( \sum_{u=r+1}^{(r+2)R} w_v(t_{k-1}) \right)^2 \right] \leq \tilde{A} R p_f, \]

(96)

Hence, by Equation (96) and Theorem 37,

\[
\mathbb{P} \left( \sum_{r=t_{k-1}}^{t_k} \sum_{u=r+1}^{(R+r)} w_v(u)w_v(r) p_f \geq 2\tilde{A} R p_f + \frac{1}{i^{1/3}} \right)
\]

\[
\leq \mathbb{P} \left( p_f \sum_{r=t_{k-1}}^{t_k} \left( \sum_{u=r+1}^{(r+2)R} w_v(u) \right)^2 \geq 2\tilde{A} R p_f + \frac{1}{i^{1/3}} \right) \]

\[
\leq \mathbb{P} \left( \sum_{r=t_{k-1}}^{t_k} \left( \sum_{u=r+1}^{(r+2)R} w_v^2(u) \right) \geq \tilde{A} + \frac{1}{2\sqrt{\tilde{A} R p_f}} \right) \]

\[
\leq A'' \exp \left( -\frac{-i\sqrt{T}}{2A''} \right). \]

By the union bound between Equation (96) and Equation (65):

\[
\mathbb{P} \left( \sum_{r=t_{k-1}}^{t_k} \sum_{u=r+1}^{(R+r)} w_v(u)w_v(r) p_f \geq 2\tilde{A} R p_f + \frac{1}{i^{1/3}} \right)
\]

\[
\leq \mathbb{P} \left( \sum_{r=t_{k-1}}^{t_k} \sum_{u=r+1}^{(R+r)} w_v(u)w_v(r) p_f \geq 2\tilde{A} R p_f + \frac{1}{i^{1/3}} \right) + \mathbb{P}(\mathcal{M}) \]

(98)

We know that for any \( \epsilon > 0 \):

\[
\mathbb{P}(\text{Exc}_k^i \geq f' | F_n) \leq \mathbb{P} \left( \sum_{r=t_{k-1}}^{t_k} \sum_{u=r+1}^{(L_r+r-1)} Y(v(r), v(u)) \geq f' \right| F_n) \mathbb{I}(\mathcal{M}) + \mathbb{I}(\mathcal{M}).
\]

(99)

Since we are dealing with a sum of Bernoulli random variables, this sum is larger than \( f' \) if and only if there are more than \( f' \) Bernoulli variables equal to 1. Let \( S \) be the random set of subsets of size \( f' \) (suppose that \( f' \) is an integer for simplicity) composed of couples \( (r, u) \) that appear as indices in the sum in the
right-hand side of Equation \([99]\), and let \(S'\) be the deterministic set of subsets of size \(f^r\) composed of couples \((r, u)\) that appear as indices in the sum in the right-hand side of Equation \([99]\), when we replace \(L_u\) by \(R\) for all \(t_k \leq r \leq t_k\). Then for \(f\) large enough:

\[
P \left( \sum_{r=t_k-1}^{t_k} \sum_{u=r+1}^{(L_u+r-1)} Y(v(r), v(u)) \geq f^r \right) \leq P \left( \bigcup_{M \in S' (r,u) \in M} \{ Y(v(r), v(u)) = 1 \} \right) \leq \sum_{M \in S' (r,u) \in M} \prod (1 - e^{-w_v(r)w_v(u)P_f}) \leq \sum_{M \in S' (r,u) \in M} \prod (w_v(r)w_v(u)P_f) \leq \left( \sum_{r=t_k-1}^{(R+r)} \sum_{u=r+1}^{(L_u+r-1)} w_v(r)w_v(u)P_f \right)^{f^r+1}.
\]

By this fact and Equation \([98]\):

\[
P(\text{Exc}_k^i \geq f^r) \leq \left( A\tilde{R}P_f + \frac{1}{i} \right)^{f^r+1} + A' \exp \left( \frac{-i\sqrt{\tilde{T}}}{A'} \right) + A \exp \left( \frac{-n^{1/8}}{A} \right) + A' \exp \left( \frac{-\sqrt{T}}{A'} \right) \leq \left( A'' + \frac{1}{i^2} \right)^{f^r+1} + A' \exp \left( \frac{-i\sqrt{\tilde{T}}}{A'} \right) + A \exp \left( \frac{-n^{1/8}}{A} \right) + A' \exp \left( \frac{-\sqrt{T}}{A'} \right)
\]

\[
\leq \exp \left( f^r + 1 \right) \left( \ln \left( \frac{A''}{i^2} \right) + \ln \left( 1 + \frac{i\sqrt{T}}{A''} \right) \right) + A' \exp \left( \frac{-i\sqrt{\tilde{T}}}{A'} \right)
\]

\[
+ A \exp \left( \frac{-n^{1/8}}{A} \right) + A' \exp \left( \frac{-\sqrt{T}}{A'} \right) \leq \exp \left( \frac{-f^r \ln(i\sqrt{T})}{A''} \right) + A' \exp \left( \frac{-i\sqrt{\tilde{T}}}{A'} \right) + A \exp \left( \frac{-n^{1/8}}{A} \right) + A' \exp \left( \frac{-\sqrt{T}}{A'} \right).
\]

If \(t_k^i \geq \ell^{11/12}\), then by definition \(i = \tilde{i}\). And we obtain similarly:

\[
P(\text{Exc}_k^i \geq f^r) \leq A \exp \left( \frac{-f^r \ln(i\sqrt{T})}{A} \right) + A \exp \left( \frac{-i\sqrt{T}}{A} \right) + A \exp \left( \frac{-n^{1/8}}{A} \right) + A \exp \left( \frac{-\sqrt{T}}{A} \right).
\]

This finishes the proof. \(\Box\)

In Theorem \([39]\) the term \(A \exp \left( \frac{-\sqrt{T}}{A} \right)\) comes from applying Theorem \([38]\) and that theorem gives a bound for all the connected components discovered after the giant connected component. Using this remark, we can sum over \(i\). And using simple computations, we obtain the concentration of the total surplus of the tail.

**Theorem 40.** Suppose that Conditions \([2]\) hold. There exists \(A > 0\), such that for any \(\epsilon > 0\), for \(f\) and \(n\) large enough, the probability that a connected component discovered after \(H_f^r\) has excess more than \(f^*\) is at most:

\[
A \exp \left( \frac{-f^* \ln(i\sqrt{T})}{A} \right) + A \exp \left( \frac{-i\sqrt{T}}{A} \right) + A \exp \left( \frac{-n^{1/8}}{A} \right).
\]
As a Corollary of the work done here we obtain a natural global upper bound on $L$.

**Corollary 40.1.** Suppose that Conditions 1 hold. There exists a constant $A > 0$ large enough, such that:

$$
\mathbb{P} \left( \sup_{t_i^0 \leq t \leq n} (L_t) \geq L_n^{1/3} \right) \leq A \exp \left( -\frac{\sqrt{f}}{A} \right) + A \exp \left( -\frac{n^{1/8}}{A} \right).
$$

**Proof.** Let $1 \leq i \leq \tilde{i}$, and denote the event "no connected component discovered after time $t_i^0$ has size larger $\frac{L_n^{1/3}}{\sqrt{fC}}"$ by $G_i$. When $G_i$ holds, $L$ visits 0 in any interval of size $\frac{L_n^{1/3}}{\sqrt{fC}}$ after $t_i^0$. In that case:

$$
\sup_{t_i^0 \leq t \leq t_{i+1}} L_t \leq \sup_{t_i^0 \leq u \leq t_{i+1}} (L_0^u - L_u^0).
$$

Moreover, by Equation (102):

$$
\mathbb{P} \left( \sup_{t_i^0 \leq u \leq t_{i+1}} (L_0^u - L_u^0) \geq \frac{L_n^{1/3}}{\sqrt{fC}} \right) \leq A \exp \left( -\frac{\sqrt{f}}{A} \right),
$$

with $A > 0$ a large constant independent of $i$. By summing this equation over $1 \leq k < \tilde{k}_i - 1$ for every $i$, and then over $1 \leq i \leq \tilde{i}$ we obtain directly:

$$
\mathbb{P} \left( \sup_{t_i^0 \leq r \leq t_n} (L_r) \geq \frac{L_n^{1/3}}{\sqrt{fC}}, \cap_{i \leq i} G_i \right) \leq A' \exp \left( -\frac{\sqrt{f}}{A'} \right) + A \exp \left( -\frac{n^{1/8}}{A} \right). \quad (101)
$$

With $A' > 0$ a large constant. By Theorem 38 there exists a large constant $A > 0$ such that:

$$
\mathbb{P}(\cup_{i \leq i} G_i) \leq A \exp \left( -\frac{\sqrt{f}}{A} \right) + A \exp \left( -\frac{n^{1/8}}{A} \right). \quad (102)
$$

By Equations (101) and (102) there exists a large constant $A > 0$ such that:

$$
\mathbb{P} \left( \sup_{t_i^0 \leq r \leq t_n} (L_r) \geq \frac{L_n^{1/3}}{\sqrt{fC}} \right) \leq \mathbb{P}(\cup_{i \leq i} G_i) + \mathbb{P} \left( \sup_{t_i^0 \leq r \leq t_n} (L_r) \geq \frac{L_n^{1/3}}{\sqrt{fC}}, \cap_{i \leq i} G_i \right)
\leq A \exp \left( -\frac{\sqrt{f}}{A} \right) + A \exp \left( -\frac{n^{1/8}}{A} \right),
$$

which finishes the proof.

This upper bound alongside Theorem 26 gives an upper bound for the whole process $L$. However, it can be refined, and it is not hard to show that $L$ gets smaller the further we advance in the exploration. We elect to stop here and as a last result we use this upper bound on $L$ and the theorems we showed in this paper to give an upper bound on the number of connected components discovered in parts of the exploration of the graph.

58
Corollary 40.2. Suppose that Conditions[7] hold. Recall that $i^* \in \mathbb{N}$ is the time at which the exploration of $H_f^*$ ends. There exists a constant $A > 0$ such that the following is true:

The probability that there exists an $i > i \geq 0$ and $k_i > k \geq 0$, such that the number of connected components discovered between times $t_k' - t + i^*$ and time $t_k' - t + i^*$, is more than $100i^3 f^2 \ell_n^{1/3}$, is at most:

$$A \exp \left( -\frac{x}{A} \right) + A \exp \left( -\frac{y}{A} \right).$$

Proof. Let $r = \frac{2f^2 \ell_n^{1/3}}{A}$, $t_1 = t_{0_0}$, and $t_2 = t_{k_1} + r$.

In order to prove this theorem we need to bound the number of times a new minimum of $L$ is reached in the interval $[t_1, t_2]$. Since $L'$ can only go down by 1, the number of new minimums created in the interval $[t_1, t_2]$ is smaller than

$$\inf_{t_1 \leq m \leq t_2} L_m' - L_i.'$$

Choose $x = -50i^3 f^2 \ell_n^{1/3}$ and. Then $(t_2, t_1, 0, x)$ verifies Conditions[9] Hence, by Theorem[27] we have:

$$P \left( \sup_{t_1 \leq u \leq w \leq t_2} \left| L_w' - L_u' - \tilde{L}_w - \tilde{L}_u \right| \geq -x \right) \leq A \exp \left( -\frac{x^2}{A(xn^{1/3} + (t_2 - t_1))} \right) \leq A' \exp \left( -\frac{x^2}{A'} \right).$$

For any $h > 0$, if $L_k < h$ for any $k \leq t_2$ then deterministically $\tilde{L}_m - \tilde{L}_l \geq \tilde{L}_m^h - \tilde{L}_l^h$ for any $1 \leq l \leq m \leq t_2$.

Hence, if $\tilde{L}_m - \tilde{L}_l \leq x$ for some $t_1 \leq t \leq t_2$ then one of the following events happens:

- There exists $0 \leq j \leq t_2$ such that $L_j \geq h$.
- There exists $t_1 \leq l \leq t_2$ such that $\tilde{L}_m^h - \tilde{L}_l^h \leq \tilde{L}_m^h - \tilde{L}_l^h \leq x$.

Let $h = \frac{10f^2 \ell_n^{1/3}}{e}$. Then for the first event, by Theorem[26] and Corollary[40.1]:

$$P \left( \sup_{1 \leq j \leq t_2} L_j \geq h \right) \leq A \exp \left( -\frac{x}{A} \right) + A \exp \left( -\frac{y}{A} \right).$$

For the second event, Conditions[3] are verified for $(t_2, t_1, h, -x)$. By Corollary[23.1] and a quick computation, for any $t_2 \geq w \geq u \geq t_1$, we have

$$-E[\tilde{L}_w^h - \tilde{L}_u^h] \leq -E[\tilde{L}_w^h - \tilde{L}_u^h] \leq x/2.$$

We can thus apply Lemma[24] to obtain:

$$P \left( \inf_{t_1 \leq u \leq w \leq t_2} \tilde{L}_w^h \leq \tilde{L}_u^h \leq x \right) \leq P \left( \inf_{t_1 \leq u \leq w \leq t_2} \tilde{L}_w^h - \tilde{L}_u^h \leq E[\tilde{L}_w^h - \tilde{L}_u^h] \leq \frac{x}{2} \right) \leq A \exp \left( -\frac{x^2}{A(xn^{1/3} + (t_2 - t_1))} \right) \leq A' \exp \left( -\frac{x^2}{A'} \right).$$
with $A' > 0$ a large constant that does not depend on $i$.

Recall that $i^* \in \mathbb{N}$ is the time at which the exploration of $H^*_f$ ends. By Theorem 28

$$
\mathbb{P} \left( \frac{3 f \ell_n \ell_3/3}{C} \geq i^* \geq \frac{f \ell_n \ell_3/3}{C} \right) \geq 1 - A \exp \left( -\frac{\sqrt{T}}{A} \right). 
$$

(106)

When this event holds, we have $[t_i^i - t + i^*, t_k^i - t + i^*] \subset [t_1, t_2]$. Hence, summing Equations (105) and (103) for $i > i \geq 1$, and using the union bound with Equation (104) and (106) finishes the proof. \qed
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