Integrable boundary impurities in the $t - J$ model with different gradings

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Abstract

We investigate the generalized supersymmetric $t - J$ model with boundary impurities in different gradings. All three different gradings: fermion, fermion, boson (FFB), boson, fermion, fermion (BFF) and fermion, boson, fermion (FBF), are studied for the generalized supersymmetric $t - J$ model. Boundary K-matrix operators are found for the different gradings. By using the graded algebraic Bethe ansatz method, we obtain the eigenvalues and the corresponding Bethe ansatz equations for the transfer matrix.

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1 Introduction

There has been extensive interests in the investigation of the impurity problems. The Anderson and Kondo lattice models describe the physics of conduction electrons in extended orbitals interacting with strongly correlated electrons in localized orbitals (impurities), see Ref.[1] for a review. For superconductivity, the nonmagnetic impurities are generally believed to have little effect on the superfluid density and the transition temperature in conventional superconductivity[2]. For the high-temperature cuprate superconductors, both magnetic and nonmagnetic impurities, for example, nonmagnetic impurity Zn and magnetic impurity Ni in cuprate LSCO, play an important role for the interpretation of experiments[3].

The study of the effects due to the presence of impurities in 1-dimensional quantum chains in the framework of integrable models has a long successful history[4, 5, 6]. In the framework of the Quantum Inverse Scattering Method (QISM), Andrei and Johannesson[4] studied an arbitrary spin S embedded in a spin-1/2 Heisenberg chain by the algebraic Bethe ansatz method[7]. This method was later generalized to other cases including the integrable supersymmetric model[8, 9].

The $t-J$ model proposed by Zhang and Rice[10] is one of the most widely accepted models to describe the high-temperature Cu-oxide superconductors. The Hamiltonian of the $t-J$ model includes the nearest neighbour hopping term ($t$) and the antiferromagnetic exchange term ($J$). The Hamiltonian of 1-dimensional $t-J$ model is written as

$$H = \sum_{j=1}^{N} \left\{ -t P \sum_{\sigma = \pm 1} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) P + J (S_j S_{j+1} - \frac{1}{4} n_j n_{j+1}) \right\}, \quad (1)$$

where the Gutzwiller’s projection operator $P$ ensures the exclusion of double occupancy in one lattice site. It is known that this model is supersymmetric and integrable for $J = \pm 2t$[11, 12]. The supersymmetric $t-J$ model was also studied in Refs.[13, 14, 15]. Using the graded algebraic Bethe ansatz method, the integrable supersymmetric $t-J$ model was studied for all three different gradings[14] for periodic boundary conditions. The generalized supersymmetric $t-J$ model with open boundary conditions for three different gradings is studied in Ref.[16].

Traditionally, the integrable models are studied for periodic boundary conditions where the Yang-Baxter equation play a key role[17]. In the last decade, the study of integrable models with open (reflecting) boundary conditions have been attracting a great deal of interests[18]. Besides the Yang-Baxter equation, the reflection equations are also important for the study of the open boundary conditions. Applying the method used in[11] and the reflection equation, the Heisenberg spin chain with boundary impurities is studied[19]. And the same method is applied to the integrable supersymmetric $t-J$ model[20, 21], see also[22] for related works. The boundary impurities for the generalized ($q$-deformed) $t-J$ model are also studied by the graded algebraic Bethe ansatz method[23, 24]. In the previous works[21, 20, 23, 22], the boundary impurities refer to the boundary impurity spins coupled with the original $t-J$ spin chain, so these impurities should correspond to the magnetic impurities. In this paper, we shall extend our previous results in[23]. Besides the FFB grading, we shall deal with BFF and FBF gradings. It seems that this kind of boundary impurities correspond to the nonmagnetic impurities. Thus all three possible gradings are studied for the generalized $t-J$ model with boundary impurities. Using the projecting method, integrable $t-J$ model with boundary impurities for different gradings are also studied in Ref.[22].
The paper is organized as follows: We introduce the model in Section 2. In section 3, we present briefly the results of FFB grading. In section 4, using the nested algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix with grading BFF. In section 5, results for FBF grading are presented. Section 6 includes a brief summary and discussions.

2 Description of the Model

In this paper, we shall study the generalized (q-deformed) supersymmetric $t - J$ model. The Hamiltonian of the model takes the following form:

$$ H = \sum_{j=1}^{N} \sum_{\sigma = \pm} \left[ \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{j+1,\sigma} + \hat{c}_{j+1,\sigma}^{\dagger} \hat{c}_{j,\sigma} \right] - 2 \sum_{j=1}^{N} \left[ \frac{1}{2} (S_{j}^{\dagger} S_{j+1} + S_{j+1} S_{j}^{\dagger}) + \cos(\eta) S_{j}^{z} S_{j+1}^{z} \right] - \frac{\cos(\eta)}{4} n_{j} n_{j+1} $$

$$ + i \sin(\eta) \sum_{j=1}^{N} [S_{j}^{z} n_{j+1} - S_{j+1}^{z} n_{j}] . \tag{2} $$

where $\hat{c}_{j,\sigma} = (1 - n_{j,\sigma}) \hat{c}_{j,\sigma}$, $\hat{c}_{j,\sigma} = c_{j,\sigma}(1 - n_{j,\sigma})$. When the anisotropic parameter $\eta = 0$, this Hamiltonian reduces to an equivalent form of the original Hamiltonian (1) with $J = 2t$. The operators $c_{j,\sigma}$ and $c_{j,\sigma}^{\dagger}$ are annihilation and creation operators of electron with spin $\sigma$ on a lattice site $j$, and we assume the total number of lattice sites is $N$, $\sigma = \pm$ represent spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations $\{c_{j,\sigma}^{\dagger}, c_{j,\tau}\} = \delta_{ij} \delta_{\sigma\tau}$. We denote by $n_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma}$ the number operator for the electron on a site $j$ with spin $\sigma$, and by $n_{j} = \sum_{\sigma = \pm} n_{j,\sigma}$ the number operator for the electron on a site $j$. The Fock vacuum state $|0> \equiv 0$ is defined as $c_{j,\sigma} |0> = 0$. Due to the exclusion of double occupancy, there are altogether three possible electronic states at a given lattice site $j$, two are fermionic and one is bosonic,

$$ |0> , \quad |\uparrow>_{j} = c_{j,\uparrow}^{\dagger} |0> , \quad |\downarrow>_{j} = c_{j,\downarrow}^{\dagger} |0> . \tag{3} $$

Here $S_{j}^{x}, S_{j}^{y}, S_{j}^{z}$ are spin operators satisfying $su(2)$ algebra and can be expressed as

$$ S_{j} = c_{j,1}^{\dagger} c_{j,-1} , \quad S_{j}^{\dagger} = c_{j,-1}^{\dagger} c_{j,1} , \quad S_{j}^{z} = \frac{1}{2} (n_{j,1} - n_{j,-1}) . \tag{4} $$

The above Hamiltonian can be obtained from the logarithmic derivative of the transfer matrix at zero spectral parameter. In the framework of QISM, the transfer matrix is constructed by the trigonometric R-matrix of the Perk-Schultz model $^{[20]}$. The non-zero entries of the R-matrix are given by

$$ \tilde{R}(\lambda)^{aa}_{aa} = \sin(\eta + \epsilon_{a}\lambda) , $$

$$ \tilde{R}(\lambda)^{ab}_{ab} = (-1)^{\epsilon_{a}\epsilon_{b}} \sin(\lambda) , $$

$$ \tilde{R}(\lambda)^{ab}_{ba} = e^{i \text{sign}(a - b) \lambda} \sin(\eta) , a \neq b , \tag{5} $$

where $\epsilon_{a}$ is the Grassman parity, $\epsilon_{a} = 0$ for boson and $\epsilon_{a} = 1$ for fermion, and

$$ \text{sign}(a - b) = \begin{cases} 1 , & \text{if } a > b \\ -1 , & \text{if } a < b . \end{cases} \tag{6} $$


This R-matrix satisfies the usual Yang-Baxter equation:

\[ \tilde{R}_{12}(\lambda - \mu)\tilde{R}_{13}(\lambda)\tilde{R}_{23}(\mu) = \tilde{R}_{23}(\mu)\tilde{R}_{13}(\lambda)\tilde{R}_{12}(\lambda - \mu) \]  \hspace{1cm} (7)

In this paper, we shall concentrate our discussion only on the two fermion and one boson case, that means

for Grassmann parities \( \epsilon_i, i = 1, 2, 3 \), two of them equal to 1, and the last one equal to zero. For example,

we let \( \epsilon_1 = \epsilon_2 = 1 \), \( \epsilon_3 = 0 \) for FFB grading, and \( \epsilon_2 = \epsilon_3 = 1 \), \( \epsilon_1 = 0 \) for BFF grading. We shall use the graded formulæ to study this model. For supersymmetric \( t - J \) model, the spin of the electrons and the

topology ‘hole’ degrees of freedom play a very similar role forming a graded superalgebra with two fermions

and one boson. The holes obey boson commutation relations, while the spinons are fermions. The graded

approach has an advantage of making clear distinction between bosonic and fermionic degrees of freedom.

Introducing a diagonal matrix \( \Pi_{ac} = (-)^{a+c} \delta_{ac} \), we change the original R-matrix to the following form,

\[ R(\lambda) = \Pi \tilde{R}(\lambda). \]  \hspace{1cm} (8)

From the non-zero elements of the R-matrix \( R_{ab}^{cd} \), we see that \( \epsilon_a + \epsilon_b + \epsilon_c + \epsilon_d = 0 \). One can show that the R-matrix satisfies the graded Yang-Baxter equation,

\[ R(\lambda - \mu)^{b_1b_2}_{a_1a_2} R(\mu)^{c_1c_2}_{b_2b_3} R(\lambda)^{a_2a_3}_{c_2c_3} (-)^{(\epsilon_{c_1} + \epsilon_{c_2})\epsilon_{d_2}} = R(\mu)^{b_1b_2}_{a_1a_2} R(\lambda)^{b_2b_3}_{a_1a_3} R(\lambda - \mu)^{c_1c_2}_{b_2b_3} (-)^{(\epsilon_{c_1} + \epsilon_{c_2})\epsilon_{d_2}}. \]  \hspace{1cm} (9)

In the framework of the QISM, we can construct the \( L \) operator from the R-matrix as \( L_{aq}(\lambda) = R_{aq}(\lambda) \),

where the subscript \( a \) represents auxiliary space, and \( q \) represents quantum space. Thus we can rewrite the graded Yang-Baxter equation \( 4 \) as the following (graded) Yang-Baxter relation,

\[ R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu). \]  \hspace{1cm} (10)

Here the tensor product is in the sense of super-tensor product defined as

\[ (F \otimes G)_{ac}^{bd} = F_{a}^{\delta_{d}}G_{c}^{\epsilon_{c}}(-)^{(\epsilon_{b} + \epsilon_{c})\epsilon_{d}}. \]  \hspace{1cm} (11)

Hereafter, all tensor products in this paper are in the sense of super-tensor products.

It is standard that the row-to-row monodromy matrix \( T_N(\lambda) \) is defined as a matrix product over the \( N \) operators on all sites of the lattice,

\[ T_a(\lambda) = L_{aN}(\lambda)L_{aN-1}(\lambda) \cdots L_1(\lambda), \]  \hspace{1cm} (12)

where the subscript \( a \) represents the auxiliary space, and \( 1, \ldots, N \) represent the quantum spaces in which the tensor product is in the graded sense. Explicitly we write \( 14 \)

\[ \{[T(\lambda)]^{ab}_{\beta_1, \ldots, \beta_N}^{\alpha_1, \ldots, \alpha_N} = L_N(\lambda)^{c_1\beta_N}_{\alpha_N}L_{N-1}(\lambda)^{c_{N-1}\beta_{N-1}}_{\alpha_{N-1}} \cdots L_1(\lambda)^{b\beta_1}_{c_2\alpha_1}(-1)^{\sum_{j=2}^{N}(\epsilon_{c_j} + \epsilon_{\beta_j})\sum_{i=1}^{j-1}(\epsilon_{\alpha_i})}. \]  \hspace{1cm} (13)

This definition is different from the non-graded case because we have the graded Yang-Baxter equation \( 4 \). By repeatedly using the Yang-Baxter relation \( 10 \), one can prove easily that the monodromy matrix also satisfies the Yang-Baxter relation,

\[ R(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda - \mu). \]  \hspace{1cm} (14)
For periodic boundary condition, the transfer matrix $\tau_{peri}(\lambda)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space,

$$\tau_{peri}(\lambda) = str T(\lambda) = \sum (-1)^{s_a} T(\lambda)_{aa}.$$  

(15)

As a consequence of the Yang-Baxter relation [14] and the unitarity property of the R-matrix, we can prove that the transfer matrix commutes with each other for different spectral parameters,

$$[\tau_{peri}(\lambda), \tau_{peri}(\mu)] = 0.$$  

(16)

In this sense we say that the model is integrable. Expanding the transfer matrix in the powers of $\lambda$, we can find conserved operators. And the Hamiltonian is defined as

$$H = \sin(q) \frac{d \ln [\tau_{peri}(\lambda)]}{d \lambda} |_{\lambda=0} = \sum_{j=1}^{N} H_{j,j+1} = \sum_{j=1}^{N} P_{j,j+1} L'_{j,j+1}(0),$$  

(17)

where $P_{ij}$ is the graded permutation operator expressed as $P_{ac}^{bd} = \delta_{ad}\delta_{bc}(-1)^{s_a s_c}$. The explicit expression of the Hamiltonian has already been presented in equation (2).

In this paper, we consider the reflecting boundary condition case. In addition to the Yang-Baxter equation, a reflection equation should be used in proving the commutativity of the transfer matrix with boundaries. The reflection equation takes the form [18],

$$R_{12}(\lambda - \mu)K_{1}(\lambda)R_{21}(\lambda + \mu)K_{2}(\mu) = K_{2}(\mu)R_{12}(\lambda + \mu)K_{1}(\lambda)R_{21}(\lambda - \mu).$$  

(18)

For the graded case, the reflection equation remains the same as the above. We only need to change the usual tensor product to the graded tensor product. We write it explicitly as

$$R(\lambda - \mu)^{b_{1}b_{2}}_{a_{1}a_{2}} K(\lambda)^{c_{1}}_{b_{1}} R(\lambda + \mu)^{c_{2}d_{2}}_{b_{2}c_{2}} K(\mu)^{d_{1}}_{c_{2}c_{1}} = K(\mu)^{b_{2}}_{a_{2}} R(\lambda + \mu)^{b_{1}c_{1}}_{a_{1}b_{1}} K(\lambda)^{c_{1}}_{b_{1}} R(\lambda - \mu)^{d_{2}d_{1}}_{c_{2}c_{1}} (-)^{(c_{1} + c_{2})r_{a_{2}}}. $$  

(19)

Instead of the monodromy matrix $T(\lambda)$ for periodic boundary conditions, we consider the double-row monodromy matrix

$$T(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda)$$  

(20)

for the reflecting boundary conditions. Using the Yang-Baxter relation, and considering the boundary $K$-matrix which satisfies the reflection equation, one can prove that the double-row monodromy matrix $T(\lambda)$ also satisfies the reflection equation,

$$R(\lambda - \mu)^{b_{1}b_{2}}_{a_{1}a_{2}} T(\lambda)^{c_{1}}_{b_{1}} R(\lambda + \mu)^{c_{2}d_{2}}_{b_{2}c_{2}} T(\mu)^{d_{1}}_{c_{2}c_{1}} (-)^{(c_{1} + c_{2})r_{a_{2}}} = T(\mu)^{b_{2}}_{a_{2}} R(\lambda + \mu)^{b_{1}c_{1}}_{a_{1}b_{1}} T(\lambda)^{c_{1}}_{b_{1}} R(\lambda - \mu)^{d_{2}d_{1}}_{c_{2}c_{1}} (-)^{(c_{1} + c_{2})r_{a_{2}}}. $$  

(21)

Next, we study the properties of the R-matrix. We define the super-transposition $st$ as

$$(A^{st})_{ij} = A_{ji}(-1)^{(c_{i} + 1)c_{j}}.$$  

(22)

We also define the inverse of the super-transposition $\tilde{st}$ as $\{A^{st}\}^{\tilde{st}} = A$. 

One can prove directly that the R-matrix satisfy the following unitarity and cross-unitarity relations:

\[ R_{12}^t(\lambda)R_{21}(-\lambda) = \rho(\lambda) \cdot I_d, \quad \rho(\lambda) = \sin(\eta + \lambda)\sin(\eta - \lambda), \]
\[ R_{12}^{st_1}(\eta - \lambda)M_1 R_{21}^{st_1}(\lambda)M_1^{-1} = \tilde{\rho}(\lambda) \cdot I_d, \quad \tilde{\rho}(\lambda) = \sin(\lambda)\sin(\eta - \lambda). \]  

(23)

(24)

Here the matrix \( M \) is determined by the R-matrix. For three different gradings, the forms of \( M \) are different. We have \( M = \text{diag}(e^{2i\eta}, 1, 1) \) for FFB grading, \( M = \text{diag}(1, 1, e^{-2i\eta}) \) for BFF grading, and \( M = 1 \) for FBF grading. We also have a property

\[ [R_{12}^{st_1 st_2}(\lambda), M \otimes M] = 0. \]  

(25)

The cross-unitarity relation can also be written as follows,

\[ \{ M_1^{-1}R_{12}^{st_1 st_2}(\eta - \lambda)M_1 \}^{st_2} R_{21}^{st_1}(\lambda) = \tilde{\rho}(\lambda), \]
\[ R_{12}^{st_1}(\lambda) \{ M_1 R_{21}^{st_1 st_2}(\eta - \lambda)M_1^{-1} \}^{st_2} = \tilde{\rho}(\lambda). \]  

(26)

(27)

In order to construct the commuting transfer matrix with boundaries, besides the reflection equation, we need the dual reflection equation. In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the R-matrix takes different forms for different models. For the models considered in this paper, we can write the dual reflection equation in the following form:

\[ R_{21}^{st_1 st_2}(\mu - \lambda)K_1^{+ st_1}(\lambda)M_1^{-1} R_{12}^{st_1 st_2}(\eta - \mu)M_1^{-1} K_2^{+ st_2}(\mu) = K_2^{+ st_2}(\mu)M_1 R_{21}^{st_1 st_2}(\eta - \mu)M_1^{-1} K_1^{+ st_1}(\lambda)R_{12}^{st_1 st_2}(\mu - \lambda). \]  

(28)

Then the transfer matrix with boundaries is defined as

\[ t(\lambda) = \text{str}K^+(\lambda)T(\lambda). \]  

(29)

The commutativity of \( t(\lambda) \) can be proved by using the unitarity and cross-unitarity relations, the reflection equation and the dual reflection equation. With a normalization \( K(0) = I_d \), the Hamiltonian can be obtained as

\[ H = \frac{1}{2} \sin(\eta) \left. \frac{d \ln t(\lambda)}{d\lambda} \right|_{\lambda=0} = \sum_{j=1}^{N-1} P_{j,j+1} L_{j,j+1}'(0) + \frac{1}{2} \sin(\eta)K_1'(0) + \frac{\text{str}_a K_0^+(0)P_{Na}L_{Na}'(0)}{\text{str}_a K_a^+(0)}. \]  

(30)

### 3 Integrable boundary impurities for \( t-J \) model with FFB grading

Boundary higher spin impurities for the supersymmetric \( t-J \) model was considered in Ref.\cite{20,21}. In the previous work\cite{13}, we studied boundary higher spin impurities for the generalized supersymmetric \( t-J \) model. In the calculation, the FFB grading is used, i.e. \( \epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0 \). In order to obtain the Hamiltonian from the transfer matrix, we use the following representations:

\[ S_k = e_k^{21}, \quad S_k^i = e_k^{12}, \quad S_k^z = \frac{1}{2}(e_k^{22} - e_k^{11}), \]  

(31)
Inserting this matrix into the reflection equation (19), we find the following solutions,

\[ Q_{k,1} = (1 - n_k, -1)c_{k,1} = e_{32}^k, \quad Q_{k,1}^\dagger = (1 - n_k, -1)c_{k,1}^\dagger = e_{23}^k, \quad Q_{k,-1} = (1 - n_k, 1)c_{k,-1} = e_{31}^k, \quad Q_{k,-1}^\dagger = (1 - n_k, 1)c_{k,-1}^\dagger = e_{13}^k, \]

where \( e_{ij}^k \) is a 3 \times 3 matrix acting on the \( k \)-th space with elements \( (e_{ij}^k)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} \).

We present some results about the boundary higher spin impurities for the generalized supersymmetric \( t - J \) model. The detailed calculations can be found in Ref. [23]. We suppose that K-matrix takes the form,

\[ K(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) & 0 \\ C(\lambda) & D(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(33)

Inserting this matrix into the reflection equation [19], we find the following solutions,

\[
\begin{align*}
A(\lambda) &= g(\lambda) \left( e^{-4i\lambda \sin(\lambda + c - s)\sin(\lambda + c + \eta + s) - \sin(2\lambda)\sin(\lambda + c - \mathbf{S}^+\eta) e^{-i(3\lambda + c + \eta + \mathbf{S}^+\eta)}} \right), \\
B(\lambda) &= g(\lambda) \sin(\eta)\sin(2\lambda) e^{-i(2\lambda - c + \mathbf{S}^+\eta)\mathbf{S}^-}, \\
C(\lambda) &= g(\lambda) \sin(\eta)\sin(2\lambda) e^{-i(2\lambda + c + \mathbf{S}^+\eta)\mathbf{S}^+}, \\
D(\lambda) &= g(\lambda) \left( \sin(\lambda + c - s)\sin(\lambda + c + \eta + s) - \sin(2\lambda)\sin(\lambda + c + \mathbf{S}^\dagger\eta) e^{-i(\lambda - c - \eta + \mathbf{S}^+\eta)} \right),
\end{align*}
\]

(34)

where \( g(\lambda) = 1/\sin(\lambda - c - \eta - s)\sin(\lambda - c + s) \). \( \mathbf{S}^x, \mathbf{S}^y \) and \( \mathbf{S}^z \) are spin-\( s \) operators satisfying the following commutation relations,

\[ [\mathbf{S}^x, \mathbf{S}^\dagger] = \pm \mathbf{S}^z, \quad [\mathbf{S}^+, \mathbf{S}^-] = \frac{\sin(2\mathbf{S}^+\eta)}{\sin(\eta)}. \]  

(35)

We suppose \( K^+ \) has the similar form as \( K \). By direct calculation, we can find \( R_1^{st_1, st_2}(\lambda) = I_1 R_2(\lambda) I_1 \) with \( I = \text{diag.}(1, -1, 1) \). For the form [23], we have \( IK(\lambda)I = K(\lambda) \). Then with the help of property \( [M_1 M_2, R(\lambda)] = 0 \), we see that there is an isomorphism between \( K \) and \( K^+ \):

\[ K(\lambda) \rightarrow K^{st}(\lambda) = K \left( \frac{n}{2} - \lambda \right) M. \]  

(36)

Given a solution to the reflection equation [19], we can also find a solution to the dual reflection equation [23]. Note that in the sense of the transfer matrix, the reflection equation and the dual reflection equation are independent of each other. For other gradings, BFF and FBF, the isomorphism (36) does not hold.

By definition in equation (21), and using the explicit form of the boundary reflecting matrices \( K \) and \( K^+ \), we can find the boundary impurity terms. The boundary impurity coupled to site 1 is written as

\[
H_1 = \frac{2}{\sin(c + \eta + s)\sin(c - s)} e^{-i\mathbf{S}^+\eta}[e^{i\mathbf{S}^-\mathbf{S}^- S_1^\dagger} + e^{-i\mathbf{S}^+\mathbf{S}^- S_1}] + (e^{-i(c + \eta)}\sin(c - \mathbf{S}^\dagger\eta) S_1^\dagger - e^{i(c + \eta)}\sin(c + \mathbf{S}^\dagger\eta) S_1^\dagger) + (e^{-i(c + \eta)}\sin(c - \mathbf{S}^\dagger\eta) + e^{i(c + \eta)}\sin(c + \mathbf{S}^\dagger\eta))(T_1 + 4i(T_1 + S_1^\dagger)).
\]  

(37)

The impurity coupled to site N is in a similar form. We remark here that in the rational limit and by some redefinition, (37) becomes the usual spin-exchange term \( \mathbf{S} \cdot \mathbf{S}_1 \) between the impurity spin and the spin in site 1.
By using the algebraic Bethe ansatz method, we can find the eigenvalue of the transfer matrix and the Bethe ansatz equations. Here we just list the results. The energy spectrum of the Hamiltonian is given by

\[
E = (N - 2) \cos(\eta) + \sum_{i=1}^{n} \frac{\sin^2(\eta)}{\sin(\mu_i + \eta)} - \sin^2(\eta) \left[ \frac{1}{\sin(c - \eta + \tilde{s}\eta)(\tilde{c} + \tilde{s}\eta)} + \frac{1}{\sin(c - \eta - \tilde{s}\eta)(\tilde{c} - 2\eta - \tilde{s}\eta)} \right],
\]

where \(\mu_1, \cdots, \mu_n\) and \(\mu_1^{(1)}, \cdots, \mu_m^{(1)}\) should satisfy the Bethe ansatz equations

\[
\frac{\sin(\mu_j^{(1)} + c + \eta + s\eta)\sin(\mu_j^{(1)} - c - \eta + s\eta)\sin(\mu_j^{(1)} - c + \eta - s\eta)}{\sin(\mu_j^{(1)} - c - \eta - s\eta)\sin(\mu_j^{(1)} + c + \eta - s\eta)\sin(\mu_j^{(1)} + c - \eta + \tilde{s}\eta)} = \prod_{i=1}^{n} \frac{\sin(\mu_j^{(1)} + \mu_i^{(1)} - \mu_i - \eta)}{\sin(\mu_j^{(1)} + \mu_i^{(1)} + \mu_i + \eta)} \prod_{i=1, i \neq j}^{m} \frac{\sin(\mu_j^{(1)} - \mu_i^{(1)} + \eta)}{\sin(\mu_j^{(1)} - \mu_i^{(1)} - \eta)},
\]

\[j = 1, \cdots, m,\]

and

\[
\frac{\sin(\mu_j + \tilde{c} - \eta - \tilde{s}\eta)\sin(\lambda + c + \eta - s\eta)}{\sin(\mu_j - \tilde{c} + 2\eta + \tilde{s}\eta)\sin(\lambda - c + s\eta)} = \frac{\sin^2N(\mu_j + \eta)}{\sin^2N(\mu_j)} \prod_{i=1}^{m} \frac{\sin(\mu_j - \mu_i^{(1)} + \eta)}{\sin(\mu_j + \mu_i^{(1)} + \eta)},
\]

\[j = 1, \cdots, n.\]

### 4 Boundary impurities for the case of BFF grading

We have \(\epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1\) for BFF grading. For a supermatrix \(X\), the Grassmann parities for its entries \(X_{ij}\) are defined as \(\epsilon_i + \epsilon_j\). We can change the representations in the last section to satisfy the grading by performing a change from 1,2,3 to 2,3,1. Explicitly, we have

\[
S_k = e_k^k, \quad S_k^\dagger = e_k^k, \quad S_k^\pm = \frac{1}{2}(e_k^k - e_k^k),
\]

\[
Q_k,1 = (1 - n_{k,-1})e_{k,1} = e_{k,1}, \quad Q_k^\dagger,1 = (1 - n_{k,-1})e_{k,1}^\dagger = e_{k,1}^\dagger, \quad Q_k,-1 = (1 - n_{k,1})e_{k,-1} = e_{k,2},
\]

\[
Q_k^\dagger,-1 = (1 - n_{k,1})e_{k,-1}^\dagger = e_{k,2}^\dagger, \quad T_k = 1 - \frac{1}{\eta} \eta_k = \frac{1}{2}(e_{k,2}^k + e_{k,3}^k) + e_{k,1}^k.
\]

For the nested algebraic Bethe ansatz method for BFF grading, the low level \(r\)-matrix is BF grading which is different from the FF grading \(r\)-matrix in the case of FFB grading, because the graded calculation for FF grading \(r\)-matrix is actually equal to the non-graded case.
4.1 Solutions to the reflection equation and the dual reflection equation

We begin with the explicit form of the R-matrix,

$$\begin{pmatrix}
  w(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & b(\lambda) & 0 & c_-(\lambda) & 0 & 0 & 0 & 0 \\
  0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 \\
  0 & c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\
  0 & 0 & 0 & 0 & -c_+(\lambda) & 0 & b(\lambda) & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda)
\end{pmatrix}$$

(42)

where we use the notations,

$$a(\lambda) = \sin(\lambda - \eta), \quad w(\lambda) = \sin(\lambda + \eta), \quad b(\lambda) = \sin(\lambda), \quad c_\pm(\lambda) = e^{\pm i\lambda}\sin(\eta).$$

(43)

We still assume that the reflecting K-matrix operator takes the form

$$K(\lambda) = \begin{pmatrix}
  A(\lambda) & B(\lambda) & 0 \\
  C(\lambda) & D(\lambda) & 0 \\
  0 & 0 & 1
\end{pmatrix}.$$

(44)

Inserting this matrix into the reflection equation (49), we can find the following non-trivial relations:

$$r(\lambda - \mu)_{a_1a_2}^{b_1b_2} K(\lambda)_{b_1}^{c_1} r(\lambda + \mu)_{b_2c_1}^{c_2d_1} K(\mu)_{c_2}^{d_2} (-1)^{(\epsilon_{c_1} + \epsilon_{c_2})r_{b_2}}$$

$$= K(\mu)_{a_2}^{b_2} r(\lambda + \mu)_{a_1b_2}^{b_1} K(\lambda)_{b_1}^{c_1} r(\lambda - \mu)_{c_2}^{d_2} (-1)^{(\epsilon_{c_1} + \epsilon_{c_2})r_{c_2}},$$

(45)

and

$$K(\lambda)_{a_1}^{b_1} K(\mu)_{b_1}^{d_1} = K(\mu)_{a_1}^{b_1} K(\lambda)_{b_1}^{d_1},$$

(46)

$$\delta_{a_1d_1} \sin(\lambda - \mu) e^{-i(\lambda + \mu)} + \sin(\lambda + \mu)e^{i(\lambda - \mu)} K(\lambda)_{a_1}^{d_1}$$

$$= e^{-i(\lambda - \mu)} \sin(\lambda + \mu) K(\mu)_{a_1}^{d_1} + e^{i(\lambda + \mu)} K(\mu)_{a_1}^{b_1} K(\lambda)_{b_1}^{d_1},$$

(47)

where all indices take values 1,2, and we have the BF grading, i.e. $\epsilon_1 = 0, \epsilon_2 = 1$. We also have introduced the notation

$$r_{12}(\lambda) = \begin{pmatrix}
  \sin(\lambda + \eta) & 0 & 0 & 0 \\
  0 & \sin(\lambda) & \sin(\eta)e^{-i\lambda} & 0 \\
  0 & \sin(\eta)e^{i\lambda} & \sin(\lambda) & 0 \\
  0 & 0 & 0 & \sin(\lambda - \eta)
\end{pmatrix}.$$

(48)

This matrix $r(\lambda)$ has the BF grading and satisfies the graded Yang-Baxter equation. To find a solution to these relations, we first construct a solution to (49) by $r$-matrix (18) in the same way as a construction of a double-row monodromy matrix. Then we check other relations (40,47). After some tedious calculations, we finally obtain the following results

$$A(\lambda) = \frac{1}{2} g(\lambda) \left[ \sin(\lambda + c + \eta)\sin(\lambda - c + \eta) + \sin(\lambda + c)\sin(\lambda - c) \right]$$
\[ B(\lambda) = g(\lambda)\sin(\eta)\sin(2\lambda)\sigma^- , \]
\[ C(\lambda) = g(\lambda)\sin(\eta)\sin(2\lambda)\sigma^+ , \]
\[ D(\lambda) = \frac{1}{2}g(\lambda) [\sin(\lambda + c - \eta)\sin(\lambda - c - \eta) + \sin(\lambda + c)\sin(\lambda - c) \\
+ \sin^2(\eta)e^{2i\lambda} + \sin(\eta)\sin(2\lambda)e^{i\eta}\sigma^+] , \]

where
\[ g(\lambda) = \frac{-e^{-2i\lambda}}{\sin(\lambda - c + \eta)\sin(\lambda - c - \eta)} , \]

\[ \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y) , \sigma^x, \sigma^y, \sigma^z \text{ are Pauli matrices, and } c \text{ is an arbitrary parameter. In the graded method, reflecting K-matrix } K^+ \text{ is a supermatrix with BFF grading. That means } B(\lambda) \text{ and } C(\lambda) \text{ are Grassmann odd, and } A(\lambda) \text{ and } D(\lambda) \text{ are Grassmann even. Therefore, } \sigma^\pm \text{ are Grassmann odd and can be represented by fermion operators } a_L, a_L^\dagger, \sigma^+ = a_L^\dagger, \sigma^- = a_L^\dagger. \text{ And } \sigma^z \text{ is Grassmann even, and can be represented as } \sigma^z = 1 - 2n_L, \text{ where we denote } n_L = a_L^\dagger a_L. \]

Next, let us solve the dual reflection equation (28) for BFF grading. For FFB grading in section 3, we have an isomorphism between \( K \) and \( K^+ \) \(^{[14]}\), this is not the case here. Considering the form of \( K \) \(^{[14]}\) we have for BFF grading, and \( R_{12}^{s_1 s_2}(\lambda) = I L_2(\lambda) I_1, \text{ where } I = \text{diag}(1, -1, -1), \text{ we do not have the relation } IK(\lambda)I = K(\lambda) \text{ now. Due to this fact, we have to solve the dual reflection equation (28) independently. The strategy is almost the same as that for reflection equation. Here we just list the final results. We assume } K^+ \text{ takes the form}

\[ K^+(\lambda) = \begin{pmatrix} A^+(\lambda) & B^+(\lambda) & 0 \\ C^+(\lambda) & D^+(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix} , \]

After some tedious calculations, we have

\[ A^+(\lambda) = \frac{1}{2}g^+(\lambda) \left[ \sin(\lambda + \tilde{c} + \frac{\eta}{2})\sin(\lambda - \tilde{c} + \frac{\eta}{2}) + \sin(\lambda + \tilde{c} - \frac{\eta}{2})\sin(\lambda - \tilde{c} - \frac{\eta}{2}) \\
- \sin^2(\eta)e^{i(2\lambda - \eta)} + \sin(\eta)\sin(2\lambda - \eta)e^{-i\eta}\tilde{\sigma}^z \right] , \]
\[ B^+(\lambda) = g^+(\lambda)e^{-i\eta}\sin(\eta)\sin(2\lambda - \eta)\tilde{\sigma}^- , \]
\[ C^+(\lambda) = g^+(\lambda)e^{-i\eta}\sin(\eta)\sin(2\lambda - \eta)\tilde{\sigma}^+ , \]
\[ D^+(\lambda) = \frac{1}{2}g^+(\lambda) \left[ \sin(\lambda + \tilde{c} - \frac{3\eta}{2})\sin(\lambda - \tilde{c} - \frac{3\eta}{2}) + \sin(\lambda + \tilde{c} - \frac{\eta}{2})\sin(\lambda - \tilde{c} - \frac{\eta}{2}) \\
- \sin^2(\eta)e^{-i(2\lambda - \eta)} + \sin(\eta)\sin(2\lambda - \eta)e^{-i\eta}\tilde{\sigma}^z \right] , \]

where \( \tilde{c} \) is an arbitrary parameter and
\[ g^+(\lambda) = \frac{-e^{i(2\lambda + \eta)}}{\sin^2(\lambda - \tilde{c} - \frac{\eta}{2})} . \]

This reflecting K-matrix \( K^+ \) is also a supermatrix with BFF grading. Therefore, \( B^+(\lambda) \text{ and } C^+(\lambda) \text{ are Grassmann odd, } A^+(\lambda), D^+(\lambda) \text{ are Grassmann even. Here we use the representation } \tilde{\sigma}^+ = a_R, \tilde{\sigma}^- = a_R^\dagger, \tilde{\sigma}^z = 1 - 2n_R, \text{ and } n_R = a_R^\dagger a_R. \]
We can thus obtain the boundary impurity terms of the Hamiltonian defined by the reflecting matrices,

\[
H_1 = \frac{2\sin(\eta)}{\sin(\eta + c)\sin(\eta - c)}(1 - n_{1,1})[a^\dagger_L c_{1,-1} + c^\dagger_{1,-1} a_L + \frac{e^{i\eta}}{2}(1 - 2n_L)]
\]

\[
+ (1 - n_{1,1}) \left( \frac{2\sin(2c)}{\sin(\eta + c)\sin(c - \eta)} - 4i \right) + \frac{\sin(\eta)e^{-i\eta}}{\sin(\eta + c)\sin(\eta - c)}(3T_1 + S^2_1 - 2).
\]

\(H_N\) has a similar form.

### 4.2 Algebraic Bethe ansatz method

We shall use the nested algebraic Bethe ansatz method to obtain the eigenvalue of the transfer matrix with open boundary conditions. We denote the double-row monodromy matrix as

\[
T(\lambda) = \begin{pmatrix}
A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\
A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\
C_1(\lambda) & C_2(\lambda) & D(\lambda)
\end{pmatrix}.
\]

(55)

For convenience, we introduce the following transformations

\[
\mathcal{A}_{ab}(\lambda) = \tilde{\mathcal{A}}_{ab}(\lambda) - \delta_{ab} \frac{e^{-2i\lambda\sin(\eta)}}{\sin(2\lambda - \eta)} D(\lambda).
\]

(56)

Note that this transformation is different from that of FFB grading. The transfer matrix \([20]\) can be rewritten as

\[
t(\lambda) = A^+(\lambda)A_{11}(\lambda) - D^+(\lambda)A_{22}(\lambda) - D(\lambda)
\]

\[
= A^+(\lambda)\tilde{A}_{11}(\lambda) - D^+(\lambda)\tilde{A}_{22}(\lambda) - U^+_3(\lambda)D(\lambda),
\]

(57)

where

\[
U^+_3(\lambda) = 1 + \frac{e^{-2i\lambda\sin(\eta)}}{\sin(2\lambda - \eta)} [A^+(\lambda) - D^+(\lambda)].
\]

(58)

We define a reference state in the \(n\)-th quantum space as \(|0 >_n = (0, 0, 1)^t\), and reference states for the boundary operators as \(\sigma^-|0 >_L = 0, \sigma^+|0 >_L = -|0 >_L, \sigma^+|0 >_L \neq 0, \text{ and } \tilde{\sigma}^-|0 >_R = 0, \tilde{\sigma}^+|0 >_R = -|0 >_R\), \(\tilde{\sigma}^+|0 >_R \neq 0\), with the help of Yang-Baxter relation, we have

\[
B_a(\lambda)|0 > = 0,
\]

\[
C_a(\lambda)|0 > \neq 0,
\]

\[
D(\lambda)|0 > = \sin^{2N}(\lambda - \eta)|0 >,
\]

\[
\tilde{A}_{ab}(\lambda)|0 > = \sin^{2N}(\lambda)[K(\lambda)]^b_c + \delta_{ab} \frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda - \eta)} |0 > = W_{ab}(\lambda)\sin^{2N}(\lambda)|0 >,
\]

(59)

where

\[
W_{12}(\lambda) = 0,
\]

\[
W_{13}(\lambda) = \sin(\lambda - \eta)e^{-2i\lambda}.
\]
\[ W_{11}(\lambda) = \frac{-e^{-2i\lambda} \sin(2\lambda)}{\sin(2\lambda - \eta) \sin(\lambda + c - \eta) \sin(\lambda - c - \eta)} [\sin(\lambda + c - \eta) \sin(\lambda - c - \eta) + \sin^2(\eta) e^{-i(2\lambda - \eta)}] \]

\[ W_{22}(\lambda) = -e^{-2\lambda} \frac{\sin(2\lambda) \sin(\lambda + c - 2\eta)}{\sin(2\lambda - \eta) \sin(\lambda + c + \eta)}. \]  

(60)

As mentioned in section 2, the double-row monodromy matrix satisfies the reflection equation (21).

We thus have the following commutation relations:

\[ C_{d_1}(\lambda)C_{d_2}(\mu) = r_{12}(\lambda - \mu)^{d_1d_2}C_{c_1}(\mu)C_{c_1}(\lambda), \]  

(61)

\[ D(\lambda)C_{d}(\mu) = \frac{\sin(\lambda + \mu) \sin(\mu - \eta) \sin(\lambda - \mu) \sin(\lambda - \mu - \eta)}{\sin(\lambda - \mu) \sin(2\lambda - \mu - \eta) \sin(\lambda - \mu - \eta)} C_{d}(\mu)D(\lambda) \]

\[ + \frac{\sin(2\mu) \sin(\eta) e^{i(\lambda - \mu)}}{\sin(\lambda - \mu) \sin(2\mu - \eta)} C_{d}(\lambda)D(\mu) + \frac{\sin(\eta) e^{i(\lambda + \mu)}}{\sin(\lambda - \mu) \sin(2\mu - \eta)} C_{b}(\mu) \tilde{A}_{b}(\lambda), \]  

(62)

Here the indices take values 1, 2, and the Grassmann parities are BF, \( \epsilon_1 = 0, \epsilon_2 = 1. \)

By use of the standard algebraic Bethe ansatz method, acting the above defined transfer matrix on the ansatz of eigenvector \( C_{d_1}(\mu_1)C_{d_2}(\mu_2) \cdots C_{d_n}(\mu_n) | 0 > F_{d_1 \cdots d_n} \), we have

\[ t(\lambda) C_{d_1}(\mu_1)C_{d_2}(\mu_2) \cdots C_{d_n}(\mu_n) | 0 > F_{d_1 \cdots d_n} = -U^+_3(\lambda) \sin^{2N}(\lambda - \eta) \prod_{i=1}^{n} \frac{\sin(\lambda + \mu_i) \sin(\lambda - \mu_i + \eta)}{\sin(\lambda + \mu_i - \eta) \sin(\lambda - \mu_i)} C_{d_1}(\mu_1) \cdots C_{d_n}(\mu_n) | 0 > F_{d_1 \cdots d_n} \]

\[ + \sin^{2N}(\lambda) \prod_{i=1}^{n} \frac{1}{\sin(\lambda - \mu_i) \sin(\lambda + \mu_i - \eta)} C_{c_1}(\mu_1) \cdots C_{c_n}(\mu_n) | 0 > t^{(1)}(\lambda)^{\epsilon_1 \cdots \epsilon_n} F_{d_1 \cdots d_n} \]

\[ + \text{u.t.}, \]  

(64)

where

\[ U^+_3 = \frac{\sin(\lambda - \eta + \frac{\eta}{2}) \sin(\lambda - \eta - \frac{3\eta}{2})}{\sin^2(\lambda - \eta - \frac{\eta}{2})}. \]  

(65)

The nested transfer matrix \( t^{(1)}(\lambda) \) is defined as

\[ t^{(1)}(\lambda)^{\epsilon_1 \cdots \epsilon_n} = (-)^{\epsilon_1} K^+(\lambda)^a \left\{ r(\lambda + \mu_1 - \eta)^{a_1c_1} r(\lambda + \mu_2 - \eta)^{a_2c_2} \cdots r(\lambda + \mu_n - \eta)^{a_nc_n} \right\} \]

\[ W_{a_1b_1}(\lambda) \left\{ r_{21}(\lambda - \mu_1)^{b_1e_1} \cdots r_{21}(\lambda - \mu_n)^{b_ne_n} F_{d_1 \cdots d_n} \right\} \]

\[ \times (-1)^{\sum_{i=1}^{n} (\epsilon_{a_i} + \epsilon_{b_i})(1 + \epsilon_{c_i})}. \]  

(66)
Thus this nested transfer matrix can still be interpreted as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case\cite{6}.

\[ t^{(1)}(\lambda) = strK^{(1)+}(\Lambda')T^{(1)}(\Lambda', \{\mu'_i\})K^{(1)}(\Lambda')T^{(1)-1}(\Lambda', \{\mu'_i\}), \]  

(67)

where we denote \( x' = x - \frac{j}{2}, x = \lambda, \mu, c, \tilde{c} \). Notice that this definition is different from that of the FFB case. The row-to-row monodromy matrix \( T^{(1)}(\Lambda', \{\mu'_i\}) \) and \( T^{(1)-1}(\Lambda', \{\mu'_i\}) \) are defined respectively as

\[
T^{(1)}_{\alpha a} (\Lambda', \{\mu'_i\})_{c_1, \ldots, c_n} = r(\lambda + \mu'_1)_{a c_1} r(\lambda + \mu'_2)_{a_2 c_2} \cdots \left[ r(\lambda + \mu'_n)_{a_n c_n} \right]^{-1} (1 + \varepsilon_n),
\]

(68)

\[
T^{(1)-1}_{b_n a} (-\Lambda', \{\mu'_i\})_{d_1, \ldots, d_n} = r_{21}(\lambda' - \mu'_n)_{b_n d_n} \cdots r_{21}(\lambda' - \mu'_2)_{b_2 d_2} r_{21}(\lambda' - \mu'_1)_{b_1 d_1}^{-1} (1 + \varepsilon_n),
\]

(69)

where we have used the unitarity relation of the \( r \)-matrix \( r_{12}(\lambda) r_{21}(-\lambda) = \sin(\eta - \lambda)\sin(\eta + \lambda) \cdot \text{id.} \)

We find that the super-tensor product in the above defined monodromy matrix differs from the original definition. Nevertheless, as in the periodic boundary condition case, we can define another graded tensor product as follows \cite{14}:

\[ F \bigotimes G_{ab}^{bd} = F^a_{c} G^d_{c} (-1)^{\varepsilon_a + \varepsilon_b} (1 + \varepsilon_c). \]

(70)

Effectively this graded tensor product switches even and odd Grassmann parities. The graded tensor product in the above monodromy matrices follows the newly defined rule.

Now, we shall find the eigenvalue of the reduced transfer matrix (67). Like the original one, we can still use the graded algebraic Bethe ansatz method. However, we should be careful about some technical points. The graded tensor product in the row-to-row monodromy matrices \cite{6,15,53} are defined by a new definition. We thus should use a newly defined graded Yang-Baxter relation, reflection equation and the dual reflection equation. We should also prove that reflecting matrices appeared in \cite{57} indeed satisfy their corresponding reflection equations.

Define another \( r \)-matrix \cite{16} by

\[ \hat{r}(\lambda)_{ab}^{bd} = (-1)^{\varepsilon_a + \varepsilon_b} r(\lambda)_{ac}. \]

(71)

This \( r \)-matrix has the cross-unitarity, relation

\[ \hat{r}_{12}(\lambda) \hat{r}_{21}(\lambda) = -\sin^2(\lambda) \cdot \text{id.} \]

(72)

With the help of the original Yang-Baxter relation, we have the following graded Yang-Baxter relation for the row-to-row monodromy matrix (68),

\[
\hat{r}(\lambda_1 - \mu_2)_{a_1 b_2}^{b_1 c_2} T^{(1)}(\Lambda_1, \{\mu_i\})_{b_1}^{c_1} T^{(1)}(\Lambda_2, \{\mu_i\})_{b_2}^{c_2} (-1)^{\varepsilon_{a_1} + \varepsilon_{c_1}} (1 + \varepsilon_{c_2}) = T^{(1)}(\Lambda_2, \{\mu_i\})_{b_2}^{c_2} T^{(1)}(\Lambda_1, \{\mu_i\})_{a_1}^{b_1} \hat{r}(\lambda_1 - \lambda_2)_{b_1 b_2}^{c_2} (-1)^{\varepsilon_{a_1} + \varepsilon_{b_1}} (1 + \varepsilon_{c_2}).
\]

(73)

The reflection equation should take the following form,

\[
\hat{r}(\lambda - \mu)_{a_1 b_2}^{b_1 c_2} K^{(1)}(\mu)_{c_1}^{b_1} \hat{r}(\lambda + \mu)_{a_2 c_2}^{d_2} K^{(1)}(\mu)_{d_2}^{c_2} (-1)^{\varepsilon_{a_1} + \varepsilon_{c_1}} (1 + \varepsilon_{c_2}) = K^{(1)}(\mu)_{a_2 c_2}^{d_2} (\lambda + \mu)_{a_1 b_2}^{b_1 c_2} K^{(1)}(\mu)_{b_2}^{d_2} \hat{r}(\lambda - \mu)_{c_1 b_2}^{c_2} (-1)^{\varepsilon_{a_1} + \varepsilon_{d_2}} (1 + \varepsilon_{c_2}).
\]

(74)
Here we find that r-matrix is the newly defined r-matrix $\hat{r}$ (71), and the super tensor-product is also in newly defined form (70). However, the reflecting K-matrix does not need to be changed, i.e., this reflecting K-matrix also satisfy the reflection equation (15). We know the reflecting K-matrix defined in (17) is written as

$$K^{(1)}(\lambda') = e^{-i\eta \sin(2\lambda' + \eta)} \begin{pmatrix} A(\lambda', \epsilon') & B(\lambda', \epsilon') \\ C(\lambda', \epsilon') & D(\lambda', \epsilon') \end{pmatrix}. \quad (75)$$

We can easily find that this K-matrix satisfy the relation (45). Thus we know that it is also a solution to the new reflection equation (74). Similarly, we can deal with the dual reflection equation. The reflecting K-matrix defined in (17)

$$K^{(1)\dagger}(\lambda') = \begin{pmatrix} A^+(\lambda' + \frac{\pi}{2}) & B^+(\lambda' + \frac{\pi}{2}) \\ C^+(\lambda' + \frac{\pi}{2}) & D^+(\lambda' + \frac{\pi}{2}) \end{pmatrix} \quad (76)$$

is shown to satisfy the corresponding dual reflection equation which is consistent with the cross-unitarity relation (72). We thus prove that the nested transfer matrix (67) is indeed a transfer matrix, and we can as previously use the graded algebraic Bethe ansatz method to find its eigenvalues and eigenvectors.

### 4.3 Nested algebraic Bethe ansatz

We show that the problem to find the eigenvalue of $t(\lambda)$ is changed into a simpler problem to find the eigenvalue of $t^{(1)}(\lambda)$. We shall still use the graded algebraic Bethe ansatz method. Denote the nested double-row monodromy matrix as

$$\mathcal{T}^{(1)}(\lambda, \{\mu_i\}) = T^{(1)}(\lambda, \{\mu_i\}) K^{(1)}(\lambda) T^{(1)\dagger}(-\lambda, \{\mu_i\}) = \begin{pmatrix} \mathcal{A}^{(1)}(\lambda) & \mathcal{B}^{(1)}(\lambda) \\ \mathcal{C}^{(1)}(\lambda) & \mathcal{D}^{(1)}(\lambda) \end{pmatrix}. \quad (77)$$

The nested double-row monodromy matrix satisfies the reflection equation,

$$\hat{r}(\lambda - \mu) b_{1i} b_{2j} \mathcal{T}^{(1)}(\lambda) c_i c_j \hat{r}(\lambda + \mu) e^{i \epsilon_1 \epsilon_2} \mathcal{T}^{(1)}(\mu) e^{-i \epsilon_1 \epsilon_2} = \mathcal{T}^{(1)}(\mu) b_{2j} b_{1i} \hat{r}(\lambda - \mu) c_j c_i \hat{r}(\lambda + \mu) e^{i \epsilon_1 \epsilon_2} \mathcal{T}^{(1)}(\lambda). \quad (78)$$

For convenience, we need the following transformation,

$$\mathcal{A}^{(1)}(\lambda) = \tilde{\mathcal{A}}^{(1)}(\lambda) - \frac{\sin (\eta) e^{2i\lambda}}{\sin(2\lambda - \eta)} \mathcal{D}^{(1)}(\lambda), \quad (79)$$

With the help of this transformation, we have the following commutation relations:

$$\mathcal{D}^{(1)}(\lambda) \mathcal{C}^{(1)}(\mu) = \frac{\sin(\lambda - \mu + \eta) \sin(\lambda + \mu)}{\sin(\lambda - \mu) \sin(\lambda + \mu - \eta)} \mathcal{C}^{(1)}(\mu) \mathcal{D}^{(1)}(\lambda) - \frac{\sin(2\mu) \sin(\eta) e^{i(\lambda - \mu)}}{\sin(\lambda - \mu) \sin(2\lambda - \eta)} \mathcal{C}^{(1)}(\lambda) \mathcal{D}^{(1)}(\mu) + \frac{\sin(\eta) e^{i(\lambda + \mu)}}{\sin(\lambda + \mu - \eta)} \mathcal{C}^{(1)}(\lambda) \tilde{\mathcal{A}}^{(1)}(\mu), \quad (80)$$

$$\tilde{\mathcal{A}}^{(1)}(\lambda) \mathcal{C}^{(1)}(\mu) = \frac{\sin(\lambda - \mu + \eta) \sin(\lambda + \mu)}{\sin(\lambda - \mu) \sin(\lambda + \mu - \eta)} \mathcal{C}^{(1)}(\mu) \tilde{\mathcal{A}}^{(1)}(\lambda) - \frac{\sin(\eta) \sin(2\lambda) e^{-i(\lambda - \mu)}}{\sin(\lambda - \mu) \sin(2\lambda - \eta)} \mathcal{C}^{(1)}(\lambda) \tilde{\mathcal{A}}^{(1)}(\mu) + \frac{\sin(2\mu) \sin(2\lambda) \sin(\eta) e^{-i(\lambda + \mu)}}{\sin(\lambda + \mu - \eta) \sin(2\lambda - \eta) \sin(2\mu - \eta)} \mathcal{C}^{(1)}(\lambda) \mathcal{D}^{(1)}(\mu), \quad (81)$$

\[\text{Page 14}\]
In the same manner as the standard algebraic Bethe ansatz method, acting the transfer matrix
on the ansatz of the eigenvector
\[ t^{(1)}(\lambda) \equiv A^+(\lambda') + \frac{\eta}{2} A^{(1)}(\lambda') - D^+(\lambda') + \frac{\eta}{2} D^{(1)}(\lambda') = U^+_1(\lambda') \hat{A}^{(1)}(\lambda') - U^+_2(\lambda') \hat{D}^{(1)}(\lambda') \]
on the ansatz of the eigenvector \( C(\hat{\mu}_1^{(1)}) C(\hat{\mu}_2^{(1)}) \cdots C(\hat{\mu}_m^{(1)}) |0 \rangle \), we find the eigenvalue of the nested transfer matrix as follows:

\[
\Lambda^{(1)}(\lambda) = U^+_1(\lambda') U_2(\lambda') \prod_{i=1}^{n} [\sin(\lambda' + \hat{\mu}_i^{(1)}) \sin(\lambda' - \hat{\mu}_i^{(1)})] \prod_{i=1}^{m} \left\{ \frac{\sin(\lambda' - \hat{\mu}_i^{(1)} + \eta) \sin(\lambda' + \hat{\mu}_i^{(1)})}{\sin(\lambda' - \hat{\mu}_i^{(1)}) \sin(\lambda' + \hat{\mu}_i^{(1)} - \eta)} \right\} 
- U^+_2(\lambda') U_1(\lambda') \prod_{i=1}^{n} [\sin(\lambda' + \hat{\mu}_i^{(1)}) \sin(\lambda' - \hat{\mu}_i^{(1)} - \eta)] \prod_{i=1}^{m} \left\{ \frac{\sin(\lambda' - \hat{\mu}_i^{(1)} + \eta) \sin(\lambda' + \hat{\mu}_i^{(1)})}{\sin(\lambda' - \hat{\mu}_i^{(1)}) \sin(\lambda' + \hat{\mu}_i^{(1)} - \eta)} \right\}
\]

where \( \hat{\mu}_1^{(1)}, \ldots, \hat{\mu}_m^{(1)} \) should satisfy the following Bethe ansatz equations,

\[
\frac{U^+_1(\hat{\mu}_j^{(1)}) U_1(\hat{\mu}_j^{(1)})}{U^+_2(\hat{\mu}_j^{(1)}) U_2(\hat{\mu}_j^{(1)})} \prod_{i=1}^{n} \frac{\sin(\hat{\mu}_j^{(1)} + \hat{\mu}_i^{(1)}) \sin(\hat{\mu}_j^{(1)} - \hat{\mu}_i^{(1)})}{\sin(\hat{\mu}_j^{(1)} + \hat{\mu}_i^{(1)} - \eta) \sin(\hat{\mu}_j^{(1)} - \hat{\mu}_i^{(1)} - \eta)} = 1, \quad j = 1, \ldots, m.
\]

The boundary parameters \( U_1, U^+_1, U_2, U^+_2 \) can be calculated from the reflecting K-matrices (75, 76). We shall present the results in the next sub-section.

### 4.4 Results for BFF grading

In this sub-section, we shall summarize the results for the case of BFF grading. The eigenvalue of the transfer matrix \( t(\lambda) \) with integrable boundary impurities is obtained as

\[
\Lambda(\lambda) = -U^+_1(\lambda) U_2(\lambda) \sin^{2N}(\lambda - \eta) \prod_{i=1}^{n} \frac{\sin(\lambda + \mu_i) \sin(\lambda - \mu_i + \eta)}{\sin(\lambda + \mu_i - \eta) \sin(\lambda - \mu_i)} 
+ \sin^{2N}(\lambda) \prod_{i=1}^{n} \frac{1}{\sin(\lambda - \mu_i) \sin(\lambda + \mu_i - \eta)} \Lambda^{(1)}(\lambda),
\]

\[
\Lambda^{(1)}(\lambda) = U^+_1(\lambda) U_1(\lambda) \prod_{i=1}^{n} [\sin(\lambda + \mu_i - \eta) \sin(\lambda - \mu_i)] \prod_{i=1}^{m} \left\{ \frac{\sin(\lambda - \mu_i^{(1)} + \eta) \sin(\lambda + \mu_i^{(1)} - \eta)}{\sin(\lambda - \mu_i^{(1)} - 2\eta) \sin(\lambda + \mu_i^{(1)} - 2\eta)} \right\} 
- U^+_2(\lambda) U_2(\lambda) \prod_{i=1}^{n} [\sin(\lambda + \mu_i - 2\eta) \sin(\lambda - \mu_i - \eta)] 
\prod_{i=1}^{m} \left\{ \frac{\sin(\lambda - \mu_i^{(1)} + \eta) \sin(\lambda + \mu_i^{(1)} - \eta)}{\sin(\lambda - \mu_i^{(1)} - 2\eta) \sin(\lambda + \mu_i^{(1)} - 2\eta)} \right\},
\]

(87)
where \( \mu_1^{(1)}, \cdots, \mu_n^{(1)} \) and \( \mu_1, \cdots, \mu_n \) should satisfy the Bethe ansatz equations:

\[
\frac{\sin(2\mu_j)}{\sin(\mu_j - 2\eta)} \prod_{i=1, i \neq j}^{n} \left\{ \frac{\sin(\mu_j + \mu_i)\sin(\mu_j - \mu_i + \eta)}{\sin(\mu_j + \mu_i - 2\eta)\sin(\mu_j - \mu_i - \eta)} \right\} \\
= \frac{\sin^{2N}(\mu_j)}{\sin^{2N}(\mu_j - \eta)} U_2^{(j)} U_2^{(\mu_j)} U_3^{(j)} U_3^{(\mu_j)} \prod_{i=1}^{m} \left\{ \frac{\sin(\mu_j - \mu_i^{(1)} + \eta)\sin(\mu_j + \mu_i^{(1)} - \eta)}{\sin(\mu_j - \mu_i^{(1)})\sin(\mu_j + \mu_i^{(1)} - 2\eta)} \right\}, \quad j = 1, \cdots, n.
\]

(88)

\[
U_1^{(j)} U_1^{(\mu_j)} \prod_{i=1}^{n} \sin(\mu_j^{(1)} + \mu_i - \eta)\sin(\mu_j^{(1)} - \mu_i) = 1, \quad j = 1, \cdots, m. \quad (89)
\]

Actually, these relations are rather general, we can take other K-matrices. For the K-matrices considered in this paper, the boundary parameters take the following form:

\[
U_1(\lambda) = -\frac{\sin(2\lambda)e^{-i(2\lambda+\eta)}}{\sin(2\lambda - 2\eta)} \frac{\sin(\lambda + c - \eta)\sin(\lambda - c)}{\sin(\lambda - c + \eta)\sin(2\lambda - \eta)},
\]

\[
U_2(\lambda) = -e^{-2i\lambda}\sin(2\lambda)\sin(\lambda + c - 2\eta),
\]

\[
U_3(\lambda) = 1,
\]

\[
U_1^{+}(\lambda) = -\frac{e^{i(2\lambda+\eta)}\sin(\lambda + c - \frac{\eta}{2})}{\sin(\lambda - c + \frac{\eta}{2})},
\]

\[
U_2^{+}(\lambda) = -\frac{e^{2i\lambda}\sin(2\lambda - \eta)\sin(\lambda - c - \frac{3\eta}{2})\sin(\lambda + c - \frac{3\eta}{2})}{\sin(2\lambda - 2\eta)\sin^{2}(\lambda - c - \frac{\eta}{2})},
\]

\[
U_3^{+}(\lambda) = \frac{\sin(\lambda - c + \frac{\eta}{2})\sin(\lambda - c - \frac{3\eta}{2})}{\sin^{2}(\lambda - c - \frac{\eta}{2})}. \quad (90)
\]

The energy of the Hamiltonian is given by

\[
E = -N\cos(\eta) - \sum_{j=1}^{n} \frac{\sin^{2}(\eta)}{\sin(\mu_j)\sin(\mu_j - \eta)} - \frac{\sin^{3}(\eta)\cos(\bar{c} + \frac{\eta}{2})}{\sin(\bar{c} - \frac{\eta}{2})\sin(\bar{c} + \frac{\eta}{2})\sin(\bar{c} + \frac{3\eta}{2})}. \quad (91)
\]

5 Results of the FBF grading

We have dealt with the gradings FFB and BFF. We shall study the last possible grading, FBF, \( \epsilon_1 = \epsilon_3 = 1 \), \( \epsilon_2 = 0 \). In this case, we choose the following representation,

\[
S_k = e_{13}^{k}, \quad S_k^{\dagger} = e_{31}^{k}, \quad S_k^{\ddagger} = \frac{1}{2}(e_{11}^{k} - e_{33}^{k}),
\]

\[
Q_{k,1} = (1 - n_{k,-1})c_{k,1} = e_{21}^{k}, \quad Q_{k,1}^{\dagger} = (1 - n_{k,-1})c_{k,1}^{\dagger} = e_{12}^{k}, \quad Q_{k,-1} = (1 - n_{k,1})c_{k,-1} = e_{23}^{k}, \quad Q_{k,-1}^{\dagger} = (1 - n_{k,1})c_{k,-1}^{\dagger} = e_{32}^{k}, \quad T_k = 1 - \frac{1}{2}n_k = \frac{1}{2}(e_{33}^{k} + e_{11}^{k}) + e_{22}^{k}. \quad (92)
\]

The calculation can be preformed in the same way as the BFF grading. Here we present some main results. We still suppose that K-matrix operators satisfying the reflection equation and the dual reflection
equation take the form (44,51). We have the following solution to the reflection equation

$$A(\lambda) = \frac{1}{2} g(\lambda) \left[ \sin(\lambda + c - \eta)\sin(\lambda - c - \eta) + \sin(\lambda + c)\sin(\lambda - c) 
+ \sin^2(\eta)e^{-2i\lambda} - \sin(\eta)\sin(2\lambda)e^{-i\eta}\sigma^z \right],$$

$$B(\lambda) = g(\lambda)\sin(\eta)\sin(2\lambda)\sigma^z,$$

$$C(\lambda) = g(\lambda)\sin(\eta)\sin(\lambda - c + \eta),$$

$$D(\lambda) = \frac{1}{2} g(\lambda) \left[ \sin(\lambda + c + \eta)\sin(\lambda - c) + \sin(\lambda + c)\sin(\lambda - c) 
+ \sin^2(\eta)e^{2i\lambda} - \sin(\eta)\sin(2\lambda)e^{-i\eta}\sigma^z \right],$$

(93)

where

$$g(\lambda) = \frac{-e^{-2i\lambda}}{\sin(\lambda - c - \eta)}. \quad (94)$$

For the case of the dual reflection equation, we have

$$A^+(\lambda) = \frac{1}{2} g^+(\lambda) \left[ \sin(\lambda + c - \frac{3\eta}{2})\sin(\lambda - c - \frac{3\eta}{2}) + \sin(\lambda + c - \frac{\eta}{2})\sin(\lambda - c - \frac{\eta}{2}) 
- \sin^2(\eta)e^{(2\lambda - \eta)} - \sin(\eta)\sin(2\lambda - \eta)e^{i\eta}\sigma^z \right],$$

$$B^+(\lambda) = g^+(\lambda)e^{i\eta}\sin(2\lambda - \eta)\tilde{\sigma}^z,$$

$$C^+(\lambda) = g^+(\lambda)e^{i\eta}\sin(2\lambda + \eta)\tilde{\sigma}^z,$$

$$D^+(\lambda) = \frac{1}{2} g^+(\lambda) \left[ \sin(\lambda + c + \frac{\eta}{2})\sin(\lambda - c + \frac{\eta}{2}) + \sin(\lambda + c + \frac{\eta}{2})\sin(\lambda - c - \frac{\eta}{2}) 
- \sin^2(\eta)e^{-i(2\lambda - \eta)} - \sin(\eta)\sin(2\lambda - \eta)e^{i\eta}\sigma^z \right],$$

(95)

where

$$g^+(\lambda) = \frac{-e^{i(2\lambda - \eta)}}{\sin^2(\lambda - c - \frac{\eta}{2})}. \quad (96)$$

The boundary impurity term in the Hamiltonian defined by the K-matrix is written as

$$H_1 = \frac{2\sin(\eta)}{\sin(\eta + c)\sin(\eta - c)} \left[ (1 - n_{1,1})[a_L^\dagger c_{1,-1} + c_{1,-1}^\dagger a_L - \frac{e^{-i\eta}}{2}(1 - 2n_L)] 
+ (1 - n_{1,1}) \left( \frac{2\sin(2c)}{\sin(\eta + c)\sin(c - \eta)} - 4i \right) + \frac{\sin(\eta)e^{i\eta}}{\sin(\eta + c)\sin(\eta - c)}(3T_1 - S_1^z - 2). \quad (97)$$

The eigenvalue of the transfer matrix with boundary impurities is obtained as

$$\Lambda(\lambda) = -U_3^+(\lambda)U_3(\lambda)\sin^{2N}(\lambda - \eta)\prod_{i=1}^n \frac{\sin(\lambda + \mu_i)\sin(\lambda - \mu_i + \eta)}{\sin(\lambda + \mu_i - \eta)\sin(\lambda - \mu_i)}$$

$$+ \sin^{2N}(\lambda) \prod_{i=1}^n \frac{1}{\sin(\lambda - \mu_i)\sin(\lambda + \mu_i - \eta)} \Lambda^{(1)}(\lambda),$$

$$\Lambda^{(1)}(\lambda) = -U_1^+(\lambda)U_1(\lambda) \prod_{i=1}^n \left[ \sin(\lambda + \mu_i - \eta)\sin(\lambda - \mu_i) \prod_{l=1}^m \left\{ \frac{\sin(\lambda - \mu_i^{(1)} - \eta)\sin(\lambda + \mu_i^{(1)} - \eta)}{\sin(\lambda - \mu_i^{(1)})\sin(\lambda + \mu_i^{(1)})} \right\} \right].$$
boundary terms in the Hamiltonian for different gradings are completely different. Each grading may
remark that K-matrix remains the same form while R-matrix changes for different gradings. Thus the
Hamiltonain with integrable boundary impurities.

The nested algebraic Bethe ansatz method, we have obtained the eigenvalues for the transfer matrix and
reflecting K-matrix operators which are solutions to the reflection equation in different gradings. Using

\[ E = -N \cos(\eta) - \sum_{j=1}^{n} \frac{\sin^2(\eta)}{\sin(\mu_j)\sin(\mu_j - \eta)} - \frac{\sin^3(\eta)\cos(\hat{c} + \frac{\pi}{2})}{\sin(\hat{c} - \frac{\pi}{2})\sin(\hat{c} + \frac{\pi}{2})\sin(\hat{c} + \frac{3\pi}{2})}. \]

6 Summary
In this paper, we have studied the integrable boundary impurity problem for the generalized (q-deformed)
supersymmetric \( t - J \) model in all possible three gradings, FFB, BFF and FBF. We have presented
reflecting K-matrix operators which are solutions to the reflection equation in different gradings. Using
the nested algebraic Bethe ansatz method, we have obtained the eigenvalues for the transfer matrix and
the Hamiltonian with integrable boundary impurities.

In all three possible gradings, we suppose that reflecting K-matrices take a similar form \( (33, 44) \). We
remark that K-matrix remains the same form while R-matrix changes for different gradings. Thus the
boundary terms in the Hamiltonian for different gradings are completely different. Each grading may

\[ + U_2^+(\lambda)U_2(\lambda) \prod_{i=1}^{n}[\sin(\lambda + \mu_i)\sin(\lambda - \mu_i + \eta)] \]
\[ \prod_{i=1}^{m} \left\{ \frac{\sin(\lambda - \mu^{(1)}_i - \eta)\sin(\lambda + \mu^{(1)}_i - \eta)}{\sin(\lambda - \mu^{(1)}_i)\sin(\lambda + \mu^{(1)}_i)} \right\}, \]
correspond to an integrable boundary impurity. It is interesting to analyze the Bethe ansatz equations and compare the ground state properties, low-lying excitations and the thermodynamic limit for different gradings. These remain as our future problems.

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