A survey on the
skew energy of oriented graphs*

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Abstract

The skew energy of an oriented graph was introduced by Adiga, Balakrishnan and So in 2010, as one of various generalizations of the energy of an undirected graph. Let $G$ be a simple undirected graph, and $G^\sigma$ be an oriented graph of $G$ with the orientation $\sigma$ and skew-adjacency matrix $S(G^\sigma)$. The skew energy of the oriented graph $G^\sigma$, denoted by $E_S(G^\sigma)$, is defined as the sum of the norms of all the eigenvalues of $S(G^\sigma)$. In this paper, we summarize most of the known results on the skew energy of oriented graphs. Some open problems are also proposed for further study.

Keywords: oriented graph, skew energy, skew-adjacency matrix, eigenvalues, extremal graph, regular oriented graph, random oriented graph, Hadamard matrix

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1 Introduction

In this introductory section, we will present the related background for introducing the concept of skew energy for oriented graphs. Some basic definitions are also given.

Let $G$ be a simple undirected graph on order $n$ with vertex set $V(G)$ and edge set $E(G)$. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the adjacency matrix of $G$ is the $n \times n$ symmetric matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if the vertices $v_i$ and $v_j$ are adjacent, and

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Let $G^\sigma$ be an oriented graph of $G$ with the orientation $\sigma$, which assigns to each edge of $G$ a direction so that the resultant graph $G^\sigma$ becomes an oriented graph or a directed graph. Then $G$ is called the underlying graph of $G^\sigma$. The skew-adjacency matrix of $G^\sigma$ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $\langle v_i, v_j \rangle$ is an arc of $G^\sigma$, otherwise $s_{ij} = s_{ji} = 0$. It is easy to see that $S(G^\sigma)$ is a skew-symmetric matrix.

Actually, this skew-adjacency matrix of an oriented graph $G^\sigma$ was first introduced by Tutte [32] in 1947, where he defined a matrix $S(G^\sigma, x) = [s_{ij}]$ with $s_{ij} = x_e$ and $s_{ji} = -x_e$ if $e = \langle v_i, v_j \rangle$ is an arc of $G^\sigma$, otherwise $s_{ij} = s_{ji} = 0$, where $x_e$ is a formal variable. Using this matrix, he showed that the graph $G$ has a perfect matching if and only if $\det(S(G^\sigma, x)) \neq 0$. Moreover, it was proved (see [15, 30] for examples) that the number of perfect matchings in a graph $G$ with $m$ edges is enumerated by

$$\frac{1}{2m} \sum_{G^\sigma \in \mathrm{Ort}(G)} \det(S(G^\sigma)),$$

where $\mathrm{Ort}(G)$ denotes the set of all oriented graphs of $G$. It is clear that when we take every $x_e = 1$, the resultant matrix is just the skew-adjacency matrix $S(G^\sigma)$.

Among the orientations of a graph $G$, the Pfaffian orientations play an important role in the enumeration of perfect matchings of $G$. A Pfaffian orientation of $G$ is such an orientation for the edges of $G$ under which every even cycle $C$ of $G$ such that $G \setminus V(C)$ has a perfect matching has the property that there are odd number of edges directed in either direction of the cycle $C$. In other words, a Pfaffian orientation of $G$ is such an orientation for the edges of $G$ under which every even cycle $C$ of $G$ such that $G \setminus V(C)$ has a perfect matching is oddly oriented. In 1961, the physicists Fisher [13], Kasteleyn [26] and Temperley [36] used the Pfaffian orientations of a graph to enumerate the number of perfect matchings in a graph. It was showed that the number of perfect matchings in $G$ is equal to the square root of $\det(S(G^\sigma))$, where $\sigma$ is a Pfaffian orientation of $G$. For more on the Pfaffian orientations, we refer to Robertson, Seymour and Thomas [33] (Ann. Math., 1999), and Thomas [37] (45min invited speech at the ICM 2006).

For an oriented graph $G^\sigma$ of an undirected graph $G$, there is the adjacency matrix $A(G^\sigma) = [a^\sigma_{ij}]$ [11], where $a^\sigma_{ij} = 1$ if $\langle v_i, v_j \rangle$ is an arc of $G^\sigma$, and $a^\sigma_{ij} = 0$ otherwise. It is clear that for the adjacency matrix of $G$, we have $A(G) = A(G^\sigma) + A^T(G^\sigma)$; whereas for the skew-adjacency matrix of $G^\sigma$, we have $S(G^\sigma) = A(G^\sigma) - A^T(G^\sigma)$. The adjacency matrix $A(G)$ of a graph $G$ has been well-studied. It has a characteristic polynomial, and it is symmetric and therefore all its eigenvalues are real. Among many topics about it, we mention the energy of a graph, which is defined as the sum of the absolute values of all its eigenvalues. The concept of the energy was introduced by Gutman in [20].
in theoretical chemistry, the energy of a given molecular graph is related to the total π-electron energy of the molecule represented by that graph. Consequently, the graph energy has some specific chemistry interests and has been extensively studied. We refer to the survey \[21\] and the book \[29\] for details. Up to now, there are various generalizations of the graph energy, such as the Laplacian energy, signless Laplacian energy, incidence energy, distance energy, and the Laplacian-energy like invariant for undirected graphs.

The skew energy of oriented graphs is another direction of generalization for the energy of graphs. First of all, we recall some definitions. The characteristic polynomial of a skew-adjacency matrix \(S(G^\sigma)\), i.e., \(\text{det}(\lambda I_n - S(G^\sigma))\) is said to be the skew-characteristic polynomial of the oriented graph \(G^\sigma\), denoted by \(\phi_s(G^\sigma, \lambda)\). From linear algebra it is known that the eigenvalues of \(S(G^\sigma)\) are just the solutions of the equation \(\phi_s(G^\sigma, \lambda) = 0\), which form the spectrum of \(S(G^\sigma)\) and are said to be the skew-spectrum of \(G^\sigma\). Since \(S(G^\sigma)\) is skew-symmetric, every eigenvalue of \(S(G^\sigma)\) is a pure imaginary number or 0.

Analogous to the definition of the energy of a simple undirected graph, the skew energy of an oriented graph \(G^\sigma\), proposed first by Adiga, Balakrishnan and So \[1\] and denoted by \(E_S(G^\sigma)\), is defined as the sum of the norms of all the eigenvalues of \(S(G^\sigma)\). There are situations when chemists use digraphs rather than graphs. One such situation is when vertices represent distinct chemical species and arcs represent the direction in which a particular reaction takes place between the two corresponding species. So, people hope that this skew energy will have similar applications as energy in chemistry.

The rest of this paper is organized as follows: In Section 2, we summarize the results about the skew-characteristic polynomial of oriented graphs. Then in Section 3, we collect some basic properties of the skew energy of oriented graphs and state the integral formulas for the skew energy. Section 4 is used to survey the results about bipartite graphs with \(Sp(G^\sigma) = Sp(G)\). General bounds of the skew energy for oriented graphs are given in Section 5 and in which the progress on characterizing the oriented graphs achieving the upper bound are described. The skew-spectra of various products of graphs are determined in Section 6 and therein their applications in skew energy are illustrated. Section 7 outlines some results on extremal oriented graphs with regard to skew energy. Section 8 is concerned with the skew energy of random oriented graphs. The last section, Section 9 is for concluding remarks. In each of the sections, we propose some open problems for further study.
2 Skew-characteristic polynomials of oriented graphs

This section is used to summarize the results about the skew-characteristic polynomials and skew-spectra of oriented graphs.

First we give some definitions. Let $G$ be an undirected graph. An $r$-matching of $G$ is an edge subset of $r$ edges such that every vertex of $G$ is incident with at most one edge in it. Denote by $m(G, r)$ the number of all $r$-matchings of $G$. Let $G^\sigma$ be an oriented graph of $G$ and $C$ be an undirected even cycle of $G$. Then $C$ is said to be \textit{evenly oriented} relative to $G^\sigma$ if it has an even number of edges oriented in clockwise direction (and now it also has an even number of edges oriented in anticlockwise direction, since $C$ is an even cycle); otherwise $C$ is \textit{oddly oriented}.

Recall that a linear subgraph $L$ of $G$ is a disjoint union of some edges and some cycles in $G$. A linear subgraph $L$ of $G$ is called \textit{even linear} if $L$ contains no odd cycle, i.e., the number of vertices of $L$ is even. Denote by $\mathcal{E}L_i(G)$ the set of all evenly linear subgraphs of $G$ with $i$ vertices. These definitions in undirected graphs are the same as that in oriented graphs. For a linear subgraph $L \in \mathcal{E}L_i$, we denote by $p_e(L)$ and $p_o(L)$ the number of evenly oriented cycles and oddly oriented cycles in $L$ relative to $G^\sigma$, respectively.

Let $G^\sigma$ be an oriented graph of a graph $G$ with the skew-adjacency matrix $S(G^\sigma)$. Let the skew-characteristic polynomial $\phi_s(G^\sigma, \lambda)$ of $G^\sigma$ be

$$\phi_s(G^\sigma, \lambda) = \det(\lambda I_n - S(G^\sigma)) = \sum_{i=0}^{n} c_i \lambda^{n-i}.$$ 

The following theorem, obtained by Hou and Lei in [24], characterizes the coefficients of skew-characteristic polynomial of an oriented graph, which is analogous to the famous Sachs Theorem [10] for an undirected graph.

\textbf{Theorem 2.1} [24] Let $G^\sigma$ be an oriented graph of a graph $G$ with the skew-characteristic polynomial $\phi_s(G^\sigma, \lambda) = \sum_{i=0}^{n} c_i \lambda^{n-i}$. Then

$$c_i = \sum_{L \in \mathcal{E}L_i} (-2)^{p_e(L)} 2^{p_o(L)},$$

where $p_e(L)$ and $p_o(L)$ are the number of evenly oriented cycles and the number of oddly oriented cycles of $L$ relative to $G^\sigma$, respectively. In particular, $c_0 = 1$ and $c_i = 0$ for all odd $i$.

In [16], Gong, Li and Xu proved that all the coefficients of a skew-characteristic polynomial are nonnegative.
Theorem 2.2 \[10\] Let $G^\sigma$ be an oriented graph of a graph $G$ with the skew-characteristic polynomial $\phi_s(G^\sigma, \lambda) = \sum_{i \geq 0} c_{2i} \lambda^{n-2i}$. Then $c_{2i} \geq 0$ for every $i$ with $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

The following results are obtained by Hou and Lei in [24] as applications of Theorem 2.1, which can be used to find recursions for the characteristic polynomials of some skew-adjacency matrices.

Corollary 2.3 [24] Let $e = \langle u, v \rangle$ be an arc of $G^\sigma$. Then

$$\phi_s(G^\sigma, \lambda) = \phi_s(G^\sigma - e, \lambda) + \phi_s(G^\sigma - u - v, \lambda) + 2 \sum_{e \in C \in Od(G^\sigma)} \phi_s(G^\sigma - C, \lambda)$$

$$- 2 \sum_{e \in C \in Ev(G^\sigma)} \phi_s(G^\sigma - C, \lambda),$$

where $Od(G^\sigma)$ and $Ev(G^\sigma)$ denote the set of all oddly oriented cycles and evenly oriented cycles of $G^\sigma$, respectively.

Corollary 2.4 [24] Let $e = \langle u, v \rangle$ be an arc of $G^\sigma$ that is on no even cycle in $G^\sigma$. Then

$$\phi_s(G^\sigma, \lambda) = \phi_s(G^\sigma - e, \lambda) + \phi_s(G^\sigma - u - v, \lambda).$$

Similarly, Xu [40] got the following recursions of the skew-characteristic polynomials by deleting a vertex.

Corollary 2.5 [40] Let $v$ be a vertex of $G^\sigma$. Then

$$\phi_s(G^\sigma, \lambda) = \lambda \phi_s(G^\sigma - v, \lambda) + \sum_{uv \in G} \phi_s(G^\sigma - u - v, \lambda) + 2 \sum_{v \in C \in Od(G^\sigma)} \phi_s(G^\sigma - C, \lambda)$$

$$- 2 \sum_{v \in C \in Ev(G^\sigma)} \phi_s(G^\sigma - C, \lambda),$$

where $Od(G^\sigma)$ and $Ev(G^\sigma)$ denote the set of all oddly oriented cycles and evenly oriented cycles of $G^\sigma$, respectively.

Corollary 2.6 [40] Let $v$ be a vertex of $G^\sigma$ that is on no even cycle in $G^\sigma$. Then

$$\phi_s(G^\sigma, \lambda) = \lambda \phi_s(G^\sigma - v, \lambda) + \sum_{uv \in G} \phi_s(G^\sigma - u - v, \lambda).$$

From Theorem 2.1 it is easy to find that the direction of an odd cycle has no effect on the coefficients of the skew-characteristic polynomial. Therefore, for an oriented graph with no even cycles, its skew-characteristic polynomial has a special form as follows.
Corollary 2.7 \cite{27} Let \( G \) be an undirected graph with no even cycles and \( G^\sigma \) be an oriented graph of \( G \). Then the skew-characteristic polynomial of \( G^\sigma \) is of the form

\[
\phi_s(G^\sigma, \lambda) = \sum_{i=0}^{\lfloor n/2 \rfloor} m(G, i)\lambda^{n-2i}.
\]

For an undirected graph \( G \) with no even cycles, the above theorem illustrates that its all oriented graphs has the same skew-characteristic polynomial and thus are all cospectral (i.e. the same skew-spectrum). In fact, the reverse also holds.

Theorem 2.8 \cite{5} The skew-adjacency matrices of a graph \( G \) are all cospectral if and only if \( G \) has no even cycles.

It should be noted that Gong and Xu \cite{18} considered weighted oriented graphs. They interpreted the coefficients of the characteristic polynomials and also established some analogues of recursions, which contains the above results as special cases. Besides, Cavers et al. \cite{5} obtained more general results by considering weighted digraphs which allows loops and dicycles of length 2.

3 Some basic properties and integral formulas

In this section, we first state some basic properties of the skew energy of oriented graphs, which can be directly derived from definition together with elementary knowledge of linear algebra. Then we state the integral formulas for the skew energy in terms of the skew-characteristic polynomial, which play a key role in determining the extremal oriented graphs for skew energy in given graph classes.

It is known that for any undirected graph with \( m \) edges, there are \( 2^m \) different orientations. The following theorem tells us that there exist some operations on orientations that keep the skew energy unchanged.

Theorem 3.1 \cite{1} Let \( G^\sigma \) be an oriented graph of \( G \), and let \( G^\sigma' \) be the oriented graph obtained from \( G^\sigma \) by reversing the orientations of all the arcs incident with a particular vertex of \( G^\sigma \). Then \( G^\sigma \) and \( G^\sigma' \) have the same spectra and hence \( E_S(G^\sigma) = E_S(G^\sigma') \).

Hou, Shen and Zhang in \cite{25} extended the above theorem. Let \( W \) be a vertex subset of an oriented graph \( G^\sigma \) and \( \overline{W} = V(G^\sigma) \setminus W \). Another oriented graph \( G^\sigma' \) of \( G \), obtained from \( G^\sigma \) by reversing the orientations of all arcs between \( W \) and \( \overline{W} \), is said to be obtained
from $G^\sigma$ by switching with respect to $W$. Two oriented graphs $G^\sigma$ and $G^{\sigma'}$ are said to be switching-equivalent if $G^{\sigma'}$ can be obtained from $G^\sigma$ by a sequence of switchings. Then we have the following result.

**Theorem 3.2** \cite{25} Let $G^\sigma$ and $G^{\sigma'}$ be two oriented graphs of a graph $G$. If $G^\sigma$ and $G^{\sigma'}$ are switching-equivalent, then $G^\sigma$ and $G^{\sigma'}$ have the same spectra and hence $\mathcal{E}_S(G^\sigma) = \mathcal{E}_S(G^{\sigma'})$.

Let $T^\sigma$ be a labeled oriented tree rooted at vertex $v$. It is showed that, through reversing the orientations of all arcs incident at some vertices other than $v$, one can transform $T^\sigma$ to another oriented tree $T^{\sigma'}$ in which the orientations of all arcs go from low labels to high labels. By using this and Theorem 3.1, Adiga, Balakrishnan and So in \cite{1} deduced some results on the skew energy of oriented trees.

**Theorem 3.3** \cite{1} The skew energy of an oriented tree is independent of its orientation.

**Corollary 3.4** \cite{1} The skew energy of an oriented tree is the same as the energy of its underlying tree.

It follows that all results about the energy of undirected trees can be immediately copied to the skew energy of oriented trees. In particular, we have the following result.

**Corollary 3.5** If $T_n^\sigma$ is an oriented tree on $n$ vertices, then

$$\mathcal{E}_S(S_n^\sigma) \leq \mathcal{E}_S(T_n^\sigma) \leq \mathcal{E}_S(P_n^\sigma)$$

where $S_n^\sigma$ and $P_n^\sigma$ denote an oriented star and an oriented path with any orientation, respectively. Equality holds if and only if the underlying tree $T_n$ satisfies that $T_n \cong S_n$ or $T_n \cong P_n$.

At the moment, we only know the above two operations that keep the skew energy unchanged. We would like to propose the following problems:

**Problem 3.6** Is there any other changing of the directions of some arcs that does not change the skew energy of an oriented graph? Is there any changing of the directions of some arcs that increases the skew energy of an oriented graph? Or more difficultly, is there a sequence of arcs such that if we successively change the directions of the arcs, then the skew energy increases every step?

Another interesting problem is the following one:
Problem 3.7 How to construct families of oriented graphs such that they have equal skew energy but they do not have the same spectra?

Next we will present the explicit expression of the skew energy for an oriented cycle $C_n^\sigma$. Fix a vertex and label the vertices of $C_n^\sigma$ successively. Reversing the arcs incident to a vertex if necessary, we obtain a new oriented cycle with arcs going from low labels to high labels possibly except one arc. Denote by $C_n^-$ the oriented cycle with the same directions for all arcs and by $C_n^+$ the oriented cycle with same directions for all arcs except one arc. The skew energies of $C_n^-$ and $C_n^+$ can be directly expressed.

**Theorem 3.8** Let $C_n^-$ be the oriented cycle with the same directions for all arcs and $C_n^+$ be the oriented cycle with same directions for all arcs except one arc. Then

$$E(S(C_n^-)) = \begin{cases} 4 \cot \frac{\pi}{n} & \text{if } n \equiv 0 \mod 2, \\ 2 \cot \frac{\pi}{2n} & \text{if } n \equiv 1 \mod 2; \end{cases}$$

$$E(S(C_n^+)) = \begin{cases} 4 \csc \frac{\pi}{n} & \text{if } n \equiv 0 \mod 2, \\ 2 \cot \frac{\pi}{2n} & \text{if } n \equiv 1 \mod 2. \end{cases}$$

We then give another two properties of the skew energy.

**Theorem 3.9** The skew energy of an oriented graph, if it is a rational number, must be an even positive integer.

**Theorem 3.10** Every even positive integer $2p$ is the skew energy of an oriented star.

Similar to the Coulson integral formula for the energy of undirected graphs [22], we next present an integral formula [1] for the skew energy which enables one to compute the skew energy of an oriented graph and compare the skew energy between two oriented graphs without actually finding out the eigenvalues.

**Theorem 3.11** Let $\phi_s(G^\sigma, \lambda)$ be the skew-characteristic polynomial of an oriented graph $G^\sigma$ on $n$ vertices. Then we have

$$E_s(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[ n + \lambda \frac{\phi_s'(G^\sigma, -\lambda)}{\phi_s(G^\sigma, -\lambda)} \right] d\lambda,$$

where $\phi_s'(G^\sigma, \lambda)$ is the derivative of $\phi_s(G^\sigma, \lambda)$. 

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Theorem 3.12 [22] Let $G^\sigma$ be an oriented graph of a graph $G$. Suppose that $\phi_s(G^\sigma, \lambda) = \sum_{i=0}^n c_i \lambda^{n-i}$ is the skew-characteristic polynomial of $G^\sigma$. Then

$$E_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\lambda^2} \ln \left( 1 + \sum_{i=1}^{\lfloor n/2 \rfloor} c_{2i} \lambda^{2i} \right) d\lambda.$$ \hspace{1cm} (3.1)

The following result, obtained by Zhu in [44], illustrates an integral formula for the difference of the skew energies of two oriented graphs of order $n$.

Theorem 3.13 [44] Let $\phi_s(G_1^\sigma, \lambda)$ and $\phi_s(G_2^\sigma, \lambda)$ be two skew-characteristic polynomials of two oriented graphs $G_1^\sigma$ and $G_2^\sigma$ of the same order. Then

$$E_S(G_1^\sigma) - E_S(G_2^\sigma) = \frac{2}{\pi} \int_{0}^{+\infty} \ln \frac{\phi_s(G_1^\sigma, \lambda)}{\phi_s(G_2^\sigma, \lambda)} d\lambda.$$ 

A new method is presented as follows to compare the skew energies of two oriented graphs whose skew-characteristic polynomials satisfy a given recurrence relation.

Theorem 3.14 [44] Let $f_n = \phi_s(G_1^\sigma, \lambda)$ and $g_n = \phi_s(G_2^\sigma, \lambda)$ be two skew-characteristic polynomials of two oriented graphs $G_1^\sigma$ and $G_2^\sigma$ of order $n$. Assume that $f_n = \lambda f_{n-1} + f_{n-2}$ and $g_n = \lambda g_{n-1} + g_{n-2}$. Write $d_n = f_n/g_n$. Let $k_0$ and $k$ be integers with $1 \leq k_0 \leq k \leq \lfloor n/2 \rfloor$. Then we have:

1. If $d_{2k_0} < d_{2k_0-1}$, then
   $$E_S(G_1^\sigma) - E_S(G_2^\sigma) = \frac{2}{\pi} \int_{0}^{+\infty} \ln d_n d\lambda < \frac{2}{\pi} \int_{0}^{+\infty} \ln d_{2k-1} d\lambda.$$ 

2. If $d_{2k_0} > d_{2k_0-1}$, then
   $$E_S(G_1^\sigma) - E_S(G_2^\sigma) = \frac{2}{\pi} \int_{0}^{+\infty} \ln d_n d\lambda > \frac{2}{\pi} \int_{0}^{+\infty} \ln d_{2k} d\lambda.$$ 

From Theorem 2.2 and the integral formula (3.1), we find that $E_S(G^\sigma)$ is a strictly monotonically increasing function of these coefficients $c_{2i}(G^\sigma)$ ($i = 1, 2, \ldots, \lfloor n/2 \rfloor$) for any oriented graph $G^\sigma$. Therefore, the method of the quasi-order relation $\preceq$, defined by Gutman and Polansky [22] on graph energy, can be generalized to the skew energy of
oriented graphs. To be specific, let
\[
\phi_s(G_{\sigma_1}, \lambda) = \sum_{i=0}^{\lfloor n/2 \rfloor} c_{2i}(G_{\sigma_1}) \lambda^{n-2i} \quad \text{and} \quad \phi_s(G_{\sigma_2}, \lambda) = \sum_{i=0}^{\lfloor n/2 \rfloor} c_{2i}(G_{\sigma_2}) \lambda^{n-2i}
\]
be the skew-characteristic polynomials of two oriented graphs \(G_{\sigma_1}\) and \(G_{\sigma_2}\) of order \(n\), respectively. If \(c_{2i}(G_{\sigma_1}) \leq c_{2i}(G_{\sigma_2})\) for all \(1 \leq i \leq \lfloor n/2 \rfloor\), then denote \(G_{\sigma_1} \preceq G_{\sigma_2}\), which implies that \(E_S(G_{\sigma_1}) \leq E_S(G_{\sigma_2})\); If \(G_{\sigma_1} \preceq G_{\sigma_2}\) and there exists at least one \(j\) such that \(c_{2j}(G_{\sigma_1}) < c_{2j}(G_{\sigma_2})\), then denote \(G_{\sigma_1} \prec G_{\sigma_2}\), which implies that \(E_S(G_{\sigma_1}) < E_S(G_{\sigma_2})\). This quasi-ordering provides an important method in comparing the skew energies of two oriented graphs. Combining this quasi-ordering with Theorem 2.1, Cui and Hou in [9] got the following result.

**Theorem 3.15** [9] If \(G\) has an orientation \(\sigma\) such that every even cycle is oddly oriented, then \(G^\sigma\) has the maximal skew energy among all orientations of \(G\).

Which graphs have an orientation \(\sigma\) such that all even cycles are oddly oriented? Fisher and Little in [14] gave a characterization for such graphs as follows.

**Theorem 3.16** [14] A graph has an orientation under which every cycle of even length is oddly oriented if and only if the graph contains no subgraph which is, after the contraction of at most one cycle of odd length, an even subdivision of \(K_{2,3}\).

Zhang and Li [43] showed that for a bipartite graph \(G\), there exists an orientation of \(G\) such that all even cycles are oddly oriented if and only if \(G\) contains no even subdivision of \(K_{2,3}\), and moreover, if it is so, it must be planar.

### 4 Graphs with \(Sp(G^\sigma) = iSp(G)\)

From Corollary 3.4 we know that for a tree \(T\), \(E_S(T^\sigma) = E(T)\). Then, Adiga, Balakrishnan and So [3] posed the question that find new families of graphs \(G\) with orientations \(\sigma\) such that \(E_S(G^\sigma) = E(G)\). Actually, for a tree \(T\), \(Sp(T^\sigma) = iSp(T)\) for any orientation \(\sigma\) of \(T\). This motivates the investigation that find new families of graphs \(G\) with orientations \(\sigma\) such that \(Sp(G^\sigma) = iSp(G)\).

From Theorem 2.1 one can deduce the following stronger result for a tree, which was also obtained by Shader and So in [34] with the method of matrix analysis.
Theorem 4.1 \([24,34]\) Let \(G\) be a graph. Then \(Sp(G^\sigma) = iSp(G)\) for any orientation \(\sigma\) of \(G\) if and only if \(G\) is a tree.

For bipartite graphs, we have the following results, which can also be obtained from Theorem 2.1.

Theorem 4.2 \([24,34]\) A graph \(G\) is bipartite if and only if there is an orientation \(\sigma\) such that \(Sp(G^\sigma) = iSp(G)\).

Corollary 4.3 \([24]\) For any bipartite graph \(G\), there is an orientation \(G^\sigma\) with \(E_S(G^\sigma) = E(G)\).

A natural and interesting question is that for which orientations \(\sigma\) of a bipartite graph \(G\), we have \(Sp(G^\sigma) = iSp(G)\). It was pointed out by Shader and So in \([34]\) that the elementary orientation of a bipartite graph \(G = G(X,Y)\), which assigns each edge the direction from \(X\) to \(Y\), is such an orientation. Recently, Cui and Hou \([9]\) gave a good characterization for an oriented bipartite graph \(G^\sigma\) with \(Sp(G^\sigma) = iSp(G)\). First recall that an even cycle \(C_{2\ell}\) is said to be oriented uniformly if \(C_{2\ell}\) is oddly (resp., evenly) oriented relative to \(G^\sigma\) when \(\ell\) is odd (resp., even).

Theorem 4.4 \([9]\) Let \(G\) be a bipartite graph and \(\sigma\) be an orientation of \(G\). Then \(Sp(G^\sigma) = iSp(G)\) if and only if every even cycle is oriented uniformly in \(G^\sigma\).

In order to ensure \(Sp(G^\sigma) = iSp(G)\) for a given oriented bipartite graph, the above theorem requires one to check that every even cycle is oriented uniformly. To save the work of checking, Chen, Li and Lian \([2]\) got the following result. Recall that a chord of a cycle \(C\) in a graph \(G\) is an edge in \(E(G) \setminus E(C)\) both of whose ends lie on \(C\). A chordless cycle is a cycle without a chord.

Theorem 4.5 \([2]\) Let \(G\) be a bipartite graph and \(\sigma\) be an orientation of \(G\). Then \(Sp(G^\sigma) = iSp(G)\) if and only if all chordless cycles are oriented uniformly in \(G^\sigma\).

Actually, it can be further simplified to check only a set of so-called generating set of cycles. For details, see Remark 2.5 of \([2]\).

It is known from Theorem 3.2 that switching-equivalence keeps the skew-spectra of an oriented graph unchanged. Therefore, Cui and Hou \([9]\) conjectured that such orientation \(\sigma\) of a bipartite graph \(G\) that \(Sp(G^\sigma) = iSp(G)\) is unique under switching-equivalence. In \([2]\) we confirmed this conjecture.
Theorem 4.6  [2] Let $G = G(X,Y)$ be a bipartite graph and $\sigma$ be an orientation of $G$. Then $Sp(G^\sigma) = iSp(G)$ if and only if $\sigma$ is switching-equivalent to the elementary orientation of $G$.

Some special families of oriented bipartite graphs with $Sp(G^\sigma) = iSp(G)$ have been constructed [9, 24, 38], one of which is obtained by considering the Cartesian product of two oriented graphs.

Let $H$ and $G$ be graphs with $m$ and $n$ vertices, respectively. The Cartesian product $H \Box G$ of $H$ and $G$ is a graph with vertex set $V(H) \times V(G)$ and there exists an edge between $(u_1, v_1)$ and $(u_2, v_2)$ if and only if $u_1 = u_2$ and $v_1v_2$ is an edge of $G$, or $v_1 = v_2$ and $u_1u_2$ is an edge of $H$. Assume that $H^\tau$ is any orientation of $H$ and $G^\sigma$ is any orientation of $G$. There is a natural way to give an orientation $H^\tau \Box G^\sigma$ of $H^\tau$ and $G^\sigma$. There is an arc from $(u_1, v_1)$ to $(u_2, v_2)$ if and only if $u_1 = u_2$ and $\langle u_1, v_2 \rangle$ is an arc of $G^\sigma$, or $v_1 = v_2$ and $\langle u_1, u_2 \rangle$ is an arc of $H^\tau$. It is easy to see that $H^\tau \Box G^\sigma$ is an oriented graph of $H \Box G$.

Theorem 4.7  [9] Let $H^\tau$ and $G^\sigma$ be oriented graphs with $Sp(H^\tau) = iSp(H)$ and $Sp(G^\sigma) = iSp(G)$, respectively. Then $Sp(H^\tau \Box G^\sigma) = iSp(H \Box G)$.

Here we notice that the orientation $\sigma$ of the hypercube $Q_k$ with $Sp(Q_k^\sigma) = iSp(Q_k)$, obtained by Tian in [38], can be viewed as a specific application of Theorem 4.7.

It also deserves to mention that Hou and Lei in [24] constructed from a tree the following interesting family of oriented bipartite graphs $G^\sigma$ with $Sp(G^\sigma) = iSp(G)$.

Let $T$ be a tree with vertex set $\{1, 2, \ldots, n\}$ and a perfect matching $M$, i.e., $T$ is nonsingular. Let $T^\sigma$ be an oriented tree of $T$. Suppose that $A(T)$ and $S(T^\sigma)$ are the adjacency matrix and the skew-adjacency matrix of $T$ and $T^\sigma$, respectively. It is known that $A(T)$ and $S(T^\sigma)$ are nonsingular since $T$ has a perfect matching. Denote by $A^{-1}(T)$ and $S^{-1}(T^\sigma)$ the inverse matrices of $A(T)$ and $S(T^\sigma)$, respectively. A path in $T$: $P(i, j) = i_1i_2 \cdots i_{2k}$ (where $i_1 = i, i_{2k} = j$) from a vertex $i$ to a vertex $j$ is said to be an alternating path if the edges $i_1i_2, i_3i_4, \ldots, i_{2k-1}i_{2k}$ are edges in the perfect matching $M$; see [4] for these definitions.

Define the inverse graph $T^{-1}$ of the nonsingular tree $T$ as the graph with vertex set $\{1, 2, \ldots, n\}$, where vertices $i$ and $j$ are adjacent in $T^{-1}$ if there is an alternating path between $i$ and $j$ in $T$. Suppose that $A(T^{-1})$ is the adjacency matrix of $T^{-1}$. It is shown in [3] that the graph $T^{-1}$ is connected and bipartite, and the inverse matrix $A^{-1}(T)$ of the adjacency matrix $A(T)$ of $T$ is similar to the adjacency matrix $A(T^{-1})$ of $T^{-1}$ via a diagonal matrix of $\pm 1$. 

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Moreover, Hou and Lei [24] found that the matrix $S^{-1}(T^\sigma)$ is also skew-symmetric with entries 0 and $\pm 1$. Thus $S^{-1}(T^\sigma)$ is the skew-adjacency matrix of some oriented graph, which is defined as the inverse oriented graph of the oriented tree $T^\sigma$, denoted by $(T^\sigma)^{-1}$. That is, the skew-adjacency matrix $S((T^\sigma)^{-1})$ of $(T^\sigma)^{-1}$ is the same as $S^{-1}(T^\sigma)$. It was shown by Hou and Lei in [24] that $(T^\sigma)^{-1}$ is just an oriented graph of $T^{-1}$ and the skew energy of $(T^\sigma)^{-1}$ equals the energy of $T^{-1}$.

**Theorem 4.8** [24] Let $T$ be a tree with a perfect matching and $T^\sigma$ be an oriented graph of $T$. Let $T^{-1}$ and $(T^\sigma)^{-1}$ be the inverse graph and inverse oriented graph of $T$ and $T^\sigma$, respectively. Then $Sp((T^\sigma)^{-1}) = iSp(T^{-1})$ and hence $E_S((T^\sigma)^{-1}) = E(T^{-1})$.

### 5 General bounds for the skew energy

Adiga, Balakrishnan and So [1] established a low bound and an upper bound for the skew energy of an oriented graph $G^\sigma$ in terms of the order and size of $G^\sigma$ as well as the maximum degree of its underlying graph.

**Theorem 5.1** [1] Let $G^\sigma$ be an oriented graph of $G$ with $n$ vertices, $m$ arcs and maximum degree $\Delta$. Then the skew energy $E_S(G^\sigma)$ of $G^\sigma$ satisfies that

$$\sqrt{2m + n(n - 1)p^{2/n}} \leq E_S(G^\sigma) \leq \sqrt{2mn} \leq n\sqrt{\Delta},$$

where $p = |\det(S(G^\sigma))| = \prod_{i=1}^{n} |\lambda_i|$.

**Corollary 5.2** Any oriented graph $G^\sigma$ satisfies that $E_S(G^\sigma) = n\sqrt{\Delta}$ if and only if its skew-adjacency matrix satisfies that $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$, where $I_n$ is the identity matrix of order $n$.

The upper bound that $E_S(G^\sigma) = n\sqrt{\Delta}$ (which, for convenience, is called the optimum skew energy, the corresponding orientation of $G$ is called the optimum orientation and the resultant oriented graph $G^\sigma$ is called the optimum skew energy oriented graph) implies that the underlying graph $G$ is a $\Delta$-regular graph. Hence a natural question was posed in [1]:

Which $k$-regular graphs on $n$ vertices have an orientation $G^\sigma$ with $E_S(G^\sigma) = n\sqrt{k}$, or equivalently $S(G^\sigma)^T S(G^\sigma) = kI_n$?

Note that $n$ must be even since any skew-symmetric matrix of odd order has a determinant 0, and it suffices to consider connected $k$-regular graphs because of the following lemma.
Lemma 5.3. Let $G_{1}^{\sigma_1}$, $G_{2}^{\sigma_2}$ be two disjoint oriented graphs of order $n_1$, $n_2$ with skew-adjacency matrices $S(G_{1}^{\sigma_1})$, $S(G_{2}^{\sigma_2})$, respectively. Then for some positive integer $k$, $S(G_{1}^{\sigma_1})^T S(G_{1}^{\sigma_1}) = kI_{n_1}$ and $S(G_{2}^{\sigma_2})^T S(G_{2}^{\sigma_2}) = kI_{n_2}$ if and only if the skew-adjacency matrix $S(G_{1}^{\sigma_1} \cup G_{2}^{\sigma_2})$ of the union $G_{1}^{\sigma_1} \cup G_{2}^{\sigma_2}$ satisfies $S(G_{1}^{\sigma_1} \cup G_{2}^{\sigma_2})^T S(G_{1}^{\sigma_1} \cup G_{2}^{\sigma_2}) = kI_{n_1+n_2}$.

Moreover, we have the following necessary condition for the complete graph $K_n$ to have the optimum orientation.

Theorem 5.4. If $K_n$ has an orientation $K_n^{\sigma}$ with $S(K_n^{\sigma})^T S(K_n^{\sigma}) = (n-1)I_n$, then $n$ is a multiple of 4.

We find that the above problem is related to the so-called weighing matrices, which are used in Combinatorial Design. A weighing matrix $W = W(n,k)$ is defined as a square matrix with entries 0, ±1 having $k$ non-zero entries per row and per column and inner product of distinct rows zero. Hence $W$ satisfies that $WW^T = kI_n$. The number $k$ is called the weight of $W$. Weighing matrices have been studied extensively, see [8, 23, 31]. Then the skew-adjacency matrix of an optimum skew energy oriented graph is a skew-symmetric weighing matrices and vice versa. Therefore, the above question is equivalent to determine the skew-symmetric weighing matrices with weighing $k$ and order $n$.

It should be pointed out that a skew-symmetric weighing matrix $W(n,n-1)$ is also called a skew-symmetric conference matrix, which is closely related to the famous Hadamard Matrix Conjecture [31], since $W(n,n-1) + I_n$ is a Hadamard matrix for each skew-symmetric weighing matrix $W(n,n-1)$. Moreover, there is also a conjecture about $W(n,n-2)$ in [23], which says that the weighing matrices $W(4t,4t-2)$ exist for any positive integer $t$. So far, we do not know whether the conjecture is true or not. We are now concerned with the skew-symmetric weighing matrix $W(4t,4t-2)$, the existence of which implies the conjecture.

For small $k$, many results have been obtained. The authors in [1] obtained that a 1-regular graph has an orientation with $S(G^{\sigma})^T S(G^{\sigma}) = I_n$ if and only if it is the graph $K_2$; while a 2-regular graph has an orientation with $S(G^{\sigma})^T S(G^{\sigma}) = 2I_n$ if and only if it is the 4-cycle $C_4$ and has an oddly orientation.

The following lemma [1] is very useful in characterizing $k$-regular graphs with optimum orientations.

Lemma 5.5. Let $S(G^{\sigma})$ be the skew-adjacency matrix of an oriented graph $G^{\sigma}$. If $S(G^{\sigma})^T S(G^{\sigma}) = kI$, then $N(u) \cap N(v)$ is even for any two distinct vertices $u$ and $v$ of $G^{\sigma}$. 

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Applying the above lemma, Gong and Xu in [17] characterized the underlying graphs of all 3-regular oriented graphs with optimum skew energy. They also gave the corresponding orientations for every underlying graph, respectively, and further proved that such orientation is unique for every underlying graph.

**Theorem 5.6** [17] Let $G^\sigma$ be a 3-regular optimum skew energy oriented graph. Then the underlying graph $G$ is either the complete graph $K_4$ or the hypercube $Q_3$.

**Theorem 5.7** [17] Let $G^\sigma$ be a 3-regular optimum skew energy oriented graph. Then $G^\sigma$ (under switching-equivalent) is either $K_4^\sigma$ or $Q_3^\sigma$ depicted in Figure 5.1.

![Figure 5.1: All 3-regular oriented graphs with optimum skew energy](image)

Recently, Chen Li and Lian in [7] determined the underlying graphs of all 4-regular oriented graphs with optimum skew energy and gave orientations of these underlying graphs such that the skew energies of the resultant oriented graphs indeed attain optimum.

**Theorem 5.8** [7] Let $G$ be a 4-regular graph. Then $G$ has an optimum orientation if and only if $G$ is a graph of $\mathcal{F}$, which consists of the underlying graphs of all oriented graphs in Figures 5.2, 5.3 and 5.4, i.e., $G_1$, $G_2$, $G_3$, the hypercube $Q_4$, the graph $G_i$ for any positive integer $i$ and the graph $H_j$ for any positive integer $j$.

**Theorem 5.9** [7] Every oriented graph in Figures 5.2, 5.3 and 5.4 has the optimum skew energy.

We point out that Gong, Zhong and Xu [19] also independently obtained the same result for the 4-regular optimum skew energy oriented graph.

For any given positive integers $n$ and $k$ with $n > k \geq 5$, it is not easy to characterize all $k$-regular graphs on order $n$ that have an orientation $G^\sigma$ such that $E_S(G^\sigma) = n\sqrt{k}$. But
Figure 5.2: The optimum orientations for $G_1$, $G_2$, $G_3$ and $Q_4$
Figure 5.3: The optimum orientation for $G_i$

Figure 5.4: The optimum orientation for $H_j$
it can be established that for any positive integer \( k \geq 3 \), there exists a connected \( k \)-regular graph \( G \) that has an orientation \( G^\sigma \) such that \( E_S(G^\sigma) = n\sqrt{k} \), namely, the hypercube \( Q_k \). Recall that the hypercube \( Q_k \) of dimension \( k \) is defined recursively in terms of the Cartesian product of graphs as follows:

\[
Q_k = \begin{cases} 
K_2, & k = 1, \\
Q_{k-1} \square Q_1, & k \geq 2.
\end{cases}
\]

Note that \( Q_k \) can also be constructed by taking two copies of \( Q_{k-1} \), and then drawing an edge between each vertex in the first copy and the corresponding vertex in the second copy. Obviously, \( Q_k \) has \( 2^k \) vertices and is a \( k \)-regular bipartite graph. Assume that the vertex set of \( Q_k \) is \( \{1, 2, \ldots, 2^{k-1}, 2^{k-1} + 1, \ldots, 2^k\} \). For the skew energy of \( Q_k \), Tian in [38] obtained the following result.

**Theorem 5.10** [38] For any positive integer \( k \), there is an orientation of the hypercube \( Q_k \) such that the resultant oriented graph \( Q_k^\sigma \) satisfies that \( E_S(Q_k^\sigma) = n\sqrt{k} \).

The orientation mentioned in above theorem was also given in [38] by the following algorithm. In fact it can be shown that the optimum orientation for \( Q_k \) is unique under switching-equivalent.

**Algorithm 1.**

1. Step 1. Give the hypercube \( Q_1 \) an orientation \( Q_1^\sigma \) such that \( \langle 1, 2 \rangle \in \Gamma(Q_1^\sigma) \).

2. Step 2. Assume that \( Q_1, Q_2, \ldots, Q_i \) have been oriented into \( Q_1^\sigma, Q_2^\sigma, \ldots, Q_i^\sigma \). For \( Q_{i+1} \), we give an orientation \( Q_{i+1}^\sigma \) using the following method:

   (i) Take two copies of \( Q_i^\sigma \), and put an edge between each vertex in the first copy and the corresponding vertex in the second copy. Assume that the vertex set of the first copy is \( \{1, 2, \ldots, 2^i\} \) and the corresponding vertex set of the second copy is \( \{2^i + 1, 2^i + 2, \ldots, 2^{i+1}\} \).

   (ii) Let \( \{i_1, i_2, \ldots, i_{2^i-1}\}, \{i_{2^i-1+1}, i_{2^i-1+2}, \ldots, i_{2^i}\} \) be a bipartition of \( \{1, 2, \ldots, 2^i\} \). Also let \( \{i_{2^i+1}, i_{2^i+2}, \ldots, i_{2^{i+1}-1}\}, \{i_{2^{i+1}+1}, i_{2^{i+1}+2}, \ldots, i_{2^{i+1}}\} \) be the corresponding bipartition of \( \{2^i + 1, 2^i + 2, \ldots, 2^{i+1}\} \). Give each edge between \( \{i_1, i_2, \ldots, i_{2^i-1}\} \) and \( \{i_{2^i+1}, i_{2^i+2}, \ldots, i_{2^{i+1}-1}\} \) an orientation such that \( \langle i_k, i_{2^i+k} \rangle \in \Gamma(Q_{i+1}^\sigma) \) for \( k = 1, 2, \ldots, 2^{i-1} \). At the same time, give each edge between \( \{i_{2^i-1+1}, i_{2^i-1+2}, \ldots, i_{2^i}\} \) and \( \{i_{2^{i+1}+2}, i_{2^{i+1}+3}, \ldots, i_{2^{i+1}}\} \) an orientation such that \( \langle i_{2^i+k}, i_k \rangle \in \Gamma(Q_{i+1}^\sigma) \) for \( k = 2^{i-1} + 1, 2^{i-1} + 2, \ldots, 2^i \).
Step 3. If \( i + 1 = d \), stop; else take \( i := i + 1 \), return to Step 2.

Besides the hypercube \( Q_k \), many other families of oriented graphs with optimum skew energy were characterized in \([2,9,28]\), some of which are presented in the following section.

![Figure 5.5: Orientations of two non-isomorphic 3-regular graphs on 6 vertices having the same skew-spectrum.](image)

From the above discussions, we know that not every \( k \)-regular oriented graph on \( n \) vertices has an orientation such that the resultant oriented graph attains optimum skew energy \( n\sqrt{k} \). For example, there are only two non-isomorphism 3-regular graphs \( G_1 \) and \( G_2 \) on 6 vertices, both of which have no optimum orientations. Figure 5.5 depicts the maximal orientations of \( G_1 \) and \( G_2 \) among all orientations of \( G_1 \) and \( G_2 \). They have the same skew-spectrum \( \pm 2i, \pm 2i, \pm i \) and thus the same skew energy. But the value of their skew energy is less than \( 6\sqrt{3} \). Based on this, we end this section by proposing a general problem as follows:

**Problem 5.11** For a given graph, how to orient it so that the oriented graph attains the minimal or maximal skew energy? One can try to think about a complete graph, a hypercube, or some other graph classes.

Moreover, since the Pfaffian orientations are important ones, the following problem is natural:

**Problem 5.12** If a graph \( G \) has a Pfaffian orientation, what can we say about the skew energy of a Pfaffian oriented graph of \( G \)?

There has been very little known on the lower bound of the skew energy, even if for the lower bound in Theorem 5.1, the extremal graphs are unknown. So we raise the following problem:

**Problem 5.13** Is the lower bound in Theorem 5.1 the best possible? or find families of graphs that attain the lower bound.
6 Skew-spectra and skew energies of various products of graphs

In this section, we consider various products of graphs, including the Cartesian product \( H \square G \), the Kronecker product \( H \otimes G \), the strong product \( H \ast G \) and the lexicographic product \( H[G] \) of \( H \) and \( G \) where \( H \) is a bipartite graph and \( G \) is an arbitrary graph. Those products are discussed in different subsections, respectively, where we first give them orientations and then show the skew-spectra of the resultant oriented graphs. As applications, new families of oriented graphs with optimum skew energy are constructed in every subsection and meanwhile some examples are given.

Let \( H \) be a graph of order \( m \) and \( G \) be a graph of order \( n \). The definition of the Cartesian product \( H \square G \) of \( H \) and \( G \) has been given in Section \[4\]. Now we recall the definitions of other graph products. The \textit{Kronecker product} \( H \otimes G \) of \( H \) and \( G \) is a graph with vertex set \( V(H) \times V(G) \) and where \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent if \( u_1 \) is adjacent to \( u_2 \) in \( H \) and \( v_1 \) is adjacent to \( v_2 \) in \( G \). The \textit{strong product} \( H \ast G \) of \( H \) and \( G \) is a graph with vertex set \( V(H) \times V(G) \); two distinct pairs \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent in \( H \ast G \) if \( u_1 \) is equal or adjacent to \( u_2 \), and \( v_1 \) is equal or adjacent to \( v_2 \). The \textit{lexicographic product} \( H[G] \) of \( H \) and \( G \) has vertex set \( V(H) \times V(G) \) where \( (u_1, v_1) \) is adjacent to \((u_2, v_2)\) if and only if \( u_1 \) is adjacent to \( u_2 \) in \( H \), or \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) in \( G \).

6.1 The orientation of \( H \square G \)

Let \( H^\tau \) and \( G^\sigma \) be any orientations of \( H \) and \( G \), respectively. Recall the natural orientation \( H^\tau \square G^\sigma \) given in Section \[4\]. There is an arc from \((u_1, v_1)\) to \((u_2, v_2)\) in \( H^\tau \square G^\sigma \) if and only if \( u_1 = u_2 \) and \((v_1, v_2)\) is an arc of \( G^\sigma \), or \( v_1 = v_2 \) and \((u_1, u_2)\) is an arc of \( H^\tau \). When \( H \) is a bipartite graph with bipartition \( X \) and \( Y \), we give another orientation of \( H \square G \) by modifying the above orientation of \( H^\tau \square G^\sigma \) with the following method. If there is an arc from \((u, v_1)\) to \((u, v_2)\) in \( H^\tau \square G^\sigma \) and \( u \in Y \), then we reverse the direction of the arc.

Theorem 6.1 [2] Let \( H^\tau \) be an oriented bipartite graph of order \( m \) and let the skew eigenvalues of \( H^\tau \) be the non-zero values \( \pm \mu_1 i, \pm \mu_2 i, \ldots, \pm \mu_t i \) and \( m - 2t \) 0’s. Let \( G^\sigma \) be an oriented graph of order \( n \) and let the skew eigenvalues of \( G^\sigma \) be the non-zero values \( \pm \lambda_1 i, \pm \lambda_2 i, \ldots, \pm \lambda_r i \) and \( n - 2r \) 0’s. Then the skew eigenvalues of the oriented graph \( (H^\tau \square G^\sigma)^o \) are \( \pm i \sqrt{\mu_j^2 + \lambda_k^2} \) with multiplicities \( 2, j = 1, \ldots, t, k = 1, \ldots, r, \pm \mu_j i \) with
Let $\lambda_k i$ with multiplicities $m - 2t$, $k = 1, \ldots, r$, and $0$ with multiplicities $(m - 2t)(n - 2r)$.

As an application of Theorem 6.1 we constructed a new family of oriented graphs with optimum skew energy in [2].

**Theorem 6.2** Let $H^r$ be an oriented $\ell$-regular bipartite graph on $m$ vertices with optimum skew energy $E_S(H^r) = mn\sqrt{\ell}$ and $G^r$ be an oriented $k$-regular graph on $n$ vertices with optimum skew energy $E_S(G^r) = n\sqrt{k}$. Then the oriented graph $(H^r \Box G^r)^o$ of $H \Box G$ has the optimum skew energy $E_S((H^r \Box G^r)^o) = mn\sqrt{\ell + k}$.

It should be noted that the special case of Theorems 6.1 and 6.2 that the bipartite graph $H$ is the path $P_m$ of length $m$ were obtained by Cui and Hou in [9].

It is known that there exists a $k$-regular graph with $n = 2^k$ vertices having an orientation $\sigma$ with optimum skew energy, which is the hypercube $Q_k$. The following examples provide new families of oriented $k$-regular graphs with optimum skew energy that have much less vertices.

**Example 6.3** Let $G_1 = K_{4,4}$, $G_2 = K_{4,4} \square G_1$, $\ldots$, $G_r = K_{4,4} \square G_{r-1}$. Because there is an orientation of $K_{4,4}$ with optimum skew energy 16; see Figure 5.4. Thus, we can get an orientation of $G_r$ with optimum skew energy $2^{3r}\sqrt{4r}$. This provides a family of $4r$-regular graphs of order $n = 2^{3r}$ having an orientation with optimum skew energy $2^{3r}\sqrt{4r}$ for $r \geq 1$.

**Example 6.4** Let $G_1 = K_4$, $G_2 = K_{4,4} \square G_1$, $\ldots$, $G_r = K_{4,4} \square G_{r-1}$. It is known that $K_4$ has an orientation with optimum skew energy; see Figure 5.1. Thus we can get an orientation of $G_r$ with optimum skew energy $2^{3r-1}\sqrt{4r-1}$. This provides a family of $4r - 1$-regular graphs of order $n = 2^{3r-1}$ having an orientation with optimum skew energy $2^{3r-1}\sqrt{4r-1}$ for $r \geq 1$.

**Example 6.5** Let $G_1 = C_4$, $G_2 = K_{4,4} \square G_1$, $\ldots$, $G_r = K_{4,4} \square G_{r-1}$. Thus, we can get an orientation of $G_r$ with optimum skew energy $2^{3r-1}\sqrt{4r-2}$. This provides a family of $4r - 2$-regular graphs of order $n = 2^{3r-1}$ having an orientation with optimum skew energy $2^{3r-1}\sqrt{4r-2}$ for $r \geq 1$.

**Example 6.6** Let $G_1 = P_2$, $G_2 = K_{4,4} \square G_1$, $\ldots$, $G_r = K_{4,4} \square G_{r-1}$. Thus, we can get an orientation of $G_r$ with optimum skew energy $2^{3r-2}\sqrt{4r-3}$. This provides a family of $4r - 3$-regular graphs of order $n = 2^{3r-2}$ having an orientation with optimum skew energy $2^{3r-2}\sqrt{4r-3}$ for $r \geq 1$. 
Moreover, if we assume that the bipartite graph $H$ is a path on two vertices, and the graph $G$ is a tree in Theorem 6.1, then the maximal result was obtained by Cui and Hou in [9], whose proof used the following lemma obtained in [42].

**Lemma 6.7** [42] Let $T$ be a tree and $\sigma$ be an arbitrary orientation of $T$. Then the oriented graph $(P_2 \square T^\sigma)^o$ has every even cycles oddly oriented.

Combining the above lemma with Theorem 3.15, the following result is immediate.

**Theorem 6.8** [9] Let $T$ be a tree. Then the oriented graph $(P_2 \square T^\sigma)^o$ has the maximal skew energy among all orientations of $P_2 \square T$.

A natural problem was proposed in [9]: for a general graph $G$, which orientations of $P_2 \square G$ yield the maximal skew energy (or minimal skew energy)? We are familiar with two orientations $P_2 \square G^\sigma$ and $(P_2 \square G^\sigma)^o$ for any orientation $G^\sigma$ of $G$. The following example illustrates that $P_2 \square G^\sigma$ may not be minimal among all orientations of $P_2 \square G$ when $G^\sigma$ is the minimal orientation of $G$. But they conjectured that $(P_2 \square G^\sigma)^o$ has the maximal skew energy among all orientations of $P_2 \square G$ when $G^\sigma$ is the maximal orientation of $G$.

**Example 6.9** [9] Let $G = P_2 \square C_4$ (in fact, $G$ is the hypercube $Q_3$) and $C_4^\sigma$ be an evenly oriented. Then $Sp(C_4^\sigma) = \{2i, -2i, 0, 0\}$ and $\mathcal{E}_S(C_4^\sigma) = 4$ has the minimal skew energy among all orientations of $C_4$. The left orientation in Figure 6.6, which is $P_2 \square C_4^\sigma$, has skew energy 12; while the right orientation has skew energy (about 11.5) less than 12.

![Figure 6.6: Two orientations of $P_2 \square C_4$](image)

### 6.2 The orientation of $H \otimes G$

Let $G_1^{\sigma_1}$, $G_2^{\sigma_2}$, $G_3^{\sigma_3}$ be the oriented graphs of order $n_1$, $n_2$, $n_3$ with skew-adjacency matrices $S_1$, $S_2$, $S_3$, respectively. Then the Kronecker product matrix $S_1 \otimes S_2 \otimes S_3$ is also skew-symmetric and is in fact the skew-adjacency matrix of an oriented graph of the Kronecker product $G_1 \otimes G_2 \otimes G_3$. Denote the corresponding oriented graph by $G_1^{\sigma_1} \otimes G_2^{\sigma_2} \otimes G_3^{\sigma_3}$. Adiga, Balakrishnan and So in [1] obtained the following result.
Theorem 6.10  [28] Let \( G_1^{\sigma_1}, G_2^{\sigma_2}, G_3^{\sigma_3} \) be the oriented regular graphs of order \( n_1, n_2, n_3 \) with optimum skew energies \( n_1\sqrt{k_1}, n_2\sqrt{k_2}, n_3\sqrt{k_3} \), respectively. Then the oriented graph \( G_1^{\sigma_1} \otimes G_2^{\sigma_2} \otimes G_3^{\sigma_3} \) has optimum skew energy \( n_1n_2n_3\sqrt{k_1k_2k_3} \).

It should be noted that the above Kronecker product of oriented graphs is naturally defined, but the product requires 3 or an odd number of oriented graphs. In what follows, we consider the Kronecker product of any number of oriented graphs. We first give the orientation of the Kronecker product \( H \otimes G \) where \( H \) is bipartite.

Let \( H \) be a bipartite graph with bipartite \((X, Y)\) and \( G \) be an arbitrary graph. For any two adjacent vertices \((u_1, v_1)\) and \((u_2, v_2)\) of \( H \otimes G \), \( u_1 \) and \( u_2 \) must be in different parts of the bipartition of vertices of \( H \) and assume that \( u_1 \in X \). Then there is an arc from \((u_1, v_1)\) to \((u_2, v_2)\) if \((u_1, u_2)\) is an arc of \( H \) and \((v_1, v_2)\) is an arc of \( G \), or \((u_2, u_1)\) is an arc of \( H \) and \((v_2, v_1)\) is an arc of \( G \); otherwise there is an arc from \((u_2, v_2)\) to \((u_1, v_1)\).

Denote by \( (H^r \otimes G^o)^o \) the resultant oriented graph. For the skew-spectrum of \( (H^r \otimes G^o)^o \), we obtained the following result in [28].

Theorem 6.11  [28] Let \( H^r \) be an oriented bipartite graph of order \( m \) and let the skew-eigenvalues of \( H^r \) be the non-zero values \( \pm \mu_1 i, \pm \mu_2 i, \ldots, \pm \mu_t i \) and \( m - 2t \) 0’s. Let \( G^o \) be an oriented graph of order \( n \) and let the skew-eigenvalues of \( G^o \) be the non-zero values \( \pm \lambda_1 i, \pm \lambda_2 i, \ldots, \pm \lambda_r i \) and \( n - 2r \) 0’s. Then the skew-eigenvalues of the oriented graph \( (H^r \otimes G^o)^o \) are \( \pm \mu_j \lambda_k i \) with multiplicities \( 2, j = 1, \ldots, t, k = 1, \ldots, r \), and 0 with multiplicities \( mn - 4rt \).

The above theorem can be used to yield a family of oriented graphs with optimum skew energy.

Theorem 6.12  [28] Let \( H^r \) be an oriented \( k \)-regular bipartite graph of order \( m \) with optimum skew energy \( m\sqrt{k} \). Let \( G^o \) be an oriented \( \ell \)-regular graph of order \( n \) and the optimum skew energy \( n\sqrt{\ell} \). Then \( (H^r \otimes G^o)^o \) is an oriented \( k\ell \)-regular bipartite graph and has the optimum skew energy \( E_S((H^r \otimes G^o)^o) = mn\sqrt{k\ell} \).

Let \( H^r \) be an oriented bipartite graph with optimum skew energy. Let \( G_1^{\sigma_1} \) and \( G_2^{\sigma_2} \) be any two oriented graphs with the optimum skew energies. By the above theorem, the oriented graph \( (H^r \otimes G_1^{\sigma_1})^o \) is bipartite and has the optimum skew energy. Therefore, the Kronecker product \( H \otimes G_1 \otimes G_2 \) can be oriented as \( ((H^r \otimes G_1^{\sigma_1})^o \otimes G_2^{\sigma_2})^o \), abbreviated as \( (H^r \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2})^o \), which is also bipartite and has the optimum skew energy. The process is valid for any number of oriented graphs. Then the following corollary is immediately implied.
Corollary 6.13 [28] Let \( H^r \) be an oriented \( k \)-regular bipartite graph of order \( m \) with optimum skew energy \( m\sqrt{k} \). Let \( G_i^\sigma \) be an oriented \( \ell_i \)-regular graph of order \( n_i \) with optimum skew energy \( n_i\sqrt{\ell_i} \) for \( i = 1, 2, \ldots, s \) and any positive integer \( s \). Then the oriented graph \( (H^r \otimes G_1^\sigma \otimes \cdots \otimes G_s^\sigma)^o \) has the optimum skew energy \( mn_1n_2\cdots n_s\sqrt{k\ell_1\ell_2\cdots \ell_s} \).

It should be noted that for any even number \( s \) in Corollary 6.13, the oriented graph obtained in Corollary 6.13 is identical to the one obtained in Theorem 6.10.

6.3 The orientation of \( H \ast G \)

Now we consider the strong product \( H \ast G \) of a bipartite graph \( H \) and a graph \( G \). Let \( H^r \) be an oriented graph of \( H \) and \( G^\sigma \) be an oriented graph of \( G \). Since the edge set of \( H \ast G \) is the disjoint-union of the edge sets of \( H \square G \) and \( H \otimes G \), there is a natural orientation of \( H \ast G \) if \( H \square G \) and \( H \otimes G \) have been given orientations.

Now we give an orientation of \( H \ast G \) such that the arc set of the resultant oriented graph is the disjoint-union of the arc sets of \( (H^r \square G^\sigma)^o \) and \( (H^r \otimes G^\sigma)^o \), which are defined in the former subsections. Denote by \( (H^r \ast G^\sigma)^o \) this resultant oriented graph. The skew-spectrum of \( (H^r \ast G^\sigma)^o \) is determined in the following theorem.

Theorem 6.14 [28] Let \( H^r \) be an oriented bipartite graph of order \( m \) and let the skew-eigenvalues of \( H^r \) be the non-zero values \( \pm \mu_1i, \pm \mu_2i, \ldots, \pm \mu_ti \) and \( m - 2t \) 0’s. Let \( G^\sigma \) be an oriented graph of order \( n \) and let the skew-eigenvalues of \( G^\sigma \) be the non-zero values \( \pm \lambda_1i, \pm \lambda_2i, \ldots, \pm \lambda_ri \) and \( n - 2r \) 0’s. Then the skew-eigenvalues of the oriented graph \( (H^r \ast G^\sigma)^o \) are \( \pm i\sqrt{(u_j^2 + 1)(\lambda_k^2 + 1) - 1} \) with multiplicities \( 2, j = 1, \ldots, t, k = 1, \ldots, r, \pm \mu_ji \) with multiplicities \( n - 2r, j = 1, \ldots, t, \pm \lambda_ki \) with multiplicities \( m - 2t, k = 1, \ldots, r, \) and 0 with multiplicities \( (m - 2t)(n - 2r) \).

Similarly, in [28] we constructed a new family of oriented graphs with the optimum skew energy by applying the above theorem.

Theorem 6.15 [28] Let \( H^r \) be an oriented \( k \)-regular bipartite graph of order \( m \) with optimum skew energy \( m\sqrt{k} \). Let \( G^\sigma \) be an oriented \( \ell \)-regular graph of order \( n \) and optimum skew energy \( n\sqrt{\ell} \). Then \( (H^r \ast G^\sigma)^o \) is an oriented \( (k + \ell + k\ell) \)-regular graph and has the optimum skew energy \( \mathcal{E}_S((H^r \ast G^\sigma)^o) = mn\sqrt{k + \ell + k\ell} \).

Comparing Theorems 6.2, 6.12 and 6.15 we find that the oriented graphs constructed from these theorems have the same order \( mn \) but different regularities, which are \( k + \ell, k\ell \) and \( k + \ell + k\ell \), respectively.
Example 6.16 Let $H = C_4$, $G_0 = K_4$, $G_1 = H \boxtimes G_0$, \ldots, $G_r = H \boxtimes G_{r-1}$. Obviously, $G_r$ is a $(2r+3)$-regular graph of order $4^{r+1}$. It is known that $H$ has the orientation with the optimum skew energy $4\sqrt{2}$ and $G_0$ has the orientation with the maximum skew energy $4\sqrt{3}$. By Theorem 6.12, $G_r$ has the orientation with the optimum skew energy $4^{r+1}\sqrt{2r+3}$.

Example 6.17 Let $H = C_4$, $G_0 = K_4$, $G_1 = H \otimes G_0$, \ldots, $G_r = H \otimes G_{r-1}$. It is obvious that $G_r$ is a $(3 \cdot 2^r)$-regular graph of order $4^{r+1}$. Then by Theorem 6.12, $G_r$ has the orientation with optimum skew energy $4^{r+1}\sqrt{3 \cdot 2^r}$.

Example 6.18 Let $H = C_4$, $G_0 = K_4$, $G_1 = H \otimes G_0$, $G_2 = G_1 \otimes G_0$, \ldots, $G_r = G_{r-1} \otimes G_0$. Note that $H$, $G_1$, $G_2$, \ldots, $G_{r-1}$ are all regular bipartite graphs and $G_r$ is a $(2 \cdot 3^r)$-regular bipartite graph of order $4^{r+1}$. Then by Theorem 6.12, $G_r$ has the orientation with optimum skew energy $4^{r+1}\sqrt{2 \cdot 3^r}$.

Example 6.19 Let $H = C_4$, $G_0 = K_4$, $G_1 = H \ast G_0$, \ldots, $G_r = H \ast G_{r-1}$. Note that $G_r$ is a $(4 \cdot 3^r - 1)$-regular graph of order $4^{r+1}$. Then by Theorem 6.12, $G_r$ has the orientation with optimum skew energy $4^{r+1}\sqrt{4 \cdot 3^r - 1}$.

From Examples 6.17, 6.18 and 6.19, one can see that for some positive integers $k$, there exist oriented $k$-regular graphs with the optimum skew energy, which has order $n \leq k^2$. It is unknown that whether for any positive integer $k$, the oriented graph exists such that its order $n$ is less than $k^2$ and it has an orientation with the optimum skew energy.

6.4 The orientation of $H[G]$

In this subsection, we consider the lexicographic product $H[G]$ of a bipartite graph $H$ and a graph $G$. All definitions and notations are the same as above. It is easy to see that the edge set $H[G]$ is the disjoint-union of the edge sets of $H \square G$ and $H \otimes K_n$, where $K_n$ is a complete graph of order $n$.

Let $H^\tau$ and $G^\sigma$ be oriented graphs of $H$ and $G$ with the skew-adjacency matrices $S_1$ and $S_2$, respectively. Let $K_n^\tau$ be an oriented graph of $K_n$ with the skew-adjacency matrix $S_3$. Then we can obtain two oriented graphs $(H^\tau \square G^\sigma)^o$ and $(H^\tau \otimes K_n^\tau)^o$. Thus it is natural to yield an orientation of $H[G]$, denoted by $H[G]^o$, such that the arc set of $H[G]^o$ is the disjoint-union of the arc sets of $(H^\tau \square G^\sigma)^o$ and $(H^\tau \otimes K_n^\tau)^o$. Let $S$ be the skew-adjacency matrix of $H[G]^o$. It follows that

$$S^TS = -\left[I_m \otimes S_2^2 + S_1^2 \otimes I_n - S_1^2 \otimes S_3^2 + S_1 \otimes (S_2S_3) - S_1 \otimes (S_3S_2)\right]$$
Suppose that $H^\tau$ is an oriented $k$-regular bipartite graph of order $m$ with optimum skew energy $m\sqrt{k}$. Let $G^\sigma$ be an oriented $\ell$-regular graph of order $n$ and optimum skew energy $n\sqrt{\ell}$. Then $S_1^T S_1 = kI_m$. Let $H[G]$ be an oriented graph of $K_n$ with optimum skew energy $n\sqrt{n-1}$. Then $S_2^T S_2 = \ell I_n$. It is obvious that $H[G]^o$ is $(kn+\ell)$-regular. Moreover, let $K_3^\varsigma$ be an oriented graph of $K_n$ with optimum skew energy $n\sqrt{n-1}$. Then $S_3^T S_3 = (n-1)I_n$, that is, $S_3$ is a skew-symmetric Hadamard matrix \cite{31} of order $n$. If another condition that $S_2 S_3 = S_3 S_2$ holds, then

$$S^T S = - \left[ I_m \otimes S^2_2 + S^1_1 \otimes I_n - S^2_1 \otimes S^2_3 \right] = (kn+\ell)I_{mn}.$$ 

Obviously, $H[G]^o$ has the optimum skew energy $mn\sqrt{kn+\ell}$.

The following example \cite{28} illustrates that the oriented graph satisfying the above conditions indeed exists.

**Example 6.20** \cite{28} Let $H^\tau$ be an arbitrary oriented $k$-regular bipartite graph of order $m$ with the optimum skew energy $m\sqrt{k}$. Let $C_4^\varsigma$ be the oriented graph of $C_4$ with optimum skew energy $4\sqrt{2}$ and the skew-adjacency matrix $S_2$, and $K_4^\varsigma$ be the oriented graph of $K_4$ with optimum skew energy $4\sqrt{3}$ and the skew-adjacency matrix $S_3$, see Figure 5.1. It can be verified that $S_2 S_3 = S_3 S_2$. It follows that $(H[G])^o$ is an oriented $(4k+2)$-regular graph of order $4m$ with optimum skew energy $4m\sqrt{4k+2}$.

There are many options for $H$, such as $P_2$, $C_4$, $K_{4,4}$, the hypercube $Q_d$ and so on, which forms a new family of oriented graphs with the optimum skew energy.

7 Extremal oriented graphs in some graph classes

One of the fundamental questions that is encountered in the study of skew energy is which oriented graphs (from a given class) have the maximum and minimum skew energy. In this section, we summarize these extremal results.

We have stated the results on the skew energy of oriented trees in Section 3. In what follows, we focus on the oriented unicyclic graphs, oriented bicyclic graphs, the oriented graphs with fixed numbers of vertices and arcs and the oriented graphs without even cycles. Most of the following results are obtained with the method of the quasi-order relation $\preceq$ introduced in Section 3.

An unicyclic graph is a connected graph with the same number of vertices and edges. Let $G(n, \ell)$ denote the set of all connected unicyclic graphs on $n$ vertices with a unique cycle of length $\ell$. Denote by $P_n$ and $C_n$ the path and cycle on $n$ vertices, respectively. Let $P^\ell_n$ be the unicyclic graph obtained by connecting a vertex of $C_\ell$ with an end-vertex.
of $P_{n-\ell}$, and $S_\ell^n$ be the graph obtained by connecting $n - \ell$ pendant vertices to a vertex of $C_\ell$, which are depicted in Figure 7.7. Note that if $\ell$ is even, by the switching-equivalence, there are only two different orientations on a unicyclic graph $G$, i.e., the unique cycle $C$ is evenly oriented or oddly oriented regardless of the orientations of other edges not on the cycle. Denote by $G^+$ and $G^-$ the oriented graphs with the unique cycle oddly oriented and evenly oriented, respectively. If $\ell$ is odd, any oriented graph of a unicyclic graph $G$ has the same skew energy, denoted by $G^\sigma$. We present some extremal results for the skew energy of unicyclic oriented graphs as follows, which were obtained in [25] by Hou, Shen and Zhang.

![Graphs $S_\ell^n$ and $P_\ell^n$ and two orientations of them](image)

**Theorem 7.1** [25] For any unicyclic graph $G$, we have $G^+ \succeq G^-$.  

**Theorem 7.2** [25] Among all orientations of unicyclic graphs on $n$ vertices, $(S_3^n)^\sigma$ has the minimal skew energy and $(S_4^n)^-$ has the second minimal skew energy for $n \geq 6$; both $(S_3^n)^\sigma$ and $(S_4^n)^-$ have the minimal skew energy, $(S_4^n)^+$ has the second minimal skew energy for $n = 5$; $C_4^-$ has the minimal skew energy, $(S_4^3)^\sigma$ has the second minimal skew energy for $n = 4$.  

**Theorem 7.3** [25] Among all orientations of unicyclic graphs, $(P_4^n)^+$ is the unique oriented graph (under switching-equivalence) with maximal skew energy.
Denote by \( U_4(a, b) \) the graph obtained by attaching two pendant paths of lengths \( a \) and \( b \) to the unique pendant vertex of \( S_4 \), and then by \( U_4^+(a, b) \) an oriented graph of \( U_4(a, b) \) with the cycle \( C_4 \) oddly oriented. Moreover, let \( B_n = \{ U_4(a, b) | 0 \leq a \leq b, a + b = n - 5 \} \), \( \mathcal{A}_n^+ = \{ U_4^+(a, b) | 0 \leq a \leq b, a + b = n - 5, a \neq 1, 3, 5 \} \) and \( B_n^+ = \{ U_4^+(a, b) | 0 \leq a \leq b, a + b = n - 5 \} \). Obviously, Theorem 7.3 implies that \( U_4^+(0, n - 5) \) has the maximal skew energy among all oriented unicyclic graphs. The first \( \lfloor \frac{n-5}{2} \rfloor \) largest skew energies among all oriented unicyclic graphs of order \( n \) were obtained by Zhu in [44].

**Theorem 7.4** [44] Let \( k = \lfloor \frac{n-5}{2} \rfloor \), \( t = \lfloor \frac{n}{2} \rfloor \) and \( \ell = \lfloor \frac{k-1}{2} \rfloor \). Then we have the following quasi-order relation in \( B_n^+ \):

\[
U_4^+(0, n - 5) \succ U_4^+(2, n - 7) \succ \cdots \succ U_4^+(2t, n - 5 - 2t) \succ U_4^+(2\ell + 1, n - 5 - 2\ell - 1) \\
\succ \cdots \succ U_4^+(7, n - 12) \succ U_4^+(5, n - 10) \succ U_4^+(3, n - 8) \succ U_4^+(1, n - 6).
\]

**Theorem 7.5** [44] Let \( G^\sigma \) be an oriented unicyclic graph with order \( n \geq 31 \). If \( G^\sigma \notin U_4^+(a, b) \), then \( E_S(G^\sigma) < E_S(U_4^+(7, n - 12)) \).

Combining Theorems 7.4 with 7.3 we have the following result.

**Theorem 7.6** [44] Let \( n \geq 31 \). The oriented unicyclic graphs of order \( n \) with the first \( \lfloor \frac{n-5}{2} \rfloor \) largest skew energies are those in \( \mathcal{A}_n^+ \).

Besides, Yang, Gong and Xu [41] considered the oriented unicyclic graphs on \( n \) vertices with fixed diameter \( d \). Note that \( 2 \leq d \leq n - 2 \) for any oriented unicyclic graph \( G^\sigma \) with order \( n \) \((n \geq 6)\). If \( d = 2 \), then \( G^\sigma \) (up to isomorphism) must be the oriented graph obtained from the oriented star \( S_n \) by adding one arc between arbitrary two pendant vertices of it. Therefore, they considered the oriented unicyclic graphs with diameter \( d \) \((3 \leq d \leq n - 2)\) and determined the oriented graph with minimal skew energy. Denote by \( U_{n,d} \) the undirected unicyclic graph obtained from the cycle \( C_4 \) by attaching a pendant vertex of the path \( P_{d-2} \) and \( n - d - 1 \) pendant edges to its two non-adjacency vertices, respectively; see Figure 7.8. Then \( U_{n,d}^- \) is the oriented graph of \( U_{n,d} \) with the cycle \( C_4 \) evenly oriented.

**Theorem 7.7** [44] Let \( n \geq 6 \) and \( d \) be an integer with \( 3 \leq d \leq n - 2 \). Then the oriented graph \( U_{n,d}^- \) is the unique oriented graph with minimal skew energy among all oriented unicyclic graphs on \( n \) vertices with diameter \( d \).

Recently, Mao and Hou [32] studied the minimal skew energy oriented unicyclic graphs with given number of pendant vertices and girth.
Recall that a bicyclic graph is a connected graph with \( n \) vertices and \( n + 1 \) edges. Denote by \( S_{n}^{3,3} \) the graph formed by joining \( n - 4 \) pendant vertices to a vertex of degree three of the \( K_{4} - e \) for \( n \geq 4 \), by \( S_{n}^{4,4} \) the graph formed by joining \( n - 5 \) pendant vertices to a vertex of degree three of the complete bipartite graph \( K_{2,3} \) for \( n \geq 5 \), and by \( P_{n}^{4,4} \) the graph formed by joining two cycles of length 4 by a path of length \( n - 7 \) for \( n \geq 8 \). For \( n = 6 \) and 7, \( P_{6}^{4,4} \) and \( P_{7}^{4,4} \) denote two new graphs. See Figure 7.9. Let \( C_{x} \) and \( C_{y} \) be two cycles in a bicyclic graph \( G \) with \( t \) common vertices. Note that if \( t \leq 1 \), then \( G \) contains exactly two cycles. If \( t \geq 2 \), then \( G \) contains exactly three cycles. The third cycle is denoted by \( C_{z} \), where \( z = x + y - 2t + 2 \). Without loss of generality, assume \( x \leq y \leq z \).

![Figure 7.8: The unicyclic graph \( U_{n,d} \)](image)

For convenience, when \( t \leq 1 \) denote by \( G^{a,b} \) the oriented bicyclic graph on which \( C_{x} \) is of orientation \( a \) and \( C_{y} \) is of orientation \( b \), where \( a, b \in \{+, -, \ast\} \) and \( \ast \) denotes any orientation; when \( t \geq 2 \), denote by \( G^{a,b,c} \) the oriented bicyclic graph on which \( C_{x} \) is of orientation \( a \), \( C_{y} \) is of orientation \( b \) and \( C_{z} \) is of orientation \( c \), where \( a, b, c \in \{+, -, \ast\} \).

Shen, Hou and Zhang \[35\] obtained the following results.

**Theorem 7.8** \[35\] Among all oriented bicyclic graphs of order \( n \), \( (S_{n}^{3,3})^{\ast,\ast,-} \) has the minimal skew energy for \( n \geq 8 \); both \( (S_{7}^{3,3})^{\ast,\ast,-} \) and \( (S_{7}^{4,4})^{-,-,-} \) have the minimal skew energy.
for \( n = 7 \); \((S^4_n)^{+}_{-,-,-} \) has the minimal skew energy for \( n = 5, 6 \).

**Theorem 7.9** [35] Among all oriented bicyclic graphs of order \( n \geq 8 \), \((P^4_n)^{+,+} \) has the maximal skew energy; \((P^4_7)^{+,+} \) have the maximal skew energy for \( n = 7 \); \((P^4_6)^{+,+} \) has the maximal skew energy for \( n = 6 \).

Next we consider the oriented graphs on \( n \) vertices with \( m \) (\( n \leq m \leq 2(n-2) \)) arcs. The oriented graphs with minimal skew energy in this oriented graph class were characterized in [16] by Gong, Li and Xu. Let \( O^+_{n,m} \) be the oriented graph on \( n \) vertices which is obtained from the oriented star \( S^\sigma_n \) with center \( v_1 \) by adding \( m - n + 1 \) arcs such that all those arcs have a common vertex \( v_2 \), where \( v_1 \) is the tail of each arc incident to it and \( v_2 \) is the head of each arc incident to it; and \( B^+_{n,m} \) be the oriented graph obtained from \( O^+_{n,m+1} \) by deleting the arc \( \langle v_1, v_2 \rangle \); see Figure 7.10.

**Theorem 7.10** [16] Let \( G^\sigma \) be an oriented graph with minimal skew energy among all oriented graphs with \( n \) vertices and \( m \) (\( n \leq m \leq 2(n-2) \)) arcs. Then, up to isomorphic, \( G^\sigma \) is

(1) \( B^+_{n,m} \) if \( m < \frac{3n-5}{2} \);

(2) either \( B^+_{n,m} \) or \( O^+_{n,m} \) if \( m = \frac{3n-5}{2} \); and

(3) \( O^+_{n,m} \) otherwise.

![Figure 7.10: Two oriented graphs \( O^+_{n,m} \) and \( B^+_{n,m} \)](image)

For the remainder of this section, we turn to the oriented graphs without even cycles. Denote by \( O_n \) the class of oriented graphs of order \( n \) with no even cycles, and by \( O_{n,m} \) the class of oriented graphs in \( O_n \) with \( m \) arcs. From corollary 2.7 it is easy to find that in the class \( O_n \) the skew-characteristic polynomial of an oriented graph is independent of its orientation, so is the skew energy. Thus we denote by \( G^* \) the oriented graph with any orientation.
In [27], we determined the minimal skew energy oriented graphs in \( O_n \) and \( O_{n,m} \) with \( n - 1 \leq m \leq \frac{3}{2}(n - 1) \). Denote by \( S_{n,m} \), \( n - 1 \leq m \leq \frac{3}{2}(n - 1) \) the graph obtained from \( S_n \) by attaching \( m - n + 1 \) edges to different pairs of pendant vertices, such that the pendant vertices of \( S_n \) is of degree no more than 2. Clearly, \( S_{n,n-1} \cong S_n \). We give the graph \( S_{n,n+2} \) for example, see Figure 7.11.

![Graph S_{n,n+2}](image)

Figure 7.11: The graph \( S_{n,n+2} \)

\[ \text{Theorem 7.11} \quad [27] \text{ Let } G^\sigma \text{ be an oriented graph in } O_n. \text{ If } G \not\cong S_n, \text{ then } E_s(G^\sigma) > E_s(S_n^*) . \]

\[ \text{Theorem 7.12} \quad [27] \text{ Let } G^\sigma \text{ be an oriented graph in } O_{n,m}, \ n - 1 \leq m \leq \frac{3}{2}(n - 1). \text{ If } G \not\cong S_{n,m}, \text{ then } E_s(G^\sigma) > E_s(S_{n,m}^*) . \]

We [27] also characterized the maximal skew energy oriented graphs in \( O_{n,n} \) and \( O_{n,n+1} \), and in the latter case we assume that \( n \) is even.

\[ \text{Theorem 7.13} \quad [27] \text{ If } n \text{ is odd, the oriented graph with maximal skew energy in } O_{n,n} \text{ is } (C_n)^*. \text{ If } n \equiv 0 \pmod{4}, \text{ the oriented graph with maximal skew energy in } O_{n,n} \text{ is } (P_n^{\ell_0})^*, \ell_0 = n/2 + 1. \text{ If } n \equiv 2 \pmod{4}, \text{ the oriented graph with maximal skew energy in } O_{n,n} \text{ is } (P_n^{\ell_1})^*, \ell_1 = n/2 \text{ or } n/2 + 2. \]

\[ \text{Theorem 7.14} \text{ let } n \text{ be an even integer, if } n \equiv 0 \pmod{4}, \text{ the oriented graph with maximal skew energy in } O_{n,n+1} \text{ is } P_n^{\ell_2}, \ell = n/2 + 1. \text{ If } n \equiv 2 \pmod{4}, \text{ the oriented graph with maximal skew energy in } O_{n,n+1} \text{ is } P_n^{\ell_1'}, \ell' = n/2. \]

8 The skew energy of random oriented graphs

In this section, we state our results [6] on the skew energy of random oriented graphs. At first let us recall the random oriented graph model and the random oriented regular
graph model. A random oriented graph $G^\sigma(n, p)$ on $n$ vertices is obtained by drawing an edge between each pair of vertices, randomly and independently, with probability $p$ and then orienting each existing edge, randomly and independently, with probability $1/2$. A random regular graph $G_{n,d}$, where $d = d(n)$ denotes the degree, is a random graph chosen uniformly from the set of all simple $d$-regular graphs on $n$ vertices. A random oriented regular graph, denoted by $G^\sigma_{n,d}$, is obtained by orienting each edge of the random regular graph $G_{n,d}$, randomly and independently, with probability $1/2$.

Given a random graph model $\mathcal{G}(n, p)$, we say that almost every graph $G(n, p) \in \mathcal{G}(n, p)$ has a certain property $\mathcal{P}$ if the probability that $G(n, p)$ has the property $\mathcal{P}$ tends to 1 as $n \to \infty$, or we say $G(n, p)$ almost surely (a.s.) satisfies the property $\mathcal{P}$.

In what follows, we state the exact estimates of the skew energy for almost all oriented graphs and almost all oriented regular graphs, respectively, where the estimate of the latter is distinguished into two cases according to the value of regular degree $d$; see [6] for details.

**Theorem 8.1** [6] For $p = \omega(\frac{1}{n})$, the skew energy $\mathcal{E}_S(G^\sigma(n, p))$ of the random oriented graph $G^\sigma(n, p)$ enjoys a.s. the following formula:

$$
\mathcal{E}_S(G^\sigma(n, p)) = n^{3/2}p^{1/2}\left(\frac{8}{3\pi} + o(1)\right).
$$

Since $p$ varies in the interval $[0, 1]$, we can get the following result.

**Corollary 8.2** It is almost sure that the skew energy is increasing as the number of arcs is getting large.

This also verifies that a tournament (an oriented complete graph) has the maximum skew energy. This reminds us to raise the following problem:

**Problem 8.3** Is it true that for every oriented graph $G^\sigma$ of a noncomplete graph $G$, there is a pair of nonadjacent vertices $u$ and $v$ such that $\mathcal{E}_S(G^\sigma + \langle u, v \rangle) \geq \mathcal{E}_S(G^\sigma)$? Or weakly speaking, for any proper spanning subgraph $G'$ of a graph $G$, can we orient $G'$ and $G$ independently such that the oriented graph of $G'$ has a skew energy smaller than that of the oriented graph of $G$?

It is known from [6,29] that the energy of an undirected random graph $G(n, p)$ enjoys the formula below:

$$
\mathcal{E}(G(n, p)) = n^{3/2}[p(1-p)]^{1/2}\left(\frac{8}{3\pi} + o(1)\right).
$$

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Since \([p(1-p)]^{1/2}\) is less than \(p^{1/2}\) for any \(0 < p < 1\), the following corollary is immediate.

**Corollary 8.4** It is almost sure that for every graph \(G\), the energy of \(G\) is less than the skew energy of an oriented graph \(G^\sigma\) of \(G\).

**Theorem 8.5** [6] For any fixed integer \(d \geq 2\), the skew energy \(\mathcal{E}_S(G^\sigma_{n,d})\) of the random oriented regular graph \(G^\sigma_{n,d}\) enjoys a.s. the following formula:

\[
\mathcal{E}_S(G^\sigma_{n,d}) = n \left( \frac{2d\sqrt{d-1}}{\pi} - \frac{d(d-2)}{\pi} \cdot \arctan \frac{2\sqrt{d-1}}{d-2} + o(1) \right).
\]

In particular, when \(d = 2\), \(\mathcal{E}_S(G^\sigma_{n,d}) = n \left( \frac{4}{\pi} + o(1) \right)\).

**Theorem 8.6** [6] For \(d = d(n) \to \infty\), the skew energy \(\mathcal{E}_S(G^\sigma_{n,d})\) of the random oriented regular graph \(G^\sigma_{n,d}\) enjoys a.s. the following formula:

\[
\mathcal{E}_S(G^\sigma_{n,d}) = nd^{1/2} \left( \frac{8}{3\pi} + o(1) \right).
\]

## 9 Concluding remarks

We conclude the survey by cleaning the following questions.

1. Corollary 8.2 holds only “almost surely”, but not “always surely”.

There are oriented graphs \(G^\sigma\) such that some of their proper subgraphs have larger skew energy than that of \(G^\sigma\). From page 57 of [29] we know that, as the energy of undirected graphs is concerned, \(\mathcal{E}(K_{n,n}) < \mathcal{E}(K_{n,n} \setminus e)\) for any edge of the complete bipartite graph \(K_{n,n}\). Then, from Theorem 4.2, Corollary 4.3 and the context thereafter, we know that there is an orientation \(\sigma\) of the bipartite graph \(K_{n,n}\) such that \(\mathcal{E}_S(K^\sigma_{n,n}) = \mathcal{E}(K_{n,n})\) and \(\mathcal{E}_S((K_{n,n} \setminus e)^\sigma) = \mathcal{E}(K_{n,n} \setminus e)\), which gives that \(\mathcal{E}_S(K^\sigma_{n,n}) < \mathcal{E}_S((K_{n,n} \setminus e)^\sigma)\).

2. Corollary 8.4 holds also only “almost surely”, but not “always surely”.

There are undirected graphs \(G\) such that some of their oriented graphs \(G^\sigma\) have smaller skew energy than the energy of \(G\). From page 26 of [29] we know that the energy of an undirected cycle \(C_n\) is as follows:

\[
\mathcal{E}(C_n) = \begin{cases} 
4 \cot \frac{\pi}{n} & \text{if } n \equiv 0 \pmod{4}, \\
4 \csc \frac{\pi}{n} & \text{if } n \equiv 2 \pmod{4}, \\
2 \csc \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{2}.
\end{cases}
\]
Comparing the energy of the undirected cycle $C_n$ and the skew energies of the two oriented cycles $C_n^+$ and $C_n^-$, we obtain the following result:

$$
E(C_n) = E_S(C_n^-) < E_S(C_n^+) \quad \text{for} \quad n \equiv 0 \pmod{4},
$$

$$
E_S(C_n^-) < E_S(C_n^+) = E(C_n) \quad \text{for} \quad n \equiv 2 \pmod{4},
$$

$$
E_S(C_n^-) = E_S(C_n^+) < E(C_n) \quad \text{for} \quad n \equiv 1 \pmod{2}.
$$

From the above one can see that there are cases that the energy of an undirected cycle is smaller than the skew energy of its oriented cycle, and there are some other cases that the skew energy of an oriented cycle is smaller than the energy of the cycle.

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