Heat flux effects on the dispersion relation for geodesic modes in rotating plasmas

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Abstract. The MHD theory of the effect of toroidal and poloidal rotations on the dynamics of Zonal Flows – ZFs and Geodesic Acoustic Modes – GAMs in axisymmetric magnetic confinement configurations is revisited. The MHD model has an arbitrariness regarding the energy conservation equation and previous works on the effect of rotation on ZFs and GAMs adopted an adiabatic law, or other simplifying assumptions, to treat this problem. However, in fusion grade plasmas, the heat transport along the magnetic field lines is rather fast and, therefore, a somewhat more appropriate model is to assume isothermal flux surfaces. This implies to take into account the heat transport equation in the model and, in the presence of rotation, this leads to an increase in the degree of the dispersion relation for these modes, giving rise to a low-frequency third branch of these modes. This has been previously obtained by Elfimov, Galvão and Sgalla [1] employing a model of circular flux surfaces from the outset. In this paper, the theoretical development is generalized by using flux coordinates, following the method of Ilgisonis et al [2]. This allows a better assessment of the applicability of the results and to investigate the relevance of the low frequency mode in non-circular tokamaks. Specific results for the TCABR tokamak are presented.

1. Introduction

Geodesic acoustic modes (GAMs) are plasma eigenmodes characterized by a perturbed electric field constrained to oscillate only in the radial direction (i.e., \( m = n = 0 \), where \( m \) and \( n \) are the poloidal and toroidal mode numbers, respectively) and by a perturbed density that varies with the first order poloidal harmonic, \( m = \pm 1 \). GAMs, whose existence was predicted in the 1960s [1], are stable oscillations over magnetic surfaces. Their generation mechanism is based on the effect of the (radial) perturbed electric field, constant and perpendicular to the magnetic surfaces, which induces a poloidal flow over the surface due to the drift velocity, \( \mathbf{E} \times \mathbf{B} \). In a toroidal magnetic surface, the geodesic curvature of the field lines gives rise to a restoring force against that flux, thus inducing an oscillatory motion. GAMs are the branch of nonzero frequency of the same dispersion relation associated with zonal flows (ZFs), which are zero-frequency modes.

The nonlinear process behind the existence of the GAMs could be described as follows. Density and temperature gradients trigger drift micro-instabilities, which give rise to a turbulent spectrum of fluctuations with large convective cells that cause the anomalous transport. After certain thresholds, the correlation between the same fluctuations triggers GAMs and ZFs by means of the Reynolds stress tensor and other parametric mechanisms. Sheared flows, associated with these modes, tend to tear the convective cells, thus reducing the anomalous transport.
So, GAMs and ZFs are a fundamental mechanism of anomalous transport self-control in high-temperature magnetized plasmas.

Although very important, the plasma rotation effects on GAMs ans ZFs have been long neglected in theoretical approaches. Recently, it has been recognized that plasma rotation can play an important role on the frequency of these modes, not only modifying the existing dispersion relation but also introducing new branches of frequencies. Even in works which attempt to consider the effects of rotation, it is usual to neglect the mass flow in the poloidal direction. It turns out that the poloidal flux seem to play a major role in L/H (tokamak confinement mode) transition. Internal transport barriers (ITB), related to the very existence of the H mode, are characterized by discontinuities in density and pressure. Curiously, in JET discharges, the poloidal velocity can reach values as high as 75 km/s inside the ITB and virtually zero immediately outside. Moreover, poloidal flow can substantially modify the equilibrium and can be a cause of the elliptic/hyperbolic regime transition of the Grad-Shafranov equation [2].

The set of MHD equations has an arbitrariness with respect to the equation of state that provides closure. In most of the situations, the energy balance equation is simplified to the adiabatic condition, i.e., the divergence of the heat flux vector is set equal to zero. It can be used to describe collisional phenomena which occur in a time scale much faster than the characteristic time of heat transport. Fusion grade plasmas, however, have a very low collision rate and are embedded in strong magnetic fields, so that the particles motion is practically parallel to the field. This generates a large phase-mixing which rapidly thermalizes the magnetic surfaces and thus isothermal magnetic surfaces, considered to be the most appropriate to model fusion plasmas, will be used as a hypothesis in the present work.

2. MHD equilibrium equations

The aim of this part is to derive in detail the equations that completely describe the equilibrium in terms of the stream functions used to express the magnetic field and the fluid velocity. For that purpose, we will follow the approach of Ilgisonis et al [3]. The complete set of unperturbed one-fluid ideal MHD equations, using the assumption of isothermal magnetic surfaces and a Gaussian system of units, reads

Momentum conservation:

$$\rho_0 (v_0 \cdot \nabla) v_0 = -\nabla p_0 + \frac{1}{c} j_0 \times B_0$$ (1)

with Ampère’s law: $j_0 = \frac{c}{4\pi} \nabla \times B_0$

Ideal Ohm’s law:

$$\nabla \times (v_0 \times B_0) = 0$$ (2)

Magnetic field with no sources or sinks:

$$\nabla \cdot B_0 = 0$$ (3)

Continuity equation:

$$\nabla \cdot (\rho_0 v_0) = 0$$ (4)

Isothermal magnetic surfaces condition ($T_0 = T_0(\psi)$):

$$\left(\frac{B_0 \cdot \nabla}{\rho_0}\right) = 0$$ (5)

During the calculations, we will use two coordinate systems. The first is the usual cylindrical one ($R, \varphi, z$) where $R$ is the distance from the axis passing through the center of the torus, $\varphi$ is the toroidal angle and $z$ is the distance from the center, along the axis of symmetry. The second
is the straight field line system \((\psi, \theta, \varphi)\), where \(\psi\) is the flux over a magnetic surface and \(\theta\) is the poloidal angle. Owing to the symmetry, the equilibrium quantities do not depend on the toroidal angle, \(\varphi\). It will be assumed that the magnetic surfaces are nested surfaces \(\psi(R, z) = \text{const}\) around the magnetic axis, i.e., \(\mathbf{B}_0 \cdot \nabla \psi = 0\). From the divergence-free requirement, we can express the magnetic field as

\[
\mathbf{B}_0 = F(R, z)\nabla \varphi + \nabla \psi \times \nabla \varphi
\]

which implies

\[
\mathbf{j}_0 = \frac{c}{4\pi} (\nabla \times \nabla \varphi - \Delta^* \psi \nabla \varphi)
\]

where \(\Delta^* \psi \equiv R^2 \nabla \cdot \left(\nabla \psi R^2\right)\). Note that

\[
B_0^2 = \frac{F^2 + |\nabla \psi|^2}{R^2}
\]

It is possible to express the fluid velocity in terms of stream functions if we integrate equation (2), i.e. \(\mathbf{v}_0 \times \mathbf{B}_0 = c\nabla \phi\), where \(\phi\) is the electrostatic potential. Hence, \(\mathbf{B}_0 \cdot \nabla \phi = 0\) and from equation (8), it follows that

\[
\mathbf{v}_0 = \lambda(R, z)\mathbf{B}_0 + R^2 \Omega(\psi) \nabla \varphi
\]

where \(\lambda\) is an arbitrary scalar function and the definition \(\Omega(\psi) \equiv c \frac{\partial \phi}{\partial \psi}\) has been used. In the forthcoming calculations, we will consider that \(F, \lambda, R^2 \Omega\) and \(\rho_0\) do not depend on the toroidal angle, \(\varphi\), i.e. \(\nabla F \cdot \nabla \varphi = \nabla \lambda \cdot \nabla \varphi = \nabla (R^2 \Omega) \cdot \nabla \varphi = \nabla \rho_0 \cdot \nabla \varphi = 0\). Substituting equation (9) in equation (4) and noting that \(\nabla^2 \varphi = 0\), we get

\[
\nabla \cdot (\rho_0 \mathbf{v}_0) = \nabla \left(\lambda \rho_0\right) \cdot \mathbf{B}_0 = 0 \implies \lambda = \frac{\kappa(\psi)}{\rho_0}
\]

where \(\kappa\) is a function depending only on \(\psi\).

2.1. Bernoulli-like equation

To obtain a Bernoulli-like equation, we project the force balance equation (1) onto the direction of \(\mathbf{B}_0\):

\[
\rho_0 \mathbf{B}_0 \cdot \nabla \left[\frac{\kappa^2 B_0^2}{2\rho_0^2} - \frac{R^2 \Omega^2}{2}\right] + \mathbf{B}_0 \cdot \nabla \rho_0 = 0
\]

Assuming quasi-neutrality, the main contribution to the mass density is due to the ions. So, from equation (5) and the ideal gases law, it follows that

\[
\frac{\rho_0}{\rho_0^*}(\psi) \simeq \frac{T_0(\psi)}{m_i}
\]

It should be noted that McClements and Hole [2] use \(\frac{\rho_0}{\rho_0^*}(\psi) \simeq \frac{T_0(\psi)}{2m_i}\) since they consider \(T_0\) to be the sum of ionic and electronic temperatures. Here we consider \(\bar{T}\) to be the average temperature. Plugging (11) into (10), we get

\[
\frac{\kappa^2 B_0^2}{2\rho_0^2} - \frac{R^2 \Omega^2}{2} + \frac{T_0}{m_i} \ln \frac{\rho_0}{\rho_0^*} = H(\psi)
\]

\[1\] In cylindrical coordinates, \(\nabla \varphi = \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)\varphi = \frac{1}{\rho} \frac{\partial}{\partial \varphi} \implies |\nabla \varphi|^2 = \frac{1}{\rho^2}.\]
where $\rho_{00}$ is an arbitrary quantity that may depend on $\psi$ and the enthalpy $H(\psi)$ is a function of $\psi$ only. Hence,

$$\frac{\rho_0(\psi, \theta)}{\rho_{00}(\psi)} = \frac{\rho_0(\psi, \theta)}{\rho_{00}(\psi)} = \exp \left[ \left( \frac{R^2 \Omega^2 - \kappa^2 B_0^2}{\rho_0^2} \right) \frac{m_i}{2T_0} \right]$$

A factor $-R^2 \Omega^2 \frac{m_i}{2T_0}$ is often introduced in the argument of the exponential, without loss of generality. $\rho_{00}$ and $p_{00}$ are arbitrary quantities that depend only on $\psi$ or are constant.

### 3. Perturbed equilibrium

Let us allow the following perturbations with respect to the equilibrium (of the form $f' = f'(\psi, \theta) \exp(-i\omega t)$)

$$\rho = \rho_0 + \rho', \quad v = v_0 + v', \quad p = p_0 + p', \quad j = j_0 + j'$$

where each of the quantities with subscript 0 depend on both $\psi$ and $\theta$. The linearized perturbed MHD equations are

$$\rho_0 \left[ -i\omega v' + (v_0 \cdot \nabla) v' + (v' \cdot \nabla) v_0 \right] + \rho'(v_0 \cdot \nabla) v_0 = -\nabla p' + j' \times B_0$$

(14)

$$v' \times B_0 = \nabla \phi'$$

(15)

$$-i\omega \rho' + \nabla \cdot (\rho_0 v' + \rho' v_0) = 0$$

(16)

$$\nabla \cdot j' = 0$$

(17)

From Ohm’s law, equation (15), the perturbed velocity can be written

$$v' = \frac{1}{B_0^2} \left( V' - F \frac{d\phi'}{d\psi} \right) B_0 + R^2 \frac{d\phi'}{d\psi} \nabla \varphi$$

(18)

where $V' = v' \cdot B_0$. In order to provide closure to the system of perturbed equations, an energy equation is needed. To obtain the perturbed equation of state, we shall use equation (6.36) of [4] in a simplified form, in which the ion thermal flux vector is regarded to be more important than dissipative effects,

$$\frac{3}{2} \frac{d}{dt} \ln \left( \frac{p}{\rho^{7/2}} \right) = -\nabla \cdot q$$

(19)

with

$$q \approx 5 \frac{b \times \nabla T}{eB}$$

(20)

where $\hat{b} = B/B$. One should note that the situation of pure toroidal flow is a degenerated case since it is not necessary to take into account the heat flow vector. This can be seen by writing the total time derivative of equation (19) in an explicit form:

$$\frac{3}{2} \left[ \frac{\partial}{\partial t} + v_\varphi \frac{\partial}{\partial \varphi} \right] \ln \left( \frac{p}{\rho^{7/2}} \right)$$

The above expression is always identically zero for stationary $\left( \frac{\partial \varphi}{\partial t} = 0 \right)$ and axisymmetric $\left( \frac{\partial \varphi}{\partial \varphi} = 0 \right)$ equilibrium. This particular situation was considered by Lahkin et al [5] for the case of purely toroidal rotation and isochoric magnetic surfaces. In their work the perturbed energy...
equation does not contain heat flux vector terms either, even though this is not an adiabatic regime. It is worth mentioning that, from the continuity equation, the validity of the hypothesis \( \rho_0 = \rho_0(\psi) \) is only assured for zero poloidal flow. Back to the general situation, with both toroidal and poloidal flows, if we perturb equations (19) and (20), the following first-order, linearized energy equation is obtained:

\[
\frac{3}{2} \rho_0^2 \left\{ \mathbf{v}' \cdot \nabla \left( \frac{p_0 \rho_0}{\gamma} \right) + \gamma \frac{p'}{\rho_0} \mathbf{v}_0 \cdot \nabla \left( \frac{p_0}{\rho_0} \right) + \left( \frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla \right) \left( \frac{p' - 2\kappa p'}{\rho_0^2} \right) \right\} = -\nabla \cdot \mathbf{q}_l \tag{21}
\]

Since \( T' = \left( \rho' - \frac{\rho' p_0}{\rho_0} \right) \frac{m_i}{eB} \), it follows that

\[
\mathbf{q}_l = \frac{5m_i}{2eB} \left\{ p_0 \mathbf{b} \times \nabla \left[ \left( \rho' - \frac{\rho' p_0}{\rho_0} \right) \frac{1}{\rho_0} \right] + p' \mathbf{b} \times \nabla \left( \frac{p_0}{\rho_0} \right) \right\} \tag{22}
\]

Substituting the perturbed velocity (18) into the perturbed continuity equation (16) leads to

\[-i\omega \rho' + \mathbf{B}_0 \cdot \nabla \left\{ \frac{\rho_0}{B_0^2} \left( V' - cF \frac{d\phi'}{d\psi} \right) + \kappa \rho' \right\} \right\] = 0 \tag{23}

The projection of (14) onto \( \mathbf{B}_0 \) is

\[-i\omega \rho_0 V' + \rho_0 \mathbf{B}_0 \cdot (\mathbf{v}_0 \cdot \nabla)\mathbf{v}' + (\mathbf{v}' \cdot \nabla)\mathbf{v}_0) \right\] \[ - \frac{\rho'}{\rho_0} \mathbf{B}_0 \cdot \nabla p_0 = -\mathbf{B}_0 \cdot \nabla \rho_0 \] \tag{24}

where the equilibrium equation \( \rho_0 \mathbf{B}_0 \cdot (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0) = -\mathbf{B}_0 \cdot \nabla \rho_0 \) has been used. The term in brackets can be transformed the same way as in Appendix A of the paper by Igison et al [3], resulting \( \mathbf{B}_0 \cdot [(\mathbf{v}_0 \cdot \nabla)\mathbf{v}' + (\mathbf{v}' \cdot \nabla)\mathbf{v}_0) = \mathbf{B}_0 \cdot \nabla \left\{ \frac{\kappa}{\rho_0} \left( V' - cF \frac{d\phi'}{d\psi} \right) - c\Omega R^2 \frac{d\phi}{d\psi} \right\} \). Using the ideal gases law \( \rho_0 \approx \frac{\rho_0 T_0(\psi)}{m_i} \) and eq. (13), the term involving the equilibrium pressure can be written as

\[
\left( \frac{\rho_0 V'}{\rho_0} \mathbf{B}_0 \cdot \nabla \rho_0 \right) = \frac{T_0 V'}{m_i T_0} \mathbf{B}_0 \cdot \nabla \rho_0 = \frac{T_0 V'}{m_i T_0} \mathbf{B}_0 \cdot \nabla \left\{ \rho_0 T_0(\psi) \exp \left[ \left( R^2 \Omega^2 - \frac{\kappa^2 B_0^2}{\rho_0^2} \right) \frac{m_i}{2 T_0} \right] \right\} = \frac{\rho_0 \exp \left[ \left( R^2 \Omega^2 - \frac{\kappa^2 B_0^2}{\rho_0^2} \right) \frac{m_i}{2 T_0} \right] \mathbf{B}_0 \cdot \nabla \left( R^2 \Omega^2 - \frac{\kappa^2 B_0^2}{\rho_0^2} \right) \frac{m_i}{2 T_0} \mathbf{B}_0 \cdot \nabla R^2 - \frac{\kappa^2 B_0^2}{\rho_0^2} \mathbf{B}_0 \cdot \nabla B_0^2 + \frac{2 \kappa^2 B_0^2}{\rho_0^2} \mathbf{B}_0 \cdot \nabla \rho_0 \right\} \]

So, the perturbed momentum equation (14) along \( \mathbf{B}_0 \) becomes

\[-i\omega \rho_0 V' + \rho_0 \mathbf{B}_0 \cdot \nabla \left\{ \frac{\kappa}{\rho_0} \left( V' - cF \frac{d\phi'}{d\psi} \right) - c\Omega R^2 \frac{d\phi}{d\psi} \right\} \hspace{1cm} \frac{\rho_0 V'}{\rho_0} \mathbf{B}_0 \cdot \nabla R^2 - \frac{\kappa^2 B_0^2}{\rho_0^2} \mathbf{B}_0 \cdot \nabla B_0^2 + \frac{2 \kappa^2 B_0^2}{\rho_0^2} \mathbf{B}_0 \cdot \nabla \rho_0 \} \right\} \hspace{1cm} - \mathbf{B}_0 \cdot \nabla \rho_0 \to P \] \tag{25}

which is clearly different from equation (53) of reference [3], in which the simple algebraic equation \( p' = \frac{c_\rho^2 p'}{\rho_0} \) (adiabatic case) was used to write the RHS simply as \( \mathbf{B}_0 \cdot \nabla \left( \frac{c_\rho^2 p'}{\rho_0} \right) \).

3.1. The average over a magnetic surface

Let us introduce the average over a magnetic surface as

\[
\langle \ldots \rangle = \frac{\int_0^{2\pi} \ldots d\theta / J}{\int_0^{2\pi} d\theta / J} \]

At this point, it is important to differentiate between a periodic quantity and a quantity with zero average. An important result is that if \( f \) is a periodic function of \( \theta \), then \( \langle \mathbf{B}_0 \cdot \nabla f \rangle = 0 \):

\[
\langle \mathbf{B}_0 \cdot \nabla f \rangle = \frac{\int_0^{2\pi} \frac{\partial f}{\partial \theta} d\theta}{\int_0^{2\pi} d\theta} = \frac{\int_0^{2\pi} df}{\int_0^{2\pi} d\theta} = \frac{f(2\pi) - f(0)}{\int_0^{2\pi} d\theta} = 0
\]

The next step is to obtain an expression for \( J' \) using the momentum conservation equation (14) and substitute it in the perturbed quasineutrality condition, eq. (17). The procedure is exactly the same of Ilgisonis et al., [3]. We have performed all the calculations but here we will just quote the final result without the whole reproduction (that can be found in Appendix B of [3]).

\[
\frac{d}{d\psi} \left\{ \int_0^{2\pi} \frac{d\theta}{J} \left( -i\omega - e\rho_0 F \frac{\partial}{\partial \psi} \right) + \frac{\rho_0}{B_0^2} \mathbf{B}_0 \cdot \nabla R^2 - \left( \frac{\rho_0^2}{B_0^2} \right)^2 \mathbf{B}_0 \cdot \nabla \left( \frac{\nabla \psi^2}{B_0^2} \right) \right\} = 0
\]

Equations (21) (with \( q_1 \) given by (22)), (23), (25) and (26) describe exactly the perturbed rotating equilibrium. In order to become tractable, they must be simplified, as shown in the following.

### 3.2. The large-aspect-ratio tokamak ordering

In order to simplify the equations, let us consider the approximations \( \epsilon \ll 1 \); \( B_0^2 = \frac{F^2}{R^2} (1 + \mathcal{O}(\epsilon^2)) \); \( f = \bar{f}(\psi) + \hat{f}(\psi, \theta) \), where \( \epsilon \) corresponds to the reciprocal of the tokamak aspect ratio and \( f \) stands for the equilibrium quantities \( \rho_0, \rho_0, F, J, \Omega_P \) and \( \Omega_T \). Before proceeding with the calculations, we should first note that\(^2\)

\[ J = (\nabla \psi \times \nabla \varphi) \cdot \nabla \theta = \mathbf{B}_0 \cdot \nabla \theta \]

and, since \( q(\psi) \equiv \frac{\mathbf{B}_0 \cdot \nabla \varphi}{\mathbf{B}_0 \cdot \nabla \theta} = \frac{F}{\bar{F} R^2} \),

\[ \Omega_P \equiv \mathbf{v}_0 \cdot \nabla \theta = \frac{\kappa(\psi)J}{\rho_0} \]

\[ \Omega_T \equiv \mathbf{v}_0 \cdot \nabla \varphi = \Omega(\psi) + \frac{\kappa(\psi)F}{\rho_0 R^2} \Rightarrow \Omega_T = \Omega(\psi) + q \Omega_P \]

where equation (9) was used. Let us also define the dimensionless coefficients \( \lambda_\rho \) and \( \lambda_F \) as

\[
\frac{1}{\rho_0} \mathbf{B}_0 \cdot \nabla \rho_0 \equiv \frac{\lambda_\rho}{R_0^2} \mathbf{B}_0 \cdot \nabla R^2
\]

\[
\frac{1}{F} \mathbf{B}_0 \cdot \nabla \bar{F} \equiv \frac{\lambda_F}{R_0^2} \mathbf{B}_0 \cdot \nabla R^2
\]

Equation (23) becomes

\[
\left( -i\omega + \Omega_P \frac{\partial}{\partial \theta} \right) \rho' + \frac{\rho_0 \bar{J}}{B_0^2} \frac{\partial V'}{\partial \theta} = \frac{c\rho_0}{F} \frac{d\phi'}{d\psi} (1 + \lambda_\rho) \mathbf{B}_0 \cdot \nabla R^2
\]

\(^2\) Note that when working with the system \( (r, \theta, \varphi) \) instead of \( (\psi, \theta, \varphi) \) one could write the Jacobian explicitly as \( J = [r (R_0 + r \cos \theta)]^{-1} \).
which is the same as equation (63) of [3]. Now let us try to simplify equation (25). Our calculations will differ form the ones of [3] because we do not have a simple expression of $p'$ in terms of $\rho'$. The LHS of equation (25) can be transformed using

$$\nabla \cdot \left( \frac{5m_i}{2eB} \left( \frac{\rho_0}{p_0} \right) \frac{d\psi}{dv} \right) \right] = \left\{ \hat{b} \times \nabla \left[ \frac{p_0}{\rho_0} \right] \right\} \cdot \nabla \left( \frac{5m_i}{2eB} \frac{p_0}{\rho_0} \right)$$

Similarly, the divergence of the second term of the RHS of equation (22) can be written as

$$\nabla \cdot q_1 \simeq \frac{5m_i R_0^2 F}{2eF} \left[ \frac{d\theta}{d\psi} \frac{\partial p_0}{\partial \theta} - \frac{1}{\rho_0} \frac{\partial p_0}{\partial \theta} \right]$$

Writing the velocities explicitly, equation (21) can be put in the form

$$\frac{3}{2} \hat{b} \rho_0 \left\{ \frac{1}{B_0} \left( V' - \frac{df}{dv} \right) B_0 \cdot \nabla \left( \frac{p_0 \rho_0 - \rho_0 - \gamma \rho_0 \kappa}{\rho_0} \right) + \gamma \rho_0 \frac{eB_0}{\rho_0} B_0 \cdot \nabla \left( \frac{p_0 \rho_0 - \rho_0 - \gamma \rho_0}{\rho_0} \right) \right\} = -\nabla \cdot q_1 \right]$$

Using the tokamak ordering, we note that $\frac{\nabla \psi^2}{B_0^2} \simeq \frac{B_{pol} R^2}{B_{tor}} = \frac{r^2}{2}$ and equation (26) can be simplified to

$$\frac{d}{d\psi} \left[ -2\pi i \omega \frac{\rho_0}{J} \frac{r^2}{q^2} \frac{df}{dv} + \frac{1}{J F} \int_0^{2\pi} d\theta \left( \rho' + \rho_0 R_0^2 V' \frac{\Omega T}{F} + \rho' R_0^2 \frac{\Omega T}{2} \right) B_0 \cdot \nabla R^2 \right] = 0$$

Equation (29) can be further simplified if we work the term $\frac{\rho_0 \rho_0 - \rho_0 - \gamma \rho_0}{\rho_0}$ out, using equation (13) (with $\rho_0(r)$ arbitrary),

$$\rho_0(r, \theta) = \rho_0(\theta) exp \left[ \left( \frac{R^2 \Omega^2 - \frac{\kappa^2 B_0^2}{\rho_0^2}}{2T_0} \right) \frac{m_i}{2T_0} \right]$$
\[ B_0 \cdot \nabla \left( \rho_0 \rho_0^{-\gamma} \right) = \frac{T_0}{m_i} B_0 \cdot \nabla \rho_0^{-\gamma+1} = \left( 1 - \gamma \right) \rho_0^{-\gamma} \frac{T_0}{m_i} B_0 \cdot \nabla \rho_0 \]

Noting that \[ B_0 \cdot \nabla \rho_0 = \rho_0 r(r) \frac{m_i}{2T_0} \left[ \Omega_2 B_0 \cdot \nabla R^2 - \kappa^2 B_0 \cdot \nabla \left( \frac{B_0^2}{\rho_0} \right) \right] \exp \left[ \left( R^2 \Omega^2 - \frac{\kappa^2 B_0^2}{\rho_0} \right) \frac{m_i}{2T_0} \right]. \]

Since \( B_0 \cdot \nabla B_0^2 \approx -\frac{F^2}{R_0} B_0 \cdot \nabla R^2, \) it follows that

\[ B_0 \cdot \nabla \left( \rho_0 \rho_0^{-\gamma} \right) = \left( \frac{1 - \gamma}{2} \right) \rho_0^{-\gamma - 1} \left[ \Omega_2 + \frac{\kappa^2 F^2}{R_0^2 \rho_0^2} (1 + 2 \lambda_\rho) \right] B_0 \cdot \nabla R^2 \]

\( B_0 \cdot \nabla R^2 \) is present in all four equation which are left to be solved. It can be simplified using

\[ B_0 \cdot \nabla R^2 = B_0 \cdot \nabla \left( R_0 + r \cos \theta \right)^2 \approx -2rR_0 \tilde{J} \sin \theta \]

So, we can write equation (29) in the form

\[ -\frac{3}{2} \frac{(1-\gamma)}{\rho_0} r R_0 J \sin \theta \frac{R_0^2}{F} \left( \frac{F^2}{F^2} \left( V' - c F \frac{d \phi'}{d \psi} \right) + \frac{\gamma^2}{\rho_0^2} \frac{\kappa}{\rho_0} \right) \left[ \Omega_2 + \frac{\kappa^2 F^2}{R_0^2 \rho_0^2} (1 + 2 \lambda_\rho) \right] + \frac{3}{2} \left( -i \omega + \frac{\gamma J}{\rho_0} \frac{\partial}{\partial \psi} \left( \frac{\rho'}{\rho} - \frac{\rho}{\rho_0} \frac{\partial \rho'}{\partial \psi} \right) \right) \]

We are left with the following four equations (they correspond to equations (16)-(19) of [3], with the difference that in this work we consider isothermal surfaces instead of isentropic ones):

\[ \left( -i \omega + \Omega_p \frac{\partial}{\partial \psi} \right) \rho' + \tilde{\rho}_0 \frac{\partial V'}{\partial \psi} = -2r R_0 J \sin \theta \frac{c \tilde{p}_0}{F} \frac{d \phi'}{d \psi} (1 + \lambda_\rho) \tag{31} \]

\[ \tilde{\rho}_0 \left( -i \omega + \Omega_p \frac{\partial}{\partial \psi} \right) V' + \tilde{\rho}_0 \frac{d \phi'}{d \psi} \left( \Omega_T - q \Omega_p - q \Omega_p \lambda_\rho \right) 2r R_0 J \sin \theta = \]

\[ = -\frac{1}{2} \left\{ \Omega_2 + \frac{\kappa^2 F^2}{R_0^2 \rho_0^2} (1 + 2 \lambda_\rho) \right\} 2r R_0 J \sin \theta - \frac{\partial \rho'}{\partial \psi} \tag{32} \]

\[ -\frac{3}{2} \frac{(1-\gamma)}{\rho_0} r R_0 J \sin \theta \frac{R_0^2}{F} \left( \frac{F^2}{F^2} \left( V' - c F \frac{d \phi'}{d \psi} \right) + \frac{\gamma^2}{\rho_0^2} \frac{\kappa}{\rho_0} \right) \left[ \Omega_2 + \frac{\kappa^2 F^2}{R_0^2 \rho_0^2} (1 + 2 \lambda_\rho) \right] + \frac{3}{2} \left( -i \omega + \frac{\gamma J}{\rho_0} \frac{\partial}{\partial \psi} \left( \frac{\rho'}{\rho} - \frac{\rho}{\rho_0} \frac{\partial \rho'}{\partial \psi} \right) \right) \]

\[ = -\frac{3}{2} \frac{(1-\gamma)}{\rho_0} r R_0 J \sin \theta \frac{R_0^2}{F} \left( \frac{F^2}{F^2} \left( V' - c F \frac{d \phi'}{d \psi} \right) + \frac{\gamma^2}{\rho_0^2} \frac{\kappa}{\rho_0} \right) \left[ \Omega_2 + \frac{\kappa^2 F^2}{R_0^2 \rho_0^2} (1 + 2 \lambda_\rho) \right] + \frac{3}{2} \left( -i \omega + \frac{\gamma J}{\rho_0} \frac{\partial}{\partial \psi} \left( \frac{\rho'}{\rho} - \frac{\rho}{\rho_0} \frac{\partial \rho'}{\partial \psi} \right) \right) \]

\[ \frac{d}{d \psi} \left[ -2 \pi \omega \frac{c \tilde{p}_0}{\tilde{J}} \frac{r^2}{q^2} \frac{d \phi'}{d \psi} - \frac{1}{F} \int_0^{2\pi} d \theta \left( \rho' + \tilde{p}_0 \frac{R_0^3}{F} \Omega_T + \rho' \frac{R_0^3 \Omega_T^2}{2} \right) 2r R_0 \sin \theta \right] = 0 \tag{34} \]

Equations (31), (32) and (33) will be used to obtain expressions for \( V', \rho' \) and \( p' \) and the last one will give the dispersion relation. In theoretical treatments with the intention of obtaining dispersion relations for GAMs, it is usual to consider only first-order effects since higher-order harmonics are responsible only for higher order (in \( \epsilon \) contributions. So, we may assume

\[ \rho' = p_s \sin \theta + p_c \cos \theta \]

\[ p' = p_s \sin \theta + p_c \cos \theta \]

\[ V' = V_s \sin \theta + V_c \cos \theta \]
4. Linear system resolution

Plugging (35) into equations (31), (32) and (33) implies the handling of considerably extensive expressions so that the aid of a symbolic software is imperative. To this end we have employed Wolfram Mathematica 8.0. The results of the $6 \times 6$ system (for $\rho_s$, $\rho_c$, $V_s$, $V_c$, $p_s$ and $p_c$) are too lengthy to be shown here. The next step is to use the aforementioned expressions in (34) in order to proceed with the obtaining of a dispersion relation. A command is used in order to bring together the whole expression in order to analyze the global numerator and denominator. Equation (34) will give the desired relation; however one must be careful if the denominator could possibly cancel some factor of the numerator, thus reducing the number of roots of $\omega^2$. To simplify the analysis, we use the following definitions:

- The acoustic speed:
  
  \[ c_s^2 = \frac{\gamma p_0}{\rho_0} = R_0^2 \omega_s^2 \]

- The poloidal and toroidal Mach numbers:
  
  \[ M_{P,T} = \frac{\omega_s}{\Omega_{P,T}} \]

- A parameter to characterize $B_0 \cdot \nabla \rho_0$ against $B_0 \cdot \nabla R^2$: $\lambda_\rho = \frac{B_0 \cdot \nabla \rho_0}{B_0 \cdot \nabla R^2} \frac{R_0^2}{R_s^0}$. After substitution in the equilibrium MHD equations, one obtains
  
  \[ \lambda_\rho = \frac{\gamma(M_s^2 - M_P M_T + M_T^2/2)}{1 - \gamma M_P^2} \]

- A parameter to characterize $B_0 \cdot \nabla R$ against $B_0 \cdot \nabla R^2$: $\lambda_F = \frac{B_0 \cdot \nabla R}{B_0 \cdot \nabla R^2} \frac{R_0^2}{R_s^0}$. After substitution in the equilibrium MHD equations, one obtains $\lambda_F \sim \lambda_p e^2$, thus being negligible.

- A parameter to quantify the contribution of the heat flow vector: $\tau = \frac{R_s^2 \omega_s p_0}{\gamma \rho_0} \frac{d\rho_0}{d\psi}$. The equivalence between this parameter and $M_d$ introduced in Ref. [6] is $M_d = \frac{\gamma}{3\gamma^2} \frac{\tau}{M_T}$. The numerator of the dispersion relation (with terms up to order $\lambda_\rho^2$) is

  \[
  8M_T^2 M_T q^2 \left(3 - (3 + 2\gamma)\Omega^2 + \Omega^4\right) - 2M_T^4 \left(2 - 3\Omega^2 + 2\Omega^3\Omega + \gamma(2\Omega + 2\Omega^3)\right) - \left(-1 + \Omega^2\right) \left(8M_T^2 q^2 \Omega^2 - 2\Omega^2 \left(-1 + 2\Omega^2 + \Omega^4\right) + M_T^4 q^2 \left(1 + \Omega(1 + \Omega)\right)\right) - 2M_T^4 M_T q^2 \times \left(4\Omega^2 \left(3 + M_d - \Omega^2\right) + M_T^2 \left(1 + 3\Omega^2 + \gamma(-1 + 3\Omega^2 - 2\Omega^4)\right)\right) - 2M_T^2 \times \left(1 + M_T^2 \left(1 + 2\rho^2\right) + 3\rho^4 + 2M_T \left(-1 + 3\Omega^2 + \rho^2(-2 + \Omega^2)\right)\right)\]

  \[
  \times (2(1 - 5\Omega^2 + \Omega^4) + M_T^2 \left(3 + \gamma - 13\Omega^2 - 7\gamma \rho^2 - 4\gamma \Omega^4\right))
  \]

while the denominator is

\[
\left[M_T^2 + \Omega - 3M_T^2 \Omega - \Omega^3 + M_P \left(-1 + M_d + 3\Omega^2\right)\right] \times \left[M_T^2 + 3M_T^2 \Omega + \Omega \left(-1 + \Omega^2\right) + M_P \left(-1 + M_d + 3\Omega^2\right)\right]
\]

We have verified that, as reported in Ref. [6], if $M_P, M_T \neq 0$ no cancellation between the numerator and the denominator takes place and the problem is analytically intractable. An insight shown in the same reference is that if $M_T \to 0$, a factor $\Omega^2 - M_T^2(1 - M_T) = 0$ is common to both parts. That limit is considered in order to simplify the equation and to obtain the other two frequencies and then the next order correction is calculated for the first root. Performing

\[\text{Curiously, only sine-terms will give nonzero contributions (in the } m = \pm 1 \text{ approach).}\]
the polynomial division of (36) (still with $M_T \neq 0$) by $\Omega^2 - M_P^2(1 - M_d)^2$, using the subroutines \texttt{PolynomialRemainder} and \texttt{PolynomialQuotient}, a fourth order (in $\Omega$) equation is obtained:

$$
\begin{align*}
\Omega^4 - 2\Omega^2[1 + M_P^2 + q^2(1 + 2M_P^2 + M_P^2 - 2M_P M_T)] + \\
+ 2q^2(1 + 2M_P^2 + 4M_P^2 - 6M_P M_T) + 1 - 2M_T^2 + S_4(\Omega) = R_4/ (\Omega^2 - M_P^2(1 - M_d)^2)
\end{align*}
$$

where the quotient $S_4(\Omega)$ and the remainder $R_4$ are terms of order $\lambda \rho^2$. Up to first order in $\lambda \rho$, we are left with a factorized equation of the form

$$
(\Omega^2 - \Omega_1^2) (\Omega^2 - \Omega_2^2) \simeq 0
$$

which has the solutions

$$
\begin{align*}
\omega_{GAM1}^2 &= \left[2 + \frac{1}{q^2} + \left(2 - \frac{1}{q^2} + \frac{2}{q^4}\right)M_P^2 + 4M_P^2 - 4 \left(1 - \frac{1}{q^2}\right)M_P M_T\right] \frac{c_s^2}{R_0} \\
\omega_{GAM2}^2 &= \left[1 + \left(3 - \frac{2}{q^2}\right)M_P^2 - 4M_P M_T\right] \frac{c_s^2}{q^2 R_0}
\end{align*}
$$

Finally we are left with the task of calculating the frequency increment of the first branch (located near $\Omega_3^2 \approx M_P^2(1 - M_d)^2$). It is necessary to work up to order $\lambda \rho^2$ in eq. (38), i.e., keeping all terms. Let us consider a frequency of the form

$$
\Omega_3 = \Omega_{d3} + \Delta_3; \Delta_3 \ll \Omega_{d3}
$$

Substituting in eq. (38), we get

$$
(\Omega_3^2 - \Omega_1^2) (\Omega_3^2 - \Omega_2^2) + S_4(\Omega_3) = \frac{R_4}{(\Omega_3^2 + \Delta_3)^2 - \Omega_{d3}^2}
$$

So, approximately (for $\Omega_1, \Omega_2 \gg \Omega_3$), we have

$$
\Omega_3^2 \Omega_2^2 \simeq \frac{R_4}{2\Delta_3 \Omega_{d3}} \Rightarrow \Delta_3 \simeq \frac{R_4}{2\Omega_3^2 \Omega_2 \Omega_{d3}}
$$

and

$$
\Omega_3^2 \simeq \Omega_{d3}^2 + 2\Delta_3 \Omega_{d3} \simeq \Omega_{d3}^2 + \frac{R_4}{\Omega_3^2 \Omega_2}
$$

which allows us to express the lowest frequency mode as

$$
\omega_{2}\omega_{PF}^2 = \left(1 - M_d^2\right) M_P^2 + \frac{q^2(\gamma - 1)}{1 + 2q^2} (M_P^2 - M_P M_T + M_T^2/2) M_T^2 \frac{c_s^2}{q^2 R_0^2}
$$

This mode is identified as a zonal flow since it has zero frequency in the case of zero rotation. Equations (39), (40) and (41) are exactly the same ones obtained in [6].

5. Analysis and discussions

It is quite instructive to calculate the frequencies obtained in eq. (39) for realistic parameters of a real device. This was performed for typical discharges of TCABR. For the sake of simplicity, we used $\gamma = 5/3$ and $Z_{eff} = 1$. The radial profiles for the temperatures were assumed to be $T_e(r) = 600eV (1 - (r/0.18)^2)^2$, $T_i(r) = 300eV (1 - (r/0.18)^2)^2$, where $r$ is measured in meters ($a = 0.18m$ being the minor radius), and the density profile was regarded as $n_{i,e}(r) = 2.2 \times 10^{19}m^{-3}(1 - (r/0.18)^2) + 10^{14}m^{-3}$ and the toroidal current density profile as
\( \propto (1 - (r/0.18)^2)^2 \cdot 3 \). The poloidal field is then calculated under the constraint that the current density integrated over the whole plasma cross section must be equal to 100 kA. These profiles, combined with a nearly constant \( B_{tor} \approx 1.1 T \), are capable of reproducing a quite reasonable safety factor \( q \), which is close to 1 at the center and grows monotonically to a value of approximately 3 at the edge. The radial poloidal and toroidal rotation profiles were obtained from the results of [7], which were then interpolated. The resulting profiles for each of the three continua are shown in figure 1. Most interesting is the measured power spectrum of acoustic modes measured in TCABR (figure 9 of [8]). It is surprising that there is strong evidence of the existence of three modes (most clearly seen in the bias regime - in Ohmic discharges the effect of higher harmonics seems to mask the lowest-order phenomena). They can be identified by characteristic frequencies of order 1 kHz, 15 kHz and 30 kHz, which could correspond to the frequencies of figure 1 close to the edge (typical locations of geodesic modes). Unfortunately, the lack of window resolution in the measurements prevents us to identify the frequency of the modes more accurately.

**Figure 1.** Estimation of the radial behavior of GAMs and ZF branches for typical TCABR tokamak parameters.

We note that the frequency of both branches of GAM modes were obtained in virtue of a factor cancellation in the numerator, which occurs in the formal limit \( M_T \to 0 \). We have analyzed in detail how fair is this approximation for TCABR tokamak rotating plasma parameters. To this purpose, the numerator (36) was solved numerically using the previously calculated safety factor \( q(r) \). Then, equation (36) was solved numerically for a number of radial positions. It turned out that the analytical expressions for GAM frequencies are rather robust, with the analytical and numerical curves virtually overlaid, with a relative difference of less than \( 10^{-7} \% \). However, with increasing \( M_T \), the error becomes more substantial. For example, if we double the toroidal velocity, the zonal flow branch obtained using the two methods present a 1% difference with respect to each other. It is interesting to note that, given the weak dependence of the GAM modes, equations (39) and (40), on the toroidal Mach number, they do not undergo severe changes when \( M_T \) is increased. However, the lowest-frequency mode, equation (41), has a forth-power dependence on \( M_T \). For instance, if TCABR toroidal mass flow is multiplied by a factor of 5 the resulting error would be of order 10%. This kind of correction should be important for the case of an additional torque source, such as a neutral beam injection.
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