THE LARGE DISTANCE LIMIT OF THE GRAVITATIONAL EFFECTIVE ACTION IN HYPERBOLIC BACKGROUNDS

A.A. BYTSENKO
Department of Theoretical Physics, State Technical University, St Petersburg 195251, Russia
S. D. ODINTSOV
Department E.C.M., Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain
S. ZERBINI
Department of Physics, University of Trento, 38050 Povo, Italy and I.N.F.N., Gruppo Collegato di Trento

Abstract

The one-loop effective action for D-dimensional quantum gravity with negative cosmological constant, is investigated in space-times with compact hyperbolic spatial section. The explicit expansion of the effective action as a power series of the curvature on hyperbolic background is derived, making use of heat-kernel and zeta-regularization techniques. It is discussed, at one-loop level, the Coleman-Weinberg type suppression of the cosmological constant, proposed by Taylor and Veneziano.

PACS number(s): 04.60 Quantum theory of gravitation

1 Introduction

From the very early investigations of quantum gravity on the de Sitter background [1, 2, 3, 4, 5], which is a reasonable candidate for the vacuum state in Einstein gravity with positive cosmological term, till quite recent works [6, 7, 8, 9, 10, 11, 12, 13, 14] it has been known that low-energy quantum gravity dynamics (in particular infrared effects) may be very important for the theory of early Universe. It is expected that an effective theory of quantum gravity, describing pre-GUT epoch of the early Universe, may lead to the resolution of some fundamental cosmological problems.

It is well known that one of the challenging problem for every reasonable quantum formulation of classical Einstein gravitational theory is that to explain why the observed cosmological constant value is so small with respect to the naive one, suggested by the quantum physics. With regard to this issue, in Refs. [13, 14], a quite interesting model of Coleman-Weinberg type suppression of the effective cosmological constant has been suggested. This model is based on the properties of large-distance limit of the effective action in 4-dimensional quantum gravity, where the logarithmic term (obtained from the corresponding ultra-violet divergence by making use of a string motivated cut-off) plays an important role. However, as it has been clearly shown by Taylor and Veneziano [14], the analysis of the one-loop quantum gravity with positive cosmological constant (De Sitter background), leads to the conclusion that, in this case, there is no actual cosmological constant suppression mechanism. For this reason, arguments have been given in Refs. [13, 14], suggesting that a theory with negative cosmological constant (i.e. with

1E-mail address: odintsoy@ebubecm1.bitnet
2E-mail address: zerbini@science.unitn.it
hyperbolic manifold as proposed ground state of the theory) may provide a real example of the existence for a cosmological constant suppression mechanism.

It is the purpose of this work to study the one-loop effective action (for a general introduction to effective action formalism, see [23]) for the D-dimensional quantum gravity on hyperbolic manifolds by means of the path-integral approach, implemented by the zeta-function regularization [15, 16]. This is equivalent to say that the ultraviolet divergences are disposed of by using of the zeta-function regularization. With regard to this, we have nothing to add to the motivations contained in Ref. [3], where it is claimed that the one-loop correction of the quantum gravity, which is an non-perturbative renormalizable theory, may provide a reasonable approximation by itself.

However, it is a well-known fact that explicit calculations of the one-loop quantum effects on compact hyperbolic manifolds are rather problematic (see [17] for a review), due to the fact that the eigenvalues of the fluctuation operators on such manifolds are, as a rule, unknown. We recall that every complete connected hyperbolic Riemannian manifold can be regarded as the quotient of $H^N$ ($N > 1$), by a discontinuous group $\Gamma$ of isometries. Here $H^N$ is the hyperbolic N-dimensional space and we will consider only the case $H^N/\Gamma$ compact, namely $\Gamma$ will be co-compact (without parabolic elements). Hence, the standard zeta-function regularization, which has successfully applied to the de Sitter background, should be implemented by using Selberg trace formula techniques [18, 19]. Furthermore, there are some technical problems associated with the extension of Selberg trace formula to transverse vector and tensor fields on $H^4/\Gamma$. In fact, so far, it has been applied to the evaluation of quantum effects of scalar fields on compact hyperbolic manifolds [20, 21]. It is known that a complete N-dimensional manifold of constant curvature is isometric to the coset space $M^N/\Gamma$, with $M^N = R^N$, $S^N$ and $H^N$. The calculations of the one-loop effective action in quantum gravity for the case $M^N = S^N$ have been done in [24, 24, 27, 28] (for a review). There, the quantum spontaneous compactification program has also been discussed in details. In this work, we are only interested in the induced D-dimensional effective cosmological and gravitational constants and we will not discuss the compactification problem.

As we have mentioned, exact one-loop effective action calculations on compact hyperbolic manifolds are difficult. However, one can use an approximation which is the analogous of inverse mass expansion. This is the large-distance limit [14]. In our case, this is equivalent to evaluate the asymptotic limit of the effective off-shell action for $a \to \infty$, $a$ being the radius of the compact hyperbolic manifold. As a result, for strictly hyperbolic elements of $\Gamma$, one is practically dealing with one-loop calculations on $H^4$ (the non-compact and simply connected case), since the topological effects get suppressed and we shall be able to make use of some recent results on this space [22].

The necessity to perform an off-shell calculation leads to the problem of the gauge dependence of the result. In this paper, we shall not pretend to solve this problematic issue. In fact, in the explicit examples, we shall employ the Landau gauge within the standard effective action, which is equivalent, in the case of constant curvature space under consideration, to the use of the gauge-fixing independent effective action (for a review of the Vilkovisky-DeWitt formalism, see, for example, [23]). We think that the actual suppression mechanism, if it exists, should not be caused by a gauge artifact. In particular, we shall show that mechanism of the cosmological constant suppression proposed in Ref. [13] is problematic as in the de Sitter case.

The paper is organised as follows. In the first section we review the general method of evaluation of the one-loop effective action in quantum gravity on the D-dimensional manifold $M^D = M^n \times M^N$, where $M^n$ is the Minkowski space-time and $M^N$ is a space of constant curvature. The zeta-regularization, implemented with heat-kernel techniques, is used in Sec. 2 in order to evaluate the determinants appearing in the one-loop Euclidean effective action. The expressions obtained are valid for a generic constant curvature background. The third section
is devoted to study of the large-distance limit of Einstein gravity on hyperbolic background. In Sec. 4, two 4-dimensional cases are investigated. Some perspectives are given in the concluding remarks. In the two Appendices, technical materials about the Selberg trace formula and the related heat-kernel expansion on compact hyperbolic manifold are reported.

## 2 The one-loop approximation of the Euclidean effective action

Let us consider the gravitational field on the D-dimensional manifold \( M^D = M^n \times M^N, \) \( D = n + N, \) where \( M^n \) is a flat manifold (in a Kaluza-Klein model, it can be identified with the Minkowski space-time) and \( M^N \) is a constant curvature space. We shall consider the Euclidean sector, thus the classical Einstein-Hilbert action, with cosmological constant \( \Lambda \) is

\[
S = -\frac{1}{16\pi G} \int d^Dz \sqrt{g} (R - 2\Lambda),
\]

where \( G \) is the gravitational constant, whose dimension is \((\text{length})^{D-2}, g\) is the determinant of the D-dimensional metric tensor \( g_{AB} \) and \( R \) is the scalar curvature. We use the following conventions: capital latin indices are related to D-dimensional space-time, greek and latin indices are related to space-time \( M^n \) and space \( M^N \) respectively, \( \nabla_A = (\nabla_\alpha, \nabla_a) \) is the covariant derivative, \( R^A_{BCD} = \nabla_C \Gamma^A_{BD} - \nabla_D \Gamma^A_{BC} + ... \) is the Riemann tensor and \( R_{AB} = R^C_{ACB}. \)

With regard to the space \( M^N \), we recall that every maximally symmetric simply connected Riemannian manifold has an isometry group of maximum dimension \( N(N + 1)/2 \) and it is isometric to one of the following constant curvature spaces: Euclidean space \( \mathbb{R}^N \) with constant curvature \( k = 0 \), the sphere \( S^N \) of radius \( a \) \((k = a^{-2})\) and the hyperbolic space \( H^N \) \((k = -a^{-2})\). For all of these spaces, the curvature tensor, the Ricci tensor and the scalar curvature have respectively the form

\[
\begin{align*}
R_{abcd} &= k (g_{ac}g_{bd} - g_{ad}g_{bc}), \\
R_{ab} &= k(N - 1)g_{ab}, \quad R = kN(N - 1).
\end{align*}
\]

In this paper, we shall mainly concentrate on the compact hyperbolic space \( M^N \). In the following, we shall briefly review the Euclidean path-integral quantization (see for example [3]). The generating functional \( W[J, \bar{g}] \) is defined by the standard way, i.e.,

\[
\exp(-W[J, \bar{g}]) = \int [Dh_{AB}] \exp \left[ -\left( S[\bar{g} + \bar{h}] + S_{GF}[\bar{g}, \bar{h}] + \int d^Dz J^{AB}h_{AB} \right) \right] \det(F^{AB}),
\]

\[
S_{GF} = \frac{c}{2} \int d^Dz \chi^A(\bar{g}, \bar{h})\chi_A(\bar{g}, \bar{h}),
\]

where, in accordance with the background-field method, \( g_{AB} = \bar{g}_{AB} + h_{AB} \), \( \chi^A(\bar{g}, \bar{h}) = 0 \) is the background gauge condition, \( c \) is an arbitrary constant and \( F^{AB} \) is the Faddeev-Popov matrix. Making use of the functional Legendre transform of \( W[J, \bar{g}] \), we arrive at the effective action, namely

\[
\Gamma_{eff} = -\log \left\{ \int [Dh_{AB}] \exp \left[ -\left( S[\bar{g} + \bar{h}] + S_{GF}[\bar{g}, \bar{h}] - \frac{\partial \Gamma_{eff}}{\partial \bar{g}_{AB}} h_{AB} \right) \right] \det(F^{AB}) \right\}.
\]

Note that the DeWitt’s gauge-invariant effective action can be obtained setting \( \bar{g}_{AB} = g_{AB} \) in Eq. (4). At one-loop level we have

\[
\Gamma_{eff} = S + \Gamma^{(1)},
\]

where the one-loop contribution to the effective action reads

\[
\Gamma^{(1)} = -\log \left\{ \int [Dh_{AB}] \exp \left[ -\left( \frac{1}{2} \int d^Dz h_{AB}K^{ABCD}h_{CD} \right) \right] \det \left( F^{AB}(h_{AB} = 0) \right) \right\},
\]
where
\[ K^{ABCD} = \frac{\delta^2}{\delta h_{AB} \delta h_{CD}} (S[\bar{g} + h] + S_{GF}[\bar{g}, h]) |_{h_{AB}=0}. \] (7)

Furthermore we shall not indicate explicitly the gauge fixing and the ghost terms. In any case, the well known procedure (see, for example, Refs. [24, 25, 26], where the effective action has been studied for the space \( S^N \)), permits to write down several general formulae at one-loop level. The functional integration over quantum fields in Eq. (6) can be done formally and, in the Euclidean section, the result is
\[ \Gamma^{(1)} = \frac{1}{2} \sum_{p,i} C_p \log \det \left( O_p^{(i)} / \mu^2 \right), \] (8)

where \( \mu^2 \) is a normalization parameter, \( i = 0,1,2 \) refer to scalars, transverse vectors and transverse symmetric traceless second rank tensors respectively and \( C_p \) are the weights associated with the Laplace-type operators \( O_p^{(i)} \). However, it is well known that, in the path-integral formulation of Euclidean quantum gravity, some of these operators are negative. In the following, we shall assume that, when necessary, the contour rotations and field redefinitions, in accordance with the prescriptions of Refs. [1, 10], have been done.

Generally speaking the effective action is divergent and one needs a regularization. In this paper we shall mainly consider \( \Lambda < 0 \) and the determinants of the operators \( O_p^{(i)} \) will be regularized by means of the zeta-function technique \[ \zeta(s|O_p) \]. The fact we are dealing with the case \( \Lambda < 0 \), implies that all \( O_p^{(i)} \) (provided the integration over imaginary field is performed when necessary) are, for a sufficiently large, positive definite. Thus, a generic Laplace-type operator on \( M^D \) can be written as
\[ O_p = -\nabla_n^2 + a^{-2} L_{N,p}. \] (9)

Here and in the following, for the sake of notational simplicity, we are leaving understood the indices \( i \). These will be restored in the final formulae. It should be noted that, in the Eq. (9), the dependence on the radius \( a \) has been factorized out. Thus, one can deal with the dimensionless operator \( L_{N,p} \) of the form
\[ L_{N,p} = L_N + X_p, \quad X_p = a^2 |\Lambda_p| + \nu_p, \] (10)

where \( L_N = -\nabla_N^2 \) is the Laplace-Beltrami operator on \( M^N \), \( \Lambda_p = b_p \Lambda \), \( b_p \) is a non-negative number, \( \nu_p \) are known constants (see Refs. [24, 25, 26]) and we assume \( X_p > 0 \). The zeta-function regularization gives
\[ \log \det \left( O_p / \mu^2 \right) = - \left[ \zeta'(0|O_p) + \log \mu^2 \zeta(0|O_p) \right], \] (11)

where the function \( \zeta(s|O_p) \) is given by
\[ \zeta(s|O_p) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} e^{-tO_p} \]
\[ = \frac{Vol(\mathbb{R}^n) \Gamma(s - \frac{n}{2})}{(4\pi)^{n/2} \Gamma(s)} a^{2s-n} \zeta(s - \frac{n}{2}|L_{N,p}). \] (12)

It may be convenient to study the dimensionless zeta function \( \zeta(z|L_{N,p}) \). The starting point is
\[ \zeta(z|L_{N,p}) = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr} e^{-tX_p} e^{-tL_N} \] (13)
On general grounds, one can write the expansion \([29, 30, 31]\), valid for \(t \to 0\)

\[
\text{Tr} \, e^{-tL_N} \simeq \sum_r K_r t^{|r-N/2|},
\]

where the coefficients \(K_r\) associated with the operator \(L_N\) can be, in principle, computed and depend on the geometry of the compact manifold. The expansion (14) is valid for a compact smooth manifold with or without boundary. We shall consider manifolds without boundary, which a possible presence of conical singularities. For boundaryless manifolds \(K_r = 0\) if \(r\) is odd. Note however, that we shall deal also with non-smooth manifolds (orbifolds) and, in this case, \(K_{2r}\) contain the familiar Seeley-DeWitt coefficients and coefficients coming from the contributions associated with conical singularities (see Appendix B).

If we make use of the general heat-kernel expansion, for \(\text{Re} \, z\) sufficiently large, we have

\[
\zeta(z | L_{N,p}) = \frac{1}{\Gamma(z)} \int_0^\infty dt \, t^{z-1} e^{-tX_p} \text{Tr} \, e^{-tL_N} \frac{d}{dt} = \frac{1}{\Gamma(z)} \int_0^\infty dt \, t^{z-1} e^{-tX_p} \left( \text{Tr} \, e^{-tL_N} - \sum_r K_{2r} t^{r-N/2} + \sum_r K_{2r} t^{r-N/2} \right) \frac{d}{dt} = \frac{1}{\Gamma(z)} \left\{ \sum_r \int_0^\infty dt \, t^{z-1} e^{-tX_p} K_{2r} t^{r-N/2} + J_p(z) \right\}.
\]

Now one may perform an analytical continuation in \(z\). Thus, from Eq. (15), one gets the following series representation

\[
\zeta(z | L_{N,p}) = \frac{1}{\Gamma(z)} \left\{ \sum_r K_{2r} \Gamma(z + r - N/2) X_p^{D/2-s-r} + J_p(z) \right\}.
\]

where \(J_p(z)\) is an entire function of \(z\). This is an example of the well known Seeley meromorphic representation for the zeta function of the elliptic operator \(L_{N,p}\). Using the Eqs. (12)-(14) we get

\[
\zeta(s | O_p) = \frac{\text{Vol}(\mathbb{R}^n) a^{-n}}{(4\pi)^{n/2}} \frac{\Gamma(s + r - D/2) X_p^{D-s-r} + J_p(s - D/2)}{\Gamma(s)}.
\]

It is convenient to deal with \(D\) odd and even \(D\) separately.

Let \(D\) be odd. The gamma functions are regular at \(s = 0\). As a result, \(\zeta(0 | O_p) = 0\) and

\[
\zeta'(0 | O_p) = \frac{\text{Vol}(\mathbb{R}^n) a^{-n}}{(4\pi)^{n/2}} \left\{ \sum_r K_{2r} \Gamma(r - D/2) X_p^{D-s-r} + J_p(-D/2) \right\}.
\]

Therefore, we have

\[
\Gamma^{(1)} = -\frac{\text{Vol}(\mathbb{R}^n) a^{-n}}{2(4\pi)^{n/2}} \sum_{p,i} C_p \left\{ \sum_r K_{2r}^{(i)} \Gamma(r - D/2) X^{(i)} p^{D-s-r} + J_{p,i}(-D/2) \right\}.
\]

Let \(D\) be even, namely \(D = 2Q\). In Eq. (16) the first gamma functions have a simple pole at \(s = 0\). So we have

\[
\zeta(0 | O_p) = \frac{\text{Vol}(\mathbb{R}^n) a^{-n}}{(4\pi)^{n/2}} \sum_{l=0}^{Q} K_{D-2l} (-X_p)^l / l!.
\]
and

$$\zeta'(0|O_p) = \frac{Vol(\mathbb{R}^n)a^{-n}}{(4\pi)^{n/2}} \left\{ \sum_{l=0}^{Q} K_{D-2l} \frac{(-X_p)^l}{l!} \left( \gamma + \Psi(l+1) + \log(a^2/X_p) \right) \right. $$

$$+ \left. \sum_{r>Q} K_{2r} \Gamma(r-Q)X_p^{Q-r} + J_p(-\frac{n}{2}) \right\}. \quad (21)$$

As a result, the one-loop contribution to the effective action reads

$$\Gamma^{(1)} = -\frac{1}{2} \sum_{p,i} C_p \left\{ \log(a^2\mu^2/X_p)\zeta(0|O_p) + \frac{Vol(\mathbb{R}^n)a^{-n}}{(4\pi)^{n/2}} \left[ \sum_{l=0}^{Q} K_{D-2l} \frac{(-X_p)^l}{l!} F(l) \right. \right.$$\n
$$+ \left. \sum_{r>Q} K_{2r} \Gamma(r-Q)(X_p)^{Q-r} + J_p(-\frac{n}{2}) \right\}, \quad (22)$$

where $F(0) = 0$ and $F(l) = \sum_{j=1}^{l} j^{-1}$.

Up to now, our results have a general form. However, in order to perform explicit computations the knowledge of the heat-kernel coefficients $K_{2r}$ and the analytical part $J_p(s)$ are necessary. It is well known that the $K_{2r}$ coefficients are, in principle, computable. On the other hand, the evaluation of the $J_p(s)$ requires an analytical continuation and this is achieved, usually, through the explicit knowledge of the spectrum of the operator $L_{N,p}$. Since the spectral properties of the Laplace operators acting on $M^N = S^N$ are well known, the one-loop corrections to pure quantum gravity in such backgrounds have been extensively studied (see for example [24, 25, 26]). In the case of a hyperbolic background, the situation is different, because the spectrum of the Laplace operator is not explicitly known. To our knowledge, only the scalar sector could be investigated by making use of Selberg trace formula techniques. However, in the framework of the one-loop approach, one may use a further approximation scheme. One of these is the large-distance limit approximation. This is particularly interesting in pure gravity and it has been recently considered in Ref.[14] for spherical 4-dimensional background.

3 The large-distance limit of the one-loop quantum gravity

One of the main reasons for the use of the large-distance limit of quantum gravity is to investigate the possible role of Coleman-Weinberg type mechanism of the cosmological constant suppression [14]. In this section, we shall analyze the asymptotic behaviour of Newton and cosmological constants (hyperbolic background). In our approach, the large distance limit is equivalent to find the asymptotics of the effective action for very large $X_p$ (note that $b_p$ must be non-vanishing and we left understood that, in the sum over $p$, the terms corresponding to some $b_p = 0$, must be omitted). From the general expressions we have found in Sec.2, it follows that this amounts to neglect the terms related to $r > Q$ and the analytical terms.

First, let us consider even-dimensional space, namely $D = 2Q$. Using the Eqs. (20) and (21), one get two leading contributions to the effective action, associated with positive self-adjoint elliptic operators $O_p$

$$\zeta(0|O_p) = \frac{Vol(\mathbb{R}^n)(-1)^Q}{(4\pi)^{Q/2}} \left\{ a^N|\Lambda_p|^Q K_0 + a^{N-2}|\Lambda_p|^{Q-1}Q(\nu_p K_0 - K_2) \right\} + O(a^{N-4}|\Lambda_p|^{Q-2}), \quad (23)$$
\[ \zeta'(0|O_p) = \frac{Vol(\mathbb{R}^n)(-1)^Q}{(4\pi)^n/2 Q!} \left\{ a^N |\Lambda_p|^Q (-\log |\Lambda_p| + F(Q)) K_0 \right\} \]

\[ + \ a^{N-2} |\Lambda|^Q [-Q \log |\Lambda_p| (\nu_p K_0 - K_2)] \]

\[ + \ \nu_p K_0 (Q F(Q) - 1) - Q F(Q - 1) K_2 + O(a^{N-4} |\Lambda_p|^{Q-2}) \right\} . \tag{24} \]

Let us introduce a physical scale by means of the following redefinition of the \( \mu^2 \) parameter

\[ \log \frac{\mu^2}{|\Lambda_p|} + F(Q) \mapsto - \log (|\Lambda| \mu^{-2}) \tag{25} \]

The leading part of \( \Gamma^{(1)} \) is given by

\[ \Gamma^{(1)} = \frac{Vol(\mathbb{R}^n)(-1)^Q \log (|\Lambda| \mu^{-2})}{2(4\pi)^n/2 Q!} \sum_{p,i} C_p \left\{ a^N |\Lambda_p|^Q K_0^{(i)} \right\} \]

\[ + \ a^{N-2} |\Lambda|^Q Q (\nu_p K_0^{(i)} - K_2^{(i)}) \right\} . \tag{26} \]

Now it is quite easy to rewrite the above one-loop corrections in terms of geometric quantities, appearing in the classical action. To this aim, it is sufficient to observe that, for constant curvature space, we have

\[ a^N = \int d^D z \sqrt{g} (Vol(\mathbb{R}^n \times \mathcal{F}_N))^{-1}, \]

\[ a^{N-2} = \int d^D z \sqrt{g} \kappa R (N(N-1)Vol(\mathbb{R}^n \times \mathcal{F}_N))^{-1}, \tag{27} \]

where \( \kappa = k a^2 \), \( \mathcal{F}_N \) is the fundamental domain (see Appendix A). For example, if \( M_N = S_N \) (\( \kappa = 1 \)), then \( V(\mathcal{F}_N) = 2\pi^{N+1}/\Gamma((N + 1)/2). \)

Thus, using the relevant part of \( \Gamma^{(1)} \) and the classical action, one can write the effective action in the form \( (D = 2Q) \)

\[ \Gamma_{eff} = S + \Gamma^{(1)} = \int d^D z \sqrt{g} \left[ \Lambda(8\pi G)^{-1} + \beta_\Lambda |\Lambda|^{D/2} \log (|\Lambda| \mu^{-2}) \right] \]

\[ - \int d^D z \sqrt{g} R \left[ (16\pi G)^{-1} + \beta_G |\Lambda|^{D/2-1} \log (|\Lambda| \mu^{-2}) \right], \tag{28} \]

where, for pure gravity, we have

\[ \beta_\Lambda = \frac{(-1)^{D/2}}{2(4\pi)^n/2(D/2)!V(\mathcal{F}_N)} \sum_{p,i} b_{p}^{D/2} C_p K_0^{(i)}, \tag{29} \]

\[ \beta_G = \frac{(-1)^{D/2-1} \kappa}{2(4\pi)^n/2(D/2 - 1)!N(N-1)V(\mathcal{F}_N)} \sum_{p,i} b_p^{D/2-1} C_p (\nu_p K_0^{(i)} - K_2^{(i)}). \tag{30} \]

In terms of the effective Newton and cosmological constants these results can be rewritten as follows

\[ \Lambda_{eff}(8\pi G_{eff})^{-1} = \Lambda(8\pi G)^{-1} + \beta_\Lambda |\Lambda|^{D/2} \log (|\Lambda| \mu^{-2}), \tag{31} \]

\[ (16\pi G_{eff})^{-1} = (16\pi G)^{-1} + \beta_G |\Lambda|^{D/2-1} \log (|\Lambda| \mu^{-2}). \tag{32} \]

As a result, the one-loop effective constants read
\[ \Lambda_{\text{eff}} = \Lambda \frac{1 + \kappa \beta_\Lambda 8 \pi G |\Lambda|^{D/2-1} \log(|\Lambda| \mu^{-2})}{1 + \beta_G 16 \pi G |\Lambda|^{D/2-1} \log(|\Lambda| \mu^{-2})}, \]  

(33) 

\[ (G \Lambda)_{\text{eff}} = (G \Lambda) \frac{1 + \kappa \beta_\Lambda 8 \pi G |\Lambda|^{D/2-1} \log(|\Lambda| \mu^{-2})}{[1 + \beta_G 16 \pi G |\Lambda|^{D/2-1} \log(|\Lambda| \mu^{-2})]^2}. \]  

(34) 

Let us discuss the asymptotic behaviour of these two effective constants. In the regime of quantum gravity, \(|\Lambda| \mu^{-2} \ll 1\). Thus, we may choose \(\mu^{-2} = \alpha'\). The string theory can provide the proof of the relevance of such choice. The square root of the inverse string tension \(\alpha'\) has been taken as the physical short-distance cut-off parameter in Ref. [14]. So, at least at the one-loop level, such string-inspired regularization is compatible with our zeta-function approach.

For spherical gravitational background (\(\Lambda > 0\)) and for \(D = N = 4\), we note that the coefficient \(\beta_\Lambda\) is positive and the one-loop corrections are kept under control by \(\Lambda\) itself. Such behaviour has been obtained in Refs. [1, 12, 14].

For hyperbolic background (\(\Lambda < 0\)) and \(D = 4M\), \(M \in \mathbb{Z}_+\), it follows that \(\beta_\Lambda\) is positive. The sign of \(\beta_G\) depends on the particular choice of linear bundles over the compact hyperbolic space, associated with inequivalent field configurations. In the large-distance limit the are no contributions to the one-loop effective action related to hyperbolic elements of the discrete group \(\Gamma\) (for detail see Appendices A and B).

In general, however, the group \(\Gamma\) may contain elliptic elements as well. In this case, the sign of \(\beta_G\) depends also on the particular choice of twisted or untwisted sectors of quantum fields of spin 0, 1 and 2. If \(\beta_G < 0\), then \(\Lambda_{\text{eff}}\) has, roughly speaking, the same order of magnitude of \(\Lambda\). On the contrary, if \(\beta_G > 0\), then \(\Lambda_{\text{eff}}\) and \((G \Lambda)_{\text{eff}}\) are increasing functions of \(\Lambda\).

For \(D = 4M - 2\), \(M \in \mathbb{Z}_+\), \(\beta_\Lambda\) is negative. In this case, if \(\beta_G < 0\), then the partial suppression of the Newton and cosmological constants can occur. On the other hand, if \(\beta_G > 0\), \(\Lambda_{\text{eff}}\) has again the same order of magnitude of \(\Lambda\). But of course, under the usual assumptions, \(G \ll \mu^2\) and all the radiative corrections remain bounded by \(G |\Lambda| \log(G |\Lambda|) \ll 1\). Note that if \(|\Lambda| \mu^{-2} \gg 1\), then for \(D = 4M\) the partial suppression holds.

Let us consider the analogous analysis valid for the odd-dimensional spaces. In this case, the conformal anomaly is absent (the \(\log \mu^2\) term is vanishing). Making use of the Eq. (19), a direct calculation gives, in the large-distance limit, the following leading expression for \(\Gamma^{(1)}\).

\[ \Gamma^{(1)} = \frac{-\text{Vol}(\mathbb{R}^n) \Gamma(-D/2)}{2(4\pi)^{n/2}} \sum_{p,i} C_p \left[ a^p |\Lambda_p|^{D/2} K_0^{(i)} \right] + D/2 a^{N-2} |\Lambda_p|^{D/2-1} (\nu_p^{(i)} K_0^{(i)} - K_2^{(i)}) \].  

(35) 

Again the one-loop effective action has the form (28), but now the coefficients \(\beta_\Lambda\) and \(\beta_G\) related to the effective action read

\[ \beta_\Lambda = \frac{(-1)^{(D-2)/2} \pi}{2(4\pi)^{n/2}(D/2)! \text{Vol}(\mathcal{F}_N)} \sum_{p,i} b_p^{(D/2-1/2)} C_p K_0^{(i)}, \]  

(36) 

\[ \beta_G = \frac{(-1)^{(D+2)/2} \kappa \pi}{2(4\pi)^{n/2}(D/2-1)! N(N-1) \text{Vol}(\mathcal{F}_N)} \sum_{p,i} b_p^{(D/2+1/2)} C_p (\nu_p^{(i)} K_0^{(i)} - K_2^{(i)}). \]  

(37)
The effective cosmological and Newton constants can be written as

\[ \Lambda_{\text{eff}} = \Lambda \frac{1 + \kappa \beta \Lambda 8 \pi G |\Lambda|^{D/2-1}}{1 + \beta G 16 \pi G |\Lambda|^{D/2-1}} \],

(38)

\[(GA)_{\text{eff}} = (GA) \frac{1 + \kappa \beta \Lambda 8 \pi G |\Lambda|^{D/2-1}}{[1 + \beta G 16 \pi G |\Lambda|^{D/2-1}]^2} \] (39)

For \( D = 4M + 1, M \in \mathbb{Z}^+\), \( \beta_\lambda \) is positive. Therefore, if \( \beta G < 0 \), then the one-loop correction remain bounded. If \( \beta G > 0 \), then a negligible suppression of the cosmological constant is possible. For \( D = 4M - 1 \) one cannot have a satisfactory suppression of the cosmological constant as well.

4 Explicit results for the 4-dimensional gravity

In this section, we shall illustrate the general results we have obtained for particular, but physically relevant, 4-dimensional case. For \( D = 4 \), we have

\[ \Lambda_{\text{eff}} = \Lambda \frac{1 - \beta \Lambda 8 \pi G |\Lambda| \log(|\Lambda|^{\mu^2})}{1 + \beta G 16 \pi G |\Lambda| \log(|\Lambda|^{\mu^2})} \].

(40)

First, let us consider the Landau gauge [12, 14]. This choice may be motivated by the fact that the standard effective action in the Landau gauge on constant curvature background, coincides with the Vilkovisky-DeWitt effective action, which is off-shell gauge-fixing independent. Generally speaking, the Vilkovisky-DeWitt effective action differs from the standard one by a term which depends on the field space metric affine connection. This correction is also different from zero in higher dimensional flat space [32, 33]. However, on 4-dimensional constant curvature space, the correction vanishes and the Vilkovisky-DeWitt effective action coincides with the standard effective action in the Landau gauge.

From Refs. [12, 14] one has

\[ \Gamma^{(1)} = \frac{1}{2} \left\{ - \log \det \left( -\nabla_0^2 + \frac{3}{a^2} \right) \log \det \left( -\nabla_1^2 + \frac{6}{a^2} \right) \right. \\
+ \left. \log \det \left( -\nabla_2^2 + 2|\Lambda| - \frac{8}{a^2} \right) \log \det \left( -\nabla_0^2 + 2|\Lambda| \right) \right\} \] (41)

in which the 0, 1, 2 are labelling the scalar, transverse vector and traceless transverse tensor respectively and the symbol (*) refers to the related ghost contribution. From the above equation, we obtain

\[ b_0 = b_2 = 2, \quad b_{0*} = b_{1*} = 0 \]
\[ C_0 = C_2 = 1, \quad C_{0*} = C_{1*} = -1 \]
\[ \nu_0 = 0, \quad \nu_2 = -8, \quad \nu_{0*} = 3, \quad \nu_{1*} = 6. \] (42)

We shall deal with two cases: \( M^4 = H^4/\Gamma \), with only strictly hyperbolic elements in \( \Gamma \) and \( M^4 = \mathbb{R} \times H^3/\Gamma \), with hyperbolic and elliptic elements in \( \Gamma \).

If \( M^4 = H^4/\Gamma \) (strictly hyperbolic elements) one may compute the coefficients \( K_0^{(p)} \), making use of the general result contained in Ref. [22]. From \( K_0^{(2)} = 5K_0^{(0)}, K_2^{(0)} = -2K_0^{(0)} \) and \( K_2^{(2)} = 10K_0^{(0)} \), a direct computation gives

\[ \beta_\Lambda = \frac{1}{4V(F_4)} \sum_{p,i} b_p^2 C_p K_0^{(i)} = \frac{6}{V(F_4)} K_0^{(0)} = \frac{6}{(4\pi)^2}, \] (43)
\[ \beta_G = \frac{1}{24V(F_4)} \sum_{p,i} b_p C_p (\nu_p K_0^{(i)} - K_2^{(i)}) = -\frac{4}{V(F_4)} K_0^{(0)} = -\frac{4}{(4\pi)^2}, \] (44)

If \( M^4 = \mathbb{R} \times H^3/\Gamma, N = 3 \) and \( n = 1 \), we get \( K_0^{(2)} = 5K_0^{(0)}, K_2^{(0)} = -K_0^{(0)} + E_3(4\pi)^{-1/2} \) and \( K_2^{(2)} = 10K_0^{(0)} + K_{2,E}^{(2)}. \) Thus, one has
\[
\beta = \frac{6}{\sqrt{4\pi V(F_3)}} K_0^{(0)} = \frac{6}{(4\pi)^2}, \quad (45)
\]
\[
\beta_G = \frac{1}{12\sqrt{4\pi V(F_3)}} \sum_{p,i} b_p C_p (\nu_p K_0^{(i)} - K_2^{(i)})
= -\frac{1}{6\sqrt{4\pi V(F_3)}} \left( 49K_0^{(0)} + K_{2,E}^{(2)} + \frac{E_3}{(4\pi)^{1/2}} \right), \quad (46)
\]
where \( E_3 \) being the elliptic number (see Appendix A) and \( K_{2,E}^{(2)} \) is the elliptic contribution to the transverse traceless tensor coefficient. If there are no elliptic elements in \( \Gamma \), then \( \beta_G = -\frac{49}{6(4\pi)^2}. \)

For the sake of completeness and for illustrative purposes let us consider the one-parameter family of covariant gauges corresponding to \( \gamma = 1 \) and \( \beta \) arbitrary in the notations of Ref. [12]. This includes as a particular case \((\beta = 1)\) the harmonic (De Donder) gauge \([34, 3, 4, 35, 26]\). From the general result of Ref. [12] one has
\[ \Gamma^{(1)}(\beta) = \frac{1}{2} \left\{ -2 \log \left( -\nabla^2 + \frac{3}{a^2} \right) - 2 \log \left( -\nabla^2 + \frac{12}{(3 - \beta)a^2} \right) + \log \left( -\nabla^2 + 2|\Lambda| - \frac{8}{a^2} \right) + \log \left( -\nabla^2 + 2|\Lambda| - \frac{3}{a^2} \right) + \log \left( -\nabla^2 + X_+ + \log \left( -\nabla^2 + X_- \right) \right) \}, \quad (47) \]
where
\[ X_\pm = \frac{B}{2} \pm \frac{1}{2} \left( B^2 - 4C \right)^{1/2} \quad (48) \]
with
\[ B = \frac{k(\beta)}{a^2} - h(\beta)|\Lambda| = -\frac{12(\beta - 1)^2}{a^2(\beta - 3)^2} - \frac{4(5 - \beta^2)}{(\beta - 3)^2}|\Lambda| \]
\[ C = c(\beta)|\Lambda|^2 = \frac{16}{(\beta - 3)^2}|\Lambda|^2. \quad (49) \]
We have to assume \( \beta < 3 \) in order to deal with non negative elliptic operator, otherwise the large distance approximation becomes problematic.

From the above equation, we obtain, in the large distance limit
\[ b_0^\pm = \frac{h(\beta)}{2} \pm \frac{1}{2} \left( h(\beta)^2 - 4c(\beta) \right)^{1/2} \quad b_1 = b_2 = 2, \quad b_{0*} = b_{1*} = 0, \]
\[ C_1 = C_2 = 1, \quad C_0^\pm = 1, \quad C_{0*} = C_{1*} = -2, \quad (50) \]
\[ \nu_0^\pm = \frac{k(\beta)}{2} \pm \frac{k(\beta)h(\beta)}{2(h(\beta)^2 - 4c(\beta))^{1/2}} \quad \nu_1 = -3, \quad \nu_2 = -8, \quad \nu_{0*} = \frac{12}{(3 - \beta)}, \quad \nu_{1*} = 3. \]

If we consider, for example, \( M^4 = H^4/\Gamma \), with a strictly hyperbolic subgroup \( \Gamma \), we have
\[ \beta_\Lambda(\beta) = \frac{32 + h(\beta)^2 - 2c(\beta)}{4(4\pi)^2} \quad (51) \]
and
\[ \beta_G(\beta) = \frac{h(\beta) \left[ 1 + 2k(\beta) \right] - 56}{12(4\pi)^2}. \]  
(52)

In particular, for \( \beta = 1 \) (harmonic gauge) we get
\[ \beta_\Lambda = \frac{10}{(4\pi)^2} \]  
(53)

and
\[ \beta_G = -\frac{13}{3(4\pi)^2}. \]  
(54)

We conclude this section with few remarks. First, the value of \( \beta_\Lambda \) is in agreement with the one obtained in Ref. [14] within the Landau gauge. The value of \( \beta_G \) depends on the choice of the manifold \( M^4 \). Furthermore its sign depends on the gauge parameter \( \beta \), limiting ourselves to strictly hyperbolic subgroups. In the harmonic gauge \( (\beta = 1) \) it is negative (opposite to the one of the corresponding quantity in a spherical background) and it remains negative for \( \beta < 1.79 \), which is within the admissible range. (Note that generally speaking this gauge parameter should not be large as this may contradict the theory of perturbations.)

The value computed in the harmonic gauge is different from the one computed in the Landau gauge, but the sign is the same. Then, we may take the viewpoint of refs. [32, 33, ?] and consider the gauge-fixing independent Vilkovisky-De Witt effective action as true off-shell effective action. In our language that also means that Landau gauge is the physical gauge, and correct physical results are obtained in this gauge. Notice also that in accordance to our previous general considerations, we note that the sign of \( \beta_G \) could depend also on the concrete choice of field configurations (twisted or untwisted fields) on the orbifold \( H^3/\Gamma \).

5 Concluding remarks

In this paper, we have discussed the one-loop effective action for D-dimensional gravity with negative cosmological constant on a hyperbolic background, by making use of zeta-function regularization and heat-kernel techniques. We have been working in the one-loop approximation and our general results are sensitive to the dimension \( D \). One of the motivations of this paper was to extend the analysis of Ref. [14] to the negative cosmological constant case. The use of large-distant limit approximation has permitted to obtain reasonable simple expressions for the effective one-loop cosmological and gravitational constants. These expressions, in the large-distance limit, depend on the heat-kernel coefficients. A novel feature, with respect to the spherical \( (\Lambda > 0) \) background, consists in the richer geometric structure one has to deal with. As a consequence, the value of the coefficient \( \beta_G \) may depends on the choice of the topological non-trivial field configurations.

We recall that the original Taylor-Veneziano suppression mechanism was based on the following bootstrap condition
\[ \Lambda_{eff} = \Lambda \left[ 1 + \beta_{TV}16\pi G|\Lambda| \log(|\Lambda_{eff}|\mu^{-2}) \right]. \]  
(55)

with \( \beta_{TV} < 0 \). Apart the argument of the logarithm, it is crucial the sign of the coefficient \( \beta_G \), which should correspond to \( \beta_{TV} \) of the toy model. In the pure gravity case and for \( \Lambda > 0 \), the sign is opposite to the right one. Thus, no significant cosmological constant suppression mechanism seems to exist, at least in the one-loop approximation, for pure quantum gravity at large-distance [14]. In the hyperbolic case, in the examples we have considered, the sign of \( \beta_G \) is negative within the Landau gauge, and we have shown the existence of a class of one-parametr
family of covariant gauges (including the harmonic one) for which the sign remains negative when the gauge parameter is less than a critical value \( \beta_c \approx 1.79 \).

Furthermore, it should be noted that the one-loop Coleman-Weinberg type suppression mechanism we have discussed here, depends strongly on the Seeley-DeWitt coefficients \( K_0 \). Since in a quantum gravitational theory with boson and fermion degrees of freedom, the contributions to \( K_0 \) occur with the opposite sign, it follows that, if the number of boson and fermion degree of freedom is the same, the coefficient \( \beta_Λ \) may vanish, due to the cancellation between fermion and boson determinants. Moreover, at one-loop level, due to presence of inequivalent field configurations, the coefficients \( K_2 \) and \( \beta_G \) might be different from zero. As a consequence choosing, for example, \( \mu^2 = |Λ| \exp (1/(G|Λ|)^b) \), \( b > 1 \), one might obtain a satisfactory cosmological constant suppression.

Finally we would like briefly to comment on the issue related to the gauge dependence. In the explicit 4-dimensional example presented in Sec.3, for illustrative purposes, we have made use of the Landau gauge and a class of one-parameter family of covariant gauges, the use of the latter Landau gauge (which we consider as the physical gauge) being justified by the fact that it reproduces the gauge-fixing independent effective action. Since the large distance limit has to be taken off-shell, this choice seems to be important.

Furthermore, many other issues concerning one-loop quantum gravity on hyperbolic background are left for further investigations (in particular, the structure and properties of graviton propagator on such background, infrared properties of quantum gravity, etc). We hope to return to these questions elsewhere.

Acknowledgments

We thank G. Cognola and L. Vanzo for discussions. A.A. Bytsenko wishes to thank INFN and the Department of Physics of Trento University for financial support and kind hospitality. S.D. Odintsov thanks MEC(Spain) and CIRIT(Generalitat de Catalunya) for financial support.

A The Selberg Trace Formula for Compact \( H^N/\Gamma \)

Here we consider an example of the Selberg trace formula valid for the \( N \)-dimensional case. For the sake of simplicity we shall limit ourselves to strictly hyperbolic subgroup of isometries of \( \Gamma \) (torsion-free subgroup of isometries). In this case \( H^N/\Gamma \) is a smooth manifold, the Laplace operator has a pure discrete spectrum, with isolated eigenvalues \( \lambda_j \), \( j = 0, 1, \ldots \) of finite multiplicity. We shall assume that the sectional curvature to be \( -1 \), therefore all the quantities will be dimensionless.

If \( h(r) \) is even and holomorphic in a strip of width greater than \( N - 1 \) about the real axis, and if \( h(r) = O(r^{-(N+\varepsilon)}) \) uniformly in this strip as \( r \to \infty \), then the Selberg trace formula holds

\[
\sum_{j=0}^{\infty} h(r_j) = \frac{V(\mathcal{F}_N)}{2} \int_{-\infty}^{\infty} h(r) \Phi_N(r) dr + \sum_{\{\psi\}} \sum_{n=1}^{\infty} \frac{\chi(P(\gamma)^n)}{S_N(n;l_\gamma)} \hat{h}(nl_\gamma),
\]

with absolute convergence on both sides. Here \( V(\mathcal{F}_N) \) is the volume of the fundamental domain \( \mathcal{F}_N \), relative to the invariant Riemannian measure, \( \hat{h} \) is the Fourier transform of \( h \), \( \gamma \in \Gamma \) is an element of the conjugacy class associated with the length of the closed geodesic \( l_\gamma \), \( \{\varphi\} \) is a set of primitive closed geodesics on the compact manifold and each \( \gamma \in \varphi \) determines the holonomy element \( P(\gamma) \) by parallel translation around \( \gamma \) and \( \chi \) is an arbitrary finite-dimensional representation of \( \Gamma \) (character of \( \Gamma \)), namely one has the homomorphism \( \chi : \Gamma \to S^1 \). Furthermore
\[ r_j^2 = \lambda_j - \rho_N, \] with \( \rho_N = (N - 1)/2, \) the sum over \( j \) include the eigenvalues and \( S_N(n; \gamma) \) is a known function of conjugacy class (see [36] for details). The density of state \( \Phi_N(r) \) is related to the Harish-Chandra function and it is given by

\[ \Phi_N(r) = \frac{\pi \frac{N}{2}}{2^{N-1} \Gamma\left(\frac{N}{2}\right)} \left| \Gamma\left(\frac{ir + \rho_N}{2}\right) \right|^2. \quad (57) \]

The function \( \Phi_N(r) \) satisfies the recurrences relation

\[ \Phi_{N+2}(r) = \frac{\rho^2_N + r^2}{2\pi N} \Phi_N(r), \quad (58) \]

in particular we have

\[ \Phi_2(r) = \frac{r}{2\pi} \tanh \pi r, \quad \Phi_3(r) = \frac{r^2}{2\pi^2}. \quad (59) \]

The above recurrence relation permits to obtain any \( \Phi_N \) starting from \( \Phi_2 \) and \( \Phi_3 \) according to whether \( N \) is even or odd. As a result, for odd dimensions we have

\[ \Phi_{2M+1}(r) = \pi^{-(M+\frac{1}{2})} \frac{M}{2^M \Gamma(M+\frac{1}{2})} \sum_{j=1}^{M} c_j r^{2j}, \quad N = 2M + 1, \quad (60) \]

while for even dimensions

\[ \Phi_{2M}(r) = \pi^{-M} r \tanh(\pi r) \sum_{j=0}^{M-1} a_j r^{2j}, \quad N = 2M, \quad (61) \]

where \( M \in \mathbb{Z}_+ \) and the set of constants \( c_j \) and \( a_j \) are defined through

\[ \sum_{j=1}^{M} c_j r^{2j} = \prod_{j=0}^{M-1} \left( r^2 + j^2 \right), \quad (62) \]

\[ \sum_{j=0}^{M-1} a_j r^{2j} = \prod_{j=0}^{M-2} \left( r^2 + \frac{(2j + 1)^2}{4} \right), \quad (M > 1). \quad (63) \]

In the following, we review two cases where the manifold is not smooth.

We start with the two dimensional case. The Lobachevsky plane \( H^2 \) can be realized as the upper half-plane in the complex plane \( C \). The Poincaré metric being \( d\bar{z}dzy^{-2}, z = x + iy \). The group of all motions without reflection of the upper half-plane \( H^2 \) coincides with the group \( PSL(2, R) = SL(2, R)/\{1, -1\} \), where 1 is the unity element of \( SL(2, R) \) (the Lobachevsky plane can be realized also as a homogeneous space \( SL(2, R)/SO(2) \) of the group \( SL(2, R) \) by its maximum compact subgroup \( SO(2) \)). The measure of the fundamental domain can be computed in terms of signature \((g, m_1, ..., m_l, h)\)

\[ V(F_2) = 2\pi \left[ 2g - 2 + \sum_{j=1}^{l} \left( 1 - \frac{1}{m_j} \right) + h \right], \quad (64) \]

where \( g \) is the genus and the numbers \( m_j \) and \( h \) are associated with elliptic and parabolic generators respectively. If we allow \( \Gamma \) to contain elliptic, but not parabolic elements, the orbifold \( H^2/\Gamma \) will be compact, but the Riemannian metric will be singular at the fixed points of the
elliptic elements. Now $S_2(n; l) = 2 \sinh(n l / 2)$ and the Selberg trace formula for scalar fields reads [18, 19]

$$
\sum_{j=0}^{\infty} h(r_j) = \frac{V(F_2)}{4\pi} \int_{-\infty}^{\infty} h(r) r \tanh(\pi r) dr + \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{\chi^n(\gamma) l}{2 \sinh(n l / 2)} h(n l) + \int_{-\infty}^{\infty} h(r) E_2(r) dr,
$$

(65)

where

$$
E_2(r) = \sum_{\{\alpha\}} \frac{1}{2 \sinh(\frac{n l}{2})} \frac{e^{-2\pi n}}{1 + e^{-2\pi r}}.
$$

(66)

The sums in the right-hand side are taken over all primitive hyperbolic $\gamma$ and elliptic $\alpha$ conjugacy classes in $\Gamma$ respectively and each number $m_\alpha$ is the order of the class with the representative $\alpha$. For strictly hyperbolic subgroup $\Gamma$, the signature of $\Gamma$ contains only hyperbolic numbers and the third term in the right-hand side of Eq. (65) is absent.

In the three dimensional case, the hyperbolic space can be realized as the upper half-space in $\mathbb{R}^3$, that is $H^3 = \{ P = (z, y) | z = x^2 + i x^2 \in \mathcal{C}, y \in (0, \infty) \}$, with the hyperbolic metric $dl^2 = (dz + i dy)^2 / y^2$. The group of isometries of $H^3$ is $SL(2, \mathcal{C}) = SL(2, \mathcal{C}) \backslash \{ -1, 1 \}$, where 1 is the unity element of $SL(2, \mathcal{C})$. It is known that all elements of $PSL(2, \mathcal{C})$ belong to one of the following conjugacy classes: elliptic, loxodromic, hyperbolic and parabolic. Since $\Gamma$ is containing elliptic elements, the space $H^3/\Gamma$ is called the associated orbifold. It is known that $H^3/\Gamma$ is always a manifold, but the Riemannian metric is singular along the axis of rotations of the elliptic elements. When the manifold $H^3/\Gamma$ is compact, which is our assumption, the discrete subgroup $\Gamma \in PSL(2, \mathcal{C})$ is co-compact and it does not contain parabolic elements. The Selberg trace formula for scalar fields [21, 37] is

$$
\sum_{j=0}^{\infty} h(r_j) = \frac{V(F_3)}{4\pi^2} \int_{-\infty}^{\infty} r^2 h(r) dr + \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{\chi^n(\gamma)}{S_3(n; l)} h(n l) + \frac{E_3}{\pi} \int_{0}^{\infty} h(r) dr,
$$

(67)

where the real number $E_3$ is called the elliptic number of $\Gamma$. For the strictly hyperbolic subgroup $m_j, h = 0$ and the third term in the right-hand side of Eq. (67) is absent.

**B** The heat-kernel expansion related to $H^N/\Gamma$

We recall that if $L_N$ be an hermitian non negative elliptic differential operator on $M^N$, the kernel of $L_N^{-h}$ can be expressed by means of an integral transform of the heat-kernel $K(t; x, x') = \exp \left( -t L_N \right)(x, x')$ and for the Laplace operator $L_N$, the asymptotic expansion of the heat kernel for small $t$, valid for a $N$-dimensional smooth manifold has the form (14).

In 2-dimensional case, choosing $h(r) = \exp \left( -(r^2 + 1/4) \right)$ and using Eq. (65), we get

$$
\text{Tr} e^{-tL_2} = \int_{-\infty}^{\infty} e^{-(r^2+1/4)t} \left[ \frac{V(F_2)r \tanh(\pi r)}{4\pi} + E_2(r) \right] dr + \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{\chi(\gamma) n l}{2 \sinh(n l / 2)} \frac{1}{\sqrt{4\pi t}} \exp \left[ -\left( \frac{t}{4} + \frac{n^2 l^2}{4t} \right) \right].
$$

(68)
We see that the presence of the elliptic elements modifies the asymptotic expansion of heat-kernel for small $t$, valid for the smooth manifolds. In fact for small $t$, we have

$$\text{Tr} e^{-tL_2} \simeq \frac{V(F_2)}{4\pi} \left( \frac{1}{t} - \frac{1}{3} \right) + E_2 + O(t),$$

where the elliptic contribution reads

$$E_2 = \sum_{\{\alpha\}} \sum_{n=1}^{m_{\alpha} - 1} \frac{\chi^n(\alpha)}{2m_{\alpha}} \sin \frac{\pi n}{m_{\alpha}} \pi \csc \left( \frac{2\pi n}{m_{\alpha}} \right).$$

Now let us focus on $H^3/\Gamma$. We shall consider a co-compact group $\Gamma$, taking into account hyperbolic (loxodromic) and elliptic elements. In 3-dimensional case, we choose $h(r) = \exp \left[ -t(r^2 + 1) \right]$. Then Eq. (67) gives

$$\text{Tr} e^{-tL_3} = \frac{V(F_3)e^{-t}}{(4\pi t)^{3/2}} + E_3 \frac{e^{-t}}{(4\pi t)^{1/2}} + \sum_{\gamma} \sum_{n=1}^{\infty} \chi^n(\gamma) \frac{1}{\sqrt{4\pi t}} \exp \left[ - \left( t + \frac{t^2}{4\pi} \right) \right].$$

Again the presence of the elliptic element modifies the asymptotic expansion for small $t$, namely

$$\text{Tr} e^{-tL_3} \simeq \frac{V(F_3)}{(4\pi)^{3/2}} t^{-3/2} + \left( \frac{V(F_3)}{(4\pi)^{1/2}} + \frac{E_3}{(4\pi)^{1/2}} \right) t^{-1/2}$$

$$+ \left( \frac{V(F_3)}{2(4\pi)^{3/2}} - \frac{E_3}{(4\pi)^{1/2}} \right) t^{1/2} + O(t^{3/2}).$$

Similar explicit results can be obtained for the strictly hyperbolic subgroup of $\Gamma$, associated with the $N$-dimensional manifold $H^N/\Gamma$, see for example [20, 21].

Finally we would like to point out that the general form of the integrated heat-kernel coefficients $K^{(1,2)}_0$ and $K^{(1,2)}_2$ for transverse vector and transverse traceless tensor fields on $H^N/\Gamma$, which we are interested in, can be computed with the help of a general algorithm, thus these coefficients are related to $K^{(0)}_0$ and $K^{(0)}_2$, related to scalar fields and discussed in this Appendix.

References

[1] G.W. Gibbons, S.W. Hawking and M.J. Perry. Nucl. Phys., B138, 141, (1978).
[2] G.W. Gibbons and M.J. Perry. Nucl. Phys., B146, 90, (1978).
[3] S.W. Hawking. The path-integral approach to quantum gravity. In General Relativity. An Einstein Centenary Survey, S.W. Hawking and W. Israel, editors. Cambridge University Press, Cambridge, (1979).
[4] S.M. Christensen and M.J. Duff. Nucl. Phys., B170, 480, (1980).
[5] S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Rocek. Phys. Rev. Lett., 45, 161, (1980).
[6] I. Antoniadis, J. Iliopulos and T.N. Tomaras. Phys. Rev. Lett., 56, 1319, (1986).
[7] T.R. Taylor and G. Veneziano. Phys. Lett., B212, 147, (1988); B213, 450 (1988).
[8] L. Ford. Phys. Rev., D31, 710 (1985).

[9] B. Allen and M. Turyn. Nucl. Phys., B292, 813, (1987).

[10] J. Polchinski. Phys. Lett., B219, 251, (1989).

[11] I. Antoniadis and E. Mottola. J. Math. Phys., 32, 1037, (1991).

[12] E.S. Fradkin and A.A. Tseytlin. Nucl. Phys., B234, 472, (1984).

[13] T.R. Taylor and G. Veneziano. Phys. Lett., B228, 311, (1989).

[14] T.R. Taylor and G. Veneziano. Nucl. Phys., B345, 210, (1990).

[15] S.W. Hawking. Commun. Math. Phys., 55, 133, (1977).

[16] J. Dowker and R. Critchley. Phys. Rev. D13, 3224, (1976).

[17] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini. Zeta-regularization with applications. World Sci., Singapore, 1994.

[18] D.A. Hejhal. The Selberg Trace Formula for PSL(2,R). Springer-Verlag, Berlin, (1976).

[19] A.B. Venkov. Spectral theory of automorphic functions and its applications. Kluwer Academic Publishers, Dordrecht, The Netherlands, (1990). Mathematics and Its Applications (Soviet Series) vol. 51.

[20] A.A. Bytsenko and S. Zerbini. Class. Quantum Grav., 9, 1365, (1992).

[21] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini. Quantum fields and extended objects in space-times with constant curvature spatial section. Preprint Trento University, UTF 325, (1994). Submitted to Phys. Rep.

[22] R. Camporesi and A. Higuchi. Phys. Rev., D47, 3339, (1993).

[23] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro. Effective action in quantum gravity. IOP Publishing, Bristol and Philadelphia, (1992).

[24] A. Chodos and E. Myers. Ann. Phys., 156, 412, (1984).

[25] C. R. Ordoñez and M. A. Rubin. Nucl. Phys., B260, 456, (1985).

[26] M.H. Sarmadi. Nucl. Phys., B263, 187, (1986).

[27] E. Myers. Phys. Rev., D33, 1663, (1986).

[28] I.L. Buchbinder, E.N. Kirillova and S.D. Odintsov. Mod. Phys. Lett., A1, 633, (1989).

[29] S. Minakshisundaram and A. Pleijel. Can. J. Math., 1, 242, (1949).

[30] B.S. DeWitt. The Dynamical Theory of Groups and Fields. Gordon and Breach, New York, (1965).

[31] P. Greiner. Arch. Rat. Mech. and Anal., 41, 163, (1971).

[32] I.L. Buchbinder, P.M. Lavrov and S.D. Odintsov. Nucl. Phys., B308, 191, (1988).

[33] S.R. Huggins, G. Kunstatter, H.P. Leivo and D.J. Toms. Nucl. Phys., B301, 627, (1988).
[34] G.’t Hooft and M. Veltman. Ann. Inst. Henri Poincaré, A20, 69, (1975).

[35] E.S. Fradkin and A.A. Tseytlin. Nucl. Phys., B227, 252, (1983).

[36] I. Chavel. *Eigenvalues in Riemannian Geometry*. Academic Press, New York, (1984).

[37] A.A. Bytsenko, G. Cognola and L. Vanzo J. Math. Phys., 33, 3108, (1992) and erratum J. Math. Phys., 34, 1614, (1993).