Spectral Density Scaling of Fluctuating Interfaces

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Covariance matrices of heights measured relative to the average height of growing self-affine surfaces in the steady state are investigated in the framework of random matrix theory. We show that the spectral density of the covariance matrix scales as $\rho(\lambda) \sim \lambda^{-\nu}$ deviating from the prediction of random matrix theory and has a scaling form, $\rho(\lambda, L) = \lambda^{-\nu} f(\lambda/L^z)$ for the lateral system size $L$, where the scaling function $f(x)$ approaches a constant for $\lambda \ll L^z$ and zero for $L^z \ll \lambda < \lambda_{\max}$. The values of exponents obtained by numerical simulations are $\nu \approx 1.67$ and $\phi \approx 1.53$ for the Edward-Wilkinson class and $\nu \approx 1.59$ and $\phi \approx 1.75$ for the Kardar-Parisi-Zhang class, respectively. The distribution of the largest eigenvalues follows a scaling form as $\rho(\lambda_{\max}, L) = 1/L^3 f_{\max}((\lambda_{\max} - \lambda)/L^f)$, which is different from the Tracy-Widom distribution of random matrix theory while the exponents $a$ and $b$ are given by the same values for the two different classes.

Over recent decades, growth phenomena of fluctuating interfaces persists as a fascinating subject of statistical physics. Fluctuating interfaces are among the most well studied non-equilibrium systems due to their simplicity as well as ubiquity in nature and fundamental science [1]. Growth of interfaces governed by local rules typically lead to the formation of self-affine surfaces with universal scaling exponents for the surface width $W(L, t)$ which is defined as the standard deviation of interface height over a system size $L$. The surface width characterizes the roughness of the interface and follows a scaling behavior $W(L, t) = L^z f(t/L^z)$, where the scaling function $f(x)$ approaches a constant for $x \gg 1$, and $f(x) \sim x^\beta$ for $x \ll 1$ with the dynamic exponent $z = \alpha/\beta$ [2]. The exponents $\alpha$, $\beta$, and $\nu$ are related to the growth exponent, respectively, which determine the universality classes of various fluctuating interfaces. The well-known universality class of the growing interfaces is the Kardar-Parisi-Zhang (KPZ) one which is predicted by the nonlinear Langevin equation [3] due to the slope dependent growth with the values of exponents $\alpha = 1/2$ and $\beta = 1/3$ for one dimension and was widely confirmed in numerical models [1]. While lacking of nonlinearity in the growth process results in another universality class called the Edward-Wilkinson (EW) class where the values of exponents are given by $\alpha = 1/2$ and $\beta = 1/4$ for one dimension [4].

The recent studies of fluctuating interfaces have dealt with other important characteristics beyond those for the scaling properties of the surface width. These include the distribution of the surface width [2], and maximal and minimal height distributions [6, 7], etc. Especially, the asymptotic distribution of KPZ height fluctuations for curved initial conditions has been computed exactly for solvable models [8] and found that it follows the Tracy-Widom distribution (TWD) [9] of a Gaussian unitary ensemble, which has been confirmed by a recent experiment on the electro-convection [10] and the simulations [11]. While for the flat initial condition, it is confirmed analytically [12] and numerically [13] that the KPZ height distribution exhibits the TWD of a Gaussian orthogonal ensemble. Although TWD has been first obtained in the statistics of the largest eigenvalues of random matrices belonging to the Gaussian ensembles [9], it has been applied to other areas [14] and its application to the growth phenomena has given the connection between the growth problem and the random matrix theory (RMT) [15].

RMT was initially proposed to explain the statistical properties of nuclear spectra [15] but the usefulness of it in understanding the statistical properties of a complex system makes RMT be applied in the various systems [16, 17]. The statistical properties of matrices with independent random elements have been well described by the RMT and one can understand the statistical properties of a system by comparing the spectral statistics of the system with the results of RMT. The empirical correlation matrices appearing in the study of various complex systems such as the price fluctuations in the stock market [18], electro encephalogram (EEG) data of brain [19], variation of various atmospheric parameters [20], biophysical issues [21] and complex network [22], have been analyzed in the framework of RMT.

The random Wishart matrix that is one of the standard tools in the RMT is defined via the product $W = \frac{1}{N}X X^\dagger$ of an $M \times N$ random matrix $X$ having its elements drawn independently from a Gaussian distribution [23]. If $X$ is a matrix whose elements represent some empirical data, then the Wishart matrix represents a empirical covariance matrix of the data, and the non-diagonal elements $W_{ij}$ of the covariance matrix have a direct interpretation.

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as cross-correlation coefficients between data $X_i$ and $X_j$. Therefore if a certain complex system shows the spectral statistics same as those of the Wishart matrix, one may think there are not significant correlations, while if it shows the different properties from that, it could be regarded as there are some correlations. However, the random and the correlated properties are too entangled in a real complex systems to be simple elucidating the properties of correlations from the RMT analysis of corresponding empirical covariance matrices. It informs just whether it is totally random or not. If the RMT is applied to the problem of fluctuating interfaces in which the strong correlation of the variables of heights has been well understood would provide important insights into the universality of RMT for such a correlated system as well as novel criteria for statistical properties of a fluctuating interface.

In this perspective, we would investigate further statistical properties of fluctuating interfaces under the framework of the RMT in this study. We construct the Wishart matrices by the product of the matrices having their elements of relative heights obtained from two models which belong to the KPZ and the EW universality classes, respectively. The distributions of eigenvalues and the largest eigenvalues for each Wishart matrix are measured respectively. The distributions of eigenvalues and the relative heights obtained from two models which matrices by the product of the matrices having their elements of relative heights obtained from two models which belong to the KPZ and the EW universality classes, respectively. The distributions of eigenvalues and the largest eigenvalues for each Wishart matrix are measured and compared with the results of the RMT. The obtained results have shown the totally different properties of those of RMT and we have found new scaling features of the distributions.

Once the actual height $h_i(t)$ at the site $i$ and the time $t$ is generated, we define the relative height, $H_{it} = h_i(t) - \langle h_i(t) \rangle$, where the spatially averaged height $\langle h_i(t) \rangle$ keeps on growing with time and by subtracting it, the relative height has the zero mean and the distribution of relative heights reaches a stationary state in the late-time regime in a finite system. We consider the $L \times T$ height matrix $H$ with elements $H_{it}$, where $L$ is the lateral system size and $T$ is the time interval we considered. We then compute the product symmetric matrix $C = \frac{1}{T} HH^\dagger$ with elements

$$C_{ij} = \frac{1}{T} \sum_{t=1}^{T} H_{it} H_{jt}, \quad (1)$$

which represents the covariance matrix of heights and contains informations about height correlations between two sites.

In the random matrix studies of eigenvalue spectra, the most popular property is the spectral density or the distribution of eigenvalues $\rho(\lambda)$. For the Wishart random matrix [23], it was shown analytically that the spectral density $\rho(\lambda)$ is given by the Marcenko-Pastur(MP) law in the limit for $M, N \to \infty$ [24]:

$$\rho_{MP}(\lambda) = \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi \sigma^2 m \lambda} \quad (2)$$

$$\lambda_{\pm} = \sigma^2 (1 \pm \sqrt{m})^2 \quad (3)$$

where $\sigma$ is a standard deviation for elements of a $M \times N$ random matrix and $m = M/N$. The spectral density $\rho_{MP}(\lambda)$ shows the important features vanishing at both edges of the MP sea and exhibiting a sharp maximum near minimum edge. For finite $M$ and $N$, the abrupt cut-off of $\rho(\lambda)$ is replaced by a rapidly-decaying edge [23].

To compare the spectral properties of the covariance matrix of heights with those of a random Wishart matrix, we have begun with the heights profiles generated by the EW model [4] which belongs to the EW universality class. We have obtained the numerical data for the one dimensional substrate with the system size $L = 128, 256, 512, 1024$. Here we have only focused on the properties in the late-time regime in which the spacial correlation of heights is dominant and the correlation is independent on the time. Thus we counted the time interval $T$ from the saturation time to the final time.

The inset of Fig. 1 (a) shows the spectral density $\rho(\lambda, L)$ of the covariance matrix of heights constructed by the EW model for $L = 1024$. The plot is the spectral density $\rho(\lambda, L)$ except for the largest eigenvalues for various $L$. (b) The distributions of the elements of the height matrices $P(H_{it}, L)$ for the various $L$. The solid line represents the Gaussian curve. (c) The height correlations $C(r, L)$ for various $L$. The solid line represents the curve of Eq. (4) (d) The plot shows the data collapse of $\rho(\lambda, L)$ and the plot of $\langle \lambda^2 \rangle / \langle \lambda \rangle$ versus $L$ is shown in the inset.
from the spectral density of the empirical covariance matrices most which have just appeared the distortion of the shapes of the bulk of spectral density [23, 24] or abnormal largest eigenvalues [13]. The power-law behavior of the spectral density has been observed empirically in some systems such as the EEG data without the value of the exponent exactly measured [19]. Analytical argument for it was proposed with the Lévy matrices [28, 29] where matrix elements are distributed according to $P(X_{ij})$ with $P(X_{ij}) \sim |X_{ij}|^{-(\mu + 1)}$. For $\mu > 2$ the distribution has finite variance while the variance diverges for $0 < \mu \leq 2$.

In the case of the Wishart Lévi matrices, the distribution of eigenvalues has fat tails unlike the prediction of the MP law [30]. In Ref. [31], the multivariate Student distribution has been used as the power-like distribution and it has been obtained that the spectral density $\rho(\lambda)$ of the Wishart Lévi matrices decays like $\lambda^{-(\mu/2+1)}$.

To check whether our spectral density follows the scheme of Lévi matrix, we have measured the distributions of the elements of the height matrix $P(H_{it}, L)$ for the various $L$. As shown in Fig. 1(b), the distributions $P(H_{it}, L)$ has a scaling form, $P(H_{it}, L) = 1/L^\alpha g(H_{it}/L^\alpha)$ and fall on a single curve, where $g(x)$ is the Gaussian curve and $\alpha = 0.5$. It results from that the only relevant scale is the roughness $W$ being consistent with the scaling description in the late-time regime [3, 32]. Thus $P(H_{it})$ has no power-law tails, which means that the origin of the power-law behavior of $\rho(\lambda)$ is not in the fat tail of the distribution of matrix elements. And it shows that the MP law might be not valid any more even when the distribution of matrix obeys the Gaussian distribution.

The elements of a Lévi matrix are uncorrelated random variables, while in the late-time regime, the elements of the height matrix, i.e., the relative heights at different sites are strongly correlated as a following equation [32]

$$C(r, L) = \langle H_{it}H_{(i+r)t}\rangle \sim L \left[1 - \frac{6r}{L}(1 - \frac{r}{L})\right]. \quad (4)$$

Figure 1(c) shows the height correlations $C(r, L)$ measured by averaging the elements of the covariance matrix $C_{ij}$, having $r = |i - j|$ which are excellent agreement with Eq. (1) for various system size $L$. It indicates that the covariance matrix of heights has elements decreasing away from the main diagonal like Eq. (1). The long-ranged correlation of the elements might give rise to the power-law behavior of eigenvalue density $\rho(\lambda)$. It is comparable to the power-law random banded matrix (PRBM) model [33] which is defined as the ensemble of matrices with elements

$$M_{ij} = G_{ij} a(|i - j|), \quad (5)$$

where the matrix $G$ runs over the GOE and $a(r) \sim r^{-\alpha}$ for large $r$. This exhibits an Anderson localization transitions at $\alpha = 1$ and allowed a detailed study of the wave function and energy-level statistics at criticality. Although the PRBM model represents ensembles of matrices with long-ranged off-diagonal disorder, its spectral properties are different from those of the covariance matrix of heights and thus it may serve as another new critical random-matrix ensemble with long-ranged off-diagonal random hopping like Eq. (1).

![FIG. 2: (a) The average of the largest eigenvalues $\langle \lambda_{\text{max}} \rangle$ as a function of the system size $L$ for the EW model. (b) The distribution of the largest eigenvalues $\rho(\lambda_{\text{max}})$. (c) The standard deviation $\sigma(L)$ of $\lambda_{\text{max}}$. (d) The data collapse of $\rho(\lambda_{\text{max}}, L)$ for the various $L$.](image)

On the other hand, we found that the finite-size distribution of eigenvalues $\rho(\lambda, L)$ for the various system size $L$ obeys a scaling form of the type,

$$\rho(\lambda, L) = \lambda^{-\nu} f \left( \frac{\lambda}{\lambda_c(L)} \right) \quad (6)$$

where $\lambda_c(L)$ is a characteristic eigenvalue which scales as $\lambda_c(L) \sim L^\phi$. $f(x)$ is a scaling function satisfying the following properties:

$$f(x) = \text{const}, \quad \text{for } \lambda \ll L^\phi$$

$$f(x) \to 0, \quad \text{for } L^\phi \ll \lambda < \lambda_{\text{max}}. \quad (7)$$

Figure 1(d) shows the data collapse of the spectral density $\rho(\lambda, L)$ with $\nu = 1.67$ and $\phi = 1.53$ and they fall on a single curve very well except for small eigenvalues.

The exponent $\phi$ can be measured by an alternative way. The n-th moment of the eigenvalue is obtained by

$$\langle \lambda^n \rangle = \int_0^{\lambda_2} \lambda^n \rho(\lambda, L) d\lambda = L^{\phi(n+1-\nu)} \int_0^{x_2} x f(x) dx \quad (8)$$

where $\lambda_2$ is the second largest eigenvalues and $x_2 = \lambda_2/L^\phi$. The integral has a finite value and we obtain

$$\langle \lambda^n \rangle \sim L^{\phi(n+1-\nu)}. \quad (9)$$

Thus we can provide that

$$\left( \frac{\lambda^2}{\lambda} \right) \sim L^\phi. \quad (10)$$
We measured $\phi = 1.53 \pm 0.03$ from it as shown in the inset of Fig. 1 (d) which is good agreement with the value used in the collapse of $\rho(\lambda, L)$.

Next, we compared the properties of the largest eigenvalues of the covariance matrix of heights with the results of the RMT. The MP law tells that the average of the largest eigenvalue $\langle \lambda_{\text{max}} \rangle$ depends on the matrix size $M$ like as $\langle \lambda_{\text{max}} \rangle \sim M$ for large $M$ and the typical fluctuations of $\lambda_{\text{max}}$ are known to be described by the TWD [3]. We measured the average of largest eigenvalue $\lambda_{\text{max}}(L)$ as a function of $L$ and found that it scales as $\langle \lambda_{\text{max}} \rangle \sim L^{a}$ with $a \approx 2.09$ (Fig. 2 (a)), which is also different from the result of the RMT. Also the distributions of the largest eigenvalue $\rho(\lambda_{\text{max}})$ deviates from the TW curve as shown in the Fig. 2 (b). Hence it does not follow the prediction of the RMT.

The fluctuations of $\lambda_{\text{max}}$ from its the average value come to be larger as $L$ increases. We measured the standard deviation $\sigma(L)$ of $\lambda_{\text{max}}$ and found that $\sigma(L)$ scales as $\sigma(L) \sim L^{b}$ with $b \approx 2.04$. We rescaled the variable $\lambda_{\text{max}}$ as $x = (\lambda_{\text{max}} - L^{a})/L^{b}$ and obtained a scaling form of $\rho(\lambda_{\text{max}}, L)$ as follows,

$$\rho(\lambda_{\text{max}}, L) = \frac{1}{L^{b}} f_{\text{max}}\left(\frac{\lambda_{\text{max}} - L^{a}}{L^{b}}\right).$$  (11)

Figure 2 (d) shows that the distributions of largest eigenvalues $\rho(\lambda_{\text{max}}, L)$ fall on a single curve with $a = 2.09$ and $b = 2.04$ for the different system size $L$. Thus, the analysis of the fluctuating interfaces belonging to the EW universality class under the framework of the RMT shows the results different from the conventional behaviors of the RMT and the new spectral scaling properties for the EW class.

![FIG. 3: (a) The spectral density $\rho(\lambda, L)$ of the covariance matrix of heights constructed by the RSOS model except for the largest eigenvalue. The inset shows $\rho(\lambda)$ including the largest eigenvalue for $L = 1024$. (b) The data collapse of $\rho(\lambda, L)$ for various $L$. The inset shows the plot of $\langle \lambda^{2} \rangle / \langle \lambda \rangle$ versus $L$.](image)

By applying the RMT to the analysis of the another universality class of the growth phenomena, the KPZ class, we would like to compare the spectral properties between two universality classes. We constructed the covariance matrix of the heights generated by the restricted-solid-on-solid (RSOS) model [3] which belongs to the KPZ universality class. The spectral density of this covariance matrix $\rho(\lambda)$ also follows a power-law behavior like the case of the EW model (Fig. 3(a)). However the value of exponent $\nu$ was obtained as $\nu = 1.59 \pm 0.01$, which is different from that of the EW model. The height correlation of the RSOS model also obeys the Eq. (6). The data collapse of the spectral density $\rho(\lambda, L)$ for various $L$ also follows the scaling form of Eq. (6). The data collapse of the spectral density $\rho(\lambda, L)$ for the RSOS model is shown in the Fig. 3(b). The values of the exponent $\nu$ and $\phi$ were taken by 1.59 and 1.75, respectively. The inset of Fig. 3(b) shows the plot of $\langle \lambda^{2} \rangle / \langle \lambda \rangle$ versus $L$ and we obtained $\phi = 1.75 \pm 0.03$ of which value is also different from that of the EW model.

The properties of the largest eigenvalues for the RSOS model are shown to be similar to those for the EW model. The average of the largest eigenvalues scales as $\langle \lambda_{\text{max}} \rangle \sim L^{a}$ with $a \approx 2.08$ (Fig. 3(a)) and the standard deviation $\sigma(L)$ of $\lambda_{\text{max}}$ scales as $\sigma(L) \sim L^{b}$ with $b \approx 2.08$ (Fig. 3(b)). Figure 3(c) shows the distribution of the largest eigenvalues which does not follow the TW curve like the EW model for the various system size $L$. The data collapse of $\rho(\lambda_{\text{max}}, L)$ by the scaling form of Eq. (11) falls on a single curve with $a = 2.08$ and $b = 2.08$ (Fig. 3(d)).

In summary, we investigated the spectral properties of covariance matrices of relative heights of fluctuating interfaces in the late-time regime. The spectral density of the covariance matrices follows the power-law behavior except for the largest eigenvalue, which is different from the MP law of RMT. It indicates that the random variables correlated like the relative height of a fluctuating interface would give rise to the power-law behavior of the spectral density of the corresponding covariance matrix. As a finite-size effect the spectral density falls zero beyond the characteristic eigenvalue depending on...
the lateral system size and has the scaling form. The values of exponents $\nu$ and $\phi$ related to the scaling form were given by the different values for the EW model and the RSOS model. It indicates that the statistical characteristics of the different classes of two models in the late-time regime are reflected to the spectral properties of them. The distribution of the largest eigenvalues also showed the different features from the TWD. The average and the standard deviation of the largest eigenvalues follow the power-law behaviors with the lateral system size. The values of the scaling exponents do not give the difference between the EW model and the RSOS model. It would be desirable if it is applied to the behavior in the short-time regime and various interface models belonging to the different universality classes and the local spectral properties of the critical random matrix ensembles using the correlated heights are further studied.

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