Inverse Magnetic Catalysis in three-flavor NJL-model with axial-vector interaction

Lang Yu\textsuperscript{1}, Jos Van Doorsselaere\textsuperscript{2}, and Mei Huang\textsuperscript{1,3}

\textsuperscript{1} Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
\textsuperscript{2} Laboratoire de Mathemathique et Physique Theorique, Universite de Tours, 37000 Tours, France and
\textsuperscript{3} Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China

(Dated: December 1, 2014)

Investigation of the QCD phase structure in the presence of strong external magnetic fields has become a major topic in both theoretical and experimental research into the physics of strongly interacting matter. This topic of paramount importance to understand the phenomenology of noncentral heavy ion collisions at Relativistic Heavy Ion Collider (RHIC) and at the Large Hadron Collider (LHC), in which a strong magnetic field reaching up to $\sqrt{eB} \sim (0.1 - 1.0) \text{ GeV}$ \textsuperscript{1–4} can be generated. In addition, strong magnetic fields could also have existed in the strong and electroweak phase transition \textsuperscript{5–6} of the early Universe, and exist in compact stars like magnetars \textsuperscript{7}.

The breaking and restoration of chiral symmetry, which is described by the behavior of the quark condensate, is one of the most intriguing nonperturbative aspects of QCD. Therefore, it is of great interest to speculate the effect of magnetic fields on the behavior of the chiral condensate in QCD at zero and finite temperatures. Since the 1990’s, a related phenomenon known as magnetic catalysis has been recognized \textsuperscript{8–11}. It refers to an enhancement of the quark condensate and a reduction of the chiral transition temperature $T_c$ under the magnetic field. This is agreed by most of earlier low-energy effective models and approximations to QCD \textsuperscript{8–10} \textsuperscript{12} \textsuperscript{23} as well as lattice QCD simulations \textsuperscript{24–28} in the past twenty years. However, recently, a lattice group \textsuperscript{29} \textsuperscript{30} revealed surprising results that the transition temperature $T_c$ decreases as a function of the external magnetic field, and the chiral condensate shows a nonmonotonic behavior as a function of the external magnetic field in the crossover region. This prediction is in contrast to the majority of previous calculations, and the partly decreasing behavior of the chiral condensate with the increasing $B$ near $T_c$ is called inverse magnetic catalysis.

There are several recent studies \textsuperscript{31–37} discussing the origin of the phenomenon of the decreasing behavior of the chiral critical temperature with increasing $B$ and the inverse magnetic catalysis around $T_c$. For example, the magnetic inhibition \textsuperscript{31}, the mass gap in the large $N_c$ limit \textsuperscript{32}, the contribution of sea quarks \textsuperscript{33} and a running scalar coupling parameter dependent on the magnetic field intensity \textsuperscript{36} are proposed to understand this puzzle. Particularly, a very natural and competitive mechanism attributes the inverse magnetic catalysis to the local chirality imbalance induced by the nontrivial topological gluon configuration, arising from a sphaleron transition \textsuperscript{34} or the instanton–anti-instanton molecule pairing \textsuperscript{37}.

As discussed in Ref. \textsuperscript{37}, the chirality imbalance, which is associated with the violation of the $P$ and $CP$ symmetry, is induced by the nonzero topological charge $Q_T$ through the axial anomaly of QCD

$$\Delta N_5 = \int d^4x \partial_\mu j_5^\mu = -2N_f Q_T,$$

(1)

where $N_f$ is the number of flavors, $\Delta N_5 = N_5(t = +\infty) - N_5(t = -\infty)$, with $N_5 = N_R - N_L$ denoting the number difference between right- and left-hand quarks, and $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ denotes the isospin singlet axial vector current. A consequence of this violation is the existence of two kinds of local domains with same quantum numbers but opposite net topological charges, which will lead to the generation of local chirality imbalances but zero average chirality, as well as the local $P$ and $CP$ violation. The modification of the QCD phase diagram by the chirality imbalance has been studied in some Refs. \textsuperscript{38–40}. Especially, the recent observation of charge azimuthal
correlations at RHIC and LHC may be resulting from the chiral magnetic effect (CME) with local $P$ and $CP$ violation, which is an interesting combined effect of both the strong magnetic field and the nontrivial topological gluon configuration of the quark-gluon plasma. Based on above discussions, the enhancement of the chiral magnetic effect (CME) with local $CP$ violation, which is an interesting combined effect of both the strong magnetic field and the nontrivial topological gluon configuration of the quark-gluon plasma. Based on above discussions, the enhancement of the chiral magnetic effect (CME) with local $CP$ violation, which is an interesting combined effect of both the strong magnetic field and the nontrivial topological gluon configuration of the quark-gluon plasma. Based on above discussions, the enhancement of the chiral magnetic effect (CME) with local $CP$ violation, which is an interesting combined effect of both the strong magnetic field and the nontrivial topological gluon configuration of the quark-gluon plasma.

In the work of Ref. [37], we presented a mechanism to generate local chirality imbalance based on the instanton–anti-instanton $(\Pi)$ molecule picture, which is regarded as one effective mechanism responsible for nonperturbative properties of QCD in the region $T \approx T_c - 2T_c$. By using an unconventional repulsive iso-scalar axial-vector interaction in a two flavor Nambu–Jona-Lasinio (NJL) model, we find that a dynamical chiral chemical potential related to the local chirality imbalance is induced spontaneously at the temperatures near $T_c$. It is also found that the increasing magnetic field helps to lower the critical temperature due to the appearance of the local chirality. Moreover, since the local chirality imbalance can only be produced around $T_c$, it gives a reasonable explanation for why inverse magnetic catalysis only appears at the temperatures around $T_c$, while magnetic catalysis still occurs at zero and low temperatures.

However, the lattice result in Ref. [30] shows that the strange quark condensate does not exhibit inverse magnetic catalysis, but simply increases with magnetic field at all temperatures. In this paper, we extend our analysis and calculations to the 2+1 flavors, and investigate the corresponding effects of the axial-vector interaction on both the local chirality imbalance and the chiral critical temperatures for $u$, $d$ and $s$ quarks. The paper is organized as follows. In Sec. II, we give a general description of the 2+1 flavor NJL model and formalism with considering the repulsive axial-vector interactions stemming from the interacting $\Pi$ molecule model (IIMM). In Sec. III, we will discuss the main results of the numerical calculations. Finally, our conclusions and perspectives are presented in Sec. IV.

II. MODEL AND FORMALISM

In this section, we present the three flavor NJL model with adding the vector and axial-vector interaction terms. The Lagrangian density of our model in the presence of an external magnetic field is given by

$$\mathcal{L} = \bar{\psi}(i\gamma_{\mu}D^{\mu} - \hat{m})\psi + \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{det}} - \mathcal{L}_{VA},$$

where $\hat{m} = \text{diag}(m_u, m_d, m_s)$ is the corresponding current mass matrix. The covariant derivative, $D_{\mu} = \partial_{\mu} - iq_{F}A_{\mu}$, couples quarks to an external magnetic field $\mathbf{B} = (0, 0, B)$ along the positive $z$ direction, via a background Abelian gauge field $A^{\mu} = (0, 0, Bx, 0)$. And $q_{F}$ is defined as the electric charge of the quark field with flavor $f$. $\mathcal{L}_{\text{sym}}$ and $\mathcal{L}_{\text{det}}$ are given by

$$\mathcal{L}_{\text{sym}} = \frac{G_{S}}{2} \sum_{a=0}^{8} \left[ (\bar{\psi}\lambda^{a}\psi)^{2} + (\bar{\psi}\lambda^{a}i\gamma^{5}\psi)^{2} \right],$$

$$\mathcal{L}_{\text{det}} = -K \left\{ \det \left[ \bar{\psi}(1 + \gamma^{5})\psi \right] + \det \left[ \bar{\psi}(1 - \gamma^{5})\psi \right] \right\},$$

where $\lambda^{a}$ are the Gell-Mann matrices in flavor space ($\lambda^{0} = \sqrt{2/3}I$) and the determinant is in flavor space also. The $\mathcal{L}_{\text{sym}}$ term corresponds to the usual four-fermion interactions of scalar and pseudoscalar channels which respect $SU(3)_{V}\otimes SU(3)_{A}\otimes U(1)_{V}\otimes U(1)_{A}$ symmetry. The $\mathcal{L}_{\text{det}}$ term corresponds to the ’t Hooft six-fermion determinant interactions which break $U(1)_{A}$ symmetry. As for the $\mathcal{L}_{VA}$ term, it represents the four-fermion interactions of vector and axial-vector channels under the invariance of $SU(3)_{V}\otimes SU(3)_{A}\otimes U(1)_{V}\otimes U(1)_{A}$ symmetry. The last two terms are added to introduce chiral interactions, which are chosen such that they correspond to effective chirally asymmetric interactions with an instanton background.

The connection between instantons and chiral symmetry breaking is well known. Essentially it is a consequence of an index theorem that shows how a topologically nontrivial gauge configuration – the instanton– gives rise to an asymmetry in occupation of left- and right chiral eigenmodes, observed through the existence of a fermion condensate. A feature of this index theorem is that all non-trivial physics happens in the zero-mode space, in a sense the price we pay for those modes not to contribute to the action, is the violation of chiral symmetry. At zero as well as low temperatures, the ’t Hooft interaction is dominant, the random instantons play an important role in chiral symmetry breaking. However, at high temperatures and near the chiral phase transition, the instantons are no longer random, but become correlated. Therefore, it was suggested in Refs. [45–47] that the growing correlations between instantons and anti-instantons near $T_c$ will lead to the decrease of random instantons but the increase of instanton–anti-instanton molecule pairs. This means that the random instantons and anti-instantons are not annihilated but paired up into the correlated $\Pi$ molecules when chiral phase transition happens. As shown in [47], in the temperature region of $T \gtrsim T_c$, the $\Pi$ molecules pairing induces a repulsive effective local four-quark interactions in the isoscalar axial-vector channel. This unconventional repulsive axial-vector interaction leads to a repulsive axial-vector mean field in the space-like components but an attractive one in the time-like components, which naturally induces a spontaneous local $CP$ violation and local chirality imbalance as shown in [37].

In Refs. [37] and [37], the instanton background only couples to light $u,d$ quarks, and the $\mathcal{L}_{VA}$ in the isoscalar
vector and axial-vector takes the form of

$$L_{VA}^{u,d} = \sum_{f=u,d} \left[ G_V (\bar{\psi}_f \gamma^\mu \psi_f)^2 + G_A (\bar{\psi}_f \gamma^\mu \gamma^5 \psi_f)^2 \right] .$$

We may generalize this term to three-flavor case as following:

$$L_{VA}^{u,d,s} = \sum_{f=u,d,s} \left[ G_V (\bar{\psi}_f \gamma^\mu \psi_f)^2 + G_A (\bar{\psi}_f \gamma^\mu \gamma^5 \psi_f)^2 \right] .$$

(6)

In reality, considering the strange quark mass is heavier than light u,d quarks, the isoscalar vector and axial-vector interaction induced by instanton background might take the form in between Eq. (5) and Eq. (6). For example, the quark propagator in a single instanton background is associated with the quark zero-modes [60]

$$\psi_{0}^{\pm}(x) = \frac{\rho}{\pi} \frac{1 \pm \gamma_5}{(r^2 + \rho^2)^{1/2}} \frac{\bar{U}}{r} ,$$

(7)

where the superscript ± corresponds to an (anti-) instanton centered at x0 and with size ρ. The spin-color matrix U satisfies ($\bar{\sigma} + \bar{r}$) U = 0 and r = x - x0. The zero mode contributions will enter into the calculation of the correlators through the leading term in the spectral representation of the quark background field propagator

$$S_{q}^{\pm}(x,y) = \frac{\psi_{0}^{\pm}(x) \psi_{0}^{\mp\dagger}(y)}{m_{q}^{2}(\rho)} + O(\rho m_{q}^{2}) .$$

(8)

Here, the flavor dependent effective quark mass $m_{q}^{2}(\rho) = m_{q} - \frac{1}{2} \rho^{2} \langle \bar{q}q \rangle$ (where q stands for up, down and strange quarks) in the denominator is generated by interactions with long-wavelength QCD vacuum fields as shown in Ref. [31]. It can be seen that the strange quark propagator in the instanton background is suppressed by 1/m2. Extend this to the instanton–anti-instanton molecule background, we can assume the actual interaction in the isoscalar vector and axial-vector channel can be written as:

$$L_{VA} = \sum_{f=u,d} \left[ G_V (\bar{\psi}_f \gamma^\mu \psi_f)^2 + G_A (\bar{\psi}_f \gamma^\mu \gamma^5 \psi_f)^2 \right] + \left[ G'_{V} (\bar{\psi}_f \gamma^\mu s)^2 + G_{A} (\bar{\psi}_f \gamma^\mu \gamma^5 s)^2 \right] .$$

(9)

with $G'_{V} \ll G_{V}$ and $G'_{A} \ll G_{A}$ ($G'_{V}/G_{V}$ or $G'_{A}/G_{A}$ ∼ $(m_{u}/m_{s})^2$ in the chirally symmetric phase by Eq. 8 where $m_{u}$ and $m_{s}$ represent the current masses of u and s quarks, respectively). A compelling consequence of this argument is that the $L_{VA}$-term, which is the essential source of the inverse magnetic catalysis in our approach, distinguishes between the isoscalar channel with light quarks and the one with strange quarks. It was found on the lattice QCD [30] that the inverse magnetic catalysis effect appears only for light quarks not for strange quarks, an implicit feature in our approach. In the following calculations, we take two cases for the isoscalar vector and axial-vector interaction:

Case I : take $L_{VA}^{u,d}$

Case II : take $L_{VA}^{u,d,s}$.

(10)

(11)

Note that it has been discussed in Ref. [37] that $G_{S}$ and $G_{V}$, are expected to be positive at the whole temperature region [54, 55, 57, 58] and are assumed to be keeping the constants fixed by the mesonic properties in QCD vacuum for simplicity, whereas $G_{A}$ is expected to be positive at zero and low temperatures [54, 55] and to be negative at the temperatures above $T_{c}$ as a result of the interacting instanton–anti-instanton molecule model [45, 47]. It has been shown in Ref. [47] that negative $G_{A}$ interaction induced by the correlated instanton–anti-instanton molecule pairs near $T_{c}$, will give rise to nontrivial influence on the chiral phase transition. Therefore, we will treat $G_{A}$ as a free parameter in the NJL model of SU(3) flavor version as we did in Ref. [37] with two light flavors.

Working at the mean field level, one gets the thermodynamical potential per unit volume $\Omega$ by integrating out the quark fields $\psi$ of the Lagrangian density of Eq. (2),

$$\Omega = \frac{1}{4G_{S}} \sum_{f=u,d,s} \sigma_{f}^{2} + \frac{K}{2G_{S}^{3}} \sigma_{u} \sigma_{d} \sigma_{s} - \frac{\tilde{\mu}_{5}^{2}}{4G_{A}} + \sum_{f=u,d,s} \Omega_{f} ,$$

(12)

where $\sigma_{f} = -2G_{S} (\bar{\psi}_{f} \gamma_{5} \psi_{f})$ (f = u, d, s) and

$$\tilde{\mu}_{5} = -2G_{A} \sum_{f=u,d} (\bar{\psi}_{f} \gamma_{5} \gamma_{5} \psi_{f}) ,$$

(13)

for Case I,

$$\tilde{\mu}_{5} = -2G_{A} \sum_{f=u,d} (\bar{\psi}_{f} \gamma_{5} \gamma_{5} \psi_{f}) ,$$

for Case II.

The contributions from the fermion loop for each flavor is given by

$$\Omega_{f} = -N_{c} \frac{|q_{f}| B}{2\pi} \sum_{s_{z},k} \alpha_{s_{z},k} \times \left[ \int_{-\infty}^{\infty} dp_{z} \int_{2\pi} f_{A}^{2}(p) \omega_{k}^{f}(p) + 2T \ln(1 + e^{-\omega_{k}^{f}/T}) \right] ,$$

(14)

where

$$\omega_{k}^{f} = \begin{cases} \sqrt{M_{f}^{2} + |p| + s_{z} \tilde{\mu}_{5} \text{sgn}(p_{z})}^{2} & \text{for } f = u, d \vspace{1em} \\
\sqrt{M_{f}^{2} + |p|^{2}} & \text{for } f = s, \end{cases}$$

(15)

are dispersion relation for the thermal eigenfrequencies with spin factors $s_{z} = \pm 1$ for Case I, and

$$\omega_{k}^{f} = \sqrt{M_{f}^{2} + |p| + s_{z} \tilde{\mu}_{5} \text{sgn}(p_{z})}^{2}$$

(16)

are dispersion relation for the thermal eigenfrequencies with spin factors $s_{z} = \pm 1$ for Case II. The gap equations
for quark mass take the following form:

\[
M_u = m_u + \sigma_u + \frac{K}{2G_S^2} \sigma_s \sigma_d, \\
M_d = m_d + \sigma_d + \frac{K}{2G_S^2} \sigma_s \sigma_u, \\
M_s = m_s + \sigma_s + \frac{K}{2G_S^2} \sigma_u \sigma_d.
\]

(17)

The 3-momentum \( p \) in a magnetic field is given by

\[
p^2 = p_z^2 + 2|q_f B| k,
\]

(18)

and \( k = 0, 1, 2, \ldots \) is a non-negative integer number labeling the Landau levels. The spin degeneracy factor is expressed as

\[
\alpha_{s,k} = \begin{cases} 
\delta_{s,z} + 1 & \text{for } k = 0, qB > 0, \\
\delta_{s,z} - 1 & \text{for } k = 0, qB < 0, \\
1 & \text{for } k \neq 0.
\end{cases}
\]

(19)

Following Ref. [38] we use a smooth regularization form factor

\[
f_{\Lambda}(p) = \sqrt{\frac{\Lambda^{2N}}{\Lambda^{2N} + |p|^{2N}}},
\]

(20)

where we take \( N = 5 \). Now, by making use of Eq. (12), \( \sigma_f \) and \( \tilde{\mu}_5 \) can be determined self-consistently as solutions to the saddle point equations

\[
\frac{\partial \Omega}{\partial \sigma_f} = \frac{\partial \Omega}{\partial \tilde{\mu}_5} = 0.
\]

(21)

Numerically one can obtain these solutions by a minimisation and moreover find the true vacuum by looking at the global minimum. This will prove essential for our model which has multiple minima breaking chiral symmetry spontaneously.

The parameters of our model, the cutoff \( \Lambda \), the coupling constants \( G_S \) and \( K \), and the current quark masses \( m_u = m_d \) and \( m_s \) are determined by fitting \( f_\pi, m_\pi, m_K \) and \( m_\eta \) to their empirical values by using the smooth regularization method. We obtain \( \Lambda = 604.5 \text{MeV}, m_u = m_d = 5.1 \text{MeV}, m_s = 133.0 \text{MeV}, G_S \Lambda^2 = 3.250 \) and \( K \Lambda^5 = 10.58 \).

\( G_A \) will be treated as a free parameter, and we will perform our calculations over a limited range of ratios

\[
r_A = G_A / G_S.
\]

(22)

It is discussed in Ref. [37] that when \( r_A \geq 0 \) we always obtain the ordinary magnetic catalysis effect and only when \( r_A < 0 \) inverse magnetic catalysis effect can be seen.

III. NUMERICAL RESULTS AND DISCUSSION

Considering the interaction in strange quark channel is suppressed, for our numerical calculations, we mostly take the isoscalar axial-vector interaction of Case I, i.e. take the form in Eq.(5).

We study the chiral phase transition at finite temperature by using Eq. (21) for several different values of \( eB \). (a) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.2 \text{GeV}^2 \) at \( r_A = 0 \). (b) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.4 \text{GeV}^2 \) at \( r_A = 0 \). (c) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.6 \text{GeV}^2 \) at \( r_A = 0 \).

FIG. 1. (color online). The quark condensate \( \sigma_f \) (f=u,d,s) (solid curves) and dynamical chiral chemical potential \( \tilde{\mu}_5 \) (green, dashed curve) as a function of \( T \) at \( r_A = 0 \) for several different values of \( eB \). (a) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.2 \text{GeV}^2 \) at \( r_A = 0 \). (b) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.4 \text{GeV}^2 \) at \( r_A = 0 \). (c) \( \sigma_f \) and \( \tilde{\mu}_5 \) for \( eB = 0.6 \text{GeV}^2 \) at \( r_A = 0 \).
which allow us to efficiently find the transition temperatures as functions of the parameters $r_A$ and $eB$. Actually, each of the condensates has thus its own specific transition temperature, which is defined by the temperature at the inflection point of the $\sigma - T$ diagram for each flavor, i.e., the maximum point of the quantity $\sigma \partial T / \partial T$. Here we will use $\sigma_u + \sigma_d$ to determine the transition temperature $T_c$ for the chiral phase transition of QCD.

In Fig. 1 we display the quark condensates of $\sigma_u$, $\sigma_d$ and $\sigma_s$ as well as dynamical chiral chemical potential $\tilde{\mu}_5$ as functions of $T$ for several different values of $eB$ without considering additional axial-vector couplings, i.e., $r_A = 0$ and $\tilde{\mu}_5 \equiv 0$, and the ordinary magnetic catalysis effect can be exactly found from the plots. In fact, even if we choose positive values for the parameter $r_A$, fitted by the conventional SU(3) flavor NJL model, we will acquire the same results. This is because that when $r_A > 0$, the potential energy density $\Omega$ can only has a local maximum at nonzero $\tilde{\mu}_5$, which forces $\langle \bar{\psi} \gamma^0 \gamma^5 \psi \rangle$ to be zero so that $\tilde{\mu}_5 = 0$ accordingly.

Next, as discussed in Ref. [37], by switching on a negative $r_A$, we introduce a non-trivial dependence of the thermodynamical potential $\Omega$ on the dynamic chiral chemical potential $\tilde{\mu}_5$, which adds an extra dimension to the mean field surface of $\Omega$. As is shown in Fig. 2, there are two local minima for the potential $\Omega$. When the temperature is low, the original local minimum representing nonzero quark condensates is dominant, since it is the global minimum. However, sufficient heating of the QCD system makes the local minimum for non-trivial $\tilde{\mu}_5$ energetically more favourable while the chiral condensates is weakened to the trivial state. As a result, the vacuum tunnels to a state with local chirality imbalance between right- and left-handed quarks as this metastable state becomes the new global minimum. It is important to point out that the potential $\Omega$ in Eq. (12) is even in $\tilde{\mu}_5$, so one can get separated local domains with chiral densities of both signs. Moreover, another important consequence of the competition between these two local minima is that no ‘mixed’ state appears, so one has either nonzero quark condensates but no chirality imbalance or an instability towards the formation of nonzero dynamic chiral chemical potential but no presence of condensates. This is clear from the fact that the minima in Fig. 2 appear on either of the axes and is a consistent feature of all our simulations.

In fact, the magnitude of the unconventional negative $G_A$, which leads to an attractive mean field in the timelike components of the axial-vector channel, reflects the coupling strength of the attraction for $\tilde{\mu}_5$. In Figs. 3 and 4 we compare our numerical results of quark condensates $\sigma_f$ ($f = u, d, s$) and dynamic chiral chemical potential $\tilde{\mu}_5$ as functions of $T$ for several magnetic fields at $r_A = -0.3, -0.5$ and $-0.7$. One can distinguish two distinct cases as a result of the magnitude of negative $r_A$: (i) $T_c(eB = 0; r_A = 0) < T_{5c}(eB = 0)$ and (ii) $T_c(eB = 0; r_A = 0) > T_{5c}(eB = 0)$.

If magnitude of $G_A$ is small, approximately $-0.5$ <}

![FIG. 2. A 2D minimal surface of the potential $\Omega$ for Case I as a function of $\sigma_d$ and $\tilde{\mu}_5$ at $T = 0$, for $(a)$ $T = 100$ MeV, (b) $T = 150$ MeV, and (c) $T = 200$ MeV.](image-url)
$r_A < 0$ for our model, it is easy to found that we are in the case (i), i.e. $T_{5c} > T_c(r_A = 0)$ at $eB = 0$. For example, as shown by Fig. 3 of $r_A = -0.3$ and Fig. 4 of $r_A = -0.5$, when the external magnetic field is not strong enough, the ordinary phase transition into the chirally restored phase takes place at a lower temperature and is the dominant effect in destroying the quark condensates; whereas a local $CP$-odd first order phase transition for $\tilde{\mu}_5$ is spontaneously generated at a higher critical temperature $T_{5c} > T_c$. As the magnetic field grows, both critical temperatures, $T_{5c}$ and $T_c$, approach each other and two local minima in the thermodynamic potential $\Omega$ co-exist like in the example of Fig. 2. At some critical value of magnetic field $B_c$ for a given $r_A$, these two critical temperatures meet with each other and the first order transition for nonzero dynamical chiral chemical potential $\tilde{\mu}_5$ becomes dominant effect, which makes $\sigma_u$ and $\sigma_d$ drop to zero at $T_c = T_{5c}$. Therefore, we find that, for $u$
FIG. 5. (color online). For Case I, the quark condensate $\sigma_f$ (f=u,d,s) (solid curves) and dynamical chiral chemical potential $\tilde{\mu}_5$ (green, dashed curve) as a function of $T$ at $r_A = -0.7$ for several different values of $eB$. (a) $\sigma_f$ and $\tilde{\mu}_5$ for $eB = 0.2$ GeV$^2$ at $r_A = -0.7$. (b) $\sigma_f$ and $\tilde{\mu}_5$ for $eB = 0.4$ GeV$^2$ at $r_A = -0.7$. (c) $\sigma_f$ and $\tilde{\mu}_5$ for $eB = 0.6$ GeV$^2$ at $r_A = -0.7$.

and d quarks in the case (i), the critical temperature $T_{5c}$ decreases with $eB$, while $T_c$ increases at first and then decreases as the magnetic field grows (see Fig. 7). As for s quark condensate, it shows a slight jump because of $\tilde{\mu}_5$ background and then continues to dissolve with increasing temperature. The critical temperature $T_{c}(\sigma_s)$ will increase with $eB$ always, depicted by Fig. 6, which is consistent with the lattice results in Ref. [30] in some sense.

If magnitude of $G_A$ is large enough ($r_A < -0.5$), that is in the case (ii), the light quark condensates are destroyed at $T_c = T_{5c}$ because the QCD ground state with chirality imbalanced density becomes more favorable around the critical temperature for any values of $eB$, before $\sigma_u$ and $\sigma_d$ are dissolved at their original critical temperature $T_c$ without considering $\tilde{\mu}_5$, e.g., shown by Fig. 5 at

FIG. 6. (color online). For Case I, the quark condensates $\sigma_s$, $\sigma_u$ and $\sigma_d$ as a function of $T$ at $r_A = -0.5$ for several different values of $eB$. (a) $\sigma_s$ at $r_A = -0.5$ for several different values of $eB$. (b) $\sigma_u$ at $r_A = -0.5$ for several different values of $eB$. (c) $\sigma_d$ at $r_A = -0.5$ for several different values of $eB$. 
The results are for Case I. The behaviors of strange quark condensate $\sigma$ and the critical temperature $T_c(\sigma_s)$ are similar to that in the case (i).

We can now put the data upon the critical temperatures obtained from Figs. 3, 4 and 5 into one $T_c - eB$ phase diagram of QCD. Adding more data points from identical simulations with other values of the background parameters $r_A$ and $eB$, we can find that the middle diagram of Fig. 7 shows two different possible types of dependence of $T_c$ on the magnetic field as a result of the free parameter $r_A$, which has been discussed explicitly above. As a consequence, a reasonable strength of $r_A$, approximately between $-0.5$ and $-0.55$ by the simulations of our model, will naturally explain the decreasing dependence of $T_c$ on $eB$ obtained in a recent lattice QCD study [29]. If the magnitude of $r_A$ is too small, less than 0.5, we can not find a monotonously decreasing dependence of $T_c$ on $eB$; If the magnitude of $r_A$ is too big, more than 0.55, the value of $T_c$ at $eB = 0$ will deviate from the lattice QCD result greatly. Furthermore, we can distinguish case (i) and case (ii) easily by the $T_{sc}(B)$ function from the top diagram of Fig. 7, the separation given by the thick black line for the critical temperature $T_c$ at $r_A = 0$. As for the critical temperature $T_c(\sigma_s)$, depicted by the bottom diagram of Fig. 7, one can find that its behavior at different negative values of $r_A$ is similar to that at $r_A = 0$, showing a slightly increasing dependence on $eB$.

![Diagram](Image)

**FIG. 7.** (a), The critical temperature $T_{sc}$ as a function of $eB$ for several different values of $r_A$ and $T_c$ as a function of $eB$ at $r_A = 0$. (b), The critical temperature $T_c$ as a function of $eB$ for several different values of $r_A$. (c), The critical temperature $T_c(\sigma_s)$ as a function of $eB$ for several different values of $r_A$. The results are for Case I.

$r_A = -0.7$. Hence, the critical temperatures both $T_c$ and $T_{sc}$ for u and d quark condensates decreases with $eB$ starting from $eB = 0$, which is just the decreasing $T_c$ dependence on $B$ predicted by Ref. [29]. On the other hand, the condensates $\sigma_u$ and $\sigma_d$ increase with the magnetic field at zero and low temperatures still, which is the ordinary magnetic catalysis effect validated in Ref. [30].

The behaviors of strange quark condensate $\sigma_s$ and the critical temperature $T_c(\sigma_s)$ are similar to that in the case (i).

The above calculations are based on Case I, where the isoscalar axial-vector interaction only involves light u,d quarks. The isoscalar nature of the interaction $\mathcal{L}_{VA}$ is essential for the nature of the phase transition. With little extra effort we were able to simulate the Case II where the four-fermion chiral attraction treats all flavors equally as given by Eq.(6). It can be seen that the results of $\sigma_u$ and $\sigma_d$ are very similar to what we found before, but rather than a small shift in the value of the heavy strange quark condensate, $\sigma_s$ undergoes the same first order phase transition as the other two light flavors and vanishes at the transition temperature $T_{sc}$ as shown in Fig. 8. As we argued in the previous section, this kind of equal coupling with negative $G_A$ to all three quark flavors is not to be expected for an axial-vector coupling induced by an instanton–anti-instanton molecule background, and unsurprisingly it does not reproduce the qualitative lattice result.

Before drawing our final conclusions, it is important to realize we can only trust our results qualitatively. Since the new minimum and the corresponding phase transition shown in Fig. 2 are in fact at a scale well beyond the cut-off of our theory, exact quantitative prediction are beyond the scope of the NJL framework. Qualitatively, however, we can be sure that the instability will emerge, and a new vacuum state will appear that is more favored than the chiral condensate when increasing magnetic fields around $T_c$ and thus give rise to inverse magnetic catalysis effect. In that sense we think that our model is a good representation of the effect of instanton–anti-instanton molecule background on the chiral condensates, but we cannot produce accurate predictions for the large chiral densities involved.
IV. CONCLUSIONS

In this paper we extend our study to the QCD phase diagram as well as the behavior of quark condensates at finite temperature under an external magnetic field within the three-flavor NJL model including additional isoscalar vector and axial-vector channels. Note that an important and unconventional feature of our model is the isoscalar axial-vector interaction with a negative coupling constant depending mostly on the up and down quarks, while the interaction in the strange quark sector is suppressed due to its heavier mass, which can be derived from the instanton–anti-instanton molecule model [47].

In this scenario, we have shown that a new way of destroying chiral condensates appears around $T_c$, replacing them by dynamical chiral chemical potential $\tilde{\mu}_5$ in a first order phase transition, which corresponds to a spontaneous generation of local $\mathcal{P}$ and $\mathcal{CP}$ violation and local chirality imbalance. Moreover, the critical temperature of this first order phase transition shows inverse magnetic catalysis, meaning that it decreases with increasing magnetic field.

The dominant features of destroying the light quark condensates depends on the parameters of the model, a tunable axial-vector coupling constant $G_A$ and the background magnetic field. When increasing the magnitude of $G_A$, we can find that it will decrease the critical temperature for the first order phase transition of $\tilde{\mu}_5$. And the increase of the magnetic field at a given $G_A$ will also decrease the critical temperature $T_{5c}$. It means that, in the generic case, the increase of the magnitude of both parameters, $G_A$ and $eB$, will catalyze the appearance of the local chirality imbalance. Therefore, a reasonable value of $G_A$, making the ordinary chiral phase transition meet with the newly found first order phase transition at $eB = 0$ (however, this is not in agreement with previous lattice results at finite temperature, and possible reasons are discussed in Ref. [37]), the inverse magnetic catalysis effect can be naturally explained and a phase diagram is reproduced in Fig. 7, consistent with lattice QCD results [29].

On the other hand, since the lattice results of Ref. [30] indicated that strange quark condensate experience magnetic catalysis only, we investigate the behavior of strange quark condensate in our model also. When we introduce a negative axial-vector interaction channel including light-quark currents only, it is found that the critical temperature $T_c(\sigma_s)$ exhibits little modification as a result of the instanton–anti-instanton molecule backgroud, in some sense consistent with lattice results, although the strange condensate shows a slight jump arising from the appearance of $\tilde{\mu}_5$ in the $\sigma - T$ diagrams. This might be improved by considering the spatial structure of the topological density distribution, which is one of our future plan. In reality, the interaction in the isoscalar vector and axial-vector channel might look like Eq. (9), with a dominant isoscalar axial-vector interaction in the light quark sector and with a suppressed interaction in the strange quark sector. In this case, if the interaction in the strange sector is small enough, the result of chiral phase transitions will be the same as that in Case I, and we can get the inverse magnetic catalysis for light quark condensate but magnetic catalysis for strange quark condensate around critical temperature, which is in agreement with lattice results.

ACKNOWLEDGEMENT

We thank valuable discussions with M. Chernodub, J.Y. Chao and I. Shovkovy. This work is supported by the NSFC under Grant No. 11275213, and 11261130311(CRC 110 by DFG and NSFC), CAS key project KJCX2-EW-N01, and Youth Innovation Promotion Association of CAS. L. Yu is partially supported by China Postdoctoral Science Foundation under Grant No. 2014M550841. The work of JVD was supported by a grant from La Region Centre (France) and the Chinese-French Cai Yuanpei 2013 grant.

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