Sensitivity of Quantum Walks with Perturbation

Chen-Fu Chiang

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Abstract

Quantum computers are susceptible to noises from the outside world. We investigate the effect of perturbation on the hitting time of a quantum walk and the stationary distribution prepared by a quantum walk based algorithm. The perturbation comes from quantizing a transition matrix $Q$ with perturbation $E$ (errors). We bound the perturbed quantum walk hitting time from above by applying Szegedy’s work and the perturbation bounds with Weyl’s perturbation theorem on classical matrix. Based on an efficient quantum sample preparation approach invented in speed-up via quantum sampling and the perturbation bounds for stationary distribution for classical matrix, we find an upper bound for the total variation distance between the prepared quantum sample and the true quantum sample.

1 Introduction

Markov chains and random walks have been useful tools in classical computation. One can use random walks to obtain the final stationary distribution of a Markov chain to sample from. In such an application the time the Markov chain takes to converge, i.e., convergence time, is of interest because shorter convergence time means lower cost in generating a sample. Sampling from stationary distributions of Markov chains combined with simulated annealing is the core of many clever classical approximation algorithms. For instance, approximating the volume of convex bodies [1], approximating the permanent of a non-negative matrix [2], and the partition function of statistical physics models such as the Ising model [3] and the Potts model [4]. In addition, one can also use the random walks to search for the marked state in the Markov chain, in which the hitting time is of interest. It is because hitting time indicates the time it requires to find the marked state.

The Markov Chain Monte Carlo (MCMC) method is a centerpiece of many efficient classical algorithms. It allows us to approximately sample from a particular distribution $\pi$ over a large state space $\Omega$. Perturbations of classical Markov chains are widely studied with respect to hitting time and stationary distribution. The variation of a stationary distribution caused by perturbation would deteriorate the accuracy of our sampling. With the above facts, we are interested to know what effect perturbation has on currently existing quantum walk based algorithms.

The note is organized as follows. In section 2.1 we present the deviation effect of perturbation on spectral gap of Markov chain and we apply it to the hitting time of a quantum walk in section

*School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816, USA. Email: cchiang@eecs.ucf.edu
In section 2.3 we state the result of total variation distance of classical stationary distributions. By using an efficient algorithm [5] for preparing quantum samples, in section 2.4 we obtain the total variation distance between the prepared quantum sample and the true quantum sample. Finally in section 3 we make our conclusion and suggest future works.

2 The Perturbation

Given a stochastic symmetric matrix $P \in \mathbb{C}^{n \times n}$, we can quantize the Markov chain [6] and we showed that the implementation of one step of quantum walk [7] can be achieved efficiently. However, the above settings always are under the assumption of perfect scenarios. In real life there are many sources of errors that would perturb the process. Noises might be propagated along with the input source or they might be introduced during the process. Here we solely look at the noises that are introduced at the beginning of the process.

The noises can be introduced due to the precision limitation and the noisy environment. For instance, not all numbers have a perfect binary representation and the approximated numbers would cause perturbation. Suppose our input decoding mechanism can always take the input matrix and represent it in a symmetric transition matrix $Q$ where $Q$ can be perfectly represented by the limited precision and it is the matrix closes to the original matrix $P$ that the system can prepare.

Let $E$ be the noise that is introduced because of system’s precision limitation and the environment, we can express the transition matrix as

$$Q = P + E.$$  

2.1 Classical Spectral Gap Perturbation

Classically many researches [8, 9, 10, 11, 12, 13] are focused on the spectral gaps and stationary distributions of the matrix with perturbation. In a recent work by Ipsen and Nadler [8], they refined the perturbation bounds for eigenvalues of Hermitians. Throughout the rest of the note, $\| \cdot \|$ always denotes $l_2$ norm, unless otherwise specified. Based on their result, we summarized the following:

**Corollary 1.** Suppose $P$ and $Q \in \mathbb{C}^{n \times n}$ are Hermitian symmetric transition matrices with respective eigenvalues

$$\lambda_n(P) \leq \ldots \leq \lambda_1(P) = 1, \quad \lambda_n(Q) \leq \ldots \leq \lambda_1(Q) = 1,$$

and $Q = P + E$, then

$$\max_{1 \leq i \leq n} |\lambda_i(P) - \lambda_i(Q)| \leq \|E\|.$$  

Furthermore, the spectral gap $\delta$ of $P$ and the spectral gap $\Delta$ of $Q$ have the following relationship

$$\delta - \|E\| \leq \Delta \leq \delta + \|E\|.$$  

**Proof.** Eq. (3) is a direct result from the Weyl’s Perturbation Theorem. The *Weyl’s Perturbation Theorem* bounds the worst-case absolute error between the $i$th exact and the $i$th perturbed eigenvalues of Hermitian matrices in terms of the $l_2$ norm [10] [11]. And since

$$1 - \lambda_2(P) = \delta, \quad 1 - \lambda_2(Q) = \Delta,$$

$$\delta - \|E\| \leq \Delta \leq \delta + \|E\|.$$
by eq. (3) we have $|\delta - \Delta| \leq \|E\|$. Therefore, in general we can bound the perturbed spectral gap $\Delta$ as

$$\delta - \|E\| \leq \Delta \leq \delta + \|E\|.$$  

(6)

Generally speaking, the global norm of $E$ might be very large when the dimensions $n >> 1$. However, in our case because $E$ is the difference between two very close stochastic symmetric matrices, its global norm would never become large.

### 2.2 Quantum Hitting Time Perturbation

Given two Hermitian stochastic matrices, $P$ and $Q$, we explore the difference between walk operators, $W(P)$ and $W(Q)$, with respect to their hitting time. Denote the set of marked elements as $|M|$. Based on the result from Corollary 7 we have the following:

**Corollary 2.** Given two symmetric reversible ergodic transition matrices $P$ and $Q \in \mathbb{C}^{n \times n}$, where $Q = P + E$, let $W(P)$ and $W(Q)$ be quantum walks based on $P$ and $Q$, respectively. Let $M$ be the set of marked elements in the state space. Denote $HT(P)$ the hitting time of walk $W(P)$ and $HT(Q)$ the hitting time of walk $W(Q)$. Suppose $|M| = \epsilon N$. If the second largest eigenvalues of $P$ and $Q$ are at most $1 - \delta$ and $1 - \Delta$, respectively, then in general

$$HT(P) = O\left(\sqrt{\frac{1}{\delta \epsilon}}\right), \quad HT(Q) = O\left(\sqrt{\frac{1}{(\delta - \|E\|)\epsilon}}\right)$$

(7)

where $\delta - \|E\| \leq \Delta \leq \delta + \|E\|$.

**Proof.** Suppose the Markov chain $P$, $Q$ and matrix $E$ are in the following block structure

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

(8)

where we order the elements such that the marked ones come last, i.e., $P_4$, $Q_4$ and $E_4 \in \mathbb{C}^{|M| \times |M|}$. The corresponding modified Markov chains [6] would be

$$\tilde{Q} = \begin{pmatrix} Q_1 & 0 \\ Q_3 & I \end{pmatrix} = \begin{pmatrix} P_1 + E_1 & 0 \\ P_3 + E_3 & I \end{pmatrix}.$$  

(9)

By [6], we have $HT(P) = O(\sqrt{\frac{1}{1 - \|P_1\|}})$ and $HT(Q) = O(\sqrt{\frac{1}{1 - \|Q_1\|}})$. Since we know

$$\|P_1\| \leq 1 - \frac{\delta \epsilon}{2} \quad \text{and} \quad \|Q_1\| \leq 1 - \frac{\Delta \epsilon}{2}$$

(10)

by [6] and by Cauchy’s interlacing theorem we have $\|E\| \geq \|E_1\|$ [15, Cor.III.1.5], we then obtain

$$\|Q_1\| \leq \min \left\{ \|P_1\|, \|E\|, 1 - \frac{(\delta - \|E\|)\epsilon}{2} \right\}$$

(11)

as $\delta - \|E\| \leq \Delta \leq \delta + \|E\|$. Therefore, the hitting times for $P$ and $Q$ are derived. $lacksquare$
2.3 Classical Sample Perturbation

In this section we adapt the results from the work [9] to bound the stationary distribution $\pi(Q)$ of a perturbed matrix $Q$ with respect to the perturbation $E$ and the true stationary distribution $\pi(P)$, i.e.,

$$ Q \cdot \pi(Q) = \pi(Q), \quad P \cdot \pi(P) = \pi(P). \tag{12} $$

Let $\Omega$ be the state space and $\Omega' = \Omega \cup \{0\}$. The total variation distance between two probability distributions over $\Omega$ is defined as

$$ D(\pi(P), \pi(Q)) = \frac{1}{2} \sum_{x \in \Omega} \| \pi(P)_x - \pi(Q)_x \|_1 = \max_{S \subseteq \Omega} | \pi(P)_S - \pi(Q)_S |. \tag{13} $$

Here $\pi(P)$ denotes the stationary distribution of matrix $P$, $\pi(P)_x$ is the $x$th element of $\pi(P)$ and $\pi(P)_S$ denotes the sum of $\pi(P)_x$ where $x \in S$, i.e., $\sum_{x \in S} \pi(P)_x = \pi(P)_S$.

In [9] it is assumed that the transition matrix is row-wise stochastic. Our matrix is column-wise stochastic (see eqn. (12)) but since it is symmetric, it is also row-stochastic. By choosing condition number $\kappa_5$ in [9], the ergodicity coefficient, using the $l_p$ norm, is defined as

$$ \tau_p(P) = \sup_{\|v\|_p=1, v^T e = 0} \| v^T P \|_p \tag{14} $$

where $e$ is a column vector of all ones. Since $P$ is a stochastic matrix, the ergodic coefficient satisfies $0 \leq \tau_1(P) \leq 1$. In case of $\tau_1(P) < 1$, we have a perturbation bound in terms of the ergodic coefficient of $P$:

$$ D(\pi(P), \pi(Q)) = \frac{1}{2} \| \pi(P) - \pi(Q) \|_1 \leq \frac{1}{2(1 - (\tau_1 P))} \| E \|_\infty. \tag{15} $$

2.4 Quantum Sample Perturbation

While there are several methods that make use of Szegedy’s quantum walk operators to prepare quantum samples [5, 16, 17], we choose [5] as the main approach to analyze as it leads to an overall speed-up in the general case. The other approaches [16, 17] take advantage of the quantum Zeno effect but the problem is that the quantum Zeno effect would result in an exponential slow-down in the general case.

The work by Wocjan and Abeyesinghe [5] showed an approach to prepare the coherent stationary distribution of a Markov Chain via modified quantum walk and Grover’s $\frac{\pi}{3}$-amplitude amplification techniques. The theorem listed below is the main theorem in Speed-up via Quantum Sampling. We refer the interested readers to [5] for details on this algorithm for the construction techniques and the computational complexity.

**Theorem 1** (Speed-up via quantum sampling [5]). Let $Q_0, Q_1, \ldots, Q_r = Q$ be a sequence of classical Markov chains with stationary distributions $\pi_0, \pi_1, \ldots, \pi_r$ and spectral gap $\delta_0, \ldots, \delta_r$. Assume that the stationary distributions of adjacent Markov chains are close to each other in the sense that $| \langle \pi_i | \pi_{i+1} \rangle |^2 \geq c$ where $c$ is some constant, for $i = 0, \ldots, r - 1$. Then for any $\eta > 0$, there is an efficient quantum sampling algorithm, making it possible to sample according to a probability distribution $\tilde{\pi}_r$ that is close to $\pi_r$ with respect to the total variation distance, i.e., $D(\tilde{\pi}_r, \pi_r) \leq \eta$. 


Based on the theorem above, we can immediately conclude the following corollary:

**Corollary 3.** When the coherent quantum sample based on the perturbed Markov chain is prepared by using techniques of [5] with precision $\eta$, the total variation distance between the prepared quantum sample $\tilde{\pi}(Q)$ and the true quantum sample $\pi(P)$ is less than $\eta + \frac{1}{2(1-(\tau_1 P))}\|E\|_\infty$.

**Proof.** By Theorem 1 we can efficiently construct a quantum sample $\tilde{\pi}(Q)$ that is $\eta$ close to $\pi(Q)$. Then by triangle inequality we obtain

$$D(\pi(P), \tilde{\pi}(Q)) \leq D(\pi(P), \pi(Q)) + D(\pi(Q) + \tilde{\pi}(Q)) \leq \frac{1}{2(1-(\tau_1 P))}\|E\|_\infty + \eta.$$ (16)

\[\Box\]

### 3 Conclusion and Discussion

We apply the existing classical results from matrix perturbation to quantum walk based algorithms. With the noise introduced at the input, as quantum system is susceptible to the outside world and some other precision limitation problem, we bounded the quantum hitting time with perturbation from the above that the perturbed quantum walk hitting time is

$$HT(Q) = O\left(\sqrt{\frac{1}{(\delta - \|E\|)\epsilon}}\right).$$ (17)

In the meanwhile, we also showed that how the quantum sample prepared by using the approach in [5] would fluctuate from the true quantum sample when perturbation is present. The analysis is based on the assumption that we have a series of Markov chains $Q_1, \ldots, Q_r = Q$. Hence, we have

$$D(\pi(P), \tilde{\pi}(Q)) \leq \frac{1}{2(1-(\tau_1 P))}\|E\|_\infty + \eta.$$ (18)

Intuitively from the analysis we can see that the total variation distance for $D(\pi(Q), \pi(Q))$ is simply additive and $D(\pi(P), \pi(Q))$ cannot be eliminated. However, if the matrix $P = Q_i$ is inside the sequence $Q_1, \ldots, Q_r$ where $1 < i < r$, can we invent a procedure to detect to avoid such overshoot? Future study is to find the relation between quantum mixing time, the time it takes to get $\eta$-close to the true stationary distribution, and the quantum hitting time as it was studied so classically. Furthermore, another possible analysis approach would be assuming that we have a series of Markov chains $P_1, \ldots, P_r = P$ (without the noise). We can adapt the analysis in [5] to study how the noise would affect (i) accuracy when blindly preparing the quantum sample without acknowledging the existence of noise or (ii) complexity when the noise is acknowledged and desired accuracy must be achieved.

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