$q$-Whittaker functions, finite fields, and Jordan forms

Slides available at snkarp.github.io

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A **partition** \( \lambda \) is a weakly-decreasing sequence of nonnegative integers.

E.g. \( \lambda = (4, 4, 1) = \begin{array}{cccc}
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{array} \)

A **semistandard tableau** \( T \) is a filling of \( \lambda \) with positive integers which is weakly increasing across rows and strictly increasing down columns.

**Definition (Schur function)**

\[
s_\lambda(x_1, x_2, \ldots) := \sum_T x^T,
\]

where the sum is over all semistandard tableaux \( T \) of shape \( \lambda \).

\( s_\lambda(x) \) is symmetric in the variables \( x_i \).
Schur functions

- e.g. \( s_{(2,1)}(x_1, x_2, x_3) = \)
  \[
  x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2
  \]

Schur functions appear in many contexts; for example, they:

- form an \textit{orthonormal basis} for the algebra of symmetric functions in \( x \);
- are characters of the \textit{irreducible polynomial representations} of \( \text{GL}_n(\mathbb{C}) \);
- give the values of the \textit{irreducible characters} of the symmetric group \( S_n \), when expanded in terms of power sum symmetric functions;
- are representatives for \textit{Schubert classes} in the cohomology ring of the Grassmannian \( \text{Gr}_{k,n}(\mathbb{C}) \);
- define the \textit{Schur processes} of Okounkov and Reshetikhin (2003).
Cauchy identity

**Theorem (Cauchy)**

\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s^\lambda(x)s^\lambda(y)
\]

- The identity is equivalent to the orthonormality of the Schur functions. It also gives the partition function for the Schur processes.
- The left-hand side counts *nonnegative-integer matrices*, and the right-hand side counts *pairs of semistandard tableaux of the same shape*.
- e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
1 + 1 = 1 + 1
\]

\[
12 + 21 = (1, 2) \times (1, 2)
\]
The Burge correspondence (also known as column Robinson–Schensted–Knuth) is a bijection

\[ M \mapsto (P(M), Q(M)) \]

between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- \( P(M) \) is obtained via column insertion and \( Q(M) \) via recording.

- e.g. \( w = 25143 \)

\[
\begin{array}{ccc}
2 & 2 & 1 & 2 \\
5 & 5 & 4 & 5 \\
\end{array}
\begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 \\
\end{array}
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
\end{array}
\]

\( P(w) \) and \( Q(w) \)
Nilpotent matrices

- An \( n \times n \) matrix \( N \) over \( \mathbb{k} \) is \textit{nilpotent} if some power of \( N \) is zero. Such an \( N \) can be conjugated over \( \mathbb{k} \) into \textit{Jordan form}. Let \( JF^\top(N) \) be the \textit{transpose} of the partition given by the sizes of the Jordan blocks.

  \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
  \end{bmatrix} \rightarrow \begin{bmatrix}
  \\
  \\
  \\
  \end{bmatrix} \quad \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \rightarrow \begin{bmatrix}
  \\
  \\
  \end{bmatrix} \quad \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \rightarrow \begin{bmatrix}
  \\
  \\
  \\
  \end{bmatrix}
  \]

  e.g.

- Algebraically, \( JF^\top(N) \) is the partition \( \lambda \) given by
  \[
  \lambda_1 + \lambda_2 + \cdots + \lambda_i = \dim(\ker(N^i)) \quad \text{for all } i.
  \]

**Theorem (Gansner (1981))**

Let \( N \) be a generic \( n \times n \) strictly upper-triangular matrix, where \( N_{i,j} = 0 \) for all inversions \((i, j)\) of \( w^{-1} \). Then \( P(w) \) and \( Q(w) \) can be read off from the Jordan forms of the leading submatrices of \( N \) and \( w^{-1}Nw \).
**Burge correspondence via Jordan forms**

- e.g. $w = 25143$

$$N = \begin{bmatrix} 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (a, b, c, d, e \in \mathbb{k} \text{ generic})$$

**P(w):**

- $1 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $3 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $1 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $3 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $4 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$
- $5 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$

**Q(w):**

- $2 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $5 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $2 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $1 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $3 \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$

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Flag variety

- A complete flag $F$ in $\mathbb{k}^n$ is a sequence of nested subspaces
  \[0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbb{k}^n, \quad \dim(F_i) = i \text{ for all } i.\]

- An $n \times n$ (nilpotent) matrix $N$ is strictly compatible with $F$ if
  \[N(F_i) \subseteq F_{i-1} \quad \text{for all } i.\]

- The matrix $N$ in Gansner’s theorem is precisely one which is strictly compatible with two complete flags $F$ and $F'$ defined by
  \[F_i := \langle e_1, e_2, \ldots, e_i \rangle \quad \text{and} \quad F'_j := \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(j)} \rangle.\]

  The two sequences of matrices in the theorem are $(N|_{F_i})_{i=1}^n$ and $(N|_{F'_j})_{j=1}^n$.

- More generally, we can take any pair of flags $(F, F')$ with relative position $w$, denoted $F \xrightarrow{w} F'$. The relative position records $\dim(F_i \cap F'_j)$ for all $i$ and $j$, or alternatively, the Schubert cell of $F'$ relative to $F$. 
Burge correspondence via flags

Theorem (Steinberg (1976, 1988), Spaltenstein (1982), Rosso (2012))

Fix partial flags $F$ and $F'$ with $F \xrightarrow{M} F'$. Let $N$ be a generic nilpotent matrix strictly compatible with both $F$ and $F'$. Then

$$P(M) = JF^T(N; F) \quad \text{and} \quad Q(M) = JF^T(N; F').$$

- If $F \xrightarrow{w} F'$, then $F' \xrightarrow{w^{-1}} F$. This implies the symmetry

$$P(w^{-1}) = Q(w).$$

- What happens when $k$ is a finite field, and we consider all choices of $N$ (not necessarily generic)?
**q-Whittaker functions**

Define \([n]_q := 1 + q + q^2 + \cdots + q^{n-1}\) and \([n]_q! := [n]_q[n - 1]_q \cdots [1]_q\).

**Definition (q-Whittaker function)**

\[
W_\lambda(x_1, x_2, \ldots; q) := \sum_T \text{wt}_q(T)x^T,
\]
where the sum is over all semistandard tableaux \(T\) of shape \(\lambda\).

- \(W_\lambda(x; q)\) is symmetric in the variables \(x_i\), and specializes to \(s_\lambda(x)\) when \(q = 0\). We obtain the \(\mathfrak{gl}_n\)-Whittaker functions as a certain \(q \to 1\) limit.

- e.g. \(T = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 7 \\
6
\end{array}\) \(\text{wt}_q(T) = [1]_q[2]_q[1]_q[2]_q[2]_q[1]_q[2] = (1 + q)^4\)

- We have the following specializations:
  \[
  W_\lambda(x; q) = P_\lambda(x; q, 0) = q^{\deg(\widetilde{H}_\lambda)}\omega(\widetilde{H}_\lambda(x; 1/q, 0)), \quad W_\lambda(x; 1) = e_\lambda^\top(x).
  \]
Theorem (Macdonald (1995))

\[
\prod_{i,j \geq 1} \prod_{d \geq 0} \frac{1}{1 - x_i y_j q^d} = \sum_{\lambda} \frac{(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}] q!} W_{\lambda}(x; q) W_{\lambda}(y; q)
\]

- This gives the partition function for the \textit{q-Whittaker processes}, a special case of the \textit{Macdonald processes} of Borodin and Corwin (2014).
- e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
(1 - q)^{-2} + (1 - q)^{-2} = (1 - q)^{-1} + (1 - q)^{-2}(1 + q)
\]

\[
\begin{align*}
12 & \quad 21 \\
\left(\frac{1}{2}, \frac{1}{2}\right) & (1 2, 1 2)
\end{align*}
\]
$q$-Burge correspondence

- e.g. $w = 12$  \quad $N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$  \quad ($a \in \mathbb{F}_{1/q}$)

\[
\begin{align*}
P(w): & \quad 1 \begin{bmatrix} \ 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ Q(w): & \quad 1 \begin{bmatrix} \ 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}
\end{align*}
\]

$P = 1 - q$

$P = q$

- e.g. $w = 21$  \quad $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

\[
\begin{align*}
P(w): & \quad 1 \begin{bmatrix} \ 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ Q(w): & \quad 2 \begin{bmatrix} \ 2 \\ 0 \end{bmatrix} \quad 2 \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]

$P = 1$
Let $1/q$ be a prime power, and fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. For semistandard tableaux $T$ and $T'$ of the same shape, define

$$p_M(T, T') := \mathbb{P}(JF^T(N; F) = T \text{ and } JF^T(N; F') = T'),$$

where $N$ is a uniformly random nilpotent matrix strictly compatible with both $F$ and $F'$. (This does not depend on the choice of $(F, F')$.)

**Theorem (Karp, Thomas (2022))**

(i) The maps $p_M(\cdot, \cdot)$ define a probabilistic bijection proving the Cauchy identity for $q$-Whittaker functions, called the $q$-Burge correspondence.

(ii) The bijection converges to the classical Burge correspondence as $q \to 0$.

- The inverse probabilities are also given by ($\ast$), but where $N$ is fixed and $(F, F')$ is uniformly random.
- Two other probabilistic bijections were given by Matveev and Petrov (2017), using $q$-analogues of row and column insertion.
Proof outline

**Theorem (Borho, MacPherson (1983); Karp, Thomas (2022))**

Fix a nilpotent matrix $N$ over $\mathbb{F}_{1/q}$ with Jordan type $\lambda$. The coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $W_\lambda(x; q)$ equals $q \sum_i \binom{\lambda_i}{2} - \binom{\alpha_i}{2}$ times the number of partial flags $F$ over $\mathbb{F}_{1/q}$ strictly compatible with $N$ satisfying

$$\dim(F_i) = \alpha_1 + \cdots + \alpha_i \quad \text{for all } i.$$ 

- e.g. $\lambda = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the coefficient of $x_1 x_2$ in $W_\lambda(x; q)$ is $q^1 \cdot \#(\text{complete flags in } \mathbb{F}_{1/q}^2) = q(1 + 1/q) = q + 1$.

- This is similar to a formula for the modified Hall–Littlewood functions $\tilde{H}_\lambda(x; q, 0)$ in terms of weakly compatible flags over $\mathbb{F}_q$.

- A key step to proving both theorems is enumerating an arbitrary double coset of $P_\alpha \backslash \text{GL}_n(\mathbb{F}_{1/q})/P_\beta$, where $P_\alpha$ and $P_\beta$ are standard parabolic subgroups of $\text{GL}_n(\mathbb{F}_{1/q})$. 

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Combinatorics of the $q$-Burge correspondence

Problem

Is $p_M(T, T')$ a rational function of $q$? (If so, it is a polynomial.)

- We have an explicit formula when $M$ is a diagonal matrix (i.e. $F = F'$).

Problem

Is there a recursive combinatorial rule for calculating $p_M(T, T')$?

- Unlike insertion-based deformations of RSK, the $q$-Burge correspondence does not admit Fomin-style local growth rules. For example, the diagram

\[
\begin{bmatrix}
F'_{j-1} & F'_j \\
F'_{i-1} & F_i \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\]

$M_{i,i} = 0$

\[
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & ? \\
\end{bmatrix}
\]

can be completed to either \[
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\]

or \[
\begin{bmatrix}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{bmatrix}
\].
Consider a path quiver with a unique sink:

\[ Q = \]

A representation \( V \) of \( Q \) is an assignment of a vector space to each vertex and a linear map to each arrow, e.g.,

\[ V = \]

We will only consider \( V \) where every linear map is injective. Isomorphism classes of such \( V \) are indexed by nonnegative-integer matrices \( M \).

We now decorate \( V \) with a linear map for the reverse of each arrow, such that a relation holds for every vertex:

\[ \alpha \circ \gamma \pm \beta \circ \delta = 0 \]

This defines a module \( V^\# \) over the preprojective algebra of \( Q \).
Up to isomorphism, $V^\#$ is given (non-uniquely) by a triple $(F, F', N)$:

$V^\# = \begin{array}{cccc}
F_1 & \text{id} & F_2 & \text{id} \\ 
N & & N & \text{id} \\ 
F_3 = F_3' & \text{id} & F_2' & \text{id} \\ 
- N & - N & - N & \end{array}$

The \textit{socle filtration} of $V^\#$ corresponds precisely to the pair of tableaux

$(T, T') = (JF^T(N; F), JF^T(N; F'))$.

e.g.

$V^\# = \langle e_1 \rangle \xrightarrow{\text{id}} \mathbb{K}^3 \xrightarrow{\text{id}} \langle e_1 \rangle \quad \leftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ \end{pmatrix}$
Counting isomorphism classes

- The $q$-Burge correspondence implies enumerative results about such modules $V^\#$. For example:

**Theorem (Karp, Thomas (2022))**

Let $(T, T')$ be a pair of semistandard tableaux of shape $\lambda$, and let $d$ be a dimension vector of $Q$. Then

$$\sum_{[V^\#]} \frac{1}{|\text{Aut}(V^\#)|} = \frac{q^{c(d)}(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}]_q!} \cdot \text{wt}_q(T) \cdot \text{wt}_q(T'),$$

where the sum is over all isomorphism classes $[V^\#]$ of modules $V^\#$ over $\mathbb{F}_{1/q}$ with dimension vector $d$ and socle filtration corresponding to $(T, T')$.

Thank you!