Exact soliton solution of Spin Chain with a external magnetic field in linear wave background

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Employing a simple, straightforward Darboux transformation we construct exact N-soliton solution for anisotropic spin chain driven by an external magnetic field in linear wave background. As a special case the explicit one- and two-soliton solution dressed by the linear wave corresponding to magnon in quantum theory is obtained analytically and its property is discussed in detail. The dispersion law, effective soliton mass, and the energy of each soliton are investigated as well. Our result show that the stability criterion of soliton is related with anisotropic parameter and the amplitude of the linear wave.

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I. INTRODUCTION

The concept of soliton in spin chain which exhibits both coherent and chaotic structures depending on the nature of the magnetic interactions has received considerable attention for decades. Soliton in quasi one-dimensional magnetic systems is no longer a theoretical concept but can be probed by neutron inelastic scattering, Mossbauer linewidth measurements, and electron spin resonance. One of the integrable models for spin chain is Landau-Lifshitz equation, which has been studied by a variety of techniques such as inverse scattering transformation, Darboux transformation, Riemann-Hilbert approach, etc. Exact soliton solutions are reported for the isotropic and anisotropic spin chains as well. The other integrable model is a type of Nonlinear-Schrödinger equation studied in Ref. It should be noted that all
these solutions are obtained in the ground state background. The exact soliton solutions in linear wave background corresponding to magnon in quantum theory \cite{20} has received no attention yet. The main goal of this paper is to search for new solutions of spin chain driven by external magnetic field in linear wave background. We obtain exact solution of N-soliton train in terms of a simple, straightforward Darboux transformation \cite{21, 22}. Particularly the time-evolution of one and two-soliton is analyzed in terms of the general solution.

The outline of this paper is organized as follows: In Sec. II the Darboux transformation is explained in detail and the general N-soliton solution is obtained. In Sec. III we discuss a special case. Exact one-soliton solution in linear wave background is obtained. The dispersion law and soliton energy are investigated as well. Sec. IV is devoted to general two-soliton solution and soliton collisions. Finally, our concluding remarks are given in Sec. V.

\section{Exact Solution of N-Soliton Train}

Our starting Hamiltonian describing the anisotropic spin chain with an external magnetic field can be written as

\[ \hat{H} = -g \mu_B B \sum_j \hat{S}_j^z - J' \sum_{j, \epsilon} \hat{S}_j^z \hat{S}_{j+\epsilon}^z - \frac{1}{2} J \sum_{j, \epsilon} \left( \hat{S}_j^+ \hat{S}_{j+\epsilon}^- + \hat{S}_j^- \hat{S}_{j+\epsilon}^+ \right), \]

where \( \hat{S}_j \equiv (\hat{S}_j^x, \hat{S}_j^y, \hat{S}_j^z) \) with \( j = 1, 2, \ldots, N \) are spin operators, \( J' > J > 0 \) is the pair interaction parameter, \( g \) the Lande factor and \( \mu_B \) is the Bohr magneton, \( B \) is the external magnetic field. With the help of Holstein-Primokoff \cite{23} transformation \( \hat{S}_j^z = S - a_j^+ a_j \), \( \hat{S}_j^+ \approx \sqrt{2S} \left( 1 - \frac{1}{4S} a_j^+ a_j \right) a_j \), \( \hat{S}_j^- \approx \sqrt{2S} a_j^\dagger \left( 1 - \frac{1}{4S} a_j^+ a_j \right) \), the Hamiltonian in Eq. (1) reduces to

\[ \hat{H} = -2J'S^2N - g \mu_B BSN + g \mu_B B \sum_j a_j^+ a_j \\
+ S \sum_{j, \epsilon} \left\{ J'(a_j^+ a_j + a_{j+\epsilon}^+ a_{j+\epsilon}) - J(a_j a_{j+\epsilon}^+ + a_j^+ a_{j+\epsilon}) \right\} \\
- J' \sum_{j, \epsilon} a_j^+ a_{j+\epsilon} a_j a_{j+\epsilon} + \frac{1}{4} J \sum_{j, \epsilon} \left[ a_j^+ a_{j+\epsilon} a_j a_{j+\epsilon} + a_j^+ a_{j+\epsilon} a_j a_{j+\epsilon}^+ + a_j^+ a_{j+\epsilon} a_j a_{j+\epsilon} \right] \]
where the boson operators $a_j$ are assumed to satisfy the usual commutation relation $[a_j, a_j^\dagger] = \delta_{jj}$, $[a_j^\dagger, a_j^\dagger] = [a_j, a_j] = 0$. The equation of motion for the operator $a_j$ on the $n$th site is $i\hbar \frac{\partial}{\partial t} a_j = [a_j, \hat{H}]$. At low temperatures, the operator $a_j$ can be treated as a classical vector such that $a_j \to a(x, t)$. So that the equation of motion in a continuum spin chain under a magnetic field can be obtained as a Nonlinear-Schrödinger type:

$$
-i \frac{\partial}{\partial t} a_j = \beta_0 \frac{\partial^2}{\partial x^2} a_j + 2\beta_0 \beta_1 a_j |a_j|^2 + 2\beta_2 a_j,
$$

(2)

where

$$
\beta_0 = \frac{2JS}{\hbar}, \beta_1 = \sqrt{\frac{J' - J}{JS}}, \beta_2 = \frac{-g \mu_B B + 4 (J' - J) S}{2\hbar},
$$

here $J' > J > 0$ (easy-axis). In this paper we will present a systematic method to construct general expressions of one- and two-soliton solutions embedded in a linear wave background for Eq. (2) and their novel properties.

By employing Ablowitz-Kaup-Newell-Segur technique one can construct the linear eigenvalue problem for Eq. (2) as follows

$$
\psi_x = U \psi, \quad \psi_t = F \psi,
$$

(3)

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $U$ and $F$ can be given in the forms

$$
U = \lambda \sigma_3 + P,
$$

$$
F = i (2\lambda^2 \beta_0 + \beta_2) \sigma_3 + i2\lambda \beta_0 P - i\beta_0 [P^2 + P_x] \sigma_3,
$$

with

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P(x, t) = \begin{pmatrix} 0 & \beta_1 a \\ -\beta_1 a & 0 \end{pmatrix},
$$

and the overbar denotes the complex conjugate. Thus Eq. (2) can be recovered from the compatibility condition $U_t - F_x + [U, F] = 0$. Based on the Lax pair (3), we can obtain general one- and two-soliton solution embedded in a linear wave background by using a straightforward Darboux transformation[21, 22].

Consider the following transformation

$$
\Psi = (\Lambda I - K) \psi, \quad K = \hat{H} \Lambda \Lambda^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2),
$$

(4)
where $H$ is a nonsingular matrix which satisfies

$$H_x = \sigma_3 H \Lambda + PH.$$  \hspace{1cm} (5)

Letting

$$\Psi_x = U_1 \Psi,$$  \hspace{1cm} (6)

where $U_1 = \lambda \sigma_3 + P_1$, $P_1 = \begin{pmatrix} 0 & \beta_1 a_1 \\ -\beta_1 \bar{a}_1 & 0 \end{pmatrix}$, and with the help of Eqs. (3), (4) and (5), we obtain the Darboux transformation for Eq. (2) from Eq. (6) in the form

$$P_1 = P + [\sigma_3, K].$$  \hspace{1cm} (7)

It is easy to verify that, if $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a eigenfunction of Eq. (3) with eigenvalue $\lambda = \lambda_1$, then $\begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}$ is also the eigenfunction, however with eigenvalue $-\bar{\lambda}_1$. Thus if taking

$$H = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\bar{\lambda}_1 \end{pmatrix},$$  \hspace{1cm} (8)

which ensures that Eq. (5) is held, we can obtain

$$K_{st} = -\bar{\lambda}_1 \delta_{st} + (\lambda_1 + \bar{\lambda}_1) \frac{\psi_s \psi_l}{\psi^T \psi}, \quad s, l = 1, 2,$$  \hspace{1cm} (9)

where $\psi^T \psi = |\psi_1|^2 + |\psi_2|^2$, and Eq. (7) becomes

$$a_1 = a + 2 \beta_1 (\lambda_1 + \bar{\lambda}_1) \frac{\psi_1 \psi_2}{\psi^T \psi},$$  \hspace{1cm} (10)

where $\psi = (\psi_1, \psi_2)^T$ is the eigenfunction of Eq. (3) corresponding to the eigenvalue $\lambda_1$ for the solution $a$. Thus by solving the Eq. (3) which is a first-order linear differential equation, we can generate a new solution $a_1$ of the Eq. (2) from a known solution $a$ which is usually called “seed” solution.

To obtain exact $N$-order solution of Eq. (2), we firstly rewrite the Darboux transformation in Eq. (10) as in the form

$$a_1 = a + 2 \beta_1 (\lambda_1 + \bar{\lambda}_1) \frac{\psi_1 [1, \lambda_1] \bar{\psi}_2 [1, \lambda_1]}{\psi [1, \lambda_1]^T \bar{\psi} [1, \lambda_1]},$$  \hspace{1cm} (11)
where $\psi[1, \lambda] = (\psi_1[1, \lambda], \psi_2[1, \lambda])^T$ denotes the eigenfunction of Eq. (3) corresponding to eigenvalue $\lambda$. Then repeating above the procedure for $N$ times, we can obtain the exact $N$-order solution

$$a_N = a + 2 \sum_{m=1}^{N} (\lambda_m + \overline{\lambda}_m) \frac{\psi_1[m, \lambda_m] \overline{\psi}_2[m, \lambda_m]}{\psi[m, \lambda_m]^T \overline{\psi}[m, \lambda_m]},$$

(12)

where

$$\psi[m, \lambda] = (\lambda - K[m-1]) \cdots (\lambda - K[1]) \psi[1, \lambda],$$

$$K_{st}[j'] = -\overline{\lambda}_{j'} \delta_{st} + (\lambda_{j'} + \overline{\lambda}_{j'}) \frac{\psi_s[j', \lambda_{j'}] \overline{\psi}_1[j', \lambda_{j'}]}{\psi[j', \lambda_{j'}]^T \overline{\psi}[j', \lambda_{j'}]},$$

here $\psi[j', \lambda]$ is the eigenfunction corresponding to $\lambda_{j'}$ for $a_{j'-1}$ with $a_0 \equiv a$ and $s, l = 1, 2, j' = 1, 2, \cdots, m-1, m = 2, 3, \cdots, N$. Thus if choosing a “seed” as the basic initial solution, by solving linear characteristic equation system (3), one can construct a set of new solutions for Eq. (2) by employing the formula (12).

As an example, we give the exact expression of one- and two-soliton solutions in a linear wave background for Eq. (2) respectively and analyze its properties. For this propose, we take the initial “seed” as $a = A_c e^{ik_c x - i\omega_c t}$ which is a linear-wave solution and satisfies the nonlinear dispersion relation $\omega_c = \beta_0 (k_c^2 - 2\beta_1^2 A_c^2) - 2\beta_2$, where $A_c$ and $\omega_c$ are the arbitrary real constants. Substituting the initial seed into (3) and solving the linear equation, by tedious calculation we obtain the expression of eigenfunction corresponding to eigenvalue $\lambda$ in the form

$$\psi_1 = -\beta_1 A_c C_1 \exp \Theta_1 + L_1 C_2 \exp \Theta_2,$$

$$\psi_2 = L_1 C_1 \exp (-\Theta_2) - \beta_1 A_c C_2 \exp (-\Theta_1),$$

(13)

where

$$\Theta_1 = \frac{1}{2} i \varphi + i M (x - \gamma t), \quad \Theta_2 = \frac{1}{2} i \varphi - i M (x - \gamma t),$$

$$L = \lambda - i \frac{k_c}{2} - i M, \quad M = \frac{1}{2} \sqrt{(k_c + i2\lambda)^2 + 4\beta_1^2 A_c^2},$$

$$\gamma = (k_c - i2\lambda) \beta_0, \quad \varphi = k_c x - \omega_c t.$$
III. ONE-SOLITON SOLUTION EMBEDDED IN LINEAR WAVE BACKGROUND

Taking the spectral parameter $\lambda = \lambda_1 = \frac{A_{s,1}}{\beta_1} + i\frac{k_{s,1}}{2}$ in Eq. (13) and substituting them into Eq. (11), we obtain the one-soliton solution in the linear wave background as follows

$$a_1 = e^{i\varphi} \left[ A_c + \frac{A_{s,1}}{\beta_1 \Delta_1} \left[ \beta_1^2 A_c^2 e^{i\Phi_1} + |L_1|^2 e^{-i\Phi_1} - 2\beta_1 A_c (\text{Re} L_1 \cosh \theta_1 + i \text{Im} L_1 \sinh \theta_1) \right] \right]$$

(14)

where

$$\Delta_1 = (|L_1|^2 + \beta_1^2 A_c^2) \cosh \theta_1 - 2\beta_1 A_c (\text{Re} L_1) \cos \Phi,$$

$$\theta_1 = 2 \text{Im} M_1 (x - V_1 t) - x_0,$$

$$\Phi_1 = 2 \text{Re} M_1 (x - V_2 t) - \varphi_0,$$

$$V_1 = \frac{\text{Im} (M_1 \gamma_1)}{\text{Im} M_1}, \quad V_2 = \frac{\text{Re} (M_1 \gamma_1)}{\text{Re} M_1}, \quad \varphi = k_c x - \omega_c t,$$

with the parameters $M_1 = \frac{1}{2} (k_c + i2\lambda_1)^2 + 4\beta_1^2 A_c^2$, $L_1 = \lambda_1 - ik_c/2 - iM_1$, and $\gamma_1 = (k_c - i2\lambda_1) \beta_0$. The parameters $x_0$ and $\varphi_0$ represent the initial center position and initial phase, which are determined by $x_0 = -\ln |C_2/C_1|$, and $\varphi_0 = \arg (C_2/C_1)$, respectively, where $C_1, C_2$ are the arbitrary complex constants. It is worth to note that because $x_0$ and $\varphi_0$ are determined by the value $C_2/C_1$, they in fact depend on only one arbitrary complex parameter. Not loss of generality, we take $C_1 = 1$.

The exact solution $a_1$ in Eq. (14) describes a soliton solution of anisotropic spin chain embedded in a linear-wave background with the soliton amplitude $\frac{A_{s,1}}{\beta_1}$, the width $\frac{1}{2 \text{Im} M_1}$, the wavenumber $k_1 = 2 \text{Re} M_1$, the frequency $\Omega_1 = 2 \text{Re} (M_1 \gamma_1)$, and the envelope velocity $V_1 = \frac{\text{Im} (M_1 \gamma_1)}{\text{Im} M_1}$. As the linear-wave amplitude vanishes, namely $A_c = 0$, this solution in Eq. (14) reduces to the solution in the from

$$a_{1-\text{sol}} = \frac{A_{s,1} e^{i(k_{s,1} x - \Omega_{s,1} t + \varphi_0)}}{\beta_1 \cosh [A_{s,1} (x - V_{s,1} t - x_0')]}.$$

(15)

where $x_0'$ is determined by $x_0' = \frac{-1}{A_{s,1}} \ln |C_2|$. The solution $a_{1-\text{sol}}$ in Eq. (15) describes a bright soliton solution with maximal amplitude $\frac{A_{s,1}}{\beta_1}$, the width $\frac{1}{A_{s,1}}$, envelope velocity $V_{s,1} = 2\beta_0 k_{s,1}$, and center position $x_0'$. The frequency $\Omega_{s,1} = \frac{1}{\hbar} [g\mu_B B + 4 \left( J' - J \right) S - 2JS A_{s,1}^2 + \frac{1}{2} \frac{k_c^2}{4JS} V_{s,1}^2]$ and wavenumber $k_{s,1} = \frac{V_{s,1}}{2\omega_0}$ of the “carrier wave” are related by the dispersion law $\Omega_{s,1} =$
$$\beta_0 (k_{s,1}^2 - A_{s,1}^2) - 2\beta_2 = \beta_0 (k_{s,1}^2 - A_{s,1}^2) + \frac{1}{\hbar} [g \mu_B B + 4 (J' - J) S].$$ It is shown that the external magnetic field $B$ change the frequency but not the amplitude. Then the soliton energy is seen to be

$$E_1 = \hbar \Omega_{s,1} = g \mu_B B + 4 \left( J' - J \right) S - 2JSA_{s,1}^2 + \frac{1}{2} m^* V_{s,1}^2$$

where an effective mass $m^*$ of soliton is $\frac{\hbar^2}{4JS}$. We also notice that the velocity of the “carrier wave”, that is the phase velocity of the soliton $\frac{\Omega_{s,1}}{k_{s,1}} = \frac{V_{s,1}}{2} - \frac{\beta_0 A_{s,1}^2 + 2\beta_2}{k_{s,1}}$, has a negative correction $\frac{\beta_0 A_{s,1}^2 + 2\beta_2}{k_{s,1}}$ for the half of envelope velocity, whereas the soliton group velocity $\frac{d\Omega_{s,1}}{dk_{s,1}} = V_{s,1}$ coincides with the envelope velocity. On the other hand when the soliton amplitude vanishes, namely $A_{s,1} = 0$, the solution $a_1$ in Eq. (14) reduces to the linear-wave solution $a = A_c e^{i\varphi}$, where $\varphi = k_c x - \omega_c t$ and group velocity $V_c = \frac{d\omega_c}{dk_c} = 2\beta_0 k_c$ coming from the nonlinear dispersion relation $\omega_c = \beta_0 (k_c^2 - 2\beta_1 A_{c}^2) - 2\beta_2$.

From the expression of $M_1$ we can directly see that $M_1$ is the pure real number when $k_c = k_{s,1}$ and $A_{s,1}^2 < 4\beta_1^2 A_{c}^2$. The condition $A_{s,1}^2 < 4\beta_1^2 A_{c}^2 = \frac{4n(J' - J)}{JS} A_{c}^2$ is a stability criterion which is related to anisotropic parameter $(J' - J)$ and the amplitude $A_c$ of the linear wave. It is worth to point out that the condition $k_c = k_{s,1}$ implies the equal group velocities for both soliton and linear wave.

IV. TWO-SOLITON SOLUTION EMBEDDED IN LINEAR WAVE BACKGROUND

According to the general formalism in section II it is easy to construct the two-soliton solution in the linear wave background as follows:

$$a_2 = a_1 + e^{i\varphi} \frac{A_{s,2}}{\beta_1 \cosh \Gamma} e^{-i \arg h_2},$$

where

$$h_2 = \frac{(\lambda_2 + \overline{\lambda_1}) \exp (i\varphi) - (\lambda_1 + \overline{\lambda_1}) \rho_1 + (\lambda_2 - \lambda_1) |\rho_1|^2 \rho_2}{(\lambda_2 - \lambda_1) + (\lambda_2 + \overline{\lambda_1}) |\rho_1|^2 + (\lambda_1 + \overline{\lambda_1}) \overline{\rho_1} \rho_2},$$

$$\Gamma = \ln |h_2|, \quad \rho_n = \frac{L_n - \beta_1 A_c e^{i\Phi_n}}{-\beta_1 A_c + L_n e^{i\Phi_n}},$$
with the notations

\[ \theta_n = 2 \text{Im} M_n (x - V_{1,n} t) - x_{0,n}, \]
\[ \Phi_n = 2 \text{Re} M_n (x - V_{2,n} t) - \varphi_{0,n}, \]
\[ V_{1,n} = \frac{\text{Im} (M_n \gamma_n)}{\text{Im} M_n}, V_{2,n} = \frac{\text{Re} (M_n \gamma_n)}{\text{Re} M_n}, \]
\[ \varphi = k_c x - \omega_c t. \]

here the parameters \( \lambda_n = -\mu A_{s,n}/2 + ik_{s,n}/2, \ L_n = \lambda_n - ik_c/2 - iM_n, \ M_n = \frac{1}{2} \sqrt{(k_c + i2\lambda_n)^2 + 4\beta_1^2 A_c^2} \) and \( \gamma_n = (k_c - i2\lambda_n) \beta_0. \) The parameters \( x_{0,n} \) and \( \varphi_{0,n} \) represent the initial center position and initial phase, which are determined by \( x_{0,n} = -\ln |C_{2,n}/C_{1,n}|, \) and \( \varphi_{0,n} = \arg (C_{2,n}/C_{1,n}), \) respectively, where \( C_{1,n}, C_{2,n} \) are the arbitrary complex constants with \( n = 1, 2. \) With the similar reason as in the case of one-soliton solution, we often set \( C_{1,n} = 1 \) and \( C_{2,n} \) are the arbitrary complex constants.

As the linear wave amplitude vanishes \( A_c = 0, \) we have the general two-bright soliton solution from Eq. (16) in the form

\[ a_{2-\text{sol}} = \frac{2}{\beta_1 \Delta_2} (G_1 e^{i(k_{s,2} x - \Omega_{s,2} t + \varphi_{0,2})} + G_2 e^{i(k_{s,1} x - \Omega_{s,1} t + \varphi_{0,1})}), \tag{17} \]

where

\[ G_1 = [(\zeta_1 - \text{Re} \zeta_3) \cosh \theta_1 + i \text{Im} \zeta_3 \sinh \theta_1], \]
\[ G_2 = [(\zeta_2 - \text{Re} \zeta_3) \cosh \theta_2 - i \text{Im} \zeta_3 \sinh \theta_2], \]
\[ \Delta_2 = \zeta_4 \cosh \theta_1 \cosh \theta_2 \]
\[ - A_{s,1} A_{s,2} [\cosh (\theta_1 + \theta_2) + \cos (\Phi_1 - \Phi_2)], \]

here \( \zeta_1 = A_{s,2} |\lambda_2 + \overline{\lambda}_1|^2, \ \zeta_2 = A_{s,1} |\lambda_2 + \overline{\lambda}_1|^2, \ \zeta_3 = A_{s,1} A_{s,2} (\overline{\lambda}_2 + \lambda_1), \ \zeta_4 = 2 |\lambda_2 + \overline{\lambda}_1|^2. \)

The solution \( a_{2-\text{sol}} \) in Eq. (17) is the general form of two-bright soliton solution for Eq. (2) which describes the interaction of two one-bright soliton solutions with the maximal amplitudes \( A_{s,n}/\beta_1, \) the width \( A_{s,n}, \) envelope velocity \( V_{s,n} = 2\beta_0 k_{s,n}, \) \( n = 1, 2, \) respectively. The frequency \( \Omega_{s,n} = \frac{1}{\hbar} [g\mu_B B + 4 \left( J' - J \right) S - 2 J S A_{s,n}^2 + \frac{1}{2} \frac{\hbar^2}{4 J S} V_{s,n}^2] \) and wavenumber \( k_{s,n} = \frac{V_{s,n}}{2\beta_0} \) of each “carrier wave” are related by the dispersion law \( \Omega_{s,n} = \beta_0 (k_{s,n}^2 - A_{s,n}^2) - 2\beta_2. \) Then the energy of each soliton is seen to be

\[ E_n = \hbar \Omega_{s,n} = g\mu_B B + 4 \left( J' - J \right) S - 2 J S A_{s,n}^2 + \frac{1}{2} m^* V_{s,n}^2, \]
with \( n = 1, 2 \), where \( m^* = \frac{\hbar^2}{4JS} \) denotes the effective mass of soliton. We also notice that the velocity of each “carrier wave”, that is the phase velocity of each soliton \( \Omega_{s,n} = \frac{V_{s,n}}{2} - \frac{\beta_0 A^2_{s,n} + 2 \beta_2}{k_{s,n}} \), has a negative correction \( \frac{\beta_0 A^2_{s,n} + 2 \beta_2}{k_{s,n}} \) for the half of envelope velocity, whereas the group velocity of each soliton \( \frac{d\Omega_{s,n}}{dk_{s,n}} = V_{s,n} \) coincides with the envelope velocity of each soliton. When the amplitude \( A_{s,n} = 0, n = 1, 2 \), the solution \( a_2 \) in Eq. (16) reduces to the linear wave solution. Therefore, in general, the solution \( a_2 \) in Eq. (16) represents the interaction of two one-soliton solution in a linear wave background.

As the discussion for Eq. (14), we consider the case of \( k_c = k_{s,1} = k_{s,2} \). Hence we have \( M_n = \frac{1}{2} \sqrt{4 \beta_1^2 A^2_c - A^2_{s,n}}, n = 1, 2 \). From this expression, we are easy to see that the condition \( A^2_{s,n} < 4 \beta_1^2 A^2_c = \frac{4\eta(J' - J)}{JS} A^2_c, n = 1, 2 \), becomes a stability criterion which is related to anisotropic parameter \( (J' - J) \) and the amplitude \( A_c \) of the linear wave. It is also worth to see that the condition \( k_c = k_{s,1} = k_{s,2} \) implies the equal group velocities both for each soliton and linear wave.

V. CONCLUSION

In this paper we obtain exact N-soliton solution for anisotropic spin chain driven by a external magnetic field in linear wave background in terms of a simple, straightforward Darboux transformation. As a special case the explicit one- and two-soliton solution dressed by the linear wave corresponding to magnon in quantum theory is obtained analytically and its property is discussed in detail. The frequency \( \Omega_{s,n} \), wavenumber \( k_{s,n} \), and the dispersion law of each “carrier wave” are also studied. We obtain explicitly the energy \( E_n \) of each soliton and the effective mass \( m^* \) of soliton. Our result show that the stability criterion of soliton is related with anisotropic parameter and the amplitude of the linear wave.

VI. ACKNOWLEDGMENT

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