Relative Leray Numbers via Spectral Sequences

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Abstract

Let \( F \) be a fixed field and let \( X \) be a simplicial complex on the vertex set \( V \). The Leray number \( L(X; F) \) is the minimal \( d \) such that for all \( i \geq d \) and \( S \subset V \), the induced complex \( X[S] \) satisfies \( \tilde{H}_i(X[S]; F) = 0 \). Leray numbers play a role in formulating and proving topological Helly type theorems. For two complexes \( X, Y \) on the same vertex set \( V \), define the relative Leray number \( L_Y(X; F) \) as the minimal \( d \) such that \( \tilde{H}_i(X[V \setminus \sigma]; F) = 0 \) for all \( i \geq d \) and \( \sigma \in Y \). In this paper we extend some results on Leray numbers to the relative setting. In particular, we give an alternative characterization of \( L_Y(X; F) \) in terms of links, and prove a relative version of the topological colorful Helly theorem. Our main tools are the Zeeman spectral sequence and a Mayer-Vietoris type spectral sequence.

1 Introduction

Let \( F \) be a fixed field and let \( X \) be a simplicial complex on the vertex set \( V \). All homology and cohomology groups appearing in the sequel, will be with \( F \) coefficients. The induced subcomplex of \( X \) on a subset \( S \subset V \) is \( X[S] = \{ \sigma \in X : \sigma \subset S \} \).

Definition 1.1. The Leray Number \( L(X) = L(X; F) \) of \( X \) over \( F \) is the minimal \( d \) such that \( \tilde{H}_i(X[S]) = 0 \) for all \( S \subset V \) and \( i \geq d \). The complex \( X \) is \( d \)-Leray over \( F \) if \( L(X) \leq d \).

First introduced by Wegner [14], the family \( L^d \) of \( d \)-Leray complexes over the field \( F \), has the following relevance to Helly type theorems. Let \( F \) be a family of sets. The Helly number \( h(F) \) is the minimal positive integer \( h \) such that if a finite subfamily \( \mathcal{G} \subset F \) satisfies \( \bigcap \mathcal{G}' \neq \emptyset \) for all \( \mathcal{G}' \subset \mathcal{G} \) of cardinality \( \leq h \), then \( \bigcap \mathcal{G} \neq \emptyset \). Let \( h(F) = \infty \) if no such finite \( h \) exists. For example, Helly’s classical theorem asserts that the Helly number of the family of convex sets in \( \mathbb{R}^d \) is \( d + 1 \). Helly type theorems can often be formulated as properties of the associated nerves. Recall that the nerve of a family of sets \( F \) is the simplicial complex \( N(F) \) on the vertex set \( F \), whose simplices are all subfamilies \( \mathcal{G} \subset F \) such that \( \bigcap \mathcal{G} \neq \emptyset \). A simple link between the Helly and Leray numbers is the inequality \( h(F) \leq L(N(F)) + 1 \) (see e.g. (1.2) in [11]). A simplicial complex \( X \) is \( d \)-representable if \( X = N(\mathcal{K}) \) for a family \( \mathcal{K} \) of convex sets in \( \mathbb{R}^d \). Let \( \mathcal{K}^d \) be the set of all \( d \)-representable complexes. Helly’s theorem can then be stated as follows: If \( X \in \mathcal{K}^d \) contain the full \( d \)-skeleton of its vertex set, then \( X \) is a simplex. The nerve lemma (see e.g. [4]) implies that \( \mathcal{K}^d \subset L^d \), but the later family is

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much richer, and there is substantial interest in understanding to what extent Helly type statements for $K^d$ remain true for $L^d$. A basic example is the following. A finite family $\mathcal{F}$ of compact sets in a topological space is a good cover if for any $\mathcal{F}' \subset \mathcal{F}$, the intersection $\bigcap \mathcal{F}'$ is either empty or contractible. If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, then by the nerve lemma $N(\mathcal{F})$ is homotopic to $\bigcup \mathcal{F}$ and therefore $L(N(\mathcal{F})) \leq d$. Hence follows the Topological Helly’s Theorem: If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, then $h(\mathcal{F}) \leq L(N(\mathcal{F})) + 1 \leq d + 1$. For other examples of topological Helly type theorems see e.g. [9, 11, 6].

In this paper we consider the following relative version of Leray numbers. Let $X$ and $Y$ be two complexes on the same vertex set $V$.

**Definition 1.2.** The relative Leray number of $X$ with respect to $Y$ are

$$L_Y(X) = L_Y(X; F) = \min\{d : \tilde{H}_i(X[V \setminus \sigma]) = 0 \text{ for all } i \geq d \text{ and } \sigma \in Y\}.$$

Our first result is an alternative characterization of $L_Y(X)$. We recall a few definitions. The star, link and costar of a simplex $\tau \in X$ are given by

$$\begin{align*}
st(X, \tau) &= \{\sigma \in X : \sigma \cup \tau \in X\} \\
lk(X, \tau) &= \{\sigma \in st(X, \tau) : \sigma \cap \tau = \emptyset\} \\
cost(X, \tau) &= \{\sigma \in X : \sigma \not\supset \tau\}.
\end{align*}$$

It is well known (see e.g. Proposition 3.1 in [10]) that $L(X) \leq d$ iff $\tilde{H}_i(lk(X, \sigma)) = 0$ for all simplices $\sigma \in X$ and $i \geq d$. The relative version of this fact is the following

**Theorem 1.3.**

$$L_Y(X) = \tilde{L}_Y(X) := \min\{d : \tilde{H}_i(lk(X, \sigma)) = 0 \text{ for all } i \geq d \text{ and } \sigma \in Y\}.$$ 

The proof of Theorem 1.3 is an application of a spectral sequence due to Zeeman [15].

The Colorful Helly Theorem due to Bárány and Lovász [1] is a fundamental result with a number of important applications in discrete geometry.

**Theorem 1.4** ([1]). Let $K_1, \ldots, K_{d+1}$ be $d+1$ finite families of convex sets in $\mathbb{R}^d$, such that $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$ for all choices of $K_1 \in K_1, \ldots, K_{d+1} \in K_{d+1}$. Then there exists an $1 \leq i \leq d+1$ such that $\bigcap_{K \in K_i} K \neq \emptyset$.

In [9] we showed that the $d$-representability of $X = N(\bigcup_{i=1}^{d+1} K_i)$ can be replaced by the weaker assumption that $X$ is $d$-Leray.

**Theorem 1.5** ([9]). Let $V = \bigcup_{i=1}^{d+1} V_i$ be a partition of $V$, and let $X$ be a $d$-Leray complex on $V$. View each $V_i$ as a 0-dimensional complex and suppose that $X$ contains the join $V_1 \ast \cdots \ast V_{d+1}$. Then there exists an $1 \leq i \leq d + 1$ such that $V_i$ is a simplex of $X$.

In fact, the transversal matroid $V_1 \ast \cdots \ast V_{d+1}$ in the statement of Theorem 1.5, can be replaced by an arbitrary matroid. In the sequel we identify a matroid with the simplicial complex of its independent sets.

**Theorem 1.6** ([9]). Let $M$ be a matroid with a rank function $\rho_M$, and let $X$ be a $d$-Leray complex over some field $\mathbb{F}$, both on the same vertex set $V$. If $M \subset X$, then there exists a $\sigma \in X$ such that $\rho_M(V \setminus \sigma) \leq d$. 

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Here we prove the following relative version of Theorem 1.6. Let $X, Y$ be simplicial complexes on the vertex set $V$. Let $Y^\vee = \{ A \subset V : V \setminus A \not\subset Y \}$ denote the Alexander dual of $Y$.

**Theorem 1.7.** Let $M$ be a matroid such that $Y^\vee \subset M \subset X$. Then there exists a simplex $\sigma \in X$ such that $\rho_M(Y \setminus \sigma) \leq L_Y(X)$.

The paper is organized as follows. In section 2 we recall the construction of the Zeeman spectral sequence (Theorem 2.1), and deduce a extension (Theorem 2.4) that may be viewed as a spectral sequence generalization of the combinatorial Alexander duality. In section 3 we prove Theorem 1.3. In section 4 we construct a Mayer-Vietoris type spectral sequence (Proposition 4.1), and use it to establish a homological non-vanishing criterion (Corollary 4.2) for certain families of complexes indexed by a geometric lattice. This result is the main ingredient in the proof of Theorem 1.7 given in section 5.

## 2 The Zeeman Spectral Sequence

Let $K$ be a simplicial complex on the vertex set $V$, with a partition $V = V_1 \cup V_2$. For $i = 1, 2$ let $K_i = K[V_i]$. We first recall the construction of a spectral sequence due to Zeeman [15] that converges to the relative homology $H_*(K, K_2)$ (see also McCrory’s paper [7]).

We need some preliminaries. For $p \geq 0$ let $F_p$ be the cohomological local system on $K$ whose value on $\tau$ in $K$ is given by $F_p(\tau) = H_p(K, \text{cost}(K, \tau))$. For $\tau \subset \tau' \in K$, the connecting map $\rho_{\tau, \tau'} : F_p(\tau) \to F_p(\tau')$ is induced by the projection $C_p(K, \text{cost}(K, \tau)) \to C_p(K, \text{cost}(K, \tau'))$. Note that $\text{st}(K, \tau) \cap \text{cost}(K, \tau) = \partial \tau * \text{lk}(K, \tau)$. Hence, by excision

$$F_p(\tau) = H_p(K, \text{cost}(K, \tau)) \cong H_p(\text{st}(K, \tau), \partial \tau * \text{lk}(K, \tau)) \cong \tilde{H}_{p-|\tau|}(\text{lk}(K, \tau)).$$

(1)

The supporting subcomplex of a chain $x = \sum_{\sigma \in K} a_\sigma \sigma \in C_*(K)$ is given by

$$\overline{x} = \{ \tau \in K : \tau \subset \sigma \text{ for some } \sigma \in K \text{ such that } a_\sigma \neq 0 \}.$$

For a subset $U$ of an abelian group $G$, let $\langle U \rangle$ denote the subgroup of $G$ generated by $U$. For a cochain $y \in C^*(K)$ and a subcomplex $K' \subset K$, let $y|_{K'}$ denote the restriction of $y$ to $K'$. Let $\tilde{C}(K) = C_*(K) \otimes C^*(K)$ and let

$$A = \langle x \otimes y \in \tilde{C}(K) : y|_{\overline{x}} = 0 \rangle, \quad B = \langle x \otimes y \in \tilde{C}(K) : y|_{K_1} = 0 \rangle.$$

For $p, q \geq 0$ let

$$\tilde{C}_{pq} = C_p(K) \otimes C^q(K), \quad A_{p,q} = A \cap \tilde{C}_{pq}, \quad B_{p,q} = B \cap \tilde{C}_{pq}.$$

Let

$$D_{-p,q} = \frac{\tilde{C}_{pq}}{A_{p,q} + B_{p,q}}.$$

Then

$$D := \frac{\tilde{C}(K)}{A + B} \cong \bigoplus_{p,q \geq 0} D_{-p,q}.$$

Let $\partial_p : C_p(K) \to C_{p-1}(K)$ and $\delta_q : C_q(K) \to C_{q+1}(K)$ be the usual boundary and coboundary maps and let $d', d'' : \tilde{C}(K) \to \tilde{C}(K)$ be given by $d'(c_p \otimes c^q) = \partial_p c_p \otimes c^q$ and $d''(c_p \otimes c^q) = (-1)^q c_p \otimes \delta_q c^q$. As $d'(A + B) + d''(A + B) \subset A + B$, we may view $d', d''$ as the horizontal and vertical maps of the double complex $D$. Let $d = d' + d''$ be the total differential of $D$. Let $\{1E_r\}$ and $\{1^1E_r\}$ be the fourth quadrant spectral sequences arising from the row, respectively column, filtrations of $D$. 

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Theorem 2.1 (Zeeman). The spectral sequence \( \{ E_r \} := \{ H^r \} \) satisfies

(i) \( \{ E_r^{p,q} \} \) converges to \( H_{p-q}(K, K_2) \).

(ii) \( E_2^{p,q} = H^q(K_1; F_p) \).

Proof. Fix an arbitrary linear order \( \prec \) on the vertex set \( V \), and let \( K(q) \) denote the set of \( q \)-simplices of \( K \). For \( \tau = \{ u_0, \ldots, u_q \} \in X(q) \) such that \( u_0 \prec \cdots \prec u_q \), let \( 1_\tau \in C^q(X) \) be the \( q \)-cochain given by \( 1_\tau([u_{\pi(0)}, \ldots, u_{\pi(q)}]) = \text{sgn}(\pi) \) and \( 1_\tau([v_0, \ldots, v_q]) = 0 \) if \( \{ v_0, \ldots, v_q \} \neq \tau \). Define a homomorphism

\[
\phi : \hat{C}_{p,q} \to \bigoplus_{\sigma \in K(p)} C_p(\sigma) \otimes C^q(\sigma \cap V_1)
\]

as follows. For

\[
\alpha = \sum_{(\sigma, \tau) \in (K(p) \times K(q))} \lambda_{\sigma, \tau} \sigma \otimes 1_\tau \in \hat{C}_{p,q}
\]

let

\[
\phi(\alpha) = \sum_{\sigma \in K(p)} \sigma \otimes \left( \sum_{\tau \in K(q)} \lambda_{\sigma, \tau} (1_\tau |_{\sigma \cap V_1}) \right).
\]

Claim 2.2.

ker \( \phi = A_{p,q} + B_{p,q} \).

Hence \( \phi \) induces an isomorphism

\[
E_0^{-p,q} = D_{-p,q} \cong \bigoplus_{\sigma \in K(p)} C_p(\sigma) \otimes C^q(\sigma \cap V_1).
\]

Proof. The inclusion \( A_{p,q} + B_{p,q} \subset \text{ker} \phi \) is clear. For the other direction, assume that \( \alpha \) as in (2) satisfies \( \phi(\alpha) = 0 \). By the definition of \( \phi \), if \( \lambda_{\sigma, \tau} \neq 0 \), then either \( \tau \not\subset \sigma \) or \( \tau \not\subset V_1 \). It follows that \( \alpha = \alpha_1 + \alpha_2 \) where

\[
\alpha_1 = \sum_{(\sigma, \tau) \in (K(p) \times K(q))} \lambda_{\sigma, \tau} \sigma \otimes 1_\tau \in A_{p,q}
\]

and

\[
\alpha_2 = \sum_{(\sigma, \tau) \in (K(p) \times K(q))} \lambda_{\sigma, \tau} \sigma \otimes 1_\tau \in B_{p,q}.
\]

Let \( \pi_{p, \tau} : C_p(K) \to C_p(K, \text{cost}(K, \tau)) \) denote the projection map. Define

\[
\psi : \hat{C}_{p,q} \to \bigoplus_{\tau \in K_1(q)} C_p(K, \text{cost}(K, \tau)) \otimes \langle 1_\tau \rangle
\]

as follows. For \( \alpha \) as in (2) let

\[
\psi(\alpha) = \sum_{\tau \in K_1(q)} \left( \sum_{\sigma \in K(p)} \lambda_{\sigma, \tau} \pi_{p, \tau}(\sigma) \right) \otimes 1_\tau.
\]
Claim 2.3.

\[ \ker \psi = A_{p,q} + B_{p,q}. \]

Hence \( \psi \) induces an isomorphism

\[ E_0^{-p,q} = D_{-p,q} \cong \bigoplus_{\tau \in K_1(q)} C_p(K, \text{cost}(K, \tau)) \otimes (1_\tau). \]  

(4)

**Proof.** The inclusion \( A_{p,q} + B_{p,q} \subset \ker \psi \) is again clear. For the other direction, assume that \( \alpha \) as in (2) satisfies \( \psi(\alpha) = 0 \). Note that if \( \tau \in K_1(q) \) and \( \lambda_{\sigma,\tau} \neq 0 \), then \( \pi_{p,\tau}(\sigma) = 0 \), i.e. \( \tau \nsubseteq \sigma \). It follows that \( \alpha = \alpha_1 + \alpha_2 \) where

\[ \alpha_1 = \sum_{(\sigma,\tau) \in K(p) \times K(q) \cap V_1} \lambda_{\sigma,\tau} \tau \otimes 1_\tau \in A_{p,q} \]

and

\[ \alpha_2 = \sum_{(\sigma,\tau) \in K(p) \times K(q) \setminus V_1} \lambda_{\sigma,\tau} \tau \otimes 1_\tau \in B_{p,q}. \]

\( \square \)

The isomorphism (3) implies that

\[ 1^E_{-p,q} = \frac{\ker[d'': D_{-p,q} \to D_{-p,q+1}]}{\text{Im}[d'': D_{-p,q-1} \to D_{-p,q}]} = \left\{ \begin{array}{ll} \bigoplus_{\sigma \in K(p)} C_p(\sigma) = C_p(K, K_2) & q = 0, \\ 0 & q \geq 1. \end{array} \right. \]

It follows that

\[ 1^E_{-p,q} = 1^E_{-p,q} = \left\{ \begin{array}{ll} H_p(K, K_2) & q = 0, \\ 0 & q \geq 1. \end{array} \right. \]  

(5)

On the other hand, (4) implies that

\[ E_1^{-p,q} = \frac{\ker[d'': D_{-p,q} \to D_{-(p-1),q}]}{\text{Im}[d'': D_{-(p+1),q} \to D_{-p,q}]} = \bigoplus_{\tau \in K_1(q)} H_p(K, \text{cost}(K, \tau)) \otimes (1_\tau). \]

Therefore

\[ E_2^{-p,q} = H^q(K_1; F_p) \]  

(6)

where \( F_p \) is the cohomological local coefficient system on \( K_1 \) given in (1). Theorem 2.1 now follows from (5) and (6).

\( \square \)

More generally, let \( L \subset K \) be a not necessarily induced subcomplex of \( K \). Let \( W \) denote the set of nonempty simplices of \( K \), let \( W_2 \subset W \) be the set of nonempty simplices of \( L \), and let \( W_1 = W \setminus W_2 \). The barycentric subdivisions of \( K \) and \( L \) satisfy \( \text{sd}(L) = \text{sd}(K)[W_2] \). For a cohomological local system \( \mathcal{G} \) on \( K \), let \( \text{sd}(\mathcal{G}) \) be the cohomological local system on \( \text{sd}(K) \) whose value on the \( k \)-simplex \( \tau = [\tau_0 \subset \cdots \subset \tau_k] \in \text{sd}(K)(k) \) is given by \( \text{sd}(\mathcal{G})(\tau) = \mathcal{G}(\tau_k) \).

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Theorem 2.4. There exists a fourth quadrant spectral sequence \( \{ E_r^{p,q} \} \) converging to \( H_{p-q}(K,L) \), with \( E_2 \) term given by

\[
E_2^{p,q} \cong H^q(\mathrm{sd}(K)[W_1]; \mathcal{F}_p)).
\]

Proof. Let \( \{ E_r^{p,q} \} \) be the Zeeman spectral sequence that converges to the relative homology \( H_{p-q}(\mathrm{sd}(K), \mathrm{sd}(L)) \). By Theorem 2.1, the \( E_2 \) term of the sequence satisfies

\[
E_2^{p,q} = H^q(\mathrm{sd}(K)[W_1], \mathcal{H}_p),
\]

where \( \mathcal{H}_p \) is the local system on \( \mathrm{sd}(K) \), whose value on the simplex \( \tau \in \mathrm{sd}(K) \) is given by \( \mathcal{H}_p(\tau) = \tilde{H}_{p-|\tau|}(\mathrm{lk}(\mathrm{sd}(K), \tau)) \). For a poset \( P \), let \( \Delta(P) \) denote the order complex of \( P \). For two simplices \( \alpha \subset \beta \in K \), let \( (\alpha, \beta) \) denote the open interval \( \{ \gamma : \alpha \subsetneq \gamma \subsetneq \beta \} \). Clearly \( \Delta(\alpha, \beta) \) is homotopic to \( S^{(|\beta|-|\alpha|)-2} \). Let \( \tau = [\tau_0 \subset \cdots \subset \tau_k] \in \mathrm{sd}(K)(k) \), then

\[
\mathrm{lk}(\mathrm{sd}(K), \tau) = \mathrm{lk}(\mathrm{sd}(K), [\tau_0, \ldots, \tau_k])
\]

\[
\cong \Delta(\emptyset, \tau_0) \ast \Delta(\tau_0, \tau_1) \ast \cdots \ast \Delta(\tau_{k-1}, \tau_k) \ast \mathrm{lk}(K, \tau_k)
\]

\[
\cong S^{|\tau_0|-2} \ast S^{|\tau_1|-|\tau_0|-2} \ast \cdots \ast S^{|\tau_k|-|\tau_{k-1}|-2} \ast \mathrm{lk}(K, \tau_k)
\]

\[
\cong S^{|\tau_k|-k-2} \ast \mathrm{lk}(K, \tau_k).
\]

It follows that

\[
\mathcal{H}_p(\tau) \cong \tilde{H}_{p-k-1}(\mathrm{lk}(\mathrm{sd}(K), \tau))
\]

\[
\cong \tilde{H}_{p-k-1}(S^{k+|\tau_k|-k-2} \ast \mathrm{lk}(K, \tau_k))
\]

\[
\cong \tilde{H}_{p-k-1}([\mathrm{lk}(K, \tau_k)]
\]

\[
= \mathcal{F}_p(\tau_k) = \mathrm{sd}(\mathcal{F}_p)(\tau).
\]

Therefore \( \mathcal{H}_p = \mathrm{sd}(\mathcal{F}_p) \), and hence (7) follows from (8).

Let \( \Delta_{n-1} \) denote the \((n-1)\)-simplex on the vertex set \([n] = \{1, \ldots, n\} \), and let \( X \) be a subcomplex of \( \Delta_{n-1} \). The homology of \( X \) is related to the cohomology of its Alexander dual \( X^\vee \) by the following

Theorem 2.5 (Combinatorial Alexander Duality). For all \( 0 \leq q \leq n-1 \)

\[
\tilde{H}_{n-2-q}(X) \cong \tilde{H}^{q-1}(X^\vee).
\]

Theorem 2.5 is a well-known and widely used consequence of the classical Alexander duality, see e.g. Section 6 in [8] and Theorem 2 in [5]. For a simple direct proof see Section 2 in [2]. Here we derive it as a special case of Theorem 2.4.

Proof of Theorem 2.5. Let \( K = \Delta_{n-1} \) and \( L = X \). Let \( \{ E_r^{p,q} \} \) be the spectral sequence of Theorem 2.4. Then

\[
E_\infty^{p,q} \cong H_{p-q}(K,L) = H_{p-q}(\Delta_{n-1}, X).
\]

Keeping the notations of Theorem 2.4, let \( W_1 \) denote the set of simplices of \( K \setminus L = \Delta_{n-1} \setminus X \). Let \( \tau = [\tau_0 \subset \cdots \subset \tau_q] \in \mathrm{sd}(\Delta_{n-1})(q) \). Then

\[
\mathrm{sd}(\mathcal{F}_p)(\tau) = \mathcal{F}_p(\tau_q) = \tilde{H}_{p-|\tau_q|}(\mathrm{lk}(\Delta_{n-1}, \tau_q))
\]

\[
= \begin{cases} 
F & \tau_q = [n] \ & p = n-1, \\
0 & \text{otherwise}. 
\end{cases}
\]
It follows that
\[ E^p_{1,q} = \begin{cases} C^{q-1}(\partial \Delta_{n-1})[W_1] & p = n - 1, \\ 0 & \text{otherwise}, \end{cases} \]
and therefore
\[ E^p_{\infty,q} = E^0_{\infty,q} = \begin{cases} \hat{H}^{q-1}(\partial \Delta_{n-1})[W_1] & p = n - 1, \\ 0 & \text{otherwise}. \end{cases} \] (11)
Next note that map \( \sigma \to [n] \setminus \sigma \) induces a simplicial isomorphism
\[ \text{sd}(\partial \Delta_{n-1})[W_1] \cong \text{sd}(X^\vee). \] (12)
Combining (10), (11) and (12), it follows that
\[ \tilde{H}_{n-2,q}(X) \cong H_{n-1,q}(\Delta_{n-1}, X) \cong E^{-(n-1),q}_{\infty} \]
\[ \cong \hat{H}^{q-1}(\text{sd}(\partial \Delta_{n-1})[W_1]) \cong \hat{H}^{q-1}(X^\vee). \]
\[ \Box \]

**Remark:** The above argument carries over without change to the case of integer coefficients.

### 3 Relative Leray Numbers

**Proof of Theorem 1.3.** Clearly, \( L_Y(X) = \max\{L_\sigma(X) : \sigma \in Y\} \), and \( \tilde{L}_Y(X) = \max\{\tilde{L}_\sigma(X) : \sigma \in Y\} \). It therefore suffices to show that \( L_\sigma(X) = \tilde{L}_\sigma(X) \) for a simplex \( \sigma \). We argue by induction on \( s = \dim \sigma \). If \( s = 1 \) then \( \sigma = \{\emptyset\} \) and \( X[V \setminus \sigma] = \text{lk}(X, \sigma) = X \). Suppose now that \( s \geq 1 \) and let \( L_\sigma(X) = d \), \( \tilde{L}_\sigma(X) = \tilde{d} \). Consider the Zeeman spectral sequence \( \{E^r_{p,q}\} \) that converges to \( H_{p-q}(X, X[V \setminus \sigma]) \). Then
\[ E^r_{1,q} = \bigoplus_{\tau \in X[V \setminus \sigma](q)} \hat{H}_{r-q-1}(\text{lk}(X, \tau)). \] (13)
We first show that \( d \leq \tilde{d} \). Let \( i \geq \tilde{d} \). If \( p - q = i + 1 \), then \( \tilde{H}_{p-q-1}(\text{lk}(X, \tau)) = \hat{H}_i(\text{lk}(X, \tau)) = 0 \) for any \( \tau \subset \sigma \). It follows from (13) that \( E^r_{1,p} = 0 \). Therefore \( H_{i+1}(X, X[V \setminus \sigma]) = 0 \). The short exact sequence
\[ 0 = H_{i+1}(X, X[V \setminus \sigma]) \to \hat{H}_i(X[V \setminus \sigma]) \to \hat{H}_i(X) = \hat{H}_i(\text{lk}(X, \emptyset)) = 0 \]
implies that \( \hat{H}_i(X[V \setminus \sigma]) = 0 \) and hence \( d \leq \tilde{d} \). For the other direction, assume that \( i \geq d \). We first show that
\[ E^{-(i+s+1),s}_{\infty}(\text{lk}(X, \sigma)). \] (14)
Indeed, Eq. (13) implies that \( E^{r,p}_{1,q} = 0 \) for \( q > s \) and \( E^{-(i+s+1),s}_{r}(\text{lk}(X, \sigma)) \). If \( r \leq \sigma \), then by induction \( L_\tau(X) = \tilde{L}_\tau(X) \), hence \( \hat{H}_j(\text{lk}(X, \tau)) = 0 \) for any \( j \geq d = L_\sigma(X) \geq L_\tau(X) \). In particular, for any \( r \geq 1 \)
\[ E^{-(i+s+2-r),s-r}_{r} = \bigoplus_{\tau \in X[\sigma](s-r)} \hat{H}_{r+1}(\text{lk}(X, \tau)) = 0, \]
hence \( E^{-(i+s+2-r),s-r}_{r} = 0 \). It follows that the differentials
\[ d_r : E^{-(i+s+2-r),s-r}_{r} \to E^{-(i+s+1),s}_{r} \]
are trivial for \( r \geq 1 \), and therefore
\[
E_{\infty}^{-(i+s+1),s} = E_1^{-(i+s+1),s} = \tilde{H}_i(\lk(X,\sigma)).
\]

On the other hand, the exact sequence
\[
0 = \tilde{H}_{i+1}(X) \to \tilde{H}_{i+1}(X, X[V \setminus \sigma]) \to \tilde{H}_i(X[V \setminus \sigma]) = 0
\]
implies that \( H_{i+1}(X, X[V \setminus \sigma]) = 0 \), and in particular
\[
E_{\infty}^{-(i+s+1),s} = 0.
\]

Finally, \( \tilde{H}_i(\lk(X,\sigma)) = 0 \) follows from (14) and (15), and hence \( d \geq \tilde{d} \).

\[\square\]

4 Empty Intersections and Non-Vanishing Homology

Let \( M \) be a matroid with rank function \( \rho_M \) on the ground set \( V \). Let \( \mathcal{K}(M) \) denote the poset of all nontrivial flats \( K \neq \emptyset, V \) of \( M \), ordered by inclusion. For a poset \( P \) and an element \( x \in \mathcal{P} \), let \( P_{\geq x} = \{ y \in P : y \geq x \} \) and \( P_{> x} = \{ y \in P : y > x \} \). It is classically known (see e.g. [3]) that \( \tilde{H}_j(\Delta(\mathcal{K}(M))) = 0 \) for \( j \neq \rho_M(V) - 2 \). Let \( K \in \mathcal{K}(M) \) and let \( B_K \) be an arbitrary basis of \( K \). The contraction of \( K \) from \( M \) is the matroid on \( V \setminus K \) defined by \( M/K = \{ A \subset V \setminus K : B_K \cup A \subset \mathcal{M} \} \) (see e.g. [12]). The matroid \( M/K \) satisfies \( \rho_{M/K}(V \setminus K) = \rho_M(V) - \rho_M(K) \) and \( \mathcal{K}(M/K) \cong \mathcal{K}(M)_{>K} \).

Let \( \{ Y_K : Y \in \mathcal{K}(M) \} \) be a family of simplicial complexes such that \( Y_K \subset Y_{K'} \) whenever \( K \subset K' \in \mathcal{K}(M) \). Let \( Y = \bigcup_{K \in \mathcal{K}(M)} Y_K \). The proof of the following result is a standard application of the method of simplicial resolutions (see e.g. Vassiliev’s paper [13]).

**Proposition 4.1.** There exists a first quadrant spectral sequence \( \{ E_{p,q}^r \} \) converging to \( H_*(Y) \) whose \( E^1 \) term satisfies
\[
E_{p,q}^1 \cong \bigoplus_{K \in \mathcal{K}(M), \rho_M(K) = \rho_M(V) - p - 1} H_q(Y_K) \otimes \tilde{H}_{p-1}(\mathcal{K}(M/K)).
\]

**Proof.** Let \( \rho_M(V) = m \). For \( 0 \leq p \leq m - 2 \) let
\[
F_p = \bigcup_{K \in \mathcal{K}(M), \rho_M(K) \geq m - p - 1} Y_K \times \Delta(\mathcal{K}(M)_{\geq K}).
\]

The filtration \( F_0 \subset \cdots \subset F_{m-2} \) gives rise to a first quadrant spectral sequence \( \{ E_{p,q}^r \} \) that converges to
\[
H_*(F_{m-2}) \cong H_*(\bigcup_{K \in \mathcal{K}(M), \rho_M(K) = m - p - 1} Y_K \times \Delta(\mathcal{K}(M)_{\geq K})) \cong H_*(Y).
\]

Let
\[
G_p = \bigcup_{K \in \mathcal{K}(M), \rho_M(K) = m - p - 1} Y_K \times \Delta(\mathcal{K}(M)_{\geq K}).
\]
Then $F_p = G_p \cup F_{p-1}$ and

$$G_p \cap F_{p-1} = \bigcup_{K \in \mathcal{K}(M)} Y_K \times \Delta(\mathcal{K}(M)_{> K}).$$

Additionally, if $K \neq K' \in \mathcal{K}(M)$ satisfy $\rho_M(K) = \rho_M(K') = m - p - 1$, then

$$\left( Y_K \times \Delta(\mathcal{K}(M)_{\geq K}) \right) \cap \left( Y_{K'} \times \Delta(\mathcal{K}(M)_{\geq K'}) \right) \subset Y_K \times \Delta(\mathcal{K}(M)_{> K})$$

Using excision, Eq. (17), and the Künneth formula, it follows that

$$E^1_{p,q} = H_{p+q}(F_p, F_{p-1})$$

$$\cong H_{p+q}(G_p, G_p \cap F_{p-1})$$

$$= H_{p+q} \left( \bigcup_{\rho_M(K)=m-p-1} Y_K \times \Delta(\mathcal{K}(M)_{\geq K}), \bigcup_{\rho_M(K)=m-p-1} Y_K \times \Delta(\mathcal{K}(M)_{> K}) \right)$$

$$\cong \bigoplus_{\rho_M(K)=m-p-1} H_{p+q} \left( Y_K \times \Delta(\mathcal{K}(M)_{\geq K}), Y_K \times \Delta(\mathcal{K}(M)_{> K}) \right)$$

$$\cong \bigoplus_{\rho_M(K)=m-p-1} H_i(Y_K) \otimes H_j \left( \Delta(\mathcal{K}(M)_{\geq K}), \Delta(\mathcal{K}(M)_{> K}) \right)$$

$$\cong \bigoplus_{\rho_M(K)=m-p-1} H_i(Y_K) \otimes \tilde{H}_{j-1}(K(M)_{> K})$$

$$\cong \bigoplus_{\rho_M(K)=m-p-1} H_i(Y_K) \otimes \tilde{H}_{j-1}(K(M/K)).$$

As $\tilde{H}_{j-1}(K(M/K)) = 0$ for $j - 1 \neq r_{M/K}(V \setminus K) - 2 = \rho_M(V) - \rho_M(K) - 2 = p - 1$, it follows from (18) that

$$E^1_{p,q} \cong \bigoplus_{\rho_M(K)=m-p-1} H_q(Y_K) \otimes \tilde{H}_{p-1}(K(M/K)).$$

Let $\{Z_K : Z \in \mathcal{K}(M)\}$ be a family of simplicial complexes such that $Z_{K'} \subset Z_K$ whenever $K \subset K' \in \mathcal{K}(M)$. Proposition 4.1 implies the following

**Corollary 4.2.** Suppose that $\bigcap_{K \in \mathcal{K}(M)} Z_K = \{\emptyset\}$. Then there exist $0 \leq p \leq \rho_M(V) - 2$ and $K \in \mathcal{K}(M)$ of rank $\rho_M(K) = \rho_M(V) - p - 1$, such that $\tilde{H}_{p-1}(Z_K) \neq 0$.

**Proof.** Let $\rho_M(V) = m$. We may assume that all $Z_K$’s are subcomplexes of the simplex $\Delta_{N-1}$ for some $N > m$. Let $Y_K = Z_K^\vee$ be the Alexander dual of $Z_K$ in $\Delta_{N-1}$. Then $Y_K \subset Y_{K'}$ for $K \subset K' \in \mathcal{K}(M)$, and

$$Y = \bigcup_{K \in \mathcal{K}(M)} Y_K = \bigcup_{K \in \mathcal{K}(M)} Z_K^\vee = \left( \bigcap_{K \in \mathcal{K}(M)} Z_K \right)^\vee = \{\emptyset\}^\vee = \partial \Delta_{N-1} \cong S^{N-2}.$$

By (16) there exist $0 \leq p \leq m - 2$ and $q \geq 0$ such that $p + q = N - 2$, and a flat $K \in \mathcal{K}(M)$ of rank $\rho_M(K) = m - p - 1$ such that $H_q(Y_K) \neq 0$. Note that $q = N - 2 - p > m - 2 - p \geq 0$. By Alexander duality we obtain

$$0 \neq H_q(Y_K) = \tilde{H}_q(Y_K) = \tilde{H}_q(Z_K^\vee) \cong \tilde{H}_{N-3-q}(Z_K) = \tilde{H}_{p-1}(Z_K).$$

□
5 A Relative Topological Colorful Helly Theorem

Proof of Theorem 1.7. Let \( M^* = \{ \sigma \subset V : \rho_M(V \setminus \sigma) = \rho_M(V) \} \) be the dual matroid of \( M \). The rank function of \( M^* \) satisfies \( \rho_{M^*}(A) = |A| - \rho_M(V) + \rho_M(V \setminus A) \). For \( K \in \mathcal{K}(M^*) \), we view the simplices of \( X^\vee \setminus X^\vee[K] \) as a poset ordered by inclusion, and consider its order complex \( Z_K = \Delta (X^\vee \setminus X^\vee[K]) \). The inclusion

\[
\text{sd} \left( X^\vee[V \setminus K] \right) = \Delta \left( X^\vee[V \setminus K] \right) \to \Delta \left( X^\vee \setminus X^\vee[K] \right) = Z_K,
\]

induces a homotopy equivalence

\[
Z_K \simeq X^\vee[V \setminus K]. \tag{19}
\]

Let \( \sigma \in X^\vee \). Then \( V \setminus \sigma \not\in X \), and hence \( V \setminus \sigma \not\in M \). In particular, \( \sigma \) does not contain a basis of \( M^* \), and thus \( \sigma \subset K \) for some \( K \in \mathcal{K}(M^*) \). Hence \( \sigma \not\in Z_K \). It follows that

\[
\bigcap_{K \in \mathcal{K}(M^*)} Z_K = \emptyset.
\]

By Corollary 4.2, there exists a \( K \in \mathcal{K}(M^*) \) such that

\[
\tilde{H}_{p-1}(Z_K) \neq 0 \tag{20}
\]

and

\[
\rho_{M^*}(K) = \rho_{M^*}(V) - p - 1. \tag{21}
\]

As \( K \in \mathcal{K}(M^*) \), it follows in particular that \( V \setminus K \not\in M \). The assumption \( Y^\vee \subset M \) then implies that \( K \in Y \). Furthermore, it follows by (21) that

\[
\rho_M(V \setminus K) = |V| - |K| - p - 1. \tag{22}
\]

Using (20),(19), Alexander duality and (22), we obtain

\[
0 \neq \tilde{H}_{p-1}(Z_K) \cong \tilde{H}_{p-1} \left( X^\vee[V \setminus K] \right) = \tilde{H}_{p-1} \left( \text{lk}(X,K)^\vee \right) \cong \tilde{H}_{|V| - |K| - p - 2} \left( \text{lk}(X,K) \right) = \tilde{H}_{\rho_{M}(V\setminus K) - 1} \left( \text{lk}(X,K) \right).
\]

As \( K \in Y \), it follows from Theorem 1.3 that \( \rho_M(V \setminus K) \leq L_Y(X) \). Finally, \( K \in X \) since \( \tilde{H}_* \left( \text{lk}(X,K) \right) \neq 0 \).

References

[1] I. Bárány, A generalization of Carathéodory’s theorem, Discrete Math., 40(1982) 141–152.

[2] D. Bayer, H. Charalambous and S. Popescu, Extremal Betti numbers and applications to monomial ideals, J. Algebra, 221(1999) 497-512.

[3] A. Björner, The homology and shellability of matroids and geometric lattices. Matroid applications, 226-283, Encyclopedia Math. Appl., 40, Cambridge Univ. Press, Cambridge, 1992.
[4] A. Björner, Nerves, fibers and homotopy groups, *J. Combin. Theory Ser. A*, 102(2003) 88-93.

[5] A. Björner, L. Butler and A. Matveev, Note on a combinatorial application of Alexander duality, *J. Combin. Theory Ser. A*, 80(1997) 163-165.

[6] E. Colin de Verdière, G. Ginot and X. Goaoc, Helly numbers of acyclic families, *Adv. Math.*, 253(2014) 163-193.

[7] C. McCrory, Zeeman’s filtration of homology, *Trans. Amer. Math. Soc.*, 250(1979) 147-166.

[8] G. Kalai, Enumeration of $\mathbb{Q}$-acyclic simplicial complexes, *Israel J. Math.*, 45(1983) 337–351.

[9] G. Kalai and R. Meshulam, A topological colorful Helly Theorem, *Adv. Math.*, 191(2005) 305–311.

[10] G. Kalai and R. Meshulam, Intersections of Leray complexes and regularity of monomial ideals, *J. Combin. Theory Ser. A*, 113(2006) 1586–1592.

[11] G. Kalai and R. Meshulam, Leray numbers of projections and a topological Helly type theorem, *Journal of Topology*, 1(2008) 551–556.

[12] J. Oxley, Matroid theory. Second edition. Oxford Graduate Texts in Mathematics, 21. Oxford University Press, Oxford, 2011.

[13] V. A. Vassiliev, Topology of plane arrangements and their complements, *Russian Math. Surveys*, 56(2001) 365-401.

[14] G. Wegner, $d$-Collapsing and nerves of families of convex sets, *Arch. Math. (Basel)*, 26(1975) 317–321.

[15] E. C. Zeeman, A generalization of the Poincaré duality for manifolds III, *Proc. London Math. Soc.*, 13(1963) 155-183.