Homological interpretation of extensions and biextensions of 1-motives

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HOMOLOGICAL INTERPRETATION OF EXTENSIONS AND BIEXTENSIONS OF 1-MOTIVES

CRISTIANA BERTOLIN

ABSTRACT. Let $k$ be a separably closed field. Let $K_i = [A_i \rightarrow^u B_i]$ (for $i = 1, 2, 3$) be three 1-motives defined over $k$. We define the geometrical notions of extension of $K_1$ by $K_3$ and of biextension of $(K_1, K_2)$ by $K_3$. We then compute the homological interpretation of these new geometrical notions: namely, the group $\text{Biext}^0(K_1, K_2; K_3)$ of automorphisms of any biextension of $(K_1, K_2)$ by $K_3$ is canonically isomorphic to the group $\text{Ext}^0(K_1 \otimes K_2, K_3)$, and the group $\text{Biext}^1(K_1, K_2; K_3)$ of isomorphism classes of biextensions of $(K_1, K_2)$ by $K_3$ is canonically isomorphic to the group $\text{Ext}^1(K_1 \otimes K_2, K_3)$.

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INTRODUCTION

Let $k$ be a separably closed field and let $S = \text{Spec}(k)$. A 1-motive $K = [u : A \rightarrow B]$ over $S$ consists of an $S$-group scheme $A$ which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, an extension $B$ of an abelian $S$-scheme by an $S$-torus, and a morphism $u : A \rightarrow B$ of $S$-group schemes. Since the field $k$ is separably closed, remark that $A = \mathbb{Z}r$ with $r \geq 0$.

Let $S$ be the big fpqc site over $S$. A 1-motive $K = [u : A \rightarrow B]$ can be viewed also as a complex of abelian sheaves on $S$ concentrated in two consecutive degrees. A morphism of 1-motives is a morphism of complexes of commutative $S$-group schemes (see [R], in particular Lemma 2.3.2)

Let $K_i = [u_i : A_i \rightarrow B_i]$ (for $i = 1, 2, 3$) be three 1-motives defined over $S$. In this paper we introduce the geometrical notions of extension of $K_1$ by $K_3$ and of biextension of $(K_1, K_2)$ by $K_3$. We then compute the homological interpretation of

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these new geometrical notions. More precisely, if \( \text{Biext}^0(K_1, K_2; K_3) \) is the group of automorphisms of any biextension of \((K_1, K_2)\) by \(K_3\), \( \text{Biext}^1(K_1, K_2; K_3) \) is the group of isomorphism classes of biextensions of \((K_1, K_2)\) by \(K_3\), \( \text{Ext}^0(K_1, K_3) \) is the group of automorphisms of any extension of \(K_1\) by \(K_3\), and \( \text{Ext}^1(K_1, K_3) \) is the group of isomorphism classes of extensions of \(K_1\) by \(K_3\), then we prove

**Theorem 0.1.** We have the following canonical isomorphisms

(a) \( \text{Biext}^1(K_1, K_2; K_3) \cong \text{Ext}^1(K_1 \otimes K_2, K_3) = \text{Hom}_{D(S)}(K_1 \otimes K_2, K_3[1]) \),

(b) \( \text{Biext}^0(K_1, K_2; K_3) \cong \text{Ext}^0(K_1 \otimes K_2, K_3) = \text{Hom}_{D(S)}(K_1 \otimes K_2, K_3) \),

(c) \( \text{Ext}^1(K_1, K_3) \cong \text{Ext}^1(K_1, K_3) = \text{Hom}_{D(S)}(K_1, K_3[1]) \),

(d) \( \text{Ext}^0(K_1, K_3) \cong \text{Ext}^0(K_1, K_3) = \text{Hom}_{D(S)}(K_1, K_3) \),

where \( K_1 \otimes K_2 \) is the derived functor of the functor \( K_2 \to K_1 \otimes K_2 \) in the derived category \( D(S) \) of complexes of abelian sheaves on \( S \).

The homological interpretation (c)-(d) of extensions of 1-motives is a special case of the homological interpretation (a)-(b) of biextensions of 1-motives: in fact, if \( K_2 = [0 \to \mathbb{Z}] \)

(1) the category of biextensions of \((K_1, [0 \to \mathbb{Z}])\) by \(K_3\) is equivalent to the category of extensions of \(K_1\) by \(K_3\), and

(2) in the derived category \( \text{Ext}^i(K_1 \otimes [0 \to \mathbb{Z}], K_3) \cong \text{Ext}^i(K_1, K_3) \) for \( i = 0, 1 \).

Applications of Theorem 0.1 are given by the isomorphism

\[
\text{Biext}^1(K_1, K_2; K_3) \cong \text{Ext}^1(K_1 \otimes K_2, K_3) = \text{Hom}_{D(C)}(K_1 \otimes K_2, K_3[1])
\]

which makes explicit the link between biextensions and bilinear morphisms. A classical example of this link is given by the Poincaré biextension of an abelian variety which defines the Weil pairing on the Tate modules. Other examples are furnished by [BOS] and [BM], where we prove that

- the group of isomorphism classes of biextensions of \((K_1, K_2)\) by \(K_3\) is isomorphic to the group of morphisms of the category \( MH \) of mixed Hodge structures from the tensor product \( T_H(K_1) \otimes T_H(K_2) \) of the Hodge realizations of \(K_1\) and \(K_2\) to the Hodge realization \( T_H(K_3) \) of \(K_3\):

\[
\text{Biext}^1(K_1, K_2; K_3) \cong \text{Hom}_{MH}(T_H(K_1) \otimes T_H(K_2), T_H(K_3)).
\]

- modulo isogenies the group of isomorphism classes of biextensions of \((K_1, K_2)\) by \(K_3\) is isomorphic to the group of morphisms of the category \( MR_{\mathbb{Z}}(k) \) of mixed realizations with integral structure from the tensor product \( T(K_1) \otimes T(K_2) \) of the realizations of \(K_1\) and \(K_2\) to the realization \( T(K_3) \) of \(K_3\):

\[
\text{Biext}^1(K_1, K_2; K_3) \otimes \mathbb{Q} \cong \text{Hom}_{MR_{\mathbb{Z}}(k)}(T(K_1) \otimes T(K_2), T(K_3)).
\]

Following Deligne’s philosophy of motives described in [D89] 1.11, this isomorphism means that the notion of biextensions of 1-motives furnishes the geometrical origin of the morphisms of \( MR_{\mathbb{Z}}(k) \) from the tensor product of the realizations of two 1-motives to the realization of another 1-motive, which are therefore motivic morphisms.

- modulo isogenies the group of isomorphism classes of biextensions of \((K_1, K_2)\) by \(K_3\) is isomorphic to the group of morphisms of Voevodsky’s triangulated category \( D\text{M}_{gm}^{\text{eff}} \) of effective geometrical motives with rational coefficients
from the tensor product $\mathcal{O}(K_1) \otimes \mathcal{O}(K_2)$ of the images of $K_1$ and $K_2$ in the category $\text{DM}^{\text{eff}}_{\text{gm}}$ to the image $\mathcal{O}(K_3)$ of $K_3$ in $\text{DM}^{\text{eff}}_{\text{gm}}$.

$\text{Biext}^i(K_1, K_2; K_3) \otimes \mathbb{Q} \cong \text{Hom}_{\text{DM}^{\text{eff}}_{\text{gm}}(k, \mathbb{Q})}(\mathcal{O}(K_1) \otimes \mathcal{O}(K_2), \mathcal{O}(K_3))$.

In [BM] we have used Theorem 0.1(a) in order to show the above isomorphism.

In [Be11] and [Be] we have introduced the notions of extension and biextension for arbitrary length 2 complexes of abelian sheaves and we have computed their homological interpretation. The definitions and the results of [Be11] and [Be] are a generalization of the definitions and the results of this paper (in particular of Theorem 0.1) to arbitrary length 2 complexes of abelian sheaves.

The idea of the proof of Theorem 0.1 is the following one: Let $K = [A \rightarrow B]$ be a 1-motives and let $L_\cdot$ be a complex of 1-motives $R \rightarrow Q \rightarrow P \rightarrow 0$. To the complex $K$ and to the bicomplex $L_\cdot$ we associate a category $\Psi_{L_\cdot}(K)$ which has the following homological description:

$\Psi^0_{L_\cdot}(K) \cong \text{Ext}^1(\text{Tot}(L_\cdot), K) \quad (i = 0, 1)$

where $\Psi^0_{L_\cdot}(K)$ is the group of automorphisms of any object of $\Psi_{L_\cdot}(K)$ and $\Psi^1_{L_\cdot}(K)$ is the group of isomorphism classes of objects of $\Psi_{L_\cdot}(K)$. Then, to any 1-motive $K = [A \rightarrow B]$ we associate a canonical flat partial resolution $L_\cdot(K)$ whose components are direct sums of objects of the kind $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$. Here “partial resolution” means that we have an isomorphism between the homology groups of $K$ and of this partial resolution only in degree 1 and 0. This is enough for our goal since only the groups $\text{Ext}^1$ and $\text{Ext}^0$ are involved in the statement of Theorem 0.1. Consider now three 1-motives $K_i$ (for $i = 1, 2, 3$). The categories $\Psi_{L_\cdot(K_i)}(K_3)$ and $\Psi_{L_\cdot(K_1) \otimes L_\cdot(K_2)}(K_3)$ admit the following geometrical description:

$\Psi_{L_\cdot(K_1)}(K_3) \cong \text{Ext}(K_1, K_3)$

$\Psi_{L_\cdot(K_1) \otimes L_\cdot(K_2)}(K_3) \cong \text{Biext}(K_1, K_2; K_3)$

Putting together this geometrical description (0.3) with the homological description (0.2), we get the proof of Theorem 0.1.

**Notation**

In this paper, $k$ is a separably closed field, $S = \text{Spec}(k)$ and $\mathbf{S}$ is the big fppf site over $S$. If $I$ is a sheaf on $\mathbf{S}$, we denote by $\mathbb{Z}[I]$ the free $\mathbb{Z}$-module generated by $I$ (see [173] Exposé IV 11).

Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K = (K^i)$, such that $K^i = 0$ for $i \neq -1$ or 0. The good truncation $\tau_{\leq n} K$ of a complex $K$ of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leq n} K)^i = K^i$ for $i < n, (\tau_{\leq n} K)^n = \ker(d^n)$ and $(\tau_{\leq n} K)^i = 0$ for $i > n$.

For any $i \in \mathbb{Z}$, the shift functor $[i] : \mathcal{K}(\mathbf{S}) \rightarrow \mathcal{K}(\mathbf{S})$ acts on a complex $K = (K^n)_n$ as $(K[i])^n = K^{i+n}$ and $d^n_{K[i]} = (-1)^i d^n_K$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes $K$ such that $\text{H}^i(K) = 0$ for $i \neq -1$ or 0. If $K$ and $K'$ are complexes of $\mathcal{D}(\mathbf{S})$, the group $\text{Ext}^i(K, K')$ is by definition $\text{Hom}_{\mathcal{D}(\mathbf{S})}(K, K'[i])$ for any $i \in \mathbb{Z}$. Let $\text{RHom}(-, -)$ be
the derived functor of the bifunctor $\text{Hom}(-, -)$. The cohomology groups
$H^i(R\text{Hom}(K, K'))$ of $R\text{Hom}(K, K')$ are isomorphic to $\text{Hom}_{\mathcal{D}(S)}(K, K'[i])$.

1. Extensions and biextensions of 1-motives

Let $G$ be abelian sheaf on $S$. A $G$-torsor is a sheaf on $S$ endowed with an action
of $G$, which is locally isomorphic to $G$ acting on itself by translation.

Let $P, G$ be abelian sheaves on $S$. An extension of $P$ by $G$ is an exact sequence
$$0 \rightarrow G \rightarrow E \rightarrow P \rightarrow 0.$$ 

Since in this paper we consider only commutative extensions, $E$ is in fact an abelian
sheaf on $S$. We denote by $\text{Ext}(P, G)$ the category of extensions of $P$ by $G$. It is a
classical result that the Baer sum of extensions defines a group law for the objects of
the category $\text{Ext}(P, G)$, which is therefore a strictly commutative Picard category.

Let $P, G$ be abelian sheaves on $S$. Denote by $m : P \times P \to P$ the group law
of $P$ and by $pr_i : P \times P \to P$ with $i = 1, 2$ the two projections of $P \times P$ in $P$.
According to [G] Exposé VII 1.1.6 and 1.2, the category of extensions of $P$ by $G$ is
equivalent to the category of 4-tuples $(P, G, E, \varphi)$, where $E$ is a $G_P$-torsor over $P$,
and $\varphi : pr_i^* E \wedge pr_2^* E \to m^* E$ is an isomorphism of torsors over $P \times P$ satisfying some
associativity and commutativity conditions (see [G] Exposé VII diagrams (1.1.4.1)
and (1.2.1)):

$$\text{Ext}(P, G) \simeq \left\{ (P, G, E, \varphi) \left| E = G_P - \text{torsor over } P \right. \right. \text{ and }$$

$$\varphi : pr_i^* E \wedge pr_2^* E \cong m^* E \text{ with ass. and comm. conditions} \right\}. \tag{1.1}$$

Here $pr_i^* E$ is the pull-back of $E$ via the projection $pr_i : P \times P \to P$ for $i = 1, 2$
and $pr_i^* E \wedge pr_2^* E$ is the contracted product of $pr_i^* E$ and $pr_2^* E$ (see 1.3 Chapter III
[G71]). It will be useful in what follows to look at the isomorphism of torsors $\varphi$ as
an associative and commutative group law on the fibres:

$$+ : E_p \times E_{p'} \to E_{p+p'}$$

where $p, p'$ are points of $P(U)$ with $U$ an $S$-scheme.

Let $I$ be a sheaf on $S$ and let $G$ be an abelian sheaf on $S$. Concerning extensions
of free commutative groups, by [G] Exposé VII 1.4 the category of extensions of
$\mathbb{Z}[I]$ by $G$ is equivalent to the category of $G_I$-torsors over $I$:

$$\text{Ext}(\mathbb{Z}[I], G) \simeq \text{Tors}(I, G_I). \tag{1.2}$$

Let $P, Q$ and $G$ be abelian sheaves on $S$. A biextension of $(P, Q)$ by $G$ is a
$G_{P \times Q}$-torsor $B$ over $P \times Q$, endowed with a structure of commutative extension
of $Q_P$ by $G_P$ and a structure of commutative extension of $P_Q$ by $G_Q$, which are
compatible one with another (for the definition of compatible extensions see [G]
Exposé VII Définition 2.1). If $m_{P, P_1, P_2}$ (resp. $m_{Q, Q_1, Q_2}$) denote the three mor-
phisms $P \times P \times Q \to P \times Q$ (resp. $P \times Q \times Q \to P \times Q$) deduced from the three
morphisms $P \times P \to P$ (resp. $Q \times Q \to Q$) group law, first and second projection,
the equivalence of categories (1.1) furnishes the following equivalent definition: a
biextension of $(P, Q)$ by $G$ is a $G_{P \times Q}$-torsor $B$ over $P \times Q$ endowed with two
isomorphisms of torsors

$$\varphi : p_1^* E \to m_{P}^* E$$

and

$$\psi : q_1^* E \to m_{Q}^* E.$$
over $P \times P \times Q$ and $P \times Q \times Q$ respectively, satisfying some associativity, commutativity and compatible conditions (see [C] Exposé VII diagrams (2.0.5),(2.0.6),(2.0.8), (2.0.9), (2.1.1)). As for extensions, we will look at the isomorphisms of torsors $\varphi$ and $\psi$ as two associative and commutative group laws on the fibres which are compatible with one another:

$$+1 : E_{p,q} E_{p',q} \rightarrow E_{p+p',q} \quad +2 : E_{p,q} E_{q,q'} \rightarrow E_{p,q+q'}$$

where $p, p'$ (resp. $q, q'$) are points of $P(U)$ (resp. of $Q(U)$) with $U$ any sheaf on $S$.

Let $K_i = [u_i : A_i \rightarrow B_i]$ (for $i = 1, 2$) be two 1-motives defined over $S$.

**Definition 1.1.** An extension $(E, \beta, \gamma)$ of $K_1$ by $K_2$ consists of

1. an extension $E$ of $B_1$ by $B_2$;
2. a trivialization $\beta$ of the extension $u_1^* E$ of $A_1$ by $B_2$ obtained as pull-back of the extension $E$ via $u_1 : A_1 \rightarrow B_1$;
3. a trivial extension $T = (T, \gamma)$ of $A_1$ by $A_2$ (i.e. an extension $T$ of $A_1$ by $A_2$ endowed with a trivialization $\gamma$) and an isomorphism of extensions $\Theta : u_2^* T \rightarrow u_1^* E$ between the push-down via $u_2 : A_2 \rightarrow B_2$ of $T$ and $u_1^* E$.

Through this isomorphism the trivialization $u_2 \circ \gamma$ of $u_2^* T$ is compatible with the trivialization $\beta$ of $u_1^* E$.

Condition (3) can be rewritten as

$$(3')$$ an homomorphism $\gamma : A_1 \rightarrow A_2$ such that $u_2 \circ \gamma$ is compatible with $\beta$.

Note that to have a trivialization $\beta : A_1 \rightarrow u_1^* E$ of $u_1^* E$ is the same thing as to have a lifting $\tilde{\beta} : A_1 \rightarrow E$ of $u_1 : A_1 \rightarrow B_1$. In fact, if we denote $p : E \rightarrow B_1$ the canonical surjection of the extension $E$, a morphism $\tilde{\beta} : A_1 \rightarrow E$ such that $p \circ \tilde{\beta} = u_1$ induces a splitting $\beta : A_1 \rightarrow u_1^* E$ that composes with $u_1^* E \rightarrow E \xrightarrow{\rho} B_1$ to $u_1 : A_1 \rightarrow B_1$, and vice versa.

**Remark 2.** We can summarize the above definition with the following diagram with exact rows:

$$\begin{array}{c}
0 \rightarrow B_2 \rightarrow E \xrightarrow{u_1} B_1 \rightarrow 0 \\
0 \rightarrow B_2 \xrightarrow{u_2} u_1^* E \cong u_2^* T \xrightarrow{\beta} A_1 \rightarrow 0 \\
0 \rightarrow A_2 \xrightarrow{T} \xrightarrow{\gamma} A_1 \rightarrow 0
\end{array}$$

In particular, we observe that the short sequence of complexes in $\mathcal{K}(S)$

$$0 \rightarrow K_2 \rightarrow \xrightarrow{[T \rightarrow E]} K_1 \rightarrow 0$$

is exact. On the other hand if $0 \rightarrow K_2 \rightarrow G \rightarrow K_1 \rightarrow 0$ is a short exact sequence of $\mathcal{K}(S)$, then the complex $G$ is an extension of 1-motives of $K_1$ by $K_2$ as defined in Definition 1.1 i.e. $G$ is a complex of the kind $[T \rightarrow E]$, with $T$ a trivial extension of $A_1$ by $A_2$ and $E$ an extension of $B_1$ by $B_2$. In fact, over a separably closed field the groups $\text{Ext}^1(A_1, A_2)$ and $\text{Ext}^1(A_1, B_2)$ are trivial.

Let $K_i = [A_i \xrightarrow{\delta_i} B_i]$ and $K_i' = [A_i' \xrightarrow{\delta_i'} B_i']$ (for $i = 1, 2$) be 1-motives defined over $S$. Let $(E, \beta, \gamma)$ be an extension of $K_1$ by $K_2$ and let $(E', \beta', \gamma')$ be an extension of $K_1'$ by $K_2'$.
Definition 1.3. A morphism of extensions

\[(F, \Upsilon, \Phi) : (E, \beta, \gamma) \to (E', \beta', \gamma')\]

consists of

1. a morphism \(F = (F, f_1, f_2) : E \to E'\) from the extension \(E\) to the extension \(E'\). In particular, \(F : E \to E'\) is a morphism of the sheaves underlying \(E\) and \(E'\), and

\[f_1 : B_1 \to B'_1 \quad f_2 : B_2 \to B'_2\]

are morphisms of abelian sheaves on \(S\);

2. a morphism of extensions \(\Upsilon = (\Upsilon, g_1, f_2) : u_1^*E \to u_1'^*E'\) compatible with the morphism \(F = (F, f_1, f_2)\) and with the trivializations \(\beta\) and \(\beta'\). In particular, \(\Upsilon : u_1^*E \to u_1'^*E'\) is a morphism of the sheaves underlying \(u_1^*E\) and \(u_1'^*E'\), and

\[g_1 : A_1 \to A'_1\]

is a morphism of abelian sheaves on \(S\);

3. a morphism of extensions \(\Phi = (\Phi, g_1, g_2) : T \to T'\) compatible with the morphism \(\Upsilon = (\Upsilon, g_1, f_2)\) and with the trivializations \(\gamma\) and \(\gamma'\). In particular, \(\Phi : T \to T'\) is a morphism of the sheaves underlying \(T\) and \(T'\), and

\[g_2 : A_2 \to A'_2\]

is an morphism of abelian sheaves on \(S\).

Condition (3) can be rewritten as

\[(\text{3}') \quad \text{a morphism } g_2 : A_2 \to A'_2 \text{ of abelian sheaves on } S \text{ compatible with } u_2 \text{ and } u'_2 \text{ (i.e. } u'_2 \circ g_2 = f_2 \circ u_2\text{) and such that}\]

\[\gamma' \circ g_1 = g_2 \circ \gamma.\]

Explicitly, the compatibility of \(\Upsilon\) with \(F\), \(\beta\) and \(\beta'\) means that the following diagram is commutative:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\beta} & u_1^*E \\
\downarrow{g_1} & & \downarrow{\Upsilon} \\
A'_1 & \xrightarrow{\beta'} & u_1'^*E'.
\end{array}
\]

The compatibility of \(\Phi\) with \(\Upsilon\), \(\gamma\) and \(\gamma'\) means that the following diagram is commutative:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\gamma} & T \\
\downarrow{g_1} & & \downarrow{\Phi} \\
A'_1 & \xrightarrow{\gamma'} & T'.
\end{array}
\]

\[
\begin{array}{ccc}
T & \xrightarrow{\Theta} & u_2*T \\
\downarrow{\Upsilon} & & \downarrow{\Upsilon} \\
u_1^*E & \xrightarrow{\Theta} & u_1'^*E'.
\end{array}
\]

We denote by \(\text{Ext}(K_1, K_2)\) the category of extensions of \(K_1\) by \(K_2\). As for extensions of abelian sheaves, it is possible to define the Baer sum of extensions of 1-motives. This notion of sum furnishes a group law for the objects of the category \(\text{Ext}(K_1, K_2)\) which is therefore a strictly commutative Picard category (see [G] Exposè VII 2.5). The zero object \((E_0, \beta_0, \gamma_0)\) of \(\text{Ext}(K_1, K_2)\) with respect to this group law consists of

- the trivial extension \(E_0 = B_1 \times B_2\) of \(B_1\) by \(B_2\), i.e. the zero object of \(\text{Ext}(B_1, B_2)\), and
Let \( \mathcal{E} \) denote by \( (\mathcal{E}, \beta, \gamma) \) the couple \((f \text{ isomorphism classes of objects of } \text{Ext}(E, \beta, \gamma)_{K} \). Let \( (B) \) be a biextension, \( (1) \) a trivialization \( T \) of \( (K) \) and \( (2) \) a trivialization \( T \) of \( (K) \) such that, via the isomorphism of extensions \( \Theta_0 = u_2 \circ T_0 \rightarrow u_1 \circ T_0 \), the push-down \( u_2 \circ f_1 \) of the automorphism \( f_1 \) of \( T_0 \) is compatible with the pull-back \( u_1 \circ f_0 \) of the automorphism \( f_0 \) of \( E_0 \), i.e. \( u_2 \circ f_1 = f_0 \circ u_1 \).

We have therefore the canonical isomorphism
\[
\text{Ext}^0(K_1, K_2) \cong \text{Hom}_{K(S)}(K_1, K_2).
\]

The group law of the category \( \text{Ext}(K_1, K_2) \) induces a group law on the set of isomorphism classes of objects of \( \text{Ext}(K_1, K_2) \) which we denote by \( \text{Ext}^1(K_1, K_2) \).

Let \( K_i = [u_i : A_i \rightarrow B_i] \)  (for \( i = 1, 2, 3 \) ) be three 1-motives defined over \( S \).

**Definition 1.4.** A biextension \((\mathcal{B}, \Psi_1, \Psi_2, \lambda)\) of \((K_1, K_2)\) by \( K_3 \) consists of

1. a biextension \( \mathcal{B} \) of \((B_1, B_2)\) by \( B_3 \);
2. a trivialization \( \Psi_1 : A_1 \times B_2 \rightarrow (u_1, id_{B_2})^* \mathcal{B} \) of the biextension \((u_1, id_{B_2})^* \mathcal{B} \) of \((A_1, B_2)\) by \( B_3 \) obtained as pull-back of \( \mathcal{B} \) via \((u_1, id_{B_2}) : A_1 \times B_2 \rightarrow B_1 \times B_2 \), and a trivialization \( \Psi_2 : B_1 \times A_2 \rightarrow (id_{B_1}, u_2)^* \mathcal{B} \) of the biextension \((id_{B_1}, u_2)^* \mathcal{B} \) of \((B_1, A_2)\) by \( B_3 \) obtained as pull-back of \( \mathcal{B} \) via \((id_{B_1}, u_2) : B_1 \times A_2 \rightarrow B_1 \times B_2 \). These two trivializations \( \Psi_1 \) and \( \Psi_2 \) have to coincide over \( A_1 \times A_2 \);
3. a trivial biextension \( T_1 = (T_1, \lambda_1) \) of \((A_1, B_2)\) by \( A_3 \), an isomorphism of biextensions \( \Theta_1 : u_3 \circ T_1 \rightarrow (u_1, id_{B_2})^* \mathcal{B} \) between the push-down via \( u_3 : A_3 \rightarrow B_3 \) of \( T_1 \) and \((u_1, id_{B_2})^* \mathcal{B} \), a trivial biextension \( T_2 = (T_2, \lambda_2) \) of \((B_1, A_2)\) by \( A_3 \) and an isomorphism of biextensions \( \Theta_2 : u_3 \circ T_2 \rightarrow (id_{B_1}, u_2)^* \mathcal{B} \) between the push-down via \( u_3 : A_3 \rightarrow B_3 \) of \( T_2 \) and \((id_{B_1}, u_2)^* \mathcal{B} \). Through the isomorphism \( \Theta_1 \) the trivialization \( u_3 \circ \lambda_1 \) of \((u_3 \circ T_1) \) is compatible with the trivialization \( \Psi_1 \) of \((u_1, id_{B_2})^* \mathcal{B} \), and through the isomorphism \( \Theta_2 \) the trivialization \( u_3 \circ \lambda_2 \) of \((u_3 \circ T_2) \) is compatible with the trivialization \( \Psi_2 \) of
\((\text{id}_{B_1}, u_2)^*\mathcal{B}\). The two trivializations \(\lambda_1\) and \(\lambda_2\) have to coincide over \(A_1 \times A_2\), i.e. \((\text{id}_{A_1}, u_2)^*\mathcal{T}_1 = (u_1, \text{id}_{A_2})^*\mathcal{T}_2\) (we will denote this biextension by \(\mathcal{T} = (\mathcal{T}, \lambda)\) with \(\lambda\) the restriction of the trivializations \(\lambda_1\) and \(\lambda_2\) over \(A_1 \times A_2\)). Moreover, we require an isomorphism of biextensions \[
\Theta : u_3^*\mathcal{T} \longrightarrow (u_1, u_2)^*\mathcal{B}
\]
which is compatible with the isomorphisms \(\Theta_1\) and \(\Theta_2\) and through which the trivialization \(u_3 \circ \lambda\) of \(u_3^*\mathcal{T}\) is compatible with the restriction \(\Psi\) of the trivializations \(\Psi_1\) and \(\Psi_2\) over \(A_1 \times A_2\).

Condition (3) can be rewritten as

\((3')\) an morphism \(\lambda : A_1 \otimes A_2 \to A_3\) such that \(u_3 \circ \lambda\) is compatible with \(\Psi\).

Let \(K_i = [u_i : A_i \to B_i]\) and \(K'_i = [u'_i : A'_i \to B'_i]\) (for \(i = 1, 2, 3\)) be 1-motives defined over \(S\). Let \((\mathcal{B}, \Psi_1, \Psi_2, \lambda)\) be a biextension of \((K_1, K_2)\) by \(K_3\) and let \((\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')\) be a biextension of \((K'_1, K'_2)\) by \(K'_3\).

**Definition 1.5. A morphism of biextensions**

\[
(\mathcal{E}, \Upsilon_1, \Upsilon_2, \Phi) : (\mathcal{B}, \Psi_1, \Psi_2, \lambda) \longrightarrow (\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')
\]

consists of

1. a morphism \(\mathcal{E} = (F, f_1, f_2, f_3) : \mathcal{B} \to \mathcal{B}'\) from the biextension \(\mathcal{B}\) to the biextension \(\mathcal{B}'\). In particular, \(F : \mathcal{B} \to \mathcal{B}'\) is a morphism of the sheaves underlying \(\mathcal{B}\) and \(\mathcal{B}'\), and
   \[
   f_1 : B_1 \to B'_1 \quad f_2 : B_2 \to B'_2 \quad f_3 : B_3 \to B'_3
   \]
   are morphisms abelian sheaves on \(S\).

2. a morphism of biextensions
   \[
   \Upsilon_1 = (\Upsilon_1, g_1, f_2, f_3) : (u_1, \text{id}_{B_2})^*\mathcal{B} \longrightarrow (u'_1, \text{id}_{B'_2})^*\mathcal{B}'
   \]
   compatible with the morphism \(\mathcal{E} = (F, f_1, f_2, f_3)\) and with the trivializations \(\Psi_1\) and \(\Psi_1'\), and a morphism of biextensions
   \[
   \Upsilon_2 = (\Upsilon_2, f_1, g_2, f_3) : (\text{id}_{B_1}, u_2)^*\mathcal{B} \longrightarrow (\text{id}_{B'_1}, u'_2)^*\mathcal{B}'
   \]
   compatible with the morphism \(\mathcal{E} = (F, f_1, f_2, f_3)\) and with the trivializations \(\Psi_2\) and \(\Psi_2'\). In particular, \(\Upsilon_1 : (u_1, \text{id}_{B_2})^*\mathcal{B} \to (u'_1, \text{id}_{B'_2})^*\mathcal{B}'\) is a morphism of the sheaves underlying \((u_1, \text{id}_{B_2})^*\mathcal{B}\) and \((u'_1, \text{id}_{B'_2})^*\mathcal{B}'\), \(\Upsilon_2 : (\text{id}_{B_1}, u_2)^*\mathcal{B} \to (\text{id}_{B'_1}, u'_2)^*\mathcal{B}'\) is a morphism of the sheaves underlying \((\text{id}_{B_1}, u_2)^*\mathcal{B}\) and \((\text{id}_{B'_1}, u'_2)^*\mathcal{B}'\), and
   \[
   g_1 : A_1 \longrightarrow A'_1 \quad g_2 : A_2 \longrightarrow A'_2
   \]
   are morphisms abelian sheaves on \(S\). By pull-back, the two morphisms \(\Upsilon_1 = (\Upsilon_1, g_1, f_2, f_3)\) and \(\Upsilon_2 = (\Upsilon_2, f_1, g_2, f_3)\) define a morphism of biextensions
   \[
   \Phi = (\Upsilon, g_1, g_2, f_3) : (u_1, u_2)^*\mathcal{B} \longrightarrow (u'_1, u'_2)^*\mathcal{B}'
   \]
   compatible with the morphism \(\mathcal{E} = (F, f_1, f_2, f_3)\) and with the trivializations \(\Psi\) (restriction of \(\Psi_1\) and \(\Psi_2\) over \(A_1 \times A_2\)) and \(\Psi'\) (restriction of \(\Psi'_1\) and \(\Psi'_2\) over \(A'_1 \times A'_2\)).
(3) a morphism of biextensions
\[ \Phi_1 = (\Phi_1, g_1, f_2, g_3) : T_1 \rightarrow T'_1 \]
compatible with the morphism \( \Upsilon_1 = (\Upsilon_1, g_1, f_2, f_3) \) and with the trivializations \( \lambda_1 \) and \( \lambda'_1 \), and a morphism of biextensions
\[ \Phi_2 = (\Phi_2, f_1, g_2, g_3) : T_2 \rightarrow T'_2 \]
compatible with the morphism \( \Upsilon_2 = (\Upsilon_2, f_1, g_2, f_3) \) and with the trivializations \( \lambda_2 \) and \( \lambda'_2 \). In particular, \( \Phi_1 : T_1 \rightarrow T'_1 \) is a morphism of the sheaves underlying \( T_1 \) and \( T'_1 \), \( \Phi_2 : T_2 \rightarrow T'_2 \) is a morphism of the sheaves underlying \( T_2 \) and \( T'_2 \), and
\[ g_3 : A_3 \rightarrow A'_3 \]
is an morphism abelian sheaves on \( S \). By pull-back, the two morphisms \( \Phi_1 = (\Phi_1, g_1, f_2, g_3) \) and \( \Phi_2 = (\Phi_2, f_1, g_2, g_3) \) define a morphism of biextensions
\[ \Phi = (\Phi, g_1, g_2, g_3) : T \rightarrow T' \]
compatible with the morphism \( \Upsilon = (\Upsilon, g_1, g_2, f_3) \) and with the trivializations \( \lambda \) (restriction of \( \lambda_1 \) and \( \lambda_2 \) over \( A_1 \times A_2 \)) and \( \lambda' \) (restriction of \( \lambda'_1 \) and \( \lambda'_2 \) over \( A_1 \times A_2 \)).

Condition (3) can be rewritten as

(3') an morphism \( g_3 : A_3 \rightarrow A'_3 \) abelian sheaves on \( S \) compatible with \( u_3 \) and \( u'_3 \) (i.e. \( u'_3 \circ g_3 = f_3 \circ u_3 \)) and such that
\[ \lambda' \circ (g_1 \times g_2) = g_3 \circ \lambda. \]

Explicitly, the compatibility of \( \Upsilon_1 \) with \( E \), \( \Psi_1 \) and \( \Psi'_1 \) means that the following diagram is commutative:
\[ \begin{array}{ccc}
A_1 \times B_2 & \xrightarrow{\Psi_1} & (u_1, id_{B_2})^* B \\
\phi_{1} & \xrightarrow{\tau_1} & (u_1, id_{B_2})^* B \\
\phi_{1}' & \xrightarrow{\tau_1'} & (u_1', id_{B_2'})^* B'
\end{array} \]
The compatibility of \( \Upsilon_2 \) with \( E \), \( \Psi_2 \) and \( \Psi'_2 \) means that the following diagram is commutative:
\[ \begin{array}{ccc}
B_1 \times A_2 & \xrightarrow{\Psi_2} & (id_{B_1}, u_2)^* B \\
\phi_{2} & \xrightarrow{\tau_2} & (id_{B_1}, u_2)^* B \\
\phi_{2}' & \xrightarrow{\tau_2'} & (id_{B_1'}, u_2')^* B'
\end{array} \]
The compatibility of \( \Upsilon \) with \( E \), \( \Psi \) and \( \Psi' \) means that the following diagram is commutative:
\[ \begin{array}{ccc}
A_1 \times A_2 & \xrightarrow{\Psi} & (u_1, u_2)^* B \\
\phi_{3} & \xrightarrow{\tau} & (u_1, u_2)^* B \\
\phi_{3}' & \xrightarrow{\tau'} & (u_1', u_2')^* B'
\end{array} \]
The compatibility of \( \Phi \) with \( \Upsilon_1 \), \( \lambda_1 \) and \( \lambda'_1 \) means that the following diagram is commutative:
\[ \begin{array}{ccc}
A_1 \times B_2 & \xrightarrow{\lambda_1} & T_1 \\
\phi_{1} & \xrightarrow{\tau} & T_1 \\
\phi_{1}' & \xrightarrow{\tau'} & T_1
\end{array} \]
\[ \begin{array}{ccc}
A'_1 \times B'_2 & \xrightarrow{\lambda'_1} & T'_1 \\
\phi_{1}' & \xrightarrow{\tau'} & T'_1 \\
\phi_{1}' & \xrightarrow{\tau'} & T'_1
\end{array} \]
We denote by $\text{Biext}(0, B_1, B_2)$ the category of biextensions of $(B_1, B_2)$ by $B_3$. The compatibility of $\Phi$ with $\Upsilon$, $\lambda$ and $\lambda'$ means that the following diagram is commutative:

$$A_1 \times A_2 \xrightarrow{\lambda} T \xrightarrow{\Phi} u_3 \ast T \cong (u_1, u_2)^{\ast}B \xrightarrow{\Upsilon} \downarrow T$$

Finally, the compatibility of $\Phi$ with $\Upsilon$, $\lambda$ and $\lambda'$ means that the following diagram is commutative:

$$A_1 \times A_2 \xrightarrow{\lambda} T \xrightarrow{\Phi} u_3 \ast T \cong (u_1, u_2)^{\ast}B \xrightarrow{\Upsilon} \downarrow T$$

We denote by $\text{Biext}(K_1, K_2; K_3)$ the category of biextensions of $(K_1, K_2)$ by $K_3$. The Baer sum of extensions defines a group law for the objects of this category which is therefore a strictly commutative Picard category (see [G] Exposé VII 2.5).

The zero object $(B_0, \Psi_{01}, \Psi_{02}, \lambda_0)$ of $\text{Biext}(K_1, K_2; K_3)$ with respect to this group law consists of

- the trivial biextension $B_0 = B_1 \times B_2 \times B_3$ of $(B_1, B_2)$ by $B_3$, i.e. the zero object of $\text{Biext}(B_1, B_2; B_3)$, and
- the trivialization $\Psi_{01} = (id_{A_1}, id_{B_2}, 0)$ (resp. $\Psi_{02} = (id_{B_1}, id_{A_2}, 0)$) of the biextension $(u_1, id_{B_2})^{\ast}B_0 = A_1 \times B_2 \times B_3$ of $(A_1, B_2)$ by $B_3$ (resp. of the biextension $(id_{B_1}, u_2)^{\ast}B_0 = B_1 \times A_2 \times B_3$ of $(B_1 \times A_2)$ by $B_3$),
- the trivial biextension $T_{10}$ of $(A_1, B_2)$ by $A_3$ (i.e. $T_{10} = (T_{10}, \lambda_{10})$ with $T_{10} = A_1 \times B_2 \times A_3$ and $\lambda_{10} = (id_{A_1}, id_{B_2}, 0)$), the isomorphism of biextensions $\Theta_{10} = (id_{A_1}, id_{B_2}, id_{B_3}) : u_3 \ast T_{10} \rightarrow (u_1, id_{B_2})^{\ast}B_0$, the trivial biextension $T_{01}$ of $(B_1, A_2)$ by $A_3$ (i.e. $T_{01} = (T_{01}, \lambda_{01})$ with $T_{01} = B_1 \times A_2 \times A_3$ and $\lambda_{01} = (id_{B_1}, id_{A_2}, 0)$) and the isomorphism of biextensions $\Theta_{01} = (id_{B_1}, id_{A_2}, id_{B_3}) : u_3 \ast T_{01} \rightarrow (id_{B_1}, u_2)^{\ast}B_0$. In particular the restriction of $\lambda_{10}$ and $\lambda_{01}$ over $A_1 \times A_2$ is $\lambda_0 = (id_{A_1}, id_{A_2}, 0)$.

We denote by $\text{Biext}^0(K_1, K_2; K_3)$ the group of automorphisms of any object of $\text{Biext}(K_1, K_2; K_3)$. It is canonically isomorphic to the group of automorphisms of the zero object $(B_0, \Psi_{01}, \Psi_{02}, \lambda_0)$. Explicitly, $\text{Biext}^0(K_1, K_2; K_3)$ consists of the couple $(f_0, (f_{10}, f_{01}))$ where

- $f_0 : B_1 \otimes B_2 \rightarrow B_3$ is an automorphism of the trivial biextension $B_0$ (i.e. $f_0 \in \text{Biext}^0(B_1, B_2; B_3)$), and
- $f_{10} : A_1 \otimes B_2 \rightarrow A_3$ is an automorphism of the trivial biextension $T_{10}$ (i.e. $f_{10} \in \text{Biext}^0(A_1, B_2; A_3)$) and $f_{01} : B_1 \otimes A_2 \rightarrow A_3$ is an automorphism of the trivial biextension $T_{01}$ (i.e. $f_{01} \in \text{Biext}^0(B_1, A_2; A_3)$) such that, via the isomorphisms of biextensions $\Theta_{10} : u_3 \ast T_{10} \rightarrow (u_1, id_{B_2})^{\ast}B_0$ and $\Theta_{01} : u_3 \ast T_{01} \rightarrow (id_{B_1}, u_2)^{\ast}B_0$, the push-down $u_3 \ast f_{10}$ of $f_{10}$ is compatible with the pull-back $(u_1, id_{B_2})^{\ast}f_0$ of $f_0$, and the push-down $u_3 \ast f_{01}$ of $f_{01}$ is compatible with the pull-back $(id_{B_1}, u_2)^{\ast}f_0$ of $f_0$, i.e. such that the
following diagram commute
\[
\begin{array}{ccc}
A_1 \otimes B_2 + B_1 \otimes A_2 & \xrightarrow{(u_1, \text{id}) + (\text{id}, u_2)} & B_1 \otimes B_2 \\
\downarrow f_{10} + f_{01} & & \downarrow f_0 \\
A_3 & \xrightarrow{u_3} & B_3.
\end{array}
\]

We have therefore the canonical isomorphism

\[
\text{Biext}^0(K_1, K_2; K_3) \cong \text{Hom}_{K(S)}(K_1 \otimes K_2, K_3).
\]

The group law of the category \text{Biext}(K_1, K_2; K_3) induces a group law on the set of isomorphism classes of objects of \text{Biext}(K_1, K_2; K_3), that we denote by \text{Biext}^1(K_1, K_2; K_3).

Remark 1.6. According to the above geometrical definitions of extensions and biextensions of 1-motives, we have the following equivalence of categories

\[
\text{Biext}(K_1, [0 \to \mathbb{Z}]; K_3) \simeq \text{Ext}(K_1, K_3).
\]

Moreover we have also the following isomorphisms

\[
\text{Biext}^i(K_1, [\mathbb{Z} \to 0]; K_3) = \begin{cases} 
\text{Hom}(B_1, A_3), & i = 0; \\
\text{Hom}(K_1, K_3), & i = 1.
\end{cases}
\]

Note that we get the same results applying the homological interpretation of biextensions furnished by our main Theorem 0.1.

2. Review on strictly commutative Picard stacks

Let \(\mathcal{S}\) be a site. For the notions of \(\mathcal{S}\)-pre-stack, \(\mathcal{S}\)-stack and morphisms of \(\mathcal{S}\)-stacks we refer to [G71] Chapter II 1.2.

A strictly commutative Picard \(\mathcal{S}\)-stack is an \(\mathcal{S}\)-stack of groupoids \(\mathcal{P}\) endowed with a functor \(+ : \mathcal{P} \times \mathcal{S} \mathcal{P} \to \mathcal{P}\), \((a, b) \mapsto a + b\), and two natural isomorphisms of associativity \(\sigma\) and of commutativity \(\tau\), which are described by the functorial isomorphisms

\[
\begin{align*}
\sigma_{a,b,c} : (a + b) + c & \xrightarrow{\cong} a + (b + c) \quad \forall \ a, b, c \in \mathcal{P}, \\
\tau_{a,b} : a + b & \xrightarrow{\cong} b + a \quad \forall \ a, b \in \mathcal{P};
\end{align*}
\]

such that for any object \(U\) of \(\mathcal{S}\), \((\mathcal{P}(U), +, \sigma, \tau)\) is a strictly commutative Picard category (i.e. it is possible to make the sum of two objects of \(\mathcal{P}(U)\) and this sum is associative and commutative, see [D73] 1.4.2 for more details). Here ”strictly” means that \(\tau_{a,a}\) is the identity for all \(a \in \mathcal{P}\). Any strictly commutative Picard \(\mathcal{S}\)-stack admits a global neutral object \(e\) and the sheaf of automorphisms of the neutral object \(\text{Aut}(e)\) is abelian.

Let \(\mathcal{P}\) and \(\mathcal{Q}\) be two strictly commutative Picard \(\mathcal{S}\)-stacks. An additive functor \((F, \sum) : \mathcal{P} \to \mathcal{Q}\) between strictly commutative Picard \(\mathcal{S}\)-stacks is a morphism of \(\mathcal{S}\)-stacks (i.e. a cartesian \(\mathcal{S}\)-functor, see [G71] Chapter I 1.1) endowed with a natural isomorphism \(\sum\) which is described by the functorial isomorphisms

\[
\sum_{a,b} : F(a + b) \xrightarrow{\cong} F(a) + F(b) \quad \forall \ a, b \in \mathcal{P}
\]

and which is compatible with the natural isomorphisms \(\sigma\) and \(\tau\) of \(\mathcal{P}\) and \(\mathcal{Q}\). A morphism of additive functors \(u : (F, \sum) \to (F', \sum')\) is an \(\mathcal{S}\)-morphism of
cartesian $S$-functors (see [G71] Chapter I 1.1) which is compatible with the natural isomorphisms $\sum$ and $\sum'$ of $F$ and $F'$ respectively.

An **equivalence of strictly commutative Picard $S$-stacks** between $P$ and $Q$ is an additive functor $(F, \sum) : P \to Q$ with $F$ an equivalence of $S$-stacks. Two strictly commutative Picard $S$-stacks are **equivalent as strictly commutative Picard $S$-stacks** if there exists an equivalence of strictly commutative Picard $S$-stacks between them.

To any strictly commutative Picard $S$-stack $P$, we associate the sheafification $\pi_0(P)$ of the pre-sheaf which associates to each object $U$ of $S$ the group of isomorphism classes of objects of $P(U)$, and the sheaf $\pi_1(P)$ of automorphisms $\text{Aut}(e)$ of the neutral object of $P$.

In [D73] §1.4 Deligne associates to each complex $K$ of $\mathcal{K}[-1,0](S)$ a strictly commutative Picard $S$-stack $st(K)$ and to each morphism of complexes $g : K \to L$ an additive functor $st(g) : st(K) \to st(L)$. Moreover, if $\text{Picard}(S)$ denotes the category whose objects are small strictly commutative Picard $S$-stacks and whose arrows are isomorphism classes of additive functors, Deligne proves the following equivalence of category

\begin{equation}
\text{st} : \mathcal{D}[-1,0](S) \longrightarrow \text{Picard}(S)
\end{equation}

\begin{align*}
K & \mapsto st(K) \\
K \xrightarrow{f} L & \mapsto st(K) \xrightarrow{st(f)} st(L).
\end{align*}

constructing explicitly the inverse equivalence of $\text{st}$, that we denote by $[\ ]$.

**Example 2.1.** Let $P$, $Q$ and $G$ be three strictly commutative Picard $S$-stacks.

I) Let

$\text{HOM}(P, Q)$

be the strictly commutative Picard $S$-stack defined as followed: for any object $U$ of $S$, the objects of the category $\text{HOM}(P, Q)(U)$ are additive functors from $P|_U$ to $Q|_U$ and its arrows are morphisms of additive functors. According [D73] 1.4.18 we have the equality $[\text{HOM}(P, Q)] = \tau_{\leq 0} \text{RHom}(|P|, |Q|)$ in the derived category $\mathcal{D}[-1,0](S)$.

II) A **biadditive functor** $(F, l, r) : P \times Q \to G$ is a morphism of $S$-stacks $F : P \times Q \to G$ endowed with two natural isomorphisms, which are described by the functorial isomorphisms

\begin{align*}
l_{a,b,c} : F(a + b, c) & \xrightarrow{\sim} F(a, c) + F(b, c) \quad \forall a, b \in P, \forall c \in Q \\
r_{a,c,d} : F(a, c + d) & \xrightarrow{\sim} F(a, c) + F(a, d) \quad \forall a \in P, \forall c, d \in Q,
\end{align*}

such that

- for any fixed $a \in P$, $F(a, -)$ is compatible with the natural isomorphisms $\sigma$ and $\tau$ of $P$ and $G$,
- for any fixed $c \in Q$, $F(-, c)$ is compatible with the natural isomorphisms $\sigma$ and $\tau$ of $Q$ and $G$,
According to D73 1.4.20, in the derived category

$\mathcal{P} \times \mathcal{Q}$

be the strictly commutative Picard $\mathcal{S}$-stack defined as followed: for any object $U$ of $\mathcal{S}$, the objects of the category $\text{HOM}(\mathcal{P}, \mathcal{Q}; \mathcal{G})(U)$ are biadditive functors from $\mathcal{P}_U \times \mathcal{Q}_U$ to $\mathcal{G}_U$ and its arrows are morphisms of biadditive functors.

III) Let

$\mathcal{P} \otimes \mathcal{Q}$

be the strictly commutative Picard $\mathcal{S}$-stack endowed with a biadditive functor

$\otimes : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \otimes \mathcal{Q}$

such that for any strictly commutative Picard $\mathcal{S}$-stack $\mathcal{G}$, the biadditive functor $\otimes$ defines the following equivalence of strictly commutative Picard $\mathcal{S}$-stacks:

\[(2.4) \quad \text{HOM}(\mathcal{P} \otimes \mathcal{Q}, \mathcal{G}) \cong \text{HOM}(\mathcal{P}, \mathcal{Q}; \mathcal{G}).\]

According to D73 1.4.20, in the derived category $\mathcal{D}^{[−1,0]}(\mathcal{S})$ we have the equality $[\mathcal{P} \otimes \mathcal{Q}] = \tau_{−1}(\mathcal{P} \otimes \mathcal{Q})$.

By §2 BeII we have the following operations on strictly commutative Picard $\mathcal{S}$-stacks:

1. The product of two strictly commutative Picard $\mathcal{S}$-stacks $\mathcal{P}$ and $\mathcal{Q}$ is the strictly commutative Picard $\mathcal{S}$-stack $\mathcal{P} \times \mathcal{Q}$ defined as followed:

- for any object $U$ of $\mathcal{S}$, an object of the category $\mathcal{P} \times \mathcal{Q}(U)$ is a pair $(p, q)$ of objects with $p$ an object of $\mathcal{P}(U)$ and $q$ an object of $\mathcal{Q}(U)$;

- for any object $U$ of $\mathcal{S}$, if $(p, q)$ and $(p', q')$ are two objects of $\mathcal{P} \times \mathcal{Q}(U)$, an arrow of $\mathcal{P} \times \mathcal{Q}(U)$ from $(p, q)$ to $(p', q')$ is a pair $(f, g)$ of arrows with $f : p \to p'$ an arrow of $\mathcal{P}(U)$ and $g : q \to q'$ an arrow of $\mathcal{Q}(U)$.

2. Let $G : \mathcal{P} \to \mathcal{Q}$ and $F : \mathcal{P}' \to \mathcal{Q}$ be additive functors between strictly commutative Picard $\mathcal{S}$-stacks. The fibered product of $\mathcal{P}$ and $\mathcal{P}'$ over $\mathcal{Q}$ via $F$ and $G$ is the strictly commutative Picard $\mathcal{S}$-stack $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ defined as followed:

- for any object $U$ of $\mathcal{S}$, the objects of the category $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ are triplets $(p, p', f)$ where $p$ is an object of $\mathcal{P}(U)$, $p'$ is an object of $\mathcal{P}'(U)$ and $f : G(p) \overset{\sim}{\to} F(p')$ is an isomorphism of $\mathcal{Q}(U)$ between $G(p)$ and $F(p')$;

- for any object $U$ of $\mathcal{S}$, if $(p_1, p'_1, f)$ and $(p_2, p'_2, g)$ are two objects of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$, an arrow of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ from $(p_1, p'_1, f)$ to $(p_2, p'_2, g)$ is a pair $(f, g)$ of arrows with $\alpha : p_1 \to p_2$ of arrow of $\mathcal{P}(U)$ and $\beta : p'_1 \to p'_2$ an arrow of $\mathcal{P}'(U)$ such that $g \circ G(\alpha) = F(\beta) \circ f$.

The fibered product $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is also called the pull-back $F^*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{P}' \to \mathcal{Q}$ or the pull-back $G^*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{P} \to \mathcal{Q}$.

3. Let $G : \mathcal{Q} \to \mathcal{P}$ and $F : \mathcal{Q} \to \mathcal{P}'$ be additive functors between strictly commutative Picard $\mathcal{S}$-stacks. The fibered sum of $\mathcal{P}$ and $\mathcal{P}'$ under $\mathcal{Q}$ via $F$ and $G$ is the
strictly commutative Picard $S$-stack $\mathcal{P} + \mathcal{Q} \mathcal{P}'$ generated by the following strictly commutative Picard $S$-pre-stack $\mathcal{D}$:

- for any object $U$ of $S$, the objects of the category $\mathcal{D}(U)$ are pairs $(p, p')$ with $p$ an object of $\mathcal{P}(U)$ and $p'$ an object of $\mathcal{P}'(U)$;
- for any object $U$ of $S$, if $(p_1, p'_1)$ and $(p_2, p'_2)$ are two objects of $\mathcal{D}(U)$, an arrow of $\mathcal{D}(U)$ from $(p_1, p'_1)$ to $(p_2, p'_2)$ is an equivalence class of triplets $(q, \alpha, \beta)$ with $q$ an object of $\mathcal{Q}(U)$, $\alpha : p_1 + G(q) \to p_2$ an arrow of $\mathcal{P}(U)$ and $\beta : p'_1 + F(q) \to p'_2$ an arrow of $\mathcal{P}'(U)$. Two triplets $(q_1, \alpha_1, \beta_1)$ and $(q_2, \alpha_2, \beta_2)$ are equivalent if there is an arrow $\gamma : q_1 \to q_2$ in $\mathcal{Q}(U)$ such that $\alpha_2 \circ (id + G(\gamma)) = \alpha_1$ and $(F(\gamma) + id) \circ \beta_1 = \beta_2$.

The fibered sum $\mathcal{P} + \mathcal{Q} \mathcal{P}'$ is also called the push-down $F_*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{Q} \to \mathcal{P}'$ or the push-down $G_*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{Q} \to \mathcal{P}$.

We have analogous operations on complexes of $\mathcal{K}[-1,0](S)$:

1. The product of two complexes $P = [d^P : P^{-1} \to P^0]$ and $Q = [d^Q : Q^{-1} \to Q^0]$ of $\mathcal{K}[-1,0](S)$ is the complex $P + Q = [(d^P, d^Q) : P^{-1} + Q^{-1} \to P^0 + Q^0]$. Via the equivalence of category $\mathcal{K}[-1,0](S)$ we have that $st(P + Q) = st(P) \times st(Q)$.

2. Let $P = [d^P : P^{-1} \to P^0]$, $Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}[-1,0](S)$ and let $f : P \to G$ and $g : Q \to G$ be morphisms of complexes. The fibered product $P \times_G Q$ of $P$ and $Q$ over $G$ is the complex $[d_{P \times_G} = d^P \times_{dG} d^Q : P^{-1} \times_G Q^{-1} \to P^0 \times_G Q^0]$. where for $i = -1,0$ the abelian sheaf $P^i \times_{G_i} Q^i$ is the fibered product of $P^i$ and of $Q^i$ over $G_i$ and the morphism of abelian sheaves $d_{P \times_G} = d^P \times_{dG} d^Q$ is given by the universal property of the fibered product $P^0 \times_G Q^0$. The fibered product $P \times_G Q$ is also called the pull-back $g^*P$ of $P$ via $g : Q \to G$ or the pull-back $f^*Q$ of $Q$ via $f : P \to G$. Remark that $st(P \times_G Q) = st(P) \times st(G) \times st(Q)$ via the equivalence of category $\mathcal{K}[-1,0](S)$.

3. Let $P = [d^P : P^{-1} \to P^0]$, $Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}[-1,0](S)$ and let $f : G \to P$ and $g : G \to Q$ be morphisms of complexes. The fibered sum $P + G Q$ of $P$ and $Q$ under $G$ is the complex $[d_{P + G} = d^P \times_{dG} d^Q + G : P^{-1} + G^{-1} Q^{-1} \to P^0 + G^0 Q^0]$. where for $i = -1,0$ the abelian sheaf $P^i + G^i Q^i$ is the fibered sum of $P^i$ and of $Q^i$ under $G_i$ and the morphism of abelian sheaves $d_{P + G} = d^P \times_{dG} d^Q$ is given by the universal property of the fibered sum $P^{-1} + G^{-1} Q^{-1}$. The fibered sum $P + G Q$ is also called the push-down $g_*P$ of $P$ via $g : Q \to G$ or the push-down $f_*Q$ of $Q$ via $f : G \to P$. We have $st(P + G Q) = st(P) + st(G) \times st(Q)$ via the equivalence of category $\mathcal{K}[-1,0](S)$.

If $\mathcal{P}$ and $\mathcal{G}$ are strictly commutative Picard $S$-stacks, by §3 [Be11] an extension $\mathcal{E} = (\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{G}$ consists of a strictly commutative Picard $S$-stack $\mathcal{E}$, two additive functors $I : \mathcal{G} \to \mathcal{E}$ and $J : \mathcal{E} \to \mathcal{P}$, and an isomorphism of additive functors $J \circ I \cong 0$, such that the following equivalent conditions are satisfied:

(a) $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ is surjective and $I$ induces an equivalence of strictly commutative Picard $S$-stacks between $\mathcal{G}$ and $\ker(J)$;

(b) $\pi_1(I) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{E})$ is injective and $J$ induces an equivalence of strictly commutative Picard $S$-stacks between $\coker(I)$ and $\mathcal{P}$.

In terms of complexes of $\mathcal{K}[-1,0](S)$, an extension $E = (E, i, j)$ of $\mathcal{P}$ by $\mathcal{G}$ consists of a complex $E$ of $\mathcal{K}[-1,0](S)$, two morphisms of complexes $i : G \to E$ and $j : E \to P$ of $\mathcal{K}[-1,0](S)$, and an homotopy between $j \circ i$ and $0$, such that the following equivalent conditions are satisfied:
(a) $H^0(j) : H^0(E) \to H^0(P)$ is surjective and $i$ induces a quasi-isomorphism between $G$ and $\tau_{\ge 0}(MC(j)[-1]);$
(b) $H^{-1}(i) : H^{-1}(G) \to H^{-1}(E)$ is injective and $j$ induces a quasi-isomorphism between $\tau_{\ge -1}MC(i)$ and $P.$

As recalled in the introduction we can see 1-motives as complexes of abelian sheaves on $S$ concentrated in two consecutive degrees. Hence via (2.3) to each 1-motives is associated a strictly commutative Picard $S$-stack and in particular, we can apply all what we have recalled in this section to 1-motives. Moreover, since a short exact sequence in $K^{[-1,0]}(S)$ is an extension of complexes in the above sense (see [Be11] Remark 3.6.), extensions of 1-motives are also extensions of complexes in the above sense, i.e. they furnishes extensions of strictly commutative Picard $S$-stacks (see Remark [1.2]).

3. PROOF OF THEOREM 0.1 (b)

Proof of Theorem 0.1 (b). Via the equivalence of category (2.3), to the trivial biextension of $(K_1, K_2)$ by $K_3$ corresponds the trivial biextension $B_0 = st(K_3) \times st(K_1) \times st(K_2)$ of $(st(K_1), st(K_2))$ by $st(K_3)$ (see [16] Definition 5.1). In particular $B_0$ is a Picard stack and so the group of isomorphism classes of arrows from $B_0$ to itself is the cohomology group $H^0((\text{HOM}(B_0, B_0)))$, where $\text{HOM}(B_0, B_0)$ is the strictly commutative Picard stack of additive functors from $B_0$ to itself. Therefore, in order to compute $\text{Biext}^0(K_1, K_2; K_3)$ it is enough to compute the complex $[\text{HOM}(B_0, B_0)].$

Let $F : B_0 \to B_0$ be an additive functor. Since $F$ is first of all an arrow from the $st(K_3)$-torsor over $st(K_1) \times st(K_2)$ underlying $B_0$ to itself, $F$ is given by the formula

$$F(b) = b + IF'J(b) \quad \forall b \in B_0$$

where $F' : st(K_1) \times st(K_2) \to st(K_3)$ is an additive functor and $J : B_0 \to st(K_1) \times st(K_2)$ and $I : st(K_3) \to B_0$ are the additive functors underlying the structure of $st(K_3)$-torsor over $st(K_1) \times st(K_2)$ of $B_0.$ Now $F : B_0 \to B_0$ must be compatible with the structures of extension of $st(K_3)$ by $st(K_1) \times st(K_2)$ and of extension of $st(K_1) \times st(K_2)$ by $st(K_3)$, therefore $F'$ must be a biadditive functor, i.e. an object of $\text{HOM}(st(K_1), st(K_2); st(K_3)).$ Hence $\text{HOM}(B_0, B_0)$ is equivalent as Picard stack to $\text{HOM}(st(K_1), st(K_2); st(K_3))$ via the following additive functor

$$\text{HOM}(st(K_1), st(K_2); st(K_3)) \longrightarrow \text{HOM}(B_0, B_0)$$

$$F' \quad \mapsto \quad (b \mapsto b + IF'J(b)).$$

By (2.4), $\text{HOM}(st(K_1), st(K_2); st(K_3)) \cong \text{HOM}(st(K_1) \otimes st(K_2), st(K_3))$ and so

$$[\text{HOM}(B_0, B_0)] = \tau_{\le 0}\text{RHom}\left(\tau_{\ge -1}(K_1 \otimes^L K_2), K_3\right),$$

and in particular the group of isomorphism classes of additive functors from $B_0$ to itself is isomorphic to the group

$$\text{Hom}_{D(S)}(K_1 \otimes^L K_2, K_3).$$

This implies that $\text{Biext}^0(K_1, K_2; K_3) \cong \text{Hom}_{D(S)}(K_1 \otimes^L K_2, K_3).$
In Section 6 we give another proof of Theorem 4.1. Remark that by (3.1) $H^{-1}(\text{Hom}(B_0, B_0)) \cong \text{Hom}_{\mathcal{D}^b(S)}(K_1 \otimes^L K_2, K_3[-1])$. Since $K_i = [A_i \to B_i]$ are 1-motives, $\text{Hom}(B_j, A_i) = 0$ for $i, j = 1, 2, 3$ (see [B09] Lemma 1.1.1), and hence the group $H^{-1}(\text{Hom}(B_0, B_0))$ is trivial.

4. THE CATEGORY $\Psi_{L^*}(G)$ AND ITS HOMOLOGICAL INTERPRETATION

Consider the following complex of 1-motives defined over $S$

\begin{equation}
R \xrightarrow{d^R} Q \xrightarrow{d^Q} P \to 0
\end{equation}

Explicitly, $R = [d^R : R^{-1} \to R^0], Q = [d^Q : Q^{-1} \to Q^0], P = [d^P : P^{-1} \to P^0]$ and $D^R = (d^R_{-1}, d^R_0), D^Q = (d^Q_{-1}, d^Q_0)$. This complex is a bicomplex $L^*$ of abelian sheaves on $S$,

\[
\begin{array}{c}
R^{-1} \xrightarrow{d^{R,-1}} Q^{-1} \xrightarrow{d^{Q,-1}} P^{-1} \to 0 \\
Q^0 \xrightarrow{d^Q} P^0 \to 0
\end{array}
\]

where $P^0, P^{-1}, Q^0, Q^{-1}, R^0, R^{-1}$ are respectively in degrees $(0, 0), (0, -1), (-1, 0), (-1, -1), (-2, 0), (-2, -1)$. Denote by $\text{Tot}(L^*)$ its total complex. Let $G = [d^G : G^{-1} \to G^0]$ be a 1-motive defined over $S$.

**Definition 4.1.** Denote by $\Psi_{L^*}(G)$ the category

1. whose objects are pairs $(E, I)$ with $E$ an extension of 1-motives of $P$ by $G$ and $I$ a trivialization of the extension $(D^Q)^*E$ of $Q$ by $G$ obtained as pull-back of $E$ by $D^Q$. Moreover we require that the corresponding trivialization $(D^R)^*I$ of $(D^R)^*(D^Q)^*E$ is the trivialization arising from the isomorphism of transitivity $(D^R)^*(D^Q)^*E \cong (D^Q \circ D^R)^*E$ and the relation $D^Q \circ D^R = 0$.

2. whose arrows $F : (E, I) \to (E', I')$ are morphisms of extensions $F : E \to E'$ of 1-motives compatible with the trivializations $I, I'$, i.e. we have an isomorphism of additive functors $F \circ I \cong I'$.

In order to compute the homological interpretation of the category $\Psi_{L^*}(G)$, the language of Picard stacks will be very useful. Hence now we translate the construction of the category $\Psi_{L^*}(G)$ in terms of Picard stacks : Let $\mathcal{R} = \text{st}(R), \mathcal{Q} = \text{st}(Q), \mathcal{P} = \text{st}(P), \mathcal{G} = \text{st}(G), D^R = \text{st}(D^R)$ and $D^Q = \text{st}(D^Q)$. The complex of 1-motives (4.1) furnishes the following complex of strictly commutative Picard $S$-stacks

\[
\mathcal{L} : \quad \mathcal{R} \xrightarrow{D^R} \mathcal{Q} \xrightarrow{D^Q} \mathcal{P} \xrightarrow{D^P} 0
\]

with $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ in degrees 0, -1 and -2 respectively. Via the equivalence of categories (2.5), to the category $\Psi_{L^*}(G)$ is associated the category $\Psi_{\mathcal{L}^*}(\mathcal{G})$.

1. whose objects are pairs $(\mathcal{E}, I)$ with $\mathcal{E}$ an extension of $\mathcal{P}$ by $\mathcal{G}$ and $I$ a trivialization of the extension $(D^Q)^*\mathcal{E}$ of $\mathcal{Q}$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ by $D^Q$. Moreover we require that the corresponding trivialization $(D^R)^*I$ of $(D^R)^*(D^Q)^*\mathcal{E}$ is the trivialization arising from the isomorphism of transitivity $(D^R)^*(D^Q)^*\mathcal{E} \cong (D^Q \circ D^R)^*\mathcal{E}$ and the relation $D^Q \circ D^R \cong 0$. 

}
Note that to have such a trivialization \( I \) is the same thing as to have a lifting \( I : \mathcal{Q} \to \mathcal{E} \) of \( D^Q : \mathcal{Q} \to \mathcal{P} \) such that \( I \circ D^R \simeq 0 \);

(2) whose arrows \( F : (\mathcal{E}, I) \to (\mathcal{E}', I') \) are morphisms of extensions \( F : \mathcal{E} \to \mathcal{E}' \) compatible with the trivializations \( I, I' \), i.e. we have an isomorphism of additive functors \( F \circ I \simeq I' \).

As observed at the end of section 2, extensions of 1-motives furnishes extensions of strictly commutative Picard stacks and so the sum of extensions of strictly commutative Picard stacks introduced in \cite{Be11} defines a group law on the set of isomorphism classes of objects of \( \Psi_L (G) \). We denote this group by \( \Psi^1_L (G) \). The neutral object of \( \Psi_L (G) \) is the object \((E_0, I_0)\) where \( E_0 \) is the trivial extension \( G \times P \) of \( P \) by \( G \) and \( I_0 \) is the trivialization \((Id_Q, 0)\) of the extension \((D^Q)^*E_0 = G \times Q\) of \( Q \) by \( G \). We can consider \( I_0 \) as the lifting \((D^Q, 0)\) of \( D^Q : Q \to P \).

The monoid of automorphisms of an object \((E, I)\) of \( \Psi_L (G) \) is canonically isomorphic to the monoid of automorphisms of \((E_0, I_0)\): to an automorphism \( F : (E_0, I_0) \to (E_0, I_0) \) the canonical isomorphism associates the automorphism \( F + Id_{(E, I)} \) of \((E_0, I_0) + (E, I) \cong (E, I) \). The monoid of automorphisms of \((E_0, I_0)\) is a commutative group via the composition law \((F, G) \to F + G\) (here \( F + G \) is the automorphism of \((E_0, I_0) + (E_0, I_0) \cong (E_0, I_0)\)). Hence we can conclude that the set of automorphisms of an object of \( \Psi_L (G) \) is a commutative group that we denote by \( \Psi^0_L (G) \).

We can now state the homological interpretation of the groups \( \Psi^i_L (G) \).

**Theorem 4.2.**

\[ \Psi^i_L (G) \cong \text{Ext}^i(\text{Tot}(L^-), G) = \text{Hom}_{D(S)}(\text{Tot}(L^-), G[i]) \quad i = 0, 1. \]

**Proof of the case \( i=0 \).** For this proof we will work with the category \( \Psi_L (\mathcal{G}) \). As observed above, \( \Psi^0_L (\mathcal{G}) \) is canonically isomorphic to the group of isomorphism classes of arrows from the neutral object \((\mathcal{E}_0, I_0)\) of \( \Psi_L (\mathcal{G}) \) to itself. By definition of arrows in the category \( \Psi_L (\mathcal{G}) \), the additive functor \( F : \mathcal{E}_0 \to \mathcal{E}_0 \) is an arrow from \((\mathcal{E}_0, I_0)\) to itself if we have an isomorphism of additive functors \( F \circ D^Q \simeq 0 \), i.e. if \( F \) is an object of the strictly commutative Picard \( S \)-stack

\[ \mathcal{K} = \ker (\text{HOM}(\mathcal{P}, \mathcal{G}) \to \text{HOM}(\mathcal{Q}, \mathcal{G})). \]

Therefore we have the equality

\[ \Psi^0_L (\mathcal{G}) = H^0([\mathcal{K}]) \]

and in order to conclude, it is enough to compute the complex \([\mathcal{K}]\) of \( \mathcal{K}^{[-1,0]}(S) \). By \cite{Be11} Lemma 3.4 we have

\[ [\mathcal{K}] = r_{\leq 0} \left( MC(r_{\leq 0} \text{RHom}(\mathcal{P}, \mathcal{G})) \xrightarrow{\partial_{R_*^{-1}} \partial_{R^0_*}} \tau_{\leq 0} \text{RHom}(\mathcal{Q}, \mathcal{G}))[-1] \right). \]

Explicitly, we get

\[ [\mathcal{K}] = \left[ \text{Hom}(P^0, G^{-1}) \xrightarrow{(\partial d^R, d^Q)} K_1 + K_2 \right] \]

where

\[ K_1 = \ker (\text{Hom}(P^0, G^0) + \text{Hom}(P^{-1}, G^{-1}) \xrightarrow{(\partial d^0, d^0}) \text{Hom}(Q^0, G^0) + \text{Hom}(Q^{-1}, G^{-1})) \]

\[ K_2 = \ker (\text{Hom}(Q^0, G^{-1}) \xrightarrow{d^Q} \text{Hom}(Q^0, G^0) + \text{Hom}(Q^{-1}, G^{-1})). \]
In order to simplify notation let $C^\cdot : C^{-3} \to C^{-2} \to C^{-1} \to C^{0}$ be the total complex $\text{Tot}(\mathcal{L})$. In particular $C^0 = P^0, C^{-1} = P^{-1} + Q^0$ and $C^{-2} = Q^{-1} + R^0$.

The stupid filtration of the complexes $C^\cdot$ and $G$ furnishes the spectral sequence

$$E_1^{pq} = \bigoplus_{p_2 - p_1 = p} \text{Ext}^q(C^{p_1}, G^{p_2}) \implies \text{Ext}^*(C^\cdot, G).$$

This spectral sequence is concentrated in the region of the plane defined by $-1 \leq p \leq 3$ and $q \geq 0$. We are interested on the total degrees -1 and 0. The rows $q = 1$ and $q = 0$ are

$$\text{Ext}^1(C^0, G^{-1}) \to \text{Ext}^1(C^0, G^0) \oplus \text{Ext}^1(C^{-1}, G^{-1}) \to \text{Ext}^1(C^{-1}, G^0) \oplus \text{Ext}^1(C^{-2}, G^{-1}) \to \ldots$$

$$\text{Hom}(C^0, G^{-1}) \xrightarrow{d_1^{-10}} \text{Hom}(C^0, G^0) \oplus \text{Hom}(C^{-1}, G^{-1}) \xrightarrow{d_0^{10}} \text{Hom}(C^{-1}, G^0) \oplus \text{Hom}(C^{-2}, G^{-1}) \to \ldots$$

Since $\text{Ext}^1(C^0, G^{-1}) = 0$, i.e. the only extension of $[G^{-1} \to 0]$ by $[0 \to C^0]$ is the trivial one, we obtain

$$\text{Hom}_{\mathcal{D}(S)}(C^\cdot, G[-1]) = \text{Ext}^{-1}(C^\cdot, G) = E_2^{-10} = \ker(d_1^{-10}),$$

$$\text{Hom}_{\mathcal{D}(S)}(C^\cdot, G) = \text{Ext}^0(C^\cdot, G) = E_2^{00} = \ker(d_1^{00})/\text{im}(d_1^{-10}).$$

Comparing the above equalities with the explicit computation (1.3) of the complex $[\mathcal{K}]$, we get

$$\text{Ext}^i(C^\cdot, G) = H^i([\mathcal{K}]) \quad i = -1, 0.$$

These equalities together with equality (1.2) give the expected statement.

**Remark 4.3.** In the computation (1.3) the term $\text{Hom}(P^{-1}, G^0)$ does not appear because we work with the good truncation $\tau_{\leq 0}\text{RHom}([\mathcal{P}], [G])$. In the spectral sequence (1.4) this term appear but we are interested in elements which become zero in $\text{Hom}(P^{-1}, G^0)$.

**Remark 4.4.** If $\mathcal{H}(S)$ denotes the category of complexes of abelian sheaves on $S$ modulo homotopy, by equality (1.6) we have $\text{Hom}_{\mathcal{D}(S)}(C^\cdot, G) = \text{Hom}_{\mathcal{H}(S)}(C^\cdot, G)$. Moreover, since $P$ and $G$ are 1-motives we have that $\text{Hom}(C^0, G^{-1}) = 0$ (B09 Lemma 1.1.1) and so $\text{Hom}_{\mathcal{D}(S)}(C^\cdot, G) = \text{Hom}_{\mathcal{H}(S)}(C^\cdot, G)$.

**Remark 4.5.** The category $\Psi_{\mathcal{L}}(\mathcal{G})$ should be a 2-category, but it is just a category because we are working with strictly commutative Picard stacks defined by 1-motives. In fact, if $A$ is a group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module and $B$ is an extension of an abelian scheme by a torus, then the group $\text{Hom}(B, A)$ is trivial (B09 Lemma 1.1.1). Because of (1.2), (1.5), (1.7), this implies that the group $\Psi_{\mathcal{L}}^{-1}(\mathcal{G})$ of automorphisms of arrows from an object of $\Psi_{\mathcal{L}}(\mathcal{G})$ to itself is trivial:

$$\Psi_{\mathcal{L}}^{-1}(\mathcal{G}) \cong H^{-1}([\mathcal{K}]) \cong \text{Ext}^{-1}(C^\cdot, G) = \ker(d_1^{-10}) = 0.$$

**Proof of the case $i=1$.** First we show how an object $(E, I)$ of $\Psi_{\mathcal{L}}(\mathcal{G})$ defines a morphism $\text{Tot}(\mathcal{L}^\cdot) \to G[1]$ in the derived category $\mathcal{D}(S)$. Recall that $E$ is an extension of 1-motives of $P$ by $G$. Denote $j : E \to P$ the surjective morphism underlying the extension $E$. Since the trivialization $I$ can be seen as a lifting $Q \to E$ of $D^Q : Q \to P$ such that $I \circ D^R = 0$, we have the following diagram in the
category $\mathcal{K}(S)$ of complexes of abelian sheaves on $S$

\[
\begin{array}{ccccccccc}
L^- : & R & \xrightarrow{D_R} & Q & \xrightarrow{D_Q} & P & \rightarrow & 0 \\
& i & & id_P & & & & \\
MC(j) : & 0 & \xrightarrow{E} & j & \xrightarrow{P} & P & \rightarrow & 0
\end{array}
\]

where $i \circ D_R = 0$ and $j \circ i = id_P \circ D_Q$. Putting the complex $P$ in degree 0, the above diagram gives an arrow

\[c(E, I) : \text{Tot}(L^-) \rightarrow MC(j)\]

in the derived category $\mathcal{D}(S)$. The complex $E$ is an extension of 1-motives of $P$ by $G$ and so as observed at the end of section 2, $G$ is quasi-isomorphic to $\tau_{\leq 0}(MC(j)[-1])$. Hence we have constructed a canonical arrow

\[
(4.9) \quad c : \Psi^1_{L^-}(G) \rightarrow \text{Hom}_{\mathcal{D}(S)}(\text{Tot}(L^-), G[1])
\]

\[
(E, I) \mapsto c(E, I).
\]

Now we will show that this arrow is bijective. The proof that this bijection is additive, i.e. that $c$ is an isomorphism of groups, is left to the reader.

**Injectivity:** Let $(E, I)$ be an object of $\Psi^1_{L^-}(G)$ such that the morphism $c(E, I)$ that it defines in $\mathcal{D}(S)$ is the zero morphism. The hypothesis that $c(E, I)$ is zero in $\mathcal{D}(S)$ implies that there exists a resolution of $G$

\[
V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow ...
\]

and a quasi isomorphism

\[\begin{array}{ccccccccc}
0 & \xrightarrow{j} & E & \xrightarrow{k} & V^0 & \rightarrow & 0 \\
& \downarrow{v^0} & & \downarrow{v^1} & & & & \\
0 & \rightarrow & V^0 & \rightarrow & V^1 & \rightarrow & V^2 & \rightarrow & ...
\end{array}\]

such that the composite

\[
\begin{array}{ccccccccc}
R & \xrightarrow{D_R} & Q & \xrightarrow{D_Q} & P & \rightarrow & 0 \\
& i & & id_P & & & & \\
0 & \xrightarrow{E} & j & \xrightarrow{P} & 0 \\
& v^0 & & v^1 & & & & \\
0 & \rightarrow & V^0 & \rightarrow & V^1 & \rightarrow & V^2 & \rightarrow & ...
\end{array}
\]

is homotopic to zero. We can assume $V^i \in \mathcal{K}[-1,0](S)$ for all $i$ and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$, consider its good truncation in degree 1). The complex of complexes $(V^i)_i$ is a resolution of $G$, and so the short sequence

\[0 \rightarrow G \rightarrow V^0 \rightarrow V^1 \rightarrow 0\]

is exact, i.e. $V^0$ is an extension of $W$ by $G$. Since the quasi-isomorphism\[\begin{array}{ccccccccc}
0 & \rightarrow & G & \rightarrow & V^0 & \rightarrow & V^1 & \rightarrow & 0
\end{array}\]

induces the identity on $G$, the extension $E$ is in fact the fibred product $P \times_{V^1} V^0$
of $P$ and $V^0$ over $V^1$. Therefore, the morphism $s : P \to V^0$ inducing the homotopy $(v^0, v^1) \circ c(\mathcal{E}, I) \sim 0$, i.e. satisfying $k \circ s = v^1 \circ id_P$, factorizes through a morphism

$$h : P \longrightarrow E = P \times_{V^1} V^0$$

satisfying

$$j \circ h = id_P \quad h \circ D^Q = i.$$

These two equalities mean that $h$ splits the extension $E$, which is therefore the trivial extension of $P$ by $G$, and that $h$ is compatible with the trivializations $I$. Hence we can conclude that the object $(E, I)$ lies in the isomorphism class of the zero object of $\Psi_L(G)$.

**Surjectivity:** Now we show that for any morphism $f$ of $\text{Hom}_{\mathcal{D}(S)}(\text{Tot}(L^-), G[1])$, there is an element of $\Psi^1_L(G)$ whose image via $c$ is $f$. The hypothesis that $f$ is an element of $\mathcal{D}(S)$ implies that there exists a resolution of $G$

$$V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow \ldots$$

such that the morphism $f$ can be described in the category $\mathcal{H}(S)$ of complexes modulo homotopy via the following diagram

$$\begin{array}{cccccc}
R & \overset{D^R}{\longrightarrow} & Q & \overset{D^Q}{\longrightarrow} & P & \longrightarrow 0 \\
\downarrow v^0 & & \downarrow v^1 & & \downarrow k & \\
0 & \longrightarrow & V^0 & \overset{id}{\longrightarrow} & V^1 & \longrightarrow V^2 \longrightarrow \ldots
\end{array}$$

We can assume $V^i \in k^{[-1,0]}(S)$ for all $i$ and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$, consider its good truncation in degree 1). Since the complex of complexes $(V^i)_i$ is a resolution of $G$, the short sequence of complexes

$$0 \longrightarrow G \longrightarrow V^0 \longrightarrow V^1 \longrightarrow 0$$

is exact, i.e. $V^0$ is an extension of $V^1$ by $G$. Consider the extension of $P$ by $G$

$$Z = (v^1)^* V^0 = V^0 \times_{V^1} P$$

obtained as pull-back of $V^0$ via $v^1 : P \to V^1$. The pull-back of a short exact sequence is again a short exact sequence, and so $0 \to G \to Z \to P \to 0$ is exact. Moreover, as observed in Remark 1.2, since $P$ and $G$ are 1-motives the complex $Z$ is an extension of 1-motives. The condition $v^1 \circ D^Q = k \circ v^0$ implies that $v^0 : Q \to V^0$ factorizes through a morphism

$$z : Q \to Z$$

satisfying $l \circ z = D^Q$, with $l : Z \to P$ the canonical surjection of the extension $Z$. Moreover the equalities $v^0 \circ D^R = D^Q \circ D^R = 0$ furnish $z \circ D^R = 0$. Therefore the datum $(Z, z)$ is an object of the category $\Psi_L(G)$. Consider now the morphism $c(Z, z) : \text{Tot}(L^-) \to G[1]$ associated to $(Z, z)$. By construction, the morphism $f$ is the composite of the morphism $c(Z, z)$

$$\begin{array}{cccccc}
R & \overset{D^R}{\longrightarrow} & Q & \overset{D^Q}{\longrightarrow} & P & \longrightarrow 0 \\
\downarrow z & & \downarrow id_P & & \downarrow & \\
0 & \longrightarrow & Z & \longrightarrow & P & \longrightarrow 0
\end{array}$$
with the morphism

\[
\begin{array}{ccc}
0 & \rightarrow & Z \\
\downarrow h & & \downarrow \phi \\
0 & \rightarrow & V^0
\end{array}
\]

where \(h : Z = (v^1)^*V^0 \rightarrow V^0\) is the canonical projection underlying the pull-back \(Z\). Since this last morphism is a morphism of resolutions of \(G\) (inducing the identity on \(G\)), we can conclude that in the derived category \(D(S)\) the morphism \(f : \text{Tot}(L^\cdot) \rightarrow G[1]\) \((4.11)\) is the morphism \(c(Z, z)\).

Using the above homological description of the groups \(\Psi^i_L(G)\) for \(i = 0, 1\) we can study how the category \(\Psi^i_L(G)\) varies with respect to the bicomplex \(L^\cdot\). Let \(R' \rightarrow Q' \rightarrow P' \rightarrow 0\) be another complex of 1-motives defined over \(S\). Denote by \(L'\) its total bicomplex. Consider a morphism of bicomplexes \(F : L' \rightarrow \rightarrow \) given by the following commutative diagram

\[\begin{array}{ccc}
R' & \overset{D_{R'}}{\rightarrow} & Q' \\
\downarrow F^{-2} & & \downarrow F^{-1} \\
R & \overset{D_{R}}{\rightarrow} & Q \\
\downarrow F^0 & & \downarrow P \\
0 & & 0
\end{array}\]

The morphism \(F\) defines a canonical functor

\[F^* : \Psi^i_L(G) \rightarrow \Psi^i_{L'}(G)\]

as follows: if \((E, I)\) is an object of \(\Psi^i_L(G)\), \(F^*(E, I)\) is the object \((E', I')\) where

- \(E'\) is the extension \((F^0)^*E\) of \(P'\) by \(G\) obtained as pull-back of \(E\) via \(F^0 : P' \rightarrow P\);
- \(I'\) is the trivialization \((F^{-1})^*I\) of \((DQ')^*E'\) induced by the trivialization \(I\) of \((DR')^*E\) via the commutativity of the first square of \(4.12\).

The commutativity of the diagram \(4.12\) implies that \((E', I')\) is in fact an object of \(\Psi^i_{L'}(G)\) (the condition \(I' \circ DQ' = 0\) is easily deducible from the corresponding conditions on \(I\) and from the commutativity of the diagram \(4.12\)).

**Proposition 4.6.** Let \(F : L' \rightarrow \rightarrow L\) be morphism of bicomplexes. The corresponding functor \(F^* : \Psi^i_L(G) \rightarrow \Psi^i_{L'}(G)\) is an equivalence of categories if and only if the homomorphisms

\[H^i(\text{Tot}(F)) : H^i(\text{Tot}(L')) \rightarrow H^i(\text{Tot}(L^\cdot))\]

are isomorphisms.

**Proof.** The functor \(F^* : \Psi^i_L(G) \rightarrow \Psi^i_{L'}(G)\) induces the homomorphisms

\[\Psi^i_L(G) \rightarrow \Psi^i_{L'}(G)\]

On the other hand the morphism of bicomplexes \(F : L' \rightarrow \rightarrow L\) defines the homomorphisms

\[\text{Ext}^i(\text{Tot}(L^\cdot), -) \rightarrow \text{Ext}^i(\text{Tot}(L'), -)\]

\(i \in \mathbb{Z}\).
Since the homomorphisms (4.13) and (4.14) are compatible with the canonical isomorphisms obtained in Theorem 4.2, the following diagrams (with \(i = 0, 1\)) are commutative:

\[
\begin{array}{ccc}
\Psi^i_L (G) & \to & \varepsilon^i (\text{Tot}(L^\cdot), G) \\
\downarrow & & \downarrow \\
\Psi^i_{L'} (G) & \to & \varepsilon^i (\text{Tot}(L'^\cdot), G).
\end{array}
\]

The functor \(F^* : \Psi_L (G) \to \Psi_{L'} (G)\) is an equivalence of categories if and only if the homomorphisms (4.13) are isomorphisms, and so using the above commutative diagrams we are reduced to prove that the homomorphisms (4.14) are isomorphisms if and only if the homomorphisms \(H^i (\text{Tot}(F)) : H^i (\text{Tot}(L'^\cdot)) \to H^i (\text{Tot}(L^\cdot))\) are isomorphisms. This last assertion is clearly true. \(\Box\)

5. Geometrical description of \(\Psi_L (G)\)

In this section we switch from cohomological notation to homological.

Let \(K = [u : A \to B]\) be a 1-motive defined over \(S\) with \(A\) in degree 1 and \(B\) in degree 0. We start constructing a canonical flat partial resolution \(L..(K)\) of the complex \(K\). But before, we introduce the following notations: if \(P\) is an abelian sheaf on \(S\), we denote by \([p]\) the point of \(\mathbb{Z}[P](U)\) defined by the point \(p\) of \(P(U)\) with \(U\) an \(S\)-scheme. In an analogous way, if \(p, q\) and \(r\) are points of \(P(U)\) we denote by \([p, q]\), \([p, q, r]\) the elements of \(\mathbb{Z}[P \times P](U)\) and \(\mathbb{Z}[P \times P \times P](U)\) respectively.

Consider the following complexes of \(\mathcal{D}[1,0](S)\)

\[
P = \mathbb{Z}[A] \xrightarrow{D_{00}} \mathbb{Z}[B]
\]

\[
Q = [0 \to \mathbb{Z}[B \times B]]
\]

\[
R = [0 \to \mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B]]
\]

(5.1)

and the following morphisms of complexes

\[
(\epsilon_1, \epsilon_0) : P \to K
\]

\[
(0, d_{00}) : Q \to P
\]

\[
(0, d_{01}) : R \to Q
\]

where for any \(U\) and for any \(a \in A(U), b_1, b_2, b_3 \in B(U)\), we set

\[
\begin{align*}
\epsilon_0[b] & = b \\
\epsilon_1[a] & = a \\
d_{00}[b_1, b_2] & = [b_1 + b_2] - [b_1] - [b_2] \\
d_{01}[b_1, b_2] & = [b_1, b_2] - [b_2, b_1] \\
d_{01}[b_1, b_2, b_3] & = [b_1 + b_2, b_3] - [b_1, b_2 + b_3] + [b_1, b_2] - [b_2, b_3] \\
D_{00}[a] & = [u(a)].
\end{align*}
\]
These data define the bicomplex $L_\ast(K)$

\[
\begin{array}{ccc}
L_{3\ast}(K) & L_{2\ast}(K) & L_{1\ast}(K) & L_{0\ast}(K) \\
\{ & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \to 0 & \to \mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B] & \to 0 \\
\downarrow & \downarrow_{d_{01}} & \downarrow_{d_{00}} \\
& & & \\
0 \to 0 & \to \mathbb{Z}[B \times B] & \to 0 \\
\downarrow_{e_1} & \downarrow_{e_0} \\
& & & \\
0 \to A & \to B & \to 0 \\
\end{array}
\]

which satisfies $L_{ij}(K) = 0$ for $(ij) \neq (00), (01), (02), (10)$ and which is endowed with an augmentation map $\epsilon = (\epsilon_1, \epsilon_0) : P \to K$. Note that the relation $\epsilon_0 \circ d_{00} = 0$ is just the group law on $B$, and the relation $d_{00} \circ d_{01} = 0$ decomposes in two relations which express the commutativity and the associativity of the group law on $B$. This augmented bicomplex $L_\ast(K)$ depends functorially on $K$: in fact, any morphism $f : K \to K'$ of 1-motives furnishes a commutative diagram

\[
\begin{array}{ccc}
L_\ast(K) & \xrightarrow{L_\ast(f)} & L_\ast(K') \\
\epsilon \downarrow & & \downarrow \epsilon \\
K & \xrightarrow{f} & K'.
\end{array}
\]

Moreover the components of the bicomplex $L_\ast(K)$ are flat since they are free $\mathbb{Z}$-modules. In order to conclude that $L_\ast(K)$ is a canonical flat partial resolution of $K$ we need the following Lemma. Let $K' = [u' : A' \to B']$ be a 1-motive defined over $S$.

**Lemma 5.1.** The category $\text{Ext}(K, K')$ of extensions of $K$ by $K'$ is equivalent to the category $\Psi_{L_\ast(K)}(K')$:

\[
\text{Ext}(K; K') \simeq \Psi_{L_\ast(K)}(K').
\]

**Proof.** In order to describe explicitly the objects of the category $\Psi_{L_\ast(K)}(K')$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $\mathbb{Z}[B]$ by $B'$ is a $(B')_B$-torsor,
- an extension of $\mathbb{Z}[A]$ by $B'$ is a $(B')_A$-torsor,
- an extension of $\mathbb{Z}[B \times B]$ by $B'$ is a $(B')_{B \times B}$-torsor, and finally
- an extension of $\mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B]$ by $B'$ consists of a couple of a $(B')_{B \times B}$-torsor and a $(B')_{B \times B \times B}$-torsor.

According to these considerations an object $(E, I)$ of $\Psi_{L_\ast(K)}(K')$ consists of:

1. an extension $E$ of $P = [D_{00} : \mathbb{Z}[A] \to \mathbb{Z}[B]]$ by $K' = [u' : A' \to B']$, i.e.
   - a $B'$-torsor $E$ over $B$,
   - a trivializations $\beta$ of the $B'$-torsor $D_{00}^\ast E$ over $A$ obtained as pull-back of $E$ via $D_{00} : \mathbb{Z}[A] \to \mathbb{Z}[B]$,
   - an homomorphism $\gamma : A \to A'$ such that the composite $u' \circ \gamma$ is compatible with $\beta$;
2. a trivialization $I$ of the extension $(0, d_{00})^\ast E$ of $Q$ by $K'$ obtained as pull-back of $E$ by $(0, d_{00}) : Q \to P$, i.e. a trivialization $I$ of the $B'$-torsor $d_{00}^\ast E$.
over \(B \times B\) obtained as pull-back of \(E\) via \(d_{00} : \mathbb{Z}[B \times B] \to \mathbb{Z}[B]\). This trivialization can be interpreted as a group law on the fibres of the \(B'\)-torsor \(E\):

\[
+ : E_{b_1} E_{b_2} \to E_{b_1 + b_2}
\]

where \(b_1, b_2\) are points of \(B(U)\) with \(U\) an \(S\)-scheme. The compatibility of \(I\) with the relation \((0, d_{00}) \circ (0, d_{01}) = 0\) imposes on the datum \((E, +)\) two relations through the two torsors over \(B \times B\) and \(B \times B \times B\). These two relations are the relations of commutativity and of associativity of the group law +, which mean that + defines over \(E\) a structure of commutative extension of \(B\) by \(B'\).

Hence the object \((E, +, \beta, \gamma)\) of \(\Psi_{L_0(K)}(K')\) is an extension of \(K\) by \(K'\) and we can conclude that the category \(\Psi_{L_0(K)}(K')\) is equivalent to the category \(\text{Ext}(K, K')\). \(\square\)

**Proposition 5.2.** The augmentation map \(\epsilon_\ast : L_0(K) \to K\) induces the isomorphisms \(H_1(\text{Tot}(L_0(K))) \cong H_1(K)\) and \(H_0(\text{Tot}(L_0(K))) \cong H_0(K)\).

**Proof.** Applying Proposition 4.6 to the augmentation map \(\epsilon_\ast : L_0(K) \to K\), we just have to prove that for any 1-motive \(K' = [u' : A' \to B']\) the functor

\[
\epsilon_\ast : \Psi_K(K') \to \Psi_{L_0(K)}(K')
\]

is an equivalence of categories (in the symbol \(\Psi_K(K')\), \(K\) is seen as a bicomplex whose only non trivial entries are \(A\) in degree \((10)\) and \(B\) in degree \((00)\)). According to definition 4.1 it is clear that the category \(\Psi_K(K')\) is equivalent to the category \(\text{Ext}(K, K')\) of extensions of \(K\) by \(K'\). On the other hand, by Lemma 5.1 also the category \(\Psi_{L_0(K)}(K')\) is equivalent to the category \(\text{Ext}(K, K')\). Hence we can conclude. \(\square\)

Let \(K_i = [u_i : A_i \to B_i]\) (for \(i = 1, 2, 3\)) be 1-motives defined over \(S\) and let \(L_0(K_i)\) be its canonical flat partial resolution. Denote by \(L_0(K_1, K_2)\) the bicomplex \(L_0(K_1) \otimes L_0(K_2)\).

**Theorem 5.3.** The category \(\text{Biext}(K_1, K_2; K_3)\) of biextensions of \((K_1, K_2)\) by \(K_3\) is equivalent to the category \(\Psi_{\tau_{\leq 1}}(L_0(K_1, K_2))\) :

\[
\text{Biext}(K_1, K_2; K_3) \simeq \Psi_{\tau_{\leq 1}}(L_0(K_1, K_2))(K_3)
\]
Proof. Explicitly, the non trivial components of $L_{ij}(K_1, K_2)$ are

\[
\begin{align*}
L_{00}(K_1, K_2) & = L_{00}(K_1) \otimes L_{00}(K_2) \\
& = \mathbb{Z}[B_1 \times B_2] \\
L_{01}(K_1, K_2) & = L_{00}(K_1) \otimes L_{01}(K_2) + L_{01}(K_1) \otimes L_{00}(K_2) \\
& = \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2] \\
L_{02}(K_1, K_2) & = L_{00}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{00}(K_2) + L_{01}(K_1) \otimes L_{01}(K_2) \\
& = \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_2 \times B_2 \times B_2] + \\
& = \mathbb{Z}[B_1 \times B_1 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2 \times B_2] + \\
& = \mathbb{Z}[B_1 \times B_1 \times B_2 \times B_2] \\
L_{03}(K_1, K_2) & = L_{01}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{01}(K_2) \\
L_{04}(K_1, K_2) & = L_{02}(K_1) \otimes L_{02}(K_2) \\
L_{10}(K_1, K_2) & = L_{10}(K_1) \otimes L_{00}(K_2) + L_{00}(K_1) \otimes L_{10}(K_2) \\
& = \mathbb{Z}[A_1 \times B_2] + \mathbb{Z}[B_1 \times A_2] \\
L_{10}(K_1, K_2) & = L_{10}(K_1) \otimes L_{01}(K_2) + L_{01}(K_1) \otimes L_{10}(K_2) \\
& = \mathbb{Z}[A_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times A_2] \\
L_{12}(K_1, K_2) & = L_{10}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{10}(K_2) \\
L_{20}(K_1, K_2) & = L_{10}(K_1) \otimes L_{10}(K_2) \\
& = \mathbb{Z}[A_1 \times A_2]
\end{align*}
\]

The differential operators of $L_{..}(K_1, K_2)$ can be computed from the below diagram, where we don’t have written the identity homomorphisms in order to avoid too heavy notation (for example instead of $(id \times D_{00}^{K_2}, D_{00}^{K_1} \times id)$ we have written just $(D_{00}^{K_2}, D_{00}^{K_1})$):

\[
\begin{array}{ccc}
L_{02}(K_1, K_2) & \rightarrow & L_{01}(K_1, K_2) \\
\downarrow & & \downarrow \\
L_{10}(K_1, K_2) & \rightarrow & L_{11}(K_1, K_2) \\
\downarrow & & \downarrow \\
L_{20}(K_1, K_2) & \rightarrow & L_{12}(K_1, K_2)
\end{array}
\]

These operators have to satisfy the well-known conditions on differential operators of bicomplexes that we recall explicitly here:

- the following sequences are exact:

\[
\begin{align*}
(5.5) & \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_2 \times B_2 \times B_2] \xrightarrow{d_{00}^{K_2}} \mathbb{Z}[B_1 \times B_2 \times B_2] \xrightarrow{d_{00}^{K_2}} \mathbb{Z}[B_1 \times B_2] \\
(5.6) & \mathbb{Z}[B_1 \times B_1 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2 \times B_2] \xrightarrow{d_{00}^{K_1}} \mathbb{Z}[B_1 \times B_1 \times B_2] \xrightarrow{d_{00}^{K_1}} \mathbb{Z}[B_1 \times B_2]
\end{align*}
\]
the following diagrams are anticommutative:

\[
\begin{align*}
Z[B_1 \times B_1 \times B_2 \times B_2] & \xrightarrow{d_{K_0}^{K_2}} Z[B_1 \times B_1 \times B_2] \\
Z[B_1 \times B_2 \times B_2] & \xrightarrow{d_{K_0}^{K_1}} Z[B_1 \times B_2]
\end{align*}
\]

(5.7)

\[
\begin{align*}
Z[A_1 \times B_2 \times B_2] & \xrightarrow{d_{K_0}^{K_1}} Z[B_1 \times B_2 \times B_2] \\
Z[A_1 \times B_2] & \xrightarrow{d_{K_0}^{K_1}} Z[B_1 \times B_2]
\end{align*}
\]

(5.8)

\[
\begin{align*}
Z[B_1 \times B_1 \times A_2] & \xrightarrow{d_{K_0}^{K_2}} Z[B_1 \times B_1 \times B_2] \\
Z[B_1 \times A_2] & \xrightarrow{d_{K_0}^{K_2}} Z[B_1 \times B_2]
\end{align*}
\]

(5.9)

\[
\begin{align*}
Z[A_1 \times A_2] & \xrightarrow{d_{K_0}^{K_2}} Z[A_1 \times B_2] \\
Z[B_1 \times A_2] & \xrightarrow{d_{K_0}^{K_2}} Z[B_1 \times B_2]
\end{align*}
\]

(5.10)

The bicomplex \(\tau_{\leq (1+)}L_{\leq (1+)}(K_1, K_2)\) is furnished by the bicomplex (5.4) where instead of \(L_{10}(K_1)\) we have

\[
L'_{10}(K_1, K_2) = L_{10}(K_1, K_2) / ((D_{K_0}^{K_2} : D_{K_0}^{K_1}) L_{20}(K_1, K_2)
\]

(5.11) \[= Z[A_1 \times B_2] + Z[B_1 \times A_2] / (id \times u_2) + (u_1 \times id) Z[A_1 \times A_2]\]

In order to describe explicitly the objects of \(\Psi_{\tau_{\leq (1+)}L_{\leq (1+)}(K_1, K_2)}(K_3)\) we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of \(L_{00}(K_1, K_2)\) by \(B_3\) is a \((B_3)_{B_1 \times B_2}\)-torsor,
- an extension of \(L'_{10}(K_1, K_2)\) by \(B_3\) consists of a \((B_3)_{B_1 \times B_2}\)-torsor and a \((B_3)_{B_1 \times A_2}\)-torsor,
- an extension of \(L_{02}(K_1, K_2)\) by \(B_3\) consists of a system of 5 torsors under the groups deduced from \(B_3\) by base change over the bases \(B_1 \times B_2 \times B_2, B_1 \times B_1 \times B_2, \ldots, B_1 \times B_1 \times B_2\).

By these considerations an object \((E, I)\) of \(\Psi_{\tau_{\leq (1+)}L_{\leq (1+)}(K_1, K_2)}(K_3)\) consists of

(1) an extension \(E\) of \([D_{K_0}^{K_1} + D_{K_0}^{K_2} : L'_{10}(K_1, K_2) \to L_{00}(K_1, K_2)]\) by \(K_3\), i.e.
(a) a \(B_3\)-torsor \(E\) over \(B_1 \times B_2\),
(b) a couple of trivializations \((\Psi_1, \Psi_2)\) of the couple of \(B_3\)-torsors \(((D_{K_0}^{K_1} \times id)^* E, (id \times D_{K_0}^{K_2})^* E)\) over \(A_1 \times B_2\) and \(B_1 \times A_2\) respectively, which are the pull-back of \(E\) via
\[
(D_{K_0}^{K_1} \times id) + (id \times D_{K_0}^{K_2}) : Z[A_1 \times B_2] + Z[B_1 \times A_2] \to Z[B_1 \times B_2];
\]

We consider the factor \(L'_{10}(K_1, K_2)\) (5.11) instead of \(L_{10}(K_1, K_2)\) and this means that the restriction of the trivializations \((\Psi_1, \Psi_2)\) have to coincide over \(A_1 \times A_2\),
(c) a homomorphism \( \gamma : \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \rightarrow A_3 \) such that the composite
\[
\mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \xrightarrow{\sim} \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \xrightarrow{\sim} B_3
\]
is compatible with the restriction of the trivializations \( \Psi_1, \Psi_2 \) over \( \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \).

(2) a trivialization \( \mathcal{I} \) of the extension \( (d_{K_1}^{K_2} + d_{00}^{K_1}, d_{K_2}^{K_1} + d_{00}^{K_1})^* E \) of \( [D_{K_1}^{K_2} + D_{00}^{K_1} : L_{11}(K_1, K_2) \rightarrow L_{00}(K_1, K_2)] \) by \( K_3 \) obtained as pull-back of \( E \) via
\[
(d_{K_2}^{K_1} + d_{00}^{K_1}, d_{K_2}^{K_1} + d_{00}^{K_1}) : [L_{11}(K_1, K_2) \rightarrow L_{01}(K_1, K_2)] \rightarrow [L_{10}'(K_1, K_2) \rightarrow L_{00}(K_1, K_2)],
\]
i.e. a couple of trivializations \( \alpha = (\alpha_1, \alpha_2) \) of the couple of \( B_3 \)-torsors over \( B_1 \times B_2 \times B_2 \) and \( B_1 \times B_1 \times B_2 \) which are the pull-back of \( E \) via
\[
(id \times d_{00}^{K_1}) + (d_{00}^{K_1} \times id) : \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2] \rightarrow \mathbb{Z}[B_1 \times B_2].
\]
The trivializations \( (\alpha_1, \alpha_2) \) can be viewed as two group laws on the fibres of the \( B_3 \)-torsor \( E \) over \( B_1 \times B_2 \):
\[
+ : E_{b_1, b_2} E_{b_1', b_2'} \rightarrow E_{b_1, b_2 + b_2'} \quad + : E_{b_1, b_2} E_{b_1', b_2} \rightarrow E_{b_1 + b_1', b_2}
\]
where \( b_2, b_2' \) (resp. \( b_1, b_1' \)) are points of \( B_2(U) \) (resp. of \( B_1(U) \)) with \( U \) any \( S \)-scheme.

The trivialization \( \mathcal{I} \), i.e. the two group laws, must be compatible with the trivializations \( \Psi_1, \Psi_2 \) underlying the trivialization \( E \). This compatibility is expressed through the 2 torsors arising from the factors \( L_{11}(K_1, K_2) \):

- the anticommutative diagram \([5.8]\) furnishes a relation of compatibility between the group law \( +_2 \) of \( E \) and the trivialization \( \Psi_1 \) of the pull-back \( (D_{K_1}^{K_2} \times id)^* E \) of \( E \) over \( A_1 \times B_2 \), which means that \( \Psi_1 \) is an additive section;
- the anticommutative diagram \([5.9]\) furnishes a relation of compatibility between the group law \( +_1 \) of \( E \) and the trivialization \( \Psi_2 \) of the pull-back \( (id \times D_{00}^{K_1})^* E \) of \( E \) over \( B_1 \times A_2 \), which means that also \( \Psi_2 \) is an additive section.

Finally, the compatibility of \( \mathcal{I} \) with the relation
\[
(d_{K_2}^{K_1} + d_{00}^{K_1}, d_{K_2}^{K_1} + d_{00}^{K_1} + (d_{00}^{K_1} + d_{00}^{K_1})) = 0
\]
impedes on the datum \((E, +_1, +_2)\) 5 relations of compatibility through the system of 5 torsors over \( B_1 \times B_2 \times B_2 \), \( B_1 \times B_2 \times B_2 \times B_2 \), \( B_1 \times B_1 \times B_2 \), \( B_1 \times B_1 \times B_2 \times B_2 \), \( B_1 \times B_1 \times B_1 \times B_2 \), \( B_1 \times B_1 \times B_2 \times B_2 \) arising from \( L_{02}(K_1, K_2) \):

- the exact sequence \([5.5]\) furnishes the two relations of commutativity and of associativity of the group law \( +_2 \), which mean that \( +_2 \) defines over \( E \) a structure of commutative extension of \((B_2)_B \), by \((B_3)_B \);
- the exact sequence \([5.6]\) expresses the two relations of commutativity and of associativity of the group law \( +_1 \), which mean that \( +_1 \) defines over \( E \) a structure of commutative extension of \((B_1)_B \) by \((B_3)_B \);
- the anticommutative diagram \([5.7]\) means that these two group laws are compatible.

Therefore these 5 conditions implies that the \( B_3 \)-torsor \( E \) is endowed with a structure of biextension of \((B_1, B_2)\) by \( B_3 \).

The object \((E, \Psi_1, \Psi_2, \gamma)\) of \( \Psi_{\tau(1_1^\perp, L_{\perp}(K_1, K_2))}(K_3) \) is therefore a biextension of \((K_1, K_2)\) by \( K_3 \).

\( \square \)

In the above proof we have not used diagram \([5.10]\) because we work with the truncated bicomplex \( \tau_{\tau(1_1^\perp, L_{\perp}(K_1, K_2))}(K_3) \) (see \([5.11]\)).
6. Proof of Theorem 0.1 (a)

Let $K_i = [A_i \rightarrow B_i]$ (for $i = 1, 2, 3$) be three 1-motives defined over $S$. Denote by $L.(K_i)$ (for $i = 1, 2$) the canonical flat partial resolution of $K_i$ introduced in §5. According to Proposition 5.2 there exists an arbitrary flat resolution $L'.(K_i)$ (for $i = 1, 2$) of $K_i$ such that the groups $\text{Tot}(L.(K_i))$ and $\text{Tot}(L'.(K_i))$ are isomorphic for $j = 0, 1$. We have therefore two canonical homomorphisms of bicomplexes

$$L.(K_1) \rightarrow L'.(K_1) \quad L.(K_2) \rightarrow L'.(K_2)$$

inducing a canonical homomorphism between the corresponding total complexes

$$\text{Tot}(L.(K_1) \otimes L.(K_2)) \rightarrow \text{Tot}(L'.(K_1) \otimes L'.(K_2))$$

which is an isomorphism in degrees 0 and 1. Denote by $L.(K_1, K_2)$ (resp. $L'.(K_1, K_2)$) the bicomplex $L.(K_1) \otimes L.(K_2)$ (resp. $L'.(K_1) \otimes L'.(K_2)$). Remark that $\text{Tot}(L'.(K_1, K_2))$ represents $K_1 \hat{\otimes} K_2$ in the derived category $\mathcal{D}(S)$:

$$\text{Tot}(L'.(K_1, K_2)) = K_1 \hat{\otimes} K_2.$$ 

By Proposition 4.6 we have the equivalence of categories

$$\Psi_{\tau \leq (1_+)} L.(K_1, K_2)(K_3) \cong \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(K_3).$$

Hence applying Theorem 0.3 which furnishes the following geometrical description of the category $\Psi_{\tau \leq (1_+)} L.(K_1, K_2)(K_3)$:

$$\text{Biext}(K_1, K_2; K_3) \cong \Psi_{\tau \leq (1_+)} L.(K_1, K_2)(K_3),$$

and applying Theorem 1.2 which furnishes the following homological description of the groups $\Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(K_3)$ for $i = 0, 1$:

$$\Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(K_3) \cong \text{Ext}^i(\text{Tot}(\tau \leq (1_+) L'.(K_1, K_2)), K_3) \cong \text{Ext}^i(K_1 \hat{\otimes} K_2, K_3),$$

we get Theorem 0.1, i.e.

$$\text{Biext}^i(K_1, K_2; K_3) \cong \text{Ext}^i(K_1 \hat{\otimes} K_2, K_3) \quad (i = 0, 1).$$

Remark 6.1. From the exact sequence $0 \rightarrow B_3 \rightarrow K_3 \rightarrow A_3[1] \rightarrow 0$ we get the long exact sequence

$$0 \rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(B_3) \rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(K_3) \rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(A_3[1])$$

$$\rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(B_3) \rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(K_3) \rightarrow \Psi_{\tau \leq (1_+)} L'.(K_1, K_2)(A_3[1]).$$

The homological interpretation of this long exact sequence is

$$0 \rightarrow \text{Hom}(T, B_3) \rightarrow \text{Hom}(T, K_3) \rightarrow \text{Hom}(T, A_3[1])$$

$$\rightarrow \text{Ext}^1(T, B_3) \rightarrow \text{Ext}^1(T, K_3) \rightarrow \text{Ext}^1(T, A_3[1]),$$

where we set $T = \text{Tot}(\tau \leq (1_+) L'.(K_1, K_2))$, and its geometrical interpretation is

$$0 \rightarrow \text{Hom}(B_1 \otimes B_2, B_3) \rightarrow \text{Hom}(K_1 \hat{\otimes} K_2, K_3) \rightarrow \text{Hom}(A_1 \otimes B_2 + B_1 \otimes A_2, A_3)$$

$$\rightarrow \text{Biext}^1(K_1, K_2; B_3) \rightarrow \text{Biext}^1(K_1, K_2; K_3) \rightarrow \text{Hom}(A_1 \otimes A_2, A_3).$$
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