Rational \( D(q) \)-quintuples

Goran Dražić

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Abstract

For a nonzero rational number \( q \), a rational \( D(q) \)-\( n \)-tuple is a set of \( n \) distinct nonzero rationals \( \{a_1, a_2, \ldots, a_n\} \) such that \( a_i a_j + q \) is a square for all \( 1 \leq i < j \leq n \). We investigate for which \( q \) there exist infinitely many rational \( D(q) \)-quintuples. We show that assuming the Parity Conjecture for the twists of several explicitly given elliptic curves, the density of such \( q \) is at least \( \frac{295026}{296010} \approx 99.5\% \).

Keywords  
Diophantine \( m \)-tuples · \( D(q) \)-\( m \)-tuples · Elliptic curves · Twists · Rank · Parity conjecture

Mathematics Subject Classification  
Primary 11D09; Secondary 11G05

1 Introduction

Let \( q \in \mathbb{Q} \) be a nonzero rational number. A set of \( n \) distinct nonzero rationals \( \{a_1, a_2, \ldots, a_n\} \) is called a rational \( D(q) \)-\( n \)-tuple if \( a_i a_j + q \) is a square for all \( 1 \leq i < j \leq n \). \( D(1) \)-\( n \)-tuples are called Diophantine \( n \)-tuples. If \( \{a_1, a_2, \ldots, a_n\} \) is a rational \( D(q) \)-\( n \)-tuple, then for all nonzero \( r \in \mathbb{Q} \), \( \{ra_1, ra_2, \ldots, ra_n\} \) is a \( D(qr^2) \)-\( n \)-tuple, since \((ra_1)(ra_2) + qr^2 = (a_1 a_2 + q)r^2\). With this in mind, we restrict to square-free integers \( q \).

The first example of a rational Diophantine quadruple was the set

\[
\left\{ \frac{1}{16}, \frac{33}{16}, \frac{17}{16}, \frac{105}{16} \right\}
\]

found by Diophantus, while the first example of an integer Diophantine quadruple, the set

\( \{1, 3, 8, 120\} \)

is due to Fermat.

In the case of integer Diophantine \( n \)-tuples, it is known that there are infinitely many Diophantine quadruples (e.g. \( \{k - 1, k + 1, 4k, 16k^3 - 4k\}, \text{ for } k \geq 2 \)). Dujella [7] showed there are no Diophantine sextuples and only finitely many Diophantine quintuples, while
recently He et al. [20] proved there are no integer Diophantine quintuples, which was a long standing conjecture.

Gibbs [18] found the first example of a rational Diophantine sextuple using a computer, and Dujella et al. [13] constructed infinite families of rational Diophantine sextuples. Dujella and Kazalicki parametrized Diophantine quadruples with a fixed product of elements using triples of points on a specific elliptic curve, and used that parametrization for counting Diophantine quadruples over finite fields [12] and for constructing rational sextuples [11]. There is no known rational Diophantine septuple.

Several papers have studied $n$-tuples which are $D(q)$-$n$-tuples for more than one $q$.

Adžaga et al. [1] constructed infinite families of integer Diophantine triples which are $D(n)$-triples for two other different integers $n$. Dujella and Petričević [16] proved that there are infinitely many essentially different integer quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$, while in [17] they constructed infinitely many essentially different quadruples of perfect squares which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples where $n_1$ and $n_2$ are distinct nonzero perfect squares. Dujella et al. [14] constructed infinitely many rational Diophantine quintuples which are also $D(0)$-quintuples.

Diophantine $m$-tuples have also been used to obtain high rank elliptic curves, i.e. by Dujella and Peral in [15].

For more detail on Diophantine $m$-tuples and rational $D(q)$-$m$-tuples, we refer the reader to [8], [9, Sections 14.6 and 16.7.], as well as the webpage of Andrej Dujella.\footnote{https://web.math.pmf.unizg.hr/~duje/dtuples.html.}

The goal of this paper is to find squarefree integers $q$ for which there exist infinitely many rational $D(q)$-quintuples. In [6], Dujella proved there exist infinitely many rational $D(q)$-quadruples for every rational $q$, and in [4], Dražić and Kazalicki, for given $q \in \mathbb{Q}$, parametrized all $m \in \mathbb{Q}$ such that there exists a rational $D(q)$-quadruple $(a, b, c, d)$ with $abcd = m$. Dujella and Fuchs [10] proved that there exist infinitely many squarefree rationals $q$ for which there exist infinitely many rational $D(q)$-quintuples and, assuming the Parity Conjecture for the twists of an explicitly given elliptic curve (isomorphic to $E(7)$, details in Tables 1, 2), the density of $q \in \mathbb{Q}$ such that there exist infinitely many rational $D(q)$-quintuples is at least 1/2. In this paper, also assuming the Parity Conjecture for twists of explicit elliptic curves, we improve the density bound to at least 295026/296010 $\approx 99.5\%$.

In [5], Dujella constructed rational $D(q)$-quintuples of the form $\{A, B, C, D, x^2\}$, with $q = \alpha x^2$. In Sect. 2 we expand his construction. In Sect. 3, we define the curve $C/\mathbb{Q}(u)$ by

$$C: \quad z_1^2 = f_4(u) c^4 + f_3(u) c^3 + f_2(u) c^2 + f_1(u) c + f_0(u),$$

where $f_i(u)$ are rational functions in $\mathbb{Q}(u)$ explicitly stated at (22).

The curve $C$ has a rational point when $c = 1$, so it is birationally equivalent to an elliptic curve $E/\mathbb{Q}(u)$. The Mordell-Weil group $E(\mathbb{Q}(u))$ has rank at least five, as we found five independent rational points, which we list at (23).

Let $q(u)$ be a rational function in variable $u$, not identically zero. We call a set of $n$ distinct, not identically zero rational functions $\{a_1(u), a_2(u), \ldots, a_n(u)\}$ a $D(q(u))$-$n$-tuple with elements in $\mathbb{Q}(u)$, if $a_i(u)a_j(u) + q(u) = h_{i,j}(u)$, $h_{i,j} \in \mathbb{Q}(u)$ for all $1 \leq i < j \leq n$. We will refer to such quintuples more briefly as $D(q(u))$-quintuples.

Every rational point on $E$ determines a $D(\alpha(u)x(u)^2)$-quintuple $\{A(u), B(u), C(u), D(u), x^2(u)\}$, provided that no two elements of the quintuple are equal and that no element nor $\alpha(u)$ is identically zero. This connection is explained in Sect. 3.

\footnote{https://web.math.pmf.unizg.hr/~duje/dtuples.html.}
Fix a squarefree $q \in \mathbb{Z}$ and assume for a moment that $\alpha(u_1)x(u_1)^2 = q\frac{s_1^2}{x(u_1)}$ for some rationals $u_1, s_1$ such that $s_1 \neq 0$. Then $\{A(u_1)/s_1, B(u_1)/s_1, C(u_1)/s_1, D(u_1)/s_1, x^2(u_1)/s_1\}$ is a rational $D(\alpha(u_1) \cdot (x(u_1)/s_1)^2)$-quintuple, that is, a rational $D(q)$-quintuple. The following reasoning was used by Dujella and Fuchs in [10]: If we find infinitely many rationals $(u_1, s_1)$ such that
\[
\alpha(u_1) = q \left( \frac{s_1}{x(u_1)} \right)^2
\]then there are infinitely many rational $D(q)$-quintuples.

Let $P(u)$ be the squarefree polynomial such that
\[
P(u) \equiv \alpha(u)x(u)^2 \mod (\mathbb{Q}(u))^2.
\] $P(u)$ is uniquely determined up to scaling by a rational square. Solving (2) is the same as finding a rational solution $(u_1, s_1)$ of
\[
P(u) = qs^2.
\]If $\deg(P(u)) \geq 5$, then Eq. (4) defines a curve of genus at least two, which by Faltings’ theorem has only finitely many solutions. Thus, only if $\deg(P(u)) \in \{1, 2, 3, 4\}$ we can hope to find infinitely many solutions $(u_1, s_1)$ of (4) and therefore, in this way, infinitely many rational $D(q)$-quintuples.

We found eight points $Q_i \in E(\mathbb{Q}(u)), i \in \{1, \ldots, 8\}$, details in Table 1, each of them determining a $D(q(u))$-quintuple, such that the polynomial $P_{Q_i}(u)$ arising from the $D(q(u))$-quintuple is of degree three or four.

Define the curves
\[
E_q^{(i)}: \quad P_{Q_i}(u) = qs^2,
\]for a fixed squarefree $q \in \mathbb{Z}$ and $i \in \{1, \ldots, 8\}$. If $q = 1$ we write $E^{(i)}$ instead of $E_1^{(i)}$. Each $E_q^{(i)}$ is a quadratic twist by $q$ of the curve $E^{(i)}$.

We want to find rational points on the curves $E_q^{(i)}$. Let us look at a concrete example when $i = 6$, for which we have $P_{Q_6}(u) = 4u^4 - 20u^3 + 13u^2 + 12u$. For each $q$, the curve $E_q^{(6)}$ has a rational point since $P_{Q_6}(u)$ has a rational zero $u = 0$. It follows that each $E_q^{(6)}$, as it is a curve of genus one, is birationally equivalent to an elliptic curve over $\mathbb{Q}$.

We want to classify squarefree $q \in \mathbb{Z}$ for which the rank of $E_q^{(6)}(\mathbb{Q})$ is positive. For such $q$, Eq. (4) has infinitely many rational solutions.

Let $E/\mathbb{Q}$ be an elliptic curve. The root number $W(E)$ is defined as the product of the local root numbers $W_p(E) \in \{\pm 1\}$:
\[
W(E) = \prod_{p \leq \infty} W_p(E),
\]where $p$ is a finite or infinite place of $\mathbb{Q}$. The local factors have the property that $W_p(E) = 1$, for all but finitely many $p$. The definition of the local root number and their properties are explained in detail in e.g. [23]. Rohrlich [22] provides an explicit formula for $W_p(E)$ when $p$ is not equal to 2 or 3 in terms of reduction types of $E$. The remaining cases when $p = 2$ or $p = 3$ were covered by Halberstadt [19]. Rizzo [21] gave a complete overview in English while removing some minimality conditions from the tables in [19].

The Birch and Swinnerton-Dyer conjecture implies the following
Conjecture 1 (The Parity Conjecture) Let \( E / \mathbb{Q} \) be an elliptic curve, then \((-1)\text{rank}_{E(\mathbb{Q})} = W(E)\).

An immediate consequence of this conjecture is that the rank of \( E(\mathbb{Q}) \) is positive whenever \( W(E) = -1 \), in which case we have infinitely many rational points on \( E \).

Assume the Parity Conjecture holds for all twists of the curves \( E^{(i)}, i \in \{1, \ldots, 8\} \). Using Desjardins [3], we obtain results for squarefree \( q \mod N_i \) in the form of the following theorem.

Theorem 2 The functions \( q \mapsto W\left(E^{(i)}_q\right) \) and \( q \mapsto W\left(E^{(i)}_{-q}\right) \) are periodic on squarefree \( q \in \mathbb{N} \) with period \( N_i \). Consequently, assuming the Parity Conjecture, the functions \( q \mapsto \text{Rank}\left(E^{(i)}_q\right) \mod 2 \) and \( q \mapsto \text{Rank}\left(E^{(i)}_{-q}\right) \mod 2 \) are periodic on squarefree \( q \in \mathbb{N} \) with period \( N_i \).

Each point \( Q_i \) in Table 1 leads to a different polynomial \( P_{Q_i}(u) \). The period \( N_i \) will depend on the periods of the local root numbers \( W_p\left(E^{(i)}_q\right) \) with respect to \( q \), for each fixed prime \( p \) dividing the conductor of \( E^{(i)} \). We will explicitly calculate \( N_i \) for each curve \( E^{(i)} \) using [3], with the help of tables in [19] and [21].

Combining results from all curves \( E^{(i)} \), with the assumption of the Parity Conjecture, we prove the following theorem.

Theorem 3 Assuming the Parity Conjecture the following holds:
(a) For each squarefree \( q \in \mathbb{N} \) in at least 295026 residue classes \( \mod 394680 \) there exist infinitely many rational \( D(q) \)-quintuples.
(b) For each squarefree \( q \in -\mathbb{N} \) in at least 295435 residue classes \( \mod 394680 \) there exist infinitely many rational \( D(q) \)-quintuples.

Remark 4 There are 296010 residue classes \( \mod 394680 \) which contain squarefree integers.

Theorem 3 shows that we cover more than 99.5% of classes \( \mod 394680 \). We conjecture that Theorem 3 holds for all squarefree \( q \in \mathbb{Z} \), that is for all \( q \in \mathbb{Q} \), but are unable to prove it using this method. For each squarefree \( q \in \mathbb{Z} \) such that \(|q| < 1000\) and \( q \neq 19, 341 \) at least one rank of the curves \( E^{(i)}_q \) is odd, therefore positive. In particular, for \( q = 341 \) we have \( \text{Rank}(E^{(1)}_q) = \text{Rank}(E^{(3)}_q) = \text{Rank}(E^{(8)}_q) = 2 \), while the other ranks are zero. For \( q = 19 \), Petričević has experimentally found a great number of \( D(q) \)-quintuples. The smallest positive \( q \) for which we do not know any rational \( D(q) \)-quintuples is 1579.

2 Initial grunt work, constructing quintuples

Following Dujella [5], we wish to find \( D(q) \)-quintuples of the form \( \{A, B, C, D, x^2\} \) with \( q = \alpha \cdot x^2 \). Dujella started from the \( D(q) \)-pair \( \{B, C\} \), with \( BC + \alpha x^2 = k^2 \). The numbers \( A = B + C - 2k \) and \( D = B + C + 2k \) both extend the pair \( \{B, C\} \) to a regular \( D(q) \)-triple. The quadruple \( \{A, B, C, D\} \) is an almost rational \( D(q) \)-quadruple, missing the condition \( AD + \alpha x^2 = \Box \). To obtain a rational \( D(q) \)-quintuple \( \{A, B, C, D, x^2\} \) we also need to satisfy that \( Y \cdot x^2 + \alpha x^2 = (Y + \alpha)x^2 = \Box \), for \( Y = A, B, C \) and \( D \).

Proposition 5 Let \( \{A, B, C, D, x^2\} \) be a rational \( D(\alpha x^2) \)-quintuple with the properties

\[
A + \alpha = a^2, \quad B + \alpha = b^2, \quad C + \alpha = c^2, \quad D + \alpha = d^2, \quad (5)
\]

\[
BC + \alpha x^2 = k^2, \quad A = B + C - 2k, \quad D = B + C + 2k. \quad (6)
\]
If we denote $p = \frac{d+a}{2}$, $r = \frac{d-a}{2}$ then
\[ b^2 = p^2 + r^2 - x^2 + \frac{(p^2 - x^2)(r^2 - x^2)}{p^2 + r^2 - x^2}. \]

**Proof** Subtracting the two rightmost equations in (6), we have
\[ 4k = D - A = (D + \alpha) - (A + \alpha) = d^2 - a^2 = (d - a)(d + a) = 2r \cdot 2p. \]

It is easy to see that
\[ k = pr, \quad a = p - r, \quad d = p + r. \]  
(7)

The second equation from (6), using (5) and (7), gives us
\[ (a^2 - \alpha) = (b^2 - \alpha) + (c^2 - \alpha) - 2k \quad \Rightarrow \quad b^2 + c^2 = p^2 + r^2 + \alpha \]  
(8)

The first equation in (6) gives us
\[ k^2 = (b^2 - \alpha)(c^2 - \alpha) + \alpha x^2. \]

Substituting $k = pr$ and manipulating using (8), we obtain
\[ 4b^2c^2 = 4 \cdot (p^2r^2 + \alpha(p^2 + r^2) - \alpha x^2). \]

Using the previous equality and (8), we have
\[
(b^2 - c^2)^2 = (b^2 + c^2)^2 - 4b^2c^2 = (p^2 + r^2 + \alpha)^2 - 4 \cdot (p^2r^2 + \alpha(p^2 + r^2) - \alpha x^2).
\]
\[
= (p^2 + r^2 - \alpha)^2 - 4p^2r^2 + 4\alpha x^2
\]
\[
= (p^2 + r^2 - \alpha - 2x^2)^2 - 4x^2(p^2 + r^2 - \alpha) - 4p^2r^2 + 4\alpha x^2
\]
\[
= (\alpha - (p^2 - x^2 + r^2 - x^2))^2 - 4(p^2r^2 - x^2p^2 - x^2r^2 + x^4).
\]

Factoring the expression in the right parentheses and switching sides leads to
\[ 4(p^2 - x^2)(r^2 - x^2) = (\alpha - (p^2 - x^2 + r^2 - x^2))^2 - (b^2 - c^2)^2. \]

The right hand side of the last equation is a difference of squares. Denoting
\[ 2v = \alpha - (p^2 - x^2 + r^2 - x^2) - (b^2 - c^2), \]  
(9)

we have
\[ \frac{2(p^2 - x^2)(r^2 - x^2)}{v} = \alpha - (p^2 - x^2 + r^2 - x^2) + (b^2 - c^2). \]  
(10)

Adding (9) and (10), then diving by two, leads to
\[ \alpha = v + \frac{(p^2 - x^2)(r^2 - x^2)}{v} + (p^2 - x^2) + (r^2 - x^2) = \frac{1}{v}(p^2 - x^2 + v)(r^2 - x^2 + v). \]  
(11)

Subtracting (9) from (10) and dividing by two gives us
\[ b^2 - c^2 = \frac{1}{v}((p^2 - x^2)(r^2 - x^2) - v^2). \]  
(12)

Eliminating $\alpha$ from (8) and (11) gives us
\[ b^2 + c^2 = p^2 + r^2 + \frac{1}{v}((p^2 - x^2 + v)(r^2 - x^2 + v)). \]  
(13)
Lastly, adding (12) and (13), as well as subtracting (12) from (13) and dividing by two, we have

\[ b^2 = p^2 + r^2 - x^2 + \frac{1}{v}(p^2 - x^2)(r^2 - x^2), \quad (14) \]
\[ c^2 = p^2 + r^2 - x^2 - v. \quad (15) \]

Substituting \( v \) into (14) using (15), we finish with

\[ b^2 = p^2 + r^2 - x^2 + \frac{(p^2 - x^2)(r^2 - x^2)}{p^2 + r^2 - c^2 - x^2}. \]

\[ \square \]

The previous proposition can be partially reversed.

**Proposition 6** Let \( p, r, c, x, b \in \mathbb{Q} \) such that

\[ b^2 = p^2 + r^2 - x^2 + \frac{(p^2 - x^2)(r^2 - x^2)}{p^2 + r^2 - c^2 - x^2}. \]

Define

\[ a = p - r, \quad d = p + r, \quad k = pr, \quad \alpha = \frac{(c^2 - r^2)(c^2 - p^2)}{c^2 + x^2 - p^2 - r^2}, \]
\[ A = a^2 - \alpha, \quad B = b^2 - \alpha, \quad C = c^2 - \alpha, \quad D = d^2 - \alpha. \]

Then \( \{A, B, C, D, x^2\} \) is a \( D(\alpha x^2) \)-quintuple provided that

(i) no two elements of the quintuple are equal or equal to zero,
(ii) \( \alpha \) is not equal to zero,
(iii) \( AD + \alpha x^2 = \square \).

**Proof** One can check by calculation that the numbers \( AB + \alpha x^2, AC + \alpha x^2, BC + \alpha x^2, BD + \alpha x^2, CD + \alpha x^2, Ax^2 + \alpha x^2, Bx^2 + \alpha x^2, Cx^2 + \alpha x^2, Dx^2 + \alpha x^2 \) are squares. This proves the proposition. \( \square \)

We now focus on rationality, and handle degeneracy issues in the proof of Theorem 9.

### 3 Reducing the number of parameters

To find rational \( D(q) \)-quintuples, according to Proposition 6, we need rational solutions of the pair of equations:

\[ b^2 = p^2 + r^2 - x^2 + \frac{(p^2 - x^2)(r^2 - x^2)}{c^2 + x^2 - p^2 - r^2} = p^2 + r^2 + \alpha - c^2, \]
\[ z^2 = AD + \alpha x^2 = (p^2 - r^2)^2 + \alpha(x^2 - 2(p^2 + r^2) + \alpha), \]

where \( \alpha \) is defined as

\[ \alpha = \frac{(c^2 - r^2)(c^2 - p^2)}{c^2 + x^2 - p^2 - r^2}. \]
We notice that $\alpha, b^2$ and $z^2$ are equal to homogeneous rational functions in $p, r, c, x$ so we start by setting $r = 1$. After that, the expressions for $\alpha, b^2, z^2$ simplify to

\[
\alpha = \frac{(c^2 - 1)(c^2 - p^2)}{c^2 + x^2 - p^2 - 1}, \\
b^2 = p^2 + 1 + \alpha - c^2, \\
z^2 = (p^2 - 1)^2 + \alpha(x^2 - 2(p^2 + 1) + \alpha).
\]

(16)

We would like to specialize one of the parameters $c, p, x$ using the other two, since we do not know how to completely solve the pair of Eqs. (16), (17). This specialization should keep the squarefree part of $\alpha$ as simple as possible.

Define the surfaces $S_1$ and $S_2$ over $\mathbb{Q}$ by the following equations:

\[
S_1: (c^2 - 1)(c^2 - p^2) = 0, \quad S_2: c^2 + x^2 - p^2 - 1 = 0,
\]

which are the zero sets of the numerator and denominator of $\alpha$. The surface $S_1$ is the union of the four planes $c = \pm p$ and $c = \pm 1$, while $S_2$ is a hyperboloid and their intersection is the union of the eight lines

\[
l_{1,2,3,4}: c = \pm 1, x = \pm p, \quad l_{5,6,7,8}: c = \pm p, x = \pm 1.
\]

A heuristic derived from Section 3, Lemma 5 in [14] tells us we could find a good specialization if we find a low degree surface in variables $c, p, x$ which intersects both $S_1$ and $S_2$ at exactly the lines $l_i$. The logical first choice are planes which contain two lines $l_i$. Such planes have equations $x = \pm 1 \pm c \pm p$, so we set $x = c + p + 1$ (changes of signs do not change anything relevant). In practice, the author came across this specialization when examining the family of $D(q(u))$-quintuples (26), found by Dujella. After the specialization, the equations for $\alpha, b^2, z^2$ are

\[
\alpha = \frac{1}{2}(c - p)(c - 1), \\
b^2 = p^2 + \frac{1 - c}{2}p - \frac{1}{2}(c^2 + c) + 1, \\
z^2 = p^4 + \frac{c - 1}{2}p^3 - \frac{5c^2 + 3}{4}p^2 + \frac{c^2 - 1}{2}p + \frac{3c^4 - 5c^2 + 2c + 4}{4}.
\]

(18)

(19)

We further reduce the number of parameters. Equation (18) is a conic in variables $b, p$ with a rational point $(1, c)$. Using standard techniques, rational points on (18) can be parametrized by

\[
p = \frac{u^2c + c/2 + 1/2 - 2u}{u^2 - 1}, \quad b = \frac{u^2 - 3uc/2 - u/2 + 1}{u^2 - 1}, \quad u \in \mathbb{Q}.
\]

(20)

Plugging the expression for $p$ from (20) into (19) makes the right hand side a polynomial of degree four in variable $c$ with coefficients in $\mathbb{Q}(u)$. Multiplying both sides by $\left(\frac{(u^2 - 1)^2}{u^2 - 1/4}\right)^2$ leads to

\[
C: z_1^2 = z^2 \cdot \left(\frac{(u^2 - 1)^2}{u^2 - 1/4}\right)^2 = f_4(u)c^4 + f_3(u)c^3 + f_2(u)c^2 + f_1(u)c + f_0(u),
\]

(21)
where \( f_i(u) \) are rational functions in variable \( u \) given by

\[
\begin{align*}
    f_4(u) &= u^4 + u^2 + 7, \\
    f_3(u) &= -3 \cdot \frac{(u^3 + 3u - 1)(2u^2 + 1)}{u^2 - 1/4}, \\
    f_2(u) &= -16u^8 + 16u^7 + 242u^6 - 76u^5 + 199u^4 - 166u^3 + 47u^2 + 10u - 13, \\
    f_1(u) &= 3 \cdot \frac{(u^3 + 3u + 1/2)(u^4 - 11/2u^3 + 4u^2 - 3/2u + 1/2)}{(u^2 - 1/4)^2}, \\
    f_0(u) &= \frac{16u^8 + 16u^7 - 116u^6 + 40u^5 + 409u^4 - 308u^3 + 25u^2 - 20u + 19}{16(u^2 - 1/4)^2}. \tag{22}
\end{align*}
\]

The curve \( C \), defined by (21), is birationally equivalent to an elliptic curve over \( \mathbb{Q}(u) \) since it has a rational point \((c, z_1) = \left(1, \frac{-4u(u - 1)^2}{u^2 - 1/4}\right)\). It is birational to the curve in Weierstrass form

\[
E : y^2 = x^3 - 27 \cdot (256u^8 + 64u^7 - 1280u^6 + 1216u^5 + 3265u^4 - 2372u^3 + 310u^2 - 32u + 169)x
\]

\[
+ 54 \cdot (4096u^{12} + 1536u^{11} - 30624u^{10} - 18400u^9 + 74448u^8 + 125568u^7 - 59313u^6
\]

\[
- 165978u^5 + 154773u^4 - 40360u^3 + 5187u^2 - 6474u + 2197).
\]

The points

\[
S_1 = [48u^4 + 168u^3 - 9u^2 - 138u + 39, -1944u^5 - 1944u^4 + 4374u^3 + 486u^2 - 972u],
\]

\[
S_2 = \left[ \frac{48u^6 + 588u^5 + 753u^4 - 1014u^3 + 24u^2 - 6u + 39}{u^2 + 2u + 1}, \right.
\]

\[
\left. \frac{-5832u^8 - 25596u^7 - 6156u^6 + 48438u^5 - 8100u^4 + 324u^3 - 3240u^2 + 162u}{u^3 + 3u^2 + 3u + 1} \right].
\]

\[
S_3 = \left[ \frac{48u^6 + 204u^5 - 855u^4 + 78u^3 + 2028u^2 - 1098u + 27}{u^2 - 6u + 9}, \right.
\]

\[
\left. \frac{-5832u^8 + 21060u^7 + 792u^6 - 94446u^5 + 102384u^4 + 34020u^3 - 67392u^2 + 486u + 8748}{u^3 - 9u^2 + 27u - 27} \right].
\]

\[
S_4 = [48u^4 + 492u^3 - 693u^2 - 84u - 69, -5832u^5 - 19764u^4 - 15228u^3 + 3402u^2 + 2754u - 324],
\]

\[
S_5 = \left[ \frac{48u^6 + 12u^5 - 291u^4 + 66u^3 + 600u^2 + 66u - 69}{u^2 + 2u + 1}, \right.
\]

\[
\left. \frac{-1080u^8 - 2484u^7 + 6480u^6 + 17550u^5 - 1512u^4 - 18468u^3 - 3348u^2 + 2538u + 324}{u^3 + 3u^2 + 3u + 1} \right]. \tag{23}
\]

are independent points in the Mordell-Weil group \( E(\mathbb{Q}(u)) \). We used Magma [2] to prove the independence of the points \( S_i \) by checking that the elliptic regulator of these points is nonzero.

Each rational point on \( E \) determines a rational point \((c(u), z_1(u)) \in C \). From (20) we obtain \( p(u) \) and \( b(u) \). We set \( r(u) = 1 \) and \( x(u) = c(u) + p(u) + 1 \). According to Proposition 6, each \( c(u) \) defines a \( D(\alpha(u)x(u)^5) \)-quintuple \( \{A(u), B(u), C(u), D(u), x^5(u)\} \), unless a degeneracy occurs (two elements of the quintuple might be equal, some element or \( \alpha(u) \) might be identically zero). The condition \( A(u)D(u) + \alpha(u)x(u)^2 \in (\mathbb{Q}(u))^2 \) is satisfied because the pair of rational functions \((c(u), z_1(u)) \) satisfies Eq. (21).
For each point on $E$ of the form $\sum_{i=1}^{5} k_i S_i$ with $k_i \in \{-6, \ldots, 6\}$, assuming it defines a $D(\alpha(u)x(u)^2)$-quintuple, we calculate the degree of the polynomial $P(u)$ defined at (3) using Magma. We did not obtain any polynomials of degree one or two. Every polynomial of degree four turned out to be reducible, some had a rational zero and some were products of two irreducible square polynomials in $\mathbb{Q}[u]$. We call polynomials of degree three and of degree four with a rational zero, such that the quintuple associated to them is a non degenerate $D(q(u))$-quintuple good polynomials.

Each good polynomial $P_0(u)$ defines an elliptic curve by the equation $y^2 = P_0(u)$, and every quadratic twist $Q_3 y^2 = P_0(u)$ of such a curve is an elliptic curve over $\mathbb{Q}$ as well (the twists of curves, where $P_0$ is of degree four, have a rational point with $y = 0$.) If $P_0(u)$ were a degree four polynomial with no rational zero, then we do not know whether the quadratic twist $Q_3 y^2 = P_0(u)$ has points over $\mathbb{Q}$, which is precisely why we discarded such polynomials.

For any two different good polynomials which define elliptic curves with the same $j$-invariant, there is a $q_0 \in \mathbb{Z}$ such that the quadratic $q_0$-twist of one curve is isomorphic over $\mathbb{Q}$ to the other curve. This is true because the $j$-invariant of all our curves is not equal to 0 or 1728 [24, Chapter X, Prop. 5.4]. We only count one representative of each class of polynomials which define elliptic curves with the same $j$-invariant.

The following points on $E$ determine a $D(q(u))$-quintuple such that the polynomial $P(u)$ is good, and all of the associated polynomials $P_{Q_i}(u)$ define elliptic curves $E^{(i)}$ which have different $j$-invariants:

### 4 Periodicity of root numbers of twists

For $E/\mathbb{Q}$ and $0 \neq t \in \mathbb{Z}$, let $E_t$ denote its quadratic twist by $t$. We also introduce some non-standard notation from Desjardins [3].

Given an integer $\beta \in \mathbb{Z}$ and a prime $p$, let $v_p(\beta)$ denote the greatest exponent of $p$ dividing $\beta$. By $\beta_{(p)}$ we denote the number such that

$$\beta = \beta_{(p)} \cdot p^{v_p(\beta)}.$$
Similarly, if \( d = \prod_{i} p_i^{e_i} \), we define \( \beta(d) \) to be the integer such that

\[
\beta = \beta(d) \cdot \prod_{i} p_i^{\nu_{p_i}(\beta)}.
\]

Desjardins [3, Theorem 1.2 b)], proved that the function

\[
t \mapsto W(E_t)
\]

is periodic on squarefree \( t \) of constant sign, assuming \( j(E) \neq 0, 1728 \). We calculate these periods, as well as give explicit formulae for \( W\left(E_t^{(i)}\right) \) for the curves \( E^{(i)} \) using [3] and tables from [21]. Note that none of the curves in our calculations have \( j \)-invariant equal to 0 or 1728.

[3, Theorem 1.2 a)] gives an explicit formula for the root number of a twist of an elliptic curve, whose \( j \)-invariant is not 0, 1728:

\[
W(E_t) = -W_2(E_t) \cdot W_3(E_t) \cdot \left(\frac{-1}{|I(6\Delta)|}\right) \cdot \left(\prod_p W_p(E_t)\right),
\]

where (·) is the Jacobi symbol.

Each factor in the previous equation is periodic on squarefree \( t \) of constant sign. This is a consequence of the properties of the Jacobi symbol, and [3, Lemma 3.2], which states that the function \( t \mapsto W_p(E_t) \) is periodic on squarefree \( t \), for every prime \( p \). Moreover, the same lemma proves that for \( p \geq 5 \), the period of \( W_p(E_t) \) divides \( p^2 \), and for \( p = 2 \) or 3, the period is \( p^{\nu_p} \), for a nonnegative integer \( \gamma_p \). For explicit curves \( E \) we can calculate \( \gamma_p \) using tables in [19] or [21].

We now state and prove an expanded version of Theorem 2 describing the curve \( E^{(6)} \). A similar version of the following theorem (with similar proofs) can be made for every curve in Table 1.

**Theorem 7** The curve \( E^{(6)} \) has Weierstrass form \( y^2 = x^3 - 24003x + 1296702 \), conductor \( C = 30 \) and its discriminant is \( \Delta = 2^{14}3^{18}5^2 \).

a) The periods of the functions \( W_2(E_t^{(6)}) \), \( W_3(E_t^{(6)}) \) and \( W_5(E_t^{(6)}) \) on squarefree \( t \) are 8, 3 and 5, respectively. The period of the function \( \left(\frac{-1}{I(30)}\right) \) on positive squarefree \( t \) is 24.

b) If \( t > 0 \), \( t' < 0 \), both \( t \) and \( t' \) are squarefree and \( t \equiv t' \pmod{120} \), then \( \left(\frac{-1}{I(30)}\right) = -\left(\frac{-1}{I(30)}\right) \). In particular, \( W(E_t^{(6)}) = -W(E_{t'}^{(6)}) \).

c) The period of \( W(E_t^{(6)}) = -W_2(E_t^{(6)})W_3(E_t^{(6)})W_5(E_t^{(6)})\left(\frac{-1}{I(30)}\right) \) on squarefree \( t \) of constant sign is 120.

d) Assuming the Parity Conjecture, if \( q \) is positive, squarefree and in one of 47 classes \( \pmod{120} \), then the rank of \( E_q^{(6)}(\mathbb{Q}) \) is positive.

If \( q \) is negative, squarefree and in one of 43 classes \( \pmod{120} \), then the rank of \( E_q^{(6)}(\mathbb{Q}) \) is positive.

**Proof** a) Assume \( t \) is positive. We first prove \( \left(\frac{-1}{I(30)}\right) = \left(\frac{-1}{I(6)}\right) \).
If $5 \nmid t$, then obviously $t_{(30)} = t_{(6)}$ so \( \left( \frac{-1}{t_{(30)}} \right) = \left( \frac{-1}{t_{(6)}} \right) \). Assume $t = 5t'$ where $5 \nmid t'$. Then
\[
\left( \frac{-1}{t_{(30)}} \right) = \left( \frac{-1}{t_{(6)}} \right) = \left( \frac{-1}{t_{(6)}} \right) \left( \frac{-1}{5t'_{(6)}} \right) = \left( \frac{-1}{t_{(6)}} \right) .
\]

To calculate \( \left( \frac{-1}{n} \right) = (-1)^{(n-1)/2} \), for an odd number $n$, we only need to know $n \mod 4$.

Therefore, to prove \( \left( \frac{-1}{n} \right) \) is periodic with period 24 (on squarefree $t$) we check several cases.

If $t = 6t'$ then $t_{(6)} = t'$ since $t$ is squarefree. \( \left( \frac{1}{7} \right) \) has period 4 so the total period is 24.

Cases $t = 3t'$, $t = 2t'$ and $t = t'$ where in each case $(t', 6) = 1$ are handled similarly.

24 is the smallest period since $1 = \left( \frac{-1}{3(6)} \right) \neq \left( \frac{-1}{(11)(6)} \right) = -1$ and $1 = \left( \frac{-1}{(2)(6)} \right) \neq \left( \frac{-1}{(14)(6)} \right) = -1$.

We can calculate $W_5(E_t)$ using [3, Prop 3.1]. In our case, if $5 \nmid t$, the reduction of $E_t$ at 5 is of type $I_2$, while if $5 \mid t$, the reduction is of type $I_2^\circ$, calculated by Magma [2]. We conclude
\[
W_5(E_t) = \begin{cases} 
1, & 5 \mid t \\
\left( \frac{5}{t} \right), & 5 \nmid t '
\end{cases}
\]

When $p = 2$ or 3, things get more complicated. According to [3, Prop 3.1], or [21, 1.1] we need to find the smallest vector $(a', b', c')$ with nonnegative entries such that
\[
(a', b', c') = (v_p(c_4), v_p(c_6), v_p(\Delta)) + k(4, 6, 12),
\]
for $k \in \mathbb{Z}$, where $c_4, c_6, \Delta$ are the usual quantities associated to the Weierstrass equation of an elliptic curve.

For $p = 3$ and $t = 1$, we have $(a', b', c') = (4, 6, 18) - 1(4, 6, 12) = (0, 0, 6)$. Twisting by $t \equiv 0 \pmod{3}$ does not change $(a', b', c')$. Per [21, Table II, row 3] we have that:
\[
t \equiv 1 \pmod{3} \Rightarrow (c_6)(3) = 2 \pmod{3}, \quad \text{so } W_3(E_t) = -1.
\]
\[
t \equiv 2 \pmod{3} \Rightarrow (c_6)(3) = 1 \pmod{3}, \quad \text{so } W_3(E_t) = 1.
\]

Twisting by $t \equiv 3, 6 \pmod{9}$ (note that $t$ is squarefree so it cannot be $\equiv 0 \pmod{9}$) we get in both cases $(a', b', c') = (6, 9, 24) - 1(4, 6, 12) = (2, 3, 12)$. Per [21, Table II, row 13], we have $W_3(E_t) = -1$. In short,
\[
W_3(E_t) = \begin{cases} 
-1, & t \equiv 0, 1 \pmod{3} \\
1, & t \equiv 2 \pmod{3}
\end{cases}
\]

For $p = 2$ and $t = 1$, we have $(a', b', c') = (4, 6, 14) - 1(4, 6, 12) = (0, 0, 2)$. Twisting by an odd $t$ will not change $(a', b', c')$. Per [21, Table III, rows 2 and 9] $W_2(E_t) = 1$ if and only if $(c_6)(2) \equiv 3 \pmod{8}$, which happens exactly when $t \equiv 1 \pmod{8}$.

Twisting by $t \equiv 2, 6 \pmod{8}$ makes $(a', b', c') = (6, 9, 20) - 1(4, 6, 12) = (2, 3, 8)$. Per [21, Table III, row 17] we have:
\[
t \equiv 2 \pmod{8} \Rightarrow (c_6)(2) = 3 \pmod{4}, \quad \text{so } W_2(E_t) = 1.
\]
\[
t \equiv 6 \pmod{8} \Rightarrow (c_6)(2) = 1 \pmod{4}, \quad \text{so } W_2(E_t) = -1.
\]

For $p = 2$ we have
\[
W_2(E_t) = \begin{cases} 
1, & t \equiv 1, 2 \pmod{8} \\
-1, & t \equiv 3, 5, 6, 7 \pmod{8}
\end{cases}
\]

This concludes the proof of part a).

b) \( \left( \frac{-1}{t_{(30)}} \right) = \left( \frac{-1}{t'_{(30)}} \right) \Leftrightarrow \left( \frac{-1}{t_{(30)}} \right) \cdot \left( \frac{-1}{t'_{(30)}} \right) = -1 \Leftrightarrow \left( -\frac{1}{t_{(30)}t'_{(30)}} \right) = -1 \).
Since $t$ and $t'$ are squarefree and since $t \equiv t' \pmod{120}$, we know that $t - t' = d \cdot (t_{(30)} - t'_{(30)})$, where $d \in \mathbb{N}$ such that $d | 30$. From this we easily conclude that $t_{(30)} \equiv t'_{(30)} \pmod{4}$, and $-t_{(30)}t'_{(30)} \equiv -\left(t'_{(30)}\right)^2 \pmod{4}$.

Now, \[
\frac{-1}{-t_{(30)}t'_{(30)}} = \frac{-1}{4k - \left(t'_{(30)}\right)^2} = -1, \text{ since } -1 \text{ is not a quadratic residue modulo a number of the form } 4k' - 1. 4k - \left(t'_{(30)}\right)^2 \equiv -1 \pmod{4} \text{ because } \left(t'_{(30)}\right)^2 \text{ is an odd square.}
\]

c) Follows easily from a) and b).

d) If $t \mod 120$ is 6, 7, 9, 11, 14, 15, 22, 26, 30, 35, 39, 41, 43, 50, 51, 53, 54, 58, 59, 61, 65, 66, 67, 71, 73, 74, 75, 77, 81, 82, 85, 86, 89, 90, 93, 95, 97, 99, 103, 105, 109, 110, 111, 114, 117, 118 or 119 and $t$ is positive and squarefree, then $W(E_t) = -1$. Assuming the Parity Conjecture, the rank of $E_t(\mathbb{Q})$ is odd, therefore positive.

If $t \mod 120$ is 1, 2, 3, 5, 10, 13, 17, 18, 19, 21, 23, 25, 27, 29, 31, 33, 34, 37, 38, 42, 45, 46, 47, 49, 55, 57, 62, 63, 69, 70, 78, 79, 83, 87, 91, 94, 98, 101, 102, 106, 107, 113 or 115 and $t$ is negative and squarefree, then $W(E_t) = -1$. Assuming the Parity Conjecture, the rank of $E_t(\mathbb{Q})$ is odd, therefore positive. \hfill \Box

**Remark 8** For every curve $E^{(i)}$, the period of $W(E^{(i)})$ is divisible by 8. We use this to prove $W(E^{(i)}_t) = -W(E^{(i)}_{t'})$, for each curve $E^{(i)}$. Each curve $E^{(i)}$ has a version of Theorem 7 similar to the one stated. We list important facts in Tables 2 and 3.

**Theorem 9** Let $q$ be a squarefree integer such that the rank of $E^{(i)}_q(\mathbb{Q})$ is positive, for at least one $i \in \{1, \ldots, 8\}$. Then there exist infinitely many rational $D(q)$-quintuples.

**Proof** Assume $i = 6$ so that the rank of $E^{(6)}_q(\mathbb{Q})$ is positive. If $i$ is any other index, the proof is similar. The quintuple

\[
\left(9(u - 1)^3(4u - 1)(u + 1), \right.
\]
\[
(u^4 - 6u^3 + 5u + 27/4)(u^4 - 6u^3 + 8u^2 - 3u + 3/4),
\]
\[
u^8 - 16u^6 + 32u^5 - 39/2u^4 + 10u^3 + 6u^2 - 9u + 9/16,
\]
\[
(4u^4 - 16u^3 + 14u^2 + 4u + 3)(u^4 - 2u^3 + 5/2u^2 + 3/4),
\]
\[
9(u^2 - 2u - 1/2)^2(u^2 + 1/2)^2
\]

is a $D(q(u))$-quintuple for

\[
q(u) = (4u^4 - 20u^3 + 13u^2 + 12u) \cdot (3(u - 1)(u^2 + 1/2)(u^2 - 2u - 1/2))^2.
\]

Evaluating the elements and $q(u)$ at $u_1 \in \mathbb{Q}$ we obtain a rational $D(q(u_1))$-quintuple for all but finitely many exceptions $u_1$. The possible exceptions are rationals $u_1$ such that $q(u_1) = 0$, or any element of the function quintuple evaluated at $u_1$ is equal to zero, or any two elements of the function quintuple evaluated at $u_1$ are equal. Such $u_1$ are roots of finitely many polynomials in one variable so the set of exceptions is finite.

Since the rank of $E^{(6)}_q(\mathbb{Q})$ is assumed positive, we know there exist infinitely many pairs of rationals $(y_1, u_1)$ that satisfy the equation

\[
y^2 = 4u^4 - 20u^3 + 13u^2 + 12u.
\]
| $E^{(i)}$ | Weierstrass form | $C$ | $\Delta$ | $N_I = \text{period of } W(E^{(i)}_I)$ |
|--------|----------------|-----|--------|---------------------------------|
| $E^{(1)}$ | $y^2 = x^3 - 33210675x + 6964980750$ | $2 \cdot 3 \cdot 5^2 \cdot 11$ | $2^{20}3^{18}5^{8}11^{4}$ | $6600 = 2^3 \cdot 3 \cdot 5^2 \cdot 11$ |
| $E^{(2)}$ | $y^2 = x^3 - 24651x + 1453194$ | $2^4 \cdot 3 \cdot 13$ | $2^{10}3^{20}13$ | $312 = 2^3 \cdot 3 \cdot 13$ |
| $E^{(3)}$ | $y^2 = x^3 - 97227x + 10789254$ | $2 \cdot 3 \cdot 5 \cdot 11$ | $2^{16}3^{16}5^{2}11^2$ | $1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$ |
| $E^{(4)}$ | $y^2 = x^3 - 7155x + 187650$ | $2^4 \cdot 5^2 \cdot 11$ | $-2^{10}3^{12}5^{3}11^2$ | $88 = 2^3 \cdot 11$ |
| $E^{(5)}$ | $y^2 = x^3 + 274725x + 126596250$ | $2^4 \cdot 3 \cdot 5^2 \cdot 11^2$ | $-2^{10}3^{18}5^611^3$ | $264 = 2^3 \cdot 3 \cdot 11$ |
| $E^{(6)}$ | $y^2 = x^3 - 24003x + 1296702$ | $2 \cdot 3 \cdot 5$ | $2^{14}3^{18}5^2$ | $120 = 2^3 \cdot 3 \cdot 5$ |
| $E^{(7)}$ | $y^2 = x^3 - 132867x + 17106174$ | $2 \cdot 3 \cdot 5 \cdot 11$ | $2^{16}3^{14}5^411^2$ | $1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$ |
| $E^{(8)}$ | $y^2 = x^3 - 1196883x + 46619118$ | $2 \cdot 3 \cdot 5 \cdot 23$ | $2^{18}3^{22}5^223^2$ | $2760 = 2^3 \cdot 3 \cdot 5 \cdot 23$ |
Table 3  Decomposition of $W(E_i^{(i)})$ into a product of local factors with periods of local factors

| $E^{(i)}$ | $W(E_i^{(i)})$ given by local $W_p$ | periods of $W_p(E_i^{(i)})$ and $\left(\frac{-1}{l_i(m)}\right)$ |
|-----------|-----------------------------------|---------------------------------|
| $E^{(1)}$ | $-W_2 \left( E_i^{(1)} \right) \cdot W_3 \left( E_i^{(1)} \right) \cdot W_5 \left( E_i^{(1)} \right) \cdot W_{11} \left( E_i^{(1)} \right) \cdot \left(\frac{-1}{l_{330}}\right)$ | 8, 3, 25, 11, 132 |
| $E^{(2)}$ | $-W_2 \left( E_i^{(2)} \right) \cdot W_3 \left( E_i^{(2)} \right) \cdot W_{13} \left( E_i^{(2)} \right) \cdot \left(\frac{-1}{l_{78}}\right)$ | 8, 3, 13, 24 |
| $E^{(3)}$ | $-W_2 \left( E_i^{(3)} \right) \cdot W_3 \left( E_i^{(3)} \right) \cdot W_5 \left( E_i^{(3)} \right) \cdot W_{11} \left( E_i^{(3)} \right) \cdot \left(\frac{-1}{l_{330}}\right)$ | 8, 3, 5, 11, 132 |
| $E^{(4)}$ | $-W_2 \left( E_i^{(4)} \right) \cdot W_3 \left( E_i^{(4)} \right) \cdot W_5 \left( E_i^{(4)} \right) \cdot W_{11} \left( E_i^{(4)} \right) \cdot \left(\frac{-1}{l_{330}}\right)$ | 8, 3, 1, 11, 132 |
| $E^{(5)}$ | $-W_2 \left( E_i^{(5)} \right) \cdot W_3 \left( E_i^{(5)} \right) \cdot W_5 \left( E_i^{(5)} \right) \cdot W_{11} \left( E_i^{(5)} \right) \cdot \left(\frac{-1}{l_{330}}\right)$ | 8, 3, 1, 1, 132 |
| $E^{(6)}$ | $-W_2 \left( E_i^{(6)} \right) \cdot W_3 \left( E_i^{(6)} \right) \cdot W_5 \left( E_i^{(6)} \right) \cdot \left(\frac{-1}{l_{30}}\right)$ | 8, 3, 5, 12 |
| $E^{(7)}$ | $-W_2 \left( E_i^{(7)} \right) \cdot W_3 \left( E_i^{(7)} \right) \cdot W_5 \left( E_i^{(7)} \right) \cdot W_{11} \left( E_i^{(7)} \right) \cdot \left(\frac{-1}{l_{330}}\right)$ | 8, 3, 5, 11, 132 |
| $E^{(8)}$ | $-W_2 \left( E_i^{(8)} \right) \cdot W_3 \left( E_i^{(8)} \right) \cdot W_5 \left( E_i^{(8)} \right) \cdot W_{23} \left( E_i^{(8)} \right) \cdot \left(\frac{-1}{l_{690}}\right)$ | 8, 3, 5, 23, 552 |
For fixed \( y_1 \) and \( q \), the previous equation has at most four different solutions in variable \( u \), so there are infinitely many different \( y_1 \) (and in a similar manner, infinitely many different \( u_1 \)) among the pairs \((y_1, u_1)\) which satisfy (24).

For each such pair \((y_1, u_1)\), let \( \eta = \frac{1}{y_1 \cdot 3(u_1 - 1)(u_1^2 + 1/2)(u_1^2 - 2u_1 - 1/2)} \). It holds that \( q(u_1) \cdot \eta^2 = q \).

Then the quintuple

\[
\left(9(u_1 - 1)^3(4u_1 - 1)(u_1 + 1)\eta, \right.
\]

\[
(u_1^4 - 6u_1^3 + 5u_1 + 27/4)(u_1^4 - 6u_1^3 + 8u_1^2 - 3u_1 + 3/4)\eta,
\]

\[
(u_1^8 - 16u_1^6 + 32u_1^5 - 39/2u_1^4 + 10u_1^3 + 6u_1^2 - 9u_1 + 9/16)\eta,
\]

\[
(4u_1^4 - 16u_1^3 + 14u_1^2 + 4u_1 + 3)(u_1^4 - 2u_1^3 + 5/2u_1^2 + 3/4)\eta,
\]

\[
9(u_1^2 - 2u_1 - 1/2)^2(u_1^2 + 1/2)\eta
\]

(25)

is a rational \( D(q) \)-quintuple for all but finitely many exceptions of pairs \((y_1, u_1)\). The last thing left to argue is that the collection of rational \( D(q) \)-quintuples just described is not finite. For each such quintuple \((A, B, C, D, E)\) we look at the square quintuple \((A^2, B^2, C^2, D^2, E^2)\).

If the described collection of rational \( D(q) \)-quintuples were finite, then the collection of associated square quintuples would also be finite. Elements of square quintuples are rational functions in variable \( u_1 \). It is an easy exercise to show that only finitely many different \( u_1 \) occur if there are only finitely many square quintuples. Since this is false, so is the assumption that there are only finitely many rational \( D(q) \)-quintuples described by (25). \( \square \)

**Proof of Theorem 2** Theorem 7 c) implies Theorem 2 for the curve \( E^{(6)} \). The proofs for the other curves \( E^{(i)}, i \neq 6 \) are similar and omitted. The periods \( N_i \) are listed in Table 2. \( \square \)

**Proof of Theorem 3** The least common denominator of the periods \( N_i \) from Theorem 2 is 394680. The proof for negative \( q \) is conducted in the same way as the proof for positive \( q \), so we assume \( q \) is a squarefree positive integer. Theorem 7 d) implies that if \( q \) mod 120 is in one of forty seven residue classes the rank of \( E^{(6)}_q( \mathbb{Q} ) \) is positive. Combining results for other curves \( E^{(i)} \) we conclude that if \( q \) is in one of 295026 residue classes mod 394680 at least one \( E^{(i)}_q( \mathbb{Q} ) \) has positive rank. Theorem 9 concludes our proof. \( \square \)

For completeness, we list the \( D(q(u)) \)-quintuples for all \( E_i, i \in \{1, \ldots, 8\} \).

\[
\left(900u^4 + 4320u^3 - 1161u^2 - 3438u + 1404, \right.
\]

\[
1600u^4 - 1600u^3 + 1100u^2 - 920u + 396, \right.
\]

\[
100u^4 + 1760u^3 - 1201u^2 - 542u + 324, \right.
\]

\[
2500u^4 - 4000u^3 + 959u^2 + 514u + 36, \right.
\]

\[
3600u^4 - 2880u^3 - 1584u^2 + 864u + 324 \right) \quad (26)
\]
is a \( D \left( (-1200u^3 + 1645u^2 - 410u - 35) \cdot [6(10u^2 - 4u - 3)]^2 \right) \)-quintuple,

\[
\left( 378u^2 - 405u + 108, \\
32u^4 - 64u^3 + 122u^2 - 117u + 36, \\
32u^4 - 16u^3 + 80u^2 - 78u + 18, \\
128u^4 - 160u^3 + 26u^2 + 15u, \\
288u^4 - 288u^3 + 90u^2 - 9u \right)
\]

is a \( D \left( (-80u^4 + 148u^3 - 65u^2 - 12u + 9) \cdot [3(4u - 1)]^2 \right) \)-quintuple,

\[
\left( 352u^4 - 244u^3 - 129u^2 + 122u - 20, \\
4u^6 + 16u^5 + 48u^4 + 48u^3 - 164u^2 + 104u - 20, \\
4u^6 - 24u^5 + 112u^4 - 120u^3 + 47u^2 - 14u + 4, \\
16u^6 - 16u^5 - 32u^4 + 100u^3 - 105u^2 + 58u - 12, \\
36u^6 - 96u^5 + 112u^4 - 88u^3 + 48u^2 - 16u + 4 \right)
\]

is a \( D \left( (-28u^4 - 44u^3 + 157u^2 - 106u + 21) \cdot [2(3u^2 - u + 1) \cdot (u - 1)]^2 \right) \)-quintuple,

\[
\left( -54u^2 + 171u - 90, \\
32u^4 - 96u^3 - 6u^2 + 127u - 30, \\
32u^4 + 144u^3 - 24u^2 - 26u - 18, \\
128u^4 + 96u^3 - 6u^2 + 31u - 6, \\
288u^4 + 576u^3 - 54u^2 - 117u - 18 \right)
\]

is a \( D \left( (112u^4 - 100u^3 - 93u^2 + 92u - 11) \cdot [3 \cdot (4u + 1)]^2 \right) \)-quintuple,

\[
\left( 450u^4 - 1665u^3 + 2052u^2 - 909u + 72, \\
50u^4 - 545u^3 + 1092u^2 - 317u + 44, \\
800u^4 - 350u^3 + 30u^2 - 158u + 2, \\
1250u^4 - 125u^3 + 192u^2 - 41u + 20, \\
4050u^4 - 405u^3 - 648u^2 - 81u \right)
\]

is a \( D \left( (300u^3 - 65u^2 + 16u + 1) \cdot [9 \cdot (5u + 1) \cdot (u - 1)]^2 \right) \)-quintuple,

\[
\left( 576u^5 - 1296u^4 + 288u^3 + 1152u^2 - 864u + 144, \\
16u^8 - 192u^7 + 704u^6 - 736u^5 - 72u^4 - 80u^3 + 624u^2 - 264u + 81, \\
16u^8 - 256u^6 + 512u^5 - 312u^4 + 160u^3 + 96u^2 - 144u + 9, \\
64u^8 - 384u^7 + 896u^6 - 1024u^5 + 528u^4 - 128u^3 + 288u^2 + 48u + 36, \\
144u^8 - 576u^7 + 576u^6 - 288u^5 + 504u^4 + 144u^3 + 144u^2 + 72u + 9 \right)
\]
is a $D \left( (4u^4 - 20u^3 + 13u^2 + 12u) \left[ 12 \cdot (2u^2 + 1) \cdot (2u^2 - 4u - 1) \cdot (u - 1) \right]^2 \right)$-quintuple,

\[
\begin{align*}
25u^2 + 30u + 20, \\
4u^2 + 24u + 20, \\
9u^2 - 2u - 4, \\
u^2 + 14u + 12, \\
16u^2 - 4
\end{align*}
\]

is a $D \left( (-40u^3 - 19u^2 + 38u + 21) \cdot 2^2 \right)$-quintuple, and

\[
\begin{align*}
324u^4 + 423u^2 - 198u + 180, \\
64u^4 + 320u^3 - 52u^2 - 248u + 60, \\
100u^4 - 256u^3 + 239u^2 + 106u + 36, \\
4u^4 + 128u^3 - 49u^2 - 86u + 12, \\
144u^4 - 576u^3 + 432u^2 + 288u + 36
\end{align*}
\]

is a $D \left( (-144u^3 + 61u^2 + 94u - 11) \cdot 6 \cdot (2u^2 - 4u - 1)^2 \right)$-quintuple.

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