On Expectation Correlate System and Chaotic Dynamics in Time-Series

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ABSTRACT

This paper suggests a new system of time-series called Expectation Correlate System (ECS) that are good at detecting the behavior of dynamical systems (both deterministic and stochastic systems) and the dependence on initial values. A new measure on sensitivity to initial values can be monitored by the newly defined Lyaponov Correlate, so ECS can be a signal to chaotic property. In a stochastic systems, small shifts in some initial value can lead to error in prediction, this property and a new measure to nonlinear systems are study by using the conditional variance of ECS. All results are computed by using Matlab.

Keywords: time-series, dynamical system, Lyaponov Correlate, chaotic, stochastic systems, prediction, Matlab.

حول نظام الارتباط المتوقع ودینامیک ی جیشان فی السلاسل الزمنیة

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الملخص

في هذا البحث تم اقتراح نظام جديد للسلاسل الزمنية يدعي بنظام الارتباط المتوقع الذي يمكن أن يكون مؤشرًا جيدًا لسلوك النظام الديناميكية (محددة كانت أم تصادفية) فضلاً عن أن خاصية الاعتماد على القيم الابتدائية. مقياس جديد منذر للحساسية بالنسبة إلى القيم الابتدائية باستخدام التعريف الجديد ارتباط لابنوف (Lyaponov Correlate) على هذا الأساس فإن نظام الارتباط المتوقع يكون

كإشارة إلى الخاصية الجيشفانية. في النظم التصادفية، التغيرات الطفيفة في بعض القيم الابتدائية يمكن أن تؤدي إلى خطأ في عملية التنبؤ، هذه الخاصية ومقياس جديد للنظام غير الخطية درست باستخدام التباني الشرطي لنظام الارتباط المتوقع. جميع النتائج تم احتسابها باستخدام برامجات ال

كليمات المفتاحية: السلاسل الزمنية، النظام الديناميكية، ارتباط لابنوف، الجيشفان،

النظام التصادفية، التنبؤ، ماتلاب.

1-Introduction

In the past few years a large literature on chaos and non-linear science has appeared in economics and many scientists and technologists from diverse disciplines including mathematics (both pure and applied) because chaos is useful as a lens through which to view the word in epidemiology, biology and ecology, not because it helps so much in prediction but because it is suggestive of pathways to complex dynamics [1]. It is associated with complex and unpredictable behavior of phenomena over time and nonlinear science studies stochastic and deterministic dynamical systems that lead to “complex” dynamics. In [6], York who proposed the word ‘chaos’ as a label for a kind of dynamical behavior characterized by the triad: infinite number of periodic trajectories; uncountable number of no periodic trajectories; hyperbolicity (instability) of all (or the overwhelming majority of; as is proposed here) trajectories in the regime.
A simple deterministic system may be defined as follows: For a discrete time index, \( T=\{0,1,2,\ldots\} \), consider a time-series \( \{X_t; t \in T\} \). Assume that \( X_0 = x_0 \) is an initial condition and that

\[
X_t = F(X_{t-1}) \text{..............................................}(1)
\]

for \( t>1 \), where \( X_t \) denoted a state vector in \( \mathbb{R}^d \), \( F \) is a real-vector function (bounded continuous first derivatives ) see [2]. In a deterministic system, it is agreed that the sensitive dependence on initial condition is a typical feature of chaotic system, which is characterized by the well-known Lyapunov exponent [4].

A discrete time stochastic system can be described by the equation

\[
X_t = F(X_{t-1}, \epsilon_t) \text{..................................}(2)
\]

Where \( \{\epsilon_t\} \) is a noise process which satisfies the equation

\[
E(\epsilon_t | X_0, X_1, \ldots, X_{n-1}) = 0.
\]

If the noise is additive, equation (2) can be written as follows:

\[
X_t = F(X_{t-1}) + \epsilon_t \text{........................................}(3)
\]

Just as in deterministic systems, there has been no general accepted definition of chaos in stochastic systems. Stochastic chaotic system sometimes means a system with a deterministically chaotic skeleton [7].

The plan of the paper is as follow: Section (2) provides a brief sketch of a new system called Expectation Correlate System (ECS) and the correlates of the trajectories are studies. The concept of Lyapunov exponent has been developed to characterize the sensitivity dependence on the initial value of

175
system (1) in section (3). Section (4) presents a quantitative description of how small noise can be amplified rapidly in a ECS if the corresponding system is chaotic. A new simple procedure is suggested to measure the non-linear property in section (5). In section (6) the method is illustrated with example.

2- The Expectation Correlate System and Deterministic Systems.

By the Expectation Correlate System (ECS) of the system (1), mean the suggested system defined as:

\[ C_n^{(s)} = X_n - X_{n-s} \] .....................................(4)

\( n > s \), \( s \) is called the step correlate. Let us consider \( s=1 \), so the ECS has the form

\[ C_n^{(1)} = X_n - X_{n-1} \] .....................................(5)

In a chaotic system, small changes of parameters can change the dynamical behavior from stable periodic cycle or limit point into a strange attracted system [5]. The suggested system (5) will try to study the correlate behavior of the trajectories which dependence on the parameters of the system and how small change in this parameter can change the correlate of the trajectories.

By using the deterministic system (1) (let us consider the one dimension case \( d=1 \)), starting at the initial points \( X_o = x_o \) and \( X_o = x_o + \delta \), respectively, for small \( \delta > 0 \), let

\[ \mu_n(x_0) = E[C_n^{(1)} | X_0 = x_o + \delta] - E[C_n^{(1)} | X_0 = x_o] \]

\[ = E[X_n - X_{n-1} | X_0 = x_o + \delta] - E[X_n - X_{n-1} | X_0 = x_o] \]

Where \( E[C_n^{(1)} | x_0 = x_0] \) is the conditional mean given \( x_0 \). For \( n = 1 \), we have

176
\[ \mu_1(x_0) = E[X_1 - X_0 \mid X_0 = x_0 + \delta] - E[X_1 - X_0 \mid X_0 = x_0] \\
= E[F(X_0) - X_0 \mid X_0 = x_0 + \delta] - E[F(X_0) - X_0 \mid X_0 = x_0] \\
= F(x_0 + \delta) - F(x_0) - \delta \\
\approx \dot{F}(x_0) \delta - \delta \]

(6)

Also, for \( n=2 \), we can see that
\[ \mu_2(x_0) \approx \dot{F}^{(2)}(x_0) \delta - \dot{F}(x_0) \delta \]

(7)

Here, \( F^{(2)}(.) \) means the 2-component of the function \( F \) and the over-dot denotes the differential operator. From (6) and (7), we have
\[ \mu_2(x_0) = \dot{F}^{(2)}(x_0) \delta - \mu_1(x_0) - \delta \]

For \( n>1 \), the general form as:
\[ \mu_n(x_0) = \dot{F}^{(n)}(x_0) \delta - \sum_{i=1}^{n-1} \mu_i(x_0) - \delta \]

(8)

For the value of \( x_0 \) such that \( |\mu_n(x_0)| > 0 \), a small shift \( \delta \) in the initial value can lead to a considerable divergence in the ECS. This means that the ECS depends on \( x_0 \) sensitivity when \( |\mu_n(x_0)| > 0 \).

\( |\mu_n(x_0)| \) Can be a signal to the correlate of the trajectories of system (1) and can determine the bifurcation parameter in a dynamical system that leads to chaotic system.

3- Lyapunov Correlate of ECS.

Lyapunov exponent is one of the most popular measures of chaos which is defined as:
\[ \dot{F}^{(n)}(x_0) \approx e^{n\lambda(x_0)} \]

(9)

where
\[ \lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \ln|\dot{F}^{(n)}(x_0)| \]

(10)
We call $\lambda(x_0)$ the Lyapunov exponent at $x_0$ [3]. To modify a new measure of chaos, we use ESC as follows: from equation (8) and (9) we have

$$\mu_n(x_0) \approx e^{n\lambda(x_0)}\delta - \sum_{i=1}^{n-1} \mu_i(x_0) - \delta$$

therefore

$$\lambda_c(x_0) = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{1}{\delta} \sum_{i=1}^{n} \mu_i(x_0) + 1 \right| \text{.................................................(11)}$$

assuming that the limit exists, $\lambda_c(x_0)$ is called Lyapunov Correlate, which characterize the average rate of exponential divergence of nearby trajectories. The magnitude of the positive of the Lyapunov correlate for all $n$ is a measure of the degree of chaos. Zero Lyapunov correlate characterize the average of the rate of convergence of trajectories.

4-The Expectation Correlate System and Stochastic Systems:

In this section, the ECS measure another property of nonlinear stochastic systems by comparing the conditional variance (given the initial condition $X_0=x$) with the variance $\epsilon_i$. Consider the ECS of system (3) when $n=1$, thus

$$C_1^{(I)} = X_1 - X_0 = F(x) + \epsilon_1 - x$$

And

$$C_2^{(I)} = X_2 - X_1 = F(F(x) + \epsilon_1) + \epsilon_2 - F(x) - \epsilon_1$$

By using Taylor expansion to the function $F$, we have

$$C_2^{(I)} = F^{(2)}(x) + \epsilon_2 + \frac{\partial F}{\partial x}(x) \epsilon_1 - F(x) - \epsilon_1$$

For $n=3$, get
On Expectation Correlate System…

\[ C^{(1)}_j = X_j - X_2 = F(X_2) + \varepsilon_j - F(X_1) - \varepsilon_2 \]

\[ = [F(F^{(2)}(x) + \hat{F}(F(x))) \varepsilon_1 + \varepsilon_2] - [F(F(x) - \varepsilon_1) + \varepsilon_2] \]

\[ = [F^{(3)}(x) + \varepsilon_3 + \hat{F}(F^{(2)}(x))\hat{F}(F(x)) \varepsilon_1 + \hat{F}(F^{(2)}(x)) \varepsilon_2] \]

Then, the general form of ECS as follows:

\[ C^{(1)}_n = F^{(n)}(x) - F^{(n-1)}(x) + \varepsilon_n - \varepsilon_{n-1} + \sum_{j=1}^{n-1} \left\{ \prod_{k=j}^{n-1} \hat{F}(F^{(k)}(x)) \right\} \varepsilon_j \]

Suppose that, \( \sigma_n^2(x) = \text{Var}(C^{(1)}_n \mid X_0 = x) \) and \( \sigma^2 = \text{Var}(\varepsilon_n) = \text{Var}(\varepsilon_i \mid X_i) \), from equation (12), take the conditional variance given \( X_0 = x \) we have

\[ \sigma_n^2(x) = \sigma^2 \sum_{j=1}^{n-1} \left\{ \prod_{k=j}^{n-1} \hat{F}(F^{(k)}(x)) \right\}^2 + \sigma^2 \sum_{j=1}^{n-2} \left\{ \prod_{k=j}^{n-2} \hat{F}(F^{(k)}(x)) \right\}^2 \ldots (13) \]

If \( |\hat{F}(x)| > 1 \) for a large of value of \( x \), \( \sigma_n^2(x) \) can be very large for moderate \( n \). It is easy to see from equation (13) that \( \sigma_n^2(x) \) also depends on the initial value sensitivity when \( |\hat{F}(x)| \) is greater then 1, which indicates the dependence of the correlate prediction on initial value, this is a typical feature of nonlinear (but not necessarily chaotic) systems. If the system (3) is stochastically chaotic, equation (13) indicates that small noise can be amplified quickly when the system starts at some initial value, which mean that the n-step prediction based on these initial values could be unreliable even for small \( n \).
5-Non-linear ECS

By using the idea developed in section (4), suppose that
\[ \Lambda_n(x) = \sum_{j=1}^{n-1} \left( \prod_{k=j}^{n-1} \hat{F}(F^{(k)}(x)) \right)^2 \] .................................(14)

Equation (13) takes the form
\[ \sigma_n^2(x) = \sigma^2 \Lambda_n(x) + \sigma^2 \Lambda_{n-1}(x) \]

By a simple change in equation (14), it is easy to see that for \( n>1 \)
\[ \sigma_n^2(x) = \left\{ (\hat{F}(F^{(n-1)}(x)))^2 + 1 \right\} \sigma^2 \Lambda_{n-1}(x) \] .............(15)

We now turn the general form of ECS which is consider in equation (4), by using the idea in equation (12), we get for \( n>s \)
\[ C_n^{(x)} = F^{(n)}(x) - F^{(n-s)}(x) + \varepsilon_n - \varepsilon_{n-s} + \sum_{j=1}^{n-1} \left( \prod_{k=j}^{n-1} \hat{F}(F^{(k)}(x)) \right) \varepsilon_j \]
\[ - \sum_{j=1}^{n-s-1} \left( \prod_{k=j}^{n-s-1} \hat{F}(F^{(k)}(x)) \right) \varepsilon_j \] .............................(16)

Using the conditional variance of equation (16), we have
\[ \sigma_{n_s}^2(x) = \sigma^2 \Lambda_n(x) + \sigma^2 \Lambda_{n-s}(x) \]
\[ = \left\{ \prod_{k=l}^{s} (\hat{F}(F^{(n-s)}(x)))^2 + 1 \right\} \sigma^2 \Lambda_{n-s}(x) \]

Where, \( \sigma_{n_s}^2(x) = \text{Var}(C_n^{(x)} \mid X_0 = x) \) and \( \sigma_{n_s}^2(x) = \sigma^2_n(x) \). The dependence of \( \sigma_{n_s}^2(x) \) to initial condition means that, the prediction depends on initial condition, which is a typical feature of non-linear but not necessarily chaotic systems. When \( F(.) \) is linear, \( \hat{F}(x) \) is a constant, \( \sigma_{n_s}^2(x) \) does not depend on \( x \) and is monotonically increasing as \( n \) increases.
6-Example

Consider one dimensional stochastic system

\[ X_t = rX_{t-1} - rX^2_{t-1} + \varepsilon_t \]

(17)

Where \( \varepsilon_t, t \geq 1 \) are independent random variable with the same distribution as a normal random variable with mean 0 and variance 0.1\( ^2 \). \( r \) is the parameter of system (17). The skeleton of system (17) is transformed logistic map with parameter \( r \). To show the behavior of \( |\mu_n(x_0)| \), take the deterministic part. By using Matlab language, we write a program to presentation of the way in which the behavior of our iteration depends on the value of the growth parameter \( r \). For each value of \( r \) in the input interval \( 1 \leq r \leq 4 \) in the horizontal axis of figure (2 a-b). We see a single limiting population until \( r = 3 \) then cycle with period 2, then a cycle of period 4, then one of period 8, the corresponding trajectories is approach to zero. If \( |\mu_n(x_0)| \) increasing this mean there are no correlation between the trajectories as show in the range \( 3.6 \leq r \leq 4 \), and \( |\mu_n(x_0)| \) has maximum value when \( r = 4 \), hence Logistic map has a strong attractors in this value.

We conclude that if \( |\mu_n(x_0)| \rightarrow 0 \), this means that there are correlations between the trajectories and the system has a limit point or cycle of finite periodic, and if \( |\mu_n(x_0)| > 0 \) there are un correlated trajectories and the strong chaos depends on the value of \( |\mu_n(x_0)| \).

Table (1) shows the resulting of \( |\mu_n(x_0)| \) in a different value of \( r \), which can be measured when the trajectories of logistic map are un correlated (complex trajectories), and table (2) shows that \( |\mu_n(x_0)| \) has sensitivity to a small shift in initial value compared to table (2). The bifurcation parameter can be also determined by this table which has \( |\mu_n(x_0)| \neq 0 \) as \( n \rightarrow \infty \).
Table (3) shows the conditional variance and the divergently resulting from a small shift in initial values, also the amplification of noise is shown.

| N | r=1.9 | r=2.7 | r=3  | r=3.4 | r=3.7 | r=4  |
|---|-------|-------|------|-------|-------|------|
|   | δ = 0.01 | δ = 0.01 | δ = 0.01 | δ = 0.01 | δ = 0.01 | δ = 0.01 |
| 1 | 0.0100 | 0.0100 | 0.0100 | 0.0100 | 0.0100 | 0.0100 |
| 2 | 0.0130 | 0.0095 | 0.0087 | 0.0067 | 0.0056 | 0.0044 |
| 3 | 0.0040 | 0.0146 | 0.0188 | 0.0332 | 0.0448 | 0.0590 |
| 4 | 0.0022 | 0.0077 | 0.0157 | 0.0448 | 0.0586 | 0.0489 |
| 5 | 0.0006 | 0.0056 | 0.0140 | 0.0646 | 0.1048 | 0.3275 |
| 6 | 0.0001 | 0.0036 | 0.0120 | 0.0837 | 0.0369 | 0.0638 |
| 7 | 0.0000 | 0.0026 | 0.0108 | 0.1214 | 0.1572 | 0.2265 |
| 8 | 0.0000 | 0.0018 | 0.0094 | 0.1413 | 0.0328 | 0.5253 |
| 9 | 0.0000 | 0.0013 | 0.0084 | 0.1921 | 0.4863 | 0.5217 |
| 10| 0.0000 | 0.0009 | 0.0074 | 0.1730 | 0.5033 | 0.2155 |
| 11| 0.0000 | 0.0006 | 0.0067 | 0.1446 | 0.5439 | 1.4807 |
| 12| 0.0000 | 0.0004 | 0.0059 | 0.0894 | 0.5365 | 1.0873 |
| 13| 0.0000 | 0.0003 | 0.0053 | 0.0345 | 0.4936 | 0.3062 |
| 14| 0.0000 | 0.0002 | 0.0047 | 0.0168 | 0.5995 | 0.1522 |
| 15| 0.0000 | 0.0001 | 0.0043 | 0.0245 | 0.8153 | 1.5032 |
| 16| 0.0000 | 0.0001 | 0.0038 | 0.0114 | 0.8392 | 0.9061 |
| 17| 0.0000 | 0.0001 | 0.0035 | 0.0191 | 0.5623 | 0.9014 |
| 18| 0.0000 | 0.0001 | 0.0031 | 0.0091 | 0.8523 | 0.0039 |
| 19| 0.0000 | 0.0000 | 0.0028 | 0.0140 | 0.7957 | 0.8674 |
| 20| 0.0000 | 0.0000 | 0.0025 | 0.0066 | 0.3831 | 0.0124 |
| 21| 0.0000 | 0.0000 | 0.0022 | 0.0108 | 0.5896 | 0.4020 |
| 98| 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0160 | 0.1189 |
| 99| 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0032 | 0.2751 |
| 100| 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0306 | 0.0953 |
### Table (2)

| n  | \(r=1.9\) \(\delta = 0.001\) | \(r=2.7\) \(\delta = 0.001\) | \(r=3\) \(\delta = 0.001\) | \(r=3.4\) \(\delta = 0.001\) | \(r=3.7\) \(\delta = 0.001\) | \(r=4\) \(\delta = 0.001\) |
|-----|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1   | 0.000100                      | 0.000100                      | 0.000100                      | 0.000100                      | 0.000100                      | 0.000100                      |
| 2   | 0.000132                      | 0.000092                      | 0.0000840                     | 0.000100                      | 0.000100                      | 0.00023                       |
| 3   | 0.000034                      | 0.000147                      | 0.001089                      | 0.00030                       | 0.00055                       | 0.00068                       |
| 4   | 0.000020                      | 0.000073                      | 0.000154                      | 0.00050                       | 0.00060                       | 0.00056                       |
| 5   | 0.000008                      | 0.000053                      | 0.00137                       | 0.00070                       | 0.00110                       | 0.00060                       |
| 6   | 0.000002                      | 0.000034                      | 0.00116                       | 0.00090                       | 0.00060                       | 0.00150                       |
| 7   | 0.000009                      | 0.000024                      | 0.00104                       | 0.00130                       | 0.00080                       | 0.00450                       |
| 8   | 0.000000                      | 0.000016                      | 0.00089                       | 0.00160                       | 0.00040                       | 0.00300                       |
| 9   | 0.000000                      | 0.000011                      | 0.00081                       | 0.00240                       | 0.00280                       | 0.03730                       |
| 10  | 0.000000                      | 0.000008                      | 0.00070                       | 0.00270                       | 0.00480                       | 0.01930                       |
| 11  | 0.000000                      | 0.000005                      | 0.00063                       | 0.00370                       | 0.00800                       | 0.04240                       |
| 12  | 0.000000                      | 0.000004                      | 0.00056                       | 0.00300                       | 0.01400                       | 0.09430                       |
| 13  | 0.000000                      | 0.000002                      | 0.00050                       | 0.00190                       | 0.02270                       | 0.27960                       |
| 14  | 0.000000                      | 0.000002                      | 0.00045                       | 0.00080                       | 0.04080                       | 0.08040                       |
| 15  | 0.000000                      | 0.000001                      | 0.00040                       | 0.00170                       | 0.05920                       | 1.44170                       |
| 16  | 0.000000                      | 0.000001                      | 0.00036                       | 0.00090                       | 0.11030                       | 1.13060                       |
| 17  | 0.000000                      | 0.000007                      | 0.00032                       | 0.00110                       | 0.09060                       | 0.40560                       |
| 18  | 0.000000                      | 0.000005                      | 0.00029                       | 0.00050                       | 0.05030                       | 0.39020                       |
| 19  | 0.000000                      | 0.000003                      | 0.00026                       | 0.00090                       | 0.02400                       | 0.86670                       |
| 20  | 0.000000                      | 0.000000                      | 0.00023                       | 0.00050                       | 0.17400                       | 0.27320                       |
| 21  | 0.000000                      | 0.000000                      | 0.00021                       | 0.00060                       | 0.27900                       | 0.56210                       |
| 22  | 0.000000                      | 0.000000                      | 0.00019                       | 0.00030                       | 0.49210                       | 0.72380                       |
| 23  | 0.000000                      | 0.000000                      | 0.00017                       | 0.00050                       | 0.67050                       | 0.46030                       |
| 24  | 0.000000                      | 0.000000                      | 0.00015                       | 0.00020                       | 0.98730                       | 0.15780                       |
| 25  | 0.000000                      | 0.000000                      | 0.00013                       | 0.00040                       | 1.02910                       | 0.47120                       |
| 26  | 0.000000                      | 0.000000                      | 0.00012                       | 0.00020                       | 0.44340                       | 0.24870                       |
| 27  | 0.000000                      | 0.000000                      | 0.00011                       | 0.00030                       | 0.17390                       | 1.54650                       |
| 28  | 0.000000                      | 0.000000                      | 0.00010                       | 0.00010                       | 0.22450                       | 0.77800                       |
| 29  | 0.000000                      | 0.000000                      | 0.00009                       | 0.00020                       | 0.38800                       | 0.30790                       |
| 30  | 0.000000                      | 0.000000                      | 0.00000                       | 0.00010                       | 0.16290                       | 0.16670                       |
| 31  | 0.000000                      | 0.000000                      | 0.00000                       | 0.00000                       | 0.41670                       | 0.91280                       |
Table (3a)

| n  | $\sigma^2_n(x)$ | $\sigma^2_0(x)$ |
|----|----------------|----------------|
|    | $r = 3, x_0 = 0.4$ | $r = 4, x_0 = 0.4$ |
| 2  | 1.6498 e-001    | 2.1552 e-001    |
| 3  | 1.6081 e-001    | 1.8293 e+000    |
| 4  | 1.3760 e-001    | 1.2168 e+002    |
| 5  | 9.4139 e-002    | 3.3636 e+002    |
| 6  | 7.4985 e-002    | 6.2536 e+003    |
| 7  | 5.3312 e-002    | 2.3578 e+006    |
| 8  | 4.4795 e-002    | 8.2847 e+009    |
| 9  | 3.4394 e-002    | 2.0201 e+013    |
| 10 | 3.0050 e-002    | 3.4465 e+016    |
| 11 | 2.4145 e-002    | 1.6978 e+020    |
| 12 | 2.1554 e-002    | 1.1865 e+024    |
| 13 | 1.7829 e-002    | 3.5202 e+027    |
| 14 | 1.6138 e-002    | 3.3947 e+031    |
| 15 | 1.3635 e-002    | 3.4302 e+031    |
| 16 | 1.2465 e-002    | 4.1079 e+029    |
| 17 | 1.0707 e-002    | 1.1676 e+029    |
| 18 | 9.8635 e-003    | 6.8458 e+028    |
| 19 | 8.5856 e-003    | 7.9180 e+027    |
| 20 | 7.9580 e-003    | 6.2190 e+028    |
| 21 | 7.0042 e-003    | 1.1221 e+031    |
| 22 | 6.5250 e-003    | 1.2883 e+034    |
| 23 | 5.7971 e-003    | 1.0417 e+037    |
| 24 | 5.4237 e-003    | 6.2552 e+040    |
| 25 | 4.8576 e-003    | 5.0153 e+043    |
| 26 | 4.5615 e-003    | 3.0802 e+046    |
| 27 | 4.1142 e-003    | 1.6723 e+050    |
| 28 | 3.8759 e-003    | 7.6873 e+053    |
| 29 | 3.5175 e-003    | 2.1088 e+057    |
| 30 | 3.0324 e-003    | 2.1803 e+058    |
Table (3b)

| n  | \( \sigma_n^2(x) \) \( x_0 = 0.3 \) | \( \sigma_n^2(x) \) \( x_0 = 0.31 \) |
|----|-------------------------------|-------------------------------|
| 2  | 5.8672 e-001                  | 5.3808 e-001                  |
| 3  | 1.1541 e+001                  | 1.0378 e+001                  |
| 4  | 1.4453 e+001                  | 1.8782 e+001                  |
| 5  | 2.0262 e+001                  | 1.3244 e+002                  |
| 6  | 1.2653 e+003                  | 2.5426 e+004                  |
| 7  | 1.2185 e+006                  | 6.0232 e+007                  |
| 8  | 8.5390 e+009                  | 5.6168 e+011                  |
| 9  | 8.5232 e+012                  | 9.7841 e+013                  |
| 10 | 2.4040 e+016                  | 2.0321 e+017                  |
| 11 | 2.1665 e+020                  | 1.9891 e+021                  |
| 12 | 4.1357 e+022                  | 6.5377 e+021                  |
| 13 | 8.1155 e+025                  | 1.5746 e+023                  |
| 14 | 2.8845 e+029                  | 5.7871 e+025                  |
| 15 | 2.3724 e+033                  | 5.8720 e+028                  |
| 16 | 2.3911 e+033                  | 3.2980 e+032                  |
| 17 | 2.0033 e+031                  | 3.2993 e+035                  |
| 18 | 1.0398 e+030                  | 1.1333 e+038                  |
| 19 | 3.2630 e+028                  | 3.5552 e+041                  |
| 20 | 3.6054 e+028                  | 8.2941 e+044                  |
| 21 | 1.0151 e+029                  | 2.3966 e+048                  |
| 22 | 2.6141 e+030                  | 1.5881 e+052                  |
| 23 | 5.4074 e+032                  | 3.0865 e+055                  |
| 24 | 9.2186 e+034                  | 1.2584 e+059                  |
| 25 | 1.8355 e+038                  | 1.0003 e+063                  |
| 26 | 1.4899 e+042                  | 4.4830 e+065                  |
| 27 | 1.6612 e+045                  | 1.1413 e+069                  |
| 28 | 1.2163 e+049                  | 1.8207 e+072                  |
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