A novel approach to consider triaxial tensile stresses within the framework of a failure criterion

Karl Roetsch | Thomas Horst

Institute for Production Technology, Westsächsische Hochschule Zwickau—University of Applied Sciences, Saxony, Germany

Correspondence
K. Roetsch, Institute for Production Technology, Westsächsische Hochschule Zwickau—University of Applied Sciences, Saxony, Germany.
Email: karl.roetsch@fh-zwickau.de

Present Address
K. Roetsch, Kornmarkt 1, 08056 Zwickau, Germany.

Funding information
Sächsisches Staatsministerium für Wissenschaft und Kunst, Grant/Award Number: 2057400300; European Social Fund

For use in micromechanical simulations of continuous fiber reinforced polymers, a more general form of the paraboloid failure criterion by Stassi-D’Alia for matrix failure was developed with explicit consideration of the hydrostatic tension strength. Regarding polymers, limits for hydrostatic tensile strength based on isotropic linear elasticity could be derived. The comparison of the newly developed extended paraboloid criterion with experimental data for yielding as well as for material separation (fracture) shows good agreement.

1 INTRODUCTION

The direction-dependent strength of continuous fiber-reinforced polymers is largely determined by the strength of the composite components. Especially the transverse strengths of a fiber reinforced composite (FRP) are strongly influenced by the polymer matrix. Even with simple transverse loads on the composite, a complicated multiaxial stress state occurs locally in the matrix [4–7]. Such local effects were investigated within the framework of numerical micromechanical models [8–10]. Measurement data from the World Wide Failure Excercise I [11] showed that the transverse tensile strength of the unidirectional (UD) layer was almost halved compared to the undisturbed matrix, while the transverse shear and compressive strengths were only slightly affected. Triaxial stress states within the matrix as a result of mismatch in Young’s modulus, Poisson’s ratios, and thermal expansion coefficients of fiber and matrix are seen as the reason for the reduced transverse tensile strength [12, 13]. To consider this behavior in numerical micromechanical models, a strength criterion for the polymeric matrix is needed that accounts for the material behavior under triaxial tension.

Strength hypothesis are used since it is impossible to determine all possible load combinations experimentally. These criteria usually have the form

\[ \Phi = 1 \] (1)
where \( \Phi \) represents the failure of the material. The failure of a material is usually context related. In the most general definition, failure can be seen as a limit (e.g., stress or strain) that cannot be exceeded without consequences. In engineering practice, failure is usually associated with material flow or material separation (fracture). Since material flow and material separation are different phenomena, they must theoretically also be defined by independent criteria. For the sake of simplicity, the criterion for yielding will be referred to as the yield criterion and that for material separation as the strength criterion. For the introduction of known criteria, the more general term “failure criterion” will be used further. Generally, Equation (1) is expressed using the stress tensor \( \sigma_{ij} \). In particular, the representation via the invariants of the stress tensor has proven to be useful. Therefore, the invariants are first defined using Einstein’s summation convention (summation over double occurring indices) as follows:

\[
I_1 = \sigma_{ii},
\]

\[
J_2 = \frac{1}{2} s_{ij} s_{ij},
\]

where

\[
s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}.
\]

The first invariant \( I_1 \) of the second order stress tensor \( \sigma_{ij} \) represents the hydrostatic part and \( J_2 \) is the second invariant of the stress deviator \( s_{ij} \) representing the shape-changing part of the stress tensor. Any stress state can be reduced to three normal stresses, the so called principal stresses, by a principal axis transformation

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_I & 0 & 0 \\
0 & \sigma_{II} & 0 \\
0 & 0 & \sigma_{III}
\end{bmatrix},
\]

whereby

\[
\sigma_I \geq \sigma_{II} \geq \sigma_{III}.
\]

The invariants of the stress tensor can thus also be expressed in terms of the principal stresses

\[
I_1 = \sigma_I + \sigma_{II} + \sigma_{III},
\]

\[
J_2 = \frac{1}{6} \left[ (\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 \right].
\]

For the visualization of failure criteria, the use of octahedral stresses is useful, which are defined as

\[
\sigma_{os} = \sqrt{\frac{2}{3} J_2},
\]

\[
\sigma_{on} = \frac{1}{3} I_1.
\]

Furthermore, quasi-static loads are assumed in the following, whereby the focus is on thermoplastic and thermoset polymers which can therefore be described with a linear-elasto-plastic material law. Under this assumption, the yield criterion represents the limit of the linear-elastic range. With occurrence of plastic deformation, the yield criterion transforms into the final strength criterion due to hardening processes. By formulating Equation (1) using the invariants, it can be seen directly from the strength criterion which part of the volume and shape change is responsible for the failure.

Christensen [3] formulated a polynomial approach based on the calculation of the elastic strain energy density to determine \( \Phi \)

\[
\Phi = a_1 I_1 + a_2 I_1^2 + b_1 J_2 + \cdots = 1.
\]
The ability of Equation (11) to form a failure criteria has been demonstrated by various independent formulations for strength criteria. Some well-known failure criteria which can be related to Equation (11) are the von Mises, the Beltrami, and the Stassi-D’Alia criteria.

The von Mises criterion (1921) predicts the onset of yielding when a critical elastic distortion energy density is exceeded. Since hydrostatic effects are not considered, the von Mises criterion is formulated only in terms of $J_2$ (cf. Equation 11).

\[ \Phi = b_1 J_2 = 1 \]  

Developing the parameter $b_1$ in Equation (12) using the tensile strength $T$, the von Mises criterion becomes

\[ \Phi = \frac{3}{T^2} J_2 = 1 \]  

or in the widely known representation

\[ \Phi = \frac{1}{T} \sqrt{3J_2} = 1. \]  

However, this criterion, which is often used for ductile metals, neglects the influence of hydrostatic stresses and is therefore not applicable to pressure-sensitive materials. A failure criterion that takes into account the influence of hydrostatic pressure was formulated by Beltrami [14]. It predicts failure when the elastic strain energy density exceeds a critical value. The calculation of the elastic strain energy density follows the same form as Equation (11), which leads to

\[ U = a_2 I_1^2 + b_1 J_2, \]  

where $U$ represents the linear elastic strain energy density for the isotropic case. The coefficients $a_2$ and $b_1$ can now be determined via the elasticity constants, from which it follows

\[ U = \frac{1}{2E} \left[ \frac{(1-2\nu)}{3} I_1^2 + 2(1+\nu)J_2 \right], \]  

where $E$, $\nu$ being the Young’s modulus and the Poisson’s ratio of the material, respectively. If a critical elastic strain energy density $U^{\text{crit}}$ under tensile stress is used as a failure criterion, the Beltrami criterion becomes [14]

\[ \Phi = \frac{(1-2\nu)}{3T^2} I_1^2 + \frac{2(1+\nu)}{T^2} J_2 = 1. \]  

Independently, Stassi-D’Alia [1] and Tschoegl [2] developed the paraboloid criterion which is an improvement of the widely known Drucker–Prager criterion. To avoid the mathematical discontinuity of the Drucker–Prager criterion, a criterion in the form of a paraboloid was proposed. Christensen [3] related the equation of the paraboloid to the approach according to Equation (11).

\[ 1 = a_1 I_1 + b_1 J_2 \]  

Using the compression strength $-C$ and the tension strength $T$ to determine $a_1$ and $b_1$, the paraboloid criterion follows

\[ \Phi = \frac{(C-T)}{CT} I_1 + \frac{3}{CT} J_2 = 1. \]  

Christensen [3] added fracture criteria to Equation (19) if the ratio of tension and compression strength $T/C$ is less than $1/2$.

\[ \sigma_1 \leq T \quad \text{for} \quad \frac{T}{C} \leq \frac{1}{2} \]  

\[ \Phi = \frac{(C-T)}{CT} I_1 + \frac{3}{CT} J_2 = 1. \]
2 | DEVELOPMENT OF THE STRENGTH CRITERION

2.1 | Basic conditions

The suitability of the paraboloid criterion as a yield criterion has been demonstrated several times [3, 15, 16]. However, as a strength criterion, a more general way of considering hydrostatic tension strength should be available. The intention is less to develop a new failure criterion, but rather to extend the applicability of the paraboloid criterion as a strength criterion for use in a numerical elasto-plastic material model. In particular, the mechanical behavior of epoxy resin will be considered, since it is widely used as a matrix material.

Investigations into the failure of epoxy resins led to the conclusion that brittle fracture always occurs in these materials regardless of the type of loading [16–18]. It has also been shown that multiaxial tensile stresses increase cavity formation within the resin [19] and lead to brittle material behavior [20, 21]. The increased cavity formation under multiaxial tension causes an extremely reduced strength of the material compared to uniaxial load. In investigations by Asp et al. [12] and Kim et al. [21], thin slices of epoxy resin were glued between two steel pipes and ripped using a so-called poker-chip test. The gluing limited the transverse contraction of the specimens, resulting in a multiaxial stress state. For different resin systems, it was found that the hydrostatic tensile strength was dramatically lower than the uniaxial tensile strength (see Table 2). A different method was used by Gross et al. [22]. Here, an epoxy resin was cooled from curing temperature inside a quartz tube. As a result of the cooling, the epoxy resin contracts which was hindered by the adhesion of the epoxy resin to the walls of the quartz tube. The results show similar reduced strengths to those observed in the poker chip test [22].

Thus, it is shown that in a FRP under transverse tension, a multiaxial tensile stress state is formed in the polymer matrix and that polymers under such a stress state exhibit significantly lower strengths than under uniaxial loads. None of the criteria presented sufficiently limits the hydrostatic tensile strength. The von-Mises criterion shows no dependence on hydrostatic pressure and thus predicts infinite hydrostatic tensile strength. This is also the case for the Beltrami criterion, which changes to the von Mises criterion for incompressible material with \( \nu = 0.5 \). The paraboloid criterion also predicts infinitely high hydrostatic tensile strengths when the tensile and compressive strengths are equal. This leads to a dramatic overestimation of the hydrostatic tensile strength of epoxy [23].

2.2 | Formulation of the strength criterion

As already demonstrated (cf. Section 1), the additive decomposition into volume and shape changing parts is suitable for the formulation of a failure criterion. The more general form of Equation (11) follows with

\[
\Phi = \sum \left( a_n I_1^n + b_n J_2^n \right) = 1.
\] (21)

In order to decide which experimental parameters should be used for the determination of the coefficients \( a_n, b_n \) the parameters \( T, C \) used by Stassi-D’Alia, Tschogl, and Christensen for the paraboloid criterion are considered first. In a uniaxial tensile test, a fracture surface perpendicular to the direction of force is obtained for an isotropic material. Thus, in this case, it can be concluded that a tensile normal stress is the critical portion of the load. In a uniaxial compression test, fracture surfaces usually deflect in the direction of the force or inclined to it. This observation can be understood if the uniaxial compressive stress state is decomposed into its volumetric and deviatoric components

\[
C = \begin{bmatrix}
-p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{bmatrix}^\text{vol} + \begin{bmatrix}
-2p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{bmatrix}^\text{dev},
\] (22)

with

\[
p = \frac{1}{3} \sigma_{kk}.
\] (23)

Under pure all-axis compressive stresses, the formation of cracks is limited. Thus, material separation (fracture) under uniaxial compressive stress is caused by the deviatoric component and thus finally by shear stresses, with the volumetric
component having a crack-inhibiting effect. Thus, the uniaxial tensile strength $T$ and the shear strength $S$ instead of $C$ are used to define the coefficients. Since the hydrostatic tensile strength $H$ is to be explicitly taken into account, the three material parameters $T$, $S$, and $H$ are used for the determination of Equation (21). To consider these three parameters in Equation (21), it follows with $n = 2$

$$1 = a_1 I_1 + b_1 J_2 + a_2 I_1^2 + b_2 J_2^2$$

(24)

Since isotropic materials do not fail under hydrostatic pressure, it follows that $a_2 = 0$ [3]. Substituting $S$, $T$, and $H$ into Equation (24) yields

$$1 = a_1 3H,$$

(25)

$$1 = b_1 S^2 + b_2 S^4,$$

(26)

$$1 = a_1 T + b_1 \frac{1}{3} T^2 + b_2 \frac{1}{9} T^4.$$  

(27)

Thus Equations (25)–(27) represent the uniaxial and hydrostatic tensile test as well as the torsional test. Solving this system of equations yields the extended paraboloid criterion

$$\Phi = \frac{1}{3H} I_1 + \left( \frac{3}{T^2} + \frac{1}{S^2} - \frac{3S^2}{HT(3S^2 - T^2)} \right) \cdot J_2 - 3 \left( \frac{1}{S^2 T^2} - \frac{1}{HT(3S^2 - T^2)} \right) \cdot J_2^2 = 1.$$  

(28)

Equation (28) allows both a convex and a partially concave failure surface. This can be formulated as a dependence on the hydrostatic tensile strength $H$. A convex surface results when

$$\frac{3S^4 T}{9S^4 - T^4} \leq H \leq \frac{S^2 T}{3S^2 - T^2}.$$  

(29)

In the general formulation of Equation (28) regardless of its use as a yield or strength criterion, the condition

$$H \leq \frac{S^2 T}{3S^2 - T^2}$$  

(30)

must hold. Otherwise, there is no uniqueness in the shear strengths. For some experimental data, no shear strengths are available, which are needed for scaling the new criterion. In this case, compressive strength $C$ is used for scaling as it is done for the standard paraboloid criterion. This is possible if reliable data for the compressive strength are available. The procedure is identical to the one already presented (cf. Equations 24–28). The determination of the parameters $a_1, b_1, b_2$ using $-C, T, H$ leads to

$$1 = \frac{1}{3H} I_1 - \frac{C^4(T - 3H) + T^4(C + 3H)}{C^2T^2H(C^2 - T^2)} \cdot J_2 + \frac{3(3HT + CT - 3HC)}{C^2T^2H(C - T)} \cdot J_2^2.$$  

(31)

The limits for $H$ to satisfy a convex yield surface are then given by

$$\frac{(C^3 + T^3) \cdot CT}{3(C^4 - T^4)} \leq H \leq \frac{CT}{3(C - T)}.$$  

(32)

The right term of Equation (32) represents the hydrostatic tensile strength implied by the standard paraboloid criterion.

Figure 1 shows the different forms of the strength criterion. A summary of these different criteria is shown in Table 1.
While \( T \) and \( S \) can be determined by relatively simple uniaxial tensile and torsional tests, the determination of the hydrostatic tensile strength is extremely difficult. According to Christensen [24], regardless of the material, yielding is not possible under hydrostatic tension. In investigations on the yield behavior of epoxy resin under hydrostatic tension by Kim et al. [21] approximately nonlinear material behavior could only be observed near the glass transition temperature. Thus, assuming a purely linear elastic material behavior under hydrostatic tension, the elastic strain energy density (Equation \( 16 \)) or the Beltrami failure criterion (Equation \( 17 \)) can be used as a limit. For hydrostatic tensile strength, Beltrami’s criterion leads to

\[
H^y = \sqrt{\frac{T^{yo}}{3(1-2\nu)}},
\]

(33)

where \( H^y \) does not represent a yield stress, but simply indicates the use of the initial tensile yield stress \( T^{yo} \) for calculation. The use of the initial tensile yield stress \( T^{yo} \) follows from the assumption that no plastic deformation and thus no hardening mechanisms are possible under hydrostatic tensile stresses. For epoxy material, usual values for \( \nu \) are in the range 0.3 ... 0.4 leading to values for \( H^y = 0.9T^{yo} \ldots 1.29T^{yo} \). However, this possibility of calculation alone is insufficient for hydrostatic tensile strength. This can be easily demonstrated if incompressible material behavior is assumed. This leads to

\[
H^y \lim \nu \rightarrow 0.5 = \infty,
\]

(34)

which is not acceptable. A limit value is therefore introduced via the uniaxial tensile stress at break

\[
H^f = \frac{T^f}{3}.
\]

(35)

\[
T^f = \frac{1}{2} \left( \frac{2}{S_0} + \frac{1}{S_1} \right)
\]

The use of the initial tensile yield stress \( T^{yo} \) follows from the assumption that no plastic deformation and thus no hardening mechanisms are possible under hydrostatic tensile stresses. For epoxy material, usual values for \( \nu \) are in the range 0.3 ... 0.4 leading to values for \( H^y = 0.9T^{yo} \ldots 1.29T^{yo} \). However, this possibility of calculation alone is insufficient for hydrostatic tensile strength. This can be easily demonstrated if incompressible material behavior is assumed. This leads to

\[
H^y \lim \nu \rightarrow 0.5 = \infty,
\]

(34)

which is not acceptable. A limit value is therefore introduced via the uniaxial tensile stress at break

\[
H^f = \frac{T^f}{3}.
\]

(35)
TABLE 2 Comparison of the poker-chip test data with calculated equitriaxial strength

| Material                      | ν [-] | T^f [MPa] | C^f [MPa] | H^exp [MPa] | H^{calc\,\text{ext\,paraboloid\,Eq.\,(36)}} [MPa] | H^{calc\,\text{paraboloid\,Eq.\,(32)}} [MPa] |
|-------------------------------|-------|-----------|-----------|-------------|---------------------------------------------|---------------------------------------------|
| Epoxy/APTA [23]               | 0.318 | 78 ± 1.2  | 91.4 ± 2  | 33.5        | 26                                          | 177.34                                      |
| Epoxy/DETA [23]               | 0.345 | 83        | 113 ± 5   | 27.5        | 27.66                                       | 104.2                                       |
| DGEBA/MHPA [23]               | –     | 84        | 115       | 26.6        | 28                                          | 103.87                                      |
| TGDDM/DDS [23]                | 0.328 | 59.9 [12] | 207       | 30.8        | 19.96                                       | 28.09                                       |
| EPON825-33DDS [21]            | 0.35  | 91.6 [25] | 138.9 [25]| 33          | 30.53                                       | 89.66                                       |

TABLE 3 Overview of experimental strength data for different epoxy-systems

| Material                      | ν [-] | C/T ratio [-] | H/T ratio [-] |
|-------------------------------|-------|---------------|---------------|
| Epoxy DGEBA/APTA [12, 26]    | 0.318 | 1.17          | 0.43          |
| Epoxy DGEBA/DETA [12, 26]    | 0.345 | 1.37          | 0.33          |
| Epoxy DGEBA/MHPA [23]        | –     | 1.37          | 0.32          |
| Epoxy TGDDM/DDS [23]         | 0.328 | 3.45          | 0.51          |
| Epoxy EPON825-33DDS [21]     | –     | 1.52          | 0.36          |

This limit value results from both the Beltrami criterion for ν = −1 and the paraboloid criterion for T/C → 0. It is therefore valid that the more critical value for the hydrostatic tensile strength is used in each case

\[ H = \min(H^y, H^f) \]  

Table 2 compares the hydrostatic tensile strengths of various epoxy resins taken from the investigations of Asp et al. [12] and Kim et al. [21] with the calculations of Equation (36) and the predictions by the paraboloid criterion. The hydrostatic tensile strength predicted by the paraboloid criterion is given via the upper limit defined in Equation (32). For high tension/compression asymmetry (T/C → 0), the hydrostatic tensile strength predicted by the paraboloid criterion equals one third of the uniaxial tensile strength (H = T/3). It is postulated that this is one meaningful value for H, which is included in Equation (36). It is obvious that Equation (36) makes much better predictions than the paraboloid criterion. It should be noted that the extended paraboloid criterion should ideally be scaled with existing measured values for hydrostatic tensile strength, so that Equation (36) should be considered as an estimate only.

3 | VALIDATION OF THE PROPOSED FAILURE CRITERION

In order to validate the applicability of the new failure criterion, biaxial failure data of different polymers are used in the following. A comparison of the biaxial strengths in the σ^1, σ^2 principal stress plane, as well as in the σ^oσ, σ^oν octahedral stress plane is made to show the effects of the explicit consideration of H. The criterion is tested both as a yield criterion and as a strength criterion. In the absence of experimental data, Equation (36) is used to calculate the hydrostatic tensile strength.

3.1 | Validation as a strength criterion for Epoxy

First, the applicability of the extended paraboloid criterion for calculating the strengths of various epoxy resins is shown, for which characteristic values for the hydrostatic tensile strength are available. Table 3 shows the characteristic values needed to calculate the strength criterion. Here, the specified limit values for H (Equations 29 and 32) must be taken into account. If experimental values for H exceed that limit they could not be considered within the strength criterion. In such a case, the upper limit of Equations (29) and (32) needs to be used.

Figures 2 and 3 show the comparison between the measured data of Asp et al. [12, 26] for different epoxy resins with different hardener systems using the extended and the standard paraboloid criterion. From Figure 2a, it can be seen that
FIGURE 2 Comparison of the extended paraboloid strength criterion and the standard paraboloid criterion with fracture data for Epoxy DGEBA/APTA. Criteria scaled with $C/T = 1.17$ and $H/T = 0.43$.

| Symbol | Source of Data | Material            | $C/T$ Ratio [-] | $H/T$ Ratio [-] |
|--------|----------------|---------------------|-----------------|-----------------|
| ◆      | Asp et al.[12,26] | Epoxy DGEBA/APTA    | 1.17            | 0.43            |

FIGURE 3 Comparison of the extended paraboloid strength criterion and the standard paraboloid criterion with fracture data for Epoxy DGEBA/DETA and DGEBA/MHPA. Criteria scaled with $C/T = 1.37$ and $H/T = 0.33$.

| Symbol | Source of Data | Material            | $C/T$ Ratio [-] | $H/T$ Ratio [-] |
|--------|----------------|---------------------|-----------------|-----------------|
| ◆      | Epoxy DGEBA/DETA | 1.37                | 0.33            |
| ▼      | Asp et al.[12,26] | Epoxy DGEBA/MHPA    | 1.37            | 0.32            |

there are only slight differences between the two criteria in the first and third quadrants, with the extended paraboloid criterion making a more conservative prediction. Figure 2b demonstrates the performance of the new criterion. Without much loss of accuracy in the biaxial stress range, it is possible to consider the measured value for the hydrostatic tensile strength. This is also confirmed by comparison with the measured data in Figure 3. In the biaxial stress space (Figure 3a) the new criterion leads to more conservative estimates in the first and third quadrants and thus a better approximation to the measured values than the standard criterion. In the octahedral stress space (Figure 3b), the explicit consideration of the measured hydrostatic tensile strength becomes visible without loss of accuracy for uni- and biaxial measured values.
Figure 4 shows the comparison between measured data of Asp et al. for a different epoxy system (TGDDM/DDS). Here it is not possible to scale the extended paraboloid criterion via the measured values for $H$, because in this case the condition (32) is exceeded. Thus, the upper limit which results from Equation (32) with $H = 0.469$ is used at this point. A strong deviation from the biaxial measured values can be seen, since these suggest an increasing strength with increasing hydrostatic tension. An inaccuracy of the measured data can therefore not be excluded at this point.

Figure 5 shows the comparison between the measured data of Kim et al. [21] for EPON825-33DDS and the predictions made by the extended and standard paraboloid criteria. Since only the characteristic values which are needed for the scaling of the criterion are available, it is not surprising that the extended paraboloid criterion shows good agreement. However, the difference to the standard paraboloid criterion can also be clearly shown here.
### 3.2 Potential for use as yield criterion

At this point, no data for $H$ are available. To demonstrate the possibilities of the extended paraboloid criterion, this value is calculated using Equation (33). The result of this calculation must satisfy the convex conditions. Otherwise, the respective limit values from the convex conditions are used to determine $H$ in case of exceeding or missing boundary values. Thus, the standard paraboloid defines the upper limit for the hydrostatic tension (yield) strength. It should further be noted that the extended paraboloid criterion also includes the standard paraboloid criterion. When using the new criterion as a yield criterion, care must be taken to ensure a convex yield surface. This limits the hydrostatic tensile strength by the convex constraints given in Equations (29) and (32). The measured values for verifying the criterion are taken from Raghava et al. [15]. An overview of the experimental values used and the calculated hydrostatic tensile strength can be found in Table 4.

In agreement with Raghava et al. [15], the standard paraboloid criterion is defined with a $C_y/T_y$ ratio of 1.30. This results from the fact that the respective $C_y/T_y$ ratios of the investigated polymers differ only slightly and thus a comparison of the criterion with each individual polymer proves to be obsolete. For the $H/T_y$ ratio, the value 1.05 is used. The extended paraboloid criterion is scaled by a mean value for $S_y/T_y = 0.64$. Figure 6 shows the measured data from Raghava [15] compared to the newly defined extended paraboloid as well as the standard paraboloid criterion. Figure 6a shows the excellent agreement of both criteria with the yield data. This shows that for PS in the shear region (second and fourth quadrant) as well as in the pressure region (third quadrant), an even better agreement of the extended paraboloid criterion compared to the standard paraboloid can be observed. This agreement for PS would be further improved by using $H/T_y = \ldots$  

| Material | $v$ [-] [27] | $C_y/T_y$ ratio [-] | $H/T_y$ ratio [-] | $S_y/T_y$ ratio [-] |
|----------|---------------|---------------------|------------------|-------------------|
| PC       | 0.35          | 1.20                | 1.05             | 0.57              |
| PS       | 0.32          | 1.30                | 0.96             | 0.70              |
| PMMA     | 0.35          | 1.30                | 1.05             | –                 |
| PVC      | 0.375         | 1.33                | 1.15             | –                 |

**TABLE 4** Overview about used data for yield criteria

---

**FIGURE 6** Comparison of the extended paraboloid yield criterion and the standard paraboloid criterion with yield data for various polymers. Criteria scaled with $C_y/T_y = 1.30$, $S_y/T_y = 0.64$, and $H/T_y = 1.05$
TABLE 5 Overview of used data for strength criteria

| Material | v [-] | C/T ratio [-] | H/T ratio [-] | S/T ratio [-] |
|----------|-------|---------------|---------------|--------------|
| PMMA [28] | 0.35  | 1.25          | 0.33          | 0.60         |
| Epoxy [28] | –     | 1.56          | 0.33          | 0.66         |

FIGURE 7 Comparison of the extended paraboloid strength criterion and the standard paraboloid criterion with fracture data for PMMA. Criteria scaled with $C/T = 1.25$, $S/T = 0.64$, and $H/T = 0.33$.

0.96 exactly in place of the mean, since a decreasing $H/T$ ratio would compress the ellipse in the first and third quadrants (along a 45° line) and stretch it in the second and fourth quadrants.

The differences between the two criteria are shown in the octahedral stress space in Figure 6b. At this point, it should be pointed out again that even when used within a yield criterion, $H$ is not a yield stress but a fracture stress. The standard paraboloid criterion for $C/T = 1.30$ leads to $H/T = 1.44$. For PC, this would even result in a value of $H/T = 2.0$ showing the sensitivity of the standard paraboloid for low $C/T$ ratios. Due to the lack of experimental data for $H$, it cannot be conclusively determined which criterion makes better predictions. Since $H$ is not a yield stress and the final strength criterion and the initial yield criterion must contact at this point, higher $H/T$ ratios would be possible within a yield criterion, which does not exclude the standard paraboloid criterion. However, the extended paraboloid criterion shows greater flexibility in fitting to measured values and also includes the standard paraboloid criterion.

3.3 Potential as a strength criterion in case of missing hydrostatic tensile strengths

When using the extended paraboloid criterion as a strength criterion, the formulation in terms of shear strength (Equation 28) is preferred, insofar as shear strengths are available as experimental data. The calculation of $H$ is done using Equation (35). This results from the fact that no complete data set is available which provides both yield stresses and ultimate stresses in absolute values. This leads to the situation that no calculation of $H$ via the initial yield stress $T^o$ is possible. Table 5 summarizes the characteristic values required to formulate the strength criterion.

In Figure 7, the preferred variant (using of $S$ instead of $C$) of the extended paraboloid criterion is used for comparison with measured data for PMMA. The findings from the previous comparisons can be confirmed here as well. However, it should be noted that no measured values for the hydrostatic tensile strength are available.

Major deviations of the extended paraboloid criterion and the standard paraboloid criterion are obtained in comparison with the measured data of Ol’khovik et al. [28] for an epoxy resin, which is not described in detail. Here, the sensitivity of the extended paraboloid criterion as a function of the shear strength $S$ can be clearly seen. This indicates that great accuracy is required in determining of $S$ for use in the extended paraboloid criterion. For comparison, the new criterion has
been formulated using $C/T$. It is found that the new criterion using $S$ together with the standard paraboloid shows good agreement in the first, second, and fourth quadrants of Figure 8a. In the fourth quadrant, a significant underestimation of the biaxial compressive strength occurs. In general, in the third quadrant, the agreement of the new criterion using $C$ is best. However, a significant underestimation of the biaxial tensile strengths can be seen. This could be an indication that the hydrostatic tensile strength is higher than indicated by Equation (36).

4 | CONCLUSION

For the micromechanical simulation of FRP, elasto-plastic material models are often used to model the matrix behavior. Many of these models are based on the paraboloid failure criterion of Stassi-D’Alia [8, 9, 29–34]. It was shown that the standard paraboloid criterion significantly overestimates the hydrostatic tensile strength of epoxy resin. Even for simple transverse loads on a composite, multiaxial stress states occur locally in the matrix [4–7]. Such triaxial stress states inside the matrix material reduce the transverse tensile strength of an FRP [12, 13]. Following this, the overestimation of the hydrostatic tensile strength can lead to an overestimation of the transverse tensile strength of an FRP or an incorrect localization of matrix failure in the context of micromechanical simulations. Therefore, a more general form of the paraboloid failure criterion with explicit consideration of the hydrostatic tensile strength $H$ was developed. Furthermore, limits for $H$ were developed in the context of isotropic linear elasticity. Here, attention was paid to the different use of $H$ within the yield and strength criterion. Two different formulations via shear strength $S$ or compressive strength $C$ for the extended paraboloid criterion were presented. Furthermore, in agreement with the fracture modes known from fracture mechanics, it was shown that the tensile and shear strengths $T$ and $S$ are the most fundamental material parameters for scaling the criterion. When compared with experimental data for different polymers, the new criterion generally shows good agreement for both yielding and material separation (fracture). Great attention must be paid to the determination of $S$, since it has been shown that the newly developed criterion exhibits a high sensitivity in the third quadrant (in the biaxial stress space) at this point. For a further validation of the extended paraboloid criterion, characteristic values for the hydrostatic tensile strength have to be determined on different polymers.

ACKNOWLEDGMENTS

The research was financed by the Saxon State Ministry for Science and the Arts within Project No. 2057400300 and by the European Social Fund. Financial support is gratefully acknowledged.

Open access funding enabled and organized by Projekt DEAL.
REFERENCES

[1] Stassi-D'Alia, F.: Flow and fracture of materials according to a new limiting condition of yielding. Meccanica 2(3), 178–195 (1967). https://doi.org/10.1007/BF02128173

[2] Tschoegl, N.W.: Failure surfaces in principal stress space. J. Polym. Sci., Part C: Polym. Symp. 32(1), 239–267 (1971). https://doi.org/10.1002/polc.5070320113

[3] Christensen, R.M.: The Theory of Materials Failure. Oxford University Press, Oxford (2016)

[4] Jin, K.K., Oh, J.H., Ha, S.K.: Effect of fiber arrangement on residual thermal stress distributions in a unidirectional composite. J. Compos. Mater. 41(5), 591–611 (2007). https://doi.org/10.1177/0021998306065290

[5] Jin, K.-K., Huang, Y., Lee, Y.-H., Ha, S.K.: Micro-mechanics of failure (MMF) for continuous fiber reinforced composites. J. Compos. Mater. 42(18), 1825–1849 (2008). https://doi.org/10.1177/0021998308093909

[6] Huang, Y., Jin, K.K., Ha, S.K.: Effects of fiber arrangement on mechanical behavior of unidirectional composites. J. Compos. Mater. 42(18), 1851–1871 (2008). https://doi.org/10.1177/0021998308093910

[7] Hojo, M., Mizuno, M., Hobbiebrunken, T., Adachi, T., Tanaka, M., Ha, S.K.: Effect of fiber array irregularities on microscopic interfacial normal stress states of transversely loaded UD-CFRP from viewpoint of failure initiation. Compos. Sci. Technol. 69(11–12), 1726–1734 (2009). https://doi.org/10.1016/j.compscitech.2008.08.032

[8] Melro, A.R., Camanho, P.P., Andrade Pires, F.M., Pinho, S.T.: Micromechanical analysis of polymer composites reinforced by unidirectional fibres: part I—constitutive modelling. Int. J. Solids Struct. 50(11–12), 1897–1905 (2013). https://doi.org/10.1016/j.ijsolstr.2013.02.009

[9] Melro, A.R., Camanho, P.P. Andrade Pires, F.M., Pinho, S.T.: Micromechanical analysis of polymer composites reinforced by unidirectional fibres: part II Micropolymer analyses. Int. J. Solids Struct. 50, 1906–1915 (2013). https://doi.org/10.1016/j.ijsolstr.2013.02.007

[10] Sun, Q., Zhou, G., Meng, Z., Chen, Z., Liu, H., Kang, H., Keten, S., Su, X.: Failure criteria of unidirectional carbon fiber reinforced polymer composites informed by a computational micromechanics model. Compos. Sci. Technol. 172, 81–95 (2019). https://doi.org/10.1016/j.compscitech.2019.01.012

[11] Soden, P.D., Hinton, M.I., Kaddour, A.S.: Biaxial test results for strength and deformation of a range of E-glass and carbon fibre reinforced composite laminates: failure exercise benchmark data. Compos. Sci. Technol. 62(12–13), 1489–1514 (2002). https://doi.org/10.1016/S0266-3538(02)00093-3

[12] Asp, L.E., Berglund, L.A., Gudmundson, P.: Effects of a composite-like stress state on the fracture of epoxies. Compos. Sci. Technol. 53(1), 27–37 (1995). https://doi.org/10.1016/0266-3538(94)00075-1

[13] Fiedler, B., Hojo, M., Ochiai, S., Schulte, K., Ochi, M.: Finite-element modeling of initial matrix failure in CFRP under static transverse tensile load. Compos. Sci. Technol. 61(1), 95–105 (2001). https://doi.org/10.1016/S0266-3538(00)00198-6

[14] Kolupaev, V.A.: Equivalent Stress Concept for Limit State Analysis, vol. 86. Springer International Publishing, Cham (2018).

[15] Raghava, R., Caddell, R.M., Yeh, G.S.Y.: The macroscopic yield behaviour of polymers. J. Mater. Sci. 8(2), 225–232 (1973). https://doi.org/10.1007/BF00550671

[16] Fiedler, B., Hojo, M., Ochiai, S., Schulte, K., Ando, M.: Failure behavior of an epoxy matrix under different kinds of static loading. Compos. Sci. Technol. 61(11), 1615–1624 (2001). https://doi.org/10.1016/S0266-3538(01)00057-4

[17] Chevalier, J., Morelle, X.P., Bailly, C., Camanho, P.P., Pardoen, T., Lani, F.: Micro-mechanics of failure (MMF) for highly crosslinked epoxy resins. Eng. Fract. Mech. 158, 1–12 (2016). https://doi.org/10.1016/j.engfracmech.2016.02.039

[18] Morelle, X.P., Chevalier, J., Bailly, C., Pardoen, T., Lani, F.: Micromechanical characterization and modeling of the deformation and failure of the highly crosslinked RTM6 epoxy resin. Mech. Time-Depend. Mater. 21, 419–454 (2017). https://doi.org/10.1007/s11043-016-9336-6

[19] Neogi, A., Mitra, N., Talejara, R.: Cavitation in epoxies under composite-like stress states. Compos. Part A: Appl. Sci. Manuf. 106, 52–58 (2018). https://doi.org/10.1016/j.compositesa.2017.12.003

[20] Morelle, X., Lani, F., Melchior, M., André, S., Bailly, C., Pardoen, T., ECCM 2012—composites at Venice. In: Proceedings of the 15th European Conference on Composite Materials (2012)

[21] Kim, J.W., Medvedev, G.A., Caruthers, J.M.: Observation of yield in triaxial deformation of glassy polymers. Polymer 54(11), 2821–2823 (2013). https://doi.org/10.1016/j.polymers.2013.03.042

[22] Gross, T.S., Jafari, H., Kusch, J., Tsukrov, I., Bayraktar, H., Goering, J.: Measuring failure stress of RTM6 epoxy resin under purely hydrostatic tensile stress using constrained tube method. Exp. Tech. 41(1), 417–424 (2015). https://doi.org/10.1002/pen.21587

[23] Asp, L.E., Berglund, L.A., Talejara, R.: A criterion for crack initiation in glassy polymers subjected to a composite-like stress state. Compos. Sci. Technol. 56(11), 1291–1301 (1996). https://doi.org/10.1016/S0266-3538(96)00090-5

[24] Christensen, R.M.: Exploration of ductile, brittle failure characteristics through a two-parameter yield/failure criterion. Mater. Sci. Eng. A 394(1–2), 417–424 (2005). https://doi.org/10.1016/j.msea.2004.11.053

[25] Vu-Bac, N., Bessa, M.A., Rabczuk, T., Liu, W.K.: A multiscale model for the quasi-static thermo-plastic behavior of highly cross-linked glassy polymers. Macromolecules 48(18), 6713–6723 (2015). https://doi.org/10.1021/acs.macromol.5b01236

[26] Asp, L.E., Berglund, L.A.: A biaxial thermomechanical disk test for glassy polymers. Exp. Mech. 37(1), 96–101 (1997). https://doi.org/10.1007/BF02328755

[27] Arndt, K.-F., Lechner, M.D.: Polymer Solids and Polymer Melts—Mechanical and Thermomechanical Properties of Polymers. Springer, Berlin, Heidelberg (2014)

[28] O’khovik, O.E.: Apparatus for testing the strength of polymers in a three-dimensional stressed state. Polymer Mech. 19(2), 270–275 (1983). https://doi.org/10.1007/bf00604237
[29] Arteiro, A., Catalanotti, G., Melro, A.R., Linde, P., Camanho, P.P.: Micro-mechanical analysis of the effect of ply thickness on the transverse compressive strength of polymer composites. Compos. Part A: Appl. Sci. Manuf. 79, 127–137 (2015). https://doi.org/10.1016/j.compositesa.2015.09.015

[30] Bai, X., Bessa, M.A., Melro, A.R., Camanho, P.P., Guo, L., Liu, W.K.: High-fidelity micro-scale modeling of the thermo-visco-plastic behavior of carbon fiber polymer matrix composites. Compos. Struct. 134, 132–141 (2015). https://doi.org/10.1016/j.compstruct.2015.08.047

[31] Varandas, L.F., Catalanotti, G., Melro, A.R., Tavares, R.P., Falzon, B.G.: Micromechanical modelling of the longitudinal compressive and tensile failure of unidirectional composites: the effect of fibre misalignment introduced via a stochastic process. Int. J. Solids Struct. 203, 157–176 (2020). https://doi.org/10.1016/j.ijsolstr.2020.07.022

[32] Tavares, R.P., Melro, A.R., Bessa, M.A., Turon, A., Liu, W.K., Camanho, P.P.: Mechanics of hybrid polymer composites: analytical and computational study. Comput. Mech. 57(3), 405–421 (2016). https://doi.org/10.1007/s00466-015-1252-0

[33] Ghayoor, H., Marsden, C.C., Hoa, S.V., Melro, A.R.: Numerical analysis of resin-rich areas and their effects on failure initiation of composites. Compos. Part A: Appl. Sci. Manuf. 117, 125–133 (2019). https://doi.org/10.1016/j.compositesa.2018.11.016

[34] Mehdikhani, M., Petrov, N.A., Straumit, I., Melro, A.R., Lomov, S.V., Gorbatikh, L.: The effect of voids on matrix cracking in composite laminates as revealed by combined computations at the micro- and meso-scales. Compos. Part A: Appl. Sci. Manuf. 117, 180–192 (2019). https://doi.org/10.1016/j.compositesa.2018.11.009

How to cite this article: Roetsch, K., Horst, T.: A novel approach to consider triaxial tensile stresses within the framework of a failure criterion. Z. Angew. Math. Mech. 102, e202100232 (2022). https://doi.org/10.1002/zamm.202100232

APPENDIX A: LIST OF SYMBOLS AND INDICES (TABLE A1)

| \(a_n, b_n\) | Parameters for adapting the failure criteria |
| \(C\) | Uniaxial compressive strength |
| \(E\) | Young’s modulus |
| \(H\) | Hydrostatic tensile strength |
| \(I_1\) | First invariant of the stress tensor |
| \(J_2\) | Second invariant of the stress deviator |
| \(p\) | Pressure |
| \(s_{ij}\) | Stress deviator |
| \(S\) | Shear strength |
| \(T\) | Uniaxial tensile strength |
| \(U\) | Linear elastic strain energy density |
| \(\nu\) | Poisson’s ratio |
| \(\sigma_{ij}\) | Cauchy stress tensor |
| \(\sigma_{os}, \sigma_{on}\) | Octahaedral shear and normal stress |
| \(\Phi\) | Function which represents the failure of material |
| \(O_{ij}\) | Belonging to orthonormal base vector system |
| \(O_l\) | Belonging to a orthonomal base vector system with axes oriented to the principal directions |
| \(O^*, O^f\) | Belonging to yield or fracture behavior |
| \(\sigma^{dev}, \sigma^{vol}\) | Belonging to the deviatoric and volumetric portion of the stress tensor |
| \(\sigma^{calc}, \sigma^{exp}\) | Belonging to calculated and experimental values |