ON THE LOCAL DOUBLING $\gamma$-FACTOR FOR CLASSICAL GROUPS OVER FUNCTION FIELDS

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Abstract. In this paper, we give a precise definition of an analytic $\gamma$-factor of an irreducible representation of a classical group over a local function field of odd characteristic so that it satisfies some notable properties which are enough to define it uniquely. We use the doubling method to define the $\gamma$-factor, and the main theorem extends works of Lapid-Rallis, Gan, Yamana, and the author to a classical group over a local function field of odd characteristic.

CONTENTS

1. Introduction 1
2. Settings and notations 3
3. The doubling method 8
4. Statement of the main theorem 11
5. Proof of the main theorem 15
6. The $\gamma$-factor of the trivial representation for the minimal cases 16
References 22

1. Introduction

Let $F$ be a local field and let $G$ be a classical group (i.e. a symplectic group, a special orthogonal group, a unitary group, or a quaternionic unitary group) over $F$. In this paper, we apply the doubling method of Piatetski-Shapiro, Rallis (\cite{4,13,16}) to irreducible representations of $G(F) \times F^\times / (G(F) \times E^\times$ if $G$ is a unitary group equipped with a quadratic extension $E/F$) in the case $\text{ch}(F) \neq 0,2$, and give a precise definition of a $\gamma$-factor $\gamma^V(s, \pi \boxtimes \omega, \psi)$ for an irreducible representation of $G(F)$, a character $\omega$ of $F^\times$, and a nontrivial additive character $\psi$ of $F$. We note here that it has been established in the case $\text{ch}(F) = 0$ \cite{2,11,13,20}. Let $G^o$ denote the Zariski connected component of $G$. It is expected that

\begin{equation}
\gamma^V(s, \pi \boxtimes \omega, \psi) = \gamma(s, \text{std} \circ \phi_{\pi \boxtimes \omega}, \psi) \tag{1.1}
\end{equation}

where $\pi^o$ is an irreducible component of the restriction of $\pi$ to $G^o$ (see Remark 2.11 for more detail), $\phi_{\pi \boxtimes \omega}$ is the $L$-parameter of $\pi \boxtimes \omega$, std is the standard homomorphism of the $L$-group $L(G^o \times \text{GL}_1) / L(G^o \times \text{Res}_{E/F} \text{GL}_1)$ if $G$ is a unitary group) into $\text{GL}_N(\mathbb{C})$, and the right hand side is the $\gamma$-factor of \cite{7}. We check some properties of $\gamma^V(s, \pi \boxtimes \omega, \psi)$, which are based on \cite{1,11}. Moreover, we prove that some fundamental properties are enough to define it uniquely. This allows us to say that the definition is “precise”.

Let $F$ be a field, let $E$ be a division $F$-algebra with $[E : F] = 1,2,4$, and let $\epsilon \in \{\pm 1\}$. Then, a pair $V = (V, h)$ is said to be an $\epsilon$-Hermitian space if

- $V$ is a vector space over over $E$ where $E$ is a division $F$ algebra with $[E : F] = 1,2,4$ (in the later, we also consider split algebras (Sect. 2)),
\begin{itemize}
  \item $h$ is a map $V \times V \to E$ satisfying
    \[ h(y, x) = \epsilon \cdot h(x, y)^*, \quad h(x, ya + zb) = h(x, y)a + h(x, z)b \]
    for $x, y, z \in V$ and $a, b \in E$.
  \end{itemize}

We assume that either $h$ is zero or non-degenerate. Put $G(V)$ the algebraic group
\[ \{ g \in GL(V) | h(gx, gy) = h(x, y) \text{ for all } x, y \in V \} \]

We also write $G(V)$ for the group of its $F$-rational points $G(V)(F)$ if there is no confusion.

Now we explain our main result. We assume that $F$ is a local field of $\text{ch}(F) \neq 0, 2$. For simplicity, in the introduction, we exclude the odd orthogonal cases and unitary cases. For an irreducible representation $\pi$ of $G(V)$, a character $\omega$ of $F^\times$, and a non-trivial additive character $\psi$ of $F$, we define the $\gamma$-factor as in [11] [13]:
\[ \gamma^V(s + \frac{1}{2}, \pi \boxtimes \omega, \psi) = \Gamma^V(s, \pi, \omega, A, \psi) c_\pi(-1) R(s, \pi, A, \psi) \]

where
\begin{itemize}
  \item $A$ is some element of the Lie algebra of $G(V)^G$ (for definition, see Sect. 3);
  \item $\Gamma^V(s, \pi, \omega, A, \psi)$ is a “normalized $\Gamma$-factor”, which is defined in Sect. 3.3. This factor is obtained from a functional equation of doubling zeta integrals;
  \item $c_\pi$ is the central character of $\pi$;
  \item $R(s, \omega, A, \psi)$ is a correction term, which is defined in Sect. 4.1.
\end{itemize}

Since the analysis in the case of $\text{ch}(F) \neq 0$ has been discussed well (e.g. [18]), we do not need to worry about the existence of $\Gamma^V(s, \pi, \omega, A, \psi)$. Moreover, we have some properties of the $\gamma$-factor as in [13].

1. (multiplicativity): Let $W$ be a totally isotropic subspace of $V$, let $W_1 = (W^\perp / W, h)$. Then the stabilizer $P(W)$ of $W$ is a parabolic subgroup of $G(V)$ which has the Levi subgroup isomorphic to $\text{GL}(W) \times G(W_1)$. Then, for irreducible representations $\sigma_0, \sigma_1$ of $\text{GL}(W)$, $G(W_1)$, we have
\[ \gamma^V(s, \pi \boxtimes \omega, \psi) = \gamma_{\text{GL}(W)}^G(s, \sigma_0 \boxtimes \omega, \psi) \gamma_{\text{GL}(W)}^S(s, \sigma_0^\vee \boxtimes \omega, \psi) \gamma^V_{\text{GL}(W)}(s, \sigma_1 \boxtimes \omega, \psi) \]

if $\pi$ is isomorphic to a subquotient of $\text{Ind}_{W_1}^{GL(W)} \sigma_0 \boxtimes \sigma_1$. Here, $\sigma_0^\vee$ is the contragredient representation of $\sigma_0$. $\gamma_{\text{GL}(W)}^G(s, -, \psi)$ is the $\gamma$-factor of Godement-Jacquet (see Sect. 4.3).

2. (global functional equation): Let $F$ be a global field of $\text{ch}(F) \neq 0, 2$, let $E$ be a division $F$-algebra with $[E : F] = 1, 4$, and let $\Sigma$ be an $\epsilon$-hermitian space over $E$. We denote $\mathbb{A}$ by the ring of adeles of $F$. Thanks to the argument of Eisenstein series, for an irreducible automorphic cuspidal representation $\Pi$ of $G(1)(\mathbb{A})$, a Hecke character $\omega$ of $\mathbb{A}^\times / F^\times$, and a non-trivial additive character $\psi$ of $\mathbb{A}/F$, we have
\[ L_S(s, \text{std} \circ \phi_{\Pi, \omega, \psi}) = \prod_{v \in S} \gamma^V(s, \Pi \boxtimes \omega, \psi) \cdot L_S(1 - s, \text{std} \circ \phi_{\Pi, \omega, \psi}) \]

where $S$ is a finite set of places of $F$ containing all places where either $\Sigma, \Pi, \omega, \psi$ is ramified (Sect. 2.7), std is the standard homomorphism of $\text{GL}_N$ into $\text{GL}_N$ and $L_S(s, \text{std} \circ \phi_{\Pi, \omega, \psi})$ is the partial standard $L$-factor, which is the product of the local $L$-functions of [1] over all places not contained in $S$.

Put $\delta$ be the integer satisfying $\delta^2 = [E : Z(E)]$ where $Z(E)$ is the center of $E$. Our main theorem (Theorem 4.4) says that
• if $G$ is anisotropic, the semisimple rank $r(G)$ of $G$ is at most $2/\delta$, then we have (1.1) with $\pi = 1_V$ and $\omega = 1$ where $1_V$ denotes the trivial representation of $G(V)$, $1$ denotes the trivial character of $F_1^\times$, $\phi_0$ denotes the principal parameter of $G(V)$ (see Sect. 2.6).

• the $\gamma$-factor $\gamma^\gamma(s, \pi \boxtimes \omega, \psi)$ is characterized by (1), (2), and the equation (1.1) for unramified cases and for minimal cases (noted above), and some basic properties.

The latter is proved by combining (1), (2), and the globalization results of [9, 3]. The former is proved by some explicit computations (Sect. 6).

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2. Settings and notations

In this section, we fix some notations.

2.1. Fields. Let $F$ be a local field of characteristic $p \neq 0, 2$, and let $E$ be either $F$ itself, a 2 dimensional semisimple $F$-algebra, or a quaternion algebra over $F$. Although our interest is primarily when $E$ is division, we allow $E$ to split (i.e. $E = F \times F$ or $E = M_2(F)$) since they appear as a localization of global division algebras.

Then, we fix some notations associated to the fields:

- $\varpi$ denotes a uniformizer of $F$, $\mathcal{O}$ denotes the integer ring of $F$, and $q$ denotes the cardinality of $\mathcal{O}/\varpi \mathcal{O}$,

- for $a \in F^\times$, $\chi_a$ denotes the character of $F^\times$ defined by $\chi_a(x) = (a, x)_F$ for $x \in F^\times$ where $(\ , )_F$ is the Hilbert symbol,

- when $[E : F] = 2$, $\chi_E$ denotes a quadratic character of $F^\times$ associated to $E/F$,

- $\text{ord}_F : F^\times \to \mathbb{Z}$ denotes the order of $F$,

- $Z(E)$ denotes the center of $E$,

- $*: E \to E : x \mapsto x^*$ denotes the main involution over $F$.

- We also denote $*: F \times F \to F \times F : (a, b) \mapsto (b, a)$.

- if $E$ is a division algebra, $\text{ord}_E : E^\times \to \mathbb{Z}$ denotes the order of $E$,

- $T_{E/F} : E \to F$ denotes

$$\begin{cases}
\text{the trace} & \text{if } [E : F] \leq 2, \\
\text{the reduced trace} & \text{if } [E : F] = 4,
\end{cases}$$

- $N_{E/F} : E^\times \to F^\times$ denotes

$$\begin{cases}
\text{the norm} & \text{if } [E : F] \leq 2, \\
\text{the reduced norm} & \text{if } [E : F] = 4,
\end{cases}$$

- $E_0 := \{ x \in E \mid T_{E/F}(x) = 0, E^1 := \{ x \in E^\times \mid N_{E/F}(x) = 1 \},$

- $|\cdot|_F$ and $|\cdot|_E$ denotes the normalized absolute values: i.e.

$$|\varpi|_F = q^{-1}, \ |\cdot|_E := |N_{E/F}(\cdot)|_F.$$
We will refer to the cases in which $h$ and we denote by $G$ and any Levi subgroup of $G$. We denote by $\eta$ where $G$.

Then, we define $\eta$ such that $h(y, x) = e(h(x, y)^* \times \eta)$ for $x, y \in V$.

2.2. Groups. Let $V = (V, h)$ where $V$ is a free right $E$ module of rank $n$ and $h$ is a map $V \times V \to E$ such that

- $h$ is either 0 or non-degenerate,
- $h(x, ya + zb) = h(x, y)a + h(x, z)b$ for $x, y \in V$ and $a, b \in E$,
- there is an $\epsilon = \pm 1$ such that $h(y, x) = e(h(x, y)^*)$ for $x, y \in V$.

We denote by $G(V)$ the algebraic group

$$\{ g \in \text{GL}(V) \mid h(gx, gy) = h(x, y) \text{ for all } x, y \in V \},$$

and we denote by $G(V)^\circ$ the Zariski connected component of $G(V)$. We also write $G(V)$ (resp. $G(V)^\circ$) for the group of its $F$-rational points $G(V)(F)$ (resp. $G(V)^\circ(F)$) if there is no confusion. We will refer to the cases in which

$$\begin{align*}
E &= F \text{ and } h = 0 \quad \text{as (GL)}, \\
E &= F, \ h \text{ is non-degenerate and } \epsilon = 1 \quad \text{as (O)}, \\
E &= F, \ h \text{ is non-degenerate and } \epsilon = -1 \quad \text{as (Sp)}, \\
[E : F] &= 2 \text{ and } h = 0 \quad \text{as (qGL)}, \\
[E : F] &= 2, \ h \text{ is non-degenerate and } \epsilon = 1 \quad \text{as (U)}, \\
[E : F] &= 4 \text{ and } h = 0 \quad \text{as (QGL)}, \\
[E : F] &= 4, \ h \text{ is non-degenerate and } \epsilon = 1 \quad \text{as (Q)}, \ \\
[E : F] &= 4, \ h \text{ is non-degenerate and } \epsilon = -1 \quad \text{as (Q-1)}.
\end{align*}$$

For a totally isotropic subspace (if $h = 0$, this means just a subspace) $W$ of $V$, we denote by $P_{G(V)}(W)$ the parabolic subgroup stabilizing $W$. We write it $P(W)$ if there is no confusion. We denote by $U(W)$ the unipotent radical of $P(W)$. We denote $W_0 = (W, 0)$ and $W_1 = (W^\perp/W, h)$, where

$$W^\perp = \{ v \in V \mid h(v, x) = 0 \text{ for all } x \in W \}.$$ 

Then there is the exact sequence

$$1 \to U(W) \to P(W) \to \text{GL}(W) \times G(W_1) \to 1$$

and any Levi subgroup of $P(W)$ is isomorphic to $\text{GL}(W) \times G(W_1)$.

Since we use the globalization, we must consider the split cases: i.e. the cases $E = F \times F$ or $M_2(F)$. Put

$$e := \begin{cases} 
(1, 0) & \text{in the case } E = F \times F, \\
(1, 0) & \text{in the case } E = M_2(F).
\end{cases}$$

Then, we define $V^\perp = (V^\perp, h^\perp)$ consisting of the vector space $V^\perp = Ve$ over $F$ and a map $h^\perp : V^\perp \times V^\perp \to F$ so that

$$\begin{align*}
h^\perp &= 0 \quad \text{in the case (U)}, \\
h(xe, ye) &= \begin{pmatrix} 0 & 0 \\
h^\perp(xe, ye) & 0 \end{pmatrix} (x, y \in V) \quad \text{in the cases (QGL),(Q),(Q-1).}
\end{align*}$$
Moreover in the case (qGL), we define \( Y_{\mathfrak{a}}' = (V_{\mathfrak{a}}', h_{\mathfrak{a}}') \) consisting of the vector space \( V_{\mathfrak{a}}' = V e^* \) over \( F \) and the zero-map \( h_{\mathfrak{a}}' : V_{\mathfrak{a}}' \times V_{\mathfrak{a}}' \to F \). Then, we have isomorphisms
\[
\begin{align*}
\iota : G(V) &\to G(V^\mathfrak{a}) \\
\iota : G(V) &\to G(V^\mathfrak{a}) : g \mapsto (g|_{V^2}, g|_{V^2}) \quad \text{in the case (qGL)}, \\
\iota : G(V) &\to G(V^\mathfrak{a}) : g \mapsto g|_{V^2} \quad \text{in the other cases}.
\end{align*}
\]

We fix some notations associated to \( V \):
- \( \mathfrak{d}(V) \) denotes the discriminant of \( V \),
- \( F_V \) denotes the field \( F(\sqrt{\mathfrak{d}}) \) where \( d \in F^\times \) such that \( \mathfrak{d}(V) \) agree with the image of \( d \),
- in the case (O), \( c(V) \) denotes the Hasse invariant of \( V \),
- in the case (U), \( \epsilon(V) \) denotes the symbol defined by \( \chi_E(\mathfrak{d}(V)) \).

2.3. **Explicit notations.** Let \( R \in \text{GL}_n(E) \) so that \( ^t R^* = \epsilon R \). We define the map
\[
\langle R \rangle : E^n \times E^n \to E : (\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle) \mapsto (x_1, \ldots, x_n)^* \cdot R \cdot (y_1, \ldots, y_n).
\]
Then, the pair \((E^n, \langle R \rangle)\) satisfies the condition of Sect. 2.1 and the group \( G((E^n, \langle R \rangle)) \) is realized as the subgroup
\[\{g \in \text{GL}_n(E) | ^t g^* \cdot R \cdot g = \epsilon R\}\]
of \( \text{GL}_n(E) \) where \( N : \text{GL}_n(E) \to Z(E)^\times \) is the reduced norm over \( Z(E) \). Note that
\[
\mathfrak{d}((E^n, \langle R \rangle)) = N(R) \times \begin{cases} (-1)^{(n-1)}2^{-n} & \text{in the cases (O), (Sp),} \\ (-1)^{(n-1)} & \text{in the case (U),} \\ (-1)^n & \text{in the cases (Q_1), (Q_{-1}).} \end{cases}
\]

2.4. **Representations.** In this paper, by “a representation of a reductive group” we mean a complex smooth representation of the group. Let \( \pi \) be a representation of \( G(V) \). Note that we often identify the representation space of \( \pi \) with \( \pi \) itself. We denote by \( \pi^\ast \) the contragredient representation. Moreover, in the cases (GL), (QGL), (QGL) we denote by \( \pi^\star \) the contragredient representation of
\[
\begin{cases}
\pi & \text{in the cases (GL), (QGL),} \\
\pi^\ast & \text{in the case (qGL)}
\end{cases}
\]

where \( \pi^\ast(g) := \pi(g^*) \) for \( g \in \text{GL}_n(E) \).

For a parabolic subgroup \( P \) of \( G(V) \) and a representation \( \sigma \) of the Levi subgroup of \( P \), we denote by \( \text{Ind}_P^G(V)\sigma \) the normalized induced representation: i.e. \( \text{Ind}_P^G(V)\sigma \) is the representation by the right translation on the space of smooth functions \( f : G(V) \to \sigma \) satisfying
\[
f(pg) = \delta_P(p)^{\frac{1}{2}}\sigma(p)(f(g))
\]
for \( p \in P, g \in G(V) \) where \( \delta_P \) is the modular function of \( P \).

Now suppose that \( E \) is split. In the cases (U), (Q_1), (Q_{-1}), (QGL), for a representation \( \pi \) of \( G(V) \), we denote by \( \pi^\mathfrak{a} \) the representation of \( G(V^\mathfrak{a}) \) satisfying \( \pi^\mathfrak{a} = \pi^\ast \circ \iota \). In the case (qGL), for a representation \( \pi \) of \( G(V) \), we denote by \( \pi^\mathfrak{a} \) the representation of \( G(V^\mathfrak{a}) \) and by \( \pi^\mathfrak{a}' \) the representation of \( G(V^\mathfrak{a}) \) satisfying \( (\pi^\mathfrak{a} \boxtimes \pi^\mathfrak{a}') \circ \iota \equiv \pi \).

2.5. **The standard local factors.** In this subsection, we explain the standard homomorphism of the complex \( L \)-dual group of
\[
\begin{cases}
G(V)^\circ \times \text{GL}_4 & \text{in the cases (O), (Sp), (Q_1), (Q_{-1}),} \\
G(V) \times \text{Res}_{E/F}^E(\text{GL}_4) & \text{in the cases (U), (qGL),} \\
G(V) \times \text{GL}_1 \times \text{GL}_4 & \text{in the cases (GL), (QGL),}
\end{cases}
\]
and we explain the standard local factors of $L$-parameters of them. Note that in the case $(GL)$, the necessity of twisting by $GL_1 \times GL_1$ is caused when one consider local components of a globalization of $G(V \times Res_{E/F}(GL_1)$ in the case $(U)$.

Put

$$N = \begin{cases} 2n & \text{in the case (GL), (U), (Q_{-1}),} \\ 2\lfloor \frac{n}{2} \rfloor & \text{in the case (O),} \\ n + 1 & \text{in the case (Sp),} \\ 4n & \text{in the case (qGL), (QGL),} \\ 2n + 1 & \text{in the case (Q_1).} \end{cases}$$

In this paper, by standard homomorphisms we mean the homomorphisms defined as follows:

- In the cases $(GL), (QGL)$, we have $l(G(V) \times GL_1) = (GL_n(C) \times C^x \times C^x) \times W_F$, and we define std by

$$\text{std} : (GL_n(C) \times C^x \times C^x) \times W_F \rightarrow GL_N(C) : (g, z_1, z_2, w) \mapsto \begin{pmatrix} z_1 g & 0 \\ 0 & z_2 \cdot g^{-1} \end{pmatrix}.$$

- In the cases $(O, n: even), (Sp), (Q_1), (Q_{-1})$, we have $l(G(V)^{\circ} \times GL_1) = (O_N(C) \times C^x) \times W_F$, and we define std by

$$\text{std} : (O_N(C) \times C^x) \times W_F \rightarrow GL_N(C) : (g, z, w) \mapsto zg \cdot w_N$$

where

$$w_N = \begin{cases} I_N & (w \in W_{Fv}) \\ \begin{pmatrix} I_{N/2-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{N/2-1} \end{pmatrix} & (w \in W_F \setminus W_{Fv}). \end{cases}$$

- In the case $(O, n: odd)$, we have $l(G(V)^{\circ} \times GL_1) = (Sp_N(C) \times C^x) \times W_F$, and we define std by

$$\text{std} : (Sp_N(C) \times C^x) \times W_F \rightarrow GL_N(C) : (g, z, w) \mapsto zg$$

- In the case $(U)$, we have $l(G(V) \times Res_{E/F} GL_1) = (GL_n(C) \times C^x \times C^x) \times W_F$, and we define std by

$$\text{std} : (GL_n(C) \times C^x \times C^x) \times W_F \rightarrow GL_N(C) : (g, z_1, z_2, w) \mapsto \begin{pmatrix} z_1 g & 0 \\ 0 & z_2J \cdot g^{-1} \cdot J^{-1} \end{pmatrix} \tilde{w}.$$

where

$$J = \begin{pmatrix} (-1) \ldots 1 \\ (-1)^{n-1} \ldots \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} I_N \\ 0 \\ -I_{N/2} \\ 0 \end{pmatrix} (w \in W_E)$$

$$\text{std} : (GL_n(C) \times GL_n(C) \times C^x \times C^x) \times W_F \rightarrow GL_N(C) : (g_1, g_2, z_1, z_2, w) \mapsto \begin{pmatrix} g_1 z_1 & g_2 z_1 \\ g_2^{-1} z_2 & g_1^{-1} z_2 \end{pmatrix} \tilde{w}.$$
Let $\phi$ be an $L$-parameter of the group $G(V)^\circ$, and let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character. Then we define the **standard $\gamma$-factor** of $\phi$ by

$$\gamma_F(s, \text{std} \circ \phi, \psi)$$

where $\gamma_F(s, -, \psi)$ is the $\gamma$-factor in the sense of [I]. Define the standard $L$- and $\epsilon$-factors in the same way.

**Remark 2.1.** We note here the local Langlands correspondence for classical groups (concluding orthogonal groups). Let $\pi$ be an irreducible representation of $G(V)$ and let $\omega$ be a character of $F^\times$. We denote by $\pi^\sigma$ an irreducible component of the restriction of $\pi$ to $G(V)^\circ$. Consider the case (O). Then the restriction of $\pi$ to $G(V)^\circ$ may become reducible. If $\pi_1$ and $\pi_2$ are irreducible components of $\pi|_{G(V)^\circ}$, considering the analogue of the argument in [I] §§3.4-3.6, we can expect that

$$\text{std} \circ \phi_{\pi_1, \omega} \cong \text{std} \circ \phi_{\pi_2, \omega}$$

as representations of $W_F \times \text{SL}_2(\mathbb{C})$. Here we denotes by $\phi_{\pi, \omega}$ the $L$-parameter of $\pi$, $\boxtimes \omega$ for $i = 1, 2$.

**Remark 2.2.** We allow $n$ to be zero. In the case $N = 0$, we set $\gamma_F(s, \text{std} \circ \phi, \psi) = 1$ for all $L$-parameter $\phi$.

### 2.6. The principal parameter

In this subsection, we explain the principal parameter, which is expected to be associated with the Steinberg representation via the Langlands correspondence.

Let $G$ be a connected reductive group over $F$, let $T$ be a maximal torus of $\hat{G}$, and let $R(\hat{G}, T)$ be the root system of $\hat{G}$ with respect to $T$, let $\Delta$ be a positive system of $R(\hat{G}, T)$, then $\Delta^1$ be a set of simple roots of $\Delta$, and for each $\alpha^\vee \in \Delta^1$, and let $X_{\alpha^\vee}$ be a fixed non-zero element of Lie($\hat{G}$) such that $\text{ad}(h)X_{\alpha^\vee} = \alpha^\vee(h)X_{\alpha^\vee}$ for $h \in \text{Lie}(T)$. Put

$$N_0 := \sum_{\alpha^\vee \in \Delta^1} X_{\alpha^\vee} \in \text{Lie}(\hat{G}) = \text{Lie}(\hat{G}).$$

Then, we define the **principal parameter** as the $L$-homomorphism $\phi_0 : \text{SL}_2(\mathbb{C}) \times W_F \to \mathbb{L}G$ such that

$$d\phi_0\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = N_0 \in \text{Lie}(\mathbb{L}G), \quad \phi_0((I_2, w)) = (1, w) \ (w \in W_F).$$

One can show that the image of $\phi_0$ is not contained in any proper parabolic subgroup of $\mathbb{L}G$. Thus $\phi_0$ is an $L$-parameter of $G$.

### 2.7. Global notations

Let $F$ be a global field of $\text{ch}(F) \neq 0, 2$, let $E$ be a division $F$-algebra of $[E : F] = 1, 2, 4$, and let $V = (V, h)$ where $V$ is an $n$-dimensional right $E$ vector space and $h$ is a map $V \times V \to E$ such that

- $h$ is either 0 or non-degenerate,
- $h(x, ya + zb) = h(x, y)a + h(x, z)b$ for $x, y, z \in V$ and $a, b \in E$,
- there is an $\epsilon = \pm 1$ such that $h(y, x) = \epsilon h(x, y)^* \ (x, y \in V)$.

We say that $V$ is unramified at a place $v$ of $F$ when $E_v/F$ is unramified at $v$ and

- $V_v$ is isometric to the space

\[
\begin{cases} 
(\mathbb{F}_v^n, (0)) & \text{in the case (GL)}, \\
(\mathbb{E}_v^n, \text{diag}(1, -1, \cdots, (-1)^{n-1})) & \text{in the case (O), (U)}, \\
(\mathbb{F}_v^n, \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}) & \text{in the case (Sp)}
\end{cases}
\]

if $E_v$ is a field,
• \( V^\circ_k \) is unramified over \( \mathbb{F}_v \) if \( \mathbb{E} \neq \mathbb{F} \) and \( \mathbb{E}_v \) is split over \( \mathbb{F}_v \).

We denote by \( \mathbb{A} \) the ring of adeles of \( \mathbb{F} \), and we denote by \( G(\mathbb{V})(\mathbb{A}) \) the restricted product \( \prod_v G(\mathbb{V}_v) \) with respect to the canonical open compact subgroups \( K_v \) for unramified places \( v \). We say that an irreducible automorphic representation \( \Pi \) of \( G(\mathbb{V})(\mathbb{A}) \) is unramified at \( v \) if \( \mathbb{V} \) is unramified at \( v \) and \( \Pi_v \) has a non-zero \( K_v \)-fixed vector. For a finite extension field \( \mathbb{F}' \) over \( \mathbb{F} \), we say that a Hecke character \( \omega \) of \( (\mathbb{A} \otimes \mathbb{F}')^\times /\mathbb{F}^\times \) is unramified at \( v \) if \( \mathbb{F}'/\mathbb{F} \) is unramified at \( v \) and \( \omega_v = |v|^s \) for some \( s_0 \in \mathbb{C} \). We say that a non-trivial additive character \( \psi \) of \( \mathbb{A}/\mathbb{F} \) is unramified at \( v \) if the order of \( \psi_v \) is zero.

3. The doubling method

3.1. Doubled spaces and unitary groups. Let \( V^\circ = (V^\circ, h^\circ) \) be a pair, where \( V^\circ = V \times V \) and \( h^\circ = h \oplus (-h) \), that is, the map defined by

\[
h^\circ((x_1, x_2), (y_1, y_2)) = h(x_1, y_1) - h(x_2, y_2)
\]

for \( x_1, x_2, y_1, y_2 \in V \). Then \( G(\mathbb{V}) \times G(\mathbb{V}) \) acts on \( V \times V \) by

\[
(g_1, g_2) \cdot (x_1, x_2) = (g_1 x_1, g_2 x_2),
\]

so that \( G(\mathbb{V}) \times G(\mathbb{V}) \) can be embedded naturally in \( G(V^\circ) \). Consider the maximal totally isotropic subspaces

\[
V^\Delta = \{(x, x) \in V^\circ \mid x \in V\},
\]

\[
V^{\nabla} = \{(x, -x) \in V^\circ \mid x \in V\}.
\]

Note that \( V = V^\Delta \oplus V^{\nabla} \). Then \( P(V^\Delta) \) is a maximal parabolic subgroup of \( G(V^\circ) \) and its Levi subgroup is isomorphic to

\[
\begin{cases}
\text{GL}(V^\Delta) \times \text{GL}(V^\circ/V^\Delta) & \text{in the cases (GL), (qGL), (QGL)} \\
\text{GL}(V^\circ/V^\Delta) & \text{in the other cases.}
\end{cases}
\]

3.2. Zeta integrals and intertwining operators. In this subsection, we assume that \( n \neq 0 \). Denote by \( \Lambda_1 : P(V^\Delta) \to Z(E^\times) \) and \( \Lambda_2 : P(V^\Delta) \to Z(E^\times) \) the characters given by

\[
\Lambda_1(g) = N_{V^\Delta}(g), \quad \Lambda_2(g) = N_{V^{\nabla}/V^\Delta}(g)
\]

Here, \( N_{V^\Delta}(g) \) (resp. \( N_{V^{\nabla}/V^\Delta}(g) \)) is the reduced norm of the image of \( g \) in \( \text{End}_E V^\Delta \) (resp. \( \text{End}_E V^{\nabla}/V^\Delta \) over \( Z(E) \)). Moreover, we define

\[
\Delta_{V^\Delta} = \begin{cases}
(\Lambda_1, \Lambda_2^{-1}) : P(V^\Delta) \to F^\times \times F^\times & \text{in the cases (GL, QGL)}, \\
(\Lambda_1(\Lambda_2)^{-1}) : P(V^\Delta) \to E^\times & \text{in the case (qGL),} \\
\Lambda_1 : P(V^\Delta) \to Z(E^\times) & \text{in the other cases.}
\end{cases}
\]

Let \( \omega \) be a character of

\[
\begin{cases}
F^\times \times F^\times & \text{in the cases (GL), (QGL),} \\
Z(E^\times) & \text{in the other cases.}
\end{cases}
\]

We denote by \( \omega^* \) the character defined by the composition of \( \omega \) and \( * \). For \( s \in \mathbb{C} \), put \( \omega_s = \omega \cdot |.|^s \). Choose a compact subgroup \( K \) of \( G(V^\circ) \) such that \( G(V^\circ) = P(V^\Delta)K \). Denote by \( I(s, \omega) \) the degenerate principal series representation

\[
\text{Ind}_{P(V^\Delta)}^{G(V^\circ)}(\omega_s \circ \Delta_{V^\Delta})
\]

consisting of smooth right \( K \)-finite functions \( f : G(V^\circ) \to \mathbb{C} \) satisfying

\[
f(pg) = \delta_{P(V^\Delta)}^\omega(p, \omega_s(\Delta_{V^\Delta}(p))) \cdot f(g)
\]
for \( p \in P(V^\triangle) \) and \( g \in G(V^\square) \), where \( \delta_{P(V^\triangle)} \) is the modular function of \( P(V^\triangle) \). We may extend \( |\Delta_{V^\triangle}| \) to a right \( K \)-invariant function on \( G(V^\square) \) uniquely. For \( f \in I(0, \omega) \), put \( f_s = f \cdot |\Delta_{V^\triangle}|^s \in I(s, \omega) \). Let \( A_1 \) be the center of \( P(V^\triangle) \). Define an intertwining operator \( M(s, \omega) : I(s, \omega) \to I(-s, (\omega^*)^{-1}) \) by

\[
(M(s, \omega)f_s)(g) = \int_{U(V^\triangle)} f_s(w_1 u g) \, du
\]

where \( w_1 \in G(V^\square) \) is a representative of the longest element of the Weyl group \( W(A_1, G(V^\square)) \) with respect to \( A_1 \). This integral defining \( M(s, \omega) \) converges absolutely for \( \Re s > 0 \) and admits a meromorphic continuation to \( \mathbb{C} \) (e.g. [18, Theorem IV.1.1]).

Let \( \pi \) be an irreducible representation of \( G(V) \). For a matrix coefficient \( \xi \) of \( \pi \), and for \( f \in I(\omega, 0) \), define the zeta integral by

\[
Z^V(f_s, \xi) = \int_{G(V)} f_s((g, 1))\xi(g) \, dg.
\]

Then the zeta integral satisfies the following basic properties as in the characteristic zero case.

**Theorem 3.1.**

1. The integral \( Z^V(f_s, \xi) \) converges absolutely for \( \Re s > 0 \) and extends to a meromorphic function in \( s \). Moreover, the function \( Z^V(f_s, \xi) \) is a rational function of \( q^{-s} \).
2. There is a meromorphic function \( \Gamma^V(s, \pi, \omega) \) such that

\[
Z^V(M(s, \omega)f_s, \xi) = \Gamma^V(s, \pi, \omega)Z^V(f_s, \xi)
\]

for all matrix coefficient \( \xi \) of \( \pi \) and \( f_s \in I(s, \omega) \).

**Proof.** The boundedness of the function \( |\Delta_{V^\triangle}| \) on \( U(V^\vee) \) in our case can be seen in [18, Lemma II.3.4]. Then the proof of [13, Theorem 3] still works. \( \square \)

### 3.3. Normalization of the intertwining operators

In this subsection, we assume that \( n \neq 0 \) unless stated otherwise. Note that \( E \) is possibly split. We denote by \( u(V^\triangle) \) the Lie algebra of \( U(V^\triangle) \). We regard \( u(V^\triangle) \) as a subspace of \( \text{End}_E(V^\square) \) and for \( r = 0, \ldots, n \) we denote by \( u(V^\triangle)_r \) the set of \( A \in u(V) \) of rank \( r \). We exclude the case \((O, n : \text{odd})\) for a while so that \( u(V^\triangle)_n \neq \emptyset \).

Fix a non-trivial additive character \( \psi : F \to \mathbb{C}^\times \) and \( A \in u(V^\triangle)_n \). We define

\[
\psi_A : U(V^\vee) \to \mathbb{C}^\times : u \mapsto \psi(\text{Tr}_{Z(E)/F} \circ \text{Tr}_{V^\square}(uA))
\]

where \( \text{Tr}_{V^\square} \) denotes the reduced trace of \( \text{End}_E(V^\square) \) over \( Z(E) \). For \( f \in I(\omega, 0) \) we define

\[
l_{\psi_A}(f_s) = \int_{U(V^\vee)} f_s(u)\psi_A(u) \, du.
\]

Then the integral defining \( l_{\psi_A} \) converges for \( \Re s > 0 \) and admits a holomorphic continuation to \( \mathbb{C} \) ([12, §3.3]).

The functional \( l_{\psi_A} \) is called a degenerate Whittaker functional, which is a \( (U(V^\vee), \psi_A) \)-equivariant functional on \( I(s, \omega) \). On the other hand, the space of \( (U(V^\vee), \psi_A) \)-equivariant functionals on \( I(s, \omega) \) is one dimensional for all \( s \in \mathbb{C} \) ([12, Theorem 3.2]). Hence we have the following proposition.

**Proposition 3.2.** There is a meromorphic function \( c(s, \omega, A, \psi) \) of \( s \) such that

\[
l_{\psi_A} \circ M(s, \omega) = c(s, \omega, A, \psi)l_{\psi_A}.
\]

Then we define the normalized intertwining operator

\[
M^*(s, \omega, A, \psi) = c(s, \omega, A, \psi)^{-1} M(s, \omega)
\]

and put

\[
\Gamma^V(s, \pi, \omega, A, \psi) = c(s, \omega, A, \psi)^{-1} \Gamma^V(s, \pi, \omega).
\]
Clearly, $\Gamma^V(s, \pi, \omega, A, \psi)$ is a meromorphic function of $s$ satisfying
\begin{equation}
Z(M^*(s, \omega, A, \psi)f_s, \xi) = \Gamma^V(s, \pi, \omega, A, \psi)Z(f_s, \xi)
\end{equation}
for any $f \in I(\omega, 0)$ and any coefficient $\xi$ of $\pi$.

**Remark 3.3.** In the case $n = 0$, we set $\Gamma^V(s, \pi, \omega, A, \psi) = 1$.

**Remark 3.4.** Thanks to the normalization, the function $\Gamma^V(s, \pi, \omega, A, \psi)$ does not depend on the choices of $K, \omega_1$, and measures on $G(V), U(V^\triangle), U(V^\triangledown)$.

### 3.4. Odd orthogonal cases

In the case $(O, n : \text{odd})$, the set $u(V^\triangle)_n$ is empty. However, Lapid-Rallis found an appropriate normalization of the intertwining operator $M(s, \omega)$, which yields the $\gamma$-factor in this case [13 §6]. We follow their construction.

Fix a nontrivial additive character $\psi : F \rightarrow \mathbb{C}^*$ and $A \in u(V^\triangle)_{n-1}$. We define $\psi_A$ and $l_{\psi_A}$ as Sect. 3.3. Moreover, we define another Whittaker functional $l'_{\psi_A, L}$ as follows. We denote by $p^\triangledown$ the projection of $V^\triangle$ onto $V^\triangledown$ along the decomposition $V^\triangle = V^\downarrow \oplus V^\triangledown$. Let $K = \ker A \cap V^\triangledown$, let $L$ be an anisotropic line of $V^\triangle$ such that $p^\triangledown(L) = K$, and let $\sigma_L \in G(V^\triangle)$ be the orthogonal reflection around $L$. We can identify $U(V^\triangledown) \cap \sigma_L P(V^\triangle)\sigma_L$ with $(V^\downarrow \cap K^\perp) \otimes K$ as in [13 Lemma 11]. We define a bilinear form

\[ h_{A,L} : (V^\downarrow \cap K^\perp) \otimes K \times (V^\downarrow \cap K^\perp) \otimes K \rightarrow F \]

by
\[ h_{A,L}(x \otimes a, y \otimes b) = h^\triangle(x', y)h^\triangledown(\sigma_L(a), b) \]

for $x, y \in V^\downarrow \cap K^\perp$ and $a, b \in K$, where $x' \in V^\triangle$ with $Ax' = x$. Then we fix the self dual Haar measure $d''u$ on $U(V^\triangledown) \cap \sigma_L P(V^\triangle)\sigma_L$ with respect to $h_{A,L}$. Now, we define the functional $l'_{\psi_A, L}$ by
\[ l'_{\psi_A, L}(f_s) := \int_{U(V^\triangledown) \cap \sigma_L P(V^\triangle)\sigma_L \backslash U(V^\triangledown)} f_s(u)\psi_A(u) d''u \]

where $d''u$ is the Haar measure on $U(V^\triangledown) \cap \sigma_L P(V^\triangle)\sigma_L \backslash U(V^\triangledown)$ which is compatible with $du$ on $U(V^\triangledown)$ and $d''u$ on $U(V^\triangledown) \cap \sigma_L P(V^\triangle)\sigma_L$. Then the integral defining $l'_{\psi_A, L}$ converges for $\Re s > 0$ and admits a holomorphic continuation to $\mathbb{C}$.

**Proposition 3.5.** There is a meromorphic function $c(s, \omega, A, L, \psi)$ of $s$ such that
\[ l'_{\psi_A, L} \circ M(s, \omega) = c(s, \omega, A, L, \psi)l_{\psi_A} \]

Then we define the normalized intertwining operator
\[ M^*(s, \omega, A, L, \psi) = c(s, \omega, A, L, \psi)^{-1}M(s, \omega) \]

and put
\[ \Gamma^V(s, \pi, \omega, A, L, \psi) = c(s, \omega, A, L, \psi)^{-1}\Gamma^V(s, \omega) \]

Clearly, $\Gamma^V(s, \pi, \omega, A, L, \psi)$ is a meromorphic function of $s$ satisfying
\begin{equation}
Z(M^*(s, \omega, A, L, \psi)f_s, \xi) = \Gamma^V(s, \pi, \omega, A, L, \psi)Z(f_s, \xi)
\end{equation}
for any $f \in I(\omega, 0)$ and any coefficient $\xi$ of $\pi$.

**Remark 3.6.** As in the other cases, the function $\Gamma^V(s, \pi, \omega, A, \psi)$ does not depend on the choices of $K, \omega_1$, and measures on $G(V), U(V^\triangle), U(V^\triangledown)$. 
4. Statement of the main theorem

4.1. Definition of the $\gamma$-factor. In the cases other than $(O, n : odd)$, we can define the invariants $N_V(A)$ and $\mathfrak{d}(A)$ for $A \in \mathfrak{u}(V^\Delta)_n$ as follows. The nilpotent element $A$ defines an isomorphism from $V^\triangledown$ to $V^\Delta$. Consider an endomorphism $\phi_A \in \text{End}_E(V)$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
V^\triangledown & \xrightarrow{A} & V^\Delta \\
p \downarrow & & \downarrow p \\
V & \xrightarrow{\phi_A} & V
\end{array}
$$

where $p$ is the first projection of $V \times V$ to $V$. Then we define

$$N_V(A) := N_V(\phi_A), \quad \mathfrak{d}(A) := (-1)^n N_V(A) \in F^\times/F^\times 2.$$

In the case $n = 0$, we set $\mathfrak{d}(A) = N_V(A) = 1$.

In the case $(O, n : odd)$, we define the invariant $N_V(\tilde{A}_L)$ for $A \in \mathfrak{u}(V^\Delta)_n$ and for $L \subset V^\triangledown$ of Sect. 3.4 as follows. We have the decomposition

$$V^\triangledown = K \oplus (V^\triangledown \cap K^\perp)$$

where $K' = \sigma_L K$. We can choose an isomorphism $\tilde{A}_L : V^\triangledown \rightarrow V^\Delta$ so that

- $\tilde{A}_L|_K = \sigma_L K$,
- $\tilde{A}_L|_{K^\perp \cap V^\triangledown} = A|_{K^\perp \cap V^\triangledown}$.

Consider an endomorphism $\phi_{\tilde{A}_L} \in \text{End}_E(V)$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
V^\triangledown & \xrightarrow{\tilde{A}_L} & V^\Delta \\
p \downarrow & & \downarrow p \\
V & \xrightarrow{\phi_{\tilde{A}_L}} & V
\end{array}
$$

where $p$ is the first projection of $V \times V$ to $V$. Then we can define

$$N_V(\tilde{A}_L) := N_V(\phi_{\tilde{A}_L}).$$

Definition 4.1. Let $\pi$ be an irreducible representation of $G(V)$, let $\omega$ be a character of

$$\begin{cases}
F^\times \times F^\times & \text{in the cases of } (GL), (QGL), \\
Z(E^\times) & \text{in the other cases},
\end{cases}$$

and let $\psi$ be a non-trivial character of $F$.

(1) In the cases other than $(O, n : odd)$, we define the $\gamma$-factor of $\pi$ by

$$\gamma^V(s + \frac{1}{2}, \pi \boxtimes \omega, \psi) = \Gamma^V(s, \pi, \omega, A, \psi) \cdot c_\pi(-1) \cdot R(s, \omega, A, \psi)$$

where $\Gamma^V(s, \pi, \omega, A, \psi)$ is the meromorphic function defined by (3.1), $c_\pi$ is the central character of $\pi$, and

$$R(s, \omega, A, \psi) = \begin{cases}
\omega_{\pi}(N_V(\frac{1}{2} A), N_V(-\frac{1}{2} A))^{-1} & \text{in the cases of } (GL), (QGL), \\
\omega_{\pi}(N_V(\frac{1}{2} A))^{-1} \omega_{\pi}(N_V(-\frac{1}{2} A)^*)^{-1} & \text{in the case of } (qGL), \\
\omega_{\pi}(N_V(\omega(\chi_{\mathfrak{d}}(A), \psi)))^{-1} & \text{in the cases of } (Sp), (Q_1), \\
\omega_{\pi}(N_V(\omega(\chi_{\mathfrak{d}}(V), \psi)))^{-1} & \text{in the cases of } (O, n : even), (Q_{-1}), \\
\omega_{\pi}(N_V(\omega(V)))^{-1} & \text{in the case of } (U).
\end{cases}$$
(2) In the case $(O, n : odd)$, we define the $\gamma$-factor of $\pi$ by

$$\gamma^V(s + \frac{1}{2}, \pi \boxtimes \omega, \psi) = \Gamma^V(s, \pi, \omega, A, L, \psi) \cdot \omega_s(N_\psi(A_L))^{-1} \cdot c(V).$$

where $\Gamma^V(s, \pi, \omega, A, L, \psi)$ is the meromorphic function defined by (3.2) and $c(V)$ is the Hasse invariant attached to $V$ (cf. §5.3).

**Remark 4.2.** This agrees with the analogue of the corrected definition of $\gamma$-factor over a local field of characteristic zero (for a precise discussion, see [11] §5.3).

**Lemma 4.3.** The definition of $\gamma^V(s, \pi \boxtimes \omega, \psi)$ does not depend on the choice of $A$ (and $L$ in the case $(O, n : odd)$).

**Proof.** Let $M$ be the Levi subgroup $P(V^\alpha) \cap P(V^\gamma) \cap P(V^\beta)$. Note that $M$ is isomorphic to $GL_n(E)$. At first, we consider the cases other than $(O, n : odd)$, $(Sp, n : even)$, $(Q_1)$. In these cases, the adjoint action of $M$ on $u(V^\alpha)_n$ is transitive. Take $A, A' \in u(V^\alpha)_n$ and $m \in M$ so that $A' = mAm^{-1}$. Then, we have

$$c(s, \omega, A', \psi) = \omega_s(\Delta_{V^\alpha}(m^{-1})) \omega_s(\Delta_{V^\alpha}(m^{-1})^*) \cdot c(s, \omega, A, \psi)$$

([13] Lemma 10). On the other hand, we have

$$\begin{cases}
N_\psi(A') = \Delta_{V^\alpha}(m) \Delta_{V^\alpha}(m)^* N_\psi(A) & (U), (\text{qGL}) \\
N_\psi(A') = \Lambda_1(m) \Lambda_2^{-1}(m) N_\psi(A) & (GL), (QGL)
\end{cases}$$

Hence, we have the lemma.

Second, in the case $(O, n : odd)$, the Levi subgroup $M$ acts transitively on the set

$$\mathcal{X} = \{(A, L) \mid A \in u(V^\alpha)_{n-1}, \ L : \text{anisotropic line in } V^\square, \ p\nabla(L) = V^\gamma \cap \ker A\}$$

by $m \cdot (A, L) = (mAm^{-1}, mL)$. Let $(A, L), (A', L') \in \mathcal{X}$ and $m \in M$ so that $(A', L') = m \cdot (A, L)$. Then, we have

$$c(s, \omega, A', L', \psi) = \omega_s(\Delta_{V^\alpha}(m))^{-2} c(s, \omega, A, L, \psi)$$

([13] Lemma 12). On the other hand, we have

$$N_\psi(\tilde{A}_L') = \omega_s(\Delta_{V^\alpha}(m))^2 N_\psi(\tilde{A}_L).$$

Hence, we have the lemma.

Finally, in the cases $(Sp, n : even), (Q_1)$, the adjoint action of $M$ on $u(V^\alpha)_n$ is not transitive. However, there are explicit formulas of $c(s, \omega, A, \psi)$ in these cases [10] [21], and we have the lemma (see [11] p.9).

**4.2 Minimal cases.** Before stating the main theorem, we list the “minimal cases”;

- The trivial cases: $(O, n = 0, 1), (Sp, n = 0), (U, n = 0), (Q_1, n = 0), (Q_2, n = 0)$;
- The anisotropic orthogonal cases: $(O^{(a)}, n = 2, 3, 4)$;
- The one dimensional quaternionic unitary cases equipped with a division quaternion algebra: $(Q_{-1}, n = 1), (Q_1, n = 1)$;
- The one dimensional unitary cases equipped with a quadratic extension field: $(U, n = 1)$;
- The two dimensional anisotropic unitary cases equipped with a ramified quadratic extension field: $(U^{(a)}, n = 2)$. 

\[\square\]
4.3. **Main theorem.** For a non-trivial character \( \psi \) of \( F \) and an irreducible representation \( \rho \) of \( \text{GL}_m(D) \) where \( D \) is the division central algebra over \( F \), we can attach the “Godement-Jacquet \( \gamma \)-factor” as

\[
\gamma^\text{GJ}(s, \rho, \psi) = \varepsilon^\text{GJ}(s, \rho, \psi) \frac{L^\text{GJ}(1-s, \rho^\prime)}{L^\text{GJ}(s, \rho)}.
\]

where \( L^\text{GJ}(s, \rho) \) (resp. \( \varepsilon^\text{GJ}(s, \rho, \psi) \)) is the \( L \)-factor (resp. the \( \varepsilon \)-factor) defined in [6] Theorems 3.3.8.7.

Now we state our main theorem:

**Theorem 4.4 (Main).** The factor \( \gamma^\psi(s, \pi \boxtimes \omega, \psi) \) satisfies the following properties:

1. (unramified twisting) \( \gamma^\psi(s, \pi \boxtimes \omega_{s_0}, \psi) = \gamma^\psi(s + s_0, \pi \boxtimes \omega, \psi) \) for \( s_0 \in \mathbb{C} \).

2. (multiplicativity) Let \( W \) be a totally isotropic subspace of \( V \), and let \( \sigma = \sigma_0 \boxtimes \sigma_1 \) be an irreducible representation of \( \text{GL}(W) \times G(W) \) (Sect. 2.7). If \( \pi \) is isomorphic to a subquotient of \( \text{Ind}^\text{G}_{P(W)}(\sigma) \), then

\[
\gamma^\psi(s, \pi \boxtimes \omega, \psi) = \gamma^{W_0}(s, \sigma_0 \boxtimes \omega, \psi) \gamma^{W_1}(s, \sigma_1 \boxtimes \omega, \psi).
\]

3. (unramified cases) Suppose that \( E \) is \( F \) itself or an unramified quadratic extension field of \( F \). If \( V, \pi, \omega \) and \( \psi \) are unramified in the sense of Sect. 2.7, then

\[
\gamma^\psi(s, \pi \boxtimes \omega, \psi) = \frac{L(1-s, \pi^\psi \boxtimes \omega^{-1}, \text{std})}{L(s, \pi \boxtimes \omega, \text{std})}.
\]

where \( L(s, -, \text{std}) \) is the standard \( L \)-function (see Sect. 2.5).

4. (split cases) Suppose that \( E \) is split, then

\[
\gamma^\psi(s, \pi \boxtimes \omega, \psi) = \begin{cases} 
\gamma^\psi(s, \pi^2 \boxtimes \omega_1 \boxtimes \omega_2, \psi) \gamma^\psi(s, \pi^2 \boxtimes \omega_2 \boxtimes \omega_1, \psi) & \text{in the case (qGL),} \\
\gamma^\psi(s, \pi^2 \boxtimes \omega, \psi) & \text{in the other cases,}
\end{cases}
\]

where \( \omega_1 \) and \( \omega_2 \) are the character of \( F^\times \) such that \( \omega_1 \boxtimes \omega_2 = \omega : F^\times \times F^\times \rightarrow \mathbb{C}^\times \) in the case (qGL).

5. (functional equation)

\[
\gamma^\psi(s, \pi \boxtimes \omega, \psi) \gamma^\psi(1-s, \pi^\psi \boxtimes \omega^{-1}, \psi^{-1}) = 1.
\]

6. (self duality)

\[
\gamma^\psi(s, \pi^\psi \boxtimes \omega, \psi) = \gamma^\psi(s, \pi \boxtimes \omega^*, \psi).
\]

7. (dependence on \( \psi \)) Denote by \( \psi_a \) the additive character \( x \mapsto \psi(ax) \) of \( F \) for \( a \in F^\times \). Then

\[
\gamma^\psi(s, \pi \boxtimes \omega, \psi_a) = T_N(s, \omega, a) \cdot \gamma^\psi(s, \pi \boxtimes \omega, \psi)
\]

where

\[
T_N(s, \omega, a) = \begin{cases} 
\omega_{s-\frac{1}{2}}(a) \frac{\chi_{\text{std}}(s, \psi)}{\chi_{\text{std}}(s, \omega)} & \text{in the cases (GL), (qGL), (QGL),} \\
\omega_{s-\frac{1}{2}}(a)^N & \text{in the cases (Sp), (U), (Q),} \\
\omega_{s-\frac{1}{2}}(a)^N \chi_{\text{std}}(s, \psi) & \text{in the cases (O), (Q_{-1}).}
\end{cases}
\]
(8) (minimal cases) Suppose that $V$ is a minimal case in the sense of Sect. 2.4. Then
\[ \gamma^V(s, 1_V \boxtimes 1, \psi) = \gamma_F(s, \text{std} \circ \phi_{\mathcal{D}_L}) \]
where $1_V$ is the trivial representation of $G(\mathcal{V})$, $1$ is the trivial character of $Z(E^\times)$, and the right hand side is the $\gamma$-factor in the sense of [7].

(9) (GL$_n$-factors) In the cases (GL), (qGL), (QGL),
\[ \gamma^V(s, \pi \boxtimes \omega, \psi) = \gamma^{GJ}(s, \pi \otimes \omega_1, \psi) \gamma^{GJ}(s, \pi^* \otimes \omega_2, \psi), \]
where $\pi^*$ is the representation of $G(V)$ defined in Sect. 2.4 and
\[
\begin{cases}
\omega = \omega_1 \boxtimes \omega_2 : F^\times \times F^\times \to \mathbb{C}^\times, & \text{in the cases (GL), (QGL),} \\
\omega_1 = \omega_2 = \omega : E^\times \to \mathbb{C}^\times, & \text{in the case (qGL).}
\end{cases}
\]

(10) (global functional equation) Let $F$ be a global field, let $\mathbb{E}$ be a division algebra over $F$ of $[\mathbb{E} : F] = 1, 2, 4$, and let $\mathcal{V}$ be one of the spaces of Sect. 2.4 over $\mathbb{E}$. Let $\Pi$ be an irreducible cuspidal automorphic representation of $G(\mathcal{V})(A)$. Then for finite set $S$ of places of $F$ containing all places where $\mathcal{V}$ and $\Pi$ are ramified, the functional equation
\[
L_S(s, \text{std} \circ \phi_{\mathcal{D}_L}) = \prod_{v \in S} \frac{\gamma^V(s, \pi \boxtimes \omega, \psi) L_v(1 - s, \text{std} \circ \phi_{\mathcal{D}_L})}{L_v(s, \text{std} \circ \phi_{\Pi_{\mathcal{D}_L}})}
\]
holds, where
\[
L_S(s, \text{std} \circ \phi_{\Pi_{\mathcal{D}_L}}) = \prod_{v \in S} L_v(s, \text{std} \circ \phi_{\Pi_{\mathcal{D}_L}})
\]
is the partial standard $L$-factor.
Moreover, the properties (1) (2), (3), (4), (5), (6), (9) and (10) determine $\gamma^V(s, \pi \boxtimes \omega, \psi)$ uniquely.

4.4. $L$- and $\epsilon$-factors. In this subsection we discuss about the $L$- and $\epsilon$-factors. We define the $L$- and $\epsilon$-factors as in [13] §10. Moreover, as in [20], we have

Proposition 4.5. The $\epsilon$-factor $\epsilon^V(s, \pi \boxtimes \omega, \psi)$ is a monomial of $q^{-s}$.

Note that Yamana gave another definition of the $L$-factor by g.c.d property [20] Theorem 5.2]. Moreover, [20] Lemma 7.2] implies that both $L$-factors coincide. Although he assume $\text{ch}(F) = 0$ in [20], the results above still hold in the case $\text{ch}(F) \neq 0$.

4.5. Relation with other types of $\gamma$-factors. In this subsection, we write the relation with other types of $\gamma$-factors. First, we compare $\gamma^V(s, \pi \boxtimes \omega, \psi)$ to the Langlands-Shahidi $\gamma$-factor. Then, we compare $\gamma^V(s, \pi \boxtimes \omega, \psi)$ to the $\gamma$-factor defined by Genestier-Lafforgue.

Before stating them, we define the doubling $\gamma$-factor of irreducible representations of $G(V)^\circ \times \text{Res} \text{GL}_1$. Let $\pi$ be an irreducible representation of $G(V)^\circ$, let $\omega$ be a character of $E^\times$, and let $\psi$ be a non-trivial character of $F$. Then we define the doubling $\gamma$-factor $\gamma^V(s, \pi \times \omega, \psi)$ by $\gamma^V(s, \pi \times \omega, \psi)$ where $\pi$ is an irreducible representation of $G(V)$ so that the restriction of $\pi$ to $G(V)^\circ$ contains $\pi$. By the discussion of [11] §3.4, §3.6, we can prove that the definition of $\gamma^V(s, \pi \times \omega, \psi)$ does not depend on the choice of $\pi$.

Now we state the relation with the Langlands-Shahidi $\gamma$-factor. The theory of Langlands-Shahidi $\gamma$-factor is extended to a quasi-split connected reductive group over a function field by Lomeli [14] [15]. Suppose that $\mathcal{V}$ is of the type (O), (Sp), (U) and $G(\mathcal{V})^\circ$ is quasi-split. For a generic irreducible representation of $G(V)^\circ$, we denote by $\gamma^{L,S}(\pi, r_1, \psi)$ the first Langlands-Shahidi $\gamma$-factor. Note that $r_1$ is associated with the standard homomorphism std defined in Sect. 2.4 as follows: We define a representation std of $\ell(G(V)^\circ \times \text{Res}_{E/F} \text{GL}_1)$ by
\[
\text{std}(g, z') = \text{std}(g, 1)^{-1} \cdot \text{std}(1, z')
\]
for \( g \in \hat{G(V)} \) and \( z' \in L(Res_{E/F} GL_1) \). Then \( r_1 \) is equivalent to \( \text{std}' \) as representations of \( L(G(V)^o \times Res_{E/F} GL_1) \).

**Corollary 4.6.** Suppose that \( V \) is of the type \((O), (Sp), (U)\) and \( G(V)^o \) is quasi-split. Let \( \pi \) be a generic irreducible representation of \( G(V)^o \), let \( \omega \) be a character of \( E^\times \), and let \( \psi \) be a non-trivial additive character of \( F \). Then we have

\[
\gamma^{LS}(\pi \boxtimes \omega, s, r_1, \psi) = \gamma^{V}(s, \pi \boxtimes \omega, \psi).
\]

*Proof.* To characterize the Langlands-Shahidi \( \gamma \)-factor \( \gamma^{LS}(\pi \boxtimes \omega, s, r_1, \psi) \), it suffice to check the properties \((i)-(vi)\) and \((ix)\) of [14, Theorems 5.1, 7.3]. On the other hand, Theorem [4.4] implies that \( \gamma^{V}(s, \pi \boxtimes \omega, \psi) \) satisfies above properties. \( \square \)

Finally, we state the relation with the \( \gamma \)-factor defined by Genestier-Lafforgue. Put

\[
H = \begin{cases} 
G(V)^o \times GL_1 & \text{in the cases } (O), (Sp), (Q_1), (Q_{-1}), \\
G(V) \times Res_{E/F}(GL_1) & \text{in the cases } (U), (qGL), \\
G(V) \times GL_1 \times GL_1 & \text{in the cases } (GL), (QGL).
\end{cases}
\]

Fix an isomorphism \( \iota : \mathbb{C} \rightarrow \mathbb{Q}_{l}^{\times} \) of field. By \( \iota \), we can regard \( \pi, \omega, \psi \) as an irreducible representation with coefficient in \( \mathbb{Q}_{l} \), a \( \mathbb{Q}_{l}^{\times} \) valued character, and a \( \mathbb{Q}_{l}^{\times} \) valued non-trivial additive character respectively. One can show that \( \pi, \omega \) are defined over a finite extension field over \( \mathbb{Q}_l \). Hence, according [5, Théorème 8.1], we can associate a semisimple \( L \)-parameter \( \sigma_{\pi \boxtimes \omega} \), that is, an \( L \)-parameter \( \sigma_{\pi \boxtimes \omega} : W_{E} \times SL_{2}(\mathbb{Q}_{l}) \rightarrow LH(\mathbb{Q}_{l}) \) such that \( \sigma_{\pi \boxtimes \omega}|_{1 \times SL_{2}(\mathbb{Q}_{l})} \) is trivial. Note that \( \sigma_{\pi \boxtimes \omega} \) is expected to be the semisimplification of the \( L \)-parameter \( \phi_{\pi \boxtimes \omega} \) of \( \pi \boxtimes \omega \). Thus, we can expect that \( \gamma_F(s, \text{std} \circ \sigma_{\pi \boxtimes \omega}, \psi) \) coincides with the standard \( \gamma \)-factor \( \gamma_F(s, \phi_{\pi \boxtimes \omega}, \psi) \) where \( \text{std} \) is the standard homomorphism of \( LH(\mathbb{Q}_{l}) \) defined in Sect. 2.3.

**Corollary 4.7.** Assume that \( \gamma_F(s, \text{std} \circ \sigma_{1 \boxtimes 1}, \psi) = \gamma^V(s, 1 \boxtimes 1, \psi) \) in the minimal cases. Then, we have

\[
\gamma_F(s, \text{std} \circ \sigma_{\pi \boxtimes \omega}, \psi) = \gamma^V(s, \pi \boxtimes \omega, \psi).
\]

5. **Proof of the main theorem**

In this section, we explain the proof of Theorem [4.4] which will be completed in Sect. 6.

5.1. **Uniqueness.** The uniqueness can be proved standard global argument (see. e.g. [13, 11]) with the globalization results [14, Appendix I] and [3, Theorem 1.1].

5.2. **Formal properties.** The properties [11] unramified twisting, [2] multiplicativity, [3] functional equation, [5] self duality, and [7] dependence of \( \psi \), are deduced from the framework of the doubling method. One can prove them in the same line with [13] (see also [11, Remark 6.3]). By using the argument of the Eisenstein series (see for example [8]), we have [10] global functional equation as in [13].

5.3. **Split cases.** [11 §6.3].

5.4. **Unramified cases.** [13 §7].

5.5. **Minimal cases.** This property is proved by explicit computation of the \( \gamma \)-factor and standard \( \gamma \)-factor respectively. The computation is done in Sect. 6 below.

5.6. **GL_\sigma-factors.** [10 Appendix].
6. THE $\gamma$-FACTOR OF THE TRIVIAL REPRESENTATION FOR THE MINIMAL CASES

In this section, we compute the standard $\gamma$-factor of the principal parameter and the $\gamma$-factor of the trivial representation in the minimal cases in the sense of Sect. 4.2, and we complete the proof of Theorem 4.4. We begin with the explanation of the standard $\gamma$-factor of the principal parameter in each case (Sect. 6.1). Then we give notations to compute the analytic $\gamma$-factor (Sect. 6.2). The computation consists of three parts: the cases $(Q_1)$, $(U, n = 1)$, $(Q_{-1}, n = 1)$ (Sect. 6.3), the case $(Q_1, n = 1)$ (Sect. 6.4), and the cases $(O^{(a)}, n = 3), (O^{(a)}, n = 4), (U_1^{(a)}, n = 2)$ (Sect. 6.5).

6.1. THE STANDARD $\gamma$-FACTOR OF THE PRINCIPAL PARAMETERS. In this subsection, we give the irreducible decomposition of the representation $\text{std} \circ \phi_0$, and give the formula of standard $\gamma$-factor in the minimal cases.

Proposition 6.1. We denote by $r_m$ the unique $m$-dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$. Then the irreducible decomposition of the representation $\text{std} \circ \phi_0$ is given by the following:

1. In the cases $(O^{(a)}, n = 2), (Q_{-1}, n = 1)$, we have $\text{std} \circ \phi_0 = (r_1 \boxtimes r_1) \oplus (r_2 \boxtimes r_1')$ as representations of $\text{SL}_2(\mathbb{C}) \times W_F$, where $r_1'$ is the character of $W_F$ as associated with $r_1$ via the local class field theory.

2. In the cases $(O^{(a)}, n = 3), (Q_1, n = 1)$, we have $\text{std} \circ \phi_0 = r_N \boxtimes 1$ as representations of $\text{SL}_2(\mathbb{C}) \times W_F$.

3. In the cases $(O^{(a)}, n = 4)$, we have $\text{std} \circ \phi_0 = (r_3 \boxtimes 1) \oplus (r_1 \boxtimes 1)$ as representations of $\text{SL}_2(\mathbb{C}) \times W_F$.

4. In the cases $(U, n = 1), (U_1^{(a)}, n = 2)$, we have $\text{std} \circ \phi_0 = (r_n \boxtimes 1) \oplus (r_2 \boxtimes \chi_E)$ as representations of $\text{SL}_2(\mathbb{C}) \times W_F$.

Therefore, the $\gamma$-factor is calculated as follows:

Proposition 6.2. (1) In the cases $(O^{(a)}, n = 2), (Q_{-1}, n = 1)$, 
$$\gamma_F(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = \frac{\zeta_F(-s + \frac{1}{2})}{\zeta_F(s + \frac{1}{2})} \gamma_F(s + \frac{1}{2}, \chi_0(\mathfrak{v}), \psi),$$

(2) In the case $(O^{(a)}, n = 3)$,
$$\gamma_F(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = \frac{\zeta_F(-s) \zeta_F(-s + 1)}{\zeta_F(s)}.$$

(3) In the case $(O^{(a)}, n = 4)$,
$$\gamma_F(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = \frac{\zeta_F(-s + \frac{1}{2}) \zeta_F(-s + \frac{1}{2}) \zeta_F(-s + \frac{1}{2})}{\zeta_F(s + \frac{1}{2}) \zeta_F(s + \frac{1}{2}) \zeta_F(s + \frac{1}{2})}.$$

(4) In the case $(Q_1, n = 1)$,
$$\gamma_F(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = \frac{\zeta_F(-s + \frac{1}{2}) \zeta_F(-s + \frac{1}{2}) \zeta_F(-s + \frac{1}{2})}{\zeta_F(s + \frac{1}{2}) \zeta_F(s + \frac{1}{2}) \zeta_F(s + \frac{1}{2})}.$$

(5) In the case $(U, n = 1)$,
$$\gamma_E(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = \frac{\zeta_F(-s + \frac{1}{2})}{\zeta_F(s + \frac{1}{2})} \gamma_F(s + \frac{1}{2}, \chi_E, \psi).$$

(6) In the case $(U_1^{(a)}, n = 2)$,
$$\gamma_E(s + \frac{1}{2}, \text{std} \circ \phi_0, \psi) = -q^{-\frac{1}{2}} \frac{\zeta_F(-s + 1)}{\zeta_F(s + 1)} \zeta_F(s + \frac{1}{2}, \chi_E, \psi)^2.$$
6.2. Preliminaries. Let $v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$ be a basis of $V = E^n$ over $E$. Then we fix the canonical basis of $V^\mathbb{C}$ by $e_1, \ldots, e_{2n}$ defined by

$$e_i = \begin{cases} (v_i, v_i), & 1 \leq i \leq n \\ (v_{i-n}, -v_{i-n}), & n + 1 \leq i \leq 2n \end{cases}$$

We have a decomposition

$$G(V^\mathbb{C}) = P(V^\Delta)(G(V) \times G(V))$$

by [3, Lemma 2.1]. Thus, $I(s, 1) \cong C(G(V))$ as $G(V) \times G(V)$ modules, and $\Gamma^V(s, 1\nu, 1, A, \psi)$ is the eigenvalue of $M^*(s, 1, A, \psi)$ on $C(G(V))^G(V) = 1_G \cdot C$. We compute them case by case.

6.3. The cases $(O^{(a)}, n = 2), (U, n = 1), (Q_{-1}, n = 1)$. Note that we assume $E$ is a division algebra. In each case, we may suppose that

$$\begin{cases} V = F^2, h = \langle \text{diag}(1, -a) \rangle & \text{in the case } (O^{(a)}, n = 2), \\ V = E, E = F(\alpha), h = \langle 1 \rangle & \text{in the case } (U, n = 1), \\ V = E, E = (a, b/F), h = \langle \alpha \rangle & \text{in the case } (Q_{-1}, n = 1) \end{cases}$$

for some $a, b \in F^\times \backslash F^\times 2$, $\alpha \in E_0$ satisfying

- $\text{ord}_F(a) = 0, 1$ and $\text{ord}_F(b) = 0, 1$,
- $\text{ord}_F(a) \neq \text{ord}_F(b)$,
- $\alpha^2 = a$.

We take $K := G(V) \times G(V) \subset G(V^\mathbb{C})$, $w_1 = (1, -1) \in K$, and $A \in u(V^\mathbb{C})$ by

$$A = \begin{pmatrix} 0 & X^{-1} \\ 0 & 0 \end{pmatrix}$$

where

$$X = \begin{cases} 0 & \alpha \\ 1 & 0 \end{cases}$$

in the case $(O^{(a)}, n = 2), (U, n = 1), (Q_{-1}, n = 1)$.

**Proposition 6.3.** Let $f$ be the $K$ invariant section of $I(0, 1)$ with $f(1) = 1$. Then, we have

$$l_{\psi_A}(f_s) = \zeta_{F(\sqrt{\nu})}(s + \frac{1}{2})^{-1}.$$ 

**Proof.** At first, we note that

$$\begin{pmatrix} 1 \\ xX \end{pmatrix} = \begin{pmatrix} -x^{-1}X^{-1} \\ 0 \\ xX \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & x^{-1}X^{-1} \end{pmatrix}$$

gives the Iwasawa decomposition in $G(V^\mathbb{C})$ for $x \in F^\times$ if $\text{ord}_F(x) < 0$.

If $\text{ord}_F(a) = 0$,

$$\zeta_F(2s + 1)f_s(\begin{pmatrix} 1 \\ xX \end{pmatrix}) = \begin{cases} \int_{F^\times} 1_{\mathcal{O}(t)}|t|^{2s+1} d^\times t & \text{ord}_F(x) \geq 0, \\ \int_{F^\times} 1_{\mathcal{O}(t)}|t|^{2s+1} d^\times t & \text{ord}_F(x) < 0 \end{cases}$$

and

$$\int_{F^\times} 1_{\mathcal{O}(t)}|t|^{2s+1} d^\times t.$$
Since \( \zeta_1(1) \) Proposition 6.4.

Thus, \( \zeta_F(2s + 1)I_{\psi_4}(f_s) = \int_{F^\times} 1_{\sigma}(t) \left( \int_F 1_{t-1_\sigma}(x) \psi(2x) \, dx \right) d^\times t \)

\[ = \int_{F^\times} 1_{\sigma}(t) \, d^\times t = 1. \]

Since \( \zeta_F(\sqrt{q}) (s + \frac{1}{2}) = \zeta_F(2s + 1) \), we have the claim in this case.

If \( \text{ord}_F(a) = 1, \)

\[ \zeta_F(2s + 1)I_{\psi_4}(f_s)(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) = \begin{cases} \int_{F^\times} 1_{\sigma}(t)|t|^{s + \frac{1}{2}} \, d^\times t & \text{ord}_F(x) \geq 0 \\ q^{s + \frac{1}{2}} \int_{F^\times} 1_{t-1_\sigma}(x)|t|^{s + \frac{1}{2}} \, d^\times t & \text{ord}_F(x) < 0 \end{cases} \]

\[ = q^{s + \frac{1}{2}} \int_{F^\times} 1_{\sigma}(t)1_{t-1_\sigma}(x)|t|^{s + \frac{1}{2}} \, d^\times t + 1_{\sigma}(x)(1 - q^{s + \frac{1}{2}})\zeta_F(2s + 1) \]

\[ = q^{s + \frac{1}{2}} \int_{F^\times} 1_{\sigma}(t)1_{t-1_\sigma}(x)|t|^{s + \frac{1}{2}} \, d^\times t - 1_{\sigma}(x)q^{s + \frac{1}{2}} \frac{1}{1 + q^{-s - \frac{1}{2}}}. \]

Thus,

\[ \zeta_F(2s + 1)I_{\psi_4}(f_s) = q^{s + \frac{1}{2}} \int_{F^\times} 1_{\sigma}(t)|t|^{s + \frac{1}{2}} \, d^\times t - \frac{q^{s + \frac{1}{2}}}{1 + q^{-s - \frac{1}{2}}}. \]

Since \( \zeta_F(\sqrt{q}) (s + \frac{1}{2}) = (1 + q^{-s - \frac{1}{2}})\zeta_F(2s + 1) \), we have the claim in this case. \( \Box \)

Thus we have:

**Proposition 6.4.**

1. In the case \((O_{(1)}, n = 2), (Q_{-1}, n = 1)\), we have

\[ \gamma^V(s + \frac{1}{2}, 1_{V} \otimes 1, \psi) = \frac{\zeta_{F}(s + \frac{1}{2})}{\zeta_{F}(\sqrt{q}) (s + \frac{1}{2})} \zeta_{F}(s + \frac{1}{2}, \chi_{\Delta_{V}}, \psi). \]

2. In the case \((U, n = 1)\), we have

\[ \gamma^V(s + \frac{1}{2}, 1_{V} \otimes 1, \psi) = \frac{\zeta_{F}(s + \frac{1}{2})}{\zeta_{F}(s + \frac{1}{2})} \zeta_{E}(s + \frac{1}{2}, \chi_{\Delta_{E}}, \psi). \]

6.4. The case \((Q_{1}, n = 1)\). Note that we assume that \( E \) is a division quaternion algebra over \( F \). In this case, \( \dim_F(V_{\Delta_{2}}) = 3 \).

**Lemma 6.5.** Take \( \alpha, \beta \in E_{0} \) so that \( \text{ord}_F(\alpha) = 0, \text{ord}_F(\beta) = 1, \alpha \beta + \beta \alpha = 0, \) and \( E \) is spanned by \( 1, \alpha, \beta, \alpha \beta \) over \( F \). We denote \( \alpha^2 \) (resp. \( \beta^2 \)) by \( a \) (resp. \( b \)). Then, we have

\[ 1_{t-1_\sigma}(ax^2 + by^2 - abz^2) = 1_{t-1_\sigma}(ax^2)1_{t-1_\sigma}(by^2)1_{t-1_\sigma}(abz^2) \]

for \( t \in F^\times \) and \( x, y, z \in F \).

**Proof.** At first, we note that \( 1, \alpha, \beta, \alpha \beta \) is a basis of \( \mathcal{O}_E \) over \( \mathcal{O} \). If \( \text{ord}_F(t) = 2m \) for some \( m \in \mathbb{Z} \), then

\[ ax^2 + by^2 - abz^2 \in t^{-1}\mathcal{O} \]

\[ \iff ax + \beta y + \alpha \beta z \in \beta^{-2m}\mathcal{O}_E \]

\[ \iff abm_x + \beta b^m y + \alpha \beta m z \in \mathcal{O}_E \]

\[ \iff ax^2 \in t^{-1}\mathcal{O}, by^2 \in t^{-1}\mathcal{O}, abz^2 \in t^{-1}\mathcal{O}. \]
Thus we have:

\[ \text{If } \text{ord}_F(t) = 2m + 1 \text{ for some } m \in \mathbb{Z}, \text{ then} \]

\[ ax^2 + by^2 - abz^2 \in t^{-1}O \]

\[ \Leftrightarrow ax + \beta y + \alpha \beta z \in \beta^{-2m}O_E \]

\[ \Leftrightarrow -\alpha \beta b^m x + b^{m+1}y - ab^{m+1}z \in O_E \]

\[ \Leftrightarrow ax^2 \in t^{-1}O, by^2 \in t^{-1}O, abz^2 \in t^{-1}O. \]

Hence we have the claim. \(\square\)

**Proposition 6.6.** Take \(\alpha, \beta \in E_0\) as in Lemma 6.3, a compact subgroup \(K = G(V) \times G(V)\), a \(K\) invariant section \(f \in I(0, 1)\) with \(f(1) = 1\), and

\[ A = \begin{pmatrix} 0 & \alpha^{-1} \\ 0 & 0 \end{pmatrix}. \]

Then, we have

\[ l_{\psi_A}(f_s) = \zeta_F(s + \frac{3}{2})^{-1}(1 + q^{-s + \frac{1}{2}}). \]

**Proof.** For \(x, y, z \in F\),

\[ \zeta_F(s + \frac{3}{2})l_{\psi_A}(f_s) = \begin{cases} 
\int_{F^2} 1_O(t)|t|^{s + \frac{1}{2}} d^xy & \text{ord}_F(ax^2 + by^2 - abz^2) \geq 0 \\
\int_{F^2} 1_O(t)|t|^{s + \frac{1}{2}}|ax^2 + by^2 - abz^2|^{s + \frac{1}{2}} d^xy & \text{ord}_F(ax^2 + by^2 - abz^2) < 0
\end{cases} \]

\[ = \int_{F^2} 1_O(t)1_{t^{-1}O}(ax^2 + by^2 - abz^2)|t|^{s + \frac{1}{2}} d^x t \]

By Lemma 6.3, this is equal to

\[ \int_{F^2} 1_O(t)1_{t^{-1}O}(ax^2)1_{t^{-1}O}(by^2)1_{t^{-1}O}(abz^2)|t|^{s + \frac{1}{2}} d^x t. \]

Therefore,

\[ \zeta_F(s + \frac{3}{2})l_{\psi_A}(f_s) = \int_{F^2} 1_O(t)|t|^{s + \frac{1}{2}} \left( \int_{F} 1_{t^{-1}O}(ax^2)\psi(2x) dx \right) \left( \int_{F} 1_{t^{-1}O}(by^2) dy \right)^2 d^x t = 1 + q^{-s + \frac{1}{2}}. \]

We note that \(\chi_{0(A)}\) is unramified, \(\chi_{0(A)}(\varpi_F) = -1\), and

\[ R(s, 1, A, \psi) = \frac{1 + q^{-s + \frac{1}{2}}}{1 + q^{-s - \frac{3}{2}}}. \]

Thus we have:

**Proposition 6.7.**

\[ \gamma^V(s + \frac{1}{2}, 1 \boxtimes 1, \psi) = \frac{\zeta_F(-s + \frac{3}{2}) \zeta_F(-s + \frac{1}{2}) \zeta_F(-s - \frac{1}{2})}{\zeta_F(s + \frac{1}{2}) \zeta_F(s + \frac{3}{2}) \zeta_F(s - \frac{1}{2})}. \]
6.5. The cases $(O^{(a)}, n = 3), (O^{(a)}, n = 4), (U, n = 2), (Q_{-1}, n = 2), (Q_{-1}, n = 3)$. We may suppose that
\[
\begin{align*}
V &= F^3, h = \langle \text{diag}(1, -a, -b) \rangle & \text{in the case } (O^{(a)}, n = 3), \\
V &= F^4, h = \langle \text{diag}(1, -a, -b, ab) \rangle & \text{in the case } (O^{(a)}, n = 4), \\
E &= F(\sqrt{b}), V = E^2, h = \langle \text{diag}(1, -a) \rangle & \text{in the case } (U^{(a)}, n = 2).
\end{align*}
\]
for some $a, b \in F^\times$ satisfying
\[
\text{ord}_F(a) = 0, \text{ord}_F(b) = 1, (a, b)_F = -1.
\]
In each case, we associate an isotropic orthogonal (resp. hermitian) space $V'$ to the space $V$ as follows:
\[
\begin{align*}
\text{the orthogonal space } (F^3, \langle \text{diag}(1, -1, -b) \rangle) & \text{ in the case } (O^{(a)}, n = 3), \\
\text{the orthogonal space } (F^4, \langle \text{diag}(1, -1, -b, b) \rangle) & \text{ in the case } (O^{(a)}, n = 4), \\
\text{the hermitian space } (E^2, \langle \text{diag}(1, -1) \rangle) & \text{ over } E \text{ in the case } (U^{(a)}, n = 2).
\end{align*}
\]
Put
\[
R := \begin{cases} 
\text{diag}(1, a, 1) & \text{in the case } (O^{(a)}, n = 3) \\
\text{diag}(1, a, 1, a) & \text{in the case } (O^{(a)}, n = 4) \\
\text{diag}(1, a) & \text{in the case } (U^{(a)}, n = 2)
\end{cases}
\]
then we have an isometry
\[
\mathcal{D} : V^\square \to V'^\square : v \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} v,
\]
and thus we have an isomorphism
\[
\mathcal{D} : G(V^\square) \to G(V'^\square) : g \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix}.
\]

**Lemma 6.8.** In the case $(O^{(a)}, n = 4), (U^{(a)}, n = 2)$, the following diagram is commutative:
\[
\begin{array}{ccc}
I^V(s, 1) & \xrightarrow{M^V(s, 1, A', \psi)} & I^V(-s, 1) \\
\mathcal{D} & & \mathcal{D} \\
I^V(s, 1) & \xrightarrow{M^V(s, 1, A, \psi)} & I^V(-s, 1)
\end{array}
\]
where $A' \in u(V'^\Delta)_n$ and $A = R^{-1}A' \in u(V^\Delta)_n$.

Choose a compact subgroup $K' := G(V'^\square) \cap GL_{2n}(O_F)$ of $G(V'^\square)$. Then $\mathcal{D}(G(V) \times G(V)) \subset K'$. Note that we have $P(V'^\Delta)K' = G(V')$ since $P(V^\Delta)G(V) \times G(V))$ by [4] Lemma 2.1. Let $f'$ be the $K'$ invariant section of $I^V(s, 1)$ with $f'(1) = 1$. Note that $\mathcal{D}(f'_s)$ is the $G(V) \times G(V)$ invariant section of $I^V(s, 1)$. Hence we have
\[
(I^V(s, 1) \xrightarrow{M^V(s, 1, A', \psi)} I^V(-s, 1) \xrightarrow{\mathcal{D}} I^V(-s, 1))
\]
(6.1)
\[
l_{\psi, A}(f'_s) = \Gamma^V(s, 1, \psi)l_{\psi, A}(f'_{-s}).
\]

Similar statement hold for the case $(O^{(a)}, n = 3)$. We may rewrite (6.1) as
\[
\gamma^V(s + \frac{1}{2}, 1_Y \boxtimes 1, \psi) = c \gamma^V(s + \frac{1}{2}, 1_Y \boxtimes 1, \psi) Z^V(f'_s, \xi') Z^V(f'_{-s}, \xi')
\]
where $\xi'$ is the non-zero coefficient of the trivial representation $G(V')$, and
\[
c = \begin{cases} 
1 & \text{in the case } (O^{(a)}, n = 4), \\
-1 & \text{in the case } (O^{(a)}, n = 3), (U^{(a)}, n = 2).
\end{cases}
\]
Then, by using the global functional equation \[10\] for the trivial representation of a special orthogonal (or a unitary) group over \(F\) which is anisotropic precisely at two places, we find that
\[
\left( \frac{Z^V(f'_s, \xi')}{Z^V(f''_s, \xi'')} \right)^2 = 1.
\]

Therefore, the meromorphic function
\[
\frac{Z^V(f'_s, \xi')}{Z^V(f''_s, \xi'')}
\]
is a constant function whose value is 1 or \(-1\). To determine the signature, we focus on the behavior of \(Z^V(f'_s, \xi')\) at \(s = 0\).

**Lemma 6.9.** Put
\[
\Omega(s) = \begin{cases} 
Z^V(f'_s, \xi') \zeta_F(s)^{-1} & \text{in the cases } (O^{(a)}, n = 3), (U^{(a)}_1, n = 2), \\
Z^V(f'_s, \xi') & \text{in the case } (O^{(a)}, n = 4).
\end{cases}
\]

Then, the meromorphic function \(\Omega(s)\) has neither a pole nor a zero at \(s = 0\).

**Proof.** Put
\[
v'_1 = (1, 0, \ldots, 0), \quad \ldots, \quad v'_n = (0, \ldots, 0, 1)
\]
a basis of \(V'\) over \(E\), and put \(V''\) the subspace of \(V'\) spanned by the isotropic vector \(v'_1 + v'_2\). Then there is the exact sequence
\[
1 \to U(W'^G) \to P(W'^\Delta) \to \mathrm{GL}_2(E) \times G(W''_0) \to 1.
\]
We denote by \(K'' \times K'''\) the image of \(P(W'^G) \cap K'\) in \(\mathrm{GL}_2(E) \times G(W''_0)\).

Now consider the trivial representation of \(G'\). It is a subrepresentation of \(\operatorname{Ind}_{P(W')}^G \delta_{P(W')}^{-1/2}\).

Note that \(\delta_{P(W')}^{-1/2} = |\lambda| : \mathrm{GL}_1(E) \times G(W''_0) \to \mathbb{C}^\times\) where
\[
\lambda = \begin{cases} 
\frac{1}{2} & \text{in the cases } (O^{(a)}, n = 3), (U^{(a)}_1, n = 2), \\
1 & \text{in the case } (O^{(a)}, n = 4).
\end{cases}
\]

Moreover, we have
\[
Z^V(f'_s, \xi') = J(s) \cdot Z^{W_1}(f''_s, \xi'') Z^{W_0}(f''_s, \xi''')
\]
where \(f''_s\) (resp. \(f'''_s\)) is the unique non-zero section in \(I^{W_1}(s, 1)^{K''}\) (resp. in \(I^{W_0}(s, 1)^{K'''}\)), \(\xi''\) (resp. \(\xi'''\)) is a non-zero coefficient of the representation \(|\lambda|\) of \(\mathrm{GL}_1(E)\) (resp. the representation 1 of \(G(W''_0)\)), and
\[
J(s) = \int_{U(W'^G) \cap P(W'^\Delta)} f''_s(u) \, du.
\]
The integral defining \(J(s)\) converges in \( \Re s > -\frac{1}{2} \) (\[20\] Lemma 5.1)), and the integrand \(f''_s(u)\) is positive for all \(u \in U(W'^G)\). Hence \(J(s)\) has neither pole nor zero at \(s = 0\). On the other hand, we have
\[
Z^{W_1}(f''_s, \xi'') Z^{W_0}(f''_s, \xi''') = \begin{cases} 
\frac{\xi_F(s) \zeta_F(s+1)}{\zeta_F(2s+1) \zeta_F(s+\frac{1}{2})} & \text{in the case } (O^{(a)}, n = 3), (U^{(a)}_1, n = 2), \\
\frac{\xi_F(s-\frac{1}{2}) \zeta_F(s+\frac{1}{2})^2 \zeta_F(s+\frac{3}{2})}{\xi_F(2s+1) \xi_F(2s+3)} & \text{in the case } (O^{(a)}, n = 4)
\end{cases}
\]
by \[13\] Proposition 3. Therefore we can conclude that \(\Omega(s)\) has neither pole nor zero at \(s = 0\). Hence we have the lemma. \(\square\)
From this lemma, we have
\[
\left. \frac{Z^V(f_s^r, \xi')}{Z^V(f_{-s}^r, \xi')} \right|_{s=0} = \begin{cases} 
-1 & (O^{(a)}, n = 3), (U_1^{(a)}, n = 2), \\
1 & (O^{(a)}, n = 4).
\end{cases}
\]

Summarizing, we have:

**Proposition 6.10.**

1. In the case \((O^{(a)}, n = 3)\),
   \[
   \gamma^V(s + \frac{1}{2}, 1_V \boxtimes 1, \psi) = \frac{\zeta_F(-s) \zeta_F(-s + 1)}{\zeta_F(s) \zeta_F(s + 1)}.
   \]

2. In the case \((O^{(a)}, n = 4)\),
   \[
   \gamma^V(s + \frac{1}{2}, 1_V \boxtimes 1, \psi) = \frac{\zeta_F(-s + \frac{3}{2}) \zeta_F(-s + \frac{1}{2}) \zeta_F(-s - \frac{1}{2})}{\zeta_F(s + \frac{3}{2}) \zeta_F(s + \frac{1}{2}) \zeta_F(s - \frac{1}{2})}.
   \]

3. In the case \((U_1^{(a)}, n = 2)\),
   \[
   \gamma^V(s + \frac{1}{2}, 1_V \boxtimes 1, \psi) = \gamma_F(s, 1, \psi) \gamma_F(s, \chi_E, \psi) \gamma_F(s + 1, 1, \psi) \gamma_F(s + 1, \chi_E, \psi).
   \]

Finally, comparing Propositions 6.4, 6.7, 6.10 to Proposition 6.2, we have Theorem 4.4 (8) and we complete the proof of Theorem 4.4.

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