On the middle convolution of local systems.

With an Appendix by Stefan Reiter and Michael Dettweiler.

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Abstract

We study the middle convolution of local systems in the setting of singular and étale cohomology. We give a motivic interpretation of the middle convolution in the étale case and prove an independence-of-$\ell$-result which yields a description of the determinant. We employ these methods to realize special linear groups regularly as Galois groups over $\mathbb{Q}(t)$. In an appendix to this article, written jointly with S. Reiter, we prove the existence of a new motivic local system whose monodromy is dense in the exceptional simple group of type $G_2$.

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Introduction

On the affine line $\mathbb{A}^1$ over an algebraically closed field or a finite field $k$ one has the derived category $D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell)$ of constructible $\mathbb{Q}_\ell$-sheaves with bounded cohomology, cf. [1] and [20]. This category supports Grothendieck’s six operations, so one has the notion of a higher direct image and a higher direct image with compact support. Consider the addition map

$$\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1.$$ 

If $\mathcal{K}$ and $\mathcal{L}$ are elements in $D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell)$, then one can consider their $\ast$-convolution as the higher direct image

$$\mathcal{K} \ast \mathcal{L} := R\pi_!(\mathcal{K} \boxtimes \mathcal{L}) \in D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell),$$

where $\mathcal{K} \boxtimes \mathcal{L} \in D^b_c(\mathbb{A}^1 \times \mathbb{A}^1, \mathbb{Q}_\ell)$ denotes the external tensor product of $\mathcal{K}$ and $\mathcal{L}$. One can also define the $!$-convolution of $\mathcal{K}$ and $\mathcal{L}$ as the higher direct image with compact supports

$$\mathcal{K} \ast ! \mathcal{L} := R\pi_!(\mathcal{K} \boxtimes \mathcal{L}) \in D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell).$$

For more details, the reader should consult the book *Rigid Local Systems* by N. Katz [18]. The structure of the convolution is very complicated in general, so it is convenient to restrict the above construction to smaller subcategories of $D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell)$. A natural candidate to work with is the abelian category of perverse sheaves $\text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell) \subseteq D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell)$, cf. [1] and [20]. This category is not stable under convolution in general, but we can consider perverse sheaves $\mathcal{K} \in \text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell)$ which have the property that for any other $\mathcal{L} \in \text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell)$, the convolutions $\mathcal{K} \ast ! \mathcal{L}$ and $\mathcal{K} \ast \mathcal{L}$ are perverse. In this case we say that $\mathcal{K}$ has the property $\mathcal{P}$.
For example, an object \( K \in \text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell) \) has the property \( \mathcal{P} \) if it is irreducible and if its isomorphism class is not translation invariant, see [18], Cor. 2.6.10.

If \( K \in \text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell) \) has the property \( \mathcal{P} \), then one can define the \textit{middle convolution} of \( K \) and \( \mathcal{L} \in \text{Perv}(\mathbb{A}^1, \mathbb{Q}_\ell) \) as

\[
(0.0.1) \quad K_{\ast \text{mid}} \mathcal{L} = K \ast \mathcal{L} = \text{Im} \left( (K \ast!) \mathcal{L} \to K_{\ast} \mathcal{L} \right),
\]

see [18], Chap. 2.6 (we mostly omit the subscript \text{mid} for notational reasons). This is a perverse sheaf which has again the property \( \mathcal{P} \), see [18], Cor. 2.6.17. Let

\[
j : \mathbb{A}^1_x \times \mathbb{A}^1_y \hookrightarrow \mathbb{P}_x^1 \times \mathbb{A}^1_y, \quad pr_2 : \mathbb{P}_x^1 \times \mathbb{A}^1_y \to \mathbb{A}^1_y,
\]

and

\[
d : \mathbb{A}^1_x \times \mathbb{A}^1_y \to \mathbb{A}^1_{y-x}, \quad (x, y) \mapsto y - x.
\]

It is more or less tautological that the middle convolution can then be interpreted in terms of middle direct images as follows (see [18], Prop. 2.8.4):

\[
(0.0.2) \quad K \ast \mathcal{L} = Rpr_2^*(j_! (pr_1^* K \otimes d^* \mathcal{L})),
\]

where \( j_! (pr_1^* K \otimes d^* \mathcal{L}) \) denotes the middle direct image (sometimes called intermediate extension) of \( pr_1^* K \otimes d^* \mathcal{L} \). One reason why one is interested in the middle convolution is that \( K \ast \mathcal{L} \) is often irreducible, while the convolutions \( K_{\ast!} \mathcal{L} \) and \( K \ast! \mathcal{L} \) are not irreducible. A striking application of the concept of middle convolution is Katz’ existence algorithm for irreducible rigid local systems, see [18], Chap. 6.

In analogy to Formula (0.0.2), one can give a definition of the middle convolution of local systems and of lisse étale sheaves (sometimes called étale local systems), simply by working with suitably small open subsets of \( \mathbb{A}^2 \) and \( \mathbb{A}^1 \) over which all constructions yield again local systems, see Sections 1.2 and 2.2. Our approach is strongly motivated by the definitions and the results of [18], Chap. 8.

The restriction of the middle convolution of perverse sheaves to a suitably small dense open subset of \( \mathbb{A}^1 \) is often given by the middle convolution of local systems, see Section 2.4. Therefore, the concept of middle convolution of local systems can also be seen as a powerful tool for the study of the monodromy of middle convoluted perverse sheaves.

We then give some applications to the inverse Galois problem and the construction problem of interesting motivic local systems. The philosophy is that by convoluting elementary objects, like the local systems associated to cyclic or dihedral Galois covers of open subsets \( S \subseteq \mathbb{A}^1 \), one obtains highly non-trivial
Galois representations and motives.

Let us now describe the content of this paper in more detail: We start in Section 1 by working on the analytic affine line $\mathbb{A}^1(\mathbb{C})$. In Section 1.2 we give the definition of the middle convolution of local systems as the parabolic cohomology of a variation of local systems. The main results are the following:

- We derive a formula for the rank of the middle convolution $\mathcal{V}_1 \ast \mathcal{V}_2$, see Prop. 1.4.1.
- We study the important case of convoluting local systems with Kummer sheaves and relate the monodromy of such convolutions to the tuple transformation $MC_\lambda$ of [7] and [8].
- An irreducibility criterion for the convolution of some local systems is given in Thm. 1.7.1.
- The effect of the middle convolution on the local monodromy is determined in some cases, see Section 1.8.

In Section 2, we study the middle convolution of lisse étale sheaves. We derive the following results:

- If the ground ring $R$ is contained in $\mathbb{C}$, then the geometric monodromy of the middle convolution of lisse sheaves can be computed using the concept of analytification, see Prop. 2.3.3.
- We show how the middle convolution of perverse sheaves is related to the middle convolution of lisse sheaves, see Section 2.4.
- Analogous to [18], Thm. 5.5.4, we prove that independence-of-$\ell$ is preserved by convolution, see Thm. 2.5.1. The theory of Hecke characters and a result of Henniart [15] imply then that over $\mathbb{Q}$, the determinant of the middle convolution is the product of the geometric determinant with a finite character and a power of the cyclotomic character.
- We give a motivic interpretation of the middle convolution, see Thm. 2.6.1. This is close to the results in [18], Chap. 8 (but not contained in them).

We then give two applications of the above methods. The first one relies on the irreducibility criterion of Thm. 1.7.1 and the above mentioned application of
the theory of Hecke characters to the determinant (see Cor. 3.2.2):

**Theorem II.** Let $\mathbb{F}_q$ be the finite field of order $q = \ell^k$, where $k \in \mathbb{N}$. Then the special linear group $\text{SL}_{2n+1}(\mathbb{F}_q)$ occurs regularly as Galois group over $\mathbb{Q}(t)$ if

$$\quad q \equiv 5 \mod 8 \quad \text{and} \quad n > 6 + 2\varphi((q - 1)/4)$$

($\varphi$ denoting Euler’s $\varphi$-function).

The theorem implies that, under the above restrictions, the simple groups $\text{PSL}_{2n+1}(\mathbb{F}_q)$ occur regularly as Galois group over $\mathbb{Q}(t)$. The latter result is the first result on regular Galois realizations of the groups $\text{PSL}_n(\mathbb{F}_q)$ over $\mathbb{Q}(t)$, where

$$(n, q - 1) = [\text{PGL}_n(\mathbb{F}_q) : \text{PSL}_n(\mathbb{F}_q)] > 2.$$  

We also realize the underlying profinite special linear groups regularly as Galois groups over $\mathbb{Q}(t)$ (Thm. 3.2.1).

In the appendix to this article, written jointly by S. Reiter and the author, we prove the existence of a new motivic lisse sheaf $\mathcal{H}$ whose monodromy is dense in the exceptional algebraic group of type $G_2$. This result relies on our results on the monodromy of the middle convolution with Kummer sheaves (Section 1.7) and on the motivic interpretation of the middle convolution (Section 2.6). The lisse sheaf $\mathcal{H}$ is not rigid, contrary to the local systems considered in [6], [9]. The explicit determination of the monodromy seems to be necessary in this case, since there exist other local systems with the same local monodromy whose monodromy is dense in the special orthogonal group $\text{SO}_7$ (see the remark at the end of the appendix). It seems to be remarkable, that the weight of (a certain arithmetic extension of) $\mathcal{H}$ is 4, contrary to the systems of [9], [6], where the weight is 6.

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The appendix has grown out from a question of P. Deligne about a possible motivic interpretation of the other $G_2$-rigid (but non-$GL_7$-rigid) local systems that exist besides the $G_2$-rigid local systems considered in [9]. The authors are indebted to Professors P. Deligne and N. Katz for their interest and several valuable remarks and discussions on the subject.

1 Convolution of local systems

Throughout this section we will write $\mathbb{A}^1$, $\mathbb{P}^1$, . . . instead of $\mathbb{A}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C})$, . . . and view these objects equipped with their associated topological and complex analytic structures. Let $X$ be a connected topological manifold. The multiplication in the fundamental group $\pi_1(X,x)$ is induced by the path product for which we use the following convention: Let $\gamma, \gamma'$ be two closed paths at $x \in X$. Then their product $\gamma \gamma'$ is given by first walking along $\gamma'$ and then walking along $\gamma$. Endomorphisms of vector spaces act from the right throughout the article.

1.1 Some notation. Let $U_0 = \mathbb{A}^1 \setminus u$, where $u := \{u_1, \ldots, u_r\}$ is a finite subset of $\mathbb{A}^1$. It is well known that there exist generators $\alpha_1, \ldots, \alpha_{r+1}$ of $\pi_1(U_0, u_0)$ which are the homotopy classes of simple loops around the points $u_1, \ldots, u_r, \infty$ (resp.) and which satisfy the product relation $\alpha_1 \cdots \alpha_{r+1} = 1$. Let $R$ be a commutative ring with a unit and let $X$ be a connected topological manifold. A local system of $R$-modules is a sheaf $\mathcal{V}$ which is locally isomorphic to $R^n$ for some $n \in \mathbb{N}$. The number $n$ is called the rank of $\mathcal{V}$ and is denoted by $\text{rk}(\mathcal{V})$. Let $\text{LS}_R(X)$ denote the category of local systems of $R$-modules on $X$. Any local system $\mathcal{V} \in \text{LS}_R(X)$ gives rise to its monodromy representation

$$\rho_\mathcal{V} : \pi_1(X,x) \rightarrow \text{GL}(\mathcal{V}), \gamma \mapsto \rho_\mathcal{V}(\gamma),$$

where $\mathcal{V} = \mathcal{V}_x$ denotes the stalk of $\mathcal{V}$ at $x$, cf. [3].

1.1.1 Definition. Let $U_0 := \mathbb{A}^1 \setminus u$ be as above and fix generators $\alpha_1, \ldots, \alpha_{r+1}$ of $\pi_1(U_0, u_0)$ as above. Let $\mathcal{V}$ be a local system on $U_0$ with monodromy representation $\rho_\mathcal{V}$. Then $\rho_\mathcal{V}$ is uniquely determined by the monodromy tuple of $\mathcal{V}$ (with respect to $\alpha_1, \ldots, \alpha_r$):

$$T_\mathcal{V} = (T_1, \ldots, T_{r+1}) \in \text{GL}(\mathcal{V})^{r+1}, \quad T_i = \rho_\mathcal{V}(\alpha_i).$$

The following definition is motivated by the results in [18], Chap. 5:

1.1.2 Definition. A local system $\mathcal{V} \in \text{LS}_K(U_0)$ has the property $T$, if it is irreducible and if there exist at least two components $T_i, T_j$, $i < j \leq r$, of the
monodromy tuple \( T_V = (T_1, \ldots, T_r, T_{r+1}) \in \text{GL}(V) \) which are not the identity. Let \( T_K(U_0) \) be the category of local systems in \( \text{LS}_K(U_0) \) having the property \( T \).

1.1.3 Definition. If \( \mathcal{V} \) is a local system on \( U_0 \) and if \( j : U_0 \to \mathbb{P}^1 \) is the natural inclusion, then the parabolic cohomology group of \( \mathcal{V} \) is defined to be \( H^1_p(U_0, \mathcal{V}) := H^1(\mathbb{P}^1, j_* \mathcal{V}) \), cf. [10], Section 1.

1.2 The middle convolution of local systems. For \( u := \{x_1, \ldots, x_p\} \subseteq \mathbb{A}^1 \) and \( v := \{y_1, \ldots, y_q\} \subseteq \mathbb{A}^1 \), set

\[
u * v := \{x_i + y_j \mid i = 1, \ldots, p, \ j = 1, \ldots, q\} \subseteq \mathbb{A}^1.
\]

Let \( U_1 := \mathbb{A}^1 \setminus u, U_2 := \mathbb{A}^1 \setminus v \) and \( S := \mathbb{A}^1 \setminus u * v \). Set

\[
\hat{f}(x, y) := \prod_{i=1}^{p} (x - x_i) \prod_{j=1}^{q} (y - x - y_j) \prod_{i,j} (y - (x_i + y_j))
\]

and let \( f \in \mathbb{C}[x, y] \) be the associated reduced polynomial. If the cardinality of \( u * v \) is equal to \( p \cdot q \), we call \( u * v \) generic. Let \( w := \{(x, y) \in \mathbb{A}^2 \mid f(x, y) = 0\} \) and let \( U := \mathbb{A}^2 \setminus w \). The space \( U \) is equipped with the two projections

\[
\text{pr}_1 : U \longrightarrow U_1, \ (x, y) \longmapsto x, \ \text{pr}_2 : U \longrightarrow S, \ (x, y) \longmapsto y,
\]

and the subtraction map

\[
d : U \longrightarrow U_2, \ (x, y) \longmapsto y - x.
\]

Let \( j : U \longrightarrow \mathbb{P}^1_{\mathbb{S}} \) denote the canonical inclusion. We further fix a basepoint \((x_0, y_0)\) of \( U \) and let \( U_0 := \text{pr}_2^{-1}(y_0) \).

1.2.1 Definition. For \( \mathcal{V}_1 \in \text{LS}_R(U_1) \) and \( \mathcal{V}_2 \in \text{LS}_R(U_2) \) consider the tensor product

\[
\mathcal{V}_1 \boxtimes \mathcal{V}_2 = \text{pr}_1^* \mathcal{V}_1 \otimes \text{d}^* \mathcal{V}_2 \in \text{LS}_R(U).
\]

Then the middle convolution of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) is the higher direct image

\[
\mathcal{V}_1 * \mathcal{V}_2 := R^1(\text{pr}_2)_*(j_!(\mathcal{V}_1 \boxtimes \mathcal{V}_2)).
\]

The following proposition follows from the local triviality of \( \text{pr}_2 \) (using the arguments of [14], Prop. 8.1):

1.2.2 Proposition. The middle convolution \( \mathcal{V}_1 * \mathcal{V}_2 \) is a local system on \( S \) whose stalk \((\mathcal{V}_1 * \mathcal{V}_2)_{y_0}\) at \( y_0 \) is canonically isomorphic to the parabolic cohomology \( H^1_p(U_0, \mathcal{V}_1 \boxtimes \mathcal{V}_2|_{U_0}) \).
1.3 The basic topological setup Throughout this and the following subsections, we assume that the coefficient ring $R$ is a field $K$ and that $u \ast v$ is generic. The first projection yields an identification of $U_0 = \text{pr}_2^{-1}(y_0)$ with

$$\mathbb{A}^1 \setminus d,$$

where $d = \{x_1, \ldots, x_p, y_0 - y_1, \ldots, y_0 - y_q\}$.

Using a suitable homeomorphism $\mathbb{P}^1 \to \mathbb{P}^1$ we can assume that we are in the following situation: The sets $u = \{x_1, \ldots, x_p\}$, $v = \{y_1, \ldots, y_q\}$, $\{y_0\}$ are element-wise real and

$$0 < x_1 < x_2 < \ldots < x_p < y_0 - y_1 < y_0 - y_2 < \ldots < y_0 - y_q.$$  

Moreover, we can assume that

\begin{equation}
|x_p - x_1| < |y_{i+1} - y_i| \quad \text{for} \quad i = 1, \ldots, q - 1.
\end{equation}

We choose generators $\alpha_1, \ldots, \alpha_{p+q}$ of $\pi_1(\mathbb{A}^1 \setminus d, x_0)$ as follows:

We also choose generators $\delta_{i,j}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$ of $\pi_1(S, y_0)$ as follows:

The long exact homotopy sequence associated to the locally trivial fibration $\text{pr}_2 : U \to S$ yields a split short exact sequence

$$1 \longrightarrow \pi_1(\mathbb{A}^1 \setminus d, x_0) \longrightarrow \pi_1(U, (x_0, y_0)) \longrightarrow \pi_1(S, y_0) \longrightarrow 1.$$  

By embedding $S \setminus \{y_1 + x_0, \ldots, y_q + x_0\}$ and $\mathbb{A}^1 \setminus d = U_0$ into $U$ we can think of the paths $\alpha_k$ and $\delta_{i,j}$ as loops in $U$. Then the path $\delta_{i,j} \alpha_k := \delta_{i,j} \alpha_k \delta_{i,j}^{-1}$ becomes a loop in $U$ and it is in the kernel of the third arrow of the above exact sequence. Therefore it has a unique inverse image under the second arrow; this defines $\delta_{i,j} \alpha_k$.
as an element of the fundamental group of the fiber. This gives us an action of \( \pi_1(S, y_0) \) on \( \pi_1(U_0, (x_0, y_0)) \). It is standard, that for \( i = 1, \ldots, p \), the following formula holds (cf. [9], Section 4.1):

\[
(\delta_i \alpha_1, \ldots, \delta_i \alpha_{p+1}) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i^{\alpha p+1}, [\alpha_i, \alpha_{p+1}], \ldots, \alpha_p^{[\alpha_i, \alpha_{p+1}]}, \alpha_i, \ldots, \alpha_p),
\]

where \([\alpha_i, \alpha_j] = \alpha_i^{-1} \alpha_j^{-1} \alpha_i \alpha_j\) and

\[
\alpha_i^{\alpha p+1} = \alpha_{p+1}^{\alpha_i} \alpha_i, \quad [\alpha_i, \alpha_{p+1}] = (\alpha_i, \alpha_{p+1})^{-1} \alpha_{k+1}[\alpha_i, \alpha_{p+1}] \quad \text{etc.}
\]

1.4 The rank of the middle convolution Let \( V_1 \in LS_R(U_1) \) and \( V_2 \in LS_R(U_2) \). The homotopy base \( \alpha_1, \ldots, \alpha_{p+q} \) on \( U_0 \) given in Section 1.3 induces homotopy bases on \( U_1 \) and \( U_2 \) by looking at the images of the maps \( pr_1|_{U_0} \) and \( d|_{U_0} \) (resp.). Let the monodromy tuples of \( V_1 \) and \( V_2 \) with respect to these homotopy bases be

\[
T_{V_1} = (A_1, \ldots, A_{p+1}) \in GL(V_1)^{p+1}
\]

and

\[
T_{V_2} = (B_1, \ldots, B_{q+1}) \in GL(V_2)^{q+1}
\]

(resp.). Then the monodromy tuple of \( V_1 \otimes V_2|_{U_0} \) is of the following form:

\[
(1.4.1) \quad T_{V_1 \otimes V_2|_{U_0}} = (C_1 = A_1 \otimes 1_{V_2}, \ldots, C_p = A_p \otimes 1_{V_2}, C_{p+1} = 1_{V_1} \otimes B_1, \ldots, C_{p+q} = 1_{V_1} \otimes B_q, C_{p+q+1} = A_{p+1} \otimes B_{q+1}) \in GL(V_1 \otimes V_2)^{p+q+1}.
\]

1.4.1 Proposition. Suppose that \( R = K \) is a field and that \( V_1 \in LS_K(U_1) \) has no global sections. Let \( \dim_K V_i = n_i, i = 1, 2 \). Then

\[
(1.4.2) \quad \text{rk}(V_1 \ast V_2) = (p + q - 1)n_1n_2 - \sum_{i=1}^{p} n_2 \dim_K \ker(A_i - 1_{V_1})
\]

\[
- \sum_{j=1}^{q} n_1 \dim_K \ker(B_j - 1_{V_2}) - \dim_K \ker(A_{p+1} \otimes B_{q+1} - 1_{V_1} \otimes V_2).
\]

Proof: We have to determine the dimension of the stalk \( (V_1 \ast V_2)|_{y_0} \) which is isomorphic to the parabolic cohomology group \( H^2_{LS}(U_0, V_1 \otimes V_2|_{U_0}) \) by Prop. 1.2.2. It follows from (1.4.1) and the properties of the tensor product that

\[
\dim_K \ker(C_i - 1_{V_1} \otimes V_2) = n_2 \dim_K \ker(A_i - 1_{V_1}), \quad \text{for } i = 1, \ldots, p,
\]
and
\[ \dim_K \ker(C_i - 1_{V_i \otimes V_2}) = n_1 \dim_K \ker(B_{i-p} - 1_{V_2}), \quad \text{for} \quad i = p + 1, \ldots, p + q. \]

Let \( \mathcal{V} = \mathcal{V}_1 \boxtimes \mathcal{V}_2|_{U_0} \). The Euler-Poincaré Formula implies that
\[
\chi(\mathbb{P}^1, j_* \mathcal{V}) = h^0(\mathbb{P}^1, j_* \mathcal{V}) - h^1(\mathbb{P}^1, j_* \mathcal{V}) + h^2(\mathbb{P}^1, j_* \mathcal{V})
\]
\[
= \chi(U_0) \cdot \rk(\mathcal{V}) + \sum_{i=1}^{p+q} \dim(\ker(C_i - 1)) + \dim(\ker(C_{p+q+1} - 1))
\]
\[
= (1 - p - q)n_1n_2 + \sum_{i=1}^{p} n_2 \dim_K \ker(A_i - 1_{V_1})
\]
\[
+ \sum_{j=1}^{q} n_1 \dim_K \ker(B_j - 1_{V_2}) + \dim_K \ker(A_{p+1} \otimes B_{q+1} - 1_{V_1 \otimes V_2}).
\]

Since \( \mathcal{V}_1 \) has no global sections, the local system \( \mathcal{V} \) has no global sections. Therefore,
\[
0 = h^0(U_0, \mathcal{V}) = h^0(\mathbb{P}^1, j_* \mathcal{V}) = h^2(\mathbb{P}^1, j_* \mathcal{V}),
\]
which implies the claim (the third equality follows from Poincaré duality, using the same arguments for the dual of \( \mathcal{V} \)). \qed

1.5 **The twisted evaluation map**  As in the last section, let
\[
\rho := \rho_{\mathcal{V}_1 \boxtimes \mathcal{V}_2|_{U_0}} : \pi_1(S, y_0) \longrightarrow \GL(V_1 \otimes V_2)
\]
be the monodromy representation of \( \mathcal{V}_1 \boxtimes \mathcal{V}_2|_{U_0} \). The Hochschild-Serre spectral sequence implies that \( H^1(U_0, \mathcal{V}_1 \boxtimes \mathcal{V}_2|_{U_0}) \simeq H^1(\pi_1(U_0), V_1 \otimes V_2) \). The group cohomology \( H^1(\pi_1(U_0), V_1 \otimes V_2) \) is the quotient of \( C^1(\pi_1(U_0), V_1 \otimes V_2) \) by \( B^1(\pi_1(U_0), V_1 \otimes V_2) \), where
\[
C^1(\pi_1(U_0), V_1 \otimes V_2) := \{ (\delta : \pi_1(U_0) \to V_1 \otimes V_2) \mid \delta(\alpha \beta) = \delta(\alpha) \rho(\beta) + \delta(\beta) \quad \forall \alpha, \beta \in \pi_1(U_0) \}
\]
\[
(1.5.1)
\]
and
\[
B^1(\pi_1(U_0), V_1 \otimes V_2) := \{ \delta_v \mid v \in V_1 \otimes V_2, \quad \delta_v(\gamma) = v(1 - \rho(\gamma)) \quad \forall \gamma \in \pi_1(U_0) \}.
\]

1.5.1 **Definition.** The linear map
\[
\tau : C^1(\pi_1(U_0), V_1 \otimes V_2) \longrightarrow (V_1 \otimes V_2)^{pq},
\]
\[
\delta \longmapsto (\delta([\alpha_1, \alpha_{p+1}]), \cdots, \delta([\alpha_p, \alpha_{p+1}]), \cdots, \delta([\alpha_1, \alpha_{p+q}]), \cdots, \delta([\alpha_p, \alpha_{p+q}]),
\]
is called the twisted evaluation map.
1.5.2 Lemma. Assume that $V_1 \in T_K(U_1)$ (see Def. 1.1.2) with $T_{V_1} = (A_1, \ldots, A_{p+1})$, and assume that $V_2 \in \text{LS}_K(U_2)$ is a rank one system with $T_{V_2} = (\lambda_1, \ldots, \lambda_{q+1}) \in \text{GL}_1(K)^{q+1}$ and $\lambda_i \neq 1$ for $i = 1, \ldots, q$.

Then the kernel of the twisted evaluation map coincides with the coboundaries $B^1(\pi_1(U_0), V)$, where $V = V_1 \otimes V_2 \simeq V_1$. Hence, the twisted evaluation map $\tau$ induces an embedding

$$\tau : H^1(\pi_1(U_0), V_1 \otimes V_2) \longrightarrow (V)^{pq}.$$ 

Proof: The cocycle relation (1.5.1) implies

$$\delta([\alpha_i, \alpha_{p+j}]) = \delta(\alpha_i)(\lambda_j - 1) + \delta(\alpha_{p+j})(1 - A_i), \quad \text{for } i = 1, \ldots, p, j = 1, \ldots, q.$$ 

By definition, the coboundaries $B^1(\pi_1(U_0), V)$ are therefore contained in $\ker(\tau)$. Conversely, assume that $\delta \in \ker(\tau)$. Since $\lambda_j$ is assumed to be $\neq 1$ one has

$$\delta(\alpha_i) = \frac{1}{1 - \lambda_j} \delta(\alpha_{p+j})(1 - A_i), \quad \text{for } i = 1, \ldots, p, j = 1, \ldots, q.$$ 

Set $v_j := \frac{1}{1 - \lambda_j} \delta(\alpha_{p+j})$. By (1.5.3),

$$v_j(1 - A_i) = v_{j'}(1 - A_i) \quad \text{for } i = 1, \ldots, p, \quad \text{and } j, j' = 1, \ldots, q.$$ 

If $v_j \neq v_{j'}$ for some $j \neq j'$, then the above equality shows that the vector $v_j - v_{j'}$ spans a trivial $\langle A_1, \ldots, A_p \rangle$-submodule of $V_1$. But since $V_1$ was assumed to be contained in $T_K(U_1)$ this is impossible. Thus $v_j = v_{j'}$ for $j \neq j'$ and

$$\delta(\alpha_i) = \frac{1}{1 - \lambda_q} \delta(\alpha_{p+q})(1 - \rho(\alpha_i)).$$ 

Consequently, the element $\delta$ is an element of $B^1(\pi_1(U_0), V)$. \hfill \Box

1.6 Convolution with Kummer sheaves. We work in the setup of the last sections with $U_1 = \mathbb{A}^1 \setminus \{u \} (u = \{x_1, \ldots, x_p\})$ and $U_2 = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. Let $K$ be a field and let $\mathcal{V}_\chi \in \text{LS}_K(\mathbb{G}_m)$ be the Kummer sheaf associated to a character

$$\chi : \pi_1(\mathbb{G}_m) \longrightarrow K^\times, \quad \alpha \mapsto \lambda \neq 1,$$

where $\alpha$ is a generator of $\pi_1(\mathbb{G}_m)$ moving counterclockwise around the origin. Let $\mathcal{V} \in T_K(U_1)$. By Prop. 1.2.2, the middle convolution $\mathcal{V} \ast \mathcal{V}_\chi$ is a local system on
$S = \Delta^1_y \setminus u$, and can therefore be seen as a local system on $U_1$. In this way, we obtain a functor

$$MC_\chi : LS_K(U_1) \rightarrow LS_K(U_1), \quad \mathcal{V} \mapsto MC_\chi(\mathcal{V}) := \mathcal{V} \ast \mathcal{V}_\chi.$$ 

The middle convolution $MC_\chi(\mathcal{V})$ is a sub-local system of $R^1(pr_2)_*(\mathcal{V} \boxtimes \mathcal{V}_\chi)$. We have

$$R^1(pr_2)_*(\mathcal{V} \boxtimes \mathcal{V}_\chi)|_{y_0} \simeq H^1(U_0, \mathcal{V} \boxtimes \mathcal{V}_\chi|_{U_0}) \simeq H^1(\pi_1(U_0, (x_0, y_0)), V),$$

where $V = V_1 \otimes V_2 \simeq V_1$ is the stalk of $\mathcal{V} \boxtimes \mathcal{V}_\chi|_{U_0}$ at $(x_0, y_0)$ with its natural action of $\pi_1(U_0, (x_0, y_0))$. The monodromy operation of $\delta \in \pi_1(S, y_0)$ on the stalk $R^1(pr_2)_*(\mathcal{V} \boxtimes \mathcal{V}_\chi)|_{y_0}$ is given by conjugating the arguments of the 1-cocycles in $C^1(\pi_1(U_0), V)$ with the inverse of $\delta$, cf. [10], Lemma 3.3. Under the twisted evaluation map $\tau : C^1(\pi_1(U_0), V) \rightarrow V^p$, this action of $\pi_1(S, y_0)$ on $H^1(\pi_1(U_0), V)$ defines an action of $\pi_1(S, y_0)$ on $V^p$. Let $\tilde{D}_i (i = 1, \ldots, p) \in GL(V^p)$ be the automorphisms induced in this way by the homotopy generators $\delta_i := \delta_{i,1}$ of $\pi_1(S) (i = 1, \ldots, p)$, where the $\delta_{i,1}$ are as in Section 1.3.

**1.6.1 Lemma.** Let $(A_1, \ldots, A_{p+1}) \in GL(V)^{p+1}$ be the monodromy tuple of $\mathcal{V}$. Then for $i = 1, \ldots, p$, the automorphism $\tilde{D}_i \in GL(V^p)$ is a block matrix which is the identity block matrix (the block structure induced by $V^p$) outside the $i$-th block row and the $i$-th block row is as follows:

$$(\lambda(A_1 - 1), \ldots, \lambda(A_{i-1} - 1), \lambda A_i, (A_{i+1} - 1), \ldots, (A_p - 1)).$$

**Proof:** The cocycle relation (1.5.1) implies that

$$\delta(\alpha[[\beta, \gamma]|\epsilon]) = \delta(\alpha\epsilon) + \delta([\beta, \gamma])\rho(\epsilon)$$

and $\delta(\alpha^{-1}) = -\delta(\alpha)\rho(\alpha)^{-1}$. If $m < i$, then these formulas together with (1.3.2) yield

$$\delta[\alpha_m, \alpha_{p+1}]\tilde{D}_i = \delta[\alpha_m^{(\delta_i)^{-1}}, \alpha_{p+1}^{(\delta_i)^{-1}}]$$

$$= \delta[\alpha_m, \alpha_{p+1}^{\alpha_p\alpha_{p+1}}]$$

$$= \delta(\alpha_m^{-1}\alpha_{p+1}^{-1}[\alpha_i, \alpha_{p+1}]\alpha_m[\alpha_{p+1}, \alpha_i]\alpha_{p+1})$$

$$= \delta[\alpha_i, \alpha_{p+1}]\lambda_1(A_m - 1) + \delta[\alpha_m, \alpha_{p+1}].$$

If $m = i$ then

$$\delta[\alpha_i, \alpha_{p+1}]\tilde{D}_i = \delta[\alpha_i^{\alpha_{p+1}}, \alpha_{p+1}^{\alpha_i\alpha_{p+1}}]$$

$$= \delta(\alpha_{p+1}^{-1}\alpha_i^{-1}[\alpha_i, \alpha_{p+1}]\alpha_i\alpha_{p+1})$$

$$= \delta[\alpha_i, \alpha_{p+1}]A_i\lambda_1.$$
where in the last equation we have used (1.6.1) and $\delta(1) = 0$. If $m > i$, then
\[
\delta[\alpha_m, \alpha_{p+1}] \tilde{D}_i = \delta[\alpha_m^{[\alpha_i, \alpha_{p+1}]}, \alpha_{p+1}] = \delta[\alpha_i, \alpha_{p+1}](A_m - 1) + \delta[\alpha_m, \alpha_{p+1}].
\]
This proves the claim. □

Let
\[ K = \{ (w_1, \ldots, w_p) \mid w_i \in \text{Im}(A_i - 1) \} \subseteq V^p, \]
\[ L = \{ (w_1 A_2 \cdots A_p, w_2 A_3 \cdots A_p, \ldots, w_p) \mid w_i \in \text{Im}(A_1 \cdots A_p \lambda_1 - 1) \} \subseteq V^p, \]
and let $W := K \cap L \subseteq V^p$. The automorphisms $\tilde{D}_i (i = 1, \ldots, p)$ can easily be seen to stabilize the subspaces $K$ and $L$ of $V^p$ and hence $W$. Let $A = (A_1, \ldots, A_{p+1})$, and let
\[ \text{MC}_\chi(A) := (D_1, \ldots, D_{p+1}) \in \text{GL}(W)^{p+1} \]
be the tuple of linear transformations on $W$ induced by $\tilde{D}_1, \ldots, \tilde{D}_p$ and $\tilde{D}_{p+1} := (D_1 \cdots D_p)^{-1}$ (resp.).

1.6.2 Theorem. Let $V \in T_K(U_1)$ and let $A \in \text{GL}(V)^{p+1}$ be the monodromy tuple of $V$ with respect to $\alpha_1, \ldots, \alpha_p$. Then the stalk of $\text{MC}_\chi(V)$ at $y_0$ is canonically isomorphic to $W$ and the monodromy tuple of $\text{MC}_\chi(V)$ with respect to the homotopy base $\delta_1, \ldots, \delta_p$ is given by $\text{MC}_\chi(A) \in \text{GL}(W)^{p+1}$.

Proof: The parabolic cohomology $H^1_p(U_0, V \otimes V_\chi|_{U_0}) = \text{MC}_\chi(V)_{y_0}$ can be seen as the subspace of $H^1(\pi_1(U_0), V)$ consisting of the elements $[\delta] \in H^1(\pi_1(U_0), V)$, where $\delta \in C^1(\pi_1(U_0), V)$ is a parabolic cocycle, i.e.,
\[ \delta(\gamma) \in \text{Im}(\rho_{V \otimes V_\chi|_{U_0}}(\gamma) - 1), \quad \forall \gamma \in \pi_1(U_0), \]
see [10], Lemma 1.2. Let $W' \subseteq V^p$ be the image of the parabolic cohomology group $H^1_p(U_0, V \otimes V_\chi|_{U_0})$ in $V^p$ under the twisted evaluation map and let $(w_1, \ldots, w_p) \in W'$. We have $w_i = \delta([\alpha_i, \alpha_{p+1}])$, where $\delta$ is some parabolic cocycle and $\alpha_i, i = 1, \ldots, p + 1$, is the homotopy base of $\pi_1(U_0)$ chosen according to the basic topological setup. By (1.6.2) we have $\delta(\alpha_i) \in \text{Im}(A_i - 1)$. The formula in (1.5.2) implies therefore that $w_i \in \text{Im}(A_i - 1)$ for $i = 1, \ldots, p$, so $W' \subseteq K$. Similarly, the condition
\[ \delta(\alpha_1 \cdots \alpha_{p+1}) \in \text{Im}(A_1 \cdots A_p \cdot \lambda_1 - 1) \text{ for all } [\delta] \in H^1_p(U_0, V) \]
forces $W' \subseteq L$ and hence $W' \subseteq K \cap L$. By Prop. 1.4.1, the rank of $\text{MC}_\chi(V)$ is equal to
\[ pn - \sum_{i=1}^p \dim(\ker(A_i - 1)) - \dim(\ker(A_{p+1} \lambda^{-1} - 1)). \]
Let $V^*$ denote the dual space of $V$ and consider the column space $(V^*)^p$ as dual space of $V^p$. Let $K^* \subseteq (V^*)^p$ and $L^* \subseteq (V^*)^p$ be the annihilators of the spaces $K$ and $L$. It is not hard to verify that since $\lambda \neq 1$, one has $K^* \cap L^* = 0$, see [7], Lemma 2.7. This implies that

$$\dim(W) = \dim(K \cap L) = \dim(\text{Ann}_{V^p}(K^* \oplus L^*)) = pn - \dim(K^*) - \dim(L^*),$$

compare to [7], Lemma 2.7. It follows therefore from dimension reasons that $W' = W$, which implies the claim. 

1.6.3 Remark. One can switch from the homotopy base $\delta_1, \ldots, \delta_p$ of $\pi_1(S, y_0)$ back to the original homotopy base $\alpha_1, \ldots, \alpha_p$ of $\pi_1(S, x_0)$ by connecting $x_0$ with $y_0$ using a path $\gamma$ in the lower half-plane. In this way, one obtains a well defined transformation $MC_\chi$ on the level of representations of $\pi_1(S, x_0)$.

By construction, the tuple transformation $MC_\lambda$ considered above is the dual of the tuple transformation $MC_\lambda$ considered in [8] and [7] (we use right-actions on row spaces, while [8] and [7] use left-actions on column spaces). Since irreducibility is preserved under dualization, Theorem 2.4 of [8] implies the following:

1.6.4 Corollary. If $V \in T_K(U_1)$, then the local system $MC_\chi(V)$ is again irreducible.

1.7 Irreducibility criteria for the middle convolution.

1.7.1 Theorem. Let $V_1 \in T_K(U_1)$ with $T_{\tilde{V}_1} = (A_1, \ldots, A_{p+1}) \in \text{GL}(V)^{p+1}$ and assume that $V_2 \in \text{LS}_K(U_2)$ is a rank-one system with

$$T_{V_2} = (\lambda_1, \ldots, \lambda_{q+1}) \in \text{GL}(K)^{q+1} \quad \text{and} \quad \lambda_i \neq 1 \quad \text{for} \quad i = 1, \ldots, q.$$

Assume that $u \ast v$ is generic. Then the local system $V_1 \ast V_2 \in \text{LS}_K(S)$ is irreducible if

$$(p - 2)n - \sum_{i=1}^{p} \dim_{K}(\ker(A_i - 1)) > 0.$$ 

Proof: This follows from induction on $q$. For $q = 1$ this is Cor. 1.6.4. If $q > 1$, then we can assume that $V_1 \ast \tilde{V}_2$ is irreducible, where

$$\tilde{V}_2 \in \text{LS}_K(A_1 \setminus \{y_2, \ldots, y_q\}) \quad \text{with} \quad T_{\tilde{V}_2} = (\lambda_2, \ldots, \lambda_q, \lambda_1 \cdot \lambda_{q+1}).$$

Let

$$p_1 : (V^p)^q \rightarrow V^p, (v_1, \ldots, v_q) \mapsto v_1,$$
and

$$p_2 : (V^p)^q \rightarrow (V^p)^{q-1}, (v_1, \ldots, v_q) \mapsto (v_2, \ldots, v_q).$$

Let

$$G_1 := \langle \delta_{i,1}, i = 1, \ldots, p \rangle \leq \pi_1(S,y_0)$$

and

$$G_2 := \langle \delta_{i,j}, i = 1, \ldots, p, j = 2, \ldots, q \rangle \leq \pi_1(S,y_0).$$

By Lemma 1.5.2, there is an embedding

$$\tau : (V_1 \ast V_2)_{y_0} = H^1_p(U_0, V_1 \boxtimes V_2 | U_0) \rightarrow (V^p)^q.$$

We can further assume that $$y_1 = 0.$$ By the assumptions defined in the basic topological setup, the $$G_1$$-module $$\text{Im}(p_1 \circ \tau)$$ is then isomorphic to the monodromy representation of $$\text{MC}_{\lambda_1}(V_1)$$ and is hence irreducible by Cor. 1.6.4. Since $$\pi_1(S,y_0)$$ is the free product of $$G_1$$ and $$G_2,$$ and since the rank of $$\text{MC}_{\lambda_1}(V_1) \geq pn - \sum_{i=1}^p \text{dim}(\ker(A_i - 1))$$ by Prop. 1.4.1, it follows that $$(V_1 \ast V_2)_{y_0}$$ contains an irreducible $$\pi_1(S,y_0)$$-submodule $$W_1$$ of rank greater than or equal to

$$n_1 = pn - \sum_{i=1}^p \text{dim}(\ker(A_i - 1)).$$

By the induction hypothesis, the $$G_2$$-module

$$(V_1 \ast \tilde{V}_2)_{y_0} = H^1_p(A^1 \setminus (\{ u \cup \{ y_0 - y_2, \ldots, y_0 - y_q \} \}, V_1 \boxtimes \tilde{V}_2 | A^1 \setminus (\{ u \cup \{ y_0 - y_2, \ldots, y_0 - y_q \} \}))$$

is irreducible. Moreover, the image of the $$G_2$$-module-homomorphism

$$p_2 \circ \tau : (V_1 \ast V_2)_{y_0} \rightarrow V^{p(q-1)}$$

coincides with the image of of the $$G_2$$-module-homomorphism

$$\tau : (V_1 \ast \tilde{V}_2)_{y_0} \rightarrow V^{p(q-1)}.$$ 

By Prop. 1.4.1, the $$\pi_1(S,y_0)$$-module $$(V_1 \ast V_2)_{y_0}$$ contains therefore an irreducible submodule $$W_2$$ of rank greater than or equal to

$$n_2 := (p + q - 3)n - \sum_{i=1}^p \text{dim}_K(\ker(A_i - 1)).$$

By Prop. 1.4.1, the rank of $$V_1 \ast V_2$$ is smaller than or equal to

$$n_3 = (p + q - 1)n - \sum_{i=1}^p \text{dim}_K(\ker(A_i - 1)).$$
One has
\[ n_1 + n_2 = 2p n + q n - 3 n - 2 \cdot \sum_{i=1}^{p} \dim_K(\ker(A_i - 1)) \]

Thus
\[ n_1 + n_2 - n_3 \geq (p - 2)n - \sum_{i=1}^{p} \dim_K(\ker(A_i - 1)) . \]

By assumption, \((p - 2)n - \sum_{i=1}^{p} \dim_K(\ker(A_i - 1)) > 0\), thus \(n_1 + n_2 > n_3\). It follows that the intersection of the irreducible submodules \(W_1\) and \(W_2\) is non-trivial and hence \(\mathcal{V}_1 * \mathcal{V}_2\) is irreducible (the spaces \(W_1\) and \(W_2\) clearly generate \((\mathcal{V}_1 * \mathcal{V}_2)_{y_0}\) as a vector space). \(\square\)

1.8 The local monodromy of the middle convolution. In the following, an expression \(J(\alpha, l)(\alpha \in K, l \in \mathbb{N})\) denotes a Jordan block of length \(l\) and eigenvalue \(\alpha\).

1.8.1 Lemma. Let \(K\) be algebraically closed. Let \(\mathcal{V}_1 \in \mathcal{T}_K(U_1)\) with \(T\mathcal{V}_1 = (A_1, \ldots, A_{p+1}) \in \text{GL}(V)^{p+1}\) and assume that \(\mathcal{V}_2 \in \mathcal{L}_K(U_2)\) is a rank-one system with \(T\mathcal{V}_2 = (\lambda_1, \ldots, \lambda_{q+1}) \in \text{GL}(K)^{q+1}\) and \(\lambda_i \neq 1\) for \(i = 1, \ldots, q\). Assume that \(u \ast v\) is generic. Let \(D_{i,j} := \rho_{\mathcal{V}_1 \ast \mathcal{V}_2}(\delta_{i,j})\), where the \(\delta_{i,j}\) are the generators of \(\pi_1(S)\) as above. Then the following holds: Every \(\lambda_j\) and every Jordan block \(J(1, 1)\) occurring in the Jordan decomposition of \(A_i\) contribute a Jordan block \(J(\alpha \lambda_j, l')\) to the Jordan decomposition of \(D_{i,j}\), where

\[
l' := \begin{cases} 
  l & \text{if } \alpha \neq 1, \lambda_j^{-1}, \\
  l - 1 & \text{if } \alpha = 1, \\
  l + 1 & \text{if } \alpha = \lambda_j^{-1}. 
\end{cases}
\]

The only other Jordan blocks which occur in the Jordan decomposition of \(D_{i,j}\) are blocks of the form \(J(1, 1)\).

Proof: We can assume that \(y_j = 0\). Let \(S^\circ := \{y \in S \mid 0 < |y| < x_p + \epsilon\}\), where \(\epsilon \in \mathbb{R}_+\) is very small. The restriction of \(pr_2\) to \(pr_2^{-1}(S^\circ)\) gives then rise to a variation of parabolic cohomology groups which is essentially \(\text{MC}_\chi(\mathcal{V}_1)\), where \(\chi\) is the character defined by sending a counterclockwise generator of \(\pi_1(\mathbb{G}_m)\) to \(\lambda = \lambda_j\) (by the conventions defined in the basic topological setup, the other components \(\lambda_k \neq \lambda_k\) do not contribute to the variation, although they contribute to the rank of \(\mathcal{V}_1 \ast \mathcal{V}_2\)). Let \(D_i\) be the \(i\)-th component of the monodromy tuple.
MC_\lambda(T_{V_1}) of MC_\chi(V_1), compare to Thm. 1.6.2. Then D_{i,j} is conjugated to a matrix of the form

\[ D_i \oplus \left( \bigoplus_{k=1}^t (J,1,1) \right), \text{ where } t = \text{rk}(V_1 \ast V_2) - \text{rk}(MC_\lambda(V_1)). \]

Therefore it suffices to prove the claim for D_i instead of D_{i,j}. It is not hard to see that one has an isomorphism

\[ \phi : \text{Im}(D_i - 1) \longrightarrow \text{Im}(A_i - 1), \quad (0, \ldots, w, 0, \ldots, 0) \longrightarrow w(A_i - 1), \]

such that \( \phi \circ D_i = \lambda A_i \circ \phi \) (use [7], Rem. 3.1, and the same arguments as [7], Lemma 4.1). The claim follows from this by an induction on the number of Jordan blocks.

The following lemma gives the monodromy at infinity in the case of the convolution with Kummer sheaves:

1.8.2 Lemma. Let \( V \in T_K(U_1) \) with \( T_V = (A_1, \ldots, A_{p+1}) \in \text{GL}(V)^{p+1} \) and let \( V_\chi \) be a Kummer sheaf associated to a non-trivial character \( \chi : \pi_1(G_m(\mathbb{C})) \rightarrow K^\times, \gamma \mapsto \lambda \). Let \((D_1, \ldots, D_{p+1})\) be the monodromy tuple of \( MC_\chi(V) \). Then the following holds: Every Jordan block \( J(\alpha, l) \) occurring in the Jordan decomposition of \( A_{p+1} \) contributes a Jordan block \( J(\alpha \lambda^{-1}, l') \) to the Jordan decomposition of \( D_{p+1} \), where

\[ l' = \begin{cases} 
   l, & \text{if } \alpha \neq 1, \lambda, \\
   l - 1, & \text{if } \alpha = \lambda, \\
   l + 1, & \text{if } \alpha = 1.
\end{cases} \]

The only other Jordan blocks which occur in the Jordan decomposition of \( D_{p+1} \) are blocks of the form \( J(\lambda^{-1}, 1) \).

Proof: The claim follows from Thm. 1.6.2 and [7], Lemma 4.1 (b), using an induction on the number of Jordan blocks. \( \square \)

2 Convolution of étale sheaves

2.1 Étale sheaves. Let \( R \) be either a field or a normal integral domain which is of finite type over \( \mathbb{Z} \) and let \( S \) be an regular integral scheme of finite type over \( R \). If \( s \) is a closed point of \( S \), then \( \bar{s} \) denotes a geometric point which extends \( s \). If \( U \) is a scheme over \( S \) and if \( \phi : S' \rightarrow S \) is a morphism of schemes, then the basechange of \( U \) which is defined by \( \phi \) is denoted by \( \bar{U}_{S'} \).
Let $\ell$ be an invertible prime in $R$, and let $F$ be either a subfield of $\overline{\mathbb{F}}_\ell$, or a finite extension of $\mathbb{Q}_\ell$, or $\overline{\mathbb{Q}}_\ell$. The category of constructible $F$-sheaves on $S$ is denoted by $\text{Constr}_F(S)$. If $\pi : X \to S$ is smooth and if $\mathcal{V} \in \text{Constr}(X)$, then one has the notion of the higher direct image $R^i\pi_*(\mathcal{V}) \in \text{Constr}(S)$ and the notion of the higher direct image with compact support $R^i\pi_!(*)(\mathcal{V}) \in \text{Constr}(S)$, see [23].

The category of constructible $F$-sheaves on $S$ which are lisse in the sense of [12] is denoted by $\text{LS}^\text{ét}_F(S)$. Any $\mathcal{V} \in \text{LS}^\text{ét}_F(S)$ corresponds to its monodromy representation

$$\rho_\mathcal{V} : \pi_1(S, \bar{\eta}) \to \text{GL}(\mathcal{V}_\bar{\eta}),$$

see [12], V, VI. An object $\mathcal{V} \in \text{LS}^\text{ét}_F(S)$ is called tamely ramified if $\rho_\mathcal{V}$ factors over the tame fundamental group $\pi_1^{\text{tm}}(S, \bar{\eta})$. The category of tamely ramified local systems is denoted by $\text{LS}^\text{tm}_F(S)$. We call $\mathcal{V} \in \text{LS}^\text{tm}_F(S)$ geometrically (absolutely) irreducible, if $\rho_\mathcal{V}|_{\pi_1(S, \bar{s})}$ is (absolutely) irreducible for some (and hence for every, see [17], 8.17.13) geometric point $\bar{s}$ of $S$.

Let $F$ be either a finite extension of $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$, and let $S/R$ as above, where $R$ is finitely generated over $\mathbb{Z}$. A sheaf $\mathcal{V} \in \text{Constr}_F(S)$ is called pure of weight $k \in \mathbb{Z}$ if for any closed point $s$ of $S$, the eigenvalues $\alpha \in \bar{F}^\times$ of the geometric Frobenius element $F_s$ under the natural action of $F_s$ on $\mathcal{V}_s$ satisfy the following condition: The image of $\alpha$ under any embedding $F \to \mathbb{C}$ is an algebraic number which is of complex absolute value $N(s)^{k/2}$, see [5]. A sheaf $\mathcal{V} \in \text{Constr}_F(S)$ is called mixed of weight $\leq k$ if it is an iterated extension of sheaves which are pure of weight $\leq k$. If $\mathcal{V} \in \text{Constr}_F(S)$ is mixed of weight $\leq k$, then the largest quotient of $\mathcal{V}$ which is pure of weight $k$ is denoted by $W^k(\mathcal{V})$.

### 2.2 The middle convolution of étale sheaves

In this section, let $R$ be a field or a normal integral domain which is of finite type over $\mathbb{Z}$. Let $u, v \subset \mathbb{A}^1_R = \text{Spec}(R[x])$ be reduced subschemes, defined by the vanishing of the polynomials

$$g_i(x) = \prod_j (x - x_{ji}) \in R[x], \quad i = 1, 2 \quad \text{(respectively)},$$

with $x_{ji}$ étale over $R$. Let $g_1 \ast g_2(x) \in R[x]$ be the reduced polynomial associated to the polynomial

$$\prod_{j_1, j_2} (x - (x_{j_1} + x_{j_2})) \in R[x]$$

and let $u \ast v \subset \mathbb{A}^1_R$ be the subscheme defined by the vanishing of $g_1 \ast g_2$. Let $U_1 := \mathbb{A}^1_R \setminus u$, $U_2 := \mathbb{A}^1_R \setminus v$ and $S := \mathbb{A}^1_R \setminus u \ast v$. Let further

$$U := \text{Spec}(R[x, y, \frac{1}{g_1(x) \cdot g_2(y - x) \cdot g_1 \ast g_2(y)})].$$
Define

\[ \text{pr}_1 : U \longrightarrow U_1, \quad \text{by} \quad (x,y) \mapsto x, \]
\[ \text{pr}_2 : U \longrightarrow S, \quad \text{by} \quad (x,y) \mapsto y, \]
\[ d : U \longrightarrow U_2, \quad \text{by} \quad (x,y) \mapsto y-x. \]

Let \( j : U \rightarrow \mathbb{P}^1_S = \mathbb{P}^1 \times S \) be the natural inclusion. The second projection \( \mathbb{P}^1_S \rightarrow S \) is denoted by \( \text{pr}_2 \).

**2.2.1 Definition.** (i) The *middle convolution* of \( V_1 \in \text{LS}^\text{ét}_F(U_1) \) and \( V_2 \in \text{LS}^\text{ét}_R(U_2) \) is the constructible sheaf

\[ V_1 \ast V_2 := R^1(\text{pr}_2)_*(j_*(V_1 \boxtimes V_2)) \]
on \( S = \mathbb{A}^1 \setminus u_1 \ast u_2 \), where \( V_1 \boxtimes V_2 := \text{pr}_1^*(V_1) \otimes d^*(V_2) \).

(ii) The *naive-convolution* of \( V_1 \in \text{LS}^\text{ét}_F(U_1) \) and \( V_2 \in \text{LS}^\text{ét}_R(U_2) \) is the constructible sheaf

\[ V_1 \ast_{\text{naive}} V_2 := R^1(\text{pr}_2)_!(V_1 \boxtimes V_2) \in \text{Constr}_F(S). \]

(iii) Let \( \mathcal{V} \in \text{LS}_{\mathbb{Q}_\ell}(U_1) \) and let \( \mathcal{L}_\chi \) be the lisse Kummer sheaf on \( U_2 = \mathbb{G}_m \) associated to a nontrivial character \( \chi : \pi_1^\text{tm}(\mathbb{G}_m) \rightarrow \widehat{\mathbb{Q}}_\ell^\times \). Then we set

\[ \text{MC}_\chi(\mathcal{V}) := \mathcal{V} \ast \mathcal{L}_\chi. \]

**2.3 First properties** Assumptions and notation as in Section 2.1 and 2.2.

**2.3.1 Proposition.** Let \( V_i \in \text{LS}^\text{tm}_F(U_i) \), \( i = 1, 2 \). Then the following holds:

(i) If \( R \) is a subfield of \( \mathbb{C} \), then the middle convolution \( V_1 \ast V_2 \) is lisse on \( \mathbb{A}^1_R \setminus u_1 \ast u_2 \).

(ii) Let \( R \) be a normal integral domain which is of finite type over \( \mathbb{Z} \) and which has a generic point of characteristic zero. Then the naive convolution \( V_1 \ast_{\text{naive}} V_2 \) and the middle convolution \( V_1 \ast V_2 \) are lisse and tame on \( \mathbb{A}^1_R \setminus u_1 \ast u_2 \).

(iii) If \( R \) is an normal integral domain which is of finite type over \( \mathbb{Z} \) and if \( V_1 \) and \( V_2 \) are pure of weight \( n_1 \) and \( n_2 \) (resp.), then

\[ V_1 \ast_{\text{naive}} V_2 \text{ is mixed of weights } \leq n_1 + n_2 + 1 \]

and

\[ V_1 \ast V_2 = W^{n_1+n_2+1}(V_1 \ast_{\text{naive}} V_2). \]
**Proof:** The first claim follows from [10], Thm. 3.2. Let $\mathcal{F} = \mathcal{V}_1 \boxtimes \mathcal{V}_2$. Since the generic point of $R$ has characteristic zero, the sheaf $R^1(pr_2)_!(\mathcal{F})$ is also tame by [16], 4.7.1 (i). It follows further from [16], 4.7.1 (ii), that $\mathcal{V}_1 *_{naive} \mathcal{V}_2 = R^1(pr_2)_!(\mathcal{F})$ is lisse on $A^1_R \setminus u_1 * u_2$. The excision sequence yields an exact sequence of constructible sheaves:

$$R^0(pr_D)_*((j_* \mathcal{F})|_D) \rightarrow R^1(pr_2)_!(\mathcal{F}) \rightarrow R^1(\overline{pr}_2)_*(j_* \mathcal{F}) \rightarrow 0. \tag{2.3.1}$$

By [16], 4.7.1 (iii), the formation of $j_* \mathcal{F}$ and of $Rj_* \mathcal{F}$ commutes with arbitrary base change on $S$ and $j_* \mathcal{F}|_D$ is again lisse and tame. Hence $R^0(pr_D)_*((j_* \mathcal{F})|_D)$ is lisse and tame on $A^1_R \setminus u_1 * u_2$. It follows therefore from (2.3.1) that the sheaf $R^1(pr_2)_*(j_* \mathcal{F})$ is also contained in $L_{\mathcal{F}}(A^1_R \setminus u_1 * u_2)$, proving (ii).

The sheaf $\mathcal{F} = \mathcal{V}_1 \boxtimes \mathcal{V}_2$ on $U$ is mixed of weight $\leq n_1 + n_2$, so the sheaf $j_* \mathcal{F}$ is mixed of weights $\leq n_1 + n_2$ by [5], Corollaire 1.8.9. This implies that also $R^0(pr_D)_*((j_* \mathcal{F})|_D)$ is mixed of weights $\leq n_1 + n_2$. Again, by [5], Thm. 3.3.1, the sheaf $R^1(pr_2)_!(\mathcal{F})$ is mixed of weights $\leq n_1 + n_2 + 1$. Moreover, by loc. cit. Thm. 3.2.3, the sheaf $R^1(\overline{pr}_2)_*(j_* \mathcal{F})$ is pure of weight $n_1 + n_2 + 1$. Thus, by (2.3.1), the sheaf $R^1(pr_2)_*(j_* \mathcal{F})$ is the weight-$n_1 + n_2 + 1$-quotient of $R^1(pr_2)_!(\mathcal{F})$. \qed

**2.3.2 Proposition.** Let $\mathcal{V}_1 \in L_{\mathcal{F}}^m(U_1)$ and $\mathcal{V}_2 \in L_{\mathcal{F}}^m(U_2)$ and assume that either $\mathcal{V}_1$ or $\mathcal{V}_2$ is geometrically irreducible and nonconstant. Then

$$R^i(pr_2)_!(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = 0 \quad \text{if} \quad i \neq 1$$

and

$$R^i(\overline{pr}_2)_*(j_* (\mathcal{V}_1 \boxtimes \mathcal{V}_2)) = 0 \quad \text{if} \quad i \neq 1. \quad \text{Proof:} \quad \text{The map} \ pr_2 \text{ is affine and smooth of relative dimension one. Since formation of} \ R^i(pr_2)_! \text{ commutes with arbitrary base change by [16], 4.7.1 (ii), proper base change shows that} \ R^i(pr_2)_!(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = 0 \text{ for } i \neq 1, 2. \text{ If } R^2(pr_2)_!(\mathcal{V}_1 \boxtimes \mathcal{V}_2) \neq 0, \text{ then there exists a geometric point } \bar{s} \text{ of } s \text{ for which} \quad (R^2(pr_2)_!(\mathcal{V}_1 \boxtimes \mathcal{V}_2))_{\bar{s}} = H^2(U_{\bar{s}}, (\mathcal{V}_1 \boxtimes \mathcal{V}_2)|_{U_{\bar{s}}}) \neq 0.$$

Consider the maps $pr_1 = pr_1|_{U_{\bar{s}}} : U_{\bar{s}} \rightarrow U_1$ and $d = d|_{U_{\bar{s}}} : U_{\bar{s}} \rightarrow U_2$. By smooth base change, $(\mathcal{V}_1 \boxtimes \mathcal{V}_2)|_{U_{\bar{s}}} \simeq pr_1^!(\mathcal{V}_1) \otimes d^*(\mathcal{V}_2)$. The lisse sheaf $(\mathcal{V}_1 \boxtimes \mathcal{V}_2)|_{U_{\bar{s}}}$ corresponds thus to a tensor product representation

$$\rho_1 \otimes \rho_2 : \pi_1(U_{\bar{s}}) \rightarrow GL(\mathcal{V}_1 \otimes \mathcal{V}_2),$$

where $\rho_i : \pi_1(U_{\bar{s}}) \rightarrow GL(\mathcal{V}_i)$, $i = 1, 2$, is the monodromy representation associated to $pr_1^!(\mathcal{V}_1)$, resp. $d^*(\mathcal{V}_2)$. We have to show that the group of coinvariants of $\rho_1 \otimes \rho_2$
vanishes. Assume that $V_2$ is geometrically irreducible and nonconstant. Let $G_2 \leq \pi_1(U_s)$ be the subgroup generated by the inertia subgroups at the points of $v$. Since $V_2$ is geometrically irreducible and since $V_2$ is lisse at $u$, the properties of the tensor product imply that the restriction of $\rho_1 \otimes \rho_2$ to $G_2$ decomposes as a direct sum of copies of an irreducible and nontrivial representation of $G_2$. Thus the group of coinvariants of $\rho_1 \otimes \rho_2$ vanishes and $R^2(pr_2)_!(V_1 \boxtimes V_2) = 0$. If $V_1$ geometrically irreducible and nonconstant, then repeat the above arguments for $V_1$.

The map $pr_2$ is smooth projective of relative dimension one. Proper basechange shows that $R^i(pr_2)_*(j_*(V_1 \boxtimes V_2)) = 0$ for $i \neq 0, 1, 2$. The excision sequence yields an isomorphism

$$R^2(pr_2)_!(V_1 \boxtimes V_2) \cong R^2(pr_2)_*(j_*(V_1 \boxtimes V_2)).$$

By what was proven above, $R^2(pr_2)_*(j_*(V_1 \boxtimes V_2)) = 0$. On the other hand, Poincaré duality implies that $R^0(pr_2)_*(j_*(V_1 \boxtimes V_2))$ is dual to $R^2(pr_2)_*(j_*(V_1^* \boxtimes V_2^*))$, where $V_i^*$, $i = 1, 2$, denotes the dual of $V_i$ (resp.). Thus, by repeating the above arguments with $V_i^*$, $i = 1, 2$, one sees that $R^2(pr_2)_*(j_*(V_1^* \boxtimes V_2^*)) = 0$ and, consequently, $R^0(pr_2)_*(j_*(V_1 \boxtimes V_2)) = 0$.

Suppose that $R$ is contained in $\mathbb{C}$. If $V \in LS^\dR(U), U = \mathbb{A}^1_R \setminus u$, then the composition of the monodromy representation $\rho_V : \pi_1^d(U, s) \rightarrow GL_n(F)$ with the canonical homomorphism $\pi_1(U(\mathbb{C}), s) \rightarrow \pi_1^d(U, s)$ defines a local system $V^\an$ on $U(\mathbb{C})$. We call $V^\an$ the analytification of $V$.

2.3.3 Proposition. Suppose that $R \subseteq \mathbb{C}$. Let $V_i \in LS^\dR(U_i), U_i = \mathbb{A}^1_R \setminus u_i, i = 1, 2$ and let $V_i^\an \in LS_F(U_i(\mathbb{C})), i = 1, 2$, be the analytifications of $V_i$. Then

$$V_1^\an \ast V_2^\an = (V_1 \ast V_2)^\an.$$

Especially, the monodromy representation $\rho_{V_1 \ast V_2}^{\an}$ of $(V_1 \ast V_2)^\an$ is the composition of $\rho_{V_1 \ast V_2}$ with the canonical homomorphism $\pi_1(S(\mathbb{C})) \rightarrow \pi_1^d(S)$.

Proof: Consider the basechange map $S_C \rightarrow S$, given by $R \rightarrow \mathbb{C}$. Then, by definition, $V_1 \ast V_2|_{S_C}$ is the parabolic cohomology of a variation of lisse sheaves in the sense of [10], Def. 3.1 (namely, the variation given by the local system $pr_1^*V_1 \otimes d^*V_2$). The claim follows then from [10], Thm. 3.2. □

2.4 The relation to the middle convolution of perverse sheaves. Let $\ell$ be a prime number and let $k$ be an algebraically closed field of characteristic 0. In
Define the \(-\)-convolution (resp. the \(!\)-convolution) as in the introduction of this article or as in \([18]\), Chap. 2.5. If \(K \in \text{Perv}(\mathbb{A}^1_k, \mathbb{Q}_\ell)\) has the property \(\mathcal{P}\) mentioned in the introduction of this article, define the middle convolution of perverse sheaves \(K_{\ast \text{mid}} L \in \text{Perv}(\mathbb{A}^1_k, \mathbb{Q}_\ell)\) to be

\[K_{\ast \text{mid}} L = K \ast L = \text{Im} \left( K \ast ! L \to K \ast * L \right) \]

Recall also that for any lisse sheaf \(V\) on the punctured affine line \(\mathbb{A}^1_k \setminus \mathbf{u}\), one obtains a perverse sheaf \(j_* \mathbb{V}[1]\) on \(\mathbb{A}^1_k\) by placing the constructible sheaf \(j_* \mathbb{V}\) in degree \(-1\), where \(j\) denotes the inclusion of \(\mathbb{A}^1_k \setminus \mathbf{u}\) into \(\mathbb{A}^1_k\).

**2.4.1 Proposition.** Let \(V_i \in \text{LS}^\text{et} \mathbb{Q}_\ell(U_i)\), let \(i_i : U_i \to \mathbb{A}^1_i\), \(i = 1, 2\), denote the inclusion maps, and let \(K_i := (i_i)_*(V_i)[1] \in \text{Perv}(\mathbb{A}^1_i, \mathbb{Q}_\ell)\). Suppose that \(V_i\) is irreducible and nonconstant. Then \(K_1\) has the property \(\mathcal{P}\) and \(V_1 \ast V_2 = (K_1 \ast \text{mid} K_2)[-1]|_S\), where \(S = \mathbb{A}^1_y \setminus \mathbf{u} \ast \mathbf{v}\).

**Proof:** That \(K_1\) has the property \(\mathcal{P}\) follows from the fact that \(V_1\) is irreducible and not translation invariant, see \([18]\), Cor. 2.6.17. By \([18]\), Prop. 2.9.2, one has

\[(2.4.1) \quad K_1 \ast \text{mid} K_2 = R(\overline{pr}_2)_* (j_{* \iota} (pr_1^* K_1 \otimes d^* K_2)),\]

where \(j_{\ast \iota}\) denotes the middle extension functor (cf. \([18]\), Sections 2.3.3 and 2.7, or \([20]\), Chap. III.5) associated to \(j : \mathbb{A}^1_x \times \mathbb{A}^1_y \hookrightarrow \mathbb{P}^1_x \times \mathbb{A}^1_y\) and

\[
\overline{pr}_2 : \mathbb{P}^1_x \times \mathbb{A}^1_y \to \mathbb{A}^1_y, \quad d : \mathbb{A}^1_x \times \mathbb{A}^1_y \to \mathbb{A}^1_y, \quad (x, y) \mapsto y - x.
\]

Let \(U \subseteq \mathbb{A}^1_x \times \mathbb{A}^1_y\) be as in Section 2.2 and let \(\overline{j} : U \to \mathbb{P}^1 \times S\) the inclusion map. Since the divisor \(D = \mathbb{P}^1 \times S \setminus U\) is smooth over \(S\) via \(\overline{pr}_2\), the middle extension \(j_{* \iota} (pr_1^* K_1 \otimes d^* K_2)\) restricted to \(\mathbb{P}^1 \times S\) coincides with \(j_{* \iota} (pr_1^* V_1 \otimes d^* V_2)[2]\) (this follows from \([18]\), Lemma 4.3.8 and Cor. 2.8.5.2a). It follows therefore from (2.4.1) that

\[K_1 \ast \text{mid} K_2|_S = R(\overline{pr}_2)_* (\overline{j}_{\ast \iota} (V_1 \boxtimes V_2)[2]).\]

But it follows from Prop. 2.3.2 that the cohomology sheaves

\[R^i (\overline{pr}_2)_* (\overline{j}_{\ast \iota} (V_1 \boxtimes V_2)) = \mathcal{H}^{i-2} (K_1 \ast \text{mid} K_2|_S)\]

vanish for \(i \neq 1\), proving the claim. \(\square\)
2.4.2 Remark. For any algebraically closed field, let $\mathcal{T}_\ell$ denote the category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on $\mathbb{A}^1$ which satisfy the following conditions:

(i) There exist a dense open subset $j : U \to \mathbb{A}^1$ such that $\mathcal{F}|_U$ is irreducible, contained in $\text{LS}^\text{ét}_{\mathbb{Q}_\ell}(U)$, and $\mathcal{F} \simeq j_* j^* \mathcal{F}$.

(ii) There are at least two distinct points of $\mathbb{A}^1$ at which $\mathcal{F}$ is not lisse.

In [18], Chap. 5, Katz defines a middle convolution functor

$$\text{MC}_\chi : \mathcal{T}_\ell \longrightarrow \mathcal{T}_\ell, \quad \mathcal{F} \mapsto (\mathcal{F}[1] *_{\text{mid}} j_\ast \mathcal{L}_\chi[1])[-1],$$

where $\mathcal{L}_\chi$ is the Kummer sheaf associated to $\chi : \pi_1^{\text{ét}}(\mathbb{G}_m) \to \bar{\mathbb{Q}}_\ell^\times$. Let $U$ be a dense open subset of $\mathbb{A}^1$ such that $\mathcal{V} = \mathcal{F}|_U$ is lisse. Let $k$ be a subfield of $\mathbb{C}$. It follows then from Prop. 2.4.1 that

$$\text{MC}_\chi(\mathcal{V}) = \text{MC}_\chi(\mathcal{F})|_U.$$

By Thm. 1.6.2, the monodromy tuple of $(\text{MC}_\chi(\mathcal{F})|_U)^{\text{an}}$ is the tuple $\text{MC}_\lambda(\mathcal{T}_\mathcal{V})$.

2.5 Independence of $\ell$. Let $E$ be a number field and let $\Sigma(E)$ denote the set of finite primes of $E$. Let $S$ be a smooth variety over $R$, where $R \subseteq \bar{\mathbb{Q}}$ is normal and finitely generated over $\mathbb{Z}$. Suppose that for any $\lambda \in \Sigma(E)$ one has given a lisse sheaf $\mathcal{V}_\lambda \in \text{LS}^\text{ét}_{E_\lambda}(S)$ such that the following holds:

Let $s : \text{Spec}(\mathbb{F}_q) \to S$ be a closed point of $S$, inducing a map

$$G_{\mathbb{F}_q} := \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) = \langle \text{Frob}_s \rangle \to \pi_1(S, \bar{s}),$$

where $\bar{s}$ is a geometric point which extends $s$, and let

$$\rho_{\mathcal{V}_\lambda} : \pi_1(S, \bar{s}) \longrightarrow \text{GL}((\mathcal{V}_\lambda)_{\bar{s}})$$

be the monodromy representations of the $\mathcal{V}_\lambda$, where $\lambda \in \Sigma(E)$. Then, for all $\lambda$ of residue characteristic prime to $q$, the characteristic polynomials of $\rho_{\mathcal{V}_\lambda}(\text{Frob}_s)$ have coefficients in $E$ and are independent of $\lambda$.

If this condition is fulfilled, then we say that the collection $(\mathcal{V}_\lambda)_{\lambda \in \Sigma(E)}$ is $E$-rational and independent of $\lambda$. The following result is close to [18], Thm. 5.5.4:

2.5.1 Theorem. Suppose that for any finite prime $\lambda \in \Sigma(E)$ one has given two nonconstant irreducible pure lisse sheaves

$$\mathcal{V}_{i,\lambda} \in \text{LS}^\text{ét}_{E_\lambda}(U_i), \quad U_i = \mathbb{A}^1_R \setminus u_i \quad (i = 1, 2),$$

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such that the collections \( (V_{i,\lambda})_{\lambda \in \Sigma(E)} \), \( i = 1, 2 \), are \( E \)-rational and independent of \( \lambda \). Then the collection of middle convolutions
\[
(V_{1,\lambda} \ast V_{2,\lambda} \in \mathcal{L}_{E_\lambda}^\text{c} (S))_{\lambda \in \Sigma(E)}
\]
is again \( E \)-rational and independent of \( \lambda \).

**Proof:** It suffices to show that the collection of naive convolutions
\[
V_{\lambda} := V_{1,\lambda} \ast \text{naive} V_{2,\lambda}
\]
is independent of \( \lambda \) (since the middle convolution \( V_{1,\lambda} \ast V_{2,\lambda} \) is the highest-weight-quotient of \( V_{\lambda} \) by Prop. 2.3.1). Let
\[
\mathcal{F}_{s,\lambda} := (\text{pr}_s^*(V_{1,\lambda}) \otimes d^*(V_{2,\lambda})) |_{U_s}
\]
be the restriction of the local system \( \text{pr}_s^*(V_{1,\lambda}) \otimes d^*(V_{2,\lambda}) \) on \( U/S \) (where \( U \) is as in Section 2.2) to the fibre \( U_s \) over \( s \in S(\overline{F}_q) \). By the Lefschetz trace formula, if \( \text{char}(\lambda) \neq \text{char}(\overline{F}_q) \), then the following holds (compare to [4], Rapport):
\[
\sum_{i=0,1,2} (-1)^i \cdot \text{Tr} (\text{Frob}_s, H^i_c(X_s, \mathcal{F}_{s,\lambda})) = \sum_{x \in U_s(\overline{F}_q)} \text{Tr}(\text{Frob}_x, \mathcal{F}_{x,\lambda}).
\]
(2.5.1)

Since \( U \) is affine, the cohomology group \( H^0_c(X_s, \mathcal{F}_{s,\lambda}) \) vanishes (compare to [11], Thm. 9.1). The cohomology group \( H^2_c(X_s, \mathcal{F}_{s,\lambda}) \) vanishes by Prop. 2.3.2. Moreover, by the multiplicativity of traces under tensoring, the right hand side of Equation (2.5.1) can be seen to satisfy
\[
\sum_{x \in U_s(\overline{F}_q)} \text{Tr}(\text{Frob}_x, \mathcal{F}_{x,\lambda}) = \sum_{x \in U_s(\overline{F}_q)} \text{Tr}(\text{Frob}_x, (V_{1,\lambda})_{\hat{s}}) \cdot \text{Tr}(\text{Frob}_{s-x}, (V_{2,\lambda})_{\overline{s-x}}).
\]

By combining the latter arguments with Equation (2.5.1), one concludes that
\[
\text{Tr} (\text{Frob}_s, H^1_c(X_s, \mathcal{F}_{s,\lambda})) = - \sum_{x \in U_s(\overline{F}_q)} \text{Tr}(\text{Frob}_x, (V_{1,\lambda})_{\hat{s}}) \cdot \text{Tr}(\text{Frob}_{s-x}, (V_{2,\lambda})_{\overline{s-x}}).
\]

By the \( E \)-rationality and independence of \( \lambda \) of the collections \( (V_{i,\lambda})_{\lambda \in \Sigma(E)} \), this expression is contained in \( E \) and is independent of \( \lambda \). By definition, it is the trace of \( \text{Frob}_s \) on the stalk of the naive convolution \( (V_{1,\lambda} \ast \text{naive} V_{2,\lambda})_{\hat{s}} \). By a similar reasoning, one sees that the traces of the powers of \( \text{Frob}_s^k \) satisfy
\[
\text{Tr} \left( \text{Frob}_s^k, (V_{1,\lambda} \ast \text{naive} V_{1,\lambda})_{\hat{s}} \right) = \sum_{x \in U_s(\overline{F}_q^k)} \text{Tr}(\text{Frob}_x, (V_{1,\lambda})_{\hat{s}}) \cdot \text{Tr}(\text{Frob}_{s-x}, (V_{2,\lambda})_{\overline{s-x}}).
\]
and are thus contained in \( E \) and are independent of \( \lambda \). Since one can express the coefficients of the characteristic polynomial of \( \text{Frob}_k \) as linear combinations of the traces of \( \text{Frob}_k^k \) (by Newton’s identities), it follows that the collections \((\mathcal{V}_{1,\lambda} \ast_{\text{naive}} \mathcal{V}_{1,\lambda})_{\lambda \in \Sigma(E)}\) and hence \((\mathcal{V}_{1,\lambda} \ast \mathcal{V}_{1,\lambda})_{\lambda \in \Sigma(E)}\) are in fact \( E \)-rational and independent of \( \lambda \). \( \square \)

**2.5.2 Theorem.** Suppose that \( R = \mathbb{Z}[\frac{1}{N}] \), where \( N \in \mathbb{N}_{>0} \). Let \( E \) be a number field and suppose that for any finite prime \( \lambda \in \Sigma(E) \) one has given two nonconstant irreducible local systems \( \mathcal{V}_i,\lambda \in \text{LS}_E^\text{ét}(U_i) \), where \( U_i = \mathbb{A}_R^1 \setminus u_i \) \((i = 1, 2)\), such that the collections \((\mathcal{V}_i,\lambda)_{\lambda \in \Sigma(E)}\), \( i = 1, 2 \), are \( E \)-rational and independent of \( \lambda \). Let \( S := \mathbb{A}_R^1 \setminus u_1 \ast u_2 \) and let \( \mathcal{V}_\lambda := \mathcal{V}_{1,\lambda} \ast \mathcal{V}_{2,\lambda} \in \text{LS}_E^\text{ét}(S) \). For \( s \in S(\mathbb{Q}) \), let

\[
(\rho^s_\lambda : G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \simeq \pi_1(s, \bar{s}) \rightarrow \text{Aut}( (\mathcal{V}_\lambda)_s))_{\lambda \in \Sigma(E)}
\]

be the induced collection of Galois representations. Then there exists a finite character \( \epsilon : G_\mathbb{Q} \rightarrow E^\times \) and an integer \( m \in \mathbb{Z} \) such that

\[
\det(\rho^s_\lambda) = \epsilon \otimes \chi_\ell^m,
\]

where \( \ell = \text{char}(\lambda) \) and \( \chi_\ell \) denotes the \( \ell \)-adic cyclotomic character.

**Proof:** Let \( s = \frac{A}{B} \in S(\mathbb{Q}) \) with coprime natural numbers \( A, B \in \mathbb{Z} \) and let \( \bar{s} \) be a complex point extending \( s \). The representation \( \rho^s_\lambda \) is unramified outside the prime divisors of \( B \cdot N \cdot \ell \) and the finite set of primes \( p \) such that \( s \) reduces to an element in the reduction of \( u_1 \ast u_2 \) modulo \( p \). Thus, by Thm. 2.5.1, the system of Galois representations \((\det(\rho^s_\lambda))_{\lambda \in \Sigma(E)}\) is a compatible system of \( E \)-rational Galois representations of \( G_\mathbb{Q} \) in the sense of Serre [25]. By [24], Prop. 1.4 in Chap. 1, any such compatible system arises from an algebraic Hecke character (the Prop. 1.4 in loc. cit. is a consequence of Henniart’s result on the algebraicity of one-dimensional compatible systems, see [15] and [19]). Any Hecke character of \( \mathbb{Q} \) (or, more generally, a totally real field) with values in \( E \) is equal to a power of the norm character times a finite order character with values in \( E \), see [24], Chap. 0.3. Therefore, the associated Galois representation is of the desired form. \( \square \)

**2.6 Motivic interpretation of the middle convolution.** Let \( R \subseteq \mathbb{C} \) be normal and finitely generated over \( \mathbb{Z} \). For \( i = 1, 2 \), let \( U_i = \mathbb{A}_R^1 \setminus u_i \), and let
and nonconstant. Then $P_i \leq E[G_i]$ idempotent elements. The homomorphism $G_i \to \text{Aut}(X_i/U_i)$ induces a map $E[G_i] \to \text{End}(R^k \pi_i!(E_\lambda))$ for any $k \in \mathbb{N}$. Thus the idempotent $P_i$ cuts out a subsheaf $P_i(R^k \pi_i!(E_\lambda))$ of $R^k \pi_i!(E_\lambda)$. The product $G_1 \times G_2$ acts on the scheme

$$\pi : X_1 \boxtimes X_2 := (X_1 \times U_1) \times_U (X_2 \times U_2) \longrightarrow U$$

and induces a homomorphism $G_1 \times G_2 \to \text{Aut}(X_1 \boxtimes X_2/U)$. Thus the product idempotent

$$P_1 \times P_2 \in E[G_1 \times G_2] \simeq E[G_1] \times E[G_2]$$
cuts out subsheaves of the higher direct images $R^k \pi_1!(E_\lambda)$, $k \in \mathbb{N}$, and on the higher direct images $R^l(\text{pr}_2 \circ \pi)_!(E_\lambda)$, $l \in \mathbb{N}$.

Call a constructible $E_\lambda$-sheaf on $U$ motivic, if it is a subfactor of a higher direct image $R^lf_!(E_\lambda)$, where $f : X \to U$ is some smooth map.

2.6.1 Theorem. Let $\mathcal{F}_i \in \text{LS}^{\text{mot}}_{E_\lambda}(U_i)$, $i = 1, 2$, be lisse sheaves which are mixed of weights $\leq k_i$ and which are of the following form:

$$\mathcal{F}_i = P_i \left(R^{k_i} \pi_i!(E_\lambda)\right), \quad i = 1, 2,$$

where the $P_i \in E[G_i]$ are idempotents which act on $X_i \xrightarrow{\pi_i} U_i$ as above. Assume that $P_i(R^l \pi_i!(E_\lambda)) = 0$ for all $l_i \neq k_i$, and assume that $\mathcal{F}_1$ or $\mathcal{F}_2$ is irreducible and nonconstant. Then

$$\mathcal{F}_1 \ast \text{naive} \mathcal{F}_2 = P_1 \times P_2 \left(R^{k_1+k_2+1}(\text{pr}_2 \circ \pi)_!(E_\lambda)\right)$$

and $P_1 \times P_2 \left(R^l(\text{pr}_2 \circ \pi)_!(E_\lambda)\right) = 0$ for $l \neq k_1 + k_2 + 1$. Moreover,

$$W^{k_1}(\mathcal{F}_1) \ast W^{k_2}(\mathcal{F}_2) = W^{k_1+k_2+1}(\mathcal{F}_1 \ast \text{naive} \mathcal{F}_2).$$

Especially, the naive convolution $\mathcal{F}_1 \ast \text{naive} \mathcal{F}_2$ and the middle convolution $W^{k_1}(\mathcal{F}_1) \ast W^{k_2}(\mathcal{F}_2)$ are motivic.

Proof: The Künneth-Formula implies that

$$\sum_{i+j=n} R^i \pi_1!(E_\lambda) \otimes R^j \pi_2!(E_\lambda) \simeq R^n \pi!(E_\lambda).$$
Thus, the assumption $P_i \left( R^{k_i} \pi_{1,!} (E_\lambda) \right) = 0$ for $l_i \neq k_i$ implies that

\[(2.6.1) \quad P_1 \left[ R^{l_1} \pi_{1,!} (E_\lambda) \right] \boxtimes P_2 \left[ R^{l_2} \pi_{2,!} (E_\lambda) \right] \simeq \begin{cases} 0 & \text{if } (i, j) \neq (k_1, k_2) \\ P_1 \times P_2 \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) & \text{if } (i, j) = (k_1, k_2) \end{cases}\]

The Leray spectral sequence $E_2 = ( R^p \text{pr}_{2,!} R^q \pi_{1,!} (E_\lambda) \Rightarrow R^{p+q} (\text{pr}_2 \circ \pi)_{!} (E_\lambda) )$ is a first quadrant spectral sequence whose only nonzero entries are at $p = 1, 2$ (since $\text{pr}_2$ is affine, the sheaf $R^0 \text{pr}_{2,!}$ vanishes). Thus it degenerates at $E_2$. It follows that $R^n(\text{pr}_2 \circ \pi)_{!} (E_\lambda) \simeq E^n$ has a descending filtration

\[R^n(\text{pr}_2 \circ \pi)_{!} (E_\lambda) \simeq E^n_1 \supseteq E^n_2 \supseteq 0,\]

where

\[E^n_1 / E^n_2 \simeq R^1 \text{pr}_{2,!} R^{n-1} \pi_{1,!} (E_\lambda) \quad \text{and} \quad E^n_2 = R^2 \text{pr}_{2,!} R^{n-2} \pi_{1,!} (E_\lambda).\]

It follows therefore from (2.6.1) and Prop. 2.3.2 that $P_1 \times P_2 \left[ R^n(\text{pr}_2 \circ \pi)_{!} (E_\lambda) \right] = 0$ for $n \neq k_1 + k_2 + 1$. It follows that

\[\mathcal{F}_1 \ast_{\text{naive}} \mathcal{F}_2 \simeq R^1 \text{pr}_{2,!} \left( P_1 \left( R^{l_1} \pi_{1,!} (E_\lambda) \right) \boxtimes P_2 \left( R^{l_2} \pi_{2,!} (E_\lambda) \right) \right) \]

\[\simeq R^1 \text{pr}_{2,!} \left( P_1 \times P_2 \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \]

\[\simeq P_1 \times P_2 \left( R^{k_1+k_2+1} (\text{pr}_2 \circ \pi)_{!} (E_\lambda) \right).\]

By [5], the sheaf $R^{k_1+k_2} \pi_{1,!} (E_\lambda)$ is mixed of weights $\leq k_1 + k_2$. The exact sequence

\[0 \rightarrow K \rightarrow P_1 \times P_2 \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \rightarrow \]

\[P_1 \times P_2 \left( W^{k_1+k_2} \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \simeq W^{k_1} (\mathcal{F}_1) \otimes W^{k_2} (\mathcal{F}_2) \rightarrow 0\]

and Prop. 2.3.2 imply an exact sequence

\[0 \rightarrow R^1 \text{pr}_{2,!} (K) \rightarrow R^1 \text{pr}_{2,!} \left( P_1 \times P_2 \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \rightarrow \]

\[R^1 \text{pr}_{2,!} \left( P_1 \times P_2 \left( W^{k_1+k_2} \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \right) \rightarrow 0.\]

Since, again by [5], the sheaf $R^1 \text{pr}_{2,!} (K)$ is mixed of weights $\leq k_1 + k_2$ it follows that $R^1 \text{pr}_{2,!} \left( P_1 \times P_2 \left( W^{k_1+k_2} \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \right)$ is pure of weight $k_1 + k_2 + 1$. It follows then from Prop. 2.3.1 (ii) that

\[W^{k_1} (\mathcal{F}_1) \ast W^{k_2} (\mathcal{F}_2) \simeq W^{k_1+k_2+1} \left( R^1 \text{pr}_{2,!} \left( P_1 \times P_2 \left( R^{k_1+k_2} \pi_{1,!} (E_\lambda) \right) \right) \right) \]

\[\simeq W^{k_1+k_2+1} \left( P_1 \times P_2 \left( R^{k_1+k_2+1} (\text{pr}_2 \circ \pi)_{!} (E_\lambda) \right) \right) \]

\[\simeq W^{k_1+k_2+1} (\mathcal{F}_1 \ast_{\text{naive}} \mathcal{F}_2).\]
3 Applications to the inverse Galois problem

3.1 Galois covers and fundamental groups. Let $R$ be a subfield of $\mathbb{C}$, let $X$ be a smooth and geometrically irreducible variety over $R$, and let $x$ be a geometric point of $X$. Any finite étale Galois cover $f : Y \to X$ with Galois group $G = G(Y/X)$ corresponds to a surjective homomorphism $\Pi_f : \pi^\text{ét}_1(X, x) \to G$. Let $F$ be as in Section 2.1, let $\chi : G \to \text{GL}(V)$ ($V \cong F^n$) be a representation, and let $\mathcal{L}_{(f, \chi)} \in \text{LS}^\text{ét}_F(X)$ denote the lisse sheaf associated to the composition $\chi \circ \Pi_f : \pi^\text{ét}_1(X, x) \to \text{GL}(V)$. If $X = \mathbb{A}_R^1 \setminus \{u\}$, where $u$ is defined by the vanishing of $\prod_{i=1}^r (x - x_i) \in \mathbb{R}[x]$, let $g_{f, \chi} \in \text{GL}(V)^{r+1}$ denote the monodromy tuple which is associated to the analytification $\mathcal{L}_{(f, \chi)}^\text{an}$ of $\mathcal{L}_{(f, \chi)}$. For $r \in \mathbb{N} > 0$, we set

$$O_r(Q) := \{ P = \{x_1, \ldots, x_r\} \subseteq \bar{Q} \mid |P| = r \text{ and } P \text{ is fixed setwise by } G_Q \}.$$

3.1.1 Proposition. Let $m \in \mathbb{N} > 1$ and let $E_\lambda$ be the completion of a number field at a finite prime $\lambda$ which admits an embedding $\chi : \mathbb{Z}/m \mathbb{Z} \hookrightarrow E_\lambda^\times$. Let $D_m = \mathbb{Z}/m \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = \langle \sigma, \rho \mid \rho^m = \sigma^2 = 1, \rho^\sigma = \rho^{-1} \rangle$ denote the dihedral group of order $2m$. Then the following holds:

(i) Let $v := \{x_1, x_2\}, x_1 \neq x_2,$ where $x_1, x_2 \in \mathbb{P}^1(Q)$. Then there exists an étale Galois cover $f : \mathfrak{F} \to \mathbb{P}^1_Q \setminus v$ with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(ii) Let $m \in \mathbb{N} > 2$ and let $\varphi$ denote Euler’s $\varphi$-function. Then there exist infinitely many elements $v \in O_{\varphi(m)}(Q)$ such that there is an étale Galois cover $f : \mathfrak{F} \to \mathbb{A}^1_Q \setminus v$ with Galois group isomorphic to $\mathbb{Z}/m \mathbb{Z} = \langle \rho \rangle$ whose monodromy tuple is

$$g_{f, \chi} = \left(\rho^{m_1}, \ldots, \rho^{m_{\varphi(m)}}, 1\right),$$

where $m_i \in \mathbb{Z}/m \mathbb{Z}^\times$.

(iii) Let $m \in \mathbb{N} > 2$, let $\mathbb{Z}/m \mathbb{Z} = \langle \rho \rangle$, and let $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$. Then there exist infinitely many

$$v = \{x_1, \ldots, x_{2+\varphi(m)}\} \in O_{2+\varphi(m)}(Q)$$

for which there exists an étale Galois cover $f : \mathfrak{F} \to \mathbb{A}^1_Q \setminus v$ with Galois group isomorphic to $D_{2m}$ with

$$g_{f, \chi} = (g_1, g_2, \rho^{m_1}, \ldots, \rho^{m_{\varphi(m)}}, 1),$$

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where $g_1, g_2$ are not contained in $(\rho)$ and where $m_i \in \mathbb{Z}/m\mathbb{Z}^\times$. Moreover, $x_1$ and $x_2$ can be assumed to be $\mathbb{Q}$-rational points.

(iv) In the situation of (iii) with $m$ odd. For any $r \geq 2 + \varphi(m)$ and (if $r > 2 + \varphi(m))$ for any choice of elements $x_{3+\varphi(m)}, \ldots, x_r \in \mathbb{A}^1(\mathbb{Q})$ such that

$$v' := v \cup \{x_{3+\varphi(m)}, \ldots, x_r\} \in \mathcal{O}_r(\mathbb{Q}),$$

there exists an étale Galois cover $f : \mathfrak{F} \to \mathcal{A}_1 \setminus v'$ with Galois group isomorphic to $D_{2m} = \langle \delta \rangle D_m$ (where $\delta$ is the central involution) such that

$$\mathfrak{g}_{f,\tilde{\chi}} = (g_1, g_2, \rho^{m_1}, \ldots, \rho^{m_{\varphi(m)}}, \delta, \ldots, \delta) \in D_{2m}^{r+1}.$$

**Proof:** See [26], Chap. 7, or [22], Chap. I.5.1, for (i) and (ii). For claim (iii), note that the dihedral group $D_m$ is a factor group of the wreath product $H = \langle \rho \rangle \wr \langle \sigma \rangle$, see [22], Prop. IV.2.3. Let $v = \{x_1, \ldots, x_{\varphi(m)}\}$ be as in (ii) and let

$$U := \{(x, y) \in \mathbb{C}^2 \mid y \neq -xx_i - x_i^2, i = 1, \ldots, \varphi(m), \text{ and } x^2 \neq 4y\}.$$

The construction given in [22], proof of Prop. IV.2.1, shows that there exists an étale Galois cover $\tilde{f} : X \to U$ with Galois group isomorphic to $H$ which is defined over $\mathbb{Q}$. By factoring out the kernel of the surjection $H \to D_m$, one sees that there exists a Galois cover $f' : X' \to U$, with Galois group isomorphic to $D_m$. Claim (iii) now follows from restricting the cover $f'$ to a suitable (punctured) line in $U$ which is defined over $\mathbb{Q}$. Claim (iv) follows from (i) and (iii) by taking fibre products of covers. \hfill \Box

3.1.2 Remark. For any Galois cover $f : \mathfrak{F} \to \mathbb{A}^1_{\mathbb{Q}} \setminus u$ as in Prop. 3.1.1 there exists an $R = \mathbb{Z}[1/r]$, $r \in \mathbb{Z}$, and a Galois cover $f_R : \mathfrak{F}_R \to \mathbb{A}^1_R \setminus u_R$ such that $f$ is the basechange of $f_R$ induced by the inclusion $R \subseteq \mathbb{Q}$.

3.2 Special linear groups as Galois groups. Let $H$ be a profinite group. One says that $H$ occurs regularly as Galois group over $\mathbb{Q}(t)$ if there exists a continuous surjection $\kappa : \text{Gal}(\mathbb{Q}(t)/\mathbb{Q}(t)) \to H$ such that $\mathbb{Q}$ is algebraically closed in the fixed field of the kernel of $\kappa$.

3.2.1 Theorem. Let $\ell$ be a prime, let $q = \ell^s$ ($s \in \mathbb{N}$), and let $n \in \mathbb{N}$. Assume that

$$q \equiv 5 \mod 8 \quad \text{and that} \quad n > 6 + 2\varphi(m), \quad \text{where} \quad m := (q - 1)/4.$$
Let
\[ E := \mathbb{Q}(\zeta_m + \zeta_m^{-1}, \zeta_4), \]
where \( \zeta_i (i \in \mathbb{N}) \) denotes a primitive \( i \)-th root of unity. Let \( \lambda \) be a prime of \( E \) with \( \text{char}(\lambda) = \ell \) and let \( O_\lambda \) be the valuation ring of the completion \( E_\lambda \). Then the special linear group \( \text{SL}_{2n+1}(O_\lambda) \) occurs regularly as Galois group over \( \mathbb{Q}(t) \).

Before giving the proof of the theorem let us mention the following corollary which immediately follows from reduction modulo \( \lambda \):

3.2.2 Corollary. The special linear group \( \text{SL}_{2n+1}(\mathbb{F}_q) \) occurs regularly as Galois group over \( \mathbb{Q}(t) \) if

\[ q \equiv 5 \mod 8 \quad \text{and} \quad n > 6 + 2\varphi((q - 1)/4). \]

Proof of the theorem: First we consider the case where \( m \geq 3 \) and \( n = 2r - 4 \), where \( r \geq 2 + \varphi(m) \): Let

\[ f_1 : \mathcal{F}_1 \to \mathbb{A}^1_{\mathbb{Q}} \setminus \{ u_1 \}, \quad u_1 = \{ x_1, \ldots, x_r \}, \]

be a \( D_{2m} \)-cover as in Prop. 3.1.1 (iv), where we assume that the points \( x_1, x_2 \) are \( \mathbb{Q} \)-rational. Let \( \chi_1 : D_{2m} \hookrightarrow \text{GL}_2(E_\lambda) \) be an orthogonal embedding of \( D_{2m} \). Thus

\[ \mathcal{F}_1 := \mathcal{L}_{(f_1, \chi_1)} \in \mathcal{L}_{E_\lambda}^{d_1} (\mathbb{A}^1_{\mathbb{Q}} \setminus \{ u_1 \}) \]

is a lisse sheaf of rank two (compare to Section 3.1 for the notation of \( \mathcal{L}_{(f_1, \chi_1)} \)). The monodromy tuple is

\[ T_{\mathcal{F}_1} = (A_1, \ldots, A_{r+1}) \in \text{GL}_2(E_\lambda)^{r+1}, \]

where \( A_1, A_2 \) are reflections, and \( A_3, \ldots, A_{r+1} \) are diagonal matrices with eigenvalues

\[ (\zeta_m^{m_1}, \zeta_m^{-m_1}), \ldots, (\zeta_m^{\varphi(m)}, \zeta_m^{-\varphi(m)}), (-1, -1), \ldots, (-1, -1), \]

where \( \zeta_m^{m_1}, \ldots, \zeta_m^{\varphi(m)} \) are the primitive powers of \( \zeta_m \) (compare to our convention in Section 2.1 for the notion of an monodromy tuple).

Let \( f_2 : \mathfrak{F}_2 \to \mathbb{A}^1_{\mathbb{Q}} \setminus \{ u_2 \}, \quad u_2 := \{ 0 \} \), be a double cover as in Prop. 3.1.1 (i), let \( \chi_2 : \mathbb{Z}/2\mathbb{Z} \hookrightarrow E_\lambda^\times \), and let

\[ \mathcal{F}_2 := \mathcal{L}_{(f_2, \chi_2)} \in \mathcal{L}_{E_\lambda}^{d_1} (\mathbb{A}^1_{\mathbb{Q}} \setminus \{ u_2 \}). \]

Thus \( \mathcal{F}_2 \) is a Kummer sheaf with \( T_{\mathcal{F}_2} = (-1, -1) \).
The middle convolution $\mathcal{F}_1 \ast \mathcal{F}_2$ is an étale local system on $\mathbb{A}^1_\mathbb{F} \setminus u_1$ (since $u_1 \ast u_2 = u_1$). By Prop. 2.3.3 and Prop. 1.4.1, the rank of $\mathcal{F}_1 \ast \mathcal{F}_2$ is $2r - 4$ and

$$T_{\mathcal{F}_1 \ast \mathcal{F}_2} = (B_1, \ldots, B_{r+1}) \in \text{GL}_{2r-4}^r,$$

where (by Lemma 1.8.1 and Prop. 2.3.3) the matrices $B_1, B_2$ are transvections, the matrices $B_3, \ldots, B_{2+\varphi(m)}$ are biperspectivities with non-trivial eigenvalues

$$(-\zeta_m^{m_1}, -\zeta_m^{-m_1}), \ldots, (-\zeta_m^{m_\varphi(m)}, -\zeta_m^{-m_\varphi(m)})$$

(respectively), the matrices $B_{3+\varphi(m)}, \ldots, B_r$ are unipotent biperspectivities, and where the last matrix is equal to $-1$ by Lemma 1.8.2.

Let

$$\mathcal{G}_1 := \mathcal{L}_{(g, \xi)}|_{\mathbb{A}^1_\mathbb{Q} \setminus u_1} \in \text{LS}^q_{E_\lambda}(\mathbb{A}^1_\mathbb{Q} \setminus u_1)$$

be the restriction of the local system $\mathcal{L}_{(g, \xi)}$, where $g : \mathfrak{G} \to \mathbb{A}^1_\mathbb{Q} \setminus \{x_1\}$ is as in Prop. 3.1.1 (i) and $\xi : \mathbb{Z}/2\mathbb{Z} \to E_\lambda^\times$. Then the monodromy tuple of $(\mathcal{F}_1 \ast \mathcal{F}_2) \otimes \mathcal{G}_1$ is

$$T_{(\mathcal{F}_1 \ast \mathcal{F}_2) \otimes \mathcal{G}_1} = (-B_1, B_2, \ldots, B_r, -B_{r+1} = 1).$$

Let $f_3 : \mathfrak{S}_3 \to \mathbb{A}^1_\mathbb{Q} \setminus u_3$ be a $\mathbb{Z}/4\mathbb{Z}$-cover as in Prop. 3.1.1 (ii), where we have chosen $u_3$ such that $u_1 \ast u_3$ is generic in the sense of Section 1.2. Let $\chi_3 : \mathbb{Z}/4\mathbb{Z} \to E_\lambda^\times$ be an embedding and set $\mathcal{F}_3 := \mathcal{L}_{(f_3, \chi_3)} \in \text{LS}^q_{E_\lambda}(\mathbb{A}^1_\mathbb{Q} \setminus u_3).$ Thus

$$T_{\mathcal{F}_3} = (i, -i, 1), \text{ where } i := \zeta_4.$$}

Then the middle convolution

$$\mathcal{V} := ((\mathcal{F}_1 \ast \mathcal{F}_2) \otimes \mathcal{G}_1) \ast \mathcal{F}_3$$

is a lisse sheaf on $S = \mathbb{A}^1_\mathbb{Q} \setminus u_1 \ast u_3$. The rank of $\mathcal{V}$ is $4r - 7 = 2n + 1$ by Prop. 2.3.3. Let $T_{\mathcal{V}} = (C_1, \ldots, C_{2r+1})$. By Lemma 1.8.1,

$$(3.2.1) \quad C_1 \sim J(-\zeta_4, 2) \oplus_{k=3}^{2r-4} J(-\zeta_4, 1) \oplus_{2r-4}^{\varphi(1)} J(1, 1)$$

(where $J(\zeta, k)$ denotes a Jordan Block of length $k$ with eigenvalue $\zeta$), $C_2$ is a homology of order four, the elements $C_3, \ldots, C_{2+\varphi(m)}$ are semisimple biperspectivities with non-trivial eigenvalues

$$(-i\zeta_m^{m_1}, -i\zeta_m^{-m_1}), \ldots, (-i\zeta_m^{m_\varphi(m)}, -i\zeta_m^{-m_\varphi(m)}),$$

the elements $C_{3+\varphi(m)}, \ldots, C_m$ are biperspectivities with non-trivial eigenvalues $(i, i)$, and the Jordan forms of the matrix $C_{i+r}$ is the Galois conjugate of the
matrix $C_i$ ($i = 1, \ldots, r$). Especially, one sees that $\langle C_1, \ldots, C_{2r} \rangle \leq \text{SL}_{4r-7}(E_\lambda) \times \langle \zeta_4 \rangle$.

It follows from Remark 3.1.2 that there exists an $N \in \mathbb{N}_{>0}$ such that $\mathcal{V}$ extends to a lisse sheaf $\tilde{V}$ on $S_R = \mathbb{A}^1_R \setminus \{u_{1,R} + u_{3,R}\}$, where $R = \mathbb{Z}[\frac{1}{\mathcal{N}_\ell}]$. Choose a $\mathbb{Q}$-rational point $s_0$ of $S_R$ and hence of $S$. By Thm. 2.5.2 and the fact that $\tilde{V}$ is pure of weight 2 (this follows from Prop. 2.3.1),

$$\det(\rho_V) = \det(\rho_V|_{\pi_1^{\text{geo}}(S,s_0)}) \otimes \chi_{\ell}^{-(4r-7)} \otimes \epsilon,$$

where $\epsilon : G_\mathbb{Q} \to \langle \zeta_4 \rangle \subseteq E^\times$ and where $G_\mathbb{Q}$ is embedded in

$$\pi_1^{\text{et}}(S,s_0) = \pi_1^{\text{geo}}(S,s_0) \rtimes G_\mathbb{Q}$$

by the choice of $s_0$. Thus

$$\text{Im}(\det(\rho_{V(1)})) \leq \langle \zeta_4 \rangle,$$

where $V(1)$ stands for the Tate twist of $\mathcal{V}$. Let

$$\rho := \rho_{V(1)} \otimes \delta : \pi_1^{\text{geo}}(S,s_0) \rtimes G_\mathbb{Q} \to \text{GL}_{4r-7}(E_\lambda),$$

where $\delta = \det(\rho_{V(1)})^{-1}$. It follows that

$$(3.2.2) \quad \text{Im}(\rho) \leq \text{SL}_{4r-7}(E_\lambda).$$

Let $\mathbb{F}_q$ be the residue field of $E_\lambda$ and let $\bar{\rho} : \pi_1^{\text{et}}(S,s_0) \to \text{GL}_{4r-7}(\mathbb{F}_q)$ denote the the residual representation of $\rho$. Let $H = \text{Im}(\rho)$ and $\bar{H} := \text{Im}(\bar{\rho})$. Let further $H^{\text{geo}} := \text{Im}(\rho^{\text{geo}})$ and $\bar{H}^{\text{geo}} := \text{Im}(\bar{\rho}^{\text{geo}})$. By Thm. 1.7.1 and Prop. 2.3.3, $\bar{H}^{\text{geo}}$ is absolutely irreducible. Let $V_1 \oplus \cdots \oplus V_l$ be a $\bar{H}^{\text{geo}}$-invariant decomposition of the underlying module $V := \mathbb{F}_q^{4r-7}$. By the eigenvalue structure of the above elements $C_1, C_r \in H^{\text{geo}}$ given in Formula (3.2.1), the reduction modulo $\lambda$ of $C_1$ and $C_r$ does not permute any of the spaces $V_1, \ldots, V_l$. Similarly, for $i = 2, \ldots, r$, the reduction modulo $\lambda$ of the elements $C_i$ and $C_{r+i}$ fix the spaces $V_1, \ldots, V_l$, if $\dim(V_i) \geq 3$. It follows thus that $\dim(V_i) \leq 2$. Since the spaces $V_1, \ldots, V_l$ are permuted transitively by the irreducibility of $\bar{H}^{\text{geo}}$ and since the dimension of $V$ is odd, we have $\dim(V_i) = 1$. But this is in contradiction to the Jordan block of length 2 which occurs in the reduction modulo $\lambda$ of $C_1$. Therefore, the group $\bar{H}^{\text{geo}}$ acts primitively on $V$. Thus, the existence of the homology $C_2$ and the classification of primitive subgroups of $\text{GL}_n(\mathbb{F}_q)$ which contain a homology of order greater than 2 ([27]) imply that $\text{SL}_{4r-7}(\mathbb{F}_q) \leq \bar{H}^{\text{geo}}$. By (3.2.2),

$$\bar{H}^{\text{geo}} = \text{SL}_{4r-7}(\mathbb{F}_q).$$
Since $\pi_{et}^\dagger(S, \bar{s}_0)$ is compact, the image of $\rho$ can be assumed to be contained in the group $\text{GL}_{4r-7}(O_\lambda)$ (see [25]) and hence in the group $\text{SL}_{4r-7}(O_\lambda)$. Since $E_\lambda$ is unramified over $Q_\ell$, the residual map $\text{SL}_{4r-7}(O_\lambda) \to \text{SL}_{4r-7}(\mathbb{F}_q)$ is Frattini (see [28], Cor. A). Therefore

$$H^{geo} = \text{SL}_{4r-7}(O_\lambda) = H.$$ 

The fundamental group $\pi_{et}^\dagger(S, \bar{s}_0)$ is a factor of $G_{Q(t)}$ and the image of $\pi_{et}^{geo}(S, \bar{s}_0)$ coincides with the image of $G_{\hat{Q}(t)}$ in $\pi_{et}^\dagger(S, \bar{s}_0)$. Thus the group $\text{SL}_{4r-7}(O_\lambda)$ occurs regularly as Galois group over $Q(t)$. This proves the claim in the case $m \geq 3$ and $n = 2r-4$.

The proof for $m \geq 3$ and $n = 2r-3$ uses the following sheaves: Let $r \geq 4 + \varphi(m)$. Let

$$\mathcal{F}_1 := \mathcal{L}_{(f_1, \chi_1)} \in \text{LS}^\dagger_{E_\lambda}(A^1_Q \setminus u_1)$$

be defined as above, where we assume that the points $x_{\varphi(m)+3}$ and $x_{\varphi(m)+4}$ coincide with $i, -i$ and that $x_1, x_2$ and $x_r$ are rational points. By a suitable tensor operation, one obtains a local system $\mathcal{F}'_1 \in \text{LS}^\dagger_{E_\lambda}(A^1_Q \setminus u_1)$ whose monodromy tuple is

$$T_{\mathcal{F}'_1} = (A'_1, \ldots, A'_{r+1}) \in \text{GL}_2(E_\lambda)^{r+1},$$

where $A'_1, A'_2$ are reflections, and $A'_3, \ldots, A'_{r+1}$ are diagonal matrices with eigenvalues

$$(3.2.3) \quad (\zeta_m^{m_1}, \zeta_m^{-m_1}), \ldots, (\zeta_m^{m_{\varphi(m)}}, \zeta_m^{-m_{\varphi(m)}}, (i, i), (-i, -i), (-1, -1), \ldots, (1, 1)).$$

Let $\mathcal{F}_2 = \mathcal{L}_{(f_2, \chi_2)}$ be the Kummer sheaf as above and let $\mathcal{F}'_3 := \mathcal{F}_2$. Let $\mathcal{G}' = \mathcal{L}_{(g', \xi')}$, where $g' : G' \to A^1_Q \setminus \{x_1, x_r\}$ is the double cover ramified at $x_1$ and $x_r$ and $\xi'$ is the embedding of the Galois group into $E_\lambda$. Now continue as above, using the sheaf

$$\mathcal{V} = ((\mathcal{F}'_1 * \mathcal{F}_2) \otimes \mathcal{G}'|_{A^1_Q \setminus u_1}) * \mathcal{F}'_3$$

which has rank $4r - 5$.

The case where $m = 1$ follows from the same arguments as above using the dihedral group $D_3$ instead of $D_m$. \qed
Appendix (by S. Reiter and M. Dettweiler): A new motivic local system of $G_2$-type

Let $J(n_1, \ldots, n_k)$ denote a unipotent matrix in Jordan canonical form which decomposes into blocks of length $n_1, \ldots, n_k$ and let $\zeta_3 \in \overline{\mathbb{Q}}_\ell$ denote a fixed primitive third root of unity. Let us call a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on a scheme $X$ to be motivic is it is a subfactor of a higher direct image sheaf of a morphism $Y \to X$. We obtain the following result:

**Theorem:** Let $k \subseteq \mathbb{C}$ be an algebraically closed field and let $x_1, x_2, x_3 \in \mathbb{A}^1(k)$ be pairwise distinct. Then there exists a motivic lisse sheaf $\mathcal{H}$ on $\mathbb{P}^1_k \setminus \{x_1, x_2, x_3, \infty\}$ with the following properties:

(i) The Jordan canonical forms of the local monodromies of $\mathcal{H}$ at $x_1, x_2, x_3, \infty$ are as follows (resp.):

\[ J(2, 2, 1, 1, 1), \ J(2, 2, 1, 1, 1), \ \text{diag}(1, \zeta_3, \zeta_3, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_3^{-1}), \ J(3, 3, 1). \]

(ii) The monodromy of $\mathcal{H}$ is dense in $G_2(\overline{\mathbb{Q}}_\ell)$.

(iii) The lisse sheaf $\mathcal{H}$ is not cohomologically rigid in the sense of [18], Chap. 5.

**Proof:** Let $X = \mathbb{P}^1_k \setminus \{x_1, x_2, x_3, \infty\}$. Remember from Section 3.1 that to any étale Galois cover $f : Y \to X$ with Galois group $G$ and to any homomorphism $\alpha : G \to \text{GL}_n(\overline{\mathbb{Q}}_\ell)$ we have associated a lisse sheaf $\mathcal{L}_{f, \alpha}$. Let us also fix generators $\alpha_1, \ldots, \alpha_4$ of the topological fundamental group $\pi_1(X(\mathbb{C}))$ and generators $\gamma_1, \gamma_2$ of $\pi_1(\mathbb{G}_m(\mathbb{C}))$ as in the Definition of the monodromy tuple (Definition 1.1.1). Let $\mathcal{L}_i = \mathcal{L}_{f_i, \alpha} \in \text{LS}_{\overline{\mathbb{Q}}_\ell}(X)$, $i = 1, 2, 3$, be as follows:

- The Galois cover $f_1 : Y_1 \to X$ is the cover of $X$ defined by $y^3 = (x - x_1)(x - x_3)$.

  The Galois group of $f_1$ is $C_3 = \langle \sigma \mid \sigma^3 = 1 \rangle$. The homomorphism $\alpha : C_3 \to \overline{\mathbb{Q}}_\ell^*$ is given by sending $\sigma$ to $\zeta_3$. The monodromy tuple of the analytification $\mathcal{L}_{f_1}^\text{an} \in \text{LS}(X(\mathbb{C}))$ is $g_1 = (\zeta_3, 1, \zeta_3, \zeta_3)$.

- The Galois cover $f_2 : Y_2 \to X$ is the cover of $X$ defined by $y^3 = (x - x_2)(x - x_3)$.

  and $\alpha$ is as above. The monodromy tuple of $\mathcal{L}_{f_2}^\text{an} \in \text{LS}(X)$ is equal to $g_2 = (1, \zeta_3, \zeta_3, \zeta_3)$.  

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The Galois cover \( f_3 : Y_3 \to X \) is the cover of \( X \) defined by \( y^3 = (x - x_3) \) and \( \alpha \) is as above. The monodromy tuple of \( \mathcal{L}^{an}_2 \in \text{LS}(X) \) is equal to \( \mathfrak{g}_3 = (1, 1, \zeta_3, \zeta_3^{-1}) \).

Let \( \chi : \pi_1(\mathbb{G}_m) \to \bar{\mathbb{Q}}_\ell^\times \) be the character, which sends the generator \( \gamma_1 \) of \( \pi_1(\mathbb{G}_m) \) to \( \zeta_3 \) and let \( \chi^{-1} \) be its dual character. Consider the following sequence of tensor operations and middle convolutions:

\[
\mathcal{H} := \mathcal{L}_3^{-1} \otimes (\text{MC}_\chi^{-1}(\mathcal{L}_3 \otimes (\text{MC}_\chi(\text{MC}_\chi^{-1}(\mathcal{L}_1) \otimes \text{MC}_\chi^{-1}(\mathcal{L}_2)))))) \in \text{LS}^{an}_{\bar{\mathbb{Q}}_\ell}(X),
\]

where \( \mathcal{L}_3^{-1} \) denotes the dual sheaf of \( \mathcal{L}_3 \). It follows from Prop. 2.3.3 (i) that then analytification \( \mathcal{H}^{an} \in \text{LS}_{\bar{\mathbb{Q}}_\ell}(X(\mathbb{C})) \) is of the form

\[
\mathcal{H}^{an} = (\mathcal{L}_3^{-1})^{an} \otimes (\text{MC}_\chi^{-1}(\mathcal{L}_3^{an} \otimes (\text{MC}_\chi(\text{MC}_\chi^{-1}(\mathcal{L}_1^{an}) \otimes \text{MC}_\chi^{-1}(\mathcal{L}_2^{an}))))).
\]

Using the explicit recipe for the computation of the middle convolution of local systems given in Thm. 1.6.2, one finds explicit matrices for the monodromy tuple \( h = (h_1, \ldots, h_4) \) of \( \mathcal{H}^{an} \). These are given in Table 1. The claim on the local monodromy of \( \mathcal{H} \) follows then from the Jordan forms of the matrices \( h_1, \ldots, h_4 \).

Since the monodromy tuples of \( \text{MC}_\chi^{-1}(\mathcal{L}_i^{an}) \), \( i = 1, 2 \), are contained in the group \( \text{Sp}_2(\mathbb{Q}_\ell) = \text{SL}_2(\mathbb{Q}_\ell) \) and generate irreducible subgroups, the monodromy tuples of the tensor product \( \text{MC}_\chi^{-1}(\mathcal{L}_1^{an}) \otimes \text{MC}_\chi^{-1}(\mathcal{L}_2^{an}) \) can be seen to generate an irreducible subgroup of the orthogonal group \( \text{SO}_4(\mathbb{Q}_\ell) \). An iterative application of Thm. 1.7.1 implies then that the local system \( \mathcal{H} \) is irreducible.

Since the elements of the monodromy tuple of \( \text{MC}_\chi^{-1}(\mathcal{L}_i^{an}) \otimes \text{MC}_\chi^{-1}(\mathcal{L}_j^{an}) \) are contained in \( \text{SO}_4(\mathbb{Q}_\ell) \), it follows from Poincaré duality that the elements of the monodromy tuple of \( \mathcal{H}^{an} \) are contained in the group \( \text{SO}_7(\mathbb{Q}_\ell) \). By a computation using MAGMA [21], one can check that the matrices stabilize a one-dimensional subspace of the third exterior power \( \Lambda^3(\mathbb{Q}_7) \). Thus, by the results of [2], the image of the monodromy representation \( \rho^{an} \) of \( \mathcal{H}^{an} \) is contained in \( G_2(\mathbb{Q}_7) \). By the classification of bireflection groups given in [13], Thm. 7.1 and Thm. 8.3, the Zariski closure of the image of \( \rho^{an} \) can be seen to coincide with \( G_2(\mathbb{Q}_\ell) \). Since \( \pi_1^\text{et}(\overline{\mathbb{Q}}_\ell) \) is the profinite closure of \( \pi_1(X(\overline{\mathbb{Q}}_\ell)) \), the Zariski closure of \( \rho^{an} \) coincides with the Zariski closure of the monodromy representation of \( \mathcal{H} \). It follows from the numerical criterion of physical rigidity given in [18], Thm. 1.1.2, and the structure of the local monodromy of \( \mathcal{H}^{an} \) that \( \mathcal{H}^{an} \) is not physically rigid. Therefore, \( \mathcal{H} \) not cohomologically rigid in the sense of [18], Chap. 5.

To prove that \( \mathcal{H} \) is motivic we can work over \( \mathbb{Z}[\zeta_3, \frac{1}{3}] \) instead of \( k \) and use an iterative application of the Künneth-formula and the motivic interpretation of the middle convolution given in Thm. 2.6.1. \( \square \)
\[
\begin{pmatrix}
1 & -3 & \zeta_3 - 1 & 0 & \zeta_3 - 4 & 0 & 2\zeta_3 + 4 \\
0 & 3\zeta_3 + 1 & 2\zeta_3 + 1 & 0 & 2\zeta_3 + 1 & -2\zeta_3 - 1 & 0 \\
0 & -3\zeta_3 & -2\zeta_3 & 0 & -2\zeta_3 - 1 & 2\zeta_3 + 1 & 0 \\
0 & 3\zeta_3 + 3 & \zeta_3 + 2 & 1 & \zeta_3 + 2 & -\zeta_3 - 2 & 0 \\
0 & 3\zeta_3 + 6 & 3 & 0 & 4 & -3 & 0 \\
0 & 6 & -2\zeta_3 + 2 & 0 & -2\zeta_3 + 2 & 2\zeta_3 - 2 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\zeta_3 - 1 & 1 & 0 & 2\zeta_3 + 1 & 0 & 0 & 0 \\
3 & 0 & 1 & -2\zeta_3 - 1 & -3 & 0 & 2\zeta_3 + 4 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta_3 & 0 & 0 & 0 & 0 \\
\zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 & 0 & 0 \\
0 & \zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 & 0 \\
0 & 3\zeta_3 + 3 & \zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 \\
0 & 0 & 0 & 0 & \zeta_3 - 1 & 0 & 1
\end{pmatrix}
\]

Table 1: The monodromy generators of $\mathcal{H}$
Remark: Let $\zeta_6$ denote a primitive 6-th root of unity and let $g_1 := (\zeta_6, 1, \zeta_6, \zeta_6)$, $g_2 = (1, \zeta_6, \zeta_6, \zeta_6)$ and $g_3 = (1, 1, \zeta_6, \zeta_6^{-1})$. Let $L_{g_i}, i = 1, 2, 3$, be the local systems on $X(\mathbb{C})$ associated to $g_i$ and let $\chi : \pi_1(G_m(\mathbb{C})) \to \bar{\mathbb{Q}}_{\ell}^\times$ be the character, which sends the generator $\gamma_1$ of $\pi_1(G_m)$ to $\zeta_6$. Then the sequence of tensor operations and middle convolutions

$$L_{g_3}^{-1} \otimes (MC_{\chi^{-1}}(L_{g_3} \otimes (MC_{\chi}(MC_{\chi^{-1}}(L_{g_1}) \otimes MC_{\chi^{-1}}(L_{g_2}))))))$$

leads to a local system on $X(\mathbb{C})$ whose local monodromy coincides with the one of $\mathcal{H}$ but whose monodromy is Zariski dense in $SO_6(\bar{\mathbb{Q}}_{\ell})$.

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