Towards a Unified Theory of Light Spanners I: Fast (Yet Optimal) Constructions

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Abstract

Seminal works on light spanners over the years provide spanners with optimal or near-optimal lightness in various graph classes, such as in general graphs [21], Euclidean spanners [33] and minor-free graphs [12]. Two shortcomings of all previous works on light spanners are: (1) The techniques are ad hoc per graph class, and thus can’t be applied broadly (e.g., some require large stretch and are thus suitable to general graphs, while others are naturally suitable to stretch 1 + \( \epsilon \)). (2) The runtimes of these constructions are almost always sub-optimal, and usually far from optimal.

This work aims at initiating a unified theory of light spanners by presenting a single framework that can be used to construct light spanners in a variety of graph classes. This theory is developed in two papers. The current paper is the first of the two — it lays the foundations of the theory of light spanners and then applies it to design fast constructions with optimal lightness for several graph classes. Our new constructions are significantly faster than the state-of-the-art for every examined graph class; moreover, our runtimes are near-linear and usually optimal.

Specifically, this paper includes the following results (for simplicity assume \( \epsilon > 0 \) is fixed):

- In general graphs: A nearly linear-time algorithm for constructing light spanners. Specifically, for any \( k \geq 2 \), we construct a \((2k - 1)(1 + \epsilon)\)-spanner with lightness \( O(n^{1/k}) \) in \( O(m \alpha(m,n)) \) time, where \( \alpha(\cdot, \cdot) \) is the inverse-Ackermann function; the lightness bound is optimal assuming Erdös’ girth conjecture (up to the \( \epsilon \)-dependency). Since \( O(m \alpha(m,n)) = O(m + n \log^* n) \), the runtime is linear in \( m \) when \( m = \Omega(n \log^* n) \). The previous state-of-the-art runtime of such a spanner is super-quadratic in \( n \) [21, 1].

- In low-dimensional Euclidean spaces: An \( O(n \log n) \)-time construction of \((1 + \epsilon)\)-spanners with lightness and degree both bounded by constants in the basic algebraic computation tree (ACT) model. This construction is optimal (up to dependencies on \( \epsilon \) and the dimension) with respect to all the involved quality measures — runtime, lightness and degree — and it resolves a major problem in the area of geometric spanners, which was open for three decades (cf. [19, 6, 50, 63]).

- In unit-disk graphs in \( \mathbb{R}^2 \): An \( O(n \log n) \)-time construction of \((1 + \epsilon)\)-spanners with constant lightness and degree, in the ACT model. This construction too is optimal with respect to all the involved quality measures. This is the first \( o(n^2) \)-time spanner construction for unit disk graphs with a nontrivial lightness bound.

- In minor-free graphs: A linear-time algorithm for constructing \((1 + \epsilon)\)-spanners with constant lightness. This is the first \( o(n^2) \)-time spanner construction for minor-free graphs (and even for some of its sub-classes, such as bounded treewidth graphs), with a nontrivial lightness bound.

Remark. Our follow-up paper builds on the foundations of the theory laid in the current paper, aiming to achieve lightness bounds with optimal dependencies on the involved parameters, most notably \( \epsilon \), but also others such as the dimension (in Euclidean spaces) or the minor size (in minor-free graphs).

\textsuperscript{1}The lightness is a normalized notion of weight: a graph’s lightness is the ratio of its weight to the MST weight.
1 Introduction

For a weighted graph $G = (V, E, w)$ and a stretch parameter $t \geq 1$, a subgraph $H = (V, E')$ of $G$ is called a $t$-spanner if $d_H(u, v) \leq t \cdot d_G(u, v)$, for every $e = (u, v) \in E$, where $d_G(u, v)$ and $d_H(u, v)$ are the distances between $u$ and $v$ in $G$ and $H$, respectively. Graph spanners were introduced in two celebrated papers from 1989 [63, 65] for unweighted graphs, where it is shown that for any $n$-vertex graph $G = (V, E)$ and integer $k \geq 1$, there is an $O(k)$-spanner with $O(n^{1+1/k})$ edges. Since then, graph spanners have been extensively studied, both for general weighted graphs and for restricted graph families, such as Euclidean spaces and minor-free graphs. In fact, spanners for Euclidean spaces—Euclidean spanners—were studied implicitly already in the pioneering SoCG’86 paper of Chew [23], who showed that any 2-dimensional Euclidean space admits a spanner of $O(n)$ edges. This bound is tight for the greedy spanner. The best running time for the same lightness bound is due to [44]. The best running time for the same lightness bound is due to [44].

The results of [64, 65] for general graphs were strengthened in [2], where it was shown that for every $n$-vertex weighted graph $G = (V, E, w)$ and integer $k \geq 1$, there is a greedy algorithm for constructing a $(2k - 1)$-spanner with $O(n^{1+1/k})$ edges, which is optimal under Erdős’ girth conjecture. The results of [64, 65] for general graphs were strengthened in [2], where it was shown that for every $n$-vertex weighted graph $G = (V, E, w)$ and integer $k \geq 1$, there is a greedy algorithm for constructing a $(2k - 1)$-spanner with $O(n^{1+1/k})$ edges, which is optimal under Erdős’ girth conjecture. The stretch is $O(k)$ and the lightness is $O(k)$.

As with the sparsity parameter, its weighted variant—lightness—has been extremely well-studied; the lightness is the ratio of the weight of the spanner to the size of a spanning tree, namely $n - 1$.) Moreover, there is an $O(m)$-time algorithm for constructing $(2k-1)$-spanners in general graphs with stretch $O(n^{1/k})$ [51]. Therefore, not only is the stretch-sparsity tradeoff in general graphs optimal (up to Erdős’ girth conjecture), one can achieve it in optimal time. For weighted graphs, one can construct $(2k - 1)$-spanners with sparsity $O(k n^{1/k})$ within time $O(km)$ [6, 72].

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The gap in our understanding of light spanners is most prominent when considering the spanner construction time. To exemplify this statement, we next survey results on light spanners in several basic graph classes, focusing mostly on the construction time.

General weighted graphs. Althöfer et al. [2] showed that the lightness of the greedy spanner is $O(n/k)$. Chandra et al. [19] improved this lightness bound to $O(k \cdot n^{(1+\epsilon)/(k-1)} \cdot (1/\epsilon)^2)$, for any $\epsilon > 0$; another, somewhat stronger, form of this tradeoff from [19], is stretch $(2k - 1) \cdot (1 + \epsilon)$, $O(n^{1+1/k})$ edges and lightness $O(k \cdot n^{1/k} \cdot (1/\epsilon)^2)$. In a sequence of works from recent years [39, 21, 44], it was shown that the lightness of the greedy spanner is $O(1/k \cdot (1/\epsilon)^3+2/k)$ (this lightness bound is due to 21); the fact that this bound holds for the greedy spanner is due to 44). The best running time for the same lightness bound in prior work is super-quadratic in $n$: $O(n^{2+1/k+\epsilon'})$ [1] for any fixed constant $\epsilon' < 1$.

Question 1. Can one construct a $(2k - 1) \cdot (1 + \epsilon)$-spanner in general weighted graphs with lightness $O(n^{1/k})$, within (nearly) linear time for any fixed $\epsilon < 1$?

Euclidean spanners in the algebraic computation tree (ACT) model. Spanners have had special success in geometric settings, especially in low-dimensional Euclidean spaces. The reason Euclidean spanners have been extensively studied over the years — in both theory and practice — is that one can achieve stretch arbitrarily close to 1 together with constant sparsity and lightness (ignoring dependencies on $\epsilon$ and the dimension $d$). In general metrics, on the other hand, a stretch better than 3 requires sparsity and lightness of $\Omega(n)$. The algebraic computation tree (ACT) model is used extensively in computational geometry, and in the area of Euclidean spanners in particular; this model, introduced by Ben-Or [9], is intimately related (and equivalent, if ignoring uniformity issues) to the real random access machine (real RAM) model. The reader can refer to 9 and Chapter 3 in the book [63] for a detailed description of ACT model; for completeness, we provided a brief description of this model in Appendix [B]. In the ACT model; for completeness, we provided a brief description of this model in Appendix [B].
model, computing \((1 + \epsilon)\)-spanners for point sets in \(\mathbb{R}^d\), \(d = O(1)\), requires \(\Omega(n \log n)\) time \(22\). There are various algorithms (see, e.g., \([17, 52, 70, 82]\)) for computing \((1 + \epsilon)\)-spanners with constant sparsity, which achieve an optimal runtime of \(O(n \log n)\) in this model, for any fixed \(\epsilon < 1\) and in any constant-dimensional Euclidean space. However, the lightness of the spanners produced by those algorithms is unbounded.

Starting in the late 80s, there has been a large body of work on light Euclidean spanners \([58, 19, 30, 33, 31, 32, 3, 71, 50, 63, 40, 57]\). Light Euclidean spanners are not only important in their own right, but they also find applications in other contexts. In particular, the breakthrough result of Rao and Smith \([71]\) gave an \(O(n \log n)\)-time approximation scheme for the Euclidean TSP, assuming that a \((1 + \epsilon)\)-spanner with constant lightness can be computed within time \(O(n \log n)\). Also, Czumaj and Lingas \([29]\) gave approximation schemes for Euclidean minimum-cost multiconnectivity problems under the same assumption. The assumption used in the results of \([71, 29]\) was made by relying on a spanner construction due to Arya et al. \([3]\), which was later shown to be flawed. Gudmundsson, Levcopoulos, and Narasimhan (hereafter, GLN) \([50]\), building on and improving over several previous works \([58, 19, 30, 33, 31, 3, 71, 50, 63, 40, 57]\), gave the first (correct) algorithm for constructing Euclidean \((1 + \epsilon)\)-spanners with constant lightness in \(O(n \log n)\) runtime, but their algorithm assumes indirect addressing. A variation of the GLN algorithm, which applies to the ACT model, takes time \(O(n \log^2 n / \log \log n)\); this is the state-of-the-art runtime for constructing \((1 + \epsilon)\)-spanners with constant lightness in the ACT model, even in \(\mathbb{R}^2\), and even allowing a super-constant lightness bound (of at most \(o(\log n)\)). The question of whether one can compute such a spanner in optimal \(O(n \log n)\) time in the ACT model was asked explicitly several times, including in the GLN paper \([50]\) and in the spanner book by Narasimhan and Smid \([63]\).

**Question 2.** Can one construct a Euclidean \((1 + \epsilon)\)-spanner with constant lightness within the optimal time of \(O(n \log n)\) in the ACT model for any fixed \(\epsilon < 1\)?

Constant lightness does not imply any sparsity bound. A stronger result would be to achieve a constant bound on both the lightness and sparsity, and even further, one could try to achieve a constant bound on the degree too, where the spanner’s degree is the maximum degree of any vertex in it. Euclidean spanners of bounded degree have been studied since the early 90s, and they too found various applications. In compact routing schemes low degree spanners give rise to routing tables of small size (see, e.g., \([18, 49, 15]\)), and more generally, the degree of the spanner determines the local memory constraints when using spanners also for other purposes, such as constructing network synchronizers and efficient broadcast protocols. Moreover, in some applications the degree of a vertex (or processor) represents its load, hence a low degree spanner guarantees that the load on all the processors in the network will be low.

**Question 3** (Question 22 in \([63]\)). Can one construct a Euclidean \((1 + \epsilon)\)-spanner with constant lightness and constant degree (and thus constant sparsity) within the optimal time of \(O(n \log n)\) in the ACT model for any fixed \(\epsilon < 1\)?

**Unit disk graphs** Given a set of \(n\) points \(P \subseteq \mathbb{R}^d\), a unit ball graph for \(P\), denoted by \(U = U(P)\), is the geometric graph with vertex set \(P\), where there is an edge between two points \(p \neq q \in P\) (with weight \(\|p, q\|\)) if and only if \(\|p, q\| \leq 1\). When \(d = 2\), we call \(U\) a unit disk graph (UDG); for convenience, we shall use the term unit disk graph also for \(d > 2\). (Refer to Section 2 for a more detailed discussion on geometric graphs.)

There is a large body of work on spanners for UDGs; see \([59, 61, 60, 47, 85, 70, 68, 46, 11]\), and the references therein. One conclusion that emerges from the previous work (see \([68]\) in particular) is that, if

\(^2\)If one allows the usage of indirect addressing, then the lower bound of \(O(n \log n)\) no longer applies.
one does not care about the running time, then constructing \((1+\epsilon)\)-spanners for unit disk graphs is just as easy as constructing \((1+\epsilon)\)-spanners for the entire Euclidean space. Moreover, the greedy \((1+\epsilon)\)-spanner for the Euclidean space, after removing from it all edges of weight larger than 1, provides a \((1+\epsilon)\)-spanner for the underlying unit disk graph. The greedy \((1+\epsilon)\)-spanner in \(\mathbb{R}^d\) has constant sparsity and lightness for constant \(\epsilon\) and \(d\), specifically, sparsity \(\Theta(\epsilon^{-d+1})\) and lightness \(\Theta(\epsilon^{-d})\) (cf. \cite{57}). The drawback of the greedy spanner is its runtime: The state-of-the-art implementation in Euclidean low-dimensional spaces is \(O(n^2 \log n)\) \cite{14}. There is a much faster variant of the greedy algorithm, sometimes referred to as “approximate-greedy”, with runtime \(O(n \log n)\) \cite{50}. Alas, removing the edges of weight larger than 1 from the approximate-greedy \((1+\epsilon)\)-spanner of the Euclidean space does not provide a \((1+\epsilon)\)-spanner for the underlying unit disk graph; in fact, the stretch of the resulting spanner may be arbitrarily poor. Instead of simply removing the edges of weight larger than 1 from the approximate-greedy spanner, one can replace them by appropriate replacement edges, as proposed in \cite{68}, but the runtime of this process will be at least linear in the size of the unit disk graph, which is \(\Omega(n^2)\) in general.

Fürer and Kasiviswanathan \cite{15} showed that sparse \((1+\epsilon)\)-spanners for UDGs can be built in nearly linear time when \(d = 2\), and in subquadratic time when \(d\) is a constant of value at least 3.

**Lemma 1.1** (Corollary 1 in \cite{40}). Given a set of \(n\) points \(P\) in \(\mathbb{R}^d\), there is an algorithm that constructs a \((1+\epsilon)\)-spanner of the unit ball graph for \(P\) with \(O(n \epsilon^{1-d})\) edges. For \(d = 2\), the running time is \(O(n(\epsilon^{-2} \log n))\), and for \(d \geq 3\), the running time is \(O(n^2 \frac{2^d}{(d/2+1)+\delta} \epsilon^{-d+1} + n \epsilon^{-d})\) for any constant \(\delta > 0\).

Thus, there is a significant gap between the fastest constructions of sparse versus light spanners in UDGs. In particular, no \(o(n^2)\)-time \((1+\epsilon)\)-spanner construction for UDGs with a nontrivial lightness bound is known, even for \(d = 2\). The question of closing this gap naturally arises.

**Question 4.** Can one construct within \(o(n^2)\) time a \((1+\epsilon)\)-spanner for UDGs with constant lightness for a fixed \(\epsilon < 1\)? Is it possible to achieve a near-linear runtime for \(d = 2\)?

**Minor-free graphs.** Althöfer et al. \cite{2} showed that the greedy \((1+\epsilon)\)-spanner in planar graphs has lightness \(O(1/\epsilon)\). Klein \cite{53} gave a fast construction of \((1+\epsilon)\)-spanners with constant lightness (albeit with a worse dependence on \(\epsilon\)). It is known that the technique of \cite{53} can be extended to bounded genus graphs, provided that an embedding into a surface of the required genus is given as input; the time for computing such an embedding is linear in the graph size and exponential in the genus.

A natural goal would be to extend the results to minor-free graphs.\footnote{A graph \(H\) is called a minor of graph \(G\) if \(H\) can be obtained from \(G\) by deleting edges and vertices and by contracting edges. A graph \(G\) is said to be \(K_r\)-minor-free, if it excludes \(K_r\) as a minor for some fixed \(r\), where \(K_r\) is the complete graph on \(r\) vertices. We shall omit the prefix \(K_r\) in the term “\(K_r\)-minor-free”, when the value of \(r\) is not important.} Borradaile, Le, and Wulff-Nilsen \cite{12} showed that the greedy \((1+\epsilon)\)-spanners of \(K_r\)-minor-free graphs have lightness \(\tilde{O}_\epsilon(r^{1/\epsilon})\), where the notation \(\tilde{O}_\epsilon(.)\) hides polylog factors of \(r\) and \(\frac{1}{\epsilon}\). However, the fastest implementation of the greedy spanner requires quadratic time \cite{2}, even in graphs with \(O(n)\) edges; more generally, the runtime of the greedy algorithm from \cite{2} on a graph with \(m = \tilde{O}_\epsilon(nr)\) edges is \(\tilde{O}_\epsilon(n^2 r^2)\). Moreover, the same situation occurs even in sub-classes of minor-free graphs, particularly bounded treewidth graphs.

**Question 5.** Can one construct in linear or nearly-linear time a \((1+\epsilon)\)-spanner for minor-free graphs with constant lightness?

### 1.1 Research Agenda: From Sparse to Light Spanners

Thus far we tried to exemplify the statement that the stretch-lightness tradeoff is not as well-understood as the stretch-sparsity tradeoff, especially when considering the spanner construction time. In fact, even...
when ignoring the construction time, there are still significant gaps between these tradeoffs when considering the fine-grained dependencies in the parameter $\epsilon$ (which primarily controls the stretch parameter) on the lightness versus sparsity, as well as other parameters, such as the dimension (in Euclidean spaces) or the minor size (in minor-free graphs). This statement is not to underestimate in any way the exciting line of work on light spanners, but rather to call for attention to the important research agenda of narrowing this gap and ideally closing it.

**Fast constructions.** All the aforementioned questions — from Question 1 to Question 5 — share one common theme: Can one achieve fast constructions of light spanners that match the corresponding constructions of sparse spanners?

**Goal 1.** Achieve fast constructions of light spanners that match the corresponding constructions of sparse spanners. In particular, achieve (nearly) linear-time constructions of light spanners for basic graph families, such as the ones covered in the aforementioned questions.

**“Truly optimal” lightness.** Achieving truly optimal stretch-lightness tradeoffs, i.e., including the dependence on $\epsilon$ (which appears in both the stretch and lightness bounds) as well as any of the other involved parameters – is a highly challenging goal. For planar graphs, the aforementioned result [2] on the greedy $(1 + \epsilon)$-spanner with lightness $O(1/\epsilon)$ provides a truly optimal lightness bound. For constant-dimensional Euclidean spaces, a truly optimal tradeoff of stretch $1 + \epsilon$ versus lightness $O(\epsilon^{-d})$ was achieved recently by the authors [57]. Can one achieve truly optimal light spanners for other well-studied graph families, such as general graphs and minor-free graphs?

**Goal 2.** Achieve truly optimal light spanners for basic graph families.

**Unification.** Some of the papers on light spanners employ inherently different techniques than others, e.g., the technique of [21] requires large stretch while others are naturally suitable to stretch $1 + \epsilon$. Since the techniques in this area are ad hoc per graph class, they can’t be applied broadly. A unified framework for light spanners would be of both theoretical and practical merit.

**Goal 3.** Achieve a unified framework of light spanners.

Establishing a thorough understanding of light spanners by meeting (some of) the above goals is not only of theoretical interest, but is also of practical importance, due to the wide applicability of spanners. Goal 1 (achieving truly optimal spanners) is of particular importance for graph families that admit light spanners with stretch $1 + \epsilon$, in spanner applications where precision is a necessity. Indeed, in such applications, the precision is basically determined by $\epsilon$, hence if it is a tiny (sub-constant) parameter, then improving the $\epsilon$-dependence on the lightness could lead to significant improvements in the performance. Perhaps the most prominent applications of light spanners are to efficient broadcast protocols in the message-passing model of distributed computing [4, 5], to network synchronization and computing global functions [7, 65, 4, 5, 69], and to the TSP [53, 54, 71, 50, 12]. There are many more applications, such as to data gathering and dissemination tasks in overlay networks [16, 83, 34], to VLSI circuit design [26, 27, 28, 75], to wireless and sensor networks [84, 10, 77], to routing [86, 65, 69, 81], to compute almost shortest paths [25, 74, 88, 41, 43], and to computing distance oracles and labels [67, 80, 73].

1.2 Our Contribution

This work aims at initiating a unified theory of light spanners by presenting a single framework that can be used and adapted to construct light spanners in a variety of graph classes. Basically, we strive to
translate results — in a unified manner — from sparse spanners to light spanners, without significant loss in any of the parameters. One of our results is particularly surprising (Theorem 1.2 for general graphs), since the new bounds for light spanners outperform the best known bounds for sparse spanners.

This theory is developed in two papers. The current paper is the first of the two — it lays the foundations of the theory of light spanners and then applies it to design fast constructions with optimal lightness for several graph classes, thereby resolving all the aforementioned questions.

Next, we elaborate on the contribution of this paper, and put it into context with previous work. For the simplicity, we shall assume here that $0 < \epsilon < 1$ is fixed; the exact dependencies of $\epsilon$ will be explicated in subsequent sections of this paper.

General graphs. For general graphs we provide a nearly linear-time spanner construction with lightness that is optimal assuming Erdős’ girth conjecture (and up to the $\epsilon$-dependency), thus answering Question 1.

Theorem 1.2. For any edge-weighted graph $G(V,E)$, a stretch parameter $k \geq 2$ and an arbitrary small fixed $\epsilon < 1$, there is a deterministic algorithm that constructs a $(2k-1)(1+\epsilon)$-spanner of $G$ with lightness $O(n^{1/k})$ in $O(ma(m,n))$ time, where $a(\cdot,\cdot)$ is the inverse-Ackermann function.

We remark that $\alpha(m,n) = O(1)$ when $m = O(n \log^* n)$; in fact, $\alpha(m,n) = O(1)$ even when $m = \Omega(n \log^*(c) n)$ for any constant $c$, where $\log^{\ell}(\cdot)$ denotes the iterated log-star function with $\ell$ stars. Thus the running time in Theorem 1.2 is linear in $m$ in almost the entire regime of graph densities, i.e., except for very sparse graphs. The previous state-of-the-art runtime for the same lightness bound is super-quadratic in $n$, namely $O(n^{2+1/k+c})$, for any constant $\epsilon' < 1$.

Surprisingly, the result of Theorem 1.2 outperforms the analog result for sparse spanners in weighted graphs: for stretch $2k-1$, the only spanner construction with sparsity $O(n^{1/k})$ is the greedy spanner, whose running time is $O(mn^{1+\frac{1}{k}})$. Other results [1, 35] with stretch $(2k-1)(1+\epsilon)$ have (nearly) linear running time, but the sparsity is $O(n^{1/k} \log(k))$, which is worse than our lightness bound by a factor of $\log(k)$.

Informally, the reason we can achieve light spanners that outperform the state-of-the-art for sparse spanners stems from the fact that our framework essentially reduces the problem of constructing light spanners in weighted graphs to that of constructing sparse spanners in unweighted graphs. (And in unweighted graphs, one can construct a $(2k-1)$-spanner with $O(n^{1/k})$ sparsity within $O(m)$ time.)

Euclidean spanners in the ACT model and unit disk graphs. We prove the following theorems, which resolve Question 3 (and thus also Question 2) as well as Question 4. Note that in all cases below, we achieve constant lightness in conjunction with the state-of-the-art bounds on all the other quality measures (sparsity and/or degree and running time).

Theorem 1.3. For any set $P$ of $n$ in $R^d$, any $d = O(1)$ and any fixed $\epsilon > 0$, one can construct in the ACT model a $(1+\epsilon)$-spanner for $P$ with constant degree and lightness within optimal time $O(n \log n)$.

Theorem 1.4. For any set $P$ of $n$ points in $R^d$, any $d = O(1)$ and any fixed $\epsilon > 0$, one can construct a $(1+\epsilon)$-spanner of the UDG for $P$ with constant sparsity and lightness. For $d = 2$, the construction runtime is $O(n \log n)$, and for $d \geq 3$ it is $O(n^{2-\frac{3}{4}+\frac{1}{2d}+\delta})$ for any constant $\delta > 0$.

Our follow-up paper (which is partially a companion paper; some of the results are in [56]) builds on the foundations of the theory laid in the current paper, aiming to achieve lightness bounds with optimal dependencies on the involved parameters, most notably $\epsilon$, but also all the other involved parameters as mentioned above.
Let \( O \) \( H \) subgraphs (1) with a potential value \( \Phi(\cdot) \) whose weights sum to 1. Let \( \tilde{O} \) takes this hierarchical partition approach to the next level, by proposing a unified framework (developed by most if not all of the works on light spanners (see, e.g., \([37, 39, 21, 12, 13, 57]\)). The current paper Our starting point is a basic hierarchical partition, which dates back to the early 90s \([6, 19]\), and was used construction then focuses on each set \( E_{\sigma,i} \), every edge in \( E_{\sigma,i} \) or having weight (strictly) larger than 1. By property (3), the diameter of clusters in \( \mathcal{C}_{i} \) is roughly \( \Omega(\frac{1}{\epsilon}) \) times the length of level \( i \) edges. The subdivision step will make the presentation of the construction simpler, though it is not required. Let \( \tilde{\text{MST}} \) be the subdivided MST.

Our spanner construction is based on a hierarchy of clusters \( \mathcal{H} = \{ \mathcal{C}_1, \mathcal{C}_2, \ldots \} \) with three properties: (1) for any \( i \geq 1 \), each \( \mathcal{C}_i \) is a partition of \( V \) into subsets, called clusters, (2) \( \mathcal{C}_i \) is a \( \Omega(\frac{1}{\epsilon}) \)-refinement of \( \mathcal{C}_{i+1} \): every cluster \( C \in \mathcal{C}_{i+1} \) is the union of \( \Omega(\frac{1}{\epsilon}) \) clusters in \( \mathcal{C}_{i} \), and (3) each cluster \( C \in \mathcal{C}_i \) has diameter \( O(\epsilon)L_i \). By property (3), the diameter of clusters in \( \mathcal{C}_i \) is roughly \( O(\epsilon) \) times the length of level \( i \) edges.

We also have a potential function \( \Phi : 2^V \rightarrow \mathbb{R}^+ \) that associates each cluster \( C \) in the hierarchy with a potential value \( \Phi(C) \), such that the total potential of clusters at level 1 satisfies \( \sum_{C \in \mathcal{C}_1} \Phi(C) = O(1)w(\text{MST}) \). Let

\[
\Phi_i = \sum_{C \in \mathcal{C}_i} \Phi(C) \quad \& \quad \Delta_i = \Phi_{i-1} - \Phi_i
\] (2)

We call \( \Phi_i \) the potential at level \( i \) and \( \Delta_i \) the potential change at level \( i \). By definition, \( \Phi_1 = O(1)w(\text{MST}) \), and for notational convenience, we set \( \Delta_1 = 0 \). The following lemma is the key in our framework.

**Lemma 1.6.** Let \( \psi \in (0, 1], t \geq 1, \epsilon > 0 \) be parameters such that \( \epsilon < 1 \) and \( E^\sigma = \bigcup_{i \in \mathbb{N}^+} E^\sigma_i \) be the set of edges defined in Equation (1). Let \( \{a_i\}_{i \in \mathbb{N}^+} \) be a sequence of positive real numbers such that \( \sum_{i \in \mathbb{N}^+} a_i \leq A \cdot w(\text{MST}) \) for some \( A \in \mathbb{R}^+ \). Let \( H_0 = \text{MST} \). For any level \( i \geq 1 \), if we can compute all subgraphs \( H_1, \ldots, H_i \subseteq G \) as well as the cluster sets \( \{\mathcal{C}_1, \ldots, \mathcal{C}_{i+1}\} \) in total runtime \( O(\sum_{j=1}^i (|\mathcal{C}_j| + |E^\sigma_j|)^{f(n,m)} + m) \) for some function \( f(\cdot, \cdot) \) such that:

\footnote{The number of clusters is not defined beforehand; it depends on the construction.}
Then we can construct a $t(1+\epsilon)$-spanner for $G(V,E)$ with lightness $O\left(\frac{\lambda+A+1}{\psi} \log \frac{1}{\epsilon} + \frac{1}{\epsilon} + 1\right)$ in time $O\left(\frac{m_f(n,m)}{\psi} \log \frac{1}{\epsilon} + T_{\text{MST}}\right)$.

Remark 1.7. In Lemma 1.6, we construct spanners for edges of $G$ level by level, starting from level 1. By Item (2), when constructing spanners for edges in $E_i^0$, we could assume by induction that all edges at level at most $i-1$ already have stretch $t(1+\epsilon)$ in the spanner constructed so far. Nonetheless, in some applications of our framework, in particular, to Euclidean spanners and spanners for unit disk graphs, this assumption is insufficient; to this end we use a stronger induction hypothesis: every edge of length less than $L_i/(1+\psi)$ (which includes edges in $E \setminus E_i^0$) already has stretch $t(1+\epsilon)$ in the spanner constructed before the construction of level $i$. On the other hand, the weaker assumption suffices for the spanner construction in general and minor-free graphs.

In all cases, we use a sparse spanner on top of a graph constructed from level-$i$ clusters and level-$i$ edges $E_i^0$, called a cluster graph, and then we translate the sparse spanner edges to the edges of the input graph. For general and minor-free graphs, this translation is straightforward. (For general graphs we use the Halperin-Zwick spanner [51] as the sparse spanner whereas for minor-free graphs we do not use any spanner, as the entire cluster graph is minor-free and thus it is automatically sparse.) On the other hand, for Euclidean and UDG graphs, the translation of the sparse spanner edges to the edges of the input graph is more involved. This translation requires the usage of edges from previous levels of the construction, and we don’t use there a sparse spanner as a black-box but rather as a white-box. Since the lightness of the spanner depends on $\lambda$, we would like to have a small $\lambda$. Thus, in our applications of the framework, we try to maximize the potential reduction $\Delta_i$. Indeed, as we will see in the applications of the framework, at each step of the construction, we perform greedy choices that aim at maximizing the potential reduction of the current step. However, in many cases, the potential reduction is 0 or (slightly) negative, while we still need to construct a non-empty graph $H_i$ to satisfy the stretch requirement imposed by Item (2) of Lemma 1.6. The role of the $\{a_i\}_{i \in \mathbb{N}^+}$ is to handle these cases by providing an additive slack: we set $a_i$ to be as large as $w(H_i)$ whenever $\Delta_i = 0$, and as a result, pay an additive factor $A$ in the lightness. Parameter $\psi$ in our specific applications of the framework is either $\epsilon$ or a fixed constant. In the following section, we show how our framework, in particular, Lemma 1.6 can be applied to obtain the results stated above. Figure 1 highlights how our unified framework is applied.

Our framework reduces the problem of efficiently constructing a light spanner to the conjunction of two problems: (1) efficiently constructing a hierarchy of clusters with several carefully chosen properties, and (2) efficiently constructing a sparse spanner. These two problems are intimately related, in the sense that the “carefully chosen properties” of the clusters are set so that we are able to efficiently apply the sparse spanner construction. Ultimately, we achieve in this way a transformation tool from sparsity to lightness; this tool, alas, cannot be applied as a black-box to any family of graphs, and some fine-tunings are needed when going from one graph family to another. On the bright side, these tunings are easily applicable, and we do not need to employ any special structure of the input graph for this task to succeed.

The main focus of the current paper, besides unification, is on the construction runtime. A nontrivial technical contribution of our paper in this context is in introducing the notion of augmented diameter of a cluster. The definition of augmented diameter appears in Section 2 but at a high level, the idea is to consider weights on both nodes and edges in a cluster, where the node weights are determined by the aforementioned potential values of clusters computed (via simple recursion) in previous levels of the hierarchy. The main advantage of augmented diameter over (ordinary) diameter is that it can be computed efficiently, while the computation of diameter is much more costly. Informally, the augmented
diameter can be computed efficiently since (i) we can upper bound the hop-diameter of clusters, and (ii) the clusters at each level are computed on top of some underlying tree; roughly speaking, that means that all the distance computations are carried out on top of subtrees of bounded hop-diameter (or depth), hence the source of efficiency. We demonstrate the applicability of our framework in designing fast (and usually optimal-time) constructions of light spanners in a variety of graph classes.

We next argue that our approach is inherently different than previous ones. To this end, we highlight one concrete result — on Euclidean spanners in the ACT model — which breaks a longstanding barrier in the area of geometric spanners, by using an inherently non-geometric approach. All the previous algorithms for light Euclidean spanners were achieved via the greedy and approximate-greedy spanner constructions. The greedy algorithm is non-geometric but slow, whereas the approximate-greedy algorithm is geometric and can be implemented much more efficiently. The analysis of the lightness in both algorithms is done via the so-called leapfrog property [30, 33, 31, 32, 50, 63], which is a geometric property. The fast spanner construction of GLN [50] implements the approximate-greedy algorithm by constructing a hierarchy of clusters with $O\left(\frac{\log n}{\log \log n}\right)$ levels and, for each level, Dijkstra’s algorithm is used for the construction of clusters for the next level. As a result, the GLN construction incurs an additional $O(\log n)$ factor for each level, which ultimately leads to a runtime of $O\left(\frac{\log^2 n}{\log \log n}\right)$. By employing indirect addressing and exploiting geometric properties, GLN design an implementation of Dijkstra’s algorithm with a runtime of $O(n)$, after a preprocessing time of $O(n \log n)$. The resulting algorithm with indirect addressing takes time $O(n \log n)$. Our approach is inherently different, and in particular we do not need to run Dijkstra’s algorithm or any other single-source shortest (or approximately shortest) path algorithm.

The key to our efficiency is in a careful usage of the new notion of augmented diameter, and its interplay with the potential function argument and the hierarchical partition that we use. We stress again that our approach is non-geometric, and the only usage of geometry is in the sparse spanner construction that we apply (which could be geometric of course, but we apply it as a black box).

1.3.1 Applications to Fast Constructions

In Section 4.1, we show that:

**Theorem 1.8.** Let $\psi = \epsilon$ and $G = (V, E, w)$ be a $(1 + \epsilon)$-spanner either for a set of $n$ points $P$ or for the unit ball graph $U$ of $P$ in $\mathbb{R}^d$. There is an algorithm that can compute all subgraphs $H_1, \ldots, H_{\ell} \subseteq G$ as well as the cluster sets $C_1, \ldots, C_i, C_{i+1}$ in total runtime $O_\epsilon \left(\sum_{j=1}^{\ell} |C_j| + |E_j^{\sigma}|\right) \alpha(m, n) + m$. Furthermore, $H_i$ satisfies Lemma 1.6 with $t = 1 + \epsilon$, $\lambda = O(\epsilon^{-d+1})$, and $A = O(\epsilon^{-d+1})$.

(Here and throughout, the $O_\epsilon$ notation suppresses a polynomial factor of $1/\epsilon$.) It is known that a Euclidean $(1 + \epsilon)$-spanner for a set of $n$ points $P$ in $\mathbb{R}^d$ with degree $O(\epsilon^{-d})$ can be constructed in $O(n \log n)$ time (cf. Theorems 10.1.3 and 10.1.10 in [63]). Furthermore, when $m = O(n \epsilon^{-d})$, we have that:

$$\alpha(m, n) = \alpha(n \epsilon^{-d}, n) = O(\alpha(n) + \log(\epsilon^{-d})) = O(\alpha(n) + d \log(1/\epsilon)).$$

Thus, Theorem 1.3 follows from Theorem 1.8 and Lemma 1.6 by observing that $T_{\text{MST}} = O(n \alpha(n))$ by Chazelle’s algorithm [20].

Fürer and Kasiviswanathan [15] constructed sparse $(1 + \epsilon)$-spanners for unit ball graphs in $O(n \epsilon^{-2} \log n)$ when $d = 2$ and in subquadratic time when $d$ is a constant of value at least 3 (see Lemma 1.1). Thus, Theorem 1.4 follows from Theorem 1.8 and Lemma 1.6.

In Section 4.2, we show that:

**Theorem 1.9.** Let $\psi = \epsilon$ and $G = (V, E, w)$ be a general graph. There is an algorithm that can compute all subgraphs $H_1, \ldots, H_{\ell} \subseteq G$ as well as the cluster sets $C_1, \ldots, C_i, C_{i+1}$ in total runtime $O_\epsilon \left(\sum_{j=1}^{\ell} |C_j| + |E_j^{\sigma}|\right) \alpha(m, n) + m$.


Figure 1: Applications of our framework in obtaining fast constructions of light spanners. See Figure 8 in the appendix for the bigger picture, where we include results of both this work and the follow-up (both of which rely on the framework developed in the current paper).

Furthermore, $H_i$ satisfies Lemma 1.6 with $t = 2k - 1$, $\lambda = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$, and $A = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$.

Since a minimum spanning tree of $G$ can be computed in $T_{MST} = O(m\alpha(m,n))$ time by Chazelle’s algorithm [20], Theorem 1.2 follows from Theorem 1.9 and Lemma 1.6.
In Section 4.3 we show that

**Theorem 1.10.** Let $\psi = 1$ and $G = (V, E, w)$ be a $K_r$-minor-free graph. There is an algorithm that can compute all subgraphs $H_1, \ldots, H_i \subseteq G$ as well as the cluster sets $C_1, \ldots, C_i, C_{i+1}$ in total runtime $O_t\left(\sum_{j=1}^{i+1}(|C_j|)\sqrt{\log r}\right)$. Furthermore, $H_i$ satisfies Lemma 1.6 with $t = (1 + \epsilon)$, $\lambda = O(\sqrt{\log r} + \frac{1}{\epsilon^{t}})$, and $A = O(\sqrt{\log r} + \frac{1}{\epsilon^{t}})$.

Since a minimum spanning tree in a $K_r$-minor-free graph can be computed in $O(nr\sqrt{\log r})$ time 62, $T_{\text{MST}} = O(nr\sqrt{\log r})$. Thus, Theorem 1.5 follows from Theorem 1.10 and Lemma 1.6.

## 2 Preliminaries

Let $G$ be an arbitrary weighted graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. We denote by $w : E(G) \to \mathbb{R}^+$ the weight function on the edge set. Sometimes we write $G = (V, E)$ to clearly explicate the vertex set and edge set of $G$, and $G = (V, E, w)$ to further indicate the weight function $w$ associated with $G$. We use MST$(G)$ to denote a minimum spanning tree of $G$; when the graph is clear from context, we simply use MST as a shorthand for MST$(G)$.

For a subgraph $H$ of $G$, we use $w(H) \triangleq \sum_{e \in E(H)} w(e)$ to denote the total edge weight of $H$. The distance between two vertices $p, q$ in $G$, denoted by $d_G(p, q)$, is the minimum weight of a path between them in $G$. The diameter of $G$, denoted by $\text{diam}(G)$, is the maximum pairwise distance in $G$. A diameter path of $G$ is a shortest (i.e., of minimum weight) path in $G$ realizing the diameter of $G$, that is, it is a shortest path between some pair $u, v$ of vertices in $G$ such that $\text{diam}(G) = d_G(u, v)$.

Sometimes we shall consider graphs with weights on both edges and vertices. We define the augmented weight of a path to be the total weight of all edges and vertices along the path. The augmented distance between two vertices in $G$ is defined as the minimum augmented weight of a path between them in $G$. The diameter of $G$, denoted by $\text{adm}(G)$, is the maximum pairwise augmented distance in $G$; since we will focus on non-negative weights, the augmented distance and augmented diameter are no smaller than the (ordinary notions of) distance and diameter. An augmented diameter path of $G$ is a path of minimum augmented weight realizing the augmented diameter of $G$.

Given a subset of vertices $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$: $G[X] = \{(u, v) \in E(G) \mid u, v \in X\}$. Let $F \subseteq E(G)$ be a subset of edges of $G$. We denote by $G[F]$ the subgraph of $G$ with $V(G[F]) = V(G)$ and $E(G[F]) = F$.

Let $S$ be a spanning subgraph of $G$; weights of edges in $S$ are inherited from $G$. The stretch of $S$ is given by $\max_{x, y \in V(G)} \frac{d_S(x, y)}{d_G(x, y)}$, and it is realized by some edge $e$ of $G$. Throughout we will use the following known observation, which implies that stretch of $S$ is equal to $\frac{d_S(u, v)}{w(u, v)}$ for some edge $(u, v) \in E(G)$.

**Observation 2.1.** $\max_{x, y \in V(G)} \frac{d_S(x, y)}{d_G(x, y)} = \max_{(x, y) \in E(G)} \frac{d_S(x, y)}{d_G(x, y)}$.

We say that $S$ is a $t$-spanner of $G$ if the stretch of $S$ is at most $t$. There is a simple greedy algorithm, called path greedy (or shortly greedy), to find a $t$-spanner of a graph $G$: Examine the edges $e = (x, y)$ in $G$ in nondecreasing order of weights, and add to the spanner edge $(x, y)$ iff the distance between $x$ and $y$ in the current spanner is larger than $t \cdot w(x, y)$.

We say that a subgraph $H$ of $G$ is a $t$-spanner for a subset of edges $X \subseteq E$ if $\max_{(u, v) \in X} \frac{d_H(u, v)}{d_G(u, v)} \leq t$.

In the context of minor-free graphs, we denote by $G/e$ the graph obtained from $G$ by contracting $e$, where $e$ is an edge in $G$. If $G$ has weights on edges, then every edge in $G/e$ inherits its weight from $G$.

In addition to general and minor-free graphs, this paper studies geometric graphs. Let $P$ be a set of $n$ points in $\mathbb{R}^d$. We denote by $\|p, q\|$ the Euclidean distance between two points $p, q \in \mathbb{R}^d$. A geometric
graph $G$ for $P$ is a graph where the vertex set corresponds to the point set, i.e., $V(G) = P$, and the edge weights are the Euclidean distances, i.e., $w(u, v) = ||u, v||$ for every edge $(u, v)$ in $G$. Note that $G$ need not be a complete graph. If $G$ is a complete graph, i.e., $G = (P, (\frac{1}{2}), ||||)$, then $G$ is equivalent to the Euclidean space induced by the point set $P$. For geometric graphs, we use the term vertex and point interchangeably.

We use $[n]$ and $[0, n]$ to denote the sets $\{1, 2, \ldots, n\}$ and $\{0, 1, \ldots, n\}$, respectively.

3 Unified Framework

Recall that we ignored the dependencies of $\epsilon$ in the introduction, by assuming it is fixed. In this section and onwards, we shall omit this assumption and explicate the exact dependencies of $\epsilon$ on the running time and lightness. Although the $\epsilon$-dependencies are not a central part of this work, they are central to our companion paper on spanners with truly optimal lightness; in particular, this will allow us to compare the lightness bounds in the two papers.

3.1 The Framework

In this section, we present in detail the framework that was highlighted briefly in Section 1.3. Let MST be a minimum spanning tree of the input $n$-vertex $m$-edge graph $G = (V, E, w)$. Let $T_{\text{MST}}$ be the running time needed to construct MST. By scaling, we shall assume w.l.o.g. that the minimum edge weight is 1. Let $\bar{w} = \frac{w(\text{MST})}{m}$. We remove from $G$ all edges of weight larger than $w(\text{MST})$; such edges do not belong to any shortest path, hence removing them does not affect the distances between vertices in $G$. We define two sets of edges, $E_{\text{light}}$ and $E_{\text{heavy}}$, as follows:

$$E_{\text{light}} = \{ e \in E : w(e) \leq \frac{\bar{w}}{\epsilon} \} \quad \& \quad E_{\text{heavy}} = E \setminus E_{\text{light}}$$  

(3)

It is possible that $\frac{\bar{w}}{\epsilon} < 1$; in this case, $E_{\text{light}} = \emptyset$. The next observation is implied by the definition of $\bar{w}$.

Observation 3.1. $w(E_{\text{light}}) \leq \frac{w(\text{MST})}{\epsilon}$.

Recall that the parameter $\epsilon$ is in the stretch $t(1 + \epsilon)$ in Lemma 1.6. It controls the stretch blow-up in Lemma 1.6 and ultimately, the stretch of the final spanner. There is an inherent trade-off between the stretch blow-up (a factor of $1 + \epsilon$) and the blow-up of the other parameters, including runtime and lightness, by at least a factor of $1/\epsilon$.

By Observation 3.1, we can safely add $E_{\text{light}}$ to our final spanner, while paying only an additive $+\frac{1}{\epsilon}$ factor to the lightness bound. Hence, by Observation 2.1, in the spanner construction that follows, it suffices to focus on the stretch for edges in $E_{\text{heavy}}$. Next, we partition the edge set $E_{\text{heavy}}$ into subsets of edges, such that for any two edges $e, e'$ in the same subset, their weights are either almost the same (up to a factor of $1 + \psi$) or they are far apart (by at least a factor of $\frac{1}{\epsilon(1 + \psi)}$, where $\psi$ is a parameter to be optimized later. In the applications of our framework in the following sections, we either choose $\psi = 1$ or $\psi = \epsilon$, depending on whether we aim for a stretch of $1 + \epsilon$ or for larger stretch (as in general graphs).

Definition 3.2 (Partitioning $E_{\text{heavy}}$). Let $\psi$ be any parameter in the range $(0, 1]$. Let $\mu_\psi = [\log_{1+\psi} \frac{1}{\epsilon}]$. We partition $E_{\text{heavy}}$ into subsets $\{E_\sigma\}_{\sigma \in [\mu_\psi]}$ such that $E^\sigma = \bigcup_{i \in \mathbb{N}^+} E_i^\sigma$ where:

$$E_i^\sigma = \left\{ e : \frac{L_i}{1+\psi} \leq w(e) \leq L_i \right\} \quad \text{with} \quad L_0 = L_0/\epsilon^i, L_0 = (1 + \psi)^\sigma \bar{w}.$$  

(4)
By definition, we have $L_i = L_{i-1}/\epsilon$ for each $i \geq 1$.

Readers may notice that if $\log_{\frac{1}{1+\psi}} \frac{1}{\epsilon}$ is not an integer, by the definition of $E^\sigma$, it could be that $E^\psi \cap E^1 \neq \emptyset$, in which case $\{E^\sigma\}_{\sigma \in [\mu_\psi]}$ is not really a partition of $E^\text{heavy}$. This can be fixed by taking to $E^\psi$ edges that are not in $\cup_{1 \leq \sigma \leq \mu_\psi-1} E^\sigma$. We henceforth assume that $\{E^\sigma\}_{\sigma \in [\mu_\psi]}$ is a partition of $E^\text{heavy}$.

The following lemma shows that it suffices to focus on the stretch of edges in $E^\sigma$, for an arbitrary $\sigma \in [\mu_\psi]$; see Appendix A.1 for a proof.

**Lemma 3.3.** If for every $\sigma \in [\mu_\psi]$, we can construct a $k$-spanner $H^\sigma \subseteq G$ for $E^\sigma$ with lightness at most $\text{Light}_{H^\sigma}$ in time $\text{Time}_{H^\sigma}(m,n)$ (where $\text{Light}_{H^\sigma}$ and $\text{Time}_{H^\sigma}(m,n)$ do not depend on $\sigma$), then we can construct a $k$-spanner for $G$ with lightness $O\left(\frac{\text{Light}_{H^\sigma} \log(1/\epsilon)}{\psi} + 1 / \epsilon\right)$ in time $O\left(\frac{\text{Time}_{H^\sigma}(m,n) \log(1/\epsilon)}{\psi} + T_{\text{MST}}\right)$.

In the applications of our framework in Section 4, we either choose $\psi = 1$ or $\psi = \epsilon$. Thus, the lightness and construction time of the final spanner in Lemma 3.3 are larger than the lightness and construction time of the spanner for a single edge set $E^\sigma$ by a factor that only depends on $\epsilon$. We shall henceforth focus on constructing a spanner for $E^\sigma$, for an arbitrarily fixed $\sigma \in [\mu_\psi]$.

Fix an arbitrary $\sigma \in [\mu_\psi]$. In what follows we present a clustering framework for constructing a spanner $H^\sigma$ for $E^\sigma$ with stretch $t(1 + \epsilon)$. We will assume that $\epsilon$ is sufficiently smaller than 1.

**Subdividing MST.** We subdivide each edge $e \in \text{MST}$ of weight more than $\bar{w}$ into $\lceil \frac{w(e)}{\bar{w}} \rceil$ edges of weight (of at most $\bar{w}$ and at least $\bar{w}/2$ each) that sums to $w(e)$. (New edges do not have to have equal weights.) Let $\tilde{\text{MST}}$ be the resulting subdivided MST. We refer to vertices that are subdividing the MST edges as virtual vertices. Let $\tilde{V}$ be the set of vertices in $V$ and virtual vertices; we call $\tilde{V}$ the extended set of vertices. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the graph that consists of the edges in $\tilde{\text{MST}}$ and $E^\sigma$.

**Observation 3.4.** $|\tilde{E}| = O(m)$.

**Proof:** It suffices to show that $|E(\tilde{\text{MST}})| = O(m)$. Indeed, since $w(\tilde{\text{MST}}) = w(\text{MST})$ and each edge of $\tilde{\text{MST}}$ has weight at least $\bar{w}/2$, we have $|E(\tilde{\text{MST}})| \leq 2m$. \hfill $\square$

The $t(1 + \epsilon)$-spanner that we construct for $E^\sigma$ is a subgraph of $\tilde{G}$ containing all edges of $\tilde{\text{MST}}$; we can enforce this assumption by adding the edges of $\tilde{\text{MST}}$ to the spanner. By replacing the edges of $\tilde{\text{MST}}$ by those of MST, we can transform any subgraph of $\tilde{G}$ that contains the entire tree MST to a subgraph of $G$ that contains the entire tree MST. We denote by $\tilde{H}^\sigma$ the $t(1 + \epsilon)$-spanner of $E^\sigma$ in $\tilde{G}$; by abusing the notation, we will write $H^\sigma$ rather than $\tilde{H}^\sigma$ in the sequel, under the understanding that in the end we transform $H^\sigma$ to a subgraph of $G$.

Recall that $E^\sigma = \cup_{i \in \mathbb{N}^+} E^\sigma_i$ where $E^\sigma_i$ is the set of edges defined in Equation (4). We refer to edges in $E^\sigma_i$ as level-$i$ edges. We say that a level $i$ is empty if the set $E^\sigma_i$ of level-$i$ edges is empty; in the sequel, we shall only consider the nonempty levels.

**Claim 3.5.** The number of (nonempty) levels is $O(\log m)$.

**Proof:** Note that every edge of $E^\sigma$ has weight at least $\bar{w}$ and at most $w(MST) = m\bar{w}$. Furthermore, the length of any edge in $E^\sigma_{i+1}$ is at least $\frac{1}{(1+\psi)^i}$ times the length of any edge in $E^\sigma_i$ for any $i \in \mathbb{N}^+$. Note that $\frac{1}{(1+\psi)^i} \geq \frac{1}{2^i}$ since $\psi \leq 1$. Therefore, the number of levels is $O\left(\frac{\log(m\bar{w})}{\log(1/(2^i))}\right) = O(\log(m))$ for any $\epsilon \leq 1/2$. \hfill $\square$

Our construction crucially relies on a hierarchy of clusters, which dates back to works of [6, 19, 36], and then used by many subsequent works, e.g., [37, 39, 21, 12, 13, 57]. A cluster in a graph is simply a subset of vertices in the graph. Nonetheless, as will become clear soon, we care also about edges connecting vertices in the cluster, and of the properties that these edges possess. Our hierarchy of clusters, denoted by $H = \{C_1, C_2, \ldots\}$ satisfies the following properties:
Lemma 1.6. For any \( i \geq 1 \), each \( C_i \) is a partition of \( \tilde{V} \). When \( i \) is large enough, \( C_i \) contains a single set \( \tilde{V} \) and \( C_{i+1} = \emptyset \).

(P2) \( C_i \) is an \( \Omega\left(\frac{1}{\epsilon}\right) \)-refinement of \( C_{i+1} \), i.e., every cluster \( C \in C_{i+1} \) is obtained as the union of \( \Omega\left(\frac{1}{\epsilon}\right) \) clusters in \( C_i \) for \( i \geq 1 \).

(P3) For each cluster \( C \in C_i \), we have \( \text{Dm}(H^\sigma[Z]) \leq gL_{i-1} \), for a sufficiently large constant \( g \) to be determined later. (Recall that \( L_i \) is defined in Equation (1).)

Remark 3.6. (1) We construct \( H^\sigma \) along with the cluster hierarchy. Suppose that at some step \( s \) of the algorithm, we construct a level-\( i \) cluster \( C \). Let \( H^\sigma_s \) be \( H^\sigma \) at step \( s \). We shall maintain (P3) by maintaining the invariant that \( \text{Dm}(H^\sigma_s[Z]) \leq gL_{i-1} \); indeed, adding more edges in later steps of the algorithm does not increase the diameter of the subgraph induced by \( C \).

(2) It is time-expensive to compute the diameter of a cluster exactly. Thus, we explicitly associate with each cluster \( C \in C_i \) a proxy parameter of the diameter during the course of the construction. This proxy parameter has two properties: (a) it is at least the diameter of the cluster, and (b) it is lower-bounded by \( \Omega(L_{i-1}) \). Property (a) is crucial in arguing for the stretch of the spanner. Property (b) is crucial to have an upper bound on the number of level-\( i \) clusters contained in a level-\( (i+1) \) cluster, which speeds up its (the level-\( (i+1) \) cluster’s) construction.

When \( \epsilon \) is sufficiently small, specifically smaller than the constant hiding in the \( \Omega \)-notation in property (P2) by at least a factor of 2, it holds that \( |C_{i+1}| \leq |C_i|/2 \), yielding a geometric decay in the number of clusters at each level of the hierarchy. This geometric decay is crucial to our fast constructions.

Our construction of the cluster hierarchy \( \mathcal{H} \) will be carried out level by level, starting from level 1. After we construct the set of level-\( (i+1) \) clusters, we compute a subgraph \( H^\sigma_i \subseteq G \) as stated in Lemma 1.6. The final spanner \( H^\sigma \) is obtained as the union of all subgraphs \( \{H^\sigma_i\}_{i \in \mathbb{N}^+} \). To bound the weight of \( H^\sigma \), we rely on a potential function \( \Phi \) that is formally defined as follows:

Definition 3.7 (Potential Function \( \Phi \)). We use a potential function \( \Phi : 2\tilde{V} \rightarrow \mathbb{R}^+ \) that maps each cluster \( C \) in the hierarchy \( \mathcal{H} \) to a potential value \( \Phi(C) \), such that the total potential of clusters at level 1 satisfies:

\[
\sum_{C \in C_1} \Phi(C) \leq w(\text{MST}) .
\]

(5)

Level-\( i \) potential is defined as \( \Phi_i = \sum_{C \in C_i} \Phi(C) \) for any \( i \geq 1 \). The potential change at level \( i \), denoted by \( \Delta_i \) for every \( i \geq 1 \), is defined as:

\[
\Delta_i = \Phi_{i-1} - \Phi_i .
\]

(6)

The key to our framework is Lemma 1.6 for convenience, we restate this lemma below.

Lemma 1.6. Let \( \psi \in (0, 1] \), \( t \geq 1 \), \( \epsilon > 0 \) be parameters such that \( \epsilon \ll 1 \) and \( E^\sigma = \bigcup_{i \in \mathbb{N}^+} E^\sigma_i \) be the set of edges defined in Equation (1). Let \( \{a_i\}_{i \in \mathbb{N}^+} \) be a sequence of positive real numbers such that \( \sum_{i \in \mathbb{N}^+} a_i \leq A \cdot w(\text{MST}) \) for some \( A \in \mathbb{R}^+ \). Let \( H_0 = \text{MST} \). For any level \( i \geq 1 \), if we can compute all subgraphs \( H_1, \ldots, H_i \subseteq G \) as well as the cluster sets \( \{C_1, \ldots, C_i, C_{i+1}\} \) in total runtime \( O(\sum_{j=1}^{i} (|C_j| + |E^\sigma_j|)f(n,m) + m) \) for some function \( f(\cdot, \cdot) \) such that:

(1) \( w(H_i) \leq \lambda \Delta_{i+1} + a_i \) for some \( \lambda \geq 0 \),

(2) for every \( u, v \in E^\sigma_i \), \( d_{H_{\leq i}}(u, v) \leq t(1 + \epsilon)w(u, v) \), where \( H_{\leq i} \) is the spanner constructed for edges of \( G \) of length up to \( L_i \).

Then we can construct a \( t(1+\epsilon) \)-spanner for \( G(V, E) \) with lightness \( O(\frac{\lambda + A + 1}{\psi} \log \frac{1}{\epsilon} + \frac{1}{t}) \) in time \( O(\frac{mf(n,m)}{\psi} \log \frac{1}{\epsilon} + T_{\text{MST}}) \).
Proof: Let $H^\sigma = \bigcup_{i \in \mathbb{N}} H_i$. Note that $w(MST) = w(MST)$, since $MST$ is simply a subdivision of MST. By condition (1) of Lemma 1.6

$$w(H^\sigma) \leq \lambda \sum_{i \in \mathbb{N}^+} \Delta_i + \sum_{i \in \mathbb{N}^+} a_i + w(MST) \leq \lambda \cdot \Phi_1 + A \cdot w(MST) + w(MST)$$

$$\leq (\lambda + A + 1)w(MST) \quad \text{(by Equation (5))}$$

Equation (7) and Lemma 3.3 implies the lightness upper bound; here $Light_{H^\sigma} = (O(\lambda) + A + 1)$. The stretch bound $t(1 + \epsilon)$ follows directly from the fact that $E^\sigma = \bigcup_{i \in \mathbb{N}^+} E_i^\sigma$ and Lemma 3.3.

To bound the running time, we note that $\sum_{i \in \mathbb{N}^+} |E_i^\sigma| \leq m$ and by property (P2), we have $\sum_{i \in \mathbb{N}^+} |C_i| = \sum_{i \in \mathbb{N}^+} \frac{O(1)}{\epsilon + 1} = O(|C_1|) = O(m)$. Thus, by the assumption of Lemma 1.6, the total running time to construct $H^\sigma$ is:

$$Time_{H^\sigma}(m, n) = O\left(\left(\sum_{i \in \mathbb{N}^+} (|C_i| + |E_i|)f(m, n) + m\right)\right) = O\left(mf(m, n)\right).$$

Plugging this runtime bound on top of Lemma 3.3 yields the required runtime bound in Lemma 1.6. \qed

In summary, two important components in our spanner construction is a hierarchy of clusters and a potential function as defined in Definition 3.7. In Section 3.2 we present a construction of level-1 clusters and a general principle for assigning potential values to clusters. In Section 3.3, we outline an efficient construction of clusters at any level $i + 1$ for $i \geq 1$. The details of the construction are deferred to Section 5. In Section 3.4 we present a general approach for constructing $H_i$. Our construction of $H_i$ assumes the existence of Algorithm $A$ that constructs a sparse spanner for a specific graph (of clusters in $C_i$) arising from our framework. In the incarnations of our framework in Section 4, we show the detailed implementation of Algorithm $A$ in each setting, which takes into account the structure of the input graph $G$. This is the only place where the structure of the input plays a role in our framework, which demonstrates the flexibility and wide applicability of our framework.

Assumptions on $\epsilon$ and the stretch. In the incarnations of our framework, we will construct spanners with stretch $t(1 + c_g \epsilon)$ for a sufficiently large constant $c_g$ that depends on $g$ only; here $g$ is the constant in property (P3). We then can recover the stretch $t(1 + \epsilon)$ by setting $\epsilon \leftarrow \epsilon/c_g$. We all also assume that $\epsilon$ is a sufficiently small constant, and in particular, $\epsilon \leq 1/c_g$.

3.2 Designing A Potential Function

In this section, we present in detail the underlying principle used to design the potential function $\Phi$ in Definition 3.7. We start by constructing and assigning potential values for level-1 clusters.

Lemma 3.8. In time $O(m)$, we can construct a set of level-1 clusters $C_1$ such that, for each cluster $C \in C_1$, the subtree $MST[C]$ of $MST$ induced by $C$ satisfies $L_0 \leq Dm(MST[C]) \leq 14L_0$.

Proof: We apply a simple greedy construction to break $MST$ into a set $S$ of subtrees of diameter at least $L_0$ and at most $5L_0$ as follows. (1) Repeatedly pick a vertex $v$ in a component $T$ of diameter at least $4L_0$, break a minimal subtree of radius at least $L_0$ with center $v$ from $T$, and add the minimal subtree to $S$. (2) For each remaining component $T'$ after step (1), there must be an $MST$ edge $e$ connecting $T'$ and a subtree $T \in S$ formed in step (1); we add $T'$ and $e$ to $T$. Finally, we form $C_1$ by taking the vertex set of each subtree in $S$ to be a level-1 cluster. The running time bound follows directly from the construction.
We now bound the diameter of each subtree in \( \mathcal{S} \). In step (1), by the description of the procedure and the fact that each \( \tilde{\text{MST}} \) edge is of weight at most \( \tilde{w} \), we have that each subtree \( T \) has a radius of at most \( L_0 + \tilde{w} \) and hence a diameter of at most \( 2(L_0 + \tilde{w}) \). In step (2), each subtree \( T \) is augmented by subtrees of diameter at most \( 4L_0 \) via \( \text{MST} \) edges in a star-like way. Thus, the augmentation in step (2) increases the diameter of \( T \) by at most \( 2(4L_0 + \tilde{w}) \). Since \( \tilde{w} \leq L_0 \), we conclude that the diameter of the resulting subtrees is upper bounded by \( 2(L_0 + \tilde{w}) + 2(4L_0 + \tilde{w}) \leq 14L_0 \), as required. \( \square \)

By choosing \( g \geq 14 \), clusters in \( \mathcal{C}_1 \) satisfy properties (P1) and (P3). Note that (P2) is not applicable to level-1 clusters by definition. As for (P3), \( \text{Dm}(H^\sigma[C]) \leq 14L_0 \), for each \( C \in \mathcal{C}_1 \), since \( H^\sigma \) includes all edges of \( \text{MST} \).

Next, we assign a potential value for each level-1 cluster as follows:

\[
\Phi(C) = \text{Dm}(\tilde{\text{MST}}[C]) \quad \forall C \in \mathcal{C}_1
\]  

(8)

We now claim that the total potential of all clusters at level 1 is at most \( w(\text{MST}) \) as stated in Definition 3.7.

Lemma 3.9. \( \Phi_1 \leq w(\text{MST}) \).

Proof: By definition of \( \Phi_1 \), we have:

\[
\Phi_1 = \sum_{C \in \mathcal{C}_1} \Phi(C) = \sum_{C \in \mathcal{C}_1} \text{Dm}(\tilde{\text{MST}}[C]) \leq \sum_{C \in \mathcal{C}_1} w(\tilde{\text{MST}}[C]) \leq w(\tilde{\text{MST}}) = w(\text{MST}) .
\]

The penultimate inequality holds since level-1 clusters induce vertex-disjoint subtrees of \( \tilde{\text{MST}} \). \( \square \)

While the potential of a level-1 cluster is the diameter of the subtree induced by the cluster, the potential assigned to a cluster at level at least 2 need not be the diameter of the cluster. Instead, it is an overestimate of the cluster’s diameter, as imposed by the following potential-diameter (PD) invariant.

PD Invariant: For every cluster \( C \in \mathcal{C}_i \) and any \( i \geq 1 \), \( \text{Dm}(H_{\leq i-1}[C]) \leq \Phi(C) \). (Recall that \( H_{\leq i-1} \) is the spanner constructed for edges of \( G \) of length up to \( L_{i-1} \), as defined in Lemma 1.6)

Remark 3.10. As discussed in Remark 3.6, it is time-expensive to compute the diameter of each cluster. By the PD Invariant, we can use the potential \( \Phi(C) \) of a cluster \( C \in \mathcal{C}_i \) as an upper bound on the diameter of \( H_{\leq i-1}[C] \). As we will demonstrate in the sequel, \( \Phi(C) \) can be computed efficiently.

To define potential values for clusters at levels at least 2, we introduce a cluster graph, in which the nodes correspond to clusters. We shall derive the potential values of clusters via their structure in the cluster graph, as described next.

Definition 3.11 (Cluster Graph). A cluster graph at level \( i \geq 1 \), denoted by \( \mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i^*, \omega) \), is a simple graph where each node corresponds to a cluster in \( \mathcal{C}_i \) and each inter-cluster edge corresponds to an edge between vertices that belong to the corresponding clusters. We assign weights to both nodes and edges as follows: for each node \( \varphi_C \in \mathcal{V}_i \) corresponding to a cluster \( C \in \mathcal{C}_i \), \( \omega(\varphi_C) = \Phi(C) \), and for each edge \( e = (\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i^* \) corresponding to an edge \((u, v)\) of \( \tilde{G} \), \( \omega(e) = w(u, v) \).

In our framework, we want the cluster graph \( \mathcal{G}_i \) to have the following properties.

Definition 3.12 (Properties of \( \mathcal{G}_i \)). (1) The edge set \( \mathcal{E}_i^* \) of \( \mathcal{G}_i \) is the union \( \tilde{\text{MST}}_i \cup \mathcal{E}_i \), where \( \tilde{\text{MST}}_i \) is the set of edges corresponding to edges in \( \text{MST} \) and \( \mathcal{E}_i \) is the set of edges corresponding to a subset of edges in \( \tilde{E}_i^* \).
(2) \( \text{MST}_i \) induces a spanning tree of \( G_i \). We abuse notation by using \( \text{MST}_i \) to denote the induced spanning tree.

(3) \( G_i \) has no removable edge: an edge \((\varphi_{C_u}, \varphi_{C_v}) \in E_i \) is removable if (3a) the path \( \text{MST}_i[\varphi_{C_u}, \varphi_{C_v}] \) between \( \varphi_{C_u} \) and \( \varphi_{C_v} \) only contains nodes in \( \text{MST}_i \) of degree at most 2 and (3b) \( \omega(\text{MST}_i[\varphi_{C_u}, \varphi_{C_v}]) \leq t(1+6g)\omega(\varphi_{C_u}, \varphi_{C_v}) \).

As we will show in the sequel, if an edge \((\varphi_{C_u}, \varphi_{C_v}) \) satisfies property (3b), there is a path of stretch at most \( t(1+6g) \) in \( H_{\leq i-1} \) between \( u \) and \( v \) and hence, we do not need to consider edge \((u, v) \) in the construction of \( H_i \). To meet the required lightness bound, it turns out that it suffices to remove edges satisfying both properties (3a) and (3b), rather than removing all edges satisfying property (3b).

At the outset of the construction of level-\((i+1)\) clusters, we construct a cluster graph \( \tilde{G}_i \). We assume that the spanning tree \( \text{MST}_i \) of \( G_i \) is given, as we construct the tree by the end of the construction of level-\(i\) clusters. After we complete the construction of level-\((i+1)\) clusters, we construct \( \text{MST}_{i+1} \) for the next level.

**Observation 3.13.** At level 1, both \( \mathcal{V}_1 \) and \( \text{MST}_1 \) can be constructed in \( O(m) \) time.

**Proof:** Edges of \( \text{MST}_1 \) correspond to the edges of \( \text{MST} \) that do not belong to any level-1 cluster, i.e., to any \( \text{MST}[C] \) where \( C \in \mathcal{C}_1 \). Thus, the observation follows from Observation \( 3.4 \) and Lemma \( 3.8 \). \( \square \)

**The structure of level-\((i+1)\) clusters.** Next, we describe how to construct the level-\((i+1)\) clusters via the cluster graph \( \tilde{G}_i \). We shall construct a collection of subgraphs \( X \) of \( \tilde{G}_i \), and then map each subgraph \( X \in \mathcal{X} \) to a cluster \( C_X \in \mathcal{C}_{i+1} \) as follows:

\[
C_X = \cup_{\varphi_C \in \mathcal{V}(X)} C.
\]

That is, \( C_X \) is the union of all level-\(i\) clusters that correspond to nodes in \( X \).

For any subgraph \( X \) in a cluster graph, we denote by \( \mathcal{V}(X) \) and \( \mathcal{E}(X) \) the vertex and edge sets of \( X \), respectively. To guarantee properties \([\text{P1}'-\text{P3}']\) defined before Remark \( 3.6 \) for clusters in \( \mathcal{C}_{i+1} \), we will make sure that subgraphs in \( \mathcal{X} \) satisfy the following properties:

- **(P1')**. \( \mathcal{V}(X) \) is a partition of \( \mathcal{V}_i \).
- **(P2')**. \( |\mathcal{V}(X)| = \Omega(\frac{1}{\epsilon}) \).
- **(P3')**. \( L_i \leq \text{Adm}(X) \leq g L_i \).

Recall that \( \text{Adm}(X) \) is the augmented diameter of \( X \), a variant of diameter defined for graphs with weights on both nodes and edges, see Section \( 2 \). Recall that the augmented diameter of \( X \) is at least the diameter of the corresponding cluster \( C_X \).

We then set the potential of cluster \( C_X \) corresponding to subgraph \( X \) as:

\[
\Phi(C_X) = \text{Adm}(X).
\]

Thus, the augmented diameter of any such subgraph \( X \) will be the weight of the corresponding node in the level-\((i+1)\) cluster graph \( \tilde{G}_{i+1} \). Our goal is to construct \( H_i \) along with \( C_{i+1} \) as guaranteed by Lemma \( 1.6 \). \( H_i \) consists of a subset of the edges in \( E_i \); we can assume that the vertex set of \( H_i \) is just the entire set \( V \). Up to this point, we have not explained yet how \( H_i \) is constructed, since the exact construction of \( H_i \) depends on specific incarnations of our framework, which may change from one graph class to another.
While properties \( [P1] \) and \( [P2] \) directly imply properties \( [P1'] \) and \( [P2'] \) of \( C_X \), property \( [P3'] \) does not directly imply property \( [P3] \); although the diameter of any weighted subgraph (with edge and vertex weights) is upper bounded by its augmented diameter, we need to guarantee that the (corresponding) edges of \( X \) belong to \( H_{\leq i} \). Indeed, without this condition, the diameter of \( H_{\leq i} \) could be much larger than the augmented diameter of \( X \).

**Lemma 3.14.** Let \( X \in \mathbb{X} \) be a subgraph of \( G_i \) satisfying properties \( [P1] - [P3'] \). Suppose that for every edge \((\varphi_{C_u}, \varphi_{C_v}) \in E(X)\), \((u,v) \in H_{\leq i} \). By setting the potential value of \( C_X \) to be \( \Phi(C_X) = \text{Adm}(X) \) for every \( X \in \mathbb{X} \), the two nodes in \( \text{PD Invariant} \) is satisfied, and that \( C_X \) satisfies all properties \( [P1] - [P3] \).

**Proof:** It can be seen directly that properties \( [P1] \) and \( [P2] \) of \( X \) directly imply properties \( [P1] \) and \( [P2] \) of \( C_X \), respectively. We prove, by induction on \( i \), that property \( [P3] \) holds and that the (corresponding) \( \text{PD Invariant} \) is satisfied. The basis \( i = 1 \) is trivial. For the induction step, we assume inductively that for each cluster \( C \in \mathcal{C}_i \), \( \text{Dm}(H_{\leq i-1}[C]) \leq gL_{i-1} \) and that the \( \text{PD Invariant} \) is satisfied: \( \Phi(C) \geq \text{Dm}(H_{\leq i-1}[C]) \). Consider any level-(\( i + 1 \)) cluster \( C_X \) corresponding to a subgraph \( X \in \mathbb{X} \). Let \( H_{C_X} \) be the graph obtained by first taking the union \( \bigcup_{\varphi \in V(X)} H_{\leq i-1}[C] \) and then adding in the edge set \( \{(u,v) \} | (\varphi_{C_u}, \varphi_{C_v}) \in E(X) \). Observe that \( H_{C_X} \) is a subgraph of \( H_{\leq i} \) by the assumption that \((u,v) \in H_{\leq i} \) for every edge \((\varphi_{C_u}, \varphi_{C_v}) \in E(X) \). We now show that \( \text{Dm}(H_{C_X}) \leq \text{Adm}(X) \), which is at most \( gL_i \) by property \( [P3'] \). This would imply both property \( [P3] \) and the \( \text{PD Invariant} \) for \( C_X \) since \( \Phi(C_X) = \text{Adm}(X) \), which would complete the proof of the induction step.

Let \( u, v \) be any two vertices in \( H_{C_X} \) whose shortest distance in \( H_{C_X} \) realizes \( \text{Dm}(H_{C_X}) \). Let \( \varphi_{C_u}, \varphi_{C_v} \) be the two nodes in \( X \) that correspond to two clusters \( C_u, C_v \) containing \( u \) and \( v \), respectively. Let \( p_{u,v} \) a path in \( G_i \) of minimum augmented weight between \( \varphi_{C_u} \) and \( \varphi_{C_v} \). Observe that \( \omega(p_{u,v}) \leq \text{Adm}(X) \). We now construct a path \( P_{u,v} \) between \( u \) and \( v \) in \( H_{C_X} \) as follows. We write \( P_{u,v} \equiv (\varphi_{C_u} = \varphi_{C_1}, e_1, \varphi_{C_2}, e_2, \ldots, \varphi_{C_\ell} = \varphi_{C_v}) \) as an alternating sequence of nodes and edges. For every \( 1 \leq p \leq \ell - 1 \), let \( (u_p, v_p) \) be the edge in \( E_i^p \) that corresponds to \( e_p \). We then define \( v_0 = u, u_\ell = v \) and

\[
P_{u,v} = Q_{H_{\leq i-1}[C_1]}(v_0, u_1) \circ (u_1, v_1) \circ Q_{H_{\leq i-1}[C_2]}(v_1, u_2) \circ (u_2, v_2) \circ \ldots \circ Q_{H_{\leq i-1}[C_\ell]}(v_\ell-1, u_\ell),
\]

where \( Q_{H_{\leq i-1}[C_p]}(v_p-1, u_p) \) for \( 1 \leq p \leq \ell \) denotes the shortest path in the corresponding subgraph between the endpoints of the respective edge, as specified in all the subscripts), and \( \circ \) is the path concatenation operator. By the induction hypothesis for the \( \text{PD Invariant} \) and \( i \), \( \omega(Q_{H_{\leq i-1}[C_p]}(v_p-1, u_p)) \leq \omega(\varphi_{C_p}) \) for each \( 1 \leq p \leq \ell \). Thus, \( \omega(P_{u,v}) \leq \omega(P_{u,v}) \leq \text{Adm}(X) \). It follows that \( \text{Dm}(H_{C_X}) \leq \omega(P_{u,v}) \leq \text{Adm}(X) \) as desired.

**Local potential change.** For each subgraph \( X \in \mathbb{X} \), we define the **local potential change** of \( X \), denoted by \( \Delta_i+1(X) \) as follows:

\[
\Delta_i+1(X) \overset{\text{def}}{=} \left( \sum_{\varphi \in V(X)} \Phi(C) \right) - \Phi(C) = \left( \sum_{\varphi \in V(X)} \omega(\varphi) \right) - \text{Adm}(X). \tag{11}
\]

**Claim 3.15.** \( \Delta_{i+1} = \sum_{X \in \mathbb{X}} \Delta_i+1(X) \).

**Proof:** By property \( [P1] \), subgraphs in \( \mathbb{X} \) are vertex-disjoint and cover the vertex set \( V_i \), hence \( \sum_{X \in \mathbb{X}} (\sum_{\varphi \in V(X)} \Phi(C)) = \sum_{C \in \mathcal{C}_i} \Phi(C) = \Phi_i \). Additionally, by the construction of level-(\( i + 1 \)) clusters, \( \sum_{X \in \mathbb{X}} \Phi(C) = \sum_{C' \in \mathcal{C}_{i+1}} \Phi(C') = \Phi_{i+1} \). Thus, we have:

\[
\sum_{X \in \mathbb{X}} \Delta_i+1(X) = \sum_{X \in \mathbb{X}} \left( \left( \sum_{\varphi \in V(X)} \Phi(C) \right) - \Phi(C) \right) = \Phi_i - \Phi_{i+1} = \Delta_i+1.
\]
The decomposition of the (global) potential change into local potential changes makes the task of analyzing the spanner weight (Item (1) in Lemma 1.6) easier as we can do so locally. Specifically, we often construct $H_i$ by considering each node in $V_i$ and taking a subset of (the corresponding edges of) the edges incident to the node to $H_i$. We then calculate the number of edges taken to $H_i$ incident to all nodes in $X$, and bound their total weight by the local potential change of $X$. By summing up over all $X$, we obtain a bound on $w(H_i)$ in terms of the (global) potential change $\Delta_{i+1}$.

### 3.3 Constructing Level-$(i + 1)$ Clusters

To obtain a fast spanner construction, we will maintain for each cluster $C \in C_i$ a representative vertex $r(C) \in C$. If $C$ contains at least one original vertex, then $r(C)$ is one original vertex in $C$; otherwise, $r(C)$ is a virtual vertex. (Recall that virtual vertices are those subdividing MST edges.) For each vertex $v \in C$, we designate $r(C)$ as the representative of $v$, i.e., we set $r(v) = r(C)$ for each $v \in C$. We use the UNION-FIND data structure to maintain these representatives. Specifically, the representative of $v$ will be given as FIND$(v)$. Whenever a level-$(i + 1)$ cluster is formed from level-$i$ clusters, we call UNION (sequentially on the level-$i$ clusters) to construct a new representative for the new cluster.

Recall that, throughout this paper, we assume that $\epsilon > 0$ is sufficiently small, i.e., $\epsilon \ll 1$.

**A careful usage of the Union-Find data structure.** We will use the UNION-FIND data structure [78] for grouping subsets of clusters to larger clusters (via the UNION operation) and checking whether two given vertices belong to the same cluster (via the FIND operation). The amortized running time of each UNION or FIND operation is $O(\alpha(a, b))$, where $a$ is the total number of UNION and FIND operations and $b$ is the number of vertices in the data structure. Note, however, that our graph $\tilde{G}$ has $n$ original vertices but $O(m)$ virtual vertices, which subdivide MST edges. Thus, if we keep both original and virtual vertices in the UNION-FIND data structure, the amortized time of an operation will be $O(\alpha(m, m)) = O(\alpha(m))$ rather than $O(\alpha(m, n))$, as the total number of UNION and FIND operations is $O(m)$, and will be super-constant for any super-constant value of $m$.

To reduce the amortized time to $O(\alpha(m, n))$, we only store original vertices in the UNION-FIND data structure. To this end, for each virtual vertex, say $x$, which subdivides an edge $(u, v) \in$ MST, we store a pointer, denoted by $p(x)$, which points to one of the endpoints, say $u$, in the same cluster with $x$, if there is at least one endpoint in the same cluster with $x$. In particular, any virtual vertex has at most two optional clusters that it can belong to at each level of the hierarchy. Hence, we can apply every UNION-FIND operation to $p(x)$ instead of $x$. For example, to check whether two virtual vertices $x$ and $y$ are in the same cluster, we compare $r(p(x)) = r(p(y))$ via two FIND operations. The total number of UNION and FIND operations in our construction remains $O(m)$ while the number of vertices that we store in the data structure is reduced to $n$. Thus, the amortized time of each operation reduces to $O(\alpha(m, n))$ and the total runtime due to all these operations is $O(m + \alpha(m, n))$.

If no endpoint of $(u, v)$ belongs to the same cluster with $x$, then the level-$i$ cluster containing $x$ is a path of virtual vertices subdivided from $(u, v)$. In this case, we simply let FIND$(x)$ operation return $x$. That is, we will not maintain $x$ in the UNION-FIND data structure, but instead use a flag to mark if $x$ is in the same cluster with one of the endpoints $\{u, v\}$ or not. Also, we maintain virtual clusters, those that only have virtual vertices, in a regular list data structure, and UNION operations can be implemented as the concatenation of two lists in $O(1)$ time. Once a virtual cluster is merged with a non-virtual cluster, all the virtual vertices need to update their flag and change their pointer $p(\cdot)$ accordingly.

Following the approach in Section 3.2, we construct a graph $G_i$ satisfying all properties in Definition 3.12. Then we construct a set $X$ of subgraphs of $G_i$ satisfying the three properties (P1)-(P3) and a
subgraph $H_i$ of $G$ (and of $\tilde{G}$ as well). Each subgraph $X \in X$ is then converted to a level-$(i + 1)$ cluster by Equation (9).

**Constructing $G_i$.** We shall assume inductively on $i$, $i \geq 1$ that:

- The set of edges $\tilde{\text{MST}}_i$ is given by the construction of the previous level $i$ in the hierarchy; for the base case $i = 1$ (see Section 3.2), $\tilde{\text{MST}}_1$ is simply a set of edges of MST that are not in any level-1 cluster.
- The weight $\omega(\varphi_C)$ on each node $\varphi_C \in V_i$ is the potential value of cluster $C \in C_i$; for the base case $i = 1$, the potential values of level-1 clusters were computed in $O(m)$ time, as discussed in Section 3.2.

By the end of this section, we will have constructed the edge set $\tilde{\text{MST}}_{i+1}$ and the weight function on nodes of $G_{i+1}$, in time $O(|V_i|\alpha(m,n))$. Computing the weight function on nodes of $G_{i+1}$ is equivalent to computing the augmented diameter of $X$, which in turn, is related to the potential function. The fact that we can compute all the weights efficiently in almost linear time is the crux of our framework.

Note that we make no inductive assumption regarding the set of edges $E_i^\sigma$, which can be computed once in $O(m)$ overall time at the outset for all levels $i \geq 1$, since the edge sets $E_1^\sigma, E_2^\sigma, \ldots$ are pairwise disjoint and the number of levels is $O(m)$ by Claim 3.5.

**Lemma 3.16.** $G_i = (V_i, E_i \cup \tilde{\text{MST}}_i, \omega)$ can be constructed in $O(\alpha(m,n)(|V_i| + |E_i^\sigma|))$ time, where $\alpha(\cdot, \cdot)$ is the inverse-Ackermann function.

**Proof:** Recall that any edge in $\tilde{\text{MST}}_i$ (of weight at most $\tilde{w}$) is of strictly smaller weight than that of any edge in $E_i^\sigma$ (of weight at least $\frac{\tilde{w}}{1+\epsilon}$) for any $i \geq 1$ and $\epsilon \ll 1$. Note that $\tilde{\text{MST}}_i$ and $E_i^\sigma$ are given at the outset of the construction of $G_i$. To construct the edge set $E_i$, we do the following. For each edge $e = (u, v) \in E_i^\sigma$, we compute the representatives $r(u), r(v)$; this can be done in $O(\alpha(m,n))$ amortized time over all the levels up to $i$ using the UNION-FIND data structure. This is because the total number of UNION/FIND operations is bounded by $O(\sum_{1 \leq j \leq i} |V_j| + |E_i^\sigma|) = O(m)$. Equipped with the representatives, it takes $O(1)$ time to check whether $e$’s endpoints lie in the same level-$i$ cluster (equivalently, whether edge $e$ forms a self-loop in the cluster graph)—by checking whether $r(u) = r(v)$. In the same way, we can check in $O(1)$ time whether edges $e = (u, v)$ and $e' = (u', v')$ are parallel in the cluster graph—by comparing the representatives of their endpoints. Among parallel edges, we only keep the edge of minimum weight in $G_i$.

Next, we remove all removable edges from $G_i$ as specified by property (3b) in Definition 3.12. First we find in $O(|V_i|)$ time a collection $P$ of maximal paths in $\tilde{\text{MST}}_i$ that only contain degree-2 vertices. By the maximality, paths in $P$ are node-disjoint. We then find for each path $P \in P$ a subset of edges $E_P \subseteq E_i$ whose both endpoints belong to $P$; this can be done in $O(|V_i| + |E_i^\sigma|)$ total time for all paths in $P$. Finally, for each path $P \in P$ and each edge $(\varphi_{C_u}, \varphi_{C_v}) \in E_P$, we can compute $\omega(P[\varphi_{C_u}, \varphi_{C_v}])$ in $O(1)$ time, after an $O(|V(P)|)$ preprocessing time, as follows. Fix an endpoint $\varphi_C \in P$ and for every node $\varphi_{C'} \in P$, we compute $\omega(P[\varphi_C, \varphi_{C'}])$ in total $O(|V(P)|)$ time. Then, we can compute in $O(1)$ time:

$$
\omega(P[\varphi_{C_u}, \varphi_{C_v}]) = \begin{cases} \\
\omega(P[\varphi_{C_u}, \varphi_{C_v}]) - \omega(P[\varphi_{C}, \varphi_{C_u}]) + \omega(\varphi_{C_v}) & \text{if } \omega(P[\varphi_{C}, \varphi_{C_u}]) \geq \omega(P[\varphi_{C}, \varphi_{C_v}]) \\
\omega(P[\varphi_{C_v}, \varphi_{C_u}]) - \omega(P[\varphi_{C}, \varphi_{C_v}]) + \omega(\varphi_{C_u}) & \text{otherwise}
\end{cases}
$$

Given $\omega(P[\varphi_{C_u}, \varphi_{C_v}])$, we can check in $O(1)$ time whether $(\varphi_{C_u}, \varphi_{C_v})$ is removable and if so, we remove it from $E_i$. The total running time to remove all removable edges is $O(|V_i| + |E_i^\sigma|)$.  \[\square\]
The following key lemma states all the properties of clusters constructed in our framework; the details of the construction are deferred to Section 5. Recall that \( \mathcal{V}(\mathcal{X}) \) and \( \mathcal{E}(\mathcal{X}) \) are the vertex set and edge set of \( \mathcal{X} \), respectively.

**Lemma 3.17.** Given \( G_i \), we can construct in time \( O((|V_i| + |E_i|)\epsilon^{-1}) \) (i) a partition of \( V_i \) into three sets \( \{V_i^{\text{high}}, V_i^{\text{low}+}, V_i^{\text{low}^-}\} \) and (ii) a collection \( \mathcal{X} \) of subgraphs of \( G_i \) and their augmented diameters, such that:

1. For every node \( \varphi_C \in V_i \): If \( \varphi_C \in V_i^{\text{high}} \), then \( \varphi_C \) is incident to \( \Omega(1/\epsilon) \) edges in \( E_i \); otherwise (\( \varphi_C \in V_i^{\text{low}+} \cup V_i^{\text{low}^-} \)), the number of edges in \( E_i \) incident to \( \varphi_C \) is \( O(1/\epsilon) \).

2. If a subgraph \( \mathcal{X} \) contains at least one node in \( V_i^{\text{low}^-} \), then every node of \( \mathcal{X} \) is in \( V_i^{\text{low}^-} \). Let \( X_i^{\text{low}^-} \subseteq \mathcal{X} \) be a set of subgraphs whose nodes are in \( V_i^{\text{low}^-} \) only.

3. Let \( \Delta_{i+1}^+(\mathcal{X}) = \Delta(\mathcal{X}) + \sum_{e \in \text{MST}(\mathcal{X})} w(e) \). Then, \( \Delta_{i+1}^+(\mathcal{X}) \geq 0 \) for every \( \mathcal{X} \in \mathcal{X} \), and

\[
\sum_{\mathcal{X} \in \mathcal{X} \setminus X_i^{\text{low}^-}} \Delta_{i+1}^+(\mathcal{X}) = \sum_{\mathcal{X} \in \mathcal{X} \setminus X_i^{\text{low}^-}} \Omega(|\mathcal{V}(\mathcal{X})|\epsilon^2 L_i). \quad (12)
\]

4. There is no edge in \( E_i \) between a node in \( V_i^{\text{high}} \) and a node in \( V_i^{\text{low}^-} \). Furthermore, if there exists an edge \( (\varphi_{C_u}, \varphi_{C_v}) \in E_i \) such that both \( \varphi_{C_u} \) and \( \varphi_{C_v} \) are in \( V_i^{\text{low}^-} \), then \( V_i^{\text{low}^-} = V_i \) and \( |E_i| = O(1/\epsilon) \); we call this case the degenerate case.

5. For every subgraph \( \mathcal{X} \in \mathcal{X} \), \( \mathcal{X} \) satisfies the three properties \([P1]-[P3]\) with constant \( g = 31 \), and \( |\mathcal{E}(\mathcal{X}) \cap E_i| = O(|\mathcal{V}(\mathcal{X})|) \).

Furthermore, the construction of \( \mathcal{X} \) can be constructed in the pointer-machine model with the same running time.

We remark the following points regarding subgraphs in \( \mathcal{X} \) constructed by Lemma 3.17:

**Remark 3.18.**

1. It is possible for a subgraph \( \mathcal{X} \in \mathcal{X} \) to contain nodes in \( V_i^{\text{high}} \) and in \( V_i^{\text{low}+} \).

2. \( \Delta_{i+1}(\mathcal{X}) \) could be negative, but \( \Delta_{i+1}^+(\mathcal{X}) \) is always non-negative by Item 2) in Lemma 3.17. We could view \( \sum_{e \in \text{MST}(\mathcal{X})} w(e) \) as a corrective term to \( \Delta_{i+1}(\mathcal{X}) \) (to make it non-negative). We call \( \Delta_{i+1}^+(\mathcal{X}) \) corrected potential change.

3. Equation \([12]\) implies that the average amount of corrected potential change per subgraph \( \mathcal{X} \in \mathcal{X} \setminus X_i^{\text{low}^-} \) is \( \Omega(|\mathcal{V}(\mathcal{X})|\epsilon^2 L_i) \). On the other hand, there is no guarantee, other than non-negativity, on the corrected potential change of \( \mathcal{X} \) if \( \mathcal{X} \in X_i^{\text{low}^-} \).

We make the following observations on subgraphs of \( \mathcal{X} \) that satisfy all the properties stated in Lemma 3.17:

**Observation 3.19.** If a subgraph \( \mathcal{X} \in \mathcal{X} \) has \( \mathcal{V}(\mathcal{X}) \cap (V_i^{\text{high}} \cup V_i^{\text{low}+}) \neq \emptyset \), then \( \mathcal{V}(\mathcal{X}) \subseteq (V_i^{\text{high}} \cup V_i^{\text{low}+}) \).

**Proof:** Follows from Item 2) in Lemma 3.17 and the fact that \( \{V_i^{\text{high}}, V_i^{\text{low}+}, V_i^{\text{low}^-}\} \) is a partition of \( V_i \).

**Observation 3.20.** Unless the degenerate case happens, for every edge \( (\varphi_{C_u}, \varphi_{C_v}) \) with one endpoint in \( V_i^{\text{low}^-} \), w.l.o.g. \( \varphi_{C_v} \), the other endpoint \( \varphi_{C_u} \) must be in \( V_i^{\text{low}+} \). As a result, \( \mathcal{E}(\mathcal{X}) \cap E_i = \emptyset \) if \( \mathcal{X} \in X_i^{\text{low}^-} \).
Proof: If the degenerate case does not happen, by Item (4) in Lemma 3.17, any edge incident to a node in \( V_i^{\text{low}} \) must be incident to a node in \( V_i^{\text{low}+} \). By Item (2), if \( \mathcal{X} \in X^{\text{low}} \), then \( \mathcal{V}(\mathcal{X}) \subseteq V_i^{\text{low}} \) and hence, there is no edge between two nodes in \( \mathcal{X} \). Thus, \( \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_i = \emptyset \). \( \square \)

Next, we show how to construct \( \tilde{\text{MST}}_{i+1} \) for the construction of the next level.

Lemma 3.21. Given the collection of subgraphs \( X \) of \( G_i \) and their augmented diameters, we can construct the set of nodes \( V_{i+1} \), and their weights, and the cluster tree \( \tilde{\text{MST}}_{i+1} \) of \( G_{i+1} \) in \( O(|V_i|\alpha(m,n)) \) time.

Proof: For each subgraph \( \mathcal{X} \in X \), we call \( \text{UNION} \) operations sequentially on the set of clusters corresponding to the nodes of \( \mathcal{X} \) to create a level-(\( i + 1 \)) cluster \( C_{\mathcal{X}} \in C_{i+1} \). Then we create a set of nodes \( V_{i+1} \) for \( G_{i+1} \): each node \( \varphi_{C_{\mathcal{X}}} \) corresponds to a cluster \( C_{\mathcal{X}} \in C_{i+1} \) (and also subgraph \( \mathcal{X} \in X \)). Next, we set the weight \( \omega(\varphi_{C_{\mathcal{X}}}) = \text{Adm}(\mathcal{X}) \). The total running time of this step is \( O(|V_i|\alpha(m,n)) \).

We now construct \( \tilde{\text{MST}}_{i+1} \). Let \( \tilde{\text{MST}}_{i}^\text{out} = \text{MST}_i \setminus (\cup_{\mathcal{X} \in X}(\mathcal{E}(\mathcal{X}) \cap \text{MST}_i)) \) be the set of \( \text{MST}_i \) edges that are not contained in any subgraph \( \mathcal{X} \in X \). Let \( \tilde{\text{MST}}_{i+1}' \) be the graph with vertex set \( V_{i+1} \) and there is an edge between two nodes \( (\mathcal{X}, \mathcal{Y}) \) in \( V_{i+1} \) of there is at least one edge in \( \tilde{\text{MST}}_{i}^\text{out} \) between two nodes in the two corresponding subgraphs \( \mathcal{X} \) and \( \mathcal{Y} \); \( \tilde{\text{MST}}_{i+1}' \) can be constructed in time \( O(|V_i|) \). Note that \( \tilde{\text{MST}}_{i+1}' \) could have parallel edges (but no self-loop). Since \( \tilde{\text{MST}}_i \) is a spanning tree of \( G_i \), \( \tilde{\text{MST}}_{i+1}' \) must be connected. \( \tilde{\text{MST}}_{i+1} \) is then a spanning tree of \( \tilde{\text{MST}}_{i+1}' \), which can be constructed in time \( O(|V_i|) \) since \( \tilde{\text{MST}}_{i+1}' \) has at most \( |V_i| \) edges. The lemma now follows. \( \square \)

In the following section, we present a general approach to construct \( H_i \), and to bound the weight of \( H_i \) in terms of the potential change. The key idea is that we can reduce the construction of \( H_i \) to the construction of a sparse spanner for (the corresponding edges of) the edges whose both endpoints have high-degree (in \( V_i^{\text{high}} \)). As we will show in Section 4, the constructions of the sparse spanners in different settings are simple, which reinforces the strength of our framework: simplicity and wide applicability.

3.4 Constructing \( H_i \)

In this section, we assume that we are given the collection \( X \) of subgraphs as described in Lemma 3.17. We define:

\[
\begin{align*}
X^{\text{high}} &= \{ \mathcal{X} \in X : \mathcal{V}(\mathcal{X}) \cap V_i^{\text{high}} \neq \emptyset \} \\
X^{\text{low}+} &= \{ \mathcal{X} \in X : \mathcal{V}(\mathcal{X}) \cap V_i^{\text{low}+} \neq \emptyset \}
\end{align*}
\]

It could be that \( X^{\text{high}} \cap X^{\text{low}+} \neq \emptyset \). By Observation 3.19, \( \{X^{\text{high}}, X^{\text{low}+}, X^{\text{low}+}\} \) is a partition of \( X \).

Construction overview. Equipped with a set of subgraphs \( X \) satisfying the properties stated in Lemma 3.17, our general approach to construct \( H_i \) is as follows. First, we add to \( H_i \) (the corresponding edge of) every edge \( e \) contained in some subgraph \( \mathcal{X} \): \( e \in \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_i \). Edges added to \( H_i \) in this step are incident to nodes in \( V_i^{\text{low}+} \cap V_i^{\text{high}} \). By Item (5) of Lemma 3.17, we only add \( O(|\mathcal{V}(\mathcal{X})|) \) edges per subgraph \( \mathcal{X} \), and hence, we can bound the total weight of these edges by \( (O(\frac{1}{\epsilon^2})) \) times the corrected potential changes of subgraphs in \( X \setminus X^{\text{low}} \), due to Item (3) of Lemma 3.17. Next, we add to \( H_i \) all edges incident to all nodes in \( V_i^{\text{low}+} \cup V_i^{\text{low}+} \). Unless we are in the degenerate case, edges added to \( H_i \) in the second step are incident to nodes in \( V_i^{\text{low}+} \) (see Observation 3.20), and hence, their total weight can be bounded by \( (O(\frac{1}{\epsilon^2})) \) times the corrected potential changes of subgraphs in \( X \setminus X^{\text{low}} \); to this end we apply both Item...
(3) of Lemma 3.17 and the fact that any node in $\mathcal{V}_{\text{low}}^+ \cup \mathcal{V}_{\text{low}}^-$ has at most $O(1/\epsilon)$ incident edges in $\mathcal{E}_i$. Now we are left with edges whose both endpoints are in $\mathcal{V}_{\text{high}}^i$, denote by $\mathcal{E}_{\text{high}}^i$. In the third step, we select a subset of (the corresponding edges of) these edges to add to $H_i$ by assuming that there is an algorithm (Algorithm $\mathcal{A}$ below) that constructs a sparse spanner for the corresponding edges of $\mathcal{E}_{\text{high}}^i$. The (small) sparsity of the spanner allows us to bound the total weight of the edges added to $H_i$ in the third step by the corrected potential changes of subgraphs in $\mathcal{X} \setminus X_{\text{low}}^-$. The existence of Algorithm $\mathcal{A}$ is the key to ensure the wide applicability of our framework.

Algorithm $\mathcal{A}$: This algorithm takes a subgraph $\mathcal{K}_i = (\mathcal{V}_{\text{high}}^i, \mathcal{E}_{\text{high}}^i, \omega)$ of $\mathcal{G}_i$ as an input, where $\mathcal{E}_{\text{high}}^i \subseteq \mathcal{E}_i$ is the set of edges whose both endpoints are in $\mathcal{V}_{\text{high}}^i$, and outputs a subset $\mathcal{E}_{\text{pruned}}^i \subseteq \mathcal{E}_{\text{high}}^i$ such that:

1. $|\mathcal{E}_{\text{pruned}}^i| \leq \xi |\mathcal{V}_{\text{high}}^i|$ for some $\xi > 0$.
2. $d_{\mathcal{K}_i}(u, v) \leq t (1 + s_A \epsilon) w(u, v)$ for every edge $(u, v)$ corresponding to an edge $(\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_{\text{high}}^i$. Here $s_A$ is some constant that depends on $g$ only (see property (P3)) and $K_i$ is the graph obtained by adding in the corresponding edges of $\mathcal{E}_{\text{pruned}}^i$ to the spanner constructed so far for every edge of length less than $L_i/(1 + \psi)$ (see Remark 3.23).

Let $\text{Time}_A = O((m' + n') \kappa(m', n'))$, when applied on an $m'$-edge $n'$-vertex subgraph.

Recall that each edge $(\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i$ has a corresponding edge $(u, v) \in \mathcal{E}_{\text{high}}^i$ that has size at most $n$ since every level-$i$ cluster corresponding to a node in $\mathcal{V}_{\text{high}}^i$ contains at least one original vertex in $G$. Furthermore, $|\mathcal{E}_{\text{high}}^i|$ is bounded by $m$ and hence, $\kappa(|\mathcal{E}_{\text{high}}^i|, |\mathcal{V}_{\text{high}}^i|) \leq \kappa(m, n)$.

Remark 3.23. If $A$ can be implemented in the ACT model in time $O((|\mathcal{V}_{\text{high}}^i| + |\mathcal{E}_{\text{high}}^i|) \kappa(m, n))$, then the construction of $H_i$ can be implemented in the ACT model in time $O((|\mathcal{V}_{\text{high}}^i| + |\mathcal{E}_{\text{high}}^i|) \kappa(m, n))$.

Constructing $H_i$. We construct $H_i$ in three steps, as briefly described in the construction overview above. Initially $H_i$ contains no edges.

- **(Step 1).** For every subgraph $\mathcal{X} \in \mathcal{X}$ and every edge $e = (\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}(\mathcal{X})$ such that $e \in \mathcal{E}_i$, we add the corresponding edge $(u, v)$ to $H_i$. (Note that if $e \notin \mathcal{E}_i$, it is in MST$_i$ and hence $(u, v)$ belongs to $H_0$).
- **(Step 2).** For each node $\varphi_{C_u} \in \mathcal{V}_{\text{low}}^+ \cup \mathcal{V}_{\text{low}}^-$, and for each edge $(\varphi_{C_u}, \varphi_{C_v})$ in $\mathcal{E}_i$ incident to $\varphi_{C_u}$, we add the corresponding edge $(u, v)$ to $H_i$.
- **(Step 3).** Let $\mathcal{E}_{\text{high}}^i \subseteq \mathcal{E}_i$ be the set of edges whose both endpoints are in $\mathcal{V}_{\text{high}}^i$, and $\mathcal{K}_i = (\mathcal{V}_{\text{high}}^i, \mathcal{E}_{\text{high}}^i, \omega)$ be a subgraph of $\mathcal{G}_i$. We run Algorithm $\mathcal{A}$ on $\mathcal{K}$ to obtain $\mathcal{E}_{\text{pruned}}^i$. For every edge $(\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_{\text{pruned}}^i$, we add the corresponding edge $(u, v)$ to $H_i$.
For every edge

Claim 3.25. edges by storing for each edge (Here we assume that we have a constant time translation from the cluster graph edges to the original

\[ O A \]

\[ \rho \]

hence Equation (14) holds. Otherwise, (Claim 3.24.

H

Proof: Let

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Proof:

Let \( F_i^\sigma = \{(u,v) \in E_i^\sigma : \exists (\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i \} \) be the set of edges in \( E_i^\sigma \) that correspond to the edges in \( \mathcal{E}_i \). We first show that:

\[
    d_{H_{\leq i}}(u,v) \leq t(1 + \max\{s_A, 6g\} \epsilon)w(u,v) \quad \forall (u,v) \in F_i^\sigma. \tag{14}
\]

To that end, let \( (\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i \) be the edge corresponding to \((u,v)\) where \((u,v) \in F_i^\sigma\). If at least one of the endpoints of \((\varphi_{C_u}, \varphi_{C_v})\) is in \( \mathcal{Y}_i^{low+} \cup \mathcal{Y}_i^{low-} \), then \((u,v) \in H_i \) by the construction in Step 2, hence Equation \[14\] holds. Otherwise, \( \{\varphi_{C_u}, \varphi_{C_v}\} \subseteq \mathcal{Y}_i^{high} \), which implies that \((\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i^{high} \). Since we add all edges of \( \mathcal{E}_i^{pruned} \) to \( H_i \), by property (2) of Algorithm \( \mathcal{A} \) the stretch of edge \((u,v)\) in \( K_i \) is at most \( t(1 + s_A \epsilon) \); note that \( K_i \subseteq H_{\leq i} \).

It remains to bound the stretch of any edge \((u',v') \in E_i^\sigma \setminus F_i^\sigma \). Recall that \((u',v')\) is not added to \( \mathcal{E}_i \) because (a) both \( u' \) and \( v' \) are in the same level-\( i \)-cluster in the construction of the cluster graph in Lemma \[3.16\], or (b) \((u',v')\) is parallel with another edge \((u,v)\) in \( \mathcal{E}_i \) also in Lemma \[3.16\] or (c) the edge \((\varphi_{C_{u'}}, \varphi_{C_{v'}})\) corresponding to \((u',v')\) is a removable edge (see Definition \[3.12\]).

In case (a), since the level-\( i \) cluster containing both \( u' \) and \( v' \) has diameter at most \( gL_{i-1} \) by property (P3), we have a path from \( u' \) to \( v' \) in \( H_{\leq i} \) of diameter at most \( gL_{i-1} = geL_i \leq \frac{L_i}{1+\psi} \leq w(u,v) \) when \( \epsilon < \frac{1}{\max\{s_A+4g,10g\}} \). Thus, the stretch of edge \((u,v)\) is 1. For case (c), the stretch of \((u',v')\) in \( H_{\leq i} \) is \( t(1 + 6ge) \) since \( \epsilon \leq 1 \). Thus, it remains to consider case (b).

Let \( C_u \) and \( C_v \) be two level-\( i \)-clusters containing \( u \) and \( v \), respectively. W.l.o.g. we assume that \( u' \in C_u \) and \( v' \in C_v \). Since we only keep the edge of minimum weight among all parallel edges, \( w(u,v) \leq w(u',v') \).

Since the level-\( i \) clusters that contain \( u \) and \( v \) have diameters at most \( gL_{i-1} = geL_i \) by property (P3), it follows that \( \text{Dm}(H_{\leq i}(C_u)) \leq g\epsilon L_i \). We have:

\[
    d_{H_{\leq i}}(u',v') \leq d_{H_{\leq i}}(u,v) + \text{Dm}(H_{\leq i}(C_u)) + \text{Dm}(H_{\leq i}(C_v))
\]

\[
    \leq \left(1 + \max\{s_A, 6g\} \epsilon\right)w(u,v) + 2g\epsilon L_i \quad \text{(by Equation } 14\text{)}
\]

\[
    \leq t\left(1 + \max\{s_A, 6g\} \epsilon\right)w(u',v') + 2g\epsilon L_i
\]

\[
    \leq t\left(1 + \max\{s_A, 6g\} \epsilon\right)w(u',v') + 4g\epsilon w(u',v') \quad \text{(since } w(u',v') \geq L_i/(1+\psi) \geq L_i/2\text{)}
\]

\[
    = t\left(1 + \max\{s_A, 4g, 10g\} \epsilon\right)w(u',v') \quad \text{(since } t \geq 1\text{)}.
\]

The lemma now follows.

Claim 3.26. Let \( \text{MST}^{in}_i = \bigcup_{X \in \mathcal{X}} (\mathcal{E}(X) \cap \text{MST}_i) \) be the set of \( \text{MST}_i \) edges that are contained in subgraphs in \( X \). Then, \( w(H_i) \leq \lambda \Delta_{i+1} + a_i \) for \( \lambda = O(\xi\epsilon^{-2} + \epsilon^{-3}) \) and \( a_i = (\xi\epsilon^{-2}) \cdot w(\text{MST}^{in}_i) + O(L_i/\epsilon^2) \).
Proof: Let \( \text{MST}_i^\text{in}(\mathcal{X}) = \mathcal{E}(\mathcal{X}) \cap \text{MST}_i \) for each subgraph \( \mathcal{X} \in \mathbb{X} \). By the definition of \( \mathbb{X}^{\text{low}^+} \) and \( \mathbb{X}^{\text{high}} \) (see Equation (13)), it holds that:

\[
|V_i^\text{high}| \leq \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}}} |\mathcal{V}(\mathcal{X})| \quad \text{and} \quad |V_i^{\text{low}^+}| \leq \sum_{\mathcal{X} \in \mathbb{X}^{\text{low}^+}} |\mathcal{V}(\mathcal{X})| \tag{15}
\]

First, we consider the non-degenerate case where \( V_i^{\text{low}^+} \neq \emptyset \). By Observation 3.20, any edge in \( E_i \) incident to a node in \( V_i^{\text{low}^+} \) is also incident to a node in \( V_i^{\text{low}^+} \). We bound the total weight of the edges added to \( H_i \) by considering each step in the construction of \( H_i \) separately. Let \( F_i^{(a)} \subseteq E_i \) be the set of edges added to \( H_i \) in the construction in Step \( a \), \( a \in \{1, 2, 3\} \).

By Observation 3.20, \( \mathcal{E}(\mathcal{X}) \cap E_i = \emptyset \) if \( \mathcal{X} \in \mathbb{X}^{\text{low}^+} \). Recall that \( \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+} = \mathbb{X} \setminus \mathbb{X}^{\text{low}^+} \). By Item (5) in Lemma 3.17, the total weight of the edges added to \( H_i \) in Step 1 is:

\[
w(F_i^{(1)}) = \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+}} O(|\mathcal{V}(\mathcal{X})|) L_i \quad \text{Eq.} 12 \quad O\left(\frac{1}{\epsilon^2}\right) \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+}} \Delta_{i+1}^+(\mathcal{X})
\]

\[
= O\left(\frac{1}{\epsilon^2}\right) \sum_{\mathcal{X} \in \mathbb{X}} \left(\Delta_{i+1}(\mathcal{X}) + w(\text{MST}_i^\text{in}(\mathcal{X}))\right)
\]

\[
= O\left(\frac{1}{\epsilon^2}\right)(\Delta_{i+1} + w(\text{MST}_i^\text{in})) \quad \text{by Claim 3.15}.
\]

Next, we bound \( w(F_i^{(2)}) \). Let \( (u, v) \) be an edge added to \( H_i \) in Step 2 and let \( (\varphi_{C_u}, \varphi_{C_v}) \) be the corresponding edge of \( (u, v) \). Since \( V_i^{\text{low}^+} \neq \emptyset \), at least one of the endpoints of \( (\varphi_{C_u}, \varphi_{C_v}) \), w.l.o.g. \( \varphi_{C_u} \), is in \( V_i^{\text{low}^+} \) by Observation 3.20. Recall by Item (1) of Lemma 3.17 that all nodes in \( V_i^{\text{low}^+} \) have low degree, i.e., incident to \( O(1/\epsilon) \) edges in \( E_i \). Thus, \( |F_i^{(2)}| = O\left(\frac{1}{\epsilon}\right)|V_i^{\text{low}^+}| \). We have:

\[
w(F_i^{(2)}) = O\left(\frac{1}{\epsilon}\right)|V_i^{\text{low}^+}| L_i \quad \text{Eq.} 15 \quad O\left(\frac{1}{\epsilon}\right) \sum_{\mathcal{X} \in \mathbb{X}^{\text{low}^+}} |\mathcal{V}(\mathcal{X})| L_i = O\left(\frac{1}{\epsilon}\right) \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+}} |\mathcal{V}(\mathcal{X})| L_i
\]

\[
eq O\left(\frac{1}{\epsilon^3}\right) \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+}} \Delta_{i+1}^+(\mathcal{X}) = O\left(\frac{1}{\epsilon^3}\right) \sum_{\mathcal{X} \in \mathbb{X}} \Delta_{i+1}^+(\mathcal{X})
\]

\[
= O\left(\frac{1}{\epsilon^3}\right)(\Delta_{i+1} + w(\text{MST}_i^\text{in})) \quad \text{by Claim 3.15}.
\]

By property (1) of Algorithm \( A \), the number of edges added to \( H_i \) in Step 3 is at most \( \xi |V_i^{\text{high}}| \). Thus:

\[
w(F_i^{(3)}) \leq \xi |V_i^{\text{high}}| L_i \quad \text{Eq.} 15 \quad \xi \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}}} |\mathcal{V}(\mathcal{X})| L_i \leq \xi \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}} \cup \mathbb{X}^{\text{low}^+}} |\mathcal{V}(\mathcal{X})| L_i
\]

\[
eq O\left(\xi \epsilon^{-2}\right) \sum_{\mathcal{X} \in \mathbb{X}^{\text{high}}} \Delta_{i+1}^+(\mathcal{X}) = O\left(\xi \epsilon^{-2}\right) \sum_{\mathcal{X} \in \mathbb{X}} \Delta_{i+1}^+(\mathcal{X})
\]

\[
= O\left(\xi \epsilon^{-2}\right)(\Delta_{i+1} + w(\text{MST}_i^\text{in})) \quad \text{by Claim 3.15}.
\]
By Equations (16) to (18), we conclude that:

\[ w(H_i) = O(\xi \varepsilon^{-2} + \varepsilon^{-3})(\Delta_i+1 + w(\tilde{\text{MST}}^\text{in}_i)) \leq \lambda(\Delta_i+1 + w(\tilde{\text{MST}}^\text{in}_i)) \]

(19)

for some \( \lambda = O(\xi \varepsilon^{-2} + \varepsilon^{-3}) \).

It remains to consider the degenerate case where \( V^\text{low}_i = V_i \). Even if we add every single edge that corresponds to an edge in \( E_i \) to \( H_i \), Item (3) in Lemma 3.17 implies that the number of such edges is at most \( O(1/\varepsilon^2) \). Thus, we have:

\[ w(H_i) = O\left( \frac{L_i}{\varepsilon^2} \right) \leq \lambda \cdot (\Delta_i+1 + w(\tilde{\text{MST}}^\text{in}_i)) + O\left( \frac{L_i}{\varepsilon^2} \right) \]

(20)

where in the last equation, we use the fact that:

\[ \Delta_i+1 + w(\tilde{\text{MST}}^\text{in}_i) = \sum_{X \in \mathcal{X}} (\Delta_i+1 + \text{MST}^\text{in}_i(X)) = \sum_{X \in \mathcal{X}} \Delta_i+1(X) \geq 0 \]

by Item (3) of Lemma 3.17. Thus, the claim follows from Equations (19) and (20).

\[ \square \]

We are now ready to prove Lemma 3.22.

**Proof:** [Proof of Lemma 3.22] The running time follows from Claim 3.24. By Claim 3.25, the stretch is \( t(1 + \max\{s_A + 4g, 10g\} \varepsilon) \). By Claim 3.26, we have \( \sum_{i \in N^+} a_i = \sum_{i \in N^+} (\lambda \text{MST}^\text{in}_i + O(L_i/\varepsilon^2)) \). Observe by the definition that the sets of corresponding edges of \( \text{MST}^\text{in}_i \) and \( \text{MST}^\text{in}_j \) are disjoint for any \( i \neq j \geq 1 \).

Thus, \( \sum_{i \in N^+} \text{MST}^\text{in}_i \leq w(\text{MST}) \). Observe that:

\[ \sum_{i \in N^+} O\left( \frac{L_i}{\varepsilon^2} \right) \leq O\left( \frac{1}{\varepsilon^2} \right) \sum_{i=1}^{i_{\text{max}}} \frac{L_i}{\varepsilon^{\text{max}-i}} = O\left( \frac{L_{i_{\text{max}}}}{\varepsilon^2(1-\varepsilon)} \right) = O\left( \frac{1}{\varepsilon^2} \right) w(\text{MST}) ; \]

here \( i_{\text{max}} \) is the maximum level. The last equation is due to that \( \varepsilon \leq 1/2 \) and every edge has weight at most \( w(\text{MST}) \) (by the removal step in the construction of \( \tilde{G} \)). Thus, \( A = \lambda + O(\varepsilon^{-2}) = O(\xi \varepsilon^{-2} + \varepsilon^{-3}) + O(\varepsilon^{-2}) = O(\xi \varepsilon^{-2} + \varepsilon^{-3}) \) as claimed.

\[ \square \]

### 3.5 Summary of Notation

| Notation       | Meaning                                                                 |
|----------------|-------------------------------------------------------------------------|
| \( E^{\text{light}} \) | \( \{e \in E(G) : w(e) \leq w/\varepsilon \} \)                       |
| \( E^{\text{heavy}} \) | \( E \setminus E^{\text{light}} \)                                    |
| \( E^\sigma \)   | \( \bigcup_{i \in N^+} E^\sigma_i \)                                   |
| \( E^\sigma_i \)  | \( \{e \in E(G) : \frac{L_i}{1 + \varepsilon} < w(e) \leq L_i \} \)    |
| \( g \)          | constant in property (P3), \( g = 31 \)                                |
| \( \tilde{G}_i = (V_i, \text{MST}_i \cup E_i, \omega) \) | cluster graph; see Definition 3.11                                         |
| \( E_i \)        | corresponds to a subset of edges of \( E^\sigma_i \)                   |
| \( \mathcal{X} \) | a collection of subgraphs of \( \tilde{G}_i \)                        |
| \( \mathcal{X}, \mathcal{V}(\mathcal{X}), \mathcal{E}(\mathcal{X}) \) | a subgraph in \( \mathcal{X} \), its vertex set, and its edge set |
4 Applications of the Unified Framework

In this section, we prove all theorems stated in Section 1 by implementing Algorithm $A$ for each graph family. We refer readers to Figure 2 for a pictorial view of how different theorems/lemmas fit together.

![Figure 2: A road map for proving theorems in Section 1.](image)

4.1 Euclidean Spanners and UDG Spanners

In this section, we give a detailed implementation of Algorithm $A$ for Euclidean and UDG spanners. The idea is to use Yao-graph like construction: for each cluster $C$ corresponds to a node in $V_{i}^{\text{high}}$, we construct a collection of cones of angle $\epsilon$ around the representative $r(C)$ of the cluster. Then for each cone, we look...
at all the representatives of the neighbors (in \(K_i\)) of \(C\) that fall into the cone, and pick to \(K_i\) the edge corresponds to the representative that is closest to \(r(C)\).

If we only aim to resolve the basic open question on Euclidean spanners in the ACT model, i.e., Question 2 then we could apply a black-box sparse spanner construction on top of the representatives of clusters corresponding to nodes in \(V_i^{\text{high}}\). The running time of the sparse spanner construction is \(O(|V_i^{\text{high}}| \log(|V_i^{\text{high}}|)) = O(|V_i^{\text{high}}| \log(n))\) in the ACT model; it is easy to show that the final running time of our algorithm remains \(O(n \log n)\), assuming that \(\epsilon\) and \(d\) are constant. (Indeed, this approach is the approach we take for the light spanner construction for general graphs, using the sparse spanner construction of Halperin-Zwick [51] as a black box; see Section 4.2.) Since we apply the black-box sparse spanner in the Euclidean space, the spanner for representatives may use some edges that are not in the input spanner, i.e., \(K_i\) will no longer be a subgraph of the input spanner. As a result, the final spanner may not have bounded degree. This is different than what we do for general graphs, because the Halperin-Zwick construction works for any subgraph.

Instead, we take a more intricate approach, as mentioned above, where we use the Yao graph like construction, but we use only edges of the input spanner. The advantage is two-fold. First, we can use as the input sparse spanner a bounded degree spanner, and as a corollary get a light spanner of bounded degree, which enables us to resolve not only Question 2 but also Question 3. Second, this advantage comes into play in the UDG case, where the same approach is being used, as we need to guarantee that every edge in the spanner must have length at most 1. The detail of the algorithm, denoted by \(A_{\text{Geom}}\), is given below.

**Algorithm \(A_{\text{Geom}}\) (Euclidean and UDG):** The input is a subgraph \(K_i = (V_i^{\text{high}}, \xi_i^{\text{high}}, \omega)\) of \(G_i\). The output is \(\xi_i^{\text{pruned}}\); initially, \(\xi_i^{\text{pruned}} = \emptyset\).

We apply the construction to each node in \(V_i^{\text{high}}\); fix an arbitrary node \(\varphi_{C_u} \in V_i^{\text{high}}\).

Let \(N_i(\varphi_{C_u})\) be the set of neighbors of \(\varphi_{C_u}\) in \(K_i\). We construct a collection of \(\tau = O(\epsilon^{1-d})\) cones \(\text{Cone}(C_u) = \{Q_1, Q_2, \ldots, Q_\tau\}\) that partition \(\mathbb{R}^d\), each of angle \(\epsilon\) and with apex at \(r(C_u)\), the representative of the cluster \(C_u\). It is well known (see, e.g. Lemma 5.2.8 in [63]) that we can construct \(\text{Cone}(C_u)\) in time \(O(\epsilon^{1-d})\) in the ACT model.

Next, for each \(j \in [\tau]\), let \(R_j = \{r(C') : \varphi_{C'} \in N_i(\varphi_{C_u}) \land (r(C') \in Q_j)\}\) be the set of representatives that belong to the cone \(Q_j \in \text{Cone}(C_u)\). Let \(r^*_j = \arg \min_{r \in R_j} ||r(C_u), r||\) be the representative in \(R_j\) that is closest to \(r(C_u)\). Let \(\varphi_{C_j}\) be the node of \(K_i\) whose cluster \(C_j\) has \(r^*_j\) as the representative. By the definition of \(R_j\), \((\varphi_{C_u}, \varphi_{C_j})\) is an edge in \(\xi_i\). We then add \((\varphi_{C_u}, \varphi_{C_j})\) to \(\xi_i^{\text{pruned}}\). In this way, we add at most one edge to \(\xi_i^{\text{pruned}}\) incident on \(\varphi_{C_u}\) for each of the \(\tau\) cones.

We now show that Algorithm \(A_{\text{Geom}}\) has all the properties as described in the abstract. Recall that we scale the input graph so that the minimum edge weight is 1. For the case of UDG, we denote by \(\rho\) the distance threshold between the centers of the disks (or balls) after scaling; parameter \(\rho\) determines the existence of edges in the input graph. Recall that \(K_i\) is the graph obtained by adding in the corresponding edges of \(\xi_i^{\text{pruned}}\) to the spanner constructed for every edge of length less than \(L_i/(1+\psi)\). Since our goal is to ultimately prove Theorem 1.8, we may assume that \(\psi = \epsilon\).

One important note is that, when arguing for the stretch of \(A_{\text{Geom}}\), we inductively assume that edges of weight less than \(\frac{L_i}{1+\psi}\) have stretch at most \((1 + c_\psi)\epsilon\) for some constant \(c_\psi\) that is sufficiently bigger than \(g\), where \(\epsilon\) is at most \(1/c_\psi\). Specifically, we will set \(c_\psi\) to be at least \(\max\{s_{A_{\text{Geom}}} + 4g, 10g\}\), which is the constant in the stretch of Lemma 3.22. With that assumption, we will show below that \(s_{A_{\text{Geom}}} = 2(19g+7)\),
and hence, we only need $c_g$ to be at least $2(21g + 7)$.

Figure 3: Illustration for the stretch bound proof of Lemma 4.1. Black dashed curves represent three level-i clusters $C_y, C_v, C_y$. The solid red edge $(x, y)$ is added to $H_i$ while the dashed red edge $(u, v)$ is not. The green shaded region represents the cone $Q_j$ of angle $\epsilon$ with apex $r_u$.

**Lemma 4.1.** Algorithm $A_{\text{Geom}}$ can be implemented in $O((|V_i^{\text{high}}| + |E_i^{\text{high}}|)\epsilon^{1-d})$ time in the ACT model. Furthermore, (1) $E_i^{\text{pruned}} = O(\epsilon^{1-d} |V_i|)$ and (2) $d_{K_i}((u, v)) \leq (1 + s_{A_{\text{Geom}}})w(u, v)$ for every edge $(u, v)$ corresponding to an edge in $E_i$, where $s_{A_{\text{Geom}}} = 2(19g + 7)$ and $\epsilon \leq 1/c_g$ for a sufficiently large constant $c_g$.

**Proof:** We first analyze the running time. We observe that, since we can construct $\text{Cone}(C_u)$ for a single node $\varphi_{C_u}$ in $O(\epsilon^{1-d})$ time in the ACT model, the running time to construct all sets of cones $\{\text{Cone}(C_u)\}_{\varphi_{C_u} \in V_i^{\text{high}}}$ is $O(|V_i^{\text{high}}|\epsilon^{1-d})$. Now consider a specific node $\varphi_{C_u}$. For each neighbor $\varphi_{C'} \in N_i(\varphi_{C_u})$ of $\varphi_{C_u}$, finding the cone $Q_j \in \text{Cone}(C_u)$ such that $r(C') \in Q_j$ takes $O(\tau) = O(\epsilon^{1-d})$ time. Thus, $\{R_j\}_{j=1}^{\tau}$ can be constructed in $O(|N_i(\varphi_{C_u})|\epsilon^{1-d})$ time. Finding the set of representatives $\{r_j\}_{j=1}^{\tau}$ takes $O(|N_i(\varphi_{C_u})|)$ time. Thus, the total running time to implement Algorithm $A_{\text{Geom}}$ is:

$$O(|V_i^{\epsilon^{1-d}}|) + \sum_{\varphi_{C_u} \in V_i^{\text{high}}} O(|N_i(\varphi_{C_u})|\epsilon^{1-d}) = O((|V_i^{\text{high}}| + |E_i^{\text{high}}|)\epsilon^{1-d})$$

as claimed.

By the construction of the algorithm, for each node $\varphi_C \in V_i^{\text{high}}$, we add at most $\tau = O(\epsilon^{1-d})$ incident edges in $E_i^{\text{high}}$ to $E_i^{\text{pruned}}$; this implies Item (1).

It remains to prove Item (2): the stretch of every edge $(u, v)$ corresponding to an edge $(\varphi_{C_u}, \varphi_{C_v}) \in E_i^{\text{high}}$ in $K_i$ is at most $(1 + s_{A_{\text{Geom}}})$ with $s_{A_{\text{Geom}}} = 4(16g + 9)$. Let $r_u^{\text{def}} = r(C_u)$ and $r_v^{\text{def}} = r(C_v)$ be the representatives of $C_u$ and $C_v$, respectively. Let $Q_j$ be the cone in $\text{Cone}(C_u)$ such that $r_v \in Q_j$ for some $j \in [\tau]$ (we are using the notation in Algorithm $A_{\text{Geom}}$). If $r_v = r_j$, then $(u, v) \in K_i$ by the construction in Algorithm $A_{\text{Geom}}$; thus, the stretch is 1. Otherwise, let $C_y$ be the level-i cluster that contains the representative $r_j$. By the construction in Algorithm $A_{\text{Geom}}$, there is an edge $(x, y) \in K_i$ where $x \in C_u$ and $y \in C_y$. (See Figure 3) By property $[3]$, $\max\{|\text{Dm}(K_i[C_u]), \text{Dm}(K_i[C_v], \text{Dm}(K_i[C_y])\}} \leq geL_i$. By
Recall that
\[\|r_u, r_v\| \leq \|u, v\| + 2g\epsilon L_i \leq (1 + 2g\epsilon) L_i\]
\[\|r_u, r_j^*\| \leq \|x, y\| + 2g\epsilon L_i \leq (1 + 2g\epsilon) L_i\]
\[\|u, v\| \leq \|r_u, r_v\| + 2g\epsilon L_i\]
\[\|x, y\| \leq \|r_u, r_j^*\| + 2g\epsilon L_i\]

Furthermore, since \(\psi = \epsilon, \frac{L_i}{1+\epsilon} \leq \|u, v\|, \|x, y\| \leq L_i\). It follows that:
\[\|u, v\| \leq (1 + \epsilon)\|x, y\|\]
\[\|x, y\| \leq (1 + \epsilon)\|u, v\|\]

**Claim 4.2.** \(\|r_v, r_j^*\| \leq (8g + 3)\epsilon L_i\).

**Proof:** Recall that \(\|r_u, r_j^*\| \leq \|r_u, r_v\|\). Let \(p\) be the projection of \(r_j^*\) on the segment \(r_u r_v\) (see Figure 3). Since \(\angle r_v r_u r_j^* \leq \epsilon\), \(\|r_j^*, p\| \leq \sin(\epsilon)\|r_u, r_j^*\| \leq \sin(1)\|r_u, r_j^*\| \leq (1 + 2g\epsilon) L_i\). We have:
\[\|r_v, r_j^*\| \leq \|p, r_j^*\| + \|r_v, p\| = \|p, r_j^*\| + \|r_u, r_v\| - \|p, r_u\|
\leq \|p, r_j^*\| + \|r_u, r_v\| - (\|r_u, r_j^*\| - \|r_j^*, p\|))\]
\[\leq (\|r_u, r_v\| - \|r_u, r_j^*\|) + 2(1 + 2g\epsilon) L_i\]

We now bound \((\|r_u, r_v\| - \|r_u, r_j^*\|)\). By Equation (21) and Equation (22), it holds that:
\[\|r_u, r_v\| - \|r_u, r_j^*\| \leq \|u, v\| + 2g\epsilon L_i - (\|x, y\| - 2g\epsilon L_i)\]
\[= \|u, v\| - \|x, y\| + 4g L_i \leq \epsilon\|x, y\| + 4g L_i \leq (4g + 1)\epsilon L_i\]

Plugging Equation (24) into Equation (23), we get \(\|r_v, r_j^*\| \leq (4g + 1)\epsilon L_i + 2\epsilon(1 + 2g) L_i \leq (8g + 3)\epsilon L_i\); here we use the fact that \((1 + 2g\epsilon) \leq 2\).
\[\square\]

By Claim 4.2, \(\|r_v, r_j^*\| < \frac{L_i}{1 + \epsilon}\) when \(\epsilon \leq 1/c_g\) for a sufficiently large constant \(c_g\). If the input graph is a UDG, then \(L_i^\gamma \neq \emptyset\) if and only if \(L_i/(1 + \epsilon) \leq \rho\) where \(\rho\) is the distance threshold (after scaling) determining edges of the UDG. Thus, \(\|r_v, r_j^*\| \leq \rho\) and hence, there is an edge \((r_v, r_j^*)\) of length \(\|r_v, r_j^*\|\) in the input UDG. (This is the only place, other than the fact that we start our construction with a \((1 + \epsilon)\)-spanner, where we take into account of the fact that the input graph is a UDG.)

Since \(\|r_v, r_j^*\| < \frac{L_i}{1 + \epsilon}\), the distance between \(r_v\) and \(r_j^*\) is preserved up to a factor of \((1 + c_g\epsilon)\) in the spanner constructed for every edge of length less than \(L_i/(1 + \epsilon)\); see Remark 1.7. That is, \(d_{K_i}(r_v, r_j^*) \leq (1 + c_g\epsilon)\|r_v, r_j^*\|\).

Note that \(r_u, r_v, r_j^*\) are all in the input point set \(P\) by the definition of representatives. By the triangle inequality, it follows that:
\[d_{K_i}(u, v) \leq d_{K_i}(u, x) + \|x, y\| + d_{K_i}(y, r_j^*) + d_{K_i}(r_j^*, r_v) + d_{K_i}(r_v, v)\]
\[\leq g\epsilon L_i + \|x, y\| + g\epsilon L_i + (1 + c_g\epsilon)\|r_v, r_j^*\| + g\epsilon L_i\]
\[\leq \|x, y\| + 3g\epsilon L_i + \underbrace{(1 + c_g\epsilon)}_{\leq 2} (8g + 3)\epsilon L_i\] (by Claim 4.2)
\[\leq \|x, y\| + (19g + 6)\epsilon L_i\]
By Equation \([22]\), we have that \(\|x, y\| \leq (1 + \epsilon)\|u, v\| \leq \|u, v\| + \epsilon L_i\). Thus, by Equation \([25]\), we have:

\[
d_{K_i}(u, v) \leq \|u, v\| + (19g + 7)\epsilon L_i \leq \frac{\|u, v\| + (1 + 2(19g + 7)\epsilon)\|u, v\|}{L_i}.
\]

That is, the stretch of \((u, v)\) in \(K_i\) is at most \(1 + s_{\text{AGeom}}\epsilon\) with \(s_{\text{AGeom}} = 2(19g + 7)\), as claimed. 

**Remark 4.3.** We remark that Algorithm \(\text{A}_{\text{Geom}}\) can be implemented slightly faster, with a running time \(O(|V_{\text{high}}|^\psi \cdot s_{\text{high}}^d + |E_{\text{high}}|^\psi \cdot \log(1/\epsilon))\) by using a data structure that allows us to search for the cone that a representative belongs to in \(O(\log(1/\epsilon))\) time. Such a data structure is described in Theorem 5.3.2 in the book by Smid and Narasimhan \([63]\).

We now ready to prove Theorem 1.8, which we restate below.

**Theorem 1.8.** Let \(\psi = \epsilon \text{ and } G = (V, E, w)\) be a \((1 + \epsilon)\)-spanner either for a set of \(n\) points \(P\) or for the unit ball graph \(U\) of \(P\) in \(\mathbb{R}^d\). There is an algorithm that can compute all subgraphs \(H_1, \ldots, H_t \subseteq G\) as well as the cluster sets \(C_1, \ldots, C_i, C_{i+1}\) in total runtime \(O((\sum_{j=1}^t (|C_j| + |E_j^\sigma|)\alpha(m, n) + n))\). Furthermore, \(H_i\) satisfies Lemma 1.6 with \(t = 1 + \epsilon\), \(\lambda = O(\epsilon^{-(d+1)})\), and \(A = O(\epsilon^{-(d+1)})\).

**Proof:** By Lemma 3.22 and Lemma 4.1, the stretch of \(H_{\leq i}\) is \(t(1 + \max\{2(19g + 7) + 4g, 10g\})\epsilon = t(1 + 2(21g + 7)\epsilon)\); we then rescale \(\epsilon = \frac{\epsilon}{2(21g + 7)}\) to get the stretch \(t(1 + \epsilon)\).

By Lemma 4.1, \(\xi = O(\epsilon^{1-d})\) where \(\xi\) is the parameter defined in Algorithm \(\text{A}\). Thus, by Lemma 3.22, \(\lambda = O(\epsilon^{-(d+1) + \epsilon^{-d}}) = O(\epsilon^{-(d+1)})\) when \(d \geq 2\), and \(A = O(\epsilon^{-(d+1)})\).

It remains to bound the running time of the construction. Note that \(|V_t| = |C_i|\) and \(|E_t^\sigma| \leq |E_i^\sigma| \leq |E_t^\sigma|\). The total running time consists of:

- the running time to construct \(S_i\), which is \(O((|V_t| + |E_t^\sigma|)\alpha(m, n))\) by Lemma 3.16 and the running time to construct \(\text{MST}_{i+1}\), which is \(O(|V_t|\alpha(m, n))\) by Lemma 3.21.
- the running time to construct \(X\), which is \(O((|V_t| + |E_t^\sigma|)\epsilon^{-1})\) by Lemma 3.17.
- the running time to construct \(H_i\), which is \(O((|V_t| + |E_t^\sigma|)\epsilon^{1-d})\) by Lemma 3.22.

Thus, the total running time is \(O((|C_i| + |E_t^\sigma|)\alpha(m, n))\). Summing the running time over all levels up to \(i\), we obtain the claimed running time. 

4.2 General Graphs

In this section, we give the detailed implementation of Algorithm \(\text{A}\) for general graphs, which we denote by \(\text{A}_{\text{Gen}}\). Here we have \(t = 2k - 1\) for an integer parameter \(k \geq 2\). We will use as a black-box the linear-time construction of sparse spanners in general unweighted graphs by Halperin and Zwick \([51]\).

**Theorem 4.4** (Halperin-Zwick \([51]\)). Given an unweighted \(n\)-vertex graph \(G\) with \(m\) edges, a \((2k - 1)\)-spanner of \(G\) with \(O(n^{1+\frac{k}{2}})\) edges can be constructed deterministically in \(O(n + m)\) time, for any \(k \geq 2\).
Lemma 4.5. Let $\psi = \epsilon$. Algorithm $A_{Gen}$ can be implemented in $O(|V^\text{high}_i| + |E^\text{high}_i|)$ time. Furthermore, (1) $|E^\text{pruned}_i| = O(n^{1/k}|V^\text{high}_i|)$ and (2) $d_{K_i}(u,v) \leq (1 + s_{A_{Gen}}\epsilon)w(u,v)$ for every edge $(u,v)$ corresponding to an edge in $E_i$, where $s_{A_{Gen}} = (4g + 1)$ and $\epsilon \leq 1$.

Proof: The running time of $A_{Gen}$ follows directly from Theorem 4.4. Also by Theorem 4.4, $|E^\text{pruned}_i| = O(n^{1/k}|V^\text{high}_i|)$; this implies Item (1).

It remains to prove Item (2): the stretch in $K_i$ (constructed as described in Algorithm $A$) for every edge $(u,v)$ corresponding to an edge $(\varphi_{C_u}, \varphi_{C_v}) \in E^\text{high}_i$ is at most $(2k - 1)(1 + (4g + 1)\epsilon)$. Recall that $K_i$ is obtained from $H_{\leq i - 1}$ by adding in the corresponding edges of all edges in $E^\text{pruned}_i$.

Let $(u_i, v_i)$ be the edge in $E_J$ that corresponds to the edge $(\varphi_{C_u}, \varphi_{C_v})$. By Theorem 4.4, there is a path $P$ between $u_i$ and $v_i$ in $J$ such that $P$ contains at most $2k - 1$ edges. We write $P = (u_1 = x_0, x_0, x_1, x_1, x_2, \ldots, x_p = v_1)$ as an alternating sequence of vertices and edges. Let $P = (\varphi_0, (\varphi_0, \varphi_1), \varphi_1, (\varphi_1, \varphi_2), \ldots, \varphi_p)$ be a path of $K_i$, written as an alternating sequence of vertices and edges, that is obtained from $P$ where $\varphi_j$ corresponds to $x_j$, $1 \leq j \leq p$. Note that $\varphi_1 = \varphi_{C_u}$ and $\varphi_p = \varphi_{C_v}$. The weight of $P$ is:

$$
\omega(P) \leq (2k - 1)L_i + (2k)g\epsilon L_i \leq (2k - 1)(1 + 2g\epsilon)L_i \\
\leq (2k - 1)(1 + 2g\epsilon)(1 + \psi)w(u,v) \quad \text{(since } w(u,v) \geq L_i/(1 + \psi) = L_i/(1 + \epsilon)) \\
\leq (2k - 1)(1 + (4g + 1)\epsilon)w(u,v) \quad \text{(since } \epsilon \leq 1)
$$

Since edges of $P$ are in $E^\text{pruned}_i$, there is a path in $K_i$ between $u$ and $v$ of weight at most $\omega(P)$, which is at most $(2k - 1)(1 + (4g + 1)\epsilon)w(u,v)$ as desired. \(\square\)

We are now ready to prove Theorem 1.9 for general graphs, which we restate below.

Theorem 1.9. Let $\psi = \epsilon$ and $G = (V,E,w)$ be a general graph. There is an algorithm that can compute all subgraphs $H_1, \ldots, H_i \subseteq G$ as well as the cluster sets $C_1, \ldots, C_i, C_{i+1}$ in total runtime $O_i(\sum_{j=1}^{i}(|C_j| + |E_j^\epsilon|)\alpha(m, n) + m)$. Furthermore, $H_i$ satisfies Lemma 1.6 with $t = 2k - 1$, $\lambda = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$, and $A = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$.

Proof: By Lemma 3.22 and Lemma 4.5, the stretch of $H_{\leq i}$ is $(2k - 1)(1 + \max\{(4g + 1) + 4g, 10g\})\epsilon = (2k - 1)(1 + (10g + 1)\epsilon)$; we then rescale $\epsilon \leftarrow \frac{\epsilon}{10g+1}$ to get the stretch $(2k - 1)(1 + \epsilon)$.

By Lemma 4.5, $\xi = O(n^{1/k})$ where $\xi$ is the parameter defined in Algorithm $A$. Thus, by Lemma 3.22, $\lambda = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$, and $A = O(n^{1/k}\epsilon^{-2} + \epsilon^{-3})$. 32
It remains to bound the running time of the construction. Observe that $|V_i| = |C_i|$ and $|E_{i}^\text{high}| \leq |E_i| \leq |E_{i}^\tau|$. Thus the running time to (1) construct $G_i$ is $O(|C_i| + |E_{i}^\tau|)\alpha(m,n)$ due to Lemma 3.16. (2) construct MST$_{i+1}$ is $O(|C_i|\alpha(m,n))$ by Lemma 3.21. (3) construct $X$ is $O(|C_i| + |E_{i}^\tau|)$ by Lemma 3.17 and (4) construct $H_i$ is $O(|C_i| + |E_{i}^\tau|)$ by Lemma 3.22. Thus, the total running time is $O(|C_i| + |E_{i}^\tau|)\alpha(m,n)$. Summing the running time over all levels up to $i$, we obtain the claimed running time.

4.3 Minor-free Graphs

Algorithm $A_{\text{Minor}}$ for minor-free graphs, which we denote by $A_{\text{Minor}}$, simply outputs the edge set $E_{i}^\text{high}$. Note that the stretch in this case is $t = 1 + \epsilon$.

Algorithm $A_{\text{Minor}}$ (Minor-free Graphs): The input is a subgraph $K_i = (V_{i}^\text{high}, E_{i}^\text{high}, \omega)$ of $G_i$. The output is $E_{i}^\text{pruned} = E_{i}^\text{high}$. The parameter $\psi$ is set to 1 in this setting. Our proof uses the following result:

Lemma 4.6 (Kostochka [55] and Thomason [79]). Any $K_r$-minor-free graph with $n$ vertices has $O(r\sqrt{\log n})$ edges.

Lemma 4.7. Algorithm $A_{\text{Minor}}$ can be implemented in $O((|V_{i}^\text{high}| + |E_{i}^\text{high}|))$ time. Furthermore, (1) $E_{i}^\text{pruned} = O(r\sqrt{\log r}|V_i|)$ and (2) $s_{A_{\text{Minor}}} = 0$.

Proof: The running time of $A_{\text{Minor}}$ follows trivially from the construction. Observe that $K_i$ is a minor of the input graph $G$, and hence, $K_i$ is $K_r$-minor-free. Thus, $|E_{i}^\text{high}| = O(r\sqrt{\log r}|V_i|)$ by Lemma 4.6. This implies Item (1). Since we take every edge of $E_{i}^\text{high}$ to $E_{i}^\text{pruned}$, the stretch is 1 and hence $s_{A_{\text{Minor}}} = 0$; this implies Item (2).

Obtaining a truly linear running time. Given Lemma 4.7, we can obtain Theorem 1.9 but with the running time containing a factor of $\alpha(m,n) = O(\alpha(n) + \log(r))$ due to the UNION-FIND data structure; here $m = O(r\sqrt{\log n})$. To remove this factor, we do not use UNION-FIND. Instead, we follow the idea of Mareš [62] that was applied to construct a minimum spanning tree for $K_r$-minor-free graphs. Specifically, after the construction of level-($i+1$) clusters, we prune the set of edges that are involved in the construction of levels at least $i + 1$, which is $\bigcup_{j \geq i+1} E_j^\tau$, as follows.

Let $E_{\geq i}^\tau = \bigcup_{j \geq i} E_j^\tau$. We inductively maintain a set of edges $E_{\geq i}$, where each edge in $E_{\geq i}$ corresponds to an edge in $E_{\geq i}^\tau$. (Note that only those in $E_i$ are involved in the construction of spanner at level $i$.) Furthermore, we inductively guarantee that $|E_{\geq i}| = O(r\sqrt{\log r}|V_i|)$; we call this the size invariant. Upon completing the construction of level-($i+1$) clusters, we construct the set of nodes $V_{i+1}$. We now consider the set of edges $E_{\geq i+1} = E_{\geq i} \setminus E_i$. Let $\hat{E}_{\geq i+1}$ be obtained from $E_{\geq i+1}$ by removing parallel edges: two edges $(\varphi_1, \varphi_2)$ and $(\varphi_1', \varphi_2')$ are parallel if there exist two subgraphs $X, Y \in \mathcal{X}$ such that, w.l.o.g., $\varphi_1, \varphi_1' \in \mathcal{V}(X)$ and $\varphi_2, \varphi_2' \in \mathcal{V}(Y)$. (Among all parallel edges, we keep the edge with minimum weight in $\hat{E}_{\geq i+1}$.) We construct the edge set $E_{\geq i+1}$ (between vertices in $V_{i+1}$) at level ($i + 1$) from $\hat{E}_{\geq i+1}$ by creating one edge $(\mathcal{X}, \mathcal{Y}) \in E_{\geq i+1}$ for each corresponding edge $(\varphi_x, \varphi_y) \in \hat{E}_{\geq i+1}$ where $\varphi_x \in \mathcal{V}(\mathcal{X})$ and $\varphi_y \in \mathcal{V}(\mathcal{Y})$; $\omega(\mathcal{X}, \mathcal{Y}) = \omega(\varphi_x, \varphi_y)$. 33
Observe that $E_{i+1}$ corresponds to a subset of edges of $E_{2i+1}$ since $E_{2i+1}$, by definition, corresponds to a subset of edges of $E_{2i+1+1}$. The stretch is in check (at most $(1 + O(\epsilon))$), since we only remove parallel edges and that level-$(i + 1)$ clusters have diameter $O(\epsilon)$ times the weight of level-$(i + 1)$ edges by property (P3). Furthermore, since $E_{2i} = O(r\sqrt{\log r}|V_i|)$ by the size invariant, the construction of $E_{i+1}$ can be done in $O(|V_i|)$ time. Since the graph $(\mathcal{V}_{i+1}, \mathcal{E}_{2i-1})$ is a minor of $G$ and hence, is $K_r$-minor-free, we conclude that $|E_{2i+1}| = O(r\sqrt{\log r}|V_i|)$ by Lemma 4.6 which implies the size invariant for level $i + 1$.

By the size invariant, we do not need UNION-FIND data structure, as $E_{2i}$ now has $O(r\sqrt{\log r}|V_i|) = O(r\sqrt{\log r}|C_i|)$ edges. Thus, the running time to construct $G_i$ in Lemma 3.16 becomes $O_*(|C_i| + |E_i|) = O_*(r\sqrt{\log r}|C_i|)$, and the running time to construct $\text{MST}_{i+1}$ in Lemma 3.21 also becomes $O((r\sqrt{\log r}|C_i|$).

We are now ready to prove Theorem 1.9 for minor-free graphs, which we restate below.

**Theorem 1.10.** Let $\psi = 1$ and $G = (V,E,w)$ be a $K_r$-minor-free graph. There is an algorithm that can compute all subgraphs $H_1, \ldots, H_i \subseteq G$ as well as the cluster sets $C_1, \ldots, C_i, C_{i+1}$ in total runtime $O_*(\sum_{i=1}^{\psi}(|C_i|)\sqrt{\log r})$. Furthermore, $H_i$ satisfies Lemma 1.6 with $t = (1 + \epsilon)$, $\lambda = O((r\sqrt{\log r})$, and $A = O((r\sqrt{\log r} + \epsilon^3)).$

**Proof:** By Lemma 3.22 and Lemma 4.7, the stretch of $H_{<i}$ is $t(1 + \max\{4g, 10g\})\epsilon = t(1 + 10\epsilon)$; we then rescale $\epsilon \leftarrow \frac{\epsilon}{\sqrt{\log r}}$ to get the stretch $t(1 + \epsilon)$.

By Lemma 4.7, $\xi = O(r\sqrt{\log r})$ where $\xi$ is the parameter defined in Algorithm $\mathcal{A}$. Thus, by Lemma 3.22, $\lambda = O(r(\sqrt{\log r} - \epsilon^2 + \epsilon^{-3})$, and $A = O(r\sqrt{\log r} - \epsilon^2 + \epsilon^{-3} + \epsilon)^3$. It remains to bound the running time of the construction. Observe that $|V_i| = |C_i|$ and $|E_{i+1}^{\text{high}}| \leq |E_{i+1}| \leq |E_{i+1}^r|$. Thus, the running time to (1) construct $G_i$ and $\text{MST}_{i+1}$ is $O_*(r\sqrt{\log r}|C_i|)$ as discussed above, (2) construct $X$ is $O_*(|V_i| + |E_{i+1}^r|) = O_*(r\sqrt{\log r}|C_i|)$ by Lemma 3.17, and (3) construct $H_i$ is $O_*(|V_i| + |E_{i+1}^r|) = O_*(r\sqrt{\log r}|C_i|)$ by Lemma 3.22 and Lemma 4.5 here $\kappa(m, n) = O(1)$. Thus, the total running time is $O_*(r\sqrt{\log r}|C_i|)$; summing the running time over all levels up to $i$, we obtain the claimed running time.

5 Clustering: Proof of Lemma 3.17

In this section, we construct the set of subgraph $X$ of the cluster graph $G_i = (V_i, \text{MST}_i \cup E_i, \omega)$ as claimed in Lemma 3.17. Our construction builds upon the construction of Borradaile, Le and Wulff-Nilsen (BLW) [12]. However, unlike their construction, which is inefficient, our main focus here is on having a linear-time construction. Using the augmented diameter, we could bound the size of subgraphs (specifically in the construction of Step 4) arising during the course of our algorithm, and compute the augmented diameters of clusters efficiently. We note that in Borradaile, Le and Wulff-Nilsen [12], the efficiency of the construction is not relevant since they use the cluster hierarchy to analyze the greedy algorithm. That is, their clusters are not used in the construction of the spanner.

Our construction has five main steps (Steps 1-5). In Step 1, we group all vertices of $V_{i}^{\text{high}}$ and their neighbors into subgraphs of $X$; the construction of Step 1 is described in Lemma 5.1. In Step 2, we deal with branching nodes of MST; the construction of Step 2 is described in Lemma 5.2. In Step 3, we augment existing subgraphs formed in Steps 1 and 2, to guarantee a special structure of the ungrouped nodes. In Step 4, we group subpaths of MST into clusters; the construction of Step 4 is described in Lemma 5.4. Finally, in Step 5, we deal with the remaining ungrouped nodes of $V_i$.

Recall that $g$ is a constant defined in property (P3) and that $\text{MST}_i$ is a spanning tree of $G_i$ by Item (2) in Definition 3.12. We refer readers to Table 1 for a summary of the notation introduced in Section 3.
Lemma 5.1 (Step 1). Let $\mathcal{V}_i^{\text{high}} = \{ \varphi_C \in \mathcal{V} : \varphi_C \text{ is incident to at least } 2g \text{ edges in } E_i \}$. Let $\mathcal{V}_i^{\text{high}+}$ be obtained from $\mathcal{V}_i^{\text{high}}$ by adding all neighbors that are connected to nodes in $\mathcal{V}_i^{\text{high}}$ via edges in $E_i$. We can construct in $O(|\mathcal{V}_i| + |E_i|)$ time a collection of node-disjoint subgraphs $X_1$ of $G_i$ such that:

1. Each subgraph $X \in X_1$ is a tree.
2. $\cup_{X \in X_1} \mathcal{V}(X) = \mathcal{V}_i^{\text{high}+}$.
3. $L_i \leq \text{Adm}(X) \leq 13L_i$, assuming that $\epsilon \leq 1/g$.
4. $|\mathcal{V}(X)| \geq \frac{2g}{e}$.

Proof: Let $J = (\mathcal{V}_i, E_i)$ be the subgraph of $G_i$ with the same vertex set and with edge set $E_i$. Let $N_{J}(\varphi)$ be the set of neighbors of a node $\varphi$ in $J$, and $N_{J}[\varphi] = N_{J}(\varphi) \cup \{ \varphi \}$. We construct $X_1$ in three steps; initially, $X_1 = \emptyset$.

1. Let $I$ be a maximal set of nodes in $\mathcal{V}_i^{\text{high}}$ such that for any two nodes $\varphi_1, \varphi_2 \in I$, $N_{J}[\varphi_1] \cap N_{J}[\varphi_2] = \emptyset$. For each node $\varphi \in I$, we form a subgraph $X$ that consists of $\varphi$, its neighbors $N_{J}[\varphi]$, and all incident edges in $E_i$ of $\varphi$. We then add $X$ to $X_1$.

2. We iterate over all nodes of $\mathcal{V}_i^{\text{high}} \setminus I$ that are not grouped yet to any subgraph. For each such node $\varphi \in \mathcal{V}_i^{\text{high}} \setminus I$, there must be a neighbor $\varphi'$ that is already grouped to a subgraph, say $X \in X_1$; if there are multiple such neighbors, we pick one of them arbitrarily. We add $\varphi$ and the edge $(\varphi, \varphi')$ to $X$. Observe that every node in $\mathcal{V}_i^{\text{high}}$ is grouped to some subgraph at the end of this step.

3. For each node $\varphi$ in $\mathcal{V}_i^{\text{high}+}$ that has not grouped to a subgraph in steps (1) and (2), there must be at least one neighbor, say $\varphi''$, of $\varphi$ that is grouped in step (1) or step (2) to a subgraph $X \in X_1$; if there are multiple such nodes, we pick one of them arbitrarily. We then add $\varphi$ and the edge $(\varphi, \varphi'')$ to $X$.

This completes the construction of $X_1$. We now show that subgraphs in $X_1$ have all desired properties.

Observe that Items (1) and (2) follow directly from the construction. For Item (4), we observe that every subgraph $X \in X_1$ is created in step (1) and hence, contains a node $\varphi \in \mathcal{V}_i^{\text{high}}$ and all of its neighbors (in $J$) by the definition of $I$. Thus, $|\mathcal{V}(X)| \geq \frac{2g}{e}$ since $\varphi$ has at least $2g/e$ neighbors.

For Item (3), we observe that each subgraph $X \in X_1$ after step (3) has hop-diameter at least 2 and at most 6. Recall that every edge $e \in E_i$ has a weight of at most $L_i$, and every node has a weight of at most $geL_i$, which is at most $L_i$ since $\epsilon \leq 1/g$. Thus, $\text{Adm}(X) \leq 7geL_i + 6L_i \leq 13L_i$. Furthermore, since every edge $e \in E_i$ has a weight of at least $L_i/(1 + \psi) \geq L_i/2$ and $X$ has at least two edges in $E_i$, $\text{Adm}(X) \geq 2(L_i/2) = L_i$; this implies Item (3).

For the construction time, first note that $I$ can be constructed via a greedy linear-time algorithm, hence step (1) can be carried out in $O(|\mathcal{V}_i| + |E_i|)$ time. Steps (2) and (3) can be implemented within this time in a straightforward way; this implies the claimed running time.

Given a tree $T$, we say that a node $x \in T$ is $T$-branching if it has degree at least 3 in $T$. For brevity, we shall omit the prefix $T$ in “$T$-branching” whenever this does not lead to confusion. Given a forest $F$, we say that $x$ is $F$-branching if it is $T$-branching for some tree $T \subseteq F$. The construction of Step 2 is described in the following lemma.

Lemma 5.2 (Step 2). Let $F_i^{(2)}$ be the forest obtained from $\tilde{\text{MST}}_i$ by removing every node in $\mathcal{V}_i^{\text{high}+}$ (defined in Lemma 5.1). We can construct in $O(|\mathcal{V}_i|)$ time a collection $X_2$ of subtrees of $F_i^{(2)}$ such that for every $X \in X_2$:

---

6 The hop-diameter of a graph is the maximum hop-distance over all pairs of vertices, where the hop-distance between a pair of vertices is the minimum number of edges over all paths between them.
(1) $X$ is a tree and has an $X$-branching node.

(2) $L_i \leq \text{Adm}(X) \leq 2L_i$.

(3) $|\mathcal{V}(X)| = \Omega(\frac{1}{\epsilon})$ when $\epsilon \leq 2/g$.

(4) Let $\tilde{F}_i^{(2)}$ be obtained from $\tilde{F}_i^{(2)}$ by removing every node contained in subgraphs of $X$. Then, for every tree $\tilde{T} \subseteq \tilde{F}_i^{(2)}$, either (4a) $\text{Adm}(\tilde{T}) \leq 6L_i$ or (4b) $\tilde{T}$ is a path.

**Proof:** We say that a tree $\tilde{T} \in \tilde{F}_i^{(2)}$ is long if $\text{Adm}(\tilde{T}) \geq 6L_i$ and short otherwise. We construct $X_2$, initially empty, as follows:

- Pick a long tree $\tilde{T}$ of $\tilde{F}_i^{(2)}$ that has at least one $\tilde{T}$-branching node, say $\varphi$. We traverse $\tilde{T}$ starting from $\varphi$ and truncate the traversal at nodes whose augmented distance from $\varphi$ is at least $L_i$, which will be the leaves of the subtree. (The exact implementation details are delayed until the end of this proof.) As a result, the subtrees induced by the visited (non-truncated) nodes is at least $L_i$ and at most $L_i + \bar{w} + gL_i$. We then form a subgraph, say $X$, from the subtree induced by the visited nodes; add $X$ to $X_2$, remove every node of $X$ from $\tilde{T}$, and repeat this step until it no longer applies.

We observe that Item (1) follows directly from the construction. Since the algorithm only stops when every long tree has no branching node, meaning that it is a path, Item (4) is satisfied. We now show Items (2) and (3).

By construction, $X$ is a tree of augmented radius at least $L_i$ and at most $L_i + gL_i + \bar{w}$, hence $L_i \leq \text{Adm}(X) \leq 2(L_i + gL_i + \bar{w}) \leq 6L_i$ since $\bar{w} < L_i$ and $\epsilon < \frac{1}{g}$; this implies Item (2).

Let $D$ be an augmented diameter path of $X$; $\text{Adm}(D) \geq L_i$ by construction. Note that every edge has a weight of at most $\bar{w} \leq L_{i-1}$ and every node has a weight of in $[L_i, gL_{i-1}]$ by property (P3). Thus, $D$ has at least $\frac{\text{Adm}(D)}{2gL_{i-1}} \geq \frac{L_i}{2gL_i} = \Omega(\frac{1}{\epsilon})$ nodes; this implies Item (3).

It remains to show that the construction of $X_2$ can be implemented efficiently. First, we construct $\tilde{F}_i^{(2)}$ by simply going through every node in $\mathcal{V}_i$ and remove nodes that are grouped in $\mathcal{V}_i^{\text{high}}$. We maintain a list $B$ of branching nodes of $\tilde{F}_i^{(2)}$; all branching nodes can be found in $O(|\mathcal{V}(\tilde{F}_i^{(2)})|) = O(|\mathcal{V}_i|)$ time. Note that $\tilde{F}_i^{(2)}$ changes during the course of the construction. Initially, nodes in $B$ are unmarked. We then repeatedly apply the following three steps:

1. Pick a node $\varphi \in B$; if $\varphi$ is marked or no longer a branching node (of some tree in current $\tilde{F}_i^{(2)}$), remove $\varphi$ from $B$ and repeat until we find a branching, unmarked node. Let $\tilde{T}$ be the tree containing the picked node, say $\varphi$.

2. We traverse $\tilde{T}$ starting from $\varphi$ until the augmented radius of the subtree induced by visited nodes of $\tilde{F}_i^{(2)}$, denoted by $\tilde{T}_\varphi$, is at least $L_i$. It is possible that all nodes of the tree $\tilde{T}$ are visited before the radius gets to be $L_i$, in which case we have $\tilde{T}_\varphi = \tilde{T}$.

3. Mark every node of $\tilde{T}_\varphi$, remove every node in $\tilde{T}_\varphi$ from $\tilde{F}_i^{(2)}$, and repeat these three steps.

Clearly, maintaining the list $B$ throughout this process can be carried out in $O(|\mathcal{V}(\tilde{F}_i^{(2)})|)$ time. Other than that, each iteration of these three steps can be implemented in time linear in the number of nodes visited during that iteration plus the number of edges in $\tilde{F}_i^{(2)}$ incident to those nodes; also note that once a node is visited, it will no longer be considered in subsequent iterations. It follows that the total running time is $O(|\mathcal{V}_i|)$. \hfill $\square$
The goal of constructing a subgraph from a branching node \( \varphi \) is to guarantee that there must be at least one neighbor, say \( \varphi' \), of \( \varphi \) that does not belong to the augmented diameter path of \( \mathcal{X} \). Thus, we could show that the amount of corrected potential change \( \Delta_{i+1}^+(\mathcal{X}) \) is at least \( \omega(\varphi') \geq L_{i-1} = \epsilon L_i \). This will ultimately help us show that the corrected potential change \( \Delta_{i+1}^+(\mathcal{X}) \) is \( \Omega(\epsilon^2 |\mathcal{V}(\mathcal{X})| L_i) \).

**Step 3: Augmenting \( \mathcal{X}_1 \cup \mathcal{X}_2 \).** Let \( \tilde{F}_{i}^{(3)} \) be the forest obtained in Item (4b) in Lemma \[5.2\] Let \( \mathcal{A} \) be the set of all nodes \( \varphi \) in \( \tilde{F}_{i}^{(3)} \) such that \( \varphi \) is in a tree \( \tilde{T} \in \tilde{F}_{i}^{(3)} \) of augmented diameter at least \( 6L_i \) and \( \varphi \) is a branching node in \( \text{MST}_i \). For each node \( \varphi \in \mathcal{A} \) such that \( \varphi \) is connected to a node, say \( \varphi' \), in a subgraph \( \mathcal{X}' = \mathcal{X}_1 \cup \mathcal{X}_2 \) via an MST edge \( e \), we add \( \varphi \) and \( e \) to \( \mathcal{X} \). We note that \( \varphi' \) exists since \( \varphi \) has degree at least 3 in MST\( _i \). (If there are many such nodes \( \varphi' \), we choose an arbitrary one.)

**Lemma 5.3.** The augmentation in Step 3 can be implemented in \( O(|\mathcal{V}_i|) \) time, and increases the augmented diameter of each subgraph in \( \mathcal{X}_1 \cup \mathcal{X}_2 \) by at most \( 4L_i \) when \( \epsilon \leq 1/g \).

Furthermore, let \( \tilde{F}_{i}^{(4)} \) be the forest obtained from \( \tilde{F}_{i}^{(3)} \) by removing every node in \( \mathcal{A} \). Then, for every tree \( \tilde{T} \subseteq \tilde{F}_{i}^{(4)} \), either:

1. \( \text{Adm}(\tilde{T}) \leq 6L_i \) or
2. \( \tilde{T} \) is a path such that (2a) every node in \( \tilde{T} \) has degree at most 2 in \( \text{MST}_i \) and (2b) at least one endpoint \( \varphi \) of \( \tilde{T} \) is connected via an MST edge to a node \( \varphi' \) in a subgraph of \( \mathcal{X}_1 \cup \mathcal{X}_2 \), unless \( \mathcal{X}_1 \cup \mathcal{X}_2 = \emptyset \).

**Proof:** Since every \( \text{MST}_i \) edge has a weight of at most \( \tilde{w} \leq L_i \) and every node has a weight of at most \( geL_i \leq L_i \) when \( \epsilon \leq 1/g \), the augmentation in Step 3 increases the augmented diameter of each subgraph in \( \mathcal{X}_1 \cup \mathcal{X}_2 \) by at most \( 2(\tilde{w} + 2\epsilon g L_i) \leq 4L_i \).

For the implementation, we first find the set \( \mathcal{A} \) in \( O(|\mathcal{V}_i|) \) time in a straightforward way. Then for each node \( \varphi \in \mathcal{A} \), we can check its neighbors in \( \text{MST}_i \) to find a node \( \varphi' \) as described in Step 3; indeed, we only need to check at most three neighbors of \( \varphi \). Thus, the running time of Step 3 is \( O(|\mathcal{V}_i|) \).

Items (1) and (2a) follow directly from the construction. For Item (2b), we note that since \( \text{MST}_i \) is a spanning tree (and hence connected), \( \tilde{T} \) must be connected via an MST edge, say \( e \), to another node not in \( \tilde{T} \), assuming that \( \mathcal{X}_1 \cup \mathcal{X}_2 \neq \emptyset \). Since every node in \( \tilde{T} \) has at most degree 2, the endpoint of \( e \) in \( \tilde{T} \) must be one of the two endpoints of \( \tilde{T} \), as claimed. \( \square \)

The main intuition behind Step 3 is to guarantee properties (2a) and (2b) for every long path \( \tilde{T} \subseteq \tilde{F}_{i}^{(4)} \). Recall that in Item (3) of Definition \[3.12\] we guarantee that \( \mathcal{G}_i \) has no removable edge. Thus, any edge between two nodes in \( \tilde{T} \) is not removable. Later, we use this property to argue that the corrected potential change \( \Delta_{i+1}^+(\mathcal{X}) \) is non-trivial for every subgraph \( \mathcal{X} \) formed in the construction of Step 4 below.

**Required definitions/preparations for Step 4.** Let \( \tilde{F}_{i}^{(4)} \) be the forest obtained from \( \tilde{F}_{i}^{(3)} \) as described in Lemma \[5.3\] By Item (2b) in Lemma \[5.3\] every tree of augmented diameter at least \( 6L_i \) of \( \tilde{F}_{i}^{(4)} \) is a simple path, which we call a long path.

**Red/Blue Coloring.** Given a path \( \tilde{P} \subseteq \tilde{F}_{i}^{(4)} \), we color their nodes red or blue. If a node has augmented distance at most \( L_i \) from at least one of the path’s endpoints, we color it red; otherwise, we color it blue. Observe that each red node belongs to the suffix or prefix of \( \tilde{P} \); the other nodes are colored blue.

The construction of Step 4 is described by the following lemma. We include the proof of all claimed properties except Item (4), which will be delayed to Section \[5.3\] as its proof is more complicated.
Lemma 5.4 (Step 4). Let $\hat{F}_i^{(4)}$ be the forest obtained from $\hat{F}_i^{(3)}$ as described in Lemma 5.3. We can construct in $O((|V_i| + |E_i|)\epsilon^{-1})$ time a collection $X_4$ of subgraphs of $G_i$ such that every $X \in X_4$:

1. $|X| \leq \text{Adm}(X) \leq 5L_i$.
2. $|\nu(X)| = \Theta(\epsilon^2)$ when $\epsilon \ll \frac{1}{9}$.
3. $\Delta_{L_i+1}(X) = \Omega(\epsilon^2 |\nu(X)| L_i)$.
4. Let $\hat{F}_i^{(5)}$ be obtained from $\hat{F}_i^{(4)}$ by removing every node contained in subgraphs of $X_4$. If we apply Red/Blue Coloring to each path of augmented diameter at least $6L_i$ in $\hat{F}_i^{(5)}$, then there is no edge in $E_i$ that connects two blue nodes in $\hat{F}_i^{(5)}$.

Proof: We only apply the construction to paths of augmented diameter at least $6L_i$ in $\hat{F}_i^{(4)}$, called long paths.

Let $\hat{P}$ be a long path. For each blue node $\nu \in \hat{P}$, we assign a subpath $\nu(\hat{P})$ of $\hat{P}$, called the interval of $\nu$, which contains every node within an augmented distance (in $\hat{P}$) at most $L_i$ from $\nu$. By definition, we have:

Claim 5.5. For any blue node $\nu$, it holds that

(a) $(2 - (3g + 2)\epsilon)L_i \leq \text{Adm}(\nu) \leq 2L_i$.
(b) Denote by $\nu_1$ and $\nu_2$ the two subpaths obtained by removing $\nu$ from the path $\nu$. Each of these subpaths has $\Theta(\epsilon)$ nodes and augmented diameter at least $(1 - 2(g + 1)\epsilon)L_i$.

Proof: (a) The upper bound on the augmented diameter of $\nu$ follows directly from the construction. Thus, it remains to prove the lower bound on $\text{Adm}(\nu)$. Let $\hat{P}$ be the path containing $\nu$. Let $\mu$ be an endpoint of $\nu$. Let $\mu'$ be the neighbor of $\mu$ in $\hat{P} \setminus \nu$; $\mu'$ exists since $\nu$ is a blue node (see Figure 4). Observe that $\text{Adm}(\hat{P}[\nu, \mu']) \geq 1$. Thus, we have:

$$\text{Adm}(\hat{P}[\nu, \mu]) \geq L_i - \bar{w} - \omega(\mu') \geq (1 - (g + 1)\epsilon)L_i$$

(27)

The first inequality in the above equation is because we count $\omega(\nu)$ twice in the sum of the augmented diameters of two paths from $\nu$ to each endpoint of $\nu$.

(b) We focus on bounding $\text{Adm}(\nu_1)$; the same bound applies to $\text{Adm}(\nu_2)$. We assume w.l.o.g. that $\nu_1 \subseteq \hat{P}[\nu, \mu]$ and hence $\text{Adm}(\nu_1) \geq \text{Adm}(\hat{P}[\nu, \mu]) - \bar{w} - \omega(\nu) \geq (1 - 2(g + 1)\epsilon)L_i$.

We now bound $|\nu(\nu_1)|$. The upper bound on the number of nodes of $\nu_1$ follows from the fact that $\nu_1$ has augmented diameter at most $2L_i$ (see Item (a)) and each node has a weight of at least $L_i \epsilon$ by property (P3'). To show the lower bound on the number of nodes of $\nu_1$, we observe that $\nu_1$ has augmented diameter at least $(1 - (g + 1)\epsilon)L_i$, which is at least $L_i/2$ when $\epsilon \ll \frac{1}{9}$ while each edge in $\nu_1$ has a weight of at most $L_i$ and each node has a weight of at most $gL_i$. It follows that $|\nu(\nu_1)| \geq \frac{\text{Adm}(\nu_1)}{(1+g)L_i} = \Omega(\frac{1}{\epsilon})$. The same bound holds for $|\nu(\nu_2)|$. 

![Figure 4: Nodes in the green shaded region belong to $\nu$.]
We keep track of a list $B$ of edges in $E_i$ with both blue endpoints. We then construct $X_4$, initially empty, as follows:

- Pick an edge $(v, \mu)$ with both blue endpoints, form a subgraph $\mathcal{X} = \{(v, \mu) \cup \mathcal{I}(v) \cup \mathcal{I}(\mu)\}$, and add $\mathcal{X}$ to $X_4$. We then remove all nodes in $\mathcal{I}_v \cup \mathcal{I}_\mu$ from the path or two paths containing $v$ and $\mu$, update the color of nodes in the new paths to satisfy Red/Blue Coloring and the edge set $B$, and repeat this step until it no longer applies.

We observe that Items (1) and (5) follow directly from the construction. For Item (2), we observe by Claim 5.5 that $\mathcal{I}(v)$ has augmented diameter at most $2L_i$ and at least $L_i$ when $\epsilon \ll \frac{1}{\ell}$, and the weight of the edge $(\mu, v)$ is at most $L_i$. Thus, $L_i \leq \text{Adm}(\mathcal{X}) \leq L_i + 2 \cdot 2L_i = 5L_i$, as claimed. Item (3) follows directly from Claim 5.5 since $|\mathcal{I}(v)| = \Theta\left(\frac{1}{\ell}\right)$ and $|\mathcal{I}(\mu)| = \Theta\left(\frac{1}{\ell}\right)$. The proof of Item (4) is delayed to Section 5.3. In a nutshell, the proof is divided into two cases: (a) $\mathcal{I}(v) \cap \mathcal{I}(\mu) = \emptyset$ and (b) $\mathcal{I}(v) \cap \mathcal{I}(\mu) \neq \emptyset$. In the former case, we show that $\Delta_{\epsilon i+1}^+(\mathcal{X}) = \Omega(|\mathcal{X}|\epsilon L_i)$; the proof is by a straightforward calculation. In the latter case, we show that $\Delta_{\epsilon i+1}^+(\mathcal{X}) = \Omega(|\mathcal{X}|\epsilon^2 L_i)$; the proof crucially uses the fact that $G_i$ has no removable edge (see Item (3) in Definition 3.12).

Finally, we show that the construction of $X_4$ can be implemented efficiently. Observe that for each long path $\tilde{P}$, coloring all nodes of $\tilde{P}$ can be done in $O(|V(\tilde{P})| = |V_i|)$ time. Since the interval $\mathcal{I}(v)$ assigned to each blue node $v$ consists of $O\left(\frac{1}{\ell}\right)$ nodes by Claim 5.5(b), listing intervals for all blue nodes can be carried out within time $O(|\mathcal{I}(\tilde{P})|\epsilon^{-1}) = O(|V_i|\epsilon^{-1})$. For each edge $(v, \mu) \in E_i$, we can check whether both endpoints are blue in $O(1)$ time. Thus, it takes $O(|E_i|\epsilon^{-1})$ time to construct $B$.

For each edge $(v, \mu) \in B$ picked in the construction of $X_4$, forming $\mathcal{X} = \{(v, \mu) \cup \mathcal{I}(v) \cup \mathcal{I}(\mu)\}$ takes $O(1)$ time. When removing any such interval $\mathcal{I}(v)$ from a path $\tilde{P}$, we may create two new sub-paths $\tilde{P}_1, \tilde{P}_2$, and then need to recolor the nodes following Red/Blue Coloring. Specifically, some blue nodes in the prefix and/or suffix of $\tilde{P}_1, \tilde{P}_2$ are colored red; importantly, a node’s color may only change from blue to red, but it may not change in the other direction.

Since the total number of nodes to be recolored as a result of removing such an interval $\mathcal{I}(v)$ is $O\left(\frac{1}{\ell}\right)$, the total recoloring running time is $O(|\mathcal{I}(\tilde{F}_i^{(5)})|\epsilon^{-1}) = O(|V_i|\epsilon^{-1})$. To bound the time required for updating the edge set $B$ throughout this process, we note that edges are never added to $B$ after its initiation. Specifically, when a blue node $v$ is recolored as red, we remove all incident edges of $v$ from $B$, and none of these edges will be considered again; this can be done in $O\left(\frac{1}{\ell}\right)$ time per node $v$, since $\epsilon$ is incident to at most $\frac{28}{\ell} = O\left(\frac{1}{\ell}\right)$ edges in $E_i$ due to the construction of Step 1 (Lemma 5.1). Once a node is added to $\mathcal{X}$, it will never be considered again. It follows that the total running time required for implementing Step 3 is $O(|V_i|\epsilon^{-1})$, as claimed.

\begin{remark}
Item (5) of Lemma 5.4 implies that for every edge $(\varphi_C, \varphi_C') \in E_i$ with both endpoints in $V(\tilde{F}_i^{(5)})$, at least one of the endpoints must belong to a low-diameter tree of $\tilde{F}_i^{(5)}$ or in a (red) suffix of a long path in $\tilde{F}_i^{(5)}$.
\end{remark}

\begin{observation}
For every tree $\tilde{T} \subseteq \tilde{F}_i^{(5)}$ such that $\text{Adm}(\tilde{T}) \leq 6L_i$, then $\tilde{T}$ is connected via $\text{MST}_i$ edge to a node in some subgraph $\mathcal{X} \in X_1 \cup X_2 \cup X_4$, unless there is no subgraph formed in Steps 1-4, i.e, $X_1 \cup X_2 \cup X_4 = \emptyset$.
\end{observation}

We call the case where $X_1 \cup X_2 \cup X_4 = \emptyset$ the degenerate case. When the degenerate case happens, $G_i$ has a very special structure, which will be described later (in Lemma 5.10); for now, we focus on the construction of the last step.
Step 5. Let \( \tilde{T} \) be a path in \( F_i^{(5)} \) obtained by Item (5) of Lemma 5.4. We construct two sets of subgraphs, denoted by \( X_5^{\text{intrnl}} \) and \( X_5^{\text{pref}} \), of \( G_i \). The construction is broken into two steps. Step 5A is only applicable when the degenerate case does not happen; Step 5B is applicable regardless of the degenerate case.

- (Step 5A) If \( \tilde{T} \) has augmented diameter at most \( 6L_i \), let \( e \) be an \( \text{MST}_i \) edge connecting \( \tilde{T} \) and a node in some subgraph \( X \in X_1 \cup X_2 \cup X_4 \); \( e \) exists by Observation 5.7. We add both \( e \) and \( \tilde{T} \) to \( X \).

- (Step 5B) Otherwise, the augmented diameter of \( \tilde{T} \) is at least \( 6L_i \) and hence, it must be a path by Item (4) in Lemma 5.2. In this case, we greedily break \( \tilde{T} \) into subpaths of augmented diameter at least \( L_i \) and at most \( 2L_i \). Let \( \tilde{P} \) be a subpath broken from \( \tilde{T} \). If \( \tilde{P} \) is connected to a node in a subgraph \( X \) via an edge \( e \in \text{MST}_i \), we add \( \tilde{P} \) and \( e \) to \( X \). If \( \tilde{P} \) contains an endpoint of \( \tilde{T} \), we add \( \tilde{P} \) to \( X_5^{\text{pref}} \); otherwise, we add \( \tilde{P} \) to \( X_5^{\text{intrnl}} \).

Lemma 5.8. We can implement the construction of \( X_5^{\text{intrnl}} \) and \( X_5^{\text{pref}} \) in \( O(|V_i|) \) time. Furthermore, every subgraph \( X \in X_5^{\text{intrnl}} \cup X_5^{\text{pref}} \) satisfies:

1. \( X \) is a subpath of \( \text{MST}_i \).
2. \( L_i \leq \text{Adm}(X) \leq 2L_i \).
3. \( |V(X)| = \Theta(\frac{1}{\epsilon}) \).

Proof: Items (1) and (2) follow directly from the construction. For Item (3), we observe the following facts: \( \text{Adm}(X) \geq L_i \), each edge has a weight of at most \( L_{i-1} \), and each node has a weight of at most \( gL_{i-1} \). Thus, \( |V(X)| \geq \frac{L_i}{1+gL_{i-1}} = \Omega(\frac{1}{\epsilon}) \). By the same argument, since each node has a weight at least \( L_{i-1} \) by property (P3'), \( |V(X)| \leq \frac{2L_i}{L_{i-1}} = O(1/\epsilon) \); this implies Item (3).

We now focus on the construction time. We observe that for every tree \( \tilde{T} \in F_i^{(5)} \), computing its augmented diameter can be done in \( O(|V(\tilde{T})|) \) time. Thus, we can identify all trees of \( F_i^{(5)} \) of augmented diameter at least \( 6L_i \) to process in Step 5B in \( O(|V(F_i^{(5)})|) = O(|V_i|) \) time. Breaking each path \( \tilde{T} \) in Step 5B into a collection of subpaths \( \{\tilde{P}_1, \ldots, \tilde{P}_k\} \) greedily can be done in \( O(|V(\tilde{T})|) \) time. For each \( j \in [k] \), to check whether \( \tilde{P}_j \) is connected by an \( \text{MST}_i \) edge to subgraph in \( X_1 \cup X_2 \cup X_4 \), we examine each node \( \varphi \in \tilde{P}_j \) and all \( \text{MST}_i \) edges incident to \( \varphi \). In total, there are at most \( |V(F_i^{(5)})| \) nodes and \( |\text{MST}_i| = |V_i| - 1 \) edges to examine; this implies the claimed time bound. \( \square \)

Finally, we construct the collection \( \mathcal{X} \) of subgraphs of \( G_i \) as follows:

\[
\mathcal{X} = X_1 \cup X_2 \cup X_4 \cup X_5^{\text{intrnl}} \cup X_5^{\text{pref}}.
\]

To complete the proof of Lemma 3.17, we need to:

1. show that subgraphs in \( \mathcal{X} \) satisfies three properties: (P1'), (P2'), and (P3'), and that \( |\mathcal{E}_i \cap \mathcal{E}(X)| = O(|V(X)|) \). This implies Item (5) of Lemma 3.17. We present the proof in Section 5.1.

2. construct a partition \( \{V_i^{\text{high}}, V_i^{\text{low}}, V_i^{\text{low}^+}, V_i^{\text{low}^-}\} \) of \( V_i \), show Items (1)-(4) and the running time bound as claimed by Lemma 3.17. We present the proof in Section 5.2.

5.1 Properties of \( \mathcal{X} \)

In this section, we prove the following lemma.
Lemma 5.9. Let $\mathcal{X}$ be the set of subgraph as defined in Equation (28). For every subgraph $\mathcal{X} \in \mathcal{X}$, $\mathcal{X}$ satisfies the three properties $(P1)-(P3)$ with $g = 31$ and $|E(\mathcal{X}) \cap E_i| = O(|V(\mathcal{X})|)$. Furthermore, $\mathcal{X}$ can be constructed in $O((|V_i| + |E_i|)\epsilon^{-1})$ time.

Proof: We observe that property $(P1)$ follows directly from the construction. Additionally, property $(P2)$ follows from Item (4) of Lemma 5.1, Items (3) of Lemma 5.2, Lemma 5.4, and Lemma 5.8. The lower bound $L_i$ on the augmented diameter of a subgraph $\mathcal{X} \in \mathcal{X}$ follows from Item (3) of Lemma 5.1, Items (2) of Lemma 5.2, Lemma 5.4, and Lemma 5.8. Thus, to complete the proof of property $(P3')$, it remains to show that $\text{Adm}(\mathcal{X}) \leq g L_i$ with $g = 31$.

If $\mathcal{X}$ is formed in Step 5B, that is $\mathcal{X} \in \mathcal{X}_{\text{intrnl}} \cup \mathcal{X}_{\text{pref}}$, then $\text{Adm}(\mathcal{X}) \leq 2L_i$ by Lemma 5.8. Otherwise, excluding any augmentation to $\mathcal{X}$ due to Step 5, Lemma 5.1, Lemma 5.2, and Lemma 5.3 yield $\text{Adm}(\mathcal{X}) \leq 13L_i + 4L_i \leq 17L_i$ where $4L_i$ is due to the augmentation in Step 3 (see Lemma 5.3). By Lemma 5.4, $\text{Adm}(\mathcal{X}) \leq \max (17L_i, 5L_i) = 17L_i$.

We then may augment $\mathcal{X}$ with trees of diameter at most $6L_i$ (Step 5A) and/or with subpaths of diameter at most $2L_i$ (Step 5B). A crucial observation is that any augmented tree or subpath is connected by an $\text{MST}_i$ edge to a node that was grouped to $\mathcal{X} \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4$, hence all the augmented trees and subpaths are added to $\mathcal{X}$ in a star-like way via $\text{MST}_i$ edges. If we denote the resulting subgraph by $\mathcal{X}^+$, then

$$\text{Adm}(\mathcal{X}^+) \leq \text{Adm}(\mathcal{X}) + 2\bar{w} + 12L_i \leq \text{Adm}(\mathcal{X}) + 14L_i \leq 31L_i.$$ 

In the above equation, term $2\bar{w}$ is from the two $\text{MST}_i$ edges connecting two augmented trees (or paths), and $12L_i$ is the upper bound on the sum of the augmented diameters of two augmented trees (or paths). Property $(P3')$ now follows.

The fact that $|E(\mathcal{X}) \cap E_i| = O(|V(\mathcal{X})|)$ and the running time bound follow directly from Lemma 5.1, Lemma 5.2, Lemma 5.3, Lemma 5.4, and Lemma 5.8. Recall that the augmentation in Step 3 is in a star-like way and hence, no cycle is formed in subgraphs of $\mathcal{X}_1 \cup \mathcal{X}_2$ after the augmentation. \hfill $\square$

5.2 Constructing a Partition of $V_i$

We first consider the degenerate case where $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 = \emptyset$.

Lemma 5.10 (Structure of Degenerate Case). If $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 = \emptyset$, then $\tilde{F}_i^{(5)} = \text{MST}_i$, and $\text{MST}_i$ is a single (long) path. Moreover, every edge $e \in E_i$ must be incident to a node in $\tilde{P}_1 \cup \tilde{P}_2$, where $\tilde{P}_1$ and $\tilde{P}_2$ are the prefix and suffix subpaths of $\text{MST}_i$ of augmented diameter at most $L_i$. Consequently, we have that $|E_i| = O(1/\epsilon^2)$.

Proof: By the assumption of the lemma, no subgraph is formed in Steps 1-4.

Since no subgraph is formed in Step 1, $\tilde{F}_i^{(2)} = \text{MST}_i$. Since no subgraph is formed in Step 2, there is no branching node in $\tilde{F}_i^{(2)}$, thus $\tilde{F}_i^{(3)} = \tilde{F}_i^{(2)}$ and it is a single (long) path. Since $\mathcal{X}_1 \cup \mathcal{X}_2 = \emptyset$, there is no augmentation in Step 3. Since no subgraph is formed in Step 4, $\tilde{F}_i^{(5)} = \tilde{F}_i^{(4)}$ and both are equal to $\text{MST}_i$, which is a long path (see Figure 5).

By Item (5) in Lemma 5.4, any edge $e \in E_i$ must be incident to a red node. The augmented distance from any red node to at least one endpoint of $\text{MST}_i$ is at most $L_i$ by the definition of Red/Blue Coloring.

![Figure 5: Red edges are edges in $E_i$; every edge is incident to at least one red node.](image)
and hence every red node belongs to $\tilde{P}_1 \cup \tilde{P}_2$. Since each node has a weight of at least $L_{i-1}$ by property (P3) we have:

\[
|\mathcal{V}(\tilde{P}_1 \cup \tilde{P}_2)| \leq \frac{2L_i}{L_{i-1}} = \frac{2}{\epsilon}
\]

Since each node of $\tilde{P}_1 \cup \tilde{P}_2$ is incident to at most $\frac{2g}{\epsilon}$ edges in $\mathcal{E}_i$ (as there is no subgraph formed in Step 1; $V_{i}^{\text{high}} = \emptyset$), it holds that $|\mathcal{E}_i| = O(1/\epsilon^2)$, as desired. $\square$

We are now ready to describe the construction of the partition \{\(V_{i}^{\text{high}}, V_{i}^{\text{low}+}, V_{i}^{\text{low}^-}\)\} of $\mathcal{V}_i$

**Construct Partition** \(\{V_{i}^{\text{high}}, V_{i}^{\text{low}+}, V_{i}^{\text{low}^-}\}\): If the degenerate case happens, we define $V_{i}^{\text{low}^-} = V_{i}$ and $V_{i}^{\text{high}} = V_{i}^{\text{low}+} = \emptyset$. Otherwise, we define $V_{i}^{\text{high}}$ to be the set of all nodes that are incident to at least $2g/\epsilon$ edges in $\mathcal{E}_i$, $V_{i}^{\text{low}^+} = \mathcal{V}_{\text{intrnl}}(X_i)$ and $V_{i}^{\text{low}^-} = V_{i} \setminus (V_{i}^{\text{high}} \cup V_{i}^{\text{low}^-})$.

We show the following property of \(\{V_{i}^{\text{high}}, V_{i}^{\text{low}+}, V_{i}^{\text{low}^-}\}\), which is equivalent to Item (4) in Lemma 3.17

**Lemma 5.11.** (1) If $X$ contains a node in $V_{i}^{\text{low}^-}$, then $\mathcal{V}(X) \subseteq V_{i}^{\text{low}^-}$.

(2) There is no edge in $\mathcal{E}_i$ between a node in $V_{i}^{\text{high}}$ and a node in $V_{i}^{\text{low}^-}$.

(3) If there exists an edge $(\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i$ such that both $\varphi_{C_u}$ and $\varphi_{C_v}$ are in $V_{i}^{\text{low}^-}$, then the degenerate case happens.

**Proof:** Item (1) follows directly from the construction. We now show Item (2). By the construction of Step 1 (Lemma 5.1), any neighbor, say $\varphi$, of a node in $V_{i}^{\text{high}}$ is in $V_{i}^{\text{high}+}$. Thus, $\varphi$ will not be considered after Step 1. It follows that there is no edge between a node in $V_{i}^{\text{high}}$ and a node in $V_{i}^{\text{low}^-}$ since nodes in $V_{i}^{\text{low}^-}$ are in Step 5.

To show Item (3), we observe by the construction that every node, say $\varphi_{C_u}$, in $V_{i}^{\text{low}^-}$ is a blue node of some long path $\tilde{P}$ in $F_i^{(5)}$. If the degenerate case does not happen, then by Item (5) of Lemma 5.4 every edge $(\varphi_{C_u}, \varphi_{C_v})$ must have the node $\varphi_{C_v}$ being a red node of $\tilde{P}$. But then by the construction of Step 5B $\varphi_{C_v}$ belongs to some subgraph of $X_5^{\text{pref}}$ and hence is not in $V_{i}^{\text{low}^-}$. $\square$

Next, we focus on bounding the corrected potential change $\Delta_{i+1}^+(X)$ of every cluster $X \in \mathbb{X}$. Specifically, we show that:

- if $X \in \mathbb{X}_1$, then $\Delta_{i+1}^+(X) = \Omega(|\mathcal{V}(X)|L_i \epsilon)$; the proof is in Lemma 5.12.
- if $X \in \mathbb{X}_2$, then $\Delta_{i+1}^+(X) = \Omega(|\mathcal{V}(X)|L_i \epsilon^2)$; the proof is in Lemma 5.13.
- if $X \in \mathbb{X}_1$, then $\Delta_{i+1}^+(X) = \Omega(|\mathcal{V}(X)|L_i \epsilon)$; the proof is in Lemma 5.14.
- the corrected potential change is non-negative and we provide a lower bound of the average corrected potential change for subgraphs in $\mathbb{X} \setminus \mathbb{X}^{\text{low}^-}$ in Lemma 5.15.

**Lemma 5.12.** For every subgraph $X \in \mathbb{X}_1$, it holds that $\Delta_{i+1}^+(X) \geq \frac{|\mathcal{V}(X)|L_i \epsilon}{2}$.

**Proof:** Let $X \in \mathbb{X}_1$ be a subgraph formed in Step 1. By Item (4) of Lemma 5.1, $|\mathcal{V}(X)| \geq \frac{2g}{\epsilon}$. By definition of $\Delta_{i}^+(X)$ (Lemma 3.17), we have:

\[
\Delta_{i+1}^+(X) \geq \sum_{\varphi \in \mathcal{V}(X)} \omega(\varphi) - \text{Adm}(\varphi) \geq \sum_{\varphi \in X} L_{i-1} - gL_i = \frac{|\mathcal{V}(X)|L_{i-1}}{2} + \left(\frac{|\mathcal{V}(X)|L_{i-1}}{2} - gL_i\right) \geq 0 \text{ since } |\mathcal{V}(X)| \geq (2g)/\epsilon
\]

\[
\geq \frac{|\mathcal{V}(X)|L_{i-1}}{2} = \frac{|\mathcal{V}(X)|\epsilon L_i}{2},
\]
as claimed.

When analyzing the corrected potential change, it is instructive to keep in mind the worst-case example, where the subgraph is a path of $\text{MST}_i$; in this case, it is not hard to verify (see Lemma 5.15) that the corrected potential change is 0. However, the key observation is that the worst-case example cannot happen for subgraphs formed in Step 2, as any such subgraph (a subtree of $\text{MST}_i$) is a branching node; such a node has at least three neighbors. Consequently, we can show that any subgraph formed in Step 2 has a sufficiently large corrected potential change, as formally argued next.

**Lemma 5.13.** For every subgraph $\mathcal{X} \in \mathcal{X}_2$, it holds that $\Delta_{i+1}^+(\mathcal{X}) = \Omega\left(|\mathcal{V}(\mathcal{X})|L_i\epsilon^2\right)$.

**Proof:** Let $\mathcal{X}$ be a subgraph that is initially formed in Step 2 and could possibly be augmented in Steps 3 and 5. Recall that in the augmentation done in Step 3, we add to $\mathcal{X}$ nodes of $\mathcal{V}_i$ via $\text{MST}_i$ edges, and in the augmentation done in Step 5, we add to $\mathcal{X}$ subtrees of $\text{MST}_i$ via $\text{MST}_i$ edges. Thus, the resulting subgraph after the augmentation remains, as prior to the augmentation, a subtree of $\text{MST}_i$. That is, $\mathcal{E}(\mathcal{X}) \subseteq \text{MST}_i$. Letting $\mathcal{D}$ be an augmented diameter path of $\mathcal{X}$, we have by definition of augmented diameter that

$$\text{Adm}(\mathcal{X}) = \sum_{\varphi \in \mathcal{D}} \omega(\varphi) + \sum_{e \in \mathcal{E}(\mathcal{D})} \omega(e).$$

Let $\mathcal{Y} = \mathcal{V}(\mathcal{X}) \setminus \mathcal{V}(\mathcal{D})$. Then $|\mathcal{Y}| > 0$ since $\mathcal{X}$ has a $\mathcal{X}$-branching node by Item (1) of Lemma 5.2 and that

$$\Delta_{i+1}^+(\mathcal{X}) = \left(\sum_{\varphi \in \mathcal{X}} \omega(\varphi) + \sum_{e \in \mathcal{E}(\mathcal{X})} \omega(e)\right) - \text{Adm}(\mathcal{X}) \geq \sum_{\varphi \in \mathcal{Y}} \omega(\varphi) \geq |\mathcal{Y}|L_{i-1}$$  \hspace{1cm} (30)

Note that $\mathcal{E}(\mathcal{X}) \subseteq \text{MST}_i$. By property (P3') $\text{Adm}(\mathcal{D}) \leq gL_i$ while each node has a weight of at least $L_{i-1}$. Thus, we have:

$$|\mathcal{V}(\mathcal{D})| \leq \frac{gL_i}{L_{i-1}} = O\left(\frac{1}{\epsilon}\right) = O\left(\frac{|\mathcal{Y}|}{\epsilon}\right),$$  \hspace{1cm} (31)

since $|\mathcal{Y}| \geq 1$. By combining Equation (30) and Equation (31), we have

$$\Delta_{i+1}^+(\mathcal{X}) \geq \frac{|\mathcal{Y}|L_{i-1}}{2} + \Omega(\epsilon|\mathcal{V}(\mathcal{D})|L_{i-1}) = \Omega(\epsilon|\mathcal{V}(\mathcal{X})|e^2L_i),$$

as claimed. \hfill \square

**Lemma 5.14.** For every subgraph $\mathcal{X} \in \mathcal{X}_4$, it holds that $\Delta_{i+1}^+(\mathcal{X}) = \Omega\left(|\mathcal{V}(\mathcal{X})|L_i\epsilon^2\right)$.

**Proof:** Let $\mathcal{X} \in \mathcal{X}_4$ be a subgraph initially formed in Step 4; $\mathcal{X}$ is possibly augmented in Step 5. Let $\mathcal{X}^+$ be $\mathcal{X}$ after the augmentation (if any). Let $\mathcal{D}^+$ be the augmented diameter path of $\mathcal{X}^+$ and $\mathcal{D} = \mathcal{D}^+ \cap \mathcal{X}$. Since the augmentation in Step 5 is by a star-like way, $\mathcal{D}$ is a path in $\mathcal{X}$. (Note that $\mathcal{X}$ might contain a cycle, and if there is a cycle, the cycle must contain the single edge of $\mathcal{X}$ in $\mathcal{E}_i$.) First, we observe that $|\mathcal{V}(\mathcal{D}^+)| = O\left(\frac{1}{\epsilon}\right)$ by the same argument in Equation (31). Furthermore, $|\mathcal{V}(\mathcal{X})| = \Omega\left(\frac{1}{\epsilon}\right)$ by Item (3) in Lemma 5.4. Thus, $|\mathcal{D}^+| = O(|\mathcal{V}(\mathcal{X})|)$. Let $\mathcal{Y} = \mathcal{V}(\mathcal{X}^+) \setminus \mathcal{V}(\mathcal{D}^+)$. Since $|\mathcal{D}^+| = O(|\mathcal{V}(\mathcal{X})|)$, by Item (4) in Lemma 5.4 it holds that:

$$\Delta_{i+1}^+(\mathcal{X}) = \Omega(|\mathcal{V}(\mathcal{X})|\epsilon^2L_i) = \Omega(|\mathcal{V}(\mathcal{X}) \cup \mathcal{V}(\mathcal{D}^+)|\epsilon^2L_i).$$  \hspace{1cm} (32)
Furthermore,
\[ \Delta_{i+1}^+(\mathcal{X}^+) = \sum_{\varphi \in \mathcal{X}^+} \omega(\varphi) + \sum_{e \in \mathcal{E}(\mathcal{X}^+) \cap \text{MST}_i} \omega(e) - \omega(D^+) \]
\[ \geq \sum_{\varphi \in \mathcal{Y}} \omega(\varphi) + \sum_{\varphi \in \mathcal{X}^+} \omega(\varphi) + \sum_{e \in \mathcal{E}(\mathcal{X}) \cap \text{MST}_i} \omega(e) - \omega(D) \]
\[ \geq \Omega(L_i|\mathcal{Y}|) + \Delta_{i+1}^+(\mathcal{X}) \overset{\text{Eq. (32)}}{=} \Omega(|\mathcal{Y}|\epsilon L_i) + \Omega(|\mathcal{X} \cup \mathcal{V}(D^+)|\epsilon^2 L_i) \]
\[ = \Omega(|\mathcal{X} \cup \mathcal{V}(D^+) \cup \mathcal{Y}|\epsilon^2 L_i) = \Omega(|\mathcal{X}|\epsilon^2 L_i), \]
as claimed. □

Next, we show Item (3) of Lemma 3.17 regarding the corrected potential changes of subgraphs in \( \mathcal{X} \).

**Lemma 5.15.** \( \Delta_{i+1}^+(\mathcal{X}) \geq 0 \) for every \( \mathcal{X} \in \mathcal{X}_i \), and
\[ \sum_{\mathcal{X} \in \mathcal{X} \setminus \mathcal{X}_i^{\text{low}}} \Delta_{i+1}^+(\mathcal{X}) = \sum_{\mathcal{X} \in \mathcal{X} \setminus \mathcal{X}_i^{\text{low}}=-} \Omega(|\mathcal{X}|\epsilon^2 L_i). \]

**Proof:** If \( \mathcal{X} \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 \), then \( \Delta_{i+1}^+(\mathcal{X}) \geq 0 \) by Lemmas 5.12 to 5.14. Otherwise, \( \mathcal{X} \in \mathcal{X}_i^{\text{pref}} \cup \mathcal{X}_i^{\text{intrnl}} \), and hence is a subpath of MST. Thus, by definition, \( \Delta_{i+1}^+(\mathcal{X}) = \sum_{\varphi \in \mathcal{X}} \omega(\varphi) + \sum_{e \in \mathcal{E}(\mathcal{X}) \cap \text{MST}_i} \omega(e) - \text{Adm}(\mathcal{X}) = 0 \). That is, \( \Delta_{i+1}^+(\mathcal{X}) \geq 0 \) in every case.

We now show a lower bound on the average potential change of subgraphs in \( \mathcal{X} \setminus \mathcal{X}_i^{\text{low}} \). We assume that the degenerate case does not happen; otherwise, \( \mathcal{X} \setminus \mathcal{X}_i^{\text{low}} = \emptyset \) and there is nothing to prove. By Item (1) of Lemma 5.11, \( \mathcal{X}_i^{\text{low}} = \mathcal{X}_i^{\text{intrnl}} \) and only subgraphs in \( \mathcal{X}_i^{\text{pref}} \) may not have positive potential change. By Lemmas 5.12 to 5.14 on average, each node \( \varphi \) in any subgraph \( \mathcal{X} \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 \) has \( \Omega(\epsilon^2 L_i) \) corrected potential change, denoted by \( \overline{\Delta}(\varphi) \).

By construction, a subgraph in \( \mathcal{X}_i^{\text{pref}} \) is a prefix (or suffix), say \( \bar{P}_1 \), of a long path \( \bar{P} \). The other suffix, say \( \bar{P}_2 \), of \( \bar{P} \) is augmented to a subgraph, say \( \mathcal{X} \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 \) by the construction of Step 3B and Item (2) Lemma 5.3. Since \( |\mathcal{V}(\bar{P}_2)| = \Omega(1/\epsilon) \) by Item (3) of Lemma 5.8 \( \sum_{\varphi \in \bar{P}_2} \overline{\Delta}(\varphi) = \Omega(1/\epsilon)(\epsilon^2 L_i) = \Omega(\epsilon L_i) \).

We distribute half this corrected potential change to all the nodes in \( \bar{P}_1 \), by Item (3) of Lemma 5.8 each gets \( \Omega(L_i/|\mathcal{X}|) = \Omega(\epsilon^2 L_i) \). This implies:
\[ \sum_{\mathcal{X} \in \mathcal{X} \setminus \mathcal{X}_i^{\text{low}}} \Delta_{i+1}^+(\mathcal{X}) = \sum_{\varphi \in \mathcal{V}_i \setminus \mathcal{V}_i^{\text{low}} \setminus \mathcal{Y}} \Omega(\epsilon^2 L_i) = \sum_{\mathcal{X} \in \mathcal{X} \setminus \mathcal{X}_i^{\text{low}} \setminus \mathcal{Y}} \Omega(|\mathcal{X}|\epsilon^2 L_i), \]
as desired. □

We are now ready to prove Lemma 3.17 that we restate below.

**Lemma 3.17.** Given \( \mathcal{G}_i \), we can construct in time \( O((|\mathcal{V}_i| + |\mathcal{E}_i|)\epsilon^{-1}) \) (i) a partition of \( \mathcal{V}_i \) into three sets \( \{\mathcal{V}_i^{\text{high}}, \mathcal{V}_i^{\text{low}+}, \mathcal{V}_i^{\text{low}^{-}}\} \) and (ii) a collection \( \mathcal{X} \) of subgraphs of \( \mathcal{G}_i \) and their augmented diameters, such that:

1. For every node \( \varphi_C \in \mathcal{V}_i \): If \( \varphi_C \in \mathcal{V}_i^{\text{high}} \), then \( \varphi_C \) is incident to \( \Omega(1/\epsilon) \) edges in \( \mathcal{E}_i \); otherwise (\( \varphi_C \in \mathcal{V}_i^{\text{low}+} \cup \mathcal{V}_i^{\text{low}^{-}} \)), the number of edges in \( \mathcal{E}_i \) incident to \( \varphi_C \) is \( O(1/\epsilon) \).

2. If a subgraph \( \mathcal{X} \) contains at least one node in \( \mathcal{V}_i^{\text{low}^{-}} \), then every node of \( \mathcal{X} \) is in \( \mathcal{V}_i^{\text{low}^{-}} \). Let \( \mathcal{X}^{\text{low}^{-}} \subseteq \mathcal{X} \) be a set of subgraphs whose nodes are in \( \mathcal{V}_i^{\text{low}^{-}} \) only.
(3) Let $\Delta^+_{i+1}(\mathcal{X}) = \Delta(\mathcal{X}) + \sum_{e \in \text{MST}_i \cap \mathcal{E}(\mathcal{X})} w(e)$. Then, $\Delta^+_{i+1}(\mathcal{X}) \geq 0$ for every $\mathcal{X} \in \mathbb{X}$, and

$$\sum_{\mathcal{X} \in \mathbb{X} \setminus \mathcal{X}^{\text{low}}} \Delta^+_{i+1}(\mathcal{X}) = \sum_{\mathcal{X} \in \mathbb{X} \setminus \mathcal{X}^{\text{low}}} \Omega(|\mathcal{V}(\mathcal{X})|\epsilon^2 |L_i|).$$

(4) There is no edge in $\mathcal{E}_i$ between a node in $\mathcal{V}_i^{\text{high}}$ and a node in $\mathcal{V}_i^{\text{low}}$. Furthermore, if there exists an edge $(\varphi_{C_u}, \varphi_{C_v}) \in \mathcal{E}_i$ such that both $\varphi_{C_u}$ and $\varphi_{C_v}$ are in $\mathcal{V}_i^{\text{low}}$, then $\mathcal{V}_i^{\text{low}} = \mathcal{V}_i$ and $|\mathcal{E}_i| = O\left(\frac{1}{\epsilon}\right)$; we call this case the degenerate case.

(5) For every subgraph $\mathcal{X} \in \mathbb{X}$, $\mathcal{X}$ satisfies the three properties $[P1]-[P3]$ with constant $g = 31$, and $|\mathcal{E}(\mathcal{X}) \cap \mathcal{E}_i| = O(|\mathcal{V}(\mathcal{X})|)$.

Furthermore, the construction of $\mathcal{X}$ can be constructed in the pointer-machine model with the same running time.

**Proof**: We observe that Items (1), (2) and (4) follow directly Lemma 5.10 and Lemma 5.11. Item (3) follows from Lemma 5.9. Item (5) follows from Lemma 5.15. The construction time is asymptotically the same as the construction time of $\mathcal{X}$, which is $O((|\mathcal{V}| + |\mathcal{E}_i|)\epsilon^{-1})$ by Lemma 5.9.

Finally, we compute the augmented diameter of each subgraph $\mathcal{X} \in \mathbb{X}$. We observe that the augmentations in Step 3 and Step 5 do not create any cycle. Thus, if $\mathcal{X}$ is initially formed in Steps 1, 2 or 5B, then finally $\mathcal{X}$ is a tree. It follows that the augmented diameter of $\mathcal{X}$ can be computed in $O(|\mathcal{V}(\mathcal{X})|\epsilon^{-1})$ time by a simple tree traversal. If $\mathcal{X}$ is formed in Step 4, then it has exactly one edge $e$ not in MST$_i$ by Item (1) in Lemma 5.4 and that $\mathcal{X}$ contains at most one cycle. Let $\mathcal{Z}$ be such a cycle (if any); $\mathcal{Z}$ has $O(1/\epsilon)$ edges by Item (3) in Lemma 5.4. Thus, we can reduce computing the diameter of $\mathcal{X}$ to computing the diameter of trees by guessing an edge of $\mathcal{Z}$ that does not belong to the diameter path of $\mathcal{X}$ and remove this edge from $\mathcal{X}$; the resulting graph is a tree. There are $O\left(\frac{1}{\epsilon}\right)$ guesses and each for each guess, computing the diameter takes $O(|\mathcal{V}(\mathcal{X})|\epsilon^{-1})$ time, which implies $O(|\mathcal{V}(\mathcal{X})|\epsilon^{-1})$ time to compute $\text{Adm}(\mathcal{X})^{\Gamma}$. Thus, the total running time to compute the augmented diameter of every subgraph $\mathcal{X} \in \mathbb{X}$ is $\sum_{\mathcal{X} \in \mathbb{X}} O(|\mathcal{V}(\mathcal{X})|\epsilon^{-1}) = O(|\mathcal{V}_i|\epsilon^{-1})$.

### 5.3 Completing the Proof of Lemma 5.4

In this section, we complete the proof of Item (4) in Lemma 5.4. We consider two cases: (Case 1) $\mathcal{I}(\nu) \cap \mathcal{I}(\mu) = \emptyset$ and (Case 2) $\mathcal{I}(\nu) \cap \mathcal{I}(\mu) \neq \emptyset$. We reuse the notation in Lemma 5.4 here.

**Case 1**: $\mathcal{I}(\nu) \cap \mathcal{I}(\mu) = \emptyset$. Let $\mathcal{X} = (\nu, \mu) \cup \mathcal{I}(\nu) \cup \mathcal{I}(\mu)$ where $e = (\nu, \mu)$ is the only edge in $\mathcal{E}_i$ contained in $\mathcal{X}$. For any subgraph $\mathcal{Z}$ of $\mathcal{X}$, we define:

$$\Phi^+(\mathcal{Z}) = \sum_{\alpha \in \mathcal{Z}} \omega(\alpha) + \sum_{e' \in \text{MST}_i \cap \mathcal{E}(\mathcal{Z})} \omega(e') \quad (33)$$

It is possible to compute the augmented diameter of $\mathcal{X}$ in $O(|\mathcal{V}(\mathcal{X})|)$ time using a more involved approach.

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Figure 6: $\mathcal{D}$ is the diameter path and enclosed trees are augmented to a Step-4 subgraph in Step 5A. The green shaded regions contain nodes in $\mathcal{D}$. (a) $\mathcal{D}$ does not contain $e$. (b) $\mathcal{D}$ contains $e$. 

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to be the total weight of nodes and $\text{MST}_i$ edges in $\mathcal{Z}$. Let $\mathcal{D}$ be an augmented diameter path of $\mathcal{X}$, and $\mathcal{Y} = \mathcal{X} \setminus \mathcal{V}(\mathcal{D})$ be the subgraph obtained from $\mathcal{X}$ by removing nodes on $\mathcal{D}$. Let $\mathcal{I}(\nu)$ and $\mathcal{I}(\mu)$ be two intervals in the construction on Step 4 that are connected by an edge $e = (\nu, \mu)$.

**Claim 5.16.** $\Phi^+(\mathcal{Y}) = \frac{5L_i}{4} + \Omega((|\mathcal{V}(\mathcal{Y})|eL_i))$.

**Proof:** Let $A = \mathcal{Y} \setminus (\mathcal{I}(\nu) \cup \mathcal{I}(\mu))$ be the subgraph of $\mathcal{Y}$ obtained by removing every node in $\mathcal{I}(\nu) \cup \mathcal{I}(\mu)$ from $\mathcal{Y}$, and $B = \mathcal{Y} \cap (\mathcal{I}(\nu) \cup \mathcal{I}(\mu))$ be the subgraph of $\mathcal{Y}$ induced by nodes of $\mathcal{Y}$ in $(\mathcal{I}(\nu) \cup \mathcal{I}(\mu))$. Since every node has a weight of at least $L_{i-1}$ by property (P3'), we have

$$\Phi^+(A) \geq |\mathcal{V}(A)|eL_i$$

(34)

We consider two cases:

- **Case 1:** $\mathcal{D}$ does not contain the edge $(\nu, \mu)$. See Figure 6(a). In this case, $\mathcal{D} \subseteq \text{MST}_i$, and that $\mathcal{I}(\nu) \cap \mathcal{D} = \emptyset$ or $\mathcal{I}(\mu) \cap \mathcal{D} = \emptyset$ since $\mathcal{I}(\nu)$ and $\mathcal{I}(\mu)$ are connected only by $e$. Focusing on $\mathcal{I}(\nu)$ (w.l.o.g), since $\mathcal{I}(\nu) \subseteq \text{MST}_i$, $\Phi^+(B) \geq \text{Adm}(\mathcal{I}(\nu)) \geq (2 - (3g + 2)e)L_i$ by Claim 5.5.

- **Case 2:** $\mathcal{D}$ contains the edge $(\nu, \mu)$. See Figure 6(b). In this case at least two sub-intervals, say $I_1, I_2$, of four intervals $\{I(\nu) \setminus \nu, I(\mu) \setminus \mu\}$ are disjoint from $\mathcal{D}$. By Claim 5.5, then $\Phi^+(B) \geq \text{Adm}(I_1) + \text{Adm}(I_2) \geq (2 - 4(g + 1)e)L_i$ by Claim 5.5.

In both cases, $\Phi^+(B) \geq (2 - 4g + 1)e)L_i \geq \frac{3L_i}{2}$ when $\epsilon \ll \frac{1}{g}$.

By Claim 5.5, $|\mathcal{V}(B)| = O(\frac{1}{\epsilon})$. This implies that:

$$\Phi^+(\mathcal{Y}) = \Phi^+(A) + \Phi^+(B) \geq \Phi^+(A) + \frac{3L_i}{2} = \frac{5L_i}{4} + |\mathcal{V}(A)|(eL_i) + \frac{L_i}{4}$$

$$= \frac{5L_i}{4} + |\mathcal{V}(A)|(eL_i) + \Omega((|\mathcal{V}(B)|eL_i))$$

$$\geq \frac{5L_i}{4} + \Omega((|\mathcal{V}(A)| + |\mathcal{V}(B)|eL_i)) = \frac{5L_i}{4} + \Omega(|\mathcal{V}(\mathcal{Y})|eL_i),$$

which concludes the proof of Claim 5.16.  

Note that $\mathcal{V}(\mathcal{D}) \leq \frac{gL_i}{L_{i-1}} = O(\frac{1}{\epsilon})$ since every node has weight at least $L_{i-1}$ by property (P3'). Thus, we have:

$$\Delta^+(\mathcal{X}) = \Phi^+(\mathcal{D}) + \Phi^+(\mathcal{Y}) - \text{Adm}(\mathcal{X}) = \Phi(\mathcal{Y}) - \omega(e)$$

$$\geq L_i/4 + \Omega((|\mathcal{V}(\mathcal{Y})|eL_i)) \quad \text{(by Claim 5.16)}$$

$$= \Omega((|\mathcal{V}(\mathcal{D})|eL_i) + \Omega(|\mathcal{V}(\mathcal{Y})|eL_i) = \Omega(|\mathcal{V}(\mathcal{X})|eL_i).$$

Thus, Item (4) of Lemma 5.4 follows.

**Case 2:** $\mathcal{I}(\nu) \cap \mathcal{I}(\mu) \notin \emptyset$. Let $\mathcal{D}$ be a diameter path of $\mathcal{X}$, and $\mathcal{Y} = \mathcal{X} \setminus \mathcal{V}(\mathcal{D})$. Recall that $\mathcal{X}$ contains only one edge $e = (\nu, \mu) \in E_i$. Let $\mathcal{P}_e = (\nu, e, \mu)$ be the path that consists of only edge $e$ and its endpoints. Let $\mathcal{P}[\nu, \mu]$ be the subpath of $\text{MST}_i$ between $\nu$ and $\mu$.

We observe that $e$ is not removable by Item (3) of Definition 3.12 and by the fact that the path $\mathcal{P}[\nu, \mu]$ is a path in $\text{MST}_i$ that contains every node of degree at most 2 in $\text{MST}_i$ (see (2a) in Lemma 5.3), $\omega(\mathcal{P}[\nu, \mu]) \geq t(1 + 6ge)\omega(e) \geq (1 + 6ge)\omega(e)$ since $t \geq 1$. Then it follows that:

$$\omega(\mathcal{P}[\nu, \mu] - \omega(\mathcal{P}_e)) > 6ge \cdot \omega(e) - \omega(\nu) - \omega(\mu)$$

$$> 6geL_i/2 - 2geL_i = geL_i$$

(35)

In particular, this means that $\omega(\mathcal{P}(\nu, \mu)) \geq \omega(e)$. 46
Thus, if $\mathcal{D}$ contains both $\nu$ and $\mu$, then it must contain $e$, since otherwise, $\mathcal{D}$ must contain $\mathcal{P}[\nu, \mu]$ and by replacing $\mathcal{P}[\nu, \mu]$ by $\mathcal{P}_e$ we obtain a shorter path by Equation (35) (see Figure 7).

**Claim 5.17.** $|\mathcal{V}(\mathcal{P}[\nu, \mu])| \leq \frac{4}{\epsilon}$ and $|\mathcal{V}(\mathcal{D})| \leq \frac{9}{\epsilon}$.

**Proof:** Observe that $\text{Adm}(\mathcal{P}[\nu, \mu]) \leq 4L_i$ since $\mathcal{P}[\nu, \mu] \subseteq \mathcal{I}(\nu) \cup \mathcal{I}(\mu)$. Thus, $|\mathcal{V}(\mathcal{P}[\nu, \mu])| \leq \frac{4L_i}{L_{i-1}} = \frac{4}{\epsilon}$ since each node of $\mathcal{P}[\nu, \mu]$ has a weight of at least $L_{i-1}$ by property (P3'). Similarly, $\text{Adm}(\mathcal{D}) \leq gL_i$ by property (P3') while each node has a weight at least $L_{i-1}$. Thus, $|\mathcal{V}(\mathcal{D})| \leq \frac{gL_i}{L_{i-1}} = \frac{9}{\epsilon}$. $\square$

We consider two cases:

- **Case 1** If $\mathcal{D}$ does not contain edge $e$, then (a) $\mathcal{D} \subseteq \widehat{\text{MST}}_i$ and (b) $|\{\nu, \mu\} \cap D| \leq 1$. From (a), we have:
  \[
  \Delta_{i+1}^+(\mathcal{X}) \geq \text{Adm}(\mathcal{D}) + \Phi^+(\mathcal{Y}) - \text{Adm}(\mathcal{X}) = \Phi^+(\mathcal{Y}) \\
  \geq \text{Adm}(\mathcal{P}[\mu, \nu]) + \Phi^+(\mathcal{Y} \setminus \mathcal{P}[\mu, \nu]) \\
  \geq w(e) + |\mathcal{V}(\mathcal{Y} \setminus \mathcal{P}[\mu, \nu])|L_{i-1} \geq L_i/2 + |\mathcal{V}(\mathcal{Y} \setminus \mathcal{P}[\mu, \nu])|eL_i \\
  = \Omega((|\mathcal{V}(\mathcal{P}[\mu, \nu])| + |\mathcal{V}(\mathcal{D})|)eL_i) \quad \text{(by Claim 5.17)} \\
  = \Omega(|\mathcal{V}(\mathcal{Y})|eL_i)
  \]

- **Case 2** If $\mathcal{D}$ contains $e$, then $\mathcal{D} \cap \mathcal{P}(\nu, \mu) = \emptyset$; here $\mathcal{P}(\nu, \mu)$ is the path obtained from $\mathcal{P}[\nu, \mu]$ by removing its endpoints. It follows that
  \[
  \Delta_{i+1}^+(\mathcal{X}) \geq \text{Adm}(\mathcal{D}) + \Phi^+(\mathcal{Y}) - \text{Adm}(\mathcal{X}) = \Phi(\mathcal{Y}) - w(e) \\
  \geq \text{Adm}(\mathcal{P}[\mu, \nu]) + \Phi(\mathcal{Y} \setminus \mathcal{P}[\mu, \nu]) - w(e) \\
  \geq geL_i + |\mathcal{V}(\mathcal{Y} \setminus \mathcal{P}[\mu, \nu])|L_{i-1} \quad \text{(by Equation 35)} \\
  = \Omega((|\mathcal{V}(\mathcal{P}[\mu, \nu])| + |\mathcal{V}(\mathcal{D})|)e^2L_i) \quad \text{(by Claim 5.17)} \\
  = \Omega(|\mathcal{V}(\mathcal{X})|e^2L_i)
  \]

In both cases, we have $\Delta_{i+1}^+(\mathcal{X}) = \Omega(|\mathcal{V}(\mathcal{X})|e^2L_i)$ as claimed in Item (4) of Lemma 5.4.

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A Omitted Proofs

A.1 Proof of Lemma 3.3

Let $H$ be a graph with $V(H) = V(G)$ and $E(H) = E_{\text{light}} \cup \left( \bigcup_{\sigma \in [\mu, \psi]} H^\sigma \right)$. The fact that $H$ is a $k$-spanner of $G$ follows directly from Observation 2.1.

To show the lightness bound of $H$, we observe $\mu \psi = O\left(\frac{\log(1/\epsilon)}{\log(1+\psi)}\right)$. Thus, it holds that:

$$w\left(\bigcup_{\sigma \in [\mu, \psi]} H^\sigma\right) \leq (\text{Light}_{H^\sigma} \mu \psi) w(MST) = O\left(\frac{\text{Light}_{H^\sigma} \log(1/\epsilon)}{\log(1 + \psi)}\right) w(MST) = O\left(\frac{\text{Light}_{H^\sigma} \log(1/\epsilon)}{\psi}\right) w(MST).$$

Here we use the fact that $\log(1+x) \geq x$ when $x \in (0,1]$. The lightness bound of $H$ now follows from Observation 3.1.

To bound the running time, we observe that the time needed to construct $E_{\text{light}}$ is $T_{\text{MST}} + O(m) = O(T_{\text{MST}})$. Since we remove edges of weight at least MST from $G$ and every edge in $E_{\text{heavy}}$ has a weight at least $\frac{\psi}{\epsilon} = \frac{w(MST)}{\epsilon m}$, the number of sets that each $E^\sigma$ is partitioned to is $O\left(\log((1+\psi)/\epsilon) (\epsilon m)\right) = O(\log(m))$ for any $\epsilon \leq 1/2$. Thus, the partition of $E_{\text{heavy}}$ can be constructed in $O(m)$ time by a straightforward implementation. It follows from the assumption of Appendix A.1 that the running time to construct $H$ is:

$$\mu \psi \text{Time}_{H^\sigma}(n, m) + O(T_{\text{MST}}) + O(m) = O\left(\frac{\text{Time}_{H^\sigma}(n, m) \log(1/\epsilon)}{\log(1 + \psi)} + T_{\text{MST}}\right)$$

$$= O\left(\frac{\text{Time}_{H^\sigma}(n, m) \log(1/\epsilon)}{\psi} + T_{\text{MST}}\right),$$

as desired. □

B The Algebraic Computation Tree Model

In this appendix we give a brief description of the algebraic computation tree (ACT) model. (Refer to [9] and Chapter 3 in the book [63] for a more detailed description.)

An ACT is a binary computation tree where each leaf is associated with an output and each internal node is either (i) labeled with a variable $f_x$ determined by $f_x = a_1 \circ a_2$ or $f_x = \sqrt{a_1}$ where $\circ \in \{+, -, \times, \div\}$ and each $a_i, i \in \{1, 2\}$, is either a value of a proper ancestor of $x$, an input element or a constant in $\mathbb{R}$, or (ii) labeled with a comparison $a \succ 0$, where $a$ is either a value of a proper ancestor of $x$, or an input element, and the left (resp. right) child is labeled with “$\leq$” (resp. “$>$”). An ACT tree $T$ corresponds to an algorithm $A_T$, which traverses a path down the tree starting at the root and either (i) evaluates $f_x$ if the node has one child, or (ii) selects left or right child depending on the outcome of the comparison.
Figure 8: Applications of our framework in obtaining fast constructions of light spanners (on the left), which will be presented in this paper, and spanners with truly optimal lightness (on the right), which will be presented in our follow-up work.

When $A_T$ reaches the leaf, it evaluates the expression by replacing all the variables with the input values and terminates. It is required that no input lead to undefined behavior, such as division by 0, or taking
a square root of a negative number; furthermore, for each leaf $w$ there should be an input on which $A_T$ terminates in $w$. A problem $P$ is solvable in the ACT model if there exists an ACT tree $T$ such that for any valid input for $P$, $A_T$ returns the value of $P$ on that input.

The ACT model is particularly convenient for proving lower bounds, as it encompasses and represents explicitly all possible execution paths of an algorithm. For most algorithmic purposes, one can alternatively consider the real RAM model; one can prove an equivalence between the two models, which holds up to some subtle issues of uniformity that lie outside the scope of the current paper.