On the spherical magnetic trajectories

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Abstract. We consider spherical indicatrix magnetic trajectories of a magnetic field in Euclidean 3−space. From classical formulation of Killing magnetic flow equations, we derive the differential equation systems for tangent spherical indicatrix magnetic trajectories in Euclidean 3−space. Then we solve these equations by using Jacobi elliptic functions. Finally, we make similar calculations for curves whose principal normal and binormal spherical indicatrix are magnetic curves.

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1. Introduction

Any magnetic vector field is known divergence zero vector field in three dimensional spaces. A magnetic trajectory of a magnetic flow created by magnetic vector field are curves called as magnetic curves. Although the problem of investigating magnetic trajectories appears to be physical problem, recent studies show that the characterization of magnetic flow in a magnetic field have brought variational perspective in more geometrical manner [2, 8]. Let S be a surface in Euclidean 3−space R3 and F denote a complete differential 2−form in a open subset U of S. Then we can write F = dω for some potential 1−form ω. If we define Γ as smooth curves that connect two fixed point of U, the Lorentz force equation is known a minimizer of the functional \( \mathcal{L}: \Gamma \rightarrow \mathbb{R} \)

\[
\mathcal{L}(\gamma) : = \frac{1}{2} \int_{\gamma} \left< \gamma', \gamma'' \right> dt + \omega(\gamma') dt.
\] (1.1)

The Euler-Lagrange equation of the functional \( \mathcal{L} \) is derived as

\[
\phi(\gamma') = \nabla_{\gamma'} \gamma',
\] (1.2)

where \( \phi \) is the skew-symmetric operator. The critical point of the functional \( \mathcal{L} \) corresponds to the Lorentz force equation [2, 4].

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Spherical magnetic trajectories

Any function defined from a space curve to a suitable sphere in Euclidean 3-space is called the spherical indicatrix (spherical image) of the curve. The spherical indicatrix of a curve in Euclidean 3-space emerges in three types: the tangent indicatrix (tangential indicatrix or tangent spherical indicatrix), the principle normal indicatrix and the binormal indicatrix of the curve. The spherical indicatrix is a nice way to envision the motion of the curve on a sphere by using components of the Frenet Frame. Furthermore, the movement of a spherical indicatrix describes the changes in the original direction of the curve [6, 7].

In this paper we consider the magnetic trajectories which are the tangent, principal and binormal indicatrices, separately. We first investigate the tangent indicatrix magnetic trajectories and we derive the Killing magnetic flow equations for tangent indicatrix magnetic vector field. Then we solve these equations by using elliptic functions. Then we apply this method the other imagine types of curves by using same calculations. But we do not dwell on variational and differential calculations of the problem of finding curves whose principle normal and binormal indicatrix are magnetic since the same procedure would repeat.

2. Preliminaries

We consider a regular curve \( \gamma \) in Euclidean 3-space \( \mathbb{R}^3 \), parametrized by arc length \( s \), \( 0 \leq s \leq \ell \). Let \( T = \gamma' (s) \) denote the unit tangent vector field, \( N (s) \) the unit principle normal vector field and \( B = T \times N \) binormal vector field at point \( \gamma (s) \). Then we have the Frenet frame \( \{ T, N, B \} \) along the curve \( \gamma \) and Frenet equations given by

\[
\begin{pmatrix}
T' \\
N' \\
B'
\end{pmatrix} = \begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B
\end{pmatrix},
\]

where \( \kappa > 0 \) and \( \tau \) are respectively curvature and torsion of \( \gamma \) [6].

If the Frenet frame of the tangent indicatrix \( \gamma_t = T \) of a space curve \( \gamma \) is \( \{ T_t, N_t, B_t \} \), then we have the following Frenet equations

\[
\begin{pmatrix}
T_t' (s_t) \\
N_t' (s_t) \\
B_t' (s_t)
\end{pmatrix} = \begin{pmatrix}
0 & \kappa_t & 0 \\
-\kappa_t & 0 & \tau_t \\
0 & -\tau_t & 0
\end{pmatrix}
\begin{pmatrix}
T_t \\
N_t \\
B_t
\end{pmatrix},
\]

where

\[
T_t = N, \ N_t = \frac{-T + fB}{\sqrt{1 + f^2}}, \ B_t = \frac{fT + B}{\sqrt{1 + f^2}}
\]

and

\[
s_t = \int \kappa (s) \, ds, \ \kappa_t = \sqrt{1 + f^2}, \ \tau_t = \sigma \sqrt{1 + f^2},
\]

where

\[
f (s) = \frac{\tau (s)}{\kappa (s)} \quad \text{and} \quad \sigma = \frac{f' (s)}{\kappa (s) (1 + f^2)^{3/2}} = \frac{\tau_t}{\kappa_t}.
\]

\( \sigma \) is the geodesic curvature of the principal image of the principal normal indicatrix of the curve \( \gamma \), \( s_t \) is natural representation of the tangent indicatrix of the curve \( \gamma \) and equal the total curvature of the curve \( \gamma \) and \( \kappa_t \) and \( \tau_t \) are the curvature and torsion of \( \gamma_t \).

If the Frenet frame of the normal indicatrix \( \gamma_n = N \) of a space curve \( \gamma \) is \( \{ T_n, N_n, B_n \} \), then we have the following Frenet equations

\[
\begin{pmatrix}
T_n' (s_n) \\
N_n' (s_n) \\
B_n' (s_n)
\end{pmatrix} = \begin{pmatrix}
0 & \kappa_n & 0 \\
-\kappa_n & 0 & \tau_n \\
0 & -\tau_n & 0
\end{pmatrix}
\begin{pmatrix}
T_n \\
N_n \\
B_n
\end{pmatrix},
\]

\( T_n = \gamma, \ N_n = -fB, \ B_n = fT + B \).
where

\[
T_n = \frac{-T + fB}{\sqrt{1 + f^2}}, \quad N_t = \frac{\sigma}{\sqrt{1 + \sigma^2}} \left[ \frac{-T + fB}{\sqrt{1 + f^2}} - \frac{N}{\sigma} \right],
\]

\[
B_t = \frac{1}{\sqrt{1 + \sigma^2}} \left[ \frac{fT + B}{\sqrt{1 + f^2}} + \sigma N \right],
\]

and

\[
s_n = \int \kappa(s) \left( \sqrt{1 + f^2(s)} \right) ds, \quad \kappa_n = \sqrt{1 + \sigma^2}, \quad \tau_t = \Gamma \sqrt{1 + \sigma^2},
\]

where

\[
\Gamma = \frac{\sigma'(s)}{\kappa(s) \sqrt{(1 + f^2)(1 + \sigma^2)^{3/2}}} = \frac{\tau_k}{\kappa_n},
\]

where \( s_n \) is natural representation of the principal normal indicatrix of the curve \( \gamma \) and \( \kappa_n \) and \( \tau_n \) are the curvature and torsion of \( \gamma_n \).

If the Frenet frame of the binormal indicatrix \( \gamma_b = N \) of a space curve \( \gamma \) is \( \{T_b, N_b, B_b\} \), then we have Frenet formula:

\[
\begin{pmatrix}
T_b'(s_b) \\
N_b'(s_b) \\
B_b'(s_b)
\end{pmatrix}
= \begin{pmatrix}
0 & \kappa_b & 0 \\
-\kappa_b & 0 & \tau_b \\
0 & -\tau_b & 0
\end{pmatrix}
\begin{pmatrix}
T_b \\
N_b \\
B_b
\end{pmatrix},
\]

(2.7)

where

\[
T_b = -N, \quad N_t = \frac{T - fB}{\sqrt{1 + f^2}}, \quad B_t = \frac{fT + B}{\sqrt{1 + f^2}},
\]

and

\[
s_b = \int \tau(s) ds, \quad \kappa_b = \frac{\sqrt{1 + f^2}}{f}, \quad \tau_b = -\sigma \frac{\sqrt{1 + f^2}}{f},
\]

where

\[
\sigma = \frac{\tau_b}{\kappa_b},
\]

where \( s_b \) is the natural representation of the binormal indicatrix of the curve \( \gamma \) and \( \kappa_b \) and \( \tau_b \) are the curvature and torsion of \( \gamma_b \) [1].

### 3. Spherical indicatrix magnetic fields

Let \( V \) be a divergence-free vector field in Euclidean 3-space \( \mathbb{R}^3 \). Then it defines a magnetic vector field. Given a differential 2–form \( F \) is a magnetic field on \( \mathbb{R}^3 \). The Lorentz force of \( F \) is defined to be the skew-symmetric operator \( \phi \) given by

\[
< \phi(X), Y > = F(X, Y)
\]

for all vector field \( X, Y \in \chi(\mathbb{R}^3) \). The associated magnetic trajectories are curves \( \gamma \) on \( \mathbb{R}^3 \) that satisfies the Lorentz force equation (1.2). On the other hand the Lorentz force \( \phi \) can be write as follows

\[
\phi(X) = V \times X,
\]

(3.2)

that is, the Lorentz force \( \phi \) of \( V \) is defined via cross product on \( \mathbb{R}^3 \). Combining (1.2) and (3.2), the Lorentz equation can be written by

\[
\phi(\gamma') = \nabla_\gamma \gamma' = V \times \gamma'
\]

for a curve \( \gamma \) on \( \mathbb{R}^3 \).

By means of these structures defined on \( \mathbb{R}^3 \), the Killing magnetic flow equations corresponding to spherical indicatrix for a unit-speed curve \( \gamma \) on \( \mathbb{R}^3 \) will be found.
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Let \( \gamma : I \subset \mathbb{R} \to \mathbb{R}^3 \) be a reparametrized curve in Euclidean 3-space and \( \{T_t, N_t, B_t\} \) is the Frenet frame along \( \gamma_t \). Then the Lorentz force in the frame \( \{T_t, N_t, B_t\} \) is written as

\[
\phi(T_t) = \kappa_t N_t, \tag{3.3}
\]

\[
\phi(N_t) = -\kappa_t T_t + \omega_t B_t \tag{3.4}
\]

and

\[
\phi(B_t) = -\omega_t B_t, \tag{3.5}
\]

where the function \( \omega_t (s_t) \) associated with each tangent indicatrix magnetic curve is quasislope measured with respect to the magnetic field \( V_t \).

Then we can give the following propositions.

**Proposition 3.1.** The tangential indicatrix \( \gamma_t \) is a magnetic trajectory of a magnetic field \( V_t \) if and only if \( V_t \) can be written along \( \gamma_t \) as

\[
V_t = \omega_t T_t + \kappa_t B_t. \tag{3.6}
\]

**Proof.** Assume that \( \gamma_t \) is a magnetic curve along a magnetic field \( V_t \) and the orthogonal frame along \( \gamma_t \) is given by \( \{T_t, N_t, B_t\} \). Then, \( V_t \) can be written as

\[
V_t = \langle V_t, T_t \rangle T_t + \langle V_t, N_t \rangle N_t + \langle V_t, B_t \rangle B_t.
\]

To find coefficient of \( V_t \), we use the Lorentz force in orthogonal frame equations (3.3), (3.4) and (3.5):

\[
\omega_t = \langle \phi(N_t), B_t \rangle = \langle V_t \times N_t, B_t \rangle = \langle V_t, T_t \rangle,
\]

\[
0 = \langle \phi(T_t), B_t \rangle = \langle V_t \times T_t, B_t \rangle = -\langle V_t, B_t \rangle
\]

and

\[
\kappa_t = \langle \phi(T_t), N_t \rangle = \langle V_t \times T_t, N_t \rangle = \langle V_t, B_t \rangle.
\]

**Proposition 3.2.** The principle indicatrix \( \gamma_n \) is a magnetic trajectory of a magnetic field \( V_n \) if and only if \( V_n \) can be written along \( \gamma_n \) as

\[
V_n = \omega_n T_n + \kappa_n B_n.
\]

**Proposition 3.3.** The binormal indicatrix \( \gamma_b \) is a magnetic trajectory of a magnetic field \( V_b \) if and only if \( V_b \) can be written along \( \gamma_b \) as

\[
V_b = \omega_b T_b + \kappa_b B_b.
\]

4. Killing magnetic flow equations for spherical indicatrix magnetic curves

Let \( \gamma_t \) be a tangential indicatrix of \( \gamma \) in \( \mathbb{R}^3 \) and \( V_t \) be a vector field along that curve. One can take a variation of \( \gamma_t \) in the direction of \( V_t \), say a map

\[
\Gamma : [0, 1] \times (-\varepsilon, \varepsilon) \to S^2 \quad (s, w) \to \Gamma(s, w)
\]

which satisfies

\[
\Gamma'(s, 0) = \gamma_t(s), \quad \left( \frac{\partial \Gamma(s, w)}{\partial w} \right)_{w=0} = V_t(s),
\]

and

\[
\left( \frac{\partial \Gamma(s, w)}{\partial s} \right)_{w=0} = \gamma_t'(s).
\]

One can write the speed function \( v_t(s, w) = \left\| \frac{\partial \Gamma(s, w)}{\partial s} \right\| \), the curvature function \( \kappa_t(s, w) \) and the torsion function \( \tau_t(s, w) \) [2, 5].
Lemma 4.1 (see [2, 3]). Let \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^3 \) be a curve in \( \mathbb{R}^3 \), \( \gamma_t \) denote the tangent indicatrices and \( V_t \) be a vector field along the curve \( \gamma_t \). Then we have the following equalities

\[
V_t (v_t) = \left( \frac{\partial v_t(s, w)}{\partial w} \right)_{w=0} = \nabla_{T_t} V_t, T_t > v_t, \tag{4.1}
\]

\[
V_t (\kappa_t) = \left( \frac{\partial \kappa_t(s, w)}{\partial w} \right)_{w=0} = \frac{1}{\kappa_t} \left< \nabla^2_{T_t} V_t, \nabla_{T_t} T_t > - 2\kappa_t < \nabla_{T_t} V_t, T_t > \right. \tag{4.2}
\]

and

\[
V_t (\tau_t) = \left( \frac{\partial \tau_t(s, w)}{\partial w} \right)_{w=0} = \left( \frac{1}{\kappa_t^2} \left< \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t > \right. \right) \right.
\]

\[
+ \tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > . \tag{4.3}
\]

Proposition 4.1. Let \( V_t \) be the restriction to the tangent indicatrix \( \gamma_t \) of a Killing vector field, say \( V_t \) of \( \mathbb{R}^3 \); then

\[
V_t (v_t) = V_t (\kappa_t) = V_t (\tau_t) = 0. \tag{4.4}
\]

Then we can give the Killing magnetic flow equations of the tangential indicatrix.

**Theorem 4.1.** Let \( \gamma_t \) be the tangential indicatrix of a regular curve \( \gamma \). Suppose that \( V_t = \omega_t T_t + \kappa_t B_t \) is a Killing vector field along \( \gamma_t \). Then the tangential indicatrix magnetic trajectories are curves on \( S^2 \) satisfying following differential equations

\[
\kappa_t^2 \left( \frac{1}{2} \omega_t - \tau_t \right) = A_1. \tag{4.5}
\]

and

\[
\kappa_t'' + \kappa_t \tau_t (\omega_t - \tau_t) + C \kappa_t + \frac{1}{2} \kappa_t^3 - A_2 \kappa_t = 0, \tag{4.6}
\]

where \( A_1, A_2 \) and \( C \) are undetermined constants.

**Proof.** Assume that \( V_t \) is a Killing vector field along \( \gamma_t \) on \( S^2 \). Along any spherical magnetic trajectory \( \gamma_t \), we have \( V_t = \omega_t T_t + \kappa_t B_t \). If \( V_t \) is Killing vector field, we calculate

\[
\omega_t' = 0,
\]

that is \( \omega_t \) is a constant, and

\[
\nabla_{T_t} V_t = \kappa_t (\omega_t - \tau_t) N_t + \kappa_t' B_t. \tag{4.7}
\]

By using the first derivative of (4.7), (4.2) and (4.4), we get

\[
\left( \kappa_t^2 \left( \frac{1}{2} \omega_t - \tau_t \right) \right)' = 0.
\]

Similarly, from (4.2) and (4.4), we find to \( V (\tau) \) as follows

\[
V_t (\tau_t) = \left( \frac{\partial \tau_t(s, w)}{\partial w} \right)_{w=0} = \left( \frac{1}{\kappa_t^2} \left< \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t > \right. \right)
\]

\[
+ \tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > . \tag{4.8}
\]

**Definition 4.1.** Any tangent indicatrix of a Euclidean curve is called the tangent indicatrix magnetic trajectory of a magnetic field \( V_t \) if it satisfies the differential equation system (4.5) and (4.6).

We can combine Eqs. (4.5) and (4.6) as follows

\[
\kappa_t'' + \frac{1}{2} \kappa_t^3 + \left( C - A_2 + \frac{1}{4} \omega_t^2 \right) \kappa_t - \frac{A_1^2}{\kappa_t^3} = 0.
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This equation admits an obvious first integral. In fact, just multiply by \(2\kappa'_t\) and integrate to get

\[
(\kappa'_t)^2 + \frac{1}{4}\kappa^4 + \left(C - A_2 + \frac{1}{4}\omega^2\right)\kappa^2 - \frac{A_1^2}{\kappa^2} = A_3.
\]

Since this equation is of the type \((u')^2 = P(h)\), where \(P\) is a polynomial of degree 3 in \(u\), it can be solved using elliptic functions as follows

\[
\kappa_t(s) = \sqrt{a_3 \left(1 - q^2 \text{sn}(rs, p)\right)},
\]

\[
\tau_t(s) = \frac{1}{2}\omega_t - \frac{A_1}{\kappa_t^2},
\]

when \(\kappa_t \neq \text{const.}\), where

\[
(u'_t)^2 + (u - a_1)(u - a_2)(u - a_3) = 0, \quad u = \kappa_t^2,
\]

\[
p = \frac{a_3 - a_2}{a_3 - a_1}, \quad q^2 = \frac{a_3 - a_2}{a_3} \quad \text{and} \quad r = \frac{1}{2}\sqrt{a_3 - a_1}.
\]

So, the curvature and the curvature of \(\gamma\) must satisfy the equations

\[
\frac{\tau}{\kappa} = \sqrt{a_3 \left(1 - q^2 \text{sn}(rs, p)\right)} - 1,
\]

and

\[
\left(\frac{\tau}{\kappa}\right)' = \kappa_t \left(1 + \frac{\tau^2}{\kappa^2}\right) \left(1 + \omega_t - \frac{1}{\sqrt{a_3 \left(1 - q^2 \text{sn}(rs, p)\right)}}\right).
\]

Making similar calculations we can give the Killing magnetic flow equations of the principle normal and binormal indicatrix.

**Theorem 4.2.** Let \(\gamma_n\) be the normal indicatrix of a regular curve \(\gamma\). Suppose that \(V_n = \omega_nT_n + \kappa_nB_n\) is a Killing vector field along \(\gamma_n\). Then the normal indicatrix magnetic trajectories are curves on \(S^2\) satisfying following differential equations

\[
\kappa_n^2 \frac{\left(1}{2}\omega_n - \tau_n\right) = A_4. \quad (4.8)
\]

and

\[
\kappa''_n + \kappa_n\tau_n \left(\omega_n - \tau_n\right) + C_1\kappa_n + \frac{1}{2}\kappa^3_n - A_5\kappa_n = 0, \quad (4.9)
\]

where \(A_4, A_5\) and \(C_1\) are undetermined constants.

**Theorem 4.3.** Let \(\gamma_b\) be the binormal indicatrix of a regular curve \(\gamma\). Suppose that \(V_b = \omega_bT_b + \kappa_bB_b\) is a Killing vector field along \(\gamma_b\). Then the binormal indicatrix magnetic trajectories are curves on \(S^2\) satisfying following differential equations

\[
\kappa_b^2 \frac{\left(1}{2}\omega_b - \tau_b\right) = A_6. \quad (4.10)
\]

and

\[
\kappa''_b + \kappa_b\tau_b \left(\omega_b - \tau_b\right) + C_2\kappa_b + \frac{1}{2}\kappa^3_b - A_7\kappa_b = 0, \quad (4.11)
\]

where \(A_6, A_7\) and \(C_2\) are undetermined constants.

**Definition 4.1.** Any principle normal (binormal) indicatrix of a Euclidean curve is called the principle (binormal) indicatrix magnetic trajectory of a magnetic field \(V_n\) (\(V_b\)) if it satisfies the differential equation system (4.8) and (4.9) (resp., (4.10) and (4.11)).

**Example 4.1.** We consider a unit-speed circular helix \(\beta(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, -\frac{s}{\sqrt{2}}\right)\) [2]. The curve \(\beta(s)\) can be seen on Fig. 1.
Curvature and torsion of $\beta (s)$ are found as $\kappa = -\tau = \frac{1}{2}$. Then, tangent indicatrix of the circular helix is

$$\beta_t \approx \beta'(s) = \left( -\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

$\beta_t$ is a circle cut from unit sphere by the plane $z = -\frac{1}{\sqrt{2}}$. The curvature and the torsion of the tangent indicatrix of the circular helix are found as $\kappa_t = \sqrt{2}, \tau_t = 0$. We can see from (4.5) and (4.6), the tangent indicatrix $\beta_t$ of $\beta$ is a tangent indicatrix magnetic trajectory with $A_2 = C + 1$ of the Killing magnetic field $V_t = A_1 T_t + \sqrt{2} B_t$.

The principal normal indicatrix of the circular helix is

$$\beta_n \approx N(s) = \left( -\cos \frac{s}{\sqrt{2}}, -\sin \frac{s}{\sqrt{2}}, 0 \right),$$

that is, $\beta_n$ lies on the great circle lines on the sphere with $\kappa_n = 1$ and $\tau_n = 0$. From (4.8) and (4.9), we show that $\beta_n$ is a principle normal indicatrix magnetic trajectory with $A_5 = C_1 + \frac{1}{2}$ of the Killing magnetic field $V_t = 2A_1 T_n + B_n$. Finally, the binormal indicatrix of the circular helix is

$$\beta_b \approx B(s) = \left( -\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sqrt{2} \right),$$

that is, $\beta_b$ is a circle cut from unit sphere by the plane $y = \frac{1}{\sqrt{2}}$. From (4.10) and (4.11), $\beta_b$ is a binormal indicatrix magnetic trajectory with $A_7 = C_2 + 1$ of the Killing magnetic field $V_b = A_6 T_b + \sqrt{2} B_b$. The graphs of $\beta_t, \beta_n$ and $\beta_b$ are given as follows.
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