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Algebraic Rainich conditions for the fourth rank tensor V

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Abstract. Algebraic conditions on the Ricci tensor in the Rainich-Misner-Wheeler unified field theory are known as the Rainich conditions. Penrose and more recently Bergqvist and Lankinen made an analogy from the Ricci tensor to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$, a certain fourth rank tensor quadratic in the Weyl curvature, which also satisfies algebraic Rainich-like conditions. However, we found that not only does the tensor $B_{\alpha\beta\mu\nu}$ fulfill these conditions, but so also does our recently proposed tensor $V_{\alpha\beta\mu\nu}$, which has many of the desirable properties of $B_{\alpha\beta\mu\nu}$. For the quasilocal small sphere limit restriction, we found that there are only two fourth rank tensors, $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$, which form a basis for good energy expressions. Both of them have the completely trace free and causal properties, these two form necessary and sufficient conditions. Surprisingly either completely traceless or causal is enough to fulfill the algebraic Rainich conditions.

1. Introduction

In 1925 Rainich proposed a unified field theory for source-free electromagnetism and gravitation [1]. Misner and Wheeler [2] 32 years later proposed a geometrically unified theory, based on the Rainich idea, now called the Rainich-Misner-Wheeler theory. The necessary conditions are called the Rainich conditions. The algebraic Rainich conditions refer to the Ricci tensor, but this tensor can be replaced by other tensors. Penrose [3] and more recently Bergqvist and Lankinen [4, 5] made an analogy from the Ricci tensor to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$, a certain fourth rank tensor quadratic in the Weyl curvature, which also satisfies algebraic Rainich-like conditions. We found that not only does the tensor $B_{\alpha\beta\mu\nu}$ fulfill these conditions, but so also does our recently proposed tensor $V_{\alpha\beta\mu\nu}$, which has many of the desirable properties of $B_{\alpha\beta\mu\nu}$. We define the Bel-Robinson tensor as follows

$$B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma}R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma}R_{\beta\mu}^{\lambda\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\mu\nu}R^2,$$  \hspace{1cm} (1)

where $R^2 = R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}$. Here we propose [6] an unique alternative 4th rank tensor for the non-negative gravitational energy in the quasilocal limit

$$V_{\alpha\beta\mu\nu} := B_{\alpha\beta\mu\nu} + W_{\alpha\beta\mu\nu} \neq B_{\alpha\beta\mu\nu},$$ \hspace{1cm} (2)

in general, where

$$W_{\alpha\beta\mu\nu} := \frac{3}{2}(R_{\alpha\mu\lambda\sigma}R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}^{\lambda\sigma}) + \frac{1}{8}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - 2g_{\alpha\beta}g_{\mu\nu})R^2.$$ \hspace{1cm} (3)
Analog of the electric $E_{ab}$ and magnetic part $H_{ab}$ are defined in terms of the Weyl tensor

$$ E_{ab} := C_{a0b0}, \quad H_{ab} := *C_{a0b0}, \quad a, b = 1, 2, 3. \quad (4) $$

The positive gravitation energy density for both $B_{\alpha \beta \mu \nu}$ and $V_{\alpha \beta \mu \nu}$ are

$$ B_{0000} \equiv V_{0000} = E_{ab} E^{ab} + H_{ab} H^{ab} \geq 0. \quad (5) $$

In the small region $P_\mu \propto B_{\mu000} = V_{\mu000} = (E_{ab}^2 + H_{ab}^2, 2\epsilon_{abc} E^{ad} H^b_d)$. Since we want $P^0 \geq |\vec{P}| \geq 0$, then $E_{ab}^2 + H_{ab}^2 \geq 2\epsilon_{abc} E^{ad} H^b_d$, we call this the causal condition.

2. Algebraic Rainich conditions

In 1925 Rainich proposed a unified field theory for the geometrization of the electromagnetic field. The basic idea is from the Einstein-Maxwell field equations:

$$ R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = -8\pi \left( F_{\alpha \lambda} F^\lambda_\beta - \frac{1}{4} g_{\alpha \beta} F_{\rho \lambda} F^{\rho \lambda} \right). \quad (6) $$

where $R_{\alpha \beta}$ is the 2nd rank Ricci tensor and $F_{\alpha \beta}$ is the electromagnetic field tensor. Taking the trace of (6), one finds $R^\alpha_\alpha = R = 0$. Then rewrite (6)

$$ R_{\alpha \beta} = -8\pi \left( F_{\alpha \sigma} F^\sigma_\beta - \frac{1}{4} g_{\alpha \beta} F_{\rho \sigma} F^{\rho \sigma} \right). \quad (7) $$

Using this, one can obtain $R_{\alpha \lambda} R^\lambda_\beta = \frac{1}{4} g_{\alpha \beta} R_{\rho \lambda} R^{\rho \lambda}$. Therefore the algebraic Rainich conditions can be written as

$$ R_{\alpha \lambda} R^\lambda_\beta = \frac{1}{4} g_{\alpha \beta} R_{\rho \lambda} R^{\rho \lambda}, \quad R^\alpha_\alpha = 0, \quad R_{00} \geq 0, \quad (8) $$

where the last condition is to ensure positive energy.

3. Fourth rank Algebraic Rainich conditions

Penrose replaced the Ricci tensor by stress tensor $T_{\alpha \beta}$ using the Einstein field equation $R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \kappa T_{\mu \nu}$. Rewrite the algebraic Rainich conditions as follows

$$ T_{\alpha \lambda} T^\lambda_\beta = \frac{1}{4} g_{\alpha \beta} T_{\rho \lambda} T^{\rho \lambda}, \quad T^\lambda_\lambda = 0, \quad T_{00} \geq 0, \quad (9) $$

Furthermore, analog from 2nd to 4th rank tensor, the Bel-Robinson tensor is the first tensor satisfying the algebraic Rainich conditions [3, 4],

$$ B_{\alpha \lambda \sigma \tau} B^\lambda_\sigma^\tau = \frac{1}{4} g_{\alpha \beta} B_{\rho \lambda \sigma \tau} B^{\rho \lambda \sigma \tau}, \quad 0 = B^\rho_\rho \sigma \tau = B^\rho_\sigma \rho \tau = \ldots, \quad B_{0000} \geq 0, \quad (10) $$

where $B_{\alpha \lambda \sigma \tau} \equiv B_{(\alpha \lambda \sigma \tau)}$. Moreover, our recently proposed $V_{\alpha \beta \mu \nu}$ also satisfies these algebraic Rainich conditions

$$ V_{\alpha \lambda \sigma \tau} V^\lambda_\sigma^\tau = \frac{1}{4} g_{\alpha \beta} V_{\rho \lambda \sigma \tau} V^{\rho \lambda \sigma \tau}, \quad 0 = V^\rho_\rho \sigma \tau = V^\rho_\sigma \rho \tau = \ldots, \quad V_{0000} \geq 0, \quad (11) $$

where $V_{\alpha \lambda \sigma \tau} \equiv V_{(\alpha \lambda)(\sigma \tau)} \equiv V_{\sigma \tau \alpha \lambda}$ and $V^\rho_\rho \sigma \tau \equiv 0 \equiv V^\rho_\sigma \rho \tau$, but $V_{\alpha \lambda \sigma \tau} \neq V_{(\alpha \lambda \sigma \tau)}$. We will show in section (IV) that the first condition indicated in (10) and (11) is an identity.
4. Two basic components of the fourth rank tensor
Edgar and Wingbrant [7] found the one-quarter identity

\[ R_{\alpha\xi\lambda\kappa}R^{\xi}_{\quad \tau \gamma}R^{\lambda}_{\quad \pi \tau} = \frac{1}{4} g_{\alpha\beta} R_{\phi\xi\lambda\kappa}R^{\phi}_{\quad \tau \gamma}R^{\lambda}_{\quad \pi \tau}, \]  
(12)

\[ R_{\alpha\xi\lambda\kappa}R^{\xi}_{\quad \tau \gamma}R^{\lambda}_{\quad \pi \gamma} = \frac{1}{4} g_{\alpha\beta} R_{\phi\xi\lambda\kappa}R^{\phi}_{\quad \tau \gamma}R^{\lambda}_{\quad \pi \gamma}. \]  
(13)

We found that the above two are not the only cases that satisfy the one-quarter property, but for any combination

\[ M_{\alpha\lambda\sigma \rho} M^{\lambda\sigma \tau} = \frac{1}{4} g_{\alpha\beta \mu \nu} M_{\rho \lambda \sigma \tau} M^{\rho \lambda \sigma \tau}, \]  
(14)

where \( M_{\alpha\lambda\sigma \rho} \) is any tensor quadratic in the Riemann curvature, non-vanishing in vacuum. For any 4th rank tensor, Deser [8] stated that there are only two basic components. For example, \((R_{\rho \lambda \sigma \tau} R^{\phi \lambda \sigma \tau})^2 \) and \((R_{\rho \lambda \sigma \tau} * R^{\phi \lambda \sigma \tau})^2 \). Iihoshi [9] found

\[ 4B_{\rho \lambda \sigma \tau} B^{\phi \lambda \sigma \tau} = (R_{\rho \lambda \sigma \tau} R^{\phi \lambda \sigma \tau})^2 + (R_{\rho \lambda \sigma \tau} * R^{\phi \lambda \sigma \tau})^2. \]  
(15)

Moreover, we found

\[ 2B_{\rho \lambda \sigma \tau} B^{\phi \lambda \sigma \tau} = (R_{\rho \lambda \sigma \tau} R^{\phi \lambda \sigma \tau})^2 - 2R_{\rho \lambda \xi \kappa} R_{\sigma \tau} \xi R^{\phi \lambda \mu \nu} R^{\sigma \tau \mu \nu}, \]  
(16)

which does not contain ‘2’ at the last term in [8].

5. Completely traceless imply positivity or causal
There are 4 basic quadratic Riemann tensors in the small region limit [10], we use our own notation

\[ \tilde{B}_{\alpha\beta \mu \nu} = R_{\alpha \lambda \mu \nu} R_{\beta}^{\quad \lambda \sigma} + R_{\alpha \lambda \sigma \nu} R_{\beta}^{\quad \mu \lambda}, \quad \tilde{K}_{\alpha \beta \mu \nu} = R_{\alpha \lambda \beta \sigma} R_{\mu \rho}^{\quad \lambda \sigma} + R_{\alpha \lambda \beta \sigma} R_{\nu \rho}^{\quad \lambda \sigma}, \]
\[ \tilde{S}_{\alpha \beta \mu \nu} = R_{\alpha \mu \lambda \sigma} R_{\beta \nu}^{\quad \lambda \sigma} + R_{\alpha \nu \lambda \sigma} R_{\beta \mu}^{\quad \lambda \sigma}, \quad \tilde{T}_{\alpha \beta \mu \nu} = -\frac{1}{8} g_{\alpha \beta \mu \nu} R_{\rho \lambda \sigma \tau} R^{\rho \lambda \sigma \tau}. \]  
(17)

We let

\[ A_{\alpha \beta \mu \nu} := a_1 \tilde{B}_{\alpha \beta \mu \nu} + a_2 \tilde{S}_{\alpha \beta \mu \nu} + a_3 \tilde{K}_{\alpha \beta \mu \nu} + a_4 \tilde{T}_{\alpha \beta \mu \nu}, \]  
(18)

and consider all the traces. It turns out because of the symmetry there are only two distinct traces

\[ 0 = 2 A_{\alpha \mu \nu} (a_1 + a_2 - a_4) g_{\mu \nu}, \]  
(19)

\[ 0 = 8 A_{\alpha \mu \nu \sigma} (a_1 - 2a_2 + 3a_3 - a_4) g_{\mu \nu}, \]  
(20)

where \( a_1 \) to \( a_4 \) are constants. The solutions are \( a_4 = a_1 + a_2 \) and \( a_2 = a_3 \). Consider the linear combination of these 4 fundamental tensors, we found \( B_{\alpha \beta \mu \nu} \) and \( V_{\alpha \beta \mu \nu} \) are the unique tensors that have the completely traceless property and imply positivity

\[ A_{\alpha \beta \mu \nu} = a_1 (\tilde{B}_{\alpha \beta \mu \nu} + \tilde{T}_{\alpha \beta \mu \nu}) + a_2 (\tilde{S}_{\alpha \beta \mu \nu} + \tilde{K}_{\alpha \beta \mu \nu} + \tilde{T}_{\alpha \beta \mu \nu}) \]
\[ = a_1 B_{\alpha \beta \mu \nu} + a_2 V_{\alpha \beta \mu \nu}, \]  
(21)

using \( B_{\alpha \beta \mu \nu} = \tilde{B}_{\alpha \beta \mu \nu} + \tilde{T}_{\alpha \beta \mu \nu} \) and \( V_{\alpha \beta \mu \nu} = \tilde{S}_{\alpha \beta \mu \nu} + \tilde{K}_{\alpha \beta \mu \nu} + \tilde{T}_{\alpha \beta \mu \nu} \).
6. Positive or causal implies completely traceless

Counter example for satisfying positive energy but not fulfilling completely traceless,

\[
X_{\alpha\beta\mu\nu} = R_{\alpha\lambda\beta\sigma} R_{\mu\lambda\nu\sigma}, \quad X_{0000} = E_{ab}E^{ab} \geq 0, \quad X^{\alpha}_{\mu\alpha\nu} = \frac{1}{4} g_{\mu\nu} R^2 \neq 0. \quad (22)
\]

Returning back to causal, consider the energy-momentum integral in a quasilocal small sphere with constant time evolution of the hypersurface. The fourth rank tensor \(X_{\alpha\beta\mu\nu}\) needs to be symmetric at the last two indices because of the small sphere limit

\[
N^\mu P_\mu = \int_{t=0}^{t} N^\mu \int x^i x^j dV = \int_{t=0}^{t} N^\mu \int x^i x^j dV = \int_{t=0}^{t} N^\mu X_{\mu li} \frac{r^2}{3} dV
\]

where we made the assumption that \(X_{0\mu\alpha}\) vanishes and fulfills causal. Consider the requirement for the energy-momentum being future pointing and non-spacelike (i.e., causal) in the small sphere limit:

\[
A_{\mu0l} = (-2a_1 - 4a_2 + 2a_3 + 3a_4)E_{ab}E^{ab} + (4a_1 + 4a_2 - 3a_4)H_{ab}H^{ab} + (2a_1 + 2a_3) \epsilon_{cab}E^{ad}H_{d}^b. \quad (24)
\]

Causal requires the magnitude of \(E_{ab}E^{ab}\) and \(H_{ab}H^{ab}\) to be the same and the energy is greater than or equal to the momentum. After some simple algebra, we found

\[
A_{\mu0l} = a_1 B_{\mu000} + a_3 V_{\mu000} = (a_1 + a_3) B_{\mu000}, \quad (25)
\]

where we require \(a_1 + a_3 \geq 0\). Hence, the completely traceless and causal properties form necessary and sufficient conditions. This means we can further simplify the algebraic Rainich conditions for a fourth rank tensor; as far as the quasilocal small sphere limit is concerned, we only need the completely trace free condition or positivity (i.e., causal). This is an interesting result which is valid in the quasilocal small sphere region.

7. Conclusion

Either completely traceless or causal is enough to fulfill the algebraic Rainich conditions.

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