Blocks of category $\mathcal{O}$ for rational Cherednik algebras 
and of cyclotomic Hecke algebras of type $G(r, p, n)$

Kentaro Wada

Abstract. We classify blocks of category $\mathcal{O}$ for rational Cherednik algebras and 
of cyclotomic Hecke algebras of type $G(r, p, n)$ by using the “residue equivalence” 
for multi-partitions.

§ 0. Introduction

Let $V$ be a finite dimensional vector space over $\mathbb{C}$, and $W \subset \text{GL}(V)$ be a 
finite complex reflection group. The rational Cherednik algebra $\mathcal{H} = \mathcal{H}(W)$ over 
$\mathbb{C}$ associated to $W$ was introduced by [EG]. It is known that the category $\mathcal{O}$ of $\mathcal{H}$ 
is a highest weight category with standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$, where $\Lambda^+$ is 
an index set of pairwise non-isomorphic simple $W$-modules over $\mathbb{C}$ ([Gu], [GGOR]).

Let $\mathcal{H} = \mathcal{H}(W)$ be the cyclotomic Hecke algebra associated to $W$ with appropriate 
parameters. Let $KZ : \mathcal{O} \to \mathcal{H}$-mod be the Knizhnik-Zamolodchikov functor defined 
in [GGOR]. It is known that $\mathcal{O}$ is a quasi-hereditary cover (highest weight cover) of $\mathcal{H}$ in the sense of [Ro]. Put $S(\lambda) = KZ(\Delta(\lambda))$. We see that there exists a one-to-one correspondence between the blocks of $\mathcal{O}$ and of $\mathcal{H}$ thanks to the double centralizer property. Moreover, we see that the classification of blocks of $\mathcal{O}$ and of $\mathcal{H}$ is given by the linkages classes on $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ or on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ (see §1 for details). Hence, it is enough to classify the blocks of $\mathcal{O}$ and of $\mathcal{H}$ that we determine the 
linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$.

In the case where $W$ is the complex reflection group of type $G(r, 1, n)$, $\mathcal{H}$ is 
also called an Ariki-Koike algebra. In this case, $\Lambda^+$ is the set of $r$-partitions of size $n$ denoted by $\mathcal{P}_{n,r}$. Then the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ is given by 
the equivalent relation “$\sim^\mathcal{H}$”, so called residue equivalence, on $\mathcal{P}_{n,r}$ by [LM]. (Note that the Specht module $S^\lambda (\lambda \in \Lambda^+)$ being considered in [LM] does not coincides with $S(\lambda)$ in general. However, one sees that the linkage classes on $\{S^\lambda \mid \lambda \in \Lambda^+\}$ coincides with the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. See §3.)

Our purpose is to classify the blocks of $\mathcal{O}$ and of $\mathcal{H}$ in the case where $W$ is 
the complex reflection group of type $G(r, p, n)$. As seen in the above, we should 
determine the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. Let $W^\dagger$ be the complex reflection 
group of type $G(r, 1, n)$, and we denote by adding the superscript $\dagger$ for objects of 
type $G(r, 1, n)$. It is known that $W$ is a normal subgroup of $W^\dagger$ with the index $p$, 
and that $\mathcal{H}$ is a subalgebra of $\mathcal{H}^\dagger$. An index set $\Lambda^+$ of pairwise non-isomorphic simple $W$-modules over $\mathbb{C}$ (thus, $\Lambda^+$ is also an index set of standard modules of $\mathcal{O}$) 
is given as the equivalent classes of $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ under a certain equivalent relation

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“∼∗” on $P_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ (see 4.3 for details). We denote by $\lambda(i) \in \Lambda^+$ the equivalent class containing $(\lambda, i) \in P_{n,r} \times \mathbb{Z}/p\mathbb{Z}$.

Some relations between representations of $\mathcal{K}$ and of $\mathcal{K}^\dagger$ have been studied in [A2], [GJ], [H1], [H2], [H3], [H4] and [H5] by using Clifford theory. Combining these results with some fundamental properties of quasi-hereditary covers, and with the classification of blocks of $\mathcal{K}^\dagger$ by using the residue equivalence “∼$R$”, we give the classification of the blocks of $O$ and of $\mathcal{K}$ by using a certain equivalent relation “≈” on $P_{n,r}$ as follows.

Let “≈” be the equivalent relation on $P_{n,r}$ defined by $\lambda \approx \mu$ if $\lambda \sim_R \mu[j]$ for some $j \in \mathbb{Z}$, where $\mu[j] \in P_{n,r}$ is obtained from $\mu \in P_{n,r}$ by a certain permutation of components of $\mu$ (see 4.3 for the precise definition of $\mu[j]$). Put $\Gamma = \{ \lambda \in P_{n,r} | \lambda \not\sim_R \mu \}$ for any $\mu \in P_{n,r}$ such that $\mu \neq \lambda$. Then our main theorem is the following.

**Theorem 4.11**

(i) If $\lambda \in \Gamma$, then $\Delta(\lambda(i))$ (resp. $S(\lambda(i))$) is a simple object of $O$ (resp. simple $\mathcal{K}$-module) for any $i \in \mathbb{Z}$. Moreover, $\Delta(\lambda(i))$ (resp. $S(\lambda(i))$) is a block of $O$ (resp. of $\mathcal{K}$) itself.

(ii) For $\lambda, \mu \in P_{n,r} \setminus \Gamma$ and $i, j \in \mathbb{Z}$, we have that

Both of $\Delta(\lambda(i))$ and $\Delta(\mu(j))$ belong to the same block of $O$

$\Leftrightarrow$ Both of $S(\lambda(i))$ and $S(\mu(j))$ belong to the same block of $\mathcal{K}$

$\Leftrightarrow \lambda \approx \mu$.

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**Notations:** For an algebra $\mathcal{A}$, we denote by $\mathcal{A}$-mod the category of finitely generated $\mathcal{A}$-modules, and denote by $\mathcal{A}$-proj the full subcategory of $\mathcal{A}$-mod consisting of projective objects. Let $K_0(\mathcal{A}$-mod) be a Grothendieck group of $\mathcal{A}$-mod. We denote by $[M]$ the image of $M$ in the $K_0(\mathcal{A}$-mod) for $M \in \mathcal{A}$-mod. For $M \in \mathcal{A}$-mod and simple object $L$ of $\mathcal{A}$-mod, we denote by $[M : L]_{\mathcal{A}}$ the multiplicity of $L$ in the composition series of $M$. We also denote by $\mathcal{A}^{\text{opp}}$ the opposite algebra of $\mathcal{A}$.

**§ 1. Some properties of quasi-hereditary covers**

In this section, we recall some notions of quasi-hereditary covers from [Ro], and prepare some fundamental properties.

**1.1.** Let $\mathcal{A}$ be a quasi-hereditary algebra over a field. Take a projective object $P$ in $\mathcal{A}$-mod, and put $B = \text{End}_{\mathcal{A}}(P)^{\text{opp}}$. Then we have the exact functor $F = \text{Hom}_{\mathcal{A}}(P, -) : \mathcal{A}$-mod $\to B$-mod. Let $X$ be a progenerator of $\mathcal{A}$-mod such that $X = P \oplus P'$ for some projective object $P'$ in $\mathcal{A}$-mod. Then $\text{End}_{\mathcal{A}}(X)^{\text{opp}}$ is Morita equivalent to $\mathcal{A}$. We may suppose that $\text{End}_{\mathcal{A}}(X)^{\text{opp}} \cong \mathcal{A}$ without loss of generality, and we suppose that $\text{End}_{\mathcal{A}}(X)^{\text{opp}} \cong \mathcal{A}$.

Throughout this section, we assume the following condition.

(A1): The functor $F$ is fully faithful when we restrict to $\mathcal{A}$-proj.
Hence, $\mathcal{A}$ is a quasi-hereditary cover of $\mathcal{B}$ in the sense of [Ro]. Since $X \in \mathcal{A}$-proj, we have that

\[(1.1.1) \quad \mathcal{A} \cong \text{End}_{\mathcal{A}}(X)^{\text{opp}} \cong \text{End}_{\mathcal{B}}(F(X))^{\text{opp}}.\]

Note that $X = P \oplus P'$. Let $\varphi_P^0 \in \text{End}_{\mathcal{A}}(X)$ be such that $\varphi_P^0$ is the identity map on $P$, and 0-map on $P'$. We denote by $\varphi_P$ the element of $\mathcal{A} \cong \text{End}_{\mathcal{A}}(X)^{\text{opp}}$ corresponding to $\varphi_P^0$. It is clear that $\varphi_P$ is an idempotent. Since $F(X) \cong \text{Hom}_{\mathcal{A}}(P, P) \oplus \text{Hom}_{\mathcal{A}}(P, P') \cong \text{End}_{\mathcal{A}}(X)\varphi_P^0 \cong \varphi_P\mathcal{A}$ as right $\mathcal{A}$-modules, we have following isomorphisms of algebras:

\[
\text{End}_{\mathcal{A}}^{\text{opp}}(F(X)) \cong \text{End}_{\mathcal{A}}^{\text{opp}}(\varphi_P\mathcal{A}) \\
\cong \varphi_P\mathcal{A} \varphi_P \\
\cong (\varphi_P^0 \text{End}_{\mathcal{A}}(X)\varphi_P^0)^{\text{opp}} \\
\cong \text{End}_{\mathcal{A}}(P)^{\text{opp}} \\
= \mathcal{B}.
\]

Thus, we have the double centralizer property:

\[(1.1.2) \quad \mathcal{A} \cong \text{End}_{\mathcal{B}}(F(X))^{\text{opp}}, \quad \mathcal{B} \cong \text{End}_{\mathcal{A}}^{\text{opp}}(F(X)).\]

This double centralizer property implies the isomorphism $Z(\mathcal{A}) \to Z(\mathcal{B})$, where $Z(\mathcal{A})$ (resp. $Z(\mathcal{B})$) is the center of $\mathcal{A}$ (resp. $\mathcal{B}$). Thus, there exists a bijection between blocks of $\mathcal{A}$ and of $\mathcal{B}$.

1.2. Recall that $\mathcal{A}$ is a quasi-hereditary algebra. Let $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ be the set of standard modules, and $\{\nabla(\lambda) \mid \lambda \in \Lambda^+\}$ be the set of costandard modules of $\mathcal{A}$. For $\lambda \in \Lambda^+$, let $L(\lambda)$ be the unique simple top of $\Delta(\lambda)$, and $P(\lambda)$ be the projective cover of $L(\lambda)$. Then $\{L(\lambda) \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic simple $\mathcal{A}$-modules.

For $\lambda \in \Lambda^+$, put $S(\lambda) = F(\Delta(\lambda))$, $D(\lambda) = F(L(\lambda))$ and $\Lambda^+ = \{\lambda \in \Lambda^+ \mid D(\lambda) \neq 0\}$. Since $\mathcal{B} \cong \varphi_P\mathcal{A} \varphi_P$ and $F = \text{Hom}_{\mathcal{A}}(P, -) = \text{Hom}_{\mathcal{A}}(\mathcal{A} \varphi_P, -)$, the following lemma is well-known (see e.g. [D, Appendix]).

Lemma 1.3.

(i) For $\lambda \in \Lambda^+_0$, we have $D(\lambda) \cong \text{Top} F(P(\lambda)) \cong \text{Top} S(\lambda)$.

(ii) $\{F(P(\lambda)) \mid \lambda \in \Lambda^+_0\}$ gives a complete set of non-isomorphic indecomposable projective $\mathcal{B}$-modules.

(iii) $\{D(\lambda) \mid \lambda \in \Lambda^+_0\}$ gives a complete set of non-isomorphic simple $\mathcal{B}$-modules.
For \( \lambda, \mu \in \Lambda^+ \), we denote by \( P(\lambda) \sim P(\mu) \) if there exists a sequence \( \lambda = \lambda_1, \lambda_2, \ldots, \lambda_{k+1} = \mu \) such that \( P(\lambda_i) \) and \( P(\lambda_{i+1}) \) have a common composition factor for any \( i = 1, \ldots, k \). Then “\( \sim \)” gives an equivalent relation on \( \{P(\lambda) \mid \lambda \in \Lambda^+\} \). It is well-known that \( P(\lambda) \sim P(\mu) \) if and only if \( P(\lambda) \) and \( P(\mu) \) belong to the same block of \( \mathcal{A} \). Similarly, we define an equivalent relation “~” on \( \{F(P(\lambda)) \mid \lambda \in \Lambda_0^+\} \), and we have that \( F(P(\lambda)) \sim F(P(\mu)) \) if and only if \( F(P(\lambda)) \) and \( F(P(\mu)) \) belong to the same block of \( \mathcal{B} \). We have the following lemma.

**Lemma 1.5.** For \( \lambda, \mu \in \Lambda_0^+ \), we have

\[
P(\lambda) \sim P(\mu) \quad \text{if and only if} \quad F(P(\lambda)) \sim F(P(\mu)).
\]

**Proof.** Let \( 1_{\mathcal{A}} \) be the unit element of \( \mathcal{A} \), and \( 1_{\mathcal{A}} = e_1 + \cdots + e_k \) be the decomposition of \( 1_{\mathcal{A}} \) to the sum of primitive central idempotents. Then we have the decomposition \( X = X_1 \oplus \cdots \oplus X_k \), where \( X_i = e_i X \) belong to a block of \( \mathcal{A} \). Then we have the decomposition of \( \mathcal{A} \) to blocks:

\[
\mathcal{A} \cong \text{End}_{\mathcal{B}}(F(X))\text{opp} = \text{End}_{\mathcal{B}}(F(X_1))\text{opp} \oplus \cdots \oplus \text{End}_{\mathcal{B}}(F(X_k))\text{opp}
\]

since \( \text{Hom}_{\mathcal{B}}(F(X_i), F(X_j)) \cong \text{Hom}_{\mathcal{A}}(X_i, X_j) = 0 \) if \( i \neq j \). Assume that

\[
F(X_i) \cong C_1 \oplus C_2 \text{ as } \mathcal{B}\text{-modules}
\]

such that \( C_1 \) and \( C_2 \) have no common composition factor. Then we have that

\[
\text{End}_{\mathcal{B}}(F(X_i)) = \text{End}_{\mathcal{B}}(C_1) \oplus \text{End}_{\mathcal{B}}(C_2) \text{ as two sided ideals of } \mathcal{A}.
\]

However, this contradicts to the decomposition (1.5.1). Thus we have that all composition factors of \( F(X_i) \) belong to the same block of \( \mathcal{B} \). This implies that

\[
P(\lambda) \sim P(\mu) \text{ only if } F(P(\lambda)) \sim F(P(\mu)) \quad \text{for } \lambda, \mu \in \Lambda_0^+.
\]

On the other hand, for \( \lambda, \mu \in \Lambda_0^+ \), assume that \( F(P(\lambda)) \) and \( F(P(\mu)) \) have a common composition factor \( D(\nu) \). Then we have

\[
\dim \text{Hom}_{\mathcal{B}}(F(P(\nu)), F(P(\lambda))) = [F(P(\lambda)) : D(\nu)] \neq 0,
\]

\[
\dim \text{Hom}_{\mathcal{B}}(F(P(\nu)), F(P(\mu))) = [F(P(\mu)) : D(\nu)] \neq 0.
\]

By the assumption (A1), we have that

\[
\text{Hom}_{\mathcal{A}}(P(\nu), P(\lambda)) = \text{Hom}_{\mathcal{B}}(F(P(\nu)), F(P(\lambda))) \neq 0,
\]

\[
\text{Hom}_{\mathcal{A}}(P(\nu), P(\mu)) = \text{Hom}_{\mathcal{B}}(F(P(\nu)), F(P(\mu))) \neq 0.
\]
Thus, we have that

\[ P(\lambda) \sim P(\mu) \text{ if } F(P(\lambda)) \sim F(P(\mu)) \text{ for } \lambda, \mu \in \Lambda_0^+. \]

Now we proved the Lemma. \( \square \)

**Remark 1.6.** In the proof of the above lemma, we also showed that, for \( \lambda \in A^+ \setminus \Lambda_0^+ \), a composition factor of \( F(P(\lambda)) \) is isomorphic to \( D(\mu) \) for some \( \mu \in \Lambda_0^+ \) such that \( P(\lambda) \sim P(\mu) \).

As a corollary, we have the following.

**Corollary 1.7.** For each \( \lambda \in A^+ \), all composition factors of \( S(\lambda) \) belong to the same block of \( \mathcal{B} \).

**Proof.** Since \( \Delta(\lambda) \) is an indecomposable \( \mathcal{A} \)-module, all composition factors of \( \Delta(\lambda) \) belong to the same block of \( \mathcal{A} \). Thus, Lemma 1.5 implies the corollary since a composition series of \( S(\lambda) \) as \( \mathcal{B} \)-modules is obtained from a composition series of \( \Delta(\lambda) \) as \( \mathcal{A} \)-modules by applying the functor \( F \). \( \square \)

1.8. From now on, we assume the following additional condition:

\( \text{(A2): } [\Delta(\lambda)] = [\nabla(\lambda)] \text{ in } K_0(\mathcal{A} \text{-mod}) \text{ for any } \lambda \in A^+. \)

By the general theory of quasi-hereditary algebras, for \( \lambda \in A^+ \), \( P(\lambda) \) has a \( \Delta \)-filtration such that \( (P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)]_{\mathcal{A}} \), where \( (P(\lambda) : \Delta(\mu)) \) is the multiplicity of \( \Delta(\mu) \) in a \( \Delta \)-filtration of \( P(\lambda) \). Combining with the assumption (A2), we have that

\[ (P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]_{\mathcal{A}}. \] (1.8.1)

This implies the following lemma.

**Lemma 1.9.** For \( \lambda, \mu \in \Lambda_0^+ \), we have

\[ [F(P(\lambda)) : D(\mu)]_{\mathcal{A}} = \sum_{\nu \in A^+} [S(\nu) : D(\lambda)]_{\mathcal{A}} [S(\nu) : D(\mu)]_{\mathcal{A}}. \]

**Proof.** Let

\[ P(\lambda) = P_k \supset P_{k-1} \supset \cdots \supset P_1 \supset P_0 = 0 \text{ such that } P_i/P_{i-1} \cong \Delta(\nu_i) \]

be a \( \Delta \)-filtration of \( P(\lambda) \). By applying the functor \( F \) to this filtration, we have a filtration of \( F(P(\lambda)) \) (as \( \mathcal{B} \)-modules)

\[ F(P(\lambda)) = F(P_k) \supset F(P_{k-1}) \supset \cdots \supset F(P_1) \supset F(P_0) = 0 \]

such that \( F(P_i)/F(P_{i-1}) \cong S(\nu_i) \). In this filtration, for \( \nu \in A^+ \), \( S(\nu) \) appears \( [\Delta(\nu) : L(\lambda)]_{\mathcal{A}} = [S(\nu) : D(\lambda)]_{\mathcal{A}} \) times by (1.8.1). Thus, we have

\[ [F(P(\lambda)) : D(\mu)]_{\mathcal{A}} = \sum_{\nu \in A^+} [S(\nu) : D(\lambda)]_{\mathcal{A}} [S(\nu) : D(\mu)]_{\mathcal{A}}. \] \( \square \)
1.10. For $\lambda, \mu \in \Lambda^+$, we denote by $S(\lambda) \sim S(\mu)$ if there exists a sequence $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_{k+1} = \mu$ $(\lambda_i \in \Lambda^+)$ such that $S(\lambda_i)$ and $S(\lambda_{i+1})$ have a common composition factor for any $i = 1, \ldots, k$. Then “$\sim$” gives an equivalent relation on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. Similarly, we define an equivalent relation “$\sim'$” on $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$.

Note that all composition factors of $\Delta(\lambda)$ belong to the same block of $\mathfrak{A}$ since $\Delta(\lambda)$ is indecomposable.

Corollary 1.7 and Lemma 1.9 imply the following proposition.

**Proposition 1.11.** For $\lambda, \mu \in \Lambda^+$ we have the following.

(i) $S(\lambda) \sim S(\mu)$ if and only if $S(\lambda)$ and $S(\mu)$ belong to the same block of $\mathfrak{B}$.

(ii) $\Delta(\lambda) \sim \Delta(\mu)$ if and only if $\Delta(\lambda)$ and $\Delta(\mu)$ belong to the same block of $\mathfrak{A}$.

**Proof.** (i) The “only if” part is clear by Corollary 1.7. We prove the “if” part. Suppose that $S(\lambda)$ and $S(\mu)$ belong to the same block of $\mathfrak{B}$. Let $D(\lambda')$ (resp. $D(\mu')$) be a composition factor of $S(\lambda)$ (resp. $S(\mu)$). Then $D(\lambda')$ and $D(\mu')$ belong to the same block of $\mathfrak{B}$. This implies that $F(P(\lambda')) \sim F(P(\mu'))$. Thus there exists a sequence $\lambda' = \lambda_1, \lambda_2, \ldots, \lambda_{k+1} = \mu'$ $(\lambda_i \in \Lambda^+)$ such that $\mathfrak{B}(P(\lambda_i))$ and $\mathfrak{B}(P(\lambda_{i+1}))$ have a common composition factor $D(\nu_i)$ for any $i = 1, \ldots, k$. By Lemma 1.9, this implies that there exist

$$
\tau_i \in \Lambda^+ \text{ such that } [S(\tau_i) : D(\lambda_i)]_{\mathfrak{B}} \neq 0 \text{ and } [S(\tau_i) : D(\nu_i)]_{\mathfrak{B}} \neq 0,
$$

$$
\tau_i' \in \Lambda^+ \text{ such that } [S(\tau_i') : D(\lambda_{i+1})]_{\mathfrak{B}} \neq 0 \text{ and } [S(\tau_i') : D(\nu_i)]_{\mathfrak{B}} \neq 0
$$

for any $i = 1, \ldots, k$. Thus we have

$$
S(\lambda_i) \sim S(\tau_i) \sim S(\nu_i) \sim S(\tau_i') \sim S(\lambda_{i+1})
$$

for any $i = 1, \ldots, k$. This implies that

$$
S(\lambda) \sim S(\lambda') = S(\lambda_1) \sim S(\lambda_2) \sim \cdots \sim S(\lambda_{k+1}) = S(\mu') \sim S(\mu).
$$

(ii) is proven in a similar way by using (1.8.1) instead of Lemma 1.9.

1.12. From now on, we assume the following additional condition:

**(A3):** $S(\lambda) = F(\Delta(\lambda)) \neq 0$ for any $\lambda \in \Lambda^+$.

Thanks to Proposition 1.11, we can classify blocks of $\mathfrak{B}$ (resp. blocks of $\mathfrak{A}$) by equivalent classes of $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ (resp. $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$) with respect to the relation “$\sim$”. The following proposition gives a relation between blocks of $\mathfrak{A}$ and of $\mathfrak{B}$.

**Proposition 1.13.** For $\lambda, \mu \in \Lambda^+$, we have

$$
\Delta(\lambda) \sim \Delta(\mu) \text{ if and only if } S(\lambda) \sim S(\mu).
$$

**Proof.** By Lemma 1.5, we have that $\Delta(\lambda)$ and $\Delta(\mu)$ belong to the same block of $\mathfrak{A}$ if and only if $S(\lambda)$ and $S(\mu)$ belong to the same block of $\mathfrak{B}$. Thus Proposition 1.11 implies this proposition.
§ 2. RATIONAL CHEREDNIK ALGEBRAS

2.1. Let $V$ be a finite dimensional vector space over $\mathbb{C}$, and $W \subset \text{GL}(V)$ be a finite complex reflection group. Let $A$ be the set of reflecting hyperplanes of $W$, and $A/W$ be the set of $W$-orbits of $A$. For $H \in A$, let $W_H$ be the subgroup of $W$ fixing $H$ pointwise, and put $e_H = |W_H|$. Take a set

$$\Omega = \{k_{H,i} \in \mathbb{C} | H \in A/W, 0 \leq i \leq e_H \text{ such that } k_{H,0} = k_{H,e_H} = 0\}.$$ 

Let $\mathcal{H}$ be the rational Cherednik algebra associated to $W$ with parameters $\Omega$ (see [GGOR, 3.1] for definitions). By [EG], it is known that $\mathcal{H}$ has the triangular decomposition

$$\mathcal{H} \cong S(V^*) \otimes_{\mathbb{C}} CW \otimes_{\mathbb{C}} S(V)$$

as vector spaces, where $S(V)$ (resp. $S(V^*)$) is the symmetric algebra of $V$ (resp. the dual space $V^*$), and $CW$ is the group ring of $W$ over $\mathbb{C}$.

Let $\mathcal{O}$ be the category of finitely generated $\mathcal{H}$-modules which are locally nilpotent for the action of $S(V) \setminus \mathbb{C}$. Let $\text{Irr} \, \mathcal{W} = \{E^\lambda | \lambda \in \Lambda^+\}$ be a complete set of non-isomorphic simple $\mathbb{C}W$-modules. For $\lambda \in \Lambda^+$, put

$$\Delta(\lambda) = \mathcal{H} \otimes_{S(V) \times \mathbb{C} \times \mathbb{C}} E^\lambda,$$

where $S(V) \times \mathbb{C} \times \mathcal{W} \cong S(V) \otimes_{\mathbb{C}} \mathbb{C}W$ is a subalgebra of $\mathcal{H}$, and we regard $E^\lambda$ as an $S(V) \times \mathbb{C}W$-module through the natural surjection $S(V) \times \mathbb{C} \to \mathbb{C}W$. It is known that $\mathcal{O}$ turns out to be a highest weight category with standard modules $\{\Delta(\lambda) | \lambda \in \Lambda^+\}$ ([GGOR], [Gu]). Let $L(\lambda)$ be the unique simple top of $\Delta(\lambda)$, then $\{L(\lambda) | \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic simple objects in $\mathcal{O}$. For $\lambda \in \Lambda^+$, we denote by $P(\lambda)$ the projective cover of $L(\lambda)$.

2.2. Let $\mathcal{H}$ be the cyclotomic Hecke algebra of $W$ corresponding $\mathcal{H}$ (see [GGOR, 5.2.5] for the choice of parameters). Then the Knizhnik-Zamolodchikov functor (simply, KZ functor) $\text{KZ} : \mathcal{O} \to \mathcal{H} \text{-mod}$ is defined in [GGOR, 5.3]. KZ functor is a exact functor, and represented by a projective object

$$P_{\text{KZ}} = \bigoplus_{\lambda \in \Lambda^+} P(\lambda)^{\oplus \text{dim}KZ(L(\lambda))} \in \mathcal{O},$$

namely, we have $\text{KZ} = \text{Hom}_\mathcal{O}(P_{\text{KZ}}, -)$. Moreover, by [GGOR, Theorem 5.15], we have

$$\mathcal{H} \cong (\text{End}_\mathcal{O}(P_{\text{KZ}}))^{\text{opp}}.$$

By [GGOR, Theorem 5.16], KZ functor is fully faithful when we restrict to projective objects in $\mathcal{O}$. Thus, $\mathcal{O}$ is a quasi-hereditary cover of $\mathcal{H}$.

Put $A = \text{End}_\mathcal{O}(X)$, $B = \mathcal{H}$ and $F = \text{KZ}$, where $X$ is a progenerator of $\mathcal{O}$ such that $X = P_{\text{KZ}} \oplus P'$ for some projective object $P'$ in $\mathcal{O}$. Then, these satisfy assumptions (A1),(A2),(A3) by [GGOR]. Thus, all results in §1 hold for this setting. In particular, we put $S(\lambda) = \text{KZ}(\Delta(\lambda))$ and $D(\lambda) = \text{KZ}(L(\lambda))$ for $\lambda \in \Lambda^+$. Let $A^+_0 = \{\lambda \in \Lambda^+ | D(\lambda) \neq 0\}$, then $\{D(\lambda) | \lambda \in A^+_0\}$ gives a complete set of non-isomorphic simple $\mathcal{H}$-modules.
2.3. In the rest of this section, we recall a modular system and a decomposition map described in [GGOR]. Let \( \mathbb{C}[\widehat{\Omega}] \) be the polynomial ring over \( \mathbb{C} \) with indeterminates \( \widehat{\Omega} = \{k_{H,i} | H \in \mathcal{A}/W, 1 \leq i \leq e_H - 1\} \). We have a homomorphism \( \varphi : \mathbb{C}[\widehat{\Omega}] \rightarrow \mathbb{C} \) of \( \mathbb{C} \)-algebras such that \( k_{H,i} \mapsto k_{H,i} \). Put \( m = \text{Ker} \varphi \). Let \( R \) be the completion of \( \mathbb{C}[\widehat{\Omega}] \) at the maximal ideal \( m \). Then \( R \) is a regular local ring with the unique maximal ideal \( \widehat{m} = ((k_{H,i} - k_{H,i})_{H \in \mathcal{A}/W, 1 \leq i \leq e_H - 1}) \). We have the canonical homomorphism \( R \rightarrow \mathbb{C} \) such that \( k_{H,i} \mapsto k_{H,i} \). Let \( K \) be the quotient field of \( R \).

Let \( \mathcal{H}_R \) be the rational Cherednik algebra of \( W \) over \( R \) with parameters \( \widehat{\Omega} \) (put \( k_{H,0} = k_{H,e_H} = 0 \)), and \( \mathcal{H}_R \) be the cyclotomic Hecke algebra over \( R \) associated to \( \mathcal{H}_R \). Then we have \( \mathcal{H} = \mathbb{C} \otimes_R \mathcal{H}_R \) and \( \mathcal{H} = \mathbb{C} \otimes_R \mathcal{H}_R \). Put \( \mathcal{H}_K = K \otimes_R \mathcal{H}_R \) and \( \mathcal{H}_K = K \otimes_R \mathcal{H}_R \). We denote objects over \( X = R \) or \( K \) by adding subscript \( X \), e.g. \( \mathcal{O}_X, \Delta(\lambda)_X, KZ_X, S(\lambda)_X \cdots \).

Under the modular system \((K, R, \mathbb{C})\), we can define the decomposition map

\[
d_{K,\mathbb{C}} : K_0(\mathcal{H}_K \text{-mod}) \rightarrow K_0(\mathcal{H}_R \text{-mod})
\]

by \([M] \mapsto [\mathbb{C} \otimes_R N]\), where \( N \) is an \( \mathcal{H}_R \)-lattice of \( M \). Thanks to [GGOR, Theorem 5.13], we have the following lemma.

**Lemma 2.4.** For \( \lambda \in \Lambda^+ \), we have

\[
d_{K,\mathbb{C}}([S_K(\lambda)]) = [S(\lambda)].
\]

\section{3. Case of type \( G(r, 1, n) \)}

In this section, we consider the complex reflection group \( W \) of type \( G(r, 1, n) \), i.e. \( W = \mathcal{G}_n \rtimes (\mathbb{Z}/r\mathbb{Z})^n \). In this case, \( \mathcal{H} \) is often called the Ariki-Koike algebra, and many results for representations of \( \mathcal{H} \) are known by several authors.

3.1. In this section, we use the modular system \((K, R, \mathbb{C})\) given in the previous section, and we take parameters as follows.

Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{C} \) with a basis \( \{e_1, \cdots, e_n\} \). Then we have \( W \subset \text{GL}(V) \). Let \( s_1, t_1 \in W \) be reflections such that

\[
s_1(e_k) = \begin{cases} e_2 & \text{if } k = 1, \\ e_1 & \text{if } k = 2, \\ e_k & \text{otherwise,}
\end{cases} \quad t_1(e_k) = \begin{cases} \zeta e_1 & \text{if } k = 1, \\ e_k & \text{otherwise,} \\
\end{cases} \quad (\zeta = \exp(2\pi \sqrt{-1}/r)),
\]

and \( H_{s_1} \) (resp. \( H_{t_1} \)) be the reflecting hyperplane corresponding to \( s_1 \) (resp. \( t_1 \)). Then \( \{H_{s_1}, H_{t_1}\} \) gives a complete set of representatives of \( W \)-orbits of \( \mathcal{A} \), and we have \( e_{H_{s_1}} = 2 \) and \( e_{H_{t_1}} = r \). Hence, we can take parameters \( \{h, k_1, \cdots, k_{r-1}\} \) (resp. \( \{h, k_1, \cdots, k_{r-1}\} \) of \( \mathcal{H} \) (resp. \( \mathcal{H}_X (X = R \text{ or } K) \)) such that \( h = k_{H_{s_1},1} \) (resp. \( h = k_{H_{t_1},1} \)) and \( k_j = k_{H_{s_1},j} \) (resp. \( k_j = k_{H_{t_1},j} \)) for \( 1 \leq j \leq r - 1 \). Then \( \mathcal{H} \) (resp. \( \mathcal{H}_R, \mathcal{H}_K \)) is the associative algebra over \( \mathbb{C} \) (resp. \( R, K \)) defined by generators...
$T_0, T_1, \cdots, T_{n-1}$ with defining relations:

\begin{align*}
(3.1.2) \quad & (T_0 - 1)(T_0 - Q_1) \cdots (T_0 - Q_{r-1}) = 0, \\
& (T_0 - 1)(T_0 + q) = 0, \\
& T_0 T_i T_0 T_1 = T_1 T_0 T_i T_0, \\
& T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-1), \\
& T_i T_j = T_j T_i \quad (|i - j| > 1),
\end{align*}

where $Q_i = \exp(2\pi \sqrt{-1}(k_i + \frac{j}{r}))$, $q = \exp(2\pi \sqrt{-1}h)$ (resp. $Q_i = \exp(2\pi \sqrt{-1}(k_i + \frac{j}{r}))$, $q = \exp(2\pi \sqrt{-1}h)$).

3.2. Let

$$
P_{n,r} = \left\{ \lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)}) \mid \lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots) \in \mathbb{Z}_{\geq 0}^r \text{ with } \lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \right\}
$$

be the set of $r$-partitions of size $n$. It is well-known that the isomorphism classes of simple $\mathbb{C}W$-modules are indexed by $P_{n,r}$, thus we have $A^+ = P_{n,r}$.

3.3. By [DJM], it is known that $\mathcal{H}_X$ ($X = K, R$ or $\mathbb{C}$, we may omit the subscript $X$ when $X = \mathbb{C}$) is a cellular algebra with respect to a poset $(A^+, \triangleright)$, where “$\triangleright$” is the dominance order on $A^+$. We denote by $S^\lambda_X$ the Specht (cell) module for $\lambda \in A^+$ constructed by using the cellular basis in [DJM].

It is known that $\mathcal{H}_K$ is semi-simple, and $\{S^\lambda_K \mid \lambda \in A^+\}$ gives a complete set of non-isomorphic simple $\mathcal{H}_K$-modules.

By the general theory of cellular algebras (see [GL] or [M2]), we can define the canonical bilinear form $\langle , \rangle : S^\lambda \times S^\mu \to \mathbb{C}$ by using the cellular basis. Put $\text{Rad } S^\lambda = \{ x \in S^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in S^\lambda \}$ and $D^\lambda = S^\lambda / \text{Rad } S^\lambda$. Let $\mathcal{K}_{n,r}$ be the set of Kleshchev multi-partitions containing in $A^+$ (see e.g. [A4] [M1] for the definition). Then it is known that $\{D^\lambda \mid \lambda \in \mathcal{K}_{n,r}\}$ gives a complete set of non-isomorphic simple $\mathcal{H}$-modules by [A4].

It is known that all composition factor of $S^\lambda$ belong to the same block of $\mathcal{H}$. Let “$\sim$” be an equivalent relation on $\{S^\lambda \mid \lambda \in A^+\}$ defined in a similar way as the equivalent relation “$\sim$” on $\{S(\lambda) \mid \lambda \in A^+\}$ in the previous section. Then it is known that

\begin{align*}
(3.3.1) \quad & S^\lambda \sim S^\mu \quad \text{if and only if } S^\lambda \text{ and } S^\mu \text{ belong to the same block of } \mathcal{H}.
\end{align*}

By (3.3.1), we can classify the blocks of $\mathcal{H}$ by the equivalent classes of $\{S^\lambda \mid \lambda \in A^+\}$ with respect to “$\sim$”, and such equivalent classes are described by using some combinatorics in [LM] as follows. For $\lambda \in A^+$, put

$$
|\lambda| = \{(i, j, k) \in \mathbb{Z}_{>0}^3 \mid 1 \leq j \leq \lambda_i^{(k)}, 1 \leq k \leq r\}.
$$
For $x = (i, j, k) \in [\lambda]$, we define
\[
\text{res}(x) = \begin{cases} 
q^{j-i}Q_{k-1} & \text{if } q \neq 1 \text{ and } Q_{k-1} \neq 0, \\
(j-i, Q_{k-1}) & \text{if } q = 1 \text{ and } Q_{l-1} \neq Q_{k-1} \text{ for } k \neq l, \\
Q_{k-1} & \text{otherwise},
\end{cases}
\]
where we put $Q_0 = 1$. Put $\text{Res}(\Lambda^+) = \{\text{res}(x) \mid x \in [\lambda] \text{ for some } \lambda \in \Lambda^+\}$. Then, we define an equivalent relation (called residue equivalence) "\(\sim_R\)" on \(\Lambda^+\) by
\[
\lambda \sim_R \mu \text{ if } \#\{x \in [\lambda] \mid \text{res}(x) = a\} = \#\{y \in [\mu] \mid \text{res}(y) = a\} \text{ for all } a \in \text{Res}(\Lambda^+).
\]

**Theorem 3.4** ([LM, Theorem 2.11]). For \(\lambda, \mu \in \Lambda^+\), we have that
\[
S^\lambda \sim S^\mu \text{ if and only if } \lambda \sim_R \mu.
\]

3.5. We take \(\text{Irr } W = \{E^\lambda \mid \lambda \in \Lambda^+\}\) such that \(K \otimes_C E^\lambda \cong S^\lambda_K\) via the isomorphism \(\mathcal{H}_K \cong K \otimes_C \mathbb{C}W\). Since \(S^\lambda_K = K \otimes_R S^\lambda_R\) and \(S^\lambda = \mathbb{C} \otimes_R S^\lambda_R\), we have that
\[
(3.5.1) \quad d_{K,C}(S^\lambda_K) = [S^\lambda].
\]
It is also well-known that \(K \otimes_C E^\lambda \cong S_K^\lambda\) via the isomorphism \(\mathcal{H}_K \cong K \otimes_C \mathbb{C}W\) (see before [GGOR, Theorem 5.13]). Thus, we have \(S^\lambda_K \cong S_K^\lambda\) as \(\mathcal{H}_K\)-modules. Then Lemma 2.4 together with (3.5.1) implies that
\[
(3.5.2) \quad [S(\lambda)] = [S^\lambda] \text{ in } K_0(\mathcal{H}\text{-mod}) \text{ for } \lambda \in \Lambda^+.
\]
Note that \(S(\lambda) \not\cong S^\lambda\) as \(\mathcal{H}\)-modules in general. Hence, \(\text{Top } S(\lambda) \not\cong \text{Top } S^\lambda\) in general. Moreover, \(\Lambda^+_0 \neq \mathcal{K}_{n,r}\) in general. Let
\[
\theta : \Lambda^+_0 \rightarrow \mathcal{K}_{n,r}
\]
be the bijection such that \(D(\lambda) \cong D^\theta(\lambda)\) as \(\mathcal{H}\)-modules. Then we have the following proposition.

**Proposition 3.6.** For \(\lambda \in \Lambda^+\) and \(\mu \in \Lambda^+_0\), we have
\[
[\Delta(\lambda) : L(\mu)]_{\mathcal{O}} = [S(\lambda) : D(\mu)]_{\mathcal{H}} = [S^\lambda : D^\theta(\mu)]_{\mathcal{H}}.
\]

*Proof.* The first equality is clear since the KZ functor is exact. By (3.5.2), we have \([S(\lambda)] = [S^\lambda]\) in \(K_0(\mathcal{H}\text{-mod})\), and \(D(\mu) \cong D^\theta(\mu)\). Thus, we have the second equality. \(\Box\)

The following theorem gives a relation between blocks of \(\mathcal{O}\) and blocks of \(\mathcal{H}\). In particular, we obtain the classification of blocks of \(\mathcal{O}\) by using the residue equivalence.
Theorem 3.7. For \( \lambda, \mu \in \Lambda^+ \), we have
\[
\Delta(\lambda) \sim \Delta(\mu) \iff S(\lambda) \sim S(\mu) \iff S^\lambda \sim S^\mu \iff \lambda \sim_R \mu.
\]

Proof. The first equivalence is Proposition 1.13. The second equivalence follows from (3.5.2). The third equivalence is Theorem 3.4. □

Remark 3.8. By [Ro], under a certain condition for parameters, it is known that \( O \) is equivalent to \( \mathcal{S}_{n,r} \)-mod as highest weight categories, where \( \mathcal{S}_{n,r} \) is the cyclotomic \( q \)-Schur algebra associated to \( \mathcal{H} \) defined in [DJM]. In this case, we have \( S(\lambda) \cong S^\lambda \), and \( \theta \) is the identity map (in particular, \( A_0^\lambda = K_{n,r} \)). So, the above theorem is known by [LM]. However, the above theorem claim that a classification of blocks of \( O \) is given by the equivalent relation "\( \sim_R \)" on \( \Lambda^+ \) (residue equivalence) even if the case where \( O \) is not equivalent to \( \mathcal{S}_{n,r} \)-mod.

§ 4. Case of type \( G(r, p, n) \)

In this section, we consider the case where \( W \) is the complex reflection group of type \( G(r, p, n) \), where \( r = pd \) for some \( d \geq 1 \). It is well-known that the complex reflection group of type \( G(r, p, n) \) is a normal subgroup of the complex reflection group of type \( G(r, 1, n) \) with the index \( p \), and we will study some relations between type \( G(r, 1, n) \) and type \( G(r, p, n) \). Hence, we denote by \( W^\dagger \) the complex reflection group of type \( G(r, 1, n) \), and we use the results in the previous section for \( W^\dagger \). In this section, we use the notations in §2 for corresponding objects of type \( G(r, p, n) \), and we denote by adding the superscript \( \dagger \) for corresponding objects of type \( G(r, 1, n) \), e.g. \( \mathcal{H}^\dagger, \mathcal{H}^\dagger, \Delta^\dagger(\lambda), KZ^\dagger, S^\dagger(\lambda), \ldots \). Let \( \text{Irr} W^\dagger = \{ E^\dagger| \lambda \in \mathcal{P}_{n,r} \} \) be a complete set of non-isomorphic simple \( CW^\dagger \)-modules considered in the previous section.

4.1. Let \( V \) be an \( n \) dimensional vector space over \( \mathbb{C} \) with a basis \( \{ e_1, \ldots, e_n \} \). Then we have \( W \subset \text{GL}(V) \). Recall that \( s_1, t_1 \in \text{GL}(V) \) is a reflection defined in (3.1.1). Then \( s_1 \) (resp. \( t_1^p \) in the case where \( p \neq r \)) is a reflection contained in \( W \), and let \( H_{s_1} \) (resp. \( H_{t_1^p} \)) be the reflecting hyperplane corresponding to \( s_1 \) (resp. \( t_1^p \)). In the case where \( p \neq r \), \( \{ H_{s_1}, H_{t_1^p} \} \) gives a complete set of representatives of \( W \)-orbits of \( \mathcal{A} \), and we have \( e_{H_{s_1}} = 2 \) and \( e_{H_{t_1^p}} = d \). Hence, we can take parameters \( \{ h, k_1, \ldots, k_{d-1} \} \) (resp. \( \{ h, k_1, \ldots, k_{d-1} \} \) of \( \mathcal{H} \) (resp. \( \mathcal{H}_X (X = R \text{ or } K) \)) such that \( h = k_{H_{s_1}} \) (resp. \( h = k_{H_{s_1}} \) and \( k_j = k_{H_{s_1}} \) (resp. \( k_j = k_{H_{s_1}} \)) for \( 1 \leq j \leq d-1 \). On the other hand, in the case where \( r = p \), \( \mathcal{A} \) is the \( W \)-orbit of \( \mathcal{A} \) itself. Hence \( \mathcal{H} \) (resp. \( \mathcal{H}_X (X = R \text{ or } K) \)) has a parameter \( \{ h \} \) (resp. \( \{ h \} \)).

Then \( \mathcal{H} \) (resp. \( \mathcal{H}_R, \mathcal{H}_K \)) is the associative algebra over \( \mathbb{C} \) (resp. \( R, K \)) defined by generators \( a_0, a_1, a_1, a_2, \ldots, a_{n-1} \) with defining relations:
\[
(a_0 - 1)(a_0 - x_1) \cdots (a_0 - x_{d-1}) = 0,
(a'_0 - 1)(a'_0 + q) = 0, \quad (a_i - 1)(a_i + q) = 0 \quad (1 \leq i \leq n - 1),
\]
\[
a_0a_1a_1 = a'_1a_1a_0, \quad a'_1a_2a_1 = a_2a'_1a_2, \quad (a_2a'_1a_1)^2 = (a'_1a_1a_2)^2,
\]
\[
a_0a_i = a_ia_0 \quad (2 \leq i \leq n - 1), \quad a'_1a_j = a_ia'_1 \quad (3 \leq j \leq n - 1),
\]
\[
\begin{align*}
\underbrace{a_1 a_0 a'_1 a_1 a'_1 a_1 \cdots}_{p+1 \text{ factors}} &= \underbrace{a_0 a'_1 a_0 a'_1 a_0 a'_1 \cdots}_{p+1 \text{ factors}}, \\
\quad a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} \quad (1 \leq i \leq n-2), \\
\quad a_i a_j &= a_j a_i \quad (1 \leq i < j - 1 \leq n-2),
\end{align*}
\]

where \(x_i = \exp(2\pi \sqrt{-1}(k_i + \frac{i^2}{q}))\), \(q = \exp(2\pi \sqrt{-1}h)\) (resp. \(x_i = \exp(2\pi \sqrt{-1}(k_i + \frac{i^2}{q}))\)), \(\phi \in \mathfrak{C}_{\mathfrak{H}^\dagger} \) (resp. \(\phi \in \mathfrak{C}_{\mathfrak{H}_{\mathfrak{X}}^\dagger} \)) (see [BMR] or [A2] for braid relations).

4.2. Put
\[
\begin{align*}
k^\dagger_{c,p+j} &= k_c + \frac{c}{d} \frac{j}{p} - \frac{c \cdot p - j}{r} \quad (0 \leq c \leq d - 1, 0 \leq j \leq p - 1), \\
k_c^\dagger &= k_c + \frac{c}{d} \frac{j}{p} - \frac{c \cdot p + j}{r} \quad (0 \leq c \leq d - 1, 0 \leq j \leq p - 1),
\end{align*}
\]

where we set \(k_0 = k_0 = 0\).

Throughout this section, let \(\mathfrak{H}^\dagger \) (resp. \(\mathfrak{H}_{\mathfrak{X}}^\dagger \) \((X = R \text{ or } K)\)) be the rational Cherednik algebra associated to \(W^\dagger\) with parameters \(\{h, k_1^\dagger, \ldots, k_r^\dagger\} \) (resp. \(\{h, k_1^\dagger, \ldots, k_r^\dagger\} \)) such that \(h = k_{H_{s11}}^\dagger \) (resp. \(h = k_{H_{s11}}^\dagger \)) and \(k_j^\dagger = k_{H_{t1j}}^\dagger \) (resp. \(k_j^\dagger = k_{H_{t1j}}^\dagger \)) for \(1 \leq j \leq r - 1\). Since
\[
\exp \left(2\pi \sqrt{-1}(k_{c,p+j}^\dagger + \frac{c \cdot p + j}{r})\right) = \exp \left(2\pi \sqrt{-1}(k_c + \frac{c}{d} \frac{j}{p})\right) = x_c \xi^j \quad (\xi = \exp(2\pi \sqrt{-1}/p)),
\]

where we put \(x_0 = 1\) (similar for \(k_{c,p+j}^\dagger\)), the defining relation (3.1.2) of \(\mathfrak{H}^\dagger \) (resp. \(\mathfrak{H}_{\mathfrak{X}}^\dagger \)) replaced by
\[
(T_{0}^p - 1)(T_{0}^p - x_1^p) \cdots (T_{0}^p - x_{d-1}^p) = 0.
\]

Since \(\mathfrak{H}_{K}^\dagger \) is semi-simple (thus, \(\mathfrak{O}_{K}^\dagger \) is also semi-simple) by [A1] , we can obtain any results for type \(G(r, 1, n)\) in the previous sections even if the case of these parameters.

By [A2, Proposition 1.6], there is the injective algebra homomorphism \(\varphi : \mathfrak{H}_{\mathfrak{X}}^\dagger \to \mathfrak{H}_{\mathfrak{X}}^\dagger \) \((X = \mathbb{C}, R \text{ or } K)\) such that \(\varphi(a_0) = T_{0}^p, \varphi(a_1^\dagger) = T_{0}^{-1} T_{1} T_{0}, \varphi(a_i) = T_{i} \) \((1 \leq i \leq n - 1)\). Under this injective homomorphism \(\varphi\), we regard \(\mathfrak{H}_{\mathfrak{X}}^\dagger \) as a subalgebra of \(\mathfrak{H}_{\mathfrak{X}}^\dagger \).

4.3. For \(M^\dagger \in \mathbb{C}W^\dagger \)-mod, we denote by \(M^\dagger \downarrow \) the restriction of the action to \(\mathbb{C}W \). For \(\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \) and \(i \in \mathbb{Z}\), we define \(\lambda[i] = (\lambda[i]^{(1)}, \ldots, \lambda[i]^{(r)}) \in \mathcal{P}_{n,r} \) by
\[
\lambda[i]^{(c-p+j)} = \lambda^{(c-p+k)} \quad (0 \leq c \leq d - 1, 1 \leq j \leq p),
\]
where \( c \cdot p < c \cdot p + k \leq (c + 1) \cdot p \) such that \( k \equiv j + i \mod p \). For an example, if \( r = 6 \) and \( p = 3 \), we have

\[
\lambda[1] = (\lambda^{(2)}, \lambda^{(3)}, \lambda^{(1)}, \lambda^{(5)}, \lambda^{(6)}, \lambda^{(4)}),
\]
\[
\lambda[2] = (\lambda^{(3)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(6)}, \lambda^{(4)}, \lambda^{(5)}),
\]
\[
\lambda[3] = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}, \lambda^{(5)}, \lambda^{(6)}) = \lambda.
\]

Let \( \mathfrak{t}_\lambda \) be the minimum positive integer such that \( \lambda[\mathfrak{t}_\lambda] = \lambda \). It is clear that \( \mathfrak{t}_\lambda \mid p \). Put \( \mathfrak{d}_\lambda = p/\mathfrak{t}_\lambda \). Then we have that \( \lambda[i + \mathfrak{t}_\lambda] = \lambda[i] \). Let \( \sim_s \) be the equivalent relation on \( P_{n,r} \times \mathbb{Z}/p\mathbb{Z} \) defined by

\[
(\lambda, j) \sim_s (\lambda[i], c \cdot \mathfrak{d}_\lambda + j) \quad (i, c \in \mathbb{Z}),
\]

where we denote by \( \overline{m} \) the image of \( m \in \mathbb{Z} \) in \( \mathbb{Z}/p\mathbb{Z} \). Let \( \Lambda^+ \) be the set of equivalent classes of \( P_{n,r} \times \mathbb{Z}/p\mathbb{Z} \) with respect to the relation \( \sim_s \), and we denote by \( \lambda(j) \in \Lambda^+ \) the equivalent class containing \( (\lambda, j) \in P_{n,r} \times \mathbb{Z}/p\mathbb{Z} \). Thus we have that \( \lambda(j) = \lambda[i][j] = \lambda[c \cdot \mathfrak{d}_\lambda + j] = \lambda[i][c \cdot \mathfrak{d}_\lambda + j] \) for \( i, c \in \mathbb{Z} \). Then it is known that,

\[
E^{[\lambda][1]} \cong E^{[\lambda][1]} \cong E^{[\lambda][1]} \oplus \cdots \oplus E^{[\lambda][1]} \quad (i \in \mathbb{Z}) \quad \text{as } CW\text{-modules},
\]

for some simple \( CW\text{-modules} \ E^{[\lambda][1]} (1 \leq j \leq \mathfrak{d}_\lambda) \), and \( \{E^{[\lambda][1]} | \lambda(j) \in \Lambda^+\} \) gives a complete set of pairwise non-isomorphic simple \( CW\text{-modules} \). Hence, we have that

\[
\text{Irr } CW = \{E^{[\lambda][1]} | \lambda(j) \in \Lambda^+\}.
\]

Moreover, we have that

\[
E^{[\lambda][1]} \cong E^{[\lambda][1]} \oplus \cdots \oplus E^{[\lambda][1]} \quad \text{as } W^\dagger\text{-modules } (1 \leq j \leq \mathfrak{d}_\lambda).
\]

4.4. For \( M^\dagger \in \mathcal{H}_X^\dagger\text{-mod} \), we denote by \( M^\dagger \downarrow \) the restriction of the action to \( \mathcal{H}_X \). On the other hand, for \( N \in \mathcal{H}_X\text{-mod} \), we denote by \( N^\dagger \) the induced module \( \mathcal{H}_X^\dagger \otimes \mathcal{H}_X N \).

Then, by (4.3.1), we have that

\[
S^\dagger_K(\lambda) \downarrow \cong S_K(\lambda(1)) \oplus \cdots \oplus S_K(\lambda(\mathfrak{d}_\lambda)) \quad \text{for } \lambda \in P_{n,r}
\]

and, by (4.3.2), we have that,

\[
S^\dagger_K(\lambda(j)) \uparrow \cong S^\dagger_K(\lambda[1]) \oplus \cdots \oplus S^\dagger_K(\lambda[\mathfrak{t}_\lambda]) \quad \text{for } \lambda(j) \in \Lambda^+.
\]

We define the group homomorphism \( \text{Res}_X : K_0(\mathcal{H}_X^\dagger\text{-mod}) \to K_0(\mathcal{H}_X\text{-mod}) \) by \([M^\dagger] \mapsto [M^\dagger] \downarrow\). We also define the group homomorphism \( \text{Ind}_X : K_0(\mathcal{H}_X\text{-mod}) \to K_0(\mathcal{H}_X^\dagger\text{-mod}) \) by \([N] \mapsto [N]^\dagger\). Since \( \mathcal{H}_X^\dagger \) is a free right \( \mathcal{H}_X\text{-module} \), induced functor
Lemma 4.5.

(i) For $\lambda \in \mathcal{P}_{n,r}$, we have
\[ [S^\dagger(\lambda)\downarrow] = [S(\lambda(1))] + \cdots + [S(\lambda(\mathcal{D}_\lambda))] \quad \text{in } K_0(\mathcal{H}^\dagger-\text{mod}). \]

(ii) For $\lambda(j) \in \Lambda^+$, we have
\[ [S(\lambda(j))\dagger] = [S^\dagger(\lambda[1])] + \cdots + [S^\dagger(\lambda[\mathcal{D}_\lambda])] \quad \text{in } K_0(\mathcal{H}^\dagger-\text{mod}). \]

Proof. (i) By Lemma 2.4 and (4.4.1), we have
\[ d_{K,C}([S_K^\dagger(\lambda)\downarrow]) = d_{K,C}([S_K(\lambda(1))] + \cdots + [S_K(\lambda(\mathcal{D}_\lambda))]) \]
\[ = [S(\lambda(1))] + \cdots + [S(\lambda(\mathcal{D}_\lambda))]. \]

On the other hand, by the definition of decomposition maps, we have
\[ d_{K,C}([S_K^\dagger(\lambda)\downarrow]) = [S^\dagger(\lambda)\downarrow]. \]

Then (i) was proven. By using (4.4.2) together with Lemma 2.4, we have (ii) in a similar way as in (i). \hfill \square

4.6. We recall some relations between simple $\mathcal{H}$-modules and simple $\mathcal{H}^\dagger$-modules which have been studied in [GJ] and [H5]

Let $\{S^\lambda | \lambda \in \mathcal{P}_{n,r}\}$ be the set of Specht modules of $\mathcal{H}^\dagger$ constructed in [DJM] as seen in the previous section. Then $\{D^{\lambda} | \lambda \in \mathcal{K}_{n,r}\}$ is a complete set of simple $\mathcal{H}^\dagger$-modules.

Let $\sigma$ be the algebra automorphism of $\mathcal{H}^\dagger$ defined by $\sigma(T_0) = \xi T_0$ ($\xi = \exp(2\pi \sqrt{-1}/p))$, $\sigma(T_i) = T_i$ for $i = 1, \ldots, n - 1$. Then we see that the restriction $\sigma|_{\mathcal{H}}$ of $\sigma$ to $\mathcal{H}$ is the identity map on $\mathcal{H}$. We also define the algebra automorphism $\tau$ of $\mathcal{H}^\dagger$ by $\tau(x) = T_0^{-1} x T_0$ for $x \in \mathcal{H}^\dagger$. Then we have that $\tau(\mathcal{H}) = \mathcal{H}$.

For $M^\dagger \in \mathcal{H}^\dagger$-mod, let $(M^\dagger)^{\sigma}$ be the twisted $\mathcal{H}^\dagger$-module of $M$ via $\sigma$. Since $\sigma|_{\mathcal{H}}$ is identity map, we have that $(M^\dagger)^{\sigma} \downarrow \cong M^\dagger \downarrow$ as $\mathcal{H}$-modules. Similarly, for $N \in \mathcal{H}$-mod, let $N^\tau$ be the twisted $\mathcal{H}$-module of $N$ via $\tau$.

For $\lambda \in \mathcal{K}_{n,r}$ and $i \in \mathbb{Z}$, we define $\lambda[i]^\dagger$ by $(D^{\lambda i})^{\sigma} \cong D^{\lambda[i]^\dagger}$. Let $\mathcal{P}_\lambda^i$ be the minimum positive integer such that $\lambda[\mathcal{P}_\lambda^i] = \lambda$ (thus $(D^{\lambda i})^{\sigma \mathcal{P}_\lambda^i} \cong D^{\lambda i}$), and put $\mathcal{D}_\lambda^i = p/\mathcal{P}_\lambda^i$. Let $D$ be a simple $\mathcal{H}$-submodule of $D^{\lambda \dagger}$. Then by [GJ, Lemma 2.2], $\mathcal{D}_\lambda^i$ is the minimum positive integer such that $D^{\mathcal{D}_\lambda^i} \cong D$. Moreover we have that, for $\lambda \in \mathcal{K}_{n,r}$ and $i = 1, \ldots, \mathcal{P}_\lambda^i$,

$$D^{\lambda \dagger} \cong D^{\lambda[i]^\dagger} \cong D \oplus D^r \oplus \cdots \oplus D^{\rho^i_{\lambda - 1}} \quad \text{as } \mathcal{H}\text{-modules}. \quad (4.6.1)$$
Let $\sim_*$ be the equivalent relation on $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ defined by

$$(\lambda, j) \sim_* (\lambda[i]^p, c \cdot \vartheta^\circ + j) \quad (i, c \in \mathbb{Z}).$$

We denote by $(\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$ the set of equivalent classes of $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ with respect to the relation $\sim_*$, and we denote by $\lambda(j)^p \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$ the equivalent class containing $(\lambda, j) \in \mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$. Then, by [GJ, Lemma 2.2] (see also [H5, Proposition 2.4]),

$$\{ D^{\lambda(j)^p} \mid \lambda(j)^p \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_* \}$$

gives a complete set of pairwise non-isomorphic simple $\mathcal{H}$-modules, where we put $D^{\lambda(j)^p} = D^{\tau_j^\circ}$ for some simple $\mathcal{H}$-submodule $D$ of $D^{1_{\mathcal{H}}} \downarrow$ (see (4.6.1)).

By [GJ, Lemma 2.2], we also have that, for $\lambda(j)^p \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$,

$$D^{\lambda(j)^p} \uparrow \cong D^{1_{\mathcal{H}}} \uparrow \oplus \cdots \oplus D^{1_{\mathcal{H}}} \uparrow$$

as $\mathcal{H}$-modules.

**Remarks 4.7.**

(i) For $\lambda \in \mathcal{K}_{n,r}, \lambda[i]^p (1 \leq i \leq \ell \lambda)$ is described in [H2] (the case of type D), [H4] (the case of type $G(r, r, n)$) and [GJ], [H5] (general case).

(ii) Recall that $\{ D(\lambda(i)^t) \mid \lambda(i)^t \in \Lambda^+_0 \}$ gives a complete set of non-isomorphic simple $\mathcal{H}$-modules (Lemma 1.3). Hence, there exists the bijection $\eta : \Lambda^+_0 \rightarrow (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$ such that $D(\lambda(i)^t) \cong D^{\eta(\lambda(i)^t)}$.

Now we have the following proposition.

**Proposition 4.8.** For $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \mathcal{K}_{n,r}$, we have the following.

\begin{enumerate}
  \item \[
  \sum_{s=1}^{\vartheta^\circ_\lambda} [S(\lambda(s)) : D^{\mu(i)^t}]_{\mathcal{H}} = \sum_{t=1}^{\vartheta^{\circ \mu}_\lambda} [S^{t_{\mathcal{H}}} : D^{1_{\mathcal{H}}} \downarrow]_{\mathcal{H}} \quad (1 \leq i \leq \vartheta^\circ_\mu, 1 \leq j \leq \ell \lambda).
  \]
  \item \[
  \sum_{s=1}^{\vartheta_\lambda} [S(\lambda(i)) : D^{\mu(i)^t}]_{\mathcal{H}} = \sum_{t=1}^{\vartheta_\mu} [S_{t_{\mathcal{H}}} : D^{1_{\mathcal{H}}} \downarrow]_{\mathcal{H}} \quad (1 \leq i \leq \vartheta_\lambda, 1 \leq j \leq \ell \mu).
  \]
\end{enumerate}

**Proof.** Let

$$S^{t_{\mathcal{H}}} = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

be a composition series of $S^{t_{\mathcal{H}}}$ in $\mathcal{H}$-mod such that $M_l / M_{l-1} \cong D^{1_{\mathcal{H}}}$. Applying the restriction functor, we have the filtration

$$S^{t_{\mathcal{H}}} \downarrow = M_k \downarrow \supset M_{k-1} \downarrow \supset \cdots \supset M_1 \downarrow \supset M_0 \downarrow = 0$$

such that $M_l \downarrow / M_{l-1} \downarrow \cong D^{1_{\mathcal{H}}} \downarrow$ in $\mathcal{H}$-mod. Thus, by (4.6.1), we have

$$[S^{t_{\mathcal{H}}} : D^{\mu(i)^t}]_{\mathcal{H}} = \sum_{t=1}^{\vartheta_\mu} [S^{t_{\mathcal{H}}} : D^{1_{\mathcal{H}}} \downarrow]_{\mathcal{H}}.$$
On the other hand, by (3.5.2) and Lemma 4.5 (i) together with $S^j \downarrow S^j$, (4.8.1) and (4.8.2) imply (i). Next we prove (ii). Let

\[(4.8.2)\]

\[S(\lambda(i)) = N_k \supset N_{k-1} \supset \cdots \supset N_1 \supset N_0 = 0\]

be a composition series of $S(\lambda(i))$ in $\mathcal{H}$-mod such that $N_i/N_{i-1} \cong D^\mu(i)^\mu$. Applying the induced functor, we have the filtration

\[S(\lambda(i))^\uparrow = N_k^\uparrow \supset N_{k-1}^\uparrow \supset \cdots \supset N_1^\uparrow \supset N_0^\uparrow = 0\]

such that $N_i^\uparrow /N_{i-1}^\uparrow \cong D^\mu(i)^\mu$ in $\mathcal{H}^\uparrow$-mod. Thus, by (4.6.2), we have

\[(4.8.3)\]

\[\big[ S(\lambda(i))^\uparrow : D^{i_1^\mu(j_1)^\mu} \big]_{\mathcal{H}^\uparrow} = \sum_{s=1}^{\delta_{\lambda}} \big[ S(\lambda(s)) : D^{\mu(s)^\mu} \big]_{\mathcal{H}^\uparrow}.\]

On the other hand, by (3.5.2) and Lemma 4.5 (ii), we have

\[(4.8.4)\]

\[\big[ S(\lambda(i))^\uparrow : D^{i_2^\mu(j_2)^\mu} \big]_{\mathcal{H}^\uparrow} = \sum_{t=1}^{\tau_{\lambda}} \big[ S^{j_2^\lambda(t)} : D^{i_2^\mu(j_2)^\mu} \big]_{\mathcal{H}^\uparrow}.\]

(4.8.3) and (4.8.4) imply (ii).

**4.9.** Recall that "$\sim_R$" is the residue equivalence on $\mathcal{P}_{n,r}$ defined in the previous section. We define an equivalent relation "$\approx$" on $\mathcal{P}_{n,r}$ by $\lambda \approx \mu$ if $\lambda \sim_R \mu[j]$ for some $j \in \mathbb{Z}$. Put

\[\Gamma = \{ \lambda \in \mathcal{P}_{n,r} \mid \lambda \not\sim_R \mu \text{ for any } \mu \in \mathcal{P}_{n,r} \text{ such that } \mu \neq \lambda \}.\]

We see easily that $\lambda \sim_R \mu$ if and only if $\lambda[i] \sim_R \mu[i]$ for any $i \in \mathbb{Z}$. Thus, we have that $\lambda[i] \in \Gamma$ if $\lambda \in \Gamma$. Then we have the following proposition.

**Proposition 4.10.** For $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$, we have that

\[S(\lambda(1)) \sim S(\lambda(2)) \sim \cdots \sim S(\lambda(\varphi_{\lambda})).\]

**Proof.** If $\varphi_{\lambda} = p$, there is nothing to prove since $\varphi_{\lambda} = 1$. Hence, we assume that $\varphi_{\lambda} \neq p$. First, we show the following claim.

**(claim):** For $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$ such that $\varphi_{\lambda} \neq p$, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$, and that $\varphi_{\mu} = p$ (thus $\varphi_{\mu} = 1$).

Since $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$ and $\mu \neq \lambda$. By [LM, Theorem 2.11], it is known that $\lambda \sim_R \mu$ if and only if $\lambda \sim_J \mu$, where "$\sim_J$"
is the Jantzen equivalence on $\mathcal{P}_{n,r}$ (see [LM, Definition 2.8] for definitions). By the
definition of the Jantzen equivalence, we may assume that $\mu$ obtained by unwrapping
a rim hook $r_\lambda^\mu$ from $\lambda$, and wrapping another rim hook $r_\mu^\nu$ from $[\lambda] \setminus r_\lambda^\mu$. Namely,
we have $[\lambda] \setminus r_\lambda^\mu = [\mu] \setminus r_\mu^\nu$ (See [LM] for notations here). Suppose that $x \in \lambda(i)$ and
$y \in \mu(j)$. Then $[\lambda] \setminus r_\lambda^\mu = [\mu] \setminus r_\mu^\nu$ implies that

\begin{equation}
(4.10.1)\quad \lambda(i) \neq \mu(i), \lambda(j) \neq \mu(j) \text{ and } \lambda(l) = \mu(l) \text{ for } l \neq i, j.
\end{equation}

Note that $\mu(i) \neq \mu(j)$ if $\lambda(i) = \lambda(j)$ and $i \neq j$. Thus, we have that $\mu(i) \neq \mu(l)$ for any
$l \neq i$ such that $l \equiv i \mod \varv$, and $c \cdot p < l \leq (c+1) \cdot p$ when $c \cdot p < i \leq (c+1) \cdot p$.
This implies that

\begin{equation}
(4.10.2)\quad \varv_\lambda \nmid \varv_\mu \text{ unless } \varv_\mu = p.
\end{equation}

In the case where $p$ is a prime number, (4.10.2) implies $\varv_\mu = p$ since $\varv_\lambda = 1$ by
$\varv_\lambda \mid p$ and $\varv_\lambda \neq p$. In the case where $p = 4$, one can easily check that $\varv_\mu = p$ directly.
Let $p \geq 6$ be not a prime number. Assume that $\varv_\mu \neq p$. Then we have $\varv_\lambda \nmid \varv_\mu$ by
(4.10.2). In a similar way as in the above arguments, we have $\varv_\mu \nmid \varv_\lambda$ (note that
$\varv_\lambda \neq p$). By the conditions $p \geq 6$, $\varv_\lambda \nmid \varv_\mu$ and $\varv_\mu \nmid \varv_\lambda$, one sees that there are at least
three integers $x_1, x_2, x_3$ such that $\lambda(x_i) \neq \mu(x_i)$ ($l = 1, 2, 3$). However, this contradicts
to (4.10.1). Thus we have $\varv_\mu = p$, and the claim was proved.

Thanks to the claim, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$, and that $\varv_\mu = 1$.
Then we can take a sequence $\lambda = \lambda_0, \cdots, \lambda_k = \mu$ satisfying the following two
conditions:

- $S^{1\lambda_{i-1}}$ and $S^{1\lambda_i}$ have a common composition factor $D^{1\varv_{\lambda_i}}$.
- There exists an integer $l$ such that $\varv_{\lambda_i} \neq 1$ for any $i < l$, and that $\varv_{\lambda_i} = 1$.

By Proposition 4.8 (i), one sees that $S(\lambda_i(1))$ has a composition factor $D^{\nu(i)}_{\lambda_i(1)}$ for
any $i \in \{1, \cdots, \lambda_{i-1}\}$ (Note that $\varv_{\lambda_i(1)} = 1$). On the other hand, by Proposition 4.8
(ii), one sees that $S(\lambda_{i-1}(j)) (1 \leq j \leq \varv_{\lambda_{i-1}})$ has a composition factor $D^{\nu(i)}_{\lambda_{i-1}(j)}$
for some $i \in \{1, \cdots, \lambda_{i-1}\}$. Thus, we have that $S(\lambda_i(1)) \sim S(\lambda_{i-1}(j))$ for any $j = 1, \cdots, \varv_{\lambda_{i-1}}$. This implies that $S(\lambda_{i-1}(1)) \sim S(\lambda_{i-1}(2)) \sim \cdots \sim S(\lambda_{i-1}(\varv_{\lambda_{i-1}}))$. By
using the (backward) inductive argument combined with Proposition 4.8, we have the
proposition.

\begin{theorem}

\begin{enumerate}
\item[(i)] For $\lambda \in \Gamma$ and $i = 1, \cdots, \varv_{\lambda}$, we have that $S(\lambda(i))$ (resp. $\Delta(\lambda(i))$) is
a simple $\mathcal{H}$-module (resp. a simple object of $\mathcal{O}$). Moreover, $S(\lambda(i))$ (resp. $\Delta(\lambda(i))$) is a block of $\mathcal{H}$ (resp. of $\mathcal{O}$) itself.
\item[(ii)] For $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$ and $i, j \in \mathbb{Z}$, we have that

\[ \Delta(\lambda(i)) \sim \Delta(\mu(j)) \iff S(\lambda(i)) \sim S(\mu(j)) \iff \lambda \approx \mu. \]
\end{enumerate}
\end{theorem}

\begin{proof}
Suppose that $S(\lambda(i))$ and $S(\mu(j))$ have a common composition factor $D^{\nu(j)}_{\lambda(j)}$.
Then, by Proposition 4.8 (ii), $S^{1\lambda(j)}$ and $S^{1\mu(j)}$ have a common composition factor
for some \( i', j' \). This implies that

\[
S(\lambda(i)) \sim S(\mu(j)) \quad \text{only if } \lambda \approx \mu.
\]

(i) Suppose that \( \lambda \in \Gamma \), then \( S^{t \lambda} \) is a simple \( \mathcal{R}^{t} \)-module from the definition of \( \Gamma \). If \( S(\lambda(i)) \sim S(\mu(j)) \) for some \( \mu(j) \in \Lambda^{+} \), we have that \( \lambda \approx \mu \) by (4.11.1). This implies that there exists an integer \( l \) such that \( \lambda = \mu[l] \) since \( \lambda \in \Gamma \). Thus, we have that \( \mu(j) = \mu[l]j = \lambda(j) \) from the definition of \( \Lambda^{+} \). Now we may assume that \( S(\lambda(i)) \) and \( S(\lambda(j)) \) have a common composition factor \( D^{\mu(k)} \). If \( \lambda(i) \neq \lambda(j) \) (i.e. \( i \not\equiv j \mod \mathfrak{d}_{\lambda} \)), we have \( \sum_{s=1}^{\mathfrak{d}_{\lambda}} [S(\lambda(s)) : D^{\mu(k)}] \geq 2 \). On the other hand, we have \( \sum_{s=1}^{\mathfrak{d}_{\lambda}} [S^{t \lambda} : D^{\mu(l)}] \leq 1 \) since \( S^{t \lambda} \) is simple. These contradict to Proposition 4.8 (i). Thus we have \( \lambda(i) = \lambda(j) = \mu(j) \). This implies (i).

Next we prove (ii). For \( \lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma \), suppose that \( S^{t \lambda} \) and \( S^{t \mu} \) have a common composition factor \( D^{t \nu} \). Then, by Proposition 4.8 (i), \( S(\lambda(i)) \) and \( S(\mu(j)) \) have a common composition factor \( D^{\nu(l)} \) for some \( i, j \) (and for any \( l \)). Thus, \( S(\lambda(i)) \sim S(\mu(j)) \). Combining Proposition 4.10 and (4.11.1), we obtain the theorem.

\[\square\]

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Graduate School of Mathematics Nagoya University, Furocho, Chikusaku, Nagoya, Japan 464-8602

E-mail address: kentaro-wada@math.nagoya-u.ac.jp