GENERALIZED ORBITAL VARIETIES FOR MIRKOVIĆ–VYBORNOV SLICES AS AFFINIZATIONS OF MIRKOVIĆ–VILONEN CYCLES

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ABSTRACT. We show that generalized orbital varieties for Mirković–Vybornov slices can be indexed by semi-standard Young tableaux. We also check that the Mirković–Vybornov isomorphism sends generalized orbital varieties to (dense subsets of) Mirković–Vilonen cycles, such that the (combinatorial) Lusztig datum of a generalized orbital variety, which it inherits from its tableau, is equal to the (geometric) Lusztig datum of its MV cycle.

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1. Introduction

1.1. Main result. In this paper, we show that generalized orbital varieties for Mirković–Vybornov slices are in bijection with semi-standard Young tableaux, and, via the Mirković–Vybornov isomorphism [MV07], can be identified with MV cycles.

Let $T(\lambda)_\mu$ denote the set of semi-standard Young tableaux of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ and weight $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m)$. Let $N = \sum_1^\ell \lambda_i = \sum_1^m \mu_i$. We order repeated entries of a tableau from left to right, so that the first occurrence of a given entry is its leftmost. Then, for $\tau \in T(\lambda)_\mu$ and $(i, k) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, \mu_i\}$, we denote by $\lambda^{(i,k)}_\tau$ (respectively, by $\mu^{(i,k)}_\tau$) the shape (respectively, the weight) of the tableau obtained from $\tau$ by deleting all $j > i$ and all but the first $k$ occurrences of $i$.

For ease of notation we identify $\lambda^{(i)}_\tau \equiv \lambda^{(i,\mu_i)}_\tau$, $\mu^{(i)}_\tau \equiv \mu^{(i,\mu_i)}_\tau$ and $\tau^{(i)} \equiv \tau^{(i,\mu_i)}$ and when there is no confusion we omit the subscript $\tau$.

Example 1. Let $\tau = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$. Then $\tau^{(2)} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$ has shape $\lambda^{(2)} = (3, 1)$ and weight $\mu^{(2)} = (2, 2)$, while $\tau^{(2,1)} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ has shape $\lambda^{(2,1)} = (2, 1)$ and weight $\mu^{(2,1)} = (2, 1)$.

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1The weight of a tableau in the alphabet $\{1, 2, \ldots, m\}$ is the $m$-tuple of non-negative integers whose $i$th entry is the number of times $i$ appears in the tableau.
The array \((\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})\) is called the GT-pattern of \(\tau\) (see [BZ88 Section 4]). Note that \(\tau\) can be reconstructed from its GT-pattern. Moreover, as we now describe, \(\tau\) defines a matrix variety through its GT-pattern.

Let \(\text{Mat}(N)\) denote the algebra of \(N \times N\) complex matrices. Let \(\mathcal{O}_\lambda\) denote the conjugacy class of the Jordan normal form \(J_\lambda\) associated to \(\lambda\), and let \(T_\mu\) denote the Mirković–Vybornov slice through the Jordan normal form \(J_\mu\) associated to \(\mu\). Elements of \(T_\mu\) take the form \(J_\mu + T\) for \(T \in \text{Mat}(N)\) any \(\mu \times \mu\)-block matrix with zeros everywhere except perhaps the first \(\min(\mu_1, \mu_j)\) columns of the last row of the \(\mu_i \times \mu_j\) block, for \(1 \leq i, j \leq m\). For example, elements of \(T_{(3,2,1)}\) take the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{pmatrix}
\]

with \(\ast\)s denoting unconstrained entries. Note, \(\dim T_\mu = \sum_{i=1}^m (2i - 1) \mu_i\).

Let \(\beta_\mu = (e_1^{(1)}, \ldots, e_1^{(\mu_1)}, \ldots, e_m^{(1)}, \ldots, e_m^{(\mu_m)})\) be a \(\mu\)-enumeration of the standard basis of \(\mathbb{C}^N\), and let \(V^{(i,k)}\) denote the span of the first \(\mu_1 + \cdots + \mu_{i-1} + k\) vectors of \(\beta_\mu\) for \((i, k) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, \mu_i\}\). We’ll also identify \(V^{(i,\mu_i)} = V^{(i,\mu)}\) for all \(i\).

Let \(n \subset \text{Mat}(N)\) denote the subalgebra of upper-triangular matrices and consider

\[
X_\tau = \{ A \in T_\mu \cap n : A|_{V^{(i,\mu)}} \in \mathcal{O}_{\lambda^{(i)}} \text{ for } 1 \leq i \leq m \}.
\]

Here, we are identifying \(A|_{V^{(i,\mu)}}\) with the top left \(N_i \times N_i\) submatrix of \(A\) for \(N_i = \sum_{j=1}^i \mu_j = \sum_{j=1}^i \lambda^{(i)}_j = \sum_{j=1}^i \lambda^{(i)}_j\).

**Example 2.** Let \(\tau = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & \end{pmatrix}\) as before. Then

\[
X_\tau = \begin{cases} 
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & d \\
0 & 0 & 0 & 0 
\end{pmatrix} : a, d = 0 \text{ and } b, c \neq 0 \end{cases}.
\]

**Theorem A.** \(X_\tau\) has one component \(X_\tau^d\) of maximum dimension \(d\) which can be computed from \(\tau\), or (independent of \(\tau\)) from \(\lambda\) and \(\mu\). Moreover, the closure \(Z_\tau = X_\tau^d\) is an irreducible component of \(\mathcal{O}_\lambda \cap T_\mu \cap n\), and conversely, every irreducible component of \(\mathcal{O}_\lambda \cap T_\mu \cap n\) is of this form.

The latter half of this claim is due to [ZJ15] where it is stated without proof. We call \(Z_\tau\) a **generalized orbital variety** for the Mirković–Vybornov slice \(T_\mu\), for when \(\tau\) is a standard Young tableau, so \(\mu = (1, \ldots, 1)\), \(T_\mu = \text{Mat}(N)\) and the decomposition \(\mathcal{O}_\lambda \cap n = \cup_{\sigma \in S(\lambda)} Z_\sigma\) recovers the ordinary orbital varieties of [Jos84] by [Spa76].

Now, \(\lambda\) and \(\mu\) can also be viewed as coweights of \(G = GL(m, \mathbb{C})\) parametrizing MV cycles via their images \(L_\lambda\) and \(L_\mu\) in the affine Grassmannian \(\text{Gr} = G(K)/G(O)\) of \(G\). Here \(O = \mathbb{C}[t]\) and \(K = \mathbb{C}(t)\).

Let \(T \subset G\) be a maximal torus and consider the homomorphism which identifies \(z_i \in X_\mu(T) = \text{Hom}(\mathbb{C}^X, T)\) and \(e_i \in \mathbb{Z}^m\) such that \(z_i(t) = t^{e_i}\), is the diagonal matrix with \((k, k)\) entry equal to \(t\) if \(k = i\) and \(1\) if \(k \neq i\). Thus \(\nu \in \mathbb{Z}^m\) defines \(t^\nu \in G(K)\) which in turn defines \(L_\nu = t^\nu G(O) \subset \text{Gr}\).

Let \(G(O)\) denote the \(G(O)\) orbit of \(L_\lambda\) and let \(S_\lambda^d\) denote the \(U_-(K)\) orbit of \(L_\mu\). Here, \(U_- \subset G\) denotes the subgroup of invertible lower-triangular matrices. **MV cycles of coweight** \((\lambda, \mu)\) are defined as the irreducible components of \(G(O) \cap S_\mu^d\). By [MV07] they give a basis of the \(\mu\)-weight space of the highest weight \(\lambda\) irreducible representation \(L(\lambda)_\mu\) of \(G\). In this case \(\nu^d\) for \(\nu^d\geq 1\).

Let \(\Phi^+\) denote the set of positive coroots \(\{\alpha_i + \cdots + \alpha_j : 1 \leq i < j \leq m\}\) for \(\alpha_i = z_i - z_{i-1}\) in \(\mathbb{Z}^m\). By [Kar90 Theorem 4.2] the MV cycles are parametrized by their \(i\)-Lusztig data, which are \(\Phi^+\)-tuples of non-negative integers ordered by a choice of reduced word \(w_0\) for the longest element \(w_0\) in the Weyl group of \(G\), and computed intrinsically in \(\text{Gr}\).
Elements of $T(\lambda)_\mu$ also acquire i-Lusztig data in $\mathbb{N}^{\mathfrak{b}^+}$ from their GT-patterns (see Section 3, Equation (6)). Let us fix the parametrization $i = (12\ldots m121)$ inducing the order

$$z_1 - z_2 < \cdots < z_1 - z_m < z_2 - z_3 < \cdots < z_2 - z_m < \cdots < z_m - z_{m-1} < z_m,$$

and henceforth omit $i$ from the notation.

**Theorem B.** The Mirković–Vybornov isomorphism, restricts to an isomorphism $\psi$ of $\mathfrak{O}_\lambda \cap \mathfrak{T}_\mu \cap \mathfrak{n}$ and $\text{Gr}^\lambda \cap S^n$ such that $\psi(\mathcal{Z}_\tau)$ is dense in an MV cycle with Lusztig datum equal to the Lusztig datum of $\tau$.

In particular, the Mirković–Vybornov isomorphism induces a Lusztig data preserving bijection between MV cycles of coweight $(\lambda, \mu)$ and semi-standard Young tableaux of shape $\lambda$ and weight $\mu$.

1.2. Applications and relation to other work.

1.2.1. Measures of MV cycles. By [MV07], MV cycles yield a basis in representations of $G$. In [BKK19], the authors show that, combinatorially, this basis is the same as Lusztig’s dual semi-canonical basis. In the appendix to [BKK19], the appendix authors show that, geometrically, these bases are different. Our comparison relies on Theorems A and B together with results of [BKK19] on what geometric equality would entail. In particular, from Equation (1) we can determine the ideal $I_\tau$ of $X_\tau$. In turn, by normalization we can obtain from $I_\tau$ the ideal of $\psi(\mathcal{Z}_\tau)$. Thus the title of this paper.

1.2.2. Big Springer fibres. Set $\text{GL}(N) \equiv \text{GL}(N, \mathbb{C})$. Given a partition $\nu \vdash N$, let $P_\nu \subset \text{GL}(N)$ be the corresponding parabolic subgroup and denote by $\mathfrak{p}_\nu$ its Lie algebra. We’ll view elements of the partial flag variety $X_\nu := \text{GL}(N)/P_\nu$ interchangeably as parabolic subalgebras of $\text{Mat}(N)$ which are conjugate to $\mathfrak{p}_\nu$ and as flags $0 = V_0 \subset V_1 \subset \cdots \subset V_\nu = \mathbb{C}^N$ such that $\dim V_i/V_{i-1} = (\nu_i^T)i$ for $i = 1, \ldots, \nu_1$. Here $\nu_i^T$ denotes the conjugate partition of $\nu$.

Shimomura, in [Shi80], establishes a bijection between components of big Springer fibres $(X_\mu)^u$, for fixed $u - 1 \in \mathfrak{O}_\lambda$, and $T(\lambda)_\mu$, generalizing Spaltenstein’s decomposition in [Spa76] in case $\mu = (1, \ldots, 1)$, and implying that big Springer fibres also have the same number of top-dimensional components as $\mathfrak{O}_\lambda \cap \mathfrak{T}_\mu \cap \mathfrak{n}$.

We conjecture that the coincidence is evidence of a correspondence implying a bijection between the top-dimensional irreducible components of $\mathfrak{O}_\lambda \cap \mathfrak{p}_\mu$ and $\mathfrak{O}_\lambda \cap \mathfrak{T}_\mu \cap \mathfrak{n}$.

Let $\mathcal{N}$ denote the nilpotent cone in $\text{Mat}(N)$. Let $\tilde{\mathfrak{g}}_\mu = \{(A, V_\mu) \in \mathcal{N} \times X_\mu : AV_i \subset V_i \text{ for } i = 1, \ldots, (\mu^T)_1 \}$. Equivalently, $\tilde{\mathfrak{g}}_\mu = \{(A, p) \in \mathcal{N} \times X_\mu : A \in \mathfrak{p} \}$. Let $A = u - 1 \in \mathfrak{O}_\lambda$ and consider the restriction of $\text{pr}_1 : \tilde{\mathfrak{g}}_\mu \to \mathcal{N}$ defined by $\text{pr}_1(A, p) = A$ to $\tilde{\mathfrak{g}}_\mu^\lambda$. We conjecture that the (resulting) diagram

$$\begin{array}{c}
\mathfrak{O}_\lambda \cap \mathfrak{p}_\mu & \longleftarrow & \{p_\mu\} \\
\downarrow & & \downarrow \\
(X_\mu)^u & \longleftarrow & \tilde{\mathfrak{g}}_\mu^\lambda & \longleftarrow & X_\mu \\
\downarrow & & & & \downarrow \\
\{A\} & \longleftarrow & \mathfrak{O}_\lambda
\end{array}$$

has an orbit-fibre duality (generalizing the bijections established in [CG09] §6.5) when $\mu = (1, \ldots, 1)$ and $\mathfrak{O}_\lambda \cap \mathfrak{p}_\mu = \mathfrak{O}_\lambda \cap \mathfrak{n}$ such that the maps $\mathfrak{O}_\lambda \cap \mathfrak{p}_\mu \to \tilde{\mathfrak{g}}_\mu^\lambda \leftarrow (X_\mu)^u$ give bijections on top dimensional irreducible components.

1.2.3. Symplectic duality of small Springer fibres. By [Web17, Theorem 5.37], the restriction of the parabolic analogue of the Grothendieck–Springer resolution $\pi : T^*X_\mu \to \mathfrak{O}_\mu \cap \mathfrak{T}_\mu$ to $X_\mu^\lambda = \pi^{-1}(\mathfrak{O}_\mu \cap \mathfrak{T}_\nu) = \{(A, V_\mu) \in \mathfrak{O}_\mu \cap \mathfrak{T}_\mu : AV_i \subset V_i \text{ for all } i = 1, \ldots, (\mu^T)_1 \}$ is symplectic dual to $\pi^! : X_\mu^\lambda \to \mathfrak{O}_\lambda \cap \mathfrak{T}_\mu$. A hard consequence of this is that $H^{top}(\pi^!(J_{\lambda^T})) = H^{top}(\mathfrak{O}_\mu \cap \mathfrak{T}_\mu \cap \mathfrak{n})$ where $J_{\lambda^T} \in \mathfrak{O}_{\lambda^T} \cap \mathfrak{T}_{\lambda^T}$ denotes the Jordan normal form associated to $\lambda^T$. Haines, in [Hai06], establishes a bijection between components of $\pi^{-1}(J_{\lambda^T})$ and $T(\lambda)_\mu$. Thus, symplectic duality indirectly predicts that components of $\mathfrak{O}_\lambda \cap \mathfrak{T}_\mu \cap \mathfrak{n}$ are in bijection with $T(\lambda)_\mu$.

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2. Generalized orbital varieties for Mirković–Vybornov slices

We begin by giving a more tractable description of the sets defined by Equation (1).

2.1. A boxy description of $X_{\tau}$.

**Lemma 1.** Let $B$ be an $(N - 1) \times (N - 1)$ matrix of the form

$$[C \ v]$$

for some $(N - 2) \times (N - 2)$ matrix $C$ and column vector $v$. Let $A$ be an $N \times N$ matrix of the form

$$[C \ v \ w]$$

for some column vector $w$. Let $p \geq 2$. If rank $C^p < \text{rank } B^p$, then rank $B^p < \text{rank } A^p$.

**Proof of Lemma 1.** Let

$$B = \begin{bmatrix} C & v \\ 0 & 0 \end{bmatrix}$$

and let

$$A = \begin{bmatrix} C & v & w \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} B & w \\ 0 & 0 \end{bmatrix}.$$

Suppose rank $B^p > \text{rank } C^p$ for $p \geq 0$. Clearly rank $A^p > \text{rank } B^p$ for $p = 0, 1$ independent of the assumption. Suppose $p \geq 2$. Since

$$B^p = \begin{bmatrix} C^p & C^{p-1}v \\ 0 & 0 \end{bmatrix}$$

this means $C^{p-1}v \notin \text{Im } C^p$. So $C^{p-2}v \notin \text{Im } C^{p-1}$ and $C^{p-2}v + C^{p-1}w \notin \text{Im } C^{p-1}$. Since

$$A^p = \begin{bmatrix} C^p & C^{p-1}v & C^{p-1}w + C^{p-2}v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

it follows that rank $A^p > \text{rank } B^p$ as desired. \hfill \Box

Now fix $A \in X_{\tau}$, with $\tau \in T(\lambda)_m$ as above. Recall that $V^{(i,k)}$ denotes the span of the first $\mu_1 + \cdots + \mu_i - 1 + k$ vectors of $\beta = (e_1^1, \ldots, e_1^{\mu_1}, \ldots, e_m^1, \ldots, e_m^{\mu_m})$.

**Lemma 2.** $A|_{V^{(m,\mu_m - 1)}} \in \Omega_{(m,\mu_m - 1)}$.

**Proof.** Let $b = A|_{V^{(m-1)}}$ and $B = A|_{V^{(m,\mu_m - 1)}}$. Assume $\mu_m > 1$ or else $b = B$ and there is nothing to show. Let $C = A|_{V^{(m,\mu_m - 2)}}$.

By definition of $X_{\tau}$, $A \in \Omega_{\lambda}$ and $b \in \Omega_{\lambda(m-1)}$. Let $\lambda(B)$ denote the the Jordan type of $B$ and $\lambda(C)$ the Jordan type of $C$. Since dim $V/V^{(m-1)} = \mu_m$ is exactly the number of boxes by which $\lambda$ and $\lambda^{(m-1)}$ differ, $\lambda(B)$ must contain one less box than $\lambda$, and $\lambda(C)$ must contain one less box than $\lambda(B)$. Let $c(A)$ denote the column coordinate of the box by which $\lambda$ and $\lambda(B)$ differ, and let $c(B)$ denote the column coordinate of the box by which $\lambda(B)$ and $\lambda(C)$ differ. Then

$$\text{rank } B^p - \text{rank } C^p = \begin{cases} 1 & p < c(B) \\ 0 & p \geq c(B) \end{cases},$$

so we can apply Lemma 1 to our choice of $(A, B, C)$ to conclude that rank $A^p > \text{rank } B^p$ for $p < c(B)$. At the same time,

$$\text{rank } A^p - \text{rank } B^p = \begin{cases} 1 & p < c(A) \\ 0 & p \geq c(A) \end{cases}$$

implies that $c(A) > c(B)$. We conclude that $B \in \Omega_{\lambda(m,\mu_m - 1)}$ as desired. \hfill \Box

The blocky rank conditions defining $X_{\tau}$ in Equation (1) can thus be refined to boxy rank conditions.
Proposition 1.  
(2) \[ X_r = \{ A \in T_{\mu} \cap n : A|_{V(\mu, \lambda \cap k)} \in \otimes_{\lambda \cap k} \} \text{ for } 1 \leq k \leq \mu_i \text{ and } 1 \leq i \leq m \}. \]

Proof. The non-obvious direction of containment is an immediate consequence of Lemma 2. \[ \square \]

2.2. Irreducibility of \( X_r \). We now prove that our matrix varieties are irreducible in top dimension. For \( 1 \leq i \leq m \), let \( \rho(i) = i z_1 + (i - 1) z_2 + \cdots + z_i \) in \( \mathbb{Z}' \) be the familiar half sum of positive coroots for \( G_i = GL(i, \mathbb{C}) \), set \( \rho \equiv \rho(m) \) and let \( \langle , \rangle \) denote the dot product.

Proposition 2. \( X_r \) has one irreducible component \( X_r \) of maximum dimension \( d = \langle \lambda - \mu, \rho \rangle \).

Set \( \tau - \lceil m \rceil \equiv \tau^{(m, \mu_m - 1)} \), and let \( r \) equal to the row coordinate of the last \( m \), aka the row coordinate of the box by which \( \tau \) and \( \tau - \lceil m \rceil \) differ.

Lemma 3. The map \( X_r \to X_{\tau - \lceil m \rceil} : A \mapsto A|_{V(\mu, \mu_m - 1)} \)
has irreducible fibres of dimension \( m - r \).

Proof. Let \( B \in X_{\tau - \lceil m \rceil} \) and let \( F_B \) denote the fibre over \( B \). We'll show that
\[ F_B \cong (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \cap L \setminus (B^{\lambda, -2})^{-1} \text{Im} B^{\lambda, -1} \cap L \]
for \( L = \text{Span}_C(e_1^\mu, \ldots, e_{m-1}^\mu) \). The dimension count will then follow by Lemma 4 below.
Assume \( \mu_m > 1 \) and let \( A \in F_B \) take the form
\[ A = \begin{bmatrix} B^v & e^\mu & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} B^v & 1 \\ 0 & 0 \end{bmatrix} \]
with \( v \in L \). Let \( u \in \text{Ker} A^{\lambda, r} \setminus \text{Ker} A^{\lambda, -1} \) and suppose without loss of generality \( u = e_{m}^\mu + w \) for some \( w \in V(\mu, \mu_m - 1) \). Then
\[ 0 = A^{\lambda, r}(u) = A^{\lambda, r}(e_{m}^\mu + w) = B^{\lambda, -1}(v + e_{m}^\mu) + B^{\lambda, r}(w). \]
That is \( v + e_{m}^\mu \in (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \) is uniquely specified by an element in \( (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} + e_{m}^\mu \) which is isomorphic to \( (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \setminus V(\mu, \mu_m - 1) \) since \( V(\mu, \mu_m - 1) = (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \).

In turn, the isomorphism
\[ (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \setminus (B^{\lambda, -2})^{-1} \text{Im} B^{\lambda, -1} \cap L \to F_B : w \mapsto \begin{bmatrix} B^w & e_{m}^\mu + w \\ 0 & 0 \end{bmatrix}, \]
of \( F_B \) and the locally closed set on the lefthand side, where note \( (B^{\lambda, -2})^{-1} \text{Im} B^{\lambda, -1} \) is excluded, since \( A^{\lambda, -1}(u) \neq 0 \), proves that \( F_B \) irreducible. By Lemma 4 below, \( F_B \) has dimension \( m - r \). \[ \square \]

Lemma 4. Let \( B \in X_{\tau - \lceil m \rceil} \) Then
(3a) \( \dim (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} = N - r \), and
(3b) \( \dim (B^{\lambda, -1})^{-1} \text{Im} B^{\lambda, -1} \cap L = m - r \).

Proof. By Lemma 2, \( B \) has Jordan type \( \lambda^{(m, \mu_m - 1)} \) which differs from \( \lambda \) by a single box in position \( (r, \lambda_r) \). Let \( J = \{ f_i^1, \ldots, f_i^{\lambda_r}, f_r^1, \ldots, f_r^{\lambda_r - 1}, \ldots, f_1^1, \ldots, f_1^{\lambda_r - 1} \} \) be a Jordan basis for \( B \). Then, with respect to \( J \),
\[ \text{Im} B^{\lambda, -1} = \text{Span}_C(f_c^1, \ldots, f_c^{\lambda_r - 1}, 1 \leq c \leq \ell) = \text{Span}_C(f_c^1, \ldots, f_c^{\lambda_r - 1}, 1 \leq c \leq \ell) + \text{Span}_C(f_c^{\lambda_r - 1}, 1 \leq c \leq \ell) = \text{Im} B^{\lambda, -1} + \text{Span}_C(f_c^{\lambda_r - 1}, 1 \leq c \leq \ell), \]
where we understand $f_p^p \equiv 0$ for $p \leq 0$. In particular, $f_c^{\lambda_c - \lambda_r + 1}$ is equal to $B^{\lambda_r - 1}(f_c^{\lambda_r})$ and is nonzero for $c > r$. Thus $\dim (B^{\lambda_r - 1})^{-1} B^{\lambda_r} = N - r$.

Let $A \in F_B$. We claim that

$$V^{(m, \mu, m - 1)} = (A^{\lambda_r - 1}) |_{V^{(m, \mu, m - 1)}} \text{Im} A^{\lambda_r} |_{V^{(m, \mu, m - 1)}} + L.$$

Let $e_a^b \in \beta_r$. If $b \leq \mu_a - 1$ then $e_a^b = A |_{V^{(m, \mu, m - 1)}} (e_a^{b+1}) - v$ for some $v \in L$. Since $A |_{V^{(m, \mu, m - 1)}} (e_a^{b+1})$ is clearly in $(A |_{V^{(m, \mu, m - 1)}})^{-1} \text{Im} A^c |_{V^{(m, \mu, m - 1)}}$ for any $c$ it follows that $e_a^b \in (A |_{V^{(m, \mu, m - 1)}})^{-1} \text{Im} A^c |_{V^{(m, \mu, m - 1)}} + L$ for any $1 \leq b \leq \mu_a$ and $1 \leq a \leq m$ except of course for $(a, b) = (m, \mu_m)$.

We can therefore apply the elementary fact that the codimension of $V'$ in $V' + V''$ is equal to the codimension of $V' \cap V''$ in $V''$ for any two vector spaces $V'$ and $V''$ to $V' = (A |_{V^{(m, \mu, m - 1)}})^{-1} \text{Im} A^{\lambda_r} |_{V^{(m, \mu, m - 1)}}$ and $V'' = L$.

Together with Equation (3a) this gives Equation (3b):

$$\dim V' \cap V'' = \dim V' + \dim V'' - \dim (V' + V'') = (N - r) + (m - 1) - (N - 1) = m - r.$$ 

\[ \square \]

We would like to use Lemma 3 to establish Proposition 2. To do so we will need the following proposition.

**Proposition 3.** Let $f : X \to Y$ be surjective, with irreducible fibres of dimension $d$. Assume $Y$ has a component of dimension $m$ and all other components of $Y$ have smaller dimension. Then $X$ has unique component of dimension $m + d$ and all other components of $X$ have smaller dimension.

To prove Proposition 3 will need [Mum88, I, §8, Theorem 2] and [Sta18, Lemma 005K] which we now recall.

**Theorem 1.** [Mum88, I, §8, Theorem 2] Let $f : X \to Y$ be a dominating morphism of varieties and let $r = \dim X - \dim Y$. Then there exists a nonempty open $U \subset Y$ such that:

1. $U \subset f(X)$
2. (for all irreducible closed subsets $W \subset Y$ such that $W \cap U \neq \emptyset$, and for all components $Z$ of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$, dim $Z = \dim W + r$.

**Lemma 5.** [Sta18, Lemma 005K] Let $X$ be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. $E$ is constructible). The following are equivalent:

1. The intersection $E \cap Z$ contains an open dense subset of $Z$.
2. The intersection $E \cap Z$ is dense in $Z$.

**Proof of Proposition 3.** Let $X = \bigcup_{\text{irr} X C} C$ be a (finite) decomposition of $X$. Consider the restriction $f_C : C \to \overline{f(C)}$ of $f$ to an arbitrary component. It is a dominant morphism of varieties, with irreducible fibres of dimension $d$.

We apply Theorem 1. Let $U \subset \overline{f(C)}$ be such that for all irreducible closed subsets $W \subset \overline{f(C)}$ such that $W \cap U \neq \emptyset$, and for all components $Z$ of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$, dim $Z = \dim W + \dim C - \dim \overline{f(C)}$. Then, taking $W = \{y\} \subset U$ for some $y \in U \subset \overline{f(C)}$, we get that dim $f^{-1}(y) = \dim C - \dim \overline{f(C)}$. Since all fibres have dimension $d$, the difference $\dim C - \dim \overline{f(C)}$ is constant and equal to $d$, independent of the component we’re in.

Since $f$ is surjective, it is in particular dominant, so we have that

$$Y = f(X) = f(\bigcup_{\text{irr} X C} C) = \bigcup_{\text{irr} X C} f(C) = \bigcup_{\text{irr} X} \overline{f(C)} = \bigcup_{\text{irr} X \overline{f(C)}}.$$

Let $C_0$ be such that $\dim \overline{f(C_0)} = m$. Then $C_0 = d + \dim f(C_0) = d + m$.

Let $f_1 = f |_{C_1}$, and let $U_i \subset f(C_i)$ be the open sets supplied by Theorem 1 or Lemma 5 for the constructible sets $E_j = f(C_i)$. Take $U = U_0 \cap U_1$ and let $y \in U$. Since $V_i = f_i^{-1}(U)$ contains $f_i^{-1}(y) = f^{-1}(y) \cap C_i = f^{-1}(y)$ the set $V = V_0 \cap V_1$ is nonempty. That’s a nonempty open set contained in $C_0 \cap C_1$. Conclude $C_0 = C_1$. Note $V_i = f_i^{-1}(U) \cap C_i$. \[ \square \]
We are now ready to prove Proposition 2.

**Proof of Proposition 2.** Consider the restriction map
\[ X\tau \to X_{\tau}(m-1). \]

By induction on \( m \), we can assume that \( X_{\tau(m-1)} \) has one irreducible component of dimension \( d \), and apply Proposition 3 in conjunction with Lemma 3 to conclude that \( X\tau \) has one irreducible component of dimension \( d + \sum_{1}^{\mu(m)} (m - r_{m,k}) \) for \( r_{m,k} \) equal to the row coordinate of the \( k \)th \( m \) in \( \tau \). Note \( X_{\tau(1)} = \{ J_{\mu_i} \} \).

We now check that \( \sum_{1}^{\mu(m)} (m - r_{m,k}) = \langle \lambda - \mu, \rho \rangle - \langle \lambda^{(m-1)} - \mu^{(m-1)}, \rho^{(m-1)} \rangle \). We start by expanding the difference on the right-hand side.

\[
\begin{align*}
\langle \lambda - \mu, \rho \rangle - \langle \lambda^{(m-1)} - \mu^{(m-1)}, \rho^{(m-1)} \rangle &= \langle \lambda_1 - \mu_1 + (m - 1) (\lambda_1 - \lambda_1^{(m-1)}) + \lambda_2 - \mu_2 + (m - 2) (\lambda_2 - \lambda_2^{(m-1)}) + \cdots + \lambda_{m-1} - \mu_{m-1} + (\lambda_{m-1} - \lambda_{m-1}^{(m-1)}) + \lambda_m - \mu_m \rangle \\
&= |\lambda| - |\mu| + \sum_{i=1}^{m-1} (m - i) (\lambda_i - \lambda_i^{(m-1)})
\end{align*}
\]

We recognize that \( \lambda_i - \lambda_i^{(m-1)} = n(\tau)_{z_i - z_m} \) and re-sum, setting \( n_{(a,b)} = n(\tau)_{z_a - z_b} \) for convenience.

\[
\begin{align*}
\sum_{i=1}^{m-1} (m - i) (\lambda_i - \lambda_i^{(m-1)}) &= n_{(1,m)} + n_{(2,m)} + \cdots + n_{(m,m)} \\
&= |\lambda| - |\mu| + \sum_{i=1}^{m-1} (m - i) (\lambda_i - \lambda_i^{(m-1)})
\end{align*}
\]

Observe that, since \( n_{(a,b)} \) is just the number of \( m \) in row \( i \), \( n_{(1,m)} + \cdots + n_{(i,m)} = \mu_m - \) the number of \( m \) in the last \( m - i + 1 \) rows. Summing the latter terms counts the number of \( m \) in row \( i \) exactly \( i \) times, for \( i = 1, \ldots, m \). But

\[
\sum_{k=1}^{\mu(m)} r_{m,k} = \sum_{i=1}^{\lambda_i} i \cdot (\text{the number of boxes numbered } m \text{ in row } i)
\]

too. Thus upon adding \( 0 = \mu_m - \mu_m \) to the re-summation we get \( m \mu_m - \sum_{k=1}^{\mu(m)} r_{m,k} \) as expected.

Similarly, the fibres of the maps \( X_{\tau(i+1)} \to X_{\tau(i)} \) for \( i = 1, \ldots, m - 2 \) have dimension

\[
(i + 1)\mu_{i+1} - \sum_{k=1}^{\mu_{i+1}} r_{i+1,k} = \langle \lambda^{(i+1)} - \mu^{(i+1)}, \rho^{(i+1)} \rangle - \langle \lambda^{(i)} - \mu^{(i)}, \rho^{(i)} \rangle.
\]

Since these are the differences making up the telescoping sum

\[
\langle \lambda - \mu, \rho \rangle = \langle \lambda - \mu, \rho \rangle - \langle \lambda^{(m-1)} - \mu^{(m-1)}, \rho^{(m-1)} \rangle + \langle \lambda^{(m-1)} - \mu^{(m-1)}, \rho^{(m-1)} \rangle - \langle \lambda^{(m-2)} - \mu^{(m-2)}, \rho^{(m-2)} \rangle + \cdots
\]

it follows that \( \dim X_{\tau} = \langle \lambda - \mu, \rho \rangle. \]

**Conjecture 1.** The map in Lemma 3 is a trivial fibration. Consequently \( X^{d}_{\tau} = X_{\tau} \) and \( Z_{\tau} = X_{\tau}^{d} \) is defined by a recurrence \( Z_{\tau} \cong Z^{d}_{\tau} \times \mathbb{C}^{m-r} \) for \( r \) equal to the row coordinate of the last \( m \) in \( \tau \).
2.3. Decomposing $\Box_{\lambda} \cap T_{\mu} \cap n$. We conclude the first part of this paper with a proof of Theorem A.

**Theorem 2.** The map $\tau \mapsto Z_\tau$ is a bijection of $T(\lambda)_\mu$ and irreducible components of $\Box_{\lambda} \cap T_{\mu} \cap n$. Moreover, $\dim Z_\tau = \dim X_\tau = \langle \lambda - \mu, \rho \rangle$.

**Proof.** Let $A \in \Box_{\lambda} \cap T_{\mu} \cap n$ be generic for a component and consider the tableau $\tau$ obtained from the GT-pattern of Jordan types of submatrices $A_{Y_{1i}}$ for $1 \leq i \leq m$. Then $\tau \in T(\lambda)_\mu$ and $A \in Z_\tau$. \hfill \Box

3. Equations of Mirković–Vilonen cycles

Let $G(K) \to \text{Gr} : g \mapsto gG(O)$ denote the quotient map, and when there is no confusion, set $[g] = gG(O)$.

Let $\text{Gr}_\mu = G_1[[t^{-1}]]L_{\mu}$ for $G_1[[t^{-1}]] = \text{Ker}(G[[t^{-1}]] \xrightarrow{t=\infty} G)$.

3.1. The Mirković–Vybornov isomorphism.

**Theorem 3.** [CK18] The map $\phi : T_{\mu} \cap N \to G_1[[t^{-1}]]t^{\mu}$ defined by

$$\phi(A) = t^\mu + a(t)$$

$$a_{ij}(t) = -\sum A_{ij}^k t^{k-1}$$

$$A_{ij}^k = k\text{th entry from the left of the } \mu_j \times \mu_i \text{ block of } A$$

yields the Mirković–Vybornov isomorphism of type $(\lambda, \mu) : \Box_{\lambda} \cap T_{\mu} \cap n \to \text{Gr}_\lambda \cap \text{Gr}_\mu$ defined by $\Psi = [\phi(A)]$.

**Proof.** The reader can check that, given $[g] \in \text{Gr}$, the map

$$[g] \mapsto \left[ t|_{O/KO} \right]_{\mu}$$

for $B = (|e_m|, \ldots, |e_m t^{\mu-1}|, \ldots, |e_1|, \ldots, |e_1 t^{\mu-1}|)$ is a two-sided inverse. \hfill \Box

**Example 3.** Let $\tau = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 \end{bmatrix}$ as before. Then

$$\phi(Z_\tau) = \left\{ g = \begin{pmatrix} t^2 & 0 & 0 \\ -a - bt & t^2 & 0 \\ -c & -d & t \end{pmatrix} : a, d = 0 \right\}$$

**Corollary 1.** The restriction $\psi = \Psi|_{\Box_{\lambda} \cap T_{\mu} \cap n}$ is an isomorphism of $\Box_{\lambda} \cap T_{\mu} \cap n$ and $\text{Gr}_\lambda \cap S_\mu$ which we’ll refer to as the restricted Mirković–Vybornov isomorphism of type $(\lambda, \mu)$.

**Proof.** Suppose $A \in T_{\mu} \cap n$. Then $\phi(A) \in G_1[[t^{-1}]]t^{\mu} \cap N_{-}(K)t^{\mu} = N_{-}(K)t^{\mu}$ so $\psi(A) \in S_\mu$. Conversely, if $\phi(A) \in N_{-}(K)t^{\mu}$, then $A \in T_{\mu} \cap n$. Since $\dim \Box_{\lambda} \cap T_{\mu} \cap n = \dim \text{Gr}_\lambda \cap S_\mu$ and $\Psi$ is onto, we can conclude that $\text{Im } \psi = \text{Gr}_\lambda \cap S_\mu$. \hfill \Box

3.2. Equal Lusztig data. Let $V$ be a finite dimensional complex vector space. In [Kam10] the author works in the right quotient $\text{Gr}^T = G(O) \backslash G(K)$ where he defines the Lusztig datum of an MV cycle using the valuation

$$\text{val} : V \otimes K \to Z : v \mapsto k$$

if $v \in V \otimes t^kO \setminus V \otimes t^{k+1}O$.

Let $w \in W$. Let $\omega_i \in X_i(T)$ denote the $i$th fundamental weight, $\omega_i = z_1 + \cdots + z_i$ for $1 \leq i \leq m$. Fix a highest weight vector $v_{\omega_i}$ in the $i$th fundamental irreducible representation $L(\omega_i)$ of $G$ and consider

$$D_{\omega_{\mu_i}}^T : \text{Gr}^T \to Z : G(O)g \mapsto \text{val}(g \tilde{w} \cdot v_{\omega_i})$$

with $\tilde{w}$ denoting the lift of $w$ to $G$. These functions cut out the semi-infinite cells $S_{\mu_i}^T = G(O)t^{\mu_i}U_w(K) \subset \text{Gr}^T$ for $U_w = \tilde{w}U \tilde{w}^{-1}$ as follows.

**Lemma 6** ([Kam10] Lemma 2.4). $D_{\omega_{\mu_i}}^T$ is constructible and

$$S_{\mu_i} = \{ L \in \text{Gr}^T : D_{\omega_{\mu_i}}^T(L) = \langle \mu, \omega_{\mu_i} \rangle \}$$

By considering the transpose map $\text{Gr} \to \text{Gr}^T : gG(O) \mapsto G(O)g^T$ we derive an analogous result for $S_{\mu_i} = U_w(K)t^{\mu_i}G(O) \subset \text{Gr}$.
**Lemma 7.** If \( gG(\mathcal{O}) \in S^\nu_w \subset Gr \), then \( G(\mathcal{O})g^T \in \nu^w_{w^0_\nu} \subset Gr^T \). In particular, the order on the vertices of the MV polytope of the MV cycle to which \( gG(\mathcal{O}) \) belongs is reversed, with the datum \( \nu_\bullet = (\nu_w)_{w \in W} \) for \( gG(\mathcal{O}) \), defining \( \nu^T_\bullet = (\nu^T_w)_{w \in W} \) for \( G(\mathcal{O})g^T \) and \( S'_w = \{ L \in Gr : D^T_{w^0_\nu \omega_\nu}(L^T) = \langle \nu, w^0_\nu \omega_\nu \rangle \} \).

**Proof.** Let \( gG(\mathcal{O}) \in S^\nu_w \subset Gr \). Then \( g = \nu w w^{-1} \nu^\nu \) for some \( n \in U \). Then

\[
g^T = t^\nu(w^{-1})^T \nu^T w^T
\]

The computation

\[
g^T w_0 \cdot v_\omega = t^\nu w_0 n \cdot v_\omega = t^\nu w_0 \cdot (v_\omega) = t^\nu w_0 \cdot v_\omega
\]

checks that \( D^T_{w^0_\nu \omega_\nu}(G(\mathcal{O})g^T) = \text{val}(g^T w_0 \cdot v_\omega) \) is equal to \( \langle \nu, w^0_\nu \omega_\nu \rangle \) agreeing with Lemma 6.

We define \( D_{w^0_\nu} \) on \( Gr \) by

\[
D_{w^0_\nu}(gG(\mathcal{O})) = D^T_{w^0_\nu \omega_\nu}(G(\mathcal{O})g^T)
\]

and rewrite

\[
S'_w = \{ L \in Gr : D_{w^0_\nu}(L) = \langle \nu, w^0_\nu \omega_\nu \rangle \text{ for all } i \}.
\]

The Lusztig datum \( n_\bullet \) of an MV cycle in \( Gr^T \) is defined by

\[
n( ) = n_\bullet = D^T_{[a \cdots b]}( ) - D^T_{[a+1 \cdots b]}( ) - D^T_{[a \cdots b-1]}( ) + D^T_{[a+1 \cdots b-1]}( )
\]

on generic elements. Here \([a \cdots b]\) is shorthand for the permutation of \( \omega_{-a+1} \) that has 1s in positions \( a \) through \( b \) and zeros elsewhere. If \( b < a \) we understand \( D^T_{[a \cdots b]} \equiv 0 \).

Given \( 1 \leq b \leq m \), let \( F_b \) be \( \text{Span}_O(e_1, \ldots, e_b) \subset K^m \), let \( w^0_\nu \) be the longest element in the Weyl group of \( G_b \), and let \( \omega^b_\nu \) denote the ith fundamental weight in \( \mathbb{Z}^b \), the weight lattice of \( G_b \). Let \( g \in U_\nu(\nu) \). Then \( g^T \in U(\nu) \) and

\[
D^T_{[a \cdots b]}(G(\mathcal{O})g^T) = D^T_{[a \cdots b]}(G_b(\mathcal{O})(g^T \mid F_b)) = D^T_{w^0_\nu \omega_{b-\nu} \omega_{b-\nu+1}}(G_b(\mathcal{O})(g^T \mid F_b)).
\]

In turn

\[
D^T_{w^0_\nu \omega_{b-\nu} \omega_{b-\nu+1}}(G_b(\mathcal{O})(g^T \mid F_b)) = D_{\omega^b_{b-\nu} \omega_{b-\nu+1}}((g^T \mid F_b)^T G_b(\mathcal{O})).
\]

In particular, if \( (g^T \mid F_b)^T G_b(\mathcal{O}) \in S^\nu_w \cap S^\nu_b \subset G_b(\mathcal{O}) / G_b(\mathcal{O}) \), then Lemma 7 implies that

\[
D_{\omega^b_{b-\nu} \omega_{b-\nu+1}}((g^T \mid F_b)^T G_b(\mathcal{O})) = \left\{ \begin{array}{ll} \langle \eta, \omega^b_{b-\nu} \omega_{b-\nu+1} \rangle & w = e \\ \langle \nu, \omega^b_{b-\nu} \omega_{b-\nu+1} \rangle & w = w^0_\nu \end{array} \right.
\]

Now fix \( \tau \in T(\lambda)_\mu \) and \( d = \langle \lambda - \mu, \rho \rangle \), and consider the inclusion \( i : Gr^A \cap S^w \to Gr^A \cap S_b^w \). Note that \( Gr^A \cap S^w = Gr^A \cap S_b^w \sqcup X \) where every component of \( X \) has dimension strictly less than \( d \). (This follows from decompositions of \( Gr^A \) and \( S_b^w \) afforded by [MV07].)

**Lemma 8.** For a dense subset of \( A \in Z_\tau, \psi(A) \in S^\lambda \cap S^\nu_b \).

**Proof.** By Corollary 1, \( \psi(Z_\tau) \) is dense in a component of \( \overline{Gr^A} \cap S_b^w \) which has pure dimension \( d \) by [MV07] Theorem 3.2]. Thus \( Z = \iota(Z_\tau) \) is an MV cycle. By [Kam10, §3.3], \( Z \cap S^\lambda \) is dense in \( Z \), so \( \psi^{-1}(i^{-1}(Z \cap S^\lambda)) \) is dense in \( Z_\tau \).

For the purpose of the next lemma, let \( \psi_b : \overline{\mathfrak{T}}_\Lambda(\psi) \cap \mathfrak{T}_\mu(\mu) \cap \mathfrak{n}_b \to \overline{Gr^A} \cap S_b^w \) denote the restricted Mirković–Vybornov isomorphism of type \( (\lambda(b), \mu(b)) \) with \( \mathfrak{n}_b \subset \text{Mat}(N_b, \mathbb{C}) \) denoting the subalgebra of \( N_b \times N_b \) upper-triangular matrices for \( N_b = \mu_1 + \cdots + \mu_b \) and let \( \pi_b : \overline{\mathfrak{T}}_\Lambda \cap \mathfrak{T}_\mu \cap \mathfrak{n} \to \overline{\mathfrak{T}}_\Lambda(b) \cap \mathfrak{T}_\mu(b) \cap \mathfrak{n}_b \) denote the restriction \( \pi_b(A) = A \mid_{\mathfrak{T}(b)} \) with \( 1 \leq b \leq m \).

**Lemma 9.** For \( 1 \leq a < b \leq m \), the generic value of \( D_{w^0_\nu \omega_{b-\nu} \omega_{b-\nu+1}} \circ \iota \circ \psi \) on \( Z_\tau \) is \( \langle \lambda(b), w^0_\nu \omega_{b-\nu} \rangle \).
Proof. Let $1 \leq a < b \leq m$. We apply Lemma 8 to $\psi_b$ and let $W_b$ be the dense subset of $B \in Z_\tau(b)$ for which $\psi_b(B) \in S^{(b)} \cap S^{\mu(b)}$. Then $\pi_b^{-1}(W_b)$ is dense in $Z_\tau$. Moreover, for $A \in \pi_b^{-1}(W_b)$, $\phi(A)^T|_{F_b} = \phi_b(\pi_b(A))^T$, so by Lemma 7, $G_b(O)\phi_b(\pi_b(A))^{T} \in S^{\lambda(b)} \cap S^{\mu(b)}$ and $D_{w_0\omega_{b-a+1}}(\tau(\psi(A))) = \langle \lambda(b), w_0\omega_{b-a+1}\rangle$.

We are just about ready to verify Theorem B. Following [BZ88], define the Lusztig datum $n_\bullet$ of $\tau \in T(\lambda)_\mu$ by the zig-zag differences
\begin{equation}
\tau_{za-zb} = \lambda_a(b) - \lambda_{a-1}(b)
\end{equation}
for $1 \leq a < b \leq m$.

**Theorem 4.** $Z = \tau(\psi(Z))$ is an MV cycle of coweight $(\lambda, \mu)$ having Lusztig datum $n_\bullet$ given by Equation (6) above.

**Proof.** Let $1 \leq a < b \leq m$. By Lemma 9, the Lusztig datum of $Z$, as defined by Equation (5), is given by
\begin{equation}
n(Z)_{za-zb} = \langle \lambda(b), u_0^b\omega_{b-a+1}\rangle - \langle \lambda(b), u_0^b\omega_{b-a}\rangle - \langle \lambda(b-1), u_0^{b-1}\omega_{b-a}\rangle + \langle \lambda(b-1), u_0^{b-1}\omega_{b-a-1}\rangle
\end{equation}
\begin{equation*}
= \langle \lambda(b), u_0^b\omega_{b-a+1}\rangle - \langle \lambda(b), u_0^{b-1}\omega_{b-a}\rangle
\end{equation*}
\begin{equation*}
= \lambda_a(b) - \lambda_{a-1}(b).
\end{equation*}

**REFERENCES**

[BKK19] Pierre Baumann, Joel Kamnitzer, and Allen Knutson. The Mirković–Vilonen basis and Duistermaat–Heckman measures (with an appendix by Anne Dranowsk, Joel Kamnitzer, and Calder Morton-Ferguson). In preparation, 2019.

[BZ88] A.D. Berenstein and A.V. Zelevinsky. Tensor product multiplicities and convex polytopes in partition space. Journal of Geometry and Physics, 5(3):453–472, 1988.

[CG09] Neil Chriss and Victor Ginzburg. Representation theory and complex geometry, Springer Science & Business Media, 2009.

[CK18] Sabin Cautis and Joel Kamnitzer. Categorical geometric symmetric Howe duality. Selecta Mathematica, 24(2):1593–1631, 2018.

[Haï06] Thomas J Haines. Equidimensionality of convolution morphisms and applications to saturation problems. Advances in Mathematics, 207(1):297–327, 2006.

[Jos84] Anthony Joseph. On the variety of a highest weight module. Journal of Algebra, 88(1):238–278, 1984.

[Kam10] Joel Kamnitzer. Mirković-Vilonen cycles and polytopes. Annals of Mathematics, 171(1):245–294, 2010.

[Mum88] David Mumford. The red book of varieties and schemes. Lecture notes in mathematics, 1358, 1988.

[MV07] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. Annals of mathematics, pages 95–143, 2007.

[Shi80] Naohisa Shimomura. A theorem on the fixed point set of a unipotent transformation on the flag manifold. Journal of the Mathematical Society of Japan, 32(1):55–64, 1980.

[Spa76] Nicolas Spaltenstein. The fixed point set of a unipotent transformation on the flag manifold. In Indagationes Mathematicae (Proceedings), volume 79, pages 452–456. North-Holland, 1976.

[Web17] Ben Webster. On generalized category $O$ for a quiver variety. Mathematische Annalen, 368(1-2):483–536, 2017.

[ZJ15] Paul Zinn-Justin. Quiver varieties and the quantum Knizhnik–Zamolodchikov equation. Theoretical and Mathematical Physics, 185(3):1741–1758, 2015.

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