DARK MATTER HALO MASS PROFILES

Dan Coe
Draft

ABSTRACT

I provide notes on the NFW, Einasto, Sérsic, and other mass profiles which provide good fits to simulated dark matter halos (I). I summarize various published $c(M)$ relations: halo concentration as a function of mass (II). The definition of the virial radius is discussed and relations are given to convert $c_{\text{vir}}$, $M_{\text{vir}}$, and $r_{\text{vir}}$ between various defined values of the halo overdensity (II).

Subject headings: cosmology: dark matter — galaxies: clusters: general — methods: data analysis — gravitational lensing

1. MASS-CONCENTRATION RELATIONS

The mass profiles of galaxy clusters appear to be more centrally concentrated than realized in simulations (Broadhurst et al. 2008; Oguri et al. 2009; Sereno et al. 2010). If true, this may be evidence for Early Dark Energy (see e.g., Grossi & Springel 2009). Or perhaps there is a less exciting explanation (e.g., Bartelmann & Loeb 2009; Lapin & Cavaliere 2009; Meneghetti et al. 2010). For more details, see my discussion in Coe et al. (2010). More conclusive results are expected from the CLASH HST MCT project and perhaps LoCuSS (Okabe et al. 2009; Richard et al. 2009).

More massive halos generally have lower concentrations than less massive halos. This is seen in both simulations and observations, though less clearly so far in the latter (see below). More massive halos form later, resulting in lower concentrations reflecting the lower background density at the time of formation (Navarro et al. 1996).

For a given radial mass profile (see §3), the concentration is defined as:

$$c_{\text{vir}} = r_{\text{vir}}/r_{-2},$$

(1)

a mishmash ratio of the virial radius $r_{\text{vir}}$ and the radius $r_{-2}$ at which $\rho \propto r^{-2}$. For an NFW profile, $r_{-2} = r_s$. The definition of the virial radius $r_{\text{vir}}$ is discussed at length in [2] but it is typically approximated as the region within which there is an average overdensity of a certain value ($\Delta_c \sim 100$ or 200) above $\rho_{\text{crit}}$. For clarity, one may quote the exact value of $\Delta_c$ used: $c_{200}$, for example.

In principle concentrations could be derived using any radial fitting profile (§3). However the choice does matter as the profiles behave differently between $r_{-2}$ and $r_{\text{vir}}$. Concentrations derived from NFW and Einasto fits to the same halos (Duffy et al. 2008) are compared in Fig. 8. Einasto $c(M)$ relations have been derived for the Millennium simulation relaxed (Gao et al. 2008) and all Hayashi & White (2008) halos. Below we focus on $c(M)$ relations derived from NFW fits.

1.1. Current $c(M)$ measurements from NFW profile fits

The current best estimates for $c(M,z)$ are probably those given by Duffy et al. (2008) and Macciò et al. (2008). Their findings are similar. Both analyze simulations which use the WMAP5 cosmology, resulting in ~20% lower concentrations than WMAP1 (Table 8) as used in the Millennium simulation (Neto et al. 2007), for example. Duffy et al. (2008) find that present-day ($z = 0$) halos follow the following mass-concentration relation:

$$c_{200} \simeq 5.74 \left( \frac{M_{200}}{2 \times 10^{12} h^{-1} M_{\odot}} \right)^{-0.097}.$$  (2)

They provide a separate relation for relaxed clusters which are more symmetric and thus better fit by radial profiles such as NFW. These have 15 – 20% higher concentrations (Fig. 7):

$$c_{200} \simeq 6.67 \left( \frac{M_{200}}{2 \times 10^{12} h^{-1} M_{\odot}} \right)^{-0.092}.$$  (3)

Intrinsic scatters are $\log_{10}(c_{200}) \simeq 0.15$. These relations are plotted in Fig. 2 along with corresponding relations from Macciò et al. (2008). Duffy et al. (2008) also supply fitted functions to halos spanning the redshift range $z = 0–2$. Full:

$$c_{200} \simeq \frac{5.71}{(1+z)^{0.47}} \left( \frac{M_{200}}{2 \times 10^{12} h^{-1} M_{\odot}} \right)^{-0.097}.$$  (4)

and relaxed:

$$c_{200} \simeq \frac{6.71}{(1+z)^{0.44}} \left( \frac{M_{200}}{2 \times 10^{12} h^{-1} M_{\odot}} \right)^{-0.092}.$$  (5)

In their Table 1, they provide uncertainties for these fit parameters as well as corresponding values for $c_{\text{vir}}$ and $M_{\text{vir}}$. These $c_{200}(M_{200}, z)$ relations are plotted in Figs. 3 & 4. Also plotted are the corresponding $c_{\text{vir}}(M_{\text{vir}}, z)$ relations provided by Duffy et al. (2008). In Fig. 4 we plot the Bullock et al. (2001) $c \propto (1+z)^{-1}$ scaling for comparison. Note that Duffy08 find weaker dependencies on redshift: $c_{200} \propto (1+z)^{-0.45}$; $c_{\text{vir}} \propto (1+z)^{-0.70}$.

1. Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr, MS 169-327, Pasadena, CA 91109
2. Cluster Lensing And Supernova survey with Hubble, http://www.stsci.edu/~postman/CLASH/
Halo concentrations are sensitive to cosmology. A higher $\sigma_8$ causes halos to form earlier, resulting in higher concentrations. This was the case in the Millennium simulations which used the WMAP 1-year cosmology, including $\sigma_8 = 0.9$. This yields concentrations $\sim 20\%$ higher than found in simulations which use WMAP5’s $\sigma_8 = 0.796$ [Duffy et al. 2008]. The effect of cosmology was explored in more detail by Macciò et al. (2008). These effects are shown in Fig. 5.

Various derived $c(M)$ relations (for $z = 0$) are plotted in Fig. 6. In our Tables 3 and 4 we provide $c_{200}(M_{200})$, $c_{vir}(M_{vir})$, and $c(M, z)$ relations, respectively, as derived by Duffy et al. (2008), Macciò et al. (2008), Neto et al. (2007), Bullock et al. (2001), Hennawi et al. (2007), and Gentile et al. (2007). The latter is the original NFW $c(M)$ prescription updated to the WMAP3 cosmology.

The various simulations considered here are outlined in Table 7. The relevant details of their adopted cosmologies ($\Omega_m, \sigma_8$) are given in Table 8. We provide the range of halo masses produced in each simulation. The dangers of extrapolating $c(M)$ relations beyond these ranges have been cited by Zhao et al. (2003), for example. Neto et al. (2007) find that 10,000 particles within the virial radius are required to yield robust concentration measurements. They note that using fewer particles introduces scatter but does not appear to introduce bias in their concentration measurements.

Hennawi et al. (2007) measure significantly larger concentrations for galaxy clusters in their simulations. Their cluster concentrations are $\sim 50\%$ and $\sim 80\%$ larger than found by Duffy et al. (2008) and Macciò et al. (2008), respectively (Fig. 6, right). Their use of $\sigma_8 = 0.95$ probably only results in concentrations inflated by $\sim 20\%$ compared to the WMAP5 $\sigma_8 = 0.796$ simulations. The remaining disagreement may be a result of their halo density fitting procedure which they claim is better for comparison with lensing measurements. Specifically, they assign large uncertainties to radial bins with large subhalos. This may bias the fitted profiles to be low at large radius (where large subhalos typically reside) resulting in higher concentrations. These results may considerably ease tensions between observed and simulated halo concentrations. The differences in fitting procedure should be better studied and understood.

1.2. Care in citing concentration expectations

A concern often noted is that the concentration measured for A1689 (in every study to date) is higher than that found in simulations for a halo of A1689’s mass. The concentration found in simulations has been cited loosely as $c \sim 5$ or $c \sim 5.5$ using a relation given by Bullock et al. (2001):

$$c_{vir} \sim \frac{9}{1 + z} \left( \frac{M_{vir}}{1.3 \times 10^{13} h^{-1} M_\odot} \right)^{-0.13},$$

(6)

This value is in excellent agreement with the WMAP 7-year maximum likelihood value $\sigma_8 = 0.803$ (Komatsu et al. 2010).

I’ll take a crack at completeness, and mention other papers with $c(M)$ relations neglected here (for no particular reason): Eke et al. (2001), Wechsler et al. (2002), Allen et al. (2002), Zhao et al. (2003), Dolag et al. (2004), Kunz et al. (2005), Lu et al. (2006), Shaw et al. (2006), Dutton et al. (2007), Giocoli et al. (2007). And I have probably missed still others!

But the expected concentration is actually lower ($c_{200} \sim 3.0$), exacerbating the disagreement with observations) for four reasons:

- $c_{200} < c_{vir}$
- $z > 0$
- WMAP7 vs. WMAP1
- $M_{vir} \approx 1.4 \times 10^{15} M_\odot / h > 10^{15} M_\odot / h$

We also note that Bullock et al. (2001) did not simulate halos as massive as A1689, with their most massive halos weighing in at $M \sim 10^{14} M_\odot / h$.

1.3. The observed $c(M)$ relation

Based on a compilation of 62 published measurements (including 10 new measurements) of halo $c_{vir}$ and $M_{vir}$, Comerford & Natarajan (2007) find the following relation (with a large scatter):

$$c_{vir} \sim \frac{14.5 \pm 6.4}{1 + z} \left( \frac{M_{vir}}{1.3 \times 10^{13} h^{-1} M_\odot} \right)^{-0.15 \pm 0.13}.$$  

(7)

For clusters as massive as A1689, Comerford’s relation converges toward that of Hennawi et al. (2007), the former being only slightly higher.

This and other observed $c(M)$ relations are shown in Fig. 6 and detailed in Table 6. The Comerford & Natarajan (2007) compilation includes both lensing and X-ray determinations of $c$ and $M$, including the X-ray samples presented by Buote et al. (2007) and Schmidt & Allen (2007). Each of these papers presented their own $c(M)$ relation. A recent $c(M)$ relation from weak lensing of individual halos was presented by Okabe et al. (2009). And $c(M)$ derived from stacked weak lensing analyses were presented by Johnston et al. (2007) and Mandelbaum et al. (2008). It seems apparent that one should study a wide enough range of halo masses to obtain a confident $c(M)$ relation.

2. OVERDENSITY WITHIN THE VIRIAL RADIUS

Various conventions are used to define the virial mass and radius. We explain and show how to convert between different definitions.

2.1. OVERdensity Definitions

The virial radius $r_{vir}$ designates the edge of the halo. Within this radius, objects are supposed to be “virialized”: gravitationally bound and settled into regular orbits. Outside this radius, objects are not in orbit although they may still be infalling. In practice, there is no sharp dividing line the two regions. And even if there were, it would be extremely difficult to discern observationally for a given massive body. Meanwhile, the objects we study are not always virialized. In fact, galaxy clusters are the largest bodies which have had time to virialize given the age of the universe. Thus some of the clusters we observe have virialized just recently, but many are still in the process of doing so.

Despite these complications, we can define a virial radius for a massive body based on theory and simulations. Early theoretical work (Peebles 1980) predicted...
that a sphere of material will collapse if its density exceeds 1.686(1 + z) times that of the background. After it collapses and virializes, the sphere will obtain an average density
\[
\Delta_c \approx 18\pi^2 \approx 178
\]
times the critical density \( \rho_{\text{crit}}(z) \) at that redshift, where
\[
\rho_{\text{crit}} = \frac{3H^2(z)}{8\pi G}.
\]

Cole & Lacey (1996) cited this as a theoretical result and then confirmed it in simulations.\(^5\)

Navarro et al. (1996) adopted the nice round number of \( \Delta_c = 200 \), which has been used commonly ever since to allow for easy comparison between papers. But the \( \Delta_c \approx 178 \) result was obtained in an Einstein de-Sitter cosmology of \( (\Omega_m, \Omega_\Lambda) = (1, 0) \). In the concordance cosmology \( (\Omega_m, \Omega_\Lambda) = (0.3, 0.7) \), we find a much lower value of \( \Delta_c \approx 100 \), as we describe next.

At least three different forms have been given for \( \Delta_c \) as a function of cosmology. For a flat universe \( (\Omega_m + \Omega_\Lambda = 1) \), Bryan & Norman (1998) give
\[
\Delta_c \approx 18\pi^2 - 82\Omega_\Lambda - 39\Omega^2 \approx 100.3.
\]
An approximation to this is given as (Eke et al. 1998; see also Hénry 2000) their Eq. A17) give
\[
\Delta_c \approx 178\Omega^0.45_m.
\]
And Nakamura & Suto (1997) their Eq. C19; see also Cole & Lacey (1996) cited this as a theoretical result and then confirmed it in simulations.\(^5\)

While results from simulations are most often reported for present-day halos, Nature provides us observers with images of clusters as they were in the past. Thus in the expressions above, we should replace the present day values of \( \Omega_{m,0} \) and \( \Omega_{\Lambda,0} \) (here “0” subscripts have been added for clarity) with:
\[
\Omega_m(z) = \frac{1}{1 + (\Omega_{\Lambda,0}/\Omega_{m,0})(1+z)^{-3}}
\]
and \( \Omega_\Lambda(z) = 1 - \Omega_m(z) \). For the massive galaxy cluster A1689 at \( z = 0.1862 \) and adopting \( \Omega_m = 0.3 \), the widely used Bryan & Norman (1998) expression yields \( \Delta_c \approx 116.6 \) and the Nakamura & Suto (1997) expression yields \( \Delta_c = 115 \). The latter was adopted by Broadhurst et al. (2005a) [private communication] so we adopt it as well for consistency in Cole et al. (2010).

In Figs. 13 and 14 we plot \( \Omega_m(z) \), \( \Delta_c(z) \), and \( \Delta_{\text{vir}}(z) \).

2.2. Conversion over density values \( \Delta_c \)

If the mass profile is well described by an NFW profile, then it is straightforward to convert \( c_{\text{vir}} \), \( r_{\text{vir}} \), and \( M_{\text{vir}} \) between different conventions of \( \Delta_c \) (c.f., Fig. 8. Converting from \( c_{200} \) \( \Delta_c = 200 \)) to \( c_{189} \) \( \Delta_c = 189 \), for example, simply involves finding that value of \( \Delta_c \) which yields the same value of \( \Omega_m \) (Eq. 38) as did \( c_{200} \). This can be accomplished by a simple root finding program, but the relation is very linear as shown in Fig. 9. Here we provide expressions
\[
ce_{94} \approx 1.328c_{200} + 0.272
\]
\[
c_{100} \approx 1.298c_{200} + 0.246
\]
\[
c_{115} \approx 1.232c_{200} + 0.189
\]
which are accurate to within 0.5\% for \( 2 < c_{200} < 25 \).

These factors may be generalized:
\[
c_{\text{vir}} \approx a c_{200} + b
\]
\[
a \approx -1.119 \log_{10} \Delta_c + 3.537
\]
\[
b \approx -0.967 \log_{10} \Delta_c + 2.181
\]
to yield \( c_{\text{vir}} \) within 1\% for \( 3 < c_{200} < 70 \) and \( 75 < \Delta_c < 140 \).

These factors are plotted in Fig. 10.

An alternate expression gives \( c_{\text{vir}} \) as a function of \( c_{200} \) and \( \Delta_c \):
\[
c_{\text{vir}} = c_{200} + c_{200}^{0.9}10^p
\]
\[
p = -(8.683 \times 10^{-5})\Delta_c^{1.82}
\]
to within 1\% for \( 3 < c_{200} < 35 \) and \( 85 < \Delta_c < 165 \) (\( z < 1 \) or so).

See also Hu & Kravtsov (2003) Appendix C).

2.3. Virial Mass

Virial mass (the mass within \( r_{\text{vir}} \)) is given by:
\[
M_{\text{vir}} = \frac{4}{3} \pi r_{\text{vir}}^3 \Delta_c \rho_{\text{crit}}(z) = \frac{r_{\text{vir}}^3 \Delta_c H^2(z)}{2G}
\]
In Fig. 12 we plot this simple relation between \( r_{\text{vir}} \) and \( M_{\text{vir}} \).
Quoted values for $M_{200}$ and $r_{200}$ can also be converted to $M_{\text{vir}}$ and $r_{\text{vir}}$ as a function of $c_{200}$ and $\Delta_c$ as plotted in Fig. 11. For a given NFW curve (with fixed $r_s$ and $\rho_s$), $r_{\text{vir}}$ simply scales with $c_{\text{vir}}$, since $r_s$ stays fixed. So $r_{\text{vir}}/r_{200} = c_{\text{vir}}/c_{200}$. As for the $M_{\text{vir}}$ conversions, we solved for those numerically.

3. MASS PROFILES

3.1. Double Power Laws

In dark matter simulations, galaxy and cluster halos (Navarro et al. 1996; Zhao 1996; Gao et al. 2008) were all shown to have mass density profiles well approximated by the NFW profile:

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}. \quad (24)$$

This profile behaves as $\rho \propto r^{-1}$ in the core, $\rho \propto r^{-2}$ at $r = r_s$, and steepens to $\rho \propto r^{-3}$ in the outskirts. The two fit parameters $\rho_s$ and $r_s$ were shown to be related and a function of halo mass. This “universal” profile is still a good approximation to today’s simulated halos. However the higher resolution does reveal subtle differences.

Deviations were sought for using a generalized version of the NFW profile (Hernquist 1990; see also Zhao 1996; Wytth et al. 2001):

$$\rho(r) = \frac{2^{3-\gamma} \rho_s}{(r/r_s)\gamma[1 + (r/r_s)\gamma]^{3-\gamma}}. \quad (25)$$

This profile behaves as $\rho \propto r^{-\gamma}$ in the core, and $\rho \propto r^{-3}$ in the outskirts. The rate of transition is governed by $\alpha$. Where NFW found $(\alpha, \beta, \gamma) = (1, 3, 1)$, Moore et al. (1999) instead found best fits of $(1.5, 3, 1.5)$. Importantly, the inner profile was steeper: $\gamma = 1.5$, $\rho \propto r^{-1.5}$. There were many other attempts to accurately resolve and measure this inner slope, including Diemand et al. (2005) who found $\rho \propto r^{-1.2}$.

The fully generalized form in Equation 25 proves a bit too general with large degeneracies between the free parameters (Klypin et al. 2001). Thus, in their efforts to determine the central slope $\gamma$, authors often use one of two constrained versions of Equation 25: either a “generalized NFW” profile with $(\alpha, \beta, \gamma) = (1, 3, \gamma)$:

$$\rho(r) = \frac{2^{3-\gamma} \rho_s}{(r/r_s)\gamma[1 + (r/r_s)^\gamma]^{3-\gamma}}, \quad (26)$$

or what we might call a “generalized Moore” profile, with $(\alpha, \beta, \gamma) = (3 - \gamma, 3, \gamma)$:

$$\rho(r) = \frac{2^{3-\gamma} \rho_s}{(r/r_s)\gamma[1 + (r/r_s)^{3-\gamma}]}. \quad (27)$$

Meanwhile, Dehnen & McLaughlin (2005) found $(\alpha, \beta, \gamma) = (4/9, 31/9, 7/9)$:

$$\rho(r) = \frac{2^{3-\gamma} \rho_s}{(r/r_s)^{7/9}[1 + (r/r_s)^{4/9}]^{3-\gamma}}. \quad (28)$$

This latter form is also often referred to as a “generalized NFW” profile, although strictly speaking it can only exactly reproduce the Moore profile and not that of NFW.

and when accounting for anisotropy, the more general $(\alpha, \beta, \gamma) = \{(3-\gamma)/5, (18-\gamma)/5, \gamma \}$:

$$\rho(r) = \frac{2\rho_s^6}{(r/r_s)^6[1 + (r/r_s)^{(3-\gamma)/5}]^6}. \quad (29)$$

3.2. Continuously Varying Power Laws

The original NFW proponents proposed a new profile which gradually flattens all the way toward the center (Navarro et al. 2004). This profile was found (Navarro et al. 2004; Merritt et al. 2005; 2006) to yield better fits to a wide range of simulated dark matter halos than did the generalized NFW profile (Eq. 26), which has an equal number (3) of free parameters, including the central slope. Inner slopes as steep as $\rho(r) \propto r^{-1.2}$ are clearly ruled out by recent simulations (Navarro et al. 2010).

The new Navarro et al. (2004) fitting form was quickly recognized (Merritt et al. 2005) as the Sérsic (1968) profile generally applied to fitting the light distributions of elliptical galaxies. The implications are intriguing: that the collapse of massive bodies, be they luminous or dark matter, may lead to similar profiles.

However to be precise, Navarro et al. (2004) fit a Sérsic-like profile to 3-D density distributions, where the Sérsic profile was fit to 2-D surface density distributions (of light). Einasto (1965) was first to use such a density law to describe a 3-D distribution, namely the spatial distribution of old stars within the Milky Way.

Today we distinguish between the “Einasto” and “Sérsic” mass profiles. The former is fit to 3-D mass density $\rho(r)$ while the latter is fit to 2-D projected mass distributions $\Sigma(R)$. Projected and deprojected approximations to the Einasto and Sérsic profiles, respectively, have also been derived (see Table 1).

The Sérsic (1968) profile is given by:

$$\Sigma(R) = \Sigma_c \exp \left\{-b_n \left[ \left( \frac{R}{R_c} \right)^{1/n} - 1 \right] \right\}. \quad (30)$$

There are three free parameters: $\Sigma_c$, $R_c$, and $n$, with $b_n$ being a function of $n$ (given below) such that half the mass is contained within $R_c$. Note that the total mass of a Sérsic profile is finite, unlike that for an NFW profile. A 3-D deprojected approximation is given by Prugniel & Simien (1997).

The Einasto mass profile is a similar function but of 3-D mass density $\rho(r)$:

$$\rho(r) = \rho_{-2} \exp \left( -\frac{2}{\alpha} \left( \left( \frac{r}{r_{-2}} \right)^{\alpha} - 1 \right) \right) \quad (31)$$

where $\rho_{-2}$ and $r_{-2}$ are the density and radius at which $\rho(r) \propto r^{-2}$. The concentration is defined as $c_{\text{vir}} = r_{\text{vir}}/r_{-2}$. Navarro et al. (2010) found $\alpha \approx 0.17$ for galaxy-sized halos in the Aquarius simulation. Gao et al. (2008) concur and found $\alpha$ increases to $\sim 0.3$ for the most massive clusters in the Millennium simulation. Duffy et al. (2008) reduce the Einasto profile to two free parameters by using the “peak height” $\alpha(\nu)$ relation from Gao et al. (2008). A 2-D projected approximation of Einasto is given by Dhar & Williams (2010).
3.4. Profile Details

Here we provide useful expressions derived from the NFW and Sérsic profiles.

3.4.1. NFW Profile

Simulated galaxy and cluster halos Navarro et al. (1996, 1997) were shown to all have mass density profiles well approximated by the NFW profile:

\[ \rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2} \]  

(33)

The two fit parameters \( \rho_s \) and \( r_s \) were shown to be related and a function of halo mass, as we discuss below. But the parameter making all the buzz these days is the central mass concentration:

\[ c_{\text{vir}} = r_{\text{vir}}/r_s \]  

(34)

where \( r_{\text{vir}} \) is the virial radius of the mass halo. As discussed above, the virial radius is estimated as that which contains an average density \( \Delta_c \rho_{\text{crit}} \), for a total virial mass of

\[ M_{\text{vir}} = \frac{4}{3} \pi \Delta_c \rho_{\text{crit}} r_{\text{vir}}^3 \]  

(35)

For an NFW halo, the mass within a sphere with radius \( r = x r_s \) can be found by simply integrating the NFW profile (Eq. 33):

\[ M(r) = 4\pi r_s^3 \int_0^x dx' x'^2 \frac{\rho_s}{x'(1 + x'^2)} \]  

(36)

\[ = 4\pi \rho_s r_s^3 \left( \ln(1 + x) - \frac{x}{1 + x} \right) \]  

(37)

Combining Eqs. 34 and 35, we find that the concentration parameter \( c \) can be obtained from the following expression, as given in Navarro et al. (1996):

\[ \rho_s/\rho_{\text{crit}} \equiv \delta_c = \frac{\Delta_c}{3} \ln(1 + c) - c/(1 + c) \]  

(38)

To fit the NFW profile to our gravitational lensing mass maps which measure projected surface density, we integrate the NFW profile along the line of sight (e.g., Golse & Kneib 2002) to find the projected surface density:

\[ \Sigma(R) = 2 \rho_s r_s F(X) \]  

(39)

with \( R = X r_s \) and

\[ F(X) = \begin{cases} 
\frac{1}{X^2 - 1} \left( 1 - \frac{1}{\sqrt{1 - X^2}} \cosh^{-1} \frac{1}{X} \right) & (X < 1) \\
\frac{1}{3} & (X = 1) \\
\frac{1}{X^2 - 1} \left( 1 - \frac{1}{\sqrt{X^2 - 1}} \cos^{-1} \frac{1}{X} \right) & (X > 1) 
\end{cases} \]  

(40)
the total mass within a cylinder of radius $R$
\[ M(R) = 4\pi r_s^3 \rho_s G(X) \quad (41) \]
with
\[
G(X) = \ln \frac{X}{2} + \begin{cases} 
\frac{1}{\sqrt{1-X^2}} \cosh^{-1} \frac{1}{X} & (X < 1) \\
1 & (X = 1) \\
\frac{1}{\sqrt{X^2-1}} \cos^{-1} \frac{1}{X} & (X > 1) 
\end{cases} \quad (42)
\]
This should not be confused with Eq. [37] which gives mass within a sphere of radius $R$.

From this we can obtain the shear due to an NFW mass profile: \( \gamma(R) = \tilde{\kappa}(R) - \kappa(R) \):
\[
\gamma(R) = 2\kappa_e \left( \frac{2G(X)}{X^2} - F(X) \right). \quad (43)
\]
The quantity measured in weak lensing studies is the reduced shear:
\[
g = \frac{(D_{LS}/D_S)\gamma}{1 - (D_{LS}/D_S)\kappa}; \quad (44)
\]
where we have finally given the redshift dependence. (All previous expressions were given for a fiducial lensed source at $z_s = \infty$.)

### 3.4.2. Sérsic Profile

We now give the Sérsic (1968) profile and quantities derived from it (e.g., Graham & Driver 2005; Terzic & Graham 2005). Note that the Sérsic profile is commonly used to describe the (projected 2-D) light profiles of elliptical galaxies. Here instead it will be discussed as describing projected mass profiles.

The Sérsic (1968) profile is given by:
\[
\Sigma(R) = \Sigma_e \exp \left\{ -b_n \left[ \left( \frac{R}{R_e} \right)^{1/n} - 1 \right] \right\}, \quad (45)
\]
There are three free parameters: $\Sigma_e$, $R_e$, and $n$, with $b_n$ being a function of $n$ (given below) such that half the mass is contained within $R_e$. (Note that the total mass of a Sérsic profile is finite, unlike that for an NFW profile.) The total projected mass within a radius $R$ is given as:
\[
M(R) = 2\pi \Sigma_e R_e^2 ne^{b_n} b_n^{-2n} \hat{\gamma}(2n, x) \quad (46)
\]
where
\[
x = b_n (R/R_e)^{1/n}, \quad (47)
\]
and \( \hat{\gamma}(a, x) = \int_0^x dt e^{-t/a-1} \) is the incomplete gamma function (with the “hat” used to distinguish $\hat{\gamma}$ from the lensing shear $\gamma$). Thus to satisfy $M(R_e) = \frac{1}{2} M(R = \infty)$, $b_n$ must obey:
\[
\Gamma(2n) = 2\hat{\gamma}(2n, b_n) \quad (48)
\]
where $\Gamma(a) = \hat{\gamma}(a, \infty)$ is the complete gamma function.

In SciPy’s “special” package, we find a routine to quickly calculate $b_n = \text{gammaincinv}(2 \ast n, 0.5)$. An approximation may also be used (Prugniel & Simien 1997):
\[
b_n \approx 2n - 1/3 + 0.009876/n \quad (49)
\]

Lensing properties of the Sérsic profile have been derived and explored in Cardone (2004) and Elíasdóttir & Möller (2007). Of special interest here is the weak shear $\gamma = \kappa - \tilde{\kappa}$. The average $\kappa$ within $R$ can be derived straightforwardly from the above expression for $M(R)$:
\[
\bar{\kappa}(R) = \frac{M(R)}{\pi R^2 \Sigma_{crit}} = 2\kappa_e ne^{b_n} x^{-2n} \hat{\gamma}(2n, x) \quad (50)
\]
where we have introduced $\kappa_e = \Sigma_e/\Sigma_{crit}$. Meanwhile, $\kappa(R) = \Sigma(R)/\Sigma_{crit}$ can be rewritten as:
\[
\kappa(R) = \kappa_e e^{(b_n-x)} \quad (51)
\]
Thus we find $\gamma(R) = \bar{\kappa}(R) - \kappa(R)$:
\[
\gamma(R) = \kappa_e e^{b_n} \left( 2nx^{-2n} \hat{\gamma}(2n, x) - e^{-x} \right), \quad (52)
\]
with the reduced shear given as $g = (\gamma D_{LS}/D_S)/(1 - \kappa D_{LS}/D_S)$.

There are fewer published fits of Sérsic profiles to simulated cluster halos. We do note that Merritt et al. (2005) found $n = 2.38 \pm 0.25$ for their cluster sample.

We thank Angelo Neto for useful conversations about the Millennium simulation and their study of halo profiles. This work was carried out at Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA.
Fig. 1.— For fixed $M_{200} = 10^{15} M_\odot$, we show what an “over-concentrated” looks like: $c_{200} = 9$ versus the expected concentration ($c_{200} = 3$). This is roughly the case for A1689 (Coe et al. 2010). Dots mark $r_s$ and dashed lines mark $r_{vir}$ and $M_{vir}$. Thicker lines are used for two decades in radius about $r_s$ to emphasize the shifting of the curves.

Fig. 2.— Expected NFW concentration $c_{200}$ and 1-$\sigma$ scatter as a function of halo mass $M_{200}$ for all clusters (left) and relaxed clusters (right).

Fig. 3.— Expected NFW concentration $c(M)$ for $z = 0$ halos and $c(M, z)$ for $z = 0, 1, 2$ halos from Duffy et al. (2008).
Fig. 4.— Expected NFW concentrations as a function of redshift for various mass halos from Duffy et al. (2008) (see Table 5). Also plotted is the $c \propto (1 + z)^{-1}$ slope expected from Bullock et al. (2001). The x axes are plotted on scales of log$(1 + z)$.

Fig. 5.— The effect of cosmology. Left: the Millennium simulation (Neto07) used WMAP1 with a higher $\sigma_8$ resulting in higher concentrations. Right: 3 different simulations with different cosmologies used.

Fig. 6.— Comparing several studies, including both simulated and observed NFW $c(M)$. 
TABLE 3
NFW $c_{200}(M_{200})$ FIT PARAMETERS ($z = 0$): $c_{200} = c_0(M_{200}/M_0)^{-\alpha} + \Delta c_{200}$

| Sample       | Cosmology | $c_0$   | $M_0 [M_\odot h]$ | $\alpha$ | $\Delta \log_{10} c_{200}$ |
|--------------|-----------|---------|-------------------|----------|-----------------------------|
| Duffy08      | all       | WMAP5   | 5.74              | $2 \times 10^{12}$ | 0.097 | 0.15 |
| Duffy08      | relaxed   | WMAP5   | 6.67              | $2 \times 10^{12}$ | 0.092 | 0.15 |
| Macci08      | all       | WMAP5   | 6.12              | $10^{12}$    | 0.110 | 0.130 |
| Macci08      | relaxed   | WMAP5   | 6.76              | $10^{12}$    | 0.098 | 0.105 |
| Macci08      | all       | WMAP3   | 5.24              | $10^{12}$    | 0.088 | 0.132 |
| Macci08      | relaxed   | WMAP3   | 5.87              | $10^{12}$    | 0.083 | 0.109 |
| Macci08      | all       | WMAP1   | 7.57              | $10^{12}$    | 0.119 | 0.129 |
| Macci08      | relaxed   | WMAP1   | 8.26              | $10^{12}$    | 0.104 | 0.111 |
| Neto07       | all       | WMAP1   | 4.67              | $10^{14}$    | 0.11  | 0.094 |
| Neto07       | relaxed   | WMAP1   | 5.26              | $10^{14}$    | 0.10  | 0.061 |

TABLE 4
NFW $c_{vir}(M_{vir})$ FIT PARAMETERS ($z = 0$): $c_{vir} = c_0(M_{vir}/M_0)^{-\alpha} + \Delta c_{vir}$

| Sample       | Cosmology | $c_0$   | $M_0 [M_\odot h]$ | $\alpha$ | $\Delta \log_{10} c_{vir}$ |
|--------------|-----------|---------|-------------------|----------|-----------------------------|
| Duffy08      | all       | WMAP5   | 7.96 ± 0.17       | $2 \times 10^{12}$ | 0.091 ± 0.007 | $\ldots$ |
| Duffy08      | relaxed   | WMAP5   | 9.23 ± 0.15       | $2 \times 10^{12}$ | 0.089 ± 0.010 | $\ldots$ |
| Macci08      | all       | WMAP5   | 8.41              | $10^{12}$    | 0.108 | $\ldots$ |
| Macci08      | relaxed   | WMAP5   | 9.35              | $10^{12}$    | 0.094 | $\ldots$ |
| Macci08      | all       | WMAP3   | 7.26              | $10^{12}$    | 0.086 | $\ldots$ |
| Macci08      | relaxed   | WMAP3   | 8.22              | $10^{12}$    | 0.080 | $\ldots$ |
| Macci08      | all       | WMAP1   | 10.26             | $10^{12}$    | 0.114 | $\ldots$ |
| Macci08      | relaxed   | WMAP1   | 11.25             | $10^{12}$    | 0.099 | $\ldots$ |
| Hennawi07    | all       | WMAP1   | 12.3              | $1.3 \times 10^{13}$ | 0.13 | 0.098 |
| Gentile07    | all       | WMAP3   | 13.6              | $10^{11}$    | 0.13 | $\ldots$ |
| Bullock01    | all       | WMAP1   | 9                 | $1.3 \times 10^{13}$ | 0.13 | 0.14$^a$ |
| Comerford07  | all observed | WMAP1 | 14.5 ± 0.64       | $1.3 \times 10^{13}$ | 0.15 ± 0.13 | 0.15 |

$^a$ [Wechsler et al. (2002) foot note 10] claim that the scatter of $\Delta \log_{10} c_{vir} = 0.18$ reported by Bullock et al. (2001) was a bit too high and should actually be 0.14, thus bringing it in line with their own measured scatter.

TABLE 5
NFW $c(M, z)$ FIT PARAMETERS FOR $0 < z < 2$: $c = c_0(M/M_0)^{-\alpha}(1 + z)^{-\beta}$

| Sample       | Cosmology | $\Delta$ | $c_0$   | $M_0 [M_\odot h]$ | $\alpha$ | $\beta$ |
|--------------|-----------|----------|---------|-------------------|----------|---------|
| Duffy08      | all       | WMAP5   | 200     | 5.71 ± 0.12       | $2 \times 10^{12}$ | 0.084 ± 0.006 | 0.47 ± 0.04 |
| Duffy08      | relaxed   | WMAP5   | 200     | 6.71 ± 0.12       | $2 \times 10^{12}$ | 0.091 ± 0.009 | 0.44 ± 0.05 |
| Duffy08      | all       | WMAP5   | vir     | 7.85 ± 0.17       | $2 \times 10^{12}$ | 0.081 ± 0.006 | 0.71 ± 0.04 |
| Duffy08      | relaxed   | WMAP5   | vir     | 9.23 ± 0.16       | $2 \times 10^{12}$ | 0.090 ± 0.009 | 0.69 ± 0.05 |

TABLE 6
OBSERVED $c(M)$ FIT PARAMETERS ($z = 0$): $c = c_0(M/M_0)^{-\alpha}(1 + z)^{-\beta}$

| Sample       | Analysis | $\Delta c$ | $c_0$   | $M_0 [h^{-1}M_0]$ | $\alpha$ | $\Delta \log_{10} c$ | $N$ | $M_{\text{min}} [h^{-1}M_0]$ | $M_{\text{max}} [h^{-1}M_0]$ | $z$ |
|--------------|----------|------------|---------|-------------------|----------|---------------------|----|---------------------------|---------------------------|----|
| Comerford07$^b$ | compilation | $\Delta c$ | 14.5 ± 6.4 | $1.3 \times 10^{13}$ | 0.15 ± 0.13 | 0.15 | 62 | $4 \times 10^{14}$ | 0.003 ± 0.089 |
| Buote07      | X-ray    | vir        | 9.0 ± 0.4 | $10^{14}$ | 0.172 ± 0.026 | $\ldots$ | 39 relaxed | $6 \times 10^{12}$ | 0.0033 ± 0.2302 |
| SchmidtAllen07 | X-ray    | vir        | 7.55 ± 0.90$^c$ | $8 \times 10^{14}$ | 0.45 ± 0.12$^c$ | $\ldots$ | 34 relaxed | $2 \times 10^{14}$ | 0.06 ± 0.7 |
| Okabe09      | WL       | vir        | 8.45 ± 3.91 | $10^{14}$ | 0.41 ± 0.19 | 0.19 | 30 | $2 \times 10^{14}$ | 1.5 ± 0.15 | 0.30 |
| Johnston07   | stacked WL | 200 | 4.1 ± 1.2 | $1.3 \times 10^{13}$ | 0.12 ± 0.04 | $\ldots$ | 130,000 | $5 \times 10^{12}$ | 5 ± 0.25 |
| Mandelbaum08$^d$ | stacked WL | 54 | 4.6 ± 0.7 | $10^{14}$ | 0.13 ± 0.07 | $\ldots$ | 222,699 | $3 \times 10^{13}$ | 6 ± 0.14 | 0.22 |
| Mandelbaum08$^e$ | stacked WL | 200 | 2.5 ± 0.4 | $6 \times 10^{14}$ | 0.13 ± 0.07 | $\ldots$ | 222,699 | $3 \times 10^{13}$ | 6 ± 0.14 | 0.22 |

Note. — SchmidtAllen07 find $\beta = 0.71 ± 0.52^c$, but all others fix $\beta = 1$ while fitting only ($c_0, \alpha$).

$^b$ Includes Buote07 and SchmidtAllen07

$^c$ Quoted uncertainties are 95% rather than 1-$\sigma$

$^d$ Mandelbaum08 used $\Delta c_{\text{vir}} = 200$ (not $\Delta c = 200$) and assumed $\Omega_m = 0.27$, corresponding to $\Delta c = \Omega_m \Delta c_{\text{vir}} = 54$.

$^e$ Converted from previous line
Halo Mass \(\log_{10}(M_{200}/M_\odot)\)

1.00
1.05
1.10
1.15
1.20
1.25

Concentration ratio: relaxed / all

Duffy08
Maccio08
Neto07

Fig. 7.— Ratio of NFW \(c_{200}\) for relaxed vs. all \(z = 0\) halos in various simulations.

Concentration \(c_{200}\)

Duffy08 (Einasto fits)
relaxed (\(z = 0\))
relaxed \(z = 0\)
relaxed \(z = 1\)
relaxed \(z = 2\)
NFW

Fig. 8.— Expected concentrations derived from Einasto profiles compared to NFW profiles. Plotted are relations for relaxed halos from Duffy et al. (2008). The yellow NFW lines were plotted in Fig. [5].

TABLE 7
Simulations

| Simulation   | Cosmology   | \(M_{\text{min}}\)  | \(M_{\text{max}}\)  | \(N\)   | \(N\) relaxed | particles within \(r_{\text{vir}}\) |
|--------------|-------------|----------------------|----------------------|--------|--------------|-------------------------------|
| Duffy08      | WMAP5       | \(10^{11}\)          | \(10^{15}\)          | 1,269  | 561          | 10,000                        |
| Maccio08     | WMAP5       | \(10^{10}\)          | \(10^{15}\)          | 9,988  | 7,060        | 500                           |
| Neto07 (Millennium) | WMAP1       | \(10^{12}\)          | \(10^{15}\)          | 53,626 | 39,330       | 10,000                        |
| Hemmawi07    | WMAP1       | \(10^{14}\)          | \(10^{15}\)          | 878    | ...          | ...                           |
| Bullock01    | WMAP1       | \(10^{11}\)          | \(10^{14}\)          | \(\sim 5,000\) | \(\sim 150-120,000\) | \(\sim 5,000-10,000\) |
| Gentile07 (NFW96) | WMAP3       | \(3 \times 10^{11}\) | \(3 \times 10^{15}\) | 19     | ...          | ...                           |

Fig. 8.— For a fixed NFW curve \((r_s, \rho_s)\), illustration of conversion of \((c, M)\) between different values of \(\Delta_c\). From Fig. [11] we find \(c_{100} \approx 1.37c_{200}\) and \(M_{100} \approx 1.3M_{200}\). The two (perfectly overlapping) black curves have identical \((r_s, \rho_s)\). Dots mark \(r_s\), while the dashed lines mark the overdensities and \(r_{\text{vir}}\) for \(\Delta_c = 200\) and 100. Note that for fixed \(c, r_s\) and \(r_{\text{vir}}\) vary with \(M_{\text{vir}}\) (black vs. yellow).

TABLE 8
Cosmological parameters

| Author        | WMAP  | \(\Omega_m\) | \(\sigma_8\) |
|---------------|-------|---------------|--------------|
| Bullock01     | WMAP1 | 0.3           | 1.0          |
| Hemmawi07     | WMAP1 | 0.3           | 0.95         |
| Neto07: Millennium | WMAP1 | 0.25         | 0.90         |
| Maccio08      | WMAP1 | 0.268         | 0.90         |
| Maccio08      | WMAP3 | 0.238         | 0.75         |
| Maccio08, Duffy08 | WMAP5 | 0.258     | 0.796        |
| ...           | WMAP7 | 0.26          | 0.803        |

Fig. 9.— Conversion from NFW \(c_{200}\) to \(c_{\text{vir}}\) for \(\Delta_c = 94, 100, 120, 140, 160, 180, 200\). The relations are extremely linear for \(c_{200} > 2\).
Fig. 10.— Parameters $a$ and $b$ for conversion of NFW $c_{\text{vir}} \approx ac_{200} + b$. Dashed lines are the relations given in Eqs. 19 & 20.

Fig. 11.— Ratios of $r_{\text{vir}}$ (left) and $M_{\text{vir}}$ (right) to the $\Delta_c = 200$ values as a function of NFW ($c$, $\Delta_c$). Note that $r_{\text{vir}}/r_{200} = c_{\text{vir}}/c_{200}$ since we hold $r_s$ (and $\rho_s$) fixed and we define $r_{\text{vir}} = c_{\text{vir}}r_s$ and $r_{200} = c_{200}r_s$.

Fig. 12.— The relation between $r_{\text{vir}}$ and $M_{\text{vir}}$ is independent of the assumed profile, only dependent on the chosen overdensity, cosmology, and halo redshift (Eq. 23). For a $z = 0$ halo with $r_{\text{vir}} = 1$ Mpc/$h$, $M_{\text{vir}} = 1.16 \times 10^{14} M_\odot/h$ ($\Delta_c/100$).
\[ \Omega_m(z) = 1 - \Omega_\Lambda(z) \] in a flat universe with a cosmological constant

\[ \Delta_c(z) \]: The average overdensity above \( \rho_{\text{crit}} \) within a collapsed halo

\[ \Delta_{\text{vir}}(z) \]: The average overdensity above \( \rho_m = \Omega_m \rho_{\text{crit}} \) within a collapsed halo
Fig. 16.— Power law slopes $\gamma'$ for 3-D density $\rho \propto r^{-\gamma'}$. Here we compare the NFW, Einasto, and S´ersic profiles. For S´ersic we use the Prugniel-Simien approximation. A realistic range of $\alpha$ and $n$ are plotted for the profiles, as found in simulations (Merritt et al. 2006; Gao et al. 2008; Navarro et al. 2010).

Fig. 17.— Power law slopes $\gamma'$ for 3-D density $\rho \propto r^{-\gamma'}$. In this figure we add the Dehnen-McLaughlin and Stadel-Moore profiles with $\gamma = 7/9$ and $\lambda = 0.10$, respectively.

Fig. 18.— Power law slopes $\gamma'$ for 3-D density $\rho \propto r^{-\gamma'}$. Here we compare various values of $\gamma$ for Dehnen-McLaughlin as observed in simulations (Merritt et al. 2006). The value marked with a * (7/9) is used in their 2-parameter fitting formula.

Fig. 19.— NFW power law slopes for projected mass within a cylinder $M(<R)$, projected mass density $\kappa(R)$, and 3-D density $\rho(r)$. 
REFERENCES

Alam, S. M. K., Bullock, J. S., & Weinberg, D. H. 2002, ApJ, 572, 34 [ADS]

Barkana, R. & Loeb, A. 2009, ArXiv e-prints [ADS]

Broadhurst, T., Benitez, N., Cole, D., Sharon, K., Zekser, K., White, R., Ford, H., Bouwens, R., et al. 2005a, ApJ, 621, 53 [ADS]

Broadhurst, T., Takada, M., Umetu, K., Kong, X., Arimoto, N., Chiba, M., & Futamase, T. 2005b, ApJ, 619, L143 [ADS]

Bullock, J. S., Kolatt, T. S., Sigad, Y., Somerville, R. S., Kravtsov, A. V., Klypin, A. A., Primack, J. R., & Dekel, A. 2001, MNRAS, 321, 559 [ADS]

Buote, D. A., Gastaldello, F., Humphrey, P. J., Zappacosta, L., Bullock, J. S., Brighenti, F., & Mathews, W. G. 2007, ApJ, 664, 124 [ADS]

Cardone, V. F. 2004, A&A, 415, 839 [ADS]

Duffy, A. R., Schaye, J., Kay, S. T., & Dalla Vecchia, C. 2008, MNRAS, 390, L64 [ADS]

Dutton, A. A., van den Bosch, F. C., Dekel, A., & Courteau, S. 2007, ApJ, 654, 27 [ADS]

Einasto, J. 1965, Trudy Inst. Astro. Alma-Ata, 51, 87 [ADS]

Eke, V. R., Navarro, J. F., & Frenk, C. S. 1998, ApJ, 503, 569 [ADS]

Eke, V. R., Navarro, J. F., & Steinmetz, M. 2001, ApJ, 554, 114 [ADS]

Eliskasté, Æ. & Möller, O. 2007, Journal of Cosmology and Astro-Particle Physics, 7, 6 [ADS]

Gao, L., Navarro, J. F., Cole, S., Frenk, C. S., White, S. D. M., Springel, V., Jenkins, A., & Neto, A. F. 2008, MNRAS, 387, 536 [ADS]

Gentile, G., Tonini, C., & Salucci, P. 2007, ArXiv Astrophysics e-prints [ADS]

Gnedin, O. Y., Weinberg, D. H., Pizagno, J., Prada, F., & Rix, H. 2007, ApJ, 671, 1115 [ADS]

Golse, G. & Kneib, J.-P. 2002, A&A, 390, 821 [ADS]

Graham, A. W. & Driver, S. P. 2005, Publications of the Astronomical Society of Australia, 22, 118 [ADS]

Graham, A. W., Merritt, D., Moore, B., Diemand, J., & Terzić, B. 2006, AJ, 132, 2701 [ADS]

Grossi, M. & Springel, V. 2009, MNRAS, 394, 1559 [ADS]

Hayashi, E. & White, S. D. M. 2008, MNRAS, 388, 2 [ADS]

Hennawi, J. F., Dalal, N., Bode, P., & Ostrikov, J. P. 2007, ApJ, 654, 714 [ADS]

Henry, J. P. 2000, ApJ, 534, 565 [ADS]

Hernquist, L. 1990, ApJ, 356, 359 [ADS]

Hu, W. & Kravtsov, A. V. 2003, ApJ, 584, 702 [ADS]

Johnston, D. E., Sheldon, E. S., Wechsler, R. H., Rozo, E., Koester, B. P., Frieman, J. A., McKay, T. A., Evrard, A. E., et al. 2007, ArXiv e-prints [ADS]

Klypin, A., Kravtsov, A. V., Bullock, J. S., & Primack, J. R. 2001, ApJ, 554, 903 [ADS]

Komatsu, E., Smith, K. M., Dunkley, J., Bennett, C. L., Gold, B., Hinshaw, G., Jarosik, N., Larson, D., et al. 2010, ArXiv e-prints [ADS]

Kuhlen, M., Strigari, L. E., Zentner, A. R., Bullock, J. S., & Primack, J. R. 2005, MNRAS, 357, 387 [ADS]

Lapi, A. & Cavaliere, A. 2009, ApJ, 695, L125 [ADS]

Lu, Y., Mo, H. J., Katz, N., & Weinberg, M. D. 2006, MNRAS, 368, 1931 [ADS]

Madau, A. V., Bullock, A. A., & van den Bosch, F. C. 2008, MNRAS, 391, 1940 [ADS]

Mandelbaum, R., Seljak, U., & Hirata, C. M. 2008, Journal of Cosmology and Astro-Particle Physics, 8, 6 [ADS]

Menechetti, M., Fedeli, C., Pace, F., Gottloeber, S., & Yepes, G. 2010, ArXiv e-prints [ADS]

Merritt, D., Graham, A. W., Moore, B., Diemand, J., & Terzić, B. 2006, AJ, 132, 2655 [ADS]

Merritt, D., Navarro, J. F., Ludlow, A., & Jenkins, A. 2005, ApJ, 624, L85 [ADS]

Moore, B., Quinn, T., Governato, F., Stadel, J., & Lake, G. 1999, MNRAS, 310, 1147 [ADS]

Nakamura, T. T. & Suto, Y. 1997, Progress of Theoretical Physics, 97, 49 [ADS]

Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563 [ADS]

2007, ApJ, 490, 493 [ADS]

Navarro, J. F., Hayashi, M., Power, C., Jenkins, A. R., Frenk, C. S., White, S. D. M., Springel, V., Stadel, J., et al. 2004, MNRAS, 349, 1030 [ADS]

Navarro, J. F., Ludlow, A., Springel, V., Wang, J., Vogelsberger, M., White, S. D. M., Jenkins, A., Frenk, C. S., et al. 2010, MNRAS, 402, 21 [ADS]

Neto, A. F., Gao, L., Bryan, G. L., Navarro, J. F., Frenk, C. S., White, S. D. M., Springel, V., et al. 2007, MNRAS, 381, 1450 [ADS]

Oguri, M., Hennawi, J. F., Gladders, M. D., Dahle, H., Natarajan, P., Dalal, N., Koester, B. P., Sharon, K., et al. 2009, ApJ, 699, 1038 [ADS]

Okabe, N., Takada, M., Umetu, K., Futamase, T., & Smith, G. P. 2009, ArXiv e-prints [ADS]

Peebles, P. J. E. 1980, The large-scale structure of the universe (Research supported by the National Science Foundation. Princeton, N.J., Princeton University Press, 1980. 435 p.) [ADS]

Prugniel, P. & Simen, F. 1997, A&A, 321, 111 [ADS]

Richard, J., Smith, G. P., Kneib, J., Ellis, R., Sanderson, A. J. R., Pei, L., Targett, T., Sand, D., et al. 2009, ArXiv e-prints [ADS]

Riess, A. G., Macri, L. M., Casertano, S., Sosey, M., Lampeitl, H., Ferguson, H. C., Filippenko, A. V., Jha, S. W., et al. 2009, ApJ, 699, 530 [ADS]

Schmidt, R. W. & Allen, S. W. 2007, MNRAS, 379, 209 [ADS]

Serey, J. L. 1968, Atlas de galaxias australes (Cordoba, Argentina: Observatorio Astronomico, 1968) [ADS]

Shaw, L. D., Weller, J., Ostriker, J. P., & Bode, P. 2006, ApJ, 646, 815 [ADS]

Spiegel, D. N., Verde, L., Peiris, H. V., Komatsu, E., Nolta, M. R., Bennett, C. L., Halpern, M., Hinshaw, G., et al. 2003, ApJS, 148, 175 [ADS]

Stadel, J., Potter, D., Moore, B., Diemand, J., Madau, P., Zemp, M., Kuhlen, M., & Quilis, V. 2009, MNRAS, 398, L21 [ADS]

Terzić, B. & Graham, A. W. 2005, MNRAS, 362, 197 [ADS]

Wechsler, R. H., Bullock, J. S., Primack, J. R., Kravtsov, A. V., & Dekel, A. 2002, ApJ, 568, 52 [ADS]

Witthae, L. D., Bower, R. S., & Spiegel, D. N. 2001, ApJ, 555, 504 [ADS]

Zhao, H. J., Jing, Y. P., Mo, H. J., & Börner, G. 2003, ApJ, 597, L9 [ADS]

Zhao, H. 1996, MNRAS, 278, 488 [ADS]