On the Geometry of Connections with Totally Skew-Symmetric Torsion on Manifolds with Additional Tensor Structures and Indefinite Metrics

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Abstract

This paper is a survey of results obtained by the authors on the geometry of connections with totally skew-symmetric torsion on the following manifolds: almost complex manifolds with Norden metric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.

Keywords: almost complex manifold, almost contact manifold, almost hypercomplex manifold, Norden metric, B-metric, anti-Hermitian metric, skew-symmetric torsion, KT-connection, HKT-connection, Bismut connection.

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Introduction

In Hermitian geometry there is a strong interest in the connections preserving the metric and the almost complex structure whose torsion is totally skew-symmetric (\cite{23, 24, 21, 1, 8, 9, 2, 3}). Such connections are called KT-connections (or Bismut connections). They find widespread application in mathematics as well as in theoretic physics. For instance, it is proved a local index theorem for non-Kähler manifolds by KT-connection in \cite{1} and the same connection is applied in string theory in \cite{21}. According to \cite{8}, on any Hermitian manifold, there exists a unique KT-connection. In \cite{3} all almost contact, almost Hermitian and $G_2$-structures admitting a KT-connection are described.

In this work\footnote{partially supported by projects IS-M-4/2008 and RS09-FMI-003 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria} we provide a survey of our investigations into connections with totally skew-symmetric torsion on almost complex manifolds with Norden met-
ric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.

In Section 1 we consider an almost complex manifold with Norden metric (i.e. a neutral metric $g$ with respect to which the almost complex structure $J$ is an anti-isometry). On such a manifold we study a natural connection (i.e. a linear connection $\nabla'$ preserving $J$ and $g$) and having totally skew-symmetric torsion. We prove that $\nabla'$ exists only when the manifold belongs to the unique basic class with non-integrable structure $J$. This is the class $W_3$ of quasi-Kähler manifolds with Norden metric. We establish conditions for the corresponding curvature tensor to be Kählerian as well as conditions $\nabla'$ to have a parallel torsion. We construct a relevant example on a 4-dimensional Lie group.

In Section 2 we consider an almost contact manifold with B-metric which is the odd-dimensional analogue of an almost complex manifold with Norden metric. On such a manifold we introduce the so-called $\varphi$KT-connection having totally skew-symmetric torsion and preserving the almost contact structure and the metric. We establish the class of the manifolds where this connection exists. We construct such a connection and study its geometry. We establish conditions for the corresponding curvature tensor to be of $\varphi$-Kähler type as well as conditions for the connection to have a parallel torsion. We construct an example on a 5-dimensional Lie group where the $\varphi$KT-connection has a parallel torsion.

In Section 3 we consider an almost hypercomplex manifold with Hermitian and anti-Hermitian metric. This metric is a neutral metric which is Hermitian with respect to the first almost complex structure and an anti-Hermitian (i.e. a Norden) metric with respect to the other two almost complex structures. On such a manifold we introduce the so-called pHKT-connection having totally skew-symmetric torsion and preserving the almost hypercomplex structure and the metric. We establish the class of the manifolds where this connection exists. We study the unique pHKT-connection $D$ on a nearly Kähler manifold with respect to the first almost complex structure. We establish that this connection coincides with the known KT-connection on nearly Kähler manifolds and therefore it has a parallel torsion. We prove the equivalence of the conditions $D$ be strong, flat and with a parallel torsion with respect to the Levi-Civita connection.

### 1. Almost complex manifold with Norden metric

Let $(M, J, g)$ be a $2n$-dimensional almost complex manifold with Norden metric, i.e. $M$ is a differentiable manifold with an almost complex structure $J$ and a pseudo-Riemannian metric $g$ such that

$$J^2 x = -x, \quad g(Jx, Jy) = -g(x, y)$$

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on $M$. Further $x, y, z, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$. 

The associated metric $\tilde{g}$ of $g$ on $M$ is defined by $\tilde{g}(x, y) = g(x, Jy)$. Both metrics are necessarily of signature $(n, n)$. The manifold $(M, J, \tilde{g})$ is also an almost complex manifold with Norden metric.

A classification of the almost complex manifolds with Norden metric is given in [4]. This classification is made with respect to the tensor $F$ of type $(0,3)$ defined by $F(x, y, z) = g((\nabla_x J)y, z)$, where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the following properties

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz). \quad (1)$$

The basic classes are $W_1$, $W_2$ and $W_3$. Their intersection is the class $W_0$ of the Kählerian-type manifolds, determined by $W_0$: $F(x, y, z) = 0 \iff \nabla J = 0$.

The class $W_3$ of the quasi-Kähler manifolds with Norden metric is determined by the condition

$$W_3: \quad F(x, y, z) + F(y, z, x) + F(z, x, y) = 0. \quad (2)$$

This is the only class of the basic classes $W_1$, $W_2$ and $W_3$, where each manifold (which is not a Kähler-type manifold) has a non-integrable almost complex structure $J$, i.e. the Nijenhuis tensor $N$, determined by $N(x, y) = (\nabla_x J)y - (\nabla_y J)x + (\nabla_{Jx} J)y - (\nabla_{Jy} J)x$ is non-zero.

The components of the inverse matrix of $g$ are denoted by $g^{ij}$ with respect to a basis $\{e_i\}$ of the tangent space $T_p M$ of $M$ at a point $p \in M$.

The square norm of $\nabla J$ is defined by $||\nabla J||^2 = g^{ij}g^{ks}(\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_s$.

**Definition 1 ([19]).** An almost complex manifold with Norden metric and $||\nabla J||^2 = 0$ is called an isotropic-Kähler manifold.

1.1. KT-connection

Let $\nabla'$ be a linear connection on an almost complex manifold with Norden metric $(M, J, g)$. If $T$ is the torsion tensor of $\nabla'$, i.e. $T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]$, then the corresponding tensor of type $(0,3)$ is determined by $T(x, y, z) = g(T(x, y), z)$.

**Definition 2 ([6]).** A linear connection $\nabla'$ preserving the almost complex structure $J$ and the Norden metric $g$, i.e. $\nabla' J = \nabla' g = 0$, is called a natural connection on $(M, J, g)$.

By analogy with Hermitian geometry we have given the following

**Definition 3 ([16]).** A natural connection $\nabla'$ on an almost complex manifold with Norden metric is called a KT-connection if its torsion tensor $T$ is totally skew-symmetric, i.e. a 3-form.

We have proved the following

**Theorem 1 ([18]).** If a KT-connection $\nabla'$ exists on an almost complex manifold with Norden metric then the manifold is quasi-Kählerian with Norden metric.
A partial decomposition of the space $T$ of the torsion $(0,3)$-tensors $T$ is valid on an almost complex manifold with Norden metric $(M, J, g)$ according to [6]:

$$T = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4,$$

where $\mathcal{T}_i (i = 1, 2, 3, 4)$ are invariant orthogonal subspaces.

**Theorem 2** ([18]). Let $\nabla'$ be a KT-connection with torsion $T$ on a quasi-Kähler manifold with Norden metric $(M, J, g) \notin \mathcal{W}_0$. Then

1) $T \in \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_4$;

2) $T$ does not belong to any of the classes $\mathcal{T}_1 \oplus \mathcal{T}_2$ and $\mathcal{T}_1 \oplus \mathcal{T}_4$;

3) $T \in \mathcal{T}_2 \oplus \mathcal{T}_4$ if and only if $T$ is determined by

$$T(x, y, z) = -\frac{1}{2} \{F(x, y, Jz) + F(y, z, Jx) + F(z, x, Jy)\}.$$  \hspace{1cm} (3)

Bearing in mind that $T$ is a 3-form, the following is valid

$$g(\nabla'_x y - \nabla'_y x, z) = \frac{1}{2} T(x, y, z).$$ \hspace{1cm} (4)

Then, by (1), (1) and (2), it follows directly that the tensor $T$, determined by (3), is the unique torsion tensor of a KT-connection, which is a linear combination of the components of the basic tensor $F$ on $(M, J, g)$ [22].

Further, the notion of the KT-connection $\nabla'$ on $(M, J, g)$ we refer to the connection with the torsion tensor determined by (3).

### 1.2. KT-connection with Kähler curvature tensor or parallel torsion

**Definition 4** ([5]). A tensor $L$ is called a Kähler tensor if it has the following properties:

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0,$$

$$L(x, y, Jz, Jw) = -L(x, y, z, w).$$

Let $R'$ be the curvature tensor of the KT-connection $\nabla'$, i.e. $R'(x, y)z = \nabla'_x (\nabla'_y z) - \nabla'_y (\nabla'_z x) - \nabla'_{[x,y]} z$. The corresponding tensor of type $(0,4)$ is determined by $R'(x, y, z, w) = g(R'(x, y)z, w)$.

We have therefore proved the following

**Theorem 3** ([16]). The following conditions are equivalent:

i) $R'$ is a Kähler tensor;

ii) $12R'(x, y, z, w) = 12R(x, y, z, w) + 2g(T(x, y), T(z, w)) - g(T(y, z), T(x, w))$;

iii) $\mathcal{S}_{x,y,z} \{g((\nabla_x J) y + (\nabla_y J) x, (\nabla_z J) w + (\nabla_J z) w)\} = 0$, where $\mathcal{S}$ denotes the cyclic sum by three arguments.
Proposition 4 ([16, 17]). Let \( \tau \) and \( \tau' \) be the scalar curvatures for \( R \) and \( R' \), respectively. Then the following is valid

i) \( 3 \| \nabla J \|^2 = 8(\tau' - \tau) \) if \( \nabla' \) has a Kähler curvature tensor;

ii) \( \| \nabla J \|^2 = 8(\tau - \tau') \) if \( \nabla' \) has a parallel torsion.

Corollary 5 ([17]). If \( \nabla' \) has a Kähler curvature tensor and a parallel torsion then \( (M, J, g) \) is an isotropic-Kähler manifold.

1.3. An example

Let \( (G, J, g) \) be a 4-dimensional almost complex manifold with Norden metric, where \( G \) is the connected Lie group with an associated Lie algebra \( \mathfrak{g} \) determined by a global basis \( \{X_i\} \) of left invariant vector fields, and \( J \) and \( g \) are the almost complex structure and the Norden metric, respectively, determined by

\[
JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2
\]

and

\[
g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,
g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j.
\]

Theorem 6 ([18]). The manifold \( (G, J, g) \) is a quasi-Kählerian with a Killing associated Norden metric \( \tilde{g} \), i.e. \( g([X_i, X_j], JX_k) + g([X_i, X_k], JX_j) = 0 \), if and only if \( g \) is defined by

\[
[X_1, X_2] = \lambda_1 X_1 + \lambda_2 X_2, \quad [X_1, X_3] = \lambda_3 X_2 - \lambda_1 X_4,
[X_1, X_4] = -\lambda_3 X_1 - \lambda_2 X_4, \quad [X_2, X_3] = \lambda_4 X_2 + \lambda_1 X_3,
[X_2, X_4] = -\lambda_4 X_1 + \lambda_2 X_3, \quad [X_3, X_4] = \lambda_3 X_3 + \lambda_4 X_4,
\]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \).

Let \( (G, J, g) \) be the manifold determined by the conditions in the last theorem.

The non-trivial components \( T_{ijk} = T(X_i, X_j, X_k) \) of the torsion \( T \) of the KT-connection \( \nabla' \) on \( (G, J, g) \) are \( T_{134} = \lambda_1, T_{234} = \lambda_2, T_{123} = -\lambda_3, T_{124} = -\lambda_4 \).

Moreover it is proved the following

Theorem 7 ([18]). The following propositions are equivalent:

i) The manifold \( (G, J, g) \) is isotropic-Kählerian;

ii) The manifold \( (G, J, g) \) is scalar flat;

iii) The KT-connection \( \nabla' \) has a Kähler curvature tensor;

iv) The equality \( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0 \) is valid.
2. Almost contact manifolds with B-metric

Let \((M, \varphi, \xi, \eta, g)\) be an almost contact manifold with B-metric (an almost contact B-metric manifold), i.e. \(M\) is a \((2n + 1)\)-dimensional differentiable manifold with an almost contact structure \((\varphi, \xi, \eta)\) which consists of an endomorphism \(\varphi\) of the tangent bundle, a vector field \(\xi\), its dual 1-form \(\eta\) as well as \(M\) is equipped with a pseudo-Riemannian metric \(g\) of signature \((n, n + 1)\), such that the following algebraic relations are satisfied

\[
\varphi \xi = 0, \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,
\]

where \(I\) denotes the identity.

Let us remark that the so-called B-metric \(g\) one can say a metric of Norden type in the odd-dimensional case, because the restriction of \(g\) on the contact distribution \(\ker \eta\) is a Norden metric with respect to the almost complex structure derived by \(\varphi\).

The associated metric \(\tilde{g}\) of \(g\) on \(M\) is defined by

\[
\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).
\]

Both metrics are necessarily of signature \((n, n + 1)\). The manifold \((M, \varphi, \xi, \eta, \tilde{g})\) is also an almost contact B-metric manifold.

A classification of the almost contact manifolds with B-metric is given in [1]. This classification is made with respect to the tensor \(F\) of type \((0, 3)\) defined by

\[
F(x, y, z) = \tilde{g}(\nabla_x \varphi y, z),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). The tensor \(F\) has the following properties

\[
F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).
\]

This classification includes eleven basic classes \(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{11}\). The special class \(\mathcal{F}_0\), belonging to any other class \(\mathcal{F}_i\) \((i = 1, 2, \ldots, 11)\), is determined by the condition \(F(x, y, z) = 0\). Hence \(\mathcal{F}_0\) is the class of almost contact B-metric manifolds with \(\nabla\)-parallel structures, i.e. \(\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = 0\).

In the present work we pay attention to \(\mathcal{F}_3\) and \(\mathcal{F}_7\), where each manifold (which is not a \(\mathcal{F}_0\)-manifold) has a non-integrable almost contact structure, i.e. the Nijenhuis tensor \(N\), determined by \(N(x, y) = [\varphi, \varphi](x, y) + d\eta(x)\xi\), is non-zero. These basic classes are characterized by the conditions

\[
\mathcal{F}_3 : \quad \bigotimes_{x, y, z} F(x, y, z) = 0, \quad F(\xi, y, z) = F(x, y, \xi) = 0,
\]

\[
\mathcal{F}_7 : \quad \bigotimes_{x, y, z} F(x, y, z) = 0, \quad F(x, y, z) = -F(\varphi x, \varphi y, z) - F(\varphi x, y, \varphi z).
\]

Let us consider the linear projectors \(h\) and \(v\) over \(T_pM\) which split (orthogonally and invariantly with respect to the structural group) any vector \(x\) into a horizontal component \(h(x) = -\varphi^2 x\) and a vertical component \(v(x) = \eta(x)\xi\).

The decomposition \(T_pM = h(T_pM) \oplus v(T_pM)\) generates the corresponding distribution of basic tensors \(F\), which gives the horizontal component \(\mathcal{F}_3\) and the vertical component \(\mathcal{F}_7\) of the class \(\mathcal{F}_3 \oplus \mathcal{F}_7\).

The square norm of \(\nabla \varphi\) is defined by \(\|\nabla \varphi\|^2 = g^{ij} g^{kl} g(\nabla_{e_i} \varphi)(\nabla_{e_j} \varphi)(e_k, e_l)\).
Definition 5 ([14]). An almost contact B-metric manifold with \( \| \nabla \phi \|^2 = 0 \) is called an isotropic-\( F_0 \)-manifold.

2.1. \( \phi \)KT-connection

Definition 6 ([14]). A linear connection \( D \) preserving the almost contact B-metric structure \((\phi, \xi, \eta, g)\), i.e. \( D\phi = D\xi = D\eta = Dg = 0 \), is called a natural connection on \((M, \phi, \xi, \eta, g)\).

Definition 7 ([14]). A natural connection \( D \) on an almost contact B-metric manifold is called a \( \phi \)KT-connection if its torsion tensor \( T \) is totally skew-symmetric, i.e. a 3-form.

The following theorem is proved.

Theorem 8 ([14]). If a \( \phi \)KT-connection \( D \) exists on an almost contact B-metric manifold \((M, \phi, \xi, \eta, g)\) then \( \xi \) is a Killing vector field and \( \mathcal{S}F = 0 \), i.e. \((M, \phi, \xi, \eta, g)\) belongs to the class \( F_3 \oplus F_7 \).

The existence of a \( \phi \)KT-connection \( D \) on a manifold in \( F_3 \oplus F_7 \) is given by the following

Proposition 9 ([14]). Let \((M, \phi, \xi, \eta, g)\) be in the class \( F_3 \oplus F_7 \). Then the connection \( D \) with a torsion tensor \( T \), determined by

\[
T(x, y, z) = -\frac{1}{2} \sum_{x, y, z} \left\{ F(x, y, \phi z) - 3\eta(x)F(y, \phi z, \xi) \right\},
\]

is a \( \phi \)KT-connection on \((M, \phi, \xi, \eta, g)\).

Further, the notion of the \( \phi \)KT-connection \( D \) on \((M, \phi, \xi, \eta, g)\) we refer to the connection with the torsion tensor determined by [5]. For this connection we have

\[
D_x y = \nabla_x y + \frac{1}{4} \left\{ 2(\nabla_x \phi) \phi y - (\nabla_y \phi) \phi x + (\nabla_{\phi y} \phi) x \\
+ 3\eta(x)\nabla_y \xi - 4\eta(y)\nabla_x \xi + 2(\nabla_x \eta) y \xi \right\}.
\]

2.2. The \( \phi \)KT-connection on the horizontal component

Let us consider a manifold from the class \( F_3 \) – the horizontal component of \( F_3 \oplus F_7 \). Since the restriction on the contact distribution of any \( F_3 \)-manifold is an almost complex manifold with Norden metric belonging to the class \( W_3 \) (known as a quasi-Kähler manifold with Norden metric), then the curvature properties are obtained in a way analogous to that in Section [II].
2.3. The $\varphi KT$-connection on the vertical component

Let $(M, \varphi, \xi, \eta, g)$ belong to the class $\mathcal{F}_7$ – the vertical component of $\mathcal{F}_3 \oplus \mathcal{F}_7$. For such a manifold the torsion of the $\varphi KT$-connection $D$ has the form

$$T(x, y) = 2 \{ \eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi + (\nabla_x \eta) y. \xi \}.$$

A tensor of $\varphi$-Kähler type we call a tensor with the properties from Definition 4 with respect to the structure $\varphi$.

We have proved the following

Theorem 10 ([14]). The curvature tensor $K$ of $D$ on a $\mathcal{F}_7$-manifold is of $\varphi$-Kähler type if and only if it has the form

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{3} \left\{ 2 (\nabla_x \eta) y (\nabla_z \eta) w - (\nabla_y \eta) z (\nabla_x \eta) w - (\nabla_z \eta) x (\nabla_y \eta) w \right\}$$

$$+ \eta(x)\eta(z) g(\nabla_y \xi, \nabla_w \xi) - \eta(x)\eta(w) g(\nabla_y \xi, \nabla_z \xi) - \eta(y)\eta(z) g(\nabla_x \xi, \nabla_w \xi) + \eta(y)\eta(w) g(\nabla_x \xi, \nabla_z \xi).$$

Theorem 11 ([14]). If $D$ has a curvature tensor $K$ of $\varphi$-Kähler type and a parallel torsion $T$ on a $\mathcal{F}_7$-manifold then

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{3} \left\{ 2 (\nabla_x \eta) y (\nabla_z \eta) w + (\nabla_x \eta) z (\nabla_y \eta) w - (\nabla_y \eta) w (\nabla_x \eta) z \right\},$$

$$\rho(K)(y, z) = \rho(y, z), \quad \tau(K) = \tau,$$

where $\rho(K)$ and $\rho$ are the Ricci tensors for $K$ and $R$, respectively, and $\tau(K)$ and $\tau$ are their corresponding scalar curvatures.

2.4. An example

Let $(G, \varphi, \xi, \eta, g)$ be a 5-dimensional almost contact manifold with B-metric, where $G$ is the connected Lie group with an associated Lie algebra $\mathfrak{g}$ determined by a global basis $\{X_i\}$ of left invariant vector fields, and $(\varphi, \xi, \eta)$ and $g$ are the almost contact structure and the B-metric, respectively, determined by

$$\varphi X_1 = X_3, \quad \varphi X_2 = X_4, \quad \varphi X_3 = -X_1, \quad \varphi X_4 = -X_2, \quad \varphi X_5 = 0;$$

$$\xi = X_5; \quad \eta(X_i) = 0 \ (i = 1, 2, 3, 4), \quad \eta(X_5) = 1;$$

$$g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = g(X_5, X_5) = 1,$$

$$g(X_i, X_j) = 0, \ i \neq j, \ i, j \in \{1, 2, 3, 4, 5\}.$$
Theorem 12 ([14]). The manifold \((G, \varphi, \xi, \eta, g)\) is a \(\mathcal{F}_7\)-manifold if and only if \(g\) is determined by the following non-zero commutators:

\[
[X_1, X_2] = -[X_3, X_4] = -\lambda_1 X_1 - \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 + 2\mu_1 X_5,
\]

\[
[X_1, X_4] = -[X_2, X_3] = -\lambda_3 X_1 - \lambda_4 X_2 + \lambda_1 X_3 - \lambda_2 X_4 + 2\mu_2 X_5,
\]

where \(\lambda_i, \mu_j \in \mathbb{R}\) (\(i = 1, 2, 3, 4; j = 1, 2\)).

Let \((G, \varphi, \xi, \eta, g)\) be the manifold determined by the conditions in the last theorem.

The non-trivial components \(T_{ijk} = T(X_i, X_j, X_k)\) of the torsion \(T\) of the \(\varphi\)KT-connection \(D\) on \((G, \varphi, \xi, \eta, g)\) are \(T_{125} = T_{345} = 2\mu_1\), \(T_{235} = T_{415} = 2\mu_2\).

Hence, using the components of \(D\), we calculate that the corresponding components of the covariant derivative of \(T\) with respect to \(D\) are zero. Thus, we have proved the following

Theorem 13 ([14]). The \(\varphi\)KT-connection \(D\) on \((G, \varphi, \xi, \eta, g)\) has a parallel torsion \(T\).

Theorem 14 ([14]). The manifold \((G, \varphi, \xi, \eta, g)\) is an isotropic-\(\mathcal{F}_0\)-manifold if and only if \(\mu_1 = \pm \mu_2\).

3. Almost hypercomplex manifolds with Hermitian and Norden metric

Let \((M, H)\) be an almost hypercomplex manifold, i.e. \(M\) is a 4\(n\)-dimensional differentiable manifold and \(H = (J_1, J_2, J_3)\) is a triple of almost complex structures with the properties:

\[
J_{\alpha} = J_{\beta} \circ J_{\gamma} = -J_{\gamma} \circ J_{\beta}, \quad J_{\alpha}^2 = -I
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\).

The standard structure of \(H\) on a 4\(n\)-dimensional vector space with a basis \(\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}\}_{k=0, 1, \ldots, n-1}\) has the form:

\[
\begin{align*}
J_1X_{4k+1} &= X_{4k+2}, & J_2X_{4k+1} &= X_{4k+3}, & J_3X_{4k+1} &= -X_{4k+4}, \\
J_1X_{4k+2} &= -X_{4k+1}, & J_2X_{4k+2} &= X_{4k+4}, & J_3X_{4k+2} &= X_{4k+3}, \\
J_1X_{4k+3} &= -X_{4k+4}, & J_2X_{4k+3} &= -X_{4k+1}, & J_3X_{4k+3} &= -X_{4k+2}, \\
J_1X_{4k+4} &= X_{4k+3}, & J_2X_{4k+4} &= -X_{4k+2}, & J_3X_{4k+4} &= X_{4k+1}.
\end{align*}
\]

Let \(g\) be a pseudo-Riemannian metric on \((M, H)\) with the properties

\[
g(x, y) = \varepsilon_\alpha g(J_\alpha x, J_\alpha y), \quad \varepsilon_\alpha = \begin{cases} 
1, & \alpha = 1; \\
-1, & \alpha = 2, 3.
\end{cases}
\]

In other words, for \(\alpha = 1\), the metric \(g\) is Hermitian with respect to \(J_1\), whereas in the cases \(\alpha = 2\) and \(\alpha = 3\) the metric \(g\) is an anti-Hermitian (i.e. Norden)
metric with respect to $J_2$ and $J_3$, respectively. Moreover, the associated bilinear forms $g_1$, $g_2$, $g_3$ are determined by

$$g_\alpha(x, y) = g(J_\alpha x, y) = -\varepsilon_\alpha g(x, J_\alpha y), \quad \alpha = 1, 2, 3.$$  

Then, we call a manifold with such a structure briefly an almost $(H, G)$-manifold $[12, 13]$.

The structural tensors of an almost $(H, G)$-manifold are the three $(0,3)$-tensors determined by

$$F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3,$$

where $\nabla$ is the Levi-Civita connection generated by $g$.

In the classification of Gray-Hervella [11] for almost Hermitian manifolds the class $G_1 = W_1 \oplus W_3 \oplus W_4$ is determined by the condition $F_1(x, x, z) = F_1(J_1 x, J_1 x, z)$.

**Theorem 15 ([15]).** If $M$ is an almost $(H, G)$-manifold which is a quasi-Kähler manifold with Norden metric regarding $J_2$ and $J_3$, then it belongs to the class $G_1$ with respect to $J_1$.

### 3.1. pHKT-connection

**Definition 8 ([15]).** A linear connection $D$ preserving the almost hypercomplex structure $H$ and the metric $g$, i.e. $DJ_1 = DJ_2 = DJ_3 = Dg = 0$, is called a natural connection on $(M, H, G)$.

**Definition 9 ([15]).** A natural connection $D$ on an almost $(H, G)$-manifold is called a pseudo-HKT-connection (briefly, a pHKT-connection) if its torsion $T$ is totally skew-symmetric, i.e. a 3-form.

For an almost complex manifold with Hermitian metric $(M, J, g)$, in [2] it is proved that there exists a unique KT-connection if and only if the Nijenhuis tensor $N_J(x, y, z) := g(N_J(x, y), z)$ is a 3-form, i.e. the manifold belongs to the class of cocalibrated structures $G_1$.

### 3.2. The class $W_{133}$

Next, we restrict the class $G_1(J_1)$ to its subclass $W_1(J_1)$ of nearly Kähler manifolds with neutral metric regarding $J_1$ defined by $F_1(x, y, z) = -F_1(y, x, z)$.

In this case $(M, H, G)$ belongs to the class $W_{133} = W_1(J_1) \cap W_3(J_2) \cap W_3(J_3)$ and dim $M \geq 8$.

We have proved the following

**Theorem 16 ([15]).** The curvature tensor $R$ of $\nabla$ on $(M, H, G) \in W_{133}$ has the following property with respect to the almost hypercomplex structure $H$:

$$R(x, y, z, w) + \sum_{\alpha=1}^3 R(x, y, J_\alpha z, J_\alpha w) = \sum_{\alpha=1}^3 \{A_\alpha(x, z, y, w) - A_\alpha(y, z, x, w)\},$$

where $A_\alpha(x, y, z, w) = g((\nabla_x J_\alpha) y, (\nabla_z J_\alpha) w), \alpha = 1, 2, 3$. 

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3.3. The pHKT-connection on a $W_{133}$-manifold

KT-connections on nearly Kähler manifolds are investigated for instance in [20]. The unique KT-connection $D^1$ for the nearly Kähler manifold $(M, J_1, g)$ on the considered almost $(H, G)$-manifold has the form

$$g(D^1_x y, z) = g(\nabla_x y, z) + \frac{1}{2} F_1(x, y, J_1 z).$$

Moreover, there exists a unique KT-connection $D^\alpha$ ($\alpha = 2, 3$) for the quasi-Kähler manifold with Norden metric $(M, J_\alpha, g)$ on the considered almost $(H, G)$-manifold such that

$$g(D^\alpha_x y, z) = g(\nabla_x y, z) - \frac{1}{4} \mathcal{S}_{x,y,z} F_\alpha(x, y, J_\alpha z).$$

In [15] we have constructed a connection $D$, using the KT-connections $D^1$, $D^2$, and $D^3$, on an almost $(H, G)$-manifold from the class $W_{133}$ and we have proved the following

Theorem 17 ([15]). The connection $D$ defined by

$$g(D_x y, z) = g(\nabla_x y, z) + \frac{1}{2} F_1(x, y, J_1 z).$$

is the unique pHKT-connection on an almost $(H, G)$-manifold from the class $W_{133}$.

Let us remark that the pHKT-connection $D$ on an almost $(H, G)$-manifold coincides with the known KT-connection $D^1$ on the corresponding nearly Kähler manifold. Then the torsion of the pHKT-connection $D$ is parallel and henceforth $T$ is coclosed, i.e. $\delta T = 0$ [3]. Moreover, the curvature tensors $K$ of $D$ and $R$ of $\nabla$ has the following relation [10]

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} A_1(x, y, z, w) + \frac{1}{4} \mathcal{S}_{x,y,z} A_1(x, y, z, w).$$

We have proved the following

Theorem 18 ([15]). Let $(M, H, G)$ be an almost $(H, G)$-manifold from $W_{133}$ and $D$ be the pHKT-connection. Then the following characteristics of this connection are equivalent:

(i) $D$ is strong ($dT = 0$);

(ii) $D$ has a $\nabla$-parallel torsion;

(iii) $D$ is flat.

Theorem 19 ([15]). Let $(M, H, G)$ be an almost $(H, G)$-manifold from $W_{133}$ and $D$ be the pHKT-connection. If $D$ is flat or strong then $(M, H, G)$ is $\nabla$-flat, isotropic-hyper-Kählerian (i.e. $\|\nabla J_\alpha\|^2 = 0$, $\alpha = 1, 2, 3$) and the torsion of $D$ is isotropic (i.e. $\|T\|^2 = 0$).
References

[1] J.-M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (4) (1989), 681–699.

[2] T. Friedrich, S. Ivanov, Vanishing theorems and string backgrounds, Class. Quantum Gravity 18 (2001), 1089–1110.

[3] T. Friedrich, S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math. 6 (2002), 303–336.

[4] G. Ganchev, A. Borisov, Note on the almost complex manifolds with a Norden metric, Compt. rend. Acad. bulg. Sci. 39 (1986), 31–34.

[5] G. Ganchev, K. Gribachev, V. Mihova, B-connections and their conformal invariants on conformally Kähler manifolds with B-metric, Publ. Inst. Math. (Beograd) (N.S.) 42 (1987), 107–121.

[6] G. Ganchev, V. Mihova, Canonical connection and the canonical conformal group on an almost complex manifold with B-metric, Ann. Univ. Sofia Fac. Math. Inform. 81 (1987), 195–206.

[7] G. Ganchev, V. Mihova, K. Gribachev, Almost contact manifolds with B-metric, Math. Balk. 7 (3-4) (1993), 261–276.

[8] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 11 (8) (1997), 257–289.

[9] G. Grantcharov, Y. Poon, Geometry of hyper-Kähler connections with torsion, Commun. Math. Phys. 213 (2000), 19–37.

[10] A. Gray, The structure of nearly Kähler manifolds, Math. Ann. 223 (1976), 233–248.

[11] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. CXXIII (IV) (1980), 35–58.

[12] K. Gribachev, M. Manev, Almost hypercomplex pseudo-Hermitian manifolds and a 4-dimensional Lie group with such structure. J. Geom. 88 (1-2) (2008), 41–52. [arXiv:0711.2798]

[13] K. Gribachev, M. Manev, S. Dimiev, On the almost hypercomplex pseudo-Hermitian manifolds, in: S. Dimiev and K. Sekigawa, (Eds.), Trends of Complex Analysis, Differential Geometry and Mathematical Physics, World Sci. Publ., Singapore, 2003, pp. 51–62. [arXiv:0809.0784]

[14] M. Manev, A connection with totally skew-symmetric torsion on almost contact manifolds with B-metric, Ann. Glob. Anal. Geom. (to appear), arXiv:1001.3800.
[15] M. Manev, K. Gribachev, A connection with parallel totally skew-symmetric torsion on a class of almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics, Int. J. Geom. Methods Mod. Phys. 8 (1) (2011) (to appear), arXiv:1003.2051.

[16] D. Mekerov, A connection with skew symmetric torsion and Kähler curvature tensor on quasi-Kähler manifolds with Norden metric, Compt. rend. Acad. bulg. Sci. 61 (2008), 1249–1256.

[17] D. Mekerov, Connection with parallel totally skew-symmetric torsion on almost complex manifolds with Norden metric, Compt. rend. Acad. bulg. Sci. 62 (12) (2009), 1501–1508.

[18] D. Mekerov, On the geometry of the connection with totally skew-symmetric torsion on almost complex manifolds with Norden metric, Compt. rend. Acad. bulg. Sci. 63 (1) (2010), 19–28.

[19] D. Mekerov, M. Manev, On the geometry of quasi-Kähler manifolds with Norden metric, Nihonkai Math. J. 16 (2) (2005), 89–93.

[20] P.-A. Nagy, Connexions with totally skew-symmetric torsion and nearly Kähler geometry, arXiv:0709.1231.

[21] A. Strominger, Superstrings with torsion, Nucl. Phys. B, 274 (2) (1986), 253–284.

[22] M. Teofilova, On the geometry of almost complex manifolds with Norden metric, Ph.D. Thesis, Plovdiv University, 2009.

[23] K. Yano, Differential geometry on complex and almost complex spaces, Pure and Applied Math. vol. 49, Pergamon Press Book, New York, 1965.

[24] K. Yano, M. Kon, Structures on manifolds, Word Scientific (1976), 601–612.