A TRACING OF THE FRACTIONAL TEMPERATURE FIELD

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Abstract. This note is devoted to a study of $L^q$-tracing of the fractional temperature field $u(t, x)$ the weak solution of the fractional heat equation $(\partial_t + (-\Delta_x)^\alpha)u(t, x) = g(t, x)$ in $L^p(\mathbb{R}_+^{1+n})$ subject to the initial temperature $u(0, x) = f(x)$ in $L^p(\mathbb{R}^n)$.

1. Introduction

Directly continuing from [7, 12], we consider the fractional heat equation in the upper-half Euclidean space $\mathbb{R}_+^{1+n} = \mathbb{R}_+ \times \mathbb{R}^n$ with $\mathbb{R}_+ = (0, \infty)$ and $n \geq 1$:

\begin{equation}
\begin{aligned}
& \left\{ \begin{array}{l}
(\partial_t + (-\Delta_x)^\alpha)u(t, x) = g(t, x) \quad \forall \, (t, x) \in \mathbb{R}_+^{1+n}; \\
u(0, x) = f(x) \quad \forall \, x \in \mathbb{R}^n,
\end{array} \right.
\end{aligned}
\end{equation}

where $(-\Delta_x)^\alpha$ denotes the fractional $(0 < \alpha < 1)$ power of the spatial Laplacian that is determined by

$$(-\Delta_x)^\alpha u(\cdot, x) = \mathcal{F}^{-1}(||\xi||^{2\alpha} \mathcal{F} u(\cdot, \xi))(x) \quad \forall \, x \in \mathbb{R}^n$$

for which $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. Specifically, we are interested in the trace of such a fractional temperature field (existing as the weak solution of (1.1))

$$u(t, x) = R_\alpha f(t, x) + S_\alpha g(t, x)$$

with

$$\begin{aligned}
R_\alpha f(t, x) &= e^{-t(-\Delta_x)^\alpha} f(x) = \int_{\mathbb{R}^n} K_t^{(\alpha)}(x - y) f(y) dy; \\
S_\alpha g(t, x) &= \int_0^t e^{-(t-s)(-\Delta_x)^\alpha} g(s, x) ds = \int_{\mathbb{R}^n} \left( \int_0^t K_t^{(\alpha)}(x - y) g(s, y) ds \right) dy,
\end{aligned}$$

where $K_t^{(\alpha)}(x)$ is the fractional heat kernel

$$K_t^{(\alpha)}(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot y - |y|^{2\alpha}} dy \quad \forall \, (t, x) \in \mathbb{R}_+^{1+n}$$

whose endpoint $\alpha = 1$ and middle-point $\alpha = 1/2$ lead to the heat kernel and Poisson kernel:

$$K_t^{(1)}(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \quad \& \quad K_t^{(\frac{1}{2})}(x) = \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right) \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}$$

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with \( \Gamma(\cdot) \) being the classical gamma function. Although there is no explicit formula for \( K^{(\alpha)}_t(x) \) under \( \alpha \in (0, 1) \setminus \{1/2\} \) (cf. [3, 10, 13, 14, 15] [8, 18, 19, 17, 22]), the following estimates are not only valid but also practical (cf. [3, 4, 5, 9, 20]):

\[
\begin{cases}
K^{(\alpha)}_t(x) \approx \min\{t^{-\frac{2}{\alpha}}, t|x|^{-(n+2\alpha)}\} \approx \frac{t}{(t^{\frac{2}{\alpha}} + |x|)^{n+2\alpha}} \quad \forall \ (t, x) \in \mathbb{R}^{1+n} _{+}; \\
\int_{\mathbb{R}^n} K^{(\alpha)}_t(x)dx = 1 \quad \forall \ t \in (0, \infty).
\end{cases}
\]

As explored in [7, 12], the regularity of \( u(t, x) \) sheds some light on the traces/restrictions of \( R_\alpha f(t, x) \) and \( S_\alpha g(t, x) \) to subsets of \( \mathbb{R}^{1+n}_{+} \) of \((1+n)\)-dimensional Lebesgue measure zero. Here \( f(x) \) and \( g(t, x) \) are arbitrary functions of the usual Lebesgue classes \( L^p(\mathbb{R}^n) \) and \( L^p(\mathbb{R}^{1+n}_{+}) \), respectively. In order to characterize the traces of \( R_\alpha f(t, x) \) and \( S_\alpha g(t, x) \) on a given compact exceptional set \( K \subset \mathbb{R}^{1+n}_{+} \), we investigate nonnegative Radon measures supported on \( K \) such that under \( 1 < p, q < \infty \) the mapping \( R_\alpha : L^p(\mathbb{R}^n) \mapsto L^q_{\mu}(\mathbb{R}^{1+n}_{+}) \) and \( S_\alpha : L^p(\mathbb{R}^{1+n}_{+}) \mapsto L^q_{\mu}(\mathbb{R}^{1+n}_{+}) \) are continuous - namely -

\[
\begin{align*}
(f) & : & (\int_{\mathbb{R}^{1+n}_{+}} |R_\alpha f(t, x)|^q d\mu(t, x))^{\frac{1}{q}} & \lesssim \|f\|_{L^p(\mathbb{R}^n)}; \\
(S) & : & (\int_{\mathbb{R}^{1+n}_{+}} |S_\alpha g(t, x)|^q d\mu(t, x))^{\frac{1}{q}} & \lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_{+})},
\end{align*}
\]

where the symbol \( A \lesssim B \) means \( A \leq cB \) for a positive constant \( c \) - moreover - \( A \approx B \) stands for both \( A \lesssim B \) and \( B \lesssim A \).

A careful examination of (1.2) and (1.3) indicates that they can be naturally unified as:

\[
\begin{align*}
(T) & : & (\int_{\mathbb{R}^{1+n}_{+}} |T_\alpha h|^q d\mu) & \lesssim \|h\|_{L^p(\mathbb{R}^n)} = \begin{cases} \|f\|_{L^p(\mathbb{R}^n)} & \mbox{as } (T_\alpha, h, \mathbb{X}) = (R_\alpha, f, \mathbb{X}); \\
\|g\|_{L^p(\mathbb{R}^{1+n}_{+})} & \mbox{as } (T_\alpha, h, \mathbb{X}) = (S_\alpha, g, \mathbb{R}^{1+n}_{+}).
\end{cases}
\end{align*}
\]

Describing such a measure \( \mu \) on \( \mathbb{R}^{1+n}_{+} \) depends on a concept of the induced capacity. For a compact set \( K \subset \mathbb{R}^{1+n}_{+} \) let

\[
C_p^{(T_\alpha)}(K) = \inf\{\|h\|^p_{L^p(\mathbb{X})} : h \geq 0 & \ T_\alpha h \geq 1_K\},
\]

where \( 1_K \) is the characteristic function of \( K \). Then, for an open subset \( O \) of \( \mathbb{R}^{1+n}_{+} \) let

\[
C_p^{(T_\alpha)}(O) = \sup\{C_p^{(T_\alpha)}(K) : \mbox{compact } K \subset O\},
\]

and hence for any set \( E \subset \mathbb{R}^{1+n}_{+} \) let

\[
C_p^{(T_\alpha)}(E) = \inf\{C_p^{(T_\alpha)}(O) : \mbox{open } O \supset E\}.
\]

According to [7, 12], if

\[
B^{(\alpha)}(t_0, x_0) \equiv \{(t, x) \in \mathbb{R}^{1+n}_{+} : t^{2\alpha} < t - t_0 < 2r^{2\alpha} & \ |x - x_0| < r\}
\]
stands for the parabolic ball with centre \((t_0, x_0) \in \mathbb{R}^{1+n}_+\) and radius \(r > 0\), then

\[
C_p^{(T_\alpha)}(B_r^{(\alpha)}(t_0, x_0)) \approx \begin{cases} r^n & \text{as } T_\alpha = R_\alpha; \\ r^{n+2\alpha(1-p)} & \text{as } T_\alpha = S_\alpha & 1 < p < 1 + \frac{n}{2\alpha}. \end{cases}
\]

Below is a tracing principle for the fractional heat equation (1.1).

**Theorem 1.1.** Let \(0 < \alpha < 1\) and \(1 < p < 1 + \frac{n}{2\alpha}\). Then

\[
\begin{cases}
\sup \left\{ \frac{\mu(B_r^{(\alpha)}(t_0, x_0))}{C_p^{(T_\alpha)}(B_r^{(\alpha)}(t_0, x_0))} : (r, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \right\} < \infty \text{ as } p < q; \\
\sup \left\{ \frac{\mu(K)}{C_p^{(T_\alpha)}(K)} : \text{compact } K \subset \mathbb{R}_+^{1+n} \right\} < \infty \text{ as } p = q;
\end{cases}
\]

\[
\int_{\mathbb{R}_+^{1+n}} \left( \sup_{t < \infty} \left( \frac{\mu(B_r^{(\alpha)}(t_0, x_0))}{C_p^{(T_\alpha)}(B_r^{(\alpha)}(t_0, x_0))} \right)^\frac{1}{q(p-1)} \right)^{\frac{q(p-1)}{p(q-1)}} \mu(t_0, x_0) < \infty \text{ as } p > q.
\]

Here, it should be noted that \(R_\alpha\)-case of Theorem 1.1 under \(p \leq q\) has been treated in [7] Theorems 3.2-3.3]. Of course, the remaining cases of Theorem 1.1 are new. Perhaps, it is worth to point out that under \(p = q\),

\[
\sup \left\{ \frac{\mu(K)}{C_p^{(T_\alpha)}(K)} : \text{compact } K \subset \mathbb{R}_+^{1+n} \right\} < \infty
\]

implies

\[
\sup \left\{ \frac{\mu(B_r^{(\alpha)}(t_0, x_0))}{C_p^{(T_\alpha)}(B_r^{(\alpha)}(t_0, x_0))} : B_r^{(\alpha)}(t_0, x_0) \subset \mathbb{R}_+^{1+n} \right\} < \infty
\]

but not conversely in general - [1] Theorem 4(ii)] and its argument might be helpful to produce a ball-based sufficient condition for (1.4) to hold. Upon \(d\mu = dt dx\) in \(S_\alpha\)-case of Theorem 1.1 we have \(\mu(B_r^{(\alpha)}(t_0, x_0)) \approx r^{n+2\alpha}\), thereby finding that (cf. [21] Theorem 1.4)) for \(g \in L^p(\mathbb{R}_+^{1+n})\) one has

\[
\| \mathcal{S}_\alpha g \|_{L^{\tilde{q}}(\mathbb{R}_+^{1+n})} \lesssim \| g \|_{L^p(\mathbb{R}_+^{1+n})} \text{ where } \tilde{q} = p \left( 1 + \frac{2\alpha p}{n + 2\alpha - 2\alpha p} \right) > p.
\]

Although \(R_\alpha\) and \(S_\alpha\) behave similarly, the argument for Theorem 1.1 will be still split into two parts - one for \(R_\alpha\) in Section 2 and another one for \(S_\alpha\) in Section 3 - this is because the subtle difference between \(R_\alpha\) and \(S_\alpha\) can be seen clearly from such a splitting arrangement.

2. \(R_\alpha\)'s Tracing

In this section we verify Theorem 1.1 for \(T_\alpha = R_\alpha\). To do so, we need three lemmas as seen below.

The first is about the dual representation of \(C_p^{(R_\alpha)}(K)\) for a given compact set \(K \subset \mathbb{R}_+^{1+n}\).

**Lemma 2.1.** Let \(\mathcal{U}^+(K)\) be the class of all nonnegative Radon measures \(\mu\) with compact support \(K \subset \mathbb{R}_+^{1+n}\) and the total variation \(\|\mu\|\). Then

\[
C_p^{(R_\alpha)}(K) = \sup \left\{ \|\mu\|_p : \mu \in \mathcal{U}^+(K) & \int_{\mathbb{R}_+^{1+n}} \left( \int_{\mathbb{R}_+^{1+n}} K_t^{\alpha}(x - y) \, d\mu(t, y) \right)^{\frac{p}{p-1}} \, dx \leq 1 \right\}.
\]
Proof. Note that
\[
\int_{\mathbb{R}_+^{1+n}} R_\alpha f(t,x) h(t,x) dt dx = \int_{\mathbb{R}^n} f(x) \left( \int_{\mathbb{R}_+^{1+n}} K_t^{(\alpha)}(x-y) h(t,y) dy \right) dx
\]
holds for all \((f,h) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}_+^{1+n})\) where \(C_0^\infty(\mathbb{R})\) stands for the class of infinitely differentiable functions with compact support in \(\mathbb{R} = \mathbb{R}^n \) or \(\mathbb{R}_+^{1+n}\). Thus, the adjoint operator of \(R_\alpha\) is defined by
\[
(R_\alpha^* h)(x) = \int_{\mathbb{R}_+^{1+n}} K_t^{(\alpha)}(x-y) h(t,y) dt dy \quad \forall \ h \in C_0^\infty(\mathbb{R}_+^{1+n}).
\]
For any nonnegative Radon measure \(\mu\) in \(\mathbb{R}_+^{1+n}\) and a continuous function \(f\) with a compact support in \(\mathbb{R}^n\), one has
\[
\left| \int_{\mathbb{R}_+^{1+n}} R_\alpha f d\mu \right| \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \|\mu\|.
\]
Therefore, the Riesz representation theorem yields a Borel measure \(\nu\) on \(\mathbb{R}^n\) such that
\[
\int_{\mathbb{R}_+^{1+n}} R_\alpha f d\mu = \int_{\mathbb{R}^n} f d\nu \quad \forall \ f \geq 0.
\]
This means that \(\nu = R_\alpha^* \mu\) can be defined by
\[
R_\alpha^* \mu(x) = \int_{\mathbb{R}_+^{1+n}} K_t^{(\alpha)}(x-y) d\mu(t,y).
\]
According to [7, Proposition 1], one gets
\[
C_p^{(R_\alpha)}(K) = \sup \left\{ \|\mu\|^p : \mu \in \mathcal{U}^+(K) \& \|R_\alpha^* \mu\|_{L^p(\mathbb{R}^n)} \leq 1 \right\}.
\]
□

The second is about \(L^p\)-boundedness of the fractional maximal operator of parabolic type.

**Lemma 2.2.** For a nonnegative Radon measure \(\mu\) on \(\mathbb{R}_+^{1+n}\) let
\[
M_\alpha \mu(x) = \sup_{r>0} r^{-\alpha} \mu(B_r^{(\alpha)}(r^{2\alpha}, x))
\]
be its fractional parabolic maximal function. Then
\[
\|M_\alpha \mu\|_{L^p(\mathbb{R}^n)} \approx \|R_\alpha^* \mu\|_{L^p(\mathbb{R}^n)} \quad \forall \ p \in (1, \infty).
\]

**Proof.** A straightforward estimation with \(x \in \mathbb{R}^n\) and \(R_\alpha^* \mu(x)\) gives
\[
R_\alpha^* \mu(x) \gtrsim \int_{B_t^{(\alpha)}(r^{2\alpha}, x)} \frac{t}{(t^{2\alpha} + |x-y|^{n+2\alpha})} d\mu(t,y) \gtrsim \frac{\mu(B_t^{(\alpha)}(r^{2\alpha}, x))}{r^n} \quad \forall \ r > 0,
\]
whence
\[
R_\alpha^* \mu(x) \gtrsim M_\alpha \mu(x).
\]
This implies
\[
\|M_\alpha \mu\|_{L^p(\mathbb{R}^n)} \lesssim \|R_\alpha^* \mu\|_{L^p(\mathbb{R}^n)}.
\]
To prove the converse inequality, we slightly modify \cite{2} (3.6.1) to get two constants \(a > 1\) and \(b > 0\) such that for any \(\lambda > 0\) and \(0 < \varepsilon \leq 1\), one has the following good-\(\lambda\) inequality
\[
|\{x \in \mathbb{R}^n : R^*_\lambda(x) > a\lambda\}| \leq b \varepsilon^\frac{n+2a}{n} |\{x \in \mathbb{R}^n : R^*_\lambda(x) > \lambda\}|
\]
+ |\{x \in \mathbb{R}^n : M_\lambda(x) > \varepsilon\lambda\}|
\) (2.1)

Inspired by \cite{2} Theorem 3.6.1, we proceed the proof by using (2.1). Multiplying (2.1) by \(\lambda^{p-1}\) and integrating in \(\lambda\), we have for any \(\gamma > 0\),
\[
\int_0^\gamma |\{x \in \mathbb{R}^n : R^*_\lambda(x) > a\lambda\}|\lambda^{p-1}d\lambda \leq b \varepsilon^\frac{n+2a}{n} \int_0^\gamma |\{x \in \mathbb{R}^n : R^*_\lambda(x) > \lambda\}|\lambda^{p-1}d\lambda
\]
+ \(\int_0^\gamma |\{x \in \mathbb{R}^n : M_\lambda(x) > \varepsilon\lambda\}|\lambda^{p-1}d\lambda\).

An equivalent formulation of the above inequality is
\[
a^{-p} \int_0^{a\gamma} |\{x \in \mathbb{R}^n : R^*_\lambda(x) > a\lambda\}|\lambda^{p-1}d\lambda \leq b \varepsilon^\frac{n+2a}{n} \int_0^\gamma |\{x \in \mathbb{R}^n : R^*_\lambda(x) > \lambda\}|\lambda^{p-1}d\lambda
\]
+ \(\varepsilon^{-p} \int_0^{\varepsilon\gamma} |\{x \in \mathbb{R}^n : M_\lambda(x) > \varepsilon\lambda\}|\lambda^{p-1}d\lambda\).

Let \(\varepsilon\) be so small that \(b \varepsilon^\frac{n+2a}{n} \leq \frac{1}{2}a^{-p}\) and \(\gamma \to \infty\). Then
\[
a^{-p} \int_{\mathbb{R}^n} (R^*_\lambda(x))^p dx \leq 2\varepsilon^{-p} \int_{\mathbb{R}^n} (M_\lambda(x))^p dx.
\]

That is
\[
\|M_\lambda\|_{L^p(\mathbb{R}^n)} \gtrsim \|R^*_\lambda\|_{L^p(\mathbb{R}^n)}.
\]

The third is about the Hedberg-Wolff potential for \(R_\alpha\):
\[
P^R_{\alpha p}\mu(t, x) = \int_0^\infty \left( \frac{\mu(B_{(t, x)}(\alpha, r))}{r^n} \right) \frac{p-1}{p} \frac{dr}{r} \quad \forall (t, x) \in \mathbb{R}^{1+n}_+.
\]

Lemma 2.3. Let \(1 < p < \infty\), \(p' = \frac{p}{p-1}\), and \(\mu\) be a nonnegative Radon measure on \(\mathbb{R}^{1+n}_+\). Then
\[
\|R^*_\lambda\|_{L^{p'}(\mathbb{R}^n)} \approx \int_{\mathbb{R}^{1+n}_+} P^R_{\alpha p}\mu d\mu.
\]

Proof. Below is a two-fold argument.

Part 1. The first task is to show
\[
\|R^*_\lambda\|_{L^{p'}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^{1+n}_+} P^R_{\alpha p}\mu d\mu.
\]
Note first that
\[
\frac{\mu(B_{(t, x)}(\alpha, r^{2\alpha}, x))}{r^n} \approx \left( \int_r^{2r} \left( \frac{\mu(B_{(s, x)}^{(\alpha)}(\alpha, r^{2\alpha})^{\alpha}, x)}{s^n} \right) \frac{p'}{p} ds \right) \lesssim \left( \int_0^\infty \left( \frac{\mu(B_{(s, x)}^{(\alpha)}(\alpha, r^{2\alpha})^{\alpha}, x)}{s^n} \right) \frac{p'}{p} ds \right) \frac{1}{p'}.
\]

□
Therefore, one has

\[ M_\alpha \mu(x) \lesssim \left( \int_0^\infty \left( \frac{\mu(B_s^{(\alpha)}(s^{2\alpha}, x))}{s^n} \right)^{\frac{p'}{p}} \frac{ds}{s} \right)^{\frac{1}{p'}}. \]

By Lemma 2.2, it is sufficient to verify

\[ \int_{\mathbb{R}^n} \int_0^\infty \left( \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, x))}{r^n} \right)^{\frac{p'}{p}} \frac{dr}{r} \, dx \lesssim \int_{\mathbb{R}^{n+1}} \int_0^\infty \left( \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, x))}{r^n} \right)^{\frac{p'}{p}} \frac{dr}{r} \, d\mu. \]

Using the Fubini theorem, one has

\[ \int_{\mathbb{R}^n} \int_0^\infty \left( \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, x))}{r^n} \right)^{\frac{p'}{p}} \frac{dr}{r} \, dx = \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, x))}{r^{np'}} \right)^{\frac{p'}{p}} \frac{dx}{r} \, dr. \]

A further application of Fubini’s theorem yields

\[ \int_{\mathbb{R}^n} \mu(B_r^{(\alpha)}(r^{2\alpha}, x))^{\frac{p'}{p}} \, dx \lesssim \int_{B_r^{(\alpha)}(r^{2\alpha}, x)} \mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dx \, d\mu \]
\[ \lesssim \int_{B_r^{(\alpha)}(r^{2\alpha}, y)} \mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dy \, d\mu \]
\[ \lesssim r^n \int_{B_r^{(\alpha)}(r^{2\alpha}, y)} \mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dy. \]

Therefore,

\[ \int_0^\infty \int_{\mathbb{R}^n} \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, x))^{\frac{p'}{p}} \, dx \, dr}{r} \approx \int_0^\infty \int_{B_r^{(\alpha)}(r^{2\alpha}, x)} \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dy \, d\mu}{r} \]
\[ \approx \int_{B_r^{(\alpha)}(r^{2\alpha}, y)} \int_0^\infty \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dy \, d\mu}{r} \]
\[ \approx \int_{\mathbb{R}^{n+1}} \int_0^\infty \frac{\mu(B_r^{(\alpha)}(r^{2\alpha}, y))^{\frac{p'}{p}} \, dy \, d\mu}{r} \, d\mu(t, x), \]

as desired.

**Part 2.** The second task is to prove

\[ \| R_\alpha^* \mu \|_{L^{p'}(\mathbb{R}^n)} \gtrsim \int_{\mathbb{R}^{n+1}} P_{\alpha, \mu}^R \, d\mu. \]

Since

\[ \| R_\alpha^* \mu \|_{L^{p'}(\mathbb{R}^n, dx)} = \int_{\mathbb{R}^n} (R_\alpha^* \mu(x))^{p'-1} (R_\alpha^* \mu(x)) \, dx \]
\[ = \int_{\mathbb{R}^{n+1}} (R_\alpha^* \mu(x))^{p'-1} M_\alpha^t(x-y) \, d\mu(t, y), \]

Upon writing

\[ K(t, x) = \int_{\mathbb{R}^n} (R_\alpha^* \mu(x))^{p'-1} M_\alpha^t(x-y) \, dx \]
we obtain a fractional heat potential analogue of the Riesz potential treatment carried in [6, Theorem 2.1].

Proof. Since (1), (2) and the left equivalence of (3) are contained in [7, Theorems 3.2&3.3] whose proofs depend on Lemma 2.1, it is enough to check the right equivalence of (3). Our approach is

Step 1. We show

$$\sup_{\lambda > 0} \frac{\lambda^2}{C_R(\mu; \lambda)} < \infty \Leftrightarrow \sup_{(t_0, x_0) \in \mathbb{R}_+^n} \frac{\mu(B_{t_0}^{(\alpha)}(t_0, x_0))}{r^m} < \infty.$$
To do so, we first denote by $Q_l^{(\alpha)}$ the $\alpha$-dyadic cube with side length $l \equiv l(Q_l^{(\alpha)})$ and corners in the set \{\$2^\alpha \mathbb{Z}^+, l^2 \mathbb{Z}^n\} with $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ - namely -

$Q_l^{(\alpha)} \equiv \{[k_0 l^{2\alpha}, (k_0 + 1) l^{2\alpha}) \times [k_1 l, (k_1 + 1) l) \times \cdots \times [k_n l, (k_n + 1) l)\}$ as $k_0 \in \mathbb{Z}_+$ & $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$. Next, we introduce the following fractional heat Hedberg-Wolff potential generated by $\mathcal{D}^\alpha$ - the family of all the above-defined $\alpha$-dyadic cubes in $\mathbb{R}^{1+n}$:

$$P^{\alpha}_R \mu(t, x) = \sum_{Q_l^{(\alpha)} \in \mathcal{D}^\alpha} \left( \frac{\mu(Q_l^{(\alpha)})}{l^n} \right)^{p' - 1} 1_{Q_l^{(\alpha)}}(t, x)$$

and then prove

$$\int_{\mathbb{R}^{1+n}_+} (P^{\alpha}_R \mu(t, x))^{\frac{q(p-1)}{q(n)}} d\mu(t, x) < \infty. \quad (2.2)$$

Indeed, by duality, (1.2) is equivalent to the following inequality

$$\|R^*_\alpha(g d\mu)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}_\alpha(\mathbb{R}^{1+n}_+)} \quad \forall \ g \in L^{q'_\alpha} = \frac{2}{q'-1}(\mathbb{R}^{1+n}_+) .$$

It is easy to check that Lemma 2.3 is also true with $P^{\alpha}_R \mu$ in place of $P^{\alpha}_R \mu$ and $g d\mu$ in place of $d\mu$. So, one has

$$\|R^*_\alpha(g d\mu)\|_{L^{p'}(\mathbb{R}^n)} \gtrsim \int_{\mathbb{R}^{1+n}_+} P^{\alpha}_R(g d\mu)(t, x)g(t, x)d\mu(t, x) \gtrsim \sum_{Q_l^{(\alpha)}} \left( \frac{\int_{Q_l^{(\alpha)}} g(t, x)d\mu(t, x)}{l^n} \right)^{p'} l^n .$$

Consequently,

$$\sum_{Q_l^{(\alpha)}} \left( \frac{\int_{Q_l^{(\alpha)}} g(t, x)d\mu(t, x)}{l^n} \right)^{p'} l^n \lesssim \|g\|_{L^{p'}_{\alpha}(\mathbb{R}^{1+n}_+)} . \quad (2.3)$$

Upon setting

$$\lambda_{Q_l^{(\alpha)}} = \left( \frac{\mu(Q_l^{(\alpha)})}{l^n} \right)^{p'},$$

one finds that (2.3) is equivalent to

$$\sum_{Q_l^{(\alpha)}} \lambda_{Q_l^{(\alpha)}} \left( \frac{\int_{Q_l^{(\alpha)}} g d\mu}{\mu(Q_l^{(\alpha)})} \right)^{p'} \lesssim \|g\|_{L^{p'}_{\mu}(\mathbb{R}^{1+n}_+)} .$$

Define the following dyadic Hardy-Littlewood maximal function

$$M^d_{\mu} h(t, x) = \sup_{(t, x) \in Q^{(\alpha)}} \frac{1}{\mu(Q^{(\alpha)})} \int_{Q^{(\alpha)}} |h(s, y)|d\mu(s, y) \quad \forall \ Q^{(\alpha)} \in \mathcal{D}^\alpha .$$

Then $M^d_{\mu}$ is bounded on $L^p_{\mu}(\mathbb{R}^{1+n}_+)$ for $1 < p < \infty$. Write

$$g(t, x) = (M^d_{\mu})^{\frac{1}{p'}} h(t, x) \quad \text{under} \ 0 \leq h \in L^{q'/p'}_{\mu}(\mathbb{R}^{1+n}_+) .$$
It is easy to check that
\[
\left( \frac{\int_{Q_1^{(\alpha)}} g(t,x) d\mu(t,x)}{\mu(Q_1^{(\alpha)})} \right)^{p'} \geq \frac{\int h(t,x) d\mu(t,x)}{\mu(Q_1^{(\alpha)})}
\]
and so that
\[
\|g\|_{L_{\mu}^p(\mathbb{R}^{1+n}_+)} \lesssim \|h\|_{L_{\mu}^{q'/p'}(\mathbb{R}^{1+n}_+)}. \]
This in turn implies
\[
\sum Q_{l_1^{(\alpha)}}^{(\alpha)} \int_{Q_{l_1^{(\alpha)}}^{(\alpha)}} h(t,x) d\mu(t,x) \lesssim \|h\|_{L_{\mu}^{q'/p'}(\mathbb{R}^{1+n}_+)} \forall h \in L_{\mu}^{q'/p'}(\mathbb{R}^{1+n}_+),
\]
and thus via duality
\[
\sum Q_{l_1^{(\alpha)}}^{(\alpha)} \frac{\lambda_{Q_{l_1^{(\alpha)}}^{(\alpha)}}}{\mu(Q_{l_1^{(\alpha)}}^{(\alpha)})} 1_{Q_{l_1^{(\alpha)}}^{(\alpha)}} \in L_{\mu}^{q/p}(\mathbb{R}^{1+n}_+),
\]
amely,
\[
\sum Q_{l_1^{(\alpha)}}^{(\alpha)} \left( \frac{\mu(Q_{l_1^{(\alpha)}}^{(\alpha)})}{l^n} \right)^{p'-1} 1_{Q_{l_1^{(\alpha)}}^{(\alpha)}} \in L_{\mu}^{q/(p-1)}(\mathbb{R}^{1+n}_+),
\]
which yields (2.2).

Next, set
\[
P_{\alpha}^{d,\tau,R} \mu(t,x) = \sum_{Q_{l_1^{(\alpha)}}^{(\alpha)} \in D_\tau} \left( \frac{\mu(Q_{l_1^{(\alpha)}}^{(\alpha)})}{l^n} \right)^{p'-1} 1_{Q_{l_1^{(\alpha)}}^{(\alpha)}}(t,x) \& D_\tau^n = D^n + \tau = \{Q_{l_1^{(\alpha)}}^{(\alpha)} + \tau \} \times 1_{Q_{l_1^{(\alpha)}}^{(\alpha)}},
\]
where \(Q_{l_1^{(\alpha)}}^{(\alpha)} + \tau = \{(t,x) + \tau : (t,x) \in Q_{l_1^{(\alpha)}}^{(\alpha)}\}\) means the \(\mathbb{R}^{1+n}_+ \ni \tau\)-shift of \(Q_{l_1^{(\alpha)}}^{(\alpha)}\). Then (2.2) implies (2.4)
\[
\sup_{\tau \in \mathbb{R}^{1+n}_+} \int_{\mathbb{R}^{1+n}_+} (P_{\alpha}^{d,\tau,R} \mu(t,x))^{q/(p-1)}(\mathbb{R}^{1+n}_+) d\mu(t,x) < \infty.
\]

Now, it remains to prove
\[
P_{\alpha}^{R} \mu \in L_{\mu}^{q/(p-1)}(\mathbb{R}^{1+n}_+).
\]

Two situations are considered in the sequel.

**Case 1.1. \(\mu\) is a doubling measure.** In this case, \(P_{\alpha}^{R} \mu \in L_{\mu}^{q/(p-1)}(\mathbb{R}^{1+n}_+)\) is a by-product of (2.2) and the following observation
\[
P_{\alpha}^{R} \mu(t,x) \lesssim \sum_{Q_{l_1^{(\alpha)}}^{(\alpha)} \times 1_{Q_{l_1^{(\alpha)}}^{(\alpha)}}(t,x),}
\]
where \(Q_{l_1^{(\alpha)}}^{(\alpha)}\) is the cube with the same center as \(Q_{l_1^{(\alpha)}}^{(\alpha)}\) and side length two times as \(Q_{l_1^{(\alpha)}}^{(\alpha)}\).

**Case 1.2. \(\mu\) is a possibly non-doubling measure.** For any \(\rho > 0\), write
\[
P_{\alpha,\rho}^{R} \mu(t,x) = \int_{0}^{\rho} \left( \frac{\mu(B_{r}^{(\alpha)}(t,x))}{r^n} \right)^{p'-1} \frac{dr}{r}.
\]
Then
\[
P^R_{\alpha, \rho} \mu(t, x) \lesssim \rho^{-(n+1)} \int_{|\tau| \leq \rho} P^d,\tau, R_{\alpha, \rho} \mu(t, x) d\tau.
\]
In fact, for a fixed \( x \in \mathbb{R}^n \) and \( \rho > 0 \) with \( 2^{i-1} \eta \leq \rho < 2^i \eta \) (where \( i \in \mathbb{Z} \) and \( \eta > 0 \) will be determined later) one has
\[
P^R_{\alpha, \rho} \mu(t, x) \lesssim \sum_{j=-\infty}^{i} \left( \frac{\mu(B_{2^j \eta}(t, x))}{(2^j \eta)^n} \right)^{p'-1}.
\]
For \( j \leq i \), let \( Q_{i,j}^{(\alpha)} \) be a cube centred at \( x \) with \( 2^{j-1} < l \leq 2^j \). Then \( B_{2^j \eta}(t, x) \subseteq Q_{i,j}^{(\alpha)} \) for sufficiently small \( \eta \). Assume not only that \( E \) is the set of all points \( \tau \in \mathbb{R}_+^{1+n} \) enjoying \( |\tau| \lesssim \rho \) with \( |E| \) being the \((1+n)\)-dimensional Lebesgue measure, but also that there exists \( Q_{i,j}^{(\alpha)} \subseteq D_{\tau}^\alpha \) satisfying \( l = 2^{j+1} \) and \( Q_{i,j}^{(\alpha)} \subseteq Q_{l}^{(\alpha)} \). A geometric consideration produces a dimensional constant \( c(n) > 0 \) such that \( |E| \geq c(n) \rho^{n+1} \forall j \leq i \). Consequently, one has
\[
\mu(B_{2^j \eta}(t, x))^{p'-1} \lesssim |E|^{-1} \int_{E} \sum_{l=2^{j+1}}^{2^{i+1}} \mu(Q_{l}^{(\alpha)})^{p'-1} 1_{Q_{l}^{(\alpha)}}(t, x) d\tau
\]
\[
\lesssim \rho^{-(n+1)} \int_{|\tau| \leq \rho} \sum_{l=2^{j+1}}^{2^{i+1}} \mu(Q_{l}^{(\alpha)})^{p'-1} 1_{Q_{l}^{(\alpha)}}(t, x) d\tau,
\]
and so that
\[
P^R_{\alpha, \rho} \mu(t, x) \lesssim \rho^{-(n+1)} \int_{|\tau| \leq \rho} \sum_{j=-\infty}^{i} \sum_{l=2^{j+1}}^{2^{i+1}} \left( \frac{\mu(Q_{l}^{(\alpha)})}{(2^j \eta)^n} \right)^{p'-1} 1_{Q_{l}^{(\alpha)}}(t, x) d\tau
\]
\[
\lesssim \rho^{-(n+1)} \int_{|\tau| \leq \rho} P^d,\tau, R_{\alpha, \rho} \mu(t, x) d\tau,
\]
whence reaching (2).

From (2), the Hölder inequality and Fubini’s theorem it follows that
\[
\int_{\mathbb{R}_+^{1+n}} \left( P^R_{\alpha, \rho} \mu(t, x) \right)^{\frac{q(p-1)}{p-q}} d\mu(t, x)
\]
\[
\lesssim \int_{\mathbb{R}_+^{1+n}} \rho^{-(n+1)} \left( \int_{|\tau| \leq C \rho} \left( P^d,\tau, R_{\alpha, \rho} \mu \right)^{\frac{q(p-1)}{p-q}} d\tau \right)^{\frac{p-q}{q(p-1)}} \left( \int_{|\tau| \leq \rho} d\tau \right)^{1-\frac{p-q}{p-q}} d\mu(t, x)
\]
\[
\lesssim \rho^{-(n+1)} \int_{|\tau| \leq \rho} \left( \int_{\mathbb{R}_+^{1+n}} \left( P^d,\tau, R_{\alpha, \rho} \mu \right)^{\frac{q(p-1)}{p-q}} d\mu(t, x) \right) d\tau
\]
\[
\lesssim \kappa(n),
\]
where the last constant \( \kappa(n) \) is independent of \( \rho \). This clearly produces
\[
P^R_{\alpha, \rho} \mu \in L^{q(p-1)/(p-q)}(\mathbb{R}_+^{1+n})
\]
via letting \( \rho \to \infty \) and utilizing the monotone convergence theorem.
Step 2. We prove
\[ P^R_{op} \mu \in L^q p-1/p-q (\mathbb{R}^{1+n}) \Rightarrow (1.2). \]

Recall that (1.2) is equivalent to the following inequality
\[ \| R_\alpha^* (g d\mu) \|_{L^p(q^n)} \lesssim \| g \|_{L^p q^n (\mathbb{R}^{1+n})} \quad \forall g \in L^q p (\mathbb{R}^{1+n}). \]

Thus, by Lemma 2.3 it is sufficient to check that \( P^R_{op} \mu \in L^q p-1/p-q (\mathbb{R}^{1+n}) \) implies
\[ \int_{\mathbb{R}^{1+n}} P^R_{op} (g d\mu) (t, x) g(t, x) d\mu \lesssim \| g \|_{L^p q^n (\mathbb{R}^{1+n}, d\mu)} \quad \forall g \in L^q p (\mathbb{R}^{1+n}). \]

There is no loss of generality in assuming \( g \geq 0 \). Since
\[ P^R_{op} (g d\mu) (t, x) \approx \int_0^\infty \left( \frac{\mu(B_{r}^{(\alpha)}(t, x))}{r^n} \right)^{p'-1} \left( \int_{B_{r}^{(\alpha)}(t, x)} g(t, x) d\mu \right)^{p'-1} \frac{dr}{r}, \]
an application of the Hölder inequality gives
\[ \int_{\mathbb{R}^{1+n}} P^R_{op} (g d\mu) (t, x) d\mu (t, x) \lesssim \int_{\mathbb{R}^{1+n}} (M_\mu g(t, x))^{p'-1} P^R_{op} \mu (t, x) g(t, x) d\mu (t, x) \]
\[ \lesssim \left( \int_{\mathbb{R}^{1+n}} (M_\mu g(t, x))^{q'} d\mu (t, x) \right)^{p'-1} \left( \int_{\mathbb{R}^{1+n}} \left( g(t, x) P^R_{op} \mu (t, x) \right)^{q'-1} d\mu (t, x) \right)^{q'/q}. \]

Here
\[ M_\mu g(t, x) = \sup_{r > 0} \frac{1}{\mu(B_{r}^{(\alpha)}(t, x))} \int_{B_{r}^{(\alpha)}(t, x)} g(s, y) d\mu (s, y) \]
is the centered Hardy-Littlewood maximal function of \( g \) with respect to \( \mu \). The fact that \( M_\mu \) is bounded on \( L^q (\mathbb{R}^{1+n}) \) (cf. [11]) and Hölder’s inequality imply
\[ \int_{\mathbb{R}^{1+n}} P^R_{op} (g d\mu) (t, x) d\mu (t, x) \lesssim \| g \|_{L^q p (\mathbb{R}^{1+n})} \left( \int_{\mathbb{R}^{1+n}} \left( P^R_{op} \mu \right)^{q (p-1)/q} d\mu \right)^{p-q}, \]
whence (2.5).

3. \( S_\alpha \)'s tracing

In this section we verify Theorem 1.1 for \( T_\alpha = S_\alpha \) and \( 1 < p < 1 + \frac{n}{2\alpha} \). Like proving Theorem 1.1 for \( T_\alpha = R_\alpha \), three lemmas are required in what follows.

The first is regarding the dual formulation of \( C^*_{p} (S_\alpha) (K) \) of a given compact set \( K \subset \mathbb{R}^{1+n} \).

Lemma 3.1. Let \( \mu \in \mathcal{U}^+ (K) \), \( 1 < p < 1 + \frac{n}{2\alpha} \), \( p' = \frac{p}{p-1} \), \( S_\alpha^* \) be the adjoint operator of \( S_\alpha \), and
\[ P^S_{op} \mu (t, x) = \int_0^\infty \left( \frac{\mu(B_{r}^{(\alpha)}(t, x))}{r^{n+2\alpha(1-p)}} \right)^{p'-1} \frac{dr}{r} \quad \forall \ (t, x) \in \mathbb{R}^{1+n}. \]
Then:
(a) \[ C^p_{\alpha}(S_\alpha)(K) = \sup\{\|\mu\|^p : \mu \in U^+(K) \& \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \leq 1\}. \]

(b) \[ \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \approx \int_{\mathbb{R}^1_+} P_s^{\alpha} \mu(t,x) \, d\mu(t,x). \]

Proof. (a) Since \( S_\alpha^{*} \) is determined by
\[ \int_{\mathbb{R}^1_+} (S_\alpha g) h \, dt \, dx = \int_{\mathbb{R}^1_+} g(t,x) \left( \int_t^\infty e^{-(s-t)(-\Delta)^{\alpha}} h(s,x) \, ds \right) \, dt \, dx \quad \forall \; g, h \in C^0_0(\mathbb{R}^1_+), \]
it follows that for any \( h \in C^0_0(\mathbb{R}^1_+) \) one has
\[ S_\alpha^{*} h(t,x) = \int_t^\infty e^{-(s-t)(-\Delta)^{\alpha}} h(s,x) \, ds = \int_{[t,\infty) \times \mathbb{R}^n} K_{s-t}^{(\alpha)}(x-y) h(s,t) \, ds \, dy. \]
The definition of \( S_\alpha^{*} \) is extended to the family of all Borel measures \( \mu \) with compact support in \( \mathbb{R}^1_+ \):
\[ S_\alpha^{*} \mu(t,x) = \int_{[t,\infty) \times \mathbb{R}^n} K_{s-t}^{(\alpha)}(x-y) \, d\mu(s,y). \]
According to [12 Proposition 2.1], we have
\[ C^p_{\alpha}(S_\alpha)(K) = \sup\{\|\mu\|^p : \mu \in U^+(K) \& \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \leq 1\}. \]

(b) This can be proved via a slight modification of the argument for Lemma [23] - in particular - by replacing the maximal function \( \mu \) by
\[ M_\alpha \mu(t,x) = \sup_{r>0} r^{-n} \int_{B_r^{(\alpha)}(t,x)} d\mu. \]

The second indicates that \( C^p_{\alpha}(S_\alpha)(K) \) of a given compact \( K \subset \mathbb{R}^1_+ \) can be realized by \( \mu_K(K) \) of an element \( \mu_K \in U^+(K) \).

Lemma 3.2. Let \( K \) be a compact subset of \( \mathbb{R}^1_+ \). Then there exists a \( \mu_K \in U^+(K) \) such that
\[ \mu_K(K) = \int_{\mathbb{R}^1_+} (S_\alpha^{*}\mu_K(t,x))' \, dt \, dx = \int_{\mathbb{R}^1_+} S_\alpha^{*}(S_\alpha^{*}\mu_K)^{p'-1} \, d\mu_K = C^p_{\alpha}(S_\alpha)(K). \]

Proof. [Lemma 3.1(a) (plus [12 Proposition 2.1])] ensures the existence of a sequence \( \{\mu_i\} \subset U^+(K) \) such that
\[ \|S_\alpha^{*}\mu_i\|_{L^p'(\mathbb{R}^1_+)} \leq 1 \& \lim_{i \to \infty} \mu_i(K) = (C^p_{\alpha}(S_\alpha)(K))^\frac{1}{p} \]
and \( \mu_i \) has a weak limit \( \mu \in U^+(K) \). Thus \( \mu(K) = (C^p_{\alpha}(S_\alpha)(K))^\frac{1}{p} \). It follows from the lower semi-continuity of \( S_\alpha^{*}\mu \) on \( U^+(K) \) that \( \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \leq 1 \). Meanwhile, the following estimation
\[ \|\mu\| \leq \int_{\mathbb{R}^1_+} S_\alpha g \, d\mu = \int_{\mathbb{R}^1_+} g(t,x)S_\alpha^{*}\mu(t,x) \, dt \, dx \leq \|g\|_{L^p(\mathbb{R}^1_+)} \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \]
gives \( \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} \geq 1 \). So, \( \|S_\alpha^{*}\mu\|_{L^p'(\mathbb{R}^1_+)} = 1 \).
Choosing $\mu_K = C_p^{(S_\alpha)}(K)^{\frac{1}{p'}} \mu$ and using $\mu(K) = (C_p^{(S_\alpha)}(K))^{\frac{1}{p}}$, one has
\[
\mu_K(K) = \int_{\mathbb{R}^{1+n}} (S_\alpha^* \mu_K(t, x))^{p'} dtdx = C_p^{(S_\alpha)}(K).
\]

Suppose that $g_0$ is the capacitary potential of $C_p^{(S_\alpha)}(K)$, i.e.,
\[
\|g_0\|_{L^p(\mathbb{R}^{1+n})}^p = C_p^{(S_\alpha)}(K) & S_\alpha g_0 \geq 1_K.
\]
Then $g_0(t, x) = (S_\alpha^* \mu_K)^{p'-1}(t, x)$. A further use of [12 Proposition 2.1] derives
\[
\mu_K(\{(t, x) \in K : S_\alpha g_0(t, x) < 1\}) = 0,
\]
whence
\[
S_\alpha(g_0) = S_\alpha(S_\alpha^* \mu_K)^{p'-1} \geq 1 \text{ a.e. } \mu_K \text{ on } K.
\]

Now, Fubini’s theorem and the Hölder inequality are utilized to derive
\[
C_p^{(S_\alpha)}(K) \leq \int_{\mathbb{R}^{1+n}} S_\alpha g_0 d\mu_K
\]
\[
= \int_{\mathbb{R}^{1+n}} \int_0^t \int_{\mathbb{R}^n} K_1^{(\alpha)}(x - y) f(s, y) dy ds d\mu_K
\]
\[
= \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} K_1^{(\alpha)}(x - y) d\mu_K f(s, y) dy ds
\]
\[
\leq \int_{\mathbb{R}^{1+n}} S_\alpha^* \mu_K(t, x) g_0(t, x) dtdx
\]
\[
\leq \|S_\alpha^* \mu_K\|_{L^{p'}}(\mathbb{R}^{1+n}) \|g_0\|_{L^p(\mathbb{R}^{1+n})}
\]
\[
= C_p^{(S_\alpha)}(K),
\]
thereby completing the proof. \hfill \square

The third is concerning the weak and strong type estimates for $C_p^{(S_\alpha)}$.

**Lemma 3.3.** Let $1 < p < \infty$ and $L^p_+(\mathbb{R}^{1+n})$ stand for the class of all nonnegative functions in $L^p(\mathbb{R}^{1+n})$. If $g \in L^p_+(\mathbb{R}^{1+n})$ and $\lambda > 0$, then:

(a) $C_p^{(S_\alpha)}(\{(t, x) \in \mathbb{R}^{1+n} : S_\alpha g(t, x) \geq \lambda\}) \leq \lambda^{-p} \|g\|_{L^p(\mathbb{R}^{1+n})}^p$;

(b) $\int_0^\infty C_p^{(S_\alpha)}(\{(t, x) \in \mathbb{R}^{1+n} : S_\alpha g(t, x) \geq \lambda\}) d\lambda^p \lesssim \|g\|_{L^p(\mathbb{R}^{1+n})}^p$.

**Proof.** (a) This follows immediately from the definition of $C_p^{(S_\alpha)}$.

(b) It is enough to check this inequality for any nonnegative function $g \in C_0^\infty(\mathbb{R}^{1+n})$. The forthcoming demonstration is a slight modification of the argument for [7 Lemma 3.1].

For each $i = 0, \pm 1, \pm 2, \ldots$ and any nonnegative function $g \in C_0^\infty(\mathbb{R}^{1+n})$, we follow the proof of [2 Theorem 7.1.1] to write
\[
K_i = \{(t, x) \in \mathbb{R}^{1+n} : S_\alpha g(t, x) \geq 2^i\}.
\]
Assume that \( \mu_i \) is the measure obtained in Lemma 3.2 for \( K_i \). Then by duality and Hölder’s inequality, one has

\[
\sum_{i=\infty}^{\infty} 2^{ip} \mu_i(\mathbb{R}_+^{1+n}) \leq \sum_{i=\infty}^{\infty} 2^{i(p-1)} \int_{\mathbb{R}_+^{1+n}} g(t, x) S^*_\alpha \mu_i(t, x) dt dx \\
\lesssim \|g\|_{L^p(\mathbb{R}_+^{1+n})} \left\| \sum_{i=\infty}^{\infty} 2^{i(p-1)} S^*_\alpha \mu_i \right\|_{L^{p'}(\mathbb{R}_+^{1+n})} \\
\equiv \|g\|_{L^p(\mathbb{R}_+^{1+n})} \|\eta\|_{L^{p'}(\mathbb{R}_+^{1+n})},
\]

where

\[
\eta(t, x) = \sum_{i=\infty}^{\infty} 2^{i(p-1)} S^*_\alpha \mu_i(t, x).
\]

For \( k = 0, \pm 1, \pm 2, \cdots \), let

\[
\eta_k(t, x) = \sum_{i=\infty}^{\infty} 2^{i(p-1)} S^*_\alpha \mu_i(t, x).
\]

Then it is easy to find that

\[
\eta_k \in L^{p'}(\mathbb{R}_+^{1+n}) \& \lim_{k \to \infty} \eta_k = \eta \text{ in } L^{p'}(\mathbb{R}_+^{1+n}).
\]

We next to prove that

\[
(3.1) \quad \|\eta\|_{L^{p'}(\mathbb{R}_+^{1+n})}^{p'} \lesssim \sum_{i=\infty}^{\infty} 2^{ip} \mu_i(\mathbb{R}_+^{1+n})
\]

according to two cases.

**Case**: \( 2 < p < \infty \). Notice first that

\[
\eta(t, x)^{p'} = p' \sum_{k=\infty}^{\infty} \eta_k(t, x)^{p'-1} 2^{k(p-1)} S^*_\alpha \mu_k(t, x) \text{ a.e. } (t, x) \in \mathbb{R}_+^{1+n}.
\]

So, the Hölder inequality yields that

\[
\|\eta\|_{L^{p'}(\mathbb{R}_+^{1+n})}^{p'} \lesssim \left( \int_{\mathbb{R}_+^{1+n}} \sum_{k=\infty}^{\infty} 2^{kp} (S^*_\alpha \mu_k)^{p'}(t, x) dt dx \right)^{2-p'} \\
\left( \int_{\mathbb{R}_+^{1+n}} \sum_{k=\infty}^{\infty} \eta_k(t, x)^{p'-1}(S^*_\alpha \mu_k)^{p'}(t, x) dt dx \right)^{1-p'}.
\]

Since Lemma 3.2 gives

\[
\int_{\mathbb{R}_+^{1+n}} \sum_{k=\infty}^{\infty} 2^{kp} (S^*_\alpha \mu_k)^{p'}(t, x) dt dx = \sum_{k=\infty}^{\infty} 2^{kp} \int_{\mathbb{R}_+^{1+n}} (S^*_\alpha \mu_k)^{p'}(t, x) dt dx = \sum_{k=\infty}^{\infty} 2^{kp} \mu_k(\mathbb{R}_+^{1+n})
\]
implies the desired inequality in (b).

\[ \int_{\mathbb{R}^{1+n}} \sum_{k=-\infty}^{\infty} \eta_k(t, x) 2^k (S_{\alpha}^* \mu_k)^{p'-1}(t, x) \, dt \, dx \]

\[ = \sum_{k=-\infty}^{\infty} \sum_{i \leq k} 2^{i(p-1)+k} \int_{\mathbb{R}^{1+n}} (S_{\alpha}^* \mu_i(t, x))(S_{\alpha}^* \mu_k(t, x))^{p'-1} \, dt \, dx \]

\[ \lesssim \sum_{k=-\infty}^{\infty} 2^{kp} C_p^{(S_{\alpha})}(K_k) \]

\[ \asymp \sum_{k=-\infty}^{\infty} 2^{kp} \mu_k(\mathbb{R}^{1+n}), \]

\[ (3.1) \] is true for \( 2 < p < \infty \).

Case: \( 1 < p \leq 2 \). A combination of (3.2) and Minkowski’s inequality gives

\[ \| \eta \|^p_{L^p(\mathbb{R}^{1+n})} = \sum_{k=-\infty}^{\infty} 2^{kp-1} \int_{\mathbb{R}^{1+n}} \left( \int_{\mathbb{R}^{1+n}} (S_{\alpha}^* \mu_i(t, x))^{p'-1} S_{\alpha}^* \mu_k(t, x) \, dt \, dx \right) \, dt \, dx \]

\[ \lesssim \sum_{k=-\infty}^{\infty} 2^{kp-1} \left( \sum_{i=-\infty}^{k} 2^{i(p-1)} \left( \int_{\mathbb{R}^{1+n}} (S_{\alpha}^* \mu_i(t, x))^{p'-1} S_{\alpha}^* \mu_i(t, x) \, dt \, dx \right)^{\frac{1}{p}} \right) \]

\[ \lesssim \sum_{k=-\infty}^{\infty} 2^{kp-1} C_p^{(S_{\alpha})}(K_k) \]

\[ \asymp \sum_{k=-\infty}^{\infty} 2^{kp-1} \mu_k(\mathbb{R}^{1+n}), \]

whence yields \((3.1)\) under \( 1 < p \leq 2 \).

As a consequence, \((3.1)\) plus

\[ \sum_{i=-\infty}^{\infty} 2^{ip} \mu_i(\mathbb{R}^{1+n}) \lesssim \sum_{i=-\infty}^{\infty} 2^{ip} C_p^{(S_{\alpha})}(K_i) \lesssim \| g \|^p_{L^p(\mathbb{R}^{1+n})}, \]

implies the desired inequality in (b). \( \square \)

Now, Theorem 3.1 for \( T_{\alpha} = S_{\alpha} \) is contained in the following assertion.

**Theorem 3.4.** For a nonnegative Radon measure \( \mu \) on \( \mathbb{R}^{1+n} \) and \( \lambda > 0 \) set

\[ C_S(\mu; \lambda) = \inf \left\{ C_p^{(S_{\alpha})}(K) : \text{compact } K \subset \mathbb{R}^{1+n} \text{ and } \mu(K) \geq \lambda \right\}. \]

(1) If \( 1 < p < \min\{ q, 1 + \frac{q}{2\alpha} \} \) then

\[ (1.3) \Leftrightarrow \sup_{\lambda > 0} \frac{\lambda^\frac{q}{p} \mu_B^{(\alpha)}(t_0, x_0)}{C_S(\mu; \lambda)} < \infty \Leftrightarrow \sup_{(r, t_0, x_0) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}} \frac{\mu_B^{(\alpha)}(t_0, x_0)}{r^{1 + \frac{q}{2\alpha}(1 - p)}} < \infty. \]
Lemma 3.2 and Hölder’s inequality gives

\[ (1.3) \iff \sup_{\lambda > 0} \frac{\lambda}{C_S(\mu; \lambda)} < \infty \quad \Rightarrow \quad \sup_{(r, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n} \frac{\mu(B_r^{(\alpha)}(t_0, x_0))}{r^{n+2\alpha(1-p)}} < \infty. \]

(3) \quad 1 < q < p < 1 + \frac{n}{2\alpha}

then

\[
\int_0^{\infty} \left( \frac{\lambda^{\frac{q}{p}}}{C_S(\mu; \lambda)} \right) \frac{d\lambda}{\lambda^q} < \infty \iff \mathcal{P}^{S}_{\alpha p} \in L^q(\mathbb{R}^{1+n}) \]}

Proof. (1) Suppose (1.3) is valid. Then, for a given compact set \( K \subset \mathbb{R}^{1+n} \), an application of Lemma 3.2 and Hölder’s inequality gives

\[
\int_{\mathbb{R}^{1+n}} g S^*_\alpha \mu_K dt dx = \int_{\mathbb{R}^{1+n}} S^*_\alpha g d\mu_K \leq \| S^*_\alpha g \|_{L^q_k(\mathbb{R}^{1+n}) \mu(K)} \| S^*_\alpha \mu_k \| \lesssim \| g \|_{L^2(\mathbb{R}^{1+n}) \mu(K)} \frac{1}{q},
\]

whence

\[
\| S^*_\alpha \mu_K \|_{L^q_k(\mathbb{R}^{1+n})} \lesssim \mu(K)^{\frac{1}{q}}.
\]

This shows that for

\[
E_\lambda(g) = \left\{ (t, x) \in \mathbb{R}^{1+n} : |S^*_\alpha g(t, x)| \geq \lambda \right\} \quad \forall \quad \lambda > 0
\]

one has

\[
\lambda \mu(E_\lambda(g)) \leq \int_{\mathbb{R}^{1+n}} |S^*_\alpha g| d\mu_{E_\lambda} \lesssim \| g \|_{L^p(\mathbb{R}^{1+n})} \| S^*_\alpha \mu_{E_\lambda} \|_{L^q(\mathbb{R}^{1+n})} \lesssim \| g \|_{L^p(\mathbb{R}^{1+n})} \mu(E_\lambda)^{\frac{1}{q}}.
\]

Therefore, we obtain

\[
\sup_{\lambda > 0} \lambda^q \mu(E_\lambda(g)) \lesssim \| g \|_{L^p(\mathbb{R}^{1+n})}^q.
\]

Picking a function \( g \in L^p(\mathbb{R}^{1+n}) \) such that \( S^*_\alpha g \geq 1 \) on a given compact \( K \subset \mathbb{R}^{1+n} \), we conclude that

\[
\mu(K)^{\frac{1}{q}} \lesssim C_p(S^*_\alpha)(K)^{\frac{1}{q}} \quad \text{and hence} \quad \lambda^q \lesssim C_S(\mu; \lambda) \quad \forall \lambda > 0.
\]

Conversely, if the last inequality is valid, then

\[
\mu(K)^{\frac{1}{q}} \lesssim C_p(S^*_\alpha)(K)^{\frac{1}{q}} \quad \forall \text{ compact } K \subset \mathbb{R}^{1+n}.
\]

Lemma 3.3 is used to derive that if \( g \in L^p(\mathbb{R}^{1+n}) \) then

\[
\int_{\mathbb{R}^{1+n}} |S^*_\alpha g|^q d\mu = \int_0^{\infty} \mu(E_\lambda) d\lambda^q \lesssim \int_0^{\infty} C_p(S^*_\alpha)(E_\lambda)^{\frac{q-p}{p}p} C_p(S^*_\alpha)(E_\lambda) \lambda^{q-p} d\lambda^p \lesssim \| g \|_{L^q(\mathbb{R}^{1+n})}^q \int_0^{\infty} C_p(S^*_\alpha)(E_\lambda) d\lambda^p \lesssim \| g \|_{L^p(\mathbb{R}^{1+n})}^q.
\]
Namely, (1.3) holds.

Next, an application of (1.5) derives that

$$\lambda^\frac{1}{p} \lesssim C_S(\mu; \lambda)^\frac{1}{p} \quad \forall \lambda > 0 \Rightarrow \mu(B_r(\alpha)(t_0, x_0)) \lesssim r^\frac{2(n+2\alpha-2\alpha p)}{p} \quad \forall r > 0.$$ 

For the reverse implication, we first note that \((t, x) \in B_r(\alpha)(t_0, x_0)\) ensures \(K(t-t_0)(x-x_0) \gtrsim r^{-n}\). 

This, along with Fubini’s theorem, yields

\[
S^*_\alpha \mu K(t_0, x_0) \approx \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{(K(t-t_0)(x-x_0))^{-\frac{1}{p}}}^\infty \frac{dr}{r^{n+1}} \right) \right) d\mu K \\
\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^\infty 1_{B_r(\alpha)(t_0, x_0)} \frac{dr}{r^{n+1}} \right) d\mu K \\
\lesssim \int_0^\infty \mu K(B_r(\alpha)(t_0, x_0)) \frac{dr}{r^{n+1}}.
\]

Therefore, for a \(\delta > 0\) to be determined later, we use the Minkowski inequality to get

\[
\|S^*_\alpha \mu K\|_{L^{p'}(\mathbb{R}^{1+n})} \lesssim \int_{\mathbb{R}^{1+n}} \left( \int_0^\infty \mu K(B_r(\alpha)(t_0, x_0)) \frac{dr}{r^{n+1}} \right)^{p'} dtdx \\
\lesssim \int_0^\infty \mu K(B_r(\alpha)(t, x)) \frac{dr}{r^{n+1}} \\
\lesssim \int_0^\delta \mu K(B_r(\alpha)(t, x)) \frac{dr}{r^{n+1}} \\
+ \int_\delta^\infty \mu K(B_r(\alpha)(t, x)) \frac{dr}{r^{n+1}} \\
\equiv I_1 + I_2.
\]

Since

\[
\|\mu K(B_r(\alpha)(t, x))\|_{L^{p'}(\mathbb{R}^{1+n})} \lesssim \mu(K)^{p'-1} \int_{\mathbb{R}^{1+n}} \mu K(B_r(\alpha)(t, x)) dtdx \lesssim \mu(K)^{p'-1} r^{n+2\alpha},
\]

it follows that

\[
I_2 \lesssim \mu(K) \int_\delta^\infty \frac{dr}{r^{n+1 - \frac{2\alpha}{p}}} \lesssim \mu(K) \delta^{2\alpha - \frac{n+2\alpha}{p}}.
\]

On the other hand,

\[
\mu(B_r(\alpha)(t_0, x_0)) \lesssim r^\frac{2(n+2\alpha-2\alpha p)}{p} \quad \forall r > 0
\]

derives

\[
\|\mu K(B_r(\alpha)(t, x))\|_{L^{p'}(\mathbb{R}^{1+n})} \lesssim r^\frac{2(n+2\alpha-2\alpha p)(p'-1)}{p} \int_{\mathbb{R}^{1+n}} \mu K(B_r(\alpha)(t, x)) dtdx \\
\lesssim \mu(K) r^\frac{2(n+2\alpha-2\alpha p)(p'-1)}{p} + n+2\alpha.
\]

This clearly forces

\[
I_1 \lesssim \mu(K)^\frac{1}{p'} \int_0^\delta r^{p'-1} \left( \frac{2(n+2\alpha-2\alpha p)(p'-1)}{p} + n+2\alpha \right) \frac{dr}{r^{n+1}} \lesssim \mu(K)^\frac{1}{p'} \delta^{\frac{(q-p)(n+2\alpha-2\alpha p)}{p^2}}.
\]
Upon choosing \( \delta = \mu(K)^{\frac{1}{n+2(p-2)p}} \), we obtain
\[
\|S_\alpha^* \mu_K\|_{L^p(\mathbb{R}_+^{1+n})} \lesssim \mu(K)^\frac{1}{\nu} \quad \text{and hence} \quad C_p^{(S_\alpha)}(K)^\frac{1}{\nu} \lesssim \mu(K)^\frac{1}{\nu}.
\]

(2) This follows from the above demonstration.

(3) Suppose \([\mathbf{I.3}]\) is valid. Then
\[
\sup_{\lambda > 0} \lambda (\mu(E_\lambda(g)))^\frac{1}{\nu} \lesssim \|g\|_{L^p(\mathbb{R}_+^{1+n})} \quad \forall \ g \in L^p(\mathbb{R}_+^{1+n}).
\]
For each integer \( i \in \mathbb{Z} \), there is a compact set \( K_i \subset \mathbb{R}_+^{1+n} \) and a function \( g_i \in L^p(\mathbb{R}_+^{1+n}) \) such that
\[
C_p^{(S_\alpha)}(K_i) \lesssim C_S(\mu; 2^i), \quad \mu(K_i) > 2^i; \quad S_\alpha g_i \leq 1_{K_i}; \quad \|g_i\|_{L^p(\mathbb{R}_+^{1+n})} \lesssim C_p^{(S_\alpha)}(K_i).
\]

Set
\[
g_{j,k} = \sup_{j \leq i \leq k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{1}{p-q}} g_i
\]
for integers \( j, k \) with \( j < k \). Then
\[
\|g_{j,k}\|_{L^p(\mathbb{R}_+^{1+n})}^p \lesssim \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{p}{p-q}} \|g_i\|_{L^p(\mathbb{R}_+^{1+n})}^p \lesssim \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{p}{p-q}} C_S(\mu; 2^i).
\]
Since for \( \forall \ (t, x) \in K_i \) and \( j \leq i \leq k \) one has
\[
|S_\alpha g_{j,k}(t, x)| \geq \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{1}{p-q}} S_\alpha g_i(t, x) \gtrsim \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{1}{p-q}},
\]
it follows that
\[
2^i < \mu(K_i) \leq \mu(E_{\frac{2^i}{C_S(\mu; 2^i)}}^{\frac{1}{p-q}}(g_{j,k})),
\]
and so that
\[
\|g_{j,k}\|_{L^p(\mathbb{R}_+^{1+n})}^q \gtrsim \int_{\mathbb{R}_+^{1+n}} |S_\alpha g_{j,k}|^q d\mu
\]
\[
\gtrsim \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{q}{p-q}} 2^i
\]
\[
\gtrsim \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{q}{p-q}} 2^i \|g_{j,k}\|_{L^p(\mathbb{R}_+^{1+n})}^q
\]
\[
\gtrsim \left( \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{q}{p-q}} C_S(\mu; 2^i) \right)^{\frac{q}{p}}
\]
\[
\gtrsim \left( \sum_{i=j}^{k} \left( \frac{2^i}{C_S(\mu; 2^i)} \right)^{\frac{q}{p-q}} \right)^{\frac{q}{p-q}} \|g_{j,k}\|_{L^p(\mathbb{R}_+^{1+n})}^q.
\]
This is the desired result thanks to
\[
\int_0^\infty \left( \frac{\lambda^\frac{p}{q}}{C_S(\mu; \lambda)} \right)^{\frac{q}{p-q}} d\lambda \lesssim \sum_{i=-\infty}^\infty \frac{2^i}{(C_S(\mu; 2^i))^{\frac{q}{p-q}}} \lesssim 1.
\]
Conversely, if

\[ \int_0^\infty \left( \frac{\chi^2}{C_S(\mu; \lambda)} \right)^{\frac{q-p}{q-\alpha}} d\lambda < \infty, \]

then setting

\[ T_{p,q}(\mu; g) = \sum_{i=-\infty}^{\infty} \frac{(\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g)))}{(C_p(S_\lambda(E_{2^i}(g)))^{\frac{q-p}{q}} \]

for each integer \( i = 0, \pm 1, \pm 2, \ldots \), and \( g \in C_0^\infty(\mathbb{R}_1^{1+n}) \), we use an integration-by-part, the Hölder inequality and Lemma 3.3 to produce

\[ \int_{\mathbb{R}_1^{1+n}} |S_\alpha g|^q d\mu = - \int_0^\infty \lambda^q d\mu(E_\lambda(g)) \]

\[ \lesssim \sum_{i=-\infty}^{\infty} (\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g))) 2^q \]

\[ \lesssim (T_{p,q}(\mu; g))^{\frac{p-q}{p}} \left( \sum_{i=-\infty}^{\infty} 2^iq C_p(S_\lambda(E_{2^i}(g)))^{\frac{q-p}{q}} \right)^{\frac{q}{p}} \]

\[ \lesssim (T_{p,q}(\mu; g))^{\frac{p-q}{p}} \left( \int_0^\infty C_p(S_\lambda) \sum_{\{t, x \in \mathbb{R}_1^{1+n} \mid |S_\alpha g(t, x)| > \lambda \}} d\lambda \right)^{\frac{q}{p}} \]

\[ \lesssim \|g\|_{L_p(\mathbb{R}_1^{1+n})}^q \]

In the last inequality we have used the following estimation:

\[ (T_{p,q}(\mu; g))^{\frac{p-q}{p}} \lesssim \left( \sum_{i=-\infty}^{\infty} \frac{(\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g)))}{(C_S(\mu; \mu(E_{2^i}(g))))^{\frac{q-p}{q}}} \right)^{\frac{p-q}{p}} \]

\[ \lesssim \left( \sum_{i=-\infty}^{\infty} \frac{(\mu(E_{2^i}(g)))^{\frac{q-p}{q}} - (\mu(E_{2^{i+1}}(g)))^{\frac{q-p}{q}}}{(C_S(\mu; \mu(E_{2^i}(g))))^{\frac{q-p}{q}}} \right)^{\frac{p-q}{p}} \]

\[ \lesssim \left( \int_0^\infty \frac{d\mu(E_{2^i}(g))^{\frac{q-p}{q}}}{(C_S(\mu; s))^{\frac{q-p}{q}}} \right)^{\frac{p-q}{p}} \]

\[ \approx \left( \int_0^\infty \frac{\lambda^\frac{2}{p}}{C_S(\mu; s)} \frac{d\lambda}{\lambda} \right)^{\frac{p-q}{p}} \]

Needless to say, the equivalence

\[ (1.3) \iff P_{\alpha p}^S \in L_<(p-1)/(p-q)(\mathbb{R}_1^{1+n}) \]

follows from Lemma 3.1(b) and a modification (cf. [6, Theorem 2.1]) of the argument for

\[ (1.2) \iff P_{\alpha p}^R \in L_<(p-1)/(p-q)(\mathbb{R}_1^{1+n}), \]

and hence the interested reader can readily work out the details.
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