Split digraphs

M. Drew LaMar

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Abstract

We generalize the class of split graphs to the directed case and show that these split digraphs can be identified from their degree sequences. The first degree sequence characterization is an extension of the concept of splittance to directed graphs, while the second characterization says a digraph is split if and only if its degree sequence satisfies one of the Fulkerson inequalities (which determine when an integer-pair sequence is digraphic) with equality.

1 Introduction

All graphs and digraphs in this article will be simple, i.e. with no self-loops or multi-edges/arcs. We consider integer-pair sequences $d = \{(d^+_i, d^-_i)\}_{i=1}^N$ and say $d$ is digraphic if there exists a digraph with degree sequence $d$. For digraphic sequences $d$, $d^+$ and $d^-$ will denote the out-degree and in-degree sequences of $d$, respectively. We denote directed graphs by $\vec{G}$, with $V(\vec{G})$ the vertex set and $A(\vec{G})$ the arc set. We will drop the reference to $\vec{G}$ when the digraph is understood through the notation $\vec{G} = (V, A)$, for example. An arc between vertices $a$ and $b$ will be denoted by $(a, b)$, with the orientation given by the ordering. Given a digraph $\vec{G} = (V, A)$ and vertex sets $X, Y \subset V$, we define the subgraph $\vec{G}[X,Y] = (X \cup Y, A[X,Y])$, where $A[X,Y] = \{(x,y) : x \in X \text{ and } y \in Y\}$. When $X = Y$, we have the usual definition of an induced subgraph and will denote this by $\vec{G}[X]$. Finally, given an index set $I$, we will denote the labeled vertices with those indices by $V_I$. Similar notation is used for undirected graphs, for example replacing digraphic with graphic, arc set $A$ with edge set $E$, and removing directional arrows.

We start with the definition of a split graph.

Definition 1.1 (Split graph [3]) A graph $G = (V, E)$ is split if and only if $V$ is a disjoint union of two sets $A$ and $B$ such that $A$ is a clique and $B$ is an independent set. The partition $S = \{A, B\}$ is called a split partition.

We will also call the degree sequence of a split graph a split degree sequence. Note that either $A$ or $B$ can be empty. It follows immediately from the definition that the complement of a split graph is split, as well as any induced subgraph. Split graphs have a forbidden induced subgraph characterization given by the exclusion of induced subgraphs $2K_2$, $C_4$, and $C_5$ [3]. They are also characterized by those graphs which are chordal and whose complements are also chordal [3].

Split digraphs are defined in a similar manner by partitioning the vertex set into a disjoint union of at most four sets $S^\pm, S^+, S^-$, and $S^0$. Similar to Definition 1.1, two of the sets, namely $S^\pm$
and $S^0$, are a clique and an independent set, respectively, with arbitrary connections between them. This shows immediately that split graphs are a special case of split digraphs. For the other two vertex sets $S^+$ and $S^-$, $\tilde{G}[S^+]$ and $\tilde{G}[S^-]$ are arbitrary induced subgraphs. The diagram in the left of Figure 1 shows the relations within and between the four different classes, with solid and dashed-dotted arrows denoting forced and allowable arcs, respectively, and the absence of an arc denoting no connections. We call dashed-dotted arrows denoting forced and allowable arcs, respectively, and the absence of an arc left of Figure 1 shows the relations within and between the four different classes, with solid and dashed-dotted arrows denoting forced and allowable arcs, respectively, and the absence of an arc right of Figure 1 using the notation of $M$-partitions [2]. In general, an $M$-partition of a digraph $\tilde{G}$ is a partition of the vertex-set $V(\tilde{G})$ into $k$ disjoint sets $\{X_1, \ldots, X_k\}$, where the arc constraints within and between sets are given by a $k \times k$ matrix $M$ with elements in $\{0, 1, *\}$. $M_{ii}$ equals 0 or 1 when $X_i$ is an independent set or clique, respectively, and is set to * when $\tilde{G}[X_i]$ is an arbitrary subgraph. Similarly, for $i \neq j$, $M_{ij}$ equal to 0, 1, or * corresponds to $\tilde{G}[X_i, X_j]$ having no arcs from $X_i$ to $X_j$, all arcs from $X_i$ to $X_j$, and no constraints on arcs from $X_i$ to $X_j$, respectively. We summarize all of this in the following definition.

**Definition 1.2 (Split digraph)** Given a digraph $\tilde{G} = (V, A)$, a vertex partition $S = \{S^\pm, S^+, S^-, S^0\}$ is called a split partition of $\tilde{G}$ if and only if $S$ defines an $M$-partition of $\tilde{G}$ with adjacency matrix $M$ given in Figure [2]. A digraph $\tilde{G}$ is a split digraph if and only if it has a non-trivial split partition.

Similar to the undirected case, we will call the degree sequence of a split digraph a split degree sequence. Note that contrary to split graphs, split digraphs do not have a forbidden subgraph characterization since $\tilde{G}[S^+]$ and $\tilde{G}[S^-]$ are arbitrary subgraphs. This paper shows that split digraphs have degree sequence characterizations analogous to the characterizations for split graphs [5], which we review in the next section.
1.1 Undirected splittance and graphicality

Hammer and Simeone [5] determine two degree sequence characterizations of split graphs: one through the splittance of a graph and the other through equality of one of the Erdős-Gallai inequalities, which give necessary and sufficient conditions for when an integer sequence is graphic. We begin by discussing the concept of splittance, whose definition is given as follows:

**Definition 1.3 (Graph splittance [5])** Define the splittance \( \sigma(G) \) of \( G \) to be the minimum number of edges to add to or remove from \( G \) in order to obtain a split graph.

Clearly a graph \( G \) is split if and only if \( \sigma(G) = 0 \). The interesting fact is that the splittance \( \sigma(G) \) can be written solely in terms of the degree sequence \( d \) of \( G \). To see this, let the degree sequence \( d \) be in non-increasing order, i.e. \( d_1 \geq \ldots \geq d_N \). If we define the corrected Durfee number \( m \) by

\[
 m = m(d) = \max\{ k : d_k \geq k - 1 \}
\]

and the splittance sequence \( \{ \sigma_k(d) \}_{k=0}^N \) by

\[
 \sigma_k(d) = \frac{1}{2} \left\{ k(k - 1) - \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{N} d_i \right\}, \tag{1}
\]

then we have

**Theorem 1.4 (Hammer and Simeone [5])** \( \sigma(G) = \min_k \sigma_k(d) = \sigma_m(d) \).

An important property of the splittance sequence (1) that leads to the second equality above is that \( \sigma_k(d) \) is non-increasing for \( 1 \leq k \leq m \) and strictly increasing for \( m < k \leq N \). Thus, \( \sigma_k(d) \) has only one region of minima, which includes the corrected Durfee number \( m \). The following corollary of Theorem 1.4 is immediate and gives us the first degree sequence characterization of split graphs.

**Corollary 1.5 (Hammer and Simeone [5])** If \( d \) is a non-increasing degree sequence, then \( d \) is split if and only if \( \sigma_m(d) = 0 \).

The sufficiency of Corollary 1.5 can be illustrated as follows. Suppose \( G \) is split with non-increasing degree sequence \( d \) and let \( m \) be the corrected Durfee number for \( d \). Then \( S = \{ A, B \} \) defines a split partition of \( G \), with the clique \( A = \{ x_1, \ldots, x_m \} \) and the independent set \( B = \{ x_{m+1}, \ldots, x_N \} \). Referring to (1), all edges from \( A \) (the term \( - \sum_{i=1}^{m} d_i \)) either go back into \( A \) (which is \( m(m-1) \)) since \( A \) is a clique) or connect to \( B \) (which equals \( \sum_{i=m+1}^{N} d_i \) since \( B \) is an independent set). Thus, \( \sigma_m(d) = 0 \). In general, the choice of \( S \) may not be unique. For example, if there is a vertex in \( A \) that has no connections with \( B \), then \( d_m = m-1 \) and we also have the split partition \( S' = (A', B') \) with \( A' = \{ x_1, \ldots, x_{m-1} \} \) and \( B' = \{ x_m, \ldots, x_N \} \).

Hammer and Simeone also highlight a very interesting relationship between the splittance and the Erdős-Gallai inequalities, which give necessary and sufficient conditions for when an integer sequence \( d \) is graphic.

**Theorem 1.6 (Erdős and Gallai [1])** Let \( d \) be a non-increasing integer sequence. Then \( d \) is graphic if and only if \( \sum_{i=1}^{N} d_i \) is even and for \( k = 1, \ldots, N \),

\[
 k(k - 1) + \sum_{i=k+1}^{N} \min\{ d_i, k \} \geq \sum_{i=1}^{k} d_i.
\]
For a non-increasing integer sequence, if we define the slack sequence \( \{ s_k(d) \} \)
then Theorem 1.6 can be restated as saying \( d \) is graphic if and only if the slack sequence \( s_k(d) \)
is non-negative. For any non-increasing integer sequence \( d \) with \( m \) it’s corrected Durfee number,

it can be easily seen that \( \min\{d_i, m\} = d_i \) for \( i = m + 1, \ldots, N \). Thus \( 2\sigma_m(d) = s_m(d) \),
which by Corollary 1.5 gives us the following second degree sequence characterization of split graphs.

**Theorem 1.7 (Hammer and Simeone [5])** If \( d \) is a non-increasing degree sequence, then \( d \)
is split if and only if \( s_m(d) = 0 \).

Thus, split sequences are somehow close to the boundary of graphicality. The remainder of the paper
in Section 2 generalizes Corollary 1.5 and Theorem 1.7 to split digraphs and is outlined as follows.

Section 2.1 defines the splittance for a directed graph \( G \) as the minimum number of arcs to add to or
remove from \( G \) to make \( G \) a split digraph. The digraph splittance can also be written solely in terms
of the degree sequence of \( G \) by defining a splittance matrix in Definition 2.10 similar to the splittance
sequence in (1), and proving in Theorem 2.12 an analogous version of Theorem 1.4 which states that
the splittance of a directed graph is equal to the minimum (taken over a specific set of entries)
of the splittance matrix. This leads to the first degree sequence characterization of split digraphs
and proving in Theorem 2.13 that \( G \) is split if and only if this minimum is zero. Since the splittance matrix
may have multiple disconnected regions of minima, in contrast to the splittance sequence [5], the
corrected Durfee number \( m \) does not have a direct extension to degree sequences for directed graphs.
However, in Definition 2.15 we define an extension which we call maximal sequences that do share
similar properties to the corrected Durfee number, and which exhibit interesting structure relative
to the splittance matrix, as can be seen in Corollary 2.17 and Lemma 2.18 in Section 2.2. This allows
us to prove the second degree sequence characterization of split digraphs in Corollary 2.20 which is
analogous to Theorem 1.7 stating that a digraph \( G \) is split if and only if it’s degree sequence satisfies
any of the Fulkerson inequalities with equality, where the Fulkerson inequalities give necessary and
sufficient conditions for an integer-pair sequence to be digraphic [4]. This is stronger than the
undirected case, which states that an undirected graph is split if and only if the \( m \)-th Erdős-Gallai
inequality is satisfied with equality, where \( m \) is the corrected Durfee number.

# 2 Degree sequence characterizations of split digraphs

In working with the degree sequence characterizations for split graphs using both the splittance as
well as the slack sequences, the degree sequence \( d \) needed to be non-increasing. In the directed case,
we need integer-pair sequences to be non-increasing as well, in particular under the lexicographical
ordering of either the first or second coordinate.

**Definition 2.1 (Positive/Negative ordering)** An integer-pair sequence \( d = \{(d_i^+, d_i^-)\}_{i=1}^N \)
is non-increasing relative to the positive lexicographical ordering if and only if \( d_i^+ \geq d_{i+1}^+ \), with
\( d_i^- \geq d_{i+1}^- \) when \( d_i^+ = d_{i+1}^+ \). In this case, we will call \( d \) positively ordered and denote the ordering
by \( d_i \geq_p d_{i+1} \). We say \( d \) is non-increasing relative to the negative lexicographical ordering by
giving preference to the second coordinate, calling \( d \) in this case negatively ordered and denoting
the ordering by \( d_i \geq_N d_{i+1} \).
For a given integer-pair sequence \( d = \{(d^+_i, d^-_i)\}_{i=1}^N \), define the sequences \( \bar{d} = \{(\bar{d}^+_i, \bar{d}^-_i)\}_{i=1}^N \) and \( \bar{d} = \{(\bar{d}^+_i, \bar{d}^-_i)\}_{i=1}^N \) to be the positive and negative orderings of \( d \), respectively. We will need one more subtle property of our degree-sequence ordering which states that if \( d_i = d_j \), then \( d_i \) is before \( d_j \) in the positive lexicographical ordering if and only if \( d_i \) is before \( d_j \) in the negative lexicographical ordering.

**Definition 2.2 (Proper ordering)** An integer-pair sequence \( d \) is said to be properly ordered with permutations \( \bar{d} \) and \( d \) when \( \bar{d} \equiv d_\pi(i) \) and \( d \equiv d_\pi(i) \) such that if \( d_i = d_j \), then \( \pi^{-1}(i) < \pi^{-1}(j) \) if and only if \( \bar{\pi}^{-1}(i) < \bar{\pi}^{-1}(j) \).

### 2.1 Splittance of a directed graph

We begin with a generalization of graph splittance in Definition 1.3 to digraphs.

**Definition 2.3 (Digraph splittance)** Define \( \sigma(G) \) to be the minimum number of arcs to be added or removed from \( G \) in order to obtain a split digraph.

The following defines two measures on arbitrary partitions \( S = \{S^+, S^+, S^-, S^0\} \), which we show in Lemma 2.6 tell us how close \( S \) is to being a split partition.

**Definition 2.4 (Split partition measures)** Let \( \bar{S} = \{S^+, S^+, S^-, S^0\} \) be an arbitrary vertex partition with \( k = |S^+ \cup S^+| \) and \( l = |S^+ \cup S^-| \). Define the split partition measures \( \bar{\sigma}(S) \) and \( \sigma(S) \) as

\[
\bar{\sigma}(S) = |S^+|(k - 1) + |S^-|l + \sum_{S^+ \cup S^-} d^-_x - \sum_{S^+ \cup S^-} d^+_x,
\]

\[
\sigma(S) = |S^+|(l - 1) + |S^-|k + \sum_{S^+ \cup S^-} d^+_x - \sum_{S^+ \cup S^-} d^-_x.
\]

By the definition of \( k \) and \( l \) in Definition 2.4, we have the equivalent formulation

\[
\bar{\sigma}(S) = kl - |S^+| + \sum_{S^+ \cup S^-} d^-_x - \sum_{S^+ \cup S^-} d^+_x,
\]

\[
\sigma(S) = kl - |S^+| + \sum_{S^- \cup S^0} d^+_x - \sum_{S^- \cup S^0} d^-_x. \tag{2}
\]

**Lemma 2.5** For an arbitrary partition \( S = \{S^+, S^+, S^-, S^0\} \), we have

\[
\bar{\sigma}(S) = \sigma(S).
\]

**Proof** Let \( S = \{S^+, S^+, S^-, S^0\} \) be a partition with \( k = |S^+ \cup S^+| \) and \( l = |S^+ \cup S^-| \). Using (2), we have

\[
\bar{\sigma}(S) - \sigma(S) = kl - |S^+| + \sum_{S^+ \cup S^-} d^-_x - \sum_{S^+ \cup S^-} d^+_x - \left[ kl - |S^+| + \sum_{S^- \cup S^0} d^+_x - \sum_{S^- \cup S^0} d^-_x \right] = \sum_V d^-_x - \sum_V d^+_x = 0.
\]
We will thus speak of the split partition measure $\sigma(S)$ and work with $\sigma \equiv \bar{\sigma}$ in all proofs that follow. We can see easily from Figure 1 that $\sigma(S) = 0$ for a split partition $S$, since all arcs from $S^\pm \cup S^+$ (the term $- \sum_{S^\pm \cup S^+} d_x^+$) include the forced arcs into $S^\pm$ and $S^-$ (the terms $|S^\pm|(k-1)$ and $|S^-|k$) and the allowed arcs into $S^+$ and $S^0$ (the term $\sum_{S^\pm \cup S^0} d_x^-$). The next lemma shows in fact that for an arbitrary partition $S = \{S^+, S^-, S^0\}$, $\sigma(S)$ gives the minimal number of arcs to add to or remove from $\bar{G}$ in order for $S$ to be a split partition. In particular, this implies $\sigma(\bar{G}) = \min_S \sigma(S)$ when minimizing over non-trivial partitions $S$. Thus, we have

**Lemma 2.6** For a partition $S = \{S^+, S^-, S^0\}$, $\sigma(S)$ gives the minimal number of arcs to add to or remove from $\bar{G}$ in order for $S$ to be a split partition.

**Proof** Given a partition $S = \{S^+, S^-, S^0\}$ with $k = |S^+ + S^-|$ and $l = |S^+ + S^-|$, it is easily seen that

$$\sum_{S^\pm \cup S^0} d_x^- = |A[S^- \cup S^0, S^+ \cup S^0]| + |A[S^\pm \cup S^+, S^+ \cup S^0]|$$

$$\sum_{S^\pm \cup S^+} d_x^+ = |A[S^\pm \cup S^+, S^+ \cup S^-]| + |A[S^\pm \cup S^+, S^+ \cup S^0]|,$n

and thus

$$\sum_{S^+ \cup S^0} d_x^- - \sum_{S^+ \cup S^+} d_x^+ = |A[S^- \cup S^0, S^+ \cup S^0]| - |A[S^\pm \cup S^+, S^+ \cup S^-]|.$$n

This leads to

$$\sigma(S) = |S^\pm|(k-1) + |S^-|k - |A[S^\pm \cup S^+, S^\pm \cup S^-]| + |A[S^- \cup S^0, S^+ \cup S^0]|.$$n

Note that the first three terms give the number of arcs to add to $\bar{G}$ for there to be all arcs from $S^\pm \cup S^+$ to $S^\pm \cup S^-$, while the last term gives the number of arcs to remove from $\bar{G}$ so that there are no arcs from $S^- \cup S^0$ to $S^+ \cup S^0$. The resulting digraph after addition and removal of these arcs will have $S$ as a split partition. □

**Corollary 2.7** A partition $S = \{S^\pm, S^+, S^-, S^0\}$ is a split partition if and only if $\sigma(S) = 0$.

**Corollary 2.8** Minimizing over non-trivial partitions $S = \{S^\pm, S^+, S^-, S^0\}$, we have

$$\sigma(\bar{G}) = \min_S \sigma(S).$$

In Theorem 2.12, we show for each fixed index pair $(k,l) \in [0,N] \times [0,N]$, there is a partition $\chi_{kl}$ such that

$$\sigma(\chi_{kl}) = \min_{|S^\pm \cup S^+| = k \atop |S^\pm \cup S^-| = l} \sigma(S).$$

This gives

$$\sigma(\bar{G}) = \min_S \sigma(S)$$

$$= \min_{(k,l) \notin \{(0,N),(N,0)\}} \min_{|S^\pm \cup S^+| = k \atop |S^\pm \cup S^-| = l} \sigma(S)$$

$$= \min_{(k,l) \notin \{(0,N),(N,0)\}} \sigma(\chi_{kl})$$

$$= \min_{(k,l) \notin \{(0,N),(N,0)\}} \Sigma_{kl}, \tag{4}$$

6
where $\Sigma_{kl} \equiv \sigma(X_{kl})$ is called the \textit{splittance matrix} and is the directed extension of the splittance sequence in (1). These special partitions $X_{kl}$ are called \textit{induced partitions}, which are defined along with a formal definition of the splittance matrix in the following way.

\textbf{Definition 2.9 (Induced partitions)} Suppose $d$ is properly ordered with $\bar{\pi}$ and $\pi$ the corresponding permutations. For $(k,l) \in [0,N] \times [0,N]$, let $A_k = \{\bar{\pi}(i)\}_{i=1}^{k}$ and $B_l = \{\pi(i)\}_{i=1}^{l}$. The partition induced by $(k,l)$ is given by $X_{kl} = \{X^\pm, X^+, X^-, X^0\}$ such that

\begin{align*}
X^\pm &= V_{A_k \cap B_l}, \\
X^+ &= V_{A_k \setminus B_l}, \\
X^- &= V_{B_l \setminus A_k}, \\
X^0 &= V_{A_k \setminus B_l}.
\end{align*}

We see immediately from this definition that $V_{A_k} = X^\pm \cup X^+$ and $V_{B_l} = X^\pm \cup X^-$, which gives

\begin{align}
    d_{X^\pm \cup X^+} \geq_p d_{X^- \cup X^0} \quad \text{and} \quad d_{X^\pm \cup X^-} \geq_n d_{X^+ \cup X^0}.
\end{align}

\textbf{Definition 2.10 (Splittance matrix)} Suppose $d$ is properly ordered. For each $(k,l) \in [0,N] \times [0,N]$ and corresponding induced partition $X_{kl}$, we define the splittance matrix $\Sigma \equiv \Sigma(d)$ such that $\Sigma_{kl} = \sigma(X_{kl})$.

The main theorem of this section is Theorem 2.12 showing

\begin{align}
\sigma(\vec{G}) = \min_{(k,l) \notin \{(N,0), (0,N)\}} \Sigma_{kl},
\end{align}

which by (4) reduces to showing (3). Note that the index-pairs $(0,N)$ and $(N,0)$ correspond to trivial split partitions and therefore $\Sigma_{0N} = \Sigma_{N0} = 0$. Thus, these corners of $\Sigma$ are not used in computing the splittance, as seen in (6).

Before proceeding to Theorem 2.12, it is instructive to give an example. Consider the following split degree sequence

\begin{align}
d = \begin{pmatrix}
d_1^+ & \cdots & d_5^+ \\
d_1^- & \cdots & d_5^-
\end{pmatrix} = \begin{pmatrix}
2 & 3 & 4 & 1 & 0 \\
1 & 2 & 2 & 2 & 3
\end{pmatrix}.
\end{align}

A particular labeled realization $\vec{G} = (V,A)$ of $d$ is given by

For a proper ordering of $d$, define the permutations $\bar{\pi} = (3 2 1 4 5)$ and $\pi = (5 3 2 4 1)$. The splittance matrix $\Sigma \equiv \Sigma(d)$ is given by

\begin{align*}
\Sigma &= \begin{pmatrix}
10 & 7 & 5 & 3 & 1 & 0 \\
6 & 4 & 2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 4 & 6 \\
0 & 1 & 3 & 5 & 7 & 10
\end{pmatrix}.
\end{align*}
There are 5 non-trivial zeros in Σ corresponding to 5 different induced split partitions. For the index pair \((k, l) = (2, 3)\), for example, we can define the sets \(\mathcal{A}_2 = \{\bar{\pi}(1), \bar{\pi}(2)\} = \{3, 2\}\) and \(\mathcal{B}_3 = \{\bar{\pi}(1), \bar{\pi}(2), \bar{\pi}(3)\} = \{5, 3, 2\}\). This index pair induces a split partition \(\chi_{23}\) as follows

\[
X^+ = V_{\mathcal{A}_2 \cap \mathcal{B}_3} = \{x_2, x_3\}, \\
X^+ = V_{\mathcal{A}_2 - \mathcal{B}_3} = \emptyset, \\
X^- = V_{\mathcal{B}_3 - \mathcal{A}_2} = \{x_3\}, \\
X^0 = V_{\mathcal{A}_2 \cap \mathcal{B}_3} = \{x_1, x_4\}.
\]

As mentioned previously, the splittance sequence \([1]\) has one region of minima, which includes the corrected Durfee number. This example illustrates, however, that there can be multiple regions of minima in Σ separated by non-zero splittance.

To prove \([6]\) in Theorem \(2.12\) we will need the following inequalities of induced partitions, which rely heavily on the proper ordering of \(d\) and give conditions on when strict inequality in \([5]\) occurs.

**Lemma 2.11** Suppose \(d\) is properly ordered and \(\chi = \{X^+, X^+, X^-, X^0\}\) is an induced partition. For \(x \in X^+, y \in X^-, z \in X^+\) and \(w \in X^0\), we have

\[
d^+_y > d^+_x, \\
d^-_y > d^-_x, \\
d^+_z = d^-_z \Rightarrow d^+_x > d^+_y, \\
d^-_y = d^-_z \Rightarrow d^-_x > d^-_x, \\
d^+_y = d^-_w \Rightarrow d^+_x > d^+_w, \\
d^+_z = d^-_w \Rightarrow d^-_x > d^-_w.
\]

**Proof** We will prove \([8a]\) and \([8c]\), with the others following similarly. For \([8a]\), let \(x \in X^+\) and \(y \in X^−\). By \([5]\), \(d_x ≥_x d_y \Rightarrow d^-_x ≥ d^+_y\). If \(d^+_x > d^+_y\), then we’re done, so suppose \(d^+_x = d^+_y\) with \(d^-_x ≥ d^-_y\). But \(d_y ≥_x d_x\) implies \(d^-_y ≥ d^-_x \Rightarrow d_x = d_y\). Thus, by the proper ordering of \(d\) we have \(\bar{\pi}^{-1}(x) < \bar{\pi}^{-1}(y) \Rightarrow \bar{\pi}^{-1}(x) < \bar{\pi}^{-1}(y) \Rightarrow x \in X^+\) since \(y \in X^−\), which is a contradiction.

Now consider \([8c]\) and let \(x \in X^+, y \in X^−\) and \(z \in X^\pm\) such that \(d^+_z = d^-_z\) with \(d^+_z ≤ d^+_z\). Since \(z \in X^\pm, x \in X^+\) and \(d^+_z = d^-_z\), we have \(d^+_z ≥ d^+_z\). But this means \(d^+_x ≤ d^+_y\), which contradicts \([8a]\). \(\square\)

**Theorem 2.12**

\[
\sigma(\bar{G}) = \min_{(k,l)\notin\{(0,N),(N,0)\}} \Sigma_{kl}
\]

**Proof** By the series of equalities in \([4]\), we only need to show

\[
\min_{|S^\pm \cup S^0| = k} \min_{|S^\pm \cup S^\pm| = l} \sigma(S) = \Sigma_{kl}.
\]

We will work with \(\Sigma_{kl} \equiv \bar{\sigma}(\chi_{kl})\) and for simplicity we drop the subscripts to \(\chi\). We thus want to show \(\sigma(S) ≥ \sigma(\chi)\) for all partitions \(S = \{S^\pm, S^+, S^−, S^0\}\) such that \(|S^\pm \cup S^0| = k\) and \(|S^\pm \cup S^\pm| = l\).

Letting \(S\) be such a partition, then by the proper ordering of \(d\) we have

\[
\sum_{S^+ \cup S^0} d^+_z ≥ \sum_{X^\pm \cup X^0} d^+_x \quad \text{and} \quad \sum_{S^\pm \cup S^+} d^+_z ≤ \sum_{X^\pm \cup X^+} d^+_x.
\]
If $|X^\pm| \geq |S^\pm|$, then by (2) and (9) we have $\sigma(S) \geq \sigma(X)$. We need to do more work however when $|X^\pm| < |S^\pm|$. We have

$$
\sigma(S) - \sigma(X) = kl - |S^\pm| + \sum_{S+X^0} d_x^- + \sum_{S\pm X^0} d_x^- - \left[kl - |X^\pm| + \sum_{X+X^0} d_x^- - \sum_{X\pm X^0} d_x^+\right]
$$

$$
= \left(\sum_{S+X^0} d_x^- - \sum_{X+X^0} d_x^-\right) + \left(\sum_{X\pm X^0} d_x^+ - \sum_{S\pm X^0} d_x^+\right) + |X^\pm| - |S^\pm|
$$

Note that the two sums in (10b) have the same number of terms since

$$
|(X^\pm \cup X^+) \cap (S^- \cup S^0)| = |X^\pm \cup X^+| - |(X^\pm \cup X^+) \cap (S^\pm \cup S^+)|
$$

$$
= |S^\pm \cup S^+| - |(X^\pm \cup X^+) \cap (S^\pm \cup S^+)|
$$

$$
= |(X^- \cup X^0) \cap (S^\pm \cup S^+)|.
$$

The same can be shown for the two sums in (10a). Now let $n = |S^\pm| - |X^\pm| > 0$ and

$$
\Omega^- = \left(\sum_{(X^\pm \cup X^-) \cap (S^\pm \cup S^0)} d_x^- - \sum_{(X^- \cup X^0) \cap (S^\pm \cup S^-)} d_x^-\right),
$$

$$
\Omega^+ = \left(\sum_{(X^\pm \cup X^+) \cap (S^- \cup S^0)} d_x^+ - \sum_{(X^- \cup X^0) \cap (S^\pm \cup S^+)} d_x^+\right),
$$

so that

$$
\sigma(S) - \sigma(X) = \Omega^- + \Omega^+ - n.
$$

We need to show $\Omega^- + \Omega^+ \geq n$. Note that by (5), $\Omega^- \geq 0$ and $\Omega^+ \geq 0$ and thus $\Omega^- + \Omega^+ \geq 0$. Since $|X^\pm \cup X^+| = |S^\pm \cup S^+|$, $|X^\pm \cup X^-| = |S^\pm \cup S^-|$ and $|S^\pm| - |X^\pm| = n > 0$, we must have at least $n$ elements in each of the sets $X^+ \cap (S^\pm)^C = X^+ \cap (S^\pm \cup S^0)$ and $X^- \cap (S^-)^C = X^- \cap (S^\pm \cup S^0)$. Let $\{x_1, \ldots, x_n\} \subset X^+ \cap (S^\pm)^C$ and $\{y_1, \ldots, y_n\} \subset X^- \cap (S^-)^C$. Our technique will be to go through all the pairings $(x_i, y_i)$ and use the strict inequalities in (8a)–(8d) to show $\Omega^- + \Omega^+ \geq n$. We have
the following four cases:

\[ x_i \in X^+ \cap S^\pm \quad \text{and} \quad y_i \in X^- \cap S^\pm, \quad (11a) \]
\[ x_i \in X^+ \cap S^\pm \quad \text{and} \quad y_i \in X^- \cap (S^+ \cup S^0), \quad (11b) \]
\[ x_i \in X^+ \cap (S^- \cup S^0) \quad \text{and} \quad y_i \in X^- \cap S^\pm, \quad (11c) \]
\[ x_i \in X^+ \cap (S^- \cup S^0) \quad \text{and} \quad y_i \in X^- \cap (S^+ \cup S^0). \quad (11d) \]

We break case (11d) into 4 sub-cases as follows:

\[ x_i \in X^+ \cap S^0 \quad \text{and} \quad y_i \in X^- \cap S^0, \quad (12a) \]
\[ x_i \in X^+ \cap S^- \quad \text{and} \quad y_i \in X^- \cap S^0, \quad (12b) \]
\[ x_i \in X^+ \cap S^0 \quad \text{and} \quad y_i \in X^- \cap S^+, \quad (12c) \]
\[ x_i \in X^+ \cap S^- \quad \text{and} \quad y_i \in X^- \cap S^+. \quad (12d) \]

All cases that we must deal with include (11a)–(11c) and (12a)–(12d). Cases (11a) and (12a) must
be dealt with differently, since in contrast to the other cases they give one term in \( \Omega^- \) and another
term in \( \Omega^+ \). First, if there is an \((x_i, y_i)\) and \((x_j, y_j)\) satisfying (11a) and (12a), respectively, then we can switch
the pairings to have \((x_i, y_j)\) and \((x_j, y_i)\) satisfying (11b) and (11c), respectively. After
such a re-pairing, we will have no pairings that satisfy (11a) and/or no pairings that satisfy (12a). So suppose there are
\( m > 0 \) pairs \((x_i, y_i)\) that satisfy (11a) and none that satisfy (12a) (the opposite
case can be proved similarly). We will define a sequence of partitions \( S = S_n, S_{n-1}, \ldots, S_{n-m} \) such
that \( |S_i^+ \cup S_i^+| = k \) and \( |S_i^\pm \cup S_i^-| = l \) with \( |S_{i-1}^\pm| = |S_i^\pm| - 1 \). Using a telescoping sum, we have

\[ \sigma(S) - \sigma(\mathcal{X}) = \sum_{i=0}^{m-1} [\sigma(S_{n-i}) - \sigma(S_{n-i-1})] + \sigma(S_{n-m}) - \sigma(\mathcal{X}). \]

We will construct the sequence of partitions such that \( \sigma(S_{n-i}) - \sigma(S_{n-i-1}) \geq 0 \) and thus

\[ \sigma(S) - \sigma(\mathcal{X}) \geq \sigma(S_{n-m}) - \sigma(\mathcal{X}), \]

thereby removing the problem cases of (11a) and (12a). If we can then show \( \sigma(S_{n-m}) - \sigma(\mathcal{X}) \geq 0 \), then
we'll be done.

For simplicity let \( x = x_i \) and \( y = y_i \). Also suppose for simplicity we are at \( S = S_n \). Similar to
above, we know \( S^0 \cap (X^0)^C = S^0 \cap (X^\pm \cup X^+ \cup X^-) \neq \emptyset \). We have three more cases for an element
\( z \in S^0 \cap (X^0)^C \), with each case and the corresponding definition of \( S_{n-1} \) defined below:

\[ z \in X^+ \cap S^0 \implies S_{n-1} = \{S^\pm - y, S^+ + z, S^- + y, S^0 - z\}, \quad (13a) \]
\[ z \in X^- \cap S^0 \implies S_{n-1} = \{S^\pm - x, S^+ + x, S^- + z, S^0 - z\}, \quad (13b) \]
\[ z \in X^\pm \cap S^0 \implies S_{n-1} = \{S^\pm - y + z, S^+ + x, S^- + y, S^0 - z\}. \quad (13c) \]

If in (10a) and (10b) we replace \( \mathcal{X} \) with \( S_{n-1} \), then for each of the cases (13a)–(13c) we have

Case (13a) : \( \sigma(S) - \sigma(S_{n-1}) = d^+_z - d^-_y - 1 \geq 0 \) by (8a),
Case (13b) : \( \sigma(S) - \sigma(S_{n-1}) = d^-_z - d^+_x - 1 \geq 0 \) by (8b),
Case (13c) : \( \sigma(S) - \sigma(S_{n-1}) = (d^-_z - d^+_x) + (d^+_z - d^-_y) - 1 \geq 0 \) by (8c).

This argument applies recursively so that \( \sigma(S_{n-i}) - \sigma(S_{n-i-1}) \geq 0 \), which gives \( \sigma(S) - \sigma(\mathcal{X}) \geq \sigma(S_{n-m}) - \sigma(\mathcal{X}) \).
For further simplicity, we will now drop the reference to \( m \) and assume that \(|S^\pm| - |X^\pm| = n\) and all pairings \((x_i, y_i)\) are in cases (11b), (11c), and (12b)–(12d). Thus, if we define the sets of pairings

\[
P^- = \{(x_i, y_i) \mid (x_i, y_i) \text{ is in case (11b), (12b), or (12d)}\},
\]

\[
P^+ = \{(x_i, y_i) \mid (x_i, y_i) \text{ is in case (11c), (12c), or (12d)}\},
\]

then we have

\[
\sigma(S) - \sigma(X) = \Omega^- + \Omega^+ - n
\]

\[
\geq \sum_{(x_i, y_i) \in P^-} (d^-_{y_i} - d^-_{x_i}) + \sum_{(x_i, y_i) \in P^+} (d^+_{x_i} - d^+_{y_i}) - n
\]

\[
\geq 0,
\]

with the last line following from (8a) and (8b). The proof is now complete. □

We immediately have the following corollary, which gives the first degree sequence characterization of split digraphs and is analogous to Theorem 1.4 for split graphs.

**Corollary 2.13** \( d \) is split if and only if there exists \((k, l) \in \{(0, N), (N, 0)\}\) such that \(\Sigma_{kl} = 0\).

The next section discusses the relationship between the splittance of a directed graph and graphicality. The section’s main result is Theorem 2.19 which is an extension of Theorem 1.7 to split digraphs.

### 2.2 Directed splittance and graphicality

Similar to Theorem 1.6, the next theorem by Fulkerson gives necessary and sufficient conditions for an integer-pair sequence to be digraphic.

**Theorem 2.14 (Fulkerson [4])** An integer-pair sequence \( d \) is digraphic if and only if \(\sum_{i=1}^{N} d^+_i = \sum_{i=1}^{N} d^-_i\) and for \( k = 1, \ldots, N \),

\[
\sum_{i=1}^{k} \min[\bar{d}^-_i, k - 1] + \sum_{i=k+1}^{N} \min[\bar{d}^-_i, k] \geq \sum_{i=1}^{k} \bar{d}^+_i
\]

(and)

\[
\sum_{i=1}^{k} \min[\bar{d}^+_i, k - 1] + \sum_{i=k+1}^{N} \min[\bar{d}^+_i, k] \geq \sum_{i=1}^{k} d^-_i.
\]

Thus, for an integer-pair sequence \( d \), we can define the slack sequences \( \{\bar{s}_k\}_{k=0}^{N} \) and \( \{s_k\}_{k=0}^{N} \) by

\[
\bar{s}_k = \sum_{i=1}^{k} \min[\bar{d}^-_i, k - 1] + \sum_{i=k+1}^{N} \min[\bar{d}^-_i, k] - \sum_{i=1}^{k} \bar{d}^+_i,
\]

\[
s_k = \sum_{i=1}^{k} \min[\bar{d}^+_i, k - 1] + \sum_{i=k+1}^{N} \min[\bar{d}^+_i, k] - \sum_{i=1}^{k} d^-_i.
\]

Note that \( \bar{s}_0 = s_0 = 0 \) and \( \bar{s}_N = s_N = 0 \).
The main theorem of this section is Theorem 2.19 which shows

$$\sigma(\bar{G}) = \min \{s_1, \ldots, s_{N-1}, \bar{s}_1, \ldots, \bar{s}_{N-1}\}.$$  \hspace{1cm} (14)

Thus, $\bar{G}$ is split if and only if $\min \{s_1, \ldots, s_{N-1}, \bar{s}_1, \ldots, \bar{s}_{N-1}\} = 0$. This is in direct contrast to the undirected case, where $s_n = 0$ for any $n$ is not a sufficient condition for a degree sequence to be split. For example, the degree sequence $d = (4 3 3 3 3)$ has slack sequence $s = (0 0 1 2 2 0)$ with $s_1 = 0$ and $s_m = 2$, where $m = 4$ is the corrected Durfee number for $d$. Therefore, by Theorem 1.7 $d$ is not split. However, if we consider the directed extension of $d$ given by

$$
\begin{pmatrix}
4 & 3 & 3 & 3 & 3 \\
4 & 3 & 3 & 3 & 3
\end{pmatrix}
$$

(15)

with slack sequences $\bar{s} = \bar{s} = s$, then a digraph $\bar{G}$ with the degree sequence (15) is split since $\min \{s_1, \ldots, s_4, \bar{s}_1, \ldots, \bar{s}_4\} = \min \{0, 1, 2, 2\} = 0$.

The splittance sequence for $d$ is $\sigma(d) = (8 \ 4 \ 2 \ 1 \ 1 \ 2)$, with the splittance matrix of (15) given by

$$
\Sigma = \begin{pmatrix}
16 & 12 & 9 & 6 & 3 & 0 \\
12 & 8 & 6 & 4 & 2 & 0 \\
9 & 6 & 4 & 3 & 2 & 1 \\
6 & 4 & 3 & 2 & 2 & 2 \\
3 & 2 & 2 & 2 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 & 4
\end{pmatrix}.
$$

It is easily seen that $\Sigma$ is symmetric and $\sigma(d) = \frac{1}{2} \text{diag}(\Sigma)$. Also, both degree sequences (15) and (7) have the property that

$$
\bar{s}_k = \min_l \Sigma_{kl} \quad \text{and} \quad \bar{s}_l = \min_k \Sigma_{kl}, \hspace{1cm} (16)
$$

which is proved in Corollary 2.17 and Lemma 2.18. Thus, the slack sequences are embedded in the splittance matrix. The following definition defines maximal sequences which we show give the precise locations in the splittance matrix where (16) is satisfied.

**Definition 2.15** Define the maximal sequences $\{m_l\}^N_{l=0}$ and $\{m_k\}^N_{k=0}$ such that

$$
m_l = \max \{i \mid d_i^+ \geq l - 1 \text{ and if } d_i^+ = l - 1, \text{ then } \bar{\pi}(i) \in B_l\},$$

$$
m_k = \max \{j \mid d_j^- \geq k - 1 \text{ and if } d_j^- = k - 1, \text{ then } \bar{\pi}(j) \in A_k\}.
$$

The maximal sequences $\{m_l\}^N_{l=0}$ and $\{m_k\}^N_{k=0}$ play a similar role to the corrected Durfee number $m$ in the undirected case, as illustrated in the following lemma.

**Lemma 2.16** For $k$ fixed ($l$ fixed), $\Sigma_{kl}$ is non-increasing for $0 \leq l \leq m_k$ ($0 \leq k \leq m_l$) and strictly increasing for $m_k < l \leq N$ ($m_l < k \leq N$).

**Proof** We will prove the assertion for a fixed row $k$. The proof for a fixed column $l$ is analogous. Let $k \geq 0$ and $l \geq 1$. We will keep track of the induced partitions $X_{kl} = \{X^+_l, X^-_l, X^-_{l-1}, X^+_0\}$ and $X_{k,l-1} = \{X^+_l, X^+_l, X^-_{l-1}, X^-_{l-1}, X^+_0\}$. We have

$$
\begin{align*}
\Sigma_{kl} - \Sigma_{k,l-1} &= kl - |X^+_l| + \sum_{B_l^+} d_i^- - \sum_{A_k} d_i^+ \\
&\quad - k(l - 1) + |X^-_{l-1}| - \sum_{B^-_{l-1}} d_i^- + \sum_{A_k} d_i^+ \\
&= k - (|X^+_l| - |X^-_{l-1}| - d^-_{\Sigma(l)}).
\end{align*}
$$
Letting \( n = \pi(l) \), we have two cases on \( x_n \). In the first case, \( x_n \in X^+_{l-1} \cap X^-_{l-1} \), which gives \( |X^+_l| - |X^-_{l-1}| = 1 \). Thus, if \( l \leq m_k \), we have

\[
\Sigma_{kl} - \Sigma_{k,l-1} = k - 1 - d^-_n \leq 0.
\]

If \( l > m_k \), then since \( x_n \in X^+_l \), \( n \in A_k \), we must have \( d^+_n < k - 1 \) and thus \( \Sigma_{kl} - \Sigma_{k,l-1} > 0 \).

In the second case, \( n \in X^0_{l-1} \cap X^-_l \), which gives \( |X^+_l| - |X^-_{l-1}| = 0 \). Thus, if \( l \leq m_k \), we have

\[
\Sigma_{kl} - \Sigma_{k,l-1} = -d^-_n \leq 0,
\]

since \( l \leq m_k \) and \( n \notin A_k \) implies \( d^-_n \geq k \). If \( l > m_k \), then \( d^-_n \leq k - 1 < k \), and thus \( \Sigma_{kl} - \Sigma_{k,l-1} > 0 \).

\( \Box \)

The following corollary is immediate.

**Corollary 2.17**

\[
\min_l \Sigma_{kl} = \Sigma_{km_k} \quad \text{and} \quad \min_k \Sigma_{kl} = \Sigma_{m_l l}
\]

Finally, we have (16) by combining Corollary 2.17 with the following lemma.

**Lemma 2.18**

\[
\bar{s}_k = \Sigma_{km_k} \quad \text{and} \quad \bar{s}_l = \Sigma_{m_l l}
\]

**Proof** We will show \( \bar{s}_k = \Sigma_{km_k} \), with the other equality proven in an analogous way. The equality is trivial for \( k = 0 \) and \( k = N \), so suppose \( 1 \leq k \leq N - 1 \). Let \( l = m_k \) and let \( X_{kl} = \{X^+, X^+, X^-, X^0\} \) be the partition induced by \((k, l)\). By the definition of \( m_k \) we have \( d^-_x \geq k - 1 \) for \( x \in X^+ \), \( d^-_x \geq k \) for \( x \in X^- \), and \( d^-_x \leq k - 1 \) for \( x \in X^+ \cup X^0 \). Thus

\[
\bar{s}_k = \sum_{i=1}^k \min_i d^-_x, k - 1 + \sum_{i=k+1}^N \min_i d^-_x, k - 1 - \sum_{i=1}^k d^+_x
\]

\[
= \sum_{X^+ \cup X^-} \min_{X^+} d^-_x, k - 1 + \sum_{X^+ \cup X^-} \min_{X^-} d^-_x, k - \sum_{X^+ \cup X^-} d^+_x
\]

\[
= |X^+| (k - 1) + |X^-| k + \sum_{X^+ \cup X^-} d^-_x - \sum_{X^+ \cup X^-} d^+_x
\]

\[
= \Sigma_{km_k}.
\]

\( \Box \)

Combining Theorem 2.12 with (16) nearly gives (14), since we have

\[
\min \{ \bar{s}_1, \ldots, \bar{s}_{N-1}, \xi_1, \ldots, \xi_{N-1} \} = \min_{(k, l) \in C} \Sigma_{kl},
\]

where \( C = [0, N] \times [0, N] \setminus \{(0, 0), (0, N), (N, 0), (N, N)\} \). Since \( \sigma(\bar{G}) = \sum_{(k, l) \notin \{0, N\} \times \{0, N\}} \Sigma_{kl} \), we will address the remaining two cases \((k, l) = (0, 0)\) or \((N, N)\) in the following theorem.

**Theorem 2.19**

\[
\sigma(\bar{G}) = \min \{ \bar{s}_1, \ldots, \bar{s}_{N-1}, \xi_1, \ldots, \xi_{N-1} \}.
\]

**Proof** If we have \((N, N) \in \arg \min_{(k, l)} \Sigma_{kl} \), then \( \bar{m}_l = m_k = N \) and we have \( d^+_N \geq N - 1 \) and \( d^-_N \geq N - 1 \), which implies \( (d^+_i, d^-_i) = (N - 1, N - 1) \) for all indices \( i \). But this means \( \bar{G} \) is a complete directed graph, and thus \( V = X^\pm \) is a split partition. Thus, \( S = \{S^+, S^0\} \), where
$S^\pm = X^\pm - \{x\}$ and $S^0 = \{x\}$ for any $x \in V$, is also a split partition, showing by Corollary 2.7 that $
abla_{N-1,N-1} \leq \sigma(S) = 0$.

For $(0,0) \in \arg \min_{(k,l)} \Sigma_{kl}$, Lemma 2.16 implies $\{(N-1,0),(0,N-1)\} \in \arg \min_{(k,l)} \Sigma_{kl}$ as well. Thus, by Theorem 2.12 and (17), we have

$$\sigma(\vec{G}) = \min_{(k,l) \notin \{(0,N),(N,0)\}} \Sigma_{kl}$$

$$= \min\{\bar{s}_1, \ldots, \bar{s}_{N-1}, \bar{s}_1, \ldots, \bar{s}_{N-1}\}$$

This leads to the second degree sequence characterization of split digraphs using slack sequences.

**Corollary 2.20** $d$ is split if and only if $\min\{\bar{s}_1, \ldots, \bar{s}_{N-1}, \bar{s}_1, \ldots, \bar{s}_{N-1}\} = 0$.

### 3 Conclusion

In this paper we have defined the class of split digraphs as a generalization of the class of split graphs and showed they have two analogous degree sequence characterizations: one in Corollary 2.13 via the concept of splittance of a digraph and the other in Corollary 2.20 in terms of the slack sequences.

Split degree sequences and their place near the top of the partially ordered set of graphic degree sequences was explored in [6]. Corollary 2.20 implies these concepts may be generalized to the directed case for degree sequences of split digraphs, with an interesting exploration into the idea of a *threshold digraph* whose degree sequences are at the boundary of graphicality.

Split graphs were also used in [7] to create a canonical decomposition of graphs into a sequence of indecomposable split graphs and one indecomposable non-split graph. A canonical decomposition of digraphs using split digraphs would be an interesting avenue of further research, with a possible application to a new characterization of unidigraphs.

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