Structure of Kernels and Cokernels of Toeplitz plus Hankel Operators

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Abstract

Toeplitz plus Hankel operators $T(a) + H(b)$, $a, b \in L^\infty$ acting on the classical Hardy spaces $H^p$, $1 < p < \infty$, are studied. If the generating functions $a$ and $b$ satisfy the so-called matching condition

$$a(t)a(1/t) = b(t)b(1/t),$$

an effective description of the structure of the kernel and cokernel of the corresponding operator is given. The results depend on the behaviour of two auxiliary scalar Toeplitz operators, and if the generating functions $a$ and $b$ are piecewise continuous, more detailed results are obtained.

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1 Introduction

Let \( X \) be a Banach space, and let \( \mathcal{L}(X) \) be the Banach algebra of all linear continuous operators on \( X \). An operator \( A \in \mathcal{L}(X) \) is called Fredholm if the range \( \text{im} A := \{ Ax : x \in X \} \) of the operator \( A \) is a closed subset of \( X \) and the null spaces \( \ker A := \{ x \in X : Ax = 0 \} \) and \( \ker A^* := \{ h \in X^* : A^* h = 0 \} \) of the operator \( A \) and the adjoint operator \( A^* \) are finite-dimensional. For the sake of convenience, the null space of the adjoint operator \( A^* \) is called the cokernel of \( A \) and it is denoted by \( \text{coker} A \). Further, if an operator \( A \in \mathcal{L}(X) \) is Fredholm, then the number

\[ \kappa := \dim \ker A - \dim \text{coker} A, \]

where \( \dim Y \) denotes the dimension of the linear space \( Y \), is referred to as the index of the operator \( A \).

The present paper is devoted to Toeplitz plus Hankel operators acting on the classical Hardy spaces. Note that the theory of Toeplitz and Hankel operators has a long and interesting history and is distinguished by exiting results and rich connections with many fields of mathematics, physics, statistical mechanics, and so on (see, for example, \([6, 18]\)). Despite the fact that Toeplitz operators and Hankel operators are quite different in their nature, they are closely related to each other. No wonder that Toeplitz plus Hankel operators \( T(a) + H(b) \) have attracted great attention, as well. Some particular points of interest are Fredholm properties, index, and even the invertibility. This is caused not only by theoretical research but also by interesting and challenging problems which arise in applications and which can be described in terms of the operators mentioned. For example, Wiener–Hopf plus Hankel operators occur in scattering theory \([10]\), whereas Toeplitz plus Hankel operators are used in the theory of random matrix ensembles \([1, 4]\). In the latter case, an interesting class of Toeplitz plus Hankel operators \( T(a) + H(b) \) has been considered. This class is defined by the condition

\[ a(t)a(1/t) = b(t)b(1/t), \quad t \in \mathbb{T}, \]

where \( \mathbb{T} := \{ t \in \mathbb{C} : |t| = 1 \} \) is the unit circle equipped with counterclockwise orientation.

At present, there is a well-developed Fredholm theory dealing for Toeplitz plus Hankel operators with piecewise continuous generating functions acting on various Banach and Hilbert spaces. For more detailed information the
The reader can consult [6, Sections 4.95-4.102], [19, Sections 4.5 and 5.7], [20], and [21]. The latter paper also contains a transparent index formula for the operators acting on the space $l^p(\mathbb{Z}_+)$. The ideas of that work, with a necessary adjustment, can also be used to obtain similar formulas for operators acting on Hardy spaces $H^p$. It is worth noting that contrary to scalar Toeplitz operators, Fredholm Toeplitz plus Hankel operators are not necessarily one-sided invertible. In a sense, their behaviour is similar to the behaviour of block Toeplitz operators. Let us recall that the kernel and cokernel dimension of a block Toeplitz operator acting on $H^p$ spaces can be expressed via partial indices of the Wiener–Hopf factorization of the corresponding generating matrix. However, as a matter of fact, for an arbitrary matrix-function there is no efficient procedure in order to obtain its Wiener–Hopf factorization. On the other hand, for Toeplitz plus Hankel operators acting on the Hardy spaces $H^p$, there is a general result [9, 10] stating that $T(a) + H(b) : H^p \to H^p$, $a, b \in L^\infty$ is Fredholm if and only if the matrix-function

$$V(a, b) := \begin{pmatrix} a - \frac{\tilde{b} \tilde{a}^{-1}}{-\tilde{b} \tilde{a}^{-1}} & \frac{\tilde{b} \tilde{a}^{-1}}{\tilde{a}^{-1}} \\ -\frac{\tilde{b} \tilde{a}^{-1}}{-\tilde{b} \tilde{a}^{-1}} & \frac{\tilde{a}^{-1}}{-\tilde{b} \tilde{a}^{-1}} \end{pmatrix},$$

where $\tilde{a} = a(1/t), \tilde{b} = b(1/t)$, admits a certain type of antisymmetric factorization. Moreover, the defect numbers $\dim \ker(T(a) + H(b))$ and $\dim \text{coker}(T(a) + H(b))$ of the operator $T(a) + H(b)$ can be expressed via partial indices of the antisymmetric factorization of the matrix $V(a, b)$. Nevertheless, for arbitrary functions $a, b \in L^\infty$ the chances of finding such a factorization and the corresponding partial indices are very slight. However, if the generating functions of the operator $T(a) + H(b)$ satisfy the additional condition (1.1), the matrix $V(a, b)$ takes the form

$$V(a, b) = \begin{pmatrix} 0 & d \\ -c & \frac{d}{\tilde{a}^{-1}} \end{pmatrix},$$

where $c := ab^{-1} = \tilde{b} \tilde{a}^{-1}$ and $d := b \tilde{a}^{-1} = \tilde{a} \tilde{b}$. Thus $V(a, b)$ becomes a triangular matrix, which is better suited for factorization theory. These ideas have been used while considering Toeplitz plus Hankel operators of the form $T(a) + H(at^{-1})$ (see [9]). On the other hand, the study of the operators $T(a) + H(a)$ in [24, 23] does not involve the factorization of the matrix $V(a, a)$. Recently, a new method to investigate the operators $T(a) + H(b) : H^p \to H^p$ with piecewise continuous generating functions $a$ and $b$ satisfying condition...
has been proposed [5]. This method is based on the antisymmetric factorization of the scalar functions $c$ and $d^{-1}$ and it leads to a complete description of Fredholm properties of the operators under consideration, including the computation of the corresponding defect numbers. In particular, it is shown that the operator $T(a) + H(b)$ is Fredholm if and only if the functions $c$ and $d^{-1}$ can be represented in a special form. Similar problems have been studied in [8], but the approach of [8] differs from [5] and does not employ any factorization theory in the defect number computation. Still, very often it is not enough to have an information about Fredholmness and index but more specific results concerning the kernel and cokernel of the corresponding operator are required. Thus one of the aims of the present work is to obtain an effective description of the kernels and cokernels of the operators in question.

In view of this, let us mention paper [13] which is the culmination of the development aimed at Fredholmness and which contains a description of kernels and cokernels of singular integral operators with some Carleman backward shifts. Although the shifts in [13] are slightly different from that appearing in Hankel operators, the approach of [13] is also based on Wiener–Hopf factorization of $2 \times 2$ matrix functions. It is also worth noting that the corresponding factorization is assumed to have factorization factors with entries from $L^\infty$. Such an assumption implies that the corresponding operator is Fredholm on any space $L^p$, $1 < p < \infty$, if and only if it is Fredholm on one single space, say on $L^2$. Besides, it turns out that most piecewise continuous generating functions are not covered by this method.

The approach used in the present paper is completely different from both [5] and [13]. For example, we try to avoid Wiener–Hopf factorization techniques as long as possible. Thus in Section 3 assuming only the right invertibility of the operator $T(c), c = ab^{-1}$ we give a description of the kernels of the operators $T(a) \pm H(b)$ for $a, b \in L^\infty$. Moreover, we show which parts of the kernels of the operators $T(c), c = ab^{-1}$ and $T(d), d = a\tilde{b}^{-1}$ make a contribution to the kernels of the operators $T(a) + H(b)$ and $T(a) - H(b)$. More precisely, there are decompositions of the kernels of the operators $T(c)$ and $T(d)$ describing these parts. It is remarkable that Fredholmness of the operator $T(a)$ plays no role in our considerations. In Section 4 some operators of the form $T(a) \pm H(t^k a), a \in L^\infty, k = -1, 0, 1$ are considered. It is shown that for such operators a version of Coburn–Simonenko theorem holds. In particular, Fredholmness of these operators implies their one-sided
invertibility. Note that the proof of these results does not use any factorization arguments. Therefore, such an approach can also be used for similar Toeplitz plus Hankel operators with discontinuous generating functions acting on $l^p$-spaces, where factorization technique is not available. However, to tell the truth, in Sections 5 and 6 we use factorizations of certain scalar functions. But these factorizations are only used in order to obtain bases for the subspaces participating in the decompositions mentioned. Such an approach gives a complete description for the structure of the null spaces of the operators $T(a) \pm H(b)$ and allows us to determine their dimensions. For this purpose, we introduce a characteristic of the factorization for functions $g \in L^\infty$ satisfying the condition $g\tilde{g} = 1$ and such that $T(g)$ is Fredholm. It is called factorization signature and takes values $+1$ and $-1$. Similar parameters occur in literature from time to time, but it seems that their importance is not truly appreciated so far. In Section 5 we also provide sufficient conditions which make possible an easy computation of the factorization signature. For piecewise continuous generating functions, a very simple formula for the factorization signature is established in Section 8. Section 6 is devoted to the description of the kernels and cokernels of the operators $T(a) + H(b)$ and $T(a) - H(b)$, $a, b \in L^\infty$ in the case where operators $T(c)$ and $T(d)$ are Fredholm. The latter condition implies Fredholmness of the operators $T(a) \pm H(b)$ and allows one to get their indices. This section also provides some results for the operators $T(a) \pm H(b)$ under additional assumption that $\text{ind } T(c) = \pm 1$ and the factorization signature is equal to one. This generalizes the corresponding results of Section 4. Section 7 is specified to the case of piecewise continuous generating functions.

The present paper has some intersection with the recent paper [5] but our approach is entirely different and leads to more general and more detailed results with an additional advantage that the kernel and cokernel of Toeplitz plus Hankel operators are completely described.

## 2 Spaces and operators

Let us introduce some operators and spaces we need. As usual, let $L^\infty(\mathbb{T})$ stand for the $C^*$-algebra of all essentially bounded Lebesgue measurable functions on $\mathbb{T}$, and let $L^p = L^p(\mathbb{T})$, $1 \leq p \leq \infty$ denote the Banach space of all
Lebesgue measurable functions \( f \) such that
\[
||f||_p := \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty,
\]
\[
||f||_\infty := \operatorname{ess sup}_{t \in \mathbb{T}} |f(t)|,
\]
is finite. Further, let \( H^p = H^p(\mathbb{T}) \) and \( \overline{H^p} = \overline{H^p(\mathbb{T})} \) refer to the Hardy spaces of all functions \( f \in L^p \) the Fourier coefficients
\[
\hat{f}_n := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})e^{-in\theta} \, d\theta
\]
of which vanish for all \( n < 0 \) and \( n > 0 \), respectively. It is a classical result that for \( p \in (1, \infty) \) the Riesz projection \( P \) defined by
\[
P : \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} \hat{f}_n e^{in\theta},
\]
is bounded on the space \( L^p \) and its range is the whole space \( H^p \). The operator \( Q := I - P \),
\[
Q : \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta} \mapsto \sum_{n=-\infty}^{-1} \hat{f}_n e^{in\theta}
\]
is also a projection and its range is a subspace of the codimension one in \( \overline{H^p} \).

We also consider the flip operator \( J : L^p \mapsto L^p \),
\[
(Jf)(t) := \overline{f(t)}, \quad t \in \mathbb{T},
\]
where the bar denotes the complex conjugation. Note that the operator \( J \) changes the orientation, satisfies the relations
\[
J^2 = I, \quad JPJ = Q, \quad JQJ = P,
\]
and for any \( a \in L^\infty \),
\[
JaJ = \overline{a}I.
\]
Further, for any \( a \in L^\infty \) consider an operator \( T(a) : H^p \mapsto H^p, 1 < p < \infty \) defined by
\[
T(a) : f \mapsto Pf.
\]
The operator $T(a)$ is obviously bounded and

$$||T(a)|| \leq c_p||a||_{\infty},$$

where $c_p$ is the norm of the Riesz projection on $L^p$. This operator is called Toeplitz operator generated by the function $a$. Toeplitz operators with matrix-valued generating functions acting on $H^p \times H^p$ are defined similarly.

For $a \in L^\infty$, the Hankel operator $H(a) : H^p \mapsto H^p$, $1 < p < \infty$ is defined by

$$H(a) : f \mapsto PaQJf.$$  

It is clear that this operator is also bounded, that is

$$||H(a)|| \leq c_p||a||_{\infty},$$

with the same constant $c_p$ as in (2.1). However, in contrast to Toeplitz operators, the corresponding generating function $a$ is not uniquely defined by the operator itself. Further, if $a$ belongs to the space of all continuous functions $C = C(\mathbb{T})$, then the Hankel operator $H(a)$ is compact on the space $H^p$. Moreover, Hankel operators are never Fredholm, whereas if a Toeplitz operator $T(a) : H^p \to H^p$ is Fredholm, then $T(a)$ is one-sided invertible.

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Recall that in the natural basis $\{t^n\}_{n \in \mathbb{Z}_+}$ of the space $H^p$, $1 < p < \infty$, Toeplitz and Hankel operators with generating function $a \in L^\infty$ can be, respectively, represented as infinite matrices $(\hat{a}_{k-j})_{k,j=0}^{\infty}$ and $(\hat{a}_{k+j+1})_{k,j=0}^{\infty}$, where $\hat{a}_k$ is the $k$-th Fourier coefficient of the function $a$. In this form, Toeplitz and Hankel operators appear on the spaces $l^p(\mathbb{Z}_+)$ (see [6, Section 2.3]). However, the study of Toeplitz plus Hankel operators on the space $l^p(\mathbb{Z}_+)$ is much more difficult since it is connected with the multiplier problem.

Let us also recall some other results concerning Toeplitz plus Hankel operators. Thus, if $T(a) + H(b) \in \mathcal{L}(H^p)$, then the adjoint operator acts on the space $H^q$, $p^{-1} + q^{-1} = 1$ and

$$\langle T(a) + H(b) \rangle^* = T(\overline{a}) + H(\overline{b}).$$

(2.2)

Suppose now that $a$ belongs to the group $GL^\infty$ of invertible elements from $L^\infty$ and

$$\hat{a}\hat{a} = \hat{b}\hat{b}.$$  

(2.3)

This relation plays an important role in what follows. It is called the matching condition, and if $a$ and $b$ satisfy (2.3), then the duo $(a, b)$ is called the
matching pair. For each matching pair \((a, b)\) one can assign another matching pair \((c, d)\), where \(c = ab^{-1}\) and \(d = b\tilde{a}^{-1}\). Such a pair \((c, d)\) is called the subordinated pair for \((a, b)\), and it is easily seen that the functions which constitutes a subordinated pair have a specific property, namely \(c\tilde{c} = 1 = d\tilde{d}\).

In passing note that these functions \(c\) and \(d\) can also be expressed in the form

\[c = \tilde{b}a^{-1}, \quad d = \tilde{b}^{-1}a.\]

Besides, if \((c, d)\) is the subordinated pair for a matching pair \((a, b)\), then \((\tilde{d}, \tilde{c})\) is the subordinated pair for the matching pair \((\tilde{a}, \tilde{b})\) defining the adjoint operator (see (2.2)). Further, a matching pair \((a, b)\) is called Fredholm, if the Toeplitz operators \(T(c)\) and \(T(d)\) are Fredholm.

In the following, any function \(g \in L^\infty\) satisfying the condition

\[g\tilde{g} = 1\]

is called matching function.

**Lemma 2.1** If \(a, b \in L^\infty\), then the following relations hold:

(i) If \((a, b)\) is a matching pair with the subordinated pair \((c, d)\), then \((at^{-n}, bt^n)\) is a matching pair with the subordinated pair \((ct^{-2n}, d)\).

(ii) If \(n \in \mathbb{N}\), then

\[T(a) + H(b) = (T(at^{-n}) + H(bt^n))T(t^n).\] (2.4)

**Proof.** Assertion (i) can be verified straightforward, and equation (2.4) is a consequence of the well–known identities

\[T(a_1a_2) = T(a_1)T(a_2) + H(a_1)H(\tilde{a}_2),\]
\[H(a_1a_2) = T(a_1)H(a_2) + H(a_1)T(\tilde{a}_2),\]

and the relation \(H(t^n)T(t^n) = 0\).

Recall that \(a\) is assumed to be invertible in \(L^\infty\) and let us point out that this is always the case when the operator \(T(a) + H(b)\) is Fredholm [5, 10]. On the other hand, the invertibility of \(a \in L^\infty\) does not automatically implies the Fredholmness either of the operators \(T(a) + H(b)\) or \(T(a)\). Nevertheless, Fredholm properties of Toeplitz operators \(T(a)\) can be described by using the Wiener–Hopf factorization of the generating function \(a\). Assume that \(p > 1, q > 1\) are real numbers such that \(p^{-1} + q^{-1} = 1\).

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Definition 2.2 We say that a function \( a \in L^\infty \) admits a weak Wiener–Hopf factorization in \( H^p \), if it can be represented in the form

\[
a = a_- t^n a_+,
\]

where \( n \in \mathbb{Z} \), \( a_+ \in H^q \), \( a_-^* \in H^p \), \( a_- \in \overline{H^p} \), and \( a_-(\infty) = 1 \).

It is well-known that the weak Wiener–Hopf factorization of a function \( a \) is unique, if it exists. The functions \( a_- \) and \( a_+ \) are called the factorization factors, and the number \( n \) is the factorization index. If \( a \in L^\infty \) and the operator \( T(a) \) is Fredholm, then the function \( a \) admits the weak Wiener–Hopf factorization with \( n = -\text{ind} T(a) \) [6, 15]. Moreover, in this case, the factorization factors possess an additional property—viz. the linear operator \( a_+^{-1} Pa_-^{-1} I \) defined on \( \text{span}\{t^k : k \in \mathbb{Z}_+\} \) can be boundedly extended on the whole space \( H^p \). Throughout this paper, such a kind of weak Wiener–Hopf factorization in \( H^p \) is called simply Wiener–Hopf factorization in \( H^p \). Note that if \( h \) is a polynomial, then the element \( Pa_-^{-1} h \) is also a polynomial. The following result is well–known.

Theorem 2.3 (see [6]) If \( a \in L^\infty \), then Toeplitz operator \( T(a) : H^p \to H^p \), \( 1 < p < \infty \) is Fredholm and \( \text{ind} T(a) = -n \) if and only if the generating function \( a \) admits the Wiener–Hopf factorization (2.5) in \( H^p \).

Recall that Fredholmness of a Toeplitz operator depends on the space where this operator acts. The reader can consult [6, 15] for more details. Another important result is given by Coburn–Simonenko theorem stating that if \( a \) is the non-zero element in \( L^\infty \), then the kernel or cokernel of the operator \( T(a) \) is trivial. Block Toeplitz operators does not possess such a property and this causes serious difficulties. For Toeplitz plus Hankel operators situation is similar to block Toeplitz operators. Therefore, the determination of the numbers \( \dim \ker (T(a) + H(b)) \) and \( \dim \text{coker} (T(a) + H(b)) \) becomes a challenging problem, and one of the aims of this work is to find them for some classes of the operators \( T(a) + H(b) \). In conclusion of this section, let us recall that one-sided inverses of a Fredholm scalar Toeplitz operator \( T(a) \) can be effectively derived. Let \( n \) be the factorization index. If \( n \geq 0 \), then \( T(a) \) is left–invertible and the operator \( T(t^{-n}) T^{-1}(a_0) \), where \( a_0 := at^{-n} \), is one of the left–inverses for \( T(a) \). On the other hand, if \( n \leq 0 \), then \( T(a) \) is right–invertible. For the sake of convenience, in this paper the notation \( T_r^{-1}(a) \) always means the operator \( T^{-1}(a_0) T(t^{-n}) \), which is one of right inverses for the operator \( T(a) \).
Besides, for $n > 0$ the kernel of the operator $T(t^{-n})$ is the linear span of the monomials $1, t, \cdots, t^{n-1}$, i.e. $\ker T(t^{-n}) = \text{span} \{1, t, \cdots, t^{n-1}\}$.

If $T(a)$ is right–invertible and $\dim \ker T(a) = \infty$, then $T_{r}^{-1}(a)$ denotes one of right inverses of $T(a)$.

3 Kernels of Toeplitz plus Hankel operators

Let $a, b \in L^\infty$. On the space $H^p$, $1 < p < \infty$ consider Toeplitz plus Hankel operators $T(a) + H(b)$ and $T(a) - H(b)$. As was already mentioned, in the present paper, the kernel spaces of these operators are studied under the additional condition (2.3), connecting the generating functions $a$ and $b$. This condition presents a unique possibility to obtain an effective description for the kernels of the operators $T(a) \pm H(b)$. Nevertheless, let us start with an auxiliary result for Toeplitz plus Hankel operators with arbitrary generating functions $a, b \in L^\infty$. By $V = V(a, b)$ we denote the matrix

$$V(a, b) := \begin{pmatrix} a - b\tilde{b}a^{-1} & d \\ c & \tilde{a}^{-1} \end{pmatrix},$$

where $c := \tilde{b}a^{-1}$, $d := b\tilde{a}^{-1}$. As usual, it is assumed that the element $a \in L^\infty(T)$ is invertible. In particular, this condition is satisfied if at least one of the operators $T(a) + H(b)$ or $T(a) - H(b)$ is Fredholm [5, 10].

Our first concern here is to introduce a formula which is known in principle – viz.

$$\begin{pmatrix} T(a) + H(b) + Q \\ 0 \\ T(a) - H(b) + Q \end{pmatrix} = A(T(V(a, b))) + \text{diag} (Q, Q))B,$$

where $A, B : L^p \times L^p \to L^p \times L^p$ are invertible operators,

$$B = \begin{pmatrix} I & 0 \\ \tilde{b}I & \tilde{a}I \end{pmatrix} \begin{pmatrix} I & I \\ J & -J \end{pmatrix},$$

and $A$ is also known but its concrete form is not important right now. Note that in the following we frequently denote the operator $aI$ of multiplication by the function $a \in L^\infty$ simply by $a$.

In order to establish formula (3.1), let us recall some well known operator identities and relations connecting Toeplitz plus Hankel operators with block
Toeplitz operators. Thus if $a$ belongs to a unital Banach algebra with the identity $e$ and if $p$ is an idempotent from the same algebra, i.e. $p^2 = p$, then

$$pa + q = (e + paq)(pa + q),$$
$$pap + q = (e - paq)(pa + q),$$

where $q = e - p$, and $e - paq$ is the inverse for the element $e + paq$.

On the other hand, one can write (see, for example, [7, 14])

$$\frac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix} \begin{pmatrix} T(a) + H(b) + Q & 0 \\ 0 & T(a) - H(b) + Q \end{pmatrix} \begin{pmatrix} I & J \\ J & -J \end{pmatrix}$$

$$= \text{diag} (P, Q) \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) \text{diag} (P, Q) + (\text{diag} (I, I) - \text{diag} (P, Q)).$$

(3.3)

Setting $p := \text{diag} (P, Q)$, $e = \text{diag} (I, I)$ and using (3.2) one can rewrite the relation (3.3) as

$$\frac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix} \begin{pmatrix} T(a) + H(b) + Q & 0 \\ 0 & T(a) - H(b) + Q \end{pmatrix} \begin{pmatrix} I & J \\ J & -J \end{pmatrix}$$

$$= p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) p + q$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( e + p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q \right) \left( p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) p + q \right)$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) + q \right)$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( \text{diag} (P, P) \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) + \text{diag} (Q, Q) \left( \begin{array}{cc} 1 & 0 \\ b & \tilde{a} \end{array} \right) \right)$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( \text{diag} (P, P) \left( \begin{array}{cc} a - \frac{bb\tilde{a} - 1}{-c} \frac{b\tilde{a} - 1}{\tilde{a} - 1} \\ \frac{b\tilde{a} - 1}{\tilde{a} - 1} \end{array} \right) + \text{diag} (Q, Q) \left( \begin{array}{cc} 1 & 0 \\ b & \tilde{a} \end{array} \right) \right)$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( \text{diag} (P, P) \left( a - \frac{bb\tilde{a} - 1}{-c} \frac{b\tilde{a} - 1}{\tilde{a} - 1} \\ \frac{b\tilde{a} - 1}{\tilde{a} - 1} \end{array} \right) + \text{diag} (Q, Q) \left( \begin{array}{cc} 1 & 0 \\ b & \tilde{a} \end{array} \right) \right)$$

$$= (e - p \left( \begin{array}{cc} a & b \\ \frac{a}{b} & \tilde{a} \end{array} \right) q) \left( \text{diag} (I, I) + \text{diag} (P, P) \left( a - \frac{bb\tilde{a} - 1}{-c} \frac{b\tilde{a} - 1}{\tilde{a} - 1} \right) \text{diag} (Q, Q) \right)$$
\[
\times \left( \text{diag}(P, P) \begin{pmatrix} a - \tilde{b} \tilde{a}^{-1} & d \\ -c & \tilde{a}^{-1} \end{pmatrix} \text{diag}(P, P) + \text{diag}(Q, Q) \right) \begin{pmatrix} 1 & 0 \\ \tilde{b} & \tilde{a} \end{pmatrix}.
\]

(3.4)

In the relation (3.4), all the operators connecting the operator diag \((T(a) + H(b) + Q, T(a) - H(b) + Q)\) and the operator

\[
\tilde{A} := \text{diag}(P, P) \begin{pmatrix} a - \tilde{b} \tilde{a}^{-1} & d \\ -c & \tilde{a}^{-1} \end{pmatrix} \text{diag}(P, P) + \text{diag}(Q, Q)
\]

are invertible and

\[
\begin{pmatrix} 1 & 0 \\ \tilde{b} & \tilde{a} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\tilde{b} \tilde{a}^{-1} & \tilde{a}^{-1} \end{pmatrix}.
\]

Thus the representation (3.1) is established.

**Remark 3.1** Relation (3.1) indicates that the operators diag \((T(a) + H(b), T(a) - H(b))\) and \(T(V(a, b))\) are simultaneously Fredholm. However, this conclusion is not always true for the operators \(T(a) + H(b)\) and \(T(a) - H(b)\) themselves. Examples can be already found among the operators \(I \pm H(b)\) with piecewise continuous generating functions (see, for example, [7]). Even if both operators \(T(a) + H(b)\) and \(T(a) - H(b)\) are Fredholm, they can have different indices. Thus, in general, the use of relation (3.1) in the study of Fredholm properties of Toeplitz plus Hankel operators is limited. Nevertheless, this relation is still very helpful in the investigation of the kernels of Toeplitz plus Hankel operators.

**Lemma 3.2** Assume that \(a, b \in L^\infty\), \(a \in GL^\infty\), and the operators \(T(a) \pm H(b)\) are considered on the space \(H^p\), \(1 < p < \infty\). Then

- If \((\varphi, \psi)^T \in \ker T(V(a, b))\), then
  \[
  (\Phi, \Psi)^T = (\varphi - JQc\varphi + JQ\tilde{a}^{-1}\psi, \varphi + JQc\varphi - JQ\tilde{a}^{-1}\psi)^T 
  \in \ker \text{diag}(T(a) + H(b), T(a) - H(b))
  \]
  (3.5)

- If \((\Phi, \Psi)^T \in \ker \text{diag}(T(a) + H(b), T(a) - H(b))\), then
  \[
  (\Phi + \Psi, P(\tilde{b}(\Phi + \Psi) + \tilde{a}JP(\Phi - \Psi)))^T \in \ker T(V(a, b)).
  \]
  (3.6)
Moreover, the operators

\[ E_1 : \ker T(V(a, b)) \to \ker \text{diag} \left( T(a) + H(b), T(a) - H(b) \right), \]

\[ E_2 : \ker \text{diag} \left( T(a) + H(b), T(a) - H(b) \right) \to \ker T(V(a, b)), \]

defined, respectively, by the relations (3.5) and (3.6) are mutually inverses to each other.

**Proof.** Consider the representation (3.1) and note that both operators \( A \) and \( B \) are invertible on the space \( L^p(\mathbb{R}) \times L^p(\mathbb{R}) \). Therefore, relation (3.1) implies that for any \( (\varphi, \psi)^T \in \ker T(V(a, b)) \), the element \( B^{-1}((\varphi, \psi)^T) \) belongs to the set

\[ \ker \text{diag} \left( T(a) + H(b) + Q, T(a) - H(b) + Q \right) = \ker \text{diag} \left( T(a) + H(b), T(a) - H(b) \right). \]

Hence

\[ \text{diag} \left( P, P \right) B^{-1}((\varphi, \psi)^T) = B^{-1}((\varphi, \psi)^T). \]

Computing the left-hand side of the last equation, one obtains the relation (3.5). Analogously, if \( (\Phi, \Psi)^T \in \ker \text{diag} \left( T(a) + H(b), T(a) - H(b) \right) \), then

\[ B((\Phi, \Psi)^T) \in \ker T(V(a, b)) \]

and

\[ \text{diag} \left( P, P \right) B((\Phi, \Psi)^T) = B((\Phi, \Psi)^T), \]

so the representation (3.6) follows.

Now let \( (\varphi, \psi) \) and \( (\Phi, \Psi) \) be as above. Then

\[ \text{diag} \left( P, P \right) B \text{diag} \left( P, P \right) B^{-1}((\varphi, \psi)^T) = BB^{-1}((\varphi, \psi)^T), \]

and

\[ \text{diag} \left( P, P \right) B^{-1} \text{diag} \left( P, P \right) B((\Phi, \Psi)^T) = B^{-1} B((\Phi, \Psi)^T), \]

which completes the proof.

From now on we also assume that the generating functions \( a, b \in L^\infty \) satisfy matching conditions (2.3). If this is the case, the kernels of Toeplitz plus Hankel operators \( T(a) + H(b) \) and \( T(a) - H(b) \) can be studied in more detail.
Note that if \((a, b)\) is a matching pair, then the matrix–function \(V(a, b)\) takes the form
\[
V(a, b) = \begin{pmatrix} 0 & d \\ -c & \tilde{a}^{-1} \end{pmatrix}.
\]
where \((c, d)\) is the corresponding subordinated pair. In addition, we also have a useful representation—viz.
\[
T(V(a, b)) = \begin{pmatrix} 0 & T(d) \\ -T(c) & T(\tilde{a}^{-1}) \end{pmatrix} = \begin{pmatrix} -T(d) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & T(\tilde{a}^{-1}) \end{pmatrix} \begin{pmatrix} -T(c) & 0 \\ 0 & I \end{pmatrix},
\]
where the operator
\[
D := \begin{pmatrix} 0 & -I \\ I & T(\tilde{a}^{-1}) \end{pmatrix}
\]
in the right–hand side of (3.7) is invertible and
\[
D^{-1} = \begin{pmatrix} T(\tilde{a}^{-1}) & I \\ -I & 0 \end{pmatrix}.
\]

Now we can establish a representation for the kernel of the block Toeplitz operator \(T(V(a, b))\).

**Proposition 3.3** Let \((a, b) \in L^\infty \times L^\infty\) be a matching pair such that the operator \(T(c), c = ab^{-1}\), is invertible from the right. Then
\[
\ker T(V(a, b)) = \Omega(c) + \hat{\Omega}(d)
\]
where
\[
\Omega(c) := \{ (\varphi, 0)^T : \varphi \in \ker T(c) \},
\]
\[
\hat{\Omega}(d) := \{ (T_r^{-1}(c)T(\tilde{a}^{-1})s, s)^T : s \in \ker T(d) \}.
\]

**Proof.** It is easily seen that \(\Omega(c)\) and \(\hat{\Omega}(d)\) are closed subspaces of the kernel of the operator \(T(V(a, b))\). It is also clear that the intersection of this two subspaces consists of the zero vector only.

If \((y_1, y_2)^T \in \ker T(V(a, b))\), then
\[
T(d)y_2 = 0, \text{ and } T(c)y_1 = T(\tilde{a}^{-1})y_2.
\]
Since $T_{r}^{-1}(c)$ is left-invertible, the space $H^{p}$ can be represented as the direct sum of the closed subspaces $\ker T(c)$ and $\operatorname{im} T_{r}^{-1}(c)$, i.e.

$$
H^{p} = \ker T(c) + \operatorname{im} T_{r}^{-1}(c).
$$

Correspondingly, the element $y_{1}$ can be written as $y_{1} = y_{10} + y_{11}$, where $y_{10} \in \ker T(c)$ and $y_{11} \in \operatorname{im} T_{r}^{-1}(c)$. Moreover, there is a unique vector $y_{3} \in H^{p}$ such that $y_{11} = T_{r}^{-1}(c)y_{3}$, so we get

$$
T(c)y_{1} = T(c)(T_{r}^{-1}(c)y_{3} + y_{10}) = y_{3} = T(\tilde{a}^{-1})y_{2}.
$$

It implies that

$$
y_{1} = T_{r}^{-1}(c)T(\tilde{a}^{-1})y_{2} + y_{10},
$$

and we get

$$(y_{1}, y_{2})^{T} = (T_{r}^{-1}(c)T(\tilde{a}^{-1})y_{2}, y_{2})^{T} + (y_{10}, 0)^{T},$$

with $(T_{r}^{-1}(c)T(\tilde{a}^{-1})y_{2}, y_{2})^{T} \in \tilde{\Omega}(d)$ and $(y_{10}, 0)^{T} \in \Omega(c)$, which completes the proof.

We already know that if an element $\varphi \neq 0$ belongs to the kernel of the operator $T(c)$, then $(\varphi, 0)^{T} \in \ker T(V(a, b))$ and Lemma 3.2 implies that

$$
\varphi - JQcP\varphi \in \ker (T(a) + H(b)),
$$

$$
\varphi + JQcP\varphi \in \ker (T(a) - H(b)).
$$

However, it is remarkable that the functions $\varphi - JQcP\varphi$ and $\varphi + JQcP\varphi$ belong to the kernel of the operator $T(c)$ as well.

**Proposition 3.4** Let $g \in L^\infty$ satisfy the relation $\tilde{g}\tilde{g} = 1$. Then

(i) If $f \in \ker T(g)$, then $JQgPf \in \ker T(g)$.

(ii) If $f \in \ker T(g)$, then $(JQgP)^{2}f = f$.

**Proof.** If $\tilde{g}\tilde{g} = 1$ and $f \in \ker T(g)$, then

$$
T(g)(JQgPf) = PgPQgPf = JQ\tilde{g}QgPf = JQ\tilde{g}gPf - JQ\tilde{g}gPf = 0,
$$

and assertion (i) follows. On the other hand, for any $f \in \ker T(g)$ one has

$$
(JQgP)^{2}f = JQgPJQgPf = P\tilde{g}QgPf = P\tilde{g}gPf = f - P\tilde{g}T(g)f = f,
$$
and we are done. \[\]

Consider now the operator \(P_g := JQgP\}_{\text{ker } T(g)}\). By Proposition 3.4, one has \(P_g : \text{ker } T(g) \to \text{ker } T(g)\) and \(P_g^2 = I\). Therefore, on the space \(\text{ker } T(g)\) the operators \(P_g^\pm := (1/2)(I \pm P_g)\) are complimentary projections, so they generate a decomposition of \(\text{ker } T(g)\).

**Corollary 3.5** Let \((c, d)\) be the subordinated pair for a matching pair \((a, b)\) \(\in L^\infty \times L^\infty\). Then the following relations hold.

\[
\begin{align*}
\text{ker } T(c) &= \text{im } P_c^- + \text{im } P_c^+,
\text{im } P_c^- &\subset \text{ker } (T(a) + H(b)), \\
\text{im } P_c^+ &\subset \text{ker } (T(a) - H(b)),
\end{align*}
\]

Corollary 3.5 shows the influence of the operator \(T(c)\) on the kernels of the operators \(T(a) + H(b)\) and \(T(a) - H(b)\). Let us now clarify the role of another operator—viz. the operator \(T(d)\), in the kernel structure of the corresponding operators \(T(a) \pm H(b)\). This problem is more involved. Assume additionally that the operator \(T(c)\) is invertible from the right. If \(s \in \text{ker } T(d)\), then the element \((T^{-1}_r(c)T(\tilde{a}^{-1})s, s)^T \in \text{ker } T(V(a, b))\). By Lemma 3.2, the element

\[
\varphi_\pm(s) := T^{-1}_r(c)T(\tilde{a}^{-1})s \pm JQcPT^{-1}_r(c)T(\tilde{a}^{-1})s \pm JQ(\tilde{a}^{-1})s \tag{3.8}
\]

belongs to the null space \(\text{ker } (T(a) \pm H(b))\) of the corresponding operator \(T(a) \pm H(b)\).

**Lemma 3.6** The map \(s \mapsto \varphi_\pm(s)\) is a one-to-one function from the space \(\text{im } P_d^\pm\) to the space \(\text{ker } (T(a) \pm H(b))\).

**Proof.** Assume that \(s \in \text{ker } T(d)\). If we show that the operator \((1/2)(PbP + P\tilde{a}JP)\) sends \(\varphi_+(s)\) into \(P_d^+s\) and the operator \((1/2)(PbP - P\tilde{a}JP)\) sends \(\varphi_-(s)\) into \(P_d^-s\), then Lemma 3.6 will follow. Consider, for example, the first case. Thus one has

\[
(PbP + P\tilde{a}JP)\varphi_+(s) = I_1 + I_2 + \cdots + I_6,
\]

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where
\[ I_1 = \tilde{P}bPT_r^{-1}(c)T(\tilde{a}^{-1})s, \]
\[ I_2 = -\tilde{P}bPJCPT_r^{-1}(c)T(\tilde{a}^{-1})s, \]
\[ I_3 = \tilde{P}bPJQ\tilde{a}^{-1}Ps, \]
\[ I_4 = P\tilde{a}JPT_r^{-1}(c)T(\tilde{a}^{-1})s \]
\[ I_5 = -P\tilde{a}JPJCPT_r^{-1}(c)T(\tilde{a}^{-1})s, \]
\[ I_6 = P\tilde{a}JPJ\tilde{a}^{-1}Ps. \]

Taking into account that \( \tilde{b} = c\tilde{a}, bc = a, JPJ = Q, J\tilde{u}J = a \) and \( P + Q = I \), we get
\[ I_1 + I_5 = \tilde{P}aPT_r^{-1}(c)T(\tilde{a}^{-1})s - \tilde{P}aQcPT_r^{-1}(c)T(\tilde{a}^{-1})s \]
\[ = \tilde{P}aPcPT_r^{-1}(c)T(\tilde{a}^{-1})s + \tilde{P}aQcPT_r^{-1}(c)T(\tilde{a}^{-1})s \]
\[ - \tilde{P}aQcPT_r^{-1}(c)T(\tilde{a}^{-1})s = \tilde{P}aP\tilde{a}^{-1}Ps, \]
which implies the relation
\[ I_1 + I_5 + I_6 = \tilde{P}aP\tilde{a}^{-1}Ps + \tilde{P}aQ\tilde{a}^{-1}Ps = s. \quad (3.9) \]

Further,
\[ I_2 + I_3 = -\tilde{P}bPJCPT_r^{-1}(c)T(\tilde{a}^{-1})s + \tilde{P}bPJQ\tilde{a}^{-1}Ps \]
\[ = -JQbQcPT_r^{-1}(c)T(\tilde{a}^{-1})s + JQbQ\tilde{a}^{-1}Ps \]
\[ = -JQbcPT_r^{-1}(c)T(\tilde{a}^{-1})s + JQbQ\tilde{a}^{-1}Ps \]
\[ + JQb\tilde{a}^{-1}Ps - JQbP\tilde{a}^{-1}Ps \]
\[ = -JQaPT_r^{-1}(c)T(\tilde{a}^{-1})s + JQdPs, \]
and consequently
\[ I_2 + I_3 + I_4 = JQdPs. \quad (3.10) \]

Comparing (3.9) and (3.10), one obtains the claim for the function \( \varphi_+(s) \).

The case of function \( \varphi_-(s) \) is considered analogously. \( \blacksquare \)

**Proposition 3.7** Let \((c, d)\) be the subordinated pair for a matching pair \((a, b) \in L^\infty \times L^\infty\). If the operator \( T(c) \) is right-invertible, then
\[
\text{ker}(T(a) + H(b)) = \varphi_+(\text{im} P_d^+) + \text{im} P_c^-, \\
\text{ker}(T(a) - H(b)) = \varphi_-(\text{im} P_d^-) + \text{im} P_c^+. \quad (3.11)
\]
Proof. As was mentioned in the proof of Lemma 3.6, for \( s \in \ker T(d) \) one has
\[
\frac{1}{2}(\tilde{P}bP + P\tilde{a}JP)\varphi^+(s) = P^+_d s,
\]
\[
\frac{1}{2}(\tilde{P}bP - P\tilde{a}JP)\varphi^-(s) = P^-_d s.
\]
On the other hand, if \( s \in \ker T(c) \), then
\[
(\frac{1}{2})(\varphi^+(s), \varphi^-(s))^T = (P^-_c s, P^+_c s)^T \in \ker \text{diag}(T(a) + H(b), T(a) - H(b)),
\]
and it is easily seen that
\[
(\frac{1}{2})(\tilde{P}bP + P\tilde{a}JP)(s - JQcPs) = 0,
\]
\[
(\frac{1}{2})(\tilde{P}bP - P\tilde{a}JP)(s + JQcPs) = 0.
\]
Using these relations one can observe that if \( s \in \text{im } P^+(d) \), then
\[
E_2((1/2)(\varphi^+(s), 0)^T) = (1/2)(\varphi^+(s), s)^T,
\]
and if \( s \in \text{im } P^-_c \), then
\[
E_2((1/2)(\varphi^+(s), 0)^T) = (1/2)(\varphi^+(s), 0)^T = (P^-_c s, 0)^T.
\]
Thus
\[
E_2(\varphi^+(\text{im } P^+_d)) \cap E_2(\varphi^+(\text{im } P^-_c)) = \{0\}.
\]
But \( E_2 \) is an isomorphism, hence
\[
\varphi^+(\text{im } P^+_d) \cap \text{im } P^-_c = \{0\},
\]
and it is clear that \( \varphi^+(\text{im } P^+_d) \) and \( \text{im } P^-_c \) are closed subspaces of \( \ker(T(a) + H(b)) \). Moreover, the direct sum of \( \varphi^+(\text{im } P^+_d) \) and \( \text{im } P^-_c \) is a closed subspace. In order to show that \( Y := \varphi^+(\text{im } P^+_d) \oplus \text{im } P^-_c \in \ker(T(a) + H(b)) \) we have to show that for \( s \in \text{im } P^-_d \), the element \( \varphi^+(s) \) belongs to \( Y \). Thus assume that \( s \in P^-_d \) and consider the element
\[
E_2((\varphi^+(s), 0)^T) = (\varphi^+(s), 0)^T \in \ker T(V(a, b)).
\]
By Proposition 3.3 we have \( \varphi^+(s) \in \ker T(c) \). Moreover, \( \varphi^+(s) \in P^-_c \) since otherwise \( E_1((\varphi^+(s), 0)^T) \notin \ker(T(a) + H(b)) \), which is a contradiction.

The related result for \( \ker(T(a) - H(b)) \) can be proved analogously.

Despite the fact that all results of this section are formulated for operators acting on \( H^p \)-spaces, they remain true for Toeplitz plus Hankel operators on \( l^p(\mathbb{Z}_+) \) and for Wiener–Hopf plus Hankel operators on \( L^p(\mathbb{R}^+) \), \( 1 \leq p < \infty \).
4 Coburn–Simonenko theorem for particular classes of Toeplitz plus Hankel operators

Coburn–Simonenko theorem for Toeplitz operators claims that if \( a \in L^\infty \) and is different from the zero element, then

\[
\min(\dim \ker T(a), \dim \operatorname{coker} T(a)) = 0.
\]

It turns out that, in general, such a statement for the operators \( T(a) + H(b) \), \( a, b \in L^\infty \) is not true. Nevertheless, the results of Section 3 allow one to single out certain classes of Toeplitz plus Hankel operators where some kind of Coburn–Simonenko theorem still remains in force.

**Theorem 4.1** Let \( a_j \in GL^\infty \), \( j = 1, 2, 3, 4 \) and let \( A \) refer to one of the four operators \( T(a_1) - H(a_1^{-1}) \), \( T(a_2) + H(a_2t) \), \( T(a_3) + H(a_3) \), \( T(a_4) - H(a_4) \). Then \( \ker A = 0 \) or \( \operatorname{coker} A = 0 \).

**Proof.** Let us start with the operator \( T(a_2) + H(ta_2) \). The duo \( (a_2, ta_2) \) constitutes a matching pair with the subordinated pair \( (t^{-1}, d) \), where \( d = a_2 \tilde{a}_2^{-1} t \). Note that the constant function \( e := e(t) = 1, t \in \mathbb{T} \), belongs to both spaces \( \ker T(t^{-1}) \) and \( \ker(T(a_2) - H(a_2t)) \). Suppose now that \( \dim \ker T(d) > 0 \). By Proposition 3.7 we get that

\[
\dim \ker(T(a_2) + H(a_2t)) = \dim \operatorname{im} P_d^+.
\]

Moreover, Coburn–Simonenko theorem gives that

\[
\operatorname{coker} T(t^{-1}) = \operatorname{coker} T(d) = \{0\}.
\]

Factorization (3.7) entails that the cokernel of \( T(V(a_2, a_2t)) \) is trivial. By (3.1) the cokernel of the diagonal operator \( \operatorname{diag}(T(a_2) + H(a_2t), T(a_2) - H(a_2t)) \) is also trivial, and hence so is the cokernel of \( T(a_2) + H(a_2t) \). On the other hand, if \( \dim \ker T(d) = 0 \), we again can use Proposition 3.7 to conclude that

\[
\ker(T(a_2) + H(a_2t)) = \{0\}.
\]

Consider now the operator \( T(a_1) - H(t^{-1}a_1) \). The representation (2.4) implies that

\[
T(a_1) - H(a_1t^{-1}) = (T(a_1t^{-1}) - H((a_1t^{-1})t)) \cdot T(t). \tag{4.1}
\]
Setting $a_2 := a_1 t^{-1}$, one rewrites the first operator in the right-hand side of (4.1) as
$$T(a_1 t^{-1}) - H((a_1 t^{-1}) t) = T(a_2) - H(a_2 t).$$
But the operators of the form $T(a_2) - H(a_2 t)$ have been already considered. In particular, the element $e$ belongs to the kernel of the operator $T(a_2) - H(a_2 t)$. Now let $\ker T(d) = \{0\}$. Since $e \notin \text{im} T(t)$, the relation (4.1) implies that
$$\ker(T(a_1) - H(a_1 t^{-1})) = \{0\}.$$ 
Now let $\dim \ker T(d) > 0$. Then
$$\ker(T(a_2) - H((a_2 t))) = \varphi_-(\text{im } P_d^-) + \text{span } \{e\}$$
and
$$\text{coker } (T(a_2) - H(a_2 t)) = \{0\}.$$ 
Note that the functions $s \in \varphi_-(\text{im } P_d^-) + \text{span } \{e\}$ can be rewritten as
$$\{(s - \hat{s}_0) + \hat{s}_0 : s \in \ker(T(a_2) - H(a_2 t))\},$$
where $\hat{s}_0$ is the zero Fourier coefficient of the function $s$. We have $s - \hat{s}_0 \in \text{im } T(t)$, so if $\hat{s}_0 \neq 0$, then $\hat{s}_0 \notin \text{im } T(t)$, and using (4.1) we get
$$\text{coker } (T(a_1) - H(a_1 t^{-1})) = \{0\}.$$ 
The remaining operators $T(a_3) + H(a_3)$ and $T(a_4) - H(a_4)$ can be considered analogously.

Corollary 4.2 If the conditions of Theorem 4.1 are satisfied and if, in addition, some of the corresponding operators is generalized invertible, then it is one sided invertible.

Recall that if a linear operator is Fredholm, then it is generalized invertible.

Remark 4.3 All of the above operators have been previously considered in literature [2, 3, 9]. Nevertheless, the proofs presented here are essentially simpler and the results of Theorem 4.1 and Corollary 4.2 are more general.
Of course, now one can ask how the operators $T(a_1) + H(a_1 t^{-1})$ and $T(a_2) - H(a_2 t)$ behave. For such operators, the situation is more complicated and the assertion of Theorem 4.1 remains valid only under additional assumptions. Consider for instance the operator $T(a_1) + H(a_1 t^{-1})$ which was previously studied in [9] in case of piecewise continuous function $a$. Analogously to the proof of Theorem 4.1 we set $a_2 = a_1 t^{-1}$ and represent this operator as the product of two operators,

$$T(a_1) + H(a_1 t^{-1}) = (T(a_2) + H(a_2 t)) T(t).$$

We already know that $\ker T(t^{-1})$ does not affect the kernel of the operator $T(a_2) + H(a_2 t)$. Indeed, one has $\ker T(t^{-1}) = \mathbb{C} e$ and $\mathbb{C} e \subset \ker (T(a_2) - H(a_2 t))$ and Proposition 3.7 implies the claim. If $\ker T(d) = \{0\}$, then using Proposition 3.7 once more we obtain that $\ker (T(a_2) + H(a_2 t)) = \{0\}$ and therefore $\ker (T(a_1) + H(a_1 t^{-1})) = \{0\}$. Assume now that $\dim \ker T(d) > 0$. By Proposition 3.7 one has $\ker (T(a_2) + H(a_2 t)) = \varphi_+(\text{im} P_d^+)$. Moreover, $\text{coker} (T(a_2) + H(a_2 t)) = \{0\}$ as it was shown in Theorem 4.1. For the sake of simplicity, suppose now that $T(a_2) + H(a_2 t)$ is an onto operator. If $\ker (T(a_2) + H(a_2 t)) \subset \text{im} T(t)$, then $\dim \text{coker} (T(a_1) + H(a_1 t^{-1})) = 1$. If $\text{im} P_d^+ \neq \{0\}$, then $T(a_1) + H(a_1 t^{-1})$ has non-trivial kernel and cokernel. If $\ker (T(a_2) + H(a_2 t)$ is not contained in $\text{im} T(t)$, we have $\text{coker} (T(a_1) + H(a_1 t^{-1})) = \{0\}$, the proof of which is similar to the proof of the assertion $\ker (T(a_1) - H(a_1 t^{-1})) = \{0\}$ in the proof of Theorem 4.1.

It is worth noting that the study of the operator $T(a_1) + H(a_1 t^{-1})$ in [9] is more involved and lengthy.

**Remark 4.4** The results of this section are also valid for related bounded Toeplitz plus Hankel operators considered on $l^p(\mathbb{Z}_+)$. The case of Wiener–Hopf plus Hankel integral operators requires more work and will be published elsewhere.

### 5 Kernel decomposition for a class of Toeplitz operators

In this section we establish certain properties of the kernels of Toeplitz operators which are needed in what follows. More precisely, we present the kernel decomposition for Fredholm Toeplitz operators $T(g)$ with symbols $g \in L^\infty$.
satisfying the relation \( g\tilde{g} = 1 \). As we already know (see Corollary 3.3), the kernel of the operator \( T(g) \) can be represented in the form

\[
\ker T(g) = \text{im} P_g^- + \text{im} P_g^+.
\]

It turns out that the spaces \( \text{im} P_g^- \) and \( \text{im} P_g^+ \) possess nice bases and this fact is actively used in the forthcoming sections. In particular, the dimensions of the subspaces \( \text{im} P_g^- \) and \( \text{im} P_g^+ \) can be determined.

If the operator \( T(g) : H^p \to H^p \) is Fredholm, then by Theorem 2.3, the function \( g \) admits a Wiener–Hopf factorization \( g = g_- t^n g_+ \) in \( H^p \). Recall that \( g_-(\infty) = 1 \).

**Proposition 5.1** Let \( g \in L^\infty \) be a function satisfying the condition \( g\tilde{g} = 1 \) and such that the operator \( T(g) : H^p \to H^p \) is Fredholm. Then

\[
g_+(0) = \pm 1.
\]

**Proof.** Without loss of generality, we can assume that the index of \( T(g) \) is equal to 0, so the operator \( T(g) \) is invertible on \( H^p \). By [6, Proposition 7.19(c)], the operator \( T(\tilde{g}) \) is invertible on \( H^q, p^{-1} + q^{-1} = 1 \). Then the function \( \tilde{g} = g^{-1} \) admits Wiener–Hopf factorization in \( H^q \),

\[
\tilde{g} = \tilde{g}_+ \tilde{g}_- = g_-^{-1} g_+^{-1}.
\]

It follows that

\[
\tilde{g}_- g_+ = g_-^{-1} \tilde{g}_+^{-1}.
\]

(5.2)

Note that the left–hand side of (5.2) belongs to the space \( H^1(\mathbb{T}) \) whereas the right–hand side is in \( \overline{H^1(\mathbb{T})} \). Taking into account the relation \( H^1(\mathbb{T}) \cap \overline{H^1(\mathbb{T})} = \mathbb{C} \), one obtains that there is a constant \( \xi \in \mathbb{C} \) such that

\[
\tilde{g}_- g_+ = g_-^{-1} g_+^{-1} = \xi,
\]

so

\[
g_+ = \xi \tilde{g}_-^{-1}, \quad g_- = \xi \tilde{g}_+.
\]

Moreover, recalling the identities \( g_-(\infty) = g_-^{-1}(\infty) = 1 \) and \( \tilde{g}_-^{-1}(0) = g_-(\infty) = 1 \) and \( \tilde{g}_+^{-1}(\infty) = g_+(0) \) we get \( g_+(0) = \xi \) and \( \xi^2 = 1 \), which implies (5.1).

\[\blacksquare\]

**Definition 5.2** Let \( g \in L^\infty \) satisfy the condition \( g\tilde{g} = 1 \) and admit the Wiener–Hopf factorization \( g = g_- t^n g_+ \), \( g_-(\infty) = 1 \) in \( H^p \). The value \( g_+(0) \) is called the factorization signature of the function \( g \) and is denoted by \( \sigma(g) \).
Corollary 5.3 If $g \in L^\infty$ is a matching function such that the operator $T(g)$ is Fredholm, then the factors $g_-$ and $g_+$ in the related Wiener–Hopf factorization of $g$ satisfy the relations

$$g_+ = \sigma(g)\tilde{g}_-^{-1}, \quad g_- = \sigma(g)\tilde{g}_+^{-1}.$$ 

Indeed, these relations immediately follow from the proof of Proposition 5.1.

The following result plays a crucial role in this paper.

Theorem 5.4 Let $g \in L^\infty$ be a matching function such that the operator $T(g) : H^p \to H^p$ is Fredholm and $n := \text{ind} T(g) > 0$. If $g = g_t^{-n}g_+$, $g_-(\infty) = 1$ is the corresponding Wiener–Hopf factorization of $g$ in $H^p$, then the following systems of functions $\mathcal{B}_\pm(g)$ form bases in the spaces $\text{im} P_g^\pm$:

(i) If $n = 2m$, $m \in \mathbb{N}$, then

$$\mathcal{B}_\pm(g) := \{g_+^{-1}(t^{m-k-1} \pm \sigma(g)t^{m+k}) : k = 0, 1, \ldots, m - 1\},$$

and

$$\dim \text{im} P_g^\pm = m.$$

(ii) If $n = 2m + 1$, $m \in \mathbb{Z}_+$, then

$$\mathcal{B}_\pm(g) := \{g_+^{-1}(t^{m+k} \pm \sigma(g)t^{m-k}) : k = 0, 1, \ldots, m - 1\} \setminus \{0\},$$

and the zero element belongs only to one of the sets $\mathcal{B}_+(g)$ or $\mathcal{B}_-(g)$. Namely, for $k = 0$ one of the terms $t^m(1 \pm \sigma(g))$ is equal to zero.

Proof. It is easily seen that the restriction of the operators $Pg_-I$ and $Pg_+^{-1}I$ on $\ker T(t^{-n}) = \text{span}\{e, \ldots, t^{n-1}\}$ map $\ker T(t^{-n})$ into $\ker T(t^{-n})$ and on the space $\ker T(t^{-n})$ the above operators are inverses to each other.

Clearly, the elements $s_j = Pg_j t^j$, $j = 0, 1, \ldots, n - 1$ are again in $\ker T(t^{-n})$ and

$$T^{-1}(g_0)s_j = g_+^{-1}P g_+^{-1}s_j = g_+^{-1}t^j,$$

where $g_0 := gt^n$. 

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Note that $T^{-1}(g_0) = g_+^{-1}Pg_-^{-1}$ and $T^{-1}(g_0)s_j \in \ker T(g)$. Moreover, the set \( \{ T^{-1}(g_0)s_j : j = 0, \cdots, n-1 \} \) constitutes a basis in $\ker T(g)$. Now one can consider the expression

$$JQgPT^{-1}(g_0)s_j = JQg_+t^{-n}g_-Pg_-^{-1}s_j$$

$$= JQt^{-n}g_+t^j = Pt^ng_+t^{-j-1} = Pt^ng_-t^{-j-1} = Pt_0^{-j}g_-.$$

Corollary 5.3 shows that $\tilde{g_-} = \sigma(g)g_+^{-1}$, which leads to the relation

$$P_g^\pm T^{-1}(g_0)s_j = \frac{1}{2}g_+^{-1}(t^j \pm \sigma(g)t^{n-j-1}), \quad j = 0, 1, \cdots, n-1.$$

Now let $n = 2m$, $m \in \mathbb{N}$. If $j \in \{0, 1, \cdots, m-1\}$, then this $j$ can be written as $j = m - k - 1$ with some $k \in \{0, 1, \cdots, m-1\}$ and vice versa. Hence

$$t^j \pm \sigma(g)t^{n-j-1} = t^{m-k-1} \pm \sigma(g)t^{m+k}, \quad j = 0, 1, \cdots, m-1.$$

On the other hand, if $j \geq m$, then $j$ can be rewritten as $j = m + k$ for a $k \in \{0, 1, \cdots, m-1\}$, and $t^j \pm \sigma(g)t^{n-j-1} = t^{m+k} \pm \sigma(g)t^{m-k-1}$. Thus one obtains, maybe up to the factor $-1$, the same function system that was found for $j \in \{0, 1, \cdots, m-1\}$. So we conclude that if $n = 2m$, then

$$\dim \ker P_g^\pm = m,$$

and assertion (i) is shown. Assertion (ii) can be proved similarly.

Let us emphasize that, in general, the determination of the factorization signature $\sigma(g)$, $g \in L^\infty$ is a difficult problem. Nevertheless, we can provide sufficient conditions which allow one to obtain $\sigma(g)$. Note that if $g \in L^\infty$ is continuous at the points $\pm 1 \in \mathbb{T}$, then the function $g$ can take only two values $-1$ or 1 at the points mentioned, i.e. $g(-1), g(1) \in \{-1, 1\}$.

**Proposition 5.5** Let $g \in L^\infty$ be a matching function such that

(i) The operator $T(g)$ is invertible on $H^p$.

(ii) The function $g$ is continuous at the point 1 or $-1$.

Then $\sigma(g) = g(1)$ or $\sigma(g) = g(-1)$, respectively.
**Proof.** Assume for definiteness that the function $g$ is continuous at the point 1. Then it admits a Wiener–Hopf factorization in $H^p$,

$$g = g_- g_+, \quad g(\infty) = 1,$$

and $g(1) = \pm 1$. By Corollary 5.3 one has

$$g = \sigma(g) \tilde{g}_+^{-1} g_+.$$

Approximate the function $g$ as follows. For a given $\varepsilon > 0$ chose an arc of $\mathbb{T}$ with endpoints $e^{i\xi_0}$ and $e^{-i\xi_0}$ such that the point 1 belongs to this arc and such that the function

$$g_\varepsilon(t) = \begin{cases} 
    g(1) & \text{if } t = e^{i\theta}, \theta \in (-\xi_0, \xi_0) \\
    g(t) & \text{otherwise}
\end{cases},$$

satisfies the condition

$$||g - g_\varepsilon|| < \varepsilon.$$

If $\varepsilon$ is small enough, then the operator $T(g_\varepsilon)$ is also invertible and

$$g_\varepsilon = \sigma(g_\varepsilon) \tilde{g}_\varepsilon^{-1} (g_\varepsilon)_+.$$  \hfill (5.3)

Since $g_\varepsilon$ is Hölder continuous in a neighbourhood of the point 1, [13, Corollary 5.15] shows that the functions $(g_\varepsilon)_+$ and $(\tilde{g}_\varepsilon)_+^{-1}$ are also Hölder continuous in a neighbourhood of the point 1 $\in \mathbb{T}$. From relation (5.3) one then obtains

$$g_\varepsilon(1) = \sigma(g_\varepsilon).$$

Additionally assume that $\varepsilon$ is so small that

$$||T^{-1}(g) - T^{-1}(g_\varepsilon)|| < 1.$$  \hfill (5.4)

The equations $T(g)h = 1$ and $T(g_\varepsilon)k = 1$ are uniquely solvable and using Wiener–Hopf factorizations

$$g = g_- g_+, \quad g_- (\infty) = 1,$$

$$g_\varepsilon = (g_\varepsilon)_- g(\varepsilon)_+, \quad (g_\varepsilon)_- (\infty) = 1,$$

one obtains

$$h = T^{-1}(g)e = g_+^{-1} P g_-^{-1} e = g_+^{-1},$$

$$k = T^{-1}(g_\varepsilon)e = (g_\varepsilon)_+^{-1} P (g_\varepsilon)_-^{-1} e = (g_\varepsilon)_+^{-1},$$

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where $e$ is the constant function $e := e(t) = 1$, $t \in \mathbb{T}$. Now the inequality (5.4) implies the estimate

$$||g^{-1}_+ - (g_\varepsilon)^{-1}_+||_p < \varepsilon.$$  

Consequently, equations $g^{-1}_+(0) = \pm 1$, $(g_\varepsilon)^{-1}_+(0) = \pm 1$ lead to the relation $g^{-1}_+(0) = (g_\varepsilon)^{-1}_+$, i.e. $\sigma(g) = \sigma(g_\varepsilon)$. Hence,

$$\sigma(g) = \sigma(g_\varepsilon) = g_\varepsilon(1) = g(1),$$

which completes the proof.

**Corollary 5.6** Let $g \in L^\infty$ be a matching function satisfying condition (ii) of Proposition 5.5. If the operator $T(g) : H^p \to H^p$ is Fredholm and $n := \text{ind} T(g)$, then $\sigma(g) = g(1)$ if $g$ is continuous at the point $1$ and $\sigma(g) = g(1)$. If $n$ is even, then $\sigma(g) = g(-1)(-1)^n$ if $g$ is continuous at the point $-1$.

### 6 Toeplitz plus Hankel operators with Fredholm matching pair

In this section the structure of the kernel and cokernel of Toeplitz plus Hankel operator $T(a) + H(b)$ is described. The operators in question are studied under the condition that their generating functions $a, b \in L^\infty$ constitute a Fredholm matching pair $(a, b)$. Recall that if a matching pair $(a, b)$ is Fredholm, then it follows from (3.1) and (3.7) that $T(a) + H(b)$ and $T(a) - H(b)$ are Fredholm operators. Set $\kappa_1 := \text{ind} T(c)$, $\kappa_2 := \text{ind} T(d)$ and let $\mathbb{Z}_-$ refer to the set of all negative integers.

**Theorem 6.1** Assume that $(a, b) \in L^\infty \times L^\infty$ is a Fredholm matching pair. Then

(i) If $(\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, then the operators $T(a) + H(b)$ and $T(a) - H(b)$ are invertible from the right and

$$\ker(T(a) + H(b)) = \text{im} P^-_c + \varphi_+ (\text{im} P^+_d),$$

$$\ker(T(a) - H(b)) = \text{im} P^+_c + \varphi_- (\text{im} P^-_d),$$

where the spaces $\text{im} P^\pm_c$ and $\text{im} P^\pm_d$ are described in Theorem 5.4 and the mappings $\varphi_\pm$ are defined by (3.8).
(ii) If \((\kappa_1, \kappa_2) \in (\mathbb{Z} \setminus \mathbb{N}) \times (\mathbb{Z} \setminus \mathbb{N})\), then the operators \(T(a) + H(b)\) and \(T(a) - H(b)\) are invertible from the left and

\[
\text{coker} \ (T(a) + H(b)) = \text{im} \ P^-_{\tilde{d}} + \varphi_+ (\text{im} \ P^+_{\tilde{d}}), \quad \text{coker} \ (T(a) - H(b)) = \text{im} \ P^+_{\tilde{d}} + \varphi_- (\text{im} \ P^-_{\tilde{d}}),
\]

and \(\text{im} \ P^\pm_{\tilde{d}} = \{0\}\) for \(\kappa_2 = 0\).

(iii) If \((\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times \mathbb{Z}_-\), then

\[
\ker(T(a) + H(b)) = \text{im} \ P^-_c, \quad \text{coker} \ (T(a) + H(b)) = \text{im} \ P^+_{\tilde{d}}, \quad \ker(T(a) - H(b)) = \text{im} \ P^+_c \quad \text{coker} \ (T(a) - H(b)) = \text{im} \ P^-_{\tilde{d}}.
\]

**Proof.** Let us note that all results concerning the kernels of the operators mentioned follow from Theorem 5.4. Considering the cokernels of the corresponding operators, we recall that \(\text{coker} \ (T(a) \pm H(b)) := \ker (T(a) \pm H(b))^*\).

Moreover, \((T(a) \pm H(b))^* = T(\sigma) \pm H(\overline{b})\) and \((\overline{d}, \overline{c})\) is again a matching pair with the subordinated pair \((\overline{d}, \overline{e})\). Further, if \(c = c_- t^{-\kappa_1} c_+\) is the Wiener–Hopf factorization of \(c\) in \(H^p\), then \(\overline{c} = (\sigma(c) \overline{c}_+)^{t^{\kappa_1}} (\sigma(c) \overline{c}_-)\) is the related Wiener–Hopf factorization of \(\overline{c}\) in \(H^q\) and \(\overline{c}_- \in H^p\), \(\overline{c}_+ \in H^q\), \(\overline{c}_- \in \overline{H}^p\), \(p^{-1} + q^{-1} = 1\), and \(\sigma(\overline{c}) = (\sigma(c))\). Of course, since the function \(\tilde{d}\) admits a similar factorization, cokernel description can be obtained directly from the previous results for the kernels of Toeplitz plus Hankel operators.

It remains to consider the case \((\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{Z}_+\). This situation is more involved and factorization \((3.7)\) already indicates that for \(\kappa_2 > 0\), the kernel dimension of \(\text{diag} \ (T(a) + H(b), T(a) - H(b))\) may be smaller than \(\kappa_2\). In order to prepare our next theorem, choose an \(n \in \mathbb{N}\) such that

\[
1 \geq 2n + \kappa_1 \geq 0.
\]

Such an \(n\) is uniquely defined and

\[
2n + \kappa_1 = \begin{cases} 
0, & \text{if } \kappa_1 \text{ is even}, \\
1, & \text{if } \kappa_1 \text{ is odd}.
\end{cases}
\]

Now the operators \((T(a) \pm H(b))\) can be represented in the form

\[
T(a) \pm H(b) = (T(at^{-n}) \pm H(bt^{-n})) T(t^{n}). \quad (6.1)
\]
Note that \((at^n, bt^n)\) is a matching pair with the subordinated pair \((ct^{-2n}, d)\). Therefore, the operators \(T(at^n) \pm H(bt^n)\) are subject to assertion (i) of Theorem 6.1. Thus they are right-invertible, and if \(\kappa_1\) is even, then

\[
\begin{align*}
\ker(T(at^n) + H(bt^n)) &= \varphi_+(\text{im } P^+_d), \\
\ker(T(at^n) - H(bt^n)) &= \varphi_-(\text{im } P^-_d),
\end{align*}
\] (6.2)

and if \(\kappa_1\) is odd, then

\[
\begin{align*}
\ker(T(at^n) + H(bt^n)) &= \frac{1 - \sigma(c)}{2} c^{-1} C + \varphi_+(\text{im } P^+_d), \\
\ker(T(at^n) - H(bt^n)) &= \frac{1 + \sigma(c)}{2} c^{-1} C + \varphi_-(\text{im } P^-_d),
\end{align*}
\] (6.3)

where the mappings \(\varphi_\pm\) depend on the functions \(at^n\) and \(bt^n\).

**Theorem 6.2** Let \((\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{Z}_+\). Then

(i) If \(\kappa_1\) is odd, then

\[
\ker(T(a) \pm H(b)) = T(t^{-n}) \left( \left\{ \frac{1 + \sigma(c)}{2} c^{-1} C + \varphi_\pm(\text{im } P^\pm_d) \right\} \cap \text{im } T(t^n) \right)
\]

\[
= \left\{ \psi \in \{T(t^{-n})u\} : u \in \left\{ \frac{1 + \sigma(c)}{2} c^{-1} C + \varphi_\pm(\text{im } P^\pm_d) \right\}, \quad \text{and } \widehat{u}_0 = \cdots = \widehat{u}_{n-1} = 0 \right\},
\]

where \(\widehat{u}_k, k = 0, 1, \cdots, n-1\) are the Fourier coefficients of the function \(u\), and the mappings \(\varphi_\pm\) depend on the functions \(at^{-n}\) and \(bt^n\).

(ii) If \(\kappa_1\) is even, then

\[
\ker(T(a) \pm H(b)) = T(t^{-n}) \left( \left\{ \varphi_\pm(\text{im } P^\pm_d) \right\} \cap \text{im } T(t^n) \right)
\]

\[
= \left\{ \psi \in \{T(t^{-n})u\} : u \in \varphi_\pm(\text{im } P^\pm_d) \text{ and } \widehat{u}_0 = \cdots = \widehat{u}_{n-1} = 0 \right\},
\]

and the mappings \(\varphi_\pm\) again depend on \(at^{-n}\) and \(bt^n\).

**Proof.** It follows immediately from representations (6.1)–(6.3). □

Theorem 6.2 can also be used to derive representations of the cokernel of the operator \(T(a) \pm H(b)\) in the situation where \((\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{Z}_+\). Indeed,
recalling that \((T(a) \pm H(b))^* = T(\overline{a}) \pm H(\overline{b})\), and \((\overline{a}, \overline{b})\) is the subordinated pair for \((a, b)\), one can note that the operators \(T(\overline{d})\) and \(T(\overline{c})\) are also Fredholm and
\[
\text{ind } T(\overline{d}) = -\kappa_2, \quad \text{ind } T(\overline{c}) = -\kappa_1,
\]
so \((-\kappa_2, \kappa_1) \in \mathbb{Z}_- \times \mathbb{Z}_+.\) Therefore, Theorem 6.2 applies and we can formulate the following result.

**Theorem 6.3** Let \((\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{Z}_+,\) and let \(m \in \mathbb{N}\) satisfy the requirement
\[
1 \geq 2m - \kappa_2 \geq 0.
\]
Then

(i) If \(\kappa_2\) is odd, then
\[
coker (T(a) \pm H(b)) = T(t^{-m}) \left\{ \frac{1 \pm \sigma(\overline{d})}{2} d^{-1} \mathbb{C} + \varphi_{\pm}(\text{im } P^\pm_{\overline{c}}) \right\} \cap \text{im } T(t^m).
\]

(ii) If \(\kappa_2\) is even, then
\[
\ker (T(a) \pm H(b)) = T(t^{-m}) \left\{ \varphi_{\pm}(\text{im } P^\pm_{\overline{c}}) \right\} \cap \text{im } T(t^m),
\]
and the mappings \(\varphi_{\pm}\) depend on \(\overline{at}^{-m}\) and \(\overline{bt}^m\).

In some cases the above approach allows one to drop the condition of Fredholmness of the operator \(T(d)\). We are not going to pursue this analysis here but rather restrict ourselves to a few special cases generalizing results of Section 4.

**Corollary 6.4** Let \((a, b) \in L^\infty \times L^\infty\) be a matching pair with the subordinated pair \((c, d)\), and let \(T(c)\) be a Fredholm operator. Then:

(i) If \(\text{ind } T(c) = 1\) and \(\sigma(c) = 1,\) then \(\ker (T(a) + H(b)) = \{0\}\) or \(\coker (T(a) + H(b)) = \{0\}.\)

(ii) If \(\text{ind } T(c) = -1\) and \(\sigma(c) = 1,\) then \(\ker (T(a) - H(b)) = \{0\}\) or \(\coker (T(a) - H(b)) = \{0\}.\)

(iii) If \(\text{ind } T(c) = 0,\) then \(\ker (T(a) \pm H(b)) = \{0\}\) or \(\coker (T(a) \pm H(b)) = \{0\}.\)
Proof. The proof of the last theorem is similar to the proof of Theorem 5.4, but in the proofs of assertions (i) and (ii) one, respectively, has to use the fact that \(c^+_1 \in \ker(T(a) + H(b))\) and \(c^+_1 \in \ker(T(at^{-1}) - H(bt))\). These inclusions can be verified by straightforward computations. Thus considering, for example, the expression \((T(at^{-1}) - H(bt))c^+_1\), one obtains

\[
(T(at^{-1}) - H(bt))c^+_1 = Pbc t^{-1}c^+_1 - Pbt Qc^+_1 = Pbc t^{-1}c^+_1 - Pbt \tilde{c}^{-1} t^{-1} = 0.
\]

Note that we have used the relations \(a = bc\) and \(c_- = \tilde{c}^{-1}\).

Corollary 6.5 Let \(b \in L^\infty\) be a matching function. If \(T(\tilde{b})\) is a Fredholm operator, then:

(i) If \(\text{ind } T(\tilde{b}) = 1\), and \(\sigma(\tilde{b}) = 1\), then \(\ker(I + H(b))\) or \(\text{coker } (I + H(b))\) is trivial.

(ii) If \(\text{ind } T(\tilde{b}) = -1\), and \(\sigma(\tilde{b}) = 1\), then \(\ker(I - H(b))\) or \(\text{coker } (I - H(b))\) is trivial.

(iii) If \(\text{ind } T(\tilde{b}) = 0\), then \(\ker(I \pm H(b))\) or \(\text{coker } (I \pm H(b))\) is trivial.

This results is a direct consequence of Corollary 6.4, since if \(b\) is a matching function, then \((1, b)\) is a matching pair with the subordinate pair \((\tilde{b}, b)\).

7 \(PC\)-generating functions

The results of the previous section can be improved if there is more information available about the generating functions \(a\) and \(b\). Thus if \(a\) and \(b\) are piecewise continuous functions, then one-sided invertibility of Toeplitz plus Hankel operators on the space \(H^p\), \(1 < p < \infty\) can be studied in more detail. Recall that a function \(a \in L^\infty\) is called piecewise continuous if for every \(t \in \mathbb{T}\) the one-sided limits \(a(t + 0)\) and \(a(t - 0)\) exist. The set of all piecewise continuous functions is denoted by \(PC(\mathbb{T})\) or simply by \(PC\). It is well-known that \(PC\) is a closed subalgebra of \(L^\infty\), and any piecewise continuous function has at most countable set of jumps. Moreover, for each \(\delta > 0\) the set \(S := \{t \in \mathbb{T} : |a(t + 0) - a(t - 0)| > \delta\}\) is finite.
Further, let us introduce the functions
\[ \nu_p(y) := \frac{1}{2} \left( 1 + \coth \left( \pi \left( y + \frac{i}{p} \right) \right) \right), \quad h_p(y) := \sinh^{-1} \left( \pi \left( y + \frac{i}{p} \right) \right), \]
where \( y \in \mathbb{R} \), and \( \mathbb{R} \) refers to the two-point compactification of \( \mathbb{R} \). Note that for given points \( u, w \in \mathbb{C} \), \( u \neq w \) the set \( \mathcal{A}_p(u, w) := \{ z \in \mathbb{C} : z = w \nu_p(y) + u(1 - \nu_p(y)), y \in \mathbb{R} \} \) forms a circular arc which starts at \( u \) and ends at \( w \) as \( y \) runs through \( \mathbb{R} \). This arc \( \mathcal{A}_p(u, w) \) has the property that from any point of the arc, the line segment \([u, w]\) is seen at the angle \( 2\pi/(\max\{p, q\}) \), \( 1/p + 1/q = 1 \). Moreover, if \( 2 < p < \infty \) \((1 < p < 2)\) the arc \( \mathcal{A}_p(u, w) \) is located on the right-hand side (left-hand side) of the straight line passing through the points \( u \) and \( w \) and directed from \( u \) to \( w \). If \( p = 2 \), the set \( \mathcal{A}_p(u, w) \) coincides with the line segment \([u, w]\).

Let \( \hat{T}_+ := \{ t \in \mathbb{T} : \text{Im} \, t > 0 \} \).

**Theorem 7.1** If \( a, b \in PC \), then the operator \( T(a) + H(b) \) is Fredholm if and only if the matrix

\[
\text{smb} \left( T(a) + H(b) \right)(t, y) := \\
\begin{pmatrix}
    a(t + 0)\nu_p(y) + a(t - 0)(1 - \nu_p(y)) & b(t + 0) - b(t - 0) \\
    \frac{b(\bar{t} - 0) - b(\bar{t} + 0)}{2i} h_p(y) & a(\bar{t} + 0)\nu_p(y) + a(\bar{t} - 0)(1 - \nu_p(y))
\end{pmatrix}
\]

is invertible for every \((t, y) \in \hat{T}_+ \times \mathbb{R}\) and the function

\[
\text{smb} \left( T(a) + H(b) \right)(t, y) := a(t + 0)\nu_p(y) + a(t - 0)(1 - \nu_p(y)) + \frac{b(t + 0) - b(t - 0)}{2} h_p(y)
\]

does not vanish on \( \{-1, 1\} \times \mathbb{R} \).

**Remark 7.2** For the first time this theorem appeared in [20] (see also [19]). An index formula can also be established similarly to [21]. If the matching condition is satisfied, an index formula can be derived from Theorem 7.1 below.
If \((a, b) \in PC \times PC\) is a matching pair, then the subordinated pair \((c, d) \in PC \times PC\), and both operators \(T(a) + H(b)\) and \(T(a) - H(b)\) are simultaneously Fredholm if and only if the matching pair \((a, b)\) is Fredholm. This follows from the fact that semi-Fredholm Toeplitz operators with piecewise continuous generating functions are indeed Fredholm operators. This assertion can be obtained immediately from \([11\text{, Proposition 3.1}]\) and \([17\text{, Lemma 1}]\) and using \((3.1)\) and \((3.7)\). Let us mention another results we need.

Let \(A\) be an operator defined on all spaces \(L^p\) for \(1 < p < \infty\). Consider the set 
\[
A_F := \{ p \in (1, \infty) \mid \text{the operator } A : H^p \to H^p \text{ is Fredholm} \}
\]

Proposition 7.3 (See \([22]\)) The set \(A_F\) is open. Moreover, for each connected component \(\gamma \in A_F\), the index of the operator \(A : L^p \to L^p, p \in \gamma\) is constant.

For Toeplitz operators the structure of the set \(A_F\) can be characterized as follows.

Proposition 7.4 (See \([23]\)) Let \(G\) be an invertible matrix–functions with entries from \(PC\), and let \(A := T(G)\). Then there is an at most countable subset \(S_A \subset (1, \infty)\) with the only possible accumulation points \(t = 1\) and \(t = \infty\) such that \(A_F = (1, \infty) \setminus S_A\).

This result can be used to describe the corresponding set \(A_F\) for Toeplitz plus Hankel operators.

Corollary 7.5 Let \(a, b \in PC\), and let \(A := \text{diag}(T(a) + H(b), T(a) - H(b)) : H^p \times H^p \to H^p \times H^p\). Then there is at most countable subset \(S_A \subset (1, \infty)\) with the only possible accumulation points \(t = 1\) and \(t = \infty\) such that \(A_F = (1, \infty) \setminus S_A\).

Proof. It follows directly from Proposition 7.4 since \(\text{diag}(T(a) + H(b), T(a) - H(b))\) is Fredholm if and only if so is the operator \(T(V(a, b))\).

Thus if \(a, b \in PC\) and the operator \(T(a) + H(b)\) is Fredholm on \(H^p\), then there is an interval \((p', p'')\) containing \(p\) such that \(T(a) + H(b)\) is Fredholm on all spaces \(H^r, r \in (p', p'')\) and the index of this operator does not depend on \(r\). Moreover, there is an interval \((p, p_0) \subset (p', p'')\), \(p < p_0\) such that \(T(a) - H(b)\) is Fredholm on \(H^r, r \in (p, p_0)\) and its index does not depend on \(r\). Now we can formulate the following result.
Proposition 7.6 Let $T(a) + H(b) : H^p \to H^p$ be a Fredholm operator and let $r \in (p, p_0)$. Then the kernel and cokernel of the operator $T(a) + H(b) : H^r \to H^r$ coincide with the kernel and cokernel of the same operator acting on the space $H^p$.

Proof. Let us first recall a result from [12]. Assume that $X_1, X_2$ are Banach spaces such that $X_1$ is continuously and densely embedded into $X_2$ and $A$ is a linear bounded operator both on $X_1$ and $X_2$. If $A$ is a Fredholm operator on each space $X_1$ and $X_2$, and

$$\text{ind } A|_{X_1 \to X_1} = \text{ind } A|_{X_2 \to X_2},$$

then

$$\ker A|_{X_1 \to X_1} = \ker A|_{X_2 \to X_2},$$

$$\coker A|_{X_1 \to X_1} = \coker A|_{X_2 \to X_2},$$

which implies the assertion. □

Let us now formulate a result concerning the kernels and cokernels of Hankel plus Toeplitz operators.

Theorem 7.7 Let $a, b \in \text{PC}$ and $(a, b)$ be a matching pair. If the operator $T(a) + H(b) : H^p \to H^p$ is Fredholm, then there is an interval $(p, p_0)$, $p < p_0$ such that for all $r \in (p, p_0)$ the pair $(a, b)$ and both operators $T(a) \pm H(b) : H^r \to H^r$ are Fredholm,

$$\ker (T(a) + H(b))|_{H^r \to H^r} = \ker (T(a) + H(b))|_{H^p \to H^p},$$

$$\coker (T(a) + H(b))|_{H^r \to H^r} = \coker (T(a) + H(b))|_{H^p \to H^p},$$

and the kernel and cokernel of the operator $T(a) + H(b) : H^r \to H^r$ are described by Theorems 6.1–6.3.

Proof. It follows immediately from Proposition 7.6 and previous considerations. □

Remark 7.8 Let $a, b \in L^\infty$ be functions which possess the property fixed in Corollary 7.5 for PC-functions. Then the results of Theorem 7.7 remain true.
8 A few remarks on factorization signature

In previous sections the factorization signature has been used to describe the kernels of Toeplitz plus Hankel operators. Therefore, the determination of this characteristic is an important problem. Let us consider this problem for piecewise continuous matching functions $c$ such that the operator $T(c)$ is Fredholm. For the sake of definiteness we assume that if $z \in \mathbb{C} \setminus \{0\}$ is a complex number, then its argument $\arg z$ is always chosen to be in the interval $(-\pi, \pi]$. Following [6, Section 5.35], for $\beta \in \mathbb{C}$ and $\tau \in \mathbb{T}$ consider a function $\varphi_{\beta,\tau}(t) \in PC$ defined by

$$\varphi_{\beta,\tau}(t) := \exp\{i\beta \arg(-t/\tau)\}, \quad t \in \mathbb{T}.$$ 

It is easily seen that $\varphi_{\beta,\tau}$ has at most one discontinuity, namely, a jump at the point $\tau$ and

$$\varphi_{\beta,\tau}(\tau + 0) = e^{-i\pi \beta}, \quad \varphi_{\beta,\tau}(\tau - 0) = e^{i\pi \beta}.$$ 

From [6] Sections 5.35 and 5.36 one obtains that:

(i) The matching function $c$ can be represented in one of the following form

$$c = \varphi_{\gamma+,c_1}, \text{ or } c = \varphi_{\gamma-,c_1} \quad (8.1)$$

where the function $c_1, c_2 \in PC$ are continuous at the points $t = +1$ and $t = -1$, respectively; $\varphi_{\gamma+} := \varphi_{\gamma+,1}, \varphi_{\gamma-} := \varphi_{\gamma-,1}$, and the parameters $\gamma_+, \gamma_-$ are defined according to [6, Section 5.36] with $\Re \gamma_+, \Re \gamma_- \in (-1/q, 1/p)$.

(ii) The operators $T(\varphi_{\gamma+})$ and $T(\varphi_{\gamma-})$ are invertible and the operators $T(c_1)$ and $T(c_2)$ are Fredholm.

It is not hard to see that

$$\varphi_{\gamma\pm} \tilde{\varphi}_{\gamma\pm} = 1, \quad \varphi_{\gamma+}(-1) = 1, \quad \varphi_{\gamma-}(1) = 1.$$ 

Proposition 5.5 then ensures that

$$\sigma(\varphi_{\gamma\pm}) = 1.$$ 

Moreover, $c_1$ and $c_2$ are also matching functions and by Proposition 5.5 and Corollary 5.6 we have

$$\sigma(c_1) = c_1(1), \quad \sigma(c_2) = c_2(-1)(-1)^n,$$ 

where $n = \text{ind} T(c_2) = \text{ind} T(c)$. 

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Theorem 8.1 If \( c \in PC \) is a matching function which admits representation (8.1), then
\[
\sigma(c) = \sigma(c_1) = \sigma(c_2).
\]

Proof. Assume for definiteness that \( c = \varphi_{\gamma} c_1 \). Rewrite the function \( c_1 \) in the form \( c_1 = gt^{-n} \) and approximate \( g \) by a function \( g_{\varepsilon} \) analogously to the corresponding function in the proof of Proposition 5.5. Recall that the operator \( T(g) \) is invertible and if \( \varepsilon \) is small enough, then
\[
\sigma(c_{\varepsilon}) = \sigma(g_{\varepsilon}),
\]
where \( c_{\varepsilon} = \varphi_{\gamma} g_{\varepsilon} t^n \). Indeed, it follows that the product of the "+"-factors of the Wiener–Hopf factorization of the factors in the representation of \( c_{\varepsilon} \) is the "+"-factor in the Wiener–Hopf factorization of \( c_{\varepsilon} \) (use [15, Corollary 5.15]). It remains to show that for sufficiently small \( \varepsilon > 0 \) one has \( \sigma(c) = \sigma(c_{\varepsilon}) \) but this can be done analogously to the proof of Proposition 5.5.

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