Qualitative behaviour of gravitational perturbations in Chaplygin Gas - Unifying Dark Matter and Dark Energy

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Abstract. Two unified models of dark matter (DM) and dark energy (DE) are analysed, namely the Generalized Chaplygin Gas and the Modified Chaplygin Gas models. We study the qualitative behaviour of gravitational perturbations in both models, analysing their dynamical systems which describe the linear density perturbations, and graphically show the wavenumber, or $k$-dependence of the solutions. Finally we compare the two models and try to conclude which of them is close to available observations.

1. Introduction
The mystery of unexplained matter and energy (dubbed, respectively, dark matter and dark energy) that dominate our universe at present epoch, and push it into an acceleration phase, together with several fine tune problems encountered with the Λ-CDM model, lead cosmologists to think about alternative models that may explain quite well the evolution of our universe. In our study we will focus on the unification of dark matter and dark energy, using Chaplygin gas models, that is, we will study the behavior of perturbations in a universe filled by a mixture of Chaplygin gas and baryons.

2. The Generalised Chaplygin Gas (GCG)
The Chaplygin gas cosmological model has been proposed in 2001\cite{1}. This model describes a fluid which is able to unify dark matter and dark energy into a unique entity, making use of the following relation between pressure ($P$) and energy density ($\rho$) \cite{2}:

$$P = -\frac{A}{\rho^\alpha}$$ \hspace{1cm} (1)

where $A$ is a positive constant, and $\alpha$ obeys the condition $0 < \alpha \leq 1$.

In order to obtain the Equation of State (EoS) for the GCG, we need the expression of the density. Using the continuity equation and the Friedmann equation, we obtain \cite{3}:
\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Evolution of the absolute value of the equation of state ($|\omega_g|$), and the square of sound speed $v_{g}^{2}$, as a function of $a$, for the values $\bar{A} = 0.75$ and $\alpha = 0.5$. For other values of these parameters the behavior is similar. In fact this kind of behavior is generic. Note that $a = a_0 = 1$ is the present epoch.}
\end{figure}

The Equation of state of GCG may be written as,

$$\omega_g = \frac{P}{\rho} = -\frac{A}{\rho^{\alpha+1}} = -\frac{\bar{A}}{\bar{A}(1 - \bar{A})a^{-3(\alpha+1)}}$$  \hspace{1cm} (3)

We notice here that the EoS is time-dependant, since the scale factor $a$ is also time-dependent. Thus, the dynamical properties of the Chaplygin gas evolve through time.

From equation (1) we can derive the sound speed equation for the GCG, and we obtain,

$$v_{g}^{2} = \frac{\partial P}{\partial \rho} = -\frac{\alpha P}{\rho} = -\alpha \omega_g$$  \hspace{1cm} (4)

From this equation, since $\alpha$ is constant, we can easily see that the square of the sound speed $v_{g}^{2}$, has the same qualitative behavior as that of the equation of state ($\omega_g$), and its value can not exceed 1 (note that we work in the natural system of units, in which $c = 1$).

In Figure 1 one illustrate the evolution of $\omega_g$. This evolution shows the transition in the dynamical behavior of GCG. As we can see $\omega_g$ evolves from 0 in past times, to a value approaching $-1$ in the far future. Therefore, it is worth note that, in the past, GCG mimics perfectly the well known CDM behavior ($\omega = 0$), and later on the universe tends to evolve dynamically as dominated by a cosmological constant $\Lambda$ ($\omega = -1$).

Thus, theoretically, GCG proves to behave dynamically as CDM or DE depending on the epoch considered, and we can interpret this as just different manifestations of the same entity,
the “GCG”, unifying in this way dark matter and dark energy.

2.1. Perturbation in Universe filled by GCG + Baryons

A set of equations that govern the dynamical evolution of n gravitationally interacting fluids is given by Veeraraghavan and Stebbins [7], where the equations describing the evolution of perturbations are written in co-moving coordinates. In our study we will analyse the evolution of perturbations in a Chaplygin gas dominated Universe with barions. In that case, i.e., adding a baryonic component to the Chaplygin gas, we have to consider a two fluid system.

We therefore get the following set of equations:

\[
\frac{d^{2}}{dt^{2}} \delta_{b} + (2 + \zeta) \frac{d}{dt} \delta_{b} - \frac{3}{2} \left( \Omega_{b} \delta_{b} + (1 + \frac{v_{m}^{2}}{a^{2}}) \Omega_{m} \delta_{m} \right) = 0
\]

\[
\frac{d^{2}}{dt^{2}} \delta_{m} + (1 + \omega_{m}) \left( \frac{\theta_{m}}{aH} - \delta_{b}' \right) + 3 \left( \frac{v_{m}^{2}}{a^{2}} - \omega_{m} \right) \delta_{m} = 0
\]

\[
\theta_{m}' + (1 - 3 \frac{v_{m}^{2}}{a^{2}}) \theta_{m} - \frac{v_{m}^{2} k^{2}}{aH(1 + \omega_{m})} \delta_{m} = 0
\]

This is a system of ordinary differential equation with three unknowns: the perturbations \( \delta_{b} \) and \( \delta_{g} \), and the velocity divergence \( \theta_{g} \). The wavenumber of a perturbation is \( k \), and \( \zeta = \frac{a v_{m}^{2}}{2 H^{2}} \).

We can transform this system into a new system of four differential equations, which in matrix notation can be written as \( x' = Ax \), where \( x = (\delta_{b}, \delta_{g}', \delta_{g}, \theta_{g}) \), so that,

\[
\begin{bmatrix}
\frac{d\delta_{b}}{dt} \\
\frac{d\delta_{g}}{dt} \\
\frac{d\delta_{g}'}{dt} \\
\frac{d\theta_{g}}{dt}
\end{bmatrix} =
\begin{bmatrix}
\frac{3}{2} \Omega_{b} & 0 & 0 & 0 \\
-(2 + \zeta) & \frac{3}{2} \Omega_{g}(1 - 3 \alpha \omega_{g}) & 0 & 0 \\
0 & 1 + \omega_{g} & 0 & 0 \\
0 & 0 & -\frac{\alpha_{g} k^{2}}{aH(1 + \omega_{g})} & 0
\end{bmatrix}
\begin{bmatrix}
\delta_{b} \\
\delta_{g} \\
\delta_{g}' \\
\theta_{g}
\end{bmatrix}
\]

It is very important to note that in this study we assume a flat universe, that is, \( \Omega_{g} + \Omega_{b} = 1 \) and we normalise the Hubble parameter for the present epoch \( (H_{0}) \). In this work we will not be interested in the exact solution of the dynamical system. Instead we will try to describe the qualitative behavior of perturbations that can lead to large scale structures in the universe. So, our first objective is to obtain an autonomous dynamical system, and for that purpose we set the epoch giving a fixed value to the scale factor \( a \) in order to compute the eigenvalues \( \lambda_{i} \) of the coefficients matrix of the dynamical system. The solution of that system is [7]:

\[
x_{i} = C_{i1} e^{\lambda_{1}t} + C_{i2} e^{\lambda_{2}t} + C_{i3} e^{\lambda_{3}t} + C_{i4} e^{\lambda_{4}t}
\]

We computed the eigenvalues, as functions of “\( k \)”, making use of “Matlab” software. We notice that after a certain scale \( k \), a non null imaginary part arises in the eigenvalues. It’s important to stress that a non null imaginary part of \( \lambda_{i} \) is associated to the existence of oscillations in the system. On the other hand, a positive real part describes a growing perturbation, while a negative real part is consequence of decaying.

Figure 2 illustrate the \( k \)-dependence of the real part of the eigenvalues for distinct epochs \( a \), which are the following 0.1, 0.4, 0.7 and 1 (present epoch). It’s important to stress out that no matter the epoch, we have at least one eigenvalue with positive real part, meaning that the perturbation grows as a consequence of gravitational instability leading to collapse. Also important is the existence of a scale range of \( k \) for which a significant decrease occurs in the values of the real part of two eigenvalues. This effect mainly appears in the CDM dominated epoch, that
Figure 2. Dependence on $k$ of the real part of $\lambda_i$ for different epochs in the case of a GCG dominated universe, for $\alpha = 0.5$ and $\bar{A} = 0.75$.

is, at early times ($a = 0.1$, for instance), and for relatively small perturbations (see the upper left plot in Figure 2). Note that a large value of $k$ corresponds to small scale perturbations, while a small wavenumber means a large scale perturbation. This scale $k$ decreases with time, being almost zero for $a = 1$. This particular behavior can be explained by the expected evolution of dark energy that leads to the accelerated expansion and thus slowing the perturbations growth. In fact, the EoS of GCG clearly anticipates this behavior through its time dependence, that shows the transition in the dynamics of GCG from CDM to DE dynamic behavior.

In Figure 3 we show the plots of the imaginary parts of the eigenvalues for the same epochs as for the real parts. The first general observation we made is that there are always two eigenvalues that have a null imaginary part, that is, those two eigenvalues are real ones. The other two are conjugates, i.e., their imaginary parts are symmetric, one is negative and the other positive. The absolute value of the imaginary part of these eigenvalues decreases from infinity (occurring at a high value of $k$, corresponding to small scale perturbations), to zero (which occurs for small $k$ or large scale perturbations). The oscillations appear, or are more important, for different scales $k$ at different epochs. This conclusion perfectly complements the results obtained in the analysis of the real part, i.e., as time grows, oscillations seems to be more important for smaller values of $k$ (large scales). This means that the collapse of perturbations, even at large scales, is more and more difficult as time grows. As we noted above, this kind of behavior is due to the relative growth of density in the dark energy component. At early times, for $a = 0.1$ for example, when dark energy gives a little contribution to the total energy density of the universe, oscillations are only important for high values of $k$ (small scale perturbations). On the other hand, at present we can see that oscillations appear also for low values of $k$ (large scale perturbations).
2.2. Critical scale

Usually, oscillations are present when pressure acts in opposition to the self gravitation of the perturbed region. However, when gravity is strong enough, pressure no longer can support the perturbation and the collapse proceeds. Therefore we can use the absence of oscillations as a criterium to set a critical scale.

Figure 4 depict the time evolution of the critical scale defined this way. In this plot we can see a minimum, occurring when the critical scale $k_c$ changes behavior, from a decreasing to an increasing function of $a$. The minimum occurs for $a = 0.75 \pm 0.05$. This peak is hard to explain. May be it is connected to the epoch when the transition from decelerated to accelerated expansion occurs. Another possibility is that the change occurs when the period of oscillations exceeds the characteristic time of collapse. However, since in this paper we are doing a pure qualitative analysis, we don’t want to make a hasty conclusion on that issue, and we prefer to leave it for a forthcoming work on the subject.

3. The Modified Chaplygin Gas

In this section we will study another unified model. It is based on the original Chaplygin gas model, but constructed on the hypothesis of a modified equation of state. We make the same steps as above in order to study the perturbations behavior.

The modified Chaplygin gas (MCG) was introduced in 2002 by H.B.Benaoum [?], with an even more exotic equation of state, which has the following form:

$$ P = A\rho - \frac{B}{\rho^\alpha} $$  \hspace{1cm} (8)
where A, B and \( \alpha \) are constant parameters.

As well as in GCG, our first objectif is to compute the EoS as a function of the free parameters. It is assumed that for \( a = a_\ast = 1 \) the MGC has no pressure [?]. Notice that \( a_\ast \) does not correspond to the present time, it's the value of the scale factor for which \( P = 0 \). Then, for \( a = a_\ast \) one has \( \rho = \rho_\ast = \rho(a_\ast) \), and we may write,

\[
B = A \rho_\ast^{\alpha + 1}
\]

Using the continuity equation we obtain the final expression for the density \( \rho \), parametrised by \( A \) and \( \alpha \), and as a function of the scale factor \( a \), i.e.:

\[
\rho = \rho_\ast \left[ \frac{A}{1 + A} + \frac{a^{-3(1+\alpha)(1+A)}}{(1 + A)^{(1+\alpha)}} \right]^{\frac{1}{1+\alpha}}
\]

Since \( A \) and \( \alpha \) are constants, we can easily obtain the EoS of MGC (\( \omega_m \)), as a function of the scale factor \( a \),

\[
\omega_m = \frac{P}{\rho} = A - \frac{A}{1 + A} + \frac{a^{-3(1+\alpha)(1+A)}}{(1 + A)^{(1+\alpha)}}
\]

and it can be easily found that,

\[
\omega_m = \begin{cases} 
-1 & \text{for } a >> a_0 \\
0 & \text{for } a = a_\ast = 1 \\
A & \text{for } a << a_0 
\end{cases}
\]

From this function we see that the EoS of MCG describes the evolution of the universe from a matter dominated era to cosmological constant dominated epoch, that is, from DM domination to DE domination of the dynamics of the universe. As in GCG model, the behavior

Figure 4. Evolution of the critical scale as function of \( a \). The values of the parameters are \( \alpha = 0.5 \) and \( \bar{A} = 0.75 \).
Figure 5. Comparison between the equation of state parameter in the two models: GCG (left panel) and MCG (right panel).

of MCG model ensures that the same entity accounts for the dominant energetic component in the universe in different epochs, DM and DE.

Notice that $a \ll a_0$ corresponds mainly to an epoch when there was no dynamical domination of neither DM nor DE. The universe was radiation dominated. Thus $A$ must be close to 0.

The behavior of the sound speed $v_m$, is important for this study. In fact it allows us to figure out if the parameters $A$ and $\alpha$ are adequate, because if the sound speed exceeds the speed of light, then the parameters values should be revisited and adjusted.

$$v_m^2 = \frac{\partial P}{\partial \rho} = A - \frac{\alpha A}{1+A} + \frac{a^{-3(1+\alpha)(1+A)}}{(1+A)} \quad (13)$$

We have chosen values of the parameters that does not cause any troubles to the sound speed value, as, for instance $A = 0.061$ and $\alpha = 0.0053$ \cite{?}. However we have found some values, from different authors, that lead to sound speeds that exceed that of light!

Since we want to compare both models, GCG and MCG, it’s important to obtain the value of the scale factor for present epoch in the MCG model. We found that for MGC, present time correspond to a scale factor $a = 3.3$ (remember that in MCG, $a = 1$ when $P = 0$).

In Figure 5 we plot the two equations of state: $\omega_g$ and $\omega_m$, for the two models, GCG and MCG respectively.

At first sight, the two plots seem to behave alike, but even if at late times the asymptotic behavior is the same, i.e., $\omega_g \equiv \omega_m \rightarrow -1$, there is a slight difference. The parameter $\omega_g$ has a faster decrease than $\omega_m$, therefore, according to the plots, dark energy becomes dynamically dominant faster in GCG, than in MCG, i.e., $\omega_g$ approaches $-1$ faster than $\omega_m$. However, qualitatively, the transition from the matter dominated epoch to the epoch of dark energy domination is approximately the same. The transition phase occurs for GCG at $a_{tr}^g \simeq 0.78$, and for MCG at $a_{tr}^m \simeq 2.4$.

3.1. Perturbations in a MGC dominated Universe with Baryons

As we have done for the GCG model, we study the behavior of perturbations in a two fluid system (MCG + Baryons), and as above we got three differential equations that we transform into the following matrix equation
Figure 6. Behaviour of real part of $\lambda_i$ dependence on $k$ through epochs for MCG.

\[
\begin{bmatrix}
\delta_b \\
\delta_b' \\
\delta_g \\
\theta_g' \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{3}{2}\Omega_b & -2 - \zeta & \frac{3}{2}\Omega_m(1 + 3\nu_m^2) & 0 \\
0 & \omega_m + 1 & -3(\nu_m^2 - \omega_m) & -\frac{1 + \omega_m}{aH} \\
0 & 0 & \frac{v_m^2 k^2}{aH(1 + \omega_m)} & 3\nu_m^2 - 1 \\
\end{bmatrix}
\begin{bmatrix}
\delta_b \\
\delta_b' \\
\delta_g \\
\theta_g' \\
\end{bmatrix}
\] (14)

All the quantities depend on the parameters $\alpha$ and $A$ (constants), and on the scale factor $a$. The indexes ‘b’ and ‘m’ stand for baryons and MCG, respectively. Notice that we keep the same conditions as in the GCG case, that is, we assume a flat universe. However the Hubble parameter $H$ is normalised for the epoch when $a = a_0 = 1$ and not for the present time (i.e., $H_0 \neq 1$). As above, $k$ is the wavenumber of the perturbation.

3.1.1. Analysis of the real part of the eigenvalues and comparison with the GCG case

The behavior of the real part of the eigenvalues at different epochs is plotted in Figure 6. As can be seen, no matter the epoch $a$ and the scale $k$, there is always an eigenvalue with positive real part, so we have always a growing or collapse mode. Moreover, we notice a decrease through time in the eigenvalue that has the positive real part. Comparing with GCG (Figure 2), in both models, collapse occurs at all epochs, however, as we can see, the scale behavior is slightly different in each of them. In the DM dominated epoch ($a = 0.1$ for GCG and $a = 0.3$ for MGC), before the transition to the accelerated phase in both models, the plots of the $k$-dependence of the real part of the eigenvalues are almost identical, we just see a slightly shift in the scale $k$, and in the magnitude of the real part of the eigenvalues. Even for present time we observe an almost null but positive eigenvalue, for both models. The differences do not seem very important in both figures (2 and 6), even if we go further in time.
3.1.2. Analysis of the imaginary part of the eigenvalues and comparison with the GCG case

From Figure 7, we see that there are always two eigenvalues that have, in some range of $k$, a non null symmetric imaginary part, and two other ones that have a null imaginary part independently of the scale $k$. We note that the magnitudes of the imaginary parts are sensitive to the value of the scale factor $a$, for a fixed scale $k$. Perturbations seem to oscillate faster as time increases, i.e., the period of oscillations decreases. This is associated to a change in the frequency of oscillations for the same scale at different epochs, which we interpret as a consequence of the dynamical change in the cosmic background evolution. In fact, as dark energy becomes more effective with time, the universe tends to be $\Lambda$-dominated, according to the behavior of $\omega_m$.

Comparing with GCG (Figure 3), the plots do not show a big difference between the two models. They have an almost identical general qualitative behavior, and we can just notice a slight shift in the scale $k$ where the oscillations seem to arise. The shift in $k$ for the real part should be due to the arising of oscillations, or a non null imaginary part in the eigenvalues. The imaginary part doesn’t give us a lot of information, except the confirmation of the previous conclusions.

3.2. Critical scale

As a critical scale can not be inferred from the behavior of the real part of the eigenvalues, it is their imaginary part that enables us to set a criterium to define a critical scale as stated above.

From Figure 8 it is evident that there is almost no difference in the behavior of the critical scale in the two models. We clearly see a ‘peak’ in both models GCG and MCG, that corresponds, for the both models, to the transition epoch, i.e., from matter domination to dark energy domination.

Figure 7. Dependence on $k$ of the imaginary part of the eigenvalues for different epochs, or different values of the scale factor $a$, in a MCG–dominated universe.
Figure 8. Comparison between the critical scales in GCG and MCG, respectively.

epoch $a^{tr}_{g} = 0.78 \pm 0.05$ and $a^{tr}_{m} = 2.2 \pm 0.4$ respectively. This represents roughly the same epoch. However, we were not able to confirm that the ‘peak’ is a consequence of the transition phase.

4. Conclusion

In this work we started with the assumption that dark matter and dark energy are the same cosmological entity that manifests itself in two different ways depending on the cosmological context. We studied the qualitative behavior of gravitational instabilities in GCG and MCG separately, using linear perturbation theory, by solving the dynamical system constituted by four differential equations describing the behavior of perturbations in the two fluid system (Chaplygin Gas + baryons).

We made a semi-analytical analysis, using, for the numerical part of the work, software provided by Matlab. We plotted the obtained results and we made a qualitative analysis of the behavior of perturbations in each of the models to finally figure out how the two models compare qualitatively. Both models describe an unifying dark matter and dark energy behavior, and that’s the distinct behavior of the equation of state parameter $\omega$ in each model that dictates the differences between them. In fact the parameter $\omega$ in both models, GCG and MCG, implies a transition to an accelerating expansion era well before the present epoch. The analysis of the evolution of perturbations shows that a collapse mode always exists at all times for all scales. However, as expected, different time scales for collapse exist, as implied by the magnitudes of the real parts of the eigenvalues of the coefficients matrix of the dynamical system. The time evolution of the critical scale $k$, which is the scale above which oscillations in the perturbations appear. In both models we observe a minimum for that critical scale. This minimum corresponds to the epoch of the transition between the decelerating to an accelerating cosmological expansion, and as we observe, it’s corresponds approximately to the same cosmological epoch in both models. Unfortunately, qualitative study alone does not enabled us to do a deeper analysis of this behavior.

Finally, after analysing and compare both models, at the qualitative level, the GCG and MCG models describe a universe, where gravitational instabilities always growth (collapse). Both models seems to be almost identical in what concerns the behaviour of their perturbations, even if their exotic equations of state are different. However, we can not state, by these results alone, which model is a better fit to today observations, and we have to keep in mind that a
quantitative study is an important complement to the analysis we carried out in this work.

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