Multidimensional integrable boundary problems

Habibullin I.T. (Ufa, Institute of mathematics of RAS, Russia)
e-mail: ihabib@imat.rb.ru

The article deals with the problem of finding integrable boundary conditions for multidimensional equations. The research is stimulated by the following problem, posed by Richard Ward: Find discrete versions of the well known generalized Toda chains, corresponding to the Lie algebras of finite growth (see R.Ward, [1]). Really, one has to find cutting off constraint for the infinite discrete Toda chain

\[ e^{f_{uv} - f_u - f_v + f} = \frac{1 + h e^{f_k - f_v}}{1 + h e^{f_u - f_k}}, \] (1)

where \( f = f(u, v, k) \) is the field variable, and sub- and super-indices denotes shifts in the corresponding arguments: \( f_u = f(u + 1, v, k), \ f_v = f(u, v + 1, k), \ f^k = f(u, v, k + 1), \ f^{-k} = f(u, v, k - 1). \)

But in the paper we study the other well known model – the Hirota equation

\[ T_u T_v - TT_{uv} = T^k v T^{-k}, \] (2)

which is also a discrete version of the Toda chain. However the algorithm proposed can be applied to any kind (discrete and continuous [2]) of integrable equations with 3 independent variables, including the Ward’s problem. Cutting off constraint for Hirota’s equation are found corresponding to the generalized Toda chains of the series \( B_n \) and \( C_n. \)

We tried also to answer the question: What is the Lax pair of the boundary value problem? It differs from that of the equation. Our observation is that boundary condition put on the field variables generates some boundary conditions for the eigenfunctions (solutions to the linear Lax system). As a rule one has to take two different Lax pairs with the eigenfunctions related to each other at the border points. The best illustration to this idea provides the example of the Kadomtsev-Petviashvili (KP) equation below in the section 4. It can be checked by direct computation that the boundary problem \( B_5, \ B_6 \) admits the Lax pair \( B_7, \ B_8, \ B_9. \) In other words the boundary problem \( B_5, \ B_6 \) is a consistency condition of the linear boundary value problem \( B_7, \ B_8 \) with the time evolution \( B_9. \) It is remarkable that passing in the aforementioned example from the eigenfunctions to the Gel’fand-Levitan-Marchenko kernel leads to a linear boundary value problem which is solved by separation of variables.

1 Algorithm of searching for BC’s

How to extract the integrable BC’s by using only the Lax representation of the equation?

To answer the question let us concentrate on the example of the Toda chain

\[ u_{2t}(n) = \exp\{u(n - 1) - u(n)\} - \exp\{u(n) - u(n + 1)\}, \] (3)
Its Lax pair is defined by Laplace transformations for the following hyperbolic equation

\[ \phi_{xt}(n) = -u_x(n) \phi_t(n) - e^{u(n-1)-u(n)} \phi(n), \]
\[ \phi(n+1) = (D_x + u_x(n)) \phi(n), \]
\[ \phi_t(n) = -e^{u(n-1)-u(n)} \phi(n-1). \]

Take one more Lax pair of the same chain essentially different from the above one

\[ \psi_{xt}(n) = -u_t(n) \psi_x(n) - e^{u(n-1)-u(n)} \psi(n), \]
\[ \psi(n+1) = (D_t + u_t(n)) \psi(n), \]
\[ \psi_x(n) = -e^{u(n-1)-u(n)} \psi(n-1). \]

Proposition 1 (see [2]). Let the operator of the form

\[ M = aD^3_x + bD^2_x + cD_x + d \]

exist such that for \( n = 0 \) for each solution \( \psi \) of the equation \( \text{(7)} \) the function \( \phi = M \psi \) solves the equation \( \text{(4)} \). Then the field variables \( u(0), u(-1), u(-2) \) are connected by one of the constraint

1) \( e^{u(-1)} = 0, \)
2) \( u(-1) = 0, \)
3) \( u(-1) = -u(0), \)
4) \( u_x(-1) = -u_t(0)e^{-u(0)-u(-1)}, \)
5) \( e^{u(-2)-u(-1)}(u_x(-2) + u_x(-1)) = e^{u(-1)-u(0)}(6u_x(-1) - 2u_x(0)). \)

The corresponding differential operators \( M \) are of the form

1) \( M_1 = a_0 e^u (D_x^3 + 2u_x D_x^2 + (u_{xx} + u^2_x) D_x) + b_0 e^u (D_x^2 + u_x D_x) + c_0 e^u D_x, \)
2) \( M_2 = e^u D_x^2 + u_x e^u D_x, \)
3) \( M_3 = e^u D_x, \)
4) \( M_4 = e^u D_x^2 + u_x e^u D_x + e^{-u}, \)
5) \( M_5 = e^u D_x^3 + 2u_x e^u D_x +
+(u_{xx} - u_{xx}(-1) + u^2_x - u^2_x(-1))e^u D_x. \)

where \( a_0, b_0, c_0 \) – are real constant parameters, and \( u = u(0). \)

Conclusion: Under integrable boundary conditions two essentially different Lax pairs (which are not related to each other by gauge transform) become conjugate along the border. The conjugation relation is nothing else but the corresponding boundary condition for the eigenfunctions.
2 Boundary conditions for the KP equation consistent with the Lax pair

Apply the method suggested above to the KP equation

\[ u_t + u_{xxx} - 6uu_x = -3\alpha^2 w_y, \quad (12) \]
\[ w_x = u_y. \]

Remind the Lax pair for it which is defined as the following system of the linear equations

\[ \phi_{xx} = \alpha \phi_y + u \phi, \quad (13) \]
\[ \phi_t = -4\phi_{xxx} + 6u \phi_x + 3(u_x + \alpha w) \phi. \quad (14) \]

The dual pair is of the form:

\[ \tilde{\phi}_{xx} = -\alpha \tilde{\phi}_y + u \tilde{\phi}, \quad (15) \]
\[ \tilde{\phi}_t = -4\tilde{\phi}_{xxx} + 6u \tilde{\phi}_x + 3(u_x - \alpha w) \tilde{\phi}. \quad (16) \]

Lemma 1. Solution \( u = u(x, y, t) \) to the KP equation satisfies the boundary condition \( w|_{y=0} = 0 \) (or the boundary condition \( u_x + \alpha w|_{y=0} = 0 \)) iff for each solution \( \phi = \phi(x, y, t) \) of the equation (14) the function \( \tilde{\phi} = \phi|_{y=0} \) (respectively the function \( \tilde{\phi} = D_x \phi|_{y=0} \)) solves the equation (16) for \( y = 0 \).

3 Zakharov-Shabat dressing and the boundary problem

Let \( u_0 \equiv 0 \) be a starting solution to the KP and let \( \phi_0 = \phi_0(x, y, t) \) be a starting eigenfunction. Find a new (dressed) eigenfunction \( \phi = \phi(x, y, t) \), corresponding to the new (dressed) potential \( u \). To this end we use the following integral operator with Volterra kernel \( K = K(x, z, y, t) \) (see [3]):

\[ \phi(x, y, t) = \phi_0(x, y, t) + \int_{-\infty}^x K(x, z', y, t) \phi_0(z', y, t) dz' \quad (17) \]

Its kernel \( K \) satisfies the Gel’fand-Levitan-Marchenko equation

\[ K(x, z, y, t) + F(x, z, y, t) + \int_{-\infty}^z K(x, z', y, t) F(z', z, y, t) dz' = 0. \quad (18) \]

The kernel of the Gel’fand-Levitan-Marchenko equation \( F \) satisfies in turn a system of linear differential equations with constant coefficients

\[ \alpha \frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} = 0, \quad (19) \]
\[ \frac{\partial F}{\partial t} + 4\left( \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} \right) = 0. \quad (20) \]
The searched solution \( u \) of the nonlinear equation is defined through the kernel \( K \) by means of the formula

\[
    u(x, y, t) = 2 \frac{\partial}{\partial x} K(x, x, y, t).
\]  

(21)

Formulate now the dressing method based on the other Lax pair, then the system \((19)-(20)\) is replaced by the following system

\[
    -\alpha \frac{\partial \tilde{F}}{\partial y} - \frac{\partial^2 \tilde{F}}{\partial x^2} + \frac{\partial^2 \tilde{F}}{\partial z^2} = 0, \quad (22)
\]

\[
    \frac{\partial \tilde{F}}{\partial t} + 4\left(\frac{\partial^3 \tilde{F}}{\partial x^3} + \frac{\partial^3 \tilde{F}}{\partial z^3}\right) = 0, \quad (23)
\]

and the kernel \( \tilde{K} \) of the transformation operator, which converts the starting eigenfunction \( \tilde{\phi}_0 \) into an eigenfunction \( \tilde{\phi} \) of the new associated equation

\[
    \tilde{\phi}(x, y, t) = \tilde{\phi}_0(x, y, t) + \int_{-\infty}^{x} \tilde{K}(x, z', y, t) \tilde{\phi}_0(z', y, t) dz'. \quad (24)
\]

will solve the similar Gel’fand-Levitan-Marchenko equation

\[
    \tilde{K}(x, z, y, t) + \tilde{F}(x, z, y, t) + \int_{-\infty}^{x} \tilde{K}(x, z', y, t) \tilde{F}(z', y, t) dz' = 0. \quad (25)
\]

**Theorem** (see [2]). Involution

\[
    \frac{\partial F}{\partial x}(x, z, 0, t) = -\frac{\partial \tilde{F}}{\partial z}(x, z, 0, t) \quad (26)
\]

corresponds to the boundary condition

\[
    u_x + \alpha w|_{y=0} = 0, \quad (27)
\]

and the involution

\[
    F(x, z, 0, t) = \tilde{F}(x, z, 0, t), \quad (28)
\]

corresponds, respectively to the boundary condition

\[
    w|_{y=0} = 0. \quad (29)
\]

Consider in more details the KP2 equation, putting in all formulae above \( \alpha = 1 \). Let the Gel’fand-Levitan-Marchenko equation have the degenerate kernel

\[
    F(x, z, y, t) = \sum_{n=1}^{N} c_n \exp((q_n^2 - p_n^2)y - 4(q_n^3 + p_n^3)t + q_n z + p_n x). \quad (30)
\]

Then evidently

\[
    \tilde{F}(x, z, y, t) = \sum_{n=1}^{N} c_n \exp((q_n^2 - p_n^2)y - 4(q_n^3 + p_n^3)t + q_n x + p_n z). \quad (31)
\]
Suppose in addition that the kernels $F$ and $\tilde{F}$ are related by the involution (26), which can be represented as
\[
\sum_{n=1}^{N} c_n p_n \exp\{q_n z + p_n x - 4(q_n^3 + p_n^3)t\} =
\]
\[
= - \sum_{n=1}^{N} c_n p_n \exp\{q_n x + p_n z - 4(q_n^3 + p_n^3)t\}.
\]
It implies immediately that
\[
p_n = q_{N-n}, \quad c_n p_n = -c_{N-n} q_n.
\] (32)
Thus, under constraint (32) the solution of the KP equation given by the formulae
\[
u(x, y, t) = -2 \frac{\partial^2}{\partial x^2} \ln \det A,
\] (33)
where $A$ is a matrix with the entries
\[
A_{nm} = \delta_{nm} + \frac{c_n}{p_n + q_m} e^{((q_n^2 - p_m^2)y + 4(q_n^3 + p_m^3)t + (p_n + q_m)x)},
\] (34)
satisfies the boundary condition (27), which is $u_x + w|_{y=0} = 0$ for the KP2. Its differential consequence with respect to $x$ looks more illustrative: $u_{xx} + u_y|_{y=0} = 0$.

### 4 Boundary problem on the stripe $0 < y < 1$

Consider the boundary problem on the stripe $0 < y < 1$,
\[
u_t + u_{xxx} - 6u u_x = -3\alpha^2 w_y,
\] (35)
\[
w_x = u_y,
\]
\[
w|_{y=0} = 0, \quad w|_{y=1} = 0,
\] (36)
which is equivalent to a compounded Lax pair. Eigenfunction solves the boundary problem
\[
D_x^2 \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} D_y \begin{pmatrix} \phi \\ \psi \end{pmatrix} + u \begin{pmatrix} \phi \\ \psi \end{pmatrix},
\] (37)
with boundary conditions imposed along the same direct lines $y = 0$ and $y = 1$
\[
\phi - \psi|_{y=0} = 0, \quad \phi - \psi|_{y=1} = 0.
\] (38)
Due to the Lemma 1 the conditions (38) are consistent with the time evolution
\[
D_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = -4D_x^2 \begin{pmatrix} \phi \\ \psi \end{pmatrix} + 6u D_x \begin{pmatrix} \phi \\ \psi \end{pmatrix} + 3u_x \begin{pmatrix} \phi \\ \psi \end{pmatrix} +
\]
\[
+ w \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.
\] (39)
According to the theorem above the Marchenko kernel which is not scalar now but a vector-function satisfies the boundary value problem

$$\alpha D_y \left( \frac{F}{\bar{F}} \right) = D_x^2 \left( \frac{F}{\bar{F}} \right) - D_z^2 \left( \frac{F}{\bar{F}} \right)$$

(40)

with the conditions

$$F - \bar{F}|_{y=0} = 0, \quad F - \bar{F}|_{y=1} = 0. \quad (41)$$

Concentrate on the case of the degenerate kernel

$$F(x, z, y, t) = \sum_{n=1}^{N} c_n \exp((q_n^2 - p_n^2)y - 4(q_n^3 + p_n^3)t + q_nz + p_nx). \quad (42)$$

By means of the constraint $\bar{F}(x, z, y, t) = F(z, x, y, t)$ one gets

$$\bar{F}(x, z, y, t) = \sum_{n=1}^{N} c_n \exp((q_n^2 - p_n^2)y - 4(q_n^3 + p_n^3)t + q_nx + p_nz). \quad (43)$$

Substituting the boundary conditions in the points $y = 0$ and $y = 1$ one gets

$$c_n = c_{N-n}, \quad q_n = p_{N-n}, \quad p_{N-n} = \bar{p}_n, \quad p_n^2 - \bar{p}_n^2 = 2ik\pi. \quad (44)$$

The last constraint shows that the quantity $p_n$ is described by an entire and a real parameters $k \in \mathbb{Z}$ and $\beta_n \in \mathbb{R}$ such that

$$p_n = p_n(k) = \frac{k\pi}{2\beta_n} + i\beta_n$$

for $n = 1, \ldots N/2$.

One can replace each of the conditions (36) (or both simultaneously) by the condition $u_x + w|_{y=y_0} = 0$ taking either $y_0 = 0$ or $y_0 = 1$. Then the boundary conditions for the Lax pair (38) will be replaced by $\phi - \psi_x|_{y=0} = 0$, or respectively, $\phi - \psi_x|_{y=1} = 0$. The Gel’fand-Levitan-Marchenko kernel will satisfy instead of (41) the following boundary condition $F_x + \bar{F}_z|_{y=0} = 0$, or respectively, $F_x + \bar{F}_z|_{y=1} = 0$.

**Remark.** The problem is open how to apply the inverse scattering transform method to the problems on the stripe.

### 5 Cutting off constraint for discrete chains

The well known Hirota equation is a reduction of the following basic equation (see, for instance, [4])

$$(D_k - 1)(\frac{t_{-k}^{-k}t^k - t^k_{-k}}{tt_{uv}}) = 0, \quad (45)$$

Here $t = t(u, v, k)$ is the dependent variable. Super- and sub-indices denote shifts with respect to discrete arguments: $t^{-k} = t(u, v, k-1)$, $t^k = t(u, v, k+1)$, $t_u = t(u+1, v, k)$, $t_v = t(u, v+1, k)$. In order to get Hirota equation one should put the expression inside of the curly brackets in
The Lax pair for the equation (45) is given as pair of chains of Laplace transformations

\[ \psi_u = \psi^k + \frac{ttu}{t_kt_k} \psi, \]
\[ \psi_v = \psi + \frac{ttk}{tv} \psi^{-k}, \]

for the following discrete hyperbolic equation

\[ \psi_{uv} = \psi_u + \frac{tv}{tk} \psi_v - \frac{tt}{tk} \psi, \]

where \( \psi_u = \psi(u+1,v,k) \) and \( \psi_v = \psi(u,v+1,k) \) etc. (about discrete Laplace transforms see survey by I.A. Dynnikov, S.P. Novikov, [5]). The iterations of Laplace transform above are enumerated by upper index. The operators forming these transforms are \( D_u \) and \( D_v \). There is one more Lax pair for the equation (45), which is based on the other kind of the Laplace chain generated by operators \( D_u^{-1} \) and \( D_v^{-1} \)

\[ \phi_{-v} = \phi^k + \frac{tv}{tk} \phi, \]
\[ \phi_{-u} = \phi + \frac{uk}{tv} \phi^{-k}, \]

serving the following hyperbolic equation

\[ \phi_{uv} = \frac{tv}{tk} \phi_u + \frac{tv}{tk} \phi_v - \frac{tt}{tk} \phi. \]

Note that discrete hyperbolic equations admit two more pairs of mutually inverse Laplace transforms based on the operators \( D_u^{-1} \), \( D_v \), and \( D_u, D_v^{-1} \). So totally they have four kinds of Laplace transforms. Two pairs define as one could see above two different Lax pairs. We show below that two others define the cutting off constraint.

Proposition 2. Let the equations (45) and (51) be connected by substitutions

\[ \psi = (aD_u + b)\phi, \]

then the following boundary condition (cutting off constraint) is satisfied

\[ (D_u - 1) \frac{tt}{(tk)^2} = 0. \]

And the substitution and its inversion are of the form

\[ \psi = \frac{tv}{tk} (1 - D_u)\phi, \quad \phi = \frac{tv}{tk} (D_v^{-1} - 1)\psi. \]

These substitutions are nothing else but the mutually inverse Laplace transforms with operators \( D_u \) and \( D_v^{-1} \), respectively. By replacing \( u \leftrightarrow v \) one would get the other kind of BC’s. Let us reduce the chain (45) into a segment \( k \in (0, N) \) by setting the BC’s at the ends \( k = 0, k = N \):

\[ t(u,v,-1) = 0; \quad (D_u - 1) \frac{tt}{(tk)^2} \bigg|_{k=N} = 0; \]
Eigenfunctions \( \phi(u,v,k) \) and \( \psi(u,v,k) \) are also restricted to the same segment \( k \in (0, N) \). At the left end the linear equations of the Lax pair are cut evidently under BC. At the right end one has to prescribe any of two equivalent equations \((52)\) taken at the point \( k = N \).

Proposition 3. Let the equations \((48)\) and \((51)\) be connected by second order substitutions

\[
\begin{align*}
\psi &= (D_u - 1)z(D_u - 1)\phi, \\
\phi &= (D_v^{-1} - 1)z(D_v^{-1} - 1)\psi,
\end{align*}
\]

\((53)\)
defined as compositions of two Laplace transforms, where \( z = \frac{t_u}{t_T} \), then the following boundary condition is satisfied

\[
(D_u - 1)\frac{t}{t^u - 1} = 0.
\]

The substitutions found allows one to cut up the infinite Lax pairs and adopt them to restricted chains.

6 Cutting off constraint for Hirota’s equation

Let us pass to the Hirota equation

\[
T_uT_v - TT_{uv} = T_k^kT_u^{-k}.
\]

\((54)\)

For this reduction the boundary conditions above can be integrated under which they take (up to the point symmetries) the following forms

i) \( T(u, v + 1, 1) = T(u + 1, v, -1) \);

ii) \( T(u, v, 0) = T(u + 1, v, -1) \).

In the continual limit the Hirota equation approaches the Toda chain (see, for instance, \([4]\)), the boundary conditions found above turn into cutting off constraint for the chain corresponding to the Lie algebras of series \( C_n \) and \( B_n \). The case corresponding to the series \( A_n \) is well known, it has been found years ago. But the case of \( D_n \) is not found yet, it is connected with rather labour consuming computations.

The discrete Toda chain \((\text{I})\) can also be studied by the method above.

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