Stability of $C_0$-quasi semigroups in Banach spaces

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Abstract. We concern on the non-autonomous abstract Cauchy problems $\dot{x}(t) = A(t)x(t)$ on Banach spaces $X$. If $A(t)$ is the infinitesimal generator of a $C_0$-quasi semigroup $R(t,s)$ on $X$, then the solution of the equation has uniquely representation $x(t) = R(0,t)x_0$. This representation shows that the stability of the quasi semigroup $R(t,s)$ influences the stability of the solution. In this paper, we investigate the stabilities of $C_0$-quasi semigroups following the existing theory of stabilities of $C_0$-semigroups $T(t)$ and bounded evolution operators $U(t,s)$. We devote the uniform, exponential, and strong stability of $C_0$-quasi semigroups in Banach spaces. The results are applicable for a large class of the time-dependent differential equations with unbounded coefficients in Banach spaces.

1. Introduction

Consider the non-autonomous abstract Cauchy problems

\[
\begin{cases}
\dot{x}(t) = A(t)x(t), & t \geq 0 \\
x(0) = x_0, & x_0 \in X
\end{cases}
\]

on Banach spaces $X$ (see [1] and [2]). The $x$ is an unknown function from the real interval $[0, \infty)$ into a Banach space $X$ and $A(t)$ is a closed linear operator in $X$ with domain $\mathcal{D}(A(t)) = \mathcal{D}$, independent of $t$ and dense in $X$. The restrictive assumptions guarantee the existence of solutions of (1) (see [1] and [2]). Let $U(t,s)$ be an evolution family of bounded linear operators on $X$. The solution of (1) is induced by evolution operators $U(t,s)$,

\[
x(t) = U(t,0)x_0.
\]

This representation shows that the stability of the family $U(t,s)$ influences the behavior of the state $x(t)$ as time $t$ evolves.

The problem (1) can be solved using another approach called quasi semigroup $R(t,s)$ on Banach space $X$. In this context $A(t)$ is the infinitesimal generator of $R(t,s)$ with domain $\mathcal{D}$ and for $x_0 \in \mathcal{D}$ the solution of (1) is given by $x(t) = R(0,t)x_0$. Definition of an infinitesimal generator and properties of quasi semigroups can be reviewed in detail in [3], [4], [5], [6], [7] and the references therein. By the following definition we verify that every $C_0$-semigroup on $X$ is a $C_0$-quasi semigroup on $X$, but it is not conversely.
Let $X$ be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators on $X$. A two-parameter commutative family $\{R(t,s)\}_{s,t \geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup, in short $C_0$-quasi semigroup, on $X$ if for each $r,s,t \geq 0$ and $x \in X$:

(a) $R(t,0) = I$, identity operator on $X$,
(b) $R(t,s + r) = R(t + r,s)R(t,r)$,
(c) $\lim_{s \to 0^+} \|R(t,s)x - x\| = 0$,
(d) there exists a continuous increasing function $M : [0, \infty) \to [0, \infty)$ such that

$$\|R(t,s)\| \leq M(s).$$

If in (d) $M(s) = 1$ for all $s \geq 0$, then $\{R(t,s)\}_{s,t \geq 0}$ is called contraction. Here we note that the norm $\| \cdot \|$ of (c) and of (d) acts on $X$ and on $\mathcal{L}(X)$, respectively. In the sequel for simplicity we denote the quasi semigroup $\{R(t,s)\}_{s,t \geq 0}$ by $R(t,s)$. Next, by (d) of the definition of $C_0$-quasi semigroup if $R(t,s)$ is a $C_0$-quasi semigroup on a Banach space $X$ and $R'(t,s)$ is the dual of $R(t,s)$ on the dual space $X'$, then for every $x \in X$ and $x' \in X'$ the map

$$t \mapsto \|R(t,s)x\| \quad \text{and} \quad s \mapsto \|R'(t,s)x'\|$$

are measurable. These properties facilitate in characterizing of stability of $C_0$-quasi semigroups.

The most noted theory of stability of linear evolution operators has been proven by Datko [8]. In a few years later Pazy [9] completed the Datko’s result and this result is well-known by Datko-Pazy Theorem. Generalizations of this result are obtained in [10], [11], [12], and [13] for exponential stability, in [14] and [15] for exponential instability, in [16] and [17] for exponential dichotomy, and in [18] the case of polynomial stability. In particular for autonomous case, the role of the family $U(t,s)$ is played by a $C_0$-semigroup $T(t)$. Some developments of the stability of $C_0$-semigroups are found in [19], [20], [21], [22], [23], [24], and [25]. The developments of concepts of stability include uniformly stable, exponentially stable, strongly stable, weakly stable, and polynomially stable. Some results use the characterization of spectrum or resolvent of the infinitesimal generator of $C_0$-semigroup in proofs. These approaches are different from the Datko-Pazy Theorem approach. The Datko’s proof use idea of a Lyapunov functional in Hilbert Space while the Pazy’s proof base on the integral of norm of evolution operator.

By a quasi semigroup approach Megan and Cuc [6] generalized the Datko’s result to investigate the exponential stability. Megan and Cuc [6] construct the sufficient and necessary conditions for exponential stability using an admissible increasing function as upper bound of integral of norm of the quasi semigroup either using an admissible increasing sequence. Moreover, Cuc [26] achieved to construct the necessary and sufficient conditions for uniform exponential dichotomy of $C_0$-quasi semigroup in Banach spaces. The obtained results generalize the similar theorem which are obtained by Datko [8] and Pazy [9] for the exponential stability and Preda and Megan [17] for the exponential dichotomy of $C_0$-quasi semigroups, respectively. Currently there has been no research that addresses the strong stability of $C_0$-quasi semigroups in Banach spaces. Besides that the results of Datko [8] and Megan and Cuc [6] give a possibility of alternative proof for the sufficiency and necessity for uniform and exponential stability.

In this paper we characterize uniform stability, exponential stability, and strong stability of $C_0$-quasi semigroups in Banach spaces. Specially, we investigate the sufficient and necessary conditions for uniform and exponential stability with a different approach to [6]. These are the main objects of this paper that is organized into the following three sequential sections. The stability and uniform stability are characterized in Section 2. Section 3 deals with the exponential stability in various types. The strong stability is characterized in Section 4.
2. Uniformly Stable

We begin with the non-uniform stability or stability of the $C_0$-quasi semigroups on a Banach space. This is the simplest stability.

**Definition 2.1.** A $C_0$-quasi semigroup $R(t, s)$ on a Banach space $X$ is said to be:

(a) stable if there exists an increasing function $N : \mathbb{R}^+ \to (0, \infty)$ such that

$$\|R(t, s)x\| \leq N(t)\|x\|$$

for all $t, s \geq 0$ and $x \in X$;

(b) uniformly stable if there exists a positive constant $N > 0$ such that

$$\|R(t, s)x\| \leq N\|x\|$$

for all $t, s \geq 0$ and $x \in X$.

The Definition 2.1 states that if $R(t, s)$ is uniformly stable, then $R(t, s)$ is stable. The converse is not valid.

**Example 2.2.** The $C_0$-quasi semigroup $R(t, s)$ defined by

$$R(t, s)x = \frac{1 + t}{1 + t + s}x$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$, is uniformly stable on $\mathbb{R}$.

The following theorem is the main result of this section.

**Theorem 2.3.** Let $R(t, s)$ be a $C_0$-quasi semigroup on a Banach space $X$. The following statements are equivalent.

(a) $R(t, s)$ is uniformly stable.

(b) There are $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $N_p$, and $N'_q$ such that

$$\sup_{t \geq 0, s > 0} \left( \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{\frac{1}{p}} \leq N_p\|x\|$$

(3)

for all $x \in X$, and

$$\sup_{t \geq 0, s > 0} \left( \frac{1}{s} \int_0^s \|R'(t, v)x'\|^q dv \right)^{\frac{1}{q}} \leq N'_q\|x'\|$$

(4)

for all $x' \in X'$.

**Proof.** (a) $\Rightarrow$ (b). By hypothesis there is a positive constant $N > 0$ such that

$$\|R(t, s)x\| \leq N\|x\|$$

for all $t, s \geq 0$ and $x \in X$. Since the inequality holds for every $t, s \geq 0$, then for $p > 1$ and $s \geq v$ it follows

$$\left( \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{\frac{1}{p}} \leq N\|x\|$$

for all $x \in X$. Hence $R(t, s)$ satisfies (3) with $N_p = N$. Since $\|R(t, s)x\| = \|R'(t, s)x'\|$, analogously we can prove that $R(t, s)$ satisfies (4) with $N'_q = N$.  

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(b) ⇒ (a). The Hölder inequality implies
\[
\left| x'(R(t, s)x) \right| = \int_0^s |R'(t, v)x'(R(t + v, s - v)x)| dv, \quad s \geq v
\]
\[
\leq \int_0^s \|R'(t, v)x'\| \|R(t + v, s - v)x\| dv
\]
\[
\leq \left( \int_0^s \|R'(t, v)x'\|^q dv \right)^{1/q} \left( \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{1/p}.
\]
Hence
\[
\left| x'(R(t, s)x) \right| \leq \left( \frac{1}{s} \int_0^s \|R'(t, v)x'\|^q dv \right)^{1/q} \left( \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{1/p}.
\]
By hypothesis there are \( N_q, N_p > 0 \) such that
\[
\left| x'(R(t, s)x) \right| \leq N_q' N_p \|x'\| \|x\|.
\]
(5)

On the other hand we have
\[
\left| x'(R(t, s)x) \right| \leq \|x'\| \|R(t, s)\| \|x\|.
\]
By definition of \( \|R(t, s)\| \) it follows
\[
\|R(t, s)\| \leq N_q' N_p := N < \infty
\]
for all \( t, s \geq 0 \). This states that \( R(t, s) \) is uniformly stable.

The implication (b) ⇒ (a) in Theorem 2.3 is not valid whenever (3) either (4) does not hold. The modifying of Example 2 of [11] with \( R(t, s)x(\xi) = T(s)x(\xi) \) describes this situation.

3. Exponentially Stable

In this section we observe the exponential stability of \( C_0 \)-quasi semigroups which is more special than the uniform stability. We begin with term of uniformly exponentially bounded of quasi semigroups.

**Definition 3.1.** A \( C_0 \)-quasi semigroup \( R(t, s) \) on a Banach space \( X \) is said to be uniformly exponentially bounded if there are \( \omega \in \mathbb{R} \) and \( N_\omega \geq 1 \) such that
\[
e^{\omega s} \|R(t, s)x\| \leq N_\omega
\]
for all \( t, s \geq 0 \).

There exists a \( C_0 \)-quasi semigroup \( R(t, s) \) which is not uniformly exponentially bounded, see Remark 1 of [11]. Specially, if \( R(t, s) = T(s) \), where \( T(s) \) is a \( C_0 \)-semigroup on Banach space \( X \), then \( R(t, s) \) is uniformly exponentially bounded on \( X \). Example 1 of [11] shows that the converse is not valid.

**Definition 3.2.** A \( C_0 \)-quasi semigroup \( R(t, s) \) on a Banach space \( X \) is said to be:
(a) exponentially stable on \( X \) if there are constant \( \alpha > 0 \) and increasing function \( N : \mathbb{R}^+ \to [1, \infty) \) such that
\[
e^{\alpha s} \|R(t, s)x\| \leq N(t) \|x\|
\]
for all \( t, s \geq 0 \) and \( x \in X \);
(b) uniformly exponentially stable on $X$ if there are constants $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha s}||R(t, s)x|| \leq N||x||$$

for all $t, s \geq 0$ and $x \in X$;

(c) exponentially stable in the Barreira-Valls sense if there are constants $N \geq 1, \beta \geq \alpha > 0$ such that

$$e^{\alpha s}||R(t, s)x|| \leq Ne^{(\beta - \alpha)t}||x||$$

for all $t, s \geq 0$ and $x \in X$.

From Definition 3.2 it is obvious that if $R(t, s)$ is uniformly exponentially stable, then $R(t, s)$ is uniformly stable and exponentially stable in the Barreira-Valls sense. If $R(t, s)$ is exponentially stable in the Barreira-Valls sense, then $R(t, s)$ is exponentially stable. If $R(t, s)$ is exponentially stable, then $R(t, s)$ is stable.

If $R(t, s)$ is uniformly exponentially bounded, then we can choose the positive constant $\omega$ such that (6) holds. Therefore, if $R(t, s)$ is uniformly exponential stable, then $R(t, s)$ is uniformly exponentially bounded. Example 2.2 describes that there exists a uniformly stable quasi semigroup but it is not exponentially stable.

**Example 3.3.** The $C_0$-quasi semigroup $R(t, s)$ defined by

$$R(t, s)x = e^{-\beta s}x, \ \beta > 0$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$, is uniformly exponentially stable on $\mathbb{R}$ (with $0 < \alpha < \beta$).

**Example 3.4.** Given a $C_0$-quasi semigroup $R(t, s)$ by

$$R(t, s)x = e^{(t+s)\sin(t+s) - t\sin t - 3s}x$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$. The quasi semigroup is exponentially stable in the Barreira-Valls sense (with $\alpha = 2$ and $\beta = 4$), but it is not uniformly exponentially stable on $\mathbb{R}$.

Supposed that $R(t, s)$ is uniformly exponentially stable on $\mathbb{R}$. There are constants $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha s}|R(t, s)x| = e^{\alpha s}e^{(t+s)\sin(t+s) - t\sin t - 3s}|x| \leq N||x||$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$. However, if we set $t = 2n\pi - \frac{\pi}{2}$ and $s = \frac{\pi}{2}$,

$$e^{2(n-1)\pi} \leq Ne^{\alpha(\pi/2)},$$

which is a contradiction for sufficiently large $n$.

**Example 3.5.** Given a function $u : \mathbb{R}^+ \rightarrow [1, \infty)$ where $u(n) = e^{n^2}$ and $u \left(n + \frac{1}{2n}\right) = 1$ for non-negative integer $n$. Let $R(t, s)$ be a $C_0$-quasi semigroup defined

$$R(t, s)x = \frac{u(t)}{u(t + s)}e^{s}x$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$. The $R(t, s)$ is exponentially stable but it is not exponentially stable in the Barreira-Valls sense on $\mathbb{R}$.
If we suppose that \( R(t,s) \) is exponentially stable in the Barreira-Valls sense, then there are constants \( N \geq 1 \) and \( \beta \geq \alpha > 0 \) such that

\[
e^{\alpha s} \| R(t,s)x \| \leq Ne^{(\beta - \alpha)t} \| x \|
\]

for all \( t, s \geq 0 \) and \( x \in \mathbb{R} \). However, for \( t = n \) dan \( s = \frac{1}{2n} \) we have

\[
e^{n(2^n - (\beta - \alpha))}e^{(\alpha - 1)\frac{1}{2^n}} \leq N
\]

which is a contradiction for sufficiently large \( n \).

Next we investigate the sufficient and necessary conditions for the uniform exponential stability of \( C_0 \)-quasi semigroup. The following results are generalization of the exponential stability of \( C_0 \)-semigroups and family of bounded evolution operators. The results are also the alternative sufficient conditions of [6].

Lemma 3.6. If \( C_0 \)-quasi semigroup \( R(t,s) \) is uniformly exponential bounded on a Banach space \( X \), then \( R(t,s) \) is uniformly exponentially stable on \( X \).

Proof. By the definition of \( C_0 \)-quasi semigroup there is a function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[
\inf_{t,s \geq 0} g(t + s) < 1 \quad \text{dan} \quad \| R(t,s) \| \leq g(s),
\]

for all \( t, s \geq 0 \). Lemma 1 of [11] implies that \( R(t,s) \) is uniformly exponentially stable on \( X \). ■

Theorem 3.7. Let \( R(t,s) \) be a uniformly exponentially bounded \( C_0 \)-quasi semigroup on a Banach space \( X \). The following statements are equivalent.

(a) \( R(t,s) \) is uniformly exponentially stable.

(b) There exist \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), \( N_p \), and \( N'_q \) such that

\[
\sup_{t \geq 0, s > 0} \left( \frac{1}{s} \int_0^s \| R(t + v, s - v)x \|^p dv \right)^{\frac{1}{p}} \leq N_p \| x \| \tag{7}
\]

for all \( x \in X \), and

\[
\sup_{t,s \geq 0} \left( \int_0^s \| R'(t,v)x' \|^q dv \right)^{\frac{1}{q}} \leq N'_q \| x' \| \tag{8}
\]

for all \( x' \in X' \).

Proof. Analogously to the proof of Theorem 2.3, the implication \( (a) \Rightarrow (b) \) is trivial.

\( (b) \Rightarrow (a) \). By hypothesis there are \( \omega \in \mathbb{R} \) and \( N_\omega \geq 1 \) such that

\[
e^{\omega s} \| R(t,s) \| \leq N_\omega
\]

for all \( t, s \geq 0 \). If \( s \geq 1 \), then for every \( x' \in X' \) implies

\[
\left( \frac{1}{s} \int_0^s \| R'(t,v)x' \|^q dv \right)^{\frac{1}{q}} \leq \left( \int_0^s \| R'(t,v)x' \|^q dv \right)^{\frac{1}{q}} \leq N'_q \| x' \|.
\]

Conversely, if \( 0 < s < 1 \), then

\[
\left( \frac{1}{s} \int_0^s \| R'(t,v)x' \|^q dv \right)^{\frac{1}{q}} \leq \left( \frac{1}{s} \int_0^s N'_q e^{\omega q} \| x' \|^q dv \right)^{\frac{1}{q}} \leq N_\omega e^{\omega} \| x' \|.
\]
From the both we conclude that if the condition (8) is valid, then the inequality

$$\sup_{t,s \geq 0} \left( \frac{1}{s} \int_0^s \| R'(t,v)x' \|^q dv \right)^\frac{1}{q} \leq N'_q \| x' \|$$

for all $x' \in X'$ is. Hence, Theorem 2.3 implies that the $C_0$-quasi semigroup $R(t,s)$ is uniformly stable on $X$. Therefore, there is a constant $N_1 > 0$ such that

$$\| R(t,s) \| \leq N_1$$  \hspace{1cm} (9)

for all $t,s \geq 0$.

On the other hand for $s > v \geq 0$ we have

$$s |x'(R(t,s)x)| \leq \left( \int_0^s \| R'(t,v)x' \|^q dv \right)^\frac{1}{q} \left( \frac{1}{s} \int_0^s \| R(t+v,s-v)x \|^p dv \right)^\frac{1}{p} \frac{1}{s^\frac{1}{p}}$$

$$\leq N_p \| x \| s^\frac{1}{p} \left( \int_0^s \| R'(t,v)x' \|^q dv \right)^\frac{1}{q} \leq N_p N'_q \| x \| \| x' \| s^\frac{1}{p}.$$

Consequently, there is a constant $N_2 > 0$ such that

$$s^\frac{1}{p} \| R(t,s) \| \leq N_2$$  \hspace{1cm} (10)

for all $t,s \geq 0$. By adding (9) and (10) we have

$$\| R(t,s) \| \leq \frac{N_1 + N_2}{1 + \frac{1}{s^\frac{1}{p}}} := g(s).$$

According to Lemma 3.6, the $R(t,s)$ is uniformly exponentially stable on $X$. \hfill \Box

**Corollary 3.8.** Let $R(t,s)$ be a uniformly stable $C_0$-quasi semigroup on a Banach space $X$. The $R(t,s)$ is exponentially stable on $X$ if and only if $R(t,s)$ satisfies (8).

By implementing the principle duality to Theorem 3.7 we obtain the following theorem.

**Theorem 3.9.** Let $R(t,s)$ be a uniformly stable $C_0$-quasi semigroup on a Banach space $X$. The following statements are equivalent.

(a) $R(t,s)$ is uniformly exponentially stable.

(b) There exist $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $N_p$, and $N'_q$ such that

$$\sup_{t,s \geq 0} \left( \frac{1}{s} \int_0^s \| R'(t,v)x' \|^q dv \right)^\frac{1}{q} \leq N'_q \| x' \|$$  \hspace{1cm} (11)

for all $x' \in X'$, and

$$\sup_{t \geq 0, s > 0} \left( \int_0^s \| R(t+v,s-v)x \|^p dv \right)^\frac{1}{p} \leq N_p \| x \|$$  \hspace{1cm} (12)

for all $x \in X$.

The following theorem shows that the uniformly stable assumption in Corollary 3.8 can be replaced by the exponentially bounded assumption.
Theorem 3.10. Let $R(t,s)$ be uniformly exponentially bounded $C_0$-quasi semigroup on a space Banach $X$. The following statements are equivalent.

(a) $R(t,s)$ is uniformly exponentially stable.

(b) There exist $q \geq 1$ and $N_q'$ such that

$$\sup_{t,s \geq 0} \left( \int_0^s \|R'(t,v)x'\|^q dv \right)^{\frac{1}{q}} \leq N_q' \|x'\|$$

for all $x' \in X'$.

Proof. Analogously to proof of Theorem 2.3, the implication (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a). By hypothesis there exist $\omega > 0$ and $N_\omega \geq 1$ such that

$$\|R(t,s)\| \leq N_\omega e^{\omega s}$$

for all $t, s \geq 0$. For $q > 1$ is fixed. If $s \geq 1$, then for every $x' \in X'$

$$\int_0^1 N_\omega^{-q} e^{-\omega qu} du |x'(R(t,s)x)|^q \leq \left( \int_0^s \|R'(t,v)x'\|^q \|R(t+s-v, t-v)\|^q N_\omega^{-q} e^{-\omega qu} dv \right) \|x\|^q \leq \left( \int_0^s \|R'(t,v)x'\|^q dv \right) \|x\|^q \leq \left( N_q' \right)^{q} \|x'\|^q \|x\|^q.$$

On the other hand we have

$$|x'(R(t,s)x)|^q \leq \|R(t,s)||^q \|x'\|^q \|x\|^q.$$

Therefore

$$\|R(t,s)\| \leq H(\omega) N_\omega N_q'$$

for all $t \geq 0$, where $H(\omega) = \left( \int_0^1 e^{-\omega qu} du \right)^{-1}$.

Conversely, if $0 \leq s < 1$, then by (13) we have

$$\|R(t,s)\| \leq N_\omega e^\omega$$

for all $t \geq 0$. Thus, $R(t,s)$ is uniformly stable on $X$. Corollary 3.8 implies that $R(t,s)$ is uniformly exponentially stable on $X$.

For $q = 1$ and the fact that $R(t,s)$ is a quasi semigroup, then for $s > v \geq 0$ we have

$$s|x'(R(t,s)x)| \leq \int_0^s \|R'(t,v)x'||R(t+v, s-v)x| dv$$

$$\leq \left( \int_0^s \|R'(t,v)x'|| dv \right) M(s) \|x\| \leq N_1' \|x'\| M(s) \|x\|.$$

Hence

$$\|R(t,s)\| \leq \frac{N_1' M(s)}{s} := g(s)$$

for all $t, s \geq 0$. Lemma 3.6 implies that $R(t,s)$ is uniformly exponentially stable on $X$. ■
4. Strongly Stable
In this section we consider a strong stability concept which is weaker than the uniform exponential stability.

**Definition 4.1.** A $C_0$-quasi semigroup $R(t,s)$ on a Banach space $X$ is said to be:
(a) strongly stable if
\[ \lim_{s \to \infty} \| R(t,s)x \| = 0 \]
for all $t \geq 0$ and $x \in X$;
(a) uniformly strongly stable if
\[ \lim_{t,s \to \infty} \| R(t,s)x \| = 0 \]
for all $x \in X$.

From the definition it is obvious that every uniformly strongly stable $C_0$-quasi semigroup is
a strongly stable $C_0$-quasi semigroup but the converse is not valid.

**Example 4.2.** Given a $C_0$-quasi semigroup by
\[ R(t,s)x = \frac{1 + t}{1 + t + s} x \]
for all $t,s \geq 0$ and $x \in \mathbb{R}$. The $R(t,s)$ is strongly stable, but it is not uniformly strongly stable on $\mathbb{R}$.

It is obvious that $R(t,s)$ is strongly stable on $\mathbb{R}$. However, for $t = s$ and $x \neq 0$ we have
\[ \lim_{t,s \to \infty} |R(t,s)x| = \lim_{s \to \infty} \frac{1 + s}{1 + 2s} |x| = \frac{1}{2} |x| \neq 0. \]

So, $R(t,s)$ is not uniformly strongly stable on $\mathbb{R}$.

**Example 4.3.** Given a $C_0$-quasi semigroup $R(t,s)$ by
\[ R(t,s)x = e^{-(s^2 + 2st)} x \]
for all $t,s \geq 0$ and $x \in \mathbb{C}$. The $R(t,s)$ is uniformly strongly stable on $\mathbb{C}$.

It follows that
\[ \lim_{t,s \to \infty} |R(t,s)x| = \lim_{t,s \to \infty} \frac{1}{e^{s^2 + 2st}} |x| = 0 \]
for all $x \in \mathbb{C}$. So, $R(t,s)$ is uniformly strongly stable on $\mathbb{R}$.

**Lemma 4.4.** If $R(t,s)$ is a uniformly strongly stable $C_0$-quasi semigroup on a Banach space $X$, then $R(t,s)$ is uniformly stable on $X$.

**Proof.** By hypothesis and fact the function $s \mapsto \| R(t,s) \|$ is bounded on every bounded interval $[0, \alpha]$, the Uniformly Boundedness Theorem (Theorem A.3.19) of [27] implies that $R(t,s)$ is uniformly stable on $X$. \[\square\]

**Theorem 4.5.** Let $R(t,s)$ be a uniformly stable $C_0$-quasi semigroup on a Banach space $X$ and $x \in X$.

(a) If there exists an unbounded sequence $(s_n) \subseteq \mathbb{R}^+$ such that $\| R(t,s_n)x \| \to 0$ for $n \to \infty$, then $\lim_{s \to \infty} \| R(t,s)x \| = 0$ for every $t \geq 0$. 

Theorem 4.7. Let $R(t,s)$ be a $C_0$-quasi semigroup of contraction, then $\lim_{s \to \infty} \|R(t,s)x\|$ exists, for every $t \geq 0$.

Proof. (a) By Lemma 4.4, $R(t,s)$ is uniformly stable. Set $N = \sup_{t,s \geq 0} \|R(t,s)\|$. By boundedness of $(s_n)_n$, for every $n \in \mathbb{N}$ there is $s$ such that $v_n = s - s_n > 0$. In this case we can choose $(v_n)$ to be an increasing sequence. The hypothesis implies that $\|R(t,v_n)x\| \to 0$ for every $t \geq 0$ as $n \to \infty$. Given any $\epsilon > 0$. There is $K \in \mathbb{N}$ such that for $n \geq K$ we have

$$\|R(t,v_n)x\| < \epsilon.$$ 

Consequently, for all $n \geq K$ we have

$$\|R(t,s)x\| \leq \|R(t,v_n,s_n)\|\|R(t,v_n)x\| \leq N\epsilon$$

for every $x \in X$ and $t \geq 0$. So, $\|R(t,s)x\| \to 0$ as $s \to \infty$ for every $t \geq 0$ and $x \in X$.

(b) For $t \geq 0$ fixed and $x \in X$. The function $s \mapsto \|R(t,s)x\|$ is decreasing. In facts, if $s_1 < s_2$, where $s_2 = s_0 + s_1, s_0 > 0$ and $R(t,s)$ is quasi semigroup of contraction, then

$$\|R(t,s_2)x\| = \|R(t+s_1,s_0)R(t,s_1)x\| \leq \|R(t+s_1,s_0)\|\|R(t,s_1)x\| \leq \|R(t,s_1)x\|.$$ 

Therefore, there is an unbounded sequence $(s_n)_n \subseteq \mathbb{R}^+$ such that $(\|R(t,s_n)x\|)$ is a bounded decreasing sequence of real numbers. The Convergence Monotone Theorem implies that for every $t \geq 0$, $\lim_{s \to \infty} \|R(t,s)x\|$ exists.

Corollary 4.6. Let $R(t,s)$ be a uniformly stable $C_0$-quasi semigroup on a Banach space $X$. The following statements are equivalent.

(a) $R(t,s)$ is strongly stable.

(b) For all $x \in X$, $t \geq 0$, and $p \geq 1$

$$\lim_{s \to \infty} \frac{1}{s} \int_0^s \|R(t,v)x\|^p dv = 0$$

Proof. (a) $\Rightarrow$ (b). This necessity is trivial.

(b) $\Rightarrow$ (a). By uniform stability we have

$$s\|R(t,s)x\|^p = \int_0^s \|R(t,s)x\|^p dv \leq \int_0^s \|R(t+s-v,v)R(t,s-v)x\|^p dv \leq Np \int_0^s \|R(t,u)x\|^p du.$$ 

Therefore,

$$\|R(t,s)x\|^p \leq Np \left( \frac{1}{s} \int_0^s \|R(t,u)x\|^p du \right). \quad (14)$$

It follows that the right hand side of (14) converges to zero as $s \to \infty$, so is the left hand side. Thus, $R(t,s)$ is strongly stable on $X$.

The following theorem is a generalization of Corollary 4.6 neglecting the uniform stability.

Theorem 4.7. Let $R(t,s)$ be a $C_0$-quasi semigroup on a Banach space $X$. The following statements are equivalent.

(a) $R(t,s)$ is strongly stable.
(b) There exist \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and a function \( N_q' : \mathbb{R}^+ \to (0, \infty) \) such that

\[
\lim_{s \to \infty} \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv = 0
\]

for all \( x \in X \) and \( t \geq 0 \), and

\[
\sup_{s > 0} \left( \frac{1}{s} \int_0^s \|R'(t, v)x'\|^q dv \right)^{\frac{1}{q}} \leq N_q'(t)\|x'\|
\]

for all \( x' \in X' \) and \( t \geq 0 \).

**Proof.** (a) \(\Rightarrow\) (b). This necessity is a direct consequence of Corollary 4.6.

(b) \(\Rightarrow\) (a). By the Hölder inequality we have

\[
s|x'(R(t, s)x)| \leq \int_0^s \|R'(t, v)x'\| \|(R(t + v, s - v)x)\| dv, \quad s \geq v
\]

\[
\leq \left( \int_0^s \|R'(t, v)x'\|^q dv \right)^{\frac{1}{q}} \left( \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{\frac{1}{p}}.
\]

Hence

\[
|x'(R(t, s)x)| \leq \left( \frac{1}{s} \int_0^s \|R'(t, v)x'\|^q dv \right)^{\frac{1}{q}} \left( \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{\frac{1}{p}}.
\]

Furthermore

\[
\|R(t, s)x\| \leq N_q'(t) \left( \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv \right)^{\frac{1}{p}}. \quad (15)
\]

Since the right hand side of (15) converges to zero as \( s \to \infty \), \( R(t, s) \) is strongly stable on \( X \). \(\blacksquare\)

By inspecting the proof of Theorem 4.7 we have the sufficiency for the uniform strong stability.

**Corollary 4.8.** Let \( R(t, s) \) be a \( C_0 \)-quasi semigroup on a Banach space \( X \). The following statements are equivalent.

(a) \( R(t, s) \) is uniformly strongly stable.

(b) There exist \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and a positive constant \( N_q' > 0 \) such that

\[
\lim_{s \to \infty} \frac{1}{s} \int_0^s \|R(t + v, s - v)x\|^p dv = 0
\]

for all \( x \in X \) and \( t \geq 0 \), and

\[
\sup_{t \geq 0, s > 0} \left( \frac{1}{s} \int_0^s \|R'(t, v)x'\|^q dv \right)^{\frac{1}{q}} \leq N_q'\|x'\|
\]

for all \( x' \in X' \).

**Remark 4.9.** We have characterized the various types of the stability for \( C_0 \)-quasi semigroups on Banach spaces which is similar to \( C_0 \)-semigroups and evolution operators. The results are applicable for the time-dependent differential equations with unbounded coefficients in Banach spaces. In particular, these results can also be applied to the investigation of the stability, controllability, and observability of the time-dependent linear systems. The next works, we shall investigate the polynomial stability of \( C_0 \)-quasi semigroups in Banach spaces which is more general than the exponential stability.
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