Exact matrix-product states for parallel dynamics: open boundaries and excess mass on the ring

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Abstract. In this paper it is shown that the steady state weights of the asymmetric simple exclusion process (ASEP) with open boundaries and parallel update can be written as a product of a scalar pair-factorized and a matrix-product state. This type of state is also obtained for an ASEP on a ring in which particles can move one or two sites. The dynamics leads to the formation of an excess hole that plays the role of a defect. In the continuous-time limit the process recovers the ASEP with a single defect particle. The phase diagram shows a first-order phase transition between two phases with different defect velocities. These are calculated exactly from the process-generating function.

Keywords: cellular automata, driven diffusive systems (theory), exact results, stationary states
1. Introduction

The asymmetric simple exclusion process (ASEP) has been used to model different dynamical systems such as traffic flow and biological processes. It is originally defined in continuous time on a discrete one-dimensional lattice. Particles on the lattice can move one site to the right at a certain rate if the target site is empty (see, e.g., [1] for a review). The model with periodic boundary conditions is known to have a uniform stationary measure [1]. However, employing open-boundary conditions (particles enter the system at one end and leave it at the other end of the lattice at certain rates) leads to so-called boundary-induced phase transitions. The steady state is of the matrix-product form [2]. There are, however, some generalizations of the ASEP with periodic boundary conditions with phase transitions. An example is the ASEP with a single defect particle that can itself move forward on empty sites and can be overtaken by normal particles [3]. The defect plays in the traffic interpretation the role of a truck that moves in an environment of cars [1]. The process with several defects can also be interpreted as a two-way traffic flow model where trucks generally move in the opposite direction as cars [4] by interchanging definitions of empty sites and defects. The defect ASEP was first introduced and studied for the case where all the hop rates are the same in which it is referred to as a second-class particle (see [5] and references therein). Since in an environment of particles (holes) it can only move to the left (right) it always finds positions with positive density gradient (0...021...1). Its dynamics was defined in this way to localize the position of a shock (that is defined as a sudden change in the density approaching two different values to the

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left and right). Through the exact solution one was then able to calculate the density profile seen from the second-class particle. This has been considered as a limiting case of a shock profile (since on a ring the density is constant, the density seen from the second-class particle far to the left and right is the same). This form of a shock can also be found in one phase of the defect ASEP and in the open-boundary ASEP along the second-order transition line [1]. Originally it described shocks with the same profile in the ASEP on the infinite line [5]. For a recent review on one-dimensional driven-diffusive systems with two species of particles see [6].

The ASEP has been extended to various discrete updating schemes rather than a random-sequential update [7,9], probably the most important of which is the parallel update which is typically used for traffic flow simulations [8]. Parallel means that all the sites are updated simultaneously and particles attempt a hop forward with probability $p$. The introduction of such a parameter is necessary to interpolate stochastically between purely deterministic movement ($p = 1$) and the continuous-time limit ($p \to 0$). We note that the parallel update on the infinite line has been used too for the numerical study of shocks in ‘lattice fluids’ [13]. For open-boundary conditions the parallel-update ASEP could be solved by two sophisticated versions of the matrix-product ansatz [14,16], see section 2. However, since then there is somehow a lack of new exact solutions for steady states of cellular automata.

In this paper we study a (totally) asymmetric exclusion process where particles can move one or two sites to the right. This type of model is motivated originally by traffic flow [23]. Two limits of hop rates have already been investigated in the literature: if the rate at which a particle moves one or two sites does not depend on its headway (the empty space in front) it turns out that the steady state is uniform [12]. If the rate at which a particle moves a single site differs if it has exactly one or more empty sites in front one finds a pair-factorized state [11,12]. Recently the general process [23] with random-sequential dynamics has been studied analytically in [10]. In this paper a special focus is spent on the case where particles are not allowed to move a single site if it would be possible to move two sites. One finds a matrix-product form of the steady state and the process is mainly equivalent to the ASEP with a single defect particle, see section 6. Here we study this special process with parallel dynamics. We focus on stochastic hopping since the completely deterministic dynamics has been studied before [17]. It will turn out that it leads to the formation of a single excess hole with a natural defect dynamics for parallel update. We find two phases with different defect velocities. The density profiles differ qualitatively in both phases. In the phase that corresponds to the power-law phase of the defect ASEP [20] we find from numerical simulations an algebraic density profile seen from the defect that reminds us of the profile in the maximum-current phase of the parallel ASEP [14]. We expect this profile to play a similar role as a limiting shock profile.

The outline of this paper is as follows. Section 2 briefly reviews the exact solution for open boundaries as obtained in [14]. Then we give an alternative form of the steady state that is a product of a scalar pair-factorized and a matrix-product state which is the most straightforward form one would expect from the knowledge of the solution for periodic boundaries and the open-boundary solutions in other update versions. It is shown how it corresponds to the solution in [16]. In section 3 we investigate a process on a ring in which particles have maximum velocity two and calculate its exact steady state. Section 4 is dedicated to the proof. Then, in section 5 the phase diagram is calculated. Phase-typical
asymptotic quantities are compared to computer simulations. Finally in section 6 we map
the process in the continuous-time case \[10\] onto the defect ASEP \[3,21\].

2. The asymmetric simple exclusion process with parallel dynamics

The ASEP is defined on a one-dimensional lattice with \( L \) sites, denoted \( l = 1, 2, \ldots, L \). Each site \( l \) may be in one of two possible states (expressed by a local state variable \( \tau \)), namely either occupied by one particle (\( \tau_l = 1 \)) or empty (\( \tau_l = 0 \)).

In the case of open boundaries the system is coupled to two boundary reservoirs, one to the left and one to the right. A particle enters onto the first site if it is empty with probability \( \alpha \) and a particle may exit from site \( L \) with probability \( \beta \). In the bulk a particle can move one site to the right with probability \( p \) if the target site is empty. Note that every site is updated simultaneously. The exact form of the stationary state can be obtained by the matrix-product ansatz originally introduced for continuous time \[2\].

2.1. The site-oriented ansatz

By introducing boundary vectors \( \langle W |, | V \rangle \) and matrices \( E \) and \( D \) for holes and particles, respectively, Evans et al \[14\] could show that

\[
P(\tau_1, \ldots, \tau_L) = Z_L^{-1} \langle W | \prod_{l=1}^{L} [(1 - \tau_l)E + \tau_l D] | V \rangle \tag{1}
\]
gives the correct steady state when the operators satisfy the bulk relations

\[
DDEE = (1 - p)DDE + (1 - p)DEE + p(1 - p)DE, \tag{2}
\]
\[
DDDE = DDD + (1 - p)DDE + pDD, \tag{3}
\]
\[
EDEE = (1 - p)EDE + EEE + pEE, \tag{4}
\]
\[
EDED = EDD + EED + pED, \tag{5}
\]
as well as relations for the right boundary

\[
DDE|V\rangle = (1 - p)DDE|V\rangle + DD|V\rangle, \tag{6}
\]
\[
EDE|V\rangle = EDD|V\rangle + EEE|V\rangle, \tag{7}
\]
\[
DD|V\rangle = \frac{p(1 - \beta)}{\beta}D|V\rangle, \tag{8}
\]
\[
ED|V\rangle = \frac{p}{\beta}E|V\rangle, \tag{9}
\]
and left boundary

\[
\langle W |DDE = (1 - p)\langle W |DE + \langle W |EE, \tag{10}
\]
\[
\langle W |DED = \langle W |DD + (1 - p)\langle W |ED, \tag{11}
\]
\[
\langle W |EE = \frac{p(1 - \alpha)}{\alpha}\langle W |E, \tag{12}
\]
\[
\langle W |ED = \frac{p}{\alpha}\langle W |D. \tag{13}
\]

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Note that—as a consequence of the particle–hole symmetry of the process—these relations are invariant under exchanging $\alpha \leftrightarrow \beta$, $E \leftrightarrow D$, $\langle W \rangle \leftrightarrow |V\rangle$ and at the same time inverting the order of the enumeration of cells (site $i$ is replaced by site $L - i + 1$). The ansatz (1) together with (2)–(13) are a notation for certain recursion relations for the weights on system size and particle number [14]. The two-dimensional ansatz

$$E = \begin{pmatrix} E_1 & gD_1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ gE_1 & 0 \end{pmatrix},$$

with appropriate boundary vectors $\langle W |$ and $| V \rangle$ and a simple function $g(p)$ reduces (2)–(13) to

$$D_1E_1 = (1 - p) [D_1 + E_1 + p],$$

$$\langle W_1 | E_1 = p \frac{(1 - \alpha)}{\alpha} \langle W_1 |,$$

$$D_1 | V_1 \rangle = p \frac{(1 - \beta)}{\beta} | V_1 \rangle.$$ 

All physical quantities can be expressed through $D_1, E_1$ and $\langle W_1 |, | V_1 \rangle$. However, using (14) the weights (1) become difficult expressions, namely a complex sum over matrix elements, that do not have an obvious physical meaning. Therefore it would be desirable to have a more easy formulation of the weights.

### 2.2. Alternative formulation of the weights

Before we present an alternative formulation of the weights, let us remember the type of solution for periodic boundary conditions which is a pair-factorized state [15]:

$$F_{ring}^{ring}(\tau_1, \ldots, \tau_L) = \mathit{\prod}_{l=1}^{L} t(\tau_l, \tau_{l+1}),$$

with simple two-site factors $t(\tau_l, \tau_{l+1})$ obeying

$$\frac{t(11)}{t(01)} = (1 - p) \frac{t(10)}{t(00)}.$$ 

A useful choice is

$$t(11) = (1 - p)t(01) \quad \text{and} \quad t(10) = t(00).$$

Equation (18) can be rewritten as a matrix-product state:

$$F_{ring}^{ring}(\tau_1, \ldots, \tau_L) = \mathsf{tr} \left( \mathsf{\prod}_{l=1}^{L} [\tau_l D + (1 - \tau_l) E] \right).$$

Of course $2 \times 2$ matrices $D$ and $E$ of the form (14) also solve the periodic case [14,12] but here we write the matrices in a vector basis in which the $t(\tau \sigma)$ become matrix elements $\langle \tau | (E + D) | \sigma \rangle$ in the style of an Ising transfer matrix:

$$E = \begin{pmatrix} t(00) & 0 \\ t(10) & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & t(01) \\ 0 & t(11) \end{pmatrix}.$$

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In this representation it is obvious that the product is self-consistent, i.e. that no terms \( t(\tau \sigma)t(\tau' \sigma') \) with \( \sigma \neq \tau' \) occur and lead to one single term (18).

Now we come back to the case of open boundaries. Inspired by (22) we take alternatively to (14)

\[
E = \begin{pmatrix}
t(00)E_1 & 0 \\
t(10)E_1 & 0
\end{pmatrix},\quad D = \begin{pmatrix}
0 & t(01)D_1 \\
0 & t(11)D_1
\end{pmatrix}.
\]  

(23)

Here \( t(\tau \sigma) \) are the two-site factors of the solution for periodic conditions (20). We set \( t(00) = t(10) = 1 \) according to (20). Since the operators \( E \) or \( D \) of the form (23) have the structure of (22) with factors \( t(\tau_{l-1}\tau_l) \) in a matrix for site \( l \), the correct connection to the operator for site \( l-1 \) is guaranteed. The boundary vectors \( \langle W | \) and \( | V \rangle \) are

\[
\langle W | = \left( \frac{p - \alpha}{\alpha} \right) \langle W_1 |,\quad | V \rangle = \left( \frac{1 - p}{1 - \beta} \right) | V_1 \rangle,
\]  

(24)

and for the bulk we find \( t(01)D_1E_1 = t(11)D_1 + E_1 + p \). So setting \( t(01) = (1 - p)^{-1} \) and \( t(11) = 1 \) recovers (15). Using the new operators (22) it is rather obvious that the ansatz (1) yields formally

\[
F(\tau_1, \ldots, \tau_L) = w(\tau_1)l(\tau_1, \tau_2) \ldots l(\tau_{L-1}, \tau_L)v(\tau_L) \times \langle W_1 | \prod_{l=1}^L [\tau_lD_1 + (1 - \tau_l)E_1] | V_1 \rangle,
\]  

(25)

i.e. a product of a pair-factorized state (reflecting the nearest-neighbor correlations of the parallel update) and a matrix state (as for other discrete-time updates such as ordered sequential and sublattice-parallel updates [7,14]). Here \( t(\tau \sigma) \) is defined through (20) and the boundary factors are

\[
w(\tau_1) = \frac{p - \alpha}{\alpha} t(01)^{\tau_1} + t(11)^{\tau_1},\quad v(\tau_L) = \frac{1 - p\tau_L}{1 - \beta\tau_L},
\]  

(26)

where we used that \( w \) is specified only up to an overall factor.

**2.3. Connection with the bond-oriented ansatz**

De Gier and Nienhuis [16] alternatively solved the parallel ASEP with open boundaries through a bond-oriented matrix ansatz:

\[
P(\tau_1, \tau_2, \ldots, \tau_L) = \langle W(\tau_1) | M(\tau_1, \tau_2)M(\tau_2, \tau_3) \cdots M(\tau_{L-1}, \tau_L) | V(\tau_L) \rangle.
\]  

(27)

The vectors and matrices \( M(\tau \sigma) \) are

\[
M(\tau \sigma) = \begin{pmatrix}
(1 - \tau)(1 - \sigma)M(00) & (1 - \tau)\sigma M(01) \\
\tau(1 - \sigma)M(10) & \tau\sigma M(11)
\end{pmatrix},
\]  

(28)

\[
\langle W(\tau) | = ((1 - \tau)\langle W(0) |, \tau\langle W(1) |),
\]  

(29)

\[
| V(\tau) \rangle = ((1 - \tau)| V(0) \rangle, \tau| V(1) \rangle)^t.
\]  

(30)
We now give a relation between the site-oriented and bond-oriented solutions. In (23) we have to take $t_{01} = 1$ and $t_{11} = 1 - p$. Then the connection is

$$
\mathcal{M}(00) = \mathcal{M}(10) = E_1, \quad \mathcal{M}(11) = (1 - p)\mathcal{M}(01) = D_1,
$$

and therefore

$$
E = M(00) + M(10), \quad D = M(01) + M(11),
$$

$$
\langle W(0) | = \langle W(1) | E_1, \quad \langle W(1) | = (1 - p)\langle W(2) | D_1,
$$

and

$$
|V(0)\rangle = |V_1\rangle, \quad |V(1)\rangle = |V_2\rangle,
$$

In fact one sees that our choice (22) is closely related to the bond-oriented solution and is just rewritten systematically in terms of a site-oriented matrix product.

3. Solution of an ASEP on a ring with excess-mass formation

We are going to consider an asymmetric exclusion process on a ring with $L$ sites, $N$ particles and periodic boundary conditions (site $L + 1 \equiv$ site 1). The system evolves under parallel dynamics according to the local update rules:

\begin{align*}
100 &\rightarrow 001, \quad \text{with probability } p, \\
101 &\rightarrow 011, \quad \text{with probability } \beta.
\end{align*}

It turns out that the stationary distribution of probabilities for the possible configurations is not ergodic, i.e. only a subspace of configurations is reached as the time increases. This stationary distribution depends strongly on the parity of the number $L - N$ of holes (unoccupied sites). We specify a certain configuration of particles by the set of gaps (number of holes) between the particles: $C = (n_1, n_2, \ldots, n_N)$. The model dynamics is such that odd-valued gaps cannot be constructed: however, they can turn into even gaps when a configuration $C(\ldots 1[\text{any odd number of 0s}]101\ldots)$ moves with conditional probability $\beta$ into a configuration with two odd-valued gaps less. These processes appear again and again until there remain either no more odd gaps ($L - N$ even) or exactly one odd gap ($L - N$ odd). In the latter case there remains, so to speak, a single excess hole (comparable to the concept of excess mass in the mathematical literature). In the following we are going to consider these two cases separately.

For an even number of holes the system arranges such that only even-length gaps remain. The weight for a configuration factorizes into $N$ factors, one for each gap. All positive even gaps have the same weight. Only the weight for zero gap is different:

$$
F(n_1, n_2, \ldots, n_N) = \prod_{\mu=1}^{N} f(n_\mu),
$$

with

$$
f(n) = \begin{cases} 
1 - p, & \text{for } n = 0, \\
1, & \text{for } n = 2, 4, \ldots, \\
0, & \text{for } n = 1, 3, \ldots.
\end{cases}
$$
In the subspace of even gaps this is equivalent to the solution of the usual ASEP on a ring (18) which can simply be written as (37), with \( f(0) = t(11) \) and \( f(n) = t(01) \), for \( n \geq 1 \) being a possible choice [18]. So this case is simple and well known and will not be discussed here further.

For \textbf{odd number of holes} there remains only one odd-valued gap (configurations with more than one odd-valued gap have probability zero in the steady state). We introduce a matrix-product ansatz for the weight of particle \( \mu \) being followed by \( 2n_\mu + \delta_{\mu,N} \) holes, \( \mu = 1, \ldots, N \). In contrast to the usual formulation where a matrix \( E \) represents a hole, the matrix \( E \) here stands for a pair of neighboring holes (00). \( D \) represents particles (1) and \( |V\rangle\langle W| = A \) stands for the excess hole together with the particle to its right (01). So we use here the convention that the position of the excess hole is always at the right end of the gap (00 00 \ldots 00 0 1) between particles \( N - 1 \) and \( N \). The ansatz is

\[
F(2n_1, 2n_2, \ldots, 2n_N + 1) = \langle W | \prod_{\mu=1}^{N-1} E^{n_\mu} D \rangle E^{n_N} |V\rangle. \tag{39}
\]

We note that it is possible [19] to write a matrix ansatz allowing for any number of odd gaps that leads finally to (39). However, from our argument above it should be clear that only one odd gap remains and so we base our solution on this simple finding. In the following we assume \( p, \beta \in (0,1) \) and discuss the special cases \( p = 1 \) and \( \beta = 0 \) later. From diagonalizing small systems we find a quartic algebra for the process related to (2)–(13). In fact, by transforming the matrices \( E \rightarrow (1 - p)^{-1}E \) and \( D \rightarrow \beta D \) almost the complete set of relations can be mapped onto the algebra of the parallel ASEP with open boundaries. To be precise, one recovers exactly (2)–(12) with \( \alpha = p \), the only exception being (13) which has to be replaced by

\[
\langle W | ED = \langle W | (D + p). \tag{40}
\]

This is in accordance with the dynamical rules (36) leading to the fact that even for \( p = \beta \) the particle–hole symmetry is broken. The matrix transformation for \( D \) and \( E \) mentioned above can be omitted in the calculation since for fixed particle and site number it leads only to an overall factor in the normalization constant

\[
Z_{N,M} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \delta_{\sum n_\mu,M} \langle W | \prod_{\mu=1}^{N-1} E^{n_\mu} D \rangle E^{n_N} |V\rangle \tag{41}
\]

for \( N \) particles and \( 2M + 1 \) holes according to (39). The initial values are

\[
\langle W | E | V\rangle = (\gamma + \beta) \langle W | V\rangle, \quad \langle W | D | V\rangle = p\gamma / \beta \langle W | V\rangle \tag{42}
\]

for some constant \( \gamma > 0 \). We just note that this leads to \( \langle W | DE | V\rangle = (1 - \beta) \langle W | D | V\rangle + (1 - p) \langle W | E | V\rangle + p\gamma \langle W | V\rangle \). We found that the algebra (2)–(12), (40) cannot be simplified by (14). However, with the alternative ansatz (23) it can. So we expect that (23) is the more robust form that holds even for broken particle–hole symmetry in probabilistic cellular automata that give rise to a matrix-product state. We take again \( t(00) = t(10) = t(11) = 1 \) and \( t(01) = (1 - p)^{-1} \). Then the weights can again be written

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as a superposition of a pair-factorized and a matrix state as (25). We can rewrite (39) as

\[
F(2n_1, 2n_2, \ldots, 2n_N + 1) = \langle W \rvert \prod_{\mu_1=1}^{N-1} E_1^{n_{\mu}} (1 - p\theta(n_{\mu}))^{-1} D_1 \rvert E_1^{n_N} \rvert V \rangle \frac{1 - p\delta(n_N, 0)}{1 - \beta \delta(n_N, 0)}.
\]  

(43)

Note that in fact \( t_0 = (1 - p)^{-1} \) are three-site dependent factors, compare [17]. However, hole pairs are treated as units and therefore the terminology pair-factorized. In our opinion this form of the weights helps us to understand the solution of this type of model. However, it is not less convenient to work directly with the matrices (23) which here are

\[
E = \begin{pmatrix} E_1 & 0 \\ E_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & (1 - p)^{-1} D_1 \\ 0 & D_1 \end{pmatrix},
\]

(44)

and lead to boundary factors

\[
\langle W \rangle = (0, \langle W_1 \rangle), \quad \rvert V \rangle = \left( \rvert V_1 \rangle, \frac{1 - p}{1 - \beta} \rvert V_1 \rangle \right)^t.
\]

(45)

This choice leads here to a ternary algebra for the indexed matrices:

\[
D_1 E_1 = (1 - p) [D_1 + E_1 + p],
\]

(46)

\[
E_1 D_1 \rvert V_1 \rangle = \frac{p(1 - \beta)}{\beta} E_1 \rvert V_1 \rangle,
\]

(47)

\[
D_1 D_1 \rvert V_1 \rangle = \frac{p(1 - \beta)}{\beta} D_1 \rvert V_1 \rangle,
\]

(48)

\[
\langle W_1 \rangle E_1 E_1 = (1 - p) \langle W_1 \rangle E_1,
\]

(49)

\[
\langle W_1 \rangle E_1 D_1 = (1 - p) \langle W_1 \rangle D_1 + p(1 - p) \langle W_1 \rangle.
\]

(50)

Translating (42) into the form with indexed matrices gives \( \langle W_1 \rvert E_1 \rvert V_1 \rangle = (\gamma + \beta)(1 - p)/(1 - \beta) \langle W_1 \rvert V_1 \rangle \) and \( \langle W_1 \rvert D_1 \rvert V_1 \rangle = p\gamma/\beta \langle W_1 \rvert V_1 \rangle \). A useful choice for \( \gamma \) (which coincides with the representation (56) that we give below) is \( \gamma = 1 - \beta \). Then one has

\[
\langle W_1 \rvert E_1 \rvert V_1 \rangle = \frac{1 - p}{1 - \beta} \langle W_1 \rvert V_1 \rangle \quad \text{and} \quad \langle W_1 \rvert D_1 \rvert V_1 \rangle = \frac{p(1 - \beta)}{\beta} \langle W_1 \rvert V_1 \rangle.
\]

(51)

The choice \( \gamma = 1 - \beta \) is useful because the algebra (46) simplifies to

\[
D_1 E_1 = (1 - p) [D_1 + E_1 + p],
\]

(52)

\[
D_1 \rvert V_1 \rangle = \frac{p(1 - \beta)}{\beta} \rvert V_1 \rangle,
\]

(53)

\[
\langle W_1 \rvert E_1 E_1 = (1 - p) \langle W_1 \rvert E_1,
\]

(54)

\[
\langle W_1 \rvert E_1 D_1 = (1 - p) \langle W_1 \rvert D_1 + p(1 - p) \langle W_1 \rangle,
\]

(55)

since here the first two rules are quadratic (and are the same as for the open-boundary case).

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For these relations we find the representation

\[
E_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
(1-p) & 0 & 0 & 0 & \ldots \\
0 & (1-p) & 0 & 0 & \ldots \\
0 & 0 & (1-p) & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(56)

\[
D_1 = \begin{pmatrix}
p(1-\beta)/\beta & p/\beta & p/\beta & p/\beta & \ldots \\
0 & (1-p) & 1 & 1 & \ldots \\
0 & 0 & (1-p) & 1 & \ldots \\
0 & 0 & 0 & (1-p) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(57)

\[
\langle W_1 | = (1-\beta, 1, 1, 1, \ldots), \quad | V_1 \rangle = (1, 0, 0, \ldots)^{t}.
\]

(58)

It should be mentioned that the known representations for the open-boundary case \[14, 16\] (which reflect the particle–hole symmetry by a symmetry in \(D_1\) and \(E_1\) as well as \(\langle W_1 |\) and \(| V_1 \rangle\)) cannot be generalized to represent the present process. However, (56)–(58) can be changed to represent the open-boundary case, namely by writing instead \(\langle W_1 | = (p(1-\alpha)/\alpha(1-p))^{n-1}(n|, for n \geq 1.

4. Proof of the steady state

Different techniques have been used previously to prove the stationary states of the ASEP with parallel update and open boundaries. The so-called canceling mechanism could be generalized for parallel dynamics \[7\]. However, it may involve more than two neighboring sites \[16\]. Here the problem remains to find representations for auxiliary matrices as well which is a difficult task in general. In \[14\] the state was proven by using the quartic algebra in a rather more mathematical language. We will prove the ansatz here in an alternative way, namely by using the quadratic and cubic rules for the matrices \(D_1\), \(E_1\) and \(A_1\) instead of the quartic rules for \(D\), \(E\) and \(A\). To do this we will derive the master equation from the local dynamical rules.

4.1. Derivation of the master equation

It is not obvious how to write down the master equation here. We now write the state of the system as the ket vector \(|n_1, n_2, \ldots, n_N\rangle\), denoting particle 1 followed by \(n_1\) holes and so on. This may formally be obtained by the tensor product of the single-particle states \(|n_\mu\rangle\). Let \(d_{jk}(n_\mu)\) be the transition probability for particle \(\mu\) to go from state \(|n_\mu + j + k\rangle\) into \(|n_\mu\rangle\) on moving \(j\) cells while particle \(\mu + 1\) moves \(k\) cells. Then the master equation can be written as (compare \[18\])

\[
\langle F | \{n_\mu\} \rangle = \langle F | \text{tr} \prod_{\mu=1}^{N} T(n_\mu),
\]

(59)
with the transfer matrix
\[
T(n_\mu) = \begin{pmatrix}
  d_{00}(n_\mu)|n_\mu\rangle & d_{01}(n_\mu)|n_\mu - 1\rangle & d_{02}(n_\mu)|n_\mu - 2\rangle \\
  d_{10}(n_\mu)|n_\mu + 1\rangle & d_{11}(n_\mu)|n_\mu\rangle & d_{12}(n_\mu)|n_\mu - 1\rangle \\
  d_{20}(n_\mu)|n_\mu + 2\rangle & d_{21}(n_\mu)|n_\mu + 1\rangle & d_{22}(n_\mu)|n_\mu\rangle
\end{pmatrix}.
\]  
(60)

The transition probabilities follow from (36) and are
\[
d_{0k}(n) = \delta_{k,n} + (1 - \beta)\delta_{k,n-1} + (1 - p)\theta(n - 1 - k),
\]  
(61)
\[
d_{1k}(n) = \beta\delta_{k,n},
\]  
(62)
\[
d_{2k}(n) = p\theta(n - k + 1).
\]  
(63)

Since we know that in the steady state there remains only one odd gap between the particles we use this to simplify the equation. Let the odd gap be between particle \(N\) and particle 1. Then we ask for the probability flow into the state \(|2n_1, 2n_2, \ldots, 2n_{N-1}, 2n_N + 1\rangle\). To obtain this state either particle \(N\) or particle 1 have been in the odd state before, since the odd gap can move only backwards. All other particles have been in an even state. Using this one finds for \(T(2n_1)\)
\[
T(2n_1) = \begin{pmatrix}
  (\delta_{n_1,0} + (1 - p)\theta(n_1))|2n_1\rangle & 0 & (\delta_{n_1,1} + (1 - p)\theta(n_1 - 1))|2n_1 - 2\rangle \\
  \beta\delta_{n_1,0}|1\rangle & 0 & \beta\delta_{n_1,1}|1\rangle \\
  p|2n_1 + 2\rangle & 0 & p\theta(n_1)|2n_1\rangle
\end{pmatrix}.
\]  
(64)

The second column vanishes because particle 2 cannot have moved one site since it had an even gap in front as claimed before. Now using the matrix ansatz this can be written as

\[
T(2n_1) = \begin{pmatrix}
  \frac{E_1^{n_1}D_1}{\beta} & 0 & \frac{\theta(n_1)E_1^{n_1-1}D_1}{\beta} \\
  1 - p)\delta_{n_1,0}A_1 & 0 & \frac{\beta\delta_{n_1,1}}{1 - \beta} A_1 \\
  \frac{p}{1 - p}E_1^{n_1+1}D_1 & 0 & \frac{p}{1 - p}\theta(n_1)E_1^{n_1}D_1
\end{pmatrix}.
\]  
(65)

Equivalently one has for the bulk
\[
T(2n_\mu) = \begin{pmatrix}
  E_1^{n_\mu}D_1 & 0 & \frac{\theta(n_\mu)E_1^{n_\mu-1}D_1}{\beta} \\
  0 & 0 & 0 \\
  \frac{p}{1 - p}E_1^{n_\mu+1}D_1 & 0 & \frac{p}{1 - p}\theta(n_\mu)E_1^{n_\mu}D_1
\end{pmatrix}, \quad \mu = 2, \ldots, N - 1,
\]  
(66)

and for \(T(2n_N + 1)\)
\[
T(2n_N + 1) = \begin{pmatrix}
  (1 - p)E_1^{n_N}A_1 & E_1^{n_N}D_1 & \theta(n_N)(1 - p)E_1^{n_N-1}A_1 \\
  0 & 0 & 0 \\
  pE_1^{n_N+1}A_1 & \frac{p}{1 - p}E_1^{n_N+1}D_1 & p\theta(n_N)E_1^{n_N}A_1
\end{pmatrix}.
\]  
(67)

Note that the component of the second row and second column, containing a factor \(d_{11}\), vanishes in every transfer matrix since it is impossible that a particle and the particle in front of it move at the same time only a single site in the steady state.
Note that only the simple structure of (23) allowed for a closed expression of the master equation. The crucial step in deriving the master equation is the following similarity transform. Take

$$L = \begin{pmatrix} (1-p)E_1 & 0 & (1-p) \\ 0 & 0 & 0 \\ -pE_1 & 0 & (1-p) \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ pE_1 & 0 & E_1 \end{pmatrix}. \quad (68)$$

Then one has the convenient expression (for $\mu = 2 \ldots N - 1$)

$$LT(2n_\mu)R = \begin{pmatrix} E_1^{n_\mu+1}D_1 + \frac{p}{1-p}\theta(n_\mu)E_1^{n_\mu}D_1E_1 & 0 & -E_1^{n_\mu+1}D_1 + \theta(n_\mu)E_1^{n_\mu}D_1E_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (69)$$

From here a straightforward calculation involving successive simplifications (without using the algebra (52)–(55)) shows that the master equation can finally be written as

$$\text{tr} A_1 \prod_{\mu=1}^{N-1} E_1^{n_\mu}D_1E_1^{n_N} = [1 - \beta n_{N,0} - p\theta(n_N)] \text{tr} A_1 \prod_{\mu=1}^{N-1} [(1-p\theta(n_\mu))E_1^{n_\mu}D_1$$

$$+ p\theta(n_\mu)E_1^{n_\mu+1}D_1E_1]E_1^{n_N} + \beta \frac{1-p\theta(n_N)}{1-\theta(n_N)} \text{tr} A_1(\delta_{n_1,0} + p\delta_{n_1,1}E_1)$$

$$\times \prod_{\mu=2}^{N-1} [(1-p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu+1}D_1E_1]E_1^{n_N}D_1$$

$$+ p\theta(n_N) \text{tr} A_1E_1 \prod_{\mu=1}^{N-1} [(1-p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu+1}D_1E_1]E_1^{n_N-1}. \quad (70)$$

Note that only the simple structure of (23) allowed for a closed expression of the master equation in terms of the primed operators.

### 4.2. Proof of the matrix-product ansatz

In the following we assume always $N \geq 2$, since the case $N = 1$ is trivial. For the proof the following simplification of the bulk terms under the product is essential:

$$(1-p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu+1}D_1E_1 = \delta_{n_\mu,0}D_1 + \theta(n_\mu)E_1^{n_\mu+1}[\{1-p\}E_1D_1 + pD_1E_1]$$

$$= \delta_{n_\mu,0}D_1 + (1-p)p\theta(n_\mu)E_1^{n_\mu+1}[E_1D_1 + p(D_1 + E_1 + p)]$$

$$= \delta_{n_\mu,0}D_1 + (1-p)p\theta(n_\mu)E_1^{n_\mu+1}[(E_1 + p)D_1 + p(E_1 + p)]$$

$$= \delta_{n_\mu,0}D_1 + (1-p)p\theta(n_\mu)E_1^{n_\mu-1}(E_1 + p)(D_1 + p). \quad (71)$$

Here we have factors $(E_1 + p)(D_1 + p)$. Note that from (52) it follows [14] that

$$(1-p)(D_1 + p)(E_1 + p) = D_1E_1. \quad (72)$$

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This can be used to simplify the following equation which turns out to be the key to the proof:

\[(D_1 + p)[\delta_{n,0} D_1 + (1 - p)\theta(n) E_1^{n-1}(E_1 + p)(D_1 + p)] = [\delta_{n,0} D_1 + (1 - p)\theta(n)(D_1 + p) E_1^{n-1}(E_1 + p)](D_1 + p)\]

\[= [\delta_{n,0} D_1 + (1 - p)\theta(n)(D_1 + p)(E_1 + p) E_1^{n-1}](D_1 + p)\]

\[= [\delta_{n,0} D_1 + \theta(n) D_1 E_1^{n-1}](D_1 + p)\]

\[= D_1 E_1^{n-1}(D_1 + p).\]  \((73)\)

Here we have used the fairly simple but essential commutation relations \(E_1(E_1 + p) = (E_1 + p)E_1, D_1(D_1 + p) = (D_1 + p)D_1\). As a consequence one has

\[(D_1 + p) \prod_{\mu}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1] = \prod_{\mu}[D_1 E_1^{n,\mu}](D_1 + p).\]  \((74)\)

In the following we consider only the case \(n_N = 0\) since the case \(n_N > 0\) can be handled in a similar fashion [19]. For \(n_N = 0\) we simplify the master equation \((70)\) on both sides using \((53)\). The result can be written as

\[p \text{ tr } A_1 E_1^{n_1} \prod_{\mu=2}^{N-1}[D_1 E_1^{n,\mu}] = \beta \text{ tr } A_1 \prod_{\mu=1}^{N-1}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1]
\]

\[+ p\beta \text{ tr } A_1(\delta_{n_1,0} + p\delta_{n_1,1} E_1) \prod_{\mu=2}^{N-1}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1].\]  \((75)\)

In the first term on the right-hand side (rhs) the factor corresponding to \(\mu = 1\) is extracted from the product. Rewriting it with the help of \((71)\) and combining terms on the rhs with \(n_1 = 0\) yields

\[\cdots = \beta \delta_{n_1,0} \text{ tr } A_1(D_1 + p) \prod_{\mu=2}^{N-1}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1]
\]

\[+ \beta(1 - p)\theta(n_1) \text{ tr } A_1 E_1^{n_1-1}(E_1 + p)(D_1 + p)
\]

\[\times \prod_{\mu=2}^{N-1}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1]
\]

\[+ p^2\beta \delta_{n_1,1} \text{ tr } A_1 E_1 \prod_{\mu=2}^{N-1}[(1 - p\theta(n)) E_1^{n,\mu} D_1 + p\theta(n) E_1^{n-1,\mu} D_1 E_1].\]  \((76)\)

In the same way one can extract the factor for \(\mu = 2\) in the third term of \((76)\) and rewriting it with \((71)\). The resulting factor \(A_1 E_1[\delta_{n_2,0} D_1 + (1 - p)\theta(n_2) E_1^{n_2,1} - (1 + p) D_1 E_1]\) is for \(n_2 = 0\), due to \((55)\), equal to \((1 - p) A_1(D_1 + p)\). For \(n_2 > 0\) it becomes \((1 - p) A_1 E_1^{n_2}(D_1 + p)\) as a consequence of \((54)\). In conclusion one finds after combining both expressions \(A_1 E_1[\delta_{n_2,0} D_1 + (1 - p)\theta(n_2) E_1^{n_2,1} - (1 + p) D_1 E_1]\) = \((1 - p) A_1 E_1^{n_2}(D_1 + p)\). Then \((76)\)
In the second term on the rhs there is a factor $(1 - p)\theta(n_1)$ which can be simplified with (55) and yields simply $(1 - p)\theta(n_1)E_{1}^{n_1-1}D_1E_1$.

$$\cdots = \beta\delta_{n_1,0} \text{tr} A_1(D_1 + p) \prod_{\mu=2}^{N-1} [(1 - p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu-1}D_1E_1]$$

$$+ \beta(1 - p)\theta(n_1) \text{tr} A_1E_{1}^{n_1-1}(E_1 + p)(D_1 + p)$$

$$\times \prod_{\mu=2}^{N-1} [(1 - p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu-1}D_1E_1]$$

$$+ p^2(1 - p)\beta\delta_{n_1,1} \text{tr} A_1E_{1}^{n_1}(D_1 + p)$$

$$\times \prod_{\mu=3}^{N-1} [(1 - p\theta(n_\mu))E_1^{n_\mu}D_1 + p\theta(n_\mu)E_1^{n_\mu-1}D_1E_1].$$  

(77)

Now use (74) and $\beta(D_1 + p)A_1 = pA_1$ which follows from (53) and combine terms with $n_1 = 1$. Then the master equation turns into

$$p \text{tr} A_1E_{1}^{n_1} \prod_{\mu=2}^{N-1} [D_1E_1^{n_\mu}] = p\delta_{n_1,0} \text{tr} A_1 \prod_{\mu=2}^{N-1} [D_1E_1^{n_\mu}]$$

$$+ p(1 - p)\delta_{n_1,1} \text{tr} A_1 [(E_1 + p)D_1 + p^2] E_1^{n_2} \prod_{\mu=3}^{N-1} [D_1E_1^{n_\mu}]$$

$$+ p^2(1 - p)\theta(n_1 - 1) \text{tr} A_1E_{1}^{n_1-1}(E_1 + p) \prod_{\mu=2}^{N-1} [D_1E_1^{n_\mu}].$$  

(78)

In the second term on the rhs there is a factor $(1 - p)A_1[(E_1 + p)D_1 + p^2]$ which can be simplified with (55) and yields simply $A_1E_1D_1$ as one can check in a few lines. Finally consider in the third term on the rhs $(1 - p)\theta(n_1 - 1)A_1E_{1}^{n_1-1}(E_1 + p)$. Since this term only for $n_1 \geq 2$ gives non-vanishing contributions, (54) can be applied and leads to $(1 - p)A_1E_{1}^{n_1-1}(E_1 + p) = A_1E_{1}^{n_1}$. Inserting these results in (78) yields the required identity.

5. Asymptotic behavior and phase transition

As claimed above the process with an even number of holes corresponds to the usual ASEP which is well studied so again we focus only on the case with a single excess hole. In contrast to the open-boundary ASEP, on a ring we have a fixed number of particles and holes. The calculation is done grand-canonical by introducing fugacities $x$ and $y$ for particles and hole pairs, respectively. Consider the grand-canonical probability $\rho_-(n)$ of finding a particle directly behind the 01 pair while there are a total number of $n$ other particles and hole pairs:

$$\rho_-(n) = \frac{x\langle W|C^{n-1}D|V\rangle}{Z_n},$$  

(79)

where

$$C = C(x, y) = xD + yE = \begin{pmatrix} yE_1 & x(1 - p)^{-1}D_1 \\ yE_1 & xD_1 \end{pmatrix}$$  

(80)

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results from (23). We note that this can be related to the corresponding expression that one would obtain from (14) by a simple similarity transform [19]. The grand-canonical normalization for an excess-hole system of a total number of \( n + 1 \) particles and hole pairs is \( Z_n = \langle W|C^n|V \rangle \). The nominator in (79) can be simplified:

\[
\langle W|C^n D|V \rangle = \beta \langle W|C^{n-1} x D|V \rangle + \beta \langle W|C^{n-1} y E|V \rangle
\]

\[
= \frac{p (1 - \beta)}{\beta} \langle W|C^{n-1} x D|V \rangle + \frac{p \beta}{\beta} \langle W|C^{n-1} y E|V \rangle
\]

\[
= \frac{p \beta}{\beta} \langle W|C^n|V \rangle - p x \langle W|C^{n-1} D|V \rangle.
\]  

(81)

We define \( S_n = \beta / p \langle W|C^n D|V \rangle \), so that \( Z_n = S_n + p x S_{n-1} \), for \( n \geq 1 \). The asymptotic form of \( S_n \) is always \( S_n \sim \lambda^{-n} \) (which follows from the theory of generating functions, see [20]), with a site-representing fugacity \( \lambda \), so that \( Z_n \sim (1 + p x \lambda)\lambda^{-n} \) and

\[
\rho_- = \frac{p x \lambda}{\beta (1 + p x \lambda)}.
\]  

(82)

Explicit derivation of the generating function \( S = \sum S_n \lambda^n \) and analyzing its singularities shows the existence of two phases (see the appendix).

- First phase
  The first singularity results from a pole. One finds a relation between the fugacities in the form \( x(\lambda) \):

\[
x = \frac{\beta}{p(1 - \beta)\lambda} \left( \beta - \frac{p(1 - p)\lambda}{\beta - p - p^2 \lambda} \right).
\]  

(83)

- Second phase
  There is also a square-root singularity in the expression for \( S \) leading to

\[
x = \left( \frac{1 - \sqrt{(1 - p)/\lambda(1 + p\lambda)}}{1 - p(1 + p\lambda)} \right)^2.
\]  

(84)

We introduce a formal asymptotic density \( \rho \sim N/(N + M) \) for a system with \( N \) particles and \( M \) hole pairs in the normal ASEP picture, where each matrix \( E \) represents a single hole. Results can be easier expressed in this form and symmetries as well as comparison with known results are more obvious. Instead of having a relation \( \lambda(x) \), we have \( x(\lambda) \) which fixes the density \( \rho \):

\[
\lambda(x) = \frac{-x(\lambda)}{p x'(\lambda)}.
\]  

(85)

Using this and equating relations (83) and (84) leads to an expression for the critical density:

\[
\rho_c = \frac{\beta(1 - \beta)}{p - \beta^2}.
\]  

(86)

This leads to the phase diagram, depicted in figure 1. The diagram shows additionally the line on which the velocity of the excess hole changes its sign which is calculated later.
At first one needs expressions for the occupations around the excess. For $\rho_-$ we find in phase 1
\begin{equation}
\rho_- = \frac{(1-p)(1-\sqrt{1-4p\rho(1-\rho)})}{(\beta-p)(1-\sqrt{1-4p\rho(1-\rho)}) + 2p(1-\beta)(1-\rho)}.
\end{equation}

In phase 2 the result in terms of $\rho$ is tedious. Parameterized in $\lambda$ it is
\begin{equation}
\rho_- = \frac{p}{\beta(1+p\lambda)} \left( \frac{1-\sqrt{(1-p)/\lambda}(1+p\lambda)}{\sqrt{(1-p)/\lambda-p}} \right)^2.
\end{equation}

As an example we take $p$ and $\beta$ such that $\rho_c = 1/2$, namely $p = 3/4$ and $\beta = 1/2$. This corresponds to the left vertical dotted line in the phase diagram 1. In figure 2 one sees how the curves corresponding to the two phases fit together to the dotted curve coming from a computer simulation with $L = 1000$. For $\rho < 1/2$ the system is in phase 2 and for $\rho > 1/2$ it is in phase 1. One sees that the exact solution for $L \to \infty$ has a kink at $\rho = 1/2$. So its derivative there has a discontinuity, indicating a first-order transition.

For $p = \beta$ the system is completely in phase 2. The comparison between computer simulation and exact solution is shown in figure 3. This case corresponds to the right vertical dotted line in figure 1. For $\beta > p$ the probability $\rho_-$ jumps at density $\rho = 1$ from $p/\beta$ to 1, see figure 3. This can be explained as follows. If there is only one hole in an infinite system then $\rho_-$ is trivially 1. But if there are three holes, both the hole pair and the single hole move backwards in space. However, with $\beta > p$, the single hole moving faster, it approaches the hole pair against the direction of their motion. Then they form a cluster of three with probability $p/\beta$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phase_diagram.png}
\caption{Phase diagram in terms of the density $\rho$ and $\beta$ for a fixed value of $p$, $p \in (0, 1)$. The thick line $\rho_c$ given by (86) shows the phase transition. On the dashed line, the velocity of the excess hole becomes zero. The special form of the curves though varies with $p$. The dotted curves serve as a guide to the eye.}
\end{figure}
Figure 2. The solid curves show the two solutions for $\rho_-(\rho)$ in the two phases in comparison with the squares coming from a computer simulation. The hopping probabilities are $p = 3/4$ and $\beta = 1/2$, so that the critical density is $\rho_c = 1/2$. The system size is $L = 1000$.

Figure 3. $\rho_-(\rho)$ for $\beta = 3/4$ and system size $L = 1000$. Dashed and continuous lines: solution of phase 2 for $p = 1/2$ and $p = 3/4$, respectively. The symbols show the computer simulation.

The probability for the occupation in front of the 01 pair is

$$\rho_+(n) = \frac{x\langle W|DC^{n-1}|V\rangle}{\langle W|C^{n}|V\rangle} = 1 - \frac{y\langle W|EC^{n-1}|V\rangle}{\langle W|C^{n}|V\rangle}. \quad (89)$$

Start with the denominator:

$$\langle W|EC^n|V\rangle = \langle W|ExDC^{n-1}|V\rangle + \langle W|EyEC^{n-1}|V\rangle = x\langle W|(D + p)C^{n-1}|V\rangle + y(1 - p)\langle W|EC^{n-1}|V\rangle = \langle W|C^n|V\rangle + px\langle W|C^{n-1}|V\rangle - py\langle W|EC^{n-1}|V\rangle. \quad (90)$$
Figure 4. The solid curves show the two solutions for $\rho_+ (\rho)$ in the two phases in comparison with the symbols coming from a computer simulation for $p = 3/4$ and $\beta = 1/2$, so that the critical density is $\rho_c = 1/2$. The system size is $L = 1000$.

Now define $T_n := \langle W|E^c_n|V \rangle$. Then one has $T_n + py T_{n-1} = Z_n + px Z_{n-1}$ and we conclude that for $n$ large $T_n$ scales as $\lambda^{-n} \cdot (1 + px \lambda)^2/(1 + p \lambda)$. Thus

$$\rho_+ = 1 - \frac{y\lambda}{1 + py\lambda}. \quad (91)$$

This leads in phase 1 to

$$1 - \rho_+ = \frac{p - \beta}{p^2(1 - \beta)} \cdot \frac{2p(1 - \rho) - 1 + \sqrt{1 - 4p\rho(1 - \rho)}}{1 - 2 \rho + \sqrt{1 - 4p\rho(1 - \rho)}}, \quad (92)$$

and in phase 2 in terms of $\lambda$ simply

$$1 - \rho_+ = \lambda^2 \left( \frac{p - \sqrt{(1 - p)/\lambda}}{1 - p(1 + p\lambda)} \right)^2. \quad (93)$$

Figure 4 shows the result in the two phases in comparison with simulations. Comparing the relations for $\rho_-$ and $1 - \rho_+$ one sees that, due to the broken particle–hole symmetry, there is no proper symmetry between the two relations.

The velocity of the excess hole in the two phases can be obtained by

$$v = p(1 - \rho_+)(1 - \beta \rho_-) - \beta \rho_-. \quad (94)$$

If it has a particle directly behind it jumps backwards with probability $\beta$ which leads to the second contribution $-\beta \rho_-$. If it has no particle in front (probability $(1 - \rho_+)$) it can jump forward with probability $p$ unless it also has a particle behind which moves forward with probability $\beta$. This leads to the first contribution $p(1 - \rho_+)(1 - \beta \rho_-)$. Note that we always argue in terms of a density $\rho$ that treats the hole pairs as single holes.

Using (82) and (91) gives rise to the following expression for $v$:

$$v = \frac{p y \lambda}{1 + py \lambda} - \frac{p x \lambda}{1 + px \lambda} = \frac{p \lambda(y - x)}{(1 + px \lambda)(1 + py \lambda)}. \quad (95)$$

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Figure 5. Exact velocity of the excess hole for $p = 3/4$ and two values of $\beta$. The regime $\rho \leq 1/2$ is independent of $\beta$. For $\rho > 1/2$: upper curve $\beta = 1/2$ and lower curve $\beta = 3/4$.

One sees that, due to the symmetry in $x$ and $y$, the average velocity of the excess hole vanishes for equal densities of particles and hole pairs ($v(\rho = 1/2) = 0$). In phase 1 (95) is rewritten as

$$v(\rho) = \frac{p(p - \beta)(1 - \rho) - p(1 - \beta)J}{p(1 - \beta)(1 - \rho) - (p - \beta)J},$$

where $J$ is the total particle current

$$J(\rho) = \frac{1 - \sqrt{1 - 4p(1 - \rho)}}{2},$$

which is expected since the flow should equal the result for an even total number of holes and it has to be phase-independent in our process. Now one can calculate the line on which the excess-hole velocity vanishes (shown in figure 1), which is given by

$$\rho(p, \beta)_{|v=0} = \begin{cases} \frac{p - \beta}{p - \beta^2}, & \text{for } 0 \leq \beta \leq 1 - \sqrt{1 - p}, \\ 1/2, & \text{for } 1 - \sqrt{1 - p} \leq \beta \leq 1. \end{cases}$$

Note that these are the values of the density in the maximum-current and high-density phase for the ASEP with open boundaries. Figure 5 shows the exact form of $v(\rho)$ for $p = 3/4$ and two different values of $\beta$. For $\rho < 1/2$ the velocity is independent of $\beta$ and the system is in phase 2. For $\rho > 1/2$ and $\beta = 3/4 (= p)$ the system remains in phase 2 (lower curve). For $\rho > 1/2$ and $\beta = 1/2$ the system is in phase 1 (upper curve). At the critical density $\rho = 1/2$ there is a discontinuity in $dv/d\rho$, indicating a first-order transition. This is expected, since the model for random-sequential dynamics shows the same type of transition [24]. This model can itself be mapped onto the ASEP with a defect as is explained in section 6.

It should be clarified that the transition is not a typical ‘jamming’ transition between condensed and fluid phases. This type of transition was already excluded for a slightly
Figure 6. Density profile for $\rho = 1/2$ and $p = \beta$. The system is in phase 2. The form of the profile shows an algebraic decay as in the model with open boundaries. In the windows the influence of changing $\beta$ is shown.

more general model by an observation in [23]. Although one finds here two different velocities of the excess hole, the overall velocity of particles remains always the same. In the infinite system the probability to have an arbitrary distance to the next particle ahead is given by the mean-field measure [10]. However, the excess hole changes the density profile. We have computed some density profiles numerically. Figure 6 shows density profiles $d(i)$ in the frame of the ‘defect’ (excess hole plus particle to its right, see section 6 for an explanation). The variable $i$ counts the distance to the right of the defect. One sees an algebraic behavior typical for phase 2. The density in front is increased and the density behind is decreased. Using the corresponding formula for the model with open boundaries $(d(i) - 1/2 \sim 1/\sqrt{i}$ to leading order in $i$) yields a good agreement with simulation data. In front of the defect one sees oscillations breaking the left–right symmetry. Figure 7 shows different density profiles for $\rho = 1/2$ and $p = 9/10$. The continuous curve is typical for phase 1 (with $\beta$ not too close to $1 - \sqrt{1-p}$). At the boundaries there are oscillations around the value of $\rho$ that decay exponentially fast, so that the overall density is realized in the bulk. This is in contrast to phase 2 where one always finds the algebraic profile which decreases almost linearly in the bulk. The other two curves correspond to phase 2 in the regions $1 - \sqrt{1-p} < \beta < p$ and $\beta > p$, see figure 1. An analytic investigation of the density profiles for a better understanding of the effects observed in computer simulations will be published elsewhere [22].

6. Limits

There are two special cases that were excluded in the solution above. Their steady state depend strongly on the initial conditions. For $\beta = 0$ the steady state is factorizable
Figure 7. Density profile for $\rho = 1/2$ and $p = 9/10$. The continuous curve shows the profile for phase 1. The other curves correspond to phase 2.

but there are absorbing states where the flow vanishes. The other case is $p = 1$. Here the matrix product as given before holds only for $\rho \geq 1/2$. For $\rho < 1/2$ there can be several odd gaps between the particles, depending on the initial condition and the state is factorizable. The probability to find two neighboring particles is zero. All other configurations are equally probable. For $\rho > 1/2$ the state is of the form (1) with (2)–(12) and (40). However, now it is natural to take $t(00) = 0$, since no neighboring hole pairs occur. Then choosing $D$ and $E$ according to (23) leads to the algebra

$$D_1 E_1 = D_1 + \mathbb{I}, \quad \langle W_1|E_1 D_1 = \langle W_1| (D_1 + \mathbb{I}), \quad D_1|V_2 = \frac{1 - \beta}{\beta}|V_2\rangle.$$

(99)

The representation given before obviously no longer holds. Here we can choose two-dimensional commuting matrices:

$$D_1 = \begin{pmatrix} 0 & \frac{1 - \beta}{\beta} \\ \frac{1 - \beta}{\beta} & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & \frac{\beta}{1 - \beta} \\ \frac{\beta}{1 - \beta} & 1 \end{pmatrix},$$

(100)

with $|V\rangle = ((1 - \beta, 1 - \beta), (1, 1))$ and, for example, $\langle W\rangle = ((0, 0), (1, 1))$. For $p$ approaching 1 one sees that the formula for the critical density (86) yields $\rho_c = \beta/(1 + \beta)$ which cannot increase $1/2$. At $\rho = 1/2$ the density profile $d(i)$ is oscillating between two purely linear profiles ($1 - x$ and $x$ in terms of a scaling function $x(i)$) for the even and the odd sub-lattice. This reflects the strong particle–hole attraction [14]. For higher values of $\rho$ the system is purely in phase 1. The formula for the occupation behind the defect becomes, for example, $\rho_+ = (2\rho - 1)/(\rho - \beta(1 - \rho))$. 

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Further we mention the continuous-time case which has been studied in [10]. When the hopping probabilities are so small that, on average, only one particle moves during a time step the parallel update turns into the random-sequential update which mimics continuous time. One has to replace $\beta \rightarrow \beta dt$, $p \rightarrow p dt$ and afterwards taking the limit $dt \rightarrow 0$. Then (36) turns into 100 $\rightarrow$ 001 at rate 1 and 101 $\rightarrow$ 011 at rate $\beta$.

The algebra (52)–(55) becomes equivalent to the DEHP algebra [2] (here with $\alpha = 1$):

$$de = d + e, \quad \langle w | e = (w | d) = \beta^{-1} | v \rangle.$$  

The non-vanishing weights can simply be written as [10] $F(2n_1, \ldots, 2n_N + 1) = \langle w | \prod_{\mu=1}^{N-1} (e^{n_\mu} d) e^{n_N} | v \rangle$. This in turn is the steady state of the ASEP with a single defect particle [3] for $\alpha = 1$. The defect ASEP is defined by the local transitions: 10 $\rightarrow$ 01 at rate 1 and for the defect particle 2: 20 $\rightarrow$ 02 at rate $\alpha = 1$. Normal particles can overtake the defect: 12 $\rightarrow$ 21 at rate $\beta$. In fact, our process can be mapped onto the defect ASEP for continuous time. An arbitrary stationary configuration has exactly one excess hole. Remember the convention that this excess hole is always localized at the right end of the cluster of holes to which it belongs. The mapping is as the matrix ansatz suggests: the 01 pair is the defect 2, the other hole pairs 00 become single holes 0 and the other particles remain normal particles [10]. Note that the particle to the right of the excess hole changes with time. However, since the particles are indistinguishable this has no effect. In the continuous-time limit the critical density (86) becomes $\rho_c = \beta$ and for the occupations around the defect one obtains the well-known results from the defect ASEP. One finds in phase 1 ($\rho > \beta$) that $\rho_- = \rho$ and $\rho_+ = 1 - (1 - \beta)(1 - \rho)$ and for the defect velocity $1 - \beta - \rho$. In phase 2 ($\rho < \beta$) one has $\rho_- = p^2 / \beta$, $\rho_+ = 1 - (1 - \rho)^2$ and $v = 1 - 2\rho$. The density profile has been calculated in [3]. The algebraic form in phase 2, where it behaves like a second-class particle, reduces to a power law and an exponential form in phase 1.

For parallel dynamics the ASEP with a single defect has no natural equivalence, since the evolution of configurations in which the pattern 120 occurs are not well defined since under parallel dynamics 1 and 2 cannot move to the right at the same time. However, the process considered in this paper solves this situation. 120 corresponds to 0100. This moves into 01100 (210) at rate $\beta(1 - p)$ (12 exchange), into 1001 (102) at rate $p(1 - \beta)$ (20 exchange) and into 01001 (201) at rate $p\beta$ (12 exchange, then 10 exchange). However, other definitions involving a different probability for two particles to move may be considered [22].

7. Conclusions

We shortly reviewed the solutions of the asymmetric simple exclusion process (ASEP) [1] and showed that the steady state weights for open boundaries can be written simply as a product of a scalar (pair-factorized) factor containing the nearest-neighbor correlations of the parallel update and a matrix-product state. In the second part we investigated a process on a ring in which particles have a maximum velocity 2, i.e. they can either move one site if they have exactly one empty site in front, or they can move two sites if they have more free sites in front. This dynamics leads to an extinction of odd-valued gaps between consecutive particles. For overall odd number of holes in the system there remains with time exactly one excess hole that leads to a natural parallel defect dynamics. The presence of the excess hole in the odd case leads again to a product of a scalar factor and a matrix-product state which we assume to be generic for this type of driven-diffusive systems. The
model exhibits a first-order phase transition separating two regimes with different defect velocities that we calculate exactly. We obtain the exact expressions for the occupations behind and in front of the defect. We have shown how the model can be mapped onto the defect ASEP in the random-sequential limit. Numerical simulated density profiles show an algebraic form in the one phase and an exponential decay around the defect in the other phase related to the form of the profiles for the ASEP with open boundaries and parallel update. For continuous time they turn into the known profiles for the defect ASEP. In the algebraic phase the defect becomes a sort of second-class particle and the profile can be regarded mainly as a limit of a shock profile with equal densities to the left and right as for continuous time. However, more details of the second-class process with parallel dynamics including an explicit form of the canonical partition function and analytical profiles are in preparation [22]. It seems to be the simplest process on a ring with one-particle species and conservative totally asymmetric dynamics with short-range interactions leading to a non-trivial steady state with phase transition. A natural generalization of the process corresponds to a simple traffic model [23] in which particles can also move one site with a different probability if it would be possible to move two sites (100 → 010) [19].

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Appendix. Derivation of the normalization-generating function

For the $S_n$ occurring in section 5 we find

$$S_n = (1 - p)\langle W_1|G_n(x, y)|V_1 \rangle + py\langle W_1|E_1 G_{n-1}(x, y)|V_1 \rangle,$$

(A.1)

where $S_0 = (1 - p)(1 - \beta)$. The functions $G_n$ obey the following recursions $G_n = C_1 G_{n-1} + pxyKG_{n-2}$ and respectively $G_n = G_{n-1}C_1 + pxyG_{n-2}K$, with $G_{-1} := 0$ and $G_0 = 1$, so that $G_1 = C_1$, $G_2 = C_1^2 + pxyK$, $G_3(x, y) = C_1^3 + pxy(C_1 K + KC_1)$ and so on. Here one has $K = (1 - p)(D_1 + E_1 + p)$ and $C_1 = xD_1 + yE_1$. Special cases of the $G(n)$ occurred in the open-boundary case [14] with $x = y = 1$. It turns out that for $x, y$ general it is difficult to work directly with $G_n(x, y)$. Instead we consider the generating function, which can be written as

$$\mathcal{F}(x, y, \lambda) = \sum_{n=0}^{\infty} \lambda^n G_n(x, y) = \sum_{n=0}^{\infty} \lambda^n (C_1 + pxy\lambda K)^n.$$

(A.2)

The term under the sum is $C_1 + pxy\lambda K = (x + pxy\lambda)D_1 + (y + pxy\lambda)E_1 + p^2 xy\lambda$. It was previously observed that it is very convenient to transform the matrices [20]. Define
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primed matrices through

\[ D_1 = \sqrt{\frac{y + pxy\lambda}{x + pxy\lambda}} [D'_1 - (1 - p)] + 1 - p, \quad (A.3) \]

\[ E_1 = \sqrt{\frac{x + pxy\lambda}{y + pxy\lambda}} [E'_1 - (1 - p)] + 1 - p. \quad (A.4) \]

One can check that these primed matrices indeed fulfill

\[ D'_1 E'_1 = (1 - p)(D'_1 + E'_1 + p) = (1 - p)K'. \]

In this notation \( C_1 + pxy\lambda K' \) becomes finally

\[ C_1 + pxy\lambda K = \sqrt{(x + pxy\lambda)(y + pxy\lambda)} K' + \omega, \quad (A.5) \]

with

\[ \omega = \omega(x, y, \lambda) = \sqrt{(x + pxy\lambda)(y + pxy\lambda)}(p - 2) + (1 - p)(x + y + 2pxy\lambda) + p^2xy. \quad (A.6) \]

After execution of the sum in (A.2) the result can be written as

\[ F(x, y, \lambda) = \frac{1}{1 - \omega \lambda} \left( 1 - \frac{\lambda \sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega \lambda} K' \right)^{-1}. \quad (A.7) \]

One further needs an expression for the action of \( D'_1 \) and \( E'_1 \) on the boundary vectors. For powers \( D'^q_1 |V\rangle \) one gets, for example,

\[ D'^q_1(x, y, \lambda)|V_1\rangle = \left( \sqrt{\frac{x + pxy\lambda}{y + pxy\lambda}} \frac{p - \beta}{\beta + 1 - p} \right)^q |V_1\rangle. \quad (A.8) \]

A calculation adaptable from the defect ASEP in [20] then yields for \( S(\lambda) = \sum_{n=0}^{\infty} \lambda^n S_n = \langle W_1|(1 - p + p\gamma\lambda E_1)F|V_1\rangle: \]

\[ S = \frac{1 - p + p\gamma}{1 - \omega \lambda} \frac{1}{1 - \gamma(1 + (p - \beta)/(\beta(1 - p))\sqrt{(x + pxy\lambda)/(y + pxy\lambda)})} \times \left[ \frac{1 + pxy\lambda}{1 - \gamma} - \frac{\beta}{1 - \gamma(1 - \sqrt{(y + pxy\lambda)/(x + pxy\lambda)})} \right], \quad (A.9) \]

with a function \( \gamma \) to be determined from

\[ \frac{\gamma(1 - \gamma)}{1 - p(1 - \gamma)} = \frac{\lambda \sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega \lambda}. \quad (A.10) \]

The singularity of (A.9) closest to the origin in phase 1 is the pole at \( \gamma^{-1} = 1 + (p - \beta)/(\beta(1 - p))\sqrt{(x + pxy\lambda)/(y + pxy\lambda)} \) and in phase 2 a square-root singularity in \( \gamma \) resulting from (A.10).
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