ABOUT ZITTERBEWEGUNG AND ELECTRON STRUCTURE†

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ABSTRACT: We start from the spinning electron theory by Barut and Zanghi, which has been recently translated into the Clifford algebra language. We “complete” such a translation, first of all, by expressing in the Clifford formalism a particular Barut–Zanghi (BZ) solution, which refers (at the classical limit) to an “internal” helical motion with a time–like speed [and is here shown to originate from the superposition of positive and negative frequency solutions of the Dirac equation].

Then, we show how to construct solutions of the Dirac equation describing helical motions with light–like speed, which meet very well the standard interpretation of the velocity operator in the Dirac equation theory (and agree with the solution proposed by Hestenes, on the basis —however— of ad-hoc assumptions that are unnecessary in the present approach).

The above results appear to support the conjecture that the Zitterbewegung motion (a helical motion, at the classical limit) is responsible for the electron spin.

0† Work partially supported by INFN, CNR, MURST and by CAPES, CNPq, FAPESP.
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**1. Introduction** – The mysterious Zitterbewegung motion, associated since long with the electron structure, seems to be responsible for the electron spin. Indeed, Schrödinger proposed the electron spin to be a consequence of a local circulatory motion, constituting the Zitterbewegung (zbw) and resulting from the interference between positive and negative energy solutions of the Dirac equation. Such an issue turned out to be of renewed interest, following recent work, e.g., by Barut et al., Hestenes, and Pavšič et al.

Let us recall that the pair of conjugate variables \((x^\mu, p_\mu)\) is not enough to characterize the spinning particle. In fact, after introducing the additional classical spinor variables \((z, i\bar{z})\) [where \(z\) is a Dirac spinor; and \(\bar{z} \equiv z^\dagger \gamma^0\)], Barut and Zanghi associated the electron spin and zbw—at the classical limit— with a canonical system [a point \(Q\) moving along a cylindrical helix] which after quantization describes the Dirac electron. In ref., Pavšič et al. presented a thoughtful study of the above results in terms of the Clifford algebra formalism; however, they left many questions still open, and aim of this note is addressing a few of them.

First of all, let us recall that Hestenes’ analysis was based on his reformulation of Dirac theory in terms of the so-called “Clifford space–time algebra (STA)” \(\mathbb{R}_{1,3}\). For details about the Dirac–Hestenes (DH) spinors and the Clifford bundle formalism (used also below), see e.g. refs. In Hestenes’ papers on his “zbw interpretation of quantum mechanics”, an ad-hoc assumption appeared, when he identified the electron velocity with the light–like vector \(u = e_0 - e_2\), with \(e_1 = \psi \gamma_i \tilde{\psi}, \ [i = 0, 2]\), where \(\psi\) is a (plane–wave) DH spinor field satisfying the Dirac–Hestenes equation (i.e., the equation representing the ordinary Dirac equation in the Clifford formalism). Then, he represented the electron internal structure by a light–like helical motion of a sub–microscopic “constituent” \(Q\), such that (for a suitable choice of the helix parameters) the helix diameter equals the electron Compton wavelength and the angular momentum of the zbw yields the correct electron spin. At last, he directly associated the complex phase factor of the electron wave–function with the zbw motion.

We are going to show, among other things, that Hestenes’ ad–hoc assumption, namely that \(u = e_0 - e_2\), is not necessary. More in general, below we shall show:

(i) how to construct in the center-of-mass (CM) frame a particular solutions of the DH equation which correspond (at the classical limit) to a helical motion with time–like velocity. [It will result to be superpositions of positive and negative energy solutions of the DH equation; that is to say (as thoroughly explained in refs., only on the basis of relativistic classical physics), superpositions of particle and antiparticle solutions of the Dirac equation. This suggests, as already pointed out in ref., the BZ theory to be indeed

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\(^0\) Not everybody may like the appearance of such a superposition. It should be clearly noticed, however, that in a generic frame our “helical” wave-functions have to be expressed as superpositions of positive and negative frequency solutions (of the Dirac equation) only when we want to regard themselves—as done in this paper—as solutions of the Dirac equation (17). On the contrary, when we regard those “helical” wave-functions as solutions of our new, non-linear, Dirac–like equation (6'), no superposition of that kind is needed: cf. ref. ; this can be considered a further point in favour of eq.(6’).
equivalent in the CM frame to Dirac’s theory;
(ii) that there exist also solutions of the DH equation that correspond at the classical
limit to a helical path with the light speed \( c \), in which case the velocity operator\(^{16} \) (as
expected in ref.\(^5\)) can actually be identified with \( u = e_0 - e_2 \).

2. Spin and electron structure – Let us start by recalling that—as shown by us in
ref.\(^7\)— the dynamical behaviour of a spinning point-like particle, that follows a world–line
\( \sigma = \sigma(\tau) \), must be individuated —besides by the canonical variables \((x^\mu, p_\mu)\)— also by
the Frenet tetrad\(^7,17\)

\[
e_\mu = R_{\nu\mu} \tilde{R} = \Lambda^\nu_\mu \gamma_\nu; \quad \Lambda^\nu_\mu \in L_+^1
\]

where \( e_0 \) is parallel to the particle velocity \( v \) (even more, \( e_0 = v \) when we can use as
parameter \( \tau \) the particle proper–time). In eq.(1), the tilde denotes the reversion operation
in the STA; namely: \( \tilde{AB} = \tilde{B} \tilde{A} \), and \( \tilde{A} = A \) if \( A \) is a scalar or a vector, while
\( \tilde{F} = -F \) when \( F \) is a 2-vector. Quantity \( R = R(\tau) \) is a “Lorentz rotation”\(^{18} \) [more
precisely, \( R \in \text{Spin}^+(1, 3) \simeq \text{SL}(2, \mathbb{C}) \), and a Lorentz transform of quantity \( a \) is given
by \( a' = Ra\tilde{R} \)]. Moreover \( R\tilde{R} = \tilde{R}R = 1 \). The Clifford STA fundamental unit–vectors
\( \gamma_\mu \), incidentally, should not be confused with the Dirac matrices \( \gamma_\mu \). Let us also recall
that, while the orthonormal vectors \( \gamma_\mu \equiv \partial/\partial x^\mu \) constitute a global tetrad in Minkowski
space–time (associated with a given inertial observer), on the contrary the Frenet tetrad
\( e_\mu \) is defined only along \( \sigma \), in such a way that \( e_0 \) is tangent to \( \sigma \). At last, it is:
\( \gamma_\mu = \eta^{\mu\nu} \gamma_\nu \), and \( \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) is the volume element of the STA.

If \( \Psi_D \in \mathbb{C}^4 \) is an ordinary Dirac spinor, in the STA it will be represented by

\[
\Psi_D \longrightarrow \Psi = \psi \varepsilon \in \mathbb{R}_{1,3}^+, \quad (2)
\]

where \( \psi \in \mathbb{R}_{1,3}^+ \) is called a Dirac–Hestenes spinor\(^8,14\) and \( \varepsilon \) is an appropriate primitive
idempotent of \( \mathbb{R}_{1,3} \). It is noticeable that the spinor field \( \psi \) carries all the essential
information contained in \( \Psi \) (and \( \Psi_D \), and —when it is nonsingular, \( \psi\tilde{\psi} \neq 0 \)— it admits\(^8,9\) a remarkable canonical decomposition in terms of a Lorentz rotation \( R \), a duality
transformation\(^{19} \) \( e^{3\gamma_5/2} \), and a dilation \( \sqrt{\rho} \):

\[
\psi = \rho^1 e^{3\gamma_5/2} R. \quad (3)
\]

In eq.(3), the normalization factor \( \rho \) belongs to \( \mathbb{R}^+ \); quantity \( \beta \) is the Takabayasi angle\(^{20} \)
and \( e^{3\gamma_5} = +1 \) for the electron (and \(-1 \) for the positron). Then, the Frenet tetrad can
also be written:
\[ \rho e_\mu = \psi \gamma_\mu \tilde{\psi} . \]  

(4)

Now, let us take as the lagrangian for a classical spinning particle, interacting with the electromagnetic potential \( A \) (a 1-vector) the expression

\[ \mathcal{L} = \langle \tilde{\psi} \dot{\psi} \gamma_2 + p(\dot{x} - \psi \gamma_0 \tilde{\psi}) + eA \psi \gamma_0 \tilde{\psi} \rangle_0 , \]

(5)

which is the translation\(^7\) of the BZ lagrangian\(^2\) into the Clifford bundle formalism [cf. also ref.\(^{21}\)]. In eq.(5), \( \langle \ldots \rangle_0 \) means “the scalar part” of the Clifford product; the dot represents the derivation with respect to the invariant time–parameter \( \tau \); and \( p \) can be regarded as a Lagrange multiplier. Let us note, moreover, that the BZ theory is also a hamiltonian system, as shown in refs.\(^{22,23}\) by means of Clifford algebras. Then, the Euler–Lagrange equations yield a system of three independent equations:

\[ \dot{\psi} \gamma_1 \gamma_2 + \pi \psi \gamma_0 = 0 \]  

(6)

\[ \dot{x} = \psi \gamma_0 \tilde{\psi} \]  

(7)

\[ \dot{\pi} = eF \cdot \dot{x} , \]  

(8)

where \( \pi \equiv p - eA \) is the kinetic momentum; \( F \equiv \partial \wedge A \) is the electromagnetic field (a bivector, in Hestenes’ language); \( \partial = \gamma^\mu \partial_\mu \) is the Dirac operator; and symbols \( \cdot \) and \( \wedge \) denote the internal and external product, respectively, in the STA. The system (6)–(8) is just that one appeared in ref.\(^2\), but written\(^7\) in terms of the STA language.

Let us pass to the free case, \( A = 0 \), for which one gets

\[ \dot{\psi} \gamma_1 \gamma_2 + p \psi \gamma_0 = 0 \]  

(9)

\[ \dot{x} = \psi \gamma_0 \tilde{\psi} \]  

(10)

\[ \dot{p} = 0 \]  

(11)

If we choose the \( \gamma_\mu \) frame in such a way that

\[ p = m \gamma_0 \]  

(12)

is a constant vector in the \( \gamma_0 \) direction, with \( p^2 = m^2 \), then for the system (9)–(11) we find the solution:

\[ \psi(\tau) = \cos(m\tau) \psi(0) + \sin(m\tau) \gamma_0 \psi(0) \gamma_0 \gamma_1 \gamma_2 , \]

(13)

where \( \psi(0) \) is a constant spinor, which translates into the Clifford language the solution\(^2\) found by BZ for their analogous system of equations. In the case of solution (13), it holds:
\[ v(\tau) \equiv \dot{x}(\tau) = \frac{pH}{m^2} + [v(0) - \frac{pH}{m^2}] \cos(2m\tau) + \frac{\dot{v}(0)}{2m} \sin(2m\tau), \]  

which clearly shows the presence of an internal helical motion (\(i.e.,\) at the classical level, of the zbw phenomenon).

In eq.(14) we have \( H = v \cdot p = \text{constant} \). If the constant is chosen to be \( m \):

\[ H = p \cdot v = m, \]  

and —more important— if now \( \psi(x) \) is a DH spinor field such that its restriction to the world–line \( \sigma \) yields \( \psi(\tau) \), namely \( \psi|_{\sigma}(x) = \psi(\tau) \), then eq.(13) writes

\[ \psi(x) = \cos(p \cdot x) \psi(0) + \sin(p \cdot x) \gamma_0 \psi(0) \gamma_0 \gamma_1 \gamma_2, \]  

which now is a quantum wave-function, solution, as we are going to see, of the Dirac equation! In fact, it is:

\[ \dot{\psi} \equiv \frac{d\psi}{d\tau} = v^\mu \partial_\mu \psi = (v \cdot \partial) \psi, \]  

and, for any eigen-spinors \( \psi \) of \( \hat{p}\psi \equiv \partial \psi \gamma_1 \gamma_2 = p\psi \), one gets\(^7\) by using the equalities \((v \cdot \partial)\psi \gamma_1 \gamma_2 = (v \cdot \hat{p})\psi = (v \cdot p)\psi = m\psi\) that eq.(9) transforms into the ordinary Dirac equation in its Dirac–Hestenes form:\(^5\)–\(^8\)

\[ \partial \psi \gamma_1 \gamma_2 + m\psi \gamma_0 = 0. \]  

Notice once more that, while eq.(9) refers to \( \psi = \psi(\tau) \), on the contrary the quantum equation (17) refers to the spinor field \( \psi = \psi(x) \) \([such\ that\ \psi|_{\sigma}(x) = \psi(\tau)]\). Then, eq.(13') is an actual solution of eq.(17); while eq.(13) —if you want— can be said to be a solution of eq.(17) \(when\ this\ equation\ is\ restricted\ along\ the\ stream–line\ \sigma\ (i.e.,\ the\ world-line\ of\ the\ “sub-microscopic”\ object \(Q\)).\) When moving from the classical to the quantum interpretation, bearing in mind the Feynman paths formalism, one has to pass from considering a single helical path to consider a congruence of helical paths.\(^2\)

Let us go back, for a moment, to the system (6)–(9). We may notice that —using eq.(16)— the [total derivative] equation (6) can be rewritten\(^7\) in the [partial derivative] noticeable form:

\(^0\#^2\) Such a congruence of helical paths can be regarded as constituting a quantum “fluid”; in such a fluid, however, we would have a flux of energy–momentum and angular momentum also along the normal to the velocity stream–lines. Therefore, this fluid would be a Weyssenhoff fluid,\(^24\) rather than a “Dirac fluid”.\(^23\)
\((\psi \gamma_0 \tilde{\psi}) \cdot \partial \psi \gamma_1 \gamma_2 + \pi \psi \gamma_0 = 0\), \hspace{1cm} (6')

which is a non-linear, Dirac–like equation.

3. Time–like helical motions – Let us prove, now, that solution \(\psi(x)\) of eq.(17), which reduces to the \(\psi(\tau)\) given by eq.(13) on the world–line \(\sigma\) of the point-like constituent \(Q\), is indeed a superposition\(^{(\#1)}\) of positive and negative energy states, i.e., of particle and antiparticle states (for a purely kinematical reinterpretation of the negative energy states in terms of antiparticles, without any recourse to a “Dirac sea”, cf. refs.\(^{15}\)). Indeed, \(\psi(\tau)\) can be written in the CM frame as:

\[\psi(\tau) = \frac{1}{2}[\psi(0) + \gamma_0 \psi(0)\gamma_0] \exp(\gamma_1 \gamma_2 m\tau) + \frac{1}{2}[\psi(0) - \gamma_0 \psi(0)\gamma_0] \exp(-\gamma_1 \gamma_2 m\tau) . \hspace{1cm} (18)\]

But quantities

\[\frac{1}{2}[\psi \pm \gamma_0 \psi\gamma_0] \equiv \Lambda_{\pm}(\psi) \hspace{1cm} (19)\]

are nothing but the positive and negative energy projection operators\(^{14}\) \(\Lambda_+, \Lambda_- \in \text{End}(\mathbb{R}_{1,3})\), respectively; so that eq.(18) can read

\[\psi(\tau) = \psi_+(0) \exp(\gamma_1 \gamma_2 m\tau) + \psi_-(0) \exp(-\gamma_1 \gamma_2 m\tau) \hspace{1cm} (20)\]

where \(\psi_{\pm}(0) \equiv \Lambda_{\pm}[\psi(0)]\), which proves our claim. It follows that any \(\psi(x)\), such that \(\psi_{\sigma}(x) = \psi(\tau)\), is a solution of the DH equation (17) in the CM frame.

It is clear that one can now construct “helical” solutions of the DH equation with time–like velocity, that manifest a zbw phenomenon. Quantity \(\tau\), let us repeat, is here an invariant time–parameter. Now, to construct a time–like solution, let us take as a concrete example:

\[\psi(0) \equiv \sqrt{\rho_+} + \sqrt{\rho_-} \gamma_1 \gamma_0 ; \quad \sqrt{\rho_\pm} \equiv \sqrt{\rho_\pm}(0) . \hspace{1cm} (21)\]

Since

\[\psi(0) \tilde{\psi}(0) = \rho_+ - \rho_- \hspace{1cm} (22)\]

we can put, for simplicity, \(\rho_+ - \rho_- = 1\). In this case, \(\psi_+(0) = \sqrt{\rho_+}\) and \(\psi_-(0) = \sqrt{\rho_-} \gamma_1 \gamma_0\). Then, by using eq.(14), from eq.(21) it follows that

\[v(0) = (\rho_+ \pm \rho_-) \gamma_0 + 2 \sqrt{\rho_+ \rho_-} \gamma_1 ; \quad \dot{v}(0) = 4m \sqrt{\rho_+ \rho_-} \gamma_2 \hspace{1cm} (23)\]
\[
H = m (\rho_+ + \rho_-),
\] (24)

and we end up with:

\[
v(\tau) = (\rho_+ + \rho_-) \gamma_0 + 2 \sqrt{\rho_+ \rho_-} \left[ \gamma_1 \cos(2m\tau) + \gamma_2 \sin(2m\tau) \right],
\] (25)

for which it is \( v^2(\tau) = 1 \) (so that we got a special time-like case). For the spin bivector \( S = \frac{\hbar}{2} \psi \gamma_2 \gamma_1 \tilde{\psi} \) and the spin vector \( s = \frac{\hbar}{2} \psi \gamma_3 \tilde{\psi} \), we have in this case

\[
S = \frac{1}{2} (\rho_+ + \rho_-) \gamma_2 \gamma_1 + \sqrt{\rho_+ \rho_-} \left[ \gamma_0 \gamma_1 \sin(2m\tau) - \gamma_0 \gamma_2 \cos(2m\tau) \right],
\] (26)

\[
s = \frac{1}{2} \gamma_3; \quad [\hbar = 1].
\] (27)

The velocity \( v(\tau) \equiv dx/d\tau \), with \( x(\tau) = x^\mu(\tau) \gamma_\mu \), is easily integrated, from eq.(25), to give:

\[
x(\tau) = (\rho_+ + \rho_-) \tau \gamma_0 + \frac{\sqrt{\rho_+ \rho_-}}{m} \left[ \gamma_1 \sin(2m\tau) - \gamma_2 \cos(2m\tau) \right] + x_0,
\] (28)

which is the parametric equation of a helix, whose diameter is \( D = 2m \sqrt{\rho_+ \rho_-} \). Equation (24) suggests to introduce a renormalized mass \( M \equiv m (\rho_+ + \rho_-) \). If one assumed the maximum diameter of that helix to be the electron Compton wave–length, one would get for the new mass \( M \) the upper limit \( M = m \sqrt{2} \).

It is worth observing that, from eq.(28), for \( L \equiv x \wedge p \) we have:

\[
L = \sqrt{\rho_+ \rho_-} \left[ \gamma_1 \gamma_0 \sin(2m\tau) - \gamma_2 \gamma_0 \cos(2m\tau) \right],
\] (29)

where we neglected the constant contribution \( mx_0 \wedge \gamma_0 \). Notice that \( \dot{L} \neq 0 \), so that \( L \) alone is not conserved; however, in view of eq.(26), we obtain that the total angular momentum \( J \) is conserved:

\[
J \equiv L + S = \frac{1}{2} (\rho_+ + \rho_-) \gamma_2 \gamma_1; \quad \dot{J} = 0;
\] (30)

which implies a nutation of the spin plane. Under the above assumption (that the maximum helix diameter be the electron Compton wave–length), one would get \( |J| \equiv [J \cdot \hat{J}]^{1/2} \leq \frac{\sqrt{2}}{2} \).

4. Light-like helical motions – Finally, let us show how one can obtain solutions of the BZ theory with speed of the helical motion equal to the light speed \( c \). To this aim, it is enough to choose \( \rho_+ = \rho_- = 1/2 \), so that \( \rho_+ - \rho_- = 0 \). In this case,
\[ \psi(0) = (1 + \gamma_1 \gamma_0) / \sqrt{2} \] is a singular spinor, actually a Majorana spinor since the charge conjugation operator \( C \) is such that \( C \psi = \psi \gamma_1 \gamma_0 \). In fact, from eq.(14) one then gets

\[ v(0) = \gamma_0 + \gamma_1 ; \quad \dot{v}(0) = 2m \gamma_1 ; \quad H = m , \] (31) (32)

which yield for the velocity \( v \):

\[ v(\tau) = \gamma_0 + \gamma_1 \cos(2m\tau) + \gamma_2 \sin(2m\tau) , \] (33)

in which it is now \( v^2(\tau) = 0 \) (light–like case). One can see that, after a convenient rotation, it is possible to write \( v \) as \( v = e_0 - e_2 \), like in Hestenes’ papers.\(^5\) Moreover, we now have:

\[ S = \frac{1}{2} \gamma_2 \gamma_1 + \frac{1}{2} [ \gamma_0 \gamma_1 \sin(2m\tau) \ - \gamma_0 \gamma_2 \cos(2m\tau) ] \] (34)
\[ s = \frac{1}{2} \gamma_3 ; \quad [ \hbar = 1 ] . \] (35)

Integration of eq.(33), now, yields:

\[ x(\tau) = \tau \gamma_0 + \frac{1}{2m} [ \gamma_1 \sin(2m\tau) - \gamma_2 \cos(2m\tau) ] + x_0 , \] (36)

and one can verify that in this case the helix diameter is actually the Compton wavelength of the electron! For \( L \equiv x \wedge p \) we obtain again

\[ L = \frac{1}{2} [ \gamma_1 \gamma_0 \sin(2m\tau) - \gamma_2 \gamma_0 \cos(2m\tau) ] ; \quad \dot{L} \neq 0 , \] (37)

whilst the conserved quantity is

\[ J \equiv L + S = \frac{1}{2} \gamma_2 \gamma_1 ; \quad \dot{J} = 0 ; \] (38)

which again implies a nutation of the spin plane.

In conclusion, we showed how to construct in the CM frame solutions of the Dirac equation associated (at the classical limit) to a helical motion with the light speed \( c \). And, in particular, we got results quite similar to Hestenes’ without his ad–hoc assumptions\(^5\) / [by making recourse, however, to a superposition\(^#1\) of particle and antiparticle\(^15\) solutions of the DH equation. Indeed, the part of the velocity \( v(\tau) \) responsible for the zbw, namely \( \sqrt{\rho_+ \rho_-} [ \gamma_1 \cos(2m\tau) + \gamma_2 \sin(2m\tau) ] \), is given by an interference term \( \psi_+ \gamma_0 \tilde{\psi}_- + \psi_- \gamma_0 \tilde{\psi}_+ \), in which quantities \( \psi_+ = \sqrt{\rho_+} \exp(\gamma_1 \gamma_2 m\tau) \) and \( \psi_- = \sqrt{\rho_-} \gamma_1 \gamma_0 \exp(-\gamma_1 \gamma_2 m\tau) \) are
the positive–energy (particle) and negative–energy (antiparticle\textsuperscript{15}) states].

5. *Acknowledgements* – The authors are grateful for discussions to A. Bugini, R. Garattini, L. Lo Monaco, G.D. Maccarrone, J.E. Maiorino, M. Pavšič, R. Pucci, F. Raciti, M. Sambataro, Q.A.G. de Souza, and particularly to E.C. de Oliveira, M. Pauri and S. Sambataro.
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