On the Fundamental Limits of Formally
(Dis)Proving Robustness in Proof-of-Learning

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Abstract—Proof-of-learning (PoL) proposes a model owner use machine learning training checkpoints to establish a proof of having expended the necessary compute for training. The authors of PoL forego cryptographic approaches and trade rigorous security guarantees for scalability to deep learning by being applicable to stochastic gradient descent and adaptive variants. This lack of formal analysis leaves the possibility that an attacker may be able to spoof a proof for a model they did not train.

We contribute a formal analysis of why the PoL protocol cannot be formally (dis)proven to be robust against spoofing adversaries. To do so, we disentangle the two roles of proof verification in PoL: (a) efficiently determining if a proof is a valid gradient descent trajectory, and (b) establishing precedence by making it more expensive to craft a proof after training completes (i.e., spoofing). We show that efficient verification results in a tradeoff between accepting legitimate proofs and rejecting invalid proofs because deep learning necessarily involves noise. Without a precise analytical model for how this noise affects training, we cannot formally guarantee if a PoL verification algorithm is robust. Then, we demonstrate that establishing precedence robustly also reduces to an open problem in learning theory: spoofing a PoL post hoc training is akin to finding different trajectories with the same endpoint in non-convex learning. Yet, we do not rigorously know if priori knowledge of the final model weights helps discover such trajectories.

We conclude that, until the aforementioned open problems are addressed, relying more heavily on cryptography is likely needed to formulate a new class of PoL protocols with formal robustness guarantees. In particular, this will help with establishing precedence. As a by-product of insights from our analysis, we also demonstrate two novel attacks against PoL.

I. INTRODUCTION

Scaling verified computing to deep learning is difficult due to the inability to express optimization problems in a format amenable for generating cryptographic proofs [1]. Thus, relatively little progress has been made in applying cryptographic primitives to verify properties (e.g., integrity, confidentiality) of training algorithms. This contrasts sharply with the progress made towards verified inference [2–4] or private inference [5, 6].

To address this bottleneck, Jia et al. [7] explored non-cryptographic solutions for verifying the correctness of stochastic gradient descent—the canonical algorithm for training deep neural networks (DNNs)—and consequently obtain a proof of computation expended towards training. They propose the Proof-of-Learning (PoL) protocol for a prover to attest to the integrity of a training run by logging the intermediate states achieved by the learner. We describe the information that is logged in §II but, for now, it is sufficient for the reader to assume that the log contains the DNN’s weights after each step of gradient descent, i.e., the training trajectory. This log becomes a proof, which can be verified by an external party; the verifier simply needs to repeat (some of) the training steps logged in the proof (i.e., duplicated execution) and compare the reproduced steps to the ones logged. This comparison takes place in the weight space. If this verification succeeds, the proof constitutes evidence that the prover ran the training algorithm correctly and obtained the model legitimately.

While the PoL protocol is simple, Jia et al. are unable to provide a formal analysis of security for the proof verification mechanism they propose. In this paper, we uncover why this is the case by disentangling the two roles the proof verification mechanism plays: (a) efficiently verifying that a sequence of model updates contained in a proof is a valid gradient descent trajectory, and (b) establishing precedence by making it more expensive to craft a proof after training completes (i.e., spoofing), than during training.

The first role of proof verification reduces verification into a detection problem. Indeed, multiple sources of noise make it impossible for the verifier to exactly reproduce each step of gradient descent contained in the prover’s proof. Avoiding noise in learning is not always possible—in fact it is sometimes desirable: accelerators like GPUs introduce noise in training but are needed to scale learning to large datasets. This means that some tolerance to noise needs to be built into the verification process, which creates a sensitivity-specificity (accept legitimate steps)-(reject invalid steps) tradeoff where the verifier must tune the threshold accordingly. However, we do not have a precise model of how this noise impacts training—specifically the learner’s trajectory over model weights. Because of this, we analytically show that we cannot formulate a proof verification mechanism that is optimal, i.e., provably minimizes an adversary’s capability to disguise their trajectory within the noise inherent to training.

These difficulties are aggravated when the verification mechanism makes approximations to improve its computational efficiency. For instance, Jia et al. propose to verify only a subset of updates in a proof. This further opens an attack surface for the adversary who is able to force the verification mechanism to verify a subset of updates of their choice.

The second role of a proof (and verification) provides a different, yet just as fundamental explanation for its lack of
formal robustness guarantees. Crafting a valid proof, once given access to a trained model, is assumed to be at least as difficult as generating the proof organically as the model is being trained. However, proving that this is the case reduces to open problems in learning theory. In the presence of a non-convex learning objective, as is the case in deep learning, multiple models corresponding to different local minima can be returned by stochastic gradient descent. Thus, what we would like to show to obtain formal guarantees with respect to this second role of the PoL protocol is that knowledge of one such minimum does not allow one to find multiple trajectories that lead to this same minimum. This is an open problem and would have implications beyond the PoL protocol, for instance towards knowledge transfer in machine learning.

From this analysis, we conclude that two classes of solutions are needed to address these limitations of the existing proof verification mechanism. For the first, we need better models of noise dynamics in learning. We for instance show how capturing both the direction and magnitude, rather than just the magnitude, of individual gradient steps helps refine such models of noise. For the second, we will need methods to guarantee that having access to the final weights does not enable spoof generation (at lower computation than training). We identify new PoL protocols that provide commitment mechanisms as a promising direction. In particular, we expect that particular care will need to be given to committing to a dataset in order to establish precedence.

Finally, we consider the practical implications of these analytical insights. Recall that Jia et al. empirically demonstrated that they could not construct an attacker capable of obtaining a spoof. Yet, this empirical assessment of security leaves the possibility that strategies not considered by Jia et al. will succeed—it creates an arms race. That is indeed what we find: our analytical insights also help us derive new attack strategies against the PoL protocol. While we are unable to instantiate a concrete attack targeting the notion of precedence in proof verification, limitations of the first role of verification, that we identify, directly lead to two novel attacks against PoL. First, we show that an adversary can exploit verification tolerance to craft a spoof made up of completely synthetic updates i.e., updates that do not result from (real) gradients computed over the model’s training data. Second, we introduce an attack that targets approximations made during verification to increase efficiency: because our adversary is able to manipulate which updates are selected for verification, they can create a spoof at a smaller computational cost by crafting a gradient descent trajectory that only needs to be partially valid.

To summarize, our contributions are:

1) We show that it is not possible to formally prove or disprove the robustness of the PoL protocol of Jia et al. without first answering several open questions. The reasons for this relate to the two roles that proof verification plays: (a) efficient verification with tolerance to noise, and (b) establishing precedence.

2) We analyze how tolerance to noise turns the verification mechanism into a detection problem: the mechanism is forced to trade-off acceptance of valid proofs with rejection of spoofs. We find this tension is exacerbated when the verification mechanism introduces approximations to improve efficiency. Despite this, we show in § VII-A the theoretical existence of optimal thresholds that address this tension. This result is stated for a class of metrics on the weights of models that are used to arbitrate the verification mechanism. This existence result highlights the need for better models of noise in learning to instantiate such an optimal verification mechanism.

3) We find that, in its current form, proof verification is not able to formally establish precedence. This is because of a limited formal understanding of non-convex learning. To address this, we expect that PoL will need to be refined with commitment mechanisms.

4) We show in § VII that insights from our analysis of noise tolerance can be exploited to mount novel practical attacks against PoL. Instead, theoretical limitations in establishing precedence with PoL do not yet result in practical attacks (see § VII).

II. BACKGROUND

Proof-of-learning (or PoL) relies on an asymmetry in the training protocol arising from the highly complex and non-linear nature of training deep neural networks (DNNs) [8]. PoL draws connections with proof-of-work [9, 10] mainly in that the authors of PoL demonstrate how gradient inversion is at least as expensive as gradient computation. Thus, the authors hypothesize that gradient calculations on data play a role similar to one-way functions [11]; we will revisit this in our manuscript. Similar to proof-of-work, PoL should (ideally) prevent an entity from claiming they have trained a model without having spent at least comparable computational effort.

In the rest of the manuscript, we use lower-case, bold-faced notation to capture random variables. See Table II for commonly used notation in Appendix A.

A. Primer on PoL

The PoL framework [7] assumes that a prover T honestly trains a machine learning model in T steps to obtain parameters W_T. The PoL (as defined in [7]) is stated below.

Definition 1. For a prover T, a proof is denoted as \( \mathcal{P}(T, W_T) = (\mathcal{W}, I, H, A) \) where all the elements of the tuple are ordered sets indexed by the training step \( t \in [T] \). In particular, (a) \( \mathcal{W} \) is a set of model-specific information that is obtained during training, (b) \( I \) denotes information about the specific data points used to obtain each state in \( \mathcal{W} \), (c) \( H \) represents cryptographic signatures of the training data, and (d) \( A \) incorporates auxiliary information that may or may not be available to an adversary \( A \), such as hyperparameters \( \mathbb{M} \), model architecture, optimizer, and loss choices.

If T logs information for every step \( t \), the exact training process (culminating at \( W_T \)) should ideally be reproducible. The memory footprint of this information, i.e., the states \( (I, H, A) \), is often small. However, storing \( \mathcal{W} \), specifically the model weights, incurs high overhead. Thus, it is common practice to log weights periodically at every \( k^{th} \) step (also known as the checkpointing interval).

Motivations for PoL include substantiating claims of ownership of a specific weight \( W_T \) or verifying the correctness of

\footnote{The source code will be released upon publication.}
delegated computations. The latter may arise in the context of distributed learning [12]. The former is motivated by the threat of model stealing [13]; the PoL protocol increases the cost of an adversary as it is not required to generate a proof for the model it has stolen (or obtained through insider access).

Verification: Without loss of generality, the verifier begins with $W_t$, and uses the information in $(I, H, A)$ to locally perform $k$ steps of training to achieve $W_{t+k}$; the difference between two such weights is termed an update. This is compared against the next stored weight $W_{t+k}$. In general, $W_{t+k} \neq W_{t+k}$ due to entropy arising from low-level components, e.g., low-level libraries and hardware [14]–[16]. We refer to such differences as noise, or stochasticity in training. This error limits the maximum $k$ that can be chosen. $W_{t+k}$ is deemed valid if $d(W_{t+k}, W_{t+k}) < \delta$, i.e., the recreated weight $W_{t+k}$ is within a $\delta$ error threshold of the original (prover-generated) weight $W_{t+k}$ using some distance function $d$ (typically an $\ell_p$ norm). The threshold $\delta$ is fixed statically by the verifier prior to proof verification and is tuned such that $\varepsilon_{\text{repr}}(t) < \delta \ll d_{\text{ref}}$, where $\varepsilon_{\text{repr}}(t)$ captures the reproduction error due to low-level randomness (at various training steps) and $d_{\text{ref}}$ is a reference distance estimated by re-running the training protocol (with varying sources of stochasticity) and recording the deviation upon completion. Jia et al. also define normalized reproduction error ($\|\varepsilon_{\text{repr}}\|$) as $\max_i \varepsilon_{\text{repr}}(t)/d_{\text{ref}}$.

As will become important when considering spoofing attacks that are cheaper than honest training, we formalize the set of valid proofs ending in a particular weight $W_T$ analogous to what was used by Thudi et al. [17]. Let $A_{D,W_T}$ be the set of $(g, d, \delta)$-proofs ending in $W_T$ generated by a specific dataset $D$, i.e., those obtained using update rules $g_i \in g$ passing thresholds $\delta$ in metric $d$. With this formalization of valid proofs, we can associate to honest training a distribution on the set of valid proofs ending in $W_T$.

Definition 2 (Honest Training). Honest training is represented by a probability measure $\mu$ on the event space $A_{D,W_T}$.

Efficient Verification: To speed up the verification procedure, it is possible to use heuristics to select which specific updates to verify. This, however, introduces a trade-off between computation savings and verification accuracy. Jia et al. utilize the top-$Q$ mechanism for probabilistic verification: the verifier selects the $Q$ largest (in their $\ell_p$ norm) updates of each epoch for verification; the intuition for this is that larger updates exist primarily in falsified proofs.

B. Creating Spoofs

Jia et al. [7] define a spoof as any proof that requires lesser computation to be generated than honest training (see Table III in Appendix A for the full list of spoof categories they introduce). One of the key contributions of this paper is to analyze why (if at all) such spoofs exists. For now, we briefly review spoofing schemes known in the literature.

1. Sequence Inversion [7]: Here, we assume an adversary has access to $W_T$ and the data used to train the model, but not to $I, A$. Such a strategy aims to invert gradient descent i.e., given $W_T$, find a corresponding $W_{T-1}$ that was used to obtain it. The authors show that such a process is difficult due to increasing entropy (as a function of training duration), and is computationally lower bounded by the cost of honest training.

2. Directed Retraining [?]?: Under the same assumptions as sequence inversion, this strategy focused on either (a) creating structurally correct proofs [4] or (b) artificially directing the weights to the final state $W_T$ quicker. The authors showed that unless the entire proof is valid, any discontinuities produced by methods in (a), e.g., concatenating proofs, would be detected by a verification mechanism that checks the largest updates first. They argued that approaches in (b) require custom training algorithms (e.g., regularizers, loss functions, etc.) with direct knowledge of the final weights $W_T$ (before it is obtained) and thus fail verification.

3. Adversarial Examples for PoL: Zhang et al. [18] introduce two techniques to generate shorter, structurally correct spoofs (see [7]) by utilizing insight from evasion [19], [20]. These strategies assume that the adversary has access to $W_T$ and the data that was used to obtain the proof, but no other information. Herein, we define $U_k(W_t, X_t)$ to represent the update to go from $W_t$ to $W_{t+k}$ with data $X_t$. We utilize $\hat{W}$ to denote a weight created by the adversary.

3.1. Synthetic Adversarial Update: The objective of the adversary is to create synthetic data $\hat{X}$ such that the following is possible (without loss of generality): $\hat{W}_{T+n} = U_n(\hat{W}_T, \hat{X})$. Such synthetic data is termed an “adversarial example” by the authors. Since generating these adversarial examples requires additional computation, the adversary would perform $T - 1$ steps of legitimate training with $T \ll T$, and utilize the adversarial update only for the update from $W_{T-1}$ to $W_T$ (ergo reducing the overall computational overhead); this is defined as “Attack 1” in Zhang et al.’s work. However, the authors only conceptually describe the attack and do not evaluate it because they find it difficult for the optimization to converge. Despite our best efforts, we also failed to have our implementation of this strategy converge and synthesize data that satisfies the adversary’s objective. Thus, given that Zhang et al. do not provide evidence that this attack strategy can succeed for the adversary, we do not consider it further in our work.

3.2. Synthetic Checkpoint Initialization: Zhang et al. propose a technique to choose intermediary weights that exploit the threshold $\delta$ picked by the verifier. Recall that this threshold allows the verifier to tolerate noise induced by stochasticity in gradient descent. The adversary can choose pairs of weights $W_t$ and $W_{t+k}$ such that $d(W_t, W_{t+k}) \ll \delta$, $\forall t, k$; this can be done by linearly interpolating between a chosen $W_0$ and the final weights $W_T$. The authors also propose techniques to minimize the distance $d(W_0, W_T)$ between the model initialization and final weights, which will further minimize the distance between the aforementioned pairs of intermediate weights. Upon initializing $\hat{X}$ with training data, $\hat{X}$ is then perturbed (using techniques described for attack 1) such that $d(W_t, U_k(W_t, \hat{X})) \approx 0 \ll \delta$; this will result in $d(W_{t+k}, U_k(W_t, \hat{X})) \ll \delta$ because $d(W_t, W_{t+k}) \ll \delta$ and

\footnote{This type of spoofs pass the verification by generating an invalid proof (i.e., with invalid updates) for the victim model, $W_T$.}

\footnote{Evasion, or creating adversarial examples is an inference-time attack against machine learning models where a perturbation is added to the inputs resulting in erroneous model predictions.}
thus pass the verification. Zhang et al. call this “Attack 2”[4].

III. Threat Model

In the rest of the paper, we consider the following threat model unless otherwise specified.

1) An adversary A has complete knowledge of W_T and the architecture used to obtain it.
2) A has knowledge about δ and Q (i.e., parameters used for verification) as well as the selection mechanism of which updates are being verified.
3) A does not have access to any other information used during training. This includes the intermediate training steps, and the sources of randomness used to obtain W_T.
4) A has access to the training dataset (or distribution).

Finally, we note that our primary analysis is on DNNs with non-convex loss landscapes since they are more computationally expensive to train and are likelier targets of model stealing.

IV. Efficient Verification of Valid Proofs

Recall that one of the two key roles played by the proof verification mechanism is efficiently verifying the validity of sequences of model updates (i.e., whether they can be obtained from a trajectory of SGD) by reproducing the updates. To understand the robustness of PoL verification, we will theoretically analyze the role it plays from two perspectives: (a) correctness (G.1 in [7]) which says that honest model trainers’ proofs should be correctly validated (even on a different hardware/software), and (b) verification efficiency (G.3 in [7]).

We first introduce the necessary conditions for these desiderata to be satisfied, namely reproducibility and representativeness. We will formalize these assumptions later in §IV-A and §IV-B respectively.

1) Reproducibility: Individual gradients are reproducible up to a small error (up to δ ≤ 1) if the per-step training data and metadata required to obtain them are logged and controlled for.
2) Representativeness: The validity of a sequence of model updates can be implied by the validity of a smaller set derived from it, so verifying the latter is equivalent to verifying the former in terms of the security guarantee.

A. On Correctness of Step-wise Verification

Assumption 1 (Reproducibility). For any sequence of intermediate model states obtained from honest training for T steps, i.e., \{W_1, \ldots, W_T\}, there exists δ such that ||W_{t+1} - U_t(W_t, D_t, M_t)|| ≤ δ \forall t \in [T-1], where U_t(W_t, D_t, M_t) results in next weight update after updating W_t (for 1 step) using (a) the training data D_t, and (b) the metadata M_t used at step t respectively.

A fundamental issue facing PoL verification is how the noise in training manifests itself (i.e., to what degree can training be reproduced). Ideally, this is to be captured by the verification threshold δ (see Assumption 1). The original work [7] proposes a step-wise verification method that relied on an oversimplification of this assumption: ||W_{t+1} - U_t(W_t, D_t, M_t)|| is i.i.d for all t. Thus, δ was assumed to have a fixed value. However, if the δ threshold is not tight, then there is a chance that the adversary can exploit it. For example, the adversary can design data and/or metadata such that the verifier-reproduced update δ_t (in yellow) points in a different direction than the one in the proof (δ_t, in black) and lies within the δ ball. The verifier will incorrectly accept this step despite it being “different” from δ_t.

Fig. 1: Attacks targeting loose noise thresholds δ. For a legitimate step, the discrepancy between g_t (update from the proof, in black) and δ_t (verifier-reproduced update, in blue) should be solely due to the noise from hardware/software, and should lie within the δ ball. However, for loose thresholds, the adversary may be able to create adversarial metadata such that the verifier-reproduced update δ_t (in yellow) points in a different direction than the one in the proof (δ_t, in black) and lies within the δ ball. The verifier will incorrectly accept this step despite it being “different” from δ_t.

To this end, let us assume that the verifier produces ACCEPT or REJECT decisions for a training step. This can be quantified using the true positive rate TPR = \frac{TP}{TP + FN} where TP is the number of accepted valid gradients and FN is the number of rejected but valid gradients. Therefore, the question becomes: “how large is the minimum threshold to retain some fixed TPR?” We formalize these requirements on the verifier in Definition 3.

Definition 3 ((δ, τ)-verification strategies). For proofs of length T, let \langle δ, τ \rangle be the set of all verification strategies that use per-step thresholds \{δ_1(W_1, M_1), \ldots, δ_T(W_T, M_T)\} (that can depend on the weights of intermediate checkpoints \{W_t\} and corresponding training metadata \{M_t\} such as hyperparameters) on a metric d over the training steps in the proof to produce verification decisions ACCEPT, REJECT with a required per-step TPR of at least τ for any weight.

\footnote{The authors also introduced “Attack 3”, which is a more computationally efficient, yet conceptually similar implementation of Attack 2. Given that the two achieve the same performance, we exclusively consider attack 2 in §VI-A.}

\footnote{We also recognize keeping the dataset secret may make the protocol more robust (refer §VIII-A). Thus our proposed attacks do not rely on the dataset access.}

\footnote{In fact, Attack 2 by Zhang et al. [18] exploits this vulnerability; however, Attack 2 is computationally heavy and does not guarantee to converge. Later in §VI we will introduce and evaluate a stronger and cheaper (only costs similar to 1 forward pass per update) attack that exploits vulnerability caused by the looseness of δ}
Our question is now reduced to the following: “does an optimal \( (d, \tau) \)-verifier exist?” Note that optimality here is defined as having the smallest per-step threshold (for each step) that maintains the \( \tau \) TPR rate.

1) Existence of Optimal Strategies: As a first step toward proving the existence of the optimal step-wise verification strategy for a given metric \( d \), we first prove its existence for per-step \( \ell_2 \) norm verification in Lemma 1. We then generalize this result to an entire class of metrics that satisfy the requirement that the boundary of their metric balls has Lebesgue measure 0. Importantly, this includes all \( \ell_p \) metrics and cosine similarity which are commonly used in ML. This is obtained for free in Corollary 1 from the proof of Lemma 1.

**Lemma 1 (Minimum Threshold).** Let \( g_i \in \mathbb{W} \) be the random variable for the \( i^{th} \) update from a given checkpoint \( W_i \), where \( \mathbb{W} = \mathbb{R}^n \) is the weight space. Assume it is absolutely continuous (see (27)) with respect to Lebesgue measure, then for a given TPR \( \tau \) there exists a minimum \( \ell_2 \) threshold \( \delta \) centered at the mean of \( g_i \), s.t. the TPR for \( \delta = \tau \) and for any \( \delta' < \delta \), the TPR is \( < \tau \).

Corollary 1 (All Metrics). In the same setup as Lemma 2, if instead the \( \ell_2 \) metric is some other metric \( d \) such that the boundary of the metric balls of \( d \) have 0 Lebesgue measure, there exists a minimum threshold \( \delta \) for \( d \) centered at the mean of \( g_i \) s.t. for any \( \delta' \leq \delta \), the TPR \( < \tau \).

The idea is that one can interpolate between different thresholds to find the thresholds with the desired TPR. However, to do so we need to check continuity properties.

**Proof:** (Proof Outline) First we will use continuity from above (for measures) to show there is some \( \delta \) s.t. the TPR \( < \tau \). Note: as we are dealing with probability measures, i.e., all sets have finite measures, we need not worry about the finiteness requirement for continuity from above. We then remark that in fact, the measure is a continuous function from a connected domain, so by intermediate value theorem, there is some \( \delta \) s.t. the TPR of \( \tau \). Moreover, the pre-image of thresholds that give TPR of \( \tau \) is a closed set (by definition of being a continuous function), and taking the subset within \([0, \delta]\) we have a compact set. Hence by extreme value theorem there exists a minimum \( \delta \) with TPR \( \tau \). This then concludes the proof.

Let \( \bar{g}_i \in \mathbb{E}(g_i) \) and consider the balls \( B_{2^n}(\bar{g}_i) \) for \( n \in \mathbb{Z} \) (i.e., balls centered at \( \bar{g}_i \) of radius \( 2^n \)). For some large \( N \), note that \( \cap_{n \in \mathbb{Z}, n \leq N} B_{2^n}(\bar{g}_i) = \bar{g}_i \) and so we have \( \mu(\cap_{n \in \mathbb{Z}, n \leq N} B_{2^n}(\bar{g}_i)) = 0 \) (a single point set so measure 0). By absolute continuity we then have \( \mu_{g_i}(\cap_{n \in \mathbb{Z}, n \leq N} B_{2^n}(\bar{g}_i)) = 0 \) and then by continuity from above \( \lim_{n \to -\infty} \mu_{g_i}(B_{2^n}(\bar{g}_i)) = 0 \). So, to conclude, we have \( \forall \tau > 0 \) for some \( n \) s.t. \( \mu_{g_i}(B_{2^n}(\bar{g}_i)) < \tau \).

Now note by continuity from above and below (for measures) taken with respect to balls \( B_r(\bar{g}_i) \) for \( \mu_{g_i} \), noting \( \mu_{g_i}(B_r(\bar{g}_i)) = \mu_{g_i}(B_r(\bar{g}_i)) \) where \( B_r(\bar{g}_i) \) is the closure of the ball (i.e., boundary of the balls have measure 0), we have \( \mu_{g_i}(B_r(\bar{g}_i)) \) is a continuous function from \([0, \infty) \to [0, \infty] \) (where the variable is the radius \( r \) of the ball). So it is a continuous function from a connected domain, so by intermediate value theorem there is some \( \delta \) s.t. the TPR is \( \tau \) (note \( \tau \leq 1 \) and we know the measure of the whole space is equal to 1, giving the upper-bound for intermediate value theorem).

Moreover, the pre-image of thresholds that give TPR = \( \tau \) is a closed (and by the previous line non-empty) set, and taking the subset within \([0, \delta]\) we have a compact set, and hence by the extreme value theorem there exists a minimum \( \delta \) with TPR equal to \( \tau \). This concludes the proof.

**Discussion:** What Lemma 1 and more generally Corollary 1 show is that for metrics such that the Lebesgue measure of the boundary of their metric balls is 0, there exists an optimal per-step verification threshold to obtain the desired TPR. The question now becomes: “how do we instantiate the optimal step-wise verification strategies?” As the first observation in this direction, we empirically demonstrate any constant threshold (i.e., same over all iterations and weights) can be loose in certain settings, allowing spoofing (in §VI-A). So the question finally becomes: “how could we leverage the hyper-parameters and weights to devise tighter thresholds?”

2) On Constructing Optimal Step-wise Verification Strategies: Constructing the optimal step-wise verification strategy (and more so proving it is indeed optimal) is an open problem (since the noise is non-zero and it varies with respect to the software and hardware used by the prover). However, we remark the following: training is successful despite the presence of noise. This suggests that despite not being able to prove optimality, one could construct a list of necessary conditions for noise that is compatible with the optimization objective of training. These are then also necessary conditions on the noise we observe when constructing proofs, and hence inform how we should design thresholds for verification mechanisms.

Intuitively, noise altering the update’s direction has the most potential to alter the convergence of a training run; if the reproduced update was to point in the opposite direction of the original update in the proof, we would be maximizing the optimization objective rather than minimizing it. Put another way, noise should have lower variance in the direction of the updates. However, this is not captured by a constant \( \ell_2 \) metric (which only computes the magnitude). Instead, it would be better to inform the design of the verification mechanism with the necessary condition we just stated: noise has to have lower variance in the direction of training. For instance, we can prove there exists a verification scheme that bounds both the error in direction and the \( \ell_2 \) error. This is given in Lemma 2. In Figure 2 we show an illustration for Lemma 2.

**Lemma 2.** For all \( \theta \in [0, 2\pi) \), \( \exists \alpha \) s.t. \( ||g - g'|| \leq \alpha \min(||g||, ||g'||) \) implies the angle between \( g \) and \( g' \) is \( \leq \theta \).

**Proof:** Take \( \alpha = \frac{1}{||g||} \) and a desired bound on the difference in angles between \( g \) and \( g' \); this defines a convex conic set \( S \) in
Lemma 2 informs how we can design step-wise verification mechanisms that bound both the difference in angle and $\ell_2$ norm. To do so, one may set the verification threshold $\delta$ to be $\alpha \cdot \min\{\|g\|, \|g'\|\}$ instead of a constant. While this analysis suggests a first step towards an improved step-wise verification strategy, the resulting verification mechanism is still not optimal. Indeed, there is no way to obtain an optimal threshold for the angle: just like the threshold on the $\ell_2$ norm of the difference, the threshold on the angle would also be impacted by implementation details (e.g., on what machine the DNN is trained, and what library is used, etc.). Constructing an optimal verification strategy remains an open problem.

**Open Question(s):** How do we instantiate the optimal verification strategy (if we can)? Moreover, can an adversary bypass the optimal verification strategy with knowledge of how the thresholds are computed?

### B. On Efficient Verification

We now turn to the efficiency perspective of proof verification and its impact on our ability to formally reason about the robustness of verification. To make verification more efficient, the verifier may either (a) only verify a subset of training steps, or (b) spend less cost on verifying individual steps. Jia et al. [7] propose selecting the $Q$ updates with largest magnitude to make verification more efficient through (a). However, recall that proof correctness is defined on a complete sequence of model updates contained in the proofs. Thus, verifying only a subset will not guarantee only correct proofs are accepted. As a counter-example, would an adversary be able to create a valid subset of model updates which is part of a larger, invalid sequence such that the larger sequence passes verification with some non-trivial probability of success? In other words, this raises the question of if an adversary can somehow control which updates are verified so as to create a valid subset of model updates from a larger, invalid sequence such that this spoof passes verification with some non-trivial probability of success.

To obtain a robust verification mechanism that is more efficient than verifying the complete sequence of model updates, we need to show the existence of a subset of updates such that its validity implies the validity of the entire training sequence. We define such a representative subset (in Definition 4) which satisfies the desired property (see Lemma 3). Note that by this definition, the set of all updates in a proof is also a representative subset of itself; it is assumed that there exists a representative subset that is smaller than the entire proof (formally stated in Assumption 2).

**Definition 4 (Representative Subset).** A subset $S$ of model updates is representative of a training process $\{g_1, \ldots, g_T\}$, if there exists any update $g_i$ such that $\|g_i - g'_i\| > \delta_i$ (i.e., the $i^{th}$ update is invalid), then there exists at least one $g_j \in S$ with $\|g_j - g'_j\| > \delta_j$, where $i$ could be equal to $j$.

**Lemma 3.** Verifying $S$ is equivalent to verifying all the training updates.

**Proof:** When verifying $S$, if all the updates pass, i.e., $\|g_j - g'_j\| < \delta_j \forall g_j \in S$, then by Definition 4 there does not exist any $g_i$ such that $\|g_i - g'_i\| > \delta_i$ in the entire sequence of training updates. On the other hand, if at least one update in the representative subset did not pass, then the proof should be rejected as it captures another update in the entire sequence that will not pass verification.

**Assumption 2 (Representativeness).** If a proof consists of $T$ training updates, then there exists a representative subset (see Definition 4) for it with size less than $T$.

Selecting a representative subset, however, is a difficult problem. It is equivalent to finding model updates that (a) are necessary for achieving the final model state, or (b) would not exist if previous updates were not computed correctly. Researchers who study optimizers for DNNs have expended significant effort in studying questions along this line, but it still remains an open problem. Nonetheless, we may leverage this definition to infer properties that an optimal selection mechanism (i.e., one that is able to select the smallest representative subset) would possess. Based on Definition 4, the representative subset must contain individual updates that can be used to infer the validity of some other updates not in the subset. This means that the metric used to select such updates must be a function of multiple training updates and take relations between updates into consideration. Thus, we believe the selection mechanism for the top-Q verification approach proposed in [7] does not satisfy Assumption 2 as it only considers the norm of individual training updates. This results in concrete attacks—we instantiate one in §VI-B to show how an adversary may create a subset of updates which seems to be valid, ergo forcing their verification (leaving the other invalid updates in the spoof untouched). Lastly, based on our findings, we frame the following open questions:

**Open Question(s):** How to select a subset of training updates to represent a training process? What properties of training updates make them more important than the others and how does this connect to optimization algorithms of DNNs?

So far, we have provided a systematic assessment of the desiderata of proof verification to correctly and robustly play its role in verifying validity of proofs; violations of these desiderata may result in spoofs passing verification. In §VI, we empirically validate our claims by introducing new attacks.
valid proofs with cheaper computational cost than honest training—especially with access to the trained model.

This is where the second role of a proof—setting a precedent for the trained model—becomes important for characterizing PoL protocol robustness. We need to ensure that given knowledge of the final weights $W_T$ obtained from honest training (but not the rest of the proof of the original model owner), it is impossible to recreate any valid sequence (defined in § II-A) resulting in $W_T$ where the sequence generation process is computationally cheaper than honest training. This leads to the following formulation for a cheapness assumption.

**Assumption 3 (Cheapness).** Given a cost function $C$, for any algorithm $F : (D, W_T) \rightarrow A_{D,W_T}$, $\mathbb{E}[C(F(D, W_T))] \geq \mathbb{E}_{P \sim A_{D,W_T}}[C(P)]$

In this section, we initiate a study of when this assumption holds, which is equivalent to understanding if an adversary can break the PoL protocol by creating a valid spoof with lesser cost than honest training. Jia et al. term such strategies as stochastic spoofing. We consider a first step towards proving when stochastic spoofing cannot exist, i.e., when the cheapness assumption cannot be violated. We do so by surfacing stability properties exhibited by stochastic spoofing adversaries but not by honest training algorithms.

We begin by defining a class of algorithms that capture the goal of stochastic spoofing: algorithms that produce valid proofs with lower expected “cost” than obtained when honestly training. This is formalized in Definition 6. Note that the non-existence of such algorithms is equivalent to the cheapness assumption, as formalized in Assumption 3.

**Definition 5 (Cost of a Proof).** We define cost as some function $C: A_{D,W_T} \rightarrow \mathbb{R}^+$, which represents the “cost” associated to computing each proof.

**Definition 6 (c-Cheap Proof Algorithm).** Consider $A_{D,W_T}$ as stated earlier in § II-A. Let $C$ be a cost function as defined in Definition 5 and $E = \mathbb{E}_{P \sim A_{D,W_T}}[C(P)]$ be the expected cost over $A_{D,W_T}$ for some (honest) distribution given by (honest) probability measure $\mu$ on event space $A_{D,W_T}$. Then an algorithm $F : (D, W_T) \rightarrow A_{D,W_T}$ which given $(D, W_T)$ outputs a proof in $A_{D,W_T}$ is c-cheap if $\mathbb{E}[C(F(D, W_T))] < cE, c \in [0, 1)$.

This definition formalizes stochastic spoofing attacks $F$ as those whose expected cost is some fraction of the expected cost of honest training; the expectations are taken using some arbitrary probability measures for generality (i.e., some measure representing honest training).

With the spoofing attacks $F$ formally defined, the main challenge in directly proving the (non-)existence of such spoofs is knowing whether or not access to a particular local minimum (i.e., a trained model’s weights) in a non-convex loss surface would enable stochastic spoofing. Without an answer for this, we cannot say if spoofing attacks exist once the weights have been stolen. Note that for convex optimization, where a unique global minimum exists, the adversary is guaranteed to reach the same minimum as the victim if the victim model is trained to convergence. However, this is usually not true under the setting of DNN training when the adversary only has access to the final state of a model (trained by an honest prover) but has no information about the rest of the proof: the loss landscape is often highly non-convex, and has many local minima that can be attained through valid paths.

However, instead of directly proving (non-)existence, we explore which properties $c$-cheap algorithms (for finding stochastic spoofs) must satisfy. This is a first step towards understanding when stochastic spoofing is possible and how one might introduce measures to prevent them.

**Lemma 4 (Stability of c-cheap algorithms).** Note: letting $\mu$ represent the distribution from honestly training, a $b$-measure subset means honest training produces a proof in that set with likelihood $b$. A $c$-cheap algorithm, with probability $\frac{1}{3}$, only produces proofs in a less than or equal to $\zeta$ measure subset of $A_{D,W_T}$ (using measure $\mu$), where $\zeta = \frac{\sqrt{3} \alpha}{|E(1-c^{-1}+a)|}$ and $\alpha = \sqrt{3} \text{Var}(C(F(D, W_T)))$.

**Proof:** First by Markov’s inequality $\mathbb{P}(\mathbb{E}[C(F(D, W_T))] - C(F)^2 > \alpha^2) \leq \frac{\text{Var}(C(F))}{\alpha^2}$, taking $a = \sqrt{3} \text{Var}(C(F))$ we get this is less than $1/3$, so with probability $2/3$, $|\mathbb{E}[C(F)] - C(F)| < a$.

Now the question is how many proofs are in $a$ ball around $F$’s mean $E_1 = \mathbb{E}[C(F)]$. This is given by $\mathbb{P}(C(F) > E_1 - a)\mathbb{P}(C(F) < E_1 + a) \leq \mathbb{P}(\mathbb{E}(C(F)) - E_1^2 \geq (E_1 - a)^2) = \mathbb{P}(\text{Var}(C(F)) \geq \frac{(E_1 - a)^2}{\text{Var}(C(F))} \leq \frac{(E_1 - a)^2}{\text{Var}(C(F))} \leq \frac{\text{Var}(C(F))}{|E(1-c^{-1}+a)|}$ where the second last inequality was just Markov’s inequality.

**Lemma 4** relates the likelihood of a stochastic spoofing attack $F$ producing a set of proofs to the likelihood honest training produces those proofs. That is, a certain set becomes more common, or more stable with the adversary’s algorithm.

Complementing Lemma 4 on enhanced stability properties of $F$, we can determine a lower bound on the query complexity needed to obtain $c$-cheap proofs with honest training. Let us define $F_{\text{honest}}$ as the algorithm sampling/querying from $A_{D,W_T}$ with honest measure $\mu$ until it obtains a $c$-cheap proof. The following lemma gives a lower bound on how many queries $F_{\text{honest}}$ needs to achieve this. This is a potentially important property to set a baseline cost any prover must capture.

**Lemma 5 (Queries).** Let $F_{\text{honest}}$ query $A_{D,W_T}$ (with probability measure $\mu$) inducing a distribution on $C(F(P), P \in A_{D,W_T}$. Then with probability $\frac{2}{3}$, $F_{\text{honest}}$ issues $\geq \frac{\text{Var}(C(F))}{(1-c^{-1}+a)^2 \log(1/\delta)}$ queries where $P = \frac{\text{Var}(C(F))}{(1-c^{-1}+a)^2 \log(1/\delta)}$ to obtain a $c$-cheap proof.

**Proof:** We drop the $\mu$ subscripts but note this is the measure for cost distribution. We have $\mathbb{P}(C(F) \leq cE) \leq \mathbb{P}((E - C(F))^2 \geq (1-c^{-1}+a)^2) \leq \frac{\text{Var}(C(F))}{(1-c^{-1}+a)^2} = P$, where the last inequality was by Markov’s inequality. Thus $\mathbb{P}(C(F) > cE) \geq 1 - P$.  

---

1We note that Jia et al. require their PoLs satisfy such a cheapness assumption in desideratum G.2 (Security) although they frame it such that a PoL is resilient to a dishonest spoof (Definition 2 in [7]).
We are interested in \( N \) s.t. \((1 - P)^N \leq 1/3\) so that with probability \( 2/3 \) we obtain a \( c \)-cheap proof after \( N \) queries if \( P(C(P) > cE) = 1 - P; \) this establishes a lower-bound as in general \( P(C(P) > cE) \geq 1 - P \). This is simply given by \( N = \frac{\log(1/3)}{\log(1-P)} \), concluding the proof.

With the previous two lemmas, we have surfaced conditions \( c \)-cheap algorithms would need to satisfy. Whether an algorithm can satisfy these properties is an open problem, and a negative to this question would also prove \( P \). We will begin our discussion with an overview of adversarial attacks against spoofing adversaries exist. It will dictate whether (or when) comprehensive defense strategies against spoofing adversaries exist.

**Open Question(s):** Definition 6 formally defines stochastic spoofing as the algorithms that can create a valid proof with lesser cost than honest training (aided by knowledge of the minimum from another training run). Do such algorithms exist? Under what conditions will such algorithms exist (or not)?

As a first step to better understanding this open problem, we will introduce and evaluate several examples of candidate (stochastic spoofing) algorithms in § VI. This will enable us to evaluate their computational costs empirically.

VI. Empirical Evaluation of Efficient Verification of Valid Proofs

In this section, our goal is to empirically explore the theoretical claims made on the robustness and efficiency of proof verification (see § IV), specifically, the reproducibility and representativeness assumptions. Arising from § IV-A, where we identify that imprecise tolerance to noise is a vulnerability in proof verification, we introduce a novel attack that outperforms prior attacks targeting this (see § II-B). After, we propose an attack against the top-\(Q\) selection mechanism (for more efficient verification), showing that it indeed does not identify representative subsets.

**Experimental Setup:** For the following experiments, we use the same setup from Jia et al. [7]: we evaluate all following experiments using CIFAR-10 and CIFAR-100 [23] datasets, which are two commonly studied image classification tasks. To ensure a fair comparison with prior work [7], we used ResNet-20 and ResNet-50 [24] as the model architecture for the two tasks respectively, and trained the models with a batch size of 128 for 200 epochs. Unless specified, all experiments are repeated 5 times and figures visualize the confidence interval.

A. On the Reproducibility in PoL

Recall that in § IV-A we analyzed one of the fundamental assumptions for the PoL protocol–reproducibility of gradient updates–and we proved the existence of optimal per-step verification thresholds. However, due to the noise encountered while performing the computations, it remains an open problem on how to construct the optimal verification strategy. Prior work by Zhang et al. [18] (discussed in § II-B) has exploited the vulnerability of static thresholds in verification to create spoofs. However, their attack is computationally costly, requiring at least \((43 \cdot n_{iter} + 1) \cdot k\) forward passes (FPs) for every update in the proof, where \( n_{iter} \) is the number of iterations for optimizing the adversarial example. More details are in Appendix B. Further, it is not guaranteed to converge. In this subsection, we will introduce a new attack that targets the same vulnerability: the infinitesimal update attack. Our new attack is more efficient and is guaranteed to succeed.

**Infinitesimal Update Attack.** First, let us re-establish our notation. Let \( g_t = W_{t+k} - W_t \) be the update obtained in the honest proof, and \( g_t = W_{t+k} - W_t \) be the update in the adversary’s spoof. Let \( g'_t = W_{t+k} - W_t \) and \( g''_t = W_{t+k} - W_t \) denote the corresponding reproduced updates by the verifier.

At a high level, the static threshold vulnerability can be exploited by utilizing updates of near-zero magnitude, as shown in Figure 3. To demonstrate this, we propose a strategy that (a) requires near-zero computational cost, (b) is guaranteed to yield updates of near-zero magnitude, and (c) is hard for the verifier to detect. The idea is simple: to obtain an update of near-zero magnitude, one can either generate a near-zero gradient, or one can use an infinitesimal learning rate. Formally, the strategy is as follows:

1) Generate model weights \( W_1, W_2, \cdots \) by linearly interpolating between \( W_0 \) and the victim model, \( W_T \) such that, \( d(W_{i+k}, W_{(i+1)k}) \ll \delta \).
2) The learning rate \( \eta \) is set to a small value (i.e., \( \eta \rightarrow 0 \)) such that the update is always smaller than \( \frac{\delta}{\epsilon} \) irrespective of the gradient value.
3) All other information logged, including the data, can be random values.

It is clear that the attack cost is low since no training is required. In fact, it is clear from the procedure above that the only cost comes from linear interpolation in step 1: for spoofing every model update, it requires 1 floating point operations for every model parameter–this is the same amount of computation needed for 1 FP, i.e., querying the model once and it is much less than the aforementioned cost of Attack 2 by Zhang et al. This attack is also hard for the verifier (with the static threshold) to detect/defend as it is always possible to create a much smaller update compared to \( \delta \); the verifier may decrease \( \delta \) to detect such updates, but valid updates will also be discarded resulting in a high false negative rate. Note that detecting linear interpolation is not sufficient either because it is not the only way to generate models: any strategy such that \( d(W_{i+k}, W_{(i+1)k}) \ll \delta \) will do.

**Evaluation & Results:** We evaluated our infinitesimal updates strategy against the original PoL framework, with the experimental setup described at the beginning of this section and the same parameters as Jia et al. [7]. We additionally assume the most powerful verifier capable of verifying all updates.\(^8\)

\(^8\)Attack 2 by Zhang et al. [18] is not guaranteed to work: we reproduced their results and found it does not work when the checkpointing interval \( k \) is small (e.g., \( k = 10 \) for CIFAR-100), since their attack relies on large \( \delta \) threshold, which is only the case when \( k \) is large. In fact, their attack is based on an incorrect assumption because \( k \) should be set by the verifier.
Fig. 3: Illustration of valid and spoofed updates in the Infinitesimal Update Attack: For a valid step, the update from the proof, \(g_t\), and the verifier-reproduced update, \(g'_t\), should both be much greater than \(\delta\). The discrepancy between \(g_t\) and \(g'_t\) should be solely caused by noise from the hardware/software. However, in the case of the infinitesimal update attack, the update from the spoof, \(\hat{g}_t\), and the verifier-reproduced update, \(\hat{g}'_t\), are both much smaller than \(\delta\), which causes their difference to also be smaller than \(\delta\). Hence, the spoof passes the verification protocol proposed by Jia et al. This illustrates how an adversary can exploit the fixed \(\delta\) to create an invalid PoL that passes verification.

![Illustration of valid and spoofed updates](image)

(a) Valid Update  
(b) Spoofed Update

As discussed earlier, an efficient verification mechanism that fails to select a representative subset of training updates may jeopardize the role of validity detection played by proof verification. This is especially true when the selection mechanism does not meet certain properties, i.e., failing to include at least one invalid update in the representative subset when there exist invalid update(s) in the proof, as discussed in § IV-B. Here we introduce an attack against the top-\(Q\) verification mechanism to illustrate how an adversary can exploit this. By doing so, we demonstrate the necessity of the aforementioned properties for correct efficient verification.

The selection mechanism for top-\(Q\) updates makes verification more efficient by decreasing the scope of verification to a subset of the updates contained in the proof. This opens an attack surface for adversaries who now only need to ensure that the updates being verified indeed pass verification. At a high level, one possible strategy is for the prover to manipulate the magnitude of updates (e.g., by using a large learning rate \(\eta\)) to control which updates are verified. More formally, the blindfold top-\(Q\) strategy we instantiate is as follows:

1. Generate the list of model checkpoints \(\hat{W}_{S}, \hat{W}_{2S}, \ldots, \hat{W}_{T-S}\) by interpolating linearly and evenly between \(W_0\) and \(W_T\), where \(S\) is the number of updates per epoch.
2. For each epoch \(i\), start with \(\hat{W}_{iS}\), make \(k \cdot Q\) valid updates by applying stochastic gradient descent (SGD) with a large learning rate \(\eta\). Store \(\hat{W}_{iS+k}, \hat{W}_{iS+2k}, \ldots, \hat{W}_{iS+Q+k}\). Record the magnitudes of these updates.
3. The rest of model states, i.e., \(\hat{W}_{(i+1)S-Q-k}, \ldots, \hat{W}_{(i+1)S-k}\) can be created by linearly interpolating between \(\hat{W}_{iS+Q-k}\) and \(\hat{W}_{(i+1)S}\), as long as these updates are smaller than any of the \(Q\) valid updates computed in the previous step. This is a similar process to the one described earlier for the infinitesimal update attack.

To analyze the per-step cost of this attack, note there are two cases: the update can be either a top-\(Q\) update or not. In the former case, besides linear interpolation, \(k\) valid gradient updates need to be computed, where each costs 1 FP and 1 backward pass (BP) or approximately 3 FPs for each update. For the latter, only linear interpolation is needed. Therefore, the expected step-wise cost is \((Q/s) \cdot (3 \cdot k + 1) + (s - Q)/s = (3 \cdot k \cdot Q)/s + 1\) FPs, where \(s\) is the number of steps per epoch.

Observe that the blindfold top-\(Q\) strategy exists only because the verifier deployed a heuristic to decrease the computational cost of verification; such heuristics provide weaker (probabilistic) guarantees in comparison to verifying the entire PoL when the updates selected for verification do not form a representative subset. The selection mechanism for top-\(Q\) updates was proposed to provide a better trade-off than a

It can be seen in Figure 4c and 4d that the proposed strategy is able to achieve a normalized reproduction error \(\|\varepsilon_{\text{repr}}\|\) significantly smaller than \(\delta\), and can thus pass verification. We also reproduce Attack 2 by Zhang et al. [18] (refer Figure 4b) and [4d] as a baseline to understand the effectiveness of our approach. We observed that our attack always outperforms that of Zhang et al. as our attack gives more consistent and significantly smaller \(\|\varepsilon_{\text{repr}}\|\).

B. On the Representativeness in PoL

As discussed earlier, an efficient verification mechanism that fails to select a representative subset of training updates may jeopardize the role of validity detection played by proof verification. This is especially true when the selection mechanism does not meet certain properties, i.e., failing to include at least one invalid update in the representative subset when there exist invalid update(s) in the proof, as discussed in § IV-B. Here we introduce an attack against the top-\(Q\) verification mechanism to illustrate how an adversary can exploit this. By doing so, we demonstrate the necessity of the aforementioned properties for correct efficient verification.

The selection mechanism for top-\(Q\) updates makes verification more efficient by decreasing the scope of verification to a subset of the updates contained in the proof. This opens an attack surface for adversaries who now only need to ensure that the updates being verified indeed pass verification. At a high level, one possible strategy is for the prover to manipulate the magnitude of updates (e.g., by using a large learning rate \(\eta\)) to control which updates are verified. More formally, the blindfold top-\(Q\) strategy we instantiate is as follows:

1. Generate the list of model checkpoints \(\hat{W}_{S}, \hat{W}_{2S}, \ldots, \hat{W}_{T-S}\) by interpolating linearly and evenly between \(W_0\) and \(W_T\), where \(S\) is the number of updates per epoch.
2. For each epoch \(i\), start with \(\hat{W}_{iS}\), make \(k \cdot Q\) valid updates by applying stochastic gradient descent (SGD) with a large learning rate \(\eta\). Store \(\hat{W}_{iS+k}, \hat{W}_{iS+2k}, \ldots, \hat{W}_{iS+Q+k}\). Record the magnitudes of these updates.
3. The rest of model states, i.e., \(\hat{W}_{(i+1)S-Q-k}, \ldots, \hat{W}_{(i+1)S-k}\) can be created by linearly interpolating between \(\hat{W}_{iS+Q-k}\) and \(\hat{W}_{(i+1)S}\), as long as these updates are smaller than any of the \(Q\) valid updates computed in the previous step. This is a similar process to the one described earlier for the infinitesimal update attack.

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Observe that the blindfold top-\(Q\) strategy exists only because the verifier deployed a heuristic to decrease the computational cost of verification; such heuristics provide weaker (probabilistic) guarantees in comparison to verifying the entire PoL when the updates selected for verification do not form a representative subset. The selection mechanism for top-\(Q\) updates was proposed to provide a better trade-off than a

It can be seen in Figure 4c and 4d that the proposed strategy is able to achieve a normalized reproduction error \(\|\varepsilon_{\text{repr}}\|\) significantly smaller than \(\delta\), and can thus pass verification. We also reproduce Attack 2 by Zhang et al. [18] (refer Figure 4b).
strategy that involved randomly selecting updates. Alternative approaches which do not understand the representativeness assumption can suffer similar pitfalls; this allows the adversary to hide invalid updates among the ones that will be verified.

**Evaluation & Results:** Following the same experimental setup as Jia et al. [7], and setting the value of $Q$ to be 5, we assumed the adversary obtained the chosen $Q$ and thus performed the blindfold top-$Q$ strategy by having 5 valid updates per epoch. As shown in Figure 5, the adversary is always able to mislead the verifier into selecting these updates.

### VII. Empirical Evaluation of the Cheapness Assumption

Recall that the second role of a proof, once verified, is to establish precedence: it should be computationally cheaper to obtain a proof from training than to create one post-hoc (even when given access to the trained model’s weights). Adversaries targeting this aspect of the PoL protocol are stochastic spoofing adversaries. We showed in §V that we cannot formally prove their non-existence due to open problems in learning theory. Therefore, in this section, we empirically analyze the difficulty in successfully constructing a stochastic spoof finding that we cannot instantiate stochastic spoofs. While it does not enable us to formally prove the non-existence of stochastic spoofing, it highlights that a better understanding of the convergence of optimizers on non-convex loss surfaces is needed to instantiate stochastic spoofing adversaries. Stochastic spoofing attacks, if ever successful, will likely follow from developments in learning theory that lead to new optimizers being invented.

Note that for our experiments we focus on DNNs for two reasons: (a) DNNs are more likely to be the target of adversaries because they are computationally expensive to train compared to convex models, and (b) it is unknown whether the same local minimum in a DNN’s non-convex loss surface can be recovered by an adversary given knowledge of that particular minimum. Instead, for convex optimization, there exists a unique global minimum and the adversary is guaranteed to achieve the same minimum as the honest trainer.

![Diagram](attachment:image.png)

**Fig. 5:** Evaluation of blindfold top-$Q$ attack (see §VI-B). We implement this attack for $Q = 5$ as shown by the dashed red line. It is assumed that the adversary knows $Q$ thus they submit only 5 large but valid updates that are constructed to pass verification. The rest of the proof is computationally cheap, and constructed to be invalid. Observe that the top-$Q$ method of Jia et al. [7] fails to detect the invalid updates (the blue curve is always 0 left of the red line) as long as the verifier does not use a larger $Q$ than what is previous claimed.

### Table I: Average distance of converged models for independent training runs on the same architecture.

| Setup            | $\ell_2$ distance | $p$-value     |
|------------------|-------------------|---------------|
| ResNet-20 CIFAR-10 | 71.472 ± 0.186    | $8.65 \times 10^{-77}$ |
| ResNet-50 CIFAR-100 | 58.056 ± 0.205    | $5.43 \times 10^{-73}$ |

**1) Why a Stochastic Spoofing Adversary Needs to Know the Final Model Weights:** To understand the difficulty of recovering the same local minimum for DNNs via honest training (i.e., without utilizing the knowledge of the final weights of another model), we conduct the following experiment: we check if two nearly identical training setups can lead to the same weights. We train models until convergence changing only the randomness in initialization and data sampling, keeping the architecture, training data, and optimization algorithm fixed. We trained multiple models independently, and compute the pairwise $\ell_2$ distance of their weights (we collected 30 data points), as shown in Table I. We find with high consistency (low standard deviation) that this distance is both large (i.e., on the order of the $\ell_2$ norm of the weights themselves (55.481 ± 0.110 for ResNet-20 and 44.624 ± 0.196 for ResNet-50) and not due to hardware noise as it is much larger than $\varepsilon_{repr}$. To determine if this is significant, we then apply a one-tailed $t$-test with null hypothesis: the distance between independent model parameters is zero. The $p$-values are summarized in Table I and we can reject the null hypothesis with significantly high confidence (low $p$-values). Since such a minimal change can consistently lead to significantly different weights, we argue that under realistic scenarios (with even larger setup differences), it is highly unlikely that one can recover the same final weights. Thus, knowledge of the final model weights is essential to a stochastic spoofing adversary.

### 2) Adversarial Reconstruction of a Proof for Known Model Weights: we now consider adversarial methods to construct a valid proof ending in $\hat{W}_T$ (s.t. $W_P = W_T$) given prior knowledge of $W_T$, the victim model weights obtained by a legitimate trainer under attack. We analyze three classes of spoofing strategies that attempt to direct the gradient updates toward the desired victim model: (a) reordering of the training data using data ordering attacks [25], (b) synthesizing data, and (c) designing adversarial update rules. In studying these strategies, we assume a strong adversarial model with access to the training dataset, as defined in §III. But we do emphasize that the protocol may be more robust by preventing the adversary from accessing the training distribution (e.g., keeping the dataset private) as mentioned in §III (which we will discuss in detail in §VII-A). Here, we assume access to the training set so as to obtain a stricter (and more realistic) assessment of security from the defender’s perspective.

### 1. Data Ordering Attacks: The adversary changes the order of the mini-batches during the training process to get a desired gradient update [25]. Through such attacks, we wish to understand how hard it is for an adversary to generate gradient updates using a new initialization ($W_0$), resulting in an unrelated training trajectory to the victim model $W_T$. In this scenario, the adversary’s goal is to minimize the distance...
an adversary is aiming to reconstruct the final weight from an unknown weight initialization.

2. Synthesizing Adversarial Data: An alternative to changing the mini-batch ordering is to change the data points themselves. To this end, the problem is to (a) find a dataset that, upon training, (b) results in final weights close to \( W_T \). One way this problem can be formulated is as below\[10\]

\[
\hat{D}^* = \arg\min_{\hat{D}} \| W_T - \arg\min_{\hat{W}} \mathcal{L}(\hat{W}, \hat{D}) \|,
\]

Note that \( \mathcal{L}(\hat{W}, \hat{D}) \) is the loss evaluated on the model weights, \( \hat{W} \) and the dataset, \( \hat{D} \). It is important to note that the adversary cannot directly optimize the distance between their weights and the stolen weights as it is not a loss for training\[7\]. We empirically evaluate this approach by synthesizing the adversarial dataset using gradient descent (i.e., \( \hat{D}^* = \arg\min_{\hat{D}} \| (\| W_T - W^* \|) \) ) and update the adversary’s model \( W^* = \arg\min_{\hat{W}} \mathcal{L}(\hat{W}, \hat{D}) \) in an alternating manner. As shown in Figure\[7\] the experiment results show that the two losses often oppose each other (i.e., one of the loss curves is increasing while the other is decreasing)\[26\]. Though this does not prove one could not further manipulate the losses to allow training, it is non-trivial and suggests possible incompatibility guarantees, as shown in Figure\[7\].

3. Designing Adversarial Update Rules: As a worst-case scenario, we can consider adversaries that use existing optimization methods that can be tuned to converge to a specific final weight \( W_T \). This attack is inspired by an existing optimizer that exploits information about the training paths for DNNs, Regularized Non-linear Acceleration (RNA)\[27\]. RNA is a convergence acceleration technique for generic optimization problems. It uses the extrapolation of trajectory path history in an iterative optimization problem to improve the convergence. Inspired by RNA, we designed an adversarial

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\[10\] This attack is different from Attack 1 of Zhang et al.\[18\] in the sense that we iteratively optimize each batch of data to decrease \( \| W_T - \hat{W} \| \) while they focus on one batch of data until \( \| W_T - \hat{W} \| \approx 0 \) for some \( \hat{W} \) created based on \( W_T \).
Effectiveness of RNA Attack

Target Model: $\mathcal{W}_T = \sum \hat{c}_i \hat{g}_i$

For each step of the attack, the RNA adversary solves for the optimal coefficients $\hat{c}_i$ that minimize the distance between the victim model and a linear combination of updates from honest training steps. The distance between the victim model and the adversary’s model generated using the RNA attack is plotted against the total number of epochs for honest training.

We observed five times to get confidence intervals (which is too small to see). The distance is decreasing but still significantly larger than 0 after 200 epochs (honest training cost). Thus, a spoof cannot be generated.

update rule that uses a linear combination of intermediate weights (or individual gradient directions) to reach the next weight such that its distance to the victim model’s weights is also minimized (as depicted in Figure 8a). For each RNA round, the adversary starts with regular training for a few training steps and records the intermediate weights $\hat{W}_i$’s (and their updates $\hat{g}_i = \hat{W}_{i+1} - \hat{W}_i$). The adversary then solves for coefficients such that $\hat{c}_i = \arg \min_{c_i} ||\hat{W}_T - \sum_i c_i \cdot \hat{g}_i||$ and obtains the weights for the next step as $\sum_i c_i \cdot \hat{g}_i$ to minimize its distance to the victim model, $\mathcal{W}_T$. In honest RNA training, the extrapolation coefficients are calculated based on the trajectory information; yet in the adversarial update rule, these coefficients are spoofed and the adversary can argue those are hyperparameters selected for this customized update rule. We call this adversarial update rule the RNA attack.

We empirically evaluate the effectiveness of this attack. We perform 10 RNA steps per epoch and the results are shown in Figure 8b. We measure the distance between the adversary’s weights generated by the RNA attack and $\mathcal{W}_T$. As we can see from the figure, as opposed to the previous two attacks we discussed, when using the RNA attack the distance between the adversary’s weights and $\mathcal{W}_T$ consistently decreases. However, after the same computational cost as honest training (200 epochs), the distance remains significantly larger than 0. Thus, the adversary cannot generate a valid spoof.

Summary: We demonstrate the importance of knowing the final model weights to be able to bootstrap a stochastic spoofing adversary. We evaluated 3 types of adversarial reconstructions of proofs based on stochastic spoofing. This allowed us to explore how hard it is to converge to $\mathcal{W}_T$ with valid gradient updates given knowledge of $\mathcal{W}_T$. We observed that it is difficult to take a model state closer to another model state (obtained from a different random initialization) by simply reordering the training data points. Hence we designed experiments to evaluate whether it is possible to make a valid training trajectory that ends at the final weights of the model from the honest prover via adversarial steps or update rules. We emphasize again that for all of the experiments above, we assumed the adversary has access to the training data and has the ability to control the internal computation of the optimizer (for the RNA attack). These capabilities are assumed to obtain an empirical assessment that is closer to the worst-case adversary. Despite these additional capabilities, it is still empirically hard for the adversary to generate a valid sequence that reaches the exact weight $\mathcal{W}_T$ while at the same time expending less computational power than a legitimate training run. This is because the adversary does not have the required knowledge from the honest prover’s training run (i.e., the initial model state, the order of training data, and all other metadata which is contained in the PoL).

Based on these empirical observations, we believe that such algorithms do not currently exist for DNN training. However, we note that any empirical study is non-comprehensive, and other attacks may exist. Their development is likely to come hand in hand with progresses in learning theory.

VIII. ENSURING SECURITY THROUGH PRECEDENCE

So far, we have formalized and empirically validated the necessary assumptions for proof verification to play its first role—efficient verification of valid proofs. We have also taken steps towards a formal understanding of the difficulties that surround ensuring cheap stochastic spoofs would not exist. As a statistical-learning answer to the latter remains elusive, in this section, we will instead rely on a security mitigation strategy to circumvent these issues, namely that of commitment, commonly used in computer security to establish precedence.

First, we revisit the two essential concepts originally defined by Jia et al. [7], namely data commitment and timestamping. We would like to emphasize the importance of these two mechanisms in terms of guaranteeing the robustness of the PoL protocol; we also want to make a note to the claim on public data release by Jia et al. [7], which we believe is unnecessary and would hurt the robustness of PoL.

A. Data Commitment

Jia et al. [7] require the prover to create signatures of data used at every step; they do so to prevent the prover from denying usage of specific data segments at a later time (i.e., serve as a cryptographic commitment). Such signatures also help circumvent spoofs that can arise from the non-uniqueness of a gradient-based update (i.e., multiple data segments can potentially result in the same gradient [17], [25]). While the original PoL paper advocates for public data release, we argue that this is unnecessary (and may result in privacy violations in scenarios where the data used to train the model is proprietary/sensitive) as the cryptographic commitment scheme binds the prover to the corresponding data segment used at a particular step; the prover can share the data at the time of verification through a secure channel. Refer to Figure 4 in Appendix D for the detailed flow of communication between the prover and the verifier.

From our experiments (which we do not present for brevity), we also observe that training data is essential for
Zhang et al.’s attacks. While it is conceivable that one can synthesize data with the aforementioned property (i.e., success in the context of the synthetic adversarial update strategy), more analysis is required to understand the cost associated with data synthesis. Data commitment schemes force adversaries to create all data before submitting the spoof, rather than only doing it at verification time and only for steps being verified.

B. Timestamping

We emphasize the need of timestamping the proof or its signature upon submission (or publishing it in a public ledger) as introduced in the original PoL protocol [7]. This will prevent (a) replay attacks where the adversary submits the exact same PoL as the victim trainer, and (b) any attacks that involve using the exact model parameters of the victim model as the final state of its spoof.

IX. The Challenge of Functionally Similar Models for PoL

One aspect of PoL robustness we have not covered yet in this paper is how to handle spoofs that return a proof for a model that is close but not exactly equal to the victim’s model. Recall (from § II-A) that our work follows Jia et al. [7] in defining the proof in the parameter/weight space, i.e., a DNN is defined by its parameters. However, DNNs can alternatively be defined as functions from a function space, i.e., all (continuous, smooth, etc.) functions from \( \mathbb{R}^n \to \mathbb{R}^m \) where \( n \) is the input dimension, and \( m \) is the output dimension.

Before we discuss concrete function spoofing adversaries, i.e., adversaries that provide valid proofs ending only in a functionally equivalent model (to the victim’s), we briefly remark on why this is more viable than spoofing exact weights. To be specific, function spoofing gives a larger threat surface; e.g., if one only considers models to be equal if they have the same training loss, this allows an adversary to reconstruct a cheaper proof (than the honest prover’s) for one of several models they can choose (that obtain the same loss). The adversary can also, depending on their goals, define functionally similar in more complex ways than simply relying on the loss of the model [7].

Proving robustness against such function spoofing adversaries reduces to an open problem in the ML community: what is the most sample efficient way to learn from an oracle? The answer to this question appears to be connected to the sample compression conjecture which states that there exists a compression scheme for every concept class that would reduce the size of the training set that is necessary to learn the concept down to its VC-dimension. This has been a major open problem for 30 years in computational learning theory [28].

In the security community, this problem is most notably encountered in model extraction [13]. Here, having only query access to the victim (oracle), an adversary needs to label a substitute dataset collected from publicly available sources and/or via synthetic generation using the oracle. For DNNs, the cost of extraction is often still several orders of magnitude more than training [29]. An efficient adversary can then either (a) minimize the number of queries, or (b) curate a more efficient (i.e., small) initial dataset which, upon annotation, can be used to train a surrogate model (detailed methods for achieving these goals can be found in Appendix C).

While the existing methods have been shown to be capable of learning functionally similar models, they are not effective attacks if their “costs” are greater than benign training. Note, however, that defining and measuring cost is ambiguous. Therefore, we have the following open problem:

Open Problem: What is the proper measure of cost for function-space attacks?

For the aforementioned strategies, assuming an agreed measure for cost, one has two sub-routines contributing to it: (a) algorithm cost: the cost of the strategy for data selection and/or generation, the (b) learning cost: the cost of training until convergence (i.e., obtaining a model which is functionally equivalent to the victim). Regardless of if the total cost associated with these function stealing strategies is less than the expected cost of training, they are still required to create a valid spoof of this stolen model. Thus, the cost of model stealing (even with insider access) is increased by having to generate a spoof. That is, the PoL protocol still provides some (albeit weaker) guarantees against function-space adversaries.

X. Conclusions

Given the generally open questions of § IV and § V, we now revisit the question we opened the paper with: can PoL be robust? While it is empirically difficult to find computationally-effective attacks against PoL, proving (or disproving) the nonexistence of spoofing adversaries remains an open problem.

One avenue to circumvent these fundamental limits in ML theory, and our understanding of optimization, is to rely more on cryptography. We noted earlier that one of the motivations for approaches like PoL is precisely to avoid using cryptography given its limited scalability when it comes to training deep neural networks; however this does not preclude us from envisioning that cryptographic primitives may be combined with ideas from PoL to provide analytical security guarantees. In particular, we have already seen how relying on data commitment mechanisms help reduce the attack surface of PoL. One could imagine extending the use of commitment mechanisms to other parts of the PoL protocol.

Moving forward, we laid down generic properties spoofing adversaries must satisfy. We believe future work can expand on these results to prove if such adversaries can or cannot exist, answering one of the open problems towards formally guaranteeing the robustness of PoL. Similarly, future work can investigate how to instantiate the optimal verification strategy, and whether adversaries can bypass this strategy.
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APPENDIX A

TABLE OF NOTATIONS AND TERMINOLOGY

We defined the notations in Table I and the categorization of spoofing in Table III.

| Variable | Purpose |
|----------|---------|
| $T$      | prover  |
| $A$      | adversary |
| $W$      | weight space of models |
| $k$      | the checkpointing interval |
| $s$      | number of steps/batches per epoch |
| $W_t$    | the $t$-th checkpoint in the honest prover’s training process |
| $W_t'$   | the reproduced model weights by the verifier for model weights $W_t$ in the PoL |
| $W_t''$  | $t$-th checkpoint generated by adversary’s spoofing process |
| $W_t'''$ | the reproduced model weights by the verifier for model weights $W_t$ in the PoL |
| $W_T$    | the final weights of the victim model |
| $W_T'$   | the final weights of the adversary’s model |
| $g_t$    | the update from step $t$ to $t+k$ in the prover’s PoL ($\hat{\delta}$) |
| $g_t'$   | the update from step $t$ to $t+k$ reproduced by the verifier ($\hat{\delta}$) |
| $g_t''$  | the update from step $t$ to $t+k$ in the adversary’s PoL ($\hat{\delta}$) |
| $g_t'''$ | the update from step $t$ to $t+k$ reproduced by the verifier ($\hat{\delta}$) |
| $D, D_t$ | the honest prover’s dataset; if with subscript $t$, then it represents the batch of data used in the $t$th training step |
| $\tilde{D}, \tilde{D}_t$ | the adversary’s dataset; if with subscript $t$, then it represents the batch of data used in the $t$th training step |
| $M_t$    | the honest prover’s training metadata (e.g., hyperparameters) at the $t$th training step |
| $\tilde{M}_t$ | the adversary’s training metadata (e.g., hyperparameters) at the $t$th training step |
| $\delta, \delta_t$ | the threshold to bound the step-wise noise for verification as described in the original Proof-of-Learning (PoL) algorithm; if with subscript $t$, then it is specific to the $t$th training step |
| $Q$      | number of updates the verifier will verify per epoch ($Q$ in the Top-$Q$ mechanism) |
| $\eta$   | the learning rate |
| $d(\cdot)$ | some distance metric |
| $\alpha$ | scaling factor for Lemma 2 |
| $\|\cdot\|$ | norm of a given vector, if not otherwise specified, then it is $\ell_2$ norm |
| $\tau$   | a specific value of true positive rate (TPR) |
| $\mathcal{U}(\cdot)$ | function of model weights, training data, and training metadata that updates the model weights |
| $C$      | cost function |
| $(g, d, \delta)$-proofs | proofs using update rules $g_i \in g$ passing thresholds $\delta$ in metric $d$ |
| $A_{D,W_T}$ | set of $(g, d, \delta)$-proofs ending in $W_T$ generated by a specific dataset $D$ |
| $\mathcal{P}$ | a proof generated by PoL |
| $F: \cdots \rightarrow \cdots$ | algorithm for creating proof e.g., Algorithm 1 of Jia et al. [7] |
| $\mathbb{S}$ | subset of model updates |
| $B_r(v)$ | balls centered at vector $v$ of radius $r$ |

TABLE II: Notations

APPENDIX B

Computational Cost Analysis for the Attacks

Here we provide a detailed analysis of the computational cost of the attacks mentioned in the paper. We use the cost of 1 forward propagation (FP) as the basic unit, which is approximately equal to $N$ floating point operations (where $N$ is the number of model parameters). Note that 1 back propagation costs approximately 2 FPs, and adding parameters of two model states together costs approximately 1 FP. Other notation used in this appendix includes: number of iterations of updating the adversarial example ($n$), and number of batches per epoch ($s$). All attacks first linearly interpolate between a random initialized state and the final stolen model state so they can spoof with the same length, so WLOG we may compare their cost step-wise (i.e., from state to state to state $t+k$).

Infinitesimal Update Attack: Apart from linear interpolation, the Infinitesimal Update attack does not require any other computation, and the linear interpolation is done once for every model parameter so the step-wise cost is 1 FP.

Attack by Zhang et al. [18]: Attack 2 by Zhang et al. needs to interpolate for every single update between state $t$ and state $t+k$ (e.g. $t$ to $t+1$, $t+1$ to $t+2$, ...), so 1 FP is required for every update (i.e., $k$ FPs in total for linear interpolation).
Besides, for each of these updates, 1 FP and 1 backward propagation (=3 FPs) are needed to compute the gradient of the model, then another backward propagation is required to differentiate the norm of the gradient with respect to the inputs to the model (this is essentially a second order gradient and the cost depends on the algorithm used to compute it; by measuring time of the code released by Zhang et al., we found empirically it takes more than 20 times than the gradient computation, so 40 FPs). Adding all these together, 43 FPs is needed for a single iteration of creating the adversarial examples, so Attack 2 costs at least \( 43 \cdot n + 1 \) · k FPs per step.

Zhang et al. tried to parallelize their Attack 2, which resulted in a different attack (their Attack 3), but it would still cost \( 43 \cdot n + 1 \) FPs per step.

**Blindfold top-\( Q \) Attack:** There are two cases here: (a) if it is one of the top-\( Q \) updates, then \( k \) valid gradient updates (3 FPs) need to be computed, so the cost is \( 3k + 1 \) FPs (1 comes from linear interpolation); (b) if it is not a top-\( Q \) update, then nothing besides linear interpolation needs to be done, so the cost is 1 FP. In every epoch, there is \( Q \) top-\( Q \) updates and \( s - Q \) non-top-\( Q \) updates, so the expected step-wise cost is \( (Q/s) \cdot (3 \cdot k + 1) + (s - Q)/s = (3 \cdot k \cdot Q)/s + 1 \) FP.

### APPENDIX C

**FUNCTION SPACE ADVERSARY**

Goal (1) can be achieved through active learning [30] where the oracle access to the victim’s model is assumed limited to model uncertainty. Given wider access to the oracle (i.e., white-box), model/knowledge distillation becomes possible [31]. Goal (2) is closely aligned with machine teaching where the objective of the “teacher” is to find the optimal (i.e., smallest) set of samples using which an algorithm (“student”) can learn a target concept thus creating a functionally-equivalent spoof. We note that using machine teaching, the cost of spoofing is now two-fold: finding the optimal teaching set, and passively learning using it. The complexity of the former is usually measured by the teaching dimension (TD)—a counter-part to the VC dimension (VCD) for passive learning [28]. While there is no general relationship between TD and VCD, it is known that for two well-defined notions, recursive and preference-based teaching, \( TD = O(VCD^2) \). Another approach to goal (2) is to generate synthetic data that captures most if not all of the information contained within an entire dataset. Known as data distillation [32] (or data condensation [33]), these methods are empirical data compression techniques that produce efficient datasets for learning with cardinality in the order of intrinsic dimensionality of a dataset—typically many orders of magnitude smaller than the original dataset. These compressed datasets could then be used to efficiently train functionally-equivalent spoofs. We again point out that although training spoofs with such optimized dataset is much more efficient compared to passive (and even active) learning, the cost of training has been shifted to compression; and whether or not the overall cost is lower is an open question [34].

### APPENDIX D

**ADDITIONAL FIGURES**

Figure 9 illustrates the communication between the verifier and the prover described in § VIII-A. Figure 10 and 11 are additional evaluation results for experiments in VII-2.

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**TABLE III: Types of spoofing:** Listed are the types of spoofing against PoL categorized by Jia et al. [7] along with pointers to where we discuss them or why they are not discussed.
Fig. 9: Overview of the communication/data exchange between the verifier and the prover: during proof creation, model checkpoints, training metadata, indices and hashes of training data are submitted to the verifier. During proof verification, the prover sends training data at step $j$ to the verifier, who then checks their hashes are consistent with the PoL. The verification is done by reproducing the model at step $j+1$ and verifying it is similar to the checkpoint at $j+1$ in the PoL.
Fig. 10: No individual update pushes the adversary’s model $\hat{W}_t$ closer to the prover’s final model $W_T$. These are plotted in the same setting as Figure 6 to illustrate finer grained information on the evolution of the updates throughout training. Note that all gradients at all intermediate steps in training push the adversary’s model away from $W_T$, as indicated by the negative values on the x-axis. We use a ResNet-20 on the CIFAR-10 dataset.
Fig. 11: No individual update pushes the adversary’s model $\hat{W}_t$ closer to the prover’s final model $W_T$. These are plotted in the same setting as Figure 10 to illustrate finer grained information on the evolution of the updates throughout training. Note that all gradients at all intermediate steps in training push the adversary’s model away from $W_T$, as indicated by the negative values on the $x$-axis. We use a ResNet-50 on the CIFAR-100 dataset.