DOES QUANTUM COSMOLOGY
PREDICT A CONSTANT DILATONIC FIELD?

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Abstract

Quantum cosmology may permit to determine the initial conditions of the Universe. In particular, it may select a specific model between many possible classical models. In this work, we study a quantum cosmological model based on the string effective action coupled to matter. The Schutz's formalism is employed in the description of the fluid. A radiation fluid is considered. In this way, a time coordinate may be identified and the Wheeler-DeWitt equation reduces in the minisuperspace to a Schrödinger-like equation. It is shown that, under some quite natural assumptions, the expectation values indicate a null axionic field and a constant dilatonic field. At the same time the scale factor exhibits a bounce revealing a singularity-free cosmological model. In some cases, the minimum value of the scale factor can be related to the value of gravitational coupling.

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1 Introduction

Quantum cosmology is, in some sense, a theory of initial conditions \([1, 2]\). A classical model depends on many initial data that must be fixed in order to have an agreement with observations. There is a hope that the quantum cosmology may lead to specific predictions concerning the values of at least some input parameters. One of the most promising candidates for a unified theory of all interactions is the string theory \([3, 4]\). In string theory, the gravitational coupling is in principle a function of the space-time coordinates, being connected with the expectation value of the dilatonic field. Moreover, the string action contains other fields like the axion. The problem we treat here is the quantum scenario coming from the string effective action in four dimensions. In particular, we are interested in which are the predictions of quantum cosmology for the evolution of the dilatonic and axionic fields, besides the scale factor.

One of the most important problems in studying quantum cosmological models is the absence of an explicit time coordinate \([5, 6]\). There are many attempts to solve this special feature of the quantum cosmological program. Here we adopt an interesting proposal: if ordinary matter is introduced in the model through the Schutz's formalism

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Quantum cosmology [7, 8], dynamical degrees of freedom are attributed to matter. Quantizing the gravity system in presence of matter described by the Schutz’s formalism, we can obtain in the minisuperspace approach a Schrödinger-like equation, where the conjugate momentum associated with matter field appears linearly. Hence, this matter field may play the role of time. This approach has been extensively employed in the literature, revealing in general the suppression of the initial singularity [9, 10, 11, 12, 13, 14]. Effectively, through this procedure, we end up with a genuine Schrödinger equation, and all the machinery of ordinary quantum mechanics can be employed in order to obtain specific predictions for the evolution of the Universe.

Hence, we consider the string effective action in four dimensions in presence of matter. More specifically, a radiative fluid is coupled to the string effective action. One of the aims in considering a radiative fluid is to keep as far as possible in the string context. There exists electromagnetic field terms in the Ramond-Ramond sector of the string action. In this way, the inclusion of the radiative fluid may not imply to get out of the pure string case. This radiative fluid is quantized through the aid of the Schutz’s variables, leading to a "time" variable. We parametrize the string action by two parameters, \( \omega \) and \( n \). The parameter \( \omega \) reflects the dilatonic coupling. In the strict string case \( \omega = -1 \). But we keep it as a free parameter in order to consider more general frameworks, like brane models. The parameter \( n \) is linked with the coupling between the dilatonic and axionic field. Again in the strict string frame, \( n = -1 \). But, we also keep it arbitrary in order to include other configurations.

Quantizing the string effective action in presence of a radiative fluid in the minisuperspace we obtain a Schrödinger-like equation. We solve it for two distinct cases: \( \omega > -3/2 \), the "normal" case, since it leads to a positive energy for the scalar field in the Einstein’s frame; \( \omega < -3/2 \), the "anomalous" case, since the energy of the scalar field becomes negative in the Einstein’s frame. In general, the eigenfunctions are not square integrable. For the normal case a superposition of them forming a wave packet can not cure this pathology, which is reflected in the infinite value of the norm of the wave function. Hence, no quantum cosmological model can be constructed for this case unless the dilatonic field is fixed as a constant from the beginning. This results in the traditional general relativity model [12].

However, in the anomalous case it is possible to have square integrable wave functions. Wave packets may be constructed. The expectation values for the scale factor, dilatonic field and the axionic field may be evaluated. Surprisingly again, the expectation value for the dilatonic field does not depend on time. At the same time, the expectation value for the axionic field is zero. Hence, the anomalous quantum cosmological model coming from string theory coupled to a radiative fluid predicts a constant gravitational coupling and no axionic field. Moreover, the scale factor exhibits a bounce. The minimum value for the scale factor can be related to the value of the gravitational coupling. Inserting the known value of \( G \), we find that the minimum of the scale factor remains well above the Planck’s length when \( \omega \to -3/2 \); however, it becomes highly transplanckian when \( \omega \to -\infty \).

In next section, we give details on the construction of the quantum system for the string effective action with a radiative fluid described by the Schutz’s formalism. In section 3, we solve the resulting Schrödinger-like equation and construct the corresponding wave
packets. In section 4, the predictions for the evolution of the dynamical variables are presented. In section 5 we analyze the generality of the results taking into account the assumptions made and the formalism employed, presenting our conclusions.

2 The quantum model

The Neveu-Schwartz sector of the string effective action is given by the Lagrangian density [15]

\[ L = \sqrt{-g} e^{-\sigma} \left\{ R + \sigma \rho \sigma - e^{2\phi} \chi \rho \right\} + L_m , \quad (1) \]

where \( \sigma \) is the dilatonic field, \( \chi \) is the axionic field and \( L_m \) is a term taking into account the presence of ordinary matter. Through the redefinition \( \phi = e^{-\sigma} \), the Lagrangian takes the form

\[ L = \sqrt{-g} \phi \left\{ R - \omega \frac{\phi^2}{\phi^2} - \frac{\phi^{n-1}}{N^2} \chi \right\} + L_m . \quad (2) \]

Two free parameters, \( \omega \) and \( n \), have been introduced in order to take into account more general frameworks besides the strict string effective action. The strict string case is characterized by \( \omega = -1 \) and \( n = -1 \). In this way, effective actions coming from \( F \) theory, supergravity theories or pure multidimensional models are included in the Lagrangian (2), as well as brane configurations [16]. In particular, some situations where \( \omega \) can be largely negative are taken into account.

The Friedmann-Robertson-Walker metric is written as

\[ ds^2 = N^2(t) dt^2 - a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] , \quad (3) \]

\( N \) being the lapse function and \( k = 1, 0, -1 \) correspond to a closed, flat and open universe, respectively. Inserting this metric in the gravitational part of the Lagrangian (2), we obtain

\[ L_G = Na^3 \phi \left\{ - \frac{6}{N^2} \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{\chi}}{\chi} \right) \right]^2 - \frac{\dot{N}}{N} \frac{\dot{a}}{a} + \frac{N^2 k}{a^2} - \frac{\omega}{N^2} \left( \frac{\dot{a}}{a} \right)^2 - \frac{\phi^{n-1}}{N^2} \dot{\chi} \right\} . \quad (4) \]

Integrating by part the second derivative of the scale factor and discarding surface terms, we end up with the following expression for the gravitational Lagrangian:

\[ L_G = 6\phi a \frac{\dddot{a}}{N} + 6a^2 \frac{\dot{\phi}}{N} - 6kNa\phi - \frac{\omega}{N} a^3 \frac{\ddot{\phi}}{\phi} - \frac{\phi^n a^3}{N} \dot{\chi}^2 . \quad (5) \]

The presence of crossing terms between the scale factor and the dilatonic field is a consequence of the non-minimal coupling in the original Lagrangian.

Following the canonical procedure, the conjugate momenta are:

\[ \Pi_a = \frac{\partial L}{\partial \dot{a}} = 12a \frac{\ddot{a}}{N} + 6a^2 \frac{\dot{\phi}}{N} ; \quad (6) \]

\[ \Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = 6a^2 \frac{\dot{\phi}}{N} - 2\omega a^3 \frac{\ddot{\phi}}{\phi} ; \quad (7) \]

\[ \Pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = -2\phi^n a^3 \frac{N \dot{\chi}}{N} . \quad (8) \]
The final expression for the gravitational Hamiltonian is:

\[ H_G = \frac{N}{3 + 2\omega} \left[ \frac{\omega}{12 a^2} \Pi_a^2 - \frac{1}{2 a^3} \Pi^2 \phi + \frac{1}{2 a^2} \Pi_a \Pi_\phi \right] - \frac{1}{4} \phi^{-n} \frac{N}{a^3} \chi^2 + 6kNa\phi \ . \] (9)

The material part of the Lagrangian can be treated using the Schutz’s formalism, which has the advantage of attributing degrees of freedom to the matter field. In this formalism this matter Lagrangian is written as

\[ L_m = \sqrt{-g} p \] (10)

where \( p \) is the pressure. We will consider a barotropic equation of state: \( p = \alpha \rho \), \( \alpha \leq 1 \). The fluid’s four-velocity is expressed in terms of five potentials \( \epsilon, \zeta, \beta, \theta \) and \( S \):

\[ U_\nu = \frac{1}{\mu} (\epsilon_\nu + \zeta \beta_\nu + \theta S_\nu) \] (11)

where \( \mu \) is the specific enthalpy. The variable \( S \) is the specific entropy, while the potentials \( \zeta \) and \( \beta \) are connected with rotation and are absent for models of the Friedmann-Robertson-Walker (FRW) type. The variables \( \epsilon \) and \( \theta \) have no clear physical meaning. The four-velocity is subject to the normalization condition

\[ U_\nu U^\nu = -1 \ . \] (12)

Using the constraint for the fluids and after some thermodynamical considerations [7, 8], the matter Lagrangian takes the form:

\[ L_m = N^{-1/\alpha} a^{3\alpha} (\alpha + 1)^{1/\alpha+1} \left( \dot{\epsilon} + \theta \dot{S} \right)^{1/\alpha+1} \exp \left( -\frac{S}{\alpha} \right) \ . \] (13)

This matter Lagrangian may be further simplified leading, by canonical methods [9], to the matter Hamiltonian

\[ H_m = p_\epsilon^{\alpha+1} a^{-3\alpha} e^S \] (14)

where \( p_\epsilon = -\rho_0 U^0 N a^3 \), \( \rho_0 \) being the rest mass density of the fluid. The canonical transformation

\[ T = -p_S e^{-S} p_\epsilon^{-(\alpha+1)} \ , \quad p_T = p_\epsilon^{\alpha+1} e^S \ , \quad \bar{\epsilon} = \epsilon - (\alpha + 1) \frac{p_S}{p_\epsilon} \ , \quad \bar{p}_\epsilon = p_\epsilon \ , \] (15)

which generalizes the one used in [9], takes the total Hamiltonian (gravitational plus matter) to the final form

\[ H = \frac{N}{3 + 2\omega} \left[ \frac{\omega}{12 a^2} \Pi_a^2 - \frac{1}{2 a^3} \Pi^2 \phi + \frac{1}{2 a^2} \Pi_a \Pi_\phi \right] - \frac{1}{4} \phi^{-n} \frac{N}{a^3} \chi^2 + 6kNa\phi + \frac{p_T}{a^{3\alpha}} \ , \] (16)

where the momentum \( p_T \) is the only remaining canonical variable associated with matter.

Quantizing this system by canonical methods, which mounts up to replace the momenta by operators and considering the constraint

\[ \hat{H} \Psi = 0 \ , \] (17)
where $\Psi$ is the wave function of the Universe, we obtain the Wheeler-DeWitt equation in the minisuperspace:

$$\left\{ -\frac{N}{3+2\omega}\left[ -\frac{\omega}{12}\frac{1}{a^3}\partial_a^2 + \frac{1}{2}\frac{\phi}{a^2}\partial_a^2 - \frac{1}{2a^2}\partial_a\partial_{\phi}\right] + \frac{1}{4}\phi^{-n}\frac{N}{a^3}\partial_{\chi}^2 + 6kN\phi - i\frac{1}{a^{3\alpha}}\partial_T \right\}\Psi = 0 \ . \quad (18)$$

The canonical transformation employed for the matter fields fixes the parametrization of the time $T$ [12]. For pressureless matter this parametrization corresponds to the cosmic time and for the radiative fluid to the conformal time.

The presence of cross derivatives makes the analysis of the Wheeler-DeWitt equation (18) quite delicate. But, we can diagonalize the Wheeler-DeWitt equation through the change of variables

$$a \rightarrow \phi^{-1/2}b \ , \quad \phi \rightarrow \phi \ . \quad (19)$$

In terms of these new variables, the Wheeler-DeWitt equation takes the form,

$$\left\{ -\frac{1}{24b}\partial_b^2 + \frac{1}{4\omega b^3}\partial_{\phi}^2 + \frac{1}{4}\phi^{1-n}b^3 b^{3\alpha} + 6k - i\phi^{(3\alpha-1)/2} \right\}\Psi = 0 \ , \quad (20)$$

where $\tilde{\omega} = \omega + 3/2$. The first derivative in the field $\phi$ is an ordering term introduced to assure the hermiticity of the effective Hamiltonian. The final results depend very weakly on this term. Remark that the change of variable (19) corresponds to perform a conformal transformation in the original Lagrangian (2), passing from the so-called Jordan’s frame to the Einstein’s frame [17]. In this sense, $\omega < -3/2$ corresponds to a scalar field with negative energy in the Einstein’s frame. When $\omega = -3/2$, the dilatonic field is null in the Einstein’s frame.

### 3 Wave functions and wave packets

The task to be addressed now is to solve the equation (20). To do so, we consider first stationary states given by

$$\Psi = \Phi e^{-iET} \ . \quad (21)$$

In order to determine the solutions of (20), the method of separation of variables is employed. This implies to write

$$\Phi(b, \phi, \chi) = X(b)Y(\phi)Z(\chi) \ . \quad (22)$$

The Wheeler-DeWitt equation reduces to

$$-\frac{1}{24b}\frac{X_{bb}}{X} + \frac{1}{4\omega b^3}\frac{Y_{\phi\phi}}{Y} + \frac{2}{4}\frac{Y_{\phi}}{Y} + \frac{1}{4}\phi^{1-n}\frac{Z_{\chi\chi}}{Z} + 6k - i\phi^{(3\alpha-1)/2} = E, \quad (23)$$

where the subscripts indicate derivative with respect to the variables $b$, $\phi$ and $\chi$. The solution for the function $Z$ is

$$\frac{Z_{\chi\chi}}{Z} = -r^2 \Rightarrow Z = Z_0 e^{\pm ir\chi} \ , \quad r \in \mathbb{R} \ . \quad (24)$$
In order to proceed, we particularize to the case \( \alpha = 1/3 \), that is, matter is a radiative fluid. This case is of special importance because an electromagnetic term is present in the effective action from string theories in the Ramond-Ramond sector. We end with two equations:

\[
X'' + \left\{ -\frac{24}{b^2} s - 144k b^2 + E \right\} X = 0 ,
\]

(25)

\[
\dot{Y} + 2\frac{\dot{Y}}{\phi} - \left\{ r^2 \bar{\omega} \phi^{(1+n)} + 4s \frac{\bar{\omega}}{\phi^2} \right\} Y = 0 ,
\]

(26)

where \( s \) is a separation constant which will play a crucial role in the analysis of the quantum model. We have also made the redefinition \( 24E \rightarrow E \).

Equations (25,26) can be solved in terms of Bessel functions when \( k = 0 \). The solution depends on the sign of \( \bar{\omega} \). The final solutions for the wave function in the spatial flat case are:

- \( \bar{\omega} > 0 \ (\omega > -3/2) \):

\[
\Psi(a, \phi, \chi, t) = A \sqrt{\frac{a}{\sqrt{\phi}}} J_\mu \left( \sqrt{E_a \phi^{1/2}} \right) K_\nu \left( r \frac{\sqrt{\bar{\omega}}}{p} \phi^p \right) e^{-iEt \pm i \chi} ;
\]

(27)

- \( \bar{\omega} < 0 \ (\omega < -3/2) \):

\[
\Psi(a, \phi, \chi, t) = A \sqrt{\frac{a}{\sqrt{\phi}}} J_\mu \left( \sqrt{E_a \phi^{1/2}} \right) J_\nu \left( r \frac{|\sqrt{\bar{\omega}}|}{p} \phi^p \right) e^{-iEt \pm i \chi} .
\]

(28)

In these expressions, \( \mu = \sqrt{24s + 1/4}, \nu = (1/2p)\sqrt{16\bar{\omega}s + 1}, p = (1 - n)/2, \) and \( J_\nu \) and \( K_\nu \) are the ordinary and modified Bessel functions. By restricting the parameter \( s \) to a convenient range of values, the wave functions (27,28) satisfies the required boundary conditions in order to guarantee the hermiticity of the Hamiltonian, but, they are not in general square integrable. So, in order to assure the consistency of the quantum model, a wave packet must be constructed. This can be achieved by performing a gaussian integration on the separation parameters \( E, r \) and \( s \). This must be done separately for each sign of \( \bar{\omega} \).

For \( \bar{\omega} > 0 \), let us consider the wave packet given by

\[
\Psi(a, \phi, \chi, T) = \int_0^\infty \int_0^\infty ds \, dx \, dy A(s) x^{\mu+1} e^{-\rho(T)x^2} y^{\mu-1} e^{-\sigma(\chi)y} \sqrt{\frac{a}{\sqrt{\phi}}} \times
\]

\[
\times J_\mu \left( x \alpha \phi^{1/2} \right) K_\nu \left( y \frac{\sqrt{\bar{\omega}}}{p} \phi^p \right) ,
\]

(29)

where we have defined \( x \equiv \sqrt{E}, y \equiv r, \rho(T) = a_1 + iT, \sigma(\chi) = a_2 + i\chi \). Moreover, \( a_{1,2} \) are positive constants and \( A(s) \) is a function to be analyzed later. Performing the
integrations on $x$ and $y$ [18], we find

$$
\Psi(a, \phi, \chi, T) = \int_{s_1}^{s_2} ds A(s) \sqrt{a} \sqrt{\phi (2 \rho(T))^{\mu+1}} e^{-\frac{a^2 \phi}{4 \rho(T)}} \left(2 \frac{\sqrt{\tilde{\omega}}}{\rho} \phi^p\right)^\nu \times \frac{1}{\left[\sigma(\chi) + \frac{\sqrt{\tilde{\omega}}}{p} \phi^p\right]^\mu+\nu} \times \frac{\Gamma(\mu + \nu) \Gamma(\mu - \nu)}{\Gamma(\mu + 1/2)} 2F_1 \left(\mu + \nu, \nu + 1/2, \mu + 1/2, \frac{\sigma(\chi) - \frac{\sqrt{\tilde{\omega}}}{p} \phi^p}{\sigma(\chi) + \frac{\sqrt{\tilde{\omega}}}{p} \phi^p}\right),
$$

(30)

where $2F_1(\alpha, \beta, \gamma, z)$ is the hypergeometric function and some unimportant constants were absorbed in the factor $A$.

For $\tilde{\omega} < 0$, we choose, with the same definitions as in the preceding case, the superposition given by

$$
\Psi(a, \phi, \chi, T) = \int_0^\infty \int_0^\infty \int_{s_1}^{s_2} ds dx dy A(s) x^{\mu+1} e^{-\rho(T)x^y} y^{\nu+1} e^{-\sigma(\chi)y} \sqrt{a} \sqrt{\phi} \times
$$

$$
\times J_\mu \left(x a \phi^{1/2}\right) J_\nu \left(y \sqrt{\frac{|\tilde{\omega}|}{p} \phi^p}\right),
$$

(31)

leading, after integration [19], to the wave function

$$
\Psi(a, \phi, \chi, T) = \int_{s_1}^{s_2} ds A(s) \sqrt{a} \sqrt{\phi (2 \rho(T))^{\mu+1}} e^{-\frac{a^2 \phi}{4 \rho(T)}} \sigma(\chi) \left(2 \frac{\sqrt{\tilde{\omega}}}{\rho} \phi^p\right)^\nu \times
$$

$$
\times \frac{\Gamma(\nu + 3/2)}{\left[\sigma^2(\chi) + \frac{|\tilde{\omega}|}{p} \phi^p\right]^\nu+3/2}
$$

(32)

4 String quantum cosmological models

In the perfect fluid employed above, a time coordinate may be identified with the matter variables. Hence, a Schrödinger-like equation has been obtained. From the solutions found, it is possible to determine a cosmological model through the computation of the expectation values for the dynamical variables $a$, $\phi$ and $\chi$. This computation follows the ordinary procedure of quantum mechanics in the Schrödinger picture:

$$
<x(T)> = \frac{\int \Psi^* x \Psi d\Omega}{\int \Psi^* \Psi d\Omega},
$$

(33)

where $d\Omega$ stands for the integration on the ensemble of variables. But, in order this computation can make sense in the spirit of usual quantum mechanics, the norm of the wave function must be time independent, otherwise there is no conservation of probability, and the unitarity is lost. Moreover, this norm must be finite. The norm of the wave function is given, as usual, by

$$
N = \int_0^\infty \int_0^\infty \int_{-\infty}^{\infty} \Psi^*(a, \phi, \chi) \Psi(a, \phi, \chi) da d\phi d\chi.
$$

(34)
We have restricted the integration on \( a \) and \( \phi \) to the positive semi-real axis due to physical grounds (anyway, negative values would give non-convergent results), while the axionic field is allowed to take values in all real axis (in any case, the boundary conditions require that the axionic field may also take negative values). We turn now to the analysis of the norm of the wave function in the two kind of situations we have: the ”normal” case \( \tilde{\omega} > 0 \) and the ”anomalous” case \( \tilde{\omega} < 0 \).

### 4.1 The normal case \( \tilde{\omega} > 0 \)

The wave function is given by (30). It presents some pathologies as we will now discuss. Let us consider

\[
\Psi^\ast \Psi = \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} ds ds' A(s)A(s') \frac{a}{\sqrt{\phi}} \frac{(a_1^{-1/2})^{\mu+\mu'}}{(2\rho(T))^{\mu+1}(2\rho^*(T))^{\mu'+1}} \exp \left[ - \frac{a_1a_2^2\phi}{2\rho(T)\rho^*(T)} \right] 
\]

\[
\times \left( 2\sqrt{\frac{\omega}{\rho}} \phi^p \right)^{\nu+\nu'} \left[ \frac{1}{\sigma(\chi) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right]^{\mu+\nu} \left[ \frac{1}{\sigma^*(\chi) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right]^{\mu'+\nu'} 
\]

\[
\times \frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)\Gamma(\mu'+\nu')\Gamma(\mu'-\nu')}{\Gamma(\mu+1/2)\Gamma(\mu'+1/2)} 
\]

\[
\times \ _2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma(T) - \sqrt{\frac{\omega}{\rho}} \phi^p}{\sigma(T) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right) 
\]

\[
\times \ _2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma^*(T) - \sqrt{\frac{\omega}{\rho}} \phi^p}{\sigma^*(T) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right) 
\]

(35)

The integration in the variable \( a \) can be easily performed, leading to

\[
\int_0^\infty \Psi^\ast \Psi \ da = \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} ds ds' A(s)A(s') \frac{1}{(2a_1)^{1+\mu+\mu'/2} \phi^{3/2}} (2\rho(T))^{\mu-\mu'} (\rho^*(T))^{\mu'-\mu} \Gamma[1 + (\mu + \mu')/2] 
\]

\[
\times \left( 2\sqrt{\frac{\omega}{\rho}} \phi^p \right)^{\nu+\nu'} \left[ \frac{1}{\sigma(\chi) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right]^{\mu+\nu} \left[ \frac{1}{\sigma^*(\chi) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right]^{\mu'+\nu'} 
\]

\[
\times \frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)\Gamma(\mu'+\nu')\Gamma(\mu'-\nu')}{\Gamma(\mu+1/2)\Gamma(\mu'+1/2)} 
\]

\[
\times \ _2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma(T) - \sqrt{\frac{\omega}{\rho}} \phi^p}{\sigma(T) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right) 
\]

\[
\times \ _2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma^*(T) - \sqrt{\frac{\omega}{\rho}} \phi^p}{\sigma^*(T) + \sqrt{\frac{\omega}{\rho}} \phi^p} \right) 
\]

(36)

Writting \( \rho(T) = Re^{i\Theta} \), where \( R = \sqrt{a_1^2 + T^2} \), \( \Theta = \arctan(T/a_1) \), the final expression is

\[
\int_0^\infty \Psi^\ast \Psi \ da = \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} ds ds' A(s)A(s') \frac{1}{(2a_1)^{1+\mu+\mu'/2} \phi^{3/2}} \cos \left[ (\mu - \mu')\Theta(T) \right] \Gamma[1 + (\mu + \mu')/2] 
\]

8
\[
\times (2 \sqrt{\omega_p} \phi_p)^{\nu+\nu'} \left[ \frac{1}{\sigma(\chi) + \sqrt{\omega_p} \phi_p} \right]^{\mu+\nu} \left[ \frac{1}{\sigma^*(\chi) + \sqrt{\omega_p} \phi_p} \right]^{\mu+\nu'} \\
\times \frac{\Gamma(\mu + \nu) \Gamma(\mu - \nu) \Gamma(\mu' + \nu') \Gamma(\mu' - \nu')}{\Gamma(\mu + 1/2) \Gamma(\mu' + 1/2)} \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma(T) - \sqrt{\omega_p} \phi_p}{\sigma(T) + \sqrt{\omega_p} \phi_p} \right) \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma^*(T) - \sqrt{\omega_p} \phi_p}{\sigma^*(T) + \sqrt{\omega_p} \phi_p} \right) \\
\times \Gamma(1 + \mu) \frac{1}{(2\mu_1)^{1+\mu} \phi^{3/2}} \left( 2 \sqrt{\omega_p} \phi_p \right)^{2\nu} \\
\times \frac{(\Gamma(\mu + \nu) \Gamma(\mu - \nu))^2}{\Gamma^2(\mu + 1/2)} \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma(T) - \sqrt{\omega_p} \phi_p}{\sigma(T) + \sqrt{\omega_p} \phi_p} \right) \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma^*(T) - \sqrt{\omega_p} \phi_p}{\sigma^*(T) + \sqrt{\omega_p} \phi_p} \right) \quad (37)
\]

The crucial point is that the expression (37) is time-dependent unless \( s = s' \). Further integrations on \( \phi \) and \( \chi \) can not change this situation. Hence, in order to assure the unitarity of the quantum model we must set

\[
A(s) = \delta(s - s_0) \quad ,
\]

where \( s_0 \) is an arbitrary number in the range of values of \( s \) which lead to the correct boundary condition. Hence, we obtain

\[
N = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \Psi^* \Psi \, da \, d\phi \, d\chi = \int_0^\infty \int_{-\infty}^\infty d\phi \, d\chi \frac{1}{(2\mu_1)^{1+\mu} \phi^{3/2}} \left( 2 \sqrt{\omega_p} \phi_p \right)^{2\nu} \Gamma(1 + \mu) \\
\times \frac{1}{\left( \sigma(\chi) + \sqrt{\omega_p} \phi_p \right)\left( \sigma^*(\chi) + \sqrt{\omega_p} \phi_p \right)} \frac{(\Gamma(\mu + \nu) \Gamma(\mu - \nu))^2}{\Gamma^2(\mu + 1/2)} \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma(T) - \sqrt{\omega_p} \phi_p}{\sigma(T) + \sqrt{\omega_p} \phi_p} \right) \\
\times 2F_1 \left( \mu + \nu + 1/2, \mu + 1/2, \frac{\sigma^*(T) - \sqrt{\omega_p} \phi_p}{\sigma^*(T) + \sqrt{\omega_p} \phi_p} \right) \quad (39)
\]

The norm is time independent. Now, in principle we could proceed and compute the expectation value for the dynamical variables. However, the norm of the wave function is divergent. The construction of the wave packet has not erased the divergences present in the norm of the eigenmodes. Even if we can not assure that this divergence appears for all wave packets, it is suggestive that others wave packets that can be constructed using more complicated expressions, disposable in integral tables, lead to the same problem.

Since the norm of the wave function is divergent, the only possible consistent situation is to fix the gravitational coupling constant from the begining. In this case, we return back to General Relativity. Moreover, since \( \phi \) is fixed from the begining, it is a free parameter. No prediction concerning the dilaton field can be obtained in this case.
4.2 The anomalous case ($\tilde{\omega} < 0$)

We compute again the norm of the wave function. It is given by

\[ N = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \Psi^*(a, \phi, \chi) \Psi(a, \phi, \chi) \, da \, d\phi \, d\chi \]

\[ = \int_0^\infty \int_0^\infty \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} A(s') A(s) \frac{a}{\sqrt{\phi^3}} \frac{(a\phi^{1/2})^\mu \rho^\mu (s - s_0)}{2\rho(T)\rho^*(T)^{\mu+1}} \]

\[ \times \exp \left[ -\frac{a_1 a^2 \phi^2}{2\rho(T)\rho^*(T)} \right] \sigma(\chi) \sigma^*(\chi) \left( 2\frac{\sqrt{\omega}}{p} \phi^p \right)^{\nu+\nu'} \]

\[ \times \Gamma(\nu + 3/2) \Gamma(\nu' + 3/2) \frac{\rho(T)^{(\mu' - \mu)/2} \rho^*(T)^{(\mu - \mu')/2}}{(2a_1)^{1+\mu/2+\mu'/2}} \int ds' ds \, d\phi \, d\chi \]  

The integration on $a$ can be performed, leading to

\[ N = \int_0^\infty \int_0^\infty \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} A(s') A(s) \frac{1}{\sqrt{\phi^3}} \frac{\rho(T)^{(\mu' - \mu)/2} \rho^*(T)^{(\mu - \mu')/2}}{(2a_1)^{1+\mu/2+\mu'/2}} \]

\[ \times \frac{\sigma(\chi) \sigma^*(\chi) \left( 2\frac{\sqrt{\omega}}{p} \phi^p \right)^{\nu+\nu'}}{\Gamma[1 + (\mu + \mu')/2] \Gamma(\nu + 3/2) \Gamma(\nu' + 3/2)} \int ds' ds \, d\phi \, d\chi \]

Writting, as before, $\rho(T) = Re^{i\Theta}$, with $R(T) = \sqrt{a_1^2 + T^2}$, $\Theta(T) = \arctan(T/a_1)$, and using the fact that, aside these terms, the rest of the integrand is symmetric by the interchange $s \leftrightarrow s'$, we end up with the integral expression

\[ N = \int_0^\infty \int_0^\infty \int_{s_1}^{s_2} \int_{s_1'}^{s_2'} A(s') A(s) \frac{1}{\sqrt{\phi^3}} \frac{\cos[(\mu' - \mu)e\Theta(T)]}{(2a_1)^{1+\mu/2+\mu'/2}} \]

\[ \times \sigma(\chi) \sigma^*(\chi) \left( 2\frac{\sqrt{\omega}}{p} \phi^p \right)^{\nu+\nu'} \]

\[ \times \frac{\Gamma[1 + (\mu + \mu')/2] \Gamma(\nu + 3/2) \Gamma(\nu' + 3/2)}{\left[ \sigma^2(\chi) + \frac{|\omega|}{p^2} \phi^{2p} \right]^{\nu+3/2}} \int ds' ds \, d\phi \, d\chi \]

The norm of the wave function is time dependent unless $A(s) = \delta(s - s_0)$ and $A(s') = \delta(s' - s_0)$. Hence,

\[ N = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{\phi^3}} \frac{1}{(2a_1)^{1+\mu}} \sigma(\chi) \sigma^*(\chi) \left( 2\frac{\sqrt{\omega}}{p} \phi^p \right)^{2\nu} \]

\[ \times \frac{\Gamma(1 + \mu) \Gamma^2(\nu + 3/2)}{\left[ \sigma^2(\chi) + \frac{|\omega|}{p^2} \phi^{2p} \right] \left[ (\sigma^*(\chi))^2 + \frac{|\omega|}{p^2} \phi^{2p} \right]^{3/2}} \int d\phi \, d\chi \]
The norm now is finite.

We proceed evaluating the expectation value of the wave function.

The expectation value for a dynamical variable $X$ is given by

$$< X > = \frac{1}{N} \int_0^\infty \int_0^\infty \int_{-\infty}^{+\infty} \Psi^* X \Psi \, da \, d\phi \, d\chi \, .$$

(44)

In general, it is possible to perform the integration in $a$, but the integration in the variables $\phi$ and $\chi$ can only be made numerically.

The result for the scale factor is:

$$< a > = \sqrt{a_1^2 + T^2 \frac{2^{2\nu+1}}{pN} \frac{1}{(2a_1)^{\mu+3/2}} \left( \frac{p}{\sqrt{|\tilde{\omega}|}} \right)^{\frac{1}{2\nu}} \int_0^\infty \int_{-\infty}^{+\infty} dz \, d\chi \, z^{-\frac{1}{2\nu}+1+2\nu}}$$

$$\times \Gamma^2(\nu + 3/2)\Gamma(\mu + 3/2)(a_2^2 + \chi^2)^{-1} \frac{1}{\left[ (a_2^2 + z^2 - \chi^2)^2 + 4a_2^2\chi^2 \right]^{\nu+3/2} \, .}$$

(45)

where $z = |\tilde{\omega}|^p$. The time behavior is given by the first term in the right hand side of (45): the scale factor exhibits a bounce and asymptotically it behaves like the FRW radiation-dominated model (remember that for the radiative fluid $T$ is the conformal time).

Performing the same calculation for the dilatonic field, we find

$$< \phi > = \frac{2^{2\nu+1}}{pN} \frac{1}{(2a_1)^{\mu+3/2}} \left( \frac{p}{\sqrt{|\tilde{\omega}|}} \right)^{\frac{1}{2\nu}} \int_0^\infty \int_{-\infty}^{+\infty} dz \, d\chi \, z^{-\frac{1}{2\nu}+1+2\nu}$$

$$\times \Gamma^2(\nu + 3/2)\Gamma(\mu + 1)(a_2^2 + \chi^2)^{-1} \frac{1}{\left[ (a_2^2 + z^2 - \chi^2)^2 + 4a_2^2\chi^2 \right]^{\nu+3/2}} \, .$$

(46)

Remark that the expectation value of $\phi$ does not depend on time. Finally, it is easy to see that

$$< \chi > = 0 \, ,$$

(47)

since the integrand is an odd function and the integration is performed in all real axis.

The expectation value of the dilatonic is time independent, but we can obtain some predictions. We have worked in units such that $\hbar = c = 1$. In this case, the Planck Mass is the inverse of the Planck length. Let us consider the situation where the scale factor carries a dimension of length. The dilatonic field has the dimension of $M_P^2$, since it is connected with the inverse of the gravitational coupling. Hence, we can construct the dimensionless quantity

$$< a >_{\text{min}} \sqrt{< \phi >} = b$$

(48)

where $b$ is a pure number and $< a >_{\text{min}}$ is the minimum value of the scale factor during the bounce. Performing the numerical integration in the expressions for the scale factor, we find that the minimum of the scale factor is of the order of $200L_{Pl}$ for $\omega = -1.6 \ (\tilde{\omega} = -0.1)$, $2L_{Pl}$ for $\omega = -3 \ (\tilde{\omega} = -1.5)$ and $10^{-5}L_{Pl}$ for $\omega = -500 \ (\tilde{\omega} \text{ essentially of}}
the same order). These results indicate that for values of $\omega$ near the critical value $-3/2$, the minimum value of the scale factor is far above the Planck length. As the value of $\omega$ grows in absolute value, the minimum scale factor becomes highly transplanckian, and in the limit $\omega \to -\infty$ the curvature singularity appears.

4.3 Brans-Dicke quantum cosmological models

The Brans-Dicke theory is a particular case of the general action (2) when $\chi = \text{constant}$. Considering the wave functions previous settled out, the wave functions for the Brans-Dicke case can be obtained as the limit $r \to 0$. Hence, the wave function for the Brans-Dicke theory coupled to perfect fluid matter is

$$\Psi(a, \phi, T) = \sqrt{a} \sqrt{\phi} J_\mu(\sqrt{E} a^{1/2}) \left\{ A\phi^{-\nu} + B\phi^{\nu} \right\} e^{-iT}.$$  \hspace{1cm} (49)

The regularity of the wave function at the origin and at infinity implies

$$\phi \to 0 \Rightarrow -\frac{1}{4} + \frac{\mu}{2} \pm \nu > 0 ;$$

$$\phi \to \infty \Rightarrow \pm \nu < 0 .$$  \hspace{1cm} (50, 51)

It is possible to obtain a range of the parameter $s$ where these conditions are satisfied. However, as in the normal string case discussed before, the wave functions (49) are not square integrable. In the computation of the norm of the wave functions we find

$$N = \int_0^\infty \int_0^\infty \Psi^* \Psi \, da \, d\phi$$

$$= \int_0^\infty \int_0^\infty \phi^{-3/2} u J_\mu(\nu u) J_\mu(\nu u) e^{-i(r^2-r'^2)} \, du \, d\phi$$

$$= \int_0^\infty \phi^{-3/2} \frac{1}{r} \delta(r-r') e^{-i(r^2-r'^2)T} \, d\phi ,$$  \hspace{1cm} (52)

with $r = \sqrt{E}$ and $u = a\phi^{1/2}$. There is a divergence when $\phi \to 0$.

The divergence pointed out occurs for any sign of $\ddot{\omega}$. The consequence is that no quantum cosmological model can be constructed in the pure Brans-Dicke case. As before, the superposition of the wave functions through the construction of the wave packet, does not alter this conclusion since we are only superposing functions that are not square integrable. The only case that makes sense is $\phi = \text{constant}$ from the beginning, that is, the gravitational coupling has no dynamics at all. This leads to the general relativity quantum cosmological model with $G \sim 1/\phi$. As in the normal string case, the value of $G$ can not be determined and it becomes a free parameter of the model.

5 Conclusions

In this work, we have studied the quantum cosmological models from the string effective action in four dimensions coupled to a radiative fluid. The radiative have been described
with the aid of the Schutz’s formalism. The quantization of this system in the minisuperspace have led to a Schrödinger-like equation where the matter variables play the role of time. Since we ended up with a genuine Schrödinger equation the predictions for the evolution of the dynamical variables were obtained through the computation of the corresponding expectation values.

The result is somehow surprising: the dilatonic field must have a value independent of time while the axionic field must be zero. However, the reason for this result is different depending if we are treating the normal case where $\omega > -3/2$ or the anomalous case where $\omega < -3/2$. In the normal case, the wave functions obtained from the Schrödinger-like equation are not square integrable. They have not led to square integrable expressions through the construction of wave packets using analytical expressions - others wave packets besides that shown here in the text have been analyzed, but they exhibit the same problems. Hence, it seems that no quantum cosmological model can be constructed unless the dilatonic field is fixed as a constant from the begining, what reduces the string action to the general relativity action. In the anomalous case, square integrable wave functions (consequently, wave packets also) can be obtained, but the expectation value of the dilatonic field leads to a time independent expression, while the expectation value for the axionic field is zero.

The fact that the imposition to have square integrables functions leads to restriction on the parameters of the quantum model is well known in ordinary quantum mechanics. In the harmonic oscillator problem the quantization of the energy results from this imposition. Here, if we want to be strict, accepting the string effective action (2), with dynamical dilatonic and axion fields, the prediction we obtain is that $\omega < -3/2$ and $< \phi > = \text{constant}$. This excludes the pure string case, $\omega = -1$. This exclusion, very strong of course, can be alleviate by considering cases where no dynamics is attributed to the dilatonic field from the begining. To consider the pure dilatonic model without the axion field does not change this situation.

The strength of any conclusion is directly related to the assumptions made. Here, one important assumption is the presence of a genuine time coordinate associated to the matter field. In other situations in quantum cosmology, such assumption has already led to very curious result: if applied to a Bianchi type I cosmological model [14], the norm of the wave packet is necessarily time dependent. One way to obtain some dynamics to the dilatonic field is to allow the norm of the wave packet to depend on time here as in the Bianchi type I case. However, as in that case, this leads to a non unitary quantum model, what is conceptually a problem, unless some techniques coming from quantum open system may be succesfully employed [20]. However, this lies outside the scope of the present work.

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