Explicit proof of equivalence of two-point functions in the two formalisms of thermal field theory

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Abstract

We give an explicit proof of equivalence of the two-point function to one-loop order in the two formalisms of thermal $\lambda \phi^3$ theory based on the expressions in the real-time formalism. It is indicated that the key-point of completing the proof is to separate carefully the imaginary part of the zero-temperature loop integral from relevant expressions and this fact will certainly be very useful for examination of the equivalence problem of the two formalisms of thermal field theory in other theories, including the one of the propagators for scalar bound states in a NJL model.

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1 Introduction

Finite temperature field theory has attracted much research interests owing to its application to phase transition of early universe and nuclear matter [1-5]. It has two formalisms: the imaginary-time and the real-time formalism [4]. However, the complete equivalence between the two formalisms has been a subtle problem, though one generally believes that the two formalisms should give identical results [6]. In a recent research on the Nambu-Goldstone mechanism of electroweak symmetry breaking at finite temperature [7], we calculate the propagators for scalar bound states in one-generation fermion condensate model [a Nambu-Jona-Lasinio (NJL)-type model [8]] and find that their denominators show different imaginary-parts in the two formalisms. By conventional inference, origin of the difference could be that either we have calculated different Green functions in the two formalisms [9], or we have missed some important technical details in the calculations. To clarify matter, we will review some similar calculations. Because the propagators for scalar bound states in a NJL model correspond to four-point amputated functions and the calculations of them could be effectively reduced to the ones of usual two-point functions, we will look back on the discussions of some two-point functions. Among them, the most typical and simple one is the two-point function in $\lambda \phi^3$ theory. It is accepted that two-point functions in the imaginary-time and the real-time formalism is equivalent. This has been discussed by a formal analysis [4] and some calculations of the imaginary parts of the Green functions [10-11]. However, as far as the whole two-point Green function is concerned, no explicit calculation of this equivalence based on the expressions in the real-time formalism has been given. In fact, for the sake of comparison with four-point amputated functions in a NJL model we need the calculations of the whole two-point function and in particular, expect to express the results in the real-time formalism so as to show correspondence between a causal Green function and a physical propagator. Following this idea, in this paper we will give an explicit proof of equivalence of the whole two-point function to one-loop order in $\lambda \phi^3$ theory in the two formalisms by means of the expressions in the real-time formalism and from it find the key-point of completing the proof. This will certainly be very useful for reexamination of the results in a NJL model.

The paper is arranged as follows. In Sect.2 we will calculate the two-point function to one-loop order in the imaginary-time formalism, then analytically continue the function defined at the Matsubara frequency

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to the physical causal propagator defined at real energy. In Sect.3 the calculations of the two-point function in the real-time formalism will be conducted. After diagonalization of the two-point function matrix by means of a thermal transformation matrix, we will obtain the physical causal propagator in the real-time formalism. In Sect.4 we will explicitly prove that the causal propagators obtained in the two formalisms are identical and indicate that the key-point of completing the proof is to separate carefully the imaginary part of the zero-temperature loop integral. Finally, in Sect.5 we come to our conclusions.

2 Two-point function in the imaginary-time formalism

In the imaginary-time formalism, Lagrangian can be expressed by

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{3!} \varphi^3.
\]  

(1)

The vertex rule will be \(- \lambda\) and the propagator for a free particle will be \(\Delta_F(i\omega_n, \vec{l}) = 1/(\omega_n^2 + \omega_l^2)\), where \(\omega_n = 2\pi n/\beta (n = 0, \pm 1, \pm 2, \ldots)\) is the Matsubara frequency, \(\beta = 1/T\) is the reciprocal of temperature and \(\omega_l^2 = l^2 + m^2\) with the three momentum \(\vec{l}\) [4,5,12]. The two-point function \(G'_F(-i\omega_m, \vec{p})\) to one-loop order submits to the equation

\[
G'_F(-i\omega_m, \vec{p}) = \Delta_F(-i\omega_m, \vec{p}) + \Delta_F(-i\omega_m, \vec{p})\Pi'(-i\omega_m, \vec{p})G'_F(-i\omega_m, \vec{p}), \quad \omega_m = \frac{2\pi m}{\beta},
\]  

(2)

which has the solution

\[
G'_F(-i\omega_m, \vec{p}) = \frac{1}{\omega_m^2 + \omega_p^2 - \Pi'(-i\omega_m, \vec{p})}, \quad \omega_p^2 = \vec{p}^2 + m^2,
\]  

(3)

where \(\Pi'(-i\omega_m, \vec{p})\) is the contribution from one-loop diagram and may be expressed by

\[
\Pi'(-i\omega_m, \vec{p}) = \lambda^2 \int \frac{d^3l}{(2\pi)^3} A(-i\omega_m, \vec{p}, \vec{l})
\]  

(4)

with

\[
A(-i\omega_m, \vec{p}, \vec{l}) = T \sum_n \frac{1}{\omega_n^2 + \omega_l^2 (\omega_m + \omega_n)^2 + \omega_p^2}, \quad \omega_{l+p}^2 = (\vec{l} + \vec{p})^2 + m^2.
\]  

(5)

The sum of the Matsubara frequency in Eq. (5) can be done by standard procedure [12]. We define the Fourier transformation

\[
\frac{1}{\omega_m^2 + \omega_k^2} = \int_0^\beta \hat{A}_F(\tau, \omega_k)e^{i\omega_m \tau}
\]  

(6)

and the inverse formula

\[
\hat{A}_F(\tau, \omega_k) = T \sum_n e^{-i\omega_n \tau} \frac{1}{\omega_n^2 + \omega_k^2}
\]  

(7)

which obeys the periodicity condition

\[
\hat{A}_F(\tau, \omega_k) = \hat{A}_F(\tau - \beta, \omega_k)
\]  

(8)

and can be calculated by the formula

\[
\hat{A}_F(\tau, \omega_k) = T \sum_n g(k_0 = i\omega_n) = T \int_{C_1 \cup C_2} \frac{dk_0}{2\pi i} g(k_0) \frac{\beta}{2} \coth \frac{\beta k_0}{2},
\]  

\[
g(k_0) = e^{-k_0 \tau} \frac{1}{\omega_k^2 - k_0^2}, \quad \text{for} \quad \tau > 0,
\]  

(9)

where \(C_1\) and \(C_2\) represent the integral paths \(\eta - i\infty \to \eta + i\infty\) and \(-\eta + i\infty \to -\eta - i\infty (\eta = 0_+)\) respectively in the complex \(k_0\) plane. The result is

\[
\hat{A}_F(\tau, \omega_k) = \frac{1}{2\omega_k} \{ n(\omega_k)e^{\omega_k \tau} + [1 + n(\omega_k)]e^{-\omega_k \tau} \},
\]  

(10)
By means of Eqs. (6),(10) and the completeness formula
\[ T \sum_n e^{i \omega_n (\tau - \tau')} = \delta(\tau - \tau'), \]
we may obtain the frequency sum (5) expressed by
\[ A(-i \omega_m, \vec{p}, \vec{l}) = \int_0^\beta d\tau' \Delta_F(\tau, \omega_l) \Delta_F(\tau, \omega_{l+p}) e^{-i \omega_m \tau'} \]
\[ = \frac{1}{4 \omega_m \omega_{l+p}} \left[ \frac{1 + n(\omega_l) + n(\omega_{l+p})}{\omega_m + \omega_l + \omega_{l+p}} + \frac{n(\omega_{l+p}) - n(\omega_l)}{\omega_m - \omega_l - \omega_{l+p}} \right. \]
\[ + \left. \frac{n(\omega_l) - n(\omega_{l+p})}{\omega_m + \omega_l - \omega_{l+p}} - \frac{1 + n(\omega_l) + n(\omega_{l+p})}{\omega_m - \omega_l - \omega_{l+p}} \right]. \]

This completes the calculation of \( G^I_F(-i \omega_m, \vec{p}) \) in Eq. (3).

To obtain physical causal Green function, we must make the analytic continuation of the energy from the discrete imaginary values to the real axis in the following way [12]
\[ -i \omega_m \rightarrow p^0 + i \varepsilon p^0, \quad \varepsilon = 0+. \]

Taking an additional factor \(-i\) into account, we may express the physical causal two-point function by
\[ G^I_F(p) = -i G^I_F(-i \omega_m, \vec{p})\big|_{-i \omega_m \rightarrow p^0 + i \varepsilon p^0} = i / [p^2 - m^2 + \Pi^I(p) + i \varepsilon], \]
noting that under the continuation (14), \( \omega_m^2 + \omega_p^2 \rightarrow -(p^2 - m^2 + i \varepsilon) \), and
\[ \Pi^I(p) = \lambda^2 \int \frac{d^3 l}{(2\pi)^3} A_F(p, \vec{l}), \]
\[ A_F(p, \vec{l}) = A(-i \omega_m, \vec{p}, \vec{l})\big|_{-i \omega_m \rightarrow p^0 + i \varepsilon p^0} \]
\[ = \frac{1}{4 \omega_m \omega_{l+p}} \left[ \frac{1 + n(\omega_l) + n(\omega_{l+p})}{-p^0 + \omega_l + \omega_{l+p} - i \varepsilon \eta(p^0)} + \frac{n(\omega_{l+p}) - n(\omega_l)}{-p^0 + \omega_l - \omega_{l+p} + i \varepsilon \eta(p^0)} \right. \]
\[ + \left. \frac{n(\omega_l) - n(\omega_{l+p})}{-p^0 - \omega_l + \omega_{l+p} - i \varepsilon \eta(p^0)} - \frac{1 + n(\omega_l) + n(\omega_{l+p})}{-p^0 - \omega_l - \omega_{l+p} + i \varepsilon \eta(p^0)} \right], \quad \eta(p^0) = \frac{p^0}{|p^0|}. \]

For making a comparison with the following results obtained in the real-time formalism, we change \( A_F(p, \vec{l}) \) into an integral representation. By the formula
\[ \frac{1}{X + i \varepsilon} = \frac{X}{X^2 + \varepsilon^2} - i \pi \delta(X) \]
and the definition
\[ n(p^0) = \theta(p^0) / (e^{\beta p^0} - 1) + \theta(-p^0) / (e^{-\beta p^0} - 1) = 1 / (e^{\beta |p^0|} - 1) = \sinh^2 \Theta(p^0), \]
we may write
\[ A_F(p, \vec{l}) = \int \frac{d l^0}{2 \pi} [L^0 - \omega_l^2 + i \varepsilon][(l^0 + p^0)^2 - \omega_{l+p}^2 + i \varepsilon] \]
\[ - \int d l_0 \left\{ \sinh^2 \Theta(p^0) P \frac{\delta(l^0 - \omega_l^2)}{(l^0 + p^0)^2 - \omega_{l+p}^2} + \sinh^2 \Theta(p^0 + p^0) P \frac{\delta(l^0 + p^0 - \omega_l^2)}{(l^0 + p^0)^2 - \omega_{l+p}^2} \right\} \]
\[ + i \eta(p^0) \pi \int d l_0 \Theta(l^0 - \omega_l^2) \delta(l^0 + p^0 - \omega_{l+p}^2) \]
\[ \text{[sinh}^2 \Theta(t^0)\eta(t^0 + p^0) + \text{sinh}^2 \Theta(t^0 + p^0)\eta(-t^0)], \] (20)

where "P" means the principal value of the integral. Substituting Eq. (20) into Eq. (16), we obtain

\[ \Pi^I(p) = \tilde{K}(p) - \tilde{H}(p) + i\tilde{S}^I(p), \] (21)

where

\[ \tilde{K}(p) = \lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{-i}{(l^2 - m^2 + i\varepsilon)[(l + p)^2 - m^2 + i\varepsilon]} = \frac{\lambda^2}{16\pi^2} \int_0^1 dx \left( \ln \frac{\Lambda^2 + M^2}{M^2} - \frac{\Lambda^2}{\Lambda^2 + M^2} \right) \] (22)

with \( M^2 = m^2 - p^2 x(1 - x) \) and the Euclidean four-momentum cutoff \( \Lambda \), is the zero-temperature loop integral,

\[ \tilde{H}(p) = 2\pi\lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{-(l + p)^2 - m^2}{[(l + p)^2 - m^2]^2 + \varepsilon^2} + (p \rightarrow -p) \delta(l^2 - m^2) \text{sinh}^2 \Theta(t^0) \] (23)

and

\[ \tilde{S}^I(p) = \eta(p^0)2\pi\lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta(l^2 - m^2)\delta[(l + p)^2 - m^2][\text{sinh}^2 \Theta(t^0)\eta(t^0 + p^0) + \text{sinh}^2 \Theta(t^0 + p^0)\eta(-t^0)]. \] (24)

Eq. (15) together with Eqs. (21)-(24) give physical causal two-point function in the imaginary-time formalism.

### 3 Two-point function in the real-time formalism

In the real-time formalism, Lagrangian can be expressed by

\[ \mathcal{L} = \frac{1}{2} \sum_{a=1,2} \partial_\mu \varphi^{(a)} \partial^\mu \varphi^{(a)} - \frac{1}{2} m^2 \sum_{a=1,2} \varphi^{(a)} \varphi^{(a)} - \frac{\lambda}{3!} \sum_{a=1,2} (-1)^{a+1} \varphi^{(a)}^3, \] (25)

where \( a = 1 \) and \( a = 2 \) respectively represent physical and ghost field. The vertex rule will be \(-i\lambda(-1)^{a+1}\) and the propagator for a free particle will become a matrix \( D^{ab}(p)(a, b = 1, 2) \) [4] with the elements expressed by

\[ D^{11}(p) = \frac{i}{p^2 - m^2 + i\varepsilon} + n(p^0)2\pi\delta(p^2 - m^2) = [D^{22}(p)]^*, \]

\[ D^{12}(p) = D^{21}(p) = e^{i\delta(p^0)}n(p^0)2\pi\delta(p^2 - m^2). \] (26)

The two-point function \( G^{Rab}(p)(a, b = 1, 2) \) obeys the equation

\[ G^{Rab}(p) = D^{ab}(p) + D^{ac}(p)i\Pi^{cd}(p)G^{Rdb}(p), \] (27)

where \( i\Pi^{cd}(p) \) is the contribution from the one-loop diagram with the vertex denotations \( c \) and \( d \). The matrix form of Eq.(27) is

\[ G^R(p) = D(p) + D(p)i\Pi(p)G^R(p). \] (28)

The propagator matrix \( D(p) \) may be diagonalized by a thermal transformation matrix \( M \) [4], i.e.

\[ D(p) = M \begin{pmatrix} \Delta_E(p) & 0 \\ 0 & \Delta_E(-p) \end{pmatrix} M, \quad \Delta_E(p) = \frac{i}{p^2 - m^2 + i\varepsilon} , \]

\[ M = \begin{pmatrix} \cosh \Theta(p^0) & \text{sinh} \Theta(p^0) \\ \text{sinh} \Theta(p^0) & \cosh \Theta(p^0) \end{pmatrix}, \quad \text{sinh}^2 \Theta(p^0) = n(p^0). \] (29)

The same matrix \( M \) may diagonalize the two-point function matrix \( G^R(p) \) and give the physical causal two-point function \( G^R_F(p) \) [4]. Thus we assume that

\[ G^R(p) = M \begin{pmatrix} G^R_F(p) & 0 \\ 0 & G^R_{\bar{F}}(p) \end{pmatrix} M \] (30)
where \( \tilde{\theta} \) then may obtain from Eq.(28) that

\[
\begin{pmatrix}
G_F^R(p) & 0 \\
0 & G_F^{R*}(p)
\end{pmatrix}
\begin{pmatrix}
\Delta_F(p) & 0 \\
0 & \Delta_F^*(p)
\end{pmatrix}
= \begin{pmatrix}
\Delta_F(p) & 0 \\
0 & \Delta_F^*(p)
\end{pmatrix}
\begin{pmatrix}
i\Pi_R(p) & 0 \\
0 & -i\Pi_R^*(p)
\end{pmatrix}
\begin{pmatrix}
G_F^R(p) & 0 \\
0 & G_F^{R*}(p)
\end{pmatrix}
\]

which has the solution

\[
G_F^R(p) = \frac{1}{\Delta_F^{-1}(p) - i\Pi_R(p)} = \frac{i}{p^2 - m^2 + i\varepsilon + \Pi_R(p)}.
\]

It is indicated that in deriving Eq.(33) we have used the diagonalization assumption (31) of \( i\Pi(p) \) and this means that

\[
\begin{pmatrix}
\Pi_R(p) & 0 \\
0 & -\Pi_R^*(p)
\end{pmatrix}
= M\Pi(p)M = M \begin{pmatrix}
\Pi^{11} & \Pi^{12} \\
\Pi^{21} & \Pi^{22}
\end{pmatrix} M.
\]

By means of the explicit expression of \( M \) in Eq. (29), Eq. (34) will be reduced to the following constraints:

\[
\begin{align*}
\Pi^{11} + \Pi^{22} + e^{\beta|p^0|/2}\Pi^{22} + e^{-\beta|p^0|/2}\Pi^{21} & = 0, \\
\Pi^{11} + \Pi^{22} + e^{-\beta|p^0|/2}\Pi^{22} + e^{\beta|p^0|/2}\Pi^{21} & = 0
\end{align*}
\]

and

\[
\Pi^R = \cosh^2\Theta\Pi^{11} + \sinh^2\Theta\Pi^{22} + \cosh\Theta\sinh\Theta(\Pi^{12} + \Pi^{21}), \\
-\Pi^{R*} = \sinh^2\Theta\Pi^{11} + \cosh^2\Theta\Pi^{22} + \sinh\Theta\cosh\Theta(\Pi^{12} + \Pi^{21}).
\]

Since the two equations in Eq.(35) exchange each other under \( |p^0| \leftrightarrow -|p^0| \), we may remove the absolute value symbol of \( p^0 \), i.e. write

\[
\begin{align*}
\Pi^{11} + \Pi^{22} + e^{\beta|p^0|/2}\Pi^{22} + e^{-\beta|p^0|/2}\Pi^{21} & = 0, \\
\Pi^{11} + \Pi^{22} + e^{-\beta|p^0|/2}\Pi^{22} + e^{\beta|p^0|/2}\Pi^{21} & = 0
\end{align*}
\]

On the other hand, Eq.(36) will lead to that

\[
\Pi^{11} = -(\Pi^{22})^*, \quad \Pi^{12} + \Pi^{21} = -(\Pi^{12} + \Pi^{21})^*.
\]

Determination of \( G_F^R(p) \) depends on calculation of \( \Pi^R(p) \) and in the same time we must check validity of the relations (37) and (38). In the one-loop approximation,

\[
i\Pi^{cd}(p) = (-1)^{c+d+1}\lambda^2 \int \frac{d^4l}{(2\pi)^4} D^{cd}(l + p) D^{dc}(l)
\]

with the following explicit expressions

\[
\begin{align*}
\Pi^{11}(p) & = \tilde{K}(p) - \tilde{H}(p) + i\tilde{S}(p), \\
\Pi^{22}(p) & = -\tilde{K}^*(p) + \tilde{H}(p) + i\tilde{S}(p), \\
\Pi^{12}(p) & = \Pi^{21}(p) = -i\tilde{R}(p),
\end{align*}
\]

where \( \tilde{K}(p) \) and \( \tilde{H}(p) \) are given by Eqs.(22) and (23) , and

\[
\tilde{S}(p) = 2\pi^2\lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta(l^2 - m^2)\delta((l + p)^2 - m^2)
\]
\[
\begin{align*}
\text{Im } \tilde{\Delta}(p) &= \tilde{\Delta}(p) = 2\pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta(p^2 - \omega^2) \delta([t^0 + p^0]^2 - \omega_{t+p}^2) \left[ \sinh(\frac{\beta p^0}{2}) \right. \\
&\left. \sinh(\beta p^0/2) \right], \\
\end{align*}
\]
Obviously, \( \Pi^{cd}(p) \) given by Eq.\,(40) satisfy Eq.\,(38) and since \( \Pi^{12} = \Pi^{21} \), Eq. \,(37) is reduced to
\[
\Pi^{11} + \Pi^{22} + \cosh(\beta p^0/2)\Pi^{12} = 0
\]
and furthermore to
\[
\tilde{S}(p) = \cosh(\beta p^0/2)\tilde{R}(p) - \text{Im } \tilde{K}(p),
\]
where Im \( \tilde{K}(p) \) represents the imaginary part of \( \tilde{K}(p) \). To verify validity of Eq.\,(43), we will use the explicit expressions \((41)\) and \((42)\). In fact, by means of the definition \((19)\), we can rewrite \( \tilde{R}(p) \) and \( \tilde{S}(p) \) by
\[
\tilde{R}(p) = \pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta([t^0 + p^0]^2 - \omega^2) \left[ \sinh(\beta p^0/2) \right. \\
&\left. \sinh(\beta p^0/2) \right], \\
\]
and
\[
\tilde{S}(p) = \cosh(\beta p^0/2)\tilde{R}(p) - \Delta(p),
\]
where
\[
\Delta(p) = 2\pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta(p^2 - \omega^2) \delta([t^0 + p^0]^2 - \omega_{t+p}^2) \left[ \sinh(\beta p^0/2) \right. \\
&\left. \sinh(\beta p^0/2) \right].
\]
On the other hand, we can calculate \( \tilde{K}(p) \) from Eq.\,(22) by the residue theorem and obtain
\[
\tilde{K}(p) = 2\pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{4\omega_{t+p}} \left[ \frac{1}{p^0 + \omega_t + \omega_{t+p} + i\epsilon} - \frac{1}{p^0 - \omega_t - \omega_{t+p} + i\epsilon} \right].
\]
Thus the imaginary part of \( \tilde{K}(p) \) may be derived and the result is
\[
\text{Im } \tilde{K}(p) = 2\pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{4\omega_{t+p}} \left[ \delta(p^0 + \omega_t + \omega_{t+p}) + \delta(p^0 - \omega_t - \omega_{t+p}) \right] = \Delta(p).
\]
This verifies that Eq. \((43)\) thus Eq.\,(37) is satisfied indeed. It should be emphasized that for verification of Eq.\,(43), separation of \(-\text{Im } \tilde{K}(p)\) from \( \tilde{S}(p) \) is essential. Substituting Eq. \((40)\) into Eq. \((36)\) and using Eqs. \((45), (48)\) and \((19)\), we obtain
\[
\Pi^R(p) = \tilde{K}(p) - \tilde{H}(p) + i[\sinh(\beta p^0/2)\tilde{R}(p) - \text{Im } \tilde{K}(p)].
\]
Eq. \((33)\) together with Eq. \((49)\) give the causal two-point function to one-loop order in the real-time formalism.

4 Equivalence of two-point functions in the two formalisms

If the causal two-point functions calculated in the two formalisms are equivalent, then we must have \( G^F_R(p) = G^R_F(p) \), and from Eqs.\,(15) and \((33)\), this means that \( \Pi^F(p) = \Pi^R(p) \), or explicitly, by Eqs.\,(21) and \((49)\),
\[
\tilde{K}(p) - \tilde{H}(p) + i\tilde{S}(p) = \tilde{K}(p) - \tilde{H}(p) + i[\sinh(\beta p^0/2)\tilde{R}(p) - \text{Im } \tilde{K}(p)].
\]
Since the real parts \([\text{Re } \tilde{K}(p) - \tilde{H}(p)\] in the two sides are the same, the problem is reduced to proving that
\[
\tilde{S}(p) + \text{Im } \tilde{K}(p) = \sinh(\beta p^0/2)\tilde{R}(p),
\]
i.e. \( \Pi^F(p) \) and \( \Pi^R(p) \) must have identical imaginary parts. To verify Eq.\,(51), we use the expression \((24)\) of \( \tilde{S}(p) \), the definition \((19)\) and Eq.\,(44) and get
\[
\tilde{S}(p) = \eta(p^0)\sinh(\beta p^0/2)\tilde{R}(p) - \tilde{\Delta}(p),
\]
where
\[
\tilde{\Delta}(p) = \pi^2 \lambda^2 \int \frac{d^4l}{(2\pi)^4} \delta(p^2 - \omega^2) \delta([t^0 + p^0]^2 - \omega_{t+p}^2) \left[ \sinh(\beta p^0/2) \right. \\
&\left. \sinh(\beta p^0/2) \right].
\]
where
\[
\tilde{\Delta}_1(p) = \eta(p^0) 2\pi^2 \lambda^2 \int \frac{dl_1}{(2\pi)^4} \delta(l^2 - m^2) \delta((l + p)^2 - m^2)[\eta(l^0 + p^0) - \eta(l^0)].
\] (53)

Owing to the factor \(\eta(l^0 + p^0) - \eta(l^0)\), in the integrand of Eq. (53) only the terms containing \(\delta(l^0 + \omega_l)\delta(l^0 + p^0 \pm \omega_{l+p})\) have non-zero contributions, we thus obtain
\[
\tilde{\Delta}_1(p) = \eta(p^0) 2\pi^2 \lambda^2 \int \frac{dl_1}{(2\pi)^4} \frac{1}{4\omega_{l+p}}[-\delta(p^0 + \omega_l + \omega_{l+p}) + \delta(p^0 - \omega_l - \omega_{l+p})]
\]
\[
= 2\pi^2 \lambda^2 \int \frac{dl_1}{(2\pi)^4} \frac{1}{4\omega_{l+p}}[\delta(p^0 + \omega_l + \omega_{l+p}) + \delta(p^0 - \omega_l - \omega_{l+p})]
\]
\[
= \text{Im} \tilde{K}(p).
\] (54)

Substituting Eq. (54) into Eq. (52) will mean that Eq. (51) is proven. Here we also indicate that separation of \(-\text{Im} \tilde{K}(p)\) from \(\hat{S}(p)\) is essential for verification of Eq. (51). This result shows the two-point function \(G^I_F(p)\) in Eq. (15) and \(G^R_F(p)\) in Eq. (33) are actually identical, hence we may omit the superscripts "\(I\)" and "\(R\)" and simply express the physical causal two-point function by
\[
G_F(p) = G^I_F(p) = G^R_F(p)
\]
\[
= i\{p^2 - m^2 + i\varepsilon + \text{Re} \tilde{K}(p) - \tilde{H}(p) + i[\hat{S}(p) + \text{Im} \tilde{K}(p)]\}
\]
\[
= i\{p^2 - m^2 + i\varepsilon + \text{Re} \tilde{K}(p) - \tilde{H}(p) + i \sinh(\beta|p^0|/2)|R(p)\}.
\] (55)

From the above demonstration we see that a key-point of completing the equivalence proof is carefully to consider and separate the imaginary part \(\text{Im} \tilde{K}(p)\) (with a minus sign) of the zero-temperature loop integral from the relevant expressions e.g. \(\hat{S}(p)\) and \(\hat{S}(p)\). Without doing so, we can not explain the diagonalization of the matrix \(\Pi(p)\) in the real-time formalism, and in particular, can not prove the identity of the imaginary parts of the denominator of \(G_F(p)\) in the two formalisms.

By means of the causal two-point function \(G_F(p)\), we can also obtain the retarded and advanced two-point functions \(G_r(p)\) and \(G_a(p)\). In fact, if starting from the imaginary-time formalism, then we may define [12]
\[
G^I_F(p) = -iG(-i\omega_m \to p^0 + i\varepsilon, \bar{p}),
\]
\[
G^I_r(p) = -iG(-i\omega_m \to p^0 + i\varepsilon, \bar{p}),
\]
\[
G^I_a(p) = -iG(-i\omega_m \to p^0 - i\varepsilon, \bar{p}).
\] (56)

From Eq. (56) we derive that
\[
G^I_F(p) = \theta(p^0)G^I_F(p) + \theta(-p^0)G^I_F(p),
\]
\[
[G^I_F(p)]^* = -G^I_a(p),
\] (57)
(58)

and then
\[
G^I_r(p) = \theta(p^0)G^I_F(p) - \theta(-p^0)[G^I_F(p)]^*.
\] (59)

Taking \(G^I_F(p)\) to be the \(G_F(p)\) in Eq. (55), we will have
\[
G_r(p) = i\{p^2 - m^2 + i\varepsilon p^0 + \text{Re} \tilde{K}(p) - \tilde{H}(p) + i\eta(p^0)[\hat{S}(p) + \text{Im} \tilde{K}(p)]\}
\]
\[
= i\{p^2 - m^2 + i\varepsilon p^0 + \text{Re} \tilde{K}(p) - \tilde{H}(p) + i \sinh(\beta|p^0|/2)|R(p)\},
\]
\[
G_a(p) = -[G_r(p)]^*,
\] (60)

which are respectively the expressions for retarded and advanced two-point functions.

## 5 Conclusions

By direct calculations, we have proven the equivalence of the whole two-point function to one-loop order including the causal, retarded and advanced one in the imaginary-time and the real-time formalism of thermal \(\lambda\phi^4\) field theory and all the results are expressed in the real-time formalism. We find that the key-point of completing the proof is carefully to consider and separate the imaginary part of the zero-temperature loop integral. The similar discussions can be generalized to the other theories e.g. \(\lambda\phi^4\) field theory, QED and the calculations to higher loop orders. In particular, it will be very useful for us to reexamine the equivalence problem of the propagators for scalar bound states in a NJL model in the two formalisms of thermal field theory.
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