Smooth Manifold Structure for Extreme Channels

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A quantum channel from a system $A$ of dimension $d_A$ to a system $B$ of dimension $d_B$ is a completely positive trace-preserving map from complex $d_A \times d_A$ to $d_B \times d_B$ matrices, and the set of all such maps with Kraus rank $r$ has the structure of a smooth manifold. We describe this set in two ways. First, as a quotient space of (a subset of) the $rd_B \times d_A$ dimensional Stiefel manifold. Secondly, as the set of all Choi-states of a fixed rank $r$. These two descriptions are topologically equivalent. This allows us to show that the set of all Choi-states corresponding to extreme channels from system $A$ to system $B$ of a fixed Kraus rank $r$ is a smooth submanifold of dimension $2rd_A - d_A^2 - r^2$ of the set of all Choi-states of rank $r$. As an application, we derive a lower bound on the number of parameters required for a quantum circuit topology to be able to approximate all extreme channels from $A$ to $B$ arbitrarily well.

I. INTRODUCTION

We describe the differential structure of the set $\mathcal{E}_{s,t,r}$ consisting of all completely positive trace-preserving (CPTP) maps from $\mathbb{C}^{s \times s}$ to $\mathbb{C}^{t \times t}$ (which we refer to as $s$ to $t$ channels) of fixed Kraus rank $r$. A linear map $\mathcal{E} : \mathbb{C}^{s \times s} \to \mathbb{C}^{t \times t}$ is called positive if it sends positive semi-definite matrices to positive semi-definite matrices. It is called completely positive (CP) if $\mathcal{E} \otimes I_p$ is positive for all $p \in \mathbb{N}$, where $I_p : \mathbb{C}^{p \times p} \to \mathbb{C}^{p \times p}$ denotes the identity channel. Choi[1] showed that a map $\mathcal{E}$ is completely positive if and only if it admits an expression $\mathcal{E}(X) = \sum_{i=1}^{r'} A_i X A_i^*$ (for all $X \in \mathbb{C}^{s \times s}$), where the $A_i \in \mathbb{C}^{t \times s}$ are called Kraus operators in quantum information theory[2]. The Kraus representation is not unique in general and the minimum number of Kraus operators $r'$, such that a representation of the form above exists, is called the Kraus rank $r$ of the map $\mathcal{E}$ (and the corresponding representation is called a ‘minimal’ Kraus representation). Note that, by Remark 4 of[1], a Kraus representation is minimal if and only if the Kraus operators $A_1, A_2, \ldots, A_{r'}$ are linearly independent. Finally, a map $\mathcal{E}$ is called trace preserving if $\text{tr} \mathcal{E}(X) = \text{tr} X$ for all $X \in \mathbb{C}^{s \times s}$, which corresponds to the requirement $\sum_{i=1}^{r'} A_i^* A_i = I$ on the Kraus operators.

CPTP maps are of interest in physics, because they describe the most general evolution a quantum system can undergo. Since the set $\mathcal{E}_{s,t}$ of all $s$ to $t$ quantum channels is convex, one can investigate the decomposition of a quantum channel into a convex combination of extreme channels. In particular, such decompositions can help to implement quantum channels in a cheaper way[3,4]. However, there are open questions about the structure of the (closure of the) set of extreme channels and finding convex decompositions into such channels. In particular, a tight bound on the number of generalized extreme channels, i.e., channels which lie in the closure of the set of all extreme channels, required for such a convex decomposition is not known[5]. The set of extreme channels has been described by Friedland and Loewy[8] using the framework of semi-algebraic geometry. In contrast to his work, we consider the set of extreme channels in the framework of differential geometry.

The paper is structured as follows. First we give an overview of the notation used in the paper. In Section III we describe the smooth manifold structure of the set $\mathcal{E}_{s,t,r}$ of $s$ to $t$ extreme channels of a fixed Kraus rank $r$: First, we adapt the characterization of unital extreme channels given by Choi[1] to trace-preserving channels in Section IIIA. Then, in Section III-B we describe the set of channels and extreme channels with the smooth structure induced by the standard smooth structure on the Kraus operators. In this picture, we find that $\mathcal{E}^{e}_{s,t,r} \subset \mathcal{E}_{s,t,r}$ is an open subset and hence a smooth submanifold. In Section III-C, we transfer this topological property (found in the Kraus representation picture) to the Choi-state picture, which will show that $\mathcal{E}^{e}_{s,t,r}$ can be considered as a smooth submanifold of the set of all Choi-states of fixed rank $r$. In Section IV, we give a rigorous proof of the known fact[5] that every channel can be decomposed into a finite convex combination of extreme channels. Finally, we look at an application to quantum information theory in Section V where we derive a lower bound on the number of parameters required for a quantum circuit topology for extreme channels, which we have used in[3].

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1 A channel $\mathcal{E}$ is unital if $\mathcal{E}(I) = I$, where $I$ denotes the identity.
II. NOTATION AND BACKGROUND

A. Notation

We use the notation $[A, B] \in \mathbb{C}^{t \times (s_1 + s_2)}$ to denote the (horizontal) concatenation of two matrices $A \in \mathbb{C}^{t \times s_1}$ and $B \in \mathbb{C}^{t \times s_2}$, i.e., the first $s_1$ columns of $[A, B]$ correspond to the columns of $A$ and the $(s_1 + 1)$th column to the $(s_1 + s_2)$th column of $B$. And we denote the vertical concatenation of the matrices $AT$ and $B^T$ by $[AT; B^T] = [A, B]^T$. For arbitrary $s, t, r \in \mathbb{N}$, we define:

- $\mathbb{C}^{s \times t}$: Complex $s \times t$ matrices
- $H_s$: Hermitian $s \times s$ matrices
- $H_+^{s \times r}$: Elements in $H_{s \times r}$ of rank $r$
- $V_{s,t}$: Set of all $V \in \mathbb{C}^{t \times s}$ s.t. $V^*V = I$ (i.e., set of all isometries from an $s$ to a $t$ dimensional system)
- $V_{s,t,r}$: Set of all $V = [A_1; A_2; \ldots; A_r] \in V_{s,t}$, such that the elements in $\{A_i\}_{i \in \{1,2,\ldots,r\}} \in \mathbb{C}^{t \times s}$ are linearly independent (over $\mathbb{C}$)
- $U(s) = V_{s,s}$: Unitary $s \times s$ matrices
- $E_{s,t}$: CPTP (C-linear) maps from $\mathbb{C}^{s \times s}$ to $\mathbb{C}^{t \times t}$
- $E_{s,t}^e$: CP and unital (C-linear) maps from $\mathbb{C}^{s \times s}$ to $\mathbb{C}^{t \times t}$
- $E_{s,t}^e$: Elements in $E_{s,t}$ that are extreme
- $E_{s,t,r}^e$: Elements in $E_{s,t}$ of Kraus rank $r$
- $E_{s,t,r}^e < r = \bigcup_{j=1}^r E_{s,t,j}^e$: Elements in $E_{s,t}$ with Kraus rank at most $r$
- $E_{s,t,r}^e = E_{s,t}^e \cap E_{s,t,r}$: Elements in $E_{s,t,r}$ that are extreme in $E_{s,t}$
- $C_{s,t}$: Set of all Choi-states corresponding to channels from an $s$ dimensional system $A$ to a $t$ dimensional system $B$, i.e., $C_{AB} \in H_{s \times t}$, such that $\text{tr}_B C_{AB} = \frac{1}{2} I$
- $C^e_{s,t}$: Elements in $C_{s,t}$ that are extreme
- $C_{s,t,r}$: Elements in $C_{s,t}$ with rank $r$
- $C_{s,t,r} = \bigcup_{j=1}^r C_{s,t,j}$: Elements in $C_{s,t}$ with rank at most $r$
- $C^e_{s,t,r}$: Elements in $C_{s,t,r}$ that are extreme in $C_{s,t}$

B. Restriction of the Domain or Image of a Smooth Map

The following statements of [3] give sufficient conditions such that a function whose domain or image is restricted is still smooth.

**Proposition 1** (Theorem 5.27 of [3]). Let $M$ and $N$ be smooth manifolds (with or without boundary). If $F : M \mapsto N$ is a smooth map and $D \subset M$ is an (immersed or embedded) submanifold, then $F|_D : D \mapsto N$ is smooth.

**Proposition 2** (Corollary 5.30 of [3]). Let $M$ and $N$ be smooth manifolds, and $S \subset N$ be an embedded submanifold. Then any smooth map $F : M \mapsto N$ whose image is contained in $S$ is also smooth as a map from $M$ to $S$.

III. SMOOTH MANIFOLD STRUCTURE FOR EXTREME CHANNELS

A. Characterization of Extreme Channels

We want to characterize the set of extreme points $E_{s,t}^e \subset E_{s,t}$. For this purpose we have to slightly modify Theorem 5 of [1]. This modification was also considered in [2].

**Theorem 3** (Characterization of extreme channels). Let $E \in E_{s,t}$ with minimal Kraus representation $E(X) = \sum_{i=1}^r A_i X A_i^*$. Then $E$ is extreme in $E_{s,t}$ if and only if all elements of the set $\{A_i^* A_j\}_{i,j \in \{1,2,\ldots,r\}}$ are linearly independent.

**Remark 1.** If the Kraus rank $r$ of the channel $E$ in Theorem 3 is bigger than $s$, then $E$ cannot be extreme, since in this case $|\{A_i^* A_j\}_{i,j \in \{1,2,\ldots,r\}}| < \dim(C^{s \times s})$.

**Proposition 4.** There exists a bijection $\Psi : E_{s,t} \mapsto E_{s,t}^e$ that sends extreme points of $E_{s,t}$ to extreme points of $E_{s,t}^e$ and vice versa.

**Proof.** Note first that $C^{t \times s}$ together with the Frobenius inner product is a (finite dimensional) Hilbert space for $l \in \mathbb{N}$. Let $E \in E_{s,t}$. Then $E$ is $\mathbb{C}$-linear by definition and bounded. By the Fréchet-Riesz representation theorem there exists an injective map $\Psi(E) = E^* : E_{s,t} \rightarrow B(C^{s \times s}, C^{s \times s})$, where $B(C^{s \times s}, C^{s \times s})$ denotes the set of linear bounded operators from $C^{s \times s}$ to $C^{s \times s}$ and $E^*$ denotes the adjoint map of $E$, i.e., for all $C \in C^{s \times s}$ and $D \in C^{t \times s}$ we have $(E(C), D) = (C, E^*(D))$. Let $\{A_i\}_{i \in \{1,2,\ldots,r\}}$ be the Kraus operators of $E$. By a direct computation one finds that the Kraus operators of $E^*$ are $\{A_i^*\}_{i \in \{1,2,\ldots,r\}}$, and therefore, $E^* \in E_{s,t}^e$. Since $(E^*)^* = E$, we can set $C = I$ in the adjoint property above to see that $\Psi^{-1}$ sends unital maps to trace-preserving maps. Therefore the map $\Psi : E_{s,t} \mapsto E_{s,t}^e$ is a bijection.

Assume that $E \in E_{s,t}$ is not extreme, i.e., there exist $E_1, E_2 \in E_{s,t}$, $E_1 \neq E$, $E_2 \neq E$ and $p \in (0,1)$ s.t.
Theorem 5 of [1] shows that $E^* = pE_1 + (1-p)E_2$. By the linearity of the adjoint map, $E^* = pE_1^* + (1-p)E_2^*$, which shows that not extremal points of $E_{r,t}$ are mapped to not extremal points of $E_{r,t}^*$. The reverse direction follows analogously.

Proof of Theorem 5

Theorem 5 of [1] shows that $E^* \in E_{r,t}$ (with linearly independent Kraus operators $A_i \in C^{s \times 1}$) is extreme if and only if the elements in $\{A_i A_j^*\}_{i,j \in \{1,\ldots,r\}}$ are linearly independent. By Proposition 3, this leads to a characterization of the extreme points in $E_{r,t}$.

A group $G$ acts freely on a set $S$ if the only element of $G$ that fixes any element of $S$ is the identity, i.e., for all $p \in S$ and $g \in G$, $g \cdot p = p$ implies $g = 1$.

Definition 2. Let $G$ be a Lie group that acts continuously on a manifold $M$. The action is said to be proper if the map $G \times M \rightarrow M \times M$ given by $(g, p) \mapsto (g \cdot p, p)$ is a proper map, i.e., the preimage of a compact set is compact.

The following Proposition gives a sufficient condition for a group action to be proper.

Proposition 7 (Corollary 21.6 of [7]). Any continuous action by a compact Lie group on a manifold is proper.

Theorem 8 (Quotient Manifold Theorem [7]). Suppose a Lie group $G$ acts smoothly, freely, and properly on a smooth manifold $M$. Then the orbit space $M/G$ is a topological manifold of dimension equal to $\dim(M) - \dim(G)$, and has a unique smooth structure with the property that the quotient map $\pi: M \rightarrow M/G$ is a smooth submersion.

Lemma 9 (Lemma 21.1 of [7]). For any continuous action of a topological group $G$ on a topological space $M$, the quotient map $\pi: M \rightarrow M/G$ is open.

Proposition 10. The Lie group $U(r)$ acts smoothly, freely and properly on the manifold $V_{s,t,r}$ by the action $U \cdot V = (U \otimes I)V$, where $U \in U(r), V \in V_{s,t,r}$ and $I$ denotes the $t \times t$ identity matrix.

Proof. We first show that $U \cdot V \in V_{s,t,r}$ for all $U \in U(r)$ and $V \in V_{s,t,r}$. Note that $V^*(U^* \otimes I)(U \otimes I)V = I$ and that the linear independence of the matrices $A_i$ (where $V = [A_1; A_2; \ldots; A_r]$) is preserved under the unitary action: Assume $i \alpha_i \left(\sum_{j=1}^r (U)_{ij} A_j \right) = 0$ for some coefficients $\alpha_i \in \mathbb{C}$. This is equivalent to $\sum_{j=1}^r (\sum_{i=1}^r \alpha_i(U)_{ij}) A_j = 0$ which implies $\sum_{j=1}^r \alpha_i(U)_{ij} = 0$ for all $i \in \{1,2,\ldots,r\}$, since the $A_j$ are linearly independent. Since $U$ is unitary, this implies $\alpha_i = 0$ for all $i \in \{1,2,\ldots,r\}$. We conclude that the group action is well defined. To show that the action is free, choose a $V \in V_{s,t,r}$ and a $U \in U(r)$ and assume that $U \cdot V = V$. Writing $V = [A_1; A_2; \ldots; A_r]$, the last equation becomes $\sum_{j=1}^r (U)_{ij} A_j = A_i$ for all $i \in \{1,2,\ldots,r\}$. Since $V \in V_{s,t,r}$, the $A_j$ are linearly independent and we conclude that $(U)_{ij} = \delta_{ij}$ or equivalently $U = I$. To see that the action is smooth, consider the map $F(U, V) = (U \otimes I)V: C^{t \times r} \times C^{s \times t} \rightarrow C^{t \times s}$. 2

3 An $r \times r$ minor of a matrix $D$ is the determinant of an $r \times r$ sub-matrix of $D$ formed by ‘deleting’ certain rows (or columns).
Since taking tensor products is a smooth operation, the map $F$ is smooth. By Propositions 3 and 4 and Lemma 5, the map $F(U, V) = (U \otimes I)V$ is smooth. Then, by Propositions 2 and 4 and Lemma 6, the map $F'(U, V) = (U \otimes I)V$ is also smooth. Since the Lie group $U(r)$ is compact, the action is proper by Proposition 7.

**Definition 3.** We define the equivalence relation $\sim$ as follows: Let $V_1, V_2 \in V_{s,t,r}$. Then $V_1 \sim V_2$ if there exists a $U \in U(r)$, such that $U \cdot V_1 = V_2$. The orbit space is $V_{s,t,r}/U(r) := \{[V] : V \in V_{s,t,r}\}$ (together with the quotient topology).

**Lemma 11.** The orbit space $V_{s,t,r}/U(r)$ is a topological manifold of dimension equal to $\dim(V_{s,t,r}) - \dim(U(r)) = 2sr - s^2 - r^2$ with a unique smooth structure such that the quotient map $\pi: V_{s,t,r} \to V_{s,t,r}/U(r)$ is a smooth submersion. Moreover, $\pi$ is an open map.

**Proof.** The first part of the theorem follows from Theorem 5 where the assumption for the theorem are satisfied because of Proposition 10. The quotient map $\pi$ is open by Lemma 9.

**Lemma 12.** There is a one-to-one correspondence between the set $E_{s,t,r}$ of channels of Kraus rank $r$ and the orbit space $V_{s,t,r}/U(r)$.

**Proof.** We define the quotient map $\pi(V) = [V]: V_{s,t,r} \to V_{s,t,r}/U(r)$ and the map $\psi: E_{s,t,r} \to V_{s,t,r}/U(r)$, by sending a channel $E \in E_{s,t,r}$ with (linearly independent) Kraus operators $\{A_i\}_{i \in \{1,2,...,r\}}$ to $\pi([A_1; A_2; \ldots; A_r])$. To show that this map is well defined, we must show that it is independent of the choice of the Kraus operators. By Remark 4 of 3, two Kraus representations $\{A_i\}_{i \in \{1,2,...,r\}}$ and $\{B_i\}_{i \in \{1,2,...,r\}}$ describe the same channel $E \in E_{s,t,r}$ if and only if there exist a unitary $U \in U(r)$, such that $B_i = \sum_{j=1}^r U_i A_j$ for all $j \in \{1,2,...,r\}$ or equivalently $V_B = (U \otimes I)V_A$, where $V_A = [A_1; A_2; \ldots; A_r] \in V_{s,t,r}$ and $V_B = [B_1; B_2; \ldots; B_r] \in V_{s,t,r}$. Therefore, the two Kraus representations describe the same channel if and only if $V_A \sim V_B$. We conclude that the map $\psi$ is well defined and injective. On the other hand, for all $W \in V_{s,t,r}/U(r)$, we can define a channel $\hat{E} \in E_{s,t,r}$ with $\psi(\hat{E}) = W$ by choosing a representative element $V \in \pi^{-1}(W)$, breaking it into blocks $V = [A_1; A_2; \ldots; A_r]$ and treating those as the channel's Kraus operators. This shows that $\psi$ is also surjective.

We are now ready to study the structure of the set $E_{s,t,r}$ of extreme channels. In the following we show that we can identify $E_{s,t,r}$ with a smooth manifold.

**Proposition 13.** The set $\hat{O} := \{[A_1; A_2; \ldots; A_r] \in V_{s,t,r} : \{A_i A_j\}_{i,j \in \{1,2,...,r\}}$ are linearly independent$\}$ is an open subset of the manifold $V_{s,t,r}$.

**Proof.** Works analogously to the proof of Proposition 6.

**Lemma 14** (Manifold structure for $E_{s,t,r}$). Let $O := \pi(\hat{O}) \subseteq V_{s,t,r}/U(r)$, where $\pi$ is the quotient map of Lemma 11. $O$ is a smooth manifold of dimension $2sr - s^2 - r^2$ and there is a one-to-one correspondence between the set $E_{s,t,r}$ of extreme channels of Kraus rank $r$ and $O$. Moreover, $E_{s,t,r}^e \neq \emptyset$.

**Proof.** Since $\pi$ is an open map and $\hat{O}$ is an open subset of $V_{s,t,r}/U(r)$, $O$ is an open subset of the orbit space $V_{s,t,r}/U(r)$. Together with Lemma 12 and Theorem 3 this implies the first part of the Lemma. For the second part, we borrow an argument from 2. Consider a channel $E$ with Kraus operators $A_i = |\psi_i\rangle\langle i|$ for $i \in \{1,2,\ldots,s\}$, where $|\psi_i\rangle \in \mathbb{C}^s$ is of unit length and $|i\rangle \in \mathbb{C}^s$ denotes the $i$th standard basis vector. Note that $\sum_{i=1}^s A_i^* A_i = I$ and that the elements in the set $\{A_i^* A_j\}_{i,j \in \{1,2,\ldots,s\}} = \{|i\rangle\langle i|\}_{i,j \in \{1,2,\ldots,s\}}$ are linearly independent. By Theorem 5 we have $E \in E_{s,t,r}^e$ and therefore $E_{s,t,r}^e \neq \emptyset$. 

Note that the above shows that the channel corresponding to the operation of discarding a system (tracing out) and then generating a new pure state is extremal.

**C. Structure of the Set of Extreme Channels in the Choi-State Representation**

We found a smooth description of the set of extreme channels $E_{s,t,r}$ in Section 11.13. This will allow us to transfer the characterization of extreme channels to the Choi-state representation.

**Lemma 15** (Manifold structure for $H^e_{s,t}$. The set $H^e_{s,t}$ is a smooth embedded submanifold of $\mathbb{R}^{5s^2}$ of dimension $2sr - r^2$.

The `real case' of Lemma 15 was shown in 9 (cf. also 10), where they considered the manifold of real symmetrical $s \times s$ matrices of rank $r$. Our proof is a straightforward generalization of the proof given in 9 to the complex case. We begin with some preparatory results.

**Definition 4.** We define $E_r$ to be a square matrix, whose first $r$ diagonal entries are equal to one and all the other entries are equal to zero.

**Proposition 16.** We have $H^e_{s,t} = \{AE_r A^* : A \in GL(C,s)\}$.

**Proof.** The inclusion “$\subseteq$” is clear. To see the inclusion “$\supseteq$”, let $H \in H^e_{s,t}$, By the spectral theorem there exists

\[\begin{align*}
A &= \begin{bmatrix}
A_{11} & \cdots & A_{1s} \\
\vdots & \ddots & \vdots \\
A_{s1} & \cdots & A_{ss}
\end{bmatrix} \\
E_r &= \begin{bmatrix}1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}
\end{align*}\]

4 Note that $\langle \phi \rangle \in \mathbb{C}^{1 \times d}$ denotes the conjugate transpose of a $d$-dimensional vector $|\phi\rangle \in \mathbb{C}^{d \times 1}$.

5 The dimension of the matrix will always be clear from the context.
a $U \in U(s)$ such that $H = UDU^*$, where $D$ is a $s \times s$ matrix with positive diagonal entries $d_1, d_2, \ldots, d_r$ and zeros elsewhere. We define $D$ as the matrix $D$ where we replace the zeros on the diagonal by ones. Then we have $D = \sqrt{DEr} \sqrt{D}$, and hence $H = U\sqrt{DEr} \sqrt{DU^*} = AE_r A^*$, where we set $A = U \sqrt{D} \in \text{GL}(C, s)$.

A sufficient condition for orbits of Lie group actions to be smooth manifolds was given in [11].

**Definition 5.** A map $f : D \to \mathbb{R}^l$ with $D \subset \mathbb{R}^s$ is semialgebraic if the graph of $f$ is semialgebraic in $\mathbb{R}^s \times \mathbb{R}^l$.

**Theorem 17 (Theorem B4 of Appendix B of [11]).** Let $\Phi : G \times S \to S$ be a smooth action of a Lie group $G$ on a smooth manifold $S$. And suppose that the action is semialgebraic. Then all the orbits are smooth embedded submanifolds of $S$.

**Proof of Lemma 17 (part 1).** We define the map $\Phi(A, H) = AHA^* : \text{GL}(C, s) \times C^{s \times s} \to C^{s \times s}$. Note that $\Phi$ describes a smooth action of the Lie group $\text{GL}(C, s)$ on the smooth manifold $C^{s \times s}$. Moreover, $\Phi$ is a semialgebraic map: The (complex) graph of $\Phi$ corresponds to the set $(A, H, H) \in (C^{s \times s})^3 : \det(A) \neq 0$ and $AH A^* - H = 0$. We can embed the complex space $(C^{s \times s})^3$ into $(\mathbb{R}^{2s^2})^3$ and rewrite the conditions as real polynomial equations. By Theorem 17 and Proposition 16 we conclude that the orbit $H_{s,t,r} = \{AE_r A^* : A \in \text{GL}(C, s)\}$ is a smooth embedded submanifold of $\mathbb{R}^{2s^2} \cong \mathbb{R}^{2s^2}$. To determine the dimension of this manifold, we need an additional result.

**Proposition 18.** Let $p \in H_{s,t,r}^+$ and write $p = A_p E_r A_p^*$ for some $A_p \in \text{GL}(C, s)$. Then the tangent space at $p$ is given by $T_p H_{s,t,r}^+ = \{\Delta E_r A_p^* + A_p E_r \Delta^* : \Delta \in C^{s \times s}\}$.

**Proof.** We define the map $\phi(A) = AE_r A^* : \text{GL}(C, s) \to H_{s,t,r}^+$ (where we used Proposition 16 to determine the image space). Note that the map $\phi$ is smooth, since from the first part of the proof of Lemma 15 the set $H_{s,t,r}^+$ is a smooth embedded submanifold (and hence we can apply Proposition 2 to the smooth map $\phi(A) = AE_r A^* : \text{GL}(C, s) \to C^{s \times s}$). Then the pushforward of $\phi$ at $A_p$ is given by $D\phi_{A_p}(\Delta) = \Delta E_r A_p^* + A_p E_r \Delta^* : T_{A_p} \text{GL}(C, s) \to T_p H_{s,t,r}^+$. The inclusion “$\supseteq$” of the claim in Proposition 18 follows, because $T_{A_p} \text{GL}(C, s) \cong C^{s \times s}$. To see the inclusion “$\subseteq$”, we show that $\phi$ has constant rank. To see this, note that the pushforward of $\phi$ at an arbitrary $A \in \text{GL}(C, s)$ is related to the pushforward at the identity in the following way $D\phi_A(\Delta A) = \Delta AE_r A^* + AE_r A^* \Delta^* = A \left( A^{-1} \Delta AE_r + E_r A^* \Delta^* (A^*)^{-1} \right)$.

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**Proof of Lemma 17 (part 1).** We define the map $\Phi(A, H) = AHA^* : \text{GL}(C, s) \times C^{s \times s} \to C^{s \times s}$. Note that $\Phi$ describes a smooth action of the Lie group $\text{GL}(C, s)$ on the smooth manifold $C^{s \times s}$. Moreover, $\Phi$ is a semialgebraic map: The (complex) graph of $\Phi$ corresponds to the set $(A, H, H) \in (C^{s \times s})^3 : \det(A) \neq 0$ and $AH A^* - H = 0$. We can embed the complex space $(C^{s \times s})^3$ into $(\mathbb{R}^{2s^2})^3$ and rewrite the conditions as real polynomial equations. By Theorem 17 and Proposition 16 we conclude that the orbit $H_{s,t,r} = \{AE_r A^* : A \in \text{GL}(C, s)\}$ is a smooth embedded submanifold of $\mathbb{R}^{2s^2} \cong \mathbb{R}^{2s^2}$. To determine the dimension of this manifold, we need an additional result.

**Proposition 18.** Let $p \in H_{s,t,r}^+$ and write $p = A_p E_r A_p^*$ for some $A_p \in \text{GL}(C, s)$. Then the tangent space at $p$ is given by $T_p H_{s,t,r}^+ = \{\Delta E_r A_p^* + A_p E_r \Delta^* : \Delta \in C^{s \times s}\}$.

**Proof.** We define the map $\phi(A) = AE_r A^* : \text{GL}(C, s) \to H_{s,t,r}^+$ (where we used Proposition 16 to determine the image space). Note that the map $\phi$ is smooth, since from the first part of the proof of Lemma 15 the set $H_{s,t,r}^+$ is a smooth embedded submanifold (and hence we can apply Proposition 2 to the smooth map $\phi(A) = AE_r A^* : \text{GL}(C, s) \to C^{s \times s}$). Then the pushforward of $\phi$ at $A_p$ is given by $D\phi_{A_p}(\Delta) = \Delta E_r A_p^* + A_p E_r \Delta^* : T_{A_p} \text{GL}(C, s) \to T_p H_{s,t,r}^+$. The inclusion “$\supseteq$” of the claim in Proposition 18 follows, because $T_{A_p} \text{GL}(C, s) \cong C^{s \times s}$. To see the inclusion “$\subseteq$”, we show that $\phi$ has constant rank. To see this, note that the pushforward of $\phi$ at an arbitrary $A \in \text{GL}(C, s)$ is related to the pushforward at the identity in the following way $D\phi_A(\Delta A) = \Delta AE_r A^* + AE_r A^* \Delta^* = A \left( A^{-1} \Delta AE_r + E_r A^* \Delta^* (A^*)^{-1} \right)$.

**Note:** The “smooth manifolds” in Theorem B4 of Appendix B of [11] correspond to a “smooth embedded submanifold” in our terminology.
To describe the set of extreme channels in the Choi-state representation, we transfer the description of Lemma 14 to $\mathcal{C}_{s,t,r}$.

**Definition 6.** We use $|\gamma\rangle_{A'\Lambda} = \sum_{i} |i\rangle_{A'} \otimes |i\rangle_{\Lambda} \in \mathbb{C}^{2}$ to denote the (unnormalized) maximally entangled state between the $s$-dimensional system $A$ and a copy of this system, denoted by $A'$. We define the Choi map $\Gamma(\mathcal{E}) = \mathcal{I}_{A'} \otimes \mathcal{E}(|\gamma\rangle_{A'\Lambda} \langle \gamma|_{A'\Lambda}) : \mathcal{E}_{s,t} \mapsto \mathbb{C}^{s \times t}$, where $\mathcal{I}_{A'}$ is the identity map on $A'$. This map sends a channel to its Choi-state.

**Definition 7 (Definition of the map $T$).** Let $V_{A\rightarrow CB} \in \mathbb{C}^{r \times s}$, where the systems $A$, $B$ and $C$ have the (complex) dimensions $s$, $t$ and $r$ respectively. We define the channel $\mathcal{E}_{V_{A\rightarrow CB}}(MA) = tr_{C} V_{A\rightarrow CB} M A V_{A\rightarrow CB}^{*} : \mathbb{C}^{s \times s} \mapsto \mathbb{C}^{t \times t}$. This allows us to define the smooth map $T(V_{A\rightarrow CB}) = \Gamma(\mathcal{E}_{V_{A\rightarrow CB}}) : \mathbb{C}^{r \times s} \mapsto \mathbb{C}^{s t \times t s}$.

Note that the map $T$ sends a Stinespring dilation $V \in V_{s,rt}$ of a channel $\mathcal{E} \in \mathcal{E}_{s,rt}$ to the Choi-state representation of $\mathcal{E}$.

**Lemma 20.** The manifolds $V_{s,t,r}/U(r)$ and $\mathcal{C}_{s,t,r}$ are homeomorphic.

**Proof.** Since the Kraus rank of a channel is equal to the rank of the corresponding Choi-state, we can consider the map $T$ as a map from $V_{s,t,r}$ to $\mathcal{C}_{s,t,r}$. This map (which we still denote by $T$) is smooth by Proposition 14 and 15. Let $\pi : V_{s,t,r} \mapsto V_{s,t,r}/U(r)$ denote the quotient map introduced in Lemma 16. For all $U \in U(r)$ we have $T((U \otimes I) V_{A\rightarrow CB}) = T(V_{A\rightarrow CB})$, because the unitary action corresponds to a change of the basis of the system $C$, which is traced out. In other words, the map $T$ is constant on the fibers of the quotient map $\pi$. By Theorem 4.30 of [2], there is a unique smooth map $\phi : V_{s,t,r}/U(r) \mapsto \mathcal{C}_{s,t,r}$, such that the following diagram commutes.

$$
\begin{array}{ccc}
V_{s,t,r} & \xrightarrow{\pi} & V_{s,t,r}/U(r) \\
& \searrow \phi & \swarrow T \\
& & \mathcal{C}_{s,t,r}
\end{array}
$$

By Lemma 17 we have a one-to-one correspondence between $\mathcal{E}_{s,t,r}$ and $V_{s,t,r}/U(r)$ (which we denote by $\mathcal{E}_{s,t,r} \leftrightarrow V_{s,t,r}/U(r)$) and by the Choi-Jamiolkowski isomorphism we have $\mathcal{E}_{s,t,r} \leftrightarrow \mathcal{C}_{s,t,r}$. Together, this implies that $\phi$ is a bijection.

We have left to show that $\phi^{-1}$ is continuous. We would like to use the fact that a bijective continuous map form a compact space to a Hausdorff space has a continuous inverse (cf. Lemma A.52 of [2]). To make our domain $V_{s,t,r}/U(r)$ compact, we enlarge it to $V_{s,rt}/U(r)$. Let $\tilde{\pi} : V_{s,rt} \mapsto V_{s,rt}/U(r)$ denote the quotient map. Since $\tilde{\pi}$ is continuous and the Stiefel manifold $V_{s,rt}$ is compact, $\tilde{\pi}(V_{s,rt}) = V_{s,rt}/U(r)$ is compact. We enlarge the domain of the map $T$ and denote this map by $\tilde{T} : V_{s,rt} \mapsto \mathcal{C}_{s,t,rt}$. Note that $\tilde{T}$ is continuous. Since $\tilde{T}$ is constant on the fibers of $\tilde{\pi}$, we can define a map $\psi : V_{s,rt}/U(r) \mapsto \mathcal{C}_{s,t,rt}$, such that the following diagram commutes.

$$
\begin{array}{ccc}
V_{s,rt} & \xrightarrow{\tilde{\pi}} & \mathcal{C}_{s,t,rt} \\
& \searrow \psi & \swarrow \tilde{T} \\
V_{s,rt}/U(r) & \xrightarrow{\tilde{\pi}} & \mathcal{C}_{s,t,rt}
\end{array}
$$

Note that $\psi$ is a bijection, because $\mathcal{E}_{s,t,rt} \leftrightarrow V_{s,rt}/U(r)$ by Remark 4 of [1], and $\mathcal{E}_{s,t,rt} \leftrightarrow \mathcal{C}_{s,t,rt}$ by the Choi-Jamiolkowski isomorphism. The map $\psi$ is continuous, because $\tilde{T}$ is continuous (and by the definition of the quotient topology). Since $V_{s,rt}/U(r)$ is compact and $\mathcal{C}_{s,t,rt}$ is Hausdorff, $\psi^{-1}$ is continuous.

To see that $\phi^{-1}$ is continuous we restrict $\psi^{-1}$ to $\mathcal{C}_{s,t,rt}$. For this purposes, we define the inclusion map $\iota(V) : V_{s,rt} \mapsto V_{s,rt}$. Note that $\iota$ is continuous and open (because $V_{s,rt}$ is an open subset of $V_{s,rt}$ by Proposition 9). Since the map $\tilde{\pi} \circ \iota$ is constant on the fibers of $\pi$, we can define a map $\bar{\iota}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
V_{s,rt} & \xrightarrow{\iota} & V_{s,rt} \\
& \searrow \tilde{\pi} \circ \iota & \swarrow \tilde{\pi} \\
V_{s,rt}/U(r) & \xrightarrow{\iota} & V_{s,rt}/U(r)
\end{array}
$$

By Lemma 9 $\tilde{\pi}$ is an open map. Then $\tilde{\pi} \circ \iota$ is an open and continuous map and hence, we can conclude that $\bar{\iota}$ is continuous and open (and injective). We are now ready to show that $\phi^{-1}$ is continuous. Note that the restriction $\tilde{\psi}^{-1} : \mathcal{C}_{s,t,rt} \mapsto V_{s,rt}/U(r)$ of $\psi^{-1}$ is still continuous. Because $\bar{\iota}$ is injective and $\psi^{-1}(\mathcal{C}_{s,t,rt}) = \bar{\iota}(V_{s,rt}/U(r))$ we can define a map $\chi : \mathcal{C}_{s,t,rt} \mapsto V_{s,rt}/U(r)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{C}_{s,t,rt} & \xrightarrow{\tilde{\psi}^{-1}} & V_{s,rt}/U(r) \\
& \searrow \iota & \swarrow \bar{\iota} \\
V_{s,rt}/U(r) & \xrightarrow{\bar{\iota}} & V_{s,rt}/U(r)
\end{array}
$$

The map $\chi$ is continuous because $\bar{\iota}$ is open and note that $\chi = \phi^{-1}$. 

**Theorem 21 (Manifold structure for $\mathcal{C}_{s,t,rt}$).** The set $\mathcal{C}_{s,t,rt}$ is an open subset of $\mathcal{C}_{s,t,rt}$. In particular it is a smooth embedded submanifold of $\mathcal{C}_{s,t,rt}$ (and of $\mathbb{R}^{2srt}$). Its dimension is $2srt - r^2 - s^2$. Moreover, $\mathcal{C}_{s,t,rt} \neq \emptyset$. 

---

7 We can also think of the unitary action as exploiting the unitary freedom on the Kraus representation, i.e., the channel itself is unchanged under the unitary action.

8 The action of the Lie group $U(r)$ on $V_{s,rt}$ is not free in general.
IV. DECOMPOSITION OF CHANNELS INTO CONVEX COMBINATIONS OF EXTREME CHANNELS

We show that every element $\mathcal{E} \in \mathcal{E}_{s,t}$ can be decomposed into a convex combination of at most $2s^2t^2 + 1$ extreme channels in $\mathcal{E}_{s,t}^e$.

Theorem 22 (Convex decomposition). For every channel $\mathcal{E} \in \mathcal{E}_{s,t}$ there exists a set $\{(p_j, \xi_j)\}_{j=1,2,\ldots,k}$, where $k \leq 2s^2t^2 + 1$, $p_j \in [0,1]$, $\sum_{j=1}^k p_j = 1$ and $\xi_j \in \mathcal{E}_{s,t}^e$, such that $\mathcal{E} = \sum_{j=1}^k p_j \xi_j$.

Remark 2. It is conjectured by Ruskai and Audenaert that $k \leq t$ if we allow convex combinations of channels $\xi_j \in \mathcal{E}_{s,t}^e$ (note that $\mathcal{E}_{s,t}^e$ is equal to the closure of the set of all $s$ to $t$ extreme channels [3]). However, as far as we know, this remains unproven.

Proof of Theorem 22. In the proof of Lemma 20 we saw that $V_{s,t}^2 / U(st)$ is compact and homeomorphic to $C_{s,t} \subseteq C_{s,t,e}$ is compact. Since $C_{s,t} \subset \mathbb{R}^{2s^2t^2}$ is also convex, by the Minkowski Theorem (see for example Theorem 2.3.4 of [12]), $C_{s,t} \subseteq C_{s,t}^e$ is the convex hull of its extreme points. By Carathéodory’s theorem (see for example Theorem 1.36 of [12]), we get that $k \leq 2s^2t^2 + 1$ in the decomposition stated in the claim.

V. APPLICATION: IMPLEMENTATION OF QUANTUM CHANNELS

Methods for implementing quantum channels from a system $A$ to a system $B$ with low experimental cost as a sequence of simple-to-perform operations were considered in [2, 3, 13, 14]. In [2], a lower bound on the number of parameters required for a quantum circuit topology that is able to perform arbitrary extreme channels from $m$ to $n$ qubits (i.e., from a system $A$ of dimension $d_A = 2^m$ to a system $B$ of dimension $d_B = 2^n$) was given. Here, we give a rigorous mathematical proof of this statement and strengthen the result by showing that a circuit topology that has fewer parameters than required by the lower bound is not able to approximate every extreme channel from $m$ to $n$ qubits arbitrarily well.

From a mathematical point of view, a quantum circuit topology can be defined as follows.

Definition 8. A quantum circuit topology is a 5-tuple $Z := (d_A, d_B, d_C, p, h)$, where $d_A, d_B, d_C \in \mathbb{N}$, $d_B d_C \geq d_A$, $p \in \mathbb{N}_0$ and $h : [0, 2\pi]^p \rightarrow V_{d_A, d_B, d_C}$ is a smooth function.

The physical interpretation is the following: We consider a quantum system $BC$ of dimension $d_{BC} := d_B d_C$, where an input state for a quantum channel is given on a subsystem $A$ of dimension $d_A$ and where the other part of the system $BC$ starts in a fixed pure state. We think of a fixed sequence of unitary operations performed on the system $BC$, where the unitaries have $p$ free parameters between them. Since the parameters corresponds to rotational angles in $\mathbb{R}^p$, we take them to lie in the interval $[0, 2\pi]$. Each choice of parameters corresponds to the implementation of a certain isometry from the system $A$ to the system $BC$. The function $h$ maps each choice of the parameters to the corresponding isometry. After performing the isometry, the system $C$ is discarded (traced out), and we read out the output of the channel on the remaining system $B$.

Definition 9. The set of quantum channels (in the Choi-state representation) that can be generated by the quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$ is defined by $H(Z) := T(h([0, 2\pi]^p))$, where the map $T$ was introduced in Definition 7 and where we take the partial trace over the first $d_C$-dimensional system, i.e., the partial trace $tr_C(\cdot)$ corresponds to $\sum_{i=1}^{d_C} (\langle i \mid \cdot \mid i \rangle \otimes I)$.

Lemma 23. Let $r \in \mathbb{N}$ be fixed and $O \subset C_{d_A, d_B, r}$ open (and non empty). A quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$ with $p < \dim(C_{d_A, d_B, r}) = 2d_A d_B r - d_A^2 - r^2$ or $d_C < r$ can only generate a set of measure zero in $O$, i.e., $H(Z) \cap O$ is of measure zero in $O$.

Proof. The idea of the proof is based on Sard’s theorem (similar to [13, 14]). Let us fix a quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$, $r \in \mathbb{N}$ and an open set $O \subset C_{d_A, d_B, r}$. We define the map $T : V_{d_A, d_B} \rightarrow C_{d_A, d_B}$ as in Definition 7 and a map $F = T \circ h$, such that the following diagram commutes.

The following diagram commutes.

\[
\begin{array}{ccc}
V_{d_A, d_B} & \xrightarrow{T} & C_{d_A, d_B} \\
\downarrow F & & \downarrow F \\
[0, 2\pi]^p & \xrightarrow{h} & O
\end{array}
\]

Case 1 ($d_C < r$): In this case, note that $H(Z) = F([0, 2\pi]^p) \subset T(V_{d_A, d_B})$. But $T(V_{d_A, d_B})$ contains only (Choi-states of) channels of Kraus rank at most $d_C < r$. Therefore, $H(Z) \cap O = \emptyset$.

Case 2 ($d_C \geq r$): Define the set $S := F([0, 2\pi]^p) \cap O$. To show that $S$ has measure zero, define the domain $D = F^{-1}(O)$ and the function $\tilde{F} = F|_D : D \rightarrow O$. By Sard’s theorem (see Appendix A) for the full technical
details) we conclude that $S = \tilde{F}(D)$ is of measure zero in the smooth submanifold $O$ if $\dim(D) \leq p < \dim(O) = \dim(C_{s,t,r})$. 

**Theorem 24** (Strong lower bound). Let $q \in \mathbb{N}$ and consider a set of quantum circuit topologies $\mathcal{R} = \{ Z_i = (d_A, d_B, d_{C_i}, p_i, h_i) \}_{i \in \{1,2,\ldots,g\}}$ where for each $i \in \{1,2,\ldots,g\}$ either $p_i < 2d_A^2(d_B - 1)$ or $d_{C_i} < d_A$. Then there exists an extreme channel $\mathcal{E}_0 \subset \mathcal{C}_{d_A, d_B, d_A}$ and a neighborhood $B(\mathcal{E}_0) \subset \mathcal{C}_{d_A, d_B, d_A}$ of $\mathcal{E}_0$, such that for all $\mathcal{E} \in B(\mathcal{E}_0)$ we have $\mathcal{E} \notin \bigcup_{i=1}^q H(Z_i)$.

This theorem considers a finite set of circuit topologies each of which either has fewer free parameters than the dimension of the set of extreme channels or discards a system whose dimension is too low to generate channels of the maximal Kraus rank for any extreme channel. The theorem says that there exist extreme channels that cannot be approximated arbitrarily well using circuit topologies from this set.

**Proof.** By Theorem 21 the set $\mathcal{C}^e_{d_A, d_B, d_A} \neq \emptyset$ is an open subset of $\mathcal{C}_{d_A, d_B, d_A}$. Hence, $H(Z_i) \cap \mathcal{C}^e_{d_A, d_B, d_A}$ is of measure zero in $\mathcal{C}_{d_A, d_B, d_A}$ by Lemma 24. Since a finite union of set of measure zero is again of measure zero, we conclude that $S := \bigcup_{i=1}^q (H(Z_i) \cap \mathcal{C}^e_{d_A, d_B, d_A})$ is of measure zero. Since $\mathcal{C}^e_{d_A, d_B, d_A} \neq \emptyset$, we can choose a channel $\mathcal{E}_0 \subset \mathcal{C}^e_{d_A, d_B, d_A} \cap S^c$ and hence $\mathcal{E}_0 \in H^c$, where we set $H := \bigcup_{i=1}^q H(Z_i)$. We have left to show that $H$ is closed in $\mathcal{C}_{d_A, d_B}$. To see this, note that the map $T_i : \mathcal{V}_{d_A, d_{C_i}} \rightarrow \mathcal{C}_{d_A, d_B}$ (as defined in Definition 4) is continuous and hence $F_i := T_i \circ h_i : [0, 2\pi]^{p_i} \rightarrow \mathcal{C}_{d_A, d_B}$ is also continuous. Since $[0, 2\pi]^{p_i}$ is compact, $H(Z_i) = F_i([0, 2\pi]^{p_i})$ is closed in $\mathcal{C}_{d_A, d_B}$ and therefore $H$ is also closed.

VI. ACKNOWLEDGEMENTS

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**Appendix A: Smoothness of $\tilde{F}$ and a version of Sard’s theorem**

The domain $D$ of the function $\tilde{F}$ in the proof of Lemma 23 might not be open. We therefore first clarify the meaning of smoothness in the case of arbitrary domains.

**Definition 10.** Let $D \subset \mathbb{R}^s$. A function $F : D \rightarrow \mathbb{R}^t$ is smooth if for all $p \in D$ there exists a neighborhood $B(p)$ of $p$ in $\mathbb{R}^s$, such that there exists an extension $\tilde{F} : B(p) \rightarrow \mathbb{R}^t$ of $F$, with $\tilde{F}$ smooth.

**Lemma 25 (Measure of the image).** Let $s < t, D \subset \mathbb{R}^s$ and $F : D \rightarrow \mathbb{R}^t$ be smooth. Then $F(D)$ has measure zero in $\mathbb{R}^t$.

**Proof.** We define $D' := D \times \{0\} \times \{0\} \times \cdots \times \{0\} \subset \mathbb{R}^t$. Note that $D'$ lies in an affine subspace of $\mathbb{R}^t$ and is therefore of measure zero (in $\mathbb{R}^t$). Let $F' := F \circ \pi : D' \rightarrow \mathbb{R}^t$, where $\pi : \mathbb{R}^t \rightarrow \mathbb{R}^s$ denotes the projection map to the first $s$ coordinates. To show that $F'$ is smooth, choose $p' \in D'$ and let $p := \pi(p')$. Because $F$ is smooth by assumption, there exists a neighborhood $B(p) \subset \mathbb{R}^s$ around the point $p$ and a smooth extension of $F$ denoted by $\tilde{F} : B(p) \rightarrow \mathbb{R}^t$. Hence $F' := F \circ \pi = B(p) \times \mathbb{R}^{t-s} \rightarrow \mathbb{R}^t$ is a smooth extension of $F'$ around $p'$. Therefore, $F'$ is a smooth map and by Proposition 6.5 of [6], we conclude that $F'(D') = F(D)$ is of measure zero.

**Lemma 26 (Restrict the range of a smooth map).** Let $D \subset \mathbb{R}^s$ be arbitrary and let $N$ be a smooth manifold. Let $N'$ be a smooth embedded submanifold of $N$ and $F : D \rightarrow N$ be a smooth map, such that $F(D) \subset N'$. Then $\tilde{F} : D \rightarrow N'$ is smooth.

**Proof.** The proof works analogously to the proof of Theorem 5.29 of [4] (see also Corollary 5.30 of [4]).

**Proof: $\tilde{F}(D)$ is of measure zero (completes the proof of case 2 of Lemma 23).** We use the notation of the proof of Lemma 23. First note that the function $F |_D : D \rightarrow \mathcal{C}^{d_A d_B \times d_A d_B}$ is smooth. By Lemma 20 the function $\tilde{F} : D \rightarrow O$ is also smooth. To show that $S := \tilde{F}(D) \subset O$ is of measure zero, we choose a collection of smooth charts $\{(U_{\alpha}, \phi_{\alpha})\}$ of the submanifold $O$ whose domains cover $S$. By Lemma 6.6 of [4] we have left to show that for all $\alpha$ the image $\phi_{\alpha}(S \cap U_{\alpha})$ is of measure zero in $\mathbb{R}^d$, where $d$ denotes the dimension of $O$. Consider the smooth map $\tilde{F}_{\alpha} := \phi_{\alpha} \circ \tilde{F} : D_{\alpha} := \tilde{F}^{-1}(U_{\alpha}) \rightarrow V_{\alpha} := \phi_{\alpha}(U_{\alpha}) \subset \mathbb{R}^d$.

By Lemma 25 the image $\tilde{F}_{\alpha}(D_{\alpha}) = \phi_{\alpha}(S \cap U_{\alpha})$ is of measure zero if $p < d$.

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