Distinguishing fuzzballs from black holes through their multipolar structure

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Introduction. Owing to the black-hole (BH) uniqueness and no-hair theorems \cite{12} (see also Refs. \cite{35}), within General Relativity (GR) any stationary BH in isolation is also axisymmetric and its multipole moments\textsuperscript{1} satisfy an elegant relation \cite{6},

\begin{equation}
\mathcal{M}_\ell^{BH} + iS_\ell^{BH} = \mathcal{M}_{\ell+1} (i\chi)^\ell,
\end{equation}

where $\mathcal{M}_\ell$ ($S_\ell$) are the Geroch-Hansen mass (current) multipole moments \cite{67}, the suffix “BH” refers to the BH metric, $\mathcal{M} = \mathcal{M}_0$ is the mass, $\chi = J/M^2$ the dimensionless spin, and $J = S_1$ the angular momentum (we use natural units throughout). Equation (1) implies that all Kerr moments with $\ell \geq 2$ can be written only in terms of the mass $\mathcal{M}$ and angular momentum $J$ of the spacetime. Introducing the dimensionless quantites $\overline{\mathcal{M}}_\ell = \mathcal{M}_\ell/M^{\ell+1}$ and $\overline{S}_\ell = S_\ell/M^{\ell+1}$, the nonvanishing moments are

\begin{equation}
\overline{\mathcal{M}}_{2n}^{BH} = (-1)^n \chi^{2n}, \quad \overline{S}_{2n+1}^{BH} = (-1)^n \chi^{2n+1}
\end{equation}

for $n = 0, 1, 2, \ldots$. The fact that $\overline{M}_1 = 0$ ($\overline{S}_1 = 0$) when $\ell$ is odd (even) is a consequence of the equatorial symmetry of the Kerr metric. Likewise, the fact that all multipoles with $\ell \geq 2$ are proportional to (powers of) the spin – as well as their specific spin dependence – is a peculiarity of the Kerr metric, that is lost for other compact-object solutions in GR \cite{8} and also for BH solutions in other gravitational theories.

Testing whether these properties hold for an astrophysical dark object provides an opportunity to perform multiple null-hypothesis tests of the Kerr metric – for example by measuring independently three multipole moments such as the mass, spin, and the mass quadrupole $M_2$ – serving as a genuine strong-gravity test of Einstein’s gravity \cite{915}. In this context it is intriguing that current gravitational-wave (GW) observations (especially the recent GW190814 \cite{18}) do not exclude the existence of exotic compact objects other than BHs and neutron stars.

In GR, BHs have curvature singularities that are conjectured to be always covered by event horizons \cite{1719}. At the quantum level, BHs behave as thermodynamical systems with the area of the event horizon and its surface gravity playing the role of the entropy and temperature, respectively \cite{2021}. In fact a BH can evaporate emitting Hawking radiation \cite{22}. This gives rise to a number of paradoxes that can be addressed in a consistent quantum theory of gravity such as string theory \cite{23}.

For special classes of extremal (charged BPS) BHs \cite{2426} one can precisely count the microstates that account for the BH entropy. In some cases one can even identify smooth horizonless geometries with the same mass, charges, and angular momentum as the corresponding BH. These geometries represent some of the microstates in the low-energy (super)gravity description. The existence of a nontrivial structure at the putative horizon scale is the essence of the fuzzball proposal \cite{2730}. In the latter, many properties of BHs in GR emerge from an averaging procedure over a large number of microstates, or as a ‘collective behavior’ of fuzzballs \cite{3035}. So far it has been hard to find a statistically significant fraction of microstate geometries both for five-dimensional (3-charge) and for four-dimensional (4-charge) BPS BHs. Yet, several classes of solutions based on a multi-center ansatz \cite{3640} have been found and their string theory origin uncovered \cite{4143}.

Although in viable astrophysical scenarios BHs are expected to be neutral, charged BPS BHs are a useful toy model to explore the properties of their microstates. Extending the fuzzball proposal to neutral, non-BPS, BHs in four dimensions and finding predictions that can be ob-

\textsuperscript{1} For a generic spacetime the multipole moments of order $\ell$ are rank-$\ell$ tensors, which reduce to scalar quantities, $M_\ell$ and $S_\ell$, in the axisymmetric case. See below for the general definition.
servationally tested so as to distinguish this from other proposals and from the standard BH picture in GR \[14\] remains an open challenge.

In this letter and in a companion paper [44], we investigate the differences in the multipolar structure between BHs and fuzzballs. As we shall argue, already at the level of the quadrupole moments the non-axisymmetric geometry of generic microstates in the four-dimensional fuzzball model leads to a much richer phenomenology and to potentially detectable deviations from GR.

**Setup.** Our method is based on Thorne’s seminal work on the multipole moments of a stationary isolated object [45]. The idea is to choose a suitable coordinate system – so called asymptotically Cartesian mass centered (ACMC) – whereby the mass and current multipole moments can be directly extracted from a multipolar expansion of the metric components. In an ACMC system, the metric of a stationary asymptotically flat object can be written as [44]

\[ ds^2 = -(1 - c_{00}) dt^2 + c_{0i} dt dx_i + (1 + c_{00}) dx_i^2 + \ldots \]  

with \( x_i = \{x, y, z\} \), and \( c_{00} \) and \( c_{0i} \) admitting a spherical-harmonic expansion \[3\] of the form [45]

\[ c_{00} = 2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{(\ell+1)!} \sqrt{\frac{4\pi}{2\ell+1}} (\mathcal{M}_{\ell m} Y_{\ell m} + (\ell^1 < \ell)) \]  

\[ c_{0i} = 2 \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{(\ell+1)!} \sqrt{\frac{4\pi}{(\ell+1)!}} (\mathcal{S}_{\ell m} Y_{\ell m} + (\ell^1 < \ell)) \]

in terms of the scalar (\( Y_{\ell m} \)) and axial vector (\( Y_{\ell m}^A \)) spherical harmonics. The expansion coefficients \( \mathcal{M}_{\ell m} \) and \( \mathcal{S}_{\ell m} \) are the mass and current multipole moments of the spacetime, respectively. They can be conveniently packed into a single complex harmonic function

\[ H = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{(\ell+1)!} \sqrt{\frac{4\pi}{2\ell+1}} (\mathcal{M}_{\ell m} + i\mathcal{S}_{\ell m}) Y_{\ell m}. \]

In the case of the Kerr metric, \( H \) is simply given by

\[ H_{\text{Kerr}} = \frac{\mathcal{M}}{\sqrt{x_1^2 + x_2^2 + (x_3 - \frac{i(J/z)}{\ell})^2}} \]

with two centers at positions \( z = \pm J/M \) along the \( z \)-axis. The harmonic expansion of Eq. [6] does not contain \( m \neq 0 \) terms, so that for each \( \ell \) the moment tensors reduce to the scalars \( \mathcal{M}_\ell \equiv \mathcal{M}_{00} \) and \( \mathcal{S}_\ell \equiv \mathcal{S}_{00} \). The same holds for more general axisymmetric metrics.

Here we consider fuzzball solutions of gravity in four dimensions minimally coupled to four Maxwell fields and three complex scalars. A general class of extremal solutions of the Einstein-Maxwell system is described by a metric of the form [47–49]

\[ ds^2 = -e^{2U}(dt + w)^2 + e^{-2U} \sum_{i=1}^{3} dx_i^2, \]

with

\[ e^{-4U} = L_1 L_2 L_3 V - K^1 K^2 K^3 M + \frac{1}{2} \sum_{l>l'}^{3} \sum_{l=1}^{3} (K^l L_{l1} L_{l1}) \]

\[ - \frac{MV}{2} \sum_{l=1}^{3} (K^l L_{l1})^2 - \frac{4}{3} \sum_{l=1}^{3} (K^l L_{l1})^2 , \]

\[ \ast dw = \frac{1}{2} (V dM - MdV + K^l dL_{l1} - L_{l1} dK^l) \],

where \( \ast \) is the Hodge dual in 3-dimensional flat space, \( \{V, L_1, K^i, M\} \) are eight harmonic functions associated to the four electric and four magnetic charges, and \( I, J = 1, 2, 3 \).

Fuzzball solutions are obtained by distributing the charges of the eight harmonic functions among \( N \) centers in such a way that the geometry near each center lift to a regular five-dimensional geometry. More explicitly, we take

\[ V = v_0 + \sum_{a=1}^{N} v_a, \quad M = m_0 + \sum_{a=1}^{N} m_a, \quad K^l = k_0^l + \sum_{a=1}^{N} k_a^l, \quad L_{l1} = \ell_{l,0} + \sum_{a=1}^{N} \ell_{l,a} \]

with \( r_a = |x - x_a| \) the distance from the \( a \)-th center.

**Results.** Comparing the metric [7] with the definition of an ACMC metric [3], one can extract the multipole moments of the fuzzball solution (details are given in Ref. [44]). The fuzzball multipole moments are encoded in the multipole harmonic function

\[ H = \sum_{a=1}^{N} \left[ V + iM + \sum_{l=1}^{3} (L_{l1} - iK^l) \right]. \]

This complex harmonic function is a generalization of the Kerr case [Eq. [6]], the latter can be interpreted as a two-center solution, with the Schwarzschild case corresponding to a single center. The above expression is instead valid for generic \( N \)-center solutions, regardless of the presence of electromagnetic and scalar fields. Expanding the harmonic function \( H \) yields the multipole moments

\[ \mathcal{M}_{\ell m} = \sum_{a=1}^{N} \left( v_a + \sum_{l} \ell_{l,a} R_{l,a}^\ell \right), \quad \ell \geq 0 \]

\[ \mathcal{S}_{\ell m} = \sum_{a=1}^{N} \left( m_a - \sum_{l} k_{l,a} R_{l,a}^\ell \right), \quad \ell \geq 1 \]

2 It can be shown that the radial (\( Y_{\ell m}^R \)) and electric (\( Y_{\ell m}^E \)) vector spherical harmonics only appear in subleading terms and do not affect the multipole moments [55].

3 The normalization of Thorne’s multipoles can be chosen in order to correspond to the Geroch-Hansen ones [6] [7] used in Eq. [1] in the axisymmetric case [44].
with $M_{00} = M$ and

$$R^a_{\ell m} = |x_a| \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}^*(\theta, \phi).$$

(12)

As in the case of axisymmetric geometries, we define dimensionless moments

$$\mathcal{M}_{\ell m} = \frac{M_{\ell m}}{M^{\ell + 1}}, \quad S_{\ell m} = \frac{S_{\ell m}}{M^{\ell + 1}}.$$

(13)

We center the coordinate system in the center-of-mass and orient the $z$-axis along the angular momentum, so that

$$\frac{1}{4} \sum_{a=1}^{N} \left( v_a + \sum_{I} k_{I,a} \right) x_a = 0 = \frac{1}{4} \sum_{a=1}^{N} \left( m_a - \sum_{I} k_{I,a} \right) x_a = \mathcal{J} e_z,$$

with $e_z$ the unit vector along $z$. With this choice $M_{1m} = 0$, $S_{1 \pm 1} = 0$, and $S_{10} = \mathcal{J}$.

Equations (11) are one of our main results, as they allow to compute the multipole moments of any multi-center microstate geometry. In fact, our method can be straightforwardly applied to any metric in ACMC form. In the following we will focus on some specific cases.

**Examples.** The simplest horizonless geometries arise from three-center solutions. We consider fuzzballs which asymptote to BHs carrying three electric ($Q_I$) and one magnetic ($P_0$) charge, obtained from orthogonal branes, so we require that $K^I$ and $M$ vanish at order $1/r$. Up to a reordering of the centers, the general solution can be written in the form

$$V = 1 + \sum_{a=1}^{3} \frac{1}{r_a}, \quad M = \kappa_1 \kappa_2 \kappa_3 \kappa_4 \left( \frac{1}{r_1} - \frac{1}{r_2} \right),$$

(15)

$$L_1 = 1 + \kappa_4 \left( \frac{\kappa_3}{r_1} - \frac{\kappa_2}{r_2} \right), \quad L_2 = 1 + \kappa_1 \kappa_4 \left( \frac{\kappa_3}{r_2} - \frac{\kappa_2}{r_1} \right),$$

$$L_3 = 1 + \kappa_1 \left( \frac{\kappa_2 \kappa_3}{r_1} + \frac{\kappa_2 \kappa_4}{r_2} + \frac{(\kappa_2 + \kappa_3)^2}{r_3} \right),$$

$$K_1 = \kappa_1 \left( \frac{\kappa_2}{r_1} - \frac{\kappa_3}{r_2} + \frac{\kappa_2 + \kappa_3}{r_3} \right),$$

$$K_2 = \kappa_3 \left( \frac{\kappa_2}{r_1} - \frac{\kappa_2}{r_2} + \frac{\kappa_3}{r_3} \right), \quad K_3 = \kappa_4 \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

with $\kappa_a$ some arbitrary integers.

Regular solutions describe microstates of a (non-rotating) BPS BH with mass

$$M = \frac{1}{4} \left( Q_1 + Q_2 + Q_3 + P_0 \right)$$

and charges

$$Q_1 = \kappa_4 (\kappa_3 - \kappa_2), \quad Q_2 = \kappa_1 \kappa_4 (\kappa_3 - \kappa_2),$$

$$Q_3 = \kappa_1 (\kappa_2^2 + 4\kappa_2 \kappa_3 + \kappa_3^2), \quad P_0 = 3.$$  

(17)

Besides the integer parameters $\kappa_a$, the solution is described in terms of some continuous parameters, namely the distances between the centers $r_{ab} = |x_a - x_b|$. These are constrained by the so-called ‘bubble equations’ [17], ensuring regularity of the five-dimensional lift and absence of closed time-like curves. In the 3-center case one has

$$r_{12} = \frac{2 \kappa_1 \kappa_4 (\kappa_2 - \kappa_3)^2}{r_{23}}$$

$$r_{13} = \frac{\kappa_1 \kappa_4 (2\kappa_2 + 5\kappa_2 \kappa_3 + 2\kappa_3^2) + \kappa_2 + \kappa_4 - \kappa_1 \kappa_3 (1 - \kappa_2 \kappa_4)}{r_{23}}$$

(18)

that allow to express $r_{12}$ and $r_{13}$ in terms of $r_{23} = L$, the surviving continuous parameter (‘modulus’) labeling the microstate. Asymptotically the solution coincides with the Kerr-Newman metric [50], whose multipolar structure is the same [51] as in the Kerr case [see Eq. (1)].

A summary of the first multipole moments for some representative cases is shown in Table I. The general expressions for the multipole moments are cumbersome so we present them in the limit of large mass ($\kappa_a \gg 1$), which is also the most interesting one from a phenomenological point of view, since it corresponds to objects with mass arbitrarily larger than the Planck mass. We consider three representative arrangements of the three centers:

- **A:** Equilateral triangle. ($\kappa_1, \kappa_2, \kappa_3, \kappa_4 = 1, 0, k, k$).

These microstate geometries fall into the class of “scaling solutions” characterized by zero angular momentum, $\mathcal{J} = 0$, equal charges $Q = (k^2, k^2, k^2)$, and mass $M = \frac{3}{4} (1 + k^2)$. Thanks to $Z_3$ symmetry around $z$, the nontrivial mass multipole moments read

$$M_{2p + 3n, 3n} = M(-L)^{2p + 3n} \sqrt{(2p + 6n)!(2p)!} \frac{\sqrt{(2p + 6n)!2^{2p + 3n}(p + 3n)!n!}}{22^{2p + 3n}(p + 3n)!n!}$$

(19)
where \( L = r_{12} = r_{23} = r_{31} \). Thus, at variance with the Kerr case, the mass quadrupole moments are not spin-induced: they can be nonzero even if the spin \( J \) vanishes. Furthermore, for \( \ell \geq 3 \) they also have \( m \neq 0 \) components of the mass moments. The large \( k \) limit of all quadrupole moments are displayed in Table II.

- **B: Isosceles triangle.** \((\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1, 0, 1, k)\). These macrostate geometries possess non vanishing angular momentum, \( J = \frac{2(k+2L)}{(k-1)kL} \), charges \( \bar{Q} = (k, k, 1) \), and mass \( \bar{M} = \frac{2+2k}{2} \). In this case \( L = r_{23} = r_{31} > r_{12} \). For \( k \to \infty \) and \( L \ll 1 \) (see Table II), the multiple moments coincide with those of the Kerr metric modulo the factors \((-1)^n\) in Eq. \( [3] \). In particular, while the Kerr metric is oblate \((\bar{M}_2 < 0)\), these solutions are prolate \((\bar{M}_2 > 0)\). However, for finite values of \( k \) the solution also displays quadrupole moments that break axial symmetry, e.g. \( \bar{M}_{22} \) and \( \bar{S}_{21} \).

- **C: Scalene triangle.** \((\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (3, 0, k, 2k)\). These macrostate geometries possess a non vanishing angular momentum \( J \) which is a complicated function of \( k \) and \( L \), with \( L = r_{23} < r_{12} < r_{31} \), charges \( \bar{Q} = (2k^2, 6k^2, 3k^2) \), and mass \( \bar{M} = \frac{3+11k^2}{4} \). For large \( k \) one finds \( J \sim \frac{\sqrt{2}kL}{4} \). Triangle inequalities require \( \nu = \frac{L}{2k^2} < 1 - \frac{1}{2\sqrt{2}} \). The multipole moments for large \( k \) are displayed in Table II. In this case both the axisymmetry and the equatorial symmetry of the Kerr metric are broken, as shown by the fact that the multipole moments \( \bar{M}_{\ell m} \) and \( \bar{S}_{\ell m} \) are generically nonzero.

It is interesting to observe that the mass and current multipole moments of these macrostate geometries are typically larger than those of a Kerr-Newman BH with same mass and angular momentum. A representative example of this property is shown in Fig. 1 where we display some ratios between multipole moments of microstate geometries of type C and those of a Kerr BH. We focus on the quadratic invariants

\[
\begin{align*}
tr \bar{M}^2_{\ell} & = \sum_{m=\ell}^{\ell} |\bar{M}_{\ell m}|^2, \\
tr \bar{S}^2_{\ell} & = \sum_{m=-\ell}^{\ell} |\bar{S}_{\ell m}|^2. \quad (20)
\end{align*}
\]

We have explored numerically a large region of the whole \((\kappa_i, L)\) parameter space and found that quadratic invariants for the microstate geometries are always larger than those of Kerr BHs for any \( \ell \). It would be interesting to find a general proof of this property, which is analogous to the fact that the Lyapunov exponent of unstable null geodesics near the photon sphere is maximum for the BH solution \[55\]. In other words, both for the multipole moments and for the Lyapunov exponent, the BH solution appears to be an extremum point in the space of the solutions.

![FIG. 1. Ratios between the quadratic invariants for the first multipole moments of a fuzzball (solution C) and a Kerr BH with the same angular momentum, as a function of \( \nu = L/(12k^2) \) with \( k = 1 \). The vertical solid line corresponds to the upper bound \( \nu_{\text{max}} = 1 - 1/\sqrt{2} \). The horizontal dotted black line refers to the fuzzball and Kerr moments being identical. In general, the fuzzball moments are always larger than the corresponding Kerr ones.](image-url)
compact objects \[^8\].

While our results suggest that very strong constraints on fuzzball geometries can be set with EMRIs, a precise analysis requires a class of neutral, nonextremal solutions, which would further imply the absence of extra emission channels (e.g., dipolar radiation). For astrophysically viable objects, we expect that the multipolar structure is the only discriminant with respect to the Kerr BH case, which can be explored with the methods presented here.

In addition to having a different quadrupole moment, microstate geometries are much less symmetric than the Kerr case, which implies the existence of multipole moments that are identically zero in the Kerr case (see also Refs. \[^8\] \[^2\]). Investigating how multipole moments that are identically zero in the Kerr case (see also Refs. \[^8\] \[^62\]) affect the GW waveform is an important task which is left for a follow-up work.

Finally, based on our results it is tempting to conjecture that the \( \ell \geq 2 \) multipole moments of microstate geometries are always larger than those of the correspond-

Kerr BH. If confirmed, this would imply that any measurement of a multipole moment smaller than those in Eq. (2) can potentially rule out the fuzzball scenario.

**Note added.** While this work was in preparation, a related work by Josif Bena and Daniel R. Mayerson appeared \[^63\]. The idea and aims of that paper are similar to ours. Ref. \[^63\] focuses on axisymmetric geometries in the BH limit, whereas our results are valid beyond axial symmetry in regions where the microstate geometries can significantly deviate from the BH metric.

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