BLOW-UP FOR STRAUSS TYPE WAVE EQUATION WITH DAMPING AND POTENTIAL

WEI DAI AND HIDEO KUBO

Abstract. We study a kind of nonlinear wave equations with damping and potential, whose coefficients are both critical in the sense of the scaling and depend only on the spatial variables. Based on the earlier works, one may think of a conjecture that the damping leads to a shift of the critical exponent of the nonlinear term, while the potential does not. We obtain a blow-up result which supports the conjecture, although the existence part is still open. What’s more, we give an upper bound of lifespan when the exponent of the nonlinear term is supposed to be critical or sub-critical.

1. Introduction

In this paper, we consider the blow-up phenomenon for the following initial value problem of semilinear wave equation with damping term and potential term:

\[ \begin{cases} \partial_t^2 u + Ar^{-1} \partial_t u - \Delta u + Br^{-2}u = |u|^p & t > 0, r := |x|, \\ u(0, x) = \varepsilon f(x), & \partial_t u(0, x) = \varepsilon g(x), & x \in \mathbb{R}^n \end{cases} \]

where \( n \geq 1, A \geq 0, B \in \mathbb{R}, p > 1 \) and \( 1 \gg \varepsilon > 0 \).

When \( A = B = 0 \), the study of (1.1) has gone through a long history (see for instance, [7, 12]), and it is well known that in this situation there exists a critical exponent which separates global in time behavior of small amplitude solutions (i.e., global existence and blow-up). The critical exponent is given by the positive root of the quadratic equation:

\[ h(n, p) := (n - 1)p^2 - (n + 1)p - 2 = 0. \]

Such result was firstly appeared in [6], and finally proved by [9] for the subcritical case, [13] for the critical case and [4] for the supercritical case.

For (1.1) only with the damping term, which means \( A > 0, B = 0 \), one related work comes from [5]. Roughly speaking, the authors proved a blow-up result when \( p \) is smaller than the critical exponent of the case \( A = B = 0 \) with \( n \) replaced by \( n + A \). This phenomenon is known as a shift of the critical exponent for the time depending damping term (see e.g., [1]). Following these results, we tend to believe that there exists a critical exponent \( p_c(n, A, B) \) in general situations, although we do not know much about the global existence part, up to now. This is partially because the damping and potential terms have singularity at the origin. In fact, for the case there exists only a regular potential term, some piecemeal results had been shown in [10], [2] and so on. The final conclusion is not clear, but these works suggest \( p_c(n, 0, B) = p_c(n, 0, 0) \) in the general situations.

Date: September 20, 2019.

2010 Mathematics Subject Classification. 35L05, 35L15, 35L70.

Key words and phrases. damping; potential; critical exponent; blow-up.
Next, we consider the exact lifespan, denoted by $T_{e,A,B}(n,p)$, of the problem (1.1). When $A = B = 0$, we have

$$ T_{e,0,0}(n,p) \lesssim \varepsilon \frac{2^{p(p-1)}}{C(n,p)} \quad 1 < p < p_c(n,0,0), $$

$$ \ln(T_{e,0,0}(n,p)) \lesssim \varepsilon^{-(p-1)} \quad p = p_c(n,0,0), $$

and the upper bounds are known to be sharp with respect to $\varepsilon$ at least in the case $3 \leq n \leq 8$. Here and throughout this paper, we denote $x \lesssim y$ and $y \gtrsim x$ mean $x \leq C y$ for some $C > 0$, which may change from line to line. Similarly, $x \approx y$ means that $x \lesssim y \lesssim x$. When $n \geq 9$ and $1 < p < p_c(n,0,0)$, the upper bound is also sharp. On the other hand, when $n = 1,2$, the sharp order in $\varepsilon$ is known according to what is assumed on the initial data. For the detailed discussion can be found in [11].

When $A > 0$, $B = 0$, some upper bound of lifespan has been shown in [5]. Among them, the upper bound for the case $p = p_c(n + A,0,0)$ coincides with the bound of $T_{e,0,0}(n + A,p_c(n + A,0,0))$, which seemed to be sharp. When $A = 0$, $B \neq 0$, we know less about the lifespan. But, by comparison principle in lower dimensions, we can derive some blow-up results, which suggests that $T_{e,0,B}(n,p) = T_{e,0,0}(n,p)$.

We remark that as for regular damping and potential terms, it has been shown in [3] that $p_c(3,2,2) = p_c(5,0,0)$ and the upper and lower bounds for $T_{e,2,2}(3,p)$ coincides with those of $T_{e,0,0}(5,p)$.

Now, we are in a position to state our main result in this paper about the upper bound of lifespan of solutions to (1.1), under the following technical requirements.

$$ B \geq -(n-2)^2/4, \quad 0 \leq A < n-1+2\rho, $$

with

$$ \rho := \frac{(2-n) + \sqrt{(n-2)^2 + 4B}}{2}. $$

**Theorem 1.1.** Assuming (1.3) are satisfied and $\text{supp}(f,g) \subset B(0,1)$ where $B(0,r)$ stand for the ball in which center at 0 and radial is $r$, let $u$ be the solution of (1.1) in the space $C^2([0,T];L^p(R^n))$ and satisfies $\text{supp} u \subset B(0,1 + t)$.

(i): When $A > 0$, max $\{\frac{n+p}{n+p+1}, \frac{1}{2} (\frac{n+p}{n+p+1})^\frac{1}{2} (1 + \sqrt{1 + \frac{\delta}{n+p+1}})\} < p < p_c(n + A,0,0)$, for any $\delta > 0$ and some constant $C_1$, we have

$$ T_{e,A,B}(n,p) \leq C_1 \varepsilon^{\frac{2(p+1)}{n+p+1}} \frac{1}{\delta}, $$

if $E_{\beta,1}$ defined in (3.2) is positive with $\beta = \frac{n-1+2\rho-A}{2} - \frac{1}{p}$.

(ii): When $A > 0$, $\frac{n+p}{n+p+1} < p \leq \frac{1}{2} \left( \frac{n+p}{n+p+1} + \sqrt{1 + \frac{\delta}{n+p+1}} \right)$, for any $\delta > 0$ and some constant $C_2$, we have

$$ T_{e,A,B} \leq C_2 \varepsilon^{\frac{1}{2p+1}} \frac{1}{\delta}, $$

if $E_{\beta,1}$ defined in (3.2) is positive with $\beta = \frac{n-1+2\rho-A}{2} - \frac{1}{p}$.

(iii): When $A = 0$, $p = p_c(n + A,0,0)$, for some constant $C_3$, we have

$$ T_{e,A,B} \leq \exp \left( C_3 \varepsilon^{-p(p-1)} \right), $$

if $E_{\beta,1}$ defined in (3.2) is positive with $\beta = \frac{n-1+2\rho-A}{2} - \frac{1}{p}$. 
(iv): When \( A = 0, \ p < p_c(n + A, 0, 0), \) for some constant \( C_4, \) we have
\[
T_{\varepsilon, A, B} \leq C_4 \varepsilon^{\frac{2p(p-1)}{(n+A)p}}
\]
if \( E_1 \) defined in (4.1) is positive.

Remark 1.1. This theorem suggests that \( p_c(n, A, B) = p_c(n + A, 0, 0) \) and \( T_{\varepsilon, A, B}(n, p) = T_{\varepsilon, 0, 0}(n + A, p) \) in general situations.

Remark 1.2. The small loss \( \delta \) in (1.5), (1.6) is also appeared in [5] for the subcritical case. On the other hand, their requirement \( A < \frac{(n - 1)^2}{(n + 1)} \) is stricter than ours (note that \( \rho = 0 \) in (1.3) for \( B = 0 \) and \( n \geq 2 \)).

This paper is organized as follows. In section 2, we introduce a different type of test functions which solve the conjugate equation of the corresponding homogeneous equation to (1.1). The blow-up result for the general case is treated in section 3. On the other hand, the non-damping case, that is, \( A = 0 \) is handled in section 4. We underline that there is no loss in the estimate of the upper bound of the lifespan in this case. Moreover, when the coefficient of the potential term is relatively small, we can eliminate the small loss appeared in the upper bound of the lifespan. This improvement is done in section 5, based on the comparison principle and the explicit representation of the fundamental solution.

2. Special solutions of the conjugate equation deduced by (1.1)

In this section, firstly we will construct a family of special solutions to
\[
P^* \Psi = \partial_t^2 \Psi - Ar^{-1} \partial_t \Psi - \Delta \Psi + Br^{-2} \Psi = 0,
\]
here \( P^* \) is the time-space conjugate operator of \( P. \) Then, we will discuss the properties of such \( \Psi, \) and use them as the test functions in the next section.

The basic idea is to reshape (2.1) and seek solutions only depend on \( t \) and \( r. \) We consider \( \Psi = r^\rho \Phi \) where \( \rho \) is defined in (1.4) in which solves
\[
\rho(\rho - 1) + (n - 1)\rho - B = 0.
\]
Here we require \( B \geq -((n - 2)^2)/4 \) so that the square root makes sense.

Lemma 2.1. Assume \( \Psi \) is spherically symmetric, then \( P^* \Psi(t, x) = 0 \) if and only if \( \Phi(t, x) \) satisfy the equation
\[
\partial_t^2 \Phi - Ar^{-1} \partial_t \Phi - \partial_r^2 \Phi - (n - 1 + 2\rho)r^{-1} \partial_r \Phi = 0, \quad t > 0, \quad r > 0
\]
with above notations.

Proof. We only need to replace \( \Psi(t, x) \) by \( r^\rho \Phi(t, x) \) and multiply \( r^{-\rho} \) in both sides. For the last two terms in \( P^* \Phi, \) we have
\[
\begin{align*}
-\rho r^{-\rho} & \partial_r (r^\rho \Phi) = \rho r^{-1} \Phi + \partial_r \Phi, \\
r^{-\rho} \partial_r^2 (r^\rho \Phi) & = \rho(\rho - 1)r^{-2} \Phi + 2\rho r^{-1} \partial_r \Phi + \partial_r^2 \Phi, \\
r^{-\rho} (-\Delta + Br^{-2}) \Phi & = -r^{-\rho}(\partial_r^2 + (n - 1)r^{-1} \partial_r - Br^{-2})(r^\rho \Phi) \\
& = -(\partial_r^2 + (n - 1 + 2\rho)r^{-1} \partial_r) \Phi,
\end{align*}
\]
where we used (2.2) in the last equality. Adding the first two terms and we finish the proof. \( \square \)
2.1. Solution for (2.3). In this section, we seek a family of homogeneous solutions of (2.3) with the form

\[ \Phi_\beta(t, x) = (t + r)^{-\beta} \phi \left( \frac{2r}{t + r} \right), \quad \beta \in \mathbb{R}. \]

From now on, we omit the subscript \( \beta \) in \( \Phi_\beta \) if it does not cause misunderstanding.

**Lemma 2.2.** With above notation, \( \Phi \) satisfies (2.3) in \( \{(t, x) : t > |x| \} \) is equivalent to that \( \phi \) satisfies

\[ 0 = z(1 - z)\phi''(z) + (\gamma - (\alpha + \beta + 1)z)\phi'(z) - \alpha \beta \phi(z), \quad z \in (0, 1) \]

with the notation

\[ \alpha = \frac{n + A - 1 + 2\rho}{2}, \quad \gamma = n - 1 + 2\rho, \]

which will be used through the paper.

**Proof.** The proof of this lemma is just some calculations and similar to the proof in [5], but for the reader’s convenience we show the details.

Set \( z = 2r(t + r)^{-1} \), we have

\[ \partial_t z = -2r(t + r)^{-2}, \quad \partial_r z = 2(t + r)^{-1} - 2r(t + r)^{-2} = 2t(t + r)^{-2}, \]

then we get

\[ \begin{align*}
\partial_t \Phi &= -\beta(t + r)^{-\beta - 1} \phi - 2r(t + r)^{-\beta - 2} \phi', \\
\partial_r^2 \Phi &= \beta(\beta + 1)(t + r)^{-\beta - 2} \phi + 4(\beta + 1) r \phi'(t + r)^{-\beta - 3} + 4 r^2 (t + r)^{-\beta - 4} \phi'', \\
\partial_r \Phi &= -\beta(t + r)^{-\beta - 1} \phi + 2t(t + r)^{-\beta - 2} \phi', \\
\partial_r^2 \Phi &= \beta(\beta + 1)(t + r)^{-\beta - 2} \phi - 4(\beta + 1) t \phi'(t + r)^{-\beta - 3} + 4 t^2 (t + r)^{-\beta - 4} \phi''.
\end{align*} \]

Then

\[ 0 = (\partial_r^2 - Ar^{-1} \partial_t - \partial_t^2 - (n - 1 + 2\rho)r^{-1} \partial_r) \Phi \]

\[ = 4(r - t)(t + r)^{-\beta - 2} \phi'' + (2Ar - 2(n - 1 + 2\rho)t + 4(\beta + 1)r)r^{-1}(t + r)^{-\beta - 2} \phi' \\
+ (A\beta + (n - 1 + 2\rho)\beta)(t + r)^{-1}(t + r)^{-\beta - 1}\phi. \]

We multiply above equation with \( (t + r)^{\beta + 2} \) in both sides, using the fact \( tr^{-1} = 2z^{-1} - 1 \) and we get

\[ 0 = 4 \left( 1 - \frac{tr^{-1}}{1 + tr^{-1}} \right) \phi'' + (2A - 2(n - 1 + 2\rho)tr^{-1} + 4(\beta + 1)) \phi' \\
+ (A\beta + (n - 1 + 2\rho)\beta)(tr^{-1} + 1) \phi \\
= 4(z - 1)\phi'' + (2A + 2(n - 1 + 2\rho) - 4(n - 1 + 2\rho)z^{-1} + 4(\beta + 1)) \phi' \\
+ (A\beta + (n - 1 + 2\rho)\beta)(2z^{-1}) \phi \\
= -4z^{-1} \left( z(1 - z)\phi'' + \left( n - 1 + 2\rho - \left( \frac{n + A - 1 + 2\rho}{2} + \beta + 1 \right) z \right) \phi' \\
- \frac{A\beta + (n - 1 + 2\rho)\beta}{2} \phi \right) \\
= -4z^{-1}(z(1 - z)\phi''(z) + (\gamma - (\alpha + \beta + 1)z)\phi'(z) - \alpha \beta \phi(z)), \]

which finishes the proof. \( \square \)
Corollary 2.3. When \( A < n - 1 + 2 \rho \), the equation (2.4) has a special solution
\[
\phi(z) = F\left(\alpha, \beta, \gamma; \frac{2r}{t + r}\right),
\]
where \( F(\alpha, \beta, \gamma; z) \) is the hypergeometric function given by
\[
F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta} \, dt
\]
for \( \gamma > \alpha > 0 \).

This corollary just followed by the property of hypergeometric differential equation. For more properties of hypergeometric differential equation and hypergeometric function, we refer the readers to, e.g., [8].

Now, we get a family of solutions of (2.1). Translating it downward by 2, and still denoted by \( \Psi \), then
\[
(2.6) \quad \Psi(t, x) = r^\rho(t + r + 2)^{-\beta} F\left(\alpha, \beta, \gamma; \frac{2r}{t + r + 2}\right)
\]
satisfies (2.1) in \( Q := \{(t, x) : 2 + t > |x|\} \). The next step is to discuss some properties of such \( \Psi \).

Lemma 2.4. For any \( \beta \in \mathbb{R} \) and \( (t, x) \in Q \), we have
\[
(2.7) \quad \partial_t \Psi_\beta(t, x) = -\beta \Psi_{\beta+1}(t, x).
\]
Moreover, for every \( (t, x) \in Q \), we have
\[
(2.8) \quad \Psi_\beta(t, x) \approx r^\rho(2 + t)^{-\beta} \cdot \begin{cases} 1, & \beta < \gamma - \alpha, \\ 1 - \ln\left(1 - \frac{2r}{2 + t}\right), & \beta = \gamma - \alpha, \\ \frac{1}{1 - \frac{2r}{2 + t}} \left(1 - \frac{\rho}{1 - \frac{2r}{2 + t}} \right)^{\gamma-\alpha-\beta}, & \beta > \gamma - \alpha. \end{cases}
\]

Proof. This lemma is similar to [5, Lemma 3.2] but more complete here. Firstly for (2.7), with \( z = \frac{2r}{t + r + 2} \) we have
\[
\partial_t \Psi_\beta(t, x) = r^\rho(t + r + 2)^{-\beta-1}(-\beta - z\partial_z)F(\alpha, \beta, \gamma; z), \quad \Psi_{\beta+1}(t, x) = r^\rho(t + r + 2)^{-\beta-1}F(\alpha, \beta + 1, \gamma; z).
\]
Using the properties
\[
\partial_z F(\alpha, \beta, \gamma; z) = \frac{\alpha z}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z),
\]
\[
\frac{\alpha z}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z) = F(\alpha, \beta + 1, \gamma; z) - F(\alpha, \beta, \gamma; z)
\]
of hypergeometric function, see e.g. [8, Section 9.2], we get (2.7) for any \( n \geq 1 \). For (2.8), we only need to estimate \( F(\alpha, \beta, \gamma; z) \) here. By the expression of \( F \), it is obvious that \( F > 0 \) when \( 0 < z < 1 \). For the rest estimates, because of that \( (1 - zt)^{-\beta} \approx 1 \) when \( 0 < z \leq 1/2, 0 < t < 1 \) and \( 1 - z \approx 1 - \frac{r}{2 + t} \), we only need to
consider $z > 1/2$. Then we have
\[ F(\alpha, \beta, \gamma; z) \approx \int_{0}^{1} t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} (1 - zt)^{-\beta} \, dt \]
\[ = \int_{1/2}^{1} t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} (1 - zt)^{-\beta} \, dt + O(1) \]
\[ \approx \int_{1/2}^{1} (1 - t)^{\gamma-\alpha-1}(1 - zt)^{-\beta} \, dt + O(1). \]

Using the change of variable $t = 1 - (1 - z)s$, we continue the calculation and get
\[ F(\alpha, \beta, \gamma; z) \approx (1 - z)^{\gamma-\alpha-\beta} \int_{0}^{(1-z)^{-1/2}} s^{\gamma-\alpha-1}(1 + zs)^{-\beta} \, ds + O(1) \]
\[ = (1 - z)^{\gamma-\alpha-\beta} \int_{1}^{(1-z)^{-1/2}} s^{\gamma-\alpha-1}(1 + zs)^{-\beta} \, ds + O(1) \]
\[ \approx (1 - z)^{\gamma-\alpha-\beta} \int_{1}^{(1-z)^{-1/2}} s^{\gamma-\alpha-\beta-1} \, ds + O(1). \]

By a fundamental calculation of this integral with different $\beta$, we finish our proof of (2.8).

2.2. Another solution for (2.3) with $A = 0$. When $A = 0$ and $n + 2\rho \in \mathbb{N}$, (2.3) looks like a linear spherically symmetric wave equation in $\mathbb{R}_+ \times \mathbb{R}^{n+2\rho}$, with solution
\[ \Phi(t, x) = \exp(-t) \int_{S^{n+2\rho-1}} \exp(x \cdot \mu) \, d\sigma_{\mu}, \]
which was firstly used in [13] when $A = B = 0$. To consider a general $\rho$, we modify it and get

**Lemma 2.5.** For any $\rho$ defined by (1.4),
\[ \Phi(t, x) := \exp(-t) \int_{-1}^{1} \exp(\lambda r) \sqrt{1 - \lambda^2}^{n-3+2\rho} \, d\lambda \]
is a solution of (2.3) with $A = 0$.

**Proof.** It is obvious that $\partial_t^2 \Phi = \Phi$, so we only need to show that
\[ \Phi - \partial_t^2 \Phi - (n - 1 + 2\rho)r^{-1} \partial_r \Phi = 0. \]
By using integration by part, we see
\[ \Phi - \partial_t^2 \Phi = \exp(-t) \int_{-1}^{1} \exp(\lambda r) \sqrt{1 - \lambda^2}^{n-3+2\rho} \, d\lambda \]
\[ = (n - 1 + 2\rho)r^{-1} \exp(-t) \int_{-1}^{1} \exp(\lambda r) \lambda \sqrt{1 - \lambda^2}^{n-3+2\rho} \, d\lambda \]
\[ = (n - 1 + 2\rho)r^{-1} \partial_r \Phi, \]
which finishes the proof.

**Lemma 2.6.** For the $\Phi$ defined above and $\Psi = r^\rho \Phi$, we have
\[ \Phi(t, x) \approx (1 + r)^{-(n - 1 + 2\rho)/2} \exp(r - t), \]
\[ \Psi(t, x) \approx r^\rho (1 + r)^{-(n - 1 + 2\rho)/2} \exp(r - t). \]
Proof. The properties of $\Psi$ follows easily from the property of $\Phi$, so we only need to show the estimate of $\Phi$. By definition of $\Phi$ and $r \geq 0$ we find
\[
\Phi(t, x) = \exp(-t) \int_0^1 \left( \exp(\lambda r) + \exp(-\lambda r) \right) \sqrt{1 - \lambda^{n-3+2\rho}} \, d\lambda \\
\approx \exp(-t) \int_0^1 \exp(\lambda r) \sqrt{1 - \lambda^{n-3+2\rho}} \, d\lambda \\
\approx \exp(r - t) \int_0^1 \exp((\lambda - 1)r) \sqrt{1 - \lambda^{n-3+2\rho}} \, d\lambda.
\]
When $r$ is small, the estimate follows easily, as for $r$ big, we set $s = (1 - \lambda)r$, then
\[
\Phi(t, x) \approx r^{-(n-1+2\rho)/2} \exp(r-t) \int_0^s \exp(-s) \sqrt{s^{n-3+2\rho}} \, ds,
\]
which lead to the estimate since the last integral is $O(1)$.

3. Proof of the blow-up phenomenon when $A \geq 0$

In this section we use $\Psi$ comes from Section 2.1 to consider the upper bound of the lifespan of solutions to (1.1) and its dependence of $\varepsilon$ under the condition (1.3).

3.1 Preliminaries for showing blowup phenomenon. Firstly we state a lemma of upper bound for lifespan to some ordinary differential inequality.

Lemma 3.1. Let $p > 1$ and $v \in C^2([t_0, \infty))$ with some $t_0 > 0$. If $v$ satisfies
\[
\begin{cases}
    v(t) \geq \delta(t+1)^a \\
v''(t) \geq k(t+1)^{-b} v(t)^p
\end{cases}
\]
with $\delta > 0$, $k > 0$, $a \geq 1$ and $(p-1)a > b - 2$, then $v$ must blows up before $C\delta^{-(p-1)/(p-1-\alpha-b+2)}$ for some $C > 0$.

If $v$ satisfies
\[t^{1-p} v(t)^p \lesssim v''(t) + 2v'(t)\]
with $v(t) \gtrsim \varepsilon^p t$ and $v'(t) \gtrsim \varepsilon^p$, then $v$ must blows up before $C' \varepsilon^{-p(p-1)}$ for some $C' > 0$.

Proof. The first result follows from [14, Lemma 2.1], and the second one follows from [5, Lemma 4.1].

For the test function (2.6), we consider the following function
\[
G_{\beta}(t) := \int_{\mathbb{R}^n} |u(t, x)|^p \Psi_{\beta}(t, x) \, dx, \quad t \geq 0.
\]

Lemma 3.2. If $\|u(t)\|_{L_x^p}$ is finite in some $[0, T]$ and $u(t, \cdot)$ is supported in $B(0, 1+t)$ for any $t$, then $G_{\beta}$ is finite in $[0, T]$ when $\beta < \gamma - \alpha$.

Proof. Noticing that for $\beta < \gamma - \alpha$, by (2.8) we have
\[
G_{\beta}(t) = \int_{\mathbb{R}^n} |u(t, x)|^p \Psi_{\beta}(t, x) \, dx \approx \int_{B(0, t+1)} |u|^p r^p (2 + t)^{-\beta} \, dx \lesssim (2 + t)^{\rho - \beta} \|u\|_{L_x^p}^p,
\]
which finishes the proof.

Remark 3.1. From the proof we can also find that $G_{\beta}(t) \approx (2 + t)^{-\beta} \|u^{p/\rho}\|_{L_x^p}^p$ when $\beta < \gamma - \alpha$. 

By Lemma 3.2, we only need to study the blow-up phenomenon of \( G_\beta(t) \) defined above. Now, we begin to construct the differential equation for \( G_\beta \) when \( \beta < \gamma - \alpha \).

**Lemma 3.3.** Let \( u \) be a solution of (1.1) with \( \text{supp} \ u \subset \{(t, x) : t \geq |x| - 1\} \). Then for every \( t \geq 0 \), we have

\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s)G_\beta \ ds = \int_{\mathbb{R}^n} u \Psi_\beta \ dx + A \int_0^t \int_{\mathbb{R}^n} r^{-1} u \Psi_\beta \ dx \ ds \\
+ 2\beta \int_0^t \int_{\mathbb{R}^n} u \Psi_{\beta+1} \ dx \ ds
\]

(3.1)

where

\[
E_{\beta,0} = \int_{\mathbb{R}^n} f(x) \Psi_\beta(0, x) \ dx,
\]

(3.2)

\[
E_{\beta,1} = \int_{\mathbb{R}^n} g(x) \Psi_\beta(0, x) + \beta f(x) \Psi_{\beta+1}(0, x) + Ar^{-1} f(x) \Psi_\beta(0, x) \ dx.
\]

**Proof.** Starting from \( G_\beta \) itself, by the property of \( u \) and integration by parts we get

\[
G_\beta(t) = \int_{\mathbb{R}^n} (\partial_t^2 u + Ar^{-1} \partial_t u + Br^{-2} u) \Psi_\beta \ dx - \int_{\mathbb{R}^n} u \Delta \Psi_\beta \ dx.
\]

By (2.1), we know

\[
u \Delta \Psi_\beta = u (\partial_t^2 \Psi_\beta - Ar^{-1} \partial_t \Psi_\beta + Br^{-2} \Psi_\beta),
\]

then

\[
G_\beta(t) = \partial_t \int_{\mathbb{R}^n} \Psi_\beta \partial_t u - u \partial_t \Psi_\beta + Ar^{-1} u \Psi_\beta \ dx.
\]

Integrating it over \([0, t] \), noticing that \( \partial_t \Psi_\beta = -\beta \Psi_{\beta+1} \), we have

\[
\varepsilon E_{\beta,1} + \int_0^t G_\beta \ ds = \int_{\mathbb{R}^n} \Psi_\beta \partial_t u - u \partial_t \Psi_\beta + Ar^{-1} u \Psi_\beta \ dx \\
= \int_{\mathbb{R}^n} \partial_t (u \Psi_\beta) + 2\beta u \Psi_{\beta+1} + Ar^{-1} u \Psi_\beta \ dx
\]

with \( E_{\beta,1} \) defined above. To eliminate the \( \partial_t \) appeared in right hand, we integrate it again, then we get (3.1). \(
\)

From now on, we set \( \beta = \gamma - \alpha - \frac{1}{q} \) with \( p \leq q < \infty \). To estimate \( G_\beta(t) \), we derive the estimate of the left hand of (3.1).

**Lemma 3.4.** Assuming \( \frac{n+p}{n+p-1} < p < \infty \), we have

(3.3)

\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s)G_\beta \ ds \\
\lesssim \left\| u \right\|_{L_p^r} (2 + t)^{\frac{n+p}{p-r} - \beta} + \begin{cases} 
\int_0^t \left\| u \right\|_{L_p^r} (2 + s)^{\frac{n+p}{p-r} - \beta - \frac{1}{q} - \frac{1}{p}} \ ds & q > p, \\
\int_0^t \left\| u \right\|_{L_p^r} (2 + s)^{\frac{n+p}{p-r} - \beta - \ln(2 + s)^{\frac{1}{p}}} \ ds & q = p.
\end{cases}
\]
Proof. We only need to estimate the three terms appeared in the right hand of (3.1). Using (2.8), for the first term with \( \beta < \gamma - \alpha \), we have
\[
\int_{\mathbb{R}^n} u \Psi_\beta \, dx \lesssim \left\| ur^\beta \right\|_{L^p_r} \left\| r^{-\frac{\beta}{p}} \Psi_\beta \right\|_{L^{p'}_r (r<t+1)} \\
\approx \left\| ur^\beta \right\|_{L^p_r} (2 + t)^{-\beta} \left\| r^{\frac{\beta}{p'}} \right\|_{L^{p'}_r (r<t+1)} \\
\approx \left\| ur^\beta \right\|_{L^p_r} (2 + t)^{\frac{n+\rho-\beta}{p'\beta}}.
\]
For the second term, noticing that \( \frac{n+\rho}{n+\rho-1} < p \), we have
\[
\int_{\mathbb{R}^n} r^{-1} u \Psi_\beta \, dx \lesssim \left\| ur^\beta \right\|_{L^p_r} (2 + s)^{-\beta} \left\| r^{\frac{\beta}{p'}} \right\|_{L^{p'}_r (r<s+1)} \approx \left\| ur^\beta \right\|_{L^p_r} (2 + s)^{\frac{n+\rho-\beta}{p'\beta} - 1}.
\]
For the third term, noticing \( \beta + 1 = \gamma - \alpha + \frac{1}{q} > \gamma - \alpha \), we have
\[
\int_{\mathbb{R}^n} u \Psi_{\beta+1} \, dx \lesssim \left\| ur^\beta \right\|_{L^p_r} (2 + s)^{-\beta-1} \left\| r^{\frac{\beta}{p'}} \left(1 - \frac{r}{2 + s}\right)^{\gamma - \alpha - 1} \right\|_{L^{p'}_r (r<s+1)}
\]
where
\[
\left\| r^{\frac{\beta}{p'}} \left(1 - \frac{r}{2 + s}\right)^{\gamma - \alpha - 1} \right\|_{L^{p'}_r (r<s+1)} = \int_{B(0,1+s)} r^\rho \left(1 - \frac{r}{2 + s}\right)^{\rho\left(\frac{\beta}{p'}\right) - 1} \, dx
\]
\[
\approx \int_0^{1+s} r^{\rho+n-1} \left(1 - \frac{r}{2 + s}\right)^{-\frac{\rho\beta}{p'}} \, dr
\]
\[
= (2 + s)^{n+\rho} \int_{(2+s)^{-1}}^1 (1 - \lambda)^{\rho+n-1} \lambda^{-\frac{\rho\beta}{p'}} \, d\lambda
\]
\[
\approx (2 + s)^{n+\rho} \left(\int_{(2+s)^{-1}}^1 \lambda^{-\frac{\rho\beta}{p'}} \, d\lambda + O(1)\right).
\]
By considering \( q > p \) and \( q = p \) separately, we get (3.3) by simple calculation and finish the proof. \( \square \)

3.2. The upper bound of lifespan for the subcritical case. In this section, we will specify \( q > p \) later for \( \frac{n+\rho}{n+\rho-1} < p < p_c(n + A, 0, 0) \). By Lemma 3.4 and Remark 3.1 we see
\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s) G_\beta \, ds \lesssim (2 + t) \frac{n+\rho-\beta}{p} G_\beta^{\frac{1}{p}} + \int_0^t (2 + s)^{n+\rho-\beta} \frac{1}{q} G_\beta^{\frac{1}{p}} \, ds,
\]
where
\[
\alpha = \frac{n + A - 1 + 2\rho}{2}, \quad \gamma = n - 1 + 2\rho, \quad \beta = \gamma - \alpha - \frac{1}{q} = \frac{n - 1 + 2\rho - A}{2} - \frac{1}{q}.
\]
Integrating this equation over \([0, t] \), for the first term in right hand we deduce that
\[
\int_0^t (2 + s)^{n+\rho-\beta} \frac{1}{p} G_\beta^{\frac{1}{p}} \, ds \lesssim \left(\int_0^t G_\beta \, ds\right)^{\frac{1}{p}} \left(\int_0^t (2 + s)^{n+\rho-\beta} \, ds\right)^{\frac{1}{p}}
\]
\[
\lesssim \left(\int_0^t G_\beta \, ds\right)^{\frac{1}{p}} (2 + t)^{\frac{n+\rho-\beta+1}{p}}.
\]
For the second term we deduce that
\[
\int_0^t (t-s)(2+s)^{\frac{n+\rho-\beta}{p}-\frac{1}{q}} G_\beta^\frac{1}{p} ds \lesssim \left( \int_0^t G_\beta ds \right)^\frac{1}{p} \left( \int_0^t (t-s)(2+s)^{n+\rho-\beta-\frac{p'}{q}-1} ds \right)^\frac{1}{p'},
\]
for some \(q\) in which ensure that \(n + \rho - \beta - \frac{p'}{q} - 1 = \frac{n+A+1}{q(p-1)} - 1 > -1\).

Since \(q > p\), mixing them together and noticing that \(\frac{n+\rho-\beta+1}{p'} q - \frac{1}{pq} > 0 + 0 - 1 = -1\), we get
\[
\varepsilon E_{\beta,0} t + \varepsilon E_{\beta,1} t^2 + \int_0^t (t-s)^2 G_\beta ds \lesssim \left( \int_0^t G_\beta ds \right)^\frac{1}{p} \left( 2 + t \right)^{\frac{h(n+A,p)}{2p} + 2 + \frac{1}{p^2} - \frac{1}{pq} + \frac{1}{p} - \frac{1}{q}}.
\]

If \(E_{\beta,1} > 0\), setting \(K_\beta(t) := \int_0^t (t-s)^2 G_\beta(s) ds\) and noticing that
\[
- \frac{h(n+A,p)}{2p} + \frac{1}{q} - \frac{1}{p} > 0 + 0 - 1 = -1,
\]
we get
\[
K_\beta(t) \geq (2 + t)^{\frac{h(n+A,p)}{2p} + 2 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{pq}} K_\beta(t)^p
\]
for all \(t > t_0\) with some \(t_0\) big enough. By Lemma 3.1, we know \(K_\beta(t)\) must blow-up before \(C\varepsilon^{\eta}\) with
\[
\eta = \frac{2pq(p-1)}{h(n+A,p)q + 2q - 2p} = \frac{1}{\frac{h(n+A,p)}{2p(p-1)} + \frac{1}{p^2} - \frac{1}{pq} - \frac{1}{q}},
\]
and so does \(G_\beta(t)\) and \(u\).

Now, we back to the choice of \(q\). By the requirements
\[
n + \rho - \beta - \frac{p'}{q} - 1 > -1, \quad q > p,
\]
we need to choose
\[
q > \max \left\{ \frac{2}{(p-1)(n+A+1)}, p \right\}.
\]
When \(p \geq \frac{2}{(p-1)(n+A+1)}\), which means \(p \geq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8}{n+A+1}} \right)\), we can choose \(q\) arbitrarily close to \(p\), such that \(E_{\beta,1} > 0\) by its continuity relates to \(\beta\), then we get (1.5) with any \(\delta\). Otherwise, we can only choose \(q\) arbitrarily close to \(\frac{2}{(p-1)(n+A+1)}\), then we get (1.6) with any \(\delta\).
3.3. The upper bound of lifespan for the critical case. When $p = p_c(n + A, 0, 0)$, things become more difficult. For writing convenience we more define

$$H_\beta(t) := \int_0^t (t - s)(2 + s)G_\beta(s) \, ds, \quad t \geq 0,$$

$$J_\beta(t) := \int_0^t (2 + s)^{-3}H_\beta(s) \, ds, \quad t \geq 0.$$

Then we have the following properties

Lemma 3.5. For every $t \geq 0$,

$$\begin{align*}
(2 + t)^2J_\beta(t) &= \frac{1}{2} \int_0^t (t - s)^2G_\beta(s) \, ds, \\
H_\beta'(t) &= (2 + t)^3J_\beta'(t) + 3(2 + t)^2J_\beta(t).
\end{align*}$$

Proof. The first relation follows from integration by parts and the fact that

$$\frac{d^2}{ds^2}(2 + s)^{-1} = \frac{2(2 + t)^2}{(2 + s)^3}.$$

The second relation follows from

$$(2 + t)^3J_\beta'(t) = H_\beta(t).$$

Differentiating on both sides of the equation and we get the result. \qed

To show the blow-up of $u$, we consider both $G_\beta$ with some $q > p$ and $G_{\beta_0}$ with $q = p$. Note $\beta \geq \beta_0$, by Lemma 3.4 and Remark 3.1 for $G_{\beta_0}$, we have

$$\int_0^t (t - s)G_{\beta_0} \, ds \lesssim (2 + t)^{n + p - \beta_0}G_{\beta_0} + \int_0^t (2 + s)^{n + p - \beta_0}1 - 1G_{\beta_0}^{-\beta} \ln(2 + s)^{\beta - p} \, ds.$$

Noticing than $\frac{n + p - \beta_0}{p} = 1 + \frac{1}{p}$ and integrating it over $[0, t]$, we have

$$\begin{align*}
\int_0^t (t - s)^2G_{\beta_0} \, ds &\lesssim \int_0^t (2 + s)^{1 + \frac{\beta}{p}}G_{\beta_0}^\frac{\beta}{p} \, ds + \int_0^t (2 + s)^{\beta}G_{\beta_0}^{-\beta} \ln(2 + s)^{\beta - p} \, ds \\
&\lesssim \left(\int_0^t (2 + s)G_{\beta_0} \, ds\right)^{\frac{\beta}{p}} \left(\left(\int_0^t (2 + s)^p \, ds\right)^{\frac{1}{p}} + \left(\int_0^t (2 + s)^{\beta - p} \ln(2 + s) \, ds\right)^{\frac{1}{p}}\right) \\
&\lesssim \left(\int_0^t (2 + s)G_{\beta_0} \, ds\right)^{\frac{\beta}{p}} (2 + t)^{\frac{2\beta - 2}{p}} \ln(2 + t)^{\frac{\beta}{p}}.
\end{align*}$$

By the definitions and properties of $H$ and $J$, we get

$$\ln(2 + t)^{1 - p}J_{\beta_0}(t) \lesssim H_{\beta_0}'(t)(2 + t)^{-1} = (2 + t)^2J_{\beta_0}'(t) + 3(2 + t)J_{\beta_0}'(t).$$

Setting $J_{\beta_0}(t) = v(\sigma)$ with $\sigma = \ln(2 + t)$, from above inequalities we obtain

$$\sigma^{1 - p}v(t) \lesssim v'(\sigma) + 2v'(\sigma)$$

with $\sigma > \sigma_0$ for some $\sigma_0$.

On the other hand, by Lemma 3.4 for $G_\beta$ and Remark 3.1 for $G_{\beta_0}$, we have

$$\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1}t \lesssim (2 + t)^{n + p - \beta_0 + \beta_0 - \beta}G_{\beta_0}^\frac{\beta}{p} + \int_0^t (2 + s)^{n + p - \beta_0 - \beta}G_{\beta_0}^{-\beta} \, ds.
Integrating it over $[0, t]$, similarly we have
\[
\varepsilon E_{\beta,0} t + \varepsilon E_{\beta,1} t^2 \\
\lesssim \left( \int_0^t (2 + s) G_{\beta_0} \, ds \right)^{\frac{1}{p}} \left( \left( \int_0^t (2 + s)^{\rho'(1 + \beta_0 - \beta)} \, ds \right)^{\frac{1}{p'}} + \left( \int_0^t (t - s)^{\rho'} \, ds \right)^{\frac{1}{p'}} \right)
\]
\[
\lesssim \left( \int_0^t (2 + s) G_{\beta_0} \, ds \right)^{\frac{1}{p}} (2 + t)^{2 - \frac{2}{p'}}.
\]
Since we assume $E_{\beta,1} > 0$, it means
\[
H'_{\beta_0}(t) \gtrsim \varepsilon p t^2 p (2 + t)^{1 - 2p} \gtrsim \varepsilon p (2 + t)
\]
for all $t > t_0$ with some $t_0$ big enough, and therefore $H_{\beta_0}(t) \gtrsim \varepsilon p (2 + t)^2$ for all $t > 2t_0$. Using the properties of $v$ and $J_\beta$, we see
\[
v'(\sigma) = (2 + t) J'_{\beta_0}(t) = (2 + t)^{-2} H_{\beta_0}(t) \gtrsim \varepsilon p
\]
for all $\sigma > \ln(2 + 2t_0)$, and therefore $v(\sigma) \gtrsim \sigma \varepsilon p$ for all $\sigma > 2 \ln(2 + 2t_0)$. Applying Lemma 3.1 here we get the upper bound of lifespan of $v$, then $J_{\beta_0}$, $H_{\beta_0}$ and finally $u$, which finishes our proof of (1.7).

4. Another proof of the blow-up phenomenon when $A = 0$

In this section, we turn to use $\Psi$ which comes from Section 2.2 as the test function. Here, we notice that (1.1) is equivalent to
\[
(\partial_t^2 - r^{-2} \Delta_{S^{n-1}}) (r^{-\rho} u) - r^{-n+1-2\rho} \partial_r (r^{-n+1+2\rho} \partial_r (r^{-\rho} u)) = r^{-\rho} |u|^p.
\]
Since it is so, we set
\[
F_0(t) := \int_{\mathbb{R}^n} r^\rho u \, dx = \int_0^\infty \int_{S^{n-1}} r^{n-1+2\rho} (r^{-\rho} u) \, d\omega \, dr
\]
\[
F_1(t) := \int_{\mathbb{R}^n} u \Psi \, dx.
\]
Firstly for $F_1(t)$, similar to the calculation of $G_\beta$, we have
\[
\varepsilon E_1 \leq \partial_t \int_{\mathbb{R}^n} \Psi u \, dx - 2 \int_{\mathbb{R}^n} u \partial_t \Psi \, dx = F'_1 + 2F_1
\]
with
\[
(4.1) \quad E_1 = \int_{\mathbb{R}^n} (f(x) + g(x)) \Psi(0, x) \, dx.
\]
Then we get that $F_1(t) \gtrsim \varepsilon$ for all $t \geq t_0$ with some $t_0$ big enough.

As for $F_0(t)$, we calculate
\[
F''_0(t) = \int_0^\infty \int_{S^{n-1}} r^{n-1+2\rho} \partial_r^2 (r^{-\rho} u) \, d\omega \, dr
\]
\[
= \int_0^\infty \int_{S^{n-1}} \partial_r (r^{-n+1+2\rho} \partial_r (r^{-\rho} u)) + r^{n-1+2\rho} \partial_r (r^{-\rho} |u|^p) \, d\omega \, dr
\]
\[
= \int_{\mathbb{R}^n} r^\rho |u|^p \, dx
\]
Using the Hölder’s inequality, we find
\[ F_0''(t) = \int_{\mathbb{R}^n} r^p |u|^p \, dx \geq \frac{|\int r^p u \, dx|^p}{(\int_{r<t} r^p \, dx)^{p-1}} \equiv (1 + t)^{-(n+\rho)(p-1)} |F_0|^p. \]
On the other hand, we also get
\[ F_0''(t) = \int_{\mathbb{R}^n} r^p |u|^p \, dx \geq \frac{|F_1|^p}{(\int_{r<t} r^{-\frac{\rho}{n}} \Psi' \, dx)^{p-1}}, \]
Using Lemma 2.6 and splitting the integral at \( r = \frac{t+1}{2} \), we get
\[ \int_{r<t+1} r^{-\frac{\rho}{n}} \Psi' \, dx \approx \int_0^{t+1} r^{n-1+p}(1 + r)^{-(n+2p)p'/2} \exp(p'r - p't) \, dr \]
\[ \lesssim (1 + t)^{n+\rho-(n+2p)p'/2}. \]
Then
\[ F_0''(t) \gtrsim \varepsilon^p(1 + t)^{(n-1+p)-(n-1)p/2}. \]
Integrating the inequality twice, we get that
\[ F_0(t) \gtrsim \varepsilon^p(1 + t)^{(n+1+p)-(n-1)p/2} \]
for all \( t \geq t_1 \) with some \( t_1 \). Applying these inequalities of \( F_0 \) to Lemma 3.1, we finish the proof of (1.8).

5. Some improvements when \( B \) is small

In this section, we want to show a bit more results under some restrictions on \( B \). For convenience, we assume
\[ f(x) \equiv 0, \quad 0 \leq g(x) \neq 0 \]
in this section.

**Theorem 5.1.** With \( f \) and \( g \) as above, \( n \geq 2 \), \( A \geq 0 \) and \( B \leq \frac{\Delta^2 + 2A-(n-1)(n-3)}{4} \), we have
\[ T_{\varepsilon,A,B}(n,p) \leq C\varepsilon^{\frac{2p(n-1)}{n+p}}, \quad p < p_c(n + A, 0, 0), \]
\[ T_{\varepsilon,A,B}(n,p) \leq (C\varepsilon^{-p(n-1)}), \quad p = p_c(n + A, 0, 0), \]
for some constant \( C \) which does not depend on \( \varepsilon \).

**Proof.** In this situation, we compare the solution of the equation (1.1) with that of the equation
\[ \begin{align*}
\left\{ \begin{array}{l}
\partial_t^2 U - \Delta + \frac{A}{4} \partial_t + \frac{\hat{B}}{4r} U = |U|^p, \\
U(0, x) = 0, \quad U_t(0, x) = \varepsilon \tilde{g}(x),
\end{array} \right.
\end{align*} \]
where \( \hat{B} = \frac{\Delta^2 + 2A-(n-1)(n-3)}{4} \) and \( \tilde{g}(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} g(rv) \, dv \). Because of that the \( \tilde{g} \) is spherically symmetric, this equation is equivalent to
\[ \begin{align*}
\left\{ \begin{array}{l}
(\partial_t - \partial_r + \frac{A}{2r}) \left( \partial_t + \partial_r + \frac{A}{2r} \right) V = F(t, r), \\
V(0, r) = V(t, 0) = 0, \quad V_t(0, r) = \varepsilon g^*(r),
\end{array} \right.
\end{align*} \]
where
\[ V = r^{\frac{n-1}{4}} U, \quad F = r^{\frac{n-1}{4}(1-p)} |V|^p, \quad g^*(r) = r^{\frac{n-1}{4}} \tilde{g}(re_1). \]
To study the blow-up of $U$, we start at (5.2) and (5.3).

**Claim 5.2.** The solution of the linear equation (5.3) is given by

$$V(t, r) = \frac{1}{2} \int_0^t \int_{|r-t|}^{r+s} \left(2^{-1}r^{-1/2}\rho^{-1/2}(\rho + r - t)\right)^{A} F(s, \rho) \, d\rho \, ds$$

$$+ \frac{\varepsilon}{2} \int_{|r-t|}^{r+t} \left(2^{-1}r^{-1/2}\rho^{-1/2}(\rho + r - t)\right)^{A} g^*(\rho) \, d\rho.$$

This claim shows that $U = r^{-\frac{n+1}{2}}V$ is nonnegative as long as $g^*$ and $F$ is nonnegative, which allows us to use the comparison principle. Meanwhile, we have

**Lemma 5.3.** Assume $V$ solves (5.3) with $p \leq p_c(n + A, 0, 0)$, then $V$ must blow-up at some $T_{c, A, \tilde{B}}(n, p)$ satisfying (5.1).

Back to $u$, we find that $\tilde{u}(t, x) := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u(t, rv) \, dv$ satisfying

$$\begin{aligned}
&\left(\partial_t^2 - \Delta + \frac{1}{r} \partial_r + \frac{\tilde{B}}{r^2}\right) \tilde{u} = \frac{\tilde{B} - B}{r^2} \tilde{u} + \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |u|^p(t, rv) \, dv, \\
&\tilde{u}(0, x) = 0, \quad \tilde{u}(0, x) = \varepsilon \tilde{g}(x),
\end{aligned}$$

where, by Jensen’s inequality,

$$\frac{\tilde{B} - B}{r^2} \tilde{u} + \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |u|^p(t, rv) \, dv \geq |\tilde{u}|^p$$

as long as $\tilde{u} \geq 0$. Then, by Claim 5.2 and $g(x) \geq 0$, we use comparison principle and find $\tilde{u} \geq U \geq 0$. By Lemma 5.3 and the relation between $U$ and $V$, we finish the proof, provided Claim 5.2 and Lemma 5.3 are valid.

Then, we only need to prove Claim 5.2 and Lemma 5.3. The former one is easy to verify by a direct calculation, so we leave it to readers. As for the latter one, we use the idea comes from [3]. To begin with, we introduce a proposition.

**Proposition 5.4** (Lemma 3.2 of [3]). Let $C_1, C_2 > 0$, $\alpha, \beta \geq 0$, $\kappa \leq 1$, $\varepsilon \in (0, 1]$, and $p > 1$. Suppose that $f(y)$ satisfies

$$f(y) \geq C_1 \varepsilon^\alpha, \quad f(y) \geq C_2 \varepsilon^\beta \int_0^y \left(1 - \frac{\eta}{y}\right) \frac{f(\eta)^p}{\eta^\kappa} \, d\eta, \quad y \geq 1.$$

Then, $f(y)$ blows up in a finite time $T_*(\varepsilon)$. Moreover, there exists a constant $C^* = C^*(C_1, C_2, p, \kappa) > 0$ such that

$$T_*(\varepsilon) \leq \begin{cases} 
\exp(C^*\varepsilon^{-((p-1)\alpha + \beta)}) & \text{if } \kappa = 1, \\
C^*\varepsilon^{-((p-1)\alpha + \beta)/(1-\kappa)} & \text{if } \kappa < 1.
\end{cases}$$

**Proof of Lemma 5.3.** Firstly, by the assumption of $g$ and the relation between $g$ and $g^*$, we know that there must exists a positive constant $c_0$ and some region $[a, b]$, such that $g^* \geq c_0$ when $r \in [a, b]$. Without loss of generality we assume $c_0 = 1$, $a = 1/2$, $b = 1$. Then, for $t < r < t + 1/2$, $t + r > 1$, by Claim 5.2 we have

$$V(t, r) \geq \varepsilon \int_{r-t}^{r+t} \left(r^{-1/2} \rho^{-1/2}(\rho + r - t)\right)^{A} g^*(\rho) \, d\rho$$

$$\geq \varepsilon r^{-\frac{1}{4}} \int_0^1 \rho^{\frac{3}{2}} g^*(\rho) \, d\rho$$

$$\geq \varepsilon r^{-\frac{1}{4}}$$.
By Claim 5.2 again, for $0 < t < 2r$, $t - r > 1$ we have

$$V(t, r) \gtrsim \int_0^t \int_{r-t+s}^{r+t-s} \left( \frac{\rho + r + s - t}{t^{1/2} \rho^{1/2}} \right)^A \rho^{\frac{1}{2} - (1-p)} |V|^p(s, \rho) \, d\rho \, ds.$$

Noticing that $\Sigma := \{(s, \rho) : 0 < \rho - s < 1/2, t - r < s + \rho < t + r\}$ is a subset of the integral region, we have

$$V(t, r) \gtrsim \varepsilon^p \int_{\Sigma} \left( \frac{\rho + r + s - t}{t^{1/2} \rho^{1/2}} \right)^A \rho^{\frac{1}{2} - (1-p)} |V|^p(s, \rho) \, d\rho \, ds$$

$$= \varepsilon^p \int_{\Sigma} \left( \frac{\rho + r + s - t}{t^{1/2} \rho^{1/2}} \right)^A \rho^{\frac{1}{2} - (1-p)} |V|^p(s, \rho) \, d\rho \, ds.$$

Using the change of variable $\xi = s + \rho$, $\eta = s - \rho$, we obtain

$$V(t, r) \gtrsim \varepsilon^p r^{-(p-\frac{1}{2})} \int_{t-r}^{t+r} \int_0^{\frac{3(t-r)}{2}} (\xi + r - t)^A \xi^{-\frac{1}{2} + \frac{1}{2}(1-p)} \eta^p \, d\eta \, d\xi$$

$$\gtrsim \varepsilon^p r^{-(p-\frac{1}{2})} (t - r)^{-\frac{1}{2}} \int_{t-r}^{t+r} (\xi + r - t)^A \xi^{-\frac{1}{2} + \frac{1}{2}(1-p)} \, d\xi.$$

Since $t < 2r$, we have $t + r > 3(t - r)$, so that

$$V(t, r) \gtrsim \varepsilon^p r^{-(p-\frac{1}{2})} \int_{t-r}^{t+r} (\xi + r - t)^A \xi^{-\frac{1}{2} + \frac{1}{2}(1-p)} \, d\xi$$

$$\gtrsim \varepsilon^p r^{-(p-\frac{1}{2})} (t - r)^{-\frac{1}{2}} \int_{t-r}^{t+r} (\xi + r - t)^A \xi^{-\frac{1}{2} + \frac{1}{2}(1-p)} \, d\xi.$$

Since it is so, we set $p^* := \frac{1}{2}((n + A - 1)p - (n + A + 1))$ and consider

$$f(y) := \inf_{(s, \rho) \in \Omega_y} \rho^p (s - \rho)^p V(s, \rho),$$

$$\Omega_y := \{(s, \rho) : 0 \leq s \leq 2\rho, s - \rho \geq y\}.$$

By the discussion before, we firstly find that $f(y) \geq C_1 \varepsilon^p$ for any $y > 1$ and some positive constant $C_1$. What’s more, by Claim 5.2 again we find that, for any $(t, r) \in \Omega_y$ with $y \geq 1$, if we set

$$\tilde{\Omega}_{z, \eta} := \{(s, \rho) : \rho \geq \eta, s + \rho \leq 3\eta, s - \rho \geq z\} \subset \Omega_z, \quad z \geq 1, \quad \eta \geq 1,$$

we have

$$V(t, r) \gtrsim \int_{\tilde{\Omega}_{1, t-r}} \left( \frac{\rho + r + s - t}{t^{1/2} \rho^{1/2}} \right)^A \rho^{\frac{1}{2} - (1-p)} |V|^p(s, \rho) \, d\rho \, ds$$

$$\gtrsim r^{-A/2} \int_{\tilde{\Omega}_{1, t-r}} \left( \frac{\rho + r + s - t}{t^{1/2} \rho^{1/2}} \right)^A f(s - \rho)^p \rho^{p^* + A + 1} \, d\rho \, ds.$$
This shows
\[ f(y) \geq C_2 \int_1^y \left( 1 - \frac{\eta}{y} \right) f(\eta)^{p_{\text{pp}^*}} d\eta \]
with \( y > 1 \) and some constant \( C_2 \). By Proposition 5.4 with \( \kappa = pp^* = \frac{h(n + A,p)}{2} + 1 \) and the relation between \( V \) and \( f \), we finish the proof.

Acknowledgment. The second author was partially supported by Grant-in-Aid for Science Research (No.19H01795 and No.16H06339), JSPS.

REFERENCES

[1] M. D’Abbicco, S. Lucente, and M. Reissig. A shift in the Strauss exponent for semilinear wave equations with a not effective damping. *J. Differential Equations*, 259:5040–5073, 2015.
[2] V. Georgiev, Ch. Heiming, and H. Kubo. Supercritical semilinear wave equation with non-negative potential. *Comm. Partial Differential Equations*, 26(11-12):2267–2303, 2001.
[3] Vladimir Georgiev, Hideo Kubo, and Kyouhei Wakasa. Critical exponent for nonlinear damped wave equations with non-negative potential in 3D. *J. Differential Equations*, 267(5):3271–3288, 2019.
[4] Vladimir Georgiev, Hans Lindblad, and Christopher D. Sogge. Weighted Strichartz estimates and global existence for semilinear wave equations. *Amer. J. Math.*, 119(6):1291–1319, 1997.
[5] M Ikeda and M Sobajima. Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping, to appear in *funkcialaj ekvacioj*. arXiv preprint arXiv:1709.04401.
[6] Fritz John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.*, 28(1-3):235–268, 1979.
[7] H. Kubo and M. Ohta. *On the global behavior of classical solutions to coupled systems of semilinear wave equations*, volume 159 of *Operator Theory Adv. and Appl.* Birkhäuser Verlag, 2005.
[8] N. N. Lebedev. *Special functions and their applications*. Dover Publications, Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication.
[9] Thomas C. Sideris. Nonexistence of global solutions to semilinear wave equations in high dimensions. *J. Differential Equations*, 52(3):378–406, 1984.
[10] W. Strauss and K. Tsutaya. Existence and blow up of small amplitude nonlinear waves with a negative potential. *Discrete Contin. Dyn. Syst.*, 3(2):175–188, 1997.
[11] Hiroyuki Takamura. Improved Kato’s lemma on ordinary differential inequality and its application to semilinear wave equations. *Nonlinear Anal.*, 125:227–240, 2015.
[12] Chengbo Wang and Xin Yu. Recent works on the Strauss conjecture. In *Recent advances in harmonic analysis and partial differential equations*, volume 581 of *Contemp. Math.*, pages 235–256. Amer. Math. Soc., Providence, RI, 2012.
[13] Borislav T. Yordanov and Qi S. Zhang. Finite time blow up for critical wave equations in high dimensions. *J. Funct. Anal.*, 231(2):361–374, 2006.
[14] Yi Zhou and Wei Han. Blow-up of solutions to semilinear wave equations with variable coefficients and boundary. *J. Math. Anal. Appl.*, 374(2):585–601, 2011.

W. Dai

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, P.R.China

E-mail address: daiwi6@zju.edu.cn

H. Kubo

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: kubo@math.sci.hokudai.ac.jp