On Functional Determinants of Laplacians in Polygons and Simplices

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Abstract

The functional determinant of an elliptic operator with positive, discrete spectrum may be defined as \(e^{-Z'(0)}\), where \(Z(s)\), the zeta function, is the sum \(\sum_{n=1}^{\infty} \lambda_n^{-s}\) analytically continued to \(s\) around the origin. In this paper \(Z'(0)\) is calculated for the Laplace operator with Dirichlet boundary conditions inside polygons and simplices with the topology of a disc in the Euclidean plane. The domains we consider are hence piece–wise flat with corners on the boundary and in the interior. Our results are complementary to earlier investigations of the determinants on smooth surfaces with smooth boundaries. The computation takes the form of a variation of the shape of the domains, which is chosen such that the coordinate transformations are conformal everywhere but at the corners. The contributions to the variation of \(Z'(0)\) in simplices then come almost exclusively from terms with singular support at the corners: in polygons there are no contributions but from the corners. We have explicit closed integrated expressions for triangles and regular polygons. Among these, there are five special cases (three triangles, the square and the circular disc), where the \(Z'(0)\) are known by other means. One special case fixes an integration constant, and the others provide four independent analytical checks.
1 Introduction

One of the basic integrals that arises in many parts in physics is

\[ \int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^{3/2}} e^{-\frac{1}{2} x_k A x} = (\text{Det}A)^{-\frac{1}{2}} \tag{1} \]

where \( A \) is a real, symmetric matrix with positive eigenvalues. For instance, let (\( \mathbb{H} \)) describe the integration of fluctuations around a classical solution in quantum mechanics, where the Lagrangian has been expanded up to second order. The dimension of \( A \) is then infinite, and (\( \mathbb{H} \)) is divergent on both sides. The equation is therefore undefined as it stands. As a basic example take a one-dimensional harmonic potential, and the fluctuations in the time interval \( 0 < \tau < L \). Then

\[ xA x = \int_{0}^{L} d\tau (\partial_{\tau} x)^2 + \omega^2 x^2, \tag{2} \]

and we have Dirichlet boundary conditions for \( x \) at \( \tau = 0 \) and \( \tau = L \). The most straightforward way, is then to go back to the Gaussian integral (\( \mathbb{H} \)), reintroduce a cut-off \( \epsilon \) in space, and modify the integration measure depending on the cut-off so that the limit when \( \epsilon \) goes to zero is finite. In quantum mechanics this is feasible: changing \( (2\pi)^{-\frac{3}{2}} \) to \( (2\pi \epsilon)^{-\frac{3}{2}} \), and including one more factor \( (2\pi \epsilon)^{-\frac{1}{2}} \), turns (\( \mathbb{H} \)) to a discrete approximation to Feynman’s sum over paths, which in the limit gives

\[ \text{Regularized}[\text{Det}A]^{-\frac{1}{2}} = \left( \frac{2\pi \sinh(\omega L)}{L} \right)^{-\frac{1}{2}}, \tag{3} \]

and this is the correct expression in the Greens function.

We may also observe that the eigenvalues of \( A \) are \( \frac{\pi^2 n^2}{L^2} + \omega^2 \), and the determinant is then formally

\[ \text{det} A \sim \prod_{n=1}^{\infty} \left( \frac{\pi^2 n^2}{L^2} + \omega^2 \right) \tag{4} \]

One way to regularize the determinant is to introduce a cut-off \( \Lambda \) in the product (\( \mathbb{H} \)), check that in the limit of large \( \Lambda \) the result separates into one finite factor and one factor divergent with \( \Lambda \), and keep the finite factor as the regularized result. For (\( \mathbb{H} \)), this gives the same result as (\( \mathbb{H} \) [16]).

A regularization can also be found from zeta function of the operator;

\[ Z_A(s) = \sum \lambda_k^{-s} \tag{5} \]

which converges when the real part of \( s \) is large enough. When this function can be analytically continued to be regular in a neighbourhood of the origin, then

\[ \text{Regularized}[\text{Det}A] = e^{-Z_A'(0)} \tag{6} \]
For (4), this again gives the same result as (3). The zeta function method was first introduced in the context of regularizing expressions like (1) by Hawking\cite{6}, to study fluctuating fields in a background of curved space.

In dimension higher than one, it is not evident that all regularizations of the determinant give the same result. In this paper we will compute $Z_A(0)$, with $A$ the Laplace operator with Dirichlet boundary conditions in two-dimensional simplicial domains. In the language of field theory this is the gaussian model, a free massless theory, albeit with unusual and obstructing boundary conditions. Even so, the direct limit is far from trivial, and the most relevant result we are aware of, works only for lattice Laplacians, discretized on rectangular $(M \times N)$ domains\cite{5}:

$$\det A \sim 2^{\frac{3}{2}} e^{\frac{G M N}{4}} (1 + \sqrt{2})^{-\frac{(M+N)}{2}} (M N)^{-\frac{1}{4}} \eta(q) \left(\frac{M}{N}\right)^{\frac{1}{4}}$$

(7)

Here, in lattice units, $MN$ is the area, $2(M+N)$ is the length of the boundary, $q = e^{-2\pi N/M}$ is the modular parameter, $G$ is Catalan’s constant and $\eta(q)$ is the modular form of Dedekind. There are now no less than three terms separately diverging with the size of the lattice. If, with hindsight, we use that for rectangular domains $Z_A(0) = \frac{1}{4}$, we can rewrite (7) in terms of an explicit lattice spacing $a$:

$$\det A \sim \frac{\text{Area}}{a^2} \frac{\text{Length}}{a L} (a^2 Z_A(0)) e^{-Z_A(0) \log \text{Area}}$$

(8)

where $e^{-B}$ are the various remaining terms in (7) which agree\cite{1} with $e^{-Z_A(0)}$\cite{2} (see Appendix 11). If nothing else, it seems likely that a discretization on a rectangular grid, of a domain which is not itself of rectangular shape, will give rise to oscillating terms in the cut–off. If (8) is to be generally valid in two dimensions, it can probably only be of a smoothened discretized determinant, where the smoothening goes over cut–off scales. Assuming that this can be done, and considering that the area, the length of the boundary, and $Z_A(0)$ are all integrals of local distributions (see section 2), it is possible to introduce local cut–off dependent counter–terms, such that the the finite remaining piece is $e^{-Z_A(0)}$.

It therefore at least makes sense to define the regularized determinant to be $e^{-Z_A(0)}$, and this is the view we take in the rest of this paper. We will use the notation $Z'_D(0)$ for our generic case: the zeta function of the Laplacian with Dirichlet boundary conditions in a simplicial domain $D$, with the topology of a disc. We will freely change the index of the zeta function to denote various special cases, and even contributions to the regularized determinant from some parts of the domain.

If we look for physical relevance, we must have fluctuating geometry, as otherwise all would disappear in an overall normalization. It is a quite old idea that the elementary excitations of non–Abelian gauge theories are string–like objects\cite{21,17,1}. In lattice gauge theories, the statistical weight of a Wilson loop, when a quark and an anti–quark are taken apart for some time, is the area of the smallest area delimited by the loop. A phenomenological model of the excitations was proposed\cite{17}, where the statistical weight

\footnote{up to a constant factor $2^{1/4}$}
of a surface is its area, and the path integral goes over imbeddings in \(d\)-dimensional external space ("\(x^\mu\)"), and over internal two-dimensional geometry ("\(g^{ab}\)\):

\[
Z \sim \int D[g^{ab}] D[x^\mu] e^{-\frac{1}{2} \left[ \int \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\mu \right]} \tag{9}
\]

Our computations are relevant to a part of the investigations of (9). With \(g^{ab}\) fixed, the integration over \(x^\mu\) is just a quadratic integral with Dirichlet boundary conditions like (1). Our calculations hence gives the finite piece of this determinant. As is well known, reparametrization invariance of the action in (9), gives rise to Faddeev–Popov determinants, which for smooth surfaces turn out to be determinants of Laplacians acting on vector fields, with modified Dirichlet boundary conditions. We have not investigated these determinants, and we are not quite sure they are relevant when we consider piecewise flat surfaces with sharp corners. At least in a class of simplices with fixed number of corners, such a surface is its own model, and we would not have any more reparametrization invariance. In this respect, an approach closer to the simplicial discretization of (9) would seem to be more appropriate[3].

Let us now see why it may be interesting to investigate the determinants on simplicial disc-like domains. Smooth disc-like manifolds with smooth boundary can always be mapped conformally onto one another. If we denote the conformal factor \(\sigma(x)\), the base metric and curvature by \(\hat{g}\) and \(\hat{R}\), and the base geodetic curvature of the boundary by \(\hat{k}\), a celebrated result[17, 1] says that

\[
Z'_{D'}(0) = Z'_{D}(0) - \frac{1}{4\pi} \int_{\partial D} d\hat{s}\hat{n} \cdot \partial\sigma + \frac{1}{6\pi} \int_{\partial D} d\hat{s}\hat{k}\sigma + \frac{1}{12\pi} \int_D d^2z \sqrt{\hat{g}} [\hat{g}^{ab} \partial_a \sigma \partial_b \sigma + \hat{R}\sigma] \tag{10}
\]

The integration constant can be computed from the upper half sphere[20]. If we disregard the boundary terms, the equations of motion of (10) are Liouville’s equation, and we refer to the combined action in (10) as the Liouville action.

In ordinary Feynman path integrals (1) the typical path is quite rough, i.e. a nowhere smooth random walk. It seems likely that the typical surface that enters in (1) is also quite rough. On two-dimensional smooth surfaces, one can open up a corner with angle \(2\pi\alpha\) (or \(\pi\alpha\) at the boundary) with a coordinate transformation which is conformal and regular everywhere but at the corner, where it instead has a logarithmic singularity. The kinetic energy term in (10) will then be logarithmically divergent at the corner. In other words, the action for smooth (and conformal) coordinate changes is infinite for these transformations. One possible procedure is then to introduce a cut-off \(r_0\) at a corner, and simply remove the quantities diverging when \(r_0\) goes to zero[7]. This is obviously dangerous; if one is not careful one easily gets a spurious finite piece from a logarithmic divergence, and \(Z'_{D'}(0)\) is a well-defined mathematical object, which has a value, the surface being smooth or not. The correct interpretation is that a conformal transformation does not change \(Z_D(0)\), but a non-conformal one does. In fact, for piece-wise flat surfaces, \(Z_D(0)\) can be written as a sum over a rational functions of the opening angles of the
corners (see section 2, Appendix 6). A typical non–conformal transformation will change the opening angles by a shear, so it will change \( Z_D(0) \). The infinity of \( (11) \) under non–conformal transformations is therefore a mirror of actually \( Z_D(0) \) changing.

Our computation give some more explicit results on determinants, to which one does not have access from smooth models. This can have some mathematical interest by itself. More speculatively, it is possible that a definition of determinants by a precise calculation of \( Z'_D(0) \), may yield a better regularization of \( (11) \) than does \( (10) \) and its Faddeev–Popov ghosts. Certainly, such a result would go far beyond what is actually done here: we have not begun to address a computation of a sum over surfaces as in \( (9) \).

The conclusions of this paper can now be stated as the following propositions:

**Proposition 1.** Domains with disc–like topology may be mapped to the upper half complex plane, the boundary being mapped to the real axis, and the corners being mapped to branch–points \( \omega_j \). If \( z \) is the coordinate in the domain, the transformation satisfies

\[
\frac{d\omega}{dz} = \phi_D \prod_j (\omega - \omega_j)^{1-\alpha_j},
\]

and the representation is determined by the angles and the lengths of the sides, up to a rational fractional transformation of the upper half plane. Choosing one parametrization the normalized area of the simplex is:

\[
\text{Area}_D = \int_{\Im \omega > 0} \frac{d\omega \wedge d\omega}{2i} \prod_j |\omega - \omega_j|^{2\alpha_j - 2},
\]

and the variation of \( Z'_D(0) \) under a general shear and dilatation can be written;

\[
\delta[Z'_D(0)|_{\text{Area} = A}] = \delta[Z'_D(0)] + \delta[Z_D(0) \log A] - \delta[Z_D(0) \log \text{Area}_D]
\]

where then value of the zeta function at the origin is;

\[
Z_D(0) = \sum_{\text{interior corners}} \frac{1}{12} \left( \frac{1}{\alpha_j} - \alpha_j \right) + \sum_{\text{boundary corners}} \frac{1}{24} \left( \frac{1}{\alpha_j} - \alpha_j \right)
\]

and the opening angles are written \( 2\pi \alpha_j \) in the interior, and \( \pi \alpha_j \) on the boundary.

**Proposition 2.** When the area is chosen \( \text{Area}_D \), the difference of \( Z'_D(0) \) between two simplices, that differ by an infinitessimal transformation, which is regular and conformal everywhere except at the corners, can be written as a sum over quantities with point mass support at the corners, and a line density on the boundary. The integral over of the boundary term is identically zero for polygons, but gives

\[
-4\pi \sum_{\text{interior corners}} \delta \alpha_i,
\]
if there are corners in the interior.

**Proposition 3.** The contribution to $\delta Z_D'(0)$ from the point mass in one corner $c$, is a sum

$$\delta Z_D'(0)|_c = \delta Z_D'(0) + Z_{\alpha_c}(0) \cdot \delta[\text{other corners}]$$  \hspace{1cm} (16)

where $\delta Z_D'(0)$ is exclusively determined locally at the corner, $Z_{\alpha_c}(0)$ is the contribution to $Z_D(0)$ from the corner, and the influence from the other corners depend on the opening angles of these, and on the lengths of the sides:

$$Z_{\alpha_c}(0) \cdot \delta[\text{other corners}] = Z_{\alpha_c}(0) \left[ \sum_{c' \neq c} \delta \alpha_{c'} \log |\omega_{c'} - \omega_c|^2 - (1 - \alpha_{c'}) \right]$$  \hspace{1cm} (17)

**Proposition 4.** The strictly local contributions may be given in integrated form, and are for a corner on the boundary:

$$Z'_{\alpha}(0) = \frac{1}{12} \left( \frac{1}{\alpha} - \alpha \right) (\gamma - \log 2) - \frac{1}{6} \left( \frac{1}{\alpha} + 3 + \alpha \right) \log \alpha + \tilde{J}(\alpha),$$  \hspace{1cm} (18)

and for a corner in the interior:

$$Z''_{\alpha}(0) = \frac{1}{6} \left( \frac{1}{\alpha} - \alpha \right) (\gamma - \log 2) - \frac{1}{6} \left( \frac{1}{\alpha} + \alpha \right) \log \alpha + 2 \tilde{J}(\alpha),$$  \hspace{1cm} (19)

where the term $\tilde{J}$ has the integral representation:

$$\tilde{J}(\alpha) = \int_0^\infty \frac{1}{x^2 - 1} \left[ \frac{1}{2x} \left( \coth \left( \frac{x}{2\alpha} \right) - \coth \left( \frac{x}{\alpha} \right) \right) - \frac{1}{12} \left( \frac{1}{\alpha} - \alpha \right) \right] dx.$$  \hspace{1cm} (20)

We have here included an integration constant in our definition of $\tilde{J}$.

Special situations are quite important to us, as they provide necessary checks. We therefore list the them separately:

**Proposition 5.** For triangles one may choose a parametrization where the branchpoints lie fixed in 0, 1 and $\infty$. The normal area of a triangle with angles $\pi \alpha_1, \pi \alpha_2, \pi \alpha_3$ is then

$$\text{Area}(\alpha_1, \alpha_2, \alpha_3) = \frac{\pi}{2} \frac{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)}{\Gamma(1 - \alpha_1) \Gamma(1 - \alpha_2) \Gamma(1 - \alpha_3)}$$  \hspace{1cm} (21)

and all terms but the strictly local in the variation of $Z_D'(0)$ vanish. The determinant of a triangular domain is thus

$$Z'_T(0) = \sum_{p=1,2,3} Z'_{\alpha_p}(0)$$  \hspace{1cm} (22)

with $Z'_{\alpha}(0)$ as in Proposition 4.
Proposition 6. For regular polygons with \( n \) corners one can choose a parametrization by mapping to the unit disc, where the corners are regularly spaced on the unit circle. The radius of the circumscribed circle is with this parametrization:

\[
R_n = \frac{\Gamma(1 + \frac{1}{n})\Gamma(1 - \frac{2}{n})}{\Gamma(1 - \frac{1}{n})} \quad (23)
\]

and the determinant in a regular polygon with radius of circumscribed circle \( R \) is;

\[
Z'_{P_n}(0) = Z_{P_n}(0) \log \frac{R^2}{R_n^2} - \frac{1}{3(n-2)} \log n + nZ'_{1 - \frac{2}{n}}(0) \quad (24)
\]

where \( Z_{P_n}(0) = \frac{n-1}{6(n-2)} \). Specializing to \( n = 2 \) we obtain the determinant in a square:

\[
Z'_{P_4}(0) = \frac{1}{4} \log \text{Area} + \frac{1}{2} \log \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{1}{4} \log \pi + \frac{5}{4} \log 2 \quad (25)
\]

Taking the limit as \( n \to \infty \) we obtain the determinant in a disc:

\[
Z'_{P_{\infty}}(0) = \frac{1}{3} \log R + \frac{5}{12} + \frac{1}{2} \log \pi + \frac{1}{6} \log 2 + 2\zeta'(-1) \quad (26)
\]

The paper is organized as follows: standard results on heat kernels and zeta functions are summarized in section 2. In section 3 a representation is found for \( \delta Z'(0) \) which treats the corners and the rest in different ways; this is the main idea. In both cases the representation is in terms of the short–times asymptotics of the heat kernel. In the interior this representation is standard, and gives the variational form of (10). That the underlying space is flat, and with disc–like topology, actually simplifies things so much that this contribution gives only the simple sum in Proposition 1. In the corners the appropriate asymptotics is the Sommerfeldt heat kernel in an infinite sector. Most of the derivations are by calculation, and most are elementary (although sometimes cumbersome) We have chosen to present those in appendices, largely in a self–contained way, but without any reference to physical arguments. The appendices can therefore be read more or less independently from the rest of the paper. Section 4 summarizes the results.

2 Heat kernel and Zeta function phenomenology

The diagonal elements of the heat kernel on smooth manifolds admit an asymptotic expansion for short times. In two dimensions the expansion goes as

\[
K_D(x,x,t) \sim \frac{c_{-1}(x)}{t} + \frac{c_{-\frac{1}{2}}(x)}{t^{\frac{1}{2}}} + c_0(x) + \ldots \quad (27)
\]
McKean and Singer proved the existence of this expansion for smooth manifolds up to terms \( t \), and for manifolds with smooth boundary up to \( t^{\frac{3}{2}} \). In the interior the coefficients \( c_i \) are polynomials in the curvature tensor, and on the boundary they are line distributions, with weight depending on the curvature of the boundary. For polygonal domains in the plane, Kac proved the expansion up to \( c_0(x) \), which in this case is made up of point masses at the corners. Since the curvature of a polygon can be said to be concentrated at the corners, the result is in some sense natural, and \( c_0(x) \) for a smooth boundary can indeed be found as the limiting case of polygons with angles coming closer and closer to \( \pi \). However, the converse is not true, that is, if one wants to compute \( Z_D(0) \) for a manifold with tips and corners, it is not possible to use a smoothened approximation.

The integrated form of (27) is

\[
\int_D dx^2 K_D(x, x, t) = \Theta_D(t) \sim \frac{c_{-1}}{t} + \frac{c_{-\frac{1}{2}}}{t^{\frac{3}{2}}} + c_0 + \ldots
\]  

The zeta function is defined by a Mellin transform of the trace of the heat kernel as

\[
Z_D(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \Theta_D(t)
\]  

The integral only converges in the lower limit if \( \Re s > -\alpha_i \) for all terms \( c_{\alpha_i} t^{\alpha_i} \) in the heat kernel, i.e. \( \Re s > 1 \) in two dimensions. One can get around the pole by a partial integration, where the boundary terms vanish for \( \Re s > 1 \):

\[
\frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \Theta_D(t) = \frac{1}{\Gamma(s)} \int_0^\infty dt \left[ \frac{t^{s-1}}{s-1} \right] \Theta_D(t)) \bigg|_0^\infty - \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{s-1} \frac{\partial}{\partial t} [t \Theta_D(t)]
\]  

The second integral now gives an analytic continuation down to \( \Re s > \frac{1}{2} \). Further on the zeta function has poles at \( s = -\alpha_i \), with residue \( \frac{c_{\alpha_i}}{\Gamma(\alpha_i)} \), except at zero and the negative integers, where the inverse gamma function has zeros, which gives finite values \( Z(-n) = (-)^n c_{-n} n! \).

For positive \( n \) greater than 1, a method that goes back to Euler [19] gives a representation of the zeta function as a convolution of electrostatic Greens functions:

\[
Z_D(n + 1) = Tr_x (\frac{1}{-\Delta})^{n+1} = \int dx \int dy_1 \ldots \int dy_n G_D(x, y_1)G_D(y_1, y_2) \ldots G_D(y_n, x)
\]  

This method can be extended to calculate the finite part of \( Z_D(s) \) at \( s = 1 \). Hence, if the expansion (28) goes on indefinitely and one knows the Greens function, one may in principle calculate the zeta function at the integers, and the residues at all the poles. But if this is true, and the zeta function does not grow too fast at infinity, then the zeta function is completely determined, which (somewhat indirectly) determines the eigenvalues. Hence all the information about the eigenvalues, and on all quantities depending on them, is
then contained in the Green’s function and the asymptotic expansion of the heat kernel for short times.

It is natural to consider what we will call a zeta function density:

\[ Z_D(x, x, s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} K_D(x, x, t) \]  

(32)

For \( \Re s < 1 \) we define the zeta function density by analytic continuation from (32). At zero, the zeta function density is determined by the asymptotics of the heat kernel for short times as in (27):

\[ Z_D(x, x, 0) = c_0(x) \]  

(33)

We may just as well consider the density associated to the derivative of the zeta function at the origin, but this quantity will depend also on the heat kernel for long times. These questions are addressed in more detail in Appendix 1.

One could also look at off–diagonal elements in (32), or equivalently, the operator \((-\Delta)^s\). Since the off–diagonal Mellin transforms are finite for all \( s \), the analytical continuation brings no off–diagonal subtractions, and effectively the diagonal and off–diagonal elements are treated differently for \( \Re s < 1 \). It would be interesting to have a proper regularization of off–diagonal terms, so that one might consider \( Z'_D(0) \) as the trace of a (regularized) operator \( Q \sim -\log(-\Delta) \), and in terms of which a regularized Laplace operator would be \( e^{-Q} \) (in the operator sense). In this paper we will only consider the diagonal elements, and we therefore refrain from calling the zeta function density the matrix elements of an operator when \( \Re s < 1 \).

Let us consider a dilation \( x \rightarrow \lambda x \) of the domain. That changes the eigenvalues in a simple way: \( E_n \rightarrow \lambda^{-2} E_n \), and the zeta function accordingly also changes simply: \( Z_{\lambda D}(s) \rightarrow \lambda^{2s} Z_D(s) \). For the quantities around the origin this implies:

\[ Z_{\lambda D}(0) = Z_D(0) \quad Z'_{\lambda D}(0) = Z_D'(0) + \log \lambda^2 Z_D(0) \]  

(34)

The invariance of \( Z_D(0) \) under dilations is a special case of invariance under regular conformal transformations. The change in \( Z'_D(0) \) logarithmically proportional to the variation of the area, is contained in Proposition 1, and was already noticeable in the asymptotic expansion of the discretized determinant in rectangular domains (8).

3 A variational formula

In this section we will write down a variational formula. It can be formulated either in terms of the asymptotics of the heat kernel for short times, or in terms of the zeta function density from section 2. Let us first motivate why we give two computations, that essentially only differ in that they work on opposite sides of the Mellin transform: the approach using the heat kernel is more traditional, but requires more delicate treatment to separate out the finite piece. Note that our formulae also differ by (a fairly simple
term) from a definition of the regularized determinant often used in field theory\[1, 17\]. The reason for this discrepancy is that we also consider transformations that are not conformal, and here the additional term matters. The approach using the zeta function is computationally somewhat simpler, once the notion of a zeta function density is admitted.

Let now \( D \) and \( D' = D + \delta D \) be two domains that differ infinitesimally. Then, for \( \Re s > 1 \),

\[
Z_{D'}(s) - Z_D(s) = \sum_{n=0}^{\infty} [E_n'' - E_n' - E_n] = -s \sum_{n=0}^{\infty} \left( \frac{\delta E_n}{E_n} \right) E_n' - s
\]

and this definition is analytically continued to lower values of \( s \). We can consider \( \{E'_n\} \) to be the eigenvalues of a modified operator \((-\Delta - \delta \Delta)\) in the domain \( D \), so that we write

\[
Z_{D'}(s) - Z_D(s) = s \text{Tr}_x[(\delta \Delta)Z_D(x, x, s + 1)]
\]

The variations we will consider map one simplex on another. They therefore leave the following sum invariant:

\[
\mathcal{S}(\alpha_j) = + \sum_{\text{boundary corners}} (1 - \alpha_j) + \sum_{\text{interior corners}} (2 - 2\alpha_j) = 2
\]

where the opening angle of a corner on the boundary (in the interior) is \( \pi \alpha \) (2\( \pi \alpha \)). The analytical expression for our variation will therefore only be determined up to the variation of a \((s\text{-dependent})\) function of \( \mathcal{S}(\alpha_i) \), that is, up to terms linear in the variations of the angles.

If we want to evaluate the variation of the derivative at zero we can do it as

\[
\frac{d}{ds} \delta Z_D(s) |_{s=0} = \lim_{s \to 0} \frac{1}{s} \text{Tr}_x[(Z_{D'}(x, x, s) - Z_D(x, x, s)) - (Z_{D'}(x, x, 0) - Z_D(x, x, 0))] = \text{Finite}
\]

which can be rewritten as

\[
\frac{d}{ds} \delta Z_D(s) |_{s=0} = \lim_{s \to 0} \frac{1}{s} \text{Tr}_x[Z_{D'}(x, x, s) - Z_D(x, x, s)]
\]

In general there will be a pole at the origin in \( s \) in (39), with a residue equal to the variation of \( Z_D(0) \). Combining (33) and (36) we have

\[
\frac{d}{ds} \delta Z_D(s) |_{s=0} = \text{Finite} \frac{1}{s} \text{Tr}_x[(\delta \Delta)Z_D(x, x, s + 1)]
\]

It is now convenient to choose the variation in a particular way. Think first of the Laplace–Beltrami operator in curved space, with metric tensor \( g^{ab} \): \( \Delta = g^{-\frac{1}{2}} \partial_a g^{ab} g^{\frac{1}{2}} \partial_b \). In two dimensions we can choose the particular form \( g^{ab} = e^{2\sigma(x)} \delta_{ab} \), and for a smooth surface with disc topology we can take \( g^{ab} \) to be the standard flat metric. In this coordinate system, the Laplacian has the following form \( \Delta = e^{-2\sigma} \partial_a^2 \). A conformal variation of the Laplace–Beltrami operator is then \( \delta \Delta = (-2\delta \sigma) \Delta \), and the final variational formula is

\[
\frac{d}{ds} \delta Z_D(s) |_{s=0} = \text{Finite} \frac{1}{s} \text{Tr}_x[2\delta \sigma(x)Z_D(x, x, s)]
\]
The formula (41) holds also when we include corners with varying opening angles, the only new effect then being that prefactor weight $\delta \sigma$ has a logarithmic singularity at the corner.

We will show in Appendix 3 that in flat domains far from the boundary, the density $Z_D(x,x,s)$ vanishes at the origin. Hence the interior of flat domains, far from the corners, give no contribution at all to $Z_D'(0)$. This is in agreement with (9), since our base space is flat (hence $\hat{R}$ is zero), and the kinetic energy term of a regular conformal variation can only give a boundary term. At straight boundaries, there is an undetermined contribution to $Z_D'(0)$, proportional to the boundary length. The contribution to the variation, $\delta Z_D'(0)$, is however fully determined following (41), and picks out the normal derivative of the variation $\delta \sigma(x)$. Here again we have exact agreement with (10).

For simplices, the boundary term integrates trivially, and we are left with only the corner contributions as the important parts. Considering that the zeta function density $Z_D(x,x,0)$, has components with point mass support at the corners, it may be surmised that integrated against the logarithm of the distance from the tip, it gives an infinite contribution to (41). This is indeed the case, that infinity is precisely the variation of $Z_D(0)$, which we subtract by taking the finite part as $s$ goes to zero. It is now important to realise that we can choose small areas around the corners, with some radius $r_0$, which we can let tend to zero at the end of the calculation. It therefore does not matter if we compute the answer up to terms proportional to $r_0$ or $r_0^2$, since these will eventually drop out. This means that we can actually substitute the true (unknown) $Z_D(x,x,s)$, with a sufficiently good approximation, computed from the Mellin transform of an approximation to the true heat kernel valid for short times. The technique for doing this is explained in Appendix 5, and the approximation to the heat kernel – the Sommerfeldt heat kernel in a sector – is described in Appendix 4. The calculations of the contributions to $Z_D'(0)$ from the corner is then done in Appendix 6.

Let us now derive an alternative formula to (41) using only the heat kernel for short times. We begin with the following equality (derived in Appendix 1):

$$Z_D'(0) = \gamma Tr_x Z_D(x,x,0) + \text{Finite}_{\epsilon \to 0} Tr_x \int_\epsilon^\infty \frac{dt}{t} K_D(x,x,t) \tag{42}$$

The heat kernel can be expanded in a complete set of states:

$$K_D(x,x,t) = \sum_v |\psi_v(x)|^2 e^{-\lambda_v t} \tag{43}$$

and the variation of (42) will be

$$\delta Z_D'(0) = \gamma \delta Tr_x Z_D(x,x,0) - \text{Finite}_{\epsilon \to 0} \int_\epsilon^\infty dt \sum_v <v|\Delta|v> e^{-\lambda_v t}$$

$$= \gamma \delta Tr_x Z_D(x,x,0) + \text{Finite}_{\epsilon \to 0} Tr_x [2\delta \sigma(x) K_D(x,x,\epsilon)] \tag{44}$$

When we are in the interior, or at a straight boundary, the zeta function density $Z_D(x,x,0)$ is zero. The remaining trace over the heat kernel in (44) then agrees with a well–known
definition of the regularized determinant in field theory, and the short-time expansion of
the heat kernel can be done by standard means [17, 1]. We then find the variational form
of (10). In piece–wise flat simplices there are further simplifications, such that the only
non–zero term is the integral over the boundary of the normal derivative δσ in (10).

Close to the corners, we cannot however use the same short–time expansion of the
heat kernel. Again we have to turn to the Sommerfeldt heat kernel, and carefully extract
the constant piece of the trace of the heat kernel for short times. Finally, we must take
into consideration that the variation of Z_D(0) is not zero at the corner, and add the first
term on the right–hand side of (44). The computation following these lines is done in
Appendix 7.

4 Calculations & Conclusions

In this section we describe the computations in the appendices, and discuss the results.

Appendix 1 is an expansion on Section 2. We establish that the zeta function density
of the derivative at zero is well–defined, but depends on the heat kernel for long times.
An alternative definition of the variation of the regularized determinant in terms on the
heat kernel for short times is derived. Appendix 2 presents a general parametrization of
variation of simplices, and computes the contribution from corners in the interior as in
Proposition 2. In Appendix 3 we treat in some detail the case of a straight boundary.
We derive that it cannot give a a term in Z_D'(x, x, 0) more singular than a line density,
and hence a contribution to Z_D'(0) proportional to boundary length. We also derive the
contribution from the line integral on the boundary to the variation of Z_D'(0). Appendix 4
presents the Sommerfeldt heat kernel in a sector, and Appendix 5 states an elementary
integral, that is convenient when one wants to compute the Mellin transform using the
Sommerfeldt heat kernel.

In Appendix 6 we compute the contribution to Z_D(0), and the strictly local contribu-
tion to Z_D'(0) from a corner with opening angle α, using the approach of the zeta function
density. We call these quantities Z_α(0) and Z_α'(0). We perform the computations for both
corners on the boundary and corners in the interior. In Appendix 7 we do the same thing,
using the slightly more involved approach of the analysis of the short–time properties of
the Sommerfeldt heat kernel. In Appendix 8 we investigate Z_α'(0) in the special cases,
where the opening angle is \frac{1}{n}. In Appendix 9 we investigate the asymptotic behaviour of
Z_α'(0) as α is large, small or close to one.

In Appendix 10 we treat a few special cases using our general variational formula. The
additional computations done here essentially boil down to taking care of the expression in
Proposition 3, that has support at one corner, but depends on the angles of all the corners,
and on the lengths of the sides. In the very special case of triangles, this additional term
is absent, but in polygons it is important. In Appendix 11 we state summarily the cases
we know to compare with, where the Z_D'(0) are known, either because the eigenvalues
are known, or because these domains are conformal images of some domain where the
eigenvalues are known. This is important information: it fixes an integration constant.
that we cannot reach; and it provides much needed analytical checks.

In conclusion we have shown that the determinants on piece–wise smooth surfaces are different from the determinants on smooth surfaces. The formula that describe variations of determinants on smooth surfaces \([10]\) give infinite results for variations that change the opening angles of the corners. This infinity is not real, it is only an imprint of actually \(Z_D(0)\) changing. We have parametrized the variations of the domains such they are conformal everywhere outside the corners, and separated the contributions to the variation of \(Z_D'(0)\) into four parts: one (Proposition 1), that expresses the variation of \(Z_D(0)\) times the logarithm of the area; one (Proposition 2), which is an integral over the boundary of the normal derivative of the conformal parameter; one (Proposition 3) which has local support at the corners, but depend on the opening angles of all the corners and on the lengths of the sides; and one (Proposition 4), which is determined strictly locally at the corners.

The first part amounts to determining how the normalized area changes with the angles and the sides. We can trivially integrate the second part, which is identical to one of the terms in the variation of determinants on smooth surfaces \([10]\). We have integrated the fourth part, both for corners on the boundary and in the interior. For general opening angles this part has an integral representation \([A6.29, A6.30]\), which for the special opening angles \(\alpha = \frac{1}{n}\), can be resolved into a finite sum \([A8.11]\).

We have been able to parametrize the area and integrate the third part, for the special cases of triangles (where the third part vanishes) and a class of polygons, which includes certain interpolations between pairs of regular polygons (Appendix 10). In general, we have not been able to integrate the third part, which therefore has to be left in variational form (Proposition 3).

The wider applicability of our approach evidently depends on better handling of the parts we have not integrated. We do not expect that one will in general be able even to express the normalized area in closed form, as a function of the angles and the sides. Perhaps one may however find a limiting form, which is valid for small opening angles. Considered as an action for simplicial surfaces, it is possible that the full expression of \(Z_D'(0)\) strongly damps out corners with small opening angles. As is well known, most investigations of sums over random surfaces in physical dimensions lead to very rough surfaces, dominated by long sharp spikes. If the full expression of \(Z_D'(0)\) as a function of one opening angle has a singularity at the origin – with the right sign of the prefactor – then surfaces with spikes will be more strongly damped by the action \(Z_D'(0)\), then by any finite expansion in local curvature. We hope to return to questions in this direction in the future.

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A1. Local and nonlocal quantities

In this appendix we collect some results on the zeta function density and compare it to other approaches in the literature. An important aspect is locality: a Mellin transform of a heat kernel, at a point, integrated over some subdomain, or over the whole domain, is said to be local, if it is determined only by the expansion of the heat kernel for short times.

We start with the definition (A1.1)

\[ Z_D(x, x, s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} K_D(x, x, t) \]

It is useful to explicitly analytically continue the integral beyond the poles at \( s = 1, s = \frac{1}{2} \) and also beyond \( s = 0 \). We will then get out a pole in \( s \) at \( s = 0 \) from the integral, which will combine with the gamma–function in front to give a regular function. We thus have a representation of the zeta function density around \( s = 0 \) in terms of a regular prefactor and a convergent integral:

\[ Z_D(x, x, s) = \frac{1}{(s - 1)(s - \frac{1}{2}) \Gamma(s)} \int_0^{\infty} dt t^{s-1} \frac{\partial}{\partial t} [t^{1/2} \frac{\partial}{\partial t} [t^{1/2} \frac{\partial}{\partial t} t K_D(x, x, t)]] \]

Using the asymptotic expansions as \( t \) is close to zero:

\[ K_D(x, x, t) \sim \frac{c_1(x)}{t} + \frac{c_{1/2}(x)}{t^{1/2}} + c_0(x) + c_{-1/2}(x) t^{1/2} + \ldots \]

\[ \frac{\partial}{\partial t} t K_D(x, x, t) \sim \frac{1}{2} \frac{c_{1/2}(x)}{t^{1/2}} + c_0(x) + \frac{3}{2} c_{-1/2}(x) t^{1/2} + \ldots \]

\[ t^{1/2} \left[ \frac{\partial}{\partial t} [t^{1/2} \frac{\partial}{\partial t} t K_D(x, x, t)] \right] \sim \frac{1}{2} c_0(x) + \frac{3}{2} c_{-1/2}(x) t^{1/2} + \ldots \]

we may evaluate as follows

\[ \text{Res} Z(x, x, s)|_{s=1} = c_1(x) \]

\[ \text{Res} Z(x, x, s)|_{s=1/2} = \pi^{1/2} c_{1/2}(x) \]

\[ Z(x, x, 0) = c_0(x) \]

We may also evaluate the derivative with respect to \( s \) at the origin:

\[ Z_D'(x, x, 0) = (3 + \gamma) c_0(x) + 2 \int_0^{\infty} dt \log t \frac{\partial}{\partial t} [t^{1/2} \frac{\partial}{\partial t} [t^{1/2} \frac{\partial}{\partial t} t K_D(x, x, t)]] \]

It is quite clear that \( Z_D'(x, x, 0) \) is a quantity which depends also on the heat kernel at large times.

An alternative formula is obtained by taking the integral from \( \epsilon \) to infinity, where \( \epsilon \) is some small positive number. As the integrand in \( \text{A1.3} \) behaves as \( t^{-1/2} \log t \) for small \( t \),
the integral is convergent, and the limit as \( \epsilon \) goes to zero is harmless. Partial integrations will however bring in divergent terms from the lower boundary. A short calculation gives

\[
Z_D'(x,x,0) = (3 + \gamma)c_0(x) + (c_0(x) \log \epsilon - 2c_1(x)\epsilon^{-\frac{1}{2}} - c_1(x)\epsilon^{-1} - 3c_0(x)) + \int_{\epsilon}^{\infty} \frac{dt}{t} K_D(x,x,t)
\]

which can be simplified to

\[
Z_D'(x,x,0) = \gamma c_0(x) + \text{Finite}_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} K_D(x,x,t)
\]

Let \( \tilde{K}_D \) be an asymptotic approximation to \( K_D \), valid for short times. We may then write the integral in (A1.5) as

\[
\text{Finite}_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{dt}{t} \tilde{K}_D(x,x,t) + \int_{0}^{1} \frac{dt}{t} [K_D(x,x,t) - \tilde{K}_D(x,x,t)] + \int_{1}^{\infty} \frac{dt}{t} K_D(x,x,t)
\]

The second and third terms are finite, if the asymptotic approximation is sufficiently good. Hence they can only contribute to \( Z_D'(x,x,0) \) as a smooth density. The only term which can give a distribution contributions, the space integral of which contains a finite piece even if taken over a vanishingly small area, is the first in (A1.6), and the term \( \gamma c_0(x) \). Both only depend on the heat kernel for short times.

When we integrate (A1.5) over the whole domain, we obtain

\[
Z_D'(0) = \gamma Z_D(0) + \text{Finite}_{\epsilon \to 0} \text{Tr}_x \int_{\epsilon}^{\infty} \frac{dt}{t} K_D(x,x,t)
\]

A2. Variation and parametrization

We normalise the areas by choosing the simplest form of the Schwarz–Cristoffel transformations that map the disc–like domains to the upper half complex plane. Let \( z \) be the variable in the domain, and \( \omega \) the variable in the upper half plane. Then we have:

\[
\frac{d\omega}{dz} = \phi_D \prod_j (\omega - \omega_j)^{1-\alpha_j}
\]

The vertices of the domain map to the branch points \( \omega_j \), which are determined up to a rational fractional transformation of the upper half plane. This redundancy may be eliminated (up to permutations) by fixing the positions of the images of three vertices. An overall phase factor \( \phi_D \) is determined by the orientation of the domain \( D \) in the \( z \)-plane. For the simplest case, triangles, the three branch points may be put in 0, 1 and \( \infty \) in the \( \omega \)-plane, and the side of the triangle between the vertices that are mapped to 0 and 1 can be placed parallel to the real axis in the \( z \)-plane. We then have the familiar formula:

\[
\frac{d\omega}{dz} = \omega^{1-\alpha_0}(1-\omega)^{1-\alpha_1}
\]
A variation of the shape of the domain, brings a simultaneous variation of all the angles, and of the positions of the branch points: the expression for the variation hence concretely depends on the parametrization chosen of the branch points in terms of the opening angles and the side lengths. Let us first consider the behaviour close to a branch point, say \( \omega_0 \). By a linear change of variables, \( \tilde{\omega} = \omega - \omega_0 \) before the variation, and \( \tilde{\omega} = \omega - \omega'_0 \) after the variation, we can consider the branch point \( \omega_0 \) to lie at the origin, and not to move under the variation. Taking a small corner around the inverse image of the branch point (in the \( z \)-plane), and ignoring terms that vary slowly over the corner, we have
\[
2\delta \sigma_{\text{corner}} = \log \left| \frac{dz'}{dz} \right|^2 = \log \left| \frac{d\tilde{z}'}{d\tilde{\omega}} \right|^2 - \log \left| \frac{dz}{d\omega} \right|^2 = \\
\delta \alpha_0 \log |\alpha_0 z|^\frac{2}{\alpha_0} + \sum_{j \neq 0} \delta \alpha_j \log |\omega_j - \omega_0|^2 - \sum_{j \neq 0} (1 - \alpha_j) \left[ \frac{\delta (\omega_j - \omega_0)}{\omega_j - \omega_0} + \text{c.c} \right] \quad (A2.3)
\]
In the bulk (i.e. away from the corners), we do not shift the \( \omega \)-coordinates, and have instead:
\[
2\delta \sigma_{\text{bulk}} = \log \left| \frac{dz'}{dz} \right|^2 = \log \left| \frac{d\tilde{z}'}{d\tilde{\omega}} \right|^2 - \log \left| \frac{dz}{d\omega} \right|^2 = \\
\sum_j \delta \alpha_j \log |\omega - \omega_j|^2 - \sum_j (1 - \alpha_j) \left[ \frac{\delta \omega_j}{\omega - \omega_j} + \text{c.c} \right] \quad (A2.4)
\]
However, by Appendix 3, in piece-wise flat domains, the only bulk contribution will arise from the normal derivative at the boundary of \( A2.4 \), so let us compute that (in the \( \omega \)-plane, that is, at the real axis):
\[
2\hat{n} \cdot \nabla \delta \sigma_{\text{bulk}} = -2 \sum_j \delta \alpha_j \frac{\Im \omega_j}{|\omega - \omega_j|^2} + 2 \sum_j (1 - \alpha_j) \Im \left[ \frac{\delta \omega_j}{(\omega - \omega_j)^2} \right] \quad (A2.5)
\]
If \( \omega_j \) is on the real axis, \( A2.5 \) vanishes identically. If \( \omega_j \) lies in the upper complex plane (an interior corner) we are to integrate the normal derivative along the boundary. The second sum in \( A2.5 \) then clearly gives zero, while the first integrates to
\[
\int_R 2\hat{n} \cdot \nabla \delta \sigma_{\text{bulk}} \, ds = -4\pi \sum_{\Im \omega_j > 0} \delta \alpha_j \quad (A2.6)
\]
For simplices with disc topology, all the contributions except the rather simple result \( A2.6 \) therefore come from the corners, and for polygons, there are no interior corners, and \( A2.3 \) gives the only relevant terms. We have not been able to put \( A2.3 \) in a form which is manifestly a total variation, but in all the special cases where we checked it, it turns out to be so. In some cases it is possible to choose the parametrization of a family of polygons such that the corners on the boundary do not move: in this case all the contributions are from the first two terms in \( A2.3 \). For triangles we have the final simplification in that we can use the parametrization (A2.2), and then the only contributions are from the first term in \( A2.3 \).
A3. Contribution of a straight boundary

On piecewise flat surfaces, away from the corners, we can always approximate the heat kernel with the free space heat kernel, or the heat kernel close to a straight boundary. The error term \((K_D - \tilde{K}_D\) in [A1.6]) is exponentially small in \(\frac{1}{t}\). In free space the diagonal element of the heat kernel is \(K_D(x,x,t) = \frac{1}{4\pi t}\). That leads to a contribution to \(Z(x,x,s)\) from \(\tilde{K}_D\) in [A1.6] which has a singularity at \(s = 1\) with residue \(\frac{1}{4\pi}\), but which is regular at lower \(s\). Inside the piecewise flat surfaces there are therefore no contributions to \(Z(0)\), or to the variation of \(Z'(0)\).

The heat kernel in the half plane with Dirichlet conditions on the \(x\)-axis is expressed as the difference between a "direct" and a "reflected" term:

\[
K((x_1, y_1), (x_2, y_2), t) = \frac{1}{4\pi t} \left( \exp\left(-\frac{(x_1-x_2)^2 + (y_1-y_2)^2}{4t}\right) - \exp\left(-\frac{(x_1-x_2)^2 + (y_1+y_2)^2}{4t}\right) \right)
\]

(A3.1)

The diagonal elements are then

\[
K((x,y), (x,y), t) = \frac{1}{4\pi t} \left( 1 - \exp\left(-\frac{y^2}{t}\right) \right)
\]

(A3.2)

We compute the Mellin transform multiplied with a convergence factor \(\exp(-\mu t)\), and \(s\) between \(\frac{1}{2}\) and 1:

\[
\frac{1}{\Gamma(s) 4\pi} \int_0^\infty dt t^{s-1} \frac{1 - \exp\left(-\frac{y^2}{t}\right)}{t} e^{-\mu t} = -\frac{\Gamma(1-s)}{4\pi \Gamma(s)} (y^2)^{s-1} + \mathcal{O}(\mu^{2-s}, \mu)
\]

(A3.3)

To investigate the contributions to \(Z(s)\) we may integrate \(y\) over a strip along the boundary with width \(\delta\) and length \(L\). We then obtain:

\[
-\frac{L \Gamma(1-s)}{4\pi \Gamma(s)} \frac{\delta^{2s-1}}{2s-1}
\]

(A3.4)

When we continue to lower values of \(s\) we find a pole at \(s = \frac{1}{2}\) with residue \(-\frac{L}{8\pi}\) which implies that the short term asymptotics of the heat kernel has one term which is a line density at the boundary, with coefficient \(-\frac{t^{-\frac{1}{2} \frac{1}{8\pi} \frac{1}{2}}}{8\pi^2}\).

At the origin, the contribution goes as \(s\), hence there is no contribution to \(Z_D(0)\) from a straight edge. We obtain contributions to \(Z'_D(0)\) as

\[
\frac{L}{4\pi \delta}
\]

(A3.5)

The interpretation of [A3.5] is, that apart from the finite and regular terms in \(Z'(x,x,0)\), there is a distribution, which is defined by the analytic continuation of \(y^{2s-2}\). If we integrate over strips parallel to the boundary, the terms as in [A3.3] would cancel pairwise, and we would be left with the outer boundary term. Hence we conclude that a straight boundary gives a contribution to \(Z'_D(x,x,0)\) which is no more singular than a line density,
and hence gives a contribution proportional to boundary length. In general one could integrate \( Z_D((x, y), (x, y), s) \) against a smooth test–function \( f(x, y) \), with \( s \) in the interval \([ \frac{1}{2}, 1] \), and then analytically continue to smaller \( s \). Such a test–function is the conformal variation \( \delta \sigma(x, y) \). Assuming an expansion normal to the boundary at \( x \) as \( 2\delta \sigma(x, y) = a_0(x) + y \cdot a_1(x) + \ldots \) we find

\[
\int_0^L dx \int_0^\delta dy 2\delta \sigma(x, y)Z((x, y), (x, y), s) = \int_0^L dx \int_0^\delta dy \left[ -\frac{\Gamma(1-s)}{4\pi \Gamma(s)} (y^2)^{s-1} [a_0(x) + y \cdot a_1(x) + \ldots] \right]
\]

The contribution to the variation of \( Z'_D(0) \) is the limit of [A3.6] as \( s \) tend to zero, which may clearly be expressed in the normal derivative at the boundary of the conformal variation:

\[
\delta Z'_{\text{straight boundary}}(0) = -\frac{1}{4\pi} \int_0^L \hat{n} \cdot \nabla(\delta \sigma) ds
\]  

(A3.7)

**A4. The Sommerfeldt kernel**

A. Sommerfeldt in 1896 solved the problem of diffraction of light by a perfectly conducting half–plane[18]. The solution takes the form of a kernel periodic in the angle variable with periodicity \( 4\pi \); the difference of one “direct” and one “reflected” wave vanishes at 0 and \( 2\pi \).

We will need the solution to the diffusion problem in a sector with opening angle \( \pi \alpha \), which is quite analogous, but for completeness given here. (The solution of diffusion problem at an interior corner with total angle \( 2\pi \alpha \) is obtained in the same way, by keeping only the “direct” term.) If the opening angle is of the form \( \frac{\pi}{n} \) the sector can be reflected in its side \( 2n \) times to precisely cover \( 2\pi \), and the solutions to both the diffusion problem and the diffraction problem are expressed by the method of images. Sommerfeldts solution is a substitute when the reflections do not make up a full turn, and is given by a certain finite number of image charges, and a correction term. It has the following integral representation:

\[
K_S(r, \phi; r', \phi'; t) = \frac{1}{4\pi t} \exp\left(-\frac{r^2 + r'^2}{4t}\right)[\nu(a)\left(\frac{rr'}{2t}, \phi - \phi'\right) - \nu(a)\left(\frac{rr'}{2t}, \phi + \phi'\right)] \quad (A4.1)
\]

where the important part is

\[
\nu(a) = \frac{1}{2\pi a} \int_{A+B} \exp(a \cos \delta) \frac{d\delta}{1 - e^{-\frac{(\delta + a)}{a}}} \quad (A4.2)
\]

\( A \) and \( B \) are paths in the plane of complex \( \delta \) that go asymptotically from \( \pi + i \cdot \infty \) to \(-\pi + i \cdot \infty \), and \(-\pi - i \cdot \infty \) to \( \pi - i \cdot \infty \), respectively. Essentially this is a superposition of
free heat kernels between $x$ and $y'$, where $|y'| = r'$ and $|x - y'|^2 = r^2 + r'^2 - 2rr'\cos(\delta)$. In the bands $\frac{(4n+1)\pi}{2} < |\text{Re}(\delta)| < \frac{(4n+3)\pi}{2}$ the contour integral can be taken to infinity since $\Re(|x - y'|^2) \to \infty$, but not in between.

$K_S$ satisfies the heat equation because it is superposition of free heat kernels, it is symmetric in $(x, y)$ and periodic in $\phi$ and $\phi'$ with period $2\pi\alpha$ by construction of $\nu_\alpha$, and it vanishes at the boundaries of the sector because it is the difference between a direct and a reflected term. Furthermore, away from the imaginary axis it is analytic in $\alpha$.

By deforming $A$ and $B$ into the straight lines $\pi + iy, -\pi + iy$ and $[-\pi, \pi], \nu_\alpha$ can be written as

$$\nu_\alpha(a, \phi) = \sum_{k: -1 < 2\alpha k - \phi < 1} \exp(a \cos(2\pi\alpha k - \phi))$$

$$-\frac{\sin(\pi/\alpha)}{2\pi\alpha} \int_{-\infty}^{\infty} \exp(-a \cosh y) \cosh(y/\alpha - i\phi/\alpha - \cos \pi/\alpha) dy$$

(A4.3)

From the image charges of the “direct” term, it follows that the normalization of the kernel is correct. If for some $k$ an image charge wanders through the line $\pm \pi$, we ought to take half the residue and the principal value of the integral in the usual fashion, but then $\alpha$ equals $\frac{\pi}{n}$, and the prefactor of the integral is zero: that is; we have just a solution by images.

**A5. A useful integral**

In this appendix we discuss an elementary but useful integral. We would like to compute a contribution to $Z_D(0)$ or $Z'_D(0)$ by substituting the true (unknown) heat kernel in a compact domain, with essentially something like a heat kernel in the full plane. This includes the case with a flat boundary, where the heat kernel can be written as the difference between a free heat kernel and an image charge reflected over the boundary, and also the case in a sector, since the Sommerfeldt heat kernel can be written as a linear superposition of free heat kernels. The first problem is that the eigenvalues in an infinite domain accumulate to zero; the Mellin transforms are therefore not convergent for any parameter $s$. We are however interested in Mellin transforms of heat kernels in compact domains, which are asymptotic to free heat kernels for short times, but fall of exponentially for large times. Furthermore, we are only interested in the singular contribution, so the problem can be resolved by limiting the integral in the Mellin transform to the interval $[0, 1]$. In practice such an integral is not analytically tractable. Instead we can introduce a convergence factor $e^{-\mu t}$, including which the Mellin transform converges for $\Re(s) > 1$, and look for the singular part in the $\mu$–independent part of the result.

We therefore consider

$$\int_0^\infty t^{s-1}e^{-\frac{\pi}{4}t}e^{-\mu t}dt = 2\left(\frac{\alpha}{\mu}\right)^{\frac{s}{2}}K_s((4\alpha\mu)^{\frac{s}{2}}) = \Gamma(s)\mu^{-s} + \Gamma(-s)(a)^s + O(\mu^{1-s}, \mu)$$

(A5.1)

by the power series expansion of the modified Bessel function.
Two variants, with \( s \) around the origin, are
\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (1 - e^{-\frac{a}{t}}) e^{-\mu t} dt = -\frac{\Gamma(-s)}{\Gamma(s)} (a)^s + O(\mu^{1-s}, \mu) \tag{A5.2}
\]
and
\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\frac{a}{t}} e^{-\mu t} dt = \frac{1}{s} \mu^{1-s} + \frac{\Gamma(1-s)}{\Gamma(s)} (a)^{s-1} + O(\mu^{2-s}, \mu) \tag{A5.3}
\]
As \( \mu \) tends to zero, in the first case, the value at zero is 1, and the derivative is \( 2\gamma + \log a \) by the expansion \( \Gamma(s) \sim 1 + \frac{\gamma}{s} - \gamma + \ldots \). In the second case the value at zero vanishes, but the derivative is \( \frac{1}{a} \).

### A6. The singular corner contribution

We will here compute the contributions to \( Z_D(0) \) and \( \delta Z'_D(0) \) from a corner. For \( Z_D(0) \) we are by Appendix 2 and Appendix 4 looking for the limit as \( s \) tends to zero of
\[
\int_{\text{corner}} d^2x Z(x, x, s) \tag{A6.1}
\]
where the zeta function density is computed from the Sommerfeldt heat kernel, and the corner is delimited by a cut-off radius \( r_0 \).

For \( \delta Z'_D(0) \) we use the variational formula (41), and look for the term finite in \( s \) as \( s \) tends to zero of
\[
\int_{\text{corner}} d^2x [2\delta \sigma(x) Z(x, x, s)] \tag{A6.2}
\]
An inspection of [A2.3] shows that the conformal factor close to the corner contains two types of terms, one which is proportional to the logarithm of the distance to the tip of the corner, and one which is constant over the corner.

We choose in the following to keep the constant factors in first term in [A2.3] so that what we compute will directly be the strictly local contribution to \( \delta Z'_D(0) \), which can then be given in integrated form. The contributions of the last two terms in [A2.3] will be proportional to the contribution to \( Z_D(0) \), and are easily included by comparison.

Our computations therefore boil down to [A6.1] and
\[
\int_0^{\pi \alpha} \int_0^{r_0} r d\phi dr \left[ \frac{\delta \alpha}{\alpha} 2 \log \alpha Z_\alpha(x, x, s) \right] \tag{A6.3}
\]
\[
\int_0^{\pi \alpha} \int_0^{r_0} r d\phi dr \left[ \frac{\delta \alpha}{\alpha} \log r^2 Z_\alpha(x, x, s) \right] \tag{A6.4}
\]
where we can directly integrate over \( \phi \) since our test functions (a constant and \( \log r^2 \)) do not depend on \( \phi \).

It will turn out that the reflected term in the Sommerfeldt kernel only contributes to \( \delta Z'_D(0) \), and only by a simple term. We can therefore in parallel compute the contributions to a corner at the boundary with opening angle \( \pi \alpha \), and a corner in the interior with opening angle \( 2\pi \alpha \), which almost only differ by a factor of two. By analyticity in \( \alpha \) we can choose the convenient range \( \frac{1}{2} < \alpha < 1 \).
The image charges

The charge from the direct term, and the image charges from the reflected term give contributions to the Mellin transform, integrated over the area of the corner, as

\[
\frac{1}{4\pi t} \int_0^{r_0} r\,dr \int_0^{\pi\alpha} d\phi [1 - \sum_{k: -1 < 2\alpha k - 2\phi/\pi < 1} \exp(-\frac{r^2}{2t}(1 - \cos(2\pi\alpha k - 2\phi)))]
\] (A6.5)

The direct term charge can be neglected, as along a straight boundary. With \(\frac{1}{2} < \alpha < 1\) the only relevant values of \(k\) in the reflected term are 0 and 1. For \(k = 0\) the integral over \(\phi\) is limited between 0 and \(\frac{\pi}{2}\), while for \(k = 1\) it goes between \(\pi\alpha - \frac{\pi}{2}\) and \(\pi\alpha\). With \(y = r\sin(\phi)\) in the first case, and \(y = r\sin(\phi - \pi\alpha)\) in the second, we find with \(s\) between \(\frac{1}{2}\) and 1:

\[
\int_0^{\frac{\pi}{2}} d\phi r Z_{\alpha}((r, \phi), (r, \phi), s) = -\frac{\Gamma(1 - s)}{4\pi \Gamma(s)} \int d\phi \sin^{2s - 2}\phi
\]

\[
= -\frac{\Gamma(1 - s)r^{2s - 1}\Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})}{4\pi \Gamma(s) \Gamma(s)}
\] (A6.6)

When we integrate up to a radius \(r_0\) and continue to lower \(s\) we find, as we should, a pole at \(s = \frac{1}{2}\) with residue \(-\frac{2\alpha}{\pi}\).

To investigate the contributions close to zero, we use the formulae

\[
\int_0^{r_0} drr^{2s - 1} = \frac{r_0^{2s}}{2s} = \frac{1}{2s} + \frac{1}{2} \log r_0^2 + O(s)
\]

\[
\int_0^{r_0} drr^{2s - 1} \log r^2 = \frac{\partial}{\partial s} \left[ \frac{r_0^{2s}}{2s} \right] = -\frac{1}{2s^2} + \frac{1}{4} (\log r_0^2)^2 + O(s)
\]

where we understand that we first integrate over \(r\), and then analytically continue to the neighbourhood of the origin.

The contribution to \(\delta Z'_{\alpha}(0)\) from the reflected image charges (hence only for boundary corners) is thus

\[
-\frac{\delta\alpha}{4\alpha}
\] (A6.7)

The reflected term: the integral

The integral part of the reflected term comes from

\[
\frac{\sin\pi/\alpha}{8\pi^2 at} \int_0^{\pi\alpha} r\,dr \int_{-\infty}^{\infty} \frac{\exp(-\frac{r^2(1+\cosh y)}{2t})}{\cosh(y/\alpha - 2i\phi/\alpha) - \cos \pi/\alpha} dy,
\] (A6.8)

where we have integrated over the angle variable \(\phi\). The \(\phi\)-integration forms part of closed contour from \(-i\infty\) to the origin, along the real axis to \(\pi\alpha\), and down parallel to the imaginary axis to \(\pi\alpha - i\infty\). The integrals over the two lines cancel. If \(y > 0\) there are two poles inside the contour, but their residues cancel. The angle integral is thus zero.
The direct term: the integral

We consider

\[- \frac{\sin \pi/\alpha}{8\pi^2 \alpha t} \int_0^{\pi \alpha} r d\phi \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{r^2 (1+\cosh y)}{2t}\right)}{\cosh y/\alpha - \cos \pi/\alpha} dy\]  \hspace{1cm} (A6.9)

Following Appendix 2, the Mellin transform of \[A6.9\], performed termwise inside the integral over \(y\), gives

\[- r^{2s-1} \frac{\Gamma(1-s)\sin(\pi/\alpha)}{\Gamma(s)8\pi} \int_{-\infty}^{\infty} \frac{1}{(\cosh y/\alpha - \cos \pi/\alpha)(\cosh^2 y/2)^{1-s}} dy\]  \hspace{1cm} (A6.10)

The integral is most tractable if considered as a correlation integral between the two factors in the denominator, then by Percival’s formula\(^2\), we have

\[r^{2s-1} \frac{\Gamma(1-s)\alpha 2^{2-2s}}{\Gamma(s)8\pi} \int_{-\infty}^{\infty} B(1-s+iy, 1-s-iy) \left[ \frac{\sinh \pi y \cosh \pi \alpha y - \sinh \pi \alpha y \cosh \pi y}{\sinh \alpha \pi y} \right] dy\]  \hspace{1cm} (A6.11)

This expression can be written more compactly as

\[r^{2s-1} \frac{\Gamma(1-s)\alpha 2^{2-2s}}{\Gamma(s)8\pi} A(s)\]  \hspace{1cm} (A6.12)

where the numerical values are determined by the integral

\[A(s) = \int_{-\infty}^{\infty} B(1-s+iy, 1-s-iy) \left[ \frac{\sinh \pi y \cosh \pi \alpha y - \sinh \pi \alpha y \cosh \pi y}{\sinh \alpha \pi y} \right] dy\]  \hspace{1cm} (A6.13)

Using the identity

\[B(1+iy, 1-iy) = \frac{\pi y}{\sinh \pi y}\]  \hspace{1cm} (A6.14)

we have

\[A(0) = \int_{-\infty}^{\infty} [\pi y \coth \alpha \pi y - \pi y \coth \pi y] dy\]  \hspace{1cm} (A6.15)

and

\[A'(0) = \int_{-\infty}^{\infty} (2\psi(2) - \psi(1+iy) - \psi(1-iy)) [\pi y \coth \alpha \pi y - \pi y \coth \pi y] dy\]  \hspace{1cm} (A6.16)

where \(\psi(x+1) = -\gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+x} \right)\).

\(^2\)This transformation brings us closer to the Lebedev–Kantorovich transform of D. Ray, that was used by McKean and Singer in [13].
Contribution to $Z_D(0)$

We integrate $A(0)$ over the radial variable:

$$\frac{r^{2s}}{2^s s!} A(0) + \mathcal{O}(s^2)$$  \hspace{1cm} (A6.17)

$A(0)$ can be computed by introducing a factor $e^{i\epsilon y}$, and closing the integral over $y$ in the upper half plane. The residues contribute

$$-2\pi \sum_{n=1}^{\infty} n(\frac{1}{\alpha^2} e^{-n\epsilon/\alpha} - e^{-n\epsilon})$$

$$= -\frac{\pi}{2} \alpha^2 \sinh^2(\epsilon/2\alpha) - \frac{1}{\sinh^2(\epsilon/2)}$$  \hspace{1cm} (A6.18)

which, in the limit as $\epsilon \to 0$, becomes $\frac{\pi}{6} (\frac{1}{\alpha^2} - 1)$

The value of $Z_R(0)$ is therefore

$$\frac{1}{24} (\frac{1}{\alpha} - \alpha)$$  \hspace{1cm} (A6.19)

a result first derived by McKean and Singer[15].

Contribution to $\delta Z_D(0)$

Using again $[A6.7]$ and the expansion of $A(s)$ and the prefactors at the origin, we obtain:

$$\left[\frac{\delta\alpha}{\alpha} \right] \left[\frac{s \alpha}{2\pi} A(0) + \frac{s^2 \alpha}{2\pi} (A'(0) + A(0)(2\gamma - 2\log 2))\right] \left[-\frac{1}{2s^2} + \frac{\log \alpha}{s}\right] + \text{h.o.t}$$  \hspace{1cm} (A6.20)

We have a pole in $s$:

$$-\frac{1}{s} \frac{\delta\alpha}{\alpha} \frac{\alpha}{4\pi} A(0)$$  \hspace{1cm} (A6.21)

Since $\frac{\delta\alpha}{4\pi} A(0)$ is the contribution to $Z_D(0)$ from the corner, we can write this variation as

$$\frac{1}{s} \frac{\delta\alpha}{\alpha} (-\frac{1}{\alpha^2} + 1) = \frac{1}{s} \delta\left[\frac{1}{24} (\frac{1}{\alpha} + \alpha)\right]$$  \hspace{1cm} (A6.22)

Let us work out that this variation is only seemingly in contradiction with the known expression of $Z_R(0)$ in $A6.19$. The variation of $Z_D(x, x, 0)$ is of course a point mass at the corner, with weight $\frac{\delta\alpha}{24} (-\frac{1}{\alpha^2} - 1)$. The variational forms (35) and (36) are however for the variation of $Z_D(s)$, not for the zeta function density. The variations we consider go between different simplices with disc–like topology, and therefore leave the sum $\sum_{i,c.} (2 - 2\alpha_j) + \sum_{b,c.} (1 - \alpha_j)$ invariant (see equation 37). The analytical expressions of the variation are then only determined up to terms linear in the variations of the angles, which is the discrepancy we have in $A6.22$. 

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The contribution finite in $s$ in $A6.20$ is

\[ \delta \alpha \left[ \frac{A(0)}{4\pi} (2 \log \alpha - 2\gamma + 2 \log 2) - \frac{A'(0)}{4\pi} \right] \quad (A6.23) \]

We compute $A'(0)$ by introducing a convergence factor $e^{i\epsilon y}$. Since $\psi(\bar{z}) = \bar{\psi}(z)$, it is enough to consider

\[ C_n = \int_{-\infty}^{\infty} \left( \frac{1}{n} - \frac{1}{n - iy} \right) [\pi y \coth \alpha \pi y - \pi y \coth \pi y] dy \quad (A6.24) \]

in terms of which $A'(0) = 2 A(0) - 2 \Re \sum_{n=1}^{\infty} C_n$. The two terms in $C_n$ have residues in the upper half plane, and sum to

\[ - \frac{\pi}{2n} \left( \frac{1}{\alpha^2 \sinh^2(\epsilon/2\alpha)} - \frac{1}{\sinh^2(\epsilon/2)} \right) + 2\pi \sum_{m=1}^{\infty} m \left( \frac{1}{\alpha^2 (n + m/\alpha)} e^{-me/\alpha} - \frac{1}{(n + m)} e^{-me} \right) = - \frac{\pi}{6n} \left( \frac{1}{\alpha^2} - 1 \right) + \pi \left( 1 - \frac{1}{\alpha} \right) \]

\[ -2\pi n e^n \int_{\epsilon}^{\infty} e^{-\mu n} \left[ \frac{1}{\alpha (e^{\mu/\alpha} - 1)} - \frac{1}{(e^{\mu} - 1)} \right] d\mu + O(\epsilon) \quad (A6.25) \]

With some care the sum in $n$ is:

\[ \sum_{n=1}^{\infty} C_n = -2\pi I(\alpha) = -2\pi \int_{0}^{\infty} \frac{1}{e^\mu - 1} \left[ \frac{1}{4 \sinh^2(\mu/2)} - \frac{1}{4\alpha^2 \sinh^2(\mu/2\alpha)} - \frac{1}{12} \left( \frac{1}{\alpha^2} - 1 \right) \right] d\mu \quad (A6.26) \]

Now we can collect all the strictly local contributions to $Z'_D(0)$ and write

\[ \delta Z'_\alpha(0) = - \frac{\delta \alpha}{4\alpha} + \delta \alpha \left[ \frac{1}{12\alpha} (\frac{1}{\alpha} - \alpha)(\log \alpha - \gamma + \log 2 - 1) - I(\alpha) \right] \quad (A6.27) \]

Here we have taken a boundary corner, and hence included the contribution from the reflected image charges.

It is clearly of interest to give $A6.27$ in integrated form, whence we introduce the primitive function of $I(\alpha)$:

\[ J(\alpha) = \int_{0}^{\infty} \frac{1}{e^\mu - 1} \left[ \frac{1}{2\mu} \coth(\frac{\mu}{2\alpha}) - \frac{\alpha}{4 \sinh^2(\mu/2)} - \frac{1}{12} \left( \frac{1}{\alpha} + \alpha \right) \right] d\mu \quad (A6.28) \]

\[ I(\alpha) = - \frac{d}{d\alpha} J(\alpha) \]

We know from Appendix 3 that the contribution to $Z'_D(0)$ (note: not the variation) of a straight boundary is proportional to the boundary length. The integrated form of $A6.27$
should therefore be zero at $\alpha = 1$. As in the variation of $Z_D(0)$, we should in addition expect undetermined linear terms in the angles.

A rather long derivation, to be given in Appendix 8 as we investigate the special values of $\alpha = \frac{1}{n}$, shows that we may integrate \[A6.27\], compare with one of the integrable cases, include the integration constant in the contribution from the corners, and simplify to:

$$Z'_\alpha(0) = \frac{1}{12}(\frac{1}{\alpha} - \alpha)(\gamma - \log 2) - \frac{1}{12}(\frac{1}{\alpha} + \frac{3}{\alpha} + \alpha) \log \alpha + \tilde{J}(\alpha) \quad (A6.29)$$

where the last term has the more symmetric integral representation

$$\tilde{J}(\alpha) = \int_0^\infty \frac{1}{e^x - 1}[\frac{1}{2x}(\coth(\frac{x}{2\alpha}) - \alpha \coth(\frac{x}{2}))-\frac{1}{12}(\frac{1}{\alpha} - \alpha)]dx \quad (A6.30)$$

**Difference at interior corners**

The only change is that the angle-independent direct term is integrated over twice as wide an angle, and the contribution from the reflected image charges are absent. The contribution to $Z_D(0)$ is then

$$Z_\alpha(0) = \frac{1}{12}(\frac{1}{\alpha} - \alpha) \quad (A6.31)$$

and the strictly local contributions to $Z_D'(0)$

$$Z'_\alpha(0) = \frac{1}{6}(\frac{1}{\alpha} - \alpha)(\gamma - \log 2) - \frac{1}{6}(\frac{1}{\alpha} + \alpha) \log \alpha + 2\tilde{J}(\alpha) \quad (A6.32)$$

**Summary of corner contributions**

For simplices with the disc topology, the contributions to the variation of the zeta function derivative are given by \[A2.3\] and a simple sum over the interior corners. There are three terms in \[A2.3\], of which the first is the only one that arises in triangular domains. It is written in final variational form valid for corners on the boundary in \[A6.27\], \[A6.26\], and in integrated form in \[A6.29\], \[A6.30\]. The last two terms in \[A2.3\] are of the same type as \[A6.3\] i.e. in variation form they will give something proportional to the contribution to $Z_D(0)$ from the corner. We give those expressions here, under the (slight) simplifying assumption that none of the branch point in the $\omega$–plane lies at infinity:

$$\delta Z'_\text{extra} = \sum_{i,j} \delta \alpha_i Z'_{\alpha_i}(0)[\delta \alpha_j \log |\omega_j - \omega_i|^{-2} - (1 - \alpha_j)[\frac{\delta(\omega_j - \omega_i)}{\omega_j - \omega_i} + c.c]] \quad (A6.33)$$

**A7. Computation using the heat kernel**

This approach was used by Dowker[4] to compute the contribution to $Z_D(0)$ from a corner. We begin with the asymptotic formulae for the trace of the heat kernel in a domain $D$:

$$Tr_x[K_D(x, x, \epsilon)] \sim \frac{c_1}{\epsilon} + \frac{c_2}{\epsilon^{\frac{1}{2}}} + Z_D(0) + O(\epsilon^{\frac{1}{2}}) \quad (A7.1)$$
\[
Tr_x[2\delta\sigma(x)K_D(x, x, \epsilon)] \sim \frac{\delta[c_1]}{\epsilon} + \frac{2\delta[c_1]}{\epsilon^2} - \log \epsilon \delta[Z_D(0)] - \gamma \delta[Z_D(0)] \\
+ \delta[Z'_D(0)] + O(\epsilon^4)
\]  
(A7.2)

As in Appendix 6 we only need to consider the image charges from the reflected term and the integral from the direct term.

**The direct term: the integral**

It is convenient to go back to the original form of the Sommerfeldt kernel (including the line integrals and the direct charge):

\[
K_{\text{direct}}(r, \phi; r, \phi; t) = \frac{1}{8\pi^2 \alpha t} \int_{A'+B'} \frac{d\delta}{1 - e^{-\frac{t}{\alpha}}} e^{-\frac{t}{4\alpha}(1 - \cos \delta)} - \text{(the pole at } \delta = 0) 
\]  
(A7.3)

We are to integrate the kernel, by itself or multiplied by \(\delta \alpha \log \alpha^2 r^2\), over the corner, and then catch the finite piece as \(t\) tends to zero. The integration contour is chosen such that the prefactor of \(r^2\) in the exponent always has negative real part. We can therefore extend the integral over all \(r\) (as in Appendix 5), without introducing significant errors. For the contribution to \(Z_D(0)\) we then find

\[
\int_0^{\pi \alpha} r d\phi \int_0^{e^{-\frac{t}{\alpha}}} K_{\text{direct}}(x, x, t) = -\frac{1}{8\pi} \int_{A'+B'} \frac{d\delta}{1 - e^{-\frac{t}{\alpha}}} \frac{1}{1 - \cos \delta} + O(e^{-C \cdot \frac{2}{4}}) 
\]  
(A7.4)

where we have afterwards moved over the integration contour to the straight lines parallel to the imaginary axis at \(\pi\) and \(-\pi\). Except for an error exponentially small in \(\frac{1}{t}\), we have only left a term independent of \(t\), that will give us what we want. By a change of variables, \(\delta \to -\delta\), the integration contour \(A'\) goes into \(-B'\), and vice versa, and the second factor in the integrand \(\frac{1}{1 - \cos \delta}\) is unchanged. It is therefore sufficient to keep in the first factor, the part odd under this reflection, and we have

\[
-\frac{1}{16\pi i} \int_{A'+B'} d\delta \cot \left( \frac{\delta}{2\alpha} \right) \frac{1}{1 - \cos \delta} 
\]  
(A7.5)

This integral is now in fact convergent at infinity, so we can close it, and evaluate it by computing the residue of the pole at the origin. Hence

\[
\int_0^{\pi \alpha} r d\phi \int_0^{e^{-\frac{t}{\alpha}}} dr K_{\text{direct}}(x, x, t) = -\frac{1}{8\pi} 2\pi i \text{Res}[\cot \left( \frac{\delta}{2\alpha} \right) \frac{1}{1 - \cos \delta}]_{\delta=0} 
\]  
\[
= \frac{1}{24} \left( \frac{1}{\alpha} - \alpha \right) 
\]  
(A7.6)

which is, of course, the right answer.

To compute the variation of \(Z'_D(0)\), it is again convenient to extend the integral over the radius to infinity. We then find

\[
\int_0^{\pi \alpha} d\phi \int_0^{\infty} \frac{1}{\alpha} \log(\alpha^2 r^2) e^{-\frac{t}{4\alpha}(1 - \cos \delta)} = \frac{\pi t}{1 - \cos \delta} (-\gamma + \log \frac{2\alpha^2 t}{1 - \cos \delta}) 
\]  
(A7.7)
(we do not write out the variational factor $\delta \alpha$, except where necessary), and hence
\[
\int_0^{\pi \alpha} r d\phi \int_0^{r_0} dr [2\delta \sigma(x)K_{\text{direct}}(x, x, t)] =
\]
\[
-\frac{1}{8\pi \alpha} \int_{A' + B'} \frac{d\delta}{1 - e^{-\frac{i \delta}{\pi}}} \frac{1}{1 - \cos \delta} (-\gamma + \log \frac{2\alpha^2 t}{1 - \cos \delta})
\]

(A7.8)

Pulling out the term divergent as $\log t$ we have
\[
\log t \frac{\delta \alpha}{\alpha} \frac{1}{24} \left( \frac{1}{\alpha} - \alpha \right)
\]
which is equal to $-\log t \delta[Z_\alpha(0)]$, up to the term linear in the angle.

Taking into account that the finite part of [A7.8] is $\delta[Z_\alpha'(0)] - \gamma \delta[Z_\alpha(0)]$ we have
\[
\delta[Z_\alpha'(0)] = \delta[Z_\alpha(0)](-\log(2\alpha^2) + 2\gamma)
\]
\[
-\frac{\delta \alpha}{8\pi \alpha} \int_{A' + B'} \frac{d\delta}{1 - e^{-\frac{i \delta}{\pi}}} \frac{1}{1 - \cos \delta} \log \frac{1}{1 - \cos \delta}
\]

(A7.10)

We choose the branch of $\log \frac{1}{1 - \cos \delta}$ symmetric under $\delta \to -\delta$. Then we can again substitute for $\frac{1}{1 - e^{-\frac{i \delta}{\pi}}}$ its anti-symmetric part, and the integral is convergent at infinity. Essentially $\log \frac{1}{1 - \cos \delta}$ is a function of $\delta^2$, so it can be chosen to have a branch cut along the negative real axis in the $\delta^2$–plane. That translates to two branch cuts in the $\delta$–plane, along the positive and negative real axis. Evenness under $\delta \to -\delta$ then determines the phase choice in the left half plane. The best we can do now is to pull both integration contours in to the imaginary axis and integrate over the branch cuts. In addition we have a singularity at the origin. We separate out a small circle around the origin with radius $\epsilon$, and an integral along the branch cuts from $\pm i\epsilon$ to $\pm i\infty$. The integrals along the branch cuts give
\[
= \frac{1}{4\alpha} - \frac{1}{e^2} - \frac{1}{12} \left( \frac{1}{\alpha} - \alpha \right) + \frac{1}{8\alpha^2} \int_\epsilon^{\infty} dy \coth \left( \frac{y}{2} \right) \frac{1}{\sinh^2 \left( \frac{y}{2\alpha} \right)}
\]

(A7.11)

Close to the origin we estimate ($\delta = \epsilon e^{i\theta}$):
\[
\log \frac{1}{1 - \cos \delta} = \log 2 - 2 \log \epsilon - 2i\theta + \frac{1}{12} \delta^2 + \ldots \quad \Re \delta > 0
\]
\[
= \log 2 - 2 \log \epsilon - 2i\theta + 2i\pi + \frac{1}{12} \delta^2 + \ldots \quad \Re \delta < 0
\]

which eventually gives
\[
\frac{1}{24} \left( \frac{1}{\alpha^2} - 1 \right)(\log 2 - 2 \log \epsilon) - \frac{1}{24} + \frac{1}{2\epsilon^2}
\]

(A7.12)
Collecting all terms from the direct integral we have

\[
\delta[Z'_\alpha(0)]_{\text{direct integral}} = \delta\alpha \left[ \frac{1}{24} \left( \frac{1}{\alpha^2} - 1 \right) (-2\gamma + 2 \log(2\alpha) - 2 \log \epsilon) 
- \frac{1}{2\epsilon^2} + \frac{1}{4\alpha} - \frac{1}{12} \left( \frac{1}{\alpha^2} + 1 \right) 
+ \frac{1}{8\alpha^2} \int_{\epsilon}^{\infty} dy \coth \left( \frac{y}{2} \right) \frac{1}{\sinh^2 \left( \frac{y}{2\alpha} \right) } \right]
\]  

(A7.13)

The image charges

We should compute the integral

\[
\delta[Z'_\alpha(0)]_{\text{image charges}} = -\delta\alpha \left[ \int_0^{\pi/2} r d\phi \int_0^{r_0} dr \frac{1}{\pi\alpha t} \log(\alpha r) e^{-\frac{\pi^2}{4}(1-\cos 2\phi)} \right]
\]  

(A7.14)

The problem is that we have to separate out a \( r_0 \)-dependent piece diverging as \( t^{-\frac{1}{2}} \), before we can have the finite value at the origin, and as it stands it is not possible to extend the integral over \( r \) to infinity. If the integral would have been over a small square, we could have separated in \( x \) and \( y \), and then the integral would be much easier. Let us first see that the segment of the square outside the quadrant is unimportant, and then compute the integral over the square. The difference is

\[
- \int_0^{r_0} dy \int_0^{r_0} dx \frac{1}{2\pi\alpha t} \log(\alpha^2 (x^2 + y^2)) e^{-\frac{\pi^2}{4}}
\]  

(A7.15)

which is essentially

\[
- \int_0^{r_0} dy \frac{y^2}{4\pi\alpha r_0} \left[ \log(\alpha^2 r_0^2) + \frac{y^2}{r_0^2} + \ldots \right] e^{-\frac{\pi^2}{4}}
\]  

(A7.16)

which only gives a contribution of order \( t^{\frac{1}{2}} \).

The integral over the square is

\[
- \int_0^{r_0} dx \int_0^{r_0} dy \frac{1}{2\pi\alpha t} \log(\alpha^2 (x^2 + y^2)) e^{-\frac{\pi^2}{4}}
\]  

(A7.17)

After first integrating over \( x \), we can extend the integral over \( y \) to infinity and find:

\[
- \int_0^{\infty} dx \frac{1}{2\pi\alpha t} (\log \alpha^2 + 2r_0 (\log r_0 - 1) + \pi y + \ldots) e^{-\frac{\pi^2}{4}} = \text{Const.} \cdot t^{-\frac{1}{2}} - \frac{1}{4\alpha}
\]  

(A7.18)

Summary

The terms diverging with the cutoff \( \epsilon \) can be brought into the integral, and then give, after some algebra,

\[
\delta[Z'_\alpha(0)] = \delta\alpha \left[ -\frac{1}{12} \left( \frac{1}{\alpha^2} - 1 \right) (1 + \gamma - \log 2\alpha) - \frac{1}{4\alpha} + J'(\alpha) \right]
\]  

(A7.19)

which is the same result as in A6.27.
A8. Corners with opening angle $\frac{1}{n}$

We proceed to simplify the general expressions of the contributions to the regularized determinants, when the angles are of the form $\frac{\pi}{n}$.

We recall that the contribution to $Z_D'(0)$ from a (boundary) corner is

$$A + B\alpha - \frac{1}{4} \log \alpha + \frac{1}{12\alpha} (\gamma - \log 2 - \log \alpha) - \frac{1}{12} \log \alpha + J(\alpha)$$

where the last term has the integral representation

$$J(\alpha) = \int_0^\infty \frac{1}{e^\mu - 1} \left[ \frac{1}{4\mu} \coth \left( \frac{x}{2\alpha} \right) - \frac{\alpha}{4 \sinh^2 \left( \frac{\mu}{2} \right)} - \frac{1}{12} \left( \frac{1}{\alpha} + \alpha \right) \right] d\mu \tag{A8.1}$$

and have included an undetermined term linear in the angles ($B\alpha$), and an integration constant, that we will not need explicitly ($A$).

The analysis proceeds by computing the integral for $\alpha$ being $\frac{1}{n}$. Let $y$ stand for $e^{2\pi i n \mu}$.

Decomposing various rational functions of $y$:

$$\log(y^n - 1) = \sum_{v=0}^{n-1} \log(y - \lambda_v)$$

$$\frac{n}{y^n - 1} = \sum_{v=0}^{n-1} \frac{1}{\lambda_v y - 1}$$

$$\frac{1}{y - 1} \left( \frac{n}{y^n - 1} + \frac{n}{2} \right) = \frac{1}{(y - 1)^2} + \frac{1}{2(y - 1)} - \sum_{v=1}^{n-1} \frac{1}{1 - \lambda_v y - \lambda_v}$$

we can absorb the terms proportional to $\frac{1}{n}$ into $B$, and write the modified integral:

$$\tilde{J}(\frac{1}{n}) = \frac{1}{n} \int_0^\infty \left[ -\frac{1}{n} \sum_{v=1}^{n-1} \frac{\lambda_v}{1 - \lambda_v y - \lambda_v} - \frac{n^2 - 1}{12(y - 1)^2} \right] dx$$

(A8.2)

The cut-off $\epsilon$ is introduced for later convenience. In fact, we can go backwards and express A8.2 as

$$\tilde{J}(\alpha = \frac{1}{n}) = \int_0^\infty d\mu \frac{1}{e^\mu - 1} \left[ (\frac{1}{2\mu}) (\coth(\frac{\mu}{2\alpha}) - \coth(\frac{\mu}{2})) - \frac{1}{12} (\frac{1}{\alpha} - \alpha) \right]$$

(A8.3)

Since up to simple terms in $\alpha$ that we keep, A8.3 and A8.1 agree on the integers, and both expressions are analytic in $\frac{1}{\alpha}$, then they must agree overall. We are therefore allowed to substitute for A8.1 the more symmetric expression A8.3. Continuing the analysis of A8.2, we use

$$\int_\epsilon^\infty \frac{dx}{x} \frac{\lambda_v}{y - \lambda_v} = \int_{\epsilon x}^\infty \frac{dx}{x} \sum_{p=1}^n \lambda_{vp} \left[ e^{(\frac{1}{n} - 1)x} \right]$$

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\[
\int_\epsilon^\infty \frac{dx}{xe^x - 1} = \frac{1}{\epsilon} - \frac{1}{2} - a)(\gamma + \log \epsilon) + \log \Gamma(a) - \frac{1}{2} \log(2\pi) + \text{small as } \epsilon
\]

\[
\sum_{v=1}^{n-1} \left[ \frac{\lambda_{uv}}{1 - \lambda_v} \right] = p - \frac{1 + n}{2}
\]

and express the integral \( \tilde{J} \) as

\[
\tilde{J}(\frac{1}{n}) = -\frac{1}{n} \sum_{v=1}^{n-1} \sum_{p} \frac{\lambda_{uv}}{1 - \lambda_v} \int_\epsilon^\infty \frac{dx}{xe^x - 1} + \frac{n^2 - 1}{12n} \log \epsilon
\]

which may finally be simplified to

\[
\tilde{J}(\frac{1}{n}) = \frac{1 - n^2}{12n}(\gamma + \log n) - \frac{1}{4n} \log n + \frac{1}{4} \left( 1 - \frac{1}{n} \right) \log(2\pi) + \sum_{p=1}^{n-1} \left( \frac{1}{2} - \frac{p}{n} \right) \log \Gamma \left( \frac{p}{n} \right)
\]

Absorbing further terms proportional to \( \frac{1}{n} \) into \( B \), we have finally

\[
Z'_\pi(0) = \hat{A} + \hat{B} \frac{1}{n} + \frac{1 - n^2}{12} \log 2 + \left( \frac{1}{4} - \frac{1}{12n} \right) \log n + \sum_{p=1}^{n-1} \left( \frac{1}{2} - \frac{p}{n} \right) \log \Gamma \left( \frac{p}{n} \right)
\]

**The integration constant**

We need one exact value to fix an integration constant. For the equilateral triangle, the regularized determinant is (see Appendix 11):

\[
Z'_\text{equilateral}(0) = \frac{1}{2} \log \pi - \frac{1}{6} \log 2 + \frac{2}{3} \log 3 + \frac{1}{2} \log \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}
\]

If we on the other hand use (A8.6), we find

\[
Z'_\text{equilateral}(0) = 3Z'_\frac{1}{3}(0) = -\frac{1}{2} \log 2 + \frac{2}{3} \log 3 + \frac{1}{2} \log \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} + 3\hat{A} + \hat{B}
\]

so we have the integration constant

\[
3\hat{A} + \hat{B} = \frac{1}{3} \log 2 + \frac{1}{2} \log \pi
\]

**Simplified general expression**

We choose to include the integration constant in the stricly local contribution to \( Z'_D(0) \), by adding

\[
(1 - \alpha)\left[ \frac{1}{6} \log 2 + \frac{1}{4} \log \pi \right]
\]
to a boundary corner (and twice this quantity to an interior corner). The final expression with an opening angle \( \frac{1}{n} \) then reads
\[
Z'_1(0) = (1 - \frac{1}{n})(\frac{1}{6} \log 2 + \frac{1}{4} \log \pi) + \frac{1-n}{12} \log 2 + \left( \frac{1}{4} - \frac{1}{12n} \right) \log n
\]
\[
+ \sum_{p=1}^{n-1} (\frac{1}{2} - \frac{p}{n}) \log \Gamma\left(\frac{p}{n}\right)
\] (A8.11)

For a general opening angle we go back to A8.3, keep track of the various terms proportional to \( \frac{1}{n} \) that had been absorbed into \( B \), and arrive at
\[
Z'_\alpha(0) = \frac{1}{12} (\frac{1}{\alpha} - \alpha)(\gamma - \log 2) - \frac{1}{12} (\frac{1}{\alpha} + 3 + \alpha) \log \alpha + \tilde{J}(\alpha)
\] (A8.12)

Throughout this appendix, we have simplified on the integral \( J \), which is the same for corners on the boundary and in the interior. The strictly local contribution to \( Z'_D(0) \) from a corner in the interior is thus
\[
Z'_\alpha(0) = \frac{1}{6} (\frac{1}{\alpha} - \alpha)(\gamma - \log 2) - \frac{1}{6} (\frac{1}{\alpha} + \alpha) \log \alpha + 2 \tilde{J}(\alpha)
\] (A8.13)

A9. Asymptotics of \( Z'_\alpha(0) \)

In this section we investigate \( Z'_\alpha(0) \) when \( \alpha \) is large or small or close to one.

Large and small \( \alpha \)

It is convenient to divide up the terms in \( Z'_\alpha(0) \) according to whether they are symmetric or antisymmetric under the transformation \( \alpha \to \frac{1}{\alpha} \). If we take the integral \( \tilde{J} \), it makes sense to first introduce a finite symmetric alternative expression
\[
J_S(\alpha) = \int_{\epsilon}^{\infty} \frac{dy}{y} \frac{1}{e^{\sqrt{\alpha} y} - 1} \frac{1}{e^{\sqrt{\alpha} y} - 1} - \frac{1}{2}\epsilon^2 + \left( \frac{1}{\sqrt{\alpha}} + \sqrt{\alpha} \right) \frac{1}{2}\epsilon + \frac{1}{12} (\alpha + 3 + \frac{1}{\alpha}) \log \epsilon
\] (A9.1)

where the limit as \( \epsilon \to 0 \) is understood. Using the elementary integrals
\[
\int_{\epsilon}^{\infty} \frac{dx}{e^{x} - 1} = -\log \epsilon + O(\epsilon)
\]
\[
\int_{\epsilon}^{\infty} \frac{dx}{e^{x} - 1 - x} = \frac{1}{\epsilon} + \frac{1}{2} \log \epsilon + \frac{\gamma}{2} - \frac{1}{2} \log 2\pi + O(\epsilon)
\]
\[
\int_{\epsilon}^{\infty} \frac{dx}{x(e^{x} - 1)^2} = \frac{1}{2\epsilon^2} - \frac{1}{2\epsilon} - \frac{1}{12} \log \epsilon + \frac{1}{12} \frac{1}{\epsilon} - \frac{\gamma}{12} - \zeta'(-1) + O(\epsilon)
\]
\[
\int_{\epsilon}^{\infty} \frac{dx}{x(e^{x} - 1)^2} = \frac{1}{2\epsilon^2} \frac{1}{12} \log \epsilon + \frac{1}{12} + \frac{\gamma}{12} + \zeta'(-1) + O(\epsilon)
\] (A9.2)
we can express the difference between $\tilde{J}$ and $J_S$ as

$$
\tilde{J}(\alpha) = J_S(\alpha) + \frac{1}{24}(\alpha + 3 + 1/\alpha) \log \alpha + \frac{1 + \alpha}{4}(\gamma - \log 2\pi) - \alpha(\frac{\gamma}{12} + \zeta'(-1)) \quad (A9.3)
$$

and divide up $Z'_\alpha(0)$ in symmetric, antisymmetric, and linear parts:

$$
Z'_\alpha(0) = [J_S(\alpha) + \frac{\gamma}{12}(\alpha + 3 + \alpha) - \frac{1}{4} \log 2\pi] - \alpha \left[ \frac{1}{4} \log 2\pi + \zeta'(-1) \right] + \left[ \frac{1}{12}(1 - \log 2) - \zeta'(-1) \right] + \frac{1}{24}(1 - \log 2 - \zeta'(-1)) + \frac{1}{4}(\gamma - \log 2 - \zeta'(-1))
$$

The behaviour of $J_S$ as $\alpha$ turns to zero is found expanding the integral in (A9.1), and subtracting the terms in the expansion divergent with $\epsilon$:

$$
J_S(\alpha) \sim_{\alpha \to 0} \frac{1}{24}(\alpha + 3 + \alpha) \log \alpha + \frac{1 - \gamma}{12} - \zeta'(-1) \frac{1}{\alpha} + \frac{1}{4}(\log 2\pi - \gamma) + \sum_{n=3}^{\infty} \frac{\zeta(n)B_{n+1}}{n(n+1)} \alpha^n \quad (A9.5)
$$

and from this we find

$$
Z'_\alpha(0) \sim_{\alpha \to 0} \frac{1}{12}(\alpha + 3 + \alpha) \log \alpha + \alpha \left[ \frac{1 - \gamma}{12} - \zeta'(-1) \right] + \frac{1}{4}(\log 2\pi - \gamma) + \sum_{n=3}^{\infty} \frac{\zeta(n)B_{n+1}}{n(n+1)} \alpha^n \quad (A9.6)
$$

Using the symmetry under $\alpha \to \frac{1}{\alpha}$ we have the asymptotic expansion for large $\alpha$:

$$
Z'_\alpha(0) \sim_{\alpha \to \infty} -\frac{1}{12}(\alpha + 3 + \alpha) \log \alpha + \alpha \left[ \frac{1 - \gamma}{12} - \zeta'(-1) \right] + \frac{1}{\alpha} \left( \frac{\gamma - \log 2}{12} \right) + \sum_{n=3}^{\infty} \frac{\zeta(n)B_{n+1}}{n(n+1)} \alpha^{-n} \quad (A9.7)
$$

**Development of $Z'_{1+\epsilon}(0)$**

It is convenient to go over to the first definition of the integral $J(\alpha)$ and write

$$
Z'_\alpha(0) = \frac{1}{12}(\alpha - \gamma - \log 2) - \frac{1}{12}(\alpha + 3 + \alpha) \log \alpha + J(\alpha) - \alpha \Delta J
$$

where $J$ has the integral representation

$$
J(\alpha) = \int_0^\infty \frac{1}{e^\mu - 1} \left[ \frac{1}{2} \coth(\frac{\mu}{2\alpha}) - \frac{\alpha}{4 \sinh^2(\frac{\mu}{2})} - \frac{1}{12} (\frac{1}{\alpha} + 3 + \alpha) \right] d\mu
$$
and the difference has the integral representation

\[
\Delta J = \int_0^\infty dx \frac{1}{x} \left( \frac{1}{(e^x - 1)^2} + \frac{1}{2(e^x - 1)} - \frac{e^x}{(e^x - 1)^3} - \frac{1}{6(e^x - 1)} \right)
\]

\[
= -\frac{1}{6} \gamma - \frac{5}{24} + \frac{1}{4} \log(2\pi) + \zeta'(-1)
\]

(A9.8)

The derivative of \( J(\alpha) \) can be expanded around \( \alpha = 1 \), and the successive terms evaluated in Mathematica, which gives

\[
J'(1 + \epsilon) = -\frac{1}{36} \epsilon + \frac{1}{16} \epsilon^2 + \mathcal{O}(\epsilon^3)
\]

(A9.9)

and putting the various terms together we have for a corner on the boundary:

\[
Z_{1+\epsilon}(0) = \left( \frac{1}{6} \log 2 - \frac{5}{24} - \frac{1}{4} \log(2\pi) - \zeta'(-1) \right) \epsilon
\]

+ \left( \frac{14}{72} + \frac{\gamma - \log 2}{12} \right) \epsilon^2 + \left( -\frac{29}{144} - \frac{\gamma - \log 2}{12} \right) \epsilon^3 + \mathcal{O}(\epsilon^4)
\]

(A9.10)

The expansion for a corner in the interior is easily obtained by adding the expansion of \( \frac{1}{4} \log(1 + \epsilon) \), and doubling that result.

A10. Special cases

Triangles

For triangles one can choose a convenient parametrization in terms of the Schwarz–Christoffel transformation that maps a point (\( z \)) in the triangle to a point (\( \omega \)) in the upper complex plane:

\[
\frac{d\omega}{dz} = \omega^{1-\alpha_0}(1-\omega)^{1-\alpha_1}
\]

(A10.1)

This fixes the area in terms of the angles to be

\[
\text{Area}(\alpha_1, \alpha_2, \alpha_3) = \frac{\pi}{2} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)\Gamma(1 - \alpha_3)}
\]

(A10.2)

One corner (with opening angle \( \pi \alpha_0 \)) will map to the origin in the \( \omega \)-plane, one corner (\( \pi \alpha_1 \)) to one, and the last corner maps to infinity. The parametrization is uniform over the space of triangles, in the sense that the branchpoints do not move. Hence the third term in (A2.3) is identically zero. The logarithm of the distance between the origin and one is zero, hence the second terms in (A2.3) are also zero. All we have left is then the strictly local contribution, so for this choice of the normalized areas, we have for triangles:

\[
Z_T'(0) = \sum_{p=1,2,3} Z_{\alpha_p}'(0)
\]

(A10.3)
An integration constant was included in the definition of \( Z'_\alpha(0) \) in Appendix 8.

By numerically solving for about the first one thousand eigenvalues of the Laplacian in isosceles triangles, and then estimating the analytic continuation of the zeta functions, Luck found that the quotient
\[
\zeta_T = \frac{Z'_T(0)}{Z_T(0)}
\] (A10.4)
varies surprisingly little over the triangles (still taking the area \( A10.2 \))\(^{14}\). We can express this \( \zeta_T \) as
\[
\zeta_T = \frac{\sum_{p=1,2,3} Z'_{\alpha p}(0)}{\sum_{p=1,2,3} Z_{\alpha p}(0)}
\] (A10.5)
It has a maximum for the equilateral triangle, where it equals 4.591151\ldots

The minimum of \( \zeta_T \) is obtained as an angle tends to zero, and the value follows from the asymptotic expansion \([\text{A9.6}]:\)
\[
\text{Limit}_{\alpha \to 0} \frac{Z'_{\alpha}(0)}{Z_{\alpha}(0)} = 2(1 - \log 2) - 24\zeta'(-1) = 4.583813\ldots
\] (A10.6)

### Polygons

In this case it is convenient to take a parametrization where the interior of the polygon \((z)\) is mapped onto the interior of a circle \((u)\) by a transformation that satisfies
\[
\frac{du}{dz} = \prod_{v}(u - e^{i\phi_v})^{1 - \alpha_v}
\] (A10.7)
where \( e^{i\phi_v} \) are the branchpoints, and the interior angles are \( \pi \alpha_v \). The important special case is a regular \( n \)–polygon, \( P_n \), that can be mapped to the unit circle by a transformation that satisfies
\[
\frac{du}{dz} = \prod_{v=0}^{n-1}(u - e^{2\pi i v/n})^2 = (u^n - 1)^{2/n}
\] (A10.8)
This parametrization fixes the radius of the circumscribed circle:
\[
R_n = \frac{\Gamma(1 + \frac{1}{n})\Gamma(1 - \frac{2}{n})}{\Gamma(1 - \frac{1}{n})}
\] (A10.9)
and the area, which is proportional to \( R_n^2 \). We can now choose a family of polygons that smoothly interpolate between a regular \( m \)–polygon (at \( a = 0 \)), and a regular \( mn \)–polygon (at \( a = 1 \)):
\[
\frac{du}{dz} = (u^m - 1)^{\frac{2}{n}(1-a)}(u^{mn} - 1)^{\frac{2}{mn}a}
\] (A10.10)
which is convenient, since the parametrization is then uniform in the family. The variation of \( Z'_D(0) \) will therefore be determined by the strictly local terms, and by the second term in \([\text{A2.3}]\). Let us express \( Z'_D(0) \) as a function of the parameter \( a \) in the family, and write out
these two terms in [A2.3] in more detail: they give the following contribution to \( \frac{d}{da} Z_{D(a)}'(0) \) from the corner \( v \), which is mapped to the branchpoint \( e^{\frac{2\pi i}{mn}} \):

\[
\frac{d}{da} Z_{D(a)}'(0) \text{from corner } v = \frac{d}{da} Z'_{\alpha_{v}(a)}(0) + Z_{\alpha_{v}(a)}(0) \frac{d\alpha_{v}(a)}{da}
\]

\[
\frac{d\alpha_{v}(a)}{da} = -\sum_{v' \neq v} \frac{d\alpha_{v'}(a)}{da} \log |u_{v'} - u_{v}|^2
\]

which can be rewritten

\[
\frac{d}{da} Z_{D(a)}'(0) \text{from corner } v = \frac{d}{da} [Z'_{\alpha_{v}(a)}(0) + \frac{1}{12} \left( \frac{1}{\alpha_{v}(a)} - \alpha_{v}(a) \right) \lambda_{v}(a)] + \frac{\lambda_{v}(a)}{6} \frac{d\alpha_{v}(a)}{da} \tag{A10.11}
\]

We have to differ between whether \( n \) does or does not divide \( v \). For the first case

\[
\lambda_{v}(a) = -\frac{2}{m} (1 - a) \log \left| \frac{u^{m} - 1}{u - 1} \right|_{u=1} - \frac{2}{mn} a \log \left| \frac{u^{mn} - 1}{u - 1} \right|_{u=1}
\]

\[
= -\frac{2}{m} (1 - a + \frac{a}{n}) \log m - \frac{2a}{mn} \log n \quad n \mid v \tag{A10.12}
\]

while for the second

\[
\lambda_{v}(a) = -\frac{2}{m} (1 - a) \log |e^{\frac{2\pi i}{n}} - 1| - \frac{2a}{mn} \log mn \quad n \not\mid v \tag{A10.13}
\]

The expression [A10.11] contains one part which is a total variation, and an extra term. Let us first do the second, summed over the corners \( v \):

\[
\frac{1}{6} \sum_{v} \lambda_{v}(a) \frac{d\alpha_{v}(a)}{da} = \frac{1}{6} \sum_{n \mid v} \left( -\frac{2}{m} (1 - a) \log m - \frac{2}{mn} \log mn \right) \left( \frac{2}{m} (1 - \frac{1}{n}) \right)
\]

\[
+ \frac{1}{6} \sum_{n \nmid v} \left( -\frac{2}{m} (1 - a) \log |e^{\frac{2\pi i}{n}} - 1| - \frac{2a}{mn} \log mn \right) \left( -\frac{2}{mn} \right)
\]

which can eventually be simplified to

\[
-\frac{2}{3} \frac{1 - a}{m} \log m + \frac{2}{3} \frac{1 - a}{mn} \log mn
\]

from which we have

\[
\int_{0}^{1} da \frac{1}{6} \sum_{v} \lambda_{v}(a) \frac{d\alpha_{v}(a)}{da} = \frac{1}{3} \left( \frac{1}{mn} \log mn - \frac{1}{m} \log m \right) \tag{A10.14}
\]

The total variation in [A10.11] is summed over the corners, and gives in integrated form:

\[
nZ'_{1 - \frac{\pi}{n}}(0) - \frac{2(n - 1)}{3n(n - 2)} \log n + \text{Const.} \tag{A10.15}
\]

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We are now ready to write down \( Z'_{P_n}(0) \) for a regular polygon with \( n \) corners and radius of circumscribed circle \( R \):

\[
Z'_{P_n}(0) = Z_{P_n}(0) \log \frac{R^2}{R_n^2} - \frac{1}{3(n - 2)} \log n + n Z'_{1 - \frac{1}{n}}(0) \tag{A10.16}
\]

where \( Z_{P_n}(0) = \frac{n-1}{6(n-2)} \) and the normal radius is as in [A10.9]. The integration constant turns out to be zero, since it has already been incorporated in the definition of \( Z'_\alpha(0) \).

**Square and disc**

For the square we obtain

\[
Z'_{P_4}(0) = \frac{1}{2} \log \left( \frac{\Gamma(\frac{3}{4})}{\pi^{\frac{1}{4}}} \right) - \frac{1}{6} \log 4 + 4\left( \frac{1}{8} \log \pi + \frac{5}{24} \log 2 \right) \\
= \frac{1}{2} \log \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right) + \frac{1}{4} \log \text{Area} + \frac{1}{4} \log \pi + \frac{5}{4} \log 2 \tag{A10.17}
\]

which agrees with the exact result from [A11.3].

The disc is obtained as the limit when \( n \) tends to infinity. We then find:

\[
Z'_{P_\infty}(0) = \frac{1}{3} \log R - 2 \frac{d}{d\alpha} Z'_\alpha(0)|_{\alpha=1} \\
= \frac{1}{3} \log R + \frac{5}{12} + \frac{1}{2} \log \pi + \frac{1}{6} \log 2 + 2\zeta'(-1) \tag{A10.18}
\]

which agrees with [A11.10].

**A11. Integrable domains.**

In this appendix we collect the cases known to us, where one can directly deduce the derivative of the zeta function at zero.

In a rectangle with side lengths \( A \) and \( B \), the eigenvalues of the Laplacian with Dirichlet boundary conditions are

\[
\lambda_{mn} = \pi^2 \left( \frac{m^2}{A^2} + \frac{n^2}{A^2} \right) \tag{A11.1}
\]

The zeta function around the origin is

\[
Z_{\text{rectangle}}(0) = \frac{1}{4} \\
Z'_{\text{rectangle}}(0) = \frac{1}{4} \log(AB) - \log[2^{\frac{3}{4}}(\frac{B}{A})^{\frac{1}{4}}\eta(q)]
\]

where \( \eta \) is the modular form of Dedekind:

\[
\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \quad q = e^{-2\pi \sqrt{\frac{T}{\pi}}} \tag{A11.2}
\]
For the square, we have the simpler expression

$$Z_{\text{square}}'(0) = \frac{1}{4} \log A^2 + \frac{1}{4} \log [\pi 2^5 \frac{\Gamma^2(\frac{3}{4})}{\Gamma^2(\frac{1}{4})}] \quad (A11.3)$$

Three triangles tile the plane by reflections in the side: the equilateral \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\); the bisected equilateral \((\frac{1}{2}, \frac{1}{3}, \frac{1}{6})\), and the right angle isosceles \((\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\). In these domains one can solve for the eigenmodes of the Laplacian by superposition of plane waves \([8, 10]\). We normalize the areas in terms of the side lengths \((a)\) of the equilateral, the sidelengths of the legs in the right angle isosceles, and the sidelength of the longest side in the bisected equilateral:

\[
E(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), n, m = (\frac{4\pi}{3a})^2 (n^2 + m^2 - nm) \quad n > m > 0
\]

\[
E(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), n, m = (\frac{\pi}{a})^2 (n^2 + m^2) \quad n > m > 0
\]

\[
E(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}), n, m = (\frac{4\pi}{3a})^2 (n^2 + m^2 + nm) \quad n > m > 0
\]

The corresponding zeta functions can be written in terms of the Riemann zeta-function and Dirichlet \(L\)-series \([10]\), which may in turn be resolved into sums of zeta functions of Hurwitz \(H(x, s)\) with different arguments \(x\). Using

\[
H(x, s) = \sum_{k=0}^{\infty} \frac{1}{(x + k)^s}
\]

\[
H(x, 0) = \frac{1}{2} - x \quad \frac{d}{ds} H(x, s) \bigg|_{s=0} = \log \frac{\Gamma(x)}{\sqrt{2\pi}}
\]

and the normal areas determined by the representation of the Schwarz–Christoffel transformation;

\[
A(\alpha_0, \alpha_1, \alpha_\infty) = \frac{\Gamma(\alpha_0) \Gamma(\alpha_1) \Gamma(\alpha_\infty)}{2 \Gamma(1-\alpha_0) \Gamma(1-\alpha_1) \Gamma(1-\alpha_\infty)} \quad (A11.4)
\]

one finds after some algebra the following expressions for the determinants:

\[
Z'(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})(0) = \frac{1}{3} \log \frac{\text{Area}}{A(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} + \log \frac{\Gamma(\frac{1}{3}) \pi \frac{3}{4} 2^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}
\]

\[
Z'(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})(0) = \frac{3}{8} \log \frac{\text{Area}}{A(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})} + \log \frac{\Gamma(\frac{1}{2}) \pi \frac{2}{3} 2^{-\frac{1}{2}}}{\Gamma(\frac{4}{3})}
\]

\[
Z'(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})(0) = \frac{5}{12} \log \frac{\text{Area}}{A(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})} + \log \frac{\Gamma(\frac{1}{2}) 3 \frac{4}{9} 2^{\frac{1}{3}} \pi^{\frac{3}{4}}}{\Gamma(\frac{2}{3})}
\]

The eigenvalues of the Laplacian in on the upper half sphere of radius one are \(l(l+1)\). If one imposes Dirichlet boundary conditions at the equator, one selects out the eigenmodes

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odd under reflection in a plane through the equator. Each eigenvalue will then be $l$ times degenerate. The zeta function of this spectrum is

$$Z_{\text{hemisphere}}(s) = \sum_{l=1}^{\infty} \frac{l}{(l(l+1))^{s}}$$  \hspace{1cm} (A11.8)

from which one can derive \[20\]

$$Z'_{\text{hemisphere}}(0) = 2\zeta'(-1) + \frac{1}{2} \log 2\pi - 4$$  \hspace{1cm} (A11.9)

The hemisphere can be mapped conformally to a disc with radius $R$. The difference of the regularized determinants on the hemisphere and on the disc can then be evaluated by computing the Liouville action (17) of the conformal factor \[20\]:

$$Z'_{\text{disc}}(0) = \frac{1}{6} \log 2 + \frac{1}{2} \log \pi + \frac{1}{3} \log R + 2\zeta'(-1) + \frac{5}{12}$$  \hspace{1cm} (A11.10)

Equation [A11.10] has also been checked numerically to great accuracy by computing the eigenvalues from the zeros of Bessel functions, and directly investigating the analytical continuation of the zeta function \[14\].
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