$p$–Adic representation of the Cuntz algebra
and the free coherent states

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Abstract

Representation of the Cuntz algebra in the space of (complex valued) functions on $p$–adic disk is introduced. The relation of this representation and the free coherent states is investigated.

1 Introduction

The present paper combines the investigations on $p$–adic mathematical physics and noncommutative probability. We introduce the representation of the Cuntz algebra in the space of (complex valued) functions on $p$–adic disk and investigate the relation of this representation and the free coherent states. The representation which we introduce will be unitarily equivalent to one of the representation considered by Bratteli and Yorgensen in [1] (without use of $p$–adic analysis).

Continuing the investigations of [2], [3], [4], we investigate the free coherent states (or shortly FCS), which are (unbounded) eigenvectors of the linear combination of annihilators in the free Fock space. In [2], [3] it was shown that the space of the free coherent states is highly degenerate for the fixed eigenvalue $\lambda$ (and infinite dimensional), and this degeneracy is naturally described by the space $D'(Z_p)$ of generalized functions on $p$–adic disk ($p$ is a number of independent creators in the free Fock space). In [1] the results of [2], [3] were reformulated using the language of rigged Hilbert spaces and the interpretation of the relation between the free coherent states and $p$–adics using noncommutative geometry was proposed (see also [5]).

$p$–Adic mathematical physics studies the problems of mathematical physics with the help of $p$–adic analysis. $p$–Adic mathematical physics was studied in [2]–[12]. For instance in the book [1] the analysis of $p$–adic pseudodifferential operators was developed. In [10] a $p$–adic approach in the string theory was proposed. In [8] a theory of $p$–adic valued distributions was investigated. In [10], [12] it was shown that the Parisi matrix used in the replica method is equivalent, in the simplest case, to a $p$–adic pseudodifferential operator. In [10] it was shown that the wavelet basis in $L^2(R)$ after the $p$–adic change of variable (the continuous map of $p$–adic numbers onto real numbers conserving the measure) maps onto the basis of eigenvectors of the Vladimirov operator of $p$–adic fractional derivation. In [5] a procedure to generate the ultrametric space used in the replica approach was proposed.

The Free (or quantum Boltzmann) Fock space has been considered in some works on quantum chromodynamics [13] and noncommutative probability [14]–[18].
Discuss in short the results of [2], [3], [4].
The free Fock space $\mathcal{F}$ over a Hilbert space $\mathcal{H}$ is the completion of the tensor algebra
$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n.$$ Creation and annihilation operators act as follows:
$$A^\dagger(f) f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n; \quad f, f_i \in \mathcal{H}$$
$$A(f) f_1 \otimes \ldots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n; \quad f, f_i \in \mathcal{H}$$
where $\langle \cdot, \cdot \rangle$ is the scalar product in the Hilbert space $\mathcal{H}$. Scalar product in the free Fock space (which we also denote $\langle \cdot, \cdot \rangle$) is defined in the standard way.

In the case when $\mathcal{H}$ is the $p$–dimensional complex Euclidean space we have $p$ creation operators $A^\dagger_i, i = 0, \ldots, p - 1$; $p$ annihilation operators $A_i, i = 0, \ldots, p - 1$ with the relations
$$A_i A^\dagger_j = \delta_{ij}. \quad (1)$$
and the vacuum vector $\Omega$ in the free Fock space satisfies
$$A_i \Omega = 0. \quad (2)$$

We will also use the following factor–algebra of the quantum Boltzmann algebra. The Cuntz algebra (with $p$ degrees of freedom) is the algebra with involution which is generated by $p$ creation operators $A^\dagger_i, i = 0, \ldots, p - 1$; $p$ annihilation operators $A_i, i = 0, \ldots, p - 1$ with commutation relations
$$A_i A^\dagger_j = \delta_{ij}; \quad (3)$$
$$\sum_{i=0}^{p-1} A^\dagger_i A_i = 1. \quad (4)$$

The free coherent states (or shortly FCS) were introduced in [2], [3] as the formal eigenvectors of the annihilation operator $A = \sum_{i=0}^{p-1} A_i$ in the free Fock space $\mathcal{F}$ for some eigenvalue $\lambda$,
$$A \Psi = \lambda \Psi. \quad (5)$$
The formal solution of (5) is
$$\Psi = \sum_{I} \lambda^{|I|} \Psi_I A^\dagger_I \Omega. \quad (6)$$
Here the multiindex $I = i_0 \ldots i_{k-1}, i_j \in \{0, \ldots, p - 1\}$ and
$$A^\dagger_I = A^\dagger_{i_{k-1}} \ldots A^\dagger_{i_0} \quad (7)$$
$\Psi_I$ are complex numbers which satisfy
$$\Psi_I = \sum_{i=0}^{p-1} \Psi_{II}. \quad (8)$$
The summation in the formula (6) runs on all sequences $I$ with finite length. The length of the sequence $I$ is denoted by $|I|$ (for instance in the formula above $|I| = k$). The formal series (6)
defines the functional with a dense domain in the free Fock space. For instance the domain of each free coherent state for \( \lambda \in (0, \sqrt{p}) \) contains the dense space \( X \) introduced below.

We define the free coherent state \( X_I \) of the form

\[
X_I = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^j \right)^k \lambda^{|I|} A_I^j \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A_I^j \Omega
\]  

(9)

The sum on \( l \) in fact contains \(|I|\) terms. For \( \lambda \in (0, \sqrt{p}) \) the coherent state \( X_I \) lies in the Hilbert space (the correspondent functional is bounded).

We denote by \( X \) the linear span of free coherent states of the form (9) and by \( X' \) we denote the space of all the free coherent states (given by (6)).

The following definitions and theorems were proposed in [2], [3], [4].

**Definition**

We define the renormalized pairing of the spaces \( X \) and \( X' \) as follows:

\[
\langle \Psi, \Phi \rangle = \lim_{\lambda \to \sqrt{p} - 0} \left( 1 - \frac{\lambda^2}{p} \right) \langle \Psi, \Phi \rangle
\]  

(10)

Here \( \Psi \in X' \), \( \Phi \in X \).

Note that the coherent states \( \Psi, \Phi \) defined by (9), (12) depend on \( \lambda \) and the product \( \langle \Psi, \Phi \rangle \) does not.

**Definition**

We denote \( \tilde{F} \) the completion of the space \( X \) of the free coherent states with respect to the norm defined by the renormalized scalar product.

The space \( \tilde{F} \) is a Hilbert space with respect to the renormalized scalar product.

**Theorem**

The space of the free coherent states

\[
X \xrightarrow{i} \tilde{F} \xrightarrow{j} X'
\]  

(11)

is a rigged Hilbert space.

Define the characteristic functions of \( p \)-adic disks

\[
\theta_k(x - x_0) = \theta(p^k |x - x_0|_p); \quad \theta(t) = 0, t > 1; \quad \theta(t) = 1, t \leq 1.
\]  

(12)

Here \( x, x_0 \in \mathbb{Z}_p \) lie in the ring of integer \( p \)-adic numbers and the function \( \theta_k(x - x_0) \) equals to 1 on the disk \( D(x_0, p^{-k}) \) of radius \( p^{-k} \) with the center in \( x_0 \) and equals to 0 outside this disk.

We compare the rigged Hilbert spaces of the free coherent states (11) and of generalized functions over \( p \)-adic disk:

\[
D(\mathbb{Z}_p) \xrightarrow{\mu} L^2(\mathbb{Z}_p) \xrightarrow{j'} D'(\mathbb{Z}_p)
\]

**Theorem**

The map \( \phi \) defined by

\[
\phi : \quad X_I \mapsto p^{|I|} \theta_{|I|}(x - I);
\]
extends to an isomorphism \( \phi \) of the rigged Hilbert spaces:

\[
\begin{array}{ccc}
X & \overset{i}{\rightarrow} & \tilde{\mathcal{F}} \\
\downarrow \phi & & \downarrow \tilde{\phi} \\
D(Z_p) & \overset{i'}{\rightarrow} & L^2(Z_p)
\end{array}
\]

between the rigged Hilbert space of the free coherent states (with the pairing given by the renormalized scalar product) and the rigged Hilbert space of generalized functions over \( p \)-adic disk.

2 The \( p \)-adic representation of the Cuntz algebra

In the present section we construct the representation of the Cuntz algebra in the space \( L^2(Z_p) \) of quadratically integrable functions on a \( p \)-adic disk. We will call this representation the \( p \)-adic representation. Equivalent representations (without application of \( p \)-adic analysis) were considered in [1].

Let us define the following operators in \( L^2(Z_p) \)

\[
A_i^\dagger \xi(x) = \sqrt{p} \theta_1(x - i)\xi([\frac{1}{p}x]);
\]
\[
A_i \xi(x) = \frac{1}{\sqrt{p}} \xi(i + px).
\]

Here

\[ [x] = x - x(\mod 1) \]

for \( x \in Q_p \) is the integer part of \( x \). \( \theta_1(x - i) \) is an indicator (or characteristic function) of the \( p \)-adic disk with the center in \( i \) and the radius \( p^{-1} \).

We have the following

Theorem The operators \( A_i^\dagger \) and \( A_i \) defined by (13) and (14) are mutually adjoint and satisfy the relations of the Cuntz algebra (1), (4):

\[
A_i A_j^\dagger = \delta_{ij}.
\]

\[
\sum_{i=0}^{p-1} A_i^\dagger A_i = 1.
\]

Proof The commutation relations of the introduced above operators look as follows

\[
A_i A_j^\dagger \xi(x) = A_i \sqrt{p} \theta_1(x - j)\xi([\frac{1}{p}x]) = \theta_1(i + px - j)\xi([\frac{1}{p}(i + px)]) = \\
= \theta_1(i + px - j)\xi([\frac{1}{p}i + \frac{1}{p}px]) = \delta_{ij} \xi(x);
\]

because \( \theta_1(i + px - j) = \delta_{ij} \). Therefore

\[
A_i A_j^\dagger = \delta_{ij}.
\]
Let us consider
\[ A^\dagger_i A_i \xi(x) = A^\dagger_i \frac{1}{\sqrt{p}} \xi(i + px) = \theta_1(x - i) \xi(i + p[\frac{1}{p}x]). \]

If \( x \neq i \pmod{p} \) then \( \theta_1(x - i) = 0 \).

Therefore the result of application of this operator can be nonzero only for \( x = i \pmod{p} \).

Therefore we can change the argument of the function \( \xi \) by \( x \). We get
\[ A^\dagger_i A_i \xi(x) = \theta_1(x - i) \xi(x); \]
\[ \sum_{i=0}^{p-1} A^\dagger_i A_i \xi(x) = \xi(x); \]

Therefore
\[ \sum_{i=0}^{p-1} A^\dagger_i A_i = 1. \]

Let us calculate the adjoint to \( A^\dagger_i \).

\[ \int_{Z_p} A^\dagger_i \xi(x) \eta(x) d\mu(x) = \int_{Z_p} \sqrt{p} \theta_1(x - i) \xi([\frac{1}{p}x]) \eta(x) d\mu(x) = \]
\[ = \sqrt{p} \int_{Z_p} \theta_1(x - i) \xi([\frac{1}{p}(i + x - i)]) \eta(i + x - i) d\mu(x) = \]
\[ = \sqrt{p} \int_{Z_p} \theta_1(x - i) \xi([\frac{1}{p}(x - i)]) \eta(i + x - i) d\mu(x) = \]
\[ = \sqrt{p} \int_{|x-i| \leq p^{-1}} \xi(\frac{x-i}{p}) \eta(i + x - i) d\mu(x) = \]
\[ = \sqrt{p} \int_{|x-i| \leq p^{-1}} \xi(\frac{x-i}{p}) \eta(i + px - i) p^{-1} d\mu \left( \frac{x-i}{p} \right) = \]
\[ = \frac{1}{\sqrt{p}} \int_{Z_p} \xi(x) \eta(i + px) d\mu(x) = \int_{Z_p} \xi(x) A_i \eta(x) d\mu(x) \]

because for \( p \)-adic Haar measure \( d\mu(px) = p^{-1} d\mu(x) \).

This finishes the proof of the theorem.

3 The \( p \)-adic representation as GNS

Let us define the linear functional \( \langle \cdot \rangle \) on the Boltzmann algebra as follows
\[ \langle A^\dagger_i A_j \rangle = p^{-\frac{1}{2}(|I|+|J|)} \]
(15)

With \( A^\dagger_i \) defined by (14) and \( A_i \) adjoint.

In the present section we prove that the considered in the previous section the \( p \)-adic representation of the Boltzmann algebra is unitary equivalent to the GNS representation generated by the state \( \langle \cdot \rangle \).
Theorem

1) The functional $\langle \cdot \rangle$ is a state.
2) In the corresponding GNS representation the condition (4) is satisfied.
3) The corresponding GNS representation is unitarily equivalent to the representation realized in the space of (quadratically integrable) functions on $p$-adic disk by the formula (14):

\[ A_i \xi(x) = \frac{1}{\sqrt{p}} \xi(i + px). \]

Proof Let us prove that for $X$ in the quantum Boltzmann algebra the functional (15) can be calculated by the integration over $p$-adic variable

\[ \langle X \rangle = (1, X 1) = \int_{Z_p} (X 1) d\mu(x) \] (16)

Here $(\cdot, \cdot)$ is the scalar product in the Hilbert space of square integrable functions on $p$–adic disk $Z_p$ and the action of $X$ at the RHS is defined by (13), (14).

The formula (16) can be proved by direct calculation. It is sufficient to prove (16) for monomials $A_I^* A_J$. We get

\[ A_I^* A_J 1 = A_I^* p^{-\frac{1}{2}|J|} = p^{\frac{1}{2}|I|} \theta_{|I|}(x - I) p^{-\frac{1}{2}|J|} \]

Therefore

\[ \langle A_I^* A_J \rangle = p^{\frac{1}{2}(|I| - |J|)} \int_{Z_p} \theta_{|I|}(x - I) d\mu(x) = p^{\frac{1}{2}(|I| - |J|)} p^{-|J|} = p^{-\frac{1}{2}(|I| + |J|)} \]

which gives (15).

Since \( \langle X \rangle = (1, X 1) \) this functional is a state. The equivalence of GNS and $p$-adic representations of Boltzmann algebra follows from (16) and the fact that 1 is a cyclic vector in $p$-adic representations of Boltzmann algebra. This finishes the proof of the theorem.

4 Representation of the Cuntz algebra in the space of FCS

In the present section we show how to construct non–Fock representation of the quantum Boltzmann algebra starting from the Fock representation. We will construct the $p$–adic representation of the Cuntz algebra by regularization of action of operators from the quantum Boltzmann algebra on the free coherent states.

Let us introduce some notations. The free coherent states are eigenvectors of the annihilation operator $\sum_{i=0}^{p-1} A_i$:

\[ \sum_{i=0}^{p-1} A_i \Phi = \lambda \Phi. \]

Here $\Phi$ is a function of $\lambda \in (0, \sqrt{p})$. Free coherent state (FCS) $\Phi$ is given by

\[ \Phi = \sum_I \Phi_I \lambda^{|I|} A_I^* \Omega. \]

Here $\Omega$ is the vacuum vector in the free Fock space, $\Phi_I$ satisfies (8).
We will also use the notation
\[ \Phi = A^\dagger \Phi \Omega. \]

The space of FCS is isomorphic to the space of generalized functions on a $p$–adic disc. Action of the generalized functions on test functions is given by the renormalized scalar product on the space of free coherent states

\[
(\Phi, \Psi) = \lim_{\lambda \to \sqrt{p} - 0} \left( 1 - \frac{\lambda^2}{p} \right) \langle \Phi, \Psi \rangle.
\]

(17)

Let us introduce the representation $T$ of the Cuntz algebra in the space of FCS.

**Lemma** The regularizations $T_i^\dagger = T(A_i^\dagger)$, $T_i = T(A_i)$ of the right shift operators on FCS

\[
T_i^\dagger \Phi = \frac{1}{\sqrt{p}} \left( \lambda A_i^\dagger \Phi + \Phi_\emptyset \right) \Omega = \frac{1}{\sqrt{p}} \left( \sum I \Phi_i \lambda^{|I|+1} A_i^\dagger + \Phi_\emptyset \right) \Omega;
\]

(18)

\[
T_i \Phi = \sqrt{p} \sum I \Phi_i \lambda^{|I|} A_i^\dagger \Omega = \sqrt{p} \lambda^{-1} \left( A_i A_i^{\text{op}} \right)^{\text{op}} \Omega.
\]

(19)

where

\[ \left( A_{i_1}^\dagger ... A_{i_k}^\dagger \right)^{\text{op}} = A_{i_k}^\dagger ... A_{i_0}^\dagger. \]

map the space of FCS into itself and define the representation of the Cuntz algebra.

**Proof** Check that the introduced operators map the space of FCS into itself. For the operator $T_i$ this follows from condition (8) for function $\Phi_i I$. For $T_i^\dagger$ this follows from the formula

\[
\sum_{i=0}^{p-1} A_i A_i^\dagger = \lambda A_i^\dagger + \Phi_\emptyset \sum_{i=0}^{p-1} A_i.
\]

(20)

Prove that $T_i T_j^\dagger = \delta_{ij}$.

\[
T_i T_j^\dagger \Phi = T_i \frac{1}{\sqrt{p}} \left( \lambda A_j^\dagger A_i^\dagger + \Phi_\emptyset \right) \Omega = \lambda^{-1} \left( A_i \left( \lambda A_j^\dagger A_i^\dagger + \Phi_\emptyset \right)^{\text{op}} \right)^{\text{op}} \Omega = \delta_{ij} \Phi.
\]

Let us check that $\sum_{i=0}^{p-1} T_i^\dagger T_i = 1$.

\[
\sum_{i=0}^{p-1} T_i^\dagger T_i \Phi = \sum_{i=0}^{p-1} T_i^\dagger \sqrt{p} \lambda^{-1} \left( A_i A_i^{\text{op}} \right)^{\text{op}} \Omega =
\]

\[
= \sum_{i=0}^{p-1} \left( \left( A_i A_i^{\text{op}} \right)^{\text{op}} A_i^\dagger + \Phi_i \right) \Omega = \Phi.
\]

Check that in the scalar product $\langle \cdot, \cdot \rangle$ the operators $T_i$, $T_i^\dagger$ are adjoint:

\[
\langle T_i \Phi, \Psi \rangle = \sqrt{p} \sum I \Phi_i \Psi_i \lambda^{|I|},
\]
\[ \langle \Phi, T_i^\dagger \Psi \rangle = \frac{1}{\sqrt{p}} \left( \Phi_0^* \Psi_0 + \lambda^2 \sum_I \Phi_{iI}^* \Psi_I \lambda^{2|I|} \right). \]

This implies that for \( \lambda \to \sqrt{p} - 0 \) one gets \( (T_i \Phi, \Psi) = (\Phi, T_i^\dagger \Psi) \).

This finishes the proof of the Lemma.

**Lemma**  
The representation of the Cuntz algebra defined by the operators \( T_i, T_i^\dagger \) is unitarily equivalent to the \( p \)-adic representation.

**Proof**  
To prove this it is enough to calculate the action of the operators \( T_i \) and \( T_i^\dagger \) on the FCS \( X_I \), which correspond to the normed indicators of \( p \)-adic disks. We get

\[ T_i^\dagger X_I = \frac{1}{\sqrt{p}} \left( \lambda A_{X_I}^i A_{X_I}^i + 1 \right) \Omega = \frac{1}{\sqrt{p}} X_{iI}. \]

\[ T_i X_I = \sqrt{p} \lambda^{-1} \left( A_i A_{X_I}^{\text{op}} \right)^{\text{op}} \Omega = \sqrt{p} \delta_{i0} X_{i_0 \ldots i_{k-1}}; \]

\[ T_i X_\emptyset = \sqrt{p} X_\emptyset; \]

where \( I = i_0 \ldots i_{k-1} \).

This finishes the proof of the Lemma.

Let us prove that the \( p \)-adic representation of the Cuntz algebra coincides with the restriction of action of antifock representation \( AF \) of the quantum Boltzmann algebra on the space of FCS.

**Theorem**  
\( p \)-Adic representation is the GNS representation generated by the state on the space of FCS

\[ \langle X \rangle = (1, X 1)_{L^2(Z_p)} = \left( \hat{1}, AF(X) \hat{1} \right) \]

(21)

Here

\[ \hat{1} = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i^{\dagger} \right)^k \Omega \]

is the coherent state corresponding to the indicator of \( Z_p \) and \( AF \) is the antifock representation.

**Proof**  
From (18), (19) one gets for the action of \( T_i, T_i^\dagger \) on free coherent state

\[ T_i^\dagger \Phi = \frac{\lambda}{\sqrt{p}} AF(A_i^{\dagger}) \Phi + \Phi_0 \Omega; \]

\[ T_i \Phi = \sqrt{p} \lambda AF(A_i) \Phi. \]

Then (17) implies the Theorem.

**Remark**  
Note that the operators \( AF(A_i^{\dagger}) \), \( AF(A_i) \) do not map the space of FCS into itself.

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