Classification of radial solutions for elliptic systems driven by the $k$-Hessian operator

Marius Ghergu* †

February 28, 2020

Abstract

We are concerned with non-constant positive radial solutions of the system

\[
\begin{align*}
S_k(D^2 u) &= |\nabla u|^m v^p & \text{in } \Omega, \\
S_k(D^2 v) &= |\nabla u|^q v^s & \text{in } \Omega,
\end{align*}
\]

where $S_k(D^2 u)$ is the $k$-Hessian operator of $u \in C^2(\Omega)$ ($1 \leq k \leq N$) and $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is either a ball or the whole space. The exponents satisfy $q > 0$, $m, s \geq 0$, $p \geq s \geq 0$ and $(k-m)(k-s) \neq pq$. In the case where $\Omega$ is a ball, we classify all the positive radial solutions according to their behavior at the boundary. Further, we consider the case $\Omega = \mathbb{R}^N$ and find that the above system admits non-constant positive radial solutions if and only if $0 \leq m < k$ and $pq < (k-m)(k-s)$. Using arguments from three component cooperative and irreducible dynamical systems we deduce the behavior at infinity of such solutions.

Keywords: Radially symmetric solutions, $k$-Hessian equation; asymptotic behavior, cooperative and irreducible dynamical systems

2010 AMS MSC: 35J47, 35B40, 70G60

1 Introduction

In this paper we study positive non-constant radially symmetric solutions of the system

\[
\begin{align*}
S_k(D^2 u) &= |\nabla u|^m v^p & \text{in } \Omega, \\
S_k(D^2 v) &= |\nabla u|^q v^s & \text{in } \Omega,
\end{align*}
\]

where $\Omega$ is either an open ball $B_R \subset \mathbb{R}^N$ ($N \geq 2$), centred at the origin and having radius $R > 0$, or $\Omega = \mathbb{R}^N$. The exponents $m, p, q, s$ are assumed to satisfy

$q > 0$, $m, s \geq 0$, $p \geq s \geq 0$

and

\[
\delta := (k-m)(k-s) - pq \neq 0.
\]
Throughout this paper, $S_k(D^2u)$ denotes the $k$-Hessian operator of $u \in C^2(\Omega)$, $1 \leq k \leq N$, defined as follows. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ be the eigenvalues of the Hessian matrix $D^2u$. Then,

$$S_k(D^2u) = P_k(\Lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k},$$

where $P_k(\Lambda)$ is the $k$-th elementary symmetric polynomial in the eigenvalues $\Lambda$. We point out that $\{S_k\}_{1 \leq k \leq N}$ is a family of operators which contains the Laplace operator ($k = 1$) and the Monge-Ampère operator ($k = N$).

For $2 \leq k \leq N$ the operators $S_k$ are fully nonlinear. Further, $S_k$ are not elliptic in general, unless they are restricted to the class

$$\Gamma_k = \{u \in C^2(\Omega) : S_i(D^2u) \geq 0 \text{ in } \Omega \text{ for all } 1 \leq i \leq k\}.$$

In this paper we study non-constant positive radial solutions of (1.1), that is, solutions $(u, v)$ which fulfill:

- $u, v \in \Gamma_k$ are positive and radially symmetric;
- $u$ and $v$ are not constant in any neighbourhood of the origin;
- $u$ and $v$ satisfy (1.1).

If $\Omega = \mathbb{R}^N$, such solutions of (1.1) will be called global radial solutions.

Throughout this paper, we identify radial solutions $(u, v)$ with their one variable representant, that is, $u(x) = u(r)$, $v(x) = v(r)$, $r = |x|$. It is now a standard argument (see, e.g., [17]) to check that any positive radial solution $(u, v)$ of (1.1) in $B_R$ satisfies

$$\begin{cases}
\left(\frac{n-1}{k-1}\right) r^{1-N} \left[ r^{N-k} |u'|^{k-1} u'' \right]' = |u'(r)|^m v^p(r) & \text{for all } 0 < r < R, \\
\left(\frac{n-1}{k-1}\right) r^{1-N} \left[ r^{N-k} |v'|^{k-1} v'' \right]' = |u'(r)|^q v^s(r) & \text{for all } 0 < r < R,
\end{cases}$$

(1.3)

$$u'(0) = v'(0) = 0, u(r) > 0, v(r) > 0$$

for all $0 < r < R$,

where $\binom{n-1}{k-1}$ stands for the binomial coefficient for the integers $n-1 \geq k-1$. A scaling argument yields easily that (1.3) is equivalent to

$$\begin{cases}
r^{1-N} \left[ r^{N-k} |u'|^{k-1} u'' \right]' = |u'(r)|^m v^p(r) & \text{for all } 0 < r < R, \\
r^{1-N} \left[ r^{N-k} |v'|^{k-1} v'' \right]' = |u'(r)|^q v^s(r) & \text{for all } 0 < r < R,
\end{cases}$$

(1.4)

$$u'(0) = v'(0) = 0, u(r) > 0, v(r) > 0$$

for all $0 < r < R$.

Partial differential equations related to the $k$-Hessian operator have been widely investigated in the last four decades. The results in Caffarelli, Nirenberg and Spruck [2] (see also Ivochina [16]) have opened up new mathematical methods in this direction. Ji and Bao [17] obtained Keller-Osserman type conditions for the existence of a solution to $S_k(D^2u) = f(u)$ in the entire space $\mathbb{R}^N$.

The study of the system (1.1) is motivated by the semilinear case

$$\begin{cases}
\Delta u = v & \text{in } \Omega, \\
\Delta v = |\nabla u|^2 & \text{in } \Omega,
\end{cases}$$

(1.5)
discussed in [5] as a steady state model of a viscous, heat conducting fluid. The time dependent version of (1.5), namely,
\[
\begin{align*}
  u_t - \Delta u &= \theta & \text{in } \Omega \times (0, T), \\
  \theta_t - \Delta \theta &= |\nabla u|^2 & \text{in } \Omega \times (0, T),
\end{align*}
\]  
(1.6)
is investigated in [6] and [7]. In the above coupled equations, \( u \) stands for the speed and \( \theta \) stands for the temperature of a unidirectional flow, independent of distance in the flow direction. Note that the steady states of (1.6) corresponds after the change \( \theta = -v \) to solutions of system (1.5).

Further extensions of (1.5) to the case of general nonlinearities appear in Singh [21], Filippucci and Vinti [9]. A recent work of Ghergu, Giacomoni and Singh [11] investigates radial solutions of the quasilinear system
\[
\begin{align*}
  \Delta \sigma u &= |\nabla u|^m v^p & \text{in } \Omega, \\
  \Delta \sigma v &= |\nabla u|^q v^s & \text{in } \Omega,
\end{align*}
\]
where \( \Delta \sigma u = \text{div}(|\nabla u|^{\sigma-2} \nabla u) \), \( \sigma > 1 \).

We should point out that the system (1.1) and its radially symmetric counterpart (1.3) is not a singular system (as for instance in [10, 13]). The difficulty in the study of (1.1) lies in the presence of the gradient terms \( |\nabla u|^m \) and \( |\nabla u|^q \) in the right-hand side of (1.1) which, as we shall see, leads to a rich structure of the solution set. In the following, for a function \( f : (0, R) \to \mathbb{R} \) we denote
\[
\lim_{r \to R} f(r) = \lim_{r \to R} f(r),
\]
provided such a limit exists. Also, \( C, c, c_1, c_2 \ldots \) stand for positive constants whose values may change on each occurrence.

Our first result is concerned with the case where \( \Omega \) is a ball.

**Theorem 1.1.** Assume \( \Omega = B_R, q > 0, m, s \geq 0, p \geq s \geq 0 \) and \( \delta \neq 0 \). Then:

(i) There are no positive radial solutions \( (u, v) \) of (1.1) with \( u(R^-) = \infty \) and \( v(R^-) < \infty \).

(ii) All positive radial solutions of (1.1) are bounded if and only if
\[
k > m \quad \text{and} \quad pq < (k - m)(k - s).
\]

(iii) There are positive radial solutions \( (u, v) \) of (1.1) with \( u(R^-) < \infty \) and \( v(R^-) = \infty \) if and only if
\[
k > m \quad \text{and} \quad pq > p(k + 1) + (k - m + 1)(k - s).
\]

(iv) There are positive radial solutions \( (u, v) \) of (1.1) with \( u(R^-) = v(R^-) = \infty \) if and only if
\[
k > m \quad \text{and} \quad (k - m)(k - s) < pq \leq p(k + 1) + (k - m + 1)(k - s).
\]

As an immediate consequence of the above result we obtain optimal conditions for the existence of boundary blow-up solutions for equations and system. In such a setting, the Keller-Osserman condition plays a crucial role (see, e.g. [8] [12] [19]).

Let us first consider the boundary blow-up system:
\[
\begin{align*}
  S_k(D^2 u) &= |\nabla u|^m v^p & \text{in } B_R, \\
  S_k(D^2 v) &= |\nabla u|^q v^s & \text{in } B_R, \\
  u = v &= \infty & \text{on } \partial B_R,
\end{align*}
\]  
(1.7)
where the boundary condition in (1.7) is understood in the following sense
\[
u(x) \to \infty, \quad \text{as } \text{dist}(x, \partial B_R) \to 0.
\]

From Theorem 1.1(iv) we find:
Corollary 1.2. Assume $q > 0$, $m, s \geq 0$, $p \geq s \geq 0$ and $\delta \neq 0$. Then, (1.1) admits positive radial solutions if and only if

$$k > m \quad \text{and} \quad (k - m)(k - s) < pq \leq p(k + 1) + (k - m + 1)(k - s).$$

Corollary 1.2 provides optimal conditions for the existence of radial solutions of the problem:

$$\left\{ \begin{array}{ll}
S_k(D^2u) = u^p|\nabla u|^q & \text{in } B_R, \\
u = \infty & \text{on } \partial B_R.
\end{array} \right.$$

(1.8)

As before, the boundary condition in (1.8) means $u(x) \to \infty$ as $\text{dist}(x, \partial B_R) \to 0$. Letting $m = q$ and $p = s$ in Corollary 1.2 we find:

Corollary 1.3. Assume $p \geq 0$, $q > 0$ and $(k - p)(k - q) \neq pq$. Then, (1.8) admits positive radial solutions if and only if

$$k > q \quad \text{and} \quad pq > (k - p)(k - q).$$

We point out that the case $q = 0$ in (1.8) is discussed in [22].

In the following we shall be concerned the case where $\Omega$ coincides with the whole space $\mathbb{R}^N$. Directly from Theorem 1.1 one has:

Corollary 1.4. Assume $\Omega = \mathbb{R}^N$, $q > 0$, $m, s \geq 0$, $p \geq s \geq 0$ and $\delta \neq 0$. Then, (1.1) admits non-constant global positive radial solutions if and only if

$$0 \leq m < k \quad \text{and} \quad pq < (k - m)(k - s).$$

(1.9)

The second condition in (1.9) reads $\delta > 0$.

We next study the exact behavior at infinity of global positive radial solutions of (1.1). Note that the system (1.1) is equivalent to (1.3) in the radial setting. For the ease of our exposition, we shall discuss the behavior at infinity of solutions to the equivalent system (1.4).

In this direction we obtain the following result:

Theorem 1.5. Assume $\Omega = \mathbb{R}^N$, $0 \leq m < k$ and $\delta > 0$. Then, any non-constant global positive solution $(u, v)$ of (1.4) satisfies

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{1 + \frac{k(k - s + 2)}{s}}} = A \quad \text{and} \quad \lim_{|x| \to \infty} \frac{v(x)}{|x|^{\frac{k(2k - 2m + q)}{s}}} = B,$$

(1.10)

where $A = A(N, m, p, q, s) > 0$ and $B = B(N, m, p, q, s) > 0$ are given by (3.27) and (3.28).

Theorem 1.5 above roughly says that any global radial solution $(u, v)$ of (1.4) stabilizes to

$$(U, V) = \left( A|x|^{1 + \frac{k(k - s + 2)}{s}}, B|x|^{\frac{k(2k - 2m + q)}{s}} \right)$$

which is in fact a singular solution of (1.4). In obtaining the exact behavior (1.10) we employ some results from three-component irreducible dynamical systems from Hirsch [14]. We recall these results in the first part of Section 3.

We point out that the requirement $\delta > 0$ in (1.2) is a classical condition on superlinearity of the system as it appears for instance in [3]. Also, the value of the limits $A$ and $B$ in (1.10) depend decreasingly on the space dimension $N \geq 2$. One may see this fact from their expressions in (3.27) and (3.28).

In our next result we show that given any pair $(a, b) \in (0, \infty) \times (0, \infty)$, there exists a unique positive global radial solutions of (1.1) that emanates from $(a, b)$. 

4
Theorem 1.6. Assume $\Omega = \mathbb{R}^N$, $0 \leq m < k$, $\delta > 0$ and $1 \leq k < N/2$. Then, for any $a > 0$, $b > 0$ there exists a unique non-constant global positive radial solution of (1.11) such that $u(0) = a$ and $v(0) = b$.

Finally, let us discuss the single equation

$$S_k(D^2u) = u^p|\nabla u|^q \quad \text{in } \mathbb{R}^N, N \geq 2,$$

(1.11)

which the prototype of our system (1.1). The case $q = 0$ was discussed in [1] and [17]. By taking $m = q$ and $p = s$ in Corollary 1.4 and Theorems 1.5 and 1.6 above we obtain:

Corollary 1.7. Assume $\Omega = \mathbb{R}^N$, $1 \leq k \leq N$, $p, q > 0$ and $k \neq p + q$. Then (1.11) has a non-constant positive radial solution if and only if $k > p + q$ and in this case, any non-constant positive radial solution $u$ of (1.11) satisfies

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{2k-q}} = C(N, p, q) > 0.$$

If, in addition, $1 \leq k < N/2$, then from any $a > 0$ there exists a unique non-constant positive radial solution $u$ of (1.11) such that $u(0) = a$.

2 Proof of Theorem 1.1

Let us argue first that if $k \leq m$ then (1.1) has no solutions in $[0, R]$.

From (1.4) we have that the mappings $r \mapsto r^\alpha|u'|^{\beta-1}u'$ and $r \mapsto r^\alpha|v'|^{\beta-1}v'$ are increasing and since $u'(0) = v'(0) = 0$, it follows that $u', v' \geq 0$ on $(0, R)$. In fact, integrating over $[r_0, r]$ in the first equation of (1.4) we find

$$r^{N-k}(u'(r))^k = r_0^{N-k}(u'(r_0))^k + \int_{r_0}^r t^{N-1}(u'(t))^m v^p(t)dt \quad \text{for all } 0 < r_0 < r < R.$$

Thus, if $u'(r) = 0$ for some $0 < r < R$, then $u' \equiv 0$ in $[0, r]$ and from the second equation of (1.4) we also get $v' \equiv 0$ on $[0, r]$, contradiction.

Integrating in the first equation of (1.4) we find

$$r^{N-k}(u'(r))^k = \int_0^r t^{N-1}(u'(t))^m v^p(t)dt \leq r^{N-k}(u'(r))^k \int_0^r t^{k-1}(u'(t))^{m-k}v^p(t)dt,$$

for all $0 < r < R$. Hence

$$1 \leq \int_0^r t^{k-1}(u'(t))^{m-k}v^p(t)dt \quad \text{for all } 0 < r < R.$$

Observe now that the integrand in the above estimate is a continuous function, so the right hand-side integral converges to zero as $r \to 0^+$, contradiction. Hence, $k > m$.

Let us rewrite the system (1.4) in the form

$$\begin{cases}
[(u')^k]'(r) + \frac{N-k}{r}(u'(r))^k = r^{k-1}(u'(r))^m v^p(r) & \text{for all } 0 < r < R, \\
[(v')^k]'(r) + \frac{N-k}{r}(v'(r))^k = r^{k-1}(u'(r))^{q} v^s(r) & \text{for all } 0 < r < R, \\
u'(0) = v'(0) = 0, u(r) > 0, v(r) > 0 & \text{for all } 0 < r < R,
\end{cases}$$

(2.1)
We rearrange the system \((2.1)\) as
\[
\begin{cases}
[(u')^{k-m}]'(r) + \frac{L}{r} (u')(r)^{k-m} = \left(1 - \frac{m}{k}\right)r^{k-1}v^p(r) & \text{for all } 0 < r < R, \\
[(v')^q]'(r) + \frac{N-k}{r}(v')(r)^q = r^{k-1}(u')(r)^q v^s(r) & \text{for all } 0 < r < R, \\
u'(0) = v'(0) = 0, u(r) > 0, v(r) > 0 & \text{for all } 0 < r < R,
\end{cases}
\] (2.2)

where
\[
L = \frac{(N-k)(k-m)}{k} > 0. \tag{2.3}
\]

From \((2.2)\) we have
\[
\begin{cases}
[r^L(u')^{k-m}]'(r) = \frac{k-m}{k} r^{k+L-1} v^p(r) & \text{for all } 0 < r < R, \\
r^{N-k}(v')^q]'(r) = r^{N-1}(u')^q v^s(r) & \text{for all } 0 < r < R.
\end{cases}
\] (2.4)

Before we proceed with the proof of Theorem 1.1 we need to establish two auxiliary results. The first lemma below provides basic estimates for solutions of \((2.2)\).

**Lemma 2.1.** Any non-constant positive radial solution \((u, v)\) of \((2.2)\) in \(B_R\) satisfies
\[
\left(N + \frac{km}{k-m}\right)(u'(r))^{k-m} < r^k v^p(r) \quad \text{for all } 0 < r < R, \tag{2.5}
\]
\[
\frac{k(k-m)}{kN - (N-k)m} r^{k-1} v^p(r) < [(u')^{k-m}]'(r) < \left(1 - \frac{m}{k}\right)r^{k-1}v^p \quad \text{for all } 0 < r < R, \tag{2.6}
\]
\[
(v'(r))^q < \frac{1}{N} r^k(u'(r))^q v^s(r) \quad \text{for all } 0 < r < R, \tag{2.7}
\]
and
\[
\frac{k}{N} v^s(r)(u'(r))^q r^{k-1} \leq [(v')^q]'(r) \leq r^{k-1} v^s(r)(u'(r))^q \quad \text{for all } 0 < r < R. \tag{2.8}
\]

**Proof.** We integrate the first equation of \((2.2)\) and using the fact that \(v\) is strictly increasing on \((0, R)\) we find
\[
r^L(u'(r))^{k-m} = \left(1 - \frac{m}{k}\right) \int_0^r t^{k+L-1}v^p(t)dt
\leq \left(1 - \frac{m}{k}\right)v^p(r) \int_0^r t^{k+L-1}dt
= \frac{k-m}{\beta(k+L)} r^{k+L}v^p(r) \quad \text{for all } 0 < r < R. \tag{2.9}
\]

This implies,
\[
(u'(r))^{k-m} < \frac{k-m}{kN - (N-k)m} r^{k} v^p(r) \quad \text{for all } 0 < r < R,
\]
which proves \((2.5)\). We next use the above estimate in the first equation of \((2.2)\) to deduce
\[
\left(1 - \frac{m}{k}\right)r^{k-1}v^p(r) \leq [(u')^{k-m}]' + \frac{L}{r} (u')(r)^{k-m}
\leq [(u')^{k-m}]' + \frac{L(k-m)}{kN - (N-k)m} r^{k-1}v^p(r),
\]
and this yields
\[
[(u')^{k-m}]'(r) > \frac{k(k-m)}{kN-(N-k)m} r^{k-1} v^p(r) \quad \text{for all } 0 < r < R.
\]

This is exactly the first half of the estimate in (2.6). The second half of (2.6) follows immediately from (2.6) since \( u' > 0 \). Let us note that from (2.6) we have that \( (u')^{k-m} \) is positive and strictly increasing so \( u' \) is also positive and strictly increasing. Using this fact, the estimates (2.7)-(2.8) are derived in a similar fashion.

Denote
\[
\Psi(r) = (u'(r))^{k-m} \quad \text{for } 0 \leq r < R.
\] (2.10)
The next result provides a refinement of the estimates in Lemma 2.1.

**Lemma 2.2.** Let \((u,v)\) be a solution of (2.4). Then, there exist constants \(c, c_1, c_2 > 0\) and \(\rho \in (0,R)\) such that
\[
c v^p(r) \leq \Psi(r) \quad \text{for all } \rho \leq r < R
\] (2.11) and
\[
c_1 \leq \Psi'(r)\Psi^{-\sigma}(r) \leq c_2 \quad \text{for all } \rho \leq r < R,
\] (2.12) where
\[
\sigma = \frac{p}{k-m} \frac{q + (k-m)(1+k)}{(p+1)k+p-s} > 0.
\] (2.13)

**Proof.** From (2.6) and (2.8) we find two positive constants \(C > c > 0\) such that
\[
c r^{k-1} v^p(r) < \Psi'(r) < C r^{k-1} v^p(r) \quad \text{for all } 0 \leq r < R,
\] (2.14) and
\[
c r^{k-1} v^s(r) \Psi^{\frac{q}{k-m}}(r) \leq [(v')^k](r) \leq C r^{k-1} v^s(r) \Psi^{\frac{q}{k-m}}(r) \quad \text{for all } 0 \leq r < R.
\] (2.15)

We multiply (2.14) by \(\Psi'(r)\) and integrate over \([0, r]\]. Since \(\Psi(0) = 0\) and \(v\) is increasing we find
\[
\Psi^{\frac{q+k-m}{k-m}}(r) \leq C \int_0^r v^{p-s}(t) [(v')^k]'(t) dt
\]
\[
\leq C v^{p-s}(r) \int_0^r [(v')^k]'(t) dt = C v^{p-s}(r)(v'(r))^k \quad \text{for all } 0 \leq r < R.
\]

Hence
\[
\Psi^{\frac{q+k-m}{k(1+m)}}(r) \leq C v^{\frac{p-s}{k}}(r)v'(r) \quad \text{for all } 0 \leq r < R.
\] (2.16)

Take \(\rho \in (0,R)\). From (2.13) we find
\[
c_1 v^p(r) < \Psi'(r) < c_2 v^p(r) \quad \text{for all } \rho \leq r < R.
\] (2.17)

We multiply (2.16) by \(\Psi'(r)\). Using \(\Psi'(r) < c_2 v^p(r)\) for all \(\rho \leq r < R\), we derive
\[
\Psi^{\frac{q+k-m}{k(1+m)}}(r) \Psi'(r) \leq C v^{\frac{p-s}{k}}(r)v'(r) \quad \text{for all } 0 \leq r < R.
\]
Integrate now the above inequality over \([\rho, r]\). Since \(v\) is continuous and positive on \([\rho, R]\), by taking a larger constant \(C > 0\) such that
\[
\Psi^{\frac{q + (k - m)(1 + k)}{k - m}}(r) \leq C v^{(p + 1)k + p - s}(r) \quad \text{for all } \rho \leq r < R.
\]
Using this last estimate together with (2.17) we write
\[
\Psi^{\frac{q + (k - m)(1 + k)}{k - m}}(r) \leq C\big(v^{p}(r)\big)^{\frac{(p + 1)k + p - s}{p}} \leq C\big(\Psi^{\prime}(r)\big)^{\frac{(p + 1)k + p - s}{p}} \quad \text{for all } \rho \leq r < R,
\]
which yields
\[
\Psi^{\sigma}(r) \leq C\Psi^{\prime}(r) \quad \text{for all } \rho \leq r < R. \quad (2.18)
\]
On the other hand, from (2.15) we have
\[
[(v^{\prime})^{k+1}]^{\prime}(r) \leq C v^{s}(r)\Psi^{\frac{q}{k - m}}(r) \quad \text{for all } \rho \leq r < R.
\]
Multiply by \(v^{\prime}\) and integrate over \([\rho, r]\) in the above inequality. We find
\[
(v^{\prime})^{k+1}(r) - (v^{\prime})^{k+1}(r_{1}) \leq C\Psi^{\frac{q}{k - m}}(r) \int_{\rho}^{r} v^{s}(t)v^{\prime}(t)dt
\]
\[
\leq C\Psi^{\frac{q}{k - m}}(r)v^{s+1}(r) \quad \text{for all } \rho \leq r < R.
\]
Again by continuity arguments and by enlarging the value of \(C > 0\), one has
\[
(v^{\prime})^{k+1}(r) \leq C\Psi^{\frac{q}{k - m}}(r)v^{s+1}(r) \quad \text{for all } \rho \leq r < R,
\]
that is,
\[
v^{\frac{q}{k+1}}(r)v^{\prime}(r) \leq C\Psi^{\frac{q}{(k+1)(k-m)}}(r) \quad \text{for all } \rho \leq r < R.
\]
Multiply the above inequality by \(\Psi^{\prime}(r)\). Using (2.17) we find
\[
v^{p+1}(r)v^{\prime}(r) \leq C\Psi^{\frac{q}{(k+1)(k-m)}}(r)\Psi^{\prime}(r) \quad \text{for all } \rho \leq r < R.
\]
A new integration over \([\rho, r]\) a continuity argument and by taking a larger constant \(C > 0\) one deduces
\[
v^{(p+1)k + p - s}(r) \leq C\Psi^{\frac{q + (k+1)(k-m)}{k - m}}(r) \quad \text{for all } \rho \leq r < R,
\]
which yields (2.11).

Using again (2.17) from which we have \(v^{p}(r) \geq c\Psi^{\prime}(r)\), we find
\[
c\left(\Psi^{\prime}(r)\right)^{\frac{(p + 1)k + p - s}{p}} \leq v^{(p+1)k + p - s}(r) \leq C\Psi^{\frac{q + (k+1)(k-m)}{k - m}}(r) \quad \text{for all } \rho \leq r < R,
\]
that is,
\[
c\Psi^{\prime}(r) \leq \Psi^{\sigma}(r) \quad \text{for all } \rho \leq r < R, \quad (2.19)
\]
where \(\sigma > 0\) is given by (2.13). We next combine (2.18) and (2.19) to deduce
\[
c_{1} \leq \Psi^{\prime}(r)\Psi^{-\sigma}(r) \leq c_{2} \quad \text{for all } \rho \leq r < R, \quad (2.20)
\]
for some \(c_{2} > c_{1} > 0\). This is exactly inequality (2.12). \(\square\)
Proof of Theorem 1.1 completed.

(i) Assume that \((u, v)\) is a solution of (1.4) with \(u(R^-) = \infty\) and \(v(R^-) < \infty\). Since \(r^{N-k}(u')^k\) is increasing (from the first equation of (1.4)) we deduce that \(u'(R^-) = \infty\). Also, from (2.10) we find

\[
[(u')^{k-m}]'(r) < C r^{k-1} v^p
\]

for some positive constants \(C > 0\). Integrating over \([0, R]\) we obtain

\[
(u'(R^-))^{k-m} < C \int_0^R t^{k-1} v^p(t) dt < \infty,
\]

which contradicts \(u'(R^-) = \infty\).

(iii)-(iv) Let \((u, v)\) be a solution of (1.4) with \(v(R^-) = \infty\) and let \(\Psi\) be defined by (2.10). From (2.11) it follows that \(\Psi(R^-) = \infty\). We integrate over \([r, R]\) in (2.12) to deduce \(\sigma > 1\) and

\[
c_1(R - r)^{-\frac{1}{k-m(\sigma-1)}} \leq u'(r) \leq c_2(R - r)^{-\frac{1}{k-m(\sigma-1)}}
\]

for all \(\rho \leq r < R\).

This shows that

\[
u(R^-) = u(\rho) + \int_\rho^R u'(r) dr < \infty \iff \int_\rho^R u'(r) dr < \infty
\]

\[
\iff \int_\rho^R (R - r)^{-\frac{1}{k-m(\sigma-1)}} dr < \infty
\]

\[
\iff \int_0^1 t^{-\frac{1}{k-m(\sigma-1)}} dt < \infty
\]

\[
\iff \sigma > \frac{k-m+1}{k-m},
\]

and

\[
u(R^-) = \infty \iff \sigma \leq \frac{k-m+1}{k-m}.
\]

Conversely, assume now that \(\sigma > 1\). The existence of a local non-constant positive solution to (2.4) in a small ball \(B_\rho\) follows from standard fixed point arguments; see e.g., [9, Proposition A1] and [4, Proposition 9]. More precisely, the mapping

\[
T : C^1[0, \rho] \times C^1[0, \rho] \to C^1[0, \rho] \times C^1[0, \rho],
\]

defined by

\[
T[u, v](r) = \begin{bmatrix}
T_1[u, v](r) \\
T_2[u, v](r)
\end{bmatrix},
\]

where

\[
\begin{cases}
T_1[u, v](r) = a + \int_0^r \left( \frac{k-m}{k} t^{-L} \int_0^t \tau^{k-L-1} v^p(\tau) d\tau \right)^{1/(k-m)} d\tau,

T_2[u, v](r) = b + \int_0^r \left( \int_0^t \tau^{k-1} v^8 |u'|^q d\tau \right)^{1/k} d\tau,
\end{cases}
\]

and \(a, b > 0\), has a fixed point in \(C^1[0, \rho] \times C^1[0, \rho]\) provided \(\rho > 0\) is small enough. Further, the scaling \((u_\lambda, v_\lambda)\) defined as

\[
u_\lambda(x) = \lambda^{1+\frac{(2k-1)+k(k-s)}{k}} u\left(\frac{x}{\lambda}\right), \quad v_\lambda(x) = \lambda^{\frac{(2k-1)(k-m)+kq}{k}} v\left(\frac{x}{\lambda}\right),
\]

for all \(\lambda > 0\), is a solution of (1.4).
provides a non-constant positive radially symmetric solution of (2.4) in the ball $B_{\lambda \rho}$. This shows that in any ball of positive radius there are non-constant positive radially symmetric solution of (1.1).

Let now $(u, v)$ be a positive non-constant solution of (1.4) in a maximum interval $[0, R_{\text{max}})$. We claim that if $\sigma > 1$ then $v(R_{\text{max}}) = \infty$. Using the estimate (2.12) in Lemma 2.2 on obtains after integrating over $[\rho, r]$ that

$$c_1(r - \rho) \leq \frac{1}{\sigma - 1} \left( \Psi_1^1(\rho) - \Psi_1^1(r) \right)$$

for all $\rho < r < R_{\text{max}}$.

Hence, by letting $r \to R_{\text{max}}$ one gets $c_1R_{\text{max}} \leq C_1\rho + \frac{1}{\sigma - 1} \Psi_1^1(\rho) < \infty$.

This implies $R_{\text{max}} < \infty$ and then, using part (i) above, one deduces that $v(R_{\text{max}}) = \infty$.

In conclusion we found

- There are positive solutions $(u, v)$ of (1.4) with $u(R^-) < \infty$ and $v(R^-) < \infty$ if and only if $\sigma > \frac{k-m+1}{k-m}$;
- There are positive solutions $(u, v)$ of (1.4) with $u(R^-) = v(R^-) < \infty$ if and only if $1 < \sigma \leq \frac{k-m+1}{k-m}$;
- All positive solutions $(u, v)$ of (1.4) are bounded if and only if $\sigma < 1$.

Note that the case $\sigma = 1$ is excluded from our analysis by the assumption (1.2). Now, the above conditions in terms of $\sigma$ are equivalent to (ii)-(iv) in the statement of Theorem 1.1. \hfill \Box

### 3 Proof of Theorem 1.5

Our approach to the study of the behaviour of solutions to (1.4) at infinity relies on some properties for three component dynamical systems obtained in Hirsch [14]. For the reader’s convenience, we shall briefly recall them below.

#### 3.1 Some results for cooperative dynamical systems

Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{R}^3$. We say that $a \leq b$ (resp. $a < b$) if $a_i \leq b_i$ (resp. $a_i < b_i$) for all $1 \leq i \leq 3$. We also define the closed interval $[a, b] = \{ u \in \mathbb{R}^3 : a \leq u \leq b \}$ and the open interval $(a, b] = \{ u \in \mathbb{R}^3 : a < u < b \}$ with endpoints at $a$ and $b$.

A set $X \subset \mathbb{R}^3$ is said to be $p$-convex if the segment line joining any two points in $X$ lies entirely in $X$. Throughout this section $X \subset \mathbb{R}^3$ is assumed to be an open $p$-convex set.

Let $g = (g_1, g_2, g_3) : X \to \mathbb{R}^3$ be a $C^1$ cooperative vector field in the sense that

$$\frac{\partial g_i}{\partial x_j} \geq 0 \quad \text{in} \quad X \quad \text{for any} \quad i, j = 1, 2, 3, \quad i \neq j.$$

For any $P \in \mathbb{R}^3$ we denote by $\Phi(t, P)$ the maximally defined solution of the differential equation

$$\frac{d\zeta}{dt} = g(\zeta) \quad (3.1)$$

subject to the initial condition $\zeta(0) = P$. The collection of maps $\{\Phi(t, \cdot)\}$ is called the flow of the differential equation (3.1).

It is well know the following comparison property for cooperative systems.
**Theorem 3.1.** (See [14]) Suppose \( g : X \to \mathbb{R}^3 \) is a \( C^1 \) cooperative vector field and let \( \zeta, \xi : [0, a] \to \mathbb{R}, a > 0, \) be two solutions of (3.1) such that

\[
\zeta(0) < \xi(0) \quad (\text{resp.} \zeta(0) \leq \xi(0)).
\]

Then

\[
\zeta(t) < \xi(t) \quad (\text{resp.} \zeta(t) \leq \xi(t)) \quad \text{for all} \ t \in [0, a].
\]

For any point \( P \in X \) we denote by \( \omega(P) \) the \( \omega \)-limit set of \( P \), that is, the set of all points \( Q \in \mathbb{R}^3 \) so that there exists \( \{t_j\}, t_j \to \infty \) (as \( j \to \infty \)) such that \( \Phi(t_j, P) \to Q \) (as \( j \to \infty \)). Let also \( E \) be the set of all equilibrium points of (3.1), that is, solutions of \( g(\zeta) = 0 \).

Hirsch [14] and then Hirsch and Smith [15] obtained that in any three component cooperative system the omega limit sets preserve the partial order between the elements of \( X \) or approach the equilibrium set \( E \). This is summarised in the result below.

**Theorem 3.2.** (Limit Set Dichotomy, see [14] Theorem 3.8, [15] Theorem 1.16)

Suppose \( g : X \to \mathbb{R}^3 \) is a \( C^1 \) cooperative vector field and let \( P, Q \in X, P < Q \). Then the following alternative holds:

(i) either \( \omega(P) \prec \omega(Q) \);

(ii) or \( \omega(P) = \omega(Q) \subset E \).

A \( C^1 \)-cooperative vector field \( g : X \to \mathbb{R}^3 \) is said to be irreducible if at any point \( P \in X \) its gradient \( \nabla g(P) \) is an irreducible matrix. Hirsch [14] showed that compact omega limit sets of cooperative and irreducible vector fields have a particular property in the sense that they approach the equilibrium set for almost all points in \( X \).

**Theorem 3.3.** (See [14] Theorem 4.1)

Suppose \( g : X \to \mathbb{R}^3 \) is a \( C^1 \) cooperative and irreducible vector field and that for all \( P \in X \) the \( \omega \)-limit set \( \omega(P) \) is compact. Then, there exists \( \Sigma \subset X \) with zero Lebesgue measure such that

\[
\omega(P) \subset E \quad \text{for all} \quad P \in X \setminus \Sigma,
\]

where \( E \) denotes the set of equilibrium point of (3.1).

### 3.2 Proof of Theorem 1.5

Let \((u,v)\) be a non-constant global positive solution of (1.4). We introduce the change of variables

\[
X(t) = \frac{ru'(r)}{u(r)}, \quad Y(t) = \frac{rv'(r)}{v(r)}, \quad Z(t) = \frac{r^k v^p(r)}{(u'(r))^{k-m}} \quad \text{and} \quad W(t) = \frac{r^{k+1}(u'(r))^q}{(v'(r))^k}.
\]

where \( t = \ln(r) \in \mathbb{R} \). Thus, a direct calculation shows that \((X,Y,Z,W)\) satisfies the system

\[
\begin{align*}
X_t &= X\left(\frac{2k-N}{k} - X + \frac{1}{k}Z\right) \quad \text{for all} \ t \in \mathbb{R}, \\
Y_t &= Y\left(\frac{2k-N}{k} - Y + \frac{1}{k}W\right) \quad \text{for all} \ t \in \mathbb{R}, \\
Z_t &= Z\left(\frac{kN-m(N-k)}{k} - \frac{k-m}{k}Z + pY\right) \quad \text{for all} \ t \in \mathbb{R}, \\
W_t &= W\left(\frac{kN-q(N-k)}{k} + sY + \frac{q}{k}Z - W\right) \quad \text{for all} \ t \in \mathbb{R}.
\end{align*}
\]
Using L’Hôpital’s rule one has
\[
\lim_{t \to \infty} X(t) = \lim_{r \to \infty} \frac{ru'(r)}{u(r)} = \lim_{r \to \infty} \left(1 + \frac{ru''(r)}{u'(r)}\right) = \lim_{t \to \infty} \left(\frac{1}{k}Z(t) + \frac{k - N}{k}\right), \tag{3.4}
\]
provided the limit \(\lim_{t \to \infty} Z(t)\) exists. Thus, it is enough to study the system consisting of the last three equations of (3.3) which we arrange in the form
\[
\zeta_t = g(\zeta) \quad \text{in} \, \mathbb{R}, \tag{3.5}
\]
where
\[
\zeta(t) = \begin{bmatrix} Y(t) \\ Z(t) \\ W(t) \end{bmatrix} \quad \text{and} \quad g(\zeta) = \begin{bmatrix} Y\left(\frac{2k-N}{k} - Y + \frac{1}{k}W\right) \\ Z\left(\frac{kN-m(N-k)}{k} - \frac{k-m}{p-1}Z + pY\right) \\ W\left(\frac{kN-q(N-k)}{p-1} + sY + \frac{q}{k}Z - W\right) \end{bmatrix}. \tag{3.6}
\]
Among all the equilibrium points of (3.5)-(3.6), only one has all components strictly positive namely
\[
\zeta_\infty = \begin{bmatrix} Y_\infty \\ Z_\infty \\ W_\infty \end{bmatrix}, \tag{3.7}
\]
where
\[
\begin{cases} 
2k-N \frac{k}{k} - Y_\infty + \frac{1}{k}W_\infty = 0, \\
kN - m(N-k) \frac{k}{k} - \frac{k-m}{k}Z_\infty + pY_\infty = 0, \\
kN - q(N-k) \frac{k}{k} + sY_\infty + \frac{q}{k}Z_\infty - W_\infty = 0.
\end{cases} \tag{3.8}
\]
Solving (3.8) we find
\[
\begin{cases} 
Y_\infty = \frac{kq + 2k(k-m)}{\delta}, \\
Z_\infty = \frac{kp}{k-m}Y_\infty + N + \frac{km}{k-m}, \\
W_\infty = kY_\infty + N - 2k.
\end{cases} \tag{3.9}
\]
**Lemma 3.4.** The equilibrium point \(\zeta_\infty\) is asymptotically stable.

**Proof.** Using the equalities in (3.8) we compute the linearized matrix of (3.5) at \(\zeta_\infty\) as follows:
\[
M_\infty = \begin{bmatrix} -Y_\infty & 0 & \frac{1}{k}Y_\infty \\ pZ_\infty & -\frac{k-m}{k}Z_\infty & 0 \\ sW_\infty & \frac{q}{k}W_\infty & -W_\infty \end{bmatrix}.
\]
Thus, the characteristic polynomial of \(M_\infty\) is
\[
P(\lambda) = \det(\lambda I - M) = \lambda^3 + a\lambda^2 + b\lambda + c,
\]
where
\[
a = Y_\infty + \frac{k-m}{k}Z_\infty + W_\infty, \tag{3.10a}
\]
\[
b = \frac{k-m}{k}Y_\infty Z_\infty + \frac{k-s}{k}Y_\infty W_\infty + \frac{k-m}{k}Z_\infty W_\infty, \tag{3.10b}
\]
\[
c = \frac{\delta}{k^2}Y_\infty Z_\infty W_\infty. \tag{3.10c}
\]
We divide our argument into two steps.

**Step 1:** $ab > 9c$. Note that from $\delta > 0$, $k > m$ and (1.2) we have $k > s$. Hence, using $Y_\infty$, $Z_\infty$, $W_\infty > 0$ we estimate (3.10a) as follows

$$a \geq \frac{k - s}{k} Y_\infty + \frac{k - m}{k} Z_\infty + \frac{k - s}{k} W_\infty.$$ 

Thus, by AM-GM inequality we find

$$a \geq 3 \left( \frac{k - m}{k} \right)^{\frac{2}{3}} \left( \frac{k - s}{k} \right)^{\frac{1}{3}} (Y_\infty Z_\infty W_\infty)^{\frac{1}{3}}.$$ 

In a similar fashion, from (3.10b) and AM-GM inequality we estimate

$$b \geq 3 \left( \frac{k - m}{k} \right)^{\frac{2}{3}} \left( \frac{k - s}{k} \right)^{\frac{1}{3}} \left( \frac{p - 1}{p} \right)^{\frac{1}{3}} (Y_\infty Z_\infty W_\infty)^{\frac{2}{3}}.$$

We now multiply the above inequalities to deduce

$$ab \geq 9 \left( \frac{k - m}{k} \right) \left( \frac{k - s}{k} \right) \left( Y_\infty Z_\infty W_\infty \right)^{\frac{1}{3}} > 9c.$$ 

**Step 2:** all three roots $\lambda_1$, $\lambda_2$ and $\lambda_3$ of the characteristic polynomial $P(\lambda)$ of $M_\infty$ have negative real part. Indeed, if $\lambda_i \in \mathbb{R}$, for all $i = 1, 2, 3$ then, since $a, b, c > 0$ it follows $P(\lambda) > 0$ for all $\lambda \geq 0$ so that $\lambda_i < 0$ for all $i = 1, 2, 3$. If $P$ has exactly one real root, say $\lambda_1 \in \mathbb{R}$, then $\Re(\lambda_2) = \Re(\lambda_3)$. Using $P(-a) = -ab + c < 0$, it follows that $\lambda_1 > -a$. Since $\lambda_1 + \lambda_2 + \lambda_3 = -a$ we easily deduce that $\Re(\lambda_2) = \Re(\lambda_3) < 0$. This proves that $\zeta_\infty$ is asymptotically stable.

**Lemma 3.5.** The following estimates hold for all $t \in \mathbb{R}$:

$$0 < Y(t) < Y_\infty, \quad N + \frac{km}{k - m} < Z(t) < Z_\infty, \quad N < W(t) < W_\infty.$$  

(3.11)

**Proof.** We proceed into four steps.

**Step 1:** Preliminary estimates:

$$Z(t) > N + \frac{km}{k - m}, \quad W(t) > N \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad \lim_{t \to -\infty} Y(t) = 0.$$  

(3.12)

The lower bounds for $Z$ and $W$ follow from (2.5a) and (2.6a) in Lemma 2.1. Since $v'(0) = 0$ and $v(0) > 0$ we have $\lim_{t \to -\infty} Y(t) = \lim_{r \to 0} \frac{v'(r)}{v(r)} = 0$.

**Step 2:** There exists $T \in \mathbb{R}$ such that $Z(t) < Z_\infty$ for all $t \in (-\infty, T]$.

It is enough to show that

$$\lim_{t \to -\infty} Z(t) = N + \frac{km}{k - m}.$$  

(3.13)

To this aim, we shall use the Generalized Mean Value Theorem\footnote{Generalized Mean Value Theorem (or Cauchy’s Theorem) states that if $f, g : [a, b] \to \mathbb{R}$ are differentiable functions on $(a, b)$ and continuous on $[a, b]$, then there exists $c \in (a, b)$ such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$.}[20, Theorem 5.9, page 107].
Let \( t \in (-\infty, 0) \) and \( r = e^t \in (0, 1) \). From the first equation of (2.2) we have

\[
\frac{d}{dr} \left[ (u')^{k-m}(r) \right] = \frac{k-m}{k} \left[ r^{k-1}v^p(r) - \frac{N-k}{r}(u'(r))^{k-m} \right].
\]

Using this fact and the Generalized Mean Value Theorem, there exists \( c \in (0, r) \) such that

\[
Z(t) = \frac{rv^m(r)}{(u')^{k-m}(r)} = \frac{d}{dr} \left[ \frac{rv^m(r)}{(u')^{k-m}(r)} \right](c) = \frac{k}{k-m} \cdot \frac{c^{k-1}v^p(c) + pc^{k-1}v^p(c) - \frac{N-k}{r}(u'(c))^{k-m}}{c^{k-1}v^p(c)}.\]

Hence

\[
Z(t) = \frac{k}{k-m} \cdot \frac{Z(\ln c)}{Z(\ln c) - (N-k)} \left( k + pY(\ln c) \right). \tag{3.14}
\]

Recall that from Step 1 above we have

\[
Z(t) > N + \frac{km}{k-m} \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.15}
\]

Thus,

\[
\frac{Z}{Z-(N-k)} < \frac{k-m}{k^2} \cdot \left( N + \frac{km}{k-m} \right).\]

It now follows from (3.14) that

\[
\limsup_{t \to -\infty} Z(t) \leq \frac{1}{k} \left( N + \frac{km}{k-m} \right) \left( k + p \lim_{t \to -\infty} Y(t) \right) = N + \frac{km}{k-m}. \tag{3.16}
\]

From (3.15), the opposite inequality is also true. Hence (3.13) holds which implies that there exists \( T \in \mathbb{R} \) such that \( Z(t) < Z_\infty \) for all \( t \leq T \).

**Step 3:** There exists a sequence \( t_j \to -\infty \) such that

\[
Y(t_j) < Y_\infty, \quad Z(t_j) < Z_\infty \quad \text{and} \quad W(t_j) < W_\infty \quad \text{for all} \quad j \geq 1. \tag{3.17}
\]

Suppose the above inequalities do not hold. By taking \( T \in \mathbb{R} \) sufficiently close to \(-\infty\) and in light of the estimates already obtained at Step 1 and 2 above, we may assume

\[
Y(t) < Y_\infty, \quad Z(t) < Z_\infty \quad \text{and} \quad W(t) \geq W_\infty \quad \text{for all} \quad t \leq T. \tag{3.18}
\]

Using these estimates in the differential equation satisfied by \( W \) in (3.3) we deduce \( W_t < 0 \) on \((-\infty, T]\). Thus, \( W \) is decreasing in a neighbourhood of \(-\infty\) and thus exists

\[
\ell := \lim_{t \to -\infty} W(t) = \lim_{r \to 0} \frac{rv^q(r)(u'(r))^q}{(v'(r))^k}.\]

From (3.18) one has

\[
\ell \geq W_\infty. \tag{3.19}
\]
On the other hand, using (2.1) one has

\[
\frac{d}{dr}[(u')^k]'(r) = r^{k-1}(u'(r))^q v^s(r) - \frac{N - k}{r}(v'(r))^k
\]

\[
\frac{d}{dr}[r^k v^s(u')^q]'(r) = k r^{k-1} v^s(r)(u'(r))^q + s r^k v'(r)(u'(r))^q + \frac{q}{k} r^k v^{s+p}(r)(u'(r))^q - k r^{k-1}(u'(r))^q v^s(r).
\]

Let \( t \in (-\infty, T] \) and \( r = e^t \). Applying the Generalized Mean Value Theorem as in the previous step and using the above equalities we find \( c \in (0, r) \) such that

\[
W(t) = \frac{W(\ln c)}{W(\ln c) - (N - k)} \left[ k - \frac{q(N - k)}{k} + s Y(\ln c) + \frac{q}{k} Z(\ln c) \right]. \tag{3.20}
\]

Recall that by (3.12) we have \( Z > N \) and \( W > N \) so that right hand side of (3.20) is positive. Letting \( t \to -\infty \) (that is, \( c \to 0 \)) in (3.20) and using \( \lim_{t \to -\infty} Z(t) < Z_\infty \) and \( \lim_{t \to -\infty} Y(t) = 0 \) we obtain

\[
\ell = \lim_{t \to -\infty} W(t) \leq \frac{\ell}{\ell - (N - k)} \left[ k - \frac{q(N - k)}{k} + \frac{q}{k} Z_\infty \right]. \tag{3.21}
\]

This yields

\[
\ell \leq N - \frac{q(N - k)}{k} + \frac{q}{k} Z_\infty.
\]

Comparing this inequality with the last equation of (3.3) we find \( \ell < W_\infty \), which contradicts (3.19). This proves that (3.17) holds.

**Step 4: Conclusion of the proof.**

We can compare now the solution \( \zeta = (Y, Z, W) \) and the equilibrium point \( \zeta_\infty = (Y_\infty, Z_\infty, W_\infty) \) on each of the intervals \([t_j, \infty)\). Thanks to Theorem 3.1 we deduce

\[
Y(t) < Y_\infty, \quad Z(t) < Z_\infty \quad \text{and} \quad W(t) < W_\infty \quad \text{for all} \quad t \geq t_j. \tag{3.22}
\]

Since \( t_j \to -\infty \) it follows that the estimates in (3.22) hold for all \( t \in \mathbb{R} \) and this together with Step 1 proves (3.11).

We are now able to complete the proof of Theorem 1.5. Let \((u, v)\) be a non-constant global positive radial solution of system (1.1) and denote by \((X, Y, Z, W)\) the corresponding solution of (3.3) as described in (3.2).

Then \( \zeta(t) = \begin{bmatrix} Y(t) \\ Z(t) \\ W(t) \end{bmatrix} \) satisfies (3.5)-(3.6). Thus, by Lemma 3.5 we have

\[
\zeta_* := \begin{bmatrix} 0 \\ N + \frac{km}{k - m} \frac{N}{N} \end{bmatrix} < \zeta(0).
\]

Denote by \( E \subset \mathbb{R}^3 \) the set of equilibrium points associated with (3.5)-(3.6). From the result in Theorem 3.3 there exists a set \( \Sigma \subset \mathbb{R}^3 \) of Lebesgue measure zero such that

\[
\omega(\bar{\zeta}) \subseteq E \quad \text{for all} \quad \bar{\zeta} \in [\zeta_*, \zeta_\infty] \setminus \Sigma. \tag{3.23}
\]
For $\bar{\zeta} \in [\zeta_*, \zeta_\infty] \setminus \Sigma$ denote by

$$
\Phi(t, \bar{\zeta}) = \begin{bmatrix} \bar{Y}(t) \\ \bar{Z}(t) \\ \bar{W}(t) \end{bmatrix}
$$

the flow of (3.5) associated with the initial data $\bar{\zeta} \in \mathbb{R}^3$. Using Theorem 3.1 and the fact that $\hat{\zeta} \geq \zeta_*$ we find

$$
\Phi(t, \bar{\zeta}) \geq \begin{bmatrix} 0 \\ N + \frac{km}{k - m} \\ 0 \end{bmatrix}
$$

for all $t \geq 0$.

Hence, $\omega(\bar{\zeta})$ is finite and consists of equilibrium points in $E$ whose second component is greater than or equal to $N + \frac{km}{k - m}$. It follows that

$$
\omega(\bar{\zeta}) \subseteq \{\zeta_1, \zeta_2, \zeta_3, \zeta_\infty\},
$$

where

$$
\zeta_1 = \begin{bmatrix} 0 \\ N + \frac{km}{k - m} \\ 0 \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} 0 \\ N + \frac{km}{k - m} \\ N + \frac{qk}{k - m} \end{bmatrix}, \quad \zeta_3 = \begin{bmatrix} \frac{2k - N}{k} \\ N + \frac{mk + p(2k - N)}{k - m} \\ 0 \end{bmatrix}
$$

and $\zeta_\infty$ is given by (3.7). Note, that $\zeta_3$ has all components non-negative if and only of $2k \geq N$.

We claim that

$$
\omega(\bar{\zeta}) = \{\zeta_\infty\} \quad \text{for all} \quad \bar{\zeta} \in [\zeta_*, \zeta_\infty] \setminus \Sigma. \tag{3.24}
$$

If $\zeta_\infty \in \omega(\bar{\zeta})$ then, using Lemma 3.4 we have that $\zeta_\infty$ is asymptotically stable, so that $\omega(\bar{\zeta}) = \{\zeta_\infty\}$ and thus, (3.24) follows.

Assume in the following that $\zeta_\infty \notin \omega(\bar{\zeta})$ so $\omega(\bar{\zeta}) \subseteq \{\zeta_1, \zeta_2, \zeta_3\}$.

If $\{\zeta_1, \zeta_2\} \subset \omega(\bar{\zeta})$ or $\{\zeta_2, \zeta_3\} \subset \omega(\bar{\zeta})$ then, along a subsequence $\bar{W}$ converges to 0 and $N + \frac{qk}{k - m}$. By the Intermediate Value Theorem we deduce that for all $\tau \in (0, N + \frac{qk}{k - m})$ there exists a sequence $t_j \to \infty$ such that $\bar{W}(t_j) = \tau$ which contradicts the fact that $\omega(\bar{\zeta})$ is finite. In a similar way, $\{\zeta_1, \zeta_3\} \subset \omega(\bar{\zeta})$ we deduce that $2k > N$ and for any $0 < \gamma < \frac{2k - N}{k}$ there exists a sequence $t_j \to \infty$ such that $\bar{Y}(t_j) = \gamma$ which again contradictions the fact that $\omega(\bar{\zeta})$ is finite.

The above arguments shows that $\omega(\bar{\zeta})$ reduces to a single element. We show in the following that this raises again a contradiction unless $\omega(\bar{\zeta}) = \{\zeta_\infty\}$. Suppose for instance that $\omega(\bar{\zeta}) = \{\zeta_2\}$, that is

$$
\lim_{t \to \infty} \bar{Y}(t) = 0, \quad \lim_{t \to \infty} \bar{Z}(t) = N + \frac{km}{k - m} \quad \text{and} \quad \lim_{t \to \infty} \bar{W}(t) = N + \frac{qk}{k - m}.
$$

But then, for large $t > 0$ we find

$$
\bar{Y}_t = \bar{Y}(\frac{2k - N}{k} - \bar{Y} + \frac{1}{k} \bar{W}) > 0.
$$

This implies that $\bar{Y}$ is increasing in a neighbourhood of infinity. Hence, for large $t > 0$ we have $\bar{Y}(t) \leq \lim_{s \to \infty} \bar{Y}(s) = 0$, contradiction. In a similar way, if $\omega(\bar{\zeta}) = \{\zeta_1\}$ or if $\omega(\bar{\zeta}) = \{\zeta_3\}$ we raise a contradiction. This finally proves (3.24).

Take now $\zeta \in [\zeta_*, \zeta_\infty] \cap \Sigma$ and let $\bar{\zeta} \in [\zeta_*, \zeta_\infty] \setminus \Sigma$ be such that $\bar{\zeta} < \zeta$. By the Limit Set Dichotomy Theorem 3.2 the following alternative holds:
\( \{ \zeta_\infty \} = \omega(\zeta) < \omega(\zeta) \);
- or \( \omega(\zeta) = \omega(\zeta) = \{ \zeta_\infty \} \).

The first alternative cannot hold. Indeed, by the comparison result in Theorem 3.1 it follows \( \omega(\zeta) \leq \zeta_\infty \) which yields \( \omega(\zeta) = \{ \zeta_\infty \} \) so,

\[ \omega(\zeta) = \{ \zeta_\infty \} \quad \text{for all} \quad \zeta \in [[\zeta_* , \zeta_\infty]]. \]

In particular,

\[ \omega(\zeta(0)) = \{ \zeta_\infty \}, \]

that is,

\[ \lim_{t \to \infty} Y(t) = Y_\infty, \quad \lim_{t \to \infty} Z(t) = Z_\infty, \quad \lim_{t \to \infty} W(t) = W_\infty. \] (3.25)

Also, from (3.4) one has

\[ X_\infty := \lim_{t \to \infty} X(t) = \frac{1}{k} Z_\infty + \frac{2k - N}{k}. \] (3.26)

A direct calculation shows that

\[ \frac{u^\delta(r)}{r^{\delta+k(k-s+2p)}} = \frac{1}{X^\delta(t)Y^k(t)Z^k(t)W(t)} \quad \text{for all} \quad r > 0. \]

Now, using (3.25)-(3.26) one has

\[ \lim_{r \to \infty} \frac{u(r)}{r^{1+k(k-s+2p)}} = A, \] (3.27)

where

\[ A = \frac{1}{Y_\infty Z_\infty W_\infty} \in (0, \infty). \]

In a similar fashion we obtain

\[ \frac{v^\delta(r)}{r^{k(2k-2m+q)}} = \frac{1}{Y^k(t)Z^k(t)W(t)} \quad \text{for all} \quad r > 0 \]

and again by (3.25)-(3.26) we derive

\[ \lim_{r \to \infty} \frac{v(r)}{r^{k(k-m)}} = B, \] (3.28)

where

\[ B = \frac{1}{Y_\infty^{1(k-m)} Z_\infty^{k-m} W_\infty} \in (0, \infty). \]

### 4 Proof of Theorem 1.6

\[
\begin{align*}
& \{ r^{1-N} \left[ r^{N-k} |u'|^{k-1} u' \right]' = |u'|^m v^p, \quad u(r) > 0 \quad \text{for all} \quad r > 0, \\
& \{ r^{1-N} \left[ r^{N-k} |v'|^{k-1} v' \right]' = |v'|^q u^q, \quad v(r) > 0 \quad \text{for all} \quad r > 0, \\
& u'(0) = v'(0) = 0, u(0) = a > 0, v(0) = b > 0. 
\end{align*}
\] (4.1)
We use the change of variable

\[ r = t^\theta, \quad \theta = -\frac{k}{N - 2k} < 0, \quad (4.2) \]

and let

\[ u(r) = U(t), \quad v(r) = V(t). \]

Then

\[ u'(r) = \frac{du}{dr}(r) = \frac{1}{\theta} \frac{d}{dr}(U(t)) \quad \text{and} \quad v'(r) = \frac{dv}{dr}(r) = \frac{1}{\theta} \frac{d}{dr}(V(t)). \]

Thus, any solution \((u, v)\) of \((4.1)\) satisfies

\[
\begin{cases}
(U_t|^{k-m-1}U_t)^t = \frac{k}{k-m} |\theta|^{k-m+1}t(1-\theta)(m-1-\theta(N-1))V^p, \quad U(t) > 0 \quad \text{for all } t > 0, \\
(V_t|^{k-1}V_t)^t = |\theta|^{k-q+1}t(1-\theta)(q-1-\theta(N-1))|U_t|^qV^s, \quad V(t) > 0 \quad \text{for all } t > 0, \\
U_t, V_t < 0, U(\infty) = u(0) > 0, V(\infty) = v(0) > 0, U_t(\infty) = V_t(\infty) = 0. \quad (4.3)
\end{cases}
\]

Set

\[ W(t) = |U_t|^{k-m-1}U_t \implies |U_t| = |W|^{\frac{1}{k-m}}, \]

and now \((4.3)\) reads

\[
\begin{cases}
W_t = f(t)V^p, \quad W(t) < 0 \quad \text{for all } t > 0, \\
(V_t|^{k-1}V_t)^t = g(t)V^s|W|^{\frac{q}{k-m}}, \quad V(t) > 0 \quad \text{for all } t > 0, \\
V_t < 0, W(\infty) = 0, V(\infty) = v(0) > 0, W(\infty) = V(\infty) = 0. \quad (4.4)
\end{cases}
\]

where

\[
\begin{cases}
f(t) = \frac{k}{k-m} |\theta|^{k-m+1}t(1-\theta)(m-1-\theta(N-1)) \quad \text{for all } t > 0, \\
g(t) = |\theta|^{k-q+1}t(1-\theta)(q-1-\theta(N-1))|V|^qW|^{\frac{q}{k-m}} \quad \text{for all } t > 0.
\end{cases}
\]

Let now \((u, v)\) and \((\tilde{u}, \tilde{v})\) be two solutions of \((4.1)\). We want to show \(u \equiv \tilde{u}\) and \(v \equiv \tilde{v}\). Define \((U, V, W)\) and \((\tilde{U}, \tilde{V}, \tilde{W})\) as above which satisfy problem \((4.3)\) and respectively

\[
\begin{cases}
\tilde{W}_t = f(t)\tilde{V}^p, \quad \tilde{W}(t) < 0 \quad \text{for all } t > 0, \\
(\tilde{V}_t|^{k-1}\tilde{V}_t)^t = g(t)\tilde{V}^s|\tilde{W}|^{\frac{q}{k-m}}, \quad \tilde{V}(t) > 0 \quad \text{for all } t > 0, \\
\tilde{V}_t < 0, \tilde{W}(\infty) = 0, \tilde{V}(\infty) = \tilde{v}(0) = b > 0, \tilde{W}(\infty) = \tilde{V}(\infty) = 0. \quad (4.5)
\end{cases}
\]

Let \(\varepsilon > 0\) be small and define

\[ u^\varepsilon(r) = (1 + \varepsilon)u(r), \quad v^\varepsilon(r) = (1 + \varepsilon)\frac{k-m}{p} v(r) \quad \text{for all } r \geq 0. \]

With the same change of variables \((4.2)\) and

\[ u^\varepsilon(r) = U^\varepsilon(t), \quad v^\varepsilon(r) = V^\varepsilon(t), \quad W^\varepsilon(t) = |U_t|^k|U_t|^{k-m-1}U_t \]

we obtain that \((W^\varepsilon, V^\varepsilon)\) satisfies

\[
\begin{cases}
W_t^\varepsilon = f(t)(V^\varepsilon)^p, \quad W^\varepsilon(t) < 0 \quad \text{for all } t > 0, \\
(V_t^\varepsilon|^{k-1}V_t^\varepsilon)^t = (1 + \varepsilon)\frac{\delta}{p} g(t)(V^\varepsilon)^s|W^\varepsilon|^{\frac{q}{k-m}}, \quad V(t) > 0 \quad \text{for all } t > 0, \\
V_t^\varepsilon < 0, W^\varepsilon(\infty) = 0, V^\varepsilon(\infty) = (1 + \varepsilon)\frac{k-m}{p} v(0) > 0, W^\varepsilon(\infty) = V^\varepsilon(\infty) = 0, \quad (4.6)
\end{cases}
\]
where $\delta$ is given by (1.2).

Observe that $V^\varepsilon(\infty) > \tilde{V}(\infty)$. Thus, from the first equation in (4.5) and (4.6) we find that

$$W^\varepsilon_t > \tilde{W}$$

in a neighbourhood of infinity. Thus, the set

$$M := \{ t > 0 : W^\varepsilon_t > \tilde{W} \text{ on } (t, \infty) \}$$

is non-empty. Set $t_0 := \inf M \geq 0$.

**Claim:** $t_0 = 0$.

Assume by contradiction that $t_0 > 0$. Then by continuity arguments one has

$$W^\varepsilon_t > \tilde{W} \text{ on } (t_0, \infty) \quad \text{and } W^\varepsilon_t(t_0) = \tilde{W}_t(t_0). \quad (4.7)$$

It follows from the first equation in (4.5) and (4.6) that

$$V^\varepsilon > \tilde{V} \text{ on } (t_0, \infty) \quad \text{and } V^\varepsilon(t_0) = \tilde{V}(t_0). \quad (4.8)$$

We next integrate over $[r, \infty]$ in (4.7) to deduce

$$|W^\varepsilon(t)| = -W^\varepsilon(t) > -\tilde{W}(t) = |\tilde{W}(t)| \quad \text{for all } t > t_0.$$ 

Using this fact in the second equation of (4.5) and (4.6) one has

$$\left( |V^\varepsilon_t|^{k-1}V^\varepsilon_t \right)_t > \left( |\tilde{V}_t|^{k-1}\tilde{V}_t \right)_t \quad \text{for all } t > t_0.$$ 

An integration over $[t, \infty]$ in the above inequality yields

$$-V^\varepsilon_t(t) = |V^\varepsilon_t(t)| > |\tilde{V}_t(t)| = -\tilde{V}(t) \quad \text{for all } t > t_0.$$ 

A further integration over $[t_0, \infty]$ in the above inequality together with the fact that $V^\varepsilon(\infty) > \tilde{V}(\infty)$ implies

$$V^\varepsilon(t_0) > \tilde{V}(t_0),$$

which contradicts the equality in (4.8). Hence, $t_0 = 0$ which proves our claim. This further yields

$$W^\varepsilon_t > \tilde{W}_t \quad \text{on } (0, \infty), \quad (4.9)$$

and integrating this inequality over $[t, \infty]$ one gets $|W^\varepsilon(t)| > |\tilde{W}(t)|$ for all $t > 0$. Hence

$$|U^\varepsilon(t)| > |\tilde{U}(t)| \quad \text{for all } t > 0.$$ 

A further integration implies $U^\varepsilon > \tilde{U}$ on $(0, \infty)$, that is,

$$U^\varepsilon(t) = (1 + \varepsilon)u(r) > \tilde{u}(r) \quad \text{for all } r > 0.$$ 

Passing to the limit with $\varepsilon \to 0^+$ one has $u \geq \tilde{u}$ on $(0, \infty)$. Also, (4.9) and the first equation in (4.5) and (4.6) imply $V^\varepsilon > \tilde{V}$ on $(0, \infty)$ which yield $v \geq \tilde{v}$ on $(0, \infty)$. Hence, we have argued that $u \geq \tilde{u}$ and $v \geq \tilde{v}$ on $(0, \infty)$. Similarly $\tilde{u} \geq u$ and $\tilde{v} \geq v$ on $(0, \infty)$ which yields $u \equiv \tilde{u}$ and $v = \tilde{v}$. 


References

[1] J. Bao, X. Ji and H. Li, Existence and nonexistence theorems for entire subsolutions of $k$-Yamabe type equations, *J. Differential Equations* **253** (2012), 2140–2160.

[2] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, *Acta Math.* **155** (1985), 261–301.

[3] M.F. Bidaut-Véron and H. Giacomini, A new dynamical approach of Emden-Fowler equations and systems, *Adv. Differential Equations* **15** (2010), 1033–1082.

[4] S. Bordoni, R. Filippucci and P. Pucci, Nonlinear elliptic inequalities with gradient terms on the Heisenberg group, *Nonlinear Anal.* **121** (2015), 262–279.

[5] J.I. Díaz, M. Lazzo and P.G. Schmidt, Large solutions for a system of elliptic equations arising from fluid dynamics, *SIAM J. Math. Anal.* **37** (2005), 490–513.

[6] J.I. Díaz, J.M. Rakotoson and P.G. Schmidt, A parabolic system involving a quadratic gradient term related to the Boussinesq approximation, *RACSAM. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **101** (2007), 113–118.

[7] J.I. Díaz, J.M. Rakotoson and P.G. Schmidt, Local strong solutions of a parabolic system related to the Boussinesq approximation for buoyancy-driven flow with viscous heating, *Adv. Differential Equations* **13** (2008), 977–1000.

[8] L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault, Entire large solutions for semilinear elliptic equations, *J. Differential Equations*, **253** (2012), 2224–2251.

[9] R. Filippucci and F. Vinti, Coercive elliptic systems with gradient terms, *Adv. Nonlinear Anal.* **6** (2017), 165–182.

[10] M. Ghergu, Steady-state solutions for Gierer-Meinhardt type systems with Dirichlet boundary condition, *Trans. Amer. Math. Soc.* **361** (2009), 3953–3976.

[11] M. Ghergu, J. Giacomoni and G. Singh, Global and blow-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids, *Nonlinearity* **32** 1546.

[12] M. Ghergu and V. Rădulescu, Singular Elliptic Problems. Bifurcation and Asymptotic Analysis, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, No 37, 2008.

[13] M. Ghergu and V. Rădulescu, A singular Gierer-Meinhardt system with different source terms, *Proc. Royal Soc. Edinburgh: Sect. A (Mathematics)* **138** (2008), 1215–1234.

[14] M.W. Hirsch, Systems of differential equations that are competitive or cooperative II: convergence almost everywhere, *SIAM J. Math. Analysis* **16** (1985), 423–439.

[15] M.W. Hirsch and H. Smith, Monotone Dynamical Systems, Handbook of Differential Equations, Vol 2 (2005), A. Cañada, P. Drabek and A. Fonda (Eds), 239–357.

[16] N.M. Ivochkina, The integral method of barrier functions and the Dirichlet problem for equations with operators of Monge-Ampère type. *Mat. Sb. (N.S.)* **112** (1980), 193–206 (Russian); *Math. USSR-Sb.* **40** (1981), 179–192 (English).
[17] X. Ji and J. Bao, Necessary and sufficient conditions on solvability for Hessian inequalities, *Proc. Amer. Math. Soc.* **138** (2010), 175–188.

[18] E. Mitidieri and S.I. Pohozaev, A priori estimates and blow up of solutions to nonlinear partial differential equations, *Proc. Steklov Inst. Math.* **234** (2001), 1-367.

[19] A. Mohammed and G. Porru, Large solutions to non-divergence structure semilinear elliptic equations with inhomogeneous term, *Adv. Nonlinear Anal.* **8** (2019), 517–532.

[20] W. Rudin, *Principles of Mathematical Analysis*, Int. Series Pure and Applied Mathematics, Third Edition, 1976.

[21] G. Singh, Classification of radial solutions for semilinear elliptic systems with nonlinear gradient terms, *Nonlinear Anal.* **129** (2015), 77–103.

[22] X. Zhang and M. Feng, Boundary blow-up solutions to the Monge-Ampère equation: Sharp conditions and asymptotic behavior, *Adv. Nonlinear Anal.* **9** (2020), 729–744.