THE ISOMORPHISM PROBLEM FOR ALMOST SPLIT KAC–MOODY GROUPS

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Abstract. In this paper, which is based on a part of the author’s Ph.D. thesis [11], we consider the isomorphism problem for almost split Kac–Moody groups, which have been constructed by Rémy via Galois descent from split Kac–Moody groups as defined by Tits. We show that under certain technical assumptions, any isomorphism between two such groups must preserve the canonical subgroup structure, i.e. the twin root datum associated to these groups, which generalizes results of Caprace in the split case.

An important technical tool we use is the existence of maximal split subgroups inside almost split Kac–Moody groups, which generalizes the corresponding result of Borel–Tits for reductive algebraic groups.

1. Introduction

Let $k$ be an algebraically closed field and $G$ a reductive algebraic group over $k$. Then $G$ is uniquely determined by its Lie algebra (which in turn is determined by a unique classical Cartan matrix $A = (a_{ij})$, the character group of a maximal torus $\Lambda \cong \mathbb{Z}^n$, and the set of its roots $c_i \in \Lambda$ and co-roots $h_i \in \Lambda^\vee$ which satisfy $h_i(c_j) = a_{ij}$.

Conversely, given a datum $D$ consisting of a generalized Cartan matrix $A = (a_{ij})$ (which uniquely determines a Kac–Moody algebra $\mathfrak{g}$), a free $\mathbb{Z}$-module $\Lambda$ and elements $c_i \in \Lambda, h_i \in \Lambda^\vee$ which satisfy $h_i(c_j) = a_{ij}$, Tits [21] associates a group functor $G_D$ on the category of commutative rings. For a field $k$, the value $G := G_D(k)$ is called a split Kac–Moody group over $k$. When $A$ is classical, $G$ is a split Chevalley group over $k$, which justifies regarding general Kac–Moody groups as infinite-dimensional Chevalley groups.

For an algebraic group $G$ defined over a field $k$, the group of rational points $G(k)$ coincides with the fixed point set of the action of $\text{Gal}(E|k)$ on $G(E)$, where $E|k$ is a Galois extension over which $G$ splits. Using this method of Galois descent, Rémy [17] started the theory of rational points of Kac–Moody groups. In this context, an almost split Kac–Moody group over a field $k$ can be thought of as an infinite-dimensional generalization of the group of $k$-rational points of a $k$-isotropic algebraic group defined over $k$.

An important structural feature of a split or almost split Kac–Moody group $G$ is the existence of certain subgroups which form a twin root datum for $G$. In the classical context, this has been used in the fundamental work by Borel–Tits [4] to describe abstract homomorphisms between isotropic algebraic groups over infinite fields. Caprace [7] could provide a similar description for isomorphisms of split Kac–Moody groups.

In this paper, we investigate the isomorphism problem for almost split Kac–Moody groups over fields of characteristic 0. Our main result is that any such isomorphism must be standard.

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2.2 Remark

Let \( \Phi = \Phi^+ \cup \Phi^- \) be a group and let \( \Phi \) be called prenilpotent if there are elements \( w, w' \in W \) such that \( w \cdot \Psi \subseteq \Phi^+ \) and \( w' \cdot \Psi \subseteq \Phi^- \). For a prenilpotent pair of roots \( \{ \alpha, \beta \} \) the closed root interval \([\alpha, \beta]\) is defined as

\[
[\alpha, \beta] := \{ \gamma \in \Phi : \gamma \supset \alpha \cap \beta \ \text{and} \ -\gamma \supset (-\alpha) \cap (-\beta) \}.
\]

Let \( [\alpha, \beta] \), \( [\alpha, \beta] \), \( (\alpha, \beta) := [\alpha, \beta] \setminus \{ \alpha \} \) and \( (\alpha, \beta) := [\alpha, \beta] \setminus \{ \beta \} \).

Definition 2.1. Let \((W, S)\) be a Coxeter system and let \( \Phi \) be the set of its roots.

Let \( G \) be a group and let \((U_\alpha)_{\alpha \in \Phi}\) be a family of non-trivial subgroups. Let \( L \subseteq \cap_{\alpha \in \Phi} N_G(U_\alpha) \) and set \( U_+ := \{ U_\alpha : \alpha > 0 \} \), \( U_- := \{ U_\alpha : \alpha < 0 \} \). Then \((H, (U_\alpha)_{\alpha \in \Phi})\) is said to be a twin root datum for \( G \) (of type \((W, S)\)) if the following conditions are satisfied:

1. (TRD 1) \( G = H \) if \( \alpha \in \Phi \).
2. (TRD 2) For each prenilpotent pair of roots \( \{ \alpha, \beta \} \), the commutator subgroup \( [U_\alpha, U_\beta] \)

is contained in \( U_{(\alpha, \beta)} := \langle U_\alpha, U_\beta, \gamma \in (\alpha, \beta) \rangle \).

3. (TRD 3) For each \( s \in S \) and each \( u \in U_{\alpha_s} \setminus \{ 1 \} \), there exist \( u', u'' \in U_{-\alpha_s} \) such that

\[
m(u) := u'u'' \ \text{conjugates} \ U_\beta \ \text{onto} \ U_{\beta_s} \ \text{for all} \ \beta \in \Phi.
\]

Moreover, for all \( u, v \in U_{\alpha_s} \setminus \{ 1 \} \), \( m(u)H = m(v)H \).

4. (TRD 4) For all \( s \in S \), \( U_{\alpha_s} \not\subseteq U_- \) and \( U_{-\alpha_s} \not\subseteq U_+ \).

Remark 2.2. This definition is due to Tits [22] who used it to axiomatize features of split Kac–Moody groups. Other examples of groups endowed with a twin root datum (which are also called groups of Kac–Moody type) include \( k \)-isotropic reductive algebraic \( k \)-groups, split Chevalley groups and certain "exotic groups" [15].

Definition 2.3. Let \( G, G' \) be two groups endowed with twin root data \((H, (U_\alpha)_{\alpha \in \Phi(W, S)})\) and \((H', (U'_\beta)_{\beta \in \Phi(W', S')})\). An isomorphism \( \varphi : G \rightarrow G' \) is said to be standard (or to preserve root groups) if there exists \( g \in G' \) such that

\[
\{ \varphi(U_\alpha) : \alpha \in \Phi \} = \{ gU'_\beta g^{-1} : \beta \in \Phi' \}.
\]
Remark 2.4. a) If $\varphi$ is standard, it follows that $\varphi(H) = gH'g^{-1}$ and that the corresponding Coxeter groups are isomorphic [8 Theorem 2.5].

b) An important intermediate step when analyzing arbitrary isomorphisms between groups of Kac–Moody type is to show that such an isomorphism is standard. Indeed, in the setting of [4] it can be first shown that an isomorphism is standard which then allows to conclude that it is "rational-by-field".

A similar factorization holds for standard isomorphism between two split Kac–Moody groups by [7].

2.2. Group combinatorics and twin buildings. Let $G$ be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W, S)})$ of type $(W, S)$. For $B_\pm := HU_\pm$, $N := H\langle m(u) : u \in U_\alpha \rangle$ and $S$ a set of representatives for the reflections with respect to the simple roots, $(B_+, B_-, N, S)$ is a twin BN-pair (see [1] Definition 6.78) for $G$. In particular, let $\Delta_\pm := G/B_\pm$ and $\Delta := (\Delta_+, \Delta_-)$. Then $\Delta$ is a twin building of type $(W, S)$ (see [1] Definition 5.133]) and there is a Bruhat decomposition of $G$

$$G = \bigcup_{w \in W} B_+WB_+ = \bigcup_{w \in W} B_-WB_-$$

and a Birkhoff decomposition

$$G = \bigcup_{w \in W} B_+WB_- = \bigcup_{w \in W} B_-WB_+.$$ 

A twin apartment $A = (A_+, A_-)$ is a subset of $\Delta$ which is isometric to the thin twin building of type $(W, S)$ (see [1] Definition 5.171]).

A subgroup $P \leq G$ containing a conjugate of $B_\varepsilon$ is called a parabolic subgroup of sign $\varepsilon$. If $P$ contains $B_\varepsilon$, there is a set $J \subseteq S$ such that $P = B_\varepsilon W_J B_\varepsilon$, where $W_J := \langle s_i : i \in J \rangle \leq W$. If $W_J$ is finite, $W_J$ (or $J$) is called spherical.

Let $J \subseteq S$. Then $L_J := H\langle U_\alpha : \alpha \in \Phi(W_J, J) \rangle$ is called a Levi factor.

2.3. Geometric realizations. One of the equivalent ways to define a building is to view it as a simplicial complex covered by subcomplexes (the apartments) which are isomorphic to the standard Coxeter complex. We briefly recall two important geometric realizations of this simplicial complex. A very good exposition of the interplay of these two constructions can be found in [13 Appendix B.4].

The CAT(0) realization. Let $W = \langle (s_i)_{i \in I} : (s_is_j)^{m_{ij}} = 1 \rangle$ be a Coxeter group. Let $A := (-\cos(\frac{\pi}{m_{ij}}))_{i,j}$. Let $V = \bigoplus_{i \in I} \mathbb{R} e_i$ and let $B_I$ denote the bilinear form induced by $A$, i.e. $B_I(e_i, e_j) := a_{ij}$. Then the representation

$$\rho : W \to \text{GL}(V), \rho(s_i)(e_j) := e_j - 2B(e_i, e_j)e_i$$

is called the standard linear representation of $W$, which can be shown to be faithful.

For a subset $J \subseteq I$, let $V_J := \bigoplus_{i \in J} \mathbb{R} e_i$ and write $B_J$ for the restriction of $B_I$ to $V_J$. For each $J \subseteq S$ such that $W_J$ is spherical, let $S_J := \{ x \in V_J : x_i \geq 0, B_J(x, x) = 1 \}$. Let $C$ be the intersection of the cone generated by these spherical cells with the half spaces $B_I(e_i, -) \leq 1$. Then $C$ serves as a model of a chamber.

For a building $\Delta$ of type $(W, S)$, this gives a geometric realization of $\Delta$ via the mirror construction (see e.g. [14 Section 4.2.1]). Moussong proved that the realization of an apartment in this realization has a natural metric which makes it a CAT(0) space. More precisely, the realization is a CAT(0) polyhedral complex with finitely many shapes of cells. By using retractions, Davis proved that the geometric realization of the entire building is CAT(0).

A point in the CAT(0) realization corresponds to a spherical residue of $\Delta$. If $\Delta = \Delta(G)$ is the building associated to a group $G$ endowed with a BN-pair, then $G$
The cone realization. Again let \((W,S)\) be a Coxeter group and let \(\rho: W \to \text{GL}(V)\) denote the standard linear representation. A root is a vector of the form \(a = wc_i\) for some \(w \in W\) and some standard basis vector \(e_i\); let \(\Phi = \Phi_+ \cup \Phi_-\) denote the set of all roots. A root \(a\) is often identified with the half-space
\[
D_a := \{ f \in V^* : f(a) \geq 0 \} \subseteq V^*
\]
it determines.
Let \(C := \{ f \in V^* : f(e_i) \geq 0 \} \) for all \(i \in I\) be the so-called fundamental chamber and let \(F_{s_i} := \{ f \in V^* : f(e_i) = 0 \}\) denote the wall associated to the simple root \(e_i\). For an arbitrary root \(a\) let \(\partial a := \{ f \in V^* : f(a) = 0 \}\) denote the wall of \(a\).
Let \(W\) act on \(V^*\) in the contragredient way, i.e. \((w \cdot f)(v) := f(w^{-1}v)\). Then
\[
\mathcal{C} := W \cdot C
\]
is called the Tits cone of \(W\). It serves as a geometric realization of the Coxeter complex of \(W\).
Let \(\Delta\) be a building of type \((W,S)\), viewed as a discrete set with a \(W\)-valued metric \(\delta\). Consider the topological space \(\Delta_{\text{cone}} := \Delta \times C / \sim\), where two points \((c,x), (d,y)\) are identified if and only if \(x = y\) and \(\delta(c,d) = \delta(x,y)\). Here \(J(x) := \{ s_i \in S : x \in F_{s_i} \}\) is the type of \(x\). For a twin building \(\Delta = (\Delta_+, \Delta_-)\) the cone realization of \(\Delta\) is defined as the link of \(\Delta_+\) and \(\Delta_-\) with the origin of both realizations identified:
\[
\Delta_{\text{cone}} := \Delta_+ \ast \Delta_- / \sim .
\]
If \(\mathcal{A}\) is a twin apartment of \(\Delta\), it turns out that its geometric realization in \(\Delta_{\text{cone}}\) is homeomorphic to the realization \(\mathcal{A}'\) of the thin twin building of type \((W,S)\), which can be viewed as two copies of the Tits cone: \(\mathcal{A}' \cong \mathcal{C} \cup -\mathcal{C} \subseteq V^*\). Note that if \(W\) is spherical, \(\mathcal{C} = V^*\), while if \(W\) is infinite the Tits cone \(\mathcal{C}\) is contained in a half-space. In both cases \(\mathcal{A} = \mathcal{C} \cup -\mathcal{C}\) makes good sense.

Let \(\mathcal{A}\) be a twin apartment of \(\Delta\) and let \(\Omega \subseteq \Delta\) be a set which is contained in \(\mathcal{A}\). Identifying \(\mathcal{A}\) with \(\mathcal{C} \cup -\mathcal{C}\), the convex hull of \(\Omega\), \(\text{conv}_{\mathcal{A}}(\Omega)\) is defined as the convex hull of \(\Omega\) in \(\mathcal{A}\), and its vectorial extension, \(\text{vect}_{\mathcal{A}}(\Omega)\) as the vector subspace spanned by \(\Omega\). The set \(\Omega\) is said to be generic if it is, viewed as a subset of \(\mathcal{C} \cup -\mathcal{C}\), the intersection of \(\mathcal{C} \cup -\mathcal{C}\) with a subspace \(L\) of \(V^*\) which meets the interior of \(\mathcal{C}\): \(\Omega = L \cap (\mathcal{C} \cup -\mathcal{C})\).
A subset \(\Omega \subseteq \Delta_{\text{cone}}\) which is contained in a twin apartment \(\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)\) is called balanced if \(\Omega \cap \mathcal{A}_+ \neq \emptyset \neq \Omega \cap \mathcal{A}_-\) and \(\Omega\) is contained in the union of a finite number of spherical facets. Here a spherical facet \(F\) is defined as
\[
F = w \cdot \left( \bigcap_{i \in I} \partial e_i \cap \bigcap_{i \notin I} D_{e_i} \right)
\]
for some \(w \in W\) and some spherical subset \(J \subseteq I\).

Two points \(x, y\) of the cone realization of a twin building are (geometrically) opposite if there is a twin apartment \(\mathcal{A} \cong \mathcal{C} \cup -\mathcal{C} \subseteq V^*\) containing \(x\) and \(y\) such that in this identification, \(x = -y\).

3. Split and almost split Kac--Moody groups

In this section we recall the definition of split and almost split Kac--Moody groups and some of their important features.
3.1 Kac–Moody algebras. Let $I$ be a finite index set, $n := |I|$ and let $A = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$ be a generalized Cartan matrix, i.e. $a_{ii} = 2$ for all $i \in I$, $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Let $\Lambda$ be a free $\mathbb{Z}$-module of finite rank and denote by $\Lambda^\vee := \text{Hom}(\Lambda, \mathbb{Z})$ its dual. For $i \in I$, let $e_i \in \Lambda$ and $h_i \in \Lambda^\vee$ be such that $h_i(e_j) = a_{ij}$. Then $\mathcal{D} = (I, A, (e_i)_{i \in I}, (h_i)_{i \in I})$ is called a Kac–Moody root datum.

The set $\Pi := \{e_i : i \in I\}$ is called the base and the set $\Pi^\vee := \{h_i : i \in I\}$ the cobase of the root datum $\mathcal{D}$.

Let $A$ be a generalized Cartan matrix. Two Kac–Moody root data involving $A$ are given by the following two examples.

The simply connected root datum $\mathcal{D}_A^\Lambda$ associated to $A$ is given by $\Lambda := \bigoplus_{i \in I} \mathbb{Z} e_i$, $\alpha_i := \sum_{j \in I} a_{ji} e_j$ and $h_i := e_i'$, where $(e_i')_{i \in I}$ is the dual basis of $(e_i)_{i \in I}$.

The minimal adjoint root datum $\mathcal{D}_A^\text{min}$ is given by $\Lambda := \bigoplus_{i \in I} \mathbb{Z} e_i$, $\alpha_i := e_i$ and $h_i := \sum_{j \in I} a_{ij} e_j$.

In general, though, neither will the family $(e_i)_{i \in I}$ be free nor generate $\Lambda$. Since for a root datum $\mathcal{D} = (I, A, (e_i)_{i \in I}, (h_i)_{i \in I})$ its dual $\mathcal{D}^\vee := (I, A^\vee, (h_i)_{i \in I}, (e_i)_{i \in I})$ is again a root datum, a similar statement holds for the family $(h_i)_{i \in I}$.

Let $K$ be a field of characteristic 0 and let $\mathcal{D}$ be a Kac–Moody root datum. The Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_\mathcal{D}$ of type $\mathcal{D}$ over $K$ is the Lie algebra generated by $\mathfrak{g}_0 := \Lambda^\vee \otimes_{\mathbb{Z}} K$ and the symbols $e_i, f_i (i = 1, \ldots, n)$ subject to the following relations:

\[
[h, e_i] = h(e_i)e_i, \quad [h, f_i] = -h(f_i)e_i \quad \text{for } h \in \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_0] = 0,
\]

\[
[e_i, f_j] = -\delta_{ij} \otimes 1, \quad [e_i, f_j] = 0 \quad \text{for } i \neq j,
\]

\[
(\text{ad} e_i)^{-a_{ij} + 1} e_j = (\text{ad} f_i)^{-a_{ij} + 1} f_j = 0.
\]

The universal enveloping algebra. Let $U_{\mathfrak{g}_\mathcal{D}}$ denote the universal enveloping algebra of $\mathfrak{g}_\mathcal{D}$. Let $Q := \mathbb{Z}^n$ with standard basis vectors $v_i$. Then there is a well-defined $Q$-grading of $U_{\mathfrak{g}_\mathcal{D}}$ by setting $h := 0$ for all $h \in \mathfrak{g}_0$, $\deg e_i := -\deg f_i := v_i$ and extending this. This means that there is a family of subspaces $(V_a)_{a \in Q}$ of $U_{\mathfrak{g}_\mathcal{D}}$ such that $U_{\mathfrak{g}_\mathcal{D}} = \bigoplus_{a \in Q} V_a$ and for $a, b \in V_a, b \in V_b$, $[a, b] \in V_{a+b}$. As $\mathfrak{g}_\mathcal{D}$ can be identified with a subalgebra of $U_{\mathfrak{g}_\mathcal{D}}$, there is an induced grading $\mathfrak{g}_q = \bigoplus_{a \in Q} \mathfrak{g}_a$. If $a$ is such that $\mathfrak{g}_a \neq 0$, $a$ is called a root and $\mathfrak{g}_a$ a nontrivial root space.

For $u \in U_{\mathfrak{g}_\mathcal{D}}$, let $u^{(n)} := \frac{1}{n!} u^n$ and $\binom{n}{i} := \frac{1}{n!} (u - 1) \cdots (u - n + 1)$.

Let $U_0$ denote the subring of $U_{\mathfrak{g}_\mathcal{D}}$ generated by all elements $\binom{n}{i}$, where $h \in \Lambda^\vee$ and $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$ let $U_i$ resp. $U_{-i}$ be the subring $\sum_{n \in \mathbb{N}} \mathbb{Z} f_i^{[n]}$ resp. $\sum_{n \in \mathbb{N}} \mathbb{Z} f_i^{[n]}$. Let $U_{\mathcal{D}}$ be the subring generated by $U_0$ and $U_i, U_{-i} (i = 1, \ldots, n)$.

It can be shown that $U_{\mathcal{D}}$ is a $\mathbb{Z}$-form of $U_{\mathfrak{g}_\mathcal{D}}$, i.e. the canonical map

\[
U_{\mathcal{D}} \otimes_{\mathbb{Z}} K \to U_{\mathfrak{g}_\mathcal{D}}
\]

is bijective.

For a subring $A$ of $U_{\mathfrak{g}_\mathcal{D}}$ and a ring $R$ let $A_R := A \otimes_{\mathbb{Z}} R$. Then $A_R$ inherits a grading. For $M \subseteq (U_{\mathcal{D}})_R$, the support of $M$ is the set of degrees which appear when decomposing elements of $M$ into their homogeneous components.

The Weyl group. From the last two sets of defining relations of $\mathfrak{g}_\mathcal{D}$ it follows that $\text{ad} e_i, \text{ad} f_i$ are locally nilpotent derivations of $\mathfrak{g}$. Then $\exp \text{ad} e_i, \exp \text{ad} f_i$ are well-defined automorphisms of $\mathfrak{g}$. Let

\[
s_i^* := \exp \text{ad} e_i \cdot \exp \text{ad} f_i \cdot \exp \text{ad} e_i
\]

and let $W^* := \{s_i^* : i \in I\} \leq \text{Aut}(\mathfrak{g})$. 

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The **Weyl group** of the generalized Cartan matrix $A$ is defined as

$$W := W_A := \{(s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1\}$$

where $m_{ii} := 1$ and for $i \neq j$, $m_{ij} := 2, 3, 4, 6$ or $\infty$ according to whether $a_{ij} a_{ji} = 0, 1, 2, 3$ or $4$. The group $W_A$ acts on $Q = \mathbb{Z}^n$ via $s_i(v_j) := v_j - a_{ij}v_i$.

The connection between $W^*$ and $W$ is as follows: It can be shown that the assignment $s_i^* \mapsto s_i$ extends to a well-defined surjective homomorphism $\pi: W^* \to W$. The action of $W^*$ permutes the root spaces of $\mathfrak{g}_D$, more precisely, we have $w^* g_a = \mathfrak{g}_{\pi(w^*)a}$.

A root $\alpha$ such that $g_a = w^* \mathfrak{g}_{\pm \alpha}$ is called a **real root**. The set of all real roots is denoted by $\Delta^r$. It can be identified with the set of roots $\Phi(W, S)$ of the Coxeter group $W$.

### 3.2. The constructive Tits functor

Let $D = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac–Moody root datum with associated Weyl group $W = W_A$ and standard generating set $S$. Let $R$ be a commutative ring with 1. For $\alpha \in \Phi = \Phi(W, S)$ let $U_\alpha$ denote a group isomorphic to $(R, +)$ and fix an isomorphism $u_\alpha : (R, +) \to U_\alpha$. Let $\mathcal{T}_D(R)$ denote the free product of $T := \text{Hom}(\Lambda, R^*)$ and the free product of all $U_\alpha$, $\alpha \in \Phi$. Then the **constructive Tits functor** $\mathcal{T}_D(R)$ is defined to be a certain quotient of $\mathcal{T}_D(R)$ such that the canonical images of $(T, (U_\alpha)_{\alpha \in \Phi(W, S)})$ embed in it and form a twin root datum of type $(W, S)$ for $\mathcal{T}_D(R)$ when $R$ is a field. (See [20] for the precise relations.) In this presentation, the torus acts on the simple root groups $U_{\alpha_i}$ via the root $c_i: t \cdot u_{\alpha_i}(r) \cdot t^{-1} = u_{\alpha_i}(t(c_i) \cdot r)$, while two reflections differ by a co-root: $m(u_{\alpha_i}(r))m(u_{\alpha_i}(1))^{-1} = r \cdot h_i$.

### 3.3. The adjoint representation

For each ring $R$, let $\text{Aut}_{fut}(\mathcal{U}_D)_R$ denote the group of $R$-automorphisms of the $R$-algebra $\mathcal{U}_D \otimes \mathbb{Z} R$ which preserve the filtration (or grading) of $(\mathcal{U}_D)_R$ inherited from $\mathcal{U}_D$ and the ideal $\mathcal{U}_D^+ \otimes \mathbb{Z} R$. Here $\mathcal{U}_D^+$ is the ideal of $\mathcal{U}_D$ generated by $\mathfrak{g}_D \mathcal{U}_D \cap \mathcal{U}_D$.

**Theorem 3.1.** Let $R$ be a ring. Then there is a homomorphism

$$\text{Ad}: \mathcal{T}_D(R) \to \text{Aut}_{fut}(\mathcal{U}_D)_R$$

characterized by the conditions

$$\text{Ad}(u_\alpha(r)) = \exp(\text{ad} e_\alpha \otimes r) = \sum_{n \geq 0} \frac{\text{ad} e_\alpha}{n!} \otimes r^n,$$

$$\text{Ad}(T(R)) \text{ fixes } (\mathcal{U}_D)_R \text{ and } \text{Ad}(h_\alpha (e_\alpha \otimes r)) = h_\alpha e_\alpha \otimes r$$

for all $h \in T(R)$, $\alpha \in \Phi$ and $r \in R$.

**Proof.** This is Theorem 9.5.3 in [17].

The homomorphism $\text{Ad}$ is called the **adjoint representation**.

Let $K$ be a field and let $G := \mathcal{T}_D(K)$ be a split Kac–Moody group. A subgroup $H \leq G$ is called **Ad-locally finite** if each $v \in (\mathcal{U}_D)_K$ is contained in a finite-dimensional $\text{Ad}$ $H$-invariant subspace. A subgroup $H \leq G$ is called **Ad-diagonalizable** if there is a basis of $(\mathcal{U}_D)_K$ in which the $H$-action is diagonal. From the explicit description of $\text{Ad}$ it follows that $T(K)$ is Ad-diagonalizable.
3.4. **Group combinatorics.** Let $K$ be a field and let $\mathcal{G}_D$ be a Tits functor. Let $G(K) := \mathcal{G}_D(K)$.

Let $X^*(T)_{\text{abs}} := \text{Hom}(T, K^*)$ denote the group of (abstract) characters of $T$ and $X_*(T)_{\text{abs}} := \text{Hom}(K^*, T)$. Then $\Lambda$ injects into $X^*(T)_{\text{abs}}$, while $\Lambda'$ injects into $X_*(T)_{\text{abs}}$. The group $\Lambda$ is called the group of algebraic characters of $T$, while the group $\Lambda'$ is called the group of algebraic cocharacters of $T$. Let $U_+ := \langle U_{\alpha} : \alpha > 0 \rangle$ and $U_- := \langle U_{\alpha} : \alpha < 0 \rangle$. Let $B_+ := TU_+$, $B_- := TU_-$. Then $B_+$ (resp. $B_-$) is called the standard positive (resp. negative) Borel subgroup, while any conjugate of $B_+$ resp. $B_-$ is called a positive resp. negative Borel group.

For $\varepsilon \in \{\pm 1\}$, the group $U_\varepsilon$ is called the unipotent radical of $B_\varepsilon$. A positive Borel group $B_1$ and a negative Borel subgroup $B_2$ are called opposite if their intersection is a Cartan subgroup.

In contrast to the theory of algebraic groups, a (positive or negative) Borel subgroup $B$ of a Kac–Moody group in general is not solvable. Indeed, $B$ is solvable if and only if $W$ is finite.

3.5. **Bounded subgroups.** We recall the close connection between Ad-locally finite groups and fixators of balanced subsets.

Let $G$ be a group endowed with a twin root datum $(T, (U_{\alpha})_{\alpha \in \Phi(W, S)})$. A subgroup $H \leq G$ is called bounded if there exists $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$ such that

$$H \subseteq B_+ \{w_1, \ldots, w_n\} B_+ \cap B_- \{w_1, \ldots, w_n\} B_-,$$

i.e. for $\varepsilon \in \{+, -\}$, $H$ is contained in a finite number of double $B_\varepsilon$-cosets.

The following is Theorem 10.2.2 in [17].

**Theorem 3.2.** Let $G = \mathcal{G}_D(K)$ be a split Kac–Moody group. For a subgroup $H \leq G$, the following conditions are equivalent.

a) $H$ is bounded.

b) $H$ fixes a point in the CAT(0)-realization of both $\Delta_+$ and $\Delta_-.$

c) $H$ is Ad-locally finite.

Now let $\Omega \subseteq \Delta_{\text{cone}}$ be a balanced subset which is contained in the standard apartment $A$. By the previous proposition, $H := \text{Fix} \Omega$ is Ad-locally finite. Rémy attaches to $H$ a certain finite-dimensional $\text{Ad} H$-invariant subspace whose construction we recall.

Let $K$ be an algebraic closure of $K$. Let $\mathcal{L}_D := \mathcal{G}_D \cap U_D$, where $U_D$ is the $K$-form of the universal enveloping algebra. Then $\mathcal{L}$ has a grading: $\mathcal{L}_D = \mathcal{L}_0 \oplus \bigoplus_{a \in \Phi} \mathcal{L}_a.$

Let $\Delta(\Omega) := \{a \in \Phi : \Omega \subseteq a\}, \Delta^+(\Omega) := \{a \in \Phi : \Omega \subseteq a, \Omega \subseteq \partial a\}$ and let $\Delta^\omega(\Omega) := \{a \in \Phi : \Omega \subseteq \partial a\}.$ Here the roots are viewed as half-spaces in the cone realization. Write $L := T(U_{\alpha} : \alpha \in \Delta^\omega(\Omega))$ and $U := \langle U_{\alpha} : \alpha \in \Delta^\omega(\Omega) \rangle.$

**Proposition 3.3.** Let $W = W_\Omega$ be the smallest $Q$-graded subspace of $(\mathcal{L}_D)_{\overline{K}}$ with the following properties:

a) $W$ contains $(\mathcal{L}_0)_{\overline{K}}$ and $(\mathcal{L}_a)_{\overline{K}}$ for all $a \in \Delta(\Omega)$.

b) The $Q$-support of $W$ contains $-\Delta^\omega(\Omega)$.

c) $W$ is stable under $H := \text{Fix} \Omega$.

Then the following properties hold:

a) $W$ is finite-dimensional and the kernel of $\text{Ad} : H \to \text{Ad} H|_W$ is precisely the center of $H$.

b) Let $\hat{H}$ (resp. $\hat{T}, \hat{L}, \hat{U}$) denote the Zariski-closure of $\text{Ad} H|_W$ (resp. $\text{Ad} T|_W$, $\text{Ad} L|_W, \text{Ad} U|_W$). Then $\hat{L}$ is a connected reductive $K$-group, $\hat{T}$ is a maximal torus of $\hat{L}$, $\hat{U}$ is unipotent and $\hat{H} = \hat{L} \times \hat{U}$ is a Levi decomposition.

**Proof.** This is [17] Lemma 10.3.1, Proposition 10.3.6. \qed
3.6. Almost split Kac-Moody groups. We recall Rémy’s construction of almost split Kac–Moody groups, cf. [17, 18]. These groups can be obtained via Galois descent, i.e. by taking the fixed points of a certain Galois group action on a split Kac–Moody group. One of the main features of an almost split Kac–Moody group is that it is again endowed with a twin root datum.

Let $K$ be a field, $\bar{K}$ an algebraic closure of $K$ and $K_{s}$ the separable closure of $K$ in $\bar{K}$. Let $D$ be a Kac–Moody root datum and let $\mathcal{G}_{D}$ be a constructive Tits functor. A prealgebraic $K$-form of $\mathcal{G}_{D}$ is a couple $(\mathcal{G}, U_{K})$, where $\mathcal{G}$ is a group functor on the category of field extensions of $K$ which coincides with $\mathcal{G}_{D}$ over extensions of $K$, and $U_{K}$ a $K$-form of the filtered algebra $(U_{D})_{\bar{K}}$ satisfying

(PA 1) The adjoint representation $\text{Ad}$ is Galois-equivariant, i.e. for each $\sigma \in \Gamma := \text{Gal}(K_{s}|K)$, the following diagram commutes, where $R_{\bar{K}} := \bar{K} \otimes_{K} R$:

\[
\begin{array}{ccc}
\mathcal{G}(R_{\bar{K}}) & \xrightarrow{\text{Ad}} & \text{Aut}_{\text{filt}}(U_{D})(R_{\bar{K}}) \\
\sigma & \downarrow & \sigma \\
\mathcal{G}(R_{\bar{K}}) & \xrightarrow{\text{Ad}} & \text{Aut}_{\text{filt}}(U_{D})(R_{\bar{K}})
\end{array}
\]

(PA 2) If $\iota : K \to L$ is an injection of fields, then $\mathcal{G}(\iota) : \mathcal{G}(K) \to \mathcal{G}(L)$ is injective, too.

Let $E$ be a field satisfying $K \subseteq E \subseteq \bar{K}$. Then a prealgebraic form $(\mathcal{G}, U_{K})$ is said to split over $E$ if it is $E$-isomorphic to the split form $(\mathcal{G}_{D}, (U_{D})_{E})$ over $E$ (see [17, 11.1.5] for a precise definition).

Convention. In this subsection, let $(\mathcal{G}, U_{K})$ always be a prealgebraic $K$-form of $\mathcal{G}_{D}$ which is assumed to split over an infinite field $E$ such that $E|K$ is a normal field extension.

Let $\Gamma := \text{Gal}(K^{\text{sep}}|K)$ be the absolute Galois group. Then for each field $L \subseteq \bar{K}$ and each $\gamma \in \Gamma$, there is an action of $\Gamma$ on $\mathcal{G}$ given by $(\gamma \cdot \mathcal{G})(L) := \mathcal{G}(\gamma \cdot L)$. Since $E|K$ is assumed to be normal, $\Gamma$ acts on $\mathcal{G}(E)$, and since $\mathcal{G}$ is assumed to split over $E$, each element of $\text{Gal}(K^{\text{sep}}|E)$ acts trivially on $\mathcal{G}(E)$, i.e. the $\Gamma$-action factors through $\text{Gal}(E|K)$.

Fix an isomorphism $\Psi : \mathcal{G}(E) \to \mathcal{G}_{D}(E)$. By abuse of notation, let $T(E) \leq \mathcal{G}(E)$ again denote the subgroup of $\mathcal{G}(E)$ which is mapped to the group $T(E) \leq \mathcal{G}_{D}(E)$. Then $\Gamma$ preserves the conjugacy class of $T(E)$ (cf. [17, 11.2.2]). For $\sigma \in \Gamma$, choose $g \in \mathcal{G}(E)$ such that the so-called rectification $\bar{\sigma} := \text{int} g^{-1} \circ \sigma$ stabilizes $T(E)$. Then $\bar{\sigma}$ induces an automorphism of $W = N(T(E))/T(E)$.

Let $(\mathcal{G}, U_{K})$ be a prealgebraic $K$-form of $G$ which splits over $E$. Then $G$ is said to satisfy (SGR) if for each $\sigma \in \Gamma$, each rectified automorphism $\bar{\sigma}$ of $G(E)$ induces a permutation of the root groups relative to $T(E)$.

Remark 3.4. By the explicit description of $\text{Aut}(\mathcal{G}_{D}(E))$ by Caprace ([7, Theorem A1]) this condition is empty: $\bar{\sigma}$ automatically preserves root groups. Indeed, by the quoted result any automorphism $\varphi$ can be written as a product $\varphi = \varphi_{2} \circ \varphi_{1}$ of an inner automorphism $\varphi_{1}$ (which can be chosen to be trivial if $\varphi(T) = T$) and an automorphism $\varphi_{2}$ which permutes the root groups: $\varphi_{2}(x_{\alpha}(r)) = x_{\iota(\alpha)}(c_{\alpha} \sigma_{\alpha}(r))$, where $\iota : \Phi \to \Phi$ is a bijection, $c_{\alpha} \in E^{\times}$ and $\sigma_{\alpha} \in \text{Aut}(E)$.

It follows that $\bar{\sigma}$ induces a permutation of the roots $\Phi$ of $W$. Moreover, $\bar{\sigma}$ induces an action on the groups $X^{+}(T(E))_{\text{abs}}$ resp. $X^{-}(T(E))_{\text{abs}}$ of abstract characters resp. cocharacters.
In this situation, $G = (\mathcal{G}, \mathcal{U})$ is called a **Kac–Moody $K$-group** if for each $\sigma$,

**(ALG 1)** $\tilde{\sigma}$ respects the $Q$-grading of $(\mathcal{U}_P)_E$ and the induced permutation of $Q$ satisfies $\bar{\sigma}(u\alpha) = n(\tilde{\sigma}(u))$ for all $n \in \mathbb{N}$.

**(ALG 2)** $\tilde{\sigma}$ stabilizes the algebraic characters $\Lambda \leq X^*(T(E))_{abs}$ resp. the algebraic cocharacters $\Lambda^* \leq X_*(T(E))_{abs}$.

Let $G = (\mathcal{G}, \mathcal{U})$ be a Kac–Moody $K$-group. Then $G$ is called **almost split** if the action of $\Gamma$ on $G(E)$ stabilizes the conjugacy classes of the standard Borel subgroups $B_+(E)$ and $B_-(E)$. The group $G$ is called **quasi-split** if there are two opposite Borel groups $B_1, B_2$ which are stable under the $\Gamma$-action. Note that a quasi-split Kac–Moody group is automatically almost split.

**Remark 3.5.** The terminology “almost split” stems from the following fact: although an almost split Kac–Moody group has an anisotropic kernel $Z(k)$, this group is **finite-dimensional**.

**Galois descent.** Let $G = (\mathcal{G}, \mathcal{U})$ be a Kac–Moody $K$-group. Then $G$ is said to be obtained via **Galois descent** if $G$ splits over the separable closure $K_s$ of $K$ in $\bar{K}$ and for each separable field sub-extension $E/K$, the group $G(E)$ is precisely the fixed point set of $\text{Gal}(K_{sep}|E)$ in $G(K_{sep})$. In this case, $\mathcal{G}$ is said to satisfy the condition (DCS).

3.7. **An explicit construction.** Rémý [17, Ch. 13.2.3] gives an explicit construction of quasi-split Kac–Moody groups as follows. Let $\mathcal{G}_D$ be a constructive Tits functor, $E|K$ a finite Galois extension, $\Gamma := \text{Gal}(E|K)$ and suppose there is a homomorphism $\ast : \Gamma \to D_A$, where $D_A$ is the Dynkin diagram associated to $A$. Then $\ast$ gives rise to an action of $\Gamma$ on $\mathcal{G}_D(E)$, and the set $G(k)$ of $\Gamma$-fixed points is a quasi-split Kac–Moody group.

**Example 3.6.** The following example is given in [18, 3.5.B]. Let $E|K$ be a separable quadratic field extension, $\text{Gal}(E|K) = \langle \sigma \rangle$ and let $\mathcal{G}_D$ be the affine Kac–Moody group $\mathcal{G}_D(K) = \text{SL}_3(K[t, t^{-1}])$. Let $\text{SU}_3(K) \leq \text{SL}_3(E)$ be the group of matrices which preserve a fixed three-dimensional $\sigma$-Hermitian form of Witt index $1$. Then the group $\text{SU}_3(K[t, t^{-1}])$ is a quasi-split Kac–Moody group obtained by the $\ast$-action where $\sigma^* \ast$ switches two nodes of the diagram associated to $\mathcal{G}_D$.

More generally, there is the following class of examples of affine quasi-split Kac–Moody groups.

**Proposition 3.7.** Let $\mathcal{G}$ be a connected simply connected almost simple algebraic group defined over $\mathbb{F}_q$ which is $\mathbb{F}_q$-isotropic. Then for any field $K$ containing $\mathbb{F}_q$, the group $G(K[t, t^{-1}])$ is an almost split Kac–Moody $\mathbb{F}_q$-group.

**Proof.** This follows from [17, Chapter 11]. A detailed proof is given in [6, Proposition 10.2].

3.8. **The Galois action on the building.** Let $K$ be a field, let $E|K$ be a normal field extension, where $E$ is infinite, and let $\Gamma := \text{Gal}(E|K)$. Let $G$ denote an almost split Kac–Moody $K$-group obtained by Galois descent which splits over $E$.

Let $\Delta = (G(E)/B_+(E), G(E)/B_-(E))$ denote the twin building associated to the group $G(E) \cong \mathcal{G}_D(E)$. The $\Gamma$-action on $G(E)$ then gives rise to an action on $\Delta$ since it preserves the respective conjugacy classes of $B_+, B_-$, cf. [17, 11.3.2].

Moreover, there is a better rectification of automorphisms available, that is, for each $\sigma \in G$ there is a $g_\sigma \in G(E)$ (well-defined up to an element in $T(E)$) such that $\sigma^* := \text{int} \ g_\sigma^{-1} \circ \sigma$ stabilizes both $B_+(E)$ and $B_-(E)$.

This gives a well-defined action of $\Gamma$ on $W$, called the $\ast$-**action**. This action stabilizes the generating set $S$, i.e. the action is by diagram automorphisms ([17, 11.3.2]).
It follows that \( \Gamma \) acts on the CAT(0)-realization of the buildings \( \Delta_+ , \Delta_- \). Although \( \Gamma \) might be infinite (there is no assumption that \( E|K \) is finite, i.e. that \( G \) splits over a finite extension of \( K \)), it can be shown that each orbit is bounded [17, 11.3.4], so by the Bruhat-Tits fixed point theorem, there are fixed points in both halves of the twin building. By the dictionary relating the building to its CAT(0)-realization, this is equivalent to saying that there are spherical residues \( R_+ , R_- \) in both buildings which are stable under the Galois group. The residues \( R_+ , R_- \) in general will not be chambers, though. Indeed, \( \Gamma \) will fix two opposite chambers if and only if \( G \) is quasi-split.

**The action on the cone realization.** Similarly, \( \Gamma \) acts on the cone realization \( \Delta_{cone} \) of \( \Delta \). Let \( \Delta^\Gamma_{cone} \) denote the set of fixed points, then it is clear that \( G(K) \) acts on \( \Delta^\Gamma_{cone} \). In what follows, certain subsets of \( \Delta^\Gamma_{cone} \) will be singled out, the stabilizers of which then will form the ingredients of a twin root datum for \( G(K) \).

To start with, a maximal generic subspace (i.e. a sub-vector space of an apartment which meets the interior of the Tits cone) which is fixed by \( \Gamma \) is called a **K-apartment**. These can be shown to exist if \( G \) splits over the separable closure of \( K \).

In the cone realization of the standard twin apartment, such a generic subspace \( L \) is given by

\[
L = \{ x \in V^* : e_i(x) = 0 \forall i : s_i \in S_0 \text{ and } e_i(x) = e_j(x) \text{ for } \Gamma^* s_i = \Gamma^* s_j \},
\]

cf. [17 Lemma 12.6.1]. Here \( S_0 \) is the type of the facet containing a maximal \( K \)-chamber \( F \), see below. Note that the type of a chamber is \( \emptyset \).

A **K-facet** is the set of \( \Gamma \)-fixed points of a \( \Gamma \)-stable facet. A maximal \( K \)-facet is a **K-chamber**. A **K-root** (resp. **K-half-apartment**, resp. **K-wall**, resp. **K-panel**) is an apartment (resp. half-apartment, resp. wall, resp. panel) relative to a \( K \)-apartment \( A_K \), i.e. the trace of the corresponding object on \( A_K \), which is assumed to be non-empty.

Two \( K \)-chambers of the same sign are called **adjacent** if they contain a common \( K \)-panel in their closure.

Two \( K \)-chambers of opposite sign are called (geometrically) **opposite** if there is a twin apartment which contains them and in which they are opposite.

For a given \( K \)-apartment \( A_K \), \( \Delta^\Gamma_K(A_K) \) is defined as the set of all real \( K \)-roots, i.e. those whose relative wall is again a generic subspace, and \( \Phi_K(A_K) \) as the set of all \( K \)-half-apartments relative to \( A_K \).

For a real \( K \)-root \( a \), let \( a^2 \) denote its restriction to \( A_K \). Then let

\[
\Delta_a := \{ b \in \Delta^{re} : \exists \lambda \geq 1 : b^2 = \lambda a^2 \}.
\]

Note that \( \Delta_a \) is a prenilpotent set of roots which is \( \Gamma \)-stable.

Finally, a **standardisation** of the cone realization \( \Delta_{cone} \) of \( G(E) \) is a triple \((A,C,−C)\) where \( A \) is a twin apartment which contains the two opposite chambers \( C \) and \( −C \) (this corresponds to fixing a maximal torus \( T \) and two opposite Borel groups \( B_1 , B_2 \) such that \( B_1 \cap B_2 = T \)). A **rational standardisation** is a triple \((A_K,F,−F)\) where \( A_K \) is a \( K \)-apartment and \( F , −F \) are two opposite \( K \)-chambers which are contained in \( A_K \). Two of these triples are called **compatible** if \( A \) contains \( A_K \) and \( C , −C \) contain \( F, −F \) respectively.

3.9. **The twin root datum of an almost split group.** Let \( K \subseteq E \subseteq K^{sep} \) be an inclusion of fields and let \( G \) be an almost split Kac–Moody \( K \)-group which is obtained by Galois descent and splits over \( E \). For a subgroup \( U \subseteq G(K^{sep}) \), let \( U(E) := G(E) \cap U \) denote the **group of \( E \)-rational points** of \( U \).
A \( \Gamma \)-invariant parabolic subgroup \( P \) of \( G \) is called a \( K \)-parabolic subgroup. Such a \( K \)-parabolic group is precisely the stabilizer of a \( K \)-facet. 

**The anisotropic kernel.** Let \( (A_K, F, -F) \) be a rational standardisation. Then \( Z := Z(A_K) := \text{Fix}_G(K^{sep})(A_K) \) is called the anisotropic kernel (with respect to \( A_K \)). Let \( Z(K) \) denote the set of its \( K \)-rational points.

Let \( \Omega := F \cup -F \). Then \( \text{Ad}_\Omega(Z(A_K)) \) is isomorphic to a semisimple algebraic \( K \)-group which is \( K \)-anisotropic. It follows that \( Z \) contains a maximal \( K \)-split torus \( T_\partial(K) \), which can be identified with the connected component of the identity of its center (cf. [17 12.5.2]). The set of all \( G(K) \)-conjugates of \( T_\partial(K) \) is in bijection with the \( K \)-apartments.

**Rational root groups.** For a real \( K \)-root \( a \), let \( V_a := \langle U_b : b \in \Delta_a \rangle(K) \). By (DCS), \( V_a \) is just the fixed point group of \( \Gamma \) acting on the \( \Gamma \)-invariant group \( U_{\Delta_a} := \langle U_b : b \in \Delta_a \rangle \).

**Rank 1 groups.** Let \( E \) be a \( K \)-panel, \( \Omega := E \cup -E \) and denote by \( M(\Omega)(K^{sep}) \) its fixator in \( G(K^{sep}) \). Then \( M(\Omega) = Z(V_\alpha, V_{-\alpha}) \) for the \( K \)-root \( \alpha \) with \( E \subseteq \partial \alpha \). The group \( M(\Omega) \) is a reductive algebraic group defined over \( K \) of split semisimple rank 1, which can be seen by considering \( \text{Ad}_\Omega(M_\Omega) \). It follows that a rational root group \( V_\alpha \) is isomorphic to a root group of a semisimple \( K \)-group (cf. [17 12.5.4]).

Let \( N(K) \) denote the stabilizer of \( A_K \) in \( G(K) \). Then \( W^2 := N(K)/Z(K) \) is called the relative Weyl group. It can be shown that \( W^2 \) is in fact a Coxeter group with generating set \( S^2 \) whose set of roots is in bijection with the half-apartments of \( A_K \), see below.

Rémy proved the following important and difficult theorem ([17 Theorem 12.4.3]).

**Theorem 3.8.** Let \( G \) be an almost split Kac–Moody \( K \)-group which is obtained by Galois descent. Let \( (A_K, F, -F) \) be a rational standardisation. Then the group of rational points \( G(K) \) is endowed with a twin root datum \((Z_A(K), (V_\alpha)_{\alpha \in \Phi(W^2, S^2)})\).

**Geometric realization of the associated twin building.** It can be checked ([17 12.4.4]) that the set of \( \Gamma \)-fixed points in \( \Delta(G(E)) \) gives a geometric realization of the twin building associated to \( G(K) \) in the sense that adjacency and opposition can be checked by looking at the fixed points in \( \Delta_{conc}(G(E)) \).

Just like in the finite-dimensional case (cf. [23 Chapter 42]), we have the following fact:

**Proposition 3.9.** Let \( G(K) \) be a quasi-split Kac–Moody group obtained via Galois descent. Then the derived group of the anisotropic kernel \( Z \) is trivial, i.e. \( Z(K) \) is abelian.

**Proof.** By definition, the Galois group \( \Gamma \) stabilizes two opposite Borel groups of \( G(E) \), where \( E \) is a splitting field of \( G \). Without loss of generality, these can be assumed to be the standard Borel groups \( B_+, B_- \). By the explicit description of the generic subspace \( A_K \) it follows that \( A_K \) is entirely contained in the cone of \( C_+ \) and \( C_- \). So any element \( g \in G(E) \) which fixes \( A_K \) will stabilize both \( B_+ \) and \( B_- \), from which it follows that \( g \in T(E) \). Thus \( Z(K) \leq T(E) \), which is abelian. \( \Box \)

### 3.10. Facts about isotropic reductive algebraic groups

Let \( G = G(k) \) be an almost split Kac–Moody group obtained via Galois descent. Let \( \Omega \) a balanced subset of \( \Delta_{conc} \) and let \( M := \text{Fix}_G(k)(\Omega) \). Then \( \text{Ad}_\Omega(M) \) can be identified with the \( k \)-points of an algebraic group defined over \( k \), and \( M \) itself is a central extension of this group.

(The fact that \( \text{Ad} M \) is defined over \( k \) is implied by the axioms that the adjoint
representation be Galois equivariant and that \( G(k) \) is obtained by Galois descent; this is one of the main motivations of introducing these two axioms.)

This is why we recall here some facts about \( k \)-rational points of algebraic groups. For the following facts see [3] or [10] Section 1.2, where a convenient summary of the results we need is given. Let \( k \) be a field, \( k \) an algebraic closure of \( k \) and \( G \) a connected reductive linear algebraic group defined over \( k \). For our purposes, we can assume that \( G \) comes with a fixed embedding, i.e. \( G \) is a Zariski-closed subgroup of some \( \text{GL}_n(k) \).

Let \( S \leq G \) be a maximal \( k \)-split torus and \( X^*(S) \) its character group. Suppose that \( G \) is isotropic over \( k \), i.e. \( S \) is non-trivial.

Let \( \Phi \subseteq X^*(S) \) be the corresponding \( k \)-root system of \( G \) with respect to \( S \), i.e. the set of weights of \( S \) acting on \( g := \text{Lie} G \) via the adjoint representation.

For \( \alpha \in \Phi \), let \( g_\alpha \leq g \) denote the corresponding root space, i.e.

\[
g_\alpha = \{ X \in g : \text{Ad} \ s(X) = \alpha(s) \cdot X \ \forall \ s \in S \}.
\]

Let \( u_\alpha := \sum_{k \geq 0} g_{k\alpha} \) and let \( U_\alpha \) be the connected unipotent subgroup of \( G \) with \( \text{Lie} U_\alpha = u_\alpha \). In fact, the only positive multiples of \( \alpha \) which could possibly belong to \( \Phi \) are \( \alpha \) and \( 2\alpha \). The group \( U_\alpha \) then is split over \( k \), cf. [3] Cor. 3.18 and normalized by the centralizer \( Z := C_G(S) \) of \( S \) in \( G \).

If \( \alpha \in \Phi \) is such that \( 2\alpha \notin \Phi \), then \( U_2 := U_\alpha \) is \( k \)-isomorphic to a vectorspace \( G_\alpha^n \).

If \( \alpha \in \Phi \) is such that \( 2\alpha \in \Phi \), then \( U_1 := U_\alpha/U_{2\alpha} \) again is isomorphic over \( k \) to a vectorspace.

In both cases, under this identification the action of \( S \) on \( U_1 \) resp. \( U_2 \) is given via the homothety induced by \( \alpha \). This means that for \( s \in S(k) \) and \( u \in U_1(k) \) or \( u \in U_2(k) \), we have

\[
s \cdot u \cdot s^{-1} = \alpha(s) \cdot u.
\]

3.11. **Further properties of Kac–Moody \( K \)-groups.** For the rest of this paper, any almost split Kac–Moody groups is understood to be obtained via Galois descent.

We briefly recall the discussion of reductive \( k \)-subgroups of \( G \) as given in [17] 12.5.2 to make the interplay of the maximal split torus and the relative root groups of an almost split Kac–Moody group explicit.

Let \( k \) be a field and let \( G = G(k) \) be an almost split Kac–Moody \( k \)-group which splits over a separable extension \( E \subseteq k^{sep} \). Let \( (A_k, F, -F) \) denote a rational standardisation.

By definition, \( F \) and \( -F \) are two minimal Galois-stable opposite spherical facets of the twin building associated to \( G(E) \). The stabilizer of \( \Omega := F \cup -F \) in \( G(E) \) can then be identified with a Levi factor \( L^J(E) := T(U_\alpha : \alpha \in \Phi(W_J)) \) where \( J \subseteq S \) is spherical. From the defining relations of the constructive Tits functor, it follows that \( L^J(E) \) is abstractly isomorphic to the \( E \)-points of a connected reductive group split over \( E \). Since \( L^J \) is invariant under the \( \Gamma \)-action, it follows that \( L^J \) is defined over \( k \). Write \( Z \) for the algebraic group \( L^J \) endowed with this \( k \)-structure. So \( Z(E) \cong L^J(E) \), while \( Z(k) \) of course is in general very different from \( L^J(k) \).

For \( \Omega \) as above, \( \text{Ad}_Z(Z) \) is a connected semisimple algebraic group defined over \( k \) which is anisotropic over \( k \). It follows that there exists a unique maximal \( k \)-split torus \( T_\alpha \) contained in \( Z \). The torus \( T_\alpha \) is central in \( Z \) and can be identified with a maximal \( k \)-split subtorus of \( T \).

More generally, let \( x \in A_k \) be a \( k \)-facet. Then for \( \Omega := x \cup -x \), the fixator of \( \Omega \) in \( G(k) \) can be identified with the \( k \)-rational points of some Levi factor \( L^{J'} \) of \( G(E) \), where the \( k \)-structure on \( L^{J'} \) again is given by the \( \Gamma \)-action. (We dealt above with
We combine this discussion with the review of rational points of algebraic groups in the previous subsection to sum up the interplay between the maximal split torus $T_d(k)$ and the root groups $V_\alpha(k)$. Let $G_a$ denote the algebraic group with $G_a(k) = (k, +)$. For a group $G$, let $\mathcal{Z}(G)$ denote the center of $G$ (which should not be confused with the anisotropic kernel $Z$ of an almost split Kac-Moody group).

**Proposition 3.10.** Let $k$ be an infinite field and let $G$ be an almost split Kac–Moody group obtained by Galois descent. Let $Z$ be a maximal split torus and $W_k$ the Weil group of $G(k)$ with $S_k$ its set of canonical generators. Let $\Phi_k = \Phi(W_k, S_k)$ denote the set of $k$-roots and $(V_\alpha(k))_{\alpha \in \Phi_k}$ the set of root groups of $G(k)$ relative to $T_d(k)$. Let $\Pi_k$ denote the set of simple roots of $\Phi_k$.

- **a)** $Z$ is a connected reductive algebraic group defined over $k$. The torus $T_d$ is a maximal $k$-split torus of $Z$ which is central in $Z$; the derived group of $Z$ is anisotropic over $k$.
- **b)** Let $J \subseteq S_k$ be such that $(W_k)_J$ is finite. Then $L^J := Z\langle V_\alpha : \alpha \in \Phi((W_k)_J) \rangle$ is a connected reductive algebraic $K$-group, in which $T_d$ is a maximal $k$-split torus. $L^J$ has split-semisimple rank $|J|$.
- **c)** Let $\alpha \in \Delta_k$. Then $X_\alpha := Z(V_\alpha, V_{-\alpha})$ is a connected reductive algebraic $K$-group of split-semisimple rank 1. $V_\alpha$ is a root group in $X_\alpha$ normalized by $Z$.

There are two possibilities:

- **i)** $V_\alpha$ is abelian and is $k$-isomorphic to $G_a^n$ for $n := \dim V_\alpha$. In this case, $V_\alpha$ is normalized by $Z$, and $T_d$ acts on $V_\alpha$ via a character $\alpha$. This means there is some $\alpha \in X^*(T_d)$ defined over $k$ such that $\alpha(t) \cdot u$ for $t \in T_d$ and $u \in V_\alpha$.

- **ii)** $V_\alpha$ is metabelian. Then $\mathfrak{Z}(V_\alpha)$ is $k$-isomorphic to $G_a^{n'}$, where $n := \dim \mathfrak{Z}(V_\alpha)$, and $V_\alpha/\mathfrak{Z}(V_\alpha)$ is $k$-isomorphic to $G_a^{n'}$, where $n' := \dim V_\alpha - \dim \mathfrak{Z}(V_\alpha)$.

The anisotropic kernel $Z$ normalizes both $V_\alpha$ and $\mathfrak{Z}(V_\alpha)$. There is a character $\alpha \in X^*(T_d)$ defined over $k$ such that $T_d$ acts on $\mathfrak{Z}(V_\alpha)$ via $2\alpha$ and on $V_\alpha/\mathfrak{Z}(V_\alpha)$ via $\alpha$.

- **d)** Let $u \in \mathfrak{Z}(V_\alpha(k)) \setminus \{1\}$ and $s_\alpha := m(u) = u''u''$ the associated $\mu$-map. Then $s_\alpha$ normalizes $T_d(k)$.

- **e)** Let $\alpha \in \Phi_k$. If $t \in T_d$ centralizes some $u \in V_\alpha \setminus \{1\}$, then $t^2$ already centralizes $V_\alpha$.

- **f)** If $\alpha, \beta \in \Phi_k$, $\alpha \neq \pm \beta$ are such that $\alpha(s_\alpha, s_\beta) < \infty$, then there is an element $t \in T_d(k)$ such that $t$ centralizes $V_\alpha$ but not $V_\beta$.

**Proof.** Part a) is clear by the above discussion; similarly, as $L^J$ is the fixator of two opposite points $x, -x$, for b) it is sufficient to check the statement about the semisimple rank of $L^J$, which follows from the fact that $\text{Ad}_{x^{-1}x}(L^J)$ is a semisimple group in which the $(V_\beta : \beta \in \Phi(W_k)_J)$ form a system of root groups in the algebraic sense.

Part c) follows from b) and the discussion of rational points of semisimple algebraic groups in the previous subsection.

For part d), note that by c) $X_\alpha$ is a reductive group with $T_d$ a maximal split torus. Then the Zariski closure of $\{ s u s^{-1} : s \in T_d \}$ is a one-dimensional subgroup of $V_\alpha$, and so is part of a maximal split reductive subgroup $F \leq X_\alpha$ which contains $T_d$, as follows from the Borel–Tits theorem (see Theorem 5.1). As $m(u)$, computed in $F$, the case when $x$ is a $k$-chamber.)
leaves \( T_d \) invariant, so must \( m(u) \), as computed in \( X_\alpha \).

Part e follows from part c) by noting that if \( V_\alpha \) is abelian, then necessarily \( \alpha(t) = 1 \) (and so already \( t \) must centralize \( V_\alpha \)). In case \( V_\alpha \) is metabelian, if \( u \in \mathcal{Z}(V_\alpha) \), then \( 2 \alpha(t) = \alpha(t^2) = 1 \) (so \( t^2 \) centralizes \( V_\alpha \)), while if \( u \notin \mathcal{Z}(V_\alpha) \), then \( \alpha(t) = 1 \), so \( t \) already centralizes \( V_\alpha \).

For part f) it follows from the assumption that \( V_\alpha, V_\beta \) are contained in some Levi factor \( L' \) with \( |J| = 2 \). Since the characters associated to \( \alpha \) and \( \beta \) are not proportional, \( C_{T_d}(V_\alpha) = \ker \alpha \) does not contain \( C_{T_d}(V_\beta) = \ker \beta \). As \( T_d(k) \) is Zariski dense in \( T_d \), the claim follows. \( \square \)

3.12. Restriction of scalars for Kac–Moody groups. We give a class of examples of quasi-split Kac–Moody groups obtained by the classical process of restriction of scalars, cf. [15, Section 2.1.2]. These examples show that an abstract isomorphism \( \psi : G_1 \rightarrow G_2 \) of two almost split Kac–Moody groups does not in general preserve the full parameter set \( \{ D, E_i | K_i, \rho_i \} \) attached to these groups.

Proposition 3.11. Let \( k \) be a field and let \( E/k \) be a finite Galois extension. Let \( G \) be a split Kac–Moody group. Then there is a quasi-split Kac–Moody group \( G' \) such that \( G(E) \) is isomorphic to \( G'(k) \).

Proof. Let \( \Gamma := \text{Gal}(E/k), n := |\Gamma| \) and let \( G_0 \) be the direct product of \( n \) copies of \( G \), indexed by the elements of \( \Gamma \). Define an action of \( \Gamma \) on \( G_0(E) \) by setting

\[
\gamma \cdot (g_{\sigma_1}, \ldots, g_{\sigma_n}) := (g_{\sigma_1}, \ldots, g_{\sigma_n}).
\]

Let \( G'(k) \) denote the fixed point set of \( \Gamma \) acting on \( G_0(E) \). Then \( G'(k) \) is precisely the diagonal subgroup of \( G_0(E) \), which is isomorphic to \( G(E) \).

It remains to be checked that this \( \Gamma \)-action is the \( * \)-action induced by a \( \Gamma \)-action on the Dynkin diagram of \( G_0 \), which allows to apply the results of §3.7. This is immediate, though, as the Dynkin diagram of \( G_0 \) is the disjoint union of \( n \) copies of the Dynkin diagram of \( G \), and \( \Gamma \) permutes these copies. \( \square \)

Remark 3.12. Let \( E/k \) be a finite Galois extension and let \( G \) be a connected almost simple \( k \)-group which is split over \( k \). Then the group \( G'(k) \cong G(E) \) provided by Proposition 3.11 is the group classically obtained by restriction of scalars. The isomorphism \( \psi : G(E) \rightarrow G'(k) \) is not covered by Borel–Tits’s theory [4] since \( G'(k) \) is not absolutely almost simple. Indeed, in this theory one restricts to absolutely almost simple groups for precisely this reason.

4. Maximal split subgroups

4.1. Split subgroups of groups of Kac–Moody type. An almost split Kac–Moody group \( G(k) \) obtained via Galois descent is by definition a subgroup of a split Kac–Moody group \( \mathcal{G}_D(E) \). On the other hand, we show in this section that \( G(k) \) possesses a maximal split subgroup \( F(k) \) of Kac–Moody type, i.e. a subgroup endowed with a twin root datum which is locally split and intersects each root group \( V_\alpha(k) \) of \( G(k) \) non-trivially.

Example 4.1. Let \( k \) be a field and let \( E/k \) be a separable extension of degree 2. Let \( h : E^3 \rightarrow E \) be a Hermitian form of Witt index 1 with associated unitary group \( SU_3 \), which can be thought of as an algebraic group defined over \( k \). Then \( SU_3(k[t, t^{-1}]) \) is an almost split Kac–Moody group obtained from the split Kac–Moody group \( SL_3(k[t, t^{-1}]) \) via Galois descent, cf. Example 3.6.

On the other hand, there is an inclusion \( SL_2(k[t, t^{-1}]) \leq SU_3(k[t, t^{-1}]) \) as for the associated root groups \( (V_\alpha(k) : \alpha \in \Phi(W,S)) \) of \( SU_3([k[t, t^{-1}])] \) it follows that \( (\mathcal{Z}(V_\alpha(k)) : \alpha \in \Phi(W,S)) \cong SL_2(k[t, t^{-1}]). \)
The twin building associated to $\text{SU}_3(\mathbb{F}_q[t, t^{-1}])$ is a semi-regular twin tree with valencies $(1 + q, 1 + q^3)$ in which the twin building associated to $\text{SL}_2(\mathbb{F}_q[t, t^{-1}])$, a regular twin tree with valency $1 + q$, embeds.

**Example 4.2.** Let $k$ be a field of characteristic $\neq 2$, $n \geq 2$ and let $q = \langle a_1, \ldots, a_n \rangle$ be a quadratic form of Witt index 1 over $k$. We may assume that $\langle a_1, a_2 \rangle = \langle 1, -1 \rangle$ and that $\langle a_3, \ldots, a_n \rangle$ is anisotropic. Let $G := \text{SO}(q)$ denote the associated special orthogonal group.

For $r = 2, \ldots, n$, let $q_r := \langle a_1, \ldots, a_r \rangle$ denote the truncated quadratic form and let $G_{q_r} := \text{SO}(q_r)$ denote the associated special orthogonal group. Note in passing that $T := G_{q_2}(k) \cong k^\times$ and $G_{q_3}(k) \cong \text{PGL}_2(k)$ – this follows from the fact that $G_{q_3}$ is a split three-dimensional semisimple group, so it is either isomorphic to $\text{SL}_2$ or $\text{PGL}_2$, and these groups can be distinguished by the torus action on the root groups.

The point of this example is that there is a chain of reductive $k$-groups

$$T = G_{q_2} \leq G_{q_3} \leq \ldots \leq G_{q_n} = G$$

which share the same maximal torus $T = G_{q_2}$ of $G$. While $G_{q_3}$ is split and contains the maximal split torus $T$, clearly it is not the only subgroup of $G_{q_3}$ with this property – any $G_i' := \text{SO}(\langle 1, -1, a_i \rangle)$ for some $i \in \{4, \ldots, n\}$ has the same property, and $G_{q_3}, G_i'$ are not conjugate over $k$ if $a_3 a_i^{-1} \not\in k^2$.

The following is a classical result by Borel-Tits ([3, Theorem 7.2]).

**Theorem 4.3.** Let $G$ be a connected reductive $k$-group. Let $S \leq G$ be a maximal $k$-split torus, $\Phi = \Phi(S, G)$ the system of $k$-roots of $G$ and $\Phi' \subseteq \Phi$ the set of non-multipliable roots. Let $\Pi$ be a set of simple roots of $\Phi'$ and for each $\alpha \in \Pi$ let $E_\alpha \leq U_\alpha$ be a $k$-subgroup which is normalized by $S$ and is $k$-isomorphic to $G_\alpha$. Then there is a unique connected $k$-split reductive $k$-subgroup $F$ which contains $S \cdot (E_\alpha : \alpha \in \Pi)$.

We prove a generalization of this result for a group $G$ endowed with a 2-spherical root datum, which might be of independent interest as it provides “many” sub-twin buildings of the twin building associated to $G$. In our context, it will be used to construct a regular diagonalizable subgroup $H \leq G$ which is mapped under any isomorphism $\varphi : G \to G'$ again to a regular diagonalizable subgroup (cf. section 5).
In a first step we define the necessary ingredients of a locally split subgroup and then go on to prove that these ingredients “integrate” to a locally split group of Kac–Moody type.

Recall that a Coxeter group $W = \langle (s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1 \rangle$ is said to be 2-spherical if $m_{ij} < \infty$ for all $i, j \in I$.

For elements $x, y \in G$ write $xy := yxy^{-1}$. For a group $G$, let $G^* := G \setminus \{1\}$.

**Definition 4.4.** Let $W = \langle (s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1 \rangle$ be a 2-spherical Coxeter group and let $G$ be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of positive simple roots.

Let $T_d$ be a subgroup of $H$ and for each $\alpha \in \Pi$ let $E_\alpha \leq U_\alpha$ be a non-trivial subgroup.

For $\alpha \in \Pi$, let $s_\alpha := m(\alpha)$ for some $\alpha \in E_\alpha$.

Then $(T_d, (E_\alpha)_{\alpha \in \Delta})$ is called a **basis for a root subdatum** if the following conditions are satisfied:

1. **(RSD 1)** For all $i, j$, $(s_\alpha, s_\beta)^{m_{ij}} \in T_d$.
2. **(RSD 2)** For all $r, t \in E^+_\alpha$, $m(r)m(t)^{-1} \in T_d$.
3. **(RSD 3)** For all $\alpha \in \Pi$, $E_\alpha$ is normalized by $T_d$ and each $s_\alpha$ normalizes $T_d$.
4. **(RSD 4)** For $v \in E^+_\alpha$ there exist $v_1, v_2 \in E^+_\alpha$ such that $m(v)(:= v'v''') = s_\alpha v_1 \cdot v \cdot s_\alpha v_2$.
5. **(RSD 5)** If $X \leq U_{[\alpha, \beta]}$ is a subgroup normalized by $T_d$ and $x = u_1u_2 \in X$ with $u_1 \in U_{[\alpha, \beta]}$, $u_2 \in U_\beta$, then $u_1, u_2 \in X$.

As the name suggests, a basis for a root subdatum gives rise to a subgroup which has a twin root datum.

**Theorem 4.5.** Let $(W, S)$ be a 2-spherical Coxeter group, let $\Phi = \Phi(W, S)$ denote the set of its roots and let $\Pi$ be the set of simple roots.

Let $G$ be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$. Let $(T_d, (E_\alpha)_{\alpha \in \Pi})$ be a basis for a root subdatum.

Let $M := T_d(s_\alpha : \alpha \in \Pi)$, $V := \langle E_\alpha : \alpha \in \Pi \rangle$ and $F := \langle M, V \rangle$. Set $F_\gamma := F \cap U_\gamma$ for $\gamma \in \Phi$.

Then $(T_d, (F_\gamma)_{\gamma \in \Phi})$ is a twin root datum of type $(W, S)$ for $F$.

The proof, which will be given after a couple of preparatory lemmas, is very much inspired by [Proof of Theorem 7.2].

**Lemma 4.6.** Let $G$ be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$.

Let $\alpha, \beta$ be two distinct positive simple roots. Then $U_{-\alpha}$ commutes with $U_\beta$.

**Proof.** The set $\Psi := \{-\alpha, \beta\}$ is a prenilpotent set of roots since $s_\Psi \subseteq \Phi^+$ and $s_\alpha \Psi \subseteq \Phi^+$. The open root interval $[-\alpha, \beta]$ is empty: Any positive root in $[-\alpha, \beta]$ must be mapped to a negative root by $s_\beta$ and hence coincides with $\beta$, while any negative root in $[-\alpha, \beta]$ must be mapped to a positive one by $s_\alpha$ and hence coincides with $-\alpha$. By the commutator axiom, $[U_{-\alpha}, U_\beta] \leq U_{[-\alpha, \beta]} = 1$. \hfill \qed

We first analyze the structure of $V$.

Let $E_{-\alpha} := s_\alpha E_\alpha$. Then $E_{-\alpha}$ is independent from the choice of $v \in E^+_\alpha$ in the definition of $s_\alpha = m(v)$ as for $v, v' \in E^+_\alpha$, $m(v)$ and $m(v')$ differ by an element of $T_d$ by (RSD 2), and $T_d$ normalizes $E_\alpha$ by (RSD 3).

For $\alpha, \beta \in \Pi$ let $E_{[\alpha, \beta]} := [E_\alpha, E_\beta]$ denote the commutator subgroup. Then $E_{[\alpha, \beta]}$ is normalized by $E_\beta$, since for $a \in E_\alpha$, $b, c \in E_\beta$,

$$c[a, b]c^{-1} = caba^{-1}b^{-1}c^{-1} = [c, a][a, cb].$$

Let $E_{(\alpha, \beta)} := E_{[\alpha, \beta]} \cdot E_\beta$.

**Lemma 4.7.** Let $\alpha, \beta \in \Pi$ be two distinct positive roots.

a) $E_{(\alpha, \beta)} = \langle u_\alpha u_\beta u_\alpha^{-1} : u_\alpha \in E_\alpha, u_\beta \in E_\beta, u_\alpha \neq 1 \rangle$. 

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b) $E_{(\alpha, \beta)}$ is normalized by $s_\alpha$.

c) Let $E'_\alpha := \langle E_{(\alpha, \gamma)} : \gamma \in \Pi, \gamma \neq \alpha \rangle$. Then $V = E_\alpha \times E'_\alpha$.

**Proof.**

a) Let $X := \{u_\alpha u_\beta u_\alpha^{-1} : u_\alpha \in E_\alpha, u_\beta \in E_\beta, u_\alpha \neq 1\}$. Then $X \leq E_{(\alpha, \beta)}$ is clear. Conversely, since $E_\alpha$ and $E_\beta$ are normalized by $T_d$, so is $X$. For $u_\alpha \in E^*_\alpha, u_\beta \in E_\beta$, note that $u_\alpha u_\beta u_\alpha^{-1} = [u_\alpha, u_\beta]u_\beta \in U_{(\alpha, \beta)}U_\beta$. By (RSD 5) it follows that $u_\beta$ and $[u_\alpha, u_\beta]$ are contained in $X$, from which the claim follows.

b) By a), it suffices to show that $s_\alpha(u_\alpha u_\beta u_\alpha^{-1})s_\alpha^{-1} \in E_{(\alpha, \beta)}$, where $u_\alpha \neq 1$. Write $s_\alpha = u_1 u_2 u_\alpha^{-1}$ for some $u_1 \in E_\alpha, u_2 \in E_{-\alpha}$ – this is legitimate as $u_\alpha \neq 1$ and $s_\alpha$ is defined only up to elements of $T_d$. Then

$$s_\alpha(u_\alpha u_\beta u_\alpha^{-1})s_\alpha^{-1} = u_1 u_2 u_\beta u_2^{-1} u_1^{-1} = u_1 u_2 u_1^{-1}$$

since $u_2 \in E_{-\alpha}$ commutes with $u_\beta$ by Lemma 4.6 from which the claim follows.

c) It is clear that $E_\alpha$ and $E'_\alpha$ are subgroups of $V$ which generate $V$. From (i) it is immediate that $E_\alpha$ normalizes $E'_\alpha$. Let $v \in E_\alpha \cap E'_\alpha$. Then by b)

$$s_\alpha v \in s_\alpha E_\alpha \cap s_\alpha E'_\alpha = E_{-\alpha} \cap E'_\alpha \leq U_- \cap U_+ = 1.$$

□

**Lemma 4.8.**

a) There is a canonical isomorphism $\pi : M/T_d \to W$.

b) Let $\alpha \in \Delta$ and $w \in W$ be such that $w\alpha$ is positive. Then $wE_\alpha \leq V$.

**Proof.**

a) Note that $T_d$ is a normal subgroup of $M$ by (RSD 3); by (RSD 1) it follows that $M/T_d \cong W$.

b) Since $T_d$ normalizes $E_\alpha$, $wE_\alpha$ is well-defined. If $l(w) = 0$, there is nothing to prove, so suppose $l(w) \geq 1$. Since $w\alpha > 0$, we can write $w = s_\beta w'$, where $\beta$ is a simple root distinct from $\alpha$ and $w'$ is such that $w'\alpha > 0$. By induction, $w'E_\alpha \leq V = E_\beta \times E'_\beta$. Since $w'E_\alpha \leq U' w'\alpha$ and $w'\alpha \neq \beta$, it follows that $w'E_\alpha \leq E'_\beta$. Then $wE_\alpha \leq s_\beta E'_\beta = E'_\beta$.

□

The next step consists of exhibiting a Bruhat decomposition for $F$.

**Lemma 4.9.** The group $F$ can be written as $F = VMV = \cup_{w \in W} VWwV$.

**Proof.** The set $V \cdot M \cdot V$ contains $V$ and $M$, is stable under inversion and closed under multiplication by elements in $V$ or $T_d$ from the right or left. To show that it coincides with $F$, it thus suffices to check that it is closed under multiplication from the right by $s_\alpha, \alpha \in \Delta$.

**First step.** For $\alpha \in \Delta, E_{-\alpha} \subseteq T_d E_\alpha \cup T_d E_\alpha s_\alpha E_\alpha$.

Indeed, $1 \in T_d E_\alpha$ while by definition each $v \in E^*_\alpha$ has the form $v = s_\alpha v_0$ for some $v_0 \in E_\alpha$. By (RSD 4), there are $v_1, v_2 \in E_\alpha$ such that $m(v_0) = s_\alpha v_1 s_\alpha v_0 s_\alpha v_2$. Then $s_\alpha^{-1} s_\alpha = v_1 v_2$, i.e. $v \in E_\alpha s_\alpha E_\alpha$, from which the claim follows.

**Second step.** Since $T_d$ normalizes $V$, we can write $VMV = \cup_{w \in W} VWwV$ unambiguously. We will show that $WwV s_\alpha \subseteq Ww s_\alpha V \cup VwV$, from which the claim will follow.

If $l(w) = 0$, i.e. $w = 1$, then by the first step and Lemma 4.7

$$V s_\alpha = E_\alpha s_\alpha s_\alpha^* E'_\alpha \subseteq E_\alpha s_\alpha V.$$
Suppose \( l(w) \geq 1 \) and the claim is proven for all \( w' \) with \( l(w') < l(w) \). Two cases can occur:
(1) \( l(ws) > l(w) \). This is the case if and only if \( w\alpha > 0 \). Then \( wE_\alpha w^{-1} \subseteq V \) by Lemma 4.3 and we calculate
\[
VwV_{s_\alpha} = VwE_\alpha s_\alpha E'_\alpha = VwE_\alpha ws_\alpha E'_\alpha \subseteq VwsaV.
\]
(2) \( l(ws) < l(w) \), i.e. \( w\alpha < 0 \). Then we can write \( w = w's_\alpha \) with \( l(w') = l(w) - 1 \geq 0 \). We calculate
\[
VwV_{s_\alpha} = Vw's_\alpha E_\alpha s_\alpha E'_\alpha = Vw'E_\alpha E'_\alpha \subseteq Vw'V \cup Vw's_\alpha V = Vw'V \cup Vw's_\alpha V.
\]
Here the last equality follows because \( w'\alpha > 0 \), which allows us to apply the first case.

We can turn to the proof of Theorem 4.5.

Proof. For \( \gamma \in \Phi \setminus \Pi \) and \( w \in W, \alpha \in \Pi \) such that \( w\alpha = \gamma \) choose some lift \( \tilde{w} \in M \) of \( w \) and set \( E_\gamma := \tilde{w}E_\alpha \tilde{w}^{-1} \). Then for each \( \gamma \in \Phi, E_\gamma \subseteq F_\gamma \). Assume for the moment that equality holds (in particular, \( E_\gamma \) will then not depend on the choice of \( \alpha \) and \( \tilde{w} \)).

Then clearly for each \( \gamma \in \Phi, F_\gamma \) is nontrivial and normalized by \( T_d \) by (RSD 3). By (RSD 4), \( s_\alpha \in \langle E_\alpha, E_\gamma \rangle \), from which it follows that \( F \) is generated by \( T_d \) and \( \langle E_\alpha, E_\gamma : \alpha \in \Delta \rangle \), i.e. (TRD 1) holds. Set \( V_\gamma := \langle F_\gamma : \gamma < 0 \rangle \). Then \( V_\gamma \cap V \leq U_\gamma \cap U_\gamma = 1 \) and therefore (TRD 4) is satisfied. Similarly, (TRD 2) holds by the definition of \( F_\gamma \) and the corresponding property for \( G \).

Axiom (TRD 3) holds for \( F_\gamma, \gamma \in \Delta \) by (RSD 2) and (RSD 4).

It remains to prove that \( F_\gamma = E_\gamma \) for \( \gamma \in \Phi \), in particular \( F_\alpha = E_\alpha \) for \( \alpha \in \Pi \) which is not clear a priori.

First step. If \( \gamma \in \Pi \), then \( F \cap U_\gamma = E_\gamma \).

By the Bruhat decomposition \( F = VMV \) it follows that \( F \cap U_\gamma = V \cap U_\gamma \). Since \( V = E_\gamma \triangleleft E_\gamma^1 \) it follows that
\[
\gamma s_\gamma (F \cap U_\gamma)s_\gamma^{-1} = s_\gamma V s_\gamma^{-1} \cap U_\gamma = E_\gamma E_\gamma' \cap U_\gamma = E_\gamma \cap U_\gamma = E_\gamma,
\]
from which it follows that \( F \cap U_\gamma = E_\gamma \).

Second step. If \( \delta \in \Phi \setminus \Pi \) is arbitrary, then \( F \cap U_\delta = E_\delta \).

Suppose first that \( \delta \in \Phi^+ \). Let \( w = \tilde{w} \in M, \alpha \in \Pi \) as in the definition of \( E_\delta \).

Then \( w(F \cap U_\delta)w^{-1} = w(V \cap U_\delta)w^{-1} = wVw^{-1} \cap (V \cap U_\alpha) \subseteq V \cap U_\alpha = E_\alpha \).

By definition, \( wE_\alpha w^{-1} \subseteq F_\delta \), and we have just shown the reverse inclusion, i.e. \( F_\delta = E_\delta \).

Clearly the same reasoning works when \( \delta \in \Phi^- \), which finishes the proof of the theorem.

Remark 4.10. The statement of Lemma 4.3 that \( F = \cup_{w \in W} VwV \) can be thought of as the fact that \( F \) is a graded subgroup of \( G \). This means that whenever \( f = b_1 wb_2 \) with \( b_1, b_2 \in B \) and \( w \in W \) is the Bruhat decomposition of an element \( f \in F \), then \( b_1, b_2 \) and \( w \) can actually be chosen to be elements of \( F \).

Remark 4.11. Let \( G \) be a group endowed with a 2-spherical twin root datum \((H, (U_\alpha)_{\alpha \in \Phi(W, S)})\). Then \((H, (U_\alpha)_{\alpha \in \Delta})\) meets conditions (RSD 1)-(RSD 4), but not necessarily (RSD 5). Indeed, if (RSD 5) is met, it follows from the proof of the preceding theorem that \( U_\alpha = (U_\alpha : \alpha \in \Delta) \). This is satisfied for isotropic reductive \( k \)-groups with \( |k| \geq 4 \), but fails e.g. for \( G_2(F_2) \).
Remark 4.12. A geometric interpretation of the theorem is as follows: Let $\Delta$ be the twin building associated to $G$, $A$ the twin apartment determined by $H$ and $C_+, C_-$ the two opposite chambers corresponding to $B_+, B_-$. On each panel $F_\alpha$ of $C_+$, fix chambers according to the action of $E_\alpha$ on $F_\alpha$. Condition (RSD 4) ensures that these form a sub-Moufang set. The remaining conditions are the necessary compatibility conditions which ensure that these chambers give rise to a sub twin building with $A$ as a twin apartment.

In particular, the twin building $\Delta(F)$ associated to $F$ embeds in $\Delta(G)$ as a closed convex subcomplex.

4.2. The case of almost split Kac–Moody groups. We apply Theorem 4.13 to almost split Kac–Moody groups.

Let $G$ be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W, S)})$. Then the twin root datum is said to be locally split (over a family of fields $(k_\alpha)_{\alpha \in \Phi}$) if $H$ is abelian and for each $\alpha \in \Phi$, $(U_\alpha, U_{-\alpha})$ is isomorphic to either $\text{SL}_2(k_\alpha)$ or $\text{PSL}_2(k_\alpha)$.

**Theorem 4.13.** Let $k$ be an infinite field and let $G(k)$ be a 2-spherical almost split Kac–Moody group obtained by Galois descent. Let $(Z(k), (V_\alpha)_{\alpha \in \Phi(W, S)})$ denote its canonical twin root datum and let $T_d(k) \leq Z(k)$ be a maximal $k$-split torus. For each simple root $\alpha \in \Pi$ let $E_\alpha \leq V_\alpha$ be a subgroup isomorphic to $(k, +)$ which is normalized by $T_d(k)$. Then there is a subgroup $F(k) \leq G(k)$ which contains $T_d(k)(E_\alpha : \alpha \in \Pi)$ and which is endowed with a locally split twin root datum.

**Proof.** Since $G$ is assumed to be 2-spherical, for each pair of simple roots $\{\alpha, \beta\} \subseteq \Delta$ the group $X_{\alpha, \beta} := Z(k)(V_{\alpha, \beta})$ can be identified with a $k$-points of a reductive algebraic $k$-group of relative rank 2. By Theorem 4.13 there is a split subgroup $Y_{\alpha, \beta} \leq X_{\alpha, \beta}$ which contains $T_d(k)$ and $E_\alpha, E_\beta$. Now $Y_{\alpha, \beta}$ is endowed with a spherical twin root datum, the properties of which imply that the axioms (RSD 1) - (RSD 4) of a root subdatum are satisfied, since these need to be checked only for rank 2 subgroups.

Since $k$ is infinite, $T_d(k)$ is Zariski dense in $T_d$. For a subgroup $X \leq V_{\alpha, \beta}$ normalized by $T_d(k)$ it follows that $X$ is normalized by $T_d$. By [3, Proposition 3.11] it follows that (RSD 5) is satisfied as well.

Theorem 4.13 gives the existence of $F$, and from the fact that the group $Y_{\alpha, \beta}$ is a split reductive group it is immediate that the twin root datum for $F$ is locally split.

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**Definition 4.14.** Let $k$ be an infinite field and let $G(k)$ be a 2-spherical almost split Kac–Moody group obtained by Galois descent. Any group $F$ obtained from $G(k)$ in this way is called a maximal split subgroup of $G$.

**Remark 4.15.** It is always possible to find subgroups $E_\alpha$ as required in Theorem 4.13: just let $E_\alpha$ be a one-dimensional $k$-subspace of $Z(\alpha)$. Then Proposition 4.10(iii) b) shows that $E_\alpha$ is normalized by $T_d(k)$. This proves Theorem 1.2.

In particular, any almost split 2-spherical Kac–Moody group is "sandwiched" between two split Kac–Moody groups: For a splitting field $E$ of $G$, one has

\[ F(k) \leq G(k) \leq G(E). \]

Here the Coxeter type of $F(k)$ is the same as the Coxeter type of $G(k)$, while the type of $G(k)$ equals the type of $G(E)$ if and only if $G$ is already split over $k$.

**Remark 4.16.** We used Theorem 4.13 to produce a locally split subgroup. The theorem is more general, though, as arbitrary sub-Moufang sets are allowed. In particular, we recover Example 4.2.
Remark 4.17. Another example of a basis for a root subdatum \((T_d, E_\alpha)_{\alpha \in \Delta}\) as required in Theorem 4.3 comes from subfields: If \(k \subseteq K\) and \(\mathcal{G}_D\) is a constructive Tits functor, then take \(T_d := T(k)\) and \(E_\alpha := U_\alpha(k)\) inside \(\mathcal{G}_D(K)\). Of course, the theorem can be applied more than once, i.e. pass first to a locally split subgroup and then to \(K\)-rational points.

Finally, just as Example 4.2 suggests, there is a chain condition on groups containing a maximal split subgroup.

**Proposition 4.18.** Let \(k\) be an infinite field, char \(k \neq 2\) and let \(G(k)\) be an almost split Kac–Moody group over \(k\) obtained by Galois descent. Let \(F \leq G\) be a maximal split subgroup. Let

\[
F = H_1 \leq H_2 \leq \ldots
\]

be a chain of subgroups which are obtained by integrating a root subdatum and such that \(H_i = H_i^\dagger\), i.e. \(H_i\) is generated by its root groups. Then the chain eventually becomes stationary.

More precisely, the length of a strictly increasing chain is bounded by \(\sum_{\Pi \in \Delta} \dim_k V_\alpha\).

**Proof.** For each simple root \(\alpha \in \Pi\) with corresponding root group \(V_\alpha(k) \leq G(k)\) let \(H_{i,\alpha} := H_i \cap V_\alpha\). Then \(H_{i,\alpha} \cap \mathcal{Z}(V_\alpha)\) is a \(k\)-sub-vectorspace since it is invariant under \(T_d(k)\), similarly for \((H_{i,\alpha} \cdot \mathcal{Z}(V_\alpha))/\mathcal{Z}(V_\alpha)\).

Recall from Proposition 3.10 that \(V_\alpha(k)\) is an extension of two finite-dimensional \(k\)-vector spaces. This implies that \((H_{i,\alpha})\) eventually becomes stationary. Since the \(H_i\) are supposed to be generated by their root groups, the first claim follows. Since in a strictly increasing chain of subgroups, in each step there is some \(\alpha \in \Pi\) such that \(H_{i,\alpha}\) is strictly contained in \(H_{i+1,\alpha}\), the second claim follows. \(\Box\)

5. The isomorphism problem

In this chapter we prove that every abstract isomorphism of two 2-spherical almost split Kac–Moody groups over fields of characteristic 0 is standard in the sense that it induces an isomorphism of the associated canonical twin root data.

5.1. Preparatory lemmas.

**Lemma 5.1.** Let \(k\) be an infinite field and let \(T\) be a \(k\)-split torus. Let \(S \leq T(k)\) be such that \(T(k)/S\) is finitely generated. Then \(S\) is Zariski dense in \(T\).

**Proof.** Since \(T\) is split over \(k\) and \(k\) is infinite, \(T(k)\) is Zariski dense in \(T\). Assume that \(S \neq T\). Then \(S\) is defined over \(k\) and so is \(S^0\), which is a \(k\)-split subtorus of \(T\) by [3, Corollary 1.9 b)]. It follows that \(\dim S^0 < \dim T\). Passing to the rational points, we find that \(X := T(k)/(S \cap S^0(k))\) contains a copy of \(k^\times\), which is not finitely generated. As \(S \cap S^0(k)\) has finite index in \(S\), \(X\) is finitely generated. This is a contradiction since any subgroup of a finitely generated abelian group is finitely generated. \(\Box\)

**Proposition 5.2.** Let \(G\) be a group endowed with a twin root datum \((H, (U_\alpha)_{\alpha \in \Phi(W,S)})\). Let \(L\) be a subgroup of \(G\) such that for each \(l \in L \setminus \{1\}\) there is a root \(\beta_l \in \Phi(W,S)\) such that \(l \in U_{\beta_l}\). Then there is some \(\beta \in \Phi(W,S)\) such that \(L \leq U_\beta\).

**Proof.** Note first that \(\beta_l = \beta_{l^{-1}}\), as \(U_{\beta_{l^{-1}}}\) is a subgroup and \(\beta_{l^{-1}}\) is uniquely determined as \(U_{\alpha} \cap U_{\beta} = 1\) for distinct roots \(\alpha \neq \beta\).

Assume for a contradiction that there are \(x, y \in L \setminus \{1\}\) such that \(\beta_x \neq \beta_y\), in particular \(xy \neq 1\). Let \(\alpha := \beta_x, \beta := \beta_y\) and \(\gamma := \beta_{xy}\).
This implies that \( U_\alpha U_\beta \cap U_\gamma \neq 1 \). Moreover, for any permutation \( \pi \) of \( \{ \alpha, \beta, \gamma \} \), it follows that \( U_{\pi(\alpha)} U_{\pi(\beta)} \cap U_{\pi(\gamma)} \neq \{1\} \): If \( u_\alpha u_\beta = u_\gamma \), then \( u_\alpha^{-1} u_\beta^{-1} = u_\gamma^{-1} \), \( u_\beta u_\gamma^{-1} = u_\alpha^{-1} \), and the permutations \((\alpha \beta),(\alpha \gamma \beta)\) generate \( \text{Sym}(\{\alpha, \beta, \gamma\}) \). This implies that if two of the three roots coincide, then all roots coincide. So we can suppose that all three roots are distinct.

If two of the three roots are positive and the remaining one is negative (or vice versa), we can assume that \( \alpha > 0, \beta > 0 \) and \( \gamma < 0 \) since the statement to be proved is invariant under permutations of the roots. But this is a contradiction as \( U_+ \cap U_- = \{1\} \). If all three roots have the same sign, say \( \alpha, \beta, \gamma \in \Phi^+ \), then choose some \( w \in W \) such that \( w_\gamma = \delta \) is a positive simple root. If \( w_\alpha \) or \( w_\beta \) is negative, this is a contradiction by the case just discussed. If \( w_\alpha, w_\beta, w_\gamma \) are all positive, then \( s_\delta w_\alpha > 0, s_\delta w_\beta > 0, s_\delta w_\gamma < 0 \), which is again a contradiction by the case just discussed.

**Proposition 5.3.** Let \( k \) be a field of characteristic 0 and let \( G \) be a connected reductive algebraic group defined over \( k \). Let \( g \in G(k) \setminus \{1\} \) be a nontrivial unipotent element. Then there exists a morphism \( \varphi : \text{SL}_2 \to G \) defined over \( k \) such that \( \varphi(u) = g \) for some unipotent element \( u \in \text{SL}_2(k) \).

**Proof.** Let \( U := \overline{\{u\}} \). As \( k \) is of characteristic 0, \( U \) is a one-dimensional unipotent group which is defined over \( k \) since \( u \in G(k) \). This implies that \( U \) is \( k \)-isomorphic to \( G_0 \). Let \( u := \text{Lie} U \). By the Jacobson-Morozov lemma (usually stated for semisimple Lie algebras over a field of characteristic 0, but holding in fact for arbitrary completely reducible subalgebras \( g \leq \mathfrak{gl}(V) \), see the original paper [13, Theorem 3]), there is a three-dimensional Lie subalgebra \( x \) which is \( k \)-isomorphic to \( \mathfrak{sl}_2 \) and contains \( u \). As \( \text{char} k = 0 \), any perfect Lie subalgebra is the Lie algebra of a closed subgroup \( X \) ([2, Corollary 7.9]). This translates into the fact that there is a closed subgroup \( X \leq G \) defined over \( k \) which is \( k \)-isomorphic to either \( \text{SL}_2 \) or \( \text{PGL}_2 \). This implies the claim.

Let \( G \) be a connected reductive group defined over \( k \) which splits over \( k \). Let \( T \) be a maximal torus and \( U \) a unipotent group which is normalized by \( T \). Then \((T,U)\) is contained in a Borel group \( B \), so there is an ordering on the set of roots \( \Phi(T,G) \) of \( T \) in \( G \) such that \( U \leq U_+ \). It is then a classical fact (cf. [4, p.65 l.7]) that \( U \) is generated by the root groups \( U_\alpha \) relative to \( T \) which are contained in \( U \).

We need an analogue of this theorem in case that \( G \) is not necessarily split over \( k \).

**Proposition 5.4.** Let \( k \) be an infinite field. Let \( G \) be a connected reductive \( k \)-group which is \( k \)-isotropic and let \( S \leq G \) be a maximal \( k \)-split torus. Let \( U \leq G \) be a unipotent subgroup defined over \( k \) which is normalized by \( S \). Then \( U \) is contained in \( (U_\alpha : \alpha > 0) \) for some ordering \( > \) of the set of roots \( \Phi(S,G) \) of \( S \) in \( G \).

**Proof.** Let \( P \) be a minimal parabolic subgroup defined over \( k \) which contains \( U \) and \( S \). Then \( P \) has a Levi decomposition \( P = Z(S)P_u \), where \( Z(S) \) is the centralizer of \( S \) and \( P_u \) is the unipotent radical of \( P \). Since \( S \) is maximal \( k \)-split, \( Z(S)(k) \) does not contain any unipotent elements. This implies that \( U(k) \leq P_u(k) \) and since \( U(k) \) is dense in \( U \), it follows that \( U \leq P_u \), which implies the claim.

Recall that a subgroup \( S \) of an almost split Kac–Moody \( k \)-group \( G(k) \) is called **diagonalizable** (over \( k \)) if there is \( g \in G(k) \) such that \( g S g^{-1} \leq T_d(k) \), where \( T_d \) is the standard maximal \( k \)-split torus of \( G(k) \).

Furthermore, a diagonalizable subgroup \( S \) is called **regular** if the fixed point set of the \( S \)-action on the associated twin building consists of a single twin apartment.
Among all diagonalizable subgroups of $G$, regular subgroups can be characterized purely group-theoretically. The following characterization can be found in [2] Proposition 5.13 for split Kac–Moody groups. We generalize this to almost split Kac–Moody groups, where care has to be taken of the anisotropic kernel.

**Lemma 5.5.** Let $k$ be a field of characteristic 0 and let $G$ be an almost split Kac–Moody $k$-group. Let $S \leq G(k)$ be a diagonalizable subgroup. Then $S$ is regular if and only if $S$ does not centralize a subgroup $X \leq G(k)$ isomorphic to either $\text{SL}_2(\mathbb{Q})$ or $\text{PSL}_2(\mathbb{Q})$.

**Proof.** Without loss of generality we may assume that $S$ is contained in the standard maximal $k$-split torus $T_d(k)$. Suppose first that $S$ is regular and centralizes $X$. As $X$ has a fixed point in both $\Delta_+$ and $\Delta_-$ and is normalized by $S$, these points can be assumed to lie in the standard twin apartment $A_k$. As $X$ normalizes $S$, it must stabilize $A_k$. Hence there is a homomorphism $\Psi : X \rightarrow W = \text{Stab}_G(A_k)/\text{Fix}_G(A_k)$. $\Psi(X)$ then is a finite group as it is a subgroup of a point stabilizer, so a finite index subgroup $X' \leq X$ is contained in the anisotropic kernel $\text{Fix}_G(A_k) = Z(k)$. Then $X' = X$ as $\text{PSL}_2(\mathbb{Q})$ is simple and $\text{SL}_2(\mathbb{Q})$ does not have a proper finite index subgroup either. (Indeed, since $U_+(\mathbb{Q})$ and $U_-(\mathbb{Q})$ are divisible, any finite index subgroup $N \leq \text{SL}_2(\mathbb{Q})$ contains $U_+(\mathbb{Q})$ and $U_-(\mathbb{Q})$, hence is equal to $\text{SL}_2(\mathbb{Q})$.) Postcomposing with the adjoint representation $A_dG$, where $(A_k, F_1, -F_1)$ is a rational standardisation and $\Omega := \{F_1, -F_1\}$, there is a homomorphism

$$X \rightarrow A_dG(Z(k))$$

which induces a representation of $\text{SL}_2(\mathbb{Q})$. This representation is rational and defined over $k$ by [2] Lemma 5.9. Since the target group is anisotropic over $k$ and therefore does not contain $k$-rational unipotent elements, this homomorphism must be trivial. Then $X \leq \ker A_dG$, which is a contradiction since the latter group is abelian.

Conversely, suppose that $S$ fixes a point $x \notin A_k$, without loss of generality suppose that $x \in \Delta_+$. Then there is a panel $E$ of $A_k$ and a chamber $C_1 \subsetneq A_k$ which has $E$ as a panel and is fixed by $S$. Indeed, let $G = (C_0, C_1, \ldots, C_n)$ be a gallery such that $C_0 \in A_k, C_n$ contains $x$ and $G$ is of minimal length among all such galleries. Then $n \geq 1$ since $x \notin A_k$, and $C_1$ is fixed by $S$ since the $S$-action is type-preserving. Let $E = C_0 \cap C_1$ and let $\alpha$ be the corresponding root of $A_k$ determined by $C_0$ and $E$. The root group $V_\alpha \leq G(k)$ parametrizes the chambers which have $E$ as a panel and which are different from $C_0$. Since $S$ fixes $A_k$ and $C_1 \not\subset A_k$, there are three chambers of the $E$-panel fixed by $S$. This means that there is some non-trivial $v \in V_\alpha$ centralized by $S$. If $v \in V_\alpha \setminus \mathcal{Z}(V_\alpha)$, this implies that $S$ centralizes the entire group $V_\alpha$, if $v \in \mathcal{Z}(V_\alpha)$, this implies at least that $S$ centralizes $\mathcal{Z}(V_\alpha)$ (recall that the action of the split torus is via a character on both $V_\alpha/\mathcal{Z}(V_\alpha)$ and $\mathcal{Z}(V_\alpha)$).

In either case, $S$ centralizes $\mathcal{Z}(V_\alpha)$ and also $\mathcal{Z}(V_{-\alpha})$. Hence $S$ centralizes the group $T_d(k)(\mathcal{Z}(V_\alpha), \mathcal{Z}(V_{-\alpha}))$. But this group contains a split semisimple group of rank 1 by Theorem 4.3, i.e. either $\text{SL}_2(k)$ or $\text{PGL}_2(k)$. In both cases the claim follows. $\square$

**Lemma 5.6.** Let $(W,S)$ be a finitely generated Coxeter group and let $A \leq W$ be a solvable subgroup. Then $A$ is finitely generated.

**Proof.** Let $X$ be the CAT(0) realization of $W$. Since $W$ is finitely generated, $X$ is finite-dimensional by construction and $W$ acts on $X$ properly and cocompactly. The conclusion follows now from [5] p. 439 Theorem I.1 (3),(4)]. $\square$

### 5.2. Proof of the main theorem.

**Setting.** Let $k, k'$ be two fields of characteristic 0 and let $G, G'$ be two 2-spherical almost split Kac–Moody groups over $k,k'$, respectively. Let $G(k), G'(k')$ denote
their rational points and suppose that \( \varphi : G(k) \to G'(k') \) is an abstract isomorphism. Let

- \( Z(k) \leq G(k), Z'(k') \leq G'(k') \) denote the respective anisotropic kernels of \( G, G' \).
- \( T_d(k) \leq Z(k), T_d'(k') \leq Z'(k') \) denote the respective maximal split tori.
- \( (W, S), (W', S') \) denote the respective Weyl groups, and \( \Phi, \Phi' \) the sets of roots with simple roots \( \Pi, \Pi' \).
- \( (U_\alpha)_{\alpha \in \Phi} \) denote the rational root groups of \( G \), and \( (V_\beta)_{\beta \in \Phi'} \) the rational root groups of \( G' \).
- \( \Delta, \Delta' \) denote the twin buildings associated to \( G \) and \( G' \).
- \( A \) and \( A' \) denote the standard twin apartments fixed by \( Z(k), Z'(k') \) respectively.

**Strategy of proof.** The proof strategy can be outlined as follows:

**Step 1.** Since \( G(k) \) is assumed to be 2-spherical, \( G(k) \) contains a maximal split subgroup \( F(k) \) containing \( T_d(k) \). A generalization of arguments from [2] can be used to exhibit a subgroup \( S(Q) \leq T_d(k) \) with the property that \( S(Q) \) fixes precisely \( A \) and \( \varphi(S(Q)) \) fixes precisely a twin apartment \( A'' \) of \( \Delta' \). By postcomposing \( \varphi \) with an inner automorphism if necessary, we assume that \( A'' = A' \).

**Step 2.** From the existence of \( S(Q) \), which is in some sense a small subgroup of the split torus, we deduce the existence of two large subgroups \( S_1 \leq T_d(k), S_2 \leq T_d'(k') \) such that \( \varphi(S_1) \leq Z'(k'), \varphi^{-1}(S_2) \leq Z(k) \). In particular, \( \varphi(S_1) \) normalizes all root groups \( V_\beta \) and \( \varphi^{-1}(S_2) \) normalizes all root groups \( U_\alpha \).

**Step 3.** We now focus on a root group \( U_\alpha \). Assume first that \( U_\alpha \) is abelian (see Step 5 for the general case). Then for \( u \in U_\alpha \), we show that \( \varphi(u) \in L^J \) for some Levi factor \( L^J \) of finite type, which depends a priori on \( u \).

Using the groups \( S_1 \) and \( S_2 \) we show that \( \varphi(u) \) actually is a unipotent element which is contained in a group \( V_{\beta_1} \cdots V_{\beta_r} \leq L^J \).

**Step 4.** Now root groups in a spherical Levi factor can be distinguished by the torus action. Again using the groups \( S_1 \) and \( S_2 \), it follows that with the above notation, \( r = 1 \), i.e. for each \( u \in U_\alpha \) there is some \( \beta_u \in \Phi' \) such that \( \varphi(v) \in V_{\beta_u} \).

Since \( U_\alpha \) is a group, it follows that \( \varphi(U_\alpha) \leq V_{\beta(\alpha)} \) for some single \( \beta(\alpha) \in \Phi' \).

**Step 5.** If \( U_\alpha \) is not abelian, the analysis of steps 3 and 4 still applies to \( \mathcal{Z}(U_\alpha) \).

Let \( u_1, \ldots, u_r \) be elements such that the canonical images of the \( u_i \) are a \( k \)-basis for \( U_\alpha/\mathcal{Z}(U_\alpha) \). Arguing as in steps 3 and 4 for the groups \( k \cdot u_i \), together with the knowledge about \( \varphi(\mathcal{Z}(U_\alpha)) \) allows to conclude that also in this case \( \varphi(U_\alpha) \) is contained in a single root group \( V_{\beta(\alpha)} \).

**Step 6.** By symmetry, each root group \( V_\beta \) satisfies \( \varphi^{-1}(V_\beta) \leq U_{\alpha(\beta)} \), so actually equality holds. This allows to conclude that \( \varphi \) maps root groups to root groups and preserves the anisotropic kernel.

The following lemma is a key step in comparing the twin root data of \( G \) and \( G' \).

**Lemma 5.7.** There exists a regular diagonalizable subgroup \( S(Q) \leq T_d(k) \) such that \( \varphi(S(Q)) \) again is regular diagonalizable.

**Proof.** Fix a maximal split subgroup \( F(k) \) of \( G(k) \) and let \( (T_d(k), (F_{\alpha})_{\alpha \in \Phi}) \) denote the associated twin root datum. Then each rank 2 subgroup \( F_{\alpha, \beta}(k) := T_d(k)(F_{\alpha,k}(k), F_{\beta,k}(k)) \) coincides with the \( k \)-points of a split reductive group of semisimple rank 2. Since these groups are defined over \( \mathbb{Z} \), it is possible to consider
$F(Q)$, the $Q$-points of $F$. More precisely, for each root $\alpha \in \Phi$ let $f_{\alpha} : (k, +) \to F_{\alpha}(k)$ denote the corresponding isomorphism and $t : (k^*)^n \to T_d(k)$ the canonical isomorphism. Then $F(Q) := t((Q^*)^n) \cdot (f_{\alpha}(Q) : \alpha \in \Phi)$.

For each simple root $\alpha \in \Pi$ let $\psi_{\alpha} : SL_2(Q) \to \langle F_{\alpha}(Q), F_{-\alpha}(Q) \rangle$ denote the canonical homomorphism. Let $D_\alpha := \langle \psi_{\alpha}(\text{diag}(x, x^{-1})) : x \in Q^* \rangle$ and let $S(Q) := \langle D_\alpha : \alpha \in \Delta \rangle$.

**Claim 1.** $S(Q)$ is regular. Note first that $S(Q)$ is invariant under the Weyl group. Indeed, it suffices to check that $s_{\alpha}(D_{\beta}) \leq S(Q)$ for two simple roots $\alpha, \beta$, and this can be verified in $F_{\alpha, \beta}$ where it follows from the explicit description of the Weyl group action on the torus in a reductive group. Assume that $S(Q)$ is not regular. Then from the proof of Lemma 5.3 it follows that there is a root $\alpha$ such that $S(Q)$ centralizes $\mathcal{Z}(V_\alpha)$, i.e. the character $2\alpha$ vanishes on $S(Q)$. Write $\alpha = u\alpha_i$ for some $w \in W$ and a simple root $\alpha_i$. Then $2\alpha_i$ vanishes on $S(Q)$ by the Weyl group invariance of $S(Q)$, but this is a contradiction since $S(Q)$ contains $D_{\alpha_i}$.

**Claim 2.** $\varphi(S(Q))$ is diagonalizable over $k'$. Since $S(Q)$ is boundedly generated by $(D_{\alpha})_{\alpha \in \Pi}$ and $\varphi(D_{\alpha}) \leq \varphi(\psi_{\alpha}(SL_2(Q)))$, it follows that $\varphi(S(Q))$ is bounded (see [12, Section 2]). Let $\Omega \subseteq \Delta'$ denote a balanced subset which is fixed by $\varphi(S(Q))$. Let $\overline{S(Q)}$ be the Zariski closure of $Ad_\Omega(\varphi(S(Q)))$. As $S(Q)$ is commutative, so is $\overline{S(Q)}$. Note that $\overline{S(Q)}$ is connected as it is generated by connected subgroups.

By [19, 3.1.1], $S := \overline{S(Q)}$ is the direct product of its semisimple and its unipotent elements: $S = S_s \times S_u$. Since the abstract representation $\rho := Ad_\Omega \circ \varphi \circ \psi_{\alpha}$ actually is rational, it follows that the image of each $S_{S_u}(Q)$ consists of semisimple elements only, i.e. is contained in $S_s$.

In particular, $S$ is a torus since it is connected and contains semisimple elements only. Clearly, $S$ is defined over $k'$. It remains to be checked that $S$ is split over $k'$. Let $g \in S(Q)$ be of infinite order. Since $g$ is contained in a $k$-split torus, the Zariski closure $S_{g}$ of $(g)$ is again a $k$-split torus by [3, Proposition 1.9 b)]. By induction, $S/S_{g}$ is a $k$-split torus, from which the result again follows by [3, Proposition 1.9 b)]. This implies the claim.

**Claim 3.** $\varphi(S(Q))$ is regular diagonalizable. This is a direct consequence of the group theoretic characterization of regular diagonalizable subgroups, Lemma 5.3.

\[ \square \]

**Remark 5.8.** If $K$ is algebraically closed and $G$ is a split Kac–Moody group over $K$, it is even possible to exhibit finite regular diagonalizable groups which are mapped to regular diagonalizable subgroups, see [3]. Still in the split case over arbitrary fields, $T' := \ker(\alpha - \beta)$ for suitably chosen roots $\alpha, \beta$ is regular. In particular, the dimension of a regular diagonalizable subgroup can vary arbitrarily.

**Remark 5.9.** The assumption that $G(k)$, $G'(k')$ be $2$-spherical is essentially only used to produce a regular subgroup $S(Q) \leq G(k)$ which is again mapped to a regular diagonalizable subgroup in $G'(k')$.

Since maximal $k'$-split tori are conjugate under $G'(k')$ [17, Theorem 12.5.3], there exists some $x \in G'(k')$ such that $(\text{int } x \circ \varphi)(S(Q)) \leq T_d'(k')$. Replacing $\varphi$ by $\text{int } x \circ \varphi$ if necessary, we assume from now on that $\varphi(S(Q)) \leq T_d'(k')$.

**Proposition 5.10.** There are subgroups $S_1 \leq T_d(k)$ and $S_2 \leq T_d'(k')$ with the property that $T_d(k)/S_1, T_d'(k')/S_2$ both are finitely generated and such that $\varphi(S_1) \leq Z'(k'), \varphi^{-1}(S_2) \leq Z(k)$.
Proof. As $T_d(k)$ normalizes $S(Q)$, $T_d(k)$ acts via $\varphi$ on the fixed point set of $\varphi(S(Q))$, which is reduced to $A'$. Let $S_1$ denote the kernel of this action, then $\varphi(S_1) \leq Z^I(k')$ by definition of the anisotropic kernel as the fixator of $A'$. As $\varphi(T_d(k))/\varphi(S_1)$ is an abelian subgroup of $W'$, it is finitely generated by Lemma 5.9. 

Similarly, as $T_d'/(k'_1)$ normalizes $\varphi(S(Q))$, $T_d'/k'_1$ acts via $\varphi^{-1}$ on the fixed point set of $S(Q)$, which is reduced to $A$. Define $S_2$ as the kernel of this action, then $S_2$ is as required by similar arguments. 

The subgroups $S_1$ and $S_2$ should be thought of as “large” as they are Zariski dense in $T_d$ and $T_d'$ respectively, by Proposition 5.11. Moreover, $S_1$ and $\varphi^{-1}(S_2)$ both normalize each root group $U_\alpha \leq G$, while $\varphi(S_1)$ and $S_2$ both normalize each root group $V_\beta \leq G'$. 

The next step consists of showing that for certain unipotent elements $u \in U_\alpha \setminus \{1\}$, $\varphi(u) \leq L^J$ for a Levi factor of spherical type containing $Z^I(k')$.

**Definition 5.11.** Let $U_\alpha \leq G$ be a root group and let $u := \text{Lie}U_\alpha = g_\alpha \oplus g_{2\alpha}$. For an element $u \in U_\alpha(k)$ let $\log u \in u$ denote the unique element such that $\exp(\log u) = u$. Then $u \in U_\alpha$ is called pure if $\log u \in g_\alpha$ or $\log u \in g_{2\alpha}$.

Note that if $U_\alpha$ is abelian, each element $u \in U_\alpha \setminus \{1\}$ is pure.

**Lemma 5.12.** Let $u \in U_\alpha(k) \setminus \{1\}$ be a pure element. Then there exists a homomorphism $\psi_u : \text{SL}_2(Q) \to G(k)$ such that

a) $\psi_u((0 1 \quad 1 0)) = u$

b) $\text{im } \psi_u$ is normalized by $S(Q)$.

**Proof.** This follows from the proof of the Proposition 5.3 or from Theorem 4.3. More precisely, since $u$ is pure, the subalgebra $k \log u$ is invariant under $\text{Ad} T_d(k)$, i.e. there is a subgroup $Y_u \leq U_\alpha$ which contains $u$ and is isomorphic to $G_\alpha$. This isomorphism can be chosen to send $u$ to 1. By Theorem 4.3, $u$ is contained in a split group which contains $T_d \cdot Y_u$. Since $Q_u$ is invariant under $S(Q)$, the claim follows. 

**Proposition 5.13.** Let $u \in U_\alpha(k) \setminus \{1\}$ be a pure element. Then $\varphi(u)$ fixes two opposite points $x, y \in A'$, i.e. $\varphi(u) \in L^J$ for a Levi factor of finite type of $G'$ relative to $T_d'$.

**Proof.** Let $\psi_u : \text{SL}_2(Q) \to G(k)$ be as in the previous lemma. Then

$$\varphi \circ \psi_u : \text{SL}_2(Q) \to G'(k')$$

is a homomorphism whose images fixes two opposite points $x, y \in \Delta'$ by [12 Proposition 2.7]. As $\psi_u$ is normalized by $S(Q)$, both $x$ and $y$ must actually be contained in $A'$ by [12 Lemma 2.1] and the fact that $\varphi(S(Q))$ fixes only $A'$. 

It remains to prove that not only $\varphi(u) \in L^J$ but actually $\varphi(u) \in V_{\beta(u)}$ for some root $\beta(u)$ depending on $u$.

The following proposition uses the trick that a unipotent element $u$ is an element of the derived group of a solvable group $B_u$, a property which is clearly preserved by a group isomorphism. This idea goes back to [4].

**Proposition 5.14.** Let $u \in U_\alpha \setminus \{1\}$ be pure and let $J \subseteq S'$ be such that $\varphi(u) \in L^J$. Then

$$\text{cl}(u) := \langle c \varphi(u)c^{-1} : c \in S_2 \rangle \leq L^J$$

is a unipotent group defined over $k'$ and normalized by $T_d'$. 

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Proof. Let \( Y_u := \langle \varphi^{-1}(c) u \varphi^{-1}(c^{-1}) : c \in S_2 \rangle \). Then \( Y_u \leq U_n \) since \( \varphi^{-1}(S_2) \) normalizes \( U_n \). Moreover \( Y_u \) is contained in \( Y'_u := \langle \alpha Y_u s^{-1} : s \in S(Q) \rangle \).

By Proposition 5.14 there is a subset \( J \subseteq S' \) such that \( \varphi(u) \in L^J \). Since \( \varphi(S(Q)) \) and \( S_2 \) are subgroups of \( T_d(k') \) it follows that \( \varphi(Y'_u) \leq L^J \).

Let \( Qu := \exp(Q \cdot \log u) \). Then \( Qu \) is a group normalized by \( S(Q) \) by Proposition 5.12. The group \( B_u := S(Q) \cdot Qu \) is solvable and \( u \) is contained in every finite index subgroup of the derived group of \( B_u \). Indeed, since \( S(Q) \) acts on \( Qu \cdot u \) via a non-trivial character, for each \( n \in \mathbb{N} \) there is some \( s \in S(Q) \) such that \( \frac{1}{n} \cdot u \in ([s, u]) \).

As \( B_u \leq Y'_u, \varphi(B_u) \leq L^J \). Since \( B_u \) is solvable, so is \( \varphi(B_u) \). By the Lie-Kolchin theorem, \( \varphi(B_u) \) has a finite index subgroup which is triangularizable, and since \( u \) is contained in the derived group, it follows that \( \varphi(u) \) is unipotent.

Note that \( \varphi(S(Q)) \) and \( \varphi^{-1}(S_2) \) commute, i.e. \( Y'_u \) is normalized by both \( S(Q) \) and \( \varphi^{-1}(S_2) \). Arguing similarly as for \( u \), it follows that \( \varphi(Y'_u) \) is unipotent since it is (up to finite index) contained in the derived group of a solvable group.

Since \( \varphi(Y'_u) \leq G'(k') \), the Zariski closure \( cl(u) \) is defined over \( k' \), and \( cl(u) \) again is unipotent (cf. [10]). By definition, \( cl(u) \) is normalized by \( S_2 \) and hence by the Zariski closure of \( S_2 \), which is \( T_d \) by Lemma 5.1.

The following step is inspired by the proof of [20] Proposition 23], which in turn is inspired by classical results.

We recall first the definition of a nibbling sequence of roots.

**Definition 5.15.** Let \((W, S)\) be a Coxeter group and let \( \alpha_1, \ldots, \alpha_n \in \Phi(W, S) \) be such that \( \{\alpha_i, \alpha_j\} \) is prenilpotent for all \( i, j \in \{1, \ldots, n\} \). Then \((\alpha_1, \ldots, \alpha_n)\) is called a nibbling sequence of roots if for all \( i < j \), \((\alpha_i, \alpha_j) \subseteq (\alpha_i+1, \alpha_j-1)\).

**Proposition 5.16.** Let \((W, S)\) be a spherical Coxeter group and let \( \Psi \subseteq \Phi(W, S) \) be a nilpotent set of roots. Then the elements of \( \Psi \) can be arranged to form a nibbling sequence of roots.

**Proof.** See [17] Section 9.1.2.

**Theorem 5.17.** Let \( u \in U_n \setminus \{1\} \) be pure and \( J \subseteq S' \) spherical such that \( \varphi(u) \in L^J \). Then \( \varphi(u) \in V_\beta \) for some \( \beta \in \Phi' \).

**Proof.** By Proposition 5.14 and Proposition 5.4 it follows that

\[
\psi(u) \in V^J := \{ V_\beta : \beta \in \Phi(W'_J), \beta > 0 \}
\]

for a suitable ordering \( >' \) of the roots of \( \Phi(W'_J) \). Since \( \Phi(W'_J) \) is finite it follows from Proposition 5.16 that \( \beta_1, \ldots, \beta_k \) is a nibbling sequence. Then

\[
\varphi(u) = v_{i_1} \cdots v_{i_r}
\]

for certain \( v_{i_j} \in V_{\beta_{i_j}}, v_{i_j} \neq 1 \).

Assume for a contradiction that \( r > 1 \).

**Claim.** In this case, there are indices \( i \neq j \) and elements \( u_i, u_j \in U_n \setminus \{1\} \) such that \( \varphi(u_i) \in V_\beta_i \) and \( \varphi(u_j) \in V_{\beta_j} \).

Since \( W'_J \) is spherical, for any two roots \( \beta_i, \beta_j \in \Phi(W'_J) \) such that \( \beta_i \neq \pm \beta_j \), there is an element \( t_{ij} \in \{ V_\alpha : \alpha \in \Phi(W'_J) \} \cap S_2 \) such \( t_{ij} \) centralizes \( V_{\beta_j} \), but not \( V_{\beta_i} \). This follows from the fact that such an element exists in \( T_d(k') \) and the fact that \( S_2 \) is Zariski dense in \( T_d \).

Consider \( v_1 := [t_{1r}, \varphi(u)] \) and \( v_2 := [t_{r1}, \varphi(u)] \).

Then \( v_1, v_2 \in \varphi(U_n) \) since \( \varphi^{-1}(t_{ij}) \) normalizes \( U_n \). Furthermore, the support of \( v_1 \) contains \( \beta_1 \) but not \( \beta_r \). Likewise, the support of \( v_2 \) does not contain \( \beta_1 \) but contains \( \beta_r \).
By repeating the process for $s_1$ and $s_2$ inductively if necessary, the claim is proven.

Now take an element $s \in S_2$ of infinite order such that $s$ centralizes $V_{\beta_i}$ but not $V_{\beta_j}$ and such that $\varphi^{-1}(s) \in T_d(k)$. The existence of such an element can be proven by appealing to the $\mathbb{Q}$-points of a split subgroup of $G'(k')$ and the fact that $\beta_i, \beta_j$ are roots in a spherical Coxeter group. Then $\varphi^{-1}(s)$ centralizes $u_i$, since $\varphi(u_i) \in V_{\beta_i}$, so $\varphi^{-1}(s^2)$ centralizes $U_{\alpha}$ since $\varphi^{-1}(s) \in T_d(k)$. But this is a contradiction, since $\varphi^{-1}(s^2)$ does not centralize $u_j$, since $\varphi(u_j) \in V_{\beta_j}$.

**Corollary 5.18.** Let $U_{\alpha}$ be a root group. Then $\varphi(\mathcal{Z}(U_{\alpha})) \leq V_{\beta}$ for some $\beta \in \Phi'$.

**Proof.** By the preceding theorem, $\varphi(\mathcal{Z}(U_{\alpha}))$ is a group which satisfies the assumptions of Lemma 5.2 since each element $u \in \mathcal{Z}(U_{\alpha})\setminus\{1\}$ is pure, so the conclusion follows.

This corollary finishes the case where all root groups are abelian. Some more effort is required when there are metabelian root groups present. These technical problems are always present when one deals with metabelian root groups, see e.g. [10] or [4].

The following lemma is inspired by the proof of [3] Theorem 2.2.

**Lemma 5.19.** Let $u \in U_{\alpha}\setminus\{1\}$ be a pure element and let $\beta \in \Phi'$ be such that $\varphi(u) \in V_{\beta}$. Then the elements $u', u'' \in U_{-\alpha}$ such that $m(u) = u'uu''$ satisfy $\varphi(u'), \varphi(u'') \in V_{-\beta}$.

**Proof.** From Lemma 5.12 it follows that $u' = u''$ and that $u'$ is pure. Let $\gamma \in \Phi'$ be such that $\varphi(u') \in V_{\gamma}$. It is clear that $\varphi(Qu) \leq V_{\beta}$ and that $\varphi(Qu') \leq V_{\gamma}$. This induces a homomorphism $\psi : SL_2(\mathbb{Q}) \to V_{\beta\gamma} := \langle V_{\beta}, V_{\gamma} \rangle$. Suppose that $\beta \neq -\gamma$. If \{\beta, \gamma\} is a prenilpotent set of roots, $V_{\beta\gamma}$ is nilpotent since each root group $V_{\alpha}$ is nilpotent, which is a contradiction since $\psi$ is nontrivial. If \{\beta, \gamma\} is not prenilpotent, the free product $V_{\beta} * V_{\gamma}$ embeds in $G'(k')$, which is a contradiction since a conjugate of $u$ in $SL_2(\mathbb{Q})$ commutes with $u'$, while this is not the case for $\varphi(u) \in V_{\beta}$ and $\varphi(u') \in V_{\gamma}$.

**Proposition 5.20.** Suppose that $U_{\alpha}$ is metabelian. Then $\varphi(U_{\alpha}) \leq V_{\beta}$ for some $\beta \in \Phi'$.

**Proof.** Let $u_1, \ldots, u_r \in U_{\alpha}$ be pure such that $\log u_1, \ldots, \log u_r$ form a basis for $\frak{g}_{\alpha}$. Let $U_i := k \cdot u_i$ and let $U_0 := \mathcal{Z}(U_{\alpha})$. Let $\gamma_0, \ldots, \gamma_r \in \Phi'$ be such that $\varphi(U_i) \leq V_{\gamma_i}$. These clearly exist, as each $U_i$ is a subgroup of $U_{\alpha}$ consisting of pure elements.

Suppose that there are $i, j$ such that $\gamma_i \neq \gamma_j$. If $w := s_{\gamma_i}s_{\gamma_j}$ has finite order, $\gamma_i$ and $\gamma_j$ are roots in a Levi factor $L'$. Then $U_{ij} := \langle U_i, U_j \rangle$ is mapped to a unipotent subgroup of $L'$ by arguments similar to those in the proof of Proposition 5.14. Arguing as in the proof of Theorem 5.14, this yields a contradiction as then there would exist a torus element $t \in T_d(k')$ such that $\varphi^{-1}(t)$ centralizes $U_i$ but not $U_j$. It follows that $w$ has infinite order. Note that $\varphi(U_{\alpha})$ is contained in the set $V' := V_{\gamma_1} \cdots V_{\gamma_r} \cdot V_{\gamma_0}$, in particular, $\varphi(U_{\alpha})$ is bounded.

Let $m_i, m_j \in G'(k')$ be such that $m_i, m_j$ stabilize $\mathcal{A}'$, act on it via $s_{\gamma_i}, s_{\gamma_j}$, and such that $\varphi^{-1}(m_i), \varphi^{-1}(m_j)$ stabilize $\mathcal{A}$. These elements can be shown to exist via e.g. invoking a split subgroup of $G'(k')$.

From the previous proposition it follows that $\varphi^{-1}(m_i), \varphi^{-1}(m_j)$ map $U_{\alpha}$ to $U_{-\alpha}$. For $t := m_im_j$ it follows that $\varphi^{-1}(t)$ normalizes $U_{\alpha}$.
Then for each \( r \in \mathbb{Z} \) there exists some \( u_r \in U_\alpha \) such that \( \varphi(u_r) \in V_{\alpha r} \). This is the desired contradiction, as this implies that \( \varphi(U_\alpha) \) is unbounded. \( \square \)

To sum up: For each \( \alpha \in \Phi \) there is a root \( i(\alpha) \in \Phi' \) such that \( \varphi(U_\alpha) \leq V_{i(\alpha)} \).

Arguing likewise for \( \varphi^{-1} \) (note that the corresponding twin apartments \( \mathcal{A}, \mathcal{A}' \) are already aligned in the right fashion) we find that for each \( \beta \in \Phi' \) there is a \( j(\beta) \in \Phi \) such that \( \varphi^{-1}(V_\beta) \leq U_{j(\beta)} \). From the inclusion
\[
U_\alpha = \varphi^{-1}(\varphi(U_\alpha)) \leq \varphi^{-1}(V_{i(\alpha)}) \leq U_{j(i(\alpha))}
\]
and the fact that \( U_\alpha \neq 1, U_\alpha \cap U_\beta = 1 \) for \( \alpha \neq \beta \), it finally follows that \( i \) and \( j \) are inverse bijections and that equality holds all along.

This discussion can be succinctly summed up by saying that any isomorphism \( \varphi : G(k) \to G'(k') \) is \textbf{standard}, cf. Definition 2.3. We have shown:

**Theorem 5.21.** Let \( G = G(k), G' = G'(k') \) be two 2-spherical almost split Kac–Moody groups over fields \( k, k' \) of characteristic 0. Let \((Z(k), (U_\alpha)_{\alpha \in \Phi(W,S)})\) and \((Z'(k'), (V_{\alpha})_{\beta \in \Phi(W',S')})\) denote the associated canonical twin root data. Suppose that \( \varphi : G(k) \to G'(k') \) is an abstract isomorphism. Then \( \varphi \) is standard.

**Proof.** By the previous discussion, there exists \( x \in G'(k') \) such that \( \varphi' := \int x \circ \varphi \) induces a bijection of the root groups. Note that \( \int x \) can be chosen to be trivial if, with the notation from above, \( \varphi(S(\mathbb{Q})) \) already fixes \( \mathcal{A}' \). Since
\[
Z(k) = \bigcap_{\alpha \in \Phi(W,S)} N_{G(k)}(U_\alpha), \quad Z'(k') = \bigcap_{\beta \in \Phi(W',S')} N_{G'(k')}(V_{\beta})
\]
there is actually an equality, not just an inclusion, see [17, Proposition 1.5.3]), it follows that \( \varphi'(Z(k)) = Z'(k') \.
\( \square \)

This proves Theorem 1.1 from the Introduction.

**Remark 5.22.** Since clearly \( \mathcal{Z}(U_\alpha) \) is mapped to \( \mathcal{Z}(V_{\beta}) \), it also follows that \( \varphi \) induces an automorphism of the refined root datum as given by Remy [17, Theorem 12.6.3].

**Corollary 5.23.** Let \( k, k' \) be two fields of characteristic 0 and let \( G(k), G'(k') \) be two 2-spherical almost split Kac–Moody groups. Let \( \varphi : G(k) \to G'(k') \) be an isomorphism rectified in such a fashion that \( \varphi \) maps root groups with respect to \( T_0(k) \) to root groups with respect to \( T_0(k') \). Let \( X_\alpha(k) := Z(k)(U_\alpha, U_{-\alpha}) \) and \( Y_{\beta}(k') := Z'(k')(V_{\beta}, V_{-\beta}) \) be two rank 1 groups such that \( \varphi(X_\alpha) = Y_{\beta} \).

Suppose that the derived groups of both \( X_\alpha \) and \( Y_{\beta} \) are absolutely almost simple and that either \( X_\alpha \) is simply connected or that \( Y_{\beta} \) is adjoint. Then there is a field isomorphism \( \sigma : k \to k' \), a rational map \( r : X_\alpha \to Y_{\beta} \) and a map \( c : X_\alpha(k) \to \mathcal{Z}(Y_{\beta}(k')) \) such that for \( x \in X_\alpha \), \( \varphi(x) = c(x) \cdot (r \circ \sigma(x)) \).

**Proof.** The assumption are made as to conform to the assumptions of Borel–Tits’s classical theorem [4, Theorem A], from which the claim follows. \( \square \)

**References**

[1] Peter Abramenko and Kenneth S. Brown. Buildings, volume 248 of Graduate Texts in Mathematics. Springer, New York, 2008.

[2] Armand Borel. Linear algebraic groups, volume 126 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.
