Bipartite qutrit local realist inequalities and the robustness of their quantum mechanical violation

Debarshi Das and Dipankar Home
Centre for Astroparticle Physics and Space Science (CAPSS), Bose Institute, Block EN, Sector V, Salt Lake, Kolkata 700 091, India

Shounak Datta, Suchetana Goswami, and A. S. Majumdar
S. N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata 700 098, India

Distinct from the type of local realist inequality (known as the CGLMP inequality) usually used for bipartite qutrit systems, we formulate a new set of local realist inequalities for bipartite qutrits by generalizing Wigner’s argument that was originally formulated for the bipartite qubit singlet state. This treatment assumes existence of the overall joint probability distributions in the underlying stochastic hidden variable space for the measurement outcomes pertaining to the relevant trichotomic observables, satisfying the locality condition and yielding the measurable marginal probabilities. Such generalized Wigner inequalities (GWI) do not reduce to Bell-CHSH type inequalities by clubbing any two outcomes, and are violated by quantum mechanics (QM) for both the bipartite qutrit isotropic and singlet states using trichotomic observables defined by six-port beam splitter as well as by the spin-1 component observables. The efficacy of GWI is then probed in these cases by comparing the QM violation of GWI with that obtained for the CGLMP inequality. This comparison is done by incorporating white noise in the singlet and isotropic qutrit states. It is found that for the six-port beam splitter observables, QM violation of GWI is more robust than that of the CGLMP inequality for singlet qutrit states, while for isotropic qutrit states, QM violation of the CGLMP inequality is more robust. On the other hand, for the spin-1 component observables, QM violation of GWI is more robust for both the type of states considered.

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I. INTRODUCTION

The foundational tenets and concepts of quantum mechanics (QM) significantly differ from classical ideas and intuitions. A seminal contribution to quantum concepts was provided by demonstrating quantum nonlocality through Bell’s inequality [1, 2] used for showing an incompatibility between quantum mechanics (QM) and the notion of local realism underpinning Bell’s inequality. Soon after the discovery of Bell’s inequality, a different formulation of local realist inequality was provided by Wigner [3]. This was based upon the assumption of the existence of joint probability distributions in the underlying stochastic hidden variable (HV) space pertaining to the occurrence of different possible combinations of outcomes for the measurements of the relevant observables, and these joint probability distributions are taken to yield all the observable marginal probabilities by satisfying the locality condition. However, Wigner’s original formulation was restricted in showing the QM incompatibility with local realism for the bipartite qubit singlet states.

Subsequently, among the few studies using Wigner’s approach are its use in the case of entangled neutral kaons [4, 5], and a study of its implication for quantum key distribution [6]. Only recently, Wigner’s formalism has been generalized for N-partite qubit states by deriving generalized Wigner inequalities (GWI) [7], and in another recent work, the temporal version of GWI, namely, Wigner’s form of the Leggett-Garg inequality has been derived [8]. Apart from these investigations, surprisingly, Wigner’s approach has remained largely unexplored.

Against this backdrop, the motivation underlying the present paper is to extend the significance of Wigner’s approach in the context of bipartite qutrit systems by developing a framework for local realist inequalities based on the assumption of the existence of joint probability distributions. Here it needs to be mentioned that investigations related to QM violations of local realist inequalities for arbitrary dimensional systems have steadily acquired much interest over the years [9–20]. In this context, we should also recall that qutrit systems are of special interest due to their experimental relevance in the areas of atomic and laser physics, as well as because of a number of foundational and information theoretic applications of qutrit systems [21–31].

For the purpose of probing quantum nonlocality of bipartite qutrit systems, particularly noteworthy is the QM incompatibility with local realism for bipartite
qutrit isotropic states as studied by Collins et. al.\textsuperscript{32} using the local realist inequality derived by them (known as the CGLMP inequality). While experimental violation of the CGLMP inequality has been demonstrated for non-maximally entangled states of bipartite qutrits\textsuperscript{32}, here it needs to be mentioned that isotropic and singlet qutrit states are regarded to be particularly relevant in quantum information processing\textsuperscript{34}. Hence in this paper, these states are used for studying the QM violation of the derived forms of GWI in the context of the six-port beam splitter and spin-1 component observables. Note that the forms of GWI for bipartite qutrits derived in this paper do not reduce to Bell-CHSH (Bell-Clauser-Horne-Shimony-Holt) type inequalities by clubbing any two outcomes.

An important point to stress is that the efficacy of any local realist inequality for demonstrating its incompatibility with QM is restricted in practical situations that are usually far from ideal. Hence the robustness of the QM violation of any local realist inequality in the presence of white noise in a given state is a key issue. The present paper provides a comparative study of the robustness of the QM violation of both the GWI and CGLMP inequality in the presence of white noise incorporated in the qutrit states considered. Results obtained in this paper demonstrate that for six-port beam splitter observables, the QM violation of CGLMP inequality is more robust against white noise for bipartite qutrit isotropic states than that obtained by using GWI. On the other hand, for bipartite qutrit singlet states, the QM violation of GWI is more robust against white noise than that pertinent to the CGLMP inequality. The corresponding calculations are also done for the spin-1 component observables. It is found that for both these types of states, the QM violation of GWI is more robust against white noise than that of the CGLMP inequality.

Interestingly, it may happen that the maximum QM violation of a local realist inequality is not obtained for maximally entangled qutrit state. In order to probe this, the maximum QM violation of GWI and the corresponding robustness against white noise present in the state has been calculated. It has been found that the maximum QM violation of GWI occurs for non-maximally asymmetric entangled qutrit state if one uses six-port beam splitter or spin-1 component observables. The maximum robustness of the QM violation of GWI against white noise present in a state is also compared with that of CGLMP inequality\textsuperscript{32}.

The plan of this paper is as follows. In the next Section II we briefly outline the original derivation by Wigner applicable for bipartite qubit singlet state. Then, in Section III, we present the derivation of GWI for bipartite qutrit systems, followed by Sections IV and V where it is shown that the derived GWI is violated by isotropic and singlet qutrit states using six-port beam splitter and spin-1 component observables respectively. In Sections VI and VIII, contingent upon using six-port beam splitter and spin-1 observables respectively, we compare the robustness of QM violations of GWI with that of CGLMP inequality for the case of isotropic and singlet qutrit states. We consider the introduction of white noise to the pure states considered in order to perform the comparative study of robustness of GWI and CGLMP inequality corresponding to the above mentioned two categories of entangled qutrit states. In Sections VII and IX we have shown the maximum QM violations of GWI using six-port beam splitter and spin-1 component observables respectively, and the corresponding maximum robustness of the QM violations of GWI against white noise present in the states. Section X contains a summary of the results obtained in this paper and we make some concluding remarks.

### II. RECAPITULATING WIGNER’S ORIGINAL DERIVATION

In the scenario considered by Wigner\textsuperscript{8}, two spin-1/2 particles are prepared in a singlet state and are then spatially separated. The spin components of the particles, respectively, are measured along three directions, say, $a, b$ and $c$. Then, in this context, considering the individual outcomes ($\pm 1$) of nine possible pairs of measurements, Wigner’s original inequality can be derived as follows.

Assuming the locality condition and an underlying stochastic HV distribution corresponding to a quantum state specified by a wave function, one can infer in the HV space, according to the reality condition, the existence of overall joint probabilities for the individual outcomes of measuring the pertinent observables, from which the observable marginal probabilities can be obtained. Thus, corresponding to an underlying stochastic HV, say $\lambda$, one can define $p_{\lambda}(v_1(a), v_1(b), v_1(c); v_2(a), v_2(b), v_2(c))$ as the overall joint probability of occurrence of the outcomes, where $v_1(a)$ represents an outcome ($\pm 1$) of the measurement of the observable $a$ for the first particle, and so on. For example, $p_{\lambda}(-, +, -; +, +, -)$ expresses the overall joint probability of occurrence of the outcomes $v_1(a) = -1, v_1(b) = +1, v_1(c) = -1$ for the first particle, and $v_2(a) = +1, v_2(b) = +1, v_2(c) = -1$ for the second particle. Then, the joint probability, say, $v_1(a) = +1$ and $v_2(b) = +1$ for the first and the second particle respectively can be written, using the perfect anti-correlation property of the singlet state, as $p_{\lambda}(a+, b+) = p_{\lambda}(+, +; -, +, -) + p_{\lambda}(+, -; +, -, +)$. Similarly, writing $p_{\lambda}(c+, b+)$ and $p_{\lambda}(a+, c+)$ as marginals, and assuming non-negativity of the overall joint probability distributions in the HV space, it can be shown that

\begin{equation}
  p_{\lambda}(a+, b+) \leq p_{\lambda}(a+, c+) + p_{\lambda}(c+, b+) \tag{1}
\end{equation}
Subsequently, by integrating over the hidden variable space for an arbitrary distribution, one can obtain the original form of Wigner’s inequality

\[ p(a+, b+) \leq p(a+, c+) + p(c+, b+) \]  

where \( p(a+, b+) \) is the observable joint probability of getting +1 for both the outcomes if the observables \( a \) and \( b \) are measured on the first and the second particle respectively, and so on.

If the respective angles between \( a \) and \( b \), and \( c \), \( b \) and \( c \) are \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \), then substituting the QM expressions for the relevant joint probabilities in the inequality given by Eq. (2), one obtains

\[ \frac{1}{2} \sin^2(\theta_{12}/2) \leq \frac{1}{2} \sin^2(\theta_{13}/2) + \frac{1}{2} \sin^2(\theta_{23}/2) \]  

- a relation which is not valid for arbitrary values of \( \theta_{12}, \theta_{13}, \theta_{23} \).

This shows an incompatibility between QM and Wigner’s form of inequality given by Eq. (2), restricted for the singlet state in the bipartite case. Note that, the above argument is within the framework of stochastic HV theory, subject to the locality condition, and the notion of determinism has not been used here.

III. GENERALIZED WIGNER INEQUALITIES FOR BIPARTITE QUTRIT SYSTEMS

Now, in order to generalise the above argument for deriving GWI for arbitrary bipartite qutrit systems, we proceed as follows. Note that in the following derivation we are not using the assumption of perfect anti-correlation embodied in the singlet states that was used in Wigner’s original derivation. Let us consider that pairs of trichotomic observables \( a \) or \( a' \) and \( b \) or \( b' \) are measured on the first and the second particle respectively.

We assume an underlying HV distribution given by \( p(\lambda) \) such that for \( 3^4 \) possible combinations of pairs of outcomes, each such pair of outcomes occurs with a certain probability in the HV space. Thus, corresponding to an underlying stochastic HV, say \( \lambda \), one can define \( p_\lambda(v_1(a), v_1(a'); v_2(b), v_2(b')) \) as overall joint probability of occurrence of the outcomes, where \( v_1(a) \) represents an outcome \((\pm 1, 0, \text{ or } -1)\) of the measurement of the observable \( a \) for the first particle, and so on. For example, \( p_\lambda(+, 0; -, +) \) expresses the overall joint probability of occurrence of the outcomes \( v_1(a) = +1 \) and \( v_1(a') = 0 \) for the first particle and \( v_2(b) = -1 \) and \( v_2(b') = +1 \) for the second particle. Then, consistent with the locality condition, the joint probability of, say \( v_1(a) = 0 \) and \( v_2(b) = - \) for the first and second particle, respectively, can be obtained as a marginal of the overall joint probabilities in the HV space, given by the following expression

\[
p_\lambda(a_0, b) = \sum_{v_1(a') = +, 0, -} \sum_{v_2(b') = +, 0, -} p_\lambda(0, v_1(a'); -, v_2(b'))
\]

\[
= p_\lambda(0, +; -, +) + p_\lambda(0, +; -, 0) + p_\lambda(0, +; -, -) + p_\lambda(0, 0; -, +) + p_\lambda(0, 0; -, 0) + p_\lambda(0, 0; -, -)
\]

\[
+ p_\lambda(0, -; -, +) + p_\lambda(0, -; -, 0) + p_\lambda(0, -; -, -)
\]

Similarly, writing \( p_\lambda(a_0, b_0), p_\lambda(a_0, b_0), p_\lambda(a_0, b_0), p_\lambda(a_0, b_0) \) as marginals, and assuming non-negativity of the overall joint probability distributions in the HV space, it can be shown that

\[
p_\lambda(a_0, b_0) - p_\lambda(a_0, b_0) - p_\lambda(a_0, b_0) - p_\lambda(a_0, b_0) \leq 0
\]

Similarly, integrating over the HV space for an arbitrary distribution, one can obtain the following form of GWI for bipartite qutrit systems:

\[
p(a_0, b_0) - p(a_0, b_0) - p(a_0, b_0) - p(a_0, b_0) \leq 0
\]

Similarly, other following forms of GWI for bipartite qutrit systems can be obtained:

\[
p(a_0, b_0) - p(a_0, b_0) - p(a_0, b_0) - p(a_0, b_0) \leq 0
\]
and all off-diagonal terms being equal to zero. These unitary operations are denoted by $U(\vec{\phi}_a)$, where $\vec{\phi}_a \equiv [\phi_a(0), \phi_a(1), \phi_a(2)]$ for the first particle and $U(\vec{\phi}_b)$, where $\vec{\phi}_b \equiv [\phi_b(0), \phi_b(1), \phi_b(2)]$ for the second particle. The freedom of choice of the measurement of both the particles is given by this unitary transformation. Then, a discrete Fourier transformation $U_{FT}$ is carried out on the first particle and $U^*_{FT}$ is carried out on the second particle. The matrix element of the discrete Fourier transformation is given by, $(U_{FT})_{jk} = \exp((j - 1)(k - 1)i2\pi/3)$ and finally measurement is done in the basis in which the initial shared state is prepared. Here the observables $a$, $a'$, $b$ and $b'$ denote unitary transformations $U(\vec{\phi}_a)$, $U(\vec{\phi}_a')$, $U(\vec{\phi}_b)$ and $U(\vec{\phi}_b')$ respectively, where $\vec{\phi}_a \equiv [\phi_a(0), \phi_a(1), \phi_a(2)]$, $\vec{\phi}_a' \equiv [\phi_a'(0), \phi_a'(1), \phi_a'(2)]$, $\vec{\phi}_b \equiv [\phi_b(0), \phi_b(1), \phi_b(2)]$ and $\vec{\phi}_b' \equiv [\phi_b'(0), \phi_b'(1), \phi_b'(2)]$.

A. QM violation of GWI for bipartite qutrit isotropic state using six-port beam splitter

Let us consider the pure isotropic qutrit state given by

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \quad (17)$$

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ are three mutually orthonormal states. In the case of six-port beam splitter, each of these states defines the state of photon passing through one of the three input ports or one of the three output ports of the six-port beam splitter. On the other hand, in the case of spin-1 component observables, $|0\rangle$, $|1\rangle$ and $|2\rangle$ are the eigenstates of spin angular momentum operator along z-direction corresponding to the eigenvalues $+1$, $0$ and $-1$ respectively (assuming $\hbar = 1$).

If measurements defined by the six-port beam splitter are performed on two particles of the state given by Eq. (17), the left hand side of the GWI given by Eq. (5) becomes

$$p(a_0, b_0) - p(a_0', b_0) - p(a_0, b_0') + p(a_0', b_0') \leq 0 \quad (12)$$
Let us assume, the observables \( a, a', b, b' \) denote spin-1 components in arbitrary directions. The matrix corresponding to the spin component in an arbitrary direction \( \hat{n} \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) for spin-1 system can be written as:

\[
\hat{n} \cdot \hat{S} = \hbar \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \sqrt{\frac{2}{3}} & 0 \\
\sin \theta e^{i\phi} \sqrt{\frac{2}{3}} & 0 & \sin \theta e^{-i\phi} \sqrt{\frac{2}{3}} \\
0 & \sin \theta e^{i\phi} \sqrt{\frac{2}{3}} & -\cos \theta
\end{pmatrix}
\]

which has three eigenvectors

\[
\begin{pmatrix}
\frac{1}{2}(1 + \cos \theta) e^{-i\phi} \\
\frac{1}{2}(1 - \cos \theta) e^{i\phi} \\
\frac{1}{2}(1 + \cos \theta) e^{i\phi}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\frac{1}{2}(1 - \cos \theta) e^{-i\phi} \\
\frac{1}{2}(1 - \cos \theta) e^{i\phi} \\
\frac{1}{2}(1 + \cos \theta) e^{-i\phi}
\end{pmatrix}
\]

corresponding to eigenvalues \( +1, 0, -1 \) respectively (assuming \( \hbar = 1 \)). Here \( \theta \) is the polar angle and \( \phi \) is the azimuthal angle. \( a \) and \( a' \) denote measurements of spin component of the first particle in the directions \( (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1) \) and \( (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2) \) respectively. Similarly, \( b \) and \( b' \) denote measurements of spin component of the second particle in the directions \( (\sin \theta_3 \cos \phi_3, \sin \theta_3 \sin \phi_3, \cos \theta_3) \) and \( (\sin \theta_4 \cos \phi_4, \sin \theta_4 \sin \phi_4, \cos \theta_4) \) respectively.

### V. QM VIOLATIONS OF GWI BY BIPARTITE QUTRIT ISOTROPIC AND SINGLET STATES USING SPIN-1 COMPONENT OBSERVABLES

If the measurements of spin-1 components in arbitrary directions are performed on the isotropic state \(|12\rangle\), the left hand side of the GWI given by Eq. (5) becomes
From numerical calculations it has been observed that for the set of measurement settings $(\theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3; \theta_4, \phi_4) = (1.09, 2.75; 0.02, 3.51; 0.52, 5.86; 0.56, 2.77)$ in radians, the maximum QM violation of the GWI occurs, and the magnitude of this maximum violation is found to be 0.12077. Similarly, one can demonstrate QM violation of the other GWIs as well by choosing appropriate settings in each case.

\section*{B. QM violation of GWI for bipartite qutrit singlet state using spin-1 component observables}

Similar to the way discussed above, it can be shown that if the trichotomic measurements using spin-1 component observables are performed on the $3 \otimes 3$-dimensional pure singlet state given by Eq.\textsuperscript{(1)}\textsuperscript{11}, the left hand side of the GWI given by Eq.\textsuperscript{(1)}\textsuperscript{6} has the maximum value 0.12077 for the measurement settings $(a; a'; b; b') \equiv (\theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3; \theta_4, \phi_4) = (1.09, 2.75; 0.02, 3.51; 0.52, 5.86; 0.56, 2.77)$ (in radian) corresponding to the maximum QM violation of the GWI. Numerical calculations show that GWI mentioned in Eq.\textsuperscript{(1)}\textsuperscript{6} gives the maximum QM violation for both bipartite qutrit isotropic state and bipartite qutrit singlet state among all the GWIs derived in this paper. Henceforth, we would, therefore, consider only the GWI given by Eq.\textsuperscript{(1)}\textsuperscript{6} in case of spin-1 component observables.

\section*{VI. COMPARISON OF GWI WITH THE CGLMP INEQUALITY FOR BIPARTITE QUTRITS CONTINGENT UPON USING SIX-PORT BEAM SPLITTER}

In order to show the efficacy of GWI derived here, we will now make a comparative analysis of the QM violation obtained through GWI with that obtained by using the CGLMP inequality for bipartite qutrits using six-port beam splitters.

The CGLMP inequality \textsuperscript{32} is derived based on a constraint that the correlations exhibited by a local realist theory must satisfy. This inequality has not been derived from the assumption of existence of a joint probability distribution (JPD) in the HV space. The CGLMP inequality for bipartite 3-dimensional system has the following form

\[ I_3 = |P(a = b) + P(b = a' + 1) + P(a' = b') + P(b' = a)| \]

\[ -|P(a = b - 1) + P(b = a') + P(a' = b' - 1) + P(b' = a - 1)| \leq 2 \] \textsuperscript{22}

The QM violation of the CGLMP inequality is quantified by \((I_3 - 2)\).

Here it may be noted that Wu et. al. had suggested another local realist inequality \textsuperscript{12} which is derived based on the assumption of a local HV model satisfying the factorizability condition, using a few algebraic theorems and basic concepts of probability theory. This inequality has the following form

\[ S = P(a + b) - P(a + b') + P(a' + b') + P(a'0, b0) \]

\[ + P(a'0, b) + P(a'0, b0) + P(a'0, b0) \leq 1 \] \textsuperscript{23}

However, an important point is that this inequality \textsuperscript{23} reduces to just a version of CHSH (Clauser-Horne-Shimony-Holt) inequality \textsuperscript{2} after one has grouped the outcomes “0” and “-” so that the inequality becomes a two outcome (“+” and “not +”) inequality. Now, it is well known that in a \(2 \times 2 \times 2\) experiment (2 parties, 2 measurement settings per party, 2 outcomes per setting), all generalised Bell inequalities are simply re-writings of the CHSH inequality, obtained by linear combinations of the CHSH inequality with the appropriate normalisation conditions. Thus, the inequality \textsuperscript{23} is equivalent to CHSH inequality. We will, therefore, not consider this inequality for probing efficacy of GWI for bipartite qutrits.
QM violations of CGLMP inequality for the bipartite qutrit isotropic and singlet states respectively using six-port beam splitter. If the left hand side of CGLMP inequality (22) is evaluated for isotropic state in terms of the four aforementioned trichotomic observables $a, a', b, b'$ denoting observables using six-port beam splitters, then the inequality is maximally violated for the choice of measurement settings $(\phi_a(0), \phi_a(1), \phi_w(2), \phi_w(1), \phi_w(2), \phi_w(1)) = (0.40, 0.64, 0.24, 0.22, -0.58, 1.11, -0.14, -0.90, 0.54, -0.08, 0.20, -0.45)$ (in radian), and the magnitude of the maximum violation is given by 0.87293 [32, 37]. On the other hand, for singlet states given by Eq. (19), CGLMP inequality (22) is not violated for arbitrary choice of measurement settings.

In order to probe the efficacy of the derived GWI, we now compare the tolerances of GWI and CGLMP inequality against white noise present in a state pertaining to measurement of observables using six-port beam splitters. For this, let us introduce the visibility parameter pertaining to a state. Consider the bipartite qutrit mixed state given by,

$$
\rho = p|\psi\rangle \langle \psi| + (1 - p) \frac{I_3 \otimes I_3}{3^2}
$$

(24)

where, $p$ is the visibility parameter which changes the pure state $|\psi\rangle$ into a mixed state $\rho$ and $(1 - p)$ denotes the amount of white noise present in the state $|\psi\rangle$ (Here we take $|\psi\rangle$ to be either the isotropic state (17) or the singlet state (19)). $p = 0$ denotes the maximally mixed separable state.

Now, we first consider Eq. (24) by taking $|\psi\rangle$ as the isotropic state given by Eq. (17), and compute respectively the left hand side of the various local realist inequalities for the pure state $|\psi\rangle$ pertaining to measurement using six-port beam splitters. Subsequently, we repeat the computation by only taking the white noise part of Eq. (24). After applying appropriate weightage using the visibility parameter, we obtain the various expressions of the left hand sides of the local realist inequalities corresponding to the mixed state $\rho$ mentioned in Eq. (24) in terms of the parameter $p$. The same procedure is followed for the case of the singlet state (19). The minimum values of $p$ for which QM violates local realist inequalities signify the maximum amounts of white noise that can be present in the given state for the persistence of the QM violation of the relevant local realist inequality, and this value of $p$ is known as the threshold visibility pertaining to the given local realist inequality. Here the threshold visibilities are calculated using measurements of six-port beam splitters as mentioned before. In the Tables (I) and (II), the threshold visibilities of GWI and CGLMP inequality pertaining to the bipartite qutrit isotropic and singlet states respectively are shown.

### Table I. Minimum value of the visibility parameter for which QM violates GWI and CGLMP inequality respectively for the bipartite qutrit isotropic state using six-port beam splitter.

| Local realist inequality | Threshold visibility |
|--------------------------|----------------------|
| GWI given by Eq. (5)     | 0.774                |
| CGLMP inequality         | 0.696                |

### Table II. Minimum value of the visibility parameter for which QM violates GWI and CGLMP inequality respectively for the bipartite qutrit singlet state using six-port beam splitter.

| Local realist inequality | Threshold visibility |
|--------------------------|----------------------|
| GWI given by Eq. (5)     | 0.774                |
| CGLMP inequality         |         |

The above mentioned Tables clearly show that for the six-port beam splitter case, the CGLMP inequality given by Eq. (24) is more robust than GWI for the persistence of the QM violation in the presence of white noise incorporated in qutrit isotropic states. On the other hand, GWI given by Eq. (24) is more robust than the CGLMP inequality given by Eq. (22) for the persistence of the QM violation in the presence of white noise incorporated in qutrit singlet states. Moreover, CGLMP inequality is not violated at all by QM for qutrit singlet states using six-port beam splitter.

### VII. MAXIMAL VIOLATIONS OF GWI AND CGLMP INEQUALITY CONTINGENT UPON USING SIX-PORT BEAM SPLITTER

It may happen that the maximum violation of a local realist inequality is not obtained for maximally entangled states, like singlet states or isotropic states, but is obtained rather for non-maximally entangled states. One can derive the Bell operator corresponding to a local realist inequality, when observable using six-port beam splitter is measured. Any typical joint probability, say, $P(a = +, b = -)$ of obtaining outcomes $+$ and $-$ respectively, when the observable $a$ is measured on the first particle and the observable $b$ in measured on the second particle, and the initial state is $|\psi\rangle \in \mathbb{C}^n$, is given by,

$$
P(a = +, b = -) = \langle \psi | \{V(\phi_a^*) \dagger \otimes V(\phi_b^*)\} | \psi \rangle
$$

(25)

where $V(\phi_a^*) = U_{PT} U(\phi_a^*)$ and $V(\phi_b^*) = U_{PT} U(\phi_b^*)$. Similarly, evaluating other joint probabilities, the left
hand side of any local realist inequality can be expressed for the initial state \( |\psi\rangle \in \mathbb{C}^n \) as \( \langle \psi | B | \psi \rangle \), where \( B \) is the Bell operator associated with the respective local realist inequality for bipartite qutrits corresponding to using six-port beam splitter. \( B \) is a \( 3 \otimes 3 \) Hermitian Matrix. Now, for the purpose of finding the maximum eigenvalue of the Bell operator associated with a particular local realist inequality, we use the Min-Max Theorem of functional analysis and linear algebra. According to Min-Max theorem, the largest and smallest eigenvalues of a Hermitian matrix \( A \in \mathbb{C}^{n \otimes n} \) can be found as, \( \lambda_{\text{max}} = \max_{x, x \neq 0} \frac{\langle x | A | x \rangle}{\langle x | x \rangle} \) and \( \lambda_{\text{min}} = \min_{x, x \neq 0} \frac{\langle x | A | x \rangle}{\langle x | x \rangle} \) respectively.

Using the above mentioned procedure it is found that, contingent upon using six-port beam splitter, the maximum QM violation of GWI given by Eq.(5) is 0.20711, which is larger than the maximum QM violations of GWI given by Eq.(5) for the state given by Eq.(26) is 0.682.

The maximum QM violation of CGLMP inequality given by Eq.(22) is 0.9149 \( \sqrt{3} \), which is a bit larger than the maximum QM violations of CGLMP inequality for bipartite qutrit isotropic and much larger than that of qutrit singlet states. Its corresponding eigenvector is a non-maximally entangled state of two qutrits, which has the following form

\[
|\psi_{\text{cglmp}}\rangle = \frac{1}{\sqrt{2 + (0.792)^2}} (|00\rangle + (0.792)|11\rangle + |22\rangle)
\]

The threshold visibility of CGLMP inequality given by Eq.(22) for the state given by Eq.(27) is 0.686.

VIII. COMPARISON OF GWI WITH THE CGLMP INEQUALITY FOR BIPARTITE QUTRITS CONTINGENT UPON USING SPIN-1 COMPONENT OBSERVABLES

Now, again in order to show the efficacy of the derived GWI, contingent upon using spin-1 component observables, we will now make a comparative analysis of the QM violation obtained through GWI with that obtained by CGLMP inequality. We perform the required comparison by inserting white noise to the pure isotropic (mentioned in Eq.(17)) and singlet (mentioned in Eq.(19)) states.

Before computing the effect of white noise incorporated in the states considered, let us first obtain the maximum QM violations of CGLMP inequality for the bipartite qutrit isotropic and singlet states respectively contingent upon using spin-1 component observables. If LHS of the CGLMP inequality (22) is evaluated in terms of the four trichotomic observables \( a, a', b, b' \) denoting spin-1 components in arbitrary directions for the isotropic state, then from numerical calculation it can be shown that for the choice of measurement settings \( \{\theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3; \theta_4, \phi_4\} = (0.45, 6.28; 1.35, 6.28; 0, 1.07; 0.90, 6.28) \) (in radian), CGLMP inequality is maximally violated and the magnitude of the maximum violation is given by 0.52951. On the other hand, it can be shown that if the trichotomic measurements labelled by \( \{a, a', b, b'\} \) denoting spin-1 components in arbitrary directions are performed on \( 3 \otimes 3 \)-dimensional singlet state given by, Eq.(19), the maximum QM violation of the CGLMP inequality is obtained to be 0.52951. This occurs for the choice of spin measurements in the direction,\( \{\theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3; \theta_4, \phi_4\} = (0.78, 5.72; 0.88, 4.47; 2.12, 3.08; 2.41, 1.92) \) (in radian).

In order to probe the efficacy of the derived GWI, we now compare the tolerances of GWI against white noise present in a state with that of CGLMP inequality, contingent upon using spin-1 component observables, in a similar way described in Section VI. In the Tables (III) and (IV), the threshold visibilities of GWI and CGLMP inequality pertaining to the bipartite qutrit isotropic and singlet states respectively are shown.

The above mentioned Tables clearly show that the GWI given by Eq.(4) is more robust than the CGLMP inequality for the persistence of the QM violation in the presence of white noise incorporated in both the qutrit isotropic and singlet states.
TABLE III. Minimum value of the visibility parameter for which QM violates GWI and CGLMP inequality respectively for the bipartite qutrit isotropic state using spin-1 component observables.

| Local realist inequality | Threshold visibility |
|--------------------------|----------------------|
| GWI given by Eq. (5)     | 0.786                |
| CGLMP inequality         | 0.791                |

TABLE IV. Minimum value of the visibility parameter for which QM violates GWI and CGLMP inequality respectively for the bipartite qutrit singlet state using spin-1 component observables.

| Local realist inequality | Threshold visibility |
|--------------------------|----------------------|
| GWI given by Eq. (5)     | 0.786                |
| CGLMP inequality         | 0.791                |

IX. MAXIMAL VIOLATIONS OF GWI AND CGLMP INEQUALITY CONTINGENT UPON USING SPIN-1 COMPONENT OBSERVABLES

As discussed in Section VII, the maximum QM violations of GWI and CGLMP inequality using spin-1 component observables can be evaluated using the Min-Max theorem as stated before. Here any typical joint probability, say, \( P(a = +, b = -) \) of obtaining outcomes + and − respectively, when the observable \( a \) is measured on the first particle and the observable \( b \) is measured on the second particle, and the initial state is \( |\psi'\rangle \in \mathbb{C}^n \), is given by,

\[
P(a = +, b = -) = \langle \psi'|(+)(\theta_1, \phi_1)(+ \otimes -)(\theta_2, \phi_2)(-)|\psi'\rangle
\]

(28)

Similarly, evaluating other joint probabilities, the LHS of any local realist inequality can be expressed for the initial state \( |\psi'\rangle \in \mathbb{C}^n \) as \( \langle \psi'|B'|\psi'\rangle \), where \( B' \) is the Bell operator associated with the respective local realist inequality for bipartite qutrits corresponding to using spin-1 component observables. \( B' \) is a \( 3 \otimes 3 \) Hermitian Matrix. The largest eigenvalue of \( B' \) will be the maximum QM violation of the corresponding local realist inequality using spin-1 component observables and using the aforementioned Min-Max theorem, one can find the largest eigenvalues of the Bell operators associated with different local realist inequalities for bipartite qutrits.

We have found that the maximum QM violation of GWI given by Eq. (5) using spin-1 component observables is 0.20711, which is larger than the maximum QM violations of GWI given by Eq. (5) for bipartite qutrit isotropic and singlet states. Its corresponding eigenvector is a non-maximally entangled state of two qutrits, which has the following form

\[
|\psi'_{gwI}\rangle = (-0.01)|00\rangle + (-0.01)|01\rangle
\]

\[
+ (0.67)|02\rangle + (-0.18)|10\rangle +
\]

\[
(-0.40)|11\rangle + (-0.19)|12\rangle +
\]

\[
(0.23)|20\rangle + (0.51)|21\rangle + (-0.13)|22\rangle
\]

(29)

Therefore, the threshold visibility of GWI given by Eq. (5) for the state given by Eq. (29) is 0.682.

The maximum QM violation of CGLMP inequality given by Eq. (22) using spin-1 component observables is 0.62877, which is a bit larger than the maximum QM violations of CGLMP inequality for bipartite qutrit isotropic and singlet states. Its corresponding eigenvector is a non-maximally entangled state of two qutrits, which has the following form

\[
|\psi'_{cglmp}\rangle = (0.51 - 0.15i)|00\rangle + (-0.22 - 0.28i)|01\rangle
\]

\[
+ (0.05 + 0.13i)|02\rangle + (-0.28 - 0.12i)|10\rangle +
\]

\[
(-0.11 - 0.31i)|11\rangle + (-0.12 + 0.23i)|12\rangle +
\]

\[
(0.21i)|20\rangle + (-0.17 + 0.22i)|21\rangle + (-0.42)|22\rangle
\]

(30)

The threshold visibility of CGLMP inequality given by Eq. (22) for the state given by Eq. (30) is 0.761.

Hence the maximum threshold visibility, contingent upon using spin-1 component observables, of GWI given by Eq. (5) corresponding to a non-maximally entangled state is much smaller than that of CGLMP inequality.

To summarize, while the maximum QM violation of GWI occurs for the state given by Eq. (20), maximum QM violation of CGLMP inequality occurs for the state given by Eq. (30) when one uses spin-1 component observables. Note that none of the states is a maximally entangled state. Interestingly, the maximum QM violations of GWI given by Eq. (5) are the same whether one uses spin-1 component observables or six-port beam splitter. But the maximum QM violations of CGLMP inequality given by Eq. (22) differ for the two different types of observables stated above.

X. CONCLUSION

In this work we have extended Wigner’s approach by deriving generalized Wigner type local realist inequalities (GWI) for bipartite qutrit systems based on the assumption of the existence of the overall joint probability distributions in the underlying stochastic HV space for the measurement outcomes pertaining to the relevant trichotomic observables, that satisfy the
locality condition, and yield the measurable marginal probabilities. An important point to stress here is that the expressions of GWIs that have been derived here do not reduce to that of Bell-CHSH inequalities by grouping any two outcomes. This feature distinguishes GWIs considered here from the other local realist inequalities for bipartite qutrits; for example, the one suggested by Wu et al. [12]. Also, note that the factorizability condition for hidden variables used in deriving Bell-CHSH inequalities is not required for deriving GWI. In this context, it should be mentioned that the role of factorizability condition for stochastic hidden variables has been subjected to a critical examination [43]. Our work based on GWI serves to validate the notion that assuming the existence of overall joint probabilities in any stochastic HV theory yielding the measurable marginal probabilities is sufficient to demonstrate for the bipartite qutrit systems an incompatibility between QM and a class of stochastic HV theories satisfying the locality condition.

Efficacy of the derived GWI for the bipartite qutrit systems has been probed by analysing the robustness of its QM violation against white noise incorporated in the states considered. A comparative study of GWI in these contexts has been performed with respect to CGLMP inequality [32], using the two widely used maximally entangled states, viz. the singlet state, and the isotropic state which are also considered to be relevant in quantum information processing. Using six-port beam splitter, we have found that CGLMP inequality has lower threshold visibility compared to GWI when white noise is incorporated in the qutrit isotropic states. On the other hand, GWI has lower threshold visibility compared to CGLMP inequality when white noise is introduced in qutrit singlet states. In fact, CGLMP inequality is not violated by QM for qutrit singlet states when six-port beam splitter is used. It is also found that, contingent upon using spin-1 component observables, GWI has a lower threshold visibility than CGLMP inequality when white noise is introduced in both isotropic and singlet states. We can, therefore, state that for showing QM incompatibility of qutrit isotropic states with local realism using six-port beam splitter, the CGLMP inequality is more efficient in terms of the robustness of its QM violation against white noise than GWI; whereas, for qutrit singlet states, GWI is more efficient than the CGLMP inequality in showing QM incompatibility with local realism using six-port beam splitter. On the other hand, if one uses spin-1 component observables, GWI is more efficient in showing QM incompatibility with local realism for both qutrit isotropic and singlet states in the presence of white noise in these states, compared to the CGLMP inequality.

Another significant result obtained in this paper is that, for both six-port beam splitter and spin-1 component observables, the maximum QM violations of GWI and the CGLMP inequality occur for non-maximally entangled states. Further, it is found that the maximum QM violation of GWI is the same whether one uses spin-1 component observables or the observables pertaining to the six-port beam splitter. On the other hand, the maximum QM violation of the CGLMP inequality differs for the two different types of observables stated above.

It requires to be studied what interesting results such comparison between GWI and the CGLMP inequality would yield when extended in the context of mixed bipartite qutrit states. Finally, it should be worth probing the possibility of any information theoretic application of GWI similar to that of the more familiar Bell-CHSH inequalities, for example, in the context of device independent quantum key generation [44], and for developing robust multipartite multilevel quantum protocols [45].

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