On approximating the rank of graph divisors

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Abstract

Baker and Norine initiated the study of graph divisors as a graph-theoretic analogue of the Riemann-Roch theory for Riemann surfaces. One of the key concepts of graph divisor theory is the rank of a divisor on a graph. The importance of the rank is well illustrated by Baker’s Specialization lemma, stating that the dimension of a linear system can only go up under specialization from curves to graphs, leading to a fruitful interaction between divisors on graphs and curves.

Due to its decisive role, determining the rank is a central problem in graph divisor theory. Kiss and Tóthmérész reformulated the problem using chip-firing games, and showed that computing the rank of a divisor on a graph is NP-hard via reduction from the Minimum Feedback Arc Set problem.

In this paper, we strengthen their result by establishing a connection between chip-firing games and the Minimum Target Set Selection problem. As a corollary, we show that the rank is difficult to approximate to within a factor of $O(2^{\log^{1-\varepsilon} n})$ for any $\varepsilon > 0$ unless $P = NP$. Furthermore, assuming the Planted Dense Subgraph Conjecture, the rank is difficult to approximate to within a factor of $O(n^{1/4-\varepsilon})$ for any $\varepsilon > 0$.

Keywords: Approximation, Chip-firing, Graph divisors, Minimum target set selection, Riemann-Roch theory

1 Introduction

Motivated by the fact that a finite graph can be viewed as a discrete analogue of a Riemann surface, Baker and Norine [3] introduced a discrete analogue of the Riemann-Roch theory for graphs. They adapted the notions of a divisor, linear equivalence, and rank to the combinatorial setting, and studied the analogy between the continuous and the discrete cases in the context of linear equivalence. They further provided several results showing that the notion of rank plays an

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important role in various problems. In particular, a graph-theoretic analogue of the Riemann-Roch theorem holds for this notion of rank. Therefore, deciding the complexity of computing the rank is a natural question that is of both algebraic geometric and combinatorial interest. This problem is attributed to H. Lenstra, and was explicitly posed by many, see [4, 13, 17].

**Previous work** From the positive side, Luo [16] introduced the notion of rank-determining sets of metric graphs, and verified the existence of finite rank-determining sets constructively. Hladký, Král, and Norine [13] confirmed a conjecture of Baker [2] relating the ranks of a divisor on a graph and on a tropical curve, and provided a purely combinatorial algorithm for computing the rank of a divisor on a tropical curve, which can be considered simply as a metric graph, see [18, 12]. For multigraphs with a constant number of vertices, Manjunath [17] gave a polynomial time algorithm that computes the rank of a divisor. Furthermore, by a corollary of the Riemann-Roch theorem, the rank can be computed in polynomial time for divisors of degree greater than $2g - 2$, where $g$ denotes the cyclotomic number of the graph (also called the genus). Baker and Shokrieh [4] provided an algorithm that can efficiently check whether the rank of a divisor on a graph is at least $c$ for any fixed constant $c$. Cori and le Borgne [10] provided a linear time algorithm for determining the rank of a divisor on a complete graph.

There are also negative results regarding the complexity of computing the rank. Amini and Manjunath [1] proposed a general Riemann-Roch theory for sub-lattices of the root lattice $A_n^1$, thus giving a geometric interpretation of the work of Baker and Norine and generalising it to sub-lattices of $A_n$. They also showed that in the general model deciding whether the rank of a divisor is at least zero is already NP-hard. The complexity of computing the rank of a divisor on a graph was settled by Kiss and Tóthmérész [15] who showed that the problem is NP-hard even in simple graphs.

**Our results** As computing the rank of a divisor is NP-hard in general, it is natural to ask whether it can be approximated within any reasonable factor. Our main contribution is establishing a connection between computing the rank and finding a so-called minimum target set in an undirected graph, a central problem in combinatorial optimization that is notoriously hard to approximate. A recent paper by Bérczi, Boros, Čepek, Kučera, and Makino [5] showed a similar connection between finding a minimum key in a directed hypergraph and finding a minimum target set in an undirected graph, therefore, roughly speaking, our result places these three classical problems on the same level of complexity. As a corollary, we deduce strong lower bounds on the approximability of the rank. In particular, we show that the rank is difficult to approximate to within a factor of $O(2^{\log^{1-\varepsilon} n})$ for any $\varepsilon > 0$ unless $P = NP$. Furthermore, assuming the Planted Dense Subgraph Conjecture, we show that the rank is difficult to approximate to within a factor of $O(n^{1/4-\varepsilon})$ for any $\varepsilon > 0$, thus

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1. $A_n$ is the lattice of $(n + 1)$-dimensional integer vectors whose components sum to zero.
providing the first polynomial hardness result for the problem. By [16, Theorem 1.6], our results also apply to computing the rank of a divisor on a tropical curve.

The rest of the paper is organized as follows. Basic definitions and notation are introduced in Section 2 together with a brief introduction into graph divisor theory. The hardness of approximation of the rank of divisors is then discussed in Section 3.

2 Preliminaries

Basic notation

We denote the sets of integer and nonnegative integer numbers by $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively. Let $V$ be a ground set and $X \subseteq V$ be a subset of $V$. For a function $f : V \to \mathbb{Z}$, we use $f(X) := \sum_{v \in X} f(v)$.

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Throughout the paper, we allow multiple edges between a pair of vertices. Given $Z \subseteq V$ and $F \subseteq E$, the degree of $Z$ in $F$ is the number of edges in $F$ going between $Z$ and $V \setminus Z$ and is denoted by $d_F(Z)$. In particular, for a vertex $v \in V$, we denote by $d_F(v)$ the degree of $v$ in $F$. The degree vector or degree sequence of $G$ is then the vector $d_G \in \mathbb{Z}_V^+$ with $d_G(v) := d_E(v)$. For two vertices $u, v \in V$, $d_G(v, u)$ refers to the number of edges connecting $v$ and $u$ in $G$.

Chip-firing

We give a brief overview of the basic definitions of graph divisor theory due to Baker and Norine [3]. For a graph $G = (V, E)$, a divisor is an assignment $f : V \to \mathbb{Z}$ of an integer number to each vertex. We usually refer to the value $f(v)$ as the number of chips on $v$, even though $f(v)$ might be negative, and hence we also refer to $f$ as a chip configuration. The degree of a divisor $f$ is defined as $\text{deg}(f) := f(V) = \sum_{v \in V} f(v)$. The set of divisors admits an equivalence-relation, called linear equivalence, which is defined based on the operation of firing a vertex. For an arbitrary vertex $v$, the firing of $v$ modifies the divisor $f$ to $f'$ with

$$f'(u) = \begin{cases} f(v) - d_G(v) & \text{if } u = v, \\ f(u) + d_G(v, u) & \text{if } u \neq v. \end{cases}$$

That is, $v$ sends a chip along each edge incident to it, and as a result, the number of chips on $v$ decreases by the degree of $v$. Note that the degree of the divisor is not changed by the firing.

Two divisors $f$ and $g$ are called linearly equivalent if $f$ can be transformed into $g$ by a sequence of firings, denoted by $f \sim g$. Note that linear equivalence is indeed an equivalence relation, since a firing of $v$ can be ‘reversed’ by firing every other vertex once. A divisor $f$ is called effective if $f(v) \geq 0$ for each $v \in V$. A divisor is called winnable if it is linearly equivalent to an effective divisor. A basic quantity associated to a divisor is its rank, defined as

$$\text{rank}_G(f) := \min\{\text{deg}(g) \mid g : V \to \mathbb{Z} \text{ is effective, } f - g \text{ is not winnable}\} - 1.$$
In other words, because of the minus one in the definition, the rank of a divisor measures the maximum number of chips that can be removed arbitrarily from $f$ so that the resulting divisor is winnable.

We say that a vertex $v$ is active with respect to a divisor $f$ if $f(v) \geq d_G(v)$. The chip-firing game is a one player game, where a step consists of choosing an active vertex, if there exists any, and firing it. The game is non-trivial if at least one vertex gets fired. We say that the game halts if we get to a divisor that has no active vertex; such a divisor is called stable. The milestone result of Björner, Lovász, and Shor states that starting from a given divisor, either every game halts and ends with the same stable divisor, or every game continues indefinitely. That is, the choices of the player do not influence whether the game halts or not, as it only depends on the starting divisor. Based on this theorem, we call a divisor $f$ halting if the chip-firing game starting from $f$ eventually halts, and non-halting otherwise.

Recall that $d_G$ denotes the divisor that for any vertex $v$ has $d_G(v)$ chips. Let us denote by $1$ the divisor that has 1 chip on every vertex. The winability of a divisor can be characterized by the chip-firing game the following way, see [3, Corollary 5.4].

**Proposition 1** (Baker and Norine). A divisor $f$ on a graph $G$ is winnable if and only if $d_G - 1 - f$ is a halting divisor.

**Remark 2.** In [3] it is assumed that $d_G - 1 - f$ has a nonnegative amount of chips on each vertex. However, the theorem remains true without this assumption (see for example [15, Remark 2.6]), hence we do not suppose nonnegativity of divisors in the chip-firing game.

To measure the deviation of a divisor $f$ of $G$ from being non-halting, we introduce its distance from a non-halting state

$$\text{dist}_{nh}^{G}(f) := \min\{\deg(g) \mid g \text{ is an effective divisor of } G, f + g \text{ is non-halting}\}.$$  

Using this notation, the rank of a divisor can be formulated as $\text{rank}_{G}^{nh}(f) = \text{dist}_{nh}^{G}(d_G - 1 - f) - 1$. Hence computing the rank of a divisor is equivalent to computing the distance of a related divisor, that is, the minimum number of chips that needs to be added to the related divisor to make the chip-firing game non-halting.

**DIST-NONHALT**

**Input:** A graph $G = (V, E)$ and a divisor $f : V \to \mathbb{Z}$.

**Goal:** Compute the distance $\text{dist}_{nh}^{G}(f)$ of $f$ from a non-halting state.

The following result provides a sufficient condition for a divisor $f$ being non-halting, see [19, Lemma 4].

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2In some works, non-active vertices are also allowed to fire, resulting in a negative number of chips on them. In such a setting, a firing of a vertex $v$ is called legal if $v$ is active, and games consisting of legal firings are called legal as well. As we only consider legal games in this work, we dismiss the word ‘legal’ throughout.
Proposition 3 (G. Tardos). Let $G = (V, E)$ be a graph and $f : V \to \mathbb{Z}$ be a divisor. If there is a chip-firing game starting from $f$ where each vertex fires at least once, then $f$ is a non-halting configuration.

A divisor $f$ is called recurrent if there is a non-trivial chip-firing game starting from $f$ that leads back to $f$. Clearly, a recurrent divisor $f$ is also non-halting. Besides the distance of a divisor from a non-halting state, its distance from a recurrent state also plays a central role in our investigations, defined as

$$\text{dist}_G^r(f) := \min \{\deg(g) \mid g \text{ is an effective divisor of } G, f + g \text{ is recurrent}\}.$$ 

**DIST-Rec**

**Input:** A graph $G = (V, E)$ and a divisor $f : V \to \mathbb{Z}$.

**Goal:** Compute the distance $\text{dist}_G^r(f)$ of $f$ from a recurrent state.

We will use the following characterization of recurrence, due to Biggs [6, Lemma 3.6]. As [6] uses a slightly different model, we include a short proof here.

**Proposition 4.** [6, Lemma 3.6] Let $G = (V, E)$ be a connected graph and $f : V \to \mathbb{Z}$ be a divisor. Then $f$ is recurrent if and only if there is a game starting from $f$ in which each vertex fires exactly once.

**Proof.** The ‘if’ direction is obvious, hence we concentrate on the ‘only if’ direction. Assume that $f$ is a recurrent divisor. Then, by definition, there is a non-trivial game that transforms $f$ to itself. We claim that in such a game each vertex fires the same number of times. Indeed, otherwise let $Z \subset V$ denote the proper subset of vertices that fired a minimum number of times. As every vertex in $V \setminus Z$ fired more times than the vertices in $Z$ and $G$ is connected, $Z$ necessarily gains chips compared to $f$, a contradiction. By [7, Lemma 4.3], there also exists a game starting from $f$ in which each vertex fires exactly once, concluding the proof.

Minimum Target Set Selection In the Minimum Target Set Selection problem (MIN-TSS), introduced by Kempe, Kleinberg, and É. Tardos [14], we are given a simple graph $G = (V, E)$ with $|V| = n$, together with a threshold function $\tau : V \to \mathbb{Z}_+$, and we consider the following activation process. As a starting step, we can activate a subset $S \subseteq V$ of vertices. Then, in every subsequent round, a vertex $v$ becomes activated if at least $\tau(v)$ of its neighbors are already active. Note that once a vertex is active, it remains active in all future rounds. The goal is to find a minimum sized initial set $S$ of active vertices (called a target set) so that the activation spreads to the entire graph (i.e., all vertices are eventually activated). For ease of notation, we measure the size of an optimal solution by

$$\text{ts}_G(\tau) := \min \{|S| \mid S \subseteq V, S \text{ is a target set of } G \text{ with respect to } \tau\}.$$
**MIN-TSS**

**Input:** A simple graph \( G = (V, E) \) and thresholds \( \tau : V \rightarrow \mathbb{Z}_+ \).

**Goal:** Find a minimum sized target set attaining \( t_{G}(\tau) \).

MIN-TSS has long been a focus of research due to its wide applications in real life scenarios. Unfortunately, not only finding a smallest target set but also approximating the minimum value is difficult. Under the assumption \( P \neq NP \), Feige and Kogan [11, Theorem 3.5] showed that the problem is difficult even in 3-regular graphs.

**Proposition 5** (Feige and Kogan). *Unless \( P = NP \), MIN-TSS cannot be approximated to within a factor of \( O(2^{\log^{1-\epsilon} n}) \) for any \( \epsilon > 0 \) even in 3-regular graphs with \( \tau(v) \in \{1, 2\} \) for \( v \in V \).*

Also, Charikar, Naamad, and Wirth [9] proved that, assuming the Planted Dense Subgraph Conjecture, MIN-TSS is difficult to approximate within a polynomial factor.

**Proposition 6** (Charikar, Naamad, and Wirth). *Assuming the Planted Dense Subgraph Conjecture, MIN-TSS cannot be approximated to within a factor of \( O(n^{1/2-\epsilon}) \) by a probabilistic polynomial-time algorithm for any \( \epsilon > 0 \).*

## 3 Hardness of approximating the rank

Now we turn to the proof of the main result of the paper, and show that approximating the rank of a graph divisor within reasonable bounds is hard. The high level idea of the proof is the following. First, we show that the Minimum Target Set Selection problem reduces to computing the distance of a divisor on an auxiliary undirected graph from a recurrent state (Section 3.1). Then we show that computing the distance of a divisor from a recurrent state reduces to computing the distance of a divisor on a slightly modified graph from a non-halting state (Section 3.2). As the computation of the rank is equivalent to computing the distance from a non-halting state for an appropriately modified divisor, the lower bounds on the approximability of the rank will follow by Propositions 5 and 6; see Theorem 14.

### 3.1 Reduction of MIN-TSS to DIST-REC

Consider an instance \( G = (V, E) \) and \( \tau : V \rightarrow \mathbb{Z}_+ \) of MIN-TSS. We construct a DIST-REC instance \( G' = (V', E') \) and \( x : V' \rightarrow \mathbb{Z} \) such that \( t_{G}(\tau) = \text{dist}_{G'}(x) \).

We define \( G' \) as follows. For each \( v \in V \), we add three vertices \( v_i, v_c \) and \( v_o \) to \( V' \). In addition, for each edge \( e = uv \in E \), we add two vertices \( p_{uv} \) and \( p_{vu} \) to \( V' \). Now let \( N = |V| + 2 \). For each vertex \( v \in V \), we add \( N \) parallel edges between \( v_i \) and \( v_c \) and a single edge between \( v_c \) and \( v_o \). Furthermore, for each edge \( e = uv \in E \), we add \( N \) parallel edges between \( u_o \) and \( p_{uv} \), a single edge between \( p_{uv} \) and \( v_i \), \( N \) parallel edges between \( v_o \) and \( p_{vu} \), and a single edge
(a) An instance of Min-TSS. The numbers denote the thresholds at the vertices. The set $S = \{u, v\}$ is a target set; it is not difficult to check that $S$ has minimum size.

(b) The subgraph of $G'$ induced by the vertices $\{v_i, v_o, u_i, u_o\} \subseteq V'_o$ and $\{v_c, u_c, p_{uv}, p_{vu}\} \subseteq V'_c$. Thick edges denote a bundle of $N = |V| + 2 = 6$ parallel edges.

(c) The corresponding Dist-Rec instance. The numbers denote the initial chip distribution $x$ where $x(v) = 1$ for each $v \in V'_c$. Adding one chip on vertices with double border results in a recurrent divisor.

Figure 1: An illustration of the reduction of Min-TSS to Dist-Rec.
Hence eventually, there is a moment when Min-TSS activation process in the results in sending one chip to $v$, firing $z$ connected to $v$. Let

$$\text{Proof.}$$

Observe that $\tau(v)$ neighbors in $S$, then $v_i$ received at least $\tau(v)$ chips up to this point, that is, the total number of chips on $v_i$ is at least $d_{G'}(v_i)$. Thus, $v_i$ can fire as well. Firing $v_i$ sends $N$ chips to $v$, hence at this point $v$ can also fire, which results in sending one chip to $v$ and making a firing at $v$ possible.

Continuing this way, we get that if a vertex $v \in V$ becomes active in the activation process in the Min-TSS instance, then $v_o$ can fire at a certain point. Hence eventually, there is a moment when $v_o$ has fired once for every $v \in V$. This means that for each edge $uv \in E$ and any of the vertices $p_{uv}$ and $p_{wu}$, either it fired once or has enough chips to do so; in the latter case, let it fire. After these, for each vertex $v \in V$, the vertex $v_i$ has either fired once or has enough chips to do so; in the latter case, let it fire, resulting in a chip configuration for which $v_i$ can fire as well. We conclude that eventually each vertex fires (exactly) once, hence $(x + y)$ is recurrent by Proposition $4$ and $y$ is effective.

By the above, $t_{sG}(\tau) = |S| = \deg(y) \geq \text{dist}_{G'}^r(x)$, concluding the proof of the claim.

**Claim 8.** $N > \text{dist}_{G'}^r(x) + 1$.

**Proof.** Observe that $t_{sG}(\tau) \leq |V|$ clearly holds. By Claim $7$ we get $N = |V| + 2 \geq t_{sG}(\tau) + 2 > \text{dist}_{G'}^r(x) + 1$ as stated.

**Claim 9.** $t_{sG}(\tau) \leq \text{dist}_{G'}^r(x)$.

**Proof.** Let $y$ be a chip configuration such that $y \geq 0$, $x + y$ is recurrent, and $\deg(y) = \text{dist}_{G'}^r(x)$. By Proposition $4$, there is a game starting from $x + y$ in which each vertex fires exactly once; from now on, we consider such a sequence of firings.

Consider a vertex $z \in V'_o$ and let $w$ be the unique vertex in $V'_o$ that is connected to $z$ by $N$ parallel edges. By definition, we have $x(z) = 1$. If $z$ fires before $w$ in the game, then $z$ can receive at most one chip from its other neighbor before firing. That is, $y(z) \geq N - 1$ as otherwise $z$ has not enough chips for firing before $w$. By Claim $8$ we have $\text{dist}_{G'}^r(x) < N - 1$, a contradiction. Hence $z$ necessarily fires after $w$ in the game, meaning that it receives $N$ chips from $w$. That is, it has enough chips to fire even if $y(z) = 0$. As $z$ only fires once and $\deg(y)$ is as small as possible, we conclude that $y(z) = 0$. Moreover, $z$ fires later than $w$ for each $z \in V'_o$, where $w$ is the unique neighbor of $z$ that is connected to $z$ by $N$ parallel edges.

Now consider a vertex $v_i$ for some $v \in V$. We show that we may assume $y(v_i) = 0$. Indeed, suppose that $y(v_i) = k > 0$. Let $\{u^1, \ldots, u^d\}$ be the neighbors of $v$ in $G$. We create a new $y'$ by replacing the chips of $y$ from $v_i$ to some $u^d_i$ such that $y'$ remains effective with $\deg(y') \leq \deg(y)$, $x + y'$ is also recurrent, but $y'(v_i) = 0$. (The minimality of $\deg(y)$, then of course implies $\deg(y) = \deg(y')$.) Take a game started from $x + y$ in which each vertex of $G'$ fires exactly once. We know that $v_i$ fires after $v_i$ in this game. We have $(x + y)(v_i) = \deg_{G'}(v_i) = \tau(v) + k$. Hence by the time $v_i$ fires, at least $\max\{0, \tau(v) - k\}$ vertices among $\{p_{u^1v}, \ldots, p_{u^dv_i}\}$ fired. Choose $k$ vertices

\[ u \in E, \text{ therefore, } p_{uv} \text{ can fire. Fire them when we can. If a vertex } v \in V \text{ has at least } \tau(v) \text{ neighbors in } S, \text{ then } v_i \text{ received at least } \tau(v) \text{ chips up to this point, that is, the total number of chips on } v_i \text{ is at least } d_{G'}(v_i). \text{ Thus, } v_i \text{ can fire as well. Firing } v_i \text{ sends } N \text{ chips to } v, \text{ hence at this point } v \text{ can also fire, which results in sending one chip to } v \text{ and making a firing at } v, \text{ possible.} \]
configuration with at least $d_i$.

If we add max

Proof.

Claim 10.

Consider a

Dist-Rec

3.2 Reduction of Dist-Nonhalt

Instance $G' = (V', E')$ and $f' : V' \rightarrow \mathbb{Z}$ such that $\text{dist}_{G'}(f) = \text{dist}^{\text{dist-Nonhalt}}_{G'}(f')$. Note the following.

Claim 10. $\text{dist}_{G'}(f) \leq 2 \cdot |E| + \sum_{v \in V : f(v) < 0} |f(v)|$.

Proof. If we add max$\{0, d_G(v) - f(v)\}$ chips to vertex $v$, then we have a chip configuration with at least $d_G(v)$ chips on each vertex $v$, which is certainly a

Note that the size of $G'$ is polynomial in the size of $G$. In particular, we have $|V'| = 3 \cdot |V| + 2 \cdot |E| = O(|V|^2)$ as $G$ is simple.
By Claim 10, recurrent configuration. Moreover, this addition needs at most the above stated number of chips. □

We define $G'$ as follows. Let $M$ be an integer large enough such that

$$\text{dist}_{G}(f) + \max \left\{ 0, \max_{v \in V} \{ f(v) - d_G(v) \} \right\} < M.$$ 

By Claim 10, $M := 2|E| + \sum_{v \in V : f(v) < 0} |f(v)| + \max \{ 0, \max_{v \in V} \{ f(v) - d_G(v) \} \} + 1$ is sufficient. Let $V' := V \cup \{ v_{\text{new}} \}$ and let $E'$ consist of $E$ together with $M$ parallel edges between $v$ and $v_{\text{new}}$ for each vertex $v \in V$. That is, $|E'| = |E| + M \cdot |V|$. Finally, we define $f'(v) := f(v) + M$ for $v \in V$ and $f'(v_{\text{new}}) := 0$.

We claim that $\text{dist}_{G}(f) = \text{dist}_{G'}(f')$. We prove the two directions separately.

**Claim 11.** $\text{dist}_{G}(f) \geq \text{dist}_{G'}(f')$.

**Proof.** Let $g$ be an effective divisor of minimum degree such that there is a chip-firing game on $G$ starting from $f + g$ in which each vertex fires exactly once. We extend $g$ to $V'$ by setting $g(v_{\text{new}}) = 0$. We claim that if $\{ v_1, \ldots, v_k \}$ is a sequence of firings starting from $f + g$ on $G$ in which each vertex occurs at most once, then these vertices can also fire in the same order starting from $f' + g$ on $G'$. Indeed, for each vertex $v \in V$, we have $d_{G'}(v) = d_G(v) + M$ and $f'(v) + g(v) = f(v) + g(v) + M$. Firing a vertex $v \in V$ modifies the chip-configuration on other vertices in $V$ the same way in $G$ and in $G'$. Hence if $v_i$ has not yet been fired and it has enough chips in the game on $G$, then it also has enough chips in the game on $G'$. This implies that there is a game on $G'$ starting from $f' + g$ in which each vertex of $V$ fires exactly once. At this point, $v_{\text{new}}$ received $M$ chips from each vertex in $V$, hence it has $M \cdot |V|$ chips. As its degree is also $M \cdot |V|$, at this point $v_{\text{new}}$ can fire, resulting in a game on $G'$ in which each vertex has fired exactly once.

By Proposition 3, the above implies that $f' + g$ is non-halting on $G'$. Hence we have $\text{dist}_{G}(f) = \text{deg}(g) \geq \text{dist}_{G'}(f')$, concluding the proof of the claim. □

**Claim 12.** $M > \text{dist}_{G'}(f')$.

**Proof.** We choose $M$ such that $M > \text{dist}_{G}(f)$, hence the claim follows by Claim 11. □

**Claim 13.** $\text{dist}_{G}(f) \leq \text{dist}_{G'}(f')$.

**Proof.** Let $g'$ be an effective divisor such that $f' + g'$ is non-halting on $G'$ and $\text{deg}(g') = \text{dist}_{G'}(f')$. We show that $g'(v_{\text{new}}) = 0$, and that for the divisor $g := g'|_V$ obtained by restricting $g'$ to the set of original vertices, the divisor $f + g$ is recurrent on $G$.

Observe that $f'(v_{\text{new}}) = 0$ and $v_{\text{new}}$ receives $M$ chips each time a vertex in $V$ fires. Moreover, $d_{G'}(v_{\text{new}})$ is divisible by $M$, hence adding less than $M$ chips on $v_{\text{new}}$ has no effect on whether it is able to fire or not. As $\text{dist}_{G'}(f') < M$, we conclude that $g'(v_{\text{new}}) = 0$. 

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Now take any game on \( G' \) starting from \( f' + g' \). We claim that no vertex \( v \in V \) can fire twice before the first firing of \( v_{new} \). Suppose to the contrary that some vertex fires twice before the first firing of \( v_{new} \), and let \( v \in V \) be the first such vertex. Then before its second firing, \( v \) received at most \( d_{G'}(v) \) chips from its neighbors, and it lost \( d_{G'}(v) \) chips upon its first firing. Since, by our assumption, \( v \) is now capable of firing for the second time, we get that \( f'(v) + g'(v) + d_{G'}(v) - d_{G'}(v) \geq d_{G'}(v) \). As \( d_{G'}(v) = d_{G}(v) + M \), this implies \( f'(v) + g'(v) \geq d_{G}(v) + 2M \). As \( f'(v) = f(v) + M \), the choice of \( M \) implies \( g'(v) \geq d_{G}(v) - f(v) + M > \text{dist}_{G'}(f') = \text{dist}_{G'}^M(f') = \text{deg}(g'), \) a contradiction.

Hence each vertex \( v \in V \) fires at most once before the first firing of \( v_{new} \). As \( v_{new} \) has to collect \( M \cdot |V| \) extra chips to be able to fire, each vertex \( v \in V \) fires exactly once before the first firing of \( v_{new} \). These firings would also be possible when starting from \( f + g \) on \( G \), hence \( f + g \) is recurrent by Proposition [11] \( \square \)

Note that the size of \( G' \) might not be polynomial in the size of \( G \) as the value of \( M \) can be large. However, the above reduction will be applied to divisors \( f \) with \( 0 \leq f(v) \leq d_{G}(v) \) for \( v \in V \), in which case the construction is polynomial. In particular, observe that \( |V'| = |V| + 1 \).

### 3.3 Lower bounds

By combining the observations of Sections 3.1 and 3.2 with Propositions 5 and 6, we are ready to prove the main result of the paper.

**Theorem 14.** Unless \( P = NP \), Dist-Nonhalt cannot be approximated to within a factor of \( O(2^{\log^{3+\epsilon} n}) \) for any \( \epsilon > 0 \). Assuming the Planted Dense Subgraph conjecture, Dist-Nonhalt cannot be approximated to within a factor of \( O(n^{1/4-\epsilon}) \) for any \( \epsilon > 0 \).

**Proof.** Let \( G = (V, E) \) be a graph and \( \tau : V \to \mathbb{Z}_+ \) be a threshold function. According to Section 3.1, computing the value \( \text{ts}_{G}(\tau) \) of the Min-TSS instance on \( G \) and \( \tau \) can be reduced to computing \( \text{dist}_{G'}(x) \) for a graph \( G' = (V', E') \) having \( |V'| = 3 \cdot |V| + 2 \cdot |E| \) vertices and \( |E'| = 2 \cdot (|V| + 3) \cdot |E| + |V| \) edges, and a divisor \( x : V' \to \mathbb{Z}_+ \) with \( 0 \leq x(v) \leq d_{G'}(v) \) for each \( v \in V' \).

As \( \text{ts}_{G}(\tau) \leq |V| \), we have \( \text{dist}_{G'}(x) \leq |V| \). Hence we can use the reduction of Section 3.2 with \( M = |V| + 1 \). Thus, we have a graph \( G'' = (V'', E'') \) with \( |V''| = 3 \cdot |V| + 2 \cdot |E| + 1 \) vertices and \( |E''| = 2 \cdot (|V| + 3) \cdot |E| + 2 \cdot |V| + |V|^2 \) edges, and a divisor \( x'' : V'' \to \mathbb{Z}_+ \) such that \( \text{ts}_{G}(\tau) = \text{dist}_{G''}^M(x'') \).

Combining the above with Propositions 5 and 6, the statements of the theorem follow. \( \square \)

**Remark 15.** The above reduction of Min-TSS to Dist-Nonhalt uses an auxiliary graph that contains parallel edges. Therefore, the stated complexity results hold for general (i.e. not necessarily simple) graphs. However, in [13, Corollary 22], Hladký, Král, and Norine proved that if one subdivides each edge of a graph with a new vertex and places 0 chips on the new vertices, then the rank of the obtained divisor on the new graph is the same as the rank of the
original divisor on the original graph. Using this result, our construction can be modified in such a way that eventually we get a \textsc{Dist-Nonhalt} instance on a simple graph, at the expense of having a higher number of vertices. Such a reduction, assuming the Planted Dense Subgraph Conjecture, still implies a polynomial hardness to the problem.

\textbf{Remark 16.} The proofs of the paper are based on a construction that resulted in a graph with large degrees. Deciding the complexity of computing the rank in degree-bounded graphs remains an interesting open problem.

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