Light hadron masses with a tadpole-improved next-nearest-neighbour lattice fermion action

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Abstract

Calculations of hadron masses are done in quenched approximation using gauge field and fermion actions which are both corrected for discretization errors to $O(a^2)$ at the classical level and which contain tadpole improvement factors. The fermion action has both nearest-neighbour and next-nearest-neighbour couplings in the kinetic and Wilson terms. Simulations done at lattice spacings of 0.27 and 0.4fm yield hadron masses which are already quite close to experimental values. The results are compared to Wilson action calculations done at comparable lattice spacings.
1 Introduction

During the past several years there has been renewed interest in the use of improved lattice actions. Many calculations have been done using the so-called clover action\cite{1,2,3} motivated by the pioneering work of Sheikholeslami and Wohlert\cite{4}. Recently the move toward improved actions has been given even more impetus by the work of Lepage and co-workers\cite{5,6} which suggests that with tadpole improvement\cite{7}, calculations can be done quite accurately even on rather coarse lattices. In this note we report on calculations done with a simple tadpole-improved next-nearest-neighbour fermion action which support this suggestion.

The essential idea of improved actions is that by including terms that are nonleading (in powers of lattice spacing) one can reduce discretization errors. Of course, the choice of action is not unique. The approach of Sheikholeslami and Wohlert\cite{4} is to impose the minimal on-shell improvement condition\cite{8} and they showed that $O(a)$ errors could be removed from physical observables by the use of the so-called clover action. An advantage of this action is that to $O(a)$ the familiar Wilson plaquette action may be used for the gauge field. However, as Lepage and co-workers\cite{5,6} have shown, a significant gain can be made in improving the gauge field actions by incorporating tadpole factors\cite{7} in the weighting coefficients of the nonleading terms. This, for
example, leads to restoration of rotational invariance\[3\] of the static potential even at lattice spacings of order 0.4fm. In ref.[3] Alford et al extended the tadpole improvement program to the light quark sector, introducing the $D_{234}$ fermion action which, at the classical level, is corrected to $O(a^2)$. A feature of the $D_{234}$ action is that it contains both clover and next-nearest-neighbour terms in addition to the terms appearing in the Wilson action.

In this work we consider the use of a simple alternative to the $D_{234}$ action which dispenses with the clover term altogether. This is the next-nearest-neighbour fermion action\[1] in which both kinetic and Wilson terms have been corrected at tree-level to $O(a^2) \[10\]$. In addition, tadpole factors are included in the next-nearest-neighbour terms. These work to remove, in a mean field sense, discretization errors due to tadpole-like couplings induced by the lattice description of the gauge field\[7\]. In conjunction with this fermion action we use a gauge field action that has been analogously improved, that is, $O(a^2)$ tree-level improvement plus tadpole factors.

Here we report some results for light-quark (u, d and s) meson and baryon masses calculated on lattices with lattice spacings of 0.4 and 0.27fm. In general, our results are fairly close to experimental values and are compatible with Wilson action calculations (see, for example, Ref.[11] done at lattice

\[1\] The use of this action with tadpole improvement has also been considered independently by Lee and Leinweber\[8\].
For purposes of comparison simulations have also been carried out with the Wilson action at lattice spacings matched to those of the improved action calculations. As expected, on our coarse lattices, the improved action results are much closer to those obtained with the Wilson action at small lattice spacing. What we observe is that the major difference between the improved action and the Wilson action is in the relative scale between the meson and baryon sectors. Mass ratios within the meson sector and within the baryon sector depend much less on the choice of action.

\section{Method}

The $SU(3)$ gauge fields are described by an action that contains both 4-link square plaquettes (pl) and planar 6-link rectangular plaquettes (rt). As shown in \cite{12} the 6-link rectangles are sufficient to remove $O(a^2)$ errors at the classical level. In addition tadpole factors are introduced into the weighting of the 6-link term. The action is

$$S_G(U) = \beta \left[ \sum_{pl} (1 - \frac{1}{3} \text{Re} \text{Tr} U_{pl}) + C_{rt} \sum_{rt} (1 - \frac{1}{3} \text{Re} \text{Tr} U_{rt}) \right]$$

(1)

where $U_{pl}$ are the square plaquettes and $U_{rt}$ are the planar 6-link plaquettes. The coefficient $C_{rt} = -1/20U_0^2$ includes the tadpole factor

$$U_0 = \left( \frac{1}{3} \text{Re} \text{Tr} U_{pl} \right)^{1/4}.$$  

(2)
The first term of (1) is just the Wilson action.

For the fermions, the Wilson action augmented by next-nearest-neighbour couplings [10] in both the kinetic and Wilson terms is used. Including tadpole factors the action is

\[ S_F(\bar{\psi}, \psi; U) = \sum_{x,\mu} \frac{4}{3} \kappa \left[ \bar{\psi}(x)(1 - \gamma_\mu)U_\mu(x)\psi(x + \mu) + \bar{\psi}(x + \mu)(1 + \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right] \]

\[ \quad - \sum_{x,\mu} \frac{1}{6} \kappa \bar{\psi}(x)(2 - \gamma_\mu)U_\mu(x)U_\mu(x + \mu)\psi(x + 2\mu) + \bar{\psi}(x + 2\mu)(2 + \gamma_\mu)U_\mu^\dagger(x + \mu)U_\mu^\dagger(x)\psi(x) \]

\[ \quad - \sum_x \bar{\psi}(x)\psi(x). \quad (3) \]

With the coefficients as in (3), \( \kappa_{\text{critical}} = 1/8 \) at the tree level, the same as for the Wilson fermion action. The Wilson action is recovered by replacing the coefficient 4/3 by 1 and dropping next-nearest-neighbour terms.

A feature of next-nearest-neighbour action (3), which it shares with the \( D234 \) action, is the presence of unphysical states in the free quark propagator with a massless dispersion relation very similar to that given by Alford et al [6]. One might wonder about the effect of such unphysical singularities. In fact, the near identity in the results reported by Alford et al [6] and by Collins et al [13] who use a clover action which has no doublers suggests to us that the singularity structure of the tree-level propagator may not be very crucial in determining the ability of an action to describe hadron masses. A
posteriori the hadron masses which we calculate show no obvious effect that can be linked to the unphysical states of the free propagator although this is something that merits further study.

Calculations were carried out in quenched approximation at two different values of $\beta$, 6.25 and 6.8 for the improved action and 4.5 and 5.5 for the Wilson action. These values were chosen so that lattice spacings determined from the string tension would match for the two actions\cite{14}. These lattice spacings are 0.4fm and 0.27fm for the smaller and larger $\beta$-values respectively. The lattice sizes used were $6^3 \times 12$ and $8^3 \times 14$.

Gauge field updating was done using the Cabbibo-Marinari pseudo-heat-bath. Periodic boundary conditions were used for the gauge field in all directions. The lattice was thermalized for 4000 sweeps then configurations were used every 250 sweeps in the case of the improved action and every 200 sweeps for the Wilson action.

Quark propagators were calculated for a range of $\kappa$ values in each simulation. A stabilized biconjugate gradient algorithm\cite{15} was used for these calculations. Periodic boundary conditions were imposed on the quark fields in spatial directions but in the time direction a Dirichlet or fixed boundary condition was used. This allows mass measurements to be made further from the source than with periodic boundary conditions. This is an important consideration given the relatively small number of time slices. The
source position was fixed to be two time steps in from the boundary in all simulations.

Meson and baryon correlators were calculated using standard local interpolating fields.

\[ \chi^{(\Gamma)}(x) = \bar{\psi}(x)\Gamma\psi(x) \]  
\[ \chi^{(N)}_i(x) = \epsilon_{abc}\psi_{a,i}(x)\left[\psi_T^c(x)C\gamma_5\psi_c(x)\right] \]  
\[ \chi^{(\Delta)}_{ijk}(x) = \epsilon_{abc}\psi_{a,i}(x)\psi_{b,j}(x)\psi_{c,k}(x) \]

for the nucleon, where \( C \) is the charge conjugation matrix, and

for the isobar.

As is well known, smeared operators can be used to enhance the overlap of the interpolating field with the ground state. This allows ground state masses to be extracted closer to the source point where statistical fluctuations are less severe. In fact, a smeared sink, although maybe less effective than a smeared source, can be implemented at very little cost. Therefore, correlators were constructed for both local and smeared sinks with local sources. Gaussian smearing\[16\] was used. The smearing function is

\[ \Phi = (1 + \alpha H)^n \]
where
\[
H(\vec{x}, \vec{y}; t) = \sum_{i=1}^{3} \left[ U_i(\vec{x}, t) \delta_{\vec{x}, \vec{y} - \hat{i}} + U_i^\dagger(\vec{x} - \hat{i}, t) \delta_{\vec{x}, \vec{y} + \hat{i}} \right].
\]
(8)

The smearing parameters were fixed at \( n = 6 \) and \( \alpha = 2 \) for all simulations. No attempt was made to optimize the smearing parameters for this exploratory calculation.

The lattice details for the calculations are summarized in Table 1.

3 Results and Conclusion

Masses were calculated using an analysis procedure motivated by Bhat-\texttt{tacharya} et al.[17]. For each channel the correlation functions \( G(t) \) were configuration averaged and the effective mass function \( M_{\text{eff}}(t) = \ln(G(t)/G(t + 1)) \) was calculated. Then a combined effective mass function was computed by a weighted average of the effective mass functions obtained from local-local and local-smeared correlators. Since the local-local correlator overestimates the ground state mass and the local-smeared correlator underestimates it, this average helps to enhance the plateau of the effective mass. The mass is then determined by averaging the combined mass function over some time interval. Except for the baryon spin-3/2 channel, it was found that compatible masses could be obtained using time averages starting 2 or 3 time steps away from the source.

For each simulation the masses were extrapolated as a function of pion
mass to the chiral limit $M_\pi = 0$. The choice of extrapolation function was either

$$M = M_0 + cM_\pi^2$$  \hspace{1cm} (9)

or

$$M = M_0 + cM_\pi^2 + dM_\pi^3.$$  \hspace{1cm} (10)

The criterion was that the cubic form (which is motivated by chiral perturbation theory\cite{18}) was used whenever the coefficient $d$ could be determined to be nonzero within the statistical errors. If the mass data showed no evidence of a cubic term then the quadratic form was used.

The errors in masses and in mass ratios were estimated using a bootstrap procedure. For each simulation 500 bootstrap samples were chosen from the original sample and analyzed for masses and mass ratios. The quoted errors on observables are one half the difference between the 16th and 84th percentile values found in the bootstrap distribution for that observable.

In addition to the u,d sector we are also interested in the strange quark sector. To fix $\kappa_s$, (i.e., the strange quark mass) the condition $K^*/K$ equals the experimentally observed value 1.8 was used. Generally speaking $\kappa_s$ does not coincide with one of our chosen $\kappa$ values. This requires an interpolation (or extrapolation) which was done linearly in $\kappa$ using the two $\kappa$ values nearest $\kappa_s$. 

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As a representative sample of our results we show the $\rho$-meson, nucleon and delta masses as a function of $M^2_\pi$ in Fig. 1. Also shown are the extrapolations to the chiral limit. Mass ratios extrapolated to the chiral limit are given in Table 2. Meson masses are given with respect to the $\rho$-meson mass and baryon masses with respect to the nucleon mass. The ratio $M_N/M_\rho$ then sets an overall scale of baryon masses relative to meson masses and this seems to be the quantity most effected by discretization errors. By 0.27fm the improved action results are fairly close to experiment and are compatible with Wilson action calculations done at small lattice spacing[11].

The quantity $J = M_V dM_V/dM_P^2$ at $M_V/M_P = 1.8$ was introduced by Lacock and Michael[19] as a measure of the relative quark mass dependence of pseudoscalar and vector mesons. Empirically this value is very close to 0.5 reflecting the fact that $M_V^2 - M_P^2$ is almost constant. Quenched lattice QCD simulations at small lattice spacings tend to give values around 0.37 which was interpreted in [19] as a failure of the quenched approximation. Our improved action results are consistent with the previous determinations.

In the continuum limit of full lattice QCD it is expected that the lattice spacing determined from all physical quantities will be the same. In a quenched calculation there is no reason why this should also be true. Nonetheless it is still expected that there will still be a scaling region in which the ratio of lattice spacings is constant. In Fig. 2 we have compiled
some results for the ratio of lattice spacing extracted from the string tension to the lattice spacing determined by the $\rho$-meson mass. The improved action results include our calculations as well as the values reported by Alford et al\cite{6} for the $D234$ action and by Collins et al\cite{13} using a tadpole-improved clover fermion action with an $O(a^2)$ tadpole-improved gauge field action. The improved action and Wilson action results show the same qualitative behaviour only shifted in lattice spacing by about a factor of 3. Unfortunately it is only the last two points at the smallest value of $M_\rho a$ (i.e., the largest $\beta$) which show a hint of scaling. If this really is the onset of scaling it would correlate very well with the onset of the weak coupling region as shown, for example, by the behaviour of the average plaquette. A remarkable feature seen in Fig. 2 is that different improved fermion actions exhibit a high degree of universality even in the non-scaling region.

If simulations done with improved actions on coarse lattices are to be useful the results should extrapolate smoothly to the continuum limit. Calculations in the light hadron sector using tadpole improved actions of the type used in this work are still too scarce to be able to make definitive statements. However, the ratio of nucleon to $\rho$-meson mass has been calculated a number of times. The results of tadpole improved actions\cite{6,13} and a sample of Wilson action results\cite{11,17,22} are presented in Fig. 3. It is encouraging that our improved action values at 0.27 and 0.4fm are compatible with the
Wilson action results at smaller lattice spacing. However, it is somewhat disconcerting that in this case they do not agree with the values given by Alford et al[6] and by Collins et al[13]. At present, the results of Ref.[6, 13] would suggest a continuum limit value for \( M_N/M_\rho \) different from the Wilson action value which is not a very palatable conclusion. It is clear that the improved action calculations have to be pushed to smaller lattice spacing to clarify this situation.

In this work we consider the use of a next-nearest-neighbour fermion action, corrected for discretization errors to \( O(a^2) \) at the classical level, for use with the tadpole improvement program of Lepage and co-workers [5, 6, 7]. An analogously improved gauge field action is used. Quenched calculations done at lattice spacings of 0.27 and 0.4fm yield hadron masses in the light quark sector which are comparable to those obtained with the Wilson action at much smaller (\( a<\sim 0.1\)fm) lattice spacing and which are quite close to experimental values. Comparison with Wilson action calculations shows very clearly the positive effect of improvement.

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| Action  | Lattice | $N_U$ | $\beta$ | $a_{st}$ | $\kappa$         | $\kappa_s$ |
|---------|---------|-------|---------|---------|------------------|-------------|
| Improved  | $6^3 \times 12$ | 160   | 6.25    | 0.4 fm  | 0.162, 0.165, 0.168, 0.171, 0.174 | 0.166 |
| Improved  | $8^3 \times 14$ | 60    | 6.8     | 0.27 fm | 0.148, 0.150, 0.152, 0.154, 0.156, 0.158 | 0.1558 |
| Wilson   | $6^3 \times 12$ | 160   | 4.5     | 0.4 fm  | 0.189, 0.193, 0.197, 0.201, 0.205, 0.209, 0.213 | 0.205 |
| Wilson   | $8^3 \times 14$ | 90    | 5.5     | 0.27 fm | 0.164, 0.168, 0.172, 0.176, 0.180 | 0.178 |

Table 1: Lattice Details. $N_U$ is the number of gauge configurations and $a_{st}$ is the lattice spacing determined from the string tension [14]. $\kappa_s$ is the hopping parameter corresponding to the strange quark mass.
|                  | Improved |             | Wilson | \(\beta = 6.25\) | \(\beta = 6.8\) | \(\beta = 4.5\) | \(\beta = 5.5\) | Exp. |
|------------------|----------|-------------|--------|-------------------|-----------------|-----------------|-----------------|------|
| \(M_\rho a_\rho\) | 1.19(5)  | 0.90(5)     | 0.90(2) | 0.71(3)          |
| \(a_\rho^{-1}\)  | 648(27)MeV | 855(45)MeV | 858(15)MeV | 1085(46)MeV |
| \(a_{st}/a_\rho\) | 1.31(5)  | 1.17(7)     | 1.73(3) | 1.50(6)          |
| \(J\)            | 0.43(8)  | 0.38(7)     | 0.31(2) | 0.32(7)          |
| \(K/\rho\)       | 0.65(2)  | 0.65(4)     | 0.61(1) | 0.61(4) | 0.64 |
| \(K^*/\rho\)     | 1.17(3)  | 1.16(4)     | 1.10(1) | 1.13(4) | 1.16 |
| \(\phi/\rho\)    | 1.31(4)  | 1.30(6)     | 1.20(2) | 1.24(6) | 1.32 |
| \(N/\rho\)       | 1.55(6)  | 1.36(9)     | 2.05(5) | 1.73(14) | 1.22 |
| \(\Delta/N\)     | 1.34(4)  | 1.38(11)    | 1.07(2) | 1.24(10) | 1.31 |
| \(\Sigma/N\)     | 1.15(2)  | 1.20(4)     | 1.05(1) | 1.10(5) | 1.27 |
| \(\Xi/N\)        | 1.23(3)  | 1.32(4)     | 1.09(1) | 1.15(7) | 1.40 |
| \(\Lambda/N\)    |          | 1.17(4)     |         | 1.08(5) | 1.19 |
| \(\Omega^-/N\)   | 1.58(4)  | 1.67(10)    | 1.19(2) | 1.38(10) | 1.78 |

Table 2: Results of the calculations extrapolated to the limit \(M_\pi = 0\).
Figure Captions

1. Nonstrange hadron masses in lattice units versus pion mass squared (squares = $\rho$-meson, circles = nucleon, triangles = Delta). The lines are the extrapolations to the chiral limit. (a) Improved action, $\beta = 6.25$, (b) Improved action, $\beta = 6.8$, (c) Wilson action, $\beta = 4.5$, (d) Wilson action, $\beta = 5.5$.

2. The ratio of the lattice spacing determined by the string tension to the lattice spacing determined by the $\rho$-meson mass $a_{st}/a_\rho$ versus $M_\rho a$ for the Wilson (solid symbols) and Improved actions (open symbols). For the Wilson action points below $M_\rho a = 0.6$, the string tension results of Bali and Schilling$^{[2]}$ were used.

3. The ratio $M_N/M_\rho$ versus $M_\rho a$ for the Wilson (solid symbols) and Improved actions (open symbols).
