On Fuzzy Regular Volterra Spaces
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Abstract

The aim of this paper is to introduce the concepts of regular $G_\delta$-sets, regular $F_\sigma$-sets and regular Volterra spaces in fuzzy setting are introduced and studied. Several characterizations of fuzzy regular Volterra spaces in terms of fuzzy regular $F_\sigma$-sets, fuzzy first category sets, fuzzy residual sets and fuzzy $\sigma$-nowhere dense sets are also established in this paper.

Key words: Fuzzy open set, fuzzy dense set, fuzzy nowhere dense set, fuzzy $\sigma$-nowhere dense set, fuzzy $G_\delta$-set, fuzzy $F_\sigma$-set, fuzzy first category set, fuzzy residual set, fuzzy $\beta$-open set and fuzzy Volterra spaces.

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1. Introduction

In 1970, J. Mack [6] introduced the concepts of regular $G_\delta$-sets and regular $F_\sigma$-sets in classical topology. K.K.Azad [1] introduced fuzzy regular open and fuzzy regular closed sets in 1981. The concepts of regular $G_\delta$-sets and regular $F_\sigma$-sets in fuzzy setting are introduced and studied in this paper. By using fuzzy regular $G_\delta$-sets, the concept of fuzzy regular Volterra spaces is introduced in this paper. Several characterizations of fuzzy regular Volterra spaces in terms of fuzzy regular $F_\sigma$-sets, fuzzy first category sets, fuzzy residual sets and fuzzy $\sigma$-nowhere dense sets are also established in this paper.

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2. Preliminaries

In 1965, L.A. Zadeh [10] introduced the concept of fuzzy set $\lambda$ on a base set $X$ as a function from $X$ into the unit interval $I = [0, 1]$. This function is also called a membership function. A membership function is a generalization of a characteristic function.

**Definition 2.1** [5] Let $\lambda$ and $\mu$ be fuzzy sets in $X$. Then for all $x \in X$,

1. $\lambda = \mu \iff \lambda(x) = \mu(x)$,
2. $\lambda \leq \mu \iff \lambda(x) \leq \mu(x)$,
3. $\psi = \lambda \lor \mu \iff \psi(x) = \max\{\lambda(x), \mu(x)\}$,
4. $\delta = \lambda \land \mu \iff \delta(x) = \min\{\lambda(x), \mu(x)\}$,
5. $\eta = \lambda^c \iff \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in $X$, the union $\psi = \bigvee_i \lambda_i$ and intersection $\delta = \bigwedge_i \lambda_i$ are defined by $\psi(x) = \sup_i \{\lambda_i(x), x \in X\}$, and $\delta(x) = \inf_i \{\lambda_i(x), x \in X\}$.

The fuzzy set $0_X$ is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set $1_X$ defined as $1_X(x) = 1$, for all $x \in X$.

**Definition 2.2** [5] A fuzzy topology is a family $T$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $\Phi, X \in T$,
2. If $A, B \in T$, then $A \cap B \in T$,
3. If $A_i \in T$, for each $i \in I$, then $\bigcup_{i \in I} A_i \in T$.

$T$ is called a fuzzy topology for $X$ and the pair $(X, T)$ is a fuzzy topological space or fts in short. Every member of $T$ is called a $T$-open fuzzy set. A fuzzy set is $T$-closed if and only if its complement is $T$-open. When no confusion is likely to arise, we shall call a $T$-open ($T$-closed) fuzzy set simply an open (closed) fuzzy set.

**Lemma 2.3** [1] For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space $X$. Then, $\bigvee cl \lambda_\alpha \leq cl(\bigvee \lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\bigvee cl \lambda_\alpha = cl(\bigvee \lambda_\alpha)$. Also $\bigvee int \lambda_\alpha \leq int(\bigvee \lambda_\alpha)$.

**Definition 2.4** [2] A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy $F_\sigma$-set in $(X, T)$ if $\lambda = \bigvee_{i=1}^\infty (\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$. 

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Definition 2.5 [2] A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy $G_\delta$-set in $(X, T)$ if $\lambda = \bigwedge_{i=1}^{\infty} (\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.6 [7] A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy dense set if there exists no fuzzy closed set $\mu$ in $(X, T)$ such that $\lambda < \mu < 1$.

Definition 2.7 [7] Let $(X, T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X, T)$ is called a fuzzy nowhere dense set if there exists no non-zero fuzzy open set $\mu$ in $(X, T)$ such that $\mu < \text{cl}(\lambda)$. That is, $\text{int cl}(\lambda) = 0$.

Definition 2.8 [7] Let $(X, T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X, T)$ is called a fuzzy first category set if $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$, where $(\lambda_i)$’s are fuzzy nowhere dense sets in $(X, T)$. Any other fuzzy set in $(X, T)$ is said to be of fuzzy second category.

Definition 2.9 [7] Let $\lambda$ be a fuzzy first category set in a fuzzy topological space $(X, T)$. Then $1 - \lambda$ is called a fuzzy residual set in $(X, T)$.

Definition 2.10 [8] Let $(X, T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X, T)$ is called a fuzzy $\sigma$-nowhere dense set if $\lambda$ is a fuzzy $F_\sigma$-set in $(X, T)$ such that $\text{int cl}(\lambda) = 0$.

Definition 2.11 A fuzzy set $\lambda$ in a fuzzy topological space $X$ is called

(1) fuzzy pre-open if $\lambda \leq \text{int cl}(\lambda)$ and fuzzy pre-closed if $\text{cl int}(\lambda) \leq \lambda$ [4],
(2) fuzzy semi-open if $\lambda \leq \text{cl int}(\lambda)$ and fuzzy semi-closed if $\text{int cl}(\lambda) \leq \lambda$ [4],
(3) fuzzy $\beta$-open if $\lambda \leq \text{cl int cl}(\lambda)$ and fuzzy $\beta$-closed if $\text{int cl int}(\lambda) \leq \lambda$ [3],
(4) fuzzy regular open if $\text{int cl}(\lambda) = \lambda$ and fuzzy regular closed if $\text{cl int}(\lambda) = \lambda$ [1].

Definition 2.12 [9] A fuzzy topological space $(X, T)$ is called a fuzzy Volterra space if $\text{cl} \left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 1$, where $(\lambda_i)$’s are fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$.

Theorem 2.13 [1] In a fuzzy topological space $(X, T)$,

(a). The closure of a fuzzy open set is a fuzzy regular closed set
(b). The interior of a fuzzy closed set is a fuzzy regular open set.

3. Fuzzy regular $G_\delta$-sets

Definition 3.1 A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy regular $G_\delta$-set if $\lambda = \bigwedge_{i=1}^{\infty} (\text{int}(\lambda_i))$, where $1 - \lambda_i \in T$. 

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Definition 3.2 A fuzzy set $\mu$ in a fuzzy topological space $(X, T)$ is called a fuzzy regular $F_{\sigma}$-set if $\mu = \vee_{i=1}^{\infty}(cl(\mu_i))$, where $\mu_i \in T$.

Proposition 3.3 If $\lambda$ is a fuzzy regular $G_{\delta}$-set in a fuzzy topological space $(X, T)$ if and only if $1 - \lambda$ is a fuzzy regular $F_{\sigma}$-set in $(X, T)$.

Proof: Let $\lambda$ be a fuzzy regular $G_{\delta}$-set in $(X, T)$. Then $\lambda = \bigwedge_{i=1}^{\infty}(int(\lambda_i))$, where $1 - \lambda_i \in T$. Now $1 - \lambda = 1 - \bigwedge_{i=1}^{\infty}(int(\lambda_i)) = \bigvee_{i=1}^{\infty}(cl(1 - \lambda_i))$. Let $\mu_i = 1 - \lambda_i$. Then $\mu_i \in T$. Hence $1 - \lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))$, $\mu_i \in T$. Therefore $1 - \lambda$ is a fuzzy regular $F_{\sigma}$-set in $(X, T)$.

Conversely, let $\lambda$ be a fuzzy regular $F_{\sigma}$-set in $(X, T)$. Then $\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))$, where $\mu_i \in T$. Now $1 - \lambda = 1 - \bigvee_{i=1}^{\infty}(cl(\mu_i)) = \bigwedge_{i=1}^{\infty}(1 - cl(\mu_i)) = \bigwedge_{i=1}^{\infty}(int(1 - \mu_i))$. Let $1 - \mu_i = \lambda_i$. Then implies that $\mu_i = 1 - \lambda_i$ and $1 - \lambda_i \in T$. Hence $1 - \lambda = \bigwedge_{i=1}^{\infty}(int(\lambda_i))$, where $1 - \lambda_i \in T$. Therefore $1 - \lambda$ is a fuzzy regular $G_{\delta}$-set in $(X, T)$.

Proposition 3.4 Let $(X, T)$ be a fuzzy topological space.

(1). If $\lambda$ is a fuzzy regular $G_{\delta}$-set in $(X, T)$, then $\lambda = \bigwedge_{i=1}^{\infty}(\delta_i)$, where $(\delta_i)$’s are fuzzy regular open sets in $(X, T)$.

(2). If $\lambda$ is a fuzzy regular $F_{\sigma}$-set in $(X, T)$, then $\lambda = \bigvee_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)$’s are fuzzy regular closed sets in $(X, T)$.

Proof: (1). Let $\lambda$ be a fuzzy regular $G_{\delta}$-set in $(X, T)$. Then $\lambda = \bigwedge_{i=1}^{\infty}(int(\lambda_i))$, where $1 - \lambda_i \in T$. Now $1 - \lambda_i \in T$ implies that $\lambda_i$ is a fuzzy closed set in $(X, T)$. By theorem 2.13, $int(\lambda_i)$ is a fuzzy regular open set in $(X, T)$. Let $\delta_i = int(\lambda_i)$. Then $\lambda = \bigwedge_{i=1}^{\infty}(\delta_i)$, where $(\delta_i)$’s are fuzzy regular open sets in $(X, T)$.

(2). Let $\lambda$ be a fuzzy regular $F_{\sigma}$-set in $(X, T)$. Then $\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))$, where $\mu_i \in T$. Now $\mu_i \in T$. By theorem 2.13, $cl(\mu_i)$ is a fuzzy regular closed set in $(X, T)$. Let $\eta_i = cl(\mu_i)$. Then $\lambda = \bigvee_{i=1}^{\infty}(\eta_i)$, where $(\eta_i)$’s are fuzzy regular closed sets in $(X, T)$.

Proposition 3.5 If $\lambda$ is a fuzzy regular $G_{\delta}$-set in a fuzzy topological space $(X, T)$, then $\lambda$ is a fuzzy $G_{\delta}$-set in $(X, T)$.

Proof: Let $\lambda$ be a fuzzy regular $G_{\delta}$-set in $(X, T)$. Then by proposition 3.4, $\lambda = \bigwedge_{i=1}^{\infty}(\delta_i)$, where $(\delta_i)$’s are fuzzy regular open sets in $(X, T)$. Since every fuzzy regular
open set is a fuzzy open set in \((X,T)\), \((\delta_i)\)'s are fuzzy open sets in \((X,T)\). Hence, \(\lambda = \wedge_{i=1}^{\infty}(\delta_i)\), where \(\delta_i \in T\). Therefore \(\lambda\) is a fuzzy \(G_\delta\)-set in \((X,T)\).

**Proposition 3.6** If \(\lambda\) is a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X,T)\), then \(\lambda\) is a fuzzy \(F_\sigma\)-set in \((X,T)\).

**Proof:** Let \(\lambda\) be a fuzzy regular \(F_\sigma\)-set in \((X,T)\). Then by proposition 3.4, \(\lambda = \vee_{i=1}^{\infty}(\eta_i)\), where \((\eta_i)\)'s are fuzzy regular closed sets in \((X,T)\). Since every fuzzy regular closed set is a fuzzy closed set in \((X,T)\), \((\eta_i)\)'s are fuzzy closed sets in \((X,T)\). Hence \(\lambda = \vee_{i=1}^{\infty}(\eta_i)\), where \(1 - \eta_i \in T\). Therefore \(\lambda\) is a fuzzy \(F_\sigma\)-set in \((X,T)\).

**Proposition 3.7** If \(cl(\wedge_{i=1}^{\infty}int(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy closed sets in a fuzzy topological space \((X,T)\), then \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\).

**Proof:** Suppose that \(cl(\wedge_{i=1}^{\infty}int(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy closed sets in \((X,T)\). But \(cl(\wedge_{i=1}^{\infty}int(\lambda_i)) \leq \wedge_{i=1}^{\infty}clint(\lambda_i)\). Then, \(1 \leq \wedge_{i=1}^{\infty}clint(\lambda_i)\). That is, \(\wedge_{i=1}^{\infty}clint(\lambda_i) = 1\). This implies that \(clint(\lambda_i) = 1,...,1\). Since \((\lambda_i)\)'s are fuzzy closed sets in \((X,T)\), \(cl(\lambda_i) = \lambda_i\). Then \(clintcl(\lambda_i) = 1\). From (1), \(\lambda_i \leq clintcl(\lambda_i)\). Therefore, \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\).

**Proposition 3.8** If a fuzzy regular \(G_\delta\)-set \(\lambda\) is a fuzzy dense set in a fuzzy topological space \((X,T)\), then \(\lambda = \wedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\).

**Proof:** Let \(\lambda\) be a fuzzy regular \(G_\delta\)-set in \((X,T)\) such that \(cl(\lambda) = 1\). Then \(\lambda = \wedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy closed sets in \((X,T)\) and \(cl[\wedge_{i=1}^{\infty}int(\lambda_i)] = cl(\lambda) = 1\). Then, by proposition 3.7, \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\). Therefore, \(\lambda = \wedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\).

**Proposition 3.9** If \(int(\mu) = 0\), where \(\mu\) is a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X,T)\), then \(\mu = \vee_{i=1}^{\infty}cl(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-closed sets in \((X,T)\).

**Proof:** Let \(\mu\) be a fuzzy regular \(F_\sigma\)-set in \((X,T)\) such that \(int(\mu) = 0\). Then, by proposition 3.3, \(1 - \mu\) is a fuzzy regular \(G_\delta\)-set in \((X,T)\) and \(cl(1 - \mu) = 1 - int(\mu) = 1 - 0 = 1\). Now, by proposition 3.8, \(1 - \mu = \wedge_{i=1}^{\infty}int(\mu_i)\), where \((\mu_i)\)'s are fuzzy \(\beta\)-open sets in \((X,T)\). Hence \(\mu = 1 - \wedge_{i=1}^{\infty}int(\mu_i) = \vee_{i=1}^{\infty}(1 - int(\mu_i)) = \)
\[ \forall_{i=1}^{\infty} \text{cl}(1 - \mu_i). \] Since \((\mu_i)\)'s are fuzzy \(\beta\)-open sets, \((1 - \mu_i)\)'s are fuzzy \(\beta\)-closed sets in \((X, T)\). Let \(\lambda_i = 1 - \mu_i\). Therefore, \(\mu = \forall_{i=1}^{\infty} \text{cl}(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-closed sets in \((X, T)\).

### 4. Fuzzy regular Volterra spaces

**Definition 4.1** A fuzzy topological space \((X, T)\) is called a fuzzy regular Volterra space if \(\text{cl}(\wedge_{i=1}^{N} (\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\).

**Proposition 4.2** If \(\text{int}(\forall_{i=1}^{N} (\mu_i)) = 0\) where \((\mu_i)\)'s are fuzzy regular \(F_\sigma\)-sets with \(\text{int}(\mu_i) = 0\) in a fuzzy topological space \((X, T)\), then \((X, T)\) is a fuzzy regular Volterra space.

**Proof:** Suppose that \(\text{int}(\forall_{i=1}^{N} (\mu_i)) = 0\), where \((\mu_i)\)'s are fuzzy regular \(F_\sigma\)-sets with \(\text{int}(\mu_i) = 0\). Now \(1 - \text{int}(\forall_{i=1}^{N} (\mu_i)) = 1\). Then, \(\text{cl}(1 - \forall_{i=1}^{N} (\mu_i)) = 1\). This implies that \(\text{cl}(\wedge_{i=1}^{N} (1 - \mu_i)) = 1\). Since \((\mu_i)\)'s are fuzzy regular \(F_\sigma\)-sets in \((X, T)\), by proposition 3.3, \((1 - \mu_i)\)'s are fuzzy regular \(G_\delta\)-sets in \((X, T)\). Also, \(\text{int}(\mu_i) = 0\) implies that \(1 - \text{int}(\mu_i) = 1\). Then, \(\text{cl}(1 - \mu_i) = 1\). Let \(\lambda_i = 1 - \mu_i\). Then \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Hence, \(\text{cl}(\wedge_{i=1}^{N} (\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Therefore \((X, T)\) is a fuzzy regular Volterra space.

**Remark:** In view of the propositions 3.9 and 4.2, one will have the following result: “If \(\text{int}(\forall_{i=1}^{\infty} (\eta_i)) = 0\), where \((\eta_i)\)'s are fuzzy \(\beta\)-closed sets in a fuzzy topological space \((X, T)\), then \((X, T)\) is a fuzzy regular Volterra space”.

**Proposition 4.3** If a fuzzy topological space \((X, T)\) is a fuzzy Volterra space, then \((X, T)\) is a fuzzy regular Volterra space.

**Proof:** Let \((X, T)\) be a fuzzy Volterra space. Let \(\lambda = \text{cl}(\wedge_{i=1}^{N} (\lambda_i))\).....(1), where \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). By proposition 3.5, the fuzzy regular \(G_\delta\)-sets \((\lambda_i)\)'s are fuzzy \(G_\delta\)-sets in \((X, T)\). Since \((X, T)\) is a fuzzy Volterra space, \(\text{cl}(\wedge_{i=1}^{N} (\lambda_i)) = 1\).....(2), where \((\lambda_i)\)'s are fuzzy dense and fuzzy \(G_\delta\)-sets in \((X, T)\). Hence, from (1) and (2), \(\lambda = 1\). Therefore \((X, T)\) is a fuzzy regular Volterra space.

**Proposition 4.4** If a fuzzy topological space \((X, T)\) is a fuzzy regular Volterra space, then \(\text{int}(\forall_{i=1}^{N} (\mu_i)) = 0\), where \((\mu_i)\)'s are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\).
Proof: Let \((X, T)\) be a fuzzy regular Volterra space. Then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Now \(1 - cl(\bigwedge_{i=1}^{N}(\lambda_i)) \leq 0\) implies that \(int(\bigvee_{i=1}^{N}(1 - \lambda_i)) = 0\). Since \((\lambda_i)\)'s are fuzzy regular \(G_\delta\)-sets, by proposition 3.3, \((1 - \lambda_i)\)'s are fuzzy regular \(F_\sigma\)-sets in \((X, T)\). By proposition 3.6, \((1 - \lambda_i)\)'s are fuzzy \(F_\sigma\)-sets in \((X, T)\). Also, \(cl(\lambda_i) = 1\) implies that \(1 - cl(\lambda_i) = 0\) and hence \(int(1 - \lambda_i) = 0\). Let \(\mu_i = 1 - \lambda_i\). Then \((\mu_i)\)'s are fuzzy \(F_\sigma\)-sets with \(int(\mu_i) = 0\). Then, by the definition of fuzzy \(\sigma\)-nowhere dense sets, \((\mu_i)\)'s are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\). Hence \(int(\bigvee_{i=1}^{N}(\mu_i)) = 0\), where \((\mu_i)\)'s are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\).

**Proposition 4.5** If \(int(\lambda) = 0\) for a fuzzy regular \(F_\sigma\)-set \(\lambda\) in a fuzzy topological space \((X, T)\), then \(\lambda\) is a fuzzy first category set in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(F_\sigma\)-set in \((X, T)\). Then \(\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))\), where \(\mu_i \in T\). Now \(int(\lambda) = 0\) implies that \(int(\bigvee_{i=1}^{\infty}(cl(\mu_i))) = 0\). But \(\bigvee_{i=1}^{\infty}(int(cl(\mu_i))) \leq int(\bigvee_{i=1}^{\infty}(cl(\mu_i))) = 0\). Then, \(\bigvee_{i=1}^{\infty}(int(cl(\mu_i))) = 0\). This implies that \(int(cl(\mu_i)) = 0\). Hence \(\mu_i\) is a fuzzy nowhere dense set in \((X, T)\). Also \(int(cl(\mu_i)) = int(cl(\mu_i)) = 0\) implies that \(cl(\mu_i)\) is a fuzzy nowhere dense set in \((X, T)\). Hence \(\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))\), where \((cl(\mu_i))\)'s are fuzzy nowhere dense sets in \((X, T)\). Therefore \(\lambda\) is a fuzzy first category set in \((X, T)\).

Remark: In view of the propositions 3.9 and 4.5 one will have the following result: “If \(int(\lambda) = 0\), for a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X, T)\), then \(\lambda = \bigvee_{i=1}^{\infty}(cl(\lambda_i))\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-closed sets in \((X, T)\), is a fuzzy first category set in \((X, T)\)”.

**Proposition 4.6** If a fuzzy regular \(G_\delta\)-set \(\lambda\) is a fuzzy dense set in a fuzzy topological space \((X, T)\), then \(\lambda\) is a fuzzy residual set in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(G_\delta\)-set with \(cl(\lambda) = 1\). Then \(1 - \lambda\) is a fuzzy regular \(F_\sigma\)-set with \(1 - cl(\lambda) = 0\). That is, \(1 - \lambda\) is a fuzzy regular \(F_\sigma\)-set with \(int(1 - \lambda) = 0\). Then by proposition 4.5, \(1 - \lambda\) is a fuzzy first category set in \((X, T)\). Therefore \(\lambda\) is a fuzzy residual set in \((X, T)\).

**Proposition 4.7** If a fuzzy topological space \((X, T)\) is a fuzzy regular Volterra space, then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\) where \((\lambda_i)\)'s are fuzzy residual sets in \((X, T)\).

Proof: Let \((X, T)\) be a fuzzy regular Volterra space. Then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). By proposition 4.6
$(\lambda_i)$’s are fuzzy residual sets in $(X,T)$. Hence $cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1$, where $(\lambda_i)$’s are fuzzy residual sets in $(X,T)$.

**Proposition 4.8** If a fuzzy topological space $(X,T)$ is a fuzzy regular Volterra space, then $int(\bigvee_{i=1}^{N}(\mu_i)) = 0$, where $(\mu_i)$’s are fuzzy first category sets in $(X,T)$.

Proof: Let $(X,T)$ be a fuzzy regular Volterra space. Then $cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1$, where $(\lambda_i)$’s are fuzzy dense and fuzzy regular $G_\delta$-sets in $(X,T)$. Now $1-cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 0$ implies that $int(1-\bigwedge_{i=1}^{N}(\lambda_i)) = 0$. Then, $int(\bigvee_{i=1}^{N}(1-\lambda_i)) = 0$.

Now $(\lambda_i)$’s are fuzzy regular $G_\delta$-sets in $(X,T)$ implies that $(1-\lambda_i)$’s are fuzzy regular $F_\sigma$-sets in $(X,T)$. Also $cl(\lambda_i) = 1$ implies that $1-cl(\lambda_i) = 0$. Then $int(1-\lambda_i) = 0$. Hence, $(1-\lambda_i)$’s are fuzzy regular $F_\sigma$-sets with $int(1-\lambda_i) = 0$. Therefore by proposition 4.5, $(1-\lambda_i)$’s are fuzzy first category sets in $(X,T)$. Let $\mu_i = 1-\lambda_i$. Hence if $(X,T)$ is a fuzzy regular Volterra space, then $int(\bigvee_{i=1}^{N}(\mu_i)) = 0$, where $(\mu_i)$’s are fuzzy first category sets in $(X,T)$.

**5. Conclusion**

In this paper, the concepts of fuzzy regular $G_\delta$-sets, fuzzy regular $F_\sigma$-sets and fuzzy regular Volterra spaces have introduced and studied. Several characterizations of fuzzy regular Volterra spaces have established in this paper.

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