THE EXCEPTIONAL ZERO PHENOMENON FOR ELLIPTIC UNITS

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Abstract. The exceptional zero phenomenon has been widely studied in the realm of $p$-adic $L$-functions, where the starting point lies in the foundational work of Mazur, Tate and Teitelbaum. This phenomenon also appears in the study of Euler systems, which comes as no surprise given the interaction between these two settings. When this occurs, one is led to study higher order derivatives of the Euler system in order to extract the arithmetic information which is usually encoded in the explicit reciprocity laws. In this work, we focus on the elliptic units of an imaginary quadratic field and study this exceptional zero phenomenon, proving an explicit formula relating the logarithm of a derived elliptic unit either to special values of Katz’s two variable $p$-adic $L$-function or to its derivatives. Further, we interpret this fact in terms of an $L$-invariant, and relate this result to other approaches to the exceptional zero phenomenon concerning Heegner points and Beilinson–Flach elements.

1. Introduction

Since the introduction of the exceptional zero phenomenon for the $p$-adic $L$-function attached to an elliptic curve by Mazur, Tate and Teitelbaum [MTT], a lot of progress has been made in that direction. The main goal of this article is to study exceptional zero phenomena for Katz’s two-variable $p$-adic $L$-function at points lying outside the region of classical interpolation, where the Euler system of elliptic units vanishes. Hence, our setting departs notably from loc. cit. and is closer in spirit to e. g. [Cas1] and [RiRo1].

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Fix once for all a prime \( p \) and a quadratic imaginary field \( K \) in which \( p \) splits. Let \( p \) and \( \bar{p} \) be the two prime ideals lying over \( p \) (assume in the introduction for notational simplicity that both are principal, and with a slight abuse of notation we denote by \( p \) and \( \bar{p} \) the generators of these ideals). Let \( K^\text{cy} \) and \( K^\text{ac} \) be the cyclotomic and anticyclotomic \( \mathbb{Z}_p \)-extensions of \( K \), respectively, and set \( K_\infty = K^\text{ac}K^\text{cy} \). Denote \( \Gamma_K = \text{Gal} \left( \frac{K_\infty}{K} \right) \) and \( \Lambda_K = \mathbb{Z}_p[[\Gamma_K]] \). Weight space is defined as the formal spectrum \( \text{Spf}(\Lambda_K) \) of the two-variable Iwasawa algebra \( \Lambda_K \). Let \( \psi \) stand for a Hecke character of finite order, conductor \( n \) and taking values in a number field \( L \), and let \( \mathbb{N} \) denote the norm character of \( K \). We denote by \( N \) the norm of \( n \), and assume that \( (N,p) = 1 \).

As an additional piece of notation, let \( H_\infty \) denote the unique \( \mathbb{Z}_p \)-extension of \( K \) in which \( \bar{p} \) is unramified (therefore, the prime \( p \) is ramified in \( H_\infty/K \)). We choose a character \( \lambda \) so that it factors through \( \text{Gal} \left( \frac{H_\infty}{K} \right) \), defining an isomorphism

\[
\text{Gal} \left( \frac{H_\infty}{K} \right) \to 1 + p\mathbb{Z}_p.
\]

The choice of \( \lambda \) is unique once we require that it be the Galois representation corresponding to a Grössencharacter for \( K \) of infinity type \((1,0)\). Although characters of \( \Gamma_K \) are elements of \( \text{Hom}_{\text{cont}} (G_K, \mathbb{Q}_p^\times) \), we are interested in the restriction to

\[
\{ \psi|_{\mathbb{N}^\infty} \lambda \text{ such that } (s,t) \in \mathbb{Z}_p^2 \},
\]

where \( \xi \) is a finite order character of \( p \)-power conductor.

One can naturally associate, via the theory of elliptic units of [Yag] and Kummer maps, a global cohomology class to \( \psi \)

\[
\kappa_{\psi,\infty} \in H^1 \left( K, \Lambda_K(\psi^{-1}) \right),
\]

which can be characterized by its specializations at characters of the form \( \psi|\mathbb{N} \), where \( \xi \) is a \( p \)-power order character of \( \Gamma_K \). We denote by \( \kappa_{\psi} \) and \( \kappa_{\psi|\mathbb{N}} \) the specializations of \( \kappa_{\psi,\infty} \) at \( \psi \) and \( \psi|\mathbb{N} \), respectively. In section 4, se prove that \( \kappa_{\psi} \) never vanishes, while \( \kappa_{\psi|\mathbb{N}} \) vanishes if and only if \( \psi(p) = 1 \) or \( \psi(\bar{p}) = 1 \). In this case, there exists a notion of derived cohomology class along the line \( \mathcal{C} \), that we define as the Zariski closure of the points \( \psi|\mathbb{N} \lambda^t \) with \( t \in \mathbb{Z}_{\geq 0} \),

\[
\kappa'_{\psi,\infty} \in H^1 \left( K, \Lambda_K(\psi^{-1})|_{\mathcal{C}} \right).
\]

The specialization of this cohomology class at \( \psi|\mathbb{N} \) encodes the arithmetic information which is usually carried out by \( \kappa_{\psi|\mathbb{N}} \) in a non-exceptional situation. To be more explicit, denote by \( u_{a,n} \) the classical Siegel unit, and to lighten notations, define

\[
(1) \quad u_\psi = \prod_{a=1}^{N-1} u_{\psi^{-1}(a)}.
\]

We expect a relation between \( \kappa'_{\psi|\mathbb{N}} \) and \( u_\psi \), and this is the content of the main result of this note, which we now state.

Assume that \( \psi(\bar{p}) = 1 \), and let us define the following \( \mathcal{L} \)-invariant

\[
(2) \quad \mathcal{L}(\psi) = -(1 - \psi(p)) \cdot \text{log}(p).
\]

Then, we have the following.

**Theorem 1.1.** Suppose that \( \psi(\bar{p}) = 1 \). Then,

\[
(3) \quad \kappa'_{\psi|\mathbb{N}} = \mathcal{L}(\psi) \cdot u_\psi.
\]

Although the previous result does not require any explicit mention to the theory of \( p \)-adic \( L \)-functions, it is fair to say that Katz’s two variable \( p \)-adic \( L \)-function plays a prominent role in our results. The interplay between the Euler system of elliptic units and Katz’s two variable \( p \)-adic \( L \)-function can be set as a very particular case of a wider theory. One may distinguish two main approaches to construct a \( p \)-adic \( L \)-function.
(a) Firstly, interpolating the algebraic parts of the special values of the classical $L$-function along the so-called critical region. This requires, as a starting point, the proof of certain algebraicity results.

(b) Secondly, as the image under a certain Perrin-Riou map of a family of cohomology classes, constructed along the so-called geometric region. These classes are typically obtained as the image under certain regulators of distinguished elements arising in the geometry of algebraic varieties.

In both approaches, the $p$-adic $L$-function is completely characterized by the value at the points lying either at the critical or at the geometric region. Moreover, some Euler factors arise, and the vanishing of these factors lead us to consider exceptional zero phenomena. In the case of the Perrin-Riou map, the shape of these factors is $\frac{1-p^j\phi}{1-p^{j+1}\phi}$, where $j$ is related with the Hodge–Tate type of the character at which we are specializing, and $\phi$ refers to a Frobenius eigenvalue. As it is suggested for instance in [KLZ, §8] or [LZ], there are two kinds of Euler factors in the usual Perrin-Riou maps: those appearing in the numerator (which typically lead to an exceptional vanishing of the $p$-adic $L$-function via explicit reciprocity laws) and those appearing in the denominator (which lead to an exceptional vanishing of the cohomology class). While the former phenomenon has been widely studied, as far as we know the latter has only been discussed in the setting of Heegner points in [Cas1] and for Beilinson–Flach elements in [RiRo1].

Katz’s two-variable $p$-adic $L$-function, $L_p(K)(\cdot)$, is defined on the domain $\text{Hom}_\text{cont}(G_K, \bar{\mathbb{Q}}_p^\times)$, but we can consider the restriction to

$$\{\psi|\mathfrak{N}^s\mathfrak{M}^t\text{ such that } (s, t) \in \mathbb{Z}_p^2\},$$

where $\xi$ is a finite order character of $p$-power conductor. In this case, we write $L_p(K, \psi)(\cdot)$ and denote $L_p(K, \psi)(\chi_{\text{triv}}) := L_p(K)(\psi)$ and $L_p(K, \psi)(\mathfrak{N}) := L_p(K)(\psi)\mathfrak{N}$.

We can now describe the main steps involved in the proof of Theorem 1.1:

(a) An explicit reciprocity law for Katz’s two-variable $p$-adic $L$-function. This expresses the special value $L_p(K, \psi)(\mathfrak{N})$ in terms of the image under a Perrin-Riou map of the cohomology class $\kappa_{\psi}\mathfrak{N}$, and directly gives us that $\log_p(\kappa_{\psi}\mathfrak{N}) = 0$, due to the vanishing of an Euler factor. The explicit description of the localization-at-$p$ map shows that we can conclude that $\kappa_{\psi}\mathfrak{N} = 0$ and consider the derived cohomology class. We refer the reader to Sections 3.3 and 1.2 for details on that.

(b) A derived reciprocity law, expressing the logarithm of the derived class in terms of $L_p(K, \psi)(\mathfrak{N})$. This requires an explicit description of the Perrin-Riou map, which at the norm character interpolates the Bloch–Kato logarithm and gives a map

$$\log_{BK} : H^1(K_p, L_p(\psi^{-1})(1)) \to \mathbb{D}_{\text{dR}}(L_p(\psi^{-1})(1)) \simeq L_p.$$ 

Under the identification $H^1(K_p, L_p(\psi^{-1})(1)) \simeq (L_p^\times)^{\psi^{-1}}$, this map corresponds to the usual Iwasawa logarithm. Then, we have the following result, whose proof is given along Section 4.2

**Proposition 1.2.** Assume that $\psi(\bar{p}) = 1$. Then,

$$\log(p) \cdot L_p(K, \psi)(\mathfrak{N}) = -(1 - p^{-1}) \cdot \log_{BK}(\log_p(\kappa'_{\psi}\mathfrak{N})).$$

(c) The functional equation for Katz’s two-variable $p$-adic $L$-function, which asserts that

$$L_p(K, \psi)(\mathfrak{N}) = L_p(K, \bar{\psi}^{-1})(\chi_{\text{triv}}).$$

(d) Katz’s $p$-adic version of Kronecker limit formula, expressing the special value $L_p(K, \psi^{-1})(\chi_{\text{triv}})$ in terms of the unit $u_{\psi}$

$$L_p(K, \bar{\psi}^{-1})(\chi_{\text{triv}}) = (1 - \psi(p))(1 - \psi^{-1}(\bar{p})p^{-1}) \cdot \log_p(u_{\psi}).$$
In Section 3.2 we properly discuss the main features of Katz’s two-variable $p$-adic $L$-function.

As a by-product of the previous discussion, along the text we also deal with other instances of the exceptional zero phenomenon. More precisely, the results of Section 4 encompass two main situations: the exceptional vanishing of $\kappa_\psi N$ and how to prove a reciprocity law for the derived class in this setting; and the exceptional vanishing of the $p$-adic $L$-function, which is a more well-established phenomenon that has been widely studied in the literature.

Once these results have been developed, the last section of the article serves to analyze how our results fit with similar statements concerning exceptional zero phenomena. In particular, we emphasize the parallelism, but also the differences, with the theory of Heegner points, as well as the fact that these elliptic units may be seen as a particular case inside the theory of Beilinson–Flach elements, where different instances of the exceptional zero phenomena also appear. In particular, when $g$ is a theta series of an imaginary quadratic field where $p$ splits and we take the pair of modular forms $(g, g^*)$, [RiRo1] describes a connection between a derived Beilinson–Flach element, an elliptic unit and an special value of the Hida–Rankin $p$-adic $L$-function attached to $(g, g^*)$. The regularity assumptions excluded the possibility of elliptic units presenting an exceptional zero, so in a certain way these results can be thought as a degenerate case inside the theory of Beilinson–Flach elements. While our main can be seen as the counterpart of [RiRo1, Theorem B] in the framework of elliptic units, we point out there is another exceptional phenomenon related to the vanishing of the numerator of the Perrin-Riou map, which in this case leads to a trivial zero of Katz’s two variable $p$-adic $L$-function (see Section 4.1) and which in the setting of Beilinson–Flach elements has been studied in [LZ2].

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2. Circular units

Circular units constitute one of the easiest examples of Euler systems, and they play a key role in the proof of the classical Iwasawa main conjecture. We recall here some of their more relevant features because of the parallelism they keep with the theory of elliptic units. We discuss what the exceptional zero phenomenon represents in this case, and then we compare this setting with that of elliptic units.

2.1. Leopoldt’s formula. Along this section, we denote by $\Lambda_{cyc} := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$, and let $W := \text{Spf}(\Lambda_{cyc})$. We fix a primitive, non-trivial even Dirichlet character of conductor $N$,

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{L}^\times$$

where $\mathbb{L}$ is a number field. In particular, for our applications to exceptional zero phenomena, we are interested in the case in which $\chi(p) = 1$. 
Definition 2.1. A classical point is a pair \((k, \xi)\), where \(k\) is an integer and \(\xi\) is a Dirichlet character of \(p\)-power conductor, corresponding to the homomorphism 
\[ z \mapsto z^k \xi(z). \]

The Kubota-Leopoldt \(p\)-adic \(L\)-function is a rigid analytic function 
\[ L_p(\chi, \cdot) : W \to \mathbb{C}_p \]
deﬁned in terms of an interpolation property for the classical points. This interpolation property concerns those classical points of the form \((k, \xi)\), with \(k \leq 0\) and \((\chi \xi)(-1) = (-1)^{k+1}\), or \(k \geq 0\) and \((\chi \xi)(-1) = (-1)^k\). In this deﬁnition, some Euler factors and periods appear. See [Das] for a more detailed exposition.

Special values of \(L\)-series are encoded in terms of the so-called Siegel units 
\[ g_a \in \mathcal{O}_Y^\times(N) \otimes \mathbb{Q} \]
attached to a ﬁxed choice of primitive \(N\)-th root of unity \(\zeta\) and a parameter \(1 \leq a \leq N - 1\). Its \(q\)-expansion is given by
\[ g_a(q) = q^{1/12} (1 - \zeta^a) \prod_{n>0} (1 - q^n \zeta^a)(1 - q^n \zeta^{-a}). \]

Let \(H_\chi\) denote the ﬁeld cut out by \(\chi\), \(H_{\chi,n} = H_\chi(\mu_{p^n})\), and \(\mathbb{Z}_\chi\) the ring generated by its values. Deﬁne the units 
\[ u_{\chi,n} := \prod_{a=1}^{N-1} (1 - \zeta_{p^n}^a)^{-1} \in (\mathcal{O}_{H_{\chi,n}}^\times \otimes \mathbb{Z}_\chi)^\chi, \]
that behave under the norm maps as predicted by the theory of Euler systems:
\[ H_{\chi,n}^{H_{\chi,n+1}}(u_{\chi,n+1}) = \begin{cases} u_{\chi,n} & \text{if } n \geq 1, \\ u_\chi \otimes (\chi(p) - 1) & \text{if } n = 0. \end{cases} \]

Hence, one can construct a coherent family of cohomology classes considering the image under the Kummer map, but taking care of the Euler factor arising at the bottom layer of the system (in concrete, this leads to a trivial zero when \(\chi(p) = 1\)). More precisely, we consider 
\[ \kappa_{\chi,n} := \delta u_{\chi,n} \in H^1(H_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^\chi = H^1(H_{\chi,n}, \mathbb{Z}_{p,\chi}(1)(\chi^{-1})). \]

These classes can be patched all together taking the projective limit for \(n \geq 1\), resulting in an element \(\kappa_{\chi,\infty}\) in 
\[ \lim_{\leftarrow} H^1(H_{\chi,n}, \mathbb{Z}_{p,\chi}(1)(\chi^{-1})) = H^1(Q, \Lambda_{\text{cyc}} \otimes \mathbb{Z}_{p,\chi}(1)(\chi^{-1})). \]

As usual, \(\Lambda_{\text{cyc}}\) can be understood as a \(p\)-adic interpolation of the Tate twists. Let \(Q_{p,\chi} := Q_p \otimes \mathbb{Z}_\chi\). The specialization maps 
\[ \nu_{k,\xi} : \Lambda_{\text{cyc}} \to Q_{p,\xi}(k-1)(\xi^{-1}) \]
give rise to a collection of global cohomology classes 
\[ \kappa_{k,\chi,\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \in H^1(Q, Q_{p,\xi}(k)(\chi \xi)^{-1}). \]

The Gauss sum associated to the character \(\eta\) is deﬁned as 
\[ g(\eta) = \sum_{a=1}^{m-1} \zeta_a^m \otimes \eta(a). \]

From now on, \(\exp^*\) stands for the Bloch–Kato dual exponential map and \(\log_{\text{BK}}\) for the Bloch–Kato logarithm, which agrees with the usual Iwasawa logarithm when \(k = 1\).
Proposition 2.2. There exists a morphism of \( \Lambda \)-modules (referred to as the Perrin-Riou map)
\[
\mathcal{L}_p : H^1(\mathbb{Q}_p, \Lambda_{\text{cyc}} \otimes \mathbb{Z}_p \mathbb{Z}_{p,\chi}(1)(\chi^{-1})) \to \Lambda_{\text{cyc}}
\]
satisfying that for all classical points \((k, \xi)\), the specialization map \(\nu_{k,\chi}(\mathcal{L}_p)\) is the homomorphism
\[
\nu_{k,\chi}(\mathcal{L}_p) : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\chi}(k)(\chi\xi)^{-1}) \to \mathcal{D}_{dR}(\mathbb{Q}_{p,\chi}(k)((\chi\xi)^{-1})) \simeq \mathbb{Q}_{p,\chi}
\]
given by
\[
\nu_{k,\chi}(\mathcal{L}_p) = \frac{1}{g((\chi\xi)^{-1})} \cdot \frac{1 - \chi\xi(p)p^{-k}}{1 - (\chi\xi)^{-1}(p)p^{k-1}} \cdot \left( \frac{(\xi^{-1})^k}{(k-1)!} \log_{\text{BK}} \text{ if } k \geq 1 \right) \cdot \left( (-k)! \exp^* \text{ if } k < 1, \right)
\]
where the target of both the Bloch–Kato logarithm and the dual exponential map is identified with \(\mathbb{Q}_{p,\chi}\).

We finally relate the image of the previous class \(\kappa\) under the Perrin-Riou regulator with the Kubota-Leopoldt \(p\)-adic \(L\)-function. See for instance [BCDDPR] for a more detailed treatment of this result.

Theorem 2.3. The cohomology class \(\kappa_{\chi,\infty} \in H^1(\mathbb{Q}, \Lambda_{\text{cyc}} \otimes \mathbb{Z}_p \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))\) satisfies
\[
L_p(\chi, s) = \mathcal{L}_p(\text{loc}_p(\kappa_{\chi,\infty})).
\]

The previous theorem must be seen as an equality in \(\Lambda\), and we may consider the specialization maps at both sides for any element in \(\mathcal{W}\) (and in particular, for any classical point).

From the previous results, it turns out that one has the equality
\[
L_p(\chi, 1) = -\frac{(1 - \chi\xi(p)p^{-1})}{(1 - (\chi\xi)^{-1}(p))} \times \frac{\log(\text{loc}_p(\kappa_{1,\chi}))}{g((\chi\xi)^{-1})}
\]
whenever \((\chi\xi)(p) \neq 1\); in that case, both the denominator and the numerator vanish.

However, since when \(\chi\) is non-trivial \(L_p(\chi, 1) \in \mathbb{Q}_p^\times\), one has that \(\kappa_{1,\chi} = 0\) if \(\chi(p) = 1\). This suggests the existence of a derived cohomology class \(\kappa_{1,\chi}'\), which is the content of the following section. We recover this idea along the article, but anyway it is good to keep in mind that in this setting one also has a \(p\)-adic Kronecker’s limit formula expressing the value of \(L_p(\chi, 1)\) in terms of a unit in the number field cut out by the character
\[
L_p(\chi, 1) = -\frac{(1 - \chi(p)p^{-1})}{g(\chi^{-1})} \cdot \log_p \left( \prod_{a=1}^{N-1} (1 - \zeta_a^a)^{\chi^{-1}(a)} \right).
\]

The quantity \(\prod_{a=1}^{N-1} (1 - \zeta_a^a)^{\chi^{-1}(a)}\) is typically referred as the circular unit attached to \(\chi\) and we denote it as \(c_\chi\). The previous identity is called Leopoldt’s formula.

Further, recall that a very interesting object of study in the theory of \(L\)-invariants is \(L'_p(\chi, 0)\), when \(\chi(p) = 1\) and therefore \(L_p(\chi, 0) = 0\). One may see that the derivative of \(L'_p(\chi, 0, s)\) at \(s = 0\) encodes information about the dual exponential map of \(\kappa_{0,\chi}\). Furthermore, in [Was], Washington provides an alternative formula for this value. More precisely, if \(\Gamma_p(x)\) stands for the Morita’s \(p\)-adic Gamma function and \(\omega\) for the Teichmüller character, one has that
\[
L'_p(\chi, 0) = \log_p \left( \prod_{a=1}^{N} \Gamma_p(a/N)\chi(a) \right) + L_p(\chi, 0) \log_p(N),
\]
and hence in the situation of exceptional vanishing \(\chi(p) = 1\), one has that
\[
L'_p(\chi, 0) = \log_p(v_\chi),
\]
where

\[ v_\chi = \prod_{a=1}^{N} \Gamma_p(a/N)^{\chi(a)} \, . \]

Finally, we point out that of course the determination of \( L'_p(\chi, 0) \) is a particular case of Gross’ conjectures, as studied by Darmon–Dasgupta–Pollack for arbitrary totally real fields.

### 2.2. Exceptional zeros and circular units

Suppose from now that \( \chi(p) = 1 \). Then, the arguments of the previous section show that the specialization at \( \chi \) vanishes, \( \kappa_{1, \chi} = 0 \). Of course, this can be interpreted in terms of the vanishing of the denominator of the Perrin-Riou map. This section, where no claim of originality is made, explain how to overcome this situation following the work of [Buy1] and explain also how the vanishing of the numerator of the Perrin-Riou map at \( s = 0 \) can be studied inside the framework developed in [Veri].

After fixing a topological generator \( \gamma \) of \( 1 + p\mathbb{Z}_p \), one may consider the short exact sequence of \( \mathbb{Z}_p \)-modules

\[ 0 \rightarrow \Lambda_{cyc} \otimes \mathbb{Z}_{p, \chi}(1)(\chi^{-1}) \xrightarrow{\gamma - 1} \Lambda_{cyc} \otimes \mathbb{Z}_{p, \chi}(1)(\chi^{-1}) \rightarrow \mathbb{Z}_{p, \chi}(1)(\chi^{-1}) \rightarrow 0 \]

which induces a long exact sequence in cohomology. Since \( H^0(\mathbb{Q}_p, \mathbb{Z}_{p, \chi}(1)(\chi^{-1})) = 0 \),

\[ 0 \rightarrow H^1(\mathbb{Q}_p, \Lambda_{cyc} \otimes \mathbb{Z}_{p, \chi}(1)(\chi^{-1})) \xrightarrow{\gamma - 1} H^1(\mathbb{Q}_p, \Lambda_{cyc} \otimes T) \xrightarrow{N} H^1(\mathbb{Q}_p, \mathbb{Z}_{p, \chi}(1)(\chi^{-1})). \]

The image of \( \kappa_{\chi, \infty} \) under the map \( \mathbb{N} \) vanishes, and hence there exists a unique \( \kappa'_{\chi, \infty} \in H^1(\mathbb{Q}_p, \Lambda_{cyc} \otimes \mathbb{Z}_{p, \chi}(1)(\chi^{-1})) \) such that

\[ \frac{\gamma - 1}{\log_p(\gamma)} \times \kappa'_{\chi, \infty} = \kappa_{\chi, \infty}. \]

Then, we have the following result.

**Proposition 2.4.** If \( \chi(p) = 1 \), the class \( \kappa_{\chi, \infty} \) vanishes at the character \( \chi \) and there exists a derived cohomology class \( \kappa'_{\chi, \infty} \in H^1(\mathbb{Q}_p, \Lambda_{cyc} \otimes \mathbb{Z}_{p, \chi}(1)(\chi^{-1})) \) such that

\[ \kappa_{\chi, \infty} = \frac{\gamma - 1}{\log_p(\gamma)} \times \kappa'_{\chi, \infty}. \]

**Remark 2.5.** It may be tempting to look for a relation between \( \kappa'_{\chi} \) and the special value \( L_p(\chi, 1) \), or alternatively, the derivative of the Kubota-Leopoldt \( p \)-adic \( L \)-function \( L'_p(\chi, 0) \). However, the fact that the Euler factor \( 1 - p^{-k-1} \) is not analytic precludes the possibility of directly taking derivatives in the reciprocity law.

In [Buy1] the author makes a connection between the value of \( L_p(\chi, 1) \) and Nekovar’s pairings, also encoded in terms of the more familiar Tate’s pairings. With the previous notations, let \( T = \mathbb{Z}_{p, \chi}(1) \otimes \chi^{-1} \) and \( T^* = \mathbb{Z}_{p, \chi}(\chi) \), viewed as representations of \( G_\mathbb{Q} \). We denote with a tilde the extended Nekovar’s groups. In [Buy1] Corollary 2.11 it is shown that

\[ \tilde{H}^1(\mathbb{Q}, T) = (\mathbb{Z}_{p, \chi}[1/p]^{\times})^\chi, \]

where \( \tilde{H}^1(\mathbb{Q}, T) \) is the Bloch–Kato Selmer group, a two-dimensional space where we may explicitly construct a basis. Indeed, define

\[ c_n = N_{\mathbb{Q}(\mu_{Nn})/F_{\chi,n}}(\zeta_{Nn} - 1) \in (\mathcal{O}_{H_{\chi,n}}^{\chi} \otimes \mathbb{Z}_n)^\chi, \]

and consider its \( \chi \)-part, \( c_\chi^{\chi} \). The element \( c_\chi^{\chi} \) is called the *tame cyclotomic unit*. For a finite abelian extension \( H' \) of \( \mathbb{Q} \) of conductor \( m \), we also define

\[ \xi_{H'} = N_{\mathbb{Q}(\mu_m)/H'}(\zeta_m - 1). \]
It turns out that the collection
\[ \xi = \xi_\infty^\chi := \{ e_\chi \xi_{H, n} \text{ for } n \geq 1 \} \in \lim_{\leftarrow} H^1(\mathbb{Q}_n, T), \]
where \( e_\chi \) is the \( \chi \)-projector, satisfies the Euler system distribution relation and \( \xi_H = 1 \). This gives another element \( z_\infty^\chi \) satisfying
\[ \frac{\gamma - 1}{\log_p(\gamma)} \times z_\infty^\chi = \xi. \]

We call \( z_0^\chi \) the cyclotomic \( p \)-unit, and \( \{ c_1^\chi, z_0^\chi \} \) is a basis of \( \tilde{H}_1^1(\mathbb{Q}, T) \). In [Sol], it is proved that \( \log_p(c_1^\chi) = \text{ord}_p(z_0^\chi) \in \mathbb{Z}_{p, \chi} \), once we fix a prime of \( \mathbb{Z}_{p, \chi} \) lying above \( p \).

The local Tate pairing induces a map
\[ \tau_\infty : H^1(\mathbb{Q}_p, T \otimes \Lambda_{\text{cyc}}) \to \text{Hom}(\lim_{\leftarrow} H^1(\mathbb{Q}_{n, p}, T^s), \mathbb{Z}_{p, \chi}). \]

We can define the element \( \mathcal{L}_\xi \) as the image of \( \xi \) under localization at \( p \) followed by the map \( \tau_\infty \). We define in the same way \( \mathcal{L}'_\xi \) as the image of \( z_0^\chi \) under
\[ \tau_0 : H^1(\mathbb{Q}_p, T) \to \text{Hom}(H^1(\mathbb{Q}_p, T^s), \mathbb{Z}_{p, \chi}). \]

Then,
\[ L_p(\chi, s) = \varepsilon_{\text{cyc}}^{1-s}(\mathcal{L}_\chi), \]
where \( \varepsilon_{\text{cyc}} \) stands for the usual cyclotomic character (this is a slight variation of Theorem 2.3 since the values of the Kubota-Leopoldt \( p \)-adic \( L \)-function at \( s \) and \( 1-s \) may be related via a functional equation). Given a prime \( p \), let \( U_{n, p} \) denote the local units inside \( (F_{\chi, n})_p \), and let \( U_n := \prod_{p \mid p} U_{n, p} \). Define also \( \mathcal{V}_n = (F_{\chi, n} \otimes \mathbb{Q}_p)^\times = \prod_{p \mid p} (F_{\chi, n})_{p}^\times \). Recall that
\[ H^1(\mathbb{Q}_p, T \otimes \Lambda_{\text{cyc}}) \xrightarrow{\sim} \mathcal{V}_\infty. \]

Coleman defines a \( \Lambda \)-module homomorphism
\[ \text{Col}^\chi_{\infty} : \mathcal{U}_\infty \to \Lambda_{\text{cyc}} \]
sending \( \xi_\infty^\chi \) to \( \mathcal{L}_\chi \). In the same way we can also define an improved Coleman map
\[ \text{Col}^\chi_{\infty} : \mathcal{V}_\infty \to \Lambda_{\text{cyc}}, \]
sending \( z_\infty^\chi \) to \( p\mathcal{L}_\chi \). Let \( \text{Col}^\chi_0 : H^1(\mathbb{Q}_p, T) \to \tilde{H}_1(\mathbb{Q}_p, \mathbb{Z}_{p, \chi}) \sim \mathbb{Z}_{p, \chi} \).

Let \( \pi_{\infty} \in H^1_{\infty} \) be a local uniformizer and set \( \alpha(\pi_{\infty}) = \text{Col}^\chi_0(\pi_{\infty}) \in \mathbb{Z}_{p, \chi} \). Let \( \text{Col}^\chi_0 \in \tilde{H}_1(\mathbb{Q}, T^s) \) be the element which maps to \( \alpha(\pi_{\infty}) \) under the isomorphism \( \tilde{H}_1(\mathbb{Q}, T^s) \xrightarrow{\sim} \mathbb{Z}_{p, \chi} \). Finally, let
\[ \hat{\mathcal{L}}_\chi := \frac{p \cdot (\gamma - 1)}{\log_p(\varepsilon_{\text{cyc}}(\gamma))} \times \mathcal{L}_\chi \in \Lambda_{\text{cyc}}, \]
and define
\[ \hat{L}_p(\chi, s) = \varepsilon_{\text{cyc}}^{1-s}(\hat{\mathcal{L}}_\chi). \]

The main result of [Buy] is the computation, via the theory of Nekovar’s pairings, of a formula for \( L_p(\chi, 1) \), which asserts that
\[ L_p(\chi, 1) = p \cdot L_p(\chi, 1) = \text{Col}^\chi_0(\pi_{\infty}) \cdot \text{ord}_p(z_0) = \text{Col}^\chi(\pi_{\infty}) \cdot \text{log}_p(c_1). \]

This also works for the case of an imaginary quadratic field when considering only the \( \mathbb{Z}_p \)-extension which is only ramified over a fixed prime \( p \) above \( p \).

We can also analyze what happens for \( L_p(\chi\omega, s) \) at \( s = 0 \), where we may follow the approach of [Ven, §3] to analyze the vanishing of the numerator in the Perrin-Riou map. To ease notations, let \( \psi = \chi\omega \). In particular, we know that \( L_p(\psi, 0) = 0 \). Using Shapiro’s lemma we identify \( H^1(\mathbb{Q}_p, \Lambda_{\text{cyc}} \otimes \mathbb{Z}_{p, \psi}(\psi^{-1})) \) with the Iwasawa cohomology \( H^1_{\text{Iw}}(\mathbb{Q}_p, \psi^{-1}) \) of the
elements in the Iwasawa cohomology may be seen as a compatible collection of classes along the cyclotomic tower.

Since the numerator of the Perrin-Riou regulator vanishes, we can consider some kind of derivative. Let $I$ stand for the augmentation ideal of $\Lambda_{cyc}$. Then, we define the derivative of the Perrin-Riou map $L_p$ as the application

$$L'_p : H^1_{Iw}(\overline{\mathbb{Q}_p}, \mathbb{Z}_{p,\psi}(\psi^{-1})) \rightarrow I/I^2,$$

i.e. the composition of $L_p$ with the projection $\{\cdot\} : I \to I/I^2$.

Let $\kappa_\psi = (\kappa_{n,\psi}) \in H^1_{Iw}(\overline{\mathbb{Q}_p}, \mathbb{Z}_{p,\psi}(\psi^{-1}))$ be the cohomology class we have previously introduced. Following the same strategy as in [Ven Prop. 3.6], and identifying $\kappa_{0,\psi}$ with an element in $\text{Hom}(\mathbb{Q}_p^\times, \mathbb{Q}_p) \otimes \mathbb{Z}_{p,\psi}(\psi^{-1})$, it yields that

$$L'(\kappa_\psi) = -g(\psi^{-1})^{-1}(1-p^{-1})^{-1}\frac{\kappa_{0,\psi}(p^{-1})}{\log_p(\gamma)} \cdot \gamma = -g(\psi^{-1})^{-1}(1-p^{-1})^{-1}\frac{\exp^*(\text{loc}_p(\kappa_{0,\psi}))}{\log_p(\gamma)} \cdot \gamma.$$

As in [Ven §5], we can relate the derivative of $L_p$ with the derivative of $L_p'(\psi, s)$ and obtain this way a formula for $L_p'(\psi, 0)$ in terms of $\exp^*(\text{loc}_p(\kappa_{0,\psi}))$.

3. Elliptic units

In this section we introduce Katz’s two-variable $p$-adic $L$-function and the theory of elliptic units, following [dCS, Yag] and the survey [BCDDPR]. We also present the Perrin-Riou big logarithm and recast Yager’s theorem, which gives an explicit reciprocity law in this setting. We recover the notations of the introduction, where $K$ is an imaginary quadratic field and we fix a prime $p$ which splits on $K$, i.e. $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}$. Let $h$ denote the class number of $K$. Then, let $\pi_p \in \mathcal{O}_K$ be such that $p^h = \pi_p \mathcal{O}_K$, and define $\varpi_p = \pi_p / \pi_p$. For simplicity, we assume that $\mathcal{O}_K^\times = \pm 1$ and that the discriminant of $K$ is an odd number $D < 0$.

Fix also a non-trivial Hecke character of finite order $\psi$, of conductor $n$. In the particular case that $\chi$ is a Dirichlet character of conductor $N := \mathbb{N}_{K/\mathbb{Q}}(n)$, the Dirichlet character may be seen as an example of the Hecke characters we are interested in, provided that $K$ is a quadratic field where all primes dividing $N$ split. Let $L$ stand for the field cut out by the character.

3.1. Elliptic units. Elliptic units are the result of evaluating modular units at CM points; they give rise to units in abelian extensions of an imaginary quadratic field $K$ and are the counterpart of circular units for cyclotomic fields. They constitute one of the key ingredients for the proof of the Iwasawa main conjecture for imaginary quadratic fields [Rub].

Let $\tau_n = \frac{1 + \sqrt{-D}}{2n}$, where $n = \mathbb{Z}N + \frac{1 + \sqrt{-D}}{2}$. The classical and $p$-adic Siegel units, which constitute one of the main ingredients of this work, are defined as

$$u_{n,a} := g_a(\tau_n), \quad u_{n,a}^{(p)} := g_{a}^{(p)}(\tau_{pn}),$$

being $g_a$ the power series of (11) and $g_{a}^{(p)} := g_{pa}(q^p)g_{a}(q)^{-p}$. As we did with circular units, we may define

$$u_\psi := \prod_{\sigma \in \text{Gal}(K_n/K)} (\sigma u_{1,n})^{-1}(\sigma),$$

where $K_n$ is the ray class field of $K$ of conductor $n$. These units are the bottom layer of a norm-coherent family of elliptic units over the two-variable $\mathbb{Z}_p$-extension $K_\infty$ of $K$.

Performing a similar construction to that of (6) and (7), the work of Yager [Yag] (in the framework of [LZ]) gives a cohomology class

$$\kappa_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})).$$
In particular, if \( \eta \) is a Hecke character of infinity type \((k_1, k_2)\) arising as a specialization of \( \Lambda_K(\psi) \), the global class

\[
\kappa_{\psi,\eta} \in H^1(K, \mathbb{Q}_p, \eta) \otimes L_p(\psi^{-1})
\]

obtained by specializing \( \kappa_{\psi,\infty} \) at \( \eta \), although it arises from elliptic units, encodes information about a Galois representation \( V_{\psi,\eta} \) of \( K \) attached to a Hecke character. Recall that the construction of the \( \Lambda \)-adic cohomology class may be described in terms of a projective limit, but neglecting the bottom layers (as with circular units), where some extra Euler factors arise.

### 3.2. Katz’s two-variable \( p \)-adic \( L \)-function of an imaginary quadratic field

The classical two-variable \( L \)-function attached to \( K \) and \( \psi \) is defined as

\[
L(K, \psi, k_1, k_2) := \sum_{\alpha \in \mathcal{O}_K} \psi(\alpha)\alpha^{-k_1}\overline{\alpha}^{-k_2},
\]

where the sum is over the set of non-zero ideals of \( \mathcal{O}_K \). This \( L \)-series allows us to recover the more familiar \( p \)-adic \( L \)-function attached to a character \( \psi \) of an imaginary quadratic field, via the relation

\[
L(K, \psi, s) = \frac{1}{2}L(K, \psi, s, s).
\]

Towards introducing Katz’s two-variable \( p \)-adic \( L \)-function of an imaginary quadratic field, we follow [DLR, Section 3]. Let \( \mathfrak{c} \subset \mathcal{O}_K \) be an integral ideal of \( K \), and let \( \Sigma \) be the set of Hecke characters of \( K \) of conductor dividing \( \mathfrak{c} \). Define \( \Sigma_K = \Sigma_K^{(1)} \cup \Sigma_K^{(2)} \subset \Sigma \) to be the disjoint union of the sets

\[
\Sigma_K^{(1)} = \{ \psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2), \text{ with } \kappa_1 \leq 0, \kappa_2 \geq 1 \},
\]

\[
\Sigma_K^{(2)} = \{ \psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2), \text{ with } \kappa_1 \geq 1, \kappa_2 \leq 0 \}.
\]

For all \( \psi \in \Sigma_K \), the complex argument \( s = 0 \) is a critical point for \( L(\psi^{-1}, s) \), and Katz’s \( p \)-adic \( L \)-function is constructed interpolating the algebraic part of \( L(\psi^{-1}, 0) \), as \( \psi \) ranges over \( \Sigma_K^{(2)} \).

Let \( \tilde{\Sigma}_K \) be the completion of \( \Sigma_K^{(2)} \) with respect to the compact open topology on the space of \( \mathcal{O}_{L_p} \)-valued functions on a certain subset of \( \mathbb{A}_K^\times \). By the work of Katz, there exists a \( p \)-adic analytic function

\[
L_p(K) : \tilde{\Sigma}_K \rightarrow \mathbb{C}_p,
\]

uniquely determined by the interpolation property that for all \( \xi \in \Sigma_K^{(2)} \) of infinity type \((\kappa_1, \kappa_2)\),

\[
L_p(K)(\xi) = a(\xi) \times c(\xi) \times f(\xi) \times \frac{\Omega_p^{\kappa_1-\kappa_2}}{\Omega_p^{\kappa_1-\kappa_2}} \times L_\mathfrak{c}(\xi^{-1}, 0),
\]

where

1. \( a(\xi) = (\kappa_1 - 1)!\pi^{-\kappa_2} \),
2. \( c(\xi) = (1 - \xi(p)p^{-1})(1 - \xi^{-1}(\overline{p})) \),
3. \( f(\xi) = D^{2/2-\kappa_2} \),
4. \( \Omega_p \in \mathbb{C}_p^\times \) is a \( p \)-adic period attached to \( K \),
5. \( \Omega \in \mathbb{C}^\times \) is the complex period associated to \( K \),
6. \( L_\mathfrak{c}(\xi^{-1}, s) \) is Hecke’s \( L \)-function associated to \( \xi^{-1} \) with the Euler factors at primes dividing \( \mathfrak{c} \) removed.

Observe that the definition is not symmetric with respect to the primes \( p \) and \( \overline{p} \) above \( p \), and hence we can also consider the function \( L_p(K)(\cdot) \). The \( p \)-adic \( L \)-function \( L_p(K)(\cdot) \) satisfies a functional equation

\[
L_p(K)(\xi) = L_p(K)((\xi')^{-1}N).
\]
It is possible to obtain an expression for the value of $L_p(K, \psi)(\omega)$ at finite order characters. This is usually referred to as the $p$-adic Kronecker’s limit formula,

\begin{equation}
L_p(K)(\psi) = \begin{cases} 
\frac{1}{2} \left( \frac{1}{p} - 1 \right) \cdot \log(\pi_p^{1/h}) & \text{if } \psi = 1; \\
(1 - \psi^{-1}(\bar{p}))(1 - \psi(p)p^{-1}) \cdot \log_p(u_\psi) & \text{if } \psi \neq 1. 
\end{cases}
\end{equation}

(20)

Here, $h$ stands for the class number of $K$ and $\pi_p$ for a generator of the $\mathcal{O}_K$-ideal $p^h$. Via the functional equation, we also have an expression for the value at the $\psi N$

\begin{equation}
L_p(K)(\psi \mathbb{N}) = \begin{cases} 
\frac{1}{2} \left( \frac{1}{p} - 1 \right) \cdot \log(\pi_p^{1/h}) & \text{if } \psi = 1; \\
(1 - \psi(p))(1 - \psi^{-1}(\bar{p})p^{-1}) \cdot \log_p(u_\psi) & \text{if } \psi \neq 1. 
\end{cases}
\end{equation}

(21)

We restrict this definition to characters of the form $\psi \xi^k \chi^{k_2}$, where $\xi$ is a character of $p$-power conductor and $\lambda$ is the character of infinity type $(1,0)$ presented at the Introduction. This is because we want to restrict to the $\mathbb{Z}_p^2$ extension of $K$ (it occurs a similar phenomenon for elliptic units, which can be defined for a wider class of extensions). Once we do this restriction, we denote the $p$-adic $L$-function as $L_p(K, \psi)(\omega)$; in particular, we write $L_p(K, \psi)(\chi_{triv}) := L_p(K)(\psi)$ and $L_p(K, \psi)(\mathbb{N}) := L_p(K)(\psi \mathbb{N})$.

Remark 3.1. Depending on the normalizations we choose for the two variable $p$-adic $L$-function, the value $L_p(K, \psi)(\psi)$ may be affected by multiplication by a non-zero explicit rational number. Further, this number can depend on the conductor of $\psi$; however, since we are restricting the function to characters of the form $\psi \xi$, where $\xi$ has $p$-power conductor, we can adopt a suitable normalization in such a way that our special value formulas always work.

Further, observe that the $p$-adic $L$-function of [Buy1] Theorem 6.3 also differs from this one in the factor $(1 - \xi^{-1}(\bar{p}))$.

3.3. Eisenstein series and improved p-adic L-functions. The Eisenstein series $E_{k,\psi}$, viewed as a function of a variable $\tau$ in the upper half plane $\mathbb{H}$ is given by the well-known expression

\begin{equation}
E_{k,\psi}(\tau) := N^k g(\bar{\psi})^{-1}(k - 1)! \sum_{(m, n) \in \mathbb{N} \times \mathbb{Z}} \frac{\psi^{-1}(n)}{(m\tau + n)^k},
\end{equation}

(22)

the sum being over the set of non-zero pairs $(m, n) \in \mathbb{N} \times \mathbb{Z}$. Special values of this function encode information about special values of the classical $L$-series

\begin{equation}
E_{k,\psi}(\tau_n) = N^k g(\bar{\chi})^{-1}(k - 1)! \frac{L(K, \psi, k, 0)}{(2\pi i)^k}.
\end{equation}

(23)

In the $p$-adic setting there are natural analogues of these objects. We want to introduce two different $p$-adic avatars of the Eisenstein series and see how their difference allows us to establish a factorization of the Katz’s two-variable $p$-adic $L$-function. The $p$-stabilized Eisenstein series is defined as

\begin{equation}
E_{k,\psi}(A, t_n, \omega_A) = (1 - \psi^{-1}(p)p^{-k/h}/p) \times n^k g(\bar{\psi})^{-1} \frac{(k - 1)!}{(2\pi i) \cdot \Omega_K^k} L(K, \psi, k, 0),
\end{equation}

(24)

while the $p$-depletion of $E_{k,\psi}$ is

\begin{equation}
E_{k,\psi}^{(p)}(\tau) = E_{k,\psi}(\tau) - (1 + \psi(p)p^{-k-1})E_{k,\psi}(p\tau) + \psi(p)p^{-k-1}E_{k,\psi}(p^2 \tau).
\end{equation}

(25)

These definitions lead to the $p$-adic counterpart of Formula (17). Let $L_p(K, \psi, k)$ denote the usual $p$-adic $L$-function attached to the imaginary quadratic field $K$ and $\psi$. We say that a character is analytic if it is of the form $\psi \chi^t$, with $t \in \mathbb{Z}$. The reason for this terminology...
is that they correspond to the subvariety along which the Euler factors appearing in the Perrin-Riou map are analytic. Restriction of $L_p(K, \psi)$ to those characters yields the equation

$$L_p(K, \psi)(\lambda^k) = (1 - \psi(p)\pi_p^{-k/h})L_p(K, \psi, k).$$

The ratio of the two $p$-adic $L$-series is a $p$-adic analytic function of $k$, since $\pi_p$ belongs to $O_{K_p}^\times$. This ratio measure the difference between working with the ordinary $p$-stabilization $E_{p, \bar{\psi}}^{(p)}$ and the $p$-depletion $E_{p, \bar{\psi}}^{[p]}$.

By a result of Katz [Ka] one has a relation between the $p$-adic $L$-function of a quadratic imaginary field and Siegel units,

$$L_p(K, \psi, 0) = -\frac{(1 - \psi(p)^{p-1})}{g(\psi)} \sum_{a=1}^{N-1} \psi^{-1}(a) \log_p u_{a,n}.$$

### 3.4. A reciprocity law for elliptic units.

In this section, we establish the existence of a Perrin-Riou map interpolating both the dual exponential map and the Bloch–Kato logarithm, as we did with circular units. We follow closely the treatment of [CH] and [LZ]. In the former, the authors develop a quite general framework that works for any pair $(V, G)$, where $V$ is a $p$-adic Galois representation and $G$ is a commutative compact $p$-adic Lie group. Then, [CH, Theorem 5.1] establishes the existence of a map whose source is the Iwasawa cohomology of $G$ and which interpolates the Bloch–Kato logarithm and the dual exponential map, depending on the Hodge–Tate type of the character at which we specialize.

Here, we are interested in a more down-to-earth version of that result, concerning the $\Lambda$-adic representation coming from a Hecke character of an imaginary quadratic field. These results have been recovered by Loeffler and Zerbes in [LZ], in the setting of two-variable Perrin-Riou regulators. In particular, Theorem 4.15 of loc. cit. gives an analogue to the previous result in our setting. We restrict here to character of $K$ of the form $\psi \lambda^{k_1} \Lambda^{k_2}$, where $\lambda$ is the character of infinity type $(1, 0)$ presented at the Introduction and $\lambda$ is its complex conjugate. Of course we may also consider twists by characters $\xi$ of $p$-power order, but we neglect this possibility.

Recall that $\Lambda_K$ is the two-variable Iwasawa algebra attached to $K$. As with circular units, for any character $\eta$ as above there exists a specialization map

$$\nu_\eta : \Lambda_K \otimes L_p(\psi^{-1}) \to K_{p, \eta}(\eta) \otimes L_p(\psi^{-1}).$$

We are again identifying the target of both the dual exponential map and the Bloch–Kato logarithm with $L_p$.

**Proposition 3.2.** There exists a morphism

$$\mathcal{L}_p : H^1(K_p, \Lambda_K \otimes L_p(\psi^{-1})) \to \Lambda_K$$

interpolating both the dual exponential map and the Bloch–Kato logarithm, and such that for any point $\eta$ of infinity type $(k_1, k_2)$ and trivial finite order character, the specialization of $\mathcal{L}$ at $\eta$ is the homomorphism

$$\nu_\eta(\mathcal{L}_p) : H^1(K_p, \mathbb{Q}_{p, \eta}(\eta) \otimes L_p(\psi^{-1})) \to \mathbb{D}_{\text{dR}}(\mathbb{Q}_{p, \eta}(\eta) \otimes L_p(\psi^{-1})) \simeq L_{p, \eta}$$

given by

$$\nu_\eta(\mathcal{L}_p) = \frac{1 - \psi(p)\pi_p^{-k_2/h}\pi_p^{-k_1/h}}{1 - \psi^{-1}(p)\pi_p^{k_2/h} \pi_p^{-k_1/h}/p} \left\{ \begin{array}{ll} (-k_2)! \exp & \text{if } k_2 \leq 0 \\ (-1)^{k_2}/k_2! \log_{BK} & \text{if } k_2 > 0 \end{array} \right.$$
Proposition 3.5. The following result is the main theorem of [Yag] and also appears as a consequence of the results of [LZ, Section 6]. To follow this parallelism, let \( k = k_1 + k_2 \) and \( r = -k_2 \). Then,
\[
1 - \psi(\bar{p})^{p^{k_2/h} - p^{k_1/h} \frac{1}{\pi_p}} = 1 - \psi(\bar{p})^{p^{k_2/h} - p^{k_1/h} \frac{1}{\pi_p}}
\]

Our results concerning elliptic units can be seen as a counterpart of those for Heegner points when the cuspidal Hida family is replaced by an Eisenstein series and the role of \( \nu_p(a_p) \) is played by \( \psi(\bar{p})^{p^{k_1/h}} \).

Remark 3.3. It is interesting to analyze the shape of the Euler factors and compare it with those of [Cas1] Theorem 3.5. To follow this parallelism, let \( k = k_1 + k_2 \) and \( r = -k_2 \). Then,
\[
1 - \psi(\bar{p})^{p^{k_2/h} - p^{k_1/h} \frac{1}{\pi_p}} = 1 - \psi(\bar{p})^{p^{k_2/h} - p^{k_1/h} \frac{1}{\pi_p}}
\]

\( (28) \)

Remark 3.4. As we discuss in the last section, this also fits well with [RiRo1] Proposition 3.2, which is a reformulation of [KLZ] Theorem 8.1.7; with the notations of loc. cit., if we fix \( s = 0 \) and identify the modular forms \( g \) and \( h \) with two theta series attached to the imaginary quadratic field \( K \), we recover the map of Proposition 3.2. Further, observe that the numerology is coherent, and the condition \( m > s \) defining the region of interpolation of the Bloch–Kato logarithm becomes in this case \( k_2 > 0 \).

The following result is the main theorem of [Yag] and also appears as a consequence of the results of [LZ, Section 6].

Proposition 3.5. The cohomology class \( \kappa_{\psi, \infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})) \) satisfies
\[
L_p(K, \psi) = L_p(\log_p(\kappa_{\psi, \infty})).
\]

Again, the denominator of the Perrin-Riou regulator may vanish. Assume that \( \psi(\bar{p}) = 1 \). Then, we have the following:
(i) If \( k_1 = k_2 = 0 \), then the numerator vanishes and the denominator equals \( 1 - p^{-1} \).
(ii) If \( k_1 = k_2 = 1 \), the numerator equals \( 1 - p^{-1} \) and the denominator vanishes.

Remark 3.6. For a fixed \( k_2 \), both the numerator and the denominator are analytic functions on the variable \( k_1 \).

4. Exceptional zeros and elliptic units

We analyze different instances of exceptional zero phenomena and discuss the existence of derived cohomology classes and some of their properties. Our main result, stated as Theorem 1.1 in the Introduction, is about the exceptional vanishing of \( \kappa_{\psi, \infty} \), but for the sake of convenience we also study the exceptional vanishing of Katz’s two variable \( p \)-adic \( L \)-function at \( \psi \) in Section 4.1. Then, in Section 4.2 we discuss the different cases of exceptional zeros at \( \psi \mathbb{N} \) and prove the main result of the note. Finally, Section 4.3 discusses the case where \( \psi \) arises as the restriction of a Dirichlet character.

4.1. Specialization at the character \( \psi \). We assume that the condition \( \psi(\bar{p}) = 1 \) is satisfied. We begin this section by establishing the vanishing of Katz’s two variable \( p \)-adic \( L \)-function at the character \( \psi \) under this hypothesis. Indeed, from (26), it is straightforward that \( \psi(\bar{p}) = 1 \) is a necessary and sufficient condition for the vanishing of \( L_p(K, \psi)(\chi_{\text{triv}}) \).

Until otherwise stated, derivatives are considered along the character \( \lambda \).

Proposition 4.1. Assume that \( \psi(\bar{p}) = 1 \). Then, \( L_p(K, \psi)(\chi_{\text{triv}}) = 0 \) and if moreover \( \psi \) comes from a Dirichlet character,
\[
L_p(K, \psi)(\chi_{\text{triv}}) = -\log(\pi_p^{1/h}) \cdot (1 - \psi(\bar{p})p^{-1}) \times \log_p(\nu_p).
\]

Proof. This follows combining (26) and (27).
The Euler factors arising in the Perrin-Riou map are analytic once the value of \( k_2 \) is fixed. In particular, here we consider the Perrin-Riou map with \( k_2 = 0 \) fixed. Then,

\[
(1 - \pi_p^{k_1/h}/p) \cdot L_p(K, \psi)(\chi_{\text{triv}}) = (1 - \pi_p^{-k_1/h}) \cdot \exp_B^\ast (\text{loc}_p(\kappa_{k, \psi})),
\]

for all \( k \geq 0 \). Taking derivatives with respect to \( k_1 \) at both sides and evaluating at \( k_1 = 0 \), we get that

\[
(1 - p^{-1})L'_p(K, \psi)(\chi_{\text{triv}}) = -(\log(\pi_p^{1/h})) \cdot \exp_B^\ast(\text{loc}_p(\kappa_\psi)).
\]

Here, under the identification of class field theory,

\[
H^1(K_p, L_p(\psi^{-1})) \simeq \text{Hom}_{\text{cont}}(K_p^\times, K_p) \otimes L_p(\psi^{-1}),
\]

and \( \text{loc}_p(\kappa_\psi) \) corresponds to the evaluation at the inverse of a local uniformizer, \( \pi_p^{-1/h} \). In particular, there is a non-canonical isomorphism with \( L_p \). Under this isomorphism, the dual exponential map corresponds to the ord map.

Combining Proposition 4.1 with (30), and identifying an element of \( \text{Hom}_{\text{cont}}(K_p^\times, K_p) \otimes L_p(\psi^{-1}) \) with its image under evaluation at \( \pi_p^{-1/h} \), we get the following.

**Proposition 4.2.** Assume that \( \psi(\overline{p}) = 1 \) and that \( \psi \) comes from a Dirichlet character. Then,

\[
\text{loc}_p(\kappa_\psi) = (1 - p^{-1})(1 - \psi(p)p^{-1}) \cdot \text{log}_p(u_\psi),
\]

4.2. Specialization at the character \( \psi N \). The motivation for this section comes from this fact.

**Lemma 4.3.** The following relation between the unit \( u_\psi \) and the specialization of \( \kappa_{\psi, \infty} \) at the character \( \psi N \) holds:

\[
\text{loc}_p(\kappa_{\psi N}) = (1 - \psi^{-1}(p)) \cdot (1 - \psi^{-1}(\overline{p})) \cdot \text{loc}_p(u_\psi).
\]

In particular, \( \text{loc}_p(\kappa_{\psi N}) \) vanishes if and only if \( \psi(p) = 1 \) or \( \psi(\overline{p}) = 1 \).

**Proof.** This follows after combining the \( p \)-adic Kronecker limit formula given in (21) with Proposition 3.5. \( \square \)

We distinguish three different situations.

(a) If \( \psi(p) = 1 \), the cohomology class \( \kappa_{\psi N} \) is 0. One can construct a derived class whose derivative along the direction \( \lambda \) is computed and expressed in terms of the unit \( u_\psi \). If \( \psi(p) \neq 1 \), the special value \( L_p(K, \psi)(\N) \) is non-zero and this value is related with the Bloch–Kato logarithm of the derived class.

(b) If \( \psi(p) = 1 \), both \( \kappa_{\psi N} = 0 \) and \( L_p(K, \psi)(\N) = 0 \). The logarithm of the derived local class \( \text{loc}_p(k_{\psi N}) \) along the direction \( \lambda \) can be expressed in terms of \( L'_p(K, \psi)(\N) \).

(c) When \( \psi(p) = \psi(\overline{p}) = 1 \), both \( \kappa_{\psi N} = 0 \) and \( L_p(K, \psi)(\N) = 0 \). Then, \( k_{\psi N} = 0 \) too, and there is a notion of second derivative of the cohomology class, whose logarithm is related with \( L'_p(K, \psi)(\N) \).

**Case (a).** Suppose that \( \psi(\overline{p}) = 1 \). Let \( C \) be the line of weight space obtained by taking the Zariski closure of all the points of the form \( \psi N \lambda' \) (alternatively, we are fixing \( k_2 = 1 \)).

In the situation of exceptional zero, the Euler factor in the denominator of the Perrin-Riou map vanishes, and hence one must carry out some new constructions to obtain a meaningful reciprocity law in this setting. We can argue the existence of a derived cohomology class arising from elliptic units, which is directly related with the special value of the derivative of Katz’s two variable \( p \)-adic \( L \)-function. Observe that derivatives must be taken along \( C \), since elsewhere the Euler factors are not analytic (and not even continuous!).
Proposition 4.4. It holds that $\kappa_{\psi|\mathbb{N}} = 0$, and there exists a derived cohomology class along $C$

$$\kappa'_{\gamma,\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})|_C)$$

satisfying that

$$\kappa_{\psi|C} = (\gamma - 1)\kappa'_{\gamma,\psi,\infty},$$

where $\gamma$ is a fixed topological generator of $1 + p\mathbb{Z}_p$.

Proof. The vanishing of the local class $\text{loc}_p(\kappa_{\psi|\mathbb{N}})$ directly follows from (32). Alternatively, we may observe that

$$(1 - p^{-1}) \log_{BK}(\text{loc}_p(\kappa_{\psi|\mathbb{N}})) = \nu_{\psi|\mathbb{N}}(\tilde{\mathcal{L}}(\kappa_{\psi,\infty})) = \left(1 - \frac{\pi_p}{p}\frac{k_2/h \cdot k_1/h}{p}\right)_{k_1=k_2=1} \cdot \nu_{\psi|\mathbb{N}}(\mathcal{L}(\kappa_{\psi,\infty})) = 0.$$ 

Since $\psi$ was taken to be non-trivial, the Bloch–Kato logarithm is an isomorphism and it holds that $\log_{BK}(\kappa_{\psi|\mathbb{N}}) = 0$. Then, since the localization map corresponds to the injection of global units inside local units, we may conclude that the global class $\kappa_{\psi|\mathbb{N}}$ also vanishes.

The construction of the derived class follows the same argument than in the case of circular units explained in Section 2.

Remark 4.5. If we normalize dividing by $\log_{p}(\gamma)$ the specialization of the resulting class at the character $\psi\mathbb{N}$ does not depend on $\gamma$ (see [Buy1, Section 3]). We define $\kappa'_{\psi,\infty} := \frac{\kappa_{\psi,\infty}}{\log_{p}(\gamma)}$.

We can relate the cohomology class with the derivative of $L_p(K)(\cdot)$ at $\psi\mathbb{N}$. Define a new function $\mathcal{E}_p(K)(\cdot)$ over $C$ given by

$$(33) \quad \mathcal{E}_p(K) = \left(1 - \frac{\pi_p}{p}\frac{k_2/h \cdot k_1/h}{p}\right) L_p(K, \psi).$$

The function satisfies that for a character $\eta$ of infinity type $(k_1, 1)$

$$(34) \quad \mathcal{E}_p(K)(\psi\eta) = \left(1 - \frac{1}{\pi_p} \cdot \frac{k_1}{h} \cdot \frac{1}{p}\right) \log_{BK}(\text{loc}_p(\kappa_{\psi|\mathbb{N}})).$$

The function $\mathcal{E}_p$ vanishes at $\psi\mathbb{N}$, due to the vanishing of the specialization of the cohomology class at the character $\psi\mathbb{N}$. Recall that for characters of infinity type $(k_1, 1)$ the Bloch–Kato logarithm interpolates the usual logarithm map.

This gives the following result and proves Theorem 1.1.

Theorem 4.6. It holds that

$$\log(\pi_p^{1/h}) \cdot L_p(K, \psi)(\mathbb{N}) = (1 - p^{-1}) \cdot \log_{BK}(\text{loc}_p(\kappa'_{\psi|\mathbb{N}})).$$

Moreover,

$$\kappa'_{\psi|\mathbb{N}} = - \log(\pi_p^{1/h}) \cdot (1 - \psi(p)) \cdot \Delta_p,$$

where $\text{loc}_p$ stands here for the composition of the Kummer map with localization at $p$.

Proof. The first part follows by considering the derivative of the function $\mathcal{E}_p(K)$ using both (33) and (34). For the second part, combine the functional equation for Katz’s two variable $p$-adic $L$-function with Proposition 4.4.

Observe that we are implicitly using the equality $(\log(\kappa))' = \log(\kappa')$. This can be easily seen by considering the natural isomorphism between weight space and $\mathbb{Z}_p[[X]]$. Then, the class $\kappa$ corresponds to a function $f$ vanishing at 0, and hence there is another function $g$ such that $f = X \cdot g$. The Bloch–Kato logarithm is a linear morphism between a $\mathbb{Z}_p[[X]]$-module and a field embedded in $C_p$, that we may denote with the letter $\Phi$. Then,

$$\Phi(f)|_{X=0} = \Phi(g)|_{X=0},$$

as desired.
Case (b). Assume now that $\psi(p) = 1$. Observe that via the functional equation for Katz’s two-variable $L$-function, this leads to

$$L_p(K, \psi)(\mathbb{N}) = 0,$$

and $L'_p(K, \psi)(\mathbb{N})$ can be related with the derived cohomology class constructed in the previous section.

**Proposition 4.7.** Assume that $\psi(p) = 1$. Then,

$$\left(1 - \psi(p)\right) \cdot L'_p(K, \psi)(\mathbb{N}) = \left(1 - \frac{\psi^{-1}(p)}{p}\right) \cdot \log_{BK}(\log_{p}(\kappa'_{\psi\mathbb{N}})).$$

**Proof.** This directly follows by considering the derivative in the reciprocity law. \qed

Unfortunately, $L'_p(K, \psi)(\mathbb{N})$ is related, via the functional equation, with the derivative at the trivial character along the direction $\lambda$, and we do not know any expression for that in terms of $u_\psi$, which would allow us to prove an analogue of Theorem [4] in this setting. In the framework of circular units, Gross’ factorization formula [Gr], combined with the results of the previous section, allows us to express $\kappa'_{\psi\mathbb{N}}$ as a linear combination of a circular unit and an elliptic unit. We discuss this in Section [4.3].

**Remark 4.8.** The results we have underlined in this part of the work point out how we can extend our computations for the derivative of the cohomology class to arbitrary directions. Indeed, we may consider the function $L_\bar{p}(K)(\cdot)$.

- When $\psi(\bar{p}) = 1$, $L_\bar{p}(K)(\cdot)$ vanishes at $\psi\mathbb{N}$ and in this case we can relate the derivative of $\log_{p}(\kappa'_{\psi\mathbb{N}})$ along the $(0, 1)$ direction with the derivative of $L_\bar{p}(K)$ at $\psi$ along the $(-1, 0)$ direction.
- When $\psi(p) = 1$, the derivative of $\log_{p}(\kappa'_{\psi\mathbb{N}})$ along the $(0, 1)$ direction can be explicitly expressed in terms of $L_\bar{p}(\psi\mathbb{N})$.

Case (c). When $\psi(p) = \psi(p) = 1$, the cohomology class $\kappa'_{\psi\mathbb{N}}$ vanishes since $L_p(K, \psi)(\mathbb{N}) = 0$. Then, we may take a second derivative of the cohomology class

$$\kappa''_{\gamma, \psi, \infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1}|_C)).$$

Now, we normalize the class dividing by $\log_{p}(\gamma^2)$ in such a way that its value at the character $\psi\mathbb{N}$, $\kappa''_{\psi\mathbb{N}}$, does not depend on $\gamma$. Proceeding in the same way as before, we have the following.

**Proposition 4.9.** Assume that $\psi(\bar{p}) = \psi(p) = 1$. Then,

$$\log(p) \cdot L'_p(K, \psi)(\mathbb{N}) = -(1 - p^{-1}) \cdot \log_{BK}(\log_{p}(\kappa''_{\psi\mathbb{N}})).$$

**Remark 4.10.** Another interesting instance of the exceptional zero phenomenon can be observed in [BDP2] Prop.3.5, which asserts that a self-dual character of infinity type $(1 + j, -j)$ with $j \geq 0$ satisfies that the evaluation of Katz’s two-variable $p$-adic $L$-function at $\nu$ agrees, up to multiplication by some periods and gamma factors, with a classical $L$-value times $(1 - \nu^{-1}(\bar{p}))^2$. Again, if $\nu(\bar{p}) = 0$ we observe the presence of an exceptional zero.

In the special case where we consider characters of infinity type $(1, 0)$, Agboola studies a variant of the $p$-adic BSD conjecture for CM elliptic curves concerning special values of Katz’s two-variable $p$-adic $L$-function. Here, we are again in a situation where our same condition leads to an exceptional vanishing.
4.3. A particular case coming from the theory of circular units. Observe that in the case where \( \psi(p) = 1 \) we have described the derived cohomology class \( \kappa'_{\psi,N} \) as an explicit multiple of the elliptic unit \( u_\psi \). However, when \( \psi(p) = 1 \) this is no longer possible, since we cannot express in terms of \( u_\psi \) the derivative of the Katz’s two variable \( p \)-adic \( L \) function at \( \psi \) along the \( \lambda \)-direction.

In general, one may wish to determine the derivatives of the \( p \)-adic \( L \) function along the different directions of weight space. We know the derivative along the \( \lambda \)-direction, and it turns out that in some particular cases we can further determine the derivative along the \textit{norm} direction.

This is the case when \( \psi \) arises as the restriction to \( G_K \) of a Dirichlet character, and hence one can invoke Gross’ factorization formula \([Gr]\): indeed,

\[
L_p(K, \psi)(\mathbb{N}^s) = L_p(\bar{\psi}, s) \cdot L_p(\psi \chi_K \omega^{-1}, 1 - s),
\]

where \( \chi_K \) stands for the quadratic character attached to \( K \). Then, the derivative of \( L_p(K, \psi)(\cdot) \) at \( \psi \) along the \textit{norm} character is given by

\[
L_p'(\psi, 0) \cdot L_p(\psi \chi_K \omega^{-1}, 1),
\]

which can be expressed as the product of certain circular units attached to both \( \psi \) and \( \psi \chi_K \omega^{-1} \).

Then, one can clearly determine the derivative along any direction: for \( \eta = \psi \lambda^a \mathbb{N}^b \), the derivative of \( L_p(K, \psi) \) at \( \psi \) along the direction \( \eta \) is given by

\[
(37) \quad - a \cdot \log(\pi_p^{1/k}) \cdot \frac{1 - \psi(p)p^{-1}}{g(\psi)} \cdot \log_p(u_\psi) + b \cdot L_p'(\psi, 0) \cdot L_p(\psi \chi_K \omega^{-1}, 1).
\]

\textbf{Remark 4.11.} There is only one direction along which this value is zero; as discussed in the inspiring presentation \([Gr]\), this has significant applications towards Iwasawa theory.

Suppose now that \( \psi(p) = 1 \). Combining \((37)\) when \( a = 1 \) and \( b = -1 \) with Proposition \(4.7\) it yields

\[
\kappa'_{\psi,N} = \lambda u_\psi + \mu c_\phi,
\]

where \( \phi = \bar{\psi}^{-1} \chi_K \omega^{-1} \) and \( \lambda \) and \( \mu \) can be determined combining \((37)\) with \((11)\):

\[
\lambda = - \log(\pi_p^{1/k}) g(\bar{\psi}) g(\psi^{-1})^{-1}(1 - \psi(\bar{p})),
\]

\[
\mu = - L_p'(\phi, 0) g(\bar{\psi}) g(\phi^{-1})^{-1}(1 - \phi(p)p^{-1})(1 - \psi(\bar{p})p^{-1})^{-1}(1 - \psi(\bar{p})).
\]

If we further assume that \( \psi(\bar{p}) = 1 \), Proposition \(4.9\) gives a new expression for the second derivative of the cohomology class

\[
\kappa''_{\psi,N} = \lambda' u_\psi + \mu' c_\phi,
\]

where now

\[
\lambda' = \log(\pi_p^{1/k})^2 g(\bar{\psi}) g(\psi^{-1})^{-1},
\]

\[
\mu' = \log(\pi_p^{1/k}) L_p'(\phi, 0) g(\bar{\psi}) g(\phi^{-1})^{-1}(1 - \phi(p)p^{-1})(1 - p^{-1})^{-1}.
\]

5. The exceptional zero phenomenon: Beilinson–Flach elements and elliptic units

In this last section, we emphasize the interplay between the results we have presented until now and other related works in this direction. In particular, we study how the phenomena we have developed arise in some of the other Euler systems discussed in the survey \([BCDDPR]\), focusing on two main aspects:

(a) Elliptic units can be seen as the natural substitute of Heegner points when the weight \( k \) cusp form is replaced by the Eisenstein series \( E_{k,\chi} \).
(b) Elliptic units can be recast in terms of Beilinson–Flach elements, when we take two weight one modular forms corresponding to theta series of the same imaginary quadratic field where the prime $p$ splits.

As it has been extensively discussed in the literature, there is a striking parallelism between the theory of Heegner points and that of elliptic units. Following this analogy, this note may be read as the counterpart of [Cas1] when the cusp form $f$ is replaced by an Eisenstein series. With the notations introduced in loc. cit., where weight space is modeled by a weight variable that we denote with the letter $k$, an exceptional zero arises at the point $(k, t) = (2, 0)$ when $a_p(f) = 1$ (the elliptic curve has split multiplicative reduction at $p$) and we specialize at the character $\psi/N$. There is a clear interplay between that setting and ours, and the same drawback concerning analyticity of the Euler factors is present. However, we would like to point out some of the differences:

- In [Cas1] the author extends the $p$-adic Gross-Zagier formula of [BDP1] and finds an explicit expression for its value at the norm character, which is not zero. However, in our setting it may occur that the $p$-adic $L$-function vanishes both at $\psi/N$ and at $\psi$. This simultaneous vanishing of the Euler factor and the $p$-adic $L$-function gives rise to a higher order vanishing of the derivative class $\kappa_{\psi, \infty}$ at the character $\psi/N$. This can be understood via [Cas1, Eq. 0.2], where the specialization of the higher dimensional Heegner cycle (whose role is now played by the unit $u_\psi$) is related by the factor

$$\left(1 - \frac{p^{k/2-1}}{\nu_k(a_p)}\right)^2.$$  

In our case, however, the link is via the factor

$$(1 - \psi^{-1}(\bar{p})p^{(k_1-1)/h}\bar{p}^{(k_2-1)/h}) \cdot (1 - \psi^{-1}(\bar{p})p^{(k_2-1)/h}\bar{p}^{(k_1-1)/h}),$$

and hence there are two possible sources of vanishing.

- In our setting there are two points where the exceptional zero phenomenon emerges: the character $\psi$ and the character $\psi/N$. In [Cas1] the vanishing of the numerator in the Perrin-Riou map would occur at $(k, t) = (0, -1)$, where there is not a clear geometric meaning of this phenomenon.

At the same time, there is a great parallelism, and the similitude between his main result and ours is evident, expressing a derived cohomology class as a certain $L$-invariant encoding information both about the elliptic curve and the imaginary quadratic field $K$.

5.1. Elliptic units and Beilinson–Flach elements. Elliptic units can be understood as a degenerate case of the theory of Beilinson–Flach elements. To make this statement more precise, let

$$g = \sum_{n \geq 1} a_n q^n \in S_1(N_g, \chi_g), \quad h = \sum_{n \geq 1} b_n q^n \in S_1(N_h, \chi_h)$$

be two normalized newforms, and let $V_g$ and $V_h$ denote the Artin representations attached to them via modularity. Consider also $V_{gh} := V_g \otimes V_h$, and let $H$ be the smallest number field cut out by this representation. We fix a prime number $p$ which does not divide $N_g N_h$. Label the roots of the $p$-th Hecke polynomial of $g$ and $h$ as

$$X^2 - a_p(g)X + \chi_g(p) = (X - \alpha_g)(X - \beta_g) \quad X^2 - a_p(h)X + \chi_h(p) = (X - \alpha_h)(X - \beta_h).$$

Let

$$g_\alpha(q) = g(q) - \beta_g q(q^p), \quad h_\alpha(q) = h(q) - \beta_h h(q^p)$$

be the smallest number field attachments to $V_{gh}$. We have

$$a_p(g) = \alpha_\beta, \quad a_p(h) = \alpha_\beta.$$
denote the $p$-stabilization of $g$ (resp. $h$) on which the Hecke operator $U_p$ acts with eigenvalue $\alpha_g$ (resp. $\alpha_h$). Let $L$ be a number field containing both the Fourier coefficients of $g$ and $h$ and the eigenvalues for the $p$-th Hecke polynomials. We can attach in a natural way two canonical differentials $\omega_p$ and $\eta_{g_h}$ to the weight one modular form $g$, as it is recalled in [RiRo1] §2 and §3. The reinterpretation of the main results of [KLZ] in [RiRo1] and [RiRo2] establishes the existence of cohomology classes

$$\kappa(g_h,h_\alpha), \kappa(g_h,h_\beta), \kappa(g_\beta,h_\alpha), \kappa(g_\beta,h_\beta) \in H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)),$$

and also give an explicit reciprocity law which in slightly rough terms asserts that

$$\left(1 - \frac{1}{p\alpha_g \beta_h}\right) \cdot \log^+(\kappa_p(g_h,h_\alpha)) = (1 - \alpha_g \beta_h) \cdot L_p(g,h,1) \pmod{L^X}.$$

Here, $L_p(g,h,s)$ stands for the Hida–Rankin $p$-adic $L$-function attached to the convolution $g \otimes h$, $\kappa_p(g_h,h_\alpha)$ is the restriction of the cohomology class to a decomposition group at $p$, and $\log^+$ is the result of applying the Bloch–Kato logarithm to a certain projection of the local class followed by the pairing with some canonical differentials. The proof of this result is based on considering Hida families $g$, $h$ interpolating $g_h$, $h_\alpha$ and proving the corresponding equality over a dense set of point of weight space. We refer the reader to [RiRo1] for a complete discussion of the results.

Now, let $g^*$ stand for the twist of $g$ by the inverse of its nebentus. When $h = g^*$, Proposition 3.12 of loc. cit. establishes that both $\kappa(g_h,g_1^*)$ and $\kappa(g_\beta,g_1^*)$ vanishes and moreover the authors establish the existence of a derived cohomology class $\kappa'(g_h,g_1^*) \in H^1(\mathbb{Q}_p, V_{gh} \otimes L_p(1))$ satisfying

$$\log^+(\kappa'(g_h,g_1^*)) = \mathcal{L}(\text{ad}^0(g_h)) \cdot L_p(g,h,1),$$

being $\mathcal{L}(\text{ad}^0(g_h))$ the $\mathcal{L}$-invariant of the adjoint of the weight one modular form $g_h$.

Suppose now that both $g$ and $h$ are theta series attached to the same quadratic imaginary field where the prime $p$ splits. In [RiRo1] §6, the authors prove a formula establishing an explicit connection between the Hida–Rankin $p$-adic $L$-function attached to the pair of modular forms $(g,g^*)$ and Katz’s two variable $p$-adic $L$-function. Indeed, let $g = \theta(\psi)$; then [RiRo1] Theorem 6.2] asserts that for any $s \in \mathbb{Z}_p$ the following equality holds up to multiplication in $L^X$:

$$L_p(g,g^*,s) = \frac{1}{\log_p(u_{\psi_{\text{ad}}})} \cdot \zeta_p(s) \cdot L_p(\chi_{K\omega},s) \cdot L_p^{\text{Katz}}(\psi_{\text{ad}},s),$$

being $\psi_{\text{ad}} = \psi/\psi'$. Note that $\psi_{\text{ad}}$ is a ring class character, regardless of whether $\psi$ is so or not. Then, according to [RiRo1],

$$L_p(g,g^*,0) = \log_p(v_1) \pmod{L^X},$$

where $v_1$ is the norm of a generator of the one-dimensional space $(\mathcal{O}_H^1[1/p] \otimes L)^{\mathbb{Q}}$.

Observe that this gives a relation between Beilinson–Flach elements and the cohomology classes coming from elliptic units via the Kummer map,

$$\kappa'(g_h,g_1^*) = \log_p(v_1) \cdot v \pmod{L^X}.$$

Additionally, the derived Beilinson–Flach element is also directly related with the cohomology class $\kappa_{\psi\mathbb{N}}$ via the factorization formula [39] and the results of the preceding sections.

As sketched in [RiRo2] §5.2, the factorization formula [39] admits a counterpart in the case where $g$ and $h$ are no longer self-dual. In this case,

$$L_p(g,h,0) = \log^+(\kappa_p(g_h,h_\alpha)) = \frac{\log_p(u_{\psi_1}) \cdot \log_p(u_{\psi_2})}{\log_p(u_{g_h})} \pmod{L^X},$$
Hida’s $\Lambda$-adic Galois representations afforded by $g$ parameterized by points $(y,z,s)$ is a finite flat extensions of the Iwasawa algebra $\Lambda_{\text{cyc}}$ and $W$.

Then, $\kappa(g_\alpha, h_\alpha) = \mathcal{C} \cdot u_2$, with $\mathcal{C}$ an explicit constant involving $u_\psi^1$, $u_g$, and the periods of $[\text{RiRo2}]$, and $u_2 = u_\psi^1 u_{g_2}$.

An interesting observation is the case where the Euler system of elliptic presents an exceptional zero never arises in the setting of $[\text{RiRo1}]$, due to the regularity assumptions which are assumed in loc. cit. Hence, our results may be seen as a degenerate case of the theory of Beilinson–Flach elements for weight one modular forms.

Let us be more precise in this last sentence. Let $g$ stand for the Hida family of CM theta series whose weight $k_1$ specialization has characteristic Hecke polynomial at $p$ given by

$$(x - \psi^{-1}(p)p^{(k_1-1)/h})(x - \psi^{-1}(p)p^{(k_1-1)/h}).$$

Similarly, let $h$ be the canonical Hida family of CM forms, such that its weight $k_2$ specialization has characteristic Hecke polynomial at $p$

$$(x - p^{(k_2-1)/h})(x - p^{(k_2-1)/h}).$$

Then, as we had already anticipated, Proposition 5.2 in $[\text{RiRo1}]$ and $[\text{LZ2}]$ may be seen as a degenerate case of $[\text{RiRo1}]$ Proposition 3.2, where we do not consider the twist by the cyclotomic character (we fix $s = 0$). Observe that there, the role played by Katz’s two-variable $p$-adic $L$-function is done not exactly by the Hida–Rankin $p$-adic $L$-function, but by its product with the factor

$$(1 - \chi_\mathfrak{g}(c)^{-1}\chi_h(c)^{-1}),$$

where $c$ is a fixed integer number relatively prime to $6pN_fN_q$.

Remark 5.1. The connection between Beilinson–Flach elements and units (in this case circular units) is also exploded in $[\text{Das}]$, where the proof of the main result, a factorization formula for the Rankin-Selberg $p$-adic $L$-function, rests on an explicit comparison between a certain unit constructed via the theory of Beilinson–Flach elements and a circular unit. However, the approach used in loc. cit. is quite different, since the unit is constructed via the specialization of the Beilinson–Flach class at a point of weight $(2,2,1)$.

5.2. Beilinson–Flach elements and exceptional zeros. As we have pointed out, elliptic units may be understood as a special case inside the theory of Beilinson–Flach elements, where the two modular forms are theta series attached to the same imaginary quadratic field. Hence, it is reasonable to expect that the two phenomena we have described concerning exceptional zeros also reproduce in this setting. This section serves to recall the main characteristics of the exceptional zero phenomenon for weight one modular forms.

With the notations of the previous section, let $g$ and $h = \mathfrak{g}^*$ Hida families interpolating two self-dual modular forms $g_\alpha$ and $h_\alpha = g_{\epsilon_{1/\beta}}^\ast$, respectively. We assume that $g \in \Lambda_{\mathfrak{g}}[[q]]$, where $\Lambda_{\mathfrak{g}}$ is a finite flat extensions of the Iwasawa algebra $\Lambda_{\text{cyc}} = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. Write $W = \text{Spf}(\Lambda_{\text{cyc}})$ and $W_\mathfrak{g} = \text{Spf}(\Lambda_{\mathfrak{g}})$. Let $y_0$ be a weight one point of $\Lambda_{\mathfrak{g}}$ such that $g_{y_0} = g_\alpha$ and $g_{y_0}^\ast = g_{\epsilon_{1/\beta}}^\ast$.

The work of $[\text{KLZ}]$ attaches to $(g, g^*)$ a three-variable family of cohomology classes $\kappa(g,g^*)$ parameterized by points $(y, z, s) \in W_\mathfrak{g} \times W_\mathfrak{g} \times W$. More precisely, if $V_\mathfrak{g}$ and $V_{\mathfrak{g}}^\ast$ stand for Hida’s $\Lambda$-adic Galois representations afforded by $g$ and $g^*$, respectively, and $z_{\text{cyc}}$ is the $\Lambda$-adic cyclotomic character,

$$\kappa(g,g^*) \in H^1(Q, V_\mathfrak{g} \hat{\otimes} V_{\mathfrak{g}}^\ast \hat{\otimes} \Lambda_{\text{cyc}}(z_{\text{cyc}} - z_{\text{cyc}}^{-1})).$$

As it follows from the discussion of $[\text{KLZ}]$ §8.10, the three-variable Hida–Rankin $p$-adic $L$-function $L_p(g,g^*)$ is the image of the class $\kappa(g,g^*)$ under a Perrin-Riou map. Again, the numerator or the denominator may vanish in some cases.
For the precise statements concerning Beilinson–Flach classes, we refer the reader to the notations of [RiRo1]. The main results we want to discuss here are the following ones:

(i) When we specialize both \( g \) and \( g^* \) at weight one, and the cyclotomic variable \( s \) is set as \( s = 0 \), the denominator of the Perrin-Riou map is zero and the cohomology class \( \kappa(g, g^*)(y_0, y_0, 0) \) vanishes. Then, the explicit reciprocity law of [KLZ] is replaced by a derived reciprocity law relating the derived cohomology class with \( L_p(g, g^*, 0) \), up to multiplication by an \( \mathcal{L} \)-invariant. This is precisely equation (38).

(ii) When we specialize both \( g \) and \( g^* \) at weight one, and the cyclotomic variable \( s \) is set as \( s = 1 \) the numerator of the Perrin-Riou map is zero but \( L_p(g, g^*, 1) \) does not vanish. This is because [KLZ] Thm. B] contains the correction factor of (44) at the \( L \)-function side, which vanishes in this case.

**Exceptional vanishing of the denominator of the Perrin-Riou big logarithm.** The denominator of the Perrin-Riou regulator introduced in [RiRo1] Prop.3.2 vanishes at all points \((y, y, \ell - 1)\) of weight \((\ell, \ell, \ell - 1)\). In particular, specializing \( g \) and \( g^* \) at the weight one modular forms \( g \) and \( g^* \) respectively, it turns out that

\[
\log^{-+}(\kappa^*(g, g^*)(y_0, y_0, 0)) = L(g_0) \cdot L_p(g, h, 0) = L(g_0) \cdot L_p(g, h, 1) \pmod{L^\times}.
\]

We would like to emphasize the similitude with our main result for elliptic units, which also relates the logarithm of the derived cohomology class with a special value of Katz’s two-variable \( p \)-adic \( L \)-function, up to multiplication by a certain \( \mathcal{L} \)-invariant.

Then, it turns out that the special value \( L_p(g, h)(y, y, 0) \) is related via the functional equation with \( L_p(g, h)(y, y, 1) \) and here, to determine an explicit expression for this value, we follow the same approach than in this note: over the line corresponding to those points of weight \((\ell, \ell, \ell)\), the \( p \)-adic \( L \)-function factors due to the analyticity of an Euler factor (this is properly developed in [Hi]), and this allows us to obtain an explicit computation of the special value \( L_p(g, h, 1) \) via Galois deformation techniques. This expression involves units and \( p \)-units in the field cut out by the Galois representation \( V_{gh} \).

**Exceptional vanishing of the numerator of the Perrin-Riou big logarithm.** The results we present now follow [LZ2] and are the counterpart of those developed in [RiRo1]. We include it here for the sake of completeness, and to illustrate how this exceptional phenomenon arises in a setting which is germane to ours.

Consider specializations of \( \kappa(g, h) \) at weights \((y, z, m)\), where \( \text{wt}(z) = m \). Then, if \( \alpha_{gy} \) and \( \beta_{gy} \) stand for the eigenvalues of the \( p \)-th Hecke polynomial of \( g_y \), with \( \text{ord}(\alpha_{gy}) \leq \text{ord}(\beta_{gy}) \), the Euler factor in the numerator of the Perrin-Riou map is

\[
1 - \frac{\alpha_{gy}}{\alpha_{gy}}.
\]

This factor is zero for all points of weight \((\ell, \ell, \ell)\). But this does not mean that the \( p \)-adic \( L \)-function vanishes at those points, since the explicit reciprocity law of [KLZ] Thm. B] contains the factor

\[
c^2 - c^{2s+2-\ell-m} = c^2(1 - c^{m-\ell})
\]

multiplying the value of \( L_p(g, h) \), where \( c \) is a fixed positive integer coprime with both \( 6p \) and the levels of \( h \) and \( h \). At these points, the Perrin-Riou map interpolates the Bloch–Kato dual exponential map, and

\[
c^2(1 - c^{m-\ell})(1 - \frac{\alpha_{gy}}{p\alpha_{gy}}) L_p(g, g^*)(y, z, m) = \left(1 - \frac{\alpha_{gy}}{\alpha_{gy}}\right) \cdot \exp^{\leftrightarrow}_{BK}(\kappa_p(g, g^*)(y, z, s)).
\]

where \( \exp^{\leftrightarrow} \) stands for the composition of the projection to a certain subspace of \( V_{gh} \otimes L_p(1) \) followed by the dual exponential map and the pairing with the canonical differentials.
Since at the previous equation both sides vanish along the line $y = z$, wt($z$) = $m$, we may consider the derivative at a point $(y, y, \ell)$, obtaining the expression

$$c^2 (1 - p^{-1}) \log_p(c) L_p(\mathbf{g}, \mathbf{h})(y, y, \ell) = L(g_y) \cdot \exp_{\mathrm{BK}}^{\ast +} (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y, y, \ell)) \pmod{L^\times}.$$  

Additionally, invoking Hida’s result on the existence of an improved $p$-adic $L$-function $[H]$, we get that

$$\log_p(c) = \exp_{\mathrm{BK}}^{\ast +} (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y, y, \ell)) \pmod{L^\times}.$$  

Observe now that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell) = L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) \pmod{L^\times},$$  

and hence

$$L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) = L(g_y)^{-1} \cdot \log^+ ((\mathbf{g}, \mathbf{g}^*) (y, y, \ell - 1)). \pmod{L^\times}.$$  

Consequently, up to multiplication by $L^\times$, we have the equality

$$\log_p(c) \left( \frac{L(g_y)}{2} \right) \log^+ (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y, y, \ell - 1)) = \exp_{\mathrm{BK}}^{\ast +} (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y, y, \ell))$$  

In particular,

$$\frac{\log_p(c)}{L(g)^2} \cdot \log^+ (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y_0, y_0, 0)) = \exp_{\mathrm{BK}}^{\ast +} (\kappa_p(\mathbf{g}, \mathbf{g}^*) (y_0, y_0, 1)) \pmod{L^\times}.$$  

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