On a possible new type of a T odd skewon field linked to electromagnetism*

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Abstract

In the framework of generally covariant (pre-metric) electrodynamics ("charge & flux electrodynamics"), the Maxwell equations can be formulated in terms of the electromagnetic excitation $H = (D, H)$ and the field strength $F = (E, B)$. If the spacetime relation linking $H$ and $F$ is assumed to be linear, the electromagnetic properties of (vacuum) spacetime are encoded into 36 components of the vacuum constitutive tensor density $\chi$. We study the propagation of electromagnetic waves and find that the metric of spacetime emerges eventually from the principal part $(\chi)$ of $\chi$ (20 independent components). In this article, we concentrate on the remaining skewon part $(\chi)$ (15 components) and the axion part $(\chi)$ (1 component). The skewon part, as we'll show for the first time, can be represented by a 2nd rank traceless tensor $S_{ij}$. By means of the Fresnel equation, we discuss how...
this tensor disturbs the light cones. Accordingly, this is a mechanism for violating Lorentz invariance and time symmetry. In contrast, the (abelian) axion part \( (3) \chi \) does not interfere with the light cones.

Keywords: Electrodynamics, light cone, metric, skewon, abelian axion.

1 The constitutive tensor density \( \chi \) of vacuum spacetime and its irreducible decomposition

In pre-metric electrodynamics \[5\] the axioms of electric charge and of magnetic flux conservation manifest themselves in the Maxwell equations for the excitation \( H = (\mathcal{D}, \mathcal{H}) \) and the field strength \( F = (E, B) \):

\[
dH = J, \quad dF = 0.
\]

If local coordinates \( x^i \) are given, with \( i, j, \ldots = 0, 1, 2, 3 \), we can decompose the excitation and field strength 2-forms into their components according to

\[
H = \frac{1}{2} H_{ij} \, dx^i \wedge dx^j, \quad F = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j.
\]

Then a linear spacetime relation, see \[13\], reads

\[
H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl} = \frac{1}{4} \epsilon_{ijkl} \chi_{klmn} F_{mn}.
\]

The constitutive tensor density \( \chi \) has 36 independent components. If we decompose it with respect to the 4-dimensional linear group into irreducible pieces, then we find

\[
\chi = (1) \chi + (2) \chi + (3) \chi, \quad \text{with} \quad 36 = 20 \oplus 15 \oplus 1
\]

independent components, respectively. In components, we have the following definitions for the irreducible pieces of \( \chi \):

\[
(2) \chi_{ijkl} := \frac{1}{2} \left( \chi_{ijkl} - \chi_{klij} \right), \quad (3) \chi_{ijkl} := \chi^{ijkl},
\]

\[
(1) \chi_{ijkl} := \chi_{ijkl} - (2) \chi_{ijkl} - (3) \chi_{ijkl}.
\]
Then, by explicit substitution of the definitions, it is straightforward to show that the irreducible piece $a$ of the irreducible piece $b$, for $a, b = 1, 2, 3$, with the Kronecker $\delta^{ab}$, behaves as follows:

$$\begin{align*}
(a) \left[ \begin{array}{c}
(b) \chi^{ijkl} \\
\end{array} \right] &= \delta^{ab} (a) \chi^{ijkl}, \\
\text{no sum over } a.
\end{align*}$$

(7)

More explicitly, we have, e.g.,

$$(1) \chi^{ijkl} = (1) \chi^{klij}, \quad (2) \chi^{ijkl} = -(2) \chi^{klij}.$$  \hspace{1cm} (8)

A simple way to see the correctness of this irreducible decomposition is to recall that $\chi^{ijkl}$ is antisymmetric in the first and in the last pair of indices; then it is possible to map it to a $6 \times 6$ matrix. And this matrix can be decomposed in its symmetric tracefree piece corresponding to $(1) \chi^{ijkl}$, the antisymmetric piece $(2) \chi^{ijkl}$, and its trace $(3) \chi^{ijkl}$, see (4).

1.1 Principal piece $(1)\chi$ and light cone structure

Let us try to get a rough picture of the physical meaning of these different irreducible pieces. From $(1)\chi^{ijkl}$ alone, if electric/magnetic reciprocity is assumed additionally to hold for $(3)$, then, up to an arbitrary conformal factor, the Lorentzian metric of spacetime can be derived \cite{12, 6, 11, 4, 16}. One can think of this reduction in the way that electric/magnetic reciprocity cuts the 20 components of $(1)\chi^{ijkl}$ into half, that is, only 10 components for the metric are left. Modulo the undetermined function, we have then 9 remaining components.

These 9 components determine the light cone at each point of spacetime. Accordingly, in $(1)\chi^{ijkl}$ the light cone of spacetime is hidden and thereby conventional Maxwell-Lorentzian vacuum electrodynamics as well. To put it more geometrically, the first irreducible piece of $\chi$, via electric/magnetic reciprocity, yields the conformal structure of spacetime. In this sense, there is no doubt that $(1)\chi^{ijkl}$ is the principal part of the constitutive tensor density $\chi$ of the vacuum.

1.2 Abelian axion $\alpha$

Thus one could be inclined to believe that it is best to require the vanishing of the second and the third irreducible piece of $\chi$. But this would appear to be premature before one has inquired into the possible meanings of these
pieces. In fact, the *abelian axion* \( \alpha(x) \), introduced by Ni \([4,8]\) in 1973, is represented by \( (3) \chi^{ijkl} \). This field is P odd (P stands for parity), i.e., it is a pseudo- or axial-scalar in conventional language. Experimentally, this field hasn’t been found so far \([1,3,2,17]\). Nevertheless, as we shall see below, the axion does not interfere with the light cone structure of spacetime at all. Therefore, this chapter is not yet closed, the abelian axion remains a serious option for a particle search in experimental high energy physics.

### 1.3 Skewon piece \( S_{i,j} \) and dissipation

If two irreducible pieces of a quantity don’t vanish possibly, it is not too far-fetched to reflect on the remaining piece \( (2) \chi^{ijkl} \) and on what its existence may mean. The conventional argument for discarding \( (2) \chi \) runs as follows, see Post \([13]\). Suppose a Lagrangian 4-form \( L \) exists for the electromagnetic field. In general \( H \sim \partial L/\partial F \). If \( H \) is assumed to be linear in \( F \), as is done in \( (3) \), then \( L \) reads

\[
L \sim H \wedge F = \chi \cdot F \wedge F = (1) \chi \cdot F \wedge F + (3) \chi \cdot F \wedge F. \tag{9}
\]

The term with \( (1) \chi \) eventually becomes the Maxwell Lagrangian, the term with \( (3) \chi \) part of the axion Lagrangian. The piece with \( (2) \chi \) drops out of the Lagrangian \( L \) because of the antisymmetry of \( (2) \chi \) according to \( (2) \chi^{ijkl} = - (2) \chi^{klij} \). Since we conventionally assume that all information of a physical system is collected in its Lagrangian, we reject \( (2) \chi^{ijkl} \neq 0 \) as being unphysical. This presents the state of the art.

However, we would like to point out that the argument with the Lagrangian does *not* forbid the existence of a non-vanishing \( (2) \chi \neq 0 \). It only implies that \( L \) is ‘insensitive’ to \( (2) \chi \). In other words, if \( (2) \chi \neq 0 \), then not all information about the system is contained in the Lagrangian.

Remember that pre-metric electrodynamics is based on the conservation laws of electric charge and magnetic flux and on an axiom about the (kinematic) electromagnetic energy-momentum current \( k \Sigma_{\alpha} \). No Lagrangian is needed nor assumed. But, of course, the proto-Lagrangian \( \Lambda := -H \wedge F/2 \) exists anyway and has indeed been introduced in the context of the discussion of \( k \Sigma_{\alpha} \). Accordingly, in pre-metric electrodynamics, even when linearity is introduced according to \( (3) \), \( \Lambda \) has no decisive meaning — and that \( \Lambda \) does not depend on \( (2) \chi \) is interesting to note but no reason for a headache.

This reminds us of a complementary property of the axion \( \alpha \) or of \( (3) \chi \). It features in \( \Lambda \), see \( (3) \), but it drops out of \( k \Sigma_{\alpha} \), as we will see below. Should we
be alarmed that the axion doesn’t contribute to the electromagnetic energy-momentum current? No, not really. As has been shown by Ni, in spite of this ‘insensitivity’ of $k\Sigma_\alpha$ against $\alpha$, one can set up a reasonable theory of the axion.

Consequently, in linear pre-metric electrodynamics, it is not alarming that $(2)\chi$ drops out from the proto-Lagrangian $\Lambda$, and in future we will take the possible existence of $(2)\chi_{ijkl}$ for granted.

What is then the possible physical meaning of $(2)\chi$? In pre-metric electrodynamics [5], the energy-momentum current reads

$$k\Sigma_\alpha := \frac{1}{2} [F \wedge (e_\alpha H) - H \wedge (e_\alpha F)].$$

(10)

We can specify a certain vector field $\xi = \xi^\alpha e_\alpha$, with the basis $e_\alpha$ of the tangent vector space at each point of spacetime. Then we can transvect the energy-momentum current with $\xi^\alpha$:

$$Q := \xi^\alpha k\Sigma_\alpha = \frac{1}{2} [F \wedge (\xi H) - H \wedge (\xi F)].$$

(11)

The scalar-valued 3-form $Q$ is expected to be related to conserved quantities provided we can find suitable (Killing type) vector fields $\xi$. Therefore we determine its exterior derivative and find after some algebra, see Rubilar [13],

$$dQ = (\xi F) \wedge J + \frac{1}{2} (F \wedge L_\xi H - H \wedge L_\xi F),$$

(12)

or, in holonomic components, with $Q^i := \epsilon_{ijkl}Q_{jkl}/6$, $J^i := \epsilon_{ijkl}J_{jkl}/6$, and $H^{ij} := \epsilon_{ijkl}H_{kl}/2$,

$$\partial_i Q^i = \xi^k F_{kl} J^l + \frac{1}{4} \left( F_{kl} L_\xi H^{kl} - H^{kl} L_\xi F_{kl}\right).$$

(13)

Here $L_\xi$ denotes the Lie derivative along $\xi$. Now we substitute the linear relation (3), or $H^{kl} = \chi^{klmn}F_{mn}/2$, and find

$$\partial_i Q^i = \xi^k F_{kl} J^l + \frac{1}{8} \left[ F_{kl} L_\xi (\chi^{klmn}F_{mn}) - \chi^{klmn}F_{mn} L_\xi F_{kl}\right].$$

(14)

We apply the Leibniz rule of the Lie derivative and rearrange a bit:

$$\partial_i Q^i = \xi^k F_{kl} J^l + \frac{1}{8} \left[ (L_\xi \chi^{ijkl}) F_{ij} F_{kl} + (\chi^{ijkl} - \chi^{klij}) F_{ij} L_\xi F_{kl}\right].$$

(15)
We substitute the irreducible pieces of $\chi^{ijkl}$. Then we have

$$\partial_i Q^i = \xi^k F_{kl} J^l + \frac{1}{8} \mathcal{L}_\xi \left( (1) \chi^{ijkl} + (3) \chi^{ijkl} \right) F_{ij} F_{kl} + \frac{1}{4} (2) \chi^{ijkl} F_{ij} \mathcal{L}_\xi F_{kl}. \quad (16)$$

If $(1)\chi$ and $(3)\chi$ carry a reasonable symmetry, namely $\mathcal{L}_\xi (1)\chi = \mathcal{L}_\xi (2)\chi = 0$, and are thus well-behaved, then, in vacuum, i.e., for $J^i = 0$, we have non-conservation of energy, for example, because of the offending term $(2)\chi F \dot{F}$. Here the dot symbolizes the ‘time’ derivative along $\xi$.

In any case we see that $(2)\chi$ induces a dissipative term with first ‘time’ derivative. This is what we might have expected since dissipative phenomena in general cannot be described in a Lagrangian framework.

It is then our hypothesis that $(2)\chi$ can represent fields which are odd under $T$ transformations. Of course, we must investigate how these skewons, as we may call them in a preliminary way, disturb the light cone and whether there is perhaps a viable subclass of the skewons.

## 2 The skewon field $S_{i}^{j}$

The skewon piece of the constitutive tensor density $\chi^{ijkl}$ is defined in (5)\textsuperscript{1}. Therefrom we can read off the algebraic symmetries

$$(2)\chi^{ijkl} = -(2)\chi^{klij}, \quad (2)\chi^{[ijkl]} = 0. \quad (17)$$

From $\chi^{ijkl}$, the skewon piece inherits the antisymmetry in the first and the second pair of indices:

$$(2)\chi^{(ij)kl} = 0, \quad (2)\chi^{(ij)(kl)} = 0. \quad (18)$$

Thus $(2)\chi^{ijkl}$ can be mapped to an antisymmetric (or skew symmetric) $6 \times 6$ matrix. For this reason we called it the skewon piece of $\chi^{ijkl}$. Since this matrix has 15 independent components, we expect that it is equivalent to a 2nd rank tensor in 4 dimensions (16 components) with vanishing trace (1 component). Accordingly, we define the skewon field by

$$S_{i}^{j} := \frac{1}{4} \hat{\epsilon}_{iklm} (2)\chi^{klmj}. \quad (19)$$

Because of (17)\textsuperscript{2}, its trace vanishes, indeed,

$$S_{n}^{n} = \frac{1}{4} \hat{\epsilon}_{nkln} (2)\chi^{[klmn]} = 0. \quad (20)$$
If we define the tracefree part of $S_{i}^{j}$ by

$$S_{i}^{j} := S_{i}^{j} - \frac{1}{4} S_{k}^{k} \delta_{i}^{j}, \quad (21)$$

then for our $S_{i}^{j}$, we have $S_{i}^{j} = S_{i}^{j}$.

Let us invert (19). We multiply by $\epsilon^{inpq}$ and find

$$\epsilon^{inpq} S_{i}^{j} := \frac{1}{4} \epsilon^{inpq} \hat{\epsilon}_{iklm} \chi^{klmj} = \frac{1}{4} \delta_{klm}^{npq} \chi^{[klm]j} \quad (22)$$

or

$$\chi^{[ijkl]} = -\frac{2}{3} \epsilon^{ijkl} S_{m}^{l}. \quad (23)$$

We expand the bracket:

$$\chi^{ijkl} + 2\chi^{[klij]} = -2 \epsilon^{ijkl} S_{m}^{l}. \quad (24)$$

The second term on the left hand side of this equation, by means of the symmetries (17) and (18), can be rewritten as $\chi^{ijkl} = -\chi^{lijk}$. Thus,

$$\chi^{ijkl} + 2\chi^{[klij]} = -2 \epsilon^{ijkl} S_{m}^{l} \quad (25)$$

or, because of (18)2,

$$\chi^{ijkl} = 2 \epsilon^{ijm[k} S_{m}^{l]}. \quad (26)$$

In order to make the symmetry (17) manifest, we rename the indices

$$\chi^{klij} = 2 \epsilon^{klm[i} S_{m}^{j]}. \quad (27)$$

and subtract (27) from (26). This yields the final result

$$\chi^{ijkl} = \epsilon^{ijm[k} S_{m}^{l]} - \epsilon^{klm[i} S_{m}^{j]}. \quad (28)$$

For (28), all the symmetries (17) and (18) can be verified straightforwardly. In (19), we chose the conventional factor as $1/4$ in order to find in (28) a formula free of inconvenient factors.
Decomposing the “dual” constitutive tensor density $\kappa_{ij}^{\ kl}$ and recovering the skewon field

Let us recall that the starting point for the discussion of the constitutive (spacetime) relation is the $\kappa$-map. Namely, we have the tensor density $\kappa_{ij}^{\ kl}$ with 36 components, see [5] Eq.(D.1.11). One can decompose this object into its irreducible pieces. Obviously, contraction is the only tool for such a decomposition. Following Post [14], we can define the contacted tensor of type (1,1),

$$\kappa_{i}^{\ k} := \kappa_{il}^{\ kl}, \quad (29)$$

with 16 independent components. The second contraction yields the pseudo-scalar function

$$\kappa := \kappa_{k}^{\ k} = \kappa_{kl}^{\ kl}. \quad (30)$$

The traceless piece

$$\kappa_{i}^{\ kl} := \kappa_{i}^{\ k} - \frac{1}{4} \kappa \delta_{l}^{k}, \quad (31)$$

has 15 independent components. These pieces can now be subtracted from the original constitutive tensor. Then,

$$\kappa_{ij}^{\ kl} = (1)\kappa_{ij}^{\ kl} + (2)\kappa_{ij}^{\ kl} + (3)\kappa_{ij}^{\ kl} \quad (32)$$

$$= (1)\kappa_{ij}^{\ kl} + 2\kappa_{[i}^{\ k} \delta_{j]}^{l} + \frac{1}{6} \kappa \delta_{[i}^{k} \delta_{j]}^{l}. \quad (33)$$

By construction, $(1)\kappa_{ij}^{\ kl}$ is the totally traceless part of the constitutive map:

$$(1)\kappa_{il}^{\ kl} = 0. \quad (34)$$

Thus, we split $\kappa$ according to $36 = 20 + 15 + 1$, and the (2,2) tensor $(1)\kappa_{ij}^{\ kl}$ is subject to the 16 constraints (34) and carries $20 = 36 - 16$ components.

Now we are prepared to proceed with the analysis of the $\chi$-picture. By definition, we have

$$\chi_{ijkl} := \frac{1}{2} \epsilon_{ijmn} \kappa_{mn}^{\ kl}. \quad (35)$$

Substituting here the decomposition (33), we find

$$\chi_{ijkl} = (1)\chi_{ijkl} + (2)\chi_{ijkl} + (3)\chi_{ijkl}. \quad (36)$$
In correspondence with (33), we have the irreducible pieces:

\begin{align}
(1) \chi_{ijkl} &= \frac{1}{2} \epsilon^{ijmn} (1) \kappa_{mn}^{\quad kl}, \\
(2) \chi_{ijkl} &= \frac{1}{2} \epsilon^{ijmn} (2) \kappa_{mn}^{\quad kl} = - \epsilon^{ijm[k} \kappa_{m]}^{\quad l]}, \\
(3) \chi_{ijkl} &= \frac{1}{2} \epsilon^{ijmn} (3) \kappa_{mn}^{\quad kl} = \frac{1}{12} \epsilon^{ijkl} \kappa.
\end{align}

Let us identify the skewon and the axion fields by

\[ S_{t}^{j} = -\frac{1}{2} \kappa_{t}^{\quad j}, \quad \alpha = \frac{1}{12} \kappa. \]

Using the S-identity (44), we have

\[ (2) \chi_{ijkl} = 2 \epsilon^{ijm[k} S_{m}^{\quad l]} = - 2 \epsilon^{klm[i} S_{m}^{\quad j]} \]

or

\[ (2) \chi_{ijkl} = \epsilon^{ijm[k} S_{m}^{\quad l]} - \epsilon^{klm[i} S_{m}^{\quad j]} \]

Thus, the S-identity (44) guarantees the skew-symmetry of \( (2) \chi \) under exchange of the first with the second index pair:

\[ (2) \chi_{ijkl} = - (2) \chi_{klij}. \]

On the other hand, the K-identity (104) provides the symmetry of the \( (1) \chi \):

\[ (1) \chi_{ijkl} = (1) \chi_{klij}. \]

This holds true because of the tracelessness property (36).

It is thus very satisfactory to find the one-to-one correspondence of the irreducible decomposition (33) of \( \kappa_{ij}^{\quad kl} \) [based on the trace extraction] and the irreducible decomposition (36) of \( \chi_{ijkl} \) [based on the separation into symmetric and skew-symmetric parts].

4 The skewon field as 6 \times 6 matrix

It is convenient to put the \( S_{t}^{j} \) also into the conventional 6 \times 6 matrix since this provides the interpretation of the spacetime relation in terms of the 3-dimensional electromagnetic field \( D, H, B, E \). Therefore we compute the
$3 \times 3$ matrices with the help of (28) in a fairly messy but straightforward way, see [15]:

\[(2) A_{ab} := (2) \chi_{0ab} = \epsilon^{abc} S_c^0, \quad (45)\]
\[(2) B_{ba} := \frac{1}{4} \hat{\epsilon}_{acd} \hat{\epsilon}_{bef} (2) \chi^{cdef} = -\hat{\epsilon}_{abc} S^0_c, \quad (46)\]
\[(2) C_{ab} := \frac{1}{2} \hat{\epsilon}_{bed} (2) \chi^{cd0a} = -S^a_b + \delta^a_b S^c_c, \quad (47)\]
\[(2) D^{ab} := \frac{1}{2} \hat{\epsilon}_{acd} (2) \chi^{0bed} = S^a_b - \delta^a_b S^c_c. \quad (48)\]

Quite generally, we have

\[
\chi^{IJ} = \begin{pmatrix}
B_{ab} & D^{ab} \\
C_{ab} & A_{ab}
\end{pmatrix}.
\quad (49)
\]

For the skewon piece, we have specifically

\[
(2) \chi^{IJ} = \frac{1}{2} (C_{[ab]} - D_{[ba]}) \begin{pmatrix}
(2) B_{ab} & (2) D^{ab} \\
(2) C_{ab} & (2) A_{ab}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\hat{\epsilon}_{abc} S^c_0 + S^a_b + \delta^a_b S^c_c \\
-S^a_b + \delta^a_b S^c_c + \epsilon^{abc} S^0_c
\end{pmatrix}.
\quad (52)\]

The rest is done easily. We substitute (53) into the spacetime relation

\[
\begin{pmatrix}
\mathcal{H}_a \\
\mathcal{D}^a
\end{pmatrix} = \begin{pmatrix}
C^b_a & B_{ba} \\
A_{ba} & D^b_a
\end{pmatrix} \begin{pmatrix}
-E_b \\
B^b
\end{pmatrix}
\quad (54)
\]

and find quite generally for the skewon contributions of the 3D excitations,

\[
(2) \mathcal{D}^a = -\epsilon^{abc} S^0 c E_b + (\delta^a_b S^c_c + S^a_b B^b),
\quad (55)\]
\[
(2) \mathcal{H}_a = (\epsilon^b_a S^c_c + \delta^b_a B^b) E_b - \hat{\epsilon}_{abc} S^0_c B^b.
\quad (56)\]

\footnote{For future reference we display here also the principal and the axion pieces as 6 $\times$ 6 matrices, respectively, namely

\[
(1) \chi^{IJ} = \begin{pmatrix}
B_{(ab)} & \frac{1}{2} (\mathcal{D}^{[b} + \mathcal{D}^{a])} \\
\frac{1}{2} (\mathcal{C}^{[a} + \mathcal{C}^{b)}) & (1) \mathcal{A}^{[ab]}
\end{pmatrix} = \begin{pmatrix}
(1) B_{(ab)} & (1) D^{ab} \\
(1) C_{[ab]} & (1) \mathcal{A}_{ab}
\end{pmatrix},
\quad (50)\]

here we used the notation $\mathcal{M}^{a b} := M^{a b} - M^c_{b c} \delta^b_a / 3$, and

\[
(3) \chi^{IJ} = \frac{1}{6} (C^c_c + D^c_c) \begin{pmatrix}
0_3 & 1_3 & 1_3 \\
1_3 & 0_3 & 0_3
\end{pmatrix} = \frac{1}{6} (C^c_c + D^c_c) \epsilon^{IJ}.
\quad (51)\]}

10
The diagonal terms of the 3D tensor $(-\delta^b_a S^c_c + S^b_a)$ are of the type as those postulated by Nieves and Pal \cite{10} for describing a “...third electromagnetic constant of an isotropic medium”. However, our “medium” is spacetime, i.e., the vacuum itself.

We stress that for the derivation of (55) and (56) we neither specialized the skewon field $S^{ij}$ nor did we apply any metric distilled from $(1)\chi^{ijkl}$. Therefore the 1+3 decompositions in (55) and (56) are generally valid for any linear spacetime relation.

5 Spatially isotropic skewon field and the ansatz of Nieves and Pal

If we specialize first to 3-dimensional isotropy, then we have, with the 3D pseudo-scalar function $s=s(x)$,

$$S^b_a = \frac{s}{2} \delta^b_a, \quad S^0_a = 0, \quad S^0_b = 0. \quad (57)$$

It was possible to formulate isotropy for the skewon piece without taking recourse to a metric tensor since $S^{ij}$ is a mixed variant tensor of 2nd rank. Thus,

$$S^{ij} = \frac{s}{2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (58)$$

and

$$-\delta^b_a S^c_c + S^b_a = -s \delta^b_a. \quad (59)$$

Consequently, equations (53) and (56) become

(2) $D^a = -s B^a$, \quad (2) $H_a = -s E_a$, \quad (60)

exactly what Nieves and Pal had postulated and discussed subsequently \cite{10}. Accordingly, the spacetime relations (55) and (56) are anisotropic generalizations of the Nieves and Pal ansatz. The off diagonal terms with $S^a_0$ and $S^b_0$ lead, respectively, to magnetic and electric Faraday type of effects of the spacetime under consideration, i.e., these terms rotate the polarization of a wave propagating in such a spacetime.
6 On the four electromagnetic constants for vacuum spacetime with spatial isotropy

Since the axion field also contributes to the spacetime relation of the type (60), we are going to determine it. We have

\[(3)\chi^{ijkl} := \alpha \hat{\epsilon}^{ijkl}, \quad \alpha = \frac{1}{4!} \hat{\epsilon}^{ijkl} (3)\chi^{ijkl},\quad (61)\]

\[(3)\chi^{IJ} = \alpha \begin{pmatrix} 0_3 & 1_3 \\ 1_3 & 0_3 \end{pmatrix} = \alpha \epsilon^{IJ}.\quad (62)\]

Because of (64), we find

\[(3) D^a = +\alpha B^a,\quad (63)\]

\[(3) H_a = -\alpha E_a.\quad (64)\]

If we take (80) from below in Cartesian coordinates, i.e., we have a Lorentz metric \(\sigma_{ij} = (c^2, -1, -1, -1)\) with \(\sigma_{ab} = -\delta_{ab}\), then the spacetime relation becomes

\[D^a = \varepsilon_0 \delta^{ab} E_b + (-s + \alpha) B^a,\quad (65)\]

\[H_a = (-s - \alpha) E_a + \mu_0 \delta_{ab} B^b.\quad (66)\]

In the special case when skewon and axion become constant fields, we can say that we found 4 electromagnetic constants for a spacetime (vacuum) with spatial isotropy: The electric constant \(\varepsilon_0\), the magnetic constant \(\mu_0\), the isotropic part \(s\) of the skewon \(S_{ab}\), and the axion \(\alpha\).

7 How does the skewon field affect light propagation [16, 15]?

For any linear spacetime relation, the Fresnel equation can be written as

\[G^{ijkl} q_i q_j q_k q_l = 0,\quad (67)\]

where \(q_i\) is the wave covector. The fourth order tensor density of weight +1, the Fresnel tensor, as we may call it, is defined by

\[G^{ijkl} := \frac{1}{4!} \hat{\epsilon}_{mnpq} \hat{\epsilon}_{rstu} \chi_{mnr} \chi^{jp[s} \chi^{k]tu}.\quad (68)\]
First, we recall \[16\] that the Fresnel equation is independent of the axion piece \((3)\chi\) of the constitutive tensor:

\[
\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{ijkl}(^{(1)}\chi + ^{(2)}\chi).
\] (69)

Thus, for arbitrary \(^{(1)}\chi\) and \(^{(2)}\chi\), we have

\[
\mathcal{G}^{ijkl}(^{(3)}\chi) = 0.
\] (70)

Furthermore, due to the skewsymmetry \([17]_1\) of \(^{(2)}\chi\), we have

\[
\mathcal{G}^{ijkl}(^{(2)}\chi) = 0.
\] (71)

By explicit calculations, we used computer algebra for it, we find

\[
\mathcal{G}^{ijkl}(^{(2)}\chi + ^{(3)}\chi) = 0.
\] (72)

This identity is non-trivial since \(\mathcal{G}\) depends cubically on the constitutive tensor \(\chi\). The identity (72) shows that the symmetric piece \(^{(1)}\chi\) is indispensable in order to obtain well behaved wave properties: If \(^{(1)}\chi = 0\), the Fresnel equation is trivially satisfied and thus no light cone structure could be induced.

Furthermore, since in general

\[
\mathcal{G}^{ijkl}(^{(1)}\chi + ^{(2)}\chi) \neq \mathcal{G}^{ijkl}(^{(1)}\chi),
\] (73)

the skewon field does influences the Fresnel equation, and therefore, eventually the light cone structure. An example of this general result can be found in the asymmetric constitutive tensor studied by Nieves and Pal \([9, 10]\). Actually, after some algebra one finds

\[
\mathcal{G}^{ijkl}(^{(1)}\chi + ^{(2)}\chi) = \mathcal{G}^{ijkl}(^{(1)}\chi)
+ \frac{2}{4!} \hat{\epsilon}_{mnpq} \hat{\epsilon}_{rstu} ^{(1)}\chi ^{mnr} i^{(2)\chi} j^{op}\kappa^{(2)\chi} l^{qt} u
+ \frac{1}{4!} \hat{\epsilon}_{mnpq} \hat{\epsilon}_{rstu} ^{(2)}\chi ^{mnr} i^{(1)\chi} j^{op}\kappa^{(2)\chi} l^{qt} u
\] (74)

or, in a (more or less) obvious notation, see the definition \((68)\),

\[
\mathcal{G}^{ijkl}(\chi, \chi, \chi) = \mathcal{G}^{ijkl}(^{(1)}\chi, ^{(1)}\chi, ^{(1)}\chi) + 2 \mathcal{G}^{ijkl}(^{(1)}\chi, ^{(2)}\chi, ^{(2)}\chi)
+ \mathcal{G}^{ijkl}(^{(2)}\chi, ^{(1)}\chi, ^{(2)}\chi).
\] (75)
The other terms vanish due to the symmetry properties of each irreducible piece.

Take now (74) and substitute the parametrization of \( (2) \chi \) in terms of \( S_i^j \), see (13). After some lengthy but straightforward algebra, one finds that the two last contributions to the right hand side of (74) are actually equal, namely

\[
G^{ijkl}(1, (2) \chi, (2) \chi) = \frac{1}{3} (1) \chi^{m(i|n|j} S_m^k S_n^l) .
\]

(77)

Therefore, the final result reads

\[
G^{ijkl}(\chi) = G^{ijkl}(1, (2) \chi) = G^{ijkl}(1) + (1) \chi^{m(i|n|j} S_m^k S_n^l) ,
\]

(78)

a very simple expression, indeed.

For the particular case of Nieves and Pal, we will use the ansatz

\[
S_i^j = S_i^j = \omega_i v^j - \frac{1}{4} \omega_k v^k \delta_i^j
\]

(79)

and assume additionally the existence of a metric \( g_{ij} \) (resulting from \( (1) \chi \)) for raising and lowering indices: \( \omega_i = g_{il} v^l \). This determines \( (2) \chi \) via (28). Furthermore, we assume for the principal part the usual metric dependent expression for the vacuum in a Riemannian spacetime, namely

\[
(1) \chi^{ijkl} = 2 \sqrt{\frac{\varepsilon_0}{\mu_0}} \sqrt{-g} g^{i[k} g^{j]l} .
\]

(80)

Because of (69), the axion piece is not required. Accordingly, we substitute \( (1) \chi \) and \( (2) \chi \) into (68). Then,

\[
G^{ijkl} q_i q_j q_k q_l = - \sqrt{\frac{\varepsilon_0}{\mu_0}} \sqrt{-g} \times
\]

\[
\left[ \frac{\varepsilon_0}{\mu_0} (q \cdot q) (v \cdot q) (v \cdot v) (v \cdot q)^2 + (v \cdot q)^4 \right] = 0 . \]

(81)

We now use \( q_i = (\omega, -k) \), \( v^i = (v, 0, 0, 0) \), \( g_{ij} = \alpha_{ij} = (c^2, -1, -1, -1) \), and \( c^2 = \frac{1}{\varepsilon_0 \mu_0} \). Thus,

\[
- \frac{1}{\varepsilon_0^3} G^{ijkl} q_i q_j q_k q_l = \left[ \omega^2 - (c \kappa)^2 \right]^2 + \left[ \frac{cv^2}{\varepsilon_0} \omega c \kappa \right]^2 = 0 . \]

(82)
This equation describes how the skewon piece, via $v$, affects light propagation. In general, for $v \neq 0$, the Fresnel equation will have complex solutions. This is again a manifestation of the dispersive properties described by the skewon piece $(2)\chi$ of the constitutive tensor. Modulo different conventions, our result (82) agrees with that of Nieves and Pal [10] Eq.(5.7).

8 Discussion

The skewon and the axion part of $\chi$ are explicitly known:

\begin{align*}
(2)\chi_{ijkl} &= \epsilon^{ijm[k} S_{m} l] - \epsilon^{klm[i} S_{m} j], \\
(3)\chi_{ijkl} &= \alpha \epsilon^{ijkl}.
\end{align*}

Accordingly, we found a traceless 2nd rank tensor field $S_{ij} = S_{ji}$ and a pseudoscalar $\alpha$. For the principal part $(1)\chi$ with its 20 independent components things are more difficult.

We can tentatively assume

\begin{align*}
(1)\chi_{ijkl} &\sim g_{[i[k} h_{l]}^j] + g_{[k[i} h_{j]}^l] + \epsilon^{ijm[k} a_{m} l] + \epsilon^{klm[i} a_{m} j],
\end{align*}

with two symmetric $g^{ij} = g^{ji}$, $h^{ij} = h^{ji}$ and a traceless tensor $a_{i}^{j}$, with $a_{k}^{k} = 0$. However, a second look convinces us that in (85) the two last terms (with $a$) should be deleted. The reason is the S-identity, see Appendix. In view of (94), a traceless (1,1) tensor can only contribute to the $(2)\chi$, not to $(1)\chi$. So, the structure of $(1)\chi$ is most probably determined only by the two symmetric tensors $g^{ij}$ and $h^{ij}$ with $10 + 10$ independent components.

Furthermore, it is clear that the two terms on the r.h.s. of (85) are equal: $g_{[i[k} h_{l]}^j] = g_{[k[i} h_{j]}^l]$. Indeed:

\begin{align*}
4g_{[i[k} h_{l]}^j] &= g^{ik} h_{lj} - g^{il} h_{kj} - g^{jk} h_{li} + g^{lj} h_{ki} = 4g_{[k[i} h_{j]}^l] \quad (86) \\
&= g^{ki} h_{lj} - g^{kj} h_{il} - g^{il} h_{jk} + g^{lj} h_{ki}. \quad (87)
\end{align*}

Since both tensors are symmetric, $g^{ij} = g^{ji}$ and $h^{ij} = h^{ji}$, the two expressions are equivalent. Thus only one term is left over:

\begin{align*}
(1)\chi_{ijkl} &\sim g_{[i[k} h_{l]}^j].
\end{align*}

By the way, if we turn to the $\kappa$-representation, we have:

\begin{align*}
(1)\kappa_{ij}^{kl} &\sim \hat{\epsilon}_{ijmn} g_{m[k} h_{l]}^n. \quad (89)
\end{align*}
For the symmetric $g^{ij} = g^{ji}$ and $h^{ij} = h^{ji}$, a contraction is automatically zero, $(^1){\kappa}_{ik}^{kl} \equiv 0$. This means that such a term belongs indeed to the first irreducible part, in accordance with Sec.3. Such a structure looks very much as the most general parametrization of the first irreducible part, but a proof is still not available.

Summarizing: it seems that the general structure of the first irreducible part reads

$$
(^1)\chi_{ijkl} \sim g^{[i} [k} h^{l][j]}.
$$

It would be desirable to find a corresponding proof.

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**A Two identities**

We can rederive our results (19) and (28) in an alternative way in order to get more insight in the relevant structure. In 4D, any object with five completely antisymmetrized indices is zero, $Z^{[ijkl]} \equiv 0$. When 4 of these 5 indices belong to the Levi-Civita symbol, we have the identity:

$$
\epsilon^{ijkl} Z_{\ldots} \equiv \epsilon^{ijkl} Z^i_{\ldots} + \epsilon^{ilmk} Z^j_{\ldots} + \epsilon^{iljk} Z^m_{\ldots} + \epsilon^{ijml} Z^k_{\ldots}.
$$

Applying this for the case when $Z = S_m^l$, we find the identity

$$
\epsilon^{ijkl} S_m^l \equiv \epsilon^{ijkl} S_m^i + \epsilon^{ilmk} S_m^j + \epsilon^{iljk} S_m^m + \epsilon^{ijml} S_m^k.
$$

Suppose that $S_m^l$ is a *traceless* tensor, i.e. $S_m^m = 0$. Then a simple rearrangement of the terms in the above identity yields:

$$
\epsilon^{ijkl} S_m^l - \epsilon^{ijml} S_m^k \equiv -\epsilon^{klmi} S_m^j + \epsilon^{klmj} S_m^i.
$$
Summarizing, we proved the **S-identity**: Every (1,1) tensor $S_{ij}$ which is traceless, $S_k^k = 0$, has the property

$$
\epsilon^{ijm[k} S_{m\,l]} = - \epsilon^{klm[i} S_{m\,j]}.
$$

This identity always holds true in four dimensions, just because of the properties of the Levi-Civita symbol. Eq. (94) can also be found by adding (26) and (27). The S-identity underlies the possibility to express $(2)_\chi$ in terms of the skewon tensor.

Let us now demonstrate another identity which holds for a (2,2) tensor $K_{ijkl}$. Consider the contraction

$$
\epsilon^{ijmn} K_{mn\,[kl]} = 1/2 \left( \epsilon^{ijmn} K_{mn\,kl} - \epsilon^{ijmn} K_{mn\,lk} \right).
$$

(95)

Apply the identity (91) to the indices $[ijmnk]$ in the first term on the r.h.s. and to the indices $[ijmnl]$ in the second term:

$$
\epsilon^{ijmn} K_{mn\,kl} = \epsilon^{kijmn} K_{mn\,il} + \epsilon^{ikjmn} K_{mn\,jl} + \epsilon^{ijkmn} K_{mn\,ml} + \epsilon^{ijmkn} K_{mn\,nl},
$$

(96)

$$
\epsilon^{ijmn} K_{mn\,lk} = \epsilon^{ijmn} K_{mn\,ik} + \epsilon^{ilmn} K_{mn\,jk} + \epsilon^{ijnm} K_{mn\,mk} + \epsilon^{jimn} K_{mn\,nk}.
$$

(97)

Suppose that the tensor $K$ is traceless: $K_m^m k = 0$. Then using (96)-(97) in (95), we find:

$$
\epsilon^{ijmn} K_{mn\,[kl]} = \epsilon^{ijmn[k} K_{mn\,l]} - \epsilon^{imn[k} K_{mn\,l]}.
$$

(98)

Now we once again apply the identity (91):

$$
\epsilon^{jmnk} K_{mn\,li} = \epsilon^{jmn\,l} i + \epsilon^{mn\,k} K_{mn\,li} + \epsilon^{imnk} K_{mn\,mi} + \epsilon^{lmnk} K_{mn\,ji}.
$$

(99)

Taking into account the tracelessness, $K_m^m n = 0$, we can rearrange the terms (move the first term from the r.h.s. to the l.h.s.) and find

$$
\epsilon^{jmn\,[k} K_{mn\,l]} = 1/2 \epsilon^{klmn} K_{mn\,ij}.
$$

(100)

Finally, using (100) in (98), we prove the **K-identity**: Every (2,2) tensor $K_{ij\,kl}$ that is traceless $K_k^i j = 0$ has the property

$$
\epsilon^{ijmn} K_{mn\,[kl]} = \epsilon^{klmn} K_{mn\,[ij]}.
$$

(101)

[Incidentally, this property applies in particular to the Weyl curvature tensor $C_{ij\,kl}$. In this case, we recover from (101) the well known anti-self double-duality of the Weyl tensor: $C_{ij\,kl} = 1/4 \epsilon^{klmn} \hat{\epsilon}_{ijpq} C_{mn\,pq}$.]
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