A class of variable coefficient elliptic equations solved using BEM

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Abstract. In this paper a BEM is used to solve a class of variable coefficient elliptic equations numerically. Some examples are considered to show the convergence, consistency, and accuracy of the numerical solutions.

1. Introduction

BEM solutions to a number of problems for homogeneous media governed by several types of equations have been obtained (see for examples [1, 2, 3, 4]). However, BEM has not been used extensively to solve problems of inhomogeneous media. Progress towards this end has been shown, works in [5, 6, 7, 8, 9, 10, 11, 12] are the examples. Specifically, the governing equation considered by Salam et. al in \cite{11} takes the form

$$\frac{\partial}{\partial x_i} \left[ \kappa_{ij} (x_1, x_2) \frac{\partial \mu (x_1, x_2)}{\partial x_j} \right] = 0 \quad (1)$$

where the coefficients $\kappa_{ij}$ depend on $x_1$ and $x_2$ and the repeated summation convention (summing from 1 to 2) is employed.

This paper is intended to extend the work by Salam et. al \cite{11} for problems with governing equation (1) to for 2D boundary value problems governed by another type of (dimensionless) elliptic equation of the form

$$\frac{\partial}{\partial x_i} \left[ \kappa_{ij} (x_1, x_2) \frac{\partial \mu (x_1, x_2)}{\partial x_j} \right] + \beta (x_1, x_2) \mu (x_1, x_2) = 0 \quad (2)$$

Equation (2) may be used to model a variety of problems such as steady infiltration problems (when $\beta < 0$, see for examples \cite{13, 14}), acoustic problems (when $\beta > 0$, see for examples \cite{15, 16}), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta = 0$).

Equation (2) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. It is necessary to place some constraint on the class of coefficients $\kappa_{ij}$ and $\beta$ for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (2).
2. The boundary value problem
Referred to a Cartesian frame \( O_{x_1,x_2} \) a solution to (2) is sought which is valid in a region \( \Omega \) in \( R^2 \) with boundary \( \partial \Omega \) which consists of a finite number of piecewise smooth closed curves. On \( \partial \Omega_1 \) the dependent variable \( \mu(x) \) \((x = (x_1,x_2))\) is specified and on \( \partial \Omega_2 \)

\[
P(x) = \kappa_{ij} \left( \frac{\partial \mu}{\partial x_j} \right) n_i
\]

is specified where \( \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \) and \( n = (n_1,n_2) \) denotes the outward pointing normal to \( \partial \Omega \).

For all points in \( \Omega \) the matrix of coefficients \( [\kappa_{ij}] \) is a real symmetric positive definite matrix so that throughout \( \Omega \) equation (2) is a second order elliptic partial differential equation. Therefore equation (2) may be written explicitly as

\[
\frac{\partial}{\partial x_1} \left( \kappa_{11} \frac{\partial \mu}{\partial x_1} \right) + 2 \frac{\partial}{\partial x_1} \left( \kappa_{12} \frac{\partial \mu}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \kappa_{22} \frac{\partial \mu}{\partial x_2} \right) + \beta \mu = 0
\]

Further, the coefficients \( \kappa_{ij} \) and \( \beta \) are required to be twice differentiable functions of the two independent variables \( x_1 \) and \( x_2 \).

The method of solution will be to obtain boundary integral equations from which numerical values of the dependent variables \( \mu \) and \( P \) may be obtained for all points in \( \Omega \). The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (2) take the form \( \kappa_{11} = \kappa_{22} \) and \( \kappa_{12} = 0 \) and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3. The boundary integral equation
The boundary integral equation is derived by transforming the variable coefficient equation (2) to a constant coefficient equation. The coefficients \( \kappa_{ij} \) and \( \beta \) are required to take the form

\[
\kappa_{ij}(x) = \pi_{ij} g(x)
\]

\[
\beta(x) = \bar{\beta} g(x)
\]

where the \( \pi_{ij} \) and \( \bar{\beta} \) are constants and \( g \) is a differentiable function of \( x \). Use of (4) and (5) and in (2) yields

\[
\pi_{ij} \frac{\partial}{\partial x_i} \left( g \frac{\partial \mu}{\partial x_j} \right) + \bar{\beta} g \mu = 0
\]

Let

\[
\mu(x) = g^{-1/2}(x) \psi(x)
\]

so that (6) may be written in the form

\[
\pi_{ij} \frac{\partial}{\partial x_i} \left[ g \frac{\partial (g^{-1/2} \psi)}{\partial x_j} \right] + \bar{\beta} g^{1/2} \psi = 0
\]

That is

\[
\pi_{ij} \left[ \left( \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] + \bar{\beta} g^{1/2} \psi = 0
\]

Use of the identity

\[
\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = - \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}
\]
permits (8) to be written in the form
\[ \frac{1}{2} \kappa_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \kappa_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \beta g^{1/2} \psi = 0 \]

It follows that if \( g \) is such that
\[ \kappa_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0 \]
then the transformation (7) carries the variable coefficients equation (6) to the constant coefficients equation
\[ \kappa_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \beta \psi = 0 \]

And if \( g \) is such that
\[ \kappa_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \beta g^{1/2} = 0 \]
then
\[ \kappa_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \]

Also, substitution of (4) and (7) into (3) gives
\[ P = -P g \psi + P_\psi g^{1/2} \]

where
\[ P_g (x) = \kappa_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \quad P_\psi (x) = \kappa_{ij} \frac{\partial \psi}{\partial x_j} n_i \]

A boundary integral equation for the solution of (10) and (12) is given in the form
\[ \eta (x_0) \psi (x_0) = \int_{\partial \Omega} \left[ \Gamma (x, x_0) \psi (x) - \Phi (x, x_0) P_\psi (x) \right] ds (x) \]

where \( x_0 = (a, b) \), \( \eta = 0 \) if \((a, b) \notin \Omega \cup \partial \Omega \), \( \eta = 1 \) if \((a, b) \in \Omega \), \( \eta = \frac{1}{2} \) if \((a, b) \in \partial \Omega \) and \( \partial \Omega \) has a continuously turning tangent at \((a, b)\).

The so called fundamental solution \( \Phi \) in (14) is any solution of the equation
\[ \kappa_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \beta \Phi = \delta (x - x_0) \]
and the \( \Gamma \) is given by
\[ \Gamma (x, x_0) = \kappa_{ij} \frac{\partial \Phi (x, x_0)}{\partial x_j} n_i \]

where \( \delta \) is the Dirac delta function. Following Azis in [17], for two-dimensional problems \( \Phi \) and \( \Gamma \) are given by

\[
\Phi (x, x_0) = \begin{cases} 
\frac{K}{2\pi} \ln R & \text{if } \beta = 0 \\
\frac{iK}{4\pi} H^{(2)}_0 (\omega R) & \text{if } \beta > 0 \\
\frac{K}{2\pi} K_0 (\omega R) & \text{if } \beta < 0 
\end{cases} \\
\Gamma (x, x_0) = \begin{cases} 
\frac{1}{2\pi} \kappa_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \beta = 0 \\
\frac{-iK}{4} H^{(2)}_1 (\omega R) \kappa_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \beta > 0 \\
\frac{K\omega}{2\pi} K_1 (\omega R) \kappa_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \beta < 0 
\end{cases}
\]
where

\[
K = \frac{\bar{\tau}}{\zeta},
\]

\[
\omega = \sqrt{|\beta|/\zeta},
\]

\[
\zeta = \left[\kappa_{11} + 2\kappa_{12}\bar{\tau} + \kappa_{22} (\bar{\tau}^2 + \bar{\tau}^2)\right]/2
\]

\[
R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2}
\]

\[
\dot{x}_1 = x_1 + \bar{\tau}x_2
\]

\[
\dot{a} = a + \bar{\tau}b
\]

\[
\dot{x}_2 = \bar{\tau}x_2
\]

\[
\dot{b} = \bar{\tau}b
\]

where \(\dot{\tau}\) and \(\bar{\tau}\) are respectively the real and the positive imaginary parts of the complex root \(\tau\) of the quadratic

\[
\kappa_{11} + 2\kappa_{12}\tau + \kappa_{22}\tau^2 = 0
\]

and \(H_0^{(2)}, H_1^{(2)}\) denote the Hankel function of second kind and order zero and order one respectively. \(K_0, K_1\) denote the modified Bessel function of order zero and order one respectively, \(i\) represents the square root of minus one. The derivatives \(\partial R/\partial x_j\) needed for the calculation of the \(\Gamma\) in (15) are given by

\[
\frac{\partial R}{\partial x_1} = \frac{1}{R} (\dot{x}_1 - \dot{a})
\]

\[
\frac{\partial R}{\partial x_2} = \bar{\tau} \left[\frac{1}{R} (\dot{x}_1 - \dot{a})\right] + \bar{\tau} \left[\frac{1}{R} (\dot{x}_2 - \dot{b})\right]
\]

Use of (7) and (13) in (14) yields

\[
\eta(x_0) g^{1/2}(x_0) \mu(x_0) = \int_{\partial \Omega} \left\{ \left[ g^{1/2}(x) \Gamma(x, x_0) - P_\eta(x) \Phi(x, x_0) \right] \mu(x) \\
- \left[ g^{-1/2}(x) \Phi(x, x_0) \right] P(x) \right\} ds(x) \quad (16)
\]

This equation provides a boundary integral equation for determining \(\mu\) and \(P\) at all points of \(\Omega\).

4. Numerical examples

Some particular boundary value problems will be solved numerically by employing the integral equation (16). The main aim is to show the validity of the analysis for deriving the boundary integral equation (16) and the appropriateness of the BEM in solving the problems through the derived boundary integral equation (16). Standard boundary element method is employed to obtain numerical results. The integrals in equation (16) are evaluated numerically using the Bode’s quadrature (see Abramowitz and Stegun in [18]).

4.1. Examples with analytical solutions

In order to see the convergence and accuracy of the BEM we will consider some examples of problems with analytical solutions. Three possible multiparameter forms of function \(g(x)\) are quadratical (satisfying (9)), exponential and trigonometrical (satisfying (11) with \(\beta > 0\) and \(\beta < 0\) respectively). For all problems considered in this section we take

\[
g(x) = [A (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]^2 \quad \text{for quadratically graded materials}
\]

\[
g(x) = [A \exp(\alpha_i x_i)]^2 \quad \text{for exponentially graded materials}
\]

\[
g(x) = \{A [\cos(\alpha_i x_i) + \sin(\alpha_i x_i)]\}^2 \quad \text{for trigonometrically graded materials}
\]
Figure 1. A quadratic inhomogeneity function $g(x) = (1 + 0.5x_1 + x_2)^2$

Figure 2. An exponential inhomogeneity function $g(x) = [\exp (0.5x_1 + x_2)]^2$

with $A = 1, \alpha_0 = 1, \alpha_1 = 0.5, \alpha_2 = 1$. Plots of $g(x)$ are shown in Figures 1–3. The geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 4. The values of the constant coefficients $\kappa_{ij}$ for the governing equation (2) are

$$\kappa_{11} = 1, \kappa_{12} = 0.5, \kappa_{22} = 1$$

Also, the parameters for the analytical solutions are taken to be

$$B = 1, \gamma_0 = 1, \gamma_1 = 0.5, \gamma_2 = 0.5$$

4.1.1. Quadratically graded media For quadratically graded media the function $g(x)$ satisfies (9). Therefore equation (10) has to be the corresponding constant variable coefficient equation in which $\beta > 0, \beta < 0$ or $\beta = 0$.

Problem 4.1.1.1: Case $\beta > 0$ in equation (10)

We take analytical solutions

$$\psi (x) = B \left[ \cos (\gamma_i x_i) + \sin (\gamma_i x_i) \right] \text{ thus } \beta = \kappa_{ij} \gamma_i \gamma_j = 0.75$$

$$\mu (x) = B \left[ \cos (\gamma_i x_i) + \sin (\gamma_i x_i) \right] / \left[ A (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) \right]$$
Figure 3. A trigonometrical inhomogeneity function 
\[ g(x) = \left[ \cos(0.5x_1 + x_2) + \sin(0.5x_1 + x_2) \right]^2 \]

Figure 4. The geometry of all problems in Section 4.1

Table 1 shows the results of the analytical and BEM solutions with 80, 160 and 320 segments of equal length. The BEM solution converges to the analytical solution as the number of segments increases.

Problem 4.1.1.2: Case \( \overline{\beta} < 0 \) in equation (10)
Analytical solutions are
\[
\psi(x) = B \exp(\gamma_i x_i) \quad \text{thus} \quad \overline{\beta} = -\kappa_{ij} \gamma_i \gamma_j = -0.75
\]
\[
\mu(x) = B \exp(\gamma_i x_i) / [A (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]
\]
The results are shown in Table 2. Again, the BEM solution converges to the analytical solution as the number of segments increases.

Problem 4.1.1.3: Case \( \overline{\beta} = 0 \) in equation (10)
### Table 1. BEM and analytical solutions for Problem 4.1.1.1

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
(x_1, x_2) & \mu & \frac{\partial \mu}{\partial x_1} & \frac{\partial \mu}{\partial x_2} & \mu & \frac{\partial \mu}{\partial x_1} & \frac{\partial \mu}{\partial x_2} \\
\hline
(1.5, 1) & 0.8070 & -0.0480 & -0.3075 & 0.8070 & -0.0477 & -0.3076 \\
(3, 1) & 0.7942 & -0.0800 & -0.3197 & 0.7942 & -0.0798 & -0.3200 \\
(5, 1) & 0.7753 & -0.1079 & -0.3289 & 0.7754 & -0.1078 & -0.3291 \\
(9, 1) & 0.7226 & -0.1536 & -0.3359 & 0.7226 & -0.1541 & -0.3397 \\
(1, 1) & 0.9261 & -0.0982 & -0.4409 & 0.9263 & -0.0986 & -0.4414 \\
(3, 3) & 0.8452 & -0.1011 & -0.3732 & 0.8453 & -0.1011 & -0.3736 \\
(5, 3) & 0.7128 & -0.1162 & -0.2984 & 0.7128 & -0.1160 & -0.2985 \\
(5, 9) & 0.6554 & -0.1246 & -0.2765 & 0.6554 & -0.1245 & -0.2766 \\
\hline
\end{array}
\]

### Table 2. BEM and analytical solutions for Problem 4.1.1.2

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
(x_1, x_2) & \mu & \frac{\partial \mu}{\partial x_1} & \frac{\partial \mu}{\partial x_2} & \mu & \frac{\partial \mu}{\partial x_1} & \frac{\partial \mu}{\partial x_2} \\
\hline
(1, 1) & 0.8716 & 0.1537 & -0.1252 & 0.8712 & 0.1541 & -0.1258 \\
(3, 1) & 0.9047 & 0.1777 & -0.0945 & 0.9044 & 0.1779 & -0.0952 \\
(5, 1) & 0.9421 & 0.2020 & -0.0659 & 0.9421 & 0.2020 & -0.0666 \\
(7, 1) & 0.9856 & 0.2270 & -0.0388 & 0.9852 & 0.2266 & -0.0394 \\
(9, 1) & 1.0336 & 0.2532 & -0.0127 & 1.0331 & 0.2523 & -0.0130 \\
(1, 1) & 0.9994 & 0.1298 & -0.2364 & 0.9997 & 0.1297 & -0.2386 \\
(3, 3) & 0.9626 & 0.1710 & -0.1373 & 0.9626 & 0.1709 & -0.1385 \\
(5, 3) & 0.9352 & 0.2277 & -0.0112 & 0.9348 & 0.2277 & -0.0116 \\
(5, 9) & 0.9375 & 0.2507 & 0.0331 & 0.9371 & 0.2506 & 0.0329 \\
\hline
\end{array}
\]
Now we choose analytical solutions

\[
\psi(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2) \quad \text{with} \quad \beta = 0
\]

\[
\mu(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2) / [A (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]
\]

The results are shown in Table 3. Once again, the BEM solution converges to the analytical solution as the number of segments increases.

### 4.1.2. Exponentially graded media

#### Problem 4.1.2.1

A BVP of anisotropic FGM will be considered as a test problem. The function of inhomogeneity \( g(x) \) satisfying equation (11) is taken to be (see Figure 2)

\[
g(x) = [A \exp(\alpha_i x_i)]^2
\]

Therefore \( \psi \) must satisfy (12). We choose analytical solution

\[
\mu(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2) / [A \exp(\alpha_i x_i)]
\]

Table 4 shows the results of the analytical and BEM solutions with 80, 160 and 320 segments of equal length. The BEM solution converges to the analytical solution as the number of segments increases.
Table 4. BEM and analytical solutions for Problem 4.1.2.1

| $(x_1, x_2)$ | $\mu$ | $\partial\mu/\partial x_1$ | $\partial\mu/\partial x_2$ | $\mu$ | $\partial\mu/\partial x_1$ | $\partial\mu/\partial x_2$ |
|-------------|-------|-----------------|-----------------|-------|-----------------|-----------------|
| BEM 80 segments | | | | BEM 160 segments | | | |
| (.1,.5) | .7502 | -.0867 | -.4610 | .7501 | -.0866 | -.4613 |
| (.3,.5) | .7310 | -.1046 | -.4691 | .7309 | -.1045 | -.4695 |
| (.5,.5) | .7087 | -.1180 | -.4717 | .7086 | -.1181 | -.4720 |
| (.7,.5) | .6840 | -.1347 | -.4697 | .6840 | -.1280 | -.4700 |
| (.9,.5) | .6578 | -.1279 | -.4697 | .6576 | -.1349 | -.4641 |
| (.5,.1) | .9157 | -.1057 | -.5613 | .9159 | -.1055 | -.5624 |
| (.5,.3) | .8077 | -.1151 | -.5179 | .8077 | -.1153 | -.5186 |
| (.5,.7) | .6190 | -.1161 | -.4252 | .6189 | -.1160 | -.4253 |
| (.5,.9) | .5385 | -.1114 | -.3797 | .5384 | -.1108 | -.3801 |

BEM 320 segments

| $(x_1, x_2)$ | $\mu$ | $\partial\mu/\partial x_1$ | $\partial\mu/\partial x_2$ | $\mu$ | $\partial\mu/\partial x_1$ | $\partial\mu/\partial x_2$ |
|-------------|-------|-----------------|-----------------|-------|-----------------|-----------------|
| (.1,.5) | .7501 | -.0866 | -.4614 | .7500 | -.0865 | -.4616 |
| (.3,.5) | .7309 | -.1044 | -.4697 | .7309 | -.1044 | -.4698 |
| (.5,.5) | .7086 | -.1181 | -.4722 | .7085 | -.1181 | -.4724 |
| (.7,.5) | .6839 | -.1281 | -.4700 | .6839 | -.1282 | -.4702 |
| (.9,.5) | .6575 | -.1352 | -.4641 | .6575 | -.1354 | -.4641 |
| (.5,.1) | .9160 | -.1056 | -.5631 | .9161 | -.1057 | -.5638 |
| (.5,.3) | .8077 | -.1153 | -.5189 | .8077 | -.1154 | -.5193 |
| (.5,.7) | .6188 | -.1160 | -.4253 | .6188 | -.1160 | -.4254 |
| (.5,.9) | .5383 | -.1108 | -.3800 | .5383 | -.1108 | -.3800 |

4.1.3. Trigonometrically graded media

Problem 4.1.3.1

The function of inhomogeneity $g(x)$ satisfying equation (11) is taken to be (see Figure 3)

$$g(x) = \{A [\cos (\alpha_i x_i) + \sin (\alpha_i x_i)]\}^2$$

The analytical solution is

$$\mu(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2) / \{A [\cos (\alpha_i x_i) + \sin (\alpha_i x_i)]\}$$

Table 5 shows the results of the analytical and BEM solutions with 80, 160 and 320 segments of equal length. The BEM solution converges to the analytical solution as the number of segments increases.

4.2. Examples without analytical solutions

In this section we will consider some examples of problems without simple analytical solutions. We setup some problems for a homogeneous isotropic material by taking $g(x) = 1$, $\kappa_{11} = \kappa_{22} = 1$, $\kappa_{12} = 0$ and with symmetrical boundary conditions. This function $g(x)$ satisfies equation (9) thus we will take $\psi(x)$ that satisfies (10) to be put into the integral equation (14). The aim is to see the consistency of the BEM of whether it produces symmetrical solutions.

Problem 4.2.1: Case $\beta > 0$ in equation (10)

For this problem we take $\beta = 1$ and the boundary conditions are as shown in Figure 5. Table 6 shows the results of the BEM solution using 80, 160, 320 and 640 segments of equal length. As
Table 5. BEM and analytical solutions for Problem 4.1.3.1

|      | BEM 80 segments | BEM 160 segments |
|------|----------------|-----------------|
|      | (x_1, x_2)     | µ               | ∂µ/∂x_1 | ∂µ/∂x_2 | µ               | ∂µ/∂x_1 | ∂µ/∂x_2 |
| (.5, .5) | .9457 | .2493 | .1383 | .9455 | .2498 | .1375 |
| (.3, .5) | .9994 | .2883 | .2221 | .9992 | .2885 | .2214 |
| (.5, .5) | 1.0616 | .3351 | .3174 | 1.0614 | .3350 | .3168 |
| (.7, .5) | 1.1341 | .3917 | .4286 | 1.1339 | .3913 | .4281 |
| (.9, .5) | 1.2191 | .4613 | .5614 | 1.2188 | .4604 | .5611 |
| (.5, 1) | 1.0134 | .1545 | .0789 | 1.0136 | .1543 | .0803 |
| (.5, 3) | 1.0180 | .2418 | .1212 | 1.0180 | .2416 | .1203 |
| (.5, 7) | 1.1474 | .4536 | .5500 | 1.1471 | .4536 | .5494 |
| (.5, 9) | 1.2874 | .6240 | .5705 | 1.2870 | .6240 | .5700 |

BEM 320 segments

|      | Analytical |
|------|------------|
| (.5, .5) | .9454 | .2500 | .1372 | .9453 | .2502 | .1369 |
| (.3, .5) | .9992 | .2887 | .2210 | .9991 | .2888 | .2207 |
| (.5, .5) | 1.0614 | .3350 | .3165 | 1.0613 | .3350 | .3162 |
| (.7, .5) | 1.1338 | .3911 | .4279 | 1.1337 | .3910 | .4276 |
| (.9, .5) | 1.2187 | .4600 | .5609 | 1.2186 | .4596 | .5608 |
| (.5, 1) | 1.0137 | .1542 | .0810 | 1.0138 | .1541 | .0817 |
| (.5, 3) | 1.0180 | .2416 | .1198 | 1.0180 | .2415 | .1194 |
| (.5, 7) | 1.1470 | .4536 | .5492 | 1.1469 | .4536 | .5489 |
| (.5, 9) | 1.2868 | .6240 | .8698 | 1.2867 | .6240 | .8695 |

Figure 5. The geometry of Problem 4.2.1 and Problem 4.2.2

expected, the results converge as the number of segments increases and also they are symmetrical about the axes x_2 = 0.5.

Problem 4.2.2: Case β < 0 in equation (10)

We take β = -1 and boundary conditions are as shown in Figure 5. Table 7 shows the results of the BEM solution using 80, 160, 320 and 640 segments of equal length. The results converge
as the number of segments increases and also they are symmetrical about the axes $x_2 = 0.5$. 

### Table 6. BEM solution for Problem 4.2.1

| $(x_1, x_2)$ | $\mu$ | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ | $\mu$ | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ |
|--------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
| $(.1,.5)$    | 1.4447 | -1.1500                       | -.0000                        | 1.4475 | -1.1502                       | -.0000                        |
| $(.3,.5)$    | 1.1875 | -1.4138                       | -.0000                        | 1.1902 | -1.4148                       | -.0000                        |
| $(.5,.5)$    | .8830  | -1.6213                       | -.0000                        | .8854  | -1.6231                       | -.0000                        |
| $(.7,.5)$    | .5433  | -1.7639                       | -.0000                        | .5453  | -1.7664                       | -.0000                        |
| $(.9,.5)$    | .1821  | -1.8358                       | -.0000                        | .1835  | -1.8392                       | -.0000                        |
| $(.5,.1)$    | .8829  | -1.6223                       | .0000                         | .8853  | -1.6235                       | -.0000                        |
| $(.5,.3)$    | .8829  | -1.6216                       | .0002                         | .8854  | -1.6232                       | .0001                         |
| $(.5,.7)$    | .8829  | -1.6216                       | -.0002                        | .8854  | -1.6232                       | -.0001                        |
| $(.5,.9)$    | .8829  | -1.6223                       | -.0000                        | .8853  | -1.6235                       | .0000                         |

### Table 7. BEM solution for Problem 4.2.2

| $(x_1, x_2)$ | $\mu$ | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ | $\mu$ | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ |
|--------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
| $(.1,.5)$    | .6628  | -9.270                        | -.0000                        | .6642  | -9.279                        | -.0000                        |
| $(.3,.5)$    | .4896  | -8.114                        | -.0000                        | .4907  | -8.125                        | -.0000                        |
| $(.5,.5)$    | .3361  | -7.289                        | -.0000                        | .3370  | -7.300                        | -.0000                        |
| $(.7,.5)$    | .1961  | -6.756                        | -.0000                        | .1968  | -6.767                        | -.0000                        |
| $(.9,.5)$    | .0640  | -6.494                        | -.0000                        | .0645  | -6.505                        | -.0000                        |
| $(.5,.1)$    | .3360  | -7.292                        | .0000                         | .3369  | -7.301                        | .0000                         |
| $(.5,.3)$    | .3360  | -7.289                        | .001                          | .3370  | -7.300                        | .0000                         |
| $(.5,.7)$    | .3360  | -7.289                        | -.0001                        | .3370  | -7.300                        | -.0000                        |
| $(.5,.9)$    | .3360  | -7.292                        | -.0000                        | .3369  | -7.301                        | -.0000                        |

for BEM $80$ segments BEM $160$ segments BEM $320$ segments BEM $640$ segments.
Problem 4.2.3: Case $\beta = 0$ in equation (10)

We consider a problem with $\beta = 0$ and the boundary conditions are as shown in Figure 6. Table 8 shows the results of the BEM solution using 80, 160, 320 and 640 segments of equal length. The results converge as the number of segments increases and also they are symmetrical about the axes $x_1 = 0.5$.

Table 8. BEM solution for Problem 4.2.3

| $(x_1,x_2)$     | $\mu$  | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ | $\mu$  | $\partial \mu / \partial x_1$ | $\partial \mu / \partial x_2$ |
|-----------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
|                 | BEM 80 segments |                                |                                | BEM 160 segments |                                |                                |
| (.1,.5)         | .5001  | .0014                         | 1.0001                        | .5001  | .0006                         | 1.0001                        |
| (.3,.5)         | .5004  | .0007                         | 1.0003                        | .5002  | .0004                         | 1.0002                        |
| (.5,.5)         | .5004  | .0000                         | 1.0004                        | .5002  | .0000                         | 1.0002                        |
| (.7,.5)         | .5004  | -.0007                        | 1.0003                        | .5002  | -.0004                        | 1.0002                        |
| (.9,.5)         | .5001  | -.0014                        | 1.0001                        | .5001  | -.0006                        | 1.0001                        |
| (.5,1)          | .1005  | -.0000                        | .9993                         | .1002  | .0000                         | .9996                         |
| (.5,3)          | .3004  | -.0000                        | .9998                         | .3002  | .0000                         | .9999                         |
| (.5,7)          | .7006  | .0000                         | 1.0012                        | .7003  | .0000                         | 1.0006                        |
| (.5,9)          | .9009  | -.0000                        | 1.0019                        | .9005  | -.0000                        | 1.0010                        |
|                 | BEM 320 segments |                                |                                | BEM 640 segments |                                |                                |
| (.1,.5)         | .5000  | .0003                         | 1.0000                        | .5000  | .0002                         | 1.0000                        |
| (.3,.5)         | .5001  | .0002                         | 1.0001                        | .5000  | .0001                         | 1.0001                        |
| (.5,.5)         | .5001  | -.0000                        | 1.0001                        | .5001  | -.0000                        | 1.0001                        |
| (.7,.5)         | .5001  | -.0002                        | 1.0001                        | .5000  | -.0001                        | 1.0001                        |
| (.9,.5)         | .5000  | -.0003                        | 1.0000                        | .5000  | -.0002                        | 1.0000                        |
| (.5,1)          | .1001  | -.0000                        | .9998                         | .1001  | -.0000                        | .9999                         |
| (.5,3)          | .3001  | -.0000                        | .9999                         | .3000  | -.0000                        | 1.0000                        |
| (.5,7)          | .7001  | -.0000                        | 1.0003                        | .7001  | -.0000                        | 1.0002                        |
| (.5,9)          | .9002  | -.0000                        | 1.0005                        | .9001  | -.0000                        | 1.0002                        |
5. Conclusion
The scalar elliptic governing equation (2) is used for modelling physical problems such as steady infiltration problems (when $\beta < 0$), acoustic problems (when $\beta > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta = 0$). The boundary integral equation (16) was derived from this governing equation (2) and straight from (16) a BEM was then constructed for calculation of numerical solutions to the problems for anisotropic functionally graded media specifically including quadratically, exponentially and trigonometrically graded media. The results show the convergence, consistency, and accuracy of the BEM solutions. Together with its ease in implementation, it may be concluded that BEM is a good numerical method for solving such kind of problems.

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