The Fractional-Order Generalization of HP Memristor-Based Chaotic Circuit with Dimensional Consistency

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Abstract: For studying the practical memristor-based chaotic circuit with fractional-order dynamic and asserting the importance of dimensional consistency awareness, the dimensional consistency aware fractional-order generalization of a Hewlett Packard (HP) memristor-based chaotic circuit with the physical meaning of fractional time component assigned has been proposed in this work. The simplest chaotic circuit based on such practical memristor has been chosen as the candidate circuit. A novel window function dedicated to HP memristor with fractional-order dynamic i.e. fractional-order HP memristor has been adopted for modelling the boundary effect. For the dynamical analysis, the revisited version of Jumarie’s modified Riemann–Liouville fractional derivative and nonlinear transformation has been used. The generalized circuit which has been found to be the simplest fractional-order HP memristor-based chaotic circuit displays a chaotic behavior with significant differences from those of its conventional integer-order prototype and its dimensional consistency ignored counterpart; thus, the importance of dimensional consistency awareness is asserted. The realization of the generalized circuit by using the fractional-order elements is indicated. The circuit emulator has also been presented.

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PUBLIC INTEREST STATEMENT

In 2008, the memristor has been physically realized by researchers in Hewlett Packard (HP) labs after being postulated by L. Chua in 1971 and has been utilized in various circuits. According to the rise of fractional-order circuits, an HP memristor-based chaotic circuit with multiple memristors has been generalized to fractional order. However, the dimensional consistency which significantly affects the dynamic of generalized chaotic circuit has been ignored. In addition, there exists a simpler single HP memristor-based chaotic circuit. Therefore, a dimensional consistency aware fractional-order generalization of such simpler circuit has been performed in this work with the physical meaning of fractional time component given. Our generalized circuit is dimensional consistent yet simpler than the previous one. This work is beneficial to the analysis and design of chaotic circuits, fractional-order circuits, HP memristor-based circuits, nonlinear circuits, etc.
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1. Introduction

Long before the invention of a novel electronic device like a wearable robot controller (Chu et al., 2020), the state of the art fourth fundamental electrical circuit element namely memristor has been postulated by Chua (Chua, 1971). For decades later yet before many recent research studies, e.g., (Omidi et al., 2017), (Ahmadi & Azhari, 2018), and (Shotorbani & Saghai, 2018) etc., have been proposed, the memristor has been physically realized by a research group in Hewlett Packard (HP) labs in 2008. The realized memristor which is often referred to as the HP memristor is based on a nanoscale TiO₂ technology (Strukov et al., 2008) and has been utilized in various circuits including the chaotic circuits. For example, the autonomous circuit with two memristors connected in an antiparallel fashioned has been proposed in Buscarino et al., (2012) where a simpler single memristor-based circuit has been later proposed by Guang-Yi et al. (Guang-Yi et al., 2013). A non-autonomous circuit has been proposed in G. Wang et al., (2015) and a circuit which is both autonomous and employs fewer elements than the previous ones including its predecessor (Wang, Sun, Jin, Mo, Song, Dong et al., 2018a, May); thus, it is the simplest HP memristor-based chaotic circuit has been proposed by Wang et al. (Wang, Sun, Jin, Mo, song, Dong et al., 2018). Apart from being the simplest one, the correct assumption on HP memristor has also been assumed in its designing process unlike the previous non-autonomous circuit which it has been incorrectly assumed in (G. Wang et al., 2015) that the HP memristor is locally active despite that this device is actually passive (Strukov et al., 2008) due to the boundary effect.

According to the rise of fractional-order circuits, on the other hand, the fractional-order generalization of chaotic circuits has been performed in many previous works where the simplest Chua’s chaotic circuit which operates in a voltage mode (Muthuswamy & Chua, 2010) has been considered in Cafagna & Grassi, (2012). The antiparallel fashioned connected HP memristor-based chaotic circuit proposed by Buscarino et al. has been generalized by Deng et al. (Deng & Wang, 2013). The simplest Chua’s circuit with fourth degree nonlinearity has been studied by Teng et al. (Teng et al., 2014). The bifurcation analysis and stabilization of fractional-order Chua’s circuit have been, respectively, performed in Abdelouahab & Lozi, (2015) and Yang et al., (2016). A fractional derivative with nonlocal-nonsingular kernel (Atangana & Baleanu, 2016) has been applied by Alkahtani (Alkahtani, 2016). Recently, both simplest Chua’s chaotic circuit and the simplest current mode chaotic circuit proposed by Jin et al. (Jin et al., 2017) have been generalized by Banchuin (Banchuin, 2020). In all of these works but Banchuin, (2020), the dimensional consistency has been ignored. Such dimensional consistency is often cited in many studies on the fractional-order generalization of linear circuit where the classical Caputo’s definition of fractional derivative has been applied in Gómez et al., (2013) and Gómez-Aguilar et al., (2014). A nonsingular kernel fractional derivative (Caputo & Fabrizio, 2015) has been applied in Atangana & Alkahtani, (2015). A nonlocal-nonsingular kernel derivative has been adopted by Gómez-Aguilar et al. (Gómez-Aguilar et al., 2017). On the other hand, the conformable fractional derivative (Khalil et al., 2014) has been applied by Martínez et al. (Martínez et al., 2018). For nonlinear circuit, on the other hand, the dimensional consistency has been mentioned only in Banchuin, (2020) which stated that such consistency significantly affects the dynamics of the circuit generalized chaotic circuit. However, the memristors assumed in this previous work are merely the hypothetical ones unlike the HP memristor which is a physically realized device. In addition, the physical meaning of fractional time component has never been assigned in Banchuin, (2020) despite that such meaning is necessary for obtaining a physically meaningful fractional-order generalization because it can be uniquely assigned depending on the circuit under consideration. As an example, it can be the fraction of time constant for an RC circuit (Gómez-Aguilar et al., 2014), (Martínez et al., 2018). For the LC and RLC circuits, the physical meaning of fractional time component can be related to the reciprocal of oscillating frequency for LC and RLC circuits (Gómez et al., 2013), (Gómez-Aguilar et al., 2014) and so on.
Therefore, a novel fractional-order generalization of a memristor-based chaotic circuit with dimensional consistency awareness has been performed in this work. Unlike (Banchuin, 2020), the HP memristor has been assumed and the physical meaning of fractional time component assigned. The simplest HP memristor-based chaotic circuit with correct memristor assumption has been chosen as our candidate circuit. Unlike Wang, Sun, Jin, Mo, Song, Dong et al. (2018a), a novel window function dedicated to the HP memristor with fractional-order dynamics, thus, can be termed the fractional-order HP memristor as stated above, proposed by Shi et al. (Shi et al., 2018) has been adopted for modelling the boundary effect in this work. For dynamical analysis of the generalized circuit, the revisited version of Jumarie’s modified Riemann–Liouville fractional derivative (Atangana & Sefer, 2013) and nonlinear transformation (Mahmoud et al., 2017) have been adopted. The simulations have been performed via MATHEMATICA. The Lyapunov exponents and dimensions have also been calculated. As a result, we have found that the generalized circuit displays a chaotic behavior with significant differences from those of its integer-order prototype proposed by Wang et al., and its dimensional consistency ignored counterpart; thus, the importance of dimensional consistency awareness has been asserted. The circuit’s eigenvalues have been solved and used for deriving the chaotic criteria. In addition, the bifurcation analysis has been performed. As a result, it has been shown that the circuit ceases to be chaotic if at least one of these criteria is violated and the circuit has undergone through a Hopf bifurcation at its equilibrium. The realization of the generalized circuit by using the fractional-order elements is indicated. Compared with the previous fractional-order circuit proposed by Deng et al., our generalized circuit is simpler apart from being dimensional consistent as merely a single fractional-order HP memristor is required. As a result, it has been found to be the simplest fractional-order HP memristor-based chaotic circuit. In addition, the emulator has also been presented.

In the following section, an overview of HP memristor will be given followed by an introduction of the candidate HP memristor-based chaotic circuit in section III. The proposed fractional-order generalization and dynamical analysis will be presented in section IV where the physical meaning of the fractional time component will be assigned and the importance of dimensional consistency awareness will be asserted. The realization of the generalized circuit and the implementation of its emulator will be outlined in section V. Finally, the conclusion will be drawn in section VI.

2. An Overview of HP Memristor
The HP memristor is a nanoscale TiO₂ based device. Its memristance \( M(t) \) can be given in terms of the minimum and maximum values of \( M(t) \) denoted by \( M_{on} \) and \( M_{off} \) and the state variable \( x(t) \) as (Strukov et al., 2008)

\[
M(t) = M_{on}x(t) + (1 - x(t))M_{off}
\]  

(1)

where \( x(t) \) which is dimensionless can be given in terms of the memristor’s current \( i(t) \) and window function \( f(x(t), i(t)) \) as follows:

\[
\frac{dx(t)}{dt} = k_i(t)f(x(t), i(t))
\]

(2)

Noted that \( k = \mu M_{on}/D^2 \) where \( \mu \) and \( D \), respectively, stand for the ion mobility and semiconductor film of thickness. Moreover, \( M_{on} \leq M(t) \leq M_{off} \) as \( 0 \leq x(t) \leq 1 \) due to the boundary effect of the device which can be modelled by \( f(x(t), i(t)) \). Since \( M_{on} > 0 \) (Strukov et al., 2008), \( M(t) > 0 \), thus, the HP memristor is passive.

3. The Candidate HP Memristor-Based Chaotic Circuit
Recently, a novel HP memristor-based chaotic circuit has been proposed by Wang et al. (Wang, Sun, et al., 2018a). Compared with the previous circuits mentioned above, this circuit employs minimum number of elements yet autonomous and assumes correct memristor assumption. Thus,
it has been chosen. The schematic diagram of this candidate circuit can be depicted in Figure 1. Instead of any $f(x(t), i(t))$, the truncate function of state variable ($\text{trunc}(x(t), i(t))$) which is given here by (3), has been used for modelling the above-mentioned boundary effects for this circuit. Unfortunately, it can be seen that $\text{trunc}(x(t), i(t))$ lacks both continuity and nonlinearity near $x(t) = 0$ and $x(t) = 1$ unlike the formal $f(x(t), i(t))$.

\[
\text{trunc}(x(t), i(t)) = \begin{cases} 
1: & 0 < x(t) < 1 \\
0: & x(t) \geq 1 \land i(t) > 0 \lor (x(t) \leq 0 \land i(t) < 0)
\end{cases}
\] (3)

4. The Fractional-Order Generalization and Dynamical Analysis

As the first step, the dynamical equations of the candidate HP memristor-based circuit must be reformulated by using the mentioned novel $f(x(t), i(t))$ instead of $\text{trunc}(x(t), i(t))$. Unlike the previous ones (Joglekar & Wolf, 2009), (Biolek et al., 2009), (Prodromakis et al., 2011), such novel $f(x(t), i(t))$ is dedicated to the fractional HP memristor as stated above beside being fully scalable with boundary lock issue resolved. According to Shi et al., (2018), this $f(x(t), i(t))$ and can be given by

\[
f(x(t), i(t)) = 1 - [a^2(x(t) - u(-i(t)))^2 + (1 - a^2)]^p
\] (4)

where $0 < a \leq 1$ and $p > 0$. Noted also that $u()$ stands for the unit step function.

As a result, the reformulated dynamical equations of the candidate HP memristor-based chaotic circuit can be obtained as given by (5) which is potentially solvable through a lie algebraic approach (Shang, 2012):

\[
\frac{d}{dt}i_1(t) = \frac{1}{L_1} [v_1(t) - (M_{on}x(t) - M_{off}(1 - x(t)))i_1(t)]
\]

\[
\frac{d}{dt}i_2(t) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t)
\] (5)

\[
\frac{d}{dt}x(t) = k(1 - [a^2(x(t) - u(-i_1(t)))^2 + (1 - a^2)]^p)i_1(t)
\]

\[
\frac{d}{dt}v_1(t) = \frac{1}{C_1} (i_2(t) - i_1(t))
\]

Secondly, we replaces all conventional derivative terms by fractional derivative ones. In order to achieve the dimensional consistency, the dimensions of conventional and fractional derivative terms must be consistent i.e. both must be given by sec$^{-1}$. Therefore, the following operation must be used

\[
\frac{d}{dt} \rightarrow \frac{1}{\sigma^\alpha} D_t^\alpha
\] (6)

where $\sigma$ denotes the fractional time component and $0 < \alpha \leq 1$. If we let $[T]$ denote the dimensional formula of time, we have $[\alpha] = [T]$ and $[D_t^\alpha] = [T^{-\alpha}]$. Thus, $[\frac{1}{\alpha} \frac{d}{dt} D_t^\alpha] = [T^{-\alpha}] = [T^{-\alpha}]$ and the dimensional consistency is now achieved. Noted also that the unit of $\frac{1}{\alpha} \frac{d}{dt} D_t^\alpha$ which is similar to that of $\frac{d}{dt}$ due to the achieved dimensional consistency and is given by sec$^{-1}$ that is physically measurable always.
On the other hand, the unit of \( D^\alpha \) which in turn is given by \( \sec^{-\alpha} \) and is physically measurable if and only if \( \alpha = 1 \).

After applying (6) to (5), we have

\[
\frac{1}{\sigma^{1-\alpha}} D^{\alpha}_t i_1(t) = \frac{1}{L_1} [v_1(t) - (M_{on} x(t) - M_{off} (1 - x(t))) i_1(t)]
\]

\[
\frac{1}{\sigma^{1-\alpha}} D^{\alpha}_t i_2(t) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t)
\]

\[
\frac{1}{\sigma^{1-\alpha}} D^{\alpha}_t x(t) = k \left(1 - |\sigma^2 x(t) - u(-i_1(t))|^2 + (1 - \sigma^2)^2\right) i_1(t)
\]

\[
\frac{1}{\sigma^{1-\alpha}} D^{\alpha}_t v_1(t) = \frac{1}{C_1} (i_2(t) - i_1(t))
\]

After some rearrangement, the following fractional-order dynamical equations can be obtained:

\[
D^{\alpha}_t i_1(t) = \frac{1}{L_1} [v_1(t) - (M_{on} x(t) - M_{off} (1 - x(t))) i_1(t)]
\]

\[
D^{\alpha}_t i_2(t) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t)
\]

\[
D^{\alpha}_t x(t) = k_u \left(1 - |\sigma^2 x(t) - u(-i_1(t))|^2 + (1 - \sigma^2)^2\right) i_1(t)
\]

\[
D^{\alpha}_t v_1(t) = \frac{1}{C_1} (i_2(t) - i_1(t))
\]

where \( k_u = \frac{1}{\pi^{\alpha-1}} C_{1u} = C_1 \sigma^{\alpha-1}, L_{1u} = L_1 \sigma^{\alpha-1}, \) and \( L_{2u} = L_2 \sigma^{\alpha-1}. \)

Since \( \sigma \) can be assigned with a unique physical meaning depending on the circuit under consideration as stated above, the next step is the assignment of such meaning. Therefore, we let our \( \sigma \) be physically the fraction of reciprocal of oscillating frequency (\( \omega \)) i.e. \( \sigma = \alpha/\omega \), where \( \omega \) stands for such frequency. This is because \( [\omega] = [T^{-1}] \). Before we proceed further, it should be mentioned here that there also exists the fractional space component whose physical meaning have been given by Shang (Shang, 2014).

After the physical meaning assignment of \( \sigma \), the next step is the derivation of its closed-form expression. Thus, the expression of \( \omega \) is necessary due to the assigned meaning of \( \sigma \). For deriving the expression of \( \omega \) and thus that of \( \sigma \), we follow the above-mentioned previous works on linear circuits by considering the dynamical equations of the candidate integer-order circuit i.e. (5). Since the circuit is nonlinear, it is convenient to consider the linear equivalence of (5). As a result, we first introduce the following theorem (Cafagna & Grassi, 2012):
Theorem 1: For an arbitrary dynamical system given by

\[ D_t^x(t) = g(x(t)) \]
where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) and \( g(x(t)) = \begin{bmatrix} g_1(x_1(t), x_2(t), \ldots, x_n(t)) \\ g_2(x_1(t), x_2(t), \ldots, x_n(t)) \\ \vdots \\ g_N(x_1(t), x_2(t), \ldots, x_n(t)) \end{bmatrix} \]

its linear equivalence can be given as follows:

\[ D_t^x(t) = J_E x(t) \]  \hspace{1cm} (10)

where \( J_E \) denotes the Jacobian matrix at \( E \) and

\[ J_E = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_N} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_N}{\partial x_1} & \frac{\partial g_N}{\partial x_2} & \cdots & \frac{\partial g_N}{\partial x_N} \end{bmatrix} |_E \]

Noted also that \( E \) stands for arbitrary equilibrium point of (9) given by \((x_{1E}, x_{2E}, \ldots, x_{NE})\). It should be mentioned here that \( J_E \) which has been widely applied in the analysis of nonlinear dynamical system (Cafagna & Grassi, 2012), (Abdelouahab et al., 2012), (Abdelouahab & Lozi, 2015) represents all slopes or differentials of \( g(x(t)) \) at \( E \). After determining the linear equivalence, the characteristic equation can be obtained via the following corollary (Cafagna & Grassi, 2012):

Corollary 1: For arbitrary linear dynamical system defined by (10), its characteristic equation can be given by

\[ \det[\lambda I - J_E] = 0 \]  \hspace{1cm} (11)

where \( \lambda \) and \( I \) denote the eigenvalue symbol and the identity matrix.

Since \( E \) of (5) can be obtained by solving

\[ g_1(i_1(t), i_2(t), x(t), v_1(t)) = \frac{1}{L_1} [v_1(t) - (M_{an} x(t) - M_{off}(1 - x(t))) i_1(t)] = 0 \]

\[ g_2(i_1(t), i_2(t), x(t), v_1(t)) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t) = 0 \]  \hspace{1cm} (12)

\[ g_3(i_1(t), i_2(t), x(t), v_1(t)) = k(1 - |a|^2(x(t) - u(-i_1(t)))^2 + (1 - a^2)) i_3(t) = 0 \]

\[ g_4(i_1(t), i_2(t), x(t), v_1(t)) = \frac{1}{C_1} (i_2(t) - i_1(t)) = 0 \]

we have \( E = (0, 0, 0, \chi) \) and where \( 0 \leq \chi \leq 1 \) and thus the linear equivalence of (5) can be obtained as

\[ \frac{d}{dt} i_1(t) = \frac{M_{off}(1 - \chi) - M_{an} \chi}{L_1} i_1(t) + \frac{1}{L_1} v_1(t) \]
\[ \frac{d}{dt} i_1(t) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t) \] (13)

\[ \frac{d}{dt} v_1(t) = \frac{1}{C_1} (i_2(t) - i_1(t)) \]

which can be recast to (10) with \( N = 4, \alpha = 1, \) and

\[
J_k = \begin{bmatrix}
- (M_{\text{on}} - M_{\text{off}} (1 - \chi))/L_1 & 0 & 0 & 1/L_1 \\
0 & R/L_2 & 0 & -1/L_2 \\
k(1 - |\alpha^2 (\chi - 1)|^2 + (1 - \alpha^2)^2) & 0 & 0 & 0 \\
-1/C_1 & 1/C_1 & 0 & 0
\end{bmatrix}.
\]

As a result, the characteristic equation of (5) can be obtained as follows:

\[ \lambda^4 + A \lambda^3 + B \lambda^2 + C \lambda = 0 \] (14)

where \( A = (L_2 M_{\text{off}} - L_1 R - L_2 M_{\text{off}} \chi + L_2 M_{\text{on}} \chi)/L_1 L_2, \) \( B = (L_1 - L_2 + C_1 M_{\text{off}} R - C_1 M_{\text{off}} \chi + C_1 M_{\text{on}} R \chi)/C_1 L_1 L_2 \) and \( C = (M_{\text{off}} - R - M_{\text{off}} \chi + M_{\text{on}} \chi)/C_1 L_1 L_2. \)

By the factorization of its LHS, (14) become

\[ \lambda (\lambda + c)(\lambda + a + \sqrt{b - \frac{\alpha^2}{4}})(\lambda + a + j \sqrt{b - \frac{\alpha^2}{4}}) = 0 \] (15)

where \( c + a = A, \) \( ac + b = B \) and \( bc = C \) i.e., \( c + a = (L_2 M_{\text{off}} - L_1 R - L_2 M_{\text{off}} \chi + L_2 M_{\text{on}} \chi)/L_1 L_2, \) \( ac + b = (L_1 - L_2 + C_1 M_{\text{off}} R - C_1 M_{\text{off}} \chi + C_1 M_{\text{on}} R \chi)/C_1 L_1 L_2 \) and \( bc = (M_{\text{off}} - R - M_{\text{off}} \chi + M_{\text{on}} \chi)/C_1 L_1 L_2. \)

Therefore, we have found that the eigenvalues are \( \lambda_1 = 0, \lambda_2 = -c \) and \( \lambda_3, \lambda_4 = -\frac{a}{2} \pm j \sqrt{b - \frac{\alpha^2}{4}}. \) From \( \lambda_3, \lambda_4, \) which refers to either \( \lambda_3 \) or \( \lambda_4, \) when either + or - sign has been, respectively, selected, it can be seen that

\[ \omega = \sqrt{b - \frac{\alpha^2}{4}} \] (16)

Thus, we have

\[ \sigma = \alpha \left( \sqrt{b - \frac{\alpha^2}{4}} \right)^{-1} \] (17)

At this point, we will analyze the dynamical behavior of the generalized circuit. For doing so, the revisited Jumarie’s modified Riemann–Liouville fractional derivative has been adopted as the basis. Such fractional derivative is the revisited version of the original Jumarie’s modified Riemann–Liouville fractional derivative which suffers the inexistence at the origin and can be defined by the following definition.
Definition 1: Let \( f(t) \) be depended on \( t \) such that \( t \in \mathbb{R} \) and \( \alpha \in \mathbb{R} \) where \( \mathbb{R} \) denotes the set of real numbers, and \( D^\alpha f(t) \) can be given by using Jumarie’s modified Riemann–Liouville definition of fractional derivative (Jumarie, 2006) as

\[
D^\alpha_t f(t) \triangleq \left\{ \begin{array}{l}
\left[ \frac{1}{\Gamma(1-\alpha)} \right]^\frac{t}{\alpha} (t-\eta)^{1-\alpha-1} (f(\eta) - f(0)) d\eta, \alpha < 0;
\\
\frac{d}{dt} \left[ (t-\eta)^{1-\alpha} (f(\eta) - f(0)) \right] d\eta, 0 < \alpha < 1;
\\
(f^n(t))^{1-\alpha}, n \leq \alpha \leq n + 1, \ n \geq 1
\end{array} \right.
\]  

(18)

In order to circumvent such inexistence at the origin, the revisited Jumarie’s modified Riemann–Liouville fractional derivative has been introduced as follows:

Definition 2: The revisited Jumarie’s modified Riemann–Liouville fractional derivative can be defined as (Atangana & Secer, 2013)

\[
D^\alpha_t f(t) \triangleq \text{FP} \left( \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\eta} \left[ (t-\eta)^{1-\alpha} (f(\eta) - f(0)) \right] d\eta \right)
\]  

(19)

where \( \text{FP} \) stands for the finite part of the fractional derivative operator (Atangana & Secer, 2013), for \( 0 < \alpha \leq 1 \).

Since the nonlinear transformation-based methodology (Mahmoud et al., 2017) has been applied, we substitute \( I_1(t) = I_1(\xi), I_2(t) = I_2(\xi), x(t) = x(\xi) \) and \( v_1(t) = v_1(\xi) \) where \( \xi = \frac{t}{t_0} \) into (8). As a result, such equation become

\[
\frac{d}{d\xi} I_1(\xi) D^\alpha_t \xi = \frac{1}{L_{1a}} \left[ V_1(\xi) - (M_{an}X(\xi) - M_{off}(1 - X(\xi)))I_1(\xi) \right]
\]

(20)

\[
\frac{d}{d\xi} I_2(\xi) D^\alpha_t \xi = \frac{R}{L_{2a}} I_2(\xi) - \frac{1}{L_{2a}} V_1(\xi)
\]

\[
\frac{d}{d\xi} X(\xi) D^\alpha_t \xi = \nu_0 (1 - \nu_0 (X(\xi) - u(-I_1(\xi))^2 + (1 - \nu_0)^2)I_1(\xi)
\]

Table 1. \( LE_1, LE_2, LE_3, LE_4 \) and \( D_L \) of the generalized circuit with dimensional consistency

| \( LE_1 \) | 0.070961 |
| \( LE_2 \) | -0.057483 |
| \( LE_3 \) | -2.492179 |
| \( LE_4 \) | -16.982975 |
| \( LE_1 + LE_2 + LE_3 + LE_4 \) | -19.461676 |
| \( D_L \) | 2.005408 |
\[ \frac{d}{d\xi} V_1(\xi) D\xi = \frac{1}{C_{1w}} (I_2(\xi) - I_1(\xi)) \]

As \( D\xi = 1 \), we have

\[ \frac{d}{d\xi} I_1(\xi) = \frac{1}{L_{1w}} [V_1(\xi) - (M_{on}X(\xi) - M_{off}(1 - X(\xi)))I_1(\xi)] \]
\[ \frac{d}{d\xi} I_2(\xi) = \frac{R}{L_{2\text{on}}} I_2(\xi) - \frac{1}{L_{2\text{on}}} V_1(\xi) \]  
(21)

\[ \frac{d}{d\xi} X(\xi) = k_u (1 - |\alpha^2 X(\xi) - u(-I_2(\xi)))^2 + (1 - \alpha^2)^p I_1(\xi) \]

\[ \frac{d}{d\xi} V_1(\xi) = \frac{1}{C_{\text{on}}} (I_2(\xi) - I_1(\xi)) \]

By using (21) and numerical simulations, the dynamical behavior of the generalized circuit can be analyzed as follows.

Example 1: we let \( C_1 = 0.4 \) F, \( L_1 = 0.25 \) H, \( L_2 = 1.333 \) H, \( R = 1.7333 \) \( \Omega \), \( M_{\text{on}} = 1.8 \) \( \Omega \) and \( M_{\text{off}} = 10 \) \( \Omega \) similarly to Wang et. al. (Wang, Sun, et al., 2018d) for ceteris paribus. We also assume that \( \alpha = 0.5 \),...
$p = 2$, $\alpha = 0.9$, $I_1(0) = -0.1$ mA, $I_2(0) = 0$ A, $X(0) = 0.1$, and $V_1(0) = 0$ V. As a result, we have $\sigma = 0.7275$ and thus $C_{1B} = 0.40665 \text{Fsec}^{-1}$, $L_{1B} = 0.25416 \text{Hsec}^{-1}$, $I_{2x} = 1.3552 \text{Hsec}^{-1}$, and $k_1 = 1.9673 \times 10^7 \text{A}^{-1} \text{sec}$. Here, we numerically solving (21) which is a nonlinear transformed version of (8), with MATHEMATICA. Based on the obtained solutions, the dynamics and phase portraits of $i_1(t)$, $i_2(t)$, $x(t)$, and $v_1(t)$ can be simulated by keeping the above relationship between $t$ and $\xi$ along with those between the normal and nonlinear transformed electrical quantities in mind as depicted in Fig. 2-11 where the axis scaling has been applied to Figs. 8, 9, and 11 for visibility. From Figure 4, the boundary effect of the memristive device on $x(t)$ can be seen. Moreover, the strange attractors which indicate the chaotic behavior can be observed from Fig. 5-11. Noted that a pinched hysteresis behavior can be from the strange attractor depicted Figure 6 where $i_1(t)$ is the current flowing through the memristive device which is passive as the attractor is located in quadrants 1 and 3, and $v_1(t)$ is partly composed of the voltage across such device. It can also be seen that those $x(t)$ related attractors depicted in Figs. 8, 9 and 11 are restricted due to the boundary effect. Unlike those of the integer circuit, these restrictions occur at $x(t) = 1$ only. This is because the minimum value of $x(t)$ is greater than 0 as can be seen from Figure 3.
Figure 7. $I_2(t)$ v.s. $v_1(t)$.

Figure 8. $I_1(t)$ v.s. $x(t)$.

Figure 9. $I_2(t)$ v.s. $x(t)$. 
Figure 10. $I_2(t)$ v.s. $i_1(t)$.

Figure 11. $V_1(t)$ v.s. $x(t)$.
For the quantitative analysis, the Lyapunov exponents and dimension must be evaluated. First, the system’s variational equation must be formulated by employing the following theorem (Hartl, 2003).

Theorem 2: For arbitrary dynamical system given by

\[
\frac{d}{d\xi}X(\xi) = F(X(\xi)) \tag{22}
\]

where \(X(\xi) = [X_1(\xi) \ X_2(\xi) \ \ldots \ X_n(\xi)]^T\) and \(F(X(\xi)) = \begin{bmatrix} F_1(X_1(\xi), X_2(\xi), \ldots, X_n(\xi)) \\ F_2(X_1(\xi), X_2(\xi), \ldots, X_n(\xi)) \\ \vdots \\ F_n(X_1(\xi), X_2(\xi), \ldots, X_n(\xi)) \end{bmatrix}\), its variational equation can be given by (23) where \(D(\xi) = [d_1(\xi) \ d_2(\xi) \ \ldots \ d_n(\xi)]^T\), \(U(\xi) = [u_1(\xi) \ u_2(\xi) \ \ldots \ u_n(\xi)]\) and \(d_j(\xi) = \left[ \frac{\partial}{\partial X_j(\xi)} F_j(X_1(\xi), X_2(\xi), \ldots, X_n(\xi)) \right. \left. \frac{\partial}{\partial X_j(\xi)} F_j(X_1(\xi), X_2(\xi), \ldots, X_n(\xi)) \right] \). Noted that \(u_j(\xi)\) where \(j = 1, 2, \ldots, n\) stands for the \(j\)th tangent vector of the system’s trajectory in an \(n\) dimensional space defined by \((X_1(\xi), X_2(\xi), \ldots, X_n(\xi))\). Moreover, \(u_j(\xi)\)'s are both linearly independent and orthonormal. \(\frac{d}{d\xi} U(\xi) = D(\xi) U(\xi) \tag{23}\)

After simultaneously solving (22) and (23), the Lyapunov exponents and dimensions can be determined from the following definitions.

Definitions 3: For arbitrary dynamical system, its \(j\)th Lyapunov exponent (LE) can be given by (Wolf et al., 1985)

\[
LE_j = \lim_{\xi \to \infty} \frac{1}{\xi} \ln \frac{\|U_j(\xi)\|}{\|U_j(0)\|} \tag{24}
\]

where \(\|\cdot\|\) denotes the Euclidian norm operator.

Definitions 4: Given the Lyapunov exponents, the Lyapunov dimension (DL) can be found as (Mahmoud et al., 2009)

\[
DL = K - \frac{\sum_{j=1}^{K} |LE_j|}{|LE_{j+1}|} \tag{25}
\]

where \(K\) is the largest integer such that \(\sum_{j=1}^{K} |LE_j| > 0\).

At this point, the quantitative analysis of generalized circuit can be performed as follows.
Example 2: Let all parameters and initial values be similar to those assumed in example 1. For the generalized circuit, there exists four Lyapunov exponents i.e. LE₁, LE₂, LE₃, and LE₄. After applying theorem 2 and corollary 2.1 to (21), LE₁, LE₂, LE₃, and LE₄ can be obtained via numerical calculation with MATHEMATICA as given in Table 1. Since LE₁ > 0 and LE₁ + LE₂ + LE₃ + LE₄ < 0, both expansion in one direction and contracting volumes in the phase space of the attractor which indicates the chaotic behavior can be observed. We have also found that the contraction outweighs the expansion because LE₁ + LE₂ + LE₃ + LE₄ < 0; therefore, the generalized circuit is dissipative. Moreover, DL can be obtained after applying corollary 2.2 to LE₁, LE₂, LE₃, and LE₄ as fractional number which indicates that manifold in the phase space is a strange attractor, as can be seen from Table 1.

For obtaining the necessary criteria for the generalized circuit to remain chaotic, the linear equivalence of (8) must be determined. Since it can be seen from (8) that

\[
g_1(i_1(t), i_2(t), x(t), v_1(t)) = \frac{1}{L_{2a}} |v_1(t) - (M_{ao}x(t) - M_{off}(1 - x(t)))i_1(t)|
\]

\[
g_2(i_1(t), i_2(t), x(t), v_1(t)) = \frac{R}{L_{2a}} i_2(t) - \frac{1}{L_{2a}} v_1(t)
\]

\[
g_3(i_1(t), i_2(t), x(t), v_1(t)) = k_v(1 - |\alpha|^2(\chi - 1)^2 + (1 - \alpha^2)^2)\bar{i}_1(t)
\]

\[
g_4(i_1(t), i_2(t), x(t), v_1(t)) = \frac{1}{C_{1a}} (i_2(t) - i_1(t))
\]

we have \(E = (0, 0, 0, \chi)\) similarly to the integer circuit; thus, the linear equivalence of (8) can be given by (10) with \(N = 4\) and \(dx(t) = (i_1(t), i_2(t), x(t), v_1(t))\). However, \(J_E = \begin{bmatrix} - \left(M_{ao}\chi - M_{off}(1 - \chi)\right)/L_{1a} & 0 & 0 & 1/L_{1a} \\ 0 & R/L_{2a} & 0 & -1/L_{2a} \\ k_v(1 - |\alpha|^2(\chi - 1)^2 + (1 - \alpha^2)^2) & 0 & 0 & 0 \\ -1/C_{1a} & 1/C_{1a} & 0 & 0 \end{bmatrix}\) must be adopted. As a result, the characteristic equation of (8) can also be given by (14) but with \(A = (L_{2a}M_{off} - L_{1a}R - L_{2a}M_{off}\chi + L_{2a} M_{an})\chi/L_{1a}L_{2a} - B = (L_{1a} - L_{2a} + C_{1a}M_{off} - C_{1a}M_{off}\chi + C_{1a}M_{an}\chi)/C_{1a}L_{1a}L_{2a} - C = (M_{off} - R - M_{off}\chi + M_{an})\chi/C_{1a}L_{1a}L_{2a} + c = (L_{2a}M_{off} - L_{1a}R - L_{2a}M_{off}\chi + L_{2a} M_{an})\chi/L_{1a}L_{2a} \) and \(bc = (L_{1a} - L_{2a} + C_{1a}M_{off}(\chi - C_{1a}M_{an}\chi)/C_{1a}L_{1a}L_{2a} + c\) and \(bc = (L_{1a} - L_{2a} + C_{1a}M_{off}(\chi - C_{1a}M_{an}\chi)/C_{1a}L_{1a}L_{2a} + c\). For the circuit to be chaotic, \(E\) must be an unstable saddle focus thus

\[
c > 0
\]

\[
a < 0
\]

\[
\alpha \geq \frac{2}{\pi} \tan^{-1} \frac{\sqrt{b - \alpha^2}}{\alpha}
\]
must be simultaneously satisfied; otherwise, it will ceases to be chaotic as illustrated in the following example.

Example 3: In this example, we will show that the generalized circuit ceases to be chaotic if the criterion on \( \alpha \) as given by (27) is violated. Since we adopt all parameters assumed in Example 1, it can be seen from (27) that \( \alpha \geq 0.728 \) must be satisfied for satisfying the criterion on \( \alpha \); thus, the circuit remains chaotic. As a result, we will demonstrate that \( E \) ceases to be an unstable saddle focus because the stable region has been enlarged so that it covers all eigenvalues; thus, the circuit ceases to be chaotic if \( \alpha < 0.728 \). In order to do so, we formulate

\[
m_1 = \frac{\alpha \pi}{2} - \|\arg(\lambda_i)\| \tag{28}
\]

where \( i = (2, 3, 4) \) as the representative of \( \lambda_2, \lambda_3 \) and \( \lambda_4 \), because the effect of \( m_1 \) to the location of \( \lambda_i \) on the complex plane is equivalent to that of \( \text{Re}(\lambda_i) \) if \( \lambda_i \) is an eigenvalue for the integer-order circuit (Banchuin, 2020), (Abdelouahab et al., 2012). Noted also that we do not need to formulate \( m_2 \) as \( \lambda_4 = 0 \); thus, its location cannot be altered by any parameter changing. By applying \( \lambda_2, \lambda_3 \) and \( \lambda_4 \) to (28), we have

\[
m_2 = \left( \frac{\alpha}{2} - 1 \right) x \tag{29}
\]

\[
m_{3,4} = \frac{\alpha \pi}{2} - \tan^{-1} \left( \frac{2}{b - \frac{a^2}{4}} \right) \tag{30}
\]

Noted that \( m_1 = m_2 \); thus they are commonly referred to as \( m_{3,4} \) for simplicity. Since the plots of \( m_i \)'s against bifurcation parameters have been used for studying the dynamic of fractional-order circuit (Banchuin, 2020) due to the aforesaid equivalence of \( m_i \)'s and \( \text{Re}(\lambda_i) \)'s (if \( \lambda_i \)'s are eigenvalues of the integer-order system) and the previous usages of the similar plots of \( \text{Re}(\lambda_i) \)'s for studying the integer-order systems (De Nino & Fanelli, 2004), (Yuan & Wang, 2016), we adopt the mentioned fractional-order circuit dedicated approach in this work. As the criterion on \( \alpha \) has been considered, we choose \( \alpha \) as our bifurcation parameters and thus the dynamics of \( m_2, m_3 \) and \( m_4 \) can be simulated as depicted in Figs. 12 and 13 which shows that \( m_3 < 0 \) always unlike \( m_2 \) and \( m_4 \). Therefore, the circuit’s stability is solely governed by \( \lambda_3 \) and \( \lambda_4 \) as \( \lambda_1 \) always be at the origin at the origin and \( \lambda_2 \) always be in the stable region of the complex plane.

From Figure 13, it can be seen that \( E \) ceases to be an unstable saddle focus; thus, the generalized circuit ceases to be chaotic when \( \alpha < 0.728 \) and vice versa as \( m_{3,4} < 0 \) and \( m_{3,4} > 0 \) which, respectively, imply that both \( \lambda_3 \) and \( \lambda_4 \) are in stable and unstable region and can be observed. In addition, we have also found that the transversality criterion is established at 0.728 because \( m_{3,4} = 0 \) where \( \frac{\partial}{\partial \alpha} m_{3,4} \big|_{\alpha=0.728} = \frac{\pi}{2} \neq 0 \). As a result, it can be stated based on the fractional-order Hopf bifurcation criteria (Abdelouahab et al., 2012) that both \( \lambda_3 \) and \( \lambda_4 \) cross the boundary between stable and unstable region with non-zero speed; thus, the circuit’s stability switches, its dynamical behavior changes and the circuit undergoes a Hopf bifurcation through \( E \) at \( \alpha = 0.728 \). If we let \( \alpha = 0.5 \), the phase portraits of \( y_2(t) \) and \( x(t) \) can be simulated as depicted in Figure 14 where an axis scaling has been applied again for visibility and a limit cycle can be observed instead of a chaotic attractor. Before we proceed further, it should be mentioned here that the generalized circuit can never be locally asymptotically stable but merely marginally stable. This is because \( \lambda_1 \) always be at the origin of the complex plane which locates on the stability boundary.

Finally, we will assert the importance of dimensional consistency awareness. In order to do so, we derive the dynamical equation of the generalized circuit without such awareness by applying
(31) to (5). As a result, (32) has been obtained then we numerically determine the associated $LE_1$, $LE_2$, $LE_3$, $LE_4$ and $D_l$ by applying the nonlinear transformation, theorem 2 and corollaries 2.1 and 2.2 to (32). The resulting $LE_1$, $LE_2$, $LE_3$, $LE_4$ and $D_l$ are different from those of the circuit with dimensional consistency awareness as can be seen from Table 2. These differences imply the different amounts of expansion and contraction along with the fractal dimension-phase space manifolds thus assert the importance of dimensional consistency awareness to the dynamical behavior of the obtained fractional-order circuit previously proposed (Banchuin, 2020).

$$\frac{d}{dt} \rightarrow D_t^\alpha$$  

(31)
Figure 14. $I_2(t)$ v.s. $x(t)$: $\alpha = 0.5$.

\[ D_0^\alpha i_1(t) = \frac{1}{L_1} \left[ v_1(t) - (M_{on}x(t) - M_{off}(1 - x(t)))i_1(t) \right] \]

\[ D_0^\alpha i_2(t) = \frac{R}{L_2} i_2(t) - \frac{1}{L_2} v_1(t) \]  

\[ D_0^\alpha x(t) = k(1 - |a^2(x(t) - u(-i_1(t)))|^2 + (1 - a^2)^3)i_1(t) \]

5. The circuit realization and emulator implementation
By inspecting (8), it has been found that the generalized circuit can be realized as a circuit with a similar structure to the integer-order circuit but with all circuit elements except the negative

| $LE_1$          | 0.027962 |
|-----------------|----------|
| $LE_2$          | -0.070488 |
| $LE_3$          | -2.549971 |
| $LE_4$          | -17.29119 |
| $D_L$           | -19.883687 |
| $D_L$           | 1.396689 |
resistor be replaced by their fractional-order counterparts i.e. fractional-order inductor, fractional-order capacitor and fractional-order HP memristor. This is because, characterizes the fractional-order HP memristor (Banchuin, 2018). Moreover, $C_{1\alpha}$ is physically the pseudo-capacitance which characterizes the fractional-order capacitor (Freeborn et al., 2013), where $L_{1\alpha}$ and $L_{2\alpha}$ are physically the inductivity which characterize the fractional-order inductor (Schäfer & Krüger, 2006). For simplicity, a fractional-order memristor emulator (Rashad et al., 2017) with proper circuit parameter adjustment could serve as the fractional-order HP memristor as it is also current controlled with quadrants 1 and 3 reside single pinched point hysteresis loop. Compared with the previous fractional-order HP memristor-based chaotic circuit, our generalized circuit is simpler as merely a single fractional-order HP memristor is required apart from being dimensional consistent. Therefore, it is the simplest fractional-order HP memristor-based chaotic circuit. By this virtue and its dimensional consistency awareness, our generalized circuit has been found to be an interesting example of fractional-order chaotic circuits. Thus, it is beneficial to those fractional-order chaotic circuit and system-related research areas.

In order to implement the circuit emulator, it is convenient to, respectively, substituting $i_1(t)$, $i_2(t)$ and $v_3(t)$ in (8) by $y(t)$, $z(t)$ and $w(t)$ which in turn do not assume specific physical meanings. Therefore, (8) can be rewritten as

$$D^\alpha_i x(t) = k_x f(x(t), y(t))y(t)$$

$$D^\gamma_y y(t) = \frac{1}{L_{1\beta}} [w(t) - (M_{on} x(t) - M_{off}(1 - x(t))) y(t)]$$

$$D^\beta_z z(t) = \frac{R}{L_{2\gamma}} z(t) - \frac{1}{L_{2\gamma}} w(t)$$

$$D^\delta_w w(t) = \frac{1}{C_{1\delta}} (z(t) - y(t))$$  \hspace{1cm} (33)

where $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $0 < \delta \leq 1$ and $f(x(t), y(t))$ can be given by (4) with $i(t) = y(t)$. Noted that the order of the fractional derivative has been allowed to be incommensurate for obtaining full degree of freedom in the implementation. As a result, $L_{1\alpha}$, $L_{2\beta}$ and $C_{1\alpha}$ become $L_{1\beta}$, $L_{2\gamma}$ and $C_{1\delta}$ which can be, respectively, given by $L_{2\beta} = L_2 \sigma^{\alpha-1}$ and $C_{1\delta} = C_1 \sigma^{\delta-1}$ for maintaining the dimensional consistency. Obviously, (33) can be recast to

$$x(t) = k_x J^\alpha_y [f(x(t), y(t))y(t)]$$

$$y(t) = \frac{1}{L_{1\beta}} J^\gamma_y [w(t) - (M_{on} x(t) - M_{off}(1 - x(t))) y(t)]$$

$$z(t) = \frac{1}{L_{2\gamma}} J^\beta_z [Rz(t) - w(t)]$$

$$w(t) = \frac{1}{C_{1\delta}} J^\delta_w [z(t) - y(t)]$$  \hspace{1cm} (34)
where \( J_i^\alpha \triangleq D_i^\alpha \), \( J_i^\beta \triangleq D_i^\beta \), \( J_i^\gamma \triangleq D_i^\gamma \) and \( J_i^\delta \triangleq D_i^\delta \). By inspecting (34), the emulator’s block diagram can be obtained as depicted in Figure 15. Based on this block diagram, the emulator circuit can be implemented by merely applying off the shelf components. An example of emulator circuits is depicted in Figure 16 where \( x(t) \), \( y(t) \), \( z(t) \) and \( w(t) \) are physically voltages. Therefore, TL084 and AD633 which operates in the voltage mode can be applied as our OPAMP and multiplier and. In addition, \( C_\alpha \), \( C_\beta \), \( C_\gamma \) and \( C_\delta \) stand for the fractional-order capacitors of order \( \alpha \), \( \beta \), \( \gamma \) and \( \delta \) with the following pseudo-capacitances

\[
C_\alpha = 0.001/k_\alpha
\]

\[
C_\beta = 0.001L_1\beta
\]  \hspace{1cm} (35)

\[
C_\gamma = 0.001L_2\gamma
\]

Figure 15. The emulator block diagram.
Figure 16. The emulator circuit.
Moreover, the resistance values can be given by

\[ r_1 = \frac{1000}{M_{\text{off}}} \]

\[ r_2 = \frac{1000}{(M_{\text{on}} + M_{\text{off}})} \]  \hspace{1cm} (36)

\[ r_3 = \frac{1000}{R} \]

For implementing the fractional-order capacitors, the RC approximated circuit of a constant phase element (CPE) (Petrzela, 2019) depicted here in Figure 17 has been found to be applicable as a fractional-order capacitor is a CPE. According to Petrzela, (2019), the parameters of such circuit must satisfy

\[ s^\chi C_x = sC_p + R_p^{-1} + \sum_{k=1}^{2} \frac{sC_k}{sC_k R_k + 1} \]  \hspace{1cm} (37)

where \( \chi = (\alpha, \beta, \gamma, \delta) \).

In the following example, the simulation results will be illustrated.

**Example 4:** Let us assume a set of parameters similar to that of Example 1 and given \( \theta = 0.8, \gamma = 0.95 \) and \( \delta = 0.85 \). The resulting parameters of the emulator circuit can be summarized in Table 3.

| Parameter | Value          |
|-----------|----------------|
| \( r_1 \) | 100 \( \Omega \) |
| \( r_2 \) | 85 \( \Omega \) |
| \( r_3 \) | 577 \( \Omega \) |
| \( C_\alpha \) | 50.832 pFsec\( ^{\alpha-1} \) |
| \( C_\beta \) | 254.16 \( \mu Fsec^{\beta-1} \) |
| \( C_\gamma \) | 1355.2 \( \mu Fsec^{\gamma-1} \) |
| \( C_\delta \) | 410.02 \( \mu Fsec^{\delta-1} \) |
| \( \alpha \) | 0.9 |
| \( \beta \) | 0.8 |
| \( \gamma \) | 0.95 |
| \( \delta \) | 0.85 |
where the resistance and pseudo-capacitance values have been evaluated via (35) and (36). The corresponding values of resistances and capacitances for the RC approximated circuits as summarized in Table 4-7-VII have been determined via (37) by fitting its coefficient to the data based on the pseudo-capacitances and orders presented in Table 3 with the least squares method. The simulated dynamics of $x(t), y(t), z(t)$ and $w(t)$ can be obtained as depicted in Fig. 18–21. Similarly to Example 1, the chaotic waveforms with boundary restriction at $x(t) = 1$ can be observed. However, these waveforms employ different frequencies as $\alpha, \beta, \gamma$ and $\delta$ are incommensurate where $z(t)$ oscillates at the highest frequency followed by $x(t)$, $w(t)$ and $y(t)$.

| Table 4. The resistance and capacitance values for approximating $C_{\alpha}$ |
|---------------------|------------------|
| $R_1$               | 1.63066 MΩ       |
| $R_2$               | 1.63605 MΩ       |
| $R_3$               | 1.64563 MΩ       |
| $R_4$               | 1.66042 MΩ       |
| $R_5$               | 1.63610 MΩ       |
| $R_6$               | 1.63589 MΩ       |
| $R_7$               | 1.63553 MΩ       |
| $R_8$               | 1.81203 MΩ       |
| $C_1$               | 0.579643 μF      |
| $C_2$               | 0.579664 μF      |
| $C_3$               | 0.579307 μF      |
| $C_4$               | 0.579500 μF      |
| $C_5$               | 0.579662 μF      |
| $C_6$               | 0.579670 μF      |
| $C_7$               | 0.579684 μF      |
| $C_p$               | 0.167286 μF      |

| Table 5. The resistance and capacitance values for approximating $C_{\beta}$ |
|---------------------|------------------|
| $R_1$               | 2.48600 Ω       |
| $R_2$               | 2.58296 Ω       |
| $R_3$               | 2.15415 Ω       |
| $R_4$               | 2.50960 Ω       |
| $R_5$               | 2.55557 Ω       |
| $R_6$               | 2.53964 Ω       |
| $R_7$               | 2.74010 Ω       |
| $R_8$               | 191.187 Ω       |
| $C_1$               | 1 μF             |
| $C_2$               | 1 μF             |
| $C_3$               | 1 μF             |
| $C_4$               | 1 μF             |
| $C_5$               | 1 μF             |
| $C_6$               | 1 μF             |
| $C_7$               | 1 μF             |
| $C_p$               | 1 μF             |
6. Conclusion
In this work, a dimensional consistency aware fractional-order generalization of HP memristor-based chaotic circuit has been performed with the physical meaning of fractional time component assigned. The simplest autonomous circuit based on such memristor with correct memristor assumption has been chosen as the candidate circuit where a novel fractional-order HP memristor dedicated window function has been adopted for modelling the boundary effect. The dynamic of the generalized circuit has been analyzed where the importance of dimensional consistency

| Table 6. The resistance and capacitance values for approximating $C_\gamma$ |
|------------------|------------------|
| $R_1$            | 1.24863 $\Omega$ |
| $R_2$            | 1.24863 $\Omega$ |
| $R_3$            | 1.24863 $\Omega$ |
| $R_4$            | 1.24863 $\Omega$ |
| $R_5$            | 1.24863 $\Omega$ |
| $R_6$            | 1.24863 $\Omega$ |
| $R_7$            | 1.24863 $\Omega$ |
| $R_p$            | 95.2000 $\Omega$ |
| $C_1$            | 1 $\mu F$        |
| $C_2$            | 1 $\mu F$        |
| $C_3$            | 1 $\mu F$        |
| $C_4$            | 1 $\mu F$        |
| $C_5$            | 1 $\mu F$        |
| $C_6$            | 1 $\mu F$        |
| $C_7$            | 1 $\mu F$        |
| $C_p$            | 1 $\mu F$        |

| Table 7. The resistance and capacitance values for approximating $C_\delta$ |
|------------------|------------------|
| $R_1$            | 133.071 $\Omega$ |
| $R_2$            | 16.2204 $\Omega$ |
| $R_3$            | 18.1372 $\Omega$ |
| $R_4$            | 15.8298 $\Omega$ |
| $R_5$            | 20.9668 $\Omega$ |
| $R_6$            | 40.8958 $\Omega$ |
| $R_7$            | 34.8016 $\Omega$ |
| $R_p$            | 93.4220 $\Omega$ |
| $C_1$            | 1 $\mu F$        |
| $C_2$            | 1 $\mu F$        |
| $C_3$            | 1 $\mu F$        |
| $C_4$            | 1 $\mu F$        |
| $C_5$            | 1 $\mu F$        |
| $C_6$            | 1 $\mu F$        |
| $C_7$            | 1 $\mu F$        |
| $C_p$            | 1 $\mu F$        |
Figure 18. $X(t)$ v.s. $t$.

Figure 19. $Y(t)$ v.s. $t$.

Figure 20. $Z(t)$ v.s. $t$.

Figure 21. $W(t)$ v.s. $t$. 
awareness has been asserted. The realization of the circuit by using the fractional-order elements is indicated. Apart from being dimensional consistent, the indicated circuit has been found to be the simplest fractional-order HP memristor-based chaotic circuit. The fractional-order capacitor-based circuit emulator has been presented where we have shown that the oscillating frequencies of the responses are depended on the orders of fractional-order capacitors. At this point, it can be seen that this work is beneficial to those research areas including the analysis and design of chaotic circuits, fractional-order circuits, HP memristor-based circuits and nonlinear circuits and so on. In addition, a similar study based on other often cited fractional derivatives e.g., the above-mentioned fractional derivatives with nonsingular and nonlocal-nonsingular kernels and conformable fractional derivative etc., has been found to be an interesting further study. A generalization of the commercially available KNOWNM memristor-based chaotic circuit (C. Volos et al., 2020, September), (C. K. Volos et al., 2020) is also a challenging open research question.

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**Data availability statement**

The data supporting the finding of this study are available within the article.

**Citation information**

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