Transit index by means of graph decomposition

K.M. Reshmi\textsuperscript{1} and Raji Pilakkat\textsuperscript{2}

Abstract
Many topological indices for graphs are defined and are widely studied. Some are distance based and some are degree based. They find applications in many fields like chemical graph theory and networking. The concept of transit of a vertex and transit index of a graph was defined by the authors in their previous work. The transit of a vertex $v$ is “the sum of the lengths of all shortest path with $v$ as an internal vertex” and the transit index of $G$ is $TI(G)$ is the sum of the transit of all vertices of $G$. In this paper we introduce the concept of majorized shortest path, transit decomposition of a graph and transit decomposition number.

Keywords
Transit Index, Majorized shortest path, Transit decomposition, Transit decomposition number.

AMS Subject Classification
05C10, 05C12.

1. Introduction

In the fields of chemical graph theory, molecular topology and mathematical chemistry, a topological index also known as a connectivity index is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound. Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. In\textsuperscript{4}, transit index of a graph was introduced and its correlation with one of the physical property -MON of octane isomers was established.

In this paper we introduce the concept of majorized shortest path and transit decomposition of a graph. We also discuss certain results which helps in computing transit index of graphs. The transit decomposition number in case of certain graphs are also determined.

Throughout $G$ denotes a simple, connected, undirected graph with vertex set $V$ and edge set $E$. For undefined terms we refer \cite{1}.

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**Definition 1.1.** [4] Let $v$ be any vertex of $G$. Then the transit of $v$ denoted by $T(v)$ “the sum of the lengths of all shortest path with $v$ as an internal vertex” and the transit index of $G$ denoted by $TI(G)$ is

$$TI(G) = \sum_{v \in V} T(v)$$

**Lemma 1.2.** [4] For a vertex $v$ of the graph $G$, $T(v) = 0$ iff $\langle N[v] \rangle$ is a clique. i.e $T(v) = 0$ iff $v$ is a simplicial vertex of $G$

**Theorem 1.3.** [4] For a path $P_n$, Transit index is

$$TI(P_n) = \frac{n(n+1)(n^2 - 3n + 2)}{12}$$

**Theorem 1.4.** [5] Let $C_n$ be a cycle with $n$ even. Then

$$i) \ TI(C_n) = \frac{n^2(n^2 - 4)}{24}$$

$$ii) \ TI(C_{n+1}) = \frac{n(n^2 - 4)(n+1)}{24}$$

**Definition 1.5.** [5] Two vertices $v_1$ and $v_2$ of a graph are called transit identical if the shortest paths passing through them are same in number and length.

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\textsuperscript{1} Department of Mathematics, Government Engineering College, Kozhikode-673005, Kerala, India.

\textsuperscript{2} Department of Mathematics, University of Calicut, Malappuram-673365, Kerala, India.

*Corresponding author: 1 reshmikm@gmail.com; 2 rajipilakkat@gmail.com

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2. Majorized shortest paths

**Definition 2.1.** A path $M$ through $v$ is called a majorized shortest path of $v$, abbreviated as $msp(v)$ or $msp$ if it satisfies the following conditions.
1. $M$ is a shortest path in $G$ with $v$ as an internal vertex.
2. There exists no path $M'$ such that, $M'$ is a shortest path in $G$ with $v$ as an internal vertex and $M$ as a subpath of it.

We denote the collection of all $msp(v)$ by $\mathcal{M}_v$ and $\bigcup_{v\in V} \mathcal{M}_v$ by $\mathcal{M}_G$.

**Example 2.2.** Consider the graph $G$ in figure [1]. Let $M_1 =\{1234\}, M_2 =\{1235\}, M_3 =\{123\}$. Then $M_1$ and $M_2$ are $msp(2)$, while $M_3$ is not a majorized shortest path of $2$.

![Graph G](image)

**Figure 1.** Graph $G$

**$\mathcal{M}_G$ for various graphs**

1. For a path $P_n$, $\mathcal{M}_G = \{ P_n \}$
2. For a star $S_n$, $\mathcal{M}_G$ is the collection of all paths of length 2 connecting two pendant vertex.

We know that there are $C(n-1, 2)$ such paths and their intersection is $\{c\}$, where $c$ is the central vertex.

$\therefore TI(G) = C(n-1, 2) \times TI(P_3) = (n-1)(n-2)$

3. For a cycle $C_n$, $n > 3$, every majorized shortest path is $d$ in length, where $d$ is the diameter. For every vertex $v \in C_n$, $|\mathcal{M}_v| = d - 1$. Hence $|\mathcal{M}_G| = n$.

**Proposition 2.3.** For a graph $G$, $\mathcal{M}_G$ is unique.

**Proposition 2.4.** Let $e = uv$ be any edge of $G$. If $e$ is not a part of any majorized shortest path in $G$, then $e \in C_3$

**Proof.** Let us assume that $e$ is not a part of any majorized shortest path in $G$. Let $v_1 \neq u$ be a neighbour of $v$ in $G$. Then the shortest path from $v_1$ to $u$ is of length $\leq 2$. If it is 2, the path $v_1vu$ will be a part of the majorized shortest path through $v$. Hence $d(u, v_1) = 1$, showing $e = uv$ is part of $C_3$.

**Proposition 2.5.** In a tree $T$, $msp(v)$ connects pendant vertices of $T$, $\forall v \in V$. Conversely every path connecting two pendant vertex is a $msp$ for every internal vertex of it.

**Proof.** Suppose $M$ be a $msp(v)$, $v \in T$. Let $M : v_1v_2 ... v_k$. Suppose if possible one of the end vertex of $M$ be a non pendant vertex of $T$. Without loss of generality let us assume $v_1$ is not a pendant vertex. Then $d(v_1) > 1$. Let $u$ be a neighbor of $v_1$ other than $v_2$. Then the path $uv_1v_2 ... v_k$ is a shortest path in $T$ with $v$ as an internal vertex and with $M$ as a subpath of it. This is a contradiction.

Conversely, let $M$ be a path connecting two pendant vertices, say $u_1$ and $u_2$ of $T$. Let $v$ be an internal vertex of $M$. We need to show that $M \subseteq \mathcal{M}_v$. Assume $M \not\subseteq \mathcal{M}_v$. Then either (i)$M$ is not a shortest path in $T$ or (ii)$M$ is a subpath of some $M'$ with $v$ as an internal vertex. Since $T$ is a tree, $u_1 - u_2$ path is unique and hence $M$ is a shortest path. So (i) does not hold.

Again $u_1, u_2$ are pendant vertices proves (ii) wrong.

Hence the proof.

**Corollary 2.6.** $|\mathcal{M}_T| = C(p, 2)$, where $T$ is a tree and $p$ the number of pendant vertices of $T$.

**Proposition 2.7.** Consider the graph $G(V,E)$. Let $v \in V$ and $\mathcal{M}_v$ be the collection of all majorized paths in $G$ with $v$ as an internal vertex. If $\mathcal{M}_v = \{M_1, M_2\}$, then $T(v) = T(M_1) + T(M_2) - T(M_1 \cap M_2)$

**Proof.** Given $\{M_1, M_2\} = msp(v)$ Let $\mathcal{S}$ be the collection of all shortest paths in $G$ with $v$ as an internal vertex. Then $T(v) =$ sum of lengths of paths in $\mathcal{S}$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the collection of all subpaths of $M_1, M_2$ with $v$ as an internal vertex, respectively. Then $T(M_i) =$ sum of the lengths of the paths in $\mathcal{S}_i$.

Consider $M_1 \cap M_2$. Either $M_1 \cap M_2 = \{v\}$ or $M_1 \cap M_2$ is a subpath of $M_1$ and $M_2$ with $v$ as an internal vertex. Let $\mathcal{S}'$ be the collection of subpaths of $M_1 \cap M_2$ with $v$ as an internal vertex. Then $\mathcal{S}' \subseteq \mathcal{S}_1$ and $\mathcal{S}' \subseteq \mathcal{S}_2$. Hence the proof.

Let $G(V,E)$ and $\mathcal{M}_v$ be the collection of all majorized path in $G$ with $v$ as an internal vertex. If $\mathcal{M}_v = \{M_i, i = 1, 2, ..., k\}$, then the result[2.7] could be extended by applying the inclusion-exclusion principle in set theory.

**Theorem 2.8.** Let $G(V,E)$ be a graph and $v \in V$. If $\mathcal{M}_v = \{M_1, M_2, ..., M_k\}$ then, $T(v) = T(M_1) + T(M_2) + ... - T(M_1 \cap M_2) + ... + (-1)^{k+1} T(M_1 \cap M_2 \cap M_3 ... M_k)$

**Proposition 2.9.** Let $G(V,E)$ and $\mathcal{M}_G$ be the collection of all majorized paths in $G$. If $\mathcal{M}_G = \{M_1, M_2\}$, then $TI(G) = TI(M_1) + TI(M_2) - TI(M_1 \cap M_2)$

**Proof.**

$$TI(G) = \sum_{v \in V} T(v)$$

$$= \sum_{v \in V} [T(M_1(v)) + T(M_2(v)) - T(M_1 \cap M_2(v))]$$

$$= \sum_{v \in M_1} T(v) + \sum_{v \in M_2} T(v) - \sum_{v \in M_1 \cap M_2} T(v)$$

$$= TI(M_1) + TI(M_2) - TI(M_1 \cap M_2)$$
Let \( G(V,E) \) be a graph and \( \mathcal{M}_G \) be the collection of all majorized path in \( G \). If \( \mathcal{M}_G = \{ M_i, i = 1, 2, \ldots, k \} \), then the result\([2.9]\) could be extended by applying the inclusion-exclusion principle in set theory.

**Theorem 2.10.** Let \( G(V,E) \) be a graph and \( \mathcal{M}_G \) be the collection of all majorized path in \( G \). If \( \mathcal{M}_G = \{ M_i, i = 1, 2, \ldots, k \} \),
\[
T(G) = T_1(M_1) + \ldots + T_v(M_k) - T_1(M_1 \cap M_2) - \ldots - T_v(M_{k-1} \cap M_k) + \ldots + (-1)^{k+1}T_v(M_1 \cap M_2 \cap M_3 \ldots \cap M_k).
\]

Hence knowing the majorized shortest paths of a graph, one could compute the transit index of a graph.

### 3. Transit decomposition

**Definition 3.1.** A decomposition of a graph \( G \) into a collection of subgraphs \( \tau = \{ T_1, T_2, \ldots, T_r \} \), where each \( T_i \) is either a chordless cycle in \( G \) or a majorized shortest path of \( G \) such that \( T(G) = \sum T(I) \cap T(J) + \ldots + \sum T_v(M_1 \cap M_2) + \ldots - T_v(M_{k-1} \cap M_k) + \ldots + (-1)^{k+1}T_v(M_1 \cap M_2 \cap M_3 \ldots \cap M_k) \) is called a Transit Decomposition of \( G \). We denote a transit decomposition of minimum cardinality by \( \tau_{\min} \).

The minimum cardinality of a transit decomposition of \( G \) is called the Transit decomposition number, denoted by \( \theta(G) \) or simply \( \theta \) if there is no other confusion. Clearly \( \mathcal{M}_G \) is a transit decomposition of \( G \). We denote \( \mid \mathcal{M}_G \mid \) by \( \theta_a(G) \) or simply \( \theta_a \).

**Example 3.2.** Consider the graph in the figure [2].

Let \( M_1 : 1234; M_2 : 1254; M_3 : 345; M_4 : 325; C_1 : 23452 \). Here \( \mathcal{M}_G = \{ M_1, M_2, M_3, M_4 \} \). Here \( \tau_{\min} = \{ M_1, M_2, C_1 \} \), the transit decomposition of minimal cardinality. Hence \( \theta = 3 \), while \( \theta_a = 4 \).

**Proposition 3.3.** If \( C_n \) is a chordless cycle in \( G \), with \( n > 3 \), then \( C_n \in \tau_{\min} \).

**Remark 3.4.** If \( e \) is an edge of \( G \) that does not belong to any cycle in \( G \), it will be a part of some \( T_i \in \tau \).

### Transit decomposition number for various graphs

- If \( G \) is a tree, \( \tau = \mathcal{M}_G \) and \( \theta = \theta_a = C(p, 2) \), where \( p \) is the number of pendant vertices.
- If \( G \) is a cycle, \( \theta = 1 \) and \( \theta_a = n, n > 3 \).
- If \( G \) is a path, \( \theta = \theta_a = 1 \).
- If \( G \) is a complete graph, \( \theta = \theta_a = 0 \).

**Theorem 3.5.** Let \( G = K_{p,q} \) be the bipartite graph. Then \( \theta = \frac{n(p-1)(q-1)}{2} \).

**Proof.** In the case of a complete bipartite graphs, every short- est path is of length \( \leq 2 \). Hence every shortest path is a majorized shortest path. \( \theta = \theta_a = \frac{n(p+1)q+1}{2} \).

The chordless cycles in \( K_{p,q} \) are of girth 4. Also every shortest path is part of some chordless cycle. Hence \( \theta = \theta_a = \frac{n(p-1)(q-1)}{2} \).

**Theorem 3.6.** Let \( G = W_n \), \( n > 4 \), be the wheel graph. Then \( \theta = \theta_a = \frac{(n-1)(n-2)}{2} \).

**Proof.** In the wheel graph every chordless cycle is \( C_3 \). Hence \( \theta = \theta_a \). Note that the diameter of the graph is 2. Hence every majorized shortest path is of length \( \leq 2 \). Since \( P_2 \) is not a majorized shortest path, all msp in \( G \) are isomorphic to \( P_3 \). On the cycle of the wheel, starting with every vertex there are 2 msp. Hence on a total (n-1) paths. Other msp are those starts and ends on the cycle of the wheel and passes through the center. With each vertex on the cycle we can associate (n-4) such paths. Hence on a total \( \frac{(n-1)(n-2)}{2} \) paths. Thus \( \theta = \theta_a = \frac{(n-1)(n-2)}{2} \).

**Theorem 3.7.** If \( G = K_{2n-I} \), \( \theta_a(G) = 2n(n-1) \) and \( \theta(G) = \frac{n(n-1)}{2}, n > 2 \), where \( I \) is the one factor of \( K_{2n-I} \).

**Proof.** In \( K_{2n-I} \), there will be \( n \) pair of vertices which are non adjacent. For a vertex \( v \), \( d(v) = n-1 \). Note that every vertex of \( G \) are transit identical. There will be \( n-1 \) number of msp of length 2. Hence \( \theta_a(G) = 2n(n-1) \).

**Theorem 3.8.** Let \( G \) be a unicyclic graph, with cycle \( C_r \). If the number of vertex of \( C_r \) with \( d(v) > 2 \) is one, then

1. \( \theta(G) = C(p, 2) + 2p + 1 \)

2. \( \theta_a(G) = \frac{1}{2} (p^2 + 3p + 2r - 4) \), where \( p \) is the number of pendant vertices of \( G \).

**Proof.** 1. When forming \( \tau_{\min} \) we first include \( C_r \). Corresponding to every pendant vertex, there will be 2 majorized shortest paths connecting it to vertices of the cycle \( C_r \). Thus including 2p paths to \( \tau \). Again there are \( C(p, 2) \) majorized shortest path connecting pendant vertices among themselves. Hence \( \theta(G) = C(p, 2) + 2p + 1 \).
2. Here \( \tau_{\min} = \mathcal{M}_G \). In the previous case if we exclude \( C_r \) and include every majorized shortest path of vertices of \( C_r \), which are \( r \) in number, we get \( \mathcal{M}_G \). Note that of these \( r \) msp, two of them forms a part of msp connecting pendant vertices to vertices of \( C_r \). Hence we get \( \theta_2(G) = \frac{1}{2}(p^2 + 3p + 2r - 4) \).

\[
\text{Theorem 3.9. Let } G \text{ be a unicyclic graph with cycle } C_r. \text{ Let } u \text{ and } v \text{ be two vertices at a distance } \left\lfloor \frac{r}{2} \right\rfloor \text{ to each other with } d(u_1), d(v_2) > 2. \text{ Let } T_1 \text{ be a tree with } p_1 \text{ pendant edges and } T_2 \text{ be a tree with } p_2 \text{ pendant edges rooted at } u \text{ and } v \text{ respectively. Then}
\]

\[
\theta(G) = \begin{cases} 
C(p_1 + p_2, 2) + p_1 + p_2 + 1 & \text{when } r \text{ is odd} \\
2p_1p_2 + C(p_1, 2) + C(p_2, 2) + 1 & \text{when } r \text{ is even}
\end{cases}
\]

\[
\text{Proof. Since } G \text{ is unicyclic with cycle } C_r, C_r \in \tau_{\min}.
\]

**Case 1**

Let \( u_1, u_2 \) and \( v_1, v_2 \) be the vertices of \( C_r \) adjacent to \( u \) and \( v \) respectively. When \( r \) is odd, the msp connecting pendant vertices of \( T_1 \) to \( T_2 \) is unique. Hence they will be \( p_1p_2 \) in number. Either of \( v_1, v_2 \) (also \( u_1, u_2 \)) lie on such msp. Without loss of generality let us assume that \( u_1 \) and \( v_1 \) lie on these msp. There will be \( p_1 \) number of msp connecting pendant vertices of \( T_1 \) to \( v_2 \) and \( p_2 \) number of msp connecting pendant vertices of \( T_2 \) to \( u_2 \). Hence we get \( \theta(G) = C(p_1 + p_2, 2) + p_1 + p_2 + 1 \)

**Case 2**

When \( r \) is even \( \left\lfloor \frac{r}{2} \right\rfloor = \frac{r}{2} \). Hence \( u \) and \( v \) are diametrically opposite vertices of \( C_r \). For every pendant vertex of \( T_1 \) there are \( 2 \) msp connecting it to a vertex of \( T_2 \). Altogether there are \( 2 \times p_1 \times p_2 \) number of msp. The number of msp connecting pendant vertices of \( T_1 \) among themselves is \( C(p_1, 2) \) and the case of \( T_2 \) is \( C(p_2, 2) \). Thus \( \theta(G) = 2p_1p_2 + C(p_1, 2) + C(p_2, 2) + 1 \).