ON RATIONAL CONNECTEDNESS OF GLOBALLY $F$-REGULAR THREEFOLDS

YOSHINORI GONGYO, ZHIYUAN LI, ZSOLT PATAKFALVI, KARL SCHWEDE, HIROMU TANAKA, AND HONG R. ZONG

Abstract. In this paper, we show that projective globally $F$-regular threefolds, defined over an algebraically closed field of characteristic $p \geq 11$, are rationally chain connected.

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0. Introduction

In the 90’s, Campana and Kollár–Miyaoka–Mori showed that smooth Fano varieties are rationally connected in characteristic zero ([KMM], [Campana]). Recently Zhang and Hacon–McKernan generalized this to log Fano varieties in characteristic zero ([Zhang], [HM]). Then,
it is natural to ask the following question: are smooth Fano varieties in positive characteristic rationally connected? This is still an open problem, but it is known that they are rationally chain connected (cf. [Kollár2, Ch V, 2.14]). In this paper, we want to consider a related problem on this result. The main theorem of this paper is as follows.

**Theorem 0.1** (Theorem 3.4). Let $k$ be an algebraically closed field of characteristic $p \geq 11$. Let $X$ be a projective globally $F$-regular threefold over $k$. Then, $X$ is rationally chain connected.

**Remark 0.2.** One can see in the proof (see Theorem 4.3) that actually over an algebraically closed field of characteristic $p \geq 11$ any projective globally $F$-regular threefold is either birational to a $Q$-Fano threefold or separably rationally connected. But it is open in positive characteristics whether $Q$-Fano threefolds are separably rationally connected or not.

Globally $F$-regular varieties were introduced by K. Smith [Sm] who drew inspiration from the tight closure theory. Global $F$-regularity is a global property of a projective variety over a field of positive characteristic as the name suggests, and they impose strong conditions on the structure of the variety.

The assumption that $X$ is globally $F$-regular is analogous to, but more restrictive than, assuming $X$ to be log Fano (for some boundary divisor $\Delta$). Indeed, there exists a del Pezzo surface in characteristic $p \leq 5$ which is not globally $F$-regular (cf. [Hara]). On the other hand, a log Fano variety $X$ in characteristic zero, is globally $F$-regular type, that is, almost all the modulo $p$ reductions $X_p$ are globally $F$-regular ([SS]).

**0.3** (The strategy for the main theorem). The methods of [Zhang] and [HM] do not work in positive characteristics since their proofs depend on semi-positivity theorems and the extension theorem, both of these results depend heavily on the Kodaira vanishing theorem. As is well-known, such vanishing theorems fail in positive characteristics ([Raynaud]). Let us overview the proof of Theorem 0.1. Let $X$ be a projective globally $F$-regular threefold. First, we use the minimal model program, which is recently developed by [CTX] and [HX]. Since a globally $F$-regular variety is log Fano, the end result of the minimal model program is a Mori fiber space. Then, after some preliminary consideration, we may assume that $X$ has a Mori fiber space structure. The reason why any consideration is necessary here is that the property of rational chain connectedness is not stable under birational model changes, e.g. the cone over elliptic curve is rationally chain connected,
but its blowup of the vertex is a $\mathbb{P}^1$-bundle over an elliptic curve, which is not rationally chain connected.

We have a Mori fiber space structure $f : X \rightarrow Y$. We only explain here the proof of the case where $\dim Y > 0$ (cf. Proposition 3.3). Then, it is enough to show that $Y$ and general fibers are rationally chain connected, and that $f$ has a section. It is easy to show the rational connectedness of $Y$ (cf. Lemma 1.3). The rational connectedness of general fibers holds by the following theorem.

**Theorem 0.4** (Theorem 2.1). Let $f : X \rightarrow Y$ be a proper morphism from a normal scheme to an $F$-finite integral scheme of characteristic $p > 0$, such that $f_*$ $\mathcal{O}_X \cong \mathcal{O}_Y$. Further let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$, such that $(X, \Delta)$ is globally $F$-regular. Then there is an open set $U \subseteq Y$, such that for every geometric point $y \in U$, $(X_y, \Delta_y)$ is globally $F$-regular.

Thus, we consider whether $f$ has a section. Actually, we only establish weaker but similar results (cf. Proposition 3.8 and Proposition 3.9). These results follow from [Hirokado] and [DS]. Since [Hirokado] needs the assumption $p \geq 11$, we also assume this condition. [DS] is a positive characteristic analogue of the result by Graber–Harris–Starr ([GHS]).

On the other hand, we also establish the following result.

**Theorem 0.5** (Theorem 5.3). Let $k$ be an algebraically closed field of characteristic zero. Let $X$ be a $\mathbb{Q}$-Gorenstein normal projective variety over $k$ which is of globally $F$-regular type. Then $X$ is rationally connected.

A log Fano variety $X$ in characteristic zero is globally $F$-regular type ([SS]). The inverse assertion is an open problem. The above theorem is a weaker result of this.

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1. **Preliminaries**

We start by the definition of globally $F$-regular varieties:
Definition 1.1. Let $X$ be a normal variety over an $F$-finite field $k$ and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. We say $(X, \Delta)$ is globally $F$-regular if for every effective $\mathbb{Q}$-divisor $E$, there exists $e \in \mathbb{Z}_{>0}$ such that the composition homomorphism

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X((p^e-1)\Delta + E)^\pi)$$

splits as an $\mathcal{O}_X$-module homomorphism where the latter map is the natural injection.

If $X$ is quasi-projective (respectively G1 and S2), but not a-priori normal, we say that $X$ is globally $F$-regular if for every effective Cartier divisor $E \geq 0$ (respectively effective Weil divisorial sheaf $E$ which is Cartier in codimension 1), there exists an $e > 0$ such that the map

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(E)$$

splits as an $\mathcal{O}_X$-module homomorphism. It follows that $X$ is then normal by [HH2, Corollary 5.11].

The following lemmas follow from the definition directly.

Lemma 1.2. If $(X, \Delta)$ is a globally $F$-regular variety over an $F$-finite field, then, for every $0 \leq \Delta' \leq \Delta$, so is $(X, \Delta')$.

Lemma 1.3. Let $f : X \to X_1$ be a small birational map or an algebraic fiber space of normal varieties over an $F$-finite field of characteristic $p > 0$. If $X$ is globally $F$-regular (resp. globally $F$-split), then so is $X_1$.

Actually a globally $F$-regular variety is a log Fano variety with some boundary.

Theorem 1.4 ([SS, Theorem 4.3]). Let $X$ be a normal projective variety over an $F$-finite field of characteristic $p > 0$. If $X$ is globally $F$-regular (resp. globally $F$-split), then $X$ is of Fano type (resp. Calabi-Yau type).

Next, we briefly explain how to reduce things from characteristic zero to characteristic $p > 0$. We use the reduction modulo positive characteristics in only Section 5. The reader is referred to [HH4, Chapter 2] and [MS, Section 3.2] for details.

Let $X$ be a normal variety over a field $k$ of characteristic zero and $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on $X$. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra $A$ of $k$, we can construct a scheme $X_A$ of finite type over $A$ and closed subschemes $D_{i,A} \subset X_A$ such that there exists
isomorphisms

\[
X \cong X_A \times_{\text{Spec } A} \text{Spec } k \\
D_i \cong D_{i,A} \times_{\text{Spec } A} \text{Spec } k.
\]

Note that we can enlarge \( A \) by localizing at a single nonzero element and replacing \( X_A \) and \( D_{i,A} \) with the corresponding open subschemes. Thus, applying the generic freeness [HH4, (2.1.4)], we may assume that \( X_A \) and \( D_{i,A} \) are flat over \( \text{Spec } A \). Enlarging \( A \) if necessary, we may also assume that \( X_A \) is normal and \( D_{i,A} \) is a prime divisor on \( X_A \). Letting \( D_A := \sum_i d_i D_{i,A} \), we refer to \((X_A, D_A)\) as a model of \((X, D)\) over \( A \).

Let \( \Gamma \) be a finitely generated group of Weil divisors on \( X \). We then refer to a group \( \Gamma_A \) of Weil divisors on \( X_A \) generated by a model of a system of generators of \( \Gamma \) as a model of \( \Gamma \) over \( A \). After possibly enlarging \( A \), we denote by \( \Gamma_\mu \) the group of Weil divisors on \( X_\mu \) obtained by restricting divisors in \( \Gamma_A \) over \( \mu \).

Given a morphism \( f : X \to Y \) of varieties over \( k \) and a model \((X_A, Y_A)\) of \((X, Y)\) over \( A \), after possibly enlarging \( A \), we may assume that \( f \) is induced by a morphism \( f_A : X_A \to Y_A \) of schemes of finite type over \( A \). Given a closed point \( \mu \in \text{Spec } A \), we obtain a corresponding morphism \( f_\mu : X_\mu \to Y_\mu \) of schemes of finite type over \( k(\mu) \). If \( f \) is projective (resp. finite), after possibly enlarging \( A \), we may assume that \( f_\mu \) is projective (resp. finite) for all closed points \( \mu \in \text{Spec } A \).

**Definition 1.5.** Use the notation as before.

(i) A projective variety (resp. an affine variety) \( X \) is said to be of **globally \( F \)-regular type** (resp. **strongly \( F \)-regular type**) if for a model of \( X \) over a finitely generated \( \mathbb{Z} \)-subalgebra \( A \) of \( k \), there exists a dense open subset \( S \subseteq \text{Spec } A \) of closed points such that \( X_\mu \) is globally \( F \)-regular (resp. strongly \( F \)-regular) for all \( \mu \in S \).

(ii) A projective variety (resp. an affine variety) \( X \) is said to be of **dense globally \( F \)-split type** (resp. **dense \( F \)-pure type**) if for a model of \( X \) over a finitely generated \( \mathbb{Z} \)-subalgebra \( A \) of \( k \), there exists a dense subset \( S \subseteq \text{Spec } A \) of closed points such that \( X_\mu \) is globally \( F \)-split (resp. \( F \)-pure) for all \( \mu \in S \).
Remark 1.6. (1) The above definition is independent of the choice of a model.
    
(2) If $X$ is of globally $F$-regular type (resp. strongly $F$-regular type), then we can take a model $X_A$ of $X$ over some $A$ such that $X_A$ is globally $F$-regular (resp. strongly $F$-regular) for all closed points $\mu \in \text{Spec } A$.

The following proposition is well-known:

Proposition 1.7. Let $X$ be a normal projective variety over a field of characteristic zero.

(1) If $X$ is $\mathbb{Q}$-Gorenstein and of globally $F$-regular type (resp. dense globally $F$-split type), then it has only log terminal singularities (resp. log canonical singularities).

(2) If $X$ is of Fano type, then $X$ is of globally $F$-regular type.

Proof. See [HW, Theorem 3.9] for (1) and [SS, Theorem 5.1] for (2).

We now turn to some preliminary results about rationally connected varieties.

Definition 1.8. Let $X$ be a variety over a field $k$. It is separably rationally connected (SRC) if there is a generically smooth family of 1–cycles with geometrically rational components:

$$
\begin{array}{ccc}
U & \xrightarrow{u} & X \\
g & \downarrow & \\
B & & \\
\end{array}
$$

over $k$ such that the double evaluation map:

$$
U \times_B U \xrightarrow{(u,u)} X \times X
$$

is generically étale and dominates $X \times X$. If we drop the étale condition, it is called rationally connected (RC). And if we do not require the generic smoothness of $U \to B$, it is called rationally chain connected (RCC).

We use the following the base change property:

Lemma 1.9 (cf. [Kollár, Ch IV, 3.2.5 Exercise]). Let $k'/k$ be the field extension of algebraic closed fields and $X$ a proper variety over $k$. If $X' = X \times_k k'$ is rationally chain connected over $k'$, then so is $X$ over $k$.

Proof. By the assumption, we can find a closed sub-scheme $B$ of Chow$(X')$ such that the corresponding family $U \to B$ satisfies that the double
evaluation map \( U \times_B U \to X' \times X' \) is surjective. We can find an intermediate finitely generated \( k \)-algebra \( R \) such that \( k \subset R \subset k' \) and, the map \( U \to B \) and the double evaluation map are defined over \( R \). We obtain \( U_R \to B_R \). Further, we may assume that \( R \) and \( B_R \) are integral. Also, by shrinking \( \text{Spec} \, R \), we may assume that, for every scheme-theoretic point \( p \in \text{Spec} \, R \), the double evaluation map

\[
U_p \times_{B_p} U_p \to X'_p \times_p X'_p
\]

is surjective. Then we claim the followings:

**Claim 1.10.** For every scheme-theoretic point \( p \in \text{Spec} \, R \), the family \( U_p \to B_p \) is a family of 1-cycles with geometrically rational components.

**Proof of Claim 1.10.** By [Kollár, Ch II, Proposition 2.2], it is sufficient to show that, for the generic point \( \xi \) of \( B_R \), \( U_\xi \) is a 1-cycle with geometrically rational components. Since \( U \to B \) is a family of 1-cycles with geometrically rational components, so is \( U_K \to B_K \) where \( K \) is the field of fractions of \( R \). Thus, it is enough to prove that the morphism

\[
B_K = B_R \times_R K \to B_R
\]

is dominant, which holds by construction. \( \square \)

Thus \( X \) is also RCC. \( \square \)

We collect the relation of separably rational connectedness and free rational curves. We use these in Section 4.

**Definition 1.11.** Let \( X \) be a smooth variety over an algebraically closed field. We call a morphism \( f : \mathbb{P}^1 \to X \) free (resp. very free) if \( f^* \mathcal{T}_X \) is non-negative (resp. ample).

In particular, we define the very free locus as:

**Definition 1.12.** Let \( X \) be a smooth quasi-projective variety over an algebraically closed field \( k \), we define the very free locus, denoted by \( X_{vf} \) to be the open subset of points \( x \in X \) which lie in a very free curve. Obviously very free locus is contained in smooth locus, \( X_{vf} \subset X_{sm} \).

The following theorem is well-known:

**Theorem 1.13** ([Kollár2, IV.2]). Let \( X \) be a smooth quasi-projective variety over an algebraically closed field \( k \), then the following are equivalent:

1. \( X \) is separably rationally connected.
2. There is a very free curve \( f : \mathbb{P}^1 \to X \) in \( X \), namely \( X_{vf} \) is non-empty.
We recall the definition of combs:

**Definition 1.14.** Let $k$ be an arbitrary field. A comb with $n$ teeth over $k$ is a projective curve with $n + 1$ irreducible components $C_0, C_1, \ldots, C_n$ over $\bar{k}$ satisfying the following conditions:

1. The curve $C_0$ is defined over $k$.
2. The union $C_1 \cup \cdots \cup C_n$ is defined over $k$ (Each individual curve may not be defined over $k$).
3. The curves $C_1, \ldots, C_n$ are smooth rational curves disjoint from each other, and each of them meets $C_0$ transversely in a single smooth point of $C_0$ (which may not be defined over $k$).

The curve $C_0$ is called the handle of the comb, and $C_1, \ldots, C_n$ the teeth.

A rational comb is a comb whose handle is a smooth rational curve.

We have a general procedure for smoothing combs using very free curves:

**Theorem 1.15** (= [TZ12, Proposition 2.4]). Given a morphism from a smooth projective curve $f_0 : C_0 \to X$ to a smooth and separably rationally connected quasi-projective variety $X$ over a field $k$, and an integer $d$. Suppose that $f_0(C) \cap X_{\text{eff}}$ is non-empty, then there are $q \gg 0$ very free rational curves $f_i : C_i \to X$, $1 \leq i \leq q$, such that

1. $C = C_0 \cup C_1 \cup \cdots \cup C_q$ is a comb with $q$ teeth. Furthermore, there is a morphism $f : C \to X$ (defined over $k$) and a smoothing of the comb $\Sigma \to T, G : \Sigma \to X$.
2. $H^1(\Sigma_t, G^*_tT_X \otimes M) = 0$ for a general member $G_t : \Sigma_t \to X$ of the smoothing and any line bundle $M$ of degree $d$.

What we actually used is a relative version of the above Theorem 1.15, which can be implied by the above theorem and is already implicitly stated in [GHS] and [DS].

**Corollary 1.16.** Let $\mathfrak{X} \to C$ be a flat family of quasi-projective varieties over a smooth projective curve $C$ over a field $k$, with separably rationally connected general fibers, then for any compactification $\tilde{\mathfrak{X}} \to C$ and multisection $C' \to \tilde{\mathfrak{X}}$ where $C'$ is a curve flat over $C$ which lies in the smooth locus of the fibration $\tilde{\mathfrak{X}} \to C$ and has non-empty intersection with

$$\bigcup_{c \in C} \tilde{\mathfrak{X}}_{c,\text{ef}}$$

And let $d$ be an integer. Then there are $q \gg 0$ rational curves $f_i : C_i \to \mathfrak{X}$, $1 \leq i \leq q$, such that

1. $C_i$’s lie in the general fiber of $\mathfrak{X} \to C$, and are very free curves in these fibers, namely with $f_i^*T_{\mathfrak{X}/C}$ ample.
(2) $\hat{C} = C' \cup C_0 \cup C_1 \cup \ldots \cup C_q$ is a comb with $q$ teeth. Furthermore, there is a morphism $f : C_t \to \bar{X}$ (defined over $k$) and a smoothing of the comb $\Sigma \to T, G : \Sigma \to \bar{X}$.

(3) $H^1(\Sigma_t, G_t^* T_{\bar{X}/C} \otimes M) = 0$ for a general member $G_t : \Sigma_t \to \bar{X}$ of the smoothing and any line bundle $M$ of degree $d$.

2. Global F-regularity and surjective morphisms

This section is devoted to the proof of the following main theorem:

**Theorem 2.1.** Let $f : X \to Y$ be a proper morphism from a normal scheme to an $F$-finite integral scheme of characteristic $p > 0$, such that $f_* O_X \cong O_Y$. Further let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$, such that $(X, \Delta)$ is globally $F$-regular. Then there is an open set $U \subseteq Y$, such that for every perfect point $y \in U$, $(X_y, \Delta_y)$ is globally $F$-regular.

**Remark 2.2.** We make the following remarks on normality. The hypothesis that $(X, \Delta)$ is globally $F$-regular implies that $X$ is globally $F$-regular and is hence normal. It follows immediately that the general fiber is also normal. The theorem then implies that the general geometric fibers are normal as well.

**Notation 2.3.** Throughout this section we only work with $F$-finite schemes and fields. The reason for this is that the notion of $F$-splitting is much better behaved under this hypothesis.

Furthermore, and just in this section, we denote the $e$-th Frobenius pushforwards by using a $1/p^e$ exponent on the structure sheaf. I.e., instead of $F^e_* O_X$ we write $O_X^{1/p^e}$. The justification for this notation is that many isomorphisms such as $O_X^{1/p^e} \otimes_k k^{1/p^{e+1}} \cong (O_X \otimes_k k^{1/p})^{1/p^e}$ are much more apparent.

We first make the observation that $F$-split proper varieties remain $F$-split after base change of the base field. We need one conceit that $H^0(X, O_X) \subseteq k$ is a separable field extension. Of course, if this is inseparable, then $H^0(X, O_X \otimes_k k^{1/p}) \cong H^0(X, O_X) \otimes_k k^{1/p}$ is clearly non-reduced so this additional hypothesis is certainly necessary (and indeed, if $X$ is $F$-split and proper over $k$, then obviously $X \times_k k$ is not $F$-split). Note that if we choose our field $k = H^0(X, O_X)$ then the separability hypothesis is automatic.

**Lemma 2.4.** Suppose that $k$ is an $F$-finite (but not necessarily perfect) field. Further suppose that $X$ is a proper scheme over $k$ and $H^0(X, O_X) = K$ is a separable field extension of $k$. If $X$ is $F$-split, then every base change $X \times_k k^{1/p^e}$ is also $F$-split. In particular $X$ is geometrically reduced [EGAIV, Proposition 4.6.1.d].
Proof. By induction on $e > 0$, it suffices to show that $X \times_k k^{1/p}$ is Frobenius split. We have the following composition of Frobenius maps.

$$\mathcal{O}_X \to \mathcal{O}_X \otimes_k k^{1/p} \to \mathcal{O}_X^{1/p} \otimes_{k^{1/p}} k^{1/p^2} \to \mathcal{O}_X^{1/p^2}.$$ 

We apply the functor $\mathcal{H}om_{\mathcal{O}_X}(\quad, \mathcal{O}_X)$ to obtain:

$$\mathcal{O}_X \leftarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_k k^{1/p}, \mathcal{O}_X) \leftarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/p} \otimes_{k^{1/p}} k^{1/p^2}, \mathcal{O}_X)$$

Notice first that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_k k^{1/p}, \mathcal{O}_X) \cong \mathcal{O}_X \otimes \mathcal{H}om_{k}(k^{1/p}, k) \cong \mathcal{O}_X \otimes_k k^{1/p}$, here $k$ is technically $f^{-1}k$ where $f : X \to \text{Spec}k$ is the structural map. Likewise

\begin{align*}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/p} \otimes_{k^{1/p}} k^{1/p^2}, \mathcal{O}_X) &
\cong \mathcal{H}om_{\mathcal{O}_X \otimes k^{1/p}}(\mathcal{O}_X^{1/p} \otimes_{k^{1/p}} k^{1/p^2}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_k k^{1/p}, \mathcal{O}_X)) \\
&\cong \mathcal{H}om_{\mathcal{O}_X \otimes k^{1/p}}(\mathcal{O}_X^{1/p} \otimes_{k^{1/p}} k^{1/p^2}, \mathcal{O}_X \otimes_k k^{1/p}) \\
&\cong \mathcal{H}om_{\mathcal{O}_X \otimes k^{1/p}}((\mathcal{O}_X \otimes_k k^{1/p})^{1/p}, \mathcal{O}_X \otimes_k k^{1/p}).
\end{align*}

Since $X$ is $F$-split, the canonical map $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{1/p^2}, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X)$ is surjective. This implies that the map

$$H^0(\beta) : \text{Hom}_{\mathcal{O}_X \otimes k^{1/p}}((\mathcal{O}_X \otimes_k k^{1/p})^{1/p}, \mathcal{O}_X \otimes_k k^{1/p}) \to H^0(X, \mathcal{O}_X \otimes_k k^{1/p})$$

is non-zero. Now $H^0(X, \mathcal{O}_X) \cong K \supseteq k$ is a separable field extension and so $H^0(X, \mathcal{O}_X \otimes_k k^{1/p}) = H^0(X, \mathcal{O}_X) \otimes_k k^{1/p} \cong K \otimes_k k^{1/p}$ is a field. Further note that $\beta$ is a $\mathcal{O}_X \otimes_k k^{1/p}$-module homomorphism, hence $H^0(\beta)$ is a $H^0(X, \mathcal{O}_X \otimes_k k^{1/p})$-module homomorphism. Thus $H^0(\beta)$ is a non-zero homomorphism over a field with the target being a one-dimensional vector space. Hence $H^0(\beta)$ is surjective. This proves that $X \times_k k^{1/p}$ is also $F$-split. \hfill \qed

As a quick corollary, we also obtain that globally $F$-split varieties are geometrically globally $F$-split.

**Corollary 2.5.** Suppose that $X$ is a proper $F$-split variety over an $F$-finite field $k$ such that $H^0(X, \mathcal{O}_X) \supseteq k$ is a separable field extension. Then $X$ is geometrically $F$-split. In other words, for every extension $k \subseteq K$, for $K$ also $F$-finite, $X \times_k K$ is $F$-split.

**Proof.** Since $\mathcal{O}_X \otimes_k k^{1/p} \to (\mathcal{O}_X \otimes_k k^{1/p})^{1/p}$ is split, so is $\mathcal{O}_X \otimes_k k^{1/p} \to \mathcal{O}_X^{1/p}$ since the latter factors the former. Thus there is a map $\phi : \mathcal{O}_X^{1/p} \to \mathcal{O}_X \otimes_k k^{1/p}$ which sends 1 to 1. It follows that

$$\left(\phi \otimes_{k^{1/p}} K^{1/p}\right) : \mathcal{O}_X^{1/p} \otimes_{k^{1/p}} K^{1/p} \cong (\mathcal{O}_X \otimes_k K)^{1/p} \to \mathcal{O}_X \otimes_k K^{1/p}$$
also sends 1 to 1. But obviously since $K \to K^{1/p}$ is split, so is the map $\mathcal{O}_X \otimes_k K \to \mathcal{O}_X \otimes_k K^{1/p}$ and so there is a $\mathcal{O}_X \otimes_k K$-linear map 

$$\beta : \mathcal{O}_X \otimes_k K^{1/p} \to \mathcal{O}_X \otimes_k K$$

which also sends 1 to 1. We compose and so obtain:

$$(\mathcal{O}_X \otimes_k K^{1/p}) / p \phi \otimes (\mathcal{O}_X \otimes_k K^{1/p}) / p \beta \to \mathcal{O}_X \otimes_k K.$$

This implies that $X \times_k K$ is $F$-split as desired. □

The previous method also extends to globally $F$-regular varieties.

**Proposition 2.6.** Suppose that $k$ is an $F$-finite (but not necessarily perfect) field. Further suppose that $X$ is a proper normal variety over $k$, $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ and $H^0(X, \mathcal{O}_X) = K$ is a separable field extension of $k$. If $(X, \Delta)$ is globally $F$-regular, then every base change 

$$(X \times_k k^{1/p^e}, \Delta \times_k k^{1/p^e})$$

is also globally $F$-regular and in particular normal.

**Proof.** By [SS, Proposition 3.12], and making $\Delta$ larger if necessary, we may assume that $(p^a - 1)\Delta$ is a $\mathbb{Z}$-divisor. By Lemma 2.4, $X$ is geometrically reduced over $k$, hence there is an effective divisor $D > 0$ on $X$ containing the support of $\Delta$ such that $X \setminus D$ is affine and smooth over $k$. Since $X$ is normal, it is geometrically G1 and S2, hence we can talk about divisors (or at worst Weil-divisorial sheaves) on base changes of $X$. We introduce the notation $\mathcal{R} := \mathcal{O}_X \otimes_k k^{1/p}$ and factor

$$X \xrightarrow{g} X' = \text{Spec} \mathcal{R} \xrightarrow{h} X$$

which is written on the structure sheaves as:

$$\mathcal{O}_{X'}^{1/p^a} \leftarrow \mathcal{O}_{X'} \cong \mathcal{O}_X \otimes_k k^{1/p} = \mathcal{R} \leftarrow \mathcal{O}_X.$$

By induction on $e > 0$, the fact that $\text{Spec} \mathcal{R}$ is $X$ as a topological space, and [SS, Theorem 3.9], it is enough to show that there is an integer $e > 0$ with $a|e$ and so the following natural inclusion is split:

$$\mathcal{R} \to \left( \mathcal{R} (h^* (p^e - 1)\Delta + h^* D) \right)^{1/p^e}.$$

Note that we may have to treat $h^* (p^e - 1)\Delta + h^* D$ as an effective Weil divisorial sheaf which is Cartier in codimension one, since we do
not know that \( \mathcal{R} \) is normal yet. Regardless, we have the following composition of Frobenius maps.

\[
\begin{align*}
\mathcal{O}_X & \rightarrow \mathcal{R} \\
& \rightarrow \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e} \\
& \rightarrow \left( \mathcal{O}_X ((p^{e+a} - 1)\Delta + p^aD) \right)^{1/p^{e+a}}
\end{align*}
\]

(1)

Here we used that \( g^*h^*D = (F^a)^*D = p^aD \) and that

\[
g^*h^*(p^e - 1)\Delta \cong (F^a)^*(p^e - 1)\Delta = (p^{e+a} - p^a)\Delta \leq (p^{e+a} - 1)\Delta
\]

We now argue as before and apply the functor \( \mathcal{H}om_{\mathcal{O}_X}(\quad, \mathcal{O}_X) \) to (1) and obtain:

\[
\begin{align*}
\mathcal{O}_X & \leftarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{O}_X) \\
& \cong \mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e}, \mathcal{O}_X \right) \\
& \cong \mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{O}_X ((p^{e+a} - 1)\Delta + p^aD) \right)^{1/p^{e+a}}, \mathcal{O}_X \right)
\end{align*}
\]

Notice that

\[
\begin{align*}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{O}_X) & = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_k k^{1/p}, \mathcal{O}_X) \\
& \cong \mathcal{O}_X \otimes \mathcal{H}om_k(k^{1/p}, k) \\
& \cong \mathcal{O}_X \otimes_k k^{1/p} \\
& = \mathcal{R},
\end{align*}
\]

here \( k \) is technically \( f^{-1}k \) where \( f : X \rightarrow \text{Spec}k \) is the structural map. Likewise

\[
\begin{align*}
\mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e}, \mathcal{O}_X \right) & \cong \mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{O}_X) \right) \\
& \cong \mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e}, \mathcal{R} \right)
\end{align*}
\]

and so we identify \( \beta \) with

\[
\mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{R} (h^*(p^e - 1)\Delta + h^*D) \right)^{1/p^e}, \mathcal{R} \right) \xrightarrow{\beta} \mathcal{R}.
\]

Since \( (X, \Delta) \) is globally \( F \)-regular, the canonical map

\[
\mathcal{H}om_{\mathcal{O}_X} \left( \left( \mathcal{O}_X ((p^e - 1)\Delta + D) \right)^{1/p^e}, \mathcal{O}_X \right) \rightarrow H^0(X, \mathcal{O}_X)
\]
is surjective for some $e > 0$ for which $a | e$ [SS, Proposition 3.8.a]. But then since $(X, \Delta)$ itself is globally $F$-regular and so globally $F$-split, we have a splitting: $(\mathcal{O}_X ((p^a - 1)\Delta))^{1/p^e} \to \mathcal{O}_X$. Twisting by the divisor $(p^e - 1)\Delta + D$, reflexifying, and taking $p^e$th roots, we obtain a splitting:

$$
(\mathcal{O}_X ((p^{e + a} - 1)\Delta + p^a D))^{1/p^{e + a}} \to (\mathcal{O}_X ((p^e - 1)\Delta + D))^{1/p^e}.
$$

We compose this with a splitting of $\mathcal{O}_X \to (\mathcal{O}_X ((p^e - 1)\Delta + D))^{1/p^e}$ to obtain a splitting of $\mathcal{O}_X \to (\mathcal{O}_X ((p^{e + a} - 1)\Delta + p^a D))^{1/p^{e + a}}$. Therefore, (2) $\text{Hom}_{\mathcal{O}_X} \left( (\mathcal{O}_X ((p^{e + a} - 1)\Delta + p^a D))^{1/p^{e + a}}, \mathcal{O}_X \right) \to H^0(X, \mathcal{O}_X)$ is surjective. This implies that the map through which (2) factors, $H^0(\beta) : \text{Hom}_{\mathcal{R}} \left( (\mathcal{R} (h^*(p^e - 1)\Delta + h^* D))^{1/p^e}, \mathcal{R} \right) \to H^0(X, \mathcal{R})$ is non-zero.

Just as before, $H^0(X, \mathcal{O}_X) \cong K \supseteq k$ is a separable field extension and so

$$
H^0(X, \mathcal{R}) = H^0(X, \mathcal{O}_X \otimes_k k^{1/p}) = H^0(X, \mathcal{O}_X) \otimes_k k^{1/p} \cong K \otimes_k k^{1/p}
$$

is a field. Further note that $\beta$ is a $\mathcal{R}$-module homomorphism, hence $H^0(\beta)$ is a $H^0(X, \mathcal{R})$-module homomorphism. Thus $H^0(\beta)$ is a non-zero homomorphism over a field with the target being a one-dimensional vector space. Hence $H^0(\beta)$ is surjective. This proves that $X \times_k k^{1/p}$ is globally $F$-regular.

**Remark 2.7.** Indeed, the proof of Proposition 2.6 in fact shows something stronger. If $D > 0$ is a divisor and $\mathcal{O}_X \to (\mathcal{O}_X ((p^e - 1)\Delta + D))^{1/p^e}$ splits for some $e$ divisible by $a$, then $\mathcal{R} \to (\mathcal{R} (h^*(p^e - 1)\Delta + h^* D))^{1/p^e}$ also splits for the same $e > 0$.

**Corollary 2.8.** Suppose that $(X, \Delta)$ is a proper, globally $F$-regular variety over an $F$-finite field $k$ such that $H^0(X, \mathcal{O}_X) \supseteq k$ is a separable field extension. Then $(X, \Delta)$ is geometrically globally $F$-regular. In other words, for every extension $k \subseteq K$ such that $K$ is also $F$-finite, we have that $(X \times_k K, \Delta \times_k K)$ is globally $F$-regular.

**Proof.** The proof is essentially the same as Corollary 2.5. We use the notation from the proof of Proposition 2.6, in particular $D$ is a divisor whose support contains the support of $\Delta$ and such that $X \setminus D$ is affine and smooth over $k$ and $e$ is such that $(p^e - 1)\Delta$ is integral. We also set
$R_e = \mathcal{O}_X \otimes_k k^{1/p^e}$ and $h_e : \text{Spec} R_e \to X$ to be the projection. Observe that

$$R_e \to (R_e(h^*(p^e - 1)\Delta + h^*D)^{1/p^e}$$

is split for some $e > 0$ by Remark 2.7.

But then $R_e \to (\mathcal{O}_X((p^e - 1)\Delta + D))^{1/p^e}$ is also split since it factors (3). Thus there is a map $\phi : (\mathcal{O}_X((p^e - 1)\Delta + D))^{1/p^e} \to R_e$ which sends $1$ to $1.$ It follows that

$$(\phi \otimes_k K^{1/p^e}) : (\mathcal{O}_X((p^e - 1)\Delta + D))^{1/p^e} \otimes K^{1/p^e} \to R_e \otimes_k \otimes K^{1/p^e}$$

also sends $1$ to $1.$ But obviously since $K \to K^{1/p^e}$ is split, so is the map $\mathcal{O}_X \otimes_k K \to \mathcal{O}_X \otimes_k K^{1/p^e}$ and so there is a map $\gamma : \mathcal{O}_X \otimes_k K^{1/p^e} \to \mathcal{O}_X \otimes_k K$ which also sends $1$ to $1.$ We compose and so obtain:

$$(\mathcal{O}_X((p^e - 1)\Delta + D) \otimes_k K)^{1/p^e} \xrightarrow{(\phi \otimes_k K^{1/p^e})} \mathcal{O}_X \otimes_k K^{1/p^e} \xrightarrow{\gamma} \mathcal{O}_X \otimes_k K.$$  

This implies that $X \times_k K$ is globally $F$-regular as desired. \hfill \Box

**Remark 2.9.** In the case that $X$ is additionally projective, we believe it is possible to prove the previous corollary with a variant of the following argument. Let $R$ denote the section ring of $X$ with respect to some ample divisor and let $S$ denote the section ring of $X \times_k K$ with respect to the pullback of the same ample divisor. We then have a map $R \to S.$ Note that $H^0(X, \mathcal{O}_X) \otimes_k K$ is the spectrum of the fiber of Spec $S$ over cone point of Spec $R.$ In particular, the fiber is regular. At this point, we can apply a variant of the argument of [SZ, Lemma 4.5], which is a generalization of [HH3, Section 7], use the strong $F$-regularity of $(R, \Delta_R)$ to conclude the strong $F$-regularity of $(S, \Delta_S).$

Now we come to the proof of our main theorem for this section.

**Proof of Theorem 2.1.** Set $\eta$ to be the generic point of $Y$ with residue field $K.$ Since $(X, \Delta)$ is globally $F$-regular, so is $(X_K, \Delta_K)$ where $X_K = X \times_Y \text{Spec} K$ and $\Delta_K$ is the pullback of $\Delta$ along the flat map $X_K \to X.$ Notice that since $f_*\mathcal{O}_X = \mathcal{O}_Y,$ we have that $f_*\mathcal{O}_{X_K} = K.$ In particular, it immediately follows from Corollary 2.8 that $(X_K, \Delta_K)$ is geometrically globally $F$-regular over $K$ and is hence geometrically normal (so that the general fibers are normal). Choose $A$ an effective divisor on $X$ containing the support of $\Delta$ such that $X \setminus A$ is affine and smooth over $K$ ($A$ can be relatively ample if $X \to Y$ is projective).
Using Proposition 2.6 and Remark 2.7, we know that for some \( e > 0 \), the map
\[
\begin{align*}
\mathcal{O}_{X^{1/p^e}} &\rightarrow \mathcal{O}_{X^K K^{1/p^e} / \mathcal{O}_{K^{1/p^e}}} \\
&\rightarrow \mathcal{O}_{X^{1/p^e} / \mathcal{O}_{K^{1/p^e}}} \\
&\rightarrow (\mathcal{O}_{X^{1/p^e}}((p^e - 1)\Delta_{K^{1/p^e}} + A_{K^{1/p^e}}))^{1/p^e}
\end{align*}
\]
splits. This map factors through the twisted relative Frobenius
\[
\mathcal{O}_{X^{1/p^e}} \rightarrow (\mathcal{O}_{X^K ((p^e - 1)\Delta_{K}) + A_{K}})^{1/p^e}
\]
which then also splits as well. By restricting the base \( Y = \text{Spec} B \), we may assume that \( B \) is affine and regular, that \( f : X \rightarrow Y \) is flat with geometrically normal fibers, and we may also assume that the map
\[
(4) \quad \mathcal{O}_{X_{X,Y,\text{Spec}B^{1/p^e}}} =: \mathcal{O}_{X_{B^{1/p^e}}} \rightarrow (\mathcal{O}_{X((p^e - 1)\Delta_{K}) + A})^{1/p^e}
\]
splits. Note that since \( X \) is normal, \( \Delta \) and \( A \) are Cartier in codimension 1 and so they remain Cartier in codimension 1 over an open set of fibers. By shrinking \( Y \) again, we may assume that \( \Delta \) and \( A \) are Cartier in codimension 1 on the fibers and in particular, we will have no trouble restricting them to the fibers if we reflexify (or equivalently take the \( \text{S}2 \)-ification).

For each perfect point \( y \in Y \), we tensor the splitting of (4) by \( \otimes_{A^{1/p^e}} k(y)^{1/p^e} \) and then reflexify with respect to \( \mathcal{O}_{X_y} \) and so obtain a splitting
\[
\mathcal{O}_{X_{k(y)^{1/p^e}}} \rightarrow (\mathcal{O}_{X_{k(y)}((p^e - 1)\Delta_y) + A_y})^{1/p^e}.
\]
Since the base field is perfect, \( X_y = X_{k(y)} = X_{k(y)^{1/p^e}} \). Also note that \( X_y \setminus A_y \) is smooth and \( \text{Supp} \Delta_y \) is contained in \( \text{Supp} A_y \) and so we have shown that \((X_y, \Delta_y)\) is strongly \( F \)-regular by [SS, Theorem 3.9].

\[ \square \]

**Remark 2.10.** One can replace the most of the proof of the above theorem with an application of [PSZ, Theorem C.(a)], at least in the case that the map \( f : X \rightarrow Y \) is projective (instead of merely proper).

### 3. The proof of the main theorem

First we show the existence of terminalization of globally \( F \)-regular threefolds.

**Proposition 3.1 (Terminalization).** Let \( k \) be an algebraically closed field \( k \) of characteristic \( p \geq 7 \). Let \( X \) be a projective globally \( F \)-regular
threefold over $k$. Then there exists a birational morphism $\pi : X' \to X$ from a projective $\mathbb{Q}$-factorial terminal globally $F$-regular threefold $X'$.

**Proof.** Take a log resolution $f : Y \to X$. Run a generalized $K_Y$-MMP over $X$. Then, we obtain a birational morphism $\pi : X' \to X$ from a projective $\mathbb{Q}$-factorial terminal threefold $X'$ such that $K_{X'}$ is nef over $X$. Let $K_{X'} + E = \pi^*K_X$. Then, $-E$ is $\pi$-nef and $\pi$-exceptional. By the negativity lemma, $E$ is effective. It follows easily that $(X', E)$ is globally $F$-regular since in fact the splitting used to prove global $F$-regularity on $X$ also extends to a splitting on $X'$ (here we crucially use that $E$ is effective, see for example [BS, Section 7.2]). In particular, $X'$ is also globally $F$-regular. \hfill $\square$

From [CTX] we can take a Mori fiber space for a variety with non pseudo-effective canonical divisor as follows:

**Theorem 3.2.** Let $k$ be an algebraically closed field of characteristic $p \geq 7$. Let $X$ be a projective $\mathbb{Q}$-factorial terminal threefold over $k$. Assume $K_X$ is not pseudo-effective. Then there exist a sequence of birational maps

$$X =: X_0 \overset{f_0}{\dasharrow} X_1 \overset{f_1}{\dasharrow} \cdots \overset{f_{N-1}}{\dasharrow} X_N =: X'$$

which satisfies the following properties.

1. Each $X_i$ is a a projective $\mathbb{Q}$-factorial terminal threefold.
2. For each $i$, there exist non-empty open subsets $\tilde{X}_i \subset X_i$ and $\tilde{X}_{i+1} \subset X_{i+1}$ such that codim$_{X_{i+1}}(X_{i+1} \setminus \tilde{X}_{i+1}) \geq 2$ and that the restriction $f_i|_{\tilde{X}_i} : \tilde{X}_i \dasharrow \tilde{X}_{i+1}$ is a projective surjective morphism.
3. There exists a surjective morphism $g : X' \to Y'$ such that

   - $Y'$ is a normal projective variety with $\dim X > \dim Y$,
   - $\rho(X'/Y') = 1$, and
   - $-K_{X'}$ is ample over $Y'$.

**Proof.** See [CTX]. \hfill $\square$

Using an MRCC fibration, we can see that Fano varieties with picard number one over uncountable fields are rationally chain connected.

**Proposition 3.3.** Let $k$ be an uncountable algebraically closed field. Let $X$ be a projective normal $\mathbb{Q}$-factorial variety over $k$. Then, the following assertions hold.

1. If $-K_X$ is big, then $X$ is uniruled.
2. If $-K_X$ is ample and $\rho(X) = 1$, then $X$ is rationally chain connected.
Proof. Let \( n := \dim X \).

(1) Taking general hyperplane sections, we can find a smooth curve \( C = H_1 \cap \cdots \cap H_{n-1} \) such that \( C \) does not intersect the singular locus of \( X \). It is sufficient to show that, for every \( c \in C \), there exists a rational curve \( c \in R \). Since \( C \) is a proper curve contained in the non-singular locus \( X_{\text{reg}} \), we see \( K_X \cdot C < 0 \). Then, the assertion follows from Bend and Break (cf. [Kollár2, Ch II, Theorem 5.8]).

(2) Take a maximally rationally chain connected (MRCC) fibration ([Kollár2, Ch IV, Section 5]):

\[
X \supset X' \xrightarrow{f} Y'.
\]

Note that \( X' \) is a non-empty open subset of \( X \) and \( f \) is proper. By (1) and [Kollár2, Ch IV, 5.2.1 Complement], we see \( \dim Y' < \dim X \). It is enough to show \( \dim Y' = 0 \). Thus, assume \( 0 < \dim Y' < \dim X \) and let us derive a contradiction. We can find a closed point \( y \in Y' \) and a non-zero effective Cartier divisor \( D_Y \) on \( Y \) such that \( y \notin \text{Supp}D_Y \). Fix a proper curve \( C_X \) in the fiber \( f^{-1}(y) \). Then, we see \( C_X \cap f^{-1}(D_Y) = \emptyset \).

Take a prime divisor \( D_X \) on \( X \) contained in the closure \( f^{-1}(D_Y) \) in \( X \). By construction, we see \( C_X \cap D_X = \emptyset \). On the other hand, since \( X \) is \( \mathbb{Q} \)-factorial and \( \rho(X) = 1 \), \( D_X \) must be a \( \mathbb{Q} \)-Cartier ample divisor. This is a contradiction. \( \square \)

The following theorem is the main theorem of this paper:

**Theorem 3.4.** Let \( k \) be an algebraically closed field of characteristic \( p \geq 11 \). Let \( X \) be a projective globally \( F \)-regular threefold over \( k \). Then, \( X \) is rationally chain connected.

**Proof.**

**Step 1.** We may assume \( k \) is uncountable by Lemma 1.9.

Moreover by Proposition 3.1, we may assume that \( X \) is \( \mathbb{Q} \)-factorial and terminal.

**Step 2.** In this step, we show that we may assume \( X \) has a Mori fiber space structure \( f : X \to Y \).

By Theorem 3.2, we obtain a birational map \( X \dasharrow X' \) where \( X' \) is also globally \( F \)-regular using property 3.2(2) and has a Mori fiber space structure \( X' \to Y' \). We claim that \( X \) is rationally chain connected if so is \( X' \).

Take a log resolution

\[
\xymatrix{ W \\
\alpha \ar[u] \quad \beta \ar[u] \ar[ll] \ar[rr] & & X' \\
X \ar[uu] & X' \ar[uu] \ar[l] \ar[r] & X'.
}\]
Then it suffices to show that $W$ is rationally chain connected. Set

$$E = \sum_{E_i: \beta\text{-exceptional}} E_i,$$

and run a $(K_W + E)$-MMP over $X'$. Since it holds that $\text{Supp} G = E$ for the equation

$$K_W + E = \beta^* K_{X'} + G,$$

this MMP terminates from the special termination theorem (see [HX, Section 5] and [Fujino, Theorem 4.2.1]). Note that log flips in this program exist by [HX, Theorem 1.1]. Since $X'$ is $\mathbb{Q}$-factorial, the end result of this MMP is isomorphic to $X'$.

Let $g : W \to W_1$ be a step of this MMP. Inductively we may assume that $W_1$ is rationally chain connected. We show the rationally connectedness of $W$ case by case:

**Case 1.** $g$ is a divisorial contraction and the exceptional divisor is contracted to a point.

**Case 2.** $g$ is a divisorial contraction and the exceptional divisor is contracted to a curve.

**Case 3.** $g$ is a flip.

Let $F$ be the exceptional prime divisor in Case 1 and Case 2 or a component of $E$ such that $C \subseteq F$ in Case 3, where $C$ is some flipping curve. By applying [HX, Theorem 3.1 and Proposition 4.1] to the pair $(W, E - \frac{1}{n}(E - F))$ for $n \gg 1$, it holds that $F$ is normal. We fix such a large integer $n \gg 1$. Note that, for $K_F + \Delta_F := (K_W + E - \frac{1}{n}(E - F))|_F$, $(F, \Delta_F)$ is klt.

Then, in Case 1, since $-(K_F + \Delta_F)$ is ample, $F$ is a rational surface, in particular it is rationally connected. Thus $W$ is rationally chain connected.

Next, for case 2, we consider the surjective morphism $h : F \to C$ to a curve $C$ which is induced by the Stein factorization theorem of $g_{|F}$. Since $-(K_F + \Delta_F)$ is $h$-ample, a general fiber $D$ of $h$ is $\mathbb{P}^1$. Thus every irreducible component of a closed fiber of $h$ is a rational curve. Therefore every irreducible component of a closed fiber of $g_{|F}$ is a rational curve. Thus $W$ is rationally chain connected.

In Case 3, let $C$ be an arbitrary flipping curve. We show $C$ is rational. It holds that $(K_F + \Delta_F) \cdot C < 0$ and $C^2 < 0$. In particular, the $\mathbb{R}_{\geq 0}[C]$ is an extremal ray of $\overline{\text{NE}}(F)$. Thus $C$ is a rational curve by [Tanaka, 3.3, 3.4]. Thus $W$ is rationally chain connected.
Step 3. By Step 2, we can assume that \( X \) has a Mori fiber structure \( f : X \to Y \).

Since \( X \) is a threefold, we will consider only when \( \dim Y = 0, 1, \) or \( 2 \). When \( \dim Y = 0 \), it holds that \( X \) is rationally chain connected by Theorem 3.3. Next we will consider the case where \( \dim Y = 1 \) and \( \dim Y = 2 \).

3.1. The case where \( \dim Y = 1 \). In this case, we use the following result of Hirokado, and also a result of de Jong–Starr for the proof:

**Theorem 3.5** ([Hirokado, Theorem 5.1(2)]). Let \( f : X \to C \) be a proper surjective morphism from a smooth threefold to a smooth curve with \( f_* \mathcal{O}_X = \mathcal{O}_C \). Suppose the following conditions.

1. \( p \geq 11 \).
2. A general fiber of \( f \) is normal.
3. The anti-canonical divisor of a general fiber is ample.

Then, general fibers are smooth.

**Theorem 3.6.** Let \( f : X \to C \) be a proper surjective morphism from a projective variety to a smooth projective curve \( C \). Suppose the general fiber of \( f \) is a smooth and separable rationally connected variety, then \( f \) has a section.

**Proof of Theorem 3.6.** Taking the flattening of \( f \), we may assume \( f \) is flat. Then the assertion follows from [DS, Theorem 5.1] (cf. [GHS, Theorem 1.1]). \( \square \)

This is the famous Graber–Harris–Starr theorem for separably rationally connected varieties. Moreover we use the fact that three dimensional terminal singularities are isolated:

**Proposition 3.7** ([Kollár3, Corollary 2.30]). Let \( X \) be a terminal threefold over an algebraically closed field of any characteristic. Then \( X \) has only isolated singularities.

**Proposition 3.8.** Let \( f : X \to Y \) be a projective surjective morphism between normal varieties with \( f_* \mathcal{O}_X = \mathcal{O}_Y \). Assume the following conditions.

1. \( p \geq 11 \).
2. \( X \) is a terminal globally \( F \)-regular threefold.
3. \( -K_X \) is \( f \)-ample.
4. \( \dim Y = 1 \).

Then, there is a section of \( f \), that is, \( s : Y \to X \) such that \( f \circ s = \text{id}_Y \).
Proof of Proposition 3.8. By Theorem 2.1, general fibers of $f$ are globally $F$-regular. In particular, they are normal. Since terminal singularities are isolated, by shrinking $Y$, we may assume $X$ is smooth. Thus, we can apply Theorem 3.5 and there is a smooth fiber $F$ of $f$. Then, Theorem 3.6 implies the assertion. □

Here we show the rational chain connectedness when $\dim Y = 1$. Indeed Lemma 1.3 implies $Y \simeq \mathbb{P}^1_k$. By Theorem 2.1, general fibers of $f$ are rational. By Proposition 3.8, $f : X \to Y$ has a section. Thus, $X$ is rationally chain connected.

3.2. The case where $\dim Y = 2$. First we show the following lemma:

Lemma 3.9. Let $f : X \to Y$ be a projective surjective morphism between normal varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.

(1) $X$ is a globally $F$-regular variety.
(2) $\dim X - \dim Y = 1$.

Then, there exists a non-empty open subset $Y' \subset Y$ which satisfies the following properties.

- Every fiber of $y \in Y'$ is $\mathbb{P}^1$.
- If $C_Y$ is a curve in $Y$ which intersects $Y'$, then $f^{-1}(C_Y) \to C_Y$ has a section.

Proof of Lemma 3.9. By shrinking $Y$, we may assume $f$ is flat. Moreover, by Theorem 2.1, we may assume that every fiber is $\mathbb{P}^1$. Thus, for every curve $C_Y$, general fibers of $S_Y := f^{-1}(C_Y) \to C_Y$ are $\mathbb{P}^1$. Take the base change of $S_Y$ to the normalization $C'_Y$ of $C_Y$. We obtain a flat projective morphism $S'_Y \to C'_Y$ whose general fibers are $\mathbb{P}^1$. Then, $S'_Y \to C'_Y$ has a section (cf. Theorem 3.6) and so does $S_Y = f^{-1}(C_Y) \to C_Y$. □

The rationally chain connectedness when $\dim Y = 2$ follows from the following proposition:

Proposition 3.10. Let $f : X \to Y$ be a surjective morphism between projective normal varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.

(1) $X$ is a globally $F$-regular threefold.
(2) $\dim Y = 2$.

Then, $X$ is rationally chain connected.

Proof of Proposition 3.10. We may assume that the base field $k$ is uncountable by the same argument as Step 1 and Corollary 2.8. Then, it
is sufficient to show that two general points \( x_1 \) and \( x_2 \) are connected by rational curves. Let \( Y' \) be a non-empty open subset as in Lemma 3.9. Fix \( x_1, x_2 \in f^{-1}(Y') \). Let \( y_1 := f(x_1) \) and \( y_2 := f(x_2) \). Since \( X \) is globally \( F \)-regular, so is \( Y' \). In particular, \( Y \) is a rational surface. Thus, there exists a rational curve \( C_Y \) on \( Y \) passing through \( y_1 \) and \( y_2 \). Then, \( x_1 \) and \( x_2 \) are connected by three rational curves: \( f^{-1}(y_1), f^{-1}(y_2) \) and a section of \( f^{-1}(C_Y) \to C_Y \). □

This completes the proof of Theorem 3.4. □

4. On separable rational connectedness

In Section 3, we treated the rational chain connectedness of globally \( F \)-regular threefolds. It is natural to ask whether they are separably rationally connected.

**Conjecture 4.1.** Let \( X \) be a globally \( F \)-regular variety over an algebraically closed field of positive characteristic. Then, \( X \) is separably rationally connected.

By MMP, we may assume that \( X \) has a Mori fiber space structure \( f : X \to Y \). Note that the property of separable rational connectedness is stable under birational maps. In this section, we treat the case where \( \dim Y = 1 \).

**Theorem 4.2.** Let \( k \) be an algebraically closed field of positive characteristic. Let \( f : X \to Y \) be a surjective morphism between projective normal varieties over \( k \) with \( f_*\mathcal{O}_X = \mathcal{O}_Y \). Assume the following conditions.

1. General fibers of \( f \) are smooth and separably rationally connected.
2. \( Y \simeq \mathbb{P}^1 \).

Then, \( X \) is separably rationally connected.

**Proof.** By Theorem 3.6, we can start with a section \( s : \mathbb{P}^1 \to X \). Then by Corollary 1.16, we can add enough very free curves \( C_1, \ldots, C_m, m \gg 0 \) in general fibers of \( f : X \to Y \) to form a comb: \( s(\mathbb{P}^1) \cup C_1 \cup \ldots \cup C_m \) which has a general smoothing to another section \( s_t : \mathbb{P}^1 \to X \) with \( H^1(s_t(\mathbb{P}^1), T_{X/\mathbb{P}^1}|_{s_t(\mathbb{P}^1)} \otimes \mathcal{O}_{\mathbb{P}^1}(d)) = 0 \) where \( |d| \leq 2 \) which means

\[ T_{X/\mathbb{P}^1}|_{s_t(\mathbb{P}^1)} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k) \]

with \( a_i > 0 \) for \( i = 1, \ldots, k \). And it is easy to see that the following sequence splits:

\[ 0 \to T_{X/\mathbb{P}^1}|_{s_t(\mathbb{P}^1)} \to T_X|_{s_t(\mathbb{P}^1)} \to T_{s_t(\mathbb{P}^1)} \to 0 \]
so $T_X|_{s_t(P^1)}$ also has positive direct summands and so $s_t(P^1)$ is a very free curve in $X$. \hfill \Box

As a consequence, we obtain the following result.

**Theorem 4.3.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $f : X \to Y$ be a projective surjective morphism between normal varieties over $k$ with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.

1. $p \geq 11$.
2. $X$ is a terminal globally $F$-regular threefold.
3. $-K_X$ is $f$-ample.
4. $\dim Y = 1$ or $\dim Y = 2$.

Then, $X$ is separably rationally connected.

**Proof.** In $\dim Y = 1$ case, since $Y$ is globally $F$-regular (Lemma 1.3), we see $Y \simeq P^1$. By Theorem 2.1, general fibers of $f$ are globally $F$-regular. In particular, they are normal and rational. Since terminal singularities are isolated, we can apply Theorem 3.5 and general fibers are smooth. Then, Theorem 4.2 implies the assertion.

In $\dim Y = 2$ case, since $Y$ is a globally $F$-regular surface, it is rational and there is a birational map $P^2 \dashrightarrow Y$. By factorization of birational maps between surfaces and base change, there is a birational map $X' \dashrightarrow X$ where $X'$ is a conic bundle over $P^2$, which is obviously separably rationally connected. \hfill \Box

## 5. Rational connectedness of varieties of globally $F$-regular type

In this section we show that a globally $F$-regular type variety is rationally connected.

**Lemma 5.1.** Let $X$ be a $\mathbb{Q}$-factorial normal projective variety of dense globally $F$-split type. Then $-K_X$ is pseudo-effective. Moreover, if $X$ is of globally $F$-regular type, $K_X$ is not pseudo-effective.

**Proof.** Assume that $-K_X$ is not pseudo-effective by contradiction. Then there exists covering curve $C$ such that $K_X.C > 0$. Taking a model $\mathcal{X} \to \text{Spec } A$

of reduction modulo positive characteristics, there exists a dense $S \subseteq \text{Spec } A$ such that $X_\mu$ is globally $F$-split and $C_\mu$ is also a movable curve of $X_\mu$ for any closed point $\mu \in S$. From Theorem 1.4, there exists an
effective \(\mathbb{Q}\)-divisor \(\Delta_\mu\) on \(X_\mu\) such that \((X_\mu, \Delta_\mu)\) is lc and \(K_{X_\mu} + \Delta_\mu \equiv 0\). We see also
\[
K_X.C = K_{X_\mu}.C_\mu = -\Delta.C_\mu \leq 0
\]
since \(C_\mu\) is movable. This is a contradiction to the assumption. For the latter statement, in particular, it holds that \(\Delta_\mu\) is big. Thus we see that \(K_X \not\equiv \mathbb{Q}\) 0. If \(K_X\) is pseudo-effective, then \(K_X \equiv 0\), moreover, \(K_X \sim_\mathbb{Q} 0\) from [Kawamata], [CKP], and [Gongyo, Theorem 1.2] or the fact that \(H^1(X, \mathcal{O}_X) = 0\). Therefore, \(K_X\) is not pseudo-effective. \(\square\)

**Corollary 5.2.** Let \(X\) be a \(\mathbb{Q}\)-factorial normal projective variety of dense globally \(F\)-split type. Then \(X\) is uniruled or a canonical variety with \(K_X \sim_\mathbb{Q} 0\).

**Proof.** Assume \(X\) is not uniruled. First we see that \(X\) has only log terminal singularities. Indeed take a dlt blow-up \(f : Y \to X\) such that \(f^*K_X = K_Y + \Gamma\). Then \(K_Y \sim_\mathbb{Q} 0\) since \(Y\) is not uniruled and \(-K_Y\) is pseudo–effective. Thus \(\Gamma = 0\) from pseudo-effectiveness of \(-(K_Y + \Gamma)\). Thus \(X\) is log terminal by the definition of dlt blow-up. Next take a terminalization. By the similar argument, we see that \(X\) is canonical. Thus we see Corollary 5.2. \(\square\)

**Theorem 5.3.** Let \(X\) be a \(\mathbb{Q}\)-Gorenstein normal projective variety of globally \(F\)-regular type. Then \(X\) is rationally connected.

**Proof.** We show by induction on dimension. Taking a small \(\mathbb{Q}\)-factorization ([BCHM]), we may assume that \(X\) is \(\mathbb{Q}\)-factorial.

We know \(K_X\) is not pseudo-effective from Lemma 5.1 and \(X\) has a log terminal singularities from Theorem 1.7 (2). By [BCHM], there exists a minimal model program terminating in a Mori fiber space
\[
f : X' \to Y.
\]
In particular, we know also \(X'\) and \(Y\) are of globally \(F\)-regular type (cf. see Proposition 1.3). We see that \(Y\) is also \(\mathbb{Q}\)-factorial globally \(F\)-regular. By the hypothesis of induction, \(Y\) is rationally connected. Thus since we apply [GHS] for \(f\), by rationally connectedness of a general fiber of \(f\), we see that \(X\) is also rationally connected. \(\square\)

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Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail address: gongyo@ms.u-tokyo.ac.jp

Department of Mathematics, Imperial College London, 180 Queen’s Gate, London SW7 2AZ, UK.
E-mail address: y.gongyo@imperial.ac.uk
Room 382-B, Department of Mathematics, Building 380, Stanford, CA 94305.
E-mail address: zli2@stanford.edu

Princeton University, Department of Mathematics, Phone: (206) 436-9676, Fine Hall, Washington Road, Princeton, NJ 08544-1000.
E-mail address: pzs@princeton.edu

McAllister Hall 318C, Department of Mathematics, Penn State University, University Park, PA 16802 USA, Office Phone: 814-865-8439.
E-mail address: schwede@math.psu.edu

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502 Japan.
E-mail address: tanakahi@math.kyoto-u.ac.jp

Room A5 Fine Hall, Washington Road, Department of Mathematics, Princeton University, Princeton NJ 08544 US.
E-mail address: rzong@math.princeton.edu