Classical and Quantum Considerations of Two-dimensional Gravity

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\textbf{Abstract}

The two-dimensional theory of gravity describing a graviton-dilaton system is considered. The graviton-dilaton coupling can be fixed such that the quantum theory remains free of the conformal anomaly for any conformal dimension of the coupled matter system, even if the dilaton does not appear as Lagrange multiplier. Interaction terms are introduced and the system is analyzed and solutions are given at the classical level and at the quantum level by using canonical quantization.

\textsuperscript{a)} Supported by the Swiss National Science Foundation
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1. Introduction

During the last few years some progress was made towards understanding and resolving the problems associated with two-dimensional gravity. (For a review see [1].) A big effort has been invested in this area after Polyakov revitalized the Liouville action by quantizing it in the light-cone gauge [2,3]. The quantum theory of the Liouville action was studied some time ago in [4] and recently its appearance at the quantum level was understood from first principles [5]. However, this approach of assuming the dynamics of the metric to be governed by the Liouville action alone had to stop at the so called $d = 1$-barrier: quantization turns out to be inconsistent if the conformal dimension of the coupled matter is $1 < d < 25$ [2,3]. This seems to indicate that some essential ingredient is missing.

Here, we want to promote the idea that the missing ingredient is the dilaton. Actually, many classical (super-)gravity theories in higher dimensions are known to be consistent only if a dilaton $\phi$ and an antisymmetric tensor field $b$ accompanies the metric $g$. In two dimensions the field strength $H = db$ vanishes and so here we need to consider only the dilaton. Indeed, in [6,7] the dilaton has been seen to solve the problem of two-dimensional gravity. There the dilaton was introduced as a Langrange multiplier that implies a constraint of constant curvature and thus takes the Liouville action on-shell. Then quantization of string theory becomes possible for any dimension $d$ and the explicit result for the partition function could be given on any genus [6]. The concept of constraining the system to constant curvature has also been treated by Jackiw and Teitelboim [8]. The constraint of constant curvature, however, is not essential for the dilaton solution of the Liouville problem. Here, we introduce dilaton self-couplings that destroy its role as a Lagrange multiplier. Nevertheless, the graviton-dilaton coupling can be fixed in such a way that the quantum theory remains free of the conformal anomaly.

Our main concern is to study the two-dimensional graviton-dilaton-matter system [6-11] in the presence of interaction terms both classically and quantum mechanically. In section 2 we review aspects of the dilaton solution to the Liouville problem and expose how dilaton self-couplings have to be introduced. In section 3 we consider the gravity action as a classical system on its own right and give solutions. In section 4 we include the Liouville contributions and consider the full quantum system by canonical quantization. We also give the Wheeler-De Witt equation and solutions.
2. The Dilaton Solution of the Liouville Problem

We begin with the classical action
\[ S(X, g, \phi) = S_g(g, \phi) + S_m(X, g) \]  
(2.1)

Here \( S_m \) describes the matter part of conformal dimension \( d \) and \( S_g \) is the gravity contribution
\[ S_g(g, \phi) = S_{EH}(g) + S_D(\phi, g) \]  
(2.2)

with Einstein-Hilbert and dilaton action:
\[ S_{EH}(g) = \frac{1}{2\pi} \int_M d^2x \sqrt{g} (\eta R_g + \mu) \]  
(2.3)
\[ S_D(\phi, g) = \frac{1}{2\pi} \int_M d^2x \sqrt{g} (b\phi R_g + \lambda e^\phi) \]  
(2.4)

where \( \eta, b, \mu \) and \( \lambda \) are constants. The first term in (2.3) gives the Euler number \( \chi(M) = 2(1 - h) \) for a closed and orientable manifold \( M \) with \( h \) handles, the second is the cosmological constant. The first term in (2.4) is the usual coupling of the dilaton \( \phi \) to curvature, the last term introduces self-couplings. Notice that with given normalization of the exponent in (2.4) the \( b \) can be fixed by quantum consistency. Moreover, with self-coupling the dilaton becomes non-trivial. Its role as a Lagrange multiplier that was essential in [6,7] for computing the partition function is lost. The dilaton will no longer trivialize the Liouville problem. Nevertheless, the \( b \) can be fixed such that the self-coupling is of weight (1,1) which implies that the theory remains free of the conformal anomaly. (For a discussion of including weight (1,1) fields into an action see [12].)

Classically, a kinetic term for \( \phi \) can easily be introduced in (2.4) by rescaling \( g_{\alpha\beta} \rightarrow e^\phi g_{\alpha\beta} \). For the quantum theory, however, it is essential that we quantize the gravity action in the form (2.4). Otherwise the usual Liouville problems will re-appear.

The appearence of the Liouville action at the quantum level is most easily seen from the path integral approach. We work in the conformal gauge, where part of the reparametrization invariance is fixed by requiring
\[ g_{\alpha\beta}(x) = e^{2\sigma(x)} \hat{g}^{\alpha\beta}(x) \]  
(2.5)

As usual the \( \hat{g} \) is a representative in the conformal class of \( g \). These classes are specified by the Teichmüller parameter \( \tau \) that will not be written in the following.
The field $\sigma$ is the Liouville mode and will be quantized while $\hat{g}_{\alpha\beta}$ is a background field (see [1]). A useful relation is

$$R_g = e^{-2\sigma} (R_{\hat{g}} + \Delta_{\hat{g}} \sigma)$$  \hspace{1cm} (2.6)$$

with $\Delta_{\hat{g}} = \frac{1}{\sqrt{g}} \partial_{\alpha} \sqrt{\hat{g}} \hat{g}^{\alpha\beta} \partial_{\beta}$.

Going to the path integral we begin with the measure expressed in terms of $g_{\alpha\beta}$. Pulling back this measure to $\hat{g}_{\alpha\beta}$ introduces the Liouville action:

$$D_g \sigma D_g \phi D_g X |det_{g}^{FP}| = D_{\hat{g}} \sigma D_{\hat{g}} \phi D_{\hat{g}} X |det_{\hat{g}}^{FP}| e^{-\frac{24-d}{12} S_L(\sigma)}$$  \hspace{1cm} (2.7)$$

where we included the Fadeev-Popov determinant arising from the gauge fixing (2.5) and the Liouville action

$$S_L(\sigma) = \frac{1}{\pi} \int_M d^2 x \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + \sigma R_{\hat{g}} + \kappa \exp 2\sigma \right)$$  \hspace{1cm} (2.8)$$

Without the dilaton the coefficient of $S_L$ in (2.7) would be the familiar $-\frac{25-d}{12}$ of Liouville theory [5]. (If the Liouville mode is not quantized, the coefficient is $-\frac{26-d}{12}$.) The change of this coefficient in the presence of the dilaton can be understood along the lines of [5]. The natural scalar product for a variation $\delta \phi(x)$ is

$$||\delta \phi(x)||_g^2 = \int_M d^2 x \sqrt{g} e^{2\sigma} \delta \phi \delta \phi$$  \hspace{1cm} (2.9)$$

Therefore in the background metric $\hat{g}_{\alpha\beta}$ the $D_g \phi$ comes with the measure $e^\sigma \delta \phi$. Going to $D_{\hat{g}} \phi$ with the measure $\delta \phi$ only does then imply

$$D_g \phi = D_{\hat{g}} \phi |J|$$  \hspace{1cm} (2.10)$$

where $|J|$ is the determinant of

$$J(x, y) = \frac{e^{\sigma(x)} \delta \phi(x)}{\delta \phi(y)} = e^{\sigma(x)} \delta^{(2)}(x - y)$$  \hspace{1cm} (2.11)$$

The $|J|$ is the Jacobian of the transformation. For to find its $\sigma$-dependence we look at the variation $\sigma \to \sigma + \delta \sigma$. Then

$$\delta \ln |J| = \delta Tr \ln J = \int_M d^2 x \delta^{(2)}(0) \sigma(x)$$  \hspace{1cm} (2.12)$$
We then need a regularization coming from the heat kernel expansion:

\[ \delta^{(2)}(0) = \sqrt{g} \left( \frac{1}{4\pi\epsilon} + \frac{1}{12\pi} R_g \right) \]

\[ = \sqrt{\hat{g}} \left( \frac{e^{2\sigma}}{4\pi\epsilon} + \frac{1}{12\pi} (R_{\hat{g}} + \Delta_{\hat{g}} \sigma) \right) \]

(2.13)

with \( \epsilon \to 0 \). Substituting this into (2.12) implies

\[ |J| = e^{\frac{1}{12} S_L(\sigma)} \]

(2.14)

and this explains the shift in the Liouville coefficient of (2.17). We see that the cosmological constant in \( S_L \) is divergent. The same holds for the contributions of the other measures that can be calculated similarly to (2.9-14). A counterterm has to be introduced that cancels this divergence.

Because of (2.7), the action (2.2) is accompanied by the Liouville terms at the quantum level to give the effective action:

\[ \frac{1}{\alpha'} S^e_f g(\gamma \sigma, \gamma^2 \phi) = \frac{1}{2\pi} \int_M d^2 x \sqrt{\hat{g}} \left( a \hat{g}^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + \frac{Q}{\gamma} \sigma R_{\hat{g}} + \frac{1}{\gamma^2} V(\gamma \sigma, \gamma^2 \phi) \right) \]

\[ + b \left( \partial_\alpha \partial_\beta \phi - \delta_{\alpha\beta} \partial^2 \phi - \gamma \partial_\alpha \phi \partial_\beta \sigma - \gamma \partial_\beta \phi \partial_\alpha \sigma \right) + \frac{\eta}{\gamma^2} \chi(M) \]

(2.15)

where

\[ V(\gamma \sigma, \gamma^2 \phi) = \mu e^{2\gamma \sigma} + \lambda e^{2\gamma \sigma + \gamma^2 \phi} \]

(2.16)

In (2.15) we kept the coefficients arbitrary. We included the inverse string tension \( \alpha' \equiv \gamma^2 \) and rescaled \( \sigma, \phi \to \gamma \sigma, \gamma^2 \phi \). Following the procedure of Kawai and Distler [3] the coefficients in (2.15) can be fixed by conformal field theory (CFT). The energy momentum tensor \( \hat{T}^g_{\alpha\beta} = \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}} \) that follows from (2.15) is

\[ \hat{T}^g_{\alpha\beta} = \delta_{\alpha\beta} \left( a \partial^\gamma \sigma \partial_\gamma \sigma + b \gamma \partial^\gamma \phi \partial_\gamma \sigma + \frac{1}{\gamma^2} V(\gamma \sigma, \gamma^2 \phi) \right) \]

\[ - 2a \partial_\alpha \sigma \partial_\beta \sigma + \frac{Q}{\gamma} \left( \partial_\alpha \partial_\beta \sigma - \delta_{\alpha\beta} \partial^2 \sigma \right) \]

\[ + b \left( \partial_\alpha \partial_\beta \phi - \delta_{\alpha\beta} \partial^2 \phi - \gamma \partial_\alpha \phi \partial_\beta \sigma - \gamma \partial_\beta \phi \partial_\alpha \sigma \right) \]

(2.17)

where we have chosen \( \hat{g}_{\alpha\beta} = \delta_{\alpha\beta} \) on a local coordinate patch. We first assume \( V = 0 \) so that the \( \sigma - \phi \) system is conformal. Going to the coordinates \( z = \frac{1}{\sqrt{2}} (x^1 + ix^2) \), \( \bar{z} = \frac{1}{\sqrt{2}} (x^1 - ix^2) \) we get the operator product expansion (OPE)

\[ \hat{T}^g(z)\hat{T}^g(w) = \frac{\frac{1}{2} c G}{(z - w)^4} + \left[ \frac{2}{(z - w)^2} + \frac{\partial_w}{z - w} \right] \hat{T}(w) + ... \]

(2.18)
where \( \widehat{T} \equiv \widehat{T}_{zz} \) and the conformal anomaly of the effective gravity system is

\[
c_G = 2 + \frac{12}{\gamma^2} (Q - a)
\]

(2.19)

If we take the value \( a \) from (2.7) and require the anomaly (2.19) to cancel the anomaly \( d - 26 \) arising from matter and gauge fixing, we obtain

\[
Q = 2a, \quad a = \alpha' \frac{24 - d}{12}
\]

(2.20)

Thus, we find the \( \sigma R_g \) coefficient not to be renormalized. This is already an improvement compared to Liouville theory without dilaton since any renormalization of the \( \sigma R_g \)-term destroys the geometrical character of the theory. Alternatively, we could have required this geometrical content not to be broken at the quantum level and (2.19) would have implied the \( a \) value given in (2.20).

For \( V \neq 0 \) the conformal anomaly still vanishes if the new terms are of weight (1,1). (For a discussion of this point see [12].) We have to look at the OPE

\[
\widehat{T}(z) : e^{\gamma \sigma + \alpha \phi} := \left[ \frac{h}{(z-w)^2} + \frac{\partial w}{z-w} \right] : e^{\gamma \sigma + \alpha \phi} : (2.21)
\]

where the conformal weight \( h \) of the exponential is

\[
h = 1 + \frac{\alpha}{b \gamma} \left( \frac{Q-2a}{2\gamma} - \gamma + \frac{a}{2b \gamma} \alpha \right)
\]

(2.22)

Using (2.20) the conformal weight of \( V \) will be (1,1) if we require

\[
b = \alpha' \frac{24 - d}{24} = \frac{a}{2}
\]

(2.23)

It is remarkable to see that the \( \phi R_g \) coupling is of order \( \alpha' \), and thus a one-loop effect. This is similar to the \( \alpha' \) expansion of the effective string theory [13] where the \( \phi R \) term is of one order higher than the matter term. The most important point to realize is that contrary to Liouville theory without dilaton, there is no gravitational dressing of the cosmological constant term and (2.22) introduces no restriction on the dimension \( d \).

Without self-couplings, \( \lambda = 0 \), the \( \phi \) can be integrated out and introduces the constraint of constant curvature. Then topological consistency requires to replace

\[
R \rightarrow R + \Lambda
\]

(2.24)

in (2.4) if \( h \neq 1 \) and we are back to the theory considered in [6,7]. From the conformal field theory point of view, the new term \( \Lambda \phi e^{2\sigma} \) can be made a field of weight (1,1)
if we also add \(4\Lambda\sigma e^{2\sigma}\) [7]. Such a \(\Lambda\)-term describes a possible renormalization after replacing (2.24), but the \(4\Lambda\sigma e^{\sigma}\) seems strange. It can, however, arise from the linear part in \(\Lambda\) of the non-local action

\[
\frac{a}{2\pi} \int d^2x \sqrt{g} (R + \Lambda) \Delta^{-1}(R + \Lambda)
\]

(2.25)

It is not clear to us whether such a combination does arise in the non-perturbative calculation of the effective action.

It is possible to compare the action (2.2) with that of an effective string theory of a two-dimensional target manifold [13], by rescaling

\[
g_{\alpha\beta} = \phi g'_{\alpha\beta}, \quad \phi = e^{-2\varphi}
\]

(2.26)

Then with \(\lambda = 0\) and replacement (2.24) in (2.4) the action (2.2) takes the form

\[
S_g = \frac{b}{2\pi} \int_M d^2x \sqrt{g} \left[ e^{-2\varphi} \left( R' + 2g'^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + \frac{\mu}{b} \right) + \Lambda e^{-4\varphi} \right] + \eta \chi(M)
\]

(2.27)

This is known to be the two-dimensional string effective action with a dilaton potential \(\frac{b}{8} e^{-2\varphi} + \Lambda e^{-4\varphi}\). Apart from quantum consistency, it should also be obvious why the rescaled action (2.27) will not be considered. For example the \(\lambda\)-term in (2.4) would become \(\lambda \exp(-2\varphi + \exp(-2\varphi))\)!

If the coupled matter system is a string theory in a \(d\)-dimensional Euclidean background:

\[
S_m(X, g) = -\frac{1}{8\pi} \int_M d^2x \sqrt{g} g^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^i
\]

(2.28)

the \(\sigma - \phi\) system leads to a \(D = d + 2\) dimensional string theory with one time direction. Absorbing the coefficient \(a\) in \(\sigma\) and \(\phi\) gives back the critical \(D = 26\) string in the limit \(d \to 24\).

In summary, we have seen that (2.1) can be consistently quantized for any \(d\), provided that \(b\) is given by (2.23). We will now analyze the system (2.1) both at the classical and the quantum level.

### 3. The Classical Analysis

Before considering the full quantum theory we analyze the system (2.1) as a classical system, i.e. without including the Liouville terms induced from quantization.
Classically, the coefficient $b$ is arbitrary. We rescale $\phi \rightarrow \frac{1}{b} \phi$. Also the case $\lambda = 0$ will be discussed and so we use the replacement (2.24). For definiteness we take the matter action to be given by (2.28).

The variation of the action with respect to $g_{\alpha\beta}$ gives the energy-momentum tensor $T_{\alpha\beta} \equiv \frac{4\pi \delta S}{\sqrt{g}} \delta g^{\alpha\beta}$:

$$T_{\alpha\beta} = \nabla_\alpha \nabla_\beta \phi - \frac{1}{2} \partial_\alpha X^i \partial_\beta X^i$$

$$+ g_{\alpha\beta}(-\nabla^\gamma \nabla_\gamma \phi + \Lambda \phi + \mu + \lambda e^{\phi/b} + \frac{1}{4} g^{\gamma\delta} \partial_\gamma X^i \partial_\delta X^i)$$

The $g_{\alpha\beta}, \phi$ and $X^i$ equations of motion are

$$T_{\alpha\beta} = 0 \quad (3.2)$$

$$R + \Lambda + \frac{\lambda}{b} e^{\phi/b} = 0 \quad (3.3)$$

$$\partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta X^i) = 0 \quad (3.4)$$

One interesting case occurs when $\lambda = 0$ and the genus of the manifold $h \neq 1$. Then for consistency $\Lambda$ must be different than zero and with a fixed sign. We work in the conformal gauge (2.5) where $\hat{g}_{\alpha\beta}$ is a background metric chosen so that

$$\hat{g}_{zz} = \frac{2}{(1 + z\bar{z})^2} \quad \text{if} \quad h = 0, R_{\hat{g}} = 1$$

$$= 1 \quad \text{if} \quad h = 1, R_{\hat{g}} = 0$$

$$= \frac{-2}{(z - \bar{z})^2} \quad \text{if} \quad h \geq 2, R_{\hat{g}} = -1$$

If $h = 0$ then $\Lambda < 0$, and the constant curvature condition in (3.3) has the solution

$$e^{2\sigma} = -\frac{1}{\Lambda} (1 + \bar{z}z)^2 \frac{f' \bar{f}'}{(1 + f\bar{f})^2} \quad (3.6)$$

A line element is then given by

$$ds^2 = -\frac{4}{\Lambda} \frac{df d\bar{f}}{(1 + f\bar{f})^2} \quad (3.7)$$

and the conformal gauge can be fixed completely by choosing $f(z) = z$, i.e. $e^{2\sigma} = -\frac{1}{\Lambda}$. Similarly when $\Lambda > 0$, $e^{2\sigma} = \frac{1}{\Lambda}$ and when $\Lambda = 0$, $\sigma = 0$. The three components of $T_{\alpha\beta}$ in eq.(3.2) (when $\Lambda < 0$) give

$$\partial \bar{\partial} \phi + \frac{2}{(1 + z\bar{z})^2} (\phi + \frac{\mu}{\Lambda}) = 0$$

$$\partial^2 \phi + \frac{2\bar{z}}{(1 + z\bar{z})^2} \partial \phi - \frac{1}{2} \partial X^i \partial X^i = 0 \quad (3.8)$$

$$\bar{\partial}^2 \phi + \frac{2z}{(1 + z\bar{z})^2} \bar{\partial} \phi - \frac{1}{2} \bar{\partial} X^i \bar{\partial} X^i = 0$$
The solution of the first equation involves an arbitrary holomorphic function $u(z)$ and its conjugate $\bar{u}$ [14]:

$$\phi = u(z) + \frac{1 - z\bar{z}}{1 + z\bar{z}} \int^z \frac{dw}{w} u(w) - \frac{\mu}{2\Lambda} + c.c. \quad (3.9)$$

The solution in (3.9) must be subject to the constraints imposed by the last two equations in (3.8), which imply

$$\partial^2 u + \frac{\partial u}{z} - \frac{u}{z\bar{z}} = \frac{1}{2} \partial X^i \partial X^i \quad (3.10)$$

and similarly for $\bar{u}$. Equation (3.4) simplifies to

$$\partial \bar{\partial} X^i = 0 \quad (3.11)$$

and the $X^i$ splits into holomorphic and antiholomorphic parts. In the absence of matter ($X^i = 0$) the solution of (3.10) can be found immediately to be

$$u = \frac{\alpha}{z} + \beta z \quad (3.12)$$

and the dilaton field $\phi(z, \bar{z})$ can be evaluated from (3.9) to be

$$\phi = -\frac{\mu}{2\Lambda} + \frac{2}{1 + z\bar{z}}(-\alpha \bar{z} + \beta z) + c.c. \quad (3.13)$$

In the presence of matter, the holomorphic part of $X^i$ is arbitrary, and the $u(z)$ can be solved in function of it from eq (3.10). The general solution is

$$u(z) = \frac{1}{c_1 + c_2} (c_1 u_1(z) + c_2 u_2(z)) \quad (3.14)$$

where

$$u_1(z) = \frac{1}{2} \int^z dz' z' \int^{z'} dz'' \partial X^i(z'') \partial X^i(z'')$$

$$u_2(z) = \frac{1}{2} \int^z dz' z'^3 \int^{z'} dz'' z''^2 \partial X^i(z'') \partial X^i(z'') \quad (3.15)$$

and $c_1$ and $c_2$ are arbitrary constants. To compare with the much simpler case when $\Lambda = 0$, the $T_{\alpha\beta}$ equations in (3.2) simplify to

$$\partial \bar{\partial} \phi = \mu$$

$$\partial^2 \phi = \frac{1}{2} \partial X^i \partial X^i$$

$$\bar{\partial}^2 \phi = \frac{1}{2} \bar{\partial} X^i \bar{\partial} X^i \quad (3.16)$$
The solution of (3.16) is easily seen to be

\[ \phi(z, \bar{z}) = \mu z \bar{z} + g(z) + \bar{g}(z) \]

\[ g(z) = \frac{1}{2} \int z \int z' \int z'' \partial X^i(z') \partial X^i(z'') \tag{3.17} \]

In the absence of matter \((X^i = 0)\), \(g(z)\) simplifies to

\[ g(z) = dz + e \tag{3.18} \]

where \(d\) and \(e\) are arbitrary constants. By shifting the coordinate \(z\) by a constant, the dilaton \(\phi\) can be put into the form [11]

\[ \phi(z, \bar{z}) = \mu z \bar{z} + c \tag{3.19} \]

This can be seen to be the familiar black hole solution [15] by recalling equation (2.26) where the metric of the effective 2d string theory is \(g'_{\alpha\beta} = \phi^{-1} g_{\alpha\beta}\) making the line element

\[ ds^2 = \frac{2dz d\bar{z}}{\mu z \bar{z} + c} \tag{3.20} \]

In this sense the solution in (3.13) can be considered as a generalization of the black hole solution with non-trivial topology. The case \(\Lambda > 0\) corresponding to \(h \geq 2\) can be treated along the same lines, and gives the solution

\[ \phi = u + \bar{u} - \frac{\mu}{\Lambda} \left[ \frac{z + \bar{z}}{z - \bar{z}} \right] - \int z \frac{dz'}{z'} u + \int \bar{z} \frac{d\bar{z}'}{\bar{z}'} \bar{u} \tag{3.21} \]

where \(u(z)\) satisfies eq. (3.10) and is solved by eqs. (3.14) and (3.15).

We now turn to the general case with \(\lambda\) and \(\Lambda \neq 0\). The constant curvature constraint is removed and we do not have to worry about the topological consistency. Notice that in this case eq. (3.3) can be solved for \(\phi\) in terms of \(R\):

\[ \phi = -b \left[ \ln(R + \Lambda) + \ln\left(-\frac{b}{\lambda}\right) \right] \tag{3.22} \]

and when this is substituted back into the action we get for the gravity part

\[ -\frac{b}{2\pi} \int d^2x \sqrt{g} \left[ (R + \Lambda)(1 + \ln(-\frac{b}{\lambda}) + \ln(R + \Lambda)) + \frac{u}{b} \right] \tag{3.23} \]

In this form the gravitational action becomes completely geometrical. In the conformal gauge it is a function of the Liouville mode. The comparison of the dynamical degrees of freedom in the actions (2.2) and (3.23) is amusing. In the first there is only
a mixed kinetic term of the form $\partial_\alpha \phi \partial^\alpha \sigma$. Perturbatively it is a scalar field-ghost combination, so the net degrees of freedom is zero. In (3.23) only the field $\sigma$ appears and has a kinetic term. However, if one expands perturbatively in $R$, the propagator corresponds to $\frac{1}{k^4}$ which can be written

$$\frac{1}{k^4 + \epsilon^2} = \frac{i}{2\epsilon} \left[ \frac{1}{k^2 + i\epsilon} - \frac{1}{k^2 - i\epsilon} \right]$$

The dynamical degree of this system is also zero.

We shall now solve the classical system (2.2) with (2.24). There is no constraint of constant curvature. First, we look for the solution of (3.2). The full solution can be found exactly in the absence of matter ($X^i = 0$). If $\phi$ is constant the classical $g_{\alpha\beta}$ solutions are of constant curvature. If $\phi$ is not constant then $\xi_\alpha = \frac{1}{\sqrt{g}} \epsilon_{\alpha\beta} \nabla^\beta \phi$ is a Killing vector [9] since (3.2) implies that $\nabla_\alpha \nabla_\beta \phi$ is proportional to $g_{\alpha\beta}$ and therefore

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

(3.24)

$$\xi_\alpha \nabla^\alpha \phi = 0$$

(3.25)

The Killing vector $\xi$ can be used to introduce new coordinates $x, y$ where $y$ is in the direction of $\xi_\alpha$ and thus ignorable while $x$ is orthogonal, with the line element:

$$ds^2 = g_{xx}(x)dx^2 + g_{yy}(x)dy^2$$

(3.26)

In these coordinates the only non-vanishing Christoffel symbols are $\Gamma^x_{xx}, \Gamma^x_{yy}$ and $\Gamma^y_{xy}$ and the $T_{xx}$ and $T_{yy}$ components of (3.2) give

$$g_{yy,x} \partial_x \phi = 2g_{xx}g_{yy}V(\phi)$$

$$g_{xx,x} \partial_x \phi = 2g_{xx}[\partial_x^2 \phi - g_{xx}V'(\phi)]$$

(3.27)

where

$$V'(\phi) = \Lambda \phi + \mu + \lambda e^{\phi/b}$$

(3.28)

while $T_{xy}$ is trivially zero. Using the freedom of a coordinate transformation $x \to x'(x)$, we can choose

$$g_{yy} = g^{xx} = h(x)$$

(3.29)

with this choice eqs. (3.27) simplify to

$$\partial_x^2 \phi = 0$$

$$\partial_x h \partial_x \phi = 2V'(\phi)$$

(3.30)
and this can be immediately solved to give
\[
\phi(x) = Ax + B
\]
\[
h(x) = \frac{2}{A} [\mu x + \Lambda (\frac{1}{2} Ax^2 + Bx) + \frac{\lambda}{A} e^\frac{1}{A}(Ax+B)]
\]
(3.31)

Moreover, the solution in (3.31) satisfies eq. (3.3) since \( R = -\frac{1}{2} \partial_x^2 h \).

Let
\[
z = F(x) e^{iy}
\]
\[
\bar{z} = F(x) e^{-iy}
\]
(3.32)
The transformation function \( F(x) \) in (3.32) and the Liouville mode \( \sigma(x) \) can be found from
\[
d s^2 = 2 e^{2\sigma} dz d\bar{z}
\]
\[
= 2 e^{2\sigma} [(\partial_x^2 F)^2 dx^2 + F^2 dy^2]
\]
(3.33)
Comparing with (3.26) and (3.29) we deduce that
\[
F(x) = F_0 e^{ \pm \int^x \frac{d\bar{z}}{k(x')}}
\]
\[
e^{2\sigma(x)} = \frac{h(x)}{2F^2(x)}
\]
(3.34)

Obviously the inverse transformation from the (x,y) coordinates to the (z,\( \bar{z} \)) coordinates is very complicated. The solutions in (3.31) and (3.34) are also a generalization of the black hole solution in eq. (3.19). It is remarkable that an exact solution for the complicated geometrical theory in (2.2) can be found. This would not have been possible without using the action in the form (2.2) where the field \( \phi(x) \) is not integrated out.

4. The Quantum Analysis

We now turn to the quantum study of two-dimensional gravity as given in (2.2). Quantum consistency requires \( b \) to be given by (2.23). The Liouville action has to be added and the effective action turns out to be (2.15) with (2.20),(2.23):
\[
S^\text{eff}_{g}(\sigma, \phi) = \eta \chi(M) + \alpha' \frac{24 - d}{12\pi} \int_M d^2 x \sqrt{g} \left( \frac{1}{2} \dot{g}^\alpha{}_{\beta} \partial_\alpha \sigma \partial_\beta \sigma \\
+ \frac{1}{4} \dot{g}^\alpha{}_{\beta} \partial_\alpha \phi \partial_\beta \sigma + (\sigma + \frac{\phi}{4}) R_{\dot{g}} + \frac{1}{2} V(\sigma, \phi) \right)
\]
(4.1)
with

\[ V(\sigma, \phi) = \mu e^{2\sigma} + \lambda e^{2\sigma+\phi} \] (4.2)

where we rescaled \( \mu, \lambda \rightarrow a\mu, a\lambda \). The action (4.1) reduces to the topological term in the limit \( \alpha' \rightarrow 0 \). The \( \phi \) and \( \sigma \) equations of motion are given by

\[ \Delta \hat{g} \sigma + R_{\hat{g}} + 2\lambda e^{2\sigma+\phi} = 0 \] (4.3)

\[ \Delta \hat{g} (\sigma + \frac{\phi}{4}) + R_{\hat{g}} + \mu e^{2\sigma} + \lambda e^{2\sigma+\phi} = 0 \] (4.4)

At this point the easiest approach to adopt is canonical quantization. The topology of \( M \) is assumed to be that of a cylinder. This allows the global choice of background metric

\[ \hat{g}_{\alpha\beta} = \text{diag}(-1, 1) \] (4.5)

with \( R_{\hat{g}} = 0 \). We shall follow the steps of Polchinski [16] in his canonical quantization of the Liouville system. Assuming the spatial coordinate \( x^1 \in [0, 2\pi] \) is periodic we can expand

\[ \sigma(0, x^1) = \sigma_0 - i \sum_{n=-\infty}^{\infty} \frac{1}{n} (\alpha_n e^{inx^1} + \tilde{\alpha}_n e^{-inx^1}) \]

\[ \partial_+ \sigma(0, x^1) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx^1} \] (4.6)

\[ \partial_- \sigma(0, x^1) = \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n e^{-inx^1} \]

where \( x^\pm = x^0 \pm x^1 \) and the prime on the summation omits the \( n = 0 \), and \( \alpha_0 = \tilde{\alpha}_0 = \frac{1}{2} \). The analog expansion for the field \( \phi \) is obtained from the replacement \( \alpha_n, \tilde{\alpha}_n \rightarrow \beta_n, \tilde{\beta}_n \) and \( \beta_0 = \tilde{\beta}_0 = \frac{1}{2} \).

The canonical momenta are

\[ \pi_\sigma = -\frac{a}{4\pi} \partial_0 \phi' \]

\[ \pi_{\phi'} = -\frac{a}{4\pi} \partial_0 \sigma \] (4.7)

where \( \phi' = \phi + 2\sigma \) and \( a \) was given in (2.20). Then the usual quantization conditions on \( \sigma, \pi_\sigma \) and \( \phi', \pi_{\phi'} \) result in the following commutators

\[ \frac{a}{2} [\phi', p_\sigma] = i \]

\[ \frac{a}{2} [\sigma_0, p_{\phi'}] = i \] (4.8)

\[ a[\alpha_n, \beta'_m] = -n \delta_{n+m,0} = a[\tilde{\alpha}_n, \tilde{\beta}'_m] \]
The energy-momentum tensor $\hat{T}_{\alpha\beta}$ has been given in (2.17) with (2.20),(2.23). The Virasoro generators $L_n^{(\phi,\sigma)}$ and $\bar{L}_n^{(\phi,\sigma)}$ are defined as the Fourier components of $\hat{T}_{++} + \frac{c}{24}$ and $\hat{T}_{--} + \frac{c}{24}$. The shift of $\frac{c}{24}$ results from the fact that $\hat{T}_{++}$ does not transform as a tensor when the coordinates are changed from the ones on the plane to those on the cylinder. Using the equations of motion in (2.17) to eliminate the second $x^0$ derivatives, we find after some algebra

\begin{align*}
L_n^{(\phi,\sigma)} &= -a \left[ \sum_{k=-\infty}^{\infty} \alpha_k \beta'_{n-k} - i n (\alpha_n + \beta'_n) \right. \\
&\quad + \left. \frac{1}{4\pi} \int_0^{2\pi} dx^1 e^{-i n x^1} V(\phi,\sigma) \right], \quad n > 0 \\
L_0^{(\phi,\sigma)} &= -a \left[ \alpha_0 \beta'_0 + \sum_{k=1}^{\infty} (\alpha_{-k} \beta'_{k} + \beta'_{-k} \alpha_k) \right. \\
&\quad + \left. \frac{1}{4\pi} \int_0^{2\pi} dx^1 V(\phi,\sigma) \right] - \frac{d - 24}{24}
\end{align*}

and corresponding expressions for $L_0^{(\phi,\sigma)}$ and $L_n^{(\phi,\sigma)}$.

Including the matter and ghosts contributions, the quantum states are

$$|\Psi> = |\Psi >_{g,m} |0 >_{gh}$$

Then the BRST condition $Q|\Psi> = 0$ implies

\begin{align*}
(L_n^{(\phi,\sigma)} + L_n^m)|\Psi >_{g,m} = 0 &= (\bar{L}_n^{(\phi,\sigma)} + \bar{L}_n^m)|\Psi >_{g,m}, \quad n > 0 \\
(L_0^{(\phi,\sigma)} + L_0^m - 1)|\Psi >_{g,m} = 0 &= (\bar{L}_0^{(\phi,\sigma)} + \bar{L}_0^m - 1)|\Psi >_{g,m}
\end{align*}

Assuming the matter fields to be in the highest weight states, and the quantum state $|\Psi >_{g,m}$ factorize, then [16]

\begin{align*}
L_n^m |\Psi >_{m} &= \bar{L}_n^m |\Psi >_{m} = 0, \quad n > 0 \\
L_0^m |\Psi >_{m} &= \bar{L}_0^m |\Psi >_{m} = \rho_m |\Psi >_{m}
\end{align*}

where $\rho_M$ is some constant.

In the following we take $M$ to be 1+1 dimensional universe where all fields are assumed to be constant in the spatial direction $x^1$. This is the mini-superspace approximation. Before considering the quantum solutions it is possible to consider the classical limit so the only conditions to be satisfied are

$$L_0^{(\phi,\sigma)} = L_0^{(\phi,\sigma)} = 1 - \rho_m$$

(4.13)
In terms of the new variables $\Sigma = e^\sigma$ and $D = e^{\phi/2}$ the condition (4.13) reads

$$a \frac{\dot{\Sigma}}{\Sigma} \left( \frac{\dot{\Sigma}}{\Sigma} + \frac{\dot{D}}{D} \right) = \rho_m - \frac{d}{24} - a \left( \mu + \lambda D^2 \right) \Sigma^2$$

(4.14)

Since the Liouville mode appears together with the dilaton, eq. (4.14) is difficult to analyze. If, however, we restrict to the classical solution so that $\Sigma, D$ have to obey (4.3), (4.4):

$$\frac{1}{\Sigma^2} \left[ \Sigma \dddot{\Sigma} - (\dot{\Sigma})^2 \right] + 2\lambda \Sigma^2 D^2 = 0$$

(4.15)

$$\frac{1}{D^2} \left[ D \dddot{D} - (\dot{D})^2 \right] + 4\mu \Sigma^2 - 4\lambda \Sigma^2 D^2 = 0$$

(4.16)

the time derivative of (4.14) implies

$$D^2 = \frac{\mu}{\lambda}$$

(4.17)

The condition (4.13) picks out the classical solution (4.17) which is simple and of particular interest, since with (4.17) the equations of motion (4.3), (4.4) reduce to the Liouville equation, describing constant curvature on $M$:

$$R_g = -2\mu$$

(4.18)

With (4.17) the condition (4.14) reduces to

$$a \left( \frac{\dot{\Sigma}}{\Sigma} \right)^2 = \rho_m - \frac{d}{24} - 2a\mu\Sigma^2$$

(4.19)

The $\Sigma$ equation of motion gives no further information since it follows from (4.19). In (4.19) we recognize the same structure of the expansion rate that was found for pure Liouville theory [16], with the difference that now the constant dilaton contributes to the vacuum energy density. For different values of $\rho_m, d$ and the cosmological constant $\mu$ we find the theory to include the de Sitter, Robertson-Walker, static and other cosmological solutions that were discussed in [16]. Eq. (4.14) describes how these solutions are disturbed in the presence of a varying dilaton, if we allow for non-classical field configurations.

At the end of section two, we mentioned that $d \to 24$ gives back the critical $D = 26$ string, where $D = d + 2$. This can also be seen from (4.14). If we let gravity decouple by $\alpha' \to 0$ then $a \to 0$ and (4.14) implies $\rho_m = \frac{d}{24}$. Since matter without gravity requires $\rho_m = 1$ we find $d = 24$ for the critical case.
We now look at the quantum theory in the mini-superspace approximation

\[ \Psi(\sigma, \phi) = \Psi(\sigma_0, \phi_0) \]  

(4.20)

We use (4.8) to replace

\[ \alpha_0 = \frac{4}{ia} \left( \frac{1}{2} \frac{\partial}{\partial \sigma_0} + \frac{\partial}{\partial \phi_0} \right) \]

(4.21)

\[ \beta'_0 = \frac{4}{ia} \frac{\partial}{\partial \phi_0} \]

Then the \( L_0 \) and \( \bar{L}_0 \) conditions imply the Wheeler-de Witt equation:

\[ \left( \frac{\partial^2}{\partial \Sigma^2} + \frac{1}{\Sigma} \frac{\partial}{\partial \Sigma} + \frac{D}{\Sigma} \frac{\partial^2}{\partial \Sigma \partial D} - \frac{a^2}{16} (\mu + \lambda D^2) - \frac{aM}{16} \frac{1}{\Sigma^2} \right) \Psi(\Sigma_0, D_0) = 0 \]

(4.22)

where \( M = \frac{d}{24} - \rho M \).

In this form (4.22) is a generalization of Bessel’s equation. Before, we have seen that the classical solution (4.17) covers the dynamics of pure Liouville theory. This can also be read from (4.22). If we restrict to the part of \((\Sigma, D)\) space that is defined by (4.17) then the \( D \) independent solution of (4.22) turns out to be of the same structure that was found in Liouville theory. For \( aM > 0, \mu < 0 \) the eq. (4.22) is then solved by

\[ \Psi = \alpha J_{-n} \left( \frac{|a|}{2} \sqrt{-\frac{1}{2}\mu \Sigma} \right) + \beta J_n \left( \frac{|a|}{2} \sqrt{-\frac{1}{2}\mu \Sigma} \right) \]

(4.23)

where \( n = \frac{1}{4}\sqrt{aM} \) and \( \alpha, \beta \) arbitrary.

In (2.9-14) the cosmological constant turned out to be regularization dependent. We may use this freedom to set \( \mu = 0 \). In that case a \( D \)-dependent solution, defined on the full \((\Sigma, D)\)-space can be found for \( aM > 0, \lambda < 0 \):

\[ \Psi = \alpha J_{-m} \left( \frac{|a|}{4} \sqrt{-\frac{1}{2}\lambda \Sigma D} \right) + \beta J_m \left( \frac{|a|}{4} \sqrt{-\frac{1}{2}\lambda \Sigma D} \right) \]

(4.24)

where \( m = \frac{1}{4}\sqrt{\frac{1}{2}aM} \).

Other solutions are obtained immediately if we set the dilaton self-coupling to zero: \( \lambda = 0 \). Since the topology of \( M \) is that of a cylinder, this does not require the replacement (2.24). Then for \( \mu = 0 \) exponential expressions are found if we set

\[ \Psi = \Sigma^\alpha D^\beta \]

(4.25)

This satisfies (4.24) if

\[ \alpha = -\frac{\beta}{2} \pm \frac{1}{2} \sqrt{\beta^2 + \frac{aM}{4}} \]

(4.26)
For $\beta = 0$, $aM < 0$ eq.(4.26) includes the classically allowed ingoing and outgoing solutions of Liouville theory. For $\lambda = 0$, $\mu \neq 0$ there are solutions independent of the dilaton:

$$\Psi = \alpha J_{-n}(\frac{|a|}{4}\sqrt{-\mu \Sigma}) + \beta J_{n}(\frac{|a|}{4}\sqrt{-\mu \Sigma})$$

which are of the type given in (4.23).

5. Summary

In conclusion, we have shown that the two-dimensional theory of gravity as described by a graviton-dilaton system can be solved both at the classical and the quantum levels. For the classical case, i.e. without including the Liouville terms induced from quantization we found generalized black hole solutions. At the quantum level the induced Liouville action has to be included. In the absence of potential terms the theory is exactly conformal and completely solvable. In the presence of interactions, the dilaton-graviton interaction can be fixed in such a way that the theory remains free of the conformal anomaly for any conformal dimension of the coupled matter system. We applied canonical quantization and gave the Wheeler-DeWitt equation. Although richer due to the presence of the dilaton, the theory is seen to include the dynamics of pure Liouville theory. From the two-dimensional cosmology point of view, it allows for de Sitter, Robertson-Walker, static and other cosmological solutions, determined by the values of matter and vacuum (cosmological constant plus dilaton) contributions.

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