Gravitoelectromagnetism in (Anti) de Sitter Spacetime

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Abstract
The presence of a non-zero cosmological term in Einstein field equations can be interpreted as the physical possibility for preferred reference frames without breaking of general covariance. This possibility is used in the process of linearizing Einstein field equations in a de Sitter background, and in formulating the resulting equations in the framework of gravitoelectromagnetism. It is proposed that this set of equations only applies to the physical vacuum and not to baryonic (normal) matter.

1 Introduction
The theory of General Relativity (GR) explains the behavior of space-time and matter on cosmologically large scales and of very dense compact astrophysical objects. It is the most accurate theory so far of the gravitational interaction. In the original formulation of GR, the metric tensor, $g_{\mu\nu}$, plays a major role, being the unknown in the Einstein Field Equations (EFE):

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

(1)

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R$ is the so-called Einstein tensor, $\Lambda$ is the Cosmological Constant (CC), $G$ is Newton’s constant, and the energy-momentum tensor is obtained through the matter Lagrangian density $L_M(\phi, A_\mu, ...)$:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial (\sqrt{-g} L_M)}{\partial g^{\mu\nu}}.$$

(2)

Field equations (1) arise from the Einstein-Hilbert action plus the Lagrangian density $L_M$ describing the matter fields

$$S[g_{\mu\nu}, \phi, A_\mu, ...] = \int d^4x \sqrt{-g} \left\{ \frac{c^4}{16\pi G} (R - 2\Lambda) + L_M(\phi, A_\mu, ...) \right\}$$

(3)

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The cosmological constant, although a major embarrassment in current theoretical physics given its observed value, is indisputably an important ingredient in accounting for the cosmological data. This arises from the study of Type Ia Supernovae which, when used as standard candles, allow to determine with some confidence important cosmological parameters. Recent analyses of the magnitude-redshift relation of about fifty Type Ia Supernovae with redshifts greater than \( z \geq 0.35 \) strongly suggest that we are living in an accelerating, low-matter density Universe in which a non-zero cosmological constant is responsible for a vacuum energy density, \( \rho_V \), often called dark energy [1, 2, 3].

\[
\rho_V \equiv \frac{\Lambda c^2}{8\pi G} \sim 10^{-29} \text{g cm}^{-3} \simeq 3.88 \text{eV/\text{mm}^3} \tag{4}
\]

The small astronomically observed value of the CC, \( \Lambda = 1.29 \times 10^{-52} \text{m}^{-2} \), and its origin remain a deep mystery. This is often call the CC problem, since with a cutoff at the Planck scale the vacuum energy density expected from quantum field theory should be larger by a factor of the order 10^{120}, in complete contradiction with the observed value.

In section 2 we consider a null CC, in this case weak gravitational fields can be treated as small perturbations of a flat Minkowski background metric. In this approximation for simple mass currents, EFE can be linearized and expressed in a form resembling Maxwell equations in terms of gravitational (gravitoelectric) and gravitomagnetic fields. In the case of a cosmological constant different from zero the background spacetime is not flat but is rather curved and described by de Sitter (dS) or Anti-de Sitter (AdS) metric (with a positive or a negative CC, respectively). In a dS background, weak gravitational fields must be treated as small perturbations to the dS metric. This turns more complex the linearization procedure, since we cannot neglect anymore second order perturbation terms, and we cannot consider harmonic gauge conditions. In section 3 we show that we can overcome these difficulties through the fact that a non-zero cosmological term in EFE can be interpreted as the possibility to define privileged coordinate systems without violating general covariance. This physical possibility together with the requirement of general covariance of the perturbed dS metric under the de Sitter group, allows in section 4 to derive locally a linearized form of EFE in a dS background around the origin of a privileged reference frame. At this particular location of the privileged coordinate frame the harmonic gauge conditions are verified and the linearized de Sitter field equations can be expressed in function of the traditional gravitational and gravitomagnetic fields used in the case of a flat background as we show in section 5. The physical interpretation of these equations is carried out in section 6, where it is argued that the linear theory of EFE in a (Anti) de Sitter background should only be valid for vacuum forms of energy.

2 Gravitoelectromagnetism in flat spacetime

GravitoElectroMagnetism (GEM) is a linear approximation of EFE, eq. (1) without a CC in a flat background and in the weak field regime, which is valid under the following assumptions:

1. the mass densities are normal (no dwarf stars), and correspond to local physical systems located in the Earth laboratory or in the solar system.
2. All motions are much slower than the speed of light, so that special relativity can be neglected. (Often special relativistic effects will hide general relativistic effects), \(v \ll c\).

3. The kinetic or potential energy of all the bodies being considered is much smaller than their energy of mass, \(T_{\mu\nu} \ll \rho c^2\).

4. The gravitational fields are always weak enough so that superposition is valid, \(\varphi \ll c^2\).

5. The distances between objects is not so large that we have to take retardation into account. (This can be ignored when we have a stationary problem where the fields have already been prescribed and are not changing with time.)

One starts by considering small perturbations, \(|h_{\alpha\beta}| \ll 1\), of Minkowsky’s metric \(\eta_{\alpha\beta}(+−−−)\) (Landau-Lifschitz “timelike convention”).

\[
g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta} \tag{5}
\]

Doing Equ. (5) into Equ. (1) with the derivation indices obeying the same rule as the covariant indices, \(f^{\mu\nu} = \eta^{\mu\nu} f_{\mu\nu}\), we obtain:

\[
- \frac{1}{2} \left( h^{\mu}_{\alpha\beta,\mu} + \eta_{\alpha\beta} h^{\mu\nu} - \bar{h}^{\mu}_{\alpha\mu,\beta} - \bar{h}^{\mu}_{\beta\mu,\alpha} \right) = \frac{8\pi G}{c^4} T_{\alpha\beta} \tag{6}
\]

As usual in order to simplify the linearization procedure we have introduced the intermediate tensor:

\[
\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h \tag{7}
\]

where \(h = h^\mu_\mu = \eta^{\mu\nu} h_{\mu\nu} = h_{00} - h_{11} - h_{22} - h_{33}\) is the trace of the perturbation tensor. Imposing the harmonic gauge condition

\[
\bar{h}^{\mu\nu}_{,\nu} = 0 \tag{8}
\]

Equ. (8) reduces to

\[
\bar{h}^{\mu}_{\alpha\beta,\mu} = -\frac{16\pi G}{c^4} T_{\alpha\beta} \tag{9}
\]

Equ. (9) can be written in function of the Dalembertian operator, \(\Box\). If \(f\) is a given function, then

\[
\Box f = f^{,\mu}_{,\mu} = \eta^{\mu\nu} f_{,\mu\nu} = \left( \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^i)^2} \right) f \tag{10}
\]

Where \(x_0 = ct\). Therefore Equ.(9) becomes

\[
\Box \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta} \tag{11}
\]

This is the approximated first order form of EFE without a cosmological constant, assuming a flat background for the metric. We will now solve these equations with the energy-momentum tensor components:

\[
T_{00} = \rho c^2, \tag{12}
\]
and
\[ T_{0i} = -\rho c v_i. \] (13)

The solution of, \[ \Box \tilde{h}_{00} = -\frac{16\pi G}{c^4} T_{00}, \] for the energy momentum tensor component given by Eq. (12) is:
\[ h_{00} = \frac{2\varphi}{c^2} . \] (14)

Where \( \varphi \) is the gravitational scalar potential. The solution of, \[ \Box \tilde{h}_{0i} = -\frac{16\pi G}{c^4} T_{0i}, \] for the energy momentum tensor component of Eq. (13) is:
\[ h_{0i} = -\frac{4A_{gi}}{c} . \] (15)

Where \( A_{gi} \) are the three components of the gravitomagnetic vector potential.

Writing the Einstein tensor in function of the intermediate tensor \( \tilde{h}_{\alpha\beta} \), and using the gauge condition of Eq. (8), we can construct the useful tensor \( G_{\alpha\beta\mu} \).

\[ G_{\alpha\beta\mu} = \frac{1}{4} \left( h_{\alpha\beta,\mu} - \tilde{h}_{\alpha\mu,\beta} \right) \] (16)

Using Eq. (16) one can re-write Eq. (11) under the following form:
\[ \frac{\partial G_{\alpha\beta\mu}}{\partial x^\mu} = -\frac{4\pi G}{c^4} T_{\alpha\beta} . \] (17)

We can also use the tensor \( G_{\alpha\beta\mu} \), Eq. (16) to express the gravitational field:
\[ g_{i} = -c^2 G_{00i} . \] (18)

Which can also be written in terms of the gravitational scalar potential \( \varphi \) and of the gravitomagnetic vector potential \( A_g \).

\[ \vec{g} = -\nabla \varphi - \frac{\partial \vec{A}_g}{\partial t} \] (19)

Similarly we formulate the gravitomagnetic field, \( \vec{B}_g \), as follows:
\[ cG_{0ij} = -(A_{gi,j} - A_{gj,i}) \] (20)

which obviously shows that the gravitomagnetic field \( \vec{B}_g \) is generated by a vectorial potential \( \vec{A}_g \).

\[ \vec{B}_g = \nabla \times \vec{A}_g \] (21)

We have now everything we need to derive Maxwell-type equations for gravity.

For the energy momentum tensor component \( T_{00} \) of Eq. (12), Eq. (17) reduces to:
\[ \frac{\partial G_{000}}{\partial x^\mu} = -\frac{4\pi G\rho}{c^2} \] (22)

Using Eq. (18), we can rearrange Eq. (22) to obtain the divergent part of the gravitational field:
\[ \nabla \cdot \vec{g} = -4\pi G\rho \] (23)
For the energy momentum tensor component $T_{0i}$ of Equ.(13), Equ.(17) reduces to:

$$\frac{\partial G_{0i\mu}}{\partial x^\mu} = \frac{4\pi G}{c^3} \rho v_i$$

(24)

Using Equ.(20), we can write Equ.(24) to obtain the rotational part of the gravitomagnetic field:

$$\nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} j_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t}$$

(25)

Where $\vec{j}_m = \rho \vec{v}$ is the mass current. The tensor $G_{\alpha\beta\mu}$, Equ.(16), has the following property:

$$G^{\alpha\beta\mu,\lambda} + G^{\alpha\lambda\beta,\mu} + G^{\alpha\mu\lambda,\beta} = 0$$

(26)

which are equivalent to the two other set of Maxwell like equations for gravity,

$$\nabla \cdot \vec{B}_g = 0$$

(27)

and

$$\nabla \times \vec{B}_g = -\frac{\partial \vec{g}}{\partial t}$$

(28)

Note also that Equ.(27) is a direct and trivial corollary of the definition of the gravitomagnetic field Equ.(21).

In summary Equs (23) (25) (27) and (28) form the set of Einstein-Maxwell equations for gravity in a flat background and in the weak field regime. They are also called GravitoElectroMagnetic (GEM) equations:

$$\nabla \cdot \vec{g} = -4\pi G \rho$$

(29)

$$\nabla \cdot \vec{B}_g = 0$$

(30)

$$\nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t}$$

(31)

$$\nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} j_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t}$$

(32)

3 General covariance, privileged reference frames and cosmological constant

Einstein field equations with and without cosmological constant, are generally covariant; that is, they preserve their form under a general coordinate transformation $x^\mu \rightarrow y^\mu$, where $y^\mu$ is an arbitrary function of $x^\mu$. As argued by J. Rayski [4], the weakest restriction of generality consists in assuming a formalism covariant only under the unimodular group of coordinate transformations satisfying the condition.

$$\text{det} \frac{\partial y^\mu}{\partial x^{\nu}} = 1$$

(33)

This may be achieved by adding to the Lagrangian of the gravitational field $L_g = \frac{c^4}{16\pi G} \sqrt{\text{det} g} R$ a new term breaking general covariance.

$$L' = -\Lambda g$$

(34)
With Λ a constant not to be subjected to any variations, whereby $g$ is denoting the determinant of the metric tensor components.

\[ g = \det g_{\mu\nu} \]  
\hspace{1cm} (35)

The new Lagrangian is not covariant anymore but it remains invariant under unimodular transformations. The corresponding equations of motion are

\[ G^{\mu\nu} - 3\Lambda g^{\mu\nu} = 0 \]  
\hspace{1cm} (36)

Taking the covariant derivative of eq.(36) we get

\[ G^{\mu\nu}_{\nu} - 3\Lambda g^{\mu\nu}_{\nu} - 3\Lambda g^{\mu\nu}\partial_{\nu}g = 0 \]  
\hspace{1cm} (37)

This equation is satisfied for the coordinate condition:

\[ g = C \]  
\hspace{1cm} (38)

$C$ denoting a constant. Substituting eq.(38) in eq.(36), we obtain:

\[ G^{\mu\nu} + \tilde{\Lambda} g^{\mu\nu} = 0 \]  
\hspace{1cm} (39)

where

\[ \tilde{\Lambda} = 3\Lambda C \]  
\hspace{1cm} (40)

From a Lagrangian containing a term violating general covariance eq.(34), we obtained Einstein field equations (for the vacuum) with a cosmological term, eq.(39), which are generally covariant. It seems that general covariance may hold true in spite of the circumstance that a certain coordinate system is privileged. Since C is arbitrary, we have a one parametric family of field equations with cosmological term, eq.(39). For the case where

\[ g = -1/3 \]  
\hspace{1cm} (41)

we obtain the usual Einstein field equations with a cosmological constant $\Lambda$:

\[ G^{\mu\nu} - \Lambda g^{\mu\nu} = 0 \]  
\hspace{1cm} (42)

Therefore we can interpret the presence of a non-zero cosmological constant in Einstein field equations as the physical possibility of privileged coordinate systems without breaking the principle of general covariance. As pointed by Rayski [4], this also indicates the possibility of producing several generally covariant but inequivalent versions of quantum theory of gravity, all of them corresponding to the same classical theory. The appearance of several inequivalent versions of quantum gravity could mean that for each one of these theories the privileged coordinate system is that one in which the observer appears to be at rest. Different coordinate systems mean different conditions of measurements, and consequently, different phenomena. Different types of phenomena may be described by different (inequivalent) mathematical formalisms without getting involved in contradiction.
4 Linearized gravity in de Sitter spacetime

We will now use this “magical” possibility, offered by the cosmological constant in EFE, of specifying a privileged coordinate system without breaking general covariance, in order to linearize EFE in a de Sitter background. In this process we closely follow the derivation of Sivaram [5].

We first re-write EFE, eqn. (1), in the following form:

$$R_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) = 0 \quad (43)$$

In the weak field approximation of Einstein field equations with a cosmological constant we cannot consider the spacetime metric $g_{\mu\nu}$ as a small perturbation, $h_{\mu\nu}$, of a flat background metric, $\eta_{\mu\nu}$, since even in the absence of matter, $T_{\mu\nu} = 0$, the Minkowski metric $\eta_{\mu\nu}$ is not solution of eq. (43). However we can consider a curved spacetime for the background, with a de Sitter metric parameterized with Poincaré coordinates:

$$\hat{g}_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu} \quad (44)$$

where

$$e^{-\sigma} = 1 - \frac{\Lambda}{12}x^2. \quad (45)$$

with $x^2 = \eta_{\mu\nu}x^\mu x^\nu$. Indices will be raised and lowered with $\eta_{\mu\nu}$. Considering the perturbations to $\hat{g}_{\mu\nu}$ to be of the form:

$$\hat{h}_{\mu\nu} = e^{2\sigma}h_{\mu\nu} \quad (46)$$

With $|\hat{h}_{\mu\nu}| << |\hat{g}_{\mu\nu}|$, we can start from the equivalent weak field conditions in flat background: $|h_{\mu\nu}| << |\eta_{\mu\nu}|$, and derive the linearized field equations in curved background by conformal mapping of the linearized theory in a flat background. Thus the overall metric $g_{\mu\nu}$, we consider in this process, is the background metric, eq. (44), added to its perturbation, eq. (46):

$$g_{\mu\nu} = e^{2\sigma}(\eta_{\mu\nu} + h_{\mu\nu}) \quad (47)$$

In this metric the Ricci tensor of the background space is:

$$R_{\mu\nu} = \frac{1}{2}((\Box h_{\mu\nu} + h_{\mu,\nu} - h_{\rho,\mu\nu}^\rho - h_{\nu,\mu\rho}^\rho)$$

$$+ (\sigma_{,\mu\nu} + \eta_{\mu\nu}\Box \sigma - 2\sigma_{,\mu\sigma\nu} + 2\eta_{\mu\nu}\sigma_{,\rho\sigma\rho})$$

$$- \sigma_{,\rho}(h_{\rho,\mu\nu}^\rho + h_{\rho,\nu\mu}^\rho - h_{\rho,\mu\nu}^\rho - h_{\rho,\nu\mu}^\rho)$$

$$+ (h_{\mu\nu}\Box \sigma - \eta_{\mu\nu}h^{\nu\lambda}\sigma_{,\rho\lambda}) \quad (48)$$

Substituting eq. (48) and eq. (47) in eq. (43) we get

$$- \Lambda e^{2\sigma}(\eta_{\mu\nu} + h_{\mu\nu}) + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T) = \frac{1}{2}((\Box h_{\mu\nu} + h_{\mu,\nu} - h_{\rho,\mu\nu}^\rho - h_{\nu,\mu\rho}^\rho)$$

$$+ (\sigma_{,\mu\nu} + \eta_{\mu\nu}\Box \sigma - 2\sigma_{,\mu\sigma\nu} + 2\eta_{\mu\nu}\sigma_{,\rho\sigma\rho})$$

$$- \sigma_{,\rho}(h_{\rho,\mu\nu}^\rho + h_{\rho,\nu\mu}^\rho - h_{\rho,\mu\nu}^\rho - h_{\rho,\nu\mu}^\rho)$$

$$+ (h_{\mu\nu}\Box \sigma - \eta_{\mu\nu}h^{\nu\lambda}\sigma_{,\rho\lambda}) \quad (49)$$
The explicit values of the derivatives of $\sigma$ in eq. (48) are:

$$\sigma_{,\mu} = \frac{\Lambda}{6} e^\sigma x_\mu$$

$$\sigma_{,\mu\nu} = \frac{\Lambda}{6} e^\sigma (\eta_{\mu\nu} + \frac{\Lambda}{6} e^\sigma x_\mu x_\nu),$$

$$\Box \sigma = \frac{\Lambda}{6} e^\sigma (4 + \frac{\Lambda}{6} e^\sigma x^2).$$

(50)

As discussed in section 3, here we take advantage of the possibility to define a privileged coordinate frame to express eq. (49) at the origin $x = 0$ of a privileged reference frame. The sigma terms in eq. (50) are simply replaced by their values at the origin

$$e^\sigma \sim 1, \quad \sigma_{,\mu} \sim 0, \quad \sigma_{,\mu\nu} \sim \frac{\Lambda}{6} \eta_{\mu\nu}, \quad \Box \sigma \sim \frac{2}{3} \Lambda$$

(51)

The only terms that survive in eq. (48) are the first line and the fourth line. At the origin, the equation becomes:

$$\Box h_{\mu\nu} - \bar{h}_{\mu\nu,\alpha} - \bar{h}_{\nu\alpha,\mu} \sim \frac{2}{3} \Lambda \left( h_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} h \right) - \frac{16\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right)$$

(52)

Where $\bar{h}_{\mu\nu}$ is given by eq. (7). When the linearization is carried out in a flat background, one usually imposes the harmonic gauge conditions, eq. (8). However, it is not possible to assume that this coordinate restriction is relevant in the present context. This would give rise to additional $\Lambda$ terms in eq. (52), the criterion we shall use for selecting the appropriate physical fields and the appropriate coordinate restriction, is the requirement of covariance under the de Sitter group.

The gauge transformations applying to eq. (49) are the infinitesimal coordinate transformations:

$$x^\mu \mapsto x^\mu + \xi^\mu$$

(53)

By carrying out such transformations on the metric eq. (47) we obtain the infinitesimal de Sitter group

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = -2\eta_{\mu\nu} \sigma_{,\rho} \xi^\rho$$

(54)

The harmonic gauge conditions are then replaced by a (Anti) de Sitter covariant version [5]:

$$\bar{h}^{\mu\nu} = \sigma^{,\nu} h - 4 \sigma_{,\mu} h^{\mu\nu}$$

(55)

Contrary to Sivaram we consider here that the field variables, which are covariant in the de Sitter group are the tensor field density:

$$\phi = e^{2\sigma} \bar{h} = -e^{2\sigma} h$$

(56)

and

$$\phi_{\mu\nu} = e^{2\sigma} \bar{h}_{\mu\nu}$$

(57)

Rewriting eq. (52) in function of the de Sitter covariant fields $\phi_{\mu\nu}$ and $\phi$, and Imposing the de Sitter gauge conditions, eq. (55) lead to the following relations at the origin of the coordinate system.

$$(\Box - \frac{4}{3} \Lambda) h \sim \frac{8\pi G}{c^4} T$$

(58)
and
\[
(\Box - \frac{2}{3} \Lambda) \tilde{h}_{\mu \nu} \sim \frac{16\pi G}{c^4} \left( T_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} T \right)
\]  
(59)

with the de Sitter gauge condition eq.(55), reducing to the harmonic gauge condition, \( \tilde{h}_{\mu \nu} = 0 \), at the origin, according to the relations eq.(51).

5 de Sitter Gravitoelectromagnetic Equations

The weak field approximation of EFE in a (Anti) de Sitter background, Eq.(59) can be written in the usual form of gravitoelectromagnetism, in terms of gravitational and gravitomagnetic fields, by using the tensor \( G_{\alpha \beta \mu \nu} \), eq.(16), together with the de Sitter gauge condition at the origin, which reduces to the harmonic gauge condition: \( \tilde{h}_{\mu \nu} = 0 \)

\[
\frac{\partial G_{\mu \nu \alpha \beta}}{\partial x^\alpha} + \frac{1}{6} \Lambda \tilde{h}_{\mu \nu} = \frac{4\pi G}{c^4} \left( T_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} T \right)
\]  
(60)

We will now solve these equations, by approximation, using the solutions of the perturbations to Minkowski’s metric we obtained in the case of linear EFE without CC, which are:

\[
\tilde{h}_{00} = \frac{4\phi}{c^2} \quad \text{and} \quad \tilde{h}_{0i} = -\frac{4A_{gi}}{c}
\]  
(61)

From the components of the energy momentum tensor given by eq.(12) and eq.(13), we deduce that the trace of the energy momentum tensor to zero order is:

\[
T = \rho c^2
\]  
(62)

Substituting the energy momentum tensor component \( T_{00} \), eq.(12), and its trace, eq.(62), into eq.(60), we obtain

\[
c^2 \frac{\partial G_{00 \alpha \beta}}{\partial x^\alpha} + \frac{1}{6} \Lambda c^2 \tilde{h}_{00} = 3\pi G \rho
\]  
(63)

Using eq.(18) and simplifying with the harmonic gauge, eq.(8), we can re-write eq.(63) in terms of the divergent part of the gravitational field:

\[
\nabla g = +3\pi G \rho - \frac{2}{3} \Lambda \phi
\]  
(64)

For the energy momentum tensor component \( T_{0i} \), eq.(13), eq.(60) reduces to

\[
c \frac{\partial G_{0i \alpha \beta}}{\partial x^\alpha} - \frac{1}{6} \Lambda c \tilde{h}_{0i} = -\frac{4\pi G}{c^2} \rho v_i
\]  
(65)

Using eq.(20), we can re-write eq.(65) in terms of the rotational part of the gravitomagnetic field:

\[
\nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} \vec{j}_m + \frac{1}{c^2} \frac{\partial \tilde{g}}{\partial t} - \frac{2}{3} \Lambda \vec{A}_g
\]  
(66)

The divergence of the gravitomagnetic field \( B_g \) and the rotational of the gravitational field \( g \) are not affected by the mass terms in eq.(18) and eq.(59).
and are derived in the same manner as in section 2 for the case of having a flat background

\[ \nabla \cdot \vec{B}_g = 0 \]  

(67)

and

\[ \nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t} \]  

(68)

In summary Equations (67), (69), (66) and (68) form the set of Gravitoelectromagnetic equations in a (Anti)de Sitter background, let us call them the de Sitter Gravitoelectromagnetic Equations:

\[ \nabla \vec{g} = +3\pi G \rho - \frac{2}{3} \Lambda \varphi \]  

(69)

\[ \nabla \vec{B}_g = 0 \]  

(70)

\[ \nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t} \]  

(71)

\[ \nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} \vec{j}_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t} - \frac{2}{3} \Lambda \vec{A}_g \]  

(72)

We stress that these equations are only valid at the origin of the coordinate system.

Substituting the trace of the energy-momentum tensor, eq. (62) into eq. (58) and assuming stationary conditions we get:

\[ \nabla^2 h - \frac{1}{L^2} h = \frac{8\pi G}{c^2} \rho \]  

(73)

where \(1/L^2 = 4\Lambda/3\). For the one-dimensional case, the solution of eq. (73) is:

\[ h = h_0 e^{-x/L} - \frac{3}{4} \rho \frac{\rho}{\rho_V} \]  

(74)

Since in general \(x << L\) the trace of the perturbation tensor \(h\) is approximately equal to the ratio between the mass density present in the energy momentum tensor and the vacuum mass density \(\rho_V\), eq. (61), coming from the CC.

\[ h \sim -\frac{3}{4} \frac{\rho}{\rho_V} \]  

(75)

6 Discussion and Conclusions

It is interesting to compare the de Sitter gravitoelectromagnetic equations (69-72) with a similar set of equations derived by Argyris to investigate the consequences of massive gravitons in general relativity [6].

\[ \nabla \vec{g} = -4\pi G \rho - \frac{1}{\lambda_{graviton}^2} \varphi \]  

(76)

\[ \nabla \vec{B}_g = 0 \]  

(77)

\[ \nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t} \]  

(78)

\[ \nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} \vec{j}_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t} - \frac{1}{\lambda_{graviton}^2} \vec{A}_g \]  

(79)
In Argyris equations the origin of the graviton mass is unknown. In de Sitter gravitoelectromagnetic equations the graviton mass is clearly resulting from a non-zero CC. Another major difference with respect to Argyris equations is the repulsive character of the gravitational field, eq. (69). Since the first term on the right hand side of eq. (69) indicates gravitational repulsion, the de Sitter Gravitoelectromagnetic equations (69-72) might exclusively hold for forms of energy associated with the vacuum (similar to the vacuum energy density resulting from the CC $\Lambda$). In contrast Argyris was assuming his equations to hold for all forms of matter. Since in the linear approximation we are considering, the condition $4 \varphi << c^2$, we can easily integrate eq. (69), for the case of $\rho = \rho_v = \frac{\Lambda c^2}{8\pi G}$, which leads directly to a repulsive gravitational field, $g = \frac{1}{8}c^2\Lambda R$, very close to the accelerated cosmological expansion, $a = \frac{1}{3}c^2\Lambda R$.

Eqs. (58) and (59) are not true field equations; they serve mainly to identify the masses of the interacting bosons, i.e., they are relations satisfied at a single point, i.e., local equations. It is easy to see from the de Sitter gravitoelectromagnetic equations (69-72) that the gravitomagnetic and gravitational fields $B_g$ and $g$ can possibly not propagate. From eq. (69) we have the gravitational permittivity, $\epsilon_g$:

$$\epsilon_g = \frac{1}{3\pi G}$$

from the first term on the r.h.s. of eq. (72) we deduce the gravitomagnetic permeability $\mu_g$:

$$\mu_g = \frac{4\pi G}{c^2}$$

The inverse of the product of eq. (80) by eq. (81) gives the propagation speed for gravitomagnetic waves $c_1$:

$$c_1 = \frac{\sqrt{3}}{2}c$$

Which is less than the speed of light, and is different from the propagation speed appearing in the second term on the r.h.s of eq. (72). a propagation speed of the fields $\vec{g}$ and $\vec{B}_g$ in the field equs. (69-72), different from the speed of light in vacuum might also be a physical manifestation of the existence of a preferred coordinate system. Although this should be understood as the impossibility for these fields to propagate, they still can define the local inertial properties of the physical vacuum. Thus indicating that vacuum energy would not "gravitate" like baryonic matter does, which is consistent with the possibility of having preferred frames without breaking the principle of general covariance.

If $\Lambda$ is negative eq. (58) and eq. (59) are in the form of Klein-Gordon equations for a massive spin zero field and a massive spin 2 field, with source terms, with Compton wavelengths $m_1c/\hbar = \sqrt{-4\Lambda/3}$ and $m_2c/\hbar = \sqrt{-2\Lambda/3}$. These mass terms clearly result from the possibility of defining a privileged reference frame without breaking the principle of general covariance.

If note also that the square of the ratio of the boson’s masses, $m_1$ and $m_2$, seems to be related with the condition that the privileged coordinate system must satisfy, eq. (41), for the case of the usual Einstein field equations with a CC.

$$-\frac{2}{3}\left(\frac{m_2}{m_1}\right)^2 = -\frac{1}{3} = g$$

(83)
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