WHEN DOES RANDOMNESS COME FROM RANDOMNESS?

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Abstract. A result of Shen says that if \( F : 2^\mathbb{N} \to 2^\mathbb{N} \) is an almost-everywhere computable, measure-preserving transformation, and \( y \in 2^\mathbb{N} \) is Martin-Löf random, then there is a Martin-Löf random \( x \in 2^\mathbb{N} \) such that \( F(x) = y \). Answering a question of Bienvenu and Porter, we show that this property holds for computable randomness, but not Schnorr randomness. These results, combined with other known results, imply that the set of Martin-Löf randoms is the largest subset of \( 2^\mathbb{N} \) satisfying this property and also satisfying randomness conservation: if \( F : 2^\mathbb{N} \to 2^\mathbb{N} \) is an almost-everywhere computable, measure-preserving map, and if \( x \in 2^\mathbb{N} \) is random, then \( F(x) \) is random.

1. Introduction

Algorithmic randomness is a branch of mathematics which gives a rigorous formulation of randomness using computability theory. The first algorithmic randomness notion, Martin-Löf randomness, was formulated by Martin-Löf [19] and has remained the dominant notion in the literature. Schnorr [34], however, felt that Martin-Löf randomness was too strong, and introduced two weaker randomness notions now known as Schnorr randomness and computable randomness.

While, historically randomness has mostly been studied on Cantor space \( 2^\mathbb{N} \) with the fair-coin measure \( \lambda \), there has been a lot of interest lately in the behavior of algorithmic randomness on other spaces and measures. Many of these results are of the form, “A point \( y \in Y \) is \( \nu \)-random (where \( \nu \) is a Borel probability measure on \( Y \)) if and only if \( y = F(x) \) for some \( \mu \)-random \( x \in X \) (where \( \mu \) is a Borel probability measure on \( X \)).”

As an example, consider von Neumann’s coin. Von Neumann showed that given a possibly biased coin with weight \( p \in (0, 1) \), one can recover the distribution of a fair coin by following this given procedure: Toss the coin twice. If the results match, start over, forgetting both results. If the results differ, use the first result, forgetting the second. Von Neumann’s procedure describes a partial computable function \( F : \subseteq 2^\mathbb{N} \to 2^\mathbb{N} \) whereby an infinite sequence of independent and identically distributed biased coin tosses \( x \in 2^\mathbb{N} \) is turned into an infinite sequence of independent and identically distributed fair coin tosses \( F(x) \).

Now, as for randomness, one can prove that for a fixed computable \( p \in (0, 1) \), a sequence \( y \in 2^\mathbb{N} \) is Martin-Löf random for the fair-coin measure if and only if \( y \) can...
be constructed via the von Neumann procedure starting with some \( x \in 2^\mathbb{N} \) which is random for the \( p \)-Bernoulli measure (\( p \)-weighted coin measure). While there are many methods available in algorithmic randomness to prove this, the easiest is to just apply the following theorem.

**Theorem 1** (See [4, Thms. 3.2, 3.5][13, Prop. 5].) Assume \( \mu \) and \( \nu \) are computable probability measures on \( 2^\mathbb{N} \) and that the map \( F: (2^\mathbb{N}, \mu) \to (2^\mathbb{N}, \nu) \) is almost-everywhere computable (\( F \) is computable on a \( \mu \)-measure-one set) and measure-preserving (\( \nu(B) = \mu(F^{-1}(B)) \) for all Borel \( B \)).

1. If \( x \) is \( \mu \)-Martin-Löf random then \( F(x) \) is \( \nu \)-Martin-Löf random.
2. If \( y \) is \( \nu \)-Martin-Löf random then \( y = F(x) \) for some \( \mu \)-Martin-Löf random \( x \).

The first half of Theorem 1 is known as randomness conservation, randomness preservation, or conservation of randomness. This result, at least in the finitary setting of Kolmogorov complexity, goes back to Levin [13, Thm. 1]. (See Gács [9].) The second half is known as no-randomness-from-nothing or no randomness ex nihilo. Bienvenu and Porter [4] attribute it as an unpublished result of Alexander Shen. Both these results, together, say that \( F \) is a surjective map from the set of \( \mu \)-Martin-Löf randoms to the set of \( \nu \)-Martin-Löf randoms. Theorem 1 also holds for other computable probability spaces with layerwise computable maps (Hoyrup and Rojas [13, Prop. 5]). (Also see Hertling and Weihrauch [11] for a randomness conservation result for partial maps between effective topological spaces.)

Theorem 1 is sufficient for proving many of the results which characterize Martin-Löf randomness for one probability space in terms of Martin-Löf randomness for another. There are many such examples in Martin-Löf random Brownian motion [13, Cor. 2], [1], [6].

Bienvenu and Porter [4] and independently Rute [30] showed that randomness conservation does not hold for computable randomness [4, Thm. 4.2][30, Cor. 9.7], but it does hold for Schnorr randomness [4, Thm. 4.1][30, Prop. 7.7]. Bienvenu and Porter asked if no-randomness-from-nothing holds for Schnorr and computable randomness.

In Section 3 we show that no-randomness-from-nothing holds for computable randomness.

In Section 4 we generalize the results of Section 3 in three ways: First, we generalize from almost-everywhere computable maps to Schnorr layerwise computable maps (a form of effectively measurable map well-suited for computable measure

\[1\] Simpson and Stephan [35] use the term “randomness preservation” for another property: if \( x \) is Martin-Löf random, then there is a PA degree \( p \) such that \( x \) is Martin-Löf random relative to \( p \).

\[2\] There is an error in the proof of no-randomness-from-nothing in [4, Thm. 3.5]. The authors say “Since \( \Phi \) is an almost total Turing functional, the image under \( \Phi \) of a \( \Pi^0_1 \) class is also a \( \Pi^0_1 \) class.” This is not true unless the \( \Pi^0_1 \) class is a subset of the domain of \( \Phi \). Fortunately, their proof only uses the \( \Pi^0_1 \) set \( 2^\mathbb{N} \setminus U_i \), where \( U_i \) is the \( i \)th level of the optimal Martin-Löf test. This set is contained in the domain of \( \Phi \). Moreover, the proof does not rely on the compactness of \( 2^\mathbb{N} \) at all, just on the effective compactness of \( K_i \). Therefore, no-randomness-from-nothing applies to all computable probability spaces, not just the compact ones, as observed by Hoyrup and Rojas [13, Prop. 5].

\[3\] In some applications (e.g. Hoyrup and Rojas [13, Cor. 2]) one may also need to apply the following theorem: if \( \nu \) is absolutely continuous with respect to \( \mu \) and the density function \( d\nu/d\mu \) is bounded from above by a constant (or by an \( L^1(\mu) \)-computable function), then every \( \nu \)-random is also \( \mu \)-random.
When does randomness come from randomness? Second, we generalize from Cantor space $2^\mathbb{N}$ to an arbitrary computable metric space. Third, we sketch how to relativize the result to an oracle, except that we use uniform relativization to which computable randomness is better suited. Section 4 is independent from the rest of the paper.

In Section 5, we give an interesting application of no-randomness-from-nothing for computable randomness. We show that if a probability measure $\mu$ is the sum of a computable sequence of measures $\mu_n$, then $x$ is $\mu$-computably random if and only if $x$ is $\mu_n$-computably random for some $n$.

In Section 6 we show no-randomness-from-nothing does not hold for Schnorr randomness. We even show something stronger. If $x$ is not computably random for $(2^\mathbb{N}, \mu)$, then there exists an almost-everywhere computable, measure-preserving map $T: (2^\mathbb{N}, \lambda) \rightarrow (2^\mathbb{N}, \mu)$ such that $T^{-1}(\{x\}) = \emptyset$.

In Section 7 we complete the picture by providing proofs of randomness conservation for difference randomness (unpublished result of Bienvenu) and 2-randomness.

Last, in Section 8, we will show how randomness conservation and no-randomness-from-nothing can be used to characterize a variety of randomness notions. The main result is that Martin-Löf randomness is the weakest randomness notion satisfying both randomness conservation and no-randomness-from-nothing. We give two different formulations of this result, one for all computable probability measures, and one for just the fair-coin probability measure. The second relies on a recent result of Petrović [28].

1.1. Conclusions on Schnorr and computable randomness. We caution the reader not to come to the hasty conclusion that Schnorr randomness and computable randomness are “unnatural” just because computable randomness does not satisfy randomness conservation and Schnorr randomness does not satisfy no-randomness-from-nothing.

Indeed there is already compelling evidence to their naturalness. Both Schnorr randomness and computable randomness have been characterized by a number of theorems in analysis [10, 27, 32]. Moreover, as argued by Schnorr [34], last paragraph] and Rute [31], Schnorr randomness is the randomness notion implicit in constructive measure theory. Last, Schnorr randomness seems to be the weakest randomness notion sufficient for working with measure theoretic objects (see, for example [10, 27, 32]).

Instead, as we will show in a future paper [29], it is randomness conservation and no-randomness-from-nothing that need to be modified. If one restricts the measure-preserving maps to those where the “conditional probability” is computable, then one recovers both randomness conservation and no-randomness-from-nothing for Schnorr and computable randomness. This class of maps is natural and covers nearly every measure-preserving map used in practice — including isomorphisms, projections on product measures, and even the von Neumann coin example above. Martin-Löf randomness also behaves better under these maps. Indeed, randomness conservation, no-randomness-from-nothing, and van Lambalgen’s theorem can be combined into one unified theorem for Schnorr and Martin-Löf randomness.

1.2. Status of randomness conservation and no-randomness-from-nothing. We end this introduction with a table summarizing the known results about randomness conservation and no-randomness-from-nothing.
| Randomness notion       | Randomness conservation | No-randomness-from-nothing |
|-------------------------|-------------------------|---------------------------|
| Kurtz random            | Yes [30, Prop. 7.7]     | No Thm. [25]              |
| Schnorr random          | Yes [4, Thm. 4.1] [30, Prop. 7.7] | No Thm. [25] |
| computable random       | No [4, Thm. 4.2] [30, Cor. 9.7] | Yes Thm. [7] |
| Martin-Löf random      | Yes [4, Thm. 3.2] [30, Prop. 7.7] | Yes Thm. [7] |
| Difference random       | Yes Prop. [28]          | Yes Prop. [28]            |
| Demuth random           | Yes Folklore            | ?                         |
| weak 2-random           | Yes [3, Thm. 5.9] [30, Prop. 7.7] | Yes [3, Thm. 6.18] |
| 2-random                | Yes Prop. [29]          | Yes Prop. [29]            |

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2. **Definitions and notation**

Let $2^N$ denote Cantor space and $2^*$ the set of all finite binary strings. Let $\varepsilon$ be the empty string, and $[\sigma]$ the cylinder set of $\sigma \in 2^*$. For a finite Borel measure $\mu$ on $2^N$ we will use the notation $\mu([\sigma]) := \mu([\sigma])$. For a finite Borel measure $\mu$ on $2^N \times 2^N$ we will use the notation $\mu([\sigma] \times [\tau]) := \mu([\sigma] \times [\tau])$. A measure $\mu$ on $2^N$ is computable if $\sigma \mapsto \mu(\sigma)$ is computable. The fair-coin measure $\lambda$ is given by $\lambda(\sigma) = 2^{-|\sigma|}$.

Given a computable map $F: 2^N \rightarrow 2^N$, the pushforward of $\mu$ along $F$ is the computable measure $\mu_F$ given by $\mu_F(\sigma) = \mu(F^{-1}([\sigma]))$.

As usual, we naturally identify the spaces $2^N \times 2^N$ and $2^N$, via the computable isomorphism $(x, y) \mapsto x \oplus y$. (Here $x \oplus y$ is the sequence $z \in 2^N$ given by $z(2n) = x(n)$ and $z(2n + 1) = y(n)$.) We also identify their computable measures, where $\mu$ on $2^N \times 2^N$ is identified with the pushforward of $\mu$ along $(x, y) \mapsto x \oplus y$.

We define Martin-Löf randomness, computable randomness, and Schnorr randomness through an integral test characterization. These characterizations are due to Levin [17], Rute [30, Thms. 5.3, 5.8], and Miyabe [23, Thm. 3.5] respectively. Recall that a lower semicomputable function $t: 2^N \rightarrow [0, \infty)$ is the sum of a computable sequence of computable functions $t_n: 2^N \rightarrow [0, \infty)$.

**Definition 2.** Let $\mu$ be a computable measure on $2^N$ and let $x \in 2^N$.

1. $x$ is $\mu$-Martin-Löf random if $t(x) < \infty$ for all lower semicomputable functions $t: 2^N \rightarrow [0, \infty]$ such that

\[
\int t d\mu \leq 1.
\]

2. $x$ is $\mu$-computably random if $t(x) < \infty$ for all lower semicomputable functions $t: 2^N \rightarrow [0, \infty]$ and all computable probability measures $\nu$ on $2^N$ such that

\[
(2.1) \quad \int_{[\sigma]} t d\mu \leq \nu(\sigma) \quad (\sigma \in 2^*).
\]

When it is true, randomness conservation is easy to prove, and in many cases well known. Therefore, the positive results in this column should probably be attributed to folklore. The citations given are for reference.
(3) \( x \) is \( \mu \)-Schnorr random if \( t(x) < \infty \) for all lower semicomputable functions \( t: 2^\mathbb{N} \to [0, \infty] \) such that
\[
\int t \, d\mu = 1.
\]

From these definitions it is obvious that Martin-Löf randomness implies computable randomness implies Schnorr randomness. (It is also known that the implications do not reverse.) Also, \( x \in 2^\mathbb{N} \) is \( \mu \)-Kurtz random if \( x \) is not in any \( \Sigma^0_2 \) \( \mu \)-null set. Every Schnorr random is Kurtz random.

Our definition of computable randomness transfers to \( 2^\mathbb{N} \times 2^\mathbb{N} \) as follows.

**Proposition 3.** Let \( \mu \) be a computable measure on \( 2^\mathbb{N} \times 2^\mathbb{N} \). A pair \((x, y) \in 2^\mathbb{N} \times 2^\mathbb{N}\) is \( \mu \)-computably random if and only if \( t(x, y) < \infty \) for all lower semicomputable functions \( t: 2^\mathbb{N} \times 2^\mathbb{N} \to [0, \infty] \) and all computable probability measures \( \nu \) on \( 2^\mathbb{N} \times 2^\mathbb{N} \) such that
\[
\int_{[\sigma \times \tau]} t \, d\mu \leq \nu(\sigma \times \tau) \quad (\sigma, \tau \in 2^\ast).
\]

**Proof.** Let \( \mu' \) denote the pushforward of \( \mu \) along \((x, y) \mapsto x \oplus y\). Given any test pair \( t, \nu \) satisfying (2.2) with \( \mu \), consider the test pair \( t', \nu' \) where \( t'(x \oplus y) = t(x, y) \) and \( \nu' \) is the pushforward of \( \nu \) under the map \((x, y) \mapsto x \oplus y\). Then \( t', \nu' \) satisfies (2.1) with \( \mu' \). Conversely, any test pair \( t, \nu \) satisfying (2.1) can be translated into a test pair \( t', \nu' \) satisfying (2.2) with \( \mu \). \( \square \)

The following more classical definition of computable randomness will be useful as well.

**Lemma 4** (See Rute \[30\] Def 2.4). If \( \mu \) is a computable measure on \( 2^\mathbb{N} \), a sequence \( x \in 2^\mathbb{N} \) is \( \mu \)-computably random if and only if both for all \( n \), \( \mu(x|_n) > 0 \) and for all computable measures \( \nu \),
\[
\lim_{n} \inf \frac{\mu(x|_n)}{\nu(x|_n)} < \infty.
\]

The ratio \( \nu(x|_n)/\mu(x|_n) \) is known as a martingale and can be thought of as a fair betting strategy. (See Rute \[30\] §2 for more discussion.) By an effective version of Doob’s martingale convergence theorem, this ratio converges on computable randoms.

**Lemma 5** (Folklore \[5\] Thm 7.1.3\[5\]). Assume \( x \in 2^\mathbb{N} \) is \( \mu \)-computably random and \( \nu \) is a computable measure. Then the following limit converges,
\[
\lim_{n} \frac{\nu(x|_n)}{\mu(x|_n)}.
\]

**Definition 6.** A partial map \( T: \subseteq 2^\mathbb{N} \to 2^\mathbb{N} \) is said to be \( \mu \)-almost-everywhere (\( \mu \)-a.e.) computable for a computable probability measure \( \mu \) if \( T \) is partial computable and \( \mu(\text{dom } T) = 1 \).

\[5\] The proof in \[5\] Thm 7.1.3 is for when \( \mu \) is the fair-coin measure, but the proof is the same for all computable measures.

\[6\] For concreteness, say \( T: \subseteq 2^\mathbb{N} \to 2^\mathbb{N} \) is partial computable if it is given by a monotone machine \( M: \subseteq 2^\ast \to 2^\ast \). Say \( x \in \text{dom } T \) if and only if there is some \( y \in 2^\mathbb{N} \) such that \( y = \lim_n M(x|_n) \). In this case, \( T(x) = y \). The domain of a partial computable map is always \( \Pi^0_2 \).
We denote $\mu$-a.e. computable maps with the notation $T: (2^N, \mu) \to 2^N$. Moreover, given a $\mu$-a.e. computable map $T: (2^N, \mu) \to 2^N$, the pushforward measure $\mu_T$ (of $\mu$ along $T$) is a well-defined probability measure computable from $T$ and $\mu$. We use the notation $T: (2^N, \mu) \to (2^N, \nu)$ to denote that $T$ is measure-preserving, that is $\nu = \mu_T$.

3. No-randomness-from-nothing for computable randomness

In this section we will prove the following.

**Theorem 7** (No-randomness-from-nothing for computable randomness.). If $\mu$ is a computable probability measure on $2^N$, $T: (2^N, \mu) \to 2^N$ is a $\mu$-a.e. computable map, and $y \in 2^N$ is $\mu_T$-computably random, then $y = T(x)$ for some $\mu$-computably random $x \in 2^N$.

The proof will be similar to that of van Lambalgen’s theorem [5, §6.9.1][26, Thm 3.4.6], which states that $(x, y)$ is Martin-Löf random if and only if $x$ is Martin-Löf random and $y$ is Martin-Löf random relative to $x$. First, however, we require a number of lemmas establishing properties of computable randomness on $2^N$ and $2^N \times 2^N$. The following lemma establishes randomness conservation for computable randomness along a.e. computable isomorphisms and will be a key tool in this proof.

**Lemma 8** (Rute [30, Prop 7.6, Thm. 7.11]). Let $\mu$ and $\nu$ be computable probability measures on $2^N$. Let $F: (2^N, \mu) \to (2^N, \nu)$ and $G: (2^N, \nu) \to (2^N, \mu)$ be almost-everywhere computable measure-preserving maps such that $G(F(x)) = x$ $\mu$-a.e. and $F(G(y)) = y$ $\nu$-a.e.

Then $F$ and $G$ both conserve computable randomness, and if $x$ is $\mu$-computably random and $y$ is $\nu$-computably random then $G(F(x)) = x$ and $F(G(y)) = y$.

However, most maps $T: (2^N, \mu) \to 2^N$ are not isomorphisms. Nonetheless, we can turn them into isomorphisms by mapping $x$ to the pair $(x, T(x))$.

**Lemma 9.** Let $\mu$ be a computable probability measure on $2^N$, let $T: (2^N, \mu) \to 2^N$ be a $\mu$-a.e. computable map, and let $(\text{id}, T)$ be the map $x \mapsto (x, T(x))$. If $(x, y)$ is $\mu(\text{id}, T)$-computably random, then $x$ is $\mu$-computably random and $y = T(x)$.

**Proof.** Clearly $(\text{id}, T)$ is a $\mu$-a.e. computable map. Moreover, notice that $(\text{id}, T)$ and its inverse $(x, y) \mapsto x$ satisfy the conditions of Lemma 8. Therefore if $(x, y)$ is $\mu(\text{id}, T)$-computably random, then $x$ is $\mu$-computably random. By the composition $(x, y) \mapsto x \mapsto (x, T(x))$, we have that $y = T(x)$. □

The main idea of the proof of Theorem 7 is as follows. The measure $\mu(\text{id}, T)$ is supported on the graph of $T$,

$$\{(x, T(x)) : x \in 2^N\}.$$
Therefore, given a $\mu_1$-computably random $y$, by this last lemma, it is sufficient to find some $x$ which makes $(x, y) \mu_{(id, \tau)}$-computably random. That is the goal of the rest of this section.

**Lemma 10.** Let $\mu$ be a computable probability measure on $2^N \times 2^N$ with second marginal $\mu_2$ (that is $\mu_2(\tau) = \mu(\varepsilon \times \tau)$). Assume $y \in 2^N$ is $\mu_2$-computably random. The following properties hold.

1. For a fixed $\tau \in 2^*$, $\mu(\cdot \times \tau)$ is a measure, that is
   $\mu(\sigma 0 \times \tau) + \mu(\sigma 1 \times \tau) = \mu(\sigma \times \tau)$ $(\sigma \in 2^*)$.
2. For a fixed $\sigma \in 2^*$, $\mu(\sigma \times \cdot)$ is a measure.
3. The following limit converges for each $\sigma \in 2^*$,
   $$\mu(\sigma | y) := \lim_n \frac{\mu(\sigma \times y|_n)}{\mu_2(y|_n)}.$$ 
4. The function $\mu(\cdot | y)$ defines a probability measure, that is $\mu(\varepsilon | y) = 1$ and $\mu(\sigma 0 | y) + \mu(\sigma 1 | y) = \mu(\sigma | y)$ $(\sigma \in 2^*)$.
5. For a continuous function $f : 2^N \times 2^N \to \mathbb{R}$, if $f | y = f(\cdot, y)$ then
   $$\int f^y d\mu(| y) = \lim_n \int_{[x] \times [y|_n]} f d\mu.$$ 
6. For a nonnegative lower semicontinuous function $t : 2^N \times 2^N \to [0, \infty]$, if $t^y = t(\cdot, y)$ then
   $$\int t^y d\mu(| y) \leq \lim_n \int_{[x] \times [y|_n]} t d\mu.$$ 

**Proof.** (1) and (2) are apparent. Then (3) follows from (2) and Lemma [5]. Then (4) follows from (1) and the definition of $\mu(\cdot | y)$.

As for (5), first consider the case where $f$ is a step function of the form $\sum_{\sigma=0}^{\bar{a}} a_\sigma 1_{[\sigma]}$. This case follows from (4). Since such step functions are dense in the continuous functions under the norm $\|f\| = \sup_f f(x)$, we have the result.

As for (6), $t = \sum_k f_k$ for a sequence of continuous nonnegative $f_k$. Then we apply the monotone convergence theorem (MCT) for integrals and Fatou’s lemma for sums,

$$\int t^y d\mu(| y) = \int \sum_k f_k^y d\mu(| y) \overset{\text{MCT}}{=} \sum_k \int f_k^y d\mu(| y)$$ 

$$\leq \lim_n \sum_k \int_{[x] \times [y|_n]} f_k d\mu\Bigg| \frac{\mu_2(y|_n)}{\mu_2(y|_n)} = \lim_n \int_{[x] \times [y|_n]} t d\mu\Bigg| \frac{\text{Fatou}}{\mu_2(y|_n)}.$$ 

**Definition 11 ([16] Def. 2.1 [2] Def. 5.37).** Let $\mu$ be a probability measure on $2^N$ which may not be computable. A sequence $x \in 2^N$ is *blind $\mu$-Martin-Löf random* (or *$\mu$-Hippocratic random*) relative to $y \in 2^N$ if $t(x) < \infty$ for all $t : 2^N \to [0, \infty]$ which are lower semicontinuous relative to $y$ such that $\int t d\mu < \infty$.

Finally, we have the tools to find some $x$ to pair with $y$. What follows is a variation of van Lambergen’s theorem, similar to that given by Takahashi [37] Thm. 5.2.
Lemma 12. Let $\mu$ be a computable measure on $2^N \times 2^N$. Let $y$ be $\mu_2$-computably random. Let $x$ be blind $\mu(\cdot|y)$-Martin-Löf random relative to $y$. Then $(x, y)$ is $\mu$-computably random.

Proof. Assume $y$ is $\mu_2$-computably random, but $(x, y)$ is not $\mu$-computably random. Then by Proposition 8 there is a lower semicomputable function $t$ and a computable measure $\nu$ such that $t(x, y) = \infty$ and \[ \int_{[\sigma] \times [\tau]} t \, d\mu \leq \nu(\sigma \times \tau). \]

Let $t^y = t(\cdot, y)$. Then $t^y$ is lower semicomputable relative to $y$. Moreover, since $t$ is lower semicontinuous, we have by Lemma 10 that
\[ \hat{t}^y d\mu(\cdot|y) \leq \lim_n \int_{\varepsilon \times y|n} t \, d\mu \leq \nu(\varepsilon \times y|n) \]
where the right-hand side converges to a finite value by Lemma 5 since $y$ is $\mu_2$-computably random. Therefore, $x$ is not blind Martin-Löf random relative to $y$ since $\int t^y d\mu(\cdot|y) < \infty$ and $t^y(x) = t(x, y) = \infty$. \qed

The proof of Theorem 7 easily follows.

Proof of Theorem 7. Let $\mu$ be a computable probability measure on $2^N$, let $T: (2^N, \mu) \to 2^N$ be an a.e. computable map, and let $y \in 2^N$ be $\mu_T$-computably random. We want to find some $\mu$-computably random $x \in 2^N$ such that $y = T(x)$.

Let $\nu = \mu(\text{id}, T)$, and let $x$ be blind $\nu(\cdot|y)$-Martin-Löf random relative to $y$ (there are $\nu(\cdot|y)$-measure-one many, so there is at least one). By Lemma 12, $(x, y)$ is $\mu(\text{id}, T)$-computably random. By Lemma 9, $x$ is computably random and $y = T(x)$. \qed

4. Three generalizations to theorem 7

Algorithmic randomness is becoming more focused around the ideas in probability theory and measure theory. Therefore the tools in algorithmic randomness need to be able to handle a larger variety of maps, a larger variety of spaces, and even a larger variety of relativizations. In this section we generalize Theorem 7 to Schnorr layerwise computable maps, arbitrary computable metric spaces, and uniform relativization. This section is independent of the later sections and may be skipped.

4.1. Generalizing the types of maps. The main results of this subsection are generalizations of Theorem 7 and Lemma 8 to Schnorr layerwise computable maps. Before proving the theorems, let us introduce Schnorr layerwise computable maps, and explain why they are important.

4.1.1. Schnorr layerwise computable maps. So far, the results in this paper have been phrased in terms of almost-everywhere computable maps. These maps are easy for a computability theorist to understand and they are sufficient for many purposes. However, almost-everywhere computable maps are only almost-everywhere continuous and therefore are not adequate computable representations of the measurable maps found in probability theory.\footnote{For example, the map $F: (2^N, \lambda) \to 2$ which takes a sequence $x \in 2^N$ and returns 1 if $\sum_{n=0}^{\infty} x(n) > 2/3$ and 0 otherwise is not almost-everywhere continuous, and therefore not almost-everywhere computable (even relative to an oracle). (We thank Bjørn Kjos-Hanssen for this example.)}
Instead, we need a notion of an “effectively $\mu$-measurable function” $F : (2^N, \mu) \to 2^N$. There are many approaches in the literature. One approach, dating back to the Russian constructivist Šanin [33, §15.4], is to use the topology of convergence in measure, which is metrizable via many equivalent metrics, including $\rho(F, G) = \int d(F(x), G(x)) \, d\mu(x)$ where $d$ is the usual metric on $2^N$. (This is a modification of the usual $L^1$-metric\textsuperscript{10}.) If $\mu$ is a Borel measure, the space $L^0(2^N, \mu; 2^N)$ of measurable functions $F : (2^N, \mu) \to 2^N$ modulo $\mu$-almost-everywhere equivalence is a Polish space.

**Definition 13.** Fix a computable probability measure $\mu$. A function $F \in L^0(2^N, \mu; 2^N)$ is **effectively measurable** if there is a computable sequence of a.e. computable functions $F_n : (2^N, \mu) \to 2^N$ such that $\rho(F_n, F_m) \leq 2^{-m}$ for all $n > m$ and $F = \lim_n F_n$ (where the limit is in the metric $\rho$).

Let $\hat{F}$ be the pointwise limit of $F_n$ (when it converges)\textsuperscript{11} Call $\hat{F}$ the **canonical representative** of $F$.

Surprisingly the canonical representative is always defined on Schnorr randoms. Moreover, if $F$ and $G$ are $\mu$-a.e. equal effectively measurable maps, then $\hat{F}(x) = \hat{G}(x)$ on $\mu$-Schnorr randoms $x$ [32, p. 41, Prop. 3.18] (also see [27, Thm. 3.9]). Finally, Rute [32, p. 41, Prop 3.23] showed that these representative functions $\hat{F}$ are equivalent to the Schnorr layerwise computable functions of Miyabe [23]. (This equivalence is an effective version of Lusin’s theorem.)

**Definition 14.** A measurable map $F : (X, \mu) \to Y$ is **Schnorr layerwise computable** if there is a computable sequence of effectively closed (that is $\Pi^0_2$) sets $C_n \subseteq X$ and a computable sequence of computable functions $F_n : C_n \to Y$ such that

1. $\mu(C_n) \geq 1 - 2^{-n}$ and $\mu(C_n)$ is computable in $n$, and
2. $F_n = F \upharpoonright C_n$ for all $n$.

The remainder of the results in this section will be expressed in terms of Schnorr layerwise computability.

4.1.2. **No-randomness-from-nothing for computable randomness and Schnorr layerwise computable functions.** In order to extend Theorem 7 to Schnorr layerwise computable functions, it suffices to prove a Schnorr layerwise computable version of Lemma 9. In order to do that, we need some lemmas.

Notice that the definitions of Schnorr randomness and computable randomness naturally extend to all computable measures on $2^N$, not just probability measures. This saves us the step of having to normalize a measure into a probability measure, as in this next lemma.

**Lemma 15.** Let $C$ be an effectively closed subset of $2^N$ such that $\mu(C)$ is computable and positive. Let $\nu$ be the measure $\nu(\sigma) = \mu(C \cap [\sigma])$. This measure is a computable measure, and for any $x \in C$, if $x$ is $\mu$-computably random then $x$ is $\nu$-computably random.

\textsuperscript{10}The space $(2^N, d)$ has bounded diameter. For a general codomain $(Y, d)$, use the metric $\rho(F, G) = \int \min\{d(F(x), G(x)), 1\} \, d\mu(x)$.

\textsuperscript{11}The sequence $F_n$ converges $\mu$-almost-everywhere since $F_n$ converges at a geometric rate of convergence in $\rho$. Also $\hat{F}$ is possibly partial, since there may be a measure zero set of $x \in 2^N$ where $\lim_n F_n(x)$ does not converge.
Proof. Since \( \mu(C) \) is computable we can compute \( \mu(C \cap \sigma]| \). By the Lebesgue density theorem, we have for almost-every \( x \in C \) that
\[
\lim_{n} \frac{\mu(C \cap [x|_n])}{\mu(x|_n)} = 1.
\]
Moreover, the set
\[
N = \left\{ x \in 2^\mathbb{N} : x \in C \text{ and } \limsup_{n} \frac{\mu(C \cap [x|_n])}{\mu(x|_n)} < 1 \right\}
\]
is a \( \Sigma^0_2 \) set of \( \mu \)-measure 0. Therefore, \( N \) does not contain any \( \mu \)-Kurtz randoms. Hence if \( x \in C \) and \( x \) is \( \mu \)-computably random (so \( \mu \)-Kurtz random), then \( \limsup_{n} \frac{\mu(C \cap [x|_n])}{\mu(x|_n)} = 1 \). \( ^{12} \)

Now, let \( \rho \) be a computable measure. Then for any \( \mu \)-computable random \( x \in C \), we have that \( \rho(x|_n)/\mu(x|_n) \) converges to a finite number (Lemma 5). Hence
\[
\liminf_{n} \frac{\rho(x|_n)}{\nu(x|_n)} = \liminf_{n} \frac{\rho(x|_n)}{\mu(C \cap [x|_n])} = \liminf_{n} \frac{\rho(x|_n)}{\mu(x|_n)} \cdot \frac{\mu(x|_n)}{\mu(C \cap [x|_n])} = \liminf_{n} \frac{\rho(x|_n)}{\mu(x|_n)} < \infty.
\]

Since \( \rho \) is arbitrary, \( x \) is \( \nu \)-computably random (Lemma 4). \( \square \)

Lemma 16. Assume \( \mu \leq \nu \) (that is \( \mu(A) \leq \nu(A) \) for all measurable sets \( A \subseteq 2^\mathbb{N} \). Then if \( x \) is \( \mu \)-computably random, then \( x \) is \( \nu \)-computably random.

Proof. Assume \( x \) is not \( \nu \)-computably random. Then there is an integral test \( t \) such that \( t(x) = \infty \) and \( \int_A t \, d\nu \leq \rho(A) \) for some computable measure \( \rho \). Then \( \int_A t \, d\mu \leq \int_A t \, d\nu \leq \rho(A) \) and therefore \( t \) is a test on \( \mu \) as well. Hence \( x \) is not \( \mu \)-computably random. \( \square \)

Lemma 17. Assume \( T : (2^\mathbb{N}, \mu) \rightarrow 2^\mathbb{N} \) is an injective Schnorr layerwise computable map. Then the pushforward measure \( \mu_T \) is computable, and \( T \) has an injective Schnorr layerwise computable inverse \( S \). Moreover, both \( T \) and \( S \) conserve computable randomness.

Proof. The computability of the pushforward is proved by Hoyrup and Rojas \( ^{12} \) Prop. 4 and Rute \( ^{32} \) p. 42, Prop. 3.25.

Let \( C_n \) and \( T_n : C_n \rightarrow 2^\mathbb{N} \) be the sequence of effectively closed sets and partial computable functions as in Definition \( ^{14} \). Since \( T_n \) is total computable on \( C_n \) (and \( 2^\mathbb{N} \) is effectively compact), the image \( D_n := T_n(C_n) \) is effectively closed in \( n \). Since \( T \) is injective, \( \mu_T(D_n) = \mu(T^{-1}(D_n)) = \mu(T_n^{-1}(D_n)) = \mu(C_n) \). For a fixed \( n \), we can compute a total computable map \( S_n : D_n \rightarrow C_n \) given as follows. For each \( y \in D_n \), we can compute the set \( P_y = \{ x \in C_n : T_n(x) = y \} \) as a \( \Pi^0_2 \) set. (To see that it is a \( \Pi^0_2 \) set, for each \( x \), just wait until \( x \notin C_n \) or \( T_n(x)|_k \neq y|_k \) for some \( k \).) Since \( T_n : C_n \rightarrow D_n \) is a bijection, \( P_y \) is a singleton \( \{ x \} \). Let \( S_n(y) = x \). This

\( ^{12} \) Actually, one can show that the limit is 1 for all \( \mu \)-Schnorr randoms \( ^{32} \) p. 51, Thm 6.3, and hence for all \( \mu \)-computable randoms, but this is not needed.
$S_n$ is computable on $D_n$ (again using the compactness of $2^N$). Combining these $S_n$ we have a Schnorr layerwise computable map $S: (2^N, \mu_T) \to (2^N, \mu)$. This map is injective (on its domain $\bigcup_n D_n$) since each of the $S_n$ are injective. Moreover, it is the inverse of $T$ since each $S_n$ is the inverse of $T_n$.

Let $x$ be $\mu$-computationally random. Since $x$ is $\mu$-Schnorr random, there is some $C_n$ from the Schnorr layerwise description of $T$ such that $x_0 \in C_n$ and $T$ is computable on $C_n$. (If $x$ is in no such $C_n$, then $x$ is in $\bigcap_n (2^N \setminus C_n)$ and cannot be Schnorr random.) Let $\mu_n$ be the measure given by $\mu_n(\sigma) = \mu(C_n \cap [\sigma])$. By Lemma 15, $\mu_n$ is computable and $x$ is $\mu_n$-computably random. Notice that $T_n: (2^N, \mu_n) \to 2^N$ is $\mu_n$-a.e. computable and therefore $(\mu_n)_T$ is computable. Moreover, $T_n: (2^N, \mu_n) \to (2^N, (\mu_n)_T)$ and $S_n: (2^N, (\mu_n)_T) \to (2^N, \mu_n)$ are a.e. computable measure-preserving inverses as in Lemma 8. Therefore $T(x) = T_n(x)$ is $(\mu_n)_T$-computably random. Since $(\mu_n)_T \leq \mu_T$, we have by Lemma 16 $T(x)$ is $\mu_T$-computably random. □

Now we can give a Schnorr layerwise computable version of Lemma 9.

Lemma 18. Let $\mu$ be a computable probability measure on $2^N$, let $T: (2^N, \mu) \to 2^N$ be a Schnorr layerwise computable map, and let $(id, T)$ be the map $x \mapsto (x, T(x))$. If $(x, y)$ is $\mu(id, T)$-computationally random, then $x$ is $\mu$-computably random and $y = T(x)$.

Proof. Clearly $(id, T)$ is an injective Schnorr layerwise computable map. By Lemma 17, $(id, T)$ has an injective Schnorr layerwise computable inverse map $S: (2^N, \mu(id, T)) \to (2^N, \mu)$. This inverse is clearly the map $(x, y) \mapsto x$ (except restricted to a smaller domain to guarantee injectivity). If $(x, y)$ is $\mu(id, T)$-computationally random, then $x = S(x, y)$ is $\mu$-computably random by the randomness conservation in Lemma 17. By the composition $(x, y) \mapsto x \mapsto (x, T(x))$, we have that $y = T(x)$. □

No-randomness-from-nothing follows just as before.

Theorem 19. Assume $(2^N, \mu)$ is a computable probability space, $T: (2^N, \mu) \to 2^N$ a Schnorr layerwise computable map, and $y \in 2^N$ a $\mu_T$-computably random. Then $y = T(x)$ for some $\mu$-computable random $x \in 2^N$.

Proof. Let $\mu$ be a computable probability measure on $2^N$, let $T: (2^N, \mu) \to 2^N$ be a Schnorr layerwise computable map, and let $y \in 2^N$ be $\mu_T$-computably random. We want to find some $\mu$-computable random $x \in 2^N$ such that $y = T(x)$.

Let $\nu = \mu(id, T)$, and let $x$ be blind $\nu(\cdot | y)$-Martin-Löf random relative to $y$ (there are $\nu(\cdot | y)$-measure-one many, so there is at least one). By Lemma 12, $(x, y)$ is $\mu(id, T)$-computably random. By Lemma 18, $x$ is computably random and $y = T(x)$. □

4.1.3. Isomorphism theorem for computable randomness and Schnorr layerwise computable maps. We end this subsection, by using Theorem 19 to give a Schnorr layerwise computable version of Lemma 6.

Theorem 20. Let $\mu$ and $\nu$ be computable probability measures on $2^N$. Let $F: (2^N, \mu) \to (2^N, \nu)$ and $G: (2^N, \nu) \to (2^N, \mu)$ be Schnorr layerwise computable measure-preserving maps such that

$$G(F(x)) = x \ \mu\text{-a.e.} \ \text{and} \ \ F(G(y)) = y \ \nu\text{-a.e.}$$

Then $F$ and $G$ both conserve computable randomness, and if $x$ is $\mu$-computably random and $y$ is $\nu$-computably random then

$$G(F(x)) = x \ \text{and} \ \ F(G(y)) = y.$$
Proof. Let $x$ be $\mu$-computably random. Then by Lemma [19], there is some $\nu$-computably random $y$ such that $G(y) = x$. It remains to prove that $F(x) = y$.

Assume not. Then $F(G(y)) = F(x) \neq y$. Since $x$ and $y$ are computably random and since $F$ and $G$ are Schnorr layerwise computable (with layerings $C_n$ and $D_n$ respectively), there is a large enough $n$ such that $x \in C_n$ and $y \in D_n$. Then consider the set

$$E_n = \{y_0 \in D_n \cap G_n^{-1}(C_n) : F_n(G_n(y_0)) \neq y_0\}.$$ 

This set is the intersection of the $\Pi_0^1$ set $D_n \cap G_n^{-1}(C_n)$ and the $\Sigma_0^1$ set

$$\{y_0 : y_0 \notin D_n \cap G_n^{-1}(C_n) \text{ or } F_n(G_n(y_0)) \neq y_0\}.$$ 

So $E_n$ is $\Sigma_2^0$. Moreover, since $F$ and $G$ are almost-surely inverses, $E_n$ has $\nu$-measure zero. Hence it contains no $\nu$-Kurtz randoms, and therefore no $\nu$-computable randoms. Since $y$ is $\nu$-computably random and in $E_n$, we have a contradiction.

The other direction follows by symmetry. □

4.2. Extending to computable metric spaces. Theorem [7] also applies to any computable metric space as we will show in this section. Randomness on computable metric spaces is well-understood; see [2, 14, 10, 32, 30, 9]. In particular, Rute [30] developed a theory of computable randomness on a computable probability measure $(X, \mu)$ where $X$ is a computable metric space and $\mu$ is a computable measure on that space. Indeed the definition of computable randomness (Definition [2]) can be extended to any computable probability space by replacing (2.1) with the following (Rute [30] Thm 5.8),

$$\int_B t \, d\mu \leq \nu(B) \quad (B \subseteq X \text{ Borel}).$$

Moreover, each computable probability space $(X, \mu)$ is isomorphic to a computable measure $\mu'$ on $2^N$, via a pair of almost-everywhere computable measure-preserving maps $I_\mu$ and $I_{\mu}^{-1}$ which commute (up to a.e. equivalence) [30] Prop. 7.9.

$$(X, \mu) \xrightarrow{I_\mu} (2^N, \mu') \xleftarrow{I_{\mu}^{-1}} (X, \mu)$$

Then any Schnorr layerwise computable measure-preserving map $T : (X, \mu) \to (Y, \nu)$ can be transferred to a Schnorr layerwise computable measure-preserving map $T' : (2^N, \mu') \to (2^N, \nu')$ such that the following diagram commutes (up to a.e. equivalence).

$$\begin{array}{ccc}
(X, \mu) & \xrightarrow{T} & (Y, \nu) \\
\downarrow I_\mu & & \downarrow I_{\nu} \\
(2^N, \mu') & \xrightarrow{T'} & (2^N, \nu')
\end{array}$$

That is, we set $T' := I_\nu \circ T \circ I_{\mu}^{-1}$. Then $T = I_{\mu}^{-1} \circ T' \circ I_\mu$ $\mu$-a.e. (Rute showed that a.e. computable maps are Schnorr layerwise computable maps [32] p. 41, Prop. 3.24] and that Schnorr layerwise computable maps are closed under composition [32] p. 42, Prop. 3.27].)

Now we can show the analogue of Theorem [7]. Let $y \in Y$ be $\nu$-random. By Theorem [20] (also Rute [30] Prop 7.6, Thm. 7.11), we have that $y' := I_\nu(y)$ is $\nu'$-computably random and $I_{\nu}^{-1}(y') = I_{\nu}^{-1}(I_\nu(y)) = y$. Applying Theorem [19] there is
some $x'$ which is $\mu'$-computably random such that $T'(x') = y'$. Last, again using
Theorem 20 we have that $x := I^{-1}_\nu(x')$ is $\mu$-computably random and $I_\nu(x) = x'$.
Since $T$ and $I^{-1}_\nu \circ T' \circ I_\nu$ are equal $\mu$-almost-everywhere, then they are equal on all
$\mu$-computable randoms.
Therefore
$$T(x) = I^{-1}_\nu(T'(I_\mu(x))) = y.$$ 

4.3. Generalizations to uniform relativization. It is well known that one can relativize most proofs in computability theory. In particular, one can straightforwardly relativize the proof of Theorem 7 to get the result that if $\mu$ is computable from $a \in \mathbb{N}^\mathbb{N}$, $T : (2^\mathbb{N}, \mu) \rightarrow 2^\mathbb{N}$ is computable from $a$, and $y$ is $\mu$-computably random relative to $a$, then there exists some $x \in \mathbb{N}^\mathbb{N}$ which is $\mu$-computably random relative to $a$ such that $T(x) = y$.

The problem with this result is that this is not the best form of relativization for computable randomness. It has been shown recently that Schnorr and computable randomness behave better under uniform relativization (which is related to truthable reducibility) \[8, \ 22, \ 24\].

The full generalization of Theorem 7 to uniform relativization would be as follows. In the remainder of this section assume that $X$ and $Y$ are computable metric spaces and that $\{\mu_a\}_{a \in \mathbb{N}^\mathbb{N}}$ is a family of probability measures on $X$ such that $\mu_a$ is uniformly computable in $a \in \mathbb{N}^\mathbb{N}$. Say that $x \in X$ is $\mu_{a_0}$-computably random uniformly relativized to $a_0 \in \mathbb{N}^\mathbb{N}$ if $t_{a_0}(x) < \infty$ for all families $\{\nu_a, t_a\}_{a \in \mathbb{N}^\mathbb{N}}$, where $\nu_a$ is a probability measure on $X$ uniformly computable in $a \in \mathbb{N}^\mathbb{N}$, and $t_a$ is uniformly lower semicomputable in $a$, such that
$$\int_A t_a \, d\mu_a \leq \nu_a(A) \quad (A \subseteq X \text{ Borel}).$$

Conjecture 21. Assume $\{T_a\}_{a \in \mathbb{N}^\mathbb{N}}$ is a family of layerwise maps $T_a : (X, \mu_a) \rightarrow Y$ where $T_a$ is Schnorr layerwise computable uniformly in $a$. Fix $a_0 \in \mathbb{N}^\mathbb{N}$. If $y \in Y$ is $(\mu_{a_0}, t_{a_0})$-computably random uniformly relativized to $a_0$, then $y = T_{a_0}(x)$ for some $x \in X$ which is $\mu_{a_0}$-computably random uniformly relativized to $a_0$.

To prove this conjecture one would need to check that each step in the proof of Theorem 19 (and its generalization to computable metrics spaces and Schnorr layerwise computable maps) can be done uniformly in $a$. We see no issue in doing this (for example, we are not aware of any nonuniform case analyses in the proof). However, verifying all the details is beyond the scope of this paper.

5. An application of no-randomness-from-nothing for computable randomness

There exist many applications of no-randomness-from-nothing in the literature for Martin-Löf randomness. Most of these are direct applications. By Theorems 7 and the generalizations in Section 4, these results also apply to computable randomness. In this section, we would like to give another less obvious application. (Recall that computable randomness can be naturally extended to computable measures which are not necessarily probability measures.)

Theorem 22. Let $\mu$ be a probability measure on $2^\mathbb{N}$ which is a countable sum of measures $\mu = \sum_n \mu_n$ where $\mu_n$ is computable in $n$. If $x$ is $\mu$-computably random, then $x$ is $\mu_n$-computably random for some $n$.

\[13\] They are even equal on all Schnorr randoms. [32, p. 41, Prop. 3.18].
Proof. Let $\nu$ be the measure which is basically a disjoint union of the $\mu_n$. Specifically, let $0^n$ denote a sequence of $n$ many 0s. Then let $\nu$ be the measure given by $\nu(0^n1\sigma) = \mu_n(\sigma)$ and $\nu(\{0^\infty\}) = 0$. This measure is computable and it is easy to see that $0^n1\sigma$ is $\nu$-computably random if and only if $x$ is $\mu_n$-computably random. Then let $T: \subseteq 2^N \to 2^N$ be the partial computable map given by $T(0^n1\sigma) = x$. This map is measure-preserving of type $T: (2^N, \nu) \to (2^N, \mu)$. By no-randomness-from-nothing, each $\mu$-computable random $x$ comes from some $\nu$-computable random $0^n1\sigma$. Therefore, $x$ is $\mu_n$-computably random for some $n$. □

The proof of Theorem 22 holds for any randomness notion satisfying no-randomness-from-nothing and satisfying the property that if $\nu(0^n1\sigma) = \mu_n(\sigma)$, $\nu(\{0^\infty\}) = 0$, and $0^n1\sigma$ is random for $\nu$ then $x$ is random for $\mu_n$. In particular, Theorem 22 holds for Martin-Löf randomness.

Question 23. Does Theorem 22 hold for Schnorr randomness?

Remark 24. The converse to Theorem 22 is as follows. If $\mu \leq \nu$ are computable measures and $x$ is $\mu$-random, then $x$ is $\nu$ random. For computable randomness, this is Lemma 16. It is also trivial to prove for many other randomness notions, including Martin-Löf and Schnorr randomness. However, notice that, just as Theorem 22 follows from no-randomness-from-nothing, its converse follows from randomness conservation. The measure $\nu$ is the sum of the measures $\mu$ and $\nu - \mu$. Take a disjoint sum of $\mu$ and $\nu - \mu$ (as in the proof of Theorem 22), map this disjoint sum to $\nu$, and apply randomness conservation.

6. NO-RANDOMNESS-FROM-NOTHING FOR SCHNORR RANDOMNESS

In this section, we will prove the following theorem.

Theorem 25. Let $\mu$ be a computable probability measure, and assume $x_0 \in 2^N$ is not computably random with respect to $\mu$. Then there exists an almost-everywhere computable measure-preserving map $T: (2^N, \lambda) \to (2^N, \mu)$ such that $T^{-1}(\{x_0\}) = \emptyset$.

As an obvious corollary we have the following.

Corollary 26. Schnorr randomness does not satisfy no-randomness-from-nothing.
where the second equality comes from
\[
\int_{\sigma} \frac{\mu(x|n)}{\nu(x|n)} \, d\nu(x) = \mu(\sigma) \quad (\text{for } n \geq |\sigma|).
\]
Notice by our construction that \( f(x_0) = 0 \). Also, the area under the curve has \( \nu \otimes \mathcal{L} \)-measure one. (Here \( \mathcal{L} \) is the Lebesgue measure on \( \mathbb{R} \).) See Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Graph of the density function \( f \) of \( \mu \) with respect to \( \nu \). The area under the curve has \( \nu \otimes \mathcal{L} \)-measure one.}
\end{figure}

Our main tool is the following lemma. (We will be working on the computable metric space \( 2^N \times [0, 2] \) for convenience. There is no loss if we replace \([0, 2]\) with \( 2^N \), where \( x \in [0, 2] \) is identified with the binary expansion of \( x/2 \).)

**Lemma 27.** Assume \( X_1 \) and \( X_2 \) are computable metric spaces (e.g. \( 2^N \) and \( 2^N \times [0, 2] \)). If \( \rho_1 \) and \( \rho_2 \) are computable probability measures on \( X_1 \) and \( X_2 \) respectively and \( \rho_1 \) is atomless, then one can construct an almost-everywhere computable measure-preserving map \( T: (X_1, \rho_1) \to (X_2, \rho_2) \) (uniformly in the codes for \( \rho_1 \) and \( \rho_2 \)).

**Proof.** Since \( \rho_1 \) is atomless, \((X_1, \rho_1)\) is isomorphic to \((2^N, \lambda)\) \([39\text{ Prop. 7.16}]\) (uniformly in \( \rho_1 \)). For any computable measure, there is an almost-everywhere computable measure-preserving map \( T: (2^N, \lambda) \to (X_2, \rho_2) \) \([39\text{ Prop. 7.9}]\). \( \square \)

6.1. **Description of the construction.** Consider the probability measure \( \rho \) which is equal to \( \nu \otimes \mathcal{L} \) restricted to the area under the curve of \( f \). (See Figure 1) Unfortunately, this is not necessarily a computable measure on \( 2^N \times [0, 2] \).

As motivation, let us consider for a moment the case where \( \rho \) is computable and \( f \) is also computable. Then it would be easy to construct the desired \( T \) as follows. Using Lemma 27 find a measure-preserving almost-everywhere computable map \( T': (2^N, \lambda) \to (2^N \times [0, 2], \rho) \). Let \( T: (2^N, \lambda) \to 2^N \) be \( T = \pi \circ T' \) where \( \pi: 2^N \times [0, 2] \to 2^N \) denotes the projection map \((x, y) \mapsto x \). The pushforward of \( \lambda \) along \( T \) is \( \lambda_T = \mu \). Last, the preimage \( \pi^{-1}(\{x_0\}) \) is contained in \( P = \{(x, y) \in 2^N \times [0, 2] : f(x) = 0\} \), which is a \( \rho \)-measure-zero \( \Pi^0_1 \) subset of \( 2^N \times [0, 2] \) (since we are assuming \( f \) is computable and \( f(x_0) = 0 \)). If we remove the set \( P \) from
the domain of $\pi$, we still have that $T$ is almost-everywhere computable\footnote{An almost-everywhere computable map is exactly a partial computable map with a $\Pi^0_2$ domain of measure one. See Rute \cite{rute}, \S7 for more details.} but now $T^{-1}\{x_0\} = \emptyset$.

In the general case where $\rho$ is not computable, we will make this construction work by approximating the density function $f$ with a step function. Namely, let $f_n(x) = \mu(x|x_0)/\nu(x|x_0)$ and let

$$\rho_n = (\nu \otimes \mathcal{L}) \langle \{(x, y) \in 2^\mathbb{N} \times [0, 2] : 0 < y < f_n(x)\}.$$

Notice for each $n$ that $\rho_n$ is a probability measure since $\int f_n \, d\nu = 1$. We will build a sequence of a.e. computable functions $T_n$, where the $n$th function, $T_n$, transitions from $\rho_n^{-1}$ to $\rho_n$. While the limit of $T_n \circ \cdots \circ T_0$ may not be a.e. computable, the first coordinate of its output will be a.e. computable, and that is enough to construct $\mu$ in the desired manner.

6.2. Requirements of the construction. Formally, $T_n$ will satisfy the following requirements.

\begin{enumerate}[label=(R\arabic*)]
\item For all $n$,
  \begin{enumerate}[label=(\alph*)]
  \item $T_{n+1} : (2^\mathbb{N} \times [0, 2], \rho_n) \to (2^\mathbb{N} \times [0, 2], \rho_{n+1})$ is measure-preserving and a.e. computable in $n$.
  \item If $(x', y') = T_{n+1}(x, y)$, then $x'\mid_n = x\mid_n$ and $0 < y \leq y' \leq f_{n+1}(x')$.
  \end{enumerate}
\end{enumerate}

\begin{enumerate}[label=(\alpha*)]
\item Let $\pi$ be the projection $\pi(x, y) = x$. Then $T := \lim_n \pi \circ T_n \circ \cdots \circ T_0$ satisfies
  \begin{enumerate}[label=(\roman*)]
  \item $T$ is well-defined, that is $\lim_n \pi \circ T_n \circ \cdots \circ T_0$ converges pointwise $\lambda$-almost-surely,
  \item $T$ is $\lambda$-almost-everywhere computable,
  \item $T$ is measure-preserving of type $(2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \mu)$ and
  \item $T^{-1}\{x_0\} = \emptyset$.
  \end{enumerate}
\end{enumerate}

6.3. Construction. (See Figure 2 for an illustration of the first three stages of the construction.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{construction.png}
\caption{First three stages of the construction. The solid line is $f_n$ and the dashed line is $f_{n-1}$.}
\end{figure}
Stage 0. Our first approximation \( f_0 \) is just the constant function 1. The measure \( \rho_0 \) mentioned above is supported on the rectangle \( 2^\mathbb{N} \times [0, 1] \). Using Lemma \( \ref{lem:approximation} \) one can compute an a.e. computable measure-preserving map \( T_0 : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N} \times [0, 2], \rho_0) \). Hence requirement \((R_0)\) is satisfied.

Stage 1. For illustration, let us handle Stage 1 explicitly. The approximation \( f_1 \), must satisfy \( \int f_1 \, d\nu = 1 \), therefore some of \( f_1 \) is below \( f_0 = 1 \) and some of \( f_1 \) is above. As seen in Subfigure \( \ref{subfig:measure} \) the measure \( \rho_1 \) is the same as \( \rho_0 \) except the mass of \( \rho_0 \) inside the rectangle labeled \( A \) has moved to the rectangle labeled \( A' \). These two rectangles must be of the same \( \nu \otimes L \) measure since \( \int f_1 \, d\nu = f_0 \). To construct \( T_1 \), divide up the support of \( \rho_0 \) into the rectangles \( A, B, \) and \( C \) as illustrated in Subfigure \( \ref{subfig:construction} \). For \( (x, y) \in B, C \), set \( T_1(x, y) = (x, y) \).

For \( A \), let \( T_1 \upharpoonright A \) be a \( \nu \otimes L \)-measure-preserving map from the interior of \( A \) to the interior of \( A' \). To construct \( T_1 \upharpoonright A \) use Lemma \( \ref{lem:approximation} \) and the following facts: the measure \( \nu \otimes L \) is atomless, the rectangles \( A \) and \( A' \) are computable, and the measures \( \nu \otimes L(A) \) and \( \nu \otimes L(A') \) are equal and computable.

The remainder of \( 2^\mathbb{N} \times [0, 2] \) (that is the set of points not included in the interiors of \( A, B, \) or \( C \)) forms an effectively closed set of \( \rho_0 \)-measure zero, and therefore we can leave it out of the domain of \( T_1 \). (In particular, \( (x, 0) \) is not in the domain of \( T_1 \) for any \( x \).)

Stage \( n+1 \). This is the same idea as Stage 1. For every \( x \), we have \( \int_{[x_1 \ldots]} f_{n+1} \, d\nu = f_n(x) \). Therefore, we know that \( \rho_{n+1} \) is the same as \( \rho_n \) except that for every \( \sigma \in 2^n \), some of the \( \rho_n \)-mass in the column \( [\sigma] \times [0, 2] \) moves between the columns \( [\sigma0] \times [0, 2] \) and \( [\sigma1] \times [0, 2] \). If it moves, it must move upward, from below \( f_n \) to above \( f_n \). Yet, none of the mass leaves the column \( [\sigma] \times [0, 2] \). See Figure \( \ref{fig:construction} \).

Construct \( T_{n+1} \) in the analogous way to \( T_1 \). In particular, \( T_{n+1} \) does not move mass out of any columns \( [\sigma] \times [0, 2] \) such that \( |\sigma| = n \). It is straightforward to verify that all the requirements of \((R_n)\) are satisfied.

6.4. Verification. We already verified requirements \((R_0)\) and \((R_n)\). Requirement \((R_\infty)\)(a), the a.e. convergence of \( T := \lim_n \pi \circ T_n \circ \cdots \circ T_0 \) is easy to verify since \( T_{n+1}(x, y) \) does not change the first \( n \) bits of \( x \). The same is true for the a.e. computability of \( T \), that is requirement \((R_\infty)\)(b).

To see that \( T : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \mu) \) is measure-preserving (requirement \((R_\infty)\)(c)) choose \( \sigma \in 2^\mathbb{N} \). Since, for all \( n, T_{n+1}(x, y) \) does not change the first \( n \) bits of \( x \), \( \omega \in T^{-1}[\sigma] \) if and only if both \( \omega \in \text{dom} T \) and \( (T_{[\sigma]} \circ \cdots \circ T_0)(\omega) \in [\sigma] \times [0, 2] \).

(That is, after stage \( n = |\sigma| \), the mass will not move between columns of that size anymore.) Therefore, for \( n = |\sigma| \),

\[
\begin{align*}
\lambda_T(\sigma) &= \lambda(T^{-1}[\sigma]) \\
&= \lambda((T_n \circ \cdots \circ T_0)^{-1}([\sigma] \times [0, 2])) \\
&= \lambda_{T_n \circ \cdots \circ T_0}([\sigma] \times [0, 2]) \\
&= \rho_n([\sigma] \times [0, 2]) \\
&= \nu \otimes L \{(x, y) \in [\sigma] \times [0, 2] : 0 < y < f_n(x)\} \\
&= \int_{[\sigma]} f_n \, d\nu = \frac{\mu(\sigma)}{\nu(\sigma)} \nu(\sigma) = \mu(\sigma).
\end{align*}
\]
Finally, for requirement \((R_\infty)(d)\), assume for a contradiction that there is some \(\omega_0\) such that \(T(\omega_0) = x_0\). Let \((x^{(n)}, y^{(n)}) = (T_n \circ \cdots \circ T_0)(\omega_0)\). By all the parts of requirement \((R_n)(b)\), we have that \(y^{(n)}\) is nonzero, nondecreasing, and bounded above by
\[
f(n) = \frac{\mu(x^{(n)} | n)}{\nu(x^{(n)} | n)} = \frac{\mu(x_0 | n)}{\nu(x_0 | n)}.
\]
However, since \(f(x_0) = 0\), this upper bound converges to 0, which contradicts that \(y^{(n)}\) is nonzero and nondecreasing.

Therefore, we have proved that there is an a.e. computable measure-preserving map \(T: (2^\mathbb{N}, \lambda) \rightarrow (2^\mathbb{N}, \mu)\) such that \(T^{-1}(\{x_0\}) = \emptyset\), proving Theorem 25.

7. Randomness conservation and non-randomness-from-nothing for difference randomness and 2-randomness

In this section, we fill in the remaining rows of the table in the introduction. While these results are not due to us, they are included for completeness.

The \(\lambda\)-difference randoms can be characterized as the \(\lambda\)-Martin-Löf randoms which do not compute \(0'\) \[7\]. For simplicity, we will take this to be our definition of \(\mu\)-difference random for any computable measure \(\mu\), as well.

**Proposition 28** (Bienvenu [personal comm.]). Let \(R\) be the class of Martin-Löf random sequences which do not compute any element in \(C \subseteq 2^\mathbb{N}\) where \(C\) is countable, then \(R\) satisfies randomness conservation and no-randomness-from-nothing.

**Proof.** Randomness conservation is easy. If \(x \in R\) and \(y = T(x)\) for a measure-preserving a.e. computable map \(T: (2^\mathbb{N}, \mu) \rightarrow (2^\mathbb{N}, \nu)\), then \(y\) is Martin-Löf random (by randomness conservation for Martin-Löf randomness), but \(y\) cannot compute an element of \(C\) (since \(x\) cannot). Hence \(y \in R\).

As for no-randomness-from-nothing, let \(K_n\) be the complement of the \(n\)th level of the universal Martin-Löf test and let \(T: (2^\mathbb{N}, \mu) \rightarrow (2^\mathbb{N}, \nu)\) be a measure-preserving a.e. computable map. Then \(K_n \subseteq \text{dom} T\) and \(K_n\) is \(\Pi^0_1\) in \(n\). Fix a Martin-Löf random \(y\). Therefore \(P^n_y := \{x \in K_n : T(x) = y\} \in \Pi^0_1\) in \(n\) and \(y\). By no-randomness-from-nothing for Martin-Löf randomness relativized to \(y\), \(P^n_y\) is nonempty for some \(n\). By a version of the cone avoidance theorem [15, Thm 2.5] relativized to \(y\), there is a member of \(P^n_y\) which does not compute a member of \(C\). \qed

Next, 2-randomness is Martin-Löf randomness relative to \(0'\).

**Proposition 29.** 2-randomness satisfies randomness conservation and no-randomness-from-nothing.

**Proof.** The proof for Martin-Löf randomness [4, Thm. 3.2, 3.5] can be relativized to \(0'\). \qed

8. Characterizing Martin-Löf randomness via randomness conservation and no-randomness-from-nothing

The concepts of no-randomness-from-nothing and randomness conservation can be used to characterize various randomness notions. For example, Theorems 7 and 25 together, characterize computable randomness as the weakest randomness notion satisfying no-randomness-from-nothing.
Our main result in this section is a new characterization of Martin-Löf randomness, but first let us consider a few other examples. Schnorr characterized Schnorr randomness via randomness conservation and the strong law of large numbers.

**Proposition 30** (Schnorr [35, Thm. 12.1]). For $x \in 2^\mathbb{N}$ the following are equivalent.

1. $x$ is $\lambda$-Schnorr random.
2. For every almost-everywhere computable, measure-preserving map $F : (2^\mathbb{N}, \lambda) \rightarrow (2^\mathbb{N}, \lambda)$, $F(x)$ satisfies the strong law of large numbers, that is
   $$\frac{1}{n} \sum_{k=0}^{n-1} y(k) = 1/2$$
   where $y = F(x)$.

Notice (1) implies (2) follows from randomness conservation since Schnorr randoms satisfy the strong law of large numbers. Gács, Hoyrup, and Rojas [10, Prop. 6] strengthened this result by showing that “measure-preserving map” can be replaced with “isomorphism.”

In Lemma 4, we characterized computable randomness via the ratio $\nu(x|_n)/\mu(x|_n)$. This function $\sigma \mapsto \nu(\sigma)/\mu(\sigma)$ is known as a martingale and represents a gambling strategy. (See, for example, [5, §6.3.1, Ch. 7][26, Ch. 7].) Kolmogorov-Loveland randomness (on the fair-coin measure $\lambda$) is a randomness notion similar to computable randomness except that one can bet on the bits out of order. Formally, call $F : 2^\mathbb{N} \rightarrow 2^\mathbb{N}$ a total computable nonmonotonic selection map if $y = F(x)$ is computed as follows.

- At stage 0, the algorithm computes an index $i_0 \in \mathbb{N}$ independently of $x$ and sets $y_0 = x(i_0)$.
- At stage 1, the algorithm computes an index $i_1 = i_1(y_0) \in \mathbb{N} \setminus \{i_0\}$ depending only on $y_0$ and sets $y_1 = x(i_1)$.
- At stage $s+1$, we have calculated $y|_{s+1}$ where $y_n = x_{i_n}$ for $0 \leq n \leq s$. The algorithm chooses a new index $i_{s+1}(y|_{s+1}) \in \mathbb{N} \setminus \{i_0, i_1, \ldots, i_n\}$ depending only on $y|_{s+1}$ and sets $y_{s+1} = x(i_{s+1})$.
- The algorithm is total in that for all $x$ and $n$, the bit $(F(x))(n)$ is calculated.

One can easily see that total computable nonmonotonic selection maps are measure-preserving maps of type $(2^\mathbb{N}, \lambda) \rightarrow (2^\mathbb{N}, \lambda)$.

A sequence $x \in 2^\mathbb{N}$ is Kolmogorov-Loveland random if given a total nonmonotonic selection map $F$, no computable $\lambda$-martingale succeeds on $F(x)$ — that is there is no computable measure $\nu$ such that $\liminf_n \nu(F(x|_n))/\lambda(F(x|_n)) < \infty$. Kolmogorov-Loveland randomness is between Martin-Löf randomness and computable randomness, and it is a major open question whether Kolmogorov-Loveland randomness equals Martin-Löf randomness. (See [5, §7.5][26, §7.6] for more about Kolmogorov-Loveland randomness.)

Bienvenu and Porter [4, Thm. 4.2] noticed the definition of Kolmogorov-Loveland randomness can be restated as in this next proposition. They noticed that this proposition, when combined with Muchnik’s result that Kolmogorov-Loveland randomness is strictly stronger than computable randomness [25], implies that computable randomness does not satisfy randomness conservation.

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15Kolmogorov-Loveland randomness can be defined via non-total selection maps as well [26 §3].
Proposition 31. For $x \in 2^\mathbb{N}$ the following are equivalent.

1. $x$ is $\lambda$-Kolmogorov-Loveland random.
2. $F(x)$ is $\lambda$-computably random for every total computable nonmonotonic selection map $F : 2^\mathbb{N} \to 2^\mathbb{N}$.

Similarly Rute [30] (based on a “martingale process” characterization of Martin-Löf randomness by Merkle, Mihailović, and Slaman [21, §4]) showed that Martin-Löf randomness and computable randomness can be characterized in terms of each other.

Proposition 32 (Rute [30] Cor. 9.6). For $x \in 2^\mathbb{N}$ the following are equivalent.

1. $x$ is $\lambda$-Martin-Löf random.
2. $F(x)$ is $\lambda_F$-computably random for every almost-everywhere computable map $F : (2^\mathbb{N}, \lambda) \to 2^\mathbb{N}$.

It is essential to Rute’s proof that $\lambda_F$ may be not be $\lambda$.

Combining Theorem 25, Proposition 32, and randomness conservation for Martin-Löf randomness we have the following. (Here $\mathcal{M}^1_{\text{comp}}$ denotes the set of computable probability measures on $2^\mathbb{N}$.)

Theorem 33. Let $A \subseteq 2^\mathbb{N} \times \mathcal{M}^1_{\text{comp}}$. The following are equivalent.

1. $(x, \mu)$ is in $A$ if and only if $x$ is $\mu$-Martin-Löf random.
2. $A$ is the largest subset of $2^\mathbb{N} \times \mathcal{M}^1_{\text{comp}}$ closed under no-randomness-from-nothing and randomness conservation.

Proof. The Martin-Löf randoms satisfy no-randomness-from-nothing and randomness conservation.

Conversely, assume $A$ is strictly larger than the Martin-Löf randoms. Then there is some $(y, \mu) \in A$ such that $y$ is not $\mu$-Martin-Löf random. Now consider some almost-everywhere computable $F : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \mu)$, which exists by Lemma 27.

By no-randomness-from-nothing for $A$, there is some $x \in 2^\mathbb{N}$ such that $(x, \lambda) \in A$ and $F(x) = y$. By randomness conservation for Martin-Löf randomness, $x$ cannot be $\lambda$-Martin-Löf random.

By Proposition 32 there is some almost-everywhere computable map $G : (2^\mathbb{N}, \lambda) \to 2^\mathbb{N}$ such that $G(x)$ is not $\lambda_C$-computably random. However, $(G(x), \lambda_G) \in A$ by randomness conservation.

By Theorem 25 there is an almost-everywhere computable map $H : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \lambda_G)$ such that $H^{-1}(\{G(x)\}) = \emptyset$, contradicting no-randomness-from-nothing for $A$. \qed

We can get a stronger result by using a recent result of Petrović [28]. He considered a variant of Kolmogorov-Loveland randomness, where instead of betting on bits of $x$, one bets on whether or not $x$ is in some clopen set $C \subseteq 2^\mathbb{N}$ (a set is clopen if and only if it is a finite union of basic open sets). Specifically, one computes a sequence $(C_n)_{n \in \mathbb{N}}$ of clopen sets such that $\lambda(C_n) = 1/2$ for all $n$ and the collection $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ is mutually $\lambda$-independent in the sense of probability theory, that is for all finite $A \subseteq \mathcal{C}$, $\lambda(\bigcap_{C \in A} C) = \prod_{C \in A} \lambda(C)$. This induces a total computable map $F(x)$ such that $(F(x))_n = 1$ if and only if $x \in C_n$. Using notation similar to Petrović’s, call such a map a sequence-set map. Since the collection $(C_n)_{n \in \mathbb{N}}$ is mutually independent, a sequence-set map is a measure-preserving map of type $F : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \lambda)$. Framed in this way, Petrović proved the following.
Theorem 34 (Petrović [28]). If $x$ is not $\lambda$-Martin-Löf random, then there is a total computable sequence-set function $F : 2^\mathbb{N} \to 2^\mathbb{N}$ such that $F(x)$ is not $\lambda$-computably random.

As a corollary, we have a strengthening of Propositions 31 and 32.

Corollary 35. For $x \in 2^\mathbb{N}$ the following are equivalent.

1. $x$ is $\lambda$-Martin-Löf random.
2. $F(x)$ is $\lambda$-computably random for every total computable measure-preserving map $F : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \lambda)$.

Now we have a stronger version of Theorem 33 by basically the same proof, replacing Proposition 32 with Corollary 35.

Theorem 36. The set of $\lambda$-Martin-Löf randoms is the largest subset of $2^\mathbb{N}$ closed under no-randomness-from-nothing and randomness conservation for a.e. computable measure-preserving maps $F : (2^\mathbb{N}, \lambda) \to (2^\mathbb{N}, \lambda)$.

Proof. Let $A \subseteq 2^\mathbb{N}$ be closed under no-randomness-from-nothing and randomness conservation. By Theorem 25 every element of $A$ must be $\lambda$-computably random. Then by Corollary 35 every element of $A$ must be $\lambda$-Martin-Löf random. Therefore the $\lambda$-Martin-Löf randoms are the largest such set. □

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