A de Casteljau Algorithm for $q$-Bernstein-Stancu Polynomials

Grzegorz Nowak

The Great Poland University of Social and Economics in Środa Wielkopolska, Paderewskiego 27, 63-000 Środa Wielkopolska, Poland

Correspondence should be addressed to Grzegorz Nowak, grzegnow2@gmail.com

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This paper is concerned with a generalization of the $q$-Bernstein polynomials and Stancu operators, where the function is evaluated at intervals which are in geometric progression. It is shown that these polynomials can be generated by a de Casteljau algorithm, which is a generalization of that relating to the classical case and $q$-Bernstein case.

1. Introduction

Let $q > 0$. For any fixed real number $q > 0$ and for $n \in \mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$, the $q$-integers of the number $[n]$ are defined by

$$[n] = \frac{(1 - q^n)}{(1 - q)}, \quad \text{for } q \neq 1, \quad [n] = n, \quad \text{for } q = 1. \quad (1.1)$$

The $q$-factorial $[n]!$, for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, is defined by

$$[n]! = [1][2] \cdots [n] \quad (n = 1, 2, \ldots), \quad [0]! = 1. \quad (1.2)$$
For the integers \( n, k \), \( (n \geq k \geq 0) \), the \( q \)-binomial or the Gaussian coefficients are defined by (see [1, page 12])

\[
\binom{n}{k} = \frac{[n]!}{[k]![n-k]!},
\]

For \( f \in C[0; 1], q > 0, \alpha \geq 0 \) and each positive integer \( n \), we introduce (see [2]) the following generalized \( q \)-Bernstein operators:

\[
B_n^{q,\alpha}(f; x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x)f\left(\frac{[k]}{n}\right),
\]

where

\[
p_{n,k}^{q,\alpha}(x) = \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} (x + \alpha[i]) \prod_{s=0}^{n-1} (1 - q^s x + \alpha[s]) \prod_{i=0}^{n-1} (1 + \alpha[i]).
\]

Note, that an empty product in (1.5) denotes 1. In the case where \( \alpha = 0 \), \( B_n^{q,\alpha}(f; x) \) reduces to the well-known \( q \)-Bernstein polynomials introduced by Phillips [3, 4] in 1997

\[
B_{n,q}(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k \prod_{i=0}^{n-1} (1 - q^i x)f\left(\frac{[k]}{n}\right).
\]

In the case where \( q = 1 \), \( B_n^{q,\alpha}(f; x) \) reduces to Bernstein-Stancu polynomials, introduced by Stancu [5] in 1968

\[
S_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k \prod_{i=0}^{n-1} (1 - x + \alpha[i]) \prod_{s=0}^{n-1} (1 + i\alpha)f\left(\frac{k}{n}\right).
\]

When \( q = 1 \) and \( \alpha = 0 \), we obtain the classical Bernstein polynomial defined by

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right).
\]

Basic facts on Bernstein polynomials, their generalizations, and applications can be found for example in [6–8]. In recent years, the \( q \)-Bernstein polynomials have attracted much interest, and a great number of interesting results related to the \( B_{n,q}(f) \) polynomials have been obtained (see [3, 4, 9–12]). Some approximation properties of the Stancu operators are presented in [5, 13–15].

Let \( \Delta^0_{q}f_j = f_j \), for \( j = 0, 1, \ldots, n \), and recursively,

\[
\Delta^1_{q}f_j = \Delta^0_{q}f_{j+1} - q^j \Delta^0_{q}f_j,
\]
for $k = 0, 1, \ldots, n - j - 1$ and $f_j = f([j]/[n])$. It is easily established by induction that $q$-differences satisfy the relation

$$\Delta_k^q f_j = \sum_{i=0}^{k} (-1)^i q^{i(i-1)/2} \binom{k}{i} f_{j+k-i}. \tag{1.10}$$

In [2], we prove that the operators $B^{q,\alpha}_{n}(f;x)$ defined by (1.4) can be expressed in terms of $q$-differences

$$B^{q,\alpha}_{n} f(x) = \sum_{k=0}^{n} \binom{n}{k} \Delta_k^q f_0 \prod_{i=0}^{k-1} \frac{x + \alpha[i]}{\alpha[i]}, \tag{1.11}$$

which generalized the well-known result [3, 4] for the $q$-Bernstein polynomial. In this paper, we show that polynomials defined by (1.4) can be generated by a de Castljau algorithm, which is a generalization of that relating to the classical case [6] and $q$-Bernstein case [4, 11].

2. Auxiliary Results

We note that $B^{q,\alpha}_{n}(f;x)$ defined by (1.4), is a monotone linear operator for any $0 < q \leq 1$ and $\alpha \geq 0$. These operators reproduces linear functions [2], that is,

$$B^{q,\alpha}_{n}(ax + b;x) = ax + b, \quad a, b \in R. \tag{2.1}$$

They also satisfy the end point interpolation conditions $B^{q,\alpha}_{n}(f;0) = f(0)$ and $B^{q,\alpha}_{n}(f;1) = f(1)$. These properties are significant in designing curves and surfaces.

Moreover, the following holds.

Lemma 2.1. Let $0 < q \leq 1$, $\alpha \geq 0$. Then,

$$\prod_{u=0}^{m-1} (q^u - q^u x + \alpha([u] - [r])) = \sum_{s=0}^{m} (-1)^s q^{(s-1)/2 + (m-s)r} \binom{m}{s} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]), \tag{2.2}$$

for all $m \in N$, $r \in N_0 = N \cup \{0\}$ and $x \in [0;1]$. 
Proof. We use induction on $m$. First, we see from equality $[-r] = -q^{-r}[r], \ (r \in \mathbb{N}),$ that (2.2) is evident for $m = 1$. Let us assume that (2.2) holds for a given $m \in \mathbb{N}$. Then, using (2.2), we obtain

$$
\prod_{u=0}^{m} (q^u - q^{m} x + \alpha([u] - [r]))
$$

$$
= (q^m - q^{m} x + \alpha([m] - [r])) \sum_{s=0}^{m} (-1)^{s} q^{s(s-1)/2+(m-s)r} \binom{m}{s} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j])
$$

$$
= \sum_{s=0}^{m} (-1)^{s} q^{s(s-1)/2+(m-s)r} (q^s + \alpha[m] - \alpha[r] + \alpha q^m [s]) \binom{m}{s} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j])
$$

$$
+ \sum_{s=1}^{m+1} (-1)^{s} q^{s(s-1)/2+(m-s+1)r+m} (1 + \alpha[s - r - 1]) \binom{m}{s-1} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j])
$$

$$
= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=r}^{m-r-1} (1 + \alpha[j])
$$

$$
+ (-1)^{m+1} q^{m(m-1)/2+mr} (1 + \alpha[m - r]) \prod_{i=0}^{m} (x + \alpha[i])
$$

$$
+ \sum_{s=1}^{m} (-1)^{s} q^{s(s-1)/2+(m+1-s)r} U_s \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]),
$$

where

$$
U_s = \binom{m}{s} (q^s + \alpha[m] - \alpha[r] + \alpha q^m [s]) q^{-r} + q^{m-s+1} \binom{m}{s-1} (1 + \alpha[s - r - 1]).
$$
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Using the obvious equalities

\[(q^r + \alpha[m] - \alpha[r])q^{-r} = 1 + \alpha[m - r],\]
\[
\left[\begin{array}{c}
m \\ s
\end{array}\right] = \left[\begin{array}{c}
m \\ s - 1
\end{array}\right] [m - s + 1],
\]

we have

\[
U_s = \left[\begin{array}{c}
m \\ s
\end{array}\right] (1 + \alpha[m - r])
\]
\[+ \left[\begin{array}{c}
m \\ s - 1
\end{array}\right] q^{m-s+1}(1 + \alpha([m - s + 1]q^{s-r-1} + [s - r - 1])).
\]

It is easy to see that

\[(m - s + 1)q^{s-r-1} + [s - r - 1] = [m - r],
\]
\[
\left[\begin{array}{c}
m \\ s
\end{array}\right] + \left[\begin{array}{c}
m \\ s - 1
\end{array}\right] q^{m-s+1} = \left[\begin{array}{c}
m + 1 \\ s
\end{array}\right].
\]

Therefore,

\[
U_s = (1 + \alpha[m - r])\left[\begin{array}{c}
m + 1 \\ s
\end{array}\right].
\]

From last equality and (2.3), we obtain

\[
\prod_{a=0}^{m} (q^r - q^r x + \alpha([u] - [r]))
\]
\[
= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=-r}^{m-r-1} (1 + \alpha[j])
\]
\[+ (-1)^{m+1} q^{m(m-1)/2+m}(1 + \alpha[m - r])\prod_{i=0}^{m}(x + \alpha[i])
\]
\[+ \sum_{s=1}^{m} (-1)^s q^{s(s-1)/2+(m+1-s)r} \left[\begin{array}{c}
m + 1 \\ s
\end{array}\right] (1 + \alpha[m - r]) \prod_{i=0}^{s-1}(x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j])
\]
\[= \sum_{s=0}^{m+1} (-1)^s q^{s(s-1)/2+(m+1-s)r} \left[\begin{array}{c}
m + 1 \\ s
\end{array}\right] \prod_{i=0}^{s-1}(x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]).
\]

This completes the proof of the lemma. \(\square\)
3. Main Result

The generalized $q$-Bernstein polynomials, defined by (1.4), may be evaluated by Algorithm 1.

In the case, where $\alpha = 0$, this is the de Casteljau algorithm for evaluating the $q$-Bernstein polynomial [3, 4]. Note that with $q = 1$ and $\alpha = 0$, we recover the original classical de Casteljau algorithm (see Hoschek and Lasser [16]). The algorithm is justified by the following theorem.

**Theorem 3.1.** Each intermediate point $f_r^{[m]}$ of the algorithm can be expressed as

$$f_r^{[m]} = \left(\prod_{i=0}^{m-1} (1 + \alpha [i])\right)^{-1} \cdot \sum_{t=0}^{m} f_{r+t} \left[ m \atop t \right] \prod_{s=0}^{t-1} (x + \alpha [r + s]) \prod_{u=0}^{m-t-1} (q^r - q^m x + \alpha ([u] - [r])), \quad (3.1)$$

and, in particular

$$f_0^{[n]} = B_n^{q,\alpha} (f; x). \quad (3.2)$$

**Proof.** We use induction on $m$. From the initial conditions in the algorithm, $f_r^{[0]} = f([r]/[n]) = f_r$, $0 \leq r \leq n$, it is clear that (3.1) holds for $m = 0$ and $0 \leq r \leq n$. Let us assume that (3.1) holds for some $m$ such that $0 \leq m < n$, and for all $r$ such that $0 \leq r \leq n - m$. Then, for $0 \leq r \leq n - m - 1$, it follows from the algorithm that

$$f_r^{[m+1]} := \left\{ (q^r - q^m x + \alpha ([m] - [r])) f_r^{[m]} + (x + \alpha [r]) f_{r+1}^{[m]} \right\} \frac{1}{1 + \alpha [m]}, \quad (3.3)$$
and using (3.1), we obtain

\[
\begin{align*}
&f_{r}[m+1]\left(\prod_{i=0}^{m}(1 + \alpha[i])\right) := (q^r - q^m x + \alpha([m] - [r])) \\
&+ \sum_{t=0}^{m} f_{r+t} \left[\prod_{s=0}^{t-1}(x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r]))\right] \\
&+ (x + \alpha[r]) \cdot \prod_{t=1}^{m} \left[\prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r]))\right] \\
&+ (x + \alpha[r]) \cdot \prod_{t=1}^{m} \left[\prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r]))\right] \\
&+ (x + \alpha[r]) \cdot \prod_{t=1}^{m} \left[\prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r]))\right] \\
&+ f_{r+m+1} \prod_{s=0}^{m} (x + \alpha[r + s]).
\end{align*}
\]
We see that

\[
\prod_{u=0}^{m-t} (q^{r+1} - q^n x + \alpha([u] - [r + 1]))
\]

\[
= \left(q^{r+1} - x - \alpha [r + 1]\right) \prod_{u=0}^{m-1} (q^{r+1} - q^{u+1} x + \alpha([u + 1] - [r + 1]))
\]

\[
= \left(q^{r+1} - x - \alpha [r + 1]\right) \prod_{u=0}^{m-1} (q^{r+1} - q^{u+1} x + \alpha([u] - [r]))
\]

\[
= \left(q^{r+1} - x - \alpha [r + 1]\right) q^{m-1} \prod_{u=0}^{m-1} (q^{r} - q^{u+1} x + \alpha([u] - [r])),
\]

and hence,

\[
f^{[m+1]}_r \left( \prod_{i=0}^{m} (1 + \alpha[i]) \right)
\]

\[
:= f_r \prod_{u=0}^{m} (q^{r} - q^{u+1} x + \alpha([u] - [r]))
\]

\[
+ \sum_{t=1}^{m} \left( \begin{bmatrix} m \\ t \end{bmatrix} (q^{r} - q^{m} x + \alpha([m] - [r])) + \begin{bmatrix} m \\ t - 1 \end{bmatrix} (q^{r+1} - x - \alpha [r + 1]) q^{m-t} \right)
\]

\[
\cdot f_{r,s} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^{r} - q^{u+1} x + \alpha([u] - [r])) + f_{r, m+1} \prod_{s=0}^{m} (x + \alpha[r+s]).
\]

It is easy to verify that

\[
\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t - 1 \end{bmatrix} = \begin{bmatrix} m + 1 \\ t \end{bmatrix},
\]

\[
\begin{bmatrix} m \\ t - 1 \end{bmatrix} + q^l \begin{bmatrix} m \\ t \end{bmatrix} = \begin{bmatrix} m + 1 \\ t \end{bmatrix}.
\]

(3.7)
Therefore,

\[
\begin{align*}
\binom{m}{t}(q^r - q^m x + \alpha([m] - [r])) + \binom{m}{t-1}(q^r x - \alpha[r+1])q^{m-t} \\
= q^r \left( \binom{m}{t} + q^{m-t+1} \binom{m}{t-1} \right) - qx^{m-t} \left( \binom{m}{t-1} + q^t \binom{m}{t} \right) \\
+ \frac{\alpha}{1-q} q^r \left( \binom{m}{t} + q^{m-t+1} \binom{m}{t-1} \right) - q^{m-t} \left( \binom{m}{t-1} + q^t \binom{m}{t} \right) \\
= \left[ \binom{m+1}{t} \right] \left( (q^r - qx^{m-t}) + \alpha([m-t] - [r]) \right). 
\end{align*}
\]  

Consequently,

\[
\begin{align*}
f_r^{[m+1]} \left( \prod_{i=0}^{m} (1 + \alpha[i]) \right) := f_r \prod_{u=0}^{m} (q^r - q^u x + \alpha([u] - [r])) \\
+ \sum_{t=1}^{m} \binom{m+1}{t} \left( (q^r - qx^{m-t}) + \alpha([m-t] - [r]) \right) \\
\cdot f_{r+s} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
+ f_{r+m+1} \prod_{s=0}^{m} (x + \alpha[r+s]) \\
= \sum_{t=0}^{m+1} \binom{m+1}{t} \cdot f_{r+s} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) 
\end{align*}
\]  

(3.8)

Thus, one has the desired result. \(\square\)

**Theorem 3.2.** For \(0 \leq m \leq n\) and \(0 \leq r \leq n - m\), we have

\[
f_r^{[m]} = \sum_{s=0}^{m} q^{m-s} \binom{m}{s} \Delta_{q}^{r} f_r \prod_{s=0}^{m-1} (x + \alpha[i]) \prod_{j=0}^{m-1} (1 + \alpha[j]),
\]  

(3.10)

for all \(x \in [0;1]\).

**Proof.** Using (2.2) and (3.1), we have

\[
f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) = \sum_{t=0}^{m} \binom{m}{t} f_{r+t} S_t(m),
\]  

(3.11)
where

\[
S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \binom{m-t}{u} \times 
\prod_{s=r}^{t+r-1} (x + \alpha[s]) \prod_{i=0}^{u-1} (x + \alpha[i]) \prod_{j=u-r}^{m-t-r-1} (1 + \alpha[j]) \quad (0 \leq t \leq m). 
\] (3.12)

First, we prove that

\[
S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \binom{m-t}{u} \cdot 
\prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) 
\] (3.13)

for all \( m \in N_0 = \{0, 1, 2, \ldots\}, t \in N_0, \) and \( x \in [0; 1]. \) Note that an empty sum denotes 0.

We use the induction on \( m. \) First, we see that (3.13) holds for \( m = 0 \) and all \( t \in N_0. \) Let us assume that (3.13) holds for a given \( m, \) and for all \( t \in N_0. \) Then, from (3.12) and (3.13), we obtain

\[
S_t(m+1) = (x + \alpha[t+r-1]) \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \binom{m-t+1}{u} \times 
\prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) 
= \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \binom{m-t+1}{u} \times 
\prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) 
+ \alpha \sum_{u=1}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \binom{m-t+1}{u} \times 
\prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]).
\] (3.14)

We see that

\[
\binom{m-t+1}{u} ([t+r-1] - [t+u+r-1]) = -q^{u+r-1} \binom{m-t-u+2}{u-1},
\] (3.15)
and hence,

\[
S_t(m + 1) = \sum_{u=0}^{m-t+1} (-1)^u q^{u(u-1)/2 + (m-t+1-u)r} \left[ \frac{m-t+1}{u} \right]
\]

\[
\cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j])
\]

\[
+ \alpha \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2 + (m-t+1-u)r} q^{u+r-1} \left[ \frac{m-t+1}{u} \right]
\]

\[
\cdot [m-t-u+1] \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j])
\]

\[
= (-1)^{m-t+1} q^{m(t+1)(m-t)/2} \prod_{i=r}^{m+r} (x + \alpha[i])
\]

\[
+ \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2 + (m-t+1-u)r}
\]

\[
\cdot \left( 1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1] \right) \left[ \frac{m-t+1}{u} \right]
\]

\[
\cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j])
\]  

Next, in view of the equality

\[
(1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1]) = 1 + \alpha[m],
\]  

we obtain (3.13). Consequently, in view of (3.11) and (3.13), we get

\[
f^{[m]}_r \prod_{i=0}^{m-1} (1 + \alpha[i]) = \sum_{t=0}^{m} \sum_{u=0}^{m-t} \left[ \frac{m-t}{u} \right] \left[ \frac{m-t}{u+t-1} \right] \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j])
\]

\[
= \sum_{t=0}^{m} \sum_{u=0}^{m-t} \left[ \frac{m-t}{u} \right] \left[ \frac{m-t}{u+t-1} \right] q^{u(t-1)/2 + (m-u)r}
\]

\[
\cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]).
\]
Next, in view of the equality

\[
\begin{bmatrix} m \\ t \end{bmatrix} \begin{bmatrix} m-t \\ u-t \end{bmatrix} = \begin{bmatrix} m \\ u \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix},
\]

(3.19)

we obtain

\[
f_r^m \prod_{i=0}^{m-1} (1 + \alpha[i]) = \sum_{u=0}^{m} \sum_{t=0}^{u} \begin{bmatrix} m \\ u \end{bmatrix} f_{r+u} (-1)^{u-t} q^{(u-t)(u-t-1)/2 + (m-u)r} \cdot \begin{bmatrix} u \\ t \end{bmatrix} \prod_{i=r}^{u-r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j])
= \sum_{u=0}^{m} \begin{bmatrix} m \\ u \end{bmatrix} q^{(m-u)r} \prod_{i=r}^{u-r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]) \cdot \sum_{t=0}^{u} \begin{bmatrix} u \\ t \end{bmatrix} (-1)^{u-t} q^{(u-t)(u-t-1)/2} f_{r+t}.
\]

(3.20)

The condition (1.10) completes the proof. \(\square\)

Theorems 3.1 and 3.2 are generalizations of Theorems 2.1 and 2.3 in [11].
Note that when \(m = n\) and \(r = 0\), (3.10) does indeed reduce to (1.11)

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