Competition between ac driving forces and Lévy flights in a nonthermal ratchet

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Abstract. Transport of overdamped particles in an asymmetrically periodic potential in the presence of Lévy noise and ac driving forces is investigated. The group velocity is used to measure the transport driven by the nonthermal noise. It is found that the Lévy flights and ac driving forces are the two different driving factors that can break thermodynamical equilibrium. The competition between the two factors will induce some peculiar phenomena. For example, multiple transport reversals occur on changing the noise intensity. Additionally, we also find that the group velocity as a function of the Lévy index is nonmonotonic for small values of the noise intensity.

Keywords: stochastic particle dynamics (theory), Brownian motion, stochastic processes (theory)
1. Introduction

In systems possessing spatial or dynamical symmetry breaking, Brownian motion combined with unbiased external input signals, deterministic and random alike, can assist directed motion of particles at submicron scales [1]. This subject was motivated by the challenge of explaining unidirectional transport in biological systems [2], as well as their potential technological applications, ranging from classical non-equilibrium models [3] to quantum systems [4]. Ratchets have been proposed to model the unidirectional motion driven by zero-mean non-equilibrium fluctuations. Broadly speaking, ratchet devices fall into three categories depending on how the applied perturbation couples to the substrate asymmetry: rocking ratchets [5], flashing ratchets [6], and correlation ratchets [7]. Additionally, entropic ratchets, in which Brownian particles move in a confined structure, instead of a potential, were also extensively studied [8]. These ratchets demand three key ingredients [9] which are (a) nonlinearity: it is necessary since the system will produce a zero-mean output from a zero-mean input in a linear system; (b) asymmetry (spatial and/or temporal): it can violate the symmetry of the response; (c) fluctuating input zero-mean force: it should break thermodynamical equilibrium.

Most studies on ratchets have referred to the consideration of normal diffusion driven by Gaussian noises. However, in the past few years, anomalous diffusion has attracted growing attention, being observed in various fields of physics and related sciences [10]–[13]. Because the Lévy flights do not possess a finite mean square displacement, their physical significance has been ignored for a long time. However, in recent years, growing experimental evidence suggests that there is a need to consider a more general type of noise than Gaussian, i.e., Lévy noise. Description of physical models in terms of Lévy flights becomes more and more popular [10]–[23]. They are actually observed in various real systems and are used to model a variety of processes such as bulk mediated surface diffusion [17], exciton and charge transport in polymers under conformational motion [18], transport in micelle systems or heterogeneous rocks [19], two-dimensional rotating flow [20], and many others [10].

Very few studies on ratchets have focused on the Lévy flights. Recently, Dybiec et al [21] studied the minimal setup for a Lévy ratchet and found that due to the nonthermal character of the Lévy noise, the net current can be obtained even in the absence of any
additional time-dependent forces. Del-Castillo-Negrete et al. [23] also found similar results for the constant force-driven Lévy ratchet for $1 < \alpha < 2$ ($\alpha$ is the Lévy index). Rosa and Beims [24] studied the optimal transport and its relation to superdiffusive transport and Lévy walks for Brownian particles in a ratchet potential in the presence of a modulated environment and external oscillating forces. In these studies the Lévy noise is an intrinsic driving factor for obtaining the net transport. In the classical forced thermal ratchets [1, 5], the driving factor is usually the external ac driving force. However, what is the difference between the intrinsic driver and the external one? How do Lévy flights compete with the ac driving forces? In order to answer these questions, in the present paper, we studied the transport of overdamped particles in an asymmetrically periodic potential in the presence of the Lévy flights and ac driving forces. Our emphasis is on finding the difference between the two driving factors and how the competition between them affects the transport.

2. Model and methods

In this study, we consider the transport of Brownian particles moving in an asymmetrically periodic potential in the presence of ac driving forces and Lévy-stable noises. The overdamped dynamics can be described by the following Langevin equation in the dimensionless form:

$$\frac{dx}{dt} = -U'(x) + A_0 \sin(\omega t) + \zeta_\alpha(t),$$

where $A_0$ and $\omega$ are the driving amplitude and frequency, respectively. The prime stands for differentiation with respect to $x$. $U(x)$ is an asymmetrically periodic potential (see figure 1)

$$U(x) = -U_0 \left[ \sin(x) + \frac{\Delta}{4} \sin(2x) \right],$$

where $U_0$ denotes the height of the potential and $\Delta$ is its asymmetric parameter.

$\zeta_\alpha(t)$ is white, symmetric Lévy-stable noise with independent increments distributed according to the stable density with the index $\alpha$. The time integral of the Lévy noise over

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an increment $\Delta t$

$$L_{\alpha,\sigma}(\Delta t) = \int_{t}^{t+\Delta t} \zeta_{\alpha}(t') \, dt'$$  \hspace{1cm} (3)

is an $\alpha$-stable process with stationary independent increments and its characteristic function (the Fourier transform of the probability density function) \[10\]–\[12\] is

$$P_L(k, \Delta t) = \exp(-\sigma |k|^\alpha \Delta t),$$  \hspace{1cm} (4)

where $\sigma$ is the intensity of the Lévy noise and $k$ is the wavenumber. The parameter $\alpha \in (0, 2]$ denotes the stability index, yielding the asymptotic long tail power law for the $\zeta$-distribution, which is of the $|\zeta|^{1-\alpha}$ type. For the special case $\alpha = 2.0$, i.e., for a Gaussian noise, we are led back to the Brownian case. From equations (3) and (4) we can obtain the discrete time representation of equation (1) for sufficiently small time step $\Delta t$:

$$x(t_{n+1}) = x(t_n) - U'(x(t_n)) \Delta t + A_0 \sin(\omega t_n) \Delta t + (2D \Delta t)^{1/\alpha} \zeta_\alpha(n),$$  \hspace{1cm} (5)

where $n = 0, 1, 2, \ldots$ and $\zeta_\alpha(n)$ is a random number possessing a Lévy-stable distribution. In order to compare with the classical forced thermal ratchets, we use $D = \sigma/2$ to describe the noise intensity. In numerical simulations, the corresponding generator is taken from \[11,12,25\]

$$\zeta_\alpha(n) = \frac{\sin(\alpha V)}{(\cos V)^{1/\alpha}} \left[ \cos((1-\alpha)V) \right]^{(1-\alpha)/\alpha},$$  \hspace{1cm} (6)

where $V$ is random number uniformly distributed on the interval $(-\pi/2, \pi/2)$, $W$ is an independent random variable distributed according to the exponential distribution with unit mean.

In this study, we mainly focus on the transport of the driven particles. In the Gaussian noise driving ratchets, due to the existence of the mean value of the noise, the displacement of particles also possesses a mean and the transport can be characterized by the average velocity. However, for the noise with distribution of a Lévy-stable law with $0 < \alpha < 1$, the mean of the noise and the overall displacement do not exist. The main feature of this distribution is that the tails cannot be cut off, or in other words, rare but large events cannot be neglected. As a consequence, the classical stochastic theory (average velocity), which is based on the ordinary central limit theorem, is no longer valid. Recently, Dybiec \textit{et al} \[21\] proposed a different approach to the Lévy ratchet problem based on the group velocity analysis for $0 < \alpha < 2$. Throughout the paper, we will use this method to describe the transport of the particles.

The median line is a very useful tool for investigation of the overall motion of the probability density of finding a particle in the vicinity of $x$ \[11\]. A median line for a stochastic process $x(t)$ is a function of $q_{0.5}(t)$ given by the relationship $\Pr(x(t) \leq q_{0.5}(t)) = 0.5$. Therefore, one can use the derivative of the median to define the group velocity of the particle packet \[21\],

$$v_g(t) = \frac{d q_{0.5}(t)}{dt},$$  \hspace{1cm} (7)
and this definition is valid even for the case of lacking average current. In the following, we mainly focus on the study of the long time group velocity,

\[ V_g = \lim_{t \to \infty} \frac{q_{0.5}(t)}{t}. \]  

(8)

In our simulations, we have considered more than \(10^5\) realizations to obtain the accurate median. In order to provide the required accuracy of the system dynamics the time step was chosen to be smaller than \(10^{-3}\). We have checked that this is sufficient for the system to obtain consistent results.

3. Numerical results and discussion

Our emphasis is on finding the median and group velocity with definitions in equation (7). In order to investigate the effects of the interplay between the ac driving forces and Lévy flights we carried out extensive numerical simulations. For simplicity we set \(U_0 = 1\) and \(\Delta = 1\) throughout the work.

Figure 2(a) shows the group velocity \(V_g\) as a function of the driving frequency \(\omega\). For the case of \(\alpha = 2.0\), the Lévy ratchet reduces to the classical forced thermal ratchet [1,5]. In this case, the only resource driving the particle current across the barrier is the ac driving force. In the adiabatic limit \(\omega \to 0\), the ac driving force can be expressed as two opposite forces \(A_0\) and \(-A_0\). The particles get enough time to cross both sides from the minima of the potential. It is easier for particles to move toward the slanted side than toward the steeper side, so the group velocity is positive. On increasing the frequency \(\omega\), due to the high frequency, the particles in one period get more time to climb the barrier from the steeper side than from the slanted side, resulting in negative group velocity. When the ac driving forces oscillate very fast, the particles will experience a time average constant force \(F = \int_0^{2\pi/\omega} F(t) \, dt = 0\), so the group velocity goes to zero. At some intermediate values of \(\omega\), the group velocity crosses zero and subsequently reverses its direction. However, when \(0 < \alpha < 2\), due to the nonthermal character of Lévy noise, Lévy noise becomes another source driving the particle current. As the index \(\alpha\) decreases, the positive group velocity decreases while the negative group velocity increases. In particular, for \(\alpha = 0.7\), the Lévy flights dominate the transport and the group velocity is even always negative. It is obvious that the transport driven by the Lévy flights is opposite to that driven by the ac driving forces for not too small values of the Lévy noise intensity. In the same potential, the ac driving forces will induce a positive group velocity, while the Lévy flights will give a negative group velocity for given index, shown in figure 2(a). Interestingly, we also found that the group velocity tends to a negative constant, instead of zero, at the fast-driving limit. In this case the effects of ac driving forces disappear and the Lévy flights will dominate the transport. In figure 2(b), we present the time dependence of the location of the median for a given index \(\alpha = 1.5\). It is found that the particles exhibit a motion toward the right direction at \(\omega = 0.1\) and the opposite direction at \(\omega = 10.0\).

Figure 3 illustrates the dependence of the group velocity \(V_g\) on the Lévy index \(\alpha\) with and without ac driving forces. From the figure we can see that the rectified transport can occur even if the external driving forces are absent. This is due to the nonthermal character of the Lévy noise which can break the thermodynamical equilibrium. Interestingly, one can see that the curves demonstrate nonmonotonic behavior for small values of the noise.

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Figure 2. (a) Group velocity $V_g$ as a function of the driving frequency $\omega$ for different values of Lévy index $\alpha$ at $D = 0.1$ and $A_0 = 1.0$. (b) Time dependence of the location of the median $q_{0.5}(t)$ for different values of the driving frequency $\omega$ at $D = 0.1$ and $A_0 = 1.0$.

These phenomena can be explained by the interplay between the potential profile and Lévy flights. Firstly, the particles stay in the minima of the potential awaiting large noise pulses to be catapulted out. For not too small values of the Lévy index, the Lévy flights are shorter, and the outliers in the Lévy noise are smaller; the distance from minima to maxima dominates the transport. In this case this distance is shorter from the steeper side (the left side) than from the slanted side (the right side). Consequently, most of the particles are thrown out from the steeper side, resulting in negative group velocity. However, for very small values of the Lévy index, the Lévy flights are longer.
and the outliers in the Lévy noise are larger. In this case, the slope of the potential dominates the transport. So it is easier for the particles to move toward the slanted side than toward the steeper side and the group velocity is positive. Therefore, there exists an intermediate value of $\alpha$ at which the group velocity takes its extra value and the group velocity as a function of the Lévy index is nonmonotonic. However, for large values of the noise intensity, the curves demonstrate monotonic behavior. This case was reported in [21]. When a large ac driving force ($A_0 = 1.0$) is added, the ac driving force dominates the transport and the group velocity is positive. However, the shape of the curve is similar to that without the ac driving forces.

In figure 4, the group velocity $V_g$ is plotted for different values of amplitude $A_0$ as a function of the noise intensity $D$. When the driving amplitude $A_0$ is small, the Lévy flights will dominate the transport and the group velocity is negative. On increasing the amplitude $A_0$, the ac driving forces gradually dominate the transport and the group velocity crosses zero and becomes positive, namely, transport reversal occurs. When $D \to 0$, the group velocity tends to zero for small driving forces ($A_0 = 0.0, 0.5, 0.65,$ and $0.75$) and goes to a finite value for large driving forces ($A_0 = 0.85, 0.9,$ and $1.0$). This is due to the fact that in the ratchets determined, the net current occurs for large driving forces and disappears for small driving forces. When $D \to \infty$, the effects of the potential and the ac driving forces disappear, so the group velocity tends to zero. For suitable amplitude $A_0$ the group velocity can change its direction on increasing the noise intensity. Remarkably, multiple transport reversals even occur at $A_0 = 0.75$ (see figure 4(b)) and the group velocity reverses its direction twice. The intensive competition between the ac driving forces and the Lévy flights leads to this peculiar phenomenon.

We next investigate the role of the amplitude $A_0$ in the transport. The results are shown in figure 5. It is found that the group velocity will tend to a negative constant for very small driving forces. As we know, in a classical forced thermal ratchet [1, 5]
the velocity will tend to zero for very small driving forces. The nonthermal character of the Lévy noise induces this different behavior. For very large values of the amplitude, the influence of the potential and the Lévy flights will become negligible and the group velocity goes to zero. At some intermediate values of amplitude $A_0$, the group velocity crosses zero and reverses its direction.

4. Concluding remarks

In this paper, we studied the transport of overdamped particles moving in an asymmetrically periodic potential in the presence of ac driving forces and Lévy flights.

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Figure 5. Dependence of group velocity $V_g$ on the driving amplitude $A_0$ for different values of the noise intensity at $\alpha = 1.5$ and $\omega = 0.1$.

Because the mean of the Lévy noise and the mean of the overall displacement do not exist, the definition of the average velocity may be invalid. Therefore, throughout the study we use the group velocity proposed by Dybiec et al [22] to measure the transport of overdamped particles. From the Langevin numerical simulations, we found that ac driving forces and Lévy flights are the two different drivers in nature that can break thermodynamical equilibrium. The competition between the two driving factors induces some peculiar phenomena. Due to the Lévy flights, the group velocity tends to a negative constant, instead of zero, for the fast-driving limit and it may be always negative for small values of the Lévy index. We also found that the relation between the group velocity and the Lévy index is nonmonotonic for small values of the noise intensity. There exists an intermediate value of the Lévy index at which the group velocity takes its extreme value. Remarkably, multiple transport reversals occur when the noise intensity changes. This is caused by the competition between the ac driving forces and Lévy flights.

Though the model presented does not pretend to be a realistic model for a real system, beyond its intrinsic theoretical interest, the results that we have presented have potential applications in many processes such as diffusive transport in plasmas, particle separation with non-Gaussian diffusion, and ratchet transport in biological systems that are intrinsically out of equilibrium.

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