Optimal trade execution in an order book model with stochastic liquidity parameters

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June 11, 2020

We analyze an optimal trade execution problem in a financial market with stochastic liquidity. To this end we set up a limit order book model in which both order book depth and resilience evolve randomly in time. Trading is allowed in both directions and at discrete points in time. We derive an explicit recursion that, under certain structural assumptions, characterizes minimal execution costs. We also discuss several qualitative aspects of optimal strategies, such as existence of profitable round trips or closing the position in one go, and compare our findings with the literature.

Keywords: optimal trade execution; limit order book; stochastic order book depth; stochastic resilience; discrete-time stochastic optimal control; long time horizon limit; profitable round trip; premature closure.

2020 MSC: Primary: 91G10; 93E20. Secondary: 60G99.

Introduction

Market liquidity describes the extent to which buying (resp. selling) an asset moves the price against the buyer (resp. seller). In an illiquid financial market large orders have a substantial adverse effect on the realized prices. Typically, this effect is not constant over time. Temporal variations of liquidity are partly driven by deterministic trends such as intra-day patterns. In addition, there exist random changes in liquidity.
such as liquidity shocks that superimpose the deterministic evolution. To benefit from
times when trading is cheap, institutional investors continuously monitor the available
liquidity and schedule their order flow accordingly. The scientific literature on optimal
trade execution problems deals with the optimization of trading schedules, when an
investor faces the task of closing a position in an illiquid market. Incorporating random
fluctuations of liquidity into models of optimal trade execution constitutes a highly active
field of research (see, e.g., [1, 6, 7, 9, 11, 12, 14, 16, 17, 18, 19, 20, 22, 24, 26] and
references therein).

In this work we analyze a trade execution problem in a financial market model with
linear stochastic price impact and stochastic resilience. To be more specific, we consider
a block-shaped limit order book, where liquidity is uniformly distributed to the left and
to the right of the mid-price. To account for stochastic liquidity, the depth of the order
book is allowed to vary randomly in time. At initial time 0 the investor observes the
current order book depth $\frac{1}{\gamma_0} > 0$ but has no precise knowledge about the order book
depth at future times (only a probabilistic assessment). If the investor executes a trade of size $\xi_0 \in \mathbb{R}$
at time 0, she incurs costs of size $\frac{\gamma_0 \xi_0^2}{2}$. Moreover, the trade of size $\xi_0$
shifts the mid-price of the order book by $\gamma_0 \xi_0$. Observe that this deviation is positive
if and only if $\xi_0 > 0$, i.e., if $\xi_0$ is a buy order. In the period from time 0 to the next
trading time 1 this deviation changes from $\gamma_0 \xi_0$ to $D_1 - \beta_1 \gamma_0 \xi_0$, where $\beta_1 > 0$ is a
positive stochastic factor (unknown to the investor at time 0). The factor $\beta_1$ describes
the resilience of the order book: if $\beta_1$ is close to 0 the order book nearly fully recovers
from the trade $\xi_0$, whereas if $\beta_1$ is close to 1 the impact of $\xi_0$ persists. We highlight
here that we do not exclude the case, where the event $\{\beta_1 > 1\}$ has positive probability,
which would reflect a possibility of self-exciting behavior of the market impact. At time
1 the value of $\beta_1$ is disclosed to the investor. Moreover, she observes the updated order
book depth $\frac{1}{\gamma_1} > 0$. Based on this information the investor executes a trade of size $\xi_1$
which generates costs $(D_1 - \frac{\gamma_1 \xi_1}{2}) \xi_1$ and moves the deviation to $D_1 - \gamma_1 \xi_1$. By
continuing this sequence of operations to arbitrary trading times $k \in \mathbb{N}$ we thus obtain
our financial market model with stochastic price impact (described by a positive process $\gamma = (\gamma_k)_{k \in \mathbb{N}_0}$) and stochastic resilience (described by a positive process $\beta = (\beta_k)_{k \in \mathbb{N}_0}$).

In this financial market we consider an investor who has to close a financial position
of size $x \in \mathbb{R}$ up to a given time $N \in \mathbb{N}$. We assume that the investor is risk-neutral and
aims at minimizing the overall trading costs. Apart from some technical integrability
conditions we do not a priori impose any restrictions on trading strategies of the investor.
In particular, even if the task is to sell a certain amount of assets (i.e., $x > 0$), we allow
for trading strategies where the investor buys assets at some points in time.

The above description of the model highlights that our setting is a certain discrete-time
formulation within the class of limit order book models, where the liquidity parameters
are stochastic (i.e., both the price impact and the resilience are positive random pro-
cesses). The approach to mathematically model liquidity via order book considerations
was initiated in [3], [4], [23] and [25]. Limit order book models with deterministically
time-varying liquidity are studied in [2], [10] and [13], while stochastic liquidity is dis-

\footnote{We allow for both buy ($\xi \geq 0$) and sell ($\xi \leq 0$) orders.}
cussed in [14]. We point out the following essential differences between our current setting and the settings in the aforementioned papers.

(a) Both in the present paper and in [14], $\beta$ and $\gamma$ are random processes, while they are deterministic functions of time in [2], [10] and [13].

(b) In [10], [13] and [14], execution strategies are constrained in one direction, while trading in both directions is allowed in the present paper and in [2].

(c) In [2], [10], [13] and [14], the resilience process (or function) $\beta$ is assumed to be $(0,1]$-valued, while we only require it to be positive in the present paper.

In our setting we encounter several new qualitative effects, which are briefly mentioned below and discussed in more detail in the main body of the paper. Moreover, for each of these effects, we identify its reason by constructing pertinent examples.

We also mention [1], which is a continuous-time counterpart of our present paper. In this connection it is worth noting that our results do not follow from the results in [1], but both papers rather concentrate on studying different questions: e.g., [1] does not study the qualitative effects mentioned in the previous paragraph (and discussed below); instead we need to work with a challenging quadratic BSDE in [1] and extend the continuous-time problem to incorporate execution strategies of infinite variation. In particular, some of the results of the present paper are required in [1] to derive, e.g., the appropriate problem formulation and the mentioned quadratic BSDE as continuous-time limits of the corresponding discrete-time objects.

In Theorem 2.1 we show that the optimal trading strategies and the minimal expected trading costs are characterized by a single stochastic process $Y = (Y_n)_{n \in \mathbb{Z} \cap (-\infty,N]}$ which is defined via a backward recursion. We prove Theorem 2.1 by means of dynamic programming. To this end we put the trade execution problem into a dynamic framework and allow for arbitrary initial times $n \in \mathbb{Z} \cap (-\infty,N]$, arbitrary initial positions $x \in \mathbb{R}$ and arbitrary initial market deviations $d \in \mathbb{R}$. In this setting we show that the minimal expected overall execution costs amount to

$$V_n(x,d) = \frac{Y_n}{\gamma_n} (d - \gamma_n x)^2 - \frac{d^2}{2 \gamma_n}.$$  \hspace{1cm} (1)

In particular, for each $n \in \mathbb{Z} \cap (-\infty,N]$ it follows that the random variable $2Y_n$ takes values in $(0,1]$ and describes to which percentage the costs of closing one unit $x = 1$ at time $n$ immediately can be reduced by executing this position optimally over $\{n, \ldots, N\}$ (given no initial market deviation $d = 0$). Accordingly, if $Y_n$ is close to $1/2$ it is nearly optimal to close the position immediately in one go, whereas if $Y_n$ is close to 0 it pays off to split the position and to put only a small fraction in the market at time $n$.

In the remainder of the article we discuss several qualitative and quantitative properties of our market model and the trade execution problem. For instance, we analyze whether our financial market admits price manipulation (in the sense of Huberman and
independent of the history up to time $t$. Thus in the case $(i.e., d = 0)$, the market does not admit price manipulation. However, for general $d \in \mathbb{R}$ we have that $V_d(0, d) = \frac{d}{n}(Y_n - \frac{1}{2})$ and thus in the case $d \neq 0$ there exist profitable round trips starting at time $n$ if and only if $Y_n < \frac{1}{2}$. We show that if the investor has a directional view on the resilience process at time $n$, (i.e., $E[\beta_{n+1} | F_n] \neq 1$, where $F_n$ represents the information available at time $n$), then she can exploit the information $d \neq 0$ and construct profitable round trips (see Corollary 4.3 and the subsequent discussion). This is in line with the results in [13] and [14], where $\beta$ is assumed to be time-homogeneous in expectation. It follows immediately from (1) to state the stochastic control problem and provide its financial interpretation. In Section 2 we introduce the mathematical setting, state the stochastic control problem and provide its financial interpretation. In Section 2 we state the stochastic control problem and provide its financial interpretation.
we solve the problem via dynamic programming, study the existence of the long-time limit \( \lim_{n \to -\infty} Y_n \) of the characterizing process \( Y \) and discuss a few technical issues. A subsetting where \( Y \) becomes deterministic is examined in Section 3. In Section 4 we study the existence of profitable round trips and in Section 5 we discuss when it is optimal to close the position prematurely; both sections describe several qualitative effects via general statements and examples. Appendix A contains the proof of Theorem 2.1. Two simple lemmas on integrability, which we often use in our arguments, are included for convenience in Appendix B.

1. A trade execution problem with stochastic market depth and stochastic resilience

In this section we introduce a financial market model where liquidity varies randomly in time. We first give the comprehensive mathematical formulation of the model and subsequently comment on its financial motivation.

**Mathematical formulation** Let \( N \in \mathbb{N} \) and let \( (\Omega, \mathcal{F}, (\mathcal{F}_k)_{k \in \mathbb{Z}}, P) \) be a filtered probability space. Denote \( L^\infty = \bigcap_{p \in [1, \infty)} L^p(\Omega, \mathcal{F}, P) \) and \( L^2_+ = \bigcup_{\varepsilon > 0} L^{2+\varepsilon}(\Omega, \mathcal{F}, P) \). Let \( \beta = (\beta_k)_{k \in \mathbb{Z}} \) and \( \gamma = (\gamma_k)_{k \in \mathbb{Z}} \) be strictly positive adapted stochastic processes, called the resilience and the price impact process, respectively. Assume that \( \beta_k, \gamma_k \in L^\infty \) for all \( k \in \mathbb{Z} \). Furthermore, it turns out to be convenient to denote the multiplicative increments of \( \gamma \) by \( \eta_n = \frac{\gamma_n}{\gamma_{n-1}}, n \in \mathbb{Z} \).

For \( n \in \mathbb{Z} \cap (-\infty, N] \) and \( x \in \mathbb{R} \), we call a real-valued adapted stochastic process \( \xi = (\xi_k)_{k \in \{n, \ldots, N\}} \) satisfying \( x + \sum_{j=n}^N \xi_j = 0 \) an execution strategy. We denote by \( \mathcal{A}_n(x) \) the set of all execution strategies \( \xi \) with \( \xi_k \in L^2_+ \) for all \( k \in \{n, \ldots, N\} \). For an execution strategy \( \xi \in \mathcal{A}_n(x) \) we call the process \( X = (X_k)_{k \in \{n, \ldots, N\}} \) satisfying \( X_k = x + \sum_{j=n}^k \xi_j \), \( k \in \{n, \ldots, N\} \) the position path associated to \( \xi \). For \( d \in \mathbb{R} \) and \( \xi \in \mathcal{A}_n(x) \) we define the deviation process \( D = (D_{k-})_{k \in \{n, \ldots, N\}} \) associated to \( \xi \) recursively by

\[
D_{n-} = d \quad \text{and} \quad D_{k-} = (D_{(k-1)-} + \gamma_{k-1}\xi_{k-1})\beta_k, \quad k \in \{n+1, \ldots, N\}. \tag{2}
\]

Note that the process \( D = (D_{k-})_{k \in \{n, \ldots, N\}} \) is adapted. The value function \( V : \Omega \times (\mathbb{Z} \cap (-\infty, N]) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) of the control problem is given by

\[
V_n(x, d) = \operatorname{ess inf}_{\xi \in \mathcal{A}_n(x)} \mathbb{E}_n \left[ \sum_{j=n}^N \left( D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \right], \quad n \in \mathbb{Z} \cap (-\infty, N], x \in \mathbb{R}, d \in \mathbb{R}, \tag{3}
\]

where the argument \( d \) is the starting point of the process \( D \) in (2), and \( \mathbb{E}_n[\cdot] \) is a shorthand notation for \( \mathbb{E}[\cdot | \mathcal{F}_n] \).

**Financial interpretation** The numbers \( N \in \mathbb{N} \) and \( n \in \mathbb{Z} \cap (-\infty, N] \) specify the end and the beginning of the trading period, respectively. The possible trading times are
given by the set \( \{n, \ldots, N\} \). The number \( x \in \mathbb{R} \) represents the initial position of the agent. A negative \( x < 0 \) means that the agent has to buy \( |x| \) shares over the trading period, while a positive \( x > 0 \) means that the agent has to sell \( x \) shares over the trading period. For an execution strategy \( \xi \in \mathcal{A}_n(x) \) the value of \( \xi_k \) specifies the number of shares bought by the agent at time \( k \in \{n, \ldots, N\} \). A negative value \( \xi_k < 0 \) means that the agent sells shares. For the associated position path \( X \) the value of \( X_k \) represents the agent’s position at time \( k \in \{n, \ldots, N\} \) directly after the trade \( \xi_k \). Observe that all position paths satisfy \( X_N = 0 \), i.e., the position is closed after the last trade at time \( N \). The process \( D \) describes the deviation of the price of a share from the unaffected price caused by the past trades of the agent. Given a deviation of size \( D_{(k-1)}+ \) directly prior to the trade at time \( k - 1 \), the deviation directly after a trade of size \( \xi_{k-1} \) equals \( D_{(k-1)}+ \gamma_{k-1} \xi_{k-1} \). In particular, the change of the deviation is proportional to the size of the trade and the proportionality factor is given by the price impact process \( \gamma \). In the language of the literature on optimal trade execution problems our model thus includes a linear price impact. This corresponds to a block-shaped limit order book, i.e., limit orders are uniformly distributed to the left and to the right of the mid-market price. The height of the order book at time \( k \) is given by \( 1/\gamma_k \). In particular, our model allows the height of the limit order book to evolve randomly in time and thereby captures stochastic market liquidity. Note that since \( \gamma \) is positive, a purchase \( \xi_k > 0 \) increases the deviation whereas a sale \( \xi_k < 0 \) decreases it. In the period after the trade at time \( k - 1 \) and before the trade at time \( k \) the deviation changes from \( D_{(k-1)}+ \gamma_{k-1} \xi_{k-1} \) to \( D_k- = (D_{(k-1)}+ \gamma_{k-1} \xi_{k-1}) \beta_k \). In the literature on optimal execution the resilience process \( \beta \) is often assumed to take values in \((0, 1)\) and describes the speed with which the deviation tends back to zero between two trades. On the contrary, we assume \( \beta \) only to be positive. A value \( \beta_k > 1 \) describes the effect when the deviation continues to move in the direction of the trade for some time after the trade. Note that the \( \beta \) factor evolves randomly in time. In particular, when making a decision about the size of the trade at time \( k - 1 \), the agent, in general, cannot predict the exact impact of this trade on the future price at time \( k \). Note, however, that the agent observes the realization of \( \beta_k \) before she makes the decision about the size of trade at time \( k \). At each time \( k \in \{n, \ldots, N\} \) the costs of a trade \( \xi_k \) amount to \( (D_k- + \gamma_k \xi_k) \xi_k \). This means that the price per share that the agent has to pay equals the mean of the deviation before the trade \( D_{(k-1)}- \) and the deviation after the trade \( D_k- + \gamma_k \xi_k \). The control problem thus corresponds to minimizing the expected costs of closing an initial position of size \( x \) within the trading period \( \{n, \ldots, N\} \) given an initial deviation \( d \).

We conclude this section with some remarks on the well-posedness of the optimal trade execution problem \((\mathcal{3})\) and a possible extension of the model.

**Remark 1.1.** Let \( n \in \mathbb{Z} \cap (-\infty, N] \), \( x, d \in \mathbb{R} \) and \( \xi \in \mathcal{A}_n(x) \). Then for the associated deviation process \( (D_k-)_{k \leq \{n, \ldots, N\}} \) it holds that \( D_k- \in L^{2+} \) for all \( k \in \{n, \ldots, N\} \).

We prove this claim by induction on \( k \). Since \( D_{n-} = d \), the claim obviously holds true for \( k = n \). Consider the step \( \{n, \ldots, N-1\} \ni k - 1 \rightarrow k \in \{n + 1, \ldots, N\} \) and note that by the Minkowski inequality and \((\mathcal{2})\), it is sufficient to show that \( D_{(k-1)-} \beta_k \in L^{2+} \) and \( \gamma_{k-1} \xi_{k-1} \beta_k \in L^{2+} \). Since \( \beta_k \in L^{\infty-} \) and, by the induction hypothesis, \( D_{(k-1)-} \in L^{2+} \) and \( \gamma_{k-1} \xi_{k-1} \beta_k \in L^{2+} \), it is sufficient to show that \( D_{(k-1)-} \beta_k \in L^{2+} \).
Remark 1.3. For $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$ and $\xi \in \mathcal{A}_n(x)$ the deviation process $D = (D_{k-})_{k \in \{n, \ldots, N\}}$ associated to $\xi$ is given explicitly by

$$D_{k-} = d \prod_{l=n+1}^{k} \beta_l + \sum_{i=n+1}^{k} \gamma_{i-1} \xi_{i-1} \prod_{l=i}^{k} \beta_l, \quad k \in \{n, \ldots, N\}.$$  

This can be established by induction on $k \in \{n, \ldots, N\}$.

Remark 1.4. One can also include an unaffected price process in the model. Indeed, if the unaffected price process is given by the square integrable martingale $S = (S_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$, then, for all $n \in \mathbb{Z} \cap (-\infty, N]$, $x \in \mathbb{R}$ and $\xi \in \mathcal{A}_n(x)$, with the notation $X_{n-1} = x$, we get

$$E_n \left[ \sum_{j=n}^{N} S_j \xi_j \right] = E_n \left[ \sum_{j=n}^{N} S_j (X_j - X_{j-1}) \right] = E_n \left[ -x S_n - \sum_{j=n}^{N-1} X_j (S_{j+1} - S_j) \right] = -x S_n.$$  

It follows that for all $n \in \mathbb{Z} \cap (-\infty, N]$ and $x, d \in \mathbb{R}$ the expected costs generated by an execution strategy $\xi \in \mathcal{A}_n(x)$ with the deviation process $(D_{k-})_{k \in \{n, \ldots, N\}}$ of (2) satisfy

$$E_n \left[ \sum_{j=n}^{N} \left( S_j + D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \xi_j \right] = -x S_n + E_n \left[ \sum_{j=n}^{N} \left( D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \xi_j \right].$$  

Hence, minimizing $E_n \left[ \sum_{j=n}^{N} \left( S_j + D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \xi_j \right]$ is equivalent to (3).

2. Characterization of minimal costs and optimal strategies

The following result provides a solution to the stochastic control problem (3). It shows that the value function and the optimal strategy in (3) are characterized by a single process $Y$ that is defined via a backward recursion.
Theorem 2.1. Assume that for all \( n \in \mathbb{Z} \cap (-\infty, N] \) we have \( \beta_n, \gamma_n, \frac{1}{\alpha_n} \in L^{\infty} \) and that for all \( n \in \mathbb{Z} \cap (-\infty, N-1) \) it holds that \( E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] < 1 \) a.s. and, with \( \alpha_n = 1 - E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] \), we have \( \frac{1}{\alpha_n} \in L^{\infty} \). Let \( (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]} \) be the process that is recursively defined by \( Y_N = \frac{1}{2} \) and

\[
Y_n = E_n[Y_{n+1}Y_{n+1}] - \frac{(E_n[Y_{n+1} (\beta_{n+1} - \eta_{n+1})])^2}{E_n[Y_{n+1} (\beta_{n+1} - \eta_{n+1})]^2 + \left(1 - \frac{\beta_{n+1}^2}{\eta_{n+1}}\right)], \quad n \in \mathbb{Z} \cap (-\infty, N-1].
\] (6)

Then it holds for all \( n \in \mathbb{Z} \cap (-\infty, N] \), \( x, d \in \mathbb{R} \) that

\[
V_n(x, d) = \frac{Y_n}{\gamma_n} (d - \gamma_n x)^2 - \frac{d^2}{2\gamma_n} \quad \text{and} \quad 0 < Y_n \leq \frac{1}{2}.
\] (7)

Moreover, for all \( x, d \in \mathbb{R} \) the (up to a \( P \)-null set) unique optimal trade size is given by

\[
\xi_n^\ast(x, d) = \frac{E_n[Y_{n+1} (\beta_{n+1} - \eta_{n+1})]}{E_n[Y_{n+1} (\beta_{n+1} - \eta_{n+1})]^2 + \left(1 - \frac{\beta_{n+1}^2}{\eta_{n+1}}\right)] \left( x - \frac{d}{\gamma_n} \right) - \frac{d}{\gamma_n}, \quad n \in \mathbb{Z} \cap (-\infty, N-1],
\] (8)

and \( \xi_n^\ast(x, d) = -x \), and we have \( \xi_n^\ast(x, d) \in L^{\infty} \) for all \( n \in \mathbb{Z} \cap (-\infty, N] \) and \( x, d \in \mathbb{R} \).

In particular, for all \( n \in \mathbb{Z} \cap (-\infty, N] \), \( x, d \in \mathbb{R} \) the process \( \xi^\ast = (\xi_k^\ast)_{k \in \{n, \ldots, N\}} \) recursively defined by \( X_{n-1}^\ast = x, D_{n-}^\ast = d, \)

\[
\xi_k^\ast = \xi_k^\ast(X_{k-1}^\ast, D_{k-}^\ast), \quad X_k^\ast = X_{k-1}^\ast + \xi_k^\ast, \quad D_{(k+1)-}^\ast = (D_{k-}^\ast + \gamma_k \xi_k^\ast) \beta_{k+1}, \quad k \in \{n, \ldots, N\}
\] (9)

is a unique optimal strategy in \( A_n(x) \) for \( \{0, 1\} \).

The proof of Theorem 2.1 is deferred to Appendix A.

We can give the following interpretation to the process \( Y \) from Theorem 2.1. Suppose that at time \( n \in \mathbb{Z} \cap (-\infty, N] \) the task is to sell \( x = 1 \) share given an initial deviation of \( d = 0 \). Then immediate execution of the share generates the costs \( \frac{d^2}{2\gamma_n} \). The optimal execution strategy incurs the expected costs \( V_n(1, 0) = \gamma_n Y_n \) (recall (7)). So, the random variable \( 2Y_n : \Omega \to [0, 1] \) describes to which percentage the costs of selling the unit immediately can be reduced by executing the position optimally.

In the next proposition we study the existence of the long-time limit \( \lim_{n \to -\infty} Y_n \).

Proposition 2.2. Let the assumptions of Theorem 2.1 be in force. Fix any \( p \in [1, \infty) \).

(i) The sequence \( (\gamma_n Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]} \) converges a.s. and in \( L^p \) as \( n \to -\infty \) to a finite nonnegative random variable.

(ii) If \( (\gamma_n)_{n \in \mathbb{Z} \cap (-\infty, N]} \) is a supermartingale, then the sequence \( (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]} \) converges a.s. and in \( L^p \) as \( n \to -\infty \) to a finite nonnegative random variable.
The assumption that \((\gamma_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is a supermartingale in (ii) means that the liquidity in the model increases in time (in average). In Lemma 3.3 below \((\gamma_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is a submartingale and \((Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) does not converge. This shows that the claim in (ii) does not in general hold in the situation when the liquidity in the model decreases in time.

\[\text{Proof. (i) It follows from (6) that for all } n \in \mathbb{Z} \cap (-\infty, N-1] \text{ it holds } Y_n \leq E_n[\gamma_{n+1}Y_{n+1}] = \frac{1}{\gamma_n} E_n[\gamma_{n+1}Y_{n+1}] \text{.} \] Thus, \((\gamma_n Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is a submartingale. Hence it converges a.s. as \(n \to -\infty\) due to the backward convergence theorem. Moreover, \((\gamma_n Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is a positive sequence in \(L^\infty\), and, by the Jensen inequality, \((\gamma_n Y_n)^p \leq E_n[(\gamma_n Y_n)^p], n \in \mathbb{Z} \cap (-\infty, N]\), hence the sequence \(((\gamma_n Y_n)^p)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is uniformly integrable. This implies the convergence in \(L^p\).

(ii) If \((\gamma_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is a supermartingale, then it converges a.s. as \(n \to -\infty\) to a \(\mathbb{R} \cup \{+\infty\}\)-valued random variable, denoted by \(\gamma_{-\infty}\), due to the backward convergence theorem. As the process \((\gamma_n)\) is positive, \(\gamma_{-\infty}\) is, in fact, \([0, +\infty]\)-valued. Furthermore, it holds:

\[0 = E[\gamma_{-\infty} 1_{\{\gamma_{-\infty} = 0\}}] \geq E[\gamma N 1_{\{\gamma_{-\infty} = 0\}}] \geq 0.\]

Together with the fact that \(\gamma_N > 0\) a.s., this implies \(\gamma_{-\infty} > 0\) a.s. It now follows from (i) that \((Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) converges a.s. as \(n \to \infty\). As the sequence \((Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}\) is bounded (being \((0, 1/2]\)-valued), it also converges in \(L^p\).

The next remark provides an improved upper bound for \(Y\).

**Remark 2.3** (Upper bound for \(Y\)). Under the assumptions of Theorem 2.1 for all \(n \in \mathbb{Z} \cap (-\infty, N]\) it holds that \(\gamma_n Y_n = Y_n(1, 0)\). For an initial position of size 1 at time \(n \in \mathbb{Z} \cap (-\infty, N]\) a possible execution strategy is to sell the whole unit at a point in time \(k \in \{n, \ldots, N\}\). If there is no initial deviation, i.e., \(d = 0\), it follows that the expected costs of such a strategy amount to \(E_n\left[\frac{2k_N}{T}\right]\). This implies that \(Y_n \leq \frac{\min_{k \in \{n, \ldots, N\}} E_n[\gamma_k]}{2\gamma_n}\), which improves the bound \(Y_n \leq \frac{1}{2}\) provided by Theorem 2.1.

Besides some integrability assumptions, Theorem 2.1 requires that \(E_n\left[\frac{\beta_n^2}{\eta_n+1}\right] < 1\) a.s. for all \(n \in \mathbb{Z} \cap (-\infty, N-1]\). The next remark discusses this assumption.

**Remark 2.4** (Discussion of the structural assumption). The assumption \(E_n\left[\frac{\beta_n^2}{\eta_n+1}\right] < 1\) a.s. for all \(n \in \mathbb{Z} \cap (-\infty, N-1]\) in Theorem 2.1 is a certain structural assumption which ensures that minimization problem \((3)\) is strictly convex. More precisely, under this assumption the coefficients \(a_n\) in front of \(\xi^2\) in \((42)\) (see Appendix A) and the random variables \(Y_n\) in \((6)\) stay positive at all times. In this remark we show that, on the one hand, this assumption is in general not necessary for that, but, on the other hand, it guarantees that the problem preserves the structure with increasing number of time steps. To this end we consider a two-period version of the problem and distinguish

\(^2\)Here we use the convention \(\infty \cdot 0 = 0\).
several cases. First, we recall that $Y_N = \frac{1}{2}$ and observe that with (42) it holds for all $x, d \in \mathbb{R}$

$$V_{N-1}(x, d) = \text{ess inf}_{\xi \in \mathcal{S}_{N-1}} \left\{ E_{N-1}[\eta_N + 1 - 2\beta_N] \frac{(\gamma_N - 1)^2}{2} + E_{N-1} \left[ (1 - \beta_N) d - \left( \frac{\beta_N}{\eta_N} - 1 \right) \gamma_N x \right] \xi 
+ E_{N-1} \left[ \frac{\gamma_N x^2}{2} - \beta_N d \xi \right] \right\}.$$  

(10)

Next, observe that the process $Y$ defined by (6) is given at time $N - 1$ by

$$Y_{N-1} = E_{N-1} \left[ \frac{\eta_N}{2} \right] - \frac{(E_{N-1}[\beta_N - \eta_N])^2}{2E_{N-1}[\eta_N - 2\beta_N + 1]} = E_{N-1}[\eta_N] - \frac{(E_{N-1}[\beta_N])^2}{2E_{N-1}[\eta_N - 2\beta_N + 1]}.$$  

(11)

Moreover, the Cauchy-Schwarz inequality ensures that $(E_{N-1}[\beta_N])^2 \leq E_{N-1} \left[ \frac{\beta_N}{\eta_N} \right] E_{N-1}[\eta_N]$ and hence it holds that

$$\frac{2E_{N-1}[\beta_N] - 1}{E_{N-1}[\eta_N]} \leq \frac{(E_{N-1}[\beta_N])^2}{E_{N-1}[\eta_N]} \leq E_{N-1} \left[ \frac{\beta_N}{\eta_N} \right].$$  

(12)

In particular, we get the following statements.

(i) On the event $\left\{ \frac{2E_{N-1}[\beta_N]}{E_{N-1}[\eta_N]} > 1 \right\}$ the minimization problem in (10) is ill-posed in the sense that it is strictly concave and one can generate infinite gains (in the limit) by choosing strategies with $|\xi| \to \infty$.

(ii) On the event $\left\{ \frac{2E_{N-1}[\beta_N]}{E_{N-1}[\eta_N]} - 1 < \frac{(E_{N-1}[\beta_N])^2}{E_{N-1}[\eta_N]} \right\}$ there exists a minimizer in (10). The random variable $Y_{N-1}$ is, however, negative. As a consequence, in view of (38), one needs to impose further conditions on $\beta_{N-1}$ and $\eta_{N-1}$ to ensure that the coefficient $a_{N-2}$ is positive and that the minimization problem at time $N - 2$ is well-posed.

(iii) On the event $\left\{ \frac{(E_{N-1}[\beta_N])^2}{E_{N-1}[\eta_N]} < 1 \right\}$, which is bigger than $\left\{ E_{N-1} \left[ \frac{\beta_N}{\eta_N} \right] < 1 \right\}$ (see (12)), there exists a minimizer in (10) and, moreover, $Y_{N-1} \in (0, \frac{1}{2})$ (see (11)).

Observe, however, that replacing the assumption $E_n \left[ \frac{\beta_{n+1}}{\eta_{n+1}} \right] < 1$ a.s. with the weaker one $\left( \frac{E_n[\beta_n]}{E_n[\eta_n]} \right)^2 < 1$ a.s. for all $n \in \mathbb{Z} \cap (-\infty, N - 1)$ does not in general allow to perform the backward induction, as the structure of the problem can be lost already on the step $N - 1 \to N - 2$. Namely, $Y_{N-1}$ can be strictly less than $\frac{1}{2}$ (in contrast to $Y_N = \frac{1}{2}$), while $E_{N-2} \left[ \frac{\beta_{N-1}}{\eta_{N-1}} \right]$ can be strictly bigger than 1 (even assuming $\left( \frac{E_{N-2}[\beta_{N-1}]}{E_{N-2}[\eta_{N-1}]} \right)^2 < 1$ a.s.), and we do not necessarily get positivity of $a_{N-2}$ (see (38)).

The next remark reveals the following property of optimal strategies: Irrespective of the position $x$ and the deviation $d$ prior to the trade at time $n$, the ratio between position and deviation after the trade $\xi_n^*(x, d)$ is given by an $\mathcal{F}_n$-measurable random variable $z_n$ (that does not depend on $(x, d)$).
Remark 2.5 (Optimal deviation-position ratio). In the setting of Theorem 2.1, the optimal position path can be characterized in terms of its ratio to the associated deviation process. More precisely, let \( z = (z_n)_{n \in \mathbb{Z} \cap (-\infty, N]} \) be the \( \mathbb{R} \cup \{ \infty \} \)-valued adapted process given by

\[
    z_n = \frac{\gamma_n E_n \left[ Y_{n+1} (\beta_{n+1} - \eta_{n+1}) \right]}{E_n \left[ (Y_{n+1} - \frac{1}{2}) \frac{\beta_{n+1}^2}{\eta_{n+1}} - Y_{n+1} \beta_{n+1} + \frac{1}{2} \right]}, \quad n \in \mathbb{Z} \cap (-\infty, N - 1], \ z_N = \infty, \quad (13)
\]

where we set \( \frac{a}{0} = \infty \) whenever \( a \in \mathbb{R} \setminus \{0\} \). Notice that the fraction defining \( z_n, n \in \mathbb{Z} \cap (-\infty, N - 1] \), a.s. does not produce \( \frac{0}{0} \) because

\[
    E_n \left[ \left( Y_{n+1} - \frac{1}{2} \right) \frac{\beta_{n+1}^2}{\eta_{n+1}} - Y_{n+1} \beta_{n+1} + \frac{1}{2} \right] = E_n \left[ Y_{n+1} (\beta_{n+1} - \eta_{n+1}) \right] = 0 \text{ a.s.}
\]

under the assumptions of Theorem 2.1. Then for all \( n \in \mathbb{Z} \cap (-\infty, N - 1], x, d \in \mathbb{R}, d \neq \gamma_n x \), the ratio between the deviation \( d + \gamma_n \xi_n^*(x, d) \) and the position \( x + \xi_n^*(x, d) \) directly after the optimal trade equals

\[
    \frac{d + \gamma_n \xi_n(x, d)}{x + \xi_n^*(x, d)} = \frac{\gamma_n E_n \left[ Y_{n+1} (\beta_{n+1} - \eta_{n+1}) \right]}{E_n \left[ Y_{n+1} (\beta_{n+1} - \eta_{n+1}) \right] + E_n \left[ \frac{1}{2} \left( 1 - \frac{\beta_{n+1}^2}{\eta_{n+1}} \right) + \frac{Y_{n+1}}{\eta_{n+1}} (\beta_{n+1} - \eta_{n+1})^2 \right]} = z_n, \quad (14)
\]

which does not depend on the pair \( (x, d) \) except the requirement \( d \neq \gamma_n x \) (the latter is to exclude the deviation-position ratio \( \frac{0}{0} \), see (8)). Likewise, for all \( x, d \in \mathbb{R}, d \neq \gamma_N x \), the deviation-position ratio after the terminal trade equals

\[
    \frac{d + \gamma_N \xi_N(x, d)}{x + \xi_N^*(x, d)} = \infty = z_N.
\]

It is worth noting that the process \( z \) can take value \( \infty \) also before the terminal time \( N \) and it is even possible that \( z \) takes finite values after being infinite (see Section 5 for more detail).

3. Processes with independent multiplicative increments

In this section we restrict attention to resilience and price impact processes that satisfy\(^3\) (PIMI) for all \( k \in \mathbb{Z} \), the random variables \( \eta_{k+1} \) and \( \beta_{k+1} \) are independent of \( \mathcal{F}_k \).

\(^3\)Recall that \( \eta_n = \frac{\gamma_n}{\gamma_{n-1}}, n \in \mathbb{Z} \).
In this case it turns out that the process $Y$ from Theorem 2.1 is deterministic.

**Corollary 3.1.** Assume (PIMI), that for all $n \in \mathbb{Z} \cap (\infty, N]$ we have $\beta_{n}, \gamma_{n}, \frac{1}{\gamma_{n}} \in L^{\infty}$ and that for all $n \in \mathbb{Z} \cap (\infty, N - 1]$ it holds that $E \left[ \beta_{n+1}^{2} \right] < 1$. Let $Y = \{Y_{n}\}_{n \in \mathbb{Z} \cap (\infty, N]}$ be the process from Theorem 2.1 that is recursively defined by $Y_{N} = \frac{1}{2}$ and $[0]$. Then $Y$ is deterministic, $(0, \frac{1}{2}]$-valued and satisfies the recursion

$$Y_{n} = E[\eta_{n+1}]Y_{n+1} - \frac{Y_{n+1}^{2}(E[\beta_{n+1}] - E[\eta_{n+1}])^{2}}{Y_{n+1}E\left(\frac{(\beta_{n+1} - \gamma_{n+1})^{2}}{\eta_{n+1}}\right)} + \frac{1}{2} \left(1 - E\left[\frac{\beta_{n+1}^{2}}{\eta_{n+1}}\right]\right), n \in \mathbb{Z} \cap (\infty, N - 1]. \quad (15)$$

Furthermore, formula (8) for optimal trade sizes in the state $(x, d) \in \mathbb{R}^{2}$ takes the form

$$\xi_{n}^{*}(x, d) = -d/n, n \in \mathbb{Z} \cap (\infty, N - 1], \quad (16)$$

and $\xi_{N}^{*}(x, d) = -x$.

**Proof.** Recursion (15) follows by a straightforward induction argument. Formula (16) is an immediate consequence of the fact that $Y$ is deterministic. \hfill $\square$

In the next proposition we discuss the long-time limit $\lim_{n \to -\infty} Y_{n}$ assuming (PIMI) and a sort of time-homogeneity (only for expectations).

**Proposition 3.2.** Suppose that the assumptions of Corollary 3.1 hold true and that

$$\tilde{\beta} = E[\beta_{n+1}], \tilde{\eta} = E[\eta_{n+1}] \text{ and } \tilde{\alpha} = E\left[\frac{\beta_{n+1}^{2}}{\eta_{n+1}}\right] \text{ do not depend on } n \in \mathbb{Z} \cap (\infty, N - 1].$$

1. If $\tilde{\beta} = 1$, we have $\tilde{\eta} > 1$, and it holds for all $n \in \mathbb{Z} \cap (\infty, N]$ that $Y_{n} = \frac{1}{2}$.

2. If $\tilde{\eta} \leq 1$, we have $\tilde{\beta} < 1$, and the sequence $Y = \{Y_{n}\}_{n \in \mathbb{Z} \cap (\infty, N]}$ converges monotonically to $0$ as $n \to -\infty$.

3. If $\tilde{\beta} \neq 1$ and $\tilde{\eta} > 1$, the sequence $Y = \{Y_{n}\}_{n \in \mathbb{Z} \cap (\infty, N]}$ converges monotonically to

$$\frac{1}{2} (1 - \tilde{\alpha}) (\tilde{\eta} - 1) \in \left(0, \frac{1}{2}\right) \quad (17)$$

as $n \to -\infty$.

**Discussion of Proposition 3.2** Suppose that at time $n$ we have $x = 1$ share to sell and the initial deviation is $d = 0$. The immediate selling of the share incurs the costs $\frac{\beta_{n}}{2}$. The optimal execution strategy produces the expected costs $V_{n}(1, 0) = \gamma_{n} Y_{n}$ (recall (17)). So, in other words, the question about the long-time limit $\lim_{n \to -\infty} Y_{n}$ is the question of how much better in comparison to the immediate selling we can perform if our time horizon is very big.

In general, dividing a large order into many small orders and executing them in consecutive time points can be profitable compared to the immediate execution because of the following reasons:
(1) the price impact process $\gamma$ penalizes trades at different times in a different way whenever $\gamma$ is nonconstant,

(2) the resilience process $\beta$ changes the deviation process $D$ between the trades whenever $\beta$ is not identically 1.

From this viewpoint the claims of Proposition 3.2, which deals with the “time-homogeneous in expectation (PIMI) case”, are naturally interpreted as follows. If the resilience is in expectation 1 ($\bar{\beta} = 1$), then neither of the above reasons suggests dividing a large order into many small orders (notice that, in this case, the price impact process $\gamma$ is increasing in average, as $\bar{\eta} > 1$). We can asymptotically get rid of the execution costs in the case of nonincreasing price impact (in the sense $\bar{\eta} \leq 1$). Notice that, in this case, the price impact is allowed to be constant, but we anyway profit from the resilience, which, in expectation, drives the deviation back to zero between two trades ($\bar{\beta} < 1$). Finally, in the remaining case of a nontrivial resilience and a geometrically increasing price impact (in the sense $\bar{\beta} \neq 1$ and $\bar{\eta} > 1$) we cannot fully get rid of the execution costs regardless of how big our time horizon is.

Proof of Proposition 3.2. From (15), we have

$$Y_n = \bar{\eta}Y_{n+1} - \frac{Y_{n+1}^2 (\bar{\beta} - \bar{\eta})^2}{Y_{n+1} (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha})}, \quad n \in \mathbb{Z} \cap (-\infty, N - 1].$$

(18)

Define $g: [0, \infty) \to \mathbb{R}$,

$$g(y) = \bar{\eta}y - \frac{y^2 (\bar{\beta} - \bar{\eta})^2}{y (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha})}, \quad y \in [0, \infty).$$

(19)

Note that $\bar{\alpha} < 1$ by assumption and that $\bar{\alpha} - 2\bar{\beta} + \bar{\eta} \geq \frac{(\bar{\beta} - \bar{\eta})^2}{\bar{\eta}} \geq 0$ because $\frac{\bar{\beta}^2}{\bar{\eta}} \leq \bar{\alpha}$ by the Cauchy-Schwarz inequality. Let $y \geq 0$. Then

$$g'(y) = \bar{\eta} - (\bar{\beta} - \bar{\eta})^2 \frac{2y (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha}) - y^2 (\bar{\alpha} - 2\bar{\beta} + \bar{\eta})}{y (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha})^2}$$

$$= \bar{\eta} - (\bar{\beta} - \bar{\eta})^2 \frac{y^2 (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + y (1 - \bar{\alpha})}{y (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha})^2}.$$

Hence, $g'(y) > 0$ is equivalent to

$$\bar{\eta} \left( y (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + \frac{1}{2} (1 - \bar{\alpha}) \right)^2 > (\bar{\beta} - \bar{\eta})^2 \left( y^2 (\bar{\alpha} - 2\bar{\beta} + \bar{\eta}) + y (1 - \bar{\alpha}) \right).$$

13
Divide by $\bar{\eta} > 0$ and note that \(\frac{(\beta-\bar{\eta})^2}{\bar{\eta}} = \frac{\beta^2}{\bar{\eta}} - 2\beta + \bar{\eta}\). This yields the equivalent statement

\[
0 < y^2 \left(\bar{\alpha} - 2\beta + \bar{\eta}\right)^2 + y \left(\bar{\alpha} - 2\beta + \bar{\eta}\right) (1 - \bar{\alpha}) + \frac{(1 - \bar{\alpha})^2}{4} - \frac{(\beta - \bar{\eta})^2}{\bar{\eta}} y^2 \left(\bar{\alpha} - 2\beta + \bar{\eta}\right)
- \frac{(\beta - \bar{\eta})^2}{\bar{\eta}} y (1 - \bar{\alpha})
= y^2 \left(\bar{\alpha} - 2\beta + \bar{\eta}\right) \left(\bar{\alpha} - \frac{\beta^2}{\bar{\eta}}\right) + y \left(\bar{\alpha} - \frac{\beta^2}{\bar{\eta}}\right) (1 - \bar{\alpha}) + \frac{(1 - \bar{\alpha})^2}{4}
= \left(y \left(\bar{\alpha} - \frac{\beta^2}{\bar{\eta}}\right) + \frac{1 - \bar{\alpha}}{2}\right)^2 + y^2 \left(\bar{\alpha} - \frac{\beta^2}{\bar{\eta}}\right) \frac{(\beta - \bar{\eta})^2}{\bar{\eta}}.
\]

Since $\bar{\alpha} < 1$ and $\frac{\beta^2}{\bar{\eta}} \leq \bar{\alpha} < 1$, this always holds true for $y \geq 0$. It follows that $g$ is strictly increasing on $[0, \infty)$.

Recall that $0 < Y_n \leq \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N - 1]$ and $Y_N = \frac{1}{2}$. In particular, $Y_{N-1} \leq Y_N$. The recursion $Y_n = g(Y_{n+1})$, $n \in \mathbb{Z} \cap (-\infty, N - 1]$ (cf. (18) and (19)), implies that the sequence $Y$ is nondecreasing. Hence, the limit $\lim_{n \to -\infty} Y_n$ exists and belongs to $[0, \frac{1}{2}]$. Moreover, it is the largest fixed point of $g$ in $[0, \frac{1}{2}]$. Indeed, since $g$ is increasing, for the largest fixed point $\bar{y}$ of $g$ in $[0, \frac{1}{2}]$, we have that $y \geq \bar{y}$ implies $g(y) \geq g(\bar{y}) = \bar{y}$. Hence, $\bar{y}$ is a lower bound of $Y$. We obtain that $\lim_{n \to -\infty} Y_n \geq \bar{y}$ and is a fixed point of $g$, which means that $\lim_{n \to -\infty} Y_n = \bar{y}$.

1. Suppose that $\bar{\beta} = 1$. The claim that $\bar{\eta} > 1$ follows from $\frac{\beta^2}{\bar{\eta}} \leq \alpha < 1$. A direct calculation shows that $g \left(\frac{1}{2}\right) = \frac{1}{2}$. Since $Y_N = \frac{1}{2}$, it follows that $Y_n = \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$.

2. Suppose that $\bar{\eta} \leq 1$. First notice that $\beta^2 \leq \bar{\eta} \bar{\alpha} < \bar{\eta} \leq 1$ and hence $\bar{\beta} < 1$. Now it follows from (19) that for all $y > 0$ we have $g(y) < y$. This yields that $0$ is the only fixed point of $g$ on $[0, \infty)$ and hence $\lim_{n \to -\infty} Y_n = 0$.

3. Suppose that $\bar{\beta} \neq 1$ and $\bar{\eta} > 1$. In this case

\[
\bar{y} = \frac{\frac{1}{2} (1 - \bar{\alpha}) (\bar{\eta} - 1)}{(1 - \bar{\alpha}) (\bar{\eta} - 1) + (\bar{\beta} - 1)^2} \in \left(0, \frac{1}{2}\right)
\]

is a further fixed point of $g$ and the only one in $(0, \infty)$. Indeed, for $y \in (0, \infty)$ the condition $g(y) = y$ is equivalent to

\[
y \left((\bar{\beta} - \bar{\eta})^2 - (\bar{\eta} - 1) (\bar{\alpha} - 2\beta + \bar{\eta})\right) = \frac{1}{2} (1 - \bar{\alpha}) (\bar{\eta} - 1).
\]

From the fact that

\[(\bar{\beta} - \bar{\eta})^2 - (\bar{\eta} - 1) (\bar{\alpha} - 2\beta + \bar{\eta}) = (1 - \bar{\alpha}) (\bar{\eta} - 1) + (\bar{\beta} - 1)^2 > (1 - \bar{\alpha}) (\bar{\eta} - 1) > 0\]

we deduce (20), which completes the proof.
The following lemma provides an example where the process \( Y = (Y_n)_{n \in \mathbb{Z}\cap(-\infty, N]} \) defined by \( Y_N = \frac{1}{2} \) and \([0]\) does not converge. In this example the price impact process \( \gamma \) is a submartingale (cf. the discussion following Proposition 2.2).

**Lemma 3.3.** Suppose that the assumptions of Corollary 3.1 hold true. Let \( \bar{\beta}_1, \bar{\beta}_2, \bar{\eta}_1, \bar{\eta}_2 \in (0, \infty) \) and \( \bar{\alpha}_1, \bar{\alpha}_2 \in (0, 1) \) such that for all \( k \in \mathbb{N}_0 \) it holds \( \bar{\beta}_1 = E[\beta_{N-2k-1}] = 1, \bar{\beta}_2 = E[\beta_{N-2k}] \neq 1, \bar{\eta}_1 = E[\eta_{N-2k-1}], \bar{\eta}_2 = E[\eta_{N-2k}] > 1, \bar{\alpha}_1 = E[\frac{\beta_{N-2k-1}}{\eta_{N-2k-1}}] \) and \( \bar{\alpha}_2 = E[\frac{\beta_{N-2k}}{\eta_{N-2k}}] \).

Then, \( \gamma \) is a submartingale and \( Y = (Y_n)_{n \in \mathbb{Z}\cap(-\infty, N]} \) does not converge as \( n \to -\infty \). In particular, the sequence \( Y \) is not monotone.

**Proof.** Note first that \( \bar{\beta}_1 = 1 \) and \( \bar{\alpha}_1 < 1 \) imply that \( \bar{\eta}_1 > 1 \) by the Cauchy-Schwarz inequality. It follows from \( 1 < \bar{\eta}_1 = E[\eta_{N-2k-1}] = E_{N-2k-2}[\eta_{N-2k-1}] = E_{N-2k-2}[\frac{\eta_{N-2k-1}}{\gamma_{N-2k-2}}] = \frac{1}{\gamma_{N-2k-2}}E_{N-2k-2}[\gamma_{N-2k-1}] \) and \( 1 < \bar{\eta}_2 = \frac{1}{\gamma_{N-2k-1}}E_{N-2k-1}[\gamma_{N-2k}] \) for all \( k \in \mathbb{N}_0 \) that \( \gamma \) is a submartingale.

For \( i \in \{1, 2\} \), denote by \( g_i \) the function defined by (19) with \( \bar{\beta} = \bar{\beta}_i, \bar{\eta} = \bar{\eta}_i \) and \( \bar{\alpha} = \bar{\alpha}_i \). Recall that \( g_1, g_2 \) are strictly increasing and note that for \( k \in \mathbb{N}_0 \), we have \( Y_{N-2k} = g_1(Y_{N-2k-1}) \) and \( Y_{N-2k} = g_2(Y_{N-2k}) \). Furthermore, the equations \( g_i(y) = y, i \in \{1, 2\} \), are (non-degenerate) quadratic ones, hence the functions \( g_i \) have at most two fixed points. We conclude that the only fixed points of \( g_1 \) are 0 and \( \frac{1}{2} \), and the only fixed points of \( g_2 \) are given by 0 and \( \bar{y} \in (0, \frac{1}{2}) \) from (20). We also notice that \( g_1(y) > y \) for \( y \in (0, \frac{1}{2}) \).

We prove by induction that \( Y_{N-m} > \bar{y} \) for all \( m \in \mathbb{N}_0 \). The case \( m = 0 \) is clear. For the induction step \( \mathbb{N}_0 \ni m \to m + 1 \in \mathbb{N} \), if \( m \) is even, we have \( Y_{N-m-1} = g_2(Y_{N-m}) > g_2(\bar{y}) = \bar{y} \). If \( m \) is odd, it holds \( Y_{N-m-1} = g_1(Y_{N-m}) > g_1(\bar{y}) > \bar{y} \).

It can further be proven inductively that \( Y_{N-m} \geq Y_{N-m-2} \) for all \( m \in \mathbb{N}_0 \) since \( g_1, g_2 \) are increasing and \( Y_{N-2} \leq \frac{1}{2} = Y_N \).

Therefore, the subsequences \( (Y_{N-2k})_{k \in \mathbb{N}_0} \) and \( (Y_{N-2k})_{k \in \mathbb{N}_0} \) of \( Y \) are decreasing in \( k \in \mathbb{N}_0 \) and bounded from below by \( \bar{y} \), which implies that the limits \( \tilde{Y}(\epsilon) = \lim_{k \to \infty} Y_{N-2k} \geq \bar{y} \) and \( Y(\epsilon) = \lim_{k \to \infty} Y_{N-2k-1} \geq \bar{y} \) exist. Taking limits on both sides of \( Y_{N-2k-1} = g_2(Y_{N-2k}) \), we obtain \( Y(\epsilon) = g_2(\tilde{Y}(\epsilon)) \) by continuity of \( g_2 \). Similarly, it holds that \( Y(\epsilon) = g_1(\tilde{Y}(\epsilon)) \). Now, if \( Y(\epsilon) \) and \( \tilde{Y}(\epsilon) \) were equal, then \( Y(\epsilon) = \tilde{Y}(\epsilon) \) would be a common fixed point of \( g_1 \) and \( g_2 \) and hence 0, which is a contradiction to \( \tilde{Y}(\epsilon) \geq \bar{y} > 0 \). We thus conclude that \( Y \) does not converge.

\( \square \)

4. Round trips

Let \( n \in \mathbb{Z}\cap(-\infty, N-1] \). Execution strategies in \( A_n(0) \) are called round trips. It follows from Theorem 2.1 that if initially the agent has no position in the asset, i.e., \( x = 0 \) at
time $n \in \mathbb{Z} \cap (-\infty, N]$, then the minimal costs amount to

$$V_n(0, d) = \frac{d^2}{\gamma_n} \left(Y_n - \frac{1}{2}\right)$$

(22)

for all $d \in \mathbb{R}$. In particular, it holds that $V_n(0, 0) = 0$, i.e., without initial deviation of the price process the agent cannot make profits in expectation. In other words, there are no profitable round trips whenever $d = 0$. The existence of profitable round trips is sometimes also referred to as price manipulation (see, e.g., [5], [15] or [21]). In this regard, if there is no initial deviation of the price process (i.e., $d = 0$), then our model does not admit price manipulation.

Below we study existence of profitable round trips when the price of a share deviates from the unaffected price, i.e., it holds $d \neq 0$. We thus assume $d \neq 0$ in this section.

Recall from (7) that the random variable $Y_n$ is $(0, \frac{1}{2}]$-valued. Together with (22), this implies the following classification:

- on $\{Y_n < \frac{1}{2}\}$ there exist profitable round trips,
- on $\{Y_n = \frac{1}{2}\}$ there are no profitable round trips.

Thus, the question reduces to finding a tractable description of the event $\{Y_n = \frac{1}{2}\}$. We first characterize this event in Proposition 4.1 and discuss several consequences of this characterization. The proof of Proposition 4.1 is postponed to Subsection 4.1.

**Proposition 4.1.** Let the assumptions of Theorem 2.1 be satisfied. Then we have

$$\left\{Y_n = \frac{1}{2}\right\} = \left\{E_n[Y_{n+1}] = \frac{1}{2}, E_n[\beta_{n+1}] = 1\right\}, \quad n \in \mathbb{Z} \cap (-\infty, N - 1],$$

where here and below we understand the equalities for events up to $P$-null sets.

**Corollary 4.2.** Under the assumptions of Theorem 2.1 it holds

$$\left\{Y_{N-1} = \frac{1}{2}\right\} = \{E_{N-1}[\beta_N] = 1\}.$$

**Proof.** The result is immediate because $Y_N = \frac{1}{2}$. \qed

**Corollary 4.3.** Under the assumptions of Theorem 2.1 we have the following inclusions for $n \in \mathbb{Z} \cap (-\infty, N - 1]$:

1. $\{Y_n = \frac{1}{2}\} \subseteq \{Y_{n+1} = \frac{1}{2}\}$ (equivalently, $\{Y_{n+1} < \frac{1}{2}\} \subseteq \{Y_n < \frac{1}{2}\}$) and
2. $\{Y_n = \frac{1}{2}\} \subseteq \{E_n[\beta_{n+1}] = 1\} \subseteq \{E_n[\beta_{n+1}] \geq 1\} \subseteq \{E_n[\eta_{n+1}] > 1\} \subseteq \{E_n[\beta_{n+1}] \neq 1\} \subseteq \{Y_n < \frac{1}{2}\}$.

The proof of Corollary 4.3 is given in Subsection 4.1.
Discussion In the literature on optimal execution it is often assumed that the resilience process $\beta$ takes values in $(0, 1)$. In this case we always have profitable round trips whenever $d \neq 0$, as we know that the deviation will go towards zero due to the resilience and we can make use of it in constructing a profitable round trip (cf. Remark 8.2 in [12] and the discussion after Model 8.3 in [14]). Formally, this fact follows from Corollary 4.3. A natural generalization of this fact to the case of (only) positive $\beta$ is the inclusion \( E_n[\beta_{n+1}] \neq 1 \subseteq \{ Y_n < \frac{1}{2} \} \) (again Corollary 4.3). The intuition is that on the event \( E_n[\beta_{n+1}] \neq 1 \) we “expect” in which direction the deviation will go in the absence of trading. A new qualitative effect in our setting is that the situation of nonexistence of profitable round trips is possible. The previous discussion explains that we necessarily need to be on the event \( \{ E_n[\beta_{n+1}] = 1 \} \) for the non-existence of profitable round trips. A somewhat unexpected effect is, however, that the inclusion \( \{ Y_n = \frac{1}{2} \} \subseteq \{ E_n[\beta_{n+1}] = 1 \} \) can be strict and hence there might exist profitable round trips on the event \( \{ E_n[\beta_{n+1}] = 1 \} \) (see Examples 4.6 and 4.8 below for a more precise discussion). In particular, we cannot distinguish \( Y_n = \frac{1}{2} \) from \( Y_n < \frac{1}{2} \) on the basis of \( E_n[\beta_{n+1}] \) alone, and, indeed, the exact characterization of the event \( \{ Y_n = \frac{1}{2} \} \) also includes \( E_n[Y_{n+1}] \) (see Proposition 4.1).

In more detail, we have the following picture. At time \( N-1 \) we distinguish between \( Y_{N-1} = \frac{1}{2} \) from \( Y_{N-1} < \frac{1}{2} \) on the basis of \( E_{N-1}[\beta_N] \) alone (Corollary 4.2). To discuss the step \( n+1 \to n \) we consider the partition of \( \Omega \) into two disjoint events (in \( \mathcal{F}_n \))

\[
\Omega = \left\{ E_n[Y_{n+1}] < \frac{1}{2} \right\} \cup \left\{ E_n[Y_{n+1}] = \frac{1}{2} \right\} =: A_n \cup B_n. \tag{23}
\]

On \( A_n \) there always exist profitable round trips when we start at time \( n \), while on \( B_n \) we distinguish between the nonexistence and the existence of profitable round trips on the basis of whether \( E_n[\beta_{n+1}] = 1 \) or \( E_n[\beta_{n+1}] \neq 1 \) holds (Proposition 4.1).

A special case, where we obtain an explicit criterion to distinguish between \( Y_n = \frac{1}{2} \) and \( Y_n < \frac{1}{2} \) for all \( n \in \mathbb{Z} \cap (-\infty, N-1) \) only in terms of the process \( \beta \) is the case of processes with independent multiplicative increments of Section 3.

**Corollary 4.4.** Let the assumptions of Corollary 3.1 be in force. We define

\[
n_0 = N \land \inf \{ n \in \mathbb{Z} \cap (-\infty, N-1) : E[\beta_k] = 1 \text{ for all } k \in \mathbb{Z} \cap [n+1, N] \}
\]

(\( \inf \emptyset = \infty \)) and notice that \( n_0 \in (\mathbb{Z} \cup \{-\infty\}) \cap [-\infty, N] \). Then, for the (deterministic) process \( Y \), we have

- \( Y_n < \frac{1}{2} \) for \( n \in \mathbb{Z} \cap (-\infty, n_0) \),
- \( Y_n = \frac{1}{2} \) for \( n \in \mathbb{Z} \cap [n_0, N] \).

**Proof.** The result follows from the previous discussion and the fact that, by Corollary 3.1, the process \( Y \) is deterministic. \( \Box \)

The next proposition contains a sufficient condition for existence of profitable round trips, which is expressed in different terms.
Proposition 4.5. Under the assumptions of Theorem 2.1 for all \( n \in \mathbb{Z} \cap (-\infty, N - 1] \) it holds

\[
\left\{ Y_n = \frac{1}{2} \right\} \subseteq \left\{ \min_{k \in \{n+1, \ldots, N\}} E_n(\gamma_k) \geq \gamma_n \right\} \tag{equivalently, \( \{ \min_{k \in \{n+1, \ldots, N\}} E_n(\gamma_k) < \gamma_n \} \subseteq \{ Y_n < \frac{1}{2} \} \).}
\]

Proof. While the result can be again inferred from the characterization of the event \( \{ Y_n = \frac{1}{2} \} \) in Proposition 4.1, the shortest proof is to recall that \( Y_n < \frac{1}{2} \) on the event \( \{ \min_{k \in \{n+1, \ldots, N\}} E_n(\gamma_k) < \gamma_n \} \) due to Remark 2.3.

We now discuss the inclusion \( \{ Y_n = \frac{1}{2} \} \subseteq \{ E_{n-2}[\beta_{n-1}] = 1 \} \) in more detail. First we present a simple example, where for \( n = N - 2 \) this inclusion is strict (cf. with Corollary 4.2).

Example 4.6. We take any deterministic sequences \( \beta \) and \( \gamma \) with \( \beta_N \neq 1 \) and \( \beta_{N-1} = 1 \) that satisfy the assumptions of Theorem 2.1. Then the process \( Y \) is deterministic. Corollary 4.2 implies that \( Y_{N-1} < \frac{1}{2} \). Hence, by Corollary 4.3 \( Y_{N-2} < \frac{1}{2} \). We thus have

\[
\left\{ Y_{N-2} = \frac{1}{2} \right\} = \emptyset \subsetneq \Omega = \{ E_{N-2}[\beta_{N-1}] = 1 \}.
\]

In other words, for \( d \neq 0 \), we have profitable round trips when we start at time \( N - 2 \), although \( E_{N-2}[\beta_{N-1}] = 1 \). This is not surprising in this example, as we see that profitable round trips are already present when we start at time \( N - 1 \) (\( Y_{N-1} < \frac{1}{2} \), which is caused by \( \beta_N \neq 1 \)). One might, therefore, intuitively expect that here all round trips do not contain a trade at time \( N - 2 \), but this is not the case! If \( d \neq 0 \), then we have for the (here, deterministic) optimal strategy \( \xi^*(0,d) \) of (8) that \( \xi^*_N(0,d) \neq 0 \). Indeed, a straightforward calculation using (8) and the fact that \( \beta, \eta, Y \) are deterministic and \( \beta_{N-1} = 1 \) reveals that \( \xi^*_N(0,d) = 0 \) if and only if it holds \((\frac{1}{2} - Y_{N-1})(1 - \frac{1}{\eta_{N-1}}) = 0\), but the latter is not true in this example because \( Y_{N-1} < \frac{1}{2} \) and \( \frac{1}{\eta_{N-1}} = \frac{\beta_{N-1}}{\eta_{N-1}} < 1 \) (recall the assumptions of Theorem 2.1).

Example 4.6 raises the question of whether profitable round trips for \( d \neq 0 \) with starting time \( n \in \mathbb{Z} \cap (-\infty, N - 2] \) can occur on the event \( \bigcap_{k=n}^{N-1} \{ E_k[\beta_{k+1}] = 1 \} \). Corollary 4.4 implies that this is impossible in the framework of (PIMI) (let alone with deterministic \( \beta \) and \( \gamma \)). But, in general, such a phenomenon is possible, and we present a specific example after the following lemma.

Lemma 4.7. Let the assumptions of Theorem 2.1 be in force and let \( n \in \mathbb{Z} \cap (-\infty, N - 1] \).

(i) We have

\[
\left\{ Y_n = \frac{1}{2} \right\} \subseteq \bigcap_{k=n}^{N-1} \{ E_k[\beta_{k+1}] = 1 \}.
\]

(24)
(ii) The inclusion in \((24)\) is strict (in the sense that the set difference has positive \(P\)-probability) if and only if
\[
\bigcap_{k=n}^{N-1} \{E_k[\beta_{k+1}] = 1\} \notin \mathcal{F}_n,
\]
where \(\mathcal{F}_n = \sigma(\mathcal{F} \cup \mathcal{N})\) with \(\mathcal{N} = \{A \in \mathcal{F} : P(A) = 0\}\).

Proof. Inclusion \((24)\) follows from Corollary 4.3. Clearly, under \((25)\), the inclusion is strict, as \(\{Y_n = \frac{1}{2}\} \in \mathcal{F}_n\). It remains to prove that, if there is \(A_n \in \mathcal{F}_n\), which is (up to a \(P\)-null set) equal to \(\bigcap_{k=n}^{N-1} \{E_k[\beta_{k+1}] = 1\}\), then \(Y_n = \frac{1}{2}\) a.s. on \(A_n\).

First, Corollary 4.2 yields \(Y_{N-1} = \frac{1}{2}\) a.s. on \(A_n\). In the case \(n = N - 2\) this concludes the proof. Let \(n \leq N - 2\). As \(A_n \in \mathcal{F}_n \subseteq \mathcal{F}_{N-2}\), we get \(E_{N-2}[Y_{N-1}] = \frac{1}{2}\) a.s. on \(A_n\). Proposition 4.1 now yields \(Y_{N-2} = \frac{1}{2}\) a.s. on \(A_n\). In the case \(n = N - 2\) this concludes the proof. If \(n \leq N - 3\), we obtain the result by iterating the same procedure. \(\square\)

We, finally, present a specific example, where for \(n = N - 2\) the inclusion in \((24)\) is strict, or, in other words, \(P(Y_{N-2} < \frac{1}{2}, E_{N-2}[\beta_{N-1}] = E_{N-1}[\beta_N] = 1) > 0\) (recall the discussion following Example 4.6).

Example 4.8. Take arbitrary \(a, p \in (0, 1)\). Let \(\mathcal{F}_n = \{\emptyset, \Omega\}\) for \(n \in \mathbb{Z} \cap (-\infty, N - 2]\), \(\mathcal{F}_{N-1} = \mathcal{F}_N = \sigma(\beta_{N-1})\) with \(\beta_{N-1}\) being distributed according to \(P(\beta_{N-1} = 1) = 1 - p\) and \(P(\beta_{N-1} = 1 \pm a) = p/2\). We set \(\beta_N = \beta_{N-1}\) and choose any process \(\gamma\) satisfying the assumptions of Theorem 2.1 (e.g., one can easily take deterministic \(\gamma\)). Then \(E_{N-2}[\beta_{N-1}] = E[\beta_{N-1}] = 1\), hence
\[
\{E_{N-2}[\beta_{N-1}] = 1\} \cap \{E_{N-1}[\beta_N] = 1\} = \{E_{N-1}[\beta_N] = 1\} = \{\beta_N = 1\},
\]
which is an event of probability \(1 - p \in (0, 1)\). We thus obtain \((25)\) for \(n = N - 2\). By Lemma 4.7, the inclusion in \((24)\) for \(n = N - 2\) is strict. As a result, we get \(P(Y_{N-2} < \frac{1}{2}, E_{N-2}[\beta_{N-1}] = E_{N-1}[\beta_N] = 1) > 0\), as required.

4.1. Proofs of Proposition 4.1 and Corollary 4.3

Proof of Proposition 4.1. Throughout the proof fix \(n \in \mathbb{Z} \cap (-\infty, N - 1]\). Let \(\nu = \frac{1}{2} - \left(\frac{1}{2} - Y_{n+1}\right) \frac{\beta_{n+1}^2}{\eta_{n+1}}\). Rewriting the definition of \(Y_n\), we obtain
\[
Y_n = E_n[\eta_{n+1} Y_{n+1}] - \frac{\left(E_n[Y_{n+1} \beta_{n+1}]\right)^2 - 2E_n[Y_{n+1} \beta_{n+1}] E_n[Y_{n+1} \eta_{n+1}] + E_n[Y_{n+1} \beta_{n+1}]^2}{E_n[\nu - 2Y_{n+1} \beta_{n+1} + Y_{n+1} \eta_{n+1}]}
\]
\[
= E_n[\nu] E_n[\nu - 2Y_{n+1} \beta_{n+1} + Y_{n+1} \eta_{n+1}] - \frac{E_n[\nu - Y_{n+1} \beta_{n+1}]^2}{E_n[\nu - 2Y_{n+1} \beta_{n+1} + Y_{n+1} \eta_{n+1}]}
\]
\[
= \frac{1}{2} - E_n \left[ \left(\frac{1}{2} - Y_{n+1}\right) \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] - \frac{\gamma_n}{a_n} \left(\frac{1}{2} - E_n \left[ \left(\frac{1}{2} - Y_{n+1}\right) \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] - E_n[Y_{n+1} \beta_{n+1}]^2 \right)
\]
with \(a_n\) from (38). Since \(\eta_{n+1}, \gamma_n, a_n > 0\) and \(Y_{n+1} \leq \frac{1}{2}\) a.s., it now follows that
\[
\left\{Y_n = \frac{1}{2}\right\} = \left\{E_n \left[ \left(\frac{1}{2} - Y_{n+1}\right) \beta_{n+1}^2 \right] \eta_{n+1} = 0, E_n[Y_{n+1} \beta_{n+1}] = \frac{1}{2}\right\}.
\]
Let \( C_n = \left\{ E_n \left[ \left( \frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] = 0 \right\} \) and denote \( B_n = \left\{ E_n[Y_{n+1}] = \frac{1}{2} \right\} \) as before. We show that \( C_n = B_n \). For the inclusion \( C_n \supseteq B_n \) note first that

\[
\int_{\{E_n[Y_{n+1}] = \frac{1}{2}\}} Y_{n+1} \, dP = \int_{\{E_n[Y_{n+1}] = \frac{1}{2}\}} E_n[Y_{n+1}] \, dP = \int_{\{E_n[Y_{n+1}] = \frac{1}{2}\}} \frac{1}{2} \, dP \tag{27}
\]

and hence that \( Y_{n+1} = \frac{1}{2} \) on \( B_n \). This together with the fact that \( B_n \in \mathcal{F}_n \) implies

\[
1_{B_n} E_n \left[ \left( \frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] = E_n \left[ 1_{B_n} \left( \frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] = 0.
\]

To prove \( C_n \subseteq B_n \), observe that \( C_n \in \mathcal{F}_n \) and that

\[
C_n \subseteq \left\{ \left( \frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\eta_{n+1}} = 0 \right\} = \left\{ Y_{n+1} = \frac{1}{2} \right\}
\]

(by an argument similar to (27)) since \( \beta_{n+1}, \eta_{n+1} > 0 \) and \( Y_{n+1} \leq \frac{1}{2} \) a.s. It thus holds that

\[
1_{C_n} E_n [Y_{n+1}] = E_n [1_{C_n} Y_{n+1}] = 1_{C_n} \frac{1}{2}.
\]

From \( C_n = B_n \) together with (26), we obtain

\[
\left\{ Y_n = \frac{1}{2} \right\} = \left\{ E_n[Y_n] = \frac{1}{2}, E_n[Y_{n+1} \beta_{n+1}] = \frac{1}{2} \right\}.
\]

Furthermore, we have

\[
1_{B_n} E_n [Y_{n+1} \beta_{n+1}] = E_n [1_{B_n} Y_{n+1} \beta_{n+1}] = 1_{B_n} \frac{1}{2} E_n [\beta_{n+1}],
\]

and hence

\[
\left\{ Y_n = \frac{1}{2} \right\} \subseteq \left\{ E_n[Y_n] = \frac{1}{2}, E_n[\beta_{n+1}] = 1 \right\}.
\]

\[ \square \]

**Proof of Corollary 4.3.** We fix \( n \in \mathbb{Z} \cap (-\infty, N - 1) \).

1. The claim follows from

\[
\left\{ Y_n = \frac{1}{2} \right\} \subseteq \left\{ E_n[Y_n] = \frac{1}{2} \right\} \subseteq \left\{ Y_{n+1} = \frac{1}{2} \right\},
\]

where the first inclusion is immediate from Proposition 4.1 and the second one follows from the facts that \( Y_{n+1} \leq \frac{1}{2} \) a.s. and (27).

2. Due to Proposition 4.1 only the inclusion \( \left\{ E_n[\beta_{n+1}] \geq 1 \right\} \subseteq \left\{ E_n[\eta_{n+1}] > 1 \right\} \) needs to be proved. By the Cauchy-Schwarz inequality and the assumption \( E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] < 1 \) a.s. we get

\[
(E_n[\beta_{n+1}])^2 \leq E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] E_n[\eta_{n+1}] < E_n[\eta_{n+1}] \text{ a.s.,}
\]

which implies the claim.

\[ \square \]
5. Closing the position in one go

Let the assumptions of Theorem 2.1 be in force. Let \( n \in \mathbb{Z} \cap (-\infty, N - 1) \). We now study when \( \xi^*_n(x, d) = -x \) for all \( x, d \in \mathbb{R} \), i.e., when it is optimal to close the whole position at time \( n < N \).

Recall that, for each \( x, d \in \mathbb{R} \), a version of the optimal trade \( \xi^*_n(x, d) \) (which is defined up to a \( P \)-null set) is given by the right-hand side of (8). We choose the versions in such a way that the random field \( (x, d) \mapsto \xi^*_n(x, d) \) is continuous (the most natural choice in view of (8)). Then we have

\[
\{ \xi^*_n(x, d) = -x \forall x, d \in \mathbb{R} \} = \bigcap_{x,d \in \mathbb{Q}} \{ \xi^*_n(x, d) = -x \},
\]

hence \( \{ \xi^*_n(x, d) = -x \forall x, d \in \mathbb{R} \} \) is an \( \mathcal{F}_n \)-measurable event (as a countable intersection of such events).

**Lemma 5.1.** Let \( n \in \mathbb{Z} \cap (-\infty, N - 1) \). Under the assumptions of Theorem 2.1 we have

\[
\{ \xi^*_n(x, d) = -x \forall x, d \in \mathbb{R} \} = \{ \mathbb{E}_n[(Y_{n+1} - \frac{1}{2}) \frac{\beta_{n+1}^2}{\eta_{n+1}} - Y_{n+1} \beta_{n+1} + \frac{1}{2}] = 0 \}, \tag{28}
\]

up to a \( P \)-null set.

**Proof.** The result follows from (8) via a straightforward calculation. \( \square \)

The next result presents a relation between the previously studied question of nonexistence of profitable round trips for \( d \neq 0 \) and the currently studied question of closing the position in one go.

**Proposition 5.2.** Let \( n \in \mathbb{Z} \cap (-\infty, N - 1) \). Under the assumptions of Theorem 2.1 we have

1. \( \{ Y_n = \frac{1}{2} \} \subseteq \{ \xi^*_n(x, d) = -x \forall x, d \in \mathbb{R} \} \).
2. \( \{ Y_n = \frac{1}{2} \} = \{ \xi^*_n(x, d) = -x \forall x, d \in \mathbb{R} \} \cap \{ \mathbb{E}_n[Y_{n+1}] = \frac{1}{2} \} \).

It is worth noting that the inclusion in part 1 can be strict in the sense that the set difference can be non-negligible, i.e., with positive probability there are profitable round trips at time \( n \) for \( d \neq 0 \) and still it is optimal to close the whole position at time \( n \) (see Example 5.5 below).

**Proof.** 1. Recall that by Proposition 4.1 and Corollary 4.3 we have

\[
\left\{ Y_n = \frac{1}{2} \right\} = \left\{ \mathbb{E}_n[Y_{n+1}] = \frac{1}{2}, \mathbb{E}_n[\beta_{n+1}] = 1 \right\} \subseteq \left\{ Y_{n+1} = \frac{1}{2} \right\}.
\]

In particular, on the event \( \{ Y_n = \frac{1}{2} \} \in \mathcal{F}_n \) it holds \( Y_{n+1} = \frac{1}{2} \) and \( \mathbb{E}_n[\beta_{n+1}] = 1 \), which implies that on the event \( \{ Y_n = \frac{1}{2} \} \in \mathcal{F}_n \) we have

\[
\mathbb{E}_n\left[\left(Y_{n+1} - \frac{1}{2}\right) \frac{\beta_{n+1}^2}{\eta_{n+1}} - Y_{n+1} \beta_{n+1} + \frac{1}{2}\right] = 0.
\]
Lemma 5.1 now yields the claim.

2. The inclusion “⊆” follows from the previous part together with Proposition 4.1. To prove the reverse inclusion “⊇” we first note that

\[
\left\{ E_n[Y_{n+1}] = \frac{1}{2} \right\} \subseteq \left\{ Y_{n+1} = \frac{1}{2} \right\}
\]

(29)
because \(Y_{n+1} \leq \frac{1}{2}\) a.s. It follows from (28) and (29) that on the \(\mathcal{F}_n\)-measurable set

\[
A_n := \{\xi_n^*(x, d) = -x \ \forall \ x, d \in \mathbb{R}\} \cap \left\{ E_n[Y_{n+1}] = \frac{1}{2} \right\}
\]

it holds \(\frac{1}{2}E_n[\beta_{n+1}] = E_n[Y_{n+1}\beta_{n+1}] = \frac{1}{2}\), i.e., \(E_n[\beta_{n+1}] = 1\). Hence,

\[
A_n \subseteq \left\{ E_n[Y_{n+1}] = \frac{1}{2}, E_n[\beta_{n+1}] = 1 \right\} = \left\{ Y_{n+1} = \frac{1}{2} \right\},
\]

where the set equality is again Proposition 4.1. This concludes the proof.

Corollary 5.3. Under the assumptions of Theorem 2.1 it holds

\[
\left\{ Y_{N-1} = \frac{1}{2} \right\} = \{\xi_{N-1}^*(x, d) = -x \ \forall \ x, d \in \mathbb{R}\}.
\]

Proof. This follows from part 2 of Proposition 5.2 because \(Y_N = \frac{1}{2}\).

We now provide more details for the case of processes with independent multiplicative increments of Section 3. We recall that in this case the process \(Y\) is deterministic. Notice, however, that the trades \(\xi_n^*(x, d)\) are still, in general, random because of the randomness in \(\gamma_n\), see (8).

Proposition 5.4. Let \(n \in \mathbb{Z} \cap (-\infty, N - 1]\). Under the assumptions of Corollary 3.1 it holds:

1. \(\{\xi_n^*(x, d) = -x \ \forall \ x, d \in \mathbb{R}\}\) is either \(\Omega\) or \(\emptyset\).

2. The following statements are equivalent:

\(i\) \(\{\xi_n^*(x, d) = -x \ \forall \ x, d \in \mathbb{R}\} = \Omega\).

\(ii\) There exist \(x, d \in \mathbb{R}\) with \(P(\gamma_n x \neq d) > 0\) such that \(\{\xi_n^*(x, d) = -x\} = \Omega\).

\(iii\) It holds that

\[
E[\beta_{n+1}] = 1 + \frac{1 - E\left[\frac{\beta_{n+1}^2}{\eta_{n+1}}\right]}{Y_{n+1}} \left(1 - Y_{n+1}\right).
\]

(30)

3. Under (30) we have that \(E[\beta_{n+1}] \geq 1\) and, if \(Y_{n+1} < \frac{1}{2}\), even that \(E[\beta_{n+1}] > 1\).
The meaning of part 3 in Proposition 5.4 is that, in the case of (PIMI) (special case: deterministic processes $\beta$ and $\gamma$), closing the position in one go is never optimal in the (usual) framework, where the resilience process $\beta$ is assumed to be $(0, 1)$-valued.

This raises the question of whether closing the position in one go can be optimal in general (that is, beyond (PIMI)) with the resilience process $\beta$ taking values in $(0, 1)$. In our setting the answer is affirmative (see Example 5.6 below). It is worth noting that in the related setting, where trading is constrained only in one direction and the process $\beta$ is $(0, 1)$-valued, the answer is negative, i.e., closing the position in one go is never optimal (see Proposition A.3 in [14] and Proposition 5.6 in [13]).

Proof. 1. Since $Y$ is deterministic and $\eta_{n+1}$ and $\beta_{n+1}$ are independent of $F_n$, Lemma 5.1 yields

$$\left\{\xi_n^*(x, d) = -x \forall x, d \in \mathbb{R}\right\} = \left\{\left(Y_{n+1} - \frac{1}{2}\right) E\left[\frac{\beta_{n+1}}{\eta_{n+1}}\right] - Y_{n+1} E[\beta_{n+1}] + \frac{1}{2} = 0\right\}, \quad (31)$$

which can be either $\Omega$ or $\emptyset$.

2. The equivalence between (i) and (ii) is a direct calculation using (8) and the fact that the factor in front of $(x - \frac{d}{\eta_n})$ on the right-hand side of (8) is deterministic under our assumptions. The equivalence between (i) and (iii) follows from (31) via a straightforward calculation.

3. The last statement is clear. \qed

We close the section with two examples announced above.

Example 5.5. Consider the processes $\beta$ and $\gamma$ satisfying the assumptions of Corollary 3.1 (in particular, (PIMI)) and, moreover, $E[\beta_N] \neq 1$ and

$$E[\beta_{N-1}] = 1 + \left(1 - E\left[\frac{\beta_{N-1}^2}{\eta_{N-1}}\right]\right) \left(\frac{1}{2} - Y_{N-1}\right) Y_{N-1}. \quad (32)$$

Below we present a specific choice of the parameters such that (32) is satisfied.

As we are in the framework of (PIMI), the process $Y$ is deterministic. Moreover, since $E[\beta_N] \neq 1$, we have $Y_{N-1} \in (0, \frac{1}{2})$ (see Corollary 4.2). Recall that on $\{Y_{N-1} < \frac{1}{2}\}$ (see $\Omega$, up to a $P$-null set) there exist profitable round trips when we start at time $N - 1$ with $d \neq 0$. In particular,

$$P(\xi_{N-1}^*(0, d) \neq 0) = 1 \quad \text{whenever } d \neq 0. \quad (33)$$

That is, even without an open position we trade at time $N - 1$ as soon as $d \neq 0$.

\footnote{For completeness we mention the explicit formula

$$\xi_{N-1}^*(0, d) = \frac{E[\beta_N] - 1}{E[\eta_N - 2\beta_N + 1]} \frac{d}{\gamma_{N-1}},$$

which can be obtained from (8) via a direct calculation and yields an alternative proof of (33).}
Moreover, notice that by Proposition 5.4 condition (32) is equivalent to
\[
\{\xi_{N-2}^*(x, d) = -x \; \forall x, d \in \mathbb{R}\} = \Omega. \tag{34}
\]
To summarize, the optimal strategy in this example is to close the position at time \( N - 2 \), to build up a new position at time \( N - 1 \) (at least if \( D_{(N-1)-} = (d - \gamma_{N-2}x) \beta_{N-1} \neq 0 \)) and to close this position at time \( N \). Interestingly, such a phenomenon can only occur if \( E[\beta_{N-1}] > 1 \), and hence it cannot happen in the (usual) framework, where the resilience process \( \beta \) is assumed to take values in \( (0, 1) \).

We, finally, remark that in this example the inclusion in part 1 of Proposition 5.2 for time \( n = N - 2 \) is strict (cf. (34) with the fact that \( \{Y_{N-2} = \frac{1}{2}\} = \emptyset \), where the latter follows from \( Y_{N-1} < \frac{1}{2} \) and part 1 of Corollary 4.3).

It remains to explain how we can satisfy (32). An easy specific example, where the requirements on \( \beta \) and \( \gamma \) listed above are satisfied, can be constructed with deterministic sequences \( \beta \) and \( \gamma \). For instance, choose arbitrary deterministic \( \gamma_{N}, \gamma_{N-1} > 0 \) and \( \beta_{N} \in (0, \sqrt{\eta_{N}}) \setminus \{1\} \). These inputs yield a deterministic \( Y_{N-1} \in (0, \frac{1}{2}) \) (see Corollary 4.2).

Take a sufficiently small \( a > 0 \) such that
\[
aY_{N-1} = \frac{1}{2} - \frac{1}{2} Y_{N-1} \in (0, 1).
\]
Finally, set \( \beta_{N-1} = 1 + a \) and choose \( \gamma_{N-2} > 0 \) to satisfy
\[
aY_{N-1} = \frac{1}{2} - \frac{1}{2} Y_{N-1} = 1 - \frac{(1 + a)^2}{\eta_{N-1}}
\]
(recall that \( \eta_{N-1} = \frac{\gamma_{N-1}}{\gamma_{N-2}} \)). This choice gives us (32) together with \( \frac{\beta_{N}^2}{\eta_{N-1}} < 1 \).

**Example 5.6.** In this example we consider a version of our model with three trading periods \( N - 2, N - 1 \) and \( N \), where the resilience process \( \beta \) is \( (0, 1) \)-valued and still it is optimal at time \( N - 2 \) to close the position in one go. To this end assume that \( \mathcal{F}_{N-2} = \{\emptyset, \Omega\} \) and \( \mathcal{F}_{N-1} = \sigma(\gamma_{N-1}) \) and that we can specify the positive random variables \( \gamma_{N-1}, \gamma_{N} \) and the \( (0, 1) \)-valued random variable \( \beta_{N} \) in such a way that \( E_{N-1} \left[ \frac{\beta_{N}^2}{\gamma_{N-1}} \right] < 1 \),
\[
\left( 1 - E_{N-1} \left[ \frac{\beta_{N}^2}{\gamma_{N}} \right] \right)^{-1} \in L^\infty \quad \text{and that } \quad Y_{N-1} \text{ and } \frac{1}{\gamma_{N-1}} \text{ are strictly negatively correlated, i.e.,}
\]
\[
E \left[ \frac{Y_{N-1}}{\gamma_{N-1}} \right] - E \left[ \frac{1}{\gamma_{N-1}} \right] E \left[ \frac{1}{\gamma_{N-1}} \right] < 0. \tag{35}
\]

\[\text{More generally, the inclusion in part 1 of Proposition 5.2 is strict whenever on a set of positive probability we have the phenomenon described in the previous paragraph. Indeed, an event, where such a phenomenon happens, is a subset of } \{\xi_{n}^*(x, d) = -x \; \forall x, d \in \mathbb{R}\} \setminus \{Y_{n} = \frac{1}{2}\} \quad \text{because on } \{Y_{n} = \frac{1}{2}\} \text{ we have } Y_{n} = Y_{n+1} = \ldots = Y_{N-1} = \frac{1}{2} \text{ (part 1 of Corollary 4.3) and hence } \xi_{n}^*(x, d) = -x \text{ for all } x, d \in \mathbb{R} \text{ and } k \in \{n, n+1, \ldots, N-1\} \text{ (part 1 of Proposition 5.2), in particular, } \xi_{k}^*(0, d) = 0 \text{ for all such } k \text{ and } d \in \mathbb{R}.\]
Below we present a specific choice such that these assumptions are satisfied. By (35) we can choose a deterministic

\[
\beta_{N-1} \in \left( \frac{E \left[ \frac{Y_{N-1}}{\gamma_{N-1}} \right]}{E \left[ Y_{N-1} \right] E \left[ \frac{1}{\gamma_{N-1}} \right]} , 1 \right) \tag{36}
\]

and then define

\[
\gamma_{N-2} = \frac{E \left[ \frac{1}{2} - Y_{N-1} \beta_{N-1} \right]}{E \left[ \left( \frac{1}{2} - Y_{N-1} \right) \frac{\beta_{N-1}^2}{\gamma_{N-1}} \right]} . \tag{37}
\]

Note that, indeed, \( \beta_{N-1} \in (0, 1) \) and \( \gamma_{N-2} > 0 \). Next, we verify that \( E \left[ \frac{\beta_{N-1}^2}{\eta_{N-1}} \right] < 1 \).

By (36) it holds \( E \left[ \beta_{N-1} Y_{N-1} \right] E \left[ \frac{1}{\gamma_{N-1}} \right] > E \left[ Y_{N-1} \right] \) . This implies

\[
E \left[ \frac{1}{2} - \beta_{N-1} Y_{N-1} \right] E \left[ \frac{\beta_{N-1}^2}{\gamma_{N-1}} \right] < E \left[ \left( \frac{1}{2} - Y_{N-1} \right) \frac{\beta_{N-1}^2}{\gamma_{N-1}} \right]
\]

and hence

\[
\gamma_{N-2} = \frac{E \left[ \frac{1}{2} - \beta_{N-1} Y_{N-1} \right]}{E \left[ \left( \frac{1}{2} - Y_{N-1} \right) \frac{\beta_{N-1}^2}{\gamma_{N-1}} \right]} < \frac{1}{E \left[ \frac{\beta_{N-1}^2}{\gamma_{N-1}} \right]} .
\]

Since \( \gamma_{N-2} \) is deterministic and \( \eta_{N-1} = \frac{\gamma_{N-2}}{\gamma_{N-2}} \), we get \( E \left[ \frac{\beta_{N-1}^2}{\eta_{N-1}} \right] < 1 \).

From (37) we obtain that

\[
E \left[ \left( Y_{N-1} - \frac{1}{2} \right) \frac{\beta_{N-1}^2}{\eta_{N-1}} - Y_{N-1} \beta_{N-1} + \frac{1}{2} \right] = 0.
\]

Therefore, it follows from Lemma 5.1 that for all \( x,d \in \mathbb{R} \) it holds that \( \bar{\xi}_{N-2}(x,d) = -x \), i.e., it is optimal to close the whole position at time \( N-2 \).

It remains to specify \( \gamma_{N-1}, \gamma_{N} \) and \( \beta_{N} \) such that \( E_{N-1} \left[ \frac{\beta_{N}^2}{\eta_{N}} \right] < 1, \left( 1 - E_{N-1} \left[ \frac{\beta_{N}^2}{\eta_{N}} \right] \right)^{-1} \in L^\infty \) and that (35) is satisfied. To this end let \( \gamma_{N-1} = \frac{1}{2}, 1 \)-valued with \( P \left( \gamma_{N-1} = 1 \right) = p \in (0, 1) \) and \( P \left( \gamma_{N-1} = \frac{1}{2} \right) = 1 - p \). Define \( \gamma_{N} = \gamma_{N-1}^2 \) and \( \beta_{N} = \gamma_{N-1}^2 \).

Note that \( \beta_{N} \) is \((0,1)\)-valued, \( \gamma_{N-1}, \gamma_{N} > 0 \) and \( \eta_{N} = \gamma_{N-1} \). Observe further that

\[
E_{N-1} \left[ \beta_{N}^2 \right] = \frac{7N-1}{4} < \gamma_{N-1} \quad \text{and hence} \quad E_{N-1} \left[ \frac{\beta_{N}^2}{\eta_{N}} \right] = \frac{7N-1}{4} \leq \frac{1}{4} \quad \text{and} \quad \left( 1 - E_{N-1} \left[ \frac{\beta_{N}^2}{\eta_{N}} \right] \right)^{-1} \in L^\infty .
\]

By definition of \( \beta_{N} \), we have \( E_{N-1} \left[ \beta_{N} \right] = \frac{7N-1}{2} \). It therefore holds

\[
Y_{N-1} = \frac{1}{2} \left( E_{N-1} \left[ \eta_{N} \right] - \left( E_{N-1} \left[ \beta_{N} \right] \right)^2 \right) \left( 1 - E_{N-1} \left[ \beta_{N} \right] + E_{N-1} \left[ \eta_{N} \right] \right) = \frac{1}{2} \left( \gamma_{N-1} - \gamma_{N-1}^2 \right).
\]

Since

\[
E \left[ \gamma_{N-1} \right] = p + \frac{1}{2} (1 - p), \quad E \left[ \gamma_{N-1}^2 \right] = p + \frac{1}{4} (1 - p) \quad \text{and} \quad E \left[ \frac{1}{\gamma_{N-1}} \right] = p + 2(1 - p),
\]

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we obtain (35):

\[
E \left[ \frac{Y_{N-1}}{\gamma_{N-1}} \right] - E \left[ \frac{1}{\gamma_{N-1}} \right] = \frac{1}{2} \left( 1 - \frac{1}{4} E [\gamma_{N-1}] \right) - \frac{1}{2} \left( E [\gamma_{N-1}] - \frac{1}{4} E [\gamma_{N-1}^2] \right) E \left[ \frac{1}{\gamma_{N-1}} \right] = \frac{1}{2} 16 (p^2 - p) < 0.
\]

For completeness, we notice that the assumptions of Theorem 2.1 which were not explicitly discussed above (e.g., \(\beta_n, \gamma_n, \frac{1}{\gamma_n} \in L^\infty\)) are trivially satisfied.

Finally, Table 1 summarizes several mentioned qualitative effects and compares our findings with the literature. In this table, the term “one-directional trading” refers to settings, where the trading is constrained in one direction, and the term “two-directional trading” refers to settings, where, like in the present paper, trading in both directions is allowed.

Table 1: We compare different settings from the viewpoint of whether premature closure is possible. Columns 2–4 briefly describe the settings, while columns 5–6 provide the answers and references to the proofs. It is worth noting that setting 1 is studied in [10] and [13], setting 2 in [14] and setting 3 in [2] (although the question of closing the position in one go is not explicitly considered in [2], hence the reference to our paper in the last column).

| One- or two-directional trading? | \(\beta\) and \(\gamma\) deterministic or stochastic? | \(\beta\) \((0,1)\)-valued? | Premature closure possible? | Reason |
|---------------------------------|--------------------------|--------------------------|-----------------------------|--------|
| 1 one-directional               | deterministic            | yes                      | no                          | Proposition 5.6 in [13] |
| 2 one-directional               | stochastic               | yes                      | no                          | Proposition A.3 in [14] |
| 3 two-directional               | deterministic            | yes                      | no                          | Proposition 5.4 in this paper |
| 4 two-directional               | deterministic            | no                       | yes                         | Example 5.5 in this paper |
| 5 two-directional               | stochastic               | yes                      | yes                         | Example 5.6 in this paper |
| 6 two-directional               | stochastic               | no                       | yes                         | trivial (follows from 4 or 5) |

A. Proof of Theorem 2.1

Proof. We first prove (7) and (8) by backward induction on \(n \in \mathbb{Z} \cap (-\infty, N]\). For the base case \(n = N\) note that for all \(x, d \in \mathbb{R}\) it holds that \(V_N(x, d) = -(d - \frac{\gamma_n}{2}) x =\)
\[
\frac{\gamma}{2} \left( \frac{d}{\gamma} - x \right)^2 - \frac{d^2}{2\gamma}.
\]
In particular, it holds that \(Y_N = \frac{1}{2} > 0\). Besides that, we have that for all \(x, d \in \mathbb{R}\), \(\xi^*_N(x, d) = -x\) is the unique element of \(A_N(x)\) and hence optimal.

Consider now the induction step \(\mathbb{Z} \cap (-\infty, N] \ni n + 1 \rightarrow n \in \mathbb{Z} \cap (-\infty, N-1]\). For all \(x, d \in \mathbb{R}\) let

\[
a_n = \gamma_n E_n \left[ \frac{Y_{n+1}}{\eta_{n+1}} (\beta_{n+1} - \eta_{n+1})^2 + \frac{1}{2} \left( 1 - \frac{\beta_{n+1}^2}{\eta_{n+1}} \right) \right],
\]

\[
b_n(x, d) = E_n \left[ d \left( 1 - \frac{\beta_{n+1}^2}{\eta_{n+1}} \right) + 2Y_{n+1} \left( \frac{\beta_{n+1}}{\eta_{n+1}} - 1 \right) (\beta_{n+1}d - \eta_{n+1}x) \right],
\]

\[
c_n(x, d) = E_n \left[ \frac{Y_{n+1}}{\gamma_{n+1}} (\beta_{n+1}d - \eta_{n+1}x)^2 - \frac{d^2 \beta_{n+1}}{2\gamma_{n+1}} \right].
\]

Note that for all \(x, d \in \mathbb{R}\) the random variables \(a_n, b_n(x, d)\) and \(c_n(x, d)\) are well-defined and finite because all factors and summands are in \(L^\infty\) due to the assumption that for all \(k \in \mathbb{Z} \cap (-\infty, N]\), it holds \(\beta_k, \gamma_k, \frac{1}{\gamma_k} \in L^\infty\), and the induction hypothesis \(0 < Y_{n+1} \leq \frac{1}{2}\). Furthermore, the induction hypothesis that \(Y_{n+1} > 0\) and the assumption that \(E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] < 1\) ensure that \(a_n > 0\). It follows from the Cauchy-Schwarz inequality and the assumption \(E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] < 1\) that

\[
Y_n = E_n[\eta_{n+1}Y_{n+1}] - \frac{E_n \left[ \sqrt{\eta_{n+1}Y_{n+1}} \sqrt{\frac{Y_{n+1}}{\eta_{n+1}} (\beta_{n+1} - \eta_{n+1})} \right]^2}{a_n/\gamma_n}
\]

\[
\geq E_n[\eta_{n+1}Y_{n+1}] - \frac{E_n[\eta_{n+1}Y_{n+1}] E_n \left[ \frac{Y_{n+1}}{\eta_{n+1}} (\beta_{n+1} - \eta_{n+1})^2 \right]}{a_n/\gamma_n}
\]

\[
= \frac{E_n[\eta_{n+1}Y_{n+1}]}{a_n/\gamma_n} \left( 1 - E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right] \right) > 0.
\]

To establish that \(Y_n \leq \frac{1}{2}\), note that

\[
\frac{1}{\gamma_n} c_n(1, 0) = \frac{1}{\gamma_n} E_n \left[ Y_{n+1} \gamma_{n+1} \right] = E_n \left[ \eta_{n+1} Y_{n+1} \right],
\]

\[
\frac{1}{\gamma_n} b_n(1, 0) = \frac{1}{\gamma_n} E_n \left[ -2Y_{n+1} \left( \frac{\beta_{n+1}}{\eta_{n+1}} - 1 \right) \gamma_{n+1} \right] = -2E_n \left[ Y_{n+1} \left( \beta_{n+1} - \eta_{n+1} \right) \right].
\]

This together with the induction hypothesis \(Y_{n+1} \leq \frac{1}{2}\) implies that

\[
Y_n = \frac{1}{\gamma_n} \left( c_n(1, 0) - \frac{b_n(1, 0)^2}{4a_n} \right) \leq \frac{1}{\gamma_n} \left( c_n(1, 0) - \frac{b_n(1, 0)^2}{4a_n} + a_n \left( \frac{b_n(1, 0)}{2a_n} - 1 \right)^2 \right)
\]

\[
= \frac{1}{\gamma_n} (a_n - b_n(1, 0) + c_n(1, 0)) = \frac{1}{2} + E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \left( Y_{n+1} - \frac{1}{2} \right) \right] \leq \frac{1}{2}.
\]
Let $\mathcal{S}_n$ be the set of all real-valued $\mathcal{F}_n$-measurable random variables $\xi \in L^{2+}$. The dynamic programming principle and the induction hypothesis ensure for all $x, d \in \mathbb{R}$ that

$$V_n(x, d) = \text{ess inf}_{\xi \in \mathcal{S}_n} \left[ \left( d + \frac{\gamma_n}{2} \right) \xi + E_n \left[ V_{n+1}(x + \xi, (d + \gamma_n \xi) \beta_{n+1}) \right] \right]$$

Furthermore, we have that

$$\text{ess inf}_{\xi \in \mathcal{S}_n} \left[ \left( d + \frac{\gamma_n}{2} \right) \xi + E_n \left[ Y_{n+1} \left( \frac{\beta_{n+1} + \gamma_n}{\gamma_{n+1}} - 1 \right) \xi + \frac{d \beta_{n+1}}{\gamma_{n+1}} - x \right]^2 - \frac{(d + \gamma_n \xi)^2 \beta_{n+1}^2}{2 \gamma_{n+1}} \right]$$

$$= \text{ess inf}_{\xi \in \mathcal{S}_n} \left[ a_n \xi^2 + b_n(x, d) \xi + c_n(x, d) \right].$$

(42)

For all $x, d \in \mathbb{R}$ we find $\xi^*_n(x, d) = -\frac{b_n(x, d)}{2a_n}$ to be the unique minimizer of $\xi \mapsto a_n \xi^2 + b_n(x, d) \xi + c_n(x, d)$. Observe further that for all $x, d \in \mathbb{R}$ it holds that

$$b_n(x, d) = \frac{2d a_n}{\gamma_n} - 2E_n \left[ Y_{n+1} \left( \beta_{n+1} \gamma_n - \gamma_{n+1} \right) \right] \left( x - \frac{d}{\gamma_n} \right),$$

(43)

which yields the representation of $\xi^*_n(x, d)$ in (8). Clearly, for all $x, d \in \mathbb{R}$ the random variable $\xi^*_n(x, d)$ is $\mathcal{F}_n$-measurable. It remains to verify that for all $x, d \in \mathbb{R}$ we have $\xi^*_n(x, d) \in L^{\infty}$. To show this we verify first that

$$E_n \left[ Y_{n+1} \left( \beta_{n+1} - \eta_{n+1} \right) \right] \in L^{\infty}.$$  

(44)

We have $\eta_{n+1} \in L^{\infty}$ as $\eta_{n+1}$ is the product of the two $L^{\infty}$-variables $\gamma_{n+1}$ and $\frac{1}{\gamma_n}$. Furthermore, we have that $\beta_{n+1} \in L^{\infty}$ by assumption and that $Y_{n+1}$ is bounded due to the induction hypothesis. Hence, by the Minkowski inequality, it holds that

$$(E \left[ \left| Y_{n+1} \left( \beta_{n+1} - \eta_{n+1} \right) \right|^p \right])^{\frac{1}{p}} \leq \left( E \left[ \left| Y_{n+1} \beta_{n+1} \right|^p \right] \right)^{\frac{1}{p}} + \left( E \left[ \left| Y_{n+1} \eta_{n+1} \right|^p \right] \right)^{\frac{1}{p}} < \infty$$  

for every $p \in [1, \infty)$, so that

$$E_n \left[ Y_{n+1} \left( \beta_{n+1} - \eta_{n+1} \right) \right] \in L^{\infty}.$$  

(46)

Next we recall that $\frac{1}{\alpha_n} \in L^{\infty}$, where $\alpha_n = 1 - E_n \left[ \frac{\beta_{n+1}^2}{\eta_{n+1}} \right]$, which implies

$$\frac{1}{E_n \left[ Y_{n+1} \left( \beta_{n+1} - \eta_{n+1} \right)^2 + \frac{1}{2} \left( 1 - \frac{\beta_{n+1}^2}{\eta_{n+1}} \right) \right]} \in L^{\infty},$$  

(47)

$^6$Note that our assumption that for all $k \in \mathbb{Z} \cap (-\infty, N]$ it holds $\beta_k, \gamma_k, \frac{1}{\gamma_k} \in L^{\infty}$ and the fact that $Y_{n+1}$ is bounded ensure that all conditional expectations in (42) are well-defined and that we can move any $\xi \in \mathcal{S}_n$, $\gamma_n$ and $\frac{1}{\gamma_n}$ outside the conditional expectations. This reasoning also applies to other calculations in this proof.
as the random variable in (47) is positive and smaller than $\frac{2}{\alpha_n}$. Together with (46) this establishes (44). Now (38) and (44) imply that $\xi_n^*(x, d) \in L^{\infty}$ for all $x, d \in \mathbb{R}$, as $x$ and $d$ are deterministic and $\frac{1}{\gamma_n} \in L^{\infty}$.

By inserting the optimal trade size $\xi_n^*(x, d) = -\frac{b_n(x, d)}{2\alpha_n}$ into (42), we obtain for all $x, d \in \mathbb{R}$ that

$$V_n(x, d) = -\frac{b_n(x, d)^2}{4\alpha_n} + c_n(x, d).$$

(48)

The dynamic programming principle ensures for all $x, d, h \in \mathbb{R}$ that

$$V_n(x, d) - \left( d + \frac{\gamma_n}{2} h \right) h = \text{ess inf}_{\xi \in S_n} \left[ (d + \frac{\gamma_n}{2} \xi) \xi - (d + \frac{\gamma_n}{2} h) h + E_n \left[ V_{n+1}(x + \xi, (d + \gamma_n \xi) \beta_{n+1}) \right] \right]$$

(49)

This implies for all $x, d \in \mathbb{R}$ that

$$(\partial_x V_n)(x, d) + \gamma_n (\partial_d V_n)(x, d) = \frac{V_n(x + h, d + \gamma_n h) - V_n(x, d)}{h} = - \left( d + \frac{\gamma_n}{2} h \right) \to -d \quad (50)$$

as $h \to 0$. In particular, we obtain that

$$(\partial^2_{xx} V_n)(0, 0) + \gamma_n (\partial^2_{dx} V_n)(0, 0) = 0 \quad \text{and} \quad (\partial^2_{xd} V_n)(0, 0) + \gamma_n (\partial^2_{dd} V_n)(0, 0) = -1. \quad (51)$$

It follows from (48) and (38) that, for almost all $\omega$, $V_n$ is a quadratic function in $(x, d) \in \mathbb{R}^2$ with $V_n(0, 0) = 0$. This together with (51) proves that

$$V_n(x, d) = \frac{(\partial^2_{xx} V_n)(0, 0)}{2} x^2 + [(\partial^2_{dx} V_n)(0, 0)] x d + \frac{(\partial^2_{dd} V_n)(0, 0)}{2} d^2$$

(52)

Moreover, it follows from (48) that

$$\frac{(\partial^2_{xx} V_n)(0, 0)}{2} = E_n \left[ \gamma_n Y_{n+1} \right] = \frac{E_n \left[ Y_{n+1} \left( \beta_{n+1} \gamma_n - \gamma_n + 1 \right) \right]^2}{a_n} = \gamma_n Y_n. \quad (53)$$

This together with (52) proves that $V_n(x, d) = \frac{\gamma_n}{\gamma_n} (d - x \gamma_n)^2 - \frac{d^2}{2\gamma_n}$ for all $x, d \in \mathbb{R}$.

In the remainder of the proof we show that for all $n \in \mathbb{Z} \cap (-\infty, N - 1]$, $x, d \in \mathbb{R}$ the process $\xi^* = (\xi_k^*)_{k \in \{n, \ldots, N}\}$ recursively defined by (9) is in $A_n(x)$. To this end we show by (forward) induction on $k \in \{n, \ldots, N\}$ that $\xi_k^*$ is $\mathcal{F}_k$-measurable and belongs to $L^2$ for all $k \in \{n, \ldots, N\}$. 29
For the base case $k = n$ we have $\xi^*_n = \xi^*_n(x, d)$ which is already known to be in $S_n$ for all $x, d \in \mathbb{R}$, i.e., $\xi^*_n$ is $F_n$-measurable and $\xi^*_n \in L^{2+}$.

Continue with the induction step ${n, \ldots, N - 2} \ni k - 1 \rightarrow k \in {n + 1, \ldots, N - 1}$. Now, the optimal trade size $\xi^*_k$ at time $k$ depends on the current value of the position path $X^*_k = x + \sum_{i=n}^{k-1} \xi^*_i$ and the current deviation $D^*_k$. By induction on $k$, it holds that $\xi^*_i$ is in $L^{2+}$ and $F_i$-measurable for all $i \in \{n, \ldots, k - 1\}$. This yields that $X^*_k$ belongs to $L^{2+}$ and is $F_k$-measurable. Furthermore, the fact that $\xi^*_i \in L^{2+}$ for all $i \in \{n, \ldots, k - 1\}$ allows us to use Remark 1.1 to obtain that $D^*_k \in L^{2+}$ as well. Besides that, it can be seen from (4) that $L$ belongs to $i$ and $\xi^*_i$ is $F_i$-measurable given that $\xi^*_i$ is $F_i$-measurable for all $i \in \{n, \ldots, k - 1\}$ and $\beta$ and $\gamma$ are adapted processes. Hence, \[ \xi^*_k(X^*_{k-1}, D^*_{k-}) = \frac{E_k [Y_{k+1} (\beta_{k+1} - \eta_{k+1})]}{E_k [\frac{Y_{k+1}}{\eta_{k+1}} (\beta_{k+1} - \eta_{k+1})^2 + \frac{1}{2} (1 - \frac{\beta_{k+1}^2}{\eta_{k+1}})]} \left( X^*_{k-1} - \frac{D^*_k}{\gamma_k} \right) - \frac{D^*_k}{\gamma_k} \] (54) is $F_k$-measurable. To prove that $\xi^*_k(X^*_{k-1}, D^*_{k-}) \in L^{2+}$, note that by the Minkowski inequality, it suffices to show that each summand is in $L^{2+}$. To begin with, it holds that $\frac{D^*_k}{\gamma_k} \in L^{2+}$ due to Lemma B.2 and $\frac{1}{\gamma_k} \in L^{\infty}$. It further follows with (44) and Lemma B.2 that \[ \frac{E_k [Y_{k+1} (\beta_{k+1} - \eta_{k+1})]}{E_k [\frac{Y_{k+1}}{\eta_{k+1}} (\beta_{k+1} - \eta_{k+1})^2 + \frac{1}{2} (1 - \frac{\beta_{k+1}^2}{\eta_{k+1}})]} \frac{D^*_k}{\gamma_k} \in L^{2+}. \] (55) Similarly, \[ \frac{E_k [Y_{k+1} (\beta_{k+1} - \eta_{k+1})]}{E_k [\frac{Y_{k+1}}{\eta_{k+1}} (\beta_{k+1} - \eta_{k+1})^2 + \frac{1}{2} (1 - \frac{\beta_{k+1}^2}{\eta_{k+1}})]} X^*_{k-1} \in L^{2+}. \] (56) This finishes the induction step ${n, \ldots, N - 2} \ni k - 1 \rightarrow k \in {n + 1, \ldots, N - 1}$.

Finally, it follows that for all $x, d \in \mathbb{R}$ it also holds true that $\xi^*_N = -X^*_N - x - \sum_{i=n}^{N-1} \xi^*_i$ is in $L^{2+}$ and $F_N$-measurable. As a result, $\xi^* \in A_n(x)$ for all $x, d \in \mathbb{R}$.

The proof of Theorem 2.1 is thus completed.

\[ \square \]

B. Integrability

Lemma B.1. Let $X, Y \in L^{\infty}$. Then, $XY$ also belongs to $L^{\infty}$.

Proof. Let $p \in [1, \infty)$. The Cauchy-Schwarz inequality yields \[ E \|XY\|^p = E \|X\|^p \|Y\|^p \leq (E \|X\|^{2p})^{\frac{1}{2}} \cdot (E \|Y\|^{2p})^{\frac{1}{2}} < \infty \] (57) since $X, Y \in L^{2p}$. Therefore, $XY \in L^p$. This is true for every $p \in [1, \infty)$, hence $XY \in L^{\infty}$.

\[ \square \]

Lemma B.2. Let $X \in L^{\infty}$ and $Y \in L^{2+}$. Then, $XY \in L^{2+}$.
Proof. Since $Y \in L^{2+}$, there exists $\varepsilon > 0$ such that $Y \in L^{2+\varepsilon}$. Let $r := 2 + \frac{\varepsilon}{2}$ and $q := 2 + \varepsilon r$. It holds that $q > 1$ and $Y \in L^{rq}$. Define $p := \frac{q}{q-1}$ and observe that $X \in L^{rp}$. By the Hölder inequality,

$$E[|XY|^r] = E[|X|^r|Y|^r] \leq (E[|X|^{rp}])^{\frac{1}{p}} \cdot (E[|Y|^{rq}])^{\frac{1}{q}} < \infty.$$  (58)

This proves that $XY \in L^{2+}$.

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