Octonionic Electrodynamics

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Abstract. Dirac’s operator and Maxwell’s equations in vacuum are derived in the algebra of split octonions. The approximations are given which lead to classical Maxwell-Heaviside equations from full octonionic equations. The non-existence of magnetic monopoles in classical electrodynamics is connected with the using of associativity limit.

PACS numbers: 03.50.De, 02.10.De, 03.30.+p
1. Introduction

Maxwell’s equations, which harbor many beautiful mathematical concepts, have been expressed in many forms since their discovery in 1873. Maxwell himself in his main book used the coordinate calculus \[1\], however in the second edition included also quaternionic representation. The original equations were a system of 16 equations, quaternionic and the familiar vector forms consist of 4 equations, and the application of bi-quaternions (or Clifford algebras) results in a version of just one equation \[2\]. Still in the literature is absent octonionic form of Maxwell’s equation. It was already mentioned that the vector algebra and Maxwell’s equations should have connections with octonions \[3\]. In this paper we want to show that in some approximation classical Maxwell-Heaviside equations can be written as the single continuity equation in the algebra of split octonions over the reals.

Octonions form the widest normed algebra after the algebras of real numbers, complex numbers, and quaternions \[4\]. Since their discovery, almost three decades before Maxwell’s equations, there have been various attempts to find appropriate uses for octonions in physics (see reviews \[5\]). One can point to the possible impact of octonions on: Color symmetry \[6\]; GUTs \[7\]; Representation of Clifford algebras \[8\]; Quantum mechanics \[9\]; Space-time symmetries \[10\]; Field theory \[11\]; Formulations of wave equations \[12\]; Quantum Hall effect \[13\]; Strings and M-theory \[14\], etc.

In our previous papers \[15\] the model where the geometry of real world was described by the split octonions was introduced. In \[16\] the octonionic version of Dirac’s equation was formulated. In this paper, except of derivation of octonionic Maxwell’s equations in vacuum, we want to show that symbolic form of Dirac’s equations is just the result of the invariance of the intervals in the octonionic geometry.

2. Octonionic Geometry

In the paper \[15\] real physical signals were associated with the elements of split octonions,

\[
s = ct + J_n x^n + j_n \hbar \lambda^n + I \chi \omega , \quad (n = 1, 2, 3)
\]

where summing by the repeated indexes is performed. In \[11\] the scalar unit is denoted as 1, the three vector-like objects as \(J_n\), the three pseudo-vectors as \(j_n\) and the pseudo-scalar as \(I\). The eight real parameters that multiply the basis units denotes the time \(t\), the special coordinates \(x^n\), and some quantities \(\lambda^n\) and \(\omega\) with the dimensions of momentum\(^{-1}\) and energy\(^{-1}\) respectively. The line element \[11\] also contains two fundamental constants of physics - the velocity of light \(c\) and Planck’s constant \(\hbar\). The appearance of these constants was connected with the existence of two different classes of zero divisors in the algebra of split octonions \[15\].

The algebra of the basis elements of split octonions can be written in the form:

\[
J_n^2 = j_n^2 = I^2 = 1 ,
\]
\[ J_n j_m = -j_m J_n = -\epsilon_{nmk} J^k , \]
\[ J_n J_m = -J_m J_n = j_n j_m = -j_m j_n = \epsilon_{nmk} j^k , \]
\[ J_n I = -IJ_n = j_n , \]
\[ j_n I = -Ij_n = J_n , \]
where \( \epsilon_{nmk} \) is the fully antisymmetric tensor and \( n, m, k = 1, 2, 3 \). From these formulas it is clear that to generate a complete 8-dimensional basis of split octonions the multiplication and distribution laws of only the three vector-like elements \( J_n \) are enough. The other two units \( j_n \) and \( I \) can be expressed as binary and triple products
\[ j_n = \frac{1}{2} \epsilon_{nmk} J^m J^k , \quad I = J_n j_n \]
(there is no summing in the second formula).

Using the conjugation rules of octonionic basis units
\[ 1^* = 1 , \quad J_n^* = -J_n , \quad j_n^* = -j_n , \quad I^* = -I , \]
one can find that the norm of (1) (interval)
\[ s^2 = ss^* = c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2 , \]
has (4+4)-signature and in general is not positively defined. However, as in the standard relativity we require
\[ s^2 \geq 0 . \]
In the classical limit \( \hbar \to 0 \) the expression (5) reduces to the ordinary 4-dimensional formula for space-time intervals.

Differentiating (1) by the proper time \( \tau \) the proper velocity of a particle can be obtained
\[ \frac{ds}{d\tau} = \frac{dt}{d\tau} \left[ c \left( 1 + I h \frac{d\omega}{dt} \right) + J_n \left( \frac{dx^n}{dt} + I h \frac{d\lambda^n}{dt} \right) \right] . \]
Then the invariance of the norm (5) gives the relation
\[ \beta = \frac{d\tau}{dt} = \sqrt{ \left[ 1 - \hbar^2 \left( \frac{d\omega}{dt} \right)^2 \right] - \frac{v^2}{c^2} \left[ 1 - \hbar^2 \left( \frac{d\lambda^n}{dx^n} \right)^2 \right] } , \]
where
\[ v_n = \frac{dx_n}{dt} \]
is the 3-velocity. So the modified Lorentz factor (8) contains extra terms and the dispersion relation in the (4+4)-space (5) has a form similar to that of double-special relativity models [17].
3. Symbolic Form of Dirac’s Equation

In [16] the octonionic form of Dirac’s equation, which in some limit is equivalent to the standard one, was obtained. Here we want to demonstrate simple derivation of Dirac’s operator from the condition of invariance of the octonionic interval [5].

For the observers with the time parameters $\tau$ and $t$ we can write the relation

$$ds = \pm cd\tau = \pm cd\beta,$$

where $\beta$ is expressed by (8). Dividing this relation by $d\tau$ and multiplying it on the particle mass $m$ we find:

$$\frac{1}{\beta} \left[ mc \left( 1 + I \hbar \frac{d\omega}{dt} \right) + J_n m \left( v^n + I \hbar \frac{d\lambda^n}{dt} \right) \right] = \pm cm. \quad (11)$$

Let us assume that

$$mc \frac{d\omega}{\beta} = - \frac{e}{c} \varphi, \quad mh \frac{d\lambda^n}{\beta} = - \frac{e}{c} A^n, \quad (12)$$

where $\varphi$ and $A^n$ are components of the electro-magnetic 4-potential. By this assumption (5) takes a form similar to intervals in Finsler-type theories with field depended metrics. For the reviews on Finsler theories see, for example [18] and references therein.

Using the assumption (12) the equation (11) takes the form:

$$\left( \frac{\varepsilon}{c} - I \frac{e}{c} \varphi \right) + J_n \left( p^n - I \frac{e}{c} A^n \right) \mp mc = 0, \quad (13)$$

where $\varepsilon = mc^2/\beta$ and $p^n = mv^n/\beta$ are energy and momentum of the particle. This equation represents one of the zero divisors in the algebra of split octonions. Importance of zero divisors in physical applications of split algebras was specially noted in [19].

The equation (13), which we receive from the invariance of the interval (10), is the symbolic form of 4-dimensional Dirac’s equation. The role of four $\gamma$-matrices here is played by the unit element of split octonions $1$ and the three vector-like elements $J_n$. Instead of ordinary complex unit $i$ in (13) the basis element $I$ is used, and the factor $\beta$ transforms to ordinary Lorentz formula if we use the limit $\hbar \to 0$ in the definition (8).

4. Maxwell’s Equations in Vacuum

The octonion that contains the electromagnetic potentials $\varphi$ and $A_n$ we write as

$$A = -\varphi + J_n A^n + j_n B^n + Ib, \quad (n = 1, 2, 3) \quad (14)$$

where $B^n$ and $b$ correspond to the extra degrees of freedom in the octonionic algebra. Here we don’t specify their meaning we only want to obtain the approximations leading us to the classical Maxwell-Heaviside equations that give successful explanation of most experiments at low energies. Examples of problems in classical electrodynamics where the fields $B^n$ and $b$ can play a role are: magnetic monopoles [20], longitudinal electrodynamic force [21], the Abraham-Minkowski controversy [22], etc.

To obtain weak field approximation in octonionic equations let us mention that, since we require positivity of norms, the elements of split octonions should have
hierarchial structure. This means that the absolute value of the scalar element should be greater than other elements and so on. From (3) it is also clear that the pseudo-vector and pseudo-scalar units are secondary since they are expressed by the fundamental vector-like elements. Appearance of Planck’s constant in the last two terms of (5) is another indication that in classical limit we can neglect the values of pseudo-vector and pseudo-scalar components. So it is natural to consider that in (14)

$$|b|, |B^n| \ll |\varphi|, |A^n| ,$$

and these components can be neglected. Invariance of octonionic intervals then will guarantee that this inequality would be preserved for different observers.

The octonionic differential operator we write as

$$\nabla = \frac{1}{c} \left( \frac{\partial}{\partial t} + I \frac{1}{\hbar} \frac{\partial}{\partial \omega} \right) + J^n \left( \frac{\partial}{\partial x^n} + I \frac{1}{\hbar} \frac{\partial}{\partial \lambda^n} \right).$$

Here we can also assume that influence of $\omega$ and $\lambda^n$ can be ignored in the classical limit. The norm of $\nabla$ when fields don’t depend on $\omega$ and $\lambda^n$ is the ordinary 4-d’Alembertian.

Assuming in (14) that $B^n$ and $b$ are small (or are constants) and $A^n$ and $\varphi$ are independent of $\omega$ and $\lambda$, we can define the electro-magnetic field in the form:

$$\nabla A = F = \left( -\frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{\partial A^n}{\partial x^n} \right) + J_n E^n + j_n H^n , \quad (n = 1, 2, 3)$$

where

$$E^n = \frac{1}{c} \frac{\partial A^n}{\partial t} - \frac{\partial \varphi}{\partial x^n} , \quad H^n = \epsilon^{nmk} \frac{\partial A_k}{\partial x^m} ,$$

are components of 3-vectors of electric and magnetic fields respectively.

Then we can postulate the Lorenz gauge (derived by L. Lorenz in 1867, and not by H. A. Lorentz, as refereed in some modern papers)

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} - \frac{\partial A^n}{\partial x^n} = 0 ,$$

or weaker condition where zero in (19) is replaced by a constant, and write continuity equation as the product of the octonions (16) and (17),

$$\nabla F = \frac{\partial E^n}{\partial x^n} + J_k \left( \frac{1}{c} \frac{\partial E^n}{\partial t} - \epsilon^{nmk} \frac{\partial H_m}{\partial x^n} \right) +$$

$$+ J_k \left( \frac{1}{c} \frac{\partial H^n}{\partial t} + \epsilon^{nmk} \frac{\partial E_m}{\partial x^n} \right) + I \frac{\partial H^n}{\partial x^n} = 0 .$$

Different signs in the second and third terms of this equation are the result of the using of the algebra (2), in particular

$$J_n J_m = -\epsilon_{nmk} J^k , \quad J_n J_m = \epsilon_{nmk} J^k .$$

Equating to zero coefficients in front of the four octonionic basis units in (20) results the full set of the homogeneous Maxwell’s equations.

We can write also octonionic current function in the form:

$$\varrho = \rho + J_n \frac{1}{c} \sigma^n ,$$
where $\rho$ is the electric charge density and $\sigma^n$ are the components of electric current vector. As before we ignored pseudo-vector and pseudo-scalar parts in (22).

Finally we can write the complete set of inhomogeneous Maxwell’s equations as one single octonionic equation

$$\nabla \mathbf{F} = \rho .$$

(23)

As it is clear from (20) and (22) in (23), as in standard electrodynamics, magnetic current is absent. This is the result of the ignoring of pseudo-vector and pseudo-scalar terms in (22), and of re-appearance of these kind of terms in (20) via octonionic products. Non-associativity of octonions is mainly governed by $j_n$ and $I$. So the non-existence of magnetic monopoles in classical electrodynamics can be explained as the use of associativity limit.

5. Conclusion and Discussion

In this paper simple octonionic forms of Dirac’s operator and Maxwell’s equations in vacuum were derived. In the classical limit there is no indication of non-associativity for the electromagnetic field and derived octonionic Maxwell’s equation (23) is similar to the bi-quaternion formulations [2]. However, split octonions that incorporate three vector-like elements should give a more successful generalization of classical electrodynamics since non-associativity (which distinguishes octonions from other normed algebras) also exists in the algebra of Euclidean 3-vectors used in the classical Maxwell-Heaviside equations. The only new feature of the octonionic formalism in the approximation used in this paper is the observation that the non-existence of magnetic monopoles in classical electrodynamics is connected with the ignoring of non-associativity. In the case of strong fields the pseudo-vector and pseudo-scalar parts of octonions can not be neglected, equations will become more complicated, and we expect to find new effects in future papers.

Acknowledgments

The author would like to acknowledge the hospitality extended during his visits at the Abdus Salam International Centre for Theoretical Physics where this work was done.

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Octonionic Electrodynamics

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