POSITIVE REPRESENTATIONS OF $C_0(X)$. I.

MARCEL DE JEU AND FREJANNE RUOFF

Abstract. We introduce the notion of a positive spectral measure on a $\sigma$-algebra, assuming values in the positive projections on a Banach lattice. Such a measure generates a bounded positive representation of the bounded measurable functions. If $X$ is a locally compact Hausdorff space, and $\pi$ is a positive representation of $C_0(X)$ on a KB-space, then $\pi$ is the restriction to $C_0(X)$ of such a representation generated by a unique regular positive spectral measure on the Borel $\sigma$-algebra of $X$. The relation between a positive representation of $C_0(X)$ on a Banach lattice and – if it exists – a generating positive spectral measure on the Borel $\sigma$-algebra is investigated; here some phenomena occur that are specific for the ordered context.

1. Introduction and overview

Suppose $X$ is a locally compact Hausdorff space, with $C_0(X)$ denoting the ordered Banach algebra of all continuous functions vanishing at infinity. Positive representations of such an algebra $C_0(X)$ on a Banach lattice $E$, i.e., representations such that positive functions act as positive operators on $E$, are quite common. Rather trivially, $C_0(X)$ acts positively by multiplication on many Banach lattices of (equivalence classes of) functions on $X$. Somewhat in disguise, since the center $Z(E)$ of an arbitrary Banach lattice is such a space for compact $X$, there is also a positive representation of this type associated with every Banach lattice. Such positive representations also occur in a very general context where ordering is not present from the start: If $E$ is an arbitrary Banach space, $X$ is compact, and $\pi$ is a bounded unital representation of $C_0(X)$ on $E$, then every cyclic closed subspace of $E$ can be supplied with an ordering and an equivalent norm so that it becomes a Banach lattice on which $C_0(X)$ acts positively. This result [4, Lemma 4.6] [13, Proposition 2.5] goes back to [11] in its earliest form.

Given the ubiquity of their occurrence, it is natural to ask which positive representations of $C_0(X)$ on a Banach lattice $E$ are generated by a (regular) positive spectral measure on the Borel $\sigma$-algebra of $X$. Here ‘positive’ refers to the fact that the measure assumes values in the positive projections on $E$. We will show that this is always the case if $E$ is a KB-space, cf. Theorem 5.7. The proof is by reduction to a known result valid for representations in suitable Banach spaces. Since the latter result requires quite some work, we also give a self-contained proof for reflexive Banach lattices, cf. Theorem 5.7.

The study of positive spectral measures and of positive representations of $C_0(X)$ (and also of representations of algebras of bounded measurable functions) appears to be new, and we use the opportunity to establish some basic features. We pay
attention to automatic continuity of positive representations (cf. Propositions 4.10 and 5.1), and also to relations that are particular to this ordered context. For example, if $P$ is a positive spectral measure that is regular in the (usual) sense of Definition 5.2 and $\Delta$ is a Borel set, then

$$P(\Delta) = \inf \{ P(V) : V \text{ open and } \Delta \subset V \}$$

in the regular operators, cf. Proposition 5.3. Therefore $P$ is outer regular as a map from the Borel $\sigma$-algebra to the regular operators on $E$. In a similar vein, if $\pi$ is a positive representation of $C_0(X)$ on $E$ that has a generating regular positive spectral measure $P$, and $V$ is open, then

$$P(V) = \sup \{ \pi(\phi) : \phi \in C_c(X), \text{ supp } \phi \subset V, \ 0 \leq \phi \leq 1_X \}$$

in the regular operators on $E$, cf. Theorem 5.12. As in the previous example, this statement, which is reminiscent of (and stems from) a well known formula in the circle of ideas around the Riesz representation theorem, is only meaningful in an ordered context.

This paper is organised as follows.

Section 2 contains the basis notation and terminology, as well as an overview of material around the Riesz representation theorem. We need a bit more than the mere existence of measures representing bounded linear functionals, and we have included what we need for the ease of reference, and also to establish terminology (which is not uniform throughout the literature).

Section 3 introduces positive spectral measures on arbitrary $\sigma$-algebras, and contains some first basic results.

In Section 4 a bounded positive representation of the bounded measurable functions on a measurable space is constructed from a positive spectral measure. Any positive representation of this algebra is, in fact, always bounded, but when is there an underlying positive spectral measure? If $E$ has $\sigma$-order continuous norm, then we can characterise the positive representations thus obtained: they are precisely the $\sigma$-order continuous ones. It is not true that an arbitrary positive representation of the bounded measurable functions has a generating positive spectral measure on the pertinent $\sigma$-algebra; this will be taken up in 10.

Section 5 focusses on the topological context of a locally compact Hausdorff space $X$. We are concerned with automatic continuity of positive representations of $C_0(X)$, and show that every positive representation of $C_0(X)$ on a KB-space has a generating regular positive spectral measure. A self-contained proof for the reflexive case is also included. Additional regularity properties of a regular positive spectral measures are investigated, as well as the relation between a positive representation of $C_0(X)$ and – if there is one – a generating possibly regular positive spectral measure for that representation.

2. Preliminaries

We start by collecting some basic notation and facts, as well as giving a precise overview of the Riesz representation theorem and the facts surrounding it.

2.1. Basics. All vector spaces in this paper are over the real numbers. If $V$ is an ordered vector space, we write $V_+$ and $V_-$ for its positive and negative cone, respectively.
If $E$ is a Banach space, then $E'$ denotes its norm dual. We write $\mathcal{L}(E)$ for the bounded linear operators on $E$, and $\text{id}_E$ for the identity map on $E$. If $E$ is a Banach lattice, then $\mathcal{L}_r(E)$ will denote the vector space of regular operators on $E$, i.e., the linear operators on $E$ that can be written as a difference of two positive linear operators on $E$. It is well known that every positive linear operator on $E$ is bounded, so that $\mathcal{L}_r(E) \subseteq \mathcal{L}(E)$, and that, for $T \geq 0$, $\|T\| = \sup \{\|Tx\| : x \in E_+, \|x\| \leq 1\}$, where $E_+$ is the positive cone of $E$ \cite[Proposition 1.3.5]{10}. Consequently the operator norm is monotone on the positive cone $\mathcal{L}_r(E)_+$ of $\mathcal{L}_r(E)$: if $T_1, T_2 \in \mathcal{L}_r(E)$ and $0 \leq T_1 \leq T_2$, then $\|T_1\| \leq \|T_2\|$.

If $E$ is a Dedekind complete Banach lattice, then $\mathcal{L}_r(E)$ is a Dedekind complete Banach lattice when supplied with the natural ordering and the regular norm $\|\cdot\|_r$, defined by $\|T\|_r = \|T\| (T \in \mathcal{L}_r(E))$, cf. \cite[Theorem 4.74]{12}. If $E \neq \{0\}$ the inclusion map $(\mathcal{L}_r(E), \|\cdot\|_r) \rightarrow (\mathcal{L}(E), \|\cdot\|)$ has norm 1 \cite[p. 255]{5}.

If $F$ is a normed space, and $E$ is a Banach lattice, then the norm of a bounded linear map $T : F \rightarrow (\mathcal{L}(E), \|\cdot\|)$ or $T : F \rightarrow (\mathcal{L}_r(E), \|\cdot\|_r)$ will be denoted by $\|T\|$. The norm of a bounded linear map $T : F \rightarrow (\mathcal{L}_r(E), \|\cdot\|_r)$ will be denoted by $\|T\|_r$.

**Definition 2.1.** If $A$ is an ordered algebra, and $E$ is a Banach lattice, then a positive representation of $A$ on $E$ is an algebra homomorphism $\pi : A \rightarrow \mathcal{L}(E)$ such that $\pi(A_+) \subseteq \mathcal{L}_r(E)_+$. If $A$ is normed, we do not require $\pi$ to be bounded. If $A$ is unital, we do not require $\pi$ to be unital.

**Remark 2.2.** In the cases of our interest, the ordered algebra $A$ is in fact a lattice, so that a positive representation of $A$ on $E$ maps $A$ into $\mathcal{L}_r(E)$. If $E$ is Dedekind complete, then $\mathcal{L}_r(E)$ is also a lattice, and it is meaningful to require that $\pi : A \rightarrow \mathcal{L}_r(E)$ is a lattice homomorphism. We emphasize that this is not required in the present paper, but in \cite{10} we will investigate to which extent this is automatically the case.

If $V$ is a not necessarily order complete ordered vector space, and $(v_n)_{n=1}^{\infty} \subseteq V$, then we will write $v_n \uparrow v$ if $v_n \leq v_{n+1}$ $(n \geq 1)$ and $v = \sup_n v_n$.

If $X$ is a set, then $1_X$ will denote the constant function 1 on $X$. If $\Omega$ is a $\sigma$-algebra of subsets of $X$, then $\mathcal{B}(X)$ will denote the bounded $\Omega$-measurable functions on $X$. It is a Banach lattice algebra, and we let $\mathcal{S}(X)$ denote the lattice subalgebra of simple functions. The order bounded subsets of $\mathcal{B}(X)$ are precisely the norm bounded ones. If $\phi_n \uparrow \phi \in \mathcal{B}(X)$, and $\phi \in \mathcal{B}(X)$, then $\phi_n \uparrow \phi$ in $\mathcal{B}(X)$ if and only if $\phi_n(\xi) \uparrow \phi(\xi)$ for all $\xi \in X$.

The supremum norm of $\phi \in \mathcal{B}(X)$ is written as $\|\phi\|$.

### 2.2. Regular Borel measures.

If $X$ is a locally compact Hausdorff space, we let $\Omega$ denote the Borel $\sigma$-algebra generated by the open sets, and $\mathcal{B}(X)$ the Banach lattice algebra of bounded Borel measurable functions on $X$. We write $\mathcal{C}_0(X)$ for the normed lattice algebra of continuous functions on $X$ with compact support, and $\mathcal{C}_0(X)$ for the Banach lattice algebra of continuous functions vanishing at infinity. If $X$ is compact, we will still write $\mathcal{C}_0(X)$ rather than $\mathcal{C}(X)$, for the sake of uniform terminology and notation.

The results in Section \cite{12} rely heavily on the Riesz representation theorem and the general theory of regular Borel measures on locally compact Hausdorff spaces. Since the terminology in this field is not entirely standardised, and we need a bit more than the bare minimum, we give precise definitions and an overview of what we need.
Adapting the terminology from [1, p. 352], we say that a measure $\mu : \Omega \to [0, \infty]$ is:

1. a Borel measure if $\mu(K) < \infty$ for all compact $K \subset X$;
2. outer regular on $\Delta \in \Omega$ if $\mu(\Delta) = \inf \{ \mu(V) : V \text{ open and } \Delta \subset V \}$;
3. inner regular on $\Delta \in \Omega$ if $\mu(\Delta) = \sup \{ \mu(K) : K \text{ compact and } K \subset \Delta \}$;
4. a regular Borel measure if it is a Borel measure that is outer regular on all $\Delta \in \Omega$ and inner regular on all open subsets of $X$.

The nomenclature is not uniform in the literature; sometimes the inner regularity on all elements of $\Omega$ rather than on just the compact subsets is incorporated in the definition of a regular Borel measure, as in [9, p. 212]. In [9, p. 212] our regular Borel measures are called Radon measures.

We let $\mathcal{M}_b(X)$ be the regular finite signed Borel measures on $\Omega$, i.e., the finite signed measures on $\Omega$ that can be written as a difference of two regular finite Borel measures. Then $\mathcal{M}_b(X)$ is a Banach lattice when supplied with the natural ordering and the total variation norm. If $\mu \in \mathcal{M}_b(X)$, then the linear functional $I_{\mu} : C_0(X) \to \mathbb{R}$, defined by $I_{\mu}(\phi) = \int_X \phi \, d\mu$ ($\phi \in C_0(X)$), is bounded.

Most of the results we need are collected in the following overview theorem. Part (1) is the combination of [1, Theorem 38.7] and [9, Theorem 7.17]; part (2) follows from [9, Corollary 7.6]; part (3) is contained in [9, Theorem 7.2]; part (4) is [9, Theorem 7.8].

Recall that a set $\Delta \in \Omega$ is $\sigma$-compact if it is the countable union of compact subsets of $X$.

**Theorem 2.3.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$. Then:

1. (Riesz Representation Theorem) The map $\mu \mapsto I_{\mu}$ is an isometric order isomorphism between the Banach lattices $\mathcal{M}_b(X)$ and $C_0(X)'$.
2. If $\mu \in \mathcal{M}_b(X)_+$, then $\mu$ is inner regular on all elements of $\Omega$.
3. If $\mu : \Omega \to [0, \infty]$ is a regular Borel measure on $\Omega$, and $V$ is an open subset of $X$, then
   
   $\mu(V) = \sup \left\{ \int_X \phi \, d\mu : \phi \in C_c(X), \supp \phi \subset V, 0 \leq \phi \leq 1_X \right\}$,

   and if $K$ is a compact subset of $X$, then

   $\mu(K) = \inf \left\{ \int_X \phi \, d\mu : \phi \in C_c(X), \phi \geq \chi_K \right\}$.

In [22], and analogously elsewhere, $\chi_K$ denotes the characteristic function of $K$.

The following fact is implied by [2, Problem 38.12].

**Lemma 2.4.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$. Let $\mu \in \mathcal{M}_b(X)$ and $\psi \in \mathcal{B}(X)$, and define $\mu^\psi : \Omega \to \mathbb{R}$ by $\mu^\psi(\Delta) = \int_\Delta \psi \, d\mu$ ($\Delta \in \Omega$). Then $\mu^\psi \in \mathcal{M}_b(X)$.

The following consequence of Lemma 2.4 will be used in the proof of Theorem 5.6.

**Lemma 2.5.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, and let $\psi \in \mathcal{B}(X)$. Suppose $\mu, \nu \in \mathcal{M}_b(X)$ are such that

$\int_X \phi \psi \, d\mu = \int \phi \, d\nu,$

In [22].
for all $\phi \in C_0(X)$. Then (2.3) holds for all $\phi \in \mathcal{B}(X)$.

Proof. If we let $\mu^\psi(\Delta) = \int_\Delta \psi \, d\mu$ ($\Delta \in \Omega$), then (2.3) implies that $\int_X \phi \, d\mu^\psi = \int_X \phi \psi \, d\mu$, for all $\phi \in C_0(X)$. Since Lemma 2.3 asserts that $\mu^\psi$ is again a regular finite signed Borel measure, the uniqueness statement in the Riesz representation theorem shows that $\mu^\psi = \nu$. But then $\int_X \phi \, d\mu^\psi = \int_X \phi \, d\nu$ for all $\phi \in \mathcal{B}(X)$, i.e., (2.3) holds for all $\phi \in \mathcal{B}(X)$.

\section{Positive spectral measures}

In this section we introduce the notion of a positive spectral measure on a general $\sigma$-algebra, and establish some basic properties.

\begin{definition}
Let $X$ be a set, $\Omega$ a $\sigma$-algebra of subsets of $X$, and $E$ a Banach lattice. A map $P : \Omega \to \mathcal{L}_c(E) \subset \mathcal{L}(E)$ is called a positive spectral measure when it has the following properties:

1. For each $\Delta$ in $\Omega$, $P(\Delta)$ is a positive projection.
2. $P(\emptyset) = 0$.
3. $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$ for $\Delta_1, \Delta_2 \in \Omega$.
4. $P$ is $\sigma$-additive for the strong operator topology on $\mathcal{L}(E)$, i.e., if $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint elements of $\Omega$, and $x \in E$, then

$$P\left(\bigcup_{n=1}^\infty \Delta_n\right)x = \sum_{n=1}^\infty P(\Delta_n)x,$$

where the series converges in the norm topology.

If $P(X) = \text{id}_E$, then $P$ is called unital.
\end{definition}

\begin{remark}
It is instrumental for the proof of Theorems 4.8 and 5.6 that, as an immediate consequence of a theorem of Pettis' \cite[Theorem IV.10.1]{Pettis} (see also \cite[Lemma III.2]{Halmos}), the combination of (1), (2), (3), and (4) is equivalent to the combination of (1), (2), (3), and (4'), where

(4') $P$ is $\sigma$-additive for the weak operator topology on $\mathcal{L}(E)$, i.e., if $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint elements of $\Omega$, and $x \in E, x' \in E'$, then

$$\langle P\left(\bigcup_{n=1}^\infty \Delta_n\right)x, x' \rangle = \sum_{n=1}^\infty \langle P(\Delta_n)x, x' \rangle.$$

It follows easily from Definition 3.1 that a positive spectral measure is finitely additive, and the following lemma on monotonicity and uniform boundedness is clear.

\begin{lemma}
Let $P : \Omega \to \mathcal{L}_c(E)$ be a positive spectral measure. Then $P$ is monotone, i.e., if $\Delta_1, \Delta_2 \in \Omega$ and $\Delta_1 \subset \Delta_2$, then $0 \leq P(\Delta_1) \leq P(\Delta_2)$. Consequently, $\|P(\Delta_1)\| \leq \|P(\Delta_2)\|$ for such $\Delta_1, \Delta_2$, and in particular $\|P(\Delta)\| \leq \|P(X)\|$, for all $\Delta \in \Omega$.

If $P : \Omega \to \mathcal{L}_c(E)$ is a positive spectral measure, and $x \in E, x' \in E'$, we define $\mu_{x,x'} : \Omega \to \mathbb{R}$ by $\mu_{x,x'}(\Delta) = \langle P(\Delta)x, x' \rangle$ ($x \in E, x' \in E'$). It is clear that $\mu_{x,x'}$ is a finite signed measure, and that $\mu_{x,x'}$ is positive when $x \in E_+, x' \in E'_+$. By a standard argument \cite[p. 97]{Halmos}, we have $\|\mu_{x,x'}\| \leq 2 \sup_{\Delta \in \Omega} |\mu_{x,x'}(\Delta)|$, so that $\|\mu_{x,x'}\| \leq 2\|P(X)\|\|x\|\|x'\|$ ($x \in E, x' \in E'$). The factor 2 can be removed here, as is stated in the following still more precise result.
\end{lemma}
Lemma 3.4. Let $P : \Omega \to \mathcal{L}_r(E)$ be a positive spectral measure, and $x \in E, x' \in E'$. Then $\|\mu_{x,x'}\| \leq \langle P(X)|x|, |x'| \rangle$. Equality holds if $x \in E_+ \cup E_-$ and $x' \in E'_+ \cup E'_-$.

For all $x \in E', x' \in E'$, $\|\mu_{x,x'}\| \leq \|P(X)\|\|x\|\|x'\|$.

Proof. Let $x \in E_+, x' \in E'_+$. If $X = \bigcup_{i=1}^n \Delta_i$ is a measurable disjoint partition of $X$, then using the fact that the $P(\Delta_i)$ are positive we see that

$$\sum_{i=1}^n |\mu_{x,x'}(\Delta_i)| = \sum_{i=1}^n \langle P(\Delta_i)x, x' \rangle$$

$$= \sum_{i=1}^n \mu_{x,x'}(\Delta_i) = \sum_{i=1}^n \langle P(\Delta_i)x, x' \rangle = \langle P(X)x, x' \rangle.$$

Hence $\|\mu_{x,x'}\| = \langle P(X)x, x' \rangle$ for $x \in E_+$ and $x' \in E'_+$. This implies that $\|\mu_{x,x'}\| = \langle P(X)|x|, |x'| \rangle$ if $x \in E_+ \cup E_-$ and $x' \in E'_+ \cup E'_-$. The rest of the lemma follows by splitting arbitrary $x \in E$ and $x' \in E'$ into their positive and negative parts. \qed

The following fact will be needed in the proof of Proposition 5.3, where it is shown that regular positive spectral measures on the Borel $\sigma$-algebra of a locally compact Hausdorff space are also regular in a natural sense that is specific for the ordered context.

Lemma 3.5. Let $\Omega$ be a $\sigma$-algebra of subsets of a set $X$, $E$ a Banach lattice, and $P : \Omega \to \mathcal{L}_r(E)$ a positive spectral measure with associated finite signed measures $\mu_{x,x'} (x \in E, x' \in E')$ on $\Omega$.

1. Suppose $\Delta, \Delta_i \in \Omega$ $(i \in I)$ are elements of $\Omega$ such that:
   a. $\Delta \subseteq \Delta_i$ $(i \in I)$;
   b. $\mu_{x,x'}(\Delta) = \inf_{i \in I} \mu_{x,x'}(\Delta_i)$, for all $x \in E_+, x' \in E'_+$.

   Then $P(\Delta) = \inf_{i \in I} P(\Delta_i)$ in $\mathcal{L}_r(E)$.

2. Suppose $\Delta, \Delta_i \in \Omega$ $(i \in I)$ are elements of $\Omega$ such that:
   a. $\Delta \supseteq \Delta_i$ $(i \in I)$;
   b. $\mu_{x,x'}(\Delta) = \sup_{i \in I} \mu_{x,x'}(\Delta_i)$, for all $x \in E_+, x' \in E'_+$.

   Then $P(\Delta) = \sup_{i \in I} P(\Delta_i)$ in $\mathcal{L}_r(E)$.

Note that we do not assume that $E$ is Dedekind complete, hence the existence of the infimum and supremum is not automatic.

Proof. We prove part (1); the proof of part (2) is similar. Since $P$ is monotone, $P(\Delta) \leq P(\Delta_i)$ $(i \in I)$. Suppose $T \in \mathcal{L}_r(E)$ and $T \leq P(\Delta_i)$ $(i \in I)$. Let $x \in E_+, x' \in E'_+$. Then $\langle Tx, x' \rangle \leq \langle P(\Delta_i)x, x' \rangle = \mu_{x,x'}(\Delta_i) (i \in I)$, so that $\langle Tx, x' \rangle \leq \inf_{i \in I} \mu_{x,x'}(\Delta_i) = \mu_{x,x'}(\Delta) = \langle P(\Delta)x, x' \rangle$. Hence $T \leq P(\Delta)$.

4. Positive $\mathcal{B}(X)$-representations generated by positive spectral measures

Since we know from Lemma 3.3 that a positive spectral measure is uniformly bounded, one can employ a standard method [8] p. 891-892] [9] p. 13-14] to construct a representation of the bounded measurable functions $\mathcal{B}(X)$ on $X$ from this
measure. In the first part of this section, we study the basic properties of the representations thus obtained. In the second part, we turn the tables and ask ourselves which positive representations of $B(X)$ arise in this fashion.

Starting with a positive spectral measure $P : \Omega \to \mathcal{L}_r(E)$ on a $\sigma$-algebra $\Omega$ of subsets of a set $X$, the associated representation $\pi_P : B(X) \to \mathcal{L}_r(E)$ is constructed as follows. If $\phi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \in B(X)$ is a simple function, with the $\Delta_i$ not necessarily disjoint, then let $\pi_P(\phi) = \sum_{i=1}^n \alpha_i P(\Delta_i) \in \mathcal{L}_r(E)$. This is well defined, and one thus obtains a representation $\pi_P$ of the simple functions $S(X)$ on $E$ that is clearly a positive representation of the ordered algebra $S(X)$. Taking the $\Delta_i$ in $\phi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$ to be a measurable disjoint partition of $X$, and invoking Lemma 3.4 in the penultimate step, we see that

$$\|\pi_P(\phi)\| = \sup\left\{\left\|\sum_{i=1}^n \alpha_i P(\Delta_i) x, x'\right\| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \left\|\sum_{i=1}^n |\alpha_i| |\langle P(\Delta_i) x, x'\rangle| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\|$$

$$\leq \left\|\phi\right\| \sup\left\{\sum_{i=1}^n |\langle P(\Delta_i) x, x'\rangle| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \left\|\phi\right\| \sup\left\{\left\|\mu_{x,x'}\right\| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \left\|\phi\right\| \left\|P(\pi_P(\phi))\right\|$$

$$= \left\|\phi\right\| \left\|\pi_P(\pi(1_X))\right\|.$$

We conclude that $\pi_P : S(X) \mapsto (\mathcal{L}_r(E), \|\cdot\|) \subset (\mathcal{L}(E), \|\cdot\|)$ is bounded with norm $\|P(\pi_P(\phi))\|$. Since $S(X)$ is dense in $B(X)$, $\pi_P$ extends uniquely to a bounded representation $\pi_P : B(X) \mapsto (\mathcal{L}(E), \|\cdot\|)$. Furthermore, since $S(X)_+$ is dense in $B(X)_+$, and $E_+$ is closed in $E$, we see that actually $\pi_P(B(X)_+) \subset \mathcal{L}_r(E)_+$. Hence $\pi_P : B(X) \mapsto (\mathcal{L}_r(E), \|\cdot\|)$ is a positive representation with norm $\|P(\pi_P(\phi))\|$.

If $E$ is Dedekind complete, then the regular norm is defined on $\mathcal{L}_r(E)$. Since $\pi_P$ is positive, $\|\cdot\|$ is monotone on $\mathcal{L}_r(E)_+$, and $|\phi| \leq \|\phi\|1_X$ for $\phi \in B(X)$, we have $\|\pi_P(\phi)\| = \left\|\|\pi_P(\phi)\|\right\| \leq \|\pi_P(\phi)\| \leq \|\pi_P(\|\phi\|1_X)\| = \|P(\phi\|1_X)\|$. We see that $\pi_P : B(X) \mapsto (\mathcal{L}_r(E), \|\cdot\|)$ is also bounded, and that $\|\pi_P(\phi)\| = \|P(\phi\|1_X)\|$. We collect our first results and a few more or less standard properties of $\pi_P$ in the following theorem.

**Theorem 4.1.** Let $\Omega$ be a $\sigma$-algebra of subsets of a set $X$, $E$ a Banach lattice, and $P : \Omega \to \mathcal{L}_r(E)$ a positive spectral measure.

1. The map $\pi_P : S(X) \mapsto (\mathcal{L}_r(E), \|\cdot\|)$, defined on the simple functions by $\pi_P(\sum_{i=1}^n \alpha_i \chi_{\Delta_i}) = \sum_{i=1}^n \alpha_i P(\Delta_i)$, extends uniquely to a bounded linear map $\pi_P : B(X) \mapsto (\mathcal{L}(E), \|\cdot\|)$. This extension is a positive representation $\pi_P : B(X) \mapsto (\mathcal{L}_r(E), \|\cdot\|)$ with norm $\|\pi_P(\phi)\| = \|P(\phi)\| = \|\pi_P(\|\phi\|1_X)\|$.

2. $\pi_P$ is unital if and only if $P$ is unital.

3. For $\phi \in B(X)$, $\pi_P(\phi) \in \mathcal{L}_r(E)$ is the unique element of $\mathcal{L}(E)$ such that, for all $x \in E$, $x' \in E'$,

$$\langle \pi_P(\phi)x, x' \rangle = \int_X \phi d\mu_{x,x'}.$$
(4) If \( \phi \in B(X) \) and \((\phi^\infty_{n=1}) \subset B(X) \) is a bounded sequence in \( B(X) \), such that 
\[ \lim_{n \to \infty} \phi_n(x) = \phi(x) \quad \text{for all } x \in X, \] 
then \( \pi_{P}(\phi) = \text{WOT-lim}_{n \to \infty} \pi_{P}(\phi_n) \). 
(5) The closed subalgebras of \((L(E), \| \cdot \|)\) generated by \( P(\Omega) \), \( \pi_{P}(S(X)) \), and 
\( \pi_{P}(B(X)) \) are equal. 
(6) The commutants \( P(\Omega)' \), \( \pi_{P}(S(X))' \), and \( \pi_{P}(B(X))' \) in \( L(E) \) are equal. 
(7) If \( E \) is Dedekind complete, then:
(a) \( \pi_{P} : B(X) \to (L_{r}(E), \| \cdot \|_{r}) \) is bounded, and \( \|\pi_{P}\|_{r} = \|P(\Omega)\| = \|\pi_{P}(1_{X})\| \). 
(b) The closed subalgebras of \((L_{r}(E), \| \cdot \|_{r})\) generated by \( P(\Omega) \), \( \pi_{P}(S(X)) \), and 
\( \pi_{P}(B(X)) \) are equal, and this common algebra is contained in the 
common algebra in part (5). 

**Proof.** For part (3), we note that \([1,1]\) holds by construction if \( \phi \in S(X) \). Since 
for fixed \( x \in E, x' \in E' \) both sides in \([1,1]\) are bounded linear functionals on \( B(X) \), 
the general case follows by continuity. The uniqueness statement in part (3) is 
clear, and part (4) is immediate from an application of the dominated convergence 
theorem. The remaining statements follow easily from the discussion preceding 
the theorem. \hfill \Box

**Remark 4.2.**

1. If \( E \) is Dedekind complete, then the positive map \( \pi : B(X) \to (L_{r}(E), \| \cdot \|_{r}) \) 
is automatically bounded (and then so is \( \pi : B(X) \to (L_{r}(E), \| \cdot \|) \)). The 
point in part (7)(a) is the value of \( \|\pi_{P}\|_{r} \).
2. If \( E \) is Dedekind complete, then there is a seemingly alternative way of 
 obtaining a positive representation of \( B(X) \) on \( E \). Indeed, one can also 
view \( \pi_{P} \) on \( S(X) \) as a bounded map \( \pi_{P} : S(X) \to (L_{r}(E), \| \cdot \|_{r}) \), 
where the codomain is now likewise a Banach space. Extending by continuity, 
we obtain a bounded positive representation \( \pi_{P} : S(X) \to (L_{r}(E), \| \cdot \|_{r}) \). 
Since the inclusion \( (L_{r}(E), \| \cdot \|_{r}) \hookrightarrow (L_{r}(E), \| \cdot \|) \) is bounded, a moment’s 
thought shows that actually \( \pi_{P} = \pi_{P}' \). Hence there is no ambiguity.

The first statement in the next result is specific for the ordered context. Note that 
we do not assume that \( E \) is Dedekind complete, hence the existence of \( \sup_{n} \pi_{P}(\phi_{n}) \) 
in the first statement is not automatic. By \([3\text{ Lemma }1.24]\), this first statement 
implies that \( \pi_{P} \) is a \( \sigma \)-order continuous map between the ordered vector spaces 
\( B(X) \) and \( L_{r}(E) \).

**Proposition 4.3.** Let \( \Omega \) be a \( \sigma \)-algebra of subsets of a set \( X \), \( E \) a Banach lattice, 
and \( P : \Omega \to L_{r}(E) \) a positive spectral measure. If \( \phi \in B(X) \) and \((\phi^\infty_{n=1} \subset B(X) \) is 
a bounded sequence in \( B(X) \) such that \( \phi_{n} \uparrow \phi \) in \( B(X) \), then \( \pi_{P}(\phi_{n}) \uparrow \pi_{P}(\phi) \) in 
\( L_{r}(E) \), and \( \pi_{P}(\phi) = \text{WOT-lim}_{n \to \infty} \pi_{P}(\phi_{n}) \).

**Proof.** Clearly \( \pi_{P}(\phi) \geq \pi_{P}(\phi_{n}) \) for all \( n \). Suppose \( T \in L_{r}(E) \) and \( T \geq \pi_{P}(\phi_{n}) \) for 
all \( n \). Then for \( x \in E_{+}, x' \in E_{+}' \) we have \( \langle Tx, x' \rangle \geq \langle \pi_{P}(\phi_{n})x, x' \rangle \) = 
\( \int_{X} \phi_{n}d\mu_{x,x'} \), for all \( n \). The dominated convergence theorem yields 
\( \langle Tx, x' \rangle \geq \int_{X} \phi d\mu_{x,x'} \) = \( \langle \pi_{P}(\phi)x, x' \rangle \). Hence \( T \geq \pi_{P}(\phi) \). We have shown that \( \pi(\phi_{n}) \uparrow \pi(\phi) \). The second 
statement follows from part (4) of Theorem 4.3 \hfill \Box

Since \( P(\Delta) = \pi_{P}(\Delta) \) (\( \Delta \in \Omega \)), the map \( P \mapsto \pi_{P} \) is injective. This validates 
the choice of the definite article in the following definition.
Definition 4.4. Let \( \Omega \) be a \( \sigma \)-algebra of subsets of a set \( X \), \( E \) a Banach lattice, \( P : \Omega \to \mathcal{L}_1(E) \) a positive spectral measure, and \( \pi_P : \mathcal{B}(X) \to \mathcal{L}_1(E) \) the positive representation of \( \mathcal{B}(X) \) on \( E \) as constructed above. Then we will say that \( \pi_P \) is generated by \( P \), and that \( P \) is the generating positive spectral measure of \( \pi_P \) on \( \Omega \).

We will now concentrate on the question which positive representations of \( \mathcal{B}(X) \) on Banach lattices have a generating positive spectral measure on \( \Omega \). This is not always the case \([10]\). A positive representation that has a generating positive spectral measure is bounded according to Theorem \([7]\), but this is not a distinguishing feature, as is shown by the next result on automatic boundedness.

Proposition 4.5. Let \( \Omega \) be a \( \sigma \)-algebra of subsets of a set \( X \), \( E \) a Banach lattice, and \( \pi : \mathcal{B}(X) \to \mathcal{L}_1(E) \) a positive representation. Then:

1. \( \pi : \mathcal{B}(X) \to (\mathcal{L}_1(E), \| \cdot \|_1) \) is bounded, and \( \| \pi \| = \| \pi(1_X) \|_1 \).
2. If \( E \) is Dedekind complete, then \( \pi : \mathcal{B}(X) \to (\mathcal{L}_1(E), \| \cdot \|_{\pi(1_X)}) \) is bounded, and \( \| \pi \| = \| \pi(1_X) \|_{\pi(1_X)} \).

Proof. As to (1), let \( \phi \in \mathcal{B}(X)_{+1} \). Since \( 0 \leq \phi \leq \| \phi \| 1_X \) and \( \pi \) is positive, we have \( 0 \leq \pi(\phi) \leq \| \phi \| \pi(1_X) \). Hence \( \| \pi(\phi) \| \leq \| \pi(1_X) \| \| \phi \| \). For general \( \phi = \phi_+ - \phi_- \in \mathcal{B}(X) \) this implies that \( \| \pi(\phi) \| \leq \| \pi(1_X) \| \| \phi_+ \| + \| \phi_- \| \) \leq 2\| \pi(1_X) \| \| \phi \| \). In particular, \( \pi : \mathcal{B}(X) \to (\mathcal{L}_1(E), \| \cdot \|_{\pi(1_X)}) \) is bounded. For the statement concerning \( \| \pi \| \) it is sufficient to show that \( \| \pi(\phi) \| \leq \| \pi(1_X) \| \| \phi \| \) for all \( \phi \) in the dense subspace \( \mathcal{S}(X) \) of \( \mathcal{B}(X) \). As to this, we first note that the map \( \Delta \mapsto \pi(\chi_\Delta) \) is a positive operator valued finitely additive measure on \( \Omega \). Proceeding as in the proof of Lemma \([3]\) one then sees that \( \sum_{i=1}^n \langle \pi(\chi_\Delta_i), x, x' \rangle \leq \langle \pi(1_X), |x|, |x'| \rangle \leq \| \pi(1_X) \| \| x \| \| x' \| \) for all measurable disjoint partitions \( \Delta = \bigcup_{i=1}^n \Delta_i \) of \( X \) and all \( x \in E, x' \in E' \). Taking the \( \Delta_i \), in \( \phi = \sum_{i=1}^n \alpha_i \chi_\Delta_i \in \mathcal{B}(X) \) to be a measurable disjoint partition of \( X \), this implies, as in the discussion preceding Theorem \([4,1]\) that

\[
\| \pi(\phi) \| = \sup \left\{ \left\| \sum_{i=1}^n \alpha_i \pi(\chi_\Delta_i), x, x' \right\| : x \in E, \| x \| \leq 1, x', x' \in E', \| x' \| \leq 1 \right\} \\
\leq \sup \left\{ \sum_{i=1}^n |\alpha_i| \left\| \langle \pi(\chi_\Delta_i), x, x' \rangle \right\| : x \in E, \| x \| \leq 1, x', x' \in E', \| x' \| \leq 1 \right\} \\
\leq \| \phi \| \sup \left\{ \sum_{i=1}^n \left\| \langle \pi(\chi_\Delta_i), x, x' \rangle \right\| : x \in E, \| x \| \leq 1, x', x' \in E', \| x' \| \leq 1 \right\} \\
\leq \| \phi \| \| \pi(1_X) \|.
\]

For part (2), let \( \phi \in \mathcal{B}(X) \). Since \( |\phi| \leq \| \phi \| 1_X \), we have \( \pi(|\phi|) \leq \| \phi \| \pi(1_X) \). Therefore \( 0 \leq \| \pi(\phi) \| \leq \| \phi \| \pi(1_X) \), which implies \( \| \pi(\phi) \|_1 = \| \pi(\phi) \|_\phi \leq \| \phi \| \pi(1_X) \) \( \| \phi \| \). This shows that \( \pi : \mathcal{B}(X) \to (\mathcal{L}_1(E), \| \cdot \|_{\pi(1_X)}) \) is bounded (which also follows from the automatic continuity of positive maps between Banach lattices), and that \( \| \pi \|_1 = \| \pi(1_X) \| \). It follows easily from the contractivity of the inclusion \( (\mathcal{L}_1(E), \| \cdot \|_1) \hookrightarrow (\mathcal{L}_1(E), \| \cdot \|_{\pi(1_X)}) \) that \( \| \pi \| = \| \pi(1_X) \| \). \( \square \)

Remark 4.6. If \( E \) is Dedekind complete, then it is well known that the operator norm and the regular norm coincide on the center \( \mathcal{Z}(E) \) of \( E \). If \( \pi(\mathcal{B}(X)) \subset \mathcal{Z}(E) \), the equality of \( \| \pi \| \) and \( \| \pi \|_1 \), as asserted in Proposition \([4,5]\), is then a priori clear. For general not necessarily central positive representations of \( \mathcal{B}(X) \) this equality is somewhat surprising, but also in this case more can be said \([10]\).
With the automatic boundedness available from Proposition 4.3 we can now give a description of the positive representations of \( B(X) \) that have a generating positive spectral measure on \( \Omega \).

**Proposition 4.7.** Let \( \Omega \) be a \( \sigma \)-algebra of subsets of a set \( X \), \( E \) a Banach lattice, and \( \pi : B(X) \to L_\sigma(E) \) a positive representation. Then the following are equivalent:

1. \( \pi \) has a generating positive spectral measure on \( \Omega \).
2. If \( \{ \Delta_n \}_{n=1}^\infty \) are pairwise disjoint elements of \( \Omega \), and \( x \in E_+ \), then
   \[
   \pi \left( \chi_{\bigcup_{n=1}^\infty \Delta_n} \right) x = \sum_{n=1}^\infty \pi (\chi_{\Delta_n}) x,
   \]
   where the series converges in the norm topology.
3. If \( \{ \Delta_n \}_{n=1}^\infty \) are pairwise disjoint elements of \( \Omega \), and \( x \in E_+ \), \( x' \in E'_+ \), then
   \[
   \langle \pi \left( \chi_{\bigcup_{n=1}^\infty \Delta_n} \right) x, x' \rangle = \sum_{n=1}^\infty \langle \pi (\chi_{\Delta_n}) x, x' \rangle.
   \]

In that case, the generating positive spectral measure \( P \) on \( \Omega \) of \( \pi \) is given by \( P(\Delta) = \pi (\chi_\Delta) \) (\( \Delta \in \Omega \)).

**Proof.** Since the only possible generating positive spectral measure \( P \) for \( \pi \) on \( \Omega \) must be given by \( P(\Delta) = \pi (\chi_\Delta) \) (\( \Delta \in \Omega \)), it is clear that (1) implies (2). Clearly (2) implies (3). Assuming (3), we first observe that the equality in (3) then holds for all \( x \in E, x' \in E' \). We define \( P(\Delta) = \pi (\chi_\Delta) \) (\( \Delta \in \Omega \)). Then \( P : \Omega \to L_\sigma(E) \) satisfies (1), (2), (3) and (4') in Definition 4.4, hence is a positive spectral measure on \( \Omega \). It is clear that the positive representation \( \pi_P \) of \( B(X) \) on \( E \) that is generated by \( P \) agrees with \( \pi \) on \( S(X) \). Since both are bounded according to Proposition 4.6, \( \pi_P = \pi \). \( \square \)

For a fairly large practical class of lattices, the criterion for the existence of a generating positive spectral measure is particularly concrete.

**Theorem 4.8.** Let \( \Omega \) be a \( \sigma \)-algebra of subsets of a set \( X \), \( E \) a \( \sigma \)-Dedekind complete Banach lattice, and \( \pi : \Omega \to L_\sigma(E) \) a positive representation. If \( E' \) consists of \( \sigma \)-order continuous linear functionals only (equivalently: if \( E \) has \( \sigma \)-order continuous norm), then the following are equivalent:

1. \( \pi \) has a generating positive spectral measure on \( \Omega \).
2. \( \pi \) is \( \sigma \)-order continuous.

The equivalence involving the \( \sigma \)-order continuity can be found in [17, p. 336].

**Proof.** We had already observed preceding Proposition 4.3 that part (1) implies part (2), even without any further assumptions on \( E \). For the converse implication, we verify the condition in part (3) of Proposition 4.7. In the pertinent notation, we let \( \psi = \chi_{\bigcup_{n=1}^\infty \Delta_n} \), and \( \psi_N = \sum_{n=1}^N \chi_{\Delta_n} \) (\( N \geq 1 \)). Then \( \psi_N \uparrow \psi \) in \( B(X) \), so \( \pi (\psi_N) \uparrow \pi (\psi) \) in \( L_\sigma(E) \) by the \( \sigma \)-order continuity of \( \pi \). By a straightforward modification of the proof of [5, Theorem 1.18], the \( \sigma \)-Dedekind completeness of \( E \) implies that \( \pi (\psi_N)x \uparrow \pi (\psi)x \) for all \( x \in E_+ \). By the assumption on \( E' \) this implies that \( \langle \pi (\psi_N)x, x' \rangle \uparrow \langle \pi (\psi)x, x' \rangle \), for all \( x \in E_+, x' \in E'_+ \). That is, the condition in part (3) of Proposition 4.7 is satisfied. \( \square \)
5. Positive $C_0(X)$-representations generated by positive spectral measures

In this section $X$ is a locally compact Hausdorff space. We will investigate the interplay between positive representations of $C_0(X)$ on a Banach lattice $E$ and $L_r(E)$-valued positive spectral measures on the Borel $\sigma$-algebra $\Omega$ of $X$. Two of our results, Theorems 5.6 and Theorem 5.7 are concerned with the existence of a generating regular positive spectral measure for such a representation. In most of the others, such as Theorem 5.1, the existence of a generating (regular) positive spectral measure is merely assumed, and the relation between the representation and the spectral measure is studied.

We start with the following result on automatic boundedness of positive representations of $C_0(X)$, in the same vein as Proposition 4.5. A conditional improvement can be found in Proposition 5.10.

**Proposition 5.1.** Let $X$ be a locally compact Hausdorff space, $E$ a Banach lattice, and $\pi : C_0(X) \to L_r(E)$ a positive representation.

1. If $X$ is compact, and $E$ is not necessarily Dedekind complete, then:
   - (a) $\|\pi(\phi)\| \leq \|\pi(1_X)\| (\|\phi_+\| + \|\phi_-\|)$ ($\phi \in C_0(X)$).
   - (b) $\pi : C_0(X) \to (L_r(E), \|\cdot\|)$ is bounded, and $\|\pi\| \leq 2\|\pi(1_X)\|\|\|.$
2. If $X$ is not necessarily compact, and $E$ is Dedekind complete, then $\pi : C_0(X) \to (L_r(E), \|\cdot\|)$ and $\pi : C_0(X) \to (L_r(E), \|\cdot\|_r)$ are both bounded, and $\|\pi\| \leq \|\pi\|_r.$
3. If $X$ is compact, and $E$ is Dedekind complete, then $\pi : C_0(X) \to (L_r(E), \|\cdot\|)$ and $\pi : C_0(X) \to (L_r(E), \|\cdot\|_r)$ are both bounded, and $\|\pi\| = \|\pi\|_r = \|\pi(1_X)\|.$

**Proof.** Part (1) follows as in the beginning of the proof of part (1) of Proposition 4.5 and part (3) follows as in the proof of part (2) of Proposition 4.5. As to part (2), if $E$ is Dedekind complete, then $(L_r(E), \|\cdot\|)$ is a Banach lattice. Hence the positive map $\pi : C_0(X) \to (L_r(E), \|\cdot\|)$ is automatically bounded. Using the contractivity of the inclusion map $(L_r(E), \|\cdot\|) \hookrightarrow (L_r(E), \|\cdot\|_r)$, all is clear.

At the time of writing we have no information for the case where $X$ is not compact and $E$ is not Dedekind complete, unless we assume more about $\pi$ (cf. Proposition 5.10).

The following definition is the usual one.

**Definition 5.2.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a Banach lattice, and $P : \Omega \to L_r(E)$ a positive spectral measure. Then $P$ is regular if the finite signed measure $\mu_{x,x'} : \Omega \to \mathbb{R}$, defined by $\mu_{x,x'}(\Delta) = \langle P(\Delta)x, x' \rangle$ ($\Delta \in \Omega$), is a regular finite signed Borel measure, for all $x \in E, x' \in E'$.

Interestingly enough, a regular positive spectral measure is also inner and outer regular on all elements of $\Omega$ in a natural sense that is only meaningful in an ordered context.

**Proposition 5.3.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a Banach lattice, and $P : \Omega \to L_r(E)$ a regular positive spectral measure. Then, for all $\Delta \in \Omega$:

1. $P(\Delta) = \inf \{ P(V) : V \text{ open and } \Delta \subset V \}$ in $L_r(E)$.
2. $P(\Delta) = \sup \{ P(K) : K \text{ compact and } K \subset \Delta \}$ in $L_r(E)$. 

Note that we do not assume that $E$ is Dedekind complete, hence the existence of the infimum and supremum is not automatic.

**Proof.** If $x \in E_{+}, x' \in E'_{+}$, then the finite measure $\mu_{x,x'}$ is a regular Borel measure by assumption. Part (2) of Theorem 2.3 shows that it is not only outer regular on all elements of $\Omega$, but also inner regular on all elements of $\Omega$. An appeal to Lemma 3.5 then finishes the proof. \[\square\]

In view of the results in Section 4, the following definition is natural.

**Definition 5.4.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a Banach lattice, and $\pi : C_{0}(X) \rightarrow L_{r}(E)$ is a positive representation of $C_{0}(X)$ on $E$. If $P : \Omega \rightarrow L_{r}(E)$ is a positive spectral measure on $\Omega$, Section 4 furnishes the positive representation $\pi_{P} : B(X) \rightarrow L_{r}(E)$ of $B(X)$ on $E$ that is generated by $P$. We say that $P$ *generates* $\pi$ if $\pi$ is the restriction of $\pi_{P}$ to $C_{0}(X)$. If $P$ is a regular positive spectral measure on $\Omega$ generating $\pi$, we say that $\pi$ has a *generating regular positive spectral measure* on $\Omega$.

**Remark 5.5.**

1. If $\pi$ has a generating regular positive spectral measure on $\Omega$, then it is unique. This is immediate from 4.1 and the uniqueness statement in the Riesz representation theorem.

2. For the sake of completeness we note that every measure $\mu : \Omega \rightarrow [0, \infty]$ that is finite on compact subsets of $X$ is automatically regular if every open subset of $X$ is $\sigma$-compact [9, Theorem 7.8]. For such spaces, every positive spectral measure on $\Omega$ is therefore automatically regular.

We will now establish two existence results for generating regular positive spectral measures. The first is a special case of the second, but since it has a considerably more straightforward proof than the second, we have included it nevertheless.

**Theorem 5.6.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a reflexive Banach lattice, and $\pi : C_{0}(X) \rightarrow L_{r}(E)$ a positive representation.

Then $\pi$ has a unique generating regular positive spectral measure $P : \Omega \rightarrow L_{r}(E)$.

**Proof.** The uniqueness of a generating regular positive spectral measure is clear from the first part of Remark 5.5. For its existence we modify a combination of ideas employed in the literature for unital representations of commutative $C^*$-algebras on Hilbert spaces [9 Theorem IX.1.4] [15 Theorem 12.22], and for bounded unital representations of $C_{0}(X)$, where $X$ is compact, on Banach spaces [14 Theorem III.3].

The strategy is to construct a positive representation $\tilde{\pi} : B(X) \rightarrow L_{r}(E)$ that extends $\pi$, and then show that $\tilde{\pi}$ has a generating regular positive spectral measure $P$ on $\Omega$, so that one can actually write $\tilde{\pi} = \pi_{P}$.

To start with the construction of $\tilde{\pi}$, we note that the reflexive Banach lattice $E$ is Dedekind complete. Therefore part (2) of Proposition 5.1 implies that $\pi : C_{0}(X) \rightarrow (L_{r}(E), \|\cdot\|)$ is bounded.

Let $x \in E, x' \in E'$, and consider the linear functional $\phi \mapsto \langle \pi(\phi)x, x' \rangle$ on $C_{0}(X)$. We have

$$\|\langle \pi(\phi)x, x' \rangle\| \leq \|\pi\| \|\phi\| \|x\| \|x'\| \quad (\phi \in C_{0}(X), x \in E, x' \in E').$$

Consequently, this functional is bounded and has norm at most $\|\pi\| \|x\| \|x'\|$. The Riesz representation theorem furnishes a regular finite signed Borel measure $\mu_{x,x'}$...
Lemma 2.5 then shows that for all \( \phi \) generating regular positive spectral measure on \( \Omega \).

Moreover, \( \| \mu_{x,x'} \| \leq \| \pi \| \| x \| \| x' \| \), and if \( x \in E_+, x' \in E'_+ \), then \( \mu_{x,x'} \geq 0 \) as a consequence of the positivity of \( \pi \). As a consequence of the uniqueness statement in the Riesz representation theorem, the map \((x, x') \mapsto \mu_{x,x'}\) is bilinear. This implies that, for fixed \( \phi \in B(X) \), the form \([\cdot, \cdot]_\phi\) on \( E \times E' \), defined by

\[
[x, x']_\phi = \int_X \phi \, d\mu_{x,x'} \quad (x \in E, x' \in E'),
\]

is also bilinear. Moreover, \( |[x, x']_\phi| \leq \| \phi \| \| \mu_{x,x'} \| \leq \| \phi \| \| \pi \| \| x \| \| x' \| \). Hence \([\cdot, \cdot]_\phi\) is a bounded bilinear form on \( E \times E' \) with norm at most \( \| \phi \| \| \pi \| \). Since \( E \) is reflexive, there exists a unique operator \( \tilde{\pi}(\phi) \in \mathcal{L}(E) \) such that \( \langle \tilde{\pi}(\phi)x, x' \rangle = [x, x']_\phi \), for all \( x \in E, x' \in E' \). Hence

\[
(5.2) \quad \langle \tilde{\pi}(\phi)x, x' \rangle = \int_X \phi \, d\mu_{x,x'} \quad (\phi \in B(X), x \in E, x' \in E').
\]

Moreover, \( \| \tilde{\pi}(\phi) \| \leq \| \pi \| \| \phi \| \), for all \( \phi \in B(X) \). Since it is clear from \( (5.2) \) that \( \tilde{\pi} \) is linear, we conclude that \( \tilde{\pi} : B(X) \to (\mathcal{L}(E), \| \cdot \|) \) is a bounded linear map. Comparing \( (5.1) \) and \( (5.2) \), we see that \( \tilde{\pi} : B(X) \to \mathcal{L}(E) \) extends \( \pi : C_0(X) \to \mathcal{L}(E) \).

We will now proceed to show that \( \tilde{\pi} \) is a positive representation of \( B(X) \) with a generating regular positive spectral measure on \( \Omega \).

Since \( \mu_{x,x'} \geq 0 \) for \( x \in E_+, x' \in E'_+ \), \( (5.2) \) implies that \( \tilde{\pi} \) is positive.

For the multiplicativity, we argue as follows. Let \( \phi, \psi \in C_0(X) \), and \( x \in E, x' \in E' \). Using \( (5.2) \) twice we have

\[
\int_X \phi \psi \, d\mu_{x,x'} = \langle \pi(\phi)\pi(\psi)x, x' \rangle = \langle \pi(\phi)\pi(\psi)x, x' \rangle = \int_X \phi \, d\mu_{\pi(\psi)x,x'}.
\]

Lemma 2.4 then shows that

\[
\int_X \phi \psi \, d\mu_{x,x'} = \int_X \phi \, d\mu_{\pi(\psi)x,x'},
\]

for all \( \phi \in B(X), \psi \in C_0(X) \) and \( x \in E, x' \in E' \). This implies, using \( (5.2) \) in the second and fourth step, that

\[
\int_X \phi \psi \, d\mu_{x,x'} = \int_X \phi \, d\mu_{\pi(\psi)x,x'} = \langle \tilde{\pi}(\phi)\pi(\psi)x, x' \rangle = \langle \pi(\psi)x, \tilde{\pi}(\phi)x' \rangle = \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)x'},
\]

for all \( \phi \in B(X), \psi \in C_0(X) \) and \( x \in E, x' \in E' \). Lemma 2.5 now shows that

\[
\int_X \phi \psi \, d\mu_{x,x'} = \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)x'}.
\]
for all $\phi, \psi \in B(X)$ and $x \in E, x' \in E'$. Using \eqref{5.2} in the second step, we then see that
\begin{equation}
\int_X \phi \psi \, d\mu_{x,x'} = \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)'x'} = \langle \tilde{\pi}(\psi) x, \tilde{\pi}(\phi)'x' \rangle = \langle \tilde{\pi}(\phi) \tilde{\pi}(\psi) x, x' \rangle,
\end{equation}
for all $\phi, \psi \in B(X)$ and $x \in E, x' \in E'$. On the other hand, \eqref{5.2} shows that
\begin{equation}
\int_X \phi \psi \, d\mu_{x,x'} = \langle \tilde{\pi}(\phi \psi) x, x' \rangle,
\end{equation}
for all $\phi, \psi \in B(X), x \in E$, and $x' \in E'$. Comparing this with \eqref{5.3}, we conclude that $\tilde{\pi}(\phi \psi) = \tilde{\pi}(\phi) \tilde{\pi}(\psi)$, for all $\phi, \psi \in B(X)$. Hence $\tilde{\pi} : B(X) \to \mathcal{L}_r(E)$ is a positive representation of $B(X)$ on $E$. To show that $\tilde{\pi}$ has a generating positive spectral measure on $\Omega$, we will verify condition (3) in Proposition 4.7. Let $x \in E_+, x' \in E_+'$, and let $\{\Delta_n\}_{n=1}^{\infty}$ be pairwise disjoint elements of $\Omega$. We must show that $\langle \tilde{\pi}(\chi_{\cup_{n=1}^{\infty} \Delta_n}) x, x' \rangle = \sum_{n=1}^{\infty} \langle \tilde{\pi}(\Delta_n) x, x' \rangle$. Using the dominated convergence theorem in the second step, we see that
\begin{align*}
\langle \tilde{\pi}(\chi_{\cup_{n=1}^{\infty} \Delta_n}) x, x' \rangle &= \int_X \chi_{\cup_{n=1}^{\infty} \Delta_n} \, d\mu_{x,x'} \\
&= \sum_{n=1}^{\infty} \int_X \chi_{\Delta_n} \, d\mu_{x,x'} \\
&= \sum_{n=1}^{\infty} \langle \tilde{\pi}(\Delta_n) x, x' \rangle.
\end{align*}
Hence Proposition 4.7 applies, and we conclude that $\tilde{\pi}$ has a generating regular positive spectral $P$ measure on $\Omega$ that is given by $P(\Delta) = \tilde{\pi}(\Delta)$. In order to conclude that $P$ is regular, we consider its associated signed measures, denoted temporarily by $\mu^P_{x,x'} (x \in E, x' \in E')$. For $\Delta \in \Omega$, we have, using \eqref{5.2} in the third step,
\begin{align*}
\mu^P_{x,x'}(\Delta) &= \langle P(\Delta) x, x' \rangle \\
&= \langle \tilde{\pi}(\chi_\Delta) x, x' \rangle \\
&= \int_X \chi_\Delta \, d\mu_{x,x'} \\
&= \mu_{x,x'}(\Delta).
\end{align*}
Hence $\mu^P_{x,x'} = \mu_{x,x'} (x \in E, x' \in E')$, which is known to be regular. \hfill \Box

We continue with the more general version of Theorem 5.6. The proof is by reduction to \cite[Theorem III.4]{14}, valid for bounded unital representations of $C_0(X)$-spaces for compact $X$ on suitable Banach spaces. The reduction is rather straightforward, but the proof of that theorem itself relies on two non-trivial results \cite[Theorems I.13 and I.14]{14}, combined with a line of reasoning as in the proof of Theorem 5.6. All in all, it is considerably more involved than that of Theorem 5.6.
Theorem 5.7. Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a KB-space, and $\pi : C_0(X) \to \mathcal{L}_r(E)$ a positive representation. Then $\pi$ has a unique generating regular positive spectral measure $P : \Omega \to \mathcal{L}_r(E)$.

Proof. Again the uniqueness is clear. As to the existence, we note that the KB-space $E$ does not contain a subspace linearly homeomorphic to $c_0$, as this property characterises the KB-spaces among the Banach lattices, cf. [10, Theorem 7.1]. Therefore [4, Theorem III.4] applies: if $X$ is a compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, and $\rho : C_0(X) \to (\mathcal{L}(E), \| \cdot \|)$ is a bounded unital representation of $X$ on $E$, then there exists a regular spectral measure $P : \Omega \to \mathcal{L}(E)$ assuming values in the projections on $E$ (i.e., it satisfies all of Definition 5.1 except possibly the positivity of the projections), such that

\[
\langle (\rho(\phi)x,x') \rangle = \int_X \phi d\mu_{x,x'}^P \quad (\phi \in C_0(X), x \in E, x' \in E'),
\]

where $\mu_{x,x'}^P(\Delta) = (P(\Delta)x,x') (\Delta \in \Omega)$ are the usual associated regular finite signed Borel measures on $\Omega$.

To reach this situation starting from our given $\pi$, we first note that by Proposition 5.1 the Dedekind completeness of the KB-space $E$ implies that $\pi : C_0(X) \to (\mathcal{L}(E), \| \cdot \|)$ is bounded. Consider the unitisation $C_0(X)_1 = \mathbb{R} \times C_0(X)$ of $C_0(X)$, with norm $\| (\lambda, \phi) \| = |\lambda| + \| \phi \| (\lambda \in \mathbb{R}, \phi \in C_0(X))$. Then the representation $\pi_\infty : C_0(X)_1 \to (\mathcal{L}(E), \| \cdot \|)$, defined by $\pi_\infty((\lambda, \phi)) = \lambda \text{id}_E + \pi(\phi) (\lambda \in \mathbb{R}, \phi \in C_0(X))$, is a unital representation of $C_0(X)_1$ on $E$ that is also bounded.

Next, we let $X_\infty = X \cup \{ \infty \}$ be the one point compactification of $X$. The algebras $C_0(X_\infty)$ and $C_0(X)_1$ are canonically isomorphic as abstract algebras, and although the pertinent isomorphism is not necessarily isometric, it is still a linear homeomorphism. Hence $\pi_\infty$ can be viewed as a bounded unital representation of $C_0(X_\infty)$ on $E$ that extends $\pi$. Consequently, there exists a regular spectral measure $P_\infty$ on the Borel $\sigma$-algebra $\Omega_\infty$ of $X_\infty$, such that

\[
\langle (\pi_\infty(\phi)x,x') \rangle = \int_{X_\infty} \phi d\mu_{x,x'}^{P_\infty} \quad (\phi \in C_0(X_\infty), x \in E, x' \in E'),
\]

where $\mu_{x,x'}^{P_\infty}(\Delta) = (P_\infty(\Delta)x,x') (\Delta \in \Omega_\infty, x \in E, x' \in E')$.

We now define $P : \Omega \to \mathcal{L}_r(E)$ by $P(\Delta) = P_\infty(\Delta) (\Delta \in \Omega \subset \Omega_\infty)$. It is routine to check that $P$ is a regular spectral measure, and we let $\mu_{x,x'} (x \in E, x' \in E')$ denote the usual associated regular finite signed Borel measures on $\Omega$. If $\phi \in C_0(X)$, then $\phi(\infty) = 0$, so (5.5) implies that

\[
\langle (\pi(\phi)x,x') \rangle = \int_X \phi d\mu_{x,x'} \quad (\phi \in C_0(X), x \in E, x' \in E').
\]

Since $\pi$ is positive, the order statement in the Riesz representation theorem implies that all measures $\mu_{x,x'} (x \in E_+, x' \in E'_+)$ are positive, which shows that $P$ assumes its values in the positive projections on $E$. Comparison with [4, 1.1] yields that the positive representation $\pi_P : \mathcal{B}(X) \to \mathcal{L}_r(E)$ that is generated by $P$ restricts to $\pi$ on $C_0(X)$, as required.

The following is now clear.

Corollary 5.8. Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, and $E$ a KB-space. Then the map $P \to \pi_P|_{C_0(X)}$, sending an $\mathcal{L}_r(E)$-valued positive spectral measure on $\Omega$ to the restriction of $\pi_P : \mathcal{B}(X) \to \mathcal{L}_r(E)$ to $C_0(X) \subset \mathcal{B}(X)$,
is a bijection between the \( \mathcal{L}_r(E) \)-valued regular positive spectral measures on \( \Omega \) and the positive representations of \( C_0(X) \) on \( E \).

If \( X \) is compact, then \( \pi_P|_{C_0(X)} \) is unital if and only if \( P \) is unital.

**Remark 5.9.** If \( X \) is compact, additional existence results for regular positive spectral measures generating \( R \)-bounded unital positive representations of \( C_0(X) \) can be obtained using [3] Proposition 2.17.

We will now concentrate on the implications of the existence of a generating positive (regular) spectral measure for \( \pi \). To start with, we have the following conditional improvement of Proposition 5.1. Note that it also covers the “missing” case in Proposition 5.1.

**Proposition 5.10.** Let \( X \) be a locally compact Hausdorff space with Borel \( \sigma \)-algebra \( \Omega \), \( E \) a Banach lattice, and \( \pi : C_0(X) \to \mathcal{L}_r(E) \) a positive representation. Suppose \( \pi \) has a generating positive spectral measure \( P \) on \( \Omega \), and let \( \pi_P : \mathcal{B}(X) \to \mathcal{L}_r(E) \) denote the generated bounded positive representation of \( \mathcal{B}(X) \) on \( E \) extending \( \pi \). Then:

1. \( \pi : C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|) \) is bounded, and \( \| \pi \| \leq \| \pi_P \| = \| P(X) \| \).
2. If \( P \) is regular, then \( \| \pi \| = \| \pi_P \| = \| P(X) \| \).
3. If \( P \) is regular and \( E \) is Dedekind complete, then the maps \( \pi : C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|), \pi : C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|_r), \pi_P : \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r) \) and \( \pi_P : \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r) \) are all bounded, and \( \| \pi \| = \| \pi_r \| = \| \pi_P \| = \| \pi_P \|_r = \| P(X) \| \).

**Proof.** We know from Theorem 4.1 that \( \| \pi_P \| = \| P(X) \| \). Certainly the restriction \( \pi \) of \( \pi_P \) to \( C_0(X) \) is also bounded, and \( \| \pi \| \leq \| \pi_P \| \). For the converse inequality in part (2) if \( P \) is regular, we use (4.1) and the isometry statement in the Riesz representation theorem to see that

\[
\| \pi_P \| = \sup \{ |\langle \pi(\phi)x, x' \rangle| : \phi \in \mathcal{B}(X), \| \phi \| \leq 1, x \in E, \| x \| \leq 1, x' \in E', \| x' \| \leq 1 \} 
\]

\[
= \sup \left\{ \left| \int_X \phi \, d\mu_{x,x'} \right| : \phi \in \mathcal{B}(X), \| \phi \| \leq 1, x \in E, \| x \| \leq 1, x' \in E', \| x' \| \leq 1 \right\} 
\]

\[
\leq \sup \{ \| \mu_{x,x'} \| : x \in E, \| x \| \leq 1, x' \in E', \| x' \| \leq 1 \} 
\]

\[
= \sup \left\{ \left| \int_X \phi \, d\mu_{x,x'} \right| : \phi \in C_0(X), \| \phi \| \leq 1, x \in E, \| x \| \leq 1, x' \in E', \| x' \| \leq 1 \right\} 
\]

\[
= \| \pi \|. 
\]

As to (3), for Dedekind complete \( E \) the positive maps \( \pi_P : \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r) \) and \( \pi_P : C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|) \) between Banach lattices are bounded. Furthermore, parts (7)(a) and (1) of Theorem 4.1 show that \( \| \pi_P \|_r = \| \pi_P \| \). We also know from part (2) of Proposition 5.1 that \( \| \pi \| \leq \| \pi_r \| \). In addition, we have \( \| \pi \|_r \leq \| \pi_P \|_r \), since \( \pi_P \) extends \( \pi \). If \( P \) is regular, then we have already established that \( \| \pi_P \| = \| \pi \| \). Combining all this, we see that, for regular \( P \) and Dedekind complete \( E \),

\[
\| \pi_P \|_r = \| \pi_P \| = \| \pi \| \leq \| \pi_r \| \leq \| \pi_P \|_r ,
\]

and the proof is complete. \( \square \)
We collect a few further consequences (some of them of course familiar from the non-ordered context) of the existence of a generating positive spectral measure for \( \pi : C_0(X) \to \mathcal{L}_r(E) \) in our next result.

**Corollary 5.11.** Let \( X \) be a locally compact Hausdorff space with Borel \( \sigma \)-algebra \( \Omega \), \( E \) a Banach lattice, and \( \pi : C_0(X) \to \mathcal{L}_r(E) \) a positive representation. Suppose \( \pi \) has a generating positive spectral measure \( P \) on \( \Omega \), and let \( \pi_P : \mathcal{B}(X) \to \mathcal{L}_r(E) \) denote the generated bounded positive representation of \( \mathcal{B}(X) \) on \( E \) extending \( \pi \). Then:

1. If \( \phi \in C_0(X) \), then there is a sequence of linear combinations of elements of \( P(\Omega) \) that converges to \( \pi(\phi) \) in \( (\mathcal{L}_r(E), \| \cdot \|_1) \). If \( E \) is Dedekind complete, then there exists such a sequence converging to \( P(\phi) \) in \( (\mathcal{L}_r(E), \| \cdot \|_1) \). If \( \phi \in C_0(X)_+ \), then the coefficients occurring in these linear combinations can be taken non-negative.
2. If \( \phi \in C_0(X) \), and \( (\phi_n)_{n=1}^\infty \subset C_0(X) \) is a norm bounded sequence such that \( \lim_{n \to \infty} \phi_n(\xi) = \phi(\xi) \) for all \( \xi \in X \), then \( \pi(\phi) = \lim \pi(\phi_n) \).
3. If \( \phi \in C_0(X) \) and \( (\phi_n)_{n=1}^\infty \subset C_0(X) \) is a norm bounded sequence in \( C_0(X) \) such that \( \phi_n(\xi) \uparrow \phi(\xi) \) for all \( \xi \in X \), then \( \pi(\phi_n) \uparrow \pi(\phi) \) in \( \mathcal{L}_r(E) \).

If \( P \) is regular, then:

4. The commutants \( P(\Omega)' \), \( \pi_P(S(X))' \), and \( \pi_P(B(X))' \) in \( \mathcal{L}(E) \) are equal. Consequently, \( P(\Omega) \subset \pi(C_0(X))'' \).

**Proof.** For part (1), we take a sequence of simple functions converging uniformly to \( \phi \) in \( \mathcal{B}(X) \) and apply parts (1) and (7)(a) of Theorem 4.11.

Part (2) is a specialisation of part (4) of Theorem 4.11

Part (3) is a specialisation of part of Proposition 4.3

As to part (3), from part (6) of Theorem 4.11 we already know that the commutants \( P(\Omega)' \), \( \pi_P(S(X))' \), and \( \pi_P(B(X))' \) in \( \mathcal{L}(E) \) are equal. We will show that \( \pi(C_0(X))' = P(\Omega)' \). Let \( T \in \mathcal{L}(E) \). Then \( T \in \pi(C_0(X))' \) if and only if \( \langle T\pi(\phi)x, x' \rangle = \langle \pi(\phi)Tx, x' \rangle \), for all \( \phi \in C_0(X), x \in E, x' \in E' \). Now note that

\[
\langle T\pi(\phi)x, x' \rangle = \langle \pi(\phi)x, T'x' \rangle = \int_X \phi d\mu_{x,T'x'},
\]

and that

\[
\langle \pi(\phi)Tx, x' \rangle = \int_X \phi d\mu_{Tx,x'}.
\]

Thus \( T \in \pi(C_0(X))' \) if and only if

\[
\int_X \phi d\mu_{x,T'x'} = \int_X \phi d\mu_{Tx,x'},
\]

for all \( \phi \in C_0(X), x \in E, x' \in E' \). By the uniqueness statement in the Riesz representation theorem, this is the case if and only if \( \mu_{x,T'x'} = \mu_{Tx,x'} \), for all \( x \in E, x' \in E' \). That is, if and only if \( \langle P(\Delta)x, T'x' \rangle = \langle P(\Delta)Tx, x' \rangle \), for all \( \Delta \in \Omega, x \in E, x' \in E' \). This, in turn, is equivalent to \( T \in P(\Omega)' \).

The folklore final statement is immediate from \( P(\Omega) \subset P(\Omega)'' = \pi(C_0(X))'' \). \( \square \)

We conclude by showing how the generating regular positive spectral measure \( P \) of \( \pi \), if it exists, can be determined directly from \( \pi \) in terms of the ordering on \( \mathcal{L}_r(E) \). By the first part of Proposition 4.3, it is sufficient to know \( P(V) \) for all open
subsets $V$ of $X$, and part (1) of the next result shows how $P(V)$ can be found from $\pi(C_0(X))$. Likewise, the second part of Proposition 5.3 shows that it is sufficient to know $P(K)$ for all compact subsets $K$ of $X$, and part (2) of the next result shows how to retrieve these from $\pi(C_0(X))$.

As in similar previous results, we do not assume that $E$ is Dedekind complete, hence the existence of the various suprema and infima in $\mathcal{L}_r(E)$ is not automatic.

**Theorem 5.12.** Let $X$ be a locally compact Hausdorff space with Borel $\sigma$-algebra $\Omega$, $E$ a Banach lattice, and $\pi : C_0(X) \to \mathcal{L}_r(E)$ a positive representation. Suppose $\pi$ has a generating regular positive spectral measure $P$ on $\Omega$.

1. Let $V$ be an open subset of $X$. Then

$$P(V) = \sup \{ \pi(\phi) : \phi \in C_c(X), \supp \phi \subset V, 0 \leq \phi \leq 1_X \}.$$ 

2. Let $K$ be an compact subset of $X$. Then

$$P(K) = \inf \{ \pi(\phi) : \phi \in C_c(X), \phi \geq \chi_K \}.$$ 

3. In addition to the expression for $P(X)$ as obtained from part (1), we also have

$$P(X) = \sup \{ \pi(\phi) : \phi \in C_0(X), 0 \leq \phi \leq 1_X \}.$$ 

4. Let $V$ be an open subset of $X$. Then $V$ is $\sigma$-compact if and only if there exists a sequence $(\phi_n)_{n=1}^{\infty}$ in $C_c(X)$ such that $\sup \phi_n \subset V$ ($n \geq 1$) and $\sup_n \phi_n = \chi_V$ in $\mathcal{B}(X)$.

In that case, there exists a sequence $(\phi_n)_{n=1}^{\infty}$ in $C_c(X)$ such that $\sup \phi_n \subset V$ and $0 \leq \phi_n \leq 1_X$ ($n \geq 1$), and $\phi_n \uparrow \chi_V$ in $\mathcal{B}(X)$.

For any norm bounded sequence $(\phi_n)$ in $\mathcal{B}(X)$ such that $\phi_n \uparrow \chi_V$ in $\mathcal{B}(X)$, we have $\pi(\phi_n) \uparrow P(V)$ and $P(V) = \operatorname{WOT-lim}_n \phi_n$.

5. $X$ is $\sigma$-compact if and only if there exists a sequence $(\phi_n)_{n=1}^{\infty}$ in $C_0(X)$ such that $0 \leq \phi_n \leq 1_X$ ($n \geq 1$) and $\sup_n \phi_n = 1_X$ in $\mathcal{B}(X)$.

In that case, there exists a sequence $(\phi_n)_{n=1}^{\infty}$ in $C_c(X)$ such that $0 \leq \phi_n \leq 1_X$ ($n \geq 1$), and $\phi_n \uparrow 1_X$ in $\mathcal{B}(X)$.

For any norm bounded sequence $(\phi_n)_{n=1}^{\infty}$ in $C_0(X)$ such that $\phi_n \uparrow 1_X$ in $\mathcal{B}(X)$, we have $\pi(\phi_n) \uparrow P(X)$ and $P(X) = \operatorname{WOT-lim}_n \phi_n$.

**Proof.** Let $\pi_P : \mathcal{B}(X) \to \mathcal{L}_r(E)$ denote the positive representation of $\mathcal{B}(X)$ on $E$ extending $\pi$ that is generated by $P$, with associated regular finite signed Borel measures $\mu_{x,x'}$ ($x \in E, x' \in E'$). Starting with part (1), if $\phi \in C_c(X)$, $\sup \phi \subset V$, and $0 \leq \phi \leq 1_X$, then $\pi(\phi) = \pi_P(\phi) \leq \pi_P(\chi_V) = P(V)$. Hence $P(V)$ is an upper bound for

$$\{ \pi(\phi) : \phi \in C_c(X), \sup \phi \subset V, 0 \leq \phi \leq 1_X \}.$$ 

If $T \in \mathcal{L}_r(E)$ is also an upper bound for this set, then, for all $\phi \in C_c(X)$ with $0 \leq \phi \leq 1_X$, $\sup \phi \subset V$, $x \in E_+, x' \in E_+^r$, we have

$$\langle Tx, x' \rangle \geq \langle \pi(\phi)x, x' \rangle$$

$$= \int_X \phi d\mu_{x,x'},$$

where (4.41) was used. Therefore, for all $x \in E_+, x' \in E_+^r$,

$$\langle Tx, x' \rangle \geq \sup \left\{ \int_X \phi d\mu_{x,x'} : \phi \in C_c(X), \sup \phi \subset V, 0 \leq \phi \leq 1_X \right\}.$$
Since by (2.2) the right hand side in this equation equals $\mu_{x,x'}(V) = \langle P(V)x, x' \rangle$, we conclude that $\langle Tx, x' \rangle \geq \langle P(V)x, x' \rangle$, for all $x \in E_+, x' \in E'_+$. Hence $T \geq P(V)$.

The proof of part (2) is similar, based on (2.2).

For part (3), the same line of reasoning shows that $P(X) \geq \pi(\phi)$ for all $\phi \in C_0(X)$ with $0 \leq \phi \leq 1_X$. If $T \geq \pi(\phi)$ for all such $\phi$, then, for all $x \in E_+, x' \in E'_+$, we find

$$\langle Tx, x' \rangle \geq \sup \left\{ \int_X \phi \, d\mu_{x,x'} : \phi \in C_0(X), \ 0 \leq \phi \leq 1_X \right\}.$$

But the right hand side is the norm of the positive functional $\phi \mapsto \int_X \phi \, d\mu_{x,x'}$ on $C_0(X)$, which by the isometry statement in the Riesz representation theorem equals $\|\mu_{x,x'}\| = \mu_{x,x'}(X) = \langle P(X)x, x' \rangle$. Therefore $T \geq P(X)$.

For part (4), if such a sequence exists, then $V = \bigcup_n \{ \xi \in X : \phi_n(\xi) > 0 \}$ is countable union of $\sigma$-compact subsets of $X$, hence $\sigma$-compact. Conversely, if $V$ is $\sigma$-compact, then we may assume that $V = \bigcup_n K_n$ where $K_n \subseteq K_{n+1}$ for all $n$. By [22] Corollary 4.32 we can choose $\psi_n \in C_c(X)$ such that $0 \leq \psi_n \leq 1_X$, $\psi_n(\xi) = 1$ for $\xi \in K_n$ and $\text{supp}(\psi_n) \subseteq V$. Let $\phi_n = \bigvee_{k=1}^n \psi_k$. Then the sequence $(\phi_n)_{n=1}^\infty$ is as required. An appeal to Proposition 4.3 concludes the proof of part (4).

The proof of part (5) is similar to that of part (4).

\section*{Acknowledgments}

The authors thank Ben de Pagter for helpful discussions.

\section*{References}

[1] Y.A. Abramovich and C.D. Aliprantis, \textit{Principles of real analysis}, 3rd Ed., Academic Press, San Diego, 1998.

[2] Y.A. Abramovich and C.D. Aliprantis, \textit{Problems in real analysis. A workbook with solutions}, 2nd Ed., Academic Press, San Diego, 1999.

[3] Y.A. Abramovich and C.D. Aliprantis, \textit{An invitation to operator theory}, American Mathematical Society, Providence, R.I., 2002.

[4] Y.A. Abramovich, E.L. Arenson and A.K. Kitover, \textit{Banach C(K)-modules and operators preserving disjointness}, Longman Scientific & Technical, Harlow, UK, 1992.

[5] C.D. Aliprantis and O. Burkinshaw, \textit{Positive operators}, Springer, Dordrecht, 2006.

[6] J.B. Conway, \textit{A course in functional analysis}, Springer, New York, 2007.

[7] N. Dunford and J.T. Schwartz, \textit{Linear operators I: general theory}, Interscience Publishers, New York, 1958.

[8] N. Dunford and J.T. Schwartz, \textit{Linear operators II: spectral theory}, Interscience Publishers, New York, 1963.

[9] G.B. Folland, \textit{Real analysis}, 2nd Ed., John Wiley, New York, 1999.

[10] M. de Jeu and X. Jiang, \textit{Positive representations of $C_0(X)$. II.}, to appear.

[11] S. Kaijser, \textit{Some representations of Banach lattices}, Ark. Math. 16 (1978), 179-193.

[12] P. Meyer-Nieberg, \textit{Banach lattices}, Springer, Berlin, 1991.

[13] B. de Pagter and W.J. Ricker, \textit{C(K)-representations and R-boundedness}, J. London Math. Soc. (2) 76 (2007), 498-512.

[14] W. Ricker, \textit{Operator algebras generated by commuting projections: a vector measure approach}, Springer, Berlin, 1999.

[15] W. Rudin, \textit{Functional analysis}, 2nd Ed., Tata McGraw-Hill, New Delhi, 1999.

[16] W. Wnuk, \textit{Banach lattices with order continuous norms}, Polish Scientific Publishers PWN, Warsaw, 1999.

[17] A.C. Zaanen, \textit{Riesz spaces: volume II}, North-Holland, Amsterdam, 1983.
