A TEMPORARY VIOLATION OF COLOR GAUGE INVARIANCE
AS A SOURCE OF THE JAFFE-WITTEN MASS GAP IN QCD

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We propose to realize a mass gap in QCD by not imposing the transversality condition on the full gluon self-energy, while preserving the color gauge invariance condition for the full gluon propagator. This is justified by the nonlinear and nonperturbative dynamics of QCD. None of physical observables/processes in low-energy QCD will be directly affected by such a temporary violation of color gauge invariance/symmetry. No truncations/approximations and no special gauge choice are made for the regularized skeleton loop integrals, contributing to the full gluon self-energy, which enters the Schwinger-Dyson equation for the full gluon propagator. In order to make the existence of a mass gap perfectly clear the corresponding subtraction procedure is introduced. All this allows one to establish the general structure of the full gluon propagator and the corresponding gluon Schwinger-Dyson equation in the presence of a mass gap. It is mainly generated by the nonlinear interaction of massless gluon modes. The physical meaning of the mass gap is to be responsible for the large-scale (low-energy/momentum), i.e., nonperturbative structure of the true QCD vacuum. In the presence of a mass gap two different types of solutions for the full gluon propagator are possible. The massive solution leads to an effective gluon mass, which explicitly depends on the gauge-fixing parameter. This solution becomes smooth at small gluon momentum in the Landau gauge. The general iteration solution is always severely singular at small gluon momentum, i.e., the gluons remain massless, and this does not depend on the gauge choice. We also formulate a general method how to restore the transversality of the gluon propagator relevant for nonperturbative QCD.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) [1, 2] is widely accepted as a realistic quantum field gauge theory of strong interactions not only at the fundamental (microscopic) quark-gluon level but at the hadronic (macroscopic) level as well. This means that in principle it should describe the properties of experimentally observed hadrons in terms of experimentally never seen quarks and gluons, i.e., to describe the hadronic word from first principles – an ultimate goal of any fundamental theory. But this is a formidable task because of the color confinement phenomenon, the dynamical mechanism of which is not yet understood, and therefore the confinement problem remains unsolved up to the present days. It prevents colored quarks and gluons to be experimentally detected as physical ("in" and "out" asymptotic) states which are colorless (i.e., color-singlets), by definition, so color confinement is permanent and absolute [1].

Today there is no doubt left that color confinement and other dynamical effects, such as spontaneous breakdown of chiral symmetry, bound-state problems, etc., being essentially nonperturbative (NP) effects, are closely related to the large-scale (low-energy/momentum) structure of the true QCD ground state and vice-versa [3, 4] (and references therein). The perturbation theory (PT) methods in general fail to investigate them. If QCD itself is a confining theory then a characteristic scale has to exist. It should be directly responsible for the above-mentioned structure of the true QCD vacuum in the same way as $\Lambda_{QCD}$ is responsible for the nontrivial perturbative dynamics there (asymptotic freedom (AF) [3]).

However, the Lagrangian of QCD [1, 2] does not contain explicitly any of the mass scale parameters which could have a physical meaning even after the corresponding renormalization program is performed. So the main goal of this paper is to show how the characteristic scale (the mass gap, for simplicity) responsible for the NP dynamics in the infrared (IR) region may explicitly appear in QCD. This becomes an imperative especially after Jaffe and Witten have formulated their theorem "Yang-Mills Existence And Mass Gap" [5]. We will show that the mass gap is dynamically generated mainly due to the nonlinear (NL) interaction of massless gluon modes.

The propagation of gluons is one of the main dynamical effects in the true QCD vacuum. It is described by the corresponding quantum equation of motion, the so-called Schwinger-Dyson (SD) equation [1] (and references therein)

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for the full gluon propagator. The importance of this equation is due to the fact that its solutions reflect the quantum-dynamical structure of the true QCD ground state. The color gauge structure of this equation is the main subject of our investigation in order to find a way how to realize a mass gap in QCD. Also we will discuss at least two possible types of solutions of the gluon SD equation in the presence of a mass gap, making no approximations/truncations and no special gauge choice for the skeleton loop integrals contributing to it. So they can be considered as the generalizations of the explicit solutions because the latter ones are necessarily based on the above-mentioned specific approximations/truncations schemes.

II. QED

It is instructive to begin with a brief explanation why a mass gap does not occur in quantum electrodynamics (QED). The photon SD equation can be symbolically written down as follows:

\[ D(q) = D^0(q) + D^0(q)\Pi(q)D(q), \]

where we omit, for convenience, the dependence on the Dirac indices, and \( D^0(q) \) is the free photon propagator. \( \Pi(q) \) describes the electron skeleton loop contribution to the photon self-energy (the so-called vacuum polarization tensor). Analytically it looks

\[ \Pi(q) \equiv \Pi_{\mu\nu}(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\mu S(p - q)\Gamma_\nu(p,q)S(p)], \]

where \( S(p) \) and \( \Gamma_\nu(p - q, q) \) represent the full electron propagator and the full electron-photon vertex, respectively. Here and everywhere below the signature is Euclidean, since it implies \( q_i \rightarrow 0 \) when \( q^2 \rightarrow 0 \) and vice-versa. This tensor has the dimensions of a mass squared, and therefore it is quadratically divergent. To make the formal existence of a mass gap (the quadratically divergent constant, so having the dimensions of a mass squared) perfectly clear, let us now, for simplicity, subtract its value at zero. One obtains

\[ \Pi^*(q) \equiv \Pi_{\mu\nu}^*(q) = \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}(0) = \Pi_{\mu\nu}(q) - \delta_{\mu\nu}\Delta^2(\lambda). \]

The explicit dependence on the dimensionless ultraviolet (UV) regulating parameter \( \lambda \) has been introduced into the mass gap \( \Delta^2(\lambda) \), given by the integral (2.2) at \( q^2 = 0 \), in order to assign a mathematical meaning to it. In this connection a few remarks are in order in advance. The dependence on \( \lambda \) (when it is not shown explicitly) is assumed in all divergent integrals here and below in the case of the gluon self-energy as well (see next section). This means that all the expressions are regularized (including photon/gluon propagator), and we can operate with them as with finite quantities. \( \lambda \) should be removed on the final stage only after performing the corresponding renormalization program (which is beyond the scope of the present investigation, of course). Whether the regulating parameter \( \lambda \) has been introduced in a gauge-invariant way (though this always can be achieved) or not, and how it should be removed is not important for the problem if a mass gap can be "released/liberated" from the corresponding vacuum. We will show in the most general way (not using the PT and no special gauge choice will be made) that this is impossible in QED and might be possible in QCD.

The decomposition of the subtracted vacuum polarization tensor into the independent tensor structures can be written as follows:

\[ \Pi_{\mu\nu}^*(q) = T_{\mu\nu}(q)q^2\Pi_1^*(q^2) + q_\mu q_\nu(q)\Pi_2^*(q^2), \]

where both invariant functions \( \Pi_n^*(q^2) \) at \( n = 1, 2 \) are, by definition, dimensionless and regular at small \( q^2 \), since \( \Pi^*(0) = 0 \); otherwise they remain arbitrary. From this relation it follows that \( \Pi^*(q) = O(q^2) \), i.e., it is always of the order \( q^2 \). Also, here and everywhere below

\[ T_{\mu\nu}(q) = \delta_{\mu\nu} - q_\mu q_\nu/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q). \]

Taking into account the subtraction (2.3), the photon SD equation becomes
\[ D(q) = D^0(q) + D^0(q)\Pi^*(q)D(q) + D^0(q)\Delta^2(\lambda)D(q). \] (2.6)

Its subtracted part can be summed up into the geometric series, so one has

\[ D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)\Delta^2(\lambda)D(q), \] (2.7)

where the modified photon propagator is

\[ \tilde{D}^0(q) = \frac{D^0(q)}{1 - \Pi^*(q)D^0(q)} = D^0(q) + D^0(q)\Pi^*(q)D^0(q) - D^0(q)\Pi^*(q)D^0(q)\Pi^*(q)D^0(q) + \ldots. \] (2.8)

Since \( \Pi^*(q) = O(q^2) \) and \( D^0(q) \sim (q^2)^{-1} \), the \( \text{IR} \) singularity of the modified photon propagator is determined by the \( \text{IR} \) singularity of the free photon propagator, i.e., \( \tilde{D}^0(q) = O(D^0(q)) \) with respect to the behavior at small photon momentum.

Similar to the subtracted photon self-energy, the photon self-energy (2.2) in terms of independent tensor structures is

\[ \Pi_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi_1(q^2) + q_\mu q_\nu \Pi_2(q^2), \] (2.9)

where again \( \Pi_n(q^2) \) at \( n = 1, 2 \) are dimensionless functions and remain arbitrary. Due to the transversality of the photon self-energy

\[ q_\mu \Pi_{\mu\nu}(q) = q_\nu \Pi_{\mu\nu}(q) = 0, \] (2.10)

which comes from the current conservation condition in QED, one has \( \Pi_2(q^2) = 0 \), i.e., it should be purely transversal

\[ \Pi_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi_1(q^2). \] (2.11)

On the other hand, from the subtraction (2.3) and the transversality condition (2.10) it follows that

\[ \Pi^*_2(q^2) = -\frac{\Delta^2(\lambda)}{q^2}, \] (2.12)

which, however, is impossible since \( \Pi^*_2(q^2) \) is a regular function of \( q^2 \), by definition. So the mass gap should be zero and consequently \( \Pi^*_2(q^2) \) as well, i.e.,

\[ \Pi^*_2(q^2) = 0, \quad \Delta^2(\lambda) = 0. \] (2.13)

Thus the subtracted photon self-energy is also transversal, i.e., satisfies the transversality condition

\[ q_\mu \Pi_{\mu\nu}(q) = q_\nu \Pi^*_\mu\nu(q) = 0, \] (2.14)

and coincides with the photon self-energy (see Eq. (2.3) at the zero mass gap). Moreover, this means that the photon self-energy does not have a pole in its invariant function \( \Pi_1(q^2) = \Pi_1^*(q^2) \). As mentioned above, in obtaining these results neither the PT has been used nor a special gauge has been chosen. So there is no place for quadratically divergent constants in QED, while logarithmic divergence still can be present in the invariant function \( \Pi_1(q^2) = \Pi_1^*(q^2) \). It is to be included into the electric charge through the corresponding renormalization program (for these detailed gauge-invariant derivations explicitly done in lower order of the PT see Refs. [2, 6, 7, 8, 9]).

In fact, the current conservation condition (2.10) lowers the quadratical divergence of the corresponding integral (2.2) to a logarithmic one. That is the reason why in QED logarithmic divergences survive only. Thus in QED there is no mass gap and the relevant photon SD equation is shown in Eq. (2.8), simply identifying the full photon propagator with its modified counterpart. In other words, in QED we can replace \( \Pi(q) \) by its subtracted counterpart \( \Pi^*(q) \)

\[ (2.9) \]
from the very beginning ($\Pi(q) \to \Pi'(q)$), totally discarding the quadratically divergent constant $\Delta^2(\lambda)$ from all the equations and relations. The current conservation condition for the photon self-energy (2.10), i.e., its transversality, and for the full photon propagator $q_\mu q_\nu D_{\mu\nu}(q) = i\xi$, where $\xi$ is the gauge-fixing parameter, are consequences of gauge invariance. They should be maintained at every stage of the calculations, since the photon is a physical state. In other words, at all stages the current conservation plays a crucial role in extracting physical information from the $S$-matrix elements in QED. For example, if some QED process includes the full photon propagator, then the corresponding $S$-matrix element is proportional to the combination $j_1^\mu(q)D_{\mu\nu}(q)j_2^\nu(q)$. The current conservation condition $j_1^\mu(q)q_\mu = j_2^\nu(q)q_\nu = 0$ implies that the unphysical (longitudinal) component of the full photon propagator does not change the physics of QED, i.e., only its physical (transversal) component is important. In its turn this means that the transversality condition imposed on the photon self-energy is important, because $\Pi_{\mu\nu}(q)$ itself is a correction to the amplitude of the physical process, for example such as electron-electron scattering.

III. QCD

Due do color confinement in QCD the gluon is not a physical state. Still, color gauge invariance should also be preserved, so the color current conservation takes place in QCD as well. However, in this theory it plays no role in the extraction of physical information from the $S$-matrix elements for the corresponding physical processes and quantities. So in QCD there is no such physical amplitude to which the gluon self-energy may directly contribute (for example, quark-quark/antiquark scattering is not a physical process). The lesson which comes from QED is that if one preserves the transversality of the photon self-energy at every stage, then there is no mass gap. Thus, in order to realize a mass gap in QCD, our proposal is not to impose the transversality condition on the gluon self-energy, but preserving the color gauge invariance condition for the full gluon propagator (see below). As mentioned above, no QCD physics will be directly affected by this. So color gauge symmetry will be violated at the initial stage (at the level of the gluon self-energy) and will be restored at the final stage (at the level of the full gluon propagator).

A. Gluon SD equation

The gluon SD equation can be symbolically written down as follows (for our purposes it is more convenient to consider the SD equation for the full gluon propagator and not for its inverse):

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q)\Pi_{\rho\sigma}(q; D)D_{\sigma\nu}(q),$$

(3.1)

where $D^0_{\mu\nu}(q)$ is the free gluon propagator. $\Pi_{\rho\sigma}(q; D)$ is the gluon self-energy, and in general it depends on the full gluon propagator due to the non-Abelian character of QCD (see below). Thus the gluon SD equation is highly NL, while the photon SD equation (2.1) is a linear one. In what follows we omit the color group indices, since for the gluon propagator (and hence for its self-energy) they are reduced to the trivial $\delta$-function, for example $D^0_{\mu\nu}(q) = D_{\mu\nu}(q)\delta^{ab}$.

Also, for convenience, we introduce $i$ into the gluon SD equation (3.1).

The gluon self-energy $\Pi_{\rho\sigma}(q; D)$ is the sum of a few terms, namely

$$\Pi_{\rho\sigma}(q; D) = -\Pi^3_{\rho\sigma}(q) - \Pi^{ab}_{\rho\sigma}(q) + \Pi^2_{\rho\sigma}(D) + \Pi_{(1)\rho\sigma}(q; D) + \Pi_{(2)\rho\sigma}(q; D) + \Pi'_{(2)\rho\sigma}(q; D),$$

(3.2)

where $\Pi^3_{\rho\sigma}(q)$ describes the skeleton loop contribution due to quark degrees of freedom (it is an analog of the vacuum polarization tensor in QED, see Eq. (2.2)), while $\Pi^{ab}_{\rho\sigma}(q)$ describes the skeleton loop contribution due to ghost degrees of freedom. Both skeleton loop integrals do not depend on the full gluon propagator $D$, so they represent the linear contribution to the gluon self-energy. $\Pi^3_{\rho\sigma}(D)$ represents the so-called constant skeleton tadpole term. $\Pi_{(1)\rho\sigma}(q; D)$ represents the skeleton loop contribution, which contains the triple gluon vertices only. $\Pi_{(2)\rho\sigma}(q; D)$ and $\Pi'_{(2)\rho\sigma}(q; D)$ describe topologically independent skeleton two-loop contributions, which combine the triple and quartic gluon vertices. The last four terms explicitly contain the full gluon propagators in different powers, that is why they form the NL part of the gluon self-energy. The explicit expressions for the corresponding skeleton loop integrals (in which the corresponding symmetry coefficients can be included) are of no importance here. Let us note that like in QED these skeleton loop integrals are in general quadratically divergent, and therefore they should be regularized (see remarks above and below).
B. A temporary violation of color gauge invariance/symmetry (TVCGI/S)

The color gauge invariance condition for the gluon self-energy (3.2) can be reduced to the three independent
transversality conditions imposed on it. It is well known that the quark contribution can be made transversal
independently of the pure gluon contributions within any regularization scheme which preserves gauge invariance, for
example such as the dimensional regularization method (DRM) [1, 2, 8, 9, 11]. So, one has
\[ q^{\rho}_{\sigma}\Pi_{\rho\sigma}(q) = q^{\sigma}_{\rho}\Pi_{\rho\sigma}(q) = 0, \] (3.3)
indeed. In the same way the sum of the gluon contributions can be done transversal by taking into account the ghost
contribution, so again one has
\[ q^{\rho}_{\sigma}\left[ \Pi_{(1)\rho\sigma}(q; D) + \Pi_{(2)\rho\sigma}(q; D) + \Pi'_{(2)\rho\sigma}(q; D) - \Pi^{gh}_{\rho\sigma}(q) \right] = 0. \] (3.4)
The role of ghost degrees of freedom is to cancel the unphysical (longitudinal) component of gauge bosons in every
order of the PT, i.e., going beyond the PT and thus being general. The previous equation just demonstrates this,
since it contains the corresponding skeleton loop integrals.

However, there is no such regularization scheme (preserving or not gauge invariance) in which the transversality
condition for the constant skeleton tadpole term could be satisfied, i.e.,
\[ q^{\rho}_{\sigma}\Pi_{\rho\sigma}(D) = q^{\sigma}_{\rho}\Pi_{\rho\sigma}(D) \neq 0, \] (3.5)
indeed. This means that in any NP approach the transversality condition imposed on the gluon self-energy may not
be valid, i.e., in general
\[ q^{\rho}_{\sigma}\Pi_{\rho\sigma}(q; D) = q^{\sigma}_{\rho}\Pi_{\rho\sigma}(q; D) \neq 0. \] (3.6)
In the PT, when the full gluon propagator is always approximated by the free one, the constant tadpole term is set to
be zero within the DRM [8, 11], i.e., \( \Pi_{\rho\sigma}(D^0) = 0 \). So in the PT the transversality condition for the gluon self-energy
is always satisfied.

The relation (3.6) justifies our proposal not to impose the transversality condition on the gluon self-energy. The relation (3.5)
emphasizes the special role of the constant skeleton tadpole term in the NP QCD dynamics. It explicitly
violates the transversality condition for the gluon self-energy (3.6). The second important observation is that now
ghosts themselves cannot automatically provide the transversality of the gluon propagator in NP QCD. However, this
does not mean that we need no ghosts at all. Of course, we need them in other sectors of QCD, for example in the
quark-gluon Ward-Takahashi identity, which contains the so-called ghost-quark scattering kernel explicitly [1].

C. Subtractions

As we already know from QED, the regularization of the gluon self-energy can be started from the subtraction its
value at the zero point (see, however, remarks below). Thus, quite similarly to the subtraction (2.3), one obtains
\[ \Pi^a_{\rho\sigma}(q; D) = \Pi_{\rho\sigma}(q; D) - \Pi_{\rho\sigma}(0; D) = \Pi_{\rho\sigma}(q; D) - \delta_{\rho\sigma}\Delta^2(\lambda; D). \] (3.7)
Let us remind once more that for our purpose, namely to demonstrate a possible existence of a mass gap \( \Delta^2(\lambda; D) \)
in QCD, it is not important how \( \lambda \) has been introduced and how it should be removed at the final stage. The mass
gap itself is mainly generated by the nonlinear interaction of massless gluon modes, slightly corrected by the linear
contributions coming from the quark and ghost degrees of freedom, namely
\[ \Delta^2(\lambda; D) = \Pi^f(D) + \sum_a \Pi^a_0(D) = \Delta^2_f(D) + \sum_a \Delta^2_0(D), \] (3.8)
where index ”a” runs as follows: \( a = -q, -gh, 1, 2, 2', \) and \( -q, -gh \) mean that both terms enter the above-mentioned
sum with minus sign (here, obviously, the tensor indices are omitted). In these relation all the divergent constants
\( \Pi'(D) \) and \( \Pi'^0(0; D) \), having the dimensions of a mass squared, are given by the corresponding skeleton loop integrals at \( q^2 = 0 \). Thus these constants summed up into the mass gap squared (3.8) cannot be discarded like in QED, since the transversality condition for the gluon self-energy is not satisfied, see Eq. (3.6). In other words, in QCD in general the quadratical divergences of the corresponding loop integrals cannot be lowered to logarithmic ones, and therefore the mass gap (3.8) should be explicitly taken into account in this theory. The transversality condition for the gluon self-energy can be satisfied partially, i.e., if one imposes it on quark and gluon (along with ghost) degrees of freedom as it follows from above. Then the mass gap is to be reduced to \( \Pi'(D) \), since all other constants \( \Pi'^0(0; D) \) can be discarded in this case (see Eq. (3.8)). However, we will stick to our proposal not to impose the transversality condition on the gluon self-energy, and thus to deal with the mass gap on account of all possible contributions.

The subtracted gluon self-energy

\[
\Pi^s_{\rho\sigma}(q; D) = \Pi^s(q; D) = \sum_a \Pi^s_a(q; D)
\]  
(3.9)

is free from the tadpole contribution, because \( \Pi^s_i(D) = \Pi_i(D) - \Pi_{ti}(D) = 0 \), by definition, at any \( D \), while in the gluon self-energy it is explicitly present through the mass gap (see Eqs. (3.8) and (3.7)). The general decomposition of the subtracted gluon self-energy into the independent tensor structures can be written down as follows:

\[
\Pi^s_{\rho\sigma}(q; D) = T_{\rho\sigma}(q) q^2 \Pi(q^2; D) + q_\rho q_\sigma \tilde{\Pi}(q^2; D),
\]  
(3.10)

where both invariant functions \( \Pi(q^2; D) \) and \( \tilde{\Pi}(q^2; D) \) are dimensionless and regular at small \( q^2 \). Since the subtracted gluon self-energy does not contain the tadpole contribution, we can now impose the color current conservation condition on it, i.e., to put

\[
q_\rho \Pi^s_{\rho\sigma}(q; D) = q_\sigma \Pi^s_{\rho\sigma}(q; D) = 0,
\]  
(3.11)

which implies \( \tilde{\Pi}(q^2; D) = 0 \), so that the subtracted gluon self-energy finally becomes purely transversal

\[
\Pi^s_{\rho\sigma}(q; D) = T_{\rho\sigma}(q) q^2 \Pi(q^2; D),
\]  
(3.12)

and it is always of the order \( q^2 \) at any \( D \), since the invariant function \( \Pi(q^2; D) \) is regular at small \( q^2 \) at any \( D \). Thus the subtracted quantities are free from the quadratic divergences, but logarithmic ones can be still present in \( \Pi(q^2; D) \) like in QED.

D. General structure of the gluon SD equation

Our strategy is not to impose the transversality condition on the gluon self-energy in order to realize a mass gap despite whether or not the tadpole term is explicitly present. To show that this works, it is instructive to substitute the subtracted gluon self-energy (3.10) (and not its transversal part (3.12)) into the initial gluon SD equation (3.1), on account of the subtraction (3.7). Then one obtains

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q)i[T_{\rho\sigma}(q) q^2 \Pi(q^2; D) + q_\rho q_\sigma \tilde{\Pi}(q^2; D)] D_{\sigma\nu}(q) + D^0_{\mu\sigma}(q)i\Delta^2(\lambda; D) D_{\sigma\nu}(q).
\]  
(3.13)

Let us now introduce the general tensor decompositions of the full and auxiliary free gluon propagators \( D_{\mu\nu}(q) = i[T_{\mu\nu}(q) d(q^2) + L_{\mu\nu}(q) d_0(q^2)]/(1/q^2) \) and

\[
D^0_{\mu\nu}(q) = i[T_{\mu\nu}(q) + L_{\mu\nu}(q) d_0(q^2)]/(1/q^2),
\]  
(3.14)

respectively. The form factor \( d_0(q^2) \) introduced into the unphysical part of the auxiliary free gluon propagator \( D^0_{\mu\nu}(q) \) is needed in order to explicitly show that the longitudinal part of the subtracted gluon self-energy \( \tilde{\Pi}(q^2; D) \) plays no role. The color gauge invariance condition imposed on the full gluon propagator

\[
q_\mu q_\nu D_{\mu\nu}(q) = i\xi,
\]  
(3.15)
implies \( d_1(q^2) = \xi \), so that the full gluon propagator becomes

\[
D_{\mu\nu}(q) = i \{ T_{\mu\nu}(q)d(q^2) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2},
\]

(3.16)

Substituting all these decompositions into the gluon SD equation (3.13), one obtains

\[
d(q^2) = \frac{1}{1 + \Pi(q^2; D) + (\Delta^2(\lambda; D)/q^2)},
\]

(3.17)

and

\[
d_0(q^2) = \frac{\xi}{1 - \xi[\Pi(q^2; D) + (\Delta^2(\lambda; D)/q^2)]},
\]

(3.18)

However, the auxiliary free gluon propagator defined in Eqs. (3.14) and (3.18) is to be equivalently replaced as follows:

\[
D^0_{\mu\nu}(q) \implies D^0_{\mu\nu}(q) + i\xi L_{\mu\nu}(q)d_0(q^2)[\Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2}] \frac{1}{q^2},
\]

(3.19)

where \( D^0_{\mu\nu}(q) \) in the right-hand-side is the standard free gluon propagator, i.e.,

\[
D^0_{\mu\nu}(q) = i \{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}.
\]

(3.20)

Then the gluon SD equation in the presence of the mass gap (3.13), on account of the explicit expression for the auxiliary free gluon form factor (3.18), and doing some tedious algebra, is also to be equivalently replaced as follows:

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q)iT_{\rho\sigma}(q)q^2\Pi(q^2; D)D_{\sigma\nu}(q)
+ D^0_{\mu\sigma}(q)i\Delta^2(\lambda; D)D_{\sigma\nu}(q) + i\xi^2 L_{\mu\nu}(q) \frac{\Delta^2(\lambda; D)}{q^4}.
\]

(3.21)

Here and below \( D^0_{\mu\nu}(q) \) is the free gluon propagator (3.20). The gluon SD equation (3.21) does not depend on \( d_0(q^2) \) and \( \Pi(q^2; D) \), i.e., they played their role and then retired from the scene. So, our derivation explicitly shows that the longitudinal part of the subtracted gluon self-energy \( \Pi(q^2; D) \) plays no role and can be put to zero without losing generality, and thus making the subtracted gluon self-energy purely transversal in accordance with Eq. (3.12).

Using now the explicit expression for the free gluon propagator (3.20) this equation can be further simplified to

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) - T_{\mu\rho}(q) \left[ \Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2} \right] D_{\sigma\nu}(q).
\]

(3.22)

It is easy to check that the full gluon propagator satisfies the color gauge invariance condition (3.15), indeed. So the full gluon propagator is the expression (3.16) with the full gluon form factor given in Eq. (3.17), which obviously satisfies Eq. (3.22). The only price we have paid by violating color gauge invariance is the gluon self-energy, while the full and free gluon propagators and the subtracted gluon self-energy always satisfy it. Let us emphasize that the expression for the full gluon form factor shown in the relation (3.17) cannot be considered as the formal solution for the full gluon propagator, since both the mass gap \( \Delta^2(\lambda; D) \) and the invariant function \( \Pi(q^2; D) \) depend on \( D \) themselves. Here it is worth noting in advance that from above it is almost clear that if one begins with the UV renormalization program, then the information on the mass gap will be totally lost. In this case instead of the regularized gluon self-energy its subtracted regularized counterpart comes into the play. In other words, in the PT limit \( \Delta^2(\lambda; D) = 0 \) one recovers the standard gluon SD equation, and the gluon self-energy coincides with its subtracted counterpart like in QED. For a more detailed explanation see below subsection C in section V.

Thus, we have established the general structure of the full gluon propagator (see Eqs. (3.16) and (3.17)) and the corresponding gluon SD equation (3.22) (which is equivalent to Eq. (3.13)) in the presence of a mass gap.
IV. MASSIVE SOLUTION

An immediate consequence of the explicit presence of the mass gap in the full gluon propagator is that a massive-type solution for it becomes possible. In other words, in this case the gluon may indeed acquire an effective mass. From Eq. (3.17) it follows that

\[
\frac{1}{q^2} d(q^2) = \frac{1}{q^2 + q^2 \Pi(q^2; \xi) + \Delta^2(\lambda, \xi)},
\]

(4.1)

where instead of the dependence on \(D\) the dependence on \(\xi\) is explicitly shown. The full gluon propagator (3.16) may have a pole-type solution at the finite point if and only if the denominator in Eq. (4.1) has a zero at this point \(q^2 = -m_g^2\) (Euclidean signature), i.e.,

\[
-m_g^2 - m_g^2 \Pi(-m_g^2; \xi) + \Delta^2(\lambda, \xi) = 0,
\]

(4.2)

where \(m_g^2 \equiv m_g^2(\lambda, \xi)\) is an effective gluon mass, and the previous equation is a transcendental equation for its determination. Excluding the mass gap, one obtains that the denominator in the full gluon propagator becomes

\[
q^2 + q^2 \Pi(q^2; \xi) + \Delta^2(\lambda, \xi) = q^2 + m_g^2 + q^2 \Pi(q^2; \xi) + m_g^2 \Pi(-m_g^2; \xi).
\]

(4.3)

Let us now expand \(\Pi(q^2; \xi)\) in a Taylor series near \(m_g^2\):

\[
\Pi(q^2; \xi) = \Pi(-m_g^2; \xi) + (q^2 + m_g^2) \Pi'(-m_g^2; \xi) + O\left((q^2 + m_g^2)^2\right).
\]

(4.4)

Substituting this expansion into the previous relation and after doing some tedious algebra, one obtains

\[
q^2 + m_g^2 + q^2 \Pi(q^2; \xi) + m_g^2 \Pi(-m_g^2; \xi) = (q^2 + m_g^2)[1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi'(-m_g^2; \xi)] [1 + \Pi^R(q^2; \xi)],
\]

(4.5)

where \(\Pi^R(q^2; \xi) = 0\) at \(q^2 = -m_g^2\); otherwise it remains arbitrary. Thus the full gluon propagator (3.16) now looks

\[
D_{\mu\nu}(q) = iT_{\mu\nu}(q) \frac{Z_3}{(q^2 + m_g^2)[1 + \Pi^R(q^2; m_g^2)]} + i\xi L_{\mu\nu}(q) \frac{1}{q^2},
\]

(4.6)

where, for future purpose, in the invariant function \(\Pi^R(q^2; m_g^2)\) instead of \(\xi\) we introduced the dependence on the gluon effective mass squared \(m_g^2\) which depends on \(\xi\) itself. The gluon renormalization constant is

\[
Z_3 = [1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi'(-m_g^2; \xi)]^{-1}.
\]

(4.7)

In the formal PT limit \(\Delta^2(\lambda, \xi) = 0\), an effective gluon mass is also zero, \(m_g^2(\lambda, \xi) = 0\), as it follows from Eq. (4.2). So an effective gluon mass is the NP effect. At the same time, it cannot be interpreted as the "physical" gluon mass, since it remains explicitly gauge-dependent quantity. The gluon renormalization constant (4.7) in this limit becomes a standard one, namely \([1 + \Pi(0; \xi)]^{-1}\). The massive-type solution (4.6) becomes smooth in the IR \((q^2 \to 0)\) in the Landau gauge \(\xi = 0\) only (the ghosts now cannot guarantee the cancellation of the longitudinal part of the full gluon propagator as mentioned above). In this connection let us point out that Landau gauge smooth (even vanishing in the IR) gluon propagator at the expense of more singular (than the free one) in the IR ghost propagator has been obtained and discussed in Refs. [12, 13] (and references therein). As mentioned above, however, these results are necessarily based on different approximations/truncations for the skeleton loop integrals contributing to the gluon self-energy.

It is interesting to note that Eq. (4.2) has a second solution in the PT limit \(\Delta^2(\lambda, \xi) = 0\). In this case, an effective gluon mass remains finite, but \(1 + \Pi(-m_g^2; \xi) = 0\). So a scale responsible for the NP dynamics is not determined by the gluon mass itself, but by the condition \(1 + \Pi(-m_g^2; \xi) = 0\). Its interpretation from a physical point of view is not clear.
V. ITERATION SOLUTION

The expression for the full gluon form factor shown in the relation (3.17) cannot be considered as the formal solution for the full gluon propagator, since both the mass gap $\Delta^2(\lambda; D)$ and the invariant function $\Pi(q^2; D)$ depend on $D$ themselves. In order to perform a formal iteration of the gluon SD equation (3.22), much more convenient to address to its "solution" for the full gluon form factor (3.17), nevertheless, and rewrite it as follows:

$$d(q^2) = 1 - \left[ \Pi(q^2; d) + \frac{\Delta^2(\lambda; d)}{q^2} \right] d(q^2) = 1 - P(q^2; d)d(q^2),$$

i.e., in the form of the corresponding transcendental (i.e., not algebraic) equation suitable for the formal nonlinear iteration procedure. Here we replace the dependence on $D$ by the equivalent dependence on $d$. For future purposes, it is convenient to introduce short-hand notations as follows:

$$\Delta^2(\lambda; d = d(0) + d(1) + d(2) + ... + d(m) + ...) = \Delta_m^2 = \Delta^2c_m(\lambda, \alpha, \xi, g^2),$$

$$\Pi(q^2; d = d(0) + d(1) + d(2) + ... + d(m) + ...) = \Pi_m(q^2),$$

and

$$P_m(q^2) = \left[ \Pi_m(q^2) + \frac{\Delta_m^2}{q^2} \right], \ m = 0, 1, 2, 3, ...$$

In these relations $\Delta_m^2$ are the auxiliary mass squared parameters, while $\Delta^2$ is the mass gap itself (see, however, remarks in Conclusions). The dimensionless constants $c_m$ via the corresponding subscripts depend on which iteration for the gluon form factor $d$ is actually done. They may depend on the dimensionless coupling constant squared $g^2$, as well as on the gauge-fixing parameter $\xi$. We also introduce the explicit dependence on the dimensionless finite (slightly different from zero) subtraction point $\alpha$, since the initial subtraction at the zero point may be dangerous [1]. The dependence of $\Delta^2$ on all these parameters is not shown explicitly, and if necessary can be restored any time. Let us also remind that all the invariant functions $\Pi_m(q^2)$ are regular at small $q^2$. If it were possible to express the full gluon form factor $d(q^2)$ in terms of these quantities then it would be the formal solution for the full gluon propagator. In fact, this is nothing but the skeleton loops expansion, since the regularized skeleton loop integrals, contributing to the gluon self-energy, have to be iterated. This is the so-called general iteration solution. No truncations/approximations and no special gauge choice have been made. This formal expansion is not a PT series. The magnitude of the coupling constant squared and the dependence of the regularized skeleton loop integrals on it is completely arbitrary.

It is instructive to describe the general iteration procedure in some details. Evidently, $d(0) = 1$, and this corresponds to the approximation of the full gluon propagator by its free counterpart in the gluon SD equation (3.22). Doing the first iteration in Eq. (5.1), one thus obtains

$$d(q^2) = 1 - P_0(q^2) + ... = 1 + d^{(1)}(q^2) + ..., \quad (5.4)$$

where obviously

$$d^{(1)}(q^2) = -P_0(q^2). \quad (5.5)$$

Doing the second iteration, one obtains

$$d(q^2) = 1 - P_1(q^2)[1 + d^{(1)}(q^2)] + ... = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + ..., \quad (5.6)$$

where

$$d^{(2)}(q^2) = -d^{(1)}(q^2) - P_1(q^2)[1 - P_0(q^2)]. \quad (5.7)$$

Doing the third iteration, one further obtains
\[ d(q^2) = 1 - P_2(q^2)[1 + d^{(1)}(q^2) + d^{(2)}(q^2)] + ... = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + d^{(3)}(q^2) + ..., \] (5.8)

where

\[ d^{(3)}(q^2) = -d^{(1)}(q^2) - d^{(2)}(q^2) - P_2(q^2)[1 - P_1(q^2)(1 - P_0(q^2))], \] (5.9)

and so on for the next iterations.

Thus up to the third iteration, one finally obtains

\[ d(q^2) = \sum_{m=0}^{\infty} d^{(m)}(q^2) = 1 - [\Pi_2(q^2) + \frac{\Delta^2_{\perp}}{q^2}]\left[1 - [\Pi_1(q^2) + \frac{\Delta^2_{\parallel}}{q^2}]\left[1 - \Pi_0(q^2) - \frac{\Delta^2_{\perp}}{q^2}\right]\right] + ... . \] (5.10)

We restrict ourselves by the iterated gluon form factor up to the third term, since this already allows to show explicitly some general features of such kind of the nonlinear iteration procedure.

### A. Splitting/shifting procedure

Doing some tedious algebra, the previous expression can be rewritten as follows:

\[ d(q^2) = [1 - \Pi_2(q^2) + \Pi_1(q^2)\Pi_2(q^2) - \Pi_0(q^2)\Pi_1(q^2)\Pi_2(q^2) + ...] \]

\[ + \frac{1}{q^2}[\Pi_2(q^2)\Delta^2_{\perp} + \Pi_1(q^2)\Delta^2_{\parallel} - \Pi_0(q^2)\Pi_1(q^2)\Delta^2_{\perp} - \Pi_0(q^2)\Pi_2(q^2)\Delta^2_{\parallel} - \Pi_1(q^2)\Pi_2(q^2)\Delta^2_{\perp} + ...] \]

\[ - \frac{1}{q^4}[\Pi_0(q^2)\Delta^2_{\perp}\Delta^2_{\parallel} + \Pi_1(q^2)\Delta^4_{\parallel} + \Pi_2(q^2)\Delta^4_{\parallel} + ...] \]

\[ - \frac{1}{q^2}[\Delta^2_{\perp} - \frac{\Delta^2_{\parallel}}{q^2} + \frac{\Delta^2_{\parallel}\Delta^2_{\perp}}{q^2} + ...], \] (5.11)

so that this formal expansion contains three different types of terms. The first type are the terms which contain only different combinations of \( \Pi_m(q^2) \) (they are not multiplied by inverse powers of \( q^2 \)); the third type of terms contains only different combinations of \( \Delta^2_{m}/q^2 \). The second type of terms contains the so-called mixed terms, containing the first and third types of terms in different combinations. The two last types of terms are multiplied by the corresponding powers of \( 1/q^2 \). Evidently, such structure of terms will be present in each iteration term for the full gluon form factor. However, any of the mixed terms can be split exactly into the first and third types of terms by keeping the necessary number of terms in the Taylor expansions in powers of \( q^2 \) for \( \Pi_m(q^2) \), which are regular functions at small \( q^2 \). Thus the IR structure of the full gluon form factor (which just is our primary goal to establish) is determined not only by the third type of terms. It gains contributions from the mixed terms as well.

Let us present the above-mentioned Taylor expansions as follows:

\[ \Pi_m(q^2) = \Pi_m(0) + (q^2/\mu^2)\Pi^{(1)}_m(0) + (q^2/\mu^2)^2\Pi^{(2)}_m(0) + O_m(q^6), \] (5.12)

since for the third iteration we need to use the Taylor expansions up to this order (here \( \mu^2 \) is some fixed mass squared (not to be mixed up with the tensor index)). For example, the mixed term \( (1/q^2)\Pi_2(q^2)\Delta^2_{\perp} \) should be split as

\[ \frac{\Delta^2_{\perp}}{q^2}\Pi_2(q^2) = \frac{\Delta^2_{\perp}}{q^2} \left[ \Pi_2(0) + (q^2/\mu^2)\Pi^{(1)}_2(0) + O(q^4) \right] = \frac{\Delta^2_{\perp}}{q^2} \Pi_2(0) + a_1\Pi^{(1)}_2(0) + O(q^2). \] (5.13)

Here and everywhere below \( a_m = (\Delta^2_{m}/\mu^2), \ m = 0, 1, 2, 3, ... \) are the dimensionless constants. The first term now is to be shifted to the third type of terms and combined with the term \( -(1/q^2)\Delta^2_{\perp} \), while the second term \( a_1\Pi^{(1)}_2(0) + O(q^2) \) is to be shifted to the first type of terms. All other mixed terms of similar structure should be treated absolutely in the same way. For the mixed term \( -(1/q^2)\Pi_0(q^2)\Delta^2_{\parallel}\Delta^2_{\perp} \), one has...
Then, for example the mixed term \(\frac{-\Delta^2 \Delta^3}{q^4} \Pi_0(q^2)\) is correctly implemented (see also Refs. [10, 16]). The mathematical formalism for this purpose, namely the distribution theory (DT) [15] to which the DRM [11] should be applied. IR singularities due to their novelty and genuine NP character. Fortunately, there already exists a well-elaborated framework for this purpose, namely the distribution theory (DT) [15] to which the DRM [11] should be applied.

The INP part of the full gluon propagator is characterized by the presence of severe power-type (or equivalently NP) IR singularities \(q^2\), \(k = 0, 1, 2, 3, \ldots\). So these IR singularities are defined as more singular than the power-type

\[
\Pi_m(q^2) \Pi_n(q^2) = \Pi_{mn}(q^2) = \Pi_{mn}(0) + (q^2/\mu^2)\Pi_{mn}^{(1)}(0) + (q^2/\mu^2)^2\Pi_{mn}^{(2)}(0) + O(q^4).
\]

(5.15)

Then, for example the mixed term \((-1/q^2)\Pi_0(q^2)\Pi_1(q^2)\Delta_2^2\) can be split as

\[
\frac{-\Delta^2}{q^2} \Pi_0(q^2) \Pi_1(q^2) = -\Delta^2 \frac{\Pi_0(q^2) + (q^2/\mu^2)\Pi_{01}^{(1)}(0) + O(q^4)}{q^2} = -\Delta^2 \Pi_{01}(0) - a_2\Pi_{01}^{(1)}(0) + O(q^4),
\]

(5.16)

so again the first term should be shifted to the third type of terms and combined with the terms containing the corresponding powers of \(1/q^2\), while other terms are to be shifted to the first type of terms.

Completing this exact splitting/shifting procedure in the expansion (5.11), one can in general represent it as follows:

\[
d(q^2) = \left(\frac{\Delta^2}{q^2}\right) B_1(\lambda, \alpha, \xi, g^2) + \left(\frac{\Delta^2}{q^2}\right)^2 B_2(\lambda, \alpha, \xi, g^2) + \left(\frac{\Delta^2}{q^2}\right)^3 B_3(\lambda, \alpha, \xi, g^2) + f_3(q^2) + \ldots,
\]

(5.17)

where we used notations (5.2), since the coefficients of the above-used Taylor expansions depend in general on the same set of parameters: \(\lambda, \alpha, \xi, g^2\). The invariant function \(f_3(q^2)\) is dimensionless and regular at small \(q^2\); otherwise it remains arbitrary. The generalization on the next iterations is almost obvious. Let us only note that in this case more terms in the corresponding Taylor expansions should be kept "alive".

B. The exact structure of the general iteration solution

Substituting the generalization of the expansion (5.17) on all iterations and omitting the tedious algebra, the general iteration solution of the gluon SD equation (3.22) for the regularized full gluon propagator (3.16) can be exactly decomposed as the sum of the two principally different terms as follows:

\[
D_{\mu\nu}(q; \Delta^2) = D_{\mu\nu}^{\text{NP}}(q; \Delta^2) + D_{\mu\nu}^{\text{PT}}(q) = i T_{\mu\nu}(q) \frac{\Delta^2}{(q^2)^2} \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \sum_{m=0}^{\infty} \Phi_{k,m}(\lambda, \alpha, \xi, g^2)
\]

\[
+ i \left[T_{\mu\nu}(q) \sum_{m=0}^{\infty} A_m(q^2) + \xi L_{\mu\nu}(q) \right] \frac{1}{q^2},
\]

(5.18)

where the superscript "NP" stands for the intrinsically NP part of the full gluon propagator. We distinguish between the two terms in Eq. (5.18) by the character of the corresponding IR singularities and the explicit presence of the mass gap (see below). Let us emphasize that the general problem of convergence of the formally regularized series (5.18) is irrelevant here. Anyway, the problem how to remove all types of the UV divergences (overlapping [14] see some remarks below as well) and overall [1, 2, 6, 7, 8, 9] is a standard one. Our problem will be how to deal with severe IR singularities due to their novelty and genuine NP character. Fortunately, there already exists a well-elaborated mathematical formalism for this purpose, namely the distribution theory (DT) [15] to which the DRM [11] should be correctly implemented (see also Refs. [10, 11]).
IR singularity of the free gluon propagator \((q^2)^{-1}\), which thus can be defined as the PT IR singularity. The INP part depends only on the transversal degrees of freedom of gauge bosons. Though its coefficients \(\Phi_{k,m}(\lambda, \alpha, \xi, g^2)\) may explicitly depend on the gauge-fixing parameter \(\xi\), the structure of this expansion itself does not depend on it. It vanishes as the mass gap goes formally to zero, while the PT part survives. The INP part of the full gluon propagator in Eq. (5.18) is nothing but the corresponding Laurent expansion in integer powers of \(q^2\) accompanied by the corresponding powers of the mass gap squared and multiplied by the sum over the \(q^2\)-independent factors, the so-called residues \(\Phi_k(\lambda, \alpha, \xi, g^2) = \sum_{m=0}^{\infty} \Phi_{k,m}(\lambda, \alpha, \xi, g^2)\). The sum over \(m\) indicates that an infinite number of iterations (all iterations) of the corresponding regularized skeleton loop integrals invokes each severe IR singularity labelled by \(k\). It is worth emphasizing that now this Laurent expansion cannot be summed up into anything similar to the initial Eq. (3.17), since its residues at poles gain additional contributions due to the splitting/shift ing procedure, i.e., they become arbitrary. However, this arbitrariness is not a problem, because severe IR singularities should be treated by the DRM correctly implemented into the DT. For this the dependence of the residues on their arguments is all that matters and not their concrete values. The PT part of the full gluon propagator, which has only the PT IR singularity, remains undetermined. In the PT part the sum over \(m\) again indicates that all iterations contribute to the PT gluon form factor \(d^{PT}(q^2) = \sum_{m=0}^{\infty} A_m(q^2)\). What we know about \(A_m(q^2)\) functions is only that they are regular functions at small \(q^2\); otherwise remaining arbitrary but \(d^{PT}(q^2)\) should satisfy AF at large \(q^2\). This is the price we have paid to fix exactly the functional dependence of the INP part of the full gluon propagator. Just this part gives rise to the dominant contributions to the numerical values of physical quantities in low-energy QCD (see below as well). In Refs. [10, 16, 17] we came to the same structure (5.18) but in a rather different way.

Both terms in Eq. (5.18) are valid in the whole energy/momentum range, i.e., they are not asymptotics. At the same time, we have achieved the exact separation between the two terms responsible for the NP (dominating in the IR \((q^2 \to 0)\)) and the nontrivial PT (dominating in the UV \((q^2 \to \infty)\)) dynamics in the true QCD vacuum. It is worth emphasizing once more that we exactly distinguish between the two terms in Eq. (5.18) by the character of the corresponding IR singularities. This first necessary condition includes the existence of a special regularization expansion for severe (i.e., NP) IR singularities, while for the PT IR singularity it does not exist [10, 15]. The second sufficient condition is the explicit presence of the mass gap (when it formally goes to zero then the PT phase survives only). So the above-mentioned separation is not only exact but unique as well. Evidently, it is only possible on the basis of the corresponding decomposition of the full gluon form factor in Eq. (3.16) as follows:

\[
d(q^2) = d(q^2) - d^{PT}(q^2) + d^{PT}(q^2) = d^{NP}(q^2) + d^{PT}(q^2).
\]  

As explained above this separation is exact and unique within the general iteration solution. Due to the character of the IR singularity the longitudinal component of the full gluon propagator should be included into its PT part, so its INP part becomes automatically transversal.

In summary, the general iteration solution (5.18) is inevitably severely singular in the IR limit \((q^2 \to 0)\), and this does not depend on the special gauge choice (see discussion below as well).

C. Remarks on overlapping divergences

The mass gap which appears first in the gluon SD equation (3.13) is the main object we have worried about to demonstrate explicitly its crucial role within our approach. Let us make, however, a few remarks in advance. As it follows from the standard gluon SD equation (3.13), the corresponding equation for the gluon self-energy looks like

\[
D^{-1}(q) = D_0^{-1}(q) - q^2 \Pi(q^2; D) - \Delta^2(\lambda; D),
\]

where we omit the tensor indices as well as the longitudinal part of the subtracted gluon self-energy, for simplicity. In order to unravel overlapping UV divergence problems in QCD, the necessary number of the differentiation with respect to the external momentum should be done first (in order to lower divergences). Then the point-like vertices, which are present in the corresponding skeleton loop integrals should be replaced by their full counterparts via the corresponding integral equations. Finally, one obtains the corresponding SD equations which are much more complicated than the standards ones, containing different scattering amplitudes, which skeleton expansions are, however, free from the above-mentioned overlapping divergences. Of course, the real procedure [14] (and references therein) is much more tedious than briefly described above. However, even at this level, it is clear that by taking derivatives with respect to the external momentum \(q\) in the SD equation for the gluon self-energy (5.20), the main initial information on the mass gap will be totally lost. Whether it will be somehow restored or not at the later stages of the renormalization program is not clear at all. Thus in order to remove overlapping UV divergences (“the water”) from the SD equations
and skeleton expansions, we are in danger to completely loose the information on the dynamical source of the mass gap ("the baby") within our approach. In order to avoid this danger and to be guaranteed that no any dynamical information are lost, we are using the standard gluon SD equation (3.13). The presence of any kind of UV divergences (overlapping and usual (overall)) in the skeleton expansions will not cause any problems in order to detect the mass gap responsible for the IR structure of the true QCD vacuum. In other words, the direct iteration solution of the standard gluon SD equation (3.13) or equivalently (3.22) is reliable to realize a mass gap, and thus to make its existence perfectly clear. The problem of convergence of such regularized skeleton loop series which appear in Eq. (5.18) is completely irrelevant in the context of the present investigation. Anyway, we keep any kind of UV divergences under control within our method, since we are working with the regularized quantities. At the same time, the existence of a mass gap responsible for the IR structure of the full gluon propagator does not depend on whether overlapping divergences are present or not in the SD equations and corresponding skeleton expansions. All this is the main reason why our starting point is the standard gluon SD equation (3.13) for the unrenormalized Green’s functions (this also simplifies notations). See discussion below as well in order to understand why the problem of overlapping divergences is not important for us. Roughly speaking, if one starts from the UV renormalized equations from the very beginning then the information about mass gap will be totally lost. So, one should start from the unrenormalized equation (but for the regularized quantities), then to release a mass gap as it was described. The next step is to perform the IR renormalized program within the general iteration solution, and on the last step to perform the UV renormalization program.

VI. DISCUSSION

It is worth recalling now that in the NP approach to QCD the ghosts are already not sufficient to guarantee the cancellation of unphysical degrees of freedom of gauge bosons. The standard way to make the full gluon propagator purely transversal is to choose the Landau gauge $\xi = 0$ from the very beginning. The system of the SD equations and the corresponding Green’s functions, which should satisfy them, is explicitly gauge-dependent. So in principle to choose gauge by hand at this level should not be a problem. The only request is that the $S$-matrix elements, describing the corresponding quantities and processes in low-energy QCD, should not depend explicitly on the gauge choice. However, as a subject for discussion let us formulate here a general method how to make the gluon propagator relevant for NP QCD to be automatically transversal.

A. The necessity of the subtractions

Many important quantities in QCD, such as the gluon and quark condensates, the topological susceptibility, the Bag constant (which is just the difference between the PT and NP vacuum energy densities, see below an example 2), etc., are defined only beyond the PT [18, 19, 20]. This means that they are determined by such $S$-matrix elements (correlation functions) from which all types of the PT contributions should be, by definition, subtracted.

It is worth emphasizing that such type of the subtractions are inevitable also for the sake of self-consistency. In low-energy QCD there exist relations between different correlation functions, for example, the Witten-Veneziano (WV) and Gell-Mann-Oakes-Renner (GMOR) formulae. The former [21, 22] relates the pion decay constant and the mass of the $\eta'$ meson to the topological susceptibility. The latter [19, 23] relates the chiral quark condensate to the pion decay constant and its mass. The famous trace anomaly relation (see, for example Refs. [19, 22] and references therein) relates the Bag constant (which is the truly NP vacuum energy density, apart from the sign) to the gluon and quark condensates. Defining thus the topological susceptibility and the gluon and quark condensates by the subtraction of all types of the PT contributions, it would not be self-consistent to retain them in the correlation function, determining the pion decay constant, and in the expressions for the pion and $\eta'$ meson masses.

A few additional remarks about the subtraction of the PT contributions are in order. Let us remind that in lattice QCD [1, 2, 23] such kind of an equivalent procedure also exists. In order to prepare an ensemble of lattice configurations for the calculation of any NP quantity or to investigate some NP phenomena, the excitations and fluctuations of gluon fields of the PT origin and magnitude should be "washed out" from the vacuum. This goal is usually achieved by using "Perfect Actions", "cooling", "cycling", etc., (see, for example, Refs. [3, 4] and references therein). Evidently, this is very similar to our method in continuous QCD (for details see below).

From QCD sum rules [19] it is well known that AF is stopped by power-type terms reflecting the growth of the coupling in the IR. Approaching the deep IR region from above, the IR sensitive contributions were parameterized in terms of a few quantities (the gluon and quark condensates, etc.), while the direct access to NP effects (i.e., to the deep IR region) was blocked by the IR divergences [19, 25]. In order to calculate the gluon condensate the corresponding subtraction of the PT gluon propagator integrated out over the deep IR region (where it certainly fails) should be also.
done (see discussion given by Shifman in Ref. [4]). In order to correctly calculate the gluon condensate by analytic methods the necessity of the subtraction of the PT part of the effective coupling constant (integrated out) has been explicitly shown in recent papers [26, 27] as well.

There also exists very serious argument in favor of inevitability of the above-discussed subtractions at all levels and all types in order to fix the gauge of truly NP QCD. In his pioneering paper [28] Gribov has investigated the quantization problem of non-Abelian gauge theories using the functional integral representation of the generating functional for non-Abelian gauge fields. It has been explicitly shown that the standard Fadeev-Popov (FP) prescription fails to fix the gauge uniquely and therefore should be modified, i.e., it is not enough to eliminate arbitrary degrees of freedom from the theory. In other words, there is an ambiguity in the gauge-fixing of non-Abelian gauge fields (the so-called Gribov ambiguity (uncertainty), which results in Gribov copies and vice versa). To resolve this problem Gribov has explicitly demonstrated that the modification reduces simply to an additional limitation on the integration range in the functional space of non-Abelian gauge fields, which consists in integrating only over the fields for which the FP determinant is positive [28] (introducing thus the so-called Gribov horizon in the functional space, see also Ref. [29]). As emphasized by Gribov, this affects the IR singularities of the PT and results in a linear increase of the charge interaction at large distances (see also remarks and discussion below in subsection D).

### B. Restoration of the transversality of gauge bosons

Anyway, to calculate correctly any truly NP quantity in low-energy QCD from first principles one has to begin with making subtractions at the fundamental quark-gluon level. First of all, it is necessary to fix a scale responsible for the NP dynamics in the system. The second step is to set it to zero in order to recover the corresponding PT phase in the system. In our case for the NP gluon propagator the formal PT limit is \( \Delta^2 = 0 \). So from Eqs. (3.16) and (3.17) the PT gluon propagator becomes

\[
D_{\mu\nu}^{PT}(q) = i \left\{ T_{\mu\nu}(q)d^{PT}(q^2) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2},
\]

where

\[
d^{PT}(q^2) = \frac{1}{1 + \Pi(q^2; D^{PT})},
\]

and from the gluon SD equation (3.22) in this limit, one recovers the corresponding gluon SD equation as follows:

\[
D_{\mu\nu}^{PT}(q) = D_{\mu\nu}^0(q) - T_{\mu\sigma}(q)\Pi(q^2; D^{PT})D_{\nu\sigma}^{PT}(q),
\]

which, of course, coincides with Eq. (3.1), since \( \Pi_{\sigma\rho}(q) = \Pi_{\rho\sigma}(q) \) in this limit (see Eq. (3.7)).

The truly NP gluon propagator is to be defined as follows:

\[
D_{\mu\nu}^{NP}(q; \Delta^2) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}(q; \Delta^2 = 0) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}^{PT}(q),
\]

so that the full gluon propagator becomes an exact sum of the two different terms

\[
D_{\mu\nu}(q; \Delta^2) = D_{\mu\nu}^{NP}(q; \Delta^2) + D_{\mu\nu}^{PT}(q).
\]

The principal difference between the full gluon propagator \( D_{\mu\nu}(q; \Delta^2) \) and the truly NP gluon propagator \( D_{\mu\nu}^{NP}(q; \Delta^2) \) is that the latter one is free from the PT "contaminations", while the former one, being also NP, is "contaminated" by the PT contributions. Since the PT limit is uniquely defined in our approach, the separation between the truly NP and PT gluon propagators is uniquely defined as well. So from Eq. (6.5) it follows that when the mass gap, which is responsible for the NP dynamics, is set to zero \( \Delta^2 = 0 \), then only the PT phase survives, i.e., the full gluon propagator is reduced to its nontrivial PT counterpart. That is a reason why we call \( \Delta^2 = 0 \) the PT limit.

Both terms in the exact decomposition (6.5) are valid in the whole energy/momentum region, i.e., they are not asymptotics. At the same time, we achieved the exact separation between the two terms responsible for the NP (dominating in the IR \( (q^2 \to 0) \)) and the nontrivial PT (dominating in the UV \( (q^2 \to \infty) \)) dynamics in the true QCD
vacuum. This is so indeed because the PT limit $\Delta^2 = 0$ is equivalent to the UV limit $q^2 \to \infty$ (see Eq. (3.17)), which means in its turn that the PT limit is in agreement with AF within our approach. Thus if in any model gluon propagator a scale responsible for the NP dynamics cannot be fixed explicitly, then in order to recover the truly NP part its behavior at infinity should be subtracted. In this case, however, the separation between the NP and PT phases may not be unique. Also, we distinguish between the two different phases in QCD not by the strength of the coupling constant, but by the presence of a mass gap (in this case the coupling constant plays no any role as it follows from our consideration).

From the definition (6.4) and all the above displayed relations, the truly NP gluon propagator finally becomes

$$D_{\mu\nu}^{NP}(q; \Delta^2) = iT_{\mu\nu}(q)d^{NP}(q^2; \Delta^2)\frac{1}{q^2},$$  \hspace{1cm} (6.6)

where the truly NP gluon form factor is

$$d^{NP}(q^2; \Delta^2) = \frac{(q^2; D^{PT}) - \Pi(q^2; D) - \frac{\Delta^2(\lambda; D)}{q^2}}{[1 + \Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2}][1 + \Pi(q^2; D^{PT})]},$$  \hspace{1cm} (6.7)

and it may be treated as the truly NP effective charge as well. Evidently, the truly NP gluon propagator is manifestly transversal, i.e., does not depend explicitly on the gauge-fixing parameter. This results in the fact that both the full gluon propagator (3.16) and the PT gluon propagator (6.1) satisfy the color gauge invariance condition (3.15) within our approach. Otherwise the truly NP gluon propagator cannot be really transversal. Let us repeat once more that Eq. (6.7) is not a solution, but it displays a general structure of the truly NP gluon form factor. To establish a possible type of solution, however, a much more convenient starting point is the full gluon form factor (see Eq. (3.17)). It will give a possible type of solution for Eq. (6.7) as well. In the PT limit $\Delta^2 = 0$ the full gluon propagator coincides with its nontrivial PT part, so the ghosts can now fulfill their role to cancel unphysical degrees of freedom of gauge bosons, since the transversality of the gluon self-energy is to be restored in this limit (see Eqs. (3.7) and (3.11)).

In summary, our prescription as how to guarantee the transversality of the gluon propagator relevant for calculations of the truly NP physical quantities and processes from first principles in low-energy QCD is to be briefly formulated as follows:

(i). The corresponding scale responsible for the NP dynamics (the mass gap) is to be fixed. The dependence of the full gluon propagator on the mass gap should be only regular. If the dependence of the full gluon propagator on a some scale parameter is singular, then it cannot be chosen as the mass gap. In particular, this means that the invariant function $\Pi(q^2; D)$ depends (if any) on the mass gap only regularly.

(ii). The exact decomposition (6.5) should be provided.

(iii). All terms, containing $D^{PT}_{\mu\nu}(q)$, should be discarded from all the relations, equations, etc. i.e., to put $D_{\mu\nu}(q; \Delta^2) = D^{NP}_{\mu\nu}(q; \Delta^2)$ everywhere. Let us note in advance that in the case of the general iteration solution the full gluon propagator should be replaced by its INP part, i.e., $D_{\mu\nu}(q; \Delta^2) = D^{INP}_{\mu\nu}(q; \Delta^2)$ (see below).

C. The truly NP massive solution

In accordance with our method how to define the truly NP gluon propagator described above, we have to establish first the scale responsible for the NP dynamics in the system under consideration and then to set it to zero. For the massive-type solution (4.6) the gluon mass may serve as a scale responsible for its NP dynamics, since in the PT limit $\Delta^2(\lambda, \xi) = 0$, the effective gluon mass is also zero, $m_g^2(\lambda, \xi) = 0$. Then the nontrivial PT part of the full gluon propagator becomes

$$D^{PT}_{\mu\nu}(q) = iT_{\mu\nu}(q)\frac{Z_3^{PT}}{q^2[1 + \Pi(q^2; 0)]} + i\xi L_{\mu\nu}(q)\frac{1}{q^2},$$  \hspace{1cm} (6.8)

Here the PT gluon propagator pole is going to zero and the gluon renormalization constant becomes a standard one, namely

$$Z_3^{PT} = [1 + \Pi(0; \xi)]^{-1}.$$  \hspace{1cm} (6.9)
The truly NP gluon propagator defined in Eq. (6.4) as the difference between the full gluon propagator (4.6) and the PT gluon propagator (4.8) becomes

\[
D^{NP}_{\mu\nu}(q; m_g^2) = iT_{\mu\nu}(q) \left[ \frac{Z_3}{(q^2 + m_g^2)[1 + \Pi^R(q^2; m_g^2)]} - \frac{Z^{PT}}{q^2[1 + \Pi^R(q^2; 0)]} \right].
\]  

(6.10)

In principle, this type of solution demonstrates the propagation of the two different purely transversal gluons: massive, i.e., the NP (first term) and massless, i.e., the nontrivial PT (second term) ones. Concluding, let us make one speculative remark. Such structure of the purely transversal gluon propagator can be a hint of a similar mechanism of a mass creating without still missing Higgs particle in Standard model. So massless gluons may correspond to the photons, while massive solutions (let us remind that there is also a second solution, see section IV) may correspond to \(Z\) and \(W^\pm\) bosons.

D. The INP iteration solution

In the general iteration solution (5.18) we distinguish first of all between its two parts by the character of the IR singularities (see above). This means the automatical presence of a mass gap, since any deviation of the behavior of the full gluon propagator from the free one requires the existence of a mass scale parameter. Thus for this solution only the explicit presence of a mass gap becomes the second sufficient condition to separate the different terms in the full gluon propagator. For all other possible types of solutions (for example, for the above-described massive-type solution) the presence of a mass gap is necessary and sufficient for this purpose. Making the subtraction in accordance with this exact criterion in Eq. (5.18), one concludes that the role of the gluon propagator responsible for the NP dynamics should be assigned to its INP part, which becomes truly NP, indeed, i.e., it vanishes in the PT limit in accordance with the general prescription. Let us recall that the separation between the INP and PT parts in the full gluon propagator is unique as well. It is based on the existence of a special regularization expansion for severe (i.e., NP) IR singularities in the DT complemented by the DRM. For the PT IR singularity such kind of the expansion does not exist \([10, 15]\). This emphasizes the special character of the nonlinear iteration solution. Thus, one obtains

\[
D^{INP}_{\mu\nu}(q; \Delta^2) = iT_{\mu\nu}(q)d^{INP}(q^2; \Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2} \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \Phi_k(\lambda, \alpha, \xi, g^2),
\]

(6.11)

where

\[
\Phi_k(\lambda, \alpha, \xi, g^2) = \sum_{m=0}^{\infty} \Phi_{k,m}(\lambda, \alpha, \xi, g^2),
\]

(6.12)

and the effective charge in this case is to be defined as follows:

\[
d^{NP}(q^2; \Delta^2) \equiv \alpha_k^{NP}(q^2; \Delta^2) = \frac{\Delta^2}{q^2} \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \Phi_k(\lambda, \alpha, \xi, g^2).
\]

(6.13)

Evidently, The INP part depends only on the transversal degrees of freedom of gauge bosons as it is required, by definition. Also, its functional dependence is uniquely fixed up to the expressions for the residues \(\Phi_k(\lambda, \alpha, \xi, g^2)\), and it is valid in the whole energy/momentum range. At large momentum it looks formally as an Operator Product Expansion (OPE) of the gluon propagator. However, it is completely suppressed in this limit \((q^2 \to \infty)\) in comparison with the PT part of the full gluon propagator. As underlined above, what we worried about is its behavior in the IR limit \((q^2 \to 0)\), which can be correspondingly treated within the DRM correctly implemented into the DT. Only this investigation will allow one to deduce whether the severe IR structure of the gluon propagator survives or not. Up to this moment, however, the INP part is a Laurent expansion in powers of \(q^2\), multiplied by the corresponding powers of the mass gap, and it starts from the simplest severe (or equivalently NP) power-type IR singularity which is only one possible in four-dimensional QCD, namely \((q^2)^{-2}\) \([10, 15]\). Let us also remind that no approximation/truncations are made for the corresponding regularized constant skeleton loop integrals, contributing to the residues \(\Phi_k(\lambda, \alpha, \xi, g^2)\) over all iterations.
The unavoidable existence of the INP part of the full gluon propagator within its general iteration solution (5.18) makes the principal difference between non-Abelian QCD and Abelian QED, where such kind of term in the full photon propagator is certainly absent (in the former theory there is direct coupling between massless gluons which finally leads to the dynamical generation of a mass gap, while in the latter one there is no direct coupling between massless photons). Precisely this term violates the cluster properties of the Wightman functions [30], and thus validates the Strocchi theorem [31], which allows for such IR singular behavior of the full gluon propagator. Contrary to QCD, the full photon propagator may have only the PT-type IR singularity (see Eq. (2.8) above).

Though the residues at poles shown in Eq. (6.2) may explicitly depend on the gauge-fixing parameter \(\xi\), the zero momentum modes enhancement (ZMME) effect itself (represented in the INP part) does not depend on it. Since we do not specify explicitly the value of the gauge-fixing parameter \(\xi\), the ZMME effect takes place at any value. This is very similar to AF. It is well known that the exponent which determines the logarithmic deviation of the full gluon propagator from the free one in the UV region \((q^2 \gg \Lambda_{QCD}^2)\) explicitly depends on the gauge-fixing parameter. At the same time, AF itself does not depend on it, i.e., it takes place at any \(\xi\).

The QCD Lagrangian does not contain a mass gap. However, we discovered that the mass scale parameter responsible for the NP dynamics in the IR region should exist in the true QCD ground state. At the level of the gluon SD equation it is hidden in the skeleton loop contributions into the gluon self-energy. Within the general iteration solution it explicitly shows up (and hence the corresponding severe IR singularities) when the gluon momentum goes to zero. At the fundamental quark-gluon (i.e., Lagrangian) level the main dynamical source of a mass gap is the self-interaction of massless gluons, i.e., the NL dynamics of QCD. The triple gluon vertex vanishes when all the gluon momenta involved go to zero \((T_3(0, 0) = 0)\), while its four-gluon counterpart survives \((T_4(0, 0, 0, 0) \neq 0)\). Then one may think that the latter one plays more important role than the former one in the IR structure of the gluon SD equation and thus in the arising of the mass gap mainly from quartic gluon potential (Feynman [32] has also arrived at the same conclusion but on a different basis). The skeleton tadpole term

\[ \Pi_t(D) = g^2 \int \frac{id^4q}{(2\pi)^4} T_4^0(q_1, 0, -q_1)D(q_1), \]  

(6.14)

(for simplicity, we omit the tensor and color indices), which explicitly violates the transversality of the gluon self-energy (see Eqs. (3.5) and (3.6)), contains only the four-gluon vertex. So there is no doubt in the important role of quartic gluon potential in NP QCD, indeed.

Thus the true QCD vacuum is really beset with severe IR singularities. Within the general iteration solution they should be summarized (accumulated) into the full gluon propagator and effectively correctly described by its structure in the deep IR domain, exactly represented by its INP part. It is worth emphasizing here that due to the arbitrariness of the above-mentioned residues \(\Phi_k(\lambda, \alpha, \xi, g^2)\) at poles \((q^2)^{-2-k}\), \(k = 0, 1, 2, 3, \ldots\), there is no smooth gluon propagator within the general iteration solution. The second step is to assign a mathematical meaning to the integrals, where such kind of severe IR singularities will explicitly appear, i.e., to define them correctly in the IR region [10, 15]. Just this IR violent behavior makes QCD as a whole an IR unstable theory, and therefore it may have no IR stable fixed point, indeed [1], which means that QCD itself might be a confining theory without involving some extra degrees of freedom [33, 34, 35, 36, 37, 38, 39, 40].

The INP part of the full gluon propagator (6.11) depends only on the transversal ("physical") degrees of freedom of gauge bosons, by construction, due to the above-described in detail subtraction procedure at the gluon propagator level. All the problems with the gauge-fixing discovered by Gribov [28] in the functional space should be attributed to the PT part of the gluon propagator within its general iteration solution. Making the above-mentioned subtraction, in order to proceed to the gluon propagator relevant for NP QCD, we thus will make it free of the gauge-fixing ambiguity in the momentum space (the implicit dependence of the residues on the gauge-fixing parameter is not dangerous). This once more emphasizes the necessity and importance of making subtraction in order to make the theory at the fundamental quark-gluon level free of this problem, which otherwise will plague the dynamics of any essentially NL gauge systems [29].

1. An example 1

It is instructive to describe the procedure of the PT subtractions of all types in more detail within the general iteration solution. For example, the gluon condensate can be formally defined (up to some unimportant for our purpose numerical factors) as the effective coupling integrated out, namely

\[ \langle 0|G^2|0\rangle \sim \int_0^\infty \alpha_s(q^2)q^2dq^2. \]  

(6.15)
In accordance with Eq. (5.19) the effective charge is to be identically decomposed as follows: $\alpha_s(q^2) = \alpha_s(q^2) - \alpha_s^{PT}(q^2) + \alpha_s^{PT}(q^2) = \alpha_s^{INP}(q^2) + \alpha_s^{PT}(q^2)$. Let us introduce further the effective scale $q_{eff}^2$, which separates the NP region from the PT one (it looks something like the above-mentioned Gribov horizon, but in the much simpler momentum space). Then the initial integral becomes the sum of four terms

$$
\langle 0\vert G^2\vert 0 \rangle \sim \int_0^\infty \alpha_s(q^2)q^2dq^2 = \int_0^{q_{eff}^2} \alpha_s^{INP}(q^2)q^2dq^2 + \int_{q_{eff}^2}^\infty \alpha_s^{INP}(q^2)q^2dq^2
$$

$$
+ \int_0^{q_{eff}^2} \alpha_s^{PT}(q^2)q^2dq^2 + \int_{q_{eff}^2}^\infty \alpha_s^{PT}(q^2)q^2dq^2.
$$

(6.16)

Within our approach the NP region includes not only the deep IR one but even more than that (it may includes some substantial part of the intermediate region as well). In the INP QCD all the three last integrals reproduce the different types of the PT contributions despite some of them might be finite numbers, nevertheless. So in our theory the "physical" gluon condensate is to be defined by subtracting all these integrals from the initial one, i.e., one has to put

$$
\langle 0\vert G^2\vert 0 \rangle_{ph} \sim \int_0^\infty \alpha_s(q^2)q^2dq^2 - \int_0^{q_{eff}^2} \alpha_s^{INP}(q^2)q^2dq^2 - \int_0^{q_{eff}^2} \alpha_s^{PT}(q^2)q^2dq^2 - \int_{q_{eff}^2}^\infty \alpha_s^{PT}(q^2)q^2dq^2
$$

$$
= \int_0^{q_{eff}^2} \alpha_s^{INP}(q^2)q^2dq^2,
$$

(6.17)

and thus it is free of all types of the PT contributions, indeed, since the right-hand-side of this equation is the INP effective charge integrated out over the NP region ($0 \leq q^2 \leq q_{eff}^2$). Of course, this expression cannot be used for actual calculation of the gluon condensate, since the INP effective charge (6.13) is not yet IR and UV renormalized. However, for the preliminary actual calculations of the gluon condensate free of all types of the PT contributions and not using a weak coupling limit solution to the $\beta$ function see our papers [18, 26] (and references therein).

In QCD sum rules the INP effective charge $\alpha_s^{INP}(q^2)$ is not known. So omitting it in Eq. (6.16), one obtains

$$
\langle 0\vert G^2\vert 0 \rangle \approx \int_0^\infty \alpha_s(q^2)q^2dq^2 - \int_0^{q_{eff}^2} \alpha_s^{PT}(q^2)q^2dq^2 - \int_{q_{eff}^2}^\infty \alpha_s^{PT}(q^2)q^2dq^2.
$$

(6.18)

Omitting further the second integral (see the discussion given by Shifman in Ref. [4]), which is nothing else but the PT effective charge integrated out over the NP region, one finally obtains

$$
\langle 0\vert G^2\vert 0 \rangle_{ph} \approx \int_0^\infty \alpha_s(q^2)q^2dq^2 - \int_{q_{eff}^2}^\infty \alpha_s^{PT}(q^2)q^2dq^2,
$$

(6.19)

i.e., in this theory the gluon condensate is again the difference between an infinite initial integral and infinite PT tail (evidently, in this integral it is justified to approximate the full effective charge by its PT counterpart, indeed). So that the above-mentioned difference is finite, and in fact it is a rather good approximation to the exact definition of the gluon condensate within our approach in Eq. (6.17), i.e., to its right-hand-side. Within this formalism the only problem here is the point of the subtraction $q_{eff}^2$ since the separation "hard vs soft" gluon momenta in this theory is not exact, while in our theory it is exactly fixed through the mass gap. Nevertheless, our and QCD sum rules values for the gluon condensate are rather close to each other [20]. Especially very good agreement is achieving when our value is recalculated at the 1 GeV scale, i.e., when we put $q_{eff}^2 = 1 GeV^2$. It is very reasonable value for the effective scale $q_{eff}^2$ separating the NP region from the PT one.

2. An example 2

In close connection with the gluon condensate is one of the main characteristics of the true QCD ground state is the above-mentioned Bag constant. It is just defined as the difference between the PT and NP vacuum energy
 densities (VED). So, we can symbolically put \( B = VED^{PT} - VED \), where \( VED \) is the NP but "contaminated" by the PT contributions (i.e., this is a full \( VED \) like the full gluon propagator). At the same time, in accordance with our method we can continue as follows: \( B = VED^{PT} - VED = VED^{PT} - [VED - VED^{PT} + VED^{PT}] = VED^{PT} - [VED^{INP} + VED^{PT}] = -VED^{INP} > 0 \), since the VED is always negative. Thus the Bag constant is nothing but the INP VED, apart from the sign, by definition, and thus is completely free of the PT "contaminations". For how to correctly define and actually calculate the Bag constant from first principles by making all necessary subtractions at all level see again our paper [26], where the relation between the Bag constant and gluon condensate is explicitly shown as well.

3. An example 3

The chiral quark condensate is formally defined as follows (Euclidean signature):

\[
<0|\bar{q}q|0>_0 = \int \frac{id^4p}{(2\pi)^4} TrS(p),
\tag{6.20}
\]

where the trace over color and Dirac matrices is understood. Here \( S(p) = i[pA(p^2) + B(p^2)] \) is the full quark propagator. After trivial derivation it becomes

\[
<0|\bar{q}q|0>_0 \sim -\int_0^\infty p^2 dp^2 B(p^2),
\tag{6.21}
\]

where and below we will omit all numerical factors as unimportant for our purpose. In accordance with our method the "physical" chiral quark condensate should be defined as follows:

\[
<0|\bar{q}q|0>_{ph} \sim -\int_0^\infty p^2 dp^2 B(p^2) + \int_{q_{eff}^2}^\infty p^2 dp^2 B(p^2) = -\int_{q_{eff}^2}^\infty p^2 dp^2 B(p^2),
\tag{6.22}
\]

where the effective scale \( q_{eff}^2 \) numerically, in principle, differs from the Yang-Mills (YM) effective scale introduced in the previous examples. However, its chosen value \( q_{eff}^2 = 1 \text{ GeV}^2 \) might be a good approximation for full QCD as well. The only thing remaining to do is to substitute for the quark running mass function \( B(p^2) \) solution of the quark SD equation in the chiral limit based on the INP gluon propagator (6.11). For actual preliminary calculations of the chiral quark condensate and other chiral QCD parameters within the INP approach to QCD see our papers [41, 42].

In summary, our consideration in general and these symbolic examples within the INP solution to QCD in particular clearly shows how to correctly calculate the physical observables (and related quantities) from first principles in low-energy QCD. The subtractions of the PT contributions of all types and at all levels are necessary to be made for this purpose. This means that INP QCD is the UV finite theory, by definition, though both renormalization programs are still needed in order to correctly define the corresponding Green’s functions and their solutions [10].

VII. CONCLUSIONS

Our consideration at this stage is necessarily formal, since the mass gap \( \Delta^2 \) remains neither IR nor UV renormalized yet. At this stage it has been only regularized, i.e., \( \Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2) \). However, there is no doubt that it will survive both multiplicative renormalization (MR) programs (which include the corresponding removal of both \( \lambda \) and \( \alpha \) parameters). The UVMR program is not our problem (it is a standard one [1, 2, 6, 7, 8, 9, 14], and anyway, as underlined above, it should follow after IRMR one is performed). Within the INP solution to QCD our problem is the IRMR program in order to render the whole theory finite, i.e., to make it free from all types of severe IR singularities parameterized in terms of the IR regularization parameter as it goes to zero at the final stage. It is not a simple task due to its novelty and really NP character. It requires much more tedious technical work how to correctly implement the DRM into the DT, and it is left to be done elsewhere (for some preliminary aspects of the IRMR program see Ref. [10]).

It is worth noting that the mass gap which appears in the gluon SD equation cannot be in principle the same one which appears in the INP part of the general iteration solution, though we have identified them, for simplicity. Let us denote the renormalized version of the mass gap \( \Delta^2(\lambda; D) \) (i.e., which appears in the gluon SD equation) as \( \Delta^2_{AW} \) and...
call it the Jaffe-Witten (JW) mass gap \( \Delta^2 \) (see theorem below as well). At the same time, the renormalized version of our mass gap \( \Delta^2(\lambda; D^{NP}) \) let us denote as \( \Lambda_{NP}^2 \), then a symbolic relation between them and \( \Lambda_{QCD}^2 \equiv \Lambda_{PT}^2 \) could be written as

\[
\Lambda_{NP}^2 \leftarrow_{\alpha_s \to 0, M_{IR}}^{\alpha_s \to 0, M_{UV} \to \infty} \Delta_{JW}^2 \to \Lambda_{PT}^2.
\]

(7.1)

Here \( \alpha_s \) is obviously the fine structure coupling constant of strong interactions, while \( M_{UV} \) and \( M_{IR} \) are the UV and IR cut-offs, respectively. The right-hand-side limit is well known as the weak coupling regime, and we know how to take it within the renormalization group equations approach \([1, 2, 6, 7, 8, 9]\). However, it would be interesting to understand within our approach how the JW mass gap \( \Delta_{JW}^2 \) may actually become \( \Lambda_{PT}^2 \). The left-hand-side limit can be regarded as the strong coupling regime, and we hope that we have explained here how to begin to deal with it, not solving the gluon SD equation directly, which is a formidable task, anyway. However, there is no doubt that the final goal of this limit, namely, the mass gap \( \Lambda_{NP}^2 \) exists, and should be renormalization group invariant in the same way as \( \Lambda_{QCD} \). It is solely responsible for the large-scale structure of the true QCD ground state, while \( \Lambda_{PT} \) is responsible for the nontrivial PT dynamics there.

Evidently, such kind of relation (7.1) is only possible due to the explicit presence of a mass gap in the gluon SD equation of motion. It leads to the exact separation between the truly NP and nontrivial PT parts (phases) at the level of a single gluon propagator. It also provides a basis for the restoration of the transversality of the gluon propagator relevant for NP QCD by formulating the above-described and demonstrated subtraction prescription. A possible relation between these two phases shown in Eq. (7.1) is a manifestation of "the problems encountered in perturbation theory are not mere mathematical artifacts but rather signify deep properties of the full theory" \[43\].

The message that we are trying to convey is that the nontrivial PT phase in the full gluon propagator indicates the existence of the truly NP one (the INP one within the general iteration solution) and the other way around.

A few years ago Jaffe and Witten have formulated the following theorem \([2]\):

**Yang-Mills Existence And Mass Gap:** Prove that for any compact simple gauge group \( G \), quantum Yang-Mills theory on \( \mathbb{R}^4 \) exists and has a mass gap \( \Delta > 0 \).

Of course, at present to prove the existence of the YM theory with compact simple gauge group \( G \) is a formidable task yet. It is rather mathematical than physical problem. However, in the case of acceptance of our proposal, one of the main results here can be then formulated similar to the above-mentioned JW theorem as follows:

**Mass Gap Existence:** If quantum Yang-Mills theory with compact simple gauge group \( G = SU(3) \) exists on \( \mathbb{R}^4 \), then it has a mass gap \( \Delta > 0 \).

It is important to emphasize that a mass gap has not been introduced by hand. It is hidden in the skeleton loop integrals, contributing to the gluon self-energy, and dynamically generated mainly due to the NL interaction of massless gluon modes. No truncations/approximations and no special gauge choice are made for the above-mentioned regularized skeleton loop integrals. An appropriate subtraction scheme has been applied to make the existence of a mass gap perfectly clear. Within the general iteration solution the mass gap shows up explicitly when the gluon momentum goes to zero. The Lagrangian of QCD does not contain a mass gap, while it explicitly appears in the gluon SD equation of motion. This once more underlines the importance of the investigation of the SD system of equations and identities \([1, 2, 6, 7, 8, 9]\) for understanding the true structure of the QCD ground state. We have established the structure of the regularized full gluon propagator (see Eqs. (3.16) and (3.17)) and the corresponding SD equation (3.22) in the presence of a mass gap.

In order to realize a mass gap, we propose not to impose the transversality condition on the gluon self-energy (see Eq. (3.6), while preserving the color gauge invariance condition (3.15) for the full gluon propagator. This proposal is justified by the NL and NP dynamics of QCD (the constant skeleton tadpole contribution to the gluon self-energy explicitly violates its transversality structure). Such a temporary violation of color gauge invariance/symmetry (TVCGI/S) is completely NP effect, since in the PT limit \( \Delta^2 = 0 \) this effect vanishes. Let us emphasize that we would propose this even if there were no explicit violation of the transversality of the gluon self-energy by the constant skeleton tadpole term. In other words, whether this term is explicitly present or not, but just color confinement (the gluon is not a physical state) gives us a possibility not to impose the transversality condition on the gluon self-energy. The existence of this term is a hint that the above-mentioned transversality might be temporary violated. Since the gluon is not a physical state because of color confinement as mentioned above, the TVCGI/S in QCD has no direct physical consequences. None of physical quantites/processes in low-energy QCD will be directly affected by this proposal.

For the calculations of physical observables from first principles in low-energy QCD we need the full gluon propagator, which transversality has been sacrificed in order to realize a mass gap (despite their general role the ghosts
cannot guarantee its transversality in this case). However, we have already pointed out how the transversality of the gluon propagator relevant for NP QCD is to be restored at the final stage. In accordance with our prescription it becomes automatically transversal, free of the PT contributions ("contaminations"), and it regularly depends on the mass gap, so that it vanishes when the mass gap goes to zero. The role of the first necessary subtraction (6.4) at the fundamental gluon propagator level or (5.19) in the case of INP QCD is to be emphasized. We also briefly described some other types of the subtractions at the hadronic level as well (i.e., when gluon and quark degrees of freedom are to be integrated out).

In QED a mass gap is always in the "gauge prison". It cannot be realized even temporarily, since the photon is a physical state. However, in QCD a door of the "color gauge prison" can be opened for a moment in order to realize a mass gap, because the gluon is not a physical state. A key to this "door" is the constant skeleton tadpole term. On the other hand, this "door" can be opened without key (as any door) by not imposing the transversality condition on the gluon self-energy. So in QED a mass gap cannot be "liberated" from the vacuum, while photons and electrons can be liberated from the vacuum in order to be physical states. In QCD a mass gap can be "liberated" from the vacuum, while gluons and quarks cannot be liberated from the vacuum in order to be physical states. In other words, there is no breakdown of $U(1)$ gauge symmetry in QED because the photon is a physical state. At the same time, a temporary breakdown of $SU(3)$ color gauge symmetry in QCD is possible because the gluon is not a physical state (color confinement).

Let us emphasize one more that no truncations/approximations and no special gauge have been made for the corresponding skeleton loop integrals within our approach, i.e., it is pure NP, by its nature. So on the general ground we have established the existence at least of two different types of solutions for the full gluon propagator in the presence of a mass gap. The so-called general iteration solution (5.18) is always severely singular in the IR ($q^2 \to 0$), i.e., the gluons always remain massless, and this does not depend on the gauge choice (this behavior of the full gluon propagator in different approximations and gauges has been earlier obtained and investigated in many papers, see, for example Ref. [10] and references therein). The massive-type solution (4.6) leads to an effective gluon mass, which explicitly depends on the gauge-fixing parameter, and it cannot be directly identified with the mass gap. Moreover, we were unable to make an effective gluon mass a gauge-invariant as a result of the renormalization, and therefore to assign to it a physical meaning. This solution becomes smooth at $q^2 \to 0$ in the Landau gauge $\xi = 0$ only. Both types of solutions are independent from each other and should be considered on equal footing, since the gluon SD equation is highly NL system. For such kind of systems the number of solutions is not fixed a priori. The UV behavior ($q^2 \to \infty$) of all solutions should be fixed by AF [1]. Due to unsolved yet confinement problem, the IR behavior ($q^2 \to 0$) is not fixed. Only solution of the color confinement problem will decide which type of formal solutions really takes place. At the present state of arts none of them can be excluded [44].

In summary, the behavior of QCD at large distances is governed by a mass gap, possibly realized in accordance with our proposal. The dynamically generated mass gap is usually related to breakdown of some symmetry (for example, the dynamically generated quark mass is an evidence of chiral symmetry breakdown). Here a mass gap is an evidence of the TVCGI/S. In the presence of a mass gap the coupling constant becomes play no role. This is also a direct evidence of the "dimensional transmutation", $g^2 \to \Delta^2(\lambda, \alpha, \xi, g^2)$ [11, 43, 44], which occurs whenever a massless theory acquires masses dynamically. It is a general feature of spontaneous symmetry breaking in field theories. The mass gap has to play a crucial role in the realization of the quantum-dynamical mechanism of color confinement [5].

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APPENDIX A: RENORMALIZATION GROUP EQUATION FOR THE EFFECTIVE CHARGE

It is instructive to make some preliminary remarks, concerning solution of the renormalization group equation for the regularized effective charge, which appears in the general iteration solution (5.18). This equation leads to the determination of the corresponding $\beta$-function, and it is

$$q^2 \frac{d\alpha_s(q^2)}{dq^2} = \beta(\alpha_s(q^2)).$$

(A1)
As we have already established, the effective charge for this solution can be uniquely decomposed as the exact sum of the two principally different terms, namely

\[
\alpha_s(q^2) = \alpha_s(q^2) - \alpha_s^{PT}(q^2) + \alpha_s^{PT}(q^2) = \alpha_s^{INP}(q^2) + \alpha_s^{PT}(q^2),
\]

(A2)

where the explicit expression for the INP part of the effective charge is given by Eq. (6.13), which is valid in the whole energy/momentum range. We also omit the dependence on the mass gap in the effective charge as unimportant. Let us remind that the PT part of the effective charge is a regular function at small \(q^2\), so it can be explicitly present as the corresponding Taylor expansion

\[
\alpha_s^{PT}(q^2) = \sum_{k=0}^{\infty} (q^2/\mu^2)^k \alpha_s^{(k)}(0),
\]

(A3)

where \(\mu^2\) is some fixed mass squared parameter introduced in Eq. (5.12). In principle the coefficients of the Taylor expansion \(\alpha_s^{(k)}(0)\) depend on the same set of parameters as the coefficients \(\Phi_k(\lambda, \alpha, \xi, g^2)\) of the Laurent expansion (6.13), i.e., \(\alpha_s^{(k)}(0) = \alpha_s^{(k)}(\lambda, \alpha, \xi, g^2)\). For convenience, let us introduce short-hand notation \(\Phi_k \equiv \Phi_k(\lambda, \alpha, \xi, g^2)\) as well. Then on account of Eq. (6.13), the effective charge (A2) explicitly becomes

\[
\alpha_s(q^2) = \alpha_s^{INP}(q^2) + \alpha_s^{PT}(q^2) = \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \Phi_k + \sum_{k=0}^{\infty} (q^2/\mu^2)^k \alpha_s^{(k)},
\]

(A4)

Substituting this expression into the renormalization group equation (A1), one obtains the formal solution for the corresponding \(\beta\)-function as follows:

\[
\beta(\alpha_s(q^2)) = -\frac{\Delta^2}{q^2} \sum_{k=0}^{\infty} (1+k)(\Delta^2/q^2)^k \Phi_k + \sum_{k=0}^{\infty} (q^2/\mu^2)^k \alpha_s^{(k)},
\]

(A5)

so the \(\beta\)-function can be also exactly and uniquely decomposed into the two different terms, namely

\[
\beta(\alpha_s(q^2)) = \beta^{INP}(\alpha_s^{INP}(q^2)) + \beta^{PT}(\alpha_s^{PT}(q^2)),
\]

(A6)

where

\[
\beta^{INP}(\alpha_s^{INP}(q^2)) = -\frac{\Delta^2}{q^2} \sum_{k=0}^{\infty} (1+k)(\Delta^2/q^2)^k \Phi_k = -\alpha_s^{INP}(q^2) - \frac{\Delta^2}{q^2} \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \Phi_k,
\]

(A7)

and

\[
\beta^{PT}(\alpha_s^{PT}(q^2)) = \sum_{k=0}^{\infty} (q^2/\mu^2)^k \alpha_s^{(k)}.
\]

(A8)

Let us note that in the NP region (i.e., at small \(q^2\)) from Eq. (A4) one obtains

\[
\alpha_s(q^2) = \alpha_s^{INP}(q^2) + O(1), \quad q^2 \to 0,
\]

(A9)

while from Eqs. (A6) and (A8) it follows

\[
\beta(\alpha_s(q^2)) = \beta^{INP}(\alpha_s^{INP}(q^2)) + O(q^2), \quad q^2 \to 0.
\]

(A10)
Thus, one can conclude that in the NP region the $\beta$-function of the general iteration solution as a function of its argument is determined by its INP part, which is always in the domain of attraction (i.e., negative, see Eq. (A7)) as it is required for the confining theory \[1\].

\[1\] W. Marciano, H. Pagels, Phys. Rep. C 36 (1978) 137.
\[2\] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory (AW, Advanced Book Program, 1995).
\[3\] Confinement, Duality, and Nonperturbative Aspects of QCD, edited by P. van Baal, NATO ASI Series B: Physics, vol. 368 (Plenum, New York, 1997).
\[4\] Non-Perturbative QCD, Structure of the QCD vacuum, edited by K-I. Aoki, O. Miymura and T. Suzuki, Prog. Theor. Phys. Suppl. 131 (1998) 1.
\[5\] A. Jaffe, E. Witten, Yang-Mills Existence and Mass Gap,
\[http://www.claymath.org/prize-problems/\] [http://www.arthurjaffe.com].
\[6\] V.N. Gribov, J. Nyiri, Quantum Electrodynamics, (Cambridge University Press, 2001).
\[7\] J.D. Bjorken, S.D. Drell, Relativistic Quantum Fields, (Mc Graw-Hill Book Company, 1978).
\[8\] C. Itzykson, J.-B. Zuber, Quantum Field Theory, (Mc Graw-Hill Book Company, 1984).
\[9\] T. Muta, Foundations of QCD, (Word Scientific, 1987).
\[10\] V. Gogohia, hep-ph/0311061.
\[11\] G. ’t Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189.
\[12\] L. von Smekal, A. Hauck, R. Alkofer, Ann. Phys. 267 (1998) 1; R. Alkofer, C.S. Fischer, F.J. Llanes-Estrada, Phys. Lett. B 611 (2005) 279.
\[13\] D.V. Shirkov, hep-ph/0208082.
\[14\] M. Baker, Ch. Lee, Phys. Rev. D 15 (1977) 2201.
\[15\] I.M. Gel’fand, G.E. Shilov, Generalized Functions, (Academic Press, New York, 1968), Vol. I.
\[16\] V. Gogohia, Phys. Lett. B 584 (2004) 225.
\[17\] V. Gogohia, Phys. Lett. B 618 (2005) 103.
\[18\] V. Gogohia, Phys. Rev. D 62 (2000) 076008.
\[19\] M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B 147 (1979) 385, 448; V.A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B 191 (1981) 301.
\[20\] I. Halperin, A. Zhitnitsky, Nucl. Phys. B 539 (1999) 166.
\[21\] E. Witten, Nucl. Phys. B 156 (1979) 269; G. Veneziano, Nucl. Phys. B 159 (1979) 213.
\[22\] V. Gogohia, Phys. Lett. B 501 (2001) 60; V. Gogohia, hep-ph/0005302; V. Gogohia, H. Toki, Phys. Rev. D 61 (2000) 036006; ibid. 63 (2001) 079901(E).
\[23\] M. Gell-Mann, R.J. Oakes, B. Renner, Phys. Rev. 175 (1968) 2195.
\[24\] C. Creutz, Quarks, Gluons and Lattice (Cambridge, 1883);
A.S. Kronfeld, hep-ph/0209321.
\[25\] V. I. Zakharov, Int. Jour. Mod. Phys. A 14 (1999) 4865.
\[26\] V. Gogohia, hep-ph/0508224.
\[27\] A.I. Alekseev, B.A. Arbuzov, hep-ph/0407056; hep-ph/0411339.
\[28\] V.N. Gribov, Nucl. Phys. B 139 (1978) 1.
\[29\] Y.L. Dokshitzer, D.E. Kharzeev, hep-ph/0404216.
\[30\] R. Streeter, A. Wightman, Spin and Statistics and all That (W.A. Benjamin, NY 1964).
\[31\] F. Strocchi, Phys. lett. B 62 (1976) 60.
\[32\] R. Feynman, Nucl. Phys. B 188 (1981) 479.
\[33\] H. Pagels, Phys. Rev. D 15 (1977) 2991.
\[34\] L. Susskind, J. Kogut, Phys. Rep. C 23 (1976) 348.
\[35\] H. Fritzsch, M. Gell-Mann, H. Leutwyler, Phys. Lett. B 47 (1973) 365.
\[36\] S. Weinberg, Phys. Rev. Lett. 31 (1973) 494.
\[37\] H. Georgi, S. Glashow, Phys. Rev. Lett. 32 (1974) 438.
\[38\] J.L. Gervais, A. Neveu, Phys. Rep. C 23 (1976) 240.
\[39\] S. Mandelstam, Phys. Rev. D 20 (1979) 3223.
\[40\] G.K. Savvidy, Phys. Lett. B 71 (1977) 133; S.G. Matynian, G.K. Savvidy, Nucl. Phys. B 134 (1978) 539.
\[41\] V. Gogohia et al., Phys. Lett. B 453 (1999) 281.
\[42\] V. Gogohia et al., Int. Jour. Mod. Phys. A 15 (2000) 45.
\[43\] F. Wilczek, Proc. Inter. Conf., QCD-20 Years Later, Aachen, June 9-13, 1992, v. 1.
\[44\] V. Gogokhia, hep-th/0604095.
\[45\] S. Coleman, E. Weinberg, Phys. Rev. D 7 (1973) 1888.
\[46\] D.J. Gross, A. Neveu, Phys. Rev. D 10 (1974) 3235.