Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in $\mathbb{R}^2$

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Abstract

We are concerned with sign-changing solutions of the following gauged nonlinear Schrödinger equation in dimension two including the so-called Chern–Simons term

$$
\begin{cases}
-\Delta u + \omega u + \left( \frac{u^2(|x|)}{|x|} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \right) u = \lambda |u|^{p-2} u & \text{in } \mathbb{R}^2, \\
u(x) = u(|x|) & \in H^1(\mathbb{R}^2),
\end{cases}
$$

where $\omega, \lambda > 0$, $p \in (4, 6)$ and

$$
h(s) = \frac{1}{2} \int_0^s \tau u^2(\tau) \, d\tau.
$$

Via a novel perturbation approach and the method of invariant sets of descending flow, we investigate the existence and multiplicity of sign-changing solutions. Moreover, energy doubling is established, i.e. the energy of sign-changing solution $w_\lambda$ is strictly larger than twice that of the ground state energy for $\lambda > 0$ large. Finally, for any sequence $\lambda_n \to \infty$ as $n \to \infty$, up to a subsequence, $\frac{1}{\lambda_n} w_{\lambda_n} \to w$ strongly in $H^1_{rad}(\mathbb{R}^2)$ as $n \to \infty$, where $w$ is a sign-changing solution of

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\[-\triangle u + \omega u = |u|^{p-2}u, \quad u \in H^{1}_{\text{rad}}(\mathbb{R}^{2}).\]

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1. Introduction and main results

1.1. Gauged Schrödinger equations

Consider the following planar gauged nonlinear Schrödinger system

\[
\begin{aligned}
-iD_{0}\phi + (D_{1}D_{1} + D_{2}D_{2})\phi &= -\lambda|\phi|^{p-2}\phi, \\
\partial_{t}A_{1} - \partial_{1}A_{0} &= -\text{Im}(\phi D_{2}\phi), \\
\partial_{t}A_{2} - \partial_{2}A_{0} &= \text{Im}(\phi D_{1}\phi), \\
\partial_{1}A_{2} - \partial_{2}A_{1} &= -\frac{1}{2}|\phi|^{2},
\end{aligned}
\]

(1.1)

where \(i\) denotes the imaginary unit, \(\partial_{0} = \frac{\partial}{\partial t},\) \(\partial_{1} = \frac{\partial}{\partial x_{1}},\) \(\partial_{2} = \frac{\partial}{\partial x_{2}}\) for \((t,x) \in \mathbb{R}^{1+2}, x = (x_{1}, x_{2})\). \(\phi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}\) is the complex scalar field, \(A_{j} : \mathbb{R}^{1+2} \rightarrow \mathbb{R}\) is the gauge field, \(D_{j} = \partial_{j} + iA_{j}\) is the covariant derivative for \(j = 0, 1, 2\), and \(\lambda\) is a positive constant representing the strength of interaction potential. One feature of this model is that this system was proposed consisting of Schrödinger equations augmented by the gauge field. And it is particularly important in studying the high-temperature superconductor, fractional quantum Hall effect and Aharonov–Bohm scattering. For more details about system (1.1), we refer the readers to [18–20]. Moreover, system (1.1) is invariant under the following gauge transformation

\[
\phi \rightarrow \phi e^{i\chi}, \quad A_{j} \rightarrow A_{j} - \partial_{j}\chi,
\]

(1.2)

where \(\chi : \mathbb{R}^{1+2} \rightarrow \mathbb{R}\) is an arbitrary \(C^\infty\) function. This system was first studied in [23–25]. The initial value problem of such a system as well as global existence and blow-up, has also been addressed in [8, 21, 22, 30] for the case \(p = 4\). We also see [31] for a global existence result in the defocusing case, and [15] for a uniqueness result to the infinite radial hierarchy.

The existence of stationary states to system (1.1) with general \(p > 2\) has been studied recently in [9]. By using the ansatz

\[
\begin{aligned}
\phi(t,x) &= u(|x|)e^{i\omega t}, \quad A_{0}(t,x) = A_{0}(|x|), \\
A_{1}(t,x) &= \frac{x_{2}}{|x|^{2}}h(|x|), \quad A_{2}(t,x) = -\frac{x_{1}}{|x|^{2}}h(|x|).
\end{aligned}
\]

Byeon, Huh and Seok found in [9] that \(u\) solves the equation

\[
-\triangle u + (\omega + \xi)u + \left(\frac{h^{2}(|x|)}{|x|^{2}} + \int_{|x|}^{+\infty} \frac{h(s)}{s}u^{2}(s)ds\right)u = \lambda|u|^{p-2}u,
\]

(1.3)

where \(h(s) = \frac{1}{2}\int_{0}^{s} \tau u^{2}(\tau)d\tau\) and \(\xi\) is an integration constant of \(A_{0}\), which takes the form

\[
A_{0}(r) = \xi + \int_{r}^{+\infty} \frac{h(s)}{s}u^{2}(s)ds.
\]

As mentioned in [9], taking \(\chi = ct\) in the gauge invariance (1.2), we obtain another stationary solution for any given stationary solution; the functions \(u(x), A_{1}(x), A_{2}(x)\) are preserved, and
\[\omega \to \omega + c, \quad A_0(x) \to A_0(x) - c.\]

That is to say, the constant \(\omega + \xi\) is a gauge invariant of the stationary solutions of the problem. Therefore, we can take \(\xi = 0\) in what follows, that is,

\[\lim_{|x| \to \infty} A_0(x) = 0,\]

which was indeed assumed in [25]. In this situation, equation (1.3) becomes

\[-\Delta u + u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right) u = \lambda |u|^{p-2} u, \quad x \in \mathbb{R}^2.\]

(1.4)

It is shown in [9] that (1.4) is indeed variational and the associated energy functional \(I_\lambda : H^1_{rad}(\mathbb{R}^2) \to \mathbb{R}\) is given by

\[I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^2} (\nabla u \nabla v + \omega uv) dx + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} h^2(|x|) dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx.\]

Here \(H^1_{rad}(\mathbb{R}^2)\) denotes the subspace of radially symmetric functions in \(H^1(\mathbb{R}^2)\) with the inner product and norm

\[\langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + \omega uv) dx, \quad \|u\| := \left(\int_{\mathbb{R}^2} (\nabla u)^2 + \omega u^2\right) dx.\]

For simplicity, in what follows, denote

\[B(u) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} h^2(|x|) dx,\]

then \(B \in C^1(H^1_{rad}(\mathbb{R}^2), \mathbb{R})\) and

\[\langle B'(u), \varphi \rangle = \int_{\mathbb{R}^2} \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right) u(x) \varphi(x) dx, \quad \forall \varphi \in H^1_{rad}(\mathbb{R}^2).\]

For any \(u, \varphi \in H^1_{rad}(\mathbb{R}^2)\), we have

\[\langle I_\lambda'(u), \varphi \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + \omega u \varphi) dx - \lambda \int_{\mathbb{R}^2} |u|^{p-2} u \varphi dx + \langle B'(u), \varphi \rangle.\]

(1.5)

And any critical point of \(I_\lambda\) in \(H^1_{rad}(\mathbb{R}^2)\) is a weak solution of equation (1.4) (see [9]).

1.2. Motivation

In the following, we summarize some relative results for equation (1.4). In contrast with Schrödinger equations, equation (1.4) is nonlocal, that is, it is not a pointwise identity with the appearance of the Chern–Simons term

\[\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right) u.\]

This nonlocal term causes some mathematical difficulties and makes the problem rough and particularly interesting. One of main obstacles is the boundedness of Palais–Smale sequences
if one tries to use directly the mountain pass theorem to get critical points of $I_{\lambda}(u)$ in $H^1_{rad}(\mathbb{R}^2)$. For $p \geq 6$, it is standard to show that the Palais–Smale condition holds for $I_{\lambda}(u)$ in $H^1_{rad}(\mathbb{R}^2)$, see [22]. For $p > 4$, the Euler–Lagrange functional is unbounded from below and exhibits a mountain-pass geometry. However, it seems hard to prove that the Palais–Smale condition holds for $p \in (4, 6)$. This problem was bypassed in [9] by using a constrained minimization taking into account both the Nehari and Pohozaev identities. Moreover, in [9], static solutions can be found for the special case $p = 4$ by passing to a self-dual equation, which leads to a Liouville equation that can be solved explicitly. But those solutions are positive. In the same paper, the case $p \in (2, 4)$ also was considered by using a minimization argument on a $L^2$-sphere. By analysing the corresponding limit equation and investigating the geometry of the Euler–Lagrange functional, Pomponio and Ruiz [38] proved the existence and non-existence of positive solutions to equation (1.4) for different values of $\lambda$ when $p \in (2, 4)$. They showed that there exists a threshold value $\omega_0 > 0$ such that the corresponding energy functional is bounded from below if $\omega > \omega_0$, and not for $\omega \in (0, \omega_0)$. Moreover, they gave an explicit expression of $\omega_0$ as follows

$$
\omega_0 = \frac{4 - p^2}{2 + p} \frac{1}{2\pi^2} \left( \frac{m^2(4 + p)}{p - 2} \right)^{-\frac{1}{2}}
$$

where

$$
m = \int_{-\infty}^{\infty} \left[ \frac{2}{p} \text{coth}^2 \left( \frac{p - 2}{2} r \right) \right] \frac{1}{2} dr
$$

is the unique positive even solution of the problem 

$$
-\nu'' + \nu = \nu^{p-1} \quad \text{in } \mathbb{R}.
$$

For $p \in (2, 4)$, boundary concentration of solutions to (1.4) was considered by Pomponio and Ruiz [39] in the case of bounded domains. As for more general nonlinearities, there have been some works in which the authors tried to weaken or eliminate the Ambrosetti–Rabinowitz condition. In the spirit of the Berestycki–Lions conditions in [5, 6], Cunha, d’Avenia, Pomponio and Siciliano [16] explored the Chern–Simons–Schrödinger system with a nonlinear term of Berestycki, Gallouët and Kavian type in [7]. In this aspect, we also would like to cite [43]. On the other hand, Li and Luo [28] also investigated the nonlocal equation (1.4) in the mass-critical case: $p = 4$ and mass-supercritical case: $p > 4$, for instance, the existence, $H^1(\mathbb{R}^2)$-bifurcation and multiplicity of normalized solutions. The existence of stationary states with a vortex point has also been considered in [10, 27]. For more existence results on the nonlinear Chern–Simons–Schrödinger equation (1.4), we refer the readers to [26, 37, 41, 42, 45] and the references therein.

1.3. Main results

To the best of our knowledge, in the literature, there are just few result on the existence of sign-changing solutions for Chern–Simons–Schrödinger systems. Combining constraint minimization method and quantitative deformation lemma, Li, Luo and Shuai [29] proved the existence and asymptotic behavior of the least energy sign-changing solutions to (1.4) when $p > 6$. Moreover, energy doubling for the sign-changing solutions was obtained. Meanwhile, Deng, Peng and Shuai [17] also established the existence and asymptotic behavior of nodal solutions to (1.4) when $p > 6$. Precisely, they obtained the existence of a sign-changing solution, which
changes signs exactly \( k \) times for any \( k \in \mathbb{N} \). Their procedure of arguments is to transform the original problem to solving a system of \((k + 1)\) equations with \((k + 1)\) unknown functions \( u_i \) with disjoint supports. Then the nodal solution is constructed through gluing \( u_i \) by matching the normal derivative at each junction point. We highlight that \( p > 6 \) plays a crucial role in [17, 29]. So, one question still remains open in these results above:

Does problem (1.4) admit sign-changing solutions in the case \( p \in (4, 6) \)?

The main interest of the present paper is to give an affirmative answer to this open question.

**Theorem 1.1 (Existence).** If \( p \in (4, 6) \), then for any \( \lambda > 0 \), equation (1.4) admits at least one least energy sign-changing solution in \( H^1_{\text{rad}}(\mathbb{R}^2) \).

Since the Chern–Simons term is involved, equation (1.4) is quite different from the following scalar field equation

\[
-\Delta u + \omega u = |u|^{p-2} u,
\]

which does not depend on the nonlocal term any more. In the literature, problem (1.6) has attracted considerable attention since 1970s. In [5], the so-called Berestycki and Lions condition was introduced to guarantee the existence of ground state solutions to a general scalar field equation \(-\Delta u = f(u)\). Moreover, the Berestycki and Lions condition is almost necessary. In this aspect, we also refer to [6, 7]. With some suitable condition on \( f \), ground state solutions have a fixed sign in general. Another topic of particular interest is sign-changing solutions, which can be sought by several variational approaches, for instance, the Nehari manifold method [2, 11, 12], Morse theory [14], heat flow method [13] and so on. Over the years, combining minimax methods, the method of invariant sets of descending flow has been a powerful tool in finding sign-changing solutions of elliptic problems. For more progress in this aspect, we refer to [3, 4, 34, 35, 46] and the references therein.

**Remark 1.1.** In contrast with problem (1.6), the Chern–Simons term makes equation (1.4) tough. In the following, we summarize some difficulties caused by the non-locality in seeking sign-changing solutions.

(1) In finding sign-changing solutions of (1.6), a crucial ingredient is the following decomposition: for any \( u \in H^1(\mathbb{R}^2) \),

\[
I(u) = I(u^+) + I(u^-),
\]

\[
\langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle, \quad \langle I'(u), u^- \rangle = \langle I'(u^-), u^- \rangle
\]

where \( I \) is the energy functional associated with (1.6) and defined by

\[
I(u) := \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.
\]

However, in applying the method of invariant sets of descending flow to equation (1.4), due to the Chern–Simons term, one of the main obstacles is that the following decompositions

\[
I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-), \quad \langle I'_\lambda(u), u^+ \rangle = \langle I'_\lambda(u^+), u^+ \rangle
\]
do not hold any more in general. Motivated by [33], via a sign-changing critical point theorem due to Liu et al [32], we attempt to seek sign-changing solutions for equation (1.4). Noting that, the spitting decomposition of the Chern–Simons term is more complicated than that of the nonlocal term involved in Schrödinger–Poisson systems (see [33]), we need some new trick. Thus, the variational framework in [32, 33] is not directly applicable due to changes of the geometric nature of the energy functional $I_{\lambda}$.

(2) The effect of the nonlocal term $\left( \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)ds \right) u$ results in two difficulties. First, it seems much complicated to find a similar auxiliary operator $A$ (see [33, section 4]), which plays a crucial role in constructing invariants sets of a descending flow associated with equation (1.4). A similar difficulty also arises in seeking sign-changing solutions of the Schrödinger–Poisson systems

\[
\begin{align*}
-\Delta u + V(x)u + \phi u &= |u|^{p-2}u &\text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 &\text{in } \mathbb{R}^3,
\end{align*}
\]

where $p \in (3, 4)$. In [33], the authors overcome this difficulty for $p \in (3, 4)$ by adding a higher order local nonlinear term than 4 and a coercive condition about $V$. Second, the fact that $p \in (4, 6)$ makes tough to get the boundedness of (PS) sequences. In [33], the authors recovered such boundedness due to the coercivity condition of $V$. However, without the coercivity condition, the method in [33] will be not applicable any more. In this paper, we develop particularly a perturbation approach by adding another nonlocal perturbation. Then we can obtain sign-changing solutions of the perturbed problems. By passing to the limit, sign-changing solutions of the original equation (1.4) are obtained.

We should point out that, this perturbation approach also works for system (1.10) when $p \in (3, 4)$ without any coercive conditions. Moreover, a similar perturbation approach also can be found in [36], but does not work in the present paper. In fact, an additional perturbation term is also involved in this paper (see section 3.1). In some sense, it can be seen as a generalization of that in [36].

Our another purpose is to establish energy doubling of sign-changing solutions to equation (1.4) with $p \in (4, 6)$. We use the local equation (1.6) to illustrate energy doubling. Set

\[
\mathcal{N} := \{u \in H^1_{rad}(\mathbb{R}^2) \setminus \{0\} : \langle I'(u), u \rangle = 0\},
\]

and

\[
c_0 = \inf \{I(u) : u \in \mathcal{N} \}.
\]

Obviously, for any sign-changing solution $w \in H^1_{rad}(\mathbb{R}^2)$ of equation (1.6), one can get that

\[
I(w) = I(w^+) + I(w^-) \geq 2c_0.
\]

In [1], Ackermann and Weth proved that for each sign-changing solution $w$ of (1.6), it holds true that

\[
I(w) > 2c_0,
\]

which is called in the literature that $w$ satisfies ‘energy doubling’ (see also [44]). Now, we give the analogue of energy doubling for problem (1.6) as follows.

**Definition 1.1.** Let $w_{\lambda} \in H^1_{rad}(\mathbb{R}^2)$ be a sign-changing solution of equation (1.4), we say $w_{\lambda}$ satisfies energy doubling, if $I_{\lambda}(w_{\lambda}) > 2c_{\lambda}$, where
\[ c_\lambda = \inf \{ I_\lambda(u), u \in \mathcal{N}_\lambda \}, \quad \mathcal{N}_\lambda := \{ u \in H^1_{rad}(\mathbb{R}^2) \setminus \{ 0 \} : I'_\lambda(u) = 0 \}. \quad (1.13) \]

Let \( w_\lambda \in H^1_{rad}(\mathbb{R}^2) \) be a sign-changing solution of equation (1.4). Since the interaction of the positive and negative parts of solutions cannot be neglected, it is known that
\[ w^\pm_\lambda \not\in \mathcal{N}_\lambda. \quad (1.14) \]
Thus, a natural open question is that whether energy doubling holds or not. Generally speaking, it is even not easy to compare \( I_\lambda(w_\lambda) \) with \( c_\lambda \). By using an approximation procedure, we give a partial answer for such an open problem, that is, energy doubling holds if \( \lambda > 0 \) large. Precisely, we have the following result.

**Theorem 1.2 (Energy doubling).** If \( p \in (4, 6) \), then there exists \( \lambda^* > 0 \) such that, for \( \lambda > \lambda^* \), any sign-changing solution \( w_\lambda \) of problem (1.4) in \( H^1_{rad}(\mathbb{R}^2) \) satisfies energy doubling. Furthermore, for any sequence \( \{ \lambda_n \} \) with \( \lambda_n \to +\infty \) as \( n \to \infty \), up to a subsequence, \( \lambda_n^\frac{1}{2} w_{\lambda_n} \to w \) in \( H^1_{rad}(\mathbb{R}^2) \), where \( w \in H^1_{rad}(\mathbb{R}^2) \) is a sign-changing solution of (1.6).

Our last purpose in the present paper is to look for infinitely many sign-changing solutions of equation (1.4) when \( \lambda \) is small enough.

**Theorem 1.3 (Multiplicity).** If \( p \in (4, 6) \), then there exists \( \lambda_* > 0 \) such that, for any \( \lambda \in (0, \lambda_*) \), equation (1.4) has infinitely many sign-changing solutions in \( H^1_{rad}(\mathbb{R}^2) \).

**Remark 1.2.** For the proof of theorem 1.3, we adopt the perturbation approach established in [32, 33]. Based on a new inequality of Sobolev type developed by Byeon, Huh and Seok in [9] and a perturbed term growing faster than 4 near infinity, the boundedness of (PS) sequences can be established. Via the method of invariants sets of a descending flow and the limiting argument, we establish the existence of multiple sign-changing solutions.

**Remark 1.3.** Here we would like to emphasize that multiple sign-changing solutions are obtained only for \( \lambda > 0 \) small. An interesting open question left is whether equation (1.4) has multiple sign-changing solutions for any \( \lambda > 0 \) and \( p \in (4, 6) \).

In the following, we want to illustrate the main reason why \( \lambda \) should be small. To use the method of invariant sets of decreasing flow, the following estimate plays an essential role (for instance, see [33, lemma 3.9])
\[ \inf \{ \| u - T_{\beta}(u) \| : u \in H^1_{rad}(\mathbb{R}^2), I_{\beta}(u) \in [c, d], \| I'_{\beta}(u) \| \geq \alpha \} > 0. \quad (1.15) \]

In lemma 5.2 below, the estimate (1.15) holds only for \( \lambda > 0 \) small enough, since only one perturbation term is given. In fact, in proving theorem 1.1, two perturbation terms help to obtain a similar estimate to (1.15) for any \( \lambda > 0 \) (see lemma 3.3 below). Then in the scheme of [33], for any \( \gamma, \beta \in (0, 1] \), one can show that the perturbed problem (3.1) (see section 3 below) admits a sequence \( \{ u^j_{\gamma, \beta} \}_{j=1}^{\infty} \) of sign-changing solutions with energy \( c^j_{\gamma, \beta} \to +\infty \) as \( j \to \infty \).

Here \( c^j_{\gamma, \beta} \) can be given similarly to \( c^j_{\gamma} \) in section 5.3. To get the sign-changing solutions of the original problem (1.4), one natural idea is to take the limit as \( \gamma, \beta \to 0^+ \). However, by the definition of \( c^j_{\gamma, \beta}, c^j_{\gamma} \) is non-decreasing in \( \gamma \). This results in difficulty in distinguishing the limits of \( \{ c^j_{\gamma, \beta} \}_{j=1}^{\infty} \) as \( \gamma, \beta \to 0^+ \). So using a similar perturbation to theorem 1.1, it seems that only one sign-changing solution can be obtained for any \( \lambda > 0 \). But in the case of \( \lambda > 0 \) small,
by Step 2 of theorem 1.3, for any \( \beta \in (0, 1] \), \( c_j^\beta \to +\infty \) as \( j \to \infty \). In contrast with \( c_j^\beta \), based on the definition of \( c_j^\beta \), it is non-increasing in \( \beta \), which yields that \( \lim_{\beta \to 0^+} c_j^\beta = c_j^\gamma \to +\infty \) as \( j \to \infty \). This guarantees the existence of infinity many sign-changing solutions of (1.4) for \( \lambda > 0 \) small.

This paper is organized as follows. In section 2, some notations and preliminaries are given. Via a perturbation argument and the method of invariant sets of decreasing flow, we prove theorem 1.1 in section 3. Section 4 is denoted to proving theorem 1.2. Moreover, the asymptotic behavior is established as the parameter \( \lambda \) goes to \( +\infty \). Finally, section 5 is denoted to proving theorem 1.3.

2. Preliminaries and functional setting

Let us fix some notation and give some preliminary results. For every \( 1 \leq s \leq +\infty \), we denote by \( \| \cdot \| \), the usual norm of the Lebesgue space \( L^s(\mathbb{R}^2) \). \( c, C \) denote (possibly different) positive constants which may change from line to line.

In the following, we introduce some properties of \( A_i (i = 0, 1, 2) \), \( B \) and \( h \), which will be frequently used later.

**Lemma 2.1 ([9]).** If \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \cap L^\infty_{\text{loc}}(\mathbb{R}^2) \), then \( A_0 \in L^\infty(\mathbb{R}^2) \). Furthermore, if \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \cap C(\mathbb{R}^2) \), then \( A_0, A_1 \) and \( A_2 \) are in \( H^1_{\text{rad}}(\mathbb{R}^2) \cap C(\mathbb{R}^2) \).

**Lemma 2.2 ([9]).**

1. If \( u_n \rightharpoonup u \) weakly in \( H^1_{\text{rad}}(\mathbb{R}^2) \) as \( n \to +\infty \), then
   - (i) \( \lim_{n \to +\infty} B(u_n) = B(u) \);
   - (ii) \( \lim_{n \to +\infty} \langle B'(u_n), u_n \rangle = \langle B'(u), u \rangle \);
   - (iii) \( \lim_{n \to +\infty} \langle B'(u_n), \varphi \rangle = \langle B'(u), \varphi \rangle \).

2. Moreover, for any \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \),
   - (iv) \( B(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds dx \);
   - (v) \( \langle B'(u), u \rangle = 6B(u) \).

The following lemma is used to estimate function \( h \).

**Lemma 2.3.** For any \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \) and \( x \in \mathbb{R}^2 \), the following statements hold:

1. for any \( r \in (2, +\infty) \), there exists \( \gamma_r > 0 \) such that \( h(|x|) \leq c_r |x|^{2(r-2)} \| u \|_{r}^2 \), \( x \in \mathbb{R}^2 \);
2. for any \( r \in (2, 4) \) and \( r' \in (4, +\infty) \), there exist \( C_r, C_{r'} > 0 \) such that
   \[
   \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \leq \| u \|_{2}^2 (C_r \| u \|_{r'}^2 + C_{r'} \| u \|_{r'}^2), \quad x \in \mathbb{R}^2.
   \]
3. for any \( r \in (6, 8) \), there exists \( C > 0 \) such that
   \[
   \int_{0}^{\infty} \frac{h(s)}{s} u^2(s) ds \leq \left[ \frac{2}{\pi} B(u) + \frac{1}{2\pi} \| u \|_{2}^2 + C (\| u \|_{4}^2 + \| u \|_{4}^2) \right].
   \]
Proof.

(i) By Hölder’s inequality, for any \( r > 2 \) and \( x \in \mathbb{R}^2 \), there exists \( c_r > 0 \) such that
\[
h(|x|) = \int_{|y| \leq |x|} \frac{1}{4\pi} u^2(y)dy \leq c_r|x|^{\frac{2(r-2)}{r}}\|u\|_r^2.
\]

(ii) It follows from (i) that if \(|x| \leq 1\), then for any \( r \in (2, 4) \) and \( r' \in (4, +\infty)\),
\[
\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)ds = \int_{|x|}^{1} \frac{h(s)}{s} u^2(s)ds + \int_{1}^{\infty} \frac{h(s)}{s} u^2(s)ds \\
\leq C\|u\|_r^2 \int_{|x|}^{1} u^2(s)ds + C\|u\|_r^2 \int_{1}^{\infty} s^{-\frac{4}{r'}} u^2(s)ds \\
\leq C\|u\|_r^2 \left( \int_{|x|}^{1} s^{-\frac{2}{r-2}}ds \right) \left( \int_{|x|}^{1} |u|^{r'}(s)ds \right)^{\frac{1}{r'}} \\
+ C\|u\|_r^2 \left( \int_{1}^{\infty} s^{-\frac{4}{r'}}ds \right) \left( \int_{1}^{\infty} u^4(s)ds \right)^{\frac{1}{4}} \\
\leq \|u\|_r^2 (C_r\|u\|_r^2 + C_r\|u\|_r^2).
\]

Based on above, the conclusion holds obviously if \(|x| > 1\). The proof of conclusion (ii) is complete.

(iii) Let
\[
\int_{0}^{\infty} \frac{h(s)}{s} u^2(s)ds = \int_{0}^{1} \frac{h(s)}{s} u^2(s)ds + \int_{1}^{\infty} \frac{h(s)}{s} u^2(s)ds := K_1 + K_2. \tag{2.1}
\]
On one hand, for any \( r \in (6, 8) \), let \( \alpha = \frac{\gamma}{8} \in (0, 1) \), then

\[
K_1 = \int_{\{s \in (0,1): h(s) \geq 1\}} \frac{h(s)}{s} u^2(s)ds + \int_{\{s \in (0,1): h(s) \leq 1\}} \frac{h(s)}{s} u^2(s)ds \\
\leq \int_{\{s \in (0,1): h(s) \geq 1\}} \frac{h^2(s)}{s} u^2(s)ds + \int_{\{s \in (0,1): h(s) \leq 1\}} \frac{h^\alpha(s)}{s} u^2(s)ds \\
\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2(x)dx + \int_{0}^{1} \frac{h^\alpha(s)}{s} u^2(s)ds \\
\leq \frac{1}{\pi} B(u) + \left( \int_{0}^{1} \frac{h^{2\alpha}(s)}{s^3}ds \right)^{\frac{1}{2}} \|u\|_4^2 \\
\leq \frac{1}{\pi} B(u) + \left( \int_{0}^{1} s^{\frac{4\alpha(r-2)}{r}} \|u\|_r^4ds \right)^{\frac{1}{2}} \|u\|_4^2 \\
\leq \frac{1}{\pi} B(u) + C(\|u\|_4^4 + \|u\|_r^2).
\tag{2.2}
\]
On the other hand,

\[ K_2 = \int_{\{s \in (1, \infty): h(s) > 1\}} \frac{h(s)}{s} u^2(s) ds + \int_{\{s \in (1, \infty): h(s) \leq 1\}} \frac{h(s)}{s} u^2(s) ds \]

\[ \leq \int_0^{\infty} \frac{h^2(s)}{s} u^2(s) ds + \int_1^{\infty} u^2(s) ds \]

\[ = \frac{1}{\pi} B(u) + \frac{1}{2\pi} \|u\|_2^2. \]  

(2.3)

Substituting (2.2) and (2.3) into (2.1), conclusion (iii) holds. □

To prove the existence and multiplicity of nodal solutions, we recall the following Pohozaev identity and an inequality of Sobolev type, which were proved in [9].

**Lemma 2.4 ([9])**. Let \( b, c \) be positive real constants and \( u \in H^1_{rad}(\mathbb{R}^2) \) be a weak solution of the equation:

\[ -\Delta u + bu + c \left( \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds \right) u = \lambda |u|^{p-1} u, \quad x \in \mathbb{R}^2, \]

then

\[ b \int_{\mathbb{R}^2} u^2 dx + 2c \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx + \frac{2\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx = 0. \]

**Lemma 2.5 ([9])**. For \( u \in H^1_{rad}(\mathbb{R}^2) \), there holds

\[ \int_{\mathbb{R}^2} u^4 dx \leq 4 \left( \int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} h^2(|x|) dx \right)^{\frac{1}{2}}. \]

Furthermore, the equality is attained by a continuum of function

\[ \left\{ u_l = \frac{\sqrt{8l}}{1 + |x|^2} \in H^1_{rad}(\mathbb{R}^2) \mid l \in (0, \infty) \right\}, \]

and

\[ \frac{1}{4} \int_{\mathbb{R}^2} |u_l|^4 dx = \int_{\mathbb{R}^2} |\nabla u_l|^2 dx = \int_{\mathbb{R}^2} \frac{u_l^2}{|x|^2} h_l^2(|x|) dx = \frac{16\pi l^2}{3}, \]

where \( h_l(|x|) := \int_0^{|x|} \frac{s}{2} u_l^2(s) ds. \)

### 3. Proof of theorem 1.1

#### 3.1. The perturbed problem

In this section, we investigate the existence of sign-changing solutions to (1.4) for \( p \in (4, 6) \) via a perturbation argument. In contrast with \( p > 6 \), the Ambrosetti–Rabinowitz condition fails for \( p \in (4, 6) \). It results in tough difficulty in getting the boundedness of Palais–Smale sequences. In [9], the authors constructed a constraint manifold coming from both deformations of range
and domain of function $u$, which was firstly used in [40]. To overcome this difficulty, we introduce a perturbation approach.

For convenience’s sake, for any $u \in H^1_{rad}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2 \setminus \{0\}$, set

$$B(u)(x) = \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)ds,$$

then

$$\langle B'(u), \varphi \rangle = \int_{\mathbb{R}^2} B(u)u \varphi dx, \quad \varphi \in H^1(\mathbb{R}^2).$$

Moreover, by lemma 2.3, $B(u)$ is well defined and $B(u)(x) \in (0, +\infty), x \in \mathbb{R}^2 \setminus \{0\}$. Taking

$$\alpha \in \left(0, \min\left\{\frac{1}{2}, \frac{p-4}{2}\right\}\right), \gamma, \beta \in (0, 1] \text{ and } q > 6,$$

we consider the modified problem

$$-\Delta u + [\omega + B(u)]u + \gamma \left(\int_{\mathbb{R}^2} |u|^4dx\right)^{\alpha} u^2 = \lambda |u|^{p-2}u + \beta |u|^{q-2}u, \quad x \in \mathbb{R}^2$$

and its associated functional is as follows

$$I_{\gamma,\beta}(u) = I_{\lambda}(u) + \gamma \left(\int_{\mathbb{R}^2} |u|^4dx\right)^{1+\alpha} - \frac{\beta}{q} \int_{\mathbb{R}^2} |u|^qdx.$$

It is easy to show that $I_{\gamma,\beta} \in C^1(H^1_{rad}(\mathbb{R}^2), \mathbb{R})$ and

$$\langle I'_{\gamma,\beta}(u), v \rangle = \langle I_{\lambda}'(u), v \rangle + \gamma \left\|u\right\|^4_{T_{4\delta}} \int_{\mathbb{R}^2} u^2vdx - \beta \int_{\mathbb{R}^2} |u|^q-2uvdx, \quad u, v \in H^1_{rad}(\mathbb{R}^2).$$

For each $u \in H^1_{rad}(\mathbb{R}^2)$, one can show that the equation

$$-\Delta v + [\omega + B(u)]v + \gamma \left(\int_{\mathbb{R}^2} |u|^4dx\right)^{\alpha} u^2v = \lambda |u|^{p-2}u + \beta |u|^{q-2}u, \quad x \in \mathbb{R}^2$$

has a unique weak solution $v := T_{\gamma,\beta}(u) \in H^1_{rad}(\mathbb{R}^2)$. Clearly, $u$ is a solution of equation (3.1) if and only if $u$ is a fixed point of $T_{\gamma,\beta}$.

**Lemma 3.1.** The operator $T_{\gamma,\beta} : u \in H^1_{rad}(\mathbb{R}^2) \mapsto v \in H^1_{rad}(\mathbb{R}^2)$ is continuous.

**Proof.** Assume that $\{u_n\} \subset H^1_{rad}(\mathbb{R}^2)$ with $u_n \to u$ strongly in $H^1_{rad}(\mathbb{R}^2)$. For simplicity, for any $n$ and $x \in \mathbb{R}^2 \setminus \{0\}$, set

$$B(u_n)(x) = \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u_n^2(s)ds,$$

where $h_n(s) = \int_0^s \frac{1}{2} u_n^2(\tau)d\tau$. Let $v = T_{\gamma,\beta}(u)$ and $v_n = T_{\gamma,\beta}(u_n)$, then we have

$$\int_{\mathbb{R}^2} (\nabla v_n \nabla w + [\omega + B(u_n)]v_n w)dx + \gamma \left\|u_n\right\|^4_{T_{4\delta}} \int_{\mathbb{R}^2} u_n^2 v_nwdx$$

$$= \lambda \int_{\mathbb{R}^2} |u_n|^{p-2}u_nwdx + \beta \int_{\mathbb{R}^2} |u_n|^{q-2}u_nwdx, \quad \forall w \in H^1_{rad}(\mathbb{R}^2).$$

(3.3)
Moreover,
\[
\int_{\mathbb{R}^2} (\nabla v \nabla w + [\omega + B(u)] vw) dx + \gamma \|u\|_4^{2\alpha} \int_{\mathbb{R}^2} u^2 v dx
= \lambda \int_{\mathbb{R}^2} |u|^{p-2} u w dx + \beta \int_{\mathbb{R}^2} |u|^{q-2} u v dx, \quad \forall w \in H^1_{rad}(\mathbb{R}^2).
\]

(3.4)

We show that \(\|v_n - v\| \to 0\) as \(n \to \infty\). Indeed, testing with \(w = v_n\) in (3.3) gives
\[
\|v_n\|^2 \leq \lambda \int_{\mathbb{R}^2} |u|^{p-2} u v_n dx + \beta \int_{\mathbb{R}^2} |u|^{q-2} u v_n dx
\]
which, together with Hölder’s inequality, implies that sequence \(\{v_n\}\) is bounded in \(H^1_{rad}(\mathbb{R}^2)\). Assume that \(v_n \rightharpoonup v^*\) weakly in \(H^1_{rad}(\mathbb{R}^2)\) and strongly in \(L^q(\mathbb{R}^2)\) for \(q \in (2, +\infty)\) after extracting a subsequence, then by (3.3) and lemmas 2.2 and 2.3 we have
\[
\int_{\mathbb{R}^2} (\nabla v^* \nabla w + [\omega + B(u)] v^* w) dx + \gamma \|u\|_4^{2\alpha} \int_{\mathbb{R}^2} u^2 v^* dx
= \lambda \int_{\mathbb{R}^2} |u|^{p-2} u v_n dx + \beta \int_{\mathbb{R}^2} |u|^{q-2} u v_n dx, \quad \forall w \in H^1_{rad}(\mathbb{R}^2),
\]
which implies that \(v^*\) is a solution of equation (3.2). By the uniqueness, we immediately get \(v = v^*\). Taking \(w = v_n - v\) in (3.3) and (3.4) and then subtracting, we have
\[
\|v_n - v\|^2 + \int_{\mathbb{R}^2} B(u_n)(v_n - v)^2 dx + \gamma \|u_n\|_4^{2\alpha} \int_{\mathbb{R}^2} u_n^2 (v_n - v)^2 dx
= \int_{\mathbb{R}^2} [B(u_n) - B(u)] (v_n - v) dx + \gamma (\|u_n\|_4^{2\alpha} - \|u\|_4^{2\alpha}) \int_{\mathbb{R}^2} u_n^2 (v_n - v) dx
+ \lambda \int_{\mathbb{R}^2} (|u_n|^{p-2} u_n - |u|^{p-2} u) (v_n - v) dx + \beta \int_{\mathbb{R}^2} (|u_n|^{q-2} u_n - |u|^{q-2} u) (v_n - v) dx.
\]

(3.6)

So it follows from lemmas 2.2 and 2.3 and Sobolev’s embedding inequality that \(v_n \to v\) in \(H^1_{rad}(\mathbb{R}^2)\) as \(n \to \infty\). Therefore, \(T_{\gamma,\beta}\) is continuous.

\[ \square \]

Lemma 3.2.

1. \(\|T'_{\gamma,\beta}(u), u - T_{\gamma,\beta}(u)\| \geq \|u - T_{\gamma,\beta}(u)\|\) for all \(u \in H^1_{rad}(\mathbb{R}^2)\);
2. \(\|T'_{\gamma,\beta}(u)\| \leq C\|u - T_{\gamma,\beta}(u)\|(1 + \|u\|^4 + \|u\|^2 + \|u\|^{2+\alpha})\) for some \(C > 0\) and all \(u \in H^1_{rad}(\mathbb{R}^2)\).

Proof.

(1) Since \(T_{\lambda,\beta}(u)\) is a solution of equation (3.2), we have
\[
\langle T'_{\gamma,\beta}(u), u - T_{\gamma,\beta}(u) \rangle = \|u - T_{\gamma,\beta}(u)\|^2 + \int_{\mathbb{R}^2} B(u)(u - T_{\gamma,\beta}(u))^2 dx
+ \gamma \|u\|_4^{2\alpha} \int_{\mathbb{R}^2} u^2 |u - T_{\gamma,\beta}(u)|^2 dx,
\]
which means \(\langle T'_{\gamma,\beta}(u), u - T_{\gamma,\beta}(u) \rangle \geq \|u - T_{\gamma,\beta}(u)\|^2\) for all \(u \in H^1_{rad}(\mathbb{R}^2)\).

(2) Notice that for all \(\varphi \in C^\infty_0(\mathbb{R}^2) \cap H^1_{rad}(\mathbb{R}^2)\),

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\[ \langle I'_{\gamma,\beta}(u), \varphi \rangle = \int_{\mathbb{R}^2} [\nabla (u - T_{\gamma,\beta}(u)) \nabla \varphi + [\omega + B(u)](u - T_{\gamma,\beta}(u)) \varphi] \, dx \\
+ \gamma \|u\|^{4\alpha} \int_{\mathbb{R}^2} u^2 (u - T_{\gamma,\beta}(u)) \varphi \, dx \]

which implies by lemma 2.3 that \( \|I'_{\gamma,\beta}(u)\| \leq C \|u - T_{\gamma,\beta}(u)\| (1 + \|u\|^2 + \|u\|^{2+\alpha}) \) for all \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \).

**Lemma 3.3.** For fixed \((\gamma, \beta) \in (0, 1) \times (0, 1)\), \( c < d \) and \( a > 0 \), there exists \( \varepsilon_{\gamma,\beta} > 0 \) such that \( \|u - T_{\gamma,\beta}(u)\| \geq \varepsilon_{\gamma,\beta} \) if \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \), \( I_{\gamma,\beta}(u) \in [c, d] \) and \( \|I'_{\gamma,\beta}(u)\| \geq a \).

**Proof.** Fix \( \mu \in (6, q) \), then for all \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \), we have

\[
\begin{align*}
I_{\gamma,\beta}(u) - \frac{1}{\mu} \langle u, u - T_{\gamma,\beta}(u) \rangle &= \frac{\mu - 2}{2\mu} \|u\|^2 + \frac{1}{\mu} \int_{\mathbb{R}^2} B(u)u(u - T_{\gamma,\beta}(u)) \, dx + \frac{\mu - 2}{\mu} B(u) \\
&+ \frac{\gamma}{\mu} \|u\|^{4\alpha} \int_{\mathbb{R}^2} u^2 (u - T_{\gamma,\beta}(u)) \, dx - \frac{1}{\mu} \int_{\mathbb{R}^2} u^2 \int_{|s|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \, dx \\
&+ \frac{(p - \mu)\lambda}{\mu p} \int_{\mathbb{R}^2} |u|^p \, dx + \frac{(q - \mu)\beta}{\mu q} \int_{\mathbb{R}^2} |u|^q \, dx + \frac{\gamma}{\lambda} \frac{\mu - 4}{4\mu} \|u\|^{4(1+\alpha)}.
\end{align*}
\] (3.7)

Then, by the conclusion (iv) of lemma 2.2, we have

\[
\begin{align*}
\|u\|^2 + 2B(u) + \beta\|u\|^q + \gamma\|u\|^{4(1+\alpha)} - \lambda\|u\|^p \\
&\leq c \left\{ |I_{\gamma,\beta}(u)| + \|u\|\|u - T_{\gamma,\beta}(u)\| + \int_{\mathbb{R}^2} B(u)\|u(u - T_{\gamma,\beta}(u))\| \, dx \right. \\
&\left. + \frac{\gamma}{\mu} \|u\|^{4\alpha} \int_{\mathbb{R}^2} |u|^2 (u - T_{\gamma,\beta}(u)) \, dx \right\},
\end{align*}
\] (3.8)

where \( c \) depends on \( \beta, \gamma, \mu, \alpha, \lambda \). On one hand, take \( r \in \left( \frac{4(1+\alpha)}{1+2\alpha}, 4 \right) \) and \( r' = q \) in lemma 2.3, then it follows from lemma 2.3, Jensen’s inequality, Hölder’s inequality and Sobolev’s inequality, we have for any \( \xi > 0 \), there exists \( C_\xi > 0 \) such that

\[
\begin{align*}
\int_{\mathbb{R}^2} |u(u - T_{\gamma,\beta}(u))| \int_{|s|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \, dx \\
&\leq \left( \int_{\mathbb{R}^2} u^2 \int_{|s|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} (u - T_{\gamma,\beta}(u))^2 \int_{|s|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \, dx \right)^{\frac{1}{2}} \\
&\leq \xi B(u) + C_\xi \|u\|^2 \|u - T_{\gamma,\beta}(u)\|^2 + C_\xi \|u\|^{4\alpha} \|u - T_{\gamma,\beta}(u)\|^{4+\alpha} + \|u\|^p \|u - T_{\gamma,\beta}(u)\|^p \\
&\leq \xi B(u) + C_\xi \left( \|u\|^{4(1+\alpha)} \|u - T_{\gamma,\beta}(u)\|^{4+\alpha} + \|u\|^p \|u - T_{\gamma,\beta}(u)\|^p \right). \\
\end{align*}
\] (3.9)

On the other hand, it follows from Hölder’s inequality and Sobolev’s inequality, we have for any \( \xi > 0 \), there exists \( C_\xi > 0 \) such that
\[ \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u(u - T_{\gamma,\beta}(u)) \, dx \leq \left( \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} |u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} (u - T_{\gamma,\beta}(u))^2 \, dx \right)^{1/2} \]  

(3.10)

Combining (3.8)–(3.10) we get

\[ ||u||^2 + B(u) + \beta||u||^4 + \gamma||u||^{4(1+\alpha)} - \lambda||u||^p \]

\[ \leq C \left( ||I_{\gamma,\beta}(u)|| + ||u|| ||u - T_{\gamma,\beta}(u)|| + ||u||^{\alpha} ||u - T_{\gamma,\beta}(u)||^{\frac{4\alpha}{1+\alpha}} \right) \]

(3.11)

Assume, on the contrary, that there exists \( \{u_n\} \subset H^1_\text{rad}(\mathbb{R}^2) \) with \( I_{\gamma,\beta}(u_n) \in [c, d] \) and \( ||u'_n, \beta(u_n)|| \geq \alpha \) such that

\[ ||u_n - T_{\gamma,\beta}(u_n)|| \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \]

Then owing to \( r \in (2, q) \), we have

\[ \int_{\mathbb{R}^2} \left( \frac{\omega}{2} |u_n|^2 + \frac{\beta}{2} |u_n|^q - C||u_n - T_{\gamma,\beta}(u_n)||^2 |u_n|^\alpha \right) \, dx \geq 0 \]  

(3.12)

for \( n \) large enough, where \( C \) has appeared in (3.11). Now we claim that sequence \( \{u_n\} \) is bounded in \( H^1_\text{rad}(\mathbb{R}^2) \). Otherwise, assume \( ||u_n|| \rightarrow \infty \). From (3.11) and (3.12), we deduce that

\[ ||u_n||^2 + B(u_n) + \beta||u_n||^4 + \gamma||u_n||^{4(1+\alpha)} - \lambda||u_n||^p \leq C \]  

(3.13)

for large \( n \). Here

\[ B(u_n) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{u_n^2}{|x|^2} h^2(|x|) \, dx \]

Note that, for any \( A_1 > 0 \), we can choose \( A_2 > 0 \) such that for any \( t \in \mathbb{R}^+ \),

\[ t^{1+\alpha} > A_1 t - A_2. \]

Applying this with \( t = ||u_n||^4 \), then by (3.13) we have

\[ ||u_n||^2 + B(u_n) + \int_{\mathbb{R}^2} \left( \beta |u_n|^q + \gamma A_1 |u_n|^4 - \lambda |u_n|^p \right) \, dx - A_2 \leq C. \]

(3.14)

Since \( 4 < p < q \), we take \( A_1 \) large enough such that the function

\[ \beta t^q + \gamma A_1 t^4 - \lambda t^p \geq 0, \quad \forall t \in \mathbb{R}^+. \]

So it is easy to get from (3.14) that \( \{u_n\} \) is bounded in \( H^1_\text{rad}(\mathbb{R}^2) \) for any fixed \( (\gamma, \beta) \in [0, 1] \times (0, 1) \). The claim combined with lemma 3.2 implies \( ||u'_n, \beta(u_n)|| \rightarrow 0 \) as \( n \rightarrow \infty \), which is a contradiction.

\[ \square \]
3.2. Invariant subsets of descending flows

Define the positive and negative cones by
\[ P^+ := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) : u \geq 0 \} \quad \text{and} \quad P^- := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) : u \leq 0 \}, \]
respectively. Set for \( \epsilon > 0, \)
\[ P^\epsilon_+ := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) : \text{dist}(u, P^+) < \epsilon \} \quad \text{and} \quad P^-_\epsilon := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) : \text{dist}(u, P^-) < \epsilon \}, \]
where \( \text{dist}(u, P^\pm) = \inf_{v \in P^\pm} \| u - v \|. \) Clearly, \( P^-_\epsilon = -P^+_\epsilon. \) Let \( W = P^+_\epsilon \cup P^-_\epsilon. \) It is not hard to check that \( W \) is an open and symmetric subset of \( H^1_{\text{rad}}(\mathbb{R}^2) \) and \( H^1_{\text{rad}}(\mathbb{R}^2) \setminus W \) contains only sign-changing functions.

The following shows that, for \( \epsilon \) small enough, all sign-changing solutions to equation (3.1) are contained in \( H^1_{\text{rad}}(\mathbb{R}^2) \setminus W. \)

**Lemma 3.4.** There exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0), \)

(1) \( T_{\gamma, \beta}(\partial P^-_\epsilon) \subset P^-_\epsilon \) and every nontrivial solution \( u \in P^-_\epsilon \) is negative,

(2) \( T_{\gamma, \beta}(\partial P^+_\epsilon) \subset P^+_\epsilon \) and every nontrivial solution \( u \in P^+_\epsilon \) is positive.

**Proof.** Since the two conclusions are similar, we only prove \( T_{\gamma, \beta}(\partial P^-_\epsilon) \subset P^-_\epsilon. \) Let \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \) and \( v = T_{\gamma, \beta}(u). \) Since \( \text{dist}(v, P^-) \leq \| v^+ \|, \) by Sobolev’s inequality we have
\[
\text{dist}(v, P^-)\| v^+ \| = \| v^+ \|^2 = \langle v, v^+ \rangle \\
\leq \lambda \int_{\mathbb{R}^2} |u|^{p-2}u v^+ \, dx + \beta \int_{\mathbb{R}^2} |u|^q - |u|^q v^+ \, dx - \gamma \| u \|^2 \int_{\mathbb{R}^2} u^2 v^+ \, dx \\
= \lambda \int_{\mathbb{R}^2} |u^+|^{p-2}u v^+ \, dx + \beta \int_{\mathbb{R}^2} |u^+|^q - |u^+|^q v^+ \, dx \\
\leq \lambda \| \text{dist}(u, P^-) \|^{p-1} + \text{dist}(u, P^-)^{q-1} \| v^+ \|,
\]
which implies that
\[ \text{dist}(v, P^-) \leq C[\text{dist}(u, P^-)^{p-1} + \text{dist}(u, P^-)^{q-1}]. \]

Then there exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0), \)
\[ \text{dist}(T_{\gamma, \beta}(u), P^-) = \text{dist}(v, P^-) < \epsilon. \]

Therefore, we have \( T_{\gamma, \beta}(u) \in P^-_\epsilon \) for any \( u \in P^-_\epsilon. \) Assume that there exists \( u \in P^-_\epsilon \) such that \( T_{\gamma, \beta}(u) = u, \) then \( u \in P^+. \) If \( u \not\equiv 0, \) by the maximum principle, \( u < 0 \) in \( \mathbb{R}^2. \)

Since the operator \( T_{\gamma, \beta} \) may not be locally Lipschitz continuous, we need to construct a locally Lipschitz continuous vector field which inherits its properties. Similar to the proof of lemma 2.1 in [4], we have

**Lemma 3.5.** There exists a locally Lipschitz continuous operator \( B_{\gamma, \beta} : H^1_{\text{rad}}(\mathbb{R}^2) \setminus K_{\gamma, \beta} \to H^1_{\text{rad}}(\mathbb{R}^2) \) such that

(i) \( T_{\gamma, \beta}(u), u - B_{\gamma, \beta}(u) \geq \frac{1}{2}\| u - T_{\gamma, \beta}(u) \|^2; \)
(ii) \( \frac{1}{2}\| u - B_{\gamma, \beta}(u) \|^2 \leq \| u - T_{\gamma, \beta}(u) \|^2 \leq 2\| u - B_{\gamma, \beta}(u) \|^2; \)
(iii) \( T_{\gamma, \beta}(\partial P^\pm_\epsilon) \subset P^\pm_\epsilon, \forall \epsilon \in (0, \epsilon_0); \)
(iv) \( B_{\gamma, \beta} \) is odd.
where $K_{\gamma, \beta} := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) : I_{\gamma, \beta}'(u) = 0 \}$ and $c_0$ has been given in lemma 3.4.

In what follows, we prove functional $I_{\gamma, \beta}$ satisfies the (PS)-condition.

**Lemma 3.6.** Let $(\gamma, \beta) \in (0, 1) \times (0, 1)$ and $c \in \mathbb{R}$. Assume $\{ u_n \} \subset H^1_{\text{rad}}(\mathbb{R}^2)$ satisfy $I_{\gamma, \beta}(u_n) \to c$ and $I_{\gamma, \beta}'(u_n) \to 0$ as $n \to \infty$, then up to a subsequence, $u_n \to u$ in $H^1_{\text{rad}}(\mathbb{R}^2)$ for some $u \in H^1_{\text{rad}}(\mathbb{R}^2)$.

**Proof.** For $\mu \in (6, q)$, we have

$$
I_{\gamma, \beta}(u_n) - \frac{1}{\mu} I'_{\gamma, \beta}(u_n), u_n) = \frac{\mu - 2}{2\mu} \| u_n \|^2 + \frac{\mu - 6}{\mu} B(u_n)
$$

$$+
\gamma \frac{\mu - 4(1 + \alpha)}{4(1 + \alpha)\mu} \| u_n \|^{4(1 + \alpha)} + \lambda \frac{\mu - p}{p\mu} \int_{\mathbb{R}^2} |u_n|^p dx + \frac{\beta q - \mu}{q\mu} \int_{\mathbb{R}^2} |u|^{q} dx,
$$

which implies that

$$
|I_{\gamma, \beta}(u_n)| + \frac{1}{\mu} |I'_{\gamma, \beta}(u_n)| \| u_n \| \geq \frac{\mu - 2}{2\mu} \| u_n \|^2 + \frac{\mu - 6}{\mu} B(u_n)
$$

$$+
\gamma \frac{\mu - 4(1 + \alpha)}{4(1 + \alpha)\mu} \| u_n \|^{4(1 + \alpha)} - \lambda \frac{\mu - p}{p\mu} \int_{\mathbb{R}^2} |u_n|^p dx + \frac{\beta q - \mu}{q\mu} \int_{\mathbb{R}^2} |u|^{q} dx.
$$

As in the proof of lemma 3.3, one sees that sequence $\{ u_n \}$ is bounded in $H^1_{\text{rad}}(\mathbb{R}^2)$. Without loss of generality, we assume that there exists $u \in H^1_{\text{rad}}(\mathbb{R}^2)$ such that

$$
u_n \to u \text{ weakly in } H^1_{\text{rad}}(\mathbb{R}^2),
u_n \to u \text{ strongly in } L^r(\mathbb{R}^2) \text{ for } r \in (2, +\infty).
$$

Note that

$$
(I'_{\gamma, \beta}(u_n) - I'_{\gamma, \beta}(u), u_n - u)
$$

$$
= \| u_n - u \|^2 + \int_{\mathbb{R}^2} B(u_n)(u_n - u)^2 dx + \int_{\mathbb{R}^2} \left[ B(u_n) - B(u) \right] u(u_n - u) dx
$$

$$-
\lambda \int_{\mathbb{R}^2} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) dx + \frac{\gamma}{4^\alpha} \int_{\mathbb{R}^2} u_n^2(u_n - u)^2 dx
$$

$$+
\gamma \left( \| u_n \|_{L^4}^4 - \| u \|_{L^4}^4 \right) \int_{\mathbb{R}^2} |u_n|^2 u - |u|^2 u (u_n - u) dx
$$

$$-
\beta \int_{\mathbb{R}^2} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) dx.
$$

(3.15)

Based on the boundedness of $\{ u_n \}$ in $H^1_{\text{rad}}(\mathbb{R}^2)$ and the conclusions of lemma 2.3 we can
deduce that
\[
\int_{\mathbb{R}^2} S(u_n)(u_n - u)^2 \, dx \to 0, \quad \int_{\mathbb{R}^2} [S(u_n) - S(u)]u(u_n - u) \, dx \to 0,
\]
\[
\int_{\mathbb{R}^2} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \to 0, \quad (I'_{\gamma,\beta}(u_n) - I'_{\gamma,\beta}(u), u_n - u) \to 0,
\]
\[
\|u_n\|_{L^4}^4 \int_{\mathbb{R}^2} u_n^2(u_n - u)^2 \, dx \to 0, \quad (\|u_n\|_{L^4}^4 - \|u\|_{L^4}^4) \int_{\mathbb{R}^2} (|u_n|^2 - |u|^2)(u_n - u) \, dx \to 0,
\]
\[
\int_{\mathbb{R}^2} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) \, dx \to 0, \quad \text{as } n \to \infty.
\]
Therefore, from (3.15) we deduce that \( u_n \to u \) in \( H^1_{rad}(\mathbb{R}^2) \) as \( n \to \infty \). \( \square \)

Now we recall a critical point theorem introduced in [32]. Let \( E \) be a Banach space, \( J \in C^1(E, \mathbb{R}) \), \( P, Q \subset E \) be open sets, \( M = P \cap Q \), \( \Sigma = \partial P \cap \partial Q \) and \( W = P \cup Q \). We denote by \( K \) the set of critical points of \( J \), that is, \( K = \{ u \in E : J'(u) = 0 \} \) and \( E_0 := E \setminus K \). For \( c \in \mathbb{R}, K_c = \{ u \in E : J(t) = c, J'(t) = 0 \} \) and \( J' = \{ u \in E : J(t) \leq c \} \).

**Definition 3.1 ([32]).** \( \{P, Q\} \) is called an admissible family of invariant sets with respect to \( J \) at level \( c \), provided that the following deformation property holds: if \( K_c \setminus W = \emptyset \), then, there exists \( \epsilon_0 > 0 \) such that for \( c \in (0, \epsilon_0) \), there exists \( \eta \in C(E, E) \) satisfying

1. \( \eta(P) \subset P, \eta(Q) \subset Q; \)
2. \( \eta|_{J \to \emptyset} = id; \)
3. \( \eta(J + \epsilon \setminus W) \subset J - \epsilon. \)

**Theorem 3.1 ([32]).** Assume that \( \{P, Q\} \) is an admissible family of invariant sets with respect to \( J \) at any level \( c \geq c_\ast := \inf_{u \in \Sigma} J(u) \) and there exists a map \( \psi_0 : \Delta \to E \) satisfying

1. \( \psi_0(\partial_1 \Delta) \subset P \) and \( \psi_0(\partial_2 \Delta) \subset Q; \)
2. \( \psi_0(\partial_0 \Delta) \cap M = \emptyset, \)
3. \( \sup_{u \in \psi_0(\partial_0 \Delta)} J(u) < c_\ast \)

where \( \Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 < 1\}, \partial_1 \Delta = \{0\} \times [0, 1], \partial_2 \Delta = [0, 1] \times \{0\}, \partial_0 \Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 = 1\}, \) Define
\[
c = \inf_{u \in \Gamma} \sup_{t \in (\Delta \setminus W)} J(u),
\]
where \( \Gamma := \{ \psi \in C(\Delta, E) : \psi(\partial_1 \Delta) \subset P, \psi(\partial_2 \Delta) \subset Q, \psi|_{\partial_0 \Delta} = \psi|_{\partial_0 \Delta} \}. \) Then \( c \geq c_\ast \) and \( K_c \setminus W = \emptyset. \)

In order to use theorem 3.1 to prove the existence of sign-changing solutions to equation (3.1), we take \( E = H^1_{rad}(\mathbb{R}^2), P = P^+_\gamma, Q = P^-_\gamma \) and \( J = I_{\gamma,\beta} \). We need to show that \( \{P^+_\gamma, P^-_\gamma\} \) is an admissible family of invariant sets for the functional \( I_{\gamma,\beta} \) at any level \( c \in \mathbb{R} \). Indeed, \( K_c \subset C \) if \( K_c \setminus W = \emptyset. \) Since the functional \( I_{\gamma,\beta} \) satisfies the (PS)-condition, \( K_c \) is compact. Thus, \( 2\delta := \text{dist}(K_c, \partial W) > 0. \) Similar to lemma 3.6 in [33], we have

**Lemma 3.7 (Deformation lemma).** Let \( S \subset H^1_{rad}(\mathbb{R}^2) \) and \( c \in \mathbb{R} \) such that
\[
\forall u \in I_{\gamma,\beta}^{-1}([c - 2\epsilon_0, c + 2\epsilon_0]) \cap S_\delta : \| I_{\gamma,\beta}(u) \| \geq \epsilon_0,
\]
where \( \epsilon_0 \) has been given in lemma 3.4 and \( S_\delta := \{ u \in S : \text{dist}(u, S) < 2\delta \}. \) Then for \( \epsilon_1 \in (0, \epsilon_0) \) there exists \( \eta \in C([0, 1] \times H^1_{rad}(\mathbb{R}^2), H^1_{rad}(\mathbb{R}^2)) \) such that
\(i\) \(\eta(t, u) = u\), if \(t = 0\) or if \(u \not\in I_{\gamma, \beta}^{-1}([c - 2\epsilon_1, c + 2\epsilon_1])\);

\(ii\) \(\eta(1, I_{\gamma, \beta}^+ \cap S) \subset I_{\gamma, \beta}^-\);

\(iii\) \(I_{\gamma, \beta}(v(\cdot, u))\) is not increasing for all \(u \in H^1_{rad}(\mathbb{R}^2)\);

\(iv\) \(\eta(t, P^\pm_S) \subset P^+_S\), \(\eta(t, P^-_S) \subset P^-_S\), \(\forall t \in [0, 1]\).

3.3. Proof of theorem 1.1

**Lemma 3.8.** For \(r \in [2, +\infty)\) there exists \(C > 0\) independent of \(\epsilon\) such that \(\|u\|_r \leq C\epsilon\) for \(u \in M = P^+_\epsilon \cap P^-_\epsilon\).

**Proof.** For any fixed \(u \in M\), we have

\[
\|u^\pm\|_r = \inf_{v \in P^\pm} \|u - v\|_r \leq C \inf_{v \in P^\pm} \|u - v\| \leq C \text{dist}(u, P^\pm).
\]

So \(\|u\|_r \leq C\epsilon\) for \(u \in M\).

**Lemma 3.9.** \(I_{\gamma, \beta}(u) \geq \frac{c}{2}\) for all \(u \in \partial P^+_\epsilon \cap \partial P^-_\epsilon\) if \(\epsilon > 0\) is small enough, that is, \(c_* \geq \frac{c}{4}\).

**Proof.** For any fixed \(u \in \partial P^+_\epsilon \cap \partial P^-_\epsilon\), we have \(\|u^+\| \geq \text{dist}(u, P^\pm) = \epsilon\). By lemma 3.8, we have

\[
I_{\gamma, \beta}(u) = \frac{1}{2} \|u\|^2 + B(u) + \frac{\gamma}{4(1 + \alpha)} \|u\|_{4(1 + \alpha)}^4 - \int_{\mathbb{R}^2} \left(\frac{\lambda}{p} |u|^p + \frac{\beta}{q} |u|^q\right) \, dx
\]

\[
\geq \frac{1}{2} \epsilon^2 - \frac{\lambda}{p} \|u\|_p^p - \frac{1}{q} \|u\|_q^q
\]

\[
\geq \frac{1}{2} \epsilon^2 - \frac{C}{p} \epsilon^p - \frac{C}{q} \epsilon^q \geq \frac{c}{4}
\]

for \(\epsilon\) small enough.

**Proof of theorem 1.1.** We use theorem 3.1 to prove the existence of sign-changing solutions to equation (3.1). Take \(S = H^1_{rad}(\mathbb{R}^2) \setminus W\) in lemma 3.7, then we can easily deduce that \(\{P^+_\epsilon, P^-_\epsilon\}\) is an admissible family of invariant sets for the functional \(I_{\gamma, \beta}\) at any level \(c \in \mathbb{R}\).

In what follows, we divide three steps to complete the proof.

**Step 1.** Choose \(v_1, v_2 \in C_0^\infty(B_1(0))\) such that \(\text{supp}(v_1) \cap \text{supp}(v_2) = \emptyset\) and \(v_1 < 0, v_2 > 0\), where \(B_1(0) := \{x \in \mathbb{R}^2 : |x| < 1\}\). For \((t, s) \in \Delta\), let

\[
\psi_0(t, s) := R[v_1(R) + sv_2(R)],
\]

where \(R > 1\) will be determined later. Clearly, for \(t, s \in [0, 1]\), \(\psi_0(0, s)(\cdot) = Rv_2(R) \in P^+\) and \(\psi_0(t, 0)(\cdot) = Rv_1(R) \in P^-\). In view of lemma 3.9, we have for small \(\epsilon > 0\), \(c_* = \inf_{\lambda, \beta \in \Sigma} I_{\gamma, \beta}(u) \geq \frac{c}{2}\) for any \((\lambda, \beta) \in (0, 1] \times (0, 1]\). Set

\[
\rho := \min\{\|v_1 + (1 - t)v_2\|_2 : 0 \leq t \leq 1\} > 0
\]
then $\|u_t\|_p^2 \geq \rho$ for $u \in \psi_0(\partial_0 \triangle)$ and it follows from lemma 3.8 that $\psi_0(\partial_0 \triangle) \cap P^+ \cap P^- = \emptyset$. Let $u_t = \psi_0(t, 1 - t)$ for $t \in [0, 1]$. A direct computation shows that

\[
\int_{\mathbb{R}^2} |\nabla u_t|^2 \, dx = R^2 \int_{\mathbb{R}^2} (t^2 |\nabla v_1|^2 + (1 - t)^2 |\nabla v_2|^2) \, dx,
\]

\[
\int_{\mathbb{R}^2} |u_t|^4 \, dx = R^{q-2} \int_{\mathbb{R}^2} (t^r |v_1|^r + (1 - t)^r |v_2|^r) \, dx \quad \text{for } r \in [2, +\infty),
\]

\[
\left( \int_{\mathbb{R}^2} |u_t|^4 \, dx \right)^{1+\alpha} = R^{2(1+\alpha)} \left( \int_{\mathbb{R}^2} (t^4 |v_1|^4 + (1 - t)^4 |v_2|^4) \, dx \right)^{1+\alpha},
\]

$B(u_t) \leq R^2 B(u_t)$, where $u_t = tv_1 + (1 - t)v_2$.

Based on the above facts, we have

\[
I_{\gamma, \beta}(u_t) \leq \frac{1}{2} \|u_t\|_p^2 + B(u_t) + \frac{\gamma}{4(1 + \alpha)} \left( \int_{\mathbb{R}^2} |u_t|^4 \, dx \right)^{1+\alpha} - \int_{\mathbb{R}^2} \left( \frac{\lambda}{p} |u_t|^p + \frac{\beta}{q} |u_t|^q \right) \, dx
\]

\[
\leq R^2 \int_{\mathbb{R}^2} (t^2 |\nabla v_1|^2 + (1 - t)^2 |\nabla v_2|^2) \, dx + R^2 B(u_t)
\]

\[
+ \frac{\gamma R^{2(1+\alpha)}}{4(1 + \alpha)} \left( \int_{\mathbb{R}^2} (t^4 |v_1|^4 + (1 - t)^4 |v_2|^4) \, dx \right)^{1+\alpha}
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^2} \omega(t^2 |v_1|^2 + (1 - t)^2 |v_2|^2) \, dx - \lambda \frac{R^{p-2}}{p} \int_{\mathbb{R}^2} (t^p |v_1|^p + (1 - t)^p |v_2|^p) \, dx.
\]

Due to $0 < \alpha < \frac{p-4}{2}$, one has $I_{\gamma, \beta}(u_t) \to -\infty$ as $R \to \infty$ uniformly for $(\gamma, \beta) \in (0, 1) \times (0, 1), t \in [0, 1]$. Hence, choosing $R$ large enough, we have

\[
\sup_{u \in \psi_0(\partial_0 \triangle)} I_{\gamma, \beta}(u) < c_* := \inf_{u \in \Sigma} I_{\gamma, \beta}(u), \quad \forall (\gamma, \beta) \in (0, 1) \times (0, 1).
\]

Since $I_{\gamma, \beta}$ satisfies the assumptions of theorem 3.1, then

\[
c_{\gamma, \beta} = \inf_{\psi \in \Psi} \sup_{u \in \psi(\triangle \setminus W)} I_{\gamma, \beta}(u)
\]

is a critical value of $I_{\gamma, \beta}$ satisfying $c_{\gamma, \beta} \geq c_*$ and there exists $u_{\gamma, \beta} \in H^1_{rad}(\mathbb{R}^2) \setminus (P^+ \cup P^-)$ such that $I_{\gamma, \beta}(u_{\gamma, \beta}) = c_{\gamma, \beta}$ and $I'_{\gamma, \beta}(u_{\gamma, \beta}) = 0$ for $(\gamma, \beta) \in (0, 1) \times (0, 1)$.

**Step 2.** Passing to the limit as $\gamma \to 0$ and $\beta \to 0$. From the definition of $c_{\gamma, \beta}$, we see that for any $(\gamma, \beta) \in (0, 1) \times (0, 1)$,

\[
c_{\gamma, \beta} \leq C_R := \sup_{u \in \psi(\triangle \setminus W)} I_{1, 0}(u) < \infty,
\]

(3.16) where $C_R$ is independent of $(\gamma, \beta) \in (0, 1) \times (0, 1)$. We may only deal with the case $\gamma = \beta$. The case $\gamma \neq \beta$ can be treated similarly. Choosing a sequence $\{\gamma_n\} \subset (0, 1)$ such that $\gamma_n \to 0^+$, then we find a sequence of sign-changing critical points $\{u_{\gamma_n}\}$ (still denoted by $\{u_{\gamma_n}\}$) of $I_{\gamma_n, \beta_n}$ and $\lim_{n \to \infty} I_{\gamma_n, \beta_n}(u_n) = \lim_{n \to \infty} c_{\gamma_n, \beta_n} = c_* \geq c_*$. Now we show that $\{u_n\}$ is a bounded sequence in $H^1_{rad}(\mathbb{R}^2)$. By the definition of $I_{\gamma, \beta}$, we have

\[
c_{\gamma_n, \beta_n} = \frac{1}{2} \|u_n\|^2 + B(u_n) + \frac{\gamma_n}{4(1 + \alpha)} \|u_n\|_{4(1+\alpha)}^4 - \int_{\mathbb{R}^2} \left( \frac{\lambda}{p} |u_n|^p + \frac{\beta_n}{q} |u_n|^q \right) \, dx
\]

(3.17)
and
\[
\|u_n\|^2 + 6B(u_n) + \gamma_n\|u_n\|^{4(1+\alpha)} - \int_{\mathbb{R}^2} (\lambda|u_n|^p + \beta_n|u_n|^q) \, dx = 0. \tag{3.18}
\]
Moreover, it follows from the Pohozaev identity that
\[
\omega\|u_n\|^2 + 4B(u_n) + \frac{\gamma_n}{2}\|u_n\|^{4(1+\alpha)} - \int_{\mathbb{R}^2} \left(\frac{2\lambda}{p}|u_n|^p + \frac{2\beta_n}{q}|u_n|^q\right) \, dx = 0.
\tag{3.19}
\]
Multiplying (3.17) by \((1 + \alpha)\), (3.18) by \(-1\) and (3.19) by \(1\) and adding them up, we get
\[
2(1 + \alpha)c_{\gamma_n, \beta_n} = \alpha \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + (1 + \alpha) \int_{\mathbb{R}^2} u_n^2 \, dx + 2\alpha B(u_n)
\]
\[
+ \frac{p-4-2\alpha}{\lambda} \int_{\mathbb{R}^2} |u_n|^p \, dx + \frac{q-4-2\alpha}{\beta_n} \int_{\mathbb{R}^2} |u_n|^q \, dx,
\tag{3.20}
\]
which, together with (3.16), implies that \(\{u_n\}\) is a bounded sequence in \(H^1_{\text{rad}}(\mathbb{R}^2)\). In view of lemma 3.9, we have
\[
\lim_{n \to \infty} I_{\lambda}(u_n) = \lim_{n \to \infty} \left( I_{\lambda, \beta_n}(u_n) - \frac{\gamma_n}{4(1+\alpha)}\|u_n\|^{4(1+\alpha)} + \frac{\beta_n}{q} \int_{\mathbb{R}^2} |u_n|^q \, dx \right)
\]
\[
= \lim_{n \to \infty} c_{\gamma_n, \beta_n} = c^* > \frac{\epsilon^2}{4}.
\]
Moreover, for any \(\psi \in C_0^\infty(\mathbb{R}^2) \cap H^1_{\text{rad}}(\mathbb{R}^2)\), we have
\[
\lim_{n \to \infty} \langle I'_{\lambda}(u_n), \psi \rangle = \lim_{n \to \infty} \left( \langle I'_{\lambda, \beta_n}(u_n), \psi \rangle - \gamma_n\|u_n\|^{4} \int_{\mathbb{R}^2} |u_n|^2 \psi \, dx + \beta_n \int_{\mathbb{R}^2} |u_n|^{q-2} u_n \psi \, dx \right) = 0.
\]
That is to say, \(\{u_n\}\) is a bounded Palais–Smale sequence for \(I_{\lambda}\) at level \(c^*\). Thus, there exists \(u^* \in H^1_{\text{rad}}(\mathbb{R}^2)\) such that \(u_n \rightharpoonup u^*\) weakly in \(H^1_{\text{rad}}(\mathbb{R}^2)\) and \(u_n \to u^*\) strongly in \(L^q(\mathbb{R}^2)\) for \(q \in (2, +\infty)\). Similarly to the arguments of lemma 3.6, we see that \(I'_{\lambda}(u^*) = 0\) and \(u_n \to u^*\) strongly in \(H^1_{\text{rad}}(\mathbb{R}^2)\) as \(n \to 0\). Thus, the fact that \(u_n \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus (P_+^c \cup P_-^c)\) yields \(u^* \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus (P_+^c \cup P_-^c)\) and then \(u^*\) is a sign-changing solution of equation (1.4).

**Step 3.** Define
\[
\bar{c} := \inf_{u \in \Theta} I_{\lambda}(u), \quad \Theta := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus \{0\}, I'_{\lambda}(u) = 0, u \not\equiv 0 \}.
\]
Based on above, we see \(\Theta \neq \emptyset\) and \(\bar{c} \leq c^*\), where \(c^*\) is given in Step 2. By the definition of \(\bar{c}\), there exists \(\{u_n\} \subset H^1_{\text{rad}}(\mathbb{R}^2)\) such that \(I_{\lambda}(u_n) \to \bar{c}\) and \(I'_{\lambda}(u_n) = 0\). Using the earlier arguments, we can deduce that \(\{u_n\}\) is a bounded sequence in \(H^1_{\text{rad}}(\mathbb{R}^2)\). Arguing as in lemma 3.6, there exists a nontrivial \(u \in H^1_{\text{rad}}(\mathbb{R}^2)\) such that \(I_{\lambda}(u) = \bar{c}\) and \(I'_{\lambda}(u) = 0\). On the other hand, we deduce from \(I'_{\lambda}(u_n), u_n^\pm = 0\) that for any \(\varepsilon > 0\), there exists \(C_\varepsilon > 0\) such that
\[
C\|u_n^\pm\|^2_p \leq \|u_n^\pm\|_p^2 \leq \lambda \int_{\mathbb{R}^2} |u_n|^{p-2} u_n u_n^\pm \, dx = \lambda \int_{\mathbb{R}^2} |u_n^\pm|^p \, dx
\]
\[^{3101}\]
which, together with the boundedness of $\{u_n\}$ in $H^1_{rad}(\mathbb{R}^2)$ implies that $\|u_n^\pm\|_p \geq C$. Hence, $\|u_k\|_p \geq C$, and then we see that $u$ is a ground state sign-changing solution of equation (1.4). The proof is complete.

\section*{4. Asymptotic behavior of nodal solutions as $\lambda \to +\infty$}

In this section we attempt to get energy double behavior of the sign-changing solution obtained in theorem 1.1 as $\lambda$ tends to infinity.

We state the following modified equation:

$$-\triangle u + \omega u + \lambda B(u)u = |u|^{p-2}u \text{ in } \mathbb{R}^2, u \in H^1_{rad}(\mathbb{R}^2),$$

where $\bar{\lambda} = \frac{l}{l^2}$ and $B(u)$ is given in section 3. The energy functional $J_{\lambda}$ corresponding to equation (4.1) is denoted by

$$J_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2)dx + \frac{\bar{\lambda}}{2} \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.$$

It is easy to check that for any $u \in H^1_{rad}(\mathbb{R}^2)$, the identity $J_{\lambda}(\bar{u}) = \lambda \frac{l}{l^2} J_{\lambda}(u)$ holds, where $\bar{u} = \lambda \frac{l}{l^2} u$. In particular, $u$ solves equation (1.4) if and only if $\bar{u}$ satisfies the modified equation (4.1).

\textbf{Lemma 4.1.} $w \in H^1_{rad}(\mathbb{R}^2)$ is a ground state sign-changing solution of equation (1.4) if and only if $w$ is a ground state sign-changing solution of equation (4.1), where $w = \lambda \frac{l}{l^2} w$.

\textbf{Proof.} It is not hard to check that $J_{\lambda}(\bar{w}) = \lambda \frac{l}{l^2} J_{\lambda}(w)$.

\textit{Necessity:} assume on the contrary that $\bar{w}$ is not a ground state sign-changing solution of equation (4.1), then there exists a sign-changing solution $\bar{v} \in H^1_{rad}(\mathbb{R}^2)$ of equation (4.1) with

$$J_{\lambda}(\bar{v}) < J_{\lambda}(\bar{w}).$$

Moreover, there exists sign-changing solution $v \in H^1_{rad}(\mathbb{R}^2)$ of equation (1.4) such that

$$J_{\lambda}(\bar{v}) = \lambda \frac{l}{l^2} J_{\lambda}(v) > \lambda \frac{l}{l^2} J_{\lambda}(w) = J_{\lambda}(\bar{w}),$$

which contradicts (4.2).

\textit{Sufficiency:} if $\bar{w}$ is a ground state sign-changing solution of equation (4.1), then we attempt to obtain a contradiction by assuming that $w := \lambda \frac{l}{l^2} w \in H^1_{rad}(\mathbb{R}^2)$ is not a ground state sign-changing solution of equation (1.4). Clearly, $w$ solves equation (1.4). Thus, there exists a sign-changing solution $v \in H^1_{rad}(\mathbb{R}^2)$ of (1.4) with

$$I_{\lambda}(v) < I_{\lambda}(w).$$

In view of equations (4.1) and (1.4) that there exists sign-changing solution $\bar{v} \in H^1_{rad}(\mathbb{R}^2)$ of equation (4.1) such that

$$I_{\lambda}(v) = \lambda \frac{l}{l^2} J_{\lambda}(\bar{v}) > \lambda \frac{l}{l^2} J_{\lambda}(\bar{w}) = I_{\lambda}(w),$$

which is a contradiction. The proof is complete.
Now we prove theorem 1.2. To emphasize the dependence on \( \lambda \), we use \( w_\lambda \in H^1_{\text{rad}}(\mathbb{R}^2) \) to denote a ground state sign-changing solution of (1.4) obtained in theorem 1.1. Then by lemma 4.1, we know that \( \bar{w}_\lambda \) is also a ground state sign-changing solution of (4.1).

**Proof of theorem 1.2.** Define

\[
m_\lambda := \inf_{u \in \Theta} J_1(u), \quad \Theta := \{ u \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus \{0\}, J'_1(u) = 0, u^+ \neq 0 \},
\]

(4.5)

In view of theorem 1.1 and lemma 4.1, we know \( \Theta \neq \emptyset \). Moreover, if we replace \( I_\lambda \) by \( J_\lambda \), then all the above calculations in section 3 can be repeated word by word. So we can also get a sign-changing solution to equation (4.1). Furthermore, we can deduce from (3.16) that

\[
m_\lambda \leq C_R := \sup_{u \in \psi_0(\Delta)} J_1(u) < \infty,
\]

(4.6)

where \( \psi_0(\Delta) \) has been defined in the proof of theorem 1.1 and \( C_R \) is independent of \( \lambda \in (0, 1] \) and the functional \( J_1 : H^1_{\text{rad}}(\mathbb{R}^2) \to \mathbb{R} \) is defined as

\[
\bar{J}_1 := \frac{1}{2} \| u \|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} h(|x|) dx + \frac{1}{4(1 + \alpha)} \| u \|_2^{4(1 + \alpha)} - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.
\]

(4.7)

We now show that, for \( \lambda \to 0^+ \), \( \{ \bar{w}_\lambda \}_{\lambda \in (0, 1)} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^2) \). Indeed,

\[
m_\lambda = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \bar{w}_\lambda|^2 dx + \frac{\bar{\lambda}}{2} \int_{\mathbb{R}^2} |\bar{w}_\lambda|^2 dx + 3 \lambda \int_{\mathbb{R}^2} \frac{\bar{h}^2(|x|)}{|x|^2} |\bar{w}_\lambda|^2 dx - \frac{1}{p} \int_{\mathbb{R}^2} |\bar{w}_\lambda|^p dx,
\]

(4.8)

where \( h(|x|) = \int_0^{|x|} \frac{\bar{w}_\lambda^2(\tau)}{2} d\tau \), and

\[
0 = \int_{\mathbb{R}^2} |\nabla \bar{w}_\lambda|^2 dx + \int_{\mathbb{R}^2} \omega \bar{w}_\lambda^2 dx + 3 \lambda \int_{\mathbb{R}^2} \frac{\bar{h}^2(|x|)}{|x|^2} |\bar{w}_\lambda|^2 dx - \int_{\mathbb{R}^2} |\bar{w}_\lambda|^p dx.
\]

(4.9)

Moreover, we have the following Pohozaev identity

\[
0 = \omega \int_{\mathbb{R}^2} \bar{w}_\lambda^2 dx + 2 \lambda \int_{\mathbb{R}^2} \frac{\bar{h}^2(|x|)}{|x|^2} |\bar{w}_\lambda|^2 dx - \frac{2}{p} \int_{\mathbb{R}^2} |\bar{w}_\lambda|^p dx.
\]

(4.10)

Multiplying (4.7) by 2, (4.8) by \(-1\) and (4.9) by 1 and adding them up, we get

\[
2m_\lambda = \int_{\mathbb{R}^2} \omega \bar{w}_\lambda^2 dx + \frac{p - 4}{p} \int_{\mathbb{R}^2} |\bar{w}_\lambda|^p dx.
\]

(4.11)

Based on conclusion (i) of lemma 2.3 and interpolation inequality, we have

\[
B(\bar{w}_\lambda) = \int_{\mathbb{R}^2} \frac{\bar{h}^2(|x|)}{|x|^2} |\bar{w}_\lambda|^2 dx \leq C \| \bar{w}_\lambda \|_4^2 \| \bar{w}_\lambda \|_2^2 \leq \| \bar{w}_\lambda \|_2 \| \bar{w}_\lambda \|_p \| \bar{w}_\lambda \|_2^2,
\]

which, together with (4.6), (4.7) and (4.10), implies that \( \{ \bar{w}_\lambda \}_{\lambda \in (0, 1)} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^2) \). Up to a subsequence, we assume that \( \bar{w}_\lambda \to \bar{w}_0 \) weakly in \( H^1_{\text{rad}}(\mathbb{R}^2) \) as \( \lambda \to 0^+ \). Note that \( \bar{w}_\lambda \) is a ground state sign-changing solution of equation (4.1), then, similarly to the arguments of lemma 3.6, we deduce that \( \bar{w}_\lambda \to \bar{w}_0 \) strongly in \( H^1_{\text{rad}}(\mathbb{R}^2) \) and \( \bar{w}_0 \) is a sign-changing solution of equation (1.6). Thus, we have
\[ m_\lambda = J_\lambda(\bar{w}_\lambda) = I(\bar{w}_0) + o(\bar{\lambda}) \geq m_0 + o(\bar{\lambda}), \]  
\tag{4.11}

where \( m_0 \) is denoted as a ground state sign-changing energy of equation (1.6). By the energy doubling property of sign-changing solution of equation (1.6), we get \( m_0 > 2c_0 \), where \( c_0 \) is defined in (1.11). In view of (4.11), we have
\[ m_\lambda = I(\bar{w}_0) + o(\bar{\lambda}) = m_0 + o(\bar{\lambda}) > 2c_0 \]  
\tag{4.12}

for \( \bar{\lambda} \) small enough.

Let \( u_\lambda \in \mathcal{N}_\lambda \) be a positive ground state solution of equation (1.4) (see [9]), then \( I_\lambda(u_\lambda) = c_\lambda \), where \( \mathcal{N}_\lambda \) and \( c_\lambda \) have been defined in (1.14) and (1.13), respectively. Similar to the argument in lemma 4.1, we can prove that there exists a positive ground state solution \( \bar{u}_\lambda := \lambda^{\frac{2-q}{2}} u_\lambda \) of equation (4.1) with \( J_\lambda(\bar{u}_\lambda) = c_\lambda \). Obviously, \( c_\lambda = \lambda^{\frac{2-q}{2}} c_\lambda \). Furthermore, taking \( \lambda \to 0^+ \), there exists \( \bar{u}_0 \in H^1_{rad}(\mathbb{R}^2) \) such that \( \bar{u}_\lambda \to \bar{u}_0 \) strongly in \( H^1_{rad}(\mathbb{R}^2) \) and \( \bar{u}_0 \) is a positive solution of equation (1.6). Hence, by the uniqueness of positive solution of (1.6), we have
\[ c_\lambda = J_\lambda(\bar{u}_\lambda) = I(\bar{u}_0) + o(\bar{\lambda}) = c_0 + o(\bar{\lambda}), \]  
\tag{4.13}

where \( c_0 \) is given in (1.11). Combining (4.12) with (4.13), we know that there exists \( \bar{\lambda}^* > 0 \) such that \( m_\lambda > 2c_\lambda \) for any \( \lambda \in (0, \bar{\lambda}^*) \). It is easy to check that \( m_\lambda \) is strictly larger than twice that of the ground state energy \( c_\lambda \) for \( \lambda \in (\lambda^*, +\infty) \), where \( \lambda^* = (\bar{\lambda}^*)^{\frac{2-q}{2}} \). The proof is complete. \( \square \)

5. Multiplicity

In this section, we prove the existence of infinitely many sign-changing solutions to equation (1.4) when \( \lambda \) is small enough.

5.1. The perturbed problem

We here employ a perturbed approach which is introduced in [32, 33] to overcome the difficulty getting the boundedness of the Palais–Smale, due to lack of the well-known Ambrosetti–Rabinowitz condition. For any fixed \( \beta \in (0, 1] \) and \( q \in (6, 8) \), we consider the modified problem
\[-\Delta u + \omega u + \mathcal{B}(u)u = \lambda |u|^{p-2} u + \beta |u|^{q-2} u, \quad x \in \mathbb{R}^2 \]  
\tag{5.1}

and its associated functional is given as below
\[ I_\beta(u) = I_\lambda(u) - \frac{\beta}{q} \int_{\mathbb{R}^1} |u|^q dx, \]

where \( \mathcal{B}(u) \) is given in section 3. It is easy to show that \( I_\beta \in C^1(H^1_{rad}(\mathbb{R}^2), \mathbb{R}) \) and
\[ \langle I'_\beta(u), v \rangle = \langle I'_\lambda(u), v \rangle - \beta \int_{\mathbb{R}^1} |u|^{q-2} uv dx, \quad u, v \in H^1_{rad}(\mathbb{R}^2). \]
Similarly to (3.2), for each \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \), the following equation
\[
-\Delta v + [\omega + B(u)]v = \lambda|u|^{p-2}u + \beta|u|^q u, \quad x \in \mathbb{R}^2
\] (5.2)
has a unique weak solution \( v \in H^1_{\text{rad}}(\mathbb{R}^2) \). In order to construct the descending flow for functional \( I_\beta \), we also introduce an auxiliary operator \( T_\beta : u \in H^1_{\text{rad}}(\mathbb{R}^2) \rightarrow v \in H^1_{\text{rad}}(\mathbb{R}^2) \), where \( v = T_\beta(u) \) is the unique weak solution of problem (3.2). Clearly, the fact that \( u \) is a solution of problem (5.2) is equivalent to that \( u \) is a fixed point of \( T_\beta \). As in section 2, one can obtain that the operator \( T_\beta : H^1_{\text{rad}}(\mathbb{R}^2) \rightarrow H^1_{\text{rad}}(\mathbb{R}^2) \) is well defined and is continuous. In the following, since the proof is similar as in section 2, we omit the details.

**Lemma 5.1.**

1. \( \langle L'_\beta(u), u - T_\beta(u) \rangle \geq \|u - T_\beta(u)\|^2 \) for all \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \);
2. \( \|L'_\beta(u)\| \leq \|u - T_\beta(u)\|(1 + \xi\|u\|^4) \) for some \( \xi > 0 \) and all \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \).

**Lemma 5.2.** For any fixed \( \beta \in (0, 1] \), \( c < d \) and \( \alpha > 0 \) (which depends on \( \beta \)) and \( \lambda_\alpha > 0 \) such that \( \|u - T_\beta(u)\| \geq \delta \) if \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \), \( I_\beta(u) \in [c, d] \) and \( \|L'_\beta(u)\| \geq \alpha \) and \( \lambda \in (0, \lambda_\alpha) \).

**Proof.** Let \( \mu \in (6, q) \). For any \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \), we have
\[
I_\beta(u) - \frac{1}{\mu} \langle u, u - T_\beta(u) \rangle = \frac{\mu - 2}{2\mu} \|u\|^2 + \frac{1}{\mu} \int_{\mathbb{R}^2} B(u) u (u - T_\beta(u)) \, dx + \frac{\mu - 2}{\mu} B(u) + \frac{(p - \mu)\beta}{\mu p} \|u\|_\infty^q + \frac{(q - \mu)\beta}{\mu q} \|u\|_\infty^q + \frac{(p - \mu)\lambda}{\mu p} \|u\|_p^p \quad \text{for some } C > 0.
\] (5.3)

Then, by the conclusion (iv) of lemma 2.2, we have
\[
\frac{\mu - 2}{2\mu} \|u\|^2 + \frac{\mu - 6}{4\mu} B(u) + \frac{(q - \mu)\beta}{\mu q} \|u\|_\infty^q + \frac{(p - \mu)\lambda}{\mu p} \|u\|_p^p \leq C \left( |I_{\gamma, \beta}(u)| + \|u\| \|u - T_{\gamma, \beta}(u)\| + \int_{\mathbb{R}^2} B(u) |u(u - T_{\gamma, \beta}(u))| \, dx \right).
\] (5.4)

By lemma 2.5, Young’s inequality and (5.4) we have
\[
\frac{\mu - 6}{8\mu} \|u\|_4^4 + \frac{\mu - 2}{4\mu} \|u\|^2 + \frac{\mu - 6}{2\mu} B(u) + \frac{(q - \mu)\beta}{\mu q} \|u\|_\infty^q + \frac{(p - \mu)\lambda}{\mu p} \|u\|_p^p \leq C \left( |I_\beta(u)| + \|u\| \|u - T_\beta(u)\| + \int_{\mathbb{R}^2} B(u) |u(u - T_\beta(u))| \, dx \right).
\] (5.5)

Set \( r = q \) in the conclusion (iii) of lemma 2.3, then by lemma 2.3 and Hölder’s inequality, for any \( \xi > 0 \), there exists \( C_\xi > 0 \) such that
\[
\int_{\mathbb{R}^2} |u(u - T_\beta(u))| \int_{|s|}^{+\infty} \frac{h(s)}{s} u^2(s) ds dx \\
= 2\pi \int_{0}^{+\infty} \frac{h(s)}{s} u^2(s) ds \int_{0}^{\infty} |u(\tau)(u(\tau) - T_\beta(u(\tau)))| \tau d\tau \\
\leq 2\pi \int_{0}^{+\infty} \frac{h(s)}{s} u^2(s) ds \left[ \xi \int_{0}^{\infty} u^2(\tau) \tau d\tau + C_\xi \int_{0}^{\infty} (u(\tau) - T_\beta(u(\tau)))^2 \tau d\tau \right] \\
\leq 2\pi \xi \int_{0}^{+\infty} \frac{h(s)}{s} u^2(s) ds + 2\pi C_\xi \int_{0}^{+\infty} \frac{h(s)}{s} u^2(s) ds \int_{0}^{\infty} (u(\tau) - T_\beta(u(\tau)))^2 \tau d\tau \\
\leq 4\pi \xi \int_{0}^{+\infty} \frac{h(s)}{s} u^2(s) ds + C_\xi \|u - T_\beta(u)\|_2^2 \left[ \frac{2}{\pi} B(u) + \frac{1}{2\pi} \|u\|_2^2 + C(\|u\|_4^4 + \|u\|_4^8) \right]. \quad (5.6)
\]

Assume, on the contrary, that there exists \( \{u_n\} \subset H^1_{\text{rad}}(\mathbb{R}^2) \) with \( I_\beta(u_n) \in [c, d] \) and \( \|I_\beta'(u_n)\| \geq \alpha \) such that

\[
\|u_n - T_\beta(u_n)\| \to 0, \quad \text{as} \ n \to \infty.
\]

Combining (3.9), (5.5) and (5.6), we have for large \( n \)

\[
\frac{\mu - 6}{8\mu} \|u_n\|_4^4 + \frac{\mu - 2}{4\mu} \|u_n\|^2_2 + \frac{\mu - 6}{2\mu} B(u_n) + \frac{(q - \mu)\beta}{\mu q} \|u_n\|^q_\beta + \frac{(p - \mu)\lambda}{\mu p} \|u_n\|^p_\lambda \\
\leq C \|I_\beta(u_n)\| + \|u_n\| \|u_n - T_\beta(u_n)\| + \|u_n\|^2 \|u_n - T_\beta(u_n)\|^2. \quad (5.7)
\]

Now we claim that sequence \( \{u_n\} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^2) \). Otherwise, we assume \( \|u_n\| \to \infty \). It follows from (5.7) that

\[
C_1 \|u_n\|_4^4 + C_2 \|u_n\|^2_2 + C_3 \|u_n\|^q_\beta + C_4 \|u_n\|^p_\lambda \leq C 
\]

for large \( n \), where \( C_i, i = 1, 2, 3, 4 \) are some positive constants. Observe that, there exists \( \lambda^* > 0 \) such that for any \( \lambda \in (0, \lambda^*) \), we have

\[
C_1 r^t + C_2 t^p - C_4 \lambda^p t^p \geq 0, \quad t \in \mathbb{R}^+, 
\]

due to \( 4 < p < q \). Applying this with \( t = |u_n| \), then we see from (5.8) that sequence \( \{u_n\} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^2) \) for any fixed \( \beta \in (0, 1] \). The claim combined with lemma 5.1 implies \( \|I_\beta'(u_n)\| \to 0 \) as \( n \to \infty \), which is a contradiction. \( \square \)

5.2. Invariant subsets of descending flows

**Lemma 5.3.** There exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \),

(1) \( T_\beta(\partial P^-_{\epsilon}) \subset P^-_{\epsilon} \) and every nontrivial solution \( u \in P^-_{\epsilon} \) is negative,

(2) \( T_\beta(\partial P^+_{\epsilon}) \subset P^+_{\epsilon} \) and every nontrivial solution \( u \in P^+_{\epsilon} \) is positive.
Lemma 5.4. There exists a locally Lipschitz continuous operator $B_\beta : H^1_{rad}(\mathbb{R}^2) \setminus K_\beta \to H^1_{rad}(\mathbb{R}^2)$ such that

(i) $\langle T_\beta'(u), u - B_\beta(u) \rangle \geq \frac{1}{2} \| u - T_\beta(u) \|^2$;

(ii) $\frac{1}{2} \| u - B_\beta(u) \|^2 \leq \| u - T_\beta(u) \|^2 \leq 2 \| u - B_\beta(u) \|^2$;

(iii) $T_\beta(\partial P^\pm) \subset P^\pm$, $\forall \epsilon \in (0, \epsilon_0)$;

(iv) $B_\beta$ is odd,

where $K_\beta := \{ u \in H^1_{rad}(\mathbb{R}^2) : T_\beta(u) = 0 \}$ and $\epsilon_0$ has been given in lemma 5.3.

Moreover, we can prove the functional $I_\beta$ satisfies the (PS)-condition with the aid of lemma 5.2.

In order to obtain infinitely many sign-changing solutions, we will make use of an abstract critical point developed by Liu et al [32], which we recall below. The notations from section 2 are still valid. Assume $G : E \to E$ be an isometric involution, that is, $G^2 = id$ and $d(Gx; Gy) = d(x, y)$ for $x, y \in E$ and $J$ be $G$-invariant on $H^1_{rad}(\mathbb{R}^2)$ in the sense that $J(Gx) = J(x)$ for any $x \in E$. Let $Q = GP$. A subset $F \subset E$ is said to be symmetric if $Gx \in F$ for any $x \in F$. The genus of a closed symmetric subset $F$ of $E \setminus \{0\}$ is denoted by $\gamma(F)$.

Definition 5.1 ([32]). $P$ is called a $G$-admissible invariant set with respect to $J$ at level $c$, if the following deformation property holds: there exist $\epsilon_0 > 0$ and a symmetric open neighborhood $N$ of $K_\epsilon \setminus W$ with $\gamma(N) < \infty$, such that for $\epsilon \in (0, \epsilon_0)$ there exists $\eta \in C(E, E)$ satisfying

1. $|\eta| P \subset P, \eta(\bar{Q}) \subset Q$.
2. $\eta \circ G = G \circ \eta$.
3. $\eta|_{\epsilon-\epsilon} = id$.
4. $\eta(\bar{J} + \epsilon) \cup (N \cup W) \subset J^{\epsilon}.

Theorem 5.1 ([32]). Assume that $P$ is a $G$-admissible invariant set with respect to $J$ at any level $c \geq c^* := \inf_{\epsilon \in E} J(u)$ and for any $n \in \mathbb{N}$, there exists a continuous map $\psi_k : B_k := \{ x \in \mathbb{R}^2 : |x| \leq 1 \} \to E$ satisfying

1. $\psi_k(0) \subset M := P \cap Q$ and $\psi_k(-t) = G\psi_k(t)$ for $t \in B_k$.
2. $\psi_k(\partial B_k) \cap M = \emptyset$.
3. $\sup_{u \in \text{Fix}_G} J(u) < c_*$ where $\text{Fix}_G := \{ u \in E; Gu = u \}$.

For $j \in \mathbb{N}$, define

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \cup W} J(u),$$

where

$$\Gamma_j := \{ B \in \{ B \in \mathbb{R}^2 : B = \psi(B_k \setminus Y) \} \text{ for some } \psi \in G_k, k \geq j, \text{ and } Y \subset B_j \text{ is open such that } -Y = Y \text{ and } \gamma(Y) \leq k - j \},$$

and

$$G_k := \{ \psi| \psi \in C(B_k, E), \psi(-t) = G\psi(t) \text{ for } t \in B_k, \psi(0) \in M \text{ and } \psi|_{\partial B_k} = \psi_k|_{\partial B_k} \}.$$
Then for \( j \geq 2 \), \( c_j \geq c_\ast \) and \( K_\ast \setminus W \neq \emptyset \) and \( c_j \to \infty \) as \( j \to \infty \).

In order to apply theorem 5.1, we set \( E = H^1_{\text{rad}}(\mathbb{R}^2) \), \( G = -i d, J = I_\beta \) and \( P = P^+_\beta \). Then \( M = P^+_\beta \cap P^-_\beta \), \( \Sigma = \partial P^+_\beta \cap \partial P^-_\beta \), and \( W = P^+_\beta \cup P^-_\beta \). Now, we show that \( P^+_\beta \) is a \( G \)-admissible invariant set for the functional \( I_\beta \) at any level \( c \). Since \( K_\ast \) is compact, there exists a symmetric open neighborhood \( N \) of \( K_\ast \setminus W \) such that \( \gamma(N) < \infty \). Similar to lemma 3.9 in [33], we have

**Lemma 5.5.** There exists \( \epsilon_0 > 0 \) such that for \( 0 < \epsilon < \epsilon' < \epsilon_0 \), there exists a continuous map \( \eta : [0, 1] \times H^1_{\text{rad}}(\mathbb{R}^2) \to H^1_{\text{rad}}(\mathbb{R}^2) \) satisfying

1. \( \eta(0, u) = u \) for \( u \in H^1_{\text{rad}}(\mathbb{R}^2) \).
2. \( \eta(t, u) = u \) for \( t \in [0, 1] \), \( u \notin I_\beta^{-1} [c - \epsilon', c + \epsilon'] \).
3. \( \eta(t, -u) = -\eta(t, u) \) for \( (t, u) \in [0, 1] \times H^1_{\text{rad}}(\mathbb{R}^2) \).
4. \( \eta(1, I_\beta^{-1} \cap (N \cup W)) \subseteq I_\beta^{-\epsilon} \).
5. \( \eta(t, P^+_\beta) \subset P^+_\beta, \eta(t, P^-_\beta) \subset P^-_\beta \) for \( t \in [0, 1] \).

### 5.3. Proof of theorem 1.3 (multiplicity)

We divide the proof into two steps.

**Step 1.** It follows from lemma 5.5 that \( P^+_\beta \) is a \( G \)-admissible invariant set for the functional \( I_\beta \) for \( \beta \in (0, 1) \) at any level \( c \). We construct \( \psi_k \) satisfying the hypotheses of theorem 5.1. For any fixed \( k \in \mathbb{N} \), we choose \( \{v_i\}_{i=1}^k \subset \{C^0_c(\mathbb{R}^2) \cap H^1_{\text{rad}}(\mathbb{R}^2)\} \setminus \{0\} \) such that \( \text{supp}(v_i) \cap \text{supp}(v_j) \) for \( i \neq j \). Define \( \psi_k \in C(B_k, H^1_{\text{rad}}(\mathbb{R}^2)) \) as

\[
\psi_k(t) = R_k \sum_{i=1}^k t_i v_i(R_k), \quad t = (t_1, t_2, ..., t_k) \in B_k.
\]

Observe that

\[
\rho_k = \min \{\|t_1 v_1 + t_2 v_2 + \cdots + t_k v_k\|_2 \geq 1\} > 0,
\]

then \( \|u_i\|_2 \geq \rho_k \) for \( u \in \psi_k(\partial B_k) \) and it follows from lemma 3.8 that \( \psi_k(\partial B_k) \cap P^+_\beta \cap P^-_\beta = \emptyset \). Similarly to the proof of theorem 1.1, we also have

\[
\sup_{u \in \psi_k(\partial B_k)} I_\beta(u) < 0 < \inf_{u \in \Sigma} I_\beta(u).
\]

Clearly, \( \psi_k(0) = \emptyset \in P^+_\beta \cap P^-_\beta \) and \( \psi_k(t) = -\psi_k(t) \) for \( t \in B_k \). For any fixed \( \beta \in (0, 1) \) and \( j \in \{1, 2, ..., k\} \), we define

\[
c_\beta^j \equiv \inf_{R \in \Gamma_j} \sup_{u \in B_k} I_\beta(u),
\]

where \( \Gamma_j \) has been defined in theorem 5.1. Based on theorem 5.1, for any fixed \( \beta \in (0, 1) \) and \( j \geq 2 \),

\[
c_\beta^1 \leq \inf_{u \in \Sigma} I_\beta(u) := c_\beta^1 \leq c_\beta^2 \to \infty, \quad \text{as} \ j \to \infty \tag{5.9}
\]

and there exists \( \{u^j_\beta\} \subset H^1_{\text{rad}}(\mathbb{R}^2) \setminus W \) such that \( I_\beta(u^j_\beta) = c_\beta^j \) and \( I_\beta'(u^j_\beta) = 0 \).

**Step 2.** Using the similar way as that arguments of theorem 1.1(existence part), for any fixed \( j \geq 2 \), \( \{u^j_\beta\}_{\beta \in (0, 1)} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^2) \), that is to say, there exists \( C > 0 \) independent of \( \beta \).
such that \( \|u_\beta\| \leq C \). Without loss of generality, we assume that \( u_\beta \rightharpoonup u_\ast \) weakly in \( H^1_{\text{rad}}(\mathbb{R}^2) \) as \( \beta \to 0^+ \). By lemma 3.9 and theorem 5.1 we define

\[
\frac{c_2^2}{4} \leq \inf_{u \in \Sigma} I_\beta(u) \leq c_\beta^j \leq c_{R_k} := \sup_{u \in \psi_k(B_k)} I(u),
\]

where \( c_{R_k} \) is independent of \( \beta \). Let \( c_\beta^j \to c_\ast^j \) as \( \beta \to 0^+ \). Then we can prove that \( u_\beta \rightharpoonup u_\ast \) strongly in \( H^1_{\text{rad}}(\mathbb{R}^2) \) as \( \beta \to 0^+ \) and \( u_\ast \in H^1_{\text{rad}}(\mathbb{R}^2) \setminus W \) such that \( I'(u_\ast) = 0 \) and \( I(u_\ast) = c_\ast^j \). We claim that \( c_\ast^j \to +\infty \) as \( j \to \infty \). Indeed, \( c_\beta^j \) is non-creasing in \( \beta \). Obviously, \( c_\beta^j \leq c_\ast^j \) and by (5.9), we have \( c_\ast^j \to +\infty \) as \( j \to \infty \). Therefore, problem (1.4) has infinitely many sign-changing solutions. The proof is complete.

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