Two Dimensional Discrete Dynamics of Integral Value Transformations

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A notion of dimension preservative map, Integral Value Transformations (IVTs) is defined over $\mathbb{N}^k$ using the set of $p$-adic functions. Thereafter, two dimensional Integral Value Transformations (IVTs) is systematically analyzed over $\mathbb{N} \times \mathbb{N}$ using pair of two variable Boolean functions. The dynamics of IVTs over $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$ is studied from algebraic perspective. It is seen that the dynamics of the IVTs solely depends on the dynamics (state transition diagram) of the pair of two variable Boolean functions. A set of sixteen Collatz-like IVTs are identified in two dimensions. Also, the dynamical system of IVTs having attractor with one, two, three and four cycles are studied. Additionally, some quantitative information of Integral Value Transformations (IVTs) in different bases and dimensions are also discussed.

**Keywords**: Boolean function, State Transition Diagram, Integral Value Transformations, Collatz-like IVTs, Attractors & Classifications.

1. Introduction

Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line \cite{Wiggins2003, Colon-Reyes2005, Aledo2013, Kovac2017}. There are plenty of works carried out over last couple of decades in discrete dynamical systems as well as in arithmetic...
What is not very well studied yet is the discrete dynamics over spaces like natural numbers. In 2010, the notion of Integral Value Transformations (IVTs) which is a new paradigm in arithmetic dynamics [Hassan et al. 2011, 2010, 2015] has been introduced. Later in 2014, a special class of one dimensional Integral Value Transformations (called Collatz-like) is studied in [Hassan 2015]. A couple of other works on IVTs and their applications in arithmetic logical circuit design and in Biology are also studied [Pal et al. 2012, Choudhury et al. 2009b, 2009a, 2008, 2011, Das et al. 2016a, Hassan et al. 2012, Das et al. 2016a]. Applications of such transformations are noteworthy in the area of Cryptology and other field of discrete mathematics [Klimov et al. 2002, Klimov & Shamir 2003, Tremblay & Manohar 1975, Selmic & Lewis 2000, Xiong & De la Torre 2013].

In this present article, an attempt has been initiated to understand the discrete dynamics of Collatz-like IVTs including other two dimensional Integral Value Transformations. It is seen that the dynamics of the IVTs are depending on the state transition diagram of two dimensional IVTs of the pair of two variable Boolean functions [Lewis & Glass 1991]. There are four types of dynamics seen with regards to attractors with different cycle length(s) 1, 2, 3 and 4. All these dynamical systems end with attractors (global and/or local) with a maximum of four length cycles. In addition, from the algebraic aspects the dynamical system of IVTs are investigated. Discrete dynamics is itself very appealing from its application viewpoint. The present study of the dynamical systems over two dimensions can be as pathways for network designing converging to the desired attractor(s).

The rest of paper is organized as follows. In Section 2, the definitions of IVT in higher dimensions and in particular in two dimensions over the set of natural numbers with zero are given. The various algebraic properties of the dynamical system of IVTs are described in this section. The Section 3 deals with the state transition diagram for the IVTs of various algebraic classes. Some important theorems including all possible type of attractors are presented in this section. The Section 4 deals with the dynamics of IVTs using their profound trajectories. The Section 4 also deals with the classifications of 256 possible IVTs in two dimensions based on their dynamics over \( \mathbb{N} \times \mathbb{N} \). The quantitative measure for the various classes in different bases and dimensions are also incorporated in this section. This Section 4 also includes the dynamics of some special class of IVTs. The Section 5 concludes the key findings of current article and future research endeavors.

2. Definitions & Basic Algebraic Structures in \( \mathbb{B}_2 \) and \( \mathbb{T} \)

2.1. Definitions and basic algebraic structures

**Definition 1.** Integral Value Transformation in the base \( p \) and dimension \( k \) (IVT\(_{i_1,i_2,\ldots,i_k}^{p,k}(m_1,m_2,\ldots,m_k) = (IVT_{i_1}^{p,k}(m_1,m_2,\ldots,m_k), IVT_{i_2}^{p,k}(m_1,m_2,\ldots,m_k),\ldots, IVT_{i_k}^{p,k}(m_1,m_2,\ldots,m_k)) \)

Here \((m_1,m_2,\ldots,m_k)\) is an element of \( \mathbb{N}^k \). The function IVT\(_{i}^{p,k}\) is defined using \( f_i \) in [Hassan et al. 2015]. \( f_i \) is a \( p \)-adic function.

**Definition 2.** Integral Value Transformation in two dimension (\( k=2 \)) with base (\( p=2 \)): (IVT\(_{i,j}^{2,2}(m,n) = (IVT_{i}^{2,2}(m,n), IVT_{j}^{2,2}(m,n)) \)

Here \((m,n)\) is an element of \( \mathbb{N} \times \mathbb{N} \) and the function \( f_i \) or \( f_j \) is the 2-variable Boolean functions and pair-wise in combining they are represented as \( f_{i,j} \) which is formally defined in the Definition 3.

**Definition 3.** The map \( f_{i,j} \) is defined on the set \( \mathbb{B}_2 = \{00, 01, 10, 11\} \) as

\[ f_{i,j}(x,y) = (f_i(x,y), f_j(x,y)) \]
Here \((x, y)\) belongs to \(\mathbb{B}_2\). In the rest of the article, we shall use \(IVT_{i,j}\) instead of \(IVT_{i,j}^{2,2}\) for notational simplicity.

Here we define a discrete dynamical system over \(\mathbb{N} \times \mathbb{N}\) as

\[
x_{n+1} = IVT_{i,j}(x_n)
\]

where \(x_n \in \mathbb{N} \times \mathbb{N}\).

**Definition 4.** For every \(x_0 \in \mathbb{N} \times \mathbb{N}\), if trajectory of the discrete dynamical system \(x_{n+1} = IVT_{i,j}(x_n)\) eventually converges to a point \((0, 0) \in \mathbb{N} \times \mathbb{N}\), then we call such dynamical system \(x_{n+1} = IVT_{i,j}(x_n)\) as Collatz-like IVT.

Let \(S\) be the set of all possible two variable Boolean functions. Clearly, there are exactly 16 such functions. Let \(T\) be the set of all possible \(IVT_{i,j}\) defined over the set \(\mathbb{N} \times \mathbb{N}\). Clearly, there are exactly 256 such IVTs. Our aim is to comprehend the dynamics of these maps. The following properties and remarks in two dimensional integral value transformation from algebraic perspective are given.

(1) **Properties:**

(a) The space \((\mathbb{B}_2, +_2, \ast_2)\) forms a vector space over \(\mathbb{F}_2 = \{0, 1\}\). Here \(+_2\) and \(\ast_2\) are modulo 2 addition and scalar multiplication. The dimension of the vector space is 2.

(b) The basis of the vector space \((\mathbb{B}_2, +_2, \ast_2)\) consists of two elements \((0, 1)\) and \((1, 0)\). The dimension of the vector space is 2.

(c) The space \((S, \oplus_2, \otimes_2)\) forms a vector space over \(\mathbb{F}_2 = \{0, 1\}\). Here \(\oplus_2\), and \(\otimes_2\) are modulo 2 point wise addition and scalar point-wise multiplication.

(d) The basis of the vector space \((S, \oplus_2, \otimes_2)\) consists of 4 elements \(f_1, f_2, f_4, f_8\). The dimension of the vector space is 4.

(e) The space \((T, \oplus_2, \otimes_2)\) forms a vector space over \(\mathbb{F}_2 = \{0, 1\}\). Here \(\oplus_2\), and \(\otimes_2\) are modulo 2 function addition and scalar function multiplication.

(f) The basis of the vector space \((T, \oplus_2, \otimes_2)\) consists of 16 elements \(IVT_{1,1}, IVT_{1,2}, IVT_{1,4}, IVT_{1,8}, IVT_{2,1}, IVT_{2,2}, IVT_{2,4}, IVT_{2,8}, IVT_{4,1}, IVT_{4,2}, IVT_{4,4}, IVT_{4,8}, IVT_{8,1}, IVT_{8,2}, IVT_{8,4}, IVT_{8,8}\). The dimension of the vector space is 16.

(g) There are 4 linear Boolean functions viz. \(f_0, f_6, f_{10}\) and \(f_{12}\) in \(S\).

(h) There are 16 linear maps in \(T\) and they are \(IVT_{0,0}, IVT_{0,6}, IVT_{0,10}, IVT_{0,12}, IVT_{6,0}, IVT_{6,6}, IVT_{6,10}, IVT_{6,12}, IVT_{10,0}, IVT_{10,6}, IVT_{10,10}, IVT_{10,12}, IVT_{12,0}, IVT_{12,6}, IVT_{12,10}\) and \(IVT_{12,12}\).

(i) Sum and Composition to two linear maps in \(T\) is linear.

(j) There are 24 bijective \(f_{i,j}\) in \(S\). The twelve of the bijective functions are \(f_{3,5}, f_{3,6}, f_{3,9}, f_{3,10}, f_{5,6}, f_{5,9}, f_{5,12}, f_{6,10}, f_{6,12}, f_{9,10}, f_{9,12}, \) and \(f_{10,12}\).

(k) There are only 6 isomorphisms (linear and bijective) exist in \(T\). They are \(f_{6,10}, f_{6,12}, f_{10,6}, f_{10,12}, f_{12,6}, \) and \(f_{12,10}\).

(2) **Remarks:**

(a) \(IVT_{i,j} \in T\) is a basis element of \(T\) if and only if \(f_i\) and \(f_j\) are also basis elements in \(S\).

(b) \(IVT_{i,j} \in T\) is linear if and only if \(f_i\) and \(f_j\) are also linear.

(c) If \(f_{i,j}\) is linear then \(f_{j,i}\) is also linear.

(d) None of the basis function in \(T\) is linear.

(e) If \(f_{i,j}\) is bijective then \(f_{j,i}\) is also bijective.

### 2.2. IVTs through STDs

In graph theory, a directed graph \(G\) is defined as \(G(V, E, F)\); where \(F\) is a function such that every edge in \(E\) an oder pair of vertices in \(V\) \[Trudeau\,2013\]. In our context, we have designed the **state transition diagram (STD)** of \(f_{i,j}\) as directed graph.
Illustration: For an example, consider the domain of 2-variable Boolean functions which have four unique possibilities i.e. 00, 01, 10 and 11 for its constituent variables as shown in Table 1.

| Variables | $f_{6,13}$ Transitions | $f_{0,12}$ Transitions |
|-----------|-------------------------|-------------------------|
| $x_2$  $x_1$ | (0,0)→(0,1) 0 0 | (0,0)→(0,0) 0 0 |
| 0 0 | 1 0 | 1 0 |
| 0 1 | 1 0 | 0 0 |
| 1 0 | 1 1 | 0 1 |
| 1 1 | 0 1 | 0 1 |

From the truth table representation of 2-variable Boolean function $f_{6,13}$, we find that $(0,0) \rightarrow (0,1)$, $(0,1) \rightarrow (1,0)$, $(1,0) \rightarrow (1,1)$ and $(1,1) \rightarrow (0,1)$ are the required transitions. Similarly, the transitions are shown for the function $f_{0,12}$. Based on these four transitions a directed graph called STD $(G = (V,E))$ can be drawn where the vertices or nodes in the graph are four possible bit pairs i.e. $V = \mathbb{B}_2$ and the directed edges $(E)$ are the transitions or mapping from the set $\mathbb{B}_2$ to $\mathbb{B}_2$. So, STD identifies the local transformations for all possible combinations of domain set to the rules, given the truth table of one or more rules of $n$-variable. The STDs of function $f_{6,13}$ and $f_{0,12}$ are shown in Fig. 1 and it is noted that there will be always four nodes each node with an out degree edge/transition, but the in degree edge for a particular node can vary from zero to maximum of four. From Fig. 1 ($f_{6,13}$), the in-degree of node (0,1) is 2, (1,1) is 1, (1,0) is 1, and (0,0) is 0. Total in-degrees are the summand of all in-degrees of four nodes that must be equal to 4.

The STDs of linear functions are given in Fig. 2. The STDs of only linear $IVT_{i,j}$ are given, we omit other STDs of linear $f_{j,i}$. The STDs of basis functions are given in Fig. 3. The STDs of only basis $IVT_{i,j}$ are given, we omit other STDs of basis $f_{j,i}$. The STDs of bijective functions are given in Fig. 4. The STDs of only bijective $IVT_{i,j}$ are given, we skip STDs of other bijective $f_{j,i}$. The STDs of isomorphisms are given in Fig. 5. The STDs of only $IVT_{i,j}$ which are isomorphic are given, we skip other STDs of isomorphism $f_{j,i}$.

3. Dynamics of $IVT_{i,j}$ Using State Transition Diagram of $f_{i,j}$

3.1. Dynamics of STDs when $p=2$, $k=2$

For $i,j \in \{0,1,\ldots,15\}$ in two dimensions, there are $2^4 \times 2^4 = 256$ functions $f_{i,j}$. From the STDs of those 256 functions $f_{i,j}$, their dynamics are pretty clear. There are specifically two kinds of dynamics possible;
one is attracting to a single attractor and other kind is periodic attractor of period \((pr), 2 \leq pr \leq 4.\) For example, the single attractor function is \(f_{0,12}\) and period 3 attractor function is \(f_{6,13}\) (Fig. 1).

**Proposition 1.** The dynamical system \(x_{n+1} = IVT_{i,j}(x_n)\) is Collatz-like if and only if \(\text{STD of } f^k_{i,j}((x, y)) = (0, 0)\) for all \((x, y) \in \mathbb{B}_2\), for some \(k \in \mathbb{N}\). Here \(f^k_{i,j}\) denotes the \(k\)-composition of the function defined as \(f^k_{i,j} = f_{i,j}(f^{k-1}_{i,j})\).
Theorem 1. There are exactly 16 Collatz-like $x_{n+1} = IVT_{i,j}(x_n)$ in $\mathbb{T}$.

Proof. As per definition of Collatz-like function, we are looking for functions $f_{i,j}$ such that for some $k$,

$$f_{i,j}^k((x, y)) = (0, 0)$$

for all $(x, y) \in \mathbb{B}_2$. We will check and count through the possible variety of STDs structure of $f_{i,j}$. 

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Fig. 4. The STDs of 12 bijective functions $f_{i,j}$.

Fig. 5. The STDs of 3 isomorphisms functions $f_{i,j}$. 

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Suppose $f_{i,j}$ has the required property and let $G_{f_{i,j}}$ be the directed graph (STD) with four vertices: $a$ is $(0,0)$, $b$ is $(0,1)$, $c$ is $(1,0)$, $d$ is $(1,1)$ and an edge $(x,y) \to (m,n)$ whenever $f_{i,j}(x,y) = (m,n)$. Each vertex of $G_{f_{i,j}}$ has out degree 1. We make the following claims, all of which are simple.

1. $G_{f_{i,j}}$ must have at least one cycle, say $C$.
2. $G_{f_{i,j}}$ must be connected, and it has only one component.
3. The cycle $C$ is the only cycle in $G_{f_{i,j}}$, since every vertex must be in the same component as $(0,0)$.
4. The cycle $C$ is a sink: for each $(x,y)$, $f^k_{i,j}(x,y)$ is in $C$ for all sufficiently large $k$ and $(0,0) \in C$.
5. If $(0,0) \neq f^k_{i,j}((0,0))$, then $f^k_{i,j}(a) \neq f^k_{i,j}(f((0,0)))$ for every $k$. And no other vertex is in $C$.

So we have $(0,0) = f_{i,j}((0,0))$. Therefore, we could have only 4 topologies (STDs) for $G_{f_{i,j}}$ as shown below. The vertex $a$ is fixed and by interchanging $b$, $c$ and $d$ in different levels/positions we can easily enumerate how many elements that have $f^k_{i,j}((x,y)) = (0,0)$.

| Topology: | Number of Functions: |
|-----------|----------------------|
|           |                      |
|           | $(f_{i,j} : (i,j))$: |
|           | $(0,0)$              |
|           | $(0,4)$, $(0,8)$, $(2,0)$, $(0,12)$, $(6,6)$, $(10,0)$ |
|           | $(2,6)$, $(2,8)$, $(4,12)$, $(6,4)$, $(8,4)$, $(10,2)$ |

Hence, the total number of Collatz-like functions in two dimensions is 16. ■

It is noted that the attractor $(0,0)$ is a global attractor for the Collatz-like IVTs, that is for every initial value taken from $\mathbb{N} \times \mathbb{N}$, the dynamical system takes all the trajectories to $(0,0)$.

**Proposition 2.** The dynamical system $x_{n+1} = IVT_{i,j}(x_n)$ possesses multiple attractor(s) of the form $(2^s - 1, 0)$, $s \in \mathbb{N}$ if and only if $f^k_{i,j}((x,y)) = (1,0)$, for all $(x,y) \in \mathbb{B}_2$, for some $k \in \mathbb{N}$.

**Theorem 2.** There are countably infinite number of local attractors of the form $(2^s - 1, 0)$, $s \in \mathbb{N}$.

**Proof.** For any $x_0 \in \mathbb{N} \times \mathbb{N}$, the dynamical system eventually proceeds to a point of the form $(2^s - 1, 0)$, according to definition of the dynamical system map $IVT_{i,j}$. It is obvious that the set of all possible points of the form $(2^s - 1, 0)$, is a subset of the $\mathbb{N} \times \mathbb{N}$. Since the set $\mathbb{N} \times \mathbb{N}$ is countable and so the subset is countable. Hence the theorem is followed. ■

**Proposition 3.** The dynamical system $x_{n+1} = IVT_{i,j}(x_n)$ possesses multiple attractor of the form $(0, 2^s - 1)$ if and only if $f^k_{i,j}((x,y)) = (0,1)$, for all $(x,y) \in \mathbb{B}_2$, $s \in \mathbb{N}$ for some $k \in \mathbb{N}$.

**Proposition 4.** There are countably infinite number of local attractors of the form $(0, 2^s - 1)$, $s \in \mathbb{N}$ for the dynamical system $x_{n+1} = IVT_{i,j}(x_n)$.

**Proposition 5.** The dynamical system $x_{n+1} = IVT_{i,j}(x_n)$ possesses multiple attractors of the form $(2^s - 1, 2^t - 1)$, $s, t \in \mathbb{N}$ if and only if $f^k_{i,j} = (1,1)$ for some $k \in \mathbb{N}$.

**Proposition 6.** There are countably infinite number of local attractors of the form $(2^s - 1, 2^t - 1)$ $s, t \in \mathbb{N}$ for the dynamical system $x_{n+1} = IVT_{i,j}(x_n)$.  


It is noted that there are STDs of some function $f_{i,j}$ such that there are periodic attractor of period $p \geq 2$. In such cases, the dynamical system $x_{n+1} = IVT_{i,j}(x_n)$ will possess to multiple attractors of form $(u, 2^t - 1), (2^t - 1, u), (0,u), (u,0)$ and $(u,v)$. There are few dynamical systems which are sensitive to the initial conditions. In the following section, we shall classify all such types of dynamical systems based on their dynamical behavior.

### 3.2. STD dynamics when $p \geq 2$, $k \geq 3$

In $p$-adic and $k$-variable functions (or $k$-dimension), we have $p^{pk}$ number of functions $f_i$ where $i \in \{0,1,\ldots,2^k - 1\}$. Depending on $k$ value, we select the number of functions in $k$ dimensions and in combining we represent as $f_{i_1,i_2,\ldots,i_k}$ and length of each function is $p^{pk}$.

In general, for any $i,j \in \{0,1,\ldots,2^k - 1\}$ in $k$ dimensions, there are $p^{pk} \times p^{pk} = p^{2pk}$ functions $f_{i_1,i_2,\ldots,i_k}$. In that case, we could have the attractor of period $(pr), 1 \leq pr \leq p^k$. The number of functions is exponentially increasing with the increment of $p$ and $k$. In principle, the discrete dynamics of such IVTs can also be obtained using STDs as we did it in two dimensions ($p = 2, k = 2$). The vertex of the STD in $p$ and $k$ dimension is $(a_1,a_2,\ldots,a_k)$; where $a_i \in \{0,1,\ldots,p^k\}$. For an example, the STDs of two Collatz-like functions $f_{128,72,160}$ ($p = 2, k = 3$) and $f_{2187,12690}$ ($p = 3, k = 2$) and a STD of function $f_{41,19230}$ ($p = 3, k = 2$) with attractors 5 and 2 is shown in Fig. 6.

![Fig. 6. The STD of $f_{128,72,160}$ ($p = 2, k = 3$), $f_{2187,12690}$ and $f_{41,19230}$ ($p = 3, k = 2$).](image)

The number of global attractors for different $p$ and $k$ value is shown in Table 2.

| Base ($p$) | Dimension ($k$) | Number of Attractors |
|-----------|----------------|---------------------|
| p=2       | k=2            | 16                  |
| p=2       | k=3            | 262144              |
| p=3       | k=2            | 4782969             |

Based on the computational evidence, the total number of Collatz-like IVTs in the base $p$ with $k$ dimension is conjectured.

**Conjecture 1.** The total number of Collatz-like IVTs in the base $p$ with $k$ dimension is $p^k(p^k - 2)$.

**Proposition 7.** If $l_k$ defines the path length from a particular point to the global attractor in the dynamics of $IVT_{i,j}^{p,k}$, then $1 \leq l_k \leq (p^k - 1)$. 


4. Dynamics of $x_{n+1} = IVT_{i,j}(x_n)$

Before getting into the detailed classifications of dynamics, a vivid example of a dynamical system $x_{n+1} = IVT_{i,j}(x_n)$, $x_n \in \mathbb{N} \times \mathbb{N}$ is adumbrated with different initial positive natural integers and their trajectories as shown in the Fig. 7. For an example, $IVT_{13,3}(0,2) = (1,3)$ i.e. $(0,2) \to (1,3)$. The trajectories (path) can be changed with the changes of $p$ value even if the taken function is fixed.

![Fig. 7. The dynamical system of $IVT_{13,3}$ for the set of initial integer pairs.](image)

4.1. Classifications of $IVT_{i,j}$ based on their dynamics when $p = 2$, $k = 2$

In this section, we have classified the 256 of $IVT_{i,j}$ into four disjoint classes ($Class I – IV$) according to the attractors of four different cycle length(s). For every class, several subclasses are identified according to the dynamic behavior of $IVT_{i,j}$ over $\mathbb{N} \times \mathbb{N}$. It can be seen that there are many $(x, y)$ components in the dynamics of $IVT_{i,j}$ where $(x, y) \in \mathbb{N} \times \mathbb{N}$. For the simplicity of graph visualization, we have taken certain limit for both $x$ and $y$ i.e. $x, y \in \{0, 1 \cdots 7\}$. For some of the $IVTs$, all the components are connected (e.g. all Collatz –like IVTs) and other cases there could be of different variety. For understanding the complete dynamics over the set of positive natural numbers in base $p$ and dimension $k$, $x, y \in \{0, 1, \cdots, p^k - 1\}$ that covering all possible transitions in their $p$-adic representations. Here, $p^k - 1$ is the minimum value that can be extended to desire limit.

4.1.1. Class-I: Dynamics of $IVT_{i,j}$: attractor with a fixed point

A total of 125 $IVT_{i,j}$ are classified into the $Class-I$ as shown in Table 3. It is noted that each of the IVTs in the $Class-I$ is having attractor with one length cycle (singleton attractor). There are 16 IVTs which are Collatz-like and rest are non-Collatz-like. For the non-Collatz-like dynamical systems, there are countable many local attractors with one cycle.

A dynamical system $IVT_{10,0}$ having an attractor with one length cycle is given in the Fig. 8. There are other 16 out of 125 which are sensitive to the initial conditions that is their trajectories are depending on initial values. Six possible $IVT_{i,j}$ are taken and for all the cases five integer pairs trajectories are shown in Fig. 9. It is noted that each of these initial condition (taken pair) provides sensitive dynamical system ending with different cyclic attractors of length one.
Table 3. Dynamics of $IVT_{i,j}$ with singleton attractor or a fixed point.

| Subclass | $IVT_{i,j} : (i, j)$ | Attractor(s) | Cardinality |
|----------|----------------------|--------------|-------------|
| 1        | (0.0), (0.4), (0.8), (0.12), (2.0), (2.2), (2.6), (2.8), (4.4), (4.12), (6.4), (6.6), (8.0), (8.4), (10.0), (10.2) | (0,0) | 16 (Collatz-like) |
| 2        | (0.3), (0.7), (0.11), (0.15), (1.14), (1.6), (1.11), (1.15), (4.7), (4.15), (5.14), (5.15), (8.3), (8.7), (9.6), (9.7) | (0, $2^n - 1$) | 16 |
| 3        | (5.0), (5.8), (6.1), (6.9), (7.0), (7.2), (7.8), (7.9), (13.0), (13.1), (14.1), (14.3), (15.0), (15.1), (15.2), (15.3) | ($2^n - 1$, 0) | 16 |
| 4        | (9.9), (9.13), (10.11), (10.15), (11.19), (11.11), (11.14), (11.15), (13.12), (13.13), (14.19), (14.15), (15.12), (15.13), (15.14), (15.15) | ($2^n - 1$, $2^n - 1$) | 16 |
| 5        | (8.11), (8.15), (9.11), (9.14), (9.15), (12.15), (13.14), (13.15) | ($x$, $2^n - 1$) | 8 |
| 6        | (13.8), (13.9), (14.9), (14.11), (15.8), (15.9), (15.10), (15.11) | ($2^n - 1$, $x$) | 8 |
| 7        | (0.2), (0.6), (0.10), (0.14), (4.6), (4.14), (8.2), (8.6) | (0, $x$) | 8 |
| 8        | (4.0), (4.8), (6.0), (6.2), (6.8), (12.0), (14.0), (14.2) | ($x$, 0) | 8 |
| 9        | (8.8), (8.12), (10.8), (10.10), (10.14), (12.12), (14.12), (14.14) | ($x$, $x$) | 8 |
| 10       | (4.11), (5.10), (5.11), (12.3), (13.2) | ($x$, $y$) | 5 |
| 11       | (4.2), (4.3), (4.10), (5.2), (8.10), (8.14), (12.2), (12.8), (12.10), (12.11), (12.14), (13.3), (13.10), (13.11), (14.8), (14.10) | Sensitive to the initial condition | 16 |

Fig. 8. The dynamical system of $IVT_{10,0}$ having an attractor with length one cycle. Here a set of 64 initial points from $\mathbb{N} \times \mathbb{N}$ are taken, which are in-circled. The point (0,0) is the global attractor.

4.1.2. Class-II: Dynamics of $IVT_{i,j}$: attractor with two length cycles

A total of 93 $IVT_{i,j}$ are classified into the Class-II as shown in Table 4. All these IVTs in the Class-II are having attractors with two length cycles. Some of these dynamical systems are having global attractor of two length cycles. There are 42 out of 93 which are sensitive to the initial conditions that is their trajectories...
A dynamical system \( IVT_{4,5} \) having an attractor with two length cycle (global attractor) is given in the Fig. 10.

4.1.3. **Class-III: Dynamics of \( IVT_{i,j} \): attractor with three length cycles**

A total of 32 \( IVT_{i,j} \) are classified into the **Class-III** as shown in Table 5. All these IVTs in the **Class-III** are having attractor with three length cycles. There are some dynamical systems with global three length cycle attractors. There are 8 out of 32 which are sensitive to the initial conditions that is their trajectories are depending on initial values. Each of these initial conditions provides sensitive dynamical systems ending three length cycle attractor.

| Subclass | \( IVT_{i,j} : (i, j) \) | Attractor(s) | Cardinality |
|----------|--------------------------|--------------|-------------|
| 1        | (0,1), (0.5), (0.9), (0.13), (4.5), (4.13), (8.1), (8.5) | \((0.0) ⇔ (0, 1)\) | 8           |
| 2        | (1.0), (1.8), (3.0), (3.2), (3.8), (9.0), (11.0), (11.2) | \((0.0) ⇔ (1, 0)\) | 8           |
| 3        | (1.1), (1.5), (3.1), (3.3), (3.7), (5.5), (7.5), (7.7) | \((0.0) ⇔ (1, 1)\) | 8           |
| 4        | (2.5), (2.13), (3.4), (3.12), (3.13), (10.5), (11.4), (11.5) | \((x, \bar{x}) ⇔ (\bar{x}, x)\) | 8           |
| 5        | (2.11), (2.15), (3.11), (3.14), (3.15), (6.15), (7.14), (7.15) | \((2^n - 1, x) ⇔ (2^n - 1, \bar{x})\) | 8           |
| 6        | (13.4), (13.5), (14.5), (14.7), (15.4), (15.5), (15.6), (15.7) | \((r, s) ⇔ (p, q)\) | 3           |
| 7        | (3.5), (3.10), (12.5) | | |
| 8        | (1.2), (1.3), (1.7), (1.10), (2.4), (2.10), (2.12), (2.14), (4.1), (4.9), (5.1), (5.3), (5.7), (6.10), (6.11), (6.14), (7.1), (7.3), (7.10), (7.11), (8.9), (8.13), (9.2), (9.8), (9.10), (10.4), (10.12), (10.13), (11.8), (11.10), (11.12), (11.13), (12.1), (12.4), (12.6), (12.7), (12.9), (12.13), (13.6), (13.7), (14.4), (14.6) | Sensitive to the initial condition | 42          |

A dynamical system \( IVT_{4,5} \) having an attractor with two length cycle (global attractor) is given in the Fig. 10.
A dynamical system $IVT_{9,1}$ having an attractor with three length cycle is shown in the Fig. 11.

4.1.4. **Class-IV: Dynamics of $IVT_{i,j}$: attractor with four length cycles**

A total of 6 $IVT_{i,j}$ are classified into the Class-IV as shown in Table 6. All these IVTs in the Class-IV are having attractors with four length cycles. These dynamical systems do possess global attractors with four length cycles.

| Subclass | $IVT_{i,j} : (i,j)$ | Attractor(s) | Cardinality |
|----------|---------------------|--------------|-------------|
| 1        | (3,6),(3,9),(5,12),(6,5),(9,5),(10,3) | (0,0) – (0,1) – (1,0) – (1,1) | 6           |

A dynamical system $IVT_{9,5}$ having an attractor with four length cycle is shown in the Fig. 12.
4.2. Classifications of $IVT_{i,j}^{p,k}$ based on their dynamics when $p \geq 2$, $k \geq 3$

In case of $p = 2$, $k = 2$, the cycle length of an attractor observed in the dynamics of $IVT_{i,j}$ does not exceed 4 which is the maximum cycle length of an STD of $f_{i,j}$. It is quite impossible to observe the dynamics of individual $IVT_{i,j}^{p,k}$ when $p$ and $k$ are sufficiently large. From the dynamics of STDs of some $f_{i,j}^{p,k}$, one can classify the corresponding $IVT_{i,j}^{p,k}$ based on their dynamics ends up with cycle length exclusively 1, 2, …, $p^k$ (Fig. 6) into $p^k$ number of classes. The cardinality of different classes based on dynamics of $IVT_{i,j}^{p,k}$ having attractors of different cycle lengths (upto $p^k$) including an exclusive outer class are presented for different $p$ and $k$ in Table 7. The exclusive outer class contains $IVT_{i,j}^{p,k}$ such that the length of the cycle of the attractors exceed the maximum cycle length ($p^k$) of the STD of $f_{i,j}^{p,k}$. Such a discrete dynamical systems is called here as chaotic dynamical system.

An example of chaotic dynamical system of $IVT_{41,19230}^{3,2}$ using the function $f_{41,19230}$ ($p = 3$, $k = 2$) is shown in Fig. 13, whose STD is given in Fig. 6 (C). From the STD of $f_{41,19230}$, we could see two types of cycles (2
and 5) is present. But, the dynamical system shows an additional cycle of length 10 (l.c.m of 2 and 5). The reason can be easily seen from the ternary representation (here, $p = 3$) of any pair $(x, y) \in \mathbb{N} \times \mathbb{N}$. Take a pair $(5, 6)$ from the cycle of length 10 from Fig. 13. The ternary representation of $(5, 6)$ is $(12, 20)$. The pair wise component are 12 (first component from left) and 20 (second component from left). These two components are from two different cycles 2 and 5 respectively from STD of $f_{41,19230}$ (Fig. 6(C)). All the STD components are distinct. So, while calculating the IVT one by one iteration, the components 12 and 20 will come back/repeat to the same initial stage after $l.c.m(5, 2)$ iterations or cycle of length $l.c.m(5, 2)$. This is true for arbitrary pair in any base $p$ and also true if we increase the dimension $k$. 

Fig. 12. The dynamical system of $IVT_{9,5}$ having an attractor with four length cycle.
Table 7. Number of $IT_{i,j}$ having attractors of different length cycles count in the respective classes.

| Category | Class-I | Class-II | Class-III | Class-IV | Class-V | Class-VI | Class-VII | Class-VIII | Class-IX | Outer |
|----------|---------|----------|-----------|----------|---------|----------|-----------|------------|----------|-------|
| $p=2,k=2$ | 125     | 93       | 24        | 6        | ×       | ×        | ×         | ×          | ×        | ×     |
| $p=2,k=3$ | 4792969 | 5752131  | 3009888   | 1692180  | 653184  | 773920   | 46080     | 5040       | ×        | 61824 |
| $p=3,k=2$ | 100000000 | 127790505 | 69554240  | 43296120 | 18144000 | 23166360 | 2073600   | 408240     | 40320   | 2947104 |

Fig. 13. The dynamical system of $IT_{41,19230}$ having an attractor with cycle length 2, 5 and 10.
Proposition 8. Number of bijective functions \( f_{i_1, i_2, \ldots, i_k}^{p, k} \) in \( p \)-adic, \( k \) dimension is \((p^k - 1)!\).

### 4.3. Dynamics of two dimensional \( IVT_{i,j} \) from algebraic perspective

Here we go with classification of two dimensional \( IVT_{i,j} \) from their algebraic perspectives such as basis, linearity, bijective and isomorphism which are shown in Table 8.

| Algebraic Character | \( IVT_{i,j} : (i, j) \) | Attractor(s) | Remark |
|---------------------|------------------------|-------------|--------|
| Basis (1,1)         | (0, 0) ⇔ (1, 1)        | Period two attractor |
| Basis (1,2)         | (0, 0) - (1, 0) - (0, 1) | Period three attractor |
| Basis (1,4)         | (0, 0) ⇔ (1, 0)        | Period two attractor |
| Basis (2,2)         | (0, 0)                 | Collatz-like |
| Basis (2,4)         | Unstable               | Sensitive to the initial condition |
| Basis (2,8)         | (0, 0)                 | Collatz-like |
| Basis (4,4)         | (x, 0)                 | Period one attractor (Non-Collatz-like) |
| Basis (8,8)         | (x, x)                 | Period one attractor (Non-Collatz-like) |
| Linear (0,0)        | (0, 0)                 | Collatz-like |
| Linear (0,6)        | (0, x)                 | Period one attractor (Non-Collatz-like) |
| Linear (0,10)       | (0, x)                 | Period one attractor (Non-Collatz-like) |
| Linear (0,12)       | (0, 0)                 | Collatz-like |
| Isomorphism (6,10)  | Unstable               | Sensitive to the initial condition |
| Linear (6,12)       | (0, 0)                 | Collatz-like |
| Isomorphism (10,12) | Unstable               | Sensitive to the initial condition |
| Linear (10,10)      | (x, x)                 | Period one attractor (Non-Collatz-like) |
| Linear (12,12)      | (x, x)                 | Period one attractor (Non-Collatz-like) |
| Bijective (3,5)     | \(|p + r| = |q + s| = 2^n - 1\) | Period two attractor |
| Bijective (3,6)     | (0, 0) - (0, 1) - (1, 0) - (1, 1) | Period four attractor |
| Bijective (3,9)     | (0, 0) - (1, 0) - (1, 1) | Period four attractor |
| Bijective (3,10)    | \(|p + r| = 2^n - 1\)  | Period two attractor |
| Bijective (5,6)     | Unstable               | Sensitive to the initial condition |
| Bijective (5,9)     | Unstable               | Sensitive to the initial condition |
| Bijective (5,12)    | Unstable               | Sensitive to the initial condition |
| Bijective (9,10)    | Unstable               | Sensitive to the initial condition |
| Bijective (9,12)    | Unstable               | Sensitive to the initial condition |

It is to be noted that the dynamics of the \( IVT_{i,j} \) is not always same as the dynamics of \( IVT_{j,i} \) although they belong to same algebraic class as previously pointed out.

### 5. Conclusions and Future Endeavors

In this manuscript, notion of dimension preservative Integral Value Transformations (IVTs) is defined over \( \mathbb{N}^k \) using the set of \( p \)-adic functions. The main focus is given in order to understand the two dimensional integral value transformation \((p = 2 \text{ and } k = 2, \text{which are computationally feasible})\) and studied the dynamics of all possible pair functions in four different kinds of attractors. Among them, a special class of IVTs (Collatz-like) is computationally identified along with some others global and local dynamical systems. In addition, from the algebraic properties of \( IVT_{i,j} \), the dynamics are critically observed. From the current analysis, it is evident that as \( p \) and \( k \) go high, the total numbers of Collatz like IVTs are increasing very fast. Consequently, the dynamical behavior of these IVTs is computationally difficult to simulate and apprehend. But, some of their dynamics can be vivid from our current analysis in two dimensions and some other cases. In this concern, we have shown some quantitative informations for other two special cases of
IVTs ($p = 2, k = 3$ and $p = 3, k = 2$). Further, the analysis also clearly evident that as $p$ and $k$ go high, we come across with some chaotic dynamical behavior. In future endeavor, one can explore with some higher $p$ and $k$ considering the limitation of simulation.

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