Catalog of Properties of the First Isodynamic Point of a Triangle

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Abstract. The first isodynamic point of a triangle is one of many notable points associated with a triangle. It is named X(15) in the Encyclopedia of Triangle Centers. This paper surveys known results about this point and gives additional properties that were discovered by computer.

Keywords. triangle geometry, first isodynamic point, computer-discovered mathematics, GeometricExplorer.

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INTRODUCTION

The first isodynamic point of a triangle is one of many notable points associated with a triangle. It is the point with trilinear coordinates

\( \left( \sin(A + \frac{\pi}{3}) : \sin(B + \frac{\pi}{3}) : \sin(C + \frac{\pi}{3}) \right) \)

with respect to the triangle. It is named \( X_{15} \) in the Encyclopedia of Triangle Centers [37].

Geometrical definition. The interior and exterior angle bisectors of angle \( A \) of \( \triangle ABC \) intersects side \( BC \) of the triangle (or its extension) in two points, \( A_1 \) and \( A_2 \). The circle with diameter \( A_1A_2 \) is called the \( A \)-Apollonius circle and is named \( C_A \). Circles \( C_B \) and \( C_C \) are defined similarly. The points in which the three Apollonius circles intersect are the isodynamic points of the triangle. The one inside \( \odot ABC \) is the 1st isodynamic point and is named \( S \). The other point of intersection is the 2nd isodynamic point and is named \( S' \).

Scope of this catalog. The mathematical literature is vast. We do not attempt to catalog every property involving the 1st isodynamic point that appears somewhere in print or on the internet. We do try to catalog any property that is simple or elegant or that can be obtained from the configuration associated with one of our top-level classifications (Triangle plus \( S \), Triangle with \( S \) and other points, Triangle plus lines through \( S \), Quadrilateral plus \( S \), etc.) by applying at most one common geometrical construction (drop a perpendicular, draw an angle bisector, construct a centroid, etc.) When analyzing triangle centers, we only look at the common ones, \( X_1 \) through \( X_{20} \).
# Classification Scheme

## Introduction

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Figures. The figures in this paper are not decorative. In order to reduce verbiage and declutter the catalog, given information appearing in the figure is not repeated in words. For example, if a figure shows line segments $WX$ and $YZ$ meeting at a point $P$, we do not state in words that $P$ is the intersection of $WX$ and $YZ$. This should make it easier for people who don’t read English to use this catalog.

A solid line or circle through multiple points means that it is given that these points lie on the same line or circle. A dashed line or circle through multiple points means that the conclusion of the theorem or result is that these points lie on the same line or circle.

| $P$ is a point on segment $XY$. | We can conclude that $X$, $P$, and $Y$ colline. |
|--------------------------------|-----------------------------------------------|
| $X$ --- $P$ --- $Y$            | $X$ --- $P$ --- $Y$                           |

It is given that $WXYZ$ is a cyclic quadrilateral. We can conclude that $W$, $X$, $Y$, and $Z$ are concyclic.

Angles that are given to be equal are marked with the same filled circle. Angles that are concluded to be equal are shaded with the same color.

| It is given that $\angle YXP = \angle PXZ$. | We can conclude that $\angle YXP = \angle QXZ$. |
|---------------------------------------------|-----------------------------------------------|
| $X$ --- $P$ --- $Z$ | $X$ --- $P$ --- $Z$ |

Two perpendicular brown lines at the point of intersection of two circles means that we can conclude that the circles are orthogonal (have perpendicular tangents at that point).

If the title of a section or subsection describes a feature of a figure, then we do not repeat this description if it is obvious from the figure. For example, in a subsection entitled “equilateral triangles”, a triangle highlighted in yellow that looks equilateral can be assumed to be an equilateral triangle.

A right-angle marker is used to indicate two lines that are given to be perpendicular. All angles are directed angles. The 1st isodynamic point of $\triangle ABC$ is always colored green. Given information that is not obvious from the associated figure is shown in brown text directly beneath the figure.
Parts of a triangle. In order to help with the classification process, we give names for various line segments associated with a triangle. A line segment from a vertex of a triangle to a non-vertex point on the opposite side is called a \textit{cevian}. A line segment joining points on two sides of the triangle is called a \textit{chord}. A chord parallel to a side of the triangle is called a \textit{parachord}. If the endpoints of a chord joining points on two sides of a triangle forms a cyclic quadrilateral with the endpoints of the third side, the chord is called an \textit{antiparallel}.

If \( P \) is a point inside a triangle, the line segment from \( P \) to a vertex is called a \textit{spoke}. The line segment from \( P \) to the foot of the perpendicular from \( P \) to a side of the triangle is called an \textit{apothem}. A line segment from \( P \) parallel to a side of the triangle ending on another side of the triangle is called a \textit{pararadius}. A line segment from \( P \) to a side of the triangle that forms an angle of \( n^\circ \) with that side is called an \textit{nº-incline}.

If \( P \) is a point inside a triangle, the line segment from a vertex passing through \( P \) and extending to the circumcircle of \( \triangle ABC \) is called a \textit{circumcevian}. The line segment from a vertex passing through \( P \) and extending to the circumcircle of \( \triangle BPC \) is called a \textit{circlecevian}. The endpoint of a cevian through \( P \) (other than a vertex) is called a \textit{trace} and the line segment along a side from that trace to a vertex of the triangle is called a \textit{trace segment}. The line segment from the midpoint of the side of a triangle extending outward to the furthest point on the circumcircle is called a \textit{sagitta}.

Discoveries. An asterisk after a property number indicates that the property was discovered by computer, either by using GeometricExplorer, Mathematica, or Geometer’s Sketchpad. If a reference is given, this means that the result was posted to an online forum in the hope that some forum member might find a geometrical proof of the property.

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Notation.

| Notation | Description |
|----------|-------------|
| $\triangle XYZ$ | Triangle $XYZ$ |
| $a, b, c$ | The lengths of the sides of $\triangle ABC$ |
| $s = (a + b + c)/2$ | |
| $\odot XYZ$ | The circle through points $X, Y, \text{ and } Z$ |
| $H$ | The orthocenter of $\triangle ABC$ |
| $K$ | The symmedian point of $\triangle ABC$ |
| $K$ | When used in an expression, $K$ denotes the area of $\triangle ABC$. |
| $I$ | The incenter of $\triangle ABC$ |
| $O$ | The circumcenter of $\triangle ABC$ |
| $r$ | The inradius of $\triangle ABC$ |
| $R$ | The circumradius of $\triangle ABC$ |
| $R(XYZ)$ | The circumradius of $\triangle XYZ$ |
| $S$ | The 1st isodynamic point of $\triangle ABC$ |
| $S(XYZ)$ | The 1st isodynamic point of $\triangle XYZ$ |
| $S'$ or $T$ | The 2nd isodynamic point of $\triangle ABC$ |
| $\Omega$ | The 1st Brocard point of $\triangle ABC$ |
| $\Omega'$ | The 2nd Brocard point of $\triangle ABC$ |
| $[F]$ | The area of figure $F$ |
| $\phi$ | The golden ratio, $(1 + \sqrt{5})/2$ |
| $X - Y - Z$ | The points $X, Y, \text{ and } Z$ collinear (i.e. are collinear). |
| $\angle XYZ$ | Directed angle $XYZ$. This is the angle through which ray $\vec{YX}$ must be rotated counterclockwise in order to coincide with ray $\vec{YZ}$. |
| $X_n$ | The $n$th Kimberling center of $\triangle ABC$ (see [36]) |
| $X_n(XYZ)$ | The $n$th Kimberling center of $\triangle XYZ$ |

Key to the property listings.

Since there are three shaded triangles, this means that the 1st isodynamic points of these triangles will be named $S_a, S_b, \text{ and } S_c$ [or $S_1, S_2, \text{ and } S_3$]. The 2nd isodynamic points will be named $T_a, T_b, \text{ and } T_c$. 
1st Isodynamic Point $S$

1. Triangle plus $S$

1.1 location of $S$

**Property 1.1.1.**

Each angle of $\triangle ABC$ has measure less than $120^\circ$.

$\quad \supset S$ lies in the interior of $\triangle ABC$.

**Property 1.1.2.**

$\angle B > 120^\circ$.

$\quad \supset S$ lies outside $\triangle ABC$.

**Property 1.1.3.**

$\supset S$ lies inside the circumcircle of $\triangle ABC$.

- Equivalently, $SO < R$.

1.2 metric properties

**Property 1.2.1.** (Tripolar Coordinates) [20] p. 106]

\[
\frac{AS}{BS} = \frac{AC}{BC}
\]

- $A$ similar property holds for $S'$. By symmetry, we have $AS \cdot BC = BS \cdot AC = CS \cdot AB$.

**Property 1.2.2.** (Spoke Length) [25]

\[
x = \frac{bc\sqrt{2}}{\sqrt{a^2 + b^2 + c^2 + 4K\sqrt{3}}}
\]

1.3 angle properties

**Property 1.3.1.** [42]

$\angle BAS + \angle SCB = 60^\circ$

**Property 1.3.2.** [35] p. 296]

$\angle ASB = \angle ACB + 60^\circ$

- A similar result holds for $S'$: $\angle ASB = \angle ACB - 60^\circ$
1.4 geometrical properties

Property 1.4.1. ‘

$A$ is the 2nd isodynamic point of $\triangle BCS$.

2. Triangle plus $S$ and constructions

2.1 angle bisectors

Property 2.1.1. ‘

$\angle BSE = \angle ESC$

Property 2.1.2.

$\angle SCE = \angle ECB$

Property 2.1.3. ‘

$AP = PE$

Property 2.1.4.

Circle with diameter $EF$ passes through $S$.

2.2 circles

Property 2.2.1.

$\theta = 60^\circ$

Property 2.2.2.

$\theta = 60^\circ$.  
- In other words, the circles meet at an angle of $60^\circ$.

Property 2.2.3.

$\theta = 60^\circ$.  
- This is a special case of Property 2.2.3
**Property 2.2.4.**

$D$ is any point on $AS$.

- $\triangle DEF$ is equilateral.
- $B, E, F, C$ are concyclic.

**Property 2.2.5.**

$P$ is any point

- Common chord of $\odot ABC$ and $\odot PST$ passes through $K$.

**Property 2.2.6.**

$P$ is any point

- $\odot ABC$ and $\odot PST$ are orthogonal.
- In particular, the Parry circle is orthogonal to the circumcircle.

**Property 2.2.7.**

$P$ is any point

- Circle with diameter $KO$ is orthogonal to $\odot PST$.
- In particular, the Parry circle is orthogonal to the Brocard circle.

**Property 2.2.8. (1st Isodynamic-Dao Triangle)**

$DEF$ is an equilateral triangle.

2.3 Euler lines

**Property 2.3.1.**

- The Euler lines of the three colored triangles concur.
- In particular, the Parry circle is orthogonal to the circumcircle or on the Neuberg cubic. See [85].
### 2.4 equilateral triangles

#### Property 2.4.1.

- $\angle BAE = \angle SAC$
- This follows from Property 4.2.10

#### Property 2.4.2.

- $P$ is the center of $\triangle XYZ$.
- $PS = PX_{13}$

#### Property 2.4.3.

- $A'$ is the reflection of $A$ about $BC$.
- $E - S - A'$

#### Property 2.5.1.

- $T$ is the reflection of $S$ about $BC$.
- $\angle BAT = \angle SAC$

#### Property 2.5.2.

- $A'$ is the reflection of $A$ about $BC$.
- $\angle SA'A = \angle SAC$
- The property is true if $S$ is replaced by any point on the $C$-Apollonian circle.
2.6 sagitta

Property 2.6.1.

\[ \alpha + \gamma = 60^\circ \]
\[ \beta - \gamma = 60^\circ \]

3. Special triangle plus \( S \)

3.1 isosceles triangle

Property 3.1.1.

\[ \theta = 30^\circ \]

3.2 isosceles right triangle

Property 3.2.1.

\[ S' \] is the orthocenter of \( \triangle ACS \);
\[ S \] is the orthocenter of \( \triangle ACS' \).

3.3 \( 30^\circ \) triangle

Property 3.3.1.

\[ B, \ X_5, \ X_{15}, \text{ and } C \text{ are concyclic.} \]

3.4 \( 60^\circ \) triangle

Property 3.4.1.

\[ [BAS] = [BSC]. \]

Property 3.4.2.

\[ \theta = 90^\circ. \]
Property 3.4.3.*

\[ \angle PAC = 30^\circ. \]

\[ A - P - Q. \]

Property 3.4.4.*

\[ A - S - M. \]

Property 3.4.5.*

\[ \odot CAT \text{ and } \odot ABS \text{ are tangent.} \]

\[ \odot CAT \text{ and } \odot BSC \text{ are tangent.} \]

Property 3.4.6.*

\[ \odot TAB \text{ and } \odot ASC \text{ are tangent.} \]

\[ \odot TAB \text{ and } \odot BSC \text{ are tangent.} \]

Property 3.4.7.*

\[ AX_{13} = AX_{15}. \]

Property 3.4.8.

\[ B, H, X_{13}, I, S, O, \text{ and } C \text{ are concyclic.} \]

\[ \text{See also } [15]. \text{ Other points on this circle (discovered by computer) are } X_{399}, X_{616}, \text{ and } X_{617}. \]

Property 3.5.1.*

\[ S \text{ lies on } AC. \]

Property 3.5.2.*

\[ BS \text{ bisects } \angle CBA. \]
3.6 \(150^\circ\) triangle

**Property 3.6.1.**

\[ ABSC \text{ is a kite.} \]

**Property 3.6.2.**

\[ B, S, C, \text{ and } H \text{ are concyclic.} \]

4. Triangle with \( S \) and other points

4.1 \( S \) and \( T \)

**Property 4.1.1.**

\[ \frac{SA}{TA} = \frac{SB}{TB} \]

**Property 4.1.2.**

\[ \angle BAC - \angle ACP = 30^\circ \]

**Property 4.1.3. (Distance Between Isodynamic Points)**

\[ x = \frac{2\sqrt{3}abc}{\sqrt{(a^2 + b^2 + c^2)^2 - 48K^2}} \]

4.2 \( S \) and one notable point

**Property 4.2.1.**

\[ S \text{ and } H \text{ lie inside } \triangle ABC. \]

\[ \angle AHB + \angle ASB = 240^\circ \]

**Property 4.2.2.**

\[ \angle CBH + \angle ASB = 150^\circ \]

**Property 4.2.3.**

\[ \angle HAS = \alpha, \angle HBS = \beta, \text{ and } \angle HCS = \gamma \]

\[ \text{Using directed angles,} \]

\[ \sin \alpha + \sin \beta + \sin \gamma = 0. \]
Property 4.2.4.*

$D$ is the orthocenter of $\triangle BCS$.

$\Rightarrow \angle BEC = 60^\circ$

Property 4.2.5.*

$H_a = X_4(BCS)$

$\Rightarrow \angle CBS = \angle SH_a C$

Property 4.2.6.*

$D = X_{14}(BCS)$

$\Rightarrow \angle BCD = \angle ACS$

Property 4.2.7.*

$D = X_{15}(BCS)$

$\Rightarrow AB \cdot SD = AS \cdot BD$

Property 4.2.8.*

$\Rightarrow B - I - S$

- This result follows from Property 3.5.2

Property 4.2.9.*

$SI < \frac{1}{4}R$.

- The smallest $k$ such that $SI < kR$ is
  
  $k \approx 0.2370406267$, where $k$ is the positive root of
  
  $4x^6 + 36x^5 + 120x^4 + 288x^3 + 513x^2 - 72x - 16$.

Property 4.2.10. (Isogonal Conjugate) [84]

$\Rightarrow \angle BAX_{13} = \angle SAC$

- In other words, the isogonal conjugate of $X_{15}$ is $X_{13}$.
  See [84], p. 296. The isotomic conjugate of $X_{15}$ is $X_{300}$.
  See [37].
4.3 \( S \) and \( T \) and one notable point

**Property 4.3.1.**

\[
\frac{(SA)^3}{TA} = \frac{SM}{TM}
\]

**Property 4.3.2.**

\[
x = \angle TMS, \alpha = \angle TAS
\]

\[
\Rightarrow x = 3\alpha \quad (\text{mod} \ 2\pi)
\]

**Property 4.3.3.**

\[
A, B, C, S, T, M \text{ lie on a hyperbola.}
\]

- The center of the hyperbola is \( X_{18334} \).

**Property 4.3.4.** (Inverse Property) p. 296]

- Inversion of \( \triangle ABC \) with respect to an isodynamic point transforms \( \triangle ABC \) into an equilateral triangle.

**Property 4.3.5.**

\[
\angle BX_{16}X_{13} = \angle X_{15}B X_{13}
\]

**Property 4.3.6.**

\[
\angle X_{14}BX_{16} = \angle X_{14}X_{15}B
\]

4.4 \( S \) and \( T \) and a point \( P \)

**Property 4.4.1.**

\[
\text{PS}/PT \text{ remains invariant as } P \text{ moves along the circumcircle.}
\]

- The common ratio is \( MX_{13}/MX_{14} \).

4.5 \( S \) and two notable points

**Property 4.5.1.**

\[
X_3 - X_{15} - X_6 \quad \text{or} \quad (X_3X_{15})^2 \cdot X_3X_6 = R^2(X_3X_{15} - X_6X_{15}).
\]

- See also [25, p. 106]. Equation was discovered by computer. Corollary: \( X_3X_{15} > X_6X_{15} \).
Property 4.5.2.

\[ X_4 - X_{17} - X_{15} \]

Property 4.5.3.

\[ X_2 - X_{15} - X_{14} \]

Property 4.5.4.

\[ X_{13}X_{15} \parallel X_2X_3 \]

In other words, \( X_{13}X_{15} \) is parallel to the Euler line of \( \triangle ABC \).

Property 4.5.5.*

\[ \angle X_{14}AX_{14} = \angle AX_{14}X_{15} \]

Property 4.5.6.*

\[ \angle S\Omega K = 60^\circ \]

Property 4.5.7.*

\[ A, B, C, X_3, X_{15}, X_{17} \text{ lie on a conic.} \]

4.6 \( S \) and three notable points

Property 4.6.1.*

\[ X_{13}X_{15} \parallel X_{14}X_{16} \]

Property 4.6.2. (Brocard Axis)

\[ X_3 - X_{15} - X_6 - X_{16} \]

A few other named points that lie on the Brocard Axis are the 3rd power point, the Brocard midpoint, the Kenmotu point, and the Taylor center. See [82].
Property 4.6.3. (Brocard Axis Metrics)

\[ \begin{align*}
\Rightarrow & \quad y(x + y + z) = xz \\
\Rightarrow & \quad z(x + y + z) = R^2 \\
\Rightarrow & \quad \frac{1}{x} + \frac{1}{x+ y + z} = \frac{2}{y} + \frac{z}{x + y} \\
\Rightarrow & \quad \frac{1}{x} + \frac{1}{x + y + z} = \frac{2}{y} + \frac{z}{x + y} \\
\Rightarrow & \quad (x + y)(y + z) = 2xz \\
\end{align*} \]

- See also [26, p. 103] and [21].

Property 4.6.4.

\[ SS' \parallel XY \]
- \( Y = X_{110} \).

Property 4.6.5.*

\[ O \Omega \cdot \Omega' = OS \cdot \Omega \Omega' \]
- The point of intersection of the two lines is \( X_{39} \), the Brocard midpoint.

Property 4.6.6.

\[ KM \text{ bisects } SX_{13} \]

Property 4.6.7.

\[ S' \text{ is the perpendicular bisector of } \Omega \Omega'. \]
- The point of intersection of the two lines is \( X_{39} \), the Brocard midpoint.

Property 4.6.8.*

\[ X_2X_{13} \text{ is tangent to } \odot X_{14}X_{14}X_{15} \]

Property 4.6.9.

\[ \odot S_aS_bS_c \text{ is tangent to } \odot SBC. \]
- By symmetry, \( \odot S_aS_bS_c \) is also tangent to \( \odot SCA \) and \( \odot SAB \). The center of \( \odot S_aS_bS_c \) is \( X_{5238} \).

4.7 \( S \) and 4 or more notable points

Property 4.7.1.

\[ F_a, F_b, F_c, F \text{ concyclic.} \]
- The result also works for \( S' \). See [30].
Property 4.7.2.

\[ \frac{1}{x} + \frac{1}{y} = \frac{1}{z} \]

Property 4.7.3. (Evans Conic)

- \( X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{18} \) lie on a conic.
- See also [23].

Property 4.7.4.

- \( X_2, X_4, X_6, X_{13}, X_{15}, X_{18} \) lie on a conic.

Property 4.7.5. (Neuberg Cubic)

- The following ten points lie on a cubic curve: \( X_1, X_3, X_4, X_{13}, X_{14}, X_{15}, X_{16}, I_a, I_b, I_c \), where \( I_a \) is the \( A \)-excenter of \( \triangle ABC \).
- A few other points that lie on the Neuberg cubic are the reflections of the vertices of the triangle about their opposite sides and the six vertices of equilateral triangles erected on the sides of \( \triangle ABC \). See [27].

5. Triangle plus lines through \( S \)

5.1 Antiparallels

Property 5.1.1.

\[ \triangle M_a M_b M_c \] is equilateral.
- See also [7].

5.2 Apothems

Property 5.2.1. (Trilinear Coordinates)

\[ \frac{SP}{SQ} = \sin(A + 60^\circ) \over \sin(B + 60^\circ) \]

Property 5.2.2. (Pedal Triangle) [26] p. 106

\[ \triangle PQR \] is an equilateral triangle.
- \[ [PQR] = \frac{2K^2 \sqrt{3}}{a^2 + b^2 + c^2 + 4K \sqrt{3}} \]
- The center of the equilateral triangle is the midpoint of \( SX_{13} \). See [13] for the area formula.
5.3 cevians

Property 5.3.1.* (Cevian Length)

\[ AE = \frac{4\sqrt{2bcK}\sqrt{a^2 + b^2 + c^2 + 4K\sqrt{3}}}{(b^2 + c^2)(4K + a^2\sqrt{3}) - \sqrt{3}(b^2 - c^2)^2} \]

Property 5.3.2.

\[ \triangle XYZ \text{ is equilateral.} \]

- See also [3].

5.4 circlecevians

Property 5.4.1.*

\[ \angle EBA = \angle ACE = 120^\circ. \]

Property 5.4.2.

\[ A - S - Sa \]

5.5 circumcevians

Property 5.5.1.*

\[ \angle EBS = \angle SCE = 60^\circ. \]

Property 5.5.2.

\[ \triangle PQR \text{ is an equilateral triangle.} \]
Property 5.5.3.

\[ L_a \] is the Simson line of \( P_a \).

- \( L_a, L_b, L_c \) bound an equilateral triangle.
- The center of the triangle is the nine-point center of \( \triangle ABC \).

Property 5.5.4.

- \( A S_a, B S_b, C S_c \) concur.
- \( P_a S_a, P_b S_b, P_c S_c \) concur.

5.6 inclines

Property 5.6.1.

\[ \angle DSA = \angle CBS. \]
- This follows from Property 5.3.1

Property 5.6.2.

\[ \triangle DEF \] is equilateral.

5.7 parachords

Property 5.7.1.

- \( A - S - T_a \)

Property 5.7.2.

- \( P_a \) is the 2nd Napoleon point of \( \triangle ASF \).
- \( D P_b, E P_c, \) and \( F P_a \) are concurrent.

Property 5.7.3.

- \( \triangle T_a T_b T_c \) is equilateral.
- The center of the equilateral triangle is \( X_{39555} \).
Property 5.7.4.

\[ T_c - T_a - T_b. \]

Property 5.7.5.

\[ \triangle T_a T_b T_c \]

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]

5.8 spokes

Property 5.8.1.

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]

\[ \triangle T_a T_b T_c \text{ is equilateral and } \triangle T_a T_b T_c \cong \triangle S_a S_b S_c. \]

Property 5.8.2.

\[ P_a \text{ is the incenter of } \triangle SBC. \]

\[ \triangle P_a P_b P_c \text{ are concurrent.} \]

\[ \text{The result remains true if } P_a \text{ is replaced by } X_n(SBC), \text{ for } n = 2, 3, 6, 13, 15, 31, 32, 36, 39, \text{ or } 50. \]

6. Triangle with \( S \) in subtriangles

6.1 formed by \( H \)

Property 6.1.1.

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]

\[ \triangle T_a T_b T_c \text{ is equilateral and } \triangle T_a T_b T_c \cong \triangle S_a S_b S_c. \]

Property 6.1.2. (3rd isodynamic-Dao triangle)

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]

\[ \text{Result is also true when using } S' \text{ instead of } S. \]

\[ \text{See also [2].} \]

Property 6.1.3.

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]
Property 6.2.1.

\[ \triangle S_d S_e S_f \text{ is equilateral.} \]

Property 6.2.2.

\[ B, C, S_b, S_c, T_b, T_c, I, I_a \text{ are concyclic} \]

Property 6.3.1.

\[ \triangle S_a S_b S_c \text{ is equilateral.} \]

Property 6.3.2.

\[ S^* = S(S_a, S_b, S_c) \]

\[ K = S^* - S. \]

\[ S^* = X_{61}. \]

Property 6.3.3.

\[ K = K(P_1, P_3, P_5) \]

\[ S_i = S(P_{i-1}P_iP_{i+1}) \]

\[ S_1, S_2, S_3, S_4, S_5, S_6 \text{ are concyclic.} \]

\[ P_1P_4, P_2P_5, P_3P_6 \text{ concur.} \]

\[ \text{Result is also true using } S'. \]
6.4 formed by $O$

**Property 6.4.1.**

$\{S_1, S_2, \ldots, S_6, T_1, T_2, \ldots, T_6\}$ lie on $\odot ABC$.

6.5 formed by interior point $P$

**Property 6.5.1.**

- $\triangle S_aS_bS_c$ is equilateral.
- $\triangle T_1T_2T_3$ is equilateral and $\triangle T_4T_5T_6 \cong \triangle S_aS_bS_c$.

See [72].

6.6 formed by $X_{13}$

**Property 6.6.1.**

Yellow point = $X_{13}$

- $S$ lies on the axis of an ellipse that passes through the six points on the perimeter of $\triangle ABC$.

Property 6.6.2. [80 Thm. 7.8.1]

- $A - X_{13} - S_a$
- $B - S_a - C$

Property 6.6.3. [80 Thm. 7.8.2]

- $\triangle AX_{14}X_{13}$ has same orientation as $\triangle ABC$
- $X_{13}, S_a, X_{14}, B$ are concyclic.

6.7 formed by similar triangles

**Property 6.7.1.** [76]

- $S_aS_bS_c$ form an equilateral triangle.
6.8 formed by excenters

**Property 6.8.1.**

- $S_aS_bS_c$ form an equilateral triangle.
- Result is also true using $S'$.

6.9 formed by six cevians

**Property 6.9.1.**

- $S_aS_bS_c$ is an equilateral triangle.
- Vertices of green angles lie on an ellipse.
- Result is also true using $S'$.

6.10 formed by a Tucker Hexagon

**Property 6.10.1. (Tucker Hexagon)**

- $C_aC_b \parallel AB$, $A_iA_c \parallel BC$, $B_iB_a \parallel CA$
- $\triangle S_aS_bS_c$ is equilateral.
- The result is true if $S$ is replaced by $S'$. The two equilateral triangles are congruent.

6.11 formed by isogonal cevians

**Property 6.11.1.**

- $B, C, S_b, S_c$ concyclic

**Property 6.11.2.**

- $S_b, S_c, T_b, T_c$ concyclic

6.12 formed by varying vertex $A$

**Property 6.12.1.**

- $L \parallel BC$, $S_1 = S(A, BC)$
- $S_1, S_2, S_b, S_4$ are concyclic.
- $r = \frac{a^2}{2h + a\sqrt{3}}$
Property 6.12.2.

\[ S_i = S(A_i BC) \]

\[ \blacktriangleright \] \( S_1, S_2, S_3, S_4 \) are concyclic.

Property 6.12.3.

\[ S_i = S(A_i BC) \]

\[ \blacktriangleright \] \( S_1, S_2, S_3, S_4 \) are concyclic.

7. Quadrilateral plus \( S \)

7.1 cyclic quadrilateral

Property 7.1.1.

\[ S_i = S(A_i BC) \]

\[ \blacktriangleright \] \( S_1, S_2, S_3, S_4 \) are concyclic.

Property 7.1.2.

\[ \triangledown \] \( \angle SBD + \angle DCS = 60^\circ \)

Property 7.1.3.

\[ \blacktriangleright \] \( \frac{1}{R(BDS)} + \frac{1}{R(CDS)} = \frac{1}{R(ADS)} \)

Property 7.1.4.

\[ T = X_{16}(ACD) \]

\[ \blacktriangleright \] \( A, S, C, T \) concyclic
Property 7.1.5.*

\[ T = X_{16}(ACD) \]

► S – E – T

Property 7.1.6.

\[ S_a = S(BCD) \]

► \( S_a, S_b, S_c, S_d \) are concyclic.

Property 7.2.1.*

\[ \frac{CS}{BS} = \phi \]

► Yellow incircles are congruent.

Property 7.2.2.*

\[ S_2 \]

is the 1st isodynamic point of \( \triangle ACD \).

► Yellow incircles are congruent.

7.3 trilateral trapezoid

Property 7.3.1.*

\[ [ADC] = \frac{1}{7}[BAS] \]

\[ [CDS] = \frac{2}{7}[BCS] \]

\[ [ACS] = 3[BDS] \]

\[ [ABC] = 7[BDS] \]

8. Pentagon plus S

8.1 regular pentagon

Property 8.1.1.

\[ \frac{CS}{BS} = \phi \]

► Yellow incircles are congruent.
Property 8.1.2.

\[ \begin{align*}
\triangleright & \ AS/CS = \phi \\
\triangleright & \ SE/CS = \sqrt{2}
\end{align*} \]

9. Hexagon plus \( S \)

9.1 regular hexagon

Property 9.1.1.

\[ P_i = S(PA_iA_{i+1}) \]

\[ \triangle S_1S_3S_5 \sim \triangle S_4S_6S_2 \]
Other Properties

Other properties of the first isodynamic point (discovered by computer) were found by Dekov [21]. A typical result is: the first isodynamic point is the isogonal conjugate of the inner Fermat point of the anticevian triangle of the outer Fermat point.

Many properties of the first isodynamic point can be found in [37]. A typical result is: $X_{15}$ is the isogonal conjugate of the isotomic conjugate of $X_{298}$. Also, lists are given for many of the lines, circles, conics, and cubics that $X_{15}$ lies on.

Many properties of the isodynamic points can be found in the dissertation [44]. Many properties of the isodynamic points can be found in [80].

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