Quantum nonlocal correlations are not dominated

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We show that no probability distribution of spin measurement outcomes on pairs of spin 1/2 particles is unambiguously more nonlocal than the quantum correlations. That is, any distribution that produces a CHSH violation larger than the quantum violation for some axis choices also produces a smaller CHSH violation for some other axis choices. In this sense, it is not possible for nature to be strictly more nonlocal than quantum theory allows.

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Local hidden variable theories (LHVT) predict that the outcomes of space-like separated measurements on particles should satisfy Bell inequalities \textsuperscript{1,2}. Quantum theory predicts that Bell inequalities are violated for suitable measurement choices on entangled particles.

Bell inequalities and measures of nonlocality for two entangled qubits are defined by considering experiments in which the corresponding particles are sent to space-like separated locations. One location is controlled by Alice, who performs measurement $A$, the other by Bob, who performs measurement $B$. We focus here on the case where $A$ and $B$ are spin measurements about given axes on spin-$\frac{1}{2}$ particles, in which case Alice’s and Bob’s outcomes $a$ and $b$ are assigned values $a, b \in \{1, -1\}$, corresponding to ‘spin up’ or ‘spin down’. We define the correlation $C(A, B)$ as the average value of the product of Alice’s and Bob’s outcomes in experiments where measurements $A$ and $B$ are chosen.

Consider for definiteness the EPR-Bohm experiment performed on spin-$\frac{1}{2}$ particles in the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$. As before Alice and Bob choose measurements $A, B$ of their particle spin projections along directions $\vec{a}_A$ and $\vec{b}_B$, respectively. In general, the vectors $\vec{a}_A$ and $\vec{b}_B$ can point along any direction in 3-dimensional Euclidean space, and the sets of their possible values define Bloch spheres $S^2$. The correlation predicted by quantum theory is $Q(\theta) = -\cos \theta$, where $\cos \theta = \vec{a}_A \cdot \vec{b}_B$. If Alice can choose between measurements $A$ and $A'$, and Bob between $B$ and $B'$, LHVT predict correlations that satisfy the CHSH inequality \textsuperscript{2}:

$$I_2 = |C(A, B) + C(A, B') + C(A', B) - C(A', B')| \leq 2.$$ 

On the other hand, sets of measurement axes can be found for which the quantum correlations violate the CHSH inequality, $I_2^{QM} > 2$, up to the Cirel’son bound $I_2^{QM} \leq 2\sqrt{2}$.

Experiments compellingly confirm quantum theory and refute the predictions of LHVT (e.g. \textsuperscript{3,9}), modulo possible loopholes (e.g. \textsuperscript{10,11}) that arise from the difficulty in carrying out theoretically ideal experiments. In this sense, quantum theory and nature are commonly said to exhibit nonlocal correlations. This is something of a misnomer, since there is a natural locality principle respected by relativistic quantum theory. A more precise statement is that quantum correlations are not locally causal, according to Bell’s definition \textsuperscript{12}. However, we will follow the common usage here, since it is adopted in most of the literature to which our work relates.

Although nonlocal in the above sense, quantum correlations do not allow superluminal signalling: neither party’s measurement choice affects the probability distribution of the other’s outcomes. In an intriguing and celebrated paper \textsuperscript{13}, Popescu and Rohrlich pointed out that quantum nonlocal correlations are not characterised by the no-signalling condition alone. They illustrated this using what they called a “superquantum” correlation function $E$ for spin measurements about given axes, defined by a probability distribution whose marginals are uniform for any spin measurement by either party, conditioned on any measurement choice of the other party. The function $E$ depends only on the relative angle $\theta$ between axes and has the form

- $E(\theta) = 1$ for $0 \leq \theta \leq \pi/4$.
- $E(\theta)$ decreases monotonically and smoothly from 1 to 0 as $\theta$ increases from $\pi/4$ to $\pi/2$.
- $E(\pi - \theta) = -E(\theta)$ for $\pi/2 \leq \theta \leq \pi$.

For coplanar axes $\vec{a}, \vec{b}, \vec{a}', \vec{b}'$ separated by successive $\pi/4$ rotations, this gives the algebraically maximal CHSH expression

$$E(\vec{a}, \vec{b}) + E(\vec{a}', \vec{b}') + E(\vec{a}, \vec{b}') - E(\vec{a}', \vec{b}) = 3E(\pi/4) - E(3\pi/4) = 4.$$ \textsuperscript{(1)}

This violates the Cirel’son bound for quantum correlations, but still follows from a non-signalling probability distribution.

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Taking this CHSH expression as a measure of nonlocality, Popescu and Rohrlich went on to ask why quantum theory is not more non-local and whether stronger forms of nonlocality might be found in nature.

This raises a question: are the Popescu-Rohrlich correlations, or any other hypothetical sets of non-signalling correlations, unambiguously more non-local than quantum correlations? To even make sense of the question, one has to accept the premise that there is at least a partial ordering of the non-locality of correlations, which is reflected by the degree of violation of Bell inequalities. Then one needs to decide which Bell inequalities to consider. It is far from obvious that there is a natural way to do this. Even for the CHSH inequality, there are infinitely many possible axis choices to consider. Moreover, the CHSH inequality is only one of an infinite number of Bell inequalities defining different possible tests of quantum non-locality.

Of course, one possible candidate measure of nonlocality is the maximum CHSH violation that a set of correlations gives, for any set of axis choices, and the Popescu-Rohrlich correlations are more non-local than quantum theory by this measure. But the maximum violation is not the only possible measure, and it is arguable whether it is the most natural.

In this paper, we answer the question above in the negative: neither the Popescu-Rohrlich correlations nor any others are unambiguously more non-local than the quantum singlet correlations.

More precisely, we consider correlation functions $C(\theta)$ defined by hypothetical probability distributions for spin measurements of two particles about randomly chosen axes separated by angle $\theta$. Here $C(\theta)$ is the correlation averaged over all pairs of axes separated by $\theta$: the actual probability distributions may depend on the axis choices as well as their angular separation. Since spin measurement outcomes correspond to a positive or negative axis vector, we take $C(\pi - \theta) = -C(\theta)$. We also make the physically motivated assumption that $C(\theta)$ depends continuously on $\theta$.

We then show that if any such $C(\theta)$ produces a larger violation of some Bell inequality than the singlet quantum correlations do, then $C(\theta)$ must produce a smaller violation (or none) of some other Bell inequality also violated by the singlet quantum correlations. In other words, any correlations that, by a measure analogous to that used by Popescu-Rohrlich, are "more nonlocal" than the singlet are also, by another such measure, "less nonlocal". This is true whether or not the correlations arise from a non-signalling probability distribution, so long as the underlying theory defines correlation functions $C(\theta)$ that depend only on the angular separation $\theta$ and not on the details of how the ensemble of measurements with given $\theta$ is produced.

We use the following CHSH inequalities. The CHSH expression for quantum spin measurements about a randomly chosen set of coplanar axes $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ separated by angles $\theta, \pi/2 - \theta, 3\pi/2 - \theta, \pi - \theta$ respectively, gives

$$I_{QM}^{CHSH1}(\theta) = |2C_{QM}(\theta) + 2C_{QM}(\pi/2 - \theta)| = 2\cos \theta + 2\cos(\pi/2 - \theta) > 2,$$

for $0 < \theta < \pi/2$, violating the CHSH inequality

$$I_{QM}^{CHSH1} \leq 2.$$  (2)

The CHSH expression for quantum spin measurements about a randomly chosen set of coplanar axes $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ separated by angles $\theta/3, 2\theta/3, 3\theta/3$ respectively gives

$$I_{QM}^{CHSH2}(\theta) = |3\cos(\theta/3) - \cos \theta| > 2,$$

for $0 < \theta < \pi/2$, violating the CHSH inequality

$$I_{QM}^{CHSH2} \leq 2.$$  (3)

Any correlation function $C(\theta)$ that is "at least as non-local" as quantum theory according to these inequalities must thus satisfy

$$|2C(\theta) + 2C(\pi/2 - \theta)| \geq I_{QM}^{CHSH1}(\theta) > 2.$$  (6)

and

$$|3C(\theta/3) - C(\theta)| \geq I_{QM}^{CHSH2}(\theta) > 2.$$  (7)

for $0 < \theta < \pi/2$.

Consider a hypothetical $C(\theta)$ with these properties.

Note first that $C(\theta)$ must have the same sign throughout the range $0 < \theta < \pi/2$. If not, then by continuity $C(\theta_0) = 0$ for some $\theta_0$ in the range, and then (6) fails at $\theta = \theta_0$.

Now consider the case in which $C(\theta) < 0$ for all $\theta$ in the range. Suppose that for some $\theta$ we have $0 > C(\theta) > -\cos \theta$. It follows from (6) that

$$C(\pi/2 - \theta) < -\cos(\pi/2 - \theta).$$  (8)

Hence either $C(\theta) = -\cos \theta$ for all $\theta$ in the range, in which case $C$ is the quantum correlation function, or $C(\theta) < -\cos \theta$ for at least one value of $\theta$ in the range.

But now, if $C(\theta_1) = -\cos \theta_1 - \delta$, for some $\delta > 0$ and some $\theta_1$ in the range, then applying (7) iteratively gives

$$-\cos(\theta_1 3^{-n}) - C(\theta_1 3^{-n}) \geq 53^{-n}.$$  (9)

However, since

$$-\cos(\theta_1 3^{-n}) \leq -1 + \theta_1^2 2^{-132 - 2n}$$

for large $n$, and $C(\theta) \geq -1$ for all $\theta$, we have

$$-\cos(\theta_1 3^{-n}) - C(\theta_1 3^{-n}) \leq \theta_1^2 2^{-132 - 2n}$$

for large $n$, contradicting (9). Hence $C(\theta) = -\cos \theta$ for all $\theta$ in the range.

Similarly, if $C(\theta) > 0$ for all $\theta$ in the range, we find $C(\theta) = \cos \theta$. This is the correlation obtained from quantum theory if one party reverses their measurement outcome. As these are the only possibilities, we see that no super-quantum correlation functions – in the sense we have defined – exist.
I. DISCUSSION

Following Popescu and Rohrlich, we have focussed on hypothetical generalisations of the quantum correlations of a pair of entangled qubits, although it would be interesting to extend the discussion further to higher dimensions and to multipartite states. We have shown that no theory can produce spin measurement correlations for pairs of spin $1/2$ particles that dominate quantum non-local correlations, in the sense that they are at least as nonlocal by every measure and more nonlocal by at least one measure.

This observation suggests another way of looking at Popescu and Rohrlich’s intriguing observations. As other recent results [14] also suggest, degrees of quantum non-locality can only be properly compared when we consider the full range of spin measurements allowed by physics, rather than restricting attention to measures of nonlocality associated with particular finite sets of measurement axis choices. However, if we look at the full set of correlations, we see that neither the correlations that Popescu and Rohrlich consider nor any other possible set of correlations are unambiguously more strongly non-local than those of quantum theory.

Quantum correlations for entangled spin $1/2$ particles take the precise form they do because they reflect the interrelation between the Bloch sphere representations of the local spin rotation group $SU(2)$ and the local spatial rotation group $SO(3)$. One reason to think that Popescu and Rohrlich’s “superquantum” correlations may not arise in nature is that they do not arise naturally from local physical symmetries in this way.

As Popescu and Rohrlich originally framed the question, the maximum quantum CHSH value of $2\sqrt{2}$ sits interestingly between the classical value of 2 and the value of 4 attainable by non-signalling correlations. From this perspective, the number $2\sqrt{2}$ seems a puzzle in need of explanation.

One intriguing line of thought suggests that the explanation is to be found in the relation between physics and the information capacity of messages. Super-quantum correlations that violate the Cirel’son bound also violate the principle of information causality [13]. If this is a fundamental principle of nature, then the puzzle is solved.

Whether the laws of nature are fundamentally information-theoretic in this or other respects, is though, presently uncertain. We are far from fully understanding the fundamental principles underlying physics, and different ways of looking at deep unresolved questions can seem to strongly suggest different answers. With that important caveat noted, our results suggest another possible perspective. The maximal quantum CHSH violation of $2\sqrt{2} = 4\cos(\pi/4)$ may suggest a puzzle when considered in isolation. However, the full set of CHSH violations given by the singlet correlation function $\text{–} \cos(\theta)$ do not seem analogously puzzling, since they are not dominated by any other correlations.

In this significant sense, the Popescu-Rohrlich correlations are not actually “superquantum”. It is true that the two-input two-output “nonlocal boxes” given by specific measurement axes are extreme points in the space of non-signalling correlations, while the quantum correlations for any state and any pairs of axis choices are not. However, if we look at the full set of correlations for all angles $\theta$, the Popescu-Rohrlich correlations are no longer distinguished from quantum singlet correlations by this criterion. From this perspective, then, there is perhaps really no fundamental puzzle about why Popescu-Rohrlich or other purportedly “super-quantum” correlations are not used by nature, since there seems no sufficiently strong theoretical reason to think they characterise singularly physically interesting generalizations of quantum theory in our space-time.

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