LOG TERMINAL ORDERS ARE NUMERICALLY RATIONAL

KENNETH CHAN

ABSTRACT. Noncommutative surfaces finite over their centres can be realised as orders over surfaces. The aim of this paper is to present a noncommutative generalisation of rational singularities, which we call numerical rationality, for such orders. We show that numerical rationality is independent of the choice of resolution. Our main result is that the log terminal orders arising from the noncommutative minimal model program [CI05], in particular, the canonical orders defined in [CHI09], are numerically rational. Both of these generalise well known facts about rational singularities in commutative algebraic geometry.

1. Introduction

The noncommutative minimal model program (c.f. [CI05]) resolves the birational classification problem for noncommutative surfaces which are finite over their centres. Such noncommutative surfaces are called orders. Many concepts from Mori theory carry over to the noncommutative setting. Of particular interest to us are the noncommutative counterparts of canonical and log terminal singularities, called canonical and log terminal orders. Canonical orders were studied extensively in [CHI09] as invariant rings and in [Cha] in the context a noncommutative version of the McKay correspondence. More generally, log terminal orders can be viewed as noncommutative quotient singularities. Absent from the noncommutative theory is a notion of rational singularities for orders, and the main objective of the present paper is to address this point.

Rational singularities are characterised by the property that their cohomology does not change after passing to the resolution. On surfaces, rational singularities were studied extensively in [Lip69]. A theorem of Artin gives a numerical characterisation of rationality of surface singularities in terms of the intersection theory on their resolutions (c.f. [Art66]). Many naturally occurring singularities are rational, examples include singularities of toric varieties and quotient singularities. The log terminal singularities of the Mori program are also rational singularities.

The distinguishing feature of our approach to studying noncommutative singularities is the use of resolutions of singularities (c.f. [CI05], Corollary 3.6). It provides the necessary technology to investigate rationality for singularities of orders. We define resolution of singularities of orders in Section 2. The most important ingredient for resolutions are terminal orders, which are
the smooth models in the noncommutative Mori program. These are defined in [CI05], in terms of discrepancies, and we review an equivalent definition in Section 1.1. Also important is the notion of a birational morphism of orders, defined in [CHI09]. If we think of an order as a noncommutative surface $X$, these concepts allow us to consider a resolution of singularities as consisting of a smooth noncommutative surface $\tilde{X}$ together with a (noncommutative) birational morphism $\tilde{X} \to X$.

To continue the discussion, we first review the definition of an order over a surface. Let $Z$ be a normal integral $k$-scheme of dimension 2. An $\mathcal{O}_Z$-order $A$ is a coherent, torsion-free sheaf of $\mathcal{O}_Z$-algebras which is generically a $k(Z)$-central simple algebra. Intuitively, we think of $A$ as a sheaf of functions on some “noncommutative space.” We restrict our attention mostly to maximal orders; this is analogous to considering normal varieties.

In section 2, we define numerical rationality (c.f. Definition 2.2) by mimicking Artin’s numerical condition for rational singularities (c.f. [Art66], Proposition 1), which states that $Z$ is a rational singularity if and only if for some resolution $\tilde{Z} \to Z$, every exceptional effective divisor on $\tilde{Z}$ has positive Euler characteristic. In the noncommutative generalisation, we are led to consider the Euler characteristic of $\tilde{A}$ restricted to divisors, and arrive at a similar condition. We show in Proposition 2.3 that numerical rationality does not depend on the choice of resolution.

Our main result, found in section 3, is that log terminal orders are numerically rational (c.f. Theorem 3.1). The proof of this theorem depends on an analysis of the dual resolution graphs of the minimal resolutions of log terminal singularities. The main tool we use is a formula which computes the Euler characteristic $\chi(\tilde{A} \otimes \mathcal{O}_E)$ where $\tilde{A}$ is a terminal order on $\tilde{Z}$ and $E$ is a divisor whose underlying variety is projective (c.f. Theorem 4.1). We learned this from an unpublished manuscript, [AdJ], of M. Artin and A. J. de Jong. We find that $\chi(\tilde{A} \otimes \mathcal{O}_E)$ has a nice expression which resembles the adjunction formula for $\chi(\mathcal{O}_E)$, thus reducing the computation of $\chi(\tilde{A} \otimes \mathcal{O}_E)$ to intersection theory on $\tilde{Z}$. A proof of this is provided in Section 4.

1.1. Definitions. We set up some basic assumptions which will be in force throughout the paper and briefly review the essential ingredients from the theory of orders on surfaces. Let $k$ be an algebraically closed field of characteristic zero. All objects below will be defined over $k$. We assume, once and for all, that all orders are normal (c.f. [CI05], Definition 2.3). We will not require the precise definition for normal orders, it is a technical condition arising from the fact that étale localisations of a maximal order are in general not maximal. The normality criterion is a relaxation of maximality which is stable under étale localisations. The important point for us is that the noncommutative Mori program is carried out for normal orders, and in particular, resolution of singularities for such orders exist.

The most useful invariant for an order in this paper is its canonical divisor. Let $A$ be an order on a normal surface $Z$ and $D$ be an irreducible
curve on $Z$. We denote by $A_D$ the localisation of $A$ at $D$ and $J(A_D)$ its Jacobson radical. The centre $Z(A_D/J(A_D))$ of $A_D/J(A_D)$ is a product of field extensions of $k(D)$. Define $e_D = \dim_{k(D)} Z(A_D/J(A_D))$ the ramification index of $A$ at $D$. Also define the ramification divisor $\Delta_A$ to be the $\mathbb{Q}$-divisor $\sum_{D \in Z^1} (1 - 1/e_D) D$ and the canonical divisor $K_A$ of $A$ by $K_Z + \Delta_A$. There is a finer invariant for an order called the ramification data, where instead of just remembering the numbers $e_D$, it remembers the extension $Z(A_D/J(A_D))$ of $k(D)$.

1.2. Acknowledgements. I take this opportunity to thank my teacher Daniel Chan, who introduced me to noncommutative algebraic geometry. He asked me to find out whether there is a good theory of rational singularities for orders, and this paper grew out of that investigation. I also thank Colin Ingalls for sending me a preprint on log terminal orders.

2. Numerical Rationality

The aim of this section is to define and study a generalisation of rational singularities for orders. We will outline some of the difficulties involved in extending such a notion noncommutatively, and hopefully convince the reader that our proposed definition is interesting. Many naturally occurring singularities in birational geometry are rational. Recall that log terminal surface singularities are simply quotients of $\mathbb{A}^2$ by a finite subgroup of $GL_2$, and these are rational singularities. Our point of view is that a version of rational singularities for orders should include the log terminal orders arising from the noncommutative minimal model program of [CI05].

The definition for rational singularity for varieties makes essential use of the existence of a resolution of singularities $\sigma : \tilde{Z} \to Z$. A resolution $\sigma : \tilde{Z} \to Z$ is rational if $\sigma_* O_{\tilde{Z}} = O_Z$ and $R^i \sigma_* O_{\tilde{Z}} = 0$ for $i > 0$; and $Z$ has rational singularities if there exists a rational resolution $\sigma : \tilde{Z} \to Z$. This definition does not depend on the resolution, since if $Z$ has a rational resolution, then all resolutions of $Z$ are rational. Orders on surfaces have resolutions of singularities (c.f. [CI05], Corollary 3.6); a resolution of an order $A$ consists of a pair $(\sigma : \tilde{Z} \to Z, \tilde{A})$, where $\sigma : \tilde{Z} \to Z$ is a resolution of varieties and $\tilde{A}$ is a terminal order on $\tilde{Z}$ with $\sigma^* A \subset \tilde{A}$ satisfying

1. for any exceptional curve $E$ of $\sigma$, the $O_{\tilde{Z},E}$-order $\tilde{A}_E$ is maximal
2. for any non-exceptional curve $D$ on $\tilde{Z}$, we have $(\sigma^* A)_D = \tilde{A}_D$.

Note that the terminal order $\tilde{A}$ is not unique, since in general, we can choose different maximal orders $\tilde{A}_E$ containing $(\sigma^* A)_E$ for each $E$, and every such choice $\{\tilde{A}_E\}_E$ produces a bona fide resolution of $A$ on $\tilde{Z}$. However, by the Artin-Mumford sequence, the ramification data of $A$ determines the ramification data of its resolutions (c.f. [CI05], Lemma 3.4). In particular, the ramification data of $\tilde{A}$ is independent of the choices of maximal orders at exceptional curves. Armed with this technology, we can explore what it means for an order to have rational singularities.
The most naïve procedure is to replace $O_{\tilde{Z}}$ by $\tilde{A}$ and say that $(\sigma, \tilde{A})$ is a rational resolution if $\sigma^* \tilde{A} = A$ and $R^i \sigma_* \tilde{A} = 0$ for $i > 0$. We see easily that this runs into problems. Firstly, as the following example shows, such a definition depends on the choice of maximal orders in blowing up, hence is not Morita invariant (c.f. [Cha], Proposition 4.1). From the point of view of noncommutative geometry, this is rather discouraging.

Example 2.1. Let $Z = \text{Spec } k[[u, v]]$ and

$$A = \begin{pmatrix} O_Z & O_Z \\ (u^3 - v^2)O_Z & O_Z \end{pmatrix}$$

be a canonical order of type $BL_1$. The order $A$ is ramified on the curve $D$ defined by the equation $u^3 - v^2$ and can be resolved by a single blowup $\sigma : \tilde{Z} \to Z$ at the cusp $p$ of $D$ (c.f. [CHI09], Figure 1). Let $\tilde{D}$ denote the strict transform of $D$ and $E$ be the exceptional curve of $\sigma$. There are three non-isomorphic terminal orders on $\tilde{Z}$

$$\tilde{A}_m = \begin{pmatrix} O_{\tilde{Z}} & O_{\tilde{Z}}(mE) \\ O_{\tilde{Z}}(-\tilde{D} - mE) & O_{\tilde{Z}} \end{pmatrix}$$

for $m = 0, 1, 2$ which are maximal at $E$ and contain $\sigma^* A$, so $R^1 \sigma_* \tilde{A}_m$ vanishes if and only if $m \neq 2$.

Moreover, the same example shows that there exists a resolution of a canonical order that is not rational in the naïve sense above. This transgresses our requirement that the canonical orders (which are log terminal) of the noncommutative Mori program should be rational. An alternative is to use the dual formulation of the definition, that is say that a resolution $(\sigma, \tilde{A})$ is rational if $\omega_{\tilde{A}}$ is a Cohen-Macaulay sheaf and $\sigma^* \omega_{\tilde{A}} = \omega_A$. Unfortunately, this too is susceptible to the same objections as above. We note here that due to the absence of a Grauert-Riemenschneider vanishing theorem, the above two formulations for rational resolutions for orders are not equivalent.

To get a good notion for rational resolutions for orders, we generalise Artin’s numerical criterion for rational singularities on varieties, which states that $Z$ has rational singularities if and only if for some resolution $\sigma : \tilde{Z} \to Z$, we have $\chi(O_E) > 0$ for all exceptional divisors $E > 0$ on $\tilde{Z}$.

**Definition 2.2.** Let $A$ be a normal order on a surface $Z$. A resolution $(\sigma : \tilde{Z} \to Z, \tilde{A})$ of $A$ is numerically rational if $\chi(\tilde{A} \otimes O_E) > 0$ holds for all exceptional divisors $E > 0$ on $\tilde{Z}$. The order $A$ is numerically rational if every resolution is numerically rational.

We prove below that the above definition has the nice property that if numerical rationality holds for some resolution, then it holds for all resolutions. This generalises the corresponding fact for rational resolutions for varieties. The adjunction formula proved in Theorem 4.1 will be used in the proof of the next proposition, and we state the result for the reader’s convenience:
let \( \tilde{A} \) be a terminal order of rank \( n^2 \) on \( \tilde{Z} \) and \( E \) be an effective exceptional divisor, then
\[
\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) = -\frac{n^2}{2}(K_{\tilde{A}} + E).
\]

We see immediately that \( \chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) \) depends only on the ramification data. In particular, if \( \sigma \) is a rational resolution, then numerical rationality is a Morita invariant property (c.f. [Cha], Proposition 4.1).

**Proposition 2.3.** Let \( (\sigma : \tilde{Z} \rightarrow Z, \tilde{A}), (\tau : \tilde{Y} \rightarrow Z, \tilde{B}) \) be resolutions of \( A \) and suppose \( \tau = \sigma \beta \) where \( \beta \) is a blowup at a point \( p \in \tilde{Z} \). Then \( (\sigma, \tilde{A}) \) is numerically rational if and only if \( (\tau, \tilde{B}) \) is numerically rational.

**Proof.** Since both \( \tilde{A} \) and \( \tilde{B} \) are terminal orders, we can use Theorem 4.1 to compute their Euler characteristics when restricted to divisors. Let \( R \) denote the ramification divisor of \( \tilde{A} \) on \( \tilde{Z} \). We have two cases to consider, depending on whether \( p \) belongs to the singular locus of \( \text{supp} \, R \).

We denote by \( E_0 = \text{Ex}(\beta) \) the exceptional curve on \( \tilde{Y} \) contracted by \( \beta \). If \( p \) is not in the singular locus of \( \text{supp} \, R \), we see that \( E_0 \) is unramified. If, in addition, \( p \in \text{supp} \, R \) then \( \beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)E_0 = \Delta_{\tilde{B}} \) where \( e_1 \) is the ramification index of the irreducible component of \( \text{supp} \, R \) containing \( p \). If \( p \notin \text{supp} \, R \), then \( \beta^*\Delta_{\tilde{A}} = \Delta_{\tilde{B}} \). For convenience, we will write this as \( \beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)E_0 = \Delta_{\tilde{B}} \) for \( e_1 = 1 \).

Now suppose \( p \in (\text{supp} \, R)_{\text{sing}} \). Note that since \( \tilde{A} \) is terminal, \( R \) only has nodal singularities. Let \( R_1, R_2 \) be irreducible components of \( R \) intersecting transversely at \( p \), and denote by \( e_i \) the ramification index of \( R_i \). Then \( e_1 = s e_2 \) for some integer \( s \). The Artin-Mumford sequence can be used to show that \( A \) is totally ramified at \( E_0 \) with \( e_0 = e_2 \) (c.f. Lemma 3.4, [CI05]). Hence \( \beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)E_0 = \Delta_{\tilde{B}} \).

We can write a general effective divisor \( E \) on \( \tilde{Y} \) as \( \beta^*\tilde{E} + m E_0 \) where \( \tilde{E} \) is some effective divisor on \( \tilde{Z} \) and \( m \in \mathbb{Z} \). Since \( \beta \) is the blowup of a smooth point, we know that \( \beta^*K_{\tilde{Z}} + E_0 = K_{\tilde{Y}} \). In each case, we get the following,
\[
\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) = \chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) + \frac{n^2}{2}m \left( m + \frac{1}{e_1} \right)
\]
for some \( e_1 > 0 \). If \( \chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0 \) for all \( E > 0 \), then putting \( m = 0 \) gives \( \chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) > 0 \) for all \( \tilde{E} > 0 \). Conversely, if \( \chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) \) is positive for all \( \tilde{E} > 0 \), then we can conclude that \( \chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) \geq 0 \) for all \( \tilde{E} > 0 \) and \( m \in \mathbb{Z} \). To see that \( \chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0 \), we find that the only nontrivial solution of \( \chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) = 0 \) occurs when \( e_1 = 1, \tilde{E} = 0 \) and \( m = -1 \). This does not correspond to an effective divisor on \( \tilde{Y} \). Hence \( \chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0 \) for all \( E > 0 \).

We say that a resolution \( (\sigma : \tilde{Z} \rightarrow Z, \tilde{A}) \) of an order \( A \) is minimal if the canonical divisor \( K_{\tilde{A}} \) is \( \sigma \)-nef (c.f. [CI05], Theorem 3.10).
Corollary 2.4. An order $A$ is numerically rational if and only if any resolution is numerically rational.

Proof. Suppose $(\sigma, \tilde{A})$ is a resolution of $A$. If $(\sigma, \tilde{A})$ is not minimal, then there exists a $K_{\tilde{A}}$-negative curve $E$ with $E^2 < 0$. By [CI05], Theorem 3.10, we can factor $\sigma = \tau' \beta'$ through a blowup $\beta'$ at a point which contracts $E$, and there exists a terminal order $A_1'$ such that $(\tau', A_1')$ is a resolution of $A$. The terminal order $A_1'$ is obtained by taking the reflexive hull of $\beta'_* \tilde{A}$. Repeating this until we reach a minimal resolution allows us to factor $\sigma = \tau \beta$ where $\beta$ is a sequence of blowups centred at closed points, and obtain a terminal order $A_1$ such that $(\tau, A_1)$ is a minimal resolution of $A$.

By Proposition 2.3 $(\sigma, \tilde{A})$ is numerically rational if and only if $(\tau, A_1)$ is numerically rational. According to Theorem 2.15 of [CHI09], minimal resolutions of $A$ have the same centres and ramification data. Since numerical rationality depends only on the ramification data, the result follows. \qed

Recall that canonical orders have crepant minimal resolutions (c.f. [CHI09], Proposition 6.1), that is, if $(\sigma, \tilde{A})$ is a minimal resolution of the canonical order $A$, then $K_{\tilde{A}} = \sigma^* K_A$. It is easy to show that canonical orders are numerically rational.

Corollary 2.5. Let $(\sigma : \tilde{Z} \to Z, \tilde{A})$ be a crepant resolution of the $\mathcal{O}_Z$-order $A$. Then $(\sigma, \tilde{A})$ is a numerically rational resolution. In particular, canonical orders are numerically rational.

Proof. If $(\sigma, \tilde{A})$ is crepant, then $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) = -n^2 E^2 / 2$, which is positive for any exceptional divisor $E$, so $(\sigma, \tilde{A})$ is numerically rational. Let $A$ be a canonical order. The minimal resolution of a canonical order $A$ is crepant, hence $A$ is numerically rational. \qed

Our definition of numerical rationality is weaker than the naïve generalisation of rational resolutions to orders: if $R^1 \sigma_* \tilde{A} = 0$, then we can see by taking the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \tilde{A}(-E) \to \tilde{A} \to \tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E \to 0$$

that $h^1(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) = 0$. Since $\mathcal{O}_E \subset \tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E$, we have $h^0(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) > 0$ hence $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) > 0$. Example 2.1 and Corollary 2.5 shows that numerical rationality is strictly weaker than the naïve generalisation of rationality.

We conclude this section with an example of an order which is not numerically rational.

Example 2.6. Consider the simple elliptic singularity of type $\tilde{E}_6$, which is given by the equation $f_\lambda = 0$ where $f_\lambda = u^3 + v^3 + w^3 + \lambda uvw$ (c.f. [Dim92], (4.9)) for some $\lambda \in k$. We construct below an order $A$ with centre $Z = \text{Spec} k[[u, v, w]]/(f_\lambda)$ whose minimal resolution is not numerically rational. Let $A$ be the $k[[u, v, w]]/(f_\lambda)$-algebra generated by $x, y$ with relations $x^2 = u, y^2 = v$ and $xy + yx = 0$. Then $A$ is a maximal order of rank 4 over $Z$. 


Let $\sigma : \tilde{Z} \to Z$ be the minimal resolution of $Z$, then $E = \text{Ex}(\sigma)$ is an elliptic curve with $E^2 = -3$. Let $(\sigma, \tilde{A})$ be a blowup of $A$. Since $A$ is maximal, $\tilde{A}$ is a maximal order on $\tilde{Z}$ containing $\sigma^* A$. A local computation shows that $(\sigma^* A)_E$ is contained in a unique maximal order, hence $\tilde{A}$ is the unique blowup of $A$ along $\sigma$. We can describe $\tilde{A}$ as follows: on the open affine set $U = \text{Spec} k[[u, v, w]][v/u, w/u]/(f_\lambda u^{-3})$, we have

$$\tilde{A}(U) = \sigma^* A(U) \langle xyu^{-1} \rangle$$

and similarly for the other standard open affine sets $V = \text{Spec} k[[u, v, w]][u/v, w/v]/(f_\lambda)$ and $W = \text{Spec} k[[u, v, w]][u/w, v/w]/(f_\lambda)$ of $\tilde{Z}$. One can check that $\tilde{A}$ is a terminal order ramified on $E$ and two divisors $D_1, D_2$ transverse to $E$, each with ramification index 2. The equations for $D_1$ and $D_2$ on $W$ are $u/w = 0$ and $v/w = 0$.

We show that the resolution $(\sigma, \tilde{A})$ is not numerically rational. The simple elliptic singularity is log canonical, so $K_{\tilde{Z}} = \sigma^* K_Z - E$. This gives $K_{\tilde{A}} = \sigma^* K_Z - E/2 + D/2$ so by the adjunction formula for orders, we have

$$\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_{mE}) = 3m(2m - 3)$$

which is negative for $m = 1$.

3. Log terminal implies numerically rational

In this section, we prove a noncommutative version of the following result: log terminal singularities are rational singularities. There is a notion of log terminal orders developed in the context of the noncommutative Mori theory of [CI05], and the analogue for rational singularities is provided by our notion of numerical rationality (c.f. Definition 2.2). The noncommutative version of the above result has the following pleasant statement.

Theorem 3.1. If $A$ is a log terminal order on $Z$, then $A$ is numerically rational.

Note that if $A$ is log terminal, the associated log pair $(Z, \Delta_A)$ of $A$ is klt ([CI05], Proposition 3.15). It follows from [KM98], Corollary 2.35 that $Z$ has log terminal singularities. So we assume below that $Z$ is the spectrum of a local ring with log terminal singularities. To prove that log terminal orders are numerically rational, by Corollary 2.4 we need only study their minimal resolutions. Since all minimal resolutions have the same ramification data ([CHI09], Theorem 2.5) we will use the following characterisation of log terminal orders, which is equivalent to the definition in [CI05]. Let $A$ be an order on $Z$ and $(\sigma : Z' \to Z, A')$ be any minimal resolution. We can write

$$K_{A'} = \sigma^* K_A + \sum_i a_i E_i$$

where the $E_i$’s range over the exceptional curves on $Z'$. Then $A$ is log terminal if and only if $\min\{a_ie_i\} > -1$, where $e_i$ is the ramification index of $A'$ at $E_i$. 
Let $A$ be a log terminal order on $Z$ and $(\sigma : \tilde{Z} \rightarrow Z, \tilde{A})$ be any resolution. Denote by $E$ the subgroup $\text{Pic} \tilde{Z}$ generated by the exceptional curves. The intersection product on exceptional curves is well defined, and it endows $E$ with the structure of a quadratic $\mathbb{Z}$-module. Given a $D \in E \otimes \mathbb{Z}$, we define the function $f_{\sigma,D} : E \rightarrow \mathbb{Q}$ by $E \mapsto -(D + E)E$. Let $E^{+} = \{ \sum a_{i}E_{i} \in E \mid a_{i} \geq 0 \}$ denote the effective cone of $E$. By Theorem 4.1, $(\sigma, \tilde{A})$ is a numerically rational resolution if and only if $f_{\sigma,K_{\tilde{A}}}(E) > 0$ for all $E \in E^{+} \setminus \{0\}$.

The function $f_{\sigma,K_{\tilde{A}}}$ is the sum of a positive definite quadratic form $q(E) = -E^{2}$ and a linear function $\ell(E) = -DE$ on $E$. The log terminal condition on $K_{\tilde{A}}$ puts constraints on the coefficients of $\ell$. To get some information out of these constraints, it is profitable to choose a different $\mathbb{Z}$-basis $\{ N_{i} \}$ for $E$. We define $N_{i}$ as follows: for an exceptional curve $E_{i}$ on $\tilde{Z}$, there is a unique factorisation $\tilde{Z} \xrightarrow{\tau_{i}} Z' \xrightarrow{\tau_{i}'} Z$ of $\sigma$ satisfying the following properties

1. $Z'$ is smooth,
2. $E_{i}$ is not contracted by $\tau_{i}$, so $\tau_{is}E_{i}$ is a curve on $Z'$, and
3. there are no $(-1)$-curves on $Z'$ except for possibly $\tau_{is}E_{i}$.

Let $N_{i} = \tau_{i}^{*}\tau_{is}E_{i}$. It is easy to see that one obtains the factorisation above by sequentially contracting $(-1)$-curves except for the pushforwards of $E_{i}$.

We will adopt the following notation for the exceptional curves on $\tilde{Z}$: the resolution $\sigma$ factors through a minimal resolution $\pi_{0} : Z_{0} \rightarrow Z$ of $Z$ so that $\sigma = \pi_{0}\pi$, we denote by $E_{1}, \ldots, E_{r}$ the exceptional curves not contracted by $\pi$ and label the rest by $E_{r+1}, \ldots, E_{r+\ell}$.

**Proposition 3.2.** Let $(\sigma : \tilde{Z} \rightarrow Z, \tilde{A})$ be a minimal resolution of an order $A$ on $Z$ and let $\sigma = \pi_{0}\pi$ be as above. Then

\[
(3.1) \quad f_{\sigma,K_{\tilde{A}}}(E) = f_{\pi_{0}\pi,K_{\tilde{A}}}((\pi_{s}E) + \left( \sum_{j=1}^{\ell} (N_{r+j}E)N_{r+j} \right)(E + K_{\tilde{A}})).
\]

**Proof.** Since $f_{\pi_{0}\pi,K_{\tilde{A}}}((\pi_{s}E) = -\pi^{*}\pi_{s}E(E + K_{\tilde{A}})$, it suffices to show that

\[
(3.2) \quad E = \pi^{*}\pi_{s}E - \left( \sum_{j=1}^{\ell} (N_{r+j}E)N_{r+j} \right).
\]

A simple computation gives

\[
N_{r+j}E_{i} = \begin{cases} 
-1 & \text{if } i = r + j \\
1 & \text{if } \tau_{i} = \alpha\tau_{r+j} \text{ where } \alpha \text{ is a blowup centered at a point on } \tau_{is}E_{i} \\
0 & \text{otherwise}
\end{cases}
\]

Let $E_{r+j_{1}}, \ldots, E_{r+j_{s}}$ be components of $N_{i}$ which intersect $E_{i}$, then $E_{i} = N_{i} - N_{r+j_{1}} - \cdots - N_{r+j_{s}}$. The curve $E_{r+j_{i}}$ is contracted by $\tau_{i}$, hence we can factor $\tau_{i} = \alpha\tau_{r+j}$ for some birational map $\alpha$. Since $E_{r+j_{i}}E_{i} \neq 0$, $\alpha$ must be a single blowup centered at a point on $\tau_{is}E_{i}$. Hence (3.2) follows from the above computation for $N_{r+j}E_{i}$. \qed
where the summation ranges over all $i$ with the minimal resolution $\pi$. 

Proposition 3.3. Suppose $A$ is log terminal. Then $0 \leq N_{r+j}K_{\tilde{A}} < 1$ for $j = 1, \ldots, \ell$.

Proof. Since $(\sigma, \tilde{A})$ is a minimal resolution, $K_{\tilde{A}}$ is $\sigma$-nef (c.f. [CI05], Theorem 3.10). Since $N_{r+j}$ is effective, we get the first inequality. Note that $N_{r+j}E_{r+j} = (\tau_{r+j}E_{r+j})^2 = -1$ and $N_{r+j}E_k = 0$ for any other exceptional curve $E_k \subseteq \text{supp}N_{r+j}$. Clearly the intersection numbers of $N_{r+j}$ with exceptional curves away from its support are non-negative, in fact, if $E_i$ is not an irreducible component of $N_{r+j}$, then $E_iN_{r+j} = 0$ or $1$. This gives

$$N_{r+j}K_{\tilde{A}} = -a_{r+j} + \sum_i a_i$$

where the summation ranges over all $i$ where $E_i$ intersects $N_{r+j}$. Now since $K_{\tilde{A}}$ is $\sigma$-nef, we have by [KM98], Lemma 3.41 that $a_i \leq 0$ for all $i$. Moreover, since $A$ is log terminal, we have $a_{r+j} > -1$, hence $N_{r+j}K_{\tilde{A}} < 1$. $\square$

The above propositions shows that log terminal orders with smooth centres are numerically rational, and that in general, a minimal resolution $(\sigma, \tilde{A})$ of a log terminal order is numerically rational if and only if $f_{\pi_0, \pi}K_{\tilde{A}}(\pi_*E) > 0$ for all $E \in E^+ \setminus \{0\}$ such that $\pi_*E > 0$. This allows us to work directly with the minimal resolution $\pi_0 : Z_0 \to Z$. Henceforth, we will drop the $\pi_*$ and refer to the exceptional curves on $Z_0$ by $E_1, \ldots, E_r$ and denote by $E_0 \subseteq \text{Pic}Z_0$ the subgroup generated by $E_1, \ldots, E_r$.

Recall that the numerical cycle $Z_{\text{num}}$ with respect to the birational morphism $\pi_0 : Z_0 \to Z$ is defined to be the minimal effective exceptional divisor $E$ on $Z_0$ such that $-E$ is $\pi_0$-nef. Given a connected effective exceptional divisor $D$, we can contract $\text{supp}D$ to get a birational morphism $\pi_D : Z_0 \to Z_D$. We define $D_{\text{num}}$ to be the numerical cycle with respect to $\pi_D$, and call it the numerical cycle of the support of $D$. We call a connected effective exceptional divisor $D$ on $Z_0$ special if $D = D_{\text{num}}$. In particular, the numerical cycle $Z_{\text{num}}$ of $Z_0$ is a special divisor.

As we shall prove in Theorem 3.5, $f_{\pi_0, \pi}K_{\tilde{A}}$ is positive for all $E \in E_0^+ \setminus \{0\}$ if its values at special divisors are positive. Theorem 3.1 follows from the next two results.

Proposition 3.4. Let $A$ be a log terminal order on $Z$. Then $f_{\pi_0, \pi}K_{\tilde{A}}(E) > 0$ for all special divisors $E \in E_0$. 

Proof. Let $E$ be a special divisor. Suppose $E_j$ is not contained in $\text{supp} E$, then $E_{\text{num}} E_j \geq 0$. Since $a_i \leq 0$ (c.f. proof of Proposition 3.3), we have

$$f_{\pi_0, \pi_* K_A}(E_{\text{num}}) \geq -E_{\text{num}} \left( E_{\text{num}} + \sum_{E_i \subseteq \text{supp} E} a_i E_i \right).$$

Now $A$ is log terminal, so $a_i > -1$ for all $i$. Moreover, by the definition of $E_{\text{num}}$, we have $-E_{\text{num}} E_i \geq 0$ for any $E_i \subseteq \text{supp} E$, so

$$f_{\pi_0, \pi_* K_A}(E_{\text{num}}) > -E_{\text{num}}(E_{\text{num}} - E_{\text{red}}),$$

where $E_{\text{red}}$ denotes the reduced exceptional divisor with the same support as $E_{\text{num}}$. Since $E_{\text{num}}$ is the numerical cycle of its support, we see that $E_{\text{num}} - E_{\text{red}}$ is an effective divisor with support contained in $\text{supp} E$. This gives $f_{\pi_0, \pi_* K_A}(E_{\text{num}}) > 0$. $\square$

**Theorem 3.5.** Let $\pi_0 : Z_0 \longrightarrow Z$ be the minimal resolution of a log terminal singularity and $g : E \longrightarrow \mathbb{Q}$ be a function $g(E) = -E^2 + \ell(E)$ with $\ell$ linear and $\ell(E_i) \leq 0$ for $i = 1, \ldots, r$. Then $g(E) > 0$ for all $E \in E_0^+ \setminus \{0\}$ if and only if $g(E) > 0$ for all special $E \in E_0$.

**Note 3.6.** Note that $f_{\pi_0, \pi_* K_A}$ satisfies the above hypotheses for $g$ if $(\sigma, \hat{A})$ is a minimal resolution. Since in this case $K_A$ is $\sigma$-nef, we have $\pi_* K_A \pi_* E_i = K_A N_i \geq 0$ for $i = 1, \ldots, r$, hence $\pi_* K_A$ is $\pi_0$-nef.

The rest of this section is devoted to the proof of the above theorem. We fix notation for the rest of the section: let $\pi_0 : Z_0 \longrightarrow Z$ be the minimal resolution of a log terminal singularity. We denote by $E_1, \ldots, E_r$ the exceptional curves and $Z_{\text{num}}$ the numerical cycle on $Z_0$. Also we will denote by $g(E) = -E^2 + \ell(E)$ a function $E_0 \longrightarrow \mathbb{Q}$ satisfying the hypothesis of Theorem 3.5.

### 3.1. Modified numerical cycle

The usual notion of numerical cycle can be modified with respect to a given effective exceptional divisor $D$ as follows: we define the numerical cycle $D'$ associated to $D$ to be the minimal effective divisor satisfying $D \leq D'$ and $-D'$ is $\pi_0$-nef. When $D = 0$, then $D'$ is just the usual numerical cycle $Z_{\text{num}}$. To see that $D'$ exists and is unique for a given $D$, pick an integer $n$ such that $D \leq nZ_{\text{num}}$. Then the divisor $D'' = \gcd\{C \mid D \leq C \leq nZ_{\text{num}}, -C$ is $\pi_0$-nef} is well defined since the gcd is taken over finitely many exceptional divisors, and clearly $D' = D''$. We can construct $D'$ inductively by the following procedure, which is modelled on the construction of $Z_{\text{num}}$ (c.f. [Rei97], Section 4.5).

We start with $D_0 = D$ and define $D_{i+1}$ recursively as follows. If $-D_i$ is $\pi_0$-nef, then we are done; otherwise there exists some irreducible exceptional curve $E$ such that $D_i \cdot E > 0$. Define $D_{i+1} = D_i + E$ and repeat. The following lemma shows that the above procedure terminates at $D'$.

**Lemma 3.7.** For each $i$, we have $D_i \leq D'$. 

LOG TERMINAL ORDERS ARE NUMERICALLY RATIONAL

Proof. Suppose $D_i \leq D'$ and let $E$ be any exceptional curve. If the effective divisor $D' - D_i$ is supported away from $E$, then $D_i E \leq D'E \leq 0$. Hence $D_{i+1} = D_i + \tilde{E}$ where $\tilde{E}$ is an exceptional curve whose multiplicity in $D_i$ is strictly less than its multiplicity in $D'$. This shows that $D_{i+1} \leq D'$. □

The following inequality will be useful for bounding $-D^2$ below.

Lemma 3.8. Let $D$ and $D'$ be as above. Then $h^0(O_D) \geq h^0(O_{D'})$.

Proof. It suffices to show that $h^0(O_{D_i}) \geq h^0(O_{D_{i+1}})$. By construction $D_{i+1} = D_i + E$ for some exceptional curve $E$ with $E \cdot D_i > 0$. Applying $\chi$ to the exact sequence $0 \rightarrow O_E(-D_i) \rightarrow O_{D_{i+1}} \rightarrow O_{D_i} \rightarrow 0$ and using the fact that $\pi_0$ is a rational resolution, we obtain

$$h^0(O_{D_{i+1}}) = \chi(O_E(-D_i)) + h^0(O_{D_i}).$$

Since $E \simeq \mathbb{P}^1$ and $E \cdot D_i > 0$, we have

$$h^0(O_{D_{i+1}}) = 1 - E \cdot D_i + h^0(O_{D_i}) \leq h^0(O_{D_i}).$$

□

3.2. Bounding $-D^2$. We gather here a few facts about numerical invariants of singularities. Let $\pi_0 : Z_0 \rightarrow Z$ be a resolution of a rational surface singularity. We denote by $b_i = -E_i^2$ for exceptional curves $E_1, \ldots, E_r$. Recall that its multiplicity can be expressed in terms of the numerical cycle by the formula $m = -Z_{\text{num}}^2$ (c.f. [Rei97], section 4.17). If, in addition, $Z$ has log terminal singularities, then $m = -Z_{\text{num}}^2$ simplifies to

$$(3.3) \quad m = 2 + \sum_{i=1}^{r} (b_i - 2).$$

This was observed in [Bri68], proof of Satz 2.11, and can be deduced from the following proposition, which we will also need for the proof of Proposition 3.12.

Proposition 3.9. Let $\pi_0 : Z_0 \rightarrow Z$ be the minimal resolution of a log terminal singularity, with exceptional curves $E_1, \ldots, E_r$ and numerical cycle $Z_{\text{num}}$. If $b_i > 2$, then the multiplicity of $E_i$ in $Z_{\text{num}}$ is 1.

Proof. We refer the reader to [Nik89], Figure 1, for the intersection graphs of the exceptional curves on minimal resolutions of log terminal singularities. Let $\Gamma$ be such a graph and let $v(i)$ denote the number of edges incident on a vertex $i$. The first observation is if $v(i) \leq b_i$ for all vertices $i$, then the numerical cycle $Z_{\text{num}}$ is reduced. In particular, the proposition holds for such graphs $\Gamma$. The graphs $\Gamma$ for which there exist vertices $i, j$ such that $v(i) < b_i$ and $b_j > 2$ have the following forms
where $b_i \geq 2$ for $i = 1, \ldots, r$.

In case 1, let $j$ be the minimal integer such that $b_i = 2$ for all $i < j$. Then the numerical cycle is $2(E_1 + \cdots + E_{j-1}) + E_j + \cdots + E_r$. For case 2, the numerical cycles are, respectively,

where $(b_1, b_2) = (2, 3), (3, 2)$.

In case 1, let $j$ be the minimal integer such that $b_i = 2$ for all $i < j$. Then the numerical cycle is $2(E_1 + \cdots + E_{j-1}) + E_j + \cdots + E_r$. For case 2, the numerical cycles are, respectively,

where the numbers above a vertex indicate its multiplicity in $Z_{\text{num}}$. We have thus shown that for any graph $\Gamma$, $b_i > 2$ implies that the multiplicity of $E_i$ in $Z_{\text{num}}$ is 1.

We denote by $\alpha = (\alpha_1, \ldots, \alpha_r)$ the vector whose entries are the discrepancies $\alpha_i$ of the exceptional curves of $\pi_0 : Z_0 \rightarrow Z$. The vector $\alpha$ can be expressed in terms of the intersection matrix $I = (E_iE_j)$ and the vector $v_I = (E_1^2 + 2, \ldots, E_r^2 + 2)$ as

$$\alpha = -I^{-1}v_I.$$  
(3.4)

Note that these $\alpha_i$'s are different from the discrepancies of the order $a_i$ introduced earlier.

**Proposition 3.10.** Let $D = \sum_{i=1}^r n_i E_i$ be an effective exceptional divisor on $Z_0$ and $s$ be the minimal integer such that $D \leq sZ_{\text{num}}$. Then

$$-D^2 \geq 2s + \sum_{i=1}^r (b_i - 2)n_i.$$  

**Proof.** The adjunction formula for a divisor on a surface gives

$$-D^2 = 2h^0(\mathcal{O}_D) + K_ZD,$$

and from equation (3.4), we compute $K_ZD = \sum_{i=1}^r (b_i - 2)n_i$. We show by induction that if $s$ is the minimal integer such that $D \leq sZ_{\text{num}}$ then $h^0(\mathcal{O}_D) \geq s$, which completes the proof. The implication is trivial for $s = 0,$
and we suppose that it holds for \( s - 1 \). By Lemma 3.8 we have \( h^0(\mathcal{O}_D) \geq h^0(\mathcal{O}_{D'}) \) where \( D' \) is the numerical cycle associated to \( D \). Now \( -D' \) is \( \sigma \)-nef, so \( D' - Z_{\text{num}} \) is effective. Taking Euler characteristics of the exact sequence, \( 0 \rightarrow \mathcal{O}_{D'} - Z_{\text{num}}(-Z_{\text{num}}) \rightarrow \mathcal{O}_{D'} \rightarrow \mathcal{O}_{Z_{\text{num}}} \rightarrow 0 \), we obtain 
\[ h^0(\mathcal{O}_{D'}) = \chi(\mathcal{O}_{D'} - Z_{\text{num}}(-Z_{\text{num}})) + h^0(\mathcal{O}_{Z_{\text{num}}}). \]

Since \( -Z_{\text{num}} \) is \( \sigma \)-nef and \( h^0(\mathcal{O}_{Z_{\text{num}}}) \geq 1 \) we have 
\[ h^0(\mathcal{O}_{D'}) \geq h^0(\mathcal{O}_{D'} - Z_{\text{num}}) + 1. \]

Now \( s \) is the minimal integer such that \( D' \leq sZ_{\text{num}} \), so \( s - 1 \) is the minimal integer such that \( D' - Z_{\text{num}} \leq (s - 1)Z_{\text{num}} \). By the induction hypothesis, we conclude that \( h^0(\mathcal{O}_D) \geq h^0(\mathcal{O}_{D'}) \geq s - 1 + 1 = s. \)

With this result, we can show that log terminal orders whose centres have canonical singularities are numerically rational.

**Proposition 3.11.** Let \( \pi_0 : Z_0 \rightarrow Z \) be a minimal resolution of a canonical surface singularity and \( g : E \rightarrow \mathbb{Q} \) be a function satisfying the hypotheses of Theorem 3.5. Then \( g(E) > 0 \) for all \( E \in E_0^+ \setminus \{0\} \) if and only if \( g(Z_{\text{num}}) > 0 \).

**Proof.** Let \( E \) be an effective divisor on \( Z_0 \) and \( s \) be the minimal integer such that \( E \leq sZ_{\text{num}} \). The above proposition applied to the case of a canonical surface singularity yields \( -E^2 \geq 2s \). For the linear term \( \ell \) in
\[ g(E) = -E^2 + \ell(E), \]
recall that \( \ell(E_i) \leq 0 \) for all \( i \), hence \( \ell(E) \geq \ell(sZ_{\text{num}}) \). It follows then 
\[ g(E) \geq 2s + \ell(sZ_{\text{num}}) = s(-Z_{\text{num}}^2 + \ell(Z_{\text{num}})) = sg(Z_{\text{num}}) > 0. \]

The final task is to generalise Proposition 3.11 for log terminal singularities with higher multiplicities. Our strategy is to decompose \( D \) as the sum of two effective divisors \( D = D_1 + D_2 \) with \( D_1D_2 \leq 0 \). Then we have \( g(D) \geq g(D_1) + g(D_2) - 2D_1D_2 \geq g(D_1) + g(D_2) \) and the problem is reduced to showing \( g(D_i) \geq 0 \) for \( i = 1, 2 \).

Let \( D = \sum_{j \in I} n_jE_j \) with \( n_j > 0 \) where \( I \subseteq [1, r] \), and we assume \( \text{supp} D \) is connected. Then \( \text{supp} D \) contracts to a log terminal singularity, and we define the multiplicity \( m(D) \) of \( D \) to be the multiplicity of the contracted singularity. The number \( m(D) \) can be computed by modifying equation (3.4) appropriately,

\[ m(D) = 2 + \sum_{j \in I} (b_j - 2), \]

and by definition of \( D_{\text{num}} \) we have \( m(D) = -D_{\text{num}}^2 \). Since \( \pi_0 \) is a minimal resolution, we have \( m(D) \geq 2 \). If \( m(D) = 2 \), then let \( D_1 = D \) and \( D_2 = 0 \). If \( m(D) > 2 \), then let \( n \) be the positive integer \( n = \min\{n_i \mid -E_i^2 > 2\} \) and define \( D_1 = \gcd(nD_{\text{num}}, D), D_2 = D - D_1 \).
Proposition 3.12. Let $D$ be a connected effective exceptional divisor on $Z_0$ and $D = D_1 + D_2$ be the decomposition above. Let $g : E \longrightarrow \mathbb{Q}$ be a function satisfying the hypotheses of Theorem 3.5. Then

1. $D_1D_2 \leq 0$
2. If $g(D_{\text{num}}) > 0$, then $g(D_1) > 0$
3. For each connected component $C$ of $D_2$, we have $m(C) < m(D)$.

Proof. The proposition is trivial if $m(D) = 2$, so we assume $m(D) > 2$. First note that the effective divisors $nD_{\text{num}} - D_1$ and $D_2$ have no common components, hence $nD_{\text{num}}D_2 \geq D_1D_2$. Since $D_2$ is supported on $\bigcup_{i \in I} E_i$ and $-D_{\text{num}}E_i \geq 0$ for any $i \in I$, we have $D_{\text{num}}D_2 \leq 0$. This proves part 1 of the proposition.

Note that $D_1 \leq nD_{\text{num}}$, and we now show that $n$ is the minimal integer with this property. By definition of $n$, there exists an irreducible component $E_s$ of $D$ of multiplicity $n$ and $-E_s^2 > 2$. By Proposition 3.9 the multiplicity of $E_s$ in $D_{\text{num}}$ is 1. Hence the multiplicities of $E_s$ in $nD_{\text{num}}$ and $D_1$ are equal, so we can conclude that $n$ is the minimal integer such that $D_1 \leq nD_{\text{num}}$. Moreover, the multiplicity of $E_j$ in $D_1$ is equal to $n$ whenever $-E_j^2 > 2$.

Now we can apply Proposition 3.10 and obtain the inequality

$$-D_1^2 \geq 2n + \sum_{j \in I} (b_j - 2)n$$

and by (3.3) the last expression is equal to $nm(D_1)$. Since $D$ is connected, the same is true for $D_1$, so $m(D_1) = -D_{\text{num}}^2$. Then by the same argument as in the proof of Proposition 3.11 we have

$$g(D_1) \geq nm(D_1) + \ell(nD_{\text{num}}) \geq n(-D_{\text{num}}^2 + \ell(D_{\text{num}})) = ng(D_{\text{num}}) > 0.$$ 

This proves part 2 of the proposition.

Since the multiplicities of $E_s$ in $D_1$ and $D$ are equal, the effective divisor $D_2$ is supported away from $E_s$. In particular, any connected component $C$ of $D_2$ is supported away from $E_s$. Since $-E_s^2 > 2$, we see from (3.5) that $m(C)$ must be strictly less than $m(D)$. This proves part 3 of the proposition. □

Note 3.13. Recall that the divisor $D_{\text{num}}$ is a special divisor.

3.3. Proof of Theorem 3.5. We assume as in the hypothesis of Theorem 3.5 that $g$ is positive on special divisors. We wish to show that for all $D \in E_n^\circ \setminus \{0\}$, $g(D) > 0$. Clearly we can assume $D$ is connected. Suppose that $m(D) = 2$, then since $g(D_{\text{num}}) > 0$ by hypothesis, we can conclude from Proposition 3.11 that $g(D) > 0$. Otherwise, let $D = D_1 + D_2$ be the decomposition from Proposition 3.12, we can conclude from parts 1 and 2 of the same proposition that $g(D) \geq g(D_1) + g(D_2)$, and $g(D_1) > 0$. It remains to show that $g(D_2) > 0$. To this end, we repeat the above argument on each connected component of $D_2$. This procedure terminates since by part 3 of Proposition 3.12 the connected components of $D_2$ have multiplicities strictly less than that of $D$. 

$\square$
4. Adjunction formula for orders

The adjunction formula for a divisor $D$ on a smooth surface $Z$ expresses the Euler characteristic of $\mathcal{O}_D$ in terms of intersection numbers involving $D$ and $K_Z$, (c.f. 4.11, [Rei97])

$$
\chi(\mathcal{O}_D) = -\frac{1}{2} (K_Z + D) D.
$$

(4.1)

Note that for the intersection product above to be well-defined, we require $\text{supp} D$ to be a projective variety, and we keep this assumption below. The aim of this section is to derive a similar adjunction formula for a terminal order $A$ on $Z$, which expresses the Euler characteristic of $A$ restricted to some divisor $D$ in terms of intersection numbers involving $D$ and $K_A$. As mentioned in the introduction, the following result appears in the unpublished work of M. Artin and A. J. de Jong.

**Theorem 4.1.** Let $A$ be a terminal order on $Z$ of rank $r^2$ and $D$ be an effective divisor whose support is projective. Then

$$
\chi(A \otimes Z \mathcal{O}_D) = -\frac{r^2}{2} (K_A + D) D
$$

(4.2)

where $K_A = K_Z + \Delta_A$ is the canonical divisor of $A$.

Note that (4.1) appears as a special case of (4.2) (where $A = \mathcal{O}_Z$), so we feel justified in calling (4.2) an adjunction formula. As we have already seen, our motivation for understanding $\chi(A \otimes Z \mathcal{O}_D)$ is to study the notion of numerical rationality. In that context, the divisor $D$ is exceptional with respect to some birational morphism, hence its support is projective. The rest of this section will be devoted to the proof of Theorem 4.1.

4.1. Setup. Let $A$ be a terminal order of rank $r^2$ on a surface $Z$ and $C$ be an irreducible curve in $Z$. Recall that $Z(A_C/J(A_C))$ is a product of field extensions of $k(C)$ which defines a union of cyclic covers of curves $\pi_C : \tilde{C} \to C$. The degree of $\pi_C$ is of course the ramification index $e_C$ of $A$ at $C$. Terminal orders can be characterised using ramification data; an order $A$ is terminal if the ramification divisor $D = \bigcup D_i$ is a normal crossing divisor on a smooth surface $Z$ and the cyclic covers $\pi_{D_i}$ ramify only at nodes $p \in D_i \cap D_j$ with $e_i | e_j$ and $\pi_{D_i}$ totally ramified at $p$.

We first compute $\chi(A \otimes Z \mathcal{O}_C)$ by filtering the sheaf $A$ as follows. Let $J_C$ be the Jacobson radical of $A \otimes k(C)$ and $J$ be its inverse image in $A$. Then $J^e = A(-C)$ where $e = e_C$ and we have a filtration $J^e = A(-C) \subset J^{e-1} \subset \cdots \subset J \subset A$, from which we obtain the exact sequences

$$0 \to J^{i-1}/J^i \to A/J^i \to A/J^{i-1} \to 0$$

for $i = 1, \ldots, e$. Hence

$$
\chi(A \otimes \mathcal{O}_C) = \chi(J^{e-1}/J^e) + \chi(J^{e-2}/J^{e-1}) + \cdots + \chi(J/J^2) + \chi(A/J).
$$

We first determine $\chi(A/J)$, and to do this, we need to know the local structure of $A/J$. 
4.2. **Local structure of** $A/J$. We can use the étale local structure of $A$ (c.f. [CI05], Definition 2.6) to determine the étale local structure of $A/J$. Let $r^2$ denote the rank of $A$ as an $\mathcal{O}_Z$-module and we assume that $A$ is ramified at $C$. Since $A$ is terminal, any other ramification curve $D$ intersect $C$ transversely at a finite number of points and the ramification indices satisfy $e_D|e$ or $e|e_D$.

(1) First suppose $p \in C$ is a nonsingular point of the ramification divisor. Let $u \in \mathfrak{m}_C$ be a uniformising parameter for $\mathcal{O}_{Z,C}$ and we denote $\mathcal{O} = \mathcal{O}_{p}^{sh}$. Then

\[
A_p^{sh} = M^{r/e \times r/e} \quad \text{where} \quad M = \begin{pmatrix}
\mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\
u \mathcal{O} & \ddots & \cdots & \\vdots & \ddots & \ddots & \mathcal{O} \\
u \mathcal{O} & \cdots & u \mathcal{O} & \mathcal{O}
\end{pmatrix} \subset \mathcal{O}^{e \times e}
\]

\[
J_p^{sh} = N^{r/e \times r/e} \quad \text{where} \quad N = \begin{pmatrix}
\mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\
u \mathcal{O} & \ddots & \cdots & \\vdots & \ddots & \ddots & \mathcal{O} \\
u \mathcal{O} & \cdots & u \mathcal{O} & u \mathcal{O}
\end{pmatrix} \subset \mathcal{O}^{e \times e},
\]

so

\[
(A/J)_p^{sh} = \left( (\mathcal{O}/u \mathcal{O})^{r/e \times r/e} \right)^e.
\]

Moreover $J_p^{sh}$ is generated, as a left (or right) $A_p^{sh}$-module by the regular normal element

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & \\vdots & \ddots & \ddots & \cdots & 0 \\
0 & \ddots & \ddots & 0 & 1 \\
u & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(2) Now suppose $p \in C \cap D$ where $D$ is a ramification curve with ramification index $e_D$. Let $v \in \mathfrak{m}_D$ be a uniformising parameter for $\mathcal{O}_{Z,D}$. Denote by $S = \mathcal{O}_{Z,p}^{sh}(x,y)/(x^e - u, y^e - v, xy - \zeta eyx)$ where $\zeta$ is a
primitive $e$-th root of unity. If $e | e_D$, then

$$A_p^\text{sh} = M^{r/e_D \times r/e_D} \text{ where } M = \begin{pmatrix} S & S & \ldots & S \\ yS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ yS & \ldots & yS & S \end{pmatrix} \subset S^{e_D \times e_D/e}$$

$$J_p^\text{sh} = N^{r/e_D \times r/e_D} \text{ where } N = \begin{pmatrix} xS & xS & \ldots & xS \\ xyS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & xS \\ xyS & \ldots & xyS & xS \end{pmatrix} \subset S^{e_D \times e_D/e}$$

Let $\overline{S} = S/xS \simeq k\{v\}[y]/(y^e - v)$, where $k\{v\}$ denotes the strict henselisation of $k[v]$ at the origin. Then

$$(A/J)^\text{sh}_p = \begin{pmatrix} \bar{S} & \bar{S} & \ldots & \bar{S} \\ y\bar{S} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{S} \\ y\bar{S} & \ldots & y\bar{S} & \bar{S} \end{pmatrix} \subset \overline{S}^{e_D \times e_D/e}.$$  

The generator for $J^\text{sh}_p$ in this case is just $x_1A_p^\text{sh}$.

(3) In the case where $e_D | e$, we denote by $S = \mathcal{O}_Z^{sh}_{x,p} (x,y) / (x^{e_D} - u, y^{e_D} - v, xy - \zeta_{e_D} yx)$. Then

$$A_p^\text{sh} = M^{r/e \times r/e} \text{ where } M = \begin{pmatrix} S & S & \ldots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & \ldots & xS & S \end{pmatrix} \subset S^{e \times e_D/e}$$

$$J_p^\text{sh} = N^{r/e \times r/e} \text{ where } N = \begin{pmatrix} xS & S & \ldots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & \ldots & xS & xS \end{pmatrix} \subset S^{e \times e_D/e}$$

so

$$(A/J)^\text{sh}_p = ((S/xS)^{e/e_D})^{r/e \times r/e}.$$
Again \( J_p^{sh} \) is generated by a regular normal element
\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & 0 & 1 \\
x & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]
Note that in each case above, the ideal \( J \) is generated locally by a regular normal element.

**Lemma 4.2.** Let \( \pi : \tilde{C} \to C \) be the cover of \( C \) determined by the ramification data. Then
\[
\chi(\mathcal{O}_{\tilde{C}}) = e\chi(\mathcal{O}_C) - \frac{e}{2} \sum_{D \in \mathbb{Z} \setminus \{C\}} \left( 1 - \frac{1}{\min\{e, e_D\}} \right).
\]

**Proof.** The cover \( \pi \) has degree \( e = e_C \), so by the Riemann-Hurwitz formula, we have
\[
\chi(\mathcal{O}_{\tilde{C}}) = e\chi(\mathcal{O}_C) - \frac{1}{2} \sum_{p \in \tilde{C}} (e_p - 1).
\]
If \( \pi(p) \) is a nonsingular point of the ramification divisor, then \( e_p = 1 \). Now suppose \( \pi(p) \in C \cap D \) where \( A \) is ramified on \( D \) with ramification index \( e_D \). If \( e_D \geq e \), then \( \pi \) is totally ramified at \( \pi(p) \), hence \( e_p = e \). If \( 1 < e_D < e \), then in the fibre \( \pi^{-1}(\pi(p)) \) there are \( e/e_D \) points each with ramification index \( e_D - 1 \). A simple calculation then yields the above formula. \( \square \)

**Lemma 4.3.** The sheaf \( A/J \) considered as a sheaf on \( C \) is a \( \pi_*\mathcal{O}_{\tilde{C}} \)-module.

**Proof.** Firstly, \( A/J \otimes k(C) \) is isomorphic to \( M_n(k(\tilde{C})) \). So it suffices to show that \( (A/J)_p\mathcal{O}^{sh}_{\tilde{C}, p} \subseteq (A/J)_p \) for all prime ideals \( p \in \text{Spec } C \) where we identify everything with their natural images in \( A/J \otimes k(C) \). From the étale local structures for \( A/J \) above, we can see that \( (A/J)_p^{sh}\mathcal{O}^{sh}_{\tilde{C}, p} \subseteq (A/J)_p^{sh} \) for all \( p \in \text{Spec } C \). Intersecting with \( A/J \otimes k(C) \) gives the desired result. \( \square \)

**Proposition 4.4.**
\[
\chi(A/J) = \frac{r^2}{2e} \left( 2\chi(\mathcal{O}_C) - C \cdot \Delta_A + \left( 1 - \frac{1}{e} \right) C^2 \right)
\]

**Proof.** The previous lemma shows that we can consider \( A/J \) as a sheaf on \( \tilde{C} \). In fact, we can see from the local structure of \( A/J \) that it is an order on \( \tilde{C} \) in the semi-simple algebra \( M_{r/e}(k(\tilde{C})) \) (semi-simple since \( k(\tilde{C}) \) is a product of fields). Hence we can embed \( A/J \) in a maximal order \( \Omega \). This gives an exact sequence of \( \mathcal{O}_C \)-modules
\[
0 \to A/J \to \pi_*\Omega \to Q \to 0
\]
where $Q$ is a torsion sheaf supported on points where $A/J$ is not a maximal order on the corresponding fibre. Since the Brauer group of a curve is trivial, $\Omega$ is a maximal order in a matrix algebra, hence is trivial Azumaya. This gives $\chi(\Omega) = (r/e)^2 \chi(\mathcal{O}_C)$ which is equal to $\chi(\pi_*\Omega)$ since $\pi$ is a finite morphism. The sheaf $Q$ is supported on points, so $\chi(Q)$ is the sum of the lengths of $Q_p$ over $p \in C$. Referring again to the local structure of $A/J$, we see that $A/J$ is nonmaximal at $p$ if and only if $p$ is a point of intersection of $C$ and a ramification curve $D$ where $e|e_D$. A simple computation shows that

$$Q_p = P^{r/e_D} \times \pi^{r/e_D}$$

where $P = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ k & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & k \\ k & \cdots & k & 0 \end{pmatrix} \subset k^{e_D/e} \times k^{e_D/e}$

hence

$$\dim(Q_p) = \frac{r^2}{e_D} \sum_{j=1}^{e_D/e-1} j = \frac{r^2}{2e} \left( \frac{1}{e} - \frac{1}{e_D} \right).$$

So

$$\chi(Q) = \sum_{D \in \mathbb{Z} \setminus \{C\}} \frac{r^2}{2e} \left( \frac{1}{e} - \frac{1}{\max\{e, e_D\}} \right) C \cdot D.$$

Combining the expressions for $\chi(\Omega)$ and $\chi(Q)$, we obtain

$$\chi(A/J) = \frac{r^2}{e} \left( \chi(\mathcal{O}_C) - \frac{1}{2} \sum_{D \in \mathbb{Z} \setminus \{C\}} \left( 1 - \frac{1}{\min\{e, e_D\}} + \frac{1}{e} - \frac{1}{\max\{e, e_D\}} \right) C \cdot D \right)$$

$$= \frac{r^2}{e} \left( \chi(\mathcal{O}_C) - \frac{1}{2} \sum_{D \in \mathbb{Z} \setminus \{C\}} \left( 1 - \frac{1}{e_D} \right) C \cdot D \right)$$

which proves the proposition. □

To finish the computation of $\chi(A \otimes Z \mathcal{O}_C)$ we need the values of $\chi(J^i/J^{i+1})$. Let $B$ be an $\mathcal{O}_C$-algebra. We say that $L$ is an invertible $(B - B)$-bimodule if there exists a $(B - B)$-bimodule $L'$ such that $L \otimes_B L' \simeq B$ as $(B - B)$-bimodules.

**Lemma 4.5.** Let $C$ be a projective curve, $B$ be an $\mathcal{O}_C$-algebra which is torsion-free as an $\mathcal{O}_C$-module, and $L, L'$ be invertible $(B - B)$-bimodules. Then

$$\chi(L \otimes_B L') = \chi(L) + \chi(L') - \chi(B).$$

**Proof.** First we assume that $L$ is generated as an $\mathcal{O}_C$-module by its sections. Suppose $p \in C$ is a closed point and $s_p \in L_p$ is a regular element, that is $B_p \longrightarrow L_p$ given by $b \longrightarrow s_p b$ is injective. Then since $L$ is generated by
sections, we can lift this to a section $s \in H^0(C, L)$. Now $B$ is torsion-free as an $\mathcal{O}_C$-module, so the map $B \rightarrow L$ given by $s$ is also injective. This gives an exact sequence $0 \rightarrow B \rightarrow L \rightarrow Q \rightarrow 0$ of right $B$-modules. Since $- \otimes_B L'$ induces an equivalence of categories, it is exact, so we have the exact sequence

$$0 \rightarrow L' \rightarrow L \otimes_B L' \rightarrow Q \otimes_B L' \rightarrow 0.$$

Note that since $L'$ is invertible, its rank as an $\mathcal{O}_C$-module is the same as the $\mathcal{O}_C$-rank of $B$. Thus the sheaf $Q$ is supported on points, and $\chi(Q \otimes_B L') = \chi(Q)$. This gives

$$\chi(L \otimes_B L') = \chi(L') + \chi(Q) + \chi(L) - \chi(B).$$

In general, let $\mathcal{O}_C(1)$ be a very ample line bundle on $C$ so that $\mathcal{O}_C(n) \otimes_C L$ is generated by sections for some $n \gg 0$. We denote by $r$ the rank of $L$ as an $\mathcal{O}_C$-module, and note that $L \otimes_B L'$ has the same rank. Then

$$\chi(\mathcal{O}_C(n) \otimes_C L \otimes_B L') = \chi(\mathcal{O}_C(n) \otimes_C L) + \chi(L') - \chi(B) = rn + \chi(L) + \chi(L') - \chi(B).$$

But $\chi(\mathcal{O}_C(n) \otimes_C L \otimes_B L') = rn + \chi(L \otimes_B L')$ so we are done. \hfill \Box

Recall that $J$ is generated locally by a regular normal element, hence so is $J^i/J^{i+1}$. The following lemma follows from the local structure of $A/J$.

**Lemma 4.6.** The sheaves $J^i/J^{i+1}$ are invertible $(A/J - A/J)$-bimodules, locally generated by a regular normal element. Moreover $J^i/J^{i+1} \otimes_{A/J} J^k/J^{k+1} \simeq J^{i+k}/J^{i+k+1}$ as $(A/J - A/J)$-bimodules.

**Corollary 4.7.**

$$\chi(A \otimes_Z \mathcal{O}_C) = \frac{r^2}{2} (2\chi(\mathcal{O}_C) - C \cdot \Delta_A)$$

**Proof.** By Lemma 4.6, we have $(J/J^2)^{\otimes e} \simeq J^e/J^{e+1} \simeq A/J \otimes_Z \mathcal{O}_Z(-C)$. By Lemma 4.5, we have $\chi(A/J \otimes_Z \mathcal{O}_Z(-C)) = \chi(J/J^2)^{\otimes e} = e\chi(J/J^2) - (e - 1)\chi(A/J)$. Note that the rank of $A/J$ as an $\mathcal{O}_C$-module is $r^2/e$, so $\chi(A/J \otimes_Z \mathcal{O}_Z(-C)) = \chi(A/J) - r^2C^2/e$. Putting these together, we get

$$\chi(J/J^2) = -\frac{r^2C^2}{e^2} + \chi(A/J).$$
Recall (4.3) from the beginning of this section,
\[
\chi(A \otimes Z \mathcal{O}_C) = \chi(J^{e-1}/J^e) + \chi(J^{e-2}/J^{e-1}) + \cdots + \chi(J/J^2) + \chi(A/J)
\]
\[
= \chi(A/J) + \sum_{i=1}^{e-1} (i\chi(J/J^2) - (i-1)\chi(A/J))
\]
\[
= \chi(A/J) + \frac{e(e-1)}{2} \left( -\frac{r^2C^2}{e^2} + \chi(A/J) \right) - \frac{(e-1)(e-2)}{2} \chi(A/J)
\]
\[
= -\frac{e-1}{2} r^2 C^2 + \left( 1 + \frac{e(e-1)}{2} - \frac{(e-1)(e-2)}{2} \right) \chi(A/J)
\]
\[
= -\frac{e-1}{2} r^2 C^2 + e\chi(A/J)
\]
\[
= \frac{r^2}{2} \left( -(1 - \frac{1}{e})C^2 + \left( 2\chi(\mathcal{O}_C) - C \cdot \Delta_A + \left( 1 - \frac{1}{e} \right) C^2 \right) \right)
\]
\[
= \frac{r^2}{2} (2\chi(\mathcal{O}_C) - C \cdot \Delta_A)
\]
\[
\square
\]

4.3. Proof of Theorem 4.1. It remains to prove Theorem 4.1 in the case of a general effective divisor \(E\).

Lemma 4.8. Let \(E = n_1 E_1 + \cdots + n_s E_s\) be an effective divisor on \(Z\) and \(V\) be a rank \(r\) vector bundle on \(Z\). Then

\[(4.4) \quad \chi(V \otimes Z \mathcal{O}_E) = -\frac{rE^2}{2} + \sum_{i=1}^{s} n_i \left( \frac{rE_i^2}{2} + \chi(V \otimes Z \mathcal{O}_{E_i}) \right).\]

Proof. We prove (4.4) by induction. For irreducible \(E\), there is nothing to prove. Suppose (4.4) holds for \(E\), we show that it holds too for \(E + E_j\).
Using the following exact sequence
\[
0 \rightarrow V \otimes Z \mathcal{O}_{E_j}(-E) \rightarrow V \otimes Z \mathcal{O}_{E+E_j} \rightarrow V \otimes Z \mathcal{O}_E \rightarrow 0
\]
we obtain
\[
\chi(V \otimes Z \mathcal{O}_{E+E_j}) = \chi(V \otimes Z \mathcal{O}_{E_j}(-E)) - \frac{rE_j^2}{2} + \sum_{i=1}^{s} n_i \left( \frac{rE_i^2}{2} + \chi(V \otimes Z \mathcal{O}_{E_i}) \right).
\]
Since \(\chi(V \otimes Z \mathcal{O}_{E_j}(-E)) = \chi(V \otimes Z \mathcal{O}_{E_j}) - rE_j \cdot E\), we get
\[
\chi(V \otimes Z \mathcal{O}_{E+E_j}) = \chi(V \otimes Z \mathcal{O}_{E_j}) + \frac{rE_j^2}{2} - \frac{r(E + E_j)^2}{2} + \sum_{i=1}^{s} n_i \left( \frac{rE_i^2}{2} + \chi(V \otimes Z \mathcal{O}_{E_i}) \right)
\]
which shows that (4.4) holds for \(E + E_j\). \(\square\)
Theorem 4.1 follows then directly from Corollary 4.7 and Lemma 4.8.

\[
\chi(A \otimes \mathcal{O}_E) = -\frac{r^2 E^2}{2} + \sum_{i=1}^{n_i} \left( \frac{r^2 E_i^2}{2} + \frac{r^2}{2} (2\chi(\mathcal{O}_{E_i}) - E_i \cdot \Delta_A) \right)
\]

\[
= \frac{r^2}{2} \left( -E^2 + \sum_{i=1}^{n_i} \left( E_i^2 - (K_A + E_i) E_i \right) \right)
\]

\[
= -\frac{r^2}{2} (K_A + E) E
\]

5. CONCLUDING REMARKS

We have shown that the notion of numerical rationality includes many interesting examples of orders which arise naturally in the context of non-commutative birational geometry. Our definition is natural in that it does not depend on the choice of resolution, nor does it depend on the choice of representative in a Morita equivalence class (if the centre has rational singularities). Moreover, the adjunction formula for orders makes it easy to check whether an order is numerically rational.

REFERENCES

[AdJ] Michael Artin and Johan de Jong. Stable orders on surfaces. Preprint.
[Art66] Michael Artin. On isolated rational singularities of surfaces. Amer. J. Math., 88:129–136, 1966.
[Bri68] Egbert Brieskorn. Rationale Singularit"aten komplexer Fl"achen. Invent. Math., 4:336–358, 1967/1968.
[Cha] Daniel Chan. McKay correspondence for canonical orders. http://arxiv.org/abs/0707.3481v1 To appear in Trans. Amer. Math. Soc.
[CHI09] Daniel Chan, Paul Hacking, and Colin Ingalls. Canonical singularities of orders over surfaces. Proc. Lond. Math. Soc. (3), 98(1):83–115, 2009.
[CI05] Daniel Chan and Colin Ingalls. The minimal model program for orders over surfaces. Invent. Math., 161(2):427–452, 2005.
[Dim92] Alexandru Dimca. Singularities and topology of hypersurfaces. Universitext. Springer-Verlag, New York, 1992.
[KM98] János Kollár and Shigefumi Mori. Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[Lip69] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. Inst. Hautes Études Sci. Publ. Math., (36):195–279, 1969.
[Nik89] V. V. Nikulin. del Pezzo surfaces with log-terminal singularities. Mat. Sb., 180(2):226–243, 304, 1989.
[Rei97] Miles Reid. Chapters on algebraic surfaces. In Complex algebraic geometry (Park City, UT, 1993), volume 3 of IAS/Park City Math. Ser., pages 3–159. Amer. Math. Soc., Providence, RI, 1997.