Disorder can enhance large self-sustained oscillations in dissipative Curie-Weiss models

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\textbf{Abstract}

We modify the spin-flip dynamics of a Curie-Weiss model with dissipative interaction potential \cite{7} by adding a site-dependent i.i.d. random magnetic field. The purpose is to analyze how disorder affects the time-evolution of the observables in the macroscopic limit. Our main result shows that, whenever the field intensity is sufficiently strong, a periodic orbit may emerge through a global bifurcation in the phase space, giving origin to a large-amplitude rhythmic behavior.

\textbf{Keywords:} collective noise-induced periodicity · disordered systems · mean-field interaction · non-equilibrium systems · random potential · saddle-node bifurcation of periodic orbits

\section{Introduction}

Large volume dynamics of noisy interacting units may display robust collective periodic behavior. Self-sustained oscillations are commonly encountered in ecology \cite{26}, neuroscience \cite{9,16} and socioeconomics \cite{27}. From a modelling point of view, great attention has been given to mean-field interacting particle systems, due to their analytical tractability. In this context, the attempt of explaining rigorously possible origins of self-organized rhythms identified various essential aspects to enhance the emergence of such coherent and structured dynamics. Seminal works \cite{21,22} have highlighted the importance of the interplay between interaction and noise. In particular, the role of noise is twofold: on the one hand, noise can lead to oscillatory states in systems whose deterministic counterparts do not display any periodic behavior (\textit{noise-induced periodicity}) \cite{21,22,25}; on the other, it can facilitate the transition from incoherence to macroscopic pulsing (\textit{excitability by noise}) \cite{5,16,17,18}. Moreover, rhythmic behaviors are intrinsically non-equilibrium phenomena, naturally in contrast with stochastic reversibility \cite{3,13}, and hence a reversibility-breaking mechanism need to enter the microscopic design of the models. Quite a number of such mechanisms have been taken into account: addition of a \textit{driving force} or of a \textit{random intrinsic frequency} in phase rotator systems \cite{4,12,23}; addition of \textit{delay} in the information transmission and/or \textit{frustration} in the interaction network in multi-population discrete particle systems \cite{11,6,8}; to name a

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A further mechanism that has been lately proposed and investigated is dissipation. Indeed, a class of irreversible models can be obtained as perturbation of classical reversible dynamics by introducing a friction term damping the interaction potential [2, 5, 7]. The simplest interacting particle system within this family is the dissipative version of the Curie-Weiss model presented in [2]: the standard spin-flip dynamics are modified so that the interaction energy undergoes a dissipative and diffusive stochastic evolution. In the infinite volume limit, for sufficiently strong interaction and zero (or sufficiently small) noise, this system exhibits stable self-sustained oscillations emerging via a Hopf bifurcation.

In the present paper we modify the noiseless version of the dissipative Curie-Weiss model introduced in [7] by adding some disorder: we embed the particle system in a site dependent, i.i.d., binary and symmetric, static random environment. Despite the simple structure of the random field we are considering, the addition of disorder makes the phase diagram of the macroscopic evolution very rich and interesting. In particular, in the parameter space there exists a tricritical point that allows to identify two half-planes for the field intensity corresponding to a small and a large-disorder regime, where the impact of the random environment is irrelevant and relevant respectively. In the small-disorder case, the system behaves as in absence of random field. As for the homogeneous model [7, Thm. 3.1], the emergence of self-sustained oscillations is due to the occurrence of a supercritical Hopf bifurcation. On the contrary, whenever the disorder becomes sufficiently strong, a stable periodic orbit arises through a saddle-node bifurcation of limit cycles rather than a Hopf bifurcation. The latter is a global phenomenon, which cannot be detected by a local analysis, and it is a genuine effect of the inhomogeneity coming from the random field.

The paper is organized as follows. In Section 2 we describe the model under consideration and we state our main results. All the proofs are postponed to Section 3.

2 Description of the model and results

We consider a simplified version of the Curie-Weiss model with dissipation in [7] and we introduce inhomogeneity in the structure of the system via a site-dependent, static, random magnetic field (acting as a random environment). Let \( \sigma = (\sigma_j)_{j=1}^N \in \{-1,+1\}^N \) denote the spin configuration and let \( \eta = (\eta_j)_{j=1}^N \in \{-1,+1\}^N \), a sequence of i.i.d. random variables with distribution \( \mu = \frac{1}{2} (\delta_{-1} + \delta_{+1}) \), denote the disorder. Given a realization of the environment \( \eta \), the stochastic process \( \{\sigma(t)\}_{t \geq 0} \) is described as follows. For \( \sigma \in \{-1,+1\}^N \), let us define \( \sigma^i \) the configuration obtained from \( \sigma \) by flipping the \( i \)-th spin. The spins will be assumed to evolve with Glauber one spin-flip dynamics: at any time \( t \), the system may experience a transition \( \sigma \rightarrow \sigma^i \) at rate \( 1 - \tanh[\sigma_i(\lambda N + h \eta_i)] \), where \( h \geq 0 \) and \( \{\lambda N(t)\}_{t \geq 0} \) is a stochastic process on \( \mathbb{R} \), driven by the stochastic differential equation

\[
d\lambda_N(t) = -\lambda_N(t)dt + \beta dm^\sigma_N(t),
\]

with \( \beta > 0 \) and \( m^\sigma_N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_j(t) \). As far as the parameters: \( \beta \) is the inverse temperature and \( h \) is the intensity of the local fields.

The two terms in the argument of the hyperbolic tangent have different effects: the first one encodes the ferromagnetic coupling between spins, while the second pushes each spin to point
the direction prescribed by the field associated with its own site. Observe that, in view of the evolution \( (1) \), the interaction is damped between two consecutive spin flips.

From a formal viewpoint, for any given realization of \( \eta \), we are considering the Markov process \((\sigma(t), \Lambda_N(t))\) on \([-1, +1]^N \times \mathbb{R}\) evolving with infinitesimal generator

\[
L_N^\eta f(\sigma, \lambda) = \sum_{j=1}^N [1 - \sigma_j \tanh(\lambda + h \eta_j)] \left[ f\left(\sigma^1, \lambda - \frac{2 \beta \sigma_j}{N}\right) - f(\sigma, \lambda)\right] - \lambda \partial_\lambda f(\sigma, \lambda).
\]  

(2)

In addition to the usual empirical magnetization, we define also the empirical averages

\[
m_N^{\sigma \eta}(t) = \frac{1}{N} \sum_{j=1}^N \sigma_j(t) \eta_j \quad \text{and} \quad \bar{\eta}_N = \frac{1}{N} \sum_{j=1}^N \eta_j.
\]

Let \( E_N \) be the image of \([-1, +1]^N \times \mathbb{R}\) under the map \( \Phi^0 : (\sigma, \lambda_N) \mapsto (m_N^{\sigma}, m_N^{\eta}, \lambda_N) \). The microscopic dynamics on the configurations, corresponding to the generator \( [2] \), induce a Markovian evolution on \( E_N \) for the process \( \{(m_N^{\sigma}(t), m_N^{\eta}(t), \Lambda_N(t))\}_{t \geq 0} \), that in turn evolves with generator

\[
\mathcal{L}_N f(m^\sigma, m^\eta, \lambda) = \sum_{j,k=-1} \lambda \bar{C}_N(j,k) \left[ f\left(m^\sigma - \frac{2}{N} j, m^\eta - \frac{2}{N} k, \lambda - \frac{2 \beta j}{N}\right) - f(m^\sigma, m^\eta, \lambda)\right] - \lambda \partial_\lambda f(m^\sigma, m^\eta, \lambda),
\]

(3)

where \( \bar{C}_N(j,k) = N \mathbb{E} \{1 + k \bar{\eta}_N + j m^\sigma + j k m^\eta\} \mathbb{E} \{1 - j \tanh(\lambda + k h)\} \), for all \( j, k \in \{-1, +1\} \). Observe that the term \( N^{-1} \sum_{i=1}^N \eta_i \) counts the number of pairs \( (\sigma_i, \eta_i) \), \( i \in \{1, \ldots, N\} \), such that \( \sigma_i = j \) and \( \eta_i = k \).

The generator \( [3] \) can be derived from \( [2] \) via the martingale problem and the property

\[
L_N^\eta (f \circ \Phi^0)(\sigma, \lambda) = (\mathcal{L}_N f) \circ \Phi^0(\sigma, \lambda) = (\mathcal{L}_N f)(m^\sigma, m^\eta, \lambda).
\]

The process \( \{\{(m_N^{\sigma}(t), m_N^{\eta}(t), \Lambda_N(t))\}_{t \geq 0} \) is an order parameter, in the sense that its dynamics completely describe the dynamics of the original system. We are going to characterize its limiting evolution. We can derive the infinite volume dynamics for our model via weak convergence in \( \mathcal{D}_{\mathbb{R}^1}(\mathbb{R}^+) \), the space of càdlàg trajectories from \( \mathbb{R}^+ \) to \( \mathbb{R}^3 \).

**Theorem 2.1** (Law of large numbers). Suppose that \( \{m_N^{\sigma}(0), m_N^{\eta}(0), \lambda(0)\} \) converges weakly to the constant \( (m^\sigma, m^\eta, \lambda) \). Then, \( \mu \)-almost surely, the stochastic process \( \{\{m_N^{\sigma}(t), m_N^{\eta}(t), \Lambda_N(t)\}\}_{t \geq 0} \) converges weakly in \( \mathcal{D}_{\mathbb{R}^1}(\mathbb{R}^+) \) to the unique solution of

\[
\begin{align*}
\dot{m}^\sigma_t &= -2m^\sigma_t + \tanh(\lambda_t + h) + \tanh(\lambda_t - h) \\
\dot{m}^\eta_t &= -2m^\eta_t + \tanh(\lambda_t + h) - \tanh(\lambda_t - h) \\
\dot{\lambda}_t &= -\lambda_t + \beta [-2m^\sigma_t + \tanh(\lambda_t + h) + \tanh(\lambda_t - h)].
\end{align*}
\]

(4)

Next we want to characterize the phase diagram for \( [4] \). Observe that the first and third equation in \( [4] \) form an independent subsystem. In other words, we are reduced to analyze the attractors on \([-1, +1] \times \mathbb{R}\) for the planar system

\[
(\dot{m}^\sigma_t, \dot{\lambda}_t) = V(m^\sigma_t, \lambda_t),
\]

(5)

with vector field given by

\[
V(x, y) = (-2x + \tanh(y + h) + \tanh(y - h), -y + \beta [-2x + \tanh(y + h) + \tanh(y - h)]).
\]

We remark that the only fixed point for \( [5] \) is the origin. Moreover, it is easy to see that \( (0, 0) \) is linearly stable whenever \( \beta < \frac{1}{2} \cosh^2(h) \); whereas, the local stability is lost for \( \beta > \frac{1}{2} \cosh^2(h) \). Much more than a local analysis can be obtained for \( [5] \). The global situation is described in the next theorem and then qualitatively illustrated in Figure \([4]\).
**Theorem 2.2** (Phase Diagram). Consider the dynamical system (5) and, for every $h \geq 0$, set

$$
\beta_c(h) := \frac{3}{2} \cosh^2(h).
$$

We have the following.

(a) Suppose $h \leq \frac{1}{2} \ln(2 + \sqrt{3})$. Then,

(i) for $\beta \leq \beta_c(h)$ the origin is a global attractor.

(ii) for $\beta > \beta_c(h)$ the system admits a unique limit cycle attracting all the trajectories except for the fixed point.

(b) Suppose $h > \frac{1}{2} \ln(2 + \sqrt{3})$. Then, there exists $0 < \beta_* (h) \leq \beta_c(h)$ such that

(i) for $\beta < \beta_* (h)$ the origin is a global attractor.

(ii) for $\beta_* (h) \leq \beta < \beta_c(h)$ the origin is locally stable and coexists with a stable periodic orbit.

(iii) for $\beta \geq \beta_c(h)$ the system admits a unique limit cycle attracting all the trajectories except for the fixed point.

\[ \text{Figure 1: Illustration of the phase portrait for the dynamical system (5). Each colored region represents a phase with attractor(s) indicated by the label: FP = fixed point; LC = limit cycle; FP+LC = coexistence of fixed point and limit cycle. The blue separation curve is the Hopf bifurcation curve; in particular, it is solid when the bifurcation is supercritical and dashed otherwise. The red line is qualitative and represents the saddle-node bifurcation of periodic orbits. The tricritical point } (h_{tc}, \beta_{tc}) \text{ corresponds to } \left( \frac{1}{2} \ln(2 + \sqrt{3}), \frac{9}{4} \right). \]

Remark. Finding, for any $h > \frac{1}{2} \ln(2 + \sqrt{3})$, the exact threshold value $\beta_*(h)$ is hard to achieve analytically. On the contrary, it is easy to get a positive lower bound for such a transition point. See Appendix B for further details.

We give a few explanations concerning the content of Theorem 2.2. All the technicalities can be found in Section 3.1. If $h \leq \frac{1}{2} \ln(2 + \sqrt{3})$ a periodic orbit appears through a local change in the stability of the fixed point. At $\beta = \beta_c(h)$ a supercritical Hopf bifurcation occurs. When $h > \frac{1}{2} \ln(2 + \sqrt{3})$ the dependence of the attractors on the parameter $\beta$ is quite nontrivial, due to the combination of a subcritical Hopf bifurcation and a saddle-node bifurcation of limit cycles. Roughly speaking, there are three possible regimes for system (5).
• Fixed point phase. For $\beta < \beta_c(h)$ the only stable attractor is the origin.

• Coexistence phase. At $\beta = \beta_c(h)$ a semistable cycle surrounding the origin is formed. By increasing the parameter $\beta$ from $\beta_c(h)$, this cycle splits into two limit cycles, the outer being stable and the inner unstable. In this phase $(0,0)$ is linearly stable. Therefore, the locally stable equilibrium coexists with a stable periodic orbit.

• Periodic orbit phase. For $\beta = \beta_c(h)$ the (subcritical) Hopf bifurcation occurs: the inner unstable limit cycle collapses at $(0,0)$ and disappears. At the same time the fixed point loses its stability and, thus, the external stable limit cycle is the only stable attractor left for $\beta > \beta_c(h)$.

Notice that even if the two scenarios depicted in Theorem 2.2 may look similar, as they both describe a transition of the type “fixed point to limit cycle”, this is not the case. They are in fact qualitatively very different. If we describe a transition of the type “fixed point to limit cycle”, this is not the case. They are in fact qualitatively very different. If $h \leq \frac{1}{2} \ln(2 + \sqrt{3})$ a small-amplitude periodic orbit bifurcates from the origin at the critical point and then it grows gradually when increasing the parameter. On the contrary, if $h > \frac{1}{2} \ln(2 + \sqrt{3})$ the stable limit cycle arises through a global bifurcation and therefore, when they originate, the oscillations are already large. As a consequence the trajectories are abruptly pushed far from the equilibrium point when the latter becomes unstable. To conclude, we can see that the addition of disorder has a significant impact as the nature of the bifurcation may be drastically changed for large enough field intensity and sudden self-sustained large-amplitude oscillations may be induced in the system.

3 Proofs

3.1 Proof of Theorem 2.1

To prove weak convergence in path-space we combine a compact containment condition with the convergence of generators.

Compact containment condition. Notice that the processes $\{m_N^*(t)\}_{t \geq 0}$ and $\{m_N(t)\}_{t \geq 0}$ are confined in $[-1, +1] \subset \mathbb{R}$ for each $N \in \mathbb{N}$. We are thus left with showing that the sequence of processes $\{m_N(t)\}_{t \geq 0}$ is contained in a compact; we do it by proving compact containment for the process $\{\beta m_N^*(t) - \lambda_N(t)\}_{t \geq 0}$.

For every $N \in \mathbb{N}$, let us define the stopping time $\tau_N^M := \inf\{t \geq 0 : |\beta m_N^*(t) - \lambda_N(t)| \geq M\}$. We study the asymptotic behavior of the sequence $\{\tau_N^M\}_{N \geq 1}$.

Lemma 3.1. For any $T > 0$ and $\epsilon > 0$, there exists $M_{\epsilon} > 0$ such that $\sup_{N \geq 1} P(\tau_N^{M_{\epsilon}} \leq T) \leq \epsilon$.

Proof. Let $M$ be an arbitrary strictly positive constant. Observe that

$$P(\tau_N^M \leq T) = P\left(\sup_{0 \leq t \leq T \wedge \tau_N^M} |\beta m_N^*(t) - \lambda_N(t)| \geq M\right).$$

We want to show that the probability in the right-hand side of the previous display can be made arbitrarily small.

Consider the function $U : [-1, +1] \times \mathbb{R} \to \mathbb{R}$, given by $U(m^\sigma, \lambda) = \frac{1}{2}(\beta m^\sigma - \lambda)^2$. Since $U(m^\sigma \pm \frac{\lambda}{N}, \lambda \pm \frac{2\beta}{N}) = U(m^\sigma, \lambda)$, the evolution of the process $\{U(m_N^*(t), \lambda_N(t))\}_{t \geq 0}$ is deterministic and driven by
the generator

\[ \mathcal{L}_N U(m^\sigma, \lambda) = -\lambda \partial_\lambda U(m^\sigma, \lambda) = -\lambda^2 + \beta m^\sigma \lambda, \]

cf. equation (3). Notice that \( \mathcal{L}_N U(m^\sigma, \lambda) \leq \frac{\beta^2}{4}. \) Therefore, we have

\[ U(m_N^\sigma(t), \lambda_N(t)) = U(m_N^\sigma(0), \lambda_N(0)) + \int_0^t \mathcal{L}_N U(m_N^\sigma(s), \lambda_N(s)) \, ds \]

\[ \leq U(m_N^\sigma(0), \lambda_N(0)) + \frac{\beta^2 t}{4}, \]

leading to

\[ P \left( \sup_{0 \leq t \leq T \wedge \tau_N} |\beta m_N^\sigma(t) - \lambda_N(t)| \geq M \right) = P \left( \sup_{0 \leq t \leq T \wedge \tau_N} U(m_N^\sigma(t), \lambda_N(t)) \geq \frac{M^2}{2} \right) \]

\[ \leq P \left( U(m_N^\sigma(0), \lambda_N(0)) \geq \frac{M^2}{2} - \frac{\beta^2 T}{4} \right). \]

The convergence in law of the initial condition implies \( P[U(m_N^\sigma(0), \lambda_N(0)) \geq c(\varepsilon)] \leq \varepsilon \) for a sufficiently large \( c(\varepsilon) > 0 \) and all \( N \in \mathbb{N} \). As a consequence, to conclude it suffices to choose the constant \( M = M_\varepsilon \) so that \( \frac{M^2}{2} - \frac{\beta^2 T}{4} \geq c(\varepsilon). \)

\[ \square \]

**Convergence of the sequence of generators.** The infinitesimal generator of the process \( ([m_N^\sigma(t), m_N^{\sigma n}(t), \lambda_N(t)])_{t \geq 0} \) is given in (3). We want to characterize the limit of the sequence \( \{\mathcal{L}_N f\}_{N \geq 1} \) for \( f \in C^2_0([-1, +1]^2 \times \mathbb{R}) \), the set of two times continuously differentiable functions that are constant outside a compact set in the interior of \([-1, +1]^2 \times \mathbb{R}\). We first Taylor expand \( f \) up to first order. For all \( j, k \in \{-1, +1\} \), we get

\[ f \left( m^\sigma - \frac{2i}{N}, m^{\sigma n} - \frac{2j}{N}, \lambda - \frac{2\beta i}{N} \right) - f(m^\sigma, m^{\sigma n}, \lambda) \]

\[ = -\frac{2}{N} \partial_{m^\sigma} f(m^\sigma, m^{\sigma n}, \lambda) - \frac{2j}{N} \partial_{m^{\sigma n}} f(m^\sigma, m^{\sigma n}, \lambda) - \frac{2\beta i}{N} \partial_\lambda f(m^\sigma, m^{\sigma n}, \lambda) + O \left( \frac{1}{N^2} \right) \]

and then, by combining the terms with \( \partial_{m^\sigma} \), with \( \partial_{m^{\sigma n}} \) and the terms with \( \partial_\lambda \), we get

\[ \mathcal{L}_N f(m^\sigma, m^{\sigma n}, \lambda) = \left\{ -2m^\sigma + (1 + \bar{\eta}_N) \tanh(\lambda + h) + (1 - \bar{\eta}_N) \tanh(\lambda - h) \right\} \partial_{m^\sigma} f(-) \]

\[ + \left\{ -2m^{\sigma n} + (1 + \bar{\eta}_N) \tanh(\lambda + h) - (1 - \bar{\eta}_N) \tanh(\lambda - h) \right\} \partial_{m^{\sigma n}} f(-) \]

\[ + \left\{ -\lambda + \beta \left[ -2m^\sigma + (1 + \bar{\eta}_N) \tanh(\lambda + h) + (1 - \bar{\eta}_N) \tanh(\lambda - h) \right] \right\} \partial_\lambda f(-) + O(1). \]

Observe that, in the limit as \( N \to \infty \), the empirical average \( \bar{\eta}_N \) converges to zero \( \mu \)-almost surely by the law of large numbers. Let \( \mathcal{L} \) be the linear generator

\[ \mathcal{L} f(m^\sigma, m^{\sigma n}, \lambda) = \left\{ -2m^\sigma + \tanh(\lambda + h) + \tanh(\lambda - h) \right\} \partial_{m^\sigma} f(-) \]

\[ + \left\{ -2m^{\sigma n} + \tanh(\lambda + h) - \tanh(\lambda - h) \right\} \partial_{m^{\sigma n}} f(-) \]

\[ + \left\{ -\lambda + \beta \left[ -2m^\sigma + \tanh(\lambda + h) + \tanh(\lambda - h) \right] \right\} \partial_\lambda f(-) + O(1). \]

Since, for every \( f \in C^2_0([-1, +1]^2 \times \mathbb{R}) \) and any compact \( K \subset \mathbb{R}^3 \), we have

\[ \lim_{N \to \infty} \sup_{(m^\sigma, m^{\sigma n}, \lambda) \in K \cap E_N} |\mathcal{L}_N f(m^\sigma, m^{\sigma n}, \lambda) - \mathcal{L} f(m^\sigma, m^{\sigma n}, \lambda)| = 0, \]
we obtain the convergence of $\mathcal{L}_N$ to $\mathcal{L}$, as $N$ tends to infinity.

To derive the weak convergence result we apply [10, Cor. 4.8.16]. We check the assumptions of the corollary are satisfied:

- By Lemma 3.1 the sequence of processes $\{(m^R_N(t), m^σ_N(t), \lambda_N(t))\}_{t \geq 0}$ satisfies the compact containment condition.
- The set $C_2([-1, +1]^2 \times \mathbb{R})$ is an algebra that separates points.
- The martingale problem for the operator $(\mathcal{L}, C_2([-1, +1]^2 \times \mathbb{R}))$ admits a unique solution by [10, Thm. 8.2.6].

The conclusion then follows.

### 3.2 Proof of Theorem 2.2

**Local analysis.** System (5) admits only the fixed point $(0, 0)$ in the phase plane $(m^σ, \lambda)$. The linearization around this point gives

$$
\begin{bmatrix}
\dot{m}^σ \\
\dot{\lambda}
\end{bmatrix} = \begin{bmatrix}
-2 & \frac{2}{\cosh^2(h)} \\
-2\beta & \frac{2\beta}{\cosh^2(h)} - 1
\end{bmatrix} \begin{bmatrix}
\bar{m}^σ \\
\lambda
\end{bmatrix}
$$

(6)

and the eigenvalues of the system are

$$
k_\pm = \frac{\beta}{\cosh^2(h)} - \frac{3}{2} \pm \sqrt{\left(\frac{\beta}{\cosh^2(h)} - \frac{3}{2}\right)^2 - 2}.
$$

These eigenvalues have both negative real part for $\beta < \frac{3}{2}\cosh^2(h)$ and both positive real part for $\beta > \frac{3}{2}\cosh^2(h)$. As a consequence, if $\beta < \frac{3}{2}\cosh^2(h)$, the origin is linearly stable; whereas, for $\beta > \frac{3}{2}\cosh^2(h)$, it loses local stability. At the critical point $\beta = \frac{3}{2}\cosh^2(h)$ a Hopf bifurcation occurs.

**Lyapunov number.** Set $\beta = \beta_c(h) = \frac{3}{2}\cosh^2(h)$. To understand whether the Hopf bifurcation is sub- or supercritical we compute the first Lyapunov number (or Lyapunov coefficient) associated with the origin. For a system cast in normal form at the bifurcation an explicit formula for such a number is given, see [14, Sect. 3.4].

The dynamical system (5) takes its normal form with respect to the new variables $x = \sqrt{2}(\lambda - \beta_c m^σ)$ and $\lambda = \lambda$. The transformation yields

$$
\begin{align*}
\dot{x}_t &= -\sqrt{2}\lambda_t \\
\dot{\lambda}_t &= \sqrt{2}x_t - g_{\beta_c,h}(\lambda_t)
\end{align*}
$$

(7)

with $g_{\beta_c,h}(\lambda) = 3\lambda - \beta_c[\tanh(\lambda + h) + \tanh(\lambda - h)]$. Given (7), we compute the first Lyapunov number $\ell_1$ by means of formula (3.4.11) in [14, Sect. 3.4]. We readily obtain

$$
\ell_1 = \frac{\beta_c [\cosh(2h) - 2]}{4\cosh^4(2h)}.
$$

By [14, Thm. 3.4.2, Sect. 3.4], we get that the Hopf bifurcation is subcritical whenever $\ell_1 > 0$ and supercritical otherwise. This corresponds to having subcriticality for $h < \frac{1}{2}\ln(2 + \sqrt{3})$ and
supercriticality for \( h > \frac{1}{2} \ln(2 + \sqrt{3}) \). At the tricritical point \((h_{tc}, \beta_{tc}) = \left( \frac{1}{4} \ln(2 + \sqrt{3}), \frac{1}{4} \right)\) we obtain \( \ell_1 = 0 \). Then we determine the second Lyapunov number \( \ell_2 \) and repeat the same reasoning as above. Since \( \ell_2 = -\frac{1}{360} < 0 \), at the tricritical point the Hopf bifurcation is supercritical. We will sketch the steps to get \( \ell_2 \) in Appendix A.

Global analysis. By performing the change of variables \( y = 2(\lambda - \beta m^0) \) and \( \lambda = \lambda \), we can transform system (5) into the Liénard system

\[
\begin{align*}
\dot{y}_t &= -2\lambda_t \\
\dot{\lambda}_t &= y_t - g_{\beta,h}(\lambda_t),
\end{align*}
\]

with \( g_{\beta,h}(\lambda) = 3\lambda - \beta [\tanh(\lambda + h) + \tanh(\lambda - h)] \). Having at hand a Liénard system is very convenient as many global properties of possible limit cycles are available and can be derived by studying the zeroes of the function \( g_{\beta,h} \), see [20, Sect. 3.8]. For the sake of readability, we collect the interesting properties of function \( g_{\beta,h} \) in Appendix B. We proceed with the analysis of the attractors.

(a) Case \( h \leq \frac{1}{4} \ln(2 + \sqrt{3}) \). In this small-disorder regime the system behaves as in absence of disorder. Exploiting properties [F1] and [F4] given in Appendix B, the phase diagram can be proven analogously to the case of the homogeneous model discussed in [7]. We refer to the proof of Theorem 3.1 therein for details.

(b) Case \( h > \frac{1}{4} \ln(2 + \sqrt{3}) \). We show that in the current regime a saddle-node bifurcation of cycles occurs.

(iii) \( \beta \geq \beta_c(h) \). In this case the function \( g_{\beta,h} \) is odd and has exactly one positive zero at \( \lambda = \lambda^* \) (see properties [F3] and [F4] in Appendix B). Moreover, \( g_{\beta,h}^{'}(0) < 0 \) and \( g_{\beta,h} \) is monotonically increasing to infinity for \( \lambda > \lambda^* \). Therefore, standard Liénard’s Theorem guarantees existence and uniqueness of a stable periodic orbit. See [20, Thm. 1, Sect. 3.8].

(ii) The crucial ingredient for proving the next two statements is the particular structure of the vector field generated by (8). It defines a semicomplete one-parameter family of negatively rotated vector fields (with respect to \( \beta \), for fixed \( h \)), see [20, Def. 1, Sect. 4.6]. For dynamical systems depending on a parameter in this peculiar way, many results concerning bifurcations, stability and global behavior of limit and separatrix cycles are known [20, Chap. 4]. In particular, we will base our analysis on the following properties:

(P1) limit cycles expand/contract monotonically as the parameter \( \beta \) varies in a fixed sense;

(P2) a limit cycle terminates either at a critical point or at a separatrix of (8);

(P3) cycles of distinct fields do not intersect.

Properties (P1) and (P2) allow to explain the rise of a separatrix cycle whose breakdown is responsible for a saddle-node bifurcation of limit cycles at \( \beta = \beta_s(h) \). We have already proved that for \( \beta \geq \beta_c(h) \) a stable periodic orbit exists. While decreasing \( \beta \) from
β_c(h) this orbit shrinks and, at the same time, the limit cycle appeared at the Hopf bifurcation point expands, until they collide producing the semistable cycle at β = β⋆(h). Looking the same process forwardly, we see what is happening in this phase. When the separatrix splits increasing β from β⋆(h), it generates two limit cycles surrounding (0,0). The inner periodic orbit is unstable (due to the subcritical Hopf bifurcation at β = β_c(h)) and represents the boundary of the basin of attraction of the origin. Moreover, the outer limit cycle inherits the stability of the exterior of semistable cycle and so it is stable. See [20, Thm. 2 and Fig. 1, Sect. 4.6] for more details.

(i) Consider the Lyapunov function

\[ U(y, \lambda) := \frac{y^2}{4} + \frac{\lambda^2}{2}. \]

Observe that its total derivative \( \dot{U}(y, \lambda) = -\lambda g_{\beta, h}(\lambda) \) is negative for every \( y \in \mathbb{R} \) and for every \( \lambda \geq 2\beta^3 \). Thus, there exists a stable domain for the flux of (8) and, in particular, the trajectories cannot escape to infinity as \( t \to +\infty \). To conclude it is sufficient to prove that in this regime the dynamical system (8) does not admit a limit cycle. Indeed, the non-existence of periodic orbits together with the existence of a stable domain for the flux guarantee that every trajectory must converge to a fixed point as \( t \to +\infty \).

Therefore, it remains to show that no limit cycle exists for \( \beta < \beta_c(h) \). From properties [P1] and [P2] it follows that, as \( \beta \) increases from \( \beta_c(h) \) to infinity, the external stable limit cycle expands and its motion covers the entire region outside the separatrix. Similarly, the inner unstable cycle contracts from it and terminates at the origin. As a consequence, for \( \beta \geq \beta_c(h) \) the whole phase space is covered by expanding or contracting periodic orbits. From property [P3] we can deduce that no periodic trajectory may exist for \( \beta < \beta_* \), as such an orbit would cross some of the cycles present for \( \beta \geq \beta_* \).

At \( \beta = 0 \) the differential system (8) has solution \( y_t = c_1 e^{-t} + c_2 e^{-2t} \), \( \lambda_t = 2c_1 e^{-t} + c_2 e^{-2t} \) (\( c_1, c_2 \in \mathbb{R} \)), excluding the possible existence of periodic solutions. Therefore \( \beta_*(h) > 0 \) for all \( h > h_{tc} \). This concludes the proof.

A Derivation of the second Lyapunov number

To determine the second Lyapunov number \( \ell_2 \) associated with the origin we follow the center manifold approach in [15, pp. 175–181]. In particular, we rely on the review of the method done in [24, Sect. 3], which is stated very clearly and in detail. We keep the same notation as therein and report here only the relevant quantities and the main steps.

Consider the dynamical system (7) and set \( h = h_{tc} = \frac{1}{4} \ln(2 + \sqrt{3}) \) and \( \beta_c = \beta_c(h_{tc}) = \beta_{tc} = \frac{9}{4} \).

We Taylor expand the second equation of (7) around \( \lambda = 0 \) up to the fifth order. It yields

\[
\begin{align*}
\dot{x}_t &= -\sqrt{2} \lambda_t \\
\dot{\lambda}_t &= \sqrt{2} x_t - \frac{4}{15} \lambda_t^5 + o(\lambda_t^5).
\end{align*}
\]

(9)

Referring to [24] eqns. (13–17), from (9) we get

\[
A = \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}, \quad B = C = D = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad E \left( \begin{pmatrix} v_1 \\ w_1 \\ v_2 \\ w_2 \\ \vdots \\ v_5 \\ w_5 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -\frac{4}{15} w_1 w_2 w_3 w_4 w_5 \end{pmatrix}.
\]
Moreover the eigenvalues of the matrix $A$ are $\pm i \sqrt{2}$ with corresponding vectors $p = q = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$. see [24], eqn. (20). Since in our case $G_{31} = 0$, cf. [24], eqn. (27), we get

$$\mathcal{H}_{32} = \mathcal{E}(q, q, q, q) = \left( -\frac{q}{15} \left( -\frac{q}{\sqrt{2}} \right)^3 \left( \frac{q}{\sqrt{2}} \right)^2 \right) = \left( 0 \frac{1}{15\sqrt{2}} \right).$$

From [24] eqn. (39) we finally obtain

$$\ell_2 = \frac{1}{12} \text{Re}(p, \mathcal{H}_{32}) = -\frac{1}{360},$$

giving the conclusion. We recall that, for any $v, w \in \mathbb{C}^2$, we have $\langle v, w \rangle := \sum_{k=1}^2 \bar{v}_k w_k$.

## B Study of the family of functions $g_{\beta, h}(\lambda)$

We are interested in computing the zeros of the function $g_{\beta, h}(\lambda) = 3\lambda - \beta [\tanh(\lambda + h) + \tanh(\lambda - h)]$. Equivalently, we look for solutions to the fixed point equation

$$\lambda = \Gamma_{\beta, h}(\lambda) \quad \text{with} \quad \Gamma_{\beta, h}(\lambda) = \frac{3}{4} [\tanh(\lambda + h) + \tanh(\lambda - h)]. \quad (10)$$

It follows from (10) that

- $\lambda \mapsto \Gamma_{\beta, h}(\lambda)$ is a continuous function for all the values of $\beta$ and $h$;
- $\lim_{\lambda \to \pm \infty} \Gamma_{\beta, h}(\lambda) = \pm \frac{3\beta}{4}$;
- $\Gamma_{\beta, h}'(\lambda) = \frac{3}{32} \left[ \frac{1}{\cosh^3(\lambda + h)} + \frac{1}{\cosh^3(\lambda - h)} \right] > 0$ for every $\lambda$, for all values of $\beta$ and $h$.

Since $\Gamma_{\beta, h}(\lambda)$ is an odd function with respect to $\lambda$, we have $\Gamma_{\beta, h}(0) = 0$ for all $\beta$ and $h$, so that (10) always admits the solution $\lambda = 0$. Now, we investigate under what conditions positive solutions $\lambda > 0$ may occur. We restrict to work in the positive half-line.

In general, if

$$\Gamma_{\beta, h}'(0) = \frac{2\beta}{3 \cosh^2(h)} > 1, \quad (11)$$

then there may be at least one positive solution. However, since $\Gamma_{\beta, h}(\lambda)$ is not always concave, there may be a positive solution even when (11) fails. In this case, there must be at least two positive solutions (corresponding to the curve $\lambda \mapsto \Gamma_{\beta, h}(\lambda)$, crossing the diagonal first from below and then from above). We study the sign of the second order derivative to have a more precise picture. We compute

$$\Gamma_{\beta, h}''(\lambda) = -\frac{2\beta}{3} \left[ \frac{\tanh(\lambda + h)}{\cosh^2(\lambda + h)} + \frac{\tanh(\lambda - h)}{\cosh^2(\lambda - h)} \right].$$

Notice that $\Gamma_{\beta, h}''(0) = 0$ for every $\beta$ and $h$. Therefore $\lambda = 0$ is an inflection point for all the values of the parameters. We search for other possible inflection points. We have

$$\Gamma_{\beta, h}(\lambda) \geq 0 \iff \sinh(\lambda + h) \cosh^3(\lambda - h) + \sinh(\lambda - h) \cosh^3(\lambda + h) \leq 0.$$

By using the definitions of hyperbolic sine and cosine, after a few algebraic manipulations, we get

$$\Gamma_{\beta, h}''(\lambda) \geq 0 \iff \cosh(2\lambda) \leq \frac{\cosh(4h) - 3}{2 \cosh(2h)}.$$
The last inequality is never satisfied if \( \frac{\cosh(4h) - 3}{2\cosh(2h)} < 1 \), while it is equivalent to \( \lambda < \frac{1}{2} \arccosh \left( \frac{\cosh(4h) - 3}{2\cosh(2h)} \right) \).

Whenever \( \frac{\cosh(4h) - 3}{2\cosh(2h)} \geq 1 \), since, by exploiting the identity \( \cosh(4h) = 2\cosh^2(2h) - 1 \), we obtain

\[
\cosh(4h) - \frac{3}{2} \cosh(2h) < 1 \iff \cosh^2(2h) - \cosh(2h) - 2 \geq 0 \iff h \geq \frac{1}{2} \ln(2 + \sqrt{3}),
\]

it follows that

- for \( h < \frac{1}{2} \ln(2 + \sqrt{3}) \), the function \( \lambda \mapsto \Gamma_{\beta,h}(\lambda) \) is strictly concave for \( \lambda > 0 \);
- for \( h > \frac{1}{2} \ln(2 + \sqrt{3}) \), there is a positive inflection point at \( \lambda_I = \frac{1}{2} \arccosh \left( \frac{\cosh(4h) - 3}{2\cosh(2h)} \right) \) such that the function \( \lambda \mapsto \Gamma_{\beta,h}(\lambda) \) is strictly convex for \( 0 < \lambda < \lambda_I \) and strictly concave for \( \lambda > \lambda_I \);
- for \( h = \frac{1}{2} \ln(2 + \sqrt{3}) \), it yields \( \lambda_I = 0 \).

To conclude, the function \( \Gamma_{\beta,h} \) has at most one inflection point and therefore it changes curvature at most once. As a consequence, we obtain the following results concerning the number of positive solutions of the fixed point equation (10).

(F1) If \( h \leq h_{tc} \) and \( \beta \leq \beta_c(h) \), then the curve \( \Gamma_{\beta,h}(\lambda) \) is strictly concave on \((0, +\infty)\) and hence there is no intersection with the diagonal.

(F2) If \( h > h_{tc} \) and \( \beta < \beta_c(h) \), the function \( \Gamma_{\beta,h} \) changes its curvature either below or above the diagonal, giving rise to none or two positive fixed points. As the mapping \( \beta \mapsto \Gamma_{\beta,h} \) is increasing, the two regions are delimited by the separation curve \( \beta_T(h) \leq \beta_c(h) \), corresponding to the choice of parameters for which there exists \( \lambda_\ast > 0 \) such that \( \Gamma_{\beta,h}(\lambda_\ast) = \lambda_\ast \) and \( \Gamma_{\beta,h}'(\lambda_\ast) = 1 \). More precisely, we have

i. for \( \beta < \beta_T(h) \) there is no intersection with the diagonal;

ii. for \( \beta_T(h) < \beta < \beta_c(h) \), the diagonal and the curve \( \Gamma_{\beta,h} \) intersect two times.

(F3) If \( h > h_{tc} \) and \( \beta = \beta_c(h) \), there is exactly one positive solution of (10).

(F4) If \( \beta > \beta_c(h) \), no matter the curvature, the function \( \Gamma_{\beta,h}(\lambda) \) crosses the diagonal at precisely one positive \( \lambda \).

Remark. Observe that the existence of two periodic orbits is consistent with property (F2ii). Moreover, the value \( \beta_T \) is a lower bound for \( \beta_\ast \), since two limit cycles exist whenever the local extrema of the function \( g_{\beta,h} \) reach a proper height/depth. See [19] for precise conditions.

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