Modified Korteweg-de Vries equation: modulated elliptic wave and a train of asymptotic solitons

Alexander Minakov

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Science of Ukraine
47 Lenin Avenue, Kharkiv, 61103, Ukraine
E-mail:minakov.ilt@gmail.com

May 10, 2014

Abstract

This paper is devoted to the long-time asymptotic analysis of the Cauchy problem for the modified Korteweg – de Vries equation with a step-like initial function, which rapidly tends to different constants as \( x \to \pm \infty \).

First investigations of this problem were done by E. Khruslov and V. Kotlyarov in 1989. By using the technique, developed by E. Khruslov for the Korteweg – de Vries equation in 1976, they studied the long-time asymptotic behavior of this problem solution in the domain \( x > 4c^2t - \text{const} \log t \), and found (discovered) that the solution breaks (splits) up into a train of so-called asymptotic solitons in the subdomain \( 4c^2t - \text{const} \log t < x \leq 4c^2t \).

Further, in 2010, V. Kotlyarov and A. Minakov studied the behavior of this problem solution in the whole \( x, t \) plane with the exception of small neighborhoods of the leading and trailing edges \( |x + 6c^2t| < \varepsilon t \) and \( -\varepsilon t < x - 4c^2t \leq 0 \). It was found that the solution of this problem in the domain \( (-6c^2 + \varepsilon)t < x < (4c^2 - \varepsilon)t \) is described by the modulated elliptic wave, as was earlier predicted by A. Gurevich and L. Pitaevskii in 1973.

In this paper the long-time asymptotic behavior of this modulated elliptic wave is studied in the small neighborhood of the leading edge \( 4c^2t - \text{const} \log t < x \leq 4c^2t \). We show, that this modulated elliptic wave also breaks up into the train of asymptotic solitons, which are similar to Khruslov solitons, but differ from them in phase.

Key words: nonlinear evolution equations, modulated elliptic wave, asymptotic solitons.

Mathematical Subject Classification 2000: 35Q15, 35B40.

1 Introduction

The history of long-time asymptotic analysis of the Cauchy problem for the modified Korteweg – de Vries equation

\[
q_t(x,t) + 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0
\]

with initial function of step-like type, i.e.

\[
q(x,0) = q_0(x) \to \begin{cases} 0 & \text{as } x \to +\infty, \\ c & \text{as } x \to -\infty \end{cases}
\]

is quite long and goes along with long-time asymptotic analysis of the Cauchy problem for the more famous Korteweg – de Vries equation. Physicists have understood qualitative description of the solution since pioneer work of A. Gurevich and L. Pitaevsky [1] (1973). Namely, \( x, t \) – plane is divided into three domains. In the left and the right domains the solution tends to constants, and in the middle domain it tends to a modulated elliptic wave. Between these domains there are transition regions. In the
transition region near the leading edge the chain of asymptotic solitons run. In this direction R. Bikbaev, R. Sharipov, V. Novokshenov actively worked (see [2]-[8]). All these works were done in the framework of a heuristic Whitham method.

There were not any rigorous mathematical papers on this theme except for papers concerning soliton domain. It was proved that near the wave front there are arise the so-called asymptotic solitons, which are generated by the edge of continuous spectrum, as opposed to usual solitons generated by discrete spectrum. For KdV equation it was done by E. Khruslov [15], [16] (1975, 1976), and for MKdV it was done by E. Khruslov, V. Kotlyarov [18] (1989). The review of results in this direction for many other integrable equations can be read in [19], [20] and in the bibliography of this papers.

From the other hand, for more than 20 years the method of Riemann – Hilbert problem and the steepest descent method ([?]!) have been being actively developed. Recently these methods were applied to study solutions in other domains of $x, t$ half-plane, not only in soliton domain ([13], [9] – [12], [21] – [23], [14]).

Let us consider, in particular, the Cauchy problem for the modified Korteweg – de Vries equation

$$q_t(x, t) + 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0$$

with initial data, which rapidly tends to 0 and some constant $c > 0$ as $x \to \pm \infty$:

$$q_0(x) = q(x, 0) \to \begin{cases} 0 & \text{as } x \to +\infty, \\ c & \text{as } x \to -\infty. \end{cases}$$

Suppose that there exists a solution of this problem $q(x, t)$, which rapidly tends to initial constants 0 and $c$ as $x \to \pm \infty$ for all $t > 0$. We interest in long-time asymptotic behavior of the solution of this problem.

This problem was considered earlier by E. Khruslov and V. Kotlyarov in 1986 [18], and also by R. Bikbaev [6] in 1995.

In [18] the authors extended Khruslov method, which was originally developed for the Korteweg – de Vries equation, to the modified Korteweg – de Vries equation. They studied the asymptotic behavior of this problem solution in the neighborhood $x > 4c^2(t - \text{const} \log t)$ and calculated explicit formulas for asymptotic solitons, which run in the domain $4c^2(t - \text{const} \log t)x < 4c^2t$. Just this result is compared with our result in this paper.

Also this problem, with more general type of initial data, was considered by R. Bikbaev in a short note [17] in 1995. He used a heuristic Whitham method and his results are based on an analysis of the Whitham equations and the theory of analytic functions on a hyperelliptic surface.

Later this problem was revisited by V. Kotlyarov and A. Minakov [17] in 2010. There the nonlinear steepest descent method for oscillatory Riemann – Hilbert problem was used and applied for rigorous establishment of the asymptotics. In this paper the asymptotics were studied in the whole $x, t$ plane with the exception of the neighborhoods of the leading and trailing edges, that is, in the domain $x < -(6c^2+\varepsilon)t$, $-(6c^2-\varepsilon)t < x < (4c^2-\varepsilon)t$, $x \geq 4c^2t$.

In particular, in the domain $-(6c^2-\varepsilon)t < x < (4c^2-\varepsilon)t$ the asymptotics of the initial Cauchy problem solution is described by a modulated elliptic wave:

**Theorem** (Kotlyarov, Minakov) Let $\xi = \frac{x}{12t}$. Then for $\xi \in \left( -\frac{c^2}{2} + \frac{\varepsilon}{12}, \frac{c^2}{3} - \frac{\varepsilon}{12} \right)$ the solution of the IBV problem ([1], [2]) takes form of a modulated elliptic wave:

$$q(x, t) = q_{el}(x, t) + o(1), \quad t \to \infty,$$

where

$$q_{el}(x, t) = \sqrt{c^2 - f^2(\xi)} \frac{\Theta(\pi i + itB_g(f(\xi)) + i\Delta(f(\xi))\tau(f(\xi)))}{\Theta(itB_g(f(\xi)) + i\Delta(f(\xi))\tau(f(\xi)))}, \quad \xi = \frac{x}{12t} \quad (1.3)$$

Here $B_g(d), \tau(d), \Delta(d)$ are defined by ([2], [14]) and the function $f(\xi)$ is defined by lemma [2.7]. It worth to mention that the point $id = if(\xi)$ is an analogue of a branch point of an Whitham zone, $\tau$ – is
the $b$ - period of the corresponding Riemannian surface, $B_g$ – is the $b$ – period of a Abelian integral of the second kind, and $\Delta$ – is an integral on a Riemannian surface.

Here $\varepsilon$ is arbitrarily small positive number. It was introduced for a technical reasons. The question is if it just a technical restriction or more significant limitation, and there must be another formula in the domain $(4c^2 - \varepsilon)t < x \leq x$.

Notice that the explicit formula, which determines $q_{el}(x, t)$, is valid up to the leading and trailing edge, i.e. in the domain $-6c^2t < x < 4c^2t$. This suggests to study the behavior of $q_{el}(x, t)$ in the neighborhood of the leading edge $x = 4c^2t$ and to compare the result with one obtained in [?].

In this paper the long-time asymptotic behavior of this modulated elliptic wave $q_{el}(x, t)$ is studied in the small neighborhood of the leading edge $4c^2t - \text{const} \log t < x \leq 4c^2t$. The main result of this paper is:

**Theorem 1.1.** Let $N \geq 1$ be any integer number, $\varepsilon > 0$ is a fixed small number. Then as $t \to \infty$

$A$ uniformly for $t$ and $x$ such that

$$4c^2t - \frac{2N + \frac{1}{2} - \varepsilon}{2c} \log t \leq x \leq 4c^2t - \frac{\frac{1}{2} + \varepsilon}{2c} \log t$$

the following is true:

$$q_{el}(x, t) = \sum_{n=1}^{N} \frac{2c}{\cosh (2c(x - 4c^2t) + (2n - \frac{1}{2}) \log t - \alpha_n(x, t))} + O \left( \frac{1}{t} \right); \quad (1.4)$$

$B$ uniformly for $t$ and $x$ such that

$$4c^2t - \frac{1 - \varepsilon}{2c} \log t \leq x < 4c^2t$$

the following is true:

$$q_{el}(x, t) = O \left( t^{-\frac{1}{2}} \right).$$

Here $\alpha_n(x, t)$ is an auxiliary well-defined function, which has the following asymptotic behavior as $t \to \infty$:

$$\alpha_n(x, t) = -\left(2n - \frac{1}{2}\right) \log \frac{1}{vt} - \frac{8c^3tv}{\log \frac{1}{v}} - \left(6n - \frac{7}{2}\right) \log 2 + O \left( \frac{\log^2 \log \frac{1}{v}}{\log v} \right), \quad (1.5)$$

where

$$v = 1 - \frac{x}{4c^2t}.$$  

For any $\gamma, \delta$ such that $0 < \delta < \gamma$, $\alpha_n(x, t)$ is bounded in the interval $4c^2t - \gamma \log t < x < 4c^2t - \delta \log t$.

In the section [4] we discuss this formula and compare it with the formula for $q(x, t)$ from [18].

---

2 Preliminaries

Let us consider the initial-value problem

$$q_t(x, t) + 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0,$$

$$q(x, 0) = q_0(x) \to \begin{cases} 0, & x \to +\infty, \\ c, & x \to -\infty, \end{cases} c > 0,$$
with initial function which rapidly tends to their limits, namely

\[\int_{-\infty}^{0} |q_0(x) - c_1 e^{2x(|l + \sigma|)}| \, dx + \int_{0}^{+\infty} |q_0(x) - c_1 e^{2x(|l + \sigma|)}| \, dx < \infty. \tag{2.6}\]

Here \(l\) satisfies the inequality \(l > c > 0\), and \(\sigma > 0\) is an arbitrary positive number. We suppose that there exists the solution of this initial value-problem and that this solution converges to its limits with the first moments: for any \(t\)

\[\int_{-\infty}^{0} |x||q(x, t) - c| \, dx + \int_{0}^{+\infty} x|q(x, t)| \, dx < \infty. \tag{2.7}\]

Let \(a^{-1}(k)\) be the standard transmission coefficient (17). It has the following properties:

1. \(a(k)\) is analytic in the upper half-plane \(\mathbb{C}_+\) with the cut \([0, i\epsilon]\), extended continuously up to the boundary with the exception of the point \(i\epsilon\), where \(a(k)\) may has a root singularity of the order \((k - i\epsilon)^{-1/4}\);

2. \(a(k)\) satisfies the symmetry conditions \(a(-k) = a(k)\);

3. \(\forall k \in (i\epsilon, 0)\ A_\pm(k) \neq 0\).

We suppose the absence of solitons, i.e. that

\[\forall k \in \mathbb{C}_+ \setminus [i\epsilon, 0] \quad a(k) \neq 0\]

and that \(a(k)\) has a root singularity at the point \(i\epsilon\) of exactly order \((k - i\epsilon)^{-1/4}\):

\[a(k) = \frac{h_1}{2} \sqrt{\frac{k + i\epsilon}{k - i\epsilon}} \left(1 + O\left(\sqrt{\frac{k - i\epsilon}{k + i\epsilon}}\right)\right), \quad k \to i\epsilon, \quad h_1 \in \mathbb{R} \setminus \{0\}. \tag{2.9}\]

The asymptotic behavior of this Cauchy problem was studied in [17]. The \(x, t\)-half-plane was divided into 3 regions with qualitatively different asymptotic behavior of the solution, namely \(x < (4c^2 - \varepsilon)t\), \((-6c^2 + \varepsilon)t < x < (4c^2 - \varepsilon)t\), \(x \geq 4c^2t\), where \(\varepsilon > 0\) is arbitrarily small positive number. The most interesting in asymptotic analysis is the domain \((-6c^2 + \varepsilon)t < x < (4c^2 - \varepsilon)t\). In this region the Cauchy problem solution is described by a modulated elliptic wave, according to formula (1.3).

Our notations

- \(\Theta\)-function \(\Theta(z|\tau)\).

In the sequel we use \(\Theta\)-function in the next form:

for any \(\tau < 0\) and \(z \in \mathbb{C}\)

\[\Theta(z|\tau) = \sum_{m=-\infty}^{\infty} \exp \left\{ \frac{1}{2} \tau m^2 + zm \right\}. \tag{2.8}\]

\(\Theta\)-function has the next property:

for any integer \(n, l\)

\[\Theta(z + 2\pi in + \tau l|\tau) = \Theta(z|\tau) \exp \left\{ -\frac{1}{2} \tau l^2 - zl \right\}. \tag{2.9}\]
• Real parameters \( d \) and \( \mu \).

Function \( d = f(\xi) : \left[-\frac{c^2}{2}, \frac{c^2}{3}\right] \to [0, c] \) is defined from the following system:

\[
\begin{align*}
\int_0^1 (\mu^2 - \lambda^2 d^2) \sqrt{\frac{1 - \lambda^2}{\xi^2 - \lambda^2 d^2}} \, d\lambda &= 0, \\
\frac{c^2}{2} + \xi &= \mu^2 + d^2.
\end{align*}
\]

(2.10)

(2.11)

**Lemma 2.1.** For \( \xi \in \left[-\frac{c^2}{2}, \frac{c^2}{3}\right] \) there exists one to one, increasing and continuous map \( f : \left[-\frac{c^2}{2}, \frac{c^2}{3}\right] \to [0, c] \) which defines the unique solution of equations (2.10) and (2.11) by formulas: \( d = f(\xi) \) and \( \mu = \mu(f(\xi)) \). The map \( \mu : [0, c] \to \left[0, \frac{c}{\sqrt{3}}\right] \) is also continuous and increasing, and their boundary values are as follows: \( f\left(\frac{c^2}{3}\right) = c, f\left(-\frac{c^2}{2}\right) = 0, \mu(0) = 0, \mu(c) = \frac{c}{\sqrt{3}} \).

The proof of lemma 2.1 is given in Appendixes.

• Functions \( B_g(d), \tau(d), \Delta(d) \).

For \( d \in (0, c) \) let \( w(k, d) = \sqrt{(k^2 + c^2)(k^2 + d^2)} \) be an analytic in \( k \) function in the domain \( k \in \mathbb{C}\setminus([ic, id] \cup [-ic, -id]) \) which is fixed by the condition \( w(0, d) = cd \). Then we define \( B_g(d), \tau(d), \Delta(d) \) as follows:

\[
B_g(d) = 24\int_{id}^{ic} \frac{(k^2 + \mu^2(d))(k^2 + d^2) \, dk}{w_+(k, d)},
\]

(2.12)

\[
\tau(d) = -\pi i \int_{id}^{ic} \frac{dk}{w_+(k, d)} \left( \int_0^{id} \frac{dk}{w(k, d)} \right)^{-1},
\]

(2.13)

\[
\Delta(d) = -\pi i \int_{id}^{ic} \frac{\log(a_+(k)a_-(k)) \, dk}{w_+(k, d)} \left( \int_0^{id} \frac{dk}{w(k, d)} \right)^{-1}.
\]

(2.14)

3 Proof of the Theorem [1.1]

Theta-functions in (1.3) are slowly-convergent as \( \xi \) tends to \( \frac{c^2}{3} \) because in this case \( \tau(f(\xi)) \) tends to 0 (see lemma 2.1 in appendix). To overcome these difficulties, i.e. to make theta series well converged, we can use Poisson summation formula and rewrite (1.3) in term of rapidly convergent theta series. Indeed, we have

\[
\Theta(z|\tau) = \Theta\left(\frac{2\pi i z}{\tau}, \frac{4\pi^2}{\tau}\right) \sqrt{\frac{2\pi}{-\tau}} \left( \exp \frac{-z^2}{2\tau} \right).
\]

(3.15)

Then the model approximation (1.3) of the solution takes the form:

\[
q_d(12t\xi, t) = \sqrt{c^2 - f^2(\xi)} \exp \left( -\frac{\tau^*(f(\xi))}{8} + \frac{\tau^*(f(\xi))}{4} (z(t, \xi) + 1) \right) \Theta\left( \frac{\tau^*(f(\xi))}{2} (z(t, \xi) + 1) \right)
\]

(3.16)
where \( z(t, \xi) = \frac{1}{\pi} (t B_0(f(\xi)) + \Delta(f(\xi))) \) and \( \tau^*(d) = \frac{4\pi^2}{\tau(d)} \). We emphasize that now \( \tau^*(f(\xi)) \to -\infty \) as \( \xi \to \frac{c^2}{3} \) and as a result theta series become well-converged.

Let us introduce a function \( F(\cdot, \cdot) \). For any \( \tau^* < 0 \) and \( z \in \mathbb{C} \) we set

\[
F(\tau^*, z) = \sqrt{c^2 - h^2(\tau^*)} \exp\left( \frac{-\tau^*}{8} + \frac{\tau^*(z + 1)}{4} \right) \Theta\left( \frac{\tau^*(z + 1)|\tau^*|}{2} \right) / \Theta\left( \frac{\tau^* z|\tau^*|}{2} \right). \tag{3.17}
\]

Here \( h(.) \) is defined by lemma 5.1. Thus we can rewrite the formula (3.16) by using the function \( F(\cdot, \cdot) \):

\[
q_{el}(x,t) = F(\tau^*(f(\xi)), z(t, \xi)).
\]

**Lemma 3.1.** For any integer \( N \geq 1 \) the following formulas is true uniformly in \( z \) as \( \tau^* \to -\infty \):

\[
F(\tau^*, z) = \sum_{n=1}^{N} \frac{2c}{\cosh \left( \frac{\tau^*(1-z+2n)}{4} \right)} + O\left(e^{\tau^*} \right), \quad 0 \leq z \leq 2N,
\]

\[
F(\tau^*, z) = O\left(e^{\frac{\tau^*}{2}} \right), \quad -\frac{1}{2} \leq z \leq \frac{1}{2}.
\]

**Proof.** We have that

\[
F(\tau^*, z) = \sqrt{c^2 - h^2(\tau^*)} \exp\left( \frac{-\tau^*}{8} + \frac{\tau^*(z + 1)}{4} \right) \Theta\left( \frac{\tau^*(z + 1)|\tau^*|}{2} \right) / \Theta\left( \frac{\tau^* z|\tau^*|}{2} \right).
\]

In the appendixes we prove (see 5.31), that \( \sqrt{c^2 - h^2(\tau^*)} \exp\left( \frac{-\tau^*}{8} \right) = 4c \left( 1 + O\left(e^{\tau^*} \right) \right) \).

Taking into account the definition of \( \Theta \) function (see formula 244), we can easily verify that for \( \tau^* \to -\infty \) the next formula is true uniformly for \(-\frac{1}{2} \leq z \leq \frac{5}{2} \):

\[
F(\tau^*, z) = 4c \left( 1 + O\left(e^{\tau^*} \right) \right) e^{\tau^*(1-z)} + O\left(e^{\tau^*(2-z)} \right) + O\left(e^{\tau^*(1-z)} \right) + O\left(e^{\tau^*(1+z)} \right).
\]

Simple calculations lead us to the next formula:

\[
F(\tau^*, z) = \frac{4c \left( 1 + O\left(e^{\tau^*} \right) \right) \left( 1 + O\left(e^{\frac{\tau^*}{2}} \right) + e^{\frac{\tau^*(2-z)}{2}} \right)}{e^{\frac{\tau^*(1-z)}{4}} + e^{\frac{\tau^*(1-z)}{4}} + O\left(e^{\frac{\tau^*(1-z)}{4}} \right) + e^{\frac{\tau^*(1+z)}{4}}},
\]

and then

\[
F(\tau^*, z) = \frac{4c \left( 1 + O\left(e^{\tau^*} \right) \right) \left( 1 + O\left(e^{\frac{\tau^*}{2}} \right) + e^{\frac{\tau^*(2-z)}{2}} \right)}{2 \cosh \left( \frac{\tau^*(1-z)}{4} \right) \left( 1 + O\left(e^{\frac{\tau^*(1-z)}{4}} \right) + e^{\frac{\tau^*(1+z)}{4}} \right)}.
\]

Now let us note that

\[
O\left(\frac{e^{\frac{\tau^*(1-z)}{4}} + e^{\frac{\tau^*(1+z)}{4}}}{\cosh \left( \frac{\tau^*(1-z)}{4} \right)} \right) = O\left(e^{\frac{\tau^*(1-z)}{2}} + e^{\frac{\tau^*(1+z)}{2}} \right).
\]
Then
\[ F(\tau^*, z) = \frac{2c}{\cosh \frac{\tau^*(1-z)}{4}} \left( 1 + O \left( e^{\frac{\tau^*}{4}} + e^{\frac{\tau^*(2-z)}{4}} + e^{\frac{\tau^*(3-z)}{4}} + e^{\frac{\tau^*(4-z)}{4}} \right) \right), \]
and
\[ F(\tau^*, z) = \frac{2c}{\cosh \frac{\tau^*(1-z)}{4}} \left( 1 + O \left( e^{\frac{\tau^*}{4}} + e^{\frac{\tau^*(2-z)}{4}} + e^{\frac{\tau^*(3-z)}{4}} + e^{\frac{\tau^*(4-z)}{4}} \right) \right). \]
Therefore
\[ F(\tau^*, z) = \frac{2c}{\cosh \frac{\tau^*(1-z)}{4}} + O \left( e^{\frac{\tau^*}{4}} \frac{\cosh \frac{\tau^*(1-z)}{4}}{\cosh \frac{\tau^*(1-z)}{4}} \right). \]
The argument of the O-estimate in the last formula may be change with:
\[
e^{\frac{\tau^*(1-z)}{4}}, \text{ for } -\frac{1}{2} \leq z \leq \frac{1}{2},
\]
\[
e^{\frac{\tau^*(2-z)}{4}}, \text{ for } \frac{1}{2} \leq z \leq 1,
\]
\[
e^{\frac{\tau^*}{4}}, \text{ for } 1 \leq z \leq \frac{3}{2},
\]
\[
e^{\frac{\tau^*(3-z)}{4}}, \text{ for } \frac{3}{2} \leq z \leq \frac{5}{2}.
\]
Therefore
\[ F(\tau^*, z) = \frac{2c}{\cosh \frac{\tau^*(1-z)}{4}} + O \left( e^{\frac{\tau^*}{4}} \right) \text{ for } 0 \leq z \leq 2,
\]
and
\[ F(\tau^*, z) = O \left( e^{\frac{\tau^*}{4}} \right) \text{ for } -\frac{1}{2} \leq z \leq \frac{1}{2}.
\]
Also we note that \( \frac{2c}{\cosh \frac{\tau^*(1-z)}{4}} = O \left( e^{\frac{\tau^*}{4}} \right) \text{ for } z \in \mathbb{R}\setminus(0, 2). \)
Besides this, due to property (2.9) of \( \Theta \)-function, we obtain that \( F(\tau^*, z) = F(\tau^*, z - 2n) \) for any integer \( n \). Therefore
\[ F(\tau^*, z) = \frac{2c}{\cosh \frac{\tau^*(1-z+2n)}{4}} + O \left( e^{\frac{\tau^*}{4}} \right) \text{ for } 2n - 2 \leq z \leq 2n,
\]
and
\[ \frac{2c}{\cosh \frac{\tau^*(1-z+2n)}{4}} = O \left( e^{\frac{\tau^*}{4}} \right) \text{ for } z \in \mathbb{R}\setminus(2n - 2, 2n). \]
So, we obtain that for any integer \( N \geq 1 \)
\[ F(\tau^*, z) = \sum_{n=1}^{N} \frac{2c}{\cosh \frac{\tau^*(1-z+2n)}{4}} + O \left( e^{\frac{\tau^*}{4}} \right), \text{ for } 0 \leq z \leq 2N.
\]
\[ F(\tau^*, z) = O \left( e^{\frac{\tau^*}{4}} \right), \text{ for } -\frac{1}{2} \leq z \leq \frac{1}{2}. \]
Now we have to express in terms of variables \( x, t \) the intervals \( z \in [0, 2N] \) and \( z \in [-\tfrac{1}{2}, \tfrac{1}{2}] \). To do this let us define small parameters \( \eta \) and \( v \):

\[
\eta = \eta(d) = 1 - \frac{d}{c}, \quad \text{and} \quad v = v(\xi) = 1 - \frac{3\xi}{c^2},
\]

which tend to zero as \( \xi \to c^2/3 \). The inverses of these functions are

\[
d(\eta) = c(1 - \eta), \quad \text{and} \quad \xi(v) = \frac{c^2(1 - v)}{3}.
\]

In the appendixes we find the connection between \( v \) and \( \eta \) (see the formulas (5.40), (5.42)), and make asymptotic expansions of \( B_\eta(d(\eta)), \Delta(d(\eta)), \tau^*(d(\eta)) \) in \( \eta \) (see the formulas (2.12), (5.34), (5.30)).

**Lemma 3.2.**

\[
z(t, \xi(v)) = \frac{8c^3 tv}{\log \frac{t}{v}} - \frac{1}{2} + O \left( \frac{1}{\log \frac{t}{v}} + \frac{t v \log \log \frac{1}{v}}{\log^2 v} \right).
\]

**Proof.** Recall that

\[
z(t, f^{-1}(d(\eta))) = \frac{t B_\eta(d(\eta)) + \Delta(d(\eta))}{\pi} = 8c^3 t \eta - \frac{1}{2} + \frac{2 \log 2}{\log \frac{1}{\eta}} + O \left( t \eta^2 \log \frac{1}{\eta} + \frac{1}{\log^2 \eta} \right).
\]

From this formula we can deduce that for any positive \( t \) and sufficiently small \( \eta ; z \geq -\frac{1}{2} \). Further we will use this remark.

By substituting in the last formula for \( z \) expression for \( \eta \) in \( v \) we obtain the equation (3.18). \( \square \)

**Correspondence of intervals in \( z \) and in \( x, t \).**

**Lemma 3.3.**

**A.** For any integer \( N \geq 1 \) and any sufficiently small positive \( \varepsilon \) there exists \( T = T(\varepsilon, N) \) such that for any \( x, t \) which satisfy the conditions \( t > T \) and

\[
4c^2 t - \frac{2N + \frac{1}{2} - \varepsilon}{2c} \log t < x < 4c^2 t - \frac{\varepsilon + \frac{1}{2}}{2c} \log t
\]

the following formula is true:

\[
0 \leq z(t, \xi) \leq 2N.
\]

**B.** For any integer \( N \geq 1 \) and any sufficiently small positive \( \varepsilon \) there exists \( T = T(\varepsilon, N) \) such that for any \( x, t \) which satisfy the conditions \( t > T \) and

\[
4c^2 t - \frac{1 - \varepsilon}{2c} \log t < x < 4c^2 t
\]

the following formula is true:

\[
- \frac{1}{2} \leq z(t, \xi) \leq 2N + \frac{1}{2}.
\]

**Proof.** Let \( \gamma \) and \( \delta \) be a some positive numbers such that \( \delta < \gamma \) and let us consider interval

\[
4c^2 t - \gamma \log t < x < 4c^2 t - \delta \log t
\]

or

\[
1 - \frac{\gamma \log t}{4c^2 t} < \frac{x}{4c^2 t} < 1 - \frac{\delta \log t}{4c^2 t}.
\]
We remember that \( v = 1 - \frac{x}{4c^2t} \) and then the last formula can be rewrite in the next form:

\[
\frac{\delta \log t}{4c^2t} < v < \frac{\gamma \log t}{4c^2t}
\]  \( (3.22) \)

We see that for \( x, t \), which lay in the interval

\[ 4c^2t - \gamma \log t < x < 4c^2t - \delta \log t, \]

if \( t \to \infty \) then \( v \to 0 \). Further we will use this fact.

For \( t \) and \( v \) such that

\[
\frac{\delta \log t}{4c^2t} < v < \frac{\gamma \log t}{4c^2t}
\]  \( (3.23) \)

also

\[
\frac{1}{\log t} \cdot \frac{1}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}} < \frac{1}{\log v} < \frac{1}{\log t} \cdot \frac{1}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}}
\]

and then

\[
\frac{2c\delta}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}} < \frac{8c^3tv}{\log \frac{1}{v}} < \frac{2c\gamma}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}}
\]

and

\[
\frac{2c\delta}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}} - \frac{8c^3tv}{\log \frac{1}{v}} + z < z < \frac{2c\gamma}{1 - \frac{\log \log t - \frac{4c^2}{\log t}}{\log t}} - \frac{8c^3tv}{\log \frac{1}{v}} + z.
\]

And then, due to formula \((3.18)\), we obtain that for any positive \( \varepsilon \) there exists \( T = T(\varepsilon, \gamma, \delta) \) such that for any \( t > T \), for any \( v \) which satisfy \((3.22)\), and uniformly for \( \delta, \gamma \) from the interval \( \frac{1}{4c} < \delta, \gamma < \frac{2N + 1}{2c} \) the following inequality holds true:

\[
2c\delta - \frac{1}{2} - \varepsilon \leq z \leq 2c\gamma - \frac{1}{2} + \varepsilon
\]

A. Now let us take \( \delta = \frac{\varepsilon + \frac{1}{2}}{2c} \), \( \gamma = \frac{2N + \frac{1}{2} - \varepsilon}{2c} \).

If we take into account that \((3.22)\) is equivalent to \((3.21)\), then we obtain that for any positive \( \varepsilon \) there exists \( T = T(\varepsilon, N) \) so that for any \( t > T \) and any \( x \) such that

\[ 4c^2t - \frac{2N + \frac{1}{2} - \varepsilon}{2c} \log t \leq x \leq 4c^2t - \frac{1}{2} + \varepsilon \cdot \frac{1}{2} + \frac{1}{2} \log t \]

the following is true:

\[ 0 \leq z \leq 2N. \]

B. If we take now \( \gamma = \frac{1 - \varepsilon}{2c} \) and take into account that always for big \( t \) : \( z \geq \frac{1}{2} \), then we obtain that for any positive \( \varepsilon \) there exists \( T = T(\varepsilon) \) so that for any \( t > T \) and any \( x \) such that

\[ 4c^2t - \frac{1 - \varepsilon}{2c} \log t < x \leq 4c^2t \]

the following is true:

\[ \frac{-1}{2} \leq z \leq \frac{1}{2}. \]
Evaluating of cosh argument in the formulas for $q_{el}$

Define $\alpha_n(x, t)$ through the relation

$$\frac{\tau^*(z - 2n + 1)}{4} \equiv 2c(x - 4c^2 t) + \left(2n - \frac{1}{2}\right) \log t - \alpha_n(x, t).$$

Then we have the following lemma

**Lemma 3.4.**

$$\alpha_n(x, t) = -\left(2n - \frac{1}{2}\right) \log \frac{1}{vt} + \frac{8c^3 t v}{\log \frac{1}{v}} - \left(6n - \frac{7}{2}\right) \log 2 + O \left(\frac{\log^2 \log \frac{1}{v}}{\log v}\right).$$

$\alpha_n(x, t)$ is bounded for $4c^2 t - \gamma \log t < x < 4c^2 t - \delta \log t$, where $0 < \delta < \gamma$ is arbitrary numbers.

**Proof.** Let us express $\frac{\tau^*(z - 2n + 1)}{4}$ in terms of $v, t$ and therefore $x, t$. We know that

$$\tau^* = -4 \log \frac{8}{\eta} + O(\eta) = 4 \log v \left(1 - \frac{\log \log \frac{1}{v} + 3 \log 2}{\log v} + \frac{\log \frac{1}{v} + \log 8c}{\log^2 v} + O \left(\frac{\log^2 \log \frac{1}{v}}{\log v}\right)\right),$$

$$z = \frac{tB_g + \Delta}{\pi},$$

$$\frac{1}{\pi} B_g = 8c^3 \eta + O(\eta^2 \log \eta) = \frac{8c^3 v}{\log \frac{1}{v}} \left(1 + \frac{\log \frac{1}{v} + \log 8c}{\log v}\right) + O \left(\frac{v \log^2 \log \frac{1}{v}}{\log^3 v}\right),$$

$$\frac{1}{\pi} \Delta = -\frac{1}{2} - 2 \log \frac{2}{\log v} + O \left(\frac{\log \log \frac{1}{v}}{\log^2 v}\right).$$

And then for $t, v$, which satisfy (3.23), in forward, but careful calculations we obtain that

$$\frac{\tau^*(z - 2n + 1)}{4} = -8c^3 t v + \left(2n - \frac{1}{2}\right) \log t + \left(2n - \frac{1}{2}\right) \log \frac{1}{vt} + \frac{8c^3 t v}{\log \frac{1}{v}} + \left(6n - \frac{7}{2}\right) \log 2 + O \left(\frac{\log^2 \log \frac{1}{v}}{\log v}\right).$$

And by remembering that $v = 1 - \frac{x}{4c^2 t}$ is function of $x, t$, we obtain that

$$\frac{\tau^*(z - 2n + 1)}{4} = 2c(x - 4c^2 t) + \left(2n - \frac{1}{2}\right) \log t - \alpha_n(x, t),$$

where

$$\alpha_n(x, t) = -\left(2n - \frac{1}{2}\right) \log \frac{1}{vt} - \frac{8c^3 t v}{\log \frac{1}{v}} - \left(6n - \frac{7}{2}\right) \log 2 + O \left(\frac{\log^2 \log \frac{1}{v}}{\log v}\right).$$

While proving lemma (3.3) we also have proved that $\alpha_n(x, t)$ is bounded for $4c^2 t - \gamma \log t < x < 4c^2 t - \delta \log t$, where $0 < \delta < \gamma$ is arbitrary numbers.

By summarizing lemmas 3.1, 3.3 and 3.4, we obtain the statements of the Theorem 1.1. Q. E. D.
4 Comparing of $q_{el}(x, t)$ and $q(x, t)$ in the domain $4c^2 t - \gamma \log t < x \leq 4c^2 t$

In this section we compare $q_{el}(x, t)$ with $q(x, t)$. In [18] the following result was obtained:

Theorem (Khruslov, Kotlyarov) Let $N \geq 1$ and $M$ be such an integer numbers that $N = \left\lfloor \frac{M + 1}{2} \right\rfloor$. Then for

\[
x > 4c^2 t - \frac{(M + 1) \log t}{2c},
\]

\[
q(x, t) = 2c \sum_{n=1}^{N} \frac{1}{\cosh \left( 2c(x - 4c^2 t) + \left( 2n - \frac{1}{2} \right) \log t - \tilde{\alpha}_n \right)} + O \left( t^{-\frac{1}{2} + \epsilon} \right),
\]

where $\tilde{\alpha}_n = \log \left[ \frac{|h_0|}{4^{2n-1}(2c)^6} \frac{\Gamma(n) \Gamma(n + \frac{1}{2})}{\Gamma_1 \Gamma_{(n-1)} \Gamma_{(n+1)}} \right]$,

\[
\Gamma^{(k)}_b = \det \left[ \Gamma(i + j + b) \right]_{i,j=0,k-1},
\]

Here $h_0$ is some constant, which is determined by $q_0(x)$.

We can rephrase the theorem by simplifying expression for $\tilde{\alpha}_n$. Also as both $M$ and $N$ are integer we can take $M = 2N$.

Theorem (Khruslov, Kotlyarov) (rephrased) Let $N \geq 1$ be an integer number. Then for

\[
x > 4c^2 t - \frac{(2N + 1) \log t}{2c},
\]

\[
q(x, t) = 2c \sum_{n=1}^{N} \frac{1}{\cosh \left( 2c(x - 4c^2 t) + \left( 2n - \frac{1}{2} \right) \log t - \tilde{\alpha}_n \right)} + O \left( t^{-\frac{1}{2} + \epsilon} \right),
\]

(4.24)

where

\[
\tilde{\alpha}_n = \log \left[ \frac{|h_0|}{4^{2n-1}(2c)^6} \frac{\Gamma(n) \Gamma(n + \frac{1}{2})}{\Gamma_1 \Gamma_{(n-1)} \Gamma_{(n+1)}} \right].
\]

(4.25)

Let us consider a curve $x = 4c^2 t - \frac{2m - \frac{1}{2}}{2c} \log t + \frac{\tilde{\alpha}_m}{2c}$, where $m \in \mathbb{N}$. All the summand in the formula (4.24) become vanishing except for one where $n = m$. Then for $x, t$ which lie on this curve

\[
q(x, t) = 2c + O \left( t^{-\frac{1}{2} + \epsilon} \right) \quad \text{as} \quad t \to \infty.
\]

Now let us consider $q_{el}(x, t)$. All the summand in the formula (1.4) become vanishing except for the one, where $n = m$. According to the formula (1.5), for $x = 4c^2 t - \frac{2m - \frac{1}{2}}{2c} \log t + \frac{\tilde{\alpha}_m}{2c}$ and $t \to \infty$ we have the following:

\[
\frac{vt}{\log \frac{1}{t}} = \frac{2m - \frac{1}{2}}{8c^3} \left( 1 + O \left( \frac{\log \log t}{\log t} \right) \right),
\]
\[
\log \frac{vt}{\log v} = \log \frac{2m - \frac{1}{2}}{8c^3} + O\left(\frac{\log \log t}{\log t}\right),
\]
and
\[
\alpha_m(x, t) = \alpha_m + O\left(\frac{\log^2 \log t}{\log t}\right),
\]
where
\[
\alpha_m = \left(2m - \frac{1}{2}\right) \log \frac{2m - \frac{1}{2}}{8c^3} - \left(6m - \frac{7}{2}\right) \log 2.
\] (4.26)
Then
\[
q_{el}(x, t) = 2c \cosh (\tilde{\alpha}_m - \alpha_m(x, t)) + O\left(\frac{t^{-\frac{1}{2}} + \varepsilon}{\log \log t \log t}\right) = 2c \cosh (\tilde{\alpha}_m - \alpha_m) + O\left(\frac{\log^2 \log t}{\log t}\right).
\]
As \(\alpha_m(4.26)\) and \(\tilde{\alpha}_m(4.25)\) are not equal, \(q(x, t) - q_{el}(x, t)\) does not tend to 0 as \(t \to \infty\).

**Remark.** In asymptotical analysis in the domain \(x < 4c^2 t\) (in [17]) we had to restrict ourselves for a smaller domain \(x < 4c^2 t - \varepsilon t\), where \(\varepsilon > 0\) is arbitrarily small number. It was done for technical reasons. We see, however, that the asymptotics from the domain \(4c^2 t - \varepsilon t < x\) with arbitrarily small \(\varepsilon\) and the asymptotics from the domain \(4c^2 t - \gamma \log t < x\) do not match. This may indicate the presence of a transition zone \(4c^2 t - \varepsilon t < x < 4c^2 t - \gamma \log t\).

5 Appendixes

**Lemma 5.1.** Let \(\tau^*(d) = \frac{4\pi^2}{\tau(d)}\), where \(\tau(.)\) is defined by the formula (2.13). Then

1. for \(d \in (0, c)\) : \(\tau^*(d) < 0\);
2. \(\tau^(.)\) is decreasing function on the interval \(d \in (0, c)\);
3. \(\tau^*(+0) = 0\), \(\tau^*(c - 0) = -\infty\).

Therefore there exists the inverse of \(\tau^*(.)\), call it \(h(.)\): \((-\infty, 0) \to (0, c)\): \(h(\tau^*(d)) = d\).

**Proof.** Indeed,
\[
\tau^*(d) = 4\pi i \int_0^d \frac{dk}{w(k, d)} \left(\int_{id}^c \frac{dk}{w_+(k, d)}\right)^{-1}.
\]
From the definition of the \(w(k, d)\) we see, that for \(k \in (id, ic)\): \(w_+(k, d) = |w_+(k, d)|\). We make change of variables \(k = iy\) in the last integrals and get
\[
\tau^*(d) = -4\pi \frac{I_1(d)}{I_0(d)},
\]
where
\[
I_1(d) = \int_0^d \frac{dy}{\sqrt{(c^2 - y^2)(d^2 - y^2)}},
\]
and
\[
I_0(d) = \int_{d}^c \frac{dy}{\sqrt{(c^2 - y^2)(y^2 - d^2)}}.
\]
Let make change of variable \( y = sd \) in the \( I_1(d) \). Then we see that \( I_1(d) \) is increasing function on \( d \). Indeed,

\[
I_1(d) = \frac{1}{c} \int_0^1 \frac{ds}{\sqrt{(1 - s^2) \left( 1 - \frac{d^2}{c^2} s^2 \right)}}.
\]

Now we make change of variable \( y = d + (c-d)s \) in the integral \( I_0(d) \). Then we see that \( I_0(d) \) is decreasing function on \( d \). Indeed,

\[
I_0(d) = \int_0^1 \frac{ds}{\sqrt{(c(1 + s) + d(1 - s))(cs + d(2 - s))}} s(1 - s).
\]

Then the statements of lemma is evident.

5.1 Proof of lemma 2.11

Proof. To show that this system has a unique solution let us define the function

\[
F(\mu, d) = \int_0^1 \left( \mu^2 - \lambda^2 d^2 \right) \sqrt{\frac{1 - \lambda^2}{c^2 - \lambda^2 d^2}} d\lambda.
\]

It is easy to see that there exists a function \( \mu = \mu(d) \) such that \( F(\mu(d), d) \equiv 0 \) and \( 0 < \mu(d) < d \). Moreover, \( \mu(d) \) is strictly increasing as \( d \in [0, c] \). Indeed, one can check that \( F(\mu, d) \) is strictly increasing in \( \mu \) and is strictly decreasing in \( d \) as \( 0 < \mu < d < c \). Now if \( 0 < d_1 < d_2 < c \) then \( F(\mu(d_1), d_1) = 0 = F(\mu(d_2), d_2) < F(\mu(d_2), d_1) \), that is \( F(\mu(d_1), d_1) \) \( < F(\mu(d_2), d_1) \) and, hence, \( \mu(d_1) < \mu(d_2) \). Furthermore \( \mu(d) \) is continuous function that follows from the representation:

\[
\mu^2(d) = \int_0^1 \lambda^2 d^2 \sqrt{\frac{1 - \lambda^2}{c^2 - \lambda^2 d^2}} d\lambda = \int_0^1 \sqrt{\frac{1 - \lambda^2}{c^2 - \lambda^2 d^2}} d\lambda,
\]

which is equivalent to equality (2.10). Equation (5.27) yields that \( \mu(0) = 0 \) and \( \mu(c) = \frac{c}{\sqrt{3}} \). Hence \( \mu^2(d) + \frac{d^2}{2} \) vary over segment \( \left[ 0, \frac{5c^2}{6} \right] \) as \( d \) vary over segment \( [0, c] \). So for any \( \xi \in \left( -\frac{c^2}{2}, \frac{c^2}{3} \right) \) there exists a single \( d \in (0, c) \) such that (2.10) and (2.11) are fulfilled. Let us denote this function by

\[
f : \left[ \frac{-c^2}{2}, \frac{c^2}{3} \right] \to [0, c], \quad f(\xi) = d.
\]

Moreover, \( f \) is increasing function. And therefore there exists an inverse function \( f^{-1} : \left[ \frac{-c^2}{2}, \frac{c^2}{3} \right] \to [0, c] \).

We can now rewrite equality (2.11) as follows:

\[
\frac{c^2}{2} + \xi = \mu^2(f(\xi)) + \frac{f^2(\xi)}{2}.
\]

The last equality and the fact that \( \mu(\cdot) \) is increasing imply that \( f(\cdot) \) is continuous function. So, system (2.10) (2.11) has a unique solution \( \mu = \mu(f(\xi)), d = f(\xi) \), and \( \mu(f(\xi)), f(\xi) \) are continuous and strictly increasing functions.
5.2 Asymptotic expansions of $\tau^*$, $B_g$, $\Delta$

In this section we don’t always write dependence of some functions on their argument to simplify the text.

Let us transform expression for $\Delta(d)$ (2.14):

$$\Delta(d) = \frac{id}{\int \log \left( a_+ (k) a_- (k) \right) \text{d}k}{w_+(k,d)} = \frac{id}{\int \log \left( a_+ (k) a_- (k) \right) \text{d}k}{w_+(k,d)} =: - \frac{I_2(d)}{I_1(d)} \quad (5.28)$$

5.3 Expansion of $I_2$ in $\eta$

First we will treat $I_2(d)$. As for $k \in (id, ic)$ $w_+(k,d) = i |w(k,d)|$, then

$$I_2(d) = |k = iy| = \int_0^{id} \frac{\log \left( a_+ (iy) a_- (iy) \right) \text{d}y}{\sqrt{(c^2 - y^2)(y^2 - d^2)}} = |y = d + (c - d)s| =$$

$$= (c - d) \int_0^1 \frac{\log \left( a_+ (i(d + (c - d)s)) a_- (i(d + (c - d)s)) \right) \text{d}s}{\sqrt{(c - d)(1 - s)(c + d + (c - d)s)(c - d)s(2d + (c - d)s)}} =$$

$$= \int_0^1 \frac{\log \left( a_+ (i(d + (c - d)s)) a_- (i(d + (c - d)s)) \right) \text{d}s}{\sqrt{s(1 - s)(c + d + (c - d)s)(2d + (c - d)s)}}.$$

and

$$I_2(d(\eta)) = \int_0^1 \frac{\log \left( a_+ (i(c(1 - \eta) + c\eta s)) a_- (i(c(1 - \eta) + c\eta s)) \right) \text{d}s}{\sqrt{s(1 - s)(c(2 - \eta) + c\eta s)(2c(1 - \eta) + c\eta s)}} =$$

$$= \frac{1}{c} \int_0^1 \frac{\log \left( a_+ (i(c(1 - \eta) + c\eta s)) a_- (i(c(1 - \eta) + c\eta s)) \right) \text{d}s}{\sqrt{s(1 - s)(2 - \eta + \eta s)(2 - 2\eta + \eta s)}}.$$

Now treat $\log a_+ a_-$. For $k \in (ic, id)$

$$\kappa_- (k) = |\kappa(k)| e^{\frac{\pi i}{4}},$$

$$\kappa_+ (k) = |\kappa(k)| e^{\frac{-\pi i}{4}}.$$
(see the definition of \( \kappa(k) \)) and then (see the definition of \( a(k) \))

\[
a_-(k) = \frac{1}{2} \left( |\kappa(k)| - \frac{i}{|\kappa(k)|} \right) \pi i e^{\frac{\pi i}{4}},
\]

\[
a_+(k) = \frac{1}{2} \left( |\kappa(k)| + \frac{i}{|\kappa(k)|} \right) e^{-\frac{\pi i}{4}},
\]

\[
a_-(k)a_+(k) = \frac{1}{4} \left( |\kappa(k)|^2 + \frac{1}{|\kappa(k)|^2} \right) = |k = iy| = \frac{1}{4} \left( \sqrt{\frac{c+y}{c-y}} + \sqrt{\frac{c-y}{c+y}} \right) = \frac{1}{4} \sqrt{\frac{c+y}{c-y}} \left( 1 + \frac{c-y}{c+y} \right) = \frac{2\sqrt{c^2 - y^2}}{c} = \frac{|y = d + (c-d)s|}{2\sqrt{(c-d)(1-s)(c+d + (c-d)s)}} = \frac{c}{2\sqrt{c\eta(1-s)(c(2 - \eta) + c\eta s)}} = \frac{1}{2\sqrt{\eta(1-s)(2 - \eta + \eta s)}}.
\]

Then

\[
\log (a_-(k)a_+(k)) = \log \frac{1}{2\sqrt{\eta(1-s)(2 - \eta + \eta s)}} = \frac{1}{2} \log \eta - \frac{1}{2} \log (1-s) - \frac{1}{2} \log (2 - \eta + \eta s) - \log 2.
\]

Then the integral \( I_2(d(\eta)) \) is equal to

\[
I_2(d(\eta)) = \int_0^1 \frac{-\frac{1}{2} \log \eta - \frac{1}{2} \log (1-s) - \frac{1}{2} \log (2 - \eta + \eta s) - \log 2}{c\sqrt{s(1-s)}\sqrt{(2 - \eta + \eta s)(2 - 2\eta + \eta s)}} ds = \int_0^1 \frac{-1}{2c} \log \sqrt{s(1-s)}\sqrt{(2 - \eta + \eta s)(2 - 2\eta + \eta s)} ds
\]

\[
+ \int_0^1 \frac{-\frac{1}{2} \log (2 - \eta + \eta s) - \log 2}{c\sqrt{s(1-s)}\sqrt{(2 - \eta + \eta s)(2 - 2\eta + \eta s)}} ds = \int_0^1 \frac{-1}{2c} \log \eta \frac{1}{\sqrt{s(1-s)}} ds + \int_0^1 \frac{-1}{2c} \log (1-s) \frac{1}{\sqrt{s(1-s)}} ds + \int_0^1 -\frac{3}{4} \log 2 + O(\eta) \frac{1}{c\sqrt{s(1-s)}} ds
\]

\[
= -\frac{1}{2c} \log \eta \left( \frac{\pi}{2} + O(\eta) \right) - \frac{1}{2c} (-\pi \log 2 + O(\eta)) + \frac{1}{c} \left( -\frac{3\pi \log 2}{4} + O(\eta) \right) = \frac{1}{2c} \log \eta \left( \frac{\pi}{2} + O(\eta) \right) + \frac{1}{2c} \left( -\pi \log 2 + O(\eta) \right) + \frac{1}{c} \left( -\frac{3\pi \log 2}{4} + O(\eta) \right).
\]
\[-\frac{\pi}{4c} \log \eta - \frac{\pi}{4c} \log 2 + O(\eta) = -\frac{\pi}{4c} \log 2\eta + O(\eta).\]

So,

\[I_2(d(\eta)) = -\frac{\pi}{4c} \log 2 \eta + O(\eta).\]  

(5.29)

The expansion of the \(I_1\) is more difficult than expansion of the \(I_2\) and is based on a \(\Theta\)-function identity.

### 5.4 Expansion of \(\tau^*(d(\eta))\) in \(\eta\)

We know (by using Poisson summation formula), that

\[
\Theta(z|\tau) = \Theta\left(\frac{2\pi iz}{\tau} \left| \frac{4\pi^2}{\tau}\right.\right) \sqrt{\frac{2\pi}{-\tau}} \left(\exp -\frac{z^2}{2\tau}\right) = \Theta\left(\frac{\tau^* z}{2\pi i}|\tau^*\right) \sqrt{\frac{-\tau^*}{2\pi}} \left(\exp -\frac{z^2 \tau^*}{8\pi^2}\right),
\]

where \(\tau^* = \frac{4\pi^2}{\tau}\) and from [17] we know that

\[
\frac{\Theta(0|\tau(d))}{\Theta(\pi i|\tau(d))} = \sqrt{\frac{c + d}{c - d}}
\]

(see formula (4.34) in [17]).

Let us recall that \(\eta(d) = 1 - \frac{d}{c}\). Then, by using the Poisson summation formula \(3.15\), we get

\[
\sqrt{\frac{2 - \eta(d)}{\eta(d)}} = \sqrt{\frac{c + d}{c - d}} = \frac{\Theta(0|\tau(d))}{\Theta(\pi i|\tau(d))} = \frac{\Theta(0|\tau^*(d))}{\Theta\left(\frac{\tau^*(d)}{2} |\tau^*(d)\right)} \exp \frac{\tau^*(d)}{8}.
\]

Now we use the inverse function \(h(.)\) of the function \(\tau^*\) (see lemma [5.1]) and rewrite the last formula.

\[
\sqrt{\frac{2 - \eta(h(\tau^*))}{\eta(h(\tau^*))}} = \frac{\Theta(0|\tau^*)}{\Theta\left(\frac{\tau^*}{2} |\tau^*\right)} \exp \frac{\tau^*}{8}.
\]

Since

\[
\Theta(z|\tau^*) = \sum_{m=-\infty}^{\infty} \exp \left\{ \frac{1}{2} \tau^* m^2 + zm \right\},
\]

we get

\[
\sqrt{\frac{2 - \eta(h(\tau^*))}{\eta(h(\tau^*))}} = \frac{\Theta(0|\tau^*)}{\Theta\left(\frac{\tau^*}{2} |\tau^*\right)} \exp \frac{\tau^*}{8}.
\]
then

\[
\eta(h(\tau^*)) = 2 \left( \frac{\Theta^2 (0|\tau^*)}{\Theta^2 (\frac{\tau^*}{2}|\tau^*) \exp \frac{\tau^*}{4}} + 1 \right)^{-1} = 2 \left( \frac{\sum_{m=-\infty}^{\infty} \exp \left( \frac{1}{2} \tau^* m^2 \right)}{(\sum_{m=-\infty}^{\infty} \exp \left( \frac{1}{2} \tau^* m + 1 \right))^2 \exp \frac{\tau^*}{4}} + 1 \right)^{-1}
\]

\[
= \left| \zeta = e^{\frac{\tau^*}{4}} \right| = 2 \left( \frac{(1 + O \left( e^{\tau^*} \right))^2}{4e^{\tau^*} (1 + O (e^{\tau^*}))^2} + 1 \right)^{-1} = 8e^{\tau^*} + O \left( e^{\tau^*} \right),
\]

Now we get

\[
\eta = 8e^{\tau^*} + O \left( e^{\tau^*} \right),
\]

\[
e^{\tau^*} = \frac{\eta}{8} + O (\eta^2),
\]

\[
\tau^* = \tau^*(d(\eta)) = -4 \log \frac{8}{\eta} + O (\eta). \quad (5.30)
\]

Also we have

\[
\sqrt{c^2 - h^2(\tau^*)} e^{-\frac{-\tau^*}{8}} = \sqrt{c^2 - c^2 (1 - \eta(h(\tau^*)))^2} e^{-\frac{-\tau^*}{8}} = 4c \left( 1 + O \left( e^{\tau^*} \right) \right). \quad (5.31)
\]

### 5.5 Expansion of \( I_1 \) in \( \eta \)

In the other way, (see the proof of lemma 5.1),

\[
\tau^*(d) = -4\pi \frac{I_1(d)}{I_0(d)},
\]

and then

\[
I_1(d) = \frac{-\tau^*(d) I_0(d)}{4\pi}. \quad (5.32)
\]

Now we get the asymptotic expansion of the \( I_0 \).

\[
I_0(d) = \int_{d}^{c} \frac{dy}{\sqrt{(c^2 - y^2)(y^2 - d^2)}} = |y = d + (c - d)s| =
\]

\[
= (c - d) \int_{0}^{1} \frac{ds}{\sqrt{(c - d)(1 - s)(c + d + (c - d)s)(c - d)s(2d + (c - d)s)}} =
\]

\[
= \int_{0}^{1} \frac{ds}{\sqrt{s(1 - s)s(2d + (c - d)s)}}.
\]

17
Let us make interchange of variables $d = d(\eta) = c(1 - \eta)$ in the last integral. Then

$$I_0(d(\eta)) = \frac{1}{c} \int_0^1 \frac{ds}{\sqrt{s(1-s)}\sqrt{(c+\eta s)(2-2\eta+\eta s)}} = \int_0^1 \frac{\left(\frac{1}{2} + O(\eta)\right)\,ds}{\sqrt{s(1-s)}\sqrt{(2-\eta+\eta s)(2-2\eta+\eta s)}} = \frac{\pi}{2c} + O(\eta).$$

And then by virtue of (5.32) and (5.30) we conclude that

$$I_1(d(\eta)) = \left(\frac{1}{2c} + O(\eta)\right) \left(\log \frac{8}{\eta} + O(\eta)\right) = \frac{1}{2c} \log \frac{8}{\eta} + O(\eta \log \eta). \quad (5.33)$$

**Remark.** Although in (5.33) we get only the first member of expansion of $I_1(kc) = \frac{1}{c} \int_0^1 \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}}$ as $k \to 1$ (here $k = \frac{d}{c}$), but in this way we can get as much as desired members of this expansion. See also [28], problem 90.

### 5.6 Expansion of $\Delta(d(\eta))$

Finally, by virtue of (5.28), (5.33) and (5.29), we conclude, that

$$\Delta(d(\eta)) = \frac{\pi}{4c} \log \frac{2\eta}{\eta} + O(\eta) = -\frac{\pi}{2c} \log \frac{1}{\eta} - 2 + O(\eta)$$

$$= \frac{\pi}{2} \left(1 - \frac{4 \log 2}{\log \frac{1}{\eta}} + O \left(\frac{1}{\log^2 \eta}\right)\right).$$

and

$$\frac{1}{\pi} \Delta(d(\eta)) = -\frac{1}{2} \left(1 - \frac{4 \log 2}{\log \frac{1}{\eta}} + O \left(\frac{1}{\log^2 \eta}\right)\right). \quad (5.34)$$

In the following three paragraphs we get the expansion of $\mu$. As we know from (5.27),

$$\mu^2(d) = \int_0^d \frac{y^2 \sqrt{d^2 - y^2}}{\sqrt{c^2 - y^2}} \, dy + \int_0^d \frac{\sqrt{d^2 - y^2}}{\sqrt{c^2 - y^2}} \, dy.$$
And then
\[ \mu^2(d) = \frac{d^2}{\int_0^d \sqrt{c^2-y^2} dy} - \frac{d}{\int_0^d \sqrt{c^2-y^2} dy} = \frac{d^2}{\int_0^d \sqrt{c^2-y^2} dy} - \frac{d}{\int_0^d \sqrt{c^2-y^2} dy} =: d^2 - \frac{I_4(d)}{I_3(d)}. \] (5.35)

5.7 Expansion of \( I_3 \) in \( \eta \)
Let us first treat \( I_3 \):

\[ I_3(d) = \int_0^d \frac{\sqrt{d^2-y^2}}{\sqrt{c^2-y^2}} dy \]

or

\[ I_3(d(\eta)) = \int_0^c \frac{\sqrt{(1-\eta)^2-y^2}}{\sqrt{c^2-y^2}} dy, \] (5.36)

where \( d(\eta) = c(1-\eta) \).

Let us differentiate \( I_3(d(\eta)) \) in \( \eta \):

\[ (I_3 \circ d)'(\eta) = \int_0^c \frac{-c^2(1-\eta)}{\sqrt{c^2(1-\eta)^2-y^2} \sqrt{c^2-y^2}} dy = |\text{see (5.28)}| = -c^2(1-\eta)I_1(d(\eta)) = \]

\[ = |\text{see (5.33)}| = -c^2(1-\eta)\left( \frac{1}{2c} \log \frac{8}{\eta} + O(\eta \log \eta) \right) = -\frac{c}{2} \log \frac{8}{\eta} + O(\eta \log \eta). \]

Then

\[ I_3(d(\eta)) = I_3(d(0)) + \int_0^\eta (I_3 \circ d)'(\tilde{\eta}) d\tilde{\eta} = c - \frac{c}{2} \int_0^\eta \left( \log \frac{8}{\tilde{\eta}} + O(\tilde{\eta} \log \tilde{\eta}) \right) d\tilde{\eta} = c - \frac{c}{2} \eta \log \frac{8c}{\eta} + O(\eta^2 \log \eta). \] (5.37)

5.8 Expansion of \( I_4 \) in \( \eta \)

Now let us treat \( I_4 \).

\[ I_4(d) = \int_0^d \frac{(d^2-y^2)^{3/2}}{\sqrt{c^2-y^2}} dy \]

and

\[ I_4(d(\eta)) = \int_0^c \frac{(c^2(1-\eta)^2-y^2)^{3/2}}{\sqrt{c^2-y^2}} dy. \]
where \( d(\eta) = c(1 - \eta) \).

Let us differentiate \( I_4(d(\cdot)) \) in \( \eta \):

\[
\frac{dI_4 \circ d}{d\eta}(\eta) = \int_0^\eta -3c^2(1 - \eta) \frac{c^2(1 - \eta)^2 - y^2}{\sqrt{c^2 - y^2}} \, dy = |\text{see } \text{above}| = -3c^3(1 - \eta) = \frac{d}{d\eta}\left(\frac{c^2}{2}\right) = \frac{c^2}{2} - \frac{c^2}{2} \eta - \frac{c^2}{2} \eta^2 + \frac{c^2}{2} \eta^3 + O(\eta^4).
\]

Then

\[
I_4(d(\eta)) = I_4(d(0)) + \int_0^\eta (1 \circ d)'(\eta) d\eta = \frac{2}{3}c^3 - 3c^3 \int_0^\eta (1 + O(\eta \log \eta)) d\eta = \frac{2}{3}c^3 - 3c^3 \eta + O(\eta^2 \log \eta).
\]

### 5.9 Expansion of \( \mu \) in \( \eta \)

Finally, by virtue of (5.35) and the fact that \( d = d(\eta) = c(1 - \eta) \), we get that

\[
\mu^2(d) = d^2 - \frac{I_4(d)}{I_3(d)}.
\]

and

\[
\mu^2(d(\eta)) = c^2(1 - \eta)^2 - \frac{2}{3}c^3 - 3c^3 \eta + O(\eta^2 \log \eta)
\]

\[
= c^2(1 - \eta)^2 - \frac{2}{3}c^3 \frac{1 - \frac{9}{2} \eta + O(\eta^2 \log \eta)}{1 - \frac{1}{2} \eta \log \frac{8e}{\eta} + O(\eta^2 \log \eta)} = c^2(1 - \eta)^2 - \frac{2}{3}c^2 \left(1 - \frac{9}{2} \eta + \frac{1}{2} \eta \log \frac{8e}{\eta} + O(\eta^2 \log^2 \eta)\right) = c^2(1 - \eta)^2 + c^2 \left(\frac{2}{3} + 3\eta - \frac{1}{3} \eta \log \frac{8e}{\eta} + O(\eta^2 \log^2 \eta)\right) = c^2 \left(\frac{1}{3} + \eta - \frac{1}{3} \eta \log \frac{8e}{\eta} + O(\eta^2 \log^2 \eta)\right) = c^2 \left(\frac{1}{3} - \frac{1}{3} \eta \log \frac{8e}{\eta}^2 + O(\eta^2 \log^2 \eta)\right).
\]

So,

\[
\mu^2(d(\eta)) = c^2 \left(\frac{1}{3} - \frac{1}{3} \eta \log \frac{8e}{\eta^2} + O(\eta^2 \log^2 \eta)\right)
\]

and

\[
3\mu^2(d(\eta)) = \frac{8e}{\eta^2} + O(\eta^2 \log^2 \eta).
\]
Let us define new variable

\[ v = v(\xi) = 1 - \frac{3\xi}{c^2}. \]

and remember the definition of the function \( d = f(\xi) \) from paragraph (5.1).

Then

\[ v(f^{-1}(d)) = 1 - \frac{3}{c^2} \left( \mu^2(d) + \frac{d^2}{2} - \frac{c^2}{2} \right) \]

and

\[ v(f^{-1}(d(\eta))) = 1 - \frac{3}{c^2} \left( \mu^2(d(\eta)) + \frac{d^2(\eta)}{2} - \frac{c^2}{2} \right) = 1 - \left( 1 - \eta \log \frac{8}{\eta e^2} + O \left( \eta^2 \log^2 \eta \right) \right) - \]

\[ - \frac{3}{c^2} \left( \frac{d^2(\eta)}{2} - \frac{c^2}{2} \right) = \eta \log \frac{8}{\eta e^2} + \frac{3}{2} \left( 1 - \frac{d^2(\eta)}{c^2} \right) + O \left( \eta^2 \log^2 \eta \right) = \eta \log \frac{8e}{\eta} + O \left( \eta^2 \log^2 \eta \right), \]

or

\[ \frac{v}{8e} = \frac{\eta}{8e} \log \frac{8e}{\eta} + O \left( \eta^2 \log^2 \eta \right). \quad (5.40) \]

5.11 \( \eta \) in \( v \)

Let us note that if \( x = x(y) \) is invertible function in some neighborhood of zero and

\[ x(y) = y + O \left( y \right), y \to 0, \]

then

\[ y(x) = x + O \left( x \right), x \to 0. \]

We have \( x = \frac{v}{8e}, y = \frac{\eta}{8e} \log \frac{8e}{\eta} \). Then

\[ \frac{\eta}{8e} \log \frac{\eta}{8e} = \frac{-v}{8e} + O \left( v^2 \right), \quad v \to 0. \]

We have got a Lambert equation

\[ we^w = z, \quad w < 0, z < 0, \]

where

\[ w = \log \frac{\eta}{8e}, \quad z = \frac{-v}{8e} + O \left( v^2 \right). \quad (5.41) \]

Following the [?], we get that this equation has the inverse for \( w << 0 \), and this inverse has the following expansion:

\[ w = -L_1 - L_2 \frac{L_2}{L_1} + O \left( \frac{L_2^2}{L_1^2} \right), \quad z \to -0, \]

where

\[ L_1 = \log \frac{1}{z}, \]

\[ L_2 = \log \log \frac{1}{z}. \]
Then
\[
e^w = e^{-L_1} e^{-L_2} \exp \left( - \frac{L_2}{L_1} \right) \exp \left( O \left( \frac{L_2^2}{L_1^2} \right) \right), \quad z \to -0,
\]
\[
e^w = -\frac{z}{\log \frac{1}{z}} \left( 1 - \frac{\log \log \frac{1}{z}}{\log \frac{1}{z}} + O \left( \frac{\log^2 \log \frac{1}{z}}{\log \frac{1}{z}} \right) \right), \quad z \to -0.
\]

And by virtue of (5.41):
\[
\frac{\eta}{8e} = \frac{v}{8e} + O \left( \frac{v^2}{8e} \right) \left( 1 - \frac{\log \log \frac{8e}{v + O(v^2)}}{\log \frac{8e}{v + O(v^2)}} + O \left( \frac{\log^2 \log \frac{8e}{v + O(v^2)}}{\log \frac{8e}{v + O(v^2)}} \right) \right), \quad z \to -0,
\]
\[
\eta = \frac{v + O(v^2)}{8e} \left( 1 - \frac{\log \log \frac{8e}{v + O(v^2)}}{\log \frac{8e}{v + O(v^2)}} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right), \quad v \to +0.
\]

As
\[
\log \log \frac{8e}{v + O(v^2)} = \log \log \left( \frac{8e}{v} \left( 1 + O(v) \right) \right) = \log \left( \log \frac{8e}{v + O(v)} \right) =
\]
\[
= \log \left( \log \frac{8e}{v} \left( 1 + O \left( \frac{v}{\log v} \right) \right) \right) = \log \log \frac{8e}{v} + O \left( \frac{v}{\log v} \right),
\]
then
\[
\eta = \frac{v}{\log \frac{8e}{v}} \left( 1 - \frac{\log \log \frac{8e}{v + O(v^2)}}{\log \frac{8e}{v + O(v^2)}} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right) =
\]
\[
= \frac{v}{\log \frac{8e}{v}} \left( 1 - \frac{\log \frac{1}{v} \left( 1 + \frac{8e}{\log v} \right)}{\log \frac{1}{v} \left( 1 + \frac{8e}{\log v} \right)} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right) =
\]
\[
= \frac{v}{\log \frac{8e}{v}} \left( 1 - \frac{\log 8e + \log \frac{1}{v} + \frac{8e}{\log v} \log \frac{1}{v}}{\log \frac{1}{v}} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right) =
\]
\[
= \frac{v}{\log \frac{8e}{v}} \left( 1 - \frac{\log 8e + \log \frac{1}{v} + \frac{8e}{\log v} \log \frac{1}{v}}{\log \frac{1}{v}} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right), \quad v \to +0.
\]
and so we get that

\[ \eta = \frac{v}{\log \frac{1}{v}} \left( 1 - \frac{\log 8e + \log \log \frac{1}{v}}{\log \frac{1}{v}} + O \left( \frac{\log^2 \log \frac{1}{v}}{\log \frac{1}{v}} \right) \right), \quad v \to +0. \] (5.42)

### 5.12 Expansion of \( B_g \)

As we know, (see (2.12),

\[ B_g(d) = 24 \int_{id}^{ic} \frac{(k^2 + \mu^2(d))(k^2 + d^2)dk}{w(k,d)}. \]

We will get the asymptotic expansion of \( B_g \) as \( d \) is tend to \( c \).

Let us do interchange \( k = iy, y \in (d, c) \) in the last integral:

\[ B_g(d) = 24(c - d) \int_{0}^{1} \sqrt{\frac{(d + (c - d)s)^2 - \mu^2(d)}{c^2 - y^2}} \frac{(c - d)s(2d + (c - d)s)}{(c - d)(1 - s)(c + d + (c - d)s)} ds = \]

\[ = 24(c - d) \int_{0}^{1} \sqrt{\frac{s}{1 - s} \frac{(d + (c - d)s)^2 - \mu^2(d)}{c + d + (c - d)s}} ds \]

Now we make integrchange \( d = d(\eta) = c(1 - \eta) \) and recall (see 5.39), that

\[ \mu^2(d(\eta)) = \frac{c^2}{3} \left( 1 + O(\eta \log \eta) \right). \]

Then

\[ B_g(d(\eta)) = 24c\eta \int_{0}^{1} \sqrt{\frac{s}{1 - s} \left( \frac{c^2(1 - \eta + \eta s)^2}{c(2 - \eta + c\eta s)} - \frac{1}{3} c^2 \left( 1 + O(\eta \log \eta) \right) \right)} ds = \]

\[ = 24c\eta \int_{0}^{1} \sqrt{\frac{s}{1 - s} \left( \frac{2}{3} c^2 \left( 1 + O(\eta \log \eta) \right) \right)} (1 + O(\eta)) ds = \]

\[ = 24c\eta \int_{0}^{1} \sqrt{\frac{s}{1 - s} \left( \frac{2}{3} c^2 \left( 1 + O(\eta \log \eta) \right) \right)} ds = 16c^3\eta \int_{0}^{1} \sqrt{\frac{s}{1 - s} (1 + O(\eta \log \eta))} ds = \]

\[ = 8\pi c^3\eta(1 + O(\eta \log \eta)), \quad \eta \to 0. \]

### Acknowledgements

The author would like to express his gratitude to V.P. Kotlyarov for the statement of the problem and valuable advices, and to I. E. Egorova and D. G. Shepelsky for useful discussions.

The research for this paper was supported in part by the Akhiezer foundation and by scholarship of National Academy of Sciences of Ukraine.
References

[1] Gurevich A.V., Pitaevskii L.P., Decay of Initial Discontinuity in the Korteweg-de Vries Equation, JETP Letters, Vol. 17, Iss. 5, p. 193, (1973)

[2] Bikbaev R. F. and Novokshenov V. Yu. mKdV equations with finite-gap boundary conditions and one-parameter Whitham solutions // In: Asymptotic Properties of Solutions of Differential Equations [in Russian]. — 1989. — P. 3—35.

[3] Bikbaev, R. F. Structure of a shock wave in the theory of the Korteweg-de Vries equation. Phys. Lett. A 141 (1989), no. 5-6, 289–293.

[4] Bikbaev, R. F.; Sharipov, R. A. The asymptotic behavior, as $t \to \infty$, of the solution of the Cauchy problem for the Korteweg-de Vries equation in a class of potentials with finite-gap behavior as $x \to \pm \infty$. (Russian) Teoret. Mat. Fiz. 78 (1989), no. 3, 345–356; translation in Theoret. and Math. Phys. 78 (1989), no. 3, 244–252

[5] Bikbaev, R. F. The Korteweg-de Vries equation with finite-gap boundary conditions, and Whitham deformations of Riemann surfaces. (Russian) Funktsional. Anal. i Prilozhen. 23 (1989), no. 4, 1–10, 96; translation in Funct. Anal. Appl. 23 (1989), no. 4, 257–266 (1990)

[6] Bikbaev, R. F. The influence of viscosity on the structure of shock waves in the MKdV model. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 199 (1992), Voprosy Kvant. Teor. Polya Statist. Fiz. 11, 37–42, 184; translation in J. Math. Sci. 77 (1995), no. 2, 3042–3045

[7] Bikbaev, R. F. Complex Whitham deformations in problems with "integrable instability". (Russian) Teoret. Mat. Fiz. 104 (1995), no. 3, 393–419; translation in Theoret. and Math. Phys. 104 (1995), no. 3, 1078–1097 (1996)

[8] Bikbaev, R. F. Modulational instability stabilization via complex Whitham deformations: nonlinear Schrodinger equation. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 215 (1994), Differentialnaya Geom. Gruppy Li i Mekh. 14, 65–76, 310; translation in J. Math. Sci. (New York) 85 (1997), no. 1, 1596–1604

[9] Boutet de Monvel A. and Kotlyarov V. P. Focusing nonlinear Schrodinger equation on the quarter plane with time-periodic boundary condition: a Riemann-Hilbert approach / J. Inst. Math. Jussieu. — 2007. — Vol. 6. — № 4. — P. 579–611.

[10] Boutet de Monvel A., Its A. R. and Kotlyarov V. P. Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition / C. R. Math. Acad. Sci. Paris. — 2007. — Vol. 345. — № 11. — P. 615–620.

[11] Boutet de Monvel A., Its A. R. and Kotlyarov V. P. Long – time asymptotics for the focusing NLS equation with time – periodic boundary condition on the half – line / Comm. Math. Phys. — 2009. — Vol. 290. — № 2. — P. 479–522.

[12] Boutet de Monvel A., Kotlyarov V. P. and Shepelsky D. G. Focusing NLS equation: long-time dynamics of step-like initial data / International Mathematics Research Notices. — 2011. — Vol. 7. — P. 1613–1653.

[13] Buckingham R. Long-time asymptotics of the nonlinear Schrodinger equation shock problem / R. Buckingham, S. Venakides // Comm. Pure Appl. Math. — 2007. — Vol. 60. — № 9. — P. 1349–1414.

[14] I. Egorova, Z. Gladka, V. Kotlyarov, G. Teschl. Long-Time Asymptotics for the Korteweg-de Vries Equation with Steplike Initial Data, 2012, arXiv:1210.7434 [nlin.SI]
[15] Khruslov, E.Ya.: Splitting of an initial step-like perturbation for the KdV equation, Letters to JETP 21, 4 (1975) 469-472

[16] E. Ya. Khruslov, “Asymptotics of the solution of the Cauchy problem for the Korteweg-de Vries equation with initial data of step type”, Mat. Sb. (N.S.), 99(141):2 (1976), 261–281

[17] V.P. Kotlyarov and A.A. Minakov, Journal of Mathematical Physics (Vol.51, Issue 9)(2010)

[18] E.Ya. Khruslov and V.P. Kotlyarov, Asymptotic solitons of the modified Korteweg-de Vries equation. Inverse problems 5 (1989), No. 6, 1075-1088.

[19] Khruslov, E. Ya.; Kotlyarov, V. P. Soliton asymptotics of nondecreasing solutions of nonlinear completely integrable evolution equations. Spectral operator theory and related topics, 129–180, Adv. Soviet Math., 19, Amer. Math. Soc., Providence, RI, 1994.

[20] Khruslov, Eugene; Kotlyarov, Vladimir Generation of asymptotic solitons in an integrable model of stimulated Raman scattering by periodic boundary data. Mat. Fiz. Anal. Geom. 10 (2003), no. 3, 366–384.

[21] Moskovchenko E. A. and Kotlyarov V. P. A new Riemann – Hilbert problem in a model of stimulated Raman scattering / J.Phys.A.: Math. Gen. — 2006. — Vol. 39. — P. 14591–14610.

[22] Moskovchenko E. A. Simple periodic boundary data and Riemann – Hilbert problem for integrable model of the stimulated Raman scattering / Journal of mathematical physics, analysis, geometry. — 2009. — Vol. 5. — № 1. — P. 82–103.

[23] Moskovchenko E. A. and Kotlyarov V. P. Periodic boundary data for an integrable model of stimulated Raman scattering: long-time asymptotic behavior / Journal of Physics A: Mathematical and Theoretical. — 2010. —Vol. 43. — № 5. — 31p.

[24] V.Yu.Novokshenov, J.Math.Sciences 125/5, 717

[25] Novoksenov, V. Ju. Asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for a nonlinear Schrodinger equation. (Russian) Dokl. Akad. Nauk SSSR 251 (1980), no. 4, 799–802.

[26] V.Yu.Novokshenov Asymptotic Formulae for the Solutions of the System of Nonlinear Schrodinger Equations, Uspekhi Matem. Nauk, 37 , N 2, p.215-216 (1982).

[27] V.Yu.Novokshenov Asymptotics as $t \to \infty$ of the Solution to a Two-Dimentional Generalisation of the Toda Lattice, Doklady AN SSSR, 265 , N 6, p.1320-1324 (1982) (Soviet Math. Dokl. 26, N 1, 264-268 (1982).)

[28] Polya, George; Szego, Gabor Problems and theorems in analysis. I. Series, integral calculus, theory of functions. Translated from the German by Dorothee Aeppli. Reprint of the 1978 English translation. Classics in Mathematics. Springer-Verlag, Berlin, 1998. xx+389 pp. ISBN: 3-540-63640-4 problem 90.