FINITENESS PROPERTIES OF AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. Affine Deligne-Lusztig varieties are closely related to the special fibre of Newton strata in the reduction of Shimura varieties or of moduli spaces of $G$-shtukas. In almost all cases, they are not quasi-compact. In this note we prove basic finiteness properties of affine Deligne-Lusztig varieties under minimal assumptions on the associated group. We show that affine Deligne-Lusztig varieties are locally of finite type, and prove a global finiteness result related to the natural group action. Similar results have previously been known for special situations.

1. Introduction

Let $F$ be a local field, $O_F$ its ring of integers, and $k_F = F_q$ its residue field, a finite field of characteristic $p$. We denote by $L$ the completion of the maximal unramified extension of $F$, and by $O_L$ its ring of integers. Then the residue field $k$ of $L$ is an algebraic closure of $F_q$. We denote by $\bar{\epsilon}$ a uniformizer of $F$, which is then also a uniformizer of $L$. Let $\sigma$ be the Frobenius of $k$ over $k_F$ and also of $L$ over $F$. We denote by $I$ the inertia group of $F$.

We consider a smooth affine group scheme $G$ over $O_F$ with reductive generic fibre. Let $P = G(O_L)$ and let $G = G_{k_F}$. We denote by $F_\ell G$ the base change to $k$ of the affine flag variety (over $k_F$) as in [PR08, § 1.c] and [BS17, Def. 9.4]. In particular, $F_\ell G$ is a sheaf on the fpqc-site of $k$-schemes (char $F = p$) resp. of perfect $k$-schemes (char $F = 0$) with $F_\ell G(k) = G(L)/P$, which is representable by an inductive limit of finite type schemes (char $F = p$) resp. of perfectly of finite type schemes (char $F = 0$). Hence we can define an underlying topological space of $F_\ell G$, which is Jacobson. Being a base change from $k_F$, we have an action of $\sigma$ on $F_\ell G$.

To define affine Deligne-Lusztig varieties we fix an element $b \in G(L)$ and a locally closed subscheme $Z$ of the loop group $LG$ which is stable under $P$-$\sigma$-conjugation.

Then for every $Z$ there is a natural action of $J_b$ on $X_Z(b)$ given by left multiplication. Our main result is

**Theorem 1.1.** Assume in addition that $Z$ is bounded.

1. The functor $X_Z(b)$ defines a locally closed reduced sub-indscheme $X_Z(b)$ of $F_\ell G$.

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(2) \(X_Z(b)\) is a scheme which is locally of finite type in the case that \(\text{char } F = p\) and locally perfectly of finite type in the case \(\text{char } F = 0\).

(3) The action of \(J_0(F)\) on the set of irreducible components of \(X_Z(b)\) has finitely many orbits.

Here, boundedness is defined in Section 3.

The first assertion follows easily from the definitions: Consider the functor \(\tilde{X}_Z(b)\) on reduced \(k\)-schemes resp. reduced perfect \(k\)-schemes with

\[
\tilde{X}_Z(b)(S) = \{ g \in LG(S) \mid g^{-1}b \sigma(g_x) \in Z(\kappa_x) \text{ for every geometric point } x \in S \}.
\]

Then \(\tilde{X}_Z(b)\) is the inverse image of \(Z\) under the morphism \(LG \to LG\) with \(g \mapsto g^{-1} b \sigma(g)\). Since \(Z\) is locally closed, also \(\tilde{X}_Z(b)\) defines a locally closed reduced sub-scheme of \(LG\). Furthermore, \(X_Z(b)\) is the image of \(\tilde{X}_Z(b)\) under the quotient map \(LG \to \mathcal{F}_L\), which is an \(L^+G\)-torsor. Hence it is again a locally closed sub-scheme.

The main tool to prove the other, main assertions of this theorem is to relate the claimed finiteness statements to finiteness properties of certain subsets of the extended Bruhat-Tits building of \(G\).

For the particular case of affine Deligne-Lusztig varieties arising as the underlying reduced subscheme of a Rapoport-Zink moduli space of \(p\)-divisible groups with additional structure, questions as in Theorem 1.1 have been considered by several people. A recent general theorem along these lines is shown by Mieda \cite{Mie}.

In the case where \(\mathcal{G}\) is reductive over \(OF\) and \(Z\) is a single \(P\)-double coset, a complete description of the set of \(J_0(F)\)-orbits of irreducible components of \(X_Z(b)\) is known. The present work was motivated by our own results in this direction in \cite{HV18}. Recently, complete descriptions were given by Zhou and Zhu \cite{ZZ} and by Nie \cite{Nie}.

2. Reduction to the parahoric case

As a first step, we reduce to the case that \(\mathcal{G}\) is a parahoric group scheme. While most assertions in the following still hold true in the general setup, the assertion that \(\mathcal{G}\) is parahoric will simplify the proofs and the notation.

By the fixed point theorem \cite[Tit79]{} 2.3.1] the group \(P \times \langle \sigma \rangle\) has a fixed point \(x\) in the Bruhat-Tits building of \(G_L\). Thus the stabiliser \(P_x\) of \(x\) is \(\sigma\)-stable and contains \(P\). We denote by \(\mathcal{G}_x\) the corresponding group scheme over \(OF\) in the sense of Bruhat and Tits.

**Lemma 2.1.** The fpqc quotient \(L^+\mathcal{G}_x/L^+\mathcal{G}\) is representable by a finitely presented (resp. perfectly finitely presented) scheme.

**Proof.** We denote \(P_{x,n} := \ker(\mathcal{G}_x(OL) \to \mathcal{G}_x(OL/e^n))\). Since the \(P_{x,n}\) form a neighbourhood basis of the unit element in \(G(L)\) we have \(P_{x,n} \subset P\) for some \(n\). Thus the positive loop group \(L^+P\) contains the kernel of the reduction map into the truncated positive loop group \(L^+_nP \to L^+_nP\). Indeed, we have just shown that this is true on geometric points and the kernel is an infinite dimensional affine space by Greenberg’s structure theorem \cite[p. 263]{Gre63}, thus in particular reduced. Hence we get \(L^+\mathcal{G}_x/L^+\mathcal{G} \cong L^+_n\mathcal{G}_x/L^+_n\mathcal{G}\). Since the latter is a quotient of linear algebraic groups over \(k_F\), the claim follows.

Since \(LG \to \mathcal{F}_L\) is an \(L^+\mathcal{G}\)-torsor, we get that \(\mathcal{F}_L\) is étale locally isomorphic to \(\mathcal{F}_L \times L^+\mathcal{G}_x/L^+\mathcal{G}\). In particular, the canonical projection \(\mathcal{F}_L \to \mathcal{F}_{L\mathcal{G}_x}\) is relatively representable and of finite type. Thus Theorem 1.1 holds true for \(\mathcal{G}\) if and only if it is true for \(\mathcal{G}_x\), as it is enough to prove the theorem after enlarging \(Z\) so that becomes stable under \(P_x\)-\(\sigma\)-conjugation. Let \(\mathcal{G}_x \subset \mathcal{G}_x\) be the parahoric group
scheme associated to \( x \). Repeating the argument above, we see that it suffices to prove Theorem 1.1 for \( \mathcal{G}^e \) instead of \( \mathcal{G} \).

Therefore we can (and will) assume from now on that \( \mathcal{G} \) is a parahoric group scheme.

3. Some properties of Bruhat-Tits buildings

We consider the following group theoretical setup. Let \( S_0 \subset G \) be a maximal \( \mathbb{L} \)-split torus defined over \( F \), let \( T_0 \) be its centraliser and let \( N_0 \) be the normaliser of \( T_0 \) in \( G \). Then \( T_0 \) is a torus because \( G \) is quasi-split over \( L \). Thus \( W = N_0(L)/T_0(L) \) is the relative Weyl group of \( G \) over \( L \). We denote by \( P_{T_0} \) the unique parahoric subgroup of \( T_0 \). The extended affine Weyl group is defined as

\[ \tilde{W} := N_0(L)/P_{T_0} \cong X_*(T)_{\mathbb{F}} \rtimes W. \]

We may choose \( S_0 \) such that \( P \) stabilises a facet in the apartment of \( S_0 \) and denote \( \tilde{W}^P = (N_0(L) \cap P)/P_{T_0} \subset \tilde{W} \). By Landvogt \cite[Appendix, Prop. 9]{LAN} the embedding \( N_0 \hookrightarrow G \) induces a bijection

\[ \tilde{W}^P \backslash \tilde{W} / \tilde{W}^P \xrightarrow{1:1} P \backslash G(L) / P. \]

We call a subset \( \tilde{X} \subset G(L) \) bounded if it is contained in a finite union of \( P \)-double cosets. The bounded subsets form a bornology on \( G(L) \), which does not depend on the choice of \( P \).

Let \( \mathcal{B}^c(G, L) \) be the extended Bruhat-Tits building of \( G \) over \( L \), that is

\[ \mathcal{B}^c(G, L) = B(G, L) \times V_0(G, L) \]

where \( B(G, L) \) is the “usual” Bruhat-Tits building of \( G \) and \( V_0(G, L) := X_0(G_{\mathbb{A}^b})_L \cong X_0(Z(G))_{\mathbb{F}^L} \) with \( Z(G) \) denoting the center of \( G \). The extended apartment \( \mathcal{A}^c(S, G) \subset \mathcal{B}^c(G, L) \) of a maximal \( \mathbb{L} \)-split torus \( S \) is defined as \( \mathcal{A}(S, G; L) \times V_0(G, L) \) where \( \mathcal{A}(S, G; L) \) denotes the apartment of \( S \). We recall from Landvogt \cite[§ 1.3]{LAN} that \( \mathcal{B}^c(G, L) \) is a polysimplicial complex with a metric \( d \) and a \( G(L) \rtimes (\sigma) \)-action by isometries. Moreover, one can canonically identify \( \mathcal{B}^c(G, F) \) with the set of \( \sigma \)-invariants \( \mathcal{B}^c(G, L)^{\sigma} \).

We consider the canonical map

\[ i : G(L) \to \text{Isom}(\mathcal{B}^c(G, L)), \]

where \( \text{Isom}(\mathcal{B}^c(G, L)) \) denotes the space of self-isometries of \( \mathcal{B}^c(G, L) \). A set \( M \subset \text{Isom}(\mathcal{B}^c(G, L)) \) is called bounded if for some (or equivalently every) non-empty bounded set \( A \subset \mathcal{B}^c(G, L) \) the set \( \{ f(x) \mid f \in M, x \in A \} \subset \mathcal{B}^c(G, L) \) is bounded.

We have the following statement about the compatibility of bornological structure.

**Proposition 3.2** \cite[Prop. 4.2.19]{BT}. A subset \( \tilde{X} \subset G(L) \) is bounded if and only if its image under \( i \) is.

We consider the following maps between extended Bruhat-Tits buildings. Let \( f : G \to G' \) be a morphism of reductive \( F \)-groups. A \( G(L) \)-equivariant map \( g : \mathcal{B}^c(G, L) \to \mathcal{B}^c(G', L) \) is called toral if for every maximal \( \mathbb{L} \)-split torus \( S \subset G_L \) there exists a maximal \( \mathbb{L} \)-split torus \( S' \subset G'_L \) such that \( f(S) \subset S' \) and \( g \) restricts to an \( X_*(S)_{\mathbb{F}^L} \)-translation equivariant map between the apartments of \( S \) and \( S' \). In \cite{LAN}, Landvogt proves that there always exists a \( G(L) \rtimes (\sigma) \)-invariant toral map, which becomes an isometry after normalising the metric on \( \mathcal{B}^c(G', L) \). However, this map depends on an auxiliary choice. We give a precise formulation of the result in the form and context that we need later on. For this consider the fixed element \( b \in G(L) \) and denote by \( \nu_b \in X_*(G)_{\mathbb{Q}} \) the Newton point of \( b \). We fix an integer \( s \gg 0 \) such that \( s \cdot \nu_b \in X_*(G) \). Denote by \( M_b \subset G \) the Levi subgroup centralising \( \nu_b \) (and thus \( s \cdot \nu_b \)). Then \( J_b \) is the inner form of \( M_b \) obtained by twisting the action
of the Frobenius by $b$. We can thus use the following result to relate the buildings of $G$ and $J_b$.

**Proposition 3.3** ([Lan00, Prop. 2.1.5], [Rou77, Lemme 5.3.2]). Let $f : M_b \hookrightarrow G$. Then there exists a toral $M_b(L) \times \langle \sigma \rangle$-equivariant injective map

$$ f_* : \mathcal{B}^e(M_b, L) \rightarrow \mathcal{B}^e(G, L). $$

Moreover, $f_*$ is injective and unique up to translation by an element of $V_0(G, L)^{\langle \sigma \rangle}$. In particular, its image is the same for every choice of $f_*$ and equal to $\mathcal{B}^e(G, L)^{(s \circ \nu_b)(O_L^e)}$. After a suitable normalisation of the metric on $\mathcal{B}^e(G, L)$, this map becomes an isometry.

**Remark 3.4.** Since $J_{b,L} \cong M_{b,L}$, we obtain an identification of $\mathcal{B}^e(J_b, L)$ with $\mathcal{B}^e(G, L)^{(s \circ \nu_b)(O_L^e)}$. However, since $J_b$ is an inner twist of a Levi subgroup of $G$, this identification will not respect the action of the Frobenius in general. In order to distinguish it from the action on $\mathcal{B}^e(J_b, L)$, we denote by $\sigma_b$. More explicitly, we have $\sigma_b = b \sigma_{|_{\mathcal{B}(J_b, L)}} \times \sigma_{|_{V_0(J_b, L)}}$. Indeed, by [Lan96, Lemma 3.3.1], the Frobenius action on on the “usual” Bruhat-Tits building $\mathcal{B}(J_b, L)$ is uniquely determined by the equation $\sigma_b \circ j = (b \sigma(j) \circ b^{-1}, \sigma_b(x))$ and thus has to be equal to $b \sigma$. It follows from the explicit description in [Lan96, (3.3.2)], that the Frobenius action on $V_0(J_b, L)$ remains the same.

Now assume that we have an embedding of reductive groups $f : G \hookrightarrow G'$. The following statement is the main result of [Lan00].

**Proposition 3.5** ([Lan00, Thm. 2.2.1]). There exists a $G(L) \times \langle \sigma \rangle$-invariant toral map $f_* : \mathcal{B}^e(G, L) \rightarrow \mathcal{B}^e(G', L)$. Furthermore the metric on $\mathcal{B}^e(G, L)$ can be normalised in a way such that $f_*$ becomes isometrical.

To simplify the notation, we identify $G$ with its image in $G'$. Now $b$, considered as element of $G'$, induces a group $J_b'$ which is an inner form of the centraliser of $\nu_b$ in $G'$. Since $f_*$ preserves the fixed points of $\nu_b(O_L^e)$, we obtain a commutative diagram by Proposition 3.3 and Remark 3.4.

\[
\begin{align*}
\mathcal{B}^e(J_b, L) & \xrightarrow{f_*} \mathcal{B}^e(J_b', L) \\
\downarrow & \\
\mathcal{B}^e(G, L) & \xrightarrow{f_*} \mathcal{B}^e(G', L).
\end{align*}
\]

**Lemma 3.7.** The restriction $f_*|_{\mathcal{B}^e(J_b, L)}$ is $\sigma_b$-equivariant.

**Proof.** We denote by $\sigma_b'$ the canonical Frobenius action on $\mathcal{B}^e(J_b', L)$. Note that the action of $b \sigma$ and the actions of $\sigma_b, \sigma_b'$ differ by the translations $t_b, t_b'$ induced by the action of $b$ on $V_0(J_b, L)$ and $V_0(J_b', L)$ respectively. Since $f_*$ is $b \sigma$-equivariant, it suffices to show that $f_* \circ t_b = t_b' \circ f_*$. 

To prove this, consider the composition of $f_*$ with the canonical projection $\mathcal{B}^e(J_b', L) \twoheadrightarrow V_0(J_b', L)$. We claim that this map factors through $V_0(J_b, L)$. This can be checked on extended apartments. Let $S \subset J_b, S' \subset J_b'$ be maximal split tori over $L$ with $f(S) \subset S'$ and $f_* (\mathcal{A}^e(S, J_b; L)) \subset \mathcal{A}^e(S', J_b'; L)$. For the intersections with the derived groups of $G, G'$ we have $S_{d\text{er}} \subset S'_{d\text{er}}$. Hence the composition $\mathcal{A}^e(S, J_b; L) \twoheadrightarrow \mathcal{A}^e(S', J_b'; L) \twoheadrightarrow V_0(J_b', L)$ is $S_{d\text{er}}(L)$-invariant and thus factors through $\mathcal{A}^e(S, J_b; L)/\mathcal{A}^e(S_{d\text{er}}, J_b_{d\text{er}}; L) = V_0(J_b, L)$.

Thus we obtain a commutative diagram...
proving $f$ through some finite union of Schubert varieties by the previous lemma, in particular since $p, p'$ of a scheme has a quasi-compact open neighbourhood. The "only if" direction follows from the previous lemma because every point $x \in X$ satisfies $\text{char } F(x) = 0$ and $\text{char } F = 0$. By \cite[Thm. 5.1]{PR08} and \cite[Prop. 1.21]{Zhu17}, the connected components of $F$ are precisely the subsets of the form $\{ \omega \}$.

For further considerations, it will be useful to fix a presentation of $\mathcal{F}_{\ell g}$ as a limit of schemes. For any $w \in \hat{W} \setminus \hat{W}/\hat{W}^p$ we denote by

$$S^0_w := P w P/P, S_w := \bigcup_{w \leq \ell(w)} S_w,$$

the Schubert cell and the Schubert variety associated with $w$, respectively. Here, $\leq$ denotes the Bruhat order on $\hat{W}$ induced by any fixed choice of an Iwahori subgroup of $P$. By \cite[§ 8]{PR08} and \cite[Thm. 9.3]{BS17} each Schubert variety (resp. cell) is a closed (resp. locally-closed) quasi-compact subscheme of $\mathcal{F}_{\ell g}$, which is of finite type in the case $\text{char } F = p$ and perfectly of finite type in the case $\text{char } F = 0$.

We equip $\mathcal{F}_{\ell g}(k)$ with the bornology induced by the canonical projection $G(L) \twoheadrightarrow G(L)/P = \mathcal{F}_{\ell g}(k)$, that is a subset $X \subset \mathcal{F}_{\ell g}(k)$ is bounded, if it is contained in a finite union of Schubert varieties. We obtain the following geometric characterisation of bounded subsets.

Lemma 4.1. A subset $X \subset \mathcal{F}_{\ell g}(k)$ is bounded if and only if it is relatively quasi-compact. In this case $X$ is even quasi-compact itself.

Proof. Since the $S_w$ are quasi-compact, any bounded subset of $\mathcal{F}_{\ell g}$ is relatively quasi-compact. The $S_w$ are Noetherian, thus their subsets are quasi-compact themselves.

On the other hand, assume that $X$ is not bounded. We prove that $X$ is not quasi-compact by constructing an infinite discrete closed subset $Y \subset X$. By definition, the set $T := \{ w \in \hat{W} \mid X \cap S^0_w \neq \emptyset \}$ is infinite. For each $w \in T$, choose an element $x_w \in X \cap S^0_w$. Then $Y := \{ x_w \mid w \in T \}$ is infinite and discrete. Its intersection with every $S_w$ for $w \in \hat{W}$ is closed, hence $Y$ is closed.

Lemma 4.2. Let $X \subset \mathcal{F}_{\ell g}$ be a locally closed reduced sub-ind-scheme. Then $X$ is a scheme if and only if every point of $X(k)$ has an open neighbourhood which is bounded as subset of $\mathcal{F}_{\ell g}(k)$. In this case $X$ is locally of finite type if $\text{char } F = p$, respectively locally of perfectly finite type if $\text{char } F = 0$.

Proof. The "only if" direction follows from the previous lemma because every point of a scheme has a quasi-compact open neighbourhood.

To prove the "if" direction, we may assume that $X$ is bounded, since its representability is a Zariski-local property. Then the embedding $X(k) \hookrightarrow \mathcal{F}_{\ell g}$ factors through some finite union of Schubert varieties by the previous lemma, in particular

$$\mathcal{B}^c(J_b, L) \xrightarrow{f} \mathcal{B}^c(J'_b, L) \xrightarrow{t} \mathcal{V}_0(J_b, L) \xrightarrow{\pi} \mathcal{V}_0(J'_b, L).$$

Since $p, p'$ and $f_*$ commute with the action of $b$, so does $f^{ab}_*$. Thus $f^{ab}_* t_b = t'_b f^{ab}_*$, proving $f_* t_b = t'_b f_*$. □
Let $X(k)$ be a locally closed subvariety of this union. Since the Schubert varieties are (perfectly) of finite type, so is $X$. 

**Remark 4.3.** The analogous assertions of Lemmas 4.1 and 4.2 in $LG(k)$, the loop group of $G$, also hold true (with the exception of the last statement of Lemma 4.2). Indeed, since a set $X \subset G(L)$ is bounded if and only if $X \cdot P$ is bounded, it suffices to prove the assertion in the case that $X$ is right $P$-invariant. Then the claim follows from the above lemmas since $LG \to \mathcal{Fl}_{\mathcal{G}}$ is an $L^+\mathcal{G}$-torsor and thus relatively representable and quasi-compact.

5. **Affine Deligne Lusztig Varieties**

We now prove the third part of the theorem. By Lemma 4.2 it is equivalent to the following proposition, which we prove below.

**Proposition 5.1.** Let $Z$ be a bounded subset of $G(L)$ and denote $\tilde{X}_Z(b) := \{g \in G(L) \mid g^{-1} b \sigma(g) \in Z\}$. Then there exists a bounded subset $\tilde{X}_0 \subset \tilde{X}_Z(b)$ such that $\tilde{X}_Z(b) = J_0(F) \cdot \tilde{X}_0$.

For the proof we need some preparation.

**Lemma 5.2.** The $\sigma$-conjugacy class of $b \in G(L)$ has a decent representative for which $B^c(J_0, F) \cap B^c(G, F) \neq \emptyset$ (viewed as subspaces of $B^c(G, L)$).

**Proof.** In Remark 4.4 we identified the extended Bruhat-Tits building $B^c(J_0, L)$ with $B^c(M_0, L)$. We fix a maximal $L$-split torus $S \subset M_0$, denote by $T$ its centraliser and by $\tilde{W}_M$ the associated extended affine Weyl group of $M_0$. Since any reductive group over $F$ is residually quasi-split by [BT87, Thm. 4.1], there exists $\sigma$-stable alcove $a$ in $A(S, M_0, L)$. The Kottwitz homomorphism maps the stabiliser $\Omega \subset \tilde{W}_M$ of $a$ isomorphically onto $\pi_1(G)_I$. Since any basic $\sigma$-conjugacy class is uniquely determined by its Kottwitz point, we may assume that $b$ (after replacing it by a $M_0(L)$-conjugate if necessary) is a representative in $M_0(L)$ of an element of $\Omega$. By [Kim, Lemma 2.2.10] we may assume this representative to be decent. It now follows from the explicit description of $\sigma_b$ in Remark 4.4 that we may take $p_0 := (p_b, p_v)$ where $p_b \in B(M_0, L)$ is the barycenter of $a$ and $p_v \in V_0(M_0, L)$ is any point fixed by $\sigma$. Then $p_0 \in B^c(J_0, F) \cap B^c(G, F)$. 

Thus after replacing $b$ by a $\sigma$-conjugate if necessary, we fix $p_0 \in B^c(J_0, F) \cap B^c(G, F)$. In order to relate the bornologies on $G(L)$ and on $B^c(G, L)$ directly, we consider the map

$$\iota: G(L) \to B^c(G, L), g \mapsto g \cdot p_0.$$ 

By the choice of $p_0$, the map $\iota$ is $G(L) \rtimes \langle \sigma \rangle$-equivariant and the restriction to $J_0(L)$ is moreover $\sigma_0$-equivariant, cf. Remark 4.4. By Proposition 4.2 for any $C' > 0$ the set $Z_{C'} := \{g \in G(L) \mid d(p_0, \iota(g)) < C'\}$ is a bounded set and for any bounded $Z \subset G(L)$ the constant $c_Z := \sup\{d(p_0, \iota(y)) \mid y \in Z\}$ is finite.

We further translate the assertion of Proposition 5.1 into a statement about boundedness properties of Bruhat-Tits buildings.

**Lemma 5.3.** Let $G$ be a reductive group over $F$ and $b \in G(L)$. Then the following are equivalent.

(a) For any $c > 0$ there exists a $C > 0$ such that if $x \in B^c(G, L)$ satisfies $d(x, b \sigma(x)) < c$ then there exists $x_0 \in B^c(J_0, F)$ with $d(x, x_0) < C$.

(b) For any $c > 0$ there exists a $C > 0$ such that if $x \in \iota(G(L))$ satisfies $d(x, b \sigma(x)) < c$ then there exists $x_0 \in \iota(J_0(F))$ with $d(x, x_0) < C$.

(c) Proposition 5.1 is true for $G$, $b$ and every bounded subset $Z \subset G(L)$. 

Proof. To see the equivalence of (a) and (b), we have to show that the distance of a point \( x \in B^r(G, L) \) to \( s(G(L)) \) is bounded above, or equivalently that there exists a bounded subset \( M \subset B^r(G, L) \) such that \( G(L) \cdot M = B^r(G, L) \) as well as the analogous assertion for \( J_b(F) \). For this, we fix an isomorphism \( X_+(Z)^! \cong \mathbb{Z}^r \), which yields an identification \( V_0(G, L) = \mathbb{R}^r \). Then we may choose \( M = a \times [0, 1]^r \), where \( a \) is any alcove of the usual Bruhat-Tits building \( B(G, L) \).

Now assume that (b) holds and let \( Z \subset G(L) \) be bounded. We fix \( g \in \tilde{X}_Z(b) \) and denote \( x := \iota(g) \).

By (b), there exist a \( C_Z > 0 \) depending only on \( Z \) and a \( j \in J_b(F) \) such that

\[
d(j^{-1} \cdot g, p_0, p_0) = d(x, j, p_0) < C_Z,
\]

i.e. \( j^{-1} \cdot g \in Z_{C_Z} \). Hence \( \tilde{X}_Z(b) = j_b(F) \cdot (X_Z(b) \cap Z_{C_Z}) \).

On the other hand, let \( c > 0 \) and assume that (c) holds. Then there exists a bounded set \( \tilde{X}_0 \subset G(L) \) such that \( \tilde{X}_Z(b) = j_b(F) \cdot \tilde{X}_0 \). Thus for any \( x = g \cdot p_0 \in \iota(\tilde{X}_Z(b)) \) there exists a \( j \in J_b(F) \) such that \( j^{-1} \cdot g \in \tilde{X}_0 \) and hence

\[
d(x, \iota(j)) = d(j^{-1} \cdot g, p_0, p_0) < C_{\tilde{X}_0}.
\]

Thus (b) holds with \( C = C_{\tilde{X}_0} \).

Proof of Proposition 5.3. Since \( Z \) is contained in a finite union of \( P \)-double cosets, it is enough to prove the theorem under the assumption that \( Z \) itself is a \( P \)-double coset.

For \( G = GL_n \) (and in fact for all \( G \) that are split over \( F \) and all \( b \)), the proposition is shown as part of the proof of [HV11] Theorem 10.1, where it is the second (and largest) part of the proof.

Next we reduce the claim to the case that \( G = GL_n \), using the equivalent condition in Lemma 5.3(a) instead of the literal statement of the proposition. The reduction step is a generalisation of the proof of the main result in [RZ99]. Let \( G \) be as in the proposition and let \( b \in G(L) \). We fix a faithful representation \( f: G \hookrightarrow G' := GL_n \). Recall the commutative diagram of equivariant toral embeddings (3.6) where the horizontal maps commute with the respective Frobenius actions, cf. Lemma 3.7.

Now let \( c > 0 \) be fixed and let \( C > 0 \) as in Lemma 5.3(a) applied to \( G' \). Thus for any \( x \in B^{r_0}(G, L) \) with \( d(x, \sigma_0(x)) < c \) there exists a point \( x'_0 \in B^{r_0}(J_b^r, F) \) such that \( d(x, x'_0) < C \). Here, we identify \( x \) and \( x'_0 \) with their images in \( B^{r_0}(G', L) \). Let \( x_0 \in B^{r_0}(G, L) \) be the closest point to \( x'_0 \) in \( B^{r_0}(G, L) \). It is the image of \( x_0 \) under the closest point mapping \( \pi_{B^{r_0}(G, L)}: B^{r_0}(G', L) \to B^{r_0}(G, L) \) and thus uniquely determined and satisfies \( d(x, x_0) \leq d(x, x'_0) < C \) ([RZ99] Lemma 1.8). By uniqueness of the closest point mapping, \( \pi_{B^{r_0}(G, L)} \) is \( (G(L) \)-equivariant. It follows by Proposition 3.3 that the point \( x_0 \) is fixed by \( s \cdot \nu_h(O_L^x) \) and is thus contained in \( B^{r_0}(J_b, L) \).

Using again the uniqueness of the closest point, we deduce that \( x_0 \) is also the image of \( x'_0 \) under the closest point mapping \( B^{r_0}(J_b^r, L) \to B^{r_0}(J_b, L) \), which is \( (\sigma_0) \)-equivariant. Thus \( x_0 \in B^{r_0}(J_b, L)^{\sigma_0} = B^{r_0}(J_b, F) \), which finishes the proof.

Thus we have proven the third part of Theorem 1.1. By Lemma 4.2 the second part of the theorem is equivalent to the following proposition.

**Proposition 5.4.** Every \( x_0 \in X_Z(b)(k) \) has a bounded open neighbourhood.

Proof. The proof is basically the same as for the last part of [HV11] Thm. 6.3. Since the situation simplifies a lot by considering only the reduced structure, and since in loc. cit. only split groups, hyperspecial \( P \), and certain \( Z \) are considered, we give the complete proof for the reader’s convenience.
Let $\omega = w_G(x_0)$ and let again $X_\omega ^{(L)} = X_\omega \cap w_G^{-1}(\omega)$. Since $X_\omega ^{(L)} \subseteq X_\omega$ is open and closed, it suffices to prove the claim for $X_\omega ^{(L)}$. We can define a $G(L)$- and $\sigma$-invariant semi-metric $d: G(L) \to \mathbb{N} \cup \{0\}$ by

$$d(g, h) \leq n \iff h^{-1}g \in P_{\rho^\vee(x^n)}P = \bigcup_{w \leq 2^n \rho^\vee} PwP$$

where $\rho^\vee$ denotes the half-sum of the positive coroots and $w \in \hat{W} \setminus \hat{W}/\hat{W}P$. Obviously this semi-metric descends to $\mathcal{R}^+_G$. Then a subset is bounded with respect to the bornology defined before Lemma 14 if and only if it is bounded with respect to $d$.

We choose $b$ as in Lemma 5.2. Let $s \in \mathbb{N}$ be as in the decency equation, i.e. $(b\sigma)^s = (s \cdot \nu)(\epsilon)$. Enlarging $s$ and $Z$ if necessary, we assume that $\omega$ and $Z$ are both $\sigma^s$-invariant. Then $X_\omega ^{(L)}$ is $\sigma^s$-stable and thus defined over the extension $k_s$ of degree $s$ of $k_F$. The closed point $x_0$ defined over some finite extension of $k_F$. By enlarging $k_s$ further if necessary, we assume that $x_0$ is a $k_s$-rational point. We denote by $\mathcal{M}$ the model of $X_\omega ^{(L)}$ over $k_s$ and for every $n \in \mathbb{N}$ we define the closed sub-ind-scheme

$$\mathcal{M}_n(k) := \{x \in \mathcal{M}(k) \mid d(x, x_0) \leq n\}.$$ 

Note that $\mathcal{M}_n$ is actually a (perfectly) finite type scheme by Lemma 5.2 and moreover defined over $k_s$ since $d$ is $\sigma$-invariant. Also note that $\mathcal{M} = \varinjlim \mathcal{M}_n$.

The decency of $b$ implies that $J_b(F_s) \subset G(F_s)$ where $F_s$ is the unramified extension of $F$ of degree $s$. Thus the $J_b(F)^0$-action stabilises $\mathcal{M}(k_s)$. Together with Proposition 5.1 we obtain that there exists an $N_0 \in \mathbb{N}$ such that for every $x \in \mathcal{M}(k)$ there exists a $y_0 \in \mathcal{M}(k_s)$ with $d(x, y_0) \leq N_0$. For every $y_0 \in \mathcal{M}(k_s)$ define the closed subscheme $\mathcal{M}_n(y_0) \subseteq \mathcal{M}_n$ by

$$\mathcal{M}_n(y_0)(k) := \{y \in \mathcal{M}_n(k) \mid d(y_0, y) \leq N_0\}.$$ 

Now consider the open subset of $\mathcal{M}_n$

$$U_n := \mathcal{M}_n(x_0) \setminus \bigcup_{y \in \mathcal{M}_n(y_0) \setminus \mathcal{M}_n} \mathcal{M}_n(y).$$

The union on the right hand side is indeed finite (and hence closed): By the triangular inequality $\mathcal{M}_n(y)$ is empty unless $y \in \mathcal{M}_{N_0+n}(k_s)$; the latter set is finite since $\mathcal{M}_{N_0+n}$ is (perfectly) of finite type. We claim that the chain $U_1 \subset U_2 \subset \cdots$ stabilises at $U_{2N_0}$ at the latest. To prove this, let $x \in U_{n}(k)$ for some $n$. We choose a rational point $y_0 \in \mathcal{M}(k_s)$ with $d(x, y_0) \leq N_0$. By definition of $U_n$, we must have $d(x, y_0) \leq N_0$. Thus $d(x, x_0) \leq d(x, y_0) + d(y_0, x_0) \leq 2N_0$, i.e. $x \in U_{2N_0}$.

Since $\mathcal{M} = \varinjlim \mathcal{M}_n$, the subset $U_{2N_0} = \varinjlim U_n$ is open in $\mathcal{M}$. It is moreover bounded and contains $x_0$. It is thus a bounded open neighbourhood of $x_0$.

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