Time-evolution of entanglement and quantum discord of bipartite systems subject to $1/f^\alpha$ noise

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Abstract—We study the dynamics of quantum correlations for two non-interacting qubits initially prepared in a maximally entangled state and then coupled with an external environment characterized by a noise spectrum of the form $1/f^\alpha$. The noise spectrum is due to the interaction of each qubit with a collection of $N_f$ classical fluctuators with fixed switching rates. We find that, depending on the characteristic of the noise spectrum considered, both entanglement and quantum discord display either a monotonic decay or the phenomena of sudden death and revivals.

I. INTRODUCTION

Quantum correlations, entanglement and quantum discord (QD), represent a valuable resource for information processing and technology [1, 2]. In turn, the dynamics of quantum correlations have been investigated for several systems, ranging from quantum optics [3, 4, 5, 6, 7, 8, 9] to nanophysics [10, 11, 12]. The major limitation of the use of entanglement and QD in practical applications is due to the unavoidable interaction of real quantum systems with their surroundings, resulting in decoherence processes, and, as a consequence, in the degradation of quantum correlations. On the other hand, the environment can even resume or preserve quantum correlations, as it happens when the backflow of information to the system becomes relevant. It is therefore of interest to understand the effect of the various kinds of environmental noise on the dynamics of quantum correlations in realistic quantum systems, where peculiar phenomena can be observed, such as sudden death and revival [13, 14]. In particular, it is crucial to investigate and compare Markovian noise, ascribed to environment with short self-correlations [15, 16], to non-Markovian noise [17], which is associated to environments with memory and may lead to a non-monotonic time dependence of entanglement and QD. Revival phenomena have been predicted both for couple of qubits interacting directly or indirectly in a common quantum reservoir [18, 19] and for non-interacting qubits in independent non-Markovian quantum environments [20].

The effect of the quantum noise on the entanglement dynamics between two quantum systems has been interpreted in terms of the transfer of the correlations back and forth from the systems themselves to the environment. This is due to the back-action of the system on the environment. On the other hand, recently, revivals of quantum correlations have been found also for quantum systems coupled to classical sources [18] and have been connected to a quantifier of non-Markovianity for the dynamics of a single-qubit. It was actually proven that a classical noise can mimic a quantum environment not affected by the system or influenced in a way that does not result in a back-action.

In this work we analyze the role played by classical environments, characterized by $1/f^\alpha$ noise spectra, on the dynamics of quantum correlations. In particular, as prototypical bipartite system, we consider two non-interacting qubits coupled to noise in different environments. Noises of the type $1/f^\alpha$ are among the main sources of decoherence in quantum solid-state devices [19, 20, 21, 22, 23, 24, 25], thus constituting a fundamental case of study for a deeper understanding of the decoherence process itself. This kind of noise spectra stem from a collection of random telegraph sources with a specific distribution of their switching rates. For $\alpha = 1$ the so-called pink $1/f$ noise is found, which is obtained from a set of random telegraph fluctuators weighted by the inverse of the switching rate. Another interesting case is the $1/f^2$ spectrum, also called brown noise from its relation to a Brownian motion. We study the dynamics of quantum correlations as a function of the parameter $\alpha$ and observe a continuous variation in their behavior, ranging from the monotonic decay characteristic of the $1/f$ noise to the phenomena of sudden death and revivals induced by the $1/f^2$ noise. While a qualitative trend is clearly identified, it is not possible to identify a precise threshold value between the decaying and the oscillating regime, since the outcomes significantly depend upon the range of selected frequencies and upon the number of random telegraph fluctuators used for the calculations. The dynamics of the two qubits is ruled by a stochastic Hamiltonian with time dependent coupling. The average of the time-evolved states over the switching parameters describes the evolution of the two-qubit state under the effect of the noise.

The paper is organized as follows: in Sec. II the definition of negativity, as a measure of entanglement, and of QD are briefly reviewed. The physical model adopted is described in Sec. III. In Sec. IV results are presented and discussed, whereas Section...
IV closes the paper with some concluding remarks.

II. MEASURES OF QUANTUM CORRELATIONS

Entanglement is evaluated by means of negativity, defined as: 
$$N = 2 \sum \lambda^-$$ 
where $\lambda^-$ are the negative eigenvalues of the partial transpose of the density matrix of the system. Negativity is zero for separable states and assumes value one for maximally entangled states.

$QD$ is defined as the difference between the total and the classical correlations in a system $Q = I - C$, where $I$ is the von Neumann entropy, and $\rho^{A(B)}$ indicates the reduced density matrix of the subsystem $A(B)$, $C$ denotes the measurement-induced quantum mutual information, namely the classical correlations, and reads $C = \max_{\{\Pi_k\}} \{S(\rho_A) - S(\rho|\{\Pi_k\})\}$, with $S(\rho|\{\Pi_k\})$ denotes the quantum conditional entropy with respect to the set of projective measurements $\{\Pi_k\}$ performed locally on subsystem $B$. Usually to compute $QD$ is not an easy task, since it involves a maximization procedure. However, for two-qubit systems described by a density matrix with a so-called “$X$” form, an analytical expression for $Q$ has been obtained [26].

III. KINEMATICS AND DYNAMICS

We consider a system of two non-interacting qubits, initially entangled, subject to a noisy classical environment with noise spectra of the type $1/f^\alpha$, with $1 \leq \alpha \leq 2$. The environment is modelled by using different configurations of bistable fluctuators. The two qubits are initially prepared in the Bell state $|\psi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Here the interaction between the qubits and the environment is assumed local, i.e. the two qubits interact with two different and independent baths. The case of a common environment affecting both qubits and its comparison with the local model is discussed in [27]. Setting $\hbar = 1$ and adopting the spin notation, the two-qubit Hamiltonian describing the interaction with a single fluctuator is given by $H(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t)$, where $H_A(t)$ is the Hamiltonian of a single qubit subject to a classical time-dependent noise which affects the transition amplitude parameter $c_{A,B}(t)$:

\begin{equation}
H_{A(B)}(t) = c_0 I_{A(B)} + \nu c_{A,B}(t) \sigma_x A(B),
\end{equation}

with $c$ the qubit energy in absence of noise (energy degeneracy is assumed), $I_{A(B)}$ the identity matrix for subspace $A(B)$, $\nu$ is the coupling constant between the system and the environment, $\sigma_x$ the Pauli matrix. If the time-dependent coefficient $c_{A,B}(t)$ can randomly flip between two values $c(t) = \pm 1$ with a fixed rate $\gamma$, then Eq. (1) describes a qubit subject to a random telegraph noise (RTN) [27], [28], [29], [30], [31]. This model has recently been extended to the case of tripartite systems [32]. The above Hamiltonian is stochastic due to the random nature of the noise parameter $c(t)$. For a specific choice of $c(t)$, the total system evolves according to the evolution operator $e^{-i \int H(t') \, dt'}$. By averaging the global state over different realizations of the sequences of $c(t)$, the two-qubit mixed state is obtained.

In order to reproduce the $1/f^\alpha$ spectrum, the single RTN frequency power density must be integrated over the switching rates $\gamma$ with a proper distribution:

\begin{equation}
S_{1/f^\alpha}(f) = \int \gamma S_{RTN}(f, \gamma) \, p_\alpha(\gamma) \, d\gamma,
\end{equation}

where $S_{RTN}(f, \gamma)$ is the random telegraph noise frequency spectral density with Lorentzian form $S_{RTN}(f, \gamma) = 4\gamma/(4\pi^2 f^2 + \gamma^2)$. The integration is performed between a minimum and a maximum value of the switching rates, respectively $\gamma_1$ and $\gamma_2$. $p(\gamma)$ is the switching rate distribution and takes a different form depending on the kind of noise:

\begin{equation}
p_\alpha(\gamma) = \begin{cases} 
\frac{1}{\gamma^\alpha} & \alpha = 1 \\
\frac{(\alpha-1)}{\gamma^{\alpha-1}} \left( \frac{\gamma_2}{\gamma_2 - \gamma_1} \right)^{\alpha-1} & 1 < \alpha \leq 2
\end{cases}
\end{equation}

It follows that, in order to simulate a frequency spectrum proportional to $1/f^\alpha$, the switching rates must be selected from a distribution proportional to $1/\gamma^\alpha$. When the integration in Eq. (2) is performed, the spectrum has the requested $1/f^\alpha$ behavior in a frequency interval, so that every frequency belonging to such interval satisfies the condition $\gamma_1 \ll f \ll \gamma_2$. Eq. (2) is obtained using a collection of sources of RTN each with a switching rate taken from the same $\gamma$-distribution.

The $1/f^\alpha$ noise spectrum is obtained from the coupling of the system with a large number of fluctuators, each characterized by a specific switching rate, picked from the distribution $p_\alpha(\gamma)$ [24] in a range $[\gamma_1, \gamma_2]$. In this case the random parameters in Eq. (1) describes a linear combination of bistable fluctuators $c(t) = \sum_{j=1}^{N_f} c_j(t)$, where $N_f$ is the number of fluctuators and we drop the subscript $A(B)$ to simplify the notation. Each $c_j(t)$ has a lorentzian power spectrum whose sum gives the power spectrum of the noise:

\begin{equation}
S(f) = \sum_{j=1}^{N_f} S_j(f; \gamma_j) = \sum_{j=1}^{N_f} \frac{\gamma_j}{\gamma_j^2 + 4\pi^2 f^2} \propto \frac{1}{f^{\alpha}}.
\end{equation}

In order to obtain a reliable $1/f^\alpha$ spectrum, it is necessary that a sufficiently large number of fluctuators is considered, and that the selected $\gamma_j$ are a representative sample of the distribution $p_\alpha(\gamma_j)$ in the range $[\gamma_1, \gamma_2]$. We assume that all the fluctuators have the same coupling constant with the environments, that is $\nu_j = \nu$ for $j = 1, ..., N_f$.

For a single fluctuator, let us say the $j^{th}$ one, the global system evolves according to the Hamiltonian $H(t)$ with a specific choice of both the parameter $c_j(t)$ and of its switching rate $\gamma_j$. The interaction with a bistable fluctuator induces a phase shift in the state of each single qubit given by $c_j(t) = -\nu \int_0^t c_j(t') dt'$ and characterized by a distribution
\[ p(\varphi_j, t) = \frac{1}{2} e^{-\gamma_j t} \times \left\{ \delta(\varphi_j + \nu t) + \delta(\varphi_j - \nu t) \right\} + \frac{\gamma_j}{\nu} [\Theta(\varphi_j + \nu t) + \Theta(\varphi_j - \nu t)] \times \left[ I_1 \left( \frac{\gamma_j t \sqrt{1 - (\varphi_j/\nu t)^2}}{\sqrt{1 - (\varphi_j/\nu t)^2}} \right) + I_0 \left( \gamma_j t \sqrt{1 - (\varphi_j/\nu t)^2} \right) \right] \]

where \( I_n(x) \) is the modified Bessel function and \( \Theta(x) \) is the Heaviside step function. For a given \( \gamma_j \) the global system is described by a X-shaped density matrix obtained by averaging over the noise the density matrix \( p(\varphi_j, \gamma_j, t) \) corresponding to a specific choice of the parameter \( \gamma_j \). \[ p(\varphi_j, \gamma_j, t) = \frac{1}{2} \left[ (1 + D_j^2(t))|\phi^+\rangle\langle\phi^+| + (1 - D_j^2(t))|\psi^+\rangle\langle\psi^+| \right] \]

where \( |\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \) is a Bell state. The function \( D_j(t) \) represents the average of the RTN phase factor i.e., \( \langle e^{2i\varphi_j(t)} \rangle = \int e^{2i\varphi_j(t)} p(\varphi_j, t) d\varphi_j = D_j(t) \) where

\[ D_j(t) = \left\{ \begin{array}{ll}
  e^{-\gamma_j t} \cosh (\kappa_\nu t) + \frac{2\nu}{\kappa_\nu} \sinh (\kappa_\nu t) & \text{for } \gamma_j \geq 2\nu \\
  e^{-\gamma_j t} \cos (\kappa_\nu t) + \frac{2\nu}{\kappa_\nu} \sin (\kappa_\nu t) & \text{for } \gamma_j \leq 2\nu
\end{array} \right. \]

with \( \kappa_\nu = \sqrt{\gamma_j^2 - (2\nu)^2} \).

In case of a collection of fluctuators, \( j = 1, \ldots, N_f \), the global evolution operator \( U(t) \propto e^{-i\sum_j \varphi_j(t)} \), for fixed values of the parameters associated to each fluctuator, permits to compute the density matrix of the global system as a function of a total noise phase \( \varphi(t) = \sum_j \varphi_j(t) \). Following the approach used in \[ \[ \text{[27]} \] \] to evaluate the dynamics of two qubits subject to a single RTN, the time-evolved density matrix of the system at time \( t \) can be expressed as: \( \rho(t) = \int \rho(\varphi, t) p(\varphi, t) d\varphi \) where \( p(\varphi, t) = \prod_j p(\varphi_j, t) \) is the global noise phase distribution. This form of the density matrix depends on the average of the phase factor \( e^{2i\varphi_j(t)} \), which can be computed in terms of the RTN coefficient \( D_j(t) \) of Eq. (7). We have

\[ \langle e^{2i\varphi(t)} \rangle = \prod_j D_j(t) \]

where the last equality holds since the RTN phase coefficients are independent. Upon inserting the above expression in \( \rho(t) \) one may evaluate the two-qubit density matrix:

\[ \rho(t; \gamma_j) = \frac{1}{2} \left[ (1 + \Gamma)|\phi^+\rangle\langle\phi^+| + (1 - \Gamma)|\psi^+\rangle\langle\psi^+| + (1 - \gamma_j)|\psi^+\rangle\langle\psi^+| \right] \]

where \( \Gamma = \prod_j D_{jA}(t) D_{jB}(t) \), where the subscripts \( A \) and \( B \) are reintroduced to underline that the result is derived from the contributions of the two subsystems. Note that the fluctuators have fixed switching rates \( \{\gamma_j\} \), \( j = 1 \ldots N_f \).

iv. Results

The density matrix in Eq. (8) preserves its X shape during time evolution. The negativity and the QD are thus given by

\[ N(t) = |\Gamma(t)| \quad Q(t) = h(\Gamma) \]

where \( h(x) = \frac{1}{2} [(1 + x) \log_2 (1 + x) + (1 - x) \log_2 (1 - x)] \). Note that the choice of a maximally entangled initial state together with the sheer dephasing nature of the interaction with the environment, makes QD a function of negativity only. In these systems, in fact, the evolved state is a mixture of Bell states.

In Fig. 1 we compare the time evolution of negativity for the case of 50 and 100 fluctuators, and for two values of the parameter \( \alpha \), i.e., \( \alpha = 1 \) pink noise (solid-black line) and \( \alpha = 2 \) brown noise (dashed-red line). 30 curves are drawn for each value of \( \alpha \), corresponding to 30 different samples of the \( \gamma_j \)'s. Different choices of the range \( [\gamma_1, \gamma_2] \) are presented. Specifically, \([10^4, 10^5]\) top panels, \([10^4, 10^5]\) bottom panels. Note that \( \tau \) is a dimensionless parameter proportional to time.
height decreases increasing the number of fluctuators. Indeed a number of peaks with periodicity $\pi/2$ and exponentially decreasing heights are present. In Fig.1 to make the picture more readable, only the first one is shown. The periodicity can be understood by analyzing the analytical expression of quantum correlations. The $D_j(t)$ functions present damped oscillations for $\gamma < 2\nu$ with periodicity $2\pi/\gamma$, and for $\gamma > 2\nu$ monotonically decay. Their weighted superposition leads to an oscillating function with periodicity $\pi/2$. Furthermore, for the $1/f^2$ noise spectrum, the selected values of the $\gamma_j$’s accumulate near the lower value of the frequency range, thus leading to a beat phenomenon with constructive interference corresponding to the above mentioned periodicity with narrow peaks.

Fig. 2 reports negativity and QD as a function of time and of the noise parameter $\alpha$, for a specific sample of the $\gamma_j$’s and for $N_f = 100$. The same qualitative behavior is found for such quantities, due to their peculiar relation as given in (2). After the initial and fast exponential decay, at increasing values of $\alpha$ the revival peaks, with $\pi/2$ periodicity, begin to appear. The height of the peaks raises with $\alpha$ and reaches its maximum for the $1/f^2$ noise spectrum.

The qualitative trend is clearly identified, while it is not possible to make precise claims about the heights of the peaks or about a threshold value, for $\alpha$, regarding the appearance of the peaks themselves. In fact, the quantitative results depend upon the range of selected frequencies, upon the number of fluctuators, and upon the $\gamma_j$’s sample.

V. CONCLUSIONS

In conclusions, we have investigated the effects of an external environment characterised by a noise spectrum of the form $1/f^\alpha$ on the dynamics of quantum correlations between two non interacting qubits initially prepared in a maximally entangled state. The environment is modelled by an ensemble of bistable classical fluctuators, with fixed switching rates chosen from a given interval and a given distribution. Our results show that, depending on the characteristic of the noise spectrum, quantum correlations display either a monotonic decay or the appearance of revivals. The qualitative trend is clearly identified, whereas the quantification of the results, and the identification of the threshold, depends explicitly on the range of selected frequencies, the number of fluctuators, and on the specific sample of the switching rates $\gamma_j$.

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