Quantum Copy-Protection from Hidden Subspaces

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Abstract

Quantum copy-protection is an innovative idea that uses the no-cloning property of quantum information to copy-protect programs and was first put forward by [Aar09]. The general goal is that a program distributor can distribute a quantum state $|\Psi\rangle$, whose classical description is secret to the users; a user can use this state to run the program $P$ on his own input, but not be able to pirate this program $P$ or create another state with the same functionality.

In the copy-protection with oracle setting, the user has access to a public oracle and can use the given quantum state and the oracle to compute on his/her own input for polynomially many times. However, the user is not able to produce an additional program (quantum or classical) that computes the same as $P$ on almost all inputs.

We present a first quantum copy protection scheme with a classical oracle for any unlearnable function families. The construction is based on membership oracles for hidden subspaces in $\mathbb{F}_2^n$, an idea derived from the public key quantum money scheme in [AC12]. We prove the security of the scheme relative to a classical oracle, namely, the subspace membership oracle with the functionality of computing the secret function we want to copy-protect. The security proof builds on the quantum lower bound for the Direct-Product problem ([AC12, BDS16]) and the quantum unlearnability of the copy-protected functions. We can show that any adversary, in order to break anti-piracy, must have implicitly done one of the following two things: prepare another state that has almost the same functionality, breaking the lower bound for the Direct-Product problem; or learn the input-output behavior of the copy-protected function.

We also show that the existence of quantum copy protection and the quantum hardness of Learning-with-Errors (LWE) will imply publicly verifiable quantum money. In the end, we point out possible directions to instantiate quantum copy protection from cryptographic primitives.

1 Introduction

Quantum copy-protection was proposed by Aaronson in [Aar09]. Similar to the more widely-studied quantum money, quantum copy-protection is also inspired by the No-Cloning property of quantum information, but it aims at a different security goal: for quantum money, we need verifiable, unclonable quantum states; for quantum copy-protection, we want some unclonable states that can also let us compute certain functions correctly.

The informal definition for quantum copy protection is as follows: given a secret function $f : \mathcal{X} \rightarrow \mathcal{Y}$ drawn from a publicly known function family $\mathcal{F}$, we want a quantum state $|\Psi_f\rangle$ that

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can be efficiently prepared given a classical description of \( f \); (2) can be used to compute \( f(x) \) efficiently and correctly for (almost) all inputs \( x \in \mathcal{X} \); (3) cannot be used to prepare more states \( \rho \) and more functions \( f' \) efficiently so that \( f' \) and \( \rho \) can compute \( f \) correctly almost anywhere.

The idea of quantum money was first introduced by Wiesner [Wie83] in around 1970. A secure quantum money should have the following properties (assuming all parties have access to some quantum resources): an efficient algorithm to prepare the quantum money state; an efficient quantum algorithm to verify the money produced by the bank with high probability; no one (except the bank) can efficiently duplicate the states accepted by the verifier except with exponentially small probability. In a secure public-key/publicly verifiable quantum money scheme, the verification part is public, i.e. anyone can verify the money efficiently.

Among the public-key quantum money schemes proposed till today, Aaronson and Christiano [AC12] constructed a simple and beautiful scheme relative to a classical oracle, namely the subspace membership oracle: the oracle has a secretly, randomly chosen \( \frac{n}{2} \)-dimensional subspace \( A \subset \mathbb{F}_2^n \) inside and upon each input vector, it outputs 1 if the vector is in \( A \) and non-zero, and outputs 0 otherwise. The oracle setting security of this scheme is proven based on quantum lower bound obtained through inner-product adversary method; later Zhandry [Zha17] instantiated the subspace membership oracle with a black-box construction from quantum-secure indistinguishability obfuscation (iO) and injective one-way functions. Indistinguishability obfuscation is an algorithm that takes in two circuits \( C_1, C_2 \) with same functionalities and produce two obfuscated circuits \( O(C_1), O(C_2) \) so that they maintain original computation functionalities, but a probabilistic polynomial-time adversary cannot distinguish between the two obfuscated circuits. [Zha17] assumes a quantum-secure version of this primitive together with injective one-way functions to construct a subspace-hiding obfuscator that achieves the subspace-hiding property needed in [AC12] quantum money scheme.

Quantum copy-protection is a less explored idea. By far we don’t have any provably secure copy-protection scheme of any class of unlearnable functions even when oracles are allowed. By unlearnable functions, we mean a function family \( F \) that is not learnable from its input/output behavior for any quantum polynomial time (QPT) adversary having only classical black-box access to the functions. [Aar09] showed that any learnable function families cannot be copy-protected. [Aar09] also showed that learnability of functions is the only obstruction to quantum copy-protection relative to a quantum oracle, but did not give a provably secure copy-protection scheme or a specific quantum oracle to build copy-protection upon. Moreover, the ultimate goal is to design an explicit and practical copy-protection scheme; therefore, this led us to raise the following open problem:

Can we design a quantum copy-protection scheme relative to classical oracle?

### 1.1 Main Results

**Quantum Copy-Protection Relative to a Classical Oracle** Our main contribution resolves the foregoing open problem with a positive answer. Namely, we present a copy-protection scheme, suggested by Paul Christiano, based on the same subspace membership oracles in [AC12] and prove the following theorem, which shows that our scheme is secure against any quantum polynomial-time adversary for a large class of functions relative to a classical oracle.

**Theorem 1.1 (Informal).** For any quantumly unlearnable family of functions \( F : \{0,1\}^n \rightarrow \{0,1\}^m, m = O(\text{poly}(n)) \), we can construct a copy-protection scheme for \( f \in F \) such that using a classical oracle:

- Any authorized user can use the program to compute \( f \) correctly and efficiently for polynomially many times.
• Any quantum polynomial time adversary (whether an authorized user or not) cannot pirate the copy-protection program of $f$, except with a negligible probability.

The high-level idea is that the copy-protection scheme requires any authorized user to query the oracle twice using an “unclonable” state in order to obtain a valid computation result. More specifically, this unclonable state $|A\rangle$ is an equal superposition over some subspace, known only to the vendor. Upon the first query, the user queries the oracle on the original state $|A\rangle$ and its own input for the function, and receives the function computation result masked with randomness; on the second query, the user uses the state after applying Quantum Fourier Transform and the same function input, and it receives the corresponding randomness. The user can then remove the randomness to obtain the correct computation result. On the other hand, any unauthorized user almost always gets useless outputs.

The anti-piracy security first relies on the quantum hardness of the Direct-Product problem in $[\text{BDS}16]$: given the membership oracle for a secret, randomly chosen $n/2$-dimensional subspace $A \subset \mathbb{F}_2^n$ and the membership oracle for its dual subspace $A^\perp$, it is hard for a quantum polynomial-time adversary to find two non-zero vectors $(u, v)$, $u \in A, v \in A^\perp$.

Next we prove that any QPT adversaries who have broken the anti-piracy security copy-protection scheme can be divided into two categories, Type I and Type II. For any Type I adversary, there is a QPT reduction algorithm which can use it to solve the Direct-Product problem, by extracting information from the produced pirate programs making queries to the oracle. For any Type II adversary, who does not achieve what the type I adversaries do, a QPT algorithm can use it to quantumly learn an unlearnable function by having only classical black-box access to the function. The quantum lower bound for direct-product problem and the unlearnable property of the copy-protected functions immediately give us the security of the scheme when the adversary can only access one piece of program., which we call mini-scheme security. Then we generalize our construction from a mini-scheme to the case when an adversary can access polynomially many programs for the same function, and prove the security in this case.

We also point out some directions to remove the oracle and implement the scheme with cryptographic primitives; the existence of quantum-secure indistinguishability obfuscation is highly likely to be necessary but not sufficient.

**Relation to Public-key Quantum Money** We show that quantum copy-protection for the decryption function from a quantum CCA secure public key encryption (PKE) scheme implies public-verifiable quantum money. If we can copy-protect the decryption function (with the secret key), we can let a quantum copy-protection program be a quantum banknote; then everyone can use the corresponding public key from the encryption scheme to verify a quantum money state. Such candidate quantum CCA secure PKE scheme can be built from Learning-with-Errors (LWE).

Similarly, we show that quantum copy-protection for certain quantum-secure injective trapdoor functions implies public-verifiable quantum money.

### 1.2 Relevant Works

**Quantum Copy-Protection** Quantum Copy-Protection was proposed by Aaronson et al in $[\text{Aar}09]$, and this paper also gave two candidate schemes for copy-protecting *point functions* without security proofs. He proved that any functions that are not quantum learnable can be quantumly copy-protected relative to a quantum oracle, based on Complexity-Theoretic No-Cloning but did not give a quantum oracle construction.

Broadbent and Lord in $[\text{BL}19]$ introduced unclonable encryption. They construct schemes for encoding classical plaintexts into quantum ciphertexts, which prevents copying of encrypted data.
Unclonable encryption can be seen as copy-protecting a unit of functional information simpler than a function.

**Quantum Money** Quantum money was first proposed by Wiesner [Wie83] in around 1970. His scheme is based on conjugate coding, consisting of a unique classical serial number and \( n \) polarized photons to determine the quantum state. However, as pointed out by Aaronson et al. [AC12], his scheme has some drawbacks, including the verifiability problem, the online attack problem and the giant database problem. After a few decades, Aaronson [Aar09] gave the first public-key quantum money and quantum copyright protection. He proved that it is possible to construct the secure public-key quantum money relative to a quantum oracle. However, his explicit scheme was broken by Lutomirski et al. [LAF*09]. Later, Aaronson and Christiano [AC12] proposed a secure public-key quantum money scheme relative to a classical oracle; they used hidden subspaces and quantum-secure digital signature to build the scheme. Its explicit scheme using conjectures about polynomials was later broken but the oracle version is proven secure through a lower bound obtained by inner-product adversary method. Zhandry [Zha17] studied the quantum lightning, a formalization of “collision-free quantum money”. He showed the relation between quantum money/quantum lightning and the security of signatures/hash functions; [Zha17] also instantiated the quantum money scheme of Aaronson and Christiano with quantum-secure indistinguishability obfuscation. More recently, Kane [Kan18] showed a new approach for public-key quantum money using modular forms. Ji et al. [JLS18] defined the pseudorandom quantum state (PRS), which is a family of quantum states such that a random member of state among this family cannot be efficiently distinguished from the state drawn according to the Haar measure. He gave efficient constructions of PRS's assuming that quantum-secure one-way functions exist. Using PRS, they can give a more generic and query secure private-key quantum money scheme.

Another interesting circumstance to consider is classically verifiable quantum money introduced in [Gav12]. The communication between the bank and the user is classical and verification is through interactive protocols between them. [RS19] gives a construction for such semi-quantum money assuming quantum-secure Message Authentication Code (MAC) and Learning-with-Errors (LWE), taking using of the noisy trapdoor claw-free function introduced by [BCM+18].

Aaronson [Aar18] showed a relation between quantum money and shadow tomography of quantum states. He proved that for any private-key quantum money scheme, a counterfeiter can produce additional bills with high probability given polynomial-many legitimate bills and exponential time, without querying to the bank.

**One-time Programs and One-time Memory** Another idea of copy-protecting softwares is through one-time program, introduced in [GKR08]. One-time programs can be executed on only one single input and nothing other than the result of this computation is leaked. One-time memory is a notion analogous to oblivious transfer but sender destroys the database after the transfer and is classically unachievable without hardware assumptions. Quantum one-time programs are discussed in [BGS13], showing that any quantum circuit can be compiled into a one-time program assuming only the same basic one-time memory devices used for classical circuits.

**Obfuscation** Obfuscation is a classical cryptographic primitive to hide the computation procedures of functions but maintain their functionalities. The most ideal and strong notion, virtual black-box obfuscation is proven to be impossible for general circuits/TMs in [BGF01]. VBB obfuscation constructions nevertheless exist for certain functionalities such as point functions; there
are also realizations of other weaker VBB notions, for example distributional VBB obfuscation based on lattice, in [GKW17] and [WZ17].

Indistinguishability obfuscation (iO) was put forward by [BGI+01, BGI+12], as a weaker substitute to the too strong notion of black-box obfuscation. What iO achieves is making two circuits with almost identical functionalities to be indistinguishable to the adversary. Garg et. al. [GGH+16] described a candidate construction for iO for NC1 circuit using multilinear maps. Recently, [AJL+19] gives iO construction from assuming bilinear maps, subexponential hardness of Learning-with-Errors (LWE), weak PRG and security amplification; the assumptions are further simplified in [JLS19]. Though difficult to construct from standard assumptions, iO as a black-box is extremely useful in building other cryptographic primitives, shown in works such as [SW14].

On the quantum side, Alagic and Fefferman [AF16] defined several notions of quantum obfuscation and proved several impossibility results.

**Watermarking** Watermarking is a different way to copy-protect softwares classically by embedding a “watermark” into the softwares’ functionalities; a verification algorithm can verify the watermark and a watermarked software cannot function properly if the watermark is removed. [BGI+01] proposed the notion of watermarking and [HMW07] gives more general and rigorous definitions for watermarking schemes. Later works, such as [CHV15, NW15, KW17, CHN+18, QWZ18] study more variants of goals such as public-key watermarking, publicly-verifiable watermarking and different watermarking schemes for cryptographic primitives, such as PRFs.

2 Preliminaries

In this paper, we use \( \mathbb{F} \) to denote \( \mathbb{F}_2 \) and use \( S(n) \) to denote all \( n/2 \) dimensional subspaces in \( \mathbb{F}^n \).

**Definition 2.1** (Dual Subspace). Given a subspace \( S \) of a vector space \( V \), let \( S^\perp \) be the orthogonal complement of \( S \): the set of \( y \in V \) such that \( x \cdot y = 0 \) for all \( x \in S \). It is not hard to show: \( S^\perp \) is also a subspace of \( V \); \((S^\perp)^\perp = S\).

2.1 Quantum Information and Query Models

In this work, we consider the quantum query model, which gives quantum circuits access to some oracles. The classical and quantum oracles are defined as follows.

**Definition 2.2** (Classical Oracle). A classical oracle \( O \) on input query \( x \) is a unitary transformation of the form \( |x⟩ \rightarrow (-1)^{h(x)}|x⟩ \) for function \( h: \{0, 1\}^* \rightarrow \{0, 1\} \).

**Definition 2.3** (Quantum Oracle). A quantum oracle is an arbitrary \( n \)-qubit unitary transformation that a quantum algorithm can apply in a black-box way. Given oracle \( U_f \) that computes a function \( f \) on input \( |x⟩ \), we usually write as \( U_f |x, y, 0⟩ \rightarrow |x, y + f(x), 0⟩ \): the first part is the quantum system over the set of possible inputs; the second part is the quantum system over a set of possible outputs; the third is the workspace register and is reset after use.

Note that a classical oracle can be queried in quantum superposition. The main difference between classical and quantum oracles is that a quantum oracle can answer with a superposition of computation results. Note that in this paper, when we apply a classical oracle as a multi-bit output function, we apply it only on classical inputs; when we apply a classical oracle in superposition on a quantum input, the oracle implements a boolean function, such as the subspace membership oracle below.
Definition 2.4 (Subspace Membership Oracles). A subspace membership oracle for a subspace \( A \subset \mathbb{F}^n \), denoted as \( U_A \), on input vector \( v \), will output 1 if \( v \in A \), \( v \neq 0 \) and will output 0 otherwise.

The “No Cloning Theorem” states that it’s impossible to clone an unknown arbitrary quantum state. This impossibility can also be characterized by query complexity, as a generalization of the No-Cloning Theorem and the BBBV lower bound for quantum search.

Theorem 2.5 (Complexity-Theoretic No-Cloning [AC12]). Given one copy of \( |\psi\rangle \), as well as oracle access to \( U_\psi \) such that \( U_\psi |\psi\rangle = -|\psi\rangle \) and \( U_\psi |\phi\rangle = |\phi\rangle \) for all \( |\phi\rangle \) orthogonal to \( |\psi\rangle \), a counterfeiter needs \( \Omega(2^{n/2}) \) queries to prepare \( |\psi\rangle \otimes 2 \) with certainty (for a worst-case \( |\psi\rangle \)).

In addition, we also use the “Gentle Measurement Lemma” or “Almost As Good As New Lemma”.

Theorem 2.6 (Gentle Measurement Lemma [Aar04]). Suppose a measurement on a mixed state \( \rho \) yields a particular outcome with probability \( 1 - \epsilon \). Then after the measurement, one can recover a state \( \tilde{\rho} \) such that \( \| \tilde{\rho} - \rho \| \leq \sqrt{\epsilon} \).

2.2 Cryptography

Definition 2.7 (Negligible Function). We call a function \( \delta(n) \) as a negligible function if for all \( c \in \mathbb{N}, \exists n_0 \in \mathbb{N} \) such that \( \delta(n) < n^{-c} \) for all \( n > n_0 \). We denote a negligible function in parameter \( n \) as \( \text{negl}(n) \).

Definition 2.8 (Quantum Unlearnability). We consider a quantum polynomial-time algorithm \( A \): a classical oracle access to function \( f \); function \( f \) is sampled from \( \mathcal{F}_n \) with an efficiently computable testing distribution \( \mathcal{D} \) over the domain \( \{0,1\}^n \); let \( \mathcal{A}^f(1^n) \to \tilde{f} = (C, \rho_f) \) be the output of \( A \), where \( C \) is a polynomial-size quantum or classical circuit and \( \rho_f \) is a (mixed) state.

A distinguisher \( \text{Dist} \) is a quantum polynomial-time algorithm having access to full description of \( f, \tilde{f} \) and testing distribution \( \mathcal{D} \): \( \text{Dist} \) samples polynomially many inputs \( x \) from \( \mathcal{D} \) and check if \( \tilde{f}(x) = f(x) \). \( \text{Dist} \) outputs 1 if and only if it finds any \( x \) such that \( \tilde{f}(x) \neq f(x) \); otherwise it outputs 0. We call a family of functions \( \mathcal{F}_n \) quantumly unlearnable if for any such quantum polynomial-time algorithm \( A \), there exists a negligible function \( \text{negl}(n) \) for all \( n \in \mathbb{N} \) such that:

\[
\Pr_{f \in \mathcal{F}_n} \left[ \text{Dist}(f, \tilde{f}, \mathcal{D}) = 0 : \tilde{f} \leftarrow \mathcal{A}^f(1^n) \right] \leq \text{negl}(n). \tag{1}
\]

Through the paper, we will sometimes refer to this probability above as the probability that \( \tilde{f} \) gets verified or the advantage of adversary \( A \).

Definition 2.9 (Public Key Quantum Money). A public-key (publicly-verifiable) quantum money should consists of the following algorithms:

- \( \text{KeyGen}(1^n) \to (sk, pk) \): takes as input a security parameter \( n \), and generates a key pair \( (sk, pk) \).
- \( \text{GenNote}(sk) \to |\$\rangle \): takes in a secret key \( sk \) and generates a quantum banknote state \( |\$\rangle \).
- \( \text{Ver}(pk, |\$\rangle) \to 0/1 \): takes as input public key \( pk \), and a claimed money state \( |\$\rangle \), and outputs either 1 for accept or 0 for reject.
A secure public-key quantum money should satisfy the following properties:

**Verification Correctness:** there exists a negligible function \( \text{negl}(n) \) so that the following holds for any \( n \in \mathbb{N} \):

\[
\Pr_{(sk,pk) \leftarrow \text{KeyGen}(1^n)} [\text{Ver}(pk, \text{GenNote}(sk)) = 1] \geq 1 - \text{negl}(n)
\]

**Unclonable Security:** Suppose a QPT adversary is given \( q = \text{poly}(n) \) number of valid bank notes \( \{\rho_i\}_{i \in [q]} \) and then gives \( q' = q + 1 \) number of claimed bank notes \( \{\rho'_j\}_{j \in [q']} \) to verification algorithm \( \text{Ver} \); for any such QPT \( A \), there exists a negligible function \( \text{negl}(n) \) for all \( n \in \mathbb{N} \) such that:

\[
\Pr_{(sk,pk) \leftarrow \text{KeyGen}(1^n)} \left[ \forall i \in [q'], \text{Ver}(pk, \rho'_j) = 1 : \{\rho'_j\}_{j \in [q']} \leftarrow A(1^n, \{\rho_i\}_{i \in [q]}) \right] \leq \text{negl}(n)
\]

Note that public key quantum money in fact refers to publicly verifiable quantum money, we will conform with the tradition of calling it public key quantum money.

**Definition 2.10 (Digital Signatures).** A (classical) public-key digital signature scheme \( D \) consists of three probabilistic polynomial-time classical algorithms:

- \( \text{KeyGen}(1^n) \rightarrow (sk, pk) \) : takes as input a security parameter \( n \), and generates a secret and public key pair \( (sk, pk) \).
- \( \text{Sign}(m, sk) \rightarrow (m, \sigma) \) : takes in secret key \( sk \) and a message \( m \), and generates a signature \( \sigma \).
- \( \text{Ver}(m, \sigma, pk) \rightarrow 0/1 \) : takes as input public key \( pk \), a message \( m \), and a claimed signature \( \sigma \), and outputs either 1 for accept or 0 for reject.

A secure digital signature scheme should satisfy the following properties:

**Verification Correctness:** For any \( n \in \mathbb{N} \), so that the following holds:

\[
\Pr_{(sk,pk) \leftarrow \text{KeyGen}(1^n)} [\text{Ver}(pk, m, \text{Sign}(sk, m)) = 1] = 1
\]

**Unforgeability:** Suppose a PPT adversary \( A \) is given \( q = \text{poly}(n) \) number of valid signatures from a signing oracle \( O_{\text{Sign}} \) on any message \( A \) queries, and then \( A \) gives a claimed signature \( (\sigma^*, m^*) \) on \( m^* \) not queried before; for any such PPT \( A \), there exists a negligible function \( \text{negl}(n) \) for all \( n \in \mathbb{N} \) such that:

\[
\Pr_{(sk,pk) \leftarrow \text{KeyGen}(1^n)} [\text{Ver}(pk, \sigma^*, m^*) = 1 : (\sigma^*, m^*) \leftarrow A^{O_{\text{Sign}}(1^n)}] \leq \text{negl}(n)
\]

More cryptography primitives and assumptions used are given in Appendix B.

### 3 Quantum Copy-Protection Relative to a Classical Oracle

We define a correct and secure quantum copy-protection as follows.
3.1 Quantum Copy-Protection Definition

Definition 3.1. Consider a family of functions $\mathcal{F}_n : \mathcal{X} \to \mathcal{Y}$, a quantum copy-protection scheme for $\mathcal{F}$ consists of the following procedures:

**Generation** ($1^n, f \to (\text{cp}(f), |\psi\rangle)$): Given $f \in \mathcal{F}_n$ and parameter $n$, the vendor can generate a copy-protected program $\text{cp}(f)$ and a quantum key $|\psi\rangle$ in $\text{poly}(n,m)$ time.

**Computation** ($1^n, \text{cp}(f), |\psi\rangle \to \{y\}_{y \in \mathcal{Y}}$): given $(|\psi\rangle, \text{cp}(f))$, a user can compute the function $f(x)$ for any $x \in \mathcal{X}$ by running the program $\text{cp}(f)$ in $\text{poly}(n)$ time.

**Correctness and Security** We assume the quantum communication channel between the vendor and the customer is secure. A quantum copy-protection scheme with parameter $n$ should satisfy the following properties:

**Correctness:** The customer with the key can compute the function $f$ by running the program $\text{cp}(f)$, i.e.,

$$\forall x \in \{0,1\}^n, \Pr[\text{cp}(f)(x, |\psi\rangle) = f(x)] \geq 1 - \text{negl}(n)$$

the probability is over the randomness used in generation of $\text{cp}(f)$.

**Anti-Piracy:** The anti-piracy security is defined through the game below.

**Setup Phase:** The challenger (vendor) samples $f \leftarrow \mathcal{F}$, and $\mathcal{D}$ is a testing distribution for $f$.
Challenger runs Generation($1^n, f$) for $k$ times to generate $k$ copies of programs:

$$\{(\text{cp}_1(f), |\psi_1\rangle), (\text{cp}_2(f), |\psi_2\rangle), \ldots, (\text{cp}_k(f), |\psi_k\rangle)\}.$$  

**Challenge Phase:** The challenger gives the $k$ copies of programs generated above to a quantum polynomial-time adversary $A$.
Afterwards, $A$ generates $k + 1$ programs $P_1, \ldots, P_{k+1}$. Each program $P_i$ consists of a polynomial-size quantum circuit $C_i$ and a (mixed) state $\rho_i$.
To verify these programs, consider the challenger now as a quantum polynomial-time distinguisher algorithm Dist which knows the full description of the function $f$, testing distribution $\mathcal{D}$ and is given the pirated programs $P_1, \ldots, P_{k+1}$. For each program $P_i$, Dist outputs 1 if and only if any input-output difference can be found between $P_i$ and $f$. We say that the $i$-th program $P_i$ is verified if $\text{Dist}(P_i, f) = 0$.
$A$ wins if all $\{P_i\}_{i \in [k+1]}$ are verified.

The scheme has anti-piracy security if for any QPT adversary $A$, there exists a negligible function $\text{negl}(n)$ such that the following holds for any $n \in \mathbb{N}$:

$$\Pr\left[ \forall i \in [k+1], P_i \text{ is verified} : \{P_i\}_{i \in [k+1]} \leftarrow A\left(\{(\text{cp}_i, |\psi_i\rangle)\}_{i \in [k]}\right) \right] \leq \text{negl}(n).$$

where the probability is taken over the choice of $f \in \mathcal{F}$ and the randomness used in the setup.

We sometimes denote the above probability as $\text{Adv}_A$ to be the advantage of $A$ in the anti-piracy security game.
Note that each pirated program $P_i$ for $i \in [k]$ should consist of a polynomial size quantum/classical circuit $C_i$ and a mixed state $\rho_i$. If the circuit $C_i$ produced is classical and does not need auxiliary quantum input, the adversary is defined to always provide a useless $\rho_i$, such as $n$ classical bits of zeros. The distinguisher will then run program $P_i$ on input $x$ as $C_i(\rho_i, x)$. Though the distinguisher is QPT, it needs quantum resources only in order to run the pirated programs; its verification criterion is solely on comparison of input-output behaviors of the pirated programs with those of $f$ and nothing else.

**Remark 3.2.** The testing distribution $D$ over the inputs is important since $D$ can be different for different $f$ sampled from the same family (for example, point functions). The role of the distinguisher $\text{Dist}$ allows the pirate programs to be tested on a polynomial number of points sampled from the distribution. For an unlearnable family of functions $\mathcal{F}$, when we copy-protect $f \in \mathcal{F}$, the distinguisher $\text{Dist}$ uses the same testing distribution $D$ corresponding to $f$ from $\mathcal{F}$ as in Definition 2.8 to verify the pirate programs.

**Remark 3.3.** In this paper, the copy-protection scheme we present is based on a classical oracle, which we will for simplicity refer to as $O$. Since the oracle $O$ is public, not only does adversary $A$ have access to the $O$, the pirate programs produced by $A$ also do. Sometimes we will use the notation $P^O$ or $C^O$ to emphasize that the program or the circuit has access to $O$.

### 3.2 Quantum Copy-Protection Mini-scheme

In this section we take use of the subspace membership oracles in $[AC12]$ to construct a quantum copy-protection scheme.

First, we give a “mini-scheme” for quantum copy-protection, which is secure when giving out a single distribution of software. In Section 5, we will show that we can generalize the mini-scheme from one to polynomial number of copies.

Let $\mathcal{F}_n$ be a family of functions: $\{0,1\}^n \rightarrow \{0,1\}^m$ where $m = \text{poly}(n)$. We assume $\mathcal{F}_n$ is quantumly unlearnable and can be computed by polynomial-size classical circuits.

The mini-scheme for quantum copy-protection of function $f \in \mathcal{F}_n$ is as follows:

**Generation:** The vendor picks a uniformly random subspace $A \subseteq \mathbb{F}_n^{n/2}$ of dimension $n/2$ and prepares a subspace state on $n$ qubits corresponding to $A$:

$$|A\rangle = \frac{1}{\sqrt{|\mathcal{F}|^{n/2}}} \sum_{v \in A} |v\rangle.$$  \hspace{1cm} (5)

as the key of the program. The classical description of $A$ is kept private.

The vendor then generates two classical (boolean function) oracles, $U_A, U_{A^\perp}$ which are membership oracles of subspace $A$ and its dual subspace $A^\perp$.

Next, it generates an oracle $\mathcal{O}$ such that

$$\mathcal{O}(x, v) = \begin{cases} 
  f(x) \oplus g(x) & \text{if } v \in A \text{ and } v \neq 0, \\
  g(x) & \text{if } v \in A^\perp \text{ and } v \neq 0, \\
  h(x) & \text{otherwise.}
\end{cases}$$ \hspace{1cm} (6)

where $g, h : \{0, 1\}^n \rightarrow \{0, 1\}$ are uniformly random functions.

$\mathcal{O}$ checks if $v$ is in $A$ or $A^\perp$ by querying $U_A, U_{A^\perp}$ on $v$.

Finally, vendor gives $|A\rangle$ via a secure quantum channel to the customer and publishes the oracle $\mathcal{O}$ as well as membership oracles $U_A, U_{A^\perp}$. 

9
Computation: Note that with subspace state $|A\rangle$, we can get the subspace state $|A^\perp\rangle$ by applying a Quantum Fourier Transform to $n$ qubits, i.e. $|A^\perp\rangle = H^\otimes_n |A\rangle$. To compute $f(x)$, customer can run the program as

$$O(x, |A\rangle) \oplus O(x, H^\otimes_n |A\rangle).$$

(7)

Note that the customer will first query $O(x, |A\rangle)$ and then $O(x, H^\otimes_n |A\rangle)$. Hence, only one piece of $|A\rangle$ is enough for the computation.

For notation we denote the copy-protected program $cp(f)$ as the oracle access to $O$ for function $f$ (as well as $U_A, U_{A^\perp}$ implicitly). The customer receives the full program as $(cp(f), |A\rangle)$. Throughout the paper, the oracle notation $O$ refers to the specific copy-protection oracle described above.

4 Analysis of The Mini-Scheme

In this section, we’ll show that the mini-scheme satisfies the Definition 3.1 of quantum copy-protection.

Correctness and Efficiency For the Generation part, as shown in [AC12], given the basis of $A$, the subspace state can be prepared in polynomial time. For the oracle, it only needs to check the membership of $A$ and $A^\perp$. Hence, assuming the truth tables of $f, g, h$ are given, the oracle $O$ can be generated in polynomial time. Therefore, the whole Generation part can be done in $\text{poly}(n, |F|)$ time.

For the Computation part, the program can compute $f(x)$ with high probability. Because for all input $x \in \{0, 1\}^n$ with a valid state $|A\rangle$, $cp(f)(x, |A\rangle) \neq f(x)$ happens with only negligible probability $\frac{1}{|F|^2}$, when $U_A(\langle |A\rangle)$ outputs 0 or $U_{A^\perp}(\langle |A^\perp\rangle)$ outputs 0.

Because of this high success probability of a single-round computation, by the gentle measurement lemma Theorem 2.6, the state $|A\rangle$ can be used for polynomial many times.

4.1 Anti-Piracy Security

Next we show that the quantum copy-protection mini-scheme for any unlearnable families of functions $\mathcal{F}_n$ has anti-piracy against any quantum polynomial-time adversaries. More formally:

**Theorem 4.1.** Given a copy-protected program for function $f \in \mathcal{F}_n : \{0, 1\}^n \rightarrow \{0, 1\}^m$, i.e. the oracle access to $cp(f)$ and a subspace state $|A\rangle$, for any QPT adversary $A$, there exists a negligible function $\text{negl}(\cdot)$ such that for any $n \in \mathbb{N}$ and any unlearnable family of functions $\mathcal{F}_n$, the following holds:

$$\Pr \left[ (P_1, P_2) \text{ are both verified : } (P_1, P_2) \leftarrow A(1^n, cp(f), |A\rangle) \right] \leq \text{negl}(n).$$

(8)

Later, for notational convenience, we denote

$$\text{Adv}_{A(\text{cp}(f), |A\rangle)} := \Pr[(P_1, P_2) \text{ are verified : } (P_1, P_2) \leftarrow A(1^n, cp(f), |A\rangle)].$$

(9)

To prove the theorem, we first need to show that the problem of finding two non-zero points in $A$ and $A^\perp$ respectively with only one copy of $|A\rangle$ is hard for any QPT adversary. This is called the “Direct-Product Problem” in [AC12]. It is clear that if the adversary $A$ is able to find two vectors $(u, v)$ where $u \in A - \{0\}$ and $v \in A^\perp - \{0\}$, $A$ can just put them together with the oracle
access to the function computation oracle $O$ to make two successfully verifies pirated programs $P_1 = (C_1^O, (u_1, v_1)), P_2 = (C_2^O, (u_2, v_2))$ where $(u_1, v_1), (u_2, v_2)$ are $(u, v)$ found by $A$. Both $P_1, P_2$ work by querying the oracle $O$ to obtain the $f(x)$ on any $x$. Then anti-piracy security is broken.

The hardness of the direct-product problem was proved by Ben-David and Sattath [BDS16]:

**Theorem 4.2** ([BDS16]). Let $\epsilon > 0$ be such that $1/\epsilon = o(2^{n/2})$. Given one copy of $|A|$ and a subspace membership oracle of $A$ and $A^\perp$, an adversary needs $\Omega(\sqrt{2n}/\epsilon)$ queries to output a pair of non-zero vectors $(u, v)$ such that $u \in A$ and $v \in A^\perp$ with probability at least $\epsilon$.

Since in later security reductions we will refer to the direct-product problem as a security game, here we briefly describe the game:

**Setup Phase:** the challenger samples a random $n/2$-dimensional subspace $A$ from $\mathbb{F}_2^n$; then prepares the membership oracle $U_A$ for $A$, $U_{A^\perp}$ for the dual subspace $A^\perp$ and a quantum state $|A\rangle$, the equal superposition of all elements in $A$.

**Query Phase:** challenger sends $|A\rangle$ to adversary; the adversary can query $U_A, U_{A^\perp}$ for polynomially many times.

**Challenge Phase:** adversary outputs two vectors $(u, v)$; challenger checks if: (1) $(u, v)$ are nonzero; (2) $u \in A, v \in A^\perp$. If these are satisfied, then adversary wins.

We review the proof for **Theorem 4.2** in Appendix A. And we immediately have a corollary for QPT adversaries:

**Corollary 4.3.** For any QPT adversary, given one copy of $|A\rangle$, where random subspace $A \subset \mathbb{F}_2^n$, $\dim(A) = n/2$ and given access to subspace membership oracles of $A$ and $A^\perp$, the probability of finding a pair of non-zero vectors $(u, v)$ such that $u \in A$ and $v \in A^\perp$ is negligible in $n$ for any $n \in \mathbb{N}$.

For the rest of the paper, when we discuss a pair of vectors $(u, v)$, we implicitly refer to non-zero vectors $u \in A$ and $v \in A^\perp$.

**Two Types of Adversary** In the next steps, we show that any adversary which breaks the copy protection scheme would either help solve the direct product problem efficiently or violate the unlearnable property of the underlying function.

For some QPT adversary $A$ which has passed verification and $P^O$ is one of the pirate programs produced by $A$, we divide the queries made by $P$ into two categories, informational and not informational.

All the queries from $P$ to $O$ are in the form of $(x, u)$, where $u$ is an element in the vector space for membership checking. If there exists at least one query that gets a reply for $f(x) \oplus g(x)$ and another one with reply for $g(x)$, for the same $x$, then we call these queries informational; one of these two queries must be on $(x, u)$, for some $u \in A$ and the other query on $(x, v)$ for some $v \in A^\perp$. Otherwise if no queries can get replies of both $f(x) \oplus g(x)$ and $g(x)$ for any $x$, they are not informational; these queries are on $(x, u)$ for $u \in A$ and $u$ in neither $A$ nor $A^\perp$, or on $(x, u)$ for $u \in A^\perp$ and $u$ in neither.

We divide the adversaries for the quantum copy protection mini-scheme into two categories and analyze them respectively:

**Type 1:** All the pirate programs produced by the adversary will make informational queries

**Type 2:** At least one pirate program produced by the adversary will not make any informational queries.
4.1.1 Type 1 Adversary

We show that if all pirate programs produced by \( \mathcal{A} \) make informational queries, then we can extract the information of \( (u,v) \) from their queries; otherwise if at least one pirate program makes no informational query or no query at all, then we can use it to quantumly learn the copy-protected function with only black-box access.

**Lemma 4.4.** For any randomly chosen \( A \subset \mathbb{F}_q^2 \) with \( \dim(A) = n/2 \), if there exists some QPT adversary \( \mathcal{A} \) in the (mini-scheme) anti-piracy security game for some \( f \in \mathcal{F}_n \) with a testing distribution \( \mathcal{D} \) and \( \mathcal{A} \) produces two successfully verified pirate programs \( \mathcal{P}^0_1, \mathcal{P}^0_2 \) with advantage \( \epsilon \), such that the queries made by \( \mathcal{P}_1, \mathcal{P}_2 \) to \( \mathcal{O} \) are informational, then there is a QPT algorithm to obtain two non-zero vectors \( (u,v) \) with probability \( \epsilon' = \epsilon/q \), where \( u \in A, v \in A^⊥ \), and \( q = \text{poly}(n) \).

**Proof.** The challenger in the copy protection security game plays as the adversary in breaking direct-product hardness, denoted as \( \mathcal{A}_1 \). In the reduction, \( \mathcal{A}_1 \) is given the membership oracle access to \( U_A, U_A^⊥ \) and state \( |A| \).

Next, we show that \( \mathcal{A}_1 \) can simulate the copy protection security game for \( \mathcal{A} \) using the information given and uses \( \mathcal{A} \) to obtain the two vectors. \( \mathcal{A}_1 \) samples \( f \in \mathcal{F} \) by itself, and simulates the anti-piracy game defined in **Definition 3.1**, specifically simulating the copy protection oracle \( \mathcal{O} \) for adversary \( \mathcal{A} \) as follows:

1) \( \mathcal{A}_1 \) gives state \( |A| \) and oracle access of \( U_A, U_A^⊥ \) to \( \mathcal{A} \).

2) On query \( (x,v) \) from \( \mathcal{A} \), \( \mathcal{A}_1 \) queries \( U_A, U_A^⊥ \) on \( v \).

3) If \( U_A(v) = 1 \), \( \mathcal{A}_1 \) computes \( f(x) \). After \( \mathcal{A}_1 \) computes \( f(x) \), it samples a random string \( g_x \) from the range of \( f \) (for example, \( g_x \leftarrow \{0,1\} \) if it is a Boolean function).

Then, \( \mathcal{B} \) sends \( f(x) \oplus g_x \) to \( \mathcal{A} \) as the query answer.

Note that \( \mathcal{A}_1 \) needs to keep a table of \( x \) and its corresponding \( g_x \). Everytime on query of \( x \), \( \mathcal{A}_1 \) first goes through the table to see if \( g_x \) has already been recorded before. Otherwise, \( \mathcal{A}_1 \) samples a \( g_x \) and adds it to the table. Since there are only polynomially many queries, \( \mathcal{A}_1 \) only needs polynomial time and space to record \( g_x \).

4) If \( U_A^⊥(v) = 1 \), \( \mathcal{A}_1 \) sends \( g_x \) to \( \mathcal{A} \). The generation of \( g_x \) is the same as above.

5) If both \( U_A^⊥(v) = 0 \) and \( U_A(v) = 0 \), \( \mathcal{A}_1 \) samples another \( h_x \) from the range and keeps a table of \( h_x \) as it does for \( g_x \). \( \mathcal{A}_1 \) sends \( h_x \) to \( \mathcal{A} \).

We can see that \( \mathcal{A}_1 \) perfectly simulates the copy-protection oracle \( \mathcal{O} \). In the end, \( \mathcal{A} \) outputs two pirate programs \( \mathcal{P}_1, \mathcal{P}_2 \) and sends \( \mathcal{A}_1 \). \( \mathcal{A}_1 \) first runs the verification algorithm by testing inputs from \( \mathcal{D} \) to verify the two quantum programs produced by adversary. If they do not pass verification, \( \mathcal{A}_1 \) aborts.

Once the adversary’s pirated programs have passed verification, \( \mathcal{A}_1 \) then runs each pirate programs \( \mathcal{P}_1, \mathcal{P}_2 \) again on a polynomial number of inputs sampled from \( \mathcal{D} \). This time, it destructively measures random two queries, one from \( \mathcal{P}_1 \) and one from \( \mathcal{P}_2 \). For each of these measurements, the reduction takes the \( n \)-bit information in the second half of the register (i.e. the \( u \)-part in \( (x,u) \)), denoted as \( u^*, v^* \); then \( \mathcal{A}_1 \) queries the membership oracles \( U_A \) and \( U_A^⊥ \) on \( u^*, v^* \) respectively to see which subspace they are in.

We require that both pirate programs need to make informational queries and the reduction measures both programs’ queries, since \( \mathcal{A} \) can always just give the entire state \( |A| \) to one of the pirate
programs such that this program makes informational query using $|A\rangle$, but making a destructive measurement to this program’s query will give us only one vector in $A$ or one in $A^\perp$.

Since each $P_1$ can make at most polynomially many queries, $A_1$ can obtain vectors $(u^*, v^*)$ that solve Direct-Product problem, with $1/poly(n)$ probability given that $P_1, P_2$ make informational queries. Since $A$ has non-negligible advantage, $A_1$ has non-negligible advantage with only a $1/poly(n)$ factor of loss.

\[ \square \]

### 4.1.2 Type 2 Adversary

Next, we analyze the case if we cannot find both $u$ and $v$ from the queries made by $P_1, P_2$ to $O$. Then it means at least one of them only gets replies with the information of $(f(x) \oplus g(x), h(x))$ or the information of $(g(x), h(x))$, for all the $x$ queried. Since both $g, h$ are random functions, these replies are random strings uncorrelated with $f(x)$. In this case, the adversary has in fact produced a pirate program $P$ that does not need to query the real oracle $O$ to get passed the verification test. All the query replies can be simulated by sampling random values and keeping a table to be consistent on the values.

**Lemma 4.5.** For any unlearnable function family $F$ and $f \in F$ with a testing distribution $D$, if there exists some QPT adversary $A$ that produces two successfully verified pirate programs with advantage $\epsilon$ in the anti-piracy security game; and at least one program $P$ makes no informational queries to $O$, then there exists a QPT algorithm that learns $f$ with probability $\epsilon/c$, where $c = poly(n)$.

**Proof.** We show the lemma above by showing the following:

\[
\text{Adv}_{A}(1^n, \rho(f), \mathcal{A}_1) \leq c \cdot \text{Adv}_{A_2}(1^n, \mathcal{O}_f)
\]

$A_2$ is a QPT adversary trying to learn $f$ with only black-box access to $f$ given in Definition 2.8; we denote this black box as a classical oracle $\mathcal{O}_f$, which on any query $x \in \mathcal{X}$, answers the query with $y = f(x)$. Here, $\text{Adv}_{A_2}(1^n, \mathcal{O}_f) = \Pr[A_2(1^n, \mathcal{O}_f) \text{ quantumly learns } f] = \Pr[P \text{ is verified : } P \leftarrow A_2(1^n, \mathcal{O}_f)]$.

The challenger in the copy protection security game plays as the adversary in learning a function $f$ using only black-box access: function $f$ along with an input distribution $D$, is sampled from a function family $F$. This adversary is denoted as $A_2$. In the reduction, $A_2$ is given the oracle access to $\mathcal{O}_f$.

Next, we show that $A_2$ can simulate the copy protection security game for $A$ using the information given and uses $A$ to quantumly learn $f$. $A_2$ samples random $n/2$-dimensional subspace $A$ over $\mathbb{F}_2$ and prepares the membership oracles (unitary matrices) $U_A, U_A^\perp$ as well as state $|A\rangle$; it simulates the copy protection oracle $\mathcal{O}$ as follows:

1. $A_2$ gives state $|A\rangle$ and oracle access of $U_A, U_A^\perp$ to $A$.
2. On query $(x, v)$ from $A$, $A_2$ applies the unitaries $U_A, U_A^\perp$ on $v$.
3. If $U_A(v) = 1$, $A_1$ queries $\mathcal{O}_f$ on input $x$. After $A_2$ obtains $f(x)$, it samples a random string $g_x$ from the range of $f$ (for example, $g_x \overset{\$}{\leftarrow} \{0, 1\}$ if it is a Boolean function).

$B$ sends $f(x) \oplus g_x$ to $A$ as the query answer.

Note that $A_2$ needs to keep a table of $x$ and its corresponding $g_x$. Every-time on query of $x$, $A_2$ first goes through the table to see if $g_x$ has already been recorded before. Otherwise, $A_2$ samples a $g_x$ and adds it to the table. Since there are only polynomially many queries, $A_2$ only needs polynomial time and space to record $g_x$.
4) If \( U_{A^\perp}(v) = 1 \), \( A_2 \) sends \( g_x \) to \( A \). The generation of \( g_x \) is the same as above.

5) If both \( U_{A^\perp}(v) = 0 \) and \( U_A(v) = 0 \), \( A_1 \) samples another \( h_x \) from the range and keeps a table of \( h_x \) as it does for \( g_x \). \( A_2 \) sends \( h_x \) to \( A \).

We can see that \( A_2 \) perfectly simulates the copy-protection oracle \( O \) for \( A \). In the end, \( A \) outputs two pirate programs \( P_1, P_2 \) and sends \( A_1 \).

\( A_2 \) randomly chooses one of the pirate programs in \( P_1, P_2 \); we denote this chosen program as \( P \) for simplicity. Importantly, an adversary that successfully learns a function in Definition 2.8 needs to produce a polynomial-size quantum circuit and a state that computes \( f \) without making any oracle queries. We show how \( A_2 \) can obtain such a circuit-state tuple from \( P \).

If \( P \) makes no query to \( O \) at all, then \( A_2 \) simply sends it to the function-learning challenger as \( A_2 \)’s own output. If \( P \) makes query only on \( u \in A \) or \( u \) in neither \( A \) or \( A^\perp \), \( A_2 \) can modify this program, a circuit-state tuple \( P = (C^O, \rho) \), into a circuit-state tuple that does not make any query to \( O \) through the following steps:

- \( A_2 \) adds an additional circuit \( \text{supp} \), together with randomness of length \( \text{poly}(n) \), in order to answer \( C \)’s queries
- If \( P \) makes queries on \( u \in A \) and \( u \) in neither \( A \) nor \( A^\perp \):
  - queries on \((x, u)\) will be answered by \( \text{supp} \) sampling a uniform random string \( s_x \) if \( u \in A \) or another uniform random string \( h_x \) if \( u \notin A \);
  - keeps a table of queried \( x \) and sampled strings to be consistent on the same query
- If \( P \) makes queries on \( u \in A^\perp \) and \( u \) in neither \( A \) nor \( A^\perp \):
  - \( \text{supp} \) samples random \( g_x \) if \( u \in A^\perp \), \( h_x \) if \( u \notin A^\perp \)
  - keeps a table of randomly sampled \( g_x \) for \( u \in A^\perp \) and \( h_x \) for \( u \notin A^\perp \) to be consistent

Note that \( \text{supp} \) can be given the information of \( A \) and \( A^\perp \) since now \( C \) does not make any query to \( O \) and \( A \), \( A^\perp \) are completely independent of any information in \( f \). Because \( P = (C^O, \rho) \) presumably does not make any informational query to \( O \), \( P' \) and \( P \) have the same functionality. And since \( C \) makes only polynomially many queries, \( \text{supp} \) is only polynomial sized.

Now, \( A_2 \) has obtained a circuit-state tuple \( P' = (C'^{\text{supp}}, \rho) \) which does not have access to oracle \( O \) and functions the same as \( P \) produced by \( A \). \( A_2 \) simply submits \( P' \) to the function-learning challenger as its own output. If \( A \)’s programs are supposed to pass the copy-protection verification, \( \Pr[\text{Dist}(C^O(\rho, \cdot), f, \mathcal{D}) = 0] \geq \epsilon \), for some non-negligible probability \( \epsilon \) and if \( A_2 \) picks the program that makes no informational query, then \( A_2 \) will successfully “learn” the function \( f \) with the same input distribution \( \mathcal{D} \) for verifying pirate programs, that is: \( \Pr[\text{Dist}(C'^{\text{supp}}(\rho, \cdot), f, \mathcal{D}) = 0] \geq \epsilon \). The program \( P' \) gets verified with probability \( \epsilon \).

Therefore, \( \text{Adv}_{A(1^n, cp(f), |A|)} \leq c \cdot \text{Adv}_{A_2(1^n, O_f)} \) for \( c \) equals one over the probability that \( A_2 \) picks the program that actually does not make any informational query. If \( A_2 \) randomly chooses one of the two programs, then \( A_2 \) has at least \( 1/2 \) probability of picking the right program given that \( A \) produces at least one program with no informational queries.
**Conclusion** With Lemma 4.5, Lemma 4.4 and Theorem 4.2, we are able to prove the security of the mini-scheme:

**Proof of Theorem 4.1.** For any QPT adversary $A$, given the copy protected program for $(f, \mathcal{D})$ sampled from a family of functions $\mathcal{F}$,

$$\text{Adv}_{A(\mathbb{R}^n, \text{cp}(f), |A|)} \leq q(n) \cdot \Pr[A_1(1^n, f, U_A, U_A^\perp, |A|)] \text{ solves direct product problem}$$

$$+ c(n) \cdot \Pr[A_2(1^n, \mathcal{O}_f) \text{ quantum learns } f]$$

where $q$ and $c$ are some polynomials of $n$, as we specified in Lemma 4.4 and Lemma 4.5.

By the definition of quantumly unlearnable functions, we have $\Pr[A_2(1^n, \mathcal{O}_f) \text{ quantum learns } f] \leq \text{negl}_1(n)$ for some negligible function $\text{negl}_1$. And by Theorem 4.2 and Corollary 4.3, we have $\Pr[A_1(1^n, f, U_A, U_A^\perp, |A|)] \text{ solves direct product problem} \leq \text{negl}_2(n)$ for some negligible function $\text{negl}_2$. Therefore, we can conclude Theorem 4.1. \qed

## 5 Generalized Construction

### 5.1 Polynomial Copies of Program Distributions

In this section, we extend the secure quantum money mini-scheme to a construction of polynomially many copy-protection programs for the same function $f \in \mathcal{F}: \{0,1\}^n \rightarrow \{0,1\}^m$. We define the generalized quantum copy-protection scheme for $f$ as follows:

- For each $i = 1, 2, \cdots, k$, the vendor runs the mini-scheme generation process for $f$:
  - Sample a random $n/2$-dimensional subspace $A_i \subset \mathbb{F}_2^n$; a random function $g_i : \{0,1\}^n \rightarrow \{0,1\}^n$ and another random function $h_i : \{0,1\}^n \rightarrow \{0,1\}^n$.
  - Generate program $P_i = (\text{cp}_i(f), |A_i|)$ as follows. Prepare a subspace state on $n$ qubits corresponding to $A_i$:
    $$|A_i\rangle = \frac{1}{\sqrt{\dim A_i}} \sum_{v \in A_i} |v\rangle.$$  \hspace{1cm} (11)

as the key for the program. The classical description of $A_i$ is kept private. Prepare membership oracles $U_{A_i}, U_{A_i^\perp}$.

Prepare an oracle $\mathcal{O}_i$ such that it computes the following:

$$\mathcal{O}_i(x,v) = \begin{cases} f(x) \oplus g_i(x) & \text{if } v \in A_i \text{ and } v \neq 0, \\ g_i(x) & \text{if } v \in A_i^\perp \text{ and } v \neq 0, \\ h_i(x) & \text{otherwise}, \end{cases}$$  \hspace{1cm} (12)

$\mathcal{O}$ checks if $v$ is in $A$ or $A^\perp$ by querying $U_{A_i}, U_{A_i^\perp}$.

$\text{cp}_i(f)$ denotes the oracle access $\mathcal{O}_i$, as well as $U_{A_i}, U_{A_i^\perp}$.

- Distribute the programs $P_1, \cdots, P_k$ to an authorized customer via a secure quantum channel.

**Remark 5.1.** The random function $g_i$ which is used to mask the value of $f(x)$ must be chosen with fresh randomness at the preparation for each program $P_i$. If we use the same $g$ for two programs $P_i, P_j$, the adversary can easily attack by creating a program that queries $\mathcal{O}_i$ with a vector in $A_i$ and $\mathcal{O}_j$ with a vector in $A_j^\perp$.  

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**Attack by Intersections of Subspaces** We analyze the (im)possibility of an obvious attack. One simple attack to the general scheme is that the adversary buys two pieces of programs \(cp_1(f), |A_1\rangle\) and \(cp_2(f), |A_2\rangle\). By measuring \(|A_1\rangle\), it gets a point \(u \in A_1\). By measuring \(|A_2\rangle\) (or \(H^{\otimes n}|A_2\rangle\)), it gets a point \(u \in A_2\) (or \(A_2^\perp\)). If \(u\) happens to be in \(A_1^\perp\), then the scheme is broken. However, we can show that the probability that \(A_2\) or \(A_2^\perp\) has nontrivial intersection (intersection of elements other than the zero element) with \(A_1\) is negligible. More generally, even for polynomial number of different randomly chosen subspaces of dimension \(n/2\), the probability that there exist any two subspaces where one has non-trivial intersection with the other (or the dual subspace of it) is negligible.

**Claim 5.2.** Given \(k = \text{poly}(n)\) uniformly random subspaces \(A_1, \ldots, A_k \subseteq \mathbb{F}^n\), each with dimension \(n/2\), the probability that there exist \(i, j \in [k]\) such that \(\dim(A_i \cap A_j) \geq n/8\) is negligible.

**Proof.** Fix two different indices \(i\) and \(j\), consider subspaces \(A_i\) and \(A_j\): \(A_i\) and \(A_j\) are chosen randomly and independently from \(S(n)\); each basis vector is selected with probability \(1/2\). The intersection \(A_i \cap A_j\) is also a subspace and let \(\dim(A_i \cap A_j)\) be the dimension of \(A_i \cap A_j\). We denote the random variable \(X_a\) as the \(a\)-th basis vector is selected to be a basis vector for both \(A_i\) and \(A_j\), for \(a \in [n]\). Clearly, \(\sum_a^n \mathbb{E}[X_a] = n/4\). We can bound the probability of obtaining an intersection with dimension \(\geq \frac{n}{8}\) using Chernoff bound:

\[
\Pr \left[ \dim(A_i \cap A_j) \geq \frac{n}{8} \right] = \Pr \left[ \sum_{a=1}^n X_a \geq \frac{1}{2} \mathbb{E}[X_a] \right] \leq e^{-\Theta(n)} \tag{13}
\]

Then for any two subspaces in \(A_1, \ldots, A_k\), we can obtain the possibility that they intersect with a larger than \(n/8\)-dimensional subspace, by union bound,

\[
\Pr \left[ \exists i, j : \dim(A_i \cap A_j) \geq \frac{n}{8} \right] \leq O(k^2) \cdot \Pr \left[ \dim(A_i \cap A_j) \geq \frac{n}{8} \right] \tag{14}
\]

\[
\leq \text{poly}(n) \cdot e^{-\Theta(n)} = \text{negl}(n).
\]

An \(8/n\)-dimensional subspace is only a negligible portion in any \(n/2\)-dimensional subspace; moreover, the probability for the existence of an intersection with dimension larger than \(8/n\) is negligible. Hence, the probability that there exists a non-negligible intersection between any two random subspaces is extremely small.

Therefore, \(A_1, A_2, \ldots, A_k\) have only negligible portions of intersections with overwhelmingly large probability, then exploiting \(A_2, \ldots, A_k\) can hardly help finding two vectors in \(A_i, A_i^\perp\) for any \(i \in [k]\). There is only quadratic improvement even for a quantum adversary using Grover search to find an element in the intersection by the Grover search lower bound. Hence, this kind of attack can be ruled out.

**Theorem 5.3.** For any quantumly unlearnable family of functions \(F : \{0, 1\}^n \rightarrow \{0, 1\}^m\), and any \(f \in F_n\), given \(k = \text{poly}(n)\) copies of programs \((cp_1(f), |A_1\rangle), \ldots, (cp_k(f), |A_k\rangle)\) constructed as above, then for any quantum polynomial-time adversary \(A\), \(A\) cannot break anti-piracy except with negligible probability.

**Proof.** If there exists a QPT adversary that successfully produces \(k + 1\) number of pirate programs \(P_1, P_2, \ldots, P_{k+1}\), then we can follow the proof for the mini-scheme security in Section 4.1 to show that these pirate programs can either be used to extract two non-zero vectors in \(A_i, A_i^\perp\) for some \(i \in [k]\), or be used to violate unlearnability of the protected function.
If all pirate programs \( \{P_i\}_{i \in [k+1]} \) make informational queries to the \( k \) oracles \( \{O_i\}_{i \in [k]} \) given in \( \{cp(f)\}_{i \in [k]} \), then by pigeonhole principle, there must be two programs \( P_i, P_j \) that make queries to the same oracle \( O_{\ell} \) for some \( \ell \in [k] \). Since both of these two programs make informational queries, i.e., they obtain both \( f(x) \oplus g(x) \) and \( g(x) \) from their queries for some \( x \) in the domain, we can then follow similar argument in Lemma 4.4 and obtain two nonzero vectors in \( A_1 \) and \( A_1^\perp \). The reduction algorithm \( A_1 \) can first guess the oracle \( O_i \) that will be queried by two pirate programs; \( A_1 \) uses the oracles it receives from a Direct-Product challenge to prepare \( O_i \) and prepares the rest of membership oracles by sampling subspaces itself. The rest of the proof is the same as in Lemma 4.4. \( A_1 \) now has a \( 1/k \) factor of loss in advantage compared to its advantage in Lemma 4.4, due to guessing.

If there exists one pirate program \( P'_i \) that makes no informational queries to any oracles in \( \{O_i\}_{i \in [k]} \), then it obtains no actual computation result \( f(x) \) for all \( x \) in the domain from the oracles; we follow the argument from Lemma 4.5 to show that a QPT reduction can use this program \( P'_i \) to quantumly learn the unlearnable function \( f \). \( A_2 \) has a \( 2/(k + 1) \) factor of loss in advantage compared to its advantage in Lemma 4.5, since it now randomly picks from \( k + 1 \) programs.

\[\square\]

5.2 Further Security through Authorization

Though the above contruction for \( k \) copies of programs is secure against QPT adversaries, if we want to further extend the generalization of any mini-scheme to polynomial copies, not necessarily relative to an oracle, we can add authorization to each copy of program. Following Aaronson and Christiano [AC12], we can enhance the security for polynomially many copy-protection programs by adding quantum-secure digital signatures. By adding authorization information, we make sure that the adversary can only attack by pirating one underlying copy-protection mini-scheme, rendering an attack by "combining" the information obtained from several programs as impossible. Then the security of polynomial-copy construction is reduced to the security of digital signature and mini-scheme.

Let \( F \) be a quantumly unlearnable family of functions and \( f \in F : \{0,1\}^n \rightarrow \{0,1\}^m \). Let \( S = (\text{KeyGen}_S, \text{Sign}_S, \text{Ver}_S) \) be a digital signature scheme. We first define the general quantum copy-protection scheme for \( f \) as follows:

- The vendor first runs the mini-scheme generation process for \( f \) and gets \( (cp(f), |A|) \). The vendor also creates a random classical serial number \( s_A \) for it. Note that \( cp(f) \) denotes the oracle access \( O \).

- The vendor runs \( \text{KeyGen}_D \) and gets the private-key \( k_{\text{private}, A} \) and public-key \( k_{\text{public}, A} \).

- The published program is \( (\tilde{cp}(f), |A|, \text{Sign}_S(s, k_{\text{private}, A})) \), where \( \tilde{cp}(f) \) computes \( f \) by querying a new oracle \( \tilde{O} \) defined as:

\[
\tilde{O}(x,v,w) = \begin{cases} 
\bot & \text{if } \text{Ver}_S(w, k_{\text{public}, A}) = 0, \\
O(x,v) & \text{otherwise.}
\end{cases}
\] (15)

**Theorem 5.4.** For any quantumly unlearnable family of functions \( F : \{0,1\}^n \rightarrow \{0,1\}^m \), and any \( f \in F \), given \( k = \text{poly}(n) \) copies of programs \( (\tilde{cp}_1(f), |A_1|, w_1), \ldots, (\tilde{cp}_k(f), |A_k|, w_k) \) and given that the quantum copy-protection mini-scheme satisfies anti-piracy and the digital signature scheme \( S \) is quantum-secure, then no quantum polynomial-time adversary \( A \) can break the standard construction except with negligible probability.
Proof. Suppose there exists a quantum polynomial time adversary that can break the general scheme with non-negligible probability, i.e., she can generate \( k + 1 \) programs \( P_1, \ldots, P_{k+1} \) such that

\[
\Pr \left[ \forall i \in [k+1] \ P_i \text{ is verified} : \{P_i\}_{i \in [k+1]} \leftarrow \mathcal{A}\left(\{(cp_j, |A_j), w_j\}_{j \in [k]}\right) \right] \geq \epsilon
\]  

(16)

For some non-negligible probability \( \epsilon \).

Let \( \sigma_i \) be the signature of \( P_i \). Since the oracle \( O \) will first check the signature of the serial number of the program, it means that

\[
\Pr \left[ \forall i \in [k+1] : \text{Ver}_D(\sigma_i, k_{\text{public},i}) = 1 \right] \geq \epsilon
\]  

(17)

By the security of the digital signature scheme \( \mathcal{D} \), it is impossible for any quantum polynomial time adversary to generate one more different signature from \( k \) signatures \( w_1, \ldots, w_k \) that can pass the verification with non-negligible probability. Hence, there exists \( i_1, i_2 \in [k+1] \) and \( j \in [k] \) such that \( \sigma_{i_1} = \sigma_{i_2} = w_j \), which means that the adversary can pirate the \( j \)-th program with non-negligible probability \( \epsilon \). Or equivalently,

\[
\Pr \left[ P_{i_1} \text{ and } P_{i_2} \text{ are verified} : (P_{i_1}, P_{i_2}) \leftarrow \mathcal{A}(cp_j(f), |A_j)) \right] \geq \epsilon.
\]  

(18)

By the security of the mini-scheme (Theorem 4.1), we know that the \( j \)-th program cannot be cloned or pirated. And by Claim 5.2, the other \( k - 1 \) programs cannot provide non-negligible advantage in copying the \( j \)-th program.

Therefore the general copy-protection scheme is secure.

Remark 5.5. The essential goal for quantum copy protection is not to prevent adversaries from creating “unauthorized” programs, but to prevent them from creating any programs that compute the copy-protected function correctly, even if they can be identified as pirate. Here we offer a way to guarantee security for any generalized copy-protection schemes, not necessarily in the oracle setting.

5.2.1 Quantum-Secure Digital Signatures

Rompel [Rom90] showed that any one-way function implies (chosen-message) secure public-key signature scheme and the security reduction is black-box. Security reductions lifted from classical to post-quantum signature schemes are also discussed in [Son14], which showed that the classical generic construction of hash-tree based signatures from one-way functions carry over to the quantum setting. Though we have the information theoretically secure quantum digital signature schemes built from quantum one-way functions in [GC01], it has the limitation that only a limited number of copies can be in circulation, or the scheme becomes insecure; thus [GC01] can not be applied to our scheme.

Boneh and Zhandry [BZ13] show that quantum chosen message queries give an adversary more power than classical chosen message queries by presenting a signature scheme that is secure under classical queries but insecure once an adversary can make quantum queries. In our case we only consider adversaries which have access to quantum resources but interact classically with oracles. Therefore, if there exists classical one-way functions secure against quantum attack, there exists a secure digital signature scheme against quantum chosen message attack that we need.

6 Public Key Quantum Money from Copy Protection

Quantum copy-protection and quantum money are closely related. In this section, we give two constructions from general quantum copy-protection scheme to build public-key quantum money:
one using a CCA2-secure public-key encryption scheme; one using a trapdoor function. The con-
struction based on CCA2-secure PKE has a reliable post-quantum PKE primitive based on LWE
problem; on the other hand, the construction from trapdoor functions may be more lightweight
due to simpler underlying primitives.

Without specification, the quantum copy-protection scheme used refers to a general quantum
copy protection for all quantum unlearnable functions.

6.1 Public-Key Quantum Money from CCA-secure Public-Key Encryption

We give the definition of a public-key encryption scheme and CCA2 security (often simply referred
to as CCA security) in Definition B.3. Quantum CCA-security was studied in [BZ13] where the
adversary can make quantum superposition queries to the encryption and decryption oracles. By
our definition of quantum copy protection security and quantum unlearnability, the oracle queries
are classical and the programs can also only output classical computation results; therefore, we
can consider a notion of “weak quantum CCA security” where the QPT adversary has quantum
resources but only queries the encryption and decryption oracles classically. The weak quantum
CCA-security definition will be the same as given in Definition B.3 except that the adversary is
QPT.

CCA-secure PKE Schemes  [BZ13] presents a proof that public-key quantum CCA can be
obtained from any identity-based encryption scheme that is selectively secure under a quantum
chosen identity attack and such an identity-based encryption scheme can be built from lattice
assumptions, specifically the Learning With Errors (LWE) assumption (Definition B.4). Since their
security requirement is the “strong” quantum CCA security where adversary can make quantum
queries, it also satisfies the weak quantum CCA-security we need.

Another candidate scheme is a CCA-secure McEliece based public-key cryptosystem in the
standard model from [DDMQN12], since McEliece public-key cryptography is yet known to be
broken by quantum adversaries either.

Let us simply assume the quantum resistance of LWE assumption, we can have:

**Theorem 6.1.** Assume that the Learning With Errors problem with certain parameters is hard for
BQP and there exists a secure quantum copy-protection scheme, then there exists a secure public-key
quantum money scheme.

6.1.1 Public-key Quantum Money Scheme I

Assume that we have an underlying public key encryption scheme called PKE, whose description
and the message space \( \mathcal{M} \) are public. The public-key quantum money scheme I is as follows:

- **KeyGen**(1^\text{n}) \rightarrow (pk, sk) : takes in security parameter \( n \); run \( \text{PKE.KeyGen}(1^n) \rightarrow (\text{PKE.pk}, \text{PKE.sk}) \).
  - It also generates a testing parameter \( k = \text{poly}(n) \).
  - Outputs \( pk = (\text{PKE.pk}, k) \) and \( sk = \text{PKE.sk} \).

- **GenNote**(sk) \rightarrow |\$\rangle : takes in the secret key; run Generation algorithm of \( cp \) for the decryption
  function \( \text{PKE.Dec}(sk, \cdot) \) to generate \( cp(\text{PKE.Dec}(sk, \cdot)) \).
  - Outputs \( |\$\rangle = \text{cp}(\text{PKE.Dec}(sk, \cdot)) \).

- **Ver**(pk, |\$\rangle) \rightarrow 0/1 : takes in the public key \( pk = (\text{PKE.pk}, k) \) and a claimed banknote state \( |\$\rangle = P \),
  i.e. a claimed copy-protection program for \( \text{PKE.Dec}(sk, \cdot) \). First, it samples \( k \) number of
uniformly random \( m_i \leftarrow \mathcal{M} \), compute \( c_i = \text{PKE.Enc}(pk, m_i) \), for each \( i \in [k] \). Next, runs \( P \) on \( c_i \) and checks if \( P(c_i) = m_i \), for each \( i \in [k] \); if all of the evaluations are equal, output 1 for accept, otherwise 0 for reject.

6.1.2 Security Analysis

Verification Correctness By the computation correctness of the underlying copy-protection \( cp \) and decryption correctness of the underlying \( \text{PKE} \), a valid banknote \( |\$\rangle = \text{cp}(\text{PKE.Enc}(pk, \cdot)) \) is supposed to compute \( \text{PKE.Dec}(sk, c) \) on a ciphertext \( c = \text{PKE.Enc}(pk, m) \) for any \( m \in \mathcal{M} \) correctly, with all but negligible probability. Therefore, verification correctness holds.

Unclonable Security We give a brief proof for the unclonable security of the quantum money scheme, whose security definition is given in Definition 2.9. We view the decryption functions \( \text{PKE.Dec}(sk, \cdot) \), where \( sk \leftarrow \text{PKE.KeyGen}(1^n) \) as a family of functions.

Claim 6.2. Assuming that \( \text{PKE} \) is quantum CCA-secure, the family of decryption functions \( \text{PKE.Dec}(sk, \cdot) \), where \( (sk, pk) \leftarrow \text{PKE.KeyGen}(1^n) \) is quantumly unlearnable (even given \( pk \) ), where the distinguisher uses a testing distribution \( \mathcal{D}_u \) over inputs that are encryptions of uniform random messages from message space using the corresponding public key \( pk \).

Proof. Suppose there exists some QPT adversary \( \mathcal{A} \) for which the decryption function family is quantumly learnable with respect to the testing distribution \( \mathcal{D}_u \), then we can construct a QPT adversary \( B \) who can break weak quantum CCA security.

After the setup phase, \( B \) passes public information it receives from the challenger to \( \mathcal{A} \). During the query phase of CCA security game, \( B \) simply passes \( \mathcal{A} \)’s computation queries on \( c \) to the decryption oracle; the oracle will return \( m \) or \( \perp \) and \( B \) sends the reply to \( \mathcal{A} \). During challenge phase, \( B \) chooses uniform random messages \( m_0, m_1 \leftarrow \mathcal{M} \) and queries the encryption oracle to get a challenge ciphertext \( ct = \text{Enc}(pk, m_b) \). \( B \) does the same thing during the query phase after challenge as the query phase before. After \( \mathcal{A} \) goes into its challenge phase, it will provide a quantum circuit \( C \) that supposedly computes \( \text{Dec}(sk, \cdot) \). \( B \) applies \( C \) on \( ct \) and gets \( m'_b \). If \( \mathcal{A} \) has non-negligible advantage \( \epsilon \) in successfully learning \( \text{PKE.Dec}(sk, \cdot) \), then we have \( m'_b = m_b \) with probability \( \epsilon \). Therefore, \( B \) breaks weak quantum CCA security with advantage \( \epsilon \).

We can then conclude that the decryption function can be copy-protected with respect to this testing distribution \( \mathcal{D}_u \), the same distribution used in \text{Ver} of the quantum money scheme.

Corollary 6.3. Assuming that quantum copy-protection scheme \( cp \) has anti-piracy security and \( \text{PKE} \) is (weak) quantum CCA-secure, then public-key quantum money scheme I has unclonable security.

Proof. By Claim 6.2, if \( \text{PKE} \) is (weak) quantum CCA-secure, \( \text{PKE.Dec} \) is quantumly unlearnable; therefore, it can be securely copy-protected since we assume the existence of a general copy-protection scheme for quantumly unlearnable functions. Suppose there is a QPT adversary \( \mathcal{A} \) that breaks unclonable security, then we can construct a QPT adversary \( B \) that breaks anti-piracy security for \( cp \).

The quantum copy protection challenger will copy-protect a decryption function of \( \text{PKE} \): it first runs the setup \( \text{PKE.KeyGen} \) and publishes the public key \( pk \); then generates \( k \) copies of copy protection program \( \{cp_i(\text{Dec}(sk, \cdot))\}_{i \in [k]} \) and gives them to \( B \). Then \( B \) sends \( pk \) to \( \mathcal{A} \) as the public key for quantum money and \( \{cp_i(\text{Dec}(sk, \cdot))\}_{i \in [k]} \) as \( k \) money states \( \{|\$\rangle\}_{i \in [k]} \). Finally, \( \mathcal{A} \) output
$k+1$ number of claimed money states $\{p_i\}_{i \in [k+1]}$ and sends to $B$. $B$ uses them as its pirate programs and passes to copy-protection challenger. Since the testing distributions for the money states and copy protection of $\text{PKE.} \text{Dec}$ are the same, if $\mathcal{A}$ has non-negligible advantage $\epsilon$, then $B$ has non-negligible advantage $\epsilon$.

6.2 Public-key Quantum Money from Trapdoor Functions

The family of trapdoor functions we need for our construction requires three more properties, in addition to Definition B.2:

- **Injective:** we need these trapdoor functions to be injective so that their inverses can be computed correctly and anyone can verify them with only negligible error.

- **Uniform testing distribution:** for an injective trapdoor function $f \in \mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$, we can verify, using a testing distribution $\mathcal{D}_u$, if a program $P$ is a copy-protection program for inversion function with trapdoor $g(t, \cdot)$, where $t$ sampled with $f$ from $\mathcal{F}$, such that $g(t, f(x)) = x, \forall x \in \mathcal{X}$. The testing inputs in $\mathcal{D}_u$ is obtained by sampling uniformly random $x \leftarrow \mathcal{X}$ and computing $y = f(x)$. Then the verification is to check if $P(y) = x$.

- **Quantum unlearnability of inverse:** for a family of injective trapdoor functions $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$, the family of its inversion functions $\{g(t, \cdot) : \forall x \in \mathcal{X}, g(t, f(x)) = x, f \in \mathcal{F}, t \text{ is a trapdoor sampled with } f\}$ is quantum unlearnable, with respect to testing distribution $\mathcal{D}_u$.

For rest of this section, we will refer to these inversion functions $\{g(t, \cdot)\}$ where the corresponding trapdoor function $\mathcal{F}$ satisfies the above properties, as inverse trapdoor functions for short. Many existing trapdoor function constructions satisfy the injective property and uniform testing distribution property. The third property is less trivial: inversion of trapdoor functions cannot be computed efficiently by their one-wayness, but giving the adversary an inversion oracle is more power; we will give some discussions on candidate trapdoor functions.

**Theorem 6.4.** If there exists a secure quantum copy-protection scheme and certain quantum-secure injective trapdoor functions, then there exists a secure public-key quantum money scheme.

6.2.1 Public Key Quantum Money Scheme II

Our public key quantum money scheme II is as follows. We assume that we already have an underlying quantum copy-protection scheme $\text{cp}$ and we have a trapdoor function family $\mathcal{F}$.

**KeyGen**$(1^n, \mathcal{F}) \rightarrow (\text{pk}, \text{sk})$: takes in security parameter $n$ and a description of a trapdoor function family $\mathcal{F} : \{0,1\}^m \rightarrow \{0,1\}^l, m,l = \text{poly}(n)$; samples $f$ and its corresponding trapdoor $t$ from $\mathcal{F}$, and generates a testing parameter $k = \text{poly}(n)$.

Outputs $\text{pk} = (k, f(\cdot))$ and $\text{sk} = g(t, \cdot)$, where $f(\cdot)$ and $g(t, \cdot)$ are presented in descriptions.

**GenNote**$(\text{sk}) \rightarrow |\$\rangle$: takes in the secret key, the inverse trapdoor function $g(t, \cdot)$ and runs $\text{Generation}$ algorithm of $\text{cp}$ to generate $\text{cp}(g(t, \cdot))$, a copy-protection program for $g(t, \cdot)$.

Outputs $|\$\rangle = \text{cp}(g(t, \cdot))$. 

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Ver(pk,|$⟩) → 0/1 : takes in the public key \( f \), testing parameter \( k \) and a claimed banknote state \(|$⟩ = P\), i.e. a claimed copy-protection program for \( g(t, \cdot) \). It samples \( k \) number of uniformly random \( x_i \leftarrow \{0, 1\}^m \), compute \( y_i = f(x_i) \), for each \( i \in [k] \). Next, runs \( P \) on \( y_i \) and checks if \( P(y_i) = x_i \) for each \( i \in [k] \); if all of the evaluations are equal, output 1 for accept, otherwise 0 for reject.

### 6.2.2 Security Analysis

**Verification Correctness:** By the computation correctness of the underlying copy-protection \( cp \), a valid banknote \(|$⟩ = cp(g(t, \cdot)) \) is supposed to compute \( g(t, y) \) on any valid \( y = f(x) \) correctly, with all but negligible probability. Therefore, verification correctness holds.

**Unclonable Security**

**Claim 6.5.** Assume that quantum copy-protection scheme \( cp \) has anti-piracy security and the copy-protected trapdoor functions satisfy the properties of injectiveness, \( D_u \) as testing distribution and quantum unlearnability of inverses, then public key quantum money scheme II has unclonable security.

Since the inverse trapdoor functions have quantumly unlearnability with respect to distribution \( D_u \), the proof idea is the same as Corollary 6.3 except by replacing the copy-protected function with \( f_t^{-1} \) and the public key with \( f \).

**Discussions on Candidate Trapdoor Functions:** A candidate trapdoor function built from the Learning With Errors (LWE) assumption is the injective trapdoor function in [GPV07] using a Type-I or a [MP12] lattice trapdoor. It satisfies the first two properties we need.

One question remains whether its inverse function has the property of being quantumly unlearnable. [Reg10] shows that if the oracle to compute the inverse of this LWE trapdoor function can answer superposition queries and give superposition outputs, then an efficient quantum attack to find out the trapdoor exists. However, the oracle in the quantum learnability challenge can be only interacted with classically; the copy protection program is also defined to have no superposition output for the underlying function copy-protected. [Reg10] also mentioned the difficulty of finding a trapdoor when no superposition queries can be made and if the inputs for the inverse function are required to be valid outputs of the trapdoor function; but no provable hardness is known.

Another option is to investigate trapdoor functions built from other post-quantum primitives such as code-based cryptography.

**Remark 6.6 (Trivial Private Quantum Money Implication).** It is easy to see that quantum copy-protection for any unlearnable functions will imply private-key quantum money: just let the money issuing authority verify a money state like verifying a pirate program. The testing distribution for many families of functions must be private (e.g. point functions or evasive functions) and otherwise important information about the functions protected would leak. But even for balanced functions with uniform testing distributions that can be public, such as PRF, the verifier still needs to have the same copy-protection program by itself in order to verify a computation, which makes much less sense. Therefore, this implication from copy-protection to public-verifiable quantum money is not easily realizable without the idea of using a trapdoor or a public key/private key pair.
7 Open Problems

7.1 Discussions of Explicit Copy-Protection Schemes

We have obtained a first provably secure copy-protection scheme based on a classical oracle. However, this oracle is very strong: it in fact computes the function for us. Therefore, we want to move forward to remove this too ideal oracle using standard cryptographic assumptions or weaker oracles, such as random oracle, to build quantum copy-protection.

The [AC12] public key quantum money scheme has been instantiated by [Zha17] assuming quantum-secure indistinguishability obfuscation(iO) and injective one-way functions. Using these assumptions, we can construct a subspace obfuscator that has the following property: after applying the subspace obfuscator \( \text{shO} \) to a subspace \( A \) to get \( \text{shO}(A) \), the original membership checking functionality is almost not affected, but the adversary cannot distinguish between \( \text{shO}(A) \) and \( \text{shO}(B) \) where \( A \subseteq B \) and \( (|F^n| - |B|) \) is still exponentially large.

However, Zhandry’s subspace obfuscator is not enough for quantum copy-protection. We need to show that, given the obfuscated membership program for subspace \( A \) and \( A^\perp \), the direct-product problem is still hard for quantum polynomial-time adversaries, a property stronger than the property realized by \( \text{shO} \). We conjecture that similar techniques may be also applied to construct obfuscators for the membership functions of \( A \) and \( A^\perp \) such that given the obfuscated programs, any QPT adversary still needs exponential time to solve the direct-product problem with non-negligible probability.

Quantum copy-protection is essentially different from quantum money, in that we need to hide the functionality \( f \) in addition to hiding the subspace. Even if we can obtain the subspace membership obfuscator with the property above, there are quite a few obstacles to building a provably secure quantum copy-protection scheme. One main obstacle for moving from the oracle setting to non-black-box setting is that we cannot use the argument about extracting information from the queries made by pirate programs, since there’s no oracle and therefore we cannot measure the queries made. On the other hand, we can also consider more specialized constructions in the cryptographic settings, to copy-protect certain families of functions such as point functions, evasive functions or PRF, instead of a scheme for general unlearnable functions.

7.2 Related Open Problems

In addition to instantiating the copy-protection scheme, the following questions highly related to quantum copy-protection are also worth investigation:

- Can we build computationally secure quantum obfuscation(VBB,VGB,diO,iO) for certain classes of classical circuits?
- Can we build quantum copy-protection/public-key quantum money relative to a (quantum) random oracle?
- What other quantum cryptography primitives can we construct by applying quantum copy-protection(either general or for specific functionalities)?

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A Proof for Direct-Product Theorem

In this section, we briefly review the proof for the direct-product lower bound. We restate the Theorem 4.2 here:

**Theorem 4.2 ([BDS16])**. Let $\epsilon > 0$ be such that $1/\epsilon = o(2^{n/2})$. Given one copy of $|A\rangle$ and a subspace membership oracle of $A$ and $A^\perp$, an adversary needs $\Omega(\sqrt{2^n/4})$ queries to output a pair of non-zero vectors $(u, v)$ such that $u \in A$ and $v \in A^\perp$ with probability at least $\epsilon$.

The main idea is to apply the inner product adversary method (Theorem A.1) from [AC12]:

**Theorem A.1** (Inner Product Adversary Method [AC12]). Let $S(n)$ denote the set of $n/2$ dimensional subspaces in $\mathbb{F}^n$. Let $R$ be a symmetric, anti-reflexive relation on $S(n)$ such that each subspace in $S(n)$ is related to at least one other subspace. Suppose that initially $|\langle \psi_{\text{init}}^A, \psi_{\text{init}}^B \rangle| \geq c$
for all \((A, B) \in \mathcal{R}\), whereas at the end we need \(\mathbb{E}_{(A, B) \in \mathcal{R}}[\|\langle \psi_f^A | \psi_f^B \rangle \|] \leq d\). Then, any algorithm achieving that must make \(\Omega((c - d)2^{n/2})\) oracle queries.

This statement is from [BDS16], a slightly stronger adaption from original theorem in [AC12]. This theorem uses the assumption that \(\mathbb{E}_{(A, B) \in \mathcal{R}}[\|\langle \psi_f^A | \psi_f^B \rangle \|] \leq d\) instead of the original assumption that \(\forall (A, B) \in \mathcal{R}, \mathcal{R}(A, B) \leq d\). But the original proof is valid for this stronger statement.

### A.0.1 Inner Product Adversary Method Overview

The idea of inner product adversary method is to bound how much progress a quantum algorithm can make at distinguishing two oracles after each query. We denote \(|\Psi_f^n\rangle\) be a quantum algorithm \(Q\)'s state after \(t\) queries to the oracle \(U\). We assume that the initial states \(|\Psi_f^0\rangle = |\psi^n\rangle\), for all oracles \(U\) and \(V\). After the final query \(T\), for all oracle pairs \((U, V)\) that algorithm \(Q\) aims to distinguish, we must have \(|\langle \Psi_f^n | \Psi_f^n \rangle| \leq 1/2\). The goal is to show that the inner product can only decrease by no more than \(\epsilon\) after a single query, hence \(Q\) must make \(\Omega(1/\epsilon)\) queries.

In the proof for quantum money in [AC12], for example, we do not have \(|\Psi_f^n\rangle = |\psi^n\rangle\) and in fact the adversary can decrease the inner product \(|\langle \Psi_f^n | \Psi_f^n \rangle|\) by a constant amount after making just one query to \(U\) or \(V\) respectively. The inner product adversary method is to choose a distribution \(D\) over oracle pairs \((U, V)\) and analyze how much the expected inner product can decrease after each query to \(U\) and \(V\):

\[
\mathbb{E}_{(U,V) \sim D}[|\langle \Psi_f^n | \Psi_f^n \rangle|].
\]

For two quantum money states \(|\phi\rangle\) and \(|\psi\rangle\) which satisfy \(\langle \phi | \psi \rangle = 1/2\), in order for the counterfeiter to perfectly counterfeit, he must map \(|\psi\rangle\) to \(|\psi\rangle^\otimes 2\) and \(|\phi\rangle\) to \(|\phi\rangle^\otimes 2\). And he would have to have \(|\langle \phi^\otimes 2 | \psi^\otimes 2 \rangle| = (|\langle \phi | \psi \rangle|)^2 = 1/4\) so he has to decrease the inner product by 1/4. The idea is to show that the average inner product can decrease by at most \(1/\exp(n)\) after each single query; the counterfeiter therefore needs to make \(2^{\Omega(n)}\) queries.

### A.1 Overview of [BDS16]'s Proof for Direct-Product Theorem

Consider the relation \(\mathcal{R}\) that contains all \((A, B)\) such that \(\dim(A \cap B) = n/2 - 1\). Then, the inner product of the initial states \(|A|B\rangle = 1/2\). Let \(\Lambda(A)\) denote all pairs of \((u, v)\) such that \(u \in A \setminus \{0\}\) and \(v \in A^\perp \setminus \{0\}\). At the end of computation, if the algorithm can output a pair \((u, v)\) \(\in \Lambda(A)\) with high probability, then the final state is close to \(|A\rangle \langle A^\perp| B\rangle = 1/4\). The constant gap implies that the algorithm needs exponential number of queries.

More specifically, they first claim that for a fixed subspace \(A\), if the algorithm can succeed with constant probability (i.e. 0.99), then it must make \(\Omega(2^{n/4})\) oracle queries. They show that the inner product of the final states can be upper bounded by:

\[
\mathbb{E}_{(A, B) \in \mathcal{R}}[|\langle \psi_f^A | \psi_f^B \rangle|] \leq 0.2 + \max_{A \in S(n), (a, b) \in \Lambda(A)} \mathbb{P}_{{B : (A, B) \in \mathcal{R}}}[(a, b) \in \Lambda(B)]
\]

\[
\leq 0.2 + 0.25 = 0.45.
\]

where \(|\psi_f^A\rangle\) is the final state of the algorithm when the oracle and initial state encodes the subspace \(A\).

The above claim follows by Theorem A.1. The first half of the inequality is obtained from first decomposing state \(|\psi_f^A\rangle\) and \(|\psi_f^B\rangle\) each as two parts, the “succeeded” and the “failed” parts, for
example $|\psi^A_f\rangle = \sum_{(u,v)\in\Lambda(A)} \beta^A_{uv} |\phi^A_{uv}\rangle |u\rangle |v\rangle + \sum_{(u,v)\notin\Lambda(A)} \beta^A_{uv} |\phi^A_{uv}\rangle |u\rangle |v\rangle$; thus they can rewrite $E_{(A,B)\in\mathbb{R}}[(|\psi^A_f|\psi^R_f)]$ into several components; next, using simple properties such as that the two sums form orthogonal vectors and applying Cauchy-Schwartz inequality, linearity of expectation, etc., one can bound the value by $\max_{A,B} \Pr_{(A,B)\in\mathbb{R}}[(a,b)\in\Lambda(B)]$.

The second line of the inequality follows from counting subspaces. The probability is over picking a vector space whose intersection with $A$ has dimension $n/2 - 1$, which can be done by picking a random basis for $A$, discarding one vector in the basis, adding a vector outside $A$ and using this set of $n/2$ vectors as a basis for $B$. We want to know the probability that $a \in B$ and $b \in B^\perp$, which means $a$ is in the span of the $n/2 - 1$ vectors from $A$ and $b$ is orthogonal to the vector chosen outside $A$. Then, we can bound both of these two events’ probabilities by $\frac{1}{2}$ and since they are independent, the total probability is upper bounded by $\frac{1}{4}$.

Next, they use similar techniques as [AC12] to show that even for exponentially small success probability $\epsilon$ and a random subspace $A$, the algorithm still needs $\Omega(\sqrt{2^{n/4}})$ queries, which completes the proof of the theorem. The first part can be proved by fixed point quantum search; the average-case to worst-case reduction can be proved by applying a random self-reduction: if the adversary can solve the direct-product problem efficiently given a uniformly random state $|A\rangle$, he can solve it given any state $|A\rangle$. Given a certain state $|A\rangle$ and the oracles $U_A, U_{A^\perp}$, apply a uniformly random invertible linear map $f : \mathbb{F}^n \to \mathbb{F}^n$ to subspace $A$. We can compose $f$ with $U_A$ and compose $f^{-T}$, which is the inverse transpose of $f$, with $U_{A^\perp}$ to respectively get the oracles for this uniformly random state $|f(A)\rangle$ and its dual subspace state. If we have an efficient adversary for a uniformly random state, we can just apply $f^{-1}$ to get a counterfeiter for state $|A\rangle$.

## B Cryptography Primitives and Assumptions

**Definition B.1** (One-Way Functions). A family of functions $F : \{0,1\}^n \to \{0,1\}^*$ is one-way if $f \in F$ can be computed in (classical) probabilistic polynomial time (PPT) and for any non-uniform PPT adversary $A$, there exists a negligible function $\negl(n)$ for all $n \in \mathbb{N}$ such that:

$$\Pr[A(f(x)) \in f^{-1}(f(x))] \leq \negl(n)$$

**Definition B.2** (Trapdoor Functions). A family of functions $F : \{0,1\}^n \to \{0,1\}^*$ is family of trapdoor functions if $F$ is a one-way function family and we can sample $f \leftarrow F$ together with a trapdoor $t$, so that for any PPT algorithm $A$:

$$\Pr[A(t, f(x)) \in f^{-1}(f(x))] = 1$$

**Definition B.3** (Public-Key Encryption). A (classical) public-key encryption scheme $PKE$ consists of three probabilistic polynomial-time algorithms:

- **KeyGen($1^n$) → (sk, pk)**: a randomized algorithm that takes as input a security parameter $n$, and generates a secret and public key pair $(sk, pk)$.

- **Enc($m \in M, pk$) → ct**: a randomized algorithm that takes in public key $pk$ and a message $m$ from message space $M$, and generates a ciphertext $ct$.

- **Dec($ct, sk$) → $m/\bot$**: takes as input a secret key $sk$ and a ciphertext $ct$, and outputs a message $m$ or a symbol $\bot$ for decryption failure.

A secure $PKE$ scheme should satisfy the following properties:
Correctness: For any \( n \in \mathbb{N} \), the following holds for any \( m \in \mathcal{M} \):

\[
\Pr[\mathsf{Dec}(sk, ct \leftarrow \mathsf{Enc}(pk, m)) = m] = 1
\]

probability taken over the choice \((sk, pk) \leftarrow \mathsf{KeyGen}(1^n)\).

**IND-CCA (CCA2) Security:** A PKE scheme is secure under (classical) indistinguishability chosen-ciphertext attack, if for every PPT adversary \( A \) there exists a negligible function \( \negl(n) \) such that for all \( n \in \mathbb{N} \), the following holds:

\[
\Pr\left[A^{\mathsf{Dec}(sk, \cdot)}(ct) = b : \quad \begin{array}{c}
(pk, sk) \leftarrow \mathsf{KeyGen}(1^n) \\
((m_0, m_1) \in \mathcal{M}) \leftarrow A^{\mathsf{Dec}(sk, \cdot)}(1^n, pk) \\
b \leftarrow \{0, 1\};
ct \leftarrow \mathsf{Enc}(pk, m_b)
\end{array}\right] \leq \frac{1}{2} + \negl(n),
\]

where \( A^{\mathsf{Dec}(sk, \cdot)} \) denotes that \( A \) has access to the decryption oracle; \( A \) can query the oracle both before and after the challenge phase. Note that \( A \) cannot query the decryption oracle on the encryptions of challenge messages \( m_0, m_1 \).

**Definition B.4 (Learning With Errors Problem).** An \( \mathsf{LWE}_{n,m,q,\chi} \) instance is as follows. Let \( \mathbb{Z}_q \) be the additive group modulo a large integer \( q \); \( A \leftarrow \mathbb{Z}^{n \times m}, s \leftarrow \mathbb{Z}^n, e \leftarrow \chi^m \) where \( \chi \) is an error distribution over \( \mathbb{Z}_q \). Given \((A, b = A^\top s + e)\), the search problem is to find \( s \).

The decisional \( \mathsf{LWE}_{n,m,q,\chi} \) is to distinguish between \((A, b = A^\top s + e)\) and \((A, u)\) where \( u \leftarrow \mathbb{Z}_q \).

**Definition B.5 (Indistinguishability Obfuscation).** An indistinguishability obfuscator \( \mathsf{iO} \) for a circuit class \( \{C_\lambda\} \) is a PPT uniform algorithm satisfying the following conditions:

- For any \( C \in \mathcal{C}_\lambda \), \( \mathsf{iO}(\lambda, C)(x) = C(x) \) for all inputs \( x \).
- For all pairs of PPT adversaries \((\mathsf{Samp}, D)\), if there exists a negligible function \( \alpha \) such that

\[
\Pr[\forall x, C_0(x) = C_1(x) : (C_0, C_1, \sigma) \leftarrow \mathsf{Samp}(\lambda)] > 1 - \alpha(\lambda)
\]

then there exists a negligible function \( \beta \) such that

\[
|\Pr[D(\sigma, \mathsf{iO}(\lambda, C_0)) = 1] - \Pr[D(\sigma, \mathsf{iO}(\lambda, C_1)) = 1]| < \beta(\lambda)
\]