A Fully Polynomial Time Approximation Scheme for Fixed-Horizon Constrained Stochastic Shortest Path Problem under Local Transitions

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Abstract

The fixed-horizon constrained stochastic shortest path problem (C-SSP) is a formalism for planning in stochastic environments under certain operating constraints. Chance-Constrained SSP (CC-SSP) is a variant that allows bounding the probability of constraint violation, which is desired in many safety-critical applications. This work considers an important variant of (C)C-SSP under local transition, capturing a broad class of SSP problems where state reachability exhibit a certain locality. Only a constant number of states can share some subsequent states. (C)C-SSP under local transition is NP-Hard even for a planning horizon of two. In this work, we propose a fully polynomial-time approximation scheme for (C)C-SSP that computes (near) optimal deterministic policies. Such an algorithm is the best approximation algorithm attainable in theory.

1 Introduction

The Markov decision process (MDP) [14] is a classical model for planning in uncertain environments. An MDP consists of states, actions, a stochastic transition function, a utility function, and an initial state. A solution of MDP is a policy that maps a state to an action that maximizes the global expected utility. The stochastic shortest path (SSP) [4] is an MDP with non-negative utility values. The problem has an interesting structure and can be formulated with a dual linear programming (LP) formulation [8] that can be interpreted as a minimum cost flow problem. Moreover, SSP admits many heuristics-based algorithms [5, 12] that utilize admissible heuristics to guide the search without exploring the whole state space.

Besides, constrained SSP (C-SSP) [1] provides the means to add mission-critical requirements while optimizing the objective function. Each requirement is formulated as a budget constraint imposed by a non-replenishable resource for which a bounded quantity is available during the entire plan execution. Resource consumption at each time step reduces the resource availability during subsequent time steps (see [7] for a detailed discussion). A stochastic policy of C-SSP is attainable using several efficient algorithms (e.g., [11]). A heuristics-based search approach in the dual LP can further improve the running time for large state spaces [19]. For deterministic policies, however, it is known that C-SSP is NP-Hard for the finite-horizon case [16] (even when the planning horizon is only 2). The problem is also NP-Hard for the discounted infinite-horizon case [10].

A special type of constraint occurs when we want to bound the probability of constraint violations by some threshold $\Delta$, which is often called a chance constrained SSP (CC-SSP). To simplify the problem, [9] proposes approximating the constraint using Markov’s inequality, which converts the problem to C-MDP. Another approach by [6] applies Hoeffding’s inequality on the sum of independent random variables to improve the bound. Both methods provide conservative policies that respect safety thresholds at the expense of the objective value (which could be arbitrarily worse than the optimal).

In the partially observable setting, the problem is called chance constrained partially observable MDP (CC-POMDP). Several algorithms address CC-POMDP under risk constraints [18, 16]. However, due to partial observability, these methods require an enumeration of histories, making the solution space exponentially large with respect to the planning horizon. To speed up the computation, [13] provides an anytime algorithm using a Lagrangian relaxation method for CC-SSP and CC-POMDP that returns feasible sub-optimal solutions and

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gradually improves the solution’s optimality when sufficient time is permitted. Unfortunately, the solution space is represented as an And-Or tree of all possible history trajectories, causing the algorithm to slow down as we increase the planning horizon.

In this work, we study a variant of (C)C-SSP in which the number of state-action pairs that share subsequent states is bounded by a constant, denoted as (C)C-SSP under local transitions. This variant captures a wide class of SSP problems where state reachability exhibit certain locality such that only a constant number of states can reach some nearby states. The main contribution of this paper is a fully polynomial time approximation scheme (FPTAS) that computes (near) optimal deterministic policies for finite-horizon (C)C-SSP under local transitions in polynomial time. Since, (C)C-SSP is shown to be NP-Hard (even under local transitions assumption [16]), our result is the best possible approximation algorithm attainable in theory.

2 Problem Definition

We provide formal definitions for C-SSP and CC-SSP as follows. A fixed-horizon constrained stochastic shortest path (C-SSP) is a tuple $M = \langle S, A, T, U, s_0, h, C, P \rangle$, where $S$ and $A$ are finite sets of discrete states and actions, respectively; $T : S \times A \times S \rightarrow [0, 1]$ is a probabilistic transition function between states, $T(s, a, s') = \Pr(s' | a, s)$, where $s, s' \in S$ and $a \in A$; $U : S \times A \rightarrow \mathbb{R}_+$ is a non-negative utility function; $s_0$ is an initial state; $h$ is the planning horizon; $C : S \times A \rightarrow \mathbb{R}_+$ is a non-negative cost function; $P \in \mathbb{R}_+$ is a positive upper bound on the cost.

A deterministic policy $\pi(\cdot, \cdot)$ is a function that maps a state and time step into an action, $\pi : S \times \{0, 1, ..., h - 1\} \rightarrow A$. For simplicity, we write $\pi(s_k)$ to denote $\pi(s_k, k)$. A run is a sequence of random states $S_0, S_1, ..., S_{h-1}, S_h$ that result from executing a policy, where $S_0 = s_0$ is known. The objective is to compute a policy that maximizes (resp. minimizes) the expected utility (resp. cost) while satisfying the constraint. More formally,

$$\text{(C-SSP)} \quad \max \pi \mathbb{E} \left[ \sum_{k=0}^{h-1} U(S_k, \pi(S_k)) \right]$$

subject to $\mathbb{E} \left[ \sum_{k=0}^{h-1} C(S_k, \pi(S_k)) \mid \pi \right] \leq P$.

The SSP problem and its constrained variants can be visualized by a direct acyclic And-Or graph (DAG), where the vertices represent the states and actions. Thus, at depth $k$ the set of state nodes are the states that are reachable from previous actions at depth $k - 1$, denoted as $S_k \subseteq S$. At each depth we have at most $|S|$ states. Fig. 1 provides a pictorial illustration of the SSP And-Or (search) graph. Not that unlike And-Or search trees obtained by history enumeration algorithms (see, e.g., [13]), with such representation, a node may have multiple parents, leading to significant reduction in the search space.

Figure 1: SSP graph where circles are state nodes and squares are actions. Circles with thick borders represent reachable states at time $k$, denoted as $S_k$. The objective function and the constraint’s left hand side can be written recursively using Bellman equation
as

\[ v_\pi(s_k) := \sum_{s_{k+1} \in S_{k+1}} T(s_k, \pi(s_k), s_{k+1}) v_\pi(s_{k+1}) + U(s_k, \pi(s_k)), \]

\[ c_\pi(s_k) := \sum_{s_{k+1} \in S_{k+1}} T(s_k, \pi(s_k), s_{k+1}) c_\pi(s_{k+1}) + C(s_k, \pi(s_k)), \]

for \( k = 0, ..., h - 1 \). A fixed-horizon chance-constrained stochastic shortest path (CC-SSP) problem is formally defined as a tuple \( M = (S, A, T, U, s_0, h, r, \Delta) \), where \( S, A, T, U, s_0, h, N \) are defined as in C-SSP, and

- \( r : S \rightarrow [0, 1] \) is the probability of failure at a given state;
- \( \Delta \) is the corresponding risk budget, a threshold on the probability of failure over the planning horizon.

Let \( R(s) \) be a Bernoulli random variable that indicates failure at state \( s \), such that \( R(s) = 1 \) if and only if \( s \) is risky state, and zero otherwise. For simplicity, we write \( R(s) \) to denote \( R(s) = 1 \). The objective of CC-SSP is to compute a deterministic policy (or a conditional plan) \( \pi \) that maximizes (or minimizes) the cumulative expected utility (or cost) while bounding the probability of failure at any time step throughout the planning horizon. More precisely,

\[
\text{(CC-SSP)} \quad \max_{\pi} \mathbb{E} \left[ \sum_{k=0}^{h-1} U(S_k, \pi(S_k)) \right] \tag{2}
\]

Subject to

\[
\Pr \left( \bigvee_{k=0}^{h} R(S_k) \mid \pi \right) \leq \Delta. \tag{3}
\]

To better understand Cons. \(3\), define the \textit{execution risk} of a run at state \( s_k \) as

\[ \text{ER}_\pi(s_k) := \Pr \left( \bigvee_{k'=k}^{h} R(S_{k'}) \mid S_k = s_k \right). \]

According to the definition, Cons. \(3\) is equivalent to \( \text{ER}_\pi(s_0) \leq \Delta \). The lemma below shows that such constraint can be computed recursively.

\textbf{Lemma 2.1} \((2)\). The execution risk of policy \( \pi \) can be written as

\[ \text{ER}_\pi(s_k) = \begin{cases} r(s_k) + (1 - r(s_k)) & \text{if } k = 0, ..., h - 1, \\ \sum_{s_{k+1} \in S} \text{ER}_\pi(s_{k+1}) \pi(s_k) T(s_k, a, s_{k+1}) & \text{if } k = h. \end{cases} \]

In this work, we study a variant of (C)C-SSP in which the number of state-action pairs that share subsequent states is bounded. Such extension denoted as (C)C-SSP under \textit{local transition}. More formally, define the set of potential next states after executing action \( a \) from state \( s_k \in S_k \) for \( k = 0, ..., h - 1 \) by,

\[ N_a(s_k) := \{ s_{k+1} \mid T(s_k, a, s_{k+1}) > 0, s_{k+1} \in S_{k+1} \}, \]

\[ N_a(s_h) := \emptyset. \]

\textbf{Definition 2.2} \((\text{Local Transition})\). There exists a constant \( \psi \in \mathbb{N}_+ \) such that

\[ |\{ s_k \mid N_a(s_k) \cap N_{a'}(s_k) \neq \emptyset, \text{ for } s_k \in S_k, a' \in A \}| \leq \psi, \]

for any \( s_k \in S_k, a \in A \).

When \( \psi = 0 \), we call our problem (C)C-SSP under \textit{disjoint transition}. The And-Or graph under disjoint transition assumption is, in fact, an And-Or tree. Such structure helps to easily obtain a dynamic programming structure that is exploited in our algorithms, shown in subsections 3.1-3.2.

To benchmark our algorithm, we relay on the notion of approximation algorithms. The subject of approximation algorithms is well-studied in the theoretical computer science community [20]. As follows, we define some standard terminology for approximation algorithms. Consider a maximization problem \( \Pi \) with non-negative objective function \( f(\cdot) \); let \( F \) be a feasible solution to \( \Pi \) and \( F^* \) be an optimal solution to \( \Pi \). \( f(F) \) denotes the objective value of \( F \). Let \( \text{OPT} = f(F^*) \) be the optimal objective value of \( F^* \). A common definition of approximate solutions is \( \alpha \)-approximation, where \( \alpha \) characterizes the approximation ratio between the approximate solution and an optimal solution.
Definition 2.3 ([20]). For \( \alpha \in [0, 1] \), an \( \alpha \)-approximation to maximization problem \( \Pi \) is an algorithm that obtains a feasible solution \( F \) for any instance such that \( f(F) \geq \alpha \cdot \text{OPT} \).

In particular, fully polynomial-time approximation scheme (FPTAS) is a \((1 - \epsilon)\)-approximation algorithm to a maximization problem, for any \( \epsilon > 0 \). The running time of a FPTAS is polynomial in the input size and for every fixed \( \frac{1}{\epsilon} \). In other words, FPTAS allows to trade the approximation ratio against the running time.

3 Algorithm

For simplicity, we first study a special variant of (C)C-SSP in which actions stochastically lead to a small number of potential states, denoted as (C)C-SSP under limited transition.

Definition 3.1 (Limited Transition). There exists a constant \( \gamma \in \mathbb{N}_+ \) such that \( N_a(s) \leq \gamma \) for all \( a \in A, s \in S \).

In subsection 3.1, we study (C)C-SSP under limited and disjoint transition, whereas in subsection 3.2, we relax the limited transition assumption. In subsection 3.3, we present an FPTAS for (C)C-SSP under local transition assumption, which generalizes the former cases.

### 3.1 FPTAS for (C)C-SSP under Limited and Disjoint Transition

**Algorithm 1:** \( \text{lim-DynSSP}[M, \epsilon] \)

**Input:** An instance of CC-SSP \( M \); a parameter \( \epsilon \) for the approximation guarantee

**Output:** A deterministic policy \( \pi \)

1. \( \text{DP}_{\text{En}}(s_k, \ell_k) \leftarrow \infty; \text{DP}_{\text{E}}(s_h, \ell_h) \leftarrow 0 \), for \( k = 0, \ldots, h \);
2. \( \text{DP}_{\text{En}}(s_h, \ell_h) \leftarrow r(s_h) \), for all \( s_h \in S_h, \ell_h \in L_h = \{0\} \)
3. for \( k = h - 1, \ldots, 0; s_k \in S_k; \ell_k \in L_k \) do
   4. \( \text{DP}_{\text{En}}(s_k, \ell_k, \ell_k) \leftarrow \text{Update}[s_k, \ell_k, (\text{DP}_{\text{En}}(\cdot, k + 1, \cdot))] \)
5. end for
6. \( \pi \leftarrow \text{Fetch-Policy}[\text{DP}] \)
7. return \( \pi \)

**Algorithm 2:** \( \text{Update}[s_k, \ell_k, \{\text{DP}(s, k + 1, \ell)\}_{s \in S_{k+1}}] \)

1. \( \text{Er}(s_k) \leftarrow \infty \)
2. \( \text{Act} \leftarrow \varnothing \)
3. \( \text{Alloc} \leftarrow 0 \)
4. for \( a \in A \) do
   5. // Find an allocation \( \mathbf{z}_{k+1} \) that achieves the minimum execution risk for action \( a \) such that the total utility value is at least \( \ell_k \)
   6. for \( \mathbf{z}_{k+1} = (\mathbf{z}_{k+1}^1, \mathbf{z}_{k+1}^2, \ldots) \in L_{k+1} \)
   7. \( \pi_a(s_k, \mathbf{z}_{k+1}) \leftarrow \frac{1}{\sum_{a_{k+1} \in N_a(s_k)} T(s_k, a, s_{k+1}) \mathbf{P}(s_k, a_{k+1})} \cdot L_k \)
   8. if \( \pi_a(s_k, \mathbf{z}_{k+1}) \geq \ell_k \) then
      9. \( \text{Er}_{\text{En}}(s_k, \mathbf{z}_{k+1}) = r(s_k) + (1 - r(s_k)) \sum_{a_{k+1} \in N_a(s_k)} T(s_k, a, s_{k+1}) \cdot \text{DP}_{\text{En}}(s_{k+1}, k + 1, \mathbf{z}_{k+1}) \)
   10. if \( \text{Er}_{\text{En}}(s_k, \mathbf{z}_{k+1}) < \text{Er}(s_k) \) then
      11. \( \text{Er}(s_k) \leftarrow \text{Er}_{\text{En}}(s_k, \mathbf{z}_{k+1}) \)
      12. \( \text{Act} \leftarrow a \)
      13. \( \text{Alloc} \leftarrow \mathbf{z}_{k+1} \)
      14. end if
   15. end if
   16. end for
17. end for
18. return \( \text{Er}(s_k), \text{Act}, \text{Alloc} \)

The procedure involves constructing a 3-dimensional dynamic programming table, \( \text{DP}(:, :, :) \), where each cell \( \text{DP}(s_k, k, \ell_k) \) corresponds to state \( s_k \in S_k \), time step \( k \), and a discrete utility value \( \ell_k \) (which we will clarify next). Each cell contains three quantities, \( \text{DP}_{\text{En}}(s_k, k, \ell_k) \in \mathbb{R}_+ \) which maintains the minimum execution risk from state \( s_k \), executing an action that achieves a total value of at least \( \ell_k \); \( \text{DP}_{\pi}(s_k, k, \ell_k) = a \), the corresponding policy action \( a \); and \( \text{DP}(s_k, k, \ell_k) \in \mathbb{R}^{N_a(s_k)} \), a value allocation for subsequent states as we see next. The main idea behind the algorithm lies in a utility discretization procedure that shrinks the set of possible values at a
Algorithm 3: Fetch-Policy[DP]

1. $C_0 \leftarrow \emptyset$ for $k = 1, \ldots, h - 1$
2. $f_0 \leftarrow$ Find the maximum $t_0 \in L_0$ such that $D_{\pi}(s_0, 0, t_0) \leq \Delta$ and $D_{\pi}(s_0, 0, t_0 + L_0) > \Delta$
3. $C_0 \leftarrow \{(s_0, t_0)\}$
4. for $k = 0, \ldots, h - 1$ do
   5.   for $(i_k, \bar{t}_k) \in C_k$ do
   6.     $\pi(i_k) \leftarrow \text{DP}_\pi(s_k', k, \bar{t}_k^i)$
   7.     $\bar{t}_{k+1} \leftarrow \text{DP}_\pi(s_k', k, \bar{t}_k^i)$
   8.     $C_{k+1} \leftarrow C_{k+1} \cup \{(s_{k+1}', \bar{t}_{k+1})\}$ where $a = \pi(i_k)$
5. end for
6. end for
7. return $\pi$

Given state into a manageable number, exploiting the limited and disjoint transition assumptions. A detailed description is provided in Algorithm lim-DynSSP (Alg. 1). The algorithm relies on two subroutines, Update (Alg. 2) and Fetch-Policy (Alg. 3). The former computes a discretized version of the Bellman equation along with the corresponding execution risk, and the latter recursively extracts the corresponding policy. Line 7 of Update computes a discretized version of the Bellman equation under discretized future rewards, and Line 9 recursively computes the execution risk based on Lemma 2.1. The pseudo-code is provided herein is for CC-SSP; however, it is also applicable to C-SSP with minor modifications. Namely, Line 9 of Update should be replaced by

$$\text{ER}_a(s_k, \bar{t}_{k+1}) \leftarrow \sum_{s'_{k+1} \in N_a(s_k)} T(s_k, a, s'_{k+1}) \cdot \text{DP}_\pi(s'_{k+1}, k+1, \bar{t}_{k+1}^j) + C(s_k, a)$$

and $\Delta$ by $P$ in Line 2 of Fetch-Policy. Thus, all results in this paper apply to C-SSP as well. (In the remaining text, the term execution risk in the context of C-SSP would refer to the total cost instead.) Let $U_{\max} := \max_{s \in S, a \in A} U(s, a)$ be the maximum utility of an action. Denote a discrete set of values $L_k$ for each time step $k = 0, \ldots, h$ as

$$L_k := \{0, L_k, 2L_k, \ldots, \left\lceil \frac{U_{\max}(h-k)}{L_k} \right\rceil L_k\}, \quad \text{where}$$

$$L_k := \frac{eU_{\max}}{(h-k)(\ln h + 1)}. \quad (4)$$

Let $\pi$ be a solution returned by lim-DynSSP, and $v_{\pi}(s_k) := \mathbb{E}\{\sum_{k'=h}^{h-1} U(S_{k'}, \pi(S_{k'}))\}$ be the corresponding value function at state $s_k$. Similarly, denote $\pi^*$ to be an optimal solution, and $v_{\pi^*}(s_k)$ be the corresponding value function. Without loss of generality, assume that $v_{\pi^*}(s_0) \geq u_{\max}$ \footnote{If $U_{\max} = U(s_k, a) > v_{\pi^*}(s_0)$, then any policy that outputs action $a$ at state $s_k$ must be infeasible. Thus, such an action can be deleted from the set of allowable actions at state $s_k$. Therefore, $U_{\max}$ can be taken as second largest utility action and so on. The procedure can be performed in polynomial-time as follows. Fix a policy $\pi(s_k) = a$, and set the rest $\pi(s_k') = a_k'$, such that action $a_k'$ achieves the minimum execution risk for $k' = h - 1, \ldots, 0$ (computed recursively using Lemma 2.1). If the solution is infeasible, repeat the procedure at different $k$. If again infeasible, one can safely drop $U(s_k, a)$, consider the next largest utility, and then repeat the procedure.}. Define $\pi^*(s_0)$ (resp., $v_{\pi^*}(s_0)$) to be a discretized objective value computed recursively by,

$$v_{\pi^*}(s_k) = \left\lfloor \frac{1}{L_k} \left( \sum_{s_{k+1} \in \bar{S}_{k+1}} T(s_k, \pi(s_k), s_{k+1}) \cdot v_{\pi^*}(s_{k+1}) + U(s_k, a) \right) \right\rfloor L_k. \quad (5)$$

The above equation corresponds to step 7 of Update.

**Lemma 3.2.** Let $\pi$ be a policy obtained by lim-DynSSP and $\pi^*$ be an optimal deterministic policy. The policy $\pi$ is feasible and satisfies $v_{\pi}(s_0) \geq v_{\pi^*}(s_0)$.

**Proof.** We show (by induction) that for some $\ell_k \in L_k$ and $\bar{t}_{k+1} \in \bar{L}_{k+1}^{N_a(s_k)}$, there exists an action $a$ such that $\ell_k = v_{\pi}(s_k, \bar{t}_{k+1}) \geq v_{\pi^*}(s_k)$, where $v_{\pi}(\cdot)$ is defined in Line 7 of Update. The algorithm enumerates all values of $\bar{t}_{k+1}$ such that it attains the minimum execution risk for every $\ell_k \in L_k$. Throughout recursion,
the procedure ensures that a feasible solution \( \pi \) can be constructed such that \( \ell_k = \ell(a, s_k, \ell_{k+1}) \geq \pi^\star(s_k) \), as shown by Fetch-Policy.

We proceed with the induction proof; for the base case, \( k = h - 1 \), we have \( \ell(a, s_{h-1}, \ell_h) = [U(s_{h-1}, a)/L_{h-1}] L_{h-1} \). Clearly, there is an action that satisfies the claim. For the inductive step, suppose the claim holds at step \( k \), we show that the claim also holds for step \( k - 1 \). Note that algorithm \text{lim-DynSSP} enumerates all discretized allocations \( \ell_k \) at step \( k - 1 \) (as per Step 6 of Update). Also note that \( \pi^\star(s_k) \in L_k \) as the largest element in set \( L_k \), defined in Eq. (4), satisfies \( |U_{\max}(h - k)/L_k| L_k \geq \pi^\star(s_k) \). Hence, there exists an allocation \( \ell_k \) such that each \( i \)-th element \( \ell_k = \ell(a, s_i, \ell_{k+1}) \geq \pi^\star(s_k) \) for some \( \ell_{k+1} \) (inductive assumption). Hence, there exists an \( \ell_{k-1} \) and an action \( a \) such that

\[
\ell_{k-1} = \ell(a, s_{k-1}, \ell_k) = \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in N_{\pi}(s_{k-1})} T(s_{k-1}, a, s_k) \ell_k + U(s_{k-1}, a) \right) \right] L_{k-1} \\
\geq \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in S_k} T(s_{k-1}, a, s_k) \pi^\star(s_k) + U(s_{k-1}, a) \right) \right] L_{k-1} \\
= \pi^\star(s_{k-1}),
\]

(6)

where the inequality follows by the inductive assumption.

It remains to show that such action \( a \) is feasible. Each cell \( \text{DP}_{\pi}(\cdot, \cdot, \cdot) \) corresponds to an action that achieves the minimum execution risk that accrues a total value of at least \( \ell_k \). Since the execution risk is a non-decreasing function (Line 9 of Update), \( \text{DP}_{\pi}(s_k, k, \ell_k) \leq \text{ER}_{\pi}(s_k) \leq \Delta \) for some \( \ell_k = \pi^\star(s_k) \). By the disjoint transition assumption, there is a unique state \( s_{k-1} \) that involves the row \( \text{DP}_{\pi}(s_k, k, \ell_k) \), \( s_k \in N_{\pi}(s_{k-1}) \), in computing \( \text{DP}_{\pi}(s_k, k-1, \ell_{k-1}) \) (Line 9 of Update). Hence, only table cells related to state \( s_{k-1} \) sets the values of \( \ell_k \) of the subsequent states \( s_k \in N_{\pi}(s_{k-1}) \). As each cell \( \text{DP}_{\pi}(s_k, k, \ell_k) \) corresponds to a single action, no two \( s_{k-1}, s'_{k-1} \) share subsequent state \( s_k \), and only one cell among row \( \text{DP}_{\pi}(s_k, k-1, \ell_{k-1}) \) is backtracked by Fetch-Policy, the policy remains consistent, i.e., it outputs a single action for each state. (Note that this is not the case if \( s_k \) has multiple parents in the AND-OR graph, which is the case under local transition assumption.) Such a action is backtracked by Fetch-Policy. Therefore, policy \( \pi \) is feasible.

\[ \square \]

Lemma 3.3. An optimal deterministic policy \( \pi^\star \) satisfies \( \pi^\star(s_k) \geq v^\star(s_k) - \sum_{k' = k}^{h-1} L_{k'} \).

Proof. We proceed with an inductive proof. For the base case, computing Eq. (5) for \( (\cdot, \cdot, \cdot) \) at \( k = h - 1 \), we have

\[
\pi^\star(s_{h-1}) = \left[ \frac{U(s_{h-1}, \pi^\star(s_{h-1}))}{L_{h-1}} \right] L_{h-1} \\
\geq U(s_{h-1}, \pi^\star(s_{h-1})) - L_{h-1} = v^\star(s_{h-1}) - L_{h-1},
\]

(7)

which follows using the property \( \frac{|x|}{y} \geq x - y \) for \( x, y \in \mathbb{R}_+ \). For the inductive step, suppose that we have, \( \pi^\star(s_{k-1}) \geq v^\star(s_{k}) - \sum_{k' = k}^{h-1} L_{k'} \). We compute the corresponding inequality for \( s_{k-1} \) as follows,

\[
\pi^\star(s_{k-1}) = \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in S_k} T(s_{k-1}, \pi^\star(s_{k-1}), s_k) \cdot \pi^\star(s_k) \\
+ U(s_{k-1}, \pi^\star(s_{k-1})) \right) \right] L_{k-1} \\
\geq \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in S_k} T(s_{k-1}, \pi^\star(s_{k-1}), s_k) \cdot \pi^\star(s_k) \\
- \sum_{k' = k}^{h-1} L_{k'} \right) \right] L_{k-1},
\]

(8)

where Eq. (9) follows by the inductive assumption. Since \( T(\cdot, \cdot, \cdot) \) is a probability function that adds up to
one, and using the property \( \lfloor \frac{x}{y} \rfloor y \geq x - y \) for \( x, y \in \mathbb{R}_+ \), we obtain,
\[
\tau_{\pi^*}(s_{k-1}) \geq \sum_{s_k \in S} T(s_{k-1}, \pi^*(s_{k-1}), s_k) \cdot v_{\pi^*}(s_k) + U(s_{k-1}, \pi^*(s_{k-1})) - \sum_{k' = k-1}^{h-1} L_{k'} \\
= v_{\pi^*}(s_{k-1}) - \sum_{k' = k-1}^{h-1} L_{k'},
\]
which completes the inductive proof.

**Corollary 3.4.** \( \text{lim-DynSSP} \) is an FPTAS for \((C)C\text{-SSP}\) under limited and disjoint transition assumptions.

**Proof.** First, observe that the algorithm maintains a dynamic programming table with minimum execution risk. Subroutine \text{Fetch-Policy} ensures that a feasible solution with such property is retrieved. The algorithm runs in \( O((h^2 \ln h + 1)^{\gamma + 1} |A||S|) \). Note that by the limited transition assumption, \( \gamma \) is a constant; thus, the running time is polynomial. By Lemma 3.3 and by the definition of \( L_k \) given in Eq. (4), we obtain

\[
\tau_{\pi^*}(s_0) \geq v_{\pi^*}(s_0) - \sum_{k = 0}^{h-1} L_k \\
= v_{\pi^*}(s_0) - \sum_{k = 0}^{h-1} \frac{\epsilon U_{\max}}{\ln h + 1} \\
= v_{\pi^*}(s_0) - \frac{\epsilon U_{\max}}{\ln h + 1} \sum_{k = 0}^{h-1} (h - k) \\
\geq v_{\pi^*}(s_0) - \frac{\epsilon U_{\max}}{\ln h + 1} (h + 1) \\
\geq (\epsilon - \epsilon) \cdot v_{\pi^*}(s_0),
\]

where Eq. (11) follows by using an upper bound on the harmonic series, \( \sum_{n = 1}^{k} \frac{1}{n} \leq \ln k + 1 \). By Lemma 3.2 and Eq. (12), \( v_{\pi}(s_0) \geq v_{\pi^*}(s_0) \geq \tau_{\pi^*}(s_0) \geq (\epsilon - \epsilon)v_{\pi^*}(s_0) \), which completes the proof.

### 3.2 FPTAS for \((C)C\text{-SSP}\) under Disjoint Transition

In this section, we relax the limited transition assumption and show how to obtain an FPTAS for \((C)C\text{-SSP}\). In other words, we assume \( \gamma \) is a polynomial in definition 3.1. The main idea behind our algorithm is to improve \text{Update} subroutine to avoid full enumeration of \( \overline{L}_{k+1} \), which is exponential in the number of subsequent states \( |N_a(s_k)| \). Such enumeration could be feasible under the limited transition assumption, but not in general. We show here how the structure of this step could be exploited. Notably, finding an allocation that achieves the minimum execution risk such that the total utility value is at least \( \ell_k \) is a slight generalization for a well-known problem called \textit{minimum Knapsack} (MinKS) [17][3]. More formally,

**Definition 3.5.** Multiple-choice minimum Knapsack problem (McMinKS) is defined as follows. Given a set of categories \( \mathcal{N} \), and a set of allowable choices \( \mathcal{M}_i \) per category \( i \in \mathcal{N} \), an item \((i,j)\) is defined by weight \( w_{i,j} \in \mathbb{R}_+ \) and value \( v_{i,j} \in \mathbb{R}_+ \) for \( i \in \mathcal{N} \) and \( j \in \mathcal{M}_i \). The goal is to select one item from the allowable choices \( \mathcal{M}_i \) per category \( i \) (hence the name multiple-choice) such that the total weight is minimized, and the total value is at least \( D \in \mathbb{R}_+ \).

The problem can be formally defined as an integer linear program (ILP) as follows.

\[(\text{McMinKS}) \quad \min \quad \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{M}_i} w_{i,j} x_{i,j}, \]

**Subject to**

\[
\sum_{j \in \mathcal{M}_i} v_{i,j} x_{i,j} \geq D, \quad \sum_{j} x_{i,j} = 1, \quad \text{for all } i \in \mathcal{N}.
\]
Algorithm 4 denoted by **KS-Update**, presents a reduction from the allocation subproblem to McMinKS in Lines 48. Indeed, finding an optimal solution for the corresponding McMinKS instance will obtain an FPTAS (Lemma 3.2 holds and hence Corollary 3.4 proof follows). However, MinKS is NP-Hard [17, 15], therefore our best bet is to find an approximate solution in polynomial time. Although there is an FPTAS for MinKS, the approximation guarantee is provided on the objective function, which in our case, following the reduction, is the constraint for the original (C)C-SSP problem. Thus, we need an algorithm that bounds the constraint violation of McMinKS (which is the objective of (C)C-SSP, following the reduction above). Some modifications are needed to the algorithm to obtain a bounded McMinKS constraint violation and handle the multiple-choice extension (as we will see next).

Algorithm 4: **KS-Update** \( s_k, \ell, \{D(s, k + 1, \ell)\}_{s \in S_{k + 1}} \)

1. \( \text{Er}(s_k) \leftarrow \infty, \text{ACT} \leftarrow \varnothing, \text{ALLOC} \leftarrow 0 \)
2. for \( a \in A \) do
   3.   // Find an allocation \( \ell_{k+1} \) that achieves the minimum execution risk for action \( a \) such that the total utility value is at least \( \ell_k \)
   4.   Let \( N' := \{1, \ldots, |N_a(s_k)|\} \)
   5.   Let \( M_a := \ell_{k+1} \) for \( i \in N' \)
   6.   Let \( w_{i,j} := T(s_k, a, s_{k+1}) \cdot D(s_{k+1}^t, k + 1, \ell_{k+1}) \) for all \( j = \ell_{k+1} \in \ell_{k+1} \) and \( s_{k+1} \in N_a(s_k) \)
   7.   Let \( v_{i,j} := T(s_k, a, s_{k+1}) \cdot T_{k+1}^l \) for all \( k = \ell_{k+1} \in M_a \) and \( i \) such that \( s_{k+1} \in N_a(s_k) \)
   8.   Let \( D := \ell_k - U(s_k, a) \)
   9.   \( \ell_{k+1} \leftarrow \text{Dyn-MinKS}(w_{i,j}, v_{i,j}) \in N', \in M_a, D \)
10.  \( \text{Er}_a(s_k, \ell_{k+1}) := r(s_k) + (1 - r(s_k)) \sum_{k=1}^{s_{k+1} \in N_a(s_k)} T(s_k, a, s_{k+1}) \cdot D(s_{k+1}^t, k + 1, \ell_{k+1}) \)
11.  if \( \text{Er}_a(s_k, \ell_{k+1}) < \text{Er}(s_k) \) then
12.     \( \text{Er}(s_k) \leftarrow \text{Er}_a(s_k, \ell_{k+1}) \)
13.     \( \text{ACT} \leftarrow a \)
14.     \( \text{ALLOC} \leftarrow \ell_{k+1} \)
15. end if
16. end for
17. return \( \text{Er}(s_k), \text{ACT}, \text{ALLOC} \)

Algorithm 5: **Dyn-MinKS** \( (w_{i,j}, v_{i,j}) \in N', j \in M_a, D \)

1. \( \ell = (\ell^l) \in \mathcal{N} \leftarrow 0 \)
2. \( \mathcal{T}(i, \rho) \leftarrow \infty \) for all \( i \in \mathcal{N} \) and \( \rho \in \mathcal{R} \)
3. \( \mathcal{T}(0, 0) \leftarrow 0 \) for all \( \rho \)
4. Let \( \mathcal{N} := \{v_{i,j} \cdot R_k \mid R_k \text{ for all } i \in \mathcal{N} \text{ and } j \in \mathcal{M}_i\} \)
5. for \( i = 1, \ldots, |\mathcal{N}| \) do
6.     for \( k \in \mathcal{R}_k = \{0, R_k, 2R_k, \ldots, D + \max_{i,j} v_{i,j} \cdot R_k\} \) do
7.         \( \mathcal{T}(i, \rho) \leftarrow \min_{\mathcal{M}_i} \mathcal{T}(i-1, [\rho - v_{i,j}]^+) + w_{i,j} \)
8.         \( \text{Alloc}(i, \rho) \leftarrow \min_{\mathcal{M}_i} \mathcal{T}(i-1, [\rho - v_{i,j}]^+) + w_{i,j} \), where \( [x]^+ = x \) if \( x \geq 0 \) and \( [x]^+ = 0 \) otherwise, for any \( x \in \mathcal{R} \)
9.     end for
10. end for
11. Find minimum \( \rho' \) such that \( \rho' \geq D \)
12. for \( i = |\mathcal{N}|, |\mathcal{N}| - 1, \ldots, 1 \) do
13.     \( j = \text{Alloc}(i, \rho'); \rho' \leftarrow \rho' - v_{i,j} \cdot \ell^l \leftarrow j \)
14. end for
15. return \( \ell \)

Algorithm 5 denoted as **Dyn-MinKS**, gives a dynamic programming procedure to solve McMinKS within a bounded constraint violation. The algorithm rounds the values into a discrete set of possible values that provably can have a bounded constraint violation (as per Lemma 3.6 below). The set of possible discretized values \( \mathcal{R}_k \) is defined as

\[
\mathcal{R}_k := \left\{ 0, 1R_k, \ldots, \frac{D + \max_{i,j} v_{i,j}}{R_k} R_k \right\},
\]

where \( R_k \) is a discretization factor defined below. Let \( \ell \) be an allocation returned by algorithm **Dyn-MinKS** and \( \ell^* \) be an optimal solution.

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Lemma 3.6. Algorithm Dyn-MinKS obtains a solution $\ell = (\ell^1, ..., \ell^{|N|})$ that satisfies

$$\sum_{i \in N} w_i,\ell^i \leq \sum_{i \in N} w_i,\ell^i,^{*} \quad \text{and} \quad \sum_{i \in N} v_i,\ell^i \geq \sum_{i \in N} v_i,\ell^i,^{*} - |N|R_k,$$

where $\ell^*$ is an optimal solution.

Proof. Define $\pi_{i,\ell^i} := [v_i,\ell^i / R_k] R_k$ as in Line 4 of Dyn-MinKS (also define $\pi_{i,\ell^i,^{*}} := [v_i,\ell^i,^{*} / R_k] R_k$). The algorithm maintains a table $\mathcal{T}B(i, \rho)$ of minimum total item weights up to category $i$ that satisfies a total value of at least $\rho$. Since the algorithm discretizes values (Line 1), and the largest element of $\mathcal{R}$ is an upper bound on $\sum_{i \in N} \pi_{i,\ell^i,^{*}}$ (by the definition of $\mathcal{R}$ in Eq. 15), then any discretized optimal total values are considered in the table. Therefore in steps 11-13 the algorithm obtains a minimum $\rho \geq D$ that accrues the least total weight, hence $\sum_{i \in N} w_i,\ell^i \leq \sum_{i \in N} w_i,\ell^i,^{*}$. By the feasibility of optimal solutions $\sum_{i \in N} \pi_{i,\ell^i,^{*}} \geq D$, and since $\rho$ is the least element that satisfies $\rho \geq D$, we have

$$\rho = \sum_{i \in N} \pi_{i,\ell^i} \geq \sum_{i \in N} \pi_{i,\ell^i,^{*}}. \quad (16)$$

Thus, by Eq. (16) and rounding values down (Line 4 of Dyn-MinKS),

$$\sum_{i \in N} v_i,\ell^i \geq \sum_{i \in N} \pi_{i,\ell^i} \geq \sum_{i \in N} \pi_{i,\ell^i,^{*}} \geq \sum_{i \in N} (v_i,\ell^i - R_k)$$

$$= \sum_{i \in N} v_i,\ell^i,^{*} - |N|R_k,$$

which completes the proof.

We define algorithm dis-DynSSP by replacing Update at Line 4 of lim-DynSSP by KS-Update, and using the following discretization factors,

$$L_k = \frac{\epsilon U_{\max}}{3(h - k)(\ln h + 1)} \quad \text{and} \quad R_k = \frac{L_k}{\gamma}. \quad (17)$$

Lemma 3.7. Let $\pi$ be a policy obtained by dis-DynSSP and $\pi^*$ be an optimal deterministic policy. The solution $\pi$ satisfies $\pi_{\pi}(s_k) \geq \pi_{\pi^*}(s_k) - 2 \sum_{k' = k}^{h - 1} L_k.$

Proof. We show (by induction) that for some $\ell_k \in L_k$ and action $a$, we have $\ell_k = \pi_{\pi}(s_k) \geq \pi_{\pi^*}(s_k) - 2 \sum_{k' = k}^{h - 1} L_k.$

We proceed with the induction proof; for the base case, $k = h - 1$, we have $\pi_{\pi}(s_{h - 1}) = [U(s_{h - 1}, a)/L_{h - 1}] L_{h - 1}.$

Clearly, there is an action that satisfies the claim. For the inductive step, suppose the claim holds at step $k$; we show that the claim also holds for step $k - 1$.

Since the dis-DynSSP considers all possible values for $\ell_{k - 1}$ at time $k - 1$, there exists an $\ell_{k - 1}$, an action $a$, and a solution $\ell_k$ (Line 5 of KS-Update) such that,

$$\ell_{k - 1} = \begin{cases} \frac{1}{L_{k - 1}} \left( \sum_{s_k \in N(s_{k - 1})} T(s_{k - 1}, a, s_k) \ell_k^i \right) + U(s_{k - 1}, a) \right) L_{k - 1} \\ \geq \begin{cases} \frac{1}{L_{k - 1}} \left( \sum_{s_k \in N(s_{k - 1})} T(s_{k - 1}, a, s_k) \ell_k^i \right) \ell_k^i,^{*} - |N_a(s_{k - 1})| R_k \\ + U(s_{k - 1}, a) \right) L_{k - 1} \end{cases} \right ) \right ) \end{cases} \quad (18)$$

where Eq. (18) follows by Lemma 3.6 (where $v_{i,j} := T(s_{k - 1}, a, s_k^i) \cdot \pi_{\pi}(s_k^i)$ and $w_{i,j} := T(s_{k - 1}, a, s_k^i) \cdot \text{DP}_{\ell_k}(s_k^i, k, \ell_k^i)$ as per Line 6 of KS-Update). By the inductive assumption and Eq. (18),

$$\ell_{k - 1} \geq \begin{cases} \frac{1}{L_{k - 1}} \left( \sum_{s_k \in S} T(s_{k - 1}, a, s_k)(\pi^*_{\pi^*}(s_k) - 2 \sum_{k' = k}^{h - 1} L_k) \\ - |N_a(s_{k - 1})| R_{k - 1} + U(s_{k - 1}, a) \right) L_{k - 1} \end{cases} \quad (19)$$
Therefore,
\[
\ell_{k-1} \geq \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in S_k} T(s_{k-1}, a, s_k)\pi^*(s_k) + U(s_{k-1}, a) \
- 2 \sum_{k' = k}^{h-1} L_{k'} - \gamma R_{k-1} \right) \right] L_{k-1}
\]
(19)

Thus, by the definition of \( R_k \) in Eq. (17), the r.h.s of Eq. (19) can be written as,
\[
\ell_{k-1} \geq \left[ \frac{1}{L_{k-1}} \left( \sum_{s_k \in S_k} T(s_{k-1}, a, s_k)\pi^*(s_k) + U(s_{k-1}, a) \
- 2 \sum_{k' = k}^{h-1} L_{k'} - L_{k-1} \right) \right] L_{k-1}
\]
(20)
\[
\geq \pi^*(s_k) - 2 \sum_{k' = k}^{h-1} L_{k'},
\]
(21)

By the disjoint transition assumption (following the feasibility argument in the proof of Lemma 3.2), policy \( \pi \) is feasible.

**Corollary 3.8.** Algorithm dis-DynSSP is an FPTAS for (C)C-SSP under disjoint transition assumption.

**Proof.** First, observe that the algorithm maintains a dynamic programming table with minimum execution risk. Lines 2 of subroutine Fetch-Policy ensures that a feasible solution with such property is constructed. The algorithm runs in polynomial time as the sizes of \( L_k \) and \( R_k \) are polynomial. By Lemma 3.3 and Lemma 3.7, expanding for \( s_0 \), and by the definition of \( L_k \) and \( R_k \), we obtain
\[
\pi^*(s_0) \geq \pi^*(s_0) - 2 \sum_{k=0}^{h-1} L_k
\]
(22)
\[
\geq v_\pi^*(s_0) - \sum_{k=0}^{h-1} L_k - 2 \sum_{k=0}^{h-1} L_k
\]
(23)

Substituting \( L_k \) obtains,
\[
= v_\pi^*(s_0) - 3 \sum_{k=0}^{h-1} \frac{eU_{\max}}{3(h-k)(\ln h + 1)}
\]
(24)
\[
= v_\pi^*(s_0) - \frac{eU_{\max}}{\ln h + 1} \sum_{k=0}^{h-1} \frac{1}{(h-k)}
\]
(25)

Using the upper bound on the harmonic series \( \sum_{n=1}^{k} \frac{1}{n} \leq \ln k + 1 \) obtains
\[
\pi^*(s_0) \geq v_\pi^*(s_0) - \frac{eU_{\max}}{\ln h + 1} (\ln h + 1)
\]
(26)
\[
\geq (1 - \varepsilon) \cdot v_\pi^*(s_0),
\]
(27)

Therefore, \( \pi^*(s_0) \geq \pi^*(s_0) \geq (1 - \varepsilon) v_\pi^*(s_0) \), which completes the proof.

### 3.3 FPTAS for (C)C-SSP under Local Transition

To deal with (C)C-SSP with local transitions, we show how to convert the allocation subproblem into a multidimensional version of McMinKS (denoted as MMCMinKS). Define a family of disjoint sets of states \( J_k \subseteq 2^{S_k} \) as
\[
J_k := \{ J_k \subseteq S_k \mid N_o(s_k) \cap N_o(s'_k) \neq \emptyset, \text{ for } s_k, s'_k \in J_k; a, a' \in \mathcal{A} \}.
\]
Theorem 3.9. Algorithm \textit{DynSSP} is an FPTAS for (C)C-SSP under local transition assumption.

A proof of the theorem can be obtained using that of Corollary 3.8 with a slight modification of Lemma 3.6 to account for higher dimensional dynamic programming table.
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