Anomalous symmetry protected topological states in interacting fermion systems

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The classification and construction of symmetry protected topological (SPT) phases have been intensively studied in interacting systems recently. To our surprise, in interacting fermion systems, there exists a new class of the so-called anomalous SPT (ASPT) states which are only well defined on the boundary of a trivial fermionic bulk. We first demonstrate the essential idea by considering an anomalous topological superconductor with time reversal symmetry $T^2 = 1$ in 2D. The physical reason is that the fermion parity might be changed locally by certain symmetry action, but is conserved if we introduce a bulk. Then we discuss the layered structure and systematical construction of ASPT states in interacting fermion systems with a total symmetry $G_f = G_b \times Z_2^t$. Finally, potential experimental realizations of ASPT states are also addressed.

Introduction – The bulk-boundary correspondence is an essential concept in the study of topological phases. In recent years, the short-range-entangled symmetry-protected topological (SPT) phases[1], e.g., topological insulators (TIs)[2–6], topological superconductors (TSCs)[5–8], topological crystalline insulators (TCIs)[9] and bosonic SPT (BSPT) phases[10–12] have been studied intensively. A hallmark of these SPT states is the existence of gapless boundary states[13] that cannot be gapped out without breaking the relevant symmetries (spontaneously or explicitly). The nonexistence of a symmetric gapped boundary (without topological orders) can be regarded as a consequence of a boundary anomaly: the symmetry action on the boundary is anomalous and cannot be realized locally (on site) by any lattice model in the same dimension. Such an anomaly is in a one-to-one correspondence with the classification of bulk SPT states[14–23]. For example, in bosonic SPT states, both the boundary anomalies and bulk SPT states are classified by (generalized) group-cohomology theory[10, 11, 24, 25].

Very recent, the concept of equivalent class of finite depth fermionic symmetric local unitary transformation (FSLU) allows us to classify and construct very general fermionic SPT(SPT) states. In particular, it has been shown that the FSLU states have a layered structure[26–33]: they can be constructed by decorating (subject to certain obstructions) 2D $(p+i p)$ topological superconductors to 2D symmetry domain walls, 1D Majorana chains to 1D symmetry domain walls or intersection lines of domain walls, complex fermion modes to 0D symmetry domain walls or intersection points of domain walls, in addition to the bosonic SPT layer.

These layers not only present a way to organize the mathematical structure describing fSPT classifications, but also distinguish physically different types of fSPT states. Most strikingly, it turns out that there exists more types of boundary anomalies when the bulk states have a richer structure. A signature phenomenon in this layered structure is the existence of the so-called anomalous SPT (ASPT) states that can only live on the boundary of a trivial bulk fSPT state. Anomalous surface states have been widely studied in the correspondence between 3D bulk SPT states and 2D long-range-entangled (LRE) surface symmetry-enriched topological (SET) states with anomalous symmetry fractionalization[34–36]. However, here both the bulk and the boundary are SRE states.

If we simply treat the bulk fSPT classification as one additive group, the bulk should be regarded as a trivial state, because its boundary can be realized as a symmetric gapped state(without topological order). Correspondingly, naively it seems that the boundary state is not anomalous as well. Nevertheless, the combination becomes nontrivial once we take into account the layered structure in fSPT classification. The anomalous boundary fSPT states are always built on a lower layer than its bulk. For example, the ASPT states studied below are built by decorating 1D Majorana chains[37] to symmetry domain walls, where its 3D bulk does not contain any Majorana chain decoration. In this setup, the Majorana chain decoration on the surface is anomalous. In this paper, we mainly consider such kinds of ASPT states which are related to fermion parity symmetry violation of FSLU transformation on the boundary. In the following, we will show how to construct this class of ASPT states systematically in 2D interacting fermion systems with a total symmetry $G_f = G_b \times Z_2^t$.

A simple example of $2D T^2 = 1$ ASPT state – It is well known that the 2D FSPT state with $T^2 = -1$ ($G_f = Z_2^{T^2} = Z_2^t \times Z_2^f$) can be constructed in the Majorana chain decoration picture[38]. However, if one wants to construct a similar state for $T^2 = 1$ ($G_f = Z_2^T \times Z_2^f$), there are some inconsistencies between the Kasteleyn orientation (fermion parity) and the symmetry action[38].

Nevertheless, we will show that the $T^2 = 1$ case with the Majorana chain decoration, although not well-
defined in pure 2D, can exist on the boundary of a trivial 3D bulk as an ASPT state. The essential difference is that, although the fermion parity of the 2D symmetric state is not conserved under FSLU transformation, the total fermion parity is conserved if we introduce additional degrees of freedom in the 3D bulk.

Mathematically, for arbitrary $G_f = G_b \times \mathbb{Z}_2^f$, the relation between the 2D boundary ASPT with Majorana decoration[characterized by $n_1 \in H^1(G_b,\mathbb{Z}_2)$] and the 3D bulk fSPT with complex fermion decoration[characterized by $n_3 \in H^3(G_b,\mathbb{Z}_2)$] will be shown to be $n_3 = s_1 \sim n_1 \sim n_1$ (Here, $s_1 \in H^1(G_b,\mathbb{Z}_2)$ indicates whether $g$ is a unitary or anti-unitary group element). Clearly such an anomaly only appears for anti-unitary symmetry group $G_b$. When $G_b = \mathbb{Z}_2^f$, the 2D ASPT state can be realized on the boundary of the 3D bulk $T^2 = 1$ fSPT state constructed in Ref. 26. Since there is a gapped, symmetric boundary state(without topological orders), we conclude that the bulk 3D $T^2 = 1$ fSPT state constructed in Ref. 26 will be trivialized.

Below, we will discuss the scheme of constructing fixed point 2D ASPT state with a total symmetry $G_f = G_b \times \mathbb{Z}_2^f$ on arbitrary triangulation, and how to introduce the 3D bulk fermion degrees of freedom to cancel the anomaly. We first try to construct a symmetric fixed point state in pure 2D. Let us consider the Majorana chain decoration following the procedure of Ref. 31. In addition to the bosonic degrees of freedom $|g\rangle$ ($g \in G_b$) on each vertex of a given triangulation $\mathcal{T}$, each link has two Majorana fermions on its two sides, an arrangement that is equivalent to spinless complex fermion $a_{ij}$, where we can split the complex fermion as $a_{ij} = (\gamma_{ij,A} + i\gamma_{ij,B})/2$. Let $|0\rangle$ be the ground state of no fermions on any of the links; then, a generating set of the Fock space is given by $\prod_{(ij) \in L} a_{ij}^\dagger |0\rangle$, where $L \subseteq L$ is a subset of all links $L$, including the empty set. Thus, the full local Hilbert space for our 2D model on a fixed triangulation $\mathcal{T}$ is $L_\mathcal{T} = \bigoplus_{L \subseteq L} \left( \prod_{(ij) \in L} a_{ij}^\dagger |0\rangle \otimes \prod_{v \in V(\mathcal{T})} C[G_b] \right)$ Here, $|G_b|_2$ is the order of the bosonic symmetry group $G_b$. We further require $a_{ij}$ to be invariant under the $G_b$ action, so the Majorana fermions transform as $U(g)\gamma_{ij,A}U(g)^\dagger = \gamma_{ij,A}$ and $U(g)\gamma_{ij,B}U(g)^\dagger = (-1)^{s_{ij}(g)}\gamma_{ij,B}$, together with $U(g)iU(g)^\dagger = (-1)^{s_{ij}(g)}i$. Given a 2D topological manifold with arbitrary triangulation $\mathcal{T}$ associated with a branching structure[39], one can construct the dual trivalent lattice denoted by $\mathcal{P}$. In order to decorate Majorana chains, we resolve each vertex of $\mathcal{P}$ by a small triangle. The new resolved lattice is called $\mathcal{P}$. We also add branching structures to $\mathcal{T}$ and Kasteleyn orientations to $\mathcal{P}$ following Ref. 31. Each vertex of $\mathcal{P}$ is occupied by a Majorana fermion (of type $A$ or $B$ depending on the Kasteleyn orientation). Given a 1-cocycle $n_1 \in H^1(G_b,\mathbb{Z}_2)$, we decorate a Majorana chain through the link $(ij)$ if and only if $n_1(g_{ij-1}g_{ij}) = 1$ [see Eq. (1) for an example of $n_1(g_{ij-1}^tg_{ij}) = n_1(g_{ij-1}g_{ij}) = 1$ and $n_1(g_{ij-1}^{-1}g_{ij}) = 0$]. Since the ant-unitary symmetry (e.g., time reversal) has nontrivial action on Majorana fermions, the decoration should be designed carefully to respect the symmetry. To be more specific, the Majorana fermions are paired according to the following rules:

(i) If $n_1(g_{ij}^{-1}g_{ij}) = 0$, then the Majorana fermions on the two sides of link $(ij)$ are in vacuum pair: $-i\gamma_{ij,A}^*\gamma_{ij,B}^* = 1$ [see $02A$ and $02B$ in Eq. (1)].

(ii) For triangle $(012)$ with three vertices labelled by the identity element $e$, $g_0^{-1}g_1$ and $g_2^{-1}g_2$, the nontrivial Majorana fermion pairing direction is according to the Kasteleyn orientation. For example, the nontrivial pairing in the left-hand-side figure of Eq. (1) is $-i\gamma_{12A}\gamma_{01A} = 1$.

(iii) For triangle $(012)$ with three vertices labelled by $g_0$, $g_1$ and $g_2$, the nontrivial pairing direction is obtained from rule (ii) by a $g_0$-action. For example, the nontrivial pairing in the right-hand-side figure of Eq. (1) is $U(g_0)(-i\gamma_{12A}\gamma_{01A}) = (1)^{-1}x_{1}(g_0)^{-1}(-i\gamma_{12A}\gamma_{01A}) = 1$. So the pairing direction is reversed if $g_0$ is an anti-unitary element in $G_b$, i.e., $g_0 = T$ when $G_b = \mathbb{Z}_2^f$ (see the blue arrow).

\[
\begin{align*}
\psi = \sum_{\text{all conf.}} \Psi \left( \begin{array}{c} \begin{array}{c} \psi_1 \\
\psi_2 \\
\psi_3
\end{array} \end{array} \right).
\end{align*}
\]

We note that the first two pairing rules are the same as Ref. 31. And the third rule is designed to respect the $G_b$ symmetry. Thus, the 2D symmetric fixed-point state can be constructed as a superposition(subject to proper algebraic conditions) of those basis states with all possible triangulations $\mathcal{T}$.

It is known that for a lattice with Kasteleyn orientations, the fluctuation of decorated Majorana chain (using the first two rules above) would not change the fermion parity of this chain [31, 40, 41]. Since the rule (iii) may violate the Kasteleyn orientations, one may wonder how the fermion parity is changed exactly. It is easy to check that the fermion parity is changed (i.e., the nontrivial pairing direction is reversed) by rule (iii) if and only if $g_0 = T \in \mathbb{Z}_2^f$ and the two Majorana fermions are of the same $A/B$ type. For a triangle $(012)$ with vertex labels
$g_0$, $g_1$ and $g_2$, we can summarize the Majorana fermion parity change as

$$\Delta P_f^2((012)) = (-1)^{s_1-n_1-n_1}(g_0,g_0^{-1}g_1g_2^{-1}g_2).$$  \hspace{1cm} (3)

According to this equation, the right-hand-side of Eq. (1) with $s_1(g_0) = n_1(g_0^{-1}g_1) = n_1(g_1^{-1}g_2) = 1$ is the only configuration in which the pairing direction (blue arrow) is reversed.

**FIG. 1: 2D ASPT on the smallest lattice – boundary of a 3D solid tetrahedron.** There is a complex fermion mode $e^{i\pi n_3(g_0,g_1,g_2,g_3)}$ (blue ball) at the center of the tetrahedron. One Majorana chain (green line) is decorated on the 2D surface.

For the whole 2D system, the fermion parity change (compare to no Majorana chain decoration) is the product of Eq. (3) for all triangles. We can first consider the smallest 2D lattice with 4 triangles (triangulation of 2-sphere) on the boundary of a 3D solid tetrahedron (see Fig. 1). The nontrivial 1-cocycle $n_1 \in H^1(G_6,\mathbb{Z}_2)$ is given by $n_1(e) = 0$ and $n_1(T) = 1$. So for the basis state with $(g_0,g_1,g_2,g_3) = (e,T,e,T)$, there is a single decorated Majorana chain going throught links (01), (12), (23) and (03) (see green line in Fig. 1). According to the rule (iii) and Eq. (3), only the pairing direction inside triangle (123) is reversed, resulting in a Majorana chain with odd fermion parity. Therefore, the desired 2D wave function

$$|\Psi\rangle_{2D} = \sum_{\{g_i\}} \psi(\{g_i\}) |\{g_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D},$$

(4)

is not legitimate for a pure 2D system, since a SRE state should be a superposition of basis states $|\{g_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D}$ with the fermion parity even.

To evade this problem, we can add a 3D bulk, and decorate a complex fermion $e^{i\pi n_3(g_0,g_1,g_2,g_3)}$ at the center of the tetrahedron (blue ball in Fig. 1). We can choose $n_3$ such that the resulting 3D wave function

$$|\Psi\rangle_{3D} = \sum_{\{g_i\}} \psi(\{g_i\}) |\{g_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D} \otimes |c(n_3)\rangle_{3D}$$

is both symmetric and fermion parity even. To find out the specific $n_3$, we observe that the total Majorana fermion parity change of a generic Majorana chain configuration on the four triangles is given by

$$\Delta P_f^2((0123)) = (-1)^{s_1-n_1-n_1}(g_0,g_0^{-1}g_1g_2^{-1}g_2, g_2^{-1}g_3).$$  \hspace{1cm} (4)

This is obtained from the product of Eq. (3) for four triangles and the fact $d(s_1-n_1-n_1) = 0$. Thus, we require the complex fermion number to be

$$n_3 = s_1-n_1-n_1,$$

(5)

such that the total fermion parity $P_f = P_f^1P_f^2$ is fixed. This equation relates the complex fermion decoration in the 3D bulk and the Majorana chain decorations of Majorana fermions on the 2D boundary. One can further show that, with the nontrivial 1-cocycles $s_1$ and $n_1$, this $n_3$ is the nontrivial 3-cocycle in $H^3(\mathbb{Z}_2^3,\mathbb{Z}_2) = \mathbb{Z}_2$. So the 3D bulk is in fact the special group super-cohomology state with $T^2 = 1$ [26].

Despite the fact that the above state is defined on one tetrahedron, one can add more and more vertices in the 3D bulk or on the 2D boundary of the tetrahedron, and finally obtain a larger lattice with arbitrary triangulation. To make the ASPT state well-defined, we only need to show that each Pachner move is $G_6$-symmetric and fermion parity even. There are two types of Pachner moves. The first type is the genuine 3D Pachner moves, which are well-defined for $dn_3 = 0$ and the complex fermions transform trivially under $G_6 = \mathbb{Z}_2^3$.

On the other hand, for the 2D Pachner moves involving the Majorana fermions on the boundary, we have for example the standard (2-2) move:

$$\Delta P_f^2(F_{2D}) = (-1)^{s_1+n_1-n_1}(g_0,g_1,g_2,g_3).$$  \hspace{1cm} (7)

which is again obtained from Eq. (3). Actually Eq. (6) is the only possible (2-2) move that changes fermion parity. Since Eq. (6) also changes the total number of Majorana loops by one, the whole wavefunction does not have a definite fermion parity(fermion parity odd/even for odd/even number of Majorana loop) and is not a legitimate 2D SRE state. Suppose the four vertices on the boundary are connected to a vertex labeled by $g_4$ in the bulk, then the 3D bulk complex fermion parity change under this $F_{2D}$ move is

$$\Delta P_f^2(F_{2D}) = (-1)^{s_3(*012)+n_3(*023)+n_3(*013)+n_3(*123)},$$

(8)

where we have used $dn_3 = 0 \pmod{2}$, and abbreviated $(g_*, g_0, g_1, g_2)$ to $(*)012$ and so on. Since $\Delta P_f^2(F_{2D}) =$
\[ \Delta P_f^i (F_{2D}) \text{, we see that the 2D } F \text{ move does not change the total fermion parity } P_f = P_f^i P_f^j. \]

\[
\begin{align*}
g_1 & \quad g_2 \\
g_0 & \quad g_1 \\
g_0 & \quad g_2
\end{align*}
\]

Similarly, we can consider the (2-0)/(0-2) moves changing the number of vertices, and it is easy to verify that both \( P_f^i \) and \( P_f^j \) are conserved. The combination of the standard (2-2) move and (2-0)/(0-2) will further induce other (2-2) Pachner move and (3-1)/(1-3) Pachner move [42]. In fact, all these 2D Pachner moves are related to the 3D tetrahedron shown in Fig. 1 by projection. So they are all symmetric and total fermion parity even.

Similar to the FSLU approach to fSPT states, the fixed point condition for the (2-2) move will give rise to a Pen- tagon equation which allows us to compute the amplitude \( \psi(\{g_i\}) \). For the case \( G_b = \mathbb{Z}_2^f \), we can choose a simple solution with \( \psi(\{g_i\}) = (\frac{1}{2})^{N_v} \) where \( N_v \) is the total number of vertices for a given triangulation \( T \).

Thus we have constructed an ASPT state with \( T^2 = 1 \) on the 2D boundary of a 3D trivial fSPT system with arbitrary triangulation lattice consistently (to be both symmetric and total fermion parity even). One may wonder whether the bulk complex fermion degrees of freedom can be moved to the 2D boundary, such that this state is defined purely in 2D. For example, for the system with only one complex fermion mode (blue ball) in the bulk in Fig. 1, we can move the complex fermion to the boundary. However, since the complex fermion mode is used to compensate the fermion parity changes for all the boundary triangles, the entanglement between them would introduce nonlocal interactions of the 2D system. So the 3D bulk is an intrinsic feature of this ASPT state.

In fact, after gauging the fermion parity, the above ASPT becomes a \( \mathbb{Z}_2 \) topologically ordered state and all the above physics can be understood as a so-called \( H^3 \) anomaly, which was discussed in the context of classifying 2D symmetry-enriched topological (SET) states [43]. The \( \mathbb{Z}_2 \) topological order has four types of anyons: the trivial anyon \( 1 \) representing bosonic excitations in the ungauged model, the fermionic anyon \( f \) representing fermionic excitations in the ungauged model, and two bosonic anyons \( e \) and \( m \), representing two types of \( \mathbb{Z}_2^f \) vortices. The two types of vortices have opposite fermion parities, indicated by the fusion rule \( m = e \times f \). Since the ASPT state has \( G_b \) symmetry in addition to the fermion-parity symmetry, the resulting state has a \( G_b \)-symmetry-enriched \( \mathbb{Z}_2 \) topological order. Correspondingly, \( n_3 \in H^1(\mathbb{Z}_2) \) becomes a piece of data describing how \( G_b \) permutes the anyons [28, 40]. In particular, the nontrivial Majorana-chain decoration \( n_1(T) = 1 \) is translated into the nontrivial symmetry action that \( T \) exchanges \( e \) and \( m \) anyons. In other words, the time-reversal symmetry flips the fermion parity of the \( \mathbb{Z}_2^f \) vertex. On the other hand, the group structure \( G = G_b \times \mathbb{Z}_2^f \) translates into the requirement that the \( f \) anyon carries a trivial symmetry fractionalization \( T^2 = +1 \).

It is well-known that this symmetry action is not compatible with the requirement that \( f \) carries \( T^2 = +1 \) [34, 36], and this incompatibility can be understood as the result of an obstruction in \( H^3(\mathbb{Z}_2) \). To see this, we recall that a symmetry-fractionalization pattern is represented by a 2-cocycle \( n_2 \in H^2(\mathbb{A}, \mathbb{A}) \) [34], where the coefficients \( \mathbb{A} \) are the fusion group of the four anyons in the \( \mathbb{Z}_2 \) topological order. Here, the choice of \( n_2 \) representing \( f \) carrying \( T^2 = +1 \) is \( n_2(T,T) = e \) or \( m \) [34]. However, neither choice satisfies the cocycle equation, because they both have the same nontrivial coboundary \( dn_2 \), indicated by the following entry,

\[ dn_2(T,T,T) = \rho_T(n_2(T,T)) - n_2(T,T) = f, \quad (10) \]

where \( \rho_T \) satisfying \( \rho_T(e) = m \) and \( \rho_T(m) = e \) denotes the nontrivial time-reversal action on the anyons. This violation of the cocycle equation indicates that this 2D SET state has an \( H^3 \) obstruction given by \( n_3 = dn_2 \), and can only be realized on the surface of a 3D SET bulk with the corresponding symmetry fractionalization given by \( n_3 \) [43]. It is straightforward to check that the cocycle \( n_3 = dn_2 \) computed in Eq. (10) is exactly the same as the \( n_3 \) computed previously using Eq. (5). Therefore, the required 3D SET bulk is the same as the result of gauging the fermion parity in the 3D SET bulk, which is a 3D \( \mathbb{Z}_2 \) topological order with point-like \( \mathbb{Z}_2 \) charges \( f \) carrying fermionic statistics. The \( n_3 \) data, describing the complex-fermion decoration in the SPT model, becomes the \( H^3 \) symmetry-fractionalization data in the SET model [44]. Therefore, the bulk-boundary correspondence between the surface and bulk SETs after gauging \( \mathbb{Z}_2^f \) provides an alternative way to understand the correspondence between the surface ASPT and the bulk trivial fSPT state.

Layered structure of ASPT phases – In addition to the ASPT phases constructed from Majorana chain decoration, the next layer of ASPT is known as the complex fermion decoration, which leads to trivialization of some BSPT when embedded into interacting fermion systems[26].
is again the $G_b = \mathbb{Z}_2^2$ case since $H^2(G_b, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\Gamma^4$ is a nontrivial cocycle in $H^4(\mathbb{Z}_2, U(1))$. After gauging fermion parity, the corresponding anomalous SET state is actually the well known $c T m T$ state which could not be realized as a pure 2D SET either [45]. Finally, we would like to mention that the decoration of $(p + ip)$ topological superconductors could also leads to another layer of ASPT states in 3D, and the full details will be discussed elsewhere.

Conclusion and discussion – In this paper, we systematically construct ASPT phases for 2D interacting fermion systems with total symmetry $G_f = G_b \times \mathbb{Z}_2^f$. We would like to stress that the layered structure and the resulting ASPT states are not just mathematical concepts but also have physical consequences. In particular, the ASPT states can in principle be constructed and detected physically. First, in realistic systems, different layers in fSPT classification can be effectively separated by energetic constraints. For instance, one can create a situation where Majorana chains are energetically expensive to create in the bulk but cheap to create on the boundary. Second, the boundary anomaly in different layers can be detected by measuring the interaction between the boundary and the bulk. It is generally understood that the anomalous boundary symmetry actions cannot be realized locally on the boundary without a bulk. For example, superconductivity with coplanar spin order can realize the $T^2 = 1$ symmetry, and it will be of great interest to study the potential anomalous time reversal symmetry action between charge($e$) and flux($m$) on the surface of these systems. To this end, it is also worthwhile to mention that ASPT also exists in 3D interacting fermion systems and the searching of 3D ASPT might tell us the evidence of extra dimension of our universe. For example, although both the Majorana chain and complex fermion decoration for $\mathbb{Z}_2$ fSPT phase (with a total symmetry $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^f$) are obstructed in purely 3D, they actually can be realized on the boundary of a 4D bulk.

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[1] Z.-C. Gu and X.-G. Wen, Phys. Rev. B 80, 155131 (2009).
[2] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[3] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[4] C. Wang, A. C. Potter, and T. Senthil, Science 343, 629 (2014), arXiv:1306.3238.
[5] C. Wang and T. Senthil, Phys. Rev. B 89, 195124 (2014).
[6] D. S. Freed and M. J. Hopkins, arXiv e-prints (2016), arXiv:1604.06527.
[7] E. Witten, Rev. Mod. Phys. 88, 035001 (2016).
[8] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, arXiv e-prints (2014), arXiv:1406.7329.
[9] L. Fu, Phys. Rev. Lett. 106, 106802 (2011).
[10] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
[11] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012).
[12] M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012).
[13] In many cases, the 2D boundary of 3D SPT states could realize the so-called anomalous topologically ordered states with ground state degeneracy on torus, and we refer these topological degeneracy also as gapless.
[14] A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).
[15] C. Wang and T. Senthil, Phys. Rev. B 87, 235122 (2013).
[16] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X 3, 041016 (2013).
[17] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Phys. Rev. X 5, 041013 (2015).
[18] P. Bonderson, C. Nayak, and X.-L. Qi, Journal of Statistical Mechanics: Theory and Experiment 2013, P09016 (2013).
[19] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B 88, 115137 (2013).
[20] X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B 89, 165132 (2014).
[21] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, ArXiv e-prints (2014), arXiv:1406.3032.
[22] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B 92, 125111 (2015).
[23] C. Wang, C.-H. Lin, and M. Levin, Phys. Rev. X 6, 021015 (2016).
[24] A. Kapustin, ArXiv e-prints (2014), arXiv:1403.1467.
[25] X.-G. Wen, Phys. Rev. B 91, 205101 (2015).
[26] Z.-C. Gu and X.-G. Wen, Phys. Rev. B 90, 115141 (2014).
[27] D. S. Freed, arXiv e-prints (2014), arXiv:1406.7278.
[28] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, Phys. Rev. B 97, 205109 (2018).
[29] D. Gaiotto and A. Kapustin, International Journal of Modern Physics A 31, 1645044 (2016).
[30] C. Wang, C.-H. Lin, and Z.-C. Gu, arXiv e-prints (2016), arXiv:1610.08478.
[31] Q.-R. Wang and Z.-C. Gu, Phys. Rev. X 8, 011055 (2018).
[32] A. Kapustin and R. Thorngren, ArXiv e-prints (2017), arXiv:1701.08264 [cond-mat.str-el].
[33] Q.-R. Wang and Z.-C. Gu, unpublished.
[34] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, ArXiv e-prints (2014), arXiv:1410.4540.
[35] C. Heinrich, F. Burnell, L. Fidkowski, and M. Levin, Phys. Rev. B 94, 235136 (2016).
[36] M. Cheng, Z.-C. Gu, S. Jiang, and Y. Qi, Phys. Rev. B 96, 115107 (2017).
[37] A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
[38] Z. Wang, S.-Q. Ning, and X. Chen, ArXiv e-prints (2017), arXiv:1708.01684 [cond-mat.str-el].
[39] F. Costantino, Mathematische Zeitschrift 251, 427 (2016).
[40] N. Tarantino and L. Fidkowski, Phys. Rev. B 94, 115115 (2016).
[41] B. Ware, J. H. Son, M. Cheng, R. V. Mishmash, J. Alicea, and B. Bauer, Phys. Rev. B 94, 115127 (2016).
[42] U. Pachner, European Journal of Combinatorics 12, 129 (1991).
[43] L. Fidkowski and A. Vishwanath, Phys. Rev. B 96, 045131 (2017).
[44] M. Cheng, N. Tantivasadakarn, and C. Wang, Phys. Rev. X 8, 011054 (2018).
[45] C. Wang and T. Senthil, Phys. Rev. B 87, 235122 (2013).