Simple Proof of Security of the BB84 Quantum Key Distribution Protocol

Peter W. Shor\textsuperscript{(1)} and John Preskill\textsuperscript{(2)}
\textsuperscript{(1)}AT&T Labs Research, Florham Park, NJ 07932, USA
\textsuperscript{(2)}Lauritsen Laboratory of High Energy Physics, California Institute of Technology, Pasadena, CA 91125, USA (February 1, 2008)

We prove that the 1984 protocol of Bennett and Brassard (BB84) for quantum key distribution is secure. We first give a key distribution protocol based on entanglement purification, which can be proven secure using methods from Lo and Chau's proof of security for a similar protocol. We then show that the security of this protocol implies the security of BB84. The entanglement-purification based protocol uses Calderbank-Shor-Steane (CSS) codes, and properties of these codes are used to remove the use of quantum computation from the Lo-Chau protocol.

Quantum cryptography differs from conventional cryptography in that the data are kept secret by the properties of quantum mechanics, rather than the conjectured difficulty of computing certain functions. The first quantum key distribution protocol, proposed in 1984\textsuperscript{(1)}, is called BB84 after its inventors (C. H. Bennett and G. Brassard). In this protocol, the participants (Alice and Bob) wish to agree on a secret key about which no eavesdropper (Eve) can obtain significant information. Alice sends each bit of the secret key in one of a set of conjugate bases which Eve does not know, and this key is protected by the impossibility of measuring the state of a quantum system simultaneously in two conjugate bases. The original papers proposing quantum key distribution\textsuperscript{(1)} proved it secure against certain attacks, including those feasible using current experimental techniques. However, for many years, it was not rigorously proven secure against an adversary able to perform any physical operation permitted by quantum mechanics.

Recently, three proofs of the security of quantum key distribution protocols have been discovered; however, none is entirely satisfactory. One proof\textsuperscript{(2)}, although easy to understand, has the drawback that the protocol requires a quantum computer. The other two\textsuperscript{[3,4]} prove the security of a protocol based on BB84, and so are applicable to near-practical settings. However, both proofs are quite complicated. We give a simpler proof by relating the security of BB84 to entanglement purification protocols\textsuperscript{[5]} and quantum error correcting codes\textsuperscript{[6]}. This new proof also may illuminate some properties of previous proofs\textsuperscript{[2,3]}, and thus give insight into them. For example, it elucidates why the rates obtainable from these proofs are related to rates for CSS codes. The proof was in fact inspired by the observation that CSS codes are hidden in the inner workings of the proof given in\textsuperscript{[2].}

We first review CSS codes and associated entanglement purification protocols. Quantum error-correcting codes are subspaces of the Hilbert space $\mathbb{C}^2^n$ which are protected from errors in a small number of these qubits, so that any such error can be measured and subsequently corrected without disturbing the encoded state. A quantum CSS code $Q$ on $n$ qubits comes from two binary codes on $n$ bits, $C_1$ and $C_2$, one contained in the other:

$$\{0\} \subset C_2 \subset C_1 \subset \mathbb{F}_2^n,$$

where $\mathbb{F}_2^n$ is the binary vector space on $n$ bits.\textsuperscript{[5]}

A set of basis states (which we call codewords) for the CSS code subspace can be obtained from vectors $v \in C_1$ as follows:

$$v \rightarrow \frac{1}{|C_2|^{1/2}} \sum_{w \in C_2} \left| v + w \right> . \quad (1)$$

If $v_1 - v_2 \in C_2$, then the codewords corresponding to $v_1$ and $v_2$ are the same. Hence these codewords correspond to cosets of $C_2$ in $C_1$, and this code protects a Hilbert space of dimension $2^{\dim C_1 - \dim C_2}$.

The above quantum code is equivalent to the dual code $Q^*$ obtained from the two binary codes

$$\{0\} \subset C_1^\perp \subset C_2^\perp \subset \mathbb{F}_2^n.$$

This equivalence can be demonstrated by applying the Hadamard transform

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

to each encoding qubit. This transformation interchanges the basis $|0\rangle, |1\rangle$ and $|+\rangle, |-\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. It also interchanges the two subspaces corresponding to the codes $Q$ and $Q^*$, although the codewords (given by Eq.\textsuperscript{(2)} of $Q$ and $Q^*$ are not likewise interchanged.

We now make a brief technical detour to define some terms. The three Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix $\sigma_x$ applies a bit flip error to a qubit, while $\sigma_z$ applies a phase flip error. We denote the Pauli matrix $\sigma_a$ acting on the $k$'th bit of the CSS code by $\sigma_{a(k)}$ for $a \in \{x,y,z\}$. For a binary vector $s$, we let

$$\sigma_{[s]}_a = \sigma_{a(1)}^{s_1} \otimes \sigma_{a(2)}^{s_2} \otimes \sigma_{a(3)}^{s_3} \otimes \ldots \otimes \sigma_{a(n)}^{s_n}$$

where $\sigma_0^0$ is the identity matrix and $s_i$ is the $i$'th bit of $s$. The matrices $\sigma_{[s]}_a (\sigma_{[s]}_a)$ have all eigenvalues $\pm 1$.\textsuperscript{[5]}
In a classical error correcting code, correction proceeds by measuring the syndrome, which is done as follows. A parity check matrix $H$ of a code $C$ is a basis of the dual vector space $C^\perp$. Suppose that we transmit a codeword $v$, which acquires errors to become $w = v + \epsilon$. The $k$th row $r_k$ of the matrix $H$ determines the $k$th bit of the syndrome for $w$, namely $r_k \cdot w \pmod{2}$. The full syndrome is thus $Hw$. If the syndrome is 0, then $w \in C$. Otherwise, the most likely value of the error $\epsilon$ can be calculated from the syndrome $[\bar{w}]$. In our quantum CSS code, we need to correct both bit and phase errors. Let $H_1$ be a parity check matrix for the code $C_1$, and $H_2$ one for the code $C_2^\perp$. To calculate the syndrome for bit flips, we measure the eigenvalue of $\sigma_z^{[r]}$ for each row $r \in H_1$ (−1’s and 1’s of the eigenvalue correspond to 0’s and 1’s of the syndrome). To calculate the syndrome for phase flips, we measure the eigenvalue of $\sigma_z^{[r]}$ for each row $r \in H_2$. This lets us correct both bit and phase flips, and if we can correct up to $t$ of each of these types of errors, we can also correct arbitrary errors on up to $t$ qubits.

The useful property of CSS codes for demonstrating the security of BB84 is that the error correction for the phases is decoupled from that for the bit values, as shown above. General quantum stabilizer codes can similarly be turned into key distribution protocols, but these appear to require a quantum computer to implement.

If one requires that a CSS code correct all errors on at most $t = \delta n$ qubits, the best codes that we know exist satisfy the quantum Gilbert-Varshamov bound. As the block length $n$ goes to infinity, these codes asymptotically protect against $\delta n$ bit errors and $\delta n$ phase errors, and encode $[1 - 2H(2\delta)]n$ qubits, where $H$ is the binary Shannon entropy $H(p) = -p \log_2(p) - (1 - p) \log_2(1 - p)$. In practice, it is better to only require that random errors are corrected with high probability. In this case, codes exist that correct $\delta n$ random phase errors and $\delta n$ random bit errors, and which encode $[1 - 2H(\delta)]n$ qubits.

We also need a description of the Bell basis. These are the four maximally entangled states

$$\Psi^\pm = \frac{1}{2 \sqrt{2}} (|01\rangle \pm |10\rangle), \quad \Phi^\pm = \frac{1}{2 \sqrt{2}} (|00\rangle \pm |11\rangle),$$

which form an orthogonal basis for the quantum state space of two qubits.

Finally, we introduce a class of quantum error correcting codes equivalent to $Q$, and parameterized by two $n$-bit binary vectors $x$ and $z$. Suppose that $Q$ is determined as above by $C_1$ and $C_2$. Then $Q_{x,z}$ has basis vectors indexed by cosets of $C_2$ in $C_1$, and for $v \in C_1$, the corresponding codeword is

$$v \mapsto \frac{1}{|C_2|^{1/2}} \sum_{w \in C_2} (-1)^{z \cdot w} |x + v + w\rangle.$$

Quantum error correcting codes and entanglement purification protocols are closely connected; we now describe the entanglement purification protocol corresponding to the CSS code $Q$. For now, we assume that the codes $C_1$ and $C_2^\perp$ correct up to $t$ errors and that $Q$ encodes $m$ qubits in $n$ qubits. Suppose Alice and Bob share $n$ pairs of qubits in a state close to $(\Phi^+)^{\otimes n}$. For the entanglement purification protocol, Alice and Bob separately measure the eigenvalues of $\sigma_x^{[r]}$ for each row $r \in H_1$ and $\sigma_x^{[r]}$ for each row $r' \in H_2$. Note that for these measurements to be performable simultaneously, they must all commute: $\sigma_x^{[r]}$ and $\sigma_x^{[r']}$ commute because the vector spaces $C_1^\perp$ and $C_2$ are orthogonal.

If Alice and Bob start with $n$ perfect EPR pairs, measuring $\sigma_x^{[r]}$ for $r \in H_1$ and $\sigma_x^{[r]}$ for $r' \in H_2$ projects each of their states onto the code subspace $Q_{x,z}$, where $x$ and $z$ are any binary vectors with $H_1 x$ and $H_2 z$ equal to the measured bit and phase syndromes, respectively. After projection, the state is $(\Phi^+)^{\otimes m}$ encoded by $Q_{x,z}$.

Now, suppose that Alice and Bob start with a state close to $(\Phi^+)^{\otimes n}$. To be specific, suppose that all their EPR pairs are in the Bell basis, with $t$ or fewer bit flips ($\Phi^+$ or $\Psi^-$ pairs) and $t$ or fewer phase flips ($\Phi^-$ or $\Psi^+$ pairs). If Alice and Bob compare their measurements of $\sigma_x^{[r]}$ ($\sigma_x^{[r]}$), the rows $r$ for which these measurements disagree give the bits which are 1 in the bit (phase) syndromes. From these syndromes, Alice and Bob can compute the locations of the bit and the phase flips, can correct these errors, and can then decode $Q_{x,z}$ to obtain $m$ perfect EPR pairs.

We will show that the following is a secure quantum key distribution protocol.

**Protocol 1: Modified Lo-Chau**

1: Alice creates $2n$ EPR pairs in the state $(\Phi^+)^{\otimes n}$.
2: Alice selects a random $2n$ bit string $b$, and performs a Hadamard transform on the second half of each EPR pair for which $b$ is 1.
3: Alice sends the second half of each EPR pair to Bob.
4: Bob receives the qubits and publicly announces this fact.
5: Alice selects $n$ of the $2n$ encoded EPR pairs to serve as check bits to test for Eve’s interference.
6: Alice announces the bit string $b$, and which $n$ EPR pairs are to be check bits.
7: Bob performs Hadamards on the qubits where $b$ is 1.
8: Alice and Bob each measure their halves of the $n$ check EPR pairs in the $|0\rangle$, $|1\rangle$ basis and share the results. If too many of these measurements disagree, they abort the protocol.
9: Alice and Bob make the measurements on their code qubits of $\sigma_x^{[r]}$ for each row $r \in H_1$ and $\sigma_x^{[r]}$ for each row $r \in H_2$. Alice and Bob share the results, compute the syndromes for bit and phase flips, and
then transform their state so as to obtain \( m \) nearly perfect EPR pairs.

10: Alice and Bob measure the EPR pairs in the \(|0\rangle, |1\rangle\) basis to obtain a shared secret key.

We now show that this protocol works. Namely, we show that the probability is exponentially small that Alice and Bob agree on a key about which Eve can obtain more than an exponentially small amount of information. We need a result of Lo and Chau \[2\] that if Alice and Bob agree on a key about which Eve can obtain more than an exponentially small amount of information, then Eve’s mutual information with the key is at most

\[
\log \left( \frac{2^m}{\delta} \right)
\]

where \( \delta \) is the fidelity of the protocol into BB84. First note that, since all Bob cares about are the bit values of the encoded key, and the string \( z \) is only used to correct the phase of the encoded qubits, Bob does not need \( z \). This is why we use CSS codes: they decouple the phase correction from the bit correct-
tion. Let $k' \in C_1$ be a binary vector that is mapped by Eq. (2) to the encoded key. Since Bob never uses $z$, we can assume that Alice does not send it. Averaging over $z$, we see that Alice effectively sends the mixed state

$$
\frac{1}{2^n|C_2|} \sum_z \left( \sum_{w_1, w_2 \in C_2} (-1)^{(w_1+w_2) \cdot z} \right. \\
\times \left. |k' + w_1 + x \rangle \langle k' + w_2 + x| \right) \\
= \frac{1}{|C_2|} \sum_{w \in C_2} |k' + w + x \rangle \langle k' + w + x|, 
$$

which is equivalently the mixture of states $|k' + x + w\rangle$ with $w$ chosen randomly in $C_2$. Let us now look at the protocol as a whole. The error correction information Alice gives Bob is $z$, and Alice sends $|k' + x + w\rangle$ over the quantum channel. Over many iterations of the algorithm, these are random variables chosen uniformly in $\mathbb{F}_2^n$ with the constraint that their difference $k' + w$ is in $C_1$. After Bob receives $k' + w + x + \epsilon$, he subtracts $x$, and corrects the result to a codeword in $C_1$, which is almost certain to be $k' + w$. The key is the coset of $k' + w$ over $C_2$.

In the BB84 protocol given below, Alice sends $|v\rangle$ to Bob, with error correction information $u + v$. These are again two random variables uniform in $\mathbb{F}_2^n$, with the constraint that $u \in C_1$. Bob obtains $v + \epsilon$, subtracts $u + v$, and corrects the result to a codeword in $C_1$, which with high probability is $u$. The key is then the coset $u + C_2$. Thus, the two protocols are completely equivalent.

**Protocol 3: BB84**

1: Alice creates $(4 + \delta)n$ random bits.
2: Alice chooses a random $(4 + \delta)n$-bit string $b$. For each bit, she creates a state in the $|0\rangle$, $|1\rangle$ basis (if the corresponding bit of $b$ is 0) or the $|+\rangle$, $|-\rangle$ basis (if the bit of $b$ is 1).
3: Alice sends the resulting qubits to Bob.
4: Bob receives the $(4 + \delta)n$ qubits, measuring each in the $|0\rangle$, $|1\rangle$ or the $|+\rangle$, $|-\rangle$ basis at random.
5: Alice announces $b$.
6: Bob discards any results where he measured a different basis than Alice prepared. With high probability, there are at least $2n$ bits left (if not, abort the protocol). Alice decides randomly on a set of $2n$ bits to use for the protocol, and chooses at random $n$ of these to be check bits.
7: Alice and Bob announce the values of their check bits. If too few of these values agree, they abort the protocol.
8: Alice announces $u + v$, where $v$ is the string consisting of the remaining non-check bits, and $u$ is a random codeword in $C_1$.
9: Bob subtracts $u + v$ from his code qubits, $v + \epsilon$, and corrects the result, $u + \epsilon$, to a codeword in $C_1$.
10: Alice and Bob use the coset of $u + C_2$ as the key.

There are a few loose ends that need to be tied up. The protocol given above uses binary codes $C_1$ and $C_2^\perp$ with large minimum distance, and thus can obtain rates given by the quantum Gilbert-Varshamov bound for CSS codes [8]. To reach the better Shannon bound for CSS codes, we need to use codes for which a random small set of phase errors and bit errors can almost always be corrected. To prove that the protocol works in this case, we need to ensure that the errors are indeed random. We do this by adding a step where Alice scrambles the qubits using a random permutation $\pi$ before sending them to Bob, and a step after Bob acknowledges receiving the qubits where Alice sends $\pi$ to Bob and he unscrambles the qubits. This can work as long as the measured bit and phase error rates are less than $11\%$, the point at which the Shannon rate $1 - 2H(\delta)$ hits 0.

For a practical key distribution protocol we need the classical code $C_1$ to be efficiently decodeable. As is shown in [6], we can let $C_2$ be a random subcode of an efficiently decodeable code $C_1$, and with high probability obtain a good code $C_2^\perp$. While known efficiently decodeable codes do not meet the Shannon bound, they come fairly close.

A weakness in both the proof given in this paper and the proofs in [2] is that they do not apply to imperfect sources; the sources must be perfect single-photon sources. A proof avoiding this difficulty was recently discovered by Michael Ben-Or [1]; it shows that any source sufficiently close to a single-photon source is still secure. However, most experimental quantum key distribution systems use weak coherent sources, and no currently known proof covers this case.

The authors thank Michael Ben-Or, Eli Biham, Hoikwong Lo, Dominic Mayers and Tal Mor for explanations of and informative discussions about their security proofs. We also thank Ike Chuang, Dan Gottesman, Alexei Kitaev and Mike Nielsen for their discussions and suggestions, which greatly improved this paper. Part of this research was done while PWS was visiting Caltech. This work has been supported in part by the Department of Energy under Grant No. DE-FG03-92-ER40701, and by DARPA through Caltech’s Quantum Information and Computation (QUIC) project administered by the Army Research Office.

[1] C. H. Bennett and G. Brassard, “Quantum cryptography: Public-key distribution and coin tossing,” in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, 1984* (IEEE Press, 1984), pp. 175–179; C.H. Bennett and G. Brassard, “Quantum public key distribution,” IBM Technical Disclosure Bulletin 28, 3153–3163 (1985).
[2] H.-K. Lo and H. F. Chau, “Unconditional security of quantum key distribution over arbitrarily long distances,” Science 283, 2050–2056 (1999), arXiv e-print quant-ph/9803006.

[3] D. Mayers, “Unconditional security in quantum cryptography,” J. Assoc. Computing Machinery (to be published), arXiv e-print quant-ph/9802025; preliminary version in D. Mayers, “Quantum key distribution and string oblivious transfer in noisy channels,” in Advances in Cryptology—Proceedings of Crypto ’96, (Springer-Verlag, New York, 1996) pp. 343–357.

[4] E. Biham, M. Boyer, P. O. Boykin, T. Mor and V. Roychowdhury, “A proof of the security of quantum key distribution,” in Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing (ACM Press, New York, in press), arXiv e-print quant-ph/9912053.

[5] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, “Mixed state entanglement and quantum error correction,” Phys. Rev. A, 54, 3824–3851 (1996), arXiv e-print quant-ph/9604024.

[6] A. R. Calderbank and P. Shor, “Good quantum error correcting codes exist,” Phys. Rev. A 54, 1098–1105 (1996), arXiv e-print quant-ph/9512032; A. M. Steane, “Multiple particle interference and error correction,” Proc. R. Soc. London A 452, 2551–2577 (1996), arXiv e-print quant-ph/9601029.

[7] This calculation may be quite difficult, but for now we ignore this practical complication.

[8] M. Ben-Or (unpublished).