A Decision Making Approach with Linguistic Weight and Unavoidable Incomparable Ranking

Yunxia Zhang¹, Degen Huang¹, Wei Gao²*, Vassilis G. Kaburlasos³

¹School of Computer Science and Technology, Dalian University of Technology, Dalian, 116081, China
²School of Computer and Information Technology, Liaoning Normal University, Dalian, 116024, China
³HUMAIN-Lab, Department of Computer and Informatics Engineering, Kavala, 65404, Greece

1. INTRODUCTION

In terms of common sense, people use linguistic terms in nature language for evaluating, reasoning and making decision, rather than crisp numbers. Hence, collection with qualitative information is always got, for example, object-attribute assessment for a car may be "very cheap" in price, "slightly comfortable" in comfort, "somewhat dangerous" in safety. To handle the qualitative information, many linguistic terms models have been proposed based on computing with words (CWW) [1,2], such as fuzzy set theory [3–5], symbolic approaches [6,7] and linguistic truth-valued mode in lattice order [8,9].

Fuzzy set theory was proposed by Zadeh in 1965 using a membership function to model the uncertain information [5]. But it is difficult to find appropriate membership function when we confronted with the circumstances where various concepts are given in various contexts. Type-2 fuzzy set is the extension of ordinary fuzzy set based on the theory of computing with words (CWW) [2]. Unlike a type-1 fuzzy set where the membership grade is a crisp number in [0, 1], the membership grade of Type-2 fuzzy set was extended to an ordinary fuzzy set [2–4]. Type-2 fuzzy set plays a role in wider applications and processes the uncertain information more flexibility [10–12].

Symbolic approach has performed well in decision making problems especially semantic interpretation of linguistic terms in nature language [15,16]. But there exists some loss of information when the results come out from the initial expression domain. Francisco Herrera and Luis Martinez proposed a 2-tuple model which are composed by a linguistic term and a numeric value assessed in [−0.5, 0.5] [17,18]. The 2-tuple model is widely used in many real problems for that it can avoid information losing in linguistic information processing and many aggregation operators of 2-tuple model were provided [19–22].

The conventional symbolic approach for linguistic value uses a linear order structure. However, in the natural language, some linguistic values are incomparable, for example, "Exactly True" and "Somewhat False." Obviously, it is difficult to describe the incomparable values utilizing a linear order structure. Lattice implication algebra (LIA) structure can imitate the uncertain and both comparable and incomparable characteristic [7]. Many researchers have studied in varied direction of LIA [23,24]. Jun Liu et al. have proposed an axiomatizable lattice ordered qualitative linguistic truth-valued logic system, which is a foundation for establishing formal linguistic truth-valued logic [25]. Xingxing He et al. have unified method for finding the structure of k-IESF in linguistic truth-valued lattice-valued propositional logic which provided a theoretical foundations and algorithms for α-resolution automated reasoning [26]. Yang Xu et al. have extended the binary α-resolution to multiary α-resolution in lattice-valued propositional logic LP(X) and lattice-valued first-order logic LF(X), obtained a result that multiary α-resolution principle in LF(X) can be equivalently transformed into that in LP(X) [27]. Yi Liu and Vassilis

CORRESPONDING AUTHOR. Email: gwlnnu@163.com

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ABSTRACT

In order to deal with the decision making problem including some linguistic values uncertainty information, we propose an approach for decision making with linguistic weighted and unavoidable incomparable ranking based on Linguistic-valued lattice implication algebra (LV-LIA). The properties of binary operations ⊗ and ⊕ are discussed in LV-LIA, and used to handle importance degree of the attributes expressed by linguistic values. We define a linguistic-real valuation function which is a positive valuation function and a linguistic-real metric distance implied by the linguistic-real valuation function is introduced, to process incomparable linguistic values in the results which need further procedure to make a certain decision. Illustrating examples show the effectiveness of the proposed approach which can rank the incomparable elements elastic.
G. Kaburlasos et al. have introduced a LIA with implication values in a complete lattice of intervals on the real number axis which followed a capacity to optimize [28]. Li Zou proposed a linguistic-valued knowledge representation mode and approximated reasoning approach with linguistic-valued credibility factors [29].

All the above researches provided the feasibility of decision making with linguistic values based on linguistic-valued lattice implication algebra (LV-LIA). Besides object-attribute assessment, the importance of the attributes is often expressed using linguistic terms. However, the linguistic weights are always transformed to numbers or values on a linear order structure in a decision making problem [30]. This paper is aiming at decision making problems not only with the linguistic evaluation set but also the linguistic important degree, i.e., linguistic weight. We will propose a decision making model based on LV-LIA which has a flexible linguistic weighted method.

LV-LIA can express both comparable and incomparable linguistic information, which is consistent with nature language characteristic. In decision making there often exist some incomparable linguistic-valued decision making results. The positive valuation function (PVF) which implies a metric distance as well as an inclusion measure function can mapping a lattice to a real set [31]. We define a linguistic-real valuation function and get a rank with incomparable linguistic values through the linguistic-real metric distance (LRMD) elastic according to the individual preference degree.

Based on the aforementioned academic ideas, we will introduce a decision making approach based on 18-element LV-LIA.

The rest of paper is organized as follows: In section 2, we briefly review the concepts of LIA and LV-LIA. In section 3, we discuss the properties of the operations $\otimes$ and $\oplus$ of LV-LIA, and propose a decision making model with linguistic weighted method. In section 4, we introduce a linguistic-real valuation function implied a metric distance to rank the incomparable decision making results. In section 5, some conclusions are summarized.

## 2. LINGUISTIC-VALUED LATTICE IMPLICATION ALGEBRA

We briefly review some concepts of linguistic truth-valued LIA. We refer to the related Ref. [7].

**Definition 1.** [7] Let $(L, \lor, \land, \rightarrow, \bot, \top)$ be a bounded lattice with universal boundaries $\bot$ (the least element) and $\top$ (the greatest element) respectively, and “$\rightarrow$” be an order-reversing involution. For any $x, y, z \in L$, if mapping $\Rightarrow: L \times L \rightarrow L$ satisfies:

1. $\Rightarrow(x \rightarrow y \rightarrow z) = x \rightarrow (y \rightarrow z)$;  
2. $\Rightarrow(x \rightarrow x = x)$;  
3. $\Rightarrow(x \rightarrow y = y' \rightarrow x')$;  
4. $\Rightarrow(x \rightarrow y = y \rightarrow x = y)$;  
5. $\Rightarrow(x \rightarrow y) = (y \rightarrow x) \rightarrow x$;  
6. $\Rightarrow(x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z)$;  
7. $\Rightarrow(x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z)$.

Then $(L, \lor, \land, \rightarrow, \bot, \top)$ is an LIA.

**Definition 2.** [7] Let $L_{n+1} = \{d_0, d_1, \ldots, d_n\}$, $d_0 < d_2 < \cdots < d_n$, $L_2 = \{b_1, b_2\}$, $b_1 < b_2$, $d_0, d_n, b_1$ and $b_2$ be Lukasiewicz implication algebra. For any $(d_i, b_j), (d_k, b_m) \in L_{n+1} \times L_2$, if

$$
\begin{align*}
(d_i, b_j) \lor (d_k, b_m) &= (d_i \lor d_k) \land b_j \land (d_k \lor b_m), \\
(d_i, b_j) \land (d_k, b_m) &= (d_i \land d_k) \lor b_j \lor (d_k \land b_m), \\
(d_i, b_j) \rightarrow (d_k, b_m) &= \left(\frac{d_i}{d_k}, b_j, b_m\right), \\
(d_n, b_j) \rightarrow (d_0, b_j \rightarrow (d_k, b_m)),
\end{align*}
$$

then $(L_{n+1} \times L_2, \lor, \land, \rightarrow, (d_1, b_1), (d_2, b_2))$ is a LIA, denote as $L_{n \times L_2}$.

**Definition 3.** [7] Let $AD_{n+1} = \{h_0, h_1, \ldots, h_n\}$ be a set of $n$ hedge operators and $h_0 < h_1 < \cdots < h_n$. $MT = \{f, t\}$ be “false($f$)” and “true($t$)”, denote $f < t$ and $L_{V((n+1)\times 2)} = AD_{n+1} \times MT$. Define the mapping $g: L_{V((n+1)\times 2)} \rightarrow L_{n \times L_2}$ as follows:

$$
g((h_i, mt)) = \begin{cases} (d_i', b_i), & mt = f, \\ (d_i, b_i), & mt = t. \end{cases}
$$

Then $g$ is bijection. Its inverse mapping is $g^{-1}$. For any $x, y \in L_{V((n+1)\times 2)}$, define

$$
x \lor y = g^{-1}(g(x) \lor g(y)),
$$

$$
x \land y = g^{-1}(g(x) \land g(y)),
$$

$$
x' = g^{-1}(g(x')), \\
x \rightarrow y = g^{-1}(g(x) \rightarrow g(y)).
$$

Then $L_{V((n+1)\times 2)} = (L_{V((n+1)\times 2)}, \lor, \land, \rightarrow, (h_0, f), (h_n, t))$ is called linguistic truth-valued LIA from $AD_{n+1}$ and $MT$ (Figure 1). $g$ is an isomorphic mapping from $(L_{V((n+1)\times 2)}, \lor, \land, \rightarrow, (h_0, f), (h_n, t))$ to $L_{n \times L_2}$.

The operation “$\rightarrow$” is

$$
\begin{align*}
(h_i, c_2) &\rightarrow (h_j, c_1) = \left(\frac{h_{\max(0,i+j-n)}}{c_2}, c_1\right), \\
(h_i, c_1) &\rightarrow (h_j, c_2) = \left(\frac{h_{\min(i+j-n)}}{c_2}, c_2\right), \\
(h_i, c_2) &\rightarrow (h_j, c_2) = \left(\frac{h_{\min(i+j-n)}}{c_2}, c_2\right). \\
\end{align*}
$$

**Figure 1** | The Hasse diagram of $L_{V((n+1)\times 2)}$. 
Note that \( n \) should be an even number when we construct the \( \mathcal{L}_{V(n+1)\times2} \).

3. THE DECISION MAKING APPROACH WITH LINGUISTIC WEIGHTED BASED ON LV-LIA

In a multi-attributes decision making problem, sometimes attribute importance is expressed with linguistic terms. We explore a linguistic weighted method expressing these linguistic weights with a linguistic lattice structure based on LV-LIA and apply it into decision making with linguistic weight in this section.

3.1. The Operations in LV-LIA

In a LIA \( L \), there exist two binary operations \( \otimes \) and \( \oplus \) as follows [7].

For any \( x, y \in L \),
\[
    x \otimes y = (x \rightarrow y')', \\
    x \oplus y = x' \rightarrow y. \tag{5}
\]

Similarly, we extend two binary operations \( \otimes \) and \( \oplus \) into LV-LIA.

\textbf{Definition 4.} For any \((h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V(n+1)\times2}\), the binary operation \( \otimes \) and \( \oplus \) are defined as follows,
\[
    (h_i, c_k) \otimes (h_j, c_l) = \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_l \right), \\
    (h_i, c_k) \oplus (h_j, c_l) = \left( h_{\max(n,n+i)}(c_k \oplus c_l), c_l \right). \tag{6}
\]

According to Eqs. (4) and (7), an equal definition is obtained.

For operation \( \otimes \), there exist
\[
    (h_i, c_k) \otimes (h_j, c_l) = \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_l \right), \tag{8}
\]
\[
    (h_i, c_k) \otimes (h_j, c_l) = \left( h_{\max(n,n+i)}(c_k \oplus c_l), c_i \right),
\]
\[
    (h_i, c_k) \otimes (h_j, c_l) = \left( h_{\min(n,n+i)}(c_k \oplus c_l), c_i \right),
\]
\[
    (h_i, c_k) \otimes (h_j, c_l) = \left( h_{\max(n,n+i)}(c_k \otimes c_l), c_i \right).
\]

And for operation \( \oplus \), there exist
\[
    (h_i, c_k) \oplus (h_j, c_l) = \left( h_{\min(n,n+i)}(c_k \oplus c_l), c_l \right), \tag{9}
\]
\[
    (h_i, c_k) \oplus (h_j, c_l) = \left( h_{\max(n,n+i)}(c_k \otimes c_l), c_i \right),
\]
\[
    (h_i, c_k) \oplus (h_j, c_l) = \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_i \right),
\]
\[
    (h_i, c_k) \oplus (h_j, c_l) = \left( h_{\max(n,n+i)}(c_k \otimes c_l), c_i \right).
\]

As \( \mathcal{L}_{V(n+1)\times2} \) is a LIA, for any \((h_i, c_k), (h_j, c_l), (h_{n'}, c_{c_2}) \in \mathcal{L}_{V(n+1)\times2}, \in \mathcal{L}_{V(n+1)\times2}\), we get the following properties,

i. \((h_i, c_k) \otimes (h_j, c_l) = (h_j, c_l) \otimes (h_i, c_k), \)
\[
    (h_i, c_k) \otimes (h_j, c_l) = (h_i, c_l) \otimes (h_j, c_i).
\]

ii. \((h_i, c_k) \otimes (h_{n'}, c_{c_2}) = (h_i, c_k), \)
\[
    (h_i, c_k) \otimes (h_{n'}, c_{c_2}) = (h_i, c_{c_2}),
\]
\[
    (h_i, c_k) \otimes (h_{n'}, c_{c_2}) = (h_{n'}, c_{c_2}),
\]
\[
    (h_i, c_k) \otimes (h_{n'}, c_{c_2}) = (h_{n'}, c_{c_2}).
\]

Some other special properties of the operations \( \otimes \) and \( \oplus \) in LV-LIA are listed as follows.

\textbf{Theorem 1.} For any \((h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V(n+1)\times2}\),
\[
    (h_i, c_k) \otimes (h_j, c_l) \leq (h_i, c_k), \]
\[
    (h_i, c_k) \otimes (h_j, c_l) \leq (h_j, c_l), \]
\[
    (h_i, c_k) \oplus (h_j, c_l) \geq (h_j, c_l), \]
\[
    (h_i, c_k) \oplus (h_j, c_l) \geq (h_i, c_i). \]

\textbf{Proof.} We prove it from four situations.

i. \( c_k = c_2 \) and \( c_i = c_1 \);

ii. \( c_k = c_1 \) and \( c_i = c_2 \);

iii. \( c_k = c_2 \) and \( c_i = c_1 \);

iv. \( c_k = c_1 \) and \( c_i = c_1 \).

Without loss of generality, we only prove situation at that \( c_k = c_2 \) and \( c_i = c_1 \). The other situations can be proved similarly.

From Eq. (8), we know that
\[
    (h_i, c_2) \otimes (h_j, c_1) = \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_l \right), \tag{10}
\]
\[
    \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_l \right) \leq (h_i, c_2), \text{ for } n - i + j \geq n - i.
\]

Hence, \((h_i, c_k) \otimes (h_j, c_l) \leq (h_i, c_i)\).

From Eq. (9), we know that
\[
    (h_i, c_2) \oplus (h_j, c_1) = \left( h_{\min(n,n+i)}(c_k \oplus c_l), c_i \right), \tag{11}
\]
\[
    \left( h_{\min(n,n+i)}(c_k \oplus c_l), c_i \right) \geq (h_i, c_2), \text{ as } n - j + i \geq i.
\]

Hence, \((h_i, c_k) \oplus (h_j, c_l) \geq (h_i, c_i)\).

It also can be proved in the other situations that
\[
    \left( h_{\min(n,n+i)}(c_i \oplus c_l), c_i \right) \leq (h_i, c_l) , \]
\[
    \left( h_{\min(n,n+i)}(c_k \otimes c_l), c_i \right) \geq (h_i, c_i) .
\]

From the properties of the operations \( \otimes \) and \( \oplus \) in LV-LIA,
\[
    (h_i, c_k) \otimes (h_j, c_l) \leq (h_i, c_k) , \]
\[
    (h_i, c_k) \otimes (h_j, c_k) \leq (h_i, c_k) .
\]

\textbf{Theorem 2.} For any \((h_i, c_k), (h_j, c_l), (h_{n'}, c_{c_2}) \in \mathcal{L}_{V(n+1)\times2}\), if \((h_i, c_k) \leq (h_j, c_l)\), then
\[
    (h_i, c_k) \otimes (h_j, c_l) \leq (h_{n'}, c_{c_2}) , \]
\[
    (h_i, c_k) \oplus (h_j, c_l) \leq (h_{n'}, c_{c_2}).
\]
It can be proved from the Eqs. (8) and (9) respectively.

According to the properties of the operations $\otimes$ and $\oplus$ in LV-LIA, we obtain

\[
(h_i, c_i) \otimes (h_j, c_j) \leq (h_i, c_i) \otimes (h_j, c_j),
\]

\[
(h_i, c_i) \oplus (h_j, c_j) \leq (h_i, c_i) \oplus (h_j, c_j).
\]

Suppose that $x_i$, $x_j$ are two assessments with respect to the $i$th and the $j$th attribute of an alternative expressed by linguistic values on LV-LIA and $\omega_i$, $\omega_j$ are two linguistic weights on LV-LIA. Then according to the Theorem 1, we have

\[
x_i \otimes \omega_i < x_i.
\]

According to the Theorem 2, if $x_i \leq x_j$ or $\omega_i \leq \omega_j$, we have

\[
\omega_i \otimes x_i \leq \omega_j \otimes x_j,
\]

\[
\omega_i \otimes x_i \leq \omega_i \otimes x_i.
\]

That shows that when the assessments of alternatives are the same, more important the attribute is, higher evaluation the alternative achieves. Accordingly, when the weights of attributes are the same, larger the assessment of alternative is, higher evaluation the alternative achieves.

### 3.2. The Decision Making Approach Based on 18LV-LIA

In generally, linguistic terms with linguistic hedges can be seen as linguistic modifiers and prime terms. In the following, we divide the linguistic terms into nine linguistic hedges (modifiers) applying to two prime terms. The linguistic modifiers of LV-LIA are \{Slightly (Sl) for short, Somewhat (So), Rather (Ra), Almost (Al), Exactly (Ex), Quite (Qu), Very (Ve), Highly (Hi), Absolutely (Ab)\} with the semantic ordering relationship $Sl < So < Ra < Al < Ex < Qu < Ve < Hi < Ab$, i.e., the hedge set is $AD_9 = \{h_0 = Sl, h_1 = So, h_2 = Ra, h_3 = Al, h_4 = Ex, h_5 = Qu, h_6 = Ve, h_7 = Hi, h_8 = Ab\}$. The prime terms are \{Dissatisfied (Ds) for short, Satisfied (Sa)\} representing a pair of opposite magnitude concept with $Ds < Sa$, i.e., the prime term set $MT = \{c_1 = Ds, c_2 = Sa\}$. Applying the linguistic modifiers of $AD_9$ to the prime terms $MT$, we obtain a partially ordered lattice $L_{V(9 \times 2)} = (V(9 \times 2), \lor, \land, \rightarrow, (h_8, c_2), (h_8, c_1))$ called 18-element linguistic-valued lattice implication algebra (18LV-LIA) (Figure 2) [32].

A decision making model is constructed based on 18LV-LIA. Weights of the attributes are expressed with linguistic weights on LV-LIA. The operation $\otimes$ can be used to calculate the linguistic weight. The attribute aggregation can be done according to the aggregation rules which can be set flexibly based on the actual cases.

There are some aggregation function about the logical relation "or" and "and".

For relation "$(h_i, c_i)$ and $(h_j, c_j)$" can be aggregated as,

\[
F_{\land}((h_i, c_i), (h_j, c_j)) = \begin{cases} 
(h_{i+j}, c_i), & \text{if } k = l \\
(h_{8+\text{sgn}(k-1.5)+\text{sgn}(j-1.5)}), & \text{if } k \neq l 
\end{cases}
\]

where $\text{sgn}(\cdot)$ is the sign function, the maximum subscript of the hedges in 18LV-LIA is 8, and 1.5 is the median point of the subscript of prime term, in fact, it can be any point in interval $(1, 2)$.

For relation "$(h_i, c_i)$ or $(h_j, c_j)$" can be aggregated as,
iii. \( F_{cr} (b_{r}, c_k), (b_{r}, c_i) = \begin{cases} (h_{u}, c_2), & \text{if } k \parallel l \\
 \max ((h_i, c_k), (h_i, c_i)), & \text{others} \end{cases} \tag{12} \)

where \( u = \max(n, 8 + i \cdot \text{sgn}(k - 1.5) + j \cdot \text{sgn}(l - 1.5)) \), and the parameters means the same with parameters in Eq. (10).

iv. \( F_{cr} ((h_i, c_k), (h_i, c_i)) = (h_i, c_k) \vee (h_i, c_i) \). \tag{13} \)

They all satisfy the associative and commutative law.

A linguistic multi-attribute decision making problem consists of a finite and non-empty alternative set \( A = \{A_1, A_2, \ldots, A_j\} \), where each alternative is defined by means of a finite set of attribute, \( C = \{C_1, C_2, \ldots, C_m\} \) which is assessed using linguistic expressions, and the corresponding weight set for attributes is \( W = (\omega_1, \omega_2, \ldots, \omega_m) \).

The decision making approach based on 18LV-LIA consists mainly of the flowing steps.

Step 1. Linguistic assessment matrix and linguistic weight transforming. The linguistic assessment matrix is

\[
E = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & \ddots & \cdots & a_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
a_{N1} & a_{N2} & \cdots & a_{Nm}
\end{pmatrix}
\]

In the linguistic assessment matrix, \( a_{ij} \in \mathcal{L}_{\mathcal{V}(9\times2)} \) is the assessment about the attribute \( C_i \) of alternative \( A_i \).

The attribute weights given by the experts are the linguistic terms in \( \mathcal{L}_{\mathcal{V}(9\times2)} \) as

\[ W = \{\omega_1, \omega_2, \ldots, \omega_m\}. \]

In this step, the attribute assessment collection of alternatives is transformed to linguistic terms on \( \mathcal{L}_{\mathcal{V}(9\times2)} \).

Step 2. Weighted evaluation matrix Construction. Weighted evaluation matrix is constructed from Eq. (6).

\[
M = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & \ddots & \cdots & b_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
b_{N1} & b_{N2} & \cdots & b_{Nm}
\end{pmatrix}
= E \otimes W,
\]

where \( b_{ij} = a_{ij} \otimes \omega_j \).

Step 3. Aggregation function setting and comprehensive evaluations aggregation.

The comprehensive evaluation

\[ b_i = F(b_{i1}, b_{i2}, \ldots, b_{im}), \]

where \( F(\cdot) \) is the aggregation function of the weighted evaluation.

Step 4. Decision making process. The comprehensive evaluations \( b_i \) (\( i = 1, 2, \ldots, N \)) are ranked with respect to each alternative according to Fig. 2.
Step 3. Suppose there is facts that a teacher who is responsible or prepare lesson fully, and professional proficiency or teaching efficiency should be with high teaching quality. It is a rule, as a teacher, IF (responsible or prepare lesson fully) and (professional proficiency or teaching efficiency) THEN (with high teaching quality).

The aggregation function for relation “or” is set as the logical operation Eq. (13). The aggregation function for relation “and” is set as the arithmetic operations Eq. (10). The weighted evaluations under W are listed in Table 2.

And by the aggregation rule, we get the assessments of four teachers with different attribute weights (Table 3).

Step 4: The results in Table 3 are ranked (Table 4).

From Table 2, we can see that the four teachers get the same assessments in attribute C1 and C3 but they get different weighted evaluations because the weights of attribute C1 and C3 are different. In fact, the two weights has the same linguistic hedges but the different prime terms, that can’t expressed utilizing a linear structure.

From Table 3, we can see that when all attributes are absolutely important, the results equal to the results without any weight. Therefore, the present method not considering weights is a special case of the proposed approach.

When the weights of attribute are different, the results will be changed. For example, Teacher A2 get higher score under the weight W2 than it under the weight W1.

From the results, we can see that when we consider all attribute absolutely unimportant, i.e., the weight vector is W3, the four teachers get the same and lowest evaluation. It means that we collected unavailable data.

From Table 4, there are incomparable values in the results. For example, there exists the relation for W2 as follows,

It is reasonable for that the incomparable cases exist in reality because of the characteristic of the linguistic value in nature language.

| Table 2 | The weighted evaluations values under W2. |
|---------|------------------------------------------|
|         | A1 | A2 | A3 | A4 |
| C1      | (h8, c1) | (h1, c2) | (h2, c2) | (h8, c1) |
| C2      | (h8, c1) | (h5, c1) | (h2, c2) | (h8, c1) |
| C3      | (h7, c1) | (h1, c2) | (h5, c2) | (h8, c1) |
| C4      | (h6, c1) | (h7, c1) | (h2, c1) | (h8, c1) |

| Table 3 | Comprehensive evaluations. |
|---------|-----------------------------|
|         | A1 | A2 | A3 | A4 |
| W1      | (h6, c2) | (h5, c2) | (h7, c2) | (h8, c2) |
| W2      | (h1, c2) | (h3, c2) | (h3, c2) | (h4, c1) |
| W3      | (h8, c1) | (h8, c1) | (h8, c1) | (h8, c1) |
| Without weights | (h6, c2) | (h5, c2) | (h7, c2) | (h8, c2) |

| Table 4 | The final results with different weights. |
|---------|------------------------------------------|
|         | W1 | W2 | W3 |
| A4 > A3 > A1 > A2, | A2 = A3 > A1, | A1 = A2 = A3 = A4. |
| A1 || A4. |
| A2 || A4. |
| A3 || A4. |

Remark: In step 3, taking the teacher A1 when the weight set is W2 as an example, we show the aggregation process.

\[
F = F_{and} (F_{or} ((h8, c1), (h1, c2)), F_{or} ((h2, c2), (h8, c1))) \\
= F_{and} (((h8, c1) \lor (h1, c2)), ((h2, c2) \lor (h8, c1))) \\
= \left( h_{\frac{1+2}{2}}, c_2 \right) \\
= (h1, c2).
\]

Example 2. To illustrate the effectiveness of the method, an example of car evaluation in Ref. [30] is considered.

Suppose that there are three kinds of cars: BMW (x1), Hyundai (x2) and Passat (x3) are under evaluation according to four attributes: safety (f1), price (f2), comfort (f3) and fuel economy (f4). The linguistic judgments for evaluating the cars on \(L_{(9\times2)}\) is as Table 5. The weights associated with the four criteria are supposed to be

\[
W_1 = [(h8, c2), (h6, c2), (h3, c2), (h5, c2)],
\]

or

\[
W_2 = [(h8, c1), (h6, c2), (h3, c1), (h5, c2)].
\]

The comprehensive evaluations is as Table 6.

From Table 6, we can see that when the weights are positive prime terms, the two methods get the same results. But when the weights consist of negative prime terms, the method in Ref. [30] is hard to handle it.

In Ref. [30], Shuwei Chen et al. used the linguistic multi-criteria decision making approach based on logical reasoning. Our method is more widely in weight setting and more flexible in comprehensive way.

In Ref. [30], weights are from the positive prime terms only. But in this paper, weights can be positive prime terms or negative prime

| Table 5 | Linguistic assessment about cars on \(L_{(9\times2)}\). |
|---------|------------------------------------------|
|         | f1 | f2 | f3 | f4 |
| x1      | (h6, c2) | (h3, c1) | (h7, c2) | (h1, c2) |
| x2      | (h1, c1) | (h5, c2) | (h1, c2) | (h4, c2) |
| x3      | (h6, c2) | (h1, c2) | (h5, c2) | (h2, c1) |
terms. And the aggregation function can be logical operators or arithmetic operators or hybrid operators in this paper. Our method can handle more widely cases. It is more flexible facing the complex problems in reality.

From the results in Table 4, incomparable linguistic values are existed rationally because of the inherent characteristic of linguistic value. However, a certain decision is demanded sometimes. Then some further works should be done in this case.

4. RANKING METHOD FOR INCOMPARABLE LINGUISTIC VALUES

From the Figure 2, we know that $(h_i, c_k)$ and $(h_{i+1}, c_l)$ are incomparable. They can describe the incomparable linguistic terms in the nature language. The characteristic brings incomparable results. We utility a PVF to make a certain ranking when it is necessary. So this section is extension and supplement of the above.

4.1. PVF for Linguistic-Value Lattice

A valuation on a crisp lattice $L$ is a real-valued function $v: L \rightarrow R$ which satisfies $v(a) + v(b) = v(a \lor b) + v(a \land b)$, $a, b \in L$. A valuation is called monotone if and only if $a \leq b$ in $L$ implies $v(a) \leq v(b)$ and positive if and only if $a < b$ implies $v(a) < v(b)$ [11].

A PVF $v(\cdot)$ in a lattice $L$ implies a metric distance $d: L \times L \rightarrow R$ given by $d(x_1, x_2) = v(x_1 \lor x_2) - v(x_1 \land x_2)$ for $L$ [11].

Utilizing a PVF, a value on a lattice can be mapped to a real number which keeping the order of these comparable elements on a lattice. Aiming to handle the incomparable problem of the linguistic values on LV-LIA, a PVF is defined called linguistic-real positive-valuation function (LR-PVF) on LV-LIA.

Definition 5. Let $\mathcal{L}_{V(\alpha+1\times2)}$ be a linguistic-value lattice, $(h_i, c_k) \in \mathcal{L}_{V(\alpha+1\times2)}$ (i = 0, 1, 2, ..., n and k = 1, 2). Let $\alpha, \beta \in [0, 1]$ be two preference parameters which represent the preference degree to positive side and negative side respectively. The LR-PVF is defined as

$$v: \mathcal{L}_{V(\alpha+1\times2)} \rightarrow R.$$  

$$v((h_i, c_k)) = \alpha (k - 1) + \beta (n \times (2 - k) + s(k) \times i),$$  

where $s(\cdot)$ is a function defined as

$$s(k) = \begin{cases} 
-1, & k = 1, \\
1, & k = 2.
\end{cases}$$

The preference parameters $\alpha, \beta$ make the LR-PVF be elastic by the individual performances. $(k - 1), (2 - k)$ and $s(k)$ are to distinguish the prime terms. An equal equipment is obtained by substituting $k$,

$$v((h_i, c_k)) = \begin{cases} 
\beta \times (n - i), & k = 1, \\
\alpha + \beta \times i, & k = 2.
\end{cases}$$

Theorem 3. The linguistic-real valuation function $v: \mathcal{L}_{V(\alpha+1\times2)} \rightarrow R$ is a PVF.

It is proved in Appendix A.

Definition 6. LRMD which implied by PVF $v(\cdot)$ in LV-LIA is defined as

$$dl: \mathcal{L}_{V(\alpha+1\times2)} \rightarrow R,$$

$$dl(x_1, x_2) = v(x_1 \lor x_2) - v(x_1 \land x_2).$$  

(16)

Theorem 4. Suppose $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V(\alpha+1\times2)}$, the LRMD between $(h_i, c_k) \land (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dc_1$, the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dc_2$. Similarly, the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dc_1$, the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dd_2$.

If $Dc_1 < Dc_2$ or $Dd_1 > Dd_2$, then $(h_i, c_k) < (h_j, c_l)$.

Proof. $Dc_1 = dl \left((h_i, c_k) \lor (h_j, c_l) \land (h_i, c_k)\right)$

$$= v((h_i, c_k) \lor (h_j, c_l) \land (h_i, c_k)) - v((h_i, c_k) \land (h_j, c_l)),$$

$$= v((h_i, c_k)) - v((h_i, c_k) \land (h_j, c_l));$$

$Dc_2 = dl \left((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)\right)$

$$= v((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)) - v((h_i, c_k) \lor (h_j, c_l)),$$

$$= v((h_i, c_k) \lor (h_j, c_l)) - v((h_i, c_k) \land (h_j, c_l));$$

If $Dc_1 < Dc_2$, i.e., $v((h_i, c_k) \land (h_j, c_l)) < v((h_i, c_k) \lor (h_j, c_l)) < v((h_i, c_k) \lor (h_j, c_l))$. Then we know $v((h_i, c_k)) < v((h_j, c_l))$. Since the LR-PVF $v(\cdot)$ is positive, we get $(h_i, c_k) < (h_j, c_l)$.

$Dd_1 = dl \left((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)\right)$

$$= v((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)) - v((h_i, c_k) \lor (h_j, c_l)),$$

$$= v((h_i, c_k) \lor (h_j, c_l)) - v((h_i, c_k));$$

$Dd_2 = dl \left((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)\right)$

$$= v((h_i, c_k) \lor (h_j, c_l) \lor (h_i, c_k)) - v((h_i, c_k) \lor (h_j, c_l)),$$

$$= v((h_i, c_k) \lor (h_j, c_l)) - v((h_i, c_k));$$

If $Dd_1 > Dd_2$, i.e., $v((h_i, c_k) \lor (h_j, c_l)) > v((h_i, c_k) \lor (h_j, c_l)) > v((h_i, c_k) \lor (h_j, c_l))$. Then we know $v((h_i, c_k)) < v((h_j, c_l))$. Since the LR-PVF $v(\cdot)$ is positive, we get $(h_i, c_k) < (h_j, c_l)$.

Definition 7. Let $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V(\alpha+1\times2)}$, the LRMD between $(h_i, c_k) \land (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dc_1$ and the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dc_2$. Similarly, the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dd_1$ and the LRMD between $(h_i, c_k) \lor (h_j, c_l)$ and $(h_i, c_k)$ is denoted as $Dd_2$.

If $Dc_1 < Dc_2$ or $Dd_1 > Dd_2$, then we say $(h_i, c_k)$ is ranked after $(h_j, c_l)$, denoted as $(h_i, c_k) < (h_j, c_l)$.

It is obvious that if $(h_i, c_k) < (h_j, c_l)$, then $(h_i, c_k) < (h_j, c_l)$. 

| Table 6 | The comprehensive evaluations from two methods. |
|---------|---------------------------------|
| Method  | $W_1$          | $W_2$          |
|---------|----------------|----------------|
|         | $x_1$ | $x_2$ | $x_3$ | $x_1$ | $x_2$ | $x_3$ |
| Our Method | $(h_6, c_2)$ | $(h_7, c_2)$ | $(h_6, c_2)$ | $(h_4, c_2)$ | $(h_3, c_2)$ | $(h_3, c_2)$ |
| Chen's method | $(h_6, c_2)$ | $(h_7, c_2)$ | $(h_6, c_2)$ | - | - | - |

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4.2. The Ranking Method Base on Linguistic-Real Valuation Function

Individual preference to the positive side and the negative side is different. We adjust the preference parameters in linguistic-real valuation function according to individual’s preference for incompa-

rable ranking.

Suppose \((h_i, c)\), \((h_j, c)\) are two incomparable lin-
guistic values in the decision making problems, we rank them utili-
zizing LRMFD from Theorem 4.

The flowchart of the developing ranking process by considering the incompa-
rable values ranking is shown in Figure 4.

**Step 1.** If two values in comprehensive results are comparable, then go to step 2; else go to step 3.

**Step 2.** The comparable values can be ranked by the lattice structure according to the Figure 1.

**Step 3.** For the incomparable values are mapped to real number by LR-PVF. For example, suppose \((h_i, c), (h_j, c)\) are incomparable, then \((h_i, c)\), \((h_j, c)\) or \((h_k, c)\) are mapped to real number according to Eq. (15),

\[
v((h_i, c)) = r_1, \quad v((h_j, c)) = r_2, \quad v((h_k, c)) = p = v(h_i, c) \lor (h_j, c)) = q.
\]

**Step 4.** Linguistic-real distance computing. According to Eqs. (17) and (18), denote the linguistic-real distance between \((h_i, c)\) and \((h_j, c)\) as \(Dc_1\), and denote the linguistic-real distance between \((h_j, c)\) and \((h_i, c)\) as \(Dc_2\) respectively. Or according to Eqs. (19) and (20), denote the linguistic-real distance between \((h_i, c)\) and \((h_j, c)\) as \(Dd_1\) and \((h_j, c)\) and \((h_i, c)\) as \(Dd_2\) respectively.

\[
Dc_1 = v((h_i, c)) - v((h_i, c) \land (h_j, c)) = r_1 - p, \\
Dc_2 = v((h_j, c)) - v((h_i, c) \land (h_j, c)) = r_2 - p.
\]

or

\[
Dd_1 = v((h_i, c)) \lor (h_j, c)) - v((h_i, c)) = q - r_1, \\
Dd_2 = v((h_i, c)) \lor (h_j, c)) - v((h_i, c)) = q - r_2.
\]

**Example 3.** In the teaching evaluation system, when the weight vector is \(W_2\), teacher \(A_2\), \(A_3\) and teacher \(A_4\) are incomparable. We show the incomparable ranking process by rank \(A_2 = A_3 = (h_3, c_2)\) and \(A_4 = (h_4, c_1)\). The preference parameters are set as \(\alpha = 0.2, \beta = 0.8\) by experience. It means that the preference degree to the positive side is lower than negative side on 18LV-LIA, then

\[
v((h_3, c_2)) = 0.8 \cdot (8 - i), \quad k = 1, \\
v((h_4, c_1)) = 0.2 + 0.8 \cdot i, \quad k = 2.
\]

**Step 5.** Incomparable ranking process. If \(Dc_1 < Dc_2\) or \(Dd_1 > Dd_2\), then \((h_i, c) < (h_j, c)\).

We also can do that in another way. In step 3, linguistic-real distance between \((h_3, c_2)\) and \((h_4, c_1)\) noted as \(Dd_1\), \((h_4, c_1)\) and \((h_3, c_2)\) noted as \(Dd_2\).

\[
Dd_1 = d((h_4, c_1), ((h_4, c_1) \land (h_4, c_1))) \\
Dd_2 = d((h_4, c_1), (h_4, c_1) \land (h_4, c_1)) = v((h_4, c_1)) - v((h_4, c_1)) = 0.8.
\]

**Step 4.** Ranking process:

Since \(Dd_1 < Dd_2\), so \((h_3, c_2) < (h_4, c_1)\). It is obvious that \(Dd_1 > Dd_2\), so \((h_3, c_2) < (h_4, c_1)\).

From above, the best one in teaching quality is teacher \(A_4\) if the system highlight the negative evaluations.

When the preference changes, the ranking results would change. Figure 5 shows \(Dc_1\) and \(Dc_2\) relation under different preference parameters.

From Figure 5, \(Dc_1\) and \(Dc_2\) is changing following preference parameters. When \(\alpha = \beta\), \(Dc_1 = Dc_2\); When \(\alpha > \beta\), \(Dc_1 < Dc_2\); When \(\beta < \alpha\), \(Dc_1 > Dc_2\). It shows that the incomparable ranking results can be adjust elastic related to people’s preference to positive side and negative side.
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5. CONCLUSIONS

LV-LIA can deal with linguistic information by manipulating directly linguistic terms without numerical approximation. We proposed a linguistic approach of decision making with linguistic weight based on LV-LIA to deal with the attribute in different importance degree. The contrast example illustrated the proposed approach is more widely applied. The linguistic aggregation function for weighted evaluations comprehensive can be logical operations or arithmetic operations or hybrid ones. It makes the proposed approach more widely applied.

Incomparable linguistic results sometimes exist and are rational. In this paper we defined a PVF named LR-PVF by which the LV-LIA can be mapped to a real set. The distance implied by LV-LIA was used to do further works when these incomparable linguistic results need to be ranked. It is elastic by adjusting the performance parameters setting by individual performance to the positive side and negative side.

The proposed approach can be extended into more fields such as pattern recognition, risk analysis, quality evaluation and so on. The data and the parameters in the example are by experience in this paper. It will lead more convincing conclusion with practical data.
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APPENDIX A

Proof of Theorem 3

The proof of Theorem 3 is as below.

Proof. There are two conditions should be proved to convince that $\nu : \mathcal{L}_{V(n+1 \times 2)} \rightarrow R$ is a positive valuation function. For any $(h_i, c_i), (h_j, c_j) \in \mathcal{L}_{V(n+1 \times 2)}$,

i. $\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \nu((h_i, c_i)) + \nu((h_j, c_j))$.

ii. If and only if $(h_i, c_i) < (h_j, c_j)$ implies $\nu((h_i, c_i)) < \nu((h_j, c_j))$.

To prove $\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j))$ as shown in Figure 1.

Then we'll prove it in three cases according to the operations $\lor$ and $\land$ as shown in Figure 1.

(a) When $j = l = 2$, according to Eq. (15),

$$\nu((h_i, c_i)) + \nu((h_j, c_j)) = (\alpha + \beta \times i) + (\alpha + \beta \times j) = 2\alpha + \beta (i + j).$$

And

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \alpha + \beta \times (i \lor j).$$

Then

$$\nu((h_i, c_i) \lor (h_j, c_j)) + \nu((h_i, c_i) \land (h_j, c_j)) = 2\alpha + \beta (i + j).$$

Therefore, we get

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - (i \lor j)).$$

(b) when $j = l = 1$, according to Eq. (15),

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - (i \land j)).$$

And

$$\nu((h_i, c_i) \lor (h_j, c_j)) \lor \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - (i \lor j)).$$

Then

$$\nu((h_i, c_i) \lor (h_j, c_j)) + \nu((h_i, c_i) \land (h_j, c_j)) = \beta (2n - i - k).$$

Therefore, we get

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - i).$$

(c) When $j = 1$, without loss of generality, let $j = 2, l = 1$,

$$\nu((h_i, c_i)) + \nu((h_j, c_j)) = (\alpha + \beta \times i) + (\alpha + \beta \times n - j) = \alpha + \beta (n + i - j).$$

And

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - (i \lor j)).$$

Then

$$\nu((h_i, c_i) \lor (h_j, c_j)) + \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - (i \lor j)).$$

Therefore, we get

$$\nu((h_i, c_i) \lor (h_j, c_j)) = \nu((h_i, c_i) \land (h_j, c_j)) = \beta (n - i).$$

The second condition is proved as the following. According to

(a) when $k = l = 2$, if $(h_i, c_i) < (h_j, c_j)$, then $i < j$

$$\nu((h_i, c_i)) = \alpha + \beta \times i < \alpha + \beta \times j = \nu((h_j, c_j)).$$

If $\nu((h_i, c_i)) < \nu((h_j, c_j))$, i.e. $\alpha + \beta \times i < \alpha + \beta \times j$, then

$$\nu((h_i, c_i)) = \beta (n - i) < \beta (n - j) = \nu((h_j, c_j)).$$

(b) When $k = l = 1$, if $(h_i, c_i) < (h_j, c_j)$, then $i > j$.

$$\nu((h_i, c_i)) = \beta (n - i) < \beta (n - j) = \nu((h_j, c_j)).$$

If $\nu((h_i, c_i)) < \nu((h_j, c_j))$, i.e. $\beta (n - i) < \beta (n - j)$, then

$$\nu((h_i, c_i)) = \beta (n - i).$$

(c) When $k = l = 1$, if $(h_i, c_i) < (h_j, c_j)$, then $i \geq n - j$.

$$\nu((h_i, c_i)) = \beta (n - i) \leq \beta j < \alpha + \beta \times j = \nu((h_j, c_j)).$$

If $\nu((h_i, c_i)) < \nu((h_j, c_j))$, that is to say, $\beta (n - i) < \alpha + \beta \times j < \beta (j + 1)$ for $\alpha < \beta$, then $n - i < j + 1$. Because $n, i, j$ are positive integers, $n - i \leq j$, i.e., $i \geq n - j$. Then $(h_i, c_i) < (h_j, c_j)$.