New congruences for $\ell$-regular overpartitions

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Abstract. Recently, Shen (2016) and Alanazi et al. (2016) studied the arithmetic properties of the $\ell$-regular overpartition function $A_\ell(n)$, which counts the number of overpartitions of $n$ into parts not divisible by $\ell$. In this note, we will present some new congruences modulo 5 when $\ell$ is a power of 5.

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1. Introduction

A partition of a natural number $n$ is a nonincreasing sequence of positive integers whose sum is $n$. For example, $6 = 3 + 2 + 1$ is a partition of 6. Let $p(n)$ denote the number of partitions of $n$. We also agree that $p(0) = 1$. It is well-known that the generating function of $p(n)$ is given by

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where we adopt the standard notation

$$(a; q)_\infty = \prod_{n \geq 0} (1 - aq^n).$$

For any positive integer $\ell$, a partition is called $\ell$-regular if none of its parts are divisible by $\ell$. Let $b_\ell(n)$ denote the number of $\ell$-regular partitions of $n$. We know that its generating function is

$$\sum_{n \geq 0} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

On the other hand, an overpartition of $n$ is a partition of $n$ in which the first occurrence of each distinct part can be overlined. Let $\overline{p}(n)$ be the number of overpartitions of $n$. We also know that the generating function of $\overline{p}(n)$ is

$$\sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty} = \frac{(q^2;q^2)^2_\infty}{(q; q)_\infty^2}.$$

Many authors have studied the arithmetic properties of $b_\ell(n)$ and $\overline{p}(n)$. We refer the interested readers to the “Introduction” part of [6] and references therein for detailed description.

In [5], Lovejoy introduced a function $\overline{A}_\ell(n)$, which counts the number of overpartitions of $n$ into parts not divisible by $\ell$. According to Shen [6], this type of
partition can be named as \( \ell \)-regular overpartition. He also obtained the generating function of \( A_\ell(n) \), that is,
\[
\sum_{n \geq 0} A_\ell(n)q^n = \frac{(q^\ell; q^\ell)_\infty^2 (q^2; q^2)_\infty}{(q; q)_\infty^2 (q^{2\ell}; q^{2\ell})_\infty}.
\] (1.1)
Meanwhile, he presented several congruences for \( A_3(n) \) and \( A_4(n) \). For \( A_3(n) \), he got
\[
A_3(4n + 1) \equiv 0 \pmod{2},
A_3(4n + 3) \equiv 0 \pmod{6},
A_3(9n + 3) \equiv 0 \pmod{6},
A_3(9n + 6) \equiv 0 \pmod{24}.
\]

More recently, Alanazi et al. [1] further studied the arithmetic properties of \( A_\ell(n) \) under modulus 3 when \( \ell \) is a power of 3. They also gave some congruences satisfied by \( A_\ell(n) \) modulo 2 and 4.

In this note, our main purpose is to study the arithmetic properties of \( A_\ell(n) \) when \( \ell \) is a power of 5. When \( \ell = 5 \) and 25, we connect \( A_\ell(n) \) with \( r_4(n) \) and \( r_8(n) \) respectively, where \( r_k(n) \) denotes the number of representations of \( n \) by \( k \) squares. The method for \( \ell = 5 \) also applies to other prime \( \ell \). When \( \ell = 125 \), we show that
\[
A_{125}(25n) \equiv A_{125}(625n) \pmod{5}.
\]
This can be regarded as an analogous result of the following congruence for overpartition function \( \overline{p}(n) \) proved by Chen et al. (see [4, Theorem 1.5])
\[
\overline{p}(25n) \equiv \overline{p}(625n) \pmod{5}.
\]
When \( \ell = 5^\alpha \) with \( \alpha \geq 4 \), we obtain new congruences similar to a result of Alanazi et al. (see [1, Theorem 3]), which states that \( \overline{A}_{3^\alpha}(27n + 18) \equiv 0 \pmod{3} \) holds for all \( n \geq 0 \) and \( \alpha \geq 3 \).

2. New congruence results

2.1. \( \ell = 5 \). One readily sees from the binomial theorem that for any prime \( p \),
\[
(q; q)_\infty^p \equiv (q^p; q^p)_\infty \pmod{p}.
\] (2.1)
Setting \( p = 5 \) and applying it to (1.1), we have
\[
\sum_{n \geq 0} A_5(n)q^n \equiv \left( \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} \right)^4 \pmod{5}.
\] (2.2)
Now let
\[
\varphi(q) := \sum_{n = -\infty}^{\infty} q^{n^2}.
\]
It is well-known that (see [2, p. 37, Eq. (22.4)])
\[
\varphi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}.
\]
We therefore have
Theorem 2.1. For any positive integer \( n \), we have

\[
\mathcal{A}_5(n) \equiv \begin{cases} 
 r_4(n) & \text{if } n \text{ is even} \\
 -r_4(n) & \text{if } n \text{ is odd}
\end{cases} \pmod{5}. \tag{2.3}
\]

We know from [3, Theorem 3.3.1] that \( r_4(n) = 8d^*(n) \) where

\[
d^*(n) = \sum_{d | n, 4 \nmid d} d.
\]

Let \( p \neq 5 \) be an odd prime and \( k \) be a nonnegative integer. One readily verifies that

\[
\sum_{i=0}^{4k+3} p^i \equiv (k+1) \sum_{i=1}^{4} i \equiv 0 \pmod{5}.
\]

Note also that \( d^*(n) \) is multiplicative. Thus we conclude

Theorem 2.2. Let \( p \neq 5 \) be an odd prime and \( k \) be a nonnegative integer. Let \( n \) be a nonnegative integer. We have

\[
\mathcal{A}_5(p^{4k+3}(pn + i)) \equiv 0 \pmod{5}, \tag{2.4}
\]

where \( i \in \{1, 2, \ldots, p - 1\} \).

Example 2.1. If we take \( p = 3, k = 0, \) and \( i = 1 \), then

\[
\mathcal{A}_5(81n + 27) \equiv 0 \pmod{5} \tag{2.5}
\]

holds for all \( n \geq 0 \).

We also note that if an odd prime \( p \) is congruent to 9 modulo 10, then \( 1 + p \equiv 0 \pmod{5} \). We therefore have \( \mathcal{A}_5(p^{4k+3}(pn + i)) \equiv 0 \pmod{5} \) for \( i \in \{1, 2, \ldots, p - 1\} \). On the other hand, if we require \( 1 + p + p^2 \equiv 0 \pmod{5} \), then \( (2p + 1)^2 \equiv -3 \pmod{5} \). However, since \( -3|5\) = -1 (here \( (*)|(*) \) denotes the Legendre symbol), such \( p \) does not exist. The above observation yields

Theorem 2.3. Let \( p \equiv 9 \pmod{10} \) be a prime and \( n \) be a nonnegative integer. We have

\[
\mathcal{A}_5(p^{4k+3}(pn + i)) \equiv 0 \pmod{5}, \tag{2.6}
\]

where \( i \in \{1, 2, \ldots, p - 1\} \).

Example 2.2. If we take \( p = 19 \) and \( i = 1 \), then

\[
\mathcal{A}_5(361n + 19) \equiv 0 \pmod{5} \tag{2.7}
\]

holds for all \( n \geq 0 \).

We should mention that this method also applies to other primes \( \ell \). In fact, if we set \( p = \ell \) in (2.1) and apply it to (1.1), then

\[
\mathcal{A}_\ell(n) \equiv \begin{cases} 
 r_{\ell-1}(n) & \text{if } n \text{ is even} \\
 -r_{\ell-1}(n) & \text{if } n \text{ is odd}
\end{cases} \pmod{\ell}. \tag{2.8}
\]

Recall that the explicit formulas of \( r_2(n) \) and \( r_6(n) \) are also known. From [3, Theorems 3.2.1 and 3.4.1], we have

\[
r_2(n) = 4 \sum_{d | n} \chi(d),
\]

where \( \chi(d) \) is the Legendre symbol.
and
\[ r_6(n) = 16 \sum_{d|n} \chi(n/d)d^2 - 4 \sum_{d|n} \chi(d)d^2, \]
where
\[ \chi(n) = \begin{cases} 1 & n \equiv 1 \pmod{4}, \\ -1 & n \equiv 3 \pmod{4}, \\ 0 & \text{otherwise}. \end{cases} \]

Through a similar argument, we conclude that

**Theorem 2.4.** For any nonnegative integers \( n, k \), odd prime \( p \), and \( i \in \{1, 2, \ldots, p-1\} \), we have
\[ A_3(p^{2k+1}(pn+i)) \equiv 0 \pmod{3} \quad \text{where } p \equiv 3 \pmod{4}, \]
\[ A_3(p^{3k+2}(pn+i)) \equiv 0 \pmod{3} \quad \text{where } p \equiv 1 \pmod{4}, \]
\[ A_7(p^{6k+5}(pn+i)) \equiv 0 \pmod{7} \quad \text{where } p \not\equiv 7. \]

**2.2. \( \ell = 25 \).** Analogous to the congruences under modulus 3 for \( A_9(n) \) obtained by Alanazi et al. [1], we will present some arithmetic properties of \( A_{25}(n) \) modulo 5. Rather than using the technique of dissection identities, we build a connection between \( A_{25}(5n) \) and \( r_8(n) \) and then apply the explicit formula of \( r_8(n) \). It is necessary to mention that this method also applies to the results of Alanazi et al. as the following relation holds
\[ A_9(n) \equiv (-1)^n r_8(n) \pmod{3}. \]

Note that
\[ \sum_{n \geq 0} A_{25}(n)q^n = \frac{(q^{25}; q^{25})_\infty^2 (q^2; q^2)_\infty}{(q^{50}; q^{50})_\infty (q; q)_\infty^2}, \]
\[ \equiv \left( \frac{(q^5; q^5)_\infty^2}{(q^{10}; q^{10})_\infty} \right)^5 \left( \sum_{n \geq 0} \overline{A}(n)q^n \right) \pmod{5}. \]

Extracting powers of the form \( q^{5n} \) from both sides and replacing \( q^5 \) by \( q \), we have
\[ \sum_{n \geq 0} A_{25}(5n)q^n = \left( \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} \right)^5 \left( \sum_{n \geq 0} \overline{A}(5n)q^n \right) \]
\[ \equiv \varphi(-q)^8 \pmod{5}. \]

Here we use the following celebrated result due to Treneer [7]
\[ \sum_{n \geq 0} \overline{A}(5n)q^n = \varphi(-q)^3 \pmod{5}. \]

It is known that the explicit formula of \( r_8(n) \) (see [3, Theorem 3.5.4]) is given by
\[ r_8(n) = 16(-1)^n \sigma_3^-(n), \]
where
\[ \sigma_3^-(n) = \sum_{d|n} (-1)^d d^3. \]

We therefore conclude that
Theorem 2.5. For any positive integer $n$, we have
\[
\overline{A}_{25}(5n) \equiv \sigma_3^-(n) \pmod{5}.
\] (2.9)

One also readily deduces several congruences from Theorem 2.5.

Theorem 2.6. Let $p \neq 5$ be an odd prime and $k$ be a nonnegative integer. Let $n$ be a nonnegative integer. We have
\[
\overline{A}_{25}(5p^{4k+3}(pn+i)) \equiv 0 \pmod{5},
\] (2.10)
where $i \in \{1, 2, \ldots, p-1\}$.

Proof. It is easy to see that
\[
\sigma_3^-(p^{4k+3}) = - \sum_{i=0}^{4k+3} p^{3i} \equiv -(k+1) \sum_{i=1}^{4} i \equiv 0 \pmod{5}.
\]

Note also that $\sigma_3^-(n)$ is multiplicative. The theorem therefore follows. \qed

Furthermore, if $p \equiv 9 \pmod{10}$ is a prime, then $1 + p^3 \equiv 0 \pmod{5}$. This yields

Theorem 2.7. Let $p \equiv 9 \pmod{10}$ be a prime and $n$ be a nonnegative integer. We have
\[
\overline{A}_{25}(5p^{3}(pn+i)) \equiv 0 \pmod{5},
\] (2.11)
where $i \in \{1, 2, \ldots, p-1\}$.

2.3. $\ell = 125$. From the generating function (1.1), we have
\[
\sum_{n \geq 0} \overline{A}_{125}(n)q^n = \frac{(q^{125}; q^{125})_{\infty}^2(q^2; q^2)_{\infty}}{(q; q^2)_\infty(q^{250}; q^{250})_\infty} = \varphi(-q^{125}) \sum_{n \geq 0} \overline{p}(n)q^n.
\]

Extracting terms of the form $q^{125n}$ and replacing $q^{125}$ by $q$, we have
\[
\sum_{n \geq 0} \overline{A}_{125}(125n)q^n = \varphi(-q) \sum_{n \geq 0} \overline{p}(125n)q^n.
\]

According to [4, Eq. (5.3)], we know that $\overline{p}(125(5n \pm 1)) \equiv 0 \pmod{5}$. We also know from [2, p. 49, Corollary (i)] that
\[
\varphi(-q) = \varphi(-q^{25}) - 2qM_1(-q^5) + 2q^4M_2(-q^5),
\]
where $M_1(q) = f(q^2, q^5)$ and $M_2(q) = f(q, q^5)$. Here $f(a, b)$ is the Ramanujan’s theta function defined as
\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}.
\]

Now we extract terms involving $q^{5n}$ from $\sum_{n \geq 0} \overline{A}_{125}(125n)q^n$ and replace $q^5$ by $q$, then
\[
\sum_{n \geq 0} \overline{A}_{125}(625n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} \overline{p}(625n)q^n \pmod{5}.
\]

Finally, we use the following congruence from [4, Theorem 1.5] for $\overline{p}(n)$
\[
\overline{p}(25n) \equiv \overline{p}(625n) \pmod{5},
\]
and obtain
\[
\sum_{n \geq 0} \overline{A}_{125}(625n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} \overline{p}(25n)q^n \pmod{5},
\]
which coincides with
\[ \sum_{n \geq 0} A_{125}(25n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} p(25n)q^n \pmod{5}. \]

We therefore conclude

**Theorem 2.8.** For any nonnegative integer \( n \), we have
\[ A_{125}(25n) \equiv A_{125}(625n) \pmod{5}. \] (2.12)

**2.4. \( \ell = 5^\alpha \) with \( \alpha \geq 4 \).** We know from (1.1) that in this case
\[ \sum_{n \geq 0} A_{5^\alpha}(n)q^n = \varphi(-q^{5\alpha}) \sum_{n \geq 0} p(n)q^n. \]

Note that for \( \alpha \geq 4 \), \( \varphi(-q^{5\alpha}) \) is a function of \( q^{625} \). Hence \( A_{5^\alpha}(625n+125) \) (resp. \( A_{5^\alpha}(625n+500) \)) is a linear combination of values of \( p(625n+125) \) (resp. \( p(625n+500) \)). Thanks to [4, Eq. (5.3)], we know that \( p(625n+125) \equiv 0 \pmod{5} \) and \( p(625n+500) \equiv 0 \pmod{5} \) hold for all \( n \geq 0 \). Hence

**Theorem 2.9.** For any nonnegative integer \( n \) and positive integer \( \alpha \geq 4 \), we have
\[ A_{5^\alpha}(625n+i) \equiv 0 \pmod{5}, \] (2.13)

where \( i = 125 \) and 500.

Note also that for \( \alpha \geq 2 \), extracting terms of the form \( q^{25n} \) and replacing \( q^{25} \) by \( q \), we obtain
\[ \sum_{n \geq 0} A_{5^\alpha}(25n)q^n = \varphi(-q^{5^\alpha-2}) \sum_{n \geq 0} p(25n)q^n. \]

On the other hand, we extract terms involving \( q^{625n} \) from \( \sum_{n \geq 0} A_{5^{\alpha+2}}(n)q^n \) and replace \( q^{625} \) by \( q \), then
\[ \sum_{n \geq 0} A_{5^{\alpha+2}}(625n)q^n = \varphi(-q^{5^{\alpha-2}}) \sum_{n \geq 0} p(625n)q^n. \]

Thanks again to [4, Theorem 1.5], which states that \( p(25n) \equiv p(625n) \pmod{5} \) for all \( n \geq 0 \), we conclude

**Theorem 2.10.** For any nonnegative integer \( n \), we have
\[ A_{5^\alpha}(25n) \equiv A_{5^{\alpha+2}}(625n) \pmod{5} \] (2.14)

for all \( \alpha \geq 2 \).

It follows from Theorems 2.9 and 2.10 that

**Theorem 2.11.** For any nonnegative integer \( n \) and positive integer \( \alpha \geq 4 \), we have
\[ A_{5^\alpha}(5^j(25n+i)) \equiv 0 \pmod{5} \] (2.15)

for all integers \( 0 \leq j \leq (\alpha - 4)/2 \), where \( i = 125 \) and 500.

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