Wilson Renormalization Group for Supersymmetric Gauge Theories and Gauge Anomalies

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Abstract

We extend the Wilson renormalization group (RG) to supersymmetric theories. As this regularization scheme preserves supersymmetry, we exploit the superspace technique. To set up the formalism we first derive the RG flow for the massless Wess-Zumino model and deduce its perturbative expansion. We then consider N=1 supersymmetric Yang-Mills and show that the local gauge symmetry –broken by the regularization– can be recovered by a suitable choice of the RG flow boundary conditions. We restrict our analysis to the first loop, the generalization to higher loops presenting no difficulty due to the iterative nature of the procedure. Furthermore, adding matter fields, we reproduce the one-loop supersymmetric chiral anomaly to the second order in the vector field.
1 Introduction

Perturbative calculations in field theories require a regularization procedure to deal with ultraviolet divergences. A powerful and simple method is that of dimensional regularization which has the remarkable feature of preserving gauge symmetry. However, it is clear that this regularization breaks supersymmetry, since fermionic and bosonic degrees of freedom match only in fixed dimensions. The only modification of dimensional regularization compatible with supersymmetry (the so-called dimensional reduction \([1]\)) turns out to be inconsistent \([2]\). The lack of a consistent regularization scheme which manifestly preserves supersymmetry implies, in particular, that superspace formalism can be used only with some care since naive manipulations may lead to ambiguities \([3]\). Although dimensional reduction does not seem to cause any practical difficulty and is extensively used \([4]\) to perform perturbative calculations, it is worthwhile to look for manifestly supersymmetric regularizations which take advantage of the superspace technique \([5]\) and are free of ambiguities.

Recently a regularization procedure based on the Wilson renormalization group (RG) has been formulated \([6, 7]\) for scalar field theories \([8, 9]\) and gauge theories \([7, 10, 11]\). With this method one introduces an ultraviolet (UV) and infrared (IR) cutoff, \(\Lambda\) and \(\Lambda_0\) respectively, in the propagator so that all Feynman diagrams become convergent in the UV region. The cutoff effective action can be obtained through the standard procedure using this modified propagator. The physical effective action is recovered by taking \(\Lambda = 0\) and the limit \(\Lambda_0 \to \infty\). The existence of such a limit is a consequence of the renormalizability (and IR finiteness) of the theory. The RG formalism provides an alternative definition of the cutoff effective action. Requiring the physical amplitudes to be independent of \(\Lambda\), one derives a differential equation (the RG flow) for the cutoff effective action, which can be solved giving a set of boundary conditions that encode both renormalizability and the renormalization conditions. Despite the presence of the cutoff \(\Lambda\) which explicitly breaks gauge invariance, one proves that, by properly fixing the boundary conditions of the RG equation, the Slavnov-Taylor (ST) identity can be satisfied when all cutoffs are removed (at least in perturbation theory). This has been shown for the pure Yang-Mills case both in terms of the “bare” couplings of the effective action at the ultraviolet scale \([7]\) and of the physical couplings \([10]\). Furthermore, as this method directly works in four space-time dimensions, it naturally extends \([12]\) to chiral gauge theories with no ambiguity in the definition of the matrix \(\gamma_5\) (contrary to dimensional regularization).

In this paper we generalize the RG method to supersymmetric theories, implementing the regularization in such a way that supersymmetry is preserved. This can be easily understood observing how the cutoffs are introduced in the RG formulation. One splits the classical action into two parts, the quadratic and the interacting one, and then multiplies the former by a cutoff function \(K_{\Lambda\Lambda_0}(p)\) which falls off sufficiently rapid for \(p^2\) outside the region \(\Lambda^2 < p^2 < \Lambda_0^2\). In the supersymmetric case it suffices to write the classical action in terms of superfields and follow the same procedure (in components this corresponds to use the same cutoff function for all fields). As our formulation works in \(d = 4\), supersymmetry is maintained and, from the very beginning, we can exploit the superspace technique, which simplifies perturbative calculations and is now unambiguous.
For a supersymmetric gauge theory, supersymmetry is preserved though gauge symmetry is explicitly broken by the regularization. As for non-supersymmetric gauge theories, we will show that by properly fixing the boundary conditions of the RG flow the ST identity associated to the gauge symmetry is recovered, when the matter representation is anomaly free. However, if the matching conditions for the anomaly cancellation are not fulfilled, we will be able to reproduce the chiral anomaly.

The paper is organized as follows. In sect. 2 we set up the RG formalism for the massless Wess-Zumino (WZ) model and, as an example of how to perform perturbative calculations, we compute the one-loop two-point function. In this section we briefly discuss the boundary conditions for the flow equation. In sect. 3 we consider the N=1 super Yang-Mills (SYM) case in this framework. In sect. 4 we formulate the effective ST identity for this theory and show how to solve the fine-tuning equation at one-loop order and at the UV scale. Sect. 5 is devoted to the computation of the one-loop chiral anomaly (up to the second order in the gauge field) and sect. 6 contains some conclusions. Finally, our conventions are given in the appendix.

2 Renormalization group flow and effective action for the Wess-Zumino model

The massless WZ model is described by the classical Lagrangian

\[ S_{cl} = S_2 + S_{int}^{(0)} \]

where

\[
S_2 = \frac{1}{16} \int_z \bar{\phi} \phi, \quad \int_z = \int d^4x \, d^2\theta \, d^2\bar{\theta}, \\
S_{int}^{(0)} = \frac{1}{48} \int d^4x \, d^2\theta \, \phi^3 + \text{h.c.}
\]

and \( \phi (\bar{\phi}) \) is a chiral (anti-chiral) superfield satisfying \( \bar{D}_\alpha \phi = 0 \) \( (D^\alpha \bar{\phi} = 0) \).

In order to quantize the theory one needs a regularization procedure of the ultraviolet divergences. We regularize these divergences by assuming that in the path integral one integrates only the fields with frequencies smaller than a given UV cutoff \( \Lambda_0 \). This procedure is equivalent to modify the free action to assume that the free propagators vanish for \( p^2 > \Lambda_0^2 \).

The generating functional is

\[
Z[J] = e^{iW[J]} = \int D\Phi \exp \left\{ \frac{1}{2} (\Phi, D^{-1}\Phi)_{0,\Lambda_0} + (J, \Phi)_{0,\Lambda_0} + S_{int}[\Phi; \Lambda_0] \right\}, \tag{2}
\]

where we have collected the fields and the sources in \( \Phi_i = (\phi, \bar{\phi}) \) and \( J_i = (J, \bar{J}) \) respectively, and we introduce the general cutoff scalar products between fields and sources

\[
\frac{1}{2} (\Phi, D^{-1}\Phi)_{\Lambda\Lambda_0} \equiv \frac{1}{16} \int_p K^{-1}_{\Lambda\Lambda_0} (p) \bar{\phi}(-p, \theta) \phi(p, \theta), \quad \int_p \equiv \int \frac{d^4p}{(2\pi)^4} \, d^2\theta \, d^2\bar{\theta} \tag{3}
\]
\[ (J, \Phi)_{\Lambda_0} = \frac{1}{16} \int_p K^{-1}_{\Lambda_0}(p) \left\{ J(-p, \theta) \frac{D^2}{p^2} \phi(p, \theta) + J(-p, \theta) \frac{D^2}{p^2} \bar{\phi}(p, \theta) \right\}, \quad (4) \]

with \( K^{-1}_{\Lambda_0}(p) \) a cutoff function which is one for \( \Lambda^2 < p^2 < \Lambda_0^2 \) and rapidly vanishes outside \( \mathbb{R}^n \). The introduction of such a cutoff function in (4) yields a regularized propagator which preserves supersymmetry, this being a global, linearly realized transformation. Hence the UV action \( S_{\text{int}}[\Phi; \Lambda_0] \) in (4) contains all possible renormalizable supersymmetric interactions, i.e. superspace integrals of superfields and their covariant derivatives which are local in \( \theta \). Dimensional analysis tells us that they are given by the monomials \( \phi \bar{\phi}, \phi, \phi^2, \phi^3, \bar{\phi}, \bar{\phi}^2, \bar{\phi}^3 \), properly integrated.

According to Wilson one integrates over the fields with frequencies \( \Lambda^2 < p^2 < \Lambda_0^2 \) and obtains \( \frac{1}{16} \)

\[ e^{iW[J]} = N[J; \Lambda] \int D\Phi \exp i \left\{ \frac{1}{2}(\Phi, D^{-1}\Phi)_{0\Lambda} + (J, \Phi)_{0\Lambda} + S_{\text{eff}}[\Phi; \Lambda] \right\}, \quad (5) \]

where \( K_{0\Lambda} = K_{0\Lambda_0} - K_{\Lambda\Lambda_0} \) and \( N[J; \Lambda] \) contributes to the quadratic part of \( W[J] \). The functional \( S_{\text{eff}}[\Phi; \Lambda] \) is the so-called Wilsonian effective action\(^1\). Since the regularization preserves supersymmetry, this functional contains all possible supersymmetric interactions. One can show that \( S_{\text{eff}}[\Phi; \Lambda] \) is the generating functional of the connected amputated cutoff Green functions — except the tree-level two-point function — in which the free propagators contain \( \Lambda \) as an infrared cutoff \( \frac{1}{16} \). In other words the functional

\[ W[J'; \Lambda] = S_{\text{eff}}[\Phi; \Lambda] + \frac{1}{2}(\Phi, D^{-1}\Phi)_{\Lambda\Lambda_0}, \quad (6) \]

with the sources \( J' \) given by

\[ J'_i(-p, \theta) = K^{-1}_{\Lambda\Lambda_0}(p) D^{-1}_{ij}(\theta) \Phi_j(-p, \theta) D_{ji}^{-1}(p), \quad (7) \]

is the generator of the cutoff connected Green functions. The matrix \( D_{ij}^{-1} \) is defined through (3) and its entries are \( D_{ij}^{-1} = 1/16 \) if \( i \neq j \) and zero otherwise. Moreover, in order to keep formulas more compact, we have introduced the two-component vector \( \varepsilon_k = (1, -1) \) and the shortened notation \( D^{-2} \equiv D^2 \) which allow to treat simultaneously chiral and anti-chiral fields.

### 2.1 Evolution equation

The requirement that the generating functional (3) is independent of the IR cutoff \( \Lambda \) gives rise to a differential equation for the Wilsonian effective action, the so-called exact RG equation (3), [7], which can be translated into an equation for \( W[J; \Lambda] \)

\[ \Lambda \partial_\Lambda W[J; \Lambda] = \frac{1}{2} \int_p \Lambda \partial_\Lambda K^{-1}_{\Lambda\Lambda_0}(p) D_{ji}^{-1}(p) \left( \frac{\delta W}{\delta J_i(-p, \theta)} \frac{\delta W}{\delta J_j(p, \theta)} - i \frac{\delta^2 W}{\delta J_i(-p, \theta) \delta J_j(p, \theta)} \right). \quad (8) \]

\(^1\)The factors \( D^2/(16p^2), D^2/(16p^2) \) are needed to write the chiral and anti-chiral superspace integral respectively, as integrals over the full superspace (see the appendix).

\(^2\)Here and in the following we explicitly write only the dependence on the cutoff \( \Lambda \), since the theory is renormalizable and we are interested in the limit \( \Lambda_0 \to \infty \).
This equation can be more easily understood taking into account that \( \Lambda \) enters as an IR cutoff in the internal propagators of the cutoff Green functions.

It is convenient to introduce the so-called “cutoff effective action” which is given by the Legendre transform of \( W[J; \Lambda] \)

\[
\Gamma[\Phi; \Lambda] = W[J; \Lambda] - \int d^4 x \ d^2 \theta \ J \dot{\phi} - \int d^4 x \ d^2 \bar{\theta} \ \bar{J} \bar{\phi}.
\]  

(9)

This functional generates the cutoff vertex functions in which the internal propagators have frequencies in the range \( \Lambda^2 < p^2 < \Lambda_0^2 \) and reduces to the physical quantum effective action in the limits \( \Lambda \to 0 \) and \( \Lambda_0 \to \infty \) [3, 9].

The evolution equation for the functional \( \Gamma[\Phi; \Lambda] \) can be derived from (8) by using (9) and inverting the functional \( \frac{\delta^2 W}{\delta J_0} \). This inversion can be performed isolating the full two-point contributions \( \Gamma_{ij} \) in the functional \( \Gamma[\Phi; \Lambda] \)

\[
(2\pi)^8 \frac{\delta^2 \Gamma}{\delta J_i(-k, \theta) \delta J_j(q, \theta_1)} = (2\pi)^4 \Gamma_{2ij}(k; \Lambda) D^{-2\varepsilon_i} (\theta) D^{-2\varepsilon_j} (\theta_1) \delta^8 (k + p) + \Gamma_{ij}^{\text{int}} [\Phi; k, p; \Lambda]
\]

and \( W_2 \) in \( W[J; \Lambda] \)

\[
(2\pi)^8 \frac{\delta^2 W}{\delta J_i(q, \theta_1) \delta J_j(q, \theta_2)} = (2\pi)^4 W_{2ik}(k; \Lambda) D^{-2\varepsilon_i} (\theta_1) D^{-2\varepsilon_k} (\theta) \delta^8 (q - k) + W_{ik}^{\text{int}} [J; q, -k; \Lambda]
\]

(10)

where the dependence on Grassmann variables in \( \Gamma^{\text{int}} \) and \( W^{\text{int}} \) is understood. Henceforth we will prefer writing all integrals in the full superspace, so that we have to cope with factors like \( \frac{D^2 (\theta)}{16 k^2} \) and \( \frac{D^2 (\theta)}{16 k^2} \) originating from chiral and anti-chiral projectors, respectively. These two factors can be simultaneously treated with the help of the vector \( \varepsilon_k \) and identifying \( \left( \frac{D^2 (\theta)}{16 k^2} \right)^{-1} \) with \( \frac{D^2 (\theta)}{16 k^2} \).

Then making use of the identity

\[
\frac{\delta \Phi_i (q, \theta_2)}{\delta \Phi_j (p, \theta_1)} = D^{-2\varepsilon_i} (\theta_1) \delta^8 (q + p) \delta_{ij}
\]

\[
= (2\pi)^8 \int_k \frac{\delta^2 W}{\delta J_k(-k, \theta) \delta J_i(q, \theta_2)} \left( \frac{D^2 (\theta)}{16 k^2} \right)^{\varepsilon_k} \frac{\delta^2 \Gamma}{\delta \Phi_j (p, \theta_1) \delta \Phi_k (k, \theta)}
\]

we can express \( W_{ij}^{\text{int}} \) in (10) as a functional of \( \Phi \) obtaining

\[
W_{ij}^{\text{int}} [J(\Phi); q, p; \Lambda] = -\Gamma_{2ij}^{-1} (p; \Lambda) \left( \frac{D^2 (\theta_2)}{16 q^2} \right)^{\varepsilon_k} \left( \frac{D^2 (\theta_1)}{16 p^2} \right)^{\varepsilon_l} \Gamma_{kl}^{\text{int}} [\Phi; q, p; \Lambda] \Gamma_{2ik}^{-1} (q; \Lambda),
\]

(11)

where the auxiliary functional \( \bar{\Gamma} \) satisfies the recursive equation

\[
\bar{\Gamma}_{ij}^{\text{int}} [\Phi; q, p; \Lambda] = (-)^{\delta_j} \Gamma_{ij}^{\text{int}} [\Phi; q, p; \Lambda] - \int_k \left( \frac{1}{16 k^2} \right)^{|\varepsilon_k|} \Gamma_{kij}^{\text{int}} [\Phi; k, p; \Lambda] \Gamma_{2ik}^{-1} (k; \Lambda) \bar{\Gamma}_{il}^{\text{int}} [\Phi; q, -k; \Lambda]
\]

(12)

which gives \( \bar{\Gamma} \) in terms of the proper vertices of \( \Gamma \). The Grassmannian parity \( \delta_j \) is zero for the (anti)chiral superfield and the factor \( (-)^{\delta_j} \) has been introduced to take into account the possible anti-commuting nature of the field (it will be needed in SYM).
Finally, inserting (10) in (8) and using (11), we obtain the evolution equation for the functional $\Gamma(\Phi)$

$$
\Lambda \partial_\Lambda \left[ \Gamma_\Lambda(\Phi) - \frac{1}{2} \int_p \mathcal{K}^{-1}_{\Lambda_0}(p) \Phi_i(-p, \theta) \mathcal{D}^{-1}_{ij}(p) \Phi_j(p, \theta) \right] = -\frac{i}{2} \int_q \Lambda \partial_\Lambda \mathcal{K}^{-1}_{\Lambda_0}(q) \times \Gamma^{-1}_{2_{ij}}(q; \Lambda) \mathcal{D}^{-1}_{ji} \Gamma^{-1}_{2_{ik}}(q, \Lambda) \left( \frac{D^2(\theta)}{16 q^2} \right)^{\varepsilon_k} \left( \frac{D^2(\theta)}{16 q^2} \right)^{\varepsilon_l} \tilde{\Gamma}_{kl}(\Phi; q, -q; \Lambda). \tag{13}
$$

This equation, together with a set of suitable boundary conditions, can be thought as an alternative definition of the theory which in principle is non-perturbative. As far as one is concerned with its perturbative solution, the usual loop expansion is recovered by solving iteratively (13). Such a solution is possible since the l.h.s. of (13) at a given loop order depends only on lower loop vertices. The RG formulation provides a very simple method to prove perturbative renormalizability, i.e. that the $\Lambda_0 \to \infty$ limit can be taken. The proof is a straightforward generalization of that given in [6]-[8] for non-supersymmetric theories.

### 2.2 Relevant couplings and boundary conditions

In order to set the boundary conditions one distinguishes between relevant couplings and irrelevant vertices according to their mass dimension. Relevant couplings have non-negative mass dimension and are defined as the value of some vertices and their derivatives at a given normalization point. Dimensional analysis tells us that they originate from the monomials $\phi\bar{\phi}$, $\phi$, $\phi^2$, $\bar{\phi}$, $\bar{\phi}^2$, $\bar{\phi}^3$, properly integrated.

The massless chiral multiplet two-point function (i.e. the $\phi\bar{\phi}$-coefficient of the cutoff effective action)

$$
\Gamma_{2_{ij}}(p; \Lambda) = \mathcal{D}^{-1}_{ij} \mathcal{K}^{-1}_{\Lambda_0}(p) + \Sigma_{2_{ij}}(p; \Lambda) \tag{14}
$$

contains the relevant coupling

$$
Z_{ij}(\Lambda) = \left. \Sigma_{2_{ij}}(p; \Lambda) \right|_{p^2 = \mu^2}
$$

where $\mu$ is some non-vanishing subtraction point, whose introduction, being $\phi$ a massless field, is required to avoid the IR divergences. We need not define the remaining relevant couplings since the corresponding monomials are not generated in perturbation theory (as can be seen from (13), only vertices with an equal number of $\phi$ and $\bar{\phi}$ receive perturbative contributions).

All the vertices appearing with a number of $\phi\bar{\phi}$ larger than one are irrelevant. A further contribution to the irrelevant part of $\Gamma$ comes from the two-point function, and is given by

$$
\Sigma_{2_{ij}}^{\text{irr}}(p; \Lambda) \equiv \Sigma_{2_{ij}}(p; \Lambda) - Z_{ij}(\Lambda).
$$

One assumes the following boundary conditions:

(i) at the UV scale $\Lambda = \Lambda_0$ all irrelevant vertices vanish. As a matter of fact $\Gamma(\Phi; \Lambda = \Lambda_0)$ reduces to the bare action, which must contain only renormalizable interactions in order to guarantee perturbative renormalizability;
(ii) the relevant couplings are fixed at the physical point $\Lambda = 0$ in terms of the physical couplings, such as the wave function normalization, the three-point coupling and the mass. In the case at hand only the first coupling evolves with $\Lambda$ whereas the remaining ones coincide with their tree-level value (e.g. $m = 0$ and $\lambda$) and for this reason are not investigated.

Hence the boundary conditions to be imposed on the relevant and irrelevant part of $\Gamma_{2}^{ij}$ are

$$Z_{ij}(\Lambda = 0) = 0, \quad \Sigma_{2}^{irr}(p; \Lambda_0) = 0,$$

respectively.

### 2.3 Loop expansion

#### (i) Tree level

The starting point of the iteration is the tree-level interaction

$$\Gamma_{ij}^{\text{int}(0)}[\Phi; q, p; \Lambda] = \frac{\lambda}{8} \delta_{ij} \int_{p'} \delta^4(\theta_1 - \theta') D(\theta')^{-2\epsilon} \delta^4(\theta_2 - \theta') \Phi_j(p') \delta^4(p + q + p')$$

(16)

together with the tree-level two-point function $\Gamma_{2}^{(0)}(p; \Lambda) = \mathcal{D}_{ij}^{-1} K_{\Lambda \Lambda_0}^{-1}(p)$. Inserting these expressions in (12) one obtains the tree-level functional $\bar{\Gamma}_{ij}^{(0)}[\Phi]$.

#### (i) One-loop calculations

The evolution equation for the functional $\Gamma[\Phi]$ at one-loop order can be derived by writing the r.h.s of (13) in terms of the known objects $\bar{\Gamma}_{ij}^{(0)}[\Phi]$ and $\Gamma_{2}^{(0)}$. For instance, the evolution equation for the two-point function is determined by the $\phi\bar{\phi}$-coefficient in (13) which, at the tree level, originates only from the second term in the r.h.s. of (12), i.e.

$$- \int_{k} \Gamma_{ml}^{\text{int}(0)}[\Phi; k, q; \Lambda] \frac{K_{\Lambda \Lambda_0}(k)}{16k^2} \mathcal{D}_{nm}(k) \Gamma_{kn}^{\text{int}(0)}[\Phi; -q, -k; \Lambda].$$

Next, substituting (16) in the expression above and carrying out some standard $D$-algebra manipulations (reported in the appendix), one finds

$$\int_{p} \tilde{\phi}(-p, \theta) \Lambda \partial_{\Lambda} \Sigma_{2}^{(1)}(p; \Lambda) \phi(p, \theta) = \frac{i}{64} \lambda^2 \int_{pq} \frac{K_{\Lambda \Lambda_0}(p + q) \Lambda \partial_{\Lambda} K_{\Lambda \Lambda_0}(q)}{q^2(p + q)^2} \times \tilde{\phi}(-p, \theta_1) \phi(p, \theta_2) \delta^4(\theta_1 - \theta_2) \bar{D}^2 D^2(q, \theta_2) \delta^4(\theta_1 - \theta_2).$$

(17)

Notice that eq. (13) describes only the evolution of the interacting part of $\Gamma$, since the tree level in (14) cancels out.

Recalling the property

$$\delta^4(\theta_1 - \theta_2) \bar{D}^2 D^2 \delta^4(\theta_1 - \theta_2) = \delta^4(\theta_1 - \theta_2),$$

(18)

one gets

$$\Lambda \partial_{\Lambda} \Sigma_{2}^{(1)}(p; \Lambda) = \frac{i}{128} \lambda^2 \int \frac{d^4q}{(2\pi)^4} \Lambda \partial_{\Lambda} \frac{K_{\Lambda \Lambda_0}(q) K_{\Lambda \Lambda_0}(p + q)}{q^2(p + q)^2}.$$

(19)
Implementing the boundary conditions (15), the solution of (19) at the physical point Λ = 0 and in the Λ₀ → ∞ limit is

$$\Sigma_2^{(1)}(p; \Lambda = 0) = \frac{i}{128} \lambda^2 \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{q^2(p+q)^2} - \frac{1}{q^2(p+q)^2} \right) \bigg|_{p^2 = \mu^2}.$$ 

Notice the crucial role of the boundary condition for \(Z_{ij}\), i.e. \(Z_{ij}^{(1)}(0) = 0\), which naturally provides the necessary subtraction to make the vertex function \(\Sigma_2^{ij}\) finite for \(\Lambda_0 \to \infty\). Conversely one can see from power counting that the remaining irrelevant vertices are finite, and no subtraction is needed. This property holds at any order in perturbation theory [8].

3 N=1 Super Yang-Mills

The super Yang-Mills (SYM) action reads [14] (the conventions are those of [15])

$$S_{SYM} = -\frac{1}{128g^2} \text{Tr} \int d^4x \, d^2\theta \, W^a \mathcal{W}_a, \quad \mathcal{W}_a = \bar{D}^2 \left( e^{-gV} D_a e^{gV} \right),$$

where \(V(x, \theta)\) is the \(N = 1\) vector supermultiplet which belongs to the adjoint representation of the gauge group \(G\). In the matrix notation \(V = V^a \tau_a\), with the matrices \(\tau_a\) satisfying \([\tau_a, \tau_b] = i f_{abc} \tau_c\), \(\text{Tr} \, \tau_a \tau_b = \delta_{ab}\). The classical action is invariant under the gauge transformation

$$e^{i\varphi} = e^{-i\bar{\chi} e^{i\chi}} \quad \bar{D}_a \chi = 0, \quad D^a \bar{\chi} = 0,$$

where \(\chi = \chi^a \tau_a\).

In order to quantize the theory one has to fix the gauge and choose a regularization procedure. From what we have seen so far it should be manifest that the introduction of the cutoff does not spoil global symmetries as long as they are linearly realized. If this is not the case the transformation of the quadratic part of the action mixes with the transformation of the rest (recall that the cutoff function multiplies only the quadratic part of the classical action). Therefore, we shall choose a supersymmetric gauge fixing instead of the familiar Wess-Zumino one in which the supersymmetry transformation is not linear.

As described in ref. [5], we add to the action a gauge fixing term which is a supersymmetric extension of the Lorentz gauge and the corresponding Faddeev-Popov term

$$S_{gf} = -\frac{1}{128\alpha} \text{Tr} \int z D^2 V \bar{D}^2 V,$$

$$S_{FP} = -\frac{1}{8} \text{Tr} \int z \left\{ \frac{1}{2} L_{gV} (c_+ + \bar{c}_+) + \frac{1}{2} (L_{gV} \coth(L_{gV}/2)) (c_+ - \bar{c}_+) \right\}$$

$$= -\frac{1}{8} \text{Tr} \int z \left\{ (c_+ + \bar{c}_-) \left[ c_+ - \bar{c}_+ + \frac{1}{2} g \left[ V, c_+ + \bar{c}_+ \right] + \cdots \right] \right\},$$

where the ghost \(c_+\) and the anti-ghost \(c_-\) are chiral fields, like the gauge parameter \(\chi\), and \(L_{gV} \cdot = [gV, \cdot]\). The classical action

$$S_{cl} = S_{SYM} + S_{gf} + S_{FP}$$

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is invariant under the BRS transformation
\[
\delta V = \eta \left[ \frac{1}{2} L_{gV}(c_+ + \bar{c}_+) + \frac{1}{2} (L_{gV} \coth(L_{gV}/2)) (c_+ - \bar{c}_+) \right], \\
\delta c_+ = -\eta c_+^2, \quad \delta \bar{c}_+ = -\eta \bar{c}_+^2, \\
\delta c_- = -\eta \frac{1}{16\alpha} \bar{D}^2 D^2 V, \quad \delta \bar{c}_- = -\eta \frac{1}{16\alpha} D^2 \bar{D}^2 V
\]
with \(\eta\) a Grassmann parameter. Introducing the sources \(\gamma_i = (\gamma_V, \gamma_{c_+}, \gamma_{\bar{c}_+})\), associated to the BRS variations of the respective superfields, the BRS action in the Fermi-Feynman gauge \((\alpha = 1)\) reads
\[
S_{\text{BRS}} = S_{\text{cl}} + \int_z \gamma_V \left[ \frac{1}{2} L_{gV}(c_+ + \bar{c}_+) + \frac{1}{2} (L_{gV} \coth(L_{gV}/2)) (c_+ - \bar{c}_+) \right] \\
- \int d^4x d^2\theta \gamma_{c_+} c_+^2 - \int d^4x d^2\bar{\theta} \gamma_{\bar{c}_+} \bar{c}_+^2
= S_2 + S^{(0)}_{\text{int}} \tag{22}
\]
with
\[
S_2 = \int_z \left[ \frac{1}{16} V \bar{D}^2 V + \frac{1}{8} (c_- \bar{c}_+ - \bar{c}_- c_+) \right].
\]
Notice that in (22) we did not introduce the BRS sources for \(c_-\) and \(\bar{c}_-\) since one can show that the effective action depends on these fields and the source \(\gamma_V\) only through the combination
\[
\tilde{\gamma}_V = \gamma_V - \frac{1}{8} (c_- + \bar{c}_-).
\]

As described in the previous section for the WZ model, we regularize the UV divergences multiplying the free propagators by a cutoff function \(K_{\Lambda_0}\), so that the generating functional \(Z[J, \gamma]\) can be written as in (2) with
\[
\Phi_i = (V, c_+, \bar{c}_-, c_-, \bar{c}_+), \quad J_i = (J_V, \xi_- + \bar{D}^2 \gamma_V, -\bar{\xi}_+, -\xi_+, \bar{\xi}_- - D^2 \gamma_V)
\]
and the cutoff scalar product between fields and sources given by
\[
(\Phi, D^{-1}\Phi)_{\Lambda_0} = \int_p K^{-1}_{\Lambda_0}(p) \left\{ -\frac{1}{16} V(-p, \theta) p^2 V(p, \theta) \\
+ \frac{1}{8} [c_-(-p, \theta) \bar{c}_+(p, \theta) - \bar{c}_-(-p, \theta) c_+(p, \theta)] \right\} \tag{23}
\]
and
\[
(J, \Phi)_{\Lambda_0} = \int_p K^{-1}_{\Lambda_0}(p) \left\{ J_V(-p, \theta) V(p, \theta) + \frac{1}{16} \left[ (\xi_- + \bar{D}^2 \gamma_V) (-p, \theta) \frac{D^2}{p^2} c_+(p, \theta) \\
+ \frac{\bar{D}^2}{p^2} \bar{c}_-(-p, \theta) \bar{\xi}_+(p, \theta) + \frac{D^2}{p^2} c_-(p, \theta) \xi_+(p, \theta) \\
+ (\bar{\xi}_- - D^2 \gamma_V) (-p, \theta) \frac{\bar{D}^2}{p^2} \bar{c}_+(p, \theta) \right] \right\}. \tag{24}
\]
The UV action $S_{\text{int}}[\Phi, \gamma; \Lambda_0]$ contains all possible relevant interactions written in terms of $\Phi_i$, $\gamma_i$ and superspace derivatives, which are invariant under Lorentz and global gauge transformations. Notice that at the tree level all quadratic contributions in the fields and sources are gathered in (23) and (24).

Afterwards one integrates over the fields with frequencies $\Lambda^2 < p^2 < \Lambda_0^2$ and the result is the analogue of (3) where the Wilsonian effective action $S_{\text{eff}}[\Phi, \gamma; \Lambda]$ depends also on the BRS sources. The generating functional of the cutoff connected Green functions $\mathcal{D}$ are $1/\nu$ sect. 2. Finally the cutoff effective action $\Gamma$ evolves according to (13) with the appropriate vertices, $\mathcal{D}_{ij}$ defined through (23). This matrix turns out to be block-diagonal and its entries are $1/8(-p^2, \varepsilon_{AB}, \varepsilon_{AB})$, $\Lambda = (+, -)$, with $\varepsilon_{AB} = -\varepsilon_{BA}$ and $\varepsilon_{+-} = 1$. The derivation of the evolution equation for the functional $W$ exactly follows that of the WZ model presented in sect. 2. Finally the cutoff effective action $\Gamma$

$$\Gamma[\Phi, \gamma; \Lambda] = W[J, \gamma; \Lambda] - \int z J_V V - \int d^4 x \, d^2 \theta \, (\xi_+ - c_+ + c_\xi)$$

$$- \int d^4 x \, d^2 \bar{\theta} \, (\bar{\xi}_+ - \bar{c}_+ + \bar{c}_\bar{\xi}) \quad (25)$$

evolves according to (13) with the appropriate vertices, $\mathcal{D}_{ij}$ and $\varepsilon_k$.

### 3.1 Matter fields

When adding matter fields to the pure super Yang-Mills action one gets SQCD, the supersymmetric generalization of QCD. Matter is described by a set of chiral superfields $\phi^i(x, \theta)$ which belong to some representation $R$ of the gauge group. Their BRS transformation reads

$$\delta \phi^i = -\eta \, c_2^i \, T_a^i \, \phi^j \equiv -\eta (c_+ \phi)^i, \quad \delta \bar{\phi}_i = \bar{\eta} \, \bar{T}_a^i \, c_+^{a} \equiv \bar{\eta} (\bar{\phi} \bar{c}_+)^i, \quad (26)$$

where the hermitian matrices $T_a$ are the generators of the gauge group in the representation $R$.

The BRS action for the matter fields is

$$S_{\text{matter}} = \frac{1}{16} \int \bar{\phi} e^{\theta \gamma_4 T_4} \phi - \int d^4 x \, d^2 \theta \, \gamma_5 \phi \, \bar{\gamma}_5 \phi + \int d^4 x \, d^2 \bar{\theta} \, \gamma_5 \phi \, \bar{\gamma}_5 \bar{\phi} \quad (27)$$

plus a possible superpotential $W$ having the general form $W(\phi) = \frac{1}{8} m_{ij} \phi^i \phi^j + \lambda_{(ijk)} \phi^i \phi^j \phi^K$, the mass matrix $m_{ij}$ and the Yukawa coupling constants $\lambda_{ijk}$ being invariant symmetric tensors in the representation $R$.

Developing the RG formalism in presence of matter fields is straightforward once one replaces the sets of fields and sources with

$$\Psi_1 = (V, c_+, \bar{c}_-, c_-, \bar{c}_+ \phi, \bar{\phi}), \quad \gamma_1 = (\gamma_V, \gamma_{c_+}, \gamma_{c_-}, \gamma_{\bar{c}_+}, \gamma_{\phi}, \gamma_{\bar{\phi}}), \quad J_1 = (J_V, \xi_+, -\xi_-, -\xi_+, -\bar{\xi}_+ - D^2 \gamma_V, J, \bar{J}). \quad (27)$$

The evolution equation for the effective action has the usual form (13), with a natural redefinition of $\varepsilon_k$ and $\mathcal{D}_{ij}^{-1}$ to take into account matter fields (e.g. $\varepsilon_k = (0, 1, 1, -1, -1, 1, -1)$).
3.2 Boundary conditions

As discussed in subsect. 2.2 we first distinguish between relevant and irrelevant vertices. The relevant part of the cutoff effective action involves full superspace integrals of monomials in the fields, sources and derivatives local in $\theta$ and with dimension not larger than two

$$\Gamma_{\text{rel}}[\Psi, \gamma; \sigma_i(\Lambda)] = \sum_i \sigma_i(\Lambda) P_i[\Psi, \gamma], \quad (28)$$

where the sum is over the monomials $P_i[\Psi, \gamma]$ invariant under Lorentz and global gauge transformations. Due to the dimensionless nature of the field $V$ this sum contains infinite terms which can be classified according to the number of gauge fields. The couplings $\sigma_i(\Lambda)$ can be expressed in terms of the cutoff vertices at a given subtraction point, generalizing the procedure used in subsect. 2.2 to define the coupling $Z_{ij}(\Lambda)$ (see also [10] for the technique of extracting the relevant part from a given functional with a non-vanishing subtraction point in the non-supersymmetric Yang-Mills case).

The remaining part of the cutoff effective action is called “irrelevant”. The boundary condition we impose on the irrelevant part of the cutoff effective action is

$$\Gamma_{\text{irr}}[\Phi, \gamma; \Lambda = \Lambda_0] = 0.$$

For $\Lambda = \Lambda_0$, then, the cutoff effective action becomes “local”, i.e. an infinite sum of local terms, and corresponds to the UV action $S_{\text{int}}[\Psi, \gamma; \Lambda_0]$, with the bare couplings given by $\sigma_i(\Lambda_0)$.

The way in which the boundary conditions for the relevant couplings $\sigma_i(\Lambda)$ are determined is not straightforward. In sect. 2 we fixed them at the physical point $\Lambda = 0$ in terms of the value of the physical couplings (such us the normalization of the chiral field). In the case of a gauge theory, as the one we are considering, there are interactions in $(28)$ which are not present in $S_{\text{BRS}}$, so that only some of the relevant couplings are connected to the physical couplings (such as the wave function normalizations and the three-vector coupling $g$ at a subtraction point $\mu$). For instance the contribution to $(28)$ with two gauge fields consists of three independent monomials

$$\int_z \text{Tr} \left[ \sigma_1 V V + \sigma_2 V D^\alpha \bar{\bar{D}} D_\alpha V + \sigma_3 V D^2 \bar{\bar{D}} V \right]$$

instead of the two in $S_{\text{BRS}}$. Therefore, in order to fix the boundary conditions for all the relevant couplings, one needs an additional fine-tuning procedure which implements the gauge symmetry at the physical point. However, this analysis involves non-local functionals and is highly not trivial. Alternatively one can discuss the symmetry at the ultraviolet scale and determine $\sigma_i(\Lambda = \Lambda_0)$. In this case the discussion is simpler, since all functionals are relevant, but one has to perform a perturbative calculation (i.e. to solve the RG equations) to obtain the physical couplings. Notice that while the physical couplings are independent of the cutoff function, the bare action, i.e. $\sigma_i(\Lambda_0)$, is generally not.

In this paper we consider the second possibility, although the wave function normalizations and the gauge coupling $g$ at a subtraction point $\mu$ are still set at $\Lambda = 0$. As a matter of fact there are combinations of the monomials in $(28)$ which are not involved in the...
fine-tuning, so that the corresponding couplings are free and can be fixed at the physical point $\Lambda = 0$. Before explaining the details of the fine-tuning procedure we recall how to implement the gauge symmetry in the RG formulation.

4 Effective ST identity

The gauge symmetry requires that the physical effective action satisfies the ST identity \(^{16}\)

$$S_{\Gamma'}[\Psi, \gamma] = 0,$$

where $\Gamma'[\Psi, \gamma] = \Gamma[\Psi, \gamma] + \frac{1}{128} \text{Tr} \int_z D^2 V \bar{D}^2 V$ and \(^{1}\)

$$S_{\Gamma'} = \int_p \left[ \left( \frac{D^2}{16p^2} \right) \frac{\delta \Gamma'}{\delta \psi_i(-p)} \frac{\delta}{\delta \gamma_i(p)} + \left( \frac{D^2}{16p^2} \right) \frac{\delta \Gamma'}{\delta \gamma_i(p)} \frac{\delta}{\delta \psi_i(-p)} \right]$$

is the Slavnov operator. The ST identity can be directly formulated for the Wilson effective action $S_{\text{eff}}$ at any $\Lambda$. Consider the generalized BRS transformation

$$\delta \psi_i(p) = K_{0\Lambda}(p) \eta \frac{\delta S_{\text{tot}}}{\delta \gamma_i(-p)}, \quad \delta c_- = -\eta \frac{1}{16} \bar{D}^2 D^2 V, \quad \delta \bar{c}_- = -\eta \frac{1}{16} D^2 \bar{D}^2 V,$$

where $\eta$ is a Grassmann parameter and $S_{\text{tot}}$ is the total action (i.e. $S_{\text{eff}}$ plus the source and the quadratic terms in (5)). Performing such a change of variable in the functional integral (5), one deduces the following identity

$$S_{JZ}[J, \gamma] = N[J, \gamma; \Lambda] \int D\Psi \exp \left\{ \frac{1}{2}(\Psi, D^{-1}\Psi)_{0\Lambda} + (J, \Psi)_{0\Lambda} + S_{\text{eff}}[\Psi; \Lambda] \right\} \Delta_{\text{eff}}[\Psi, \gamma; \Lambda],$$

where $S_J$ is the usual ST operator

$$S_J = \int_p J_i(p) \left( -\delta \frac{\delta}{\delta \gamma_i(p)} + \frac{1}{16} \int_p \left[ D^2 \xi_+(p) + \bar{D}^2 \xi_+(p) \right] \frac{\delta}{\delta J_i(p)} \right)$$

with $\delta_i$ the source ghost number, and the functional $\Delta_{\text{eff}}$ reads:

$$\Delta_{\text{eff}}[\Psi, \gamma; \Lambda] = i \int_p K_{0\Lambda}(p) \exp \left( -iS_{\text{eff}} \right) \left\{ \frac{\delta}{\delta \psi_i(p)} \frac{\delta}{\delta \gamma_i(-p)} \right\} \exp(iS_{\text{eff}})$$

$$-i \int_p \left[ \psi_i(p) D^{-1}_{ij}(p) \frac{\delta}{\delta \gamma_j(p)} + (c_+ - \bar{c}_+)(p) \frac{\delta}{\delta V(p)} - \frac{1}{16} V(p) \left( D^2 \frac{\delta}{\delta c_-(p)} + \bar{D}^2 \frac{\delta}{\delta \bar{c}_-(p)} \right) \right] S_{\text{eff}}.$$  

Whereas the l.h.s of the identity (32) arises from the variation of the source term $(J, \Psi)_{0\Lambda}$, the functional $\Delta_{\text{eff}}$ originates from the Jacobian of the transformation (31) and from the variation of the rest of $S_{\text{tot}}$. Restoration of symmetry, $S_{JZ}[J, \gamma] = 0$, translates into

$$\Delta_{\text{eff}}[\Psi, \gamma; \Lambda] = 0 \quad \text{for any } \Lambda.$$  

\(^3\)From now on the sum over the fields in $\Psi$ will not include $c_-$ and $\bar{c}_-$.  

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From a perturbative point of view, instead of studying $\Delta_{\text{eff}}$ it is convenient to introduce \cite{12,17,18} its Legendre transform $\Delta_{\Gamma}$, in which reducible contributions are absent. Recalling (6) and (7) which relate $S_{\text{eff}}[\Psi; \gamma; \Lambda]$ to $W[J; \gamma; \Lambda]$, and using (6), (23) one finds

$$
\Delta_{\Gamma}[\Psi, \gamma; \Lambda] = - \int_p \left[ K_{0\Lambda}(p) \left( \frac{D^2(\theta_1)}{16p^2} \right) \varepsilon_i \frac{\delta \Gamma'}{\delta \Psi_i(-p)} \frac{\delta \Gamma'}{\delta \gamma_i(p)} - \frac{K_{0\Lambda}(p)}{K_{\Lambda\Lambda}(p)} D^{-1}_{ij}(p) \Psi_i(p) \frac{\delta \Gamma'}{\delta \gamma_i(p)} \right] - 2i \hbar \int_{pq} \frac{K_{0\Lambda}(p)}{K_{\Lambda\Lambda}(p)} D^{-1}_{ij}(p) \left( \frac{D^2(\theta_2)}{16q^2} \right) \varepsilon_k \frac{\delta W}{\delta J_i(p) \delta J_j(q)} \times \frac{\delta^2}{\delta \Psi_k(-q) \delta \gamma_j(-p)} \left( \Gamma - \int_z \gamma \nu(c_+ - \bar{c}_+) \right),
$$

where $\delta^2W/\delta J\delta J$ is that functional of $\Psi$ and $\gamma$ appearing in the inversion (30) and (31). Finally, after performing such an inversion, the cutoff ST identity reads

$$
\Delta_{\Gamma}[\Psi, \gamma; \Lambda] \equiv \bar{\Delta}_{\Gamma} + \hat{\Delta}_{\Gamma} = 0 ,
$$

with

$$
\bar{\Delta}_{\Gamma} = - \int_p K_{0\Lambda}(p) \left( \frac{D^2(\theta_1)}{16p^2} \right) \varepsilon_i \frac{\delta \Gamma'}{\delta \Psi_i(-p)} \frac{\delta \Gamma'}{\delta \gamma_i(p)} + \int_p \frac{K_{0\Lambda}(p)}{K_{\Lambda\Lambda}(p)} D^{-1}_{ij}(p) \Psi_i(p) \frac{\delta \Gamma'}{\delta \gamma_i(p)}
$$

and

$$
\hat{\Delta}_{\Gamma} = i\hbar \int_{pq} \frac{K_{0\Lambda}(p)}{K_{\Lambda\Lambda}(p)} \left( \frac{D^2(\theta_2)}{16q^2} \right) \varepsilon_k \left\{ \frac{(-1)^{\delta_i}}{16q^2} \left[ \delta \Gamma^{-1}(q; \Lambda) \frac{\delta}{\delta \gamma_j(q)} \right]_{jl} - \delta_{jl} \delta^8(p-q) \right\} \times (\Gamma_2^{-1}(p; \Lambda) D^{-1}(p) \delta_{ij}) \frac{\delta^2}{\delta \gamma_j(q) \delta \gamma_i(p)} \left( \Gamma - \int_z \gamma \nu(c_+ - \bar{c}_+) \right).
$$

Notice that at $\Lambda = 0$ the cutoff ST identity reduces to $\Delta_{\Gamma}(0) = 0$ and, in the UV limit, becomes the usual ST identity \cite{29}. Moreover we have inserted the factor $\hbar$ in (36) to put into evidence that $\bar{\Delta}_{\Gamma}$ vanishes at the tree level.

For the sake of future analysis, we introduce the functional

$$
\Pi[\Psi, \gamma; \Lambda] = \Gamma[\Psi, \gamma; \Lambda] - \frac{1}{2} \left( \Psi; D^{-1}\Psi \right)_{\Lambda\Lambda} - \frac{1}{2} \left( \Psi; D^{-1}\Psi \right)_{0\Lambda0}
$$

differing from the cutoff effective action only in the tree-level two-point function, in which the IR cutoff has been removed. With such a definition, in the $\Lambda_0 \to \infty$ limit the tree-level contribution to $\Pi(\Lambda)$ coincides with $S_{\text{BRS}}$, whereas at the tree level $\Gamma_2(\Lambda)$ contains the IR cutoff (see \cite{14}). In terms of $\Pi$ the functional $\bar{\Delta}_{\Gamma}$ can be rewritten as

$$
\bar{\Delta}_{\Gamma}[\Psi, \gamma; \Lambda] = - \int_p K_{0\Lambda}(p) \left( \frac{D^2(\theta_1)}{16p^2} \right) \varepsilon_i \frac{\delta \Pi'[\Psi, \gamma; \Lambda]}{\delta \Psi_i(-p)} \frac{\delta \Pi'[\Psi, \gamma; \Lambda]}{\delta \gamma_i(p)},
$$

where $\Pi'$ is the expression obtained by removing the gauge fixing term in (37). Thus, in the $\Lambda_0 \to \infty$ limit, with the help of (30) one has

$$
\bar{\Delta}_{\Gamma}[\Psi, \gamma; \Lambda] \to S_{\text{IV}(\Lambda)} \Pi'[\Lambda] \quad \text{for} \quad \Lambda_0 \to \infty
$$

at any $\Lambda$. The existence of such a limit is guaranteed in perturbation theory by the UV finiteness of the cutoff effective action (perturbative renormalizability). In order to show this property holds also for $\bar{\Delta}_{\Gamma}$, it suffices to recognize that the presence of cutoff functions having almost non-intersecting supports forces the loop momenta in (30) to be of the order of $\Lambda$. Henceforth we will take the $\Lambda_0 \to \infty$ limit in $\bar{\Delta}_{\Gamma}$.  

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4.1 Perturbative solution of $\Delta \Gamma = 0$

The proof of the ST identity (34) in the RG formalism, with possible anomalies, is based on induction in the loop number and closely follows that of non-supersymmetric gauge theories discussed in [10, 17, 12]. For the sake of completeness we resume here the key issues.

One can show [17] that the evolution of the vertices of $\Delta \Gamma$ at the loop $\ell$ depends on vertices of $\Delta \Gamma$ itself at lower loop order, so that if $\Delta \Gamma^{(\ell')} = 0$ at any loop order $\ell' < \ell$, then

$$\Lambda \partial_\Lambda \Delta \Gamma^{(\ell)} = 0. \quad (39)$$

Thus one can analyse $\Delta \Gamma$ at an arbitrary value of $\Lambda$. There are two natural choices corresponding to $\Lambda = 0$ and $\Lambda = \Lambda_R$ much bigger than the subtraction scale $\mu$, i.e. $\Lambda_R = \Lambda_0$. With the former the gauge symmetry condition fixes the relevant part of the effective action in terms of the physical coupling $g(\mu)$ and provides the boundary conditions of the RG flow, whereas with the latter the gauge symmetry condition determines the cutoff dependent bare couplings. With this choice the implementation of symmetry is simplified due to the locality of the functionals involved. Although the computation of physical vertices is generally cumbersome, this second possibility is more convenient in the computation of quantities which do not evolve with the cutoff $\Lambda$, such as the gauge anomaly. Hence we will adopt the second possibility in the present paper.

We now discuss the vanishing of $\Delta \Gamma$. Also for this functional we define its relevant part, isolating all supersymmetric monomials in the fields, sources and their derivatives with ghost number one and dimension three. The rest is included in $\Delta \Gamma_{\text{irr}}$.

At the UV scale $\Delta \Gamma$ is local, or, more precisely, $\Delta \Gamma_{\text{irr}}(\Lambda_0) = O(1/\Lambda_0)$, so that the irrelevant contributions disappear in the $\Lambda_0 \to \infty$ limit. This can be understood with the following argument. From (38), $\Delta \Gamma(\Lambda_0)$ is manifestly relevant, since $\Pi(\Lambda_0) = \Pi_{\text{rel}}(\Lambda_0)$, while it is easy to convince oneself that $\Delta \Gamma(\Lambda_0) = \Delta \Gamma_{\text{rel}}(\Lambda_0) + O(1/\Lambda_0)$. As a matter of fact, from (36) one notices that irrelevant terms may arise from $\bar{\Gamma}[\Phi, \gamma; \Lambda_0]$ and the cutoff functions. At $\Lambda_0$, $\bar{\Gamma}$ is given by either a relevant vertex or a sequence of relevant vertices joint by propagators with a cutoff function $K_{\Lambda_0\infty}(q + P)$, where $P$ is a combination of external momenta (see (12)). Since the integral is damped by these cutoff functions, only the contributions with a restricted number of propagators survive in the $\Lambda_0 \to \infty$ limit. One can infer from power counting that they are of the relevant type. A similar argument holds for the possible irrelevant contributions coming from $K_{\Lambda_0\infty}(p)$. Then (39) ensures the locality of $\Delta \Gamma(\Lambda)$ at any $\Lambda$.

Once the locality of $\Delta \Gamma(\Lambda)$ is shown, the solvability of the equation $\Delta \Gamma(\Lambda) = 0$ can be proven using cohomological methods [16, 19]. This is a consequence of the $\Lambda$-independence of $\Delta \Gamma$ and the solvability of the same equation at $\Lambda = 0$, where the cohomological problem reduces to the standard one.

Henceforth we will consider the first loop, the generalization to higher loops being straightforward due to the iterative nature of the solution. Using (38), at $\Lambda = \Lambda_0$ and at

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4Here and in the following locality means that each term in the expansion of the functionals in the gauge field $V$ contains only couplings with non-negative dimension.
the first loop (34) reads
\[ S_{\Pi(0)} \Pi^{(1)}(\Lambda_0) + \hat{\Delta}^{(1)}_{\Gamma, \text{rel}}(\Lambda_0) = 0. \] (40)

This fine-tuning equation allows to fix some of the relevant couplings in \( \Pi^{(1)}(\Lambda_0) \). As a matter of fact the most general functional \( \Pi^{(1)}(\Lambda_0) \) can be cast into the form (28) and split into two contributions
\[ \Pi^{(1)}(\Lambda_0) = \Pi^{(1)}_{\text{inv}}(\Lambda_0) + \tilde{\Pi}^{(1)}(\Lambda_0), \] (41)
where \( \Pi^{(1)}_{\text{inv}} \) contains all the independent monomials which are invariant, i.e.
\[ S_{\Pi(0)} \Pi^{(1)}_{\text{inv}}(\Lambda_0) = 0. \]

The explicit form of \( \Pi^{(1)}_{\text{inv}}^{(1)} \) is obtained from \( S_{\text{BRS}} \) in (22) and (26) with the replacement
\[(V, \gamma_i, c_+, \bar{c}_+, g, \phi, \bar{\phi}) \rightarrow (\sqrt{z_1}V, \sqrt{z_2} \gamma_i, \sqrt{z_2} c_+, g, \sqrt{z_4} \phi, \sqrt{z_4} \bar{\phi}).\]

The remaining monomials contribute to \( \tilde{\Pi} \). Inserting (41) into (40), one finds
\[ S_{\Pi(0)} \tilde{\Pi}^{(1)}(\Lambda_0) = -\hat{\Delta}^{(1)}_{\Gamma}(\Lambda_0), \]
which yields the couplings in \( \tilde{\Pi}^{(1)} \) since \( \hat{\Delta}^{(1)}_{\Gamma}(\Lambda_0) \) depends only on \( S_{\text{BRS}} \). An explicit calculation shows that the only divergences are powers of \( \Lambda_0 \) according to the dimension of the relative vertex. In particular dimensionless couplings are finite, due to the presence in (36) of cutoff functions having almost non-intersecting supports [5].

As to the couplings \( z_i(\Lambda_0) \), which are not involved in the fine-tuning, one is allowed to set them equal to their physical values at \( \Lambda = 0 \), i.e. \( z_i(0) = 1 \). In the standard language this corresponds to the renormalization prescriptions.

Instead of solving the fine-tuning equation and determine the (cutoff-dependent) couplings of the UV action, in the next section we will deal with the computation of the gauge anomaly, which well illustrates how the method works and meanwhile is a cutoff independent result. At one loop such independence is guaranteed by the absence of the anomaly at the tree level and by the evolution equation (39).

5 Gauge anomaly

For \( N=1 \) SYM within the superspace approach it has been demonstrated [19] that the only possible anomaly is the supersymmetric extension of the standard Adler-Bardeen anomaly [20] and its explicit form is given in ref. [21, 22]. As well known, its structure is non-polynomial [21, 23] and can be expressed as an infinite series in the gauge field \( V \). In the following we restrict ourselves to the first term of this expansion, since higher order polynomials can be inferred [23] using the consistency condition [13] which, at this order, forces the one-loop anomaly \( \mathcal{A}^{(1)} \) to obey \( S_{\Pi(0)} \mathcal{A}^{(1)} = 0 \).

In our framework a violation of the ST identity results in the impossibility of fixing the relevant couplings \( \sigma_i(\Lambda_0) \) in \( \Pi^{(1)}(\Lambda_0) \) in such a way that (10) is satisfied. In other words, this happens when there are relevant monomials in \( \hat{\Delta}_{\Gamma} \) which are not trivial cocycles of the cohomology of the BRS operator.

5See [12] for the explicit computation of some of these couplings in non-supersymmetric QCD.
As a first step we write $\hat{\Delta}_\Gamma$ at one loop order. Performing the $\Lambda_0 \to \infty$ limit in (36) and setting $\Lambda = \Lambda_0$, one has
\[
\hat{\Delta}_\Gamma^{(1)} = i \int_{pq} K_{0\Lambda_0}(p) \left[ \left( \frac{1}{16q^2} \right)^{|\epsilon_j|} K_{\Lambda_0\infty}(q)(-)^j D_{jk}(q) \bar{\Gamma}_{ki}^{(0)}(-q,-p;\Lambda) - \delta_{ij}\delta^8(p-q) \right] \times \left( \frac{D^2(\theta_1)}{16p^2} \right)^{\epsilon_i} \frac{\delta^2}{\delta \Psi_j(q) \delta \gamma_i(p)} \left( S_{\text{BRS}} - \int_z \gamma_V(c_+ - \bar{c}_+) \right). \tag{42}
\]
Then we isolate the matter contribution in $\hat{\Delta}_\Gamma^{(1)}$ which, depending on the representation of the matter fields, can possibly give rise to the anomaly
\[
\hat{\Delta}_\Gamma^{(1)} = \hat{\Delta}_{\Gamma}^{\text{SYM}(1)} + i \int_{pq} K_{0\Lambda_0}(p) K_{\Lambda_0\infty}(q) \left[ \frac{\delta^2 \bar{\Gamma}^{(0)}}{\delta \phi(-p) \delta \bar{\phi}(-q)} \frac{D^2(\theta_1)}{16p^2} \frac{\delta^2 S_{\text{BRS}}}{\delta \phi(q) \delta \bar{\gamma}(p)} \right] \times D \to \bar{D}, \phi \to \bar{\phi}, \gamma_\phi \to \bar{\gamma}_\phi. \tag{43}
\]

Figure 1: Matter contribution to the $c_+V-V$ vertex of $\hat{\Delta}_\Gamma$. The wavy, dashed and full line denotes the vector, ghost and matter fields respectively; the double line represents the BRS source associated to the matter field. The cross denotes the insertion of the cutoff function $K_{0\Lambda_0}$ in the product of the $c_+\bar{\phi}-V-V-\bar{\phi}$ vertex of $\bar{\Gamma}$ with: (a) the irreducible $\bar{\phi}-V-V-\bar{\phi}$ vertex of $\bar{\Gamma}$; (b) the reducible $\bar{\phi}-V-V-\bar{\phi}$ vertex of $\bar{\Gamma}$. All external momenta are incoming and integration over the loop momentum is understood.

Inserting (12) in (43) and extracting the tree-level vertices of $\Gamma$ from $S_{\text{BRS}}$, one can see that the matter contribution to the $c_+V-V$ vertex of $\hat{\Delta}_\Gamma$ is made of two pieces, as shown in fig. 1. The first, originating from the irreducible part of the $\bar{\phi}-V-V-\bar{\phi}$ vertex of $\bar{\Gamma}$, is given by
\[
-\frac{ig^2}{32} \int_{pq} \text{Tr} [c_+(-p-q,\theta_1)V(p,\theta_2)V(q,\theta_2)] \int \frac{d^4k}{(2\pi)^4} K_{0\Lambda_0}(k)K_{\Lambda_0\infty}(p+q-k) \times \delta^4(\theta_1 - \theta_2) D^2D^2\delta^4(\theta_1 - \theta_2) \tag{44}
\]
and, as suggested from the graph depicted in fig. 1a which is not typically triangle-shaped, does not contribute to the anomaly. As a matter of fact, by restricting to the Yang-Mills sector, one immediately recognizes that the anti-symmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ can not be generated from such a term. Indeed using (18) and performing the loop integration, the expression in (14) becomes

$$g^2 \int \frac{d^4p d^4q d^4\theta}{(2\pi)^8} \{ (a_1 \Lambda_0^2 + a_2 (p+q)^2) \text{Tr} [c_+(-p-q,\theta)V(p,\theta)V(q,\theta)] + O((p+q)^4/\Lambda_0^2) \}$$

where the $a_i$’s are finite cutoff-dependent numbers which can be explicitly computed once the cutoff function is specified. The finiteness of such coefficients is due to the presence of cutoff functions having almost non-intersecting supports, i.e. $k^2 \lesssim \Lambda_0^2$ and $(p+q-k)^2 \gtrsim \Lambda_0^2$. These two monomials belong to the trivial cohomology of $\mathcal{S}_r$ and their coefficients, together with those stemming from analogous monomials of $\hat{\Delta}_r^{\text{SYM}}$, fix the parameters in $\hat{\Pi}^{(1)}$ via (10).

We turn now to the contribution associated to the graph represented in fig. 1b, which originates from the second term in the iterative expansion of $\hat{\Gamma}$ in vertices of $\Gamma$. It reads

$$i \frac{g^2}{256} \int_{pq} \int \frac{d^4k}{(2\pi)^4} \text{Tr} [c_+(-p-q,\theta_2)V(p,\theta_1)V(q,\theta_2)] \frac{K_{0\Lambda_0}(k-q)K_{\Lambda_0\infty}(p+k)K_{\Lambda_0\infty}(k)}{k^2(k+p)^2} \times \hat{D}^2 \hat{D}^2(k,\theta_1) \delta^4(\theta_1-\theta_2) \hat{D}^2(k+p,\theta_1) \delta^4(\theta_1-\theta_2). \quad (45)$$

After integrating the $\hat{D}^2 \hat{D}^2$ derivatives by parts and using the algebra of covariant derivatives (reported in the appendix) and (18), one finds that the only non-vanishing terms in (45) are

$$i \frac{g^2}{256} \int \frac{d^4p d^4q d^4\theta}{(2\pi)^8} \int \frac{d^4k}{(2\pi)^4} \frac{K_{0\Lambda_0}(k-q)K_{\Lambda_0\infty}(k)K_{\Lambda_0\infty}(p+k)}{k^2(k+p)^2} \times \text{Tr} [c_+(-p-q,\theta)\left((\hat{D}^2 \hat{D}^2 + 8k_{\alpha\dot{\alpha}} \hat{D}^\alpha \hat{D}^{\alpha} + 16k^2) V(p,\theta)\right) V(q,\theta)]. \quad (46)$$

By performing the loop integration one finds out that the first and the third term in the trace generate only monomials which belong to the trivial cohomology of $\mathcal{S}_r$, i.e.

$$g^2 \int \frac{d^4p d^4q d^4\theta}{(2\pi)^8} \left\{ a_3 \text{Tr} [c_+(-p-q,\theta) (\hat{D}^2 \hat{D}^2 V(p,\theta)) V(q,\theta)] \right\}$$

$$+ (a_4 \Lambda_0^2 + a_5 P^2) \text{Tr} [c_+(-p-q,\theta) V(p,\theta)V(q,\theta)] + O(P^4/\Lambda_0^2) \}$$

where $P$ is some combination of the momenta $p$ and $q$ and the $a_i$’s are finite cutoff-dependent numbers. We are now left with the second term in the trace in (45). Exploiting symmetry properties and expanding into external momenta we obtain

$$\frac{g^2}{1024 \pi^2} \int \frac{d^4p d^4q d^4\theta}{(2\pi)^8} \text{Tr} [c_+(-p-q,\theta) (\hat{D}^\alpha \hat{D}^\alpha V(p,\theta)) V(q,\theta)] (q_{a\dot{a}} I_1 + p_{a\dot{a}} I_2) \quad (47)$$

where

$$I_1 = \int_0^\infty dx K_{\Lambda_0\infty}(x) \frac{dK_{0\Lambda_0}(x)}{dx} + O(P^2/\Lambda_0^2)$$

$$I_2 = \int_0^\infty dx \left[ K_{\Lambda_0\infty}(x) \frac{dK_{0\Lambda_0}(x)}{dx} + \frac{K_{\Lambda_0\infty}(x)}{x} K_{0\Lambda_0}(x) \right] + O(P^2/\Lambda_0^2)$$
with $x = k^2/\Lambda_0^2$ and $P$ as above. Notice that in the $\Lambda_0 \rightarrow \infty$ limit $I_1$ yields a cutoff independent number, i.e. $-1/3$, since it is determined only by the values $K_{\Lambda_0 \rightarrow \infty}(0) = 0$ and $K_{\Lambda_0 \rightarrow \infty}(\infty) = 1$. On the contrary $I_2$ depends on the choice of the cutoff function.

In (47) the structure proportional to $p_\alpha \dot{\alpha}$ does not contribute to the anomaly, basically because, in the coordinate space, all derivatives act on the same superfield. On the other hand, had it played a role in determining the anomaly, like all other contributions analyzed above, our method would have led to an inconsistent result, as $I_2$ and the $a_i$’s depend on the cutoff function. Hence, only the term with $q_\alpha \dot{\alpha}$ can generate a genuine anomaly. By setting $I_1 = -1/3$ in (47) one gets

$$
\frac{g^2}{3072 \pi^2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4\theta}{(2\pi)^8} \text{Tr} [c_+(-p-q,\theta) \left(\bar{D}^\alpha D^\alpha V(p,\theta)\right) q_\alpha \dot{\alpha} V(q,\theta)]
$$

which has the true structure of the anomaly.

The $\bar{c}_+ V - V$ vertex of $\hat{\Delta}_\Gamma$ can be derived repeating the steps described above. Also in this case one identifies the anomalous contribution by isolating its cutoff independent part, which turns out to be

$$
-\frac{g^2}{3072 \pi^2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4\theta}{(2\pi)^8} \text{Tr} [\bar{c}_+(-p-q,\theta) \left(D^\alpha \bar{D}^\dot{\alpha} V(p,\theta)\right) q_\alpha \dot{\alpha} V(q,\theta)].
$$

Finally, summing up (48) and (49), and switching to the coordinate space, the anomaly has the well-known form

$$
A = \frac{g^2}{6144 \pi^2} \int \left( \text{Tr} [c_+ \bar{D}^\alpha D^\alpha V \{D_\alpha, \bar{D}_\dot{\alpha}\} V] - \text{Tr} [\bar{c}_+ D^\alpha \bar{D}^\dot{\alpha} V \{D_\alpha, \bar{D}_\dot{\alpha}\} V] \right).
$$

As a remark, we notice that in order to reproduce the standard abelian anomaly in non-supersymmetric QCD one has to perform the integration over the Grassmannian variables, identify the ghost $c$ with $c_+ + \bar{c}_+$ and replace $g$ with $2g$ to recover the usual gluon-fermion coupling (see eq. (26)). Then one finds that the coefficient of the monomial $\varepsilon^{\mu\nu\rho\sigma} \text{Tr} [\partial_\mu c \partial_\nu A_\rho A_\sigma]$ is exactly $g^2/(24\pi^2)$.

6 Conclusions

In this paper we considered supersymmetric (gauge) theories within the RG approach. Although we restricted to the WZ model and $N=1$ SYM, the formalism is developed in such a way it can be applied to any supersymmetric theory with an arbitrary field content and with extended supersymmetry.

An advantage of the RG formulation is that the regularization is implemented by introducing a cutoff in the loop momenta which makes all the Green functions UV finite. This means one need not analytically continue the Feynman integrals in the space-time dimension $d$, which is kept fixed (in our case $d = 4$). Therefore both the equality of bosonic and fermionic degrees of freedom is safe—a necessary condition for supersymmetry—and the superspace technique presents no ambiguity, for instance in handling the algebra of covariant derivatives, traces of $\sigma$ matrices and using Fierz identities.
However, in the RG approach the presence of the cutoff explicitly breaks gauge symmetry. This is an unavoidable consequence of the absence of a regularization scheme that manifestly preserves both supersymmetry and BRS invariance, which in turn is intimately related to the existence of the chiral anomaly.

In this paper we showed that the Slavnov-Taylor identity for the physical effective action of an anomaly-free theory is perturbatively recovered by solving the fine-tuning equation (34) at the UV scale. Such a procedure was sketched in subsect. 4.1. On the other hand, in case of unfulfilled matching conditions for the anomaly cancellation, we reproduced the supersymmetric chiral anomaly by a simple one-loop calculation. We performed a one-loop analysis, but the procedure systematically generalizes to higher order.

In our framework one can derive the non-renormalization theorem for the WZ model with no substantial modification with respect to the standard proof [24]. In the massless case, it is straightforward recognizing that chiral superfield interactions of the type $\int d^4x \, d^2\theta \, (\zeta \phi + m \phi^2 + \lambda \phi^3)$ do not receive any finite or infinite perturbative contributions at the first loop. As a matter of fact, inserting (12) at the tree level in (13) one can see that only vertices with an equal number of chiral and anti-chiral fields acquire one-loop corrections. At higher loops the mechanism is less obvious, since there are reducible vertices of $\Gamma_{\phi\bar{\phi}}$ with any number of $\phi$ and $\bar{\phi}$, which apparently give rise to interactions like $\phi^m \bar{\phi}^n$, even with $n \neq m$. However, by repeatedly using the rules of covariant derivative algebra, one is easily convinced that all the interactions with $n \neq m$ vanish. More generally, this argument can be extended to the massive case.

As well known, in the superspace formulation of SYM one has to face with the problem of infrared singularities, due to the appearance of the pseudoscalar field $C(x)$, the $\theta = 0$ component of the gauge superfield (this difficulty is obviously circumvented in the Wess-Zumino gauge [25], where the field $C$ is absent). To avoid this problem one can assume [15] that all fields are made massive by adding suitable supersymmetric mass terms in the action. Since these masses break BRS invariance, the corresponding Slavnov-Taylor identity holds only in the asymptotic region of momentum space.

In our formulation the presence of the IR cutoff $\Lambda$ naturally makes all cutoff vertices IR finite for $\Lambda \neq 0$. Furthermore, for a non-supersymmetric massless theory it has been proven, by induction in the number of loops [26], that the vertex functions without exceptional momenta are finite for $\Lambda \to 0$, once the relevant couplings are defined in terms of cutoff vertices evaluated at some non-vanishing subtraction points. In this proof the convergence of loop integrals for $\Lambda \to 0$ is simply controlled by the number of soft momenta in the vertices which appear in the iterative solution of the RG equation (13). Therefore we believe its generalization to the supersymmetric case presents no difficulty.

Finally, though we restricted our analysis to the perturbative regime, the RG formulation is in principle non-perturbative and provides a natural context in which to clarify the connection between exact results and those obtained in perturbation theory. In particular, it would be interesting to consider issues such as the anomaly puzzle and the violation of holomorphicity [27].
Note added: After the completion of this paper, a paper by S. Falkenberg and B. Geyer appeared in which the RG formalism in the effective average action approach is applied to N=1 SYM. In this paper an approximate solution of the RG flow for SYM within the background field method is obtained by a truncation of the average action. Nevertheless, as remarked by the authors themselves, such a truncation conflicts with BRS invariance.

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Appendix: supersymmetric conventions

The notations and conventions are those of [13]. Given a Weyl spinor $\psi_\alpha$, $\alpha = 1, 2$, indices can be raised and lowered as follows

$$\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \psi^{\dot\beta},$$

with

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \quad \varepsilon^{12} = 1, \quad \varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}, \quad \varepsilon_{\dot\alpha\dot\beta} \varepsilon_{\dot\gamma\dot\beta} = \delta^\alpha_{\dot\gamma},$$

(the same for dotted indices). The summation convention is $\psi\chi = \psi_{\alpha} \chi^\alpha$ and $\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot\alpha} \bar{\chi}^{\dot\alpha}$.

The matrices $\sigma^\mu$ with lower indices are

$$\sigma^\mu_{\alpha\dot\beta} = (1, \sigma^i)_{\alpha\dot\beta},$$

where the $\sigma^i$'s are the Pauli matrices, whereas those with upper indices are

$$\bar{\sigma}_{\dot\mu}^{\dot\alpha\dot\beta} = \sigma^\mu_{\dot\alpha\dot\beta} = \varepsilon^{\beta\dot\alpha} \varepsilon^{\dot\alpha\dot\beta} \sigma^\mu_{\dot\alpha\dot\beta}.$$

A vector superfield $V(x, \theta, \bar{\theta})$ has the following expansion

$$V(x, \theta, \bar{\theta}) = C(x) + \theta \chi(x) + \bar{\theta} \bar{\chi}(x) + \frac{1}{2} \theta^2 M(x) + \frac{1}{2} \bar{\theta}^2 \bar{M}(x) + \theta \sigma^\mu \bar{\theta} A_\mu(x) + \frac{1}{2} \theta^2 \lambda(x) + \frac{1}{2} \bar{\theta}^2 \bar{\lambda}(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 D(x),$$

(51)

where the components are ordinary space-time fields. A chiral (anti-chiral) superfield $\phi$ ($\bar{\phi}$) expanded in component fields is

$$\phi(x, \theta, \bar{\theta}) = e^{-i\theta \sigma^\mu \bar{\theta} A_\mu} \left( \phi(x) + \theta \psi(x) + \theta^2 F(x) \right)$$

$$\bar{\phi}(x, \theta, \bar{\theta}) = e^{i\bar{\theta} \sigma^\mu \theta A_\mu} \left( \bar{\phi}(x) + \bar{\theta} \bar{\psi}(x) + \bar{\theta}^2 \bar{F}(x) \right).$$

(52)
The components of a vector superfield transform under supersymmetry as

\[ \delta_\alpha C = \chi \]
\[ \tilde{\delta}_\dot{\alpha} C = \bar{\chi} \]
\[ \delta_\alpha \chi^\beta = \delta_\alpha^\beta M \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\chi}^\dot{\beta} = -\delta_\dot{\alpha}^\dot{\beta} \bar{M} \]
\[ \delta_\alpha \xi_\alpha = \sigma_{\alpha\dot{\alpha}}^\mu (A_\mu + i\partial_\mu C) \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\xi}_\dot{\alpha} = -\sigma_{\alpha\dot{\alpha}}^\mu (A_\mu - i\partial_\mu C) \]
\[ \delta_\alpha M = 0 \]
\[ \tilde{\delta}_\dot{\alpha} \bar{M} = 0 \]
\[ \delta_\alpha \bar{M} = \lambda_\alpha - i(\sigma^\mu \partial_\mu \bar{\chi})_\alpha \]
\[ \tilde{\delta}_\dot{\alpha} \bar{M} = \bar{\lambda}_{\dot{\alpha}} + i(\partial_\mu \chi \sigma^\mu)_{\dot{\alpha}} \] (53)
\[ \delta_\alpha A_\mu = \frac{1}{2} (\sigma_\mu \bar{\lambda})_\alpha - \frac{1}{2} (\sigma^\nu \sigma_\nu \partial_\mu \chi)_\alpha \]
\[ \tilde{\delta}_\dot{\alpha} \bar{A}_\mu = \frac{1}{2} (\lambda \sigma_\mu)_\dot{\alpha} + \frac{1}{2} (\partial_\mu \bar{\chi} \sigma_\mu \sigma^\nu)_\dot{\alpha} \]
\[ \delta_\alpha \lambda^\beta = \delta_\alpha^\beta D + i(\sigma^\nu \sigma^\mu)_{\alpha}^\beta \partial_\nu A_\mu \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\lambda}^\dot{\beta} = -\delta_\dot{\alpha}^\dot{\beta} \bar{D} + i(\sigma^\mu \sigma^\nu)_{\dot{\alpha}}^\dot{\beta} \partial_\nu A_\mu \]
\[ \delta_\alpha \bar{\lambda} = i\sigma_{\alpha\dot{\alpha}} \partial_\mu M \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\lambda} = i\sigma_{\alpha\dot{\alpha}} \partial_\mu \bar{M} \]
\[ \delta_\alpha D = -i(\sigma^\mu \partial_\mu \lambda)_\alpha \]
\[ \tilde{\delta}_\dot{\alpha} \bar{D} = i(\partial_\mu \lambda \sigma^\mu)_{\dot{\alpha}} \].

For the components of the chiral and anti-chiral superfields one has

\[ \delta_\alpha \phi = \psi_\alpha \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\phi} = \bar{\psi}_{\dot{\alpha}} \]
\[ \delta_\alpha \psi^\beta = 2\delta_\alpha^\beta F \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\psi}^\dot{\beta} = -2\delta_\dot{\alpha}^\dot{\beta} \bar{F} \]
\[ \delta_\alpha F = 0 \]
\[ \tilde{\delta}_\dot{\alpha} \bar{F} = 0 \] (54)
\[ \delta_\alpha \bar{F} = 0 \]
\[ \tilde{\delta}_\dot{\alpha} \bar{F} = 0 \]
\[ \delta_\alpha \bar{\psi}_\alpha = 2i\sigma_{\alpha\dot{\alpha}} \partial_\mu \bar{\phi} \]
\[ \tilde{\delta}_\dot{\alpha} \bar{\phi}_\dot{\alpha} = 2i\sigma_{\alpha\dot{\alpha}} \partial_\mu \phi \]
\[ \delta_\alpha \bar{\phi} = -i(\sigma^\mu \partial_\mu \bar{\psi})_\alpha \]
\[ \tilde{\delta}_\dot{\alpha} \bar{F} = i(\partial_\mu \sigma^\mu)_{\dot{\alpha}} \].

The covariant derivatives, defined such as to anti-commute with the supersymmetry transformation rules, are given by

\[ D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma_{a\dot{a}}^\mu \bar{\theta}^a \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^a \sigma_{a\dot{a}}^\mu \partial_\mu. \] (55)

They obey the algebra

\[ \{ D_\alpha, \bar{D}_{\dot{\alpha}} \} = 2i\sigma_{a\dot{a}}^\mu \partial_\mu \] (56)

(the other anti-commutators vanish). Useful relations these covariant derivatives satisfy are

\[ [D_\alpha, \bar{D}^2] = 4i(\sigma^\mu \bar{D})_\alpha \partial_\mu, \quad [\bar{D}_{\dot{\alpha}}, D^2] = -4i(D\sigma^\mu)_{\dot{\alpha}} \partial_\mu \]
\[ [D^2, \bar{D}^2] = 8iD\sigma^\mu \bar{D} \partial_\mu + 16\bar{D}^2 = -8i\bar{D}\sigma^\mu D \partial_\mu - 16\bar{D}^2 \]
\[ DD^2 = -\frac{1}{2} D_\alpha D^2 - \frac{1}{2} D^2 \bar{D}_{\dot{\alpha}} , \quad \bar{D} \bar{D}_{\dot{\alpha}} = -\frac{1}{2} D_\alpha \bar{D}^2 - \frac{1}{2} \bar{D}^2 D_{\dot{\alpha}}. \] (57)

The superspace integral of a superfield \( V \), or of a (anti)chiral superfield \( \phi \) (\( \bar{\phi} \)) is given by

\[ \int_z V = \int d^4 x \, D^2 \bar{D}^2 V, \quad \int d^4 x \, d^2 \theta \phi = \int d^4 x \, D^2 \phi, \quad \int d^4 x \, d^2 \bar{\theta} \bar{\phi} = \int d^4 x \, \bar{D}^2 \bar{\phi}, \] (58)
the integral with respect to the Grassmann variable $\theta$ being defined by the derivative $\partial/\partial \theta$.

The following operators

$$P^T = \frac{D\bar{D}D}{8\theta^2}, \quad P^L = -\frac{D^2\bar{D}^2 + \bar{D}^2D^2}{16\theta^2}$$

(59)

are projectors. In particular, $P^L$ can be used to write integrals of chiral (or anti-chiral) superfields as integrals over the full superspace measure (recall that only for this measure the integration by parts holds). For instance

$$\int \frac{d^4 p d^2 \theta}{(2\pi)^4} \phi = \int_{\theta^2} \frac{D^2}{16\theta^2} \phi.$$ 

The delta function is defined by

$$\delta^8(z_1 - z_2) = \delta^4(\theta_1 - \theta_2) \delta^4(x_1 - x_2) = \frac{1}{16} (\theta_1 - \theta_2)^2 (\bar{\theta}_1 - \bar{\theta}_2)^2 \delta^4(x_1 - x_2).$$

The functional derivatives are

$$\frac{\delta V(z_1)}{\delta V(z_2)} = \delta^8(z_1 - z_2), \quad \frac{\delta \phi(z_1)}{\delta \phi(z_2)} = \bar{D}^2 \delta^8(z_1 - z_2), \quad \frac{\delta \bar{\phi}(z_1)}{\delta \bar{\phi}(z_2)} = D^2 \delta^8(z_1 - z_2).$$

(60)

Finally, in order to separate the trivial cocycles from the anomaly in (43), it can be useful to switch to components. Then for the non-supersymmetric YM sector the anomaly is proportional to $\varepsilon_{\mu\nu\rho\sigma}$, which is generated by the following trace

$$\text{Tr} \left[ \sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\tau \right] = 2 \left( g^{\mu\nu} g^{\rho\tau} + g^{\mu\rho} g^{\nu\tau} - g^{\mu\rho} g^{\nu\tau} - i \varepsilon^{\mu\nu\rho\sigma} \right).$$

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