Some Remarks on Some Strongly Coupled Reaction-Diffusion Equations

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Abstract: The primary goal of this paper is to characterize solutions to coupled reaction-diffusion systems. Indeed, we use operators theory to show that under suitable assumptions, then the system given by

\[ u_t = M \Delta u + F(u) \]

have solutions. As applications, we consider a mathematical model arising in Biology and in Chemistry.

1 Introduction

The primary goal of this paper is to make an investigation on a particular type of partial differential systems, that is, the strongly coupled reaction-diffusion system. Recall that early investigations on this problem are due

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to Fichera, see, e.g., [8], and recently by Amann, see, e.g., [1, 2, 3]. Also, recall that in most of publications on the proposed problem, the diffusion matrix $M$ is supposed to be diagonal and sometimes with positive entries. Obviously such an assumption cannot be applied to some interesting cases arising in several fields such as in Biology, Chemistry, and Ecology. In this paper, we consider the general case, that is, we assume that $M$ is any matrix without any restrictions on its entries. However it will be shown that if all eigenvalues of $M$ belong to $S = \{ z \in \mathbb{C} : \Re z \geq 0 \}$, then the reaction-diffusion systems have a unique solution. The main idea of our investigation is based on operators theory, especially unbounded normal operators and related semi-groups of contraction. The strongly coupled reaction-diffusion system is defined as

$$u_t = M \Delta u + F(u),$$

where $M$ is a $d \times d$ real matrix and $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is of class $C^2$. Throughout this paper we will assume that $d$ is even; the case where $d$ is odd will be investigated elsewhere. However, the author expects to use a similar method as in this paper.

As stated above, we propose to solve Eq.(1) using unbounded normal operator method. Under appropriate hypotheses, we prove that Eq.(1) has a unique solution.

Let us consider the needed background for it. Let $A$(not necessarily bounded) be a normal operator in the (complex) space Hilbert $\mathcal{H}$ ( $A$ is densely defined, and closed such that $AA^* = A^*A$). Using the spectral theorem for unbounded normal operators, see, e.g., [4, or 11, pp. 348-355], it is well-known that $A = A_1 - iA_2$, where $A_1$ and $A_2$, are respectively the real part and minus the imaginary part of $A$. Notice that $A_k$’s are respectively self-adjoint operators. It is also well-known that if we assume
that \( A_k \)'s are nonnegative self-adjoint operators, then \( iA \) is an m-accretive operator. Thus \((iA)\) is the infinitesimal generator of a contraction semigroup, see, e.g., [4, or 10 Corollary 4.4, p. 15]. To examine Eq.(1), we study the linear part of it, that it, the diffusion operator \( L = M\Delta \) in the Hilbert space \( H = [L^2(\Omega)]^d \) where \( \Omega \) is a bounded subset of \( \mathbb{R}^p \) with a smooth boundary. In fact, we need to study the following problem

\[
(S_0) \quad \begin{cases}
  u_t = -M\Delta u \\
  u_{|\partial\Omega} = 0
\end{cases}
\]

More generally, let \( A, B, C, \) and \( E \) be unbounded normal operators in \( H \), and consider the following system

\[
(S_1) \quad \begin{cases}
  u_t = Au + Bv \\
  v_t = Cu + Ev \\
  u(0, x) = u_0(x) \\
  v(0, x) = v_0(x)
\end{cases}
\]

Such a system is equivalent to the following:

\[
(S_2) \quad \begin{cases}
  X_t = TX \text{ with } X = (u, v) \\
  X(0, x) = X_0(x) = (u_0(x), v_0(x))
\end{cases}
\]

where the unbounded matrix operator \( T \) is defined by

\[
T : [D(A) \cap D(C)] \oplus [D(B) \cap D(E)] \to H_0 \oplus H_0
\]

and

\[
T = \begin{pmatrix} A & B \\ C & E \end{pmatrix}
\]

Set \( T = -i \tilde{T} \). Thus, the unbounded operator \( \tilde{T} \) is defined by

\[
\tilde{T} = \begin{pmatrix} iA & iB \\ iC & iE \end{pmatrix}
\]

and

\[
3
\]
It follows that \((S_1) \iff (S_2) \iff (S_3)\), where \((S_3)\) is given by

\[
(S_3) \left\{ \begin{array}{l}
X_t = -i\bar{T} X \text{ with } X = (u, v) \\
X(0, x) = X_0(x) = (u_0(x), v_0(x))
\end{array} \right.
\]

The next step is to show that \(\bar{T}\) is a normal operator, which implies that \(-T\) is an m-accretive operator.

# 2 Diffusion Equation

In this section, we show that the general problem \((S_1)\) admits a unique solution under appropriate hypotheses on \(A, B, C,\) and \(E\). As particular case, the equation \((S_0)\) will be considered.

**Theorem 2.1.** Let \(A, B, C,\) and \(E\) be unbounded normal operator in the Hilbert space \(H\). Assume the following conditions hold true

1. \(N(A) = \{0\},\) and \(N(E) = \{0\}\)
2. \(D(A) \cap D(C)\) and \(D(B) \cap D(E)\) are dense in \(H\)
3. \(A^{-1}B\) and \(C^{-1}E\) are closed operators in \(H\)
4. \(A, B, C,\) and \(E\) commute each other

where \(N(A)\) and \(N(E)\) are respectively the Kernels of the operators \(A\) and \(E\). Then the matrix operator \(\bar{T}\) is normal in \(H \oplus H\).

Proof. Recall that the matrix operator \(\bar{T}\) is defined in \(H \oplus H\) by, \(D(\bar{T}) = [D(A) \cap D(B)] \oplus [D(C) \cap D(E)],\) \(\bar{T}(u, v) = \langle i(Au + Bv), i(Cu + Ev) \rangle \forall (u, v) \in D(\bar{T})\). Therefore, it is a densely defined operator in \(H \oplus H\), according to assumption (2). Let \((u_n, v_n)\) be a sequence in \(D(\bar{T})\) such that
\((u_n, v_n)\) converges to \((u, v)\) and \(\tilde{T}(u_n, v_n)\) converges to \((i\xi, i\eta)\) in \(H \oplus H\). In other words, \(Au_n + Bv_n\) and \(Cu_n + Ev_n\) converge to \(\xi\) and \(\eta\) respectively. Since the kernel \(N(A) = \{0\}\) (according to (1)), then \(A^{-1}Bv_n \longrightarrow A^{-1}\xi - u\). Now, since \(A^{-1}B\) is closed then \(v \in D(A^{-1}B) = D(B)\) and \(A^{-1}Bv = A^{-1}\xi - u\). In addition \(u \in D(A)\), since \(u = A^{-1}\xi - A^{-1}Bv \in D(A)\). Thus, it easily follows that \((u, v) \in D(C) \oplus D(E)\) and \(Au + Ev = \xi\). Using a similar argument yields \((u, v) \in D(C) \oplus D(E)\) and \(Cu + Ev = \eta\). Therefore, \(\tilde{T}\) is a closed operator. Now, we have

\[
\tilde{T}\tilde{T}^* = \begin{pmatrix}
A* A + C* B & B* A + E* B \\
A* C + C* E & B* C + E* E
\end{pmatrix}
\]

and,

\[
\tilde{T}^* \tilde{T} = \begin{pmatrix}
AA* + CB* & BA* + EB* \\
AC* + CE* & BC* + EE*
\end{pmatrix}
\]

Clearly \(AA* = A*A\), and \(EE* = E*E\) (\(A\) and \(E\) are normal operators).

Let show that \(C* B = B* C\). A similar argument can be used to show that \(BA* = AB*\), \(E* B = EB*\), and \(C* E = CE*\). Let us write \(C = C_1 - iC_2\) and \(B = B_1 - B_2\) as stated in the introduction of this paper. Thus \(C* = C_1 + iC_2\) and \(B* = B_1 + iB_2\). Therefore \(C* B = (C_1 B_1 + C_2 B_2) + i(C_2 B_1 - C_1 B_2)\); in the same way \(B* C = (B_1 C_1 + B_2 C_2) + i(B_2 C_1 - B_1 C_2)\). Now, since \(BC = CB\) (according to (4)), we have \(C_p B_q = B_q C_p\) for \(p, q = 1, 2\). It follows that \(C* B = B* C\). In summary, we have \(\tilde{T}^* \tilde{T} = \tilde{T} \tilde{T}^*\). The proof is complete.

**Corollary 2.2** Under previous assumptions. The operator \(\tilde{T}\) can be decomposed as \(\tilde{T} = T_1 - iT_2\) where \(T_1\) and \(T_2\) are respectively self-adjoint operators. Assume that both \(T_1\), \(T_2\) are nonnegative operators. Then the operator \(-T = i\tilde{T}\) is m-accretive. In addition \(S_1\) admits a unique solution.
Prof. Since $\tilde{T}$ is a normal operator and that both its real and minus imaginary parts are nonnegative self-adjoint operators, then $i\tilde{T} = -T$ is m-accretive, see, eg., [10, Corollary 4.4, p. 15]. It follows that the system $(S_1)$ admits a unique solution. Equivalently both $(S_2)$ and $(S_2)$ admit unique solutions.

We apply previous results to the problem $(S_0)$. Assume that $d = 2n$ and set

$$A = -M_1\Delta, \quad B = -M_2\Delta$$

$$C = -M_3\Delta, \quad E = -M_4\Delta$$

where $M_k$ ($k = 1, 2, 3, 4$) is a $n \times n$-matrix. Consider the following problem

$$(S_4) \begin{cases}
X_t = -M\Delta X \text{ with } X = (u, v) \\
X(0, x) = X_0(x) = (u_0(x), v_0(x))
\end{cases}$$

where $M\Delta : [H^2(\Omega) \cap H^1_0(\Omega)]^{2n} \to [L^2(\Omega)]^{2n}$ is defined as

$$M\Delta = \begin{pmatrix}
M_1\Delta & M_2\Delta \\
M_3\Delta & M_4\Delta
\end{pmatrix}$$

Set $T = -M\Delta$ and $-T = i\tilde{T}$. Therefore $\tilde{T}$ is defined as

$$\tilde{T} = \begin{pmatrix}
-iM_1\Delta & -iM_2\Delta \\
-iM_3\Delta & -iM_4\Delta
\end{pmatrix}$$

According to theorem 2.1, $\tilde{T}$ is a normal operator. We have the following result.

**Theorem 2.3** Under previous assumptions. Assume that $M$ is not the zero matrix. In addition if all eigenvalues of $M$ belong to the set $S = \{z \in \mathbb{C} : \Re z \geq 0\}$. Then the problem $(S_4)$ admits a unique solution.
Proof. The operators $A$, $B$, $C$, and $E$ given in Eq.(2) satisfy assumptions (1)–(2)–(3)–(4) of the theorem 2.2. It turns out that $\tilde{T}$ is a normal operator. Now let us show that if all eigenvalues of $M$ belong to $S = \{ z \in \mathbb{C} : \Re z \geq 0 \}$, then $i\tilde{T} = M\Delta$ is m-accretive. Let $\lambda_1, \lambda_1, ..., \lambda_r$ ($k_1 + k_2 + ... + k_r = 2n$) be eigenvalues of $M$. Following the Jordan decomposition method for $M$, it is well-known that $M\Delta$ can be decomposed as

$$M\Delta = \Pi \begin{pmatrix} J_{k_1}(\lambda_1)\Delta & 0 & ... & 0 \\ 0 & J_{k_2}(\lambda_2)\Delta & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_{k_r}(\lambda_r)\Delta \end{pmatrix} \Pi^{-1}$$

where $\Pi$ is a nonsingular matrix, and

$$J_k(\lambda)\Delta = \begin{pmatrix} \lambda\Delta & \Delta & ... & 0 \\ 0 & \lambda\Delta & \Delta & ... \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda\Delta \end{pmatrix}$$

Thus, all eigenvalues of $M$ belong to $S = \{ z \in \mathbb{C} : \Re z \geq 0 \}$ if and only if $i\tilde{T} = M\Delta$ is m-accretive. Therefore, if all eigenvalues of $M$ belong to $S$ then $i\tilde{T} = M\Delta$ generates a contraction semi-group. In such a case ($S_4$) admits a unique solution. Equivalently ($S_0$) admits a unique solution, since ($S_0$) is a particular case of ($S_4$).

### 3 Coupling Problem

Assume $d = 2n$ and consider the coupling of the diffusion with the reaction term, that is, the system given by Eq.(1). We define the following operators
\[
\begin{align*}
D(T) &= [H^1_0(\Omega) \cap H^2(\Omega)]^d \\
Tu &= M \Delta u \quad \forall u \in D(T)
\end{align*}
\]

where \( M \) is the \( d \times d \)-matrix given in the Eq.(1).

\[
\begin{align*}
D(R) &= \{u \in [L^2(\Omega)]^d : F(u) \in [L^2(\Omega)]^d\} \\
Ru(t, x) &= F(u(t, x)) \quad \text{a.e. } u \in D(R)
\end{align*}
\]

We will make the following hypotheses

\( (H_0) \) all eigenvalues of the matrix \( M \) belong to \( S = \{z \in \mathbb{C} : \Re z \geq 0\} \)

\( (H_1) \) Assume the operator \( R \) is m-accretive, and that \( 0 \in R(0) \)

We have the following.

**Theorem 3.1.** Under assumptions \((H_0)\) and \((H_0)\). Then the Eq.(1) admits a unique solution.

Proof. The main idea is to show that the nonlinear operator given by \( T + R \) is m-accretive in \( H = [L^2(\Omega)]^d \). Consider Yosida’s approximation for \( R \). It is defined as

\[
R_\lambda = \frac{1}{\lambda} [I - (I + \lambda R)^{-1}], \quad \lambda > 0 \tag{3}
\]

It is well-known that \( R_\lambda \) is m-accretive and that \( \frac{1}{\lambda} \)-Lipschitz in \( H \). Now consider the following equation

\[
\varepsilon u_\lambda + Tu_\lambda + R_\lambda u_\lambda = v, \quad \text{and } u_\lambda = 0, \quad \text{on } \partial \Omega \tag{4}
\]

Since \( T + R_\lambda \) is m-accretive, see, e.g., [5], then Eq.(4) admits a unique solution \( u_\lambda \in D(T) \) for any \( v \in H \), and \( \lambda > 0 \). We also know the family
\((u_\lambda)_{\lambda \geq 0}\) is bounded by \(\frac{1}{\varepsilon} \|v\|_H\). Using the fact \(R_\lambda\) is m-accretive, \(R_\lambda 0 = 0\), and by integration by parts it easily follows that
\[
\int_{\Omega^d} TuR_\lambda u dx \geq 0, \quad \forall u \in [H^1_0(\Omega) \cap H^2(\Omega)]^d
\]  
(5)
Thus, multiplying Eq.(4) by \(Tu_\lambda\), and from Eq.(5), it turns out that \((Tu_\lambda)\) and \((R_\lambda u_\lambda)\) are bounded. From the compactness embedding, \([H^2(\Omega)]^d \hookrightarrow [L^2(\Omega)]^d\), and the fact \((u_\lambda), (Tu_\lambda)\), and \((R_\lambda u_\lambda)\) are bounded, it turns out that: \(u_\lambda\) strongly converges to \(u\), \((Tu_\lambda)\) weakly converges to \(\xi\), and \((R_\lambda u_\lambda)\) weakly converges to \(\eta\), as \(\lambda\) approaches to 0 in \(H\). Since \(T\) is closed, then \(Tu = \xi\). Since \(R\) is m-accretive, then \(Ru = \eta\), see, e.g., [5]. In summary \(T + R\) is m-accretive under assumptions \((H_0)\) and \((H_0)\). Therefore the algebraic sum (see, e.g., [6]) \((T + R)\) generates a nonlinear contraction semi-group, that is the Eq.(1) admits a unique solution.

4 Applications

In this section, we consider a model considered in [9]. The problem we will study represents a mathematical model describing various chemical and biological phenomena. In [9], Lyapunov functionals have used to prove a global existence of unique solutions. Here, we use the method described above to prove that the given problem admits a unique solution, under suitable assumptions.

Our model is described as
\[
(M) \quad \begin{cases} 
 u_t - \alpha \Delta u - \beta \Delta v = -\sigma f(u, v) \text{ in } (0, \infty) \times \Omega \\
 v_t - \gamma \Delta u - \alpha \Delta v = \rho f(u, v) \text{ in } (0, \infty) \times \Omega \\
 u_\nu = v_\nu = 0 \text{ on } (0, \infty) \times \partial \Omega \\
 u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \text{ in } \Omega 
\end{cases}
\]
where \( u(t) \) and \( v(t) \) represent either chemical concentrations or biological population densities, \( \Omega \) is a bounded open subset of class \( C^1 \) in \( \mathbb{R}^n \), \( u_\nu \) (respectively \( v_\nu \)) denotes the outward normal derivative on \( \partial \Omega \), and \( \alpha, \beta, \gamma, \rho, \) and \( \sigma \) are positive constants. In [9], the following hypothesis is made

\[
2\alpha > (\beta + \gamma) \tag{6}
\]

The (M) can be expressed as

\[
\begin{cases}
(u_t, v_t) = M \Delta(u, v) + F(u, v) \\
(u_\nu, v_\nu) = (0, 0) \\
(u(0, x), v(0, x)) = (u_0(x), v_0(x))
\end{cases}
\tag{N}
\]

where \( F(u, v) = (-\sigma f(u, v), \rho f(u, v)) \), and

\[
M = \begin{bmatrix}
\alpha & \beta \\
\gamma & \alpha
\end{bmatrix}
\]

It is obvious to see that all eigenvalues of of the diffusion matrix \( M \) are given as, \( \text{EV}(M) = \{\alpha + \sqrt{\beta \gamma}, \alpha - \sqrt{\beta \gamma}\} \). Since all eigenvalues of \( M \) are nonnegative, then

\[
\alpha > \sqrt{\beta \gamma} \tag{7}
\]

Clearly Eq.(6) implies Eq.(7). Indeed, \( \frac{1}{2}(\beta + \gamma) \geq \sqrt{\beta \gamma} \). Therefore, instead of considering Eq.(7), we will only assume that Eq.(6) holds.

Consider the Hilbert space \( H = [L^2(\Omega)]^2 \) and set

\[
D(T) = [H^1_0(\Omega) \cap H^2(\Omega)]^2
\]

and

\[
T(u, v) = M(\Delta u, \Delta v)
\]
In the same way, define

\[ D(R) = \{(u, v) \in [L^2(\Omega)]^2 : (-\sigma f(u, v), \rho f(u, v)) \in [L^2(\Omega)]^2}\] 

and

\[ R(u, v) = (-\sigma f(u(t, x), v(t, x)), \rho f(u(t, x), v(t, x))) \text{ a.e } u, v \in D(R) \]

We will make the following assumption

\[ f(0, 0) = 0 \] (8)

For instance from the fact that \( R \) is accretive, the following holds

\[ -\sigma uf(u, v) + \rho vf(u, v) \geq 0, \ \forall (u, v) \in D(R) \] (9)

More generally, assume that \( f \) is given such that \( R \) is a nonlinear m-accretive operator in \([L^2(\Omega)]^2\). Thus, we have the following.

**Proposition 4.1.** Under Eq.(6), and Eq.(8), then the problem described in (M) admits a unique solution.

Proof. Obvious as consequences of Theorem 2.3 and Theorem 3.1.

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