On the sum of the values of a polynomial at natural numbers which form a decreasing arithmetic progression

Bakir Farhi

A tribute to Ibn al-Banna al-Marrakushi on the 700th anniversary of his death

Received: 12 November 2021 / Accepted: 20 January 2022 / Published online: 28 January 2022
© The Indian National Science Academy 2022

Abstract The purpose of this paper consists to study the sums of the type \( P(n) + P(n-d) + P(n-2d) + \ldots \), where \( P \) is a real polynomial, \( d \) is a positive integer and the sum stops at the value of \( P \) at the smallest natural number of the form \( (n-kd) \) \((k \in \mathbb{N})\). Precisely, for a given \( d \), we characterize the \( \mathbb{R} \)-vector space \( \mathcal{E}_d \) consisting of the real polynomials \( P \) for which the above sum is polynomial in \( n \). The case \( d = 2 \) is studied in more details.

In the last part of the paper, we approach the problem through formal power series; this inspires us to generalize the spaces \( \mathcal{E}_d \) and the underlying results. Also, it should be pointed out that the paper is motivated by the curious formula:

\[
\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t - 1 = \sum_{n=0}^{+\infty} \frac{B_n(X)}{n!} t^n
\]

Communicated by Sanoli Gun.

B. FARHI
Laboratoire de Mathématiques Appliquées, Faculté des Sciences Exactes, Université de Bejaia, 06000 Bejaia, Algeria
E-mail: bakir.farhi@gmail.com
and the Bernoulli numbers \( B_n \) are the values of the Bernoulli polynomials at \( X = 0 \); that is, \( B_n := B_n(0) \) \((\forall n \in \mathbb{N})\). To make the difference between the Bernoulli polynomials and the Bernoulli numbers, we always put the indeterminate \( X \) in evidence when it comes to polynomials. Similarly, the Euler polynomials \( E_n(X) \) \((n \in \mathbb{N})\) can be defined by their exponential generating function:

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{+\infty} E_n(X) \frac{t^n}{n!}.
\]

The Genocchi numbers \( G_n \) \((n \in \mathbb{N})\) can also be defined by their exponential generating function:

\[
\frac{2t}{e^t + 1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!}.
\]

The famous Genocchi theorem \([6]\) states that the \( G_n \)'s are all integers. The Bernoulli polynomials and numbers, the Euler polynomials and the Genocchi numbers have been studied by several authors (see e.g., \([3–5,7]\)) and have many important and remarkable properties; among them we just cite the followings:

\[
B_n(X + 1) - B_n(X) = nX^{n-1} \quad (\forall n \in \mathbb{N}), \tag{1.1}
\]

\[
B_n \left( \frac{1}{2} \right) = \left( \frac{1}{2^{n-1}} - 1 \right) B_n \quad (\forall n \in \mathbb{N}), \tag{1.2}
\]

\[
G_n = 2 \left( 1 - 2^n \right) B_n = nE_{n-1}(0) \quad (\forall n \in \mathbb{N}^*). \tag{1.3}
\]

In Arabic mathematics, there is a long tradition of determining closed forms for the sums of the type: \((1^k + 2^k + \cdots + n^k) \in \mathbb{N}^+\). Apart from the sum \((1 + 2 + \cdots + n)\) for which the closed form has been known since antiquity (equal to \( \frac{n(n+1)}{2} \)), we can cite

\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(al-Karaji)},
\]

\[
1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \quad \text{(al-Karaji)},
\]

\[
1^4 + 2^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} \quad \text{(Ibn al-Haytham)}.
\]

In the course of the seventeenth century, it became increasingly evident that for any nonzero polynomial \( P \in \mathbb{R}[X] \), the sum \( P(0) + P(1) + \cdots + P(n) \) \((n \in \mathbb{N})\) is polynomial in \( n \) with degree \((\deg P + 1)\). Equivalently, for any \( P \in \mathbb{R}[X] \), there exists \( Q \in \mathbb{R}[X] \) satisfying \( Q(X + 1) - Q(X) = P(X) \). More explicitly, Jacob Bernoulli \([2]\) obtained the following remarkable formula:

\[
0^k + 1^k + \cdots + n^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+1-i} + n^k \quad (\forall k, n \in \mathbb{N}).
\]

For a given positive integer \( d \) and a given polynomial \( P \in \mathbb{R}[X] \), let us write:

\[
P(n) + P(n-d) + P(n-2d) + \ldots \quad (n \in \mathbb{N}) \tag{\Sigma}
\]

to designate the sum of the values of \( P \) from \( i = n \) to the smallest natural number of the form \((n - kd) \in \mathbb{N}\), by decreasing \( i \) in each step by \( d \); that is the sum \( \sum_{0 \leq k \leq \frac{n}{d}} P(n - kd) \). Although the sums \((\Sigma)\) are polynomial in \( n \) when \( d = 1 \) (as seen above), this is not always the case for \( d \geq 2 \). Indeed, we have for example (for \( n \in \mathbb{N} \)):

\[
n + (n-2) + (n-4) + \cdots = \left\lfloor \frac{n+1}{2} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right),
\]

which is not polynomial in \( n \). In his pioneer book entitled “Raf” al-ḥijab ‘an wujuh a’mal al-ḥisab” (see e.g., \([1]\)), meaning “Lifting the veil from the faces of the workings of Arithmetic”, the great Arabic mathematician
Ibn al-Banna al-Marrakushi (1256 - 1321) found a case when the sum \( \sum \) becomes polynomial in \( n \) with \( d \neq 1 \). He precisely obtained the following curious formula:

\[
n^2 + (n-2)^2 + (n-4)^2 + \cdots = \frac{n(n+1)(n+2)}{6} \quad (\forall n \in \mathbb{N}).
\] (1.4)

More interestingly is the proof of Ibn al-Banna of his formula (1.4). A natural way to prove (1.4) consists to distinguish two cases according to the parity of \( n \). But Ibn al-Banna did not do this; he proved (1.4) in one step! To do so, he remarked that any perfect square number can be written as a sum of two consecutive triangular numbers; precisely, we have for all \( n \in \mathbb{N} \):

\[
n^2 = T_n + T_{n-1},
\]

where \( T_k := \frac{k(k+1)}{2} \) is the triangular number of order \( k \). By admitting negative numbers (one thing that Ibn al-Banna himself avoids!\(^1\)), the Ibn al-Banna method for proving (1.4) becomes as follows:

For any \( n \in \mathbb{N} \), we have

\[
n^2 + (n-2)^2 + (n-4)^2 + \cdots = (T_n + T_{n-1}) + (T_{n-2} + T_{n-3}) + (T_{n-4} + T_{n-5}) + \cdots
\]

where the last sum stops at \( T_{n-1} \) when \( n \) is even and stops at \( T_0 \) when \( n \) is odd. But because \( T_{n-1} = 0 \), we can stop the sum in question at \( T_0 \) in both cases. So we have

\[
n^2 + (n-2)^2 + (n-4)^2 + \cdots = T_0 + T_1 + T_2 + \cdots + T_n
\]

\[
= \sum_{k=0}^{n} \frac{k(k+1)}{2}
\]

\[
= \frac{n(n+1)(n+2)}{6},
\]

as required.

Looking closely at the previous Ibn al-Banna proof of Formula (1.4), we see that the fundamental property that makes it work is \( T_{n-1} = 0 \). So, by proceeding in the same way, we can establish other formulas of the same type as (1.4) and as curious as it is, with other values of \( d \). I have obtained for example the following:

\[
n \left( n^2 + 1 \right) + (n-3) \left( (n-3)^2 + 1 \right) + (n-6) \left( (n-6)^2 + 1 \right) + \cdots = \frac{n(n+1)(n+2)(n+3)}{12}.
\] (1.5)

To prove (1.5), let us observe that for any \( n \in \mathbb{N} \), we have

\[
n(n^2 + 1) = \alpha_n + \alpha_{n-1} + \alpha_{n-2},
\]

where \( \alpha_k := \frac{k(k+1)(k+2)}{3} \) (\( \forall k \in \mathbb{Z} \)). Thus we have

\[
n \left( n^2 + 1 \right) + (n-3) \left( (n-3)^2 + 1 \right) + (n-6) \left( (n-6)^2 + 1 \right) + \cdots
\]

\[
= \left( \alpha_n + \alpha_{n-1} + \alpha_{n-2} \right) + \left( \alpha_{n-3} + \alpha_{n-4} + \alpha_{n-5} \right) + \cdots
\]

\[
= \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots.
\]

\(^1\) Although his predecessor al-Samawal (1130-1180) uses negative numbers with great skill.
where the last sum stops at \( \alpha_{-2} \) when \( n \equiv 0 \mod 3 \), stops at \( \alpha_{-1} \) when \( n \equiv 1 \mod 3 \) and stops at \( \alpha_0 \) when \( n \equiv 2 \mod 3 \). But because \( \alpha_{-2} = \alpha_{-1} = 0 \) we can stop the sum in question at \( \alpha_0 \) in both cases. So we get

\[
\begin{align*}
n \left( n^2 + 1 \right) + (n - 3) \left( (n - 3)^2 + 1 \right) + (n - 6) \left( (n - 6)^2 + 1 \right) + \ldots &= \alpha_0 + \alpha_1 + \ldots + \alpha_n \\
\quad &= \sum_{k=0}^{n} k(k+1)(k+2) \\
\quad &= \frac{n(n+1)(n+2)(n+3)}{12},
\end{align*}
\]

as required. Actually, we can even find a sum of type \( \Sigma \) with \( d = 3 \) which is polynomial in \( n \) with degree only 2. To do so, it suffices to take in the previous reasoning \( \alpha_k = (k+1)(k+2) \) (instead of \( \alpha_k = \frac{1}{3} k(k+1)(k+2) \)).

We then obtain that for \( P_0(n) = 3n^2 + 3n + 2 \ (\forall n \in \mathbb{N}) \), we have

\[
P_0(n) + P_0(n-3) + P_0(n-6) + \ldots = \frac{(n+1)(n+2)(n+3)}{3} \quad (\forall n \in \mathbb{N}).
\]

However, I personally find Formula (1.5) more elegant than the last! Actually, the mystery of Formula (1.4) can be explained otherwise by using the following immediate identity which holds for any \( n \in \mathbb{N} \):

\[
n^2 + (n-2)^2 + (n-4)^2 + \ldots = \frac{1}{2} \sum_{k=0}^{n} (n-2k)^2;
\]

but unfortunately, this simple technique cannot be used to explain other identities of the same type as (1.4) (it is incapable for example to explain (1.5)).

Throughout this paper, for a given positive integer \( d \), we let \( \mathcal{E}_d \) denote the set of all polynomials \( P \in \mathbb{R}[X] \) for which the sum \( \Sigma \) is polynomial in \( n \). For example, we have (according to what is said above): \( X \notin \mathcal{E}_2 \), \( X^2 \in \mathcal{E}_2 \), \( X(X^2 + 1) \in \mathcal{E}_3 \) and \( 3X^2 + 3X + 2 \in \mathcal{E}_3 \). It is immediate that \( \mathcal{E}_d \ (d \in \mathbb{N}^+) \) is a \( \mathbb{R} \)-linear subspace of \( \mathbb{R}[X] \). This paper is devoted to studying the detailed structure of \( \mathcal{E}_d \); in particular to determine it an explicit basis and to specify its codimension in \( \mathbb{R}[X] \). The case \( d = 1 \) is trivial because we know that the sum \( \Sigma \) is always polynomial in \( n \) when \( d = 1 \); thus \( \mathcal{E}_1 = \mathbb{R}[X] \). The case \( d = 2 \) is studied in more detail by providing for \( \mathcal{E}_2 \) a basis which is close (in a some sense) to the canonical basis of \( \mathbb{R}[X] \); this will connect us with the sequence of the values of the Euler polynomials at the origin. Then, we go on to prove the generality of the Ibn al-Banna method (exposed above). Precisely, we will show that whenever the sum \( \Sigma \) is polynomial in \( n \), the Ibn al-Banna method can be used to determine its closed form. We conclude the paper by giving (briefly) another approach (using convolution products and power series) to studying the spaces \( \mathcal{E}_d \). This approach inspires us moreover a generalization of the spaces \( \mathcal{E}_d \), and some extensive results are given without proofs. It should also be noted that during the description of the second approach of studying and generalizing the spaces \( \mathcal{E}_d \), some other notations will be introduced. Especially, for more clarity and convenience, the sum \( \Sigma \) is denoted by \( S_{P,d}(n) \), highlighting each of the parameters \( P \), \( d \) and \( n \).

2 The results and the proofs

For a given positive integer \( d \), recall that \( \mathcal{E}_d \) denotes the vector space of all real polynomials \( P \) for which the sum \( P(n) + P(n-d) + P(n-2d) + \ldots \) \( (n \in \mathbb{N}) \) remains polynomial in \( n \). Our main result is the following:

**Theorem 2.1** Let \( d \) be a positive integer. Then, a polynomial \( P \in \mathbb{R}[X] \) belongs to \( \mathcal{E}_d \) if and only if it has the form:

\[
P(X) = \binom{X+d}{d} f(X+d) - \binom{X}{d} f(X),
\]

where \( f \in \mathbb{R}[X] \). Besides, if \( (2.1) \) holds then we have for any natural number \( n \):

\[
P(n-d) + P(n-2d) + P(n-3d) + \ldots = \binom{n}{d} f(n).
\]
Proof Let $P \in \mathbb{R}[X]$.

- Suppose that $P \in \mathcal{E}_d$; that is there exists $S \in \mathbb{R}[X]$ for which we have for any natural number $n$:

$$P(n) + P(n - d) + P(n - 2d) + \cdots = S(n).$$

Using (2.2), we have in particular:

$$S(0) = P(0), S(1) = P(1), \ldots, S(d - 1) = P(d - 1).$$

This shows that the real polynomial $(S(X) - P(X))$ vanishes at $0, 1, \ldots, d - 1$. Consequently, there exists $f_0 \in \mathbb{R}[X]$ such that:

$$S(X) - P(X) = X(X - 1) \cdots (X - d + 1)f_0(X) = \binom{X}{d}f(X),$$

where $f := d!f_0 \in \mathbb{R}[X]$. By specializing the obtained polynomial equality $S(X) - P(X) = \binom{X}{d}f(X)$ to $X = n \in \mathbb{N}$, we get (according to (2.2)):

$$P(n - d) + P(n - 2d) + P(n - 3d) + \cdots = \binom{n}{d}f(n) \quad (\forall n \in \mathbb{N}).$$

Next, by applying (2.3) for $(n + d)$ (instead of $n \in \mathbb{N}$), we get

$$P(n) + P(n - d) + P(n - 2d) + \cdots = \binom{n + d}{d}f(n + d) \quad (\forall n \in \mathbb{N}).$$

Then, by subtracting (2.3) from (2.4), we obtain that:

$$P(n) = \binom{n + d}{d}f(n + d) - \binom{n}{d}f(n) \quad (\forall n \in \mathbb{N}).$$

Finally, since the two sides of the last formula are both polynomials in $n$ then its validity for any $n \in \mathbb{N}$ implies its validity as a polynomial identity; that is

$$P(X) = \binom{X + d}{d}f(X + d) - \binom{X}{d}f(X),$$

as required.

- Conversely, suppose that $P$ has the form (2.1). Then, for any $n, q \in \mathbb{N}$, we have

$$\sum_{k=1}^{q} P(n - kd) = \sum_{k=1}^{q} \left( \binom{n - (k - 1)d}{d}f(n - (k - 1)d) - \binom{n - kd}{d}f(n - kd) \right)$$

which telescopic sum

$$= \binom{n}{d}f(n) - \binom{n - qd}{d}f(n - qd).$$

By taking in particular $q = \lfloor \frac{n}{d} \rfloor$ and putting $r := n - qd$ (so $r$ is the remainder of the euclidean division of $n$ by $d$, and consequently it is the smallest natural number among the integers $n, n - d, n - 2d, n - 3d, \ldots$), we find that:

$$P(n - d) + P(n - 2d) + P(n - 3d) + \cdots = \binom{n}{d}f(n) - \binom{r}{d}f(r).$$

But since $r \in \{0, 1, \ldots, d - 1\}$, we have $\binom{r}{d} = 0$; hence

$$P(n - d) + P(n - 2d) + P(n - 3d) + \cdots = \binom{n}{d}f(n),$$

showing that the sum $P(n - d) + P(n - 2d) + P(n - 3d) + \ldots$ is polynomial in $n$; thus $P \in \mathcal{E}_d$, as required. This completes the proof of the theorem.
Let \(d\) be a positive integer. Then the family of real polynomials
\[
\mathcal{B} := \left\{ \left( \begin{array}{c} X + d \\ d \end{array} \right)^k X^k \right\}_{k \in \mathbb{N}}
\]
constitutes a basis for the \(\mathbb{R}\)-vector space \(E_d\).

**Corollary 2.2** Let \(d\) be a positive integer. Then the family of real polynomials
\[
\mathcal{B} := \left\{ \left( \begin{array}{c} X + d \\ d \end{array} \right)^k X^k \right\}_{k \in \mathbb{N}}
\]
constitutes a basis for the \(\mathbb{R}\)-vector space \(E_d\).

**Proof**
- First, let us show that \(\mathcal{B}\) generates \(E_d\). For any given \(P \in E_d\), we can write (according to Theorem 2.1):
\[
P(X) = \left( \begin{array}{c} X + d \\ d \end{array} \right) f(X + d) - \left( \begin{array}{c} X \\ d \end{array} \right) f(X),
\]
for some \(f \in \mathbb{R}[X]\). Then, by expressing \(f\) in the canonical basis of \(\mathbb{R}[X]\); that is in the form:
\[
f(X) = \sum_{k=0}^{n} a_k X^k
\]
(with \(n \in \mathbb{N}\) and \(a_0, a_1, \ldots, a_n \in \mathbb{R}\)), we obtain that:
\[
P(X) = \left( \begin{array}{c} X + d \\ d \end{array} \right) \sum_{k=0}^{n} a_k (X + d)^k - \left( \begin{array}{c} X \\ d \end{array} \right) \sum_{k=0}^{n} a_k X^k
\]
\[
= \sum_{k=0}^{n} a_k \left[ \left( \begin{array}{c} X + d \\ d \end{array} \right) (X + d)^k - \left( \begin{array}{c} X \\ d \end{array} \right) X^k \right],
\]
which is an expression of \(P\) as a linear combination (with real coefficients) of elements of \(\mathcal{B}\). Therefore, \(\mathcal{B}\) generates \(E_d\).

- Now, let us show that \(\mathcal{B}\) is a free family in the \(\mathbb{R}\)-vector space \(\mathbb{R}[X]\). Let \(K\) be a finite nonempty subset of \(\mathbb{N}\) and \((\lambda_k)_{k \in K}\) be a family of real numbers such that:
\[
\sum_{k \in K} \lambda_k \left( \left( \begin{array}{c} X + d \\ d \end{array} \right) (X + d)^k - \left( \begin{array}{c} X \\ d \end{array} \right) X^k \right) = 0. \tag{2.5}
\]
So, we have to show that \(\lambda_k = 0\) for every \(k \in K\). By putting \(Q(X) := \left( \begin{array}{c} X \\ d \end{array} \right) \sum_{k \in K} \lambda_k X^k \in \mathbb{R}[X]\), we have
\[
(2.5) \iff Q(X + d) = Q(X) \iff Q \text{ is } d\text{-periodic.}
\]
But if \(Q\) is not zero, we have that \(\deg Q \geq \deg \left( \begin{array}{c} X \\ d \end{array} \right) = d \geq 1\); so \(Q\) cannot be \(d\)-periodic (because a real continuous function on \(\mathbb{R}\) which is \(d\)-periodic is inevitably bounded, while a real polynomial with degree \(\geq 1\) is never bounded). Thus \(Q\) is zero, implying that \(\lambda_k = 0\) for every \(k \in K\). Consequently, \(\mathcal{B}\) is free.

In conclusion, \(\mathcal{B}\) is a basis of the \(\mathbb{R}\)-vector space \(E_d\). The corollary is proved.

**An important example (the case \(d = 2\)):** The particular case \(d = 2\) is the one to which Ibn al-Banna’s formula corresponds, and for this reason, it requires more attention. By applying Corollary 2.2 for \(d = 2\), we find that the \(\mathbb{R}\)-vector space \(E_2\) has as a basis the family of polynomials \((e_k)_{k \in \mathbb{N}}\), defined by:
\[
e_k(X) := \left( \begin{array}{c} X + 2 \\ 2 \end{array} \right) (X + 2)^k - \left( \begin{array}{c} X \\ 2 \end{array} \right) X^k \quad (\forall k \in \mathbb{N}).
\]
The calculations give
\[
e_0(X) = 2X + 1,
\]
\[
e_1(X) = 3X^2 + 4X + 2,
\]
\[
e_2(X) = 4X^3 + 9X^2 + 10X + 4, \text{ etc.}
\]
In particular, we observe that the polynomial $X^2$ (involved in Ibn al-Banna’s formula) can be written as:

$$X^2 = \frac{1}{3}(e_1 - 2e_0),$$

which is a linear combination of the $e_k$’s, so belongs to $\mathcal{E}_2$. Next, we observe that:

$$X^3 - \frac{1}{4} = \frac{1}{4}(e_2 - 3e_1 + e_0),$$

showing that $(X^3 - \frac{1}{4}) \in \mathcal{E}_2$. More generally, we will show later (see Corollary 2.4 and Theorem 2.5) that for all natural number $k$, we have

$$(X^k - E_k(0)) \in \mathcal{E}_2.$$ 

In addition, the family of polynomials $(X^k - E_k(0))_{k \in \mathbb{N}^*}$ constitutes a basis for the $\mathbb{R}$-vector space $\mathcal{E}_2$ (simpler than $(e_k)_{k \in \mathbb{N}^*}$).

In what follows, we derive from Theorem 2.1 a “simple” complement subspace (in $\mathbb{R}[X]$) of the $\mathbb{R}$-vector space $\mathcal{E}_d$ ($d \in \mathbb{N}^*$). We have the following corollary:

**Corollary 2.3** Let $d$ be a positive integer. Then the $\mathbb{R}$-vector space $\mathbb{R}_{d-2}[X]$ is a complement subspace (in $\mathbb{R}[X]$) of $\mathcal{E}_d$; that is

$$\mathcal{E}_d \oplus \mathbb{R}_{d-2}[X] = \mathbb{R}[X].$$

In particular, we have

$$\text{codim}_{\mathbb{R}[X]} \mathcal{E}_d = d - 1.$$ 

**Proof** We have to show that $\mathcal{E}_d \cap \mathbb{R}_{d-2}[X] = \{0_{\mathbb{R}[X]}\}$ and that $\mathcal{E}_d + \mathbb{R}_{d-2}[X] = \mathbb{R}[X]$.

- Let us first show that $\mathcal{E}_d \cap \mathbb{R}_{d-2}[X] = \{0_{\mathbb{R}[X]}\}$. So, let $P \in \mathcal{E}_d \cap \mathbb{R}_{d-2}[X]$ and let us show that $P$ is necessarily the zero polynomial (the second inclusion $\{0_{\mathbb{R}[X]}\} \subset \mathcal{E}_d \cap \mathbb{R}_{d-2}[X]$ is trivial). From $P \in \mathcal{E}_d$, we deduce (by Theorem 2.1) that there is $f \in \mathbb{R}[X]$ such that:

$$P(X) = \begin{pmatrix} X + d \\ d \end{pmatrix} f(X + d) - \begin{pmatrix} X \\ d \end{pmatrix} f(X).$$

If we suppose $P \neq 0_{\mathbb{R}[X]}$, we would have $f \neq 0_{\mathbb{R}[X]}$ and

$$\deg P = \deg \left( \begin{pmatrix} X \\ d \end{pmatrix} f(X) \right) - 1 = d + \deg f - 1 \geq d - 1,$$

which contradicts the fact that $P \in \mathbb{R}_{d-2}[X]$. Hence $P = 0_{\mathbb{R}[X]}$, as required. This confirms that $\mathcal{E}_d \cap \mathbb{R}_{d-2}[X] = \{0_{\mathbb{R}[X]}\}$.

- Now, let us show that $\mathcal{E}_d + \mathbb{R}_{d-2}[X] = \mathbb{R}[X]$. So, let $P \in \mathbb{R}[X]$ and let us show the existence of two polynomials $Q \in \mathcal{E}_d$ and $R \in \mathbb{R}_{d-2}[X]$ such that $P = Q + R$ (the other inclusion $\mathcal{E}_d + \mathbb{R}_{d-2}[X] \subset \mathbb{R}[X]$ is trivial). To do so, let us consider $P^* \in \mathbb{R}[X]$ such that:

$$P^*(X + 1) - P^*(X) = P(dX)$$

(such $P^*$ exists because $P(dX) \in \mathbb{R}[X]$). Let us then consider the euclidean division (in $\mathbb{R}[X]$) of $P^*$ by the polynomial $(dX)$ (which is of degree $d$):

$$P^*(X) = \begin{pmatrix} dX \\ d \end{pmatrix} q(X) + r(X), \quad (2.6)$$

where $q, r \in \mathbb{R}[X]$ and $\deg r \leq d - 1$. Using (2.6), we have that

$$P^*(X + 1) - P^*(X) = \begin{pmatrix} dX + d \\ d \end{pmatrix} q(X + 1) - \begin{pmatrix} dX \\ d \end{pmatrix} q(X) + r(X + 1) - r(X);$$
that is

\[ P(dX) = \left( \frac{dX + d}{d} \right) q(X + 1) - \left( \frac{dX}{d} \right) q(X) + r(X + 1) - r(X). \]

By substituting in this last equality \( X \) by \( \frac{X}{d} \), we get

\[ P(X) = \left( \frac{X + d}{d} \right) q \left( \frac{X + d}{d} \right) - \left( \frac{X}{d} \right) q \left( \frac{X}{d} \right) + r \left( \frac{X}{d} + 1 \right) - r \left( \frac{X}{d} \right). \]

So, it suffices to take

\[ Q(X) := \left( \frac{X + d}{d} \right) q \left( \frac{X + d}{d} \right) - \left( \frac{X}{d} \right) q \left( \frac{X}{d} \right) \]

and

\[ R(X) := r \left( \frac{X}{d} + 1 \right) - r \left( \frac{X}{d} \right) \]

to have \( P = Q + R \), with \( Q \in \mathcal{E}_d \) (according to Theorem 2.1) and \( R \in \mathbb{R}_{d-2}[X] \) (since \( \deg R = \deg r - 1 \leq d - 2 \)). Consequently, we have \( \mathbb{R}[X] = \mathcal{E}_d + \mathbb{R}_{d-2}[X] \), as required. This completes the proof of the corollary. \( \Box \)

From Corollary 2.3, we derive the following important corollary relating to the particular case \( d = 2 \).

**Corollary 2.4** There is a unique real sequence \( (c_n)_{n \in \mathbb{N}} \) such that:

\[ (X^n - c_n) \in \mathcal{E}_2 \quad (\forall n \in \mathbb{N}). \]

Besides, we have \( c_{2n} = 0 \) for any positive integer \( n \).

**Proof** By Corollary 2.3, we have

\[ \mathbb{R}[X] = \mathcal{E}_2 \oplus \mathbb{R}_0[X]. \]

Since \( \mathbb{R}_0[X] \) is constituted by constant polynomials, this shows in particular that for any \( n \in \mathbb{N} \), the monomial \( X^n \) can be written in a unique way as \( X^n = P_n(X) + c_n \), with \( P_n \in \mathcal{E}_2 \) and \( c_n \in \mathbb{R} \). In other words, for any \( n \in \mathbb{N} \), there is a unique \( c_n \in \mathbb{R} \) such that \( (X^n - c_n) \in \mathcal{E}_2 \). This confirms the first part of the corollary. Next, for any \( n \in \mathbb{N}^* \), the immediate formula:

\[ N^{2n} + (N - 2)^{2n} + (N - 4)^{2n} + \cdots = \frac{1}{2} \sum_{k=0}^{N} (N - 2k)^{2n} \quad (\forall N \in \mathbb{N}) \]

insures that \( X^{2n} \in \mathcal{E}_2 \); hence \( c_{2n} = 0 \). This confirms the second part of the corollary and achieves the proof. \( \Box \)

For the sequel, we let \( (c_n)_{n \in \mathbb{N}} \) denote the real sequence established by Corollary 2.4. The following theorem provides explicit expressions of the \( c_n \)'s \( (n \in \mathbb{N}) \), using the Euler polynomials and the Genocchi numbers.

**Theorem 2.5** For every natural number \( n \), we have

\[ c_n = E_n(0) = \frac{G_{n+1}}{n+1}. \]

**Proof** Let \( n \) be a fixed natural number. We exactly follow the second part of the proof of Corollary 2.3 (i.e., the proof of the fact that \( \mathbb{R}[X] = \mathcal{E}_d + \mathbb{R}_{d-2}[X] \)) by taking \( d = 2 \) and \( P(X) = X^n \). We thus need a polynomial \( P^* \in \mathbb{R}[X] \) such that:

\[ P^*(X + 1) - P^*(X) = P(2X) = (2X)^n = 2^n X^n. \]

According to Formula (1.1), we can take

\[ P^*(X) = \frac{2^n}{n+1} B_{n+1}(X). \]
Then, consider the euclidean division (in \(\mathbb{R}[X]\)) of this chosen polynomial \(P^*(X)\) by the polynomial \(\left(\frac{2X}{2}\right) = X(2X - 1)\):

\[P^*(X) = X(2X - 1)q(X) + r(X),\]

where \(q, r \in \mathbb{R}[X]\) and \(\deg r \leq 1\). Expressing \(r(X)\) as \(r(X) = aX + b\) \((a, b \in \mathbb{R})\), we get the polynomial identity:

\[
\frac{2^n}{n + 1}B_{n+1}(X) = X(2X - 1)q(X) + aX + b.
\]

(2.7)

According to the second part of the proof of Corollary 2.3, we have

\[c_n = R(X) := r \left(\frac{X}{2} + 1\right) - r \left(\frac{X}{2}\right) = a \left(\frac{X}{2} + 1\right) + b - \left(a \frac{X}{2} + b\right) = a.\]

So, we have to show that \(a = E_n(0) = \frac{G_{n+1}}{n+1}\). By taking in (2.7) successively \(X = 0\) and \(X = \frac{1}{2}\), we obtain that:

\[b = \frac{2^n}{n + 1}B_{n+1}(0) = \frac{2^n}{n + 1}B_{n+1},\]

and

\[
\frac{a}{2} + b = \frac{2^n}{n + 1}B_{n+1} \left(\frac{1}{2}\right)
\]

\[= \frac{2^n}{n + 1} \left(\frac{1}{2^n} - 1\right)B_{n+1} \quad \text{(according to (1.2))}
\]

\[= \frac{1 - 2^n}{n + 1}B_{n+1}.
\]

Thus

\[
a = 2 \left[\left(\frac{a}{2} + b\right) - b\right]
\]

\[= 2 \left[\frac{1 - 2^n}{n + 1}B_{n+1} - \frac{2^n}{n + 1}B_{n+1}\right]
\]

\[= 2 \left(1 - \frac{2^n}{n + 1}\right)B_{n+1}
\]

\[= E_n(0) = \frac{G_{n+1}}{n + 1} \quad \text{(according to (1.3))},
\]

as required. This completes the proof. \(\square\)

Remarks 2.6 1. Since, as known, \(G_n = 0\) for any odd integer \(n \geq 3\), we deduce from Theorem 2.5 the fact that \(c_n = 0\) for any even integer \(n \geq 2\), which is already announced by Corollary 2.4 (but proved in a different way).

2. Using the formula \(c_n = \frac{G_{n+1}}{n+1}\) (of Theorem 2.5) together with the fact that each Genocchi number \(G_n\) \((n \in \mathbb{N^*})\) is a multiple of the odd part of \(n\) (see e.g., [5, Corollary 2.2]), we easily show that the \(c_n\)’s are all rational numbers with denominators powers of 2. Precisely, for any \(n \in \mathbb{N}\), we can write \(c_n = \frac{a}{2^e}\), with \(a \in \mathbb{Z}\), \(e \in \mathbb{N}\) and \(e \leq \vartheta_2(n + 1)\).

The following table provides the values of the first numbers \(c_n\) \((n = 0, 1, \ldots, 15)\):

| \(n\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| \(c_n\) | 1 | -\(\frac{1}{2}\) | 0 | \(\frac{1}{3}\) | 0 | -\(\frac{1}{2}\) | 0 | \(\frac{17}{8}\) | 0 | -\(\frac{31}{7}\) | 0 | \(\frac{591}{4}\) | 0 | -\(\frac{5461}{2}\) | 0 | \(\frac{920569}{16}\) |

Now, we will derive from our main theorem 2.1 that the Ibn al-Banna method (exposed in §1) for expressing in a closed form a sum of the type \(P(n) + P(n - d) + P(n - 2d) + \ldots\), when \(d \geq 2\) is an integer and \(P \in \mathcal{D}_d\), is in fact general. Precisely, we will show that in such a context, it is always possible to express \(P\) in the form \(P(X) = \alpha(X) + \alpha(X - 1) + \cdots + \alpha(X - d + 1)\), where \(\alpha \in \mathbb{R}[X]\) and \(\alpha(-1) = \alpha(-2) = \cdots = \alpha(-d + 1) = 0\).
Theorem 2.7 (Generalizability of the Ibn al-Banna method) Let \( d \geq 2 \) be an integer. Then a real polynomial \( P \) belongs to \( \mathcal{D}_d \) if and only if there exists \( \alpha \in \mathbb{R}[X] \), satisfying \( \alpha(-1) = \alpha(-2) = \cdots = \alpha(-d + 1) = 0 \), such that:

\[
P(X) = \alpha(X) + \alpha(X - 1) + \cdots + \alpha(X - d + 1).
\]

Proof Let \( P \in \mathbb{R}[X] \) be fixed.

- Suppose that \( P \in \mathcal{D}_d \) and let us show that \( P \) is of the form required by the theorem. According to Theorem 2.1, there exists \( f \in \mathbb{R}[X] \) such that:

\[
P(X) = \left( \frac{X + d}{d} \right) f(X + d) - \left( \frac{X}{d} \right) f(X).
\]

Now, consider \( \alpha \in \mathbb{R}[X] \) defined by:

\[
\alpha(X) := \left( \frac{X + d}{d} \right) f(X + d) - \left( \frac{X + d - 1}{d} \right) f(X + d - 1).
\]

So, we have clearly

\[
\alpha(-1) = \alpha(-2) = \cdots = \alpha(-d + 1) = 0.
\]

Next, we have

\[
\alpha(X) + \alpha(X - 1) + \cdots + \alpha(X - d + 1) = \sum_{k=0}^{d-1} \alpha(X - k)
\]

\[
= \sum_{k=0}^{d-1} \left\{ \left( \frac{X - k + d}{d} \right) f(X - k + d) - \left( \frac{X - k + d - 1}{d} \right) f(X - k + d - 1) \right\}
\]

[telescopic sum]

\[
= \left( \frac{X + d}{d} \right) f(X + d) - \left( \frac{X}{d} \right) f(X)
\]

\[
= P(X).
\]

Therefore \( P \) has the form required by the theorem.

- Conversely, suppose that there is \( \alpha \in \mathbb{R}[X] \), satisfying \( \alpha(-1) = \alpha(-2) = \cdots = \alpha(-d + 1) = 0 \), such that:

\[
P(X) = \alpha(X) + \alpha(X - 1) + \cdots + \alpha(X - d + 1)
\]

and let us show that \( P \in \mathcal{D}_d \). For a given \( n \in \mathbb{N} \), denoting by \( r_n \) the remainder of the euclidean division of \( n \) by \( d \) (so \( r_n \in \{0, 1, \ldots, d - 1\} \)), we have

\[
P(n) + P(n - d) + P(n - 2d) + \ldots = P(n) + P(n - d) + P(n - 2d) + \cdots + P(r_n)
\]

\[
= \alpha(n) + \alpha(n - 1) + \cdots + \alpha(n - d + 1)
\]

\[
+ \alpha(n - d) + \alpha(n - d - 1) + \cdots + \alpha(n - 2d + 1)
\]

\[
+ \alpha(n - 2d) + \alpha(n - 2d - 1) + \cdots + \alpha(n - 3d + 1)
\]

\[
+ \cdots + \alpha(r_n) + \alpha(r_n - 1) + \cdots + \alpha(r_n - d + 1)
\]

\[
= \alpha(n) + \alpha(n - 1) + \cdots + \alpha(r_n - d + 1).
\]

But since \(-d + 1 \leq r_n - d + 1 \leq 0 \) and \( \alpha(k) = 0 \) for \( k = -1, -2, \ldots, -d + 1 \), it follows that:

\[
P(n) + P(n - d) + P(n - 2d) + \cdots = \alpha(n) + \alpha(n - 1) + \cdots + \alpha(0),
\]

which is polynomial in \( n \) (since \( \alpha \in \mathbb{R}[X] \)). This concludes that \( P \in \mathcal{D}_d \), as required. The proof of the theorem is complete. \( \square \)
Study of the spaces $\mathcal{E}_d$ through formal power series and generalization

We briefly present in what follows a new approach (using convolution products and formal power series) to studying the $\mathbb{R}$-vector space $\mathcal{E}_d$ ($d \in \mathbb{N}^*$) and expressing in a closed form the sums of the type:

$$P(n) + P(n - d) + P(n - 2d) + \ldots$$

($P \in \mathbb{R}[X], n \in \mathbb{N}$).

Then, we lean on the obtained result in order to generalize the spaces $\mathcal{E}_d$. The results concerning the generalized spaces in question, which are analog to the above results on the $\mathcal{E}_d$'s, are given without proofs.

By definition, the ordinary convolution product of two real sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ is the real sequence $(a \ast b)_{n \in \mathbb{N}}$, defined by:

$$(a \ast b)_n := \sum_{k=0}^{n} a_k b_{n-k} \quad (\forall n \in \mathbb{N}).$$

It is known that the convolution product constitutes a commutative and associative law of composition on the set of the real sequences. In addition, the Dirac delta sequence $\delta = (\delta_n)_{n \in \mathbb{N}}$, defined by: $\delta_0 = 1$ and $\delta_n = 0$ ($\forall n \geq 1$) is the neutral element of $\ast$. Finally, every real sequence $(a_n)_{n \in \mathbb{N}}$ with $a_0 \neq 0$ has an inverse element for $\ast$. Further, the ordinary generating function of a given real sequence $u = (u_n)_{n \in \mathbb{N}}$ is the formal power series defined by:

$$\sigma(u) := \sum_{n=0}^{+\infty} u_n X^n.$$

When $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ are two real sequences, we have the fundamental formula:

$$\sigma(a) \cdot \sigma(b) = \sigma(a \ast b).$$

Next, for a positive integer $d$, let $\delta^{(d)} = (\delta^{(d)}_n)_{n \in \mathbb{N}}$ denote the real sequence defined by:

$$\delta^{(d)}_n := \begin{cases} 1 & \text{if } n \text{ is a multiple of } d \\ 0 & \text{otherwise} \end{cases} \quad (\forall n \in \mathbb{N}).$$

(Notice that $\delta^{(1)} \equiv 1$ and $\delta^{(+\infty)} = \delta$). So, the ordinary generating function of $\delta^{(d)}$ ($d \in \mathbb{N}^*$) is given by:

$$\sigma(\delta^{(d)}) := \sum_{n=0}^{+\infty} \delta^{(d)}_n X^n = \sum_{n \in \mathbb{N}} X^n \sum_{k=0}^{+\infty} X^{dk};$$

that is

$$\sigma(\delta^{(d)}) = \frac{1}{1 - X^d}. \quad (2.8)$$

For simplicity, for $P \in \mathbb{R}[X]$ and $d \in \mathbb{N}^*$, we let $S_{P,d}$ denote the real sequence of general term $S_{P,d}(n)$ ($n \in \mathbb{N}$) defined by:

$$S_{P,d}(n) := P(n) + P(n - d) + P(n - 2d) + \ldots$$

(recall that the sum on the right stops at $P(r)$, where $r$ is the remainder of the euclidean division of $n$ by $d$). So, given $d$ a positive integer and $P$ a real polynomial, we have for any natural number $n$:

$$S_{P,d}(n) := P(n) + P(n - d) + P(n - 2d) + \ldots = \sum_{k=0}^{n} \delta^{(d)}_k P(n - k) = \left(\delta^{(d)} \ast P\right)(n);$$

hence the fundamental formula of this approach:

$$S_{P,d} = \delta^{(d)} \ast P. \quad (2.9)$$

We also need for this approach the following well-known theorem which characterizes the ordinary generating functions of polynomial sequences.
Theorem 1 ([8, Corollary 4.3.1]) Let \((a_n)_{n \in \mathbb{N}}\) be a non-zero real sequence. Then \((a_n)_{n \in \mathbb{N}}\) is polynomial in \(n\) if and only if its ordinary generating function is a rational function of the form \(\frac{U(X)}{(1-X)^{\alpha}}\), with \(\alpha \in \mathbb{N}^*, U \in \mathbb{R}[X]\), \(\deg U < \alpha\) and \(U(1) \neq 0\). In addition, in the situation where \(\alpha (a_n) = \frac{U(X)}{(1-X)^{\alpha}}\) (with the above conditions on \(U\) and \(\alpha\)), the degree of \(a_n\) (as a polynomial in \(n\)) is equal to \((\alpha - 1)\).

According to Theorem 1, it is associated to every polynomial \(P \in \mathbb{R}[X]\) a unique polynomial \(A_P \in \mathbb{R}[X]\) satisfying the identity of formal power series:

\[
\sum_{n=0}^{+\infty} P(n)X^n = \frac{A_P(X)}{(1-X)^{\deg P + 1}}.
\] (2.10)

If, in addition, \(P \neq 0_{\mathbb{R}[X]}\) then we have \(A_P \neq 0_{\mathbb{R}[X]}\), \(A_P(1) \neq 0\) and \(\deg A_P \leq \deg P\). The correspondence \(P \mapsto A_P\) provides a simple characterization of the property of belonging to a certain space \(E_d\) for a given real polynomial. This characterization inspires us a generalization of the \(\mathbb{R}\)-vector spaces \(E_d\). First, we have the following proposition:

Proposition 2.8 Let \(d\) be a positive integer and \(P\) be a real polynomial. Then \(P\) belongs to \(E_d\) if and only if its ordinary generating function is a rational function of the form \(U(X)\) and \(U(1) = 0\).

Proof The result of the proposition is obvious for \(P = 0_{\mathbb{R}[X]}\). Suppose for the sequel of this proof that \(P \neq 0_{\mathbb{R}[X]}\) and set \(s := \deg P \in \mathbb{N}\). By using successively Formulas (2.9), (2.8) and (2.10), we get

\[
\sigma(S_{P,d}) = \sigma(\delta^{(d)} * P) = \sigma(\delta^{(d)}) \cdot \sigma(P) = \frac{1}{1 - X^d} \sum_{n=0}^{+\infty} P(n)X^n = \frac{1}{(1 - X)(1 + X + X^2 + \cdots + X^{d-1})} \cdot \frac{A_P(X)}{(1 - X)^{s+1}} = \frac{A_P(X)}{(1 + X + X^2 + \cdots + X^{d-1})(1 - X)^{s+2}}.
\] (2.11)

• Suppose that \(A_P\) is a multiple (in \(\mathbb{R}[X]\)) of the polynomial \((1 + X + X^2 + \cdots + X^{d-1})\) and let \(U(X) := \frac{A_P(X)}{1 + X + X^2 + \cdots + X^{d-1}} \in \mathbb{R}[X]\). Then \(\deg U = \deg A_P - (d - 1) \leq \deg P = s\) and \(U(1) = \frac{A_P(1)}{d} \neq 0\).

Next, Formula (2.11) becomes: \(\sigma(S_{P,d}) = \frac{U(X)}{(1 - X)^{s+2}}\), which implies (according to Theorem 1) that \(S_{P,d}(n)\) is polynomial in \(n\); that is \(P \in E_d\).

• Conversely, suppose that \(P \in E_d\); that is \(S_{P,d}(n)\) is polynomial in \(n\). By Theorem 1, the power series \(\sigma(S_{P,d})\) has the form:

\[
\sigma(S_{P,d}) = \frac{U(X)}{(1 - X)^\alpha},
\]

where \(\alpha \in \mathbb{N}^*, U \in \mathbb{R}[X]\), \(\deg U < \alpha\) and \(U(1) \neq 0\). It follows (according to (2.11)) that:

\[
\frac{U(X)}{(1 - X)^\alpha} = \frac{A_P(X)}{(1 + X + X^2 + \cdots + X^{d-1})(1 - X)^{s+2}}.
\]

Thus

\[
\left(1 + X + X^2 + \cdots + X^{d-1}\right)(1 - X)^{s+2} U(X) = (1 - X)^\alpha A_P(X).
\]

This shows in particular that \((1 + X + X^2 + \cdots + X^{d-1}) \mid (1 - X)^\alpha A_P(X)\). But since \((1 + X + X^2 + \cdots + X^{d-1})\) is coprime with \((1 - X)^\alpha\) (because \((1 + X + X^2 + \cdots + X^{d-1})\) does not vanish at \(X = 1\)), it follows (according to Gauss’s lemma) that \((1 + X + X^2 + \cdots + X^{d-1}) \mid A_P(X)\), as required. This completes the proof of the proposition. \(\square\)
By carefully observing the proof of Proposition 2.8, we see that the polynomial $(1 + X + X^2 + \cdots + X^{d-1})$ which appears there has no particularity apart from the fact that it does not vanish at $X = 1$. This inspires us a generalization of the spaces $\mathcal{E}_d$ in the following way.

**Definition 2.9** (Generalization of the spaces $\mathcal{E}_d$) Let $D \in \mathbb{R}[X]$ such that $D(1) \neq 0$. We define $\mathcal{E}_D$ as the set of all polynomials $P \in \mathbb{R}[X]$ for which the polynomial $A_P$ is a multiple of $D$ (in $\mathbb{R}[X]$).

Notice that for all positive integer $d$, we have (according to Proposition 2.8)

$$\mathcal{E}_d = \mathcal{E}_{(1+X+X^2+\cdots+X^{d-1})}.$$ 

Further, we show quite easily that for all $D \in \mathbb{R}[X]$, with $D(1) \neq 1$, the set $\mathcal{E}_D$ constitutes a $\mathbb{R}$-linear subspace of $\mathbb{R}[X]$. A basis of $\mathcal{E}_D$ (as a $\mathbb{R}$-vector space) is specified by the following theorem, the proof of which (left to the reader’s imagination) mainly uses Theorem 1.

**Theorem 2.10** Let $d$ be a positive integer and $D$ be a real polynomial of degree $(d-1)$ such that $D(1) \neq 0$. Let $h \in \mathbb{R}[X]$ defined by the formal power series identity:

$$\frac{D(X)}{(1-X)^a} = \sum_{n=0}^{+\infty} h(n) X^n.$$ 

Then, the family of polynomials $(h, h \ast 1, h \ast X, h \ast X^2, \ldots)$ constitutes a basis for the $\mathbb{R}$-vector space $\mathcal{E}_D$. □

**Remark 2.11** We can derive our main theorem 2.1 on the spaces $\mathcal{E}_d$ from the general theorem 2.10 as follows.

- First, show that we have for all positive integer $d$:

$$\frac{1 + X + X^2 + \cdots + X^{d-1}}{(1-X)^d} = \sum_{n=0}^{+\infty} \left( \binom{n+d}{d} - \binom{n}{d} \right) X^n.$$ 

- (Less easy!). Show the following statement:

For all positive integer $d$ and all polynomial $f \in \mathbb{R}[X]$, there exists a unique polynomial $g \in \mathbb{R}[X]$, which vanishes at $(-1)$, such that:

$$\left( \binom{n+d}{d} - \binom{n}{d} \right) \ast f(n) = \left( \binom{n+d}{d} \right) g(n+d) - \left( \binom{n}{d} \right) g(n).$$

In addition, the correspondence $f \mapsto g$ from $\mathbb{R}[X]$ to $\{ g \in \mathbb{R}[X] : g(-1) = 0 \}$ is bijective.

We finally derive from Theorem 2.10 the below corollary, generalizing Corollary 2.3.

**Corollary 2.12** In the context of Theorem 2.10, we have

$$\mathcal{E}_D \oplus \mathbb{R}_{d-2}[X] = \mathbb{R}[X].$$

In particular, we have

$$\text{codim } \mathbb{R}[X] \mathcal{E}_D = d - 1.$$ (□)

**References**

1. M. Aballagh. Raf al-hijab d’Ibn al-Banna: édition critique, traduction, étude philosophique et analyse mathématique. Thèse de l’université Paris I, 1988.
2. J. Bernoulli. Ars Conjectandi, Thurneysen Brothers, Bâle, 1713.
3. L. Comtet. Advanced Combinatorics: The Art of Finite and Infinite Expansions, revised and enlarged ed., D. Reidel Publ. Co., Dordrecht, 1974.
4. B. Farhi. A new generalization of the Genocchi numbers and its consequence on the Bernoulli polynomials, preprint, 2020. Available at arXiv:2012.01969.
5. B. Farhi. Arithmetic properties of the Genocchi numbers and their generalization, preprint, 2021. Available at arXiv:2103.07890.
6. A. Genocchi. Intorno all’espressioni generali di numeri Bernoulliani, Annali di scienze mat. e fisiche, compilati da Barnaba Tortolini, 3 (1852), p. 395-405.
7. N. Nielsen. Traité élémentaire des nombres de Bernoulli, Gauthier–Villars, Paris, 1923.
8. R.P. Stanley. Enumerative combinatorics, vol. 1, 2nd edition, Cambridge Studies in Advanced Mathematics 49, Cambridge Univ. Press, Cambridge, 2011.