Berge’s Maximum Theorem for Noncompact Image Sets

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Abstract

This note generalizes Berge’s maximum theorem to noncompact image sets. It is also clarifies the results from E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, “Berge’s theorem for noncompact image sets,” J. Math. Anal. Appl. 397(1)(2013), pp. 255–259 on the extension to noncompact image sets of another Berge’s theorem, that states semi-continuity of value functions. Here we explain that the notion of a $K$-inf-compact function introduced there is applicable to metrizable topological spaces and to more general compactly generated topological spaces. For Hausdorff topological spaces we introduce the notion of a $KN$-inf-compact function ($N$ stands for “nets” in $K$-inf-compactness), which coincides with $K$-inf-compactness for compactly generated and, in particular, for metrizable topological spaces.

1 Introduction

Let $X$ and $Y$ be Hausdorff topological spaces, $u : X \times Y \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ is the family of all nonempty subsets of the set $Y$. Consider a value function

$$v(x) := \inf_{y \in \Phi(x)} u(x, y), \quad x \in X. \quad (1.1)$$

A set-valued mapping $F : X \to 2^Y$ is upper semi-continuous at $x \in X$ if, for any neighborhood $G$ of the set $F(x)$, there is a neighborhood of $x$, say $O(x)$, such that $F(y) \subseteq G$ for all $y \in O(x)$; a set-valued mapping $F : X \to 2^Y$ is lower semi-continuous at $x \in X$ if, for any open set $G$ with $F(x) \cap G \neq \emptyset$, there is a neighborhood of $x$, say $O(x)$, such that if $y \in O(x)$, then $F(y) \cap G \neq \emptyset$. A set-valued mapping is called upper (lower) semi-continuous, if it is upper (lower) semi-continuous at all $x \in X$. A set-valued mapping is called continuous, if it is upper and lower semi-continuous. For a topological space $U$, we denote by $K(U)$ the family of all nonempty compact subsets of $U$.

For Hausdorff topological spaces, Berge’s well-known maximum theorem (cf. Berge [3, p. 116]) has the following formulation.

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**Berge’s Maximum Theorem.** (Hu and Papageorgiou [4, p. 84]) If \( u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R} \) is a continuous function and \( \Phi : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y}) \) is a continuous set-valued mapping, then the value function \( v : \mathbb{X} \rightarrow \mathbb{R} \) is continuous and the solution multifunction \( \Phi^* : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y}) \), defined as

\[
\Phi^*(x) = \{ y \in \Phi(x) : v(x) = u(x, y) \}, \quad x \in \mathbb{X},
\]

is upper semi-continuous and compact-valued.

This paper extends Berge’s theorem to possibly noncompact sets \( \Phi(x), x \in \mathbb{X} \). For a numerical function \( f \), defined on a nonempty subset \( \mathbb{U} \) of a topological space \( \mathbb{U} \), consider the level sets

\[
\mathcal{D}_f(\lambda; \mathbb{U}) = \{ y \in \mathbb{U} : f(y) \leq \lambda \}, \quad \lambda \in \mathbb{R}.
\]

We recall that a function \( f \) is lower semi-continuous on \( \mathbb{U} \) if all the level sets \( \mathcal{D}_f(\lambda; \mathbb{U}) \) are closed, and a function \( f \) is inf-compact (also sometimes called lower semi-compact) on \( \mathbb{U} \) if all these sets are compact.

For \( \mathbb{Z} \subseteq \mathbb{X} \), let

\[
\text{Gr}_\mathbb{Z}(\Phi) = \{ (x, y) \in \mathbb{X} \times \mathbb{Y} : y \in \Phi(x) \}
\]

**Definition 1.1.** (Feinberg et al. [6, Definition 1.1]) A function \( u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R} \) is called \( \mathbb{K} \)-inf-compact on \( \text{Gr}_\mathbb{X}(\Phi) \), if for every \( K \in \mathbb{K}(\mathbb{X}) \) this function is inf-compact on \( \text{Gr}_K(\Phi) \).

In many applications the space \( \mathbb{X} \) is compactly generated. Recall that a topological space \( \mathbb{X} \) is compactly generated (Munkres [13, p. 283] or a \( k \)-space, Kelley [10, p. 230], Engelking [4, p. 152]) if it satisfies the following property: each set \( A \subseteq \mathbb{X} \) is closed in \( \mathbb{X} \) if \( A \cap K \) is closed in \( K \) for each \( K \in \mathbb{K}(\mathbb{X}) \). In particular, all locally compact spaces (hence, manifolds) and all sequential spaces (hence, first-countable, including metrizable/metric spaces) are compactly generated; see Munkres [13, Lemma 46.3, p. 283], Engelking [4, Theorem 3.3.20, p. 152].

The following theorem and its generalization for Hausdorff topological spaces, Theorem 1.4, are the main results of this paper.

**Theorem 1.2.** Assume that:

(a) \( \mathbb{X} \) is a compactly generated topological space;

(b) \( \Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y}) \) is lower semi-continuous;

(c) \( u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R} \) is \( \mathbb{K} \)-inf-compact and upper semi-continuous on \( \text{Gr}_\mathbb{X}(\Phi) \).

Then the value function \( v : \mathbb{X} \rightarrow \mathbb{R} \) is continuous and the solution multifunction \( \Phi^* : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y}) \) is upper semi-continuous and compact-valued.

When \( \mathbb{X} \) is a compactly generated topological space, Theorem 1.2 generalizes Berge’s Maximum Theorem because, if \( \Phi \) is a compact-valued and upper semi-continuous mapping, then a lower semi-continuous function \( u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R} \) is \( \mathbb{K} \)-inf-compact on \( \text{Gr}_\mathbb{X}(\Phi) \); Feinberg et al. [6, Lemma 2.1(i)] or its generalization, Lemma 3.3(i). Note that a more particular result than Theorem 1.2 is formulated in Feinberg et al. [6, Theorem 4.1] for Hausdorff topological spaces, where upper semi-continuity of the solution multifunction
Φ* : X → K(Y) is stated for a continuous set-valued mapping Φ : X → S(Y) and for a continuous function u. However, when the topological space X is Hausdorff, it is necessary to consider a more restrictive assumption for u(·, ·) than K-inf-compactness, because K-inf-compactness of u on Gr(Φ) is not sufficient for lower semi-continuity of the value function v on X; see Example 5.1. This assumption (we call it KKN-inf-compactness) is a generalization of the K-inf-compactness property in the way it is formulated in Feinberg et al. [5] as Assumption (W∗)(ii) for metric spaces.

**Definition 1.3.** A function u : X × Y → R is called KKN-inf-compact on GrX(Φ), if the following two conditions hold:

(i) u(·, ·) is lower semi-continuous on GrX(Φ);

(ii) for any convergent net {x_i}_i∈I with values in X whose limit x belongs to X, any net {y_i}_i∈I defined on the same ordered set I with y_i ∈ Φ(x_i), i ∈ I, and satisfying the condition that the set {u(x_i, y_i) : i ∈ I} is bounded above, has an accumulation point y ∈ Φ(x).

As proved below, K-inf-compactness and KKN-inf-compactness are equivalent if X is a compactly generated topological space (Corollary 2.2). All the statements in Feinberg et al. [6] are formulated for K-inf-compact functions u. However, the statements of Feinberg et al. [6, Theorems 1.2, 3.1(a), 4.1 and Lemma 2.3] require slightly stronger assumptions if we want them to hold for a Hausdorff topological space X. Indeed, the proofs in [6] Theorems 1.2, 3.1(a), 4.1 and Lemma 2.3] rely on the claim that, for any convergent net {x_α} in a Hausdorff topological space with a limit x, the set (∪_α {x_α}) ∪ {x} is a compact set. This is true for converging sequences, but not necessarily for converging nets (cf. Example 5.1). Restricting attention to compactly generated spaces, which are more general objects than metric spaces, makes all the results in [6] valid and makes it possible to formulate Berge’s maximum theorem for K-inf-compact functions and noncompact image sets; see Theorem 1.2. For general Hausdorff topological spaces, the KKN-inf-compactness assumption is needed; see Theorems 1.4, 3.4, 3.5 and Proposition 3.6.

When X is a Hausdorff topological space, the following theorem is analogous to Theorem 1.2. In view of Lemma 3.3(i), Theorem 1.4 generalizes Berge’s Maximum Theorem to possibly noncompact sets Φ(x), x ∈ X.

**Theorem 1.4.** If a function u : X × Y → R is K-inf-compact and upper semi-continuous on GrX(Φ) and Φ : X → S(Y) is a lower semi-continuous set-valued mapping, then the value function v : X → R is continuous and the solution multifunction Φ* : X → K(Y) is upper semi-continuous and compact-valued.

The papers that extend Berge’s theorems to different directions include Ausubel and Deneckere [1], Horsley et al. [7], Leininger [11], Montes-de-Oca and Lemus-Rodríguez [12], and Walker [14]. Relations with provided above theorems are mostly superficial; most papers impose stronger topological restrictions [7], assume compact-valuedness of the feasibility multifunction Φ [11, 14], or impose restrictions on the value function v rather than on the primitives of the model [11]. None of these papers appeals to compactly generated spaces and/or K/KN-inf-compactness. Only the recent paper by Montes-de-Oca and Lemus-Rodríguez [12] contains results relevant to this paper. They restrict attention to metric spaces and use inf-compactness, rather than generalizations provided above, in addition to other restrictions to derive special
cases of presented results on the value function $v$ and solution multifunction $\Phi^*$. In particular, the main results of \cite{12}, Theorems 3.1 and 4.1, are corollaries of Theorems \ref{thm1} and \ref{thm2} above, as well as of \cite{6} Theorem 4.1] applied to metric spaces.

2 Classification of Inf-Compactness Properties

In this section we study the relation between $\mathbb{K}N$-inf-compactness and $\mathbb{K}$-inf-compactness.

Theorem 2.1. The following statements hold:

(i) if $u : X \times Y \to \overline{\mathbb{R}}$ is $\mathbb{K}N$-inf-compact on $\text{Gr}_X(\Phi)$, then the function $u(\cdot, \cdot)$ is $\mathbb{K}$-inf-compact on $\text{Gr}_X(\Phi)$;

(ii) if $u : X \times Y \to \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\text{Gr}_X(\Phi)$ and $X$ is a compactly generated topological space, then the function $u(\cdot, \cdot)$ is $\mathbb{K}N$-inf-compact on $\text{Gr}_X(\Phi)$.

Theorem 2.1 implies the following statement.

Corollary 2.2. If $X$ is a compactly generated topological space, then a function $u : X \times Y \to \overline{\mathbb{R}}$ is $\mathbb{K}N$-inf-compact on $\text{Gr}_X(\Phi)$ if and only if it is $\mathbb{K}$-inf-compact on $\text{Gr}_X(\Phi)$.

Before the proof of Theorem 2.1 we describe auxiliary properties of set-valued mappings.

2.1 Properties of Set-Valued Mappings

This subsection introduces $\mathbb{K}$-upper semi-compact and $\mathbb{K}N$-upper semi-compact set-valued mappings and relates these two objects to each other and to upper semi-continuous set-valued mappings.

Definition 2.3. A set-valued mapping $\Psi : X \to \mathcal{S}(Y)$ is $\mathbb{K}$-upper semi-compact, if for every $K \in \mathbb{K}(X)$ the set $\text{Gr}_K(\Psi)$ is compact.

Definition 2.4. The mapping $\Psi : X \to \mathcal{S}(Y)$ is $\mathbb{K}N$-upper semi-compact if for any convergent net $\{x_i\}_{i \in I}$ with values in $X$, whose limit $x$ belongs to $X$, any net $\{y_i\}_{i \in I}$, defined on the same ordered set $I$ with $y_i \in \Psi(x_i)$, $i \in I$, has an accumulation point $y \in \Psi(x)$.

Theorem 2.5. The following statements hold:

(i) a set-valued mapping $\Psi : X \to \mathcal{S}(Y)$ is $\mathbb{K}N$-upper semi-compact if and only if it is upper semi-continuous and compact-valued;

(ii) a $\mathbb{K}N$-upper semi-compact set-valued mapping $\Psi : X \to \mathcal{S}(Y)$ is $\mathbb{K}$-upper semi-compact;

(iii) if $X$ is a compactly generated topological space and $\Psi : X \to \mathcal{S}(Y)$ is a $\mathbb{K}$-upper semi-compact set-valued mapping, then $\Psi$ is compact-valued and upper semi-continuous.
Proof. (i) Let $\Psi : X \rightarrow S(Y)$ be a $Kn$-upper semi-compact set-valued mapping. Restricting attention to constant nets $\{x_i\}_{i \in I}$, it follows that $\Psi$ takes compact values. Let us prove that $\Psi$ is upper semi-continuous. Suppose, to the contrary, that $\Psi$ is not upper semi-continuous at some point $x_0 \in X$. Then there is an open neighborhood $V$ of $\Psi(x_0)$ such that for every neighborhood $U$ of $x_0$, there is an $x_U \in U$ with $\Psi(x_U) \not\subseteq V$. In particular, we can select a $y_U \in \Psi(x_U) \setminus V$. Now consider the nets $\{x_U : U \in I\}$ and $\{y_U : U \in I\}$, where $I$ is the directed set of neighborhoods of $x_0$. The net $\{x_U\}$ converges to $x_0$. Since $\Psi$ is $Kn$-upper semi-compact, the net $\{y_U\}$ has an accumulation point $y_0 \in \Psi(x_0) \subseteq V$. The net $\{y_U\}$ lies in the closed set $V^c$, which is the complement of $V$, and therefore $y_0 \in V^c$. This contradiction implies that $\Psi : X \rightarrow S(Y)$ is upper semi-continuous.

Vice versa, let $\Psi : X \rightarrow S(Y)$ be upper semi-continuous, let $\{x_i\}_{i \in I}$ be a convergent net with values in $X$ whose limit $x$ belongs to $X$ and $\{y_i\}_{i \in I}$ be a net defined on the same ordered set $I$ with $y_i \in \Psi(x_i)$, $i \in I$. Aliprantis and Border [2, Corollary 17.17, p. 564] yields that the net $\{y_i\}_{i \in I}$ has an accumulation point $y \in \Phi(x)$. Therefore, the function $u(\cdot, \cdot)$ is $Kn$-inf-compact on $Gr_X(\Phi)$.

(ii) Let $\Psi : X \rightarrow S(Y)$ be a $Kn$-upper semi-compact set-valued mapping. This mapping is $K$-upper semi-compact, because its restriction to any compact set $K$ of $X$ is $Kn$-upper semi-compact and its graph $Gr_X(\Psi)$ is compact by virtue of characterizations of compactness via nets.

(iii) Since $\Psi : X \rightarrow S(Y)$ is $K$-upper semi-compact, it is compact-valued. We prove that $\Psi : X \rightarrow S(Y)$ is upper semi-continuous. Recall (Aliprantis and Border [2, Lemma 17.4, p. 559]) that $\Psi$ is upper semi-continuous if, for each closed subset $F$ of $Y$, the set

$$\{ x \in X : \Psi(x) \cap F \neq \emptyset \}$$

is closed. Since $X$ is compactly generated, it suffices to show for each compact $K \subseteq X$ that $\Psi|_K$, the restriction of $\Psi : X \rightarrow S(Y)$ to the domain $K$, is upper semi-continuous: for each closed subset $F$ of $Y$,

$$\{ x \in K : \Psi|_K(x) \cap F \neq \emptyset \} = \{ x \in X : \Psi(x) \cap F \neq \emptyset \} \cap K$$

is closed and consequently, that the set in (2.1) is closed.

So let $K \in K(X)$. Since $\Psi|_K$ is compact-valued, its upper semi-continuity follows from compactness of $Gr_X(\Psi)$, that is, for every net $(x_\alpha, y_\alpha)$ in $Gr_X(\Psi)$ with $x_\alpha \rightarrow x$ for some $x \in K$, that net $(y_\alpha)$ has a limit point in $\Psi|_K(x)$, that is, a convergent subnet with limit $y \in \Psi|_K(x)$; see Aliprantis and Border [2, Corollary 17.17, p. 564]. 

Theorem 2.5 implies the following statement.

Corollary 2.6. Let $X$ be a compactly generated topological space. A set-valued mapping $\Psi : X \rightarrow S(Y)$ is $K$-upper semi-compact if and only if it is $Kn$-upper semi-compact.

2.2 Proof of Theorem 2.1

Proof of Theorem 2.1 (i) Let $u : X \times Y \rightarrow R$ be $Kn$-inf-compact on $Gr_X(\Phi)$, $K \in K(X)$, and $\lambda \in R$. Prove that the level set $D_{u(\cdot, \cdot)}(\lambda; Gr_K(\Phi))$ is compact, that is, any net $\{(x_i, y_i)\}_{i \in I}$ with values in
\(D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi))\) has an accumulation point \((x, y) \in D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi))\). Indeed, condition (ii) of Definition 1.3 and compactness of \(K\) implies that a net \(\{(x_i, y_i)\}_{i \in I} \subset D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi))\) has an accumulation point \((x, y) \in \text{Gr}_X(\Phi)\). Condition (i) of Definition 1.3 yields that the set \(D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi))\) is closed, that is, \((x, y) \in D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi))\). Therefore, the function \(u(\cdot, \cdot)\) is \(K\)-inf-compact on \(\text{Gr}_X(\Phi)\).

(ii) Let \(\mathcal{X}\) be a compactly generated topological space and \(u : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}\) be \(K\)-inf-compact on \(\text{Gr}_X(\Phi)\).

Fix an arbitrary \(\lambda \in \mathbb{R}\). According to the definition of \(\mathcal{K}\mathcal{N}\)-inf-compactness, it is sufficient to prove that: (a) the set \(D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_X(\Phi))\) is closed, and (b) for any convergent net \(\{x_i\}_{i \in I}\) with values in \(\mathcal{X}\) whose limit \(x\) belongs to \(\mathcal{X}\), any net \(\{y_i\}_{i \in I}\), defined on the same ordered set \(I\) with \(y_i \in \Phi(x_i), i \in I\), and satisfying the condition that the set \(\{u(x_i, y_i) : i \in I\}\) is bounded above by \(\lambda\), has an accumulation point \(y \in \Phi(x)\).

Set \(\mathcal{Y} := \mathcal{Y} \cup \{a\}\), where \(a\) is a subset of \(\mathcal{Y}\) such that \(a \notin \mathcal{Y}\) (such set exists according to Cantor’s theorem). A subset \(\mathcal{O} \subseteq \mathcal{Y}\) is called open in \(\mathcal{Y}\), if \(\mathcal{O} \setminus \{a\}\) is open in \(\mathcal{Y}\). Note that the point \(a\) is isolated and a set \(\mathcal{K} \subseteq \mathcal{Y}\) is open (closed, compact) in \(\mathcal{Y}\) if and only if the set \(\mathcal{K} \setminus \{a\}\) is open (closed, compact respectively) in \(\mathcal{Y}\). Therefore, the topological space \(\mathcal{Y}\), endowed with such topology of open subsets \(\mathcal{O}\), is Hausdorff.

According to Theorem 2.5 the set-valued mapping \(\Phi_\lambda : \mathcal{X} \to \mathcal{S}(\mathcal{Y})\),

\[\Phi_\lambda(x) := \{y \in \Phi(x) : u(x, y) \leq \lambda\} \cup \{a\}, \quad x \in \mathcal{X},\]

is compact-valued, upper semi-continuous, \(\mathcal{K}\)-upper semi-compact, and \(\mathcal{K}\mathcal{N}\)-upper semi-compact, because the topological space \(\mathcal{X}\) is compactly generated and for every \(\lambda \in \mathcal{K}(\mathcal{X})\) the set

\[\text{Gr}_K(\Phi_\lambda) = D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_K(\Phi)) \cup (K \times \{a\})\]

is compact in \(\mathcal{X} \times \mathcal{Y}\).

Since \(\Phi_\lambda : \mathcal{X} \to \mathcal{K}(\mathcal{Y})\) is upper semi-continuous, the set \(\text{Gr}_\mathcal{X}(\Phi_\lambda)\) is closed in \(\mathcal{X} \times \mathcal{Y}\). Therefore, the set \(D_{u(\cdot, \cdot)}(\lambda; \text{Gr}_\mathcal{X}(\Phi)) = \text{Gr}_\mathcal{X}(\Phi_\lambda) \setminus (\mathcal{X} \times \{a\})\) is closed in \(\mathcal{X} \times \mathcal{Y}\) and in \(\mathcal{X} \times \mathcal{Y}\), because the set \(\mathcal{X} \times \{a\}\) is open and closed simultaneously in \(\mathcal{X} \times \mathcal{Y}\) and \(\text{Gr}_\mathcal{X}(\Phi_\lambda) \cap (\mathcal{X} \times \{a\}) = \emptyset\). Statement (a) is proved.

Statement (b) follows from \(\mathcal{K}\mathcal{N}\)-upper semi-compactness of \(\Phi_\lambda : \mathcal{X} \to \mathcal{K}(\mathcal{Y})\). Indeed, if \(\{x_i\}_{i \in I}\) is a convergent net with values in \(\mathcal{X}\) whose limit \(x\) belongs to \(\mathcal{X}\), and \(\{y_i\}_{i \in I}\) is a net defined on the same ordered set \(I\) with \(y_i \in \Phi(x_i), i \in I\), and satisfying the condition that the set \(\{u(x_i, y_i) : i \in I\}\) is bounded above by \(\lambda\), then this net has an accumulation point \(y \in \Phi(x)\), because \(y_i \neq a\) for any \(i \in I\) and \(a\) is an isolated point in \(\mathcal{Y}\). Statement (b) is proved. Since statements (a) and (b) hold for any real \(\lambda\), the function \(u\) is \(\mathcal{K}\mathcal{N}\)-inf-compact on \(\text{Gr}_\mathcal{X}(\Phi)\).

\[\Box\]

3 Properties of \(\mathcal{K}\)-Inf-Compact and \(\mathcal{K}\mathcal{N}\)-Inf-Compact Functions

The following theorems state some properties of \(\mathcal{K}\)-inf-compact functions.

**Theorem 3.1.** If a function \(u : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}\) is \(\mathcal{K}\)-inf-compact on \(\text{Gr}_\mathcal{X}(\Phi)\) then:

(a) For each \(x \in \mathcal{X}\), the set \(\Phi^*(x)\) is nonempty.

(b) If \(v(x) = +\infty\), then \(\Phi^*(x) = \Phi(x)\). If \(v(x) < +\infty\), then \(\Phi^*(x)\) is compact.
(c) For each $K \in \mathbb{K}(\mathbb{X})$, the restriction $v|_K : K \to \mathbb{R}$ is lower semi-continuous.

(d) If $\mathbb{X}$ is compactly generated, the function $v : \mathbb{X} \to \overline{\mathbb{R}}$ is lower semi-continuous.

Proof. \(\Box\). Let $x \in \mathbb{X}$. If $v(x) = \inf_{y \in \Phi(x)} u(x, y) = +\infty$, then $u(x, y) = +\infty$ for all $y \in \Phi(x)$. Hence $\Phi^*(x) = \Phi(x)$ is nonempty. Next, let $v(x) \in \mathbb{R} \cup \{-\infty\}$ and let $\lambda \in (v(x), +\infty)$. Then $\lambda$ belongs to $\mathbb{R}$ and $D_{u,(\cdot)}(\lambda; \text{Gr}_x(\Phi))$ is nonempty and compact. The former follows from $v(x) < \lambda$, and the latter follows from $\mathbb{K}$-inf-compactness of $u$ applied to the compact set $K = \{x\}$. So, for $\lambda \in (v(x), +\infty)$, the level sets $D_{u,(\cdot)}(\lambda; \text{Gr}_x(\Phi))$ are nonempty, compact, and have the finite intersection property. Hence, their intersection

$$\cap_{\lambda \in (v(x), +\infty)} D_{u,(\cdot)}(\lambda, \text{Gr}_x(\Phi)) = \{(x, y) \in \text{Gr}_x(\Phi) : u(x, y) \leq v(x)\} = \{x\} \times \Phi^*(x)$$

is nonempty and compact. A fortiori, the projection $\Phi^*(x)$ onto $\mathbb{Y}$ is nonempty and compact.

(\(\Box\)). Let $K \in \mathbb{K}(\mathbb{X})$ and $\lambda \in \mathbb{R}$. To show that $D_{u,(\cdot)}(\lambda; K) = \{x \in K : v(x) \leq \lambda\}$ is closed, consider a convergent net $x_\alpha \to x$ in $K$ with $v(x_\alpha) \leq \lambda$ for all $\alpha$. By (\(\Box\)), there exists, for each $\alpha$, some $y_\alpha \in \Phi(x_\alpha)$ with $u(x_\alpha, y_\alpha) = v(x_\alpha) \leq \lambda$. So the net $(x_\alpha, y_\alpha)$ belongs to $D_{u,(\cdot)}(\lambda; \text{Gr}_K(\Phi))$, which is compact by $\mathbb{K}$-inf-compactness. Consequently, it has a convergent subnet (without loss of generality, the original net) with the limit $(x, y) \in D_{u,(\cdot)}(\lambda; \text{Gr}_K(\Phi))$. In particular, $y \in \Phi(x)$ and $v(x) \leq u(x, y) \leq \lambda$, as we had to show.

(\(\Box\)). Let $\mathbb{X}$ be compactly generated and let $\lambda \in \mathbb{R}$. By (\(\Box\)), $D_{v,(\cdot)}(\lambda; K) = D_{v,(\cdot)}(\lambda; \mathbb{X}) \cap K$ is closed for each $K \in \mathbb{K}(\mathbb{X})$. Therefore, $D_{v,(\cdot)}(\lambda; \mathbb{X})$ is closed. \(\Box\)

Corollary 3.2. (cf. Feinberg and Lewis [8 Proposition 3.1], Feinberg et al. [6 Corollary 3.2]) If a function $u : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is inf-compact on $\text{Gr}_\mathbb{X}(\Phi)$, then the function $v : \mathbb{X} \to \overline{\mathbb{R}}$ is inf-compact and the conclusions of Theorem 3.1 hold.

Proof. In view of Theorem 3.1, $\Phi^*(x) \neq \emptyset$ and the function $v(x)$ is defined for all $x \in \mathbb{X}$. For any $\lambda \in \mathbb{R}$, the level set $D_{v,(\cdot)}(\lambda; \mathbb{X})$ is compact as the projection of the compact set $D_{u,(\cdot)}(\lambda; \text{Gr}_\mathbb{X}(\Phi))$ on $\mathbb{X}$. Thus the function $v(\cdot)$ is inf-compact. \(\Box\)

For an upper semi-continuous set-valued mapping $\Phi : \mathbb{X} \to \mathbb{K}(\mathbb{Y})$, the set $\text{Gr}_\mathbb{X}(\Phi)$ is closed; Berge [3 Theorem 6, p. 112]. Therefore, for such $\Phi$, if a function $u(\cdot, \cdot)$ is lower semi-continuous on $\mathbb{X} \times \mathbb{Y}$, then it is lower semi-continuous on $\text{Gr}_\mathbb{X}(\Phi)$. Thus, Lemma 3.3(i) implies that Theorems 1.2 and 1.4 are natural generalizations of Berge’s Maximum Theorem. Lemma 3.3 generalizes [6 Lemma 2.1].

Lemma 3.3. The following statements hold:

(i) if $u : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is lower semi-continuous on $\text{Gr}_\mathbb{X}(\Phi)$ and $\Phi : \mathbb{X} \to \mathbb{K}(\mathbb{Y})$ is upper semi-continuous, then the function $u(\cdot, \cdot)$ is $\mathbb{K}$-inf-compact on $\text{Gr}_\mathbb{X}(\Phi)$;

(ii) if $u : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ is inf-compact on $\text{Gr}_\mathbb{X}(\Phi)$, then the function $u(\cdot, \cdot)$ is $\mathbb{K}$-inf-compact on $\text{Gr}_\mathbb{X}(\Phi)$ and, therefore, it is $\mathbb{K}$-inf-compact on $\text{Gr}_\mathbb{X}(\Phi)$.

Proof. (i) Let $\{x_i\}_{i \in I}$ be a convergent net with values in $\mathbb{X}$ whose limit $x$ belongs to $\mathbb{X}$ and $\{y_i\}_{i \in I}$ be a net defined on the same ordered set $I$ with $y_i \in \Phi(x_i)$, $i \in I$, and satisfying the condition that the set
\{u(x_i, y_i) : i \in I\} is bounded above by \(\lambda \in \mathbb{R}\). Let us prove that a net \(\{y_i\}_{i \in I}\) has an accumulation point \(y \in \Phi(x)\) such that \(u(x, y) \leq \lambda\). Aliprantis and Border [2, Corollary 17.17, p. 564] yields that a net \(\{y_i\}_{i \in I}\) has an accumulation point \(y \in \Phi(x)\). The lower semi-continuity of \(u\) on Gr(\(\Phi\)) implies that \(u(x, y) \leq \lambda\). Therefore, the function \(u(\cdot, \cdot)\) is \(\mathbb{K}\)-inf-compact on Gr\(_X\)(\(\Phi\)).

(ii) The function \(u\) is lower semi-continuous on Gr\(_X\)(\(\Phi\)) because the level set \(D_u(\cdot, \cdot)(\lambda, \text{Gr}_X(\Phi))\) is compact and, therefore, it is closed for any \(\lambda \in \mathbb{R}\). Let us prove that for any convergent net \(\{x_i\}_{i \in I}\) with values in \(X\) whose limit \(x\) belongs to \(X\), any net \(\{y_i\}_{i \in I}\) defined on the same ordered set \(I\) with \(y_i \in \Phi(x_i)\), \(i \in I\), and satisfying the condition that the set \(\{u(x_i, y_i) : i \in I\}\) is bounded above, has an accumulation point \(y \in \Phi(x)\). This holds because the level set \(D_u(\cdot, \cdot)(\lambda, \text{Gr}_X(\Phi))\) is compact for any \(\lambda \in \mathbb{R}\) and because of the characterizations of compactness via nets.

As explained above, Theorems 1.2, 4.1 and Lemma 2.3 from Feinberg et al. [6] are proved there, in fact, for \(\mathbb{K}\)-inf-compact functions \(u\). In addition, the proofs of [6, Theorem 3.1(a) and Corollary 3.2] use [6, Theorem 1.2 and Lemma 2.3]. If the space \(X\) is compactly generated (in particular, a metrizable topological space is compactly generated), a function \(u\) is \(\mathbb{K}\)-inf-compact if and only if it is \(\mathbb{K}\)-inf-compact; Corollary 2.2 Below we state the corrected formulations of Feinberg et al. [6, Theorem3 1.2, 3.1(a) and Lemma 2.3] for Hausdorff topological spaces \(X\) and \(Y\). We do not provide the corrected formulation of Feinberg et al. [6, Theorem 4.1] because Theorem 1.4 is a stronger statement.

**Theorem 3.4.** (cf. Feinberg et al. [6, Theorem 1.2]) If the function \(u : X \times Y \to \mathbb{R}\) is \(\mathbb{K}\)-inf-compact on Gr\(_X\)(\(\Phi\)), then the function \(v : X \to \mathbb{R}\) is lower semi-continuous.

**Lemma 3.5.** (cf. Feinberg et al. [6, Lemma 2.3]) A \(\mathbb{K}\)-inf-compact function \(u(\cdot, \cdot)\) on Gr\(_X\)(\(\Phi\)) is lower semi-continuous on Gr\(_X\)(\(\Phi\)).

**Proposition 3.6.** (cf. Feinberg et al. [6, Theorem 3.1(a)]) If the function \(u : X \times Y \to \mathbb{R}\) is \(\mathbb{K}\)-inf-compact on Gr\(_X\)(\(\Phi\)), then Gr\(_X\)(\(\Phi^*\)) is a Borel subset of \(X \times Y\).

In addition, [6, Corollary 3.2] is restated above as Corollary 3.2 with the proof that does not use [6, Theorem 1.2]. Example 5.1 demonstrate that the conclusions of Theorem 3.4 and Lemma 3.5 do not hold if \(u\) is \(\mathbb{K}\)-inf-compact on Gr\(_X\)(\(\Phi\)) and \(X\) is Hausdorff.

**4 Proofs of Theorems 1.2 and 1.4**

According to Corollary 2.2 Theorem 1.2 is a direct corollary of Theorem 1.4.

**Proof of Theorem 1.4.** The function \(u\) is continuous on Gr\(_X\)(\(\Phi\)), because it is upper semi-continuous on Gr\(_X\)(\(\Phi\)) and, according to Definition 1.3 it is lower semi-continuous on Gr\(_X\)(\(\Phi\)). In view of Theorems 3.1 and 3.4 the value function \(v : X \to \mathbb{R}\) is continuous and the solution set-valued mapping \(\Phi^* : X \to \mathbb{K}(Y)\) is compact-valued. Let us show that the solution multi-function \(\Phi^*\) is \(\mathbb{K}\)-upper semi-compact. Consider a net \(\{x_i\}_{i \in I}\) with values in \(X\) whose limit \(x\) belongs to \(X\). Then any net \(\{y_i\}_{i \in I}\), defined on the same ordered set \(I\) with \(y_i \in \Phi^*(x_i)\), \(i \in I\), has an accumulation point \(y \in \Phi^*(x)\). Indeed, \(v(x_i) = u(x_i, y_i)\) for any \(i \in I\). Since \(x_i \to x\) and \(v\) is continuous, a net \(\{v(x_i)\}\) is bounded above by a finite constant eventually
in \( I \). Therefore, \( KN \)-inf-compactness of the function \( u \) on \( Gr_X(\Phi) \) implies that the net \( \{y_i\}_{i \in I} \) has an accumulation point \( y \in \Phi(x) \). Since the functions \( u \) and \( v \) are continuous on \( Gr_X(\Phi) \) and \( X \) respectively, \( y \in \Phi^*(x) \). Thus, the solution multifunction \( \Phi^* \) is \( KN \)-upper semi-compact and, in view of Theorem 2.5(i), it is upper semi-continuous.

\[ \square \]

5 Counterexample

In the example below, \( X \) is a Hausdorff topological space, \( Y \) is a singleton, \( \Phi : X \to Y \) is a continuous mapping, \( u \) is a \( K \)-inf-compact real-valued function on \( Gr_X(\Phi) \) such that: (i) \( u \) is not lower semi-continuous on \( X \times Y \) and (ii) the value function \( v \) is not lower semi-continuous on \( X \) either.

Example 5.1. Consider a space \([0, \omega_1]\) of ordinals in the order topology, with \( \omega_1 \) the least uncountable ordinal. Each non-limit ordinal \( \alpha \) is an isolated point: \( \alpha = 0 \) is isolated, since \( \{0\} = [0, 1) \) is open. And if \( \alpha \neq 0 \), there is a \( \beta \) with \( \beta + 1 = \alpha \). Hence \( \{\alpha\} = (\beta, \alpha + 1) \) is open. Now let \( X \) be the subspace consisting of all non-limit ordinals and \( \omega_1 \).

A set in \( X \) is compact if and only if it is finite. To see why an infinite set \( X \subseteq X \) is not compact, let \( C \subseteq X \setminus \{\omega_1\} \) be a countably infinite subset. Identifying each \( c \in C \) with its countable set of predecessors \( \{x \in [0, \omega_1] : x < c\} \) and using that a countable union of countable sets is countable, it follows that the supremum (union) \( s \) of \( C \) in \([0, \omega_1]\) is countable and consequently satisfies \( s < \omega_1 \). Then \( (s, \omega_1) \) is an open set containing \( \omega_1 \). Since \( (s, \omega_1) \) fails to cover the infinitely many isolated terms \( c \in C \), collection \( \{\{x\} : x \in X \setminus \{\omega_1\}\} \cup \{(s, \omega_1)\} \) is a cover of \( X \) without a finite subcover.

Now let \( Y = \{y\} \) be a singleton set, \( \Phi(x) = \{y\} \) for each \( x \in X \), and define \( u : X \times Y \to \mathbb{R} \) by \( u(x, y) = 0 \) if \( x \neq \omega_1 \) and \( u(\omega_1, y) = 1 \). Since \( Y \) and each compact subset of \( X \) are finite, \( u \) is \( K \)-inf-compact. But \( u \) is not lower semi-continuous: the set \( A \) of non-limit ordinals is directed in its usual order and net \( (x_\alpha, y)_{\alpha \in A} \) with \( x_\alpha = \alpha \) converges to \( (\omega_1, y) \), yet

\[
\lim \inf \alpha u(x_\alpha, y) = 0 < 1 = u(\omega_1, y),
\]

contradicting lower semi-continuity.

This also shows that the value function \( v : x \mapsto \inf_{y \in \Phi(x)} u(x, y) \) need not be lower semi-continuous: in this example, \( \Phi(x) = \{y\} \), so \( v(x) = u(x, y) \) for all \( x \in X \) so that we have an analogous violation of lower semi-continuity:

\[
\lim \inf \alpha v(x_\alpha) = 0 < 1 = v(\omega_1).
\]

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