Quantum Mechanical Square Root Speedup in a Structured Search Problem

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Abstract

An unstructured search for one item out of $N$ can be performed quantum mechanically in time of order $\sqrt{N}$ whereas classically this requires of order $N$ steps. This raises the question of whether square root speedup persists in problems with more structure. In this note we focus on one example of a structured problem and find a quantum algorithm which takes time of order the square root of the classical time.

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INTRODUCTION

An unstructured search for one item out of \( N \) can be performed quantum mechanically in time of order \( \sqrt{N} \) whereas classically this requires of order \( N \) steps [1]. The \( \sqrt{N} \) is optimal [2]. This raises the question of whether square root speedup persists in problems with more structure [3]. In this note we focus on one example of a structured problem and find an (optimal) quantum algorithm which takes time of order the square root of the classical time. Some of the methods in this paper are similar to those found in [4].

I. MAIN RESULT

Consider a function \( F(x,y) \) with \( x \) and \( y \) integers, \( 1 \leq x \leq L \) and \( 1 \leq y \leq L \). The function has the property that it is 0 except at a single value of \( (x,y) \) where it takes the value 1. We imagine that a subroutine which computes \( F \) is available but we have no further knowledge of \( F \). The goal is to discover the unique \( (x_0,y_0) \) where \( F(x_0,y_0) = 1 \). As described so far, classically it is necessary to search the \( L^2 \) values of \( (x,y) \) whereas quantum mechanically, Grover's algorithm finds \( (x_0,y_0) \) with of order \( L \) calls of the (quantum) subroutine for \( F \).

Now suppose we also have available a subroutine which computes a function \( G(x) \), \( 1 \leq x \leq L \). This function \( G \) is known to have the property that it takes the value 1 on a set with \( M \) elements and is 0 otherwise. Furthermore we are guaranteed that \( G(x_0) = 1 \). The goal is to find \( (x_0,y_0) \) as fast as possible where time is measured in the total number of function calls of \( F \) and \( G \).

If \( M \) equals \( L \), the function \( G \) is identically 1 and is useless. If \( M \) equals 1, the best strategy — classically or quantum mechanically — is to use \( G \) to find \( x_0 \) and then \( F \) to find \( y_0 \). We thus restrict our attention to the case where \( 1 \ll M \ll L \). Furthermore we assume that \( M \) is known.

Classically, \( (x_0,y_0) \) can be found in of order \( ML \) steps. The strategy is to consider each \( x \) in turn. If \( G(x) = 0 \) move to the next \( x \); otherwise check \( F(x,y) \) for each \( y \). This time is optimal: the easier problem in which we are told that \( G(x) = 1, 1 \leq x \leq M \), (and \( G(x_0) = 1 \) already requires a search through \( ML \) items. The same argument shows that no quantum algorithm can succeed in fewer than \( \sqrt{ML} \) steps.

We now present a quantum algorithm that succeeds in time of order \( \sqrt{ML} \). We work in an \( L^2 \) dimensional Hilbert space with orthonormal basis elements \( |x\rangle|y\rangle \). Quantum code for \( F \) and \( G \) allows us to readily construct the unitary operators \( (-1)^F \) and \( (-1)^G \) defined by

\[
(-1)^F |x\rangle|y\rangle = (-1)^{F(x,y)} |x\rangle|y\rangle
\]

(1)

and

\[
(-1)^G |x\rangle|y\rangle = (-1)^{G(x)} |x\rangle|y\rangle
\]

(2)

Furthermore with \( |s\rangle \) defined as

\[
|s\rangle = \frac{1}{\sqrt{L}} \sum_{z=1}^{L} |z\rangle
\]

(3)

we can also construct \( \hat{U}_1 \) and \( \hat{U}_2 \) given by
\[ \hat{U}_1 = (2|s\rangle\langle s| - 1) \otimes 1 \]  
and  
\[ \hat{U}_2 = 1 \otimes (2|s\rangle\langle s| - 1) \]  

All of these are immediate generalizations of the building blocks of Grover’s algorithm \[1\].

Now consider the operator \( \hat{W} \) defined by
\[ \hat{W} = [\hat{U}_2(-1)\hat{F}]^k \]  
where \( k \) is the closest integer to \( \frac{\pi}{4} \sqrt{L} \). What is \( \hat{W} |x\rangle|s\rangle \)? If \( x \neq x_0 \), then \( F(x, y) = 0 \) for all \( y \) and \( \hat{W} \) acts as the identity. If \( x = x_0 \), then \( \hat{W} \) is executing the Grover algorithm on the \( y \) coordinate with the function \( F(x_0, y) \) and \( \hat{W}|x_0\rangle|s\rangle = |x_0\rangle|y_0\rangle \). (The algorithm actually produces a state which is \( |x_0\rangle|y_0\rangle \) plus corrections of order \( \frac{1}{\sqrt{L}} \). Throughout this paper we ignore these corrections.) Thus
\[ \hat{W} |x\rangle|s\rangle = \begin{cases} |x\rangle|s\rangle & \text{if } x \neq x_0 \\ |x_0\rangle|y_0\rangle & \text{if } x = x_0 \end{cases} \]  

Next consider
\[ \hat{V} = \hat{W}^{\dagger} (-1)^{\hat{F}} \hat{W} \]  

It follows from (7) that
\[ \hat{V} |x\rangle|s\rangle = \begin{cases} |x\rangle|s\rangle & \text{if } x \neq x_0 \\ -|x_0\rangle|s\rangle & \text{if } x = x_0 \end{cases} \]  

We will use \( \hat{V} \) as a subroutine in our overall algorithm.

We now take advantage of the function \( G \). First define the superposition of \( |x\rangle \)'s for which \( G(x) = 1 \),
\[ |\psi\rangle = \frac{1}{\sqrt{M}} \sum_{G(x)=1} |x\rangle \]  

Note that \( |\psi\rangle|s\rangle \) can be obtained from \( |s\rangle|s\rangle \) by a straightforward generalization \[5\] of the Grover algorithm to the case where the number of marked items is \( M \),
\[ [\hat{U}_1(-1)^\hat{G}]^j |s\rangle|s\rangle = |\psi\rangle|s\rangle \]  

where \( j \) is the closest integer to \( \frac{\pi}{4} \sqrt{\frac{L}{M}} \). (The \( y \) coordinate is just coming along for the ride.)

Furthermore we can construct
\[ \hat{U}_\psi = (2|\psi\rangle\langle \psi| - 1) \otimes 1 \]  

by
\[ \hat{U}_\psi = [(-1)^\hat{G}\hat{U}_1]^j \hat{U}_1 [\hat{U}_1(-1)^\hat{G}]^j \]
Using $\hat{U}_\psi$ and $\hat{V}$ we can use the Grover algorithm to find $x_0$ from a set of $M$ things instead of from a set of $L$ things. That is, with $h$ being the closest integer to $\frac{\pi}{4}\sqrt{M}$, we have that

$$[\hat{U}_\psi \hat{V}]^h |\psi\rangle s = |x_0\rangle s \ .$$  \hspace{1cm} (14)$$

To run the algorithm we start in the state $|s\rangle s\rangle$, apply (11) and then (14) obtaining $|x_0\rangle s\rangle$. How many steps does it take to produce $|x_0\rangle s\rangle$ by this method? Up to constants, it takes $\sqrt{L}$ calls of $F$ to construct $\hat{W}$ (see (3)) and accordingly $\sqrt{L}$ calls of $F$ to construct $\hat{V}$ (see (8)). Now $\hat{U}_\psi$ takes $\sqrt{\frac{L}{M}}$ calls of $G$ (see (13)), so $\hat{U}_\psi \hat{V}$ in (14) takes $\sqrt{\frac{L}{M}} + \sqrt{L}$ which is $\sqrt{L}$. Thus producing $|x_0\rangle s\rangle$ from $|\psi\rangle s\rangle$ takes $h\sqrt{L}$, that is $\sqrt{ML}$ steps. (Note that producing $|\psi\rangle s\rangle$ by (11) requires an irrelevant extra $\sqrt{\frac{L}{M}}$ steps.)

We have found $x_0$ in of order $\sqrt{ML}$ steps. Using the function $F(x_0, y)$ we can quantum mechanically search through the $L$ values of $y$ in time $\sqrt{L}$. The total time required to find $(x_0, y_0)$ remains of order $\sqrt{ML}$.

II. ANOTHER EXAMPLE

The problem just discussed is a more structured version of the following. We are given a subroutine for a function $f(z)$ with $1 \leq z \leq N$ which is guaranteed to be 0 except at a unique but unknown point $z_0$. Suppose we are also given a function $g(z)$ and we know that $g(z) = 1$ for $z$ in a set of size $M$ and that $z_0$ is in this set. A natural way to try to use $g$ to speed up the quantum search for $z_0$ is to (i) use the Grover algorithm to construct the state $|\phi\rangle$ which is a superposition of the $M$ basis states $|z\rangle$ with $g(z) = 1$ and then (ii) use the Grover algorithm on $|\phi\rangle$ with the function $f$. We now explicitly do this and find that the total time is of order $\sqrt{N}$ so that the added ability to call $g$ is of no help.

To begin we define

$$|\sigma\rangle = \frac{1}{\sqrt{N}} \sum_{z=1}^{N} |z\rangle \hspace{1cm} (15)$$

and

$$|\phi\rangle = \frac{1}{\sqrt{M}} \sum_{g(z)=1} |z\rangle \ .$$  \hspace{1cm} (16)$$

The state $|\phi\rangle$ can be obtained, as in (11), with $\ell$ applications of

$$\hat{U} = [2|\sigma\rangle\langle\sigma| - 1]|-1\rangle\hat{g}$$  \hspace{1cm} (17)$$

where $\ell$ is the integer closest to $\frac{\pi}{4}\sqrt{\frac{N}{M}}$; that is

$$\hat{U}^{\ell} |\sigma\rangle = |\phi\rangle \ .$$  \hspace{1cm} (18)$$

As in (12) and (13), we can produce
\[ 2|\phi \rangle \langle \phi| - 1 = \hat{U}^\ell (2|\sigma \rangle \langle \sigma| - 1) \hat{U}^\ell. \]  

Finally

\[ [(2|\phi \rangle \langle \phi| - 1)(-1)^f |\phi\rangle = |z_0\rangle \]  

where \( h \) is the closest integer to \( \frac{2}{\pi} \sqrt{M} \).

How many function calls does it take to produce \( |z_0\rangle \)? Since \( 2|\phi \rangle \langle \phi| - 1 \) requires of order \( \sqrt{N} \) calls of \( g \) and we must use \( 2|\phi \rangle \langle \phi| - 1 \) of order \( \sqrt{M} \) times, we need of order \( \sqrt{N} \) calls of \( g \). Thus the availability of \( g \) yields no speedup over the Grover algorithm using \( f \) alone.

It is easy to see why using \( g \) is never of computational benefit — quantum mechanically or classically. Suppose it was of benefit and an algorithm using \( f \) and \( g \) existed which was faster than any algorithm using \( f \) alone. Now given only a function \( f(z) \) which is 1 at a unique unknown \( z_0 \), one could always construct two functions \( g_1(z) \) and \( g_2(z) \) as

\[ g_1(z) = \begin{cases} 1 & \text{if } 1 \leq z \leq M - 1 \text{ or } f(z) = 1 \\ 0 & \text{otherwise} \end{cases} \]  

and

\[ g_2(z) = \begin{cases} 1 & \text{if } 1 \leq z \leq M \text{ or } f(z) = 1 \\ 0 & \text{otherwise} \end{cases}. \]  

Note that if \( 1 \leq z_0 \leq M \) then \( g_2 \) is 1 for \( M \) values of \( z \) (including \( z_0 \)) whereas if \( M \leq z_0 \leq L \) then \( g_1 \) is 1 for \( M \) values of \( z \) (including \( z_0 \)). Run the purportedly faster algorithm using first \( f \) and \( g_1 \) and then with \( f \) and \( g_2 \). This would produce \( z_0 \) as quickly as the faster algorithm, which is a contradiction.

The example in this section shows that whenever the Grover algorithm is used within the Grover algorithm [4], the time required to construct the operator \( 2|\phi \rangle \langle \phi| - 1 \) must be taken into account.

**III. DISCUSSION**

In the first section we gave an example of a structured search problem where the (best possible) quantum algorithm succeeded in a time of order the square root of the classical time. The example in Section 2 also has this feature. Here the apparent structure introduced through the function \( g \) does not allow the classical search to be done in time faster than \( N \) nor does it allow the quantum search to be done faster than \( \sqrt{N} \). These two examples give evidence that quantum square root speed up may persist in a wide range of structured problems.

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