Variational analysis of the discontinuous Galerkin time-stepping method for parabolic equations

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The discontinuous Galerkin (DG) time-stepping method applied to abstract evolution equation of parabolic type is studied using a variational approach. We establish the inf-sup condition or Babuška–Brezzi condition for the DG bilinear form. Then, a nearly best approximation property and a nearly symmetric error estimate are obtained as corollaries. Moreover, the optimal order error estimates under appropriate regularity assumption on the solution are derived as direct applications of the standard interpolation error estimates. Our method of analysis is new. It differs from previous works on the DG time-stepping method by which the method is formulated as the one-step method. We apply our abstract results to finite element approximation of the inhomogeneous heat equation in a polyhedral domain \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), and derive the optimal order error estimates in several norms, for example, in the \( H^1(0,T;H^{-1}(\Omega)) \), \( L^2(0,T;H^1_0(\Omega)) \) and \( L^\infty(0,T;L^2(\Omega)) \) norms, and so on.

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1. Introduction

The discontinuous Galerkin (DG) time-stepping method, which is designated below as the \( dG(q) \) method, is a time discretization method using piecewise polynomials of degree \( q \) with an integer \( q \geq 0 \). The \( dG(q) \) method was proposed originally by Lasaint and Raviart \[14\] for ordinary differential equations. (Galerkin time-stepping methods of other kinds were proposed earlier by \[11, 12\].) Later, the method was applied to space-time discretization method for the moving boundary problem of the heat equation by Jamet \[13\]. Standard time-discretization methods are formulated as one-step or multi-step methods: approximations are computed at nodal points. By contrast, the \( dG(q) \) method gives approximations as piecewise polynomials so that approximations at arbitrary point are available. Therefore, the method is useful to address moving boundary problems and a system composed of equations having different natures. Indeed, the method is applied actively to fluid–structure interaction problems (see \[3\]).
It was described in [13] that the dG(q) method is interpreted as an one-step method and that it is strongly A-stable of order $2q + 1$. Moreover, after applying a numerical quadrature formula, the dG(q) method to $y'(t) = \lambda y(t)$ with a scalar $\lambda$ was found to agree with the sub-diagonal $(q + 1, q)$ Padé rational approximation of $e^{-z}$ (see [14]). Particularly, the dG(0) method implies the backward Euler method. For this reason, earlier studies of stability and convergence of the dG(q) method are accomplished by formulating the method as a one-step method. However, this seems to make analysis somewhat intricate, especially for large $q$. We review those earlier studies in greater detail below.

The purpose of this paper is to present a different approach: we study the dG(q) method using a variational approach. In fact, the dG(q) method is the Galerkin approximation of the variational formulation of the equation and several techniques developed in the literature of the DG method (see [1] for example) are applicable. Consequently, the analysis becomes greatly simplified for any $q$ and optimal order error estimates in some appropriate norms are established. To clarify the variational characteristics of the dG(q) method, we apply the method to abstract evolution equations of parabolic type (the coefficient might depend on the time). Then, the finite element approximation of the heat equation is studied as an application of abstract results. We concentrate our attention to the case $q \geq 1$, because the backward Euler method is well studied so far.

We first formulate the problem to be addressed. Letting $H$ and $V$ be (real) Hilbert spaces such that $V \subset H$ are dense with the continuous injection, then the inner product and norms are denoted as $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$, $\langle \cdot, \cdot \rangle_V$, $\| \cdot \| = \| \cdot \|_H$ and $\| \cdot \|_V$. The topological dual spaces $H'$ and $V'$ are denoted respectively as $H'$ and $V'$. As usual, we identify $H$ with $H'$ and consider the triple $V \subset H \subset V'$. Moreover, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V$ represents the duality pairing between $V'$ and $V$. Let $J = (0, T)$ with $T > 0$. Assuming that, for a.e. $t \in J$, we are given a linear operator $A(t)$ of $V \rightarrow V'$, and assuming that there exist two positive constants $M$ and $\alpha$ which are independent of $t \in J$ such that

\[
\begin{align*}
\langle A(t)w, v \rangle &\leq M \|w\|_V \|v\|_V & (w, v \in V, \text{ a.e. } t \in J); 
\langle A(t)v, v \rangle &\geq \alpha \|v\|^2_V & (v \in V, \text{ a.e. } t \in J),
\end{align*}
\]

we consider the abstract evolution equation of parabolic type as

\[u' + A(t)u = F(t), \quad t \in J; \quad u(0) = u_0,\]

where $u'$ denotes $du(t)/dt$ and where $F : J \rightarrow V'$ and $u_0 \in H$ are given functions.

Several frameworks and methods can be used to establish the well-posedness (unique existence of a solution) of (2). For this study, we use the variational method of J. L. Lions ([3] Chap. XVIII and [18] Chap. IV) because it is appropriate for the analysis of the dG(q) method. To recall it, we require some additional notation. Set $X = L^2(J; V) \cap H^1(J; V')$, $Y_1 = L^2(J; V)$ and $Y = Y_1 \times H$ with

\[
\|w\|^2_X = \|w\|^2_{L^2(J; V)} + \|w'\|^2_{L^2(J; V')}, \quad \|v_1\|^2_{Y_1} = \|v_1\|^2_{L^2(J; V)}, \quad \|v\|^2_Y = \|v_1\|^2_{L^2(J; V)} + \|v_2\|^2_H
\]

for $w \in X$ and $v = (v_1, v_2) \in Y$.

The weak formulation of (2) is stated as follows. Given

\[F \in L^2(J; V'), \quad u_0 \in H,
\]

find $u \in X$ such that

\[B(u, v) = \int_J \langle F, v_1 \rangle \ dt + (u_0, v_2) \quad (\forall v = (v_1, v_2) \in Y),\]
where
\[ B(w, v) = \int_J \left[ \langle w', v_1 \rangle + \langle A(t)w, v_1 \rangle \right] \, dt + (w(0), v_2) \quad (w \in \mathcal{X}, v = (v_1, v_2) \in \mathcal{Y}). \]  

(3c)

The space \( \mathcal{X} \) is embedded continuously in the set of \( H \)-valued continuous functions on \( J \) (see [5, theorem XVIII-1], [18, theorem 25.2]). In other words, there exists a positive constant \( C_{TV,T} \) depending only on \( T \) such that
\[ \max_{t \in J} \|v(t)\|_H \leq C_{TV,T} \|v\|_X \quad (v \in \mathcal{X}). \]  

(4)

Particularly, \( w(0) \in H \) in (3c) is well-defined.

The bilinear form \( B \) is bounded in \( \mathcal{X} \times \mathcal{Y} \) as
\[ \|B\| = \sup_{w \in \mathcal{X}, v \in \mathcal{Y}} \frac{|B(w, v)|}{\|w\|_X \|v\|_Y} < \infty. \]

Moreover, it is known that (see [10, Theorem 6.6]):
\[ \exists \beta > 0, \quad \inf_{w \in \mathcal{X}, v \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{B(w, v)}{\|w\|_X \|v\|_Y} = \beta; \]  

(5a)

\[ v \in \mathcal{Y}, \quad (\forall w \in \mathcal{X}, B(w, v) = 0) \implies (v = 0). \]  

(5b)

Therefore, we can apply the Banach–Nečas–Babuška theorem or Babuška–Lax–Milgram theorem (see [10, Theorem 2.6], [16, Theorem 3] and [2, Theorem 5.2.1] for example) to conclude that there exists a unique \( u \in \mathcal{X} \) satisfying (3) and it satisfies
\[ \|u\|_X \leq C \left( \|F\|_{L^2(J; V')} + \|u_0\| \right) \]  

with a positive constant \( C \). The constant \( C \) depends only on \( M \) and \( \alpha \); it is independent of \( T \). Actually, (5a) is the necessary and sufficient condition for the well-posedness of (3). (The case \( u_0 = 0 \) is described explicitly in [10]. However, the modification to the case \( u_0 \neq 0 \) is straightforward.) Equality (5a) is commonly designated as the inf-sup condition or Babuška–Brezzi condition. Furthermore, (5) is equivalent to
\[ \exists \beta > 0, \quad \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{B(w, v)}{\|w\|_X \|v\|_Y} = \inf_{v \in \mathcal{Y}} \sup_{w \in \mathcal{X}} \frac{B(w, v)}{\|w\|_X \|v\|_Y} = \beta. \]  

(6)

This equivalence is verified by considering the associating operators with \( B \) and the operator norms of their inverse operators. Indeed, this equivalence plays an important role in the discussion below.

The dG(\( q \)) method described below (see [10]) is based on the formulation (3), which means that the dG(\( q \)) method is consistent with (3) in the sense of Lemma 1. (The consistency is also called the Galerkin orthogonality.) Therefore, it is natural to ask whether a discrete version of (5), particularly (5a), is available. If it is established, then the best approximation property and optimal order error estimates are obtained as direct consequences. Although such an approach is quite standard for elliptic problems, apparently little is done for parabolic problems.

In this paper, after describing the dG(\( q \)) method, we first prove that there exists a positive constant \( c_1 \) such that (see Theorem 1)
\[ \inf_{w_r \in \mathcal{S}_r} \sup_{v_r \in \mathcal{S}_r} \frac{B_r(w_r, v_r)}{\|w_r\|_{\mathcal{X}, r} \|v_r\|_{\mathcal{Y}, r, \#}} = c_1 \]
which is a discrete version of (52a). Herein, $B_{\tau}$ is the DG approximation of $B$. Also, $S_{\tau}$ is the set of $V$-valued piecewise polynomials of degree $q$ defined on a non-uniform partition of $J$ with size parameter $\tau > 0$. Moreover, $\| \cdot \|_{X,\tau}$ and $\| \cdot \|_{Y,\tau,\#}$ are the DG norms corresponding to $\| \cdot \|_{X}$ and $\| \cdot \|_{Y}$, respectively. (The precise definition of these symbols will be presented in Section 3.) Then, as a direct consequence, we demonstrate that there exists a positive constant $c_1'$ such that (see Theorem 2)

$$\|u - u_{\tau}\|_{X,\tau} \leq c_1' \inf_{w_{\tau} \in S_{\tau}} \|u - w_{\tau}\|_{X,\tau,*},$$

where $u_{\tau} \in S_{\tau}$ is the solution of dG($q$) method and $\| \cdot \|_{X,\tau,*}$ denotes another DG norm corresponding to $\| \cdot \|_{X}$ satisfying $\|v_{\tau}\|_{X,\tau} \leq \|v_{\tau}\|_{X,\tau,*}$ for $v_{\tau} \in S_{\tau}$. This result is neither a best approximation property nor a symmetric error estimate in the sense of [6]; it is only a nearly best approximation property and nearly symmetric error estimate. However, using this result, one can obtain optimal order error estimates under appropriate regularity of solution $u$ of (53).

We prove (see Theorem 3).

$$\left( \sum_{n=0}^{N-1} \|u'_{n} - u''_{n}\|_{L^2(J_n,J_n)} \right)^{1/2} \leq c_3\tau^q \|u(q)\|_{X}$$

and

$$\sup_{1 \leq n \leq N} \|u(t_n) - u_{\tau}(t_n)\| + \|u - u_{\tau}\|_{L^2(J_1,J_2)} \leq c_4\tau^{q+1} \|u(q+1)\|_{Y,\tau}.$$
results in the $L^\infty(J; L^2(\Omega))$ norm and super-convergence results with the log-factor at time-nodal points. Those results were extended to several directions in Thomée [17, Chapter 12]. Eriksson and Johnson [7, 8] examined adaptive algorithms and a posteriori error estimates for the dG($q$)cG($1$) method for the heat equation with $q = 0, 1$. They also proved several a priori estimates of optimal order with the log-factor in, for example, the $L^\infty(J; L^2(\Omega))$ and $L^\infty(J \times \Omega)$ norms. Chrysafinos and Walkington [3] considered the dG($q$)cG($k$) method for the heat equation. They presented a kind of symmetric error estimate with a special projection operator with no explicit convergence rate. In [4], [7], and [8], the finite element spaces might be different at each time slab. Leykekhman and Vexler [15] proved a best approximation operator. Leykekhman and Vexler [15] proved a best approximation property of the form
\[
\|u - u_{h,\tau}\|_{L^\infty(J \times \Omega)} \leq C \log(T/\tau) \log(1/h) \inf_{\chi \in X_{h,\tau}} \|u - \chi\|_{L^\infty(J \times \Omega)},
\]
where $u_{h,\tau}$ denotes the solution of the dG($q$)cG($k$) method for the heat equation in a convex polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $X_{h,\tau}$ is the dG($q$)cG($k$) finite element space. Therein, the quasi-uniformity conditions were assumed both for time and space meshes.

This paper comprises four sections with an appendix. In Section 2 the dG($q$) method for [3] and the main results, Theorems [11, 13], are stated. The proof of Theorems [2, 3] are also described there. The proof of Theorem [11] is presented in Section 3. Section 4 is devoted to the study of the dG($q$)cG($k$) method for the inhomogeneous heat equation. The proof of stability results used in Section 4 is stated in Appendix A.

2. DG time-stepping method dG($q$) and main results

Let $N$ be a positive integer. We introduce $N + 1$ distinct points $0 = t_0 < t_1 \cdots < t_n < \cdots < t_N = T$. Set $J_n = (t_n, t_{n+1}]$ and $\tau_n = t_{n+1} - t_n$ for $n = 0, \ldots, N - 1$.

We consider the partitions of $J$ as $\Delta_\tau = \{J_0, \ldots, J_{N-1}\}$, where $\tau = \max_{0 \leq n \leq N-1} \tau_n$. Without loss of generality, we assume that $\tau \leq 1$. We set
\[
C^0(\Delta_\tau; H) = \{v \in L^\infty(J; H) | v|_{J_n} \in C^0(J_n; H), \ 0 \leq n \leq N - 1\},
\]
where $C^0(J_n; H)$ denotes the set of $H$-valued continuous functions on $J_n$. Spaces $C^0(\Delta_\tau; V)$ and $C^0(J_n; V)$ are defined similarly. For arbitrary $v \in C^0(\Delta_\tau; H)$, we write
\[
v^{n,+} = \lim_{t \uparrow t_n} v(t), \quad v^{n+1} = v(t_{n+1}) \quad (n = 0, \ldots, N - 1).
\]
Let $q \geq 0$ be an integer and set
\[
S_\tau = S^q_\tau(H, V) = \{v \in C^0(\Delta_\tau; H) | v|_{J_n} \in \mathcal{P}^q(J_n; V), \ 0 \leq n \leq N - 1\},
\]
where $\mathcal{P}^q(J_n; V)$ denotes the set of $V$-valued polynomials of $t \in J_n$ with degree $\leq q$.

The DG time-stepping method dG($q$) is presented below. Find $u_\tau \in S_\tau$ such that
\[
B_\tau(u_\tau, v_\tau) = \int_J \langle F, v_\tau \rangle \ dt + (u_0, v_\tau^{0,+}) \quad (\forall v_\tau \in S_\tau),
\]
where
\[
B_\tau(u_\tau, v_\tau) = \sum_{n=0}^{N-1} \int_{J_n} [(u_n', v_\tau) + \langle A(t)u_\tau, v_\tau \rangle] \ dt + (u_\tau^{0,+}, v_\tau^{0,+}) + \sum_{n=1}^{N-1} (u_n^{n,+} - u_n^{n}, v_\tau^{n+}).
\]
An alternate expression of $B_\tau(u_\tau, v_\tau)$ is given as

$$B_\tau(u_\tau, v_\tau) = \sum_{n=0}^{N-1} \int_{J_n} \left[ -(u_\tau, v_\tau') + \langle A(t)u_\tau, v_\tau \rangle \right] dt + (u^N_\tau, v^N_\tau) + \sum_{n=1}^{N-1} (u^n_\tau, v^n_\tau - v^{n+}_\tau). \quad (10)$$

Because the solution $u$ of (3) is a function of $C^0(\overline{J}; H)$, we have $(u^{n+} - u^n, v^{n+}_\tau) = 0$ for $v_\tau \in \mathcal{S}_\tau$. Consequently, we obtain the following result.

**Lemma 1** (Consistency). If $u \in \mathcal{X}$ and $u_\tau \in \mathcal{S}_\tau$ respectively represent the solutions of (3) and (10), then we have $B_\tau(u - u_\tau, v_\tau) = 0$ for all $v_\tau \in \mathcal{S}_\tau$.

For $v_\tau \in \mathcal{S}_\tau$ and a sequence $\{k_n\} = \{k_n\}_{n=0}^{N-1}$, we set the following.

$$\nu(v_\tau, \{k_n\}) = \sum_{n=0}^{N-1} \int_{J_n} \left[ \|v_\tau\|_V^2 + \|v_\tau\|_Y^2 \right] dt + \|v^0_\tau\|^2 + \sum_{n=1}^{N-1} k_n \|v^{n+}_\tau - v^n_\tau\|^2,$$

$$\eta(v_\tau, \{k_n\}) = \sum_{n=0}^{N-1} \int_{J_n} \|v_\tau\|_Y^2 dt + \|v^0_\tau\|^2 + \sum_{n=1}^{N-1} k_n \|v^{n+}_\tau\|^2,$$

$$\nu^*(v_\tau, \{k_n\}) = \sum_{n=0}^{N-1} \int_{J_n} \left[ \|v_\tau\|_V^2 + \|v_\tau\|_Y^2 \right] dt + \|v^N_\tau\|^2 + \sum_{n=1}^{N-1} k_n \|v^{n+}_\tau - v^n_\tau\|^2,$$

$$\eta^*(v_\tau, \{k_n\}) = \sum_{n=0}^{N-1} \int_{J_n} \|v_\tau\|_Y^2 dt + \|v^N_\tau\|^2 + \sum_{n=1}^{N-1} k_n \|v^{n+}_\tau\|^2.$$

We use the following norms:

$$\|v_\tau\|_{X, \tau} = \nu(v_\tau; \{1\}); \quad \|v_\tau\|_{X, \tau,*} = \nu(v_\tau; \{\tau_n^{-1}\}); \quad \|v_\tau\|_{X, \tau,*} = \nu(v_\tau; \{\tau_n\}); \quad (11a)$$

$$\|v_\tau\|_{Y, \tau} = \eta(v_\tau; \{1\}); \quad \|v_\tau\|_{Y, \tau,*} = \eta(v_\tau; \{\tau_n^{-1}\}); \quad \|v_\tau\|_{Y, \tau,*} = \eta(v_\tau; \{\tau_n\}); \quad (11b)$$

$$\|v_\tau\|_{X, \tau} = \nu^*(v_\tau; \{1\}); \quad \|v_\tau\|_{X, \tau,*} = \nu^*(v_\tau; \{\tau_n^{-1}\}); \quad \|v_\tau\|_{X, \tau,*} = \nu^*(v_\tau; \{\tau_n\}); \quad (11c)$$

$$\|v_\tau\|_{Y, \tau} = \eta^*(v_\tau; \{1\}); \quad \|v_\tau\|_{Y, \tau,*} = \eta^*(v_\tau; \{\tau_n^{-1}\}); \quad \|v_\tau\|_{Y, \tau,*} = \eta^*(v_\tau; \{\tau_n\}). \quad (11d)$$

Because we are assuming $\tau_n \leq 1$, we have

$$\|v_\tau\|_{X, \tau,*} \leq \|v_\tau\|_{X, \tau} \leq \|v_\tau\|_{X, \tau,*}$$

for $v_\tau \in \mathcal{S}_\tau$. The same relations hold for other norms defined as (11b)–(11d).

The following lemma directly follows those definitions.

**Lemma 2.** For $w_\tau, v_\tau \in \mathcal{S}_\tau$, we have

$$|B_\tau(w_\tau, v_\tau)| \leq \begin{cases} M_0 \|w_\tau\|_{X, \tau} \|v_\tau\|_{Y, \tau} \\ M_1 \|w_\tau\|_{X, \tau,*} \|v_\tau\|_{Y, \tau,*} \\ M_2 \|w_\tau\|_{Y, \tau} \|v_\tau\|_{X, \tau} \end{cases}$$

where $M_j, j = 0, 1, 2$, are positive constants depending only on $M$.

We are now in a position to state the main results presented in this paper.
Theorem 1. Let $q \geq 1$ be an integer. Then there exist positive constants $c_1$ and $c_2$ depending only on $\alpha$, $M$ and $q$ such that

$$\inf_{u_r \in S_r} \sup_{v_r \in S_r} \frac{B_r(w_r, v_r)}{\|w_r\|_{\mathcal{X}, \tau, \#} \|v_r\|_{\mathcal{Y}, \tau, \#}} = c_1; \quad (12a)$$

$$\inf_{u_r \in S_r} \sup_{v_r \in S_r} \frac{B_r(w_r, v_r)}{\|w_r\|_{\mathcal{Y}, \tau, \#} \|v_r\|_{\mathcal{X}, \tau}} = c_2. \quad (12b)$$

The proof of Theorem 1 will be given in Section 3. The following result, Theorem 2, is a readily obtainable corollary of Theorem 1 and Lemma 2.

Theorem 2. For an integer $q \geq 1$, we have

$$\|u - u_r\|_{\mathcal{X}, \tau} \leq \left(1 + \frac{M_1}{c_1}\right) \inf_{w_r \in S_r} \|u - w_r\|_{\mathcal{X}, \tau, \#}, \quad (13a)$$

$$\|u - u_r\|_{\mathcal{Y}, \tau, \#} \leq \left(1 + \frac{M_2}{c_2}\right) \inf_{w_r \in S_r} \|u - w_r\|_{\mathcal{Y}, \tau}, \quad (13b)$$

where $u \in \mathcal{X}$ and $u_r \in S_r$ respectively denote the solutions of (3) and (10).

Proof. Let $w_r \in S_r$ be arbitrary. In view of Theorem 1, Lemmas 1 and 2, we have

$$\|w_r - u_r\|_{\mathcal{X}, \tau} \leq \frac{1}{c_1} \sup_{v_r \in S_r} \frac{B_r(w_r - u_r, v_r)}{\|v_r\|_{\mathcal{Y}, \tau, \#}} \leq \frac{1}{c_1} \sup_{v_r \in S_r} \frac{B_r(w_r - u, v_r)}{\|v_r\|_{\mathcal{Y}, \tau, \#}} \leq \frac{M_1}{c_1} \|w_r - u\|_{\mathcal{X}, \tau, \#}.$$ 

Therefore, using the triangle inequality, we obtain (13a). The proof of (13b) is the same. \qed

We derive some optimal order error estimates using Theorem 2. We set $t_{n,j} = t_n + j(t_{n+1} - t_n)/q = t_n + j\tau_n/q$ ($j = 0, \ldots, q$). For $v \in L^2(J_n; V) \cap H^1(J_n; V')$, there exists a unique $I_nv \in \mathcal{P}(J_n; V)$ such that

$$(I_nv)(t_{n,j}) = v(t_{n,j}) \quad (j = 0, \ldots, q).$$

The following error estimates for the Lagrange interpolation $I_nv$ is proved in the standard way using Taylor’s theorem (see [20, theorem 4.A] for example). We write $v^{(s)} = (d/dt)^sv$ for positive integer $s$.

Lemma 3. Letting $q \geq 1$ be an integer, then there exists an absolute positive constant $C$ such that

$$\|v - I_nv\|_{L^2(J_n; U)} \leq C\tau_n^s\|v^{(s)}\|_{L^2(J_n; U)}, \quad (14a)$$

$$\|v' - (I_nv)'\|_{L^2(J_n; U)} \leq C\tau_n^{s-1}\|v^{(s)}\|_{L^2(J_n; U)} \quad (14b)$$

for an integer $s$ with $1 < s \leq q + 1$ and $v \in H^{q+1}(J_n; U)$, where $U = V, H, V'$. Constant $C$ is independent of $U$.

Combining those results, we can deduce the following result.
Theorem 3. Letting \( q \geq 1 \) be an integer and letting \( u \in \mathcal{X} \) and \( u_r \in S_r \) respectively denote the solutions of (3) and (10), then if \( u^{(q)} \in \mathcal{X} = L^2(J;V) \cap H^1(J;V') \), it follows that

\[
\left( \sum_{n=0}^{N-1} \| u' - u'_r \|_{L^2(J_n;V')} \right)^{1/2} \leq c_3 \tau^q \| u^{(q)} \|_{\mathcal{X}}. \tag{15a}
\]

Moreover, if \( u^{(q+1)} \in \mathcal{Y}_1 = L^2(J;V) \), then

\[
\sup_{1 \leq n \leq N} \| u(t_n) - u_r(t_n) \| + \| u - u_r \|_{L^2(J;V)} + \left( \sum_{n=1}^{N-1} \tau_n \| u_r(t_n) - u_r^{n+1} \| \right)^{1/2} \leq c_4 \tau^{q+1} \| u^{(q+1)} \|_{\mathcal{Y}_1}. \tag{15b}
\]

Herein, \( c_3 \) and \( c_4 \) denote positive constants depending only on \( \alpha, M \) and \( q \).

Proof. We take \( w_r \in S_r \) defined by \( w_r|_{J_n} = I_n u \) \((n = 0,1,\ldots,N-1)\). Set \( c_1' = 1 + M_1/c_1 \). Since \( u^{n,+} - u^{n,-} = 0 \) and \( u^{n+1} - u^{n+1} = 0 \) for \( n = 0,1,\ldots,N-1 \), we have by (13a) and (14)

\[
\sum_{n=0}^{N-1} \| u' - u'_r \|_{L^2(J_n;V')}^2 \leq (c_1')^2 \| u - u_r \|_{\mathcal{X},r,*}^2 \leq (c_1')^2 \sum_{n=0}^{N-1} \| u' - u'_r \|_{L^2(J_n;V')}^2 + \sup_{1 \leq n \leq N} \| u(t_n) - u_r(t_n) \| + M \| u - u_r \|_{L^2(J_n;V)} \]

\[
\leq C_2 (c_1')^2 \sum_{n=0}^{N-1} \left( \tau_n^{2q} \| u^{(q+1)} \|_{L^2(J_n;V')}^2 + \tau_n^{2q} \| u^{(q)} \|_{L^2(J_n;V)}^2 \right).
\]

Inequality (15b) is proved similarly using (13b) and (14b). \( \square \)

3. Proof of Theorem [I]

This section is devoted to the proof of Theorem [I]. First, we collect some preliminary results. Throughout this section, we let \( n = 0,1,\ldots,N-1 \) and \( q \geq 1 \), unless otherwise stated explicitly.

The following projection is a slight modification of [17, (12.9)], For \( v \in L^2(J_n;V) \), there exists a unique \( \tilde{v} \in \mathcal{P}^q(J_n;V) \) such that

\[
\tilde{v}^{n,+} = 0; \tag{16a}
\]

\[
\int_{J_n} [v(t) - \tilde{v}(t)] (t-t_n)^l \ dt = 0 \ (l = 0,1,\ldots,q-1). \tag{16b}
\]

Projection \( \pi_n : L^2(J_n;V) \to \mathcal{P}^q(J_n;V) \) is defined as \( \tilde{v} = \pi_n v \). In fact, \( \tilde{v} \) is expressed as \( \tilde{v} = \sum_{l=1}^{q-1} a_l (t-t_n)^l \) with \( a_1,\ldots,a_q \in V \) in view of (16a). Therefore, (16b) implies the system of \( V \)-valued linear equations for unknowns \( a_1,\ldots,a_q \). The number of equations is also \( q \): it suffices to check the uniqueness to verify the existence of \( \tilde{v} \). However, it is a direct consequence of the following (17b). Alternatively, one could follow the same argument as [17] to deduce the uniqueness. Therefore, the projection \( \pi_n \) is well-defined.
Lemma 4. Letting $U = H, V$, the projection $\pi_n$ satisfies the following:

$$
\int_{J_n} (\pi_n v, \chi)_U \, dt = \int_{J_n} (v, \chi)_U \, dt \quad (\chi \in \mathcal{P}^{q-1}(J_n; V));
$$

$$
\|\pi_n v\|_{L^2(J_n, U)} \leq C_q \|v\|_{L^2(J_n; U)} \quad (v \in L^2(J_n; V)),
$$

(17a)

(17b)

where $C_q$ denotes a positive constant depending only on $q$.

Proof of (17a). Let $v \in L^2(J_n; V)$ and $\chi \in \mathcal{P}^{q-1}(J_n; V)$. Writing $\chi = \sum_{l=0}^{q-1} b_l (t - t_n)^l$ with $b_0, \ldots, b_{q-1} \in V$ and using (16b), we have

$$
\int_{J_n} (v - \pi_n v, \chi)_U \, dt = \sum_{l=0}^{q-1} \left( \int_{J_n} (v - \pi_n v)(t - t_n)^l \, dt, b_l \right)_U = 0.
$$

In fact, the first equality is justified because $v : J_n \to U$ is Bochner integrable; see [19, §V.5] for an illustrative example. \hfill \Box

Proof of (17b). Let $v \in L^2(J_n; V)$ and set $\tilde{v} = \pi_n v \in \mathcal{P}^q(J_n; V)$. Substituting $w = \tilde{v}' \in \mathcal{P}^{q-1}(J_n; V)$ into (17a), we have

$$
\int_{J_n} (\tilde{v}, \tilde{v}')_U \, dt = \int_{J_n} (v, \tilde{v}')_U \, dt.
$$

(18)

By virtue of (16a), the left-hand side is estimated as

$$
\int_{J_n} (\tilde{v}, \tilde{v}')_U \, dt = \frac{1}{2} \int_{J_n} \frac{d}{dt} \|\tilde{v}'(t)\|_U^2 \, dt \geq \frac{1}{2} \int_{t_n}^t \frac{d}{ds} \|\tilde{v}'(s)\|_U^2 \, ds = \frac{1}{2} \|\tilde{v}(t)\|_U^2
$$

for $t \in J_n$. To estimate the right-hand side of (18), we apply the inverse inequality

$$
\|w'\|_{L^2(J_n, U)} \leq \frac{C_q}{2 \tau_n} \|w\|_{L^2(J_n, U)} \quad (w \in \mathcal{P}^q(J_n; U)),
$$

where $C_q$ denotes a positive constant depending only on $q$. The proof of this inequality is exactly the same as the scalar case. In particular, $C_q$ is independent of $U$. That is,

$$
\int_{J_n} (v, \tilde{v}')_U \, dt \leq \|v\|_{L^2(J_n; U)} \|\tilde{v}'\|_{L^2(J_n, U)} \leq \frac{C_q}{2 \tau_n} \|v\|_{L^2(J_n; U)} \|\tilde{v}\|_{L^2(J_n; U)}.
$$

Summing up, we obtain

$$
\|\tilde{v}(t)\|_U^2 \leq \frac{C_q}{\tau_n} \|v\|_{L^2(J_n; U)} \|\tilde{v}\|_{L^2(J_n; U)}.
$$

Integrating the both sides in $t \in J_n$, we deduce the desired inequality (17b). \hfill \Box

We consider the trace inequality (14) for $T = 1$ and write $C_{T_k} = C_{T_k,1}$, which is an absolute constant. The scaling argument gives the following lemma.

Lemma 5. For $v \in L^2(J_n; V) \cap H^1(J_n; V')$, we have

$$
\max_{t_n \leq t \leq t_{n+1}} \|v(t)\| \leq \frac{C_{T_k}}{\tau_n^{1/2}} \left( \|v\|_{L^2(J_n; V)}^2 + \tau_n \|v'\|_{L^2(J_n; V')}^2 \right)^{1/2}.
$$

(19)
By virtue of (11), \( A(t) \) is invertible for a.e. \( t \in J \). Moreover, we have the following.

**Lemma 6.** (i) \( \langle g, A(t)^{-1}g \rangle \geq \frac{\alpha}{M^2} \|g\|_V \) for all \( g \in V' \) and a.e. \( t \in J \).

(ii) \( \|A(t)^{-1}g\|_V \leq \frac{1}{\alpha} \|g\|_{V'} \) for all \( g \in V' \) and a.e. \( t \in J \).

Now, we can state the following proof.

**Proof of Theorem (12a).** Let \( w_\tau \in S_\tau \) and set \( \phi = A^{-1}(t)w_\tau' \in L^2(J; V) \). We define \( \tilde{\phi}_\tau \in S_\tau \) as \( \tilde{\phi}_J = \pi_n(\phi|_{J_n}) \) for \( n = 0, \ldots, N - 1 \). For abbreviation, we write \( w = w_\tau \) and \( \tilde{\phi} = \tilde{\phi}_\tau \). According to (16a), Lemmas 4 and 6, we know

\[
\tilde{\phi}^{n,+} = 0;
\]

\[
\int_{J_n} (\chi, \tilde{\phi}) \, dt = \int_{J_n} (\chi, \phi) \, dt \quad (\forall \chi \in P^{q-1}(J_n; V));
\]

\[
\|\tilde{\phi}\|_{L^2(J_n; V)} \leq C_q \|A^{-1}w'\|_{L^2(J_n; V)} \leq \frac{C_q}{\alpha} \|w'\|_{L^2(J_n; V')}.
\]

Now setting \( v = \tilde{\phi} + \mu w \in S_\tau \) with \( \mu > 0 \), the value of \( \mu \) will be specified later.

First, we prove

\[
\|v\|_{Y, \tau, \#} \leq C_1 \|w\|_{X, \tau},
\]

where \( C_1 \) is a positive constant depending only on \( \mu, \alpha \) and \( q \). Using (20a), (20b) and Lemma 4, we can calculate

\[
\|v\|_{Y, \tau, \#}^2 \leq \sum_{n=0}^{N-1} \int_{J_n} 2M \left[ \|\tilde{\phi}\|_V^2 + \mu^2 \|w\|_V^2 \right] dt + \mu^2 \sum_{n=1}^{N-1} \tau_n \|w^{n,+}\|^2 + \delta \|w^{0,+}\|^2
\]

\[
\leq \sum_{n=0}^{N-1} \int_{J_n} \left[ \left( 2M \frac{C_1^2}{\alpha^2} + \mu^2 C_1^2 \tau_n \right) \|w'\|_V^2 + \mu^2 \left( 2M + C_1^2 \right) \|w\|_V^2 \right] dt + \mu^2 \|w^{0,+}\|^2,
\]

which implies (21).

We apply (20a), (20b), (20c), Lemma 6 and Young’s inequality to obtain

\[
B_\tau(w, \tilde{\phi}) = \sum_{n=0}^{N-1} \int_{J_n} 2 \left[ \langle w', A(t)^{-1}w' \rangle + \langle A(t)w, \tilde{\phi} \rangle \right] dt
\]

\[
\geq \sum_{n=0}^{N-1} \int_{J_n} 2 \left[ \frac{\alpha}{M^2} \|w'\|_V^2 - M\|w\|_V \|\tilde{\phi}\|_V \right] dt
\]

\[
\geq \sum_{n=0}^{N-1} \int_{J_n} 2 \left[ \frac{\alpha}{M^2} \|w'\|_V^2 - \frac{M^2}{2\delta} \|w\|_V^2 - \frac{\delta}{2} \|\tilde{\phi}\|_V^2 \right] dt
\]

\[
\geq \sum_{n=0}^{N-1} \int_{J_n} 2 \left[ \left( \frac{\alpha}{M^2} - \frac{C_1^2 \delta}{\alpha^2} \right) \|w'\|_V^2 - \frac{M^2}{\delta} \|w\|_V^2 \right] dt,
\]

where \( \delta > 0 \) is arbitrary. To estimate \( B_\tau(w, \mu w) \), we recall the elementary identity

\[
(\chi^{n,+} - \chi^n, \chi^{n,+}) = \frac{1}{2} \|\chi^{n,+}\|^2 - \frac{1}{2} \|\chi^n\|^2 + \frac{1}{2} \|\chi^{n,+} - \chi^n\|^2
\]
for $\chi \in S_\tau$ and $n = 0, \ldots, N - 1$. That is, we can calculate as

$$B_\tau(w, \mu w) = \sum_{n=0}^{N-1} \int_{J_n} \left[ \frac{\mu}{2} \frac{d}{dt} \|w\|^2 + \mu \langle A(t)w, w \rangle \right] dt$$

$$+ \frac{\mu}{2} \sum_{n=1}^{N-1} \left[ \|w^{n,+}\|^2 - \|w^n\|^2 + \|w^{n,+} - w^n\|^2 \right] + \mu \|w^{0,+}\|^2$$

$$\geq \sum_{n=0}^{N-1} \int_{J_n} \mu \alpha \|w\|^2_Y dt + \frac{\mu}{2} \sum_{n=1}^{N-1} \|w^{n,+} - w^n\|^2 + \mu \|w^{0,+}\|^2.$$

Summing up, we deduce

$$B_\tau(w, v) \geq \sum_{n=0}^{N-1} \int_{J_n} \left[ 2 \left( \frac{\alpha}{M^2} - \frac{C_2^2 \delta}{\alpha^2} \right) \|w'\|^2_Y + \left( \mu \alpha - \frac{M^2}{\delta} \right) \|w\|^2_Y \right] dt$$

$$+ \frac{\mu}{2} \sum_{n=1}^{N-1} \|w^{n,+} - w^n\|^2 + \mu \|w^{0,+}\|^2.$$

We take a suitably small $\delta$ and then choose a suitably large $\mu$. Consequently, there exists a positive constant $C_2$ depending only on $\alpha, M$ and $q$ such that

$$B_\tau(w, v) \geq C_2 \|w\|^2_{X,\tau}.$$

This, together with (21), implies

$$\sup_{v \in S_\tau} \frac{B_\tau(w, v)}{\|v\|^2_{Y,\tau,\#}} \geq C \|w\|_{X,\tau}$$

with $C = C_2/C_1$. Let us denote by $c_1$ the greatest number $C > 0$ such that the above inequality holds. This completes the proof of (12a). \qed

**Proof of Theorem 7, 12b.** It suffices to prove that

$$\exists c_2 > 0, \inf_{\nu \in S_\tau, \nu \in S_{\tau}} \sup_{\|w\|_{Y,\tau,\#}, \|v\|_{X,\tau}} \frac{B_\tau(w, \nu)}{\|w\|_{Y,\tau,\#}, \|v\|_{X,\tau}} = c_2; \quad (22a)$$

$$\nu \in S_{\tau}, \quad (\forall w \in S_{\tau}, B_\tau(w, \nu) = 0) \implies (\nu = 0). \quad (22b)$$

Actually, as recalled from the Introduction (i.e., the equivalence (5) and (6)), (22) is equivalent to

$$\inf_{w \in S_{\tau}} \sup_{w \in S_{\tau}} \frac{B_\tau(w, \nu)}{\|w\|_{Y,\tau,\#}, \|v\|_{X,\tau}} = \inf_{\nu \in S_{\tau}} \sup_{w \in S_{\tau}} \frac{B_\tau(w, \nu)}{\|w\|_{Y,\tau,\#}, \|v\|_{X,\tau}} = c_2.$$

First, (22b) follows (12a). Let $\nu \in S_{\tau}$ and set $\phi = A^{-1} \nu \in S_{\tau}$. We introduce $\bar{\phi} \in S_{\tau}$ as in the proof of (12a). Now we set $w = -\phi + \mu \nu$ with $\mu > 0$. Then, in exactly the same way, we deduce

$$B_\tau(w, \nu) \geq C \|v\|^2_{X,\tau}, \quad \|w\|_{Y,\tau,\#} \leq C' \|v\|_{X,\tau},$$

and consequently obtain (22a). This completes the proof of (12b). \qed
4. Application to the finite element method

This section presents application of our results, Theorems 1–3, to error analysis of the finite element method. Letting \( \Omega \) be a polyhedral domain \( \mathbb{R}^d \), \( d = 2, 3 \), with the boundary \( \partial \Omega \), we consider the heat equation for the function \( u = u(x, t) \) of \( (x, t) \in \Omega \times [0, T) \),

\[
\begin{aligned}
\partial_t u &= \Delta u + f & \text{in } \Omega \times J, \\
u &= 0 & \text{on } \partial \Omega \times J, \\
u|_{t=0} &= u_0 & \text{on } \Omega,
\end{aligned}
\]

where \( \partial_t u = \partial u/\partial t \) and \( \Delta u = \nabla \cdot \nabla u = \sum_{j=1}^d \partial^2 u/\partial x_j^2 \).

We use the Lebesgue space \( L^2 = L^2(\Omega) \) and the standard Sobolev spaces \( H^k = H^k(\Omega) \), \( k \geq 1 \). Let \( H = L^2(\Omega) \) with the inner product \( (v, w) = (v, w)_{L^2} \) and norm \( \|v\| = \|v\|_H = \|\nabla v\| \). Moreover, set \( V = H_0^1 = \{ v \in H^1 \mid v|_{\partial \Omega} = 0 \} \) with the norm \( \|v\|_V = \|v\|_{H_0^1} = \|\nabla v\| \). The space \( V' \) implies the dual space \( H^{-1} = H^{-1}(\Omega) \) of \( H_0^1 \) equipped with the norm \( \|v\|_{H^{-1}} = \sup_{w \in H_0^1} (v, w)/\|w\|_{H_0^1} \). The duality pairing between \( H_0^1 \) and \( H^{-1} \) is denoted as \( \langle w, v \rangle = H^{-1}(w, v)_{H_0^1} \).

Spaces \( X^2 = L^2(J; H_0^1) \cap H^1(J; H^{-1}) \), \( Y_1 = L^2(J; H_0^1) \) and \( Y = Y_1 \times H \) are Hilbert spaces equipped with the norms \( \|v\|_X^2 = \|v\|_{L^2(J; H_0^1)}^2 + \|\partial_t v\|_{L^2(J; H^{-1})}^2 \), \( \|v\|_{Y_1}^2 = \|v\|_{L^2(J; H_0^1)}^2 + \|v\|_{H_0^1}^2 \), respectively.

The operator \( A : H_0^1 \to H^{-1} \) and functional \( F \) on \( H_0^1 \) are defined as

\[
\langle Au, v \rangle = (\nabla u, \nabla v) \quad (u, v \in H_0^1),
\]

\[
\langle F, v \rangle = (f, v) \quad (v \in H_0^1).
\]

With these interpretations, (23) is converted into the abstract evolution equation of (2). The weak formulation of (23) is given as follows: given

\[
f \in L^2(J; L^2), \quad u_0 \in L^2,
\]

find \( u \in X \) such that

\[
B(u, v) = \int_J (f, v_1) \, dt + (u_0, v_2) \quad (\forall v = (v_1, v_2) \in \mathcal{Y}),
\]

where

\[
B(u, v) = \int_J \langle \partial_t u, v_1 \rangle + (\nabla u, \nabla v_1) \, dt + (u(0), v_2) \quad (u \in X, v = (v_1, v_2) \in \mathcal{Y}).
\]

We proceed to the presentation of the finite element approximation. Let \( \{T_h\}_h \) be a family of \textit{shape-regular} triangulation of \( \Omega \). The granularity parameter \( h \) is defined as \( h = \max_{K \in T_h} h_K \), where \( h_K \) denotes the diameter of the circumscribed ball of \( K \). For an integer \( k \geq 1 \), we introduce the conforming \( P^k \) finite element space

\[
X_h = X_h^k = \{ v \in C^0(\overline{\Omega}) \mid v|_K \in P^k(K; \mathbb{R}) \ (\forall K \in T_h), \ v|_{\partial \Omega} = 0 \} \subset H_0^1.
\]

Recall that \( S_\tau \) is defined as (9). The space–time finite element space is given as

\[
S_{h, \tau} = S_\tau^h(X_h, X_h).
\]

It is noteworthy that \( S_{h, \tau} \subset S_\tau \).
The dG(q)cG(k) method reads: find $u_{h,\tau} \in \mathcal{S}_{h,\tau}$ such that

$$B_{\tau}(u_{h,\tau}, v) = \int_J (f, v) \, dt + (u_0, v^{0,+}) \quad (\forall v \in \mathcal{S}_{h,\tau}),$$

(25a)

where

$$B_{\tau}(u_{h,\tau}, v) = \sum_{n=0}^{N-1} \int_{J_n} [(\partial_t u_{h,\tau}, v) + (\nabla u_{h,\tau}, \nabla v)] \, dt + (u_{h,\tau}^{0,+}, v^{0,+}) + \sum_{n=1}^{N-1} (u_{h,\tau}^{n,+} - u_{h,\tau}^{n-1}, v^{n,+}).$$

(25b)

Remark 7. To avoid unimportant difficulties, we set in (25) the same initial function $u_0$ as (24). This fact implies that the initial value $u_{h,\tau}(0) \in X_h$ of the solution $u_{h,\tau}$ of (25) must be

$$u_{h,\tau}(0) = P_h u_0,$$

where $P_h$ denotes the $L^2$ projection of $L^1 \to X_h$ defined as

$$(P_h w - w, v_h) = 0 \quad (v_h \in X_h).$$

(26)

In this setting, we have the consistency $B_{\tau}(u - u_{h,\tau}, v) = 0$ for all $v \in \mathcal{S}_{h,\tau}$, where $u$ and $u_{h,\tau}$ respectively represent the solutions of (24) and (25). Therefore, in exactly the same way as for the proof of Theorems 1 and 2, we obtain

$$\|u - u_{h,\tau}\|_{X,}\tau} \leq C \inf_{w \in \mathcal{S}_{h,\tau}} \|u - w\|_{X,\tau},$$

and

$$\|u - u_{h,\tau}\|_{Y,\tau} \leq C \inf_{w \in \mathcal{S}_{h,\tau}} \|u - w\|_{Y,\tau}.$$

Hereinafter, we use $C$ to represent general constants independent of $h$, $\tau$, and $T$.

Unfortunately, those inequalities are useless for deducing explicit convergence rates directly. Instead, we use a space semi-discrete scheme (27) below as an auxiliary problem. Set $X_h = H^1(J; X_h)$, $Y_{1h} = L^2(J; X_h)$ and $Y_h = Y_{1h} \times X_h$. They equip the norms $\| \cdot \|_{X_h} = \| \cdot \|_{X}$, $\| \cdot \|_{Y_{1h}} = \| \cdot \|_{Y_1}$ and $\| \cdot \|_{Y_h} = \| \cdot \|_{Y}$. It is noteworthy that $X_h \subset X$, $Y_{1h} \subset Y_1$ and $Y_h \subset Y$. We consider the problem to find $u_h \in X_h$ such that

$$B(u_h, v_h) = \int_J (f, v_{1h}) \, dt + (u_0, v_{2h}) \quad (\forall v_h = (v_{1h}, v_{2h}) \in Y_h),$$

(27a)

where

$$B(u_h, v_h) = \int_J [(\partial_t u_h, v_{1h}) + (\nabla u_h, \nabla v_{1h})] \, dt + (u_h(0), v_{2h}).$$

(27b)

Introducing the discrete Laplacian $A_h : X_h \to X_h$ by

$$(A_h w_h, v_h) = (\nabla w_h, \nabla v_h) \quad (w_h, v_h \in X_h),$$

the problem (27) is expressed equivalently as

$$\frac{d}{dt} u_h(t) + A_h u_h(t) = P_h f(t) \quad (t \in J); \quad u_h(0) = P_h u_0,$$

where $P_h$ denotes the $L^2$ projection defined as (26).

Because (25) is regarded as a time discretization scheme to (27), we can apply Theorem (4) directly to obtain the following lemma.
Lemma 8. Let $u_{h,\tau}$ and $u_h$ respectively represent the solutions of (23) and (24). If $\partial^q_t u_h \in \mathcal{X}_h$, then
\[
\left( \sum_{n=0}^{N-1} \| \partial_t u_h - \partial_t u_{h,\tau} \|_{L^2(J_n;H^{-1})} \right)^{1/2} \leq C \tau^q \| \partial^q_t u_h \|_{\mathcal{X}}.
\]
Moreover, if $\partial^{q+1}_t u_h \in \mathcal{Y}_{1h}$, then
\[
\sup_{1 \leq n \leq N} \| u_h(t_n) - u_{h,\tau}(t_n) \| + \| u_h - u_{h,\tau} \|_{L^2(J;H^q_{1h})} + \left( \sum_{n=1}^{N-1} \tau_n \| u_{h,\tau}(t_n) - u^{n+1}_{h,\tau} \| \right)^{1/2} \leq C \tau^{q+1} \| \partial^{q+1}_t u_h \|_{\mathcal{Y}_1}.
\]

Below, we study the error $u - u_h$ and stability of $u_h$ in various norms. First, the following stability result is an application of (5); the proof is postponed for Appendix A.

Lemma 9. Let $u$ and $u_h$ respectively represent the solutions of (24) and (27). If $\partial^q_t u \in \mathcal{X}$ and $\partial^q_t u_h \in \mathcal{X}_h$, then
\[
\| \partial^q_t u_h \|_{\mathcal{X}} \leq C \| \partial^q_t u \|_{\mathcal{X}}. \tag{28a}
\]
Moreover, if $\partial^{q+1}_t u \in \mathcal{Y}_1$ and $\partial^{q+1}_t u_h \in \mathcal{Y}_{1h}$, then
\[
\| \partial^{q+1}_t u_h \|_{\mathcal{Y}_1} \leq C \| \partial^{q+1}_t u \|_{\mathcal{Y}_1}. \tag{28b}
\]

For a positive integer $k$, we write
\[
|v|_{L^2(J;H^k)}^2 = \int_J \left( \sum_{|\alpha|=k} \| \partial^\alpha v \|_{L^2}^2 \right)^{1/2} dt,
\]
where $\alpha = (\alpha_1, \ldots, \alpha_d)$ denotes the multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_d = k$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$.

Lemma 10. Let $u$ and $u_h$ respectively represent the solutions of (24) and (27). If $u$ is sufficiently regular, we have
\[
\left( \sum_{n=0}^{N-1} \| \partial_t u - \partial_t u_h \|_{L^2(J_n;H^{-1})} \right)^{1/2} \leq C h^k \left( \| \partial_t u \|_{L^2(J;H^k)} + |u|_{L^2(J;H^{k+1})} \right), \tag{29a}
\]
\[
\| u - u_h \|_{L^2(J;H^k)} \leq C h^k \left( \| \partial_t u \|_{L^2(J;H^k)} + |u|_{L^2(J;H^{k+1})} \right), \tag{29b}
\]
\[
\| u - u_h \|_{L^\infty(J;L^2)} \leq C h^{k+1} \left( \| \partial_t u \|_{L^2(J;H^{k+1})} + |u|_{L^\infty(J;H^{k+1})} \right). \tag{29c}
\]

Proof. Because $\mathcal{X}_h \subset \mathcal{X}$ and $\mathcal{Y}_h \subset \mathcal{Y}$, we have the consistency
\[
B(u - u_h, v_h) = 0 \quad (v_h \in \mathcal{Y}_h). \tag{30}
\]
Furthermore, there exists $\beta' > 0$ which is independent of $h$ such that
\[
\inf_{w_h \in \mathcal{X}_h} \sup_{v_h \in \mathcal{Y}_h} \frac{B(w_h, v_h)}{\| w_h \|_{\mathcal{X}} \| v_h \|_{\mathcal{Y}}} = \beta'. \tag{31}
\]
Although Equality (31) does not follow directly from (25a), it is derived using the same method as the proof of (25a). In fact, for any \( w_h \in \mathcal{X}_h \), let \( v_{1h} = A_h^{-1} \partial_t w_h + \mu w_h \) and \( v_{2h} = \lambda w_h \) with \( \mu, \lambda > 0 \). Then, by making \( \mu \) and \( \lambda \) suitably large, we infer (31); see [10, theorem 6.6] for details. Let \( w_h \in \mathcal{X}_h \) be arbitrary. Using (30) and (31), we have

\[
|u - u_h|_\mathcal{X} \leq C \inf_{w_h \in \mathcal{X}_h} |u - w_h|_\mathcal{X}
\]

in the similar way as that used for the proof of Theorem 2. This, together with the standard interpolation error estimates, implies (29a) and (29b). The estimate (29c) is not new; see [10, Theorem 6.14, Remark 6.15] for example.

Summing up those lemmas, we obtain the following theorem as the final result of this paper. Let

\[
\begin{align*}
\|v\|_{Z_1} &= \|\partial_t^q v\|_{L^2(J;H^{-1})} + \|\partial_t^{q+1} v\|_{L^2(J;H^0_1)} + \|\partial_t v\|_{L^2(J;H^k)} + |v|_{L^2(J;H^{k+1})}, \\
\|v\|_{Z_2} &= \|\partial_t^q v\|_{L^2(J;H^0_1)} + \|\partial_t v\|_{L^2(J;H^k)} + |v|_{L^2(J;H^{k+1})}, \\
\|v\|_{Z_3} &= \|\partial_t^{q+1} v\|_{L^2(J;H^0_1)} + \|\partial_t v\|_{L^2(J;H^{k+1})} + |v|_{L^\infty(J;H^{k+1})}
\end{align*}
\]

for a sufficiently regular function \( v \).

**Theorem 4.** Letting \( k \) and \( q \) be integers \( \geq 1 \) and letting \( u \in \mathcal{X} \) and \( u_{h,\tau} \in \mathcal{S}_{h,\tau} \) be the respective solutions of (24) and (27), we assume that \( u \) is sufficiently regular that \( \|u\|_{Z_j} < \infty \), \( j = 1, 2, 3 \). Then, there exist positive constants \( c_5, \ldots, c_8 \) depending only on \( \Omega, k, \) and \( q \) such that

\[
\left( \sum_{n=0}^{N-1} \|\partial_t u - \partial_t u_{h,\tau}\|_{L^2(J;H^{-1})}^2 \right)^{1/2} \leq c_5 (h^k + \tau^q) \|u\|_{Z_1},
\]

\[
\sup_{1 \leq n \leq N} \|u(t_n) - u_{h,\tau}(t_n)\| \leq c_6 (h^{k+1} + \tau^{q+1}) \|u\|_{Z_2},
\]

\[
\left( \sum_{n=1}^{N-1} \|\tau_n \|_{u_{h,\tau}(t_n) - u_{h,\tau}^n\|_2^2 \right)^{1/2} \leq c_8 \tau^{q+1} \|\partial_t^{q+1} u\|_{L^2(J;H^0_1)}.
\]

**A. Proof of Lemma 9**

This appendix is devoted to a proof of Lemma 9. We suppose that \( u \) and \( u_h \) are, respectively, the solutions of (24) and (27). If \( \partial_t^q u \in \mathcal{X} \) and \( \partial_t^q u_h \in \mathcal{X}_h \), then we have

\[
\tilde{B}(\partial_t^q u - \partial_t^q u_h, v_{1h}) = 0 \quad (v_{1h} \in \mathcal{Y}_{1h}),
\]

where

\[
\tilde{B}(u, v_1) = \int_J \left[ (\partial_t u, v_1) + (\nabla u, \nabla v_1) \right] dt \quad (u \in \mathcal{X}, v_1 \in \mathcal{Y}_1).
\]

In fact, (31) gives \( \tilde{B}(u - u_h, \psi_h) = 0 \) for all \( \psi_h \in C_0^\infty(J;X_h) \). Substituting \( \psi_h = (-1)^q \partial_t^q \phi_h \) with \( \phi_h \in C_0^\infty(J;X_h) \) for this identity, we obtain (32) by the integration by parts and density argument. (It is noteworthy that \( C_0^\infty(J;X_h) \) is dense in \( \mathcal{Y}_{1h} \).) Moreover, in exactly the same manner as the for proof of (31), we deduce

\[
\exists \tilde{\beta} > 0, \quad \inf_{w_h \in \mathcal{X}_h} \sup_{v_{1h} \in \mathcal{Y}_{1h}} \frac{\tilde{B}(w_h, v_{1h})}{\|w_h\|_\mathcal{X} \|v_{1h}\|_{\mathcal{Y}_1}} = \tilde{\beta}.
\]
Therefore,
\[
\|\partial_t^2 u_h\|_{C^1} \leq \frac{1}{\beta} \sup_{v_1h \in \mathcal{V}_1h} \frac{\tilde{B}(\partial_t^2 u_h, v_1h)}{\|v_1h\|_{C^1}} = \frac{1}{\beta} \sup_{v_1h \in \mathcal{V}_1h} \frac{\tilde{B}(\partial_t^2 u, v_1h)}{\|v_1h\|_{C^1}} \leq C \|\partial_t^2 u\|_X,
\]
which implies (28a).

We proceed to the proof of (28b). Assume that \(\partial_t^{q+1} u \in \mathcal{Y}_1\) and \(\partial_t^{q+1} u_h \in \mathcal{Y}_1h\). In exactly the same way for deriving (32), we have
\[
B^*(\partial_t^{q+1} u - \partial_t^{q+1} u_h, v_h) = 0 \quad (v_h \in \mathcal{V}_h),
\]
where
\[
B^*(w, v) = \int_J \left[\langle w, -\partial_t v \rangle + \langle \nabla w, \nabla v \rangle \right] dt
\]
and \(\mathcal{V}_h\) denotes the completion of \(C_0^\infty(J; X_h)\) by the norm \(\|v\|_{\mathcal{V}_h}^2 = \|v\|_{L^2(J; H^1)}^2 + \|\partial_t v\|_{L^2(J; H^{-1})}^2\).

If establishing the following (34), we can obtain (28b) as shown above:
\[
\exists \beta^* > 0, \quad \inf_{w_h \in \mathcal{W}_h} \sup_{v_h \in \mathcal{V}_h} \frac{B^*(w_h, v_h)}{\|w_h\|_{\mathcal{W}_h} \|v_h\|_{\mathcal{V}_h}} = \beta^*,
\]
where \(\mathcal{W}_h = L^2(J; X_h)\) and \(\|w\|_{\mathcal{W}_h} = \|w\|_{L^2(J; H^1)}\).

However, the direct proof of (34) is apparently so difficult that we take a detour. We will show
\[
\exists \beta^* > 0, \quad \inf_{v_h \in \mathcal{V}_h} \sup_{w_h \in \mathcal{W}_h} \frac{B^*(w_h, v_h)}{\|w_h\|_{\mathcal{W}_h} \|v_h\|_{\mathcal{V}_h}} = \beta^*,
\]
(35a)
\[
\quad \forall v_h \in \mathcal{V}_h, \quad (\forall w_h \in \mathcal{W}_h, \quad B^*(w_h, v_h) = 0) \implies (v_h = 0).
\]
(35b)

Then, the general theory engenders (34). Recall the equivalence (5) and (6) described in the Introduction.

Proof of (35a). Letting \(v_h \in \mathcal{V}_h\) and setting \(w_h = -A_h^{-1} \partial_t v_h + v_h\), then we calculate
\[
B^*(w_h, v_h) \geq \int_J \left(\|\partial_t v_h\|_{H^{-1}}^2 + \frac{1}{2} \frac{d}{dt} \|v_h\|^2 - \|\partial_t v_h\|_{H^1_0}^2 \|v_h\|_{H^{-1}} + \|v\|_{H^1_0}^2 \right) dt
\]
\[
\geq \frac{1}{2} \|v_h\|_{\mathcal{V}_h}^2.
\]
Combining this with \(\|w_h\|_{\mathcal{W}_h} \leq C \|v_h\|_{\mathcal{V}_h}\), we deduce (35a). \(\square\)

Proof of (35b). Letting \(v_h \in \mathcal{V}_h\), then according to (4), we have \(v_h(T) = 0\). Substituting \(w_h = v_h \in \mathcal{W}_h\) for \(B^*(w_h, v_h) = 0\), we have

\[
0 = -\frac{1}{2} \int_J \|v_h\|^2 dt + \|v\|_{L^2(J; H^1_0)}^2 = \|v\|_{L^2(J; H^1_0)}^2,
\]
which gives \(v_h = 0\). \(\square\)

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