QUIVER APPROACHES TO QUASI-HOPF ALGEBRAS

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Abstract

We provide a quiver setting for quasi-Hopf algebras, generalizing the Hopf quiver theory. As applications we obtain some general structure theorems, in particular the quasi-Hopf analogue of the Cartier theorem and the Cartier-Gabriel decomposition theorem.

1 Introduction

The notion of quasi-Hopf algebras was introduced by Drinfeld [9] in connection with the Knizhnik-Zamolodchikov system of equations. It is obtained from that of Hopf algebras by a weakening of the coassociativity axiom. Quasi-Hopf algebras turn out to be very useful in various areas of mathematics and physics such as low-dimensional topology, number theory, integrable systems, and conformal field theory.

Quivers are oriented graphs consisting of vertices and arrows. They are widely used in many areas of mathematics and physics. In particular thanks to their combinatorial behavior, quivers are very powerful in the investigation of algebraic structures and representation theory.

We propose to carry out a systematic study of elementary quasi-Hopf algebras and pointed dual quasi-Hopf algebras [16] by taking advantage of quiver techniques (see e.g. [2]). The goal of the present paper is to provide a handy quiver setting. For a wider setting and the convenience of exposition, we work mainly with dual quasi-Hopf algebras. A standard dualisation process will give the corresponding results for quasi-Hopf algebras. To avoid too many dual’s and quasi’s we use the term “Majid algebra” for “dual quasi-Hopf algebra”, which was proposed by Shnider-Sternberg [22].
Throughout, we work over a field $k$. We show in Section 3 that the path coalgebra $kQ$ of a quiver $Q$ admits a Majid algebra structure if and only if $Q$ is a Hopf quiver \[7\], and that any coradically graded pointed Majid algebra $H$ can be embedded into a Majid algebra structure on the path coalgebra of some unique Hopf quiver $Q(H)$ determined completely by $H$. This generalizes the quiver setting for Hopf algebras \[6, 7, 11, 25\] into the broader class of quasi-Hopf algebras. As applications we obtain some general structure theorems in Section 4, namely the quasi-Hopf analogue of the Cartier theorem and the Cartier-Gabriel decomposition theorem for Hopf algebras (see e.g. \[4, 18, 24\]). In particular we show that a cocommutative connected Majid algebra over a field of characteristic zero is isomorphic to the universal enveloping algebra of a Lie algebra, which indicates that there is no cocommutative connected Majid algebra out of the usual Hopf setting.

2 Majid Algebras and Hopf Quivers

Introduction to Hopf algebras and Majid algebras can be found in the books \[24, 17, 15, 22\]. For basic knowledge of quivers and their applications to algebras and representation theory see \[2\].

2.1 Majid Algebras

A dual quasi-bialgebra, or Majid bialgebra for short, is a coalgebra $(H, \Delta, \varepsilon)$ equipped with a compatible quasi-algebra structure. Namely, there exist two coalgebra homomorphisms

$$M : H \otimes H \longrightarrow H, \quad a \otimes b \mapsto ab, \quad \mu : k \longrightarrow H, \quad \lambda : k \mapsto \lambda 1_H$$

and a convolution-invertible map $\Phi : H^\otimes 3 \longrightarrow k$ called reassociator, such that for all $a, b, c, d \in H$ the following equalities hold:

$$a_1(b_1c_1)\Phi(a_2, b_2, c_2) = \Phi(a_1, b_1, c_1)(a_2b_2)c_2, \quad (2.1)$$

$$1_H a = a = a 1_H, \quad (2.2)$$

$$\Phi(a_1, b_1, c_1d_1)\Phi(a_2b_2, c_2, d_2) = \Phi(b_1, c_1, d_1)\Phi(a_1, b_2c_2, d_2)\Phi(a_2, b_3, c_3), \quad (2.3)$$

$$\Phi(a, 1_H, b) = \varepsilon(a)\varepsilon(b). \quad (2.4)$$
Here and below we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct. $H$ is called a Majid algebra if, moreover, there exist a coalgebra antimorphism $S : H \rightarrow H$ and two functionals $\alpha, \beta : H \rightarrow k$ such that for all $a \in H$,

$$S(a_1)\alpha(a_2)a_3 = \alpha(a)1_H, \quad a_1\beta(a_2)S(a_3) = \beta(a)1_H, \quad (2.5)$$

$$\Phi(a_1, S(a_3), a_5)\beta(a_2)\alpha(a_4) = \Phi^{-1}(S(a_1), a_3, S(a_5))\alpha(a_2)\beta(a_4) = \varepsilon(a). \quad (2.6)$$

A Majid algebra $H$ is said to be pointed, if the underlying coalgebra is pointed. That is, all the simple subcoalgebras of $H$ are one-dimensional. For a given pointed Majid algebra $(H, \Delta, \varepsilon, M, \mu, \Phi, S, \alpha, \beta)$, let $\{H_n\}_{n \geq 0}$ be its coradical filtration, and $\text{gr} H = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$ the corresponding coradically graded coalgebra. It is routine to verify that $\text{gr} H$ has an induced Majid algebra structure similar to the Hopf case in [18]. The corresponding graded reassociator $\text{gr} \Phi$ satisfies $\text{gr} \Phi(\overline{a}, \overline{b}, \overline{c}) = 0$ for all $\overline{a}, \overline{b}, \overline{c} \in \text{gr} H$ unless they all lie in $H_0$. Similar condition holds for $\text{gr} \alpha$ and $\text{gr} \beta$. In particular, $H_0$ is a sub Majid algebra and turns out to be the group algebra $kG$ of the group $G = G(H)$, the set of group-like elements of $H$.

### 2.2 Hopf Quivers

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows, and $s, t : Q_1 \rightarrow Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver $Q$ is a finitely ordered sequence of $l$ arrows $a_i \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l - 1$. By convention a vertex is said to be a trivial path of length 0. The path coalgebra $kQ$ is the $k$-space spanned by the paths of $Q$ with counit and comultiplication maps defined by $\varepsilon(g) = 1$, $\Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each nontrivial path $p = a_n \cdots a_1$, $\varepsilon(p) = 0$,

$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.$$

The length of paths gives a natural gradation to the path coalgebra. Let $Q_n$ denote the set of paths of length $n$ in $Q$, then $kQ = \bigoplus_{n \geq 0} kQ_n$ and $\Delta(kQ_n) \subseteq \bigoplus_{n=i+j} kQ_i \otimes kQ_j$. Clearly $kQ$ is pointed with the set of group-
likes \( G(kQ) = Q_0 \), and has the following coradical filtration

\[
kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots .
\]

Hence \( kQ \) is coradically graded. The path coalgebras can be presented as cotensor coalgebras, so they are cofree in the category of pointed coalgebras and enjoy a universal mapping property (see e.g. [25]).

According to [7], a quiver \( Q \) is said to be a Hopf quiver if the corresponding path coalgebra \( kQ \) admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups. Let \( G \) be a group, \( C \) the set of conjugacy classes. A ramification datum \( R \) of the group \( G \) is a formal sum \( \sum_{C \in C} R_C C \) of conjugacy classes with coefficients in \( \mathbb{N} = \{0, 1, 2, \cdots \} \). The corresponding Hopf quiver \( Q = Q(G, R) \) is defined as follows: the set of vertices \( Q_0 \) is \( G \), and for each \( x \in G \) and \( c \in C \), there are \( R_C \) arrows going from \( x \) to \( cx \). For a given Hopf quiver \( Q \), the set of graded Hopf structures on \( kQ \) is in one-to-one correspondence with the set of \( kQ_0 \)-Hopf bimodule structures on \( kQ_1 \).

3 Quiver Setting for Majid Algebras

A Majid algebra \( H \) is a priori a coalgebra. If \( H \) is pointed, then by [5] there is a unique quiver \( Q(H) \) such that \( H \) can be viewed as a “large” subcoalgebra of the path coalgebra \( kQ(H) \). Here by a “large” subcoalgebra of \( kQ(H) \) is meant it contains at least the \( k \)-space \( kQ(H)_0 \oplus kQ(H)_1 \) spanned by the set of vertices and the set of arrows. The main aim of this section is to determine what quivers come up as the quivers of pointed Majid algebras, and conversely to construct Majid structures from these quivers.

Firstly we consider for what quiver \( Q \) the associated path coalgebra \( kQ \) can be endowed with a graded Majid algebra structure. It turns out that we have nothing new beyond the Hopf quivers of Cibils and Rosso [7].

**Theorem 3.1.** Let \( Q \) be a quiver. Then the path coalgebra \( kQ \) admits a graded Majid algebra structure if and only if \( Q \) is a Hopf quiver.

**Proof:** We assume first that \( Q \) is a Hopf quiver. Then by [7], there exists a graded Hopf algebra structure on the path coalgebra \( kQ \). Roughly the
graded Hopf structure is constructed as follows. First by definition there exist a group $G$ and a ramification datum $R$ such that $Q = Q(G, R)$. Then view $kQ_0$ as the group Hopf algebra $kG$ and choose a $kG$-Hopf bimodule structure on the space $kQ_1$. Finally the graded Hopf algebra is obtained by extending the bimodule structure to get a compatible algebra structure with path coalgebra $kQ$ via the universal mapping property [25]. Note that Hopf algebras can be viewed as Majid algebras with trivial reassociator. Therefore, for the Hopf quiver $Q$, the associated path coalgebra $kQ$ admits a fortiori a graded Majid algebra structure.

Conversely, we prove that if $kQ$ admits a graded Majid algebra structure then $Q$ is a Hopf quiver. Assume that $(kQ, \Delta, \varepsilon, M, \Phi, S, \alpha, \beta)$ is a graded Majid algebra. First of all we can restrict the multiplication $M$ to $Q_0$ and make it a group. Indeed, since $M$ is a coalgebra map we have $\Delta(gh) = \Delta(M(g \otimes h) = gh \otimes gh$ for all $g, h \in Q_0$. That is, the multiplication is closed inside $Q_0$. For all $g \in Q_0$, we have $\alpha(g)\beta(g) \neq 0$ by (2.6) and then $S(g)g = gS(g) = 1$ by (2.5). Denote $S(g)$ by $g^{-1}$. Note that $g^{-1} \in Q_0$ since $S$ is a coalgebra antimorphism. In addition, since $\Phi$ is convolution-invertible we have $\Phi(f, g, h) \neq 0$ for all $f, g, h \in Q_0$, and now by (2.1) we have $f(gh) = (fg)h$, the associativity. It follows that endowed with the binary operation $M$ the set of vertices $Q_0$ becomes a group. For brevity we denote the group $(Q_0, M)$ as $G$.

Let $M$ denote the space $kQ_1$. Note that $M$ is a $kG$-bicomodule with structure maps $(\delta_L, \delta_R)$ defined by

\begin{align*}
\delta_L : M &\longrightarrow kG \otimes M, \quad a \mapsto t(a) \otimes a, \\
\delta_R : M &\longrightarrow M \otimes kG, \quad a \mapsto a \otimes s(a),
\end{align*}

for all $a \in Q_1$. Let $^gM^h = \{m \in M \mid \delta_L(m) = g \otimes m, \delta_R(m) = m \otimes h\}$ be the $(g, h)$-isotypic component of $M$. Then it is in fact the $k$-span of the arrows with source $h$ and target $g$. The graded quasi-algebra structure induces the following linear $kG$-actions on $M$:

\begin{align*}
\rho_L : kG \otimes M &\longrightarrow M, \quad g \otimes m \mapsto g.m, \\
\rho_R : M \otimes kG &\longrightarrow M, \quad m \otimes g \mapsto m.g,
\end{align*}

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where \( g.m \) and \( m.g \) mean the multiplication \( M(g \otimes m) \) and \( M(m \otimes g) \) respectively in \( kQ \). It is worthy to make a remark here that in general \( \rho_L \) (resp. \( \rho_R \)) does not make \( M \) a left (resp. right) \( kG \)-module, since the associativity holds only up to a non-zero scalar by (2.1). Precisely, for all \( e, f, g, h \in G \) and \( m \in gM^h \) we have

\[
e.(f.m) = \frac{\Phi(e,f,g)}{\Phi(e,f,h)} (ef).m, \tag{3.5}
\]

\[
(m.e).f = \frac{\Phi(h,e,f)}{\Phi(g,e,f)} m.(ef), \tag{3.6}
\]

\[
(e.m).f = \frac{\Phi(e,h,f)}{\Phi(e,g,f)} e.(m.f). \tag{3.7}
\]

The axioms of graded Majid algebra ensure that \( \rho_L \) and \( \rho_R \) are \( kG \)-bicomodule morphisms. Namely, since \( M \) is a coalgebra morphism we have for all \( f \in G \) and \( m \in gM^h \),

\[
\delta_L(f.m) = fg \otimes f.m, \quad \delta_L(m.f) = gf \otimes m.f, \tag{3.8}
\]

\[
\delta_R(f.m) = f.m \otimes fh, \quad \delta_R(m.f) = m.f \otimes hf. \tag{3.9}
\]

These equalities lead to \( f. gM^h \subseteq f gM^{fh} \), \( gM^h.f \subseteq g fM^{hf} \). Notice that \( f \in G \) is invertible, it follows that \( f^{-1}. f gM^{fh} \subseteq gM^h \) and \( g fM^{hf}.f^{-1} \subseteq gM^h \), and that \( f^{-1}.(f. gM^h) = gM^h \) and \( (gM^h.f).f^{-1} = gM^h \) by (3.5)-(3.6) and (2.2). It is interesting to note that the composition of \( f^{-1} \) and \( f \) actions in the preceding two identities are given by non-zero scalars \( \frac{\Phi(f^{-1}, f, g)}{\Phi(f^{-1}, f, h)} \) and \( \frac{\Phi(h, f^{-1}, f)}{\Phi(g, f^{-1}, f)} \) respectively, which are not necessarily 1 as usual. By combining the above arguments we have the following identities of vector spaces

\[
f. gM^h = f gM^{fh}, \quad gM^h.f = g fM^{hf}. \tag{3.10}
\]

In particular it follows that for all \( x, g, c \in G \),

\[
g^{-1}cgx = g^{-1}cM^1.x = (g^{-1}. (cM^1).g))x.
\]

It is clear that

\[
\dim_k g^{-1}cgx = \dim_k cM^1.
\]

Now recall the geometric meaning of the isotypic spaces. The above equation implies that, for all \( x \in G \) and all \( c' \in C \), where \( C \) is the conjugacy class
containing \( c \), there are \( \dim_k c \times M^1 \) arrows going from \( x \) to \( c' \times x \) in \( Q \). Let \( C \) be the set of the conjugacy classes of \( G \). For each \( C \in C \), fix an element \( c \in C \) and set \( R_C = \dim_k c \times M^1 \). Take a ramification datum of \( G \) as \( R = \sum_{C \in C} R_C C \). It follows from the previous arguments that \( Q \) is exactly the Hopf quiver \( Q(G, R) \). We are done. \( \square \)

Next we investigate the construction of (non-trivial) Majid algebras from given Hopf quivers. Naturally we need a quasi-Hopf analogue of the notion of Hopf bimodules, whose axioms already appear implicitly in the preceding proof of Theorem 3.1.

**Definition 3.2.** Assume that \( H \) is a Majid algebra with reassociator \( \Phi \). A linear space \( M \) is called an \( H \)-Majid bimodule, if \( M \) is an \( H \)-bicomodule with structure maps \( (\delta_L, \delta_R) \), and there are two \( H \)-bicomodule morphisms \( \rho_L : H \otimes M \rightarrow M, \ h \otimes m \mapsto h.m, \ \rho_R : M \otimes H \rightarrow M, \ m \otimes h \mapsto m.h \) such that for all \( g, h \in H, m \in M \), the following equalities hold:

\[
1_H.m = m = m.1_H, \tag{3.11}
\]
\[
g_1.(h_1.m_0)\Phi(g_2,h_2,m_1) = \Phi(g_1,h_1,m^{-1})(g_2 h_2).m^0, \tag{3.12}
\]
\[
m_0.(g_1 h_1)\Phi(m_1,g_2,h_2) = \Phi(m^{-1},g_1,h_1)(m^0.g_2).h_2, \tag{3.13}
\]
\[
g_1.(m_0.h_1)\Phi(g_2,m_1,h_2) = \Phi(g_1,m^{-1},h_1)(g_2.m^0).h_2, \tag{3.14}
\]

where we use the Sweedler notation \( \delta_L(m) = m^{-1} \otimes m^0, \ \delta_R(m) = m_0 \otimes m_1 \) for comodule structure maps.

We remark that this notion can be rephrased by that of \( H \)-bimodules in the monoidal category of \( H \)-bicomodules, cf. [17]. Of course, if the 3-cocycle \( \Phi \) is trivial then the Majid algebra \( H \) is a usual Hopf algebra and \( H \)-Majid bimodules are the usual Hopf bimodules.

Now let \( Q \) be a Hopf quiver and assume that \((H, \Delta, \varepsilon, M, \mu, \Phi, S, \alpha, \beta)\) is a graded Majid algebra structure on the path coalgebra \( kQ \). We have proved that \( Q_0 \) is a group, denoted by \( G \), and \( kQ_0 \) is a sub Majid algebra with reassociator \( \Phi \) and quasi-antipode \((S, \alpha, \beta)\). For brevity we denote this
Majid algebra by \((kG, \Phi)\). Note that the restriction of \(\Phi\) to \(G\) is a 3-cocycle and the space \(kQ_1\) is a natural \((kG, \Phi)\)-Majid bimodule with structure maps given by (3.1)-(3.4).

Conversely, let \(Q = Q(G, R)\) be a Hopf quiver and \(\Phi\) a 3-cocycle on the group \(G\). Then \(kQ_0 = kG\) can be understood as a Majid algebra with reassociator \(\Phi\). If the space \(kQ_1\) can be endowed with a \((kG, \Phi)\)-Majid bimodule structure, then we can construct a graded Majid algebra structure on the path coalgebra \(kQ\) via these data as follows. The process is similar to [7].

Let \(M_0 : kQ \otimes kQ \rightarrow kQ_0\) be the composition of the canonical projection \(\pi_0 \otimes \pi_0 : kQ \otimes kQ \rightarrow kQ_0 \otimes kQ_0\) and the multiplication of the group algebra \(kG = kQ_0\), and \(M_1 : kQ \otimes kQ \rightarrow kQ_1\) the composition of the canonical projection \(\pi_0 \otimes \pi_1 \oplus \pi_1 \otimes \pi_0 : kQ \otimes kQ \rightarrow kQ_0 \otimes kQ_1 \oplus kQ_1 \otimes kQ_0\) and the sum of left and right quasi \(kG\)-actions. Then it is clear that \(M_0\) is a coalgebra map and \(M_1\) is a \(kQ_0\)-bicomodule map. Let \(M_n = M_1^\otimes n \circ \Delta_2^{(n-1)} : kQ \otimes kQ \rightarrow kQ_n\), where \(\Delta_2^{(n-1)}\) is the \((n-1)\)-iterated comultiplication of the tensor product coalgebra \(kQ \otimes kQ\). For any pair of paths \(p\) and \(q\) with \(l(p) + l(q) = m\), it is easy to see that \(M_n(p \otimes q) = 0\) if \(m \neq n\). Therefore by the universal mapping property \(M = \sum_{n \geq 0} M_n : kQ \otimes kQ \rightarrow kQ\) is a well-defined coalgebra map and moreover respects the length gradation. The quasi-associativity for the map \(M\) follows from the quasi-associativity (3.12)-(3.14) of the quasi-bimodule. Note that the reassociator for \(kQ\) is obtained by a trivial extension of \(\Phi\) such that \(\Phi(x, y, z) = 0\) whenever one of \(x, y, z\) lies out of \(kQ_0\). Hence the map \(M\) defines a quasi-algebra structure and we get a graded Majid bialgebra structure on \(kQ\). We can also construct a quasi-antipode \((S, \alpha, \beta)\) again via the universal mapping property. For all \(g \in Q_0\), recall that \(S(g) = g^{-1}\) and let \(\alpha(g) = 1\) and \(\beta(g) = 1/\Phi(g, g^{-1}, g)\).

Extend \(\alpha\) and \(\beta\) to the function on \(kQ\) by letting \(\alpha(p) = \beta(p) = 0\) for all nontrivial paths \(p\) in the quiver \(Q\). Let \(kQ^{\text{cop}}\) denote the coopposite coalgebra of \(kQ\). Set \(S_0 : kQ^{\text{cop}} \rightarrow kQ_0\) to be the composition of the projection \(\pi_0 : kQ^{\text{cop}} \rightarrow kQ_0\) and the map of taking inversion of the group \(G\). Set \(S_1 : kQ^{\text{cop}} \rightarrow kQ_1\) to be the composition of the projection \(\pi_1 : kQ^{\text{cop}} \rightarrow kQ_1\) and the map \(a \mapsto -\frac{\Phi(s(a), s(a), s(a)^{-1})}{\Phi(t(a), s(a), s(a)^{-1})}(t(a)^{-1}.a.s(a)^{-1}\) for all \(a \in Q_1\).
Then $S_0$ is a coalgebra map and $S_1$ is a $kQ_0$-bicomodule map, then there is a coalgebra map $S : kQ^{\text{cop}} \to kQ$ by the universal mapping property. By direct verification we can show that it is the desired quasi-antipode map.

We summarize the foregoing arguments in the following:

**Proposition 3.3.** Let $G$ be a group and $(kG, \Phi, S, \alpha, \beta)$ a Majid algebra. Let $Q = Q(G, R)$ be the Hopf quiver associated to a ramification datum $R$ of $G$. Then the path coalgebra $kQ$ admits a graded Majid algebra structure with $kQ_0 \cong (kG, \Phi, S, \alpha, \beta)$ as a sub Majid algebra if and only if $kQ_1$ admits a $(kG, \Phi)$-Majid bimodule structure. Moreover, the set of such graded Majid algebra structures on the path coalgebra $kQ$ is in one-to-one correspondence with the set of $(kG, \Phi)$-Majid bimodule structures on $kQ_1$.

We remark that a Hopf quiver can be realized by different groups with ramification data. So in order to get all the graded Majid algebra structures on the path coalgebra $kQ$ of a given quiver $Q$, we should consider all the possible realizations of $Q$ as Hopf quiver and all the possible $kQ_0$-Majid bimodule structures on $kQ_1$. Here comes up a natural question of classifying the category of $(kG, \Phi)$-Majid bimodules for general $G$ and $\Phi$. It is well-known from [20, 21] that the category of Hopf bimodules over a finite-dimensional Hopf algebra $H$ is equivalent to the module category of the Drinfeld double $D(H)$. There is an analogue for the setting of Majid algebras and Majid bimodules. In particular, when $G$ is a finite group, the category of $(kG, \Phi)$-Majid bimodules can be described by the module category of the twisted quantum double $D^\Phi(G)$ introduced in [8]. The author is grateful to the referee for suggesting that the relation between this reference and the present paper should be investigated.

Finally we consider general pointed Majid algebras. Let $H$ be a pointed Majid algebra and $\text{gr} H$ its coradically graded version as mentioned in subsection 2.1. Write the set $G(H)$ of group-like elements as $G$. Then $H_0 \cong (kG, \Phi, S, \alpha, \beta)$ as Majid algebras for some appropriate $\Phi, \alpha, \beta$ and $H_1/H_0$ is a $(kG, \Phi)$-Majid bimodule. Let $Q(H)$ be the quiver of $H$, then it must be a Hopf quiver since by construction $kQ(H)_1 \cong H_1/H_0$ which admits a $kQ_0 \cong (kG, \Phi)$-Majid bimodule structure. The Gabriel type theorem for pointed Hopf algebras in [25] can be generalized to the following for pointed
Majid algebras.

**Theorem 3.4.** Suppose that \( H \) is a pointed Majid algebra and \( \text{gr} H \) its graded version induced by the coradical filtration. Then there is a unique Hopf quiver \( Q(H) \) and a graded Majid algebra structure on the path coalgebra \( kQ(H) \) such that \( \text{gr} H \) can be embedded into it as a sub Majid algebra which contains \( kQ(H)_0 \oplus kQ(H)_1 \).

The proof in [25] can be modified to the quasi setting, so we omit the detail. This theorem enables us to construct pointed Majid algebras exhaustively on Hopf quivers. A classification program of pointed Majid algebras can be carried out in the quiver framework. The first step is to classify all \((kG, \Phi)\)-Majid bimodules for general group \( G \) and 3-cocycle \( \Phi \). This amounts to a classification of graded Majid algebras on path coalgebras. This is achieved in a subsequent work [14] by generalizing [6, 7] further to our Majid setting. The second step is to classify large sub Majid algebras of those on path coalgebras. This gives a classification of general pointed coradically graded Majid algebras. The third step is to carry out a suitable deformation process (see e.g. [23]) to get general pointed Majid algebras from the graded ones. Certainly the classification problem is very difficult. In this paper we do not intend to go very far in this program.

We conclude this section with a corollary of Theorems 3.1 and 3.4.

**Corollary 3.5.** Let \( Q \) be an arbitrary quiver. Then the path coalgebra \( kQ \) admits a Majid algebra structure (not necessarily graded) if and only if \( Q \) is a Hopf quiver.

**4 Some Structure Theorems**

We apply the quiver setting to investigate general pointed Majid algebras. In particular, some structure theorems analogous to the Cartier theorem and the Cartier-Gabriel decomposition theorem are obtained.

Let \((H, \Phi, S)\) be a pointed Majid algebra, \( G \) its set of group-likes and \( Q \) its quiver. Assume that \( Q \) is the Hopf quiver \( Q(G, R) \) associated to some ramification datum \( R = \sum_{C \in C} R_CC \). Note that the quiver \( Q \) is connected if and only if the set \( \{ c \in C \mid R_C \neq 0 \} \) generates the group \( G \). In general,
for each $g \in G$ let $Q(g)$ be the connected component of $Q$ containing $g$. Denote by $e$ the unit element of $G$. The set $N$ of vertices of $Q(e)$ is a normal subgroup of $G$, since it is generated by the set $\{c \in C \mid R_C \neq 0\}$ which is a union of conacy classes. Each connected component $Q(g)$ is identical to $Q(e)$ as graphs, and its set of vertices is exactly the coset $Ng$. The number of connected components of $Q$ is exactly the index $[G : N]$.

The graphical features of the quiver $Q$ imply similar properties for $H$. The coalgebra embedding $H \hookrightarrow kQ$ decomposes $H$ into blocks in traditional terms of algebra (see [12]), or link-indecomposable components in the sense of Montgomery (see [19]). Let $H(g)$ be the image of $H$ in $kQ(g)$, i.e., the block (or link-indecomposable component) of $H$ containing $g$. We call $H(e)$ the principal block. Obviously $G(H(e)) = N$ and the number of blocks is equal to $[G : N]$. Moreover, we have the following theorem. It is a quasi-Hopf analogue of the Theorem 3.2 of [19], which can be viewed as a generalization of the Cartier-Gabriel decomposition theorem.

**Theorem 4.1.** Keep the notations as above.

1. The map $\text{Tr}_g : H(e) \rightarrow H(g)$ defined by $p \mapsto pg$ is a coalgebra isomorphism.

2. $H(g)H(h) \subseteq H(gh)$ and $S(H(g)) \subseteq H(g^{-1})$. In particular, $H(e)$ is a Majid algebra.

3. Assume further that $H$ is coradically graded. Then there is a Majid algebra isomorphism $H \cong H(e)\#_\sigma^\Phi kG/N$, where $\sigma : G/N \times G/N \rightarrow N$ is a 2-cocycle and $H(e)\#_\sigma^\Phi kG/N$ is a crossed product twisted by $\Phi$.

**Proof:** The claims (1) and (2) are easy. We only prove the claim (3). Take a set $T$ of distinct coset representatives of $N$ in $G$. In particular, for the unit coset $N$ we take $e$ as its representative. For any $g \in G$, write $\overline{g} \in T$ as the representative of the coset in that $g$ lies. Then there is a 2-cocycle $\sigma : G/N \times G/N \rightarrow N$ such that $\overline{u} \overline{v} = \sigma(\overline{u}, \overline{v})\overline{uv}$ for any $\overline{u}, \overline{v} \in T$. It follows by (1) that $H = \bigoplus_{\overline{u} \in T} H(e)\overline{u} \cong H(e) \otimes kG/N$. Since $H$ is coradically graded, by Theorem 3.4 it can be viewed as a sub Majid algebra of $kQ$. Hence we can choose a basis for $H$ consisting of paths or linear combinations of paths.
with the same source and target. For any basis elements \( p, q \in H(e) \) and \( \bar{u}, \bar{v} \in T \), define the operation

\[
(p \otimes \bar{u})(q \otimes \bar{v}) = p(\bar{u} \triangleright q) \otimes \Theta(p, q, \bar{u}, \bar{v})\sigma(\bar{u}, \bar{v})\bar{uv},
\]

where \( \bar{u} \triangleright q = \bar{u}(q\bar{u}^{-1}) \) and \( \Theta(p, q, \bar{u}, \bar{v}) \in k \) is equal to

\[
\frac{\Phi(s(p), \bar{u}, s(q)\bar{v})\Phi(s(q), \bar{u}^{-1}, \bar{u}\bar{v})\Phi(t(p), t(q)\bar{u}^{-1}, \bar{u}\bar{v})\Phi(t(p), t(q), \bar{u}\bar{v})}{\Phi(t(p), \bar{u}, t(q)\bar{v})\Phi(t(q), \bar{u}^{-1}, \bar{u}\bar{v})\Phi(t(p), t(q), \bar{u}\bar{v})}\Phi(s(p), s(q), \bar{u}\bar{v})\Phi(\bar{u}, t(q)\bar{u}^{-1}, \bar{u}\bar{v})\Phi(t(p), t(q), \bar{u}\bar{v})\Phi(s(p), s(q), \bar{u}\bar{v})\Phi(\bar{u}, t(q)\bar{u}^{-1}, \bar{u}\bar{v})\Phi(t(p), t(q), \bar{u}\bar{v})\Phi(s(p), s(q), \bar{u}\bar{v})}.
\]

Here we use \( s(p) \) and \( t(p) \) to denote respectively the source and the target of a path \( p \). This operation defines a Majid algebra structure on the tensor coalgebra \( H(e) \otimes kG/N \), the so-called crossed product twisted by \( \Phi \). Now by direct calculation we can show that the coalgebra isomorphism

\[
H \longrightarrow H(e) \otimes kG/N, \quad p\bar{u} \mapsto p \otimes \bar{u}
\]

preserves the quasi-algebra structure. Therefore we get the desired Majid algebra isomorphism \( H \cong H(e) \#^\Phi kG/N \).

\hspace{1cm} \Box

**Remark 4.2.** Thanks to this theorem, the study of pointed Majid algebras can be reduced to their principal blocks, or equivalently to the connected case. We make the technical assumption in (3) to guarantee a simpler definition and exposition for the crossed product \( H(e) \#^\Phi kG/N \) twisted by \( \Phi \). One can handle more general situation by adjusting related works on smash products of quasi-Hopf algebras (see for instance [3]) to Majid algebras.

In the following, we consider cocommutative pointed Majid algebras. We hope to develop a quasi-Hopf analogue of the Cartier theorem. Hence from now on, the ground field \( k \) is assumed to be of characteristic zero.

Let \( H \) be a cocommutative pointed Majid algebra and \( Q \) its quiver. By the decomposition theorem, we may assume that \( H \) is link-indecomposable, or equivalently the quiver \( Q \) is connected. In this situation, \( Q \) must be a multi-loop quiver, that is, a quiver with only one vertex and with arrows starting and ending at it. If there were at least two vertices in \( Q \), then there is at least one arrow such that its source is different from its target. Let \( a \) be such an arrow in \( Q \). Since by [5] \( H \) can be regarded as a large subcoalgebra
of \( kQ \), one may assume that \( a \in H \). This leads to a contradiction with the cocommutativeness:

\[
\Delta(a) = t(a) \otimes a + a \otimes s(a) \neq a \otimes t(a) + s(a) \otimes a = \Delta^{op}(a).
\]

Now let \( g \) denote the set \( \{ x \in H \mid \Delta(x) = x \otimes 1 + 1 \otimes x \} \) of primitive elements. For any \( x, y, z \in g \), their multiplication in \( H \) is associative

\[
x(yz) = (xy)z
\]

according to the axioms (2.1) and (2.4). With bracket defined by the usual commutator, \( g \) becomes a Lie algebra. Let \( U(g) \) denote the corresponding universal enveloping algebra. The classical theorem of Cartier asserts that a cocommutative connected (=pointed and irreducible) Hopf algebra must be isomorphic to the universal enveloping algebra of the Lie algebra of its primitive elements. We show that this is also the case for Majid algebras. The key point is that, as quasi-algebra a cocommutative connected Majid algebra is generated by primitive elements.

**Theorem 4.3.** Let \( H \) be a cocommutative connected Majid algebra over a field \( k \) of characteristic zero and \( g \) the set of primitives. Then \( H \cong U(g) \). In particular, \( H \) is a usual Hopf algebra and is generated by primitives.

**Proof:** By the universal property of enveloping algebra \( U(g) \), the embedding \( g \hookrightarrow H \) can be extended to a Majid algebra map \( \phi : U(g) \to H \). The map \( \phi \) is injective, since as coalgebra map its restriction to the space of primitive elements is injective (see e.g. [18]).

On the other hand, as coalgebra \( H \) can be embedded into the path coalgebra of a multi-loop quiver. The space spanned by the loops is in fact isomorphic to \( g \). We denote the maximal cocommutative sub coalgebra of the path coalgebra by \( \text{Sc}(g) \). Since \( H \) is cocommutative, it can even be viewed as a sub coalgebra of \( \text{Sc}(g) \).

Now we get a series of coalgebra embeddings \( U(g) \hookrightarrow H \hookrightarrow \text{Sc}(g) \). By the Poincaré-Birkhoff-Witt theorem and the structural property of \( \text{Sc}(g) \), one can show that the composition of coalgebra maps actually gives rise to an isomorphism \( U(g) \cong \text{Sc}(g) \) of coalgebras. This leads to the claimed isomorphism \( H \cong U(g) \). \( \square \)
We remark that the proof is a bit sketchy. The same argument was used in [13] to provide a simple proof for the classical Cartier theorem. As corollary of the previous results, cocommutative pointed Majid algebras must be the usual Hopf algebras with possibly nontrivial 3-cocycles and isomorphic to the smash product of universal enveloping algebras and group algebras. In particular, finite-dimensional cocommutative pointed Majid algebras are finite group algebras with 3-cocycles.

5 Summary

We have built up a quiver setting for pointed Majid algebras. For the case of elementary quasi-Hopf algebras, that is all its simple modules are one-dimensional, the quiver setting can be provided dually by using path algebras of quivers instead of path coalgebras.

At present there is still a lack of abundant examples and general structure theorems in the quasi-Hopf algebra theory, let alone the classification. By taking advantage of quiver techniques, bundles of examples can be constructed easily on concrete Hopf quivers via the process showed in Section 3. The graphical information naturally indicates some general structure theorems. Further, a classification program may be carried out in the quiver framework.

The quiver setting is expected to be very useful in carrying out a systematic study for the theory of quasi-Hopf algebras. In particular, we will show in forthcoming works that the quiver techniques can help to generalize the celebrated theory of pointed Hopf algebras (see [1] and references therein) to quasi-Hopf algebras, and to construct and classify some interesting finite tensor categories, cf. [10].

Acknowledgement: The author thanks the referee for valuable comments and suggestions which improve the exposition. The research was supported by the National NSF of China under grant number 10601052. Part of the work was done in the Chern Institute of Mathematics (CIM) supported by the Visiting Scholar Program. The author thanks CIM, in particular Professor Chengming Bai, for hospitality.
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