A Path integral approach to the scattering theory of quantum transport

D. Endesfelder

Oxford University, Theoretical Physics,
1 Keble Road,
United Kingdom
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The scattering theory of quantum transport relates transport properties of disordered mesoscopic conductors to their transfer matrix T. We introduce a novel approach to the statistics of transport quantities which expresses the probability distribution of T as a path integral. The path integral is derived for a model of conductors with broken time reversal invariance in arbitrary dimensions. It is applied to the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation which describes quasi-one-dimensional wires. We use the equivalent channel model whose probability distribution for the eigenvalues of TT† is equivalent to the DMPK equation independent of the values of the forward scattering mean free paths. We find that infinitely strong forward scattering corresponds to diffusion on the coset space of the transfer matrix group. It is shown that the saddle point of the path integral corresponds to ballistic conductors with large conductances. We solve the saddle point equation and recover random matrix theory from the saddle point approximation to the path integral.

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1. INTRODUCTION

Advances in microfabrication technology led to the realization of mesoscopic electronic devices. In such devices the mean free path for inelastic electron scattering exceeds the dimension of the device. As a consequence phase coherence is maintained which leads to quantum interference effects like universal conductance fluctuations, persistent currents and Aharonov-Bohm oscillations in rings, or weak localization. The phase coherence also has serious theoretical implications. It causes large conductance fluctuations which are related to the problem of high gradient operators in the field theoretic description of the metal insulator transition. These fluctuations manifest themselves already in the metallic region as logarithmic normal tails of the conductance probability distribution. As the critical regime is approached the conductance probability distribution becomes increasingly broader until it reaches a logarithmic normal form in the insulating regime.

A common approach to transport quantities of mesoscopic conductors is the scattering theory of quantum transport. It models the conductor by a disordered region which is connected to a number of ideal leads which support propagating wave modes. The number of leads corresponds to the number of measurement terminals. Here only two terminal geometries will be considered. The scattering matrix relates the amplitudes ℏk, ℏ′k of the incoming with the amplitudes O₂k, O₂′k (k = 1, ..., N) of the scattered propagating wave modes at the Fermi energy,

\[
\begin{pmatrix}
O \\
O'
\end{pmatrix} = S \begin{pmatrix}
I \\
I'
\end{pmatrix},
\]

where

\[
S = \begin{pmatrix}
t & t' \\
\bar{r} & \bar{r}'
\end{pmatrix},
\]

t and \(\bar{r}\) are the transmission and reflection matrices for incident waves from the left, and \(t'\) and \(\bar{r}'\) are the transmission and reflection matrices for incident waves from the right. The dimensionless two-probe conductance \(g = G/\hbar^2\) in terms of the transmission eigenvalues \(T_k\) of \(tt'\) is

\[
g = \sum_{k=1}^{N} T_k.
\]

There are three universality classes which correspond to different physical situations. Conductors with time reversal invariance lie in the orthogonal universality class. The unitary universality class corresponds to conductors in which the time reversal symmetry is broken, e.g. by a magnetic field. Conductors with spin-flip scattering processes but no time reversal symmetry breaking fall into the symplectic universality class. Recently the quasi-one-dimensional wire has attracted considerable attention. The width of a quasi-one-dimensional wire is of the order of the mean free path for elastic electron scattering so that transverse diffusion can be neglected and the cross section of the wire becomes structureless. Interesting non-perturbative results which are valid for all wire lengths have been obtained for this system. Furthermore it has been the ideal playground for new ideas in the field of quantum transport.

One of these ideas is the Fokker-Planck (FP) approach to quasi-one-dimensional wires. The FP equation which describes the probability distribution for the transmission eigenvalues is known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. It has been derived
by a number of authors who started from various different models. Its form is
\[
\frac{\partial p(s; \{ \Gamma_k \})}{\partial s} = \frac{2}{\gamma} \sum_{k=1}^{N} \frac{\partial}{\partial \Gamma_k} \left( \frac{\partial p}{\partial \Gamma_k} + \beta p \frac{\partial \Omega(\{ \Gamma_k \})}{\partial \Gamma_k} \right), \tag{4}
\]
where
\[
\Omega(\{ \Gamma_k \}) = -\sum_{k<l} \ln |(\cosh \Gamma_k - \cosh \Gamma_l)/2| \nonumber \\
-1/\beta \sum_k \ln |\sinh \Gamma_k|, \tag{5}
\]
\[
\gamma = \beta N + 2 - \beta, \quad \text{and} \quad \cosh \Gamma_k = (2 - T_k)/T_k. \quad \text{The values of} \quad \beta \quad \text{are} \quad 1, 2, \quad \text{and} \quad 4 \quad \text{for the orthogonal, unitary and symplectic universality class respectively. The DMPK equation has been studied intensively in the past few years.} \quad \text{These equations} \quad \text{are more powerful than the FP approach when it can be developed into a tool for the disorder averages. The DMPK equation is}
\]
\[
\frac{\partial p(s; \{ \Gamma_k \})}{\partial s} = \frac{1}{\beta} \sum_k \ln |\sinh \Gamma_k|, \tag{5}
\]
\[
\psi_1(\{ \Gamma_k \}) \text{obeys a Schr"{o}dinger equation for} \quad N \quad \text{non-interacting particles. As a consequence the exact form of} \quad p(s; \{ \Gamma_k \}) \quad \text{could be determined. This solution has been the basis for Frahm's exact calculation of the one- and two-point correlation functions of the transmission eigenvalues.}
\]

In this paper we present a novel approach to the scattering theory of quantum transport which expresses the probability distribution of the transfer matrix as a path integral. Our motivation has been the belief that the path integral technique can be developed into a tool which is more powerful than the FP approach when it comes to the description of higher-dimensional conductors.

\section{II. SCATTERING MODEL}

We use the transfer matrix \( T \) instead of the \( S \)-matrix to model the scattering properties of the disordered conductor. The transfer matrix relates the scattering amplitudes in the left lead with the scattering amplitudes in the right lead
\[
\begin{pmatrix}
O' \\
I'
\end{pmatrix} = T \begin{pmatrix}
I \\
O
\end{pmatrix}. \tag{7}
\]
It has the advantage that it obeys the multiplication law
\[
T(L + \delta L, 0) = T(L + \delta L, L)T(L, 0) \tag{8}
\]
which leads to the simple Langevin equation
\[
\dot{T}(x) = dT(x, 0)/dx = \varepsilon(x)T(x, 0) \quad \text{for the stochastic evolution of the transfer matrix. The disorder is generated by the multiplicative noise} \quad \varepsilon.
\]
In this paper we consider only conductors in the unitary universality class. Then, \( T \) obeys the symmetry constraint
\[
\Sigma_x T^\dagger \Sigma_z T = 1 \tag{10}
\]
where \( \Sigma_x \) is a real, diagonal \( N \times N \) matrix and \( u_i \) \( (i = 1, 2, 3; 4) \) are unitary \( N \times N \) matrices.

The relation (9) implies that \( \Sigma_x \varepsilon \Sigma_z \varepsilon + \varepsilon = 0 \) leading to the symmetries
\[
\varepsilon_{11} = -\varepsilon_{11}^\dagger, \quad \varepsilon_{22} = -\varepsilon_{22}^\dagger, \quad \varepsilon_{12} = \varepsilon_{21}^\dagger \tag{13}
\]
for the noise. The stochastic properties of \( \varepsilon \) could be derived from a microscopic Hamiltonian. Here, we adopt a simple model which assumes Gaussian white noise such that
\[
\langle \varepsilon_{kl}(x) \rangle = 0, \nonumber \\
\langle \varepsilon_{kl}(x)\varepsilon_{kl'}(x') \rangle = \frac{1}{t_{kl}^f} \delta_{kk'} \delta_{ll'} \delta(x - x'), 
\]
and all other independent second moments are zero. The mean free paths \( t_{kl}^f \), \( t_{kl}^b \), and \( t_{kl}^f \) for forward and backward scattering, respectively, are defined by the limits of the disorder averages
\[
\frac{1}{t_{kl}^f} = \lim_{\delta L \to 0} \frac{\langle |t_{kl} - \delta t_{kl}|^2 \rangle_{\delta L}}{\delta L}, \nonumber \\
\frac{1}{t_{kl}^b} = \lim_{\delta L \to 0} \frac{\langle |t_{kl} - \delta t_{kl}|^2 \rangle_{\delta L}}{\delta L}, \nonumber \\
\frac{1}{t_{kl}^o} = \lim_{\delta L \to 0} \frac{\langle |t_{kl}^o|^2 \rangle_{\delta L}}{\delta L}, \tag{15}
\]
for a short piece of conductor with length $\delta L$. Note that the symmetries $[l_3]$ imply the relation $l_{kl} = l_{lk}$.

We want a path integral representation of the stochastic process $[l_1]$ in terms of the transfer matrix $T$. The derivation technique which is most suited for that purpose derives the path integral directly from the Langevin equation (see chapter 4 in Ref. [3]). The symmetry constraints $[l_0]$ on $T$ will be taken into account by $\delta$-functions which leads naturally to the invariant measure of the transfer matrix group as the path integration measure. We illustrate the essential ideas of the derivation technique with the simple example of diffusion on a circle before we deal with the transfer matrix.

### III. DIFFUSION ON THE CIRCLE AS A SIMPLE EXAMPLE

Let the angle $\varphi$ determine the position on a circle. The analogue of the Langevin equation $[l_1]$ is

$$\dot{u} \equiv \frac{du(t)}{dt} = \varepsilon(t)u(t)$$

(16)

where $u = \exp(i\varphi)$. The symmetry $\varepsilon^* = -\varepsilon$ implies $d(u\varepsilon^*)/dt = 0$ which ensures that $u$ remains a phase. Choosing Gaussian white noise for the imaginary part of $\varepsilon$ such that

$$\langle \varepsilon(t) \rangle = 0$$

$$\langle \varepsilon(t)\varepsilon(t')^* \rangle = 2D\delta(t - t')$$

(17)

leads to the FP equation

$$\frac{\partial p(t; \varphi)}{\partial t} = D\frac{\partial^2 p(t; \varphi)}{(\partial \varphi)^2}$$

(18)

which describes diffusion on the circle.

The probability distribution of $u$ can be formally expressed as

$$p(t; u) = \langle \delta(u - \bar{u}(t)) \rangle$$

(19)

where $u \equiv u^{(1)} + iu^{(2)}$, $\delta(u) \equiv \delta(u^{(1)})\delta(u^{(2)})$, and $\bar{u}(t)$ is the value of $u$ which is acquired at time $t$ for a certain realization of the noise and the initial value $\bar{u}(0) = u_0$. The brackets $\langle \cdots \rangle$ denote the average over all possible noise configurations. The path integral representation is derived by inserting a product of $\delta$-functions

$$p(t; u) = \left\langle \prod_{t' = 0}^t du(t')\delta(u(t') - \bar{u}(t'))\delta(u(t) - u) \right\rangle$$

(20)

where $du \equiv du^{(1)}du^{(2)}$. The $\delta$-function $\delta(u(t') - \bar{u}(t'))$ restricts the value of $u(t')$ to $\bar{u}(t')$. Since $\bar{u}(t')$ is not explicitly known we enforce this constraint implicitly by the relation $\dot{u}(t)u^{-1}(t) - \varepsilon(t) = 0$ which follows from the Langevin equation (10). That leads to

$$p(t; u) = \left\langle \prod_{t' = 0}^t du(t') | \det \hat{A} | \delta(\dot{u}(t')u^{-1}(t') - \varepsilon(t')) \right\rangle$$

$$\times \delta(u(t) - u)$$

(21)

where the operator $\hat{A}$ is defined by the functional derivative

$$\hat{A}_{xy}(t, t') = \frac{\delta(\dot{u}(t)u^{-1}(t) - \varepsilon(t))/\delta u^{(y)}(t')}{}.$$  

(22)

The average over the Gaussian probability measure

$$P[\varepsilon] \prod_{x = 0}^L d\varepsilon(x) = \frac{1}{N} \exp \left\{ - \int_0^L d\varepsilon(x)\varepsilon^*(x) \right\} \prod_{x = 0}^L d\varepsilon(x),$$

(23)

where

$$d\varepsilon = d\varepsilon^{(1)}d\varepsilon^{(2)}\delta(\varepsilon + \varepsilon^*)$$

yields

$$p(t; u) = \frac{1}{N} \int_0^t \prod_{t' = 0}^t du(t')\delta(\dot{u}(t')u^{-1}(t') + \dot{u}^*(t')$$

$$\times u^{-1}(t'))| \det \hat{A} | \exp \{-\hat{S}\},$$

(25)

where

$$\hat{S} = \frac{1}{4D} \int_0^t dt' \dot{u}(t')u^{-1}(t')\dot{u}(t')u^{-1}(t')^*$$

(26)

and the path summation includes all paths which start at $u_0$ and end at $u$.

The property that $\dot{u}(t)u^{-1}(t) + \dot{u}^*(t)u^{-1}(t) = \dot{u}(t)u^{-1}(t) + \dot{u}^*(t)u^{-1}(t)^*$ if $w(t) = u(t)v(t)$ and $v(t)$ is a phase, suggests that $\prod_{t' = 0}^t du(t')\delta(\dot{u}(t')u^{-1}(t') + \dot{u}^*(t')u^{-1}(t'))$ is proportional to $\prod_{t' = 0}^t d\mu(u(t'))$ where $d\mu(u)$ is the invariant measure on $U(1)$. This becomes explicit if the $\delta$-function is introduced via an auxiliary field $\kappa(t')$

$$p(t; u) = \int_0^t \prod_{t' = 0}^t du(t')d\kappa(t') | \det \hat{A} | \exp \{-\hat{S}\},$$

(27)

where

$$\hat{S} = S + i \int_0^t dt' \kappa(t'(t')u^{-1}(t') + \dot{u}^*(t')u^{-1}(t'))$$

$$= S + i \int_0^t dt' \kappa(t') \frac{d}{dt'} \ln(u(t')u^*(t')).$$

(28)

Partial integration yields
where the determinant of $\hat{A} = \hat{B}\hat{C}\hat{D}$ into a product of three operators

$[\hat{B}](t,t') = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta(t - t') & -i\delta(t - t') \\ -i\delta(t - t') & \delta(t - t') \end{pmatrix}$,  

$[\hat{C}](t,t') = \begin{pmatrix} a(t,t') & 0 \\ 0 & a^*(t,t') \end{pmatrix}$,  

$[\hat{D}](t,t') = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta(t - t') & i\delta(t - t') \\ i\delta(t - t') & \delta(t - t') \end{pmatrix}$

implies that det $\hat{A} = \text{det} \hat{C} = \text{det} \hat{\dot{a}} \text{det} \hat{a}^*$ since det $\hat{B} = \text{det} \hat{D} = 1$. The operator $\hat{a}$ can be as well factorized into $\hat{a} = \hat{a}_1\hat{a}_2\hat{a}_3$ where

$a_1(t,t') = u^{-1}(t)\delta(t - t')$  

$a_2(t,t') = \frac{d}{dt}\delta(t - t')$  

$a_3(t,t') = \delta(t - t') - \delta(t - t')\dot{u}(t')u^{-1}(t')$

The determinant of $\hat{a}_1\hat{a}_2^*$ is one since the $\delta$-function in the path integration measure enforces that $u(t)u^*(t) = 1$. The determinant of $\hat{a}_2$ is an irrelevant constant which contributes only to the normalization. Using det $= \text{exp} \text{tr} \ln$ and $\ln(1 + x) = \sum_{k=1}^{\infty}(-1)^{k+1}x^k/k$ to evaluate det $\hat{a}_3$ yields

$$S = \frac{1}{4D} \int_0^t dt' \dot{u}(t')\dot{u}^*(t').$$

To calculate det $\hat{A}$ we evaluate Eq. (22) which gives

$$A_{11}(t,t') = (a(t,t') + a^*(t,t'))/2,$$

$$A_{12}(t,t') = i(a(t,t') - a^*(t,t'))/2,$$

$$A_{21}(t,t') = -(a(t,t') - a^*(t,t'))/2,$$

$$A_{22}(t,t') = (a(t,t') + a^*(t,t'))/2,$$

The restriction to $uu^* = 1$ in the invariant measure simplifies the action $S = \frac{1}{4D} \int_0^t dt' \dot{u}(t')\dot{u}^*(t')$. The decomposition $\hat{A} = \hat{B}\hat{C}\hat{D}$ into a product of three operators

$$A_{11}(t,t') = (a(t,t') + a^*(t,t'))/2,$$

$$A_{12}(t,t') = i(a(t,t') - a^*(t,t'))/2,$$

$$A_{21}(t,t') = -(a(t,t') - a^*(t,t'))/2,$$

$$A_{22}(t,t') = (a(t,t') + a^*(t,t'))/2,$$

where

$$a(t,t') = u^{-1}(t) \left( \frac{d}{dt} \delta(t - t') - \delta(t - t')\dot{u}(t)u^{-1}(t) \right).$$

(33)

The analogue of Eq. (20) for the transfer matrix is

$$p(t;u) = \mathcal{N}^{-1} \int \prod_{t'=0}^t du(t') |\text{det} \hat{A}| \exp\{-S\}$$

(30)

since $du\delta(\ln(uu^*)) = du\delta(uu^* - 1)$ which is proportional to the invariant measure $d\mu(u)$. The restriction to $uu^* = 1$ in the invariant measure simplifies the action $S = \frac{1}{4D} \int_0^t dt' \dot{u}(t')\dot{u}^*(t')$. The decomposition $\hat{A} = \hat{B}\hat{C}\hat{D}$ into a product of three operators

$$[\hat{B}](t,t') = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta(t - t') & -i\delta(t - t') \\ -i\delta(t - t') & \delta(t - t') \end{pmatrix},$$

$$[\hat{C}](t,t') = \begin{pmatrix} a(t,t') & 0 \\ 0 & a^*(t,t') \end{pmatrix},$$

$$[\hat{D}](t,t') = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta(t - t') & i\delta(t - t') \\ i\delta(t - t') & \delta(t - t') \end{pmatrix}$$

implies that det $\hat{A} = \text{det} \hat{C} = \text{det} \hat{\dot{a}} \text{det} \hat{a}^*$ since det $\hat{B} = \text{det} \hat{D} = 1$. The operator $\hat{a}$ can be as well factorized into $\hat{a} = \hat{a}_1\hat{a}_2\hat{a}_3$ where

$$a_1(t,t') = u^{-1}(t)\delta(t - t')$$

$$a_2(t,t') = \frac{d}{dt}\delta(t - t')$$

$$a_3(t,t') = \delta(t - t') - \delta(t - t')\dot{u}(t')u^{-1}(t')$$

(35)

Performing the average over the Gaussian probability measure

$$p(t;u) = \mathcal{N}^{-1} \int \prod_{t'=0}^t du(t') \exp\{-S\}$$

(37)

IV. THE PATH INTEGRAL FOR THE TRANSFER MATRIX

The analogue of Eq. (20) for the transfer matrix is

$$p(L;T) = \int \left\langle \int \prod_{x=0}^L dT(x) \delta(T(x) - \bar{T}(x)) \right. \right.$$  

$$\times \delta(T(L) - \bar{T}) \left. \right\rangle,$$

(38)

where

$$dT \equiv \prod_{k,l} dT^{(1)}_{kl} dT^{(2)}_{kl}$$

$$\delta(T - \bar{T}) \equiv \prod_{k,l} \delta(T^{(1)}_{kl} - T^{(1)}_{kl})$$

$$\times \delta(T^{(2)}_{kl} - T^{(2)}_{kl}).$$

(39)

Enforcing $\bar{T}(x)$ by $T(x)T^{-1}(x) - \varepsilon(x) = 0$ which follows from the Langevin equation (14) yields

$$p(L;T) = \int \left\langle \int \prod_{x=0}^L dT(x) |\text{det} \hat{A}| \delta(T(x)T^{-1}(x) - \varepsilon(x)) \right. \right.$$  

$$\times \delta(T(L) - \bar{T}) \left. \right\rangle,$$

(40)

where the operator $\hat{A}$ is defined by the functional derivative

$$A^{(j)}_{kl,k'l'}(x,x') \equiv \frac{\delta[T(x)T^{-1}(x) - \varepsilon(x)]_{kl}}{\delta T^{(j)}_{k'l'}(x')}.$$  

(41)
\[ P[\epsilon] = \frac{1}{N^L} \exp \left\{ - \frac{1}{2} \int_0^L dx \left\{ \sum_{i,j,k,l} \delta \left( \epsilon_{kl}^{11} + \epsilon_{lk}^{11} \right) \delta \left( \epsilon_{ik}^{11} + \epsilon_{ki}^{11} \right) \right\} \right\} \int_{x=0}^L \prod_{l} d\epsilon(x), \]

where

\[ d\epsilon = \prod_{i,j,k,l} \delta \left( \epsilon_{kl} + \epsilon_{lk} \right) \delta \left( \epsilon_{ik} + \epsilon_{ki} \right), \]

\[ \delta S(\epsilon) \equiv \prod_{k<l} \left\{ \delta \left( \epsilon_{kl}^{(1)} + \epsilon_{lk}^{(1)} \right) \delta \left( \epsilon_{ik}^{(1)} + \epsilon_{ki}^{(1)} \right) \right\} \]

\[ \prod_{k} \delta \left( \epsilon_{kk}^{(1)} \right) \delta \left( \epsilon_{kk}^{(1)} \right) \]

\[ \prod_{k,l} \delta \left( \epsilon_{kl}^{(1)} \right) \delta \left( \epsilon_{lk}^{(1)} \right) \delta \left( \epsilon_{ik}^{(1)} \right) \delta \left( \epsilon_{ki}^{(1)} \right) \delta \left( \epsilon_{kl}^{(1)} \right) \delta \left( \epsilon_{lk}^{(1)} \right) \delta \left( \epsilon_{ik}^{(1)} \right) \delta \left( \epsilon_{ki}^{(1)} \right) \]

\[ \prod_{k}< \prod_{l} d\epsilon(x), \]

\[ (42) \]

We proceed with the calculation of \( \det A \). Using

\[ \partial / \partial T_{kl} = \partial / \partial T_{kl} + \partial / \partial T_{kl}^{*}, \]

\[ \partial / \partial T_{kl}^{*} = i(\partial / \partial T_{kl} - \partial / \partial T_{kl}^{*}), \]

and \( \partial T_{kl} / \partial T_{kl}^{*} = -T_{kl} T_{kl}^{*} \) to evaluate Eq. (1) yields

\[ \left[ \hat{A} \right]^{11} = (\hat{A} + \hat{A}^*) / 2, \]

\[ \left[ \hat{A} \right]^{12} = i(\hat{A} - \hat{A}^*) / 2, \]

\[ \left[ \hat{A} \right]^{21} = -i(\hat{A} - \hat{A}^*) / 2, \]

\[ \left[ \hat{A} \right]^{22} = (\hat{A} + \hat{A}^*) / 2, \]

\[ (47) \]

where

\[ A_{kl,k'U}(x,x') = \delta_{kl} T_{nl}^{-1}(x) \left( \frac{d}{dx} \delta(x-x') \delta_{mk'} \delta_{nl'} - \delta(x-x') \delta_{kl} \delta_{nl'} \right) \]

\[ (48) \]

The decomposition \( \hat{A} = \hat{B} \hat{C} \hat{D} \) into a product of three operators

\[ \hat{B} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \hat{1} \\ \hat{1} & i \hat{1} \end{array} \right), \]

\[ \hat{C} = \left( \begin{array}{cc} \hat{A} & 0 \\ 0 & \hat{A}^* \end{array} \right), \]

\[ \hat{D} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \hat{1} \\ i \hat{1} & 1 \end{array} \right), \]

\[ (49) \]

where \( \left[ \hat{1} \right]_{kl,k'U}(x,x') = \delta(x-x') \delta_{kl} \delta_{nl'} \), implies that \( \det \hat{A} = \det \hat{C} = \det \hat{A} \det \hat{A}^* \) since \( \det \hat{B} = \det \hat{D} = 1 \). The operator \( \hat{A} \) can be as well factorized into \( \hat{A} = \hat{A}_1 \hat{A}_2 \hat{A}_3 \) where

\[ \hat{A}_1_{kl,k'U}(x,x') = \left[ 1 \otimes (T^{-1})_{kl}(x) \right]_{kl,k'U} \delta(x-x') \]

\[ \hat{A}_2_{kl,k'U}(x,x') = \left[ -T_{kl}(x,x') \right]_{kl,k'U} \delta(x-x') \]

\[ \hat{A}_3_{kl,k'U}(x,x') = \left[ \delta(x-x') \right]_{kl,k'U} \delta(x-x') \]

\[ (50) \]

The product \( \det \hat{A}_1 \det \hat{A}_2 \) is one since the determinant of the transfer matrix is a phase. The determinant of \( \hat{A}_2 \) is an irrelevant constant which contributes only to the normalization. Using \( \det = \exp \text{tr} \ln \) and

\[ \ln(1 + x) = \sum_{k=1}^\infty (-1)^{k+1} x^k / k \]

to evaluate \( \det \hat{A}_3 \) yields

\[ \det \hat{A}_3 = \exp \left\{ -N \theta(0) \right\} \int_0^L dx \text{tr} \left( \hat{T}(x) \right) \left( T^{-1}(x) \right) \]

\[ (51) \]

The symmetries of the transfer matrix imply that \( \text{tr} \left( \hat{T}(x) T^{-1}(x) \right) + \left( \hat{T}(x) T^{-1}(x) \right)^* = 0 \) which gives \( \det \hat{A}_3 \det \hat{A}_3 = 1 \). That leads to the final form

\[ p(L; T) = N^{-1} \int_0^L T \left\{ \int_{x=0}^L d\mu(T(x)) \exp\{-S\} \right\} \]

of the path integral, where \( S \) is the action of Eq. (45).
V. THE DMPK EQUATION

We formulate the DMPK equation in terms of diffusion on the coset space of the transfer matrix group as has been done by Huffman [4]. In our context that can be achieved with the equivalent channel model (ECM). This model has been introduced by Mello and Tomsovic for the orthogonal universality class [34,35]. They showed that it is equivalent to the DMPK equation with \( \beta = 1 \), in the sense that the joint probability distributions for the space which is formed by the matrices \( M \) and \( \Gamma \) is isomorphic to the coset space of the transfer matrix group. The ECM for the unitary universality class is just the model (14) with backscattering mean free paths of the form

\[
\frac{1}{l_{mn}} = \frac{1}{L} \quad (53)
\]

and arbitrary forward scattering mean free paths. It is equivalent to the DMPK equation with \( \beta = 2 \) in the same sense. The difference between the DMPK equation and the ECM is that the unitary matrices need not be isotropically distributed and that there can be correlations between them and \( \Gamma \).

We choose forward scattering to be infinitely strong so that the mean free paths \( l_{mn} \) and \( l'_{mn} \) are zero. Then, the action (14) simplifies

\[
S = \frac{Nl}{2} \int_0^L dx \text{tr} \left\{ \left[ T T^{-1} \right]^{12} \left[ T T^{-1} \right]^{12} \right\} \\
+ \left[ T T^{-1} \right]^{21} \left[ \left[ T T^{-1} \right]^{21} \right]^\dagger \quad (54)
\]

Using that \( \dot{T} T^{-1} = -T T^{-1} \dot{T} \) and the symmetries of \( T T^{-1} \) one can simplify further

\[
S = \frac{Nl}{8} \int_0^L dx \text{tr} \left\{ \left( \dot{T} T^{-1} + \left( \dot{T} T^{-1} \right)^\dagger \right) \right\} \\
= \frac{Nl}{8} \int_0^L dx \text{tr} \left\{ 2 \dot{T} T^{-1} \left( \dot{T} T^{-1} \right)^\dagger \\
- \dot{T} T^{-1} \right\} \\
= -\frac{Nl}{8} \int_0^L dx \text{tr} \left( \dot{M} \dot{M}^{-1} \right), \quad (55)
\]

where \( M = T \) which does not depend on \( u_1 \) and \( u_3 \) anymore. The infinite strong forward scattering immediately randomizes the probability distribution of \( u_1 \) and \( u_3 \) so that they become isotropically distributed. Note that the space which is formed by the matrices \( M \) is isomorphic to the coset space of the transfer matrix group. The path integral describes diffusion on the coset space since the action is the classical action for free motion on this space.

Introducing the dimensionless length \( s = x/(NL) \) yields

\[
S = -\frac{1}{8} \int_0^{1/g_{cl}} ds \text{tr} \left( \dot{M} \dot{M}^{-1} \right), \quad (56)
\]

where the dot now stands for the derivative with respect to \( s \) and \( g_{cl} \equiv NL/L \) is the classical (bare) conductance in units of \( e^2/h \). Hence, large conductances correspond to the ‘short time’ regime of the path integral which justifies a saddle point approach for good conductors. The variation \( M(s) + \delta M(s) = \dot{T}(s) M(s) T(s), \) where \( \delta T = 1 + \varepsilon \) and \( \varepsilon \) obeys the symmetries (13) leads to the saddlepoint equation

\[
0 = \delta S = \frac{1}{g_{cl}} \int_0^{1/g_{cl}} ds \text{tr} \left( \delta T M + M \delta T M^{-1} \right) \\
- M \left( \varepsilon M^{-1} + M^{-1} \varepsilon \right). \quad (57)
\]

One can verify easily that \( M_{sp}(s) = \exp \{ s X \} \) is the solution for a path which starts at \( M(0) = 1 \) and ends at \( M = \exp \{ X/g_{cl} \} \). Evaluation of the saddle point action yields the transfer matrix probability measure in saddle point approximation

\[
p(L; T) d\mu(T) = \prod_k \exp \left\{ -\frac{NL}{4L} \Gamma_k^2 \right\} \ d\mu(T) \\
= \prod_k \left( \cosh \Gamma_k - \cosh \Gamma_l \right)^2 \prod_k \exp \left\{ -\frac{NL}{4L} \Gamma_k^2 \right\} \\
\times \prod_k \sinh \Gamma_k d\Gamma_k \prod_k \ d\mu(u_k). \quad (58)
\]

This is just the random matrix theory probability distribution measure which has been proposed for the transfer matrix \( M \) [4]. Since it is known that random transfer matrix theory describes the stochastic properties of ballistic conductors, we conclude that the saddle point of the path integral correctly describes the ballistic regime of the conductor.

VI. CONCLUSION

In summary we have presented a path integral approach to the stochastic properties of mesoscopic disordered conductors. Its application to quasi-one-dimensional wires in the ballistic regime led to the random transfer matrix theory probability distribution. We believe that known results for the quasi-one-dimensional wire could be recovered by a systematic perturbation expansion in powers of \( 1/g_{cl} \). At the moment it is not clear to us whether the ‘short time regime’ of the path integral in higher dimensions corresponds as well to conductors with large conductances. That still has to be clarified. The further development of the path integral technique also remains to be done.
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APPENDIX A: THE INARIANT MEASURE OF THE TRANSFER MATRIX GROUP

The invariant measure on the transfer matrix group does not change under multiplication with a fixed transfer matrix $T_0$ from the left or the right

$$d\mu(T) = d\mu(T_0T) = d\mu(HTT_0). \quad (A1)$$

In this appendix we prove the claim of sect. IV that \( \prod_{x=0}^{L} d\mu(T(x)) \delta_S(T(x)T^{-1}(x)) \) is proportional to \( \prod_{x=0}^{L} d\mu(T(x)) \).

Since the inverse of \( T \) in \( \delta_S \) cannot be handled as easily as \( u^{-1} \) in the example of diffusion on the circle, we show first that \( \delta_S(\varepsilon) \propto \delta_S(\Sigma z T^\dagger \Sigma z T) \) up to a Jacobian. This will allow to replace \( TT^{-1} \) in the argument of \( \delta_S \) by \( \Sigma z T^\dagger \Sigma z T \).

Writing the \( \delta \)-function in terms of its Fourier representation yields

$$\delta_S(\varepsilon) = \frac{1}{(2\pi)^{2N^2}} \int d\kappa \exp\left\{ \frac{i}{2} \text{tr}[\kappa(\varepsilon + \Sigma z \varepsilon^\dagger \Sigma z)] \right\} \quad (A2)$$

where

$$\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix}, \quad (A3)$$

$$\kappa_{11}^\dagger = \kappa_{11}, \quad \kappa_{22}^\dagger = \kappa_{22}, \quad \kappa_{12}^\dagger = -\kappa_{21}, \quad (A4)$$

and

$$d\kappa = \prod_{k<l} d\kappa_{kl}^{(1)} d\kappa_{kl}^{(2)} d\kappa_{kl}^{(2)} d\kappa_{kl}^{(2)}, \quad (A5)$$

Then the linear transformation

$$\varepsilon' = \Sigma z T^\dagger \Sigma z \varepsilon T \quad (A6)$$

of \( \varepsilon \) can be absorbed into \( \kappa \),

$$\delta_S(\varepsilon') = \frac{1}{(2\pi)^{4N^2}} \int d\kappa \exp\left\{ \frac{i}{2} \text{tr}[\kappa'(\varepsilon + \Sigma z \varepsilon^\dagger \Sigma z)] \right\}, \quad (A7)$$

where

$$\kappa' = T \kappa \Sigma z T^\dagger \Sigma z. \quad (A8)$$

Since \( \kappa' \) has the same symmetries as \( \kappa \) it follows that

$$\delta_S(\varepsilon') = \delta_S(\varepsilon)/|J(T)|, \quad (A9)$$

where \( J(T) \) is the Jacobian of the linear transformation \( (A8) \). Hence, replacement of the argument \( TT^{-1} \) in \( \delta_S \) by \( \Sigma z T^\dagger \Sigma z T \) via the linear transformation \( (A6) \) yields

$$\prod_{x=0}^{L} \delta_S(T(x)T^{-1}(x)) \propto \int d\kappa \|J(T(x))\| \exp\left\{ i \int_{x=0}^{L} dx \text{tr}\left[\kappa \frac{d}{dx}(\Sigma z T^\dagger \Sigma z T)\right] \right\}. \quad (A10)$$

Partial integration and using that \( \Sigma z T^\dagger \Sigma z T = 1 \) at the endpoints gives

$$\prod_{x=0}^{L} \delta_S(T(x)T^{-1}(x)) \propto \int d\kappa \|J(T(x))\| \exp\left\{ -i \int_{x=0}^{L} dx \text{tr}\left[\kappa (\Sigma z T^\dagger \Sigma z T - 1)\right] \right\}. \quad (A11)$$

The Jacobian of the transformation \( \tilde{\kappa} = -\kappa \) is a constant. Hence

$$\prod_{x=0}^{L} \delta_S(T(x)T^{-1}(x)) \propto \prod_{x=0}^{L} |J(T(x))| \delta_S(\Sigma z T^\dagger(x)\Sigma z T(x) - 1). \quad (A12)$$

In order to calculate \( J(T) \) we introduce the \( (4N^2) \)-vector notation

$$\tilde{\kappa}^T = (\kappa_{11}, \ldots, \kappa_{12N}, \kappa_{21}, \ldots, \kappa_{2N2N}) \quad (A13)$$
of the matrix \( \kappa \). Then \( \tilde{\kappa}' = (T \otimes (\Sigma z T \mid \Sigma z)^T) \tilde{\kappa} \). There is a complex matrix \( E \) such that \( \tilde{\kappa} = E \kappa_{\text{ind}} \), where \( \kappa_{\text{ind}} \) contains the 4N^2 real and imaginary parts of the independent matrix elements of \( \kappa \). Therefore

\[
\tilde{\kappa}'_{\text{ind}} = E^{-1} (T \otimes (\Sigma z T \mid \Sigma z)^T) E \kappa_{\text{ind}}. \quad (A14)
\]

\( J(T) \) is the determinant of this linear transformation, which is one since the \( \delta \)-functions in Eq. \( (A12) \) enforces \( \Sigma z T \mid \Sigma z \) to be the inverse of \( T \). That leads to

\[
L \prod_{x=0}^{L} \delta_S \left( \tilde{T}(x) T^{-1}(x) \right) \propto \prod_{x=0}^{L} \delta_S \left( \Sigma z T \mid \Sigma z \right) T(x) - 1).
\]

It remains to be shown that

\[
d\mu(T) \equiv dT \delta_S \left( \Sigma z T \mid \Sigma z - 1 \right) \quad (A16)
\]

has the properties \( (A1) \), and therefore is the invariant measure.

For multiplication with a transfer matrix \( T_0 \) from the left, the argument of the \( \delta \)-function does not change which leads to

\[
d\mu(T_0 T) = dT \delta_T \left( \Sigma z T \mid \Sigma z - 1 \right), \quad (A17)
\]

where \( J(T_0) \) is the Jacobian of the linear transformation \( T^* = T_0 T \). Expressing this transformation in terms of real vectors yields

\[
\left( \begin{array}{c} \tilde{T}_{(1)}' \\ \tilde{T}_{(2)}' \end{array} \right) = \left( \begin{array}{ccc} T_{(1)}'^0 & 1 \\ T_{(2)}'^0 & 1 \end{array} \right) \left( \begin{array}{c} T_0'^0 \\ 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 1 \\ 0 & 1 \end{array} \right)
\]

The Jacobian \( J(T_0) \) is the determinant of the transformation matrix which can be decomposed into the product

\[
\frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & i \end{array} \right) \left( \begin{array}{cc} T_0^0 & 0 \\ 0 & T_0^0 \end{array} \right) \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ 1 & 1 \end{array} \right)
\]

of three matrices. Since \( \Sigma z T_0^0 \Sigma z T_0 = 1 \) implies that \( \det T_0 \det T_0^0 = 1 \) one finds that \( J(T_0) = 1 \) and therefore \( d\mu(T_0 T) = d\mu(T) \).

Analogously it can be shown that the Jacobian for the multiplication with \( T_0 \) from the right is one as well which gives

\[
d\mu(T T_0) = dT \delta_S \left( \Sigma z T_0^0 \Sigma z (T T_0 - 1) T_0 \right). \quad (A20)
\]

As shown above \( \delta_S \left( \Sigma z T_0^0 \Sigma z \varepsilon T_0 \right) = \delta_S (\varepsilon) \). Hence

\[
d\mu(T T_0) = dT \delta_S \left( \Sigma z T_0^0 \Sigma z - 1 \right) = d\mu(T)
\]

which proves our claim.

* Present Address: Malteser Gasse 16, 69123 Heidelberg, Germany.

1 For reviews, see Mesoscopic Phenomena in Solids, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991) and Quantum Coherence in Mesoscopic Systems, Vol. 254 of NATO Advanced Study Institute Series B: Physics, edited by B. Kramer (Plenum, New York, 1991).

2 V. E. Kravtsov and I. V. Lerner, Sov. Phys. JETP 61, 758 (1985).

3 B. L. Al'tshuler, V. E. Kravtsov, and I. V. Lerner, Sov. Phys. JETP 64, 1352 (1986).

4 V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, Phys. Lett. A 134, 245 (1989).

5 F. J. Wegner, Z. Phys. B 78, 33 (1990).

6 in Mesoscopic Phenomena in Solids, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991).

7 R. Landauer, Philos. Mag. 21, 863 (1970).

8 M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B 31, 6207 (1985).

9 M. R. Zirnbauer, Phys. Rev. Lett. 69, 1584 (1992).

10 A. D. Mirlin, A. Müller-Groeling, and M. R. Zirnbauer, Ann. Phys. 236, 325 (1994).

11 K. Frahm, Phys. Rev. Lett. 74, 4706 (1995).

12 B. Rejaei, Phys. Rev. B 53, R13 235 (1996).

13 O. N. Dorokhov, JETP Lett. 36, 318 (1982).

14 P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) 181, 290 (1988).

15 P. A. Mello and A. D. Stone, Phys. Rev. B 44, 3559 (1991).

16 A. M. S. Macêdo and J. T. Chalker, Phys. Rev. B 46, 3559 (1991).

17 P. A. Mello and S. Tomsovic, Phys. Rev. B 46, 15963 (1992).

18 P. A. Mello, Phys. Rev. Lett. 60, 1089 (1988).

19 P. A. Mello, E. Akkermans, and B. Shapiro, Phys. Rev. B 53, 16 555 (1996).

20 A. Hüffmann, J. Phys. A 23, 5733 (1990).

21 J. T. Chalker and A. M. S. Macêdo, Phys. Rev. Lett. 71, 3693 (1993).

22 A. M. S. Macedo, Phys. Rev. B 49, 1858 (1994).

23 A. M. S. Macedo, Phys. Rev. B 49, 11 736 (1994).

24 A. M. S. Macêdo and J. T. Chalker, Phys. Rev. B 49, 4695 (1994).

25 C. W. J. Beenakker, Phys. Rev. B 49, 2205 (1994).

26 M. Caselle, Phys. Rev. Lett. 74, 2776 (1995).

27 P. W. Brouwer and K. Frahm, Phys. Rev. B 53, 1490 (1996).

28 C. W. J. Beenakker and B. Rejaei, Phys. Rev. Lett. 71, 3689 (1993).

29 C. W. J. Beenakker and B. Rejaei, Phys. Rev. B 49, 7499 (1994).

30 B. Sutherland, Phys. Rev. A 5, 1372 (1972).

31 A. Pandey and P. Shakya, J. Phys. A 24, 3907 (1991).

32 P. A. Mello and J.-L. Pichard, J. Phys. I (France) 1, 493 (1991).
33 O. N. Dorokhov, Sov. Phys. JETP 58, 606 (1983).
34 D. Endesfelder, Phys. Rev. B 53, 16 555 (1996).
35 J. T. Chalker and M.Bernhardt, Phys. Rev. Lett. 70, 982 (1993).
36 J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford University Press, Oxford, 1993).
37 P. A. Mello and S. Tomsovic, Phys. Rev. Lett. 67, 342 (1991).
38 M. A. Olshanetsky and A. M. Perelomov, Physics Reports 71, 313 (1981).
39 M. A. Olshanetsky and A. M. Perelomov, Physics Reports 94, 313 (1983).
40 K. A. Muttalib, J-L. Pichard, and A. D. Stone, Phys. Rev. Lett. 59, 2475 (1987).
41 C. W. J. Beenakker, Phys. Rev. B 47, 15763 (1993).
42 H. U. Baranger and P. A. Mello, Phys. Rev. Lett. 73, 142 (1994).