Ultrafast Long-Distance Quantum Communication with Static Linear Optics

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We propose a projection measurement onto encoded Bell states with a static network of linear optical elements. By increasing the size of the quantum error correction code, both Bell measurement efficiency and photon-loss tolerance can be made arbitrarily high at the same time. As a main application, we show that all-optical quantum communication over large distances with communication rates similar to those of classical communication is possible solely based on local state teleportations using optical sources of encoded Bell states, fixed arrays of beam splitters, and photon detectors discriminating up to two photons. As another application, generalizing state teleportation to gate teleportation for quantum computation, we find that in order to achieve universality the intrinsic loss tolerance must be sacrificed and a minimal amount of feedforward has to be added.

Introduction. Two conceptual, theoretical breakthroughs were made almost 15 years ago with regards to the implementation of quantum computation and communication. Knill, Laflamme, and Milburn (KLM) showed that, quite surprisingly at that time, linear optical elements such as beam splitters and phase shifters are sufficient for universal quantum computation [1]. Duan, Lukin, Cirac, and Zoller (DLCZ) demonstrated that also quantum communication over large distances can be, in principle, achieved by means of linear optics together with atomic-ensemble quantum memories [2].

A crucial element in either scheme is quantum error detection (QED). In the DLCZ quantum repeater, entangled states are distributed over sufficiently small channel segments and then connected by entanglement swapping, where the primary source of errors is loss, especially during the transmission of the photonic states. Such channel losses can be immediately detected during the initial entanglement distribution step, while memory losses are suppressed later during the entanglement swapping procedure (or at the latest through a final postselection step). This way long-distance quantum communication (LDQC) becomes possible, as opposed to using direct quantum communication without QED and without memories where the transmission rate would drop exponentially with distance.

Could it be useful to replace QED by quantum error correction (QEC) for LDQC? First of all, the original quantum repeater idea [3, 4] emphasized the fact that, in contrast to scalable quantum computation, scalable quantum communication is possible with the help of entanglement purification, corresponding to a form of QED against both channel losses and local gate errors, with no need for many levels of additional complex QEC codes. This huge simplification, however, comes at a price: entanglement purification is probabilistic and so are typically the entanglement distribution over lossy channel segments (including the local state preparations) and the subsequent entanglement swappings (at least when restricted to linear optics) [5]. As a consequence, this type of nested quantum repeater is extremely slow, relying on two-way classical communication and long-lived quantum memories. In fact, to scale up their scheme to larger distances, DLCZ need sufficiently good atomic-ensemble memories at every repeater station in order to store the initial entangled states until other entanglement distributions and connections succeed. While certain variations of the original DLCZ scheme lead to a significant improvement of the repeater rates [6], conceptually, this type of quantum repeater is fundamentally limited by the long waiting times for the heralding signals at every repeater node.

In recent years, various proposals have been made to employ QEC codes for LDQC. Since these codes suppress errors deterministically, long waiting times and two-way classical communication (and hence the use of quantum memories) can be, in principle, completely avoided. While one class of schemes focussed on the correction of operational errors with channel losses still suppressed through heralded distribution and purification of entanglement [7–10], another class did include QEC against transmission losses making high-rate loss-tolerant [11, 12] or even fully fault-tolerant [13, 14] LDQC possible. These latter schemes are limited only by the speed of the local gate operations and thus, they approach rates as obtainable in classical communication [15]. Our scheme also belongs to this class and allows for ultrafast LDQC, but unlike [11, 13, 14] it does so in an all-optical fashion without the use of difficult local quantum gates (implementable via local nonlinear matter-light interactions [11, 14]).

For this purpose, by employing a certain version of loss-tolerant parity-codes [11, 14, 16], we suggest to send encoded qubit states directly, which are then subject to a Bell measurement (BM) together with locally prepared, encoded Bell states after every few kilometers (see Fig. 1). These local state teleportations allow for a non-destructive loss-error syndrome detection and a qubit state recovery in one step. The use of QEC by teleportation [17] along the channel is conceptually similar to the protocol of Ref. [14]. However, in our scheme, every teleportation is performed with optical (encoded) Bell states and linear optical elements [15]. It turns out that the encoding has two positive effects: the larger the code is, the more efficient the ideal BM (despite the linear-optics con-
FIG. 1. One-way communication scheme: (a) To send a quantum state $|in\rangle^{(n,m)}$ over a long distance, repeater stations (R) at shorter distances $L_0$ are used to recover the qubit from accumulated losses (fading arrows). A classical signal (double line) defines a single Pauli correction $X$ at the receiver (not necessarily a bit flip). (b) Each repeater station consists of an encoded Bell state, not corrupted by loss, and a highly efficient, loss-resistant, logical Bell Measurement (BML) acting on the incoming signal and one half of the Bell state. The other half of the Bell state is sent to the next station along with a classical signal containing the result of the BM.

The logical BM has to distinguish between these four logical Bell states. For the simple DR encoding alone ($n = m = 1$), this can be partially achieved by combining the modes of the two DR qubits pairwise at 50 : 50 beam splitters and then counting the photon clicks. Unique click patterns are obtained for $|\psi^+\rangle$ and $|\psi^-\rangle$, whereas $|\phi^\pm\rangle$ are indistinguishable from each other. Thus, the overall BM efficiency is 50%, which is optimal for DR encoding without ancilla photons or feedforward [19]. Our approach to a BM on QPC($n,m$)-encoded qubits is to simply adopt the standard DR method and combine all modes of the two logical qubits pairwise at beam splitters followed by photon detections. It turns out that for higher values of $n$ and $m$, we obtain a better BM efficiency and also gain intrinsic loss robustness.

To see this we now analyze the logical BM in more detail. First, we derive the BM success probabilities when the input states are not subject to a loss channel; later we incorporate lossy inputs. It is useful to reorder the mode labels of the $4nm$ optical modes appearing in the BM such that the corresponding blocks of the individual logical qubits become joint bipartite blocks (see Fig. 2). We will denote this reordering by the symbol $\equiv$ in the following. Let us now first state two lemmata.

Lemma 1. [20]. After reordering, the Bell states can be written as

\begin{equation}
|\phi^\pm\rangle^{(n,m)} \approx \frac{1}{\sqrt{2^{n+1}}} \sum_{i=1}^{n} \left| \psi_{i}\right\rangle^{(m)},
\end{equation}

\begin{equation}
|\psi^\pm\rangle^{(n,m)} \approx \frac{1}{\sqrt{2^{n+1}}} \sum_{i=1}^{n} \left| \phi_{i}\right\rangle^{(m)},
\end{equation}

where the sets $C_n^\pm$ and $D_n^\pm$ are given as

\begin{equation}
C_n^\pm = \left\{ \vec{r} \in \{\chi,\sigma\}^n \mid \#\sigma^+ \in 2N_0 \right\},
\end{equation}

\begin{equation}
D_n^\pm = \left\{ \vec{r} \in \{\chi,\sigma\}^n \mid \#\sigma^+ \in 2N_0 + 1 \right\},
\end{equation}

and the states $|\chi^\pm\rangle^{(m)}$ and $|\sigma^\pm\rangle^{(m)}$ are

\begin{equation}
|\chi^\pm\rangle^{(m)} = \frac{1}{\sqrt{2}} \left( |00\rangle^{(m)} \pm |11\rangle^{(m)} \right),
\end{equation}

\begin{equation}
|\sigma^\pm\rangle^{(m)} = \frac{1}{\sqrt{2}} \left( |01\rangle^{(m)} \pm |10\rangle^{(m)} \right).
\end{equation}

In this representation, by counting the number of appearances of a $\sigma$-state (+ or −), we can discriminate the $\phi$-states from the $\psi$-states. To do so, it is necessary to distinguish between $\chi$- and $\sigma$-states. This can be achieved in our BM by looking at the number $n_{\text{odd}} - n_{\text{even}}$, where $n_{\text{odd}}$ ($n_{\text{even}}$) is the total number of photons detected in a mode with an odd (even) mode label in the relevant block (see Fig. 3). Since odd and even modes are not mixed during the BM, it suffices to consider the input states. As can be seen in Eqs. (7) and (8), $n_{\text{odd}} - n_{\text{even}}$ is

\begin{equation}
|\phi^\pm\rangle^{(n,m)} = \frac{1}{\sqrt{2}} \left( |00\rangle^{(n,m)} \pm |11\rangle^{(n,m)} \right),
\end{equation}

\begin{equation}
|\psi^\pm\rangle^{(n,m)} = \frac{1}{\sqrt{2}} \left( |01\rangle^{(n,m)} \pm |10\rangle^{(n,m)} \right).
\end{equation}
\( n = 2 \) blocks contains 2\( m = 4 \) photons. The modes on the left (red) belong to the incoming signal and are thus subject to loss, while those on the right (blue) are part of the encoded Bell state provided in the repeater station. Note that the mode labels have been rearranged as mentioned in the text to account for the block structure. The dashed ellipses highlight dual-rail pairs that are combined at the BM. (b) Optical BM setup for a single \( m = 2 \) block: here we show the proposed optical setup adapted to the common polarization encoding (equivalent optical setups are available for any other realization of a DR encoding). It consists only of standard 50 : 50 beam splitters, polarization beam splitters and photon detectors that can resolve up to 2 photons.

\( \pm 2m \) for \( \chi \)-states and 0 for \( \sigma \)-states. After discriminating \( \phi \)- and \( \psi \)-states, only the distinction between + and − remains. It turns out that the \( \chi \)-states are not useful for this, but for the \( \sigma \)-states, we obtain the following.

**Lemma 2.** [27]. The states \( |\sigma^+\rangle^{(m)} \) and \( |\sigma^-\rangle^{(m)} \) can be distinguished by the total number of photons in detectors \( 1, \ldots, 2m \). It is even for + and odd for −.

With the two lemmata above we can now calculate the BM success probability for perfect input states: the number of \( \sigma \)-states discriminates \( \phi \) and \( \psi \), while a single \( \sigma \)-state is enough to distinguish + and −. Thus, only one of the \( 2^m \) summands in Eqs. (3) and (4) leads to ambiguous outputs, namely that containing only \( \chi \)-states. This yields a success probability of \( 1 - 2^{-n} \) [28]. More important, however, is the following. When the input state is subject to loss, in many cases, we are still able to distinguish \( \chi \)-states from \( \sigma \)-states and + from −, thus a successful BM is often possible. We are interested in the BM success probability when a certain number \( l \) of photons is lost. In accordance with our one-way communication scheme, we assume that half of the physical qubits that participate in the BM (corresponding to one logical qubit) are locally created and therefore not subject to transmission loss [29].

The strategy for the BM including losses is the same as for the loss-free case: we discriminate between \( \phi \) and \( \psi \) and then distinguish + from −. To use Lemma 2 for the discrimination of \( \psi \) and \( \phi \), the ”loss derivatives” of the \( \chi \)- and \( \sigma \)-states need to be distinguishable. It turns out that a single DR pair (a pair of physical qubits combined at the BM, see Fig. 2) that is not corrupted by loss suffices. To see this, notice that for such a pair, \( n_{\text{odd}} - n_{\text{even}} \) is always \( \pm 2 \) for \( \chi \) and 0 for \( \sigma \), even when the other pairs of the same block are corrupted by loss. However, all information to discriminate + from − in a given block is destroyed, as soon as only a single photon is lost. Therefore we need at least one block that is not corrupted by loss at all. Altogether this gives our main theorem.

**Theorem 1.** [27]. If the input states of the BM are subject to loss and \( l \) physical qubits (photons) are lost, a successful BM is possible if and only if there is an intact DR pair in every block and at least one block is not corrupted by loss at all. The success probability of the BM is then given by

\[
 p_l = \frac{1}{(nm)^l} \sum_{i=0}^{l} \left( 1 - 2^{-(n-i)} \right) \left( \begin{array}{c} n \\ i \end{array} \right) \sum_{j_1, \ldots, j_l = 1}^{m-1} \prod_{k=1}^{l} \left( \begin{array}{c} m \\ j_k \end{array} \right) \sum_{i=1}^{j_k} \sum_{j_k+1}^{j_k+j_{k+1}} \left( \begin{array}{c} j_k+j_{k+1} \\ i \end{array} \right).
\]

Here the sums over the indices \( j_k \) represent the number of unique ways to distribute \( l \) photon losses among the photons of \( i \) blocks while leaving at least one DR pair intact per block. Note that, although the distance of QPC(\( n, m \)) is \( \min(n, m) \), a successful BM is possible even if much more photons are lost, up to a maximum of \( (n-1)(m-1) \). When looking at the success probabilities \( p_l \) for different \( n \) and \( m \) (some examples are given in TABLE I) it becomes clear that most of the time increasing \( n \) gives better results than increasing \( m \). As a rule of thumb, \( n \) should be chosen sufficiently larger than \( m \) (perhaps even more than linearly), because a too big value of \( m \) increases the chance that in all blocks at least one photon is lost. At the same time, however, a too small value of \( m \) means the risk of corrupting all dual-rail pairs in a block is higher. Increasing \( n \) on the other hand only gives more blocks, thus increasing the chance to get at least one without any corruption. Conceptually, this is the most important result obtained here: a larger

| \( (n, m) \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|---|---|
| (2,2)     | 75 | 50 |   |   |   |   |   |
| (3,2)     | 87.5 | 75 | 40 |   |   |   |   |
| (4,3)     | 93.75 | 87.5 | 77.27 | 61.36 | 40.91 | 20.45 | 5.84 |
| (5,3)     | 96.88 | 93.75 | 88.39 | 79.12 | 65.11 | 47.20 | 28.32 |
| (10,5)    | 99.90 | 99.80 | 99.63 | 99.31 | 98.79 | 97.96 | 96.72 |
code with a larger number of blocks $n$ results in a higher linear-optics BM efficiency and a higher loss tolerance at the same time. This is different from other BM schemes with certain encodings, where for a bigger encoding with more photons a trade-off occurs between a growing BM efficiency and a growing loss-induced error probability (where the latter may be circumvented through an additional layer of QEC codes requiring extra quantum logic gates [32] or fast feedforward operations [12]).

Long-Distance Quantum Communication. To send a QPC($n,m$)-encoded qubit state over a total distance $L$ we propose to place repeater stations after every channel segment with length $L_0$. At every station an encoded Bell state $|\psi^+\rangle^{(n,m)}$ is available on demand (i.e., created and consumed locally), and a logical BM is performed on one half of the Bell state together with the incoming encoded qubit (see Fig. 1). Every time the logical qubit must travel over a distance $L_0$ and hence its physical qubits suffer from loss according to a transmission coefficient of $\gamma = \exp(-L_0/L_{\text{att}})$ (with attenuation length $L_{\text{att}} = 22$ km). Whenever the BM succeeds, the qubit state is recovered from loss and appears at the other half of the Bell state to be sent to the next station [17, 33] [34]. The total success probability is then given as

$$p = \sum_{l=0}^{(n-1)(m-1)} p_l \frac{(n^m)}{l^m} \gamma^{nm-l} (1-\gamma)^l . \quad (9)$$

Here the coefficients $p_l$ are the success probabilities for the BM when $l$ photons are lost (cf. Theorem 1). The dependence of a qubit’s total transmission probability on the repeater spacing $L_0$ for different QPC-encodings and different total distances are shown in Fig. 3. These plots indicate that for any $(n,m)$ chosen high enough, there exists a spacing $L_0$ such that near-unit success probability can be achieved even for intercontinental distances.

For $p \approx 1$, the repeater rate becomes $R = p/t_0 \approx 1/t_0$, where $t_0$ is the elementary time needed at every repeater station until the incoming signal qubit has been processed and a fresh encoded Bell state is ready for teleporting and error-correcting the next qubit. Since our logical Bell states are assumed to be available on demand (which is a strong assumption that will be discussed below), $t_0$ corresponds only to the duration of the linear-optics processing with photon detection. Compared to those times required in a matter-based scheme with $t_0 \sim 1\mu s$ (even assuming future enhanced ion-cavity coupling strengths [14] or an all-optical scheme including feedforward [12] with $t_0 \sim 10\text{ns}$ (provided all circuits can be integrated [35]), corresponding to rates $R \sim \text{MHz}$ or $R \sim 0.1 \text{GHz}$, respectively, our static linear optical scheme allows, in principle, for GHz-rates and beyond. For our scheme to be free of feedforward at the intermediate stations, the updated (logical) Pauli frame after each teleportation must be recorded and classically communicated to the end of the channel. To do this final Pauli correction, and also to perform measurements in different logical Pauli bases for quantum key distribution (QKD), we have to ensure that at least a minimal set of logical gates on our QPC-qubits can be implemented efficiently with static linear optics. More generally, let us discuss next universal gates and gate teleportation based on our encoded BM.

Quantum Gate Teleportation and Quantum Computation. The physical Pauli operators of the QPC($n,m$) may be denoted as $X_{i,j}, Y_{i,j}, Z_{i,j}$, with $i = 1...n$ and $j = 1...m$ labelling the $(i,j)$-th DR qubit. Utilizing the stabilizer formalism [14, 36], for $nm$ physical qubits, one logical qubit is defined by the $nm-1$ stabilizers $Z_{i,j}Z_{i,j+1}$ ($i = 1...n, j = 1,..., m - 1$) and $X_{i,1}X_{i+1,1}...X_{i,m}X_{i+1,m}$ ($i = 1,..., n - 1$), with logical operators $X^{(n,m)} = X_{i,1}...X_{i,m}$ (for any $i$) and $Z^{(n,m)} = Z_{1,j}...Z_{n,j}$ (for any $j$) [14]. In other words, simple Pauli logic can be performed directly via suitable Pauli gates on the DR qubits. This is sufficient for the final Pauli frame correction in our LDQC scheme as well as for QKD applications. More generally, logical $X$- and $Z$-rotations are then given by $\exp(-iX^{(n,m)}\theta/2)$ and $\exp(-iZ^{(n,m)}\theta/2)$, respectively, and for any $\theta \notin \pi\mathbb{Z}$ and $n > 1, m > 1$, an entangling operation is needed that acts on the physical qubits. Based on our encoded linear-optics BM, we can use logical gate teleportation with suitable encoded offline resource states [37] to implement arbitrary Clifford computations (including logical two-qubit gates such as $\text{CNOT}^{(n,m)}$) in an intrinsically loss-tolerant fashion with no need for feedforward between the Clifford gates (and with only a final Pauli frame correction). This is a huge simplification compared to KLM [1] who require feedforward for every single CNOT and additional QEC codes to correct photon-loss errors. However, for universality, any single-qubit gate of KLM can be performed directly on the DR qubits, whereas in our general ($n,m$)-scheme, the logical non-Clifford gates do not allow for a static BM-based gate teleportation or a non-entangling transversal gate application. Therefore, for universality, we have to sacrifice the intrinsic loss toler-
ance and employ the most simple versions of the QPC such as \((n, 1)\) [38]. In this case, an arbitrary logical X-rotation \(\exp(-iX^{(n, 1)}\theta/2)\) can be done via the same rotation \(\exp(-iX_{1+i}\theta/2)\) directly on the \(i\)th DR qubit (for any \(i\)) and the remaining set of Clifford operations (including a single-qubit \(\pi/2\)-rotation exp\((-iZ^{(n, 1)}\pi/4)\) for universality) can be achieved through gate teleportation using the static linear-optics BM scheme. Since QPC\((n, 1)\) is enough to realize arbitrarily efficient BMs (for sufficiently high \(n\)), efficient linear-optics quantum computation is possible provided a little, simple Pauli feedforward is added every time when a sequence of Clifford gates is followed by a non-Clifford gate. In terms of feedforward, this is also a simplification compared to existing schemes, where every two-qubit gate requires Pauli corrections on randomly selected physical qubits (for KLM [11] or even non-Pauli feedforward is needed (for one-way quantum computation [22, 35]).

Discussion and Conclusions. We proposed an efficient linear-optics BM onto QPC-encoded Bell states and showed that, by incorporating protection against transmission losses, it can be used to realize ultrafast high-rate LDQC in an all-optical fashion. With no need for matter qubits (neither as quantum memories nor as local quantum processors) or feedforward operations, our communication scheme is most suitable to be integrated along an optical fiber channel via chips that contain quantum sources [39, 41], interferometers [42], and photon detectors [43]. In order to generate the encoded multi-qubit Bell states on demand at every integrated station, deterministic (e.g. semiconductor [44, 45] sources of single photons, physical Bell and GHZ states could be employed – in principle, still without memories and feedforward [46, 47]. Preliminary investigations on the robustness of our BM scheme against non-unit state fidelities are promising and are, together with other realistic imperfections, left for future work.

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[5] Alternative encodings beyond discrete qubit states such as continuous-variable Gaussian states allow for unconditional state preparation and deterministic entanglement swapping with linear optics, however, QED and entanglement purification are difficult in this case. Yet another promising alternative could be the use of both discrete and continuous variables that combines the advantages and avoids the disadvantages of the individual approaches [18, 19].
A proof of this lemma is given in the Supplemental Material.

For $n = 1$, one obtains the usual result of a 50% BM efficiency that holds, in particular, for a linear-optics BM on DR-encoded Bell states with $n = m = 1$ [19]. While it was shown only recently that this 50% bound can be exceeded by adding squeezing operations [51] or ancilla states [52, 53], it had been known already that different types of encoding compared to DR qubits can lead to an improved BM efficiency [54]. Note that most of our results here were presented at the Third International Conference on Quantum Error Correction QEC14 at ETH Zurich (http://www.qec14.ethz.ch), including the QPC-encoded BM efficiency of $1 - 2^{-n}$ for the loss-free case (independent of a similar recent result by Lee, Ralph, and Jeong for GHZ states [54]).

A generalization to photon losses on both qubits of the BM can be found in the Supplemental Material.

A proof of this theorem is given in the Supplemental Material.

A more extensive table with success probabilities of larger $n, m$ codes and losses of up to 18 photons can be found in the Supplemental Material.

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Note that without the use of a QEC code photon losses would still be detectable at every station, but those events must be completely discarded and only the no-loss events can be kept. This results in a total success probability (for an $N$-photon state to reach the end of the channel after distance $L$) of $\left\{\exp\left(\frac{-L}{L_{\text{att}}}\right)\right\}^{L/L_{\text{att}}} = \left\{\exp(-L/L_{\text{att}})\right\}^{\frac{L}{L_{\text{att}}}}$, identical to a direct transmission over the entire channel. Especially for $N = 1$ (e.g., simple DR encoding), the scheme with intermediate QED-stations to detect photon losses via local quantum teleportation never beats the direct transmission of the DR qubit over the entire channel. The finite BM efficiency with linear optics seems to make things even worse, however, in realistic scenarios including detector dark counts, the intermediate detections may indeed help [55]. Such a scheme would correspond to the one-way version of an entanglement-based quantum relay, which also does not employ quantum memories and hence does not suppress the loss-induced exponential decay.

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If feedforward is allowed, there are ways to achieve universality in a loss-tolerant or even fault-tolerant fashion on QPC-type encoding [56, 57].

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SUPPLEMENTAL MATERIAL

A. Block representation of the Bell states

Lemma 1. After reordering, the Bell states can be written as

\[ |\phi^\pm\rangle^{(n,m)} \approx \frac{1}{2^{(n-1)/2}} \sum_{\vec{r} \in C^+_n} \bigotimes_{i=1}^n |v_i\rangle^{(m)}, \quad |\psi^\pm\rangle^{(n,m)} \approx \frac{1}{2^{(n-1)/2}} \sum_{\vec{r} \in D^+_n} \bigotimes_{i=1}^n |v_i\rangle^{(m)}, \]  

where the sets \( C^+_n \) and \( D^+_n \) are given as

\[ C^+_n = \{ \vec{r} \in (\pm, \pm^\perp)^n | \#\sigma^\pm \in 2N_0 \}, \quad D^+_n = \{ \vec{r} \in (\pm^\perp, \pm)^n | \#\sigma^\pm \in 2N_0 + 1 \} \]  

and the states \( |\chi^\pm\rangle^{(m)} \) and \( |\sigma^\pm\rangle^{(m)} \) are

\[ |\chi^\pm\rangle^{(m)} = \frac{1}{\sqrt{2}} \left( |00\rangle^{(m)} \pm |11\rangle^{(m)} \right), \quad |\sigma^\pm\rangle^{(m)} = \frac{1}{\sqrt{2}} \left( |01\rangle^{(m)} \pm |10\rangle^{(m)} \right). \]

Proof. It is rather straightforward to show that

\[ |0\rangle^{(n,m)} = \frac{1}{2^{(n-1)/2}} \sum_{\vec{r} \in A_n} \bigotimes_{i=1}^n |r_i\rangle^{(m)}, \quad |1\rangle^{(n,m)} = \frac{1}{2^{(n-1)/2}} \sum_{\vec{r} \in B_n} \bigotimes_{i=1}^n |r_i\rangle^{(m)}, \]

where

\[ A_n = \{ \vec{r} \in \{0,1\}^n | \sum_{i=1}^n r_i \in 2N_0 \}, \quad B_n = \{ \vec{r} \in \{0,1\}^n | \sum_{i=1}^n r_i \in 2N_0 + 1 \}. \]

Using this notation and the reordering mentioned in the main text (i.e., collecting blocks) the Bell states become

\[ |\phi^\pm\rangle^{(n,m)} \approx \frac{1}{\sqrt{2}} \left( |\alpha\rangle^{(n,m)} \pm |\beta\rangle^{(n,m)} \right), \quad |\psi^\pm\rangle^{(n,m)} \approx \frac{1}{\sqrt{2}} \left( |\gamma\rangle^{(n,m)} \pm |\delta\rangle^{(n,m)} \right), \]

with

\[ |\alpha\rangle^{(n,m)} = \frac{1}{2^{n-1}} \sum_{\vec{r} \in A_n} \bigotimes_{i=1}^n |r_i, s_i\rangle^{(m)}, \quad |\beta\rangle^{(n,m)} = \frac{1}{2^{n-1}} \sum_{\vec{r} \in B_n} \bigotimes_{i=1}^n |r_i, s_i\rangle^{(m)}, \]

\[ |\gamma\rangle^{(n,m)} = \frac{1}{2^{n-1}} \sum_{\vec{r} \in A_n} \bigotimes_{i=1}^n |r_i, s_i\rangle^{(m)}, \quad |\delta\rangle^{(n,m)} = \frac{1}{2^{n-1}} \sum_{\vec{r} \in B_n} \bigotimes_{i=1}^n |r_i, s_i\rangle^{(m)}. \]

The lemma is proven via induction on \( n \) and with arbitrary constant \( m \). For \( n = 1 \) the lemma gives \( |\phi^\pm(1,m)\rangle \approx |\chi^\pm(m)\rangle \) and \( |\psi^\pm(1,m)\rangle \approx |\sigma^\pm(m)\rangle \), which is consistent with Eqs. (1) and (2). Furthermore, we have

\[ |\alpha\rangle^{(n+1,m)} = \frac{1}{2} \left( |\alpha\rangle^{(n,m)} \otimes |0,0\rangle^{(m)} + |\beta\rangle^{(n,m)} \otimes |1,1\rangle^{(m)} + |\gamma\rangle^{(n,m)} \otimes |0,1\rangle^{(m)} + |\delta\rangle^{(n,m)} \otimes |1,0\rangle^{(m)} \right), \]

\[ |\beta\rangle^{(n+1,m)} = \frac{1}{2} \left( |\alpha\rangle^{(n,m)} \otimes |1,1\rangle^{(m)} + |\beta\rangle^{(n,m)} \otimes |0,0\rangle^{(m)} + |\gamma\rangle^{(n,m)} \otimes |1,0\rangle^{(m)} + |\delta\rangle^{(n,m)} \otimes |0,1\rangle^{(m)} \right), \]

\[ |\gamma\rangle^{(n+1,m)} = \frac{1}{2} \left( |\alpha\rangle^{(n,m)} \otimes |0,1\rangle^{(m)} + |\beta\rangle^{(n,m)} \otimes |1,0\rangle^{(m)} + |\gamma\rangle^{(n,m)} \otimes |0,0\rangle^{(m)} + |\delta\rangle^{(n,m)} \otimes |1,1\rangle^{(m)} \right), \]

\[ |\delta\rangle^{(n+1,m)} = \frac{1}{2} \left( |\alpha\rangle^{(n,m)} \otimes |1,0\rangle^{(m)} + |\beta\rangle^{(n,m)} \otimes |0,1\rangle^{(m)} + |\gamma\rangle^{(n,m)} \otimes |1,1\rangle^{(m)} + |\delta\rangle^{(n,m)} \otimes |0,0\rangle^{(m)} \right). \]
Using this on $|\phi^{\pm}\rangle^{(n+1,m)}$ in (S6) we get

$$|\phi^{\pm}\rangle^{(n+1,m)} = \frac{1}{2} \left[ |(\alpha^{(n,m)} \pm \beta^{(n,m)})(\gamma^{(n,m)} \pm \delta^{(n,m)})|\sigma^{\pm}\rangle^{(m)} \right]$$

$$= \frac{1}{\sqrt{2}} \left[ |\psi^{\pm}\rangle^{(n,m)} |\chi^{\pm}\rangle^{(m)} + |\phi^{\pm}\rangle^{(n,m)} |\sigma^{\pm}\rangle^{(m)} \right]$$

$$= \frac{1}{2^{n/2}} \left[ \sum_{\vec{v} \in \mathbb{C}^{\pm}_n} n \prod_{i=1}^n |v_i\rangle^{(m)} |\chi^{\pm}\rangle^{(m)} + \sum_{\vec{v} \in \mathbb{C}^{\pm}_{n+1}} n \prod_{i=1}^n |v_i\rangle^{(m)} |\sigma^{\pm}\rangle^{(m)} \right] = \frac{1}{2^{n/2}} \sum_{\vec{v} \in \mathbb{C}^{\pm}_{n+1}} n+1 \prod_{i=1}^n |v_i\rangle^{(m)},$$

where in the last line we used the lemma for $n$. For the Bell states $|\psi^{\pm}\rangle^{(n+1,m)}$ the proof works analogously.

**B. Distinguishing the sign in a single block**

**Lemma 2.** The states $|\sigma^{+}\rangle^{(m)}$ and $|\sigma^{-}\rangle^{(m)}$ can be distinguished by the total number of photons in detectors $1, \ldots, 2m$. It is even for $+$ and odd for $-$. 

**Proof.** For the states $|\sigma^{\pm}\rangle^{(m)}$, at every beam splitter a single photon is mixed with vacuum. Thus, at each beam splitter the photon is either transmitted or reflected. When using phase-free beam splitters, a phase shift of $\frac{\pi}{2}$ is picked up for each reflection and no phase shift for transmission. Let $n_r$ be the number of reflections needed to convert the first term of $|\sigma^{\pm}\rangle^{(m)}$ in (S8) into a given combination of clicks. The amplitude of the corresponding eigenstate in the state entering the detectors (after the beam splitters) is given by

$$\frac{1}{2^{(m+1)/2}} \left( e^{in_r \frac{\pi}{2}} \pm e^{i(2m-n_r)\frac{\pi}{2}} \right) \propto 1 \pm (-1)^{m-n_r}.$$ 

This shows that the click patterns are unique to either $+$ or $-$ and that they are classified by the parity of $m-n_r$. The latter equals the parity of $n_{(1, \ldots, 2m)}$.

**C. Bell measurement success probability**

**Theorem 1.** If the input states of the BM are subject to loss and $l$ physical qubits (photons) are lost, a successful BM is possible if and only if there is an intact DR pair in every block and at least one block is not corrupted by loss at all. The success probability of the BM is then given by

$$p_l = \frac{1}{\binom{nm}{l}} \sum_{i=0}^l \binom{n}{i} \prod_{j=1}^{m-1} \sum_{j_k=1}^{m} \binom{m}{j_k},$$

where $N_{i,l,m}$

**Proof.** As described in the main text, the logic of the BM contains two steps: 1) Discriminating the $\psi$- from the $\phi$-states. To do so, the expansion of the Bell states in terms of $\chi$- and $\sigma$-states given in Lemma 1 is used. $\phi$-states ($\psi$-states) correspond to an even (odd) number of appearances of $\sigma$. To discriminate $\sigma$ and $\chi$ in a block, an intact dual-rail pair is necessary. Thus, we need at least one intact DR pair in every block for the discrimination of $\psi$ and $\phi$. 2) To discriminate $+$ and $-$ only an uncorrupted $\sigma$-state is useful. Thus, at least one block must be fully intact.

To calculate the success probability of the BM, given that $l$ losses are detected, observe that $N_{i,l,m}$ describes the number of unique ways to distribute $l$ photon losses on the photons of $i$ blocks while leaving at least one DR pair intact. (Recall that we assume perfect Bell states at the repeater stations, thus only one of the logical qubits is subject to loss. For a calculation of the success probability in the case, where both logical qubits are subject to the same amount of loss, see section D.) Here $j_k$ is the number of photons lost in the $k$-th of the $i$ corrupted blocks and the binomial coefficient $\binom{m}{j_k}$ is the number of unique ways to distribute these on the $m$ photons of the corrupted logical qubit in block $k$. The binomial coefficient $\binom{n}{i}$ denotes the number of possible choices of $i$ corrupted blocks while $\binom{nm}{l}$ denotes all possible ways to distribute the $l$ losses on the $nm$ physical qubits. Thus, only the weight $1 - 2^{-(n-i)}$ which
represents the success probability of a BM given that $i$ blocks are corrupted remains to be shown. If $i$ blocks are corrupted by loss, all those terms in Lemma 1 become useless for the BM, for which the remaining $n - i$ blocks are in $\chi$-states. There are $\sum_{j=0}^{i} \binom{i}{j} = 2^i$ of these terms. Thus, the success probability is given by $\frac{2^{n-2i}}{2^n} = 1 - 2^{-(n-i)}$. □

D. Bell measurement with symmetric distribution of losses

In the presented approach the resource states at the repeater stations are assumed to be perfect, thus giving a one-sided lossy Bell measurement: only one of the logical qubits is subject to loss. However, for other applications, e.g. quantum relays, the case of equal loss on both qubits will be important. The main statement in Theorem 1 remains the same, but the success probability $p_l$ changes. Since now all $2nm$ qubits are subject to loss, the normalizing binomial coefficient must be replaced by $\left(\begin{array}{c} {2nm} \\ l \end{array}\right)$. Furthermore, the number of unique ways to distribute $l$ losses on the photons of $i$ blocks is different. It is now given by

$$N_{i,l,m} = \sum_{j_1,\ldots,j_i=1}^{l} \prod_{k=1}^{i} \sum_{r=1}^{m-1} \binom{m}{r} \sum_{s_1,\ldots,s_r=1}^{2} \prod_{t=1}^{r} \binom{2}{s_t}. \quad (S13)$$

Here $j_k$ is the number of photons lost in corrupted block $k$, $r$ is the number of DR pairs corrupted by loss in the currently viewed block $k$ and $s_t$ is the number of photons lost in corrupted DR pair $t$.

E. Bell measurement efficiencies

In the following tables the success probabilities $p_l$ of the BM according to Eqs. (S12) and (S13) are given for various QPC($n,m$) encodings. In Table Ia losses occur only on one of the logical qubits, as assumed in the proposed communication scheme. In Table Ib the BM efficiencies are given for a symmetric distribution of the losses on both logical qubits.
| \((n, m)\) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| (1,1)    | 50  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (1,2)    | 50  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,1)    | 75  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,2)    | 75  | 50  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,3)    | 75  | 50  | 20  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,2)    | 87.5| 75  | 40  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,3)    | 87.5| 75  | 56.25| 32.14| 10.71 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,4)    | 87.5| 75  | 56.82| 36.82| 20.61| 9.09 | 2.60 |     |     |     |     |     |     |     |     |     |     |     |     |
| (4,3)    | 93.75| 87.5| 77.27| 61.36| 40.91| 20.45| 5.84 |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,5)    | 87.5| 75  | 57.14| 37.91| 22.80| 12.49| 5.99 | 2.33 | 0.58 |     |     |     |     |     |     |     |     |     |     |
| (5,3)    | 96.88| 93.75| 88.39| 79.12| 65.11| 47.20| 28.32| 12.59| 3.15 |     |     |     |     |     |     |     |     |     |     |
| (4,4)    | 93.75| 87.5| 77.50| 63.93| 48.46| 33.52| 20.68| 10.91| 4.48 | 1.12 |     |     |     |     |     |     |     |     |     |
| (4,5)    | 93.75| 87.5| 77.63| 64.47| 49.90| 36.28| 24.90| 16.06| 9.62 | 5.21 | 2.44 | 0.89 | 0.20 |     |     |     |     |     |     |
| (5,4)    | 96.88| 93.75| 88.49| 80.59| 70.07| 57.66| 44.58| 32.04| 21.05| 12.29| 6.06 | 2.29 | 0.51 |     |     |     |     |     |     |
| (5,5)    | 96.88| 93.75| 88.54| 80.84| 70.84| 59.41| 47.75| 36.85| 27.30| 19.35| 13.04| 8.26 | 4.83 | 2.52 | 1.12 | 0.38 | 0.08 |     |     |
| (7,4)    | 99.22| 98.44| 97.05| 94.75| 91.20| 86.10| 79.31| 70.94| 61.32| 50.96| 40.49| 30.51| 21.58| 14.13| 8.38 | 4.35 | 1.87 | 0.60 | 0.11 |
| (10,5)   | 99.90| 99.80| 99.63| 99.31| 98.79| 97.96| 96.72| 94.94| 92.53| 89.41| 85.55| 80.97| 75.75| 70.00| 63.86| 57.49| 51.05| 44.68| 38.51 |
| (13,6)   | 99.99| 99.98| 99.95| 99.91| 99.84| 99.72| 99.53| 99.23| 98.79| 98.15| 97.28| 96.11| 94.60| 92.71| 90.43| 87.74| 84.66| 81.21| 77.43 |
| (14,7)   | 99.99| 99.99| 99.98| 99.96| 99.92| 99.86| 99.76| 99.60| 99.36| 99.01| 98.52| 97.84| 96.95| 95.79| 94.35| 92.60| 90.53| 88.13| 85.42 |
| (21,6)   | 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 99.99| 99.99| 99.98| 99.97| 99.95| 99.92| 99.88| 99.82| 99.74| 99.62| 99.46 |
| (31,7)   | 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0 |
| \((n, m)\) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| (1,1)    | 50  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (1,2)    | 50  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,1)    | 75  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,2)    | 75  | 50  | 7.14|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (2,3)    | 75  | 50  | 22.73| 5.45| 0.61|     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,2)    | 87.5| 75  | 43.18| 10.91| 1.21|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| (3,3)    | 87.5| 75  | 57.35| 36.40| 18.31| 6.93| 1.89| 0.34| 0.03|     |     |     |     |     |     |     |     |     |     |     |
| (3,4)    | 87.5| 75  | 57.61| 39.43| 24.86| 14.24| 7.30| 3.23| 1.19| 0.35| 0.08| 0.01| 0.00|     |     |     |     |     |     |     |
| (4,3)    | 93.75| 87.5| 77.72| 63.44| 46.22| 29.35| 15.91| 7.21| 2.67| 0.78| 0.17| 0.03| 0.00|     |     |     |     |     |     |     |
| (3,5)    | 87.5| 75  | 57.76| 39.90| 25.94| 16.14| 9.57| 5.37| 2.80| 1.33| 0.56| 0.21| 0.06| 0.01| 0.00| 0.00| 0.00| 0.00|     |
| (5,3)    | 96.88| 93.75| 88.58| 80.08| 67.99| 53.30| 38.03| 24.38| 13.88| 6.94| 3.01| 1.11| 0.34| 0.09| 0.02| 0.00| 0.00|     |     |
| (4,4)    | 93.75| 87.5| 77.82| 65.24| 51.45| 38.35| 27.00| 17.88| 11.06| 6.32| 3.30| 1.54| 0.64| 0.22| 0.06| 0.01| 0.00| 0.00| 0.00|
| (4,5)    | 93.75| 87.5| 77.88| 65.49| 52.14| 39.78| 29.34| 21.00| 14.59| 9.82| 6.37| 3.97| 2.35| 1.32| 0.69| 0.33| 0.14| 0.05| 0.01|
| (5,4)    | 96.88| 93.75| 88.62| 81.20| 71.69| 60.82| 49.59| 38.83| 29.16| 20.94| 14.33| 9.30| 5.69| 3.25| 1.72| 0.84| 0.37| 0.14| 0.04|
| (5,5)    | 96.88| 93.75| 88.65| 81.31| 72.06| 61.73| 51.31| 41.55| 32.87| 25.45| 19.28| 14.29| 10.34| 7.29| 5.00| 3.32| 2.12| 1.30| 0.76|
| (7,4)    | 99.22| 98.44| 97.07| 94.88| 91.59| 87.02| 81.14| 74.08| 66.13| 57.64| 49.02| 40.63| 32.78| 25.71| 19.57| 14.43| 10.28| 7.06| 4.66|
| (7,5)    | 99.90| 99.80| 99.63| 99.32| 98.82| 98.05| 96.91| 95.31| 93.20| 90.52| 87.25| 83.44| 79.13| 74.41| 69.37| 64.13| 58.79| 53.45| 48.19|
| (10,5)   | 99.99| 99.98| 99.95| 99.91| 99.84| 99.73| 99.55| 99.27| 98.87| 98.30| 97.52| 96.51| 95.22| 93.63| 91.72| 89.49| 86.96| 84.13| 81.03|
| (14,7)   | 99.99| 99.99| 99.98| 99.96| 99.92| 99.86| 99.77| 99.62| 99.40| 99.08| 98.63| 98.03| 97.24| 96.24| 95.00| 93.51| 91.76| 89.74| 87.47|
| (21,6)   | 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0|
| (31,7)   | 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0| 100.0|