TRACE FORMULAS FOR TIME PERIODIC COMPLEX HAMILTONIANS ON LATTICE

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Abstract. We consider time periodic Hamiltonians with complex potentials on the lattice and determine trace formulas. As a corollary we estimate eigenvalues of the quasienergy operator in terms of the norm of potentials.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. We discuss scattering and trace formulas for the Schrödinger equation on the lattice \( Z^d \):

\[
d\frac{du(t)}{dt} = -ih(t)u(t), \quad h(t) = \Delta + V(t),
\]

(1.1)

where \( h(t) \) is the Hamiltonian, \( \tau \)-periodic in time \( t \) and \( \Delta \) is the discrete Laplacian given by

\[
(\Delta f)_x = \frac{1}{2} \sum_{|x-y|=1} (f_x - f_y), \quad f = (f_x)_{x \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \quad x = (x_j)_1^d \in \mathbb{Z}^d.
\]

(1.2)

It is known that the spectrum of the Laplacian \( \Delta \) is absolutely continuous and satisfies

\[
\sigma(\Delta) = \sigma_{ac}(\Delta) = [0, 2d].
\]

Here \( V(t) \) is \( \tau \)-periodic in time potential: \( (V(t)f)_x = V_x(t)f_x \), for all \( (t, x) \in \mathbb{R} \times \mathbb{Z}^d \). Introduce the space \( \ell^p(\mathbb{Z}^d), p \geq 1 \) of sequences \( f = (f_x)_{x \in \mathbb{Z}^d} \) equipped with the norm given by

\[
\|f\|_p = \|f\|_{\ell^p(\mathbb{Z}^d)} = \left( \sum_{x \in \mathbb{Z}^d} |f_x|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty),
\]

and let \( \|f\|_{\infty} = \|f\|_{\epsilon_{\infty}(\mathbb{Z}^d)} = \sup_{x \in \mathbb{Z}^d} |f_x| \). For a Banach space \( \mathfrak{B} \) we write \( \ell^r(\mathfrak{B}), r \geq 1 \) for the space of \( \mathfrak{B} \)-valued sequences with \( p^{th} \) power summable norms, and \( L^r(T_\tau, \mathfrak{B}) \) for the \( \mathfrak{B} \)-valued \( L^r \)-space. In the case \( f(\cdot) \in L^r(T_\tau, \ell^p(\mathbb{Z}^d)) \) we define the norm \( \|f\|_{r,p} \) by

\[
\|f\|_{r,p} = \int_{T_\tau} \|f(t)\|_{\ell^p(\mathbb{Z}^d)}^r dt, \quad p, r \geq 1.
\]

(1.3)

Note that \( \|f\|_{p,r} \leq \|f\|_{q,r} \) for all \( p \geq q \geq 1 \). We assume that potentials can be complex-valued and satisfy

**Condition V.** Let \( d \geq 3 \). The function \( V(t) \) is \( \tau \)-periodic, and satisfies

\[
\|V\|_{p,2} < \infty, \quad \begin{cases} 
1 \leq p < \frac{6}{5} & \text{if } d = 3 \\
1 \leq p < \frac{4}{3} & \text{if } d \geq 4.
\end{cases}
\]

(1.4)

We discuss trace formulas for operators with complex potentials. Recall that, in general, a trace formula is an identity connecting the integral of the potential and various sums of
eigenvalues and integrals of coefficients of S-matrix of the Schrödinger operator (or other spectral characteristics). We shortly describe results about multidimensional trace formulas:

- **Real potentials.** The first result was obtained by Buslaev [6], see also [12], [38] and references therein. Trace formulas for Stark operators and magnetic Schrödinger operators were discussed in [31], [30]. The trace formulas for Schrödinger operators on the lattice \( \mathbb{Z}^d \) with real decaying potentials were determined by Isozaki–Korotyaev [15].

- **Complex potentials.** Unfortunately, we know only few papers about the trace formulas for Schrödinger operators with complex-valued potentials decaying at infinity. Trace formulas for Schrödinger operators with complex decaying potentials were determined by Korotyaev [25] in the continuous case and in the discrete case by Korotyaev and Laptev [28], Korotyaev [27] and for the specific case \( \text{Im} V \leq 0 \) by Malamud and Neidhardt [34].

Our main goal is to determined trace formulas for time periodic Hamiltonian with complex (and for real) potentials on the lattice. The case of \( \mathbb{R}^d \) is more complicated [24]. We do not know any results about it.

For Hilbert space \( \mathcal{H} \) and \( \mathbb{T}_\tau = \mathbb{R}/(\tau\mathbb{Z}) \) we introduce the space \( \tilde{\mathcal{H}} = L^2(\mathbb{T}_\tau, \mathcal{H}) \) of functions \( f \to f(t) \) that are \( \tau \)-periodic in time with values in \( \mathcal{H} \) equipped with the norm

\[
\|f\|_{\tilde{\mathcal{H}}}^2 = \frac{1}{\tau} \int_0^\tau \|f(t)\|_{\mathcal{H}}^2 dt.
\]

The space \( \tilde{\mathcal{H}} \) can be realized as \( \ell^2(\mathcal{H}) \) via the Fourier transform \( \Phi : \tilde{\mathcal{H}} \to \ell^2(\mathcal{H}) \) defined by

\[
f \to \Phi f = (f_n)_{n \in \mathbb{Z}}, \quad f_n = \left(\Phi f\right)_n = \frac{1}{\sqrt{\tau}} \int_0^\tau e^{-i\omega t} f(t) dt, \quad \omega = \frac{2\pi}{\tau}, \quad f \in \tilde{\mathcal{H}}.
\]

Let \( \partial = -i \frac{\partial}{\partial t} \) be the self-adjoint operator in \( L^2(\mathbb{T}_\tau) \). We also denote \( \partial = -i \frac{\partial}{\partial t} \) the corresponding operator in \( \tilde{\mathcal{H}} \) with the natural domain \( \mathcal{D} = \mathcal{D}(\partial) \). We use the notation \( \langle A(t) \rangle \) to indicate multiplication by \( A(t) \) on the space \( \tilde{\mathcal{H}} \). Introduce the operators \( \tilde{h}_o \) and \( \tilde{h} \) on \( \tilde{\mathcal{H}} = L^2(\mathbb{T}_\tau, \ell^2(\mathbb{Z}^d)) \) by

\[
\tilde{h}_o = \partial + \Delta, \quad \tilde{h} = \tilde{h}_o + \langle V(t) \rangle.
\]

It is known that the spectrum \( \sigma(\Delta) = [0, 2d] \). Then the spectrum of \( \tilde{h}_o \) has the form

\[
\sigma(\tilde{h}_o) = \sigma_{ac}(\tilde{h}_o) = \bigcup_{n \in \mathbb{Z}} \sigma(\Delta + \omega n) = \bigcup_{n \in \mathbb{Z}} [\omega n, \omega n + 2d]. \tag{1.5}
\]

Note that if \( \omega > 2d \), then the spectrum of \( \tilde{h}_o \) has the band structure with the bands \( \sigma(\Delta + \omega n) = [\omega n, \omega n + 2d] \) separated by gaps. Let \( B_1 \) and \( B_2 \) be the trace and the Hilbert-Schmidt class equipped with the norm \( \| \cdot \|_{B_1} \) and \( \| \cdot \|_{B_2} \), respectively. Introduce the free resolvent \( R_o(\lambda) = (\tilde{h}_o - \lambda)^{-1}, \lambda \in \mathbb{C}_\pm \). Below we show that if \( V \) satisfies Condition \( V \), then

\[
VR_o(\lambda) \in B_2, \quad \forall \lambda \in \mathbb{C}_\pm. \tag{1.6}
\]

This yields \( \mathcal{D}(\tilde{h}) = \mathcal{D}(\tilde{h}_o) \) and

\[
\sigma_{ess}(\tilde{h}) = \sigma_{ess}(\tilde{h}_o). \tag{1.7}
\]

Thus the operator \( \tilde{h} \) has only discrete spectrum in \( \mathbb{C}_\pm \). Define the perturbed resolvent \( R(\lambda) = (\tilde{h} - \lambda)^{-1} \) for all \( \lambda \in \mathbb{C}_\pm \setminus \sigma_{disc}(\tilde{h}) \). We have the very useful identity

\[
\langle e^{i\omega t} \rangle R(\lambda) \langle e^{-i\omega t} \rangle = R(\lambda + \omega), \quad \forall \lambda \in \mathbb{C}_\pm. \tag{1.8}
\]
It means that the spectrum of $\tilde{h}$ (and $\tilde{h}_0$) is $\omega$-periodic. Thus it is sufficient to study eigenvalues of $\tilde{h}$ in a strip $\text{Re} \lambda \in [0, \omega)$. We consider the case of the half strip $\Lambda \subset \mathbb{C}_+$ defined by

$$
\Lambda = [0, \omega) \times i\mathbb{R}_+ \subset \mathbb{C}_+, \quad \omega = \frac{2\pi}{r}.
$$

The proof for the lower half strip $\overline{\Lambda}$ is similar. The operator $\tilde{h}$ has $N \leq \infty$ eigenvalues $\{\lambda_j, j = 1, \ldots, N\}$ in the strip $\Lambda$. Here and below each eigenvalue is counted according to its algebraic multiplicity. We have similar consideration for the case $\mathbb{C}_-$.  

1.2. **Main results.** We assume that a potential $V$ satisfy Condition V. We define a operator-valued function $\mathfrak{F}$ (below we show that $\mathfrak{F}(\lambda) \in \mathcal{B}_2$) and the regularized determinant $\mathcal{D}$ by

$$
\mathfrak{F}(\lambda) = |V|^\frac{1}{2}R_0(\lambda)|V|^{\frac{1}{2}}e^{i\text{arg}V}, \quad \lambda \in \mathbb{C}_+, 
$$

and

$$
\mathcal{D}(\lambda) = \det [(I + \mathfrak{F})e^{-\delta}](\lambda), \quad \lambda \in \mathbb{C}_+, 
$$

$\mathcal{D}$ is the basic function to study trace formulas. We describe main properties of $\mathcal{D}$. 

**Theorem 1.1.** Let $V$ satisfy Condition V and a constant $C_*$ be defined by (2.3).

i) Then the operator-valued function $\mathfrak{F} : \mathbb{C}_* \to \mathcal{B}_2(L^2(\mathbb{T}_r, \mathcal{L}(\mathbb{Z}^d)))$, defined by (1.1) is analytic and Hölder continuous up to the boundary. Moreover, it satisfies:

$$
|\mathfrak{F}(\lambda)|_{\mathcal{B}_2} \leq C_*|V|_{p, 2} \quad \forall \lambda \in \mathbb{C}_+, 
$$

$$
C_* = 1 + (1 + \frac{\tau d}{\pi})(C_g + \tau^{-\frac{1}{2}}) + \frac{C_g}{\tau},
$$

where $C_g = \frac{1}{2\pi} + \frac{5 + 3\pi}{8}\sqrt{2}$ and the constant $C_*$ is defined by (3.3).

ii) The modified determinant $\mathcal{D}$ is analytic in $\mathbb{C}_+$, Hölder up to the boundary and satisfies

$$
\mathcal{D}(\lambda + \omega) = \mathcal{D}(\lambda) \quad \forall \lambda \in \mathbb{C}_+, 
$$

$$
\mathcal{D}(\lambda) = 1 + \frac{O(1)}{\nu} \quad \text{as} \quad \nu := |\text{Im} \lambda| \to \infty, 
$$

$$
\sup_{\lambda \in \mathbb{C}_+} |\mathcal{D}(\lambda)| \leq e^{\frac{C_*}{2}|V|_{2, 2}}. 
$$

Moreover, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of the operator $\tilde{h}$, then

$$
\nu(1 - e^{-\tau \nu}) \leq 2|V|_{2, 2}^2. 
$$

**Remark.** 1) For complex potentials we discuss eigenvalues of $\tilde{h}$ only in the domain $[0, \omega] \times (0, i\nu_o], \nu_o = 2|V|_{2, 2}$, since the operator $\tilde{h}$ does not have zeros in the domain $\{ |\text{Im} \lambda| > \nu_o \}$.

2) If the operator $V$ is bounded, then eigenvalues of $\tilde{h}$ belong to the strip $\{ |\text{Im} \lambda| < |\text{Im} V| \}$.

We define the disc $\mathbb{D}_r \subset \mathbb{C}$ with the radius $r > 0$ by $\mathbb{D}_r = \{ z \in \mathbb{C} : |z| < r \}$, and abbreviate $\mathbb{D} = \mathbb{D}_1$. We define the Hardy space in a disk $\mathbb{D}$. We say a function $F$ belongs the Hardy space $\mathcal{H} = \mathcal{H}_\infty(\mathbb{D})$ if $F$ is analytic in $\mathbb{D}$ and satisfies

$$
\|F\|_{\mathcal{H}} := \sup_{\lambda \in \mathbb{D}} |F(z)| < \infty.
$$

For $\lambda \in \Lambda$ we define the new spectral variable $z \in \mathbb{D}$ by

$$
z = e^{i\tau \lambda} \in \mathbb{D}, \quad \lambda(z) = -\frac{i}{\tau} \ln z \in \Lambda = [0, \omega] \times \mathbb{R}_+ \subset \mathbb{C}_+.
The function $z = e^{i\tau\lambda}, \lambda \in \Lambda$ is a conformal mapping from the strip $\Lambda$ onto the unit disk $\mathbb{D}$. Theorem 1.1 shows that a function

$$\psi(z) := D(\lambda(z)), \quad z \in \mathbb{D},$$

belongs to the Hardy space $\mathcal{H}_\infty(\mathbb{D})$.

Define operators $J_1, J_2$ on $L^2(\mathbb{T})$ and operators $F_1, F_2$ on $\tilde{\mathcal{H}}$ by

$$(J_1f)(t) = i \int_0^t f(s)ds, \quad (J_2f)(t) = i \int_0^\tau f(s)ds,$$

and

$$F_1 = J_1\tilde{V}, \quad F_2 = J_2\tilde{V}, \quad \tilde{V}(t) = e^{i\Delta}V(t)e^{-i\Delta}, \quad \mathcal{R}_1 = (I + F_1)^{-1}.$$ (1.17)

Note that the operator $I + F_1$ is invertible.

**Theorem 1.2.** Let a potential $V$ satisfy Condition V and the constant $C_\bullet$ be defined by (1.11). Then the function $\psi(z) := D(\lambda(z))$ belongs to $\mathcal{H}_\infty(\mathbb{D})$ and is H"older up to the boundary and satisfies

$$\|\psi\|_{\mathcal{H}_\infty(\mathbb{D})} \leq e^{\frac{C_\bullet}{2}\|V\|^2_{p,2}}.$$ (1.18)

The zeros $\{z_j\}_{j=1}^N$ of $\psi$ in $\mathbb{D}$ satisfy $\sum_{j=1}^N (1 - |z_j|) < \infty$. Moreover, the function $\log \psi(z)$ is analytic in $\mathbb{D}$, for some $r_0 > 0$ and has the Taylor series:

$$\log \psi(z) = \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots, \quad \text{as} \quad |z| < r_0,$$ (1.19)

where the coefficients $\psi_n$ are given by

$$\psi_1 = -\text{Tr} F_1\mathcal{R}_1 e_\tau F_2, \quad \psi_2 = -\text{Tr} F_1\mathcal{R}_1 e_\tau^2 F_2 + \frac{1}{2}\text{Tr} F_1(\mathcal{R}_1 e_\tau F_2)^2, \ldots$$ (1.20)

and $e_\tau = e^{-i\tau\Delta}$ and $F_1, F_2$ are defined by (1.17) and $\mathcal{R}_1 = (I + F_1)^{-1}$.

**Remark.** 1) We transform the analytic problem from the domain $\mathbb{C}_+$ to the disk $\mathbb{D}$. The energy periodic property (1.12) of the determinant $D(\lambda + \omega) = D(\lambda)$ and asymptotics (1.13) are crucial here.

2) It is unusual that due to (1.19) the determinant $\log D(\lambda) = e^{-\tau \text{Im} \lambda}O(1)$ as $\text{Im} \lambda \to \infty$.

Recall that the operator $\tilde{h}$ has $N \leq \infty$ eigenvalues $\{\lambda_j, j = 1, \ldots, N\}$ in the domain $\Lambda$. Each point $z_j = z(\lambda_j) \in \mathbb{D}$ is a zero of $\psi(z)$. For the function $\psi$ we define the Blaschke product $B(z), z \in \mathbb{D}$ by: $B = 1$ if $N = 0$ and

$$B(z) = \prod_{j=1}^N \frac{|z_j|}{z_j} \frac{(z - z_j)}{(1 - z_j z),} \quad z_j = e^{i\tau\lambda_j} \quad \text{if} \quad N \geq 1.$$ (1.21)

It is well known that the Blaschke product $B(z), z \in \mathbb{D}$ given by (1.21) converges absolutely for $\{|z| < 1\}$ and satisfies $B \in \mathcal{H}_\infty(\mathbb{D})$ with $\|B\|_{\mathcal{H}_\infty} \leq 1$, since $\psi \in \mathcal{H}_\infty(\mathbb{D})$. The Blaschke product $B$ has the standard Taylor series at $z = 0$:

$$\log B(z) = B_0 - B_1 z - B_2 z^2 - \ldots \quad \text{as} \quad z \to 0,$$ (1.22)

where $B_0 = \log B(0) < 0$ and $B_n = \frac{1}{n} \sum_{j=1}^N \left( \frac{1}{z_j} - \overline{z}_j \right), n \geq 1$. In particular we have

$$B_0 = \log B(0) = -\tau \sum \text{Im} \lambda_j < 0.$$ (1.23)
We describe the canonical representation of the determinant $\psi(z), z \in \mathbb{D}$.

**Corollary 1.3.** Let a potential $V$ satisfy Condition V. Then the determinant $\psi$ has a canonical factorization for all $|z| < 1$ given by

$$
\psi(z) = B(z)e^{\Psi(z)}, \quad \Psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),
$$

where

$$
d\mu(t) = \ln|\psi(e^{it})|dt - dm(t),
$$

where $\ln|\psi(e^{it})| \in L^1(\mathbb{T})$ and $m \geq 0$ is some singular measure on $[0, 2\pi]$, such that $\text{supp}\ m \subset \{t \in [0, 2\pi]: \psi(e^{it}) = 0\}$. Moreover, $\Psi$ has the Taylor series at $z = 0$ in some disk $\{|z| < r\}$:

$$
\Psi(z) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \ldots,
$$

where

$$
\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t) = \int_0^{2\pi} \log|\psi(e^{it})|dt - m(\mathbb{T}), \quad \mu_n = \frac{1}{\pi} \int_0^{2\pi} e^{-int} d\mu(t), \quad n \in \mathbb{N}.
$$

We present our main result about trace formulas.

**Theorem 1.4.** Let $V$ satisfy Condition V. Then the following trace formula holds true:

$$
-\frac{i}{\tau z} \text{Tr} \left( R(\lambda) - R_0(\lambda) + R_0(\lambda)VR_0(\lambda) \right) = \sum_{\lambda \in \Lambda} \frac{(1 - |z_j|^2)}{(z - z_j)(1 - \bar{z}_j z)} + \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it} d\mu(t)}{(e^{it} - z)^2},
$$

$$
\frac{m(\mathbb{T})}{2\pi} + \tau \sum_{\lambda \in \Lambda} \text{Im} \lambda_j = \frac{1}{\pi} \int_0^{2\pi} \log|\psi(e^{it})|dt \geq 0,
$$

$$
B_n = \psi_n + \mu_n, \quad n = 1, 2, 3, \ldots
$$

where $\lambda(z) = \frac{i}{\tau} \ln z$, $z \in \mathbb{D}$ and the measure $d\mu(t) = \log|\psi(e^{it})|dt - dm(t)$, and $B_n$ are given by (1.22), and in particular,

$$
B_1 = \sum_{j=1}^{N} \left( \frac{1}{z_j} - z_j \right) = \psi_1 + \frac{1}{\pi} \int_\mathbb{T} e^{-it} d\mu(t),
$$

**Remark.** 1) The measure $d\mu(t)$ in (1.26) is some analog of the spectral shift function for complex potentials.

2) The trace formula (1.27) has the term $m(\mathbb{R})$ which is absent for real potentials. There is an open problem: when this term is absent (or there exists) for specific complex potentials.

3) Consider the $\psi(z) = \mathcal{D}(\lambda(z)), z \in \mathbb{D}$. If $z = e^{it}$, then $\lambda = \frac{\ln z}{\tau} = \frac{i}{\tau} \in [0, \omega]$. Then we obtain

$$
\int_0^{2\pi} \log|\psi(e^{it})|dt = \tau \int_0^\omega \log|\mathcal{D}(\lambda + i0)|d\lambda.
$$

Moreover, we can do the same with all integrals in Theorem 1.4.

**Corollary 1.5.** Let a potential $V$ satisfy Condition V. Then the following estimate hold true:

$$
\frac{m(\mathbb{T})}{2\pi} + \tau \sum_{\lambda \in \Lambda} \text{Im} \lambda_j \leq C^2 \|V\|_{p,2},
$$

where the constant $C_\ast$ is defined by (1.17).
Remark. 1) The measure $d\mu(t)$ in (1.26) is some analog of the spectral shift function for complex potentials.

2) If a potential $V$ does not depend on time then there are estimates of complex eigenvalues in terms of potentials, see [28], [27]. In the continuous case there are a lot of results about it, see, e.g., [9], [8] and references therein.

For time-periodic Hamiltonians many papers have been devoted to scattering mainly for operators $h(t) = -\Delta + V(t, x)$ on $\mathbb{R}^d, d \geq 1$, and to the spectral analysis of the corresponding monodromy operator. Zel’dovich [15] and Howland [13] reduced the problem with a time-dependent Hamiltonian to a problem with a time-independent Hamiltonian by introducing an additional time coordinate. Completeness of the wave operators for $\tilde{h}, \tilde{h}_0$ was established by Yajima [43]. In [13], [19] it was shown that $\tilde{h}$ has no singular continuous spectrum. Moreover, Korotyaev [19] proved in that the total number of embedded eigenvalues on the interval $[0, \omega]$, counting multiplicity, is finite. The case of Schrödinger operators with time-periodic electric and homogeneous magnetic field was discussed in [22], [23], [44], see also recent papers [1], [2], [3], [17], [35]. Moreover, scattering for three body systems was considered in [20], see also [36].

Now we discuss stationary case of discrete multidimensional Schrödinger operators on the cubic lattice $\mathbb{Z}^d, d \geq 2$, when potentials are real and do not depend on time. For Schrödinger operators with decaying potentials on the lattice $\mathbb{Z}^d$, Boutet de Monvel and Sahbani [5] used Mourre’s method to prove completeness of the wave operators, absence of singular continuous spectrum and local finiteness of eigenvalues away from threshold energies. Isozaki and Korotyaev [15] studied the direct and the inverse scattering problem as well as trace formulas. Korotyaev and Moller [29] discussed the spectral theory for potentials $V \in \ell^p, p > 1$. Isozaki and Morioka [16] and Vesalainen [42] proved that the point-spectrum of $H$ on the interval $(0, 2d)$ is absent, see also [4]. An upper bound on the number of discrete eigenvalues in terms of some norm of potentials was given by Korotyaev and Slousch [32], Rozenblum and Solomyak [40]. For closely related problems, we mention that Parra and Richard [37] re-proved the results from [5] for periodic graphs. Finally, scattering on periodic metric graphs associated with $\mathbb{Z}^d$ was considered by Korotyaev and Saburova [32].

2. Regularized determinants

2.1. Preliminary analysis. We present the concepts and facts needed below. Let $\mathcal{H}$ be a complex separable Hilbert space. The class of bounded and compact operators in $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$ respectively. Let $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ be the trace and the Hilbert-Schmidt class equipped with the norm $\| \cdot \|_{\mathcal{B}_1}$ and $\| \cdot \|_{\mathcal{B}_2}$, respectively. If it is evident which $\mathcal{H}$ is meant, we shall write simply $\mathcal{B}, \mathcal{B}_\infty,...$. We recall some well-known facts about determinants from [11].

- Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_1$. Then
  \[
  \text{Tr } AB = \text{Tr } BA, \tag{2.1}
  \]

  \[
  \det(I + AB) = \det(I + BA). \tag{2.2}
  \]

- If $A, B \in \mathcal{B}_1$, then
  \[
  | \det(I + A) | \leq e^{\|A\|_{\mathcal{B}_1}},
  \]

  \[
  | \det(I + A) - \det(I + B) | \leq \| A - B \|_{\mathcal{B}_1} e^{\|A\|_{\mathcal{B}_1} + \|B\|_{\mathcal{B}_1}}. \tag{2.3}
  \]
We define the modified determinant $\det_2(I + A)$ by

$$\det_2(I + A) = \det \left( (I + A)e^{-A} \right),$$

(2.4)

and $I + A$ is invertible if and only if $\det_2(A) \neq 0$, see Chapter IV in [11].

- If $A \in \mathcal{B}_2$, then (see [11])

$$|\det_2(I + A)| \leq e^{\frac{1}{2}\|A\|_{\mathcal{B}_2}^2}.$$  

(2.5)

- Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_2$. Then

$$\det_2(I + AB) = \det_2(I + BA).$$  

(2.6)

- Suppose a function $A(\cdot) : \Omega \to \mathcal{B}_1$ is analytic for a domain $\Omega \subset \mathbb{C}$, and the operator $(I + A(z))^{-1}$ is bounded for any $z \in \Omega$. Then the function $f(z) = \det(I + A(z))$ satisfies

$$f'(z) = f(z) \text{Tr} \left( (I + A(z))^{-1} A'(z) \right) \quad \forall z \in \Omega.$$  

(2.7)

In order to investigate the determinant $\mathcal{D}(\lambda)$ we need a following lemma.

**Lemma 2.1.** Let operators $A, B \in \mathcal{B}_2$ act on some Hilbert space $\mathcal{H}$. Then

$$e^{A}e^{B}e^{-(A-B)} - I \in \mathcal{B}_1,$$  

(2.8)

$$\det \left( e^{A}e^{-(A-B)}e^{B} \right) = \det \left( e^{B}e^{A}e^{-(A-B)} \right) = 1,$$  

(2.9)

$$\det \left( (I + A + B)e^{-(A-B)} \right) = \det \left( e^{-A}(I + A + B)e^{-B} \right).$$  

(2.10)

If in addition $I + A$ is invertible and $\mathcal{R} = (I + A)^{-1}$, then

$$\det \left( (I + A + B)e^{-(A-B)} \right) = \det \left( (I + A)e^{-A} \right) \det \left( (I + \mathcal{R}B)e^{-B} \right).$$  

(2.11)

**Proof.** Using the Taylor series $e^z = 1 + z + \frac{z^2}{2} + \ldots$ at $z = A, B, -A - B$ we obtain

$$e^Ae^B e^{-(A-B)} = (I + A + A_1)(I + B + B_1)(I - A - B + C_1) = I + G,$$

where $A_1, B_1, C_1, G$ are some trace operators, which yields (2.8). We show (2.9). Define the trace class valued function $F(t) - I$ and the determinant $D(t)$ by

$$F(t) = e^{tA}e^{tB}e^{-(A+B)}, \quad D(t) = \det F(t)$$

for $t \in \mathbb{R}$. From (2.7) we obtain the derivative

$$D'(t) = D(t) \text{Tr}(F'(t)F(t)^{-1}).$$

Using this formula, we get:

$$D'(t) = D(t) \text{Tr}(F'(t)F(t)^{-1})$$

$$= D(t) \text{Tr} \left( e^{tA}e^{tB}(e^{-tB}Ae^{tB} - A)e^{-t(A+B)} \right) e^{t(A+B)}e^{-tB}e^{-tA} = D(t) \text{Tr} \left( (e^{-tB}Ae^{tB} - A) \right) = 0,$$
which yields (2.9). We show (2.10). Using (2.9) we obtain
\[
\det \left( (I + A + B)e^{-A-B} \right) = \det \left( (I + A + B)e^{-B}e^{-A} \cdot e^A e^B e^{-A-B} \right)
\]
\[
\det \left( (I + A + B)e^{-B}e^{-A} \right) \det \left( e^A e^B e^{-A-B} \right) = \det \left( (I + A)e^{-A} \right) \det \left( (I + R B)e^{-B} \right),
\]
which yields (2.10). If in addition \( I + A \) is invertible, then (2.10) gives
\[
\det \left( (I + A + B)e^{-A-B} \right) = \det \left( (I + A)e^{-A} \right) \det \left( (I + R B)e^{-B} \right),
\]
which yields (2.11). □

Let \( \mathfrak{B} = L^2(\mathbb{T}_\tau, \mathfrak{B}) \) be the space of function \( v : \mathbb{T}_\tau \to v(t) \in \mathfrak{B} \) for some Banach space \( \mathfrak{B} \), which are measurable and satisfy \( \int_0^T \| v(t) \|^2_{\mathfrak{B}} dt < \infty \). Recall that operators \( J_1, J_2 \) act on \( L^2(\mathbb{T}_\tau) \) and are given by
\[
(J_1 f)(t) = i \int_0^t f(s) ds, \quad (J_2 f)(t) = i \int_0^t f(s) ds,
\]
(2.12)

Note that the operator \( I + J_1 \) is invertible.

**Lemma 2.2.** i) The operator \( \partial = -i \frac{\partial}{\partial t} \) acting on \( L^2(\mathbb{T}_\tau) \) has the resolvent given by
\[
((\partial - \lambda)^{-1} f)(t) = \int_0^t \frac{ie^i\lambda(t-s)}{1-z} f(s) ds + \int_0^t e^{i\lambda(t-s)} f(s) ds, \quad z = e^{i\tau},
\]
(2.13)

where \( f \in L^2(\mathbb{T}_\tau) \) and \( \lambda \in \mathbb{C} \setminus \sigma(\partial) \).

ii) Let an operator function \( V \in L^2(\mathbb{T}_\tau, \mathcal{B}_2(\mathcal{H})) \). Then operators \( J_1 V \) and \( J_2 V \) on \( \mathfrak{H} \) belong to \( \mathfrak{B}_2 \) and satisfy
\[
\| J_2 V \|^2_{\mathfrak{B}_2} \leq \tau \int_0^\tau \| V(t) \|^2_{\mathfrak{B}_2} dt \leq \tau^2 \sup_{t \in [0,\tau]} \| V(t) \|^2_{\mathfrak{B}_2},
\]
\[
\| J_1 V \|^2_{\mathfrak{B}_2} \leq \int_0^\tau \| V(t) \|^2_{\mathfrak{B}_2} (\tau - t) dt \leq \frac{\tau^2}{2} \sup_{t \in [0,\tau]} \| V(t) \|^2_{\mathfrak{B}_2}.
\]
(2.14)

**Proof.** i) We have \((\partial - \lambda)u = f\), where \( u', f \in L^2(\mathbb{T}_\tau) \). Then using \( u(\tau) = u(0) \) we obtain
\[
u' - i\lambda u = if, \quad (e^{-i\lambda t} u')' = ie^{-i\lambda t} f, \quad e^{-i\lambda t} u(t) = u(0) + \int_0^t ie^{-i\lambda s} f(s) ds
\]
\[
e^{-i\lambda \tau} u(\tau) = u(0) + \int_0^\tau ie^{-i\lambda s} f(s) ds, \quad (e^{-i\lambda t} - 1) u(0) = \int_0^\tau ie^{-i\lambda s} f(s) ds,
\]
which yields (2.13).

ii) The Gilbert-Schmidt norms of \( J_j V, j = 1, 2 \) are
\[
\| J_2 V \|^2_{\mathfrak{B}_2} = \int_0^\tau ds \int_0^t \| V(s) \|^2_{\mathfrak{B}_2} dt = \tau \int_0^\tau \| V(t) \|^2_{\mathfrak{B}_2} dt \leq \tau^2 \sup_{t \in [0,\tau]} \| V(t) \|^2_{\mathfrak{B}_2},
\]
\[
\| J_1 V \|^2_{\mathfrak{B}_2} = \int_0^\tau dt \int_0^t \| V(s) \|^2_{\mathfrak{B}_2} ds = \int_0^\tau \| V(s) \|^2_{\mathfrak{B}_2} (\tau - s) ds \leq \frac{\tau^2}{2} \sup_{t \in [0,\tau]} \| V(t) \|^2_{\mathfrak{B}_2}.
\]
Proposition 2.3. Let $h_o$ be a bounded self-adjoint operator on the separable Hilbert space $\mathcal{H}$.

i) The resolvent $R_o(\lambda) = (\partial + h_o - \lambda)^{-1}, \lambda \in \mathbb{C} \setminus \mathbb{R}$ on $f \in L^2(\mathbb{T}_\tau) \times \mathcal{H}$ has the form given by

$$R_o(\lambda) f(t) = i e^{i\tau \varphi} \int_0^\tau \left( \mathbb{I}_{t-s} + \frac{e^{i\tau \varphi}}{1 - e^{i\tau \varphi}} \right) e^{-i\varphi s} f(s) ds,$$

(2.15)

where $\mathbb{I}_t = 1, t > 0$ and $\mathbb{I}_t = 0, t < 0$.

ii) Let, in addition, an operator function $V \in L^2(\mathbb{T}_\tau, \mathcal{B}_2(\mathcal{H}))$ and let $c = \int_0^\tau \|V(s)\|_{\mathcal{B}_2}^2 ds$. Then operator $R_o(\lambda)V$ on $\mathcal{H}$ belongs to $\mathcal{B}_2$ and satisfies

$$\|R_o(\lambda)V\|_{\mathcal{B}_2}^2 \leq \frac{2c}{\nu(1 - e^{-\nu\tau})}, \quad \nu := \text{Im} \lambda > 0.$$  

(2.16)

Moreover, if $\lambda \in \mathbb{C}_+$ is an eigenvalue of the operator $\tilde{h}_o + V$, then

$$\nu(1 - e^{-\nu\tau}) \leq 2c.$$  

(2.17)

Proof. The statement i) follows from Lemma 2.2.

ii) Using (2.15) we present $R_o$ in the form $R_o = R_1 + XR_2$, where

$$R_1(\lambda) f(t) = \int_0^\tau e^{i\lambda(t-s)\varphi} f(s) ds, \quad R_2(\lambda) f(t) = \int_0^\tau e^{i\lambda(t-s)\varphi} f(s) ds, \quad X = \frac{e^{i\tau \varphi}}{1 - e^{i\tau \varphi}}.$$  

Consider the case $\nu = \text{Im} \lambda > 0$, the proof for $\nu < 0$ is similar. The Gilbert-Schmidt norm of $R_1V$ is

$$\|R_1V\|_{\mathcal{B}_2}^2 = \int_0^\tau e^{-2\nu(t-s)} dt \int_0^\tau \|V(s)\|_{\mathcal{B}_2}^2 ds = \int_0^\tau e^{2\nu s} \|V(s)\|_{\mathcal{B}_2}^2 ds \int_s^\tau e^{-2\nu t} dt = \frac{1}{2\nu} \int_0^\tau \|V(s)\|_{\mathcal{B}_2}^2 (1 - e^{-2\nu(\tau-s)}) ds \leq \frac{c}{2\nu},$$

(2.18)

and the Gilbert-Schmidt norm of $R_2V$ is

$$\|R_2V\|_{\mathcal{B}_2}^2 = \int_0^\tau e^{-2\nu t} dt \int_0^\tau e^{2\nu s} \|V(s)\|_{\mathcal{B}_2}^2 ds = \frac{1 - e^{-2\nu\tau}}{2\nu} \int_0^\tau e^{2\nu s} \|V(s)\|_{\mathcal{B}_2}^2 ds \leq \frac{e^{2\nu\tau} - 1}{2\nu} c.$$  

Then the estimate $\|X\| \leq \frac{e^{-\nu\tau}}{1 - e^{-\nu\tau}} = \frac{1}{\nu e^{\nu\tau}}$ gives

$$\|XR_2(\lambda)V\|_{\mathcal{B}_2}^2 \leq \frac{(e^{2\nu\tau} - 1)}{(e^{\nu\tau} - 1)^2} \frac{c}{2\nu} = \frac{(e^{\nu\tau} + 1)}{(e^{\nu\tau} - 1)^2} \frac{c}{2\nu},$$

(2.19)

since $\frac{(e^{\nu\tau} + 1)}{(e^{\nu\tau} - 1)^2} = 1 + \frac{2}{e^{\nu\tau} - 1} \leq 1 + \frac{2}{\nu\tau}$. Then we obtain

$$\|R_oV\|_{\mathcal{B}_2}^2 \leq 2\|R_1V\|_{\mathcal{B}_2}^2 + 2\|R_2V\|_{\mathcal{B}_2}^2 \leq \frac{2e^{\nu\tau}}{\nu(e^{\nu\tau} - 1)} = \frac{2c}{\nu(1 - e^{-\nu\tau})}.$$  

(2.20)

If $\|R_o(\lambda)V\|_{\mathcal{B}_2} < 1$, then the operator $I + R_o(\lambda)V$ has an inverse. Thus from (2.20) we have that if $\frac{2c}{\nu(1 - e^{-\nu\tau})} < 1$, then $\lambda$ is not an eigenvalue of the operator $H_o + V$. Then if $\lambda$ is an eigenvalue of the operator $H_o + V$, then $2c \geq \nu(1 - e^{-\nu})$.■
2.2. **Determinants.** The operator $\partial = -i\frac{\partial}{\partial \tau}$ on $L^2(\mathbb{T}_\tau)$ has the spectrum $\sigma(\partial) = \{\omega \mathbb{Z}\}$, where $\omega = \frac{2\pi}{r}$.

**Lemma 2.4.** Let $h_0$ be a bounded self-adjoint operator on the separable Hilbert space $\mathcal{H}$. Let an operator-valued function $V \in L^2(\mathbb{T}_\tau, \mathcal{B}_2(\mathcal{H}))$ and $R_0(\lambda) = (\partial + h_0 - \lambda)^{-1}, \lambda \in \mathbb{C}_\pm$. Then

i) Operators $R_0(\lambda)V$ and $VR_0(\lambda) \in \mathcal{B}_2(\mathcal{H})$ for any $\lambda \in \mathbb{C}_\pm$; the modified determinant $D(\lambda) = \det \left[(I + VR_0)e^{-VR_0}\right](\lambda)$ is well defined, analytic in $\mathbb{C}_\pm$ and satisfies

\[
D(\lambda) = 1 + O(1/\nu) \quad \text{as} \quad \nu := \text{Im} \lambda \to \pm \infty, \quad \lambda \in \mathbb{C}_\pm. \tag{2.21}
\]

ii) The modified determinant $D(\lambda)$ satisfies

\[
D(\lambda + \omega) = D(\lambda) \quad \forall \lambda \in \mathbb{C}_\pm, \tag{2.22}
\]

\[
\frac{D'(\lambda)}{D(\lambda)} = -\text{Tr} \left((R_0(\lambda)V^2R(\lambda))\right), \tag{2.23}
\]

\[
\log D(\lambda) = -\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \text{Tr} (R_0(\lambda)V)^n, \tag{2.24}
\]

where the traces $T_n := \text{Tr}(VR_0(\lambda))^n, n \geq 2$ satisfy

\[
|T_n(\lambda)| \leq \left(\frac{4}{\nu} \int_0^r \|V(s)\|_{\mathcal{B}_2}^2 ds\right)^{\frac{n}{2}} \quad \forall \nu \geq 1/r. \tag{2.25}
\]

**Remark.** Due to (2.21) we take the branch of $\log D$ so that $\log D(\lambda) = o(1)$ as $|\text{Im} \lambda| \to \infty$.

**Proof.** i) Lemma 2.3 gives that $VR_0(\lambda) \in \mathcal{B}_2(\mathcal{H})$ for any $\lambda \in \mathbb{C}_\pm$. We show that the determinant $D$ is well defined. The Taylor series for the entire function $e^{-E}$ and the estimate (2.16) give at $A(\lambda) = VR_0(\lambda)$

\[
[(I + A)e^{-A}] = (I + A)(1 - A + A^2O(1)) = 1 - A^2 + A^2O(1) = I + A^2O(1).
\]

Moreover, this asymptotics and (2.16), (2.23) imply (2.21).

ii) The identities (1.7) and (2.6) yield (2.22). Take $|\text{Im} \lambda| \geq r$ for $r > 0$ large enough. Then from (2.16), we have by the resolvent equation

\[
R(\lambda) = R_0(\lambda) + \sum_{n=1}^{\infty} (-1)^n \left(R_0(\lambda)V\right)^n R_0(\lambda), \tag{2.26}
\]

where the right-hand side is uniformly convergent on $\{\lambda \in \mathbb{C} : |\text{Im} \lambda| \geq r\}$. Using (2.7) and (2.1), we have the following for $\lambda \in \Lambda$:

\[
D'(\lambda) = -D(\lambda) \text{Tr} \left(e^{A(\lambda)}(I + A(\lambda))^{-1}A(\lambda)A'(\lambda)e^{-A(\lambda)}\right)
\]

\[
= -D(\lambda) \text{Tr}(I + A(\lambda))^{-1}A(\lambda)A'(\lambda) = -D(\lambda) \text{Tr} VR(\lambda)V R_0(\lambda) \tag{2.27}
\]

\[
= -D(\lambda) \text{Tr} \left(R_0(\lambda)VR(\lambda)V R_0(\lambda)\right) = -D(\lambda) \text{Tr} \left((R_0(\lambda)V)^2 R(\lambda)\right),
\]

since $R(\lambda)VR_0(\lambda) = R_0(\lambda)VR(\lambda)$, which yields (2.23). Thus (2.26) gives

\[
(\log D(\lambda))' = -\text{Tr} \sum_{n=0}^{\infty} (-1)^n \left(R_0(\lambda)V\right)^{n+2} R_0(\lambda). \tag{2.28}
\]
Then integrating we obtain (2.24) since we have the identity
\[ \frac{d}{d\lambda} \left( \text{Tr} \left( VR_0(\lambda) \right)^n \right) = n \text{Tr} \left( VR_0(\lambda) \right)^{n-1} R_0(\lambda). \]

iii) From \( \nu \tau \geq 1 \) we have \( 1 + \frac{1}{\nu \tau} \leq 2 \) and using (2.16) we obtain (2.25):
\[ |\text{Tr} T_n| = |\text{Tr} A^n(\lambda)| \leq \|VR_0(\lambda)\|_n^{\nu \tau} \leq \left( \frac{4}{\nu} \int_0^\tau \|V(s)\|^2 d\xi ds \right)^{\frac{n-1}{2}}. \]

We are ready to prove the main theorem of this section. Here we transform the presentation of the modified determinant \( D \) in the forms (2.29), (2.31) convenient for us. Via this presentation we determine the asymptotics (2.30) of \( D(\lambda) \) in terms of \( z = e^{i\tau \lambda} \) as \( \text{Im} \lambda \to +\infty \). In fact, we determine the Taylor expansion (2.33) of \( \psi(z) \) in some disk \( \{|z| < r\} \).

**Theorem 2.5.** Let \( h_o \) be a bounded self-adjoint operator on the separable Hilbert space \( \mathcal{H} \).
Let an operator function \( V \in L^2(T_o, B_2(\mathcal{H})) \). Then

i) The modified determinant \( D(\lambda) = \det \left[(I + R_o V) e^{-R_o V}\right](\lambda) \) is analytic in \( \Lambda \) and the function \( \psi(z) = D(\lambda(z)) \) is analytic in \( \mathbb{D} \) and has the following form
\[ \psi(z) = \det \left[(I + F) e^{-F}\right](z), \quad z = e^{i\tau \lambda} \in \mathbb{D}, \quad (2.29) \]
where the operator \( F(z) \) acts on \( \mathcal{H} \) and is given by
\[ F(z) = F_1 + \gamma(z) F_2, \quad \gamma(z) = \frac{z a}{1 - z a}, \quad a = e^{-i\tau h_o}, \quad (2.30) \]
where \( F_1, F_2 \in B_2(\mathcal{H}) \). Moreover, the operator \( I + F_1 \) is invertible and if \( R_1 = (I + F_1)^{-1} \), then
\[ \psi(z) = \det \left[(I + R_1 \gamma F_2) e^{-\gamma F_2}\right](z), \quad |z| < 1, \quad (2.31) \]

iii) The function \( \log \psi(z) \) is analytic in \( \mathbb{D} \), for some \( r > 0 \) and has the following form
\[ \log \psi(z) = \sum_{n=1}^\infty \frac{(-1)^n}{n} \text{Tr} F_1 (R_1 \gamma(z) F_2)^n. \quad (2.32) \]
Moreover, it has the following Taylor series
\[ \log \psi(z) = \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots, \quad |z| < r, \quad (2.33) \]
where
\[ \psi_1 = - \text{Tr} F_1 R_1 a F_2, \quad \psi_2 = - \text{Tr} F_1 R_1 a^2 F_2 + \frac{1}{2} \text{Tr} F_1 (R_1 a F_2)^2, \quad (2.34) \]

**Proof.** Due to Lemma 2.4, an operator \( R_o(\lambda)V \in B_2(\mathcal{H}) \) for any \( \lambda \in G \). Recall that \( z = e^{i\tau \lambda} \), where \( \lambda \in \Lambda \) and \( 1_t = 1, t \geq 0 \) and \( 0_t = 0, t < 0 \). Due to (2.15) the operator \( (\partial + h_o \lambda)^{-1} V, \lambda \in \mathbb{C}_+ \) on \( \mathcal{H} = L^2(T_o, \mathcal{H}) \) has the form given by
\[ (R_o(\lambda)V f)(t) = i e^{it\varphi} \int_0^\tau \left[ 1_{t-s} + \gamma(z) \right] e^{-is\varphi} V(s) f(s) ds = bF(z) b^{-1} f, \quad (2.35) \]

\[ \varphi = \lambda - h_o, \quad e^{i\tau \varphi} = za, \quad z = e^{i\tau \lambda}, \quad a = e^{-i\tau h_o}, \quad (2.35) \]
where $f \in \tilde{H}$ and $b = e^{it\varphi}$ is a multiplication operator $e^{it\varphi}f(t)$ in $\tilde{H}$. Then from (2.6) we obtain for $\psi(z) = \det D(\lambda(z))$:

$$\psi(z) = \det \left[(I + R_\varphi V)e^{-R_\varphi V}\right](\lambda(z)) = \det \left[(I + F)e^{-F}\right](z).$$

We have the decomposition $F = F_1 + \gamma F_2$, where due to (2.14) the operators $F_1, F_2 \in B_2(\tilde{H})$ and the operator $F_1$ is invertible, as the Volterra operator. Thus due to Lemma 2.1 we obtain

$$\psi(z) = \det \left[(I + F_1)e^{-F_1}\right]D_2(z), \quad D_2(z) := \det \left[(I + R_\gamma(z)F_2)e^{-\gamma(z)F_2}\right]. \quad (2.36)$$

Recall that the operator $\tilde{V}(t) = e^{ith_a}V(t)e^{-ith_a}$, where $V(t) \in B_2(\mathcal{H})$. Due to (2.14) the operator $F_2 \in B_2(\tilde{H})$ and if $z \to 0$ then we obtain

$$\|\gamma(z)F_2\|_{B_2} \leq \frac{\|z\|}{1 - |z|}\|F_2\|_{B_2} = O(|z|\|F_2\|_{B_2}),$$

which yields $D_2(z) \to 1$. From (2.21) we have $\psi(z) \to 1$. From (2.36) we obtain

$$\psi(z) = \det \left[(I + F_1)e^{-F_1}\right]D_2(z) \to 1, \quad D_2(z) \to 1,$$

which yields $\det \left[(I + F_1)e^{-F_1}\right] = 1$ and we get (2.31).

We show (2.32). Estimates in (2.14) yield

$$\|F_1\|_{B_2} \leq C_1, \quad \|F_2\|_{B_2} \leq C_2, \quad \|R\| \leq C_o, \quad (2.37)$$

for some constants $C_1, C_2, C_o$. Let $A = \gamma F_2$ and let $D(t) = \det \left[(I + R_\tau A)e^{-\tau A}\right], t \in \mathbb{R}$. From (2.7), (2.37) and $R - I = -F_1 R$ we obtain

$$\frac{D'(t)}{D(t)} = \text{Tr} \left[ \left(R A - A\right)e^{-\tau A}\right]e^{\tau A}(I + R_\tau A)^{-1} = \text{Tr}(R - I)A(I + R_\tau A)^{-1} = -\text{Tr} F_1 R A(I + R_\tau A)^{-1} = \text{Tr} F_1 \sum_{n \geq 0} (-\tau A)^{n+1} t^n \quad (2.38)$$

and the integration yields (2.32) for $z$ small enough.

Let $\zeta = za$, $a = e^{-itr_a}$. We have $\gamma(z) = \frac{z\zeta}{1 - \zeta} = \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \ldots$, and then

$$\gamma(z)F_2 = (\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \ldots)F_2, \quad (2.39)$$

From (2.7), (2.37) and (2.32) we obtain for the term with $n = 1$ and $n = 2$:

$$-\text{Tr} F_1 R \gamma(z)F_2 = -\text{Tr} F_1 R(za + z^2 a^2 + \ldots)F_2,$$

$$\text{Tr} F_1(RF_2(z))^2 = \text{Tr} F_1 \left(R(za + z^2 a^2 + \ldots)F_2\right)^2 = z^2 \text{Tr} F_1(RaF_2)^2 + O(z^3). \quad (2.40)$$

Collecting asymptotics from (2.40) we obtain (2.34). ■
3. Proof of main theorems

3.1. Laplacian on the lattice. We need results about the resolvent \( r_\alpha(\lambda) = (\Delta - \lambda)^{-1} \) on \( \ell^2(\mathbb{Z}^d) \) from [29]:

**Theorem 3.1.** Let \( d \geq 3 \). Let \( u, v \in \ell^{2p}(\mathbb{Z}^d) \) with \( 1 \leq p < \begin{cases} \frac{6}{5} - \frac{3d}{2d+1} & \text{if } d = 3 \\ \frac{6p-4}{4p} - \frac{3p-4}{4p} & \text{if } d \geq 4 \end{cases} \). Then the operator-valued function \( f: \mathbb{C} \setminus [0, 2d] \rightarrow B_2 \), defined by

\[
f(\lambda) := u(\Delta - \lambda)^{-1} v
\]

is analytic and Hölder continuous up to the boundary and satisfies for all \( \lambda \in \mathbb{C} \setminus [0, 2d] \):

\[
\|f(\lambda)\|_{B_2} \leq C_* \|u\|_{2p} \|v\|_{2p},
\]

where

\[
C_* = p^{\frac{d(p-1)}{2p}} + c_d(3 + 2d)^{\frac{d-d}{p}}, \quad c_d = \begin{cases} 16 \frac{d}{4} & \text{if } d = 3 \\ 14 \frac{d}{2d-1} & \text{if } d = 4 \\ \frac{2}{3d(p-1)} & \text{if } d \geq 5 \end{cases}, \quad c = \begin{cases} 6(p-1) - 6 \frac{d}{5p} & \text{if } d = 3 \\ 2p-4 \frac{d-1}{p} - 5 \frac{d-3}{p} & \text{if } d = 4 \\ \frac{3d(p-1)(3d-2p)}{2} & \text{if } d \geq 5 \end{cases}.
\]

Moreover, we have

\[
\|f(\lambda) - f(\mu)\|_{B_2} \leq C_\alpha |\lambda - \mu|^\alpha \|u\|_{2p} \|v\|_{2p}, \quad \forall \lambda, \mu \in \overline{\mathbb{C}},
\]

where \( \alpha, C_\alpha \) are some positive constants.

Below we need a simple corollary

**Corollary 3.2.** Under the conditions of Theorem 3.1 the operator-valued function \( f : \mathbb{C} \setminus [0, 2d] \rightarrow B_2 \) satisfies

\[
\|f(\lambda)\|_{B_2} \leq C_* \max \{1, \tau(\lambda)\} \|u\|_{2p} \|v\|_{2p}, \quad \forall \lambda \in \mathbb{C} \setminus [0, 2d],
\]

where \( \tau(\lambda) = \text{dist}\{\lambda, \sigma(\Delta)\} \), and the constant \( C_* \) is defined by (3.5).

**Proof.** Let \( \lambda \in \mathbb{C} \setminus [0, 2d] \). If \( \tau(\lambda) \leq 1 \), then from (3.2) we obtain (3.5). If \( \tau(\lambda) \geq 1 \), then we have

\[
\|f(\lambda)\|_{B_2} \leq \|u\|_{2p} \|v\|_{2p} r_\alpha(\lambda) \|r_\alpha(\lambda)\| \leq \|u\|_{2p} \|v\|_{2p} / \tau(\lambda),
\]

which yields (3.5). \( \blacksquare \)

In order to prove main theorem we need a simple estimate.

**Lemma 3.3.** Consider a function \( g(a) = \frac{1}{a} - \frac{e^{-i\alpha a}}{\sin a} \) in a domain \( \mathcal{S}_+ := \{a \in \mathbb{C}_+ : |\Re a| \leq \frac{3\pi}{4}\} \) for some parameter \( \alpha \in [-3, 1] \). Then

\[
\max_{\mathcal{S}_+} \{|g| \} \leq C_g := \frac{4}{3\pi} + \frac{4}{5} \frac{5 + 3\pi}{\sqrt{2}}.
\]

**Proof.** We have a simple decomposition

\[
g = \frac{1}{a} - \frac{e^{-i\alpha a}}{\sin a} = s + f, \quad \text{where} \quad s = \frac{1}{a} - \frac{1}{\sin a}, \quad f = \frac{1 - e^{-i\alpha a}}{\sin a}
\]
and the functions $s, f$ is analytic in the strip $\mathcal{S} = \{ |\operatorname{Re} a| < \frac{3\pi}{4} \}$. By the maximum principle, the function $s$ in the strip $\mathcal{S}$ has maximum on the lines $\operatorname{Re} a = b := \frac{3\pi}{4}$, which yields
\begin{equation}
\max_{\mathcal{S}} |s| = \max_{t \in \mathbb{R}} |s(b + it)| \leq \max_{t \in \mathbb{R}} \frac{1}{|\sin (b + it)|} + \max_{t \in \mathbb{R}} \frac{1}{|b + it|} = \sqrt{2 + \frac{4}{3\pi}}. \tag{3.8}
\end{equation}

Consider the function $f$ in the half strip $\mathcal{S}_+$. We have
\begin{equation}
\max_{\mathcal{S}_+} |f| = \max\{ f_{\pm}, f_o \}, \quad f_{\pm} = \max_{t \geq 0} |f(\pm b + it)|, \quad f_o = \max_{a \in [-b, b]} |f(a)|. \tag{3.9}
\end{equation}

Let $a = \pm b + it, \ t \geq 0$. We obtain
\begin{equation}
|f(a)| = \frac{|1 - e^{-i\alpha a}|}{|\sin a|} \leq \frac{1}{|\sin a|} + \frac{|e^{-i\alpha a}|}{|\sin a|} \leq \sqrt{2 + \frac{e^{2\alpha}}{|\sin a|}}, \tag{3.10}
\end{equation}

since $e^{2\alpha} = e^{\pm i\Delta} e^{-2t} = \mp i e^{-2t}$. This yields $f_{\pm} \leq \sqrt{2 + 2}$. Let $-b \leq a \leq b$. Then we have $|f(a)| = \frac{|1 - e^{-i\alpha a}|}{|\sin a|} = \frac{2|\sin \frac{a}{2}|}{|\sin a|}$ and we obtain
\begin{equation}
|f(a)| = \frac{2|\sin \frac{a}{2}|}{|\sin a|} \leq \frac{|x|}{|\sin \frac{x}{4}|} = \frac{2e^{(x-1)t}}{|e^{2t} - 1|} \leq 2, \tag{3.10}
\end{equation}

This yields $f_o \leq \sqrt{2} + \frac{3\pi}{4\sqrt{2}}$ and then $|g| \leq \sqrt{2} + \frac{4}{3\pi} + \frac{3\pi}{4\sqrt{2}}$. 

3.2. Proof of main theorems. Consider the operator $\tilde{h} = \tilde{h}_o + V$ on $\tilde{H} = L^2(\mathbb{T}, \ell^2(\mathbb{Z}^d))$, where $\tilde{h}_o = \partial + \Delta$ is the free operator. Due to the factorization $V = v q$, we define the operator-valued function $\mathfrak{F}(\lambda)$ on $\tilde{H}$ by
\begin{equation}
\mathfrak{F}(\lambda) = q R_o v, \quad \lambda \in \mathbb{C}_\pm, \quad \text{where} \quad R_o(\lambda) = (\tilde{h}_o - \lambda)^{-1}, \quad q = |V|^\frac{1}{4}, \quad v = q e^{i\arg V}. \tag{3.11}
\end{equation}

**Theorem 3.4.** Let $V$ satisfy Condition V and the operator $\mathfrak{F}(\lambda), \lambda \in \mathbb{C}_\pm$ be defined by (3.11). Then
\begin{equation}
\mathfrak{F}(\lambda), \quad V R_o(\lambda) \in \mathcal{B}_2 := \mathcal{B}_2(\tilde{H}), \quad \forall \ \lambda \in \mathbb{C}_\pm. \tag{3.12}
\end{equation}

Moreover, if we define $\|V\|_{p,1} = \int_0^\tau \|V(t)\|_{\ell^p(\mathbb{Z}^d)} dt$, then the operator-valued function $\mathfrak{F} : \mathbb{C}_\pm \to \mathcal{B}_2$ is analytic and Hölder continuous up to the boundary and satisfies
\begin{equation}
C_\ast = \frac{1}{\sqrt{2}} + C_g \left( C_g + \frac{\sqrt{2}}{\sqrt{\pi \tau}} + C_\tau \right), \quad C_\tau = 1 + \frac{\tau d}{\pi}, \quad C_g = \frac{4}{3\pi} + \frac{5 + 3\pi}{4} \sqrt{2}, \tag{3.13}
\end{equation}

and
\begin{equation}
\|\mathfrak{F}(\lambda) - \mathfrak{F}(\mu)\|_{\mathcal{B}_2} \leq C_{1,\alpha} |\lambda - \mu|^\alpha \|V\|_{p,1}, \quad \forall \ \lambda, \mu \in \mathbb{C}_\pm, \tag{3.14}
\end{equation}

where $\alpha, C_{1,\alpha}$ are some positive constants and the constant $C_\ast$ is defined by (3.3).
Thus we obtain

\[ F = \sum_{j} F_j(\lambda) f(t) = q(t) \int_0^\tau \vartheta(A, k) \chi_j(\Delta) v(s) f(s) ds = \sum_{j} (F_j(\lambda) f)(t), \]

where

\[ \vartheta(a, \varkappa) = -\frac{1}{2a} + g(a), \quad g(a) := \frac{1}{a} - \frac{e^{-ia\varkappa}}{\sin a}. \]

Thus we obtain

\[ F_0(\lambda) f(t) = q(t) \int_0^\tau \vartheta(A, \varkappa) \chi_0(\Delta) v(s) f(s) ds \]

\[ = \frac{q(t)}{2} \int_0^\tau \left( -\frac{1}{A} + g(A) \right) \chi_0(\Delta) v(s) f(s) ds, \]

Proof. The results in \([3.12]\) have been proved in Lemma \([2,3]\).

Due to \((1.8)\) it is enough to prove \((3.13)\) only for the case \(\lambda \in \Lambda = [0, \omega] \times i\mathbb{R}_+.\)

Using \((2.15)\) we present \(\mathfrak{F} f\), where \(f \in \mathcal{H}, \lambda \in \mathbb{C}_+\) in the form given by

\[ (\mathfrak{F}(\lambda) f)(t) = iq(t)e^{it\varphi} \int_0^\tau \left( \mathbb{1}_{t-s} + \gamma \right) e^{-is\varphi} v(s) f(s) ds = (\mathfrak{F}_1(\lambda) f)(t) + (\mathfrak{F}_2(\lambda) f)(t), \]

\[ (\mathfrak{F}_1(\lambda) f)(t) = iq(t) \int_0^t e^{i(t-s)(\lambda - \Delta)} v(s) f(s) ds, \quad (\mathfrak{F}_2(\lambda) f)(t) = q(t) \int_0^\tau \vartheta(A, \varkappa) v(s) f(s) ds, \]

where

\[ \varphi = \lambda - \Delta, \quad A = \frac{\tau \varphi}{2} = \frac{\tau(\lambda - \Delta)}{2}, \quad \gamma = e^{-i\varphi} = -\frac{e^{iA}}{2 - e^{iA}}, \]

\[ \varkappa = \frac{2(\lambda - \Delta)}{\tau} - 1 \in [-3, 1], \quad \vartheta(\alpha, \varkappa) = \frac{e^{-i\alpha\varkappa}}{2 \sin \alpha}, \quad \alpha \in \mathbb{R}, \quad \omega = \frac{2\pi}{\tau}. \]

The first term, the operator-valued function \(\mathfrak{F}_1 : \mathbb{C} \rightarrow \mathfrak{B}_2\) is entire and satisfies

\[ \|\mathfrak{F}_1\|^2_{\mathfrak{B}_2} = \int_0^\tau dt \int_0^t \|q(t)e^{i(t-s)(\lambda-\Delta)} v(s)\|^2_{\mathfrak{B}_2} ds \leq \int_0^\tau dt \int_0^t \|q(t)\|^2_{\mathfrak{B}_4} \|\chi_0(\lambda-\Delta) v(s)\|^2_{\mathfrak{B}_2} e^{-2(t-s)\Im \lambda} ds \]

\[ \leq \int_0^\tau \|q(t)\|^2_{\mathfrak{B}_4} dt \int_0^t \|\chi_0(\lambda-\Delta) v(s)\|^2_{\mathfrak{B}_2} ds = \frac{1}{2} \left[ \int_0^\tau \left( \int_0^t \|V(t)\|^2_{\mathfrak{B}_2} dt \right)^2 \right] = \frac{1}{2} \|V\|^2_{\mathfrak{B}_2}, \quad \forall \ \Im \lambda \geq 0, \]

where \(\|q(t)\|_{\mathfrak{B}_4} = \|q(t)\|_{\mathfrak{B}_4}\) is the norm in \(L^4(\mathbb{Z}^d), a \geq 1\) and \(\|q(t)\|^2_{\mathfrak{B}_2} = \|V(t)\|^2_{\mathfrak{B}_2}.

Define the interval \(I(\omega) = [\lambda_0 - \frac{\omega}{2}, \lambda_0 + \frac{\omega}{2}]\) for any fixed \(\lambda_0 \in [0, 2d] \cap [0, \omega]\). Define \(\chi\) by

\[ \chi = \sum_j \chi_j, \quad \chi_0(\mu) = \begin{cases} 1, & \mu \in I(\omega) \\ 0, & \mu \in \mathbb{R} \setminus I(\omega) \end{cases}, \quad \chi_j = \chi_0(\cdot - \omega j), \quad j \in \mathbb{Z}. \]

Note that the sum \(\chi(\Delta) = \sum_j \chi_j(\Delta)\) is finite and the number of the function \(\chi_j(\Delta) \neq 0\) is less than \(1 + \frac{2d}{\omega}\), since \(\sigma(\Delta) = [0, 2d]\).

Consider the operator-valued function \(\mathfrak{F}_2\). We rewrite one in the form

\[ (\mathfrak{F}_2(\lambda) f)(t) = q(t) \int_0^\tau \vartheta(A, k) \chi_j(\Delta) v(s) f(s) ds = \sum_j (F_j(\lambda) f)(t), \]

\[ (F_j(\lambda) f)(t) = q(t) \int_0^\tau \vartheta(A, k) \chi_j(\Delta) v(s) f(s) ds. \]

We consider smoothness and estimates of \(F_0(\lambda)\). The proof for other \(F_j, j \neq 0\) is similar. We present \(\vartheta(A, k)\) in the following form

\[ \vartheta(a, \varkappa) = -\frac{1}{2a} + g(a), \quad g(a) := \frac{1}{a} - \frac{e^{-ia\varkappa}}{\sin a}. \]
which yields $F_0 = F_{0r} + F_{0g}$, where

$$F_{0r}(\lambda)f(t) = \frac{q(t)}{\tau} \int_0^\tau r_o(\lambda)\chi_0(\Delta)v(s)f(s)ds,$$

$$F_{0g}(\lambda)f(t) = \frac{q(t)}{2} \int_0^\tau g(A)\chi_0(\Delta)v(s)f(s)ds,$$

and $r_o(\lambda) = (\Delta - \lambda)^{-1}$. In order to consider the operator-valued function $F_{0g} : \mathbb{C} \to \tilde{B}_2$ we need to discuss the function $g(A)\chi_0(\Delta)$. We have

$$A = \frac{\tau}{2}(\lambda - \Delta) = \zeta + A_o, \quad \text{where} \quad A_o = \frac{\tau}{2}(\lambda_0 - \Delta), \quad \zeta = \frac{\tau}{2}(\lambda - \lambda_o)$$

and

$$\|A_o\chi_0(\Delta)\| \leq \frac{\tau\omega}{4} = \frac{\pi}{2}, \quad \left\{ \begin{array}{l} |\zeta| < \frac{\tau\omega}{8} = \frac{\tau}{8} \quad \text{if} \quad |\lambda - \lambda_o| < \frac{\tau}{4}, \\
|\zeta| < \frac{\tau\omega}{8} = \frac{\tau}{8} \quad \text{if} \quad |\lambda - \lambda_o| < \frac{\tau}{4}. \end{array} \right.$$

Thus we have

$$F_{0g}(\lambda)f(t) = \frac{q(t)}{2} \int_0^\tau g(A_o + \zeta)\chi_0(\Delta)v(s)f(s)ds$$

where $g(a)$ is analytic in the domain $\mathfrak{G}_+ := \{a \in \mathbb{C}_+ : |\text{Re} a| \leq \frac{3\pi}{4}\}$ and due to (3.6) it satisfies $\|g(A)\chi_0(\Delta)\| \leq C_g = \frac{4}{3\tau} + \frac{\pi + 3\pi}{4}\sqrt{2}$. Then we obtain

$$\|F_{0g}(\lambda)\|_{\tilde{B}_2}^2 = \int_0^\tau dt \int_0^\tau \|q(t)\|_1^2 \|g(A_o + \zeta)\chi_0(\Delta)v_s\|_{\tilde{B}_2}^2 ds \leq \int_0^\tau \|q(t)\|_1^2 \|g(s)\|_{\tilde{B}_2}^2 ds \leq \frac{C_g^2}{2} \int_0^\tau \|q(t)\|_{\tilde{B}_2}^2 ds \leq \frac{C_g^2}{2} \|V\|_{2,1}^2. \quad (3.23)$$

Consider the operator-valued function $F_{0r} : \lambda \to \tilde{B}_2$. We have $F_{0r} = G_0 + G_1$, where

$$(G_0(\lambda)f)(t) = \frac{q(t)}{\tau} \int_0^\tau r_o(\lambda)v(s)f(s)ds, \quad (G_1(\lambda)f)(t) = \frac{q(t)}{\tau} \int_0^\tau r_o(\lambda)(\chi_0(\Delta) - 1)v(s)f(s)ds.$$}

Due to Theorem 3.1 the operator-valued function $G_0 : \mathbb{C}_+ \to \mathcal{B}_2$ is analytic and Hölder continuous up to the boundary and via (3.2) satisfies

$$\|G_0(\lambda)\|_{\tilde{B}_2}^2 = \frac{1}{\tau^2} \int_0^\tau dt \int_0^\tau \|q(t)||r_o(\lambda)v(s)\|_{\tilde{B}_2}^2 ds \leq \frac{C^2}{\tau^2} \int_0^\tau \|q(t)||_p^2 \|V(s)||_p^2 ds = \frac{C^2}{\tau^2} \|V\|_{p,1}^2.$$}

(3.24)

The operator-valued function $r_o(\lambda)(\chi_0(\Delta) - 1) : I(\frac{\pi}{2}) \times \mathbb{R}_+ \to \mathcal{B}$ is analytic and Hölder continuous up to the boundary $I(\frac{\pi}{2}) \subset I(\omega)$ and satisfies $\|r_o(\lambda)(\chi_0(\Delta) - 1)\| \leq \frac{\pi}{2}$. Moreover, we have

$$\|G_1(\lambda)\|_{\tilde{B}_2}^2 = \frac{1}{\tau^2} \int_0^\tau dt \int_0^\tau \|q(t)||r_o(\lambda)(\chi_0(\Delta) - 1)v(s)\|_{\tilde{B}_2}^2 ds \leq \frac{2}{\tau^2} \int_0^\tau dt \int_0^\tau \|q(t)||^2 \|q(s)||^2 ds \leq \frac{\sqrt{2}}{\pi\tau} \|V\|_{2,1}^2.$$}

(3.25)

Collecting all estimates we obtain

$$\|F_0(\lambda)\|_{\tilde{B}_2} \leq \left( C_g + \frac{\sqrt{2}}{\pi\tau} \right) \|V\|_{2,1} + \frac{C_\tau}{\tau} \|V\|_{p,1}.$$
Recall that the sum $\chi(\Delta) = \sum_j \chi_j(\Delta)$ is finite and the number of the function $\chi_j(\Delta) \neq 0$ is less than $C_\tau = 1 + \frac{2d}{\pi}$, since $\sigma(\Delta) = [0, 2d]$. Thus we have

$$\| \sum_j (F_j(\lambda)) \|_{\mathcal{B}_2} \leq C_\tau \left( C_g + \frac{\sqrt{2}}{\pi \tau} \right) \|V\|_{2,1} + C_\tau \frac{C_\tau}{\tau} \|V\|_{p,1}$$

and jointly with (3.17) we obtain (3.13). \[\blacksquare\]

**Proof of Theorem 1.1** Let $V$ satisfy (1.4) and a constant $C_\ast$ be defined by (3.3).

i) Due to Theorem 3.4 the operator-valued function $\hat{\Psi}_\ast : \mathbb{C}_\pm \to \mathcal{B}_2(\ell^2(\mathbb{Z}^d))$, defined by (1.9) is analytic and Hölder up to the boundary. Moreover, it satisfies (1.11).

ii) From i) we deduce that the modified determinant $D(\lambda) = \det \left[ (I + \hat{\Psi}) e^{-\hat{\Psi}} \right](\lambda)$ is analytic in $\mathbb{C}_\pm$, is Hölder up to the boundary. Identity (1.8) at $V = 0$ implies (1.12), i.e., $D(\lambda + \omega) = D(\lambda)$ for all $\lambda \in \mathbb{C}_\pm$. Asymptotics (2.21) yields (1.13). From (1.11) and (2.5) we obtain (1.14). Moreover, estimate (1.15) has been proved in Proposition 2.3. \[\blacksquare\]

**Proof of Theorem 1.2** From Theorem 1.1 we obtain that the function $\psi \in \mathcal{H}_\infty(D)$ and is Hölder up to the boundary and satisfies (1.11). The zeros $\{z_j\}_{j=1}^N$ of $\psi$ in $D$ satisfy $\sum_{j=1}^N (1 - |z_j|) < \infty$. Moreover, the function $\log \psi(z)$ is analytic in $D_{r_0}$ and has the Taylor series

$$\log \psi(z) = \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots, \quad \text{as} \quad |z| < r_0.$$  

where coefficients $\psi_n$ are given by Theorem 2.5. \[\blacksquare\]

We recall the standard facts about the canonical factorization of functions from Hardy space, see e.g. [10, 18] and in the needed specific form for us from [28].

**Theorem 3.5.** Let a function $\psi \in \mathcal{H}_\infty(D)$ and be Hölder up to the boundary. Then there exists a singular measure $m \geq 0$ on $[0, 2\pi]$, such that $\psi$ has a canonical factorization for all $|z| < 1$ given by

$$\psi(z) = B(z)e^{\Psi(z)}, \quad \Psi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$  

where the measure $d\mu(t) = \log|\psi(e^{it})|dt - dm(t)$ and $\log|\psi(e^{it})| \in L^1(\mathbb{T})$ and the measure $m$ satisfies $\text{supp } m \subset \{t \in [0, 2\pi] : \psi(e^{it}) = 0\}$. Moreover, $\Psi$ has the Taylor series

$$\Psi(z) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \ldots,$$

in some disk $\{|z| < r\}$, $r > 0$, where

$$\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t) = \int_0^{2\pi} \log|f(e^{it})|dt - m(\mathbb{T}), \quad \mu_n = \frac{1}{\pi} \int_0^{2\pi} e^{-int} d\mu(t), \quad n \in \mathbb{N}.$$  

We are ready to describe the function $\psi$.

**Proof of Corollary 1.3** The proof follows from Theorem 1.2 and Theorem 3.5. \[\blacksquare\]

We describe trace formulae.

**Proof of Theorem 1.4** Using (2.23) and the identity $R = R_o - R_o V R_o + R_o V R_o V R$ we obtain

$$\frac{D'(\lambda)}{D(\lambda)} = - \text{Tr} \left((R_o V)^2 R\right)(\lambda) = - \text{Tr} \left(R - R_o + R_o V R_o\right)(\lambda).$$
Then differentiation of $\psi(z) = D(\lambda(z))$ in (1.24) yields
\[
\frac{\psi'(z)}{\psi(z)} = \sum \frac{(1 - |z|^2)}{(z - z_j)(1 - \overline{z_j} z)} + \frac{1}{\pi} \int_{0}^{2\pi} \frac{e^{it}d\mu(t)}{(e^{it} - z)^2} = \frac{D'(\lambda(z))}{D(\lambda(z))}\lambda'(z),
\]
where $\lambda'(z) = \frac{1}{z \lambda(z)} = \frac{1}{iz \pi}$. Thus collecting the two last identities we get (1.26).

Due to the canonical representation (1.24), the function $\frac{\psi(z)}{B(z)}$ has no zeros in the disc $D$ and
\[
\Psi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i t + \frac{z}{2}} e^{i t - \frac{z}{2}} d\mu(t), \quad z \in D,
\]
(3.28)
where the measure $d\mu = \log |f(e^{it})|dt - dm(t)$. In order to show (1.27)–(1.29) we need the asymptotics of the Schwartz integral $\Psi(z)$ as $z \to 0$ from (3.27):
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i t + \frac{z}{2}} e^{i t - \frac{z}{2}} d\mu(t) = \frac{\mu(\overline{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \ldots \quad \text{as} \quad |z| < 1, \quad (3.29)
\]
where
\[
\mu(\overline{T}) = \int_{0}^{2\pi} d\mu(t) = \int_{0}^{2\pi} \log |f(e^{it})|dt - \nu(\overline{T}), \quad \mu_n = \frac{1}{\pi} \int_{0}^{2\pi} e^{-i n t} d\mu(t), \quad n \in \mathbb{Z}.
\]

We have the identity $\log \psi(z) = \log B(z) + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i t + \frac{z}{2}} d\mu(t)$ for all $z \in D_{r_0}$. Combining asymptotics (1.19), (1.22) and (3.29) we obtain (1.27)–(1.28). In particular, we have
\[-\log B(0) = \sum_{j=1}^{N} \text{Im} \tau \lambda_j \geq 0 \quad \text{and} \quad \frac{\mu(\overline{T})}{2\pi} \geq 0 \quad \text{and} \quad (1.22). \]

**Proof of Corollary 1.5.** Let a potential $V$ satisfy (1.4). Then substituting estimate (1.14) into (1.27) we obtain
\[
\frac{\nu(\overline{T})}{2\pi} + \tau \sum_{\lambda \in \Lambda} \text{Im} \lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\psi(e^{it})|dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} C^2 \frac{2}{2} \|V\|_{p, 2} dt = \frac{C^2}{2} \|V\|_{p, 2}^2,
\]
which yields (1.30). ■

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