On pattern structures of the $N$-soliton solution of the discrete KP equation over a finite field

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Received 8 September 2006, in final form 11 December 2006
Published 17 January 2007
Online at stacks.iop.org/JPhysA/40/949

Abstract
The existence and properties of coherent pattern in the multisoliton solutions of the dKP equation over a finite field are investigated. To that end, starting with an algebro-geometric construction over a finite field, we derive a ‘travelling wave’ formula for $N$-soliton solutions in a finite field. However, despite it having a form similar to its analogue in the complex field case, the finite-field solutions produce patterns essentially different from those of classical interacting solitons.

PACS numbers: 02.30.Ik, 02.10.De, 05.45.Yv

1. Introduction

There are many diverse methods in the modern theory of integrable systems. One of them, an algebro-geometric approach [8, 9], was applied to obtain solutions of discrete soliton equations over finite fields [1, 3, 5]. Within this approach efficient tools for finding algebro-geometric solutions based on hyperelliptic curves of arbitrary genus were proposed [2, 4]. These results are in direct analogy to the complex field case, but there are some peculiarities. For instance, since finite fields are never algebraically closed there are many more possibilities for the construction of breather-type solutions (for short discussion see [5]). Also other properties, such as finiteness or cyclic structure, reflect in the character of solutions.

Our aim here is to discuss the appearance of stable travelling patterns for a general $N$-soliton solution over a finite field. To do that we write a general determinant formula (8) for $N$-soliton solution in a travelling wave form (12). The formula obtained is analogous to the typical form of a soliton solution of a Hirota bilinear equation (see e.g. [14], p 23). Since

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the vacuum solution in this setting is \( \tau \equiv 1 \), it is necessary to make a slight modification of
the algebro-geometric construction presented in [3].

The determinant form of the solutions of the dKP equation was already investigated [15].
Since the determinant formula is neither a Casorati nor a discrete Gram-type determinant,
we provide a direct link to the travelling wave form without referring to the previous work.
Moreover, even though we are mainly concerned with finite fields, we point out that
the calculations performed are valid for arbitrary fields.

There is one more reason to transform the finite-field solutions into travelling wave form.
There exists a well-established systematic procedure for deriving soliton cellular automata
starting from discrete soliton equations in Hirota form [12, 16]. Since solutions over a finite
field could be interpreted also as cellular automata, we need a convenient form to investigate
relationships between them.

This paper is organized as follows. In section 2 we recall and transpose some results from
an algebro-geometric construction of finite-field-valued solutions of the discrete KP equation
with explicit determinant formula for \( N \)-soliton solutions [3] to the case of dKP with arbitrary
coefficients (2). In the next section, we prove that the \( N \)-soliton solution can be rewritten
in travelling wave form. As a remark we also give an alternative proof which explains why
only pairwise interaction terms appear in the \( N \)-soliton solutions which appear as substitution
operations for matrix elements. In section 4, we discuss the patterns produced by finite-field
solitons and give some examples. The main conclusion is that travelling wave patterns in
\( N \)-soliton solutions obtained by the algebro-geometric approach are generally absent for
\( N > 2 \). We finish with some conclusions and remarks on possible future work in the
last section.

2. Algebro-geometric construction of solutions of dKP equation over finite fields with
simple vacuum

A finite-field version of an algebro-geometric construction of solutions \( \tau : \mathbb{Z}^3 \to \mathbb{F} \) for the
discrete KP equation

\[
(T_1 \tau)(T_3 \tau) - (T_1 T_3 \tau)(T_1 \tau) + (T_1 T_3 \tau)(T_1 \tau) = 0
\]

(1)

was studied in [3]. By \( T_i \) we denoted here a shift operator in a variable \( n_i \), for example
\( T_2 (n_1, n_2, n_3) = \tau(n_1, n_2 + 1, n_3) \). If we prefer to have vacuum solution \( \bar{\tau}(n_1, n_2, n_3) \equiv 1 \)
then we need to consider the dKP equation with coefficients \( Z_i \in \mathbb{F} \), i.e.,

\[
Z_1 (T_1 \bar{\tau})(T_3 \bar{\tau}) - Z_2 (T_2 \bar{\tau})(T_1 \bar{\tau}) + Z_3 (T_3 \bar{\tau})(T_1 \bar{\tau}) = 0
\]

(2)

with the constraint

\[
Z_1 - Z_2 + Z_3 = 0.
\]

(3)

Note that a correspondence between \( \tau \) and \( \bar{\tau} \) is achieved by taking

\[
\bar{\tau} = (Z_1/\delta)^{-n_1 n_2} (Z_2/\delta)^{-n_2 n_3} (Z_3/\delta)^{-n_1 n_3} \tau
\]

(4)

for nonzero constants \( Z_1, Z_2, Z_3 \) and any nonzero \( \delta \in \mathbb{F} \).

It follows from [3], in the case of purely soliton solutions (i.e. from a curve of genus
\( g = 0 \)), that \( N \)-soliton solutions can be expressed in a determinant form (theorem 1). As a
component of this formula we need a vacuum solution of (1)

\[
t_0 = (A_1 - A_2)^{n_1 n_2} (A_1 - A_3)^{n_1 n_3} (A_2 - A_3)^{n_2 n_3},
\]

(5)

where \( A_i \in \mathbb{F} \), and auxiliary functions \( \phi_{\alpha}(t) \), \( \alpha = 1, 2, \ldots, N \), in the form

\[
\phi_{\alpha}(t) = \frac{1}{t - C_{\alpha}} \prod_{k=1}^{N} \left( \frac{t - A_k}{C_{\alpha} - A_k} \right)^{n_k}.
\]

(6)
Parameters $C_\alpha$, where $\alpha = 1, \ldots, N$, may take value in some finite algebraic extension $\mathbb{L} \supset \mathbb{K}$, but they are constrained to the $\mathbb{K}$-rationality conditions $\forall \sigma \in G(\mathbb{L}/\mathbb{K}), \sigma(C_\alpha) = C_\alpha'$. By $G(\mathbb{L}/\mathbb{K})$ we denoted the Galois group, i.e. the group of automorphisms of $\mathbb{L}$, with fixed field $\mathbb{K}$. Similarly, $N$ pairs $D_\beta, E_\beta \in \mathbb{L}$, for $\beta = 1, \ldots, N$, satisfy the $\mathbb{K}$-rationality conditions $\forall \sigma \in G(\mathbb{L}/\mathbb{K}): \sigma([D_\beta, E_\beta]) = [D_\beta', E_\beta']$. These conditions give rise to some generalization of breather-type solutions (see [5]). We assume that all parameters in the construction are distinct. Finally, denote by $\phi_A(D, E)$ the $N \times N$ matrix with element in row $\beta$ and column $\alpha$ given by

$$ \phi_A(D, E)_{\alpha\beta} = \phi_\alpha(D_\beta) - \phi_\alpha(E_\beta). $$

(7)

**Theorem 1.** The function $\tau(n_1, n_2, n_3)$ given by

$$ \tau = \tau_0 \cdot \gamma \det \phi_A(D, E), $$

(8)

where $\gamma \in \mathbb{F}$ is some constant, is the $\mathbb{F}$-valued $N$-soliton solution of the discrete KP equation (1).

In the case of dKP equation (2), it follows from (4) that a vacuum function $\bar{\tau}_0$ can be chosen in the form

$$ \bar{\tau}_0 = \left( \frac{A_1 - A_2}{Z_1} \right)^{n_{11} n_{21} n_{31}} \left( \frac{A_1 - A_3}{Z_2} \right)^{n_{11} n_{21} n_{31}} \left( \frac{A_2 - A_3}{Z_3} \right)^{n_{21} n_{31}}. $$

Then for parameters $A_1 - A_2 = Z_1, A_1 - A_3 = Z_2, A_2 - A_3 = Z_3$ we have $\bar{\tau}_0 \equiv 1$ and

$$ \bar{\tau} = \det \phi_A(D, E) $$

(9)

is the $\mathbb{F}$-valued $N$-soliton solution of equation (2).

**Remark.** The same result can also be derived using the general algebro-geometric construction of solutions of the version of the dKP equation given in (2). To do this, the wavefunction $\psi$ is given by definition 1 in [3], but with a different definition of the expansions of $\psi(n_1, n_2, n_3)$ at $A_i$, namely, for $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$,

$$ \psi(n_1, n_2, n_3) = t_i^{n_1} Z_j^{n_{1j}} Z_k^{n_{1k}} \bar{\zeta}_i^{(j)}(n_1, n_2, n_3), $$

where $t_i$ are the fixed $\mathbb{K}$-rational local parameters at $A_i$. The linear equation for $\psi$ in the general case is of the form

$$ (T_i \psi - T_j \psi) + Z_k \frac{\zeta_i^{(j)}}{\zeta_0^{(j)}} \psi = 0. $$

The remaining part of the construction is the same as in [3].

**3. Travelling wave form for the N-soliton solution**

In this section, we wish to transform the soliton solutions given in (8) into an equivalent but more convenient form. In doing this, we will use the fact, referred as a gauge invariance, that for any solution $\tau$ of the dKP equation, $\tau' = e^{\alpha A} e^{\beta A} e^{\gamma A} \tau$ is also a solution for any constant $\alpha, \beta, \gamma, \delta$. We write $\tau' \simeq \tau$ if $\tau'$ can be obtained from $\tau$ using this gauge invariance.
Lemma 1. Let $M(x, y)$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, denote the Cauchy matrix, the matrix with $(i, j)$th entry $1/(x_i - y_j)$. It is well known that

$$\det M(x, y) = \prod_{p<q}(x_p - x_q)(y_p - y_q) \prod_{p} (x_p - y_p).$$

Also we have

$$M(x, y)^{-1}M(x, z) = \text{diag} \left( \prod_p (y_p - z_i) \right) M(y, z) \text{diag} \left( \prod_p (x_p - z_i) \right).$$

Proof. Let $x_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. Then, the $(i, j)$th entry in the inverse of $M(x, y)$ is

$$[M(x, y)^{-1}]_{i,j} = (-1)^{i+j} \frac{\det M(x_j, y_i)}{\det M(x, y)} = (x_j - y_i) \prod_{p \neq j} (x_j - y_p) \prod_{p \neq i} (y_p - y_i).$$

Further, the $(i, j)$th entry in the product $M(x, y)^{-1}M(x, z)$ is

$$\prod_p (x_p - y_i) \sum_{k=1}^n \prod_{p \neq k} (x_k - z_j) \prod_p (x_p - y_i) = \prod_p (y_p - y_i) V(x) \prod_p (x_p - z_j) \sum_{k=1}^n (-1)^{n-k} V(x_k) \prod_{p \neq i} (x_p - y_p) \prod_{p \neq k} (x_p - z_j),$$

where $V(x) = \prod_{p<q}(x_p - x_q)$ is the Vandermonde determinant in variables $x_1, \ldots, x_n$ and $V(x_k)$ is the Vandermonde determinant in variables $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. In the $(i, j)$th entry, the sum is of degree $(n-1)(n-2)/2 + 2(n-1) = n(n-1)/2 + n - 1$ and we will factorize it by identifying all of its zeros. First, there are $n(n-1)/2$ coming when $x_i = x_s$ for any $r < s$ and $n - 1$ from $y_r = z_j$ for $r \neq i$. This argument uses the fact that for any polynomial function $f$ of degree less than $n - 1$ one has $\sum_{k=1}^n (-1)^{n-k} V(x_k) f(x_k) = 0$ since the left-hand side is the expansion by the final column of the vanishing determinant

$$\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-2} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & f(x_n) \end{bmatrix}.$$ 

Hence, the $(i, j)$th entry is, up to a constant factor,

$$\prod_p (x_p - y_i) \prod_{p \neq i} (y_p - z_j) \prod_p (x_p - z_j) = \prod_p (x_p - y_i) \prod_{p \neq i} (y_p - y_i) \frac{1}{y_i - z_j} \prod_p (x_p - z_j).$$

To verify that it is precisely correct, and hence complete the proof of (11), we observe that, as it should, the above equals $\delta_{ij}$, the Kronecker delta, when $z = y$. \qed

Theorem 2. Let $q$ denote any fixed generator of $F^*$, i.e., a multiplicative subgroup of the finite field $F$. The $N$-soliton solution (9) of the dKP equation over a finite field $F$ admits the following form:

$$\tau = \sum_{J \subset \{1, \ldots, N\}} (-1)^{|J|} \left( \prod_{i,j \in J, i < j} a_{ij} \right) q^{\sum_{i,j} b_{ij}},$$

where

$$\tau' = \sum_{J \subset \{1, \ldots, N\}} (-1)^{|J|} \left( \prod_{i,j \in J, i < j} a_{ij} \right) q^{\sum_{i,j} b_{ij}},$$

is a solution of the dKP equation. The N-soliton solution (9) of the dKP equation over a finite field $F$ admits the following form:

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where

$$\tau' = \sum_{J \subset \{1, \ldots, N\}} (-1)^{|J|} \left( \prod_{i,j \in J, i < j} a_{ij} \right) q^{\sum_{i,j} b_{ij}},$$

is a solution of the dKP equation.
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where the sum is taken over all subsets of \( \{1, \ldots, N\} \) and \( \#J \) denotes the cardinality of \( J \).

In \( (12) \),

\[
    a_{ij} := \frac{(D_i - D_j)(E_i - E_j)}{(D_i - E_j)(D_j - E_i)},
\]

the exponents are \( \hat{\eta}_j = \eta_j + \eta^0_j \) where

\[
    \eta_j := \sum_{k=1}^{3} p_k^j n_k,
\]

and the parameters \( p_k^j \) and phase constants \( \eta^0_j \) are defined by

\[
    q^{\eta^0} := \frac{E_i - A_k}{D_i - A_k} \quad \text{and} \quad q^{\eta^0} := \prod_{p=1}^{N} \frac{(C_p - D_i)}{(C_p - E_i)} \prod_{p=1, p \neq i}^{N} \frac{(D_p - E_i)}{(D_p - D_i)}.
\]

The name travelling wave form comes from the form of \( q^{\hat{\eta}_j} \), which are in direct analogy with the usual term \( \exp(kx - \omega t) \) of linear plane waves. Note that for any \( \tau \) satisfying \( (2) \)

\[
    \tau(1, 0, 0), \tau(0, 1, 0) \quad \text{and} \quad \tau(0, 0, 1)
\]

depend only on cross-ratio of appropriate points on the projective line. We also point out that to find \( p_k^j \) and \( \eta^0_j \) we need to solve a discrete logarithm problem.

**Proof.** The determinant in \( (9) \) is

\[
    \bar{\tau} = \det \left( -M(C, D) \ \text{diag} \left( \prod_{k=1}^{3} \left( \frac{D_i - A_k}{C_i - A_k} \right)^{n_k} \right) + M(C, E) \ \text{diag} \left( \prod_{k=1}^{3} \left( \frac{E_i - A_k}{C_i - A_k} \right)^{n_k} \right) \right),
\]

where \( M \) denotes the Cauchy matrix as defined in lemma 1, and so

\[
    \bar{\tau} \simeq \tau' := \det \left( I - M(C, D)^{-1} M(C, E) \ \text{diag} \left( \prod_{k=1}^{3} \left( \frac{D_i - A_k}{D_i - E_i} \right)^{n_k} \right) \right).
\]

Using \( (11) \) and the fact that for any matrices \( P, Q \), \( \det PQ = \det QP \), we get

\[
    \tau' = \det\left( I - M(D, E) \ \text{diag}(D_i - E_i) \ \text{diag}(q^{b_i}) \right).
\]

Now the determinant \( \det(P + Q) \), of the sum of two \( N \times N \) matrices, may be expressed as the sum over all subsets \( J \subset \{1, \ldots, N\} \) of \( \det R_J \) where the \( i \)th column of \( R_J \) equals the \( i \)th column of \( P \) if \( i \in J \) and otherwise equals the \( i \)th column of \( Q \). Using this fact,

\[
    \tau' = \sum_{J \subset \{1, \ldots, N\}} (-1)^{#J} \det M_J(D, E) \prod_{i \in J} (D_i - E_i) q^{b_i},
\]

where \( M_J \) denotes the Cauchy matrix with row and column indices restricted to \( J \). Formula \( (12) \) now follows immediately from \( (10) \).

**Remark.** We also present here an alternative proof of theorem 2. The argument presented shows why there is only pairwise interaction of solitons.

Starting from \( (9) \), for each \( i \) one divides the \( i \)th column of \( (7) \) by \( \phi_i(D_i) \) and we see that \( \bar{\tau} \simeq \tau' = \det \Phi \) where

\[
    [\Phi]_{i,j} = \left( \frac{D_i - C_j}{D_j - C_i} \right) \left( \frac{D_i - C_j}{E_j - C_i} \right)^{q^{\eta^0_i_j} n_k} \prod_{x=1}^{3} \left( \frac{D_j - A_k}{D_i - A_k} \right)^{n_k},
\]
where \( \eta_j \) is given by (14), is also a solution of (2). Further, if we define

\[
[\hat{\Phi}]_{ij} := \left( \frac{D_i - C_i}{D_j - C_i} \right) - \left( \frac{D_i - C_i}{E_j - C_i} \right) q^{\eta_j} \tag{17}
\]

it is easy to see that \( \det \Phi \simeq \det \hat{\Phi} \).

We will next show how \( \det \Phi \) can be obtained from a vacuum solution \( \det \Phi_0 \) where

\[
[\hat{\Phi}_0]_{ij} = \left( \frac{D_i - C_i}{D_j - C_i} \right) \text{ by means of substitution operations.}
\]

The key idea we use in finding coefficients of different powers of \( q \) in formula (12) is the observation that, according to (17), for each term including \( q^{\eta_j} \) we should make the substitution

\[
\left( \frac{D_i - C_i}{D_j - C_i} \right) \rightarrow \left( \frac{D_i - C_i}{E_k - C_i} \right)
\]

for each \( i \), in the \( k \)-th row of \( \Phi_0 \). Then, \( \det \Phi \) can be expressed in terms of determinants obtained by making appropriate replacements in \( \det \Phi_0 \).

First note that

\[
\det \Phi_0 = M(-C, -D) \prod_i (D_i - C_i) = \prod_{i,j=1}^N (C_i - C_j)(D_i - D_j) (C_i - D_j)(D_i - C_j) \tag{19}
\]

Denote by \( \Phi_J \) the matrix with replacements (18) in the rows \( k \in J \). We first consider the case \( J = \{ k \} \) for fixed \( k \). Since \( D_k \) appears only in the \( k \)-th row and \( k \)-th column, then \( \Phi_J \) after multiplying \( k \)-th column by \(-\frac{E_k - C_k}{D_k - C_k}\) becomes the determinant (19) with \( D_k \) replaced by \( E_k \).

Then,

\[
\det \Phi_J = -\left( \frac{D_k - C_k}{E_k - C_k} \right) \det \Phi_0 \bigg|_{D_k \rightarrow E_k} = -\left( \frac{D_k - C_k}{E_k - C_k} \right) \prod_{i,j=1}^N (C_i - C_j)(D_i - D_j) (C_i - D_j)(D_i - C_j) \bigg|_{D_k \rightarrow E_k}
\]

\[
= -\left( \frac{D_k - C_k}{E_k - C_k} \right) \prod_{j' = 1, j' \neq k}^N \left( \frac{D_k - D_{j'}}{E_k - C_{j'}} \right)^{-1} \left( \frac{E_k - D_{j'}}{E_k - C_{j'}} \right) \det \Phi_0.
\]

In the notation of the theorem, the coefficient of \( q^{\eta_j} \) is \( \det \Phi_{\{k\}} = -q^{\eta_j} \det \Phi_0 \).

Now we consider coefficient of \( q^{\sum_{k \in I} \eta_k} \). This is the determinant \( \det \Phi_J \) obtained from \( \det \Phi_0 \) by making replacements (18) in rows \( k \in J \) for general \( J \subseteq I \). The number of elements in \( J \) is denoted by \# \( J \). Repeating the procedure described above for each \( k \in J \) we arrive at

\[
\det \Phi_J = (-1)^{\# J} \left( \prod_{k \in J} \frac{D_k - C_k}{E_k - C_k} \right) \det \Phi_0 \bigg|_{D_k \rightarrow E_k, k \in J}.
\tag{20}
\]

Then, we have

\[
\det \Phi_J|_{D_k \rightarrow E_k, k \in J} = \prod_{(k, k') \subset J} \left( \frac{D_k - D_{k'}}{E_k - D_{k'}} \right) \left( \frac{E_k - D_{k'}}{E_k - E_{k'}} \right) \prod_{k \in J} \left( \prod_{j = 1, j \neq k}^N \left( \frac{D_k - D_{j'}}{D_k - C_{j'}} \right)^{-1} \left( \frac{E_k - D_{j'}}{E_k - C_{j'}} \right) \right) \det \Phi_0.
\]

The extra terms for pairs \( \{k, k'\} \in J \) are present because, since \( k_j \neq k_i \), we also need to perform the replacements \( D_{k_j} \rightarrow E_{k_j} \) in the factor \( \prod_{j = 1, j \neq k}^N \left( \frac{D_k - D_{j'}}{D_k - C_{j'}} \right)^{-1} \left( \frac{E_k - D_{j'}}{E_k - C_{j'}} \right) \). This is
done by multiplying by \((D_{h_{1}}-D_{h_{2}})(E_{j_{1}}-E_{j_{2}})\). We point out here that the extra corrections are for pairs \(\{k_{i}, k_{j}\} \notin J\) and it is this that is responsible for the existence of only pairwise interaction terms in formula (12). So finally, the coefficient of \(q^{\sum_{i} \eta_{i}}\) is

\[
\det \Phi_{J} = (-1)^{|J|} \left( \prod_{i,j \in J, i < j} a_{ij} \right) \left( \prod_{k \in J} q^{\eta_{k}} \right) \det \Phi_{0}.
\]

After dividing all terms by \(\det \Phi_{0}\) and collecting them together we obtain formula (12).

Note that our considerations are valid for any field of definition of the parameters \(C_{i}\) and \(D_{j}\).

**Remark.** The results obtained above also apply to the discrete analogue of generalized Toda equation [6]

\[
Z_{1} \sum_{m_{1}=1}^{N_{1}} f_{m_{1}+1,m_{2},m_{3}} f_{m_{1}-1,m_{2},m_{3}} + Z_{2} \sum_{m_{1},m_{2}+1,m_{3}} f_{m_{1},m_{2}-1,m_{3}} + Z_{3} \sum_{m_{1},m_{2},m_{3}+1} f_{m_{1},m_{2},m_{3}-1} = 0.
\]

The transition from dKP to DAGTE (where we assume \((m_{1} + m_{2} + m_{3}) \equiv 0 \mod 2\)) is given by

\[
f_{m_{1},m_{2},m_{3}} = \tau(n_{1}, n_{2}, n_{3}) = \tau(m_{1} - m_{2}, m_{2} - m_{3}, m_{3} - m_{1}) \text{ for } i \neq j \neq k \neq i \text{ or, equivalently, } m_{i} = -(n_{i} + n_{j}) \text{ for } i = j \neq k \neq i.
\]

In this situation, \(\tilde{f}_{0,0,0} = \tau(0,0,0)\). With the variables \(m_{1}, m_{2}, m_{3}\) the exponents of a travelling wave given by (14) have similar form, namely \(\eta_{i} = \sum_{j=1}^{3} p_{j}^{i} m_{j}\), where \(p_{1}^{i} = \sum_{k=1}^{3} (2\delta_{jk} - 1) p_{k}\) and \(\delta_{jk}\) is the Kronecker delta.

4. Pattern structures in soliton interactions

So far we have seen that the finite-field \(N\)-soliton solutions we obtained have the same structure as the complex field case. In particular, the travelling wave form (12) is completely analogous to the complex field case. Despite this correspondence, the typical soliton-like interaction pattern is not present in the finite-field case. Below we argue that one cannot expect that an arbitrary \(N\)-soliton solution over a finite field will be a collection of asymptotically separated waves interacting as in the way characteristic of the complex field case.

These differences in the interaction properties follow naturally from the differences in structure between finite fields and the complex numbers. First, finite fields have no good (total) ordering which are consistent for both addition and multiplication and so it is impossible to find in the finite field case a direct analogue of wave amplitude; rather than following a wave propagation by observing how its points of maximum amplitude move we may only trace the propagation of patterns. The appearance of a one-soliton solution however in the finite-field case mimics the usual appearance quite well, as can be seen in figure 1.

The next difference results from the replacement of exponentials in the complex case by powers of a generator \(q\) of the multiplicative group \(\mathbb{F}^{*}\) of the finite field \(\mathbb{F}\). Since \(q^{2|\mathbb{F}|-1} = 1\), this implies periodicity of \(\tau(n_{1}, n_{2}, n_{3})\) with respect to each variable \(n_{i}\). As a consequence, we cannot discuss the asymptotic behaviour of the solution in usual sense. Moreover, for any given solution one could restrict analysis to a finite base cube containing all information about the solution, but the rest is periodic repetition. The length of any edge of a base cube is at most \(|\mathbb{F}| - 1\).

Considering theorem 2, the \(i\)th one-soliton component of the \(N\)-soliton solution is unchanged by a shift in the lattice by \(\eta_{i} = (n'_{1}, n'_{2}, n'_{3})\) for any \(n'_{1}, n'_{2}, n'_{3}\) satisfying \(q^{\eta_{i}} = 1\) or, equivalently,

\[
\eta_{i} = \sum_{k=1}^{3} p_{i}^{k} n_{k}' \equiv 0 \mod (|\mathbb{F}| - 1).
\]
Figure 1. A plot of $\tau(n_1, n_2, n_3)$ function of three one-soliton solutions (see examples for details). We fix $n_1 = 0$, and $n_2, n_3 \in \{0, \ldots, 16\}$. The $n_2$ axis is directed to the right and the $n_3$ axis is directed upwards. Elements of $F_{17}$ are represented on the following scale: from 0 (dark) to 16 (light grey).

Figure 2. A plot of $\tau(n_1, n_2, n_3)$ for the three two-soliton interactions (AB, AC, BC) of one-soliton solutions presented in figure 1. We fix $n_1 = 0$, and $n_2, n_3 \in \{0, \ldots, 16\}$. In the second row we have $n_1 = 1$. The $n_2$ axis is directed to the right and the $n_3$ axis is directed upwards.

Since expression (12) contains $q^{\eta_i}$ for $i \in \{1, 2, \ldots, N\}$, a period vector $\vec{n} = (n_1, n_2, n_3)$ for this solution should be a common solution of (22) for all $i$. In general, it is impossible to find a nonzero solution for $N \geq 3$ and it means there is no additional structure within the base cube in this case. An example of such a three-soliton solution is shown in figure 3 and the structure of a base cube is discussed below in more detail.
Figure 3. A three-soliton solution \( \tau(n_1, n_2, n_3) \) being the solitonic sum of those from figure 1 for \( n_1 = 0, 1, 2, 4 \) and \( 8 \). The \( n_2 \) axis is directed to the right and the \( n_3 \) axis is directed upwards. Elements of \( F_{17} \) are represented on the following scale: from 0 (dark) to 16 (light grey).

**Examples.** Fix the finite field to be \( F = F_{17} \), i.e. the field of integers modulo 17. As a generator of \( F^* \) we choose \( q = 3 \). Let us fix \( A_1 = 7, A_2 = 4 \) and \( A_3 = 3 \) so that the coefficients in the dKP equation are

\[
A_2 - A_3 = Z_1 = 1, \quad A_1 - A_3 = Z_2 = 4, \quad A_1 - A_2 = Z_3 = 3.
\]

**One-soliton solutions.** In figure 1, we present three one-soliton solutions of the dKP equation over \( F \). For the solution A on the left, denoted by subscript 1, we have chosen \( C_1 = 10, D_1 = 12, E_1 = 14 \) and finally \( C_3 = 8, D_3 = 13, E_3 = 15 \) for the solution C on the right. The respective parameters in the exponent (14) are

\[
\vec{p}_1 = (p^1_1, p^2_1, p^3_1) = (6, 7, 14), \quad \vec{p}_2 = (p^1_2, p^2_2, p^3_2) = (6, 9, 5),
\]

\[
\vec{p}_3 = (p^1_3, p^2_3, p^3_3) = (11, 5, 10).
\]

Thus periods in variables \( n_1, n_2, n_3 \) are as follows: 8, 16, 8 for the soliton 1, 8, 16, 16 for the soliton 2 and 16, 16, 8 for 3. We fixed \( n_1 = 0 \), since increasing any variable by 1 results in shift in the other two variables and so the plots for any value of \( n_1 \) are simply a translation of the plots we present. This comes from the fact that in general there exists a non-zero solution in two variables \( n_j, n_k \) of the single equation (22) with the third variable \( n_i \) being fixed \((i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i)\). The special case is if \( p^i \) is not a linear combination of \( p^j \) and \( p^k \) with coefficients from \( \mathbb{Z} \) mod \((|F| - 1)\). (This could happen for instance if \( p^i \) and \( p^k \) are zero divisors of \( |F| - 1 \).) In the case presented in figure 1,
period vectors might be chosen to be $\vec{n}_1 = (1, 0, 3)$ for A, $\vec{n}_2 = (1, 0, 2)$ for B and $\vec{n}_3 = (1, 1, 0)$ for C. Further, the period vectors $\vec{n}$ for these solutions in the plane $n_2, n_3$ are $\vec{n}_{1a} = \vec{n}_{3a} = (0, 2, -1), \vec{n}_{1b} = \vec{n}_{3b} = (0, 0, 8), \vec{n}_{2a} = (0, 3, 1)$ and $\vec{n}_{2b} = (0, -1, 5)$.

(All periods vectors are not uniquely determined.)

**Two-soliton solutions.** In figure 2, the two-soliton solutions representing the interaction of two of the one-soliton solutions shown in figure 1 are presented. The upper line is for $n_1 = 0$ while the lower is for $n_1 = 1$. In the cases A interacting with B (called AB) and B interacting with C (AC) it is a simple shift by respective vectors $(1, -1, -4)$ and $(1, -5, -5)$. (*Note.* Since $q^{16} = 1$, the lines $n = 0$ and $n = 16$ are identical but both of them are shown on plots.) For the case AC there are no such a shift, because the two equations (22) for $i = 1$ and $i = 3$ for $n_2$ and $n_3$ have no nonzero solution for any value of $n_1$. This is because $n_1 \cdot (6, 11) \notin \mathbb{Z}_{16} \cdot (7, 5)$. Since $\text{span}(\vec{n}_{1a}, \vec{n}_{1b}) \cap \text{span}(\vec{n}_{2a}, \vec{n}_{2b}) = \{0\}$, there are no nonzero period vectors in the plane $n_2, n_3$ for either AB or BC. In contrast, for AC the period vectors in this plane are exactly the same as for soliton A and C (since they are the same). Similarly, one could examine other planes obtaining plane period vectors $(0, 0, 8)$ for AC, $(8, 0, 0)$ and $(4, 8, 0)$ for AB and none for BC. Periods for AB are $(8, 16, 16)$, for AC are $(16, 16, 8)$ and for BC are the maximal $(16, 16, 16)$. Note that BC shows that it is possible for a two-soliton solution to have no structure within the base cube.

**A three-soliton solution.** In this case, there are three equations (22) for three variables, so in general there are no nonzero solutions. The example presented in figure 3 using parameters in A, B and C is of this kind. Because of this, there are no extra period vectors and so this solution has no structure within a base cube.

It is clear that this is also the generic situation for the interaction of $N$-soliton solutions obtained by the algebro-geometric approach for $N > 2$.

**5. Concluding remarks**

In summary, we have seen that the interaction of three or more finite-field soliton solutions of the dKP equation has no more structure in the base cube than random data. Given a finite field $\mathbb{F}$ of sufficient size, it would seem to be computationally impractical to attempt to reconstruct the parameters $C_a, D_a$ and $E_a$ which define the solution, from the seemingly random data that they generate. This leads one to imagine possible applications of such solutions in encrypted data transmission. In conclusion, we note that applications of elliptic curves over finite fields have already been studied for some time [7, 13].

Examining soliton solutions over finite fields is a relatively new subject and requires further study. The approach we presented here is related to periodicity in two different ways. The first is because of the algebro-geometric construction and the second comes from the properties of finite fields. Moreover, finite extensions of finite fields are never algebraically closed and this fact is reflected in the existence of many new possibilities in the construction of breather-type solutions. In contrast with the case of the field of complex numbers, which is an algebraically closed extension of the field of real numbers of degree two, for finite fields we can have extensions of arbitrary degree. Simple examples and relevant terminology were presented in [5]. From the other side, in the classical continuous case many types of breather-type solutions are known, including ‘sophisticated’ complexitons (see [10, 11] and references therein). It should be very interesting to investigate analogues of these for the finite field case. However, it will require both substantial development of the theory discussed in this paper and clarification of some details of complexiton solutions in a discrete setting. For these reasons, we leave this topic for future research.
On pattern structures of the N-soliton solution of the discrete KP equation

Acknowledgments

The authors thank Professors T Tokihiro and R Willox for creating a friendly environment for research at the Graduate School of Mathematical Sciences, University of Tokyo and for their overall help. MB is partially supported by Grant-in-Aid of the JSPS Postdoctoral Fellowship for Foreign Researchers of Japan Society for the Promotion of Science (ID No P 05050) and by Polish Ministry of Science and Information Society Technologies (Grant No 1 P03B 01728).

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