MEAN DENSITY OF INHOMOGENEOUS BOOLEAN MODELS WITH LOWER DIMENSIONAL TYPICAL GRAIN

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The Problem

A random closed set $\Theta : (\Omega, \mathcal{F}, P) \longrightarrow (F, \sigma_F)^1$ in $\mathbb{R}^d$ with integer Hausorff dimension $n$ may induce a random Radon measure $\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$ on $\mathbb{R}^d$, and, as a consequence, an expected measure

$$E[\mu_\Theta](B) := E[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$
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Questions

1. Is $E[\mu_\Theta]$ absolutely continuous w.r.t. $\mathcal{H}^d$?

1\textsuperscript{F} = \text{closed subsets in } \mathbb{R}^d; \sigma_{\mathbb{F}} = \sigma\text{-algebra generated by the hit-or-miss topology}
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1. Is $E[\mu_\Theta]$ absolutely continuous w.r.t. $\mathcal{H}^d$?
2. If so, which is its density?

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A random closed set \( \Theta : (\Omega, \mathcal{F}, P) \rightarrow (\mathcal{F}, \sigma_\mathcal{F}) \) in \( \mathbb{R}^d \) with integer Hausdorff dimension \( n \) may induce a random Radon measure \( \mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot) \) on \( \mathbb{R}^d \), and, as a consequence, an expected measure

\[
\mathbb{E}[\mu_\Theta](B) := \mathbb{E}[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.
\]

Questions

1. Is \( \mathbb{E}[\mu_\Theta] \) absolutely continuous w.r.t. \( \mathcal{H}^d \)?
2. If so, which is its density?

Notation

If \( \mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d \) we denote by \( \lambda_\Theta \) its density and we call \( \lambda_\Theta(x) \) the mean density of \( \Theta \) at point \( x \in \mathbb{R}^d \).

1. \( \mathcal{F} \) = closed subsets in \( \mathbb{R}^d \); \( \sigma_\mathcal{F} = \sigma \)-algebra generated by the hit-or-miss topology
What we know

If $n = d$,
then $E[\mu_\Theta] \ll H^d$ with density $\lambda_\Theta(x) = P(x \in \Theta)$ for $H^d$-a.e. $x \in \mathbb{R}^d$.

(Robbins H.E. (1944). On the measure of a random set, Ann.Math.Statistics, 15, 70–74)

If $\Theta = X$ random point ($n = 0$),
then $E[\mu_X](\cdot) = P(X \in \cdot) \ll H^d$ if $X$ admits pdf $f$, and $\lambda_X = f$.

If $0 < n < d$ and $\Theta$ is stationary,
then $E[\mu_\Theta] \ll H^d$ with density $\lambda_\Theta(x) = c > 0 \forall x \in \mathbb{R}^d$.

Problem
What if $0 < n < d$ and $\Theta$ is NOT stationary?
What we know

- If $n = d$, then $E[\mu_\Theta] \ll H_d$ with density $\lambda_\Theta(x) = P(x \in \Theta)$ for $H_d$-a.e. $x \in \mathbb{R}^d$.

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- If $\Theta = \{\text{random point}\}$ ($n = 0$), then $E[\mu_X](\cdot) = P(X \in \cdot) \ll H_d$ if $X$ admits pdf $f$, and $\lambda_X = f$.

- If $0 < n < d$ and $\Theta$ is stationary, then $E[\mu_\Theta] \ll H_d$ with density $\lambda_\Theta(x) = c > 0 \forall x \in \mathbb{R}^d$.

Problem

What if $0 < n < d$ and $\Theta$ is NOT stationary?
What we know

- If $n = d$, then $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$ with density

$$\lambda_\Theta(x) = \mathbb{P}(x \in \Theta) \text{ for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$ 

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- If \( \Theta = X \) random point \( (n = 0) \), then \( \mathbb{E}[\mu_X](\cdot) = \mathbb{P}(X \in \cdot) \ll \mathcal{H}^d \) iff
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- If \( 0 < n < d \) and \( \Theta \) is stationary, then \( \mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d \) with density

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Problem

What if \( 0 < n < d \) and \( \Theta \) is NOT stationary?
Boolean model in $\mathbb{R}^d$

Definition

- $\Psi = \{x_i\}_{i \in \mathbb{N}}$: Poisson point process in $\mathbb{R}^d$ with intensity $f$;
- $\{Z_i\}_{i \in \mathbb{N}}$: sequence of IID random compact sets in $\mathbb{R}^d$, which are also independent of the Poisson process $\Psi$;
- $Z_0$: random compact set of the same distribution as the $Z_i$’s.

The random closed set

$$\Theta := \bigcup_i (x_i + Z_i)$$

is said **(inhomogeneous) Boolean model with intensity $f$ and typical grain $Z_0$**.

It is usually assumed that

$$\mathbb{E}[\text{card}\{i : (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^d.$$
We assume that the typical grain $Z_0$ is a lower dimensional random closed set in $\mathbb{R}^d$, uniquely determined by a random quantity in a suitable mark space $K$; i.e. $\forall s \in K$

$$Z_0(s) = n\text{-dimensional compact subset of } \mathbb{R}^d \text{ containing the origin.}$$
We assume that the typical grain \( Z_0 \) is a lower dimensional random closed set in \( \mathbb{R}^d \), uniquely determined by a random quantity in a suitable mark space \( K \); i.e. \( \forall s \in K \)

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Let us consider the Boolean model

\[
\Theta(\omega) := \bigcup_{(x_i,s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,
\]

with \( \Phi \) Poisson point process in \( \mathbb{R}^d \times K \) with intensity measure

\[
\Lambda(dy \times ds) = f(y)dyQ(ds).
\]
The main result

Under general regularity assumptions on $Z_0$, related to the existence of its Minkowski content, and on the intensity $f$ of the underlying Poisson point process, we can prove that

$$E[\mu_\Theta] \ll H^d$$

with density

$$\lambda_\Theta(x) = \int K \int_{Z_0(x,s)} f(y) H^n(dy) Q(ds)$$

for $H^d$-a.e. $x \in \mathbb{R}^d$, where $Z_{x,s} := x - Z_0(s)$. 
The main result

Under general regularity assumptions on $Z_0$, related to the existence of its Minkowski content, and on the intensity $f$ of the underlying Poisson point process, we can prove that

$$ \mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d \quad \text{with density} $$

$$ \lambda_\Theta(x) = \int_K \int_{Z^{x,s}} f(y)\mathcal{H}^n(dy)Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d, $$

where $Z^{x,s} := x - Z_0(s)$. 
Notation:
- $b_m = \text{volume of the unit ball in } \mathbb{R}^m$;
- $S \oplus r := S \oplus B_r(0)$.

**Definition (Minkowski content)**

The \textbf{n-dimensional Minkowski content} $\mathcal{M}^n(S)$ of a closed set $S \subset \mathbb{R}^d$ is defined by

$$\mathcal{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S \oplus r)}{b_d - n r^{d-n}}$$

whenever the limit exists finite.
Definition

We say that a compact set $S \subset \mathbb{R}^d$ is

- **$n$-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;
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**Theorem (H.Federer (1969))**

\[
\mathcal{M}^n(S) = \mathcal{H}^n(S) \text{ for any compact } n\text{-rectifiable set } S \subset \mathbb{R}^d.
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Definition

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- **$n$-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;

- is countably $\mathcal{H}^n$-rectifiable if there exist countably many Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\left(S \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n)\right) = 0.$$ 

Theorem (H. Federer (1969))

$\mathcal{M}^n(S) = \mathcal{H}^n(S)$ for any compact $n$-rectifiable set $S \subset \mathbb{R}^d$. 

Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that
\[
\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)
\] (1)
holds for some $\gamma > 0$ and some Radon measure $\eta$ in $\mathbb{R}^d$, $\eta \ll \mathcal{H}^n$. Then
\[
\mathcal{M}^n(S) = \mathcal{H}^n(S).
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Theorem (L.Ambrosio-N.Fusco-D.Pallara (2000))

Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that

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Remarks:

- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\tilde{S} \cap \cdot)$ for some closed set $\tilde{S} \supseteq S$;
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Remarks:

- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\tilde{S} \cap \cdot)$ for some closed set $\tilde{S} \supseteq S$;
- $\eta$ can be assumed to be a probability measure;
- it can be proved that $\mathcal{H}^n(S) < \infty$ and

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(S \oplus r \cap A)}{b_d r^{d-n}} = \mathcal{H}^n(S \cap A)$$

for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^n(S \cap \partial A) = 0$. 
A generalization of $\mathcal{M}^n$

**Theorem (EV (2007))**

Let $\mu$ be a positive measure in $\mathbb{R}^d$ absolutely continuous w.r.t. $\mathcal{H}^d$ with density $f$ such that

- i) $f$ is locally bounded (i.e. $\sup_{x \in K} f(x) < \infty$ for any compact $K \subset \mathbb{R}^d$);
- ii) the set of all discontinuity points of $f$ is $\mathcal{H}^n$-negligible.

Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set as in the previous theorem. Then

$$\lim_{r \downarrow 0} \mu(S \oplus r) = \int_S f(x) \mathcal{H}^d(dx).$$

This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with $f$ intensity of the underlying Poisson point process in $\mathbb{R}^d$, $S = x - Z_0(s)$, $s \in K$. 

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Then

$$\lim_{r \to 0} \mu(S \oplus r) = \int_S f(x) \, d\mathcal{H}^n(x).$$

This result applies in the proof of the formula of the mean density $\lambda_{\Theta}(x)$, with $f$ = intensity of the underlying Poisson point process in $\mathbb{R}^d$, $S = x - Z_0(s)$, $s \in K$. 

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Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set as in the previous theorem. Then

$$\lim_{r \downarrow 0} \frac{\mu(S \oplus r)}{b_{d-n} r^{d-n}} = \int_S f(x) \mathcal{H}^n(dx).$$
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This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with

- $f =$ intensity of the underlying Poisson point process in $\mathbb{R}^d$,
- $S = x - Z_0(s), \ s \in K$. 

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Assumptions

Let us consider the Boolean model $\Theta$ in $\mathbb{R}^d$

$$
\Theta(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,
$$

with $\Phi$ Poisson point process in $\mathbb{R}^d \times \mathcal{K}$ having intensity measure

$$
\Lambda(dy \times ds) = f(y)dyQ(ds)
$$

such that

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\mathbb{E}[\text{card}\{i : (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^d.
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Assumptions

Let us consider the Boolean model $\Theta$ in $\mathbb{R}^d$

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with $\Phi$ Poisson point process in $\mathbb{R}^d \times \mathbb{K}$ having intensity measure

$$\Lambda(dy \times ds) = f(y)dyQ(ds)$$

such that

$$\int_{\mathbb{K}} \int_{(-Z_0(s))\oplus R} \Lambda(dy \times ds) < \infty \quad \forall R > 0.$$
Let us assume that the following conditions on $Z_0$ and $f$ are fulfilled:

(A1) $Z_0(s)$ is a countably $H^n$-rectifiable compact set for $Q$-a.e. $s \in K$. Further there exist $\gamma > 0$ and a random closed set $\tilde{Z}_0 \supseteq Z_0$ with $E_Q[H^n(\tilde{Z}_0)] < \infty$ such that, for $Q$-a.e. $s \in K$, $H^n(\tilde{Z}_0(s) \cap B_r(x)) \geq \gamma r^n \forall x \in Z_0(s), \forall r \in (0, 1)$.

(A2) the set of all discontinuity points of $f$ is $H^n$-negligible and $f$ is locally bounded such that for any compact set $K \subset \mathbb{R}^d$ $\sup_{y \in K} \|\delta f(y)\| \leq \xi_K(\delta := \text{diam } Z_0)$ holds for some random variable $\xi_K$ with $E_Q[H^n(\tilde{Z}_0) \xi_K] < \infty$. 
Let us assume that the following conditions on $Z_0$ and $f$ are fulfilled:

(A1) $Z_0(s)$ is a countably $\mathcal{H}^n$-rectifiable compact set for $Q$-a.e. $s \in K$. Further there exist $\gamma > 0$ and a random closed set $\tilde{Z}_0 \supseteq Z_0$ with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)] < \infty$ such that, for $Q$-a.e. $s \in K$,

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(A2) the set of all discontinuity points of $f$ is $\mathcal{H}^n$-negligible and $f$ is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K \oplus \delta} f(y) \leq \xi_K \quad (\delta := \text{diam} Z_0)$$

holds for some random variable $\xi_K$ with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)\xi_K] < \infty$. 
Theorem (EV (2007))

For any Boolean model $\Theta$ as in Assumptions

- $E[\mu_\Theta]$ is locally finite and absolutely continuous w.r.t. $\mathcal{H}^d$;
- the mean density $\lambda_\Theta$ is given by

$$\lambda_\Theta(x) = \int_{K} \int_{Z^{x,s}} f(y) \mathcal{H}^n(dy) Q(ds)$$

for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

where $Z^{x,s} := x - Z_0(s)$. 

Stationary case. If
- $f \equiv c > 0$,
- $Z_0$ satisfies the assumption (A1),
then the integrability condition (IC) on $\Lambda$ and the assumption (A2) are satisfied.
Special cases

1. **Stationary case.** If

   - \( f \equiv c > 0 \),
   - \( Z_0 \) satisfies the assumption (A1),

then \( E[\mu_\Theta] \ll \mathcal{H}^d \) with density

\[
\lambda_\Theta(x) = cE_Q[\mathcal{H}^n(Z_0)] \quad \forall x \in \mathbb{R}^d.
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then $E[\mu_\Theta] \ll \mathcal{H}^d$ with density

$$\lambda_\Theta(x) = cE_Q[\mathcal{H}^n(Z_0)] \quad \forall x \in \mathbb{R}^d.$$

2. **Deterministic typical grain.** If
   - $f$ is locally bounded and such that the set of all its discontinuity points is $\mathcal{H}^n$-negligible,
   - $Z_0$ is a countably $\mathcal{H}^n$-rectifiable compact set such that

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)$$

holds for some $\gamma > 0$ and some probability measure $\eta \ll \mathcal{H}^n$,
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\]

2 **Deterministic typical grain.** If
   - \( f \) is locally bounded and such that the set of all its discontinuity points is \( \mathcal{H}^n \)-negligible,
   - \( Z_0 \) is a countably \( \mathcal{H}^n \)-rectifiable compact set such that

\[
\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)
\]

holds for some \( \gamma > 0 \) and some probability measure \( \eta \ll \mathcal{H}^n \),

then \( \mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d \) with density

\[
\lambda_\Theta(x) = \int_{Z_0} f(x - y)\mathcal{H}^n(dy) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.
\]
Main steps of the proof

$E[\mu_\Theta]$ is locally finite.

$E[\mu_\Theta] = \lambda_\Theta H_\mathbb{R}^d$ for some integrable function $\lambda_\Theta$ on $\mathbb{R}^d$.

For any bounded Borel set $A \subset \mathbb{R}^d$ with $H_\mathbb{R}^d(\partial A) = 0$,

$$\lim_{r \downarrow 0} E[H_\mathbb{R}^d(\Theta \oplus r \cap A)] = E[H_\mathbb{R}^d(\Theta \cap A)],$$

i.e.

$$\lim_{r \downarrow 0} \int_A P(x \in \Theta \oplus r) dxdy = \int_A \lim_{r \downarrow 0} P(x \in \Theta \oplus r) dxdy.$$
Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_\Theta]$ is locally finite.
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Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_\Theta]$ is locally finite.
- $\mathbb{E}[\mu_\Theta] = \lambda_\Theta \mathcal{H}^d$ for some integrable function $\lambda_\Theta$ on $\mathbb{R}^d$.
- For any bounded Borel set $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(\partial A) = 0$,
  \[
  \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta \oplus_r \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta \cap A)],
  \]
Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_\Theta]$ is locally finite.
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\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta \oplus r \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta \cap A)],
$$

i.e.

$$
\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}} \, dx = \int_A \lambda_\Theta(x) \, dx.
$$
Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_{\Theta}]$ is locally finite.
- $\mathbb{E}[\mu_{\Theta}] = \lambda_{\Theta} \mathcal{H}^d$ for some integrable function $\lambda_{\Theta}$ on $\mathbb{R}^d$.
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- By (A1) and (A2)

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}} \, dx = \int_A \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}} \, dx.$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

$$\lambda_{\Theta}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}}$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$, 

$$
\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \int_{\mathbb{K}} \frac{\int_{Z^{x,s} \oplus r} f(y) \, dy}{b_{d-n} r^{d-n}} Q(ds)
$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

$$
\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}}
$$

$$
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\lambda_{\Theta}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}}
$$

$$
= \lim_{r \downarrow 0} \int_{K} \int_{Z_{x,s} \Theta r} \frac{f(y) dy}{b_{d-n} r^{d-n}} Q(ds)
$$

$$
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$$

$$
= \int_{K} \int_{Z_{x,s}} f(y) \mathcal{H}^n(dy) Q(ds).
$$
Estimation of the mean density

By the proof of the main theorem we get that, for any Boolean model $\Theta$ as in the Assumptions,

$$\lambda_{\Theta}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}} \in \mathbb{R} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d$$

This suggest an estimator of $\lambda_{\Theta}(x)$ in terms of the empirical capacity functional of $\Theta$:

Let $\Theta_1, \ldots, \Theta_N$ be a random sample of $\Theta$; we define

$$\hat{\lambda}_N(\Theta)(x) := \sum_{i=1}^N 1_{\Theta_i \cap B_{\mathbb{R}^d}(x) \neq \emptyset} b_{d-n} r^{d-n}$$

with $R_N$ such that $R_N \to 0$ and $N R_N^{d-n} \to \infty$ for $N \to \infty$.

Then

$$\lim_{N \to \infty} \hat{\lambda}_N(\Theta)(x) = \lambda_{\Theta}(x) \text{ in probability}$$

for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$. 

Elena Villa (University of Milan)
Mean density of Boolean models
Aachen, 4 March 2008
Estimation of the mean density

By the proof of the main theorem we get that, for any Boolean model $\Theta$ as in the Assumptions,

$$
\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{P(x \in \Theta \oplus r)}{b_d - n r^{d-n}} \in \mathbb{R} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d
$$

This suggests an estimator of $\lambda_\Theta(x)$ in terms of the empirical capacity functional of $\Theta$:
Let $\Theta_1, \ldots, \Theta_N$ be a random sample of $\Theta$; we define

$$
\hat{\lambda}_\Theta^N(x) := \frac{\sum_{i=1}^N 1_{\Theta_i \cap B_{R_N}(x) \neq \emptyset}}{N b_d - n R_N^{d-n}},
$$

with $R_N$ such that $R_N \to 0$ and $NR_N^{d-n} \to \infty$ for $N \to \infty$. Then

$$
\lim_{N \to \infty} \hat{\lambda}_\Theta^N(x) = \lambda_\Theta(x) \quad \text{in probability, for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.
$$
Remark:
Even if the extreme case $n = 0$ can be handle with much more elementary tools, we may notice that if $\Theta = X$ is a random variable with pdf $f_X$, then

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Remark:
Even if the extreme case \( n = 0 \) can be handle with much more elementary tools, we may notice that if \( \Theta = X \) is a random variable with pdf \( f_X \), then

- \( \lambda_\Theta = f_X \);
- If \( X_1, \ldots, X_N \) is a random sample of \( X \), \( \hat{\lambda}_\Theta^N(x) \) becomes in this case

\[
\hat{f}_X^N(x) = \frac{\sum_{i=1}^N 1_{B_{R_N}(x)}(X_i)}{Nb_1 R_N} = \frac{\text{card}\{i : X_i \in I_x\}}{N|I_x|},
\]

where \( I_x \) is the interval in \( \mathbb{R} \) centered in \( x \) with length \( |I_x| = 2R_N \) with the usual condition

\[
|I_x| \longrightarrow 0 \quad \text{and} \quad N|I_x| \longrightarrow \infty \quad \text{as} \quad N \rightarrow \infty.
\]