Adaptive and Efficient Algorithms for
Tracking the Best Expert

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Abstract

In this paper, we consider the problem of prediction with expert advice in dynamic environments. We choose tracking regret as the performance metric and develop two adaptive and efficient algorithms with data-dependent tracking regret bounds. The first algorithm achieves a second-order tracking regret bound, which improves existing first-order bounds. The second algorithm enjoys a path-length bound, which is generally not comparable to the second-order bound but offers advantages in slowly moving environments. Both algorithms are developed under the online mirror descent framework and draw inspiration from existing algorithms that attain data-dependent bounds of static regret. The key idea is to use a clipped simplex in the updating step of online mirror descent. Finally, we extend our algorithms and analysis to online matrix prediction and provide the first data-dependent tracking regret bound for this problem.

Keywords: Prediction with Expert Advice, Tracking Regret, Adaptive Online Learning

1. Introduction

We study the problem of prediction with expert advice, where a learner makes sequential predictions by combining advice from $K$ experts. We consider the following decision-theoretic setup (Freund and Schapire, 1997): In each round $t = 1, \ldots, T$, the learner chooses a distribution $w_t$ over $K$ experts, and at the same time an adversary decides a loss vector $\ell_t$ encoding the loss of each expert: $\ell_t = (\ell_t[1], \ldots, \ell_t[K]) \in [0,1]^K$. Then, the learner observes the loss vector $\ell_t$ and suffers a weighted average loss $\hat{\ell}_t = \langle w_t, \ell_t \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. The classic metric to measure the learner’s performance is static regret, defined as the difference between the cumulative loss of the learner and that of the best single expert over $T$ rounds in hindsight:

$$SR(T) = \sum_{t=1}^{T} \hat{\ell}_t - \min_{E \in [K]} \sum_{t=1}^{T} \ell_t[E]$$

where $[K] = \{1, 2, \ldots, K\}$. During the past decades, minimizing static regret has been extensively studied, and minimax-optimal algorithms with $O(\sqrt{T \log K})$ regret bounds as well as adaptive algorithms with data-dependent regret bounds have been developed (Cesa-Bianchi and Lugosi, 2006). However, the static regret is only meaningful for stationary environments where a single expert performs well over $T$ rounds, and fails to illustrate the performance of online algorithms in changing environments where the best expert could switch over time.
To address this limitation, a more stringent metric called tracking regret has been introduced and studied in the literature under the name of “tracking the best expert” (Herbster and Warmuth, 1998; Vovk, 1999; Herbster and Warmuth, 2001; Bousquet and Warmuth, 2002). Instead of competing with a single expert, in tracking regret the learner is compared against a sequence of experts $E_1, \ldots, E_T$ with a small number of switches $E_t \neq E_{t-1}$:

$$
TR(T, S) = \sum_{t=1}^{T} \hat{\ell}_t - \min_{(E_1, \ldots, E_T) \in C(T, S)} \sum_{t=1}^{T} \ell_t[E_t] = \sum_{t=1}^{T} \hat{\ell}_t - \sum_{t=1}^{T} \ell_t[E^*_t] \tag{1}
$$

where $C(T, S)$ is the set comprised of all sequence of experts in which the expert switches at most $S - 1$ times:

$$
C(T, S) = \left\{ (E_1, \ldots, E_T) \in [K]^T \mid \sum_{t=2}^{T} \{E_t \neq E_{t-1}\} \leq S - 1 \right\} \tag{2}
$$

and $E^*_1, \ldots, E^*_T$ is the best sequence of experts in $C(T, S)$:

$$
(E^*_1, \ldots, E^*_T) = \arg\min_{(E_1, \ldots, E_T) \in C(T, S)} \sum_{t=1}^{T} \ell_t[E_t]. \tag{3}
$$

It is easy to see that the tracking regret includes the static regret as a special case by setting $S = 1$.

As early as 20 years ago, Herbster and Warmuth (1998, 2001) have developed two algorithms for tracking the best expert, namely, fixed share and projection update, both of which enjoy an $O(\sqrt{ST \log (KT/S)})$ tracking regret bound. While this bound is not improvable in general, we are interested in obtaining more favorable data-dependent bounds of tracking regret, which match the above bound in the worst case but become much smaller in benign environments. In this paper, we present two adaptive and efficient algorithms that enjoy data-dependent tracking regret bounds. The first algorithm is shown to achieve a novel second-order tracking regret bound of $O(\sqrt{SL_2 \log (KT/S)} + S \log (KT/S))$, where $L_2$ is the sum of squared loss of $E^*_1, \ldots, E^*_T$:

$$
L_2 = \sum_{t=1}^{T} (\ell_t[E^*_t])^2. \tag{4}
$$

The second algorithm attains a path-length bound of $O(\sqrt{SP_\infty \log (KT/S)} + S)$, where $P_\infty$ is the sum of the square of the difference between consecutive loss vectors $\ell_1, \ldots, \ell_T$:

$$
P_\infty = \sum_{t=1}^{T} \Vert \ell_t - \ell_{t-1} \Vert_\infty^2 \tag{5}
$$

with $\ell_0 = (0, \ldots, 0)$. While this bound has been derived by previous work (Wei et al., 2016), their algorithm is inefficient since it needs to maintain $KT$ virtual experts and update the weights of $Kt$ experts in round $t$, which makes the space and time complexities per round grow linearly with $t$. By contrast, our algorithm performs on $K$ real experts and thus its space and time complexities per round are independent of $t$. 

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The second-order and the path-length bounds are not comparable in general and each has its own advantage: The former is better in the case that the loss of the best sequence of experts is small, while the latter exhibits superiority when the loss of all experts (i.e., the loss vector) moves slowly with time. Nevertheless, our second-order bound is better than the existing first-order bound (Cesa-bianchi et al., 2012) of $O(\sqrt{S L_1 \log (KT/S)} + S \log (KT/S))$ with $L_1 = \sum_{t=1}^{T} \ell_t [e_t^*]$, since the loss of experts is in the range of $[0, 1]$.

Both of our algorithms fall into the online mirror descent (OMD) framework (Shalev-Shwartz, 2011) and are inspired by existing algorithms that enjoy data-dependent static regret bounds (Cesa-Bianchi et al., 2005; Chiang et al., 2012). The key technique is that in the updating step of OMD, we restrict the feasible set to be a clipped simplex to ensure the distribution assigned to each expert is lower bounded by a constant. While this technique can be shown as a different form of projection update, its advantage is that the intermediate distribution appearing in projection update is avoided and thus, we can analyze our algorithms under the framework of OMD. We also re-derive the Prod method (Cesa-Bianchi et al., 2005), which enjoys the second-order static regret bound, in the OMD framework so that the technique of clipped simplex can be applied. Finally, we present extensions of our algorithms and analysis to online matrix prediction and establish the first data-dependent tracking regret bound for this problem.

2. Related Work

In this section, we briefly review related work on prediction with expert advice.

2.1 Static Regret

In their seminal work, Littlestone and Warmuth (1994) and Vovk (1990) introduced the multiplicative weights update (MWU) method, also known as the exponentiated gradient (EG) algorithm (Kivinen and Warmuth, 1997) and the Hedge algorithm (Freund and Schapire, 1997). Starting from a uniform distribution $w_1 = (1/K, 1/K, \ldots, 1/K)$, at each round $t$, MWU updates the distribution as

$$w_{t+1}[i] = \frac{w_t[i] \exp(-\eta \ell_t[i])}{\sum_{j=1}^{K} w_t[j] \exp(-\eta \ell_t[j])}, \forall i \in [K]$$

where $\eta$ is the learning rate. MWU was known to enjoy the first-order bound on static regret (Freund and Schapire, 1997). Such bound is also attainable for the follow the perturbed leader (FPL) method (Hannan, 1957; Kalai and Vempala, 2003), where the distribution is chosen based on the observed past loss vectors and a random generated loss vector. Cesa-Bianchi et al. (2005) proposed the Prod algorithm where the exponential update $w_{t+1}[i] \propto w_t[i] \exp(-\eta \ell_t[i])$ in MWU is replaced with the so-called multilinear update $w_{t+1}[i] \propto w_t[i] (1 - \eta \ell_t[i])$, and showed that Prod achieves the second-order bound on static regret. While both the first-order and the second-order bounds belong to the family of small-loss bounds, there also exist other classes of data-dependent bounds. Hazan and Kale (2010) derived the variance bound which depends on the deviation of the loss vector from its average. Chiang et al. (2012) showed that a variant of MWU achieves the path-length bound. While the second-order bound is better than the first-order bound, except for the
first-order bound, the other three bounds are not comparable in general (Steinhardt and Liang, 2014).

2.2 Tracking Regret

Two classic algorithms for minimizing tracking regret are fixed share and projection update (Herbster and Warmuth, 1998, 2001), both of which are variants of the MWU method. In each round $t$, both algorithms first compute an intermediate distribution $w_{t+1}^m$ following MWU in (6). Then, the fixed share algorithm explicitly compels each expert $i \in [K]$ to share a fraction of its assigned distribution with the other experts:

$$w_{t+1}[i] = (1 - \alpha)w_{t+1}^m[i] + \sum_{j \in [K], j \neq i} \frac{\alpha}{K - 1} w_{t+1}^m[j].$$

Different from this, in projection update, sharing is implicitly performed by projecting the intermediate distribution $w_{t+1}^m$ onto a subset of the simplex $\Delta_K$:

$$w_{t+1} = \arg \min_{w \in \Delta_K \cap [\alpha, 1]^K} D_\phi(w \| w_{t+1}^m)$$

where $D_\phi(\cdot \| \cdot)$ denotes Bregman divergence with respect to the negative entropy function and will be made clear in the next section. In both algorithms, the parameter $\alpha$ controls the extent of sharing. Cesa-bianchi et al. (2012) showed that with appropriate configuration of parameters $\eta$ and $\alpha$, both algorithms enjoy the first-order tracking regret bound. Luo and Schapire (2015) developed the AdaNormalHedge method, which is parameter-free and attains a refined first-order bound on tracking regret.

3. Algorithms

In this section, we first introduce the online mirror descent framework, then propose our algorithms, and finally discuss parameter tuning for our algorithms.

3.1 Online Mirror Descent

Our algorithms are developed under the online mirror descent (OMD) framework. OMD is believed to be the gold standard for online learning (Srebro et al., 2011; Steinhardt and Liang, 2014), and a variety of algorithms such as online gradient descent and exponentiated gradient can be derived from this framework (Shalev-Shwartz, 2011). As outlined in Algorithm 1, at each round $t$, after observing the loss vector $\ell_t$, OMD (configured with learning rate $\eta$) updates the distribution as

$$w_{t+1} = \arg \min_{w \in \Delta_K} \langle w, \eta \ell_t \rangle + D_\phi(w \| w_t)$$

where $\Delta_K$ is the $K$-simplex:

$$\Delta_K = \left\{ w \in \mathbb{R}^K \mid w[i] \geq 0, \forall i \in [K]; \sum_{i=1}^K w[i] = 1 \right\}.$$
Algorithm 1 Online Mirror Descent (specialized for prediction with expert advice)

Require: learning rate $\eta > 0$
1: Initialize $w_1 = (1/K, 1/K, \ldots, 1/K)$
2: for $t = 1, \ldots, T$ do
3: Choose distribution $w_t$
4: Observe loss vector $\ell_t$ and suffer a loss $\langle w_t, \ell_t \rangle$
5: $w_{t+1} = \arg \min_{w \in \Delta_K} \langle w, \eta \ell_t \rangle + D_\phi(w \| w_t)$
6: end for

$\phi$ is the negative Shannon entropy function:
$$\phi(w) = \sum_{i=1}^{K} w[i] \log w[i]$$

and $D_\phi(\cdot \| \cdot)$ denotes the Bregman divergence with respect to $\phi$:
$$D_\phi(x \| y) = \phi(x) - \phi(y) - \langle x - y, \nabla \phi(y) \rangle.$$

Though seemingly different, Algorithm 1 is exactly identical to the classic MWU method (Shalev-Shwartz, 2011), which can achieve an $O(\sqrt{T \log K})$ static regret bound but fails to attain meaningful tracking regret bounds. However, we show that Algorithm 1 with a simple yet powerful modification—replacing the simplex $\Delta_K$ with a clipped simplex defined below—is able to achieve meaningful tracking regret bounds.

Theorem 1 Consider the following clipped simplex
$$\tilde{\Delta}_K = \left\{ w \in \Delta_K \mid w[i] \geq \frac{S}{TK}, \forall i \in [K] \right\}.$$

Let $A$ be a variant of Algorithm 1 that replaces Step 5 with
$$w_{t+1} = \arg \min_{w \in \tilde{\Delta}_K} \langle w, \eta \ell_t \rangle + D_\phi(w \| w_t).$$

For $\eta > 0$, the tracking regret of $A$ satisfies
$$\text{TR}(T, S) \leq \eta T + \frac{S \log (KT/S)}{\eta} + S.$$

Picking $\eta = \sqrt{\frac{S \log (KT/S)}{T}}$ leads to a tracking regret bound of $O(\sqrt{ST \log (KT/S)})$.

In fact, the technique of restricting the feasible set to be the clipped simplex can be shown as a different form of the projection update method (Herbster and Warmuth, 2001) as follows.

Proposition 1 Let $A$ be the variant of Algorithm 1 defined in Theorem 1 and $B$ be the projection update method defined in (7) configured with $\alpha = S/(TK)$. Let $w_t$ and $\hat{w}_t$ be the distributions chosen in round $t$ by $A$ and $B$ respectively. We have $w_t = \hat{w}_t$ for all $t \in [T]$. 

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Algorithm 2 Prod on Clipped Simplex (PCS)

Require: learning rate $\eta \in (0, 1/2]$

1: Initialize $w_1$ to be arbitrary distribution in $\tilde{\Delta}_K$

2: for $t = 1, \ldots, T$ do

3: Choose distribution $w_t$

4: Observe loss vector $\ell_t$ and suffer a loss $\langle w_t, \ell_t \rangle$

5: Update $w_t$ according to (14)

6: end for

Nevertheless, directly using clipped simplex in the updating step of OMD avoids the intermediate distribution appearing in projection update method and allows us to follow the analysis framework of OMD. In the following, we combine clipped simplex with existing algorithms that enjoy data-dependent static regret bounds to yield new algorithms with data-dependent tracking regret bounds.

### 3.2 Proposed Algorithms

Our first algorithm is a variant of the Prod method (Cesa-Bianchi et al., 2005). While Prod was known to enjoy the second-order static regret bound, we show that equipped with clipped simplex, this method can also attain similar results for tracking regret. Recall that at each round $t$, after observing the loss vector $\ell_t$, Prod performs the following computation to update the distribution:

$$w_{t+1}[i] = \frac{w_t[i](1 - \eta \ell_t[i])}{\sum_{j=1}^K w_t[j](1 - \eta \ell_t[j])}, \forall i \in [K]. \quad (13)$$

To combine Prod with clipped simplex, we first re-derive the above update in the OMD framework:

$$w_{t+1} = \arg \min_{w \in \Delta_K} \langle w, -\log (1 - \eta \ell_t) \rangle + D_\phi(w \| w_t)$$

where the $\log(\cdot)$ function is point-wise. Then, we replace the simplex $\Delta_K$ with the clipped simplex $\tilde{\Delta}_K$:

$$w_{t+1} = \arg \min_{w \in \tilde{\Delta}_K} \langle w, -\log (1 - \eta \ell_t) \rangle + D_\phi(w \| w_t). \quad (14)$$

We name the obtained algorithm as Prod on Clipped Simplex (PCS), which is summarized in Algorithm 2 and achieves the second-order bound on tracking regret as follows.

**Theorem 2** For $\eta \in (0, 1/2]$, the tracking regret of PCS satisfies

$$\text{TR}(T, S) \leq \eta L_2 + \frac{S \log (KT/S)}{\eta} + \frac{3S}{2}$$

where $L_2$ is the sum of squared loss of the best expert and is defined in (4). Picking $\eta = \min \left\{ \sqrt{\frac{S \log (KT/S)}{L_2}}, \frac{1}{2} \right\}$ leads to a tracking regret bound of $O(\sqrt{SL_2 \log (KT/S)} + S \log (KT/S))$.  

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Algorithm 3 Optimistic descent on Clipped Simplex (OCS)

Require: learning rate $\eta > 0$

1. Initialize $\tilde{w}_1$ to be arbitrary distribution in $\tilde{\Delta}_K$ and set $\ell_0 = (0, 0, \ldots, 0)$

2. for $t = 1, \ldots, T$ do
   3. Choose distribution $w_t$ according to (17)
   4. Observe loss vector $\ell_t$ and suffer a loss $\langle w_t, \ell_t \rangle$
   5. Update $\tilde{w}_t$ according to (18)

6. end for

Remark 1 To the best of our knowledge, it is the first time that the second-order bound is derived for tracking the best expert. Compared to the existing first-order bound (Cesa-Bianchi et al., 2012) of $O(\sqrt{S L_1 \log (KT/S)} + S \log (KT/S))$ with $L_1 = \sum_{t=1}^{T} \ell_t |\mathcal{E}_t^*|$, our second-order bound can be much smaller for small losses.

Our second algorithm is a variant of the optimistic mirror descent (OptMD) method (Chiang et al., 2012; Rakhlin and Sridharan, 2013). In OptMD, there exists an auxiliary sequence of distributions $\tilde{w}_1, \ldots, \tilde{w}_T$, which proceeds in the same way as online mirror descent in (8):

$$\tilde{w}_t = \arg\min_{w \in \Delta_K} \langle w, \eta \ell_{t-1} \rangle + D_\phi(w \| \tilde{w}_{t-1}) .$$

(15)

At each round $t$, based on the auxiliary distribution $\tilde{w}_t$, OptMD chooses $w_t$ as

$$w_t = \arg\min_{w \in \Delta_K} \langle w, \eta \ell_{t-1} \rangle + D_\phi(w \| \tilde{w}_t) .$$

(16)

The intuition behind OptMD, as spelled out by Chiang et al. (2012), is as follows. On one hand, if the loss vectors move slowly (i.e., $\ell_t$ is close to $\ell_{t-1}$), the chosen distribution $w_t$ in (16) can be seen as an approximation to the following imaginary perfect choice:

$$w_t = \arg\min_{w \in \Delta_K} \langle w, \eta \ell_t \rangle + D_\phi(w \| \tilde{w}_t)$$

which minimizes the loss of the $t$-th round $\langle w, \ell_t \rangle$ and thus leads to a small regret. On the other hand, even under the worst case that $\ell_t$ is far away from $\ell_{t-1}$, the Bregman divergence term $D_\phi(w \| \tilde{w}_t)$ in (16) protects $w_t$ from deviating too much from $\tilde{w}_t$ in (15) and hence prevents from incurring a large regret.

While OptMD was originally designed for static regret, we show that by combining with clipped simplex, the above intuition also translates into similar results for tracking regret. Specifically, we replace the simplex $\Delta_K$ in (15) and (16) with the clipped simplex $\tilde{\Delta}_K$:

$$w_t = \arg\min_{w \in \tilde{\Delta}_K} \langle w, \eta \ell_{t-1} \rangle + D_\phi(w \| \tilde{w}_t) ;$$

(17)

$$\tilde{w}_{t+1} = \arg\min_{w \in \tilde{\Delta}_K} \langle w, \eta \ell_t \rangle + D_\phi(w \| \tilde{w}_t) .$$

(18)

The resulting algorithm is outlined in Algorithm 3, which is referred to as Optimistic descent on Clipped Simplex (OCS) and enjoys the following path-length bound on tracking regret.
**Theorem 3** For \( \eta > 0 \), the tracking regret of OCS satisfies

\[
\text{TR}(T, S) \leq \eta P_\infty + \frac{S \log (KT/S)}{\eta} + S
\]

where \( P_\infty \) is the sum of squared difference between consecutive loss vectors and is defined in (5). Picking \( \eta = \sqrt{\frac{S \log (KT/S)}{P_\infty}} \) leads to a tracking regret bound of \( O(\sqrt{SP_\infty \log (KT/S)} + S) \).

**Remark 2** The above path-length bound is generally not comparable to the second-order bound derived in Theorem 2. Specifically, the path-length bound becomes smaller when the loss vector gradually changes, while the second-order bound is better for small losses.

**Remark 3** In each round, the main computational overhead of our two algorithms is solving the minimization problems. Thanks to the fact that the clipped simplex \( \Delta_K \) is a convex set and all the objective functions to minimize are convex, we can solve these minimization problems efficiently by using general convex optimization methods (Boyd and Vandenberghe, 2004).

### 3.3 Parameter Tuning

We note that to attain the second-order and the path-length tracking regret bounds, our algorithms PCS and OCS require prior knowledge of \( L_2 \) and \( P_\infty \) respectively for tuning the learning rates. Obtaining the second-order bound without such hindsight knowledge is highly challenging due to the non-monotonic issue (Gaillard et al., 2014) and remains open even in the context of static regret. However, by employing a variant of doubling trick (Wei and Luo, 2018), we can provide a parameter-free version of OCS achieving the path-length bound.

The main idea is to split the time horizon \([1, T]\) into a serials of epochs, and run OCS with different learning rates in different epochs. Specifically, let \( m = 1, 2, \ldots \) index the epoch. We denote the learning rate used in the \( m \)-th epoch by \( \eta_m \) and the starting round of the \( m \)-th epoch by \( \tau_m + 1 \). For every epoch \( m \), we maintain a variable \( P_m \), which is initialized to be 0 in the beginning of the \( m \)-th epoch and updated in each round \( t \) (belonging to this epoch) as \( P_m = P_m + \| \ell_t - \ell_{t-1} \|_2^2 \). In other words, at the end of round \( t \), we have \( P_m = \sum_{s=\tau_m+1}^{t} \| \ell_s - \ell_{s-1} \|_2^2 \), which reveals the fact that \( P_m \) denotes the path-length (pertaining to the \( m \)-th epoch) up to round \( t \). The role of \( P_m \) is as follows: At the end of each round (in the \( m \)-th epoch), we will check whether the inequality \( \eta_m > \sqrt{\frac{S \log (KT/S)}{P_m'}} \) holds true or not. If it is true, we conclude that the currently-used learning rate \( \eta_m \) is not suitable and hence enter into a new epoch (the \((m+1)\)-th epoch) with half the learning rate: \( \eta_{m+1} = \frac{\eta_m}{2} \). The above procedure is summarized in Algorithm 4, which is referred to as OCS+ and enjoys the following theoretical guarantee.

**Theorem 4** The tracking regret of the OCS+ algorithm satisfies

\[
\text{TR}(T, S) \leq O(\sqrt{S(P_\infty + 1) \log (KT/S)} + S)
\]

where \( P_\infty \) is defined in (5).
Algorithm 4 Optimistic descent on Clipped Simplex plus doubling trick (OCS+)

Require: Time horizon $T$, maximum number of switches $S - 1$

1: Initialize $\tilde{w}_1 \in \tilde{\Delta}_K$ arbitrarily and set $m = 1, \eta_1 = \sqrt{S \log (KT/S)}, \tau_1 = 0, \ell_0 = (0, \ldots, 0)$

2: while $t \leq T$ do

3: $P_m = 0$

4: $t = \tau_m + 1$

5: while $t \leq T$ do

6: Choose distribution $w_t = \arg \min_{w \in \tilde{\Delta}_K} \langle w, \eta_m \ell_{t-1} \rangle + D_\phi(w \| \tilde{w}_t)$

7: Observe loss vector $\ell_t$ and suffer a loss $\langle w_t, \ell_t \rangle$

8: Update $\tilde{w}_{t+1} = \arg \min_{w \in \tilde{\Delta}_K} \langle w, \eta_{m} \ell_{t} \rangle + D_\phi(w \| \tilde{w}_t)$

9: $P_m = P_m + \|\ell_t - \ell_{t-1}\|_\infty$

10: if $\eta_m > \sqrt{\frac{S \log (KT/S)}{P_m}}$ then

11: $\eta_{m+1} = \eta_m / 2$

12: $\tau_{m+1} = t$

13: $m = m + 1$

14: break

15: else

16: $t = t + 1$

17: end if

18: end while

19: end while

Finally, we would like to discuss the possibility of obtaining parameter-free algorithms with the second-order bound by the two-layer approach (Hazan and Seshadhri, 2007), where multiple copies of a base algorithm are created in different rounds, and their outputs are combined by a sleeping expert algorithm. While this approach can lead to tracking regret bounds (Adamskiy et al., 2016), it currently faces difficulties in achieving our second-order bound.

Specifically, let $[p, q] \subseteq [1, T]$ be a time interval and $E^*$ be the best expert in this interval. Then the regret of the sleeping expert algorithm in $[p, q]$ with respect to $E^*$ can be decomposed as

$$
\sum_{t=p}^{q} \langle w_t, \ell_t \rangle - \sum_{t=p}^{q} \ell_t [E^*] = \sum_{t=p}^{q} \langle w_t, \ell_t \rangle - \sum_{t=p}^{q} \langle w_t^p, \ell_t \rangle + \sum_{t=p}^{q} \langle w_t^p, \ell_t \rangle - \sum_{t=p}^{q} \ell_t [E^*]
$$

where $w_t$ and $w_t^p$ denote the outputs in round $t$ of the sleeping expert algorithm and the base algorithm created in round $p$, respectively. The term $A$ is the regret of the sleeping expert algorithm with respect to the base algorithm, and the term $B$ is the regret of the base algorithm with respect to $E^*$.
To obtain the second-order tracking regret bound, one needs to derive second-order bounds for both \( A \) and \( B \). While for \( B \) this can be easily done by picking Prod as the base algorithm, unfortunately, for \( A \) no existing sleeping expert algorithms can achieve our second-order bound. In fact, the state-of-the-art bounds for the sleeping expert problem are a refined first-order bound (Luo and Schapire, 2015) and a second-order excess loss bound (Gaillard et al., 2014), the latter of which depends on the sum of squared excess loss (i.e., the difference between the loss of the learner and that of the best expert) and is hence not comparable to our second-order bound that depends on the sum of squared loss of the best expert.

4. Extension to Online Matrix Prediction

We now extend our algorithms to online matrix prediction (Hazan et al., 2012), which can model a variety of problems such as online collaborative filtering and online max-cut. Before describing the setup, we first introduce some useful definitions and notations. Let \( A \) be a \( K \times K \) matrix,\(^1\) we use \( \| A \| \) and \( \| A \|_* \) to denote the nuclear and the spectral norms of \( A \) respectively, which are defined by

\[
\| A \| = \sum_{i=1}^{K} |\lambda_i(A)|; \quad \| A \|_* = \max_{i \in [K]} |\lambda_i(A)|
\]

where \( \lambda_i(A) \) is the \( i \)-th eigenvalue of \( A \). It is well known that the nuclear norm is the dual norm of the spectral norm, and vice versa. We use \( I_K \) to denote the \( K \times K \) identity matrix.

In matrix settings, the counterpart of the \( K \)-simplex \( \Delta_K \) is the \( K \)-spectraplex, defined as

\[
\Omega_K = \{ W \in S^K_+ | \text{Tr}(W) = 1 \}
\]

where \( S^K_+ \) is the set comprised of all \( K \times K \) positive semidefinite matrices, and \( \text{Tr}(\cdot) \) denotes the trace. Given a matrix \( W \in \Omega_K \), let \( W = V\Lambda V^T \) be the eigendecomposition of \( W \), where \( V \) is an orthogonal matrix whose columns are the eigenvectors of \( W \), and \( \Lambda \) is a diagonal matrix whose entries are the eigenvalues of \( W \). We define \( \log \Lambda \) to be a diagonal matrix with \( (\log \Lambda)_{ii} = \log (\Lambda_{ii}) \) and define \( \log W \) by

\[
\log W = V(\log \Lambda)V^T.
\]

We are now ready to describe the setup of online matrix prediction, which is taken from Steinhardt and Liang (2014): In each round \( t \), a learner chooses a prediction matrix \( W_t \in \Omega_K \), and meanwhile an adversary decides a loss matrix \( Z_t \) satisfying \( \| Z_t \|_* \leq 1 \). Then, the learner observes the loss matrix \( Z_t \) and suffers a loss \( \text{Tr}(W_tZ_t) \). Similarly to (1), we define the tracking regret as

\[
\text{TR}(T,S) = \sum_{t=1}^{T} \text{Tr}(W_tZ_t) - \min_{(U_1, \ldots, U_T) \in \mathcal{U}(T,S)} \sum_{t=1}^{T} \text{Tr}(U_tZ_t) = \sum_{t=1}^{T} \text{Tr}(W_tZ_t) - \sum_{t=1}^{T} \text{Tr}(U_t^*Z_t)
\]

\(^1\) Throughout this section, all matrices are assumed to be symmetric and real.
Algorithm 5 Prod on Clipped SPectraplex (PCSP)

**Require:** learning rate $\eta \in (0, 1/2]$

1. Initialize $W_1$ to be arbitrary matrix in $\tilde{\Omega}_K$
2. for $t = 1, \ldots, T$ do
3. Choose prediction matrix $W_t$
4. Observe loss matrix $Z_t$ and suffer a loss $\text{Tr}(W_t Z_t)$
5. $W_{t+1} = \arg \min_{W \in \tilde{\Omega}_K} \text{Tr}(-W \log (I_K - \eta Z_t)) + \mathcal{D}_\psi(W || W_t)$
6. end for

where $\mathcal{U}(T, S)$ is the set of sequences of matrices in $\Omega_K$ with switches not more than $S - 1$:

$$\mathcal{U}(T, S) = \left\{(U_1, \ldots, U_T) \in \Omega_K^T | \sum_{t=2}^{T} \mathbb{1}\{U_t \neq U_{t-1}\} \leq S - 1\right\}$$

and $U_1^*, \ldots, U_T^*$ is the best sequence in $\mathcal{U}(T, S)$:

$$(U_1^*, \ldots, U_T^*) = \arg \min_{(U_1, \ldots, U_T) \in \mathcal{U}(T, S)} \sum_{t=1}^{T} \text{Tr}(U_t Z_t).$$

As pointed out by Steinhardt and Liang (2014), the problem of prediction with expert advice can be viewed as a special case of online matrix prediction by setting $W_t = \text{diag}(w_t)$ and $Z_t = \text{diag}(\ell_t)$, where $\text{diag}(\cdot)$ denotes the diagonalization of a vector. Based on this observation, we construct the clipped spectraplex $\tilde{\Omega}_K$ as a natural extension of the clipped simplex:

$$\tilde{\Omega}_K = \left\{W \in S^K_+ | \text{Tr}(W) = 1, \lambda_{\text{min}}(W) \geq \frac{S}{TK} \right\}$$

(19)

where $\lambda_{\text{min}}(W)$ denotes the minimum eigenvalue of $W$. By the Weyl’s inequality (Weyl, 1912), it is easy to show that $\tilde{\Omega}_K$ is a convex set. Furthermore, we realize that in matrix algebra, $\text{Tr}(AB)$ plays a similar role as $\langle a, b \rangle$ for vectors $a$ and $b$ (Tsuda et al., 2005), and introduce the negative Von Neumann entropy generalizing the negative Shannon entropy:

$$\psi(W) = \text{Tr}(W \log W), \ W \in \Omega_K.$$

(20)

Finally, the Bregman divergence can also be smoothly extended to the matrix function $\psi$:

$$\mathcal{D}_\psi(A || B) = \psi(A) - \psi(B) - \text{Tr} \left( \left(A - B\right) \nabla \psi(B) \right).$$

(21)

Equipped with these, extending our algorithms to online matrix prediction is straightforward. For brevity, we only provide the extension of our first algorithm PCS in Algorithm 5 (referred to as Prod on Clipped SPectraplex, PCSP), and the extension of our second algorithm can be done in the same way. Similarly to Theorem 2, we have the following theoretical guarantee for PCSP.

**Theorem 5** For $\eta \in (0, 1/2]$, the tracking regret of PCSP satisfies

$$\text{TR}(T, S) \leq \eta M_2 + \frac{S \log (KT/S)}{\eta} + \frac{5S}{2}$$
where we define $M_2 = \sum_{t=1}^{T} \text{Tr}(U_t^* Z_t^2)$. Picking $\eta = \min\left\{ \sqrt{\frac{S \log (KT/S)}{M_2}}, \frac{1}{2} \right\}$ leads to a tracking regret bound of $O(\sqrt{SM_2 \log (KT/S) + S \log (KT/S)})$.

**Remark 4** While there exist data-independent tracking regret bounds for the problem of online matrix prediction (Gyorgy and Szepesvari, 2016), to the best of our knowledge, the bound in Theorem 5 is the first data-dependent bound on tracking regret for this problem.

5. Conclusion and Future Work

In this paper, we develop two adaptive and efficient algorithms that enjoy data-dependent bounds for the problem of tracking the best expert. The first algorithm is inspired by the Prod algorithm and attains the second-order tracking regret bound improving previous first-order bounds. The second algorithm draws inspiration from the optimistic mirror descent method and achieves the path-length bound offering advantages in slowly moving environments. We also provide an extension of our algorithms and analysis to the problem of online matrix prediction and present the first data-dependent tracking regret bound for this problem.

There are several future directions to pursue. First, in the current study, both the time horizon $T$ and the maximum number of switches $S - 1$ are assumed to be known in advance. In the future, we will try to develop more adaptive algorithms that are efficient and can adapt to unknown $T$ and $S$. Second, in the context of static regret, Steinhardt and Liang (2014) have derived a bound on the order of $O(\sqrt{P_* \log K} + \log K)$, where $P_* = \sum_{t=1}^{T}(\ell_t[E_*] - \ell_{t-1}[E_*])^2$ and $E_*$ is the best expert over $T$ rounds. This bound is better than both the second-order and the path-length bounds. It is appealing to obtain similar results for tracking regret. Finally, in light of recent advances in obtaining data-dependent static regret bounds for the multi-armed bandits problem (Wei and Luo, 2018; Bubeck et al., 2019), it would be interesting to examine whether our algorithms and analysis can be extended to the bandits setting.

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Appendix A. Proof of Proposition 1

We prove the statement $w_t = \hat{w}_t$, $\forall t \in [T]$ by mathematical induction.

(i) $w_1 = \hat{w}_1$ holds trivially as both are equal to $(1/K, 1/K, \ldots, 1/K)$.

(ii) Suppose $w_t = \hat{w}_t$ holds for some $t \geq 1$. We show that the statement is also true for $t + 1$. First, we state the expression of $\hat{w}_{t+1}$ according to (7) and (6):

$$\hat{w}_{t+1} = \arg\min_{w \in \Delta_K} D_\phi(w || \hat{w}_{t+1}^m).$$

Note that for $\alpha = S/(TK)$, we have $\Delta_K \cap [\alpha, 1]^K = \tilde{\Delta}_K$. Thus, (23) can rewritten as

$$\hat{w}_{t+1} = \arg\min_{w \in \Delta_K} D_\phi(w || \hat{w}_{t+1}^m).$$

For clarity, we here also restate $w_{t+1}$, which is defined in (12):

$$w_{t+1} = \arg\min_{w \in \Delta_K} (w, \eta \ell_t) + D_\phi(w || w_t).$$

To proceed, we define a convex function on $\tilde{\Delta}_K$:

$$f(w) = (w, \eta \ell_t) + D_\phi(w || w_t), \ w \in \tilde{\Delta}_K.$$

By (25) and (24), we have $w_{t+1} = \arg\min_{w \in \Delta_K} f(w)$ and $\hat{w}_{t+1} \in \tilde{\Delta}_K$, which implies

$$f(w_{t+1}) \leq f(\hat{w}_{t+1}).$$

It remains to show that the opposite, i.e., $f(\hat{w}_{t+1}) \leq f(w_{t+1})$, also holds. To this end, we introduce the following lemma.

**Lemma 1.** Let $u \in \mathbb{R}^K$ be any $K$-dimensional vector satisfying $\sum_{i=1}^K u[i] = 1$. We have

$$\langle u - \hat{w}_{t+1}^m, \eta \ell_t + \nabla \phi(\hat{w}_{t+1}^n) - \nabla \phi(\hat{w}_t) \rangle = 0.$$

Consider $u = \hat{w}^m + \hat{w}_{t+1} - w_{t+1}$. We have

$$\sum_{i=1}^K u[i] = \sum_{i=1}^K (\hat{w}^m_{t+1} + \hat{w}_{t+1} - w_{t+1})[i] = \sum_{i=1}^K \hat{w}^m_{t+1}[i] + \sum_{i=1}^K \hat{w}_{t+1}[i] - \sum_{i=1}^K w_{t+1}[i] = 1.$$

Thus, we can apply Lemma 1 and get

$$\langle \hat{w}_{t+1} - w_{t+1}, \eta \ell_t + \nabla \phi(\hat{w}_{t+1}^n) - \nabla \phi(\hat{w}_t) \rangle = 0.$$

On the other hand, note that $\tilde{\Delta}_K$ is a convex set. By (24) and the first order optimal condition, we have

$$\langle \hat{w}_{t+1} - \nu, \nabla \phi(\hat{w}_{t+1}) - \nabla \phi(\hat{w}_{t+1}^m) \rangle \leq 0, \ \forall \nu \in \tilde{\Delta}_K.$$
By (25), \( w_{t+1} \in \tilde{\Delta}_K \). Therefore, we can substitute \( v = w_{t+1} \) into the above inequality and obtain
\[
\langle \hat{w}_{t+1} - w_{t+1}, \nabla \phi(\hat{w}_{t+1}) - \nabla \phi(w^m_{t+1}) \rangle \leq 0. \tag{29}
\]
Adding (28) to (29) gives
\[
\langle \hat{w}_{t+1} - w_{t+1}, \eta \ell_t + \nabla \phi(\hat{w}_{t+1}) - \nabla \phi(w_t) \rangle \leq 0.
\]
By the definition of convex, we get
\[
\nabla \phi(\hat{w}_{t+1}) = \nabla f(\hat{w}_{t+1}).
\]
Combining this with the above inequality and noticing that \( f \) is convex, we get
\[
f(\hat{w}_{t+1}) - f(w_{t+1}) \leq \langle \hat{w}_{t+1} - w_{t+1}, \nabla f(\hat{w}_{t+1}) \rangle \leq 0. \tag{30}
\]
Combining (27) and (30) and recalling \( w_{t+1} = \arg \min_{w \in \tilde{\Delta}_K} f(w) \), we obtain \( f(\hat{w}_{t+1}) = f(w_{t+1}) = \min_{w \in \tilde{\Delta}_K} f(w) \). Finally, since \( \phi \) and hence \( f \) are strongly convex functions, \( f(\hat{w}_{t+1}) = f(w_{t+1}) = \min_{w \in \tilde{\Delta}_K} f(w) \) implies \( \hat{w}_{t+1} = w_{t+1} \).

Appendix B. Proof of Theorem 1

By the definition of tracking regret in (1)–(3), we can divide the time horizon \([1, T]\) into \( S \) disjoint intervals \([I_1, I_2], \ldots, [I_S, I_{S+1}]\) with \( I_1 = 1 \) and \( I_{S+1} = T + 1 \) such that in each interval \([I_s, I_{s+1})\), \( s \in [S] \), the compared expert \( \mathcal{E}^*_t \) remains the same, i.e.,
\[
\mathcal{E}^*_{I_2} = \mathcal{E}^*_{I_3} = \cdots = \mathcal{E}^*_{I_{S+1}} = \mathcal{E}^*_{I_{s+1}} - 1, \quad \forall s \in [S]. \tag{31}
\]
Fix \( s \in [S] \). We now consider the tracking regret in the \( s \)-th interval \([I_s, I_{s+1})\):
\[
\sum_{t=I_s}^{I_{s+1}} (\ell_t - \ell_t[\mathcal{E}^*_t]) = \sum_{t=I_s}^{I_{s+1}} (\langle w_t, \ell_t \rangle - \ell_t[\mathcal{E}^*_t]).
\]
To express the term \( \ell_t[\mathcal{E}^*_t] \) as an inner product between two vectors, we introduce one-hot vectors \( e_1, \ldots, e_T \) defined as
\[
e_t[i] = \begin{cases} 1, & i = \mathcal{E}^*_t, \\ 0, & \text{otherwise}, \end{cases} \quad \forall i \in [K]. \tag{32}
\]
Then, we have \( \ell_t[\mathcal{E}^*_t] = \langle e_t, \ell_t \rangle \) and
\[
\sum_{t=I_s}^{I_{s+1}} (\langle w_t, \ell_t \rangle - \ell_t[\mathcal{E}^*_t]) = \sum_{t=I_s}^{I_{s+1}} \langle w_t - e_t, \ell_t \rangle. \tag{33}
\]
We further define \( \bar{e}_t \in \tilde{\Delta}_K \) by
\[
\bar{e}_t[i] = (1 - \frac{S}{T})e_t[i] + \frac{S}{TK}, \quad \forall i \in [K] \tag{34}
\]
and decompose the right-hand side of (33) as
\[
\sum_{t=I_s}^{I_{s+1}} \langle w_t - e_t, \ell_t \rangle = \sum_{t=I_s}^{I_{s+1}} \langle w_t - \bar{e}_t, \ell_t \rangle + \sum_{t=I_s}^{I_{s+1}} \langle \bar{e}_t - e_t, \ell_t \rangle
\]
where the last term can be bounded by the following lemma.
Lemma 2 For any $\ell_t \in [0, 1]^K$ and any $e_t \in \Delta_K$, let $\tilde{e}_t$ be defined as in (34). We have

$$\langle e_t - e_t, \ell_t \rangle \leq \frac{S}{T}.$$ 

It follows that

$$\text{TR}(T, S) = \sum_{s=1}^{S} \sum_{t \in I_s} \langle \hat{e}_t - \ell_t | \ell_t^s \rangle = \sum_{s=1}^{S} \sum_{t \in I_s} \langle w_t - e_t, \ell_t \rangle \leq \sum_{s=1}^{S} \sum_{t \in I_s} \langle w_t - \tilde{e}_t, \ell_t \rangle + \sum_{s=1}^{S} \sum_{t \in I_s} \frac{S}{T} \quad (35)$$

Then, we decompose $\langle w_t - \tilde{e}_t, \ell_t \rangle$ as

$$\langle w_t - \tilde{e}_t, \ell_t \rangle = \langle w_t - w_{t+1}, \ell_t \rangle + \langle w_{t+1} - \tilde{e}_t, \ell_t \rangle. \quad (36)$$

The term $\langle w_t - w_{t+1}, \ell_t \rangle$ can be bounded by the following lemma.

Lemma 3 For any $t \in [T]$, we have

$$\langle w_t - w_{t+1}, \ell_t \rangle \leq \eta. \quad (37)$$

It remains to bound the term $\langle w_{t+1} - \tilde{e}_t, \ell_t \rangle$. To this end, we define a convex function on the clipped simplex $\bar{\Delta}_K$:

$$f(w) = \langle w, \eta \ell_t \rangle + D_{\phi}(w\|w_t), \ w \in \bar{\Delta}_K$$

and rewrite the updating step in (12) as

$$w_{t+1} = \arg \min_{w \in \bar{\Delta}_K} f(w).$$

By the first order optimal condition, we have

$$\langle w_{t+1} - u, \nabla f(w_{t+1}) \rangle \leq 0, \ \forall u \in \bar{\Delta}_K.$$

Substituting $u = \tilde{e}_t$, we get

$$\langle w_{t+1} - e_t, \nabla f(w_{t+1}) \rangle \leq 0$$

$$\langle w_{t+1} - \tilde{e}_t, \eta \ell_t + \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle \leq 0$$

$$\eta \langle w_{t+1} - e_t, \ell_t \rangle \leq \langle \tilde{e}_t - w_{t+1}, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle. $$

Thus, we have

$$\langle w_{t+1} - e_t, \ell_t \rangle \leq \frac{1}{\eta} \langle \tilde{e}_t, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle - \frac{1}{\eta} \langle w_{t+1}, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle$$

$$= \frac{1}{\eta} \langle \tilde{e}_t, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle - \frac{1}{\eta} D_{\phi}(w_{t+1}\|w_t)$$

$$\leq \frac{1}{\eta} \langle \tilde{e}_t, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle.$$
where the first equality follows from the definition of Bregman divergence in (10), and the last inequality holds since Bregman divergence is always non-negative. Summing the above inequality over $t = I_s, \ldots, I_{s+1} - 1$, we get

$$\sum_{t=I_s}^{I_{s+1} - 1} \langle w_{t+1} - \bar{e}_t, \ell_t \rangle \leq \frac{1}{\eta} \sum_{t=I_s}^{I_{s+1} - 1} \langle \bar{e}_t, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle$$

$$= \frac{1}{\eta} \sum_{t=I_s}^{I_{s+1} - 1} \langle \bar{e}_{I_s}, \nabla \phi(w_{I_{s+1}}) - \nabla \phi(w_{I_s}) \rangle$$

$$= \frac{1}{\eta} \sum_{i=1}^{K} \bar{e}_{I_s}[i] \log \frac{w_{I_{s+1}}[i]}{w_{I_s}[i]}$$

$$\leq \frac{1}{\eta} \sum_{i=1}^{K} \bar{e}_{I_s}[i] \log \left( \frac{KT}{S} \right) \frac{1}{\eta}$$

(38)

Combining (36) with (37) and (38) gives

$$\sum_{t=I_s}^{I_{s+1} - 1} \langle w_t - \bar{e}_t, \ell_t \rangle = \sum_{t=I_s}^{I_{s+1} - 1} \langle w_t - w_{t+1}, \ell_t \rangle + \sum_{t=I_s}^{I_{s+1} - 1} \langle w_{t+1} - \bar{e}_t, \ell_t \rangle$$

$$\leq \sum_{t=I_s}^{I_{s+1} - 1} \eta + \frac{\log \left( \frac{KT}{S} \right)}{\eta}$$

$$= \eta(I_{s+1} - I_s) + \frac{\log \left( \frac{KT}{S} \right)}{\eta}.$$ 

Substituting the above inequality into (35), we obtain

$$\text{TR}(T, S) \leq \sum_{s=1}^{S} \sum_{t=I_s}^{I_{s+1} - 1} \langle w_t - \bar{e}_t, \ell_t \rangle + S$$

$$\leq \sum_{s=1}^{S} \eta(I_{s+1} - I_s) + \sum_{s=1}^{S} \frac{\log \left( \frac{KT}{S} \right)}{\eta} + S$$

$$= \eta(T) + \frac{S \log \left( \frac{KT}{S} \right)}{\eta} + S$$

which concludes the proof.
Appendix C. Proof of Theorem 2

Following the proof of Theorem 1 in Appendix B, we have

$$\text{TR}(T, S) \leq \frac{1}{\eta} \sum_{s=1}^{S} \sum_{t=I_s}^{I_{s+1}-1} \langle w_t - \bar{e}_t, \eta \ell_t \rangle + S.$$  \hfill (39)

We first decompose $\langle w_t - \bar{e}_t, \eta \ell_t \rangle$ as

$$\langle w_t - \bar{e}_t, \eta \ell_t \rangle = \langle w_t - w_{t+1}, \eta \ell_t \rangle + \langle w_{t+1} - \bar{e}_t, \eta \ell_t + \log (1 - \eta \ell_t) \rangle + \langle w_{t+1} - \bar{e}_t, - \log (1 - \eta \ell_t) \rangle.$$  \hfill (40)

Let $f(w)$ be a convex function defined as

$$f(w) = \langle w, - \log (1 - \eta \ell_t) \rangle + D_{\phi}(w \| w_t), \ w \in \Delta_K.$$  

Then, Step 5 of Algorithm 2 is identical to

$$w_{t+1} = \arg \min_{w \in \Delta_K} f(w).$$

By the first order optimal condition, we have

$$\langle w_{t+1} - u, \nabla f(w_{t+1}) \rangle \leq 0, \ \forall u \in \Delta_K.$$  

Substituting $u = \bar{e}_t$ into the above inequality gives

$$\langle w_{t+1} - \bar{e}_t, - \log (1 - \eta \ell_t) \rangle \leq \langle \bar{e}_t - w_{t+1}, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle$$

which, together with the decomposition in (40), leads to

$$\langle w_t - \bar{e}_t, \eta \ell_t \rangle \leq \langle w_t - w_{t+1}, \eta \ell_t \rangle + \langle w_{t+1} - \bar{e}_t, \eta \ell_t + \log (1 - \eta \ell_t) \rangle + \langle \bar{e}_t - w_{t+1}, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle$$

$$= \langle w_t, \eta \ell_t \rangle + \langle w_{t+1}, \log (1 - \eta \ell_t) - \nabla \phi(w_{t+1}) + \nabla \phi(w_t) \rangle + \langle \bar{e}_t, \eta \ell_t + \log (1 - \eta \ell_t) \rangle$$

$$+ \langle \bar{e}_t, \nabla \phi(w_{t+1}) - \nabla \phi(w_t) \rangle.$$  \hfill (41)

Below, we bound $A_t, B_t$ and $\sum_{t=I_s}^{I_{s+1}-1} C_t$ separately.

(i) Bounding $A_t$. By the definition of $\phi$ in (9), we have

$$A_t = \langle w_t, \eta \ell_t \rangle + \sum_{i=1}^{K} w_{t+1}[i] \log \frac{(1 - \eta \ell_t[i]) w_t[i]}{w_{t+1}[i]}.$$  

Then, we introduce $p_{t+1} \in \Delta_K$ defined as

$$p_{t+1}[i] = \frac{w_t[i](1 - \eta \ell_t[i])}{\sum_{j=1}^{K} w_t[j](1 - \eta \ell_t[j])}, \ \forall i \in [K].$$
and get

\[ A_t = \langle w_t, \eta \ell_t \rangle + \sum_{i=1}^{K} w_{t+1}[i] \log \left( \frac{1 - \eta \ell_t[i]}{w_{t+1}[i]} \right) \]

\[ + \sum_{i=1}^{K} w_{t+1}[i] \log \frac{w_{t+1}[i]}{p_{t+1}[i]} - D_\phi(w_{t+1} || p_{t+1}) \]

\[ \leq \langle w_t, \eta \ell_t \rangle + \sum_{i=1}^{K} w_{t+1}[i] \log \left( \frac{1 - \eta \ell_t[i]}{w_{t+1}[i]} \right) \]

\[ = \langle w_t, \eta \ell_t \rangle + \sum_{i=1}^{K} w_{t+1}[i] \log \left( 1 - \eta \ell_t[i] \right) \]

\[ = \langle w_t, \eta \ell_t \rangle + \log \sum_{j=1}^{K} w_t[j] \left( 1 - \eta \ell_t[j] \right) \]

\[ = \langle w_t, \eta \ell_t \rangle + \log \left( \sum_{j=1}^{K} w_t[j] - \eta \sum_{j=1}^{K} w_t[j] \ell_t[j] \right) \]

\[ = \langle w_t, \eta \ell_t \rangle + \log \left( 1 - \langle w_t, \eta \ell_t \rangle \right) \]

\[ \leq 0 \]

where the first equality follows from the definition of Bregman divergence and the second equality is due to the definition of \( p_{t+1} \); the first inequality holds since Bregman divergence is always non-negative, and the last inequality holds since \( \langle w_t, \eta \ell_t \rangle \in [0, 1/2] \) and \( x + \log (1 - x) \leq 0, \forall x \in [0, 1) \).

(ii) Bounding \( B_t \). By the fact that \( \forall t \in [T], \eta \ell_t \in [0, 1/2]^K \) and the well-known inequality \( x - \log (1 - x) \leq x^2, \forall x \in (-\infty, 1/2], -x - \log (1 - x) \leq x^2 \), we have

\[ B_t = \sum_{i=1}^{K} e_t[i] \left( -\eta \ell_t[i] - \log \left( 1 - \eta \ell_t[i] \right) \right) \]

\[ \leq \sum_{i=1}^{K} e_t[i] (\eta \ell_t[i])^2 = \sum_{i=1}^{K} \left( 1 - \frac{S}{T} \right) e_t[i] (\eta \ell_t[i])^2 + \sum_{i=1}^{K} \frac{S(\eta \ell_t[i])^2}{TK} \]

\[ \leq \sum_{i=1}^{K} e_t[i] (\eta \ell_t[i])^2 + \sum_{i=1}^{K} \frac{\eta S}{2TK} \]

\[ = \eta^2 (\ell_t[\mathcal{E}_t^*])^2 + \frac{\eta S}{2T} \quad \text{(43)} \]

(iii) Bounding \( \sum_{t=I_s}^{I_{s+1}-1} C_t \). Following (38), we have

\[ \sum_{t=I_s}^{I_{s+1}-1} C_t \leq \log (KT/S). \quad \text{(44)} \]
Combining (39)–(44), we have

\[ TR(T, S) \leq \frac{1}{\eta} \sum_{s=1}^{S} \sum_{t=I_s}^{I_{s+1}-1} \left( \eta^2 (\ell_t \| E_t \|^2) + \frac{\eta S}{2T} \right) + \frac{1}{\eta} \sum_{s=1}^{S} \log (KT/S) + S \]

\[ = \eta \sum_{t=1}^{T} (\ell_t \| E_t \|^2) + \frac{S \log (KT/S)}{\eta} + \frac{3S}{2} \]

which finishes the proof.

**Appendix D. Proof of Theorem 3**

Following the proof of Theorem 1 in Appendix B, we have

\[ TR(T, S) \leq \sum_{s=1}^{S} \sum_{t=I_s}^{I_{s+1}-1} \langle w_t - \bar{e}_t, \ell_t \rangle + S. \] (45)

We start by splitting \( \langle w_t - \bar{e}_t, \ell_t \rangle \) into three terms:

\[ \langle w_t - \bar{e}_t, \ell_t \rangle = \langle w_t - \bar{w}_{t+1}, \ell_t \rangle + \langle \bar{w}_{t+1} - \bar{e}_t, \ell_t \rangle \]

\[ = \langle w_t - \bar{w}_{t+1}, \ell_t \rangle + \langle w_t - \bar{w}_{t+1}, \ell_{t-1} \rangle + \langle \bar{w}_{t+1} - \bar{e}_t, \ell_t \rangle. \] (46)

The first term can be bounded by the following lemma.

**Lemma 4** For any \( t \in [T] \), we have

\[ \langle w_t - \bar{w}_{t+1}, \ell_t \rangle \leq \eta \| \ell_t \| \ell_{t-1} \|^2. \] (47)

To bound the second and the third terms, we define two convex functions on the clipped simplex \( \bar{\Delta}_K \):

\[ f(w) = \langle w, \eta \ell_{t-1} \rangle + D_{\phi}(w \| \bar{w}_t), \quad w \in \bar{\Delta}_K; \]

\[ g(w) = \langle w, \eta \ell_t \rangle + D_{\phi}(w \| \bar{w}_t), \quad w \in \bar{\Delta}_K. \]

Then, we can rewrite Steps 3 and 5 in Algorithm 3 as

\[ w_t = \arg \min_{w \in \bar{\Delta}_K} f(w); \quad \bar{w}_{t+1} = \arg \min_{w \in \bar{\Delta}_K} g(w). \]

By the first order optimal condition, we have

\[ \langle w_t - u, \nabla f(w_t) \rangle \leq 0, \quad \forall u \in \bar{\Delta}_K; \quad \langle \bar{w}_{t+1} - v, \nabla g(\bar{w}_{t+1}) \rangle \leq 0, \quad \forall v \in \bar{\Delta}_K. \]

Substituting \( u = \bar{w}_{t+1} \) and \( v = \bar{e}_t \) into the above two inequalities respectively, we get

\[ \langle w_t - \bar{w}_{t+1}, \nabla f(w_t) \rangle \leq 0 \]

\[ \langle w_t - \bar{w}_{t+1}, \eta \ell_{t-1} \rangle + \langle \nabla \phi(w_t) - \nabla \phi(\bar{w}_t), w_t \rangle \leq 0 \]

\[ \langle w_t - \bar{w}_{t+1}, \ell_{t-1} \rangle \leq \frac{1}{\eta} \langle w_t - \bar{w}_{t+1}, \nabla \phi(w_t) - \nabla \phi(\bar{w}_t) \rangle; \] (48)
and
\[
\langle \bar{w}_{t+1} - \bar{e}_t, \nabla g(\bar{w}_{t+1}) \rangle \leq 0
\]
\[
\langle \bar{w}_{t+1} - \bar{e}_t, \eta \ell_t + \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle \leq 0
\]
\[
(\bar{w}_{t+1} - \bar{e}_t, \ell_t) \leq \frac{1}{\eta} \langle \bar{w}_{t+1} - \bar{e}_t, \nabla \phi(\bar{w}_t) - \nabla \phi(\bar{w}_{t+1}) \rangle.
\]

Combining (48) and (49) and rearranging, we have
\[
\langle w_t - \bar{w}_{t+1}, \ell_{t-1} \rangle + \langle \bar{w}_{t+1} - \bar{e}_t, \ell_t \rangle
\]
\[
\leq \frac{1}{\eta} \left( \langle w_t, \nabla \phi(\bar{w}_t) - \nabla \phi(w_t) \rangle - \langle \bar{w}_{t+1}, \nabla \phi(\bar{w}_t) - \nabla \phi(w_t) \rangle 
\right.
\]
\[
+ \langle \bar{w}_{t+1}, \nabla \phi(\bar{w}_t) - \nabla \phi(\bar{w}_{t+1}) \rangle
\]
\[
+ \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle
\]
\[
= \frac{1}{\eta} \left( \langle w_t, \nabla \phi(\bar{w}_t) - \nabla \phi(w_t) \rangle + \langle \bar{w}_{t+1}, \nabla \phi(\bar{w}_t) - \nabla \phi(\bar{w}_{t+1}) \rangle + \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle \right)
\]
\[
= \frac{1}{\eta} \left( - D_\phi(w_t || \bar{w}_t) - D_\phi(\bar{w}_{t+1} || w_t) + \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle \right)
\]
\[
\leq \frac{1}{\eta} \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle
\]

where the last equality follows from the definition of Bregman divergence in (10), and the last inequality holds since Bregman divergence is always non-negative. Substituting the above inequality and (47) into (46), we get
\[
\langle w_t - \bar{e}_t, \ell_t \rangle \leq \eta \| \ell_t - \ell_{t-1} \|_\infty^2 + \frac{1}{\eta} \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle.
\]

Summing this inequality over \( t = \mathcal{I}_s, \ldots, \mathcal{I}_{s+1} - 1 \) and following the same derivation as in (38), we have
\[
\sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \langle w_t - \bar{e}_t, \ell_t \rangle \leq \sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \eta \| \ell_t - \ell_{t-1} \|_\infty^2 + \frac{1}{\eta} \sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \langle \bar{e}_t, \nabla \phi(\bar{w}_{t+1}) - \nabla \phi(\bar{w}_t) \rangle
\]
\[
\leq \sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \eta \| \ell_t - \ell_{t-1} \|_\infty^2 + \frac{\log(KT/S)}{\eta}.
\]

Substituting the above inequality into (45), we get
\[
\text{TR}(T, S) \leq \sum_{s=1}^{S} \sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \langle w_t - \bar{e}_t, \ell_t \rangle + S
\]
\[
\leq \sum_{s=1}^{S} \left( \sum_{t=\mathcal{I}_s}^{\mathcal{I}_{s+1}-1} \eta \| \ell_t - \ell_{t-1} \|_\infty^2 + \frac{\log(KT/S)}{\eta} \right) + S
\]
\[
= \eta P_{\infty} + \frac{S \log(KT/S)}{\eta} + S.
\]
This completes the proof.
Appendix E. Proof of Theorem 4

Let $m^*$ be the last epoch such that

$$m^* = \max \{ m : \tau_m < T \}$$

and define $\tau_{m^*+1} = T$. We begin with bounding the tracking regret in each epoch $m = 1, \ldots, m^*$. Specifically, considering the $m$-th epoch, by the proof of Theorem 3 in Appendix D, we have

$$\sum_{t=\tau_m+1}^{\tau_{m^*+1}} \hat{e}_t - \sum_{t=\tau_m+1}^{\tau_{m^*+1}} \ell_t [\mathcal{E}_t^*] \leq \eta_m P_m + \frac{S \log (KT/S)}{\eta_m} + \frac{S(\tau_{m^*+1} - \tau_m)}{T}$$

$$= \eta_m \sum_{t=\tau_m+1}^{\tau_{m^*+1}} \| \ell_t - \ell_{t-1} \|_\infty^2 + \frac{S \log (KT/S)}{\eta_m} + \frac{S(\tau_{m^*+1} - \tau_m)}{T}$$

$$\leq \sum_{t=\tau_m+1}^{\tau_{m^*+1}} \| \ell_t - \ell_{t-1} \|_\infty^2 + \eta_m + \frac{S \log (KT/S)}{\eta_m} + \frac{S(\tau_{m^*+1} - \tau_m)}{T}$$

$$\leq \frac{2S \log (KT/S)}{\eta_m} + \eta_m + \frac{S \log (KT/S)}{\eta_m} + \frac{S(\tau_{m^*+1} - \tau_m)}{T}$$

where the second inequality is due to the fact that $\ell_t \in [0,1]^K$, $\forall t \in [T]$, and the last inequality holds since for each epoch, the condition in Line 10 of Algorithm 4 can be violated only at the last round of the epoch. Summing (50) over $m = 1, \ldots, m^*$, we get

$$\text{TR}(T,S) = \sum_{t=1}^{T} \hat{e}_t - \sum_{t=1}^{T} \ell_t [\mathcal{E}_t^*] = \sum_{m=1}^{m^*} \sum_{t=\tau_m+1}^{\tau_{m+1}} \hat{e}_t - \sum_{m=1}^{m^*} \sum_{t=\tau_m+1}^{\tau_{m+1}} \ell_t [\mathcal{E}_t^*]$$

$$\leq \sum_{m=1}^{m^*} \left( \frac{2S \log (KT/S)}{\eta_m} + \eta_m + \frac{S(\tau_{m^*+1} - \tau_m)}{T} \right)$$

$$= \sum_{m=1}^{m^*} \frac{2S \log (KT/S)}{\eta_m} + \sum_{m=1}^{m^*} \eta_m + \frac{S(\tau_{m^*+1} - \tau_1)}{T}$$

$$= \sum_{m=1}^{m^*} \frac{2S \log (KT/S)}{\eta_m} + \sum_{m=1}^{m^*} \eta_m + S.$$

By the update rule of $\eta_m$ (Line 11 in Algorithm 4), we have $\eta_m = \frac{\sqrt{S \log (KT/S)}}{2m-1}$ and thus

$$\text{TR}(T,S) \leq \sqrt{S \log (KT/S)} \sum_{m=1}^{m^*} 2^m + \sqrt{S \log (KT/S)} \sum_{m=1}^{m^*} \frac{1}{2m-1} + S$$

$$\leq (2^{m^*+1} - 2) \sqrt{S \log (KT/S)} + 2 \sqrt{S \log (KT/S)} + S$$

$$= 2^{m^*+1} \sqrt{S \log (KT/S)} + S.$$
Below we consider two cases:

(i) $m^* = 1$. In this case, it trivially follows that

$$\text{TR}(T, S) \leq 4\sqrt{S \log (KT/S)} + S \leq O(\sqrt{S(P_\infty + 1) \log (KT/S)} + S).$$  \tag{52}$$

(ii) $m^* > 1$. In this case, since the $(m^* - 1)$-th epoch has finished, we have

$$\eta_{m^*-1} > \sqrt{\frac{S \log (KT/S)}{P_{m^*-1}}}$$

which implies

$$\frac{\sqrt{S \log (KT/S)}}{2^{m^*-2}} > \sqrt{\frac{S \log (KT/S)}{P_{m^*-1}}}$$

$$2^{m^*-2} < \sqrt{P_{m^*-1}}$$

$$2^{m^*+1} < 8\sqrt{P_{m^*-1}}.$$

Substituting the above inequality into (51) gives

$$\text{TR}(T, S) \leq 8\sqrt{P_{m^*-1}S \log (KT/S)} + S$$

$$\leq 8\sqrt{P_\infty S \log (KT/S)} + S$$

$$\leq O(\sqrt{S(P_\infty + 1) \log (KT/S)} + S).$$  \tag{53}$$

Combining (52) and (53) completes the proof.

**Appendix F. Proof of Theorem 5**

The proof below is a generalization of the proof of Theorem 2 in Appendix C. Similarly to (31), we first divide the time horizon $[1, T]$ into $S$ disjoint intervals $[I_1, I_2], \ldots, [I_S, I_{S+1}]$ with $I_1 = 1$ and $I_{S+1} = T + 1$ such that in each interval $[I_s, I_{s+1})$, $s \in [S]$, the compared matrix $U^*_t$ remains the same, i.e.,

$$U^*_{I_s} = U^*_{I_{s+1}} = U^*_{I_{s+2}} = \cdots = U^*_{I_{S+1-1}}, \forall s \in [S].$$  \tag{54}$$

Fix $s \in [S]$. We consider the tracking regret in the $s$-th interval:

$$\sum_{t=I_s}^{I_{s+1}-1} \text{Tr}(W_t Z_t) - \sum_{t=I_s}^{I_{s+1}-1} \text{Tr}(U^*_t Z_t) = \sum_{t=I_s}^{I_{s+1}-1} \text{Tr}((W_t - U^*_t)Z_t).$$  \tag{55}$$

Let $\tilde{U}^*_t$ be defined by

$$\tilde{U}^* = (1 - \frac{S}{T})U^* + \frac{SI_K}{TK} = (1 - \frac{S}{T})U^* + \frac{SI}{TK}$$

in which (and in the following) the subscript $K$ of the $K \times K$ identity matrix $I_K$ is omitted for brevity. We decompose the right-hand side of (55) as

$$\sum_{t=I_s}^{I_{s+1}-1} \text{Tr}((W_t - U^*_t)Z_t) = \sum_{t=I_s}^{I_{s+1}-1} \text{Tr}((W_t - \tilde{U}^*_t)Z_t) + \sum_{t=I_s}^{I_{s+1}-1} \text{Tr}((\tilde{U}^*_t - U^*_t)Z_t)$$  \tag{56}$$

where $\text{Tr}((\tilde{U}^*_t - U^*_t)Z_t)$ can be bounded by the following lemma.
Lemma 5 For any $t \in [T]$, we have
\[
\text{Tr} \left( (\tilde{U}_t^* - U_t^*)Z_t \right) \leq \frac{2S}{T}.
\] (58)

Below we focus on bounding $\eta \text{Tr} \left( (W_t - \tilde{U}_t^*)Z_t \right) = \text{Tr} \left( (W_t - \tilde{U}_t^*)(\eta Z_t) \right)$ and start by splitting it into three terms:
\[
\begin{align*}
\text{Tr} \left( (W_t - \tilde{U}_t^*)(\eta Z_t) \right) \\
= \text{Tr} \left( (W_t - W_{t+1})(\eta Z_t) \right) + \text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)(\eta Z_t + \log (I - \eta Z_t)) \right) \\
+ \text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)(- \log (I - \eta Z_t)) \right).
\end{align*}
\] (59)

Then, we introduce a convex function on the clipped spectraplex $\tilde{\Omega}_K$:
\[
H(W) = \text{Tr}(-W \log (I - \eta Z_t)) + D_\psi(W||W_t), \; W \in \tilde{\Omega}_K
\]
and rewrite Step 5 of Algorithm 5 as
\[
W_{t+1} = \arg \min_{W \in \tilde{\Omega}_K} H(W).
\]

By the first order optimal condition and the fact that $\tilde{U}_t^* \in \tilde{\Omega}_K$, we have
\[
\text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)\nabla H(W_{t+1}) \right) \leq 0.
\]

Expanding $\nabla H(W_{t+1})$ and using the equality $\nabla \psi(W) = I + \log W$, we get
\[
\text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)(- \log (I - \eta Z_t) + \log W_{t+1} - \log W_t) \right) \leq 0
\]
which implies
\[
\text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)(- \log (I - \eta Z_t)) \right) \leq \text{Tr} \left( (\tilde{U}_t^* - W_{t+1})(\log W_{t+1} - \log W_t) \right).
\]

Combining the above inequality with (59) gives
\[
\begin{align*}
\text{Tr} \left( (W_t - \tilde{U}_t^*)(\eta Z_t) \right) \\
\leq \text{Tr} \left( (W_t - W_{t+1})(\eta Z_t) \right) + \text{Tr} \left( (W_{t+1} - \tilde{U}_t^*)(\eta Z_t + \log (I - \eta Z_t)) \right) \\
+ \text{Tr} \left( (\tilde{U}_t^* - W_{t+1})(\log W_{t+1} - \log W_t) \right) \\
= \text{Tr} \left( (W_t - W_{t+1})(\eta Z_t) \right) + \text{Tr} \left( W_{t+1}(\eta Z_t + \log (I - \eta Z_t) - \log W_{t+1} + \log W_t) \right) \\
+ \text{Tr} \left( -\tilde{U}_t^*(\eta Z_t + \log (I - \eta Z_t)) \right) + \text{Tr} \left( \tilde{U}_t^*(\log W_{t+1} - \log W_t) \right).
\end{align*}
\]

The following lemmas bound $A_t$ and $B_t$ respectively:

Lemma 6 For any $t \in [T]$, we have
\[
A_t = \text{Tr} \left( (W_t - W_{t+1})(\eta Z_t) \right) + \text{Tr} \left( W_{t+1}(\eta Z_t + \log (I - \eta Z_t) - \log W_{t+1} + \log W_t) \right) \leq 0.
\]
Lemma 7 For any $t \in [T]$, we have

$$B_t = \text{Tr} \left( -\bar{U}_t^* (\eta Z_t + \log (I - \eta Z_t)) \right) \leq \eta^2 \text{Tr} (U_t^* Z_t^2) + \frac{\eta S}{2T}.$$ 

It follows that

$$\sum_{t=I_s}^{I_{s+1}-1} \text{Tr} \left( (W_t - U_t^*)(\eta Z_t) \right) \leq 0 + \sum_{t=I_s}^{I_{s+1}-1} \eta^2 \text{Tr} (U_t^* Z_t^2) + \sum_{t=I_s}^{I_{s+1}-1} \frac{\eta S}{2T} + \sum_{t=I_s}^{I_{s+1}-1} \text{Tr} (\bar{U}_t^* (\log W_{t+1} - \log W_t))$$

$$= \sum_{t=I_s}^{I_{s+1}-1} \eta^2 \text{Tr} (U_t^* Z_t^2) + \frac{\eta S(I_{s+1} - I_s)}{2T} + \sum_{t=I_s}^{I_{s+1}-1} \text{Tr} (\bar{U}_t^* (\log W_{t+1} - \log W_t))$$

$$= \sum_{t=I_s}^{I_{s+1}-1} \eta^2 \text{Tr} (U_t^* Z_t^2) + \frac{\eta S(I_{s+1} - I_s)}{2T} + \text{Tr} (\bar{U}_{I_s}^* (\log W_{I_{s+1}} - \log W_{I_s}))$$

$$\leq \sum_{t=I_s}^{I_{s+1}-1} \eta^2 \text{Tr} (U_t^* Z_t^2) + \frac{\eta S(I_{s+1} - I_s)}{2T} + \log (KT/S)$$

where the first equality holds since $\bar{U}_{I_s}^* = \bar{U}_{I_{s+1}}^* = \bar{U}_{I_{s+2}}^* = \cdots = \bar{U}_{I_{s+1-1}}^*$, and the second inequality is due to the following lemma.

Lemma 8 For any $X, Y, Z \in \tilde{\Omega}_K$, we have

$$\text{Tr} \left( X (\log Y - \log Z) \right) \leq \log (KT/S).$$

Dividing both sides of (60) by $\eta$ and summing over $s = 1, \ldots, S$ leads to

$$\sum_{s=1}^S \sum_{t=I_s}^{I_{s+1}-1} \text{Tr} \left( (W_t - U_t^*)Z_t \right) \leq \sum_{s=1}^S \sum_{t=I_s}^{I_{s+1}-1} \eta \text{Tr} (U_t^* Z_t^2) + \frac{S(I_{s+1} - I_s)}{2T} + \frac{S \log (KT/S)}{\eta}$$

$$= \eta \sum_{t=1}^T \text{Tr} (U_t^* Z_t^2) + \frac{S}{2} + \frac{S \log (KT/S)}{\eta}.$$
Appendix G. Proofs of Lemmas

In this appendix, we provide the proofs of all lemmas.

G.1 Proof of Lemma 1

Define
\[ C = \sum_{j=1}^{K} \hat{w}_t[j] \exp(-\eta \ell_t[j]). \]

We can rewrite (22) as
\[ \hat{w}_{t+1}^m[i] = \frac{\hat{w}_t[i] \exp(-\eta \ell_t[i])}{C}, \forall i \in [K]. \]

By the definition of \( \phi \) in (9), for any \( i \in [K] \) we have
\[ \eta \ell_t[i] + \nabla \phi(\hat{w}_{t+1}^m)[i] - \nabla \phi(\hat{w}_t)[i] = \eta \ell_t[i] + \log \left( \frac{\hat{w}_t[i] \exp(-\eta \ell_t[i])}{C} \right) - \log (\hat{w}_t[i]) = -\log C. \]

It follows that
\[ \langle u - \hat{w}_{t+1}^m, \eta \ell_t + \nabla \phi(\hat{w}_{t+1}^m) - \nabla \phi(\hat{w}_t) \rangle = \sum_{i=1}^{K} (u[i] - \hat{w}_{t+1}^m[i]) (\eta \ell_t[i] + \nabla \phi(\hat{w}_{t+1}^m)[i] - \nabla \phi(\hat{w}_t)[i]) \]
\[ = - (\log C) \sum_{i=1}^{K} (u[i] - \hat{w}_{t+1}^m[i]) = - (\log C) \left( \sum_{i=1}^{K} u[i] - \sum_{i=1}^{K} \hat{w}_{t+1}^m[i] \right) = 0 \]
where the last inequality holds since \( \sum_{i=1}^{K} u[i] = 1 \) and \( \sum_{i=1}^{K} \hat{w}_{t+1}^m[i] = 1 \).

G.2 Proof of Lemma 2

By the definition of \( \bar{e}_t \) in (34), we have
\[ \bar{e}_t[i] - e_t[i] = (1 - \frac{S}{T})e_t[i] + \frac{S}{TK} - e_t[i] = - \frac{Se_t[i]}{T} + \frac{S}{TK}, \forall i \in [K]. \]

It follows that
\[ \langle \bar{e}_t - e_t, \ell_t \rangle = \sum_{i=1}^{K} (\bar{e}_t[i] - e_t[i]) \ell_t[i] = \sum_{i=1}^{K} \left( - \frac{Se_t[i]}{T} + \frac{S}{TK} \right) \ell_t[i] \]
\[ = - \sum_{i=1}^{K} \frac{Se_t[i]}{T} \ell_t[i] + \frac{S}{TK} \sum_{i=1}^{K} \ell_t[i] \]
\[ \leq 0 + \frac{S}{TK} \cdot K = \frac{S}{T} \]
where the inequality holds since \( 0 \leq e_t[i], \ell_t[i] \leq 1, \forall i \in [K] \).
G.3 Proof of Lemma 3

We first introduce the definition of Fenchel conjugate:

**Definition 1** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f: \mathcal{X} \mapsto \mathbb{R}$ be a convex function. The Fenchel conjugate of $f$ is a function $f^*: \mathbb{R}^n \mapsto \mathbb{R}$, defined as

$$f^*(y) = \sup_{x \in \mathcal{X}} \langle x, y \rangle - f(x), \quad y \in \mathbb{R}^n.$$  

As a powerful tool in convex analysis, the Fenchel conjugate has many properties among which we mainly utilize the following three properties, the proof of which can be found at, e.g., Shalev-Shwartz (2007).

**Theorem 6** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f: \mathcal{X} \mapsto \mathbb{R}$ be a convex function. If $f$ is further closed and $\mu$-strongly convex with respect to a norm $\| \cdot \|$, then its Fenchel conjugate function $f^*$ is everywhere differentiable and the gradient of $f^*$ satisfies

- for any $y \in \mathbb{R}^n$,
  $$\nabla f^*(y) = \arg\max_{x \in \mathcal{X}} \langle x, y \rangle - f(x);$$  

- for any $y, z \in \mathbb{R}^n$,
  $$\| \nabla f^*(y) - \nabla f^*(z) \| \leq \frac{1}{\mu} \| y - z \|_*$$  

where $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$;

- for any $x \in \mathcal{X}$,
  $$\nabla f^*(\nabla f(x)) = x.$$  

Let $\hat{\phi}$ be the negative Shannon entropy function $\phi$ with domain being the clipped simplex $\tilde{\Delta}_K$. It is easy to see that $\hat{\phi}$ is closed as $\phi$ is a continuous function and $\tilde{\Delta}_K$ is a closed set. Furthermore, it is well-known that $\hat{\phi}$ and hence $\hat{\phi}$ are 1-strongly convex with respect to the $\| \cdot \|_1$ norm (Shalev-Shwartz, 2007). Therefore, $\hat{\phi}$ meets the condition of Theorem 6 and $\nabla \hat{\phi}^*$ enjoys the three above properties, which play a key role in the analysis below.

Fix $t \in [T]$. By the updating step in (12), we have

$$w_{t+1} = \arg\min_{w \in \Delta_K} \langle w, \eta \ell_t \rangle + D_\phi(w\|w_t)$$
$$= \arg\min_{w \in \tilde{\Delta}_K} \langle w, \eta \ell_t \rangle + \phi(w) - \langle w, \nabla \phi(w_t) \rangle$$
$$= \arg\max_{w \in \tilde{\Delta}_K} \langle w, \nabla \phi(w_t) - \eta \ell_t \rangle - \phi(w)$$
$$= \arg\max_{w \in \tilde{\Delta}_K} \langle w, \nabla \phi(w_t) - \eta \ell_t \rangle - \hat{\phi}(w)$$
$$= \nabla \hat{\phi}^*(\nabla \hat{\phi}(w_t) - \eta \ell_t)$$

where the last equality follows from (61). On the other hand, by (63) we can rewrite $w_t$ as

$$w_t = \nabla \hat{\phi}^*(\nabla \hat{\phi}(w_t)) = \nabla \hat{\phi}^*(\nabla \phi(w_t)).$$
Combining the above two equalities, we get
\[
\langle w_t - w_{t+1}, \ell_t \rangle = \langle \nabla \hat{\phi}^*(\nabla \phi(w_t)) - \nabla \hat{\phi}^*(\nabla \phi(w_t) - \eta \ell_t), \ell_t \rangle \\
\leq \| \nabla \hat{\phi}^*(\nabla \phi(w_t)) - \nabla \hat{\phi}^*(\nabla \phi(w_t) - \eta \ell_t) \|_1 \| \ell_t \|_\infty \\
\leq \| (\nabla \phi(w_t)) - (\nabla \phi(w_t) - \eta \ell_t) \|_\infty \| \ell_t \|_\infty \\
= \eta \| \ell_t \|_\infty^2 \\
\leq \eta
\]
where the first inequality follows from the Cauchy-Schwarz inequality, the second inequality is due to (62) and the fact that the dual norm of \( \| \cdot \|_1 \) is \( \| \cdot \|_\infty \), and the last inequality holds since \( \ell_t \in [0, 1]^K \).

G.4 Proof of Lemma 4
The proof is similar to that of Lemma 3 in Appendix G.3. Fix \( t \in [T] \). Focusing on Step 3 of Algorithm 3 and following the same derivation as in (64), we have
\[
w_t = \arg \min_{w \in \Delta_K} \langle w, \eta \ell_{t-1} \rangle + D_\phi(w, \tilde{w}_t) \\
= \arg \min_{w \in \Delta_K} \langle w, \eta \ell_{t-1} \rangle + \phi(w) - \langle w, \nabla \phi(\tilde{w}_t) \rangle \\
= \arg \max_{w \in \Delta_K} \langle w, \nabla \phi(\tilde{w}_t) - \eta \ell_{t-1} \rangle - \phi(w) \\
= \arg \max_{w \in \Delta_K} \langle w, \nabla \phi(\tilde{w}_t) - \eta \ell_{t-1} \rangle - \hat{\phi}(w) \\
= \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_{t-1}).
\]
Similarly, by Step 5 of Algorithm 3, we also have
\[
\tilde{w}_{t+1} = \arg \max_{w \in \Delta_K} \langle w, \nabla \phi(\tilde{w}_t) - \eta \ell_t \rangle - \hat{\phi}(w) \\
= \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_t).
\]
Utilizing the above two equalities and realizing that the dual norm of \( \| \cdot \|_1 \) is \( \| \cdot \|_\infty \), we finish the proof as follows:
\[
\langle w_t - \tilde{w}_{t+1}, \ell_t - \ell_{t-1} \rangle \\
= \langle \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_{t-1}) - \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_t), \ell_t - \ell_{t-1} \rangle \\
\leq \| \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_{t-1}) - \nabla \hat{\phi}^*(\nabla \phi(\tilde{w}_t) - \eta \ell_t) \|_1 \| \ell_t - \ell_{t-1} \|_\infty \\
\leq \| (\nabla \phi(\tilde{w}_t) - \eta \ell_{t-1}) - (\nabla \phi(\tilde{w}_t) - \eta \ell_t) \|_\infty \| \ell_t - \ell_{t-1} \|_\infty \\
= \eta \| \ell_t - \ell_{t-1} \|_\infty^2
\]
where the first inequality is due to the Cauchy-Schwarz inequality, and the second inequality follows from (62).
G.5 Proof of Lemma 5

Fix \( t \in [T] \). By the definition of \( \bar{U}_t^* \) in (56), we have

\[
\text{Tr}((\bar{U}_t^* - U_t^*)Z_t) = \frac{S \text{Tr}((-KU_t^* + I)Z_t)}{TK} = \frac{S \text{Tr}(-U_t^*Z_t)}{T} + \frac{S \text{Tr}(Z_t)}{TK}. (65)
\]

Since \( U_t^* \in \Omega_K \) is positive semidefinite, the eigenvalues of \( U_t^* \) are all non-negative, which implies

\[
\|U_t^*\| = \sum_{i=1}^{K} |\lambda_i(U_t^*)| = \sum_{i=1}^{K} \lambda_i(U_t^*) = \text{Tr}(U_t^*) = 1
\]

where \( \lambda_i(\cdot) \) denotes the \( i \)-th eigenvalue. Combining this with the fact that \( \|Z_t\|_* \leq 1 \) and \( \| \cdot \| \) is the dual norm of \( \| \cdot \|_* \), by the Cauchy-Schwarz inequality, we get

\[
\text{Tr}(-U_t^*Z_t) \leq \| -U_t^* \| \|Z_t\|_* = \|U_t^*\| \|Z_t\|_* \leq 1. (66)
\]

On the other hand, we have

\[
\text{Tr}(Z_t) = \sum_{i=1}^{K} \lambda_i(Z_t) \leq K. (67)
\]

We finish the proof by combining (65), (66), and (67).

G.6 Proof of Lemma 6

Given a \( K \times K \) symmetric and real matrix \( W \), let \( W = V\Lambda V^T \) be the eigendecomposition of \( W \), where \( V \) is an orthogonal matrix whose columns are the eigenvectors of \( W \), and \( \Lambda \) is a diagonal matrix whose entries are the eigenvalues of \( W \). We define \( \exp(\Lambda) \) to be a diagonal matrix with \( (\exp(\Lambda))_{ii} = \exp(\Lambda_{ii}) \) and define \( \exp(W) \) by

\[
\exp(W) = V \exp(\Lambda)V^T.
\]

Following the proof of Theorem 2 in Appendix C, we introduce \( P_{t+1} \in \Omega_K \) defined by

\[
P_{t+1} = \frac{\exp(\log W_t + \log (I - \eta Z_t))}{\text{Tr}\left(\exp(\log W_t + \log (I - \eta Z_t))\right)}
\]

and rewrite \( A_t \) as

\[
A_t = \text{Tr}\left((W_t - W_{t+1})(\eta Z_t)\right) + \text{Tr}\left(W_{t+1}(\eta Z_t + \log (I - \eta Z_t) - \log W_{t+1} + \log W_t)\right)
\]
\[
= \text{Tr}(\eta W_t Z_t) + \text{Tr}\left(W_{t+1}(\log (I - \eta Z_t) - \log W_{t+1} + \log W_t)\right)
\]
\[
= \text{Tr}(\eta W_t Z_t) + \text{Tr}\left(W_{t+1}(\log (I - \eta Z_t) - \log P_{t+1} + \log W_t)\right)
\]
\[
+ \text{Tr}(W_{t+1}(\log P_{t+1} - \log W_{t+1}))
\]
\[
= \text{Tr}(\eta W_t Z_t) + \text{Tr}\left(W_{t+1}(\log (I - \eta Z_t) - \log P_{t+1} + \log W_t)\right) - D_\psi(W_{t+1}\|P_{t+1})
\]

where the last equality follows from the definition of Bregman divergence with respect to \( \psi \) in (21).
Define $Q_t = \log W_t + \log (I - \eta Z_t)$. Let $Q_t = V\Lambda V^T$ be the eigendecomposition of $Q_t$. We have

$$P_{t+1} = \frac{\exp(Q_t)}{\text{Tr} (\exp(Q_t))} = \frac{\exp(V\Lambda V^T)}{\text{Tr} (\exp(V\Lambda V^T))} = \frac{V \exp(\Lambda)V^T}{\text{Tr} (V \exp(\Lambda)V^T)}. \quad (69)$$

Since $(V \exp(\Lambda)V^T)V = (V \exp(\Lambda))(V^TV) = V \exp(\Lambda)$, we know that the entries of the diagonal matrix $\exp(\Lambda)$ are the eigenvalues of $V \exp(\Lambda)V^T$ and thus

$$\text{Tr}(V \exp(\Lambda)V^T) = \sum_{i=1}^{K} \lambda_i (V \exp(\Lambda)V^T) = \text{Tr}(\exp(\Lambda)) \quad (70)$$

where recall that $\lambda_i(\cdot)$ denotes the $i$-th eigenvalue. Substituting the above equality into (69), we get

$$P_{t+1} = \frac{V \exp(\Lambda)V^T}{\text{Tr}(\exp(\Lambda))} = V \frac{\exp(\Lambda)}{\text{Tr}(\exp(\Lambda))}V^T.$$

Denoting $r = \text{Tr}(\exp(\Lambda))$, we have

$$\log P_{t+1} = V \log \left( \frac{\exp(\Lambda)}{r} \right) V^T = V(\Lambda - (\log r)I)V^T$$

$$= V\Lambda V^T - (\log r)VV^T = Q_t - (\log r)I$$

which, together with the definition of $Q_t$, implies

$$\log (I - \eta Z_t) + \log W_t - \log P_{t+1} = Q_t - \log P_{t+1} = (\log r)I$$

and hence

$$\text{Tr}\left(W_{t+1}(\log (I - \eta Z_t) - \log P_{t+1} + \log W_t)\right) = (\log r) \text{Tr}(W_{t+1}) = \log r \quad (71)$$

where the last equality holds since $W_{t+1}$ belongs to the clipped spectraplex $\tilde{\Omega}_K$ defined in (19).

It remains to investigate the upper bound of $r$. To this end, by (70) and the definition of $Q_t$, we rewrite $r$ as

$$r = \text{Tr}(V \exp(\Lambda)V^T) = \text{Tr}(\exp(Q_t)) = \text{Tr}\left(\exp\left(\log W_t + \log (I - \eta Z_t)\right)\right). \quad (72)$$

To proceed, we introduce the Golden-Thompson inequality (Golden, 1965; Thompson, 1965): for any symmetric matrices $A$ and $B$,

$$\text{Tr}\left(\exp (A + B)\right) \leq \text{Tr}\left(\exp (A) \exp (B)\right).$$

Applying this inequality to (72) gives

$$r \leq \text{Tr}\left(\exp \left(\log W_t\right) \exp \left(\log (I - \eta Z_t)\right)\right) = \text{Tr}\left(W_t(I - \eta Z_t)\right) = 1 - \text{Tr}(\eta W_t Z_t) \quad (73)$$
where the last equality holds since $\text{Tr}(W_t I) = \text{Tr}(W_t) = 1$. Combining (73) with (71) and (68), we get

\[
A_t = \text{Tr}(\eta W_t Z_t) + \log r - D_\psi(W_{t+1}\|P_{t+1}) \\
\leq \text{Tr}(\eta W_t Z_t) + \log (1 - \text{Tr}(\eta W_t Z_t)) - D_\psi(W_{t+1}\|P_{t+1}) \\
\leq \text{Tr}(\eta W_t Z_t) + \log (1 - \text{Tr}(\eta W_t Z_t))
\]

where the last inequality holds since Bregman divergence is always non-negative.

Finally, note that $\|W_t\| = 1$, $\|Z_t\|_* \leq 1$ and $\| \cdot \|$ is the dual norm of $\| \cdot \|_*$. Application of the Cauchy-Schwarz inequality gives

\[
\text{Tr}(\eta W_t Z_t) = \eta \text{Tr}(W_t Z_t) \leq \eta \|W_t\|\|Z_t\|_* \leq \eta \leq \frac{1}{2}.
\]

We conclude the proof by recalling the well-known inequality: $x + \log (1 - x) \leq 0$, $\forall x < 1$.

**G.7 Proof of Lemma 7**

We start by rewriting $B_t$ as

\[
B_t = \text{Tr}\left(-U_t^*(\eta Z_t + \log (I - \eta Z_t))\right) \\
= \text{Tr}\left(-U_t^*(\eta Z_t + \log (I - \eta Z_t))\right) + \text{Tr}\left((U_t^* - U_t^*) (\eta Z_t + \log (I - \eta Z_t))\right). \quad (74)
\]

We first focus on bounding the last term. By the definition of $U_t^*$ in (56), we have

\[
\text{Tr}\left((U_t^* - U_t^*) (\eta Z_t + \log (I - \eta Z_t))\right) = \frac{S \text{Tr}\left((KU_t^* - I) (\eta Z_t + \log (I - \eta Z_t))\right)}{TK} \\
= \frac{S \text{Tr}\left(U_t^*(\eta Z_t + \log (I - \eta Z_t))\right)}{T} + \frac{S \text{Tr}\left(-\eta Z_t - \log (I - \eta Z_t)\right)}{TK}. \quad (75)
\]

Let $Z_t = V \Lambda V^T$ be the eigendecomposition of $Z_t$. We have

\[
\eta Z_t + \log (I - \eta Z_t) = \eta V \Lambda V^T + \log (I - \eta V \Lambda V^T) = \eta V \Lambda V^T + \log (V V^T - \eta V \Lambda V^T) \\
= \eta V \Lambda V^T + \log (V (I - \eta \Lambda) V^T) = \eta V \Lambda V^T + V \log (I - \eta \Lambda) V^T \\
= V \left(\eta \Lambda + \log (I - \eta \Lambda)\right) V^T.
\]

For any $i \in [K]$, let $a_i$ be the $i$-th diagonal entry of $\Lambda$. Since $\|Z_t\|_* \leq 1$, we have $|a_i| \leq 1$ and $|\eta a_i| \leq \eta \leq 1/2$. Utilizing the inequality $x + \log (1 - x) \leq 0$, $\forall x < 1$, we get

\[
\eta a_i + \log (1 - \eta a_i) \leq 0
\]

which implies the eigenvalues of $\eta Z_t + \log (I - \eta Z_t)$ are all non-positive, and $\eta Z_t + \log (I - \eta Z_t)$ is hence negative semidefinite. Combining this with the fact that $U_t^*$ is positive semidefinite, we conclude that the eigenvalues of $U_t^* (\eta Z_t + \log (I - \eta Z_t))$ are all non-positive, which
implies

\[ \text{Tr} \left( U_t^* (\eta Z_t + \log (I - \eta Z_t)) \right) = \sum_{i=1}^{K} \lambda_i \left( U_t^* (\eta Z_t + \log (I - \eta Z_t)) \right) \leq 0. \quad (76) \]

On the other hand, by the inequality 

\[ -x - \log (1 - x) \leq |x|/2, \quad \forall x \in [-1/2, 1/2], \]

we have 

\[ -\eta a_i - \log (1 - \eta a_i) \leq \frac{|\eta a_i|}{2} \leq \frac{\eta}{2} \]

and

\[ \text{Tr} \left( -\eta Z_t - \log (I - \eta Z_t) \right) = \text{Tr} \left( V (-\eta \Lambda - \log (I - \eta \Lambda)) V^T \right) \]

\[ = \sum_{i=1}^{K} \left( -\eta a_i - \log (1 - \eta a_i) \right) \]

\[ \leq \sum_{i=1}^{K} \frac{\eta}{2} = \frac{K \eta}{2}. \]

Combining the above inequality with (76) and (75) gives

\[ \text{Tr} \left( (U_t^* - \bar{U}_t^*) (\eta Z_t + \log (I - \eta Z_t)) \right) \leq \frac{\eta S}{2T}. \quad (77) \]

We now turn to bound \( \text{Tr} \left( -U_t^* (\eta Z_t + \log (I - \eta Z_t)) \right) \). To this end, we introduce the following fact (Steinhardt and Liang, 2014): for any symmetric and real matrix \( X \) satisfying 

\[ -\frac{I}{2} \preceq X \preceq \frac{I}{2}, \]

we have 

\[ -X - X^2 \preceq \log (I - X) \]

where \( A \preceq B \) means that \( B - A \) is positive semidefinite. Since \( \|Z_t\|_* \leq 1 \) and \( \eta \in (0, 1/2] \), we know that the maximum absolute eigenvalue of \( \eta Z_t \) is not more than 1/2, i.e., \( \max_{i \in [K]} |\lambda_i(\eta Z_t)| \leq 1/2 \). Therefore, we have 

\[ -\frac{I}{2} \preceq \eta Z_t \leq \frac{I}{2} \]

and

\[ -\eta Z_t - \eta^2 Z_t^2 \preceq \log (I - \eta Z_t) \]

which implies \( \log (I - \eta Z_t) + \eta Z_t + \eta^2 Z_t^2 \) is positive semidefinite. Combining this with the fact that \(-U_t^* \) is negative semidefinite, we conclude that the eigenvalues of \(-U_t^* \left( \log (I - \eta Z_t) + \eta Z_t + \eta^2 Z_t^2 \right)\) are all non-positive and hence

\[ \text{Tr} \left( -U_t^* \left( \log (I - \eta Z_t) + \eta Z_t + \eta^2 Z_t^2 \right) \right) \leq 0. \]

Rearranging the above inequality, we obtain

\[ \text{Tr} \left( -U_t^* (\eta Z_t + \log (I - \eta Z_t)) \right) \leq \eta^2 \text{Tr} \left( U_t^* Z_t^2 \right). \]

Substituting the above inequality and (77) into (74) completes the proof.
G.8 Proof of Lemma 8

We start by proving the following fact: for any \( W \in \tilde{\Omega}_K \), we have
\[
-\log \left( \frac{KT}{S} \right) \leq \lambda_i(\log W) \leq 0, \quad \forall i \in [K].
\]  
(78)

where \( \lambda_i(\cdot) \) denotes the \( i \)-th eigenvalue.

**Proof.** Fix \( W \in \tilde{\Omega}_K \). Let \( W = V \Lambda V^T \) be the eigendecomposition of \( W \). It follows that
\[
\log W = V(\log \Lambda)V^T
\]
which implies that the diagonal entries of \( \log \Lambda \) are the eigenvalues of \( \log W \). For any \( i \in [K] \), let \( a_i = \Lambda_{ii} \) denote the \( i \)-th diagonal entry of \( \Lambda \). Since \( a_i \) is the eigenvalue of \( W \in \tilde{\Omega}_K \), by the definition of \( \tilde{\Omega}_K \) in (19), we have
\[
\frac{S}{TK} \leq a_i \leq 1
\]
and hence
\[
-\log \left( \frac{KT}{S} \right) \leq \log a_i \leq 0.
\]

We finish the proof by noticing that \( \lambda_i(\log W) = (\log \Lambda)_{ii} = \log (\Lambda_{ii}) = \log a_i \). \( \square \)

We are now ready to prove Lemma 8. Fix \( X, Y, Z \in \tilde{\Omega}_K \). First, applying (78) to \( Y \) and \( Z \) indicates that \( \log Y \) is negative semidefinite and \( \log Z \) satisfies
\[
\| \log Z \|_* = \max_{i \in [K]} |\lambda_i(\log Z)| \leq \log (KT/S).
\]

Then, we expand \( \text{Tr}(X(\log Y - \log Z)) \) as
\[
\text{Tr}(X(\log Y - \log Z)) = \text{Tr}(X \log Y) + \text{Tr}(-X \log Z).
\]  
(79)

Since \( X \in \tilde{\Omega}_K \) is positive semidefinite, we conclude that the eigenvalues of \( X \log Y \) are all non-positive and thus
\[
\text{Tr}(X \log Y) = \sum_{i=1}^{K} \lambda_i(X \log Y) \leq 0.
\]  
(80)

Finally, application of the Cauchy-Schwarz inequality gives
\[
\text{Tr}(-X \log Z) \leq \| -X \| \| \log Z \|_* = \| X \| \| \log Z \|_* = \| \log Z \|_* \leq \log (KT/S).
\]  
(81)

Combining (79), (80), and (81) completes the proof.