Constructing balanced equations of motion for particles in general relativistic theories: the general case

Emanuel Gallo and Osvaldo M. Moreschi
Facultad de Matemática, Astronomía, Física y Computación (FaMAF),
Universidad Nacional de Córdoba,
Instituto de Física Enrique Gaviola (IFEG), CONICET,
Ciudad Universitaria,(5000) Córdoba, Argentina.

March 15, 2018

Abstract

We present a general approach for the formulation of equations of motion for compact objects in general relativistic theories. The particle is assumed to be moving in a geometric background which in turn is asymptotically flat. Our approach defines a model for particle like objects which emphasizes different aspects of a set of gravitating objects; by concentrating in the main contributions coming from back reaction effects due to gravitational radiation.

1 Introduction

Recently, the first direct detections of gravitational waves\cite{1,2,3,4,5,6,7} have been announced. Thus a new way to observe the Universe and to receive information from it has become operative. In particular, with current detectors we expect to learn more about astrophysical processes involving systems with presence of strong gravitational fields and high relative velocities among compact bodies like black holes or neutron stars.

Although the detectors were not operating at maximum power and sensitivity the first detected signal was strong enough to be easily observed. When in the near future, the detectors acquire the expected designed sensitivity there will be even better defined signals in which the fine physical interpretation of the data will become an active area of research. Furthermore, with the higher sensitivity it comes the bigger cosmic volume that the detectors can explore, and therefore one also expect to observe a variety of systems with many different physical situations. The work for full numerical relativity calculations will probably have to extend to unexpected initial conditions. Under this circumstances the need to estimate the most appropriate initial conditions of the system, will require for physical models that can handle strong gravity and back reaction in the dynamics.

Among the frequently approximate methods used we can mention the post-newtonian methods\cite{8,9,10}, the self-force approach\cite{11,12}, and models based on effective one body approach\cite{13} techniques, etc. In particular for the case of a binary system of black holes, it is a good starting model to think the system as consisting of point particles whose dynamics is described in a flat background with corrections due to the perturbations originated by the rest of the components of the system and the particles itself. All these approaches have their advantages and limitations, namely validity range, ease of computation, etc. However in order to arrive to these kind of models one needs to settle several issues, as for example what is the notion of particles in the general relativity framework that one should use.

Several related questions arise: what is the convenient structure that one should assume for the corresponding particle? in what way one could refer to the related equations of motion? how should one determine the corresponding equations of motion?

In reference \cite{18} we have presented a new derivation of the equation of motion of charged particles in a flat background, interacting with an non-radiating external field, which balance the electromagnetic radiation. The new equations of motion is

\[
(\dot{m} - \ddot{\alpha}) v^a + m \ddot{v}^a + \alpha \dddot{v}^a = eF(B)^b_a v^b + \frac{2}{3} e^2 \dot{v}^b \dot{v}_b v^a; \tag{1}
\]

where we are using abstract indices \(a, b, \ldots\), and beyond the standard mass \(m\), four velocity \(v^a\), charge \(e\) and electromagnetic field tensor \(F\), there appears the new degree of freedom \(\alpha\), depending on the proper time \(\tau\), a dot over a scalar means derivative with respect to \(\tau\) and over a vector means the covariant derivative in the direction of the vector \(v^a\).

This equation can also be decomposed in terms of its projection in the direction of the velocity

\[
\dot{m} - \ddot{\alpha} + \alpha \dddot{v}^a = -\frac{2}{3} e^2 \dot{v}^a; \tag{2}
\]

and its orthogonal complement

\[
m \dot{v}^a = eF(B)^b_a v^b - \alpha(\dot{v}^a - \dot{v}^a \dddot{v}^a). \tag{3}
\]

We have shown that this approach is successful for the derivation of the equations of motion, since it agrees with...
the well known Lorentz-Dirac dynamics for a particular case.

It is our goal to construct the analog to equation (1) for the gravitational case; where the variation of total momentum per unit time is balanced with radiation at infinity and some delicate issues related to this problem are solved. The present approach for building the equations of motion has in mind that one is concerned with the first order back reaction effects coming from the fact that accelerated masses radiate gravitationally. When dealing with complex systems, if one is interested in higher order effects, then one should improve the model to the appropriate order. We here focus on the presentation for the construction of the equations of motion that take into account the first order back reaction effects due to gravitational radiation, within a wide class of general relativistic theories; previous to the fixing of a particular field equation. We do assume however that the energy momentum tensor is divergent free, and that isolated bodies are well represented by asymptotically flat spacetimes.

One may ask why another work should be studied that considers the momentum balance as a starting point for the derivation of the equations of motion. However, in this article we deal with several aspects that are not usually discussed in the literature, and what is more, regardless of the final field equations.

Of course referring to particles in the context of general relativity raises several issues. To begin with, the concept of particle involves the idea of a very compact object; but when one makes an object smaller and smaller, one does not end with a point like object but with a black hole. Secondly, if one defines particles as objects with a distributional energy momentum tensor with support on a timelike world line, one faces the problem of regularity of the spacetime. In this respect in reference (20) Geroch and Traschen define a notion of regular manifolds that admit distributions; and one dimensional distributions are not allowed in 4-dimensional manifolds. But one can always consider one dimensional distributions as the sources of the linear field equation on backgrounds; and this is the approach we use here.

In particular, we will present here the formal aspect of the basic concepts inherent this paper as follows. The basic concepts of particles as isolated objects in special relativity that include center of mass, intrinsic angular momentum and distributions; with the intention to clarify in detail the fundamental concepts that we are generalizing to the dynamics of a compact object, and so prevent future explanatory discussions.

So, since we are warning of the basic level of this section, we advice, for the reader interested in getting immediately into the subject, to just jump to the third following section.

2 Preliminaries

In this section we review some basic concepts in special relativity that include center of mass, intrinsic angular momentum and distributions; with the intention to clarify in detail the fundamental concepts that we are generalizing to the dynamics of a compact object, and so prevent future explanatory discussions.

So, since we are warning of the basic level of this section, we advice, for the reader interested in getting immediately into the subject, to just jump to the third following section.

2.1 Particles as isolated objects in special relativity

The notion of a point like object in a fixed background is normally presented through an energy momentum tensor which has support only on a timelike world line. The first case one should understand is of course the flat background case.

Consider the situation in which a distribution of matter that has support within a world tube of finite size. Later we will take the limit in which the tube collapses
to a timelike line.

Having a flat background we have at our disposal the translational Killing vectors \( K_{\bar{a}} = \frac{\partial}{\partial x^\bar{a}} \); with \( \bar{a} = 0, 1, 2, 3 \), and also the rotational Killing vectors \( K_{ab}(\xi) = \eta_{\bar{a}c}(x^\bar{c} - \xi^\bar{c}) \frac{\partial}{\partial x^\bar{a}} - \eta_{\bar{b}c}(x^\bar{c} - \xi^\bar{c}) \frac{\partial}{\partial x^\bar{b}} \); which are the generator of Lorentz rotations around the point \( \xi^\bar{a} \).

The metric is represented by the energy-momentum tensor \( T_{\bar{a}b} \). Note that we are using the notation \( a, b, c \) for abstract indices while underlined indices are numeric. Then, at zero order, the energy-momentum tensor satisfies that its divergence is zero, namely:

\[
\nabla \cdot T = 0.
\]

(4)

Let \( K^a \) denote any of the Killing symmetries, then, for each symmetry one has a conserved quantity; in fact it has been explained elsewhere \( \ref{21} \) \( \ref{18} \) how these conserved quantities can be related to spheres that ‘surround’ the sources. Let us define the three form \( D_{abc}(K) = T^d_{
abla f}(K) \epsilon_{abcd} \); then its exterior derivative is \( dD_{abcd} = k \nabla f(T^l_{\bar{a}} K^\bar{c}) \epsilon_{abcd} \), where \( k \) is a constant. This exterior derivative vanishes due to the fact that the divergence of \( T \) is zero and \( K \) is a Killing symmetry. Then if \( \Sigma \) is a three dimensional region, such that the world tube goes through its interior, the quantity

\[
Q = \int_\Sigma D(K);
\]

(5)

is conserved; in the sense that if \( \Sigma' \) is any other three dimensional region, such that the world tube goes through its interior, then, the integration of \( D \) on \( \Sigma' \) also gives \( Q \). Another way of seeing this is the following: if the boundary of \( \Sigma \) is the sphere \( S \), then for any other hypersurface \( \Sigma' \) with the same boundary the calculation of the quantity \( Q \) will give the same value; so that at the end, one associates \( Q \) to a given surface \( S \). In this work, we use analogous ideas for the case of gravitating objects, and we will think of these spheres as residing at future null infinity; where the total momentum and flux of gravitational radiation are evaluated.

Using these structure one defines \( P_{\bar{a}} \) by

\[
P_{\bar{a}} = \int_\Sigma D(K_{\bar{a}});
\]

(6)
as the components of the conserved total momentum, and also

\[
J_{ab}(\xi) = \int_\Sigma D(K_{ab}(\xi));
\]

(7)
as the components of the conserved total angular momentum.

Let us observe that

\[
J_{ab}(\xi_2) - J_{ab}(\xi_1) = \eta_{\bar{a}c}(\xi^\bar{c}_2 - \xi^\bar{c}_1) P_{\bar{a}} - \eta_{\bar{b}c}(\xi^\bar{c}_2 - \xi^\bar{c}_1) P_{\bar{b}};
\]

(8)
or in other words

\[
J_{ab}(\xi_2) = J_{ab}(\xi_1) + (\xi^\bar{c}_2 - \xi^\bar{c}_1) P_{\bar{c}} - (\xi^\bar{c}_1 - \xi^\bar{c}_2) P_{\bar{c}}.
\]

(9)

It is probably worthwhile to notice that if one acts with Lorentz transformations on the initial frame \( e_\alpha \equiv \frac{\partial}{\partial x^\alpha} \), then the set of quantities \( P_{\bar{a}} \) transforms as components of a co-vector, while the set \( J_{ab} \) transforms as the components of an antisymmetric tensor.

Let us emphasize again that \( J_{ab}(\xi) \) is conserved for any \( \xi \). But one could consider, for example \( \xi_1(\tau_1) \) as a timelike world line with proper time \( \tau_1 \). Therefore, calling \( v_1^\alpha = \frac{\partial}{\partial \tau_1} \) one has

\[
\frac{d}{d\tau_1} J_{ab}(\xi_1) = v_1^\alpha P_{\bar{a}b} - v_1^b P_{\bar{a}};
\]

(10)

Note that if one chooses \( v_1 \) to be parallel to \( P \), then

\[
\frac{d}{d\tau_1} J_{ab}(\xi_1) = 0.
\]

(11)

Also, let us note that fixing \( \xi_2 \), one has

\[
J_{ab}(\xi_1)P_{\bar{b}c} = J_{ab}(\xi_2)P_{\bar{b}c} - (\xi^\bar{c}_1 - \xi^\bar{c}_2) P^2 + (\xi^\bar{c}_1 - \xi^\bar{c}_2) P_{\bar{c}} P_{\bar{c}};
\]

(12)

so that one can always pick a \( \xi^\bar{c}_2 \) so that

\[
J_{ab}(\xi_1)P_{\bar{b}c} = 0.
\]

(13)

Using a rest frame, in which the generator of time translations is aligned with the total momentum \( P_\alpha \); in other words, its spacelike components are zero; i.e., \( P^i = 0 \) with \( i = 1, 2, 3 \); one has \( J^{0i}(\xi_1) = 0 \) and so

\[
\int_\Sigma (\xi^\bar{c}_1 - \xi^\bar{c}_2) T^{0\bar{c}} \epsilon_{\bar{c}} = \int_\Sigma (\xi^\bar{c}_1 - \xi^\bar{c}_2) T^{0\bar{c}} \epsilon_{\bar{c}} = (\xi^\bar{c}_1 - \xi^\bar{c}_2) P^\alpha = 0;
\]

(14)

and therefore one arrives at the center of mass coordinates

\[
\xi^\bar{c}_m = \frac{1}{M} \int_\Sigma x^\bar{c} T^{0\bar{c}} \epsilon_{\bar{c}};
\]

(15)

where now \( \Sigma \) is an adapted hypersurface \( x^0 = \)constant, \( \epsilon_{\bar{c}} \) is the volume element, and we have used \( P^0 = M \) in the rest frame. For this reason condition \( \ref{13} \) is used \( \ref{22} \) for defining center of mass in special relativity. Then all other points that are translated parallel to \( P \) are also center of mass points; which means that the center of mass world line has velocity \( v = \frac{P}{M} \).

Now, let us consider the case in which the world tube gets smaller until it collapses to the timelike world line \( z \). In this case \( x^\bar{a} \) is compelled to be evaluated at the world line; and since \( P^0 = M \), one has that the center of mass \( \xi^\bar{c}_m \) is at the world line \( z \).

2.2 The intrinsic angular momentum

The intrinsic angular momentum \( S_{ab} \) is defined as the angular momentum evaluated at the center of mass, namely:

\[
S_{ab} = J_{ab}(\xi_m).
\]

(16)

Since any contraction of \( S_{ab} \) with \( P^\bar{a} \) gives zero, there are only three degrees of freedom involved in the intrinsic angular momentum; so that, similarly as the magnetic field is defined in terms of the electromagnetic tensor and the volume element, one defines the spin vector from

\[
S_{\bar{a}} = \frac{1}{2} \epsilon_{\bar{a}bcd} J_{bcd}(\xi) P_{\bar{d}};
\]

(17)
which by construction is orthogonal to $P^a$, and so it only involves three degrees of freedom. This is also known as the Pauli-Lubanski pseudovector.

In [17] we have emphasized the fact that the spin vector can be calculated with respect to any choice of reference center $\xi$; since the contraction with the momentum vector eliminates this dependence.

Then one can think in reconstructing the complete angular momentum tensor from the spin vector. Let us note that

$$
\epsilon^{abcd}S_{ab}P_{cd} = \frac{1}{2} \epsilon^{abcd} \rho^c\sigma^d \eta_{ab} P^c
= - \left( M^2 J^a_{\xi} + P^a_{\xi} J^a_{\xi} - P^a_{\xi} J^a_{\xi} \right)
$$

(18)

One can see then that the intrinsic angular momentum tensor, in terms of the spin vector, is given by

$$
S^{ab} = \frac{1}{M^2} \epsilon^{abcd} P_{cd} S_{\xi}.
$$

(19)

Although we have previously introduced first the concept of center of mass and then the one of intrinsic angular momentum; this last equation allows us to define the center of mass, in terms of the intrinsic angular momentum, in the following way:

- Let us first define the intrinsic spin vector $S^a$ by:
  $$
  S^a = \frac{1}{2} \epsilon^{abcd} \rho^c\sigma^d (\text{any } \xi) P^a.
  $$

- Then let us define the intrinsic angular momentum tensor $S^{ab}$ from:
  $$
  S^{ab} = \frac{1}{M^2} \epsilon^{abcd} P_{cd} S_{\xi}.
  $$

Let us emphasize that this definition only employs intrinsic properties of the system, i.e.; it does not recourse to any additional structure.

- Finally the center of mass points are defined as those that satisfy:
  $$
  S^{ab} = J_{ab}(\xi_{cm}).
  $$

We conclude then that there is a one to one relation between the concept of center of mass and intrinsic angular momentum.

Then the total angular momentum with respect to an arbitrary center $\xi$ can be expressed in terms of the intrinsic angular momentum and the center of mass by:

$$
J^{a\xi}(\xi) = S^{ab} + P^a_{\xi} \left( \xi^b - \xi^b_{cm} \right) - P^b_{\xi} \left( \xi^a - \xi^a_{cm} \right).
$$

(20)

One can see that the spin vector $S^a$ is also conserved along the world line.

It is also clear from this analysis that elementary particles can have momentum and spin but not dipole; as it has been frequently assumed in the literature. By dipole we precisely mean $d^a = S^{ab} P_b$.

We have not considered here the case of electrically charged particles since the energy momentum tensor we have considered are of finite space support.

Of course one could consider composite particles which have extra structure, beyond momentum and spin, and then one is free to include the extra dynamics for the extra degrees of freedom, compatible with the conservation of the energy momentum tensor.

### 2.3 Particles as compact objects in curved spacetimes

One can use the concept of particle when the dimension of the object and its structure is such that only the monopole nature is enough to describe its dynamics. In the language of Mathisson [23 24], the dynamical skeleton is described by a monopole.

Let us assume that in a neighborhood of the timelike curve $C$ we have at our disposal coordinates $x^a$, so that the curve is given by $x^a = z^a(\tau)$, with $t^a d\tau > 0$. At each point along the curve $C$ we can express a 3-dimensional delta function $\delta^{(3)}$, in terms of local coordinates, with support on $C$ by

$$
\delta^{(3)}(x^1 - z^1(\tau), (x^2 - z^2(\tau), (x^3 - z^3(\tau)))
= \frac{1}{\sqrt{-\det g}} \delta(x^1 - z^1(\tau)) \delta(x^2 - z^2(\tau)) \delta(x^3 - z^3(\tau));
$$

(21)

where $\det g$ is the determinant of the four metric.

Then, an energy momentum tensor for a monopole particle with support on a timelike worldline $C$ is represented [12 11] by

$$
T_{ab}(x^0 = z^0(\tau), x^1, x^2, x^3) = M v_a(\tau) v_b(\tau)
= \frac{1}{\sqrt{-\det g}} \delta(x^1 - z^1(\tau)) \delta(x^2 - z^2(\tau)) \delta(x^3 - z^3(\tau));
$$

(22)

Below we will specify further the expression that is used in this work.

In the following sections we present the construction of the model.

### 3 Local dynamics at zero order in back reaction

As mentioned above, one of the objectives of the present work is to discuss several general issues that must be taken into account in the construction of the equations of motion that can be applied to a binary system, in the dynamical situation in which the back reaction effects due to gravitational radiation are important. The model is described in terms of an asymptotically flat spacetime at future null infinity. For this reason we concentrate in one particle, to which we assign the label $A$. The rest of the system, which it could be another particle is assigned label $B$.

It is our intention to construct a model for a binary system that takes into account the leading order contributions to the equations of motion. This is enough to
force us to discuss a variety of issue that appear in this type of problems. In order to obtain a high degree of precision in the predicting power of the equations of motion, one should extend this work to higher orders. In separate works we will present what we call first order equations of motion, in the harmonic gauge, and second order equations of motion in the null gauge. This instead is the general treatment of the first order version of the balanced equations of motion.

Let the metric \( g \) be the exact metric that corresponds to an isolated binary system of compact objects. Then there are infinite ways in which it can be decomposed in a form:

\[
g = \eta + h_A + h_B + h_{AB};
\]

where \( \eta \) is a flat metric, \( h_A \) is proportional to a parameter \( M_A \), that one can think is some kind of measure of the mass of system \( A \), similarly \( h_B \) is proportional to a parameter \( M_B \), and \( h_{AB} \) is proportional to both parameters.

Since we will deal with back reactions for each body, we need to single out, at future null infinity, the gravitational radiation corresponding to each object. Therefore, we need, at this stage of the dynamics, to be able to distinguish terms in the geometry that correspond to each of the binary components.

To study the gravitational radiation emitted by the motion of particle \( A \), we will model the asymptotic structure of a sub-metric

\[
g_A = \eta + h_A;
\]

in order to calculate the reaction due to gravitational radiation.

We adjust \( g_A \) to model a compact object, which dynamics is affected by the existence of system \( B \), and therefore by the geometry determined by a sub-metric

\[
g_B = \eta + h_B.
\]

The appropriate choice of the flat metric \( \eta \) was discussed in article [25]; the basic idea is that it should be related to a local notion of center of mass frame.

For the back reaction calculations, we need to capture the leading order behavior of the metric; although taking \( h_A \) to agree in the linear regime with \( h_A \) and also \( h_B \) to agree in the leading order with \( h_B \) is enough for first order back reaction, gravitational radiation effects.

Although in this work we will mainly use the geometries (24) and (25), one can think that the model represents the combined system by a leading order approximation of the spacetime metric of the form

\[
g = \eta + h_A + h_B;
\]

which is also assumed to be asymptotically flat at future null infinity; so that one can discuss the effects of gravitational radiation, and the metric can also be expressed as (25). For the purpose of accuracy in the dynamical calculations, we need \( h_A \) and \( h_B \) to represent as close as possible the exact geometry of the binary system; and therefore one should pursue this approach to higher orders; trying to get as close as possible to the exact representation of the metric.

Although we will be studying the dynamics of system \( A \), to simplify the notation we will avoid using a subindex \( A \), whenever possible.

The expression for the momentum of a particle can be deduced from the conservation law for the energy-momentum tensor.

The limit in which one considers particle \( A \) to be ‘small’ enough so that the radiation effects can be neglected, gives rise to the zero order in \( \lambda \) approximation; where \( \lambda \) is a parameter associated to the strength of gravitational radiation, that will be clarified below.

The zero order in \( \lambda \) equations of motion for particle \( A \) comes from the zero order conservation equation for the energy momentum tensor of this particle; namely

\[
\nabla_{(B)a} T_{(A)ab} = 0.
\]

This zero order equations of motion is considered in terms of an energy-momentum which is a distribution on the background geometry \( g_{(B)} \). Then, at this order, for the energy momentum tensor of a monopole body \( A \), one must consider

\[
T_{(A)ab}(x^0 = z^0(\tau), x^1, x^2, x^3) = M_A v_{(A)a}(\tau) v_{(A)b}(\tau) 
\]

\[
\frac{1}{2} \delta(x^1 - z^1(\tau)) \delta(x^2 - z^2(\tau)) \delta(x^3 - z^3(\tau));
\]

where \( \delta g_{(B)} \) is the determinant of the metric \( g_{(B)} \) and \( (x^1, x^2, x^3) \) are local spacelike coordinates.

Consider a small world tube around the world line of the particle. Let \( e^a_b \) be a frame basis (an orthonormal frame with respect to metric \( g_{(B)} \)); where, as before, we are distinguishing between the abstract indices \( (a, b, c, \ldots) \) and the numeric frame indices \( (\alpha, \beta, \gamma, \ldots) \) and let \( \theta^\alpha_\gamma \) be its coframe. Then one can integrate, in the interior of the world tube, the expression

\[
0 = \int \nabla_{(B)a} T_{(A)ab} e^b_c dV^4 - \int \nabla_{(B)b} T_{(A)ab} e^a_c dV^4 - \int T_{(A)ab} \epsilon^b_c \epsilon^a_d dV^3 - \int T_{(A)b} \Gamma^a_{(B)bc} \epsilon^b_c dV^4;
\]

where \( \epsilon^a_c \) is the vector gradient at the future boundary, \( t^a_\alpha \) is the corresponding vector at the past boundary, of an adapted time coordinate \( t = x^0 \), namely \( t^a = g_{(B)ab} dt_b \), and \( \Gamma^a_{(B)bc} \) is the torsion free metric connection of \( g_{(B)} \).

We can always reparameterize the local time \( t \), to adapt it to this geometric construction, so that the future boundary coincide with \( t + d\tau \) and the past boundary coincide with \( t \); where \( d\tau \) is the increment in proper
time along the curve $C$, between the two intersection points of the curve with the future and past boundary hypersurfaces of the world tube. Then carrying out the integration on the right hand side, one obtains

$$
M_A g_B (v (A), e^a_c g_B (v (A), t)]_{t^+ \to t^-} =
$$

$$
- M_A g_B (v (A), e^a_c g_B (v (A), t)]_{t^+ \to t^-} - M_A v^a (A) \Gamma (B) \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} dt;
$$

where all the calculation is done with respect to the background metric $g_B$. Let us note that, from the choice of $t$, we have $g_B (v (A), t) = v (A) (t) = 1$.

Then, the local notion of momentum is obtained, from

$$
P_{(A)E} = M_A g_B (v (A), e^a_c); \tag{31}
$$

so that one has

$$
0 = P_{(A)E} [t^+ \to t^-] - M_A g_{ab} (v (A)) e^a_c e^b_c;
$$

(32)

Dividing by $dt$ and taking the limit to zero, the first two terms constitute the total derivative

$$
\frac{d}{dt} P_{(A)E} = - M_A g_{ab} (v (A)) e^a_c e^b_c \tag{33}
$$

where in the first equality we are using that the vector $v (A)$ has unit modulus with respect to $g_B$. Then subtracting the last term of $\text{[12]}$, one obtains, from $\text{[20]}$, at zero order in back reaction due to gravitational radiation, the well known equation

$$
M_A g_{ab} (v (A)) e^a_c e^b_c = 0. \tag{34}
$$

In what follows we will study the corrections to this equation coming from back reaction effects due to gravitational radiation.

In order to make a connection with the asymptotic discussion we identify the basis field vectors $e^a_c$ with four translational Killing vectors of the background metric in the asymptotic region.

4 General asymptotically flat spacetimes

4.1 Particles as isolated objects in curved spacetimes

When considering a compact object as isolated, in the framework of isolated spacetimes, one realizes that in general one can ascribe a flat background to the spacetime. More concretely, in the asymptotic region one can always write the metric as

$$
g = \eta_{\text{asy}} + h_{\text{asy}}; \tag{35}
$$

where $\eta_{\text{asy}}$ is a flat metric associated to inertial frames in the asymptotic region and $h_{\text{asy}}$ the tensor where all the physical information is encoded.

In reference $\text{[26]}$ we have indicated how to construct a family of inertial Bondi frames in a neighborhood of future null infinity. They include the determination of a null tetrad and associated null coordinate system in the asymptotic region. For each choice of inertial frame, the metric adopts a decomposition of the form $\text{[30]}$. But, the warning is that there are as many inertial frames as there are proper supertranslation generators of the asymptotic BMS\textsuperscript{27} symmetry group. Therefore, there are as many flat metrics $\eta_{\text{asy}}$ as there are proper supertranslation generators. This is the reason why this asymptotic region is not asymptotically Minkowskian; since there are too many flat background geometries. By asymptotically Minkowskian we mean a spacetime which in the asymptotic region is decomposed in terms of a unique flat metric.

Also, one knows that the difficulties in finding appropriate rest frames comes from the existence of gravitational radiation $\text{[28, 29, 30]}$. Although in the past we have found a way to select rest frames based on the notion of center of mass and intrinsic angular momentum $\text{[31, 32]}$, In other words, for each point at future null infinity, we have a way to single out a unique decomposition of the metric in the form $\text{[33]}$, with an appropriately selected flat background.

This complicated structure of rest frames in the asymptotic region and its subtle relation with corresponding symmetries of flat backgrounds in the interior has normally been neglected in works dealing with equations of motion of particles in relativistic theories; on the contrary we choose to take into account these effects at the beginning for the discussion of back reaction on the motion of compact objects.

4.2 The leading order behaviour of an adapted null tetrad

Let now $u$ denotes a null hypersurface that contains future directed null geodesics that reach future null infinity.

One can express the asymptotic geometry in terms of a complex null tetrad $(\ell^\alpha, m^\alpha, \overline{m}^\alpha, n^\alpha)$ with the properties:

$$
g_{ab} \ell^a n^b = - g_{ab} m^a \overline{m}^b = 1, \tag{36}
$$

and all other possible scalar products being zero; so that the metric can be expressed by

$$
g_{ab} = \ell_a n_b + n_a \ell_b - m_a \overline{m}_b - \overline{m}_a m_b. \tag{37}
$$

Here, and whenever dealing with null tetrads we will make use of the GHP\textsuperscript{33} notation.

Using the null polar coordinate system $(x^0, x^1, x^2, x^3) = (u, r, x^1, x^2)$ one can express the null tetrad as:

$$
\ell_a = (du)_a \tag{38}
$$
\[ l^a = \left( \frac{\partial}{\partial y} \right)^a \]  
\[ m^a = \xi^i \left( \frac{\partial}{\partial x^i} \right)^a \]  
\[ n^a = \left( \frac{\partial}{\partial u} \right)^a + U \left( \frac{\partial}{\partial r} \right)^a + X^i \left( \frac{\partial}{\partial x^i} \right)^a \]

with \( i = 2, 3 \).

4.3 The total momentum

Given any section \( S \) at future null infinity, the total momentum of a generic spacetime, in terms of an inertial (Bondi) frame \([20]\), is normally given \([31]\) by

\[ \mathcal{P}^\nu = -\frac{1}{4\pi} \int_S \hat{l}_0^\nu (\Psi_0^0 + \sigma_0 \delta_0) dS^2, \]

where the auxiliary null vector \( \hat{l}_0(x^2, x^3) \), is defined in terms of the angular coordinates \((x^2, x^3)\), by

\[ \hat{l}_0(x^2, x^3) \equiv \left( 1, \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta) \right) = \left( 1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}} \right); \]

where \( \mu, \nu, \cdots = 0, 1, 2, 3 \), and we are using either the standard sphere angular coordinates \((\theta, \phi)\) or the complex angular coordinates \( \zeta = \frac{\sin^2(\phi)}{2} \), where \( (\zeta, \bar{\zeta}) \) are complex stereographic coordinates of the sphere; which are related to the standard coordinates by \( \zeta = e^{i\varphi} \cot(\frac{\theta}{2}) \); and here a dot means \( \frac{\partial}{\partial u} \); i.e. the partial derivative with respect to the inertial asymptotic time \( u \). \( \Psi_2^0 \) is a component of the Weyl tensor in the GHP notation and \( \sigma_0 \) is the leading order behavior of the spin coefficient \( \sigma \) in terms of the asymptotic coordinate \( \bar{r} \). So the set of intrinsic inertial coordinates at future null infinity are \((\bar{u}, \bar{\theta}, \bar{\phi})\) or \((\bar{u}, \zeta, \bar{\zeta})\).

4.4 The total momentum and flux from charge integrals

The total momentum for the monopole particle can be calculated using the charge integral of the Riemann tensor technique. In this subsection we will use the notation of reference \([31]\).

Let us note that for any 2-form \( w \) defined on a sphere \( S \), one can define the charge integral \([31]\) obtaining

\[ Q_S(w) = 4 \int \left[ -\tilde{w}_0 (\Psi_1 - \Phi_{10}) + 2\tilde{w}_1 (\Psi_2 - \Phi_{11} - \Lambda) - \tilde{w}_0 (\Psi_3 - \Phi_{21}) \right] dS^2 + \text{c.c.} \]

where \( (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2) \) are the components of \( w_{AB} \) with respect to the dyad adapted to \( S \), namely for which the vectors \( m \) and \( \bar{m} \) are tangent to \( S \) and \( \ell \) is outgoing, and \( dS^2 \) is the area element induced from the spacetime metric. If \( w \) has a regular extension to future null infinity, denoted by \( \mathcal{I}^+ \), we can take the limit of \( S \) to a section of \( \mathcal{I}^+ \) obtaining

\[ Q_S(w) = 4 \int \left[ -w_0 \Psi_1^0 + 2w_1 (\Psi_2^0 - \Phi_{10}) \right] dS^2 + \text{c.c.} \]

where the upper-script 0 denotes the leading order term expansion of the Weyl spinor components. The charge integral at \( \mathcal{I}^+ \) turns out to be independent of the Ricci tensor since as we have shown in \([19]\) for any asymptotically flat spacetime, the components of the Ricci tensor go to zero faster than the accompanying Weyl components in the above terms.

The motivation for these charges, is that they reproduce the known charges in the case of linearized Einstein’s field equations. In the following, we assume that independent of the exact nature of the final field equations, they reduce in the linear regime to the same equations as those obtained starting from Einstein’s equations. Using the equations of reference \([33]\) one can obtain expressions for the asymptotic fields with respect to an inertial reference frame: the radiation field is given by

\[ \Psi_3 = \frac{\partial_0 \sigma_0'}{r^2} + O\left( \frac{1}{r^3} \right), \]

while the relationship between the leading order behavior of the shear and primed shear is

\[ \dot{\sigma}_0 = -\sigma_0'. \]

The time derivative of the leading order behavior of \( \Psi_2 \) is

\[ \dot{\Psi}_2^0 = \partial_0 \Psi_3^0 + \sigma_0 \Psi_4, \]

and the leading order behavior of the radiation component is

\[ \Psi_4^0 = \dot{\sigma}_0'. \]

In the standard presentation, in terms of an inertial frame, one has \( \Psi_3^0 = -\partial_0 \sigma_0 \), and using equation (38) of \([31]\) one can write the charge integral as

\[ Q_S(w) = 4 \int \left[ -w_2 \Psi_1^0 + 2w_1 (\Psi_2^0 + \Phi_{10}) \right] dS^2 + \text{c.c.} \]

We would rather like to emphasize the natural appearance of the \( \sigma_0' \) field in these equations, instead of the normally used \( \sigma_0 \); so that we express:

\[ Q_S(w) = 4 \int \left[ -w_2 \Psi_1^0 + 2w_1 (\Psi_2^0 - \sigma_0 \sigma_0') \right] dS^2 + \text{c.c.} \]

In the calculation for the flux of the momentum one takes
where the time derivative is with respect to an inertial time. We conclude that the natural way to encode the radiation flux of momentum is in terms of the \((\sigma'_0, \sigma'_0)\) factor.

The final form of this radiation field \(\sigma'_0\) will depend on the particular gauge one uses for the calculation; however we advance here that this field is gauge invariant in the inertial harmonic frame discussion, as we will show in a separate article. Instead, the field \(\sigma_0\) depends on natural new freedoms that appear in the harmonic gauge treatment.

It follows that the time variation of the total momentum, of a generic spacetime, can be expressed by

\[
\dot{P}^\mu = -\frac{1}{4\pi} \int_S \left[ \hat{\nabla}_0^\mu \sigma_0' \sigma_0' \right] dS^2 \equiv -F^\mu; \tag{54}
\]

that is, \(F^\mu\) is the total momentum flux.

Since in the dynamics of a particle one normally makes use of a notion of proper time; it is natural to express the dynamical equations in terms of this frame independent time parameter. This proper time will be related with a corresponding time variable at future null infinity. Let \(u = \text{constant}\) represent a general time coordinate and set of sections such that

\[
\frac{\partial \hat{u}}{\partial u} = \tilde{V}(u, \zeta, \bar{\zeta}) > 0, \tag{55}
\]

is the time derivative of the inertial time \(\hat{u}\) with respect to the non-inertial time \(u\); so that we can use the inertial coordinates or the non-inertial coordinates \((u, \zeta, \bar{\zeta})\). This means that given the initial section \(S\) one can define a new consecutive section \(S'\) so that both sections are defined by the condition \(u = \text{constant}\).

Let us remark that the total momentum can be expressed in terms of the supermomentum \(M\)

\[
\Psi_M = \Psi_2^0 - \sigma_0' \sigma_0' + 3 \hat{\sigma}^2 \bar{\sigma}_0, \tag{56}
\]

that we have presented elsewhere\cite{28}; so that the momentum is:

\[
\mathcal{P}^\mu = -\frac{1}{4\pi} \int_S \hat{\nabla}_0^\mu \Psi_M dS^2; \tag{57}
\]

where \(S\) is a general section at future null infinity. An important property of the supermomentum \(M\) is the transformation law under BMS transformations, namely\cite{29}

\[
\tilde{\Psi}_M = \frac{1}{K^2} (\Psi_M - \hat{\sigma}^2 \bar{\sigma}^2 \gamma); \tag{58}
\]

where \(K\) represents here the Lorentz factor, and \(\gamma\) the supertranslation. From this it can be easily deduced that, a supertranslation does not affect the calculation of the total momentum. Alternatively, if one uses a section that does not coincide with a section \(\hat{u} = \text{constant}\), then one can still use expression (52) to calculate the momentum.

The time variation of the total momentum with respect to the inertial time can be expressed simply in terms of the time variation of this supermomentum, since

\[
\dot{\Psi}_M = \sigma_0' \sigma_0'. \tag{59}
\]

Then when evaluating the time variation of the momentum with respect to the non-inertial time variable \(u\), one needs to consider

\[
\frac{\partial \Psi_M}{\partial u} = \tilde{V} \frac{\partial \Psi_M}{\partial u} = \tilde{V} \sigma_0' \sigma_0'; \tag{60}
\]

so that

\[
\frac{d\mathcal{P}^\mu}{du} = -\frac{1}{4\pi} \int_S \hat{\nabla}_0^\mu \sigma_0' \sigma_0' dS^2 \equiv -\mathcal{F}^\mu; \tag{61}
\]

where now \(\mathcal{F}^\mu\) is the instantaneous momentum flux with respect to the dynamical time \(u\), and \(S\) is defined by the \(u = \text{constant}\) condition.

This means that even when considering non-inertial times, one only needs to calculate the inertial primed shear to evaluate the flux of gravitational radiation.

### 4.5 Expansion of a radiating spacetime in terms of the strength of the radiation (The \(\lambda\) expansion)

One important aspect of the gravitational field is that for many interesting astrophysical systems the gravitational energy radiated is weak. Also most of the difficulties that arise at null infinity in the construction of appropriate rest reference frames\cite{29} come from the presence of gravitational radiation. It is therefore tempting to expand the structure equations of an asymptotically flat spacetime\cite{19} in terms of the strength of the gravitational radiation.

We will use the smallness parameter \(\lambda\) and assign to the momentum flux the order \(O(\lambda^2)\).

Every field at \(\mathcal{I}^+\) can be expanded as a series with different powers of \(\lambda\); that is for a field \(f(u, \zeta, \bar{\zeta})\) we will write

\[
f(\hat{u}, \zeta, \bar{\zeta}) = f_0(\zeta, \bar{\zeta}) + \lambda f_1(\zeta, \bar{\zeta}) + \lambda^2 f_2(\zeta, \bar{\zeta}) + O(\lambda^3).\]

The idea is to carry out this expansion on the ‘center of mass frame’ defined in terms of ‘nice sections’\cite{28}.

For a general asymptotically flat spacetime\cite{19} we will define the order \(\lambda\) by assigning \(\sigma'_0\) to be of order \(\lambda\), that is:

\[
\sigma'_0 \equiv O(\lambda); \tag{62}
\]

where \(\sigma'_0\) is the leading order behavior of the GHP spin coefficient \(\sigma'\), in an asymptotic null inertial frame.
5 Relation between null tetrads of the flat and curved geometries

In this section we will study the relation of the different tetrads that appear when one considers the decomposition of the metric \( g \) with respect to diverse backgrounds.

5.1 The general case

For each decomposition of the metric of the form
\[
g_{ab} = \eta_{ab} + h_{ab} ;
\]
we have at our disposal two families of null tetrads; one for each of the metrics.

Let us call \( \epsilon_{1,2,3,4} = (l,n,m,\bar{m}) \) a null tetrad of \( g \) and \( \epsilon_{1,2,3,4} = (l,n,m,\bar{m}) \) a null tetrad of \( \eta \). Since both metrics have the same asymptotic future null infinity we can take those tetrads so that they agree asymptotically.

But one should have in mind that the flat metric \( \eta \) that is used in the interior of the spacetime, need not agree with an asymptotic inertial flat metric at future null infinity.

Let us use \( G \) for the gravitational constant; then, \( h \) is order \( O(G) \) and one can express
\[
e_a = e_a + G d^b_a e_b.
\]

Then, taking all the contractions with the metric, up to second order, one obtains
\[
g(e_a, e_b) = \eta_{ab} + G d^c_a \eta_{bc} + G d^c_a \eta_{bc} + G^2 d^c_a \eta_{bc} d^d_a \eta_{cd} + h_{ab} + G d^c_a h_{bc} + G d^c_a h_{bc} + G^2 d^c_a h_{bc} d^d_a h_{cd} ;
\]
(65)

Let us use \( j^c_a \) to denote minus half the antisymmetric part of \( d^c_a \), namely, \( j_{ab} \equiv j^c_a \eta_{bc} = -j^c_a \eta_{bc} \); the one has
\[
G d^c_a \eta_{bc} = - \frac{1}{2} j^c_{ab} \eta_{bc} + O(G^2);
\]
(66)
or alternatively
\[
G d^c_a = - \frac{1}{2} j^c_{ab} \eta_{bc} - \frac{1}{2} j^c_{ab} \eta_{bc} + O(G^2).
\]
(67)

For the first basis vector, one has
\[
l = e_1 = e_1 + G d^b_1 e_b = e_1 - \frac{1}{2} (h_{l_b} + j_{l_b}) \eta_{bc} e_c
\]
\[
= 1 - \frac{1}{2} \left( h_{ln} n + h_{ln} 1 - h_{lm} \bar{m} - h_{lm} \bar{m} \right)
- \frac{1}{2} \left( j_{ln} n + j_{ln} 1 - j_{lm} \bar{m} - j_{lm} \bar{m} \right) .
\]
(68)

And similarly for the other null tetrad vectors one also has:
\[
n = n - \frac{1}{2} \left( h_{nn} n + h_{mn} \bar{m} - h_{mm} \bar{m} \right)
- \frac{1}{2} \left( j_{nn} n + j_{nm} \bar{m} - j_{mm} \bar{m} \right),
\]
(69)
\[
m = m - \frac{1}{2} \left( h_{ml} n + h_{mn} \bar{m} - h_{mm} \bar{m} \right)
- \frac{1}{2} \left( j_{ml} n + j_{mn} \bar{m} - j_{mm} \bar{m} \right),
\]
(70)
\[
m = m - \frac{1}{2} \left( h_{ml} n + h_{mn} \bar{m} - h_{mm} \bar{m} \right)
- \frac{1}{2} \left( j_{ml} n + j_{mn} \bar{m} - j_{mm} \bar{m} \right),
\]
(71)

It can be seen that this can easily be extended to second order by generalizing (61) to
\[
e_a = e_a + G d^b_a e^b + G^2 d^b_a e^b + O(G^3).
\]
(72)

We could also define the the co-basis \( f^a \) and \( f^a \) from
\[
f_a^b \equiv \eta^{ab} c^c_b g_{ca},
\]
(73)
and
\[
f_a^b \equiv \eta^{ab} c^c_b \eta_{ca}.
\]
(74)

In terms of the co-basis notation one would express the departure from the flat basis in terms of the quantity \( \Delta f^\alpha_\beta \), given by
\[
f^a = f^a + \Delta f^a_\beta f^\beta_\alpha.
\]
(75)

Then in first order one has
\[
\delta^b_\alpha = \epsilon^b_\alpha = e^b_\alpha = e^b_\alpha + G d^b_\alpha e^b + G^2 d^b_\alpha e^b + O(G^3);
\]
(76)

which means that
\[
f^a = f^a + G d^b_\alpha f^\beta_\alpha
\]
(77)

and also note that \( f^{1,2,3,4} = (n,l,\bar{m},\bar{m}) \); where we are omitting the abstract index denoting the one-form character of the expressions.

We are using the implicit notation in which, with the exception of the expressions \( \varepsilon^a_\alpha \), the indices \( \underline{a} \) refer to components with respect to the tetrad vectors \( e^a_\underline{a} \).

Then, one could also work with this co-basis for the calculations; but we will continue with the vectorial representation.

5.2 Relation with the asymptotic structure

In the study of asymptotically flat spacetimes one normally recurrs to the conformal techniques so that the conformal metric
\[
\bar{g}_{ab} = \Omega^2 g_{ab},
\]
(78)
is regular at future null infinity. One can then define the regular tetrad at infinity given by [26]
\[
\bar{e}^a = \Omega^{-2} e^a
\]
(79)
\[
\bar{n}^a = \Omega^{-1} n^a
\]
(80)
\[
\bar{\bar{n}}^a = n^a.
\]
(81)
The asymptotic structure provides several ways in which one can decompose the asymptotic metric in terms of a flat background $\eta_{\text{asy}}$ and a deviation term $h_{\text{asy}}$; so that
\[ g = \eta_{\text{asy}} + h_{\text{asy}}. \]

Let us assume for a moment that we have one such decomposition.

For those decompositions, one can also define
\[ \tilde{g}_{ab} = \Omega^2 \eta_{ab}; \] (82)
which must coincide asymptotically with $\tilde{g}_{ab}$. In other words, the components of $\Omega^2 h_{ab}$ in terms of any regular tetrad must go to zero like $\Omega$. Equivalently we may express
\[ \Omega^2 h_{ab} = \frac{1}{r} h^{(0)ab} + \frac{1}{r^2} h^{(1)ab} + \frac{1}{r^3} h^{(2)ab} + \ldots \] (83)
where the indices $\underline{a} \underline{b}$ ... refer to a regular tetrad components and $r$ is an appropriate asymptotic radial coordinate.

Some specific tetrad components are:
\[ \Omega^2 h_{tt} = \Omega^{-1} h_{tt} = \frac{1}{r} h^{(0)tt} + \ldots = \Omega^{-1} \frac{1}{r} h^{(0)tt} + \ldots, \] (84)
\[ \Omega^2 h_{lm} = \Omega^{-1} h_{lm} = \frac{1}{r} h^{(0)lm} + \ldots = \Omega^{-1} \frac{1}{r} h^{(0)lm} + \ldots, \] (85)
\[ \Omega^2 h_{\hat{m}\hat{n}} = h_{mm} = \frac{1}{r} h^{(0)mm} + \ldots = \Omega^{-1} \frac{1}{r} h^{(0)mm} + \ldots, \] (86)
\[ \Omega^2 h_{\hat{n}\hat{n}} = \Omega^2 h_{nn} = \frac{1}{r} h^{(0)nn} + \ldots = \Omega \frac{1}{r} h^{(0)nn} + \ldots; \] (87)
from which we can see that the most asymptotically divergent tetrad component is $h_{nn}$ which could grow as $\Omega^{-1}$. This means that we could also express
\[ \Omega^2 h_{ab} = \frac{1}{r} h^{(0)ab} + \frac{1}{r^2} h^{(1)ab} + \frac{1}{r^3} h^{(2)ab} + \ldots \] (88)
where the indices $\underline{a} \underline{b}$ ... refer to a standard tetrad components.

When considering the case of Einstein equations of general relativity, the natural question arises: Do the solutions to the linear problem in standard gauges have this natural asymptotic decomposition? One can check by direct calculation, of the Schwarzschild solution in the harmonic gauge that, for example, the component
\[ h_{tt} = -\frac{4MA}{r} + \mathcal{O}\left(\frac{1}{r^2}\right); \] (89)
which is even true for the Schwarzschild solution in the standard coordinate system; where in both cases we take the vector $l$ to be $l = \frac{\partial}{\partial t} + \frac{\partial}{\partial r}$, in terms of their intrinsic spherical coordinates. Therefore we conclude that the answer is negative; and so we do not expect all of the above relations, for the asymptotic behaviour of the null tetrad components of the solution $h$, to apply.

However, the Schwarzschild solution is asymptotically flat; so one could still use the assignment for the first order null tetrad, based on the given null tetrad of the flat background metric associated to the solution $h$, as given previously, to calculate all of the asymptotic quantities, including radiation field and curvature components.

### 5.3 Relation between the interior flat metric and the asymptotic flat metrics

It is also important to remark that in having a decomposition in the interior of the form
\[ g_{ab} = \eta_{ab} + h_{ab}; \] (85)
in which the flat metric $\eta_{ab}$ does not belong to the family of adapted asymptotic flat metrics $\eta_{\text{asy}}$: when one considers two points in the spacetime $(\mathcal{M}, \eta_{ab})$, the two future directed null cones define two sections at future null infinity that will be related by a BMS translation at $\mathcal{I}^+$, of the spacetime $(\mathcal{M}, \eta_{ab})$: but it will not coincide in general with a BMS translation at future null infinity $\mathcal{I}^+$, of the original spacetime $(\mathcal{M}, \eta_{ab})$. Instead, these two sections will be related by a supertranslation.

It is because of this reason that an inertial (Bondi) system with respect to the spacetime structure $(\mathcal{M}, \eta_{ab})$ might not coincide in general with inertial systems of the spacetime structure $(\mathcal{M}, \eta_{ab})$. More concretely, sections $\hat{u} = \text{constant}$ of an inertial system of $(\mathcal{M}, \eta_{ab})$, at the boundary manifold $\mathcal{I}^+$, will be defined by a family $\alpha(u_B, \zeta, \zeta)$, in terms of a inertial system of the asymptotic structure of the spacetime $(\mathcal{M}, \eta_{ab})$.

In the construction of a model for compact objects in metric gravitational theories one is normally forced to relate a choice of an interior flat metric $\eta$ with an asymptotic flat metric $\eta_{\text{asy}}$: so the question is: how does one choose the interior flat metric and the asymptotic flat metric? since the choice seems to be crucial for the construction of the model. The answer is related to the general objective of describing the dynamical evolution of the physical quantities of the system. In the interior of the spacetime, we have review in section 2.1 the basic concepts of total momentum and total angular momentum; which in turn comes along with the concept of center of mass. At future null infinity we have constructed in references [11] and [12] the two definitions of intrinsic angular momentum in general relativity which do not suffer from supertranslation ambiguities. These definitions of intrinsic angular momentum come along with the corresponding definition of center of mass frame at future null infinity. Using these center of mass frames, we can select, at each observing asymptotic point $p$ a unique inertial asymptotic flat metric $\eta_{\text{asy}}$.  


Since the model for compact objects will intend to relate the value of physical quantities in the interior with value of physical quantities in the asymptotic region, one should use in both regimes the frames which are best related to the center of mass frames. For this reason we next define an asymptotic flat metric coming from the definition of center of mass at future null infinity.

5.4 The asymptotic center of mass flat metric

We can use the definition of center of mass at future null infinity to define a new preferred flat asymptotic background using the invariance of the translations under BMS transformations. Let us see this in detail. The set of center of mass sections at future null infinity to define a new preferred flat asymptotic background using the invariance of the translations un-

5.5 The radiation fields in a linear space-time

Here we consider the metric $g_A$, which is assumed to be asymptotically flat, and therefore it can be decomposed in the form \([35]\). We will use the set of null tetrads described in subsection 5.1 and study the radiation fields for large values of affine distance along future null directions of the flat background metric.

Our calculation of the shear gives:

\[
\sigma_0 = -\frac{1}{2} \left( \partial_0 (h_{\lambda m}) + \partial_0 (j_{\lambda m}) + h_{0mm} \right).
\]

It must be emphasized that our result has an extra term involving the edth operator when compared with equation 46 of reference \([37]\). These computations have been checked with REDUCE.

We have indicated before that in these kind of approximations one should not focus on the concept of spin coefficient $\sigma$, but instead should consider as more representative of the radiation content the spin coefficient $\sigma'$. In particular, its asymptotic leading order behavior $\sigma'_0$ is independent of the gauge $j_{\lambda m}$ discussed above, and one always has

\[
\sigma'_0 = \frac{1}{2} \frac{\partial h_{0mm}}{\partial \tilde{u}};
\]

and

\[
\Psi'^0 = \frac{\partial \sigma'_0}{\partial \tilde{u}};
\]

where $\tilde{u}$ is an asymptotic inertial null retarded coordinate.

These then are the two fields that codify the appearance of gravitational radiation.
6 The appropriate dynamical times and the balanced equations of motion

In this section we gather together the main ideas and use them to build the general form of the balanced equations of motion.

Let us recall the metrics we have been using. We assume that there exists an exact metric of the spacetime of the form given in \([26]\). We study in detail the asymptotic structure of the sub-metric \(g_A\) by disregarding the effects of the complementary system \(B\). We adjust \(g_A\) to model a compact object, which dynamics is affected by the existence of system \(B\), and therefore by the geometry determined by the sub-metric \(g_B\), appearing in \([25]\).

In the calculation of the back reaction effects one models the emitted gravitational radiation by considering the asymptotic radiation fields in the metric \(g_A\); although of course in the real world, the radiation emitted by body \(A\) will be partially absorbed by system \(B\), and therefore will never reach future null infinity. But in any case we do choose to estimate the local back reaction effects by this method; since it is difficult to achieve a local definition of gravitational radiation emitted by a compact object.

6.1 Form of the dynamical equations in the interior

The basic dynamics is based on the existence of a smooth curve \(C(t)\) in the manifold, such that it is timelike with respect to the metrics \(\eta\) and \(g_B\).

In analogous studies of the dynamics of charged particles in Minkowski spacetime, we have found\([18]\) that new degrees of freedom might be expected. In that case, the dynamics could be presented in an equations of motion of the form

\[ D(v, \dot{v}, \ddot{v}, \alpha_1, \alpha_2) = \mathcal{F}(x^\mu, v, \dot{v}); \]

where the \(\alpha\)'s where new degrees of freedom, and with

\[ g(v, \mathcal{F}) \neq 0. \]

In the gravitational case, the balanced equations of motion is constructed by correcting, in first order of gravitational radiation, the dynamics dictated by \([24]\); where the new right hand side will have the information of the total radiated momentum. But observing the balanced equation \([61]\) at future null infinity one can notice that the timelike component is not vanishing.

Therefore, in the interior, the balanced equations of motion must have (at least) the structure

\[ M_A \left( v^a \nabla_{(B)\alpha} v^b + w v^b \right)(u_{\text{int}}) = F^b(u_{\text{asy}}); \]

where \(w\) shows a new possible degree of freedom, and it is emphasized that the left hand side is expressed in terms of an interior notion of time \(u_{\text{int}}\), that we will fix below, and the right hand side is expressed in terms of a corresponding asymptotic notion of dynamical time \(u_{\text{asy}}\), and where \(F\) is minus the integrated flux of gravitational radiation on an asymptotic sphere; namely

\[ F^\mu = - \mathcal{F}_V = - \frac{1}{4\pi} \int_S \hat{h}_V \sigma_0 \sigma_0 dS^2. \]

The final relation between \(u_{\text{asy}}\) and \(u_{\text{int}}\) will be settled in subsection \([52]\) below.

Although the natural dynamical time for the seed equation \([24]\) is \(\tau\), for the actual numeric computation we can also use as parameterization the proper time \(\tau_0\), with respect to the metric \(\eta\). Let \(v\) and \(\dot{v}\) be the corresponding tangent vectors to the proper times \(\tau_0\) and \(\tau\). Then the basic differential operators are \(v^a \partial_b v^a\) or \(v^b \nabla_{(B)b} v^a\); where \(\partial_b\) is the covariant derivative associated with the metric \(\eta\).

Let us note that the two velocity vectors are proportional

\[ v^b = \Upsilon v^\dot{b}. \]

So that one has

\[ 1 = g_B(v, \dot{v}) = \Upsilon^2 g_B(v, v) = \Upsilon^2 (1 + h_B(v, v)); \]

which gives \(\Upsilon\) in terms of \(v\) and \(g_B\).

Expressing the covariant derivative \(\nabla_{(B)a}\) of \(g_B\), in terms of the covariant derivative \(\partial_b\) of \(\eta\), one has

\[ \nabla_{(B)a} v^b = \partial_a v^b + \gamma_{ab} c v^c; \]

and using the relation between the vectors \(v\) and \(\dot{v}\) one also has

\[ v^a \nabla_{(B)a} v^b = \Upsilon^2 v^a \partial_a v^b + \gamma_{ab} \frac{d\Upsilon}{d\tau_0} v^b + \Upsilon^2 \gamma_{ab} c v^a v^c. \]

Let us introduce the notation:

\[ \mathbf{a}^a \equiv v^b \partial_b v^a, \]

and

\[ \mathbf{f}^b \equiv - \gamma_{ab} c v^a v^c - \frac{d\Upsilon}{\Upsilon} v^b - \frac{w}{\Upsilon} v^b; \]

so that the equations of motion can be expressed as

\[ \mathbf{a}^a = \mathbf{f}^a + f^{a}_{\lambda}; \]

with \(f^{a\lambda}\) defined by

\[ M f^{a\lambda} \equiv \frac{1}{\Upsilon^2} F^\mu. \]

We also define \(\mathbf{r}^a_{\lambda}\) by

\[ \mathbf{r}^a_{\lambda} \equiv - \gamma_{ab} c v^a v^c (\eta_{ab} - v_a v^b); \]

which it should be remarked, only depends on the background geometry \(g_B\) and \(v\).
6.2 Relation between the interior and exterior dynamics

In general, the charge integrals of the Riemann tensor give a connection between the asymptotically defined total momentum $P$ and the momentum $P$ defined in the interior for the particle. This relation is given in terms of a metric structure just determined by $g_A$. But the dynamics we are looking for involves the interaction of particle $A$ with the background metric $g_B$; which generates gravitational radiation that affects the dynamics.

In particular, the components of the momentum are calculated with respect to the generators of translations; which at infinity are the BMS translations and the momentum of the BMS translations, is not necessary in the null gauge; and it remains the equations of motion as expressed with respect to the flat background metric $g$. As we have said, in the case of the linear structure the new notation $\tilde{V} = \hat{V}(\tau, \zeta, \tilde{\zeta})$ gives

\[
\tilde{V}(u, \zeta, \tilde{\zeta}) = \frac{\partial \tilde{u}}{\partial u} = \frac{1}{\frac{d\tilde{u}}{d\tau} \partial \tau} \frac{d}{d\tau} V(\tau, \zeta, \tilde{\zeta});
\]

with the new notation

\[
V = \frac{\partial \tilde{u}}{\partial \tau}.
\]

Then, we can define the gravitational radiation force, in terms of the $\tau$ time, by

\[
F^\mu(\tau_0) = -\frac{1}{4\pi} \int_S l_0^\mu V_0 \sigma_0^\nu \tilde{\sigma}_0^\nu dS^2;
\]

where we use the definition

\[
V_0 = \frac{\partial \tilde{u}}{\partial \tau_0}.
\]

This also invites to define the gravitational radiation force, with respect to the proper time $\tau_0$; namely:

\[
F^\mu(\tau_0) = -\frac{1}{4\pi} \int_S l_0^\mu V_0 \sigma_0^\nu \tilde{\sigma}_0^\nu dS^2;
\]

where $V_0 = \frac{\partial \tilde{u}}{\partial \tau_0}$. Note that $F^\mu = \Upsilon F^\mu_0$.

After considering the general form of the balanced equations of motion in the form

\[
M_A\left(\frac{d\nu}{d\tau} + \nu^b \gamma^b_0 + w\nu^b\right)(u_{\text{int}}) = F^b(u_{\text{asy}});
\]

we choose to express the right hand side in terms of the interior dynamical time. With this choice, the left hand side is already expressed in terms of its natural dynamical time; which simplifies the algebraic expressions, and one needs only to consider the details of expressing the flux force in terms of the interior time. As explained above one should allow for a degree of freedom that relates the asymptotic time with the interior one; which it can be encoded in the quantity $\chi = \frac{d\tau}{d\tau_0}$. Then we express

\[
M_A\left(\frac{d\nu}{d\tau} + \nu^b \gamma^b_0 + w\nu^b\right)(\tau) = F^b(u_{\text{asy}}) = \frac{1}{\chi} F^\mu = \Upsilon F^\mu_0.
\]

Using the $\tau_0$ dynamical time on the let hand side, one can express the equations of motion by

\[
M_A\left(\frac{d\nu}{d\tau} + \nu^b \gamma^b_0 + w\nu^b\right)(\tau) = F^b(u_{\text{asy}}) = \frac{1}{\chi} F^\mu = \Upsilon F^\mu_0.
\]

It is important to remark that this natural consideration for a degree of freedom relating interior and asymptotic dynamical times, is not necessary in the null gauge; as we will show in a separate work.

Thus, equation (113) is the main equations of motion, as expressed with respect to $\tau_0$. Contracting this equation with $\eta_0 d^d v^d$ gives

\[
M_A\left(\gamma^b_0 v^a + \nu^a w + \frac{1}{\chi} d\nu \right)(\tau) = \frac{1}{\chi} F^\mu_0 = \Upsilon F^\mu_0.
\]

and it remains the equations of motion

\[
M_A a^b = M_A f^b + \frac{1}{\chi} F^\mu_0 \left(\eta_0^b - v_0^b v^b\right).
\]

Equation (115) is understood as an equation for $w$. The main dynamical equation is then (110), where the possible degree of freedom $\chi$ depends on the detail nature of gauge being used. As we have said, in the case of the null gauge that we will present in a separate article, one has that $\chi = 1$. 

13
6.3 The balanced equations of motion in terms of orders

Our goal is to set the equations of motion for a compact object (a particle) that takes into account the first order back reaction due to the emission of gravitational radiation. This requires to deal with the dynamics in the interior of the spacetime and also with the gravitational radiation defined at future null infinity.

In setting the interior geometry we have found a natural expansion in terms of the gravitational constant $G$.

In the asymptotic study of gravitational radiation, it appears very naturally an expansion in terms of a quantity $\lambda$ that has the information on the strength of gravitational radiation, and is defined such that the flux of momentum is of order $\lambda^2$.

In making a link between the interior study and the asymptotic study, it is then natural to make a double expansion in terms of powers of $G$ and $\lambda$. We will therefore refer to a $(p, q)$ order meaning terms up to order $(G^p, \lambda^q)$.

Someone could say that it is unnecessary to include in the discussion of orders the $\lambda$ parameter; since actually one could estimate the way in which the gravitational constant appears in the radiation field. In particular, one can deduce that since at lowest order the radiation field is calculated from the term $h_A$ of the metric, it is already of order $G$. But also, since the radiation field is generated by the accelerated motion of the particle in the background, it is also of order $M^2$: which means that $\lambda = O(M_A M_B) = O(G^2)$. Although this is true, we prefer to keep the control parameter $\lambda$ for the discussion of orders; since it better captures the incidence of the radiation fields in the dynamics, and also, because from a formal point of view, the accelerations could be from non-gravitational origin, as electromagnetic one, and we should be able to discuss the motion for those cases too. So we only assume that $\lambda = O(G)$.

The functional form of $F_0$ depends crucially on the gauge one is using to relate the interior dynamics with the asymptotic region. So, it might be that actually $F_0 = F_0(u, v, f)$; that is, the flux might also depend on time derivatives of the velocity. Because of this we next indicate the way the equations of motion must be understood.

Let us observe that at first sight, the first term on the left hand side of (14) is of order $(1, 0)$: since $M_A$ is order $G$. The other terms on the right hand side could be considered of order $(1, 0)$ in the non-gravitational case, or of order $(2, 0)$ in the binary case. The right hand side depends quadratically on the radiation and therefore is at least of order $(2, 2)$.

The idea is then, to consider first the equation up to order $G^2$, and calculate the acceleration at this order; and then we proceed to calculate the flux of gravitational radiation, that will depend on the details of the chosen gauges, and with this, we build the complete equation and integrate the operator $\nabla^2 \partial_A v^A$ at order $(2, 2)$. To work at orders of the exterior forces means that, any possible appearance of a time derivative on the velocity, in the flux, must be replaced by $f_1$; so that in the general case, $F_0$ must be understood as

$$F_0 = F_0(\tau_0, v, f).$$

As indicated before, the details of these expressions, depend on the preferred gauge choices.

Of course, for a specify use of the equations of motion one should maintain the different terms at consistent orders of approximation.

7 Final comments

When embarking on the construction of a model for compact objects within the particle paradigm in general relativistic theories, it is natural to recur to approximation schemes in terms of some order parameter. One knows from the beginning that such construction will have sense only if it is thought in terms of finite orders.

In this work we have presented the general form of the equations of motion for particles subjected to back reaction effects due to gravitational radiation by using various choices of dynamical time. We have also pointed out several of the issues one must handle when constructing balanced equations of motion, as for example the relation of generators of translations in the interior an at future null infinity. These type of problems is generally overlooked in the literature. The reasons for the usual simplifications of the descriptions is because the subject is very complex, as has been indicated in [56]. The specific equations of motion can be calculated when a definite field equation and a choice of gauges has been made.

Our main basic working assumptions have been: that the back reaction effects due to gravitational radiation can be well represented by the calculation in the leading order effects of the particle using the unaffected null cones; that the notion of center of mass at future null infinity are well described by the previous works of the authors; that to each point at future null infinity there corresponds a center of mass flat background metric, as described above, which must be related to the corresponding notion of center of mass in the interior at the needed order of approximation; and that the first correction to the zero order equations of motion comes from the balance of the change of momentum with the radiation of momentum at infinity. In relation to the structure of the particle, we have assumed that its mechanical properties are completely determined by a constant mass and we have disregarded possible spin and dipole contributions. Regarding the dipole, we have remarked in section 2.2 that at zero order, particles should not be assigned any dipole structure.

With respect to the unspecified theory of gravity that describes the structure of the spacetime, we recall here that in this presentation we are assuming: that isolated objects are well represented by asymptotically flat spacetimes; that the linear structure of the field equation is
such that allows for the identification of the total momentum at infinity with the total momentum calculated in the interior from the energy momentum tensor; that the energy momentum tensor is conserved, that is its divergence is zero.

Some aspects of the model presented here has certain similarities with the so called post-Minkowskian formalisms usually studied within the framework of Einstein’s equations\textsuperscript{[33, 34, 35]}. However, our approach differs from those formalisms in the way in which the equations of motion are obtained because our approach makes use of the energy-momentum tensor and its conservation rather than obtaining them through particular field equations. Furthermore, the post-Minkowskian formalism is usually presented in the harmonic gauge, while our presentation is independent of the chosen gauge, even thought the fine details of the equations of motion depend on each choice of gauge and field equations under consideration.

To all this, we can add that the way in which different gauges affect the final form of the equations of motion, is a subject that deserves a detailed study that we will present elsewhere. In particular, it should be stressed that our result has been calculated with the leading order geometry for the object and background; since we have been concerned with the first order back reaction, due to gravitational radiation. In order to improve in the predicting power of the equations, then one should extend the calculations to higher orders of the specific field equations, that are being used. For example, in \textsuperscript{[41]} we present the balanced equations of motion in general relativity in the harmonic gauge, up to second order in the accelerations and field equations; and in \textsuperscript{[42]} we present the corresponding balanced equations of motion in the harmonic gauge in first order of the accelerations and field equations.

It is important to remark that in analogy to what we have found in the case of charged particles in Minkowski spacetime, there are at least two further degrees of freedom involved in the dynamics. Although the $w$ degree of freedom might be associated to whether one wants to consider a constant mass parameter or not for the particle; the best choice for $w$ depends on the gauge one has chosen.

Furthermore, particular gauge choices might involve new degrees of freedom; as we will show in separate works.

The fact that the final form of the equations of motion is dependent on the choice of gauges for the calculations, should not be understood in a negative way; on the contrary, it is the natural expectation for a finite degree of freedom model intended to represent and infinite degree of freedom system. However from the astrophysical point of view there is a general believe that the main observational properties in the collapsing phase of compact object system, might be well represented by models with finite degrees of freedom. From another point of view, models with finite degrees of freedom might provide well approximated initial data to be used in full numerical calculations for the final stages of collapse.

This approach to the equations of motion can be generalized in a number of ways. The natural next topic that we plan to deal is the introduction of spin for the particles. This and other issues will be dealt in future works.

Acknowledgments

We have benefited from discussions with R. Wald, to whom we are very grateful. We acknowledge support from CONICET and SeCyT-UNC.

References

[1] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116 no. 6, (2016) 061102 arXiv:1602.03837 [gr-qc]
[2] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “Properties of the Binary Black Hole Merger GW150914,” Phys. Rev. Lett. 116 no. 24, (2016) 241102 arXiv:1602.03840 [gr-qc]
[3] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence,” Phys. Rev. Lett. 116 no. 24, (2016) 241103 arXiv:1606.04855 [gr-qc]
[4] VIRGO, LIGO Scientific Collaboration, B. P. Abbott et al., “GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2.” Phys. Rev. Lett. 118 no. 22, (2017) 221101 arXiv:1706.01812 [gr-qc]
[5] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence,” Phys. Rev. Lett. 119 no. 14, (2017) 141101 arXiv:1709.09660 [gr-qc]
[6] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,” Phys. Rev. Lett. 119 no. 16, (2017) 161101 arXiv:1710.05832 [gr-qc]
[7] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., “GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence,” arXiv:1711.05578 [astro-ph.HE]
[8] C. M. Will, “Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations.
III. Radiation reaction for binary systems with spinning bodies,” Phys. Rev. D71 (2005) 084027, arXiv:gr-qc/0502039

[9] L. Blanchet, “Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries,” Living Rev. Rel. 17 (2014) 2, arXiv:1310.1528 [gr-qc].

[10] P. Jaranowski and G. Schäfer, “Dimensional regularization of local singularities in the 4th post-Newtonian two-point-mass Hamiltonian,” Phys. Rev. D87 (2013) 081503, arXiv:1303.3225 [gr-qc].

[11] E. Poisson, A. Pound, and I. Vega, “The Motion of point particles in curved spacetime,” Living Rev. Rel. 14 (2011) 7, arXiv:1102.0529 [gr-qc].

[12] R. M. Wald, “Introduction to Gravitational Self-Force,” Fundam. Theor. Phys. 162 (2011) 253–262, arXiv:0907.0412 [gr-qc] [253(2009)].

[13] T. Damour, A. Nagar, M. Hannam, S. Husa, and B. Brugmann, “Accurate effective-one-body waveforms of inspiralling and coalescing black-hole binaries.” Arxiv:0803.3162 [gr-qc], 2008.

[14] R. P. Kerr, “The lorentz-covariant approximation method in general relativity,” Nuovo Cim. XIII no. 3, (1959) 469–491.

[15] R. P. Kerr, “On the lorentz-covariant approximation method in general relativity. ii second approximation,” Nuovo Cim. XIII no. 3, (1959) 492–502.

[16] R. P. Kerr, “On the lorentz-invariant approximation method in general relativity. iii the einstein-maxwell field,” Nuovo Cim. XIII no. 4, (1959) 673–689.

[17] R. P. Kerr, “On the quasi-static approximation in general relativity. iii the einstein-maxwell field,” Nuovo Cim. XVI no. 1, (1960) 26–60.

[18] E. Gallo and O. M. Moreschi, “New derivation for the equations of motion for particles in electromagnetism,” Phys. Rev. D85 (2012) 065005, arXiv:1112.5344 [gr-qc].

[19] O. M. Moreschi, “General future asymptotically flat spacetimes,” Class. Quantum Grav. 4 (1987) 1063–1084.

[20] R. Geroch and J. Traschen, “Strings and other distributional sources in general relativity,” Phys. Rev. D 36 (1987) 1017–1031.

[21] L. B. Szabados, “Quasi-local energy-momentum and angular momentum in general relativity,” Living Rev. Relativity 12 ((2009) 4) . http://www.livingreviews.org/lrr-2009-4.

[22] W. Dixon, “Dynamics of extended bodies in general relativity. I. Momentum and angular momentum,” Proc. Roy. Soc. Lond. A314 (1970) 499–527.

[23] M. Mathisson, “New mechanics of material systems,” Gen. Rel. Grav. 42 (2010) 1011–1048. Acta Phys. Pol. 6, 163 (1937).

[24] M. Mathisson, “The variational equation of relativistic dynamics,” Proc. Camb. Philos. Soc. 36 (1940) 331–350.

[25] E. Gallo and O. M. Moreschi, “Constructing balanced equations of motion for particles in general relativistic theories: the general case,” arXiv:1609.02110 [gr-qc].

[26] O. M. Moreschi, “On angular momentum at future null infinity,” Class. Quantum Grav. 3 (1986) 503–525.

[27] R. Sachs, “Asymptotic symmetries in gravitational theory,” Phys. Rev. 128 (1962) 2851–2864.

[28] O. M. Moreschi, “Supercenter of Mass System at Future Null Infinity,” Class. Quant. Grav. 5 (1988) 423–435.

[29] O. M. Moreschi and S. Dain, “Rest frame system for asymptotically flat space-times,” J. Math. Phys. 39 no. 12, (1998) 6631–6650.

[30] S. Dain and O. M. Moreschi, “General existence proof for rest frame system in asymptotically flat space-time,” Class. Quantum Grav. 17 (2000) 3663–3672.

[31] O. M. Moreschi, “Intrinsic angular momentum and center of mass in general relativity,” Class. Quantum Grav. 21 (2004) 5409–5425.

[32] E. Gallo and O. M. Moreschi, “Intrinsic angular momentum for radiating spacetimes which agrees with the Komar integral in the axisymmetric case,” Phys. Rev. D89 (2014) 084009, arXiv:1404.2475 [gr-qc].

[33] R. Geroch, A. Held, and R. Penrose, “A space-time calculus based on pairs of null directions,” J. Math. Phys. 14 (1973) 874–881.

[34] R. Penrose, “Asymptotic properties of fields and space-times,” Phys. Rev. Lett. 10 (1963) 66–68.

[35] L. Lehner and O. Moreschi, “On the definition of the center of mass for a system of relativistic particles,” J. Math. Phys. 36 no. 7, (1995) 3377–3394.

[36] O. M. Moreschi, “On the limitation of approximation methods around stationary backgrounds for isolated systems,” Class. Quantum Grav. 5 (1988) L53–L57.

[37] M. Walker and C. M. Will, “Relativistic Kepler problem 2. Asymptotic behavior of the field in the infinite past,” Phys. Rev. D19 (1979) 3495. [Erratum: Phys. Rev.D20,3437(1979)].
[38] N. D. J. I. J. M. Luis Bel, Thibaut Damour, “Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity,” *Gen Relat Gravit.* 13 (1981) 963.

[39] S. F. Smith and P. Havas, “Effects of gravitational radiation reaction in the general relativistic two-body problem by a lorentz-invariant approximation method,” *Phys. Rev.* 138 (Apr, 1965) B495–B508, http://link.aps.org/doi/10.1103/PhysRev.138.B495.

[40] A. Rosenblum, “Effects of gravitational radiation reaction in the general relativistic two-body problem by a lorentz-invariant approximation method,” *J. Phys. A* 16 (1983) 2575.

[41] E. Gallo and O. M. Moreschi, “Constructing balanced equations of motion for particles in general relativity: the null gauge case,” arXiv:18xx.yyyyy [gr-qc].

[42] E. Gallo and O. M. Moreschi, “Constructing balanced equations of motion for particles in general relativity: the harmonic gauge case,” arXiv:1711.08501 [gr-qc].