Pseudo-developing maps for ideal triangulations II: 
Positively oriented ideal triangulations of cone-manifolds

Alex Casella, Feng Luo and Stephan Tillmann

Abstract  We generalise work of Young-Eun Choi to the setting of ideal triangulations with vertex 
links of arbitrary genus, showing that the set of all (possibly incomplete) hyperbolic cone-manifold 
structures realised by positively oriented hyperbolic ideal tetrahedra on a given topological ideal tri-
angulation and with prescribed cone angles at all edges is (if non-empty) a smooth complex manifold 
of dimension the sum of the genera of the vertex links. Moreover, we show that the complex lengths 
of a collection of peripheral elements give a local holomorphic parameterisation of this manifold.

AMS Classification  57M25, 57N10

Keywords  3-manifold, hyperbolic structure, cone-manifold, ideal triangulation

1 Introduction

The complement $N$ of the vertices in a triangulated orientable 3–dimensional pseudo-manifold $P$ carries a complete hyperbolic cone-manifold structure, where the singular locus is contained in the 1–skeleton and each ideal tetrahedron develops into an ideal hyperbolic tetrahedron. For instance, such a structure is obtained by realising each ideal tetrahedron in $N$ as a regular ideal hyperbolic 3–simplex (see [9] and §2.4). Let $T$ denote the (topological) ideal triangulation of $N$ and $E$ the set of ideal edges. For any prescribed cone angles $\kappa: E \to \mathbb{R}$, let $\mathcal{D}^+(T, \kappa)$ be the set of all (possibly incomplete) hyperbolic cone-manifold structures with the ideal tetrahedra in $T$ realised by positively oriented hyperbolic tetrahedra and with the prescribed cone angles $\kappa$. We show that if this set is non-empty, then it is a smooth complex manifold of dimension the sum of the genera of the vertex links (Corollary 3). This generalises the first main theorem of Choi [2]. This result is deduced as a consequence of a more general result (Theorem 2), which is essentially due to Neumann [6]. Moreover, we show that the complex lengths of a collection of non-trivial peripheral elements, $g$ for each vertex link of genus $g$, give a local holomorphic parametrisation of $\mathcal{D}^+(T, \kappa)$ (Corollary 10). This is achieved through a generalisation (Theorem 9) of the second main theorem of Choi [2].

Our approach is different from Choi’s and includes new results on the interplay between tangential angle structures, the boundary map and the study of the volume function. It builds on previous work of Neumann [6] as well as Futer and Guéritaud [3].

The results of this paper show that the examples of pseudo-manifolds $M$ with spherical vertex links in [9] exhibit generic behaviour—for any prescribed cone angles at the edges, the manifold $\mathcal{D}^+(M, \kappa)$ is either 0–dimensional or empty. We conclude this paper by giving two
examples, which both have two components where one might have expected one. The first example is a once-cusped hyperbolic 3–manifold of finite volume with the property that the two discrete and faithful characters lie on different components of the $\text{PSL}_2(\mathbb{C})$–character variety. It was found via an application of the volume function explained to us by Nathan Dunfield. The second example is a manifold with two boundary components—a torus and a genus two surface. In this example, one can find distinct prescribed cone angles $\kappa_1$ and $\kappa_2$ for the edges of an ideal triangulation with the property that $\kappa_1$ and $\kappa_2$ differ by $2\pi$ at two edges. In contrast, for manifolds with only torus boundary components, a standard Euler characteristic argument shows that at each edge the total cone angle has to be exactly $2\pi$.

2 Preliminaries

2.1 Conventions for vector spaces

If $X$ is a finite set, $\mathbb{R}^X$ denotes the real vector space of all functions $X \to \mathbb{R}$ and we assume that $\mathbb{R}^X$ has the standard inner product, so that

$$X^* = \{ x^* \in \mathbb{R}^X \mid x^* \in X \text{ and for all } y \in X : x^*(y) = \delta_{xy} \}$$

is an orthonormal basis of $\mathbb{R}^X$, where Kronecker’s notation $\delta_{xy} = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$ is used.

2.2 Pseudo-manifolds and triangulations

Let $\Delta$ be a finite union of pairwise disjoint, oriented Euclidean 3–simplices, and $\Phi$ be a family of orientation-reversing affine isomorphisms pairing the facets in $\Delta$, with the properties that $\varphi \in \Phi$ if and only if $\varphi^{-1} \in \Phi$, and every codimension-one facet is the domain of a unique element of $\Phi$. The elements of $\Phi$ are termed face pairings. The quotient space $P = \Delta/\Phi$ with the quotient topology is a closed, orientable 3–dimensional pseudo-manifold, and the quotient map is denoted $p: \Delta \to P$. The triple $T = (\Delta, \Phi, p)$ is a (singular) triangulation of $P$. The adjective singular is usually omitted, and we will not need to distinguish between the cases of a simplicial or a singular triangulation. We will always assume that $P$ is connected. In the case where $P$ is not connected, the results of this paper apply to its connected components.

We will use the following notation:

$$T = \{ \sigma_i \} = \Delta^{(3)}$$
$$E = \{ e_j \} = P^{(1)}$$
$$V = \{ v_k \} = P^{(0)}$$

Note that $E$ and $V$ are equivalence classes of 1–simplices and 0–simplices of $\Delta$ respectively. The set of non-manifold points of $P$ is contained in the 0–skeleton. Denote this set $V_s \subseteq V = P^{(0)}$. The cases of interest are usually when $V_s = \emptyset$ or $V_s = V$. In the first case $P$ is a closed 3–manifold. In the second case $T$ restricts to an ideal triangulation of the topologically finite, non-compact 3–manifold $N = P \setminus P^{(0)}$ and $P$ is the end-compactification of $N$.

For each vertex $v \in V$, $\text{Lk}(v)$ is a closed orientable surface of genus $g_v \geq 0$, with a triangulation $T_v$ induced from $T$. We will repeatedly make use of the following fact, which follows from a direct Euler characteristic calculation:

Lemma 1 \[ |T| - |E| + |V| = \sum_{v \in V} g_v. \]
2.3 Quadrilateral index

Let $\sigma$ be a 3–simplex. A quadrilateral type $q$ in $\sigma$ is a partition of its set of vertices into two sets of cardinality two. The name alludes to its geometric realisation as a properly embedded quadrilateral disc separating a pair of opposite 1–simplices $e_0$ and $e_1$. See [10] for an exposition of this well-known geometric viewpoint, which goes back to Haken [4]. It will be convenient to regard a quadrilateral type as a set containing these two opposite 1–simplices, and we write

$$\{e_0, e_1\} = q < \sigma$$

and say that $e_0$ and $e_1$ face $q$.

There are precisely three quadrilateral types in $\sigma$. There is a natural action of the symmetric group $\text{Sym}(4)$ on the set of vertices of $\sigma$. Choose an orientation of $\sigma$. The alternating group $\text{Alt}(4)$ fixes the orientation and permutes the three quadrilateral types. The stabiliser of a quadrilateral type is the Klein four group $K$. So there is a natural faithful action of the cyclic group $C_3 \cong \text{Alt}(4)/K$ on the set of all quadrilateral types, and hence a natural cyclic order on that set. Throughout, $q, q', q''$ denote the three quadrilateral types in the 3–simplex $\sigma$ and the action of $C_3$ is indicated by the prime mark, so $(q')' = q''$ and $(q'')' = q$ and the natural cyclic order is therefore given by $q \rightarrow q' \rightarrow q'' \rightarrow q$.

Denote $\sq$ the set of all normal quadrilateral types in $\tilde{\Delta}$. The quotient map $p: \tilde{\Delta} \rightarrow P$ induces a natural map $p: \tilde{\Delta}^{(1)} \rightarrow P^{(1)} = E$. For any $e \in E$ and $q \in \sq$, the number of edges in the preimage $p^{-1}(e) \subset \tilde{\Delta}^{(1)}$ facing $q$ is:

$$i(q, e) = |q \cap p^{-1}(e)| \in \{0, 1, 2\}.$$

If $i(q, e) > 0$, we say that $e$ faces $q$, and write $q \sim e$.

2.4 Cone-deformation variety

The cone-deformation variety $\mathcal{D}(T; \star)$ is the set of all $(z, \xi) \in \mathbb{C}^\square \times (S^1)^E$ satisfying:

i) for each edge $e \in E$,

$$\prod_{q \in \square} z(q)^{i(q, e)} = \xi(e), \quad (1)$$

ii) for each $q \in \square$,

$$z(q') = \frac{1}{1 - z(q)} \quad (2)$$

Equation (2) is the parameter relation for $q$. Applying the cyclic ordering gives:

$$z(q'') = 1 - \frac{1}{z(q)} = \frac{z(q) - 1}{z(q)} \quad \text{and} \quad z(q)z(q')z(q'') = -1.$$  

Equation (1) is the cone-hyperbolic gluing equation for $e$. Multiplying all of these equations gives the identity

$$\prod_{e \in E} \xi(e) = 1.$$  

If $\xi(e) = 1$, then equation (1) is the usual hyperbolic gluing equation of $e$.  

3
We have the projections onto the factors
\[
s: \mathcal{D}(\mathcal{T}; \star) \to \mathbb{C}^\square \quad \text{and} \quad c: \mathcal{D}(\mathcal{T}; \star) \to (S^1)^E,
\]
where \(s\) gives the \emph{shapes} of the tetrahedra and \(c\) gives the \emph{curvature} at the edges.

Denote the upper half plane in \(\mathbb{C}\) by \(\mathbb{H}\) and for each \(\xi \in (S^1)^E\) define
\[
\mathcal{D}^+(\mathcal{T}; \xi) = \mathcal{D}(\mathcal{T}; \star) \cap (\mathbb{H}^\square \times \{\xi\}).
\]
It was observed in [9] that the cone-deformation variety is non-empty for any triangulation since one may choose each \(z(q) = \frac{1}{2}(1 + \sqrt{-3})\) to be the shape of the regular hyperbolic ideal 3-simplex, and this turns \(N\) into a complete hyperbolic cone-manifold. Hence there always exists \(\xi \in (S^1)^E\) such that \(\mathcal{D}^+(\mathcal{T}; \xi) \neq \emptyset\).

One can study the (topological) connected components of \(\mathcal{D}^+(\mathcal{T}; \xi)\) via the set of all \(\kappa \in \mathbb{R}^E\) with the property that \(z \in \mathbb{H}^\square\) satisfies the parameter relations and for each edge \(e\), we have:
\[
\sum_{q \in \square} i(q, e) \log(z(q)) = \kappa(e), \tag{3}
\]
and
\[
\xi(e) = \exp(ik(e)). \tag{4}
\]
Throughout this paper, \(\log\) is the standard branch on \(\mathbb{C} \setminus (-\infty, 0]\) unless stated otherwise. It follows from analytic continuation that on each connected component of \(\mathcal{D}^+(\mathcal{T}; \xi)\), the left-hand side of (3) is constant.

Below Corollaries 3 and 10 imply that if \(\mathcal{D}^+(\mathcal{T}; \xi)\) is non-empty, then it is a smooth complex manifold of dimension the sum of the genera of the vertex links, and each of its components has a holomorphic parametrisation by the holonomies of peripheral elements, one for each genus. This will be proved using the more general \emph{complex-curvature} and \emph{log-curvature} maps defined in §2.5, which respectively generalise the left-hand sides of (1) and (3).

### 2.5 The log-curvature map

Let \(\mathbb{H}\) be the upper half plane and \(Z = \{z \in \mathbb{H}^\square | z(q') = \frac{1}{1-z(q)}\}\).

The \emph{log-curvature map} \(G: Z \to \mathbb{C}^E\) is defined by:
\[
G(z)(e) = \sum_{q \in \square} i(q, e) \log(z(q)).
\]

The log-curvature map is, of course, closely related to the complex-curvature map \(c: Z \to \mathbb{C}^E\)
\[
c(z)(e) = \prod_{q \in \square} z(q)^{i(q, e)}.
\]
For instance, a well-known Euler characteristic argument shows that if the link of each vertex is a torus, then \(G^{-1}(2\pi i, \ldots, 2\pi i) = c^{-1}(1, \ldots, 1)\). In general, \(c^{-1}(u)\) is a countable union of sets of the form \(G^{-1}(u_k)\), and these sets are pairwise disjoint by analytic continuation. But since \(c^{-1}(u)\) is an affine algebraic set, at most finitely many of these sets will be non-empty. Hence each level set \(G^{-1}(u_k)\) of the log-curvature function is also an affine algebraic set.
3 Statements and proofs of the main results

Our central result is Theorem 9, generalising Choi’s main technical result [2, Theorem 4.13] to pseudo-manifolds with positively oriented triangulations and prescribed log-curvature.

3.1 Rank of the log-curvature map

Fixing a preferred quadrilateral type \( q < \sigma \) for every 3–simplex, there is a natural identification between \( Z \) and \( H^T \) via the projection map \( \pi: Z \to H^T \), where

\[
\pi(z)(\sigma) = z(q).
\]

Hence the log-curvature map can and will be viewed as a map \( G: H^T \to \mathbb{C}^E \). The following result is a corollary of [6, Theorem 4.1]:

**Theorem 2 (Neumann)** \( dG \) has constant rank \( |T| - \sum_{v \in V} g_v \).

**Proof** For each tetrahedron \( \sigma_i \in T \), fix a quadrilateral type \( q_i < \sigma_i \). Using the quadrilateral index, we let \( B = (b_{ij}) \) be the \( 2|T| \times |E| \) matrix, where for all even \( i \):

\[
\begin{align*}
  b_{ij} &= i(q_i, e_j) - i(q'_i, e_j), \\
  b_{i+1,j} &= i(q'_i, e_j) - i(q''_i, e_j).
\end{align*}
\]

It is well known that \( dG \) has the same rank as \( B \) (see [8]), hence it is enough to show that \( \text{rank} \ B = |T| - \sum_{v \in V} g_v \). This is the contents of [6, Theorem, 4.1]. Neumann defines a linear map \( \beta: C_1 \to J \), where \( C_1 \) is the free \( \mathbb{Z} \)–module generated by \( E \) and \( J \) is the free \( \mathbb{Z} \)–module generated by \( T^2 \), and shows that

\[
\dim(\ker \beta^*/\text{im} \beta) = \dim \bigoplus_{v \in V} H_1(Lk(v)) = \sum_{v \in V} 2g_v,
\]

where \( \beta^* \) is the dual map. Moreover, Neumann shows \( \ker \beta^* = 2|T| - |E| + |V| \), and therefore

\[
\dim(\text{im} \beta) = \dim(\ker \beta^*) - \dim \bigoplus_{v \in V} H_1(Lk(v)) = 2|T| - |E| + |V| - \sum_{v \in V} 2g_v.
\]

Using particular bases of \( C_0 \) and \( J \), \( B \) is the matrix associated to \( \beta \), giving

\[
\text{rank} \ B = \dim(\text{im} \beta) = 2|T| - |E| + |V| - \sum_{v \in V} 2g_v = T - \sum_{v \in V} g_v,
\]

where the last equality follows from Lemma 1.

Theorem 2 and the implicit function theorem imply the following result.

**Corollary 3** For all \( u \in \mathbb{C}^E \), the complex variety \( G^{-1}(u) \) is either empty or a smooth complex manifold of dimension \( \sum_{v \in V} g_v \).
3.2 Boundary Map

Let $\alpha$ be an oriented closed normal curve on $\text{Lk}(v)$, representing a non-trivial element in $H_1(\text{Lk}(v))$ and let $t \in T_v^{(2)}$ be a normal triangle contained in tetrahedron $\sigma \in T$. A segment of $\alpha$ with respect to $t$ is an oriented connected component of $\alpha \cap t$. Let $S^t_\alpha$ be the set of all segments of $\alpha$ with respect to $t$, and $S_\alpha = \bigcup_{t \in T_\partial} S^t_\alpha$ be the set of all segments of $\alpha$.

![Figure 1](image)

Figure 1: On the left: Each $s \in S_\alpha$ determines a quadrilateral type $q_s$ and isolates a vertex $x_s$. For an observer sitting in the cusp, $x_s$ here lies on the right side of $s$. On the right: $\alpha_1, \alpha_2$ encircle the endpoint of an edge $e \in E$.

Each $s \in S_\alpha$ uniquely determines a quadrilateral type $q \in \square$ such that $s \subset q$, which will be denoted by $q_s$. If $s \in S^t_\alpha$, $s$ divides $t$ in two regions and isolates one of the three vertices, say $x_s$ (Figure 1). With respect to the orientation of $s$ and the induced orientation on $\text{Lk}(v)$, it makes sense to say that $x_s$ lies on the right side or on the left side of $s$ when viewed from the cusp. We therefore define

$$\eta(s) = \begin{cases} +1 & \text{if } x_s \text{ lies on the right side of } s, \\ -1 & \text{if } x_s \text{ lies on the left side of } s, \end{cases}$$

$$\eta(q, s) = \begin{cases} \eta(s) & \text{if } q = q_s, \\ 0 & \text{otherwise} \end{cases}$$

and finally

$$\eta(q, \alpha) = \sum_{s \in S_\alpha} \eta(q, s).$$

In particular, when $\alpha$ encircles one endpoint of an edge $e \in E$, then $\eta(q, \alpha) = \pm i(q, e)$ for all $q \in \square$, where the sign depends on whether $\alpha$ is oriented anticlockwise or clockwise with respect to the cusp (Figure 1).

Hence we can write the log holonomy of $\alpha$ as $h_\alpha : \mathbb{Z} \to \mathbb{C}$,

$$h_\alpha(z) = \sum_{q \in \square} \eta(q, \alpha) \log z(q).$$

Notice that if $\alpha_1$ is normally isotopic to $\alpha_2$, then $h_{\alpha_1}(z) = h_{\alpha_2}(z)$, and if $\alpha_1$ is the union of $n$ normal curves all normally isotopic to $\alpha_2$, then $h_{\alpha_1}(z) = nh_{\alpha_2}(z)$. We therefore have
a natural extension of the log holonomy to \( \mathcal{N}_\mathbb{R} \), the \( \mathbb{R} \)-vector space with basis the normal isotopy classes of curves on \( \text{Lk}(v) \). For all \( a, b \in \mathbb{R} \) and \( \alpha, \beta \in \mathcal{N}_\mathbb{R} \) define

\[
h_{a\alpha+b\beta}(z) := a h_\alpha(z) + b h_\beta(z).
\]

Note that if \( \alpha^{-1} \) is the closed normal curve \( \alpha \) with opposite orientation, then \( h_{\alpha^{-1}}(z) = 0 \).

If \( G(z) \neq (2\pi i, \ldots, 2\pi i) \), then the value of \( h_\alpha(z) \) may not be an invariant of the homology class of \( \alpha \), but depends on the choice of normal representative since an isotopy pushing \( \alpha \) over a vertex in \( \text{Lk}(v) \) may change its value.

For each link \( \text{Lk}(v_i) \), let \( g_i \) denote its genus and choose a canonical homotopy group generating set \( \mathcal{L}_i \cup \mathcal{M}_i \), where \( \mathcal{L}_i = \{ \lambda_1^i, \ldots, \lambda_{g_i}^i \} \) is the set of longitudes and \( \mathcal{M}_i = \{ \mu_1^i, \ldots, \mu_{g_i}^i \} \) the set of meridians, indexed and oriented such that the algebraic intersection number satisfies \( \iota(\lambda_j^i, \mu_j^i) = 1 \) for all \( j \) and \( \iota(\alpha, \beta) = 0 \) for all other pairs of elements \( \alpha, \beta \in \mathcal{L}_i \cup \mathcal{M}_i \). Moreover, we assume that each element of \( \mathcal{L}_i \cup \mathcal{M}_i \) is an oriented normal curve of \( \text{Lk}(v_i) \).

Let \( \mathcal{L} = \bigcup_i \mathcal{L}_i \) and define the boundary map \( H_\mathcal{L} : Z \to \mathbb{C}^\mathcal{L} \) by

\[
H_\mathcal{L}(z)(\lambda) = h_\lambda(z) \quad \text{for all } \lambda \in \mathcal{L}.
\]

Similarly, we let \( \mathcal{M} = \bigcup_i \mathcal{M}_i \) and \( \text{Lk}(V) = \bigcup_i \text{Lk}(v_i) \). Hence \( \mathcal{L} \cup \mathcal{M} \) is a generating set for \( H_1(\text{Lk}(V)) \).

### 3.3 Tangential angle structures

Our study of derivatives naturally leads us (at least implicitly) to the tangent space of the space of all angle structures on \( \mathcal{T} \). Following [5], the space of all tangential angle structures \( \text{TAS} = \text{TAS}(\mathcal{T}) \) is the set of all \( \alpha \in \mathbb{R}^\mathcal{L} \) such that

- \( \sum_{\sigma} \alpha(q) = 0, \forall \sigma \in T \), and
- \( \sum_{q \in \Box} i(q, e) \alpha(q) = 0, \forall e \in E \).

The endgame of the proof of Theorem 9 uses a specific spanning set of

\[
\text{TAS}_\mathcal{L} = \text{TAS}_\mathcal{L}(\mathcal{T}) = \{ w \in \text{TAS} \mid \sum_{q \in \Box} \eta(q, \lambda) w(q) = 0 \ \forall \lambda \in \mathcal{L} \} \leq \text{TAS},
\]

which we will now determine. For every edge \( e \in E \) and every normal curve \( \gamma \) on a vertex link, let \( Q_e, Q_\gamma \in \mathbb{R}^\mathcal{L} \) be defined by

\[
Q_e = \sum_{q, q \sim e} (q')^* - (q'')^*,
\]

\[
Q_\gamma = \sum_{q \in \Box} \eta(q, \gamma) \left( (q')^* - (q'')^* \right).
\]

In their study of angle structure on cusped manifolds, Futer and Guéritaud [3, Section 4] introduced \( Q_e \) and \( Q_\gamma \) under the names of leading–trailing deformation around \( e \) and leading–trailing deformation along \( \gamma \) respectively. In fact, for every edge \( e \), one may choose a normal closed curve \( \rho \) about one endpoint of \( e \) and an orientation of \( \rho \) such that \( Q_e = Q_\rho \). Hence \( Q_e \) can be thought of as a leading–trailing deformation along \( \rho \). Some of the results of [3] extend directly to the more general setting of an oriented pseudo-manifold. As in [3, Lemma 4.5], we have \( Q_e, Q_\gamma \in \text{TAS} \).
Lemma 4 [3, Lemma 4.4] Let $\alpha, \beta$ be oriented closed normal curves on $\text{Lk}(V)$ that intersect transversely, if at all. Then

$$\frac{\partial}{\partial Q_\beta} \text{im}(h_\alpha) = \sum_{q \in \Box} \eta(q, \alpha) Q_\beta(q) = 2 \iota(\alpha, \beta).$$

Proof The proof in [3, Lemma 4.4] does not use the fact that $\partial M$ is a union of tori (and in particular that in this case $h_\alpha$ is independent of the choice of normal curve in a homology class) and applies verbatim.

Lemma 5 The set $\{Q_e\}_{e \in E}$ spans a subspace of dimension $|T| - \sum_i g_i$.

Proof For each tetrahedron $\sigma_j \in T$ fix a normal quadrilateral type $q_j < \sigma_j$. Let $B$ be the $(|E| \times 3|T|)$ matrix whose $i$-th row is the vector $Q_{e_i}$, namely

$$\left( Q_{e_1}(q_1), Q_{e_1}(q'_1), Q_{e_1}(q''_1), \ldots, Q_{e_1}(q_{[T]}), Q_{e_1}(q''_{[T]}), Q_{e_1}(q''_{[T]}) \right),$$

where $e_i \in E$. For every vertex $v \in V$, we define a row vector $r_v \in \mathbb{R}^{|E|}$, whose $i$-th entry is the number of endpoints that the $i$-th edge $e_i$ has at the vertex $v$.

In [3, Lemma 3.3], a matrix $A$ and vectors $r_e$ were constructed in a similar fashion, and they are related to $B$ and $r_v$ as follows:

- Let $\sigma \in S_3$ be the permutation $(123)$ and $A_i, B_i$ be the $i$-th columns of $A, B$ respectively. Then for all $0 \leq k \leq |T|$ and $1 \leq i \leq 3$, $B_{3k+i}$ is the column vector made up of the last $|E|$ entries of $A_{3k+\sigma^{-1}(i)} - A_{3k+\sigma(i)}$;
- $r_v$ is the row vector made of the last $|E|$ entries of $r_e$.

Following the proof of [3, Lemma 3.3], one checks that the vectors $r_v$ form a basis for the row null space of $B$ and therefore rank($B$) = $|E| - |V|$. The conclusion follows from Lemma 1.

The following result generalises [3, Proposition 4.6]. To simplify notation, we write $\{Q_\lambda\} = \{Q_\lambda\}_{\lambda \in \mathcal{L}}$ etc.

Lemma 6 The set $\{Q_\lambda\} \cup \{Q_\mu\}$ is linearly independent, and $\text{Span}(\{Q_\lambda\} \cup \{Q_\mu\}) \cap \text{Span}\{Q_e\} = \{0\}$. In particular,

1. $\dim \text{Span}(\{Q_\mu\}) = \sum_i g_i$,
2. $\dim \text{Span}(\{Q_e\} \cup \{Q_\lambda\}) = |T|$,
3. $\text{Span}(\{Q_e\} \cup \{Q_\lambda\} \cup \{Q_\mu\}) = \text{TAS}(T)$.

Proof For every edge $e$, choose a normal closed curve $\rho$ about one endpoint of $e$. Then we can choose an orientation on $\rho$ such that $Q_e = Q_\rho$ and think of $Q_e$ as a leading–trailing deformation along $\rho$.

Let $I : \text{Span}(\{Q_\lambda\} \cup \{Q_\mu\} \cup \{Q_\rho\}) \longrightarrow \mathbb{R}^{|L| + |M|}$ be the map defined as follows. If $X = \sum a_i Q_\lambda + \sum b_j Q_\mu + \sum c_k Q_\rho$, set $\xi = \sum a_i \lambda_i + \sum b_j \mu_j + \sum c_k \rho_k$ and define

$$I(X) = \left( \frac{\partial}{\partial Q_\lambda} \text{im}(h_\xi), \ldots, \frac{\partial}{\partial Q_{\lambda i}} \text{im}(h_\xi), \frac{\partial}{\partial Q_\mu} \text{im}(h_\xi), \ldots, \frac{\partial}{\partial Q_{\mu i}} \text{im}(h_\xi) \right).$$

The linearity of $h_\xi$ implies that $I$ is linear, and by Lemma 4, we have
• \( I(Q_{\rho}) = (0, \ldots, 0) \), i.e. \( \text{Span}\{Q_{\rho}\} \subset \text{Ker}\ I \);
• \( I(Q_{\lambda_i}) \) has 1 in the \((|L| + i)\)-th entry and 0 everywhere else, i.e. maps to the \((|L| + i)\)-th standard basis vector.
• \( I(Q_{\mu_j}) \) has 1 in the \(j\)-th entry and 0 everywhere else, i.e. maps to the \(j\)-th standard basis vector.

This implies \( \text{Span} (\{Q_{\lambda}\} \cup \{Q_{\mu}\}) \cong \text{Im} I = \mathbb{R}^{|L|+|M|} \), so \( \{Q_{\lambda}\}_{\lambda \in L} \cup \{Q_{\mu}\}_{\mu \in M} \) is linearly independent, and \( \text{Span} (\{Q_{\lambda}\} \cup \{Q_{\mu}\}) \cap \text{Span}\{Q_{\rho}\} = \{0\} \). This implies (i), and together with Lemma 5, it implies (ii). By [5, Corollary 2.3], we have

\[
\dim \text{TAS} = |V| - |E| + 2|T| = |T| + \sum_i g_i.
\]

It now follows from Lemma 5 that \( \dim \text{Span} (\{Q_{\rho}\} \cup \{Q_{\lambda}\} \cup \{Q_{\mu}\}) = \dim \text{TAS} \), which implies (iii). We also note that this shows \( \text{Span}\{Q_{\rho}\} = \text{Ker} I \).

**Theorem 7** The set \( \{Q_{\rho}\} \cup \{Q_{\lambda}\} \) is a subset of \( \text{TAS}_L \), whereas \( \text{Span}(\{Q_{\mu}\}) \cap \text{TAS}_L = \{0\} \). Therefore \( \text{Span}(\{Q_{\rho}\} \cup \{Q_{\lambda}\}) = \text{TAS}_L \).

**Proof** Since \( \{Q_{\rho}\} \cup \{Q_{\lambda}\} \) is a subset of \( \text{TAS} \), the first part follows from checking that for all \( \lambda, \lambda' \in L \) and \( e \in E \)

\[
\sum_{\eta \in \square} \eta(\eta', \lambda') Q_e(\eta) = 0, \quad (5)
\]

\[
\sum_{\eta \in \square} \eta(\eta', \lambda) Q_e(\eta) = 0. \quad (6)
\]

Observe that for all \( q, \eta \in \square \),

\[
(q')^*(\eta) - (q'')^*(\eta) = -(q'^*)^{*(q)} - (q''^*)^{*(q)}
\]

hence

\[
\sum_{\eta \in \square} \eta(\eta, \lambda') Q_e(\eta) = \sum_{\eta \in \square} \eta(\eta, \lambda') \sum_{q \in \square} i(g, e) ((q')^*(\eta) - (q'')^*(\eta))
\]

\[
= \sum_{q \in \square} i(g, e) \sum_{\eta \in \square} \eta(\eta, \lambda') ((q')^*(\eta) - (q'')^*(\eta))
\]

\[
= -\sum_{q \in \square} i(g, e) \sum_{\eta \in \square} \eta(\eta, \lambda') ((q')^*(\eta) - (q'')^*(\eta))
\]

\[
= -\sum_{q \in \square} i(g, e) Q_{\lambda'}(q) = 0,
\]

as \( Q_{\lambda'} \in \text{TAS} \). This shows \( Q_{\rho} \in \text{TAS}_L \).

Now suppose \( \lambda, \lambda' \in L \), then \( \iota(\lambda', \lambda) = 0 \) as they are disjoint by assumption. By Lemma 4, this shows that \( \{Q_{\lambda}\} \), and in particular \( \{Q_{\rho}\} \cup \{Q_{\lambda}\} \), is a subset of \( \text{TAS}_L \), and by Lemma 6(ii),

\[
\dim \text{TAS}_L \geq |T|.
\]

For the next part, let \( \sum_{i,j} a_{ij}^2 Q_{\mu_{ij}} \) be an element in \( \text{Span}(\{Q_{\mu}\}) \) and suppose by contradiction \( \sum_{i,j} a_{ij}^2 Q_{\mu_{ij}} \in \text{TAS}_L \). Then for all \( l \) and for all \( \lambda'_l \in L_l \), by Lemma 4,

\[
0 = \sum_{q \in \square} \eta(q, \lambda'_l) \left( \sum_{i,j} a_{ij}^2 Q_{\mu_{ij}}(q) \right) = \sum_{i,j} a_{ij}^2 \iota(\lambda'_l, \mu_{ij}).
\]
But \( \iota(\chi^l_k, \mu^j_k) \neq 0 \) if and only if \( i = l \) and \( k = j \), therefore \( a^l_k = 0 \) for all \( l, k \) and
\[
\text{Span}\{\{Q_{\mu}\}\} \cap \text{TAS}_L = \{0\}.
\]
It follows from Lemma 6(i) that \( \dim \text{TAS}_L \leq |T| \). Hence \( \dim \text{TAS}_L = |T| \) and so
\[
\text{Span}\{\{Q_{\epsilon}\} \cup \{Q_{\lambda}\}\} = \text{TAS}_L.
\]
\[\blacksquare\]

### 3.4 A Parametrization of \( G^{-1}(u) \)

Using the identification \( Z = \mathbb{H}^T \), we write \( H_L : \mathbb{H}^T \rightarrow \mathbb{C}^E \), giving the map
\[
(G, H_L) : \mathbb{H}^T \rightarrow \mathbb{C}^E \times \mathbb{C}^L
\]
\[
z \mapsto (G(z), H_L(z)).
\]
We already remarked that when \( G(z) \neq (2\pi i, \ldots, 2\pi i) \), the value of the boundary map \( H_L \) depends on the choice of normal curves representing homology classes of peripheral curves. The following lemma shows that the \textit{injectivity} of the differential of \( (G, H_L) \) is independent of this choice.

**Lemma 8** For each vertex \( v_i \in V \), let \( g_i \) be the genus of \( \text{Lk}(v_i) \). Let \( A_i = \{\alpha^i_1, \ldots, \alpha^i_{g_i}\} \) and \( B_i = \{\beta^i_1, \ldots, \beta^i_{g_i}\} \) be two sets of longitudes on \( \text{Lk}(v_i) \), such that \( \alpha^i_j \) and \( \beta^i_j \) are representatives of the same element in \( H_1(\text{Lk}(v_i)) \), and set \( A = \bigcup_i A_i \) and \( B = \bigcup_i B_i \). Then
\[
\text{rank} \ d(G, H_A) = \text{rank} \ d(G, H_B).
\]
In particular \( d(G, H_A) \) is injective if and only if \( d(G, H_B) \) is injective.

**Proof** As \( \alpha^i_j \) and \( \beta^i_j \) are in the same homotopy class,
\[
h_{\alpha^i_j}(z) = h_{\beta^i_j}(z) + \sum_{e \in E} a_e G(z)(e)
\]
for some \( a_e \in \mathbb{R} \).

Hence
\[
\nabla h_{\alpha^i_j}(z) = \nabla h_{\beta^i_j}(z) + \sum_{e \in E} a_e \nabla G(z)(e),
\]
i.e. each row of \( d(H_A) \) is a linear combinations of the corresponding row of \( d(H_B) \) and rows of \( d(G) \). It follows that \( d(G, H_A) \) is related to \( d(G, H_B) \) by elementary row operations, hence they have the same rank. \[\blacksquare\]

**Theorem 9** The derivative \( d(G, H_L) : T_z(\mathbb{H}^T) \rightarrow \mathbb{C}^E \times \mathbb{C}^L \) is injective for any \( z \in \mathbb{H}^T \).

**Proof** Let \( z \in Z \). For every \( q \in \Box \), \( z(q) \in \mathbb{H} \) uniquely determines three angles \( \alpha, \beta \) and \( \gamma \) of the triangle with vertices \( (0,1,z(q)) \) such that \( z(q) = \frac{\sin(\beta)}{\sin(\gamma)} e^{i\alpha} \) and
\[
\begin{align*}
z(q') &= \frac{\sin(\gamma)}{\sin(\alpha)} e^{i\beta} \\
z(q'') &= \frac{\sin(\alpha)}{\sin(\beta)} e^{i\gamma}.
\end{align*}
\]

10
Hence we can identify $Z$ with the set $X = \{ x \in \mathbb{R}^{|E|} \mid \sum_{q<\sigma} x(q) = \pi \}$ via the map $\phi^T : X \to Z$, where

$$\phi^T (x)(q) = \frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)}.$$ 

Under this identification, we can write $(G, H_L)$ as a map $(G, H_L) : X \to \mathcal{C}^E \times \mathcal{C}^L = (\mathbb{R} \times \mathbb{R})^{E \cup L}$ where, for each edge $e \in E$ and curve $\lambda \in L$,

$$G(x)(e) = \sum_{q,q\sim e} \log \left( \frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)} \right)$$

$$= \left( \sum_{q,q\sim e} \log(\sin(x(q')))) - \log(\sin(x(q''))), \sum_{q \in \Box} i(q,e)x(q) \right).$$

$$H_L(x)(\lambda) = \sum_{q \in \Box} \eta(q, \lambda) \log \left( \frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)} \right)$$

$$= \left( \sum_{q \in \Box} \eta(q, \lambda) \left( \log(\sin(x(q'))) - \log(\sin(x(q''))) \right), \sum_{q \in \Box} \eta(q, \lambda)x(q) \right).$$

Now our goal is to show that $d(G, H_L) : T_x X \to (\mathbb{R} \times \mathbb{R})^{E \cup L}$ is injective for all $x \in X$. Let $w \in T_x X = \{ w \in \mathbb{R}^{|E|} \mid \sum_{q<\sigma} w(q) = 0 \ \forall \sigma \in T \}$ be a tangent vector such that $d(G, H_L)(x)(w) = 0$.

We recall the elementary fact that $\frac{d}{dt} \log(\sin t) = \cot t$. For every edge $e$,

$$\sum_{q,q\sim e} \cot(x(q'))w(q') - \cot(x(q''))w(q'') = 0,$$  \hspace{1cm} (7)

$$\sum_{q \in \Box} i(q,e)w(q) = 0.$$ \hspace{1cm} (8)

and for every curve $\lambda$,

$$\sum_{q \in \Box} \eta(q, \lambda) \left( \cot(x(q'))w(q') - \cot(x(q''))w(q'') \right) = 0,$$ \hspace{1cm} (9)

$$\sum_{q \in \Box} \eta(q, \lambda)w(q) = 0.$$ \hspace{1cm} (10)

Observe that (8) and (10) imply $w \in T_{\Lambda L} \leq \Lambda X \subset T_x X$.

Let $F : \mathbb{R}^{|E\Box|} \to \mathbb{R}$ be the volume function

$$F(x) = \sum_{q \in \Box} \Lambda(x(q)),$$

where $\Lambda(x) = -\int_0^x \log |2\sin(u)| du$ is the Lobachevsky function. The Hessian of $F$ is the diagonal matrix with diagonal entries $-\cot(x(q'))$, and it is negative definite at each point $x \in X$ (see, for instance, [3, pg. 17]). Recall that for $e \in E$ and $\lambda \in L$, we have ,

$$Q_e = \sum_{q,q\sim e} (q')^* - (q'')^*,$$

$$Q_\Lambda = \sum_{q \in \Box} \eta(q, \lambda) ((q')^* - (q'')^*).$$
Then by (7) and (9)

\[ Q_e \cdot \text{Hess}_x(F)w = 0 \quad \forall e \in E, \]

\[ Q_\lambda \cdot \text{Hess}_x(F)w = 0 \quad \forall \lambda \in \mathcal{L}. \]

Theorem 7 shows that \( \text{TAS}_\mathcal{L} \) is spanned by \( \{Q_e\}_{e \in E} \cup \{Q_\lambda\}_{\lambda \in \mathcal{L}} \). Hence \( w \) can be written as a linear combination of the \( Q_e \) and \( Q_\lambda \), and so

\[ w \cdot \text{Hess}_x(F)w = 0. \]

But since \( \text{Hess}_x(F) \) is negative definite, we have \( w = 0 \) and hence \( d(G,H_\mathcal{L}) \) is injective. ■

Theorem 2 shows that the rank of \( dG \) is constant \( |T| - \sum_{v \in V} g_v \), hence Theorem 9 implies the following result.

**Corollary 10** For all \( u \in \mathfrak{g}^E \) such that \( G^{-1}(u) \neq \emptyset \), the restriction map

\[ H_\mathcal{L}|_{G^{-1}(u)} : G^{-1}(u) \rightarrow \mathfrak{g}^\mathcal{L} \]

is a local diffeomorphism onto its image.

## 4 Two Dehn-surgery components

Our first example of a one-cusped hyperbolic 3–manifold of finite volume illustrates Choi’s original result. It also has the property that the two discrete and faithful characters lie on different components of the \( PSL_2(\mathbb{C}) \)–character variety. It was found via the following construction due to Nathan Dunfield. Let \( N \) be a 2–cusped manifold with strong geometric isolation (see [7] for a definition). Let \( A \) be the collection of all hyperbolic Dehn fillings on the first cusp in \( N \); let \( M_1, M_2, M_3, \ldots \) be distinct manifolds in \( A \). Fix an orientation of \( N \), and let \( \chi_0 \) be the character of an associated holonomy representation. Let \( \chi_n \) be the character of the holonomy representation of \( M_n \) that is in a small open neighbourhood of \( \chi_0 \), and let \( X_n \) be the component of the character variety of \( M_n \) which contains \( \chi_n \). By strong geometric isolation, the image of \( X_n \) in the character variety of the second cusp is independent of \( n \), call it \( C \). Now the differential of the volume function on \( X_n \) pulls back from a 1–form on \( C \). Hence, the quantity

\[ \max\{\text{vol}(\chi) \mid \chi \in X_n\} - \min\{\text{vol}(\chi) \mid \chi \in X_n\} \]

is a constant \( V \) independent of \( n \). By volume rigidity, \( V \leq 2 \min(\text{vol}(M_n)) \). So if \( M_n \) has non-minimal volume (and of course \( \text{vol}(M_n) \rightarrow \text{vol}(N) \) so there are many such), it follows that \( \chi_n \) and its complex conjugate must lie on different components of the character variety of \( M_n \).

The following explicit example was found by looking in the census due to Callahan, Hildebrand and Weeks (as shipped with Regina [1]) for an example given by Neumann and Reid [7]. Let \( M \) be the manifold \( t12046 \) in this census, and let \( N \) be the manifold obtained from \( M \) by \( (2,1) \)-Dehn filling on the first cusp. We fix the following ideal triangulation \( T \) on \( N \):

Let \( z_i \) be a shape parameter of the tetrahedron \( T_i \), with respect to the edge \((01)\), with orientation such that \( z'_i \) is the parameter at \((02)\) and \( z''_i \) is at \((03)\). A standard computation provides the two solutions

\[ z^{(0)} = \left( i, \frac{1+i}{2}, i, \frac{2+i}{2}, \frac{3+i}{5}, i, \frac{1+i}{2} \right) \in \mathfrak{g}^7 \]
and its complex conjugate $\overline{z(0)}$ corresponding to discrete and faithful characters. Notice that $z(0)$ is positively oriented. The one-dimensional components $C_0$ and $C'_0$ containing them have the following rational parametrizations:

$$
\varphi_0(u) = \begin{pmatrix}
\frac{-1+i+u}{-1+(1-i)u} \\
\frac{1+i}{(2+2i)-2u} \\
\frac{(1+i) - \frac{i}{u}}{2(1+1+u)^2} \\
\frac{(-1+u)(-i+(1+i)u)}{u} \\
\frac{(1+i)+2u}{(2+2i)u-2u^2}
\end{pmatrix}
$$

and

$$
\varphi'_0(u) = \begin{pmatrix}
\frac{-1+i+u}{-1+(1+i)u} \\
\frac{1-i}{(2-2i)-2u} \\
\frac{(1-i) + \frac{i}{u}}{2(1+1+u)^2} \\
\frac{(-1+u)(i+(1-i)u)}{u} \\
\frac{(1+i)-2u}{(2-2i)u-2u^2}
\end{pmatrix}
$$

In particular, notice that $\varphi_0(u) = \varphi'_0(\overline{u})$.

Henceforth, we will refer to the $j$-th components of $\varphi_0(u)$ and $\varphi'_0(u)$ by $(\varphi_0)_j(u)$ and $(\varphi'_0)_j(u)$ respectively. The natural domain of $\varphi_0$ is $\mathbb{C} \setminus X$, where $X$ is the finite set of poles of $(\varphi_0)_j$.

A direct computation shows that $X$ contains precisely 6 elements, whose image via $\varphi_0$ are all ideal points of $C_0$, therefore $\text{Im}(\varphi_0) = \text{Im}(\varphi'_0)$.

Together with the image of $u = \infty$, they sum up to a total of 7 ideal points, hence the image of $\varphi_0$ is a 7-punctured sphere. Moreover, a direct calculation reveals that the images of $\varphi_0$ and $\varphi'_0$ are disjoint.

Denoting by $\text{im}(w)$ the imaginary part of a complex number $w \in \mathbb{C}$, we recall that the complete solution $\varphi_0(i) = z(0)$ is geometric in the sense that

$$
\text{im}((\varphi_0)_j(i)) = \text{im}(z(0)) > 0 \quad \forall j \in \{1, \ldots, 7\}.
$$

The set of all positively oriented solutions is

$$
Q = \bigcap_{j=1}^{7} \{ u \in \mathbb{C} \mid \text{im}((\varphi_0)_j(u)) > 0\}.
$$

| Tetrahedron | Face 012 | Face 013 | Face 023 | Face 123 |
|-------------|----------|----------|----------|----------|
| 0           | 4 (203)  | 2 (321)  | 6 (032)  | 5 (120)  |
| 1           | 3 (312)  | 2 (012)  | 4 (013)  | 6 (031)  |
| 2           | 1 (013)  | 4 (213)  | 5 (123)  | 0 (310)  |
| 3           | 6 (312)  | 4 (012)  | 5 (203)  | 1 (120)  |
| 4           | 3 (013)  | 1 (023)  | 0 (102)  | 2 (103)  |
| 5           | 0 (312)  | 6 (012)  | 3 (203)  | 2 (023)  |
| 6           | 5 (013)  | 1 (132)  | 0 (032)  | 3 (120)  |

Table 1: The ideal triangulation $\mathcal{T}$ on $N$. 


For $u = x + iy$, $\{\text{im}\, ((\varphi_0)_j(u)) = 0\}$ is a 1-dimensional subvariety of $\mathbb{R}^2$, which can be explicitly computed from $\varphi_0$. 

\[
\text{im}\, ((\varphi_0)_1) = (x - 1)^2 + (y - \frac{1}{2})^2 - \frac{1}{4}, \\
\text{im}\, ((\varphi_0)_2) = y - x, \\
\text{im}\, ((\varphi_0)_3) = x^2 + y^2 - x, \\
\text{im}\, ((\varphi_0)_4) = (y + \frac{1}{2})^2 - (x\sqrt{3} - \frac{1}{2\sqrt{3}})^2 - \frac{1}{6}, \\
\text{im}\, ((\varphi_0)_5) = -1 + 3x - 3x^2 + x^3 - x^2y + y^2 + xy^2 - y^3, \\
\text{im}\, ((\varphi_0)_6) = y, \\
\text{im}\, ((\varphi_0)_7) = 2x - 3x^2 - 2x^3 + 2xy + y^2 - 2xy^2.
\]

As shown in Figure 2, $Q$ is a simply connected open set of the plane and 
\[\partial Q \subset V \left(\text{im}\, ((\varphi_0)_4(u)), \text{im}\, ((\varphi_0)_5(u)), \text{im}\, ((\varphi_0)_7(u))\right).\]

We deduce that $\varphi_0(Q)$ is an ideal square contained in $\pi'(C_0)$. 

\*Figure 2: $Q$ is an ideal square.*
5 A reducible complex-curvature level set

Let $M$ be the oriented pseudo–manifold given by the ideal triangulation $\mathcal{T}$ shown in Figure 3. $M$ has two vertices $v_1$ and $v_2$, whose links are closed orientable surfaces of genus $g_1 = 1$ and $g_2 = 2$ respectively. The induced triangulations of the links $\text{lk}(v_1)$ and $\text{lk}(v_2)$ are shown in the bottom right corner of Figure 3 and in Figure 4, respectively.

Figure 3: Shown are the five tetrahedra in the triangulation of $M$ and, in the bottom right corner, the induced triangulation of the vertex link of $v_1$ (viewed from the cusp).

Figure 4: Triangulation induced on $v_2$ by $\mathcal{T}$.

We use the same notation as in the previous example, so $z_i$ is the shape parameter of tetrahedron $T_i$ with respect to edge $(12)$, and $z = (z_0, z_1, z_2, z_3, z_4)$. From the face pairings of $\mathcal{T}$ we deduce the following complex-curvature and log-curvature maps:
For instance, let $u \in \text{Im} G$, $G^{-1}(u)$ is the 3-dimensional variety generated by the ideal

$$I = (z_1 + e^{-u_0} - 1, (z_0 - 1)z_2(z_3 - 1)(z_4 - 1) - e^{u_1}z_0(1 - z_1)z_3z_4).$$

For instance, let

$$z^0 = \left( e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{\pi i}{3}} \right),$$

then $u^0 = G(z^0) = \left( \frac{\pi i}{3}, \frac{5\pi i}{3}, 2\pi i, 6\pi i \right)$ and

$$G^{-1}(u^0) = \left\{ (z_0, z_1, z_2, z_3, z_4) \in \mathbb{H}^5 \mid z_1 = e^{\frac{\pi i}{3}}, z_2 = \frac{z_0z_3z_4e^{\frac{5\pi i}{3}}}{(z_0 - 1)(z_3 - 1)(z_4 - 1)} \right\}.$$
For
\[ z^1 = \left( \frac{1}{1 - e^{\frac{5}{6}\pi i}}, \frac{1}{1 - e^{\frac{5}{6}\pi i}}, \frac{1}{1 - e^{\frac{5}{6}\pi i}}, \frac{1}{1 - e^{\frac{5}{6}\pi i}} \right), \]
\[ u^1 = G(z^1) = \left( \frac{\pi i}{3}, \frac{11}{3}\pi i, 2\pi i, 4\pi i \right), \]
so
\[ G(z^0) \neq G(z^1) \text{ but } c(z^0) = c(z^1). \]

By analytic continuation \( G^{-1}(u^0) \) and \( G^{-1}(u^1) \) are disjoint, hence the above argument shows that \( G^{-1}(u^0) \) and \( G^{-1}(u^1) \) are disjoint complex varieties contained in the complex variety \( g^{-1}(c(z^0)) \). Moreover, because all shape parameters are assumed to have positive imaginary part, it follows from (13) that \( 0 < G(z)(e_1) < 5\pi \) and hence
\[ g^{-1}(c(z^0)) = G^{-1}(u^0) \cup G^{-1}(u^1). \]

Let \( \mathcal{L} = \{\lambda_1, \lambda_2, \lambda_3\} \) be the set of three simple closed curves shown in Figures 3 and 4. The boundary map with respect to \( \mathcal{L} \) and its differential are
\[
e^{H_{\mathcal{L}}}(z) = \begin{pmatrix} z'_{1/4} \\ z_{1/2} \\ z_0^{x_1} z_0^{x_2} z_0^{x_3} \end{pmatrix} \]
\[
H_{\mathcal{L}}(z) = \begin{pmatrix} -\log(1 - z_0) + \log(z_2) - \log(z_2 - 1) \\ \log(z_1) - \log(z_4) \\ \log(z_0) + \log(1 - z_1) - \log(1 - z_2) - \log(1 - z_3) - \log(1 - z_4) \end{pmatrix}.
\]
and
\[
dH_{\mathcal{L}}(z) = \begin{pmatrix} 1 - z_0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{z_1}{z_2} & 1 & -\frac{1}{z_4} \\ \frac{1}{z_0} & \frac{1}{1 - z_1} & \frac{1}{1 - z_2} & \frac{1}{1 - z_3} & \frac{1}{1 - z_4} \end{pmatrix}.
\]

Together with the differential of the log-curvature map we obtain:
\[
d(G, H)(z) = \begin{pmatrix} 0 & \frac{-1}{z_1 - 1} & 0 & 0 & 0 \\ \frac{1}{z_0(z_0 - 1)} & \frac{z_1 - 1}{z_1 - 1} & \frac{1}{z_2} & \frac{1}{z_3(z_3 - 1)} & \frac{1}{z_4(z_4 - 1)} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{z_0(z_0 - 1)} & \frac{z_1 - 1}{z_1 - 1} & \frac{-1}{z_2} & \frac{-1}{z_3(z_3 - 1)} & \frac{-1}{z_4(z_4 - 1)} \\ \frac{1}{z_0} & \frac{-1}{z_1} & \frac{1}{z_2} & \frac{1}{z_3} & \frac{-1}{z_4} \end{pmatrix}.
\]

The determinant of the minor obtained by removing the second and third row from \( d(G, H)(z) \) is
\[
-2(-1 + z_0 + z_3 + z_4 + z_0 z_2 - z_0 z_2 z_3 - z_0 z_2 z_4 - z_3 z_4 - z_0 z_3 z_4 + z_0 z_2 z_3 z_4)
\]
\[
\frac{z_0 z_1 z_2 z_3 z_4 (1 - z_0) (1 - z_1) (1 - z_2) (1 - z_3) (1 - z_4)}{z_0 z_1 z_2 z_3 z_4 (1 - z_0) (1 - z_1) (1 - z_2) (1 - z_3) (1 - z_4)}.
\]

For all \( u \in \text{Im}(G) \), the restriction map
\[ H_{\mathcal{L}}|_{G^{-1}(u)} : G^{-1}(u) \rightarrow \Phi^3 \]
is a local diffeomorphism onto its image. In particular we can parameterize $G^{-1}(u)$ through $H_L$. For $u^0$ as above, we get the following local parameterization around $z^0$,

$$
\begin{cases}
  z_0 = 1 - z_4k; \\
  z_1 = e^{\frac{x}{3}} \\
  z_2 = \frac{-z_4k'}{1-z_4k} \\
  z_3 = z_4e^{t_2} \\
  z_4 \text{ is the solution of the equation } \frac{kk'}{1-z_4k'} = \frac{(1-z_4k)e^{t_2}+\frac{3}{2}\pi i}{(z_4e^{t_2}-1)(z_4-1)}
\end{cases}
$$

where $k = e^{\frac{5}{6}\pi i + t_2 + t_3 - t_1}$, $k' = e^{\frac{5}{6}\pi i + t_2 + t_3 + t_1}$ and for all $(t_1, t_2, t_3) \in \mathbb{C}^3$ close to $H_L(e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}) = (0, 0, \pi i)$.

**Acknowledgements.** The first author is supported by a Commonwealth of Australia International Postgraduate Research Scholarship. The research of the second author is partially supported by the United States National Science Foundation grants NSF DMS 1222663, 1207832 and 1405106. Research of the third author is partially supported by Australian Research Council grant DP140100158. The authors are grateful to the referees for comments that improved this paper.

**References**

[1] Benjamin A. Burton and Ryan Budney and William Pettersson and others, *Regina: Software for 3-manifold topology and normal surface theory*, [http://regina.sourceforge.net/], 1999–2014.

[2] Young-Eun Choi: *Positively oriented ideal triangulations on hyperbolic three-manifolds*, Topology 43, no. 6, 1345–1371 (2004).

[3] David Futer and François Guéritaud: *From angled triangulations to hyperbolic structures*, Contemp. Math., 541, 159-182 (2011).

[4] Wolfgang Haken: *Theorie der Normalflächen*, Acta Math. 105, 245–375 (1961).

[5] Feng Luo: *Volume optimization, normal surfaces, and Thurston’s equation on triangulated 3-manifolds* J. Differential Geom. 93 (2013), no. 2, 299–326.

[6] Walter D. Neumann: *Combinatorics of triangulations and the Chern Simons invariant for hyperbolic 3-manifolds*, in Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State, Berlin, 243–272, (1992)

[7] Walter D. Neumann and Alan Reid: *Rigidity of cusps in deformations of hyperbolic 3-orbifolds*, Math. Ann. 295 (1993), no. 2, 223–237.

[8] Walter D. Neumann and Don Zagier: *Volumes of hyperbolic three–manifolds*, Topology, 24, 307-332 (1985).

[9] Henry Segerman and Stephan Tillmann: *Pseudo-Developing Maps for Ideal Triangulations I: Essential Edges and Generalised Hyperbolic Gluing Equations*, Contemp. Math., 560, 85–102 (2011).

[10] Stephan Tillmann: *Normal surfaces in topologically finite 3-manifolds*, L’Enseignement Mathématique, 54, (2008), no. 2, 329–380.
Alex Casella,
School of Mathematics and Statistics F07,
The University of Sydney,
NSW 2006 Australia
(casella@maths.usyd.edu.au)

Feng Luo,
Department of Mathematics,
Rutgers University,
New Brunswick, NJ 08854, USA
(fluo@math.rutgers.edu)

Stephan Tillmann,
School of Mathematics and Statistics F07,
The University of Sydney,
NSW 2006 Australia
(tillmann@maths.usyd.edu.au)