Contravariant symbol quantization on $S^2$

A.V.Karabegov
Joint Institute for Nuclear research, Dubna, Russia
March 28, 2022

Abstract

We define an algebra of contravariant symbols on $S^2$ and give an algebraic proof of the Correspondence Principle for that algebra.

§0. Introduction.

In [1] F.A.Berezin introduced a general concept of quantization on a symplectic manifold $\Omega$. To define a quantization on $\Omega$ one needs the following data. Let $F$ be a set of positive numbers with a limit point 0. For each $h \in F$ let $A_h \subset \mathcal{C}^\infty(\Omega)$ be an algebra with multiplication $*_h$ such that for $h < h'$, $A_h \supset A_{h'}$ as linear spaces. Denote $A = \cup A_h$. Assume that for each $h \in F$ there is given a representation of $A_h$ in some Hilbert space $H_h$. These data define a quantization on $\Omega$ if the Correspondence Principle holds, i.e. for $f, g \in A$

$$\lim_{h \to 0} f *_h g = fg; \quad \lim_{h \to 0} h^{-1}(f *_h g - g *_h f) = i\{f, g\},$$

where $\{\cdot, \cdot\}$ denotes a Poisson bracket on $\Omega$. If for $f \in A_h$ $\hat{f}$ denotes a corresponding operator in $H_h$, the function $f$ is called a symbol of $\hat{f}$.

Thus to define a quantization one may start from an appropriate construction of symbols. In [2] Berezin introduced covariant and contravariant operator symbols and extensively investigated their various properties. In [3] he applied covariant symbols to quantization of Kähler manifolds. A particular example of covariant symbol quantization on $S^2$ was considered in [4].
Therein Berezin described the algebra of covariant symbols and gave an analytic proof of the Correspondence Principle for covariant symbol quantization on $S^2$.

A more advanced approach to quantization as to deformation of classical mechanics was developed in [4].

In this paper we will define algebras of co- and contravariant symbols on $S^2$, two of them in the same framework and give an algebraic proof of the Correspondence Principle both for co- and contravariant symbol quantizations.

§1. Covariant and contravariant symbols on $S^2$.

Consider a Hilbert space $L^2(C, d\alpha_n)$, with a measure

$$d\alpha_n(z, \bar{z}) = \frac{n}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^{n+1}}$$

and with the scalar product denoted by $(\cdot, \cdot)$. Let $H_n$ be the n-dimensional subspace of $L^2(C, d\alpha_n)$ of all polynomials in $z$ of degree $\leq n - 1$. For $v \in C$ the vectors $e_v(z) = (1 + z\bar{v})^{n-1} \in H_n$ have a following reproducing property, for $f \in H_n$ $f(v) = (f, e_v)$.

Definition. The covariant symbol of an operator $A \in H_n$ is the function $f(z, \bar{z}) = (Ae_{\bar{z}}, e_{\bar{z}})/(e_{\bar{z}}, e_{\bar{z}})$.

To define a contravariant symbol one needs a notion of the canonical measure $\mu_n$ on $C$,

$$d\mu_n(z, \bar{z}) = (e_{\bar{z}}, e_{\bar{z}})d\alpha_n(z, \bar{z}) = \frac{n}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}.$$  

Let $P_{z,\bar{z}}$ denote the orthogonal projection operator on $e_{\bar{z}}$ in $H_n$.

Definition. A function $f(z, \bar{z})$ is a contravariant symbol of the operator $A$ if

$$A = \int f(z, \bar{z}) P_{z,\bar{z}} d\mu_n(z, \bar{z}).$$

Let $G = SU(2)$ be the group of all unitary $2 \times 2$-matrices with the determinant 1. Let $G$ act on $C$ from the right by fractional-linear transformations, for

$$g = \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \in G,$$  

(1)
\[ z \in \mathbb{C} \colon zg = (az - \bar{b})/(bz + \bar{a}). \] Actually \( G \) acts on the widened complex plane, i.e. on the Riemann sphere \( S^2 \) by rigid rotations. The canonical measure \( \mu_n \), considered as a measure on \( S^2 \), is rotation-invariant.

For each natural \( n \) the group \( G \) has exactly one unitary \( n \)-dimensional representation up to unitary equivalence. Denote it by \( \pi_n \). There exists a realization of \( \pi_n \) in \( H_n \) as follows. For \( g \) given by (1) and \( f \in H_n \), one has \( (\pi_n(g)f)(z) = (bz + \bar{a})^{n-1}f(zg) \). Since \( \pi_n(g)e_v = (\bar{a} - \bar{b}v)^{n-1}e_{\bar{v}}^{-1} \), one immediately finds that both for covariant and contravariant symbols, the symbol — operator correspondence is \( G \)-equivariant, i.e. if \( f(z, \bar{z}) \) is a symbol of an operator \( A \) in \( H_n \) then \( f(zg, \bar{z}g) \) is a symbol of \( \pi_n(g)A\pi_n(g^{-1}) \). Thus it is natural to consider both covariant and contravariant symbols as functions on \( S^2 \). In particular to define a covariant symbol at infinity one needs to replace \( e_v \) by \( e_{\infty} = z^{n-1} \) in the definition of a covariant symbol. A nice invariant way to introduce the so called coherent states \( \{e_v\} \) and covariant symbols in terms of line bundles can be found in [5].

\[ \pi_n(X) = (-bz^2 + 2az + c)\frac{d}{dz} + (n-1)(bz - a). \] Thus for \( u \in U \) \( u_n = \pi_n(u) \) is a differential operator with polynomial coefficients in \( z \) and \( n \). Let \( s_n(u) \) denote the covariant symbol of \( u \in U \). It can be calculated as follows

\[ s_n(u) = \frac{(u_ne_\bar{z}, e_\bar{z})}{(e_\bar{z}, e_\bar{z})} = \frac{(u_ne_\bar{z})(z)}{e_\bar{z}(z)} = \frac{u_n(1 + z\bar{z})^{n-1}}{(1 + z\bar{z})^{n-1}}. \]

Observe that the symbol \( s_n(u) \) is polynomial in \( n \).
The adjoint action $Ad$ of $G$ on $su(2)$ (by rotations) can be naturally extended to $U$. Then for $u \in U$, $g \in G$ one has $\pi_n(Ad(g)u) = \pi_n(g)\pi_n(u)\pi_n(g^{-1})$. Therefore, from $G$-equivariance of covariant symbols it follows that the mapping $u \mapsto s_n(u)$ is also $G$-equivariant, i.e., for $g \in G$ $s_n(Ad(g)u)(z, \bar{z}) = s_n(u)(gz, \bar{gz})$.

Now we will give an explicit description of the mapping $s_n$ using a $G$-module structure of $U$ under adjoint action.

Consider elements of $sl(2,\mathbb{C})$ as complex linear functionals on $su(2)$ with respect to $Ad$-invariant pairing $X, Y \mapsto -\frac{1}{2}trXY$ for $X \in sl(2,\mathbb{C})$ and $Y \in su(2)$. The symmetrization mapping $Sym$ (see [6]) is a $\mathbb{C}$-linear isomorphism of the algebra $\Lambda$ of all complex polynomials on $su(2)$ onto $U$ such that if $f(Y) = -\frac{1}{2}trXY$ is a functional on $su(2)$ for an arbitrary $X \in sl(2,\mathbb{C})$ then $Sym(f^k) = X^k$ for all natural $k$. It is $G$-equivariant, i.e., $Sym$ maps $f(Ad(g^{-1})Y)$ to $Ad(g)Sym(f)$ for $f \in \Lambda$. Let $I, M$ denote the spaces of all rotation-invariant and harmonic polynomials in $\Lambda$ respectively. Then $Z = Sym(I)$ is the center of $U$. Denote $E = Sym(M)$. It is known that $\Lambda = I \otimes M$ and $U = Z \otimes E$ (in the both tensor products $x \otimes y$ corresponds to the respective product $xy$, see [6]). Thus each element $u \in U$ can be written as $u = z_1v_1 + \ldots + z_kv_k$ for some $z_i \in Z, v_i \in E$.

Since $\pi_n$ is irreducible for each $z \in Z$ the operator $\pi_n(z)$ is scalar. Denote that scalar by $\chi_n(z)$. The function $\chi_n$ is a homomorphism of $Z$ into $\mathbb{C}$ and is called a central character of $U$ corresponding to $\pi_n$.

**Lemma 1.** For $z \in Z, u \in U$

(i) the symbol $s_n(z)$ is a constant equal to $\chi_n(z)$;

(ii) $s_n(zu) = s_n(z)s_n(u)$.

**Proof.** Since the covariant symbol of the identity operator is identically $1$, the symbol of $\pi_n(z)$ is identically equal to $\chi_n(z)$, which proves (i). Now, (ii) is obvious.

In order to describe $s_n$ on $U$ it suffices to know the restrictions of $s_n$ to $Z$ and $E$.

The adjoint action of $G$ on $su(2)$ keeps invariant a square of Euclidean radius, $(r(Y))^2 = -\frac{1}{2}trY^2$, $Y \in su(2)$, which is a quadratic polynomial on $su(2)$. It is known (see, e.g. [6]) that $Z$ is a polynomial algebra in the Casimir element $z_0 = Sym(r^2)$. A direct calculation provides

**Lemma 2.** $s_n(z_0) = 1 - n^2$. 

4
Let $M_k$ denote the subspace of $M$ of harmonic polynomials of degree $k$. It is known that with respect to the action of $G$ on $\Lambda$, via a change of variables, $M_k$ is a $(2k+1)$-dimensional irreducible subspace. Let $v_0$ denote the element of $U$ corresponding to

$$V = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \in \mathfrak{sl}(2, \mathbb{C}),$$

$f_0(Y) = -\frac{1}{2}trVY$. Then for each natural $k$ $\text{Sym}(f_0^k) = v_0^k$. It is easy to check directly that $f_0^k$ is harmonic, so $f_0^k \in M_k$. Using (2) one gets that $(v_0^k)_n = (\frac{d}{d\theta})^k$ for all $n$. Now from (3) follows

Lemma 3. $s_n(v_0^k) = (n-1)(n-2)\ldots(n-k)(\frac{\theta}{1+z\bar{z}})^k$.

Consider a $G$-equivariant embedding of $S^2$ in $\mathfrak{su}(2)$ given as follows

$$S^2 \supset \mathbb{C} \ni z \mapsto \left( \begin{array}{cc} i\frac{1-z\bar{z}}{1+z\bar{z}} & -2i\frac{z}{1+z\bar{z}} \\ -2i\frac{\bar{z}}{1+z\bar{z}} & -i\frac{1-z\bar{z}}{1+z\bar{z}} \end{array} \right) \in \mathfrak{su}(2).$$

The image of $S^2$ is the unit sphere with respect to the Euclidean scalar product $X, Y \mapsto -\frac{1}{2}trXY$ in $\mathfrak{su}(2)$. Then the pullback of $f_0(Y)$ to $S^2$ is $i\frac{\bar{z}}{1+z\bar{z}}$. Thus identifying $S^2$ with the unit sphere in $\mathfrak{su}(2)$ one gets

$$s_n(\text{Sym}(f_0^k)) = s_n(v_0^k) = \frac{1}{i^k}(n-1)(n-2)\ldots(n-k)f_0^k|_{S^2}. \quad (4)$$

Since all the ingredients of (4) are $G$-equivariant, one can replace $f_0^k$ in (4) by a linear combination of its rotations by the elements of $G$. Since $G$ acts irreducibly in $M_k$ one thus obtains an arbitrary element of $M_k$.

Lemma 4. For all $f \in M_k$ $s_n(\text{Sym}(f)) = \frac{1}{i^k}(n-1)(n-2)\ldots(n-k)f|_{S^2}$.

Denote $E_k = \text{Sym}(M_k)$. Since $M = \oplus_k M_k$ then $E = \oplus_k E_k$. Therefore, an arbitrary element of $U$ may be expressed as a sum of monomials of a form $z_0^jv$ with $v \in E_k$. Combining Lemmas 1 - 4, one gets

Proposition 1. Let $v \in E_k, v = \text{Sym}(f)$ for some $f \in M_k$. Then

$$s_n(z_0^jv) = (\frac{1}{i})^{2j+k}(n^2-1)^j(n-1)(n-2)\ldots(n-k)f|_{S^2}.$$
§3. Symbol algebras.

Let $R$ denote the space of restrictions of all polynomials from $\Lambda$ to the unit sphere $S^2$. The elements of $R$ are called regular functions on $S^2$. It is known (see, e.g. [3]) that the restriction of the space $M$ of harmonic polynomials to $S^2$ is a bijection of $M$ onto $R$. Therefore each regular function on $S^2$ is a restriction of a unique harmonic polynomial. Denote by $R_k$ the restriction of $M_k$. Thus $R = \oplus_k R_k$.

Since for all $u \in U$ $s_n(u)$ is polynomial in $n$ one can consider $s_t(u)$ for arbitrary $t \in C$. It is obvious that Lemma 2 is valid for $s_t(u)$ for all complex $t$. Namely the mapping $z \mapsto s_t(z)$ is a homomorphism of $Z$ to $C$ and for $z \in Z$, $u \in U$ $s_t(zu) = s_t(z)s_t(u)$.

Denote $A_{1/t} = s_t(U)$. In the sequel $N^*$ will denote the set of all positive integers. From Proposition 1 immediately follows

Proposition 2. For $t = n \in N^*$ $A_{1/t} = \oplus_{k=0}^{k=n-1} R_k$. For all other values of $t$ $A_{1/t}$ consists of all regular functions.

We are going to show that the kernel of the mapping $s_t$ is a two-sided ideal in $U$, thus obtaining a quotient algebra structure in $A_{1/t}$.

Let $J_t$ be the two-sided ideal in $U$ generated by $Z \cap \text{Ker} s_t$.

Lemma 5. $U = E + J_t$.

Proof. For $u = zv$ with $z \in Z$, $v \in E$ one has $u = s_t(z)v + (z - s_t(z))v$ where $s_t(z)$ is identified with the respective constant in $U$. The assertion of Lemma follows from the fact that $z - s_t(z) \in Z \cap \text{Ker} s_t$.

Lemma 6. For $t \notin N^*$ Ker $s_t = J_t$.

Proof. From Lemma 5 follows that Ker $s_t = E \cap \text{Ker} s_t + J_t$. Since the restriction of the space of harmonic polynomials to a sphere is a bijection onto the space of regular functions, it follows from Lemma 4 that $E \cap \text{Ker} s_t$ is trivial for $t \notin N^*$.

Proposition 3. For all $t \in C$ Ker $s_t$ is a two-sided ideal in $U$.

Proof. For $t = n \in N^*$ Ker $s_t = \text{Ker} \pi_n \subset U$. For the rest of $t$ Lemma 6 is applied.
Now \( A_{1/t} \) carries a quotient algebra structure. Denote the corresponding multiplication in \( A_{1/t} \) by \(*_{1/t} \).

A following Lemma is obtained from direct calculations.

Lemma 7. The function \( f(z, \bar{z}) = (n+1)(n+2)\ldots(n+k)(\frac{z}{1+z\bar{z}})^k \) is a contravariant symbol of the operator \( \pi_n(v^k_0) = (\frac{d}{dz})^k \) in \( H_n \).

Proposition 4. For \( n \in \mathbb{N}^* \), \( u \in U \) the function \( s_{-n}(u)(-1/\bar{z}, -1/z) \) is a contravariant symbol of the operator \( \pi_n(u) \) in \( H_n \).

Proof. It follows from Lemma 2 that \( s_n \) coincides with \( s_{-n} \) on the center \( Z \) of \( U \). Therefore the ideals \( J_n \) and \( J_{-n} \) coincide as well. Since \( J_n \subset \text{Ker } \pi_n \) for each \( u \in J_{-n} \) both the symbol \( s_{-n}(u) \) and operator \( \pi_n(u) \) are zero. It follows from Lemma 5, that it remains to check the assertion of the Proposition for \( u \in E_k \), since \( E = \oplus E_k \). The rest follows from Lemma 7, the irreducibility of \( E_k \) with respect to the adjoint action of \( G \) and equivariance of contravariant symbols.

Thus the algebra \( A_{-1/n} \) consists of contravariant symbols of all operators in \( H_n \) up to the antipodal mapping \( z \mapsto -1/\bar{z} \) of the sphere \( S^2 \). Moreover, the mapping which maps the symbol \( s_{-n}(u) \) to the operator \( \pi_n(u) \) in \( H_n \) is a correctly defined homomorphism of \( A_{-1/n} \) onto \( \text{End } H_n \).

§4. The proof of the Correspondence Principle.

Recall now some basic facts about filtration in the universal enveloping algebra and Poisson structure in the symmetric algebra of a Lie algebra (see, e.g. [6]).

Let \( U_k \) denote the subspace of \( U \) spanned by monomials of degree \( \leq k \). Then \( \{U_k\} \) is a filtration, for \( u \in U_k \), \( v \in U_l \) both \( uv, vu \in U_{k+l} \) and \( uv-vu \in U_{k+l-1} \).

Let \( \Lambda_k \) denote the subspace of \( \Lambda \) of homogenous polynomials of degree \( k \). Then \( \text{Sym}(\Lambda_k) \subset U_k \). Moreover, \( \text{Sym} \) composed with the quotient mapping \( U_k \to U_k/U_{k-1} \) establishes an isomorphism of \( \Lambda_k \) onto \( U_k/U_{k-1} \). For \( u \in U_k \) let \( \underline{u} \) denote the unique element of \( \Lambda_k \) such that \( \text{Sym}(\underline{u}) \equiv u \mod U_{k-1} \).

There exists a natural Poisson structure on \( \Lambda \) such that if \( f_i(Y) = -\frac{1}{2} \text{tr } X_i Y \), \( i = 1, 2, 3 \) are linear functionals on \( su(2) \) corresponding to \( X_i \in sl(2, \mathbb{C}) \) with \( [X_1, X_2] = X_3 \), then \( \{f_1, f_2\} = f_3 \). The symplectic leaves of that Poisson structure are the \( G \)-orbits in \( su(2) \), i.e. the spheres. Denote by
\{\cdot, \cdot\}_{S^2}$ the restriction of the Poisson bracket to the unit sphere $S^2$. Then for $f, g \in \Lambda \{f|_{S^2}, g|_{S^2}\}_{S^2} = \{f, g\}|_{S^2}$.

For $u \in U_k, v \in U_l$ $uv = vu = u \cdot v$ while $uv - vu = \{u, v\}$.

Proposition 5. Let $u \in U_k$. Then
\[
\lim_{t \to \infty} \frac{1}{tk} s_t(u) = \frac{1}{tk} u|_{S^2}.
\]

Proof. If $f \in \Lambda_k$ and $u = \text{Sym}(f) \in U_k$, then $u = f$. In particular $z_0 \in U_2$, $z_0 = r^2$ and the restriction of $r^2$ to the unit sphere $S^2$ is identically 1. Now the proof follows from Proposition 1.

Let $f, g$ be regular functions on $S^2$. From Proposition 2 follows that for $t \notin \mathbb{N}^*$ or sufficiently big $t = n \in \mathbb{N}^*$ the product $f \ast_{1/t} g$ is defined.

Theorem. For any regular functions $f, g$ on $S^2$ holds
\[
\lim_{t \to \infty} f \ast_{1/t} g = fg; \quad \lim_{t \to \infty} t(f \ast_{1/t} g - g \ast_{1/t} f) = i\{f, g\}_{S^2}.
\]

Proof. It is enough to consider $f \in R_k$, $g \in R_l$. Let $u \in E_k$, $v \in E_l$ be such that $\frac{1}{t} u|_{U_k}$ and $\frac{1}{t} v|_{U_l}$ are the harmonic extensions of $f$ and $g$, respectively. Then, using Proposition 5 one gets
\[
f \cdot g = \frac{1}{tk} u|_{S^2}, \frac{1}{t} v|_{S^2}; \quad \frac{1}{tk+1} uv|_{S^2} = \lim_{t \to \infty} \frac{1}{tk+l} s_t(uv) = \lim_{t \to \infty} \frac{1}{tk+l} s_t(u|_{S^2}) = \lim_{t \to \infty} \frac{1}{tk+l} s_t(u|_{S^2}).
\]

Applying Lemma 4 to the last expression one finally obtains
\[
f \cdot g = \lim_{t \to \infty} \frac{(t-1) \ldots (t-k)(t-1) \ldots (t-l)}{tk+l} f \ast_{1/t} g = \lim_{t \to \infty} f \ast_{1/t} g.
\]

Proceeding in a similar manner one gets
\[
i\{f, g\}_{S^2} = \left\{ \frac{1}{tk} u|_{S^2}, \frac{1}{t} v|_{S^2} \right\}_{S^2} = \frac{1}{tk+l-1} \{u, v\}|_{S^2} = \frac{1}{tk+l-1} uv - vu|_{S^2} =
\]
\[
\lim_{t \to \infty} \frac{1}{tk+l-1} s_t(uv - vu) = \lim_{t \to \infty} \frac{(t-1) \ldots (t-k)(t-1) \ldots (t-l)}{tk+l} t(f \ast_{1/t} g - g \ast_{1/t} f) =
\]
\[
\lim_{t \to \infty} t(f \ast_{1/t} g - g \ast_{1/t} f).
\]
Let $F = \{1, 1/2, 1/3, \ldots\}$. According to the Theorem, for $h = 1/n \in F$ the algebras $A_{1/n}$ and $A_{-1/n}$ of covariant and contravariant symbols of operators in $H_n$ form the data for covariant and contravariant quantization on $S^2$ respectively.

Acknowledgements

I wish to express my gratitude to Professors R.G.Airapetyan, B.V.Fedosov and M.S.Narasimhan for helpful discussions. I am pleased to thank for kind hospitality the ICTP, Trieste, where the work has been completed.

References

[1] F.A.Berezin, General Concept of Quantization, Commun. Math. Phys. 40, 153(1975).

[2] F.A.Berezin, Covariant and contravariant symbols of operators, Soviet Math. Izvestia, 36, n.5, 1134(1972).

[3] F.A.Berezin, Quantization, Soviet Math. Izvestia, 38, n.5(1974).

[4] F.Bayen, M.Flato, C.Fronsdal, A.Lichnerovicz and D.Sternheimer, Deformation theory and quantization, Ann. Phys.111, 1 (1978).

[5] J.Rawnsley, M.Cahen, S.Gutt, Quantization of Kähler manifolds I: geometric interpretation of Berezin’s quantization, Journ. of Geom. and Phys.7,n.1, 45(1990).

[6] J.Dixmier, Algèbres Enveloppantes, Paris, Gauthier - Villars, 1974.

[7] B.Kostant, Lie group representations on polynomial rings, Amer.J. Math.85, 327(1963).

[8] R.Courant, D.Hilbert, Methods of Mathematical Physics, Vol.I, New York, Interscience, 1953.