Self-Excited Dynamics of Discrete-Time Lur'e Systems

Juan A. Paredes⁎, Syed Aseem Ul Islam⁎, Omran Kouba⁎, Dennis S. Bernstein⁎

⁎University of Michigan, Ann Arbor, Michigan, 48109, USA
⁎Higher Institute for Applied Sciences and Technology, Damascus, Syria

Abstract

Self-excited systems arise in numerous applications, such as biochemical systems, fluid-structure interaction, and combustion. This paper analyzes a discrete-time Lur’e system with a piecewise-linear saturation feedback nonlinearity. The main result provides sufficient conditions under which the Lur’e system is self-excited in the sense that its response is bounded and nonconvergent.

Keywords: self-oscillation, self-excitation, discrete-time, nonlinear feedback, Lur’e system

1. Introduction

A self-excited system has the property that the input is constant but the response is oscillatory. Self-excited systems arise in numerous applications, such as biochemical systems, fluid-structure interaction, and combustion. The classical example of a self-excited system is the van der Pol oscillator, which has two states whose asymptotic response converges to a limit cycle. A self-excited system, however, may have an arbitrary number of states and need not possess a limit cycle. Overviews of self-excited systems are given in [1, 2].

Models of self-excited systems are typically based on the relevant physics of the application. From a systems perspective, the main interest is in understanding the features of the components of the system that give rise to self-sustained oscillations. Understanding these mechanisms can illuminate the relevant physics in specific domains and provide unity across different domains.

A unifying model for self-excited systems is a feedback loop involving linear and nonlinear elements; systems of this type are called Lur’e systems. Lur’e systems have been widely studied in the classical literature on stability theory [3]. Within the context of self-excited systems, Lur’e systems under various assumptions are considered in [4, 5, 6, 7, 8, 9, 10, 11, 12]. Oscillating discrete-time systems are considered in [13, 14].

Roughly speaking, self-excited oscillations arise from a combination of stabilizing and destabilizing effects. Destabilization at small signal levels causes the response to grow from the vicinity of an equilibrium, whereas stabilization at large signal levels causes the response to decay when the state is far from an equilibrium. In particular, negative damping at low signal levels and positive damping at high signal levels is the mechanism that gives rise to a limit cycle in the van der Pol oscillator [15, pp. 103–107]. Note that, although systems with limit-cycle oscillations are self-excited, the converse need not be true since the response of a self-excited system may oscillate without the trajectory reaching a limit cycle. Alternative mechanisms exist, however; for example, time delays are destabilizing, and Lur’e models with time delay have been considered as models of self-excited systems [16].

The present paper considers a discrete-time Lur’e system with asymptotically stable linear dynamics, a zero at 1, and a piecewise-linear saturation feedback nonlinearity. For this Lur’e system, sufficiently large scalings of the asymptotically stable dynamics yield closed-loop unstable dynamics while the saturation function operates in its linear region. Under large signal levels, the saturation function yields a constant signal, which effectively breaks the loop, thus allowing the open-loop dynamics to stabilize the response. The zero at 1 acts as a high-pass
filter, which ensures that the response does not converge, whereas the saturation function yields a constant signal. Hence, while the saturation function operates in the nonsaturated region, the closed-loop system is unstable, and, while the saturation function operates in the saturated region and yields a constant signal, the closed-loop system is asymptotically stable and has a constant input. The contribution of the present paper is to prove that this model structure yields self-excited oscillations for sufficiently large scalings of the asymptotically stable dynamics. A preliminary study of self-excited oscillations in a similar discrete-time Lur'e model was performed in [15]. However, the present paper goes far beyond [15] in breadth and depth of the analysis of these systems.

The contents of the paper are as follows. Section 2 considers a simple discrete-time linear feedback system and analyzes its transfer function to study the range of values of the linear dynamics scalings for which the closed-loop system is asymptotically stable. Section 3 considers the same linear feedback system and analyzes its state space model to study the conditions under which the response of the closed-loop system does not converge and is not bounded. Section 4 extends the problem in Sections 2 and 3 by including a saturation nonlinearity. Under certain conditions, this discrete-time Lur'e system is shown to have a bounded, non-convergent response for sufficiently large values of the loop gain. Section 5 presents an example that illustrates the conditions for self-excitation presented in Section 4.

Nomenclature. $\mathbb{R} \triangleq (-\infty, \infty)$, $\mathbb{C}$ denotes the complex numbers, $\mathcal{R}$ denotes range, $\mathcal{N}$ denotes null space, $(\cdot)^\dagger$ denotes complex conjugate, $(\cdot)^*$ denotes complex conjugate transpose, $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{C}^n$, and $q$ denotes the forward shift operator. For a polynomial $p$, define $\text{spr}(p) \triangleq \max\{|z| : z \in \mathbb{C} \text{ and } p(z) = 0\}$, and for a matrix $A \in \mathbb{R}^{n \times m}$, let $\text{spec}(A)$ denote the set of eigenvalues of $A$, let $\chi_A$ denote the characteristic polynomial of $A$, and let $\text{spr}(A)$ denote the spectral radius of $A$. For $\gamma > 0$, sat$_\gamma(x)$ denotes the saturation function, such that, for all $x \in [-\gamma, \gamma]$, sat$_\gamma(x) = x$, and, for all $|x| > \gamma$, sat$_\gamma(x) = (\text{sign } x) \gamma$.

2. Transfer Function Analysis of the Linear Feedback System

Let $G$ be a strictly proper, asymptotically stable, discrete-time SISO transfer function with a zero at 1 and no other zeros on the unit circle. Let $G = N/D$, where the polynomials $N$ and $D$ are coprime, $D$ is monic, $n \triangleq \deg D$, and $m \triangleq \deg N < n$. Note that $N(1) = 0$ and $D(1) \neq 0$, and thus $G(1) = 0$. Furthermore, for all $\theta \in (-\pi, \pi) \{0\}$, $G(e^{i\theta}) \neq 0$.

For all $\alpha \in \mathbb{R}$, the closed-loop transfer function from $v$ to $y$ of the linear feedback system in Fig. 1 is given by

$$G_{yv,\alpha}(q) \triangleq \frac{\alpha G(q)}{1 - \alpha G(q)} = \frac{\alpha N(q)}{p_\alpha(q)},$$

(1)

where $p_\alpha(q) \triangleq D(q) - \alpha N(q)$. The forward shift operator $q$ accounts for both the free and forced response of the linear feedback system in Fig. 1 for pole-zero analysis, $q$ is replaced by the $Z$-transform complex variable $z$. Note that $p_\alpha = D$, and thus $\text{spr}(p_0) < 1$. However, for all $|\alpha|$ sufficiently large, it follows from the root locus asymptote rule that $p_\alpha$ has at least $n - m$ roots outside the closed unit disk, and thus $\text{spr}(p_\alpha) > 1$. Note that, since Fig. 1 has no sign change in the loop, the root locus parameter $\alpha$ plays the role of $-k$ in the standard root locus. The following result is immediate.

Figure 1: Discrete-time linear feedback system with input $v$ and output $y$.

Proposition 2.1. Let $\alpha \in \mathbb{R} \{0\}$ and $\theta \in (-\pi, \pi) \{0\}$. Then $p_\alpha(e^{i\theta}) = 0$ if and only if $\alpha = 1/G(e^{i\theta})$.

Proposition 2.1 implies that, if $\theta \in (-\pi, \pi) \{0\}$ and $G(e^{i\theta})$ is real, then $e^{i\theta}$ is a pole of $G_{yv,1/G(e^{i\theta})}$ and thus an element of either the 0-deg or 180-deg root locus of $G_{yv,\alpha}$. Now, define

$$\Theta_n \triangleq \{\theta \in (-\pi, \pi) \{0\} : G(e^{i\theta}) \in (-\infty, 0)\},$$

(2)

$$\Theta_p \triangleq \{\theta \in (-\pi, \pi) \{0\} : G(e^{i\theta}) \in (0, \infty)\},$$

(3)

so that $\Theta_n \cup \Theta_p = \{\theta \in (-\pi, \pi) \{0\} : G(e^{i\theta}) \text{ is real}\}$. Note that $\Theta_n$ is the set of angles at which the 180-deg root locus of $G_{yv,\alpha}$ crosses the unit circle, and $\Theta_p$ is the set of angles at which the 0-deg root locus of $G_{yv,\alpha}$ crosses the unit circle, which occurs in both cases for $\alpha = 1/G(e^{i\theta})$. Since the 180-deg and 0-deg
root locus plots of $G_{yv,\alpha}$ have $n - m$ asymptotes as $\alpha \to -\infty$ and $\alpha \to \infty$, respectively, it follows that
\[
\text{card}(\Theta_n) \geq n - m, \quad (4)
\]
\[
\text{card}(\Theta_p) \geq n - m. \quad (5)
\]
Furthermore, in the case where $n - m = 1$, the positive real axis is an asymptote of the 0-deg root locus plot. Since $G$ has a zero at 1, it follows that two poles must break in on the positive real axis, which implies that
\[
\text{card}(\Theta_p) \geq \min\{2, n - m\}, \quad (6)
\]
as illustrated by the following example.

**Example 2.2.** Let $\alpha \geq 0$ and $G(q) = \frac{(q-\alpha)(q-\beta)}{q^2(q-\alpha)(q-\beta)}$. Since $n = 4$ and $m = 3$, it follows that the 0-deg root locus of the closed-loop system has one asymptote, as shown in Fig. 2. However, the root locus plot crosses the unit circle at two points due to the pole break-in on the positive real axis, and thus $\text{card}(\Theta_p) = 2$.

![Figure 2: Example 2.2 0-deg root locus of the linear feedback system in Fig. 1](image)

The following result implies that the root locus plot of $G$ intersects the unit circle at a finite number of points.

**Proposition 2.3.** $\Theta_p$ and $\Theta_n$ are finite.

**Proof.** Let $r$ be a positive integer such that $h(z) = e^{|z|}$ is a polynomial. Now, let $z = e^{\theta}$, where $\theta \in \Theta_n \cup \Theta_p$. Since $G(z)$ is real and $|z| = 1$, it follows that
\[
\frac{D(z)}{N(z)} = \left(\frac{D(z)}{N(z)}\right) = \frac{D(z)}{N(z)} = \frac{D(1/z)}{N(1/z)},
\]
Hence, $h(z) = 0$. Since $h$ has a finite number of roots, it follows that $\Theta_n$ and $\Theta_p$ are finite. \hfill $\square$

**Proposition 2.4.** If $\alpha \in (\alpha_n, \alpha_p)$, then $G_{yv,\alpha}$ is asymptotically stable. Furthermore,
\[
\text{spr}(p_{\alpha_n}) = \text{spr}(p_{\alpha_p}) = 1. \quad (9)
\]

**Proof.** Suppose there exists $\alpha \in (\alpha_n, 0)$ such that spr($p_{\alpha_n}$) $\geq$ 1. Since spr is continuous and spr($p_{\alpha}$) = $\text{spr}(D) < 1$, the intermediate value theorem implies that there exists $\alpha_1 \in (\alpha_n, 0)$ such that spr($p_{\alpha_1}$) = 1. Hence, there exists $\theta_1 \in (-\pi, \pi]/\{0\}$ such that $p_{\alpha_1}(e^{\theta_1}) = 0$. Hence, $G(e^{\theta_1}) = 1/\alpha_1 < 0$, and thus $\theta_1 \in \Theta_n$. Therefore, (7) implies that
\[
\max_{\theta \in \Theta_n} 1/G(e^{\theta}) = \alpha_n < \alpha_1 = 1/G(e^{\theta_1}),
\]
which is a contradiction. Hence, for all $\alpha \in (\alpha_n, 0]$, $G_{yv,\alpha}$ is asymptotically stable. Similarly, for all $\alpha \in [0, \alpha_p)$, $G_{yv,\alpha}$ is asymptotically stable. Hence, for all $\alpha \in (\alpha_n, \alpha_p)$, $G_{yv,\alpha}$ is asymptotically stable.

Next, let $\theta_n \in \Theta_n$ satisfy $\alpha_n = 1/G(e^{\theta_n})$. Then, Proposition 2.1 implies that $p_{\alpha_n}(e^{\theta_n}) = 0$, that is, $e^{\theta_n}$ is a root of $p_{\alpha_n}$, and thus spr($p_{\alpha_n}$) $\geq$ 1. Now, suppose that spr($p_{\alpha_n}$) $>$ 1. Since spr($p_0$) $<$ 1 and spr is continuous, it follows that there exists $\alpha_1 \in (\alpha_n, 0)$ such that spr($p_{\alpha_1}$) = 1. Since $\alpha_1 \in (\alpha_n, \alpha_p)$, it follows that $G_{yv,\alpha_1}$ is asymptotically stable, and thus spr($p_{\alpha_1}$) $<$ 1, which is a contradiction. Hence, spr($p_{\alpha_n}$) $= 1$. Similarly, spr($p_{\alpha_p}$) $= 1$. \hfill $\square$

The following result is an immediate consequence of the root locus asymptote rule.

**Lemma 2.5.** There exist $\beta_n < 0$ and $\beta_p > 0$ such that, for all $\alpha < \beta_n$ and all $\alpha > \beta_p$, $p_\alpha$ has at least $n - m$ roots with absolute value greater than 1, all of which are simple.

The following example shows that $\alpha_p$ defined by (8) is not necessarily the supremum of all values of $\alpha$ such that $G_{yv,\alpha}$ is asymptotically stable. In other words, there may exist $\alpha > \alpha_p$ such that $G_{yv,\alpha}$ is asymptotically stable.
Example 2.6. Let \( G(q) = \frac{(q^2 - 0.05 + 0.88)(q - 1)}{q^2(q^2 + 0.1q + 0.7769)(q - 1)} \). Fig. (3b) shows the root locus for \( \alpha > 0 \) and \( \alpha < 0 \), and Fig. (3c) shows \( \text{spr}(p_\alpha) \) versus \( \alpha \), which indicates that there exists \( \alpha > \alpha_p \) such that \( \text{spr}(p_\alpha) < 1 \).

Recall from Lemma 2.3 that, for all \( |\alpha| \) sufficiently large, \( A + \alpha BC \) has at least \( n - m \) eigenvalues with absolute value greater than 1, all of which are simple.

Lemma 3.2. Consider the linear feedback system in Fig. 1 with \( v \equiv 0 \). Let \( \alpha \in \mathbb{R} \), assume there exists a simple eigenvalue \( \lambda \in \text{spec}(A + \alpha BC) \) such that \( |\lambda| > 1 \), let \( \xi \in \mathbb{C}^n \) be an associated eigenvector, let \( X \subset \mathbb{C}^n \) be the \( n - 1 \)-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the remaining eigenvalues of \( A + \alpha BC \), and assume that \( x_0 \notin X \). Then the following statements hold:

i) For all \( k \geq 0 \), \( x_k \notin X \).

ii) \( \limsup_{k \to \infty} |y_k| = \infty \).

Proof. Since \( (A, B) \) is controllable, Fact 7.15.10 of [18, p. 599] implies that \( A \) is cyclic. Since, in addition, \( (A + \alpha BC, B) \) is controllable, it follows that \( A + \alpha BC \) is cyclic. Therefore, each eigenvalue of \( A + \alpha BC \) has exactly one associated eigenvector.

Let \( \lambda_1, \ldots, \lambda_r \) be the distinct eigenvalues of \( A + \alpha BC \), for all \( j \in \{1, \ldots, r\} \), let \( n_j \) be the algebraic multiplicity of \( \lambda_j \) and, for all \( i \in \{1, \ldots, n_j\} \), let \( \xi_{j,i} \) be a generalized eigenvector of \( A + \alpha BC \) corresponding to \( \lambda_j \) such that \( (A + \alpha BC - \lambda_j I)^{i-1}\xi_{j,i} \neq 0 \), \( (A + \alpha BC - \lambda_j I)\xi_{j,i} = 0 \), and \( (\xi_{j,1}, \ldots, \xi_{j,n_j}) \) is a Jordan chain of \( A + \alpha BC \) associated with \( \lambda_j \). Note that, for all \( j \in \{1, \ldots, r\} \), \( \xi_{j,1} \) is an eigenvector associated with \( \lambda_j \). Without loss of generality, define \( \lambda_1 \triangleq \lambda \) and \( \xi_{1,1} \triangleq \xi \). Note that, since \( \lambda \) is simple, it follows that \( n_1 = 1 \).

Next, it follows from the Jordan decomposition and equation (7.8.5) from [10, p. 594] that

\[
(A + \alpha BC)^k = S \begin{bmatrix} J^k_1 & \cdots & J^k_r \end{bmatrix} S^{-1},
\]

where \( S \triangleq [\xi_1 \cdots \xi_r] \in \mathbb{C}^{n \times n} \), for all \( j \in \{1, \ldots, r\} \), \( \xi_j \triangleq [\xi_{j,1} \cdots \xi_{j,n_j}] \in \mathbb{C}^{n \times n_j} \), \( J_j \in \mathbb{C}^{n_j \times n_j} \) is the Jordan block associated with the eigenvalue \( \lambda_j \) of \( A + \alpha BC \), and

\[
J_j^k = \begin{bmatrix} \lambda_j^k & (\lambda_j^k\lambda_j^{-1}) & \cdots & (\lambda_j^{-n_j+1}) & \cdots & (\lambda_j^{-n_j+1}) & \cdots & \cdots & (\lambda_j^{-n_j+1}) \\
(\lambda_j^{-n_j+1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(\lambda_j^{-n_j+1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(\lambda_j^{-n_j+1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_j^k & (\lambda_j^{-n_j+1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}.
\]
where, for all $k < i$, \( \binom{k}{i} \triangleq 0 \). Since $S$ is invertible, the $n$ linearly independent generalized eigenvectors $\xi_i, \xi_{2,1}, \ldots, \xi_{2,n_2}, \ldots, \xi_{r,1}, \ldots, \xi_{r,n_r}$ comprise a basis of $C^n$. Therefore, it follows that, for all $j \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n_j\}$, there exists $\beta_{j,i} \in C$ such that

$$x_0 = \sum_{j=1}^r \sum_{i=1}^{n_j} \beta_{j,i} \xi_{j,i} = \beta_{1,1} \xi + \sum_{j=2}^r \sum_{i=1}^{n_j} \beta_{j,i} \xi_{j,i}$$

$$= S \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} 
= S \begin{bmatrix} J_1 \\ \vdots \\ J_r \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}$$

where $\beta_j \triangleq [\beta_{j,1} \ldots \beta_{j,n_j}]^T \in \mathbb{C}^{n_j \times 1}$. It thus follows from (16) that, for all $k \geq 0$,

$$x_k = S \begin{bmatrix} J_1 \\ \vdots \\ J_r \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} = S \begin{bmatrix} J_1 \beta_1 \\ \vdots \\ J_r \beta_r \end{bmatrix}$$

(17)

and, for all $j \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n_j\}$, and $k \geq 0$,

$$\gamma_{j,i}(k) \triangleq \beta_{j,i} + \sum_{l=2}^{n_j-i+1} \binom{k}{l-1} \lambda_j^{l-1} \beta_{j,i+l-1}.$$  

(18)

In particular,

$$\gamma_{1,1}(k) = \beta_{1,1}.$$  

(19)

Hence,

$$x_k = \beta_{1,1} \lambda_1^k \xi + \sum_{j=2}^r \sum_{i=1}^{n_j} \gamma_{j,i}(k) \lambda_j^k \xi_{j,i},$$

(20)

and thus it follows from (16) that

$$y_k = p_1(k) \lambda_1^k + \sum_{j=2}^r p_j(k) \lambda_j^k,$$

(21)

where

$$p_1(k) \triangleq \beta_{1,1} C \xi,$$

(22)

and, for all $j = 2, \ldots, r$,

$$p_j(k) \triangleq \sum_{l=1}^{n_j} \left( \beta_j + \sum_{i=2}^{n_j-l+1} \binom{k}{l-1} \beta_j \lambda_j^{l-i} \right) C \xi_{j,i}.$$  

(23)

Note that $p_1(k), \ldots, p_r(k)$ are polynomials in $k$ with complex coefficients. Now, the $n - 1$-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the eigenvalues $\lambda_2, \ldots, \lambda_r$ of $A + \alpha BC$ is given by

$$\mathcal{X} = \mathbb{R}((\xi_2 \cdots \xi_{2,n_2} \cdots \xi_{r,1} \cdots \xi_{r,n_r})).$$

(24)

It follows from (19) that, since $x_0 \notin \mathcal{X}$, $\beta_{1,1} \neq 0$, and thus it follows from (20) that, for all $k \geq 0$, $x_k \notin \mathcal{X}$, which proves $i$. Next, Lemma 3.1 implies that $C \xi \neq 0$. Hence, $p_1$ is not the zero polynomial. Finally, since $p_1$ is not the zero polynomial and $|\lambda| > 1$, Proposition A.3 implies that $\lim \sup_{k \to \infty} |y_k| = \infty$, which proves $ii$. \hfill \Box

The following alternative characterization of $\mathcal{X}$ is worth noting.

**Proposition 3.3.** Let $\alpha \in \mathbb{R}$, assume there exists a simple eigenvalue $\lambda \in \sigma(A + \alpha BC)$ such that $|\lambda| > 1$, define $\mathcal{X}$ as in Lemma 3.2 and define the polynomial $q(z) = \chi_{A+\alpha BC}(z)/(z - \lambda)$. Then $\mathcal{X} = \mathcal{N}(q(A + \alpha BC))$.

The following is a corollary of Lemma 3.2.

**Corollary 3.4.** Consider the linear feedback system in Fig. 3 with $v \equiv 0$. Let $\alpha \in \mathbb{R}$, assume there exists a simple eigenvalue $\lambda \in \sigma(A + \alpha BC)$ such that $|\lambda| > 1$, let $\xi \in \mathbb{C}^n$ be an associated eigenvector, let $\mathcal{X} \subset \mathbb{C}^n$ be the $n - 1$-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the remaining eigenvalues of $A + \alpha BC$, let $k_0 \geq 0$, assume that $x_{k_0} \notin \mathcal{X}$, and let $M > |y_{k_0}|$. Then, the following statements hold:

i) For all $k \geq k_0$, $x_k \notin \mathcal{X}$.

ii) There exists $k_1 > k_0$ such that the following statements hold:

a) For all $k \in \{k_0, \ldots, k_1 - 1\}$, $|y_k| < M$.

b) $|y_{k_1}| \geq M$.

4. **Analysis of the Lur'e System**

We now consider the discrete-time Lur'e system in Fig. 4 which has the closed-loop dynamics

$$x_{k+1} = Ax_k + \alpha B s(x_k),$$

(25)

$$y_k = C x_k.$$  

(26)

**Proposition 4.1.** Let $y$ be the output of the discrete-time Lur'e system in Fig. 4. Then, $y$ is bounded.
Lemma 4.3. Let $y$ be the output of the discrete-time Lur’e system in Fig. 4 with $\alpha \in \mathbb{R}\setminus\{0\}$. Consider the following statements:

a) $\lim_{k \to \infty} y_k$ exists, and $\lim_{k \to \infty} y_k = 0$.

b) $\lim_{k \to \infty} y_k$ exists, and $\lim_{k \to \infty} y_k \in (-1, 1)$.

c) $\lim_{k \to \infty} y_k$ exists, and $\lim_{k \to \infty} y_k \in [-1, 1]$.

d) $\lim_{k \to \infty} y_k$ exists.

Then $a) \iff b) \iff c) \iff d)$.

Proof: Note that $a) \implies b) \implies c) \implies d)$.

To prove $b) \implies a)$, note that $\lim_{k \to \infty} y_k \in (-1, 1)$ implies that there exists $k_1 \geq n$ such that, for all $k \geq k_1$, $|y_k| < 1$ and $S_k = S_2$. Hence, for all $k \geq k_1$ $y_k$ is given by (12) and (13) with $x_0$ replaced by $y_k$. Since $\lim_{k \to \infty} y_k = \lim_{k \to \infty} C(A + \alpha BC)^{k-k_1} x_{k_1}$ exists, Lemma 4.2 implies that $\lim_{k \to \infty} y_k = 0$.

To prove $c) \implies b)$, consider the case where $\lim_{k \to \infty} y_k = 1$, so that $\lim_{k \to \infty} y_k = 1$. Next, note that it follows from input-to-state stability for linear time-invariant discrete-time systems (see [20, Example 3.4]) that

$$\lim_{k \to \infty} \sum_{i=0}^{k-1} CA^{k-1-i} B(v_i - 1) = 0. \quad (27)$$

Since $\lim_{k \to \infty} CA^k x_0 = 0$ and $C(I - A)^{-1} B = G(1) = 0$, it follows from (27) that

$$1 = \lim_{k \to \infty} y_k = \lim_{k \to \infty} CA^k x_0 + \lim_{k \to \infty} \sum_{i=0}^{k-1} CA^{k-1-i} Bv_i = \alpha \lim_{k \to \infty} \sum_{i=0}^{k-1} CA^{k-1-i} B + \alpha \lim_{k \to \infty} \sum_{i=0}^{k-1} CA^{k-1-i} B(v_i - 1) = \alpha \lim_{k \to \infty} \sum_{i=0}^{k-1} CA^{k} B = \alpha C(I - A)^{-1} B = 0,$$

which is a contradiction. Hence, $\lim_{k \to \infty} y_k < 1$.

A similar argument implies that $\lim_{k \to \infty} y_k > -1$. Hence, $\lim_{k \to \infty} y_k \in (-1, 1)$.

To prove $d) \implies c)$, suppose that $\lim_{k \to \infty} y_k > 1$. Then there exists $k_1 \geq n$ such that, for all $k \geq k_1$, $y_k \geq 1$, and thus $S_k = S_1$. Hence, for all $k \geq k_1$, $y_k$
is given by
\[ y_k = C A^{k-k_1} x_{k_1} + \alpha \sum_{i=0}^{k-k_1-1} C A^{k-k_1-1-i} B \]
\[ = C A^{k-k_1} x_{k_1} + \alpha \sum_{i=0}^{k-k_1-1} C A^i B. \]
Since \( \lim_{k \to \infty} A^k = 0 \) and \( C(I - A)^{-1}B = G(1) = 0 \), it follows that
\[ \lim_{k \to \infty} y_k = \lim_{k \to \infty} C A^{k-k_1} x_{k_1} + \lim_{k \to \infty} \alpha \sum_{i=0}^{k-k_1-1} C A^i B = \alpha \sum_{i=0}^{\infty} C A^i B = \alpha C(I - A)^{-1}B = 0. \]

Hence, there exists \( k_2 > k_1 \) such that \( y_{k_2} < 1 \), which is a contradiction. Therefore, \( \lim_{k \to \infty} y_k \notin 1 \). Similarly, \( \lim_{k \to \infty} y_k \geq -1 \). Hence, \( \lim_{k \to \infty} y_k \in [-1, 1] \).

**Lemma 4.4.** Consider the discrete-time Lur'e system in Fig. 5. Let \( \alpha \in \mathbb{R} \), assume there exists a simple eigenvalue \( \lambda \in \text{spec}(A + \alpha BC) \) such that \( |\lambda| > 1 \), let \( \xi \in \mathbb{C}^n \) be an associated eigenvector, let \( \mathcal{X} \subseteq \mathbb{C}^n \) be the \( n-1 \)-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the remaining eigenvalues of \( A + \alpha BC \), let \( S_0 = 0 \), and assume that \( x_{k_0} \notin \mathcal{X} \) and \( S_{k_0} = S_2 \). Then there exists \( k_1 > k_0 \) such that, for all \( k \in \{k_0, k_0 + 1, \ldots, k_1 - 1\} \), \( S_k = S_2 \) and \( x_k \notin \mathcal{X} \), and such that \( x_{k_1} \notin \mathcal{X} \) and \( S_{k_1} \neq S_2 \).

**Proof:** For all \( k \geq k_0 \) such that \( S_k = S_2 \), the system dynamics are given by the discrete-time linear feedback system in Fig. 1 with \( v = 0 \). Hence, the result follows from Corollary 3.3 with \( M = 1 \).

The following result gives sufficient conditions under which the response of the discrete-time Lur'e system is bounded and nonconvergent.

**Theorem 4.5.** Consider the discrete-time Lur'e system in Fig. 5. Let \( \alpha \in \mathbb{R} \), assume there exists a simple eigenvalue \( \lambda \in \text{spec}(A + \alpha BC) \) such that \( |\lambda| > 1 \), let \( \xi \in \mathbb{C}^n \) be an associated eigenvector, and let \( \mathcal{X} \subseteq \mathbb{C}^n \) be the \( n-1 \)-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the remaining eigenvalues of \( A + \alpha BC \). Furthermore, assume that, if \( S_0 = S_2 \), then \( x_0 \notin \mathcal{X} \), and, for all \( k \geq 0 \) such that \( S_k \neq S_2 \) and \( S_{k+1} = S_2 \), it follows that \( x_{k+1} \notin \mathcal{X} \). Then \( y \) is bounded and \( \lim_{k \to \infty} y_k \) does exist.

**Proof:** Proposition 4.4 implies that \( y \) is bounded. Suppose that \( \lim_{k \to \infty} y_k \) exists, and thus Lemma 4.3 implies that \( \lim_{k \to \infty} y_k = 0 \). Now, let \( k_0 \geq 0 \) be the smallest nonnegative integer such that, for all \( k \geq k_0 \), \( |y_k| < 1 \) and thus \( S_k = S_2 \). In the case where \( k_0 = 0 \), it follows that \( x_0 \notin \mathcal{X} \), and thus Lemma 4.3 implies that there exists \( k > k_0 \) such that \( |y_k| > 1 \). Alternatively, in the case where \( k_0 > 0 \), note that \( S_{k_0 - 1} \neq S_2 \) and \( x_{k_0} \notin \mathcal{X} \), and thus Lemma 4.3 implies that there exists \( k > k_0 \) such that \( |y_k| > 1 \). In both cases, the existence of \( k > k_0 \) such that \( |y_k| > 1 \) contradicts the fact that, for all \( k \geq k_0 \), \( |y_k| < 1 \). Therefore, \( \lim_{k \to \infty} y_k \) does not exist.

The following result shows that, for almost all initial conditions \( x_0 \), the hypotheses of Theorem 4.5 are satisfied.

**Theorem 4.6.** Consider the discrete-time Lur'e system in Fig. 5. Let \( \alpha \in \mathbb{R} \), assume \( A \) is nonsingular, assume there exists a simple eigenvalue \( \lambda \in \text{spec}(A + \alpha BC) \) such that \( |\lambda| > 1 \), let \( \xi \in \mathbb{C}^n \) be an associated eigenvector, and let \( \mathcal{X} \subseteq \mathbb{C}^n \) be the \( n-1 \)-dimensional subspace spanned by the eigenvectors and generalized eigenvectors associated with the remaining eigenvalues of \( A + \alpha BC \). Then, for almost all \( x_0 \in \mathbb{R}^n \), \( y \) is bounded and \( \lim_{k \to \infty} y_k \) does not exist.

**Proof:** Define \( f: \mathbb{R}^n \to \mathbb{R}^n \) by \( f(x) = Ax + \alpha B s_{\text{sat}}(Cx) \). Letting \( \mathcal{E} \) denote a proper affine subspace of \( \mathbb{R}^n \), it follows that
\[ f^{-1}(\mathcal{E}) \subseteq A^{-1}(\mathcal{E} - \alpha B) \cup A^{-1}(\mathcal{E} + \alpha B) \]
\[ \cup (A + \alpha BC)^{-1}(\mathcal{E}). \]
Hence, the inverse image of every subset of the union of a finite number of proper affine subspaces of \( \mathbb{R}^n \) is a subset of the union of a finite number of proper affine subspaces. In particular, \( f^{-1}(\mathcal{X}) \) has measure zero. Now, for all \( k \geq 1 \), define \( f^{-k}(\mathcal{X}) \setminus f^{-k}(f^{-k}(\mathcal{X})) \). By induction, it follows that, for all \( k \geq 0 \), \( f^{-k}(\mathcal{X}) \) a subset of the union of a finite number of proper affine subspaces and thus has measure zero. Hence, \( \cup_{k \geq 1} f^{-k}(\mathcal{X}) \) is a countable union of sets with measure zero, and thus has measure zero. Therefore, for all \( x_0 \notin \cup_{k \geq 1} f^{-k}(\mathcal{X}) \), it follows that, for all \( k \geq 0 \), \( x_k \notin \mathcal{X} \).

**5. Numerical Example**
A key assumption in Theorem 4.5 is the requirement that, if, at step \( k \), the system changes either
Example 5.1. Let $G(q) = \frac{q^{-1}}{\sqrt{q^2 + 0.5}}$, so that $\Theta_n = \{\pi\}$ rad, $\Theta_p = \{\pm \cos(3/4)\}$ rad, $\alpha_n = -1.25$, and $\alpha_p = 0.5$. The root locus of the closed-loop linear system is shown in Fig. 6, along with spr for a range of $\alpha$. Note that, for all $\alpha > \alpha_p$, both eigenvalues of $A + \alpha BC$ are unstable, whereas, for all $\alpha < \alpha_n$, one eigenvalue of $A + \alpha BC$ is unstable and one eigenvalue of $A + \alpha BC$ is asymptotically stable.

![Figure 6: Example](image)

Let the minimal realization of $G$ be given by

$$A = \begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

and let $\alpha = -2.5 < \alpha_n$. Then $\text{spec}(A + \alpha BC) = \{\lambda, \lambda_2\}$, where $\lambda = -0.75 + 0.25\sqrt{41} \approx -2.35$ and $\lambda_2 = -0.75 + 0.25\sqrt{41} \approx 0.89$. Hence, $|\lambda| > 1$ and $|\lambda_2| < 1$. Furthermore, let $\xi = [0.1 -0.75 -0.25\sqrt{41}]^T$ and $\xi_2 = [0.1 -0.75 + 0.25\sqrt{41}]^T$ be eigenvectors of $A + \alpha BC$ associated with $\lambda$ and $\lambda_2$, respectively. Hence, $X = \{a \xi_2 : a \in \mathbb{R}\}$. Now, let $\Psi \in \mathbb{C}^\mathbb{C}$ satisfy $||\Psi|| = 1$ and $X = ||\Psi||^2$, that is, $\Psi^\ast \xi_2 = 0$, and define the projector $P = \hat{\Psi} \Psi^\ast$. Note that, for all $x \in \mathbb{R}^2$, $x \notin X$ if and only if $||Px|| \neq 0$.

Let $x_0$ be given by

$$x_0 = [-5.5 - 6.5\sqrt{41} - 51]^T,$$

such that $y_0 = Cx_0 \approx 3.88 > 1$. Then, it follows from the system dynamics with exact symbolic computation that

$$x_1 = Ax_0 + \alpha B = [17.5 - 6.5\sqrt{41} - 5.5 - 6.5\sqrt{41}]^T,$$

$$x_2 = Ax_1 + \alpha B = [17.5-3.25\sqrt{41} - 17.5 - 6.5\sqrt{41}]^T,$$

$$x_3 = Ax_2 + \alpha B = [6.5 17.5 - 3.25\sqrt{41}]^T,$$

$$x_4 = Ax_3 + \alpha B = [-4.875 + 1.625\sqrt{41}]^T$$

Hence, $y_1 = Cx_1 = 23$, $y_2 = Cx_2 = 0.25 + 3.25\sqrt{41} \approx 21.06 > 1$, $y_3 = Cx_3 = -11.25 + 3.25\sqrt{41} \approx 9.56 > 1$, $y_4 = Cx_4 = -11.375 + 1.625\sqrt{41} \approx -0.97 \in (-1,1)$, and thus $S_0 = S_1 = S_2 = S_3 = S_4 = S_2$. Note that $x_4 = 6.5\xi_2$.

Hence, for all $k \geq 4$,

$$C(A + \alpha BC)^{k-4}x_4 = 6.5C(A + \alpha BC)^{k-4}\xi_2 = 6.5\lambda_2^{k-4}C\xi_2.$$ 

Since $|C\xi_4| = 6.5|C\xi_2| = |-11.375 + 1.625\sqrt{41}| \approx 0.97 < 1$, it follows from (29) that, for all $k \geq 4$,

$$|C(A + \alpha BC)^{k-4}x_4| = |6.5|\lambda_2^{k-4}|C\xi_2|$$

$$\approx 0.97 (0.85)^{k-4} < 1,$$

and thus $S_k = S_2$. Hence,

$$\lim_{k \to \infty} y_k = \lim_{k \to \infty} C(A + \alpha BC)^{k-4}x_4 = 6.5C\xi_2 \lim_{k \to \infty} \lambda_2^{k-4} = 0,$$

and, for all $k \geq 4$,

$$||Px_k|| = ||P(A + \alpha BC)^{k-4}x_k|| = 6.5 ||\lambda_2^{k-4}|||P\xi_2|| = 0.$$

Therefore, $||Px_4|| = 0$, which implies that $x_4 \notin X$, and thus the assumptions of Theorem 4.3 are not satisfied. In this case, the output converges and thus the system does not have self-excited oscillations, as shown in Fig. 4.

Next, let $x_0$ be given by

$$x_0 = [-5.5 - 6.5\sqrt{41} - 51 + \varepsilon]^T,$$

where $\varepsilon \equiv 10^{-12}$, which represents a small perturbation of Fig. 4, such that $y_0 = Cx_0 \approx 3.88 > 1$. 

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With the initial condition (30), it follows that
\[ x_1 = [17.5 - 6.5 - 0.5\varepsilon - 5.5 - 6.5\sqrt{4T}]^T, \]
\[ x_2 = [17.75 - 3.25\sqrt{4T} - 0.5\varepsilon - 5.5 - 6.5\sqrt{4T}]^T, \]
\[ x_3 = [6.5 - 0.25\varepsilon - 5.5 - 6.5\sqrt{4T}]^T, \]
\[ x_4 = [-4.875 + 1.625\sqrt{4T} - 0.25\varepsilon]^T. \]

Hence, \( y_1 = Cx_1 = 23 - 0.5\varepsilon, \) \( y_2 = Cx_2 = 0.25 + 3.25\sqrt{4T} \approx 21.06 > 1, \) \( y_3 = Cx_3 = -11.25 + 3.25\sqrt{4T} + 0.25\varepsilon \approx 9.56 > 1, \) \( y_4 = Cx_4 = -11.375 + 1.625\sqrt{4T} + 0.25\varepsilon \approx -0.97 \in (-1, 1), \) and thus \( S_0 = S_1 = S_2 = S_3 = S_4 = S_5. \) Defining \( \kappa_1 \triangleq 0.5 - 3\sqrt{4T}/82 \) and \( \kappa_2 \triangleq 0.5 + 3\sqrt{4T}/82, \) it follows that
\[ \kappa_1 x_1 + \kappa_2 x_2 = \begin{bmatrix} -\sqrt{4T}/8 + 9\sqrt{4T}/328 \\ 0.5 - 3\sqrt{4T}/82 \\ \end{bmatrix}, \]
\[ \begin{bmatrix} -\sqrt{4T}/8 + 9\sqrt{4T}/328 \\ 0.5 + 3\sqrt{4T}/82 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Then, \( x_4 = 6.5\xi + [0 - 0.25\varepsilon] = -0.25\kappa_2 \xi + (6.5 - 0.25\kappa_2 \xi) = \eta_1 \xi + \eta_2 \xi_2, \) where \( \eta_1, \eta_2 \neq 0. \) Hence, for all \( k \geq 4, \)
\[ C(A + \alpha BC)^{-4} x_4 = \eta_1 C(A + \alpha BC)^{-4} \xi + \eta_2 C(A + \alpha BC)^{-4} \xi_2 = \eta_1 \xi + \eta_2 \xi_2, \]
\[ \begin{align*}
(C(A + \alpha BC)^{-4} x_4)
&= \eta_1 \xi + \eta_2 \xi_2,
&= \eta_1 \xi + \eta_2 \xi_2
\end{align*} \]

Since \( \eta_1 \xi \neq 0, \) \( \eta_2 \xi_2 \neq 0, |\lambda| > |\lambda_2|, \) and \( |\lambda| > 1, \) it follows from (31) and Proposition A.3 that
\[ \limsup_{k \to \infty} |C(A + \alpha BC)^{-4} x_4| = \limsup_{k \to \infty} |\eta_1 \xi + \eta_2 \xi_2 | = \infty. \]

Hence, there exists \( k > 4 \) such that \( S_k \neq S_2. \) Furthermore, note that
\[ ||Px_4|| = ||\eta_1 P\xi + \eta_2 P\xi_2|| = ||\eta_1 P\xi|| \neq 0, \]
which implies that \( x_4 \notin \mathcal{X}. \) Therefore, with \( x_0 \) given by (28), \( x_4 \notin \mathcal{X}, \) and thus the assumptions of Theorem 4.3 are satisfied. In this case, Fig. 7 shows that the output does not converge and is bounded, and thus the system has self-excited oscillations. In fact, the asymptotic response is periodic with period 2 steps.

6. Conclusions and Future Work

This paper analyzed a discrete-time Lur’e system that exhibits self-excited oscillations. This system involves an asymptotically stable linear system with a zero at 1 connected in feedback with a piecewise-linear saturation nonlinearity. It was shown that, for sufficiently large loop gains, the response of the system is bounded and does not converge, and thus the system has self-excited oscillations. A numerical example illustrated the conditions under which this discrete-time Lur’e system yields a self-excited response. Future work will extend the Lur’e model to sigmoidal nonlinearities as well as the use of this model for system identification.

Appendix A

Lemma A.1. Let \( z_1, \ldots, z_m \in \mathbb{C} \) be distinct, assume that, for all \( i = 1, \ldots, m, |z_i| \geq 1, \) let \( a_1, \ldots, a_m \in \mathbb{C}, \) and assume that \( \lim_{k \to \infty} \sum_{i=1}^m a_i z_i^k = 0. \) Then, \( a_1 = \cdots = a_m = 0. \)

Proof: First, consider the case where \( m = 1. \) Suppose that \( a_1 \neq 0. \) Then, \( \lim_{k \to \infty} z_1^k = 0, \) and thus \( \lim_{k \to \infty} |z_1|^k = 0, \) which contradicts the assumption that \( |z_1| \geq 1. \) Therefore, \( a_1 = 0. \)

Next, consider the case where \( m \geq 2. \) Let \( s \in \{1, \ldots, m\}, \) and define the polynomial
\[
p_s(z) \triangleq \prod_{1 \leq i \leq m} \frac{z - z_i}{z_s - z_i}.
\]
Writing \( p_s(z) = \sum_{j=0}^{m-1} b_j z^j, \) where \( b_0, \ldots, b_{m-1} \in \mathbb{C}, \) with \( b_{m-1} \neq 0, \) then
\[
p_s(z) = \sum_{j=0}^{m-1} b_{m-1-j} z^j.
\]
\( \mathbb{C} \), it follows that, for all \( k \geq 1 \),
\[
\sum_{i=1}^{m} a_i z_i^k p(z_i) = \sum_{j=0}^{m-1} b_j \sum_{i=1}^{m} a_i z_i^k + \sum_{j=0}^{m-1} b_j w_{k+j},
\]
where \( w_k \triangleq \sum_{i=1}^{m} a_i z_i^k \). Since, for all \( j = 0, \ldots, m-1 \), \( \lim_{k \to \infty} w_{k+j} = 0 \), it follows that
\[
\lim_{k \to \infty} \sum_{i=1}^{m} a_i z_i^k p(z_i) = 0.
\]
Next, note that, since \( p_s(z_s) = 1 \) and, for all \( t \in \{1, \ldots, m\} \setminus \{s\} \), \( p_s(z_i) = 0 \), it follows that
\[
\sum_{i=1}^{m} a_i z_i^k p(z_i) = a_s z_s^k.
\]
It thus follows from (32) that
\[
\lim_{k \to \infty} a_s z_s^k = 0.
\]
Now, suppose that \( a_s \neq 0 \). Then, (34) implies that \( \lim_{k \to \infty} |z_s|^k = 0 \). Hence, \( |z_s| < 1 \), which contradicts the assumption that \( |z_s| \geq 1 \). Therefore, \( a_s = 0 \), and thus \( a_1 = \cdots = a_m = 0 \). \( \square \)

The following is a corollary of Lemma A.1.

**Corollary A.2.** Let \( z_1, \ldots, z_m \in \mathbb{C} \) be distinct, assume that, for all \( i = 1, \ldots, m \), \( |z_i| \geq 1 \), and let \( a_1, \ldots, a_m \in \mathbb{C} \), at least one of which is nonzero. Then,
\[
\limsup_{k \to \infty} \left| \sum_{i=1}^{m} a_i z_i^k \right| > 0.
\]

The following result is used in the proof of Lemma 3.2.

**Proposition A.3.** Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) be distinct, and assume that \( \max_{i=1, \ldots, n} |\lambda_i| > 1 \). Furthermore, let \( p_1, \ldots, p_n \) be nonzero polynomials with complex coefficients, and, for all \( k \geq 1 \), define
\[
y_k \triangleq \sum_{i=1}^{n} p_i(k)^{\lambda_i^k}.
\]
Then,
\[
\limsup_{k \to \infty} |y_k| = \infty.
\]

**Proof:** Label \( \lambda_1, \ldots, \lambda_n \) such that \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 0 \), and define \( \rho \triangleq \max_{i \in \{1, \ldots, n\}} |\lambda_i| > 1 \) and \( s \in \{1, \ldots, n-1\} \) such that \( \rho = |\lambda_1| = \cdots = |\lambda_s| > |\lambda_{s+1}| \). Furthermore, let \( y_k = y_{k,1} + y_{k,2} \), where
\[
y_{k,1} \triangleq \sum_{i=1}^{s} p_i(k)^{\lambda_i^k}, \quad \quad y_{k,2} \triangleq \sum_{i=s+1}^{n} p_i(k)^{\lambda_i^k}, \quad (38)
\]
where, in the case \( s = n \), \( y_{k,1} \triangleq 0 \). Note that, for all \( k \geq 0 \),
\[
\left| \frac{y_{k,2}}{\rho^k} \right| = \left| \sum_{i=s+1}^{n} p_i(k) \left( \frac{\lambda_i}{\rho} \right)^k \right| \leq \left( \sum_{i=s+1}^{n} |p_i(k)| \right) \left| \frac{\lambda_{s+1}^k}{\rho} \right|,
\]
which implies that
\[
\lim_{k \to \infty} \frac{y_{k,2}}{\rho^k} = 0. \quad (39)
\]
Next, let \( \lambda_1, \ldots, \lambda_n \) be ordered such that \( d \triangleq \deg p_1 = \deg p_2 = \cdots = \deg p_m \geq \cdots \geq \deg p_{n-1} \geq \cdots \geq \deg p_s \), where \( m \in \{1, \ldots, s\} \). Thus, for all \( i \in \{1, \ldots, m\} \), there exists a nonzero complex number \( a_i \), and a polynomial \( q_i \), such that \( \deg q_i \leq d \) and, for all \( k \geq 0 \), \( p_i(k) = a_i k^d + q_i(k) \). Hence,
\[
\frac{y_{k,1}}{\rho^k} = \sum_{i=1}^{m} a_i z_i^k + \sum_{i=1}^{n} q_i(k) \frac{k^d}{k^d} z_i^k + \sum_{i=m+1}^{n} p_i(k) \frac{k^d}{k^d} z_i^k, \quad (40)
\]
where, for all \( i \in \{1, \ldots, s\} \), \( z_i \triangleq \lambda_i/\rho \). It thus follows from (40) that
\[
\lim_{k \to \infty} \left( \frac{y_{k,1}}{\rho^k} - \sum_{i=1}^{m} a_i z_i^k \right) = 0. \quad (41)
\]
Hence, (39) and (41) imply that
\[
\lim_{k \to \infty} \left( \frac{y_{k,1}}{\rho^k} - \sum_{i=1}^{m} a_i z_i^k \right) = 0. \quad (42)
\]
Next, since \( \lambda_1, \ldots, \lambda_m \) are distinct, it follows that \( z_1, \ldots, z_m \) are distinct. Corollary A.2 thus implies that there exist \( \varepsilon > 0 \) and an increasing sequence \( (k_j)_{j \geq 0} \) of positive integers such that, for all \( j \geq 0 \),
\[
\left| \sum_{i=1}^{m} a_i z_i^{k_j} \right| > 2 \varepsilon, \quad (43)
\]
and, from (42), that there exists a nonnegative integer \( j_0 \) such that, for all \( j \geq j_0 \),
\[
\left| \frac{y_{k_j}}{\rho^{k_j}} - \sum_{i=1}^{m} a_i z_i^{k_j} \right| < \varepsilon. \quad (44)
\]
Hence, \(13\) and \(14\) imply that, for all \(j \geq j_0\),
\[
\left| \frac{y_{k-j}}{\rho^{k_j}k_j} \right| \geq \sum_{i=1}^{m} a_i z_i^j - \left| \frac{y_{k-j}}{\rho^{k_j}k_j} - \sum_{i=1}^{m} a_i z_i^j \right| > \varepsilon,
\]
and thus \(|y_{k-j}| > \varepsilon \rho^{k_j}k_j^j\). Since \(\rho > 1\), it follows that
\[
\lim_{j \to \infty} |y_{k-j}| = \infty,
\]
which implies \((37)\). □

Lemma A.4. Let \(M\) be an \(n \times n\) real matrix, assume that \(1\) is not an eigenvalue of \(\text{M}\), and let \(x\) and \(y\) be \(n \times 1\) real vectors. Then the following statements hold:

i) If \(\lim_{k \to \infty} M^k\) exists, then the limit is zero.

ii) If \(\lim_{k \to \infty} M^k x\) exists, then the limit is zero.

iii) If \(\lim_{k \to \infty} y^T M^k x\) exists, then the limit is zero.

Proof: To prove ii), write \(\chi_M(z) = z^n + a_1 z^{n-1} + \cdots + a_n\), and, for all \(k \geq 0\), define \(t_k = y^T M^k x\). Then the Cayley-Hamilton implies that, for all \(k \geq 0\),
\[
t_{n+k} + a_1 t_{n+k-1} + \cdots + a_n t_k = 0. \tag{45}
\]
Since \(\lim_{k \to \infty} t_k\) exists, letting \(k \to \infty\) in \((45)\) yields
\[
0 = (1 + a_1 + \cdots + a_n) \lim_{k \to \infty} t_k = \chi_M(1) \lim_{k \to \infty} t_k. \tag{46}
\]
Since \(\chi_M(1) \neq 0\), \((10)\) implies that \(\lim_{k \to \infty} t_k = 0\). Finally, i) and ii) follow by taking \(x\) and \(y\) to be columns of the \(n \times n\) identity matrix. □

CRediT authorship contribution statement

Juan A. Paredes: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. Syed Aseem Ul Islam: Conceptualization, Writing – original draft. Omran Kouba: Formal analysis, Writing – review & editing. Dennis S. Bernstein: Writing – original draft, Writing – review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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