Gauge equivalent integrable equations on $\mathfrak{sl}(3)$ with reductions

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Abstract. We consider an auxiliary linear problem on the algebra $\mathfrak{sl}(3)$ with reductions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ type introduced recently by Gerdjikov, Mikhailov and Valchev (GMV), and discuss its relation with a Generalized Zakharov–Shabat system having the same type of reduction symmetry. We describe the gauge-equivalent hierarchies and find explicitly the analog of the first nontrivial soliton equation in the hierarchy of equations related to GMV linear problem.

1. Introduction
The concept of gauge-equivalent soliton equations originates from the paper of Mikhailov and Zakharov [1], where it was applied to reveal relations between several different models of classical field theory, such as the $n$-field, the sine-Gordon equation and the Thirring model. As far as we know the next example was the discovery of the gauge equivalence between the nonlinear Schrödinger equation (NLS),

\[ i\varphi_t + \varphi_{xx} + 2|\varphi|^2 = 0, \tag{1} \]

(here $\varphi$ is a complex-valued function going to zero when $|x|$ goes to infinity) and the Heisenberg Ferromagnet equation,

\[ S_t = -\frac{1}{2i}[S, S_{xx}], \tag{2} \]

with $S(x)$ being a $\mathfrak{sl}(2, \mathbb{C})$-valued function such that

\[ S^\dagger = -S, \quad \lim_{x \to \pm \infty} S(x) = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^2(x) = 1, \]

($^\dagger$ means Hermitian conjugation). As it is well known the NLS equations is represented in a Lax form $[L, A] = 0$, where the operator $L$ defines the auxiliary problem on $\mathfrak{sl}(2, \mathbb{C})$,

\[ L\psi = (i\partial_x + q - \lambda \sigma_3)\psi = 0, \tag{3} \]

known as classical Zakharov–Shabat system. Here the ‘potential’ $q(x)$ is a smooth function of the type

\[ \begin{pmatrix} 0 & q_+(x) \\ q_-(x) & 0 \end{pmatrix}, \]
where \( q_{\pm}(x) \) are complex-valued functions that rapidly decay when \( |x| \) tends to infinity. In order to get the NLS one must have \( q_+(x) = \varphi(x) \) and \( q_-(x) = \varphi^*(x) \), where * means complex conjugation. From its side HF equation has a Lax representation \([\tilde{L}, \tilde{A}] = 0\) with \(\tilde{L}(i\partial_x - \lambda S(x)) \tilde{\psi} = \tilde{L} \tilde{\psi} = 0\).

Zakharov and Takhtadjan [2] showed that \(L\) and \(\tilde{L}\) are gauge-equivalent, that is:

\[
\tilde{L} = \psi_0^{-1} L \psi_0,
\]

where \(\psi_0\) is the Jost solution of (3) for \(\lambda = 0\) having the properties:

\[
\lim_{x \to \pm \infty} \psi_0(x) = 1, \quad \lim_{x \to -\infty} \psi_0(x) = T^{-1}(0),
\]

and \(T(\lambda)\) is the transition matrix.

**Remark 1.1** It should be mentioned that \(T(0)\) should also satisfy the relation \(T^{-1}(0)\sigma_3 T(0) = \sigma_3\), we shall make some comments about that later.

As a result NLS and HF systems are equivalent because one has Lax representation \([L, A] = 0\) and the other has Lax representation \([\tilde{L}, \tilde{A}] = 0\) with \(\tilde{A} = \psi_0^{-1} A \psi_0\).

We have been able to extend the above result to the hierarchies of soliton systems associated with \(L\) and \(\tilde{L}\), as well known the NLS and HF are the first nontrivial equations in these hierarchies, see [3]. This was achieved through the theory of Recursion Operators \(\Lambda (\tilde{\Lambda})\) related to \(L\) and \(\tilde{L}\) respectively. Since Recursion Operators give not only the hierarchies of equations but also the hierarchies of their conservation laws, Hamiltonian structures etc., etc., we in fact established a gauge-covariant theory of the Recursion Operators for these systems.

Later we generalized the theory, see [4–6] for a system of the type

\[
(i\partial_x + q(x) - \lambda J) \psi = L \psi = 0
\]

called generalized Zakharov–Shabat system (GZS) when \(J\) is real and Caudrey–Beals–Coifman system (CBC system) when \(J\) is complex. In the above \(q(x)\) and \(J\) belong to some fixed simple Lie algebra \(g\) in some finite dimensional irreducible representation; (3) is obtained for \(g = \mathfrak{sl}(2, \mathbb{C})\). The element \(J\) is regular, that is, the kernel of \(\text{ad}_J\) (\(\text{ad}_J X \equiv [J, X], X \in g\)) is a Cartan subalgebra \(h \subset g\). The potential \(q(x)\) belongs to the orthogonal completion \(h^\perp\) of \(h\) with respect to the Killing form:

\[
\langle X, Y \rangle = \text{tr} (\text{ad}_X \text{ad}_Y), \quad X, Y \in g,
\]

\(q(x)\) is smooth and tends to zero as \(x \to \pm \infty\). We shall restrict ourselves below to the case when \(q(x)\) is of Schwartz type. This is enough for our purposes. The theory for (4) was first developed for the case \(g = \mathfrak{sl}(n, \mathbb{C})\) and it turned out the case when \(J\) is not real presents more difficulties which is the reason for the different names. It is not possible here to give a historic review, for such a review and an extended bibliography see [7]. Let us only mention that the case when we have CBC system in arbitrary representation of the algebra \(g\) has been studied in [8]. The corresponding gauge-equivalent system is of the form

\[
(i\partial_x - \lambda S(x)) \tilde{\psi} = \tilde{L} \tilde{\psi} = 0
\]

and is referred as GZS system (or CBC) system in pole gauge in contrast to (4) which is referred as GZS system (or CBC) system in canonical gauge. Here \(S(x)\) takes values in the orbit of \(J\) in the coadjoint representation of \(g\) and satisfies \(\lim_{x \to \pm \infty} S(x) = J\). Let us mention that as before
the first nontrivial systems in the hierarchies of soliton equations corresponding to (4) and (5) are called NLS and HF type equations, respectively.

The correspondence between the soliton equations associated with the GZS (CBC) system in pole and canonical gauge goes even further, to the geometric interpretation of the soliton equations as fundamental fields of a certain Poisson–Nijenhuis structure, see [7], but in this short note we just mention this issue.

2. Systems on $\mathfrak{sl}(3)$ and the Gerdjikov–Mikhailov–Valchev (GMV) system

The GZS in pole gauge on $\mathfrak{sl}(3)$ has been studied as an application of the general results about GZS and CBC system in pole gauge. The Generating Operator has been calculated and some systems of HF type with possible physical applications have been found [9]. However, the interest in it was renewed after the system that we refer below as GMV has been introduced [10,11]. At the beginning the GMV system study started independently, spectral properties were studied and Generating Operators were calculated. Later it was pointed out that GMV could be treated as $\mathfrak{sl}(3)$ GZS system in pole gauge with additional reductions of Mikhailov type, so that the Generating Operators found for the GMV system could be obtained from the Generating Operator for the general $\mathfrak{sl}(3)$ system and geometric interpretation has been clarified [12,13]. We want to take this further and describe the first nontrivial soliton system related to GMV linear problem as gauge-equivalent to some NLS type system. Now we introduce the GMV system.

By this name we shall call the auxiliary linear problem

$$\tilde{L}_0\tilde{\psi} = (i\partial_x + \lambda S_1)\tilde{\psi} = 0, \quad S_1 = \begin{pmatrix} 0 & u & v \\ v^* & 0 & 0 \\ u^* & 0 & 0 \end{pmatrix}.$$ 

In the above $u, v$ (the potentials) are smooth complex valued functions on $x$ belonging to the real line and by * we denote the complex conjugation. In addition, the functions $u$ and $v$ satisfy the relation

$$|u|^2 + |v|^2 = 1.$$

Natural asymptotic conditions for $u$ and $v$ are

$$\lim_{x \to \pm\infty} u(x) = 0, \quad \lim_{x \to \pm\infty} v(x) = v_\pm = e^{i\eta_\pm},$$

or

$$\lim_{x \to \pm\infty} u(x) = u_\pm = e^{i\mu_\pm}, \quad \lim_{x \to \pm\infty} v(x) = 0,$$

where $\eta_\pm, \mu_\pm$ are real constants.

As described in [10,11] the GMV system arises when one looks for integrable system having a Lax representation $[\tilde{L}_0, \tilde{A}] = 0$ with $\tilde{L}_0$ and $\tilde{A}$ subject to Mikhailov-type reduction requirements, see [14]. In this particular case the Mikhailov reduction group $G_R$ acting on the fundamental solutions of the system (6) is generated by the two elements $g_1$ and $g_2$ acting in the following way:

$$g_1(\tilde{\psi})(x, \lambda) = [\tilde{\psi}(x, \lambda^*)]^{-1},$$
$$g_2(\tilde{\psi})(x, \lambda) = H \tilde{\psi}(x, -\lambda) H, \quad H = \text{diag}(-1, 1, 1),$$

where $^\dagger$ denotes Hermitian conjugation. Since $g_1g_2 = g_2g_1$ and $g_1^2 = g_2^2 = \text{id}$, $G_R = \mathbb{Z}_2 \times \mathbb{Z}_2$. The map $\mathcal{H}: X \mapsto HXH^{-1}$ is involutive automorphism of $\mathfrak{sl}(3, \mathbb{C})$ which commutes with the complex conjugation $\sigma$ defining the real form $\mathfrak{su}(3)$ of $\mathfrak{sl}(3, \mathbb{C})$, $(\sigma(X) = -X^\dagger)$. Then introducing the spaces

$$\mathfrak{g}^{[j]} = \{X : \mathcal{H}(X) = (-1)^j X\}, \quad j = 0, 1,$$
we get the splittings
\[
\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]}, \quad \mathfrak{su}(3) = (\mathfrak{g}^{0} \cap \mathfrak{su}(3)) \oplus (\mathfrak{g}^{[1]} \cap \mathfrak{su}(3)).
\]
The spaces \(\mathfrak{g}^{[0]}, \mathfrak{g}^{[1]}\) consist: \(\mathfrak{g}^{[1]}\) of all off-diagonal matrices \(X = (x_{ij})\) for which \(x_{23} = x_{32} = 0\) and \(\mathfrak{g}^{[0]}\) of all traceless matrices \(X = (x_{ij})\) for which \(x_{12} = x_{21} = x_{13} = x_{31} = 0\). Of course, \(S \in \mathfrak{g}^{[1]}\). Also, since \(\mathcal{H}\) is automorphism, the spaces \(\mathfrak{g}^{[0]}\) and \(\mathfrak{g}^{[1]}\) are orthogonal with respect to the Killing form.

The invariance under the group generated by \(g_{1}, g_{2}\) means that if \(\tilde{\psi}\) is the common \(G_{R}\)-invariant fundamental solution of (6) and a linear problem of the type
\[
\tilde{A}\psi = i\partial_{t}\psi + \left(\sum_{k=0}^{n} \lambda^{k} \tilde{A}_{k}\right)\psi = 0, \quad \tilde{A}_{k} \in \mathfrak{sl}(3, \mathbb{C})
\]
we must have
\[
\tilde{A}_{2k+1} \in \mathfrak{g}^{[1]} \cap \mathfrak{su}(3), \quad \tilde{A}_{2k} \in \mathfrak{g}^{[0]} \cap \mathfrak{su}(3), \quad k = 0, 1, 2, \ldots
\]
In the same way \(S_{1} \in \mathfrak{g}^{[1]} \cap \mathfrak{su}(3)\) which forces \(S_{1}\) to be in the form we have written.

In the papers [10, 11] the study of the spectral properties of (6) has been initiated and the generating operators for the system have been calculated. We shall show now that the GMV problem is gauge-equivalent to a generalized Zakharov–Shabat problem (GZS problem) also with \(Z_{2} \times Z_{2}\) reduction.

3. **GZS type system gauge-equivalent to GMV system**

It can be checked the matrix \(S\) has constant eigenvalues. As pointed in [10], we have \(g^{-1} S g = J_{1}\), where \(g\) is unitary matrix \((g^{\dagger} = g^{-1})\) of the form
\[
g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ u^{*} & \sqrt{2} v & u^{*} \\ v^{*} & -\sqrt{2} u & v^{*} \end{pmatrix}, \quad J_{1} = \text{diag}(1, 0, -1).
\]
Since \(g(x) \in \text{SU}(3)\) the values of \(S_{1}(x)\) will be in the orbit \(\mathcal{O}_{J_{1}}(\text{SU}(3))\) of \(J_{1}\) with respect to \(\text{SU}(3)\) (it is a submanifold of \(\mathfrak{su}(3)\) of course). Thus \(S_{1}(x) \in \mathcal{O}_{J_{1}}(\text{SU}(3)) \cap \mathfrak{g}^{[1]}\). Let us note also that conversely, if we assume that \(S_{1}(x) \in \mathcal{O}_{J_{1}}(\text{SU}(3)) \cap \mathfrak{g}^{[1]}\) then first \(S_{1}\) has the form as in (6).

Next, as easily checked, the eigenvalues of the matrix \(S_{1}\) are \(\mu_{1} = 0, \mu_{2} = -\mu_{3} = \sqrt{|u|^{2} + |v|^{2}}\) but since they coincide with 0, \(\pm 1\) we must have \(|u|^{2} + |v|^{2} = 1\).

Thus we speak about a problem of GZS type in pole gauge with reductions. The above suggests that maybe we can use \(g\) in order to find the gauge-equivalent GZS system in canonical gauge. So let us make a gauge transformation \(L' = g^{-1} L g\). Then the GMV system is transformed into a system of the type
\[
L' \psi' = (i\partial_{x} + q' + \lambda J_{1}) \psi' = 0,
\]
where \(q' = i g^{-1} g_{x}\), or after expressing \(g\) with \(u\) and \(v\),
\[
q' = \frac{1}{2} \begin{pmatrix} uu_{x}^{*} + vv_{x}^{*} & \sqrt{2}(uv_{x} - vu_{x}) & uu_{x}^{*} + v^{*}v_{x} \\ \sqrt{2}(v^{*}u_{x} - u^{*}v_{x}) & 2(v^{*}v_{x} + uu_{x}^{*}) & \sqrt{2}(v^{*}u_{x}^{*} - u^{*}v_{x}^{*}) \\ uu_{x}^{*} + vv_{x}^{*} & \sqrt{2}(uv_{x} - vu_{x}) & uu_{x}^{*} + v^{*}v_{x} \end{pmatrix}.
\]
One can see that the condition $|u|^2 + |v|^2 = 1$ ensures that $q^\dagger = q'$ and that $\text{tr} q' = 0$ but $q'$ is not off-diagonal as GZS requires. We must address that but let us make first an important observation. $q'$ has a symmetry that is the counterpart to the symmetry defined by the automorphism $H$. Indeed, a simple calculation shows that

$$g^{-1} H g = K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(7)

and as a consequence

$$\text{Ad}(g^{-1}) \circ \mathcal{H} = \mathcal{K} \circ \text{Ad}(g^{-1}),$$

where $\mathcal{K}$ is the involutive automorphism $X \mapsto KXXK$ (of course $K^{-1} = K$). One checks that

$$\mathcal{K} q' = q', \quad \mathcal{K} J_1 = -J_1, \quad \mathcal{K} J_2 = J_2,$$

where

$$J_2 = J_1^2 - \frac{1}{3} \text{tr}(J_1^2) 1 = \frac{1}{3} \text{diag}(1, -2, 1).$$

Now let us consider the diagonal part of $q'$. We see that it is equal to $\frac{3i}{2}(uu_x^* + vv_x^*)J_2$ and the function $\frac{3i}{2}(uu_x^* + vv_x^*)$ is real, let us denote it by $a(x)$. We shall make now corrections to $g$ in order to obtain a gauge transformation which will give a potential having the same symmetries as $q'$ but having all the entries on its diagonal equal to zero. Let us try to find $\tilde{g} = g(x)(\exp i\xi x)J_2$, where $\xi(x)$ a real function, in such a way that the potential constructed through $\tilde{g}$ will have the symmetries of $q'$ but will be with zero diagonal elements. First we see that $\tilde{g}^{-1} S \tilde{g} = J_1$ since $J_1$ and $J_2$ are both diagonal, hence commute. Next, the new potential $q = i\tilde{g}^{-1} \tilde{g}_x$ equals

$$q = \exp (-i\xi J_2) q' \exp (i\xi J_2) - \xi_x J_2.$$ 

The diagonal part of the first term in the right-hand side is clearly the same as that of $q'$ so it will suffice to take $\xi(x)$ satisfying $\xi_x = a(x)$ in order to ensure that $q$ will have diagonal elements equal to zero. This of course can be done, one can set for example

$$\xi(x) = \int_{-\infty}^{x} a(y) dy.$$

Now the important thing is that $q$ also satisfies $\mathcal{K} q = q$. This follows from the fact that since $\mathcal{K} J_2 = J_2$ then $K$ commutes with $J_2$. Also, since $\xi$ is real we have $q^\dagger = q$. Finally, from our construction follows that $q(x) \to 0$ when $x \to \pm \infty$. How fast $q(x)$ goes to zero depends of course how fast $u(x)$ and $v(x)$ go to their constant values. Thus we obtained that our initial linear problem should be gauge-equivalent to GZS type linear problem

$$L_0 \psi = (i\partial_x + q + \lambda J_1) \psi = 0,$$

(8)

where we have the symmetry requirements

$$q^\dagger = q, \quad J_1^\dagger = J_1, \quad \mathcal{K} q = q, \quad \mathcal{K} J_1 = -J_1.$$ 

(9)

It is readily checked that if $q$ satisfies the restrictions (9) then it must have the form

$$q(x) = \begin{pmatrix} 0 & w(x) & \beta(x) \\ w^*(x) & 0 & w^*(x) \\ \beta(x) & w(x) & 0 \end{pmatrix},$$
where $\beta(x)$ is real function and $w(x)$ is complex function. Thus $q$ is parametrized by 3 real scalar functions (as is $S(x)$). The linear problem (8) is naturally a GZS type linear problem with $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction given by the elements $h_1, h_2$:

$$h_1(\psi)(x, \lambda) = [\psi(x, \lambda^*)]^{-1}, \quad (10)$$

$$h_2(\psi)(x, \lambda) = K\psi(x, -\lambda) = K\psi(x, -\lambda)K. \quad (11)$$

It is instructive to show how one can obtain from $L_0$ the GMV problem $\tilde{L}_0$. Let $\psi_0$ be the Jost solution of $L_0$ for $\lambda = 0$, that is

$$i\partial_x \psi_0 + q\psi_0 = 0, \quad \lim_{x \to -\infty} \psi_0 = 1.$$  

Since in the future using upper indexes will become complicated we shall use hat to denote the inverse element. With this notation we have that $Kq\tilde{K} = q$ immediately gives $K\psi_0\tilde{K} = \psi_0$, one also sees that $\psi_0$ is unitary matrix. Let us take a constant orthogonal matrix $R$ such that $H = RKR$. (A glance at (7) shows that such $R$ exists). Then let us take the function $\psi^H = \psi_0 R$. One readily checks that $\psi^H H = K\psi^H$. Of course, $\psi^H$ satisfies the same equation as $\psi_0$ but have different asymptotic when $x \to -\infty$. It is also a unitary matrix. If we perform a gauge transformation $L \to \tilde{\psi}^H L \psi^H$ we shall get the linear operator

$$i\partial_x + \lambda S, \quad S = \tilde{\psi}^H J_1 \psi^H. \quad (12)$$

Then obviously $S^\dagger = S$ and as it can be checked also $HS\tilde{H} = -S$ because $KJ_1\tilde{K} = -J_1$. Thus, $L_0$ defined in (8) subject to reductions (10) is gauge-equivalent to the GMV system.

Of course, there remains the question about the asymptotic of $\psi^H$ that ensure the proper asymptotic of $S$ in the GMV problem.

**Remark 3.1** The same issue arises in the gauge-equivalence of the Nonlinear Schrödinger equation and the Heisenberg Ferromagnet equation, [2], we remind that in order to ensure that

$$\lim_{x \to \pm \infty} S = \sigma_3$$

one needs to assume that $T(0)\sigma_3 T(0) = \sigma_3$, where $T(0)$ is the transition matrix for the classical Zakharov–Shabat system.

Let us define $g_{\pm} = \lim_{x \to \pm \infty} g(u(x), v(x))$. When $x \to -\infty$ we have that $S$ in (12) tends to $\tilde{R}J_1R$ so one simply can choose $\tilde{R} = g_-$ to ensure the proper limit. When $x \to +\infty$ in order to have the proper limit, just as in the $\mathfrak{sl}(2)$ case, one needs to impose the restriction:

$$\tilde{R}T(0)R = g_+, \quad \text{or equivalently,} \quad g_- T(0)\hat{g}_- = g_+.$$  

Now assume that $\chi$ is a fundamental analytic solution (FAS) of $L_0\psi = 0$ such that $\chi \to \exp iJ_1x$ when $x \to -\infty$. Then $\tilde{\chi} = \tilde{\psi}^H \chi$ is a fundamental solution of $L\psi = 0$ and when $x \to -\infty$ one has $\tilde{\chi} \to g_- \exp iJ_1x$. Thus knowing FAS for one of the systems one can find the FAS for the other. From the other hand the FAS are essential for the theory of the Recursion Operators so between the theories in canonical and in pole gauge there is a close relation, a fact reported as gauge-covariance of the theory.

We shall give a brief sketch about this in the next sections and shall outline what happens with the theory when we have reductions. We shall omit most of the detail, for them see [8] for the general case and [15] for the system with $\mathbb{Z}_p$ reductions. From the other hand we shall consider the case of a general $\mathbb{Z}_p$ reduction of the GZS systems on simple Lie algebras or more precisely Caudrey–Beals–Coifman (CBC) system, [16] since the element $J$ in front of $\lambda$ is complex, because for GZS on $\mathfrak{sl}(3)$ there are some specifics that make the things not so transparent as we would
wish. The full description of the theory for the GMV system and its gauge-equivalent will be given in a separate article.

Now we are going to give some examples of gauge-equivalent equations. As it is known for the GZS (CBC) system the evolution defined by the NLEE is linearized in terms of the transition matrix $T(\lambda)$ and has the form $i\dot{T} = [F(\lambda), T]$, where $F(\lambda)$ is a polynomial in $\lambda$ with values in the Cartan subalgebra $\mathfrak{h}$ of the diagonal matrices called dispersion. The equations corresponding to linear dependence on $\lambda$ are linear, quadratic dependence (no linear term) corresponds to Schrödinger type equations in canonical gauge and Heisenberg Ferromagnet type equation for the pole gauge etc.

The first two equations in the hierarchy related to GMV system correspond to the choices of the dispersion

(a) $F(\lambda) = \lambda J_1$,
(b) $F(\lambda) = -\lambda^2 J_2 + c\lambda J_1$ where $c = \text{const}$.

The system corresponding to the case (a) is

$$u_t - u_x = 0, \quad v_x - v_t = 0.$$

The system corresponding to the case (b) is (see [10])

$$iu_t + u_{xx} + (uu_x + vv_x^*)u_x + (uu_x^* + vv_x^*)u = 0,$$
$$iv_t + v_{xx} + (uu_x + vv_x^*)v_x + (uu_x^* + vv_x^*)v = 0.$$

In particular, the analog of the HF-type equation is the system obtained for $c = 0$

$$iu_t + u_{xx} + (uu_x + vv_x^*)u_x + (uu_x^* + vv_x^*)u = 0,$$
$$iv_t + v_{xx} + (uu_x + vv_x^*)v_x + (uu_x^* + vv_x^*)v = 0.$$

(13)

Now consider the gauge-equivalent hierarchy related to GZS system with reductions. As usual, the system corresponding to the case (a) is linear, it is

$$w_x - w_t = 0, \quad \beta_x - \beta_t = 0.$$

The system corresponding to the second choice of the dispersion law is

$$iw_t + w_{xx} - icw_x + (\beta^2 + i\beta_x + 2|w|^2)w = 0,$$
$$\beta_t + (2|w|^2 - c\beta)x = 0,$$

which is the gauge-equivalent to the second equation in the GMV hierarchy. In particular, for $c = 0$ we get the gauge-equivalence of (13) and

$$iw_t + w_{xx} + (\beta^2 + i\beta_x + 2|w|^2)w = 0,$$
$$\beta_t + 2|w|^2x = 0,$$

which is the NLS type equation gauge-equivalent to the HF-type equation (13). It is a Gross–Pitaevskii type of NLS for $w$ with potential defined by another field $\beta(x)$.  

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4. Completeness relations in canonical gauge and pole gauge for the CBC system

Consider the CBC system \( L\psi = (i\partial_x - \lambda J + q)\psi = 0 \) on some simple Lie algebra \( \mathfrak{g} \) in a typical representation. As mentioned, \( J \) is such that \( \mathfrak{h} = \ker ad_J \) defines a Cartan subalgebra (we shall call it ‘the Cartan subalgebra’) and \( q(x) \) takes values in the orthogonal complement \( \mathfrak{h}^\perp = \bar{\mathfrak{g}} \) with respect to the Killing form. Let \( \Delta \) be the root system of \( \mathfrak{g} \) (defined by \( \mathfrak{h} \)) and let \( \Sigma = \cup_{\alpha \in \Delta} \zeta_{\alpha} \) be the bunch of straight lines \( \zeta_{\alpha} \) where \( \zeta_{\alpha} = \{ \lambda : \text{Im}(\lambda(\alpha(J)) = 0) \} \). The connected components of the set \( \mathbb{C} \setminus \Sigma \) are open sectors in the \( \lambda \)-plain. We shall denote these sectors by \( \Omega_\nu \) ordering them anti-clockwise. Clearly \( \nu \) takes values from 1 to some even number \( 2M \). Then the boundary of the sector \( \Omega_\nu \) consists of two rays: \( l_\nu \) and \( l_{\nu+1} \) (\( l_\nu \) comes before \( l_{\nu+1} \) when we turn anti-clockwise) so that \( \overline{\Omega_\nu} \cap \overline{\Omega_{\nu-1}} = l_\nu \). (The ‘bar’ here denotes the closure). We shall understand the number \( \nu \) modulo \( 2M \). Naturally,

\[
\mathbb{C} \setminus \Sigma = \bigcup_{\nu=1}^{2M} \Omega_\nu, \quad \Omega_\nu \cap \Omega_\mu = \emptyset, \quad \nu \neq \mu.
\]

If \( \alpha, \beta \in \Delta, \; \alpha \neq \beta \) and \( \nu \) is fixed, then for \( \lambda \in \Omega_\nu \) the value \( \text{Im}[\lambda(\alpha - \beta)(J)] \) does not change sign. This permits to define ordering of the roots relative to the sector \( \Omega_\nu \) (\( \nu \)-ordering), we write \( \alpha >_\nu \beta \) iff \( \text{Im}(\lambda(\alpha - \beta)(J)) > 0 \). Consequently, in each \( \Omega_\nu \) we have splitting into positive and negative roots \( \Delta = \Delta^+_\nu \cup \Delta^-_\nu \).

Limiting ourselves to the typical representation of \( \mathfrak{g} \), for a large class of potentials \( q(x) \) it can be shown that in each of the sectors \( \Omega_\nu \) there exists fundamental solution (FAS) \( \chi_\nu(x,\lambda) \) with the properties:

(a) \( \chi_\nu(x,\lambda) \) is meromorphic in \( \Omega_\nu \) and has only finite number of poles. The poles define the discrete spectrum of the problem and for simplicity below we shall assume that \( \chi_\nu(x,\lambda) \) is analytic in \( \Omega_\nu \).

(b) \( \chi_\nu(x,\lambda) \) allows extension by continuity to the boundary of \( \Omega_\nu \) (to the rays \( l_\nu \) and \( l_{\nu+1} \)).

(c) For \( \lambda \in \Omega_\nu \) the function \( \chi_\nu(x,\lambda)e^{i\lambda Jx} \) is bounded and we have \( \lim_{x \to -\infty} \chi_\nu(x,\lambda)e^{i\lambda Jx} = 1 \) and

\[
\lim_{\lambda \to +\infty} \chi_\nu(x,\lambda)e^{i\lambda Jx} = 1.
\]

Below we shall have a situation when some set of functions \( f_\nu(\lambda) \) is such that each \( f_\nu(\lambda) \) is analytic in \( \Omega_\nu \) and allows extension to the boundary of this sector \( l_\nu \cup l_{\nu+1} \). Then say on \( l_\nu \) we have the extensions from the left and from the right. We shall denote the extension from the left by \( f_\nu^+(\lambda) \) and from the right by \( f_\nu^- \) \( (\lambda) \) (of course \( \lambda \in l_\nu \)). Thus for example on each \( l_\nu \) we have the solutions \( \chi_\nu^\pm(\lambda, x) \) of the CBC problem.

The fundamental solutions \( \chi_\nu(x,\lambda) \) permit to construct the resolvent of the operator \( \Psi \mapsto [L, \Psi] \) which naturally must be strongly related to the NLEEs having Lax representation \( [L, A] = 0 \). This is indeed, so, the resolvent kernel of the above operator gives rise to completeness relations. Let \( \pi_0 \) be the orthogonal projector on \( \mathfrak{g} \), let \( E_\alpha, \alpha \in \Delta \) are the root vectors and let us define the following functions, called Generalized Exponents or adjoint solutions.

\[
e^{+\nu}_\alpha(x, \lambda) = \pi_0(\chi_\nu(x, \lambda)E_\alpha \chi^{-1}_\nu(x, \lambda)), \quad \lambda \in \Omega_\nu.
\]

As we agreed, for \( \lambda \in l_\nu \) we shall write \( e^{(+\nu)}_\alpha(x, \lambda) \) if the solution is extended from the sector \( \Omega_\nu \) and \( e^{(-\nu)}_\alpha(x, \lambda) \) if the solution is extended from the sector \( \Omega_{\nu-1} \). In other words, for \( \lambda \in l_\nu \)

\[
e^{(+\nu)}_\alpha(x, \lambda) = \pi_0(\chi_{\nu-1}(x, \lambda)E_\alpha \chi^{-1}_{\nu-1}(x, \lambda)), \quad e^{(-\nu)}_\alpha(x, \lambda) = \pi_0(\chi_{\nu}(x, \lambda)E_\alpha \chi^{-1}_{\nu}(x, \lambda)).
\]
Taking all this into account, the completeness relation (we remind that there is no discrete spectrum) can be cast into the following form:

\[ \Pi_0 \delta(x - y) = \frac{1}{2\pi} \sum_{\nu=1}^{M} \int \left\{ \sum_{\alpha \in \delta^+} \left( e^{(+\nu)}_{\alpha}(x) \otimes e^{(-\nu)}_{-\alpha}(y) - e^{(-\nu)}_{-\alpha}(x) \otimes e^{(+\nu)}_{\alpha}(y) \right) \right\} d\lambda, \]

where

\[ \Pi_0 = \sum_{\gamma \in \Delta} \frac{E_{\gamma} \otimes E^{-\gamma}}{\gamma(J)}, \quad \delta^+_\nu = \Delta^+_\nu \cap \Delta\nu, \quad \delta\nu = \{ \alpha \in \Delta : \text{Im}(\lambda\alpha(J)) = 0 \text{ for } \lambda \in l_\nu \}. \]

In the above we also assumed that the rays are oriented from 0 to \( \infty \) and for shortness we have omitted the dependence on \( \lambda \). The formula itself must be understood in the following way. First, it is assumed that \( \mathfrak{g}^* \) is identified with \( \mathfrak{g} \), the pairing between them being given by the Killing form. So for example, for \( X, Y, Z \in \mathfrak{g} \) making a contraction of \( X \otimes Y \) with \( Z \) on the right we obtain \( X(Y, Z) \) and making contraction from the left we get \( (Z, X)Y \). Next, the formula for \( \Pi_0 \) implies that making a contraction with \( \Pi_0 \) from the right we get \( \Pi_0 X = \text{ad}^{-1}_J \pi_0 X \) and similarly from the left \( X \Pi_0 = -\text{ad}^{-1}_J \pi_0 X \). (On the space \( \mathfrak{g} \) the operator \( \text{ad}_J \) is invertible.)

Suppose that we have a \( L^1 \)-integrable function \( g : \mathbb{R} \to \bar{\mathfrak{g}} \). Then we have the expansions (for \( \epsilon = +1 \) and \( \epsilon = -1 \) respectively)

\[ g(x) = \frac{1}{2\pi} \sum_{\nu=1}^{2M} \int \left\{ \sum_{\alpha \in \delta^+} \left( e^{(\pm\nu)}_{\alpha}(x) \langle \langle e^{(-\nu)}_{-\alpha}, [J, g] \rangle \rangle - e^{(-\nu)}_{-\alpha}(x) \langle \langle e^{(\nu)}_{\alpha}, [J, g] \rangle \rangle \right) \right\} d\lambda. \tag{14} \]

In the above we used the following notation: for two functions \( f_1(x), f_2(x) \) with values in \( \mathfrak{g} \) we put

\[ \langle \langle f_1, f_2 \rangle \rangle = \int_{-\infty}^{+\infty} \langle f_1(x), f_2(x) \rangle dx. \]

It can be shown that for either of the choices for \( \epsilon \) the expansion (14) converges in the same sense as the Fourier expansions for \( g(x) \). Finally, if one introduces the operators

\[ \Lambda_{\pm}(X(x)) = \text{ad}^{-1}_J \left( i\partial_x X + \pi_0[q, X] + i\text{ad}_q \int_{\pm\infty}^x \right) \]

One can see that

\[ (\Lambda_- - \lambda)e^{(\pm\nu)}_{\alpha} = 0, \quad (\Lambda_- - \lambda)e^{(-\nu)}_{-\alpha} = 0, \quad \alpha \in \delta^+\nu, \]

\[ (\Lambda_+ - \lambda)e^{(+\nu)}_{\alpha} = 0, \quad (\Lambda_+ - \lambda)e^{(-\nu)}_{\alpha} = 0, \quad \alpha \in \delta^-\nu. \]

The operators \( \Lambda_{\pm} \) are the famous Generating Operators, for the expansions (14) they play the role that \( i\partial_x \) plays for the Fourier expansion. The importance of the above expansions for the theory of NLEE\( \text{s} \) (they appeared first in the seminal work of [17] for the ZS system and later were generalized) is based on the fact that when one expands over the Generalized Exponents the potential \( g(x) \) and \( \delta q(x) \) one gets as coefficients a set of minimal scattering data and its variation. A crucial fact from the theory is that the coefficients in these expansions have linear evolution for the NLEE\( \text{s} \) related to \( L \). Through the operators \( \Lambda_{\pm} \) could be obtained the NLEE\( \text{s} \), their conservation laws, the hierarchy of symplectic structures, etc, see [7].
The system, gauge-equivalent to the CBC system \( L \), is defined as \( \tilde{L}\tilde{\psi} = (i\partial_x - \lambda S)\tilde{\psi} \) where \( S(x) \) takes values in the orbit \( O_J \) of the element \( J \) in the adjoint representation of the algebra \( g \). One must write here also the limits of \( S(x) \) when \( x \to \pm \infty \) but we do not want to specify them now. The system on \( O_J \) could be considered in its own right but in case there is a gauge transformation from \( L \) to \( L = \psi_0 L\psi_0 \) with \( \psi_0 \) which is a solution of CBC for \( \lambda = 0 \) and such that it gives the proper values of \( \lim_{x \to \pm \infty} S(x) \) the expansions for \( L \) could be obtained immediately from those we know for \( L \) using a transformation \( \text{Ad}(\hat{\psi}_0(x)) \otimes \text{Ad}(\hat{\psi}_0(y)) \). Indeed, in this way we obtain

\[
\hat{\Pi}_0 \delta(x - y) = \frac{1}{2\pi} \int_{\nu=1}^{2M} \frac{\alpha \in \delta^*}{\nu} \left\{ \sum_{\alpha \in \delta^*} \left( \tilde{e}_\alpha^{(+)\nu}(x) \otimes \tilde{e}_\alpha^{(-)\nu}(y) - \tilde{e}_\alpha^{(-)\nu}(x) \otimes \tilde{e}_\alpha^{(+)\nu}(y) \right) \right\},
\]

where

\[
\hat{\Pi}_0 = \text{Ad}(\hat{\psi}_0(x)) \otimes \text{Ad}(\hat{\psi}_0(y)) \Pi_0,
\]

\[
\tilde{e}_\alpha^\nu(x) = \text{Ad}(\hat{\psi}_0)e_\alpha^\nu(x) = \tilde{\pi}_0(\chi^\nu E_\alpha \chi^\nu),
\]

\[
\tilde{\pi}_0 = \text{Ad}(\hat{\psi}_0(x))\pi_0 \text{Ad}(\psi_0(x)),
\]

and the functions \( \tilde{g}(x) \) are such that for each \( x \) we have \( \tilde{g}(x) \in \tilde{h}^\perp(x) \) where \( \tilde{h} = \ker \text{ad}_S \). The Generating Operators for \( L \) are

\[
\hat{\Lambda}_\pm = \text{Ad}(\hat{\psi}_0(x))\Lambda_\pm \text{Ad}(\psi_0(x)).
\]

Of course, there is the question how to calculate \( \hat{\Lambda}_\pm \) in terms of \( S(x) \) and its derivatives. This is not very easy, but there is a clear procedure how to do that. The above theory is referred as a gauge-covariant theory of the Recursion Operators since it permits to reformulate immediately results from canonical to pole gauge [3].

The existence of reductions complicates the picture. Let us first see what happens in canonical gauge if there are \( \mathbb{Z}_p \) reductions.

5. Completeness relations in the presence of reductions defined by automorphisms

Assume that for the CBC system on the algebra \( g \) we have a reduction group \( G_M \) generated by one element \( h \) acting as

\[
h(\psi(x, \lambda)) = K(\psi(x, \omega^{-1} \lambda)), \quad \omega = \exp \frac{2\pi i}{p},
\]

where \( K \) is automorphism of order \( p \) of the Lie group \( G \) corresponding to the algebra \( g \). It generates an automorphism of \( g \) which we shall denote by the same letter \( K \). Since this immediately leads to \( Kf = \omega f \) and \( Kq = q \) the automorphism \( K \) preserves the Cartan subalgebra. Next, since \( h^p = \text{id} \), \( G_M \) is isomorphic to \( \mathbb{Z}_p \). The automorphism \( K \) acts also on the set of roots, that action we shall denote by the same letter again, and then for the root vectors we have \( K(E_\alpha) = \zeta(\alpha)E_K\alpha \), where \( \zeta(\alpha) \) are numbers, such that \( \zeta(\alpha)\zeta(-\alpha) = 1 \) and \( \zeta(\alpha)\zeta(\beta) = \zeta(\alpha + \beta) \) if \( \alpha + \beta \in \Delta \) (see [18] for the details).

Since the automorphisms of \( g \) leave the Killing form invariant, the algebra \( g \) splits into a direct sum of eigenspaces of \( K \) (upper indexes are understood modulo \( p \)), that is:

\[
g = \bigoplus_{s=0}^{p-1} g^{[s]}, \quad [g^{[k]} \otimes g^{[l]}] \subset g^{[k+l]}.
\]

For each \( X \in g^{[s]} \) we have \( KX = \omega^s X \) and the spaces \( g^{[s]} \), \( g^{[k]} \) are orthogonal with respect to the Killing form if \( k + s \neq 0 \) (mod \( p \)). Because \( K \) is an automorphism of \( g \) leaving \( h \) invariant, it
leaves invariant also the orthogonal complement $\tilde{\mathfrak{g}} = \mathfrak{h}^\perp$ of $\mathfrak{h}$. Thus each $\mathfrak{g}^{[s]}$ splits into $\mathfrak{g}^{[s]} \oplus \mathfrak{h}^{[s]}$ and

$$\tilde{\mathfrak{g}} = \oplus_{s=0}^{p-1} \mathfrak{g}^{[s]}, \quad \mathfrak{h} = \oplus_{s=0}^{p-1} \mathfrak{h}^{[s]}.$$  

The spaces $\mathfrak{g}^{[s]}$ and $\mathfrak{h}^{[s]}$ are orthogonal with respect to the Killing form for arbitrary $k$ and $s$.

The invariance of the set of the fundamental solutions can be additionally specified if we take the FAS $\chi_\nu(x,\lambda)$ defined in $\Omega_\nu$, $\nu = 1, 2, \ldots, 2M$. Then one easily sees that $K(\chi_\nu(x,\lambda))$ must be analytic solution in the sector containing $\omega_\lambda$, one has an action of $K$ (multiplication by $\omega$ on the sectors $\Omega_\nu$ and on the rays $l_\nu$. One can see that if $\omega_\Omega_\nu = \Omega_{\nu+a}$ then $\omega_{l_\nu} = l_{\nu+a}$, $\delta_\nu^\perp = \delta_\nu^\perp$. Using that one can cast the completeness relations into the form:

$$\Pi_0 \delta(x - y) = \frac{1}{2\pi pH} \sum_{\nu=1}^{2M} \sum_{k=1}^P \left\{ \sum_{\nu, s, l=1}^{2M} \omega^{\nu, s} K^k (\varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda)) \langle \varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda), [J, h] \rangle \right\} d\lambda,$$

where

$$(K \times K)(X \otimes Y) = K(X) \otimes K(Y).$$

Note that the numbers $\zeta(\alpha)$ don’t appear any more, this occurs because we apply $K$ always on products of the type $E_\alpha \otimes E_{\alpha}$. The expansions of a function $g(x)$ over the adjoint solutions can be simplified further, if for arbitrary $x$ the value $g(x) \in \mathfrak{g}^{[s]}$. In that case

$$g(x) = \frac{\epsilon}{2\pi PH} \sum_{\nu=1}^{2M} \left\{ \sum_{\nu, s, l=1}^{2M} \omega^{\nu, s} K^k (\varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda)) \langle \varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda), [J, h] \rangle \right\} d\lambda,$$

In the above are written two expansions, one for $\epsilon = +1$ and the other for $\epsilon = -1$. Making a contraction one can see that $g(x)$ is actually expanded over the functions:

$$\varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda) = \sum_{k=1}^P \omega^{-\nu, s} K^k (\varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda)), \quad \nu = 1, 2, \ldots, a,$$

which are up to a multiplier $p$ the orthogonal projections of $\varepsilon_{\alpha}^{(\pm, \nu)}(x, \lambda)$ on $\mathfrak{g}^{[s]}$.

The operators $\Lambda_\pm$ map functions with values in $\mathfrak{g}^{[s]}$ into functions with values in $\mathfrak{g}^{[s-1]}$. In particular, for the new Generalized Exponents we have

$$\Lambda_- \varepsilon_{\alpha}^{(\pm, \nu, s-1)} = \lambda_- \varepsilon_{\alpha}^{(\pm, \nu, s)}, \quad \Lambda_- \varepsilon_{\alpha}^{(\pm, \nu, s-1)} = \lambda_- \varepsilon_{\alpha}^{(\pm, \nu, s)}, \quad \alpha \in \delta_\nu^\perp,$$

Therefore $\varepsilon_{\alpha}^{(\pm, \nu, s)}$ are not eigenfunctions of $\Lambda_\pm$. However, we obviously have:

$$\Lambda_\pm \varepsilon_{\alpha}^{(\pm, \nu, s)} = \lambda^\pm \varepsilon_{\alpha}^{(\pm, \nu, s)}, \quad \alpha \in \delta_\nu^\perp,$$

This is important, so let us formulate it separately:
Theorem 5.1 For the expansions (15) the role of the Recursion Operators are played by the \( p \)-th powers of the operators \( \Lambda \pm \).

All this can be reformulated for the gauge-equivalent system, one has in fact all these relations but with tilde and the role of \( K \) is played by \( H = \text{Ad}\left(\hat{\psi}_0\right)K\text{Ad}(\hat{\psi}_0) \). The new Generating Operators will be \( \tilde{\Lambda}^p \pm \) since the operators \( \tilde{\Lambda}^p \pm \) map functions with values in \( \tilde{g}^{[s]} \) into functions with values in \( \tilde{g}^{[s-1]} \), where \( \tilde{g}^{[s]} = \text{Ad}(\hat{\psi}_0)\tilde{g}^{[s]} \).

All the above is applicable to the GMV system. Here \( p = 2, \omega = -1 \), we have two sectors – the upper and the lower half-plane, the two rays are the the positive and the negative semi-axes. For that reason the integrals over them are written as integral on the whole axis, etc. We have the automorphism given by \( H \) which is treated according to the general theory above. Of course must remember that there is one more \( \mathbb{Z}_2 \) reduction, given by \( g_1 \), see (6). However, its effect is reduced to the requirement that the coefficients belong to \( \mathfrak{su}(3) \), see (6). In fact our pairs can be written on the real form \( \mathfrak{su}(3) \), that is ‘algebraically’ nothing serious happens.

Let us end with some comments. As one sees, the automorphism \( H \) depends \( \psi_0 \). Therefore, in principle it is \( x \)-dependent. The situation that we had for the GMV system and its gauge-equivalent when \( H \) is constant is very pleasant, since it permits to parametrize the potential \( S \) in a very nice way. This may seem quite unique, but it is not so. In fact, returning back to Section 3 we see that when \( K \) is inner automorphism given by \( X \mapsto KXK \) we can make a gauge transformation to a system having constant \( H \). Finally, we want to underline that the Mikhailov-type reductions have the nice property that they are not destroyed by the gauge transformations and the theory still remains gauge-covariant.

6. Conclusion
In this article we considered the problem of finding gauge-equivalent hierarchies of soliton equations related to Generalized Zakharov–Shabat system (GZS system) on \( \mathfrak{sl}(3) \) in canonical and in pole gauge when reductions are present. This question is of theoretical importance for the soliton equations (one could recall the impact caused by the discovery that the Nonlinear Schrödinger equation and the Heisenberg ferromagnet (HF) equation are gauge-equivalent). In particular, we found the gauge-equivalent to the system introduced recently by Gerdjikov, Mikhailov and Valchev (GMV system) and we have found the gauge-equivalent counterpart of the first two equations in the hierarchy related to the GMV system. It will be interesting to find some other low rank examples of CBC systems with reductions in pole and in canonical gauge thus increasing the pool of gauge-equivalent NLEEs.

There are hopes that the theory of the systems in general position which is put already in gauge-covariant form will be generalized when there are reductions of \( \mathbb{Z}_p \) type and we made some steps in that direction. It should be underlined that since the Generating Operator theory for the GMV system has been developed independently (and specially for it), see [10, 11], it is still not in the gauge-covariant form we sketched in the previous sections and this is a task still to be done.

There is one more aspect about our results we would like to mention. Though one can suggest that the HF-type equation related tom GMV system should have similar physical applications as HF equation (of course, related to models of interacting spin 1 particles rather than spin 1/2 particles as HF) we are not aware of such application yet. From the other side, the gauge-equivalent to this system turned out to be of the Gross–Pitaevskii type and there are strong hopes that it has physical applications since Gross–Pitaevskii type equations are used in number of models describing Bose–Einstein condensate.

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