Investigating ultra-long gravitational waves with measurements of pulsars rotational parameters

Maxim Pshirkov,
PRAO ASC LPI,
Russia
Introduction

• Search for gravitational waves (GW) is “Holy Grail” of modern physics

• Every implemented technique has its own frequency limitations:
  • Ground-based interferometers: 10-1000 Hz
  • Space-borne interferometers (future): 10^{-5}-1 Hz
  • Pulsar timing: 10^{-9} - 10^{-7} Hz
  • CMB: 10^{-18} – 10^{-16} Hz

• Our goal: extend frequency range in ultra-low region using existing pulsar data
GW interacts with electromagnetic waves from the pulsar affecting apparent frequency (Sazhin, 1978; Detweiler, 1979)

\[
\frac{\Delta \nu}{\nu} = -\frac{1}{2} \int \frac{\partial h_{zz}}{\partial t} \, dl
\]

These variations will show up in anomalous timing residuals:

\[
R(t) = \int_0^t \left[ \nu_0 - \nu(t) \right] / \nu_0 \, dt \sim \frac{h_0}{\omega} \cos(\omega t)
\]
Pulsar timing and gravitational waves detection: basics (2/2)

Prefit residuals

\[ \omega > 2\pi / T_{\text{span}} \]

Postfit residuals

\[ \ddot{\nu} = \nu \]
\[ \dot{\nu} = \dot{\nu} \]

\[ \ddot{\nu} = \nu + \delta \nu \]
\[ \dot{\nu} = \dot{\nu} + \delta \dot{\nu} \]
\[ \ddot{\nu} = \ddot{\nu} + \delta \ddot{\nu} \]
...a bit of mathematics (1/4):

GW metric:

\[ h_{ij} = h p_{ij} e^{i k_{\mu} x_{\mu}} = h p_{ij} e^{-i (k_0 c t - k_i x^i)} \]

Induced variation of rotational frequency:

\[
\frac{\delta \nu(t)}{\nu_0} = \frac{1}{2} h e^i e^j p_{ij} e^{-ikct} \left[ \frac{1 - e^{i(1 - \tilde{k}_i e^i) kD}}{1 - \tilde{k}_i e^i} \right]
\]

Variations of frequency derivatives:

\[
\frac{\delta \dot{\nu}(t)}{\nu_0} = \frac{-i k c}{2} h e^i e^j p_{ij} e^{-ikct} \left[ \frac{1 - e^{i(1 - \tilde{k}_i e^i) kD}}{1 - \tilde{k}_i e^i} \right]
\]

\[
\frac{\delta \ddot{\nu}(t)}{\nu_0} = \frac{-k^2 c^2}{2} h e^i e^j p_{ij} e^{-ikct} \left[ \frac{1 - e^{i(1 - \tilde{k}_i e^i) kD}}{1 - \tilde{k}_i e^i} \right]
\]
...a bit of mathematics (2/4):

Stochastic GWB: 

\[ h_{ij}(t, x^i) = \int d^3 k \sum_{s=1,2} \left[ h_s(k^i, t) p_{ij}(k^i) e^{ik_i x^i} + c.c. \right] \]

\[ \langle h_s(k^i) \rangle = 0 \quad \langle h_s(k^i) h_s^*(k'^i) \rangle = \frac{P_h(k)}{16\pi k^3} \delta_{ss} \delta^3(k^i - k'^i) \]

Response:

\[ \frac{\delta}{dt^{1,2}} \frac{d^{1,2} v}{v_0} = \int d^3 k \sum_{s=1,2} \left[ h_s(k^i) \tilde{R}_{1,2}(t; k^i, s) + c.c. \right] \]

\[ \tilde{R}_1(t; k^i, s) = \frac{-ic}{2} e^i e^j p_{ij} e^{-ikct} \left[ \frac{1 - e^{i(1 - \tilde{k}_i e^i)kD}}{1 - \tilde{k}_i e^i} \right] \]

\[ \tilde{R}_2(t; k^i, s) = \frac{-k^2 c^2}{2} e^i e^j p_{ij} e^{-ikct} \left[ \frac{1 - e^{i(1 - \tilde{k}_i e^i)kD}}{1 - \tilde{k}_i e^i} \right] \]
…a bit of mathematics (3/4):

Statistical properties of response:

\[
\left\langle \frac{\delta \dot{v}(t)}{v_0} \right\rangle = 0 \quad \left\langle \left( \frac{\delta \dot{v}(t)}{v_0} \right)^2 \right\rangle = \int \frac{dk}{k} P_h(k) \tilde{R}_1^2(k)
\]

\[
\left\langle \frac{\delta \ddot{v}(t)}{v_0} \right\rangle = 0 \quad \left\langle \left( \frac{\delta \ddot{v}(t)}{v_0} \right)^2 \right\rangle = \int \frac{dk}{k} P_h(k) \tilde{R}_2^2(k)
\]

Transfer functions:

\[
\tilde{R}_{1,2}^2(k) = \frac{1}{8\pi} \int d\Omega \sum_s \left| \tilde{T}_{1,2}(t;k^i,s) \right|^2
\]

\[
\tilde{R}_1^2(k) = \frac{k^2c^2}{6} - \frac{c^2}{4D^2} + \frac{\cos(kD)\sin(kD)c^2}{4kD^3}
\]

\[
\tilde{R}_2^2(k) = \frac{k^4c^4}{6} - \frac{k^2c^4}{4D^2} + \frac{\cos(kD)\sin(kD)kc^4}{4D^3}
\]
Energy density per unit logarithmic interval of frequency, $\Omega$:

$$
\Omega_{gw}(k) = \frac{1}{\rho_c} \frac{d\rho_{gw}}{d \log k} = \frac{k^2 c^2}{6H_0^2} P_h(k)
$$

$$
\Omega_{gw}(k) = \Omega_{gw}(k_0) \left( \frac{k}{k_0} \right)^{n_T}
$$

Finally, combining from bits no.1-4:

$$
\left\langle \left( \frac{\delta \dot{v}(t)}{v_0} \right)^2 \right\rangle = \begin{cases} 
\frac{c^2}{4\pi^2 n_T} \Omega_{gw}(k_0) k_H^2 k_0^{-n_T} \left[ k_{\text{max}}^{n_T} - k_{\text{min}}^{n_T} \right], & n_T \neq 0 \\
\frac{c^2}{4\pi^2 n_T} \Omega_{gw} k_H^2 \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right), & n_T = 0 
\end{cases}
$$

$$
\left\langle \left( \frac{\delta \ddot{v}(t)}{v_0} \right)^2 \right\rangle = \begin{cases} 
\frac{c^4}{4\pi^2 (n_T + 2)} \Omega_{gw}(k_0) k_H^2 k_0^{-n_T} \left[ k_{\text{max}}^{n_T+2} - k_{\text{min}}^{n_T+2} \right], & n_T \neq -2 \\
\frac{c^4}{4\pi^2 n_T} \Omega_{gw} k_H k_0^2 \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right), & n_T = 2 
\end{cases}
$$
Constraints from $\ddot{\nu}$ (1/2)

- Second derivative is more suitable for our purposes (there is a \textit{a priori} unknown spin-down term in the first derivative).

- There are a lot (~20) of pulsars with low value of
  (both MSP and ordinary)

- We can assume that magnitude of $\ddot{\nu}$ due to GW cannot exceed that value, thus limiting the energy density of GWB in ultra low-frequency range:

\[
\Omega_{gw}(f_0) \leq \frac{(n_T + 2) T_{obs}^{n_T+2}}{4\pi^2 H_0^2 f_0^{n_T}} \left( \frac{\ddot{\nu}_{obs}}{\nu} \right)^2
\]
• We can make estimations using (e.g.) properties of PSR B1937+21 (from the ATNF Pulsar Database):

\[
\begin{align*}
\nu &= 642 \text{ Hz} \\
\dot{\nu} &= 4 \times 10^{-26} \text{s}^{-3} \\
\ddot{\nu} &= 6.2 \times 10^{-29} \text{s}^{-2} \\
\frac{\ddot{\nu}}{\nu} &= \frac{10^{-26}}{10^{14}} \\

* f_0 = 10^{-2} \text{ yr}^{-1}
\end{align*}
\]

Alternative: implement stability parameter $\Delta_8$ (Arzoumanian et al., 1994);
The best results come from J0437-4715.

\[
\begin{array}{|c|c|}
\hline
n_T & \Omega(f_0^*) \\
\hline
0 & 3 \times 10^{-6} \\
-1/2 & 7 \times 10^{-6} \\
-2/3 & 9 \times 10^{-6} \\
-1 & 1.4 \times 10^{-5} \\
\hline
\end{array}
\]
• Previous method can only place more or less stringent limits, it cannot provide detection of GWB.
• For the latter, we should find unique correlation in time series of some rotational parameter for several pulsars.
• We should restrain ourselves to time series of $v$ and $\dot{v}$.
• Data are prepared as follows: total span of timing observations is sampled into shorter sub-intervals, e.g. one year long; frequency is calculated for each interval.
• The correlation coefficient between the observed values of $\Delta v$ (with subtraction of constant linear term to correct spin-down effect).

$$f(\theta) = \frac{1}{N} \sum_{i=0}^{i=N-1} \frac{\Delta v_1(t_i, e_1)}{v_{01}} \frac{\Delta v_2(t_i, e_2)}{v_{02}}$$
Due to the linearity of the problem, correlation is given by usual formula:

\[ \langle f(\theta) \rangle = \sigma_{\Delta v}^2 \zeta(\theta) \]

\[ \zeta(\theta) = \frac{3(1 - \cos(\theta))}{4} \log \frac{1 - \cos(\theta)}{2} - \frac{1 - \cos(\theta)}{8} + \frac{1}{2} + \frac{1}{2} \delta(\theta) \]
• Correlation strength $\sigma_{\Delta \nu}^2$ can be very roughly estimated as follows:

$$\sigma_{\Delta \nu}^2 = \left\langle \left( \frac{\Delta \nu}{\nu_0} \right)^2 \right\rangle \approx \left( \frac{1}{2} \frac{\delta \dot{\nu}(t)}{\nu_0^2} \left( \frac{T_{samp}}{2} \right)^2 \right)^2$$

• GWB-induced part of the second derivative was estimated earlier on. Also, we can detect fractional deviations of frequency $\Delta \nu / \nu$ at $R \sim 10^{-14}$ level (rms of residuals: $1 \, \mu$s, 1 year of observations). Combining that, we arrive at (assuming flat spectrum of GW):

$$\Omega_{gw} < \frac{32}{\pi^2} R^2 \left( \frac{T_H}{T_{samp}} \right)^2$$

$$\Omega_{gw} \approx 10^{-7}$$
Conclusions

• Influence of ultra-low frequency \((10^{-12} - 10^{-9} \text{ Hz})\) GWB can be sought in rotational parameters of PSRs.

• Precise measurement of the second derivative for >20 PSRs provide us with following constraints (depending on spectral index of GWB):

\[
\Omega_{gw} \leq 10^{-6} - 10^{-5}
\]

• We can use time series of rotational frequency of different pulsars to search for correlation due to the presence of GWB and thus detect that GWB:

\[
\Omega_{gw} \approx 10^{-7}
\]

• Problem calls for further, much more rigorous development!
• That work was supported by RFBR grants no.
  • 06-02-16816-a
  • 07-02-01034-a
  • 09-02-00922-a

THANK YOU!