On the strength of the Kerr singularity and cosmic censorship

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It has been suggested by Israel that the Kerr singularity cannot be strong in the sense of Tipler, for it tends to cause repulsive effects. We show here that, contrary to that suggestion, nearly all null geodesics reaching this singularity do in fact terminate in Tipler’s strong curvature singularity. Implications of this result are discussed in the context of an earlier cosmic censorship theorem which constraints the occurrence of Kerr-like naked singularities in generic collapse situations.

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I. INTRODUCTION

The cosmic censorship hypothesis of Penrose [1] states that a physically realistic gravitational collapse can never result in a “naked” singularity—that is, all singularities arising in such situations should always be enclosed within an event horizon and hence invisible to distant observers. This hypothesis plays a fundamental role in the theory of black holes. Unfortunately, in spite of much sustained effort, no complete proof (or convincing counterexample) to cosmic censorship has been found as yet. Many partial results, however, have been established which considerably restrict the class of possible naked singularities. A cosmic censorship theorem of this type has previously been proved by one of the present authors [2]. The aim of this paper is to answer a certain essential question closely related to that result.

There exist many exact solutions of the Einstein equations in which naked singularities do occur. Among these solutions the most important one, due to its direct astrophysical applications, is undoubtedly that of Kerr [3]. This solution depends on two parameters $m$ and $a$ (see below), and represents the exterior gravitational field of a rotating body with mass $m$ and angular momentum $ma$, as measured from infinity in geometrized units with $c = G = 1$. As is well known [4], the maximal analytic extension of the Kerr solution with $m > 0$ and $a \neq 0$ contains the ring curvature singularity that may be interpreted as the outcome of collapse of a rotating object. When $|a| \leq m$, this singularity is always hidden behind an event horizon. But if $|a| > m$, there is no event horizon and the singularity is visible for all observers; moreover, there are closed timelike curves through every point of the spacetime [5]. Clearly, the Kerr solution is highly idealized and the singularity with $|a| > m$ cannot be a counterexample to Penrose’s hypothesis. However, it is not unlikely that the pathologies occurring in the case $|a| > m$ could also arise in more general scenarios of the collapse of a rapidly rotating star.

The censorship theorem of Ref. [2] shows that, under certain reasonable assumptions, a physically realistic collapse developing from a regular initial state cannot lead to the formation of a final state resembling the Kerr solution with $|a| > m$—i.e. of a naked singularity accompanied by closed timelike curves. An important role in this result plays a certain inextendibility condition, which is assumed to hold for all (achronal) null geodesics terminating at the Kerr-like naked singularity under consideration. This condition characterizes the curvature strength of the singularity; roughly speaking, it holds for a given null geodesic $\lambda$ if the curvature near the singularity is strong enough so that at least one irrotational congruence of Jacobi fields along $\lambda$ is forced to refocus as $\lambda$ approaches the singularity (for more details on this condition, see Refs. [6,7]).

It is well known [7] that the inextendibility condition will always hold for achronal null geodesics terminating at the so-called strong curvature singularities defined by Tipler [8] (see below). These singularities have the property that all objects approaching them are crushed to zero volume. One often assumes that all singularities arising in physically realistic collapse should be of the strong curvature type (see, e.g., Refs. [9–11]). However, Israel [12] has suggested that the Kerr singularity fails to be strong in the sense of Tipler’s definition, for it tends to cause repulsive effects. Since the inextendibility condition is similar in spirit to Tipler’s definition (in both cases some Jacobi fields are refocused), the question one immediately asks is whether this condition can still be expected to hold for null geodesics terminating at the Kerr singularity—i.e. whether the censorship theorem of Ref. [2] can be applied to singularities of this type. In this paper we will obtain a result that provides some positive answer to this question. Namely, we

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will show that, in general, all null geodesics reaching the Kerr singularity with \( m > 0 \) and \( a \neq 0 \) do fact terminate, contrary to the suggestion of Israel, at Tipler’s strong curvature singularity.

II. PRELIMINARIES

To begin with, we need to recall some of the basic facts on the Kerr solution. In Boyer and Lindquist coordinates \((r, \theta, \phi, t)\) the Kerr metric is given by (cf. Ref. [13], p. 161):

\[
ds^2 = \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + (r^2 + a^2)^2 \sin^2 \theta d\phi^2 - dt^2 + \frac{2mr}{\rho^2} (a \sin^2 \theta d\phi - dt)^2,
\]

where \( \rho^2 \equiv r^2 + a^2 \cos^2 \theta \) and \( \Delta \equiv r^2 - 2mr + a^2 \). As mentioned earlier, the constant \( m \) represents the mass of the metric source while \( a \) is its angular momentum per unit mass. The Kerr spacetime is stationary and axisymmetric, with Killing vector fields \( \xi^a \equiv \frac{\partial}{\partial t} \) and \( \omega^a \equiv \frac{\partial}{\partial \phi} \). Moreover this spacetime is Ricci flat and is of Petrov type D. The ring singularity is located in the equatorial plane \( \theta = \pi/2 \) at points where \( r = 0 \). As shown by Carter [5], the only null geodesics which can reach this singularity are those lying strictly in the equatorial plane on the positive \( r \) side. The equations of motion for these geodesics are (cf. Ref. [14], p. 328):

\[
w' \equiv \frac{dr}{ds} = \pm \left[ E^2 + \frac{2m}{r^3} (L - aE)^2 - \frac{1}{r^2} (L^2 - a^2 E^2) \right]^{1/2},
\]

\[
w^\phi \equiv \frac{d\phi}{ds} = \frac{1}{\Delta} \left[ \frac{2ma}{r} E + \left( 1 - \frac{2m}{r} \right) L \right],
\]

\[
w^t \equiv \frac{dt}{ds} = \frac{1}{\Delta} \left[ \left( r^2 + a^2 \right) + \frac{2a^2m}{r} \right] E - \frac{2ma}{r} L,
\]

where \( w' \), \( w^\phi \) and \( w^t \) are the coordinate basis components of the tangent vector, \( \mathbf{u} \), to a given null geodesic \( \lambda(s) \) parametrized by an affine parameter \( s \). (Note that the corresponding component \( u^\theta \equiv d\theta/ds \) of \( \mathbf{u} \) identically vanishes as \( \lambda(s) \) lies in the equatorial plane.) The quantities \( E \) and \( L \) are the constants of the motion associated with the Killing vectors \( \xi^a \) and \( \omega^a \); they are defined as follows: \( E \equiv \xi^a u^a \) and \( L \equiv \omega^a u^a \). Physically, \( E \) and \( L \) can be interpreted, respectively, as the energy at infinity and the angular momentum about the symmetry axis, \( \theta = 0 \), of a photon moving along \( \lambda(s) \). It is also worth noting here that if \( L = aE \), then \( \lambda(s) \) must belong to one of the two principal null congruences associated with the algebraic type D of the solution (see Ref. [14], p. 329). The \( \pm \) signs in Eq. (2) correspond to outgoing and ingoing geodesics, respectively.

Let us also recall that, according to Tipler’s definition [8], an affinely parametrized null geodesic \( \lambda(s) \) is said to terminate in a strong curvature singularity at affine parameter value \( s_0 \) if the following holds (cf. Ref. [8], p. 160): Let \( \mu(s) \) be a 2-form on the normal space to the tangent vector to \( \lambda(s) \) determined by two linearly independent vorticity-free Jacobi fields \( Z_1(s) \) and \( Z_2(s) \) along \( \lambda(s) \), i.e. \( \mu(s) \equiv Z_1 \wedge Z_2 \). For all \( \mu(s) \) that vanish for at most finitely many \( s \) in some neighbourhood \((s_0, s_1)\) of \( s_0 \), we have \( \lim_{s \to s_0} ||\mu(s)|| = 0 \). Very useful criteria for determining whether a given null geodesic terminates in a strong curvature singularity have been found by Clarke and Królak [15]. One of these criteria, which will be used in proving our result, can be formulated as the following

**Proposition 2.1.** Let \( \lambda(s) \) \((0 < s \leq s_1)\) be an affinely parametrized null geodesic and let \( \{E_i\} \ (i = 1, 2, 3, 4) \) be a pseudo-orthonormal tetrad parallelly propagated along \( \lambda(s) \), with \( E_1 \cdot E_1 = E_2 \cdot E_2 = E_3 \cdot E_3 = E_4 \cdot E_4 = 1 \), all other scalar products vanishing and \( E_4 = \mathbf{u} \), where \( \mathbf{u} \) is the tangent vector to \( \lambda(s) \). Let \( C^{m}_{4n4} \) \((m, n \in \{1, 2\})\) be some component of the Weyl tensor with respect to the tetrad \( \{E_i\} \). If there exists some affine parameter value \( s_0 \in (0, s_1] \) such that \( C^{m}_{4n4} \geq K s^{-2} \) on \((0, s_0] \), where \( K \) is some positive constant, then \( \lambda(s) \) terminates in Tipler’s strong curvature singularity as \( s \to 0 \).

**Proof.** The proof follows immediately from Proposition 7 of Ref. [15].

III. THE MAIN RESULT

We are now in a position to state and prove our main result.
Theorem 3.1. Let $\lambda(s) \ (s > 0)$ be an affinely parametrized null geodesic in the Kerr spacetime with $m > 0$ and $a \neq 0$. Suppose that $\lambda(s)$ approaches the ring singularity as $s \to 0$. If $\lambda(s)$ does not belong to either of the two principal null congruences, then $\lambda(s)$ terminates in Tipler’s strong curvature singularity as $s \to 0$.

In very brief outline, the proof of our result runs as follows. We first construct a certain pseudo-orthonormal tetrad $\{E_i\}$ parallely propagated along the geodesic $\lambda(s)$. We next find one of the components of the Weyl tensor with respect to the tetrad $\{E_i\}$. It turns out that if $\lambda(s)$ is not a member of the principal null congruences, then this component must grow along $\lambda(s)$ as fast as $\sim r^{-5}$, where $r$ is the Boyer-Lindquist radial coordinate on $\lambda(s)$. Using Eq. (2), we then show that this tetrad component of the Weyl tensor must diverge along $\lambda(s)$ at least as fast as $\sim s^{-2}$, where $s$ is the affine parameter. This implies, by Proposition 2.1, that $\lambda(s)$ must then terminate in Tipler’s strong curvature singularity. The rigorous proof is as follows.

Proof. Let us assume that the spacetime is parametrized by the Boyer-Lindquist coordinates $(r, \theta, \phi, t)$. Since $\lambda(s)$ reaches the ring singularity at $r = 0$, it must lie entirely in the equatorial plane $\theta = \pi/2$ on the positive $r$ side. The coordinate $r$ will change along $\lambda(s)$ according to Eq. (2). From this equation it follows easily that there may exist at most two positive values of $r$ on $\lambda(s)$ for which $dr/ds = 0$, and so we will have $dr/ds \neq 0$ along $\lambda(s)$ for all $r$ in a sufficiently small interval about 0. It is thus clear that there must exist some affine parameter value $s_1 > 0$ such that $dr/ds > 0$ along $\lambda(s)$ for all $s \in (0, s_1]$, because $r > 0$ and $s > 0$ on $\lambda(s)$, and $r \to 0$ along $\lambda(s)$ as $s \to 0$. Let us now fix such a parameter value $s_1$. According to Proposition 2.1, in order to prove the theorem, it suffices to show that the rate of growth of the curvature along $\lambda(s)$ is strong enough for $s$ in some small interval about 0. For convenience, and without loss of generality, we can thus assume that the affine parameter on $\lambda(s)$ ranges over the interval $(0, s_1]$.

Let $u$ be the tangent vector to $\lambda(s)$. The coordinate basis components $u^\alpha$, $u^\theta$ and $u^t$ of $u$ are given by Eqs. (2)-(4), with the + sign in Eq. (2) as $dr/ds > 0$ along $\lambda(s)$. The corresponding component $u^\phi$ of $u$ identically vanishes as $\lambda(s)$ lies entirely in the equatorial plane. Given any point $p \in \lambda(s)$, we define $k$ to be the vector in the tangent space $T_p$ with the following coordinate basis components: $k^\theta = k^r = k^\phi = 0$ and $k^t = r^{-1}$, where $r$ is the radial coordinate of $p$. One can readily verify that, with respect to the scalar product given by the Kerr metric (1), $k$ is a unit spacelike vector orthogonal to $u$, i.e. $\mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{k} \cdot \mathbf{u} = 0$. In addition, $k$ is parallely transported along $\lambda(s)$ (see Appendix A). The pair $\{k, u\}$ can easily be extended to the pseudo-orthonormal tetrad $\{E_i\}$ mentioned in Proposition 2.1. To see this, let us first fix some point $q \in \lambda(s)$. Let us now take a unit spacelike vector orthogonal to the plane spanned by the vectors $u$ and $k$ in the tangent space $T_q$; we will denote this vector by $n$. (Clearly, such a vector can always be found since $u$ is null and $k$ is spacelike.) Now let $\{E_i\} \ (i = 1, 2, 3, 4)$ be some orthonormal basis in the space $T_q$, with the spacelike vectors $e_2$ and $e_3$ chosen so that $e_2 = k$ and $e_3 = n$. Using the spacelike vector $e_1$ and the timelike vector $e_4$, we now define $v_1 = \alpha(e_4 + e_1)$ and $v_2 = \alpha(e_4 - e_1)$, where $\alpha \neq 0$ is some constant. Evidently, these new vectors are null and each of them is orthogonal to both $k$ and $u$. Moreover, as $v_1 \cdot v_2 = -2\alpha^2 \neq 0$, at least one of them, say $v_1$, is not parallel to the vector $u$, i.e. we must have $u \cdot v_1 \neq 0$. By suitable choice of the constant $\alpha$ in the definition of $v_1$ one can always normalize $v_1$ so that $u \cdot v_1 = -1$. The pseudo-orthonormal tetrad $\{E_i\}$ in the space $T_q$ can now be chosen as follows: $E_1 \equiv n$, $E_2 \equiv k$, $E_3 \equiv v_1$ and $E_4 \equiv u$. By parallely transporting this tetrad along $\lambda(s)$ one obtains the pseudo-orthonormal basis at each point of $\lambda(s)$.

We shall now find one of the components $C^i_{jk}$ of the Weyl tensor with respect to the tetrad $\{E_i\}$. The components $C_{mijkl}$ of the Weyl tensor with respect to $\{E_i\}$ can be obtained from the coordinate components $C_{abcd}$ of the Weyl tensor according to

$$C_{mijkl} = C_{abcd} E^a_i E^b_j E^c_k E^d_l, \quad (5)$$

where $E^a_i$ (resp., $E^b_i$, $E^c_i$ and $E^d_i$) is the $a$th (resp., $b$th, $c$th and $d$th) coordinate basis component of the vector $E_a$ (resp., $E_i$, $E_k$ and $E_l$) of the tetrad $\{E_i\}$. Having the tetrad components $C_{mijkl}$, we can now easily find the tetrad components $C^i_{jk}$ of the Weyl tensor:

$$C^i_{jk} = \eta^{im} C_{mijkl}, \quad (6)$$

where $\eta^{im}$ is the inverse of the matrix $\eta_{im} \equiv E_i \cdot E_m$; that is, we have $\eta^{11} = \eta^{22} = -\eta^{33} = -\eta^{44} = 1$ and $\eta^{im} = 0$ in all other cases. Since the tetrad $\{E_i\}$ is chosen so that $E_2 = k$ and $E_4 = u$, and the vectors $k$ and $u$ are given in the explicit form, we can find, applying (5) and (6), an explicit expression for the component $C^2_{424}$ of the Weyl tensor with respect to $\{E_i\}$. This is done in Appendix B; the result is

$$C^2_{424} = \frac{3m(L-aE)^2}{r^5}. \quad (7)$$

The task is now to show that if $\lambda(s)$ is not a member of the principal null congruences, then there exists some affine parameter interval $(0, s_0]$ of $\lambda(s)$ on which $C^2_{424} \geq Ks^{-2}$, where $K$ is some positive constant. To do this, let
us first recall that the affine parameter $s$ and the radial coordinate $r$ on $\lambda(s)$ are related by Eq. (2), with the $+$ sign as $dr/ds > 0$ on $(0, s_1]$. Let us now rewrite this equation in the form

\[ \frac{ds}{dr} = \frac{r^{3/2}}{F(r)}, \tag{8} \]

where $F(r) \equiv [r^3E^2 - r(L^2 - a^2E^2) + 2m(L - aE)^2]^{1/2}$. As $dr/ds > 0$ on $(0, s_1]$, it is obvious that $ds/dr > 0$ on $(0, r_1)$, where $r_1$ denotes the value of the coordinate $r$ of the point $\lambda(s_1)$. By (8) it is thus clear that $F(r)$ is strictly positive on $(0, r_1]$. It is also clear, as $\lim_{r \to 0} F(r) = |L - aE|\sqrt{2m}$, that $F(r)$ is bounded on $(0, r_1]$. So there exists a positive number $F_0 \equiv \sup\{F(r)|0 < r \leq r_1\}$. Consider now the function $y(r) \equiv 2r^{5/2}(5F_0)^{-1}$ defined on $(0, r_1]$. Since $dy/dr = r^{3/2}/F_0$ and $F_0 \geq F(r) > 0$ on $(0, r_1]$, by (8) we have $ds/dr \geq dy/dr > 0$ for all $r \in (0, r_1]$. From this inequality with (7), and taking into account the fact that $m > 0$, we obtain

\[ \frac{ds}{dr} \geq \frac{3m(L - aE)^2}{r^3(s)} \geq Ks^{-2}. \tag{9} \]

for all $s \in (0, s_0]$, where $K \equiv 12m(L - aE)^2(5F_0)^{-2}$. Suppose now that $\lambda(s)$ does not belong to either of the two principal null congruences; then we have $L \neq aE$ (see Sec. 2), and hence $K > 0$. By (9) and Proposition 2.1 it is thus clear that $\lambda(s)$ must terminate in Tipler’s strong curvature singularity as $s \to 0$, which is the desired conclusion.

IV. CONCLUDING REMARKS

We have examined the curvature strength of the Kerr singularity with $m > 0$ and $a \neq 0$. We have shown that every null geodesic reaching this singularity, with the exception of those belonging to the principal null congruences, must in fact terminate in Tipler’s strong curvature singularity. This is a typical property of null geodesics approaching the Kerr singularity because the principal null geodesics are very special and can be considered to form a set of “measure zero” in the family of all null geodesics reaching the singularity (Ref. [14], pp. 328, 329). The existence of these special geodesics is due to the high symmetry of the solution and one can expect that such geodesics will not occur in more general spacetimes. Thus if one attempts to define the curvature strength of a Kerr-like naked singularity, which could possibly arise in a generic collapse, one way of doing this is to assume that all null geodesics approaching such a singularity will behave much the same as a typical null geodesic approaching the Kerr singularity with $|a| > m$—i.e. will also terminate in a singularity of the strong curvature type. Using this assumption, one may then attempt to formulate and prove a theorem which would constrain or prohibit the occurrence of Kerr-like naked singularities in generic collapse situations. A theorem of this type was established in Ref. [2].

In this context, it is worth recalling that the inextendibility condition assumed in the theorem of Ref. [2] may in fact hold for a much more general class of possible singularities than only those of the strong curvature type, for the curvature need not necessarily diverge along geodesics satisfying this condition [3]. However, we have checked that this condition fails to hold for the principal null geodesics approaching the Kerr singularity (in fact, it will always fail to hold for principal null geodesics in any Ricci-flat spacetime). Thus the theorem of Ref. [2] does not exclude the possibility that a naked Kerr singularity accompanied by closed timelike curves could develop from some nonsingular initial data. However, from the proof of that theorem it may be concluded that this can happen only if the inextendibility condition fails to hold for all null geodesics terminating in the past at the naked Kerr singularity and generating the future Cauchy horizon due to the formation of this singularity. According to our result, this is possible only if all these geodesics belong to the principal null congruences. This would be a very special case.

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APPENDIX A:

In this appendix, we demonstrate that the vector $k$ is parallelly transported along the null geodesic $\lambda(s)$ with the tangent vector $u$. To this end we only need to show that

$$
\left( \frac{\partial k^c}{\partial x^c} + \Gamma^c_{\quad ab} k^a \right) u^b = 0,
$$

where $\Gamma^c_{\quad ab}$ are the connection coefficients, which can be obtained from the metric tensor $g_{ab}$ according to

$$
\Gamma^c_{\quad ab} = \frac{1}{2} g^{cd} \left( \frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{da}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right).
$$

Since $u^0 = k^r = k^\phi = k^t = 0$ and $k^\theta = r^{-1}$, the components of Eq. (A1) in the $(r, \theta, \phi, t)$ coordinate system take the form

for $c = r$:

$$
(\Gamma^r_{\quad \theta r} u^r + \Gamma^r_{\quad \theta \phi} u^\phi + \Gamma^r_{\quad \theta t} u^t) r^{-1} = 0,
$$

for $c = \theta$:

$$
- r^{-2} u^r + (\Gamma^\theta_{\quad \phi r} u^r + \Gamma^\theta_{\quad \phi \phi} u^\phi + \Gamma^\theta_{\quad \phi t} u^t) r^{-1} = 0,
$$

for $c = \phi$:

$$
(\Gamma^\phi_{\quad \phi r} u^r + \Gamma^\phi_{\quad \phi \phi} u^\phi + \Gamma^\phi_{\quad \phi t} u^t) r^{-1} = 0,
$$

for $c = t$:

$$
(\Gamma^t_{\quad \phi r} u^r + \Gamma^t_{\quad \phi \phi} u^\phi + \Gamma^t_{\quad \phi t} u^t) r^{-1} = 0.
$$

Applying (A3) to the Kerr metric (1), we can now calculate the connection coefficients appearing in Eqs. (A3)-(A6); the result is

$$
\Gamma^r_{\quad \phi r} = - \frac{a^2 \sin \theta \cos \theta}{\rho^2}, \quad \Gamma^\theta_{\quad \phi r} = \frac{r}{\rho^2}, \quad \Gamma^\phi_{\quad \phi \phi} = \frac{(\rho^2 + 2a^2 \sin^2 \theta) \cos \theta}{\rho^4 \sin \theta}, \quad \Gamma^\phi_{\quad \phi t} = \frac{2ma r \sin^2 \theta \cos \theta}{\rho^4},
$$

$$
\Gamma^t_{\quad \phi r} = \Gamma^t_{\quad \phi \phi} = \Gamma^\theta_{\quad \phi t} = \Gamma^\phi_{\quad \phi t} = \Gamma^r_{\quad \phi t} = \Gamma^r_{\quad \phi \phi} = \Gamma^r_{\quad \phi t} = \Gamma^r_{\quad \phi \phi} = 0.
$$

Substituting these coefficients into Eqs. (A3)-(A6), and putting $\theta = \pi/2$ as $\lambda(s)$ lies in the equatorial plane, we can now readily see that Eqs. (A3)-(A6) are satisfied, as desired.

APPENDIX B:

We give in this appendix some details of the computation of the component $C^2_{\phi 24}$ of the Weyl tensor with respect to the tetrad $(E_i)$. Since $E_2 = k$ and $E_4 = u$, and $u^\theta = k^r = k^\phi = k^t = 0$ and $k^\theta = r^{-1}$, the expression (5) for $C_{2424}$ takes the form

$$
C_{2424} = r^{-2} \left[ C_{\theta \phi r} (u^r)^2 + C_{\theta \phi \phi} (u^\phi)^2 + C_{\theta t t} (u^t)^2 \right] + 2r^{-2} \left( C_{\theta \phi r} u^r u^\phi + C_{\theta \phi t} u^r u^t + C_{\theta t t} u^\phi u^t \right),
$$

where the components $u^r$, $u^\phi$ and $u^t$ of $u$ are given by Eqs. (2)-(4), with the $+$ sign in Eq. (2). We recall that the Weyl tensor $C_{abcd}$ is defined by

$$
C_{abcd} = R_{abcd} + g_{a[c} R_{d]b} + g_{b[c} R_{d]a} + \frac{1}{3} R g_{a[c} g_{d]b},
$$
where $R_{abcd}$, $R_{ab}$ and $R$ denote the curvature tensor, the Ricci tensor and the curvature scalar, respectively. Since the Kerr spacetime is Ricci flat, $R_{ab}$ and $R$ will vanish, and so $C_{abcd} = R_{abcd}$. In this case the coordinate components of the Weyl tensor can be obtained from the metric components $g_{ab}$ according to

$$
C_{abcd} = \frac{1}{2} \left( \frac{\partial^2 g_{ac}}{\partial x^d \partial x^b} + \frac{\partial^2 g_{bd}}{\partial x^c \partial x^a} - \frac{\partial^2 g_{bc}}{\partial x^d \partial x^a} - \frac{\partial^2 g_{ad}}{\partial x^c \partial x^b} \right) + g_{ef} (\Gamma^e_{ca} \Gamma^f_{bd} - \Gamma^e_{da} \Gamma^f_{bc}),
$$

where the connection coefficients are given by (A2). Applying this formula to the Kerr metric (1), we can now calculate the coordinate components of the Weyl tensor appearing in (B1); the result is

$$
C_{\theta r \theta r} = \frac{mr(3a^2 \cos^2 \theta - r^2)}{\rho^2 \Delta},
$$

$$
C_{\theta \phi \theta \phi} = -\frac{mr(3a^2 \cos^2 \theta - r^2)(a^2 \Delta \sin^2 \theta + 2(r^2 + a^2)^2 \sin^2 \theta)}{\rho^6},
$$

$$
C_{\theta \theta \theta \theta} = -\frac{mr(3a^2 \cos^2 \theta - r^2)(2a^2 \sin^2 \theta + \Delta)}{\rho^6},
$$

$$
C_{\theta r \theta \phi} = C_{\theta r \theta t} = 0.
$$

Inserting these expressions in (B1), and putting $\theta = \pi/2$ as $\lambda(s)$ lies in the equatorial plane, we get

$$
C_{2424} = \frac{3m(L - aE)^2}{r^5}.
$$

Finally, we note that $C_{2424} = C^{2424}$, which is clear from (6).

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