Max-Half-Mchart: A Simultaneous Control Chart Using a Half-Normal Distribution for Subgroup Observations

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ABSTRACT A Simultaneous control chart is a well-known tool for monitoring the process mean and process variability with a single chart. In recent decades, many researchers have been interested in developing simultaneous control charts. The Shewhart chart is the most common and simple simultaneous control chart. The Multivariate Maximum control chart (Max-Mchart) is a type Shewhart chart that simultaneously monitors the process of multivariate data. This paper proposes a new transformation using a half-normal distribution to improve the Max-chart performance for subgroup observations. The new proposed chart is called Max-Half-Mchart. The Average Run Length (ARL) results show that the proposed Max-Half-Mchart outperforms the Max-Mchart. Additionally, in real data scenarios, the proposed Max-Half-Mchart is consistent with the statistic in the Hotelling T^2 chart and the Generalized Variance (GV) chart.

INDEX TERMS Single control chart, multivariate control chart, simultaneous monitoring, subgroup observations, Max-Half-Mchart.

I. INTRODUCTION A simultaneous chart is a single chart that simultaneously monitors the process mean and process variability. Many researchers have proposed simultaneous control charts. For univariate cases, the simultaneous control chart was first introduced by White and Schroeder [1] using the boxplot method. Since then, many researchers have developed univariate simultaneous control charts. Chen and Cheng [2] developed the Maximum Chart (Max-Chart) that combined statistics $\bar{X}$ and statistics $S$. A study conducted by Chao and Cheng [3] developed the semicircle control chart using a half-circle graph. The simultaneous Exponentially Weighted Moving Average (EWMA), the so-called omnibus EWMA, was introduced by Domangue and Patch [4]. Zhang et al. [5] developed the simultaneous univariate EWMA using the Generalized Likelihood Ratio (GLR). Chen et al. [6] developed a simultaneous EWMA chart using $\bar{X}$ and statistics $S$. Chen and Thaga [7] developed the Maximum Cumulative Sum (Max-CUSUM) control chart for autocorrelated data. Thaga [8] created the Sum of Square Cumulative Sum (SS-CUSUM) chart based on the sum of squares of the maximum standard CUSUM statistics.

Researchers are extending the work on univariate cases. A multivariate simultaneous control chart was also developed. Spiring and Cheng [9] developed the Alternate Variable control chart, which is presented in a boxplot form. Chen et al. [10] developed the Maximum Multivariate EWMA (Max-MEWMA) to handle cases with changing sample sizes. The Maximum Multivariate CUSUM (Max-MCUSUM) was introduced by Cheng and Thaga [11] using a standardized mean vector and the covariance matrix for the statistics. The Max-MCUSUM for autocorrelated data was developed by Khusna et al. [12]. The bivariate Max-Chart was developed by combining the Hotelling T^2 and Generalized Variance (GV) statistics [13]. Then, Thaga and Gabrieliri [14] extended The Bivariate Max-Chart [13] to the Maximum Multivariate chart (Max-Mchart) combined the Hotelling T^2 and GV statistics using the normal standard distribution. Sabahno et al. [15] expanded Max-Mchart using the gamma distribution in GV for monitoring process variability. Max-Mchart for individual observation was developed by Kruba et al. [16] using normal standard and half-normal distributions, called the Max-Half-Mchart.

From among the studies mentioned above, Max-Mchart is well known. However, in some cases, Max-Mchart [14] produces an incorrect result. Furthermore, the Max-Mchart [14]...
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II. NEW SIMULTANEOUS CONTROL CHART

Let $X_{ij}$ denote multivariate observations, with $i$ representing the subgroup order and $j$ representing the observation order in each group. In this research, Max-Mchart carries out the following transformations [14]:

$$M_i = \Phi^{-1} \left[ H_p \left( n (\bar{x}_i - \bar{\mu}) (\bar{x}_i - \bar{\mu})' \right) \right], \quad i = 1, 2, \ldots, n$$  \hspace{1cm} (1)

And

$$V_i = \Phi^{-1} \left[ H_{2m-4} \left\{ \frac{(2n - 2) \sqrt{|S_i|}}{\sqrt{\Sigma}} \right\} \right], \quad i = 1, 2, \ldots, n$$  \hspace{1cm} (2)

$\bar{x}$ and $S_i$ can be computed by the following equation:

$$\bar{x}_i = \frac{\sum_{j}^{m} x_{ij}}{m}, \quad j = 1, 2, \ldots, m$$  \hspace{1cm} (3)

$$S_i = \frac{1}{m-1} \sum_{j}^{m} (x_{ij} - \bar{x}_i) (x_{ij} - \bar{x}_i)'$$  \hspace{1cm} (4)

The $\mu$ and $\Sigma$ are defined as follows:

$$\mu = \begin{bmatrix} 
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_p 
\end{bmatrix}$$  \hspace{1cm} (5)

$$\Sigma = \begin{bmatrix} 
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} 
\end{bmatrix}$$  \hspace{1cm} (6)

$\Phi(.)$ denotes the normal standard distribution function, $H(.)$ denotes the Chi-square distribution function, $n$ is the number of subgroups, $m$ is the order in a subgroup, and $p$ is the number of quality characteristics. The Max-Mchart is defined as:

$$C_i = \max \{|M_i|, |V_i|\}$$  \hspace{1cm} (7)

In some cases, monitoring the mean process and variability process using conventional Max-Mchart as equation (7) produces incorrect results. This fact is empirically proven by applying the conventional Max-Mchart to Woven Poly Propylene (WPP) production i.e., plastic bag production [17]. Figure 1 presents Hotelling’s $T^2$ statistic, and Figure 2 depicts the GV statistic. Figure 3 shows that, in the conventional Max-Mchart the 6th, 16th, 20th, and 46th samples are out of control, even though they are in-control in terms of the mean and variance, as shown in Figures 6(a) and 6(b). However, the 6th, the 16th, the 20th, and 46th samples are marked as $V^{++}$ because the out-of-control signals are caused only by the change in the covariance matrix. The calculations presented in Figures 1–3 are provided in Table 1.
| Obs | $T^2$  | $D$    | $H(T^2)$ | $H(D)$ | Max-Mchart | Max-Half-Mchart |
|-----|--------|--------|----------|---------|-------------|-----------------|
| 1   | 11.294 | 1.771  | 0.996    | 0.06    | 2.69        | 2.69            |
| 2   | 3.938  | 2.843  | 0.860    | 0.17    | 1.08        | 1.48            |
| 3   | 2.547  | 1.552  | 0.720    | 0.04    | 0.58        | 1.71            |
| 4   | 14.067 | 6.155  | 0.999    | 0.59    | 3.13        | 3.13            |
| 5   | 4.886  | 1.168  | 0.917    | 0.02    | 1.39        | 2.02            |
| 6   | 4.816  | 0.359  | 0.910    | 0.00    | 1.34        | 3.14            |
| 7   | 0.254  | 2.693  | 0.119    | 0.15    | -1.18       | 1.18            |
| 8   | 6.103  | 1.655  | 0.953    | 0.05    | 1.67        | 1.63            |
| 9   | 11.547 | 0.435  | 0.997    | 0.00    | 2.74        | 2.98            |
| 10  | 1.129  | 0.880  | 0.431    | 0.01    | -0.17       | 2.32            |
| 11  | 2.410  | 1.198  | 0.700    | 0.02    | 0.53        | 1.99            |
| 12  | 1.121  | 0.577  | 0.429    | 0.00    | -0.18       | 2.72            |
| 13  | 1.701  | 0.692  | 0.573    | 0.01    | 0.18        | 2.55            |
| 14  | 10.566 | 2.252  | 0.995    | 0.10    | 2.57        | 2.57            |
| 15  | 0.483  | 1.106  | 0.214    | 0.02    | -0.79       | 2.08            |
| 16  | 7.646  | 0.000  | 0.978    | 0.00    | 2.02        | 9.18            |
| 17  | 7.399  | 0.989  | 0.975    | 0.01    | 1.96        | 2.20            |
| 18  | 1.278  | 0.627  | 0.472    | 0.00    | -0.07       | 2.65            |
| 19  | 1.313  | 1.682  | 0.481    | 0.05    | -0.05       | 1.61            |
| 20  | 0.683  | 0.338  | 0.289    | 0.00    | -0.56       | 3.19            |
| 21  | 1.237  | 0.810  | 0.461    | 0.01    | -0.10       | 2.40            |
| 22  | 0.664  | 0.676  | 0.282    | 0.01    | -0.58       | 2.58            |
| 23  | 1.909  | 1.521  | 0.420    | 0.04    | -0.20       | 1.73            |
| 24  | 5.312  | 4.066  | 0.930    | 0.33    | 1.47        | 1.47            |
| 25  | 3.504  | 1.670  | 0.827    | 0.05    | 0.94        | 1.62            |
| 26  | 3.187  | 1.483  | 0.797    | 0.04    | 0.83        | 1.76            |
| 27  | 3.066  | 1.122  | 0.778    | 0.02    | 0.76        | 2.07            |
| 28  | 5.072  | 2.102  | 0.921    | 0.09    | 1.41        | 1.41            |
| 29  | 2.225  | 0.905  | 0.671    | 0.01    | 0.44        | 2.29            |
| 30  | 10.169 | 1.367  | 0.994    | 0.03    | 2.50        | 2.50            |
| 31  | 74.260 | 1.717  | 1.000    | 0.06    | 8.21        | 8.21            |
| 32  | 6.799  | 1.416  | 0.967    | 0.04    | 1.83        | 1.83            |
| 33  | 26.886 | 4.208  | 0.996    | 0.35    | 4.68        | 4.68            |
| 34  | 0.094  | 1.631  | 0.860    | 0.05    | -1.68       | 1.68            |
| 35  | 45.776 | 2.647  | 0.720    | 0.15    | 6.34        | 6.34            |
| 36  | 10.092 | 0.885  | 0.999    | 0.01    | 2.49        | 2.49            |
| 37  | 1.380  | 2.405  | 0.917    | 0.12    | 0.00        | 1.17            |
| 38  | 7.581  | 3.091  | 0.910    | 0.20    | 2.00        | 2.00            |
| 39  | 0.091  | 0.777  | 0.119    | 0.01    | -1.70       | 2.44            |
| 40  | 9.160  | 0.440  | 0.953    | 0.00    | 2.32        | 2.97            |
TABLE 1. (Continued.) Max-mchart and max-half-mchart statistics for the data set.

| Obs | $T^2$  | $D$  | $H(T^2)$ | $H(D)$ | Max-Mchart | Max-Half-Mchart |
|-----|--------|------|----------|--------|------------|-----------------|
|     |        |      |          |        | $M_i$      | $V_i$           |
|-----|--------|------|----------|--------|------------|-----------------|
| 41  | 2.956  | 1.281| 0.997    | 0.03   | 0.75       | -1.92          |
| 42  | 17.744 | 2.617| 0.431    | 0.14   | 3.63       | -1.06          |
| 43  | 1.983  | 1.042| 0.700    | 0.02   | 0.33       | -2.14          |
| 44  | 0.676  | 1.112| 0.429    | 0.02   | -0.56      | -2.08          |
| 45  | 4.471  | 1.162| 0.573    | 0.02   | 1.24       | -2.03          |
| 46  | 0.859  | 0.381| 0.995    | 0.00   | -0.39      | -3.09          |
| 47  | 8.739  | 4.395| 2.214    | 0.38   | 2.24       | -0.31          |
| 48  | 2.409  | 1.235| 0.978    | 0.02   | 0.52       | -1.96          |
| 49  | 0.934  | 2.394| 0.975    | 0.12   | -0.32      | -1.18          |
| 50  | 2.218  | 0.606| 0.472    | 0.00   | 0.44       | -2.68          |
| 51  | 4.792  | 4.145| 0.481    | 0.34   | 1.33       | -0.40          |
| 52  | 4.817  | 1.434| 0.289    | 0.04   | 1.34       | -1.80          |
| 53  | 6.934  | 1.465| 0.461    | 0.04   | 1.86       | -1.77          |
| 54  | 1.451  | 1.608| 0.282    | 0.05   | 0.04       | -1.66          |

As reported in Table 1, the 6th, 16th, 20th, and 46th samples are detected as out-of-control because the cumulative distribution function of the Chi-square is too small (close to zero). Therefore, when its value is transformed to the quantile of the standard normal distribution, it produces a large negative value. To refine this inaccurate conclusion obtained by Max-Mchart, this study adopts a quantile approach similar to Max-Half-Mchart for individual observations [16]. Kruba et al. [16] proposed a half-normal distribution approach since it has a positive domain that is the same as the cumulative distribution function of the Chi-square distribution. This new Max-Mchart is the so-called Max-Half-Mchart as it is developed with transformation using a half-normal distribution. The half-normal distribution is used because it only has the positive domain that will lead to the correct result even when the cumulative distribution function of the Chi-square is too small. The statistics can be calculated using the following equation:

$$C_i^H = \max \left\{ M_i^H, V_i^H \right\} \quad (8)$$

The difference between equation (7) and equation (8) is the absolute notation. Equation (8) has no absolute sign because the half-normal only has the positive domain.

$$M_i^H = G^{-1} \left[ H_p \left( n \left( \overline{x}_i - \overline{\mu} \right) \right) \left( \overline{x}_i - \overline{\mu} \right)' \right], \quad i = 1, 2, \ldots, n \quad (9)$$

and

$$V_i^H = G^{-1} \left[ H_{2m-4} \left\{ \frac{(2n-2) \sqrt{|S_i'|}}{\sqrt{\Sigma}} \right\} \right], \quad i = 1, 2, \ldots, n \quad (10)$$

$G(.)$ denotes the half-normal distribution function, and $H(.)$ represents the Chi-square distribution function. The following steps are required in the implementation of Max-Half-Mchart:

Algorithm 1 Plotting Procedure of the Proposed Max-Half-Mchart

1. Calculate statistic $M_i^H$ using equation (9), statistic $V_i^H$ by using equation (10) and $C_i^H$ using equation (8) for each subgroup observation.
2. Determine the upper control limit (UCL) using Algorithm 2, which is presented in the next section.
3. If $C_i^H > UCL$ then the process shift occurs according to the following details:
   - If $M_i^H > UCL$ is indicated by $M_i^+\rightarrow$, this indicates a shift in the process mean.
   - If $V_i^H > UCL$ is indicated by $V_i^+\rightarrow$, this indicates a shift in the process variability.
   - If $M_i^H > UCL$ and $V_i^H > UCL$ is indicated by $B_i^+\rightarrow$, this indicates the process mean shift as well as the process variability.

Visualization of the proposed Max-Half-Mchart ($C_i^H$) can be calculated with equation (8). To obtain equation (8), the statistic of $M_i^H$ for the process mean in equation (9) and $V_i^H$ for the process variability in equation (10) must be calculated first since the equation (8) consist of the equation (9) and equation (10). Then, calculate the UCL for control limit of the proposed Max-Half-Mchart. For the UCL will be discussed in the next section using Algorithm 2. After determining the UCL, the statistic $C_i^H$ will be compared with the UCL with criteria as point 3 at the Algorithm 1 above.

III. BOOTSTRAP CONTROL LIMIT

The control limit of the conventional Max-Mchart and proposed Max-Half-Mchart use only UCL (Upper Control Limit) since the statistics of both control charts are greater
The UCL of Max-Mchart can be calculated with an analytical formula as in Thuga and Gabaitiri [14]. However, the proposed Max-Half-Mchart has an unknown specific distribution, and the bootstrap control limit is therefore used. In this paper, the results of Max-Mchart and Max-Half-Mchart are compared. To provide an equivalent comparison for both charts, the bootstrap approach is used to calculate the UCL for the conventional Max-Mchart and the proposed Max-Half-Mchart.

In this paper, the control limit for the proposed chart is free of distribution. The bootstrap approach defines the control limit. Many types of research have been conducted using the bootstrap approach to determine the control limit. Khusna et al. [19] proposed the bootstrap for Max-MCUSUM for autocorrelated data. Ahsan et al. [20] also built a control limit in the T² control chart for detection instructions. The Bootstrap control limit for conventional Max-Mchart was also developed by Kruba et al. [21] when monitoring fertilizer. The control limit is therefore used. In this paper, the results of Max-Mchart and Max-Half-Mchart are compared. To provide an equivalent comparison for both charts, the bootstrap approach is used to calculate the UCL for the conventional Max-Mchart and the proposed Max-Half-Mchart.

Algorithm 2

Bootstrap Control Limit

1. Set α, µ and Σ.
2. For iteration \( l = 1 : 1000 \) do the following
   - Generate data following a multivariate normal distribution \( N_p (\bar{\mu}, \Sigma) \)
   - Calculate statistics using equation (9) for the process mean, equation (10) for the process variability, and statistics of the proposed Max-Half-Mchart using equation (8). Then, for the conventional Max-Mchart, calculation statistic use equation (1) for the process mean, equation (2) for the process variability, and equation (7) for the statistics of the proposed chart
   - Resampling bootstrap the statistic equation (8) and equation (7) as many as \( B \) times to obtain \( B \) values of \( C^H \)
   - Calculate the \( 100(1-\alpha) \) th percentile from the \( B \) samples using \( C^H_{(100(1-\alpha))} \) and \( C^H_{(100(1-\alpha))} \)
3. Calculate \( UCL_h = \sum_{l=1}^{1000} c^H_{(100(1-\alpha))} \) (Max-Half-Mchart) and \( UCL_s = \sum_{l=1}^{1000} c^H_{(100(1-\alpha))} \) (Max-Mchart).

The UCL can be obtained using Algorithm 2. First, set the \( \alpha \bar{\mu} \) and \( \Sigma \). In this paper, the author set \( \alpha = 0.00275 \) and the \( \bar{\mu} \) and \( \Sigma \) as in-control phase (Phase 1). Second, for 1000 iteration generate 5000 data following a multivariate normal distribution \( N_p (\bar{\mu}, \Sigma) \). Then, calculate statistic of the proposed Max-Half-Mchart as explain in Algorithm 1. For, the statistics of conventional Max-Mchart (\( C_i \)) can be calculated as equation (7). To obtain equation (7), the statistic of \( M_i \) for the process mean in equation (1) and \( V_i \) for the process variability in equation (2) must be calculated first since the equation (7) consist of the equation (1) and equation (2). Then, resampling bootstrap the statistics of the proposed Max-Half-Mchart (\( C^H_i \)) and the conventional Max-Mchart (\( C_i \)) as many as 1000 times to obtain 1000 statistics of both charts. From those statistic, calculate the percentile from 1000 samples using \( C^H_{(100(1-\alpha))} \) and \( C^H_{(100(1-\alpha))} \). The last, the control limit can be calculated as \( UCL_h = \sum_{l=1}^{1000} c^H_{(100(1-\alpha))} \) for the proposed Max-Half-Mchart and \( UCL_s = \sum_{l=1}^{1000} c^H_{(100(1-\alpha))} \) for the conventional Max-Mchart.

IV. PERFORMANCE COMPARISON OF MAX-MCHART AND MAX-HALF-MCHART FOR SUBGROUP OBSERVATIONS

ARL is the average number of observations that are first found to be out-of-control. The ARL serves to measure the effectiveness of a control chart when detecting changes in a process. In this paper, a simulation is carried out with 5000 samples that follow \( N_p (\bar{\mu}, \Sigma) \). Then, 5000 samples are assigned into 1000 groups, each of which contains five samples. The ARL is obtained by shifting the mean vector \((\bar{a}, \bar{\mu}_g)\) and shifting the covariance matrix \((b, \Sigma_g)\), where the index \( g \) indicates good conditions (in-control). Thus, ARL0 is the value of ARL when \( \bar{a} = 0 \) and \( b = 1 \). The performance of the proposed Max-Half-Mchart is compared to that of the conventional Max-Mchart using the ARL value. The calculation of the control limits for both charts uses the bootstrap approach with \( \alpha = 0.00275 \). The ARL calculation results are explained for the following scenario. The scenario is a combination of correlation \( \rho = 0.3 \) with two quality characteristics of \( p = 4 \) and \( p = 5 \). The ARL can be calculated using algorithm 3.

Algorithm 3

Average Run Length

1. Set the UCL as calculated by Algorithm 2.
2. Set \( \bar{\mu} \) and \( \Sigma \).
3. Generate data following a multivariate normal distribution \( N_p (\bar{\mu}, \Sigma) \).
4. Calculate the statistics using equation (9) for the process mean, equation (10) for the process variability, and equation (8) for the statistics of the proposed chart (\( C^H \)). Then, for the conventional Max-Mchart, calculate statistics using equation (1) for the process mean, equation (2) for the process variability, and equation (7) for statistics of the proposed chart.
5. Compare \( C^H_i \) with the UCL for the proposed Max-Mchart and comparing \( C_i \) with the UCLs for the conventional Max-Half-Mchart.
6. Calculate the Run Length (RL) until the first condition \( C^H_i > UCL_h \) (Max-Half-Mchart) and \( C_i > UCL_s \) (Max-Mchart). Then, stop and return to step 3.
7. Repeat steps 3-6 1000 times.
8. Calculate \( ARL = \frac{\sum_{l=1}^{1000} RL}{1000} \).
TABLE 2. ARL of max-half-mchart observation with $p = 0.3$ and $p = 4$.

| $a$ | 1   | 1.5 | 2   | 2.5 | 3   | 3.5 | 4   | 4.5 | 5   | 5.5 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 368.17 | 24.17 | 12.12 | 4.67 | 3.63 | 2.37 | 2.27 | 1.17 | 1.17 | 1.20 |
| 0.5 | 378.26 | 192.19 | 129.05 | 97.26 | 78.17 | 65.39 | 56.24 | 49.39 | 44.05 | 39.76 |
| 1   | 142.24 | 18.69 | 15.32 | 7.46 | 5.34 | 2.19 | 1.18 | 1.17 | 1.14 | 1.14 |
| 1.5 | 74.89 | 68.84 | 63.66 | 59.22 | 55.37 | 51.99 | 49.00 | 46.34 | 43.97 | 41.83 |
| 2   | 11.21 | 6.01 | 4.05 | 3.14 | 1.07 | 1.11 | 1.10 | 1.09 | 1.12 | 1.07 |
| 2.5 | 39.99 | 38.22 | 36.60 | 35.13 | 33.77 | 32.51 | 31.35 | 30.27 | 29.27 | 28.33 |
| 3   | 5.17 | 3.85 | 2.17 | 1.02 | 1.00 | 1.01 | 1.02 | 1.02 | 1.03 | 1.02 |
| 3.5 | 27.45 | 26.62 | 25.84 | 25.11 | 24.42 | 23.77 | 23.16 | 22.58 | 22.02 | 21.50 |
| 4   | 3.28 | 2.24 | 1.03 | 1.00 | 1.01 | 1.00 | 1.00 | 1.01 | 1.00 | 1.00 |
| 4.5 | 21.00 | 20.52 | 20.07 | 19.64 | 19.22 | 18.83 | 18.45 | 18.08 | 17.73 | 17.40 |
| 5   | 1.02 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 5.5 | 17.08 | 16.77 | 16.47 | 16.19 | 15.91 | 15.64 | 15.39 | 15.14 | 14.90 | 14.67 |
| 6   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 6.5 | 12.55 | 12.39 | 12.23 | 12.08 | 11.93 | 11.79 | 11.65 | 11.51 | 11.38 | 11.25 |
| 7   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 7.5 | 11.12 | 11.00 | 10.88 | 10.76 | 10.65 | 10.54 | 10.43 | 10.32 | 10.21 | 10.11 |
| 8   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8.5 | 10.01 | 9.91 | 9.82 | 9.72 | 9.63 | 9.54 | 9.45 | 9.37 | 9.28 | 9.20 |

The calculation of the ARL of the proposed Max-Half-Mchart and the conventional Max-Mchart start from determining of the UCL as in Algorithm 2. Then, set $\mu$ and $\Sigma$ according to the required shift for the process mean and process variability. For the 1000 iteration, from those parameters generate data following a multivariate normal. After that, calculate the statistic for the proposed Max-Half-Mchart ($C^H_i$) and the conventional Max-Mchart ($C_i$) from the data as in Algorithm 1 and Algorithm 2 above. Then calculate the Run Length (RL) until the first condition $C^H_i > UCL_h$ for the proposed Max-Half-Mchart and $C_i > UCL_s$ for the conventional Max-Mchart. Last, from those iterations calculate the ARL as

$$\text{ARL} = \frac{\sum_{i=1}^{1000} \text{RL}_i}{1000}. $$

Table 2 presents the ARL of the proposed Max-Half-Mchart with $p = 4$ and $p = 0.3$. When the mean vector shifts from $\bar{a} = 0$ to $\bar{a} = [0.5, 0.5, 0.5, 0.5]$ and the covariance matrix is in an in-control condition, the ARL reduces from 368 to 142. However, the ARL also decreases from 368 to 24. If the covariance matrix shifts from $b = 1$ to $b = 1.5$, the mean vector remains in control. This finding indicates that the proposed Max-Half-Mchart is faster when detecting the change in covariance matrix than the change in the mean shift compared to the conventional Maxchart. Meanwhile, the conventional Max-Mchart detects the change in the mean vector faster than that of the covariance matrix when compared to the proposed Max-Half-Mchart.
TABLE 3. ARL of max-mchart observation with \( \rho = 0.3 \) and \( p = 5 \).

| a       | 1     | 1.5   | 2     | 2.5   | 3     | 3.5   | 4     | 4.5   | 5     | 5.5   |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0       | 374.40| 13.34 | 5.03  | 2.77  | 2.29  | 2.07  | 1.69  | 1.71  | 1.55  | 1.51  |
| 0.5     | 361.23| 185.89| 125.60| 94.89 | 76.37 | 63.99 | 55.09 | 48.42 | 43.21 | 39.04 |
| 1       | 99.30 | 5.48  | 3.08  | 2.28  | 1.89  | 1.54  | 1.51  | 1.43  | 1.57  | 1.38  |
| 1.5     | 24.63 | 23.59 | 22.62 | 21.74 | 20.92 | 20.17 | 19.48 | 18.83 | 18.23 | 17.66 |
| 2       | 4.42  | 1.72  | 1.40  | 1.44  | 1.31  | 1.44  | 1.37  | 1.42  | 1.25  | 1.22  |
| 2.5     | 1.28  | 1.10  | 1.08  | 1.04  | 1.05  | 1.14  | 1.17  | 1.09  | 1.06  | 1.15  |
| 3       | 13.22 | 12.93 | 12.65 | 12.39 | 12.13 | 11.89 | 11.66 | 11.44 | 11.23 | 11.02 |
| 3.5     | 1.02  | 1.01  | 1.01  | 1.00  | 1.00  | 1.02  | 1.01  | 1.05  | 1.03  | 1.06  |
| 4       | 10.83 | 10.64 | 10.46 | 10.28 | 10.11 | 9.95  | 9.79  | 9.64  | 9.50  | 9.35  |
| 4.5     | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  |

FIGURE 4. ARL shift in mean vector.

FIGURE 5. ARL shift in covariance matrix.

\( \tilde{\alpha} = \hat{\alpha} = [4.5, 4.5, 4.5, 4.5, 4.5] \) and the covariance matrix is in-control \( (b = 1) \), the ARLs of the proposed Max-Half-Mchart and the conventional Max-Mchart are lower than the ARL for both charts in Table 2. It can be concluded that the number of quality characteristics affects the performance of the proposed Max-Half-Mchart and the conventional Max-Mchart when detecting a mean shift. By comparing the ARL results in Tables 2 and 3, it can be concluded that, for subgroup observations, the proposed Max-Half-Mchart performs better than the conventional Max-Mchart for a higher number of quality characteristics.

Figure 4 presents the ARL comparison for the proposed Max-Half-Mchart and the conventional Max-Mchart when there is a shift in the mean vector while the covariance matrix remains in-control. Figure 5 shows the ARL comparison of the proposed Max-Half-Mchart and conventional Max-Mchart when the mean vector remains in-control, and there is a shift in the covariance matrix. Figure 4 shows...
that both control charts exhibit better performance when the number of quality characteristics becomes higher ($p = 4$, $top = 5$) during detection of shifts in the mean vector. Figure 5 shows a significant difference between the proposed Max-Half-Mchart and the conventional Max-Mchart. The ARL of the proposed Max-Half-Mchart decreases significantly when there is a shift in the covariance matrix, while the mean vector remains in-control. Therefore, it can be concluded that the proposed Max-Half-Mchart is more sensitive than the conventional Max-Mchart when detecting the change in the covariance matrix. In addition, Figure 4 and Figure 5 illustrate that the proposed Max-Half-Mchart is more sensitive than the conventional Max-Mchart as the ARL value of the proposed Max-Half-Mchart approaches one.

### V. APPLICATION SCENARIO

#### A. APPLICATION TO SIMULATION DATA

Simulation data are used to compare the application of the proposed Max-Half-Mchart, the conventional Max-Mchart, Hotelling $T^2$ control chart and GV control chart. The random variables $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5$ are generated for the scenario summarized in Table 3. For each data set, 175 random samples are generated that follow $N_5(\mu_g + \vec{a}, \sigma^2 \Sigma_g)$, where the values of $\vec{a}$ and $\sigma^2$ determined by the scenario described in Table 4. The $\mu_g$ is the mean vector that is calculated from the in-control phase (Phase I). The $\Sigma_g$ is the covariance matrix that is also calculated from the in-control phase. The 175 random samples are divided into 35 subgroups, where each subgroup contains five observations. The first half of the data remains

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**TABLE 4.** The scenario of simulation data.

| Original data | First half of data | Second half of data | Monitoring result |
|---------------|--------------------|---------------------|-------------------|
| Dataset 1     | $a=0.0$, $b=1.0$   | In control          | In-control        |
| Dataset 2     | $a=0.0$, $b=1.0$   | Shift in mean       | $M^{++}$ start from $16^a$ sample |
| Dataset 3     | $a=0.0$, $b=1.0$   | Shift in covariance | $V^{++}$ start from $16^a$ |
| Dataset 4     | $a=0.0$, $b=1.0$   | Shift in mean and variance | $B^{++}$ start from $16^a$ except for the $20^a$, $28^a$, $32^a$, and $33^a$ samples are marked by $M^{++}$ |

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**FIGURE 6.** The plot of data set 1 (a), data set 2 (b), data set 3 (c), and data set 4 (d).
FIGURE 7. The plot of statistic \( T_i \) form Hotelling T\(^2\) control chart for data set 1 (a), data set 2 (b), data set 3 (c), and data set 4 (d).

in control (Phase I). The second half of the data (Phase II) represents the shift in the mean vector, shift in the covariance matrix, or shift in both the mean vector and the covariance matrix.

Plots of the data set used in each simulation are shown in Figure 6. If the statistics of the data simulation are shifted in the vector mean, then they are marked by \( M^+ \). If they have shifted in the covariance matrix, then they are marked with \( V^+ \). However, if they have shifted in both the vector mean and covariance matrix, then they are marked by \( B^+ \).

The upper control limit (UCL) is obtained using Algorithm 2. All data sets are generated in two sections except Data set 1, for which they are generated using the in-control vector mean \( \vec{a} = [0, 0, 0, 0] \) and in-control covariance \( \vec{b} = 1 \) for both sections. Therefore, in Figure 7(a), Figure 8(a), Figure 9(a) and Figure 10(a) no out-of-control signal is detected.

Figure 7 shows the simulation of the Hotelling T\(^2\) control chart. The Hotelling T\(^2\) control chart is a chart that is only monitoring the shift in the process mean. Figure 7(b) and Figure 7 (d) are the monitoring result for Data set 2 and Data set 4, respectively. The second half of Data set 2 is generated with the shift in the mean vector \( \vec{a} = [1.7, 1.7, 1.7, 1.7, 1.7] \) and the covariance matrix is in-control \( \vec{b} = 1 \). The second half of Data set 4 is generated by setting \( \vec{a} = [1.7, 1.7, 1.7, 1.7, 1.7] \) and \( \vec{b} = 2.7 \). The statistics of the Hoteling T\(^2\) control chart are compared with UCL = 21.09.

The out-of-control signals for both charts are detected from the 16\(^{th}\) sample since the statistic of the Hotelling T\(^2\)(\( T_i \)) exceeds the UCL. The out-of-control signals detected in Figure 7(b) and Figure 7(d) are caused only by the change in the mean vector. Consequently, in Figure 7(c) no out-of-control signal has been detected since the Data set 3 generates only shifts in covariance matrix by setting \( \vec{b} = 2.7 \).

Figure 8 illustrates the simulation of the GV control chart. In Figure 8(b), no out-of-control has been detected since the GV control chart only monitoring the shift in the covariance matrix. Figure 8(c) and Figure 8(d) are the monitoring results for Data set 3 and Data set 4, respectively. The second half of Data set 3 is generated with the shift in the covariance matrix \( \vec{b} = 2.7 \) and the mean vector is in-control. The second half of Data set 4 is generated by setting \( \vec{a} = [1.7, 1.7, 1.7, 1.7, 1.7] \) and \( \vec{b} = 2.7 \). The out-of-control signals for both charts are detected from the 16\(^{th}\) sample since the statistic of the GV control chart exceeds the UCL. As shown in Figure 8(c) and Figure 8(d), the out-of-control signals detected are caused only by the change in the covariance matrix.

Figure 9 provides the simulation of the proposed Max-Half-Mchart using the half-normal distribution. Figure 9(b)
is the monitoring result for Data set 2. The second half of Data set 2 is generated with the shift in the mean vector $\bar{\alpha} = [1.7, 1.7, 1.7, 1.7, 1.7]$ and the covariance matrix is in-control ($b = 1$). The statistics of the proposed Max-Half-Mchart are compared with $\text{UCL} = 3.75$. The first out-of-control signal is detected at the 16th sample since the statistic of the proposed Max-Half-Mchart ($C_{16}^H$) exceeds the UCL. The out-of-control signal is caused only by the change in the mean vector. Therefore, the out-of-control signals start from the 16th sample are marked with $M^{++}$.

Figure 9(c) demonstrates the proposed Max-Half-Mchart ($C_i^H$) statistic for Data set 3. The $C_i^H$ statistics are compared with $\text{UCL} = 3.75$, and the first out-of-control signal is detected at the 16th sample, when the statistic $C_{16}^H$ exceeds UCL = 3.75. The out-of-control signal is caused only by the change in the covariance matrix. Therefore, the out-of-control signals are marked with $V^{++}$. The second half of Data set 4 is generated with changes in both the mean vector and the covariance matrix by setting $\bar{\alpha} = [1.7, 1.7, 1.7, 1.7, 1.7]$ and $b = 2.7$. The monitoring result of Data set 4 is displayed in Figure 9(d). The first out-of-control signal is detected at the 16th sample. All of-out-of control are marked with $M^{++}$ since the out-of-controls are caused only by the shift in the mean vector.

Figure 10 displays the conventional Max-Mchart applied to the simulation data. Figure 10(b) is the monitoring result for Data set 2. The statistics for the conventional Max-Mchart are compared with $\text{UCL} = 3.595$, and the first out-of-control signal is detected at the 16th sample, because of the shift in both the mean vector and the covariance matrix. In addition, the 20th, 28th, 32nd, and 33rd samples are marked with $M^{++}$.
Figure 10(c) displays the Max-Mchart from ($C_i$) statistics from Data set 3. The $C_i$ statistics are compared with UCL = 3.595, and the first out-of-control signal is detected at the 16th sample since the statistic $C_{10}$ exceeds the UCL = 3.595. The out-of-control signal is caused only by the change in the covariance matrix. Therefore, the out-of-control signals are marked with $V^{++}$. However, not all samples in Phase II for Data set 3 are detected as out-of-control signals. The 29th sample is in-control because $C_{29}$ is less than UCL = 3.595. The monitoring result for Data set 4 is displayed in Figure 10(d). The first out-of-control signal is detected at the 16th sample. All out-of-control signals are marked with $B^{++}$, starting from the 16th sample, because of the shift in both the mean vector and the covariance matrix. We can conclude from this simulation scenario that the proposed Max-half-Mchart has more consistent results with the Hotelling $T^2$ and GV control charts in detecting shift both mean vector and covariance matrix than the conventional Max-Mxchart.

**B. APPLICATION TO REAL DATA**

The proposed Max-Half-Mchart is applied to Woven Poly Propylene (WPP) i.e., plastic bag production [17]. The data were measured in a day in three shifts (morning, evening, and night). Data measurements carried out include data from 2 October until 30 October insisted 54 observations and has two quality characteristics. The variables used in this paper refer to the characteristics of quality Woven Poly Propylene products that as identified by the company; these variables include the length of Woven Poly Propylene (WPP) and weight of Woven Poly Propylene [17]. In Figure 1, the Hotelling $T^2$ control chart detects seven out-of-control signals at the 1st, 4th, 9th, 31st, 33rd, 35th, 42nd observations. However, in the GV control chart, no out-of-control signals were detected.

In Figure 11, the proposed Max-Half-Mchart detected seven out-of-control signals; these are the same out-of-control signals found by the Hotelling $T^2$ control chart. These seven out-of-control signals are marked with $M^{++}$ because the out-of-control signals are caused only by the change in the mean vector. Meanwhile, the conventional Max-Mchart detected the nine out-of-control signals exhibited in Figure 3. Six out-of-control observations are marked with $M^{++}$, including the 4th, 31st, 33rd, 35th, and 42nd observations. The other three out-of-control signals are marked with $V^{++}$ at the 6th, 16th, 20th and 46th observations because they are caused only by the change in the covariance matrix. These results show that the conventional Max-Mchart does not consistently monitor Woven Poly Propylene product with the GV control chart since the GV control chart does not
VI. CONCLUSION

The proposed transformation using a half-normal distribution that produced the Max-Half-Mchart improves the conventional Max-Mchart. Based on the ARL value, the proposed Max-Half-Mchart outperforms the conventional Max-Mchart when monitoring process variability in either small or large shifts. The conventional Max-Mchart, which uses the bootstrap control limit, also demonstrates excellent performance when the mean vector shifts less than 0.5. The simulation study on four scenarios shows that the proposed Max-Half-Mchart detects out-of-control signals correctly when the process is shifting in both mean and variance. This work also proved that the proposed Max-Half-Mchart solves the drawback of the conventional Max-Mchart and the proposed Max-Half-Mchart is more consistent with the statistic in Hotelling’s $T^2$ control chart and the GV control chart than the conventional Max-Mchart.

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