Bias of Root Numbers for Modular Newforms of Cubic Level

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Abstract. Let $H_{2k}^\pm (N^3)$ denote the set of modular newforms of cubic level $N^3$, weight $2k$, and root number $\pm 1$. For $N > 1$ squarefree and $k > 1$, we use an analytic method to establish neat and explicit formulas for the difference $|H_{2k}^+(N^3)| - |H_{2k}^-(N^3)|$ as a multiple of the product of $\varphi(N)$ and the class number of $\mathbb{Q}(\sqrt{-N})$. In particular, the formulas exhibit a strict bias towards the root number $+1$. Our main tool is a root-number weighted simple Petersson formula for such newforms.

1. Introduction

Let $S_{2k}(M)$ denote the space of modular forms of level $M$, weight $2k$, and trivial nebentypus. According to the Atkin–Lehner theory of newforms (see [AL] and [Iwa] §6.6), $S_{2k}(M)$ has an orthogonal decomposition with respect to the Petersson inner product

$$S_{2k}(M) = S^0_{2k}(M) \oplus S^*_{2k}(M),$$

where $S^0_{2k}(M)$ is the space of oldforms and $S^*_{2k}(M)$ is the space spanned by newforms. A newform $f \in S^*_{2k}(M)$ is an eigenfunction of all the Hecke operators $T_n$, normalized so that the first Fourier coefficient is 1. Let $\lambda_f(n)$ denote its $n$-th Hecke eigenvalue, which is known to be real. Then we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{k-1/2}e(nz), \quad \text{Im}(z) > 0,$$

where $e(z) = e^{2\pi i z}$. Let $H^*_{2k}(M)$ denote the orthogonal basis for $S^*_{2k}(M)$ consisting of newforms of level $M$ and weight $2k$. The dimension of $S^*_{2k}(M)$ (or the cardinality $|H^*_{2k}(M)|$) is well known by the work of G. Martin (see [Mar1] Theorem 1).

For $f \in H^*_{2k}(M)$, let $\epsilon_f \in \{\pm 1\}$ denote the root number of $f$, i.e. the sign in the functional equation of its $L$-function $L(s, f)$. We would like to consider the splitting

$$H^*_{2k}(M) = H^0_{2k}(M) \cup H^*_{2k}(M)$$

according to the sign of root number. A natural question is: What can we say about $|H^0_{2k}(M)|$ or, in other words, the distribution of root numbers in $H^*_{2k}(M)$?

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When the level $M > 1$ is squarefree\footnote{The case $M = 1$ is not of interest here, for the root number is $i^{2k} = (-1)^k$, so either $H^+_{2k}(1) = H^+_{2k}(1)$ or $H^-_{2k}(1) = H^-_{2k}(1)$.} Iwaniec, Luo, and Sunark established the Petersson formulas over $H^+_2(M)$ and $H^-_{2k}(M)$ and used them to prove the asymptotic formula (see \cite[Corollary 2.14]{ILS}):\footnote{Some mistakes in \cite{Yam} were corrected by Skoruppa and Zagier \cite{SZ}.}

\begin{equation}
|H^+_k(M)| = \frac{2k-1}{24}\varphi(M) + O((kM)^{5/6}),
\end{equation}

where as usual $\varphi(M)$ is Euler’s totient function. In particular, this implies that the root numbers are equidistributed between $+1$ and $-1$ as $kM \to \infty$. Note that the formula of G. Martin (see \cite[Theorem 1]{Mar1} or \cite[Theorem 2.1]{Mar2}) in this case reads:

\begin{equation}
|H^+_k(M)| = \frac{2k-1}{12}\varphi(M) + \left(\frac{1}{4} + \frac{k}{2} - \frac{k}{2}\right)v^1(M) + \left(\frac{1}{3} + \frac{2k}{3} - \frac{2k}{3}\right)v^2(M) + \delta(k, 1)\mu(M),
\end{equation}

where $v^1(M)$ and $v^2(M)$ are multiplicative functions defined by

$v^1(p) = \chi_{-4}(p) - 1, \quad v^2(p) = \chi_{-3}(p) - 1,$

$\delta(k, 1)$ is the Kronecker $\delta$-symbol that detects $k = 1$, $\mu$ is the Möbius function, and $\chi_{-4}$ and $\chi_{-3}$ are the quadratic characters attached to $\mathbb{Q}(\sqrt{-4})$ and $\mathbb{Q}(\sqrt{-3})$, respectively.

Recently, using Yamauchi’s trace formula for Atkin–Lehner involutions on $S^+_2(M)$ in \cite{Yam}, K. Martin (\cite[§2]{Mar2}) obtained the following formula for squarefree $M > 3$ (the formula for prime $M > 3$ was essentially contained in \cite{Wak}, in which Yamauchi’s trace formula was also used):

\begin{equation}
|H^+_{2k}(M)| - |H^-_{2k}(M)| = c_M h(D_{-M}) - \delta(k, 1),
\end{equation}

where

\begin{equation}
c_M = \begin{cases} 1/2, & \text{if } M \equiv 1, 2 \pmod{4}, \\ 1, & \text{if } M \equiv 7 \pmod{8}, \\ 2, & \text{if } M \equiv 3 \pmod{8}, \end{cases}
\end{equation}

$D_{-M}$ and $h(D_{-M})$, respectively, denote the discriminant and the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-M})$. He also proved that for $M = 2$ or $3$,

\begin{equation}
|H^+_{2k}(2)| - |H^-_{2k}(2)| = \begin{cases} 1, & \text{if } k \equiv 0, 1 \pmod{4}, \\ 0, & \text{if } k \equiv 2, 3 \pmod{4}, \end{cases}
\end{equation}

and

\begin{equation}
|H^+_{2k}(3)| - |H^-_{2k}(3)| = \begin{cases} 1, & \text{if } k \equiv 0, 1 \pmod{3}, \\ 0, & \text{if } k \equiv 2 \pmod{3}. \end{cases}
\end{equation}

It is therefore clear that, in any fixed $H^+_{2k}(M)$ (with squarefree $M > 1$), there is a strict bias of root numbers towards $+1$ with magnitude on the order of the class number $h(D_{-M})$. Moreover, in view of the dimension formula for $|H^+_2(M)|$ as in \cite{ILS}, K. Martin immediately obtained explicit formulas for $|H^+_2(M)|$. For a fundamental discriminant $D < 0$, we recall Dirichlet’s class number formula

\begin{equation}
L(1, \chi_D) = \frac{2\pi h(D)}{w(D)\sqrt{|D|}}.
\end{equation}
and the well-known upper bound $L(1, \chi_D) \ll \log |D|$ (as usual $\chi_D$ is the quadratic character attached to $\mathbb{Q}(\sqrt{D})$, $L(s, \chi_D)$ is its Dirichlet $L$-function, $h(D)$ is the class number, and $w(D)$ is the number of units). From these we infer that $h(D_{-M}) = O(\sqrt{M} \log M)$ \(^3\) Consequently, the error term in (1.1) may be improved into $O(\sqrt{M} \log M)$ \(^3\)

K. Martin said in [Mar2] that “the trace formula of Yamauchi is valid for arbitrary level, but becomes considerably more complicated”, and that, “in principle”, his approach also works for non-squarefree level, “but the resulting formulas may be messy”.

**Main results.** In the present paper, we use a new analytic method due to Balkanova, Frolenkov [BF], and Shenhui Liu [Liu] to derive a still very “neat” formula for $|H^\pm_{2k}(M)| = |H^\pm_{2k}(M)|$ in the case of cubic level $M = N^3$ with $N > 1$ squarefree. As indicated in [Mar2], this would possibly be prohibitively difficult from the approach via Yamauchi’s trace formula. Instead, our approach is based on the root-number weighted ($\Delta$-type) simple Petersson formula over $H^\pm_{2k}(N^3)$ established in [PWZ] (see (2.12) in Proposition 2.1).

The following is our main theorem.

**Theorem 1.1.** Let notation be as above. Assume that $N > 1$ is squarefree and $k > 1$. When $N > 3$, we have

$$|H^\pm_{2k}(N^3)| = c_N \varphi(N) h(D_{-N}),$$

where the constant $c_N$ is given as in (1.4), $\varphi(N)$ is Euler’s totient function, and $h(D_{-N})$ is the class number of $\mathbb{Q}(\sqrt{-N})$. When $N = 2$, we have

$$|H^\pm_{2k}(8)| = \begin{cases} 0, & \text{if } k \equiv 0, 1 \pmod{4}, \\ 1, & \text{if } k \equiv 2, 3 \pmod{4}. \end{cases}$$

When $N = 3$, we have

$$|H^\pm_{2k}(27)| = \begin{cases} 1, & \text{if } k \equiv 0, 1 \pmod{3}, \\ 2, & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

This theorem manifests that the root numbers in $H^\pm_{2k}(N^3)$ have a strict bias in favor of +1 with magnitude on the order of $\varphi(N) h(D_{-N})$. Also note that there is a perfect equidistribution of root numbers only when $N = 2$ and $k \equiv 0, 1 \pmod{4}$.

According to [Mar1] Theorem 1, for $N > 1$ squarefree, we have

$$|H^\pm_{2k}(N^3)| = \frac{2k - 1}{12} \varphi(N)^2 \nu(N) + \left(\frac{1}{4} + \left\lfloor \frac{k}{2} \right\rfloor - \frac{k}{2}\right) \delta(N, 2) + \left(\frac{1}{3} + \left\lfloor \frac{2k}{3} \right\rfloor - \frac{2k}{3}\right) \delta(N, 3),$$

with

$$\nu(N) = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

Consequently,

$$|H^\pm_{2k}(N^3)| = \frac{2k - 1}{24} \varphi(N)^2 \nu(N) + \frac{c_N}{2} \varphi(N) h(D_{-N})$$

\[^3\]Note that we also have Siegel’s ineffective lower bound $h(D_{-M}) \gg M^{1/2-\varepsilon}$ for any $\varepsilon > 0$.

\[^4\]By simple considerations, one may find that the error term $O(2^\omega(M))$ in [Mar2] is not correct (\(\omega(M) \) is the number of prime factors in \(M\)). Actually, one has $|H^\pm_{2k}(M)| = (2k - 1)\varphi(M)/12 + O(2^\omega(M))$ for $M$ squarefree (see (1.2)).
if $N > 3$,

$$|H_{2k}^\pm(8)| = \left\lfloor \frac{k}{2} \right\rfloor + \begin{cases} 0, & \text{if } k \equiv 0, 1 \pmod{4}, \\ \pm \frac{1}{2}, & \text{if } k \equiv 2, 3 \pmod{4}, \end{cases}$$

(1.13)

and

$$|H_{2k}^\pm(27)| = \left\lfloor \frac{8k}{3} \right\rfloor - \frac{1}{2} + \begin{cases} \pm \frac{1}{2}, & \text{if } k \equiv 0, 1 \pmod{3}, \\ \pm 1, & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

(1.14)

From these we deduce the asymptotic formula

$$|H_{2k}^\pm(N^3)| = \frac{2k-1}{24} \varphi(N)^2 \nu(N) + O\left(N^{3/2} \log N\right).$$

(1.15)

We conclude this Introduction with the following conjecture.

**Conjecture 1.2.** For any $M > 1$ we have

$$|H_{2k}^\pm(M)| \geq |H_{2k}^\pm(M)|,$$

and

$$|H_{2k}^\pm(M)| - |H_{2k}^\pm(M)|$$

is independent of $k$ if $M$ is not divisible by 2 or 3 and $k > 1$. Moreover, we have the asymptotic formula

$$|H_{2k}^\pm(M)| = \frac{2k-1}{24} M s_0^\circ(M) + O\left(\sqrt{M} \log M\right),$$

(1.16)

in which $s_0^\circ$ is the multiplicative function in Definition 1' (A) in [Mar1], satisfying

$$s_0^\circ(p) = 1 - \frac{1}{p}, \quad s_0^\circ(p^2) = 1 - \frac{1}{p} - \frac{1}{p^2}, \quad s_0^\circ(p^\alpha) = \left(1 - \frac{1}{p}\right)^\alpha \left(1 - \frac{1}{p^2}\right) \quad (\alpha \geq 3).$$

(1.17)

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2. Preliminaries

**2.1. Notation.** Let $\varphi(n)$ be the Euler totient function, and $\mu(n)$ be the Möbius function. For complex $s$, define

$$\sigma_s(n) = \sum_{d \mid n} d^s.$$  

(2.1)

Let $\delta(m, n)$ be the Kronecker $\delta$-symbol that detects $m = n$.

Set $e(z) = e^{2\pi iz}$. We define the Kloosterman sum

$$S(m, n; c) = \sum_{a \pmod{c}}^* e\left(\frac{am+n}{c}\right),$$

where $\sum^*$ restricts the summation to the primitive residue classes, and $\mathfrak{n}$ denotes the multiplicative inverse of $a$ modulo $c$. We record here Weil’s bound for the Kloosterman sum

$$|S(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \sigma_0(c).$$  

(2.2)

Let $J_\nu(z)$ be the Bessel function of the first kind ([Wat]). Henceforth, we shall be only concerned with $J_{2k-1}(x)$ for real $x > 0$ and integer $k \geq 1$. We have the following estimates (see [Wat, §§2.11, 7.1])

$$J_{2k-1}(x) \ll_k \begin{cases} x^{2k-1}, & x \leq 1, \\ x^{-1/2}, & x > 1. \end{cases}$$

(2.3)
Moreover, the following Mellin–Barnes integral representation for the Bessel function will be used later (see [MOS §3.6.3])

\[ J_{2k-1}(x) = \frac{1}{4\pi i} \int_{(\sigma)} \frac{\Gamma\left(k - \frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(k + \frac{1}{2} - \frac{s}{2}\right)} \left(\frac{x}{2}\right)^{-s} ds, \]

for \(1 - 2k < \sigma < 1\); by Stirling’s formula, the integral is absolutely convergent if \(1 - 2k < \sigma < 0\).

For \(0 < \alpha \leq 1\), we define the Hurwitz zeta function

\[ \zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \text{Re}(s) > 1. \]

According to Theorems 12.4 and 12.6 in [Apo], \(\zeta(s, \alpha)\) has an analytic continuation to the whole complex plane except for a simple pole at \(s = 1\) with residue 1, and we have Hurwitz’s formula

\[ \zeta(1 - s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} (e(-s/4)F(\alpha, s) + e(s/4)F(-\alpha, s)), \quad \text{Re}(s) > 1, \]

with

\[ F(\beta, s) = \sum_{n=1}^{\infty} \frac{e(n\beta)}{n^s}, \]

where \(\beta\) is real and \(\text{Re}(s) > 1\).

### 2.2. Simple Petersson formulas over \(H_{2k}^*(N^3)\)

We introduce the symmetric square \(L\)-function for \(f \in H_{2k}^*(N^3)\):

\[ L(s, \text{sym}^2(f)) = \zeta^{(N)}(2s) \sum_{(n,N)=1} \frac{\lambda_f(n^2)}{n^s}, \quad \text{Re}(s) > 1, \]

where \(\lambda_f(n)\) are the Hecke eigenvalues of \(f\), and \(\zeta^{(N)}(s)\) is the partial Riemann zeta-function defined by

\[ \zeta^{(N)}(s) = \prod_{p \mid N} \frac{1}{1 - p^{-s}}, \quad \text{Re}(s) > 1. \]

It is known that \(L(s, \text{sym}^2(f))\) admits a holomorphic continuation to the entire complex plane.

We can now state the two types of simple Petersson formulas in [PWZ].

**Proposition 2.1.** Assume that \(N > 1\) is squarefree and \(k > 1\). Set

\[ \Delta_{2k,N^3}(m,n) = \sum_{f \in H_{2k}^*(N^3)} \frac{\lambda_f(m)\lambda_f(n)}{L(1, \text{sym}^2(f))}, \]

and

\[ \Delta_i_{2k,N^3}(m,n) = \sum_{f \in H_{2k}^*(N^3)} \epsilon_f \frac{\lambda_f(m)\lambda_f(n)}{L(1, \text{sym}^2(f))}. \]

For \((mn, N) = 1\), we have

\[ \Delta_{2k,N^3}(m,n) = \delta(m,n) \frac{(2k-1)N^2 \varphi(N)}{2\pi^2} \]

\[ + \frac{(-1)^k(2k-1)}{\pi} \sum_{c=1}^{\infty} \frac{A_N(c)}{c} S(m, n; N^2 c) J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{N^2 c} \right), \]
and

\[
\Delta_{2kn}^i(m, n) = \frac{(2k - 1)N^{3/2}}{\pi} \sum_{(c, N) = 1} A_N(c) J_{2k - 1} \left( \frac{4\pi \sqrt{mn}}{N^{3/2}c} \right),
\]

where \(A_N(c) = \prod_{p|N} A_p(c)\) with

\[
A_p(c) = \begin{cases} 
-1, & \text{if } p \nmid c, \\
 p - 1, & \text{if } p \mid c,
\end{cases}
\]

and \(N\) is the multiplicative inverse of \(N\) modulo \(c\).

The meaning of the adjective “simple” is threefold. First, the formulas are closely related to the simple trace formula of Deligne and Kazhdan in the early 1980’s, in which a supercuspidal matrix coefficient is used (see [Gel]). Second, for each prime \(p|N\), the local component \(\pi_p\) associated to \(f \in H_{2k}(N^4)\) is known to be a simple supercuspidal representation (see [GR, KL]). Third, the above Petersson formulas of cubic level \(M = N^3\) look quite simple compared to those of squarefree level \(M > 1\) as in [ILS].

The \(\Delta^*\)-type Petersson formula in [ILS] has been generalized in a similar manner to the case of arbitrary level by Ng, Nelson, and Barrett et al. [Ng, Nel, BBD] (see also [ILS, Rou, BM]). The \(\Delta^*\)-type formula (2.11) is deducible from theirs after simplifications, but the \(\Delta^*\)-type formula in (2.12) is novel.

Finally, we comment that the condition \(k > 1\) is required in the proof in [PWZ] so that a certain matrix coefficient of \(\pi_\infty\) is integrable. With additional efforts, their proof might still be carried out for \(k = 1\).

2.3. A Mellin–Barnes type integral and Gauss’ hypergeometric function. As in [BF 5-2] and [Liu §2.5], for \(1/2 < \text{Re}(s) < 2k\), we define the integral

\[
I_{k, s}(x) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma \left( k - \frac{1}{2} + \frac{1}{2}w \right)}{\Gamma \left( k + \frac{1}{2} - \frac{1}{2}w \right)} \Gamma(1 - s - w) \sin \left( \frac{\pi s + w}{2} \right) x^w dw, \quad (x > 0),
\]

with \(1 - 2k < \sigma < 1 - \text{Re}(s)\). We have the following explicit formulas for \(I_{k, s}(x)\) involving Gauss’ hypergeometric function. See [BF Lemmas 5.2-5.4] and [Liu Lemma 2] (the formulation of the latter is simpler but equivalent to that of the former via some identities for the gamma function).

**Lemma 2.2.** Let \(x > 0\). Assume that \(1/2 < \text{Re}(s) < 2k\). We have

\[
I_{k, s}(x) = \begin{cases} 
2(-1)^k \cos \left( \frac{\pi s}{2} \right) \frac{\Gamma(2k - s)}{\Gamma(2k)} x^{1 - 2k} F_2 \left( \begin{array}{c} k - \frac{s}{2}, k + \frac{1}{2} - \frac{s}{2}, 2k; \frac{4}{x^2} \end{array} \right), & \text{if } x > 2, \\
\frac{(-1)^k 2^{2s}}{\sqrt{\pi}} \cos \left( \frac{\pi s}{2} \right) \frac{\Gamma(2k - s) \Gamma(s - \frac{1}{2})}{\Gamma(2k + s - 1)}, & \text{if } x = 2, \\
\frac{\Gamma(k - \frac{1}{2}s)}{\Gamma(k + \frac{1}{2}s)} x^{1 - s} F_2 \left( \begin{array}{c} k - \frac{s}{2}, 1 - k - \frac{s}{2}; \frac{1}{2}, \frac{1}{4} \end{array} \right), & \text{if } x < 2,
\end{cases}
\]

in which \(F_2(\alpha, \beta; \gamma; z)\) is Gauss’ hypergeometric function.

Recall that for \(|z| < 1\), \(F_2(\alpha, \beta; \gamma; z)\) is defined by Gauss’ hypergeometric series

\[
F_2(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n) \Gamma(\beta + n)}{\Gamma(\gamma + n)!} z^n.
\]
The simple fact that $\genfrac{[}{]}{0pt}{}{2}{1}(\alpha, \beta; \gamma; z) \to 1$ as $|z| \to 0$ will be useful when verifying the convergence of certain infinite sums. It is known (see [MOS] §2.1) that

\begin{equation}
2F_1 \left( \frac{\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2}; \sin^2 \theta \right) = \cos(\alpha \theta).
\end{equation}

2.4. Zagier’s $L$-functions. For integers $c$ and $\Delta$, with $c \geq 1$, let

\[ \rho(c, \Delta) = \#\{a \pmod{2c} : a^2 \equiv \Delta \pmod{4c}\}. \]

Clearly, $\rho(c, \Delta) = 0$ if $\Delta \equiv 2, 3 \pmod{4}$. For $\text{Re}(s) > 1$, define the $L$-function

\begin{equation}
L(s, \Delta) = \frac{\zeta(2s)}{\zeta(s)} \sum_{c=1}^{\infty} \frac{\rho(c, \Delta)}{c^s}.
\end{equation}

We recollect some fundamental results of Zagier [Zag] Proposition 3 as follows.

Lemma 2.3. The $L$-function $L(s, \Delta)$ has an analytic continuation to the whole complex plane, which is entire except for a simple pole at $s = 1$ when $\Delta$ is a square. More precisely, $L(s, \Delta)$ can be expressed in terms of standard Dirichlet series:

\begin{equation}
L(s, \Delta) = \begin{cases} 
0, & \text{if } \Delta \equiv 2, 3 \pmod{4}, \\
\zeta(2s - 1), & \text{if } \Delta = 0, \\
L(s, \chi_D) \sum_{d|q} \mu(d) \chi_D(d) \frac{\sigma_{1-2s}(q/d)}{d^s}, & \text{if } \Delta \equiv 0, 1 \pmod{4}, \Delta \neq 0,
\end{cases}
\end{equation}

where in the last case we have written $\Delta = Dq^2$ with $D$ the discriminant of $\mathbb{Q}(\sqrt{\Delta})$, $\chi_D = \left( \frac{D}{\cdot} \right)$ is the Kronecker symbol, $L(s, \chi_D)$ is the Dirichlet $L$-function of $\chi_D$, and $\sigma_{1-2s}$ is defined as in (2.1).

Lemma 2.4. Let $N, m, n$ be integers, with $N \geq 1$. For $\text{Re}(s) > 3/2$, we have

\begin{equation}
\sum_{(c, N) = 1} \frac{1}{c^{1+s}} \sum_{a \pmod{c}} S(m, a^2; c)e\left(\frac{an}{c}\right) = \frac{L(N)(s, n^2 - 4m)}{\zeta(N)(2s)},
\end{equation}

with

\begin{equation}
L(N)(s, \Delta) = \frac{\zeta(N)(2s)}{\zeta(N)(s)} \sum_{(c, N) = 1} \frac{\rho(c, \Delta)}{c^s}.
\end{equation}

Proof. According to the proof of Lemma 4.1 in [BF], we have

\[ \sum_{a \pmod{c}} S(m, a^2; c)e\left(\frac{an}{c}\right) = c \sum_{d|c} \mu(d) \rho(c/d, n^2 - 4m), \]

and (2.19) follows immediately upon using the formula for a product of two Dirichlet series.

Q.E.D.

Remark 2.1. For the last case in (2.18), a simple observation is that $L(N)(s, \Delta) = L(s, \Delta)$ if $N|D$ and $(N, q) = 1$.

Finally, we have the following trivial bound for $L(s, \Delta)$ in the $\Delta$-aspect.

Lemma 2.5. Set $\sigma = \text{Re}(s)$. If $\Delta$ is not a square, then

\begin{equation}
L(s, \Delta) \leq \left|\Delta\right|^{\max\left\{0, \frac{1}{2} - \frac{3}{2}\sigma, \frac{3}{2} - \sigma\right\} + \varepsilon},
\end{equation}

for any $\varepsilon > 0$, with the implied constant uniform for $s$ in compact sets.
Proof. Consider the last formula in (2.18). By the trivial bound $|L(s, \chi_D)| \leq \zeta(\sigma)$ for $\sigma > 1$, the functional equation for $L(s, \chi_D)$, and the Phragmén–Lindelöf convex principle, we infer that $L(s, \chi_D) \ll_{\epsilon, \sigma} |D|^{\max\{0, -\frac{1}{2} \sigma - \frac{1}{2}, -\sigma\} + \epsilon}$, whereas the finite sum over $d$ in (2.18) is trivially bounded by $O(q^{\max\{0, 1 - 2\sigma\} + \epsilon})$. Then (2.21) follows immediately.

Q.E.D.

3. Proof of Theorem 1.1

We consider the mean

\begin{equation}
M^f(s) = \sum_{f \in H^*_{2k}(N^3)} L(s, \text{sym}^2(f)) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+3}{2})} \frac{N^{3/2}}{2\pi} \frac{1-s}{s} L(s, -4N),
\end{equation}

so that $|H^+_{2k}(N^3)| - |H^-_{2k}(N^3)| = M^f(1)$. We shall prove in §4 the following exact formula for $M^f(s)$.

**Theorem 3.1.** Assume that $N > 1$ is squarefree and $k > 1$. For $2 - 2k < \text{Re}(s) < 2k - 1$, we have

\begin{equation}
M^f(s) = M_0^f(s) + M_1^f(s) + M_2^f(s),
\end{equation}

with

\begin{equation}
M_0^f(s) = \frac{(2k-1)\sqrt{N} \Lambda(N)}{2\pi} \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k + \frac{3}{2})} \left( \frac{N^{3/2}}{2\pi} \right)^{1-s} L(s, -4N),
\end{equation}

\begin{equation}
M_1^f(s) = \frac{(2k-1)N^{3/2}}{\pi} \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k + \frac{3}{2})} \left( \frac{N^{3/2}}{2\pi} \right)^{1-s} \sum_{d|N} \frac{\mu(d)}{d} 
\end{equation}

\begin{equation}
\cdot \sum_{1 \leq n < 2d/N^3/2} L(s, (nN^2/d)^2 - 4N) \frac{1}{2F_1\left(k - \frac{s}{2}; 1 - k - \frac{s}{4}; \frac{1}{2}, \frac{1}{2}, \frac{n^2N^3}{4d^2}\right)},
\end{equation}

and

\begin{equation}
M_2^f(s) = \frac{(-1)^k(2\pi)^{s}N^{3(k-1)}}{\pi^2 N^{3(k-1)}} \frac{\Gamma(2k - s)}{\Gamma(2k - 1)} \sum_{d|N} \frac{\mu(d)}{d^{s-2k+1}} 
\end{equation}

\begin{equation}
\cdot \sum_{n > 2d/N^3/2} \frac{L(s, (nN^2/d)^2 - 4N)}{n^{2k-s}} \frac{1}{2F_1\left(k - \frac{s}{2}; k + \frac{1}{2}, \frac{s}{2}; 2k; \frac{4d^2}{n^2N^3}\right)}.
\end{equation}

Firstly, note that the inequality $1 \leq n < 2d/N^3/2$ in (3.4) has only two integer solutions (with $d|N$): either $N = d = 2, n = 1$, or $N = d = 3, n = 1$. Now let $s = 1$. The expressions in (3.4) and (3.5) simplify greatly because

1. $\cos(\pi s/2)$ vanishes at $s = 1$, and
2. for $N = 2$ or 3, the only remaining hypergeometric function may be evaluated explicitly by (2.10).

After some calculations, we obtain

\begin{equation}
M^f(1) = \frac{\sqrt{N} \Lambda(N)}{\pi} L(1, -4N) - \delta(N, 2) \frac{2\sqrt{2}}{\pi} L(1, -4) \cos\left(\frac{\pi}{4}\right)
\end{equation}

\begin{equation}
- \delta(N, 3) \frac{2\sqrt{3}}{\pi} L(1, -3) \cos\left(\frac{\pi}{3}\right).
\end{equation}
Here $2\sqrt{2}$ and $2\sqrt{3}$ come from $2N^{3/2}/d$ for $N = d = 2$ and $N = d = 3$ respectively. Recall that $N > 1$ is squarefree. By (2.18) in Lemma 2.3 along with Dirichlet’s class number formula (1.7), we have

$$L(1, -4N) = \begin{cases} 
\frac{\pi}{2\sqrt{N}} h(-4N), & \text{if } N \equiv 1, 2 \pmod{4}, \\
\frac{\pi}{2\sqrt{N}} h(-N) (3 - \chi_{-N}(2)), & \text{if } N \equiv 3 \pmod{4}, N \neq 3, \\
\frac{\pi}{6\sqrt{3}} (3 - \chi_3(2)), & \text{if } N = 3,
\end{cases}$$

and

$$L(1, -4) = \frac{\pi}{4}, \quad L(1, -3) = \frac{\pi}{3\sqrt{3}},$$

in which

$$\chi_{-N}(2) = \begin{cases} 
1, & \text{if } N \equiv 7 \pmod{8}, \\
-1, & \text{if } N \equiv 3 \pmod{8}.
\end{cases}$$

Note that $\sigma_{-1}(2) = 3/2$, that $h(-3) = h(-4) = h(-8) = 1$, and that $w(-3) = 6, w(-4) = 4$ and $w(D) = 2$ for every fundamental discriminant $D < -4$. Moreover,

$$\cos\left(\frac{2k - 1}{4}\right) = \begin{cases} 
1/\sqrt{2}, & \text{if } k \equiv 0, 1 \pmod{4}, \\
-1/\sqrt{2}, & \text{if } k \equiv 2, 3 \pmod{4},
\end{cases}$$

and

$$\cos\left(\frac{2k - 1}{3}\right) = \begin{cases} 
1/2, & \text{if } k \equiv 0, 1 \pmod{3}, \\
-1, & \text{if } k \equiv 2 \pmod{3}.
\end{cases}$$

Combining the foregoing results, Theorem 1.1 follows immediately.

4. Proof of Theorem 3.1

The main ideas in the proof of Theorem 3.1 below are essentially due to Balkanova, Frolenkov, and Shenhui Liu [BF, Liu].

Recall the definition of $M^5(s)$ in (3.1). We first assume that $3/2 < \text{Re}(s) < 2k - 1$. In view of (2.9) and (2.10), we infer that

$$M^5(s) = \zeta^{(N)}(2s) \sum_{n,N=1}^{n \leq N} \frac{\Delta_{2k,N}^N(1, n^2)}{n^s}.$$

By the Petersson formula (2.12) in Proposition 2.1, we have

$$M^5(s) = \frac{(2k - 1)N^{3/2}}{\pi} \zeta^{(N)}(2s) \sum_{c,n}^{c \equiv 0, \text{mod } N} \frac{S(N^3, n^2; c)}{cn^s} J_{2k-1} \left( \frac{4\pi n}{N^{3/2} c} \right).$$

By (2.22) and (2.23), the double series is absolutely convergent if $\text{Re}(s) > 3/2$. Next, we use the Mellin–Barnes formula for the Bessel function as in (2.4), obtaining

$$M^5(s) = \frac{(2k - 1)N^{3/2}}{\pi} \zeta^{(N)}(2s) \sum_{c,n} \frac{1}{c} \frac{1}{4\pi i} \int \frac{\Gamma(k - \frac{1}{2} + \frac{1}{2} w)}{\Gamma(k + \frac{1}{2} - \frac{1}{2} w)} \left( \frac{N^{3/2} c}{2\pi} \right)^w \sum_{(n,N=1)} \frac{S(N^3, n^2; c)}{n^{s+w}} dw,$$

where, by Stirling’s formula and Weil’s bound, we require $1 - \text{Re}(s) < \sigma < -1/2$ to guarantee the absolute convergence of the integral over $w$ and the sums over $c$ and $n$. Note that our assumption $\text{Re}(s) < 2k - 1$ implies $1 - \text{Re}(s) > 1 - 2k$, so the condition $\sigma > 1 - 2k$ for (2.4) is guaranteed.
For the innermost sum over \( n \), we use the Möbius function to relax the coprimality condition \((n, N) = 1\) and write \( n = a + cn \) so that the \( m \)-sum yields the Hurwitz zeta function \( \zeta(s + w, a/c) \) as in \( (2.5) \). Thus

\[
\sum_{(n, N) = 1} \frac{S(N^3, n^2; c)}{n^{s+w}} = \sum_{d|N} \mu(d) \sum_{n=1}^{\infty} \frac{S(N^3, (nd)^2; c)}{n^{s+w}}
\]

\[
= \frac{1}{c^{s+w}} \sum_{d|N} \mu(d) \sum_{a=1}^{c} S(N^3, (ad)^2; c) \zeta(s + w, \frac{a}{c}).
\]

Recall that \( \zeta(s + w, a/c) \) has a simple pole at \( w = 1 - s \) with residue 1. We shift the \( w \)-contour to the left down to \( \text{Re}(w) = \sigma_1 \), with \( 1 - 2k < \sigma_1 < -\text{Re}(s) \), crossing a simple pole at \( w = 1 - s \). Therefore

\[
M^1(s) = \frac{(2k-1)N^{3/2}}{2\pi} \zeta(N)(2s) \frac{\Gamma(k - \frac{s}{2})}{\Gamma(k + \frac{s}{2})} (\frac{N^{3/2}}{2\pi})^{1-s} \sum_{d|N} \mu(d) \sum_{(c,N) = 1} \frac{1}{c^{1+s}} \sum_{a=1}^{c} S(N^3, (ad)^2; c) \zeta(s + w, \frac{a}{c})
\]

(4.1)

For the first term in (4.1), with the observation that the \( a \)-sum may be rewritten as

\[
\sum_{a (\text{mod } c)} S(N, (aN^2d^2; c) = \sum_{a (\text{mod } c)} S(N, a^2; c),
\]

we obtain the term \( M^1_0(s) \) in \((3.3)\) by virtue of Lemma 2.4 (with \( m = N \) and \( n = 0 \)) and Remark 2.1. Applying Hurwitz’s formula \((2.9)\), we have

\[
\sum_{a=1}^{c} S(N^3, (ad)^2; c) \zeta(s + w, \frac{a}{c})
\]

\[
= 2(2\pi)^{s+w-1} \Gamma(1 - s - w) \sin\left(\frac{\pi s + w}{2}\right) \sum_{a (\text{mod } c)} S(N^3, (ad)^2; c) F_1(a/c, 1 - s - w).
\]

Substituting this into the second term in (4.1) and opening the function \( F(a/c, 1 - s - w) \) according to \((2.7)\), we arrive at

\[
\frac{(2k-1)N^{3/2}}{\pi} \zeta(N)(2s) \sum_{d|N} \mu(d) \sum_{n=1}^{\infty} \frac{I_{k,s}(nN^{3/2}/d)}{(2\pi n)^{1-s}} \sum_{(c,N) = 1} \frac{1}{c^{1+s}} \sum_{a (\text{mod } c)} S(N^3, (ad)^2; c) e\left(\frac{an}{c}\right),
\]

in which \( I_{k,s}(nN^{3/2}/d) \) is the integral defined as in \((2.13)\); the absolute convergence of the sums over \( c \) and \( n \) may be easily verified for \( 3/2 < \text{Re}(s) < 2k - 1 \). An application of Lemmas 2.2 and 2.4 (along with Remark 2.1) yields the sum of \( M^1_1(s) \) and \( M^2_1(s) \) as in \((3.4)\) and \((3.5)\). Note that

\footnote{We remark that in the case \( k = 1 \) there is an issue with the proof of \([BF]\) Lemma 5.1. When they shift the integral contour to the left, the simple pole of the gamma function at \( w = 1 - 2k = -1 \) is also crossed. Additional work is required to address the problem of the analytic continuation of the resulting expression from \( 3/2 < \text{Re}(s) < 2 \) to \( 0 < \text{Re}(s) < 1 \).}
the $a$-sum above is equal to
\[
\sum_{a \pmod{c}} S(N,(aN^2d)^2;c) e\left(\frac{an}{c}\right) = \sum_{a \pmod{c}} S(N,a^2;c) e\left(\frac{anN^2/d}{c}\right).
\]
Moreover, since $N > 1$ is squarefree, it is easy to see that $\Delta = (nN^2/d)^2 - 4N = N(nN/d)^2 - 4$ is divisible by $p$ but not $p^2$ for any odd prime $p|N$, and that $\Delta \equiv 8, 12 \pmod{16}$ when $2|N$, and, in view of Remark 2.1, it readily follows that $L^{(N)}(s,\Delta) = L(s,\Delta)$.

We have thus established the validity of (3.2) for $3/2 < \Re(s) < 2k-1$. Next, since $(nN^2/d^2 - 4N)$ could never be a square for any $n \geq 0$ (see the arguments at the end of the last paragraph), the $L(s,-4N)$ and $L(s,(nN^2/d^2 - 4N)$ in (3.3)–(3.5) are all entire according to Lemma 2.3. Finally, in view of (2.15) and the crude estimate (2.21) in Lemma 2.5, the infinite series over $n$ on the right-hand side of (3.5) is absolutely and compactly convergent for $2 - 2k < \Re(s) < 2k - 1$ and thus gives rise to an analytic function of $s$ on this domain. The proof is completed by the principle of analytic continuation.

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