One of the main problems in any quantum resource theory is the characterization of the conversions between resources by means of the free operations of the theory. In this work, we advance on this characterization within the quantum coherence resource theory by introducing the notion of coherence vector of a general quantum state. This is a probability vector that can be interpreted as a concave extension of the coherence vector defined for pure states. We show that the coherence vector completely characterizes the notions of being incoherent, as well as being maximally coherent. Moreover, using this notion and the majorization relation, we obtain a necessary condition for the conversion of general quantum states by means of incoherent operations. These results generalize the necessary conditions of conversions for pure states given in the literature, and evidence that the tools of the majorization lattice are useful also in the general case. Finally, we introduce a family of coherence quantifiers by considering concave and symmetric functions applied to the coherence vector of a general quantum state. We compare this proposal with the convex roof measure of coherence and others quantifiers given in the literature.

I. INTRODUCTION

Quantum coherence is one of the fundamental aspects of the quantum theory with practical relevance in numerous fields of quantum physics, particularly in quantum information processing. It is also considered as a quantum resource that can be converted, manipulated and quantified\cite{1,2}, being more than merely a side result of the superposition principle. Quantum coherence admits a resource-theoretic formulation in terms of incoherent states (free states), coherent states (resources) and incoherent operations (free operations).

Since coherence is a basis dependent concept, the three components of the resource-theoretic formulation have to be defined for a given incoherent basis. Incoherent states are the ones that are diagonal in the incoherent basis, whereas coherent states have off-diagonal elements in that basis. Regarding the free operations, there is not a unique definition. Several definitions, often motivated by its operational interpretations, have been introduced (see e.g.\cite{3} for a review of these definitions). In this work, we restrict our attention to the definition of incoherent operation introduced in\cite{1}, which satisfies that coherence can not be generated from any incoherent input state, not even in a probabilistic way.

One of the main problems in any resource theory is characterizing the conversion between states by means of free operations\cite{4}. In the quantum coherence case, this problem has been completely solved for incoherent transformations from pure to pure states (see Refs.\cite{5,9} or Lemma\cite{1}, as well as for transformations from pure to mixed states (see Refs.\cite{9,10} or Prop.\cite{5} for this more general case). This characterization is given in terms of the majorization relation\cite{11} between the coherence vectors of the pure states. Motivated by this fact, we propose a generalization of the coherence vector to general quantum states, and we advance on the characterization of the state conversion by means of incoherent operations by appealing to the majorization lattice theory\cite{12,13}. More precisely, given a pure state decomposition of a quantum state, we define the coherence vector of the decomposition in terms of the coherence vectors of the pure states. Then, we define the coherence vector of a general quantum state as the supremum (in terms of the majorization order relation) of the coherence vectors of all pure-state decompositions. In this way, our proposal can be interpreted as a concave roof extension of the pure state case. Alternatively, the coherence vector of a general state $\rho$ can be also defined as the supremum of all coherence vectors of the pure states that can be converted into $\rho$ by means of an incoherent operation. Hence, our definition also acquires an operational meaning.

We prove that the generalized coherence vector characterizes the notions of being incoherent, as well as being maximally coherent. In addition, we extend the necessary condition of Prop.\cite{5} (see Refs.\cite{9,10}) to the general case of initial mixed states, which is also given in terms of the majorization relation of the corresponding coherence vectors.

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This result is an step forward on the characterization of conversions between general quantum states under incoherent operations, whose complete solution is only known for the single qubit system [16, 17]. Indeed, in higher dimensions \((d \geq 4)\), it was recently shown that a finite number of conditions in terms of coherence measures are not sufficient to fully characterize coherence transformations on general quantum states [10]. Thus, the complete characterization on the general case remains open.

Another main problem in any resource theory is to quantify the amount of resource of each state [8]. There are several coherence quantifiers and each of them captures diverse operational aspects of coherence, for instance, the distillable coherence, the coherence cost [18, 19], the relative entropy of coherence and the \(\ell_1\)-norm of coherence [11], among others (see e.g. [3]). Providing new quantifiers of coherence is an ongoing topic in the resource theory of coherence. A common strategy for obtaining a coherence quantifier is to define a suitable function on the pure states and then extend it to the entire set of quantum states. The extension can be done in different ways. The most frequently used is the convex roof construction [6, 9], which was originally applied in the entanglement theory [20, 21]. A recent proposal was given in [22], based on the state conversion process from pure to mixed quantum states by means of incoherent operations. In this work, we also present a different approach to obtain a family of coherence quantifiers, based on the generalized coherence vector.

This paper is organized as follows. In Sec. II we recall the basics elements of the resource theory of quantum coherence. In particular, we review the notions incoherent and coherent states, and incoherent operations. In addition, we present some results about conversions of incoherent states, as well as its axiomatic quantification, focusing on coherence measures based on the convex roof construction and on coherence monotones recently introduced. In Sec. III we introduce the notion of generalized coherence vector, valid for arbitrary quantum states. We show that it is a good definition, since it allows to characterize the notions of being incoherent, as well as being maximally coherent. In Sec. IV we show that the generalized coherence vector allows us to provide a necessary condition in terms of a majorization relation for the conversion between general quantum states. In Sec. V we provide a family of monotones based on the coherence vector, and we compare it with the convex roof construction and other monotones introduced in the literature. In Sec. VI we applied our monotones to quantify the coherence of a qubit system and a maximally coherent qutrit going through a depolarizing channel. Finally, some concluding remarks are given in Sec. VII. For the sake of readability, auxiliaries lemmas and proofs are presented separately in the appendices A and B respectively.

II. PRELIMINARIES: RESOURCE THEORY OF QUANTUM COHERENCE

In this section, we review the resource theory of quantum coherence introduced in [11]. In what follows, we consider a quantum system represented by a \(d\)-dimensional Hilbert space \(\mathcal{H}\). Moreover, we denote as \(\mathcal{S}(\mathcal{H})\) the set of density operators and as \(\mathcal{P}(\mathcal{H})\) the set of pure states. Since the coherence of a quantum state is a basis dependent notion, it is necessary to choose an reference basis in order to formulate its resource theory, which is usually called incoherent basis. In the rest of this work, we will choose the computational basis \(\mathcal{B} = \{|i\rangle\}_{i=0}^{d-1}\) as the incoherent basis.

A. Free states, resources and free operations

Any resource theory is built from the basic notions of free states, resources and free operations. In the case of the resource theory of coherence, the free state are diagonal in the incoherent basis, i.e., a state \(\rho\) is incoherent if and only if \(\rho = \sum_{i=0}^{d-1} p_i |i\rangle\langle i|\), with \(\sum_{i=0}^{d-1} p_i = 1\) and \(p_i \geq 0\) for all \(i \in \{0, \ldots, d-1\}\). We call them incoherent states, and we denote the set of incoherent states as \(\mathcal{I}\). The resources of a theory are the states which are not free. In the coherence case, the resources are the non-diagonal states in the incoherent basis, and we call them coherent states. Regarding the free operations, several definitions have been introduced [3]. For each choice of free operations, we obtain different resource theories for coherence. In what follows, we focus on the incoherent operations introduced in [11].

In order to define the free operations, we consider completely positive and trace-preserving maps (CPTP) define on the space of density matrix \(\mathcal{S}(\mathcal{H})\). If \(\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})\) is a CPTP map, it has an operator-sum representation in terms of Kraus operators \(\{K_n\}_{1 \leq n \leq N}\) (with \(N\) an arbitrary natural number) of the form \(\Lambda(\rho) = \sum_{n=1}^{N} K_n \rho K_n^\dagger\), where Kraus operators are such that \(\sum_{n=1}^{N} K_n^\dagger K_n = I\) (with \(I\) the identity of the Hilbert space). The free operations for any resource theory of coherence have to be CPTP maps satisfying, at least, the additional condition of not creating coherence from an incoherent state. More precisely, \(\Lambda(\rho) \in \mathcal{I}\) for any \(\rho \in \mathcal{I}\). All operations of this type form the set of maximally incoherent operations (MIO).

In this work, we are interested in a subset of the maximally incoherent operations, the so-called incoherent operations (IO), which were introduced in [11]. IO can be defined in terms of Kraus operations as follows [19, 23, 24]:
Definition 1 (Incoherent operation). A CPTP map \( \Lambda \) is an incoherent operation if it admits a Kraus representation \( \{ K_n \}_{n=1}^N \), such that the Kraus operators are incoherent, that is, \( K_n |i\rangle \propto f_n(|i\rangle) \), for all \( n \in \{1, \ldots, N\} \), with \( f_n \) a relabeling function of the set \( \{0, \ldots, d-1\} \).

B. Necessary and sufficient conditions for coherent transformations

In this subsection, we recall some important results about state transformations under incoherent operations. We denote as \( \rho \rightarrow IO \sigma \) whenever there exists a suitable incoherent operation \( \Lambda \) such that \( \sigma = \Lambda(\rho) \).

First, we note that any incoherent state can be reached by any other state by means of a suitable incoherent operation, that is, for a given \( \sigma \in \mathcal{I} \) there exists a \( \rho \) such that \( \rho \rightarrow IO \sigma \). Moreover, there are some states that can be converted into any other state (not necessarily incoherent) only by means of incoherent operations. More precisely, there exist states \( \rho \) called maximally coherent state (MCS), such that \( \rho \rightarrow IO \sigma \) for any \( \sigma \in \mathcal{S}(\mathcal{H}) \).

The canonical MCS state is a pure state of the form \( \rho_{mcs} = |\Psi_{mcs}\rangle \langle \Psi_{mcs}| \) with \( |\Psi_{mcs}\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle \). The set of all MCSs can be obtained from the orbit of \( \rho_{mcs} \) under the set of unitary incoherent operations, which are given by operators of the form \( U_{IO} = \sum_{i=0}^{d-1} e^{i\theta_i} |i\rangle \langle i| \), with \( \theta_i \in \mathbb{R} \) and \( \pi \) a permutation acting on the set \( \{0, 1, \ldots, d-1\} \).

In order to address the general problem of state transformations, we need the following notions. Let \( \Delta_d \) be the set of \( d \)-dimensional probability vectors, i.e.,

\[
\Delta_d = \{ u = (u_0, \ldots, u_{d-1}) \in \mathbb{R}^d : u_i \geq 0 \text{ and } \sum_{i=0}^{d-1} u_i = 1 \},
\]

and let \( \Delta_d^+ \subseteq \Delta_d \) be the set of \( d \)-dimensional probability vectors with their entries decreasingly ordered. The coherence vector of a pure state of a \( d \)-dimensional Hilbert space is a probability vector in \( \Delta_d \) defined as follows:

Definition 2 (Coherence vector). Let \( \mathcal{B} = \{|i\rangle\}_{i=0}^{d-1} \) be the incoherent basis. The coherence vector of a pure state \( |\psi\rangle \langle \psi| \) is defined as

\[
\mu(|\psi\rangle \langle \psi|) = \left( |\langle 0|\psi|\rangle|^2, \ldots, |\langle d-1|\psi|\rangle|^2 \right).
\]

We also define the ordered coherence vector \( \mu^+ (|\psi\rangle \langle \psi|) \in \Delta_d^+ \), which is given by the entries of the vector \( \mu(|\psi\rangle \langle \psi|) \), but in a non-increasing order.

The state transformations between quantum states is related with the majorization relation of probability vectors. The majorization relation is defined as follows (see e.g. [11]):

Definition 3 (Majorization relation). Given \( u, v \in \Delta_d \), it said that \( u \) is majorized by \( v \) (denoted as \( u \preceq v \)) if, and only if, \( \sum_{i=0}^{d-1} u_{\pi_a(i)} \leq \sum_{i=0}^{d-1} v_{\pi_a(i)} \), for all \( \pi_a \), \( \pi_v \) permutations acting on the set \( \{0, \ldots, d-1\} \) that sort the entries of \( u \) and \( v \), respectively, in a non-increasing order.

The majorization relation is preorder relation on the set \( \Delta_d \) and a partial order on the set \( \Delta_d^+ \). Moreover, the set \( \Delta_d^+ \) together with the majorization relation \( \preceq \) is a complete lattice [12, 13, 15], and the algorithms to obtain the supremum and infimum can be found in [13] (see e.g. [12, 13]).

In particular, the supremum of a set \( U \subseteq \Delta_d^+ \), denoted as \( \bigvee U \), can be computed as follows. Let \( L_{U} \) be the upper envelope of the polygonal curve given by the linear interpolation of the set of points \( \{(j, S_j)\}_{0 \leq j \leq d} \), where \( S_j \) is the supremum of \( s_j(u) \) for all \( u \in U \) and \( s_j(u) = \sum_{i=0}^{j-1} u_i \), with the convention \( S_0 = 0 \). As it is shown in [13], \( L_{U} \) is the Lorenz curve associated to the probability vector \( \bigvee U \).

Therefore, \( \bigvee U = (L_{U}(1), L_{U}(2) - L_{U}(1), \ldots, L_{U}(d - 1) - L_{U}(d-1)) \).

The majorization relation is intimately related with Schur-concave functions (see e.g. [11, I.3]), which are functions that anti-preserves the preorder relation. More precisely, a function \( f : \Delta_d \rightarrow \mathbb{R} \) is Schur-concave if, for all \( u, v \in \Delta_d \) such that \( u \preceq v \), \( f(u) \geq f(v) \). Moreover, if the function \( f \) also satisfies that \( f(u) > f(v) \) whenever \( u \) is strictly

---

1 A preorder relation is a reflexive and transitive binary relation, and a partial order relation is a preorder relation that it is also antisymmetric. A set \( P \) with a partial order relation is a complete lattice if the supremum and infimum of any subset of \( P \) exist (see e.g. [27]).
2 We recall that the upper envelope of a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is defined as \( \inf \{ g : f \leq g \text{ and } g \text{ is continuos and concave} \} \) (see e.g. [28, Def.4.1.5]).
3 The Lorenz curve of a probability vector \( u \in \Delta_d \) is an increasing and concave function \( L_u : [0, d] \rightarrow [0, 1] \) formed by the linear interpolation of the points \( \{(j, s_j(u))\}_{0 \leq j \leq d} \). It can be shown that \( u \preceq v \iff L_u \leq L_v \) (see e.g. [11]).
majorized by \( v \) (i.e. when \( u \preceq v \) and \( u \neq \Pi v \), with \( \Pi \) a permutation matrix), we say that it is strictly Schur-concave. In particular, the generalized entropies, including Shannon, Rényi and Tsallis entropies, are examples of strictly Schur-convave functions (see e.g. \[20\]).

Taking into account these definitions, we present the following results about necessary and sufficient conditions for coherent transformations. The first result completely characterizes the incoherent transformations between pure states in terms of the majorization relation between their corresponding coherence vectors (see \[5\]–\[9\]).

**Proposition 4.** Let \( |\psi\rangle \langle \psi| \) and \( |\phi\rangle \langle \phi| \) be an arbitrary two pure states and \( \Lambda \) an incoherent operation. Then,

\[
|\psi\rangle \langle \psi| \xrightarrow{\text{IO}} |\phi\rangle \langle \phi| \iff \mu(|\psi\rangle \langle \psi|) \leq \mu(|\phi\rangle \langle \phi|).
\]

(3)

Notice that whether both transformations are possible we have \( |\psi\rangle \langle \psi| \xleftrightarrow{\text{IO}} |\phi\rangle \langle \phi| \iff \mu(|\psi\rangle \langle \psi|) = \Pi (\mu(|\phi\rangle \langle \phi|)) \) with \( \Pi \) a permutation matrix. As a consequence, the coherence vector \( \mu(|\psi\rangle \langle \psi|) \) of a given pure state \( |\psi\rangle \langle \psi| \) and its ordered probability vector \( \mu^1(|\psi\rangle \langle \psi|) \) are equivalent in a coherence-resource sense.

The next result, given in \[10\] Th. 4, is a generalization of the above proposition, and it provides necessary and sufficient conditions for transformation from pure states to arbitrary states by means of incoherent operations.

**Proposition 5.** Let \( |\psi\rangle \langle \psi| \) be an arbitrary pure state and \( \sigma \) be an arbitrary quantum state. Then,

\[
|\psi\rangle \langle \psi| \xrightarrow{\text{IO}} \sigma \iff \exists \{p_n, |\phi_n\rangle\} \text{ such that } \sigma = \sum_{n=1}^{N} p_n |\phi_n\rangle \langle \phi_n| \text{ and } \mu(|\psi\rangle \langle \psi|) \leq \sum_{n=1}^{N} p_n \mu^1(|\phi_n\rangle \langle \phi_n|).
\]

(4)

A related result, given in \[9\] Lemma 4], provides with the particular decomposition of the final state \( \sigma \) in the r.h.s of (4),

\[
|\psi\rangle \langle \psi| \xrightarrow{\text{IO}} \sigma \implies \mu(|\psi\rangle \langle \psi|) \leq \sum_{n=1}^{N} p_n \mu^1(|\phi_n\rangle \langle \phi_n|),
\]

(5)

where \( p_n = \text{Tr}(K_n |\psi\rangle \langle \psi| K_n^\dagger) \), \( |\phi_n\rangle \langle \phi_n| = K_n |\psi\rangle \langle \psi| K_n^\dagger/p_n \), and \( \{K_n\}_{1 \leq n \leq N} \) are the incoherent Kraus operators of the incoherent operation \( \Lambda \), which satisfies \( \sigma = \Lambda(|\psi\rangle \langle \psi|) \). The result given in Prop. 4 is a particular case of Prop. 5 but in the former the incoherent transformations are fully characterize by the majorization relation between the corresponding coherence vectors of the pure states.

### C. Coherence measures

In this subsection, we introduce the notion of coherence measures, based mainly on the axiomatic formulation given in \[1\].

**Definition 6 (Coherence measure).** A coherence measure is a function \( C : \mathcal{S}(\mathcal{H}) \to \mathbb{R} \) satisfying the following conditions:

\begin{enumerate}
  \item[(C1)] Vanishing on incoherent states: \( C(\rho) = 0 \) for any \( \rho \) incoherent.
  \item[(C2)] Monotonicity under incoherent operations: \( C(\rho) \geq C(\Lambda(\rho)) \) for any incoherent operation \( \Lambda \) and any state \( \rho \).
  \item[(C3)] Monotonicity under selective incoherent operation: \( C(\rho) \geq \sum_{n=1}^{N} p_n C(\sigma_n) \) for any incoherent operation \( \Lambda \), with incoherent Kraus operators \( \{K_n\}_{1 \leq n \leq N} \), and any state \( \rho \), where \( p_n = \text{Tr} K_n \rho K_n^\dagger \) and \( \sigma_n = K_n \rho K_n^\dagger/p_n \).
  \item[(C4)] Normalization: \( C(\rho) = 1 \) for any \( \rho \) maximally coherent.
  \item[(C5)] Convexity: \( C(\sum_{k=1}^{M} q_k \rho_k) \leq \sum_{k=1}^{M} q_k C(\rho_k) \).
\end{enumerate}

Condition \((C1)\) guarantees that the measure is well defined for the incoherent states. Condition \((C2)\) ensures that it is consistent with incoherent operations. Both are the minimal requirements for any quantity which pretends to quantify the coherence resource. Condition \((C3)\) guarantees that coherence does not increase under incoherent measurements, even if one has access to the individual measurement outcomes. When a quantifier satisfies the conditions \((C4)\)–\((C5)\), it is called coherence monotone. We have included the condition \((C4)\) because maximally coherent states \( \rho_{\text{mcs}} \) is the golden unit of the coherence resource theory with the incoherent operations given in Def. 1 (the golden unit does not necessary exist for other set of free operations, see e.g. \[3\]). The relevance of this condition is discussed in \[25\].
Finally, condition \((C_5)\) is often related with the fact that mixing states does not increases the amount of coherence. Although the convexity condition \((C_5)\) is a desirable property for coherence quantifiers, it is not considered as an essential one. Indeed, there are important quantifiers of coherence that do not satisfy it (e.g. the maximum relative entropy of coherence is not a convex monotone \(^{29}\)). Finally, we note that when conditions \((C_2)\) and \((C_5)\) are satisfied, condition \((C_3)\) is automatically satisfied.

There several quantifiers of coherence that satisfy some or all of the conditions given in Def. \(^{6}\). In this work, we are interested in families of coherence measures constructed from quantifiers of pure states. Before introducing an important result for coherence measures restricted to pure states (see e.g. \(^{5, 9}\)), we need to define the following set of functions,

\[
\mathcal{F} = \{ f : \mathbb{R}^d \to [0,1] : f \text{ is symmetric and concave, } f(1,0,\ldots,0) = 0 \text{ and } f(1/d,\ldots,1/d) = 1 \}. \tag{6}
\]

Since a symmetric and concave functions is a Schur-concave function \(^{11}\), we have that all functions in \(\mathcal{F}\) are Schur-concave.

**Proposition 7.** Given a coherence monotone \(C : \mathcal{S}(\mathcal{H}) \to \mathbb{R}\) satisfying conditions \((C_1),(C_4)\), there exists a function \(f_C \in \mathcal{F}\), such that the restriction of \(C\) to the pure states, denoted as \(C|_{\mathcal{P}(\mathcal{H})}\), can be written as

\[
C|_{\mathcal{P}(\mathcal{H})}(|\psi\rangle \langle \psi|) = f_C(\mu(|\psi\rangle \langle \psi|)), \tag{7}
\]

This result guarantees that the restriction of any coherence monotone to pure states can be written in terms of a function of \(\mathcal{F}\) evaluated on the coherence vectors of the pure state. We will called as \(f_C\) to the associated vector function of the coherence monotone \(C\).

Conversely, given a vector function \(f \in \mathcal{F}\), it is possible to define a coherence monotone. One possibility consists in appealing to convex roof construction (see e.g. \(^{29}\)), as it is done in \(^{5, 9}\). Before introducing the convex roof measure of coherence, we define the set of all pure state decompositions of a given state \(\rho\),

\[
\mathcal{D}(\rho) = \left\{ \{q_k, |\psi_k\rangle\}_1 \leq k \leq M : \rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| \right\}. \tag{8}
\]

**Definition 8 (Convex roof measure).** For any function \(f \in \mathcal{F}\), the convex roof measure of coherence \(C_f^\text{cr} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}\) is defined as

\[
C_f^\text{cr}(\rho) = \inf_{\{q_k, |\psi_k\rangle\}_1 \leq k \leq M \in \mathcal{D}(\rho)} \sum_{k=1}^{M} q_k f(\mu(|\psi\rangle \langle \psi|)). \tag{9}
\]

The infimum in \((9)\) can be replaced by a minimum, since there is always an optimal decomposition of the pure state \(\rho\) that reaches the supremum (see e.g. \(^{31}\)). The convex roof measure \(C_f^\text{cr}\) is a good quantifier of coherence since it satisfies conditions \((C_2),(C_5)\).

The name of the measure \(C_f^\text{cr}\) is based on the fact that it is the convex roof extension of any coherence monotone with associated vector function equal to \(f\). This property is stated in the following lemma.

**Proposition 9.** Let \(C : \mathcal{S}(\mathcal{H}) \to \mathbb{R}\) be a coherence measure. Then,

\[
C \leq C_f^\text{cr}, \tag{10}
\]

where \(f_C\) is a function associated to \(C\).

The convex roof construction is widely used, especially in the context of entanglement measures \(^{20, 21}\). However, it is not the only possibility in the literature, recently another construction for coherence measures was proposed \(^{22}\).

Before introducing this measure of coherence, we need to define the set of all pure state that can be converted into \(\rho\) by means of incoherent operations,

\[
\mathcal{O}(\rho) = \left\{ |\psi\rangle \langle \psi| : |\psi\rangle \langle \psi| \to \rho \right\}. \tag{11}
\]

Now, we introduce the definition of the coherence measure given in \(^{22}\). In this work we will call it top monotone of coherence.

**Definition 10 (Top monotone).** For any function \(f \in \mathcal{F}\), the top monotone of coherence \(C_f^\text{top} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}\) is defined as

\[
C_f^\text{top}(\rho) = \inf_{|\psi\rangle \langle \psi| \in \mathcal{O}(\rho)} f(\mu(|\psi\rangle \langle \psi|)). \tag{12}
\]
The top monotone $C_{f}^{\text{top}}(\rho)$ satisfies conditions \([C_2]–[C_4]\) whereas condition \([C_5]\) holds if and only if $C_{f}^{\text{top}}(\rho) = C_{f}^{\text{cr}}(\rho) \ [22, \text{Th.4}].$ The chosen name for this measure is based on the fact that, for any measure $C(\rho)$, such that $C(\psi \langle \psi \rangle) = C_{f}^{\text{top}}(\psi \langle \psi \rangle)$ for all $\psi \langle \psi \rangle \in \mathcal{P}(\mathcal{H})$, the following order relation is satisfied

$$C_{f}^{\text{top}} \geq C.$$  \hspace{1cm} (13)

As in the case of coherence measures based on the convex roof construction, the infimum in (12) can be replaced by a minimum, since there always exists a pure state that attains it. This is a consequence of the continuity of $f$ on $\Delta_d$ (concave functions on $\mathbb{R}^d$ are continuous on any subset of $\mathbb{R}^d$ \ [32, Th.10.1] \) and the compactness of the set $\mathcal{O}(\rho)$, a fact that we will show in Lemma 34. Notice that in some proofs given in \ [22 \) it is assumed the existence of the minimum in (12), but its existence is not prove in general (see e.g. the proofs of monotonicity and strong monotonicity of $C_{f}^{\text{top}}$, or the converse part of the proof of Th. 3 regarding the convexity of $C_{f}^{\text{top}}$ or the proof of Th. 7 regarding the continuity of $C_{f}^{\text{top}}$). Therefore, our Lemma 34 fills these gaps.

### III. COHERENT VECTOR FOR QUANTUM STATES: DEFINITION AND PROPERTIES

In this section, we introduce the **coherence vector for quantum states**, generalizing the definition given in \ [2 \). This definition connects the notion of coherence with the majorization lattice theory. Moreover, it allows to introduce a new family of coherence measures, alternative to $C_{f}^{\text{cr}}(\rho)$ and $C_{f}^{\text{top}}(\rho)$.

Inspired by the definitions of coherence measures $C_{f}^{\text{cr}}(\rho)$ and $C_{f}^{\text{top}}(\rho)$, we define two sets of probability vectors associated to a given mixed state $\rho$ that will be useful for studying coherence properties of $\rho$. The first one is obtained from the pure state decomposition of the mixed state, we denote it as $\mathcal{U}^{\text{psd}}(\rho)$, where the acronym psd refers to pure state decomposition of $\rho$.

**Definition 11 (Pure state decomposition set).** Let $\rho$ be a mixed state. For each pure state decomposition $(q_k, |\psi_k\rangle)_{1 \leq k \leq M}$ of $\rho$, we consider the coherence vector $\sum_{k=1}^{M} q_k \mu^k(\langle |\psi_k\rangle |\psi_k\rangle)$. Let $\mathcal{U}^{\text{psd}}(\rho)$ be the set formed by the coherence vectors of all pure state decompositions of $\rho$, i.e.,

$$\mathcal{U}^{\text{psd}}(\rho) = \left\{ \sum_{k=1}^{M} q_k \mu^k(\langle |\psi_k\rangle |\psi_k\rangle) : \{ q_k, |\psi_k\rangle \}_{1 \leq k \leq M} \in \mathcal{D}(\rho) \right\}.$$  \hspace{1cm} (14)

The second one is obtained from the set of all pure states that can be converted into the mixed state, we denote it as $\mathcal{U}^{\text{psc}}(\rho)$, where the acronym psc refers to pure states connected to $\rho$.

**Definition 12 (Connected pure states set).** Given a mixed state $\rho$, we define the set formed by the order coherence vectors of all the pure state that can be connected with $\rho$ by means of IO operations, that is,

$$\mathcal{U}^{\text{psc}}(\rho) = \left\{ \mu^k(\langle |\psi\rangle |\psi\rangle) : |\psi\rangle \langle |\psi\rangle \in \mathcal{O}(\rho) \right\}.$$  \hspace{1cm} (15)

An interesting property of these sets is that both are convex sets.

**Proposition 13.** The sets $\mathcal{U}^{\text{psd}}(\rho)$ and $\mathcal{U}^{\text{psc}}(\rho)$ are convex.

Another observation that will be often exploited to characterize quantum coherence is the following. Since, for a given $\rho$, both sets $\mathcal{U}^{\text{psd}}(\rho)$ and $\mathcal{U}^{\text{psc}}(\rho)$ are subsets of $\Delta_d$, and the majorization lattice is complete (see e.g. \ [12, 13 \), the supremum and infimum (with respect to majorization partial order) of these sets always exist. In particular, in the following proposition we show that the supremum of both sets coincide.

**Proposition 14.** $\sqrt[\text{psd}]{\mathcal{U}}(\rho) = \sqrt[\text{psc}]{\mathcal{U}}(\rho)$.

This result allows to define the coherence vector of a general quantum state, generalizing the definition given in \ [2 \), as follows.

**Definition 15 (Generalized coherence vector).** Given a quantum state $\rho$, we define the coherence vector of $\rho$, $\nu(\rho)$, as

$$\nu(\rho) = \sqrt[\text{psd}]{\mathcal{U}}(\rho),$$  \hspace{1cm} (16)

or, equivalently as $\nu(\rho) = \sqrt[\text{psc}]{\mathcal{U}}(\rho)$.
Notice that for a pure state, the generalized coherence vector is equal to the ordered coherence vector, i.e., $\nu(|\psi\rangle \langle \psi|) = \mu^4(|\psi\rangle \langle \psi|)$, which means that the Def. 15 is a suitable extension of Def. 2.

We observe that whenever $\sqrt{U^{\text{psd}}(\rho)} \in U^{\text{psd}}(\rho)$, $\sqrt{U^{\text{psd}}(\rho)}$ is a maximum. We call an optimal pure state decomposition to the ensemble that attains this maximum.

**Definition 16 (Optimal pure state decomposition).** An ensemble $\{\tilde{q}_k, |\tilde{\psi}_k\rangle\}_{1 \leq k \leq M}$ is an optimal pure state decomposition of $\rho$ if $\{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho)$ and $\sum_{k=1}^M q_k \mu^4(|\tilde{\psi}_k\rangle \langle \tilde{\psi}_k|) = \nu(\rho)$.

In Ref. [22], it is posed that it is not easy to prove whether there always exists such optimal ensemble for a general quantum state. In Sec. [11] we will provide a method to check if the supremum is a maximum. In particular, we will show that there are qutrit states for which the optimal ensemble does not exist, closing the question about the existence of the optimal pure state decomposition of a general quantum state.

On the other hand, whenever $\sqrt{U^{\text{psc}}(\rho)} \in U^{\text{psc}}(\rho)$, $\sqrt{U^{\text{psc}}(\rho)}$ is also maximum. We call optimal pure state to the state that attains this maximum.

**Definition 17 (Optimal pure state).** A pure state $|\tilde{\psi}\rangle$ is optimal if $|\tilde{\psi}\rangle \langle \tilde{\psi}| \in \mathcal{O}(\rho)$ and $\mu^4(|\tilde{\psi}\rangle \langle \tilde{\psi}|) = \nu(\rho)$.

Moreover, we have that when there exists an optimal ensemble, there also exists an optimal pure state, and vice versa.

**Proposition 18.** $\nu(\rho) \in U^{\text{psd}}(\rho) \iff \nu(\rho) \in U^{\text{psc}}(\rho)$.

In what follows, we will show that the generalized coherence vector satisfies several properties that capture the coherence of a general quantum state. The first observation is that the generalized coherence vector completely characterizes the notion of incoherent state.

**Proposition 19.** $\rho$ is incoherent $\iff \nu(\rho) = (1, 0, \ldots, 0)$.

This result justifies the Def. 15 of the coherence vector of $\rho$, since $\rho$ is coherent if and only if $\nu(\rho) \neq (1, 0, \ldots, 0)$. We also have that the coherence vector of $\rho$ fully characterizes maximally coherent states as follows.

**Proposition 20.** $\rho$ is maximally coherent $\iff \nu(\rho) = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$.

We observe that, by definition, for any mixture of pure quantum states, the following majorization relation is satisfied,

**Proposition 21.** Let $\rho = \sum_{k=1}^M p_k |\psi_k\rangle \langle \psi_k|$. Then,

$$\sum_{k=1}^M p_k \nu(|\psi_k\rangle \langle \psi_k|) \leq \nu(\rho).$$

This result allows us to interpret the definition $\nu(\rho) = \sqrt{U^{\text{psd}}(\rho)}$ as the concave roof extension of the coherence vector defined for pure states [3].

Alternatively, the equivalence $\sqrt{U^{\text{psd}}(\rho)} = \sqrt{U^{\text{psc}}(\rho)}$, provides the generalized coherent vector with an operational interpretation in terms of pure state transformations. In this sense, our definition of a generalized coherence vector (Def. 15) is a physical and mathematical suitable extension of the corresponding definition for pure states (Def. 3).

**IV. NECESSARY CONDITIONS FOR INCOHERENT TRANSFORMATIONS**

In this section, we use the coherence vector definition, given in [15] for characterizing state transformations between mixed states.

**Proposition 22.** Let $\rho$ and $\sigma$ be two arbitrary quantum states. Then,

$$\rho \xrightarrow{10} \sigma \implies \forall \{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho), \exists \{r_l, |\phi_l\rangle\}_{l \in L} \in \mathcal{D}(\sigma) : \sum_{k=1}^M q_k \mu^4(|\psi_k\rangle \langle \psi_k|) \leq \sum_{l \in L} r_l \mu^4(|\phi_l\rangle \langle \phi_l|).$$

---

4 Indeed, our proposal can be posed in a more abstract framework generalizing the notion of concave roof extension of a function. Precisely, let $\Omega$ be a compact convex set and $\Omega^{\text{pure}}$ be the set formed by its extremal points. The concave roof $f^{\text{cr}} : \Omega \rightarrow \mathbb{R}$ of the function $f : \Omega^{\text{pure}} \rightarrow \mathbb{R}$ is defined as $f^{\text{cr}}(\omega) := \sup \sum_k q_k f(\omega_k)$, where the supremum is taken in $\mathbb{R}$ over all extremal convex decompositions of $\omega = \sum_k q_k \omega_k$, $\omega_k \in \Omega^{\text{pure}}$ (see e.g. [30]). This construction can be generalized to the majorization lattice $\Delta^4$ as follows. The concave roof $f^{\text{cr}} : \Omega \rightarrow \Delta^4$ of the function $f : \Omega^{\text{pure}} \rightarrow \Delta^4$ can be defined as $f^{\text{cr}}(\omega) := \vee \sum_k q_k f(\omega_k)$, where, in this case, the supremum in taken in the complete majorization lattice $\Delta^4$. 

4
Notice that this result generalizes the necessary condition of Prop. 5. In addition, we have the following consequences.

Corollary 23. Let \( \rho \) and \( \sigma \) be two arbitrary quantum states. Then,
\[
\rho \rightarrow_\mathrm{IO} \sigma \quad \Rightarrow \quad \nu(\rho) \leq \sum_{n=1}^{N} p_n \nu(\sigma_n),
\]
(19)
with \( p_n = \mathrm{Tr} K_n \rho K_n^\dagger \) and \( \sigma_n = K_n \rho K_n^\dagger / p_n \), where \( \{ K_n \}_{1 \leq n \leq N} \) are incoherent Kraus operators such that \( \sigma = \sum_{n=1}^{N} K_n \rho K_n^\dagger \).

We now observe that the majorization relation (19) generalizes the necessary condition for incoherent transformations from pure to mixed states, given in [5], to the general case, i.e. mixed to mixed states.

Finally, another consequence of Prop. 22 is that the coherence vectors of two states \( \rho \) and \( \sigma \) satisfy a majorization whenever \( \rho \) can be transformed into \( \sigma \).

Corollary 24. Let \( \rho \) and \( \sigma \) be two arbitrary quantum states. Then,
\[
\rho \rightarrow_\mathrm{IO} \sigma \quad \Rightarrow \quad \nu(\rho) \leq \nu(\sigma).
\]
(20)
Notice that this condition is not sufficient even for single qubit systems. In fact, a qubit state \( \rho \) with Bloch vector \( (r_x, r_y, r_z) \) can be converted into another state \( \sigma \) with Bloch vector \( (s_x, s_y, s_x) \) by means of incoherent operations if and only if \( s_x^2 + s_y^2 \leq r_x^2 + r_y^2 \) and \( s_z^2 \leq 1 - (1 - r_z^2) / (r_x^2 + r_y^2) (s_x^2 + s_y^2) \) (see [10, 17]). By using the result given in Eq. (23) (or in (22)), it can be shown that only the first condition is equivalent to the r.h.s of (20). Moreover, in higher dimensions \( (d \geq 4) \), a finite number of conditions in terms of coherence measures are not enough to completely characterize the coherence transformations [10].

V. A FAMILY OF COHERENCE MONOTONES

In this section, we introduce a new family of coherence monotones, alternative to \( C_f^{\mathrm{CT}} \) and \( C_f^{\mathrm{top}} \). We adopt a different approach to the ones given in Def. 8 and 10. Our proposal is based on the coherence vector of an arbitrary state introduced in Def. 15. The fact that the coherence vector satisfies the properties given in Prop. 19–22 allows us to introduce the following family of coherence measures that we call coherence vector monotone.

Definition 25 (Coherence vector monotone). For any function \( f \in \mathcal{F} \), the coherence vector monotone \( C_f^{\mathrm{CV}} : \mathcal{S}(\mathcal{H}) \rightarrow \mathbb{R} \) is defined as
\[
C_f^{\mathrm{CV}}(\rho) = f(\nu(\rho)),
\]
(21)
where \( \nu(\rho) \) is the coherence vector of \( \rho \), given in Def. 15.

We observe that this family of quantifiers of coherence is good definition. More precisely, we have the following result.

Proposition 26. For any function \( f \in \mathcal{F} \), the coherence vector monotone \( C_f^{\mathrm{CV}} \) satisfies conditions (C1)–(C4).

In what follows, we are going to characterize the order relation between the coherence quantifiers \( C_f^{\mathrm{CT}} \), \( C_f^{\mathrm{top}} \) and \( C_f^{\mathrm{CV}} \). First, we note that \( C_f^{\mathrm{top}}(\rho) \geq C_f^{\mathrm{CV}}(\rho) \) and \( C_f^{\mathrm{top}}(\rho) \geq C_f^{\mathrm{CV}}(\rho) \), for all \( \rho \in \mathcal{S}(\mathcal{H}) \). This result is an immediate consequence of the fact that \( C_f^{\mathrm{top}}(|\psi\rangle \langle \psi|) = C_f^{\mathrm{top}}(|\psi\rangle \langle \psi|) = C_f^{\mathrm{top}}(|\psi\rangle \langle \psi|) \) for all \( |\psi\rangle \langle \psi| \in \mathcal{P}(\mathcal{H}) \), and the Ineq. 13. Moreover, for some \( \rho \in \mathcal{S}(\mathcal{H}) \), we have \( C_f^{\mathrm{top}}(\rho) = C_f^{\mathrm{CV}}(\rho) \). The following result characterizes this situation.

Proposition 27. The following statements are equivalent:

1. There exists an optimal pure state decomposition of \( \rho \), i.e., \( \nu(\rho) \in \mathcal{U}^{\mathrm{opt}}(\rho) \).
2. \( C_f^{\mathrm{CV}}(\rho) = C_f^{\mathrm{top}}(\rho) \) for all \( f \in \mathcal{F} \).
3. \( C_f^{\mathrm{CV}}(\rho) = C_f^{\mathrm{top}}(\rho) \) for some \( f \in \mathcal{F} \) and \( f \) strictly Schur-concave.
The scheme of Fig. 1 summarizes the relationships among the three families of coherence quantifiers.

In particular, this result gives us a method to address the question about the existence of an optimal ensemble of a general quantum state. In fact, by exploiting this result, we will show in Sec. VI examples of qutrit states for which the optimal ensemble does not exist.

In general, there is not an order relation between $C_f^{\text{cv}}(\rho)$ and $C_f^{\text{cr}}(\rho)$. However, if there exists an optimal pure state decomposition of a state, we obtain the following result.

**Proposition 28.** If there exists an optimal pure state decomposition of $\rho$, then $C_f^{\text{cr}}(\rho) \geq C_f^{\text{cv}}(\rho)$.

On the contrary, for linear functions, we have the opposite relation between $C_f^{\text{cr}}$ and $C_f^{\text{cv}}$.

**Proposition 29.** Let $f \in \mathcal{F}$ be such that $f|_{\Delta_d^+} = c + \ell$, where $c \in \mathbb{R}$ and $\ell : \Delta_d^+ \to \mathbb{R}$ is a linear function. Then, $C_f^{\text{cr}}(\rho) \leq C_f^{\text{cv}}(\rho) = C_f^{\text{top}}(\rho)$.

Examples of this class of functions are $f(u) = d(1 - u_1^d)/(d - 1)$, $f(u) = d \cdot u_1^d$ or $f(u) = 1 - u_1^d + u_2^d$, where $u_1^d = (u_1^d)_k$. In particular, for the former function, we have that all quantifiers coincide and are equal to the normalized geometric measure of coherence [33], that is, for $f(u) = d(1 - u_1^d)/(d - 1)$,

$$
C_f^{\text{cr}}(\rho) = C_f^{\text{cv}}(\rho) = C_f^{\text{top}}(\rho) = \frac{d}{d - 1} \min_{\{q_k : H_k \in \mathcal{D}(\rho)\}} \sum_{k=1}^{M} q_k \left( 1 - \max_{0 \leq i \leq d - 1} |\langle i | \psi_k \rangle|^2 \right). \quad (22)
$$

The scheme of Fig. 1 summarizes the relationships among the three families of coherence quantifiers.

Regarding the convexity of $C_f^{\text{cv}}$, from Prop. 29 it is clear that if there exists a state $\rho \in \mathcal{S(H)}$, such that $C_f^{\text{cr}}(\rho) > C_f^{\text{cv}}(\rho)$, then $C_f^{\text{cv}}$ is not convex. In the case $C_f^{\text{cr}} \leq C_f^{\text{cv}}$, $C_f^{\text{cr}}$ can be convex, as in the case of Eq. (22).

### VI. EXAMPLES

In this section, we calculate the coherence vector for two simple models. First, we consider a qubit state, and we obtain its generalized coherence vector. Secondly, we consider a maximally coherent qutrit going through a depolarizing channel, and we compute the value of $C_f^{\text{cr}}(\rho)$, $C_f^{\text{top}}(\rho)$ and $C_f^{\text{cv}}(\rho)$ for a given state $\rho$ and $f \in \mathcal{F}$.

#### A. Qubit case

Let us consider a qubit system in a general state $\rho = \frac{1 + \vec{r} \cdot \vec{\sigma}}{2}$, with $\vec{r} = (r_x, r_y, r_z)$, such that $||\vec{r}|| \leq 1$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector formed by the Pauli matrices.

In a previous work [22], it has been shown that the supremum of $\mathcal{U}^{\text{psd}}(\rho)$ is a maximum, and it is given by

$$
\nu(\rho) = \left( \frac{1 + r}{2}, \frac{1 - r}{2} \right), \quad (23)
$$

where $r = \sqrt{1 - r_x^2 - r_y^2}$. An optimal pure state decomposition of $\rho$ is given by $\{q, |\psi^+\rangle : 1-q, |\psi^-\rangle \}$ where $|\psi^\pm \rangle = \frac{1 + s^z \cdot \vec{\sigma}}{2}$, with $s^z = (r_x, r_y, \pm r)$ and $q = (r_x + r)/2r \in [0,1]$. As a consequence of Prop. 27, we have that $C_f^{\text{cv}} = C_f^{\text{top}}$.

On the other hand, it has been shown that for any function $f(\nu(\rho)) = \tilde{f}(r)$, such that $\tilde{f}$ is a convex function on $r$, $C_f^{\text{top}}$ is a convex monotone of coherence and $C_f^{\text{top}} = C_f^{\text{cr}}$ [22]. For the qubit case, most of the well-known coherence...
measures, like $\ell_1$-norm, relative entropy, geometric coherence, and so on, admit a formulation with a function $\tilde{f}$ convex on $r$. This means that we have the triple equivalence among the families $C_f^{\mathrm{cv}}$, $C_f^{\mathrm{top}}$ and $C_f^{\nu}$ in this case.

In order to observe a difference between $C_f^{\mathrm{cv}}$ and $C_f^{\mathrm{top}}$, due to Prop. 27 we need an example where $\nu(\rho)$ is not a maximum. According to the above discussion, this could be possible, in principle, in higher dimensions ($d \geq 3$). In what follows, we provide an example for $d = 3$.

**B. Qutrit case**

Let us consider a qutrit system in the maximally coherent state $|\psi_{\text{mcs}}\rangle = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$, going through a depolarizing channel with depolarization probability $p$, that is,

$$\rho_p = \Lambda_p(|\psi_{\text{mcs}}\rangle \langle \psi_{\text{mcs}}|) = \frac{p}{3} + (1-p) |\psi_{\text{mcs}}\rangle \langle \psi_{\text{mcs}}|,$$

where we have introduce the notation $\rho_p$ for the state after the depolarizing channel.

On the other hand, we consider the function $f(u) = 1 - u_1^3 + u_2^3$. Clearly, $f$ satisfies the conditions of Prop. 29. Therefore, we have that for this function both measures $C_f^{\mathrm{cv}}$ and $C_f^{\mathrm{top}}$ are equal.

In Fig. 2(a), we plot $C_f^{\mathrm{cv}}(\rho_p)$ (or, equivalently $C_f^{\mathrm{top}}(\rho_p)$) and $C_f^{\nu}(\rho_p)$ as functions of $p \in [0,1]$. Both functions are monotonically decreasing in terms of $p$, and in the open interval $(0,1)$, we have $C_f^{\nu}(\rho_p) = C_f^{\mathrm{top}}(\rho_p) > C_f^{\nu}(\rho_p)$. Equivalently, this means that the supremum is not a maximum (see Prop. 27).

In Fig. 2(b), we plot $\nu(\rho_p)$ (orange points) and $u_1^{\mathrm{top}}(\rho_p) = \arg\min_{u \in C_f^{\mathrm{top}}} f(u)$. It is shown that $u_1^{\mathrm{top}}(\rho_p) \leq \nu(\rho_p)$ and $u_1^{\mathrm{top}}(\rho_p) \neq \nu(\rho_p)$ for several values of $p$ in the open interval $(0,1)$. Finally, in Fig. 2(c), we consider $\rho_p$ for $p = 0.3$ and we depict $\nu(\rho_p)$ and the region $\{u \in \Delta_3^+: u \geq \nu(\rho_p)\}$. In addition, we generate $10^5$ random unitary matrices of dimensions from 3 up to $9^7$. For each unitary matrix, we use the Schrödinger theorem (see Eq. (B13) or [33]) to generate an ensemble compatible with $\rho_p$ and we plot its coherence vector. This plot evidences that the optimal ensemble of this state does not exist (which can be inferred from the left figure and Prop. 27).

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5 The upper bound 9 is not arbitrary. According to Lemma 1 in [31], the optimal $C_f^{\nu}(\rho_p)$ requires at most nine terms. It is conjectured in [35] Conjecture and Lemma 7] that three terms are enough.
VII. CONCLUDING REMARKS

In this work, we have advanced on the characterization of the quantum coherence resource theory by introducing the notion of generalized coherence vector of an arbitrary quantum state. This probability vector can be interpreted as a concave roof extension of the coherence vectors defined for pure states. We show that it is a good definition, since it allows to characterize the notions of being incoherent, as well as being maximally coherent. Using this notion and the majorization relation, we obtain a necessary condition for the conversion of general quantum states by means of incoherent operations. This generalizes the result for pure states given in the literature, and evidences that the tools of the majorization lattice are useful also in the general case.

Moreover, we have introduced a family of monotones based on the generalized coherence vector, the coherence vector monotones. In order to do this, we considered concave and symmetric functions applied to the coherence vector of a general quantum state. This family of monotone was compared with the families of the convex roof measure and the top monotone. We obtain that the coherence vector monotone is lower than or equal to the top monotone, and the equality is only satisfied when the coherence vector of the state is a maximum. In addition, we have obtained that there is no a definite order between the convex roof measure and the coherence vector monotone. We provided several examples showing that our proposed monotone can be strictly greater than, equal to or strictly lower than the convex roof measure. We have also applied the coherence vector monotone to quantify the coherence of a qubit system and a maximally coherent qutrit going through a depolarizing channel.

Finally, we stress that our framework, which is mainly based the majorization lattice theory, could also be used in other majorization-based resources theories.

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Appendix A: Auxiliaries lemmas

The following result is necessary for Prop 22. It states that convex combinations of ordered probability vectors preserves the majorization relation.

**Lemma 30.** Let \( u_0, \ldots, u_m \in \Delta_d \) and \( v_0, \ldots, v_m \in \Delta_d \) be two sequence of ordered probability vectors, such that \( u_i \preceq v_i \), for all \( 0 \leq i \leq m \). For any probability vector \( q = (q_0, \ldots, q_m) \in \Delta_{m+1} \), the vectors \( u = \sum_{\ell=0}^m q_\ell u_\ell \) and \( v = \sum_{\ell=0}^m q_\ell v_\ell \) belong to \( \Delta_d \), and \( u \preceq v \).

**Proof.** Let \( q = (q_0, \ldots, q_m) \) be an arbitrary probability vector in \( \Delta_{m+1} \). Firstly, we note that \( (u)_i = \sum_{\ell=0}^m q_\ell (u)_\ell \geq 0 \) for all \( 0 \leq i \leq d-1 \), and \( \sum_{\ell=0}^{d-1} (u)_i = \sum_{\ell=0}^{d-1} \sum_{i=0}^m q_\ell (u)_\ell = \sum_{\ell=0}^m q_\ell \sum_{i=0}^{d-1} (u)_\ell = 1 \), i.e., \( u \in \Delta_d \). Moreover, since \( (u)_i+1 \leq (u)_i \), for all \( 0 \leq i \leq d-2 \), and for all \( 0 \leq \ell \leq m \), we have \( (u)_{i+1} = \sum_{\ell=0}^m q_\ell (u)_{\ell+1} \leq \sum_{\ell=0}^m q_\ell (u)_\ell = (u)_i \).

Hence, \( u \in \Delta_d \). Analogously, \( v \in \Delta_d \).

Secondly, since \( u_\ell \preceq v_\ell \), for all \( 0 \leq \ell \leq m \), then we have \( \sum_{i=0}^k (u)_i \leq \sum_{i=0}^k (v)_i \), for all \( 0 \leq k \leq d-1 \). Therefore, for all \( 0 \leq k \leq d-1 \), we have \( \sum_{i=0}^k (u)_i = \sum_{i=0}^k \sum_{\ell=0}^m q_\ell (u)_\ell = \sum_{\ell=0}^m q_\ell \sum_{i=0}^k (u)_\ell \leq \sum_{\ell=0}^m q_\ell \sum_{i=0}^k (v)_\ell = \sum_{i=0}^k (v)_i \).

Hence, \( u \preceq v \).

The following result is necessary for Prop 27.

**Lemma 31.** Let \( f : \Delta_d \to \mathbb{R} \) be a strictly Schur-concave function, and \( u, v \in \Delta_d \). If \( f(u) = f(v) \), then either (i) \( u \preceq v \) and \( v \preceq u \) (incomparable) or (ii) \( u = \Pi v \), with \( \Pi \) a permutation matrix.

**Proof.** Given \( u, v \in \Delta_d \), we suppose that \( f(u) = f(v) \). Then, there are two options: (i) \( u \) and \( v \) are incomparable or (ii) \( u \) and \( v \) are comparable. If (i) is the case, there is nothing to prove. If (ii) is the case, without loss of generality, we can assume \( u \preceq v \). Since \( f(u) = f(v) \), and \( f \) is strictly Schur-concave, we conclude \( u = \Pi v \), with \( \Pi \) a permutation matrix.

The following two lemmas will be necessary to prove that the sets \( \mathcal{U}^{\text{top}}(\rho) \) and \( \mathcal{U}^{\text{psd}}(\rho) \) have the same supremum (see Prop 14).
Lemma 32. $U_{\text{psd}}(\rho) \subseteq U_{\text{psc}}(\rho)$.

Proof. Given an arbitrary $u \in U_{\text{psd}}(\rho)$, there exists a pure state decomposition $\{q_k, |\psi_k\rangle\}_{1 \leq k \leq M}$ of $\rho$ such that $\sum_{k=1}^{M} q_k \mu^k(|\psi_k\rangle \langle \psi_k|) = u$. Moreover, always exists a pure state $|\psi\rangle \langle \psi|$ such that $\mu^k(|\psi\rangle \langle \psi|) = u$.

Since the majorization relation is reflexive, $\mu^k(|\psi\rangle \langle \psi|) \leq \sum_{k=1}^{M} q_k \mu^k(|\psi_k\rangle \langle \psi_k|)$. Finally, from Proposition 5 we have that $|\psi\rangle \langle \psi| \rightarrow \rho$, and $\mu \in U_{\text{psc}}(\rho)$. Therefore, $U_{\text{psd}}(\rho) \subseteq U_{\text{psc}}(\rho)$.

Lemma 33. For each $u \in U_{\text{psc}}(\rho)$, there exists an element $u' \in U_{\text{psd}}(\rho)$, such that $u \leq u'$.

Proof. Given an arbitrary $u \in U_{\text{psc}}(\rho)$, there exists a pure state $|\psi\rangle \langle \psi|$ such that $\mu^k(|\psi\rangle \langle \psi|) = u$ and $|\psi\rangle \langle \psi| \rightarrow \rho$.

From Proposition 5 there exists a pure state decomposition $\{q_k, |\psi_k\rangle\}_{1 \leq k \leq M}$ of $\rho$ such that $u \leq \sum_{k=1}^{M} q_k \mu^k(|\psi_k\rangle \langle \psi_k|)$.

Since $\rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k|$, then $u' = \sum_{k=1}^{M} q_k \mu^k(|\psi_k\rangle \langle \psi_k|)$ belongs to $U_{\text{psd}}(\rho)$. Therefore, $u \leq u'$.

The next result proves the compactness of the set $O(\rho)$. We will use it in the proof of Prop. 19. On the other hand, Lemma 34 together with the continuity of $f$ allows us to replace the infimum in (12) by a minimum. This allows us to fill the gaps in some proofs of [22], where the existence of an optimal state in (12) is assumed, but not proved.

Lemma 34. The set $O(\rho) = \{|\psi\rangle \langle \psi| : |\psi\rangle \langle \psi| \rightarrow \rho\}$ is a compact set.

Proof. According to [16] it is possible to define the set $O(\rho)$ with pure states $|\psi\rangle \langle \psi|$ which satisfy $\sum_{n=1}^{N} K_n |\psi\rangle \langle \psi| K_n^\dagger = \rho$, where $K_n$ are incoherent Kraus operators and $N$ is fixed. By definition we have the two following conditions for the Kraus operators,

\[
\sum_{n} K_n^\dagger K_n = I.
\]

\[
K_n |i\rangle \propto |f_n(i)\rangle, \text{ with } f_n \text{ a relabeling of } \{0, \ldots, d-1\}.
\]

On the one hand, from Eq. (A1), it follows that each incoherent Kraus operator is bounded, i.e., $\|K_n\|_{HS} = \text{Tr}(K_n^\dagger K_n) \leq d$. On the other hand, condition (A2) is equivalent to

\[
(K_n)_{j,i} = (K_n)_{j,i} \delta_{j,f_n(i)}, \quad \forall i, j \in \{0, \ldots, d-1\}.
\]

Notice that condition (A3) are $d^2$ equations for the entries of $K_n$.

We denote the set of all relabeling functions as $R = \{f : \{0, \ldots, d-1\} \rightarrow \{0, \ldots, d-1\}\}$ and the $N$-Cartesian product as $R^N$. Given $\hat{f} = (f_1, \ldots, f_N) \in R^N$, we define the set

\[
K_{\hat{f}} = \left\{ (K_1, \ldots, K_N) \in \mathbb{C}_d^{d \times d} \times \cdots \times \mathbb{C}_d^{d \times d} : \sum_{n=1}^{N} K_n^\dagger K_n = I, \quad (K_n)_{j,i} = (K_n)_{j,i} \delta_{j,f_n(i)} \quad \forall i, j \in \{0, \ldots, d-1\} \right\},
\]

and the set

\[
V_{\hat{f}}(\rho) = \left\{ (|\psi\rangle \langle \psi|, K_1, \ldots, K_N) \in \mathcal{P}(\mathcal{H}) \times \mathbb{C}_d^{d \times d} \times \cdots \times \mathbb{C}_d^{d \times d} : (K_1, \ldots, K_N) \in K_{\hat{f}}, \quad \sum_{n=1}^{N} K_n |\psi\rangle \langle \psi| K_n^\dagger = \rho \right\}.
\]

Finally, we consider the set $V(\rho) = \bigcup_{\hat{f} \in R^N} V_{\hat{f}}(\rho)$. Since $R$ is a finite set, $V(\rho)$ is a finite union of sets. Notice that

\[
|\psi\rangle \langle \psi| \in O(\rho) \iff \exists (K_1, \ldots, K_N) \in \mathbb{C}_d^{d \times d} \times \cdots \times \mathbb{C}_d^{d \times d} : (|\psi\rangle \langle \psi|, K_1, \ldots, K_N) \in V(\rho)
\]

Since $\mathcal{P}(\mathcal{H})$ is closed and the set $V_{\hat{f}}(\rho)$ is given by a finite number of equations, we have that $V_{\hat{f}}(\rho)$ is a closed set. Moreover, $V(\rho)$ is bounded, since $\mathcal{P}(\mathcal{H})$ is bounded and each incoherent Kraus operator has $\|K_n\|_{HS} \leq d$. Therefore, $V(\rho)$ is a compact set, since it is a finite union of compact sets.

---

6 For any continuous function $h$ the set $\{x : h(x) = 0\}$ is closed.
Let us denote the projection of the set \( \mathcal{P}(H) \times \mathbb{C}^{d \times d} \times \ldots \times \mathbb{C}^{d \times d} \) onto the first coordinate as \( \Pi : \mathcal{P}(H) \times \mathbb{C}^{d \times d} \times \ldots \times \mathbb{C}^{d \times d} \rightarrow \mathcal{P}(H) \). We are going to show that \( \Pi(V(\rho)) = \mathcal{O}(\rho) \). On the one hand, let \( |\psi \rangle \langle \psi| \in \Pi(V(\rho)) \). Then, there is a an element \( (|\psi \rangle \langle \psi|, K_1, \ldots, K_N) \in V(\rho) \). Therefore, using equivalence (A1), we have \( |\psi \rangle \langle \psi| \in \mathcal{O}(\rho) \). On the other hand, if \( |\psi \rangle \langle \psi| \in \mathcal{O}(\rho) \), there exists \( (K_1, \ldots, K_N) \in \mathbb{C}^{d \times d} \times \ldots \times \mathbb{C}^{d \times d} \) such that \( (|\psi \rangle \langle \psi|, K_1, \ldots, K_N) \in V(\rho) \). Then, \( |\psi \rangle \langle \psi| \in \Pi(V(\rho)) \).

Since \( \Pi \) is a continuous function and \( V(\rho) \) is compact, then \( \Pi(V(\rho)) \) is compact. Therefore, we conclude that \( \mathcal{O}(\rho) \) is a compact set.

\[ \square \]

Appendix B: Proofs of propositions given in the main text

For the sake of readability, we repeat the statements of the propositions given in the main text and we provide their corresponding proofs.

**Proposition 9.** Let \( C : \mathcal{S}(H) \to \mathbb{R} \) be a coherence measure. Then,

\[
C \leq C_{\text{fs}}^+,
\]

where \( f_C \) is a function associated to \( C \).

**Proof.** Given an arbitrary quantum state \( \rho \), we consider a pure state decomposition \( \{q_k, |\psi_k\rangle\} \) of the state, i.e., \( \rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| \). Since \( C : \mathcal{S}(H) \to \mathbb{R} \) satisfies conditions (C1)−(C4) from Prop. 7 there exists a function \( f_C \in F \), such that

\[
C(|\psi \rangle \langle \psi|) = f_C(\mu(|\psi \rangle \langle \psi|)), \quad \forall |\psi \rangle \langle \psi| \in \mathcal{P}(H).
\]

In addition, \( C \) satisfies condition [C5] hence

\[
C(\rho) \leq \sum_{k=1}^{M} q_k C(|\psi_k\rangle \langle \psi_k|) = \sum_{k=1}^{M} q_k f_C(\mu(|\psi_k\rangle \langle \psi_k|)).
\]

The inequality [B3] is valid for any pure state decomposition of \( \rho \), then

\[
C(\rho) \leq \inf_{\{q_k, |\psi_k\rangle\} \in \mathcal{D}(\rho)} \sum_{k=1}^{M} q_k f_C(\mu(|\psi_k\rangle \langle \psi_k|)).
\]

By definition, the r.h.s of [B4] is the convex roof measure for the function \( f_C \). Therefore, we obtain \( C(\rho) \leq C_{\text{fs}}^+(\rho) \) \( \square \)

**Proposition 13.** The sets \( \mathcal{U}^{\text{psd}}(\rho) \) and \( \mathcal{U}^{\text{psc}}(\rho) \) are convex.

**Proof.** We start with the set \( \mathcal{U}^{\text{psd}}(\rho) \). Let \( u, u' \in \mathcal{U}^{\text{psd}}(\rho) \). Given \( t \in (0, 1) \), we consider the ordered probability vector \( u_t = tu + (1-t)u' \).

By definition of \( \mathcal{U}^{\text{psd}}(\rho) \), we have \( \{q_k, |\psi_k\rangle\} \) \( 1 \leq k \leq M \) and \( \{q'_k, |\psi'_k\rangle\} \) \( 1 \leq k \leq M' \), two pure state decompositions of \( \rho \), such that \( u = \sum_{k=1}^{M} q_k \mu^k (|\psi_k\rangle \langle \psi_k|) \) and \( u' = \sum_{k=1}^{M'} q'_k \mu^{k'} (|\psi'_k\rangle \langle \psi'_k|) \). Since \( \rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| = \sum_{k=1}^{M'} q'_k |\psi'_k\rangle \langle \psi'_k| \), we have

\[
\sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| + (1-t) \sum_{k=1}^{M'} q'_k |\psi'_k\rangle \langle \psi'_k| = \rho.
\]

Therefore, the join \( \{tq_k, |\psi_k\rangle\} \cup \{(1-t)q'_k, |\psi'_k\rangle\} \) \( 1 \leq k \leq M \cup M' \), is also a pure state decomposition of \( \rho \), and \( u_t = \sum_{k=1}^{M} tq_k \mu^k (|\psi_k\rangle \langle \psi_k|) + \sum_{k=1}^{M'} (1-t)q'_k \mu^{k'} (|\psi'_k\rangle \langle \psi'_k|) \in \mathcal{U}^{\text{psd}}(\rho) \). Hence, \( \mathcal{U}^{\text{psd}}(\rho) \) is a convex set.

Now, we consider the set \( \mathcal{U}^{\text{psc}}(\rho) \). Let \( u, u' \in \mathcal{U}^{\text{psc}}(\rho) \). Again, given \( t \in (0, 1) \), we consider the ordered probability vector \( u_t = tu + (1-t)u' \). Also, we consider a pure state \( |u_t \rangle \langle u_t| \), such that \( \mu^t (|u_t \rangle \langle u_t|) = u_t \).

From Lemma 33, we know that there are two probability vectors \( v, v' \in \mathcal{U}^{\text{psd}}(\rho) \), such that \( u \preceq v \) and \( u' \preceq v' \). If we define the ordered probability vector \( v_t = tv + (1-t)v' \), then, from Lemma 30 we have \( u_t = tu + (1-t)u' \preceq \)
Proposition 14. \( \bigvee \mathcal{U}^{\text{psd}}(\rho) = \mathcal{U}^{\text{psc}}(\rho) \).

Proof. From Lemma 32, we have \( \mathcal{U}^{\text{psd}}(\rho) \subseteq \mathcal{U}^{\text{psc}}(\rho) \). Then, \( \bigvee \mathcal{U}^{\text{psd}}(\rho) \subseteq \mathcal{U}^{\text{psc}}(\rho) \). In addition, from Lemma 33, we have \( \mathcal{U}^{\text{psc}}(\rho) \subseteq \bigvee \mathcal{U}^{\text{psd}}(\rho) \). Therefore, since the majorization relation is antisymmetric, we obtain \( \bigvee \mathcal{U}^{\text{psd}}(\rho) = \mathcal{U}^{\text{psc}}(\rho) \).

Proposition 18. \( \nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho) \iff \nu(\rho) \in \mathcal{U}^{\text{psc}}(\rho) \).

Proof.

(\( \Rightarrow \)) Suppose \( \nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho) \). Then, from Lemma 32, it follows that \( \nu(\rho) \in \mathcal{U}^{\text{psc}}(\rho) \).

(\( \Leftarrow \)) Suppose \( \nu(\rho) \in \mathcal{U}^{\text{psc}}(\rho) \). From Lemma 33, exists \( u' \in \mathcal{U}^{\text{psd}}(\rho) \) such that \( \nu(\rho) \preceq u' \). From Prop. 14, we also have that \( u' \preceq \nu(\rho) \). Then, \( u' = \nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho) \).

Proposition 19. \( \rho \) is incoherent \( \iff \nu(\rho) = (1, 0, \ldots, 0) \).

Proof.

(\( \Rightarrow \)) Let \( \rho \in \mathcal{S}(\mathcal{H}) \) be a pure state. By definition, \( \rho \) is diagonal in the incoherent basis, that is, \( \rho = \sum_{i=0}^{d-1} p_i |i\rangle \langle i| \). Since, \( \{p_i, |i\rangle \} \in \mathcal{D}(\rho) \) and \( \sum_1 p_i \mu_k^i (|i\rangle \langle i|) = (1, 0, \ldots, 0) \in \mathcal{U}^{\text{psd}}(\rho) \), then \( \nu(\rho) = (1, 0, \ldots, 0) \).

(\( \Leftarrow \)) Let \( \rho \in \mathcal{S}(\mathcal{H}) \) be such that \( \nu(\rho) = (1, 0, \ldots, 0) \). To prove the converse statement, we appeal to *reductio ad absurdum* by assuming that \( \rho \) is a coherent state. From Prop. 14, we have that \( \nu(\rho) = \bigvee \mathcal{U}^{\text{psc}}(\rho) \), with

\[
\mathcal{U}^{\text{psc}}(\rho) = \left\{ \mu^i (|\psi\rangle\langle\psi|) : |\psi\rangle \langle\psi| \in \mathcal{O}(\rho) \right\}.
\] (B6)

According to the formula of the supremum [13], the first entry of \( \nu(\rho) \) is given by the supremum of the first entries of the vectors of \( \mathcal{U}^{\text{psc}}(\rho) \), i.e.,

\[
(\nu(\rho))_1 = \bigvee \left\{ (\mu^i (|\psi\rangle\langle\psi|))_1 : |\psi\rangle \langle\psi| \in \mathcal{O}(\rho) \right\},
\] (B7)

where

\[
(\mu^i (|\psi\rangle\langle\psi|))_1 = \max_{0 \leq i \leq d-1} |\langle i|\psi\rangle|^2.
\] (B8)

Then,

\[
(\nu(\rho))_1 = \max_{0 \leq i \leq d-1} \bigvee \left\{ |\langle i|\psi\rangle|^2 : |\psi\rangle \langle\psi| \in \mathcal{O}(\rho) \right\}.
\] (B9)

For each \( 0 \leq i \leq d-1 \), we consider the function \( f_i : \mathcal{O}(\rho) \rightarrow \mathbb{R} \), given by \( f_i(|\psi\rangle\langle\psi|) = |\langle i|\psi\rangle|^2 \). Since \( \mathcal{O}(\rho) \) is compact (see Lemma 34) and \( f_i \) is continuous, there exists a density matrix \( |\psi_i\rangle \langle\psi_i| \in \mathcal{O}(\rho) \) which is the maximum of \( f_i \) in \( \mathcal{O}(\rho) \), i.e.,

\[
f_i(|\psi_i\rangle\langle\psi_i|) = \max \left\{ |\langle i|\psi\rangle|^2 : |\psi\rangle \langle\psi| \in \mathcal{O}(\rho) \right\}.
\] (B10)

Therefore, if we define \( f_{i^*}(|\psi_{i^*}\rangle\langle\psi_{i^*}|) = \max_{0 \leq i \leq d-1} f_{i^*}(|\psi_i\rangle\langle\psi_i|) \), we have

\[
(\nu(\rho))_1 = f_{i^*}(|\psi_{i^*}\rangle\langle\psi_{i^*}|),
\] (B11)

with \( |\psi_{i^*}\rangle \langle\psi_{i^*}| \in \mathcal{O}(\rho) \). By hypothesis, \( \nu(\rho) = (1, 0, \ldots, 0) \), then \( f_{i^*}(|\psi_{i^*}\rangle\langle\psi_{i^*}|) = |\langle i^*|\psi_{i^*}\rangle|^2 = 1 \). This implies that \( |\psi_{i^*}\rangle \langle\psi_{i^*}| \) is an incoherent pure state that can be transformed into \( \rho \) by means of an incoherent operation. Therefore, \( \rho \) has to be incoherent, which is absurd by hypothesis.
Proposition 20. \( \rho \) is maximally coherent \iff \( \nu(\rho) = \left( \frac{1}{d}, \ldots, \frac{1}{d} \right) \).

Proof.

\( \implies \) Let \( \rho \in S(\mathcal{H}) \) be an arbitrary maximally coherent state, that is, \( \rho = U_{\text{IO}} |\Psi|^\text{mcs} \langle \Psi|^\text{mcs} |U_{\text{IO}}^\dagger \), with \( |\Psi|^\text{mcs} = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle \) and \( U_{\text{IO}} = \sum_{i=0}^{d-1} e^{i\theta(i)} |i\rangle \), where \( \theta(i) \in \mathbb{R} \) and \( \pi \) is a permutation acting on the set \{0, 1, \ldots, d - 1\}. Since \( \langle i | U_{\text{IO}} |\Psi|^\text{mcs} \rangle \geq 1/d \) for all \( i \in \{0, 1, \ldots, d - 1\} \), we have \( \nu(\rho) = \mu \left( U_{\text{IO}} |\Psi|^\text{mcs} \langle \Psi|^\text{mcs} |U_{\text{IO}}^\dagger \right) = \left( 1/d, \ldots, 1/d \right) \).

\( \impliedby \) Let \( \rho \in S(\mathcal{H}) \) be such that \( \nu(\rho) = \left( \frac{1}{d}, \ldots, \frac{1}{d} \right) \). Firstly, we consider the pure state case, i.e., \( \rho = |\psi\rangle \langle \psi| \). The coherence vector of \( \rho \) is given by \( \nu(\rho) = \mu^{|\psi\rangle \langle \psi|} = \left( \frac{1}{d}, \ldots, \frac{1}{d} \right) \). From Def. 2 it follows \( \langle i |\psi\rangle \rangle^2 = 1/d \) for all \( i \in \{0, 1, \ldots, d - 1\} \). Therefore, \( |\psi\rangle = U_{\text{IO}} |\Psi|^\text{mcs} \), with \( U_{\text{IO}} = \sum_{i=0}^{d-1} e^{i\theta(i)} |i\rangle \) and \( \theta(i) \in \mathbb{R} \). This implies that \( \rho \) is a maximally coherent state.

Secondly, we are going to show that \( \rho \) has to be a pure state. We appeal to reductio ad absurdum by assuming that \( \rho \) is a mixed state. Let \{\( q_k, |\psi_k\rangle \}_{1 \leq k \leq M} \) be an arbitrary pure state decomposition of \( \rho \), i.e., \( \rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| \). On the one hand, by definition of \( \nu(\rho) \), we have \( \sum_{1 \leq k \leq M} q_k \mu^{|\psi_k\rangle \langle \psi_k|} \leq (1/d, \ldots, 1/d) \). On the other hand, since \( (1/d, \ldots, 1/d) \) is the bottom of the majorization lattice, we have \( (1/d, \ldots, 1/d) \leq \sum_{1 \leq k \leq M} q_k \mu^{|\psi_k\rangle \langle \psi_k|} \). Then, \( \sum_{1 \leq k \leq M} q_k \mu^{|\psi_k\rangle \langle \psi_k|} = (1/d, \ldots, 1/d) \). Moreover, the probability vector \( (1/d, \ldots, 1/d) \) is an extreme point of the \( d - 1 \)-simplex, which implies that \( \mu^{|\psi_k\rangle \langle \psi_k|} = (1/d, \ldots, 1/d) \) for all \( k \in \{1, \ldots, M\} \). Then, states \( |\psi_k\rangle \langle \psi_k| \) have to be maximally coherent states. Therefore, we conclude that any pure state decomposition of \( \rho \) has to be formed by maximally coherent pure states.

In particular, we consider the spectral decomposition of \( \rho \),

\[ \rho = \sum_{j=1}^{d} \lambda_j |e_j\rangle \langle e_j|, \quad (B12) \]

The eigenvectors have to be maximally coherent pure states. Since, by hypothesis \( \rho \) is a mixed state, there are at least two eigenvalues different from zero. Without loss of generality, we consider \( \lambda_1, \lambda_2 > 0 \). In terms of the incoherent basis, we have \( |e_1\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle \) and \( |e_2\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle \), with \( \alpha_i, \beta_i \in \mathbb{R} \), for all \( i \in \{0, 1, \ldots, d - 1\} \).

According to the Schrödinger mixture theorem (see e.g. [33]), any ensemble \{\( p_k, |\phi_k\rangle \)\}_{1 \leq k \leq M} is a pure state decomposition of \( \rho \) if, and only if, there exist a unitary matrix \( U \) such that

\[ |\phi_k\rangle = \frac{1}{\sqrt{p_k}} \sum_{j=1}^{d} \sqrt{\lambda_j} U_{k,j} |e_j\rangle, \quad (B13) \]

We consider a \( d \times d \) unitary matrix of the form

\[ U = \begin{pmatrix} U_{11} & U_{1,2} & 0 \\ -U_{2} & U_{1,1} & 0 \\ 0 & 0 & 1 \\ \end{pmatrix} \quad (B14) \]

with \( U_{1,1} = \sqrt{\frac{\lambda_2}{\lambda_1 + \lambda_2}} \) and \( U_{1,2} = -e^{i(\alpha_0 - \beta_0)} \sqrt{\frac{\lambda_1}{\lambda_1 + \lambda_2}} \). Then,

\[ |\phi_1\rangle = \frac{1}{\sqrt{p_1}} \left( \sqrt{\lambda_1} U_{1,1} |e_1\rangle + \sqrt{\lambda_2} U_{1,2} |e_2\rangle \right), \quad (B15) \]

and, taking into account the expression of \( |e_1\rangle \) and \( |e_2\rangle \) in the incoherent basis, we obtain

\[ \langle 0 |\phi_1\rangle = \frac{1}{\sqrt{p_1}} \left( e^{i\alpha_0} \sqrt{\lambda_1} U_{1,1} + e^{i\beta_0} \sqrt{\lambda_2} U_{1,2} \right) = 0, \quad (B16) \]

which is in contradiction with \( |\phi_1\rangle \) being a maximally coherent state. Therefore, \( \rho \) cannot be a mixed state, it has to be a pure state.
Let \( \rho = \sum_{k=1}^{M} p_k |\psi_k\rangle \langle \psi_k| \). Then,

\[
\sum_{k=1}^{M} p_k \nu \left( |\psi_k\rangle \langle \psi_k| \right) \leq \nu(\rho).
\] (B17)

**Proof.** Let \( \rho = \sum_{k=1}^{M} p_k |\psi_k\rangle \langle \psi_k| \). We have \( \sum_{k=1}^{M} p_k \mu^k \left( |\psi_k\rangle \langle \psi_k| \right) = \sum_{k=1}^{M} p_k \nu \left( |\psi_k\rangle \langle \psi_k| \right) \in U^\text{psd}(\rho) \). Then, by definition of the supremum, \( \sum_{k=1}^{M} p_k \nu \left( |\psi_k\rangle \langle \psi_k| \right) \preceq \nu(\rho) \).

---

**Proposition 22** Let \( \rho \) and \( \sigma \) be two arbitrary quantum states. Then,

\[
\rho \xrightarrow{\text{IO}} \sigma \implies \forall \{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho), \exists \{r_l, |\phi_l\rangle\}_{l \in L} \in \mathcal{D}(\sigma) : \sum_{k=1}^{M} q_k \mu^k \left( |\psi_k\rangle \langle \psi_k| \right) \preceq \sum_{l=1}^{L} r_l \mu^l \left( |\phi_l\rangle \langle \phi_l| \right). \] (B18)

**Proof.** Let \( \Lambda \) be an incoherent operation, with incoherent Kraus operators \( \{K_n\}_{1 \leq n \leq N} \), such that \( \sigma = \Lambda(\rho) = \sum_{n=1}^{N} K_n \rho K_n^\dagger \). Let \( \{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \) be an arbitrary pure state decomposition of \( \rho \), that is, \( \rho = \sum_{k=1}^{M} q_k |\psi_k\rangle \langle \psi_k| \). Then, we have \( \sigma = \Lambda(\rho) = \sum_{n=1}^{N} \sum_{k=1}^{M} q_k p_{n,k} |\phi_{n,k}\rangle \langle \phi_{n,k}| \), with \( p_{n,k} = \text{Tr} (K_n |\psi_k\rangle \langle \psi_k| K_n^\dagger) \) and \( |\phi_{n,k}\rangle = K_n |\psi_k\rangle / \sqrt{p_{n,k}} \).

In particular, for each \( |\psi_k\rangle \), we have \( |\psi_k\rangle \xrightarrow{\text{IO}} \sum_{n=1}^{N} p_{n,k} |\phi_{n,k}\rangle \). Then, according to Eq. (5) (Lemma 4),

\[
\mu^k \left( |\psi_k\rangle \langle \psi_k| \right) \preceq \sum_{n=1}^{N} p_{n,k} \mu^k \left( |\phi_{n,k}\rangle \langle \phi_{n,k}| \right). \] (B19)

Applying Lemma 30 for the sequences of ordered probability vectors \( \{\mu^k \left( |\psi_k\rangle \langle \psi_k| \right)\}_{1 \leq k \leq M} \) and \( \{\sum_{n=1}^{N} p_{n,k} \mu^k \left( |\phi_{n,k}\rangle \langle \phi_{n,k}| \right)\}_{1 \leq k \leq M} \), we obtain

\[
\sum_{k=1}^{M} q_k \mu^k \left( |\psi_k\rangle \langle \psi_k| \right) \preceq \sum_{n=1}^{N} \sum_{k=1}^{M} q_k p_{n,k} \mu^k \left( |\phi_{n,k}\rangle \langle \phi_{n,k}| \right), \] (B20)

where \( q_k \geq 0 \) and \( \sum_{k=1}^{M} q_k = 1 \). Defining \( r_l = q_k p_{n,k} \), \( |\phi_l\rangle = |\phi_{n,k}\rangle \) and \( L = \{(n,k) : 1 \leq n \leq N, 1 \leq k \leq M\} \), we can rewrite expression (B20) as

\[
\sum_{k=1}^{M} q_k \mu^k \left( |\psi_k\rangle \langle \psi_k| \right) \preceq \sum_{l \in L} r_l \mu^l \left( |\phi_l\rangle \langle \phi_l| \right), \] (B21)

with \( \{r_l, |\phi_l\rangle\}_{1 \leq l \leq L} \in \mathcal{D}(\sigma) \). Since the majorization relation (B21) is valid for any pure state decomposition of \( \rho \), we conclude that for each \( \{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho) \), there exists a pure state decomposition \( \{r_l, |\phi_l\rangle\}_{1 \leq l \leq L} \in \mathcal{D}(\sigma) \), such that relation (B21) is satisfied.

---

**Corollary 23** Let \( \rho \) and \( \sigma \) be two arbitrary quantum states. Then,

\[
\rho \xrightarrow{\text{IO}} \sigma \implies \nu(\rho) \preceq \sum_{n=1}^{N} p_n \nu(\sigma_n), \] (B22)

with \( p_n = \text{Tr} (K_n \rho K_n^\dagger) \) and \( \sigma_n = K_n \rho K_n^\dagger / p_n \), where \( \{K_n\}_{1 \leq n \leq N} \) are incoherent Kraus operators such that \( \sigma = \sum_{n=1}^{N} K_n \rho K_n^\dagger \).

**Proof.** Let \( \Lambda \) be an incoherent operation, with incoherent Kraus operators \( \{K_n\}_{1 \leq n \leq N} \), such that \( \sigma = \Lambda(\rho) = \sum_{n=1}^{N} K_n \rho K_n^\dagger \), and define \( p_n = \text{Tr} K_n \rho K_n^\dagger \) and \( \sigma_n = K_n \rho K_n^\dagger / p_n \).
For any arbitrary pure state decomposition \( \{ q_k, |\psi_k\rangle \}_{1 \leq k \leq M} \) of \( \rho \), we can write \( \sigma_n = \sum_{k=1}^{M} q_k p_{n,k} |\phi_{n,k}\rangle \langle \phi_{n,k}| / p_n \), with \( p_n = \sum_k q_k p_{n,k} \). Since \( \sum_{k=1}^{M} q_k p_{n,k} \mu^k( |\phi_{n,k}\rangle \langle \phi_{n,k}| ) / p_n \in \mathcal{U}_{psd}(\sigma_n) \), then

\[
\sum_{k=1}^{M} q_k p_{n,k} \mu^k( |\phi_{n,k}\rangle \langle \phi_{n,k}| ) \leq \nu(\sigma_n).
\] (B23)

Multiplying by \( p_n \), summing over \( n \), and using \( \text{Prop. 20} \), we obtain

\[
\sum_{k=1}^{M} q_k \mu^k( |\psi_k\rangle \langle \psi_k| ) \leq \sum_{n=1}^{N} \sum_{k=1}^{M} q_k p_{n,k} \mu^k( |\phi_{n,k}\rangle \langle \phi_{n,k}| ) \leq \sum_{n=1}^{N} p_n \nu(\sigma_n).
\] (B24)

The last majorization relation does not depend on the pure state decomposition of \( \rho \), then \( \sum_{n=1}^{N} p_n \nu(\sigma_n) \) is also an upper bound of \( \mathcal{U}_{psd}(\rho) \). Therefore, by definition of supremum, we conclude that \( \nu(\rho) \leq \sum_n p_n \nu(\sigma_n) \).

\[ \square \]

**Corollary 24.** Let \( \rho \) and \( \sigma \) be two arbitrary quantum states. Then,

\[
\rho \xrightarrow{\text{IO}} \sigma \implies \nu(\rho) \leq \nu(\sigma).
\] (B25)

**Proof.** Since \( \rho \xrightarrow{\text{IO}} \sigma \), from Prop. 22, we have that, for all \( \{ q_k, |\psi_k\rangle \}_{1 \leq k \leq M} \in \mathcal{D}(\rho) \), there is a \( \{ r_l, |\phi_l\rangle \}_{l \in L} \in \mathcal{D}(\sigma) \), such that

\[
\sum_{k=1}^{M} q_k \mu^k( |\psi_k\rangle \langle \psi_k| ) \leq \sum_{l \in L} r_l \mu^l( |\phi_l\rangle \langle \phi_l| ).
\] (B26)

Then, from the definition of the supremum we have

\[
\sum_{k=1}^{M} q_k \mu^k( |\psi_k\rangle \langle \psi_k| ) \leq \nu(\sigma).
\] (B27)

This implies that \( \nu(\sigma) \) is an upper bound of the set \( \mathcal{U}_{psd}(\rho) \). Therefore, by definition of \( \nu(\rho) \), we have \( \nu(\rho) \leq \nu(\sigma) \).

\[ \square \]

**Proposition 26.** For any function \( f \in \mathcal{F} \), the coherence vector measure \( C_f^{CV} \) satisfies conditions \( [C_1] \sim [C_4] \).

**Proof.**

(C1) By Prop. 19 if \( \rho \in \mathcal{I} \), then \( \nu(\rho) = (1, 0, \ldots, 0) \). Therefore, \( C_f^{CV}(\rho) = f(1, 0, \ldots, 0) = 0 \).

(C2) Since \( \rho \xrightarrow{\text{IO}} \Lambda(\rho) \), from Cor. 24 we obtain \( \nu(\rho) \leq \nu(\Lambda(\rho)) \). Moreover, \( f \) is symmetric and concave, then \( f \) is also Schur-concave, which implies that \( f(\nu(\rho)) \geq f(\nu(\Lambda(\rho))) \). Finally, we conclude that \( C_f^{CV}(\rho) \geq C_f^{CV}(\Lambda(\rho)) \).

(C3) Let \( \rho \in \mathcal{S}(\mathcal{H}) \) be an arbitrary quantum state and \( \Lambda \) an incoherent operation, with incoherent Kraus operators \( \{ K_n \}_{1 \leq n \leq N} \), and \( p_n = \text{Tr} K_n \rho K_n^\dagger \) and \( \sigma_n = K_n \rho K_n^\dagger / p_n \). If we define \( \sigma = \Lambda(\rho) \), from Cor. 23 Eq. 19, we obtain \( \nu(\rho) \leq \sum_{n=1}^{N} p_n \nu(\sigma_n) \). Then, we have

\[
\sum_{n=1}^{N} p_n f(\nu(\sigma_n)) \leq f\left( \sum_{n=1}^{N} p_n \nu(\sigma_n) \right) \leq f(\nu(\rho)),
\]

where in the first inequality we have used the concavity of \( f \) and in the second one the Schur-concavity. Finally, taking into account the coherence vector definition, we conclude

\[
\sum_{n=1}^{N} p_n C_f^{CV}(\sigma_n) \leq C_f^{CV}(\rho).
\]

(C4) By Prop. 20 if \( \rho \) is a maximally coherent state, then \( \nu(\rho) = (1/d, \ldots, 1/d) \). Therefore, \( C_f^{CV}(\rho) = f(1/d, \ldots, 1/d) = 1 \).
Proposition 27 \textit{The following statements are equivalent:}

1. There exists an optimal pure state decomposition of $\rho$, i.e., $\nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho)$.
2. $C_f^{cv}(\rho) = C_f^{\text{top}}(\rho)$ for all $f \in \mathcal{F}$.
3. $C_f^{cv}(\rho) = C_f^{\text{top}}(\rho)$ for some $f \in \mathcal{F}$ strictly Schur-concave.

\textit{Proof.}

$(1. \implies 2.)$ Let $f \in \mathcal{F}$. On the one hand, by relation \eqref{eq:concavity}, we have $C_f^{\text{top}}(\rho) \geq C_f^{cv}(\rho)$. On the other hand, if there exists an optimal pure state decomposition of $\rho$, then $\nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho)$. From Lemma 32, we have $\nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho)$. By definition of the top measure, $C_f^{\text{top}}(\rho) \leq f(u)$ for all $u \in \mathcal{U}^{\text{psd}}(\rho)$. In particular, $C_f^{\text{top}}(\rho) \leq f(\nu(\rho)) = C_f^{cv}(\rho)$. Finally, we conclude $C_f^{cv}(\rho) = C_f^{\text{top}}(\rho)$, which is valid for all $f \in \mathcal{F}$.

$(2. \implies 3.)$ Trivial.

$(3. \implies 1.)$ Let $f \in \mathcal{F}$ be a strictly Schur-concave function such that $C_f^{cv}(\rho) = C_f^{\text{top}}(\rho)$.

Notice that

$$C_f^{\text{top}}(\rho) = \min_{\psi \in \mathcal{O}(\rho)} f(\mu(\psi) \langle \psi |) = \inf_{u \in \mathcal{U}^{\text{psd}}(\rho)} f(u). \quad (B28)$$

We denote the probability vector where the minimum is reached as $\tilde{u}$. Then, $f(\nu(\rho)) = C_f^{cv}(\rho) = C_f^{\text{top}}(\rho) = f(\tilde{u})$. Since $f$ is strictly Schur-concave, then by Lemma 31, we have $\nu(\rho) = \tilde{u} \in \mathcal{U}^{\text{psd}}(\rho)$. Finally, by Lemma 18, $\nu(\rho) \in \mathcal{U}^{\text{psd}}(\rho)$, i.e., there exists an optimal pure state decomposition of $\rho$.

Proposition 28 \textit{If there exists an optimal pure state decomposition of $\rho$, then $C_f^{cv}(\rho) \geq C_f^{ct}(\rho)$}.

\textit{Proof.} Let $\{\tilde{q}_k, |\tilde{\psi}_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho)$ be the be an optimal pure state decomposition of $\rho$. Thus, $\nu(\rho) = \sum_{k=1}^{M} \tilde{q}_k|\tilde{\psi}_k\rangle \langle \tilde{\psi}_k|$. Let $f \in \mathcal{F}$, then

$$C_f^{cv}(\rho) = f(\nu(\rho)) = f \left( \sum_{k=1}^{M} \tilde{q}_k \mu(\tilde{\psi}_k) \langle \tilde{\psi}_k | \right) \quad (B29)$$

$$\geq \sum_{k=1}^{M} \tilde{q}_k f \left( \mu(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |) \right) \quad (B30)$$

$$\geq \inf_{\{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho)} \sum_{k=1}^{M} q_k f \left( \mu(|\psi_k \rangle \langle \psi_k |) \right) = C_f^{ct}(\rho), \quad (B31)$$

where the first inequality comes from the concavity and symmetric properties of $f$, and the second one comes from the definition of the convex roof measure.

Proposition 29 \textit{Let $f \in \mathcal{F}$ be such that $f|_{\Delta^+} = c + \ell$, where $c \in \mathbb{R}$ and $\ell : \Delta^+ \to \mathbb{R}$ is a linear function. Then, $C_f^{cv} \leq C_f^{ct} = C_f^{\text{top}}$.}

\textit{Proof.} Let $\rho \in \mathcal{S}(\mathcal{H})$. On the one hand, by definition of the convex roof measure, we have

$$C_f^{ct}(\rho) = \inf_{\{q_k, |\psi_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho)} \sum_{k=1}^{M} q_k f (\mu(|\psi_k \rangle \langle \psi_k |)) = \sum_{k=1}^{M} \tilde{q}_k f \left( \mu^+(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |) \right), \quad (B32)$$

with $\{\tilde{q}_k, |\tilde{\psi}_k\rangle\}_{1 \leq k \leq M} \in \mathcal{D}(\rho)$ the pure state decomposition of $\rho$ where the minimum is reached. Taking into account the form of $f$, we get

$$C_f^{ct}(\rho) = \sum_{k=1}^{M} \tilde{q}_k \left( c + \ell \left( \mu^+(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |) \right) \right) = c + \ell \left( \sum_{k=1}^{M} \tilde{q}_k \mu^+(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |) \right) = \sum_{k=1}^{M} \tilde{q}_k \mu^+(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |), \quad (B33)$$

where $\mu^+(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |) = \sum_{k=1}^{M} \mu(|\tilde{\psi}_k \rangle \langle \tilde{\psi}_k |)$.
where we have used the linearity of $\ell$ and the condition $\sum_{k=1}^{M} q_k = 1$.

On the other hand, by definition of $\nu(\rho)$ and Schur-concavity of $f$, we have

$$C_f^{cv}(\rho) = f(\nu(\rho)) \leq f\left(\sum_{k=1}^{M} q_k \mu^k(\psi_k)\psi_k\right), \quad \forall\{q_k, \psi_k\}_{1 \leq k \leq M} \in D(\rho). \quad \text{(B34)}$$

In particular,

$$C_f^{cv}(\rho) \leq f\left(\sum_{k=1}^{M} \tilde{q}_k \mu^k(\tilde{\psi}_k)\tilde{\psi}_k\right). \quad \text{(B35)}$$

Therefore, we conclude $C_f^{cv}(\rho) \leq C_f^{ct}(\rho)$.

In order to prove the equality part of the proposition, first we note that $C_f^{ct}(\rho) \leq C_f^{top}(\rho)$, see Ineq. (13). Moreover, by definition of the top measure, we have

$$C_f^{top}(\rho) \leq f(\mu^k(\psi)\psi), \quad \forall\psi \in O(\rho). \quad \text{(B36)}$$

Since $U^{res}(\rho) \subseteq U^{psc}(\rho)$ (see Lemma 32), we have that $\sum_{k=1}^{M} \tilde{q}_k \mu^k(\tilde{\psi}_k)\tilde{\psi}_k \in U^{psc}(\rho)$. Then, there is $\tilde{\psi} \in O(\rho)$, such that $\mu^k(\tilde{\psi})\tilde{\psi} = \sum_{k=1}^{M} \tilde{q}_k \mu^k(\tilde{\psi}_k)\tilde{\psi}_k$. Therefore,

$$C_f^{top}(\rho) \leq f(\mu^k(\tilde{\psi})\tilde{\psi}) = f\left(\sum_{k=1}^{M} \tilde{q}_k \mu^k(\tilde{\psi}_k)\tilde{\psi}_k\right). \quad \text{(B37)}$$

Then, from (B33) and (B37), we have $C_f^{top}(\rho) \leq C_f^{ct}(\rho)$. Finally, we conclude that $C_f^{top}(\rho) = C_f^{ct}(\rho)$. \hfill \qed

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