Infinite symmetric groups and combinatorial constructions of topological field theory type

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Abstract. This paper contains a survey of train constructions for infinite symmetric groups and related groups. For certain pairs (a group $G$, a subgroup $K$) categories are constructed whose morphisms are two-dimensional surfaces tiled by polygons and coloured in a certain way. A product of morphisms is a gluing together of combinatorial bordisms, and functors from the category of bordisms to the category of Hilbert spaces and bounded operators correspond to unitary representations of $G$. The construction has numerous variations: instead of surfaces there can also be one-dimensional objects of Brauer diagram type, multidimensional pseudomanifolds, and bipartite graphs.

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1. Introduction

1.1. Infinite symmetric groups and topological field theories. The representation theory of infinite symmetric groups $S_\infty$ was initiated by two ‘orthogonal’ papers. The first was Thoma’s 1964 paper [1], where he introduced analogues of characters for the group $S_\infty$ of finitely supported infinite permutations. The second was Lieberman’s 1972 paper [2], where he classified all unitary representations of the complete infinite symmetric group.

Initially, Thoma introduced characters of the group $S_\infty$ as the traces of operators of representations on type II$_1$ Murray–von Neumann factors. Below we mention the term ‘factors’ several times, but actually these objects will not be used and are not necessary for understanding the paper. Later, Vershik and Kerov [3] explicitly constructed on factors the representations of $S_\infty$ corresponding to Thoma characters (see, also [4]) and showed how to obtain Thoma characters as limits of characters of finite symmetric groups [5].

In 1989 Olshanski [6] noted that Thoma characters correspond to representations of the double $S_\infty \times S_\infty$ that have fixed vectors with respect to the diagonal subgroup. In the same paper he obtained the first example of a (one-dimensional) topological field theory from representations of infinite symmetric groups (the meaning of this remark is explained below). One recent achievement in the representation theory of infinite symmetric groups is the explicit decomposition in [7] of an analogue of regular representations for the double $S_\infty \times S_\infty$.

Now something about the term ‘topological field theory’ in the sense of the definition proposed by Atiyah in [8] (see also the recent survey [9]). Consider the category
whose objects are \((k - 1)\)-dimensional manifolds, and in which a morphism from \(M\) to \(N\) is a \(k\)-dimensional manifold \(R\) whose boundary is identified with the disjoint union \(M \sqcup N\). The product of two morphisms \(R: M \to N\) and \(Q: N \to K\) is the result of gluing \(R\) and \(Q\) together along the manifold \(N\). A topological field theory is defined to be a functor from this category to the category of linear spaces and linear operators.\(^1\)\(^2\) The content of the present paper involves the construction of various combinatorial objects of ‘topological field theory’ type from representations of infinite symmetric groups [6], [15]–[17].

This paper is more or less self-contained, though a minimal familiarity of the reader with group representation theory is assumed.

1.2. The infinite symmetric group. By \(S_\infty\) we denote the group of all finitely supported permutations of a countable set. By default we usually assume that the countable set is the set of natural numbers \(\mathbb{N}\). If we need another countable set \(\Omega\), we write \(S_\infty(\Omega)\). The group \(S_\infty\) is the inductive limit of the finite symmetric groups \(S_n\):

\[
S_\infty := \lim_{\rightarrow} S_n.
\]

Also, \(S_\infty\) can be realized as the group of infinite invertible 0-1 matrices\(^3\) \(g\) such that \(g_{kk} = 1\) for sufficiently large \(k\). We will represent permutations (finite or infinite) by pictures of the type

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

The group \(S_\infty\) is a countable discrete group, not a type \(I\) group (see, for instance, [18], [19]), and a unitary representation theory in the usual sense for this group is impossible.

Let \(\alpha = 0, 1, 2, \ldots\), and denote by \(S_\infty[\alpha]\) the subgroup of \(S_\infty\) consisting of permutations fixing the points \(1, \ldots, \alpha\) (for \(\alpha = 0\) we have \(S_\infty[0] = S_\infty\)). This subgroup consists of 0-1 matrices of the form

\[
g = \begin{pmatrix} 1_\alpha & 0 \\ 0 & r \end{pmatrix}, \quad \text{where } r \in S_\infty
\]

(1\(\alpha\) is the identity matrix of order \(\alpha\)). We get a chain of subgroups

\[
S_\infty = S_\infty[0] \supset S_\infty[1] \supset S_\infty[2] \supset \cdots
\]

\(^1\)The term ‘bordisms’ is used in this context. However, the theory of bordisms is an older area of mathematics (see, for instance, [10]) and conflicts with the well-established terminology easily arise here. Apparently, the term ‘field theory’ was introduced by analogy with ‘conformal quantum field theories’; see [11], [12]).

\(^2\)Two-dimensional ‘topological field theories’ in which surfaces are tiled by polygons were considered in [13] and [14].

\(^3\)By a 0-1 matrix we mean a matrix such that all the matrix elements in it are 0 or 1, and each column and each row contains at most one 1. Such a matrix is invertible if and only if each column and each row contains precisely one 1.
All the subgroups $S_\infty[\alpha]$ are canonically isomorphic to the group $S_\infty$, and in the notation (1.1) the isomorphism is given by $r \mapsto g$.

1.3. Complete infinite symmetric group. By $\mathcal{S}_\infty$ we denote the group of all permutations of a countable set, and by $\mathcal{S}_\infty(\Omega)$ the group of all permutations of a countable set $\Omega$. As above, we define the subgroups $\mathcal{S}_\infty[\alpha]$ consisting of matrices of the form
\[
\begin{pmatrix}
1 & 0 \\
0 & r
\end{pmatrix},
\]
where $r \in \mathcal{S}_\infty$.

We define a totally disconnected topology on $\mathcal{S}_\infty$, assuming that the subgroups $\mathcal{S}_\infty[\alpha]$ are open. We give two equivalent definitions of this topology:

- $g_j \to g$ in $\mathcal{S}_\infty$ if for any $k \in \mathbb{N}$ we have $g_j(k) = g(k)$ starting at some $j$;
- $g_j \to g$ if the sequence of corresponding 0-1 matrices converges weakly.\(^4\)

The group $\mathcal{S}_\infty$ is a complete (in the Raikov sense\(^5\)) topological group with respect to our topology.\(^6\)

In 1972 Lieberman [2] obtained a complete classification of unitary representations of the group $\mathcal{S}_\infty$. This classification is not complicated: all irreducible representations are induced from representations $\rho \otimes I$ of subgroups $S(\alpha) \times \mathcal{S}_\infty[\alpha]$, where $\rho$ is irreducible and $I$ is the trivial one-dimensional representation of $\mathcal{S}_\infty[\alpha]$ (for further details, see §2 below). On the other hand, in a certain sense this classification is not too interesting since the representation theory of $\mathcal{S}_\infty$ contains nothing essentially new in comparison with the representation theory of finite symmetric groups.

A representation theory of the infinite symmetric group does exist, however, but it is neither a unitary representation theory of $\mathcal{S}_\infty$ nor a representation theory of $\mathcal{S}_\infty$. A key step was the 1964 paper [1] of Thoma.

1.4. The Thoma theorem. Let $f$ be a complex-valued function on a countable group $G$. It is said to be central if it is constant on conjugacy classes. It is positive definite if for any finite collection $g_j \in G$ the matrix with elements $f(g_j g_k^{-1})$ is positive semidefinite. The set of central positive-definite functions is a convex cone. By $\mathcal{K}(G)$ we denote the set of such functions satisfying the condition $f(1) = 1$. The set $\mathcal{K}(G)$ is convex and compact with respect to the topology of pointwise convergence. By the Krein–Milman theorem (see, for instance, [20]), $\mathcal{K}(G)$ is the convex hull of its set of extreme points.

Definition 1.1 [1]. A character of the group $G$ is an extreme point of the set $\mathcal{K}(G)$.

Note that for finite or compact groups, ‘characters’ defined in this way have the form $\chi(g)/\dim \chi$, where $\chi(g)$ is a usual irreducible character of $G$.

\(^4\)See, for instance, [20]. In our case this means elementwise convergence of matrices, that is, stabilization.

\(^5\)That is, any two-sided fundamental sequence converges, where a sequence $g_j$ is said to be two-sided fundamental if the double sequences $g_j g_k^{-1}$ and $g_k^{-1} g_j$ both tend to 1 as $j, k \to \infty$. This definition differs from Bourbaki’s definition of completeness, which requires the convergence of left-fundamental sequences and right-fundamental sequences; see [21], [22].

\(^6\)This is the unique separable metrizable topology on $\mathcal{S}_\infty$ [23]. Recall that a topological group is said to be Polish if it is metrizable, separable, and complete (another version of the definition: if its topology is Polish). For Polish groups there is a wide collection of theorems on automatic continuity of measurable homomorphisms, on automatic continuity of arbitrary homomorphisms, and on uniqueness of Polish topologies (see [24], [25] and references in the latter).
Consider a collection of non-negative real numbers $\alpha_j, \beta_j, \gamma$ such that

$$\alpha_1 \geq \alpha_2 \geq \cdots, \quad \beta_1 \geq \beta_2 \geq \cdots, \quad \sum \alpha_j + \sum \beta_j + \gamma = 1.$$ 

For such a collection of parameters we define a function $\chi_{\alpha, \beta, \gamma}(g)$ on $S_\infty$ by the formula

$$\chi_{\alpha, \beta, \gamma}(g) = \prod_{k=2}^{\infty} \left( \sum_j \alpha_j^k + (-1)^{j-1} \sum_j \beta_j^k \right)^{r_k(g)},$$

where $r_k(g)$ is the number of cycles of length $k$ in a permutation $g$ (the product on the right-hand side is actually finite).

**Theorem 1.2** (Thoma [1]). The functions $\chi_{\alpha, \beta, \gamma}(g)$ are characters of the group $S_\infty$, and all the characters of $S_\infty$ have this form.

There are two equivalent but different ways to define a representation corresponding to a character.

The first way (which Thoma had in mind): these expressions have the form $\text{tr} \rho(g)$ for representations of a symmetric group that generate Murray–von Neumann factors of type II$_1$. We do not explain here what this means (see, for instance, [18], §§5 and 13.1, or [26]), but we note only that explicit constructions of such factor-representations are contained in [3] and [4], and we note also that Thoma characters are limits of irreducible characters of finite symmetric groups (see [5]).

The second way (our starting point in this paper) was proposed by Olshanski in [6]. Thoma characters correspond to unitary representations of the double of the symmetric group. Let us look at the double.

**1.5. The bisymmetric group.** Consider a Thoma character $\chi_{\alpha, \beta, \gamma}(g)$ and a collection of formal vectors $v_g$ labelled by elements $g$ of the group $S_\infty$. We define an inner product for these vectors by the formula

$$\langle v_g, v_h \rangle := \chi_{\alpha, \beta, \gamma}(gh^{-1}).$$

Since the function $\chi_{\alpha, \beta, \gamma}$ is positive definite, a system of vectors with such inner products can be realized in a certain Hilbert space $\mathcal{H}_{\alpha, \beta, \gamma}$. It is natural to assume that the vectors $v_g$ generate $\mathcal{H}_{\alpha, \beta, \gamma}$. Such a collection of vectors is unique in the following sense: if $v_g'$ is another collection of vectors with the same inner products, then there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that $U v_g = v_g'$ (this is the standard construction of a Hilbert space from a reproducing kernel; see, for instance, [27], §7.1).

Further, we note that for any $p, q \in S_\infty$

$$\langle v_{pg_1q^{-1}}, v_{pg_2q^{-1}} \rangle = \chi((pg_1q^{-1})(pg_2q^{-1})^{-1})^{-1} \chi((pg_1g_2^{-1}p^{-1})^{-1}) = \chi(g_1g_2^{-1}) = \langle v_{g_1}, v_{g_2} \rangle,$$

that is, the transformation $v_g \mapsto v_{pg^{-1}}$ does not change the matrix of inner products. Therefore, there exists a unitary operator $T(p, q)$ on $\mathcal{H}_{\alpha, \beta, \gamma}$ such that

$$T(p, q) v_g = v_{pq^{-1}}.$$
for all $g \in S_\infty$. It is easy to see that $T(p, q)$ is a unitary representation of the double of $S_\infty$, that is, of the group $S_\infty \times S_\infty$. Also, the vector $v_e$ is fixed by the diagonal $\text{diag}(S_\infty)$ of $S_\infty \times S_\infty$.

An explicit construction of the representations of the double $S_\infty \times S_\infty$ that correspond to Thoma characters is given below in §4.3.

**Remark.** Let $\chi$ be a character of an irreducible representation $\rho$ of a finite or compact group $G$ on a space $W$. Applying the indicated construction to $\chi$, we get a representation $T$ of the group $G \times G$ on the space $\text{Hom}(W)$ of linear operators on $W$, given by the formula

$$T(g_1, g_2)A = \rho(g_1^{-1})A\rho(g_2).$$

The inner product in $\text{Hom}(W)$ is given by $\langle A, B \rangle = \text{tr} AB^*$. The vector fixed by the diagonal subgroup is the identity operator on $W$, and $v_g$ coincides with $\rho(g) \in \text{Hom}(W)$.

We recall some definitions from representation theory.

**Definition 1.3.** Let $G$ be a group and $K$ a subgroup.

1) An irreducible unitary representation $\rho$ of a group $G$ on a Hilbert space $H$ is said to be spherical with respect to $K$ or $K$-spherical if there exists in $H$ a non-zero $K$-fixed vector that is unique up to a scalar factor.

2) With the same notation, the function

$$\phi(g) = \langle \rho(g)v, v \rangle$$

is called a spherical function.

3) If for any irreducible unitary representation $\rho$ of $G$ there exists at most one (up to a scalar factor) $K$-fixed vector, then the pair $(G, K)$ is said to be spherical.$^7$,$^8$

**Theorem 1.4.** a) The pair $(S_\infty \times S_\infty, \text{diag}(S_\infty))$ is spherical.

b) The spherical functions on $S_\infty \times S_\infty$ are the characters $\chi_{\alpha, \beta, \gamma}(g_1 g_2^{-1}).$

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$^7$For unitary representations of Lie groups the main statement about sphericity is the following theorem due to Gelfand (1950): *If* $G/K$ *is a Riemannian symmetric space, then the subgroup $K$ is spherical in $G$.* Other spherical pairs were later found, obtained from Gelfand pairs by enlarging $G$ slightly or diminishing $K$ slightly (on this topic the paper [28] contains diverse continuations).

There is a theory of infinite-dimensional spherical pairs that can be obtained as limits of Gelfand (or almost Gelfand) pairs (see Olshanski’s papers [6] and [29]). Nessonov [30] later observed that spherical subgroups of infinite-dimensional groups can be unexpectedly ‘small’, and moreover, it is possible to work with such spherical pairs. In the present paper we consider numerous spherical pairs which have no finite-dimensional analogues (on classical groups see [31], and on the group of diffeomorphisms of the circle see [32]).

$^8$Strictly speaking, we should add a further requirement to this definition: for any $g \notin K$ there is a spherical representation $\rho$ with a $K$-spherical vector $v$ such that $\rho(g)v \neq v$. Let us explain what is the matter here.

Consider an action of the group $\mathbb{Z}$ on a countable set with trivial stabilizers. Thus, we get an embedding $\mathbb{Z} \to \widetilde{S}_\infty$. The Lieberman theorem implies that a $\mathbb{Z}$-fixed vector in a unitary representation of $\widetilde{S}_\infty$ is automatically $\widetilde{S}_\infty$-fixed. Therefore, for all spherical pairs of the form $G \supset \widetilde{S}_\infty$ (this paper contains many examples of such pairs), $\mathbb{Z}$-fixed vectors are fixed by the subgroup $\widetilde{S}_\infty$. 
Remark. It is known that for many classes of groups (finite and compact groups, semisimple real and semisimple $p$-adic groups, nilpotent Lie groups) irreducible unitary representations of the direct product $G_1 \times G_2$ are precisely tensor products $\rho_1 \otimes \rho_2$ of irreducible representations of $G_1$ and $G_2$. Such theorems hold if at least one group has type I (see [18], Proposition 13.1.8). For $G_1 = G_2 = S_\infty$ this is not the case. The Thoma representations of the group $S_\infty \times S_\infty$ are not tensor products (except for the cases $\alpha_1 = \beta_1 = 1$, which correspond to one-dimensional representations). There is no pathology in this phenomenon.

Remark. Let us restrict a Thoma representation of the double to the first factor. Consider the weakly closed operator algebra generated by the operators of the representation. It turns out that this algebra is a $\text{II}_1$-factor. The trace on this factor is

$$\text{tr}(A) := \langle \rho(A) v_e, v_e \rangle.$$ 

We recall once more that the language of factors is not used in this paper.

Remark. Let $r$ be an element of the complete symmetric group $S_\infty$. Obviously, for all $g_1, g_2 \in S_\infty$ we have

$$\langle e_{rg_1r^{-1}}, e_{rg_2r^{-1}} \rangle = \langle e_{g_1}, e_{g_2} \rangle.$$ 

This means that the operator $T(r, r)$ is defined for all $r \in S_\infty$.

Define the bisymmetric group $G$ to be the subgroup of $S_\infty \times S_\infty$ consisting of the pairs $(g, h)$ such that $gh^{-1} \in S_\infty$. We get the following result.

**Proposition 1.5.** Each Thoma representation extends to a unitary representation of the bisymmetric group $G$.

It turns out that the bisymmetric group is a ‘good object’. It is a type I group (Olshanski [6]), its unitary representations have been classified (Okounkov [33]; see Theorem 4.3 below), and the corresponding harmonic analysis [7] is substantial. Also, there is the following phenomenon, which has no finite-dimensional analogues.

1.6. Multiplication of double cosets and the train category. Let $G$ be a group and $K_1, K_2$ subgroups of it. A double coset is a subset in $G$ of the form $K_1 g K_2$. By $K_1 \setminus G / K_2$ we denote the set of double cosets, that is, the quotient set of $G$ with respect to the equivalence relation $g \sim k_1 g k_2$, where $k_1 \in K_1$ and $k_2 \in K_2$.

In the diagonal $S_\infty$ of $G$ we consider the subgroups $K[\alpha] := S_\infty[\alpha]$. Now we intend to define a product

$$K[\alpha] \setminus G / K[\beta] \times K[\beta] \setminus G / K[\gamma] \rightarrow K[\alpha] \setminus G / K[\gamma].$$

To this end, consider the following sequence $\theta_j[\beta]$ in $K[\beta]$: 

$$\theta_j[\beta] = \begin{pmatrix} 1_\beta & 0 & 0 & 0 \\ 0 & 0 & 1_j & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}. \quad (1.2)$$
We take double cosets

\[ p \in K[\alpha] \setminus G/K[\beta], \quad q \in K[\beta] \setminus G/K[\gamma] \]

and representatives \( p \in p \) and \( q \in q \) of them. Consider the sequence of double cosets

\[ r_j := K[\alpha] \cdot p \theta_j [\beta] q \cdot K[\gamma] \in K[\alpha] \setminus G/K[\gamma]. \]

**Lemma 1.6.** The sequence \( r_j \) is eventually constant.

This lemma can easily be proved by multiplication of 0-1 matrices.

We can now define the product \( p \circ q \) as

\[ p \circ q = r_N \quad \text{for sufficiently large } N. \]

It is easy to verify (also by multiplication of matrices) that the multiplication is associative. In other words, we get a category \( \mathcal{S} = \mathcal{S}(G, K) \), the train of the pair \((G, K)\). Its objects are non-negative integers, and the sets of morphisms are

\[ \text{Mor}_{\mathcal{S}}(\alpha, \beta) = K[\alpha] \setminus G/K[\beta]. \]

The category \( \mathcal{S} \) admits a transparent combinatorial description (which was in fact obtained in [6]; see §4.2 below).

**1.7. Multiplicativity. Representations of the train category.** Consider a unitary representation \( \rho \) of the bisymmetric group \( G \) on a Hilbert space \( H \) (the representation is assumed to be continuous on the subgroup \( K \)). Denote by \( H[\alpha] \) the subspace of \( K[\alpha] \)-fixed vectors and by \( P[\alpha] \) the operator of orthogonal projection onto \( H[\alpha] \). We consider a double coset \( p \in K[\alpha] \setminus G/K[\beta] \) and define an operator

\[ \tilde{\rho}(p) : H[\beta] \to H[\alpha]. \]

To do this we choose a representative \( p \in p \) and let

\[ \tilde{\rho}(p) := P[\alpha] \rho(p) |_{H[\beta]}. \]

**Lemma 1.7.** The operator \( \tilde{\rho}(p) \) does not depend on the choice of a representative \( p \in p \).

**Proof.** Let \( v \in H[\alpha] \), \( w \in H[\beta] \) and let \( h_1 \in K[\alpha] \), \( h_2 \in K[\beta] \). Then

\[ \langle v, \rho(h_1ph_2)w \rangle = \langle \rho(h_1^{-1})v, \rho(p)\rho(h_2)w \rangle = \langle v, \rho(p)w \rangle. \]

It turns out that we get a representation of the category \( \mathcal{S} \).

**Theorem 1.8** (Multiplicativity theorem). For any \( \alpha, \beta, \gamma \) and any \( p \in K[\alpha] \setminus G/K[\beta] \) and \( q \in K[\beta] \setminus G/K[\gamma] \),

\[ \tilde{\rho}(p \circ q) = \tilde{\rho}(p) \circ \rho(q). \]
We note also that the operators $\rho(p)$ are contractive:

$$\|\rho(p)\| \leq 1.$$  \hspace{1cm} (1.3)

We define the *involution* $p \rightarrow p^*$,

$$K[\alpha] \setminus G/K[\beta] \rightarrow K[\beta] \setminus G/K[\alpha],$$

induced on the category $\mathcal{S}$ by the map $p \mapsto p^{-1}$. Obviously,

$$(p \circ q)^* = q^* \circ p^*.$$  \hspace{1cm} (1.4)

**Theorem 1.9.** The map $\rho \mapsto \tilde{\rho}$ is a bijection between the set of unitary representations of the bisymmetric group $G$ and the set of contractive *-representations of the category $\mathcal{S}$. 

1.8. The goal of this paper. All the topics and statements discussed above were known by the mid-1990s ([1], [3], [5], [6], [34]). The theory of the Thoma factor-representations and representations of the bisymmetric group were regarded as a limit of the classical representation theory. In [34] it was observed that multiplication of double cosets and the multiplicativity theorem are highly general phenomena for infinite-dimensional groups.\(^9\) The main obstacle was a description of double cosets, which for symmetric groups was obtained in [15]–[17]. The main topic of the present paper is a geometric description of various train categories in terms of ‘topological field theories’.

1.9. Structure of the paper. Section 2 contains a proof of the Lieberman theorem.

The main ‘pattern’ construction is described in §3. For the group $G = S_\infty \times S_\infty \times S_\infty$ and its subgroup $K = \text{diag}(S_\infty)$ we construct the category of double cosets. We show that morphisms are described by two-dimensional surfaces with checkerwise coloured triangulations and coloured edges. A product of morphisms is a gluing together of surfaces. We obtain a correspondence (in both directions) between unitary representations of $G$ and representations of the category of triangulated surfaces. A family of $K$-spherical representations of $G$ is constructed and a conjecture is formulated about a description of all spherical representations. In §3 proofs also are ‘pattern’ proofs which can be applied for a wide class of pairs $(G, K)$ without essential changes. Therefore, in §§4–7 we concentrate on combinatorial-geometric descriptions of trains (categories of double cosets).

\(^9\)A first example of such a multiplication was described by Ismagilov in [35]. Now constructions analogous to the present paper exist for infinite-dimensional classical groups (see [29], [31], [36], [37]), for infinite-dimensional $p$-adic groups (see [38]), and for groups of automorphisms of measure spaces (see [39]–[41]).
In §4 we return to representations of the bisymmetric group. In this case the combinatorial structure is reduced to one-dimensional objects which were described in [6]. Also, we consider other examples of pairs \((G, K)\) whose trains can be described in terms of ‘one-dimensional manifolds’.

For the bisymmetric group Olshanski and Okounkov obtained for unitary representations a complete classification whose perception requires a greater familiarity with representation theory than does the main text of this paper, and so the classification is expounded in an addendum to §4 (see §4.7).

Some pairs \((G, K)\) producing two-dimensional theories are discussed in §5 (here there is a wide zoo described in [16]).

Next we consider wreath products \(S_\infty \wr (S_\infty^m)\) as \(K\), and as \(G\) we take (for instance) the \(m\)-times enlarged infinite symmetric group. Then we get a category whose morphisms are described in terms of \(m\)-valent bipartite graphs.

After bisymmetric and trisymmetric groups it is natural to consider the \(n\)-fold product \(G = S_\infty \times \cdots \times S_\infty\) with the subgroup \(K = \text{diag}(S_\infty)\). The corresponding train admits (at least) two apparently different geometric descriptions. One variant is given in terms of two-dimensional surfaces tiled by \(n\)-gons, and we briefly mention it in §3.18. The same category can be described in terms of bordisms of \((n - 1)\)-dimensional pseudomanifolds: this is the content of §7.

In §8 we consider the pair consisting of the infinite symmetric group \(G\) and a Young subgroup \(K\). In this case, an application of a general technique leads to two-dimensional surfaces tiled by monogons (of course there is a description of a train in less exotic terms). This is the simplest representative of our zoo, and here we can achieve a better understanding. We present a proof of the formula obtained by Nessonov for \(K\)-spherical functions on \(G\).

1.10. Remarks concerning §1. a) Different proofs of the Thoma theorem were obtained in [1], [5], [33].

b) The problem of characters of Thoma type can be formulated for an arbitrary group \(G\). The interesting theory is obtained for infinite unitary groups, orthogonal groups, and symplectic (quaternionic unitary) groups (see [42] and [43]). Characters can be regarded as characters of factor-representations or as spherical functions on the double.

c) For a discrete group \(S_\infty\) it is easy to construct innumerable families of unitary representations (see, for instance, [44] or Appendix F.4 in the Russian edition of [34]). It seems that attempts to extend such activity further than a mass production of unitary representations have failed.

d) In the papers [1]–[6] mentioned at the beginning of this paper, the representation theory of infinite symmetric groups arises in one way or another as a limit from the representation theory of finite symmetric groups. Almost all constructions in the present paper (except for the topic with the bisymmetric group and the Thoma representations) are not such limits. Therefore, we get an informal problem about ‘descent down’: what our constructions (for instance, various ways of ‘geometrical encoding’ of symmetric groups) can give on the level of the classical theory?
2. The Lieberman theorem

In this section we prove Lieberman’s classification theorem (Theorem 2.8). For the further exposition we need not the theorem itself but Theorem 2.4 from [45] on extensions of unitary representations of \( S_\infty \) to the semigroup of all 0-1 matrices (these two theorems are easily derived from each other). Moreover, we refer to the proofs of several of the lemmas in this section as ‘pattern’ arguments.

2.1. Equivalent definitions of continuity. Consider a unitary representation \( \rho \) of the group \( S_\infty \) on a Hilbert space \( H \). As above, let \( H[\alpha] \) denote the subspace of all \( S_\infty[\alpha] \)-fixed vectors.

**Proposition 2.1.** a) The following conditions are equivalent:

1. \( \rho \) admits an extension to a continuous representation\(^{10} \) of the group \( S_\infty \);
2. \( \bigcup_\alpha H[\alpha] \) is dense in \( H \).

b) If \( \rho \) is irreducible, then these conditions are equivalent to the following condition:

3. \( \bigcup_\alpha H[\alpha] \neq 0 \).

**Proof.** a) The non-obvious statement is \((*) \Rightarrow (**)\). Let \( v \in H, \|v\| = 1 \). Denote by \( W[\alpha] \) the closed convex hull of the orbit \( S_\infty[\alpha] \cdot v \). Since the subgroups \( S_\infty[\alpha] \) form a fundamental system of neighbourhoods of the identity, for each \( \varepsilon > 0 \) there exists an \( \alpha \) such that for each \( g \in S_\infty[\alpha] \) we have \( \langle gv, v \rangle > 1 - \varepsilon \), and therefore \( \|\rho(g)v - v\| < \sqrt{2\varepsilon} \). Hence, for each \( w \in W[\alpha] \) we have \( \|w - v\| \leq \sqrt{2\varepsilon} \). On the other hand, any closed convex subset in a Hilbert space contains a unique element of minimum length. Denote such an element of \( W[\alpha] \) by \( w^\circ \). By the uniqueness, \( w^\circ \) is \( S_\infty[\alpha] \)-invariant. Thus, we can approximate \( v \) by elements of \( \bigcup_\alpha H[\alpha] \) with arbitrary precision.

\((**) \Rightarrow (*)\). Let \( v \in H, \|v\| = 1 \). Let \( w \in H[\beta] \) satisfy \( \|w - v\| \leq \delta \). Then for all \( g \in S_\infty[\beta] \) we have \( \|\rho(g)v - w\| = \|\rho(g)v - \rho(g)w\| \leq \delta \); in particular, \( \|\rho(g)v - w\| < 2\delta \).

b) Let \( S_\alpha \subset S_\infty \) be the subgroup consisting of permutations fixing \( \alpha + 1, \alpha + 2, \ldots \). If \( \beta > \alpha \), then the subgroups \( S_\alpha \) and \( S_\infty[\beta] \) commute, and therefore the subspace \( H[\beta] \) is \( S_\alpha \)-invariant. Hence, the subspace \( \bigcup H[\beta] \) is \( S_\alpha \)-invariant for each \( \alpha \). Therefore, \( \bigcup H[\beta] \) is invariant with respect to the union of the subgroups \( S_\alpha \), which is the whole of the group \( S_\infty \). By irreducibility the closure of \( \bigcup \beta H[\beta] \) coincides with \( H \). \( \square \)

2.2. Extension to the semigroup \( B_n \) of 0-1 matrices. Denote by \( B_n \) the semigroup of 0-1 matrices of order \( n \), and by \( B_\infty \) the semigroup of infinite 0-1 matrices, equipped with the weak operator topology.\(^{11} \) We note that the multiplication in \( B_\infty \) is separately continuous\(^{12} \) but not jointly continuous.

\(^{10}\)Recall that a unitary representation of a topological group is a continuous homomorphism to the group of all unitary operators, equipped with the weak operator topology (on the group of all unitary operators the weak topology coincides with the strong operator topology).

\(^{11}\)In our case a sequence \( g^{(k)} \) converges to \( g \) if for any \( i \) and \( j \) the sequence of matrix elements \( g_{ij}^{(k)} \) coincides with \( g_{ij} \) starting from some \( k \).

\(^{12}\)In the space of operators with norm \( \leq 1 \) equipped with the weak topology the multiplication is separately continuous.
The following statement can easily be verified (a formal proof was given in [34], Lemma VIII.1.1).

**Lemma 2.2.** The group $S_\infty$ is dense in $B_\infty$.

For a matrix $p \in B_\infty$, denote by
\[
\{p\}_n \in B_n
\]
its upper left corner of size $n \times n$. The following statement is obvious.

**Lemma 2.3.** The map $p \mapsto \{p\}_n$ induces a bijection
\[
S_\infty[k] \setminus S_\infty/S_\infty[k] \to B_k.
\]

Let us introduce notation for the following element of $B_\infty$:
\[
\theta[\beta] := \begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix} \infty.
\] (2.1)

We emphasize that
\[
\theta[\beta] = \lim_{j \to \infty} \theta_j[\beta],
\] (2.2)
where $\theta_j[\beta]$ is given by (1.2).

**Theorem 2.4.** a) Each unitary representation $\rho$ of the group $\overline{S}_\infty$ admits a unique extension to a continuous representation $\overline{\rho}$ of the semigroup $B_\infty$, and moreover, $\overline{\rho}(p^*) = \overline{\rho}(p)^*$.

b) $\overline{\rho}(\theta[\alpha]) = P[\alpha]$, where $P[\alpha]$ is the operator of orthogonal projection onto $H[\alpha]$.

*Proof.* a) Let $p \in B_\infty$. Choose an element $g_n \in S_\infty$ such that $\{g_n\}_n = \{p\}_n$, and denote by $g_n \in S_\infty[n] \setminus S_\infty/S_\infty[n]$ the double coset containing $g_n$. By Lemma 2.3 we have $g_{n+k} \in g_n$. Therefore,
\[
P[n]\rho(g_{n+j})P[n] = P[n]\rho(g_n) P[n]
\] (2.3)
(see the proof of Lemma 1.7). We define the operators
\[
A_n(g) := P[n]\rho(g_n)P[n].
\]
The equality (2.3) then takes the form
\[
P_n A_{n+j} P_n = A_n.
\]
Keeping in the mind that $\|A_n\| \leq 1$, we get that the sequence $A_n$ has a weak limit, which we denote by $\overline{\rho}(p)$.

It is easy to see that the map $p \mapsto \overline{\rho}(p)$ is weakly continuous from $B_\infty$ to the semigroup of operators with norm $\leq 1$. By the separate continuity of the multiplication in $B_\infty$ and that in the semigroup of contractive operators, the continuous extension of the representation is a representation.

b) First, $\theta[n]^2 = \theta[n] = \theta[n]^*$. Therefore, $\overline{\rho}(\theta[n])$ is an operator of orthogonal projection. Denote its image by $W$. 

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Second, by (2.2) and the continuity of the representation $\overline{\rho}$, the sequence $\rho(\theta_j[n])$ converges to $\overline{\rho}(\theta_j[\beta])$. But $\rho(\theta_j[n])$ is the identity operator on $H[n]$, and hence $H[n] \subseteq W$.

Third, let $g \in S_\infty[n]$ and $w \in W$. Then

$$\rho(g)w = \rho(g)\overline{\rho}(\theta[n])w = \overline{\rho}(g\theta[n])w = \overline{\rho}(\theta[n])w = w,$$

that is, $W \subseteq H[n]$. 

2.3. The action of $B_n$ in the space $H(n)$ of fixed vectors. For $r \in B_n$ we define the element $\pi(r) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in B_\infty$. Denote by $B_\infty^n$ the subsemigroup of $B_\infty$ consisting of the matrices $\pi(r)$. Obviously, $\pi(r)\theta[n] = \theta[n]\pi(r) = \pi(r)$. Substituting these matrices into $\overline{\rho}$, we get that the operator $\overline{\rho}(\pi(r))$ has the block structure

$$\overline{\rho} \begin{pmatrix} \xi_n(r) & 0 \\ 0 & 0 \end{pmatrix}: H[n] \oplus H[n] \rightarrow H[n] \oplus H[n],$$

where $\xi_n(r)$ is an operator on $H[n]$.

Corollary 2.5. a) For any unitary representation $\rho$ of the group $\overline{S}_\infty$ and any $n \geq 0$ the natural representation $\xi_n(r)$ of the semigroup $B_n$ is defined on the subspace $H[n]$.

b) In $H[n]^{\perp}$ the semigroup $B_\infty^n$ acts by zero operators.

Lemma 2.6. If $\rho$ is irreducible, then so is $\xi_n(r)$.

Proof. Assume that there exists a $B_n$-invariant subspace $V \subseteq H[n]$. We assert that for any $g \in B_\infty$ we have $P[n](\overline{\rho}(g))V \subseteq V$. Indeed, let $v \in V$. Then

$$P[n](\overline{\rho}(g))v = P[n](\overline{\rho}(g))P[n]v = (\overline{\rho}(\theta[n])\overline{\rho}(g)\overline{\rho}(\theta[n]))v = (\overline{\rho}(\theta[n]g\theta[n]))v.$$

Note that $\theta[n]g\theta[n] \in B_\infty^n$, and therefore $P[n](\overline{\rho}(g))v \in V$. Hence, $\rho(g)$ lies in a proper subspace of $H$. 

Lemma 2.7. Let $H[n]$ be $S_\infty$-cyclic, so that the system of subspaces $\rho(g)H[n]$, where $g$ ranges over $S_\infty$, is total in $H$. Then the representation $\rho$ is uniquely determined by the representation $\xi_n$ of the semigroup $B_n$ on $H[n]$.

Proof. Let $v, w \in H[n]$. Then

$$\langle \rho(g)v, \rho(h)w \rangle = \langle \rho(g)\rho(\theta[n])v, \rho(h)\rho(\theta[n])w \rangle = \langle \rho(\theta[n])\rho(h^*)\rho(g)\rho(\theta[n])v, w \rangle = \langle \rho(\theta[n]h^*g\theta[n])v, w \rangle = \langle \xi_n(\theta[n]h^*g\theta[n])v, w \rangle,$$

and we see that inner products $\langle \rho(g)v, \rho(h)w \rangle$ are uniquely determined by $\xi_n$. Hence, the system of vectors $\rho(g)v$, where $g$ ranges over $S_\infty$ and $v$ ranges over $H[n]$, is uniquely determined up to unitary equivalence. Therefore, the operators

$$\rho(h): \rho(g)v \mapsto \rho(hg)v$$

are uniquely determined.
2.4. The Lieberman theorem. We fix \( n = 0, 1, \ldots \) and an irreducible representation \( \mu \) of the finite group \( S_n \) on the space \( L = L(\mu) \). Denote by \( \iota \) the trivial one-dimensional representation of the group \( S_\infty \), and consider the representation of the group \( S_\infty \) induced from the representation \( \mu \otimes \iota \) of the subgroup \( S_n \times S_\infty \). We explain the meaning of the term ‘induced representation’\(^{13}\) in this specific case.

Consider the set \( \Omega \) of all \( n \)-element subsets of \( \mathbb{N} \) and the space \( V \) of all \( L \)-valued functions on \( \Omega \). For any \( g \in S_\infty \) and \( I \in \Omega \) we define the substitution \( \sigma(g, I) \) by the following rule. Denote the elements of \( I \) by \( i_1 < \cdots < i_n \), and those of \( gI \) by \( j_1 < \cdots < j_n \). For each \( i_m \) the element \( gi_m \) has the form \( j_l \). We set \( \sigma(g, I)m = l \) and define an action of the group \( S_\infty \) on the space \( V \) by the formula

\[
T_{n,\mu}(g)f(I) = \mu(\sigma(g, I)) f(gI).
\]

**Theorem 2.8.** \( a) \) Each irreducible unitary representation of the group \( S_\infty \) has the form \( T_{n,\mu} \).

\( b) \) Any unitary representation \( \rho \) of \( S_\infty \) is a direct sum of irreducible representations.

\( c) \) This decomposition is unique (that is, collections of summands with multiplicities are always the same).

**Proof.** By the depth of a representation we mean the smallest \( n \) for which \( H[n] \neq 0 \). Let the depth of \( \rho \) be \( n \).

\( a) \) The semigroup \( B_n \) acts in \( H[n] \), and \( \xi_n(g) = 0 \) for \( g \in B_n \setminus S_n \) (otherwise \( n \) is not the smallest). Therefore, we have an irreducible representation \( \mu \) of the group \( S_n \) on \( H[n] \), and \( \rho \) is uniquely determined by \( \mu \). Hence, \( \rho = T_{n,\mu} \).

\( b) \) We explain how to choose an irreducible subrepresentation.

First, let the representation \( \xi_n \) of the semigroup \( B_n \) on \( H[n] \) be irreducible. Consider the \( B_n \)-cyclic span \( R \) of the subspace \( H[n] \). Assume that \( R \) is reducible, \( R = R_1 \oplus R_2 \). Then the projection operators from \( H[n] \) to \( R_1 \) and \( R_2 \) are, in particular, \( B_n \)-intertwining, and their images are subrepresentations in \( H[n] \) which are irreducible. Therefore, one of the projections (say, the projection onto \( R_2 \)) is zero. But then \( H[n] \subset R_1 \), and the cyclic span of \( H[n] \) is contained in \( R_1 \). We come to a contradiction.

Let \( \xi_n \) be reducible and let \( V \) be an irreducible \( B_n \)-subrepresentation in \( H[n] \). Then (see the proof of Lemma 2.6) the \( B_\infty \)-cyclic span of the subspace \( V \) is contained in \( V \oplus H^\perp \). As we saw above, the cyclic span is an irreducible subrepresentation.

Repeating the same reasoning, we decompose \( \rho \) into a sum of irreducible subrepresentations.

\( c) \) We consider a decomposition of \( \rho \) into irreducible subrepresentations, and we decompose \( H \) into the sum \( H = H^0 := \bigoplus_{j=0}^\infty \rho_j \), where \( \rho_j \) is the sum of all irreducible summands of depth \( j \). Note that \( \rho_0 \) is the subspace of \( S_\infty \)-invariant

---

\(^{13}\)Formally, unitary induction in the sense of Mackey (see [46], §13.2) is defined for locally compact groups. In our case it can be used (see [46], §13.2, Problems 5-7) because \( S_\infty/(S_n \times S_\infty[\alpha]) \) is a countable space with a discrete measure. It is also possible to use a construction with fibre bundles (see, for instance, [46], §13.4) or a construction for finite groups in [47], §7.
vectors. It is uniquely determined by the representation \( \rho \), and this in turn determines \( H^1 := \bigoplus_{j=1}^{\infty} \rho_j \). Next, \( \rho_1 \) is the cyclic span of the subspace \( H[1] \), and therefore \( \rho_1 \) also is determined by the representation \( \rho \), and so on.

Thus, all the summands \( \rho_j \) are canonically determined. The problem of a decomposition of \( \rho_j \) into irreducible subrepresentations is equivalent to the decomposition of the representation of the group \( S_j \) on \( H[j] \), and this decomposition is unique. \( \square \)

2.5. Remarks concerning § 2. The complete infinite-dimensional unitary group, orthogonal group, and symplectic group possess similar properties \([48]\), as does the group of all measure-preserving transformations of a Lebesgue measure space (see \([34]\), Chap. VIII).

3. The trisymmetric group and two-dimensional triangulated bordisms

3.1. Multiplication of double cosets. We consider the product \( G = S_\infty \times S_\infty \times S_\infty \) of three copies of the infinite symmetric group, with elements of \( G \) denoted by \( g = (g_{\text{red}}, g_{\text{yellow}}, g_{\text{blue}}) \), where \( g_\nu \) is one of these three ‘colour’ permutations. Let \( K \cong S_\infty \) be the diagonal subgroup, that is, the subgroup consisting of triples of the form \((g, g, g)\). Let \( K[\alpha] \) be the subgroups corresponding to the subgroups \( S_\infty[\alpha] \) under the isomorphism \( S_\infty \twoheadrightarrow K \). We define the elements \( \theta_j[\beta] \in K[\beta] \) by the same formula (1.2) as above.

For any \( \alpha, \beta, \gamma \in \mathbb{Z}_+ \) we define a multiplication

\[
K[\alpha] \setminus G/K[\beta] \times K[\beta] \setminus G/K[\gamma] \to K[\alpha] \setminus G/K[\gamma]
\]

of double cosets as above. More precisely, for double cosets \( p \) and \( q \), we choose representatives \( p \in p \) and \( q \in q \) of them and consider the following sequence of double cosets:

\[
t_j = K[\alpha] \cdot p \theta_j[n] q \cdot K[\gamma]. \tag{3.1}
\]

**Lemma 3.1.** The sequence \( t_j \) is eventually constant. Its limit does not depend on the choice of the representatives \( p \) and \( q \).

**Proof.** Let \( I[\beta] = \{1, 2, \ldots, \beta\} \).

The *support* \( \text{supp}(g) \) of a permutation \( g \) is defined to be the set of all \( i \in \mathbb{N} \) such that \( gi \neq i \). Note that \( \text{supp}(g) = \text{supp}(g^{-1}) \). Let \( s(g) \) be the maximal element of the support. As a sufficiently large \( j \) in (3.1) we take any number larger than \( \max_\nu \max(s(p_\nu), s(q_\nu)) - \beta \). Then for all \( \nu \) we have

\[
(\text{supp}(p_\nu) \setminus I[\beta]) \cap (\theta_j[\beta] \text{supp}(q_\nu) \setminus I[\beta]) = \emptyset.
\]

We clarify the phenomenon of multiplication in Fig. 1. There we represent permutations \( q_\nu, \theta_j[\beta], p_\nu \) and their product. The vertical dashed lines denote the gluing together of the pictures corresponding to \( q_\nu, \theta_j[\beta], \) and \( p_\nu \). The boldface dots are elements of the sets \( I[\alpha], I[\beta], \) and \( I[\gamma] \) (in the figure, \( \alpha = 3, \beta = 2, \gamma = 3 \), and \( j = 6 \)). All the remaining elements of the corresponding copies of \( \mathbb{N} \) are said to be ‘milky’. The ‘vertical’ arcs at \( q_\nu, \theta_j[\beta], \) and \( p_\nu \) are highlighted in boldface if at least one of their endpoints is boldface. Also, for \( q_\nu \) and \( p_\nu \) we highlight in boldface
the arcs whose endpoints lie in the supports. Informally, we say that the boldface arcs correspond to the elements of the figure containing the input information.

All the other vertical arcs will be said to be ‘milky’. Under our choice of \( j \) a milky endpoint of a boldface arc of the permutation \( q_\nu \) cannot be joined (through \( \theta_j[\beta] \)) with a boldface arc of \( p_\nu \), and vice versa.

On the lowest piece of the figure, where the permutation \( p_\nu \theta_j[n] q_\nu \) is displayed, we highlight by boldface the arcs that were obtained by gluing together arcs at least one of which was boldface.

We change \( j \) to \( j + 1 \) and see what happens with the boldface elements of the figure. Milky endpoints of boldface arcs (both upper and lower) move two units to the right. Recall here that we are dealing with double cosets. We can perform a permutation of milky points in the upper row and a permutation of milky points in the lower row in such a way that we return all the endpoints of the boldface arcs to their initial positions, and moreover, these permutations are the same for all \( \nu = 1, 2, 3 \). This proves the first statement.

Now consider a product \( p_\nu h \theta_j[\beta] q_\nu \), where \( h \in K[\beta] \). For sufficiently large \( j \) we have

\[
p_\nu h \theta_j[\beta] q_\nu = p_\nu \theta_j[\beta] q_\nu (\theta_j[\beta]^{-1} h \theta_j[\beta]),
\]

and this proves the second statement. \( \square \)

**Lemma 3.2.** The multiplication of double cosets is associative.
We omit the proof. A formal proof of Lemmas 3.1 and 3.2 on the level of 0-1 matrices was given in [17].

The map \( g \mapsto g^{-1} \) induces an involution \( g \mapsto g^* \):

\[
K[\alpha] \setminus G/K[\beta] \to K[\beta] \setminus G/K[\alpha].
\]

Obviously,

\[
(p \circ q)^* = p^* \circ q^*.
\]

3.2. The group \( G \) and combinatorial data. We fix a \( g = (g_{\text{red}}, g_{\text{yellow}}, g_{\text{blue}}) \in G \) and consider a countable collection of identical triangles whose sides are coloured red, yellow, and blue clockwise. Let the triangles be enumerated by the natural numbers and mark them by the sign \( + \). Consider another countable collection of enumerated triangles whose sides are coloured red, yellow, and blue anticlockwise and mark these triangles by the sign \( - \). The triangles are regarded as oriented; see Fig. 2, (a).

![Figure 2](image-url)

Figure 2. (a) Two countable collections of coloured triangles. (b) A result of a gluing together. (c) This position of triangles is admissible. (d) A double triangle.

Suppose that the permutation \( g_{\text{red}} \) sends \( k \) to \( l \). Then we glue the red side of the \( k \)th plus-triangle together with the red side of the \( l \)th minus-triangle according to the orientations. We repeat this for all \( k \in \mathbb{N} \) and all colours (red, yellow, blue). As a result, we obtain a two-dimensional (disconnected) triangulated surface (see Fig. 2, (b)).

Note that the sense of the term ‘triangulation’ can vary slightly. We make it clear that each triangle is embedded into the surface injectively, but the intersection of two triangles is not necessarily a vertex or a side. Moreover, if \( m \) is sufficiently large, then \( g_{\nu}m = m \) for all colours, and therefore the \( m \)th plus-triangle is glued along its
perimeter together with the \( m \)th minus-triangle, producing a sphere separated by an ‘equator’ into two triangles. We call a sphere glued together from two triangles a **double triangle**.

We define an **equipped surface** \( \Xi \) to be the union of a countable family of oriented triangulated compact two-dimensional surfaces endowed with the following combinatorial structure:

- each triangle is marked by a plus or minus, and the pluses and minuses are arranged checkerwise (that is, the neighbours of plus-triangles are minus-triangles and vice versa);
- edges of the triangulation are coloured red, yellow, and blue in such a way that a triangle cannot have two edges of the same colour, and moreover, the three colours of the sides of a plus-triangle are arranged clockwise, while the colours of the sides of a minus-triangle are arranged anticlockwise;
- all the plus-triangles are enumerated by the natural numbers, as are all the minus-triangles;
- all but finitely many of the connected components of the surface \( \Xi \) are double triangles such that the numeric labels of the plus-triangles and minus-triangles coincide.

Equipped surfaces are considered up to the natural combinatorial equivalence preserving all the described structures.

**Theorem 3.3.** The map constructed from the group \( G = S_\infty \times S_\infty \times S_\infty \) to the space of all equipped surfaces is a bijection.

For a proof it is sufficient to construct the inverse map. Consider an equipped surface. For each red edge \( \zeta \) we look at the numeric label \( k(\zeta) \) on the incident plus-face and the numeric label \( l(\zeta) \) on the incident minus-face. Then let \( g_{\text{red}}k(\zeta) = l(\zeta) \). The permutations \( g_{\text{yellow}} \) and \( g_{\text{blue}} \) are defined in the same way.

**Examples and observations.** 1) If \( g = (g, g, g) \) is a diagonal element, then the surface \( \Xi \) consists of double triangles.

2) Replacing infinity by a finite number \( n \), we get a one-to-one correspondence between the group \( S_n \times S_n \times S_n \) and the set of equipped surfaces glued together from \( 2n \) triangles.

3) It is easy to see that each vertex is incident to edges of only two colours, and the colours of these edges alternate (see Fig. 2, (b)). We colour the vertex using the colour supplementary to the colours of the edges. It can readily be checked that red vertices are in a one-to-one correspondence with independent cycles of the permutation \( g_{\text{yellow}}^{-1}g_{\text{blue}} \).

4) For \( g \in G \) we consider the subgroup \( \Gamma_g \) of \( S_\infty \) generated by the elements \( g_{\text{yellow}}^{-1}g_{\text{blue}} \) and \( g_{\text{blue}}^{-1}g_{\text{red}} \). It is easy to see that the components of the surface are in a one-to-one correspondence with the orbits of the group \( \Gamma_g \).

5) Let \( g \in G \) and \( h \in K \). The right-multiplication operation \( g \mapsto gh \) reduces to a permutation of the plus-labels on the equipped surface.

### 3.3. The product of three symmetric groups and combinatorial data

The last remark implies the following corollary.
Theorem 3.4. Passing from \( g \) to the corresponding double coset \( g \in K[\alpha] \backslash G / K[\beta] \) is equivalent to forgetting the plus-labels with numbers \( \beta \) and the minus-labels with numbers \( \alpha \).

Pro forma we present a description of the combinatorial data corresponding to double cosets. Denote by \( \mathcal{L}[\alpha, \beta] \) the set of compact triangulated surfaces equipped with the following data:

- on each triangle there is a plus or minus, and plus- and minus-triangles are arranged checkerwise;
- edges of the triangulation are coloured red, yellow, or blue, on plus-triangles red, yellow, and blue edges are arranged clockwise, and on minus-triangles they are anticlockwise;
- an injective map \( \iota^+ \) from the set \( \{1, \ldots, \beta\} \) to the set of plus-triangles and an injective map \( \iota^- \) from \( \{1, \ldots, \alpha\} \) to the set of minus-triangles are given, and the triangles labeled in this way will be called entries and exits;
- each double triangle has at least one label.

Examples and observations. 1) Whether or not to keep the countable family of double triangles without labels is a matter of taste. In any case, such triangles do not carry any information.

2) Let \( K \) be a group. The double coset space

\[
\text{diag}(K) \setminus K \times K \times K / \text{diag}(K)
\]

is in a one-to-one correspondence with the conjugacy classes of \( K \times K \) with respect to diag(\( K \)).

3) In particular, we get a combinatorial description of pairs of substitutions \( \sigma_1, \sigma_2 \) described up to a common conjugation, \( (\sigma_1, \sigma_2) \sim (\tau \sigma_1 \tau^{-1}, \tau \sigma_1 \tau^{-1}) \).

3.4. Multiplication of double cosets. Let \( \mathfrak{P} \in \mathcal{L}[\alpha, \beta] \) and \( \Omega \in \mathcal{L}[\beta, \gamma] \). We define a new coset \( \mathfrak{R} \in \mathcal{L}[\alpha, \gamma] \) according to the following rule. We make holes on the place of labelled plus-triangles (entries) \( \iota^+_{\mathfrak{P}}(k) \in \mathfrak{P} \) and minus-triangles (exits) \( \iota^-_{\Omega}(k) \in \Omega \), and then we glue together the boundaries of \( \iota^+_{\mathfrak{P}}(k) \) and \( \iota^-_{\Omega}(k) \) according to the colouring of the edges. We do this for each \( k \) and get a new two-dimensional object equipped with a triangulation, a colouring of edges, pluses, minuses, and labels on triangles. The object consists of triangles glued together along edges, each edge being incident to precisely two triangles. Such an object is a two-dimensional triangulated surface, but some pairs of its vertices may be identified\(^{14} \) (see Fig. 3). We cut such gluings of vertices and get a new surface with the desired structure: the edges are also coloured, and the faces are marked by pluses and minuses checkerwise and are endowed with minus labels not exceeding \( \alpha \) and plus labels not exceeding \( \gamma \).

Proposition 3.5. The described gluing operation corresponds to multiplication of double cosets.

\(^{14}\)Such an effect can occur if two entries having a common vertex are glued together with two exits having a common vertex.
Proof. Consider two elements \( p \) and \( q \) of the group \( G \) and the associated triangulated surfaces, and glue the latter pair together according to our prescription. It is easy to see that this corresponds to a multiplication in the group.

We ‘evaluate’ in this way the product

\[
p\theta_j[\beta]q = p(\theta_j[\beta]q)
\]

for large \( j \).

Before gluing we must permute the minus-labels on the surface corresponding to \( q \). After this, the minus-triangles of the surface \( \theta_j[\beta]q \) with numbers \( > \beta \) are glued together with the double-triangles of the surface \( p \). The surface is then the same as before, except that the labels on the re-glued triangles change. Similarly, the plus-triangles of the surface \( p \) with labels \( > \beta \) are glued together with the double triangles of the surface \( \theta_j[\beta]q \). We note that this changing of the labels does not matter: these labels will be forgotten anyway after passing to double cosets.

Thus, we observe that surfaces are actually transformed only when they are glued together along triangles with labels \( \leq \beta \), and this is our operation. \( \square \)

Example. Multiplication in the semigroup \( K[0] \setminus G/K[0] \) is the operation of disjoint union of surfaces. In particular, this semigroup is commutative.

The following statement is obvious.

Proposition 3.6. The involution \( p \rightarrow p^* \) corresponds to changing the orientation of the surface and replacing all minuses by pluses and all pluses by minuses.

3.5. The multiplicativity theorem. Let \( \rho \) be a unitary representation of the group \( G \) on a Hilbert space \( H \). Let \( H[\alpha] \), \( P[\alpha] \), and \( \tilde{\rho}(g) \) be as above (see §1.7). Then the following multiplicativity theorem holds.

Theorem 3.7. For any \( \alpha, \beta, \gamma \in \mathbb{Z}_+ \) and any

\[
p \in K[\alpha] \setminus G/K[\beta] , \quad q \in K[\beta] \setminus G/K[\gamma]
\]

the equality

\[
\tilde{\rho}(p)\tilde{\rho}(q) = \tilde{\rho}(p \circ q)
\]

holds, and moreover,

\[
\tilde{\rho}(p^*) = \tilde{\rho}(p)^* , \quad \|\tilde{\rho}(p)\| \leq 1.
\]
Proof. The same arguments as in the proof of Proposition 2.1, b) show that the closure \( V \) of the subspace \( \bigcup H[\alpha] \) is \( G \)-invariant. Therefore, the orthogonal complement \( V^\perp \) is also \( G \)-invariant, and hence the operators \( \tilde{\rho}(p) \) depend only on the restriction of \( \rho \) to the subspace \( V \). We can thus assume without loss of generality that \( V = H \). By Proposition 2.1 the representation of the subgroup \( K \simeq S_\infty \) on \( H \) extends continuously to the complete symmetric group \( \overline{S}_\infty \). Using Theorem 2.4, b), we now observe that the sequence \( \rho(\theta_j[\beta]) \) converges weakly to the projection \( P[\beta] \).

We choose representatives of the double cosets: \( p \in p \) and \( q \in q \). For sufficiently large \( j \)

\[
\tilde{\rho}(p \circ q)P[\gamma] = P[\alpha] \rho(p\theta_j[\beta]q) P[\gamma] = \lim_{k \to \infty} P[\alpha] \rho(p\theta_k[\beta]q) P[\gamma]
\]

\[
= \lim_{k \to \infty} P[\alpha] \rho(p) \rho(\theta_k[\beta]) \rho(q) P[\gamma]
\]

\[
= P[\alpha] \rho(p) \left( \lim_{k \to \infty} \rho(\theta_k[\beta]) \right) \rho(q) P[\gamma] = P[\alpha] \rho(p) P[\beta] \rho(q) P[\gamma]
\]

\[
= (P[\alpha] \rho(p) P[\beta])(P[\beta] \rho(q) P[\gamma]) = \tilde{\rho}(p) \tilde{\rho}(q) P[\gamma],
\]

where the limit is the weak operator limit. This proves (3.2). The assertions (3.3) are obvious. \( \square \)

3.6. Sphericity.

**Theorem 3.8.** The pair \((G, K)\) is spherical.

**Proof.** The semigroup \( K \setminus G/K \) is Abelian. Therefore, any irreducible representation of it compatible with the involution is one-dimensional. This semigroup acts in \( H[0] \). If its representation on \( H[0] \) is reducible, then (see the proof of Lemma 2.6) the representation of the whole group \( G \) is also reducible. \( \square \)

We should single out the following result, which was just used.

**Theorem 3.9.** The \( K \)-spherical functions of the group \( G \) are homomorphisms from the semigroup \( K \setminus G/K \) to the multiplicative semigroup of complex numbers with absolute values \( \leq 1 \).

3.7. The trisymmetric group \( G \). We remark that there was a certain dissonance in the proof of Theorem 3.7. The space \( H \) was decomposed into a sum of two \( G \)-invariant subspaces, \( H = V \oplus V^\perp \). Concerning the subrepresentation on \( V \) we make a non-trivial assertion, but for \( V^\perp \) we refer to the identity \( 0 \cdot 0 = 0 \). Accordingly, it is natural to narrow the class of representations we are working with.

We define the trisymmetric group to be the subgroup \( G \) of \( \overline{S}_\infty \times \overline{S}_\infty \times \overline{S}_\infty \) consisting of triples \((g_{\text{red}}, g_{\text{yellow}}, g_{\text{blue}})\) such that \( g_{\text{red}} g_{\text{yellow}}^{-1}, g_{\text{yellow}} g_{\text{blue}}^{-1} \in S_\infty \). In other words, we take the subgroup of \( \overline{S}_\infty \times \overline{S}_\infty \times \overline{S}_\infty \) generated by the group \( G = S_\infty \times S_\infty \times S_\infty \) and the diagonal subgroup \( K \) consisting of the elements \((g, g, g)\) with \( g \in \overline{S}_\infty \). On the group \( G \) we introduce a topology, assuming that the topology on \( K \) is the standard topology of \( \overline{S}_\infty \), and the whole group is the disconnected union of the (one-sided) cosets in \( G/K \).

The following statement is tautological (see Proposition 2.1).
Lemma 3.10. a) A unitary representation \( \rho \) of \( G \) on the space \( H \) admits a continuous extension to the group \( G \) if and only if the restriction of \( \rho \) to \( K \) admits a continuous extension to \( K \).

b) A unitary representation of the group \( G \) on a space \( H \) admits a continuous extension to \( G \) if and only if the subspace \( \bigcup H[\alpha] \) is dense in \( H \).

Just as earlier, we define the family of subgroups \( K[\alpha] \) of \( K \). As above, a multiplication of double cosets is defined, and the natural map

\[
K[\alpha] \setminus G/K[\beta] \rightarrow K[\alpha] \setminus G/K[\beta]
\]

is a bijection.

Again as above, for an element of \( G \) we construct a colored triangulated surface with labels. As before, it is the disjoint union of a countable family of compact triangulated surfaces, and almost all the surfaces are double triangles. The only difference is that now we do not require coincidence of labels on the different sides of almost all the double triangles.

3.8. The approximation theorem. We consider the category \( \mathcal{K}(G, K) \), the train of the pair \( (G, K) \), whose objects are the non-negative integers and whose morphisms are the double coset spaces \( K[\alpha] \setminus G/K[\beta] \). Let \( R \) be a \( * \)-representation of it. This means that for each \( \alpha \in \mathbb{Z}_+ \) we have a Hilbert space \( V[\alpha] \), and for each morphism \( p: \beta \rightarrow \alpha \) we have a bounded linear operator \( R(p): V[\beta] \rightarrow V[\alpha] \) such that

\[
R(p \circ q) = R(p)R(q), \quad R(p^*) = R(p)^*, \quad R(Id_\alpha) = 1_{V[\alpha]},
\]

where \( Id_\alpha \) is the identity automorphism of the object \( \alpha \) and \( 1_{V[\alpha]} \) is the identity operator on \( V[\alpha] \).

Theorem 3.11. Any \( * \)-representation \( R \) of the category \( \mathcal{K}(G, K) \) is equivalent to the representation \( \tilde{\rho} \) obtained from some unitary representation \( \rho \) of the group \( G \), and moreover, the representation \( \rho \) is unique.

In the next subsection we obtain the representation \( \rho \) as a limit of representations of the semigroups \( \text{Mor}(\alpha, \alpha) \).

3.9. The inverse construction. Proof of Theorem 3.11. The following construction is a special case of the abstract Theorem VIII.1.10 in [34].

Let \( \alpha \preceq \beta \). We define the following morphisms of our category:

- \( \theta_\alpha^\beta \in \text{Mor}(\beta, \beta) \) is the collection of double triangles with the labels

\[
(1, 1), (2, 2), \ldots, (\alpha, \alpha), (\alpha + 1, \emptyset), \ldots, (\beta, \emptyset), (\emptyset, \alpha + 1), \ldots, (\emptyset, \beta);
\]

- \( \mu_\alpha^\beta \in \text{Mor}(\alpha, \beta) \) is the collection of double triangles with the labels

\[
(1, 1), (2, 2), \ldots, (\alpha, \alpha), (\emptyset, \alpha + 1), \ldots, (\emptyset, \beta);
\]

- \( \nu_\alpha^\beta \in \text{Mor}(\beta, \alpha) \) is defined as \( \nu_\alpha^\beta = (\mu_\alpha^\beta)^* \).
It is easy to see that
\[(\theta^\beta_\alpha)^* = \theta^\beta_\alpha = (\theta^\beta_\alpha)^2; \quad \mu^\beta_\alpha \circ \nu^\beta_\alpha = \theta^\beta_\alpha, \quad \nu^\beta_\alpha \circ \mu^\beta_\alpha = 1_\alpha; \]
\[\theta^\beta_\alpha \circ \mu^\beta_\alpha = \mu^\beta_\alpha, \quad \nu^\beta_\alpha \circ \theta^\beta_\alpha = \nu^\beta_\alpha.\]

Let us apply the functor \(R\) to these identities. We get that \(R(\theta^\beta_\alpha)\) is the operator of orthogonal projection on \(V[\beta]\), and \(\mu^\beta_\alpha\) is the operator of isometric embedding \(V[\alpha] \to V[\beta]\) that identifies \(V[\alpha]\) with the image of the operator \(\theta^\beta_\alpha\).

Next, we construct a semigroup embedding \(\zeta: \text{Mor}(\alpha, \alpha) \to \text{Mor}(\beta, \beta)\). To a surface in \(\text{Mor}(\alpha, \alpha)\), we add a collection of double triangles with labels \(15(\alpha+1, \emptyset), \ldots, (\beta, \emptyset), (\emptyset, \alpha+1), \ldots, (\emptyset, \beta)\).

We also set
\[\Gamma = \lim_{\to} \text{Mor}(\alpha, \alpha).\]

It is easy to see that
\[\zeta(p)\theta^\beta_\alpha = \theta^\beta_\alpha \zeta(p), \quad \zeta(p)\mu^\beta_\alpha = \mu^\beta_\alpha \mu(p).\]

We decompose the space \(V[\beta]\) into the direct sum \(\text{im } R(\theta^\beta_\alpha) \oplus (\text{im } R(\theta^\beta_\alpha))^\perp\). By the above identities, the operator \(R(\zeta(p))\) has the block structure
\[\begin{pmatrix}
R(p) & 0 \\
0 & 0
\end{pmatrix}.
\]

Next, note that for \(\alpha \leq \beta \leq \gamma\) we have
\[\mu^\gamma_\beta \circ \mu^\beta_\alpha = \mu^\gamma_\alpha.\]

We consider the chain of embeddings of Hilbert spaces
\[
\cdots \to V[\alpha] \to V[\alpha+1] \to \cdots,
\]
take the union of all the members of this chain, and denote the completion by \(V\). The semigroup \(\text{Mor}(\alpha, \alpha)\) acts in each \(V[\alpha]\), these actions are compatible, and therefore in the limit space we get a representation \(R^\circ\) of the semigroup \(\Gamma\). By \(P[\alpha]\) we denote the projection operator onto \(V[\alpha]\). Then
\[P[\alpha]|_{V[\beta]} = R^\circ(\theta^\beta_\alpha).
\]

**Lemma 3.12.** \(\|R^\circ(p)\| \leq 1\) for each \(p \in \Gamma\).

**Proof.** Let \(p \in \text{Mor}(\alpha, \alpha)\), and suppose that the corresponding surface consist of \(2N\) triangles. Arranging labels arbitrarily on empty triangles, we get a \(g \in S_N\) such that \(p = \nu^N_\alpha g \mu^N_\alpha\). Therefore, \(R^\circ(p) = R(\nu^N_\alpha)R(g)R(\mu^N_\alpha)\). But
\[R(g)R(g)^* = R(g)R(g^{-1}) = 1_{H[N]},\]
that is, \(R(g)\) is unitary. The operator \(R(\mu^N_\alpha)\) is an isometric embedding, and \(R(\nu^N_\alpha)\) is adjoint to it. Therefore, the norm of the product is at most 1. \(\square\)

---

\(^{15}\)We note that all the elements of the image are non-invertible.
Lemma 3.13. Let $p \in G$, and let $p_{\alpha} \in \text{Mor}(\alpha, \alpha)$ be the double coset containing $p$, regarded as an element of $\Gamma$. Then the sequence of operators $R^\alpha(p_{\alpha})$ has a weak limit and its norm does not exceed 1.

Proof. This follows from the compatibility

$$P[\alpha]R^\alpha(p_{\alpha+1})P[\alpha] = R^\alpha(p_{\alpha})$$

and the previous lemma. We decompose $H$ into a direct sum of subspaces

$$H = \bigoplus_{\alpha=0}^{\infty}(H[\alpha] \oplus H[\alpha - 1])$$

and represent elements of $R^\alpha(p_{\alpha})$ as block matrices. For such a matrix all blocks with at least one index $\geq \alpha$ are zero. Passing from $R^\alpha(p_{\alpha})$ to $R^\alpha(p_{\alpha-1})$ involves zeroing out the blocks with index $\alpha$. The reverse passage involves writing in these blocks, and the norm remains $\leq 1$.

For $p \in G$ let $\rho(p) := \lim_{\alpha \to \infty} \rho(p_{\alpha})$. Let $p, q \in G$ and $r = pq$. Assume that all six of the permutations $p_\nu, q_\nu$ are trivial on the elements $N+1, N+2, \ldots$. We will take only $\alpha$ and $\beta$ larger than $N$. Then

$$p_{\alpha}q_{\beta} = r_{\min(\alpha, \beta)}.$$ 

Applying the functor $R$ to both sides and passing to the weak limit

$$\lim_{\alpha \to \infty} \left( \lim_{\beta \to \infty} \ldots \right),$$

we get that $\rho(p)\rho(q) = \rho(r) = \rho(pq)$.

A representation of the group $G$ has been constructed. Its restriction to the diagonal is continuous in the topology of the complete symmetric group, and therefore it admits a continuous extension to the whole group $G$.

3.10. Countable tensor products of Hilbert spaces. (See [49].) Let $H_j$ be Hilbert spaces and let $\xi_j \in H_j$ be fixed unit vectors. In each $H_j$ we choose an orthonormal basis $e_k^{(j)}$ such that $e_1^{(j)} = \xi_j$. Consider the Hilbert space

$$\bigotimes_{j=1}^{\infty}(H_j, \xi_j),$$

with basis formed by the formal vectors

$$e_{k_1}^{(1)} \otimes e_{k_2}^{(2)} \otimes e_{k_3}^{(3)} \otimes \ldots,$$

where all the factors beginning at some number have the form $e_1^{(n)}$ (and therefore the basis is countable).

For a sequence of vectors $v^{(j)} \in H_j$, we define a decomposable vector

$$v^{(1)} \otimes v^{(2)} \otimes v^{(3)} \otimes \ldots,$$
expanding it in the basis by multiplying out the parentheses (and choosing terms
with $e_1^{(j)} = \xi_j$ from all but finitely many parentheses). The result is contained in
our space if and only if the products
\[ \prod_{j=1}^{\infty} \|v^{(j)}\|, \quad \prod_{j=1}^{\infty} \langle v^{(j)}, \xi_j \rangle \]
are convergent. Under these conditions, we have
\[ \left\langle \bigotimes_{j=1}^{\infty} v^{(j)}, \bigotimes_{j=1}^{\infty} w^{(j)} \right\rangle = \prod_{j=1}^{\infty} \langle v^{(j)}, w^{(j)} \rangle_{H_j}. \] 
(3.4)

This construction does not depend on the choice of bases in the spaces $H_j$ (the
formula (3.4) can be regarded as an invariant definition of the tensor product), but
does depend on the distinguished vectors $\xi_j$.

3.11. Constructions of spherical representations of $G$. I. We consider three
Hilbert spaces $V_{\text{red}}$, $V_{\text{yellow}}$, and $V_{\text{blue}}$ (they can be finite- or infinite-dimensional),
their tensor product
\[ X := V_{\text{red}} \otimes V_{\text{yellow}} \otimes V_{\text{blue}}, \] 
and a unit vector $\xi \in X$. Next, consider the countable tensor product
\[ H := (X, \xi) \otimes (X, \xi) \otimes \cdots = (V_{\text{red}} \otimes V_{\text{yellow}} \otimes V_{\text{blue}}, \xi) \otimes (V_{\text{red}} \otimes V_{\text{yellow}} \otimes V_{\text{blue}}, \xi) \otimes \cdots \]
and let
\[ \Xi := \xi^{\otimes \infty} \in H. \]

We define an action $\rho$ of the group $G$ in $H$. The first copy of $S_\infty$ interchanges the
red factors $V_{\text{red}}$, the second copy interchanges the yellow factors $V_{\text{yellow}}$, the third
copy the blue factors $V_{\text{blue}}$. The diagonal subgroup $K = S_\infty$ acts by permutations
of the factors $(X, \xi)$.

Remark. a) We cannot (except in very degenerate cases) extend the action of the red
copy of $S_\infty$ to an action of the complete symmetric group. Indeed, the transposition
of the two factors $V_{\text{red}}$ in
\[ (V_{\text{red}} \otimes V_{\text{yellow}} \otimes V_{\text{blue}}, \xi) \otimes (V_{\text{red}} \otimes V_{\text{yellow}} \otimes V_{\text{blue}}, \xi) \]
does not in general fix the vector $\xi \otimes \xi$, and therefore the product $(12)(34)(56)\ldots$
of elementary transpositions takes the vector $\Xi$ out of $H$.

b) The vector $\Xi$ is the unique $K$-fixed vector. Indeed, complete the vector $\xi$ to
an orthonormal basis $r_1 = \xi$, $r_2, r_3, \ldots$ in $X$. Let $h$ be a fixed vector, and let $\beta_{i_1 i_2 \ldots}$
be the coefficients of its expansion in the basis $r_{i_1} \otimes r_{i_2} \otimes \cdots$. Then the $\beta_{\ldots}$
do not change under permutations of subscripts. If not all the $i_k$ are equal to 1, then we
get a countable number of equal coefficients. Therefore, this coefficient is 0.

c) Similarly, $H[\alpha]$ is
\[ \underbrace{(X, \xi) \otimes \cdots \otimes (X, \xi)}_{\alpha \text{ times}} \otimes \xi \otimes \xi \otimes \cdots. \]
d) Denote by $U(V)$ the group of all unitary operators on a Hilbert space $V$. Consider on $X = U(V_{\text{red}}) \times U(V_{\text{yellow}}) \times U(V_{\text{blue}})$ the subgroup $Q$ fixing $\xi$. Then $Q$ acts on each factor of the product $X \otimes X \otimes \cdots$ and therefore on the whole tensor product. This action commutes with the representation of $G$.

**Proposition 3.14.** The cyclic $G$-span of the vector $\Xi$ is an irreducible representation of $G$.

**Proof.** This is a variant of Lemma 2.6. □

### 3.12. Spherical functions.

Let $e^\text{red}_i$, $e^\text{yellow}_j$, and $e^\text{blue}_k$ be orthonormal bases in $V_{\text{red}}$, $V_{\text{yellow}}$, and $V_{\text{blue}}$, respectively. Let

$$\xi = \sum \alpha_{ijk} e^\text{red}_i \otimes e^\text{yellow}_j \otimes e^\text{blue}_k.$$  

Let us write an expression for the spherical function

$$\Phi(g) := \langle \rho(g)\Xi, \Xi \rangle, \quad g \in g \in K \setminus G/K.$$  

We consider the corresponding triangulated surface $\mathcal{S} \in \mathcal{L}[0,0]$. To each edge of the surface we assign a basis vector $e^\nu_i$ of the corresponding colour $\nu$. We call such data an assignment. Let an assignment $\mathcal{S}$ be fixed. For each triangle $\Delta$ we denote by $t_{\text{red}}(\Delta)$, $t_{\text{yellow}}(\Delta)$, and $t_{\text{blue}}(\Delta)$ the indices of the basis vectors on its sides.

**Theorem 3.15.** Spherical functions are given by the formula

$$\Phi(g) = \sum_{\mathcal{S}} \prod_{\Delta \text{ a plus-triangle}} \alpha_{t_{\text{red}}(\Delta)t_{\text{yellow}}(\Delta)t_{\text{blue}}(\Delta)}$$

$$\times \prod_{\Delta \text{ a minus-triangle}} \alpha_{t_{\text{red}}(\Delta)t_{\text{yellow}}(\Delta)t_{\text{blue}}(\Delta)}, \quad (3.6)$$

where $\mathcal{S}$ ranges over all assignments.

**Proof.** Let the support of $g$ be contained in $\{1, \ldots, N\}$. Then it suffices to consider the representation of $S_N \times S_N \times S_N$ on $X^{\otimes N}$ and to evaluate its matrix element $\langle \rho(g)\xi \otimes \xi, \xi \otimes \xi \rangle$. For brevity denote the red, yellow, and blue permutations by $p$, $q$, and $r$ and the elements of the bases by $x_i$, $y_j$, and $z_k$. In this notation,

$$\Xi = \sum \alpha_{i_1j_1k_1i_2j_2k_2 \cdots} \langle x_{i_1} \otimes y_{j_1} \otimes z_{k_1}, x_{i_2} \otimes y_{j_2} \otimes z_{k_2}, \cdots \rangle,$$

$$\rho(g)\Xi = \sum \alpha_{i_1j_1k_1i_2j_2k_2 \cdots} \langle x_{i_{p(1)}} \otimes y_{j_{q(1)}} \otimes z_{k_{r(1)}}, x_{i_{p(2)}} \otimes y_{j_{q(2)}} \otimes z_{k_{r(2)}}, \cdots \rangle.$$

Consider a basis vector $\langle x_{i_1} \otimes y_{j_1} \otimes z_{k_1}, x_{i_2} \otimes y_{j_2} \otimes z_{k_2}, \cdots \rangle$. The coefficients of this vector in $\Xi$ and $\rho(g)\Xi$ are equal, respectively, to

$$\prod_m \alpha_i \cdot \prod_m \alpha_i \cdot \prod_m \alpha_i.$$ 

Now we write $e^\text{red}_i$ on each red edge between a plus-triangle with label $m$ and a minus-triangle with label $p(m)$, and we assign basis vectors to yellow and blue edges similarly. We get an assignment for the triangulated surface, and the product of the expressions (3.7) just gives a summand in (3.6). □
3.13. Constructions of spherical representations of G. II. The previous construction can be generalized. We can consider super-tensor products instead of the usual tensor products.

Let $V = V^\overline{0} \oplus V^\overline{1}$ be a linear superspace, that is, a linear space decomposed into the sum of an even part and an odd part. We call such object a superspace. We define the tensor square of $V$ in the usual way, but change the operator of transposition of summands to

\[
(v^\overline{0} \oplus v^\overline{1}) \otimes (w^\overline{0} \oplus w^\overline{1}) = (v^\overline{0} \otimes w^\overline{0}) + (v^\overline{0} \otimes w^\overline{1}) + (v^\overline{1} \otimes w^\overline{0}) + (v^\overline{1} \otimes w^\overline{1})
\]

\[
\mapsto (w^\overline{0} \otimes v^\overline{0}) + (w^\overline{0} \otimes v^\overline{1}) + (w^\overline{1} \otimes v^\overline{0}) - (w^\overline{1} \otimes v^\overline{1}). \tag{3.8}
\]

Given actions of the transpositions $(12), (23), \ldots$, we can define an action of the symmetric group $S_n$ in a tensor power $V^{\otimes n}$. We can say this is in another way: transporting an odd vector past an odd vector, we change sign. A tensor product equipped with such an action of the symmetric group is called a super-tensor product.

Now let $V_{\text{red}}, V_{\text{yellow}},$ and $V_{\text{blue}}$ be superspaces. We define a superspace structure in their tensor product $X$ in an obvious way:

\[
X^{(0)} := (V_{\text{red}}^{(0)} \otimes V_{\text{yellow}}^{(0)} \otimes V_{\text{blue}}^{(0)}) \oplus (V_{\text{red}}^{(1)} \otimes V_{\text{yellow}}^{(1)} \otimes V_{\text{blue}}^{(0)})
\]

\[
\oplus (V_{\text{red}}^{(1)} \otimes V_{\text{yellow}}^{(0)} \otimes V_{\text{blue}}^{(1)}) \oplus (V_{\text{red}}^{(0)} \otimes V_{\text{yellow}}^{(1)} \otimes V_{\text{blue}}^{(1)}),
\]

and $X^{(1)}$ is the orthogonal complement of $X^{(0)}$. Choose a unit vector $\xi \in X^{(0)}$ and consider the super-tensor product $H = (X, \xi) \otimes (X, \xi) \otimes \cdots$. The complete symmetric group $K = S_\infty$ acts in $H$ by permutations of factors. Indeed, vectors of the form $\bigotimes_{j=1}^\infty v_j$, where each $v_j$ is either even or odd and $v_j = \xi$ for all but finitely many $j$, form a total system in $H$. On the other hand, an action of an infinite permutation on such a vector is well defined. The action of the three copies of the symmetric group $S_\infty$ is defined as above.

As in the case of ordinary tensor products, $\otimes \xi^\infty$ is the unique $K$-fixed vector, and its $G$-cyclic span is an irreducible representation.

3.14. The Fock representation of the group of isometries of a Hilbert space. For details see, for instance, [34], §6.1. Let $\text{Isom}(V)$ denote the group of isometries of a real Hilbert space $V$. Obviously, isometries have the form $Rv = Uv + h$, where $U$ is an orthogonal operator (we denote the group of all orthogonal operators by $O(V)$) and $h \in V$.

Lemma 3.16. Let $V$ be a real Hilbert space. Then there exists a Hilbert space $F(V)$ (the Fock space) and a total system of vectors $\psi_v \in F(V)$, where $v$ ranges over $V$, such that

\[
\langle \psi_v, \psi_w \rangle_{F(V)} = \exp \left( -\frac{||v - w||^2}{2} \right). \tag{3.9}
\]

Proof. Let $V = \mathbb{R}^n$, and consider $\mathbb{C}^n$ with the Gaussian measure density $\pi^{-n} e^{-\|z\|^2}$. In $L^2$ (with respect to this measure) we consider the system of vectors

\[
\psi_v(z) = \exp \left( \langle z, v \rangle - \frac{\langle v, v \rangle}{2} \right).
\]
As $F(\mathbb{R}^n)$ we take the closure of the linear span of the vectors $\psi_v(z)$, $v \in \mathbb{C}^n$ (in fact, $F(\mathbb{R}^n)$ is the subspace of $L^2$ consisting of entire functions on $\mathbb{C}^n$).

Next, we consider the embeddings $\mathbb{C}^n \to \mathbb{C}^{n+1}$ and the corresponding embeddings $J_n$ of function spaces given by

$$J_n f(z_1, \ldots, z_n, z_{n+1}) = f(z_1, \ldots, z_n).$$

It is easy to see that the $J_n$ are isometric embeddings

$$L^2(\mathbb{C}^n, \pi^{-n} e^{-\|z\|^2}) \to L^2(\mathbb{C}^{n+1}, \pi^{-n-1} e^{-\|z\|^2}),$$

which induce embeddings $F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1})$. They send the vectors $\psi_{v_1,\ldots,v_n}$ to $\psi_{v_1,\ldots,v_{n,0}}$. In the limit, we get a Hilbert space with a system of vectors $\psi_{v_1,\ldots,v_{n,0}}$, where $v_j = 0$ starting from some place, and their inner products are given by (3.9). It remains to close this system in the limit Hilbert space. In this way we get the space $F(\ell_2)$, and any infinite-dimensional Hilbert space can be identified with $\ell_2$.

Let $R$ be an isometry of the space $V$, and consider the map of the system $\psi_v$ to itself given by the formula

$$\psi_v \mapsto \psi_{Rv}.$$ 

Then $\langle \psi(Rv), \psi(Rw) \rangle = \langle \psi(v), \psi(w) \rangle$, and therefore the map can be extended to a unitary operator $\sigma(R)$ on the space $F(V)$. As a result, we get a unitary representation of the group $\text{Isom}(V)$.

**Remark.** This construction is described in a slightly ‘immaterial’ way. In fact, the space $F(V)$ can be realized as a space of holomorphic functions on the complexification of the space $V$, and the operators corresponding to orthogonal transformations and shifts have the respective forms

$$f(z) \mapsto (zU), \quad f(z) \mapsto f(z + h) \exp(-\langle z, h \rangle_V).$$

It is easy to see that this representation of the group $\text{Isom}(V)$ is $O(V)$-spherical, the $O(V)$-fixed vector is $\psi_0$, and the spherical function is $\exp(-\|h\|^2/2)$.

### 3.15. Constructions of spherical representations of $G$. III.

Our next goal is to embed the group $G$ into the group of isometries of a real Hilbert space and then to obtain representations of $G$ on the Fock space by restriction of the representations of $\text{Isom}(V)$ constructed above.

As $V$ we take the tensor product $\ell_2 \otimes \ell_2 \otimes \ell_2$. On this space we have the tautological representation $\pi_1 \otimes \pi_2 \otimes \pi_3$ of the direct product $\mathcal{S}_\infty \times \mathcal{S}_\infty \times \mathcal{S}_\infty$ ($\pi_\nu$ denotes the tautological representation of the $\nu$th copy of $\mathcal{S}_\infty$ on the $\nu$th copy of $\ell_2$). We define the formal expression

$$u := \sum_j e_j \otimes e_j \otimes e_j, \quad (3.10)$$

which, of course, is not contained in the space $\ell_2 \otimes \ell_2 \otimes \ell_2$. 

We consider the transformations of $\ell_2 \otimes \ell_2 \otimes \ell_2$ given by

$$R_t(g_1, g_2, g_3)v = \pi_1(g_1) \otimes \pi_2(g_2) \otimes \pi_3(g_3)v + t \cdot [\pi_1(g_1) \otimes \pi_2(g_2) \otimes \pi_3(g_3)u - u],$$

(3.11)

where $t$ is a real parameter. First, note that the expression in square brackets vanishes if $g_1 = g_2 = g_3$. Therefore, for $(g_1, g_2, g_3) \in G$ this expression is contained in $\ell_2 \otimes \ell_2 \otimes \ell_2$, and hence the affine isometric transformations (3.11) are well defined on $\ell_2 \otimes \ell_2 \otimes \ell_2$. Second, we get an action of the group $G$ (this can easily be checked directly, but we can also see that (3.11) looks like an expression defining a linear transformation in the space whose origin is shifted to the vector $u$).

Restricting the Fock representation of the group $\text{Isom}(\ell_2 \otimes \ell_2 \otimes \ell_2)$ to the subgroup $G$, we get a series of representations of $G$; we denote them by $v^{1,1,1}_t$.

Next, we have a representation of $G$ on $\ell_2(G/K) = \ell_2(G/K)$. It is spherical, and it is natural to understand it as $v^{1,1,1}_\infty$ (in any case, this is so on the level of a limit of spherical functions).

The construction can be varied. Consider the action of the group $S_\infty \times S_\infty \times S_\infty$ on $\ell_2 \otimes \ell_2$, where the first and second factors act in the tautological way, and the third factor in the trivial way; that is, in fact we get an action of $S_\infty \times S_\infty$. We next consider the expression $\sum_j e_j \otimes e_j$ and repeat the same construction. As a result, we get a series of representations $v^{1,1,0}_s$ of the group $G$.

We can repeat this with omission of the first or the second factor in the product $S_\infty \times S_\infty \times S_\infty$. Denote the representations obtained in this way by $v^{1,0,1}_p$ and $v^{1,0,1}_q$.

### 3.16. Conjectures

1) Let $\Upsilon$ be a representation of $G$ on a super-tensor product, and consider the tensor product

$$\Upsilon \otimes v^{1,1,1}_t \otimes v^{1,1,0}_s \otimes v^{1,0,1}_p \otimes v^{1,0,1}_q.$$  

(3.12)

It contains a unique $K$-fixed vector, and its cyclic span is an irreducible representation of $G$. **CONJECTURE:** *All $K$-spherical representations of the group $G$ can be obtained in this way.*

2) Consider an irreducible unitary representation $\kappa$ of the group $\mathcal{S}_\infty \times \mathcal{S}_\infty \times \mathcal{S}_\infty$ (since $\mathcal{S}_\infty$ is a type I group, $\kappa$ is a tensor product $\kappa_1 \otimes \kappa_2 \otimes \kappa_3$ of representations of $\mathcal{S}_\infty$). **CONJECTURE:** *Any irreducible unitary representation of the group $G$ is contained in the decomposition of some representation of the form*

$$\Upsilon \otimes v^{1,1,1}_t \otimes v^{1,1,0}_s \otimes v^{1,0,1}_p \otimes v^{1,0,1}_p \otimes \kappa.$$  

(3.13)

3) **CONJECTURE:** *The group $G$ has type I, and any unitary representation of $G$ can be expressed as a direct integral of irreducible representations.*

### 3.17. Spherical characters and self-similarity

We consider the semigroup $\text{Mor}(\alpha, \alpha)$, and in it the collection of $\alpha$ double triangles with labels $(1, 1), \ldots, (\alpha, \alpha)$, together with an arbitrary surface without labels. The semigroup $\Sigma_\alpha$ of all such morphisms is the centre of the semigroup $\text{Mor}(\alpha, \alpha)$.

The semigroups $\Sigma_\alpha$ with different $\alpha$ are isomorphic, and the corresponding isomorphism $\pi^{\alpha+1}_\alpha: \Sigma_{\alpha+1} \rightarrow \Sigma_\alpha$ amounts to forgetting the two labels $\alpha + 1$. In the notation of §3.9,

$$\pi^{\alpha+1}_\alpha(p) = v^{\alpha+1}_\alpha \circ p \circ \mu^{\alpha+1}_\alpha.$$
We note that Σ₀ = Mor(0, 0) = K \ G/K.

Let ρ be an irreducible representation of the group G on a space H. Then the semigroup Σ_α acts in H[α] by scalar operators, that is, we get a homomorphism χ_α from Σ_α to the multiplicative semigroup of complex numbers with absolute value ≤ 1. For any irreducible unitary representation ρ of G,

\[ \tilde{\rho}(\pi_\alpha^{α+1}(p)) = \tilde{\rho}(\rho_\alpha^{α+1})\tilde{\rho}(p)\tilde{\rho}(\rho_\alpha^{α+1}) = χ_{α+1}(p) \cdot 1. \]

We get the following statement.

**Proposition 3.17.** The character χ does not depend on α.

We call \( χ : K \setminus G/K \to \mathbb{C} \) the spherical character of an irreducible representation.

In the group \( G \subset S_\infty \times S_\infty \times S_\infty \) we define the subgroup

\[ G[α] := G \cap (S_\infty[α] \times S_\infty[α] \times S_\infty[α]). \]

Clearly, \( G[α] \) is canonically isomorphic to \( G \).

Let us consider an irreducible unitary representation \( ρ \) of \( G \) on a space \( H \) and restrict it to the subgroup \( G[α] \). If \( α \) is sufficiently large, then \( H[α] \neq 0 \). The semigroup \( Σ_α \) of the group \( G \) acts in \( H[α] \) as scalar operators, and it coincides with the semigroup \( Σ_0 \) of the group \( G[α] \). It is easy to verify that the \( G[α] \)-cyclic span of any non-zero element of \( H[α] \) is a spherical representation of \( G[α] \). Moreover, all spherical subrepresentations of the restriction \( ρ|_{G[α]} \) are equivalent, and their spherical functions coincide with the spherical character of the representation \( ρ \) (for all \( α \)).

**3.18. Several copies of the symmetric group.** We consider the product of \( n \) copies of the symmetric group \( G = S_\infty \times \cdots \times S_\infty \) and the diagonal subgroup \( K \). In this case the constructions in the present section can be repeated literally. The only difference is that we consider \( n \)-gons instead of triangles.

Here we must fix a cyclic order on the set of copies of the symmetric group. A change of the cyclic order leads to an equivalent theory, but the surfaces corresponding to elements of the group \( G \) change.

![Figure 4. Surfaces glued together from digons and monogons.](image-url)
We can use the general construction for $n = 2$ and get a surface glued together from digons (see Fig. 4). Here though, there is a simpler language, which is used in §4.

We remark that another language, the language of $(n-1)$-dimensional complexes, is possible in the general case (see §7).

3.19. Remarks concerning §3. a) The main constructions in this section were obtained in [15].

b) The Belyi data. Consider a triangulated equipped surface $\mathfrak{P} \in K \setminus G/K$. We assert that there is a canonical map from $\mathfrak{P}$ to a double triangle $\mathfrak{T}$. More precisely, we send each plus-triangle in $\mathfrak{P}$ to a plus-triangle in $\mathfrak{T}$ according to the colouring of the sides, and we send each minus-triangle in $\mathfrak{P}$ to a minus-triangle in $\mathfrak{T}$. Further, a double triangle can be identified naturally with the Riemann sphere (the vertices with the points 0, 1, and $\infty$ and the edges with the segments $[0,1]$, $[1,\infty]$, and $[\infty,0]$). Then the complex structure can be lifted to the surface $\mathfrak{P}$, and we get an analytic map from the Riemann surface $\mathfrak{P}$ to the sphere $\mathfrak{T}$ with ramification points at the vertices of the triangulation and with three critical values at the points 0, 1, and $\infty$. According to the Belyi theorem [50], [51], a map from an algebraic complex curve to the sphere with three critical values exists precisely for curves defined over the algebraic closure $\mathbb{Q}$ of the field $\mathbb{Q}$.

The correspondence between $K \setminus G/K$ and the set of coverings over the Riemann sphere with three critical values is a strange a posteriori coincidence of two sets. What this means is not yet clear. We remark that by definition the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$ acts on the Belyi data (that is, on the set of coverings over the sphere with three critical values). This group is also not obvious in our approach.

c) Consider a group $P$ and the space of conjugacy classes of $P \times P$ with respect to the diagonal subgroup $P$. The group $\text{GL}(2, \mathbb{Z})$ acts on this space as the group of outer automorphisms of the free group with two generators (see, for instance, [52]). In our case $P = S_\infty$, and $\text{GL}(2, \mathbb{Z})$ acts on $K \setminus G/K$.

4. One-dimensional constructions (chips)

4.1. $(G, K)$-pairs. In §§4–8 we consider numerous examples of pairs (a group $G$, a subgroup $K$). The following properties hold in all the cases discussed below:

1) The pair $G \supset K$ is spherical.\(^{16}\)
2) The category of double cosets is defined.
3) The multiplicativity theorem holds (like Theorem 3.7).
4) An approximation theorem holds (like Theorem 3.11).
5) There is (except for a degenerate case discussed in §8) a three-flow construction of spherical representations as in §§3.11, 3.13, and 3.15.
6) Irreducible representations admit spherical characters (see §3.17).

The arguments in §§2 and 3 can easily be applied to all the cases discussed below. For this reason, we concentrate below on combinatorial realizations of double coset

\(^{16}\)Except for the bisymmetric group discussed in this section, the finite analogues of all pairs $(G, K)$ under discussion are not spherical. It should also be noted that a minor perturbation of our constructions produces large families of non-spherical pairs $G \supset K$ (see [16]). We will not aspire to such an increase in generality.
spaces. The notation \( G, K, G, K \) varies and at any time refers to the specific groups then under discussion.

4.2. Bisymmetric group and Olshanski chips. Let \( G \) be the product \( S_\infty \times S_\infty \) of two copies of the symmetric group, and let \( K \) be the diagonal subgroup. We will represent pairs of permutations as diagrams of the form

\[
\begin{array}{cccccccc}
\cdots & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\begin{array}{cccccccc}
\cdots & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

The two copies (left and right) of the natural numbers are identified by reflection with respect to the dashed line.

Let us explain how to describe the double coset spaces \( K[\alpha] \backslash G \backslash K[\beta] \). Consider an element \( (g_1, g_2) \in G \) and the corresponding diagram. For each \( m > \beta \) we connect by a ‘horizontal’ arc each left circle labelled by \( m \) of the upper row with the corresponding right circle. We do the same for the circles of the lower row with numbers \( > \alpha \) (see Fig. 5, where \( \beta = 3 \) and \( \alpha = 2 \)). Let us put a cross on each ‘horizontal’ arc, and glue horizontal arcs together with ‘vertical’ arcs. We call the picture obtained a chip. A chip can contain arcs of three types.

\[
\begin{array}{cccccccc}
\cdots & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\begin{array}{cccccccc}
\cdots & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

Figure 5. Construction of a chip from two permutations. After gluing the arcs we get a collection of curves. For each curve we remember only its ends (if the curve is not closed) and the number of crosses on the curve.
a) **Arcs going from up to down.** The endpoints of such an arc are located to one side of the dashed axis, and the number of crosses on the arc is even.

b) **Arcs with both endpoints in the upper row or with both endpoints in the lower row.** Then the number of crosses on the arc is odd.

c) **Cycles.** The number of crosses on a cycle is even. We note that the total number of cycles is infinite, but all but finitely many cycles carry precisely two crosses (and such cycles can be removed without loss of information).

Of course, we can replace collections of crosses on arcs by non-negative integers. Multiplication of chips is given by gluing them together (see Fig. 6). The cycles of the factors are put together to form the cycles of the products.

![Figure 6. Multiplication of chips. We identify upper circles in the lower diagram with lower circles in the upper diagram.](image)

In the language of chips, the involution $K[\alpha] \setminus G/K[\beta] \to K[\beta] \setminus G/K[\alpha]$ corresponds to reflection with respect to the horizontal line.

**4.3. Representations.** We repeat the construction in §3.11, except that we must take two factors, $V$ and $W$, instead of the three in (3.5). For a unit vector $\xi \in V \otimes W$ it is easy to verify that there are orthonormal systems $e_j \in V$ and $e'_k \in W$ such that

$$\xi = \sum \alpha_j^{1/2} e_j \otimes e'_j, \quad \sum \alpha_j = 1, \quad \alpha_j \geq 0.$$  

If $V = V^{(0)} \oplus V^{(1)}$ and $W = W^{(0)} \oplus W^{(1)}$ are superspaces as in §3.13, and $\xi \in V \otimes W$ is an even unit vector, then there are orthonormal systems

$$e_j \in V^{(0)}, \quad e'_k \in W^{(0)}, \quad f_m \in V^{(1)}, \quad f'_l \in W^{(1)}$$  

such that

$$\xi = \sum \alpha_j^{1/2} e_j \otimes e'_j + \sum \beta_m^{1/2} f_m \otimes f'_m, \quad \sum \alpha_j + \sum \beta_m = 1, \quad \alpha_j, \beta_j \geq 0.$$  

As in §3.13, we consider the super-tensor product

$$(V \otimes W, \xi) \otimes (V \otimes W, \xi) \otimes \cdots$$  

(4.3)
Infinite symmetric groups

and the action of $G$ in it. The vector $\Xi = \xi^\otimes \infty$ is a spherical vector, and the numbers $\alpha_j$ and $\beta_m$ are the Thoma parameters (and $\gamma = 0$, so not all representations are obtained).

The construction in §3.15 gives the Fock representation with

$$\alpha_1 = e^{-t^2/2}, \quad \alpha_2 = \alpha_3 = \cdots = \beta_1 = \beta_2 = \cdots = 0. \quad \text{(4.4)}$$

**Remark.** A product of two spherical functions in our case is a spherical function. Indeed, consider the tensor product of two spherical representations of $G$ with spherical vectors $v$ and $w$. Then it contains a unique $K$-fixed vector,\footnote{For more details, see [32], Theorem 4. Let $G \supset K$ be a spherical pair, and assume that $K$ does not admit non-trivial finite-dimensional representations. Then it is easy to prove that a product of spherical functions is a spherical function.} and the spherical function is obviously the product of the spherical functions. Multiplying the spherical functions with parameters $\alpha_i, \beta_i$ and $\alpha'_i, \beta'_i$, we get a function whose alpha-parameters are $\alpha_i \alpha'_j$ and $\beta_i \beta'_j$, and whose beta-parameters are $\alpha_i \beta'_j$ and $\beta_i \alpha'_j$.

Considering tensor products of representations on super-tensor powers (4.3) with Fock representations, we get all the Thoma spherical functions, except for $\gamma = 1$. The last representation is realized on $\ell_2(S_\infty)$, on which the group $S_\infty \times S_\infty$ acts by left and right shifts.

**4.4. Another example.** We now consider the set $\Omega = \mathbb{N} \times \{0, 1\}$ and the group $G = S_\infty(\Omega)$. Let the subgroup $K$ act on $\mathbb{N} \times \{0, 1\}$ by permutations of the factor $\mathbb{N}$ (in comparison with the previous example, we have the same subgroup $K$, but the group $G$ is enlarged). We represent elements of $G$ as diagrams of the form

$$\ldots 7 6 5 4 3 2 1 1 2 3 4 5 6 7 \quad \text{(4.5)}$$

(now we allow arcs going from the left side of the picture to the right side, and vice versa). To pass to double coset spaces (see Fig. 7, (a)) $K[\alpha] \backslash G/K[\beta]$, we connect as above the corresponding elements of the upper row with the same indices $> \beta$ by an arc. However, now we also draw an arrow from right to left and attach the sign $+$ to this arrow. Similarly, we draw arrows on arcs between the corresponding elements of the lower row and attach the sign $-$ to these arrows.

After gluing we get a collection of chains. Either these chains are closed or both their endpoints are black circles in the figure. The chains consist of interlacing links of two types: arrows with signs, and just segments. Segments can be contracted to points without loss of information (the only exception are chains consisting of just one segment). A piece of a chain is shown in Fig. 7, (b). Note that the pluses and minuses on the arrows alternate.

**Remark.** In this construction, pluses and minuses are essential only to distinguish the orientations of the cycles.
The rule for multiplication of double cosets is clear.

The constructions of representations also extend easily to this case. For instance, consider a Hilbert space $V$ and a unit vector $\xi \in V \otimes V$, and consider the countable tensor product

$$(V \otimes V, \xi) \otimes (V \otimes V, \xi) \otimes \cdots.$$ 

The group $G$ acts by permutations of the factors $V$, and the subgroup $K$ by permutations of the blocks $(V \otimes V, \xi)$. The action of $K$ admits an extension to an action of the complete symmetric group $K = S_\infty$.

4.5. **A more complicated example.** We now consider the union $\Omega$ of four copies of $\mathbb{N}$; it is convenient to think that $\Omega = \mathbb{N} \times \{1, 2, 3, 4\}$. Let $G = S_\infty(\Omega)$, and consider the subgroup $K_1 \subset G$ which fixes all the elements of the set $\mathbb{N} \times \{3, 4\}$, while on the product $\mathbb{N} \times \{1, 2\}$ it acts by permutations of the factor $\mathbb{N}$. The subgroup $K_2$ acts similarly on $\mathbb{N} \times \{3, 4\}$, fixing the elements of $\mathbb{N} \times \{1, 2\}$ (see Fig. 8). We display elements of this group by pictures of the form (4.1) and (4.5), but now we must draw 4 copies of $\mathbb{N}$. To be definite we arrange them in the order

$$\ldots 5_1 4_1 3_1 2_1 1_1 1_2 2_2 3_2 4_2 5_2 \ldots \quad \ldots 5_3 4_3 3_3 2_3 1_3 1_4 2_4 3_4 4_4 5_4 \ldots \ldots$$

Figure 7. Re §4.4.

Figure 8. Re §4.5. The group $K_1$ acts by permutations of columns of the upper row, the group $K_2$ permutes columns of the lower row.
Let

\[ K[\alpha_1, \alpha_2] := K_1[\alpha_1] \times K_2[\alpha_2]. \]

Thus, we get a multiplication

\[ K[\alpha_1, \alpha_2] \setminus G/K[\beta_1, \beta_2] \times K[\beta_1, \beta_2] \setminus G/K[\gamma_1, \gamma_2] \rightarrow K[\alpha_1, \alpha_2] \setminus G/K[\gamma_1, \gamma_2]. \]

Let us construct the picture corresponding to a double coset (see Fig. 9). We draw arrows joining corresponding elements of \( \mathbb{N} \times \{2\} \) and \( \mathbb{N} \times \{1\} \), and we colour these arrows, say, blue. On the upper arrows we put a +, and on the lower arrows a −. We perform the same operation with arrows from \( \mathbb{N} \times \{4\} \) to \( \mathbb{N} \times \{3\} \), except that we colour the arrows red (in the figure we mark ‘red’ by lines dashed with boldface dots). Then we glue the chains together just as above.

![Figure 9. Re §4.5. Constructing a chip. In the upper row are four copies of \( \mathbb{N} \) (that is, the set \( \mathbb{N} \times \{1, 2, 3, 4\} \) stretched out to form the line). In the lower row is another copy of the set \( \mathbb{N} \times \{1, 2, 3, 4\} \). An element of \( G \) is represented as a collection of strings going from up to down.]

It remains to explain how to construct representations. We consider a Hilbert space \( V \), two unit vectors \( \xi \in V \otimes V \) and \( \eta \in V \otimes V \), and the tensor product

\[ (V \otimes V, \xi) \otimes (V \otimes V, \eta) \otimes (V \otimes V, \xi) \otimes (V \otimes V, \eta) \otimes \cdots. \]

The group \( G \) acts by permutations of the factors \( V \), the subgroup \( K_1 \) by permutations of the factors \( (V \otimes V, \xi) \), and the group \( K_2 \) by permutations of the factors \( (V \otimes V, \eta) \). The action of \( K_1 \) (and that of \( K_2 \)) can be extended to an action of the complete symmetric group.

4.6. Remarks concerning §4. a) The Brauer category (see [53] and also [54]). The objects of the Brauer category are non-negative integers. A morphism from \( m \) to \( n \) is a diagram of the form shown in Fig. 10, (a), with \( m \) boldface dots in the upper row and \( n \) boldface dots in the lower row (\( m - n \) must be even). Forming products of morphisms \( m \rightarrow n \) and \( n \rightarrow k \) corresponds to gluing diagrams together.

Let us consider the space \( V = \mathbb{C}^p \) equipped with a symmetric non-degenerate bilinear form (and, correspondingly, with the action of the orthogonal group \( O(p) \)). From the diagram we construct an operator \( V^\otimes m \rightarrow V^\otimes n \). To vertical arcs we assign permutations of factors, to upper horizontal arcs we assign convolutions with respect to the corresponding indices, and to lower horizontal arcs we assign multiplications by invariants. To a cycle we assign multiplication by the scalar \( p \).
As a result, we get an action of the Brauer category by $O(p)$-intertwining operators on tensor powers of $V$.

Literally the same construction applies for the symplectic groups $\text{Sp}(2n, \mathbb{C})$ (in this case we assign multiplication by $(-2n)$ to a cycle), and also for the full linear groups. In the last case we take mixed tensors $V^\otimes m \otimes (V')^\otimes k$ with fixed $m - k$. The corresponding set of boldface dots then splits into two parts ($m$ points and $k$ points). Horizontal arcs can connect only points from different groups (convolutions are possible only for pairs $V, V'$), and vertical arcs connect only points from groups of the same type (see Fig. 10).

b) A construction of the spherical representations of the bisymmetric group was obtained as a result of several transformations from the construction of Vershik and Kerov [3]. The action on the space (4.3) was introduced in [6]. A general construction was proposed in [55], and we follow a construction in [56].

4.7. Addendum to §4. Classification of representations of the bisymmetric group. We return to the notation of §§4.2 and 4.3.

4.7.1. Duality. Denote by $\text{U}(V)$ and $\text{U}(W)$ the groups of all unitary operators on $V$ and $W$, respectively. Denote by $\mathcal{U}$ the subgroup of $\text{U}(V) \times \text{U}(W)$ consisting of all operators $A \otimes B$ fixing the vector $\xi$. It easy to see that this group is a direct product of unitary groups:

$$\mathcal{U} := \prod_{x \in [-1,1], x \neq 0} \text{U}(n_x),$$

where $n_x$ is the multiplicity of the occurrence of $x$ in the collection of numbers $\alpha_i, \beta_j$. In fact, this product is finite or countable (moreover, almost all the non-trivial factors have the form $\text{U}(1)$), and the group $\mathcal{U}$ obtained is compact and separable. Its irreducible representations are tensor products of representations of the factors (in fact, these tensor products are finite; see [57], §27.43). Next, we note that the pairs of operators of the form $(e^{i\theta}, e^{-i\theta})$ are contained in the centre of $\mathcal{U}$ and act trivially on $V \otimes W$.

The group $\mathcal{U}$ acts on each factor of the product (4.3). Since $\mathcal{U}$ fixes $\xi$, it acts on the whole tensor product. Obviously $\mathcal{U}$ commutes with the action of the bisymmetric group $G$. The following analogue of the Schur–Weyl duality holds (see [6]).

**Theorem 4.1.** a) The groups $G$ and $\mathcal{U}$ are dual on the tensor product (4.3) in the following sense. A representation of $G \otimes \mathcal{U}$ is a direct sum of representations
\( \rho_\nu \otimes \pi_\nu \), where the \( \rho_\nu \) are unitary representations of \( G \) and the \( \pi_\nu \) are unitary representations of \( \mathcal{U} \), and moreover, the \( \rho_\nu \) are pairwise distinct and the \( \pi_\nu \) are pairwise distinct.

b) A spherical subrepresentation of \( G \) is realized on the space of \( \mathcal{U} \)-fixed vectors.

4.7.2. Classifying unitary representations of the bisymmetric group. Let \( \rho \) be an irreducible unitary representation of the bisymmetric group \( G \) on a space \( H \), and let \( H[n] \) be the subspaces of \( K[n] \)-fixed vectors. The representation \( \rho \) has a spherical character (see §3.17), which in our case must be one of the Thoma characters (and therefore the numbers \( \alpha_i, \beta_j \), and \( \gamma \) are defined).

Figure 11. (a) The unit chip. (b) The projection \( H[n + 1] \to H[n] \). (c) An element of the semigroup \( \Gamma_1 \).

Irreducible \( \ast \)-representations of the chip semigroups \( K[n] \setminus G/K[n] \) compatible with the involution are realized on the subspaces \( H[n] \). Arguments in the proof of Lemma 2.7 show that any such representation uniquely determines the initial representation \( \rho \). Let \( n \) be the smallest positive integer for which \( H[n] \neq 0 \). It is easy to verify that the chip in Fig. 11, (b) corresponds to the projection operator from \( H[n] \) to \( H[n - 1] \). Since this operator is zero, all the operators corresponding to chips with horizontal arcs are zero. Denote the semigroup of chips without horizontal arcs by \( \Delta_n \). It splits naturally into a direct product

\[
\Delta_n = \Xi \times \Gamma_1 \times \Gamma_2
\]

of three semigroups, where \( \Xi \) is the free Abelian semigroup whose elements are the unit chip (see Fig. 11, (a)) together with a collection of cycles. The semigroups \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic, \( \Gamma_1 \) consists of diagrams of the form shown in Fig. 11, (c), where on the left-hand side there are no crosses on arcs, and \( \Gamma_2 \) consists of the bilaterally reflected diagrams.

The semigroup \( \Xi \) is isomorphic to \( K[0] \setminus G/K[0] \). Its action is determined by the spherical character of the representation.

The semigroups \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic to the semidirect product \( \Gamma := S_n \rtimes \mathbb{Z}_n^+ \).

We denote Young diagrams by \( \lambda, \mu, \ldots \). The number of cells in \( \lambda \) is denoted by \( |\lambda| \), the number of rows by \( l(\lambda) \), and the number of columns by \( h(\lambda) \). We recall that irreducible representations of the group \( S_k \) have a canonical enumeration by \( k \)-cell Young diagrams (see, for instance, [46], §16.2).

We define a diagram distribution \( \Lambda \) to be a finite subset \( \mathcal{I} \) (the support of the distribution) of the interval \([-1, 1]\) such that to each point \( x \in \mathcal{I} \) there corresponds a Young diagram \( \lambda(x) \) (see Fig. 12) and \( \sum_{x \in \mathcal{I}} |\lambda(x)| = n \).
Lemma 4.2. Irreducible \(*\)-representations of the semigroup \(\Gamma\) are enumerated by diagram distributions.

This can easily be proved using the Wigner–Mackey arguments (see [46], §13.3). Let us describe the representation corresponding to a diagram distribution \(\Lambda\). Consider the following subsemigroup of \(\Gamma\):

\[
\Gamma_\Lambda := \prod_{x \in I} S_{|\lambda(x)|} \rtimes \mathbb{Z}_+^{[\lambda(x)]}.
\]

We take the irreducible representation of \(S_{|\lambda(x)|}\) corresponding to the diagram \(\lambda(x)\) and extend it to the semigroup \(S_{|\lambda(x)|} \rtimes \mathbb{Z}_+^{[\lambda(x)]}\), assuming that the generators of the semigroup \(\mathbb{Z}_+^{[\lambda(x)]}\) all act as multiplications by \(x\). Next we consider the tensor product of such representations and induce\(^{18}\) a representation of \(\Gamma\) from the constructed representation of the semigroup \(\Gamma_\Lambda\).

Thus, an irreducible representation of the semigroup \(\Delta_n\) is determined by the Thoma parameters \(\alpha_i, \beta_j\) and two diagram distributions, say, \(\Lambda\) and \(M\). However, not all such representations are suitable for us.

Theorem 4.3 (Olshanski–Okounkov [6], [33]). The representation of \(\Delta_n\) determined by the parameters \(\alpha, \beta\) and \(\Lambda, M\) can be realized on the space \(H[n]\) of fixed vectors of an irreducible representation of the bisymmetric group \(G\) of depth \(n\) if and only if the following conditions hold.

- The supports of the distributions \(\Lambda\) and \(M\) are contained in the set \(\{\alpha_1, \alpha_2, \ldots, -\beta_1, -\beta_2, \ldots, 0\}\).

- For \(x > 0\) the number \(l(\lambda(x)) + l(\mu(x))\) does not exceed the multiplicity of the occurrence of \(x\) in the sequence \(\alpha_i\).

- For \(x < 0\) the number \(h(\lambda(x)) + h(\mu(x))\) does not exceed the multiplicity of the occurrence of \(x\) in the sequence \((-\beta_j)\).

We remark that a reconstruction of \(\rho\) from such data is constructive in a certain sense (see the proof of Lemma 2.7), however, the question of more explicit realizations is still open. Incidentally, for spherical representations of \(G\) with \(\gamma > 0\) an explicit description of the representation is apparently not known.

\(^{18}\)Formally, the induction is defined for representations of groups and not for semigroups. However, the definition in [47], §7, is applicable in our case.
5. Two-dimensional constructions

5.1. Example. Consider the set \( \Omega = \mathbb{N} \times \{1, 2, 3\} \). We treat elements of the subsets \( \mathbb{N} \times \{1\} \), \( \mathbb{N} \times \{2\} \), and \( \mathbb{N} \times \{3\} \) as coloured red, yellow, and blue, respectively. Therefore, any element of \( \Omega \) is uniquely determined by its number and its colour. Let \( G = S_\infty(\Omega) \). By \( K \) we denote the group of all finite permutations of \( \mathbb{N} \times \{1, 2, 3\} \) induced by permutations of \( \mathbb{N} \).

Remark. In comparison with the trisymmetric group in §3, the group \( G \) has been enlarged, and the group \( K \) remains the same.

To each element of the group \( G \) we will assign a triangulated surface with certain additional data. To do this, we consider a collection of identical triangles whose sides are coloured red, yellow, and blue clockwise from the inside. We enumerate these triangles and assign a plus to each of them (see Fig. 13, (a)). Next, we construct a countable collection of reflected coloured triangles, we also enumerate them, and we assign a minus to each triangle. Suppose that a \( g \in G \) sends the element of \( \Omega \) with number \( k \) and colour \( \nu \) to the element with number \( l \) and colour \( \mu \). Then we glue the \( \nu \)-edge of the \( k \)th plus-triangle together with the \( \mu \)-edge of the \( l \)th minus-triangle according to the orientation. As in §3, we get an oriented triangulated surface (see Fig. 13, (b)) such that:

- the plus-triangles and minus-triangles are arranged checkerwise;
- the plus-triangles are enumerated by \( \mathbb{N} \), as are the minus-triangles;
- on the inner sides of plus-triangles the colours are arranged clockwise, on the inner sides of minus-triangles they are anticlockwise, and the colours on an edge with two sides can be different in general;
- all the components of the surface are compact, and all but finitely many of them are glued together from a pair of triangles in such a way that the numbers on the faces coincide and the colours on the two sides of each edge of the double triangles coincide.

It is easy to see that an element \( g \) can be uniquely reconstructed from such data: we must just examine all the edges and look at the colours and labels on the two sides of an edge.
Passing to the double coset space \( K[\alpha] \backslash G / K[\beta] \) means forgetting the plus-labels > \( \beta \) and the minus-labels > \( \alpha \).

All that has been said above about the trisymmetric group extends to our case more or less literally.

5.2. Degree of generality. We consider a finite collection of countable sets \( \Omega_1, \ldots, \Omega_p \) and let

\[
G = S_\infty(\Omega_1) \times \cdots \times S_\infty(\Omega_p).
\]

Let \( K_1, \ldots, K_q \) be copies of the infinite symmetric group, and assume that each group \( K_j \) acts on each \( \Omega_i \) in such a way that all the orbits either are one-point sets or are constructed like \( S_\infty / S_\infty[1] \cong \mathbb{N} \). Suppose that the non-trivial orbits of different groups \( K_j \) are disjoint. Then we get an action of the group \( K_1 \times \cdots \times K_q \) on \( \Omega_1 \cup \cdots \cup \Omega_p \). Therefore, we get an embedding

\[
K_1 \times \cdots \times K_q \to G.
\]

Assume (although we can get around this restriction) that the group \( K_1 \times \cdots \times K_q \) has no fixed points on \( \Omega_1 \cup \cdots \cup \Omega_p \).

Then for each element of \( G \) we can construct a two-dimensional surface tiled by polygons and coloured in a certain way (as in the examples discussed above). Different types of polygons also must be given different colours\(^{19} \) corresponding to different factors \( K_j \); the number of sides of a polygon corresponds to the number of non-trivial orbits of the group \( K_j \).

We omit the statement of a formal rule; it is contained in [6].

All that was said about the trisymmetric group in §3 can be extended to this general situation.

Remark. If each group \( K_j \) has \( \leq 2 \) non-trivial orbits, then we glue surfaces together from digons, and in this case the language of chips can be used. There is also the case when each group \( K_j \) has a unique non-trivial orbit. This case is examined in §8.

5.3. Wreath products. Let \( H \) be a group, and consider the group \( H^\mathbb{N} \) of functions \( f \) on \( \mathbb{N} \) with values in \( H \) and equal to 1 for almost all \( n \in \mathbb{N} \). The product is defined pointwise. The group \( S_\infty \) acts on \( H^\mathbb{N} \) by permutations of \( \mathbb{N} \). Therefore, we can define the semidirect product \( S_\infty \rtimes H^\mathbb{N} \). This group is called the wreath product of the groups \( S_\infty \) and \( H \), and we denote it by

\[
S_\infty \rtimes H := S_\infty \ltimes H^\mathbb{N}.
\]

In fact, we encounter the cases \( H = \mathbb{Z}_k \) and \( H = S_k \) below. Let us describe the corresponding wreath products in a more convenient form.

Consider the set \( \mathbb{N} \times \{1, \ldots, k\} \). We can regard it as the set of points of the integer lattice in a horizontal strip. We divide this strip into vertical columns (see Fig. 14). The wreath product of \( S_\infty \) and \( S_k \) consists of the permutations

\(^{19}\)Since the colours of polygons distinguishing different groups \( K_j \) have an origin different from the colours of the edges, we could use some other attribute instead of ‘colour’, such as ‘perfume’, ‘melody’, ‘citizenship’, and so on.
preserving the partition into columns (that is, we can permute columns and then take an arbitrary permutation in each column).

The wreath product of $S_\infty$ and $\mathbb{Z}_k$ is the group of permutations preserving the partition into columns and preserving cyclic order inside columns.

We also introduce the completed wreath product $\overline{S_\infty \rtimes H}$, which is the semidirect product of the complete symmetric group and the group of all functions $\mathbb{N} \to H$.

### 5.4. An example with a wreath product.

Let $\Omega = \mathbb{N} \times \{1, 2, 3\}$ and $G = S_\infty(\Omega)$. As a subgroup $K$ we take the wreath product $S_\infty \rtimes \mathbb{Z}_3$. By $K[\alpha]$ we denote the subgroup of $K$ fixing all the elements of the first $\alpha$ columns.

For $g \in G$ we construct a two-dimensional surface just as in §5.1, except that there is a difference when we pass to double cosets. In that case, when removing the label from a triangle we must also remove the colouring of the inner side of its perimeter.

**Example.** An element of $K \setminus G/K$ is a triangulated surface on whose faces we arrange pluses and minuses checkerwise.

We explain what must change in passing to representations. Consider the Hilbert space $V$ and a vector $\xi \in V \otimes V \otimes V$ which is fixed under cyclic permutations of the factors, and then consider the tensor product

$$(V \otimes V \otimes V, \xi) \otimes (V \otimes V \otimes V, \xi) \otimes \cdots .$$

The group $G$ acts by permutations of the factors $V$, and the group $K$ by permutations of the blocks $(V \otimes V \otimes V, \xi)$ and cyclic permutations of the factors in each block $V \otimes V \otimes V$. The action of $K$ can be extended to an action of the completed wreath product $\overline{S_\infty \rtimes \mathbb{Z}_3}$.

### 6. Categories of bipartite graphs

Here we consider only one example (but this construction has numerous variations).

#### 6.1. The correspondence between permutations and bipartite 3-valent graphs

Let $G$ be the group of finitely supported permutations of the set $\Omega := \mathbb{N} \times \{1, 2, 3\}$, and let $K$ be the wreath product of $S_\infty$ with $S_3$. As above, we think of the elements of the sets $\mathbb{N} \times \{1\}$, $\mathbb{N} \times \{2\}$, and $\mathbb{N} \times \{3\}$ as being coloured red, yellow, and blue, respectively. We construct a countable collection of such
pictures: for a vertex from which three segments go out (we call them *semi-edges*), the segments are coloured red, yellow, and blue. To each vertex we assign a plus and a number $\in \mathbb{N}$. Then we construct another countable family of the same pictures, but to each of those vertices we assign a minus and a number (see Fig. 15, (a)). Suppose that the $g \in G$ sends an element of colour $\nu$ and number $k$ to an element of colour $\mu$ and number $l$. Then we glue the semi-edge of colour $\nu$ going out from the plus-vertex with number $k$ to the semi-edge of colour $\mu$ going out from the minus-vertex with number $l$. We do this for each element of $\Omega$ and get a bipartite graph (see Fig. 15, (b)) with the following additional structures and properties:

- on vertices there are pluses and minuses, and moreover, edges can connect only plus-vertices with minus-vertices;
- precisely three edges go out from each vertex;
- semi-edges are coloured red, yellow, and blue, and the semi-edges going out from a particular vertex are coloured in different colours;
- the plus-vertices are numbered, as are the minus-vertices;
- the graph consists of finite components, and almost all of them are two-vertex components with coinciding numerical labels on the two vertices and coinciding colours on the halves of each edge (see Fig. 15, (c)).

It is easy to see that an element of the symmetric group can be uniquely reconstructed from this picture.

Consider the larger group $\mathbf{K} \supset \mathbf{K}$ which is the completed wreath product $\overline{S}_\infty \overline{S}_3$ and the group $\mathbf{G}$ of permutations which is generated by $\mathbf{K}$ and $G$. We can apply to this group the same procedure for constructing a graph. The list of properties of the graph obtained in this way is almost the same, except that the last condition must be weakened to the following:

- the graph consists of finite components and almost all of them are two-vertex components (see Fig. 15, (d)).

### 6.2. Category of double cosets.

Let $K[\alpha]$ be the subgroup of $K$ fixing all points in the set

$$\{1, \ldots, \alpha\} \times \{1, 2, 3\} \subset \mathbb{N} \times \{1, 2, 3\}.$$
Passing to the double coset space corresponds to forgetting the plus-labels with numbers \( > \beta \) and the colours of the semi-edges going out from these vertices, and also forgetting the minus-labels with numbers \( > \alpha \) and the colours of the incident semi-edges.

**Example.** Passing to \( K[0] \setminus G/K[0] \) means forgetting all the numerical labels and all the colours. Let us consider the same construction for the finite groups \( G = S_{3N} \) and \( K = S_N \times S_3 \). Then we get that

\[
(S_N \times S_3) \setminus S_{3N}/(S_N \times S_3^N)
\]

is in a one-to-one correspondence with the set of all \( 2N \)-vertex bipartite 3-valent graphs (the plus-part and the minus-part remain distinguished).

Let us consider double cosets

\[
p \in K[\alpha] \setminus G/K[\beta] \quad \text{and} \quad q \in K[\beta] \setminus G/K[\gamma].
\]

We describe their product in the language of graphs (see Fig. 16). For \( k \leq \beta \) take a plus-vertex of \( p \) with label \( k \) and a minus-vertex of \( q \) with label \( k \). From the graphs we cut off these vertices together with their incident semi-edges, and we glue the resulting three half-cut edges of the graph \( p \) together with the resulting three half-cut edges of the graph \( q \) according to the colours of the removed semi-edges. Doing this for all \( k \), we get a morphism from \( \gamma \) to \( \alpha \).

### 6.3. Representations of the group \( G \).

These representations can be constructed just as above. For instance, we can consider a (super-)Hilbert space \( V \), an (even) vector \( \xi \in V \otimes V \otimes V \) invariant with respect to permutations of the factors, and an action of the group \( G \) in the tensor product

\[
(V \otimes V \otimes V, \xi) \otimes (V \otimes V \otimes V, \xi) \otimes \cdots.
\]

To repeat the construction of representations on the Fock space, we consider a Hilbert space \( L \) with an orthonormal basis \( e_{k,\nu} \), where \( \nu = 1, 2, 3 \) and \( k \) ranges
over \( \mathbb{N} \). The group \( G \) acts in \( L \) in the natural way. Next, we consider the tensor product \( L \otimes L \otimes L \), as \( u \) (see §3.10) we take

\[
u := \sum_{k=1}^{\infty} \sum_{\sigma \in S_3} e_{k,\sigma(1)} \otimes e_{k,\sigma(2)} \otimes e_{k,\sigma(3)},
\]

and we repeat the construction in §3.15.

6.4. Variations. A) Let \( G \) be the group of all finitely supported permutations of the set \( \Omega := \mathbb{N} \times \{1, \ldots, l\} \), and let \( K \) be the wreath product of \( S_\infty \) and \( S_l \). Then all the above considerations can be repeated in our case, with the single difference that we get bipartite \( l \)-valent graphs instead of 3-valent ones.

![Figure 17. Re §6.4. A triangulation and the dual ribbon graph.](image)

B) Let

\[
G = S_\infty(\mathbb{N} \times \{1, \ldots, 6\}), \quad K = (S_\infty \rtimes S_3) \times (S_\infty \rtimes S_3).
\]

Then we introduce two ‘colours’ (or ‘perfumes’) for colouring the vertices of a graph, and we repeat the same construction.

C) Here we can work in a wider generality as in §5.2.

D) The constructions in §5 can be presented using the language of ribbon graphs. We recall that a ribbon graph is a graph with a fixed cyclic order of the edges at each vertex. There is a standard way to assign a ribbon graph to any oriented triangulated surface (see Fig. 17).

7. Bordisms of pseudomanifolds

7.1. Pseudomanifolds. We recall the definition of a pseudomanifold.

First, we define a simplicial cell complex. Consider a disjoint union \( \bigcup \Xi_j \) of finitely many simplices \( \Xi_j \), and then the quotient-space \( \Sigma \) of the space \( \bigcup \Xi_j \) with respect to a collection of equivalences with the following properties.

a) For any simplex \( \Xi_i \) the tautological map \( \xi_i : \Xi_i \to \Sigma \) is an embedding, and therefore we can regard \( \Xi_i \) as a subset of \( \Sigma \).

b) For any pair of simplices \( \Xi_i, \Xi_j \) their intersection \( \xi_i^{-1}(\xi_i(\Xi_i) \cap \xi_j(\Xi_j)) \subset \Xi_i \) is a union of faces of \( \Xi_i \), and the partially defined map

\[
\Xi_i \xrightarrow{\xi_i} \Sigma \xrightarrow{\xi_j^{-1}} \Xi_j
\]

is affine on each face.
Such objects are called *simplicial cell complexes*.\(^{20}\)

A *pseudomanifold* (for details, see [58], [59]) of dimension \(n\) is a simplicial cell complex satisfying the following conditions.

a) Each face is contained in an \(n\)-dimensional face (we call \(n\)-dimensional faces *chambers*).

b) Each \((n - 1)\)-dimensional face is contained in precisely two chambers.\(^{21}\)

**Remark.** We recall the origins of the notion of a pseudomanifold. It is well known (R. Thom) that generally speaking a singular cycle in a manifold cannot be realized as an image of a manifold. On the other hand, any singular \(\mathbb{Z}\)- (or \(\mathbb{Z}_2\))-cycle can be realized as a pseudomanifold (by the definition of a cycle; see [58]). Also, intersection homologies were introduced by Goretski and MacPherson in terms of pseudomanifolds [60].

**Link.** Let \(\Sigma\) be a pseudomanifold and \(\Gamma\) a \(k\)-dimensional face of it. Consider all \((k + 1)\)-dimensional faces \(\Phi_j\) containing \(\Gamma\) and choose a point \(\phi_j\) in the interior of each face \(\Phi_j\). For any face \(\Psi_k \supset \Gamma\) consider the convex hull of all the points \(\phi_j\) contained in \(\Psi_k\). The link of the face \(\Gamma\) is the simplicial cell complex whose faces are such convex hulls.

A pseudomanifold is said to be *normal* if the link of any face of codimension \(\geq 2\) is connected.

A *normalization* \(\widetilde{\Sigma}\) of a pseudomanifold \(\Sigma\) (see [60]) is a normal pseudomanifold \(\widetilde{\Sigma}\) and a map \(\pi: \widetilde{\Sigma} \to \Sigma\) such that

- the restriction of \(\pi\) to each face of \(\widetilde{\Sigma}\) is an affine bijection of faces;
- the map \(\pi\) sends distinct \(n\)-dimensional and \((n - 1)\)-dimensional faces to distinct faces.

*Any pseudomanifold has a unique normalization up to a natural equivalence* [60].

---

\(^{20}\)The standard definition of a simplicial complex contains the more stringent requirement that the intersection of two simplices either is empty or is a face.

\(^{21}\)Usually, ‘strong connectedness’ is also required: the complex is connected and remains connected after removing all faces of codimension \(\geq 2\).
Then the set $\mathcal{O} \setminus \Xi$ is disconnected and consists of $m$ components; denote them by $\mathcal{O}_1, \ldots, \mathcal{O}_m$. Let $\bar{\mathcal{O}}_j$ be the closure of $\mathcal{O}_j$ in $\Sigma$, $\bar{\mathcal{O}}_j = \mathcal{O}_j \cup \Xi_j$. We replace $\mathcal{O}$ by the disjoint union of the sets $\bar{\mathcal{O}}_j$ and get a new pseudomanifold $\Sigma'$. We repeat the same operation with another face with a disconnected link. These operations increase the number of faces of codimension $\geq 2$. The faces of dimensions $n$ and $(n - 1)$ remain the same (and their inclusion relations are preserved). Therefore, the process is finite, and as a result we get a normal pseudomanifold.

### 7.2. Collections of permutations and pseudomanifolds.

As a group $G$ we take the product of $(n+1)$ copies of the group $S_\infty$, and $K$ is the diagonal subgroup.\(^{22}\)

We take $n + 1$ kinds of colours (red, orange, ..., blue) and paint copies of the group $S_\infty$ with these colours. Then we take a countable collection of identical $n$-dimensional simplices whose faces have these colours (different faces have different colours). Separately, we colour vertices in such a way that the colour of a vertex coincides with the colour of the opposite face. We enumerate these simplices, and attach a $+$ to each simplex. Now we construct another collection of simplices differing from those of the first collection by a mirror reflection.\(^ {23}\) We enumerate these simplices and attach a $-$ to each of them.

Suppose that an element of, say, a green copy of $S_\infty$ sends $k$ to $l$. Then we glue the green face of the $k$th plus-simplex together with the green face of the $l$th minus-simplex in such a way that each vertex is identified with a vertex of the same colour. We do this for all the copies of the symmetric group and all the numbers $k$. As a result, we get a simplicial cell complex which is a countable disjoint union of oriented normal pseudomanifolds. This complex satisfies the following conditions:

- chambers are marked by pluses and minuses checkerwise, that is, a neighbour (across an $(n - 1)$-dimensional wall) of a plus-chamber is a minus-chamber and vice versa;
- plus-chambers (respectively, minus-chambers) are enumerated by the natural numbers;
- the $(n - 1)$-dimensional faces are coloured with the colours indicated above, the colours of the $(n - 1)$-dimensional walls of each chamber are pairwise distinct, and the colours of the walls of plus-chambers and minus-chambers correspond to the orientation, moreover, the vertices are also coloured so that the colour of a vertex differs from all the colours of the adjacent $(n - 1)$-dimensional faces;
- almost all components are obtained by gluing together two simplices with the same labels on a plus-simplex and a minus-simplex.

It is easy to see that such collections of data are in a one-to-one correspondence with the group $G$.

### 7.3. Bordisms of pseudomanifolds.

The passage to the double coset space is as usual: we forget plus-labels with numbers $> \beta$ and minus-labels with numbers $> \alpha$.

Let $p \in K[\alpha] \setminus G/K[\beta]$ and $q \in K[\beta] \setminus G/K[\gamma]$. For any $k \leq \beta$ we take a minus-chamber of the pseudomanifold $q$ and a plus-chamber of the

\(^{22}\)Recall that this pair (a group, a subgroup) was considered above in § 3.18.

\(^{23}\)We remark that all the colourings of the $n$-faces of a chamber in $n + 1$ pairwise different colours are equivalent with respect to affine transformations of $\mathbb{R}^n$. If we consider only proper affine transformations, then we get two non-equivalent types of simplices.
pseudomanifold \( p \). We remove these chambers from the pseudomanifolds and glue the boundaries of the resulting holes together according to the colours of their \((n-1)\)-dimensional faces. Doing this for all \( k \leq \beta \), we get a new pseudomanifold with coloured \((n-1)\)-dimensional faces and vertices, with pluses and minuses arranged checkerwise, and also with numerical plus-labels \( \leq \gamma \) and numerical minus-labels \( \leq \alpha \). In general, this pseudomanifold is not normal. Its normalization corresponds to the product of double cosets \( p \circ q \).

### 7.4. Correspondence with the two-dimensional construction.
In §3.18 two-dimensional surfaces tiled by \( n \)-gons were assigned to the same double coset spaces \( K[\alpha] \setminus G/K[\beta] \). We explain how to obtain the same surface starting from the pseudomanifold. Let us number the \((n-1)\)-dimensional faces \( A_1, \ldots, A_n \) of our model simplex \( \Delta \). In the interior of each intersection \( A_1 \cap A_2, A_2 \cap A_3, \ldots, A_n \cap A_1 \) we choose a point and connect these points by a closed polygonal line. We colour each segment of the line with the colour of the face through which it passes. Consider a two-dimensional surface \( \Pi \subset \Delta \) homeomorphic to a disc whose boundary coincides with our polygonal line and whose interior is contained in the interior of the simplex. It remains to note that all the chambers of our pseudomanifold are copies of the simplex \( \Delta \), and therefore in each chamber we have a copy of the surface \( \Pi \). Taking the union of these surfaces, we get a two-dimensional surface tiled by \( n \)-gons.

### 7.5. Remarks concerning §7.
Constructions of pseudomanifolds from coloured graphs were considered in [61]–[63]. In [59] and [64] the correspondence between pseudomanifolds and collections of involutions was used. Here we follow [17].

### 8. Spherical functions with respect to a Young subgroup. Nessonov theorem

In the general language of §5 and [16], the object described below corresponds to two-dimensional surfaces glued together from monogons (see Fig. 4).

#### 8.1. The \((G, K)\)-pair with a Young subgroup.
Now let
\[
G = S_\infty(\mathbb{N} \times \{1, \ldots, m\}),
\]
and let \( K_j = S_\infty(\mathbb{N} \times \{j\}) \), \( K_j = S_\infty(\mathbb{N} \times \{j\}) \), and
\[
K = K_1 \times \cdots \times K_m, \quad K = K_1 \times \cdots \times K_m.
\]
Denote by \( G \) the group generated by \( G \) and \( K \).

For \( g \in G \) we define \( s_{\nu\mu}(g) \) to be the number of elements of colour \( \nu \) that are sent by our permutation \( g \in G \) to elements of colour \( \mu \). For any \( \mu \) these numbers satisfy the relation
\[
\sum_{\nu; \nu \neq \mu} s_{\nu\mu} = \sum_{\nu; \nu \neq \mu} s_{\mu\nu}, \quad (8.1)
\]
(the number of elements leaving the set \( \mathbb{N} \times \{\mu\} \) equals the number of incoming elements).
8.2. Constructing representations. We construct a collection of representations of the group $G$. Consider a Euclidean space $V$, $m$ unit vectors $\xi_1, \ldots, \xi_m \in V$ generating $V$, and the tensor product

$$[(V, \xi_1) \otimes (V, \xi_1) \otimes \cdots] \otimes [(V, \xi_2) \otimes (V, \xi_2) \otimes \cdots] \otimes \cdots.$$ 

The group $G$ acts by permutations of the factors $V$, and each $K_j$ acts by permutations in each group. The vector

$$\xi_1^{\otimes \infty} \otimes \cdots \otimes \xi_m^{\otimes \infty}$$

is $K$-fixed. Its cyclic span is a $K$-spherical representation of $G$. The spherical function is

$$\Phi(g) = \prod_{\nu, \mu: \nu \neq \mu} \langle \xi_\nu, \xi_\mu \rangle s_{\nu \mu}(g).$$

8.3. The Nessonov theorem.

**Theorem 8.1** [65]. *The construction described above exhausts all $K$-spherical representations of the group $G$.*

A proof is given below in §§8.5 and 8.6.

8.4. Category of double cosets. Let

$$K[\alpha_1, \ldots, \alpha_m] = K_1[\alpha_1] \times \cdots \times K_m[\alpha_m].$$

We describe the category of double coset spaces

$$K[\alpha_1, \ldots, \alpha_m] \setminus G/K[\beta_1, \ldots, \beta_m].$$

For definiteness let $m = 3$ (see Fig. 19).

**Figure 19.** Re §8.4.

Let

$$J_\nu[\gamma] := \{1, \ldots, \gamma\} \times \{\nu\} \subset \mathbb{N} \times \{1, 2, 3\}.$$

We construct a diagram with arcs. In the upper row we put the elements of the set $J_1[\beta_1] \cup J_2[\beta_2] \cup J_3[\beta_3]$, and in the lower row the elements of $J_1[\alpha_1] \cup J_2[\alpha_2] \cup J_3[\alpha_3]$. We mark different ‘colours’ ($\nu = 1, 2, 3$) by crosses, squares, and triangles. Suppose that $g$ sends $(l, \nu)$ to $(k, \mu)$. We draw the following objects depending on the positions of the pairs $(l, \nu)$ and $(k, \mu)$:

- if $(l, \nu) \in J_\nu[\beta_\nu]$ and $(k, \mu) \in J_\mu[\alpha_\mu]$, then we draw an arc from $(l, \nu)$ to $(k, \mu)$;
\begin{itemize}
  \item if \((l, \nu) \in J_{\nu}[\beta_{\nu}]\) and \((k, \mu) \notin J_{\mu}[\alpha_{\mu}]\), then we draw an arc with the origin at \((l, \nu)\) and a free lower endpoint marked by the colour \(\mu\), and we proceed similarly in the case when \((l, \nu) \notin J_{\nu}[\beta_{\nu}]\) and \((k, \mu) \in J_{\mu}[\alpha_{\mu}]\) (we draw an arc with endpoint at \((l, \nu)\) and a free upper endpoint marked by the colour \(\nu\));
  \item if \((l, \nu) \notin J_{\nu}[\beta_{\nu}], (k, \mu) \notin J_{\mu}[\alpha_{\mu}],\) and \(\mu \neq \nu\), then we draw an arc with the upper endpoint coloured \(\nu\) and the lower endpoint coloured \(\mu\) (and we distinguish the upper and lower endpoints of the arc);
  \item if \((l, \nu) \notin J_{\nu}[\beta_{\nu}], (k, \mu) \notin J_{\mu}[\alpha_{\mu}],\) and \(\mu = \nu\), then we do not draw anything (so that the final picture is finite).
\end{itemize}

Multiplication is a gluing together of the diagrams as in the case of chips.

### 8.5. The semigroup \(\Gamma := K \setminus G/K\) and its characters.

According to the construction in the previous subsection, to a double coset \(\in K \setminus G/K\) we assign a collection of arcs from up to down with coloured endpoints. Only the number \(s_{\nu\mu}\) of arcs of each colour is important. Thus, to any double coset we assign the collection of numbers \(s_{\nu\mu}\), where \(\nu, \mu \leq m\) and \(\mu \neq \nu\) (so that these numbers form an \(m \times m\) matrix \(S\) without diagonal\(^{24}\)). The numbers \(s_{\nu\mu}\) are non-negative integers satisfying the conditions (8.1). Multiplication of double cosets corresponds to addition of matrices.

As in the case of the trisymmetric group, the semigroup \(\Gamma := K \setminus G/K\) has a canonical embedding in any semigroup \(K[\alpha_1, \ldots, \alpha_m] \setminus G/K[\alpha_1, \ldots, \alpha_m]\) as the centre. Corresponding to each unitary \(K\)-spherical representation of the group \(G\) is a homomorphism \(\chi\) from \(K \setminus G/K\) to the multiplicative semigroup of complex numbers with absolute value \(\leq 1\):

\[\chi(S_1 + S_2) = \chi(S_1)\chi(S_2).\]

Also,

\[\chi(S^t) = \overline{\chi(S)}.\]

In our language, Theorem 8.1 asserts that for any unitary \(K\)-spherical representation \(\rho\) of the group \(G\) there is a Hermitian positive-semidefinite matrix \(A\) of order \(m\) with 1s on the diagonal such that the spherical character of \(\rho\) is

\[\chi(S) = \prod_{\mu, \nu; \mu \neq \nu} a_{\nu\mu}^{s_{\nu\mu}}.\]

This statement would be quite simple if the character \(\chi\) did not vanish. However, we must examine the structure of the subsemigroup of its zeros, and this makes the proof longer.

Denote by \(E_{\nu\mu}\) the corresponding matrix unit, that is, the matrix with 1 at the \(\nu\mu\) position and 0 elsewhere. We will represent matrices \(S\) as \(S = \sum s_{\nu\mu}E_{\nu\mu}\). We define cycles to be elements of \(\Gamma\) having the form

\[\sigma[l_1 \ldots l_p] = E_{l_1l_2} + E_{l_2l_3} + \cdots + E_{l_pl_1}, \quad \text{where} \ l_i \neq l_j.\]

\textbf{Lemma 8.2.} The semigroup \(\Gamma\) is generated by cycles.

\(^{24}\)It is possible to set the diagonal entries equal to \(\infty\).
Proof. Consider a matrix $S \in \Gamma$. Let $s_{p_1p_2} > 0$. By the condition (8.1) there is a $p_3$ such that $s_{p_2p_3} > 0$. If $p_3 \neq p_1$, then we take $s_{p_3p_4} > 0$, and so on, until we come to $p_i = p_j$. Then the matrix $S - \sigma[p_ip_{i+1}\ldots p_j]$ is non-negative and satisfies (8.1).

Fix a spherical representation $\rho$ and the corresponding character $\chi$. We say that colours $\mu$ and $\nu$ are contained in the same component if there exists a cycle $\sigma$ containing $\mu$ and $\nu$ such that $\chi(\sigma) \neq 0$.

Remark. The construction of representations in § 8.2 involved the space $V$ and a set of vectors $\xi_1, \ldots, \xi_n$. The colours $\mu$ and $\nu$ lie in different components if there exists an orthogonal decomposition $V = V_1 \oplus V_2$ such that all the vectors $\xi_j$ lie in these summands and, moreover, $\xi_\mu \in V_1$ and $\xi_\nu \in V_2$.

Lemma 8.3. If $\mu$ and $\nu$ are not contained in the same component, then $\chi(S) = 0$ for any matrix $S \in \Gamma$ with $s_{\mu\nu} > 0$ (or $s_{\nu\mu} > 0$).

Proof. It suffices to decompose $S$ into a sum of cycles.

Lemma 8.4. Let $\nu$ and $\mu$ lie in the same component. Then there exists a chain $\nu_1 = \nu, \nu_2, \ldots, \nu_k = \mu$ such that

$$\chi(E_{\nu_1\nu_2} + E_{\nu_2\nu_3} + \cdots + E_{\nu_{k-1}\nu_k}) \neq 0.$$ 

Note that

$$E_{\mu\nu} + E_{\nu\mu} = \sigma[\mu\nu].$$

Proof. We choose a cycle $\sigma[l_1 \ldots l_p]$ containing $\nu$ and $\mu$. Then

$$\sigma[l_1 \ldots l_p] + \sigma[l_1 \ldots l_p]^t = \sigma[l_1l_2] + \sigma[l_2l_3] + \cdots + \sigma[l_pl_1].$$

Since $\chi(\sigma + \sigma^t) = |\chi(\sigma)|^2 \neq 0$, we have $\chi(\sigma[l_kl_{k+1}]) \neq 0$. □

Lemma 8.5. If $\chi(\sigma[l_1l_2]) = 0$, then $\chi(\sigma[l_1l_2 \ldots l_p]) = 0$ for any cycle $\sigma[l_1l_2 \ldots l_p]$.

Proof. The matrix

$$(\sigma[l_1l_2 \ldots l_p] + \sigma[l_1l_2 \ldots l_p]^t) - \sigma[l_1l_2]$$

belongs to $\Gamma$. □

8.6. Proof of Theorem 8.1. We preserve the notation of the previous subsection. To be definite, consider the component containing the colour $\nu = 1$. Without loss of generality, we can assume that the component consists of consecutive colours $\nu = 1, 2, \ldots, n$. Consider the graph $\Delta$ whose vertices are the points $1, 2, \ldots, n$, while an edge connects two vertices $\mu$ and $\nu$ if $\chi(\sigma[\mu\nu]) \neq 0$. Let $\Xi$ be an arbitrary spanning tree of this graph.

We consider the group $K[1, 0, \ldots, 0]$, the subspace $V$ of $K[1, 0, \ldots, 0]$-fixed vectors, and a representation of the semigroup

$$\Gamma_1 := K[1, 0, \ldots, 0] \setminus G/K[1, 0, \ldots, 0]$$
on $V$. Consider the elements $\pi_j$ of the semigroup that are given by the pictures of the type shown in Fig. 20.

It is easy to see that $\pi_j^2 = \pi_j$. Therefore, the corresponding operators $P_j$ are orthogonal projections on $V$.

Note that

$$P_i P_j P_i = P_i \cdot \chi(\sigma[ij]).$$

On the left-hand side we have a positive operator, and therefore

$$\chi(\sigma(ij)) > 0.$$

Next, we write a slightly more general equality. For a cycle $\sigma[\nu_1 \nu_2 \ldots \nu_q]$ we have

$$P_{\nu_q} P_{\nu_1} P_{\nu_2} \ldots P_{\nu_q} = \chi(\sigma[\nu_1 \nu_2 \ldots \nu_q]) P_{\nu_q}. \quad (8.2)$$

**Lemma 8.6.** For each $j \leq n$ the range of the projection $P_j$ is one-dimensional.

**Proof.** The operator $P_1$ is the projection onto the spherical vector. Let $\chi(\sigma[1j]) = s \neq 0$. Then $P_1 P_j P_1 = sP_1$, so that $P_j \neq 0$. On the other hand, $P_j P_1 P_j = sP_j$, and therefore the rank of $P_j$ does not exceed 1. Thus, for all $j$ joined to 1 by an edge in the graph $\Delta$ we have $\text{rk} P_j = 1$. Next, we repeat the same reasoning for $P_j$ instead of $P_1$ and refer to the connectedness of the graph $\Delta$. \[\square\]

Denote by $L_\nu$ the line $P_\nu V$, and choose a unit vector $v_1$ in $L_1$. Let $\nu_1 = 1$, $\nu_2$, $\ldots$, $\nu_q = \nu$ be a path from 1 to $\nu$ along the spanning tree $\Xi$. We consider the vector

$$w_\nu := P_{\nu_q} \ldots P_{\nu_1} v_1 \in L_\nu.$$

It is non-zero, and moreover,

$$\langle w_\nu, w_\nu \rangle = \langle P_{\nu_q} \ldots P_{\nu_1} v_1, P_{\nu_q} \ldots P_{\nu_1} v_1 \rangle = \langle P_{\nu_q} \ldots P_{\nu_1} v_1, v_1 \rangle$$

$$= \langle v_1, v_1 \rangle \chi \left( \sum E_{\nu_1 \nu_{q+1}} + \sum E_{\nu_{q+1} \nu_q} \right) = \langle v_1, v_1 \rangle \prod_q \chi(\sigma[\nu_1 \nu_{q+1}]).$$

Let

$$v_\nu := \frac{w_\nu}{\|w_\nu\|}.$$

Then for all pairs $\mu$, $\nu$ which are adjacent vertices in the spanning tree we have

$$\langle v_\mu, v_\nu \rangle = \chi(\sigma[\mu \nu])^{-1/2}. \quad (8.3)$$
Let $S$ be an $n \times n$ matrix with non-negative integer elements (as above, diagonal elements are not defined; we do not impose the conditions (8.1)), and let

$$
\psi(S) = \prod_{\mu \neq \nu} \langle v_\mu, v_\nu \rangle^{s_{\mu\nu}}.
$$

**Lemma 8.7.** If $S$ satisfies the condition

$$
\sum_{\mu: \mu \leq n, \mu \neq \nu} s_{\mu\nu} = \sum_{\mu: \mu \leq n, \mu \neq \nu} s_{\nu\mu}, \quad (8.4)
$$

then $\psi(S) = \chi(S)$.

**Proof.** Recall that the semigroup $\Gamma_1$ of all $S$ satisfying (8.4) is generated by cycles. If a cycle is not contained in the graph, then we have zero on both sides of (8.4). If a cycle has the form $\sigma[\mu\nu]$, where $\mu\nu$ is an edge of the spanning tree, then the two sides coincide by (8.3). Let $\mu\nu$ be an edge of the graph $\Delta$ not contained in the tree $\Xi$. Adding $\mu\nu$ to $\Xi$, we get a graph with a unique cycle. We pass along the cycle in the direction $\mu\nu$ and take the corresponding element $\sigma[\kappa_1 \ldots \kappa_p]$ (where $\kappa_1 = \nu$, $\kappa_p = \mu$) of the semigroup $\Gamma_1$. In this case the equality (8.4) reduces to (8.2).

Now consider an arbitrary cycle $C$ passing along edges of $\Delta$. It is a linear combination, with integer coefficients, of cycles of the form $\sigma[\mu\nu]$ and the cycles $\sigma[\kappa_1 \ldots \kappa_p]$ just defined. Keeping in the mind that the values of $\psi$ on the cycles $\sigma[\mu\nu]$ and $\sigma[\kappa_1 \ldots \kappa_p]$ are invertible, we see that the assertion is valid for $C$. \[ \square \]

Applying Lemma 8.7 to all components of the set $\{1, \ldots, m\}$, we get Theorem 8.1.

**8.7. Remarks.** As we have seen above, there are many spherical pairs $G \supset K$ related to infinite symmetric groups. Completeness of the list of spherical functions (see the conjectures in §3.16) has so far been proved only in the cases of the Thoma theorem and the Nessonov theorem, and also in the following three cases relating to the bisymmetric group (these were considered by Olshanski in [6]):

1) $G = S_{2\infty}$ and $K$ is the hyperoctahedral group $S_\infty \ast \mathbb{Z}_2$;
2) $G = S_{2\infty+1}$ and $K = S_\infty \ast \mathbb{Z}_2$;
3) $G = S_{\infty+1} \times S_\infty$ and $K = \text{diag } S_\infty$.

Passing to the language of train categories, we observe that these five pairs are really simpler than other pairs. The corresponding combinatorial manifolds are one-dimensional, and an additional structure attributed to arcs is a number (a ‘length’ in [6]) and not a coloured tiling as in the general constructions.

There are other cases possessing this property. One of these is the symmetric group $S_{3\infty}$ with the subgroup $(S_\infty \ast \mathbb{Z}_2) \times S_\infty$ (we can also consider $S_{4\infty} \supset (S_\infty \ast \mathbb{Z}_2) \times S_\infty \times S_\infty$, and so on, but two hyperoctahedral factors is a complicated case).

The next level of complexity involves spherical pairs connected with chips, first of all the example in §4.4 and the pair considered by Nessonov in [66].
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Bibliography

[1] E. Thoma, “Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe”, Math. Z. 85:1 (1964), 40–61.

[2] A. Lieberman, “The structure of certain unitary representations of infinite symmetric groups”, Trans. Amer. Math. Soc. 164 (1972), 189–198.

[3] А. М. Вершик, С. В. Керов, “Характеры и фактор-представления бесконечной симметрической группы”, Докл. АН СССР 257:5 (1981), 1037–1040; English transl., A. M. Vershik and S. V. Kerov, “Characters and factor representations of the infinite symmetric group”, Soviet Math. Dokl. 23:2 (1981), 389–392.

[4] A. M. Vershik, “Totally nonfree actions and the infinite symmetric group”, Mosc. Math. J. 12:1 (2012), 193–212.

[5] A. М. Вершик, С. В. Керов, “Асимптотическая теория характеров симметрической группы”, Функц. анализ и прил. 15:4 (1981), 15–27; English transl., A. M. Vershik and S. V. Kerov, “Asymptotic theory of characters of the symmetric group”, Funct. Anal. Appl. 15:4 (1981), 246–255.

[6] Г. И. Ольшанский, “Унитарные представления (G, K)-пар, связанных с бесконечной симметрической группой S(∞)”, Алгебра и анализ 1:4 (1989), 178–209; English transl., G. I. Ol’shanskii, “Unitary representations of (G, K)-pairs that are connected with the infinite symmetric group S(∞)”, Leningrad Math. J. 1:4 (1990), 983–1014.

[7] S. Kerov, G. Olshanski, and A. Vershik, “Harmonic analysis on the infinite symmetric group”, Invent. Math. 158:3 (2004), 551–642.

[8] M. Atiyah, “Topological quantum field theories”, Inst. Hautes Études Sci. Publ. Math. 68 (1988), 175–186.

[9] C. Teleman, Five lectures on topological field theory, 2014, 36 pp., http://math.berkeley.edu/~teleman/math/barclect.pdf.

[10] R. M. Switzer, Algebraic topology – homotopy and homology, Grundlehren Math. Wiss., vol. 212, Springer-Verlag, New York–Heidelberg 1975, xii+526 pp.

[11] G. B. Segal, “The definition of conformal field theory”, Differential geometrical methods in theoretical physics (Como 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht 1988, pp. 165–171.

[12] Ю. А. Неретин, “Голоморфные продолжения представлений группы диффеоморфизмов окружности”, Матем. сб. 180:5 (1989), 635–657; English transl., Yu. A. Neretin, “Holomorphic extensions of representations of the group of diffeomorphisms of the circle”, Math. USSR-Sb. 67:1 (1990), 75–97.

[13] J. C. Baez, “An introduction to spin foam models of BF theory and quantum gravity”, Geometry and quantum physics (Schladming, 1999), Lecture Notes in Phys., vol. 543, Springer, Berlin 2000, pp. 25–93.

[14] S. M. Natanzon, “Cyclic foam topological field theories”, J. Geom. Phys. 60:6-8 (2010), 874–883.

[15] Yu. A. Neretin, “Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories”, Int. Math. Res. Not. IMRN 2012:3 (2012), 501–523.

[16] Yu. A. Neretin, Infinite symmetric group and combinatorial descriptions of semigroups of double cosets, 2011, 39 pp., arXiv:1106.1161.

[17] A. A. Gaifullin and Yu. A. Neretin, Infinite symmetric group and bordisms of pseudomanifolds, 2015, 14 pp., arXiv:1501.04062.
[18] J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, vol. XXIX, Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris 1964, xi+382 pp.

[19] E. Thoma, “Eine Charakterisierung diskreter Gruppen vom Typ Γ”, *Invent. Math.* 6:3 (1968), 190–196.

[20] А. А. Кириллов, А. Д. Гвишиани, *Теоремы и задачи функционального анализа*, Наука, М. 1979, 382 с.; English transl., A. A. Kirillov and A. D. Gvishiani, *Theorems and problems in functional analysis*, Problem Books in Math., Springer-Verlag, New York–Berlin 1982, ix+347 pp.

[21] Д. А. Райков, “О пополнении топологических групп”, *Изв. АН СССР. Сер. матем.* 10:6 (1946), 513–528. [D.A. Raikov, “On the completion of topological groups”, *Izv. Akad. Nauk SSSR Ser. Mat.* 10:6 (1946), 513–528.]

[22] N. Bourbaki, *Eléments de mathématique. Première partie. (Fascicule III.) Livre III. Topologie générale*, Chap. 3: Groupes topologiques. Chap. 4: Nombres réels, 3ème éd., rev. et augm., Actualités Sci. Indust., vol. 1143, Hermann, Paris 1960, 236 pp.; Chap. V: Groupes à un paramètre. Chap. VI: Espaces numériques et espaces projectifs. Chap. VII: Les groupes additifs $R^n$. Chap. VIII: Nombres complexes, Actualités Sci. Indust., vol. 1235, 1963, 151 pp.

[23] A. S. Kechris and C. Rosendal, “Turbulence, amalgamation, and generic automorphisms of homogeneous structures”, *Proc. Lond. Math. Soc.* (3) 94:2 (2007), 302–350.

[24] H. Becker and A. S. Kechris, *The descriptive set theory of Polish groups actions*, London Math. Soc. Lecture Note Ser., vol. 232, Cambridge Univ. Press, Cambridge 1996, xii+136 pp.

[25] T. Tsankov, “Automatic continuity for the unitary group”, *Proc. Amer. Math. Soc.* 141:10 (2013), 3673–3680.

[26] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*, Math. Sci. Res. Inst. Publ., vol. 14, Springer-Verlag, New York 1989, x+288 pp.

[27] Yu. A. Neretin, *Lectures on Gaussian integral operators and classical groups*, EMS Ser. Lect. Math., Eur. Math. Soc. (EMS), Z¨ urich 2011, xii+559 pp.

[28] M. Krämer, “Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen”, *Compositio Math.* 38:2 (1979), 129–153.

[29] G. I. Ol’shanskii, “Unitary representations of infinite dimensional pairs $(G, K)$ and the formalism of R. Howe”, *Representation of Lie groups and related topics*, Adv. Stud. Contemp. Math., vol. 7, Gordon and Breach, New York 1990, pp. 269–463.

[30] Н. И. Нессонов, “Фактор-представления группы $GL(\infty)$ и допустимые представления $GL(\infty)^X \Gamma$”, *Матем. физика, анализ, геометрия. Харьковский матем. журн.* 10:2 (2003), 167–187; “Π” 10:4, 524–556. [N. I. Nessonov, “Factor-representations of the group $GL(\infty)$ and admissible representations of $GL(\infty)^X \Gamma$”, *Mat. Fiz. Anal. Geom.* 10:2 (2003), 167–187]; “Π” 10:4, 524–556.

[31] Ю. А. Неретин, “Сферичность и умножение двойных классов смежности для бесконечномерных классических групп”, *Функциал. анализ и его прил.* 45:3 (2011), 79–96; English transl., Yu. A. Neretin, “Sphericity and multiplication of double cosets for infinite-dimensional classical groups”, *Func. Anal. Appl.* 45:3 (2011), 225–239.

[32] Yu. A. Neretin, *The subgroup $\text{PSL}_2(\mathbb{R})$ is spherical in the group of diffeomorphisms of the circle*, 2015, 6 pp., arXiv:1501.05820.

[33] А. Ю. Окуньков, “Теорема Тома и представления бесконечной бисимметрической группы”, *Функциал. анализ и его прил.* 28:2 (1994), 31–40;
Infinite symmetric groups

English transl., A. Yu. Okounkov, “Thoma’s theorem and representations of the infinite bisymmetric group”, *Funct. Anal. Appl.* **28:**2 (1994), 100–107.

[34] Ю. А. Неретин, *Категории симметрий и бесконечномерные группы*, УРСС, М. 1998, 431 с.; English transl., Yu. A. Neretin, *Categories of symmetries and infinite-dimensional groups*, London Math. Soc. Monogr. (N.S.), vol. 16, The Clarendon Press, Oxford Univ. Press, New York 1996, xiv+417 pp.

[35] Р. С. Исмагилов, “Элементарные сферические функции на группе $SL(2, P)$ над полем $P$, не являющимся локально компактным, относительно подгруппы матриц с целыми элементами”, *Изв. АН СССР. Сер. матем.* **31:**2 (1967), 361–390; English transl., R. S. Ismagilov, “Elementary spherical functions on the group $SL(2, P)$ over a field $P$, which is not locally compact, with respect to the subgroup of matrices with integral elements”, *Math. USSR-Izv.* **1:**2 (1967), 349–380.

[36] Yu. A. Neretin, “Multi-operator colligations and multivariate characteristic functions”, *Anal. Math. Phys.* **1:**2-3 (2011), 121–138.

[37] Yu. A. Neretin, *Multiplication of conjugacy classes, colligations, and characteristic functions of matrix argument*, 2012 (v2 – 2015), 20 pp., arXiv:1211.7091.

[38] Ю. А. Неретин, “Бесконечномерные $p$-адичные группы, полугруппы двойных классов смежности и внутренние функции на ансамблях Брюа–Титса”, *Изв. РАН. Сер. матем.* **79:**3 (2015), 87–130; English transl., Yu. A. Neretin, “Infinite-dimensional $p$-adic groups, semigroups of double cosets, and inner functions on Bruhat–Tits buildings”, *Izv. Math.* **79:**3 (2015), 512–553.

[39] Ю. А. Неретин, “Категории бистохастических мер и представления некоторых бесконечномерных групп”, *Матем. сб.* **183:**2 (1992), 52–76; English transl., Yu. A. Neretin, “Categories of bistochastic measures, and representations of some infinite-dimensional groups”, *Russian Acad. Sci. Sb. Math.* **75:**1 (1993), 197–219.

[40] Yu. A. Neretin, “Spreading maps (polymorphisms), symmetries of Poisson processes, and matching summation”, *Теория представлений, динамические системы, комбинаторные и алгоритмические методы. VII*, Зап. науч. сем. ПОМИ, **292**, ПОМИ, СПб. 2002, с. 62–91; also published in *J. Math. Sci. (N. Y.)* **126:**2 (2005), 1077–1094.

[41] Yu. Neretin, “Symmetries of Gaussian measures and operator colligations”, *J. Funct. Anal.* **263:**3 (2012), 782–802.

[42] Ş. Strătilă and D. Voiculescu, *Representations of AF-algebras and of the group $U(\infty)$*, Lecture Notes in Math., vol. 486, Springer-Verlag, Berlin–New York 1975, viii+169 pp.

[43] А. М. Вершик, С. В. Керов, “Характеры и фактор-представления бесконечномерной унитарной группы”, *Докл. АН СССР* **267:**2 (1982), 272–276; English transl., A. M. Vershik and S. V. Kerov, “Characters and factor representations of the infinite unitary group”, *Soviet Math. Dokl.* **26:**3 (1982), 570–574.

[44] N. Obata, “Certain unitary representations of the infinite symmetric group. I”, *Nagoya Math. J.* **105** (1987), 109–119; “II”, *Nagoya Math. J.* **106** (1987), 143–162.

[45] G. I. Olshansky, “Unitary representations of the infinite symmetric group: a semigroup approach”, *Representations of Lie groups and Lie algebras*, Pt. 2 (Budapest 1971), Akad. Kiadó, Budapest 1985, pp. 181–197.

[46] А. А. Кириллов, *Элементы теории представлений*, Наука, М. 1972, 336 с.; English transl., A. A. Kirillov, *Elements of the theory of representations*, Grundlehren Math. Wiss., vol. 220, Springer-Verlag, Berlin–New York 1976, xi+315 pp.
Yu. A. Neretin

[47] J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, Paris 1967, xii+135 pp. (not consecutively paged).

[48] Г. И. Ольшанский, “Бесконечномерные классические группы конечного R-ранга: описание представлений и асимптотическая теория”, Функц. анализ и его прил. 18:1 (1984), 28–42; English transl., G.I. Ol’shanskii, “Infinite-dimensional classical groups of finite r-rank: description of representations and asymptotic theory”, Funct. Anal. Appl. 18:1 (1984), 22–34.

[49] J. von Neumann, “On infinite direct products”, Compositio Math. 6 (1938), 1–77; reprinted in: Collected works, vol. 3: Rings of operators, Pergamon Press, New York–Oxford–London–Paris 1961, pp. 323–399.

[50] Г. В. Белый, “О расширениях Галуа максимального кругового поля”, Изв. АН СССР. Сер. матем. 43:2 (1979), 267–276; English transl., G.V. Belyi, “On Galois extensions of a maximal cyclotomic field”, Math. USSR-Izv. 14:2 (1980), 247–256.

[51] Г. В. Белый, “Новое доказательство теоремы о трех точках”, Матем. сб. 193:3 (2002), 21–24; English transl., G.V. Belyi, “Another proof of the three points theorem”, Sb. Math. 193:3 (2002), 329–332.

[52] Ю. А. Неретин, “Спектральные данные для пары матриц порядка 3 и действие группы GL(2, Z)”, Изв. РАН. Сер. матем. 75:5 (2011), 93–102; English transl., Yu. A. Neretin, “Spectral data for a pair of matrices of order three and an action of the group GL(2, Z)”, Izv. Math. 75:5 (2011), 959–969.

[53] R. Brauer, “On algebras which are connected with the semisimple continuous groups”, Ann. of Math. (2) 38:4 (1937), 857–872.

[54] С. В. Керов, “Реализации представлений полугруппы Браузера”, Дифференциальная геометрия, группы Ли и механика. IX, Зап. науч. сем. ЛОМИ, 164, Изд-во Наука, Ленингр. отд., Л. 1987, с. 189–193; English transl., S.V. Kerov, “Realizations of representations of the Brauer semigroup”, J. Soviet Math. 47:2 (1989), 2503–2507.

[55] Е. Нейманн, “Одно замечание о представлениях бесконечных симметрических групп”, Теория представлений, динамические системы, комбинаторные методы. XXI, Зап. науч. сем. ПОМИ, 403, ПОМИ, СПб. 2012, с. 103–109; English transl., Yu. A. Neretin, “A remark on representations of infinite symmetric groups”, J. Math. Sci. (N. Y.) 190:3 (2013), 464–467.

[56] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups, Grundlehren Math. Wiss., vol. 152, Springer-Verlag, New York–Berlin 1970, ix+771 pp.

[57] M. Goresky and R. MacPherson, “Intersection homology theory”, *Topology* 19:2 (1980), 135–162.

[58] M. Pezzana, “Diagrammi di Heegaard e triangolazione contratta”, Collection in memory of Enrico Bombiani, *Boll. Un. Mat. Ital.* (4) 12:3, suppl. (1975), 98–105.
[62] M. Ferri, “Una rappresentazione delle $n$-varietà topologiche triangolabili mediante grafi $(n+1)$-colorati”, *Boll. Un. Mat. Ital. B* (5) 13:1 (1976), 250–260.

[63] M. Ferri, C. Gagliardi, and L. Grasselli, “A graph-theoretical representation of PL-manifolds: a survey on crystallizations”, *Aequationes Math.* 31:1 (1986), 121–141.

[64] A. Gaifullin, “Combinatorial realisation of cycles and small covers”, *European Congress of Mathematics*, Proceedings of the 6th congress (6ECM) (Kraków 2–7 July, 2012), Eur. Math. Soc., Zürich 2013, pp. 315–330; 2012, 14 pp., arXiv: 1204.0208.

[65] Н.И. Нессонов, “Представления $\mathfrak{S}_\infty$, допустимые относительно подгрупп Юнга”, *Матем. сб.* 203:3 (2012), 127–160; English transl., N.I. Nessonov, “Representations of $\mathfrak{S}_\infty$ admissible with respect to Young subgroups”, *Sb. Math.* 203:3 (2012), 424–458.

[66] Н.И. Нессонов, “КМШ-состояния на $\mathfrak{S}_\infty$, инвариантные относительно подгрупп Юнга”, *Функц. анализ и его прил.* 47:2 (2013), 55–67; English transl., N.I. Nessonov, “KMS states on $\mathfrak{S}_\infty$ invariant with respect to the Young subgroups”, *Funct. Anal. Appl.* 47:2 (2013), 127–137.

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