Fast Global Convergence of Policy Optimization for Constrained MDPs

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Abstract

We address the issue of safety in reinforcement learning. We pose the problem in a discounted infinite-horizon constrained Markov decision process framework. Existing results have shown that gradient-based methods are able to achieve an $O(1/\sqrt{T})$ global convergence rate both for the optimality gap and the constraint violation. We exhibit a natural policy gradient-based algorithm that has a faster convergence rate $O(\log(T)/T)$ for both the optimality gap and the constraint violation. When Slater’s condition is satisfied and known a priori, zero constraint violation can be further guaranteed for a sufficiently large $T$ while maintaining the same convergence rate for the optimality gap.

1 Introduction

Policy gradient (PG) methods and their variants play an important role in recent advances of reinforcement learning (RL) [20]. These gradient-based methods are attractive due to their flexibility in being applicable to any differentiable policy parameterization and generality for extensions to function approximations [2]. Standard PG methods typically focus on optimizing a single objective. However, in many real-world applications, stringent safety constraints are imposed on a learned control policy [10, 4]. For example, an energy-efficient wireless communication system may want to consume minimum power without violating any constraint on quality of service (QoS) [15]. The model of a constrained Markov decision process (CMDP) [3, 12], where the goal is to optimize an objective while satisfying safety constraints, is a standard approach for modeling the necessary safety criteria of a control problem through constraints on safety costs.

PG methods, including the natural policy gradient (NPG) [13], have seen great success in practice in solving CMDPs. Lagrangian-based methods [21, 23, 22, 17] optimize the CMDP problem as a saddle-point problem via primal-dual methods, while constrained policy optimization methods [1, 28] calculate new dual variables from scratch at each update to maintain constraints during the learning process. Although these algorithms provide a way to iteratively optimize the learned policy, they can only guarantee local convergence, with no guarantees on the convergence rate to a globally optimal solution.

Motivated by seminal works on guaranteed global convergence rate [2, 19], Ding et al. [9] developed an NPG-based primal-dual method under the softmax policy parameterization, which provides an $O(1/\sqrt{T})$ global convergence rate for both the optimality gap and the constraint violation, if given an oracle that returns the exact performance of a policy, where $T$ is the total number of iterations that the algorithm executes. Similarly, Xu et al. [26] showed how to attain the global optimal with the same order convergence rate via NPG-based primal methods. As a comparison, NPG-based methods for MDPs enjoy $O(1/T)$ convergence rate [2] and asymptotic linear convergence [14], or even non-asymptotic linear convergence with certain regularizations [6, 32]. The following basic theoretical question is still open:

Can we design a PG-based algorithm with a global convergence rate faster than $O(1/\sqrt{T})$ in CMDPs?

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Table 1: Global convergence rates for the optimality gap and the constraint violation given an oracle for exact policy evaluation

| Algorithm           | Optimality gap | Constraint violation |
|---------------------|----------------|----------------------|
| PG/NPG [2]          | $O(1/T)$       | /                    |
| PG-Entropy [19]     | $O(\exp(-T))$ | /                    |
| NPG-Entropy [6]     | $O(\exp(-T))$ | /                    |
| NPG-PD [9]          | $O(1/\sqrt{T})$ | $O(1/\sqrt{T})$ |
| CRPO [26]           | $O(1/\sqrt{T})$ | $O(1/\sqrt{T})$ |
| PMD-PD              | $\tilde{O}(1/T)$ | $\tilde{O}(1/T)$ |
| PMD-PD-Zero         | $\tilde{O}(1/T)$ | 0                    |

We answer the above question affirmatively by proposing an NPG-based (PMD-PD) algorithm and establishing an $O(\log(T)/T)$ global convergence rate for both the optimality gap and the constraint violation, which is nearly dimension-free (specifically, depending at most logarithmically on the dimension of the action space) and is faster than other CMDP algorithms listed in Table 1. Furthermore, if Slater’s condition is satisfied, the PMD-PD-Zero algorithm can guarantee zero constraint violation after some $T$ with no change in the order of convergence rate for the optimality gap. Details about Slater’s condition and the specific $T$ will be provided in Section 5. While the tabular setting with exact policy evaluation is somewhat restricting, it is the setting in which the cleanest results have been obtained so far and is an important step towards understanding more complex problems in RL.

In work conducted concurrently with this work, Ying et al. [29] and Li et al. [16] have addressed the same question for designing PG-based algorithms in the CMDP problem with a faster convergence rate. Ying et al. [29] propose an NPG-aided dual approach, where the dual function is smoothed by entropy regularization in the objective function. They are able to ensure an $\tilde{O}(1/T)$ convergence rate to the optimal policy of the entropy-regularized CMDP, but not to the true optimal policy, for which the convergence rate is slower with an $O(1/\sqrt{T})$ rate. They also make an additional strong assumption that the initial state distribution covers the entire state space. Importantly, we do not make this strong assumption in this work. While such an assumption is initially used in the analysis of the global convergence of PG methods for MDPs [2, 19], it is not required when analyzing the global convergence of NPG methods [2, 6]. Moreover, this assumption does not necessarily hold for safe RL or CMDP, since the agent needs to avoid dangerous states even initially and the optimal policy will depend on the initial state distribution. Li et al. [16] propose a primal-dual approach with an $O(\log^2(T)/T)$ convergence rate to the true optimal policy by smoothing the Lagrangian with suitable regularization on both primal and dual variables. However, they assume that the Markov chain induced by any stationary policy is ergodic in order to ensure smoothness of the dual function. This assumption, though weaker than the assumption made by Ying et al., still may not hold in the scenario where an extremely cautious policy may sacrifice the objective while not visiting any dangerous state. In this work, we propose an algorithm with a faster $O(\log(T)/T)$ convergence rate to the true optimal policy without such assumptions. Moreover, we also present an important extension of our approach to the setting with zero constraint violation while maintaining the same rate of convergence to the optimal policy.

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1. All algorithms listed below are under the softmax policy parameterization.

2. This table is presented for $T \geq \text{poly}(|S|, |A|, \|d^*_\pi/\mu\|_\infty)$, with polynomial terms independent of $T$ omitted, where $|S|$ and $|A|$ are the number of states and actions respectively, $\mu$ is the starting state distribution for the algorithms, and $d^*_\pi$ is the state visitation distribution when executing an optimal policy $\pi$.

3. The specific convergence rate is $O(\log(T)/T)$, where $O(\cdot)$ omits logarithm terms.

4. It only holds after some $T$. Details will be provided in Section 5.
1.1 Other related works

Global convergence of PG-based methods. Recent years have witnessed a flurry of works studying the global convergence properties of policy gradient methods and their variants. It has been shown that PG methods converge sublinearly for unregularized MDPs \[2, 19\], while NPG methods with appropriate policy parameterization can converge asymptotically linearly under the same setting \[14\]. When entropy regularization is enforced, both PG and NPG methods can guarantee linear convergence rate \([19, 6]\). Moreover, Neu et al. \[21\] and Zhan et al. \[32\] provided a unified view to interpret NPG methods as mirror descent, thereby enabling the adaptation of mirror descent’s techniques to analyze NPG-based methods.

Fast convergence rate of convex constrained optimization. The conventional primal-dual subgradient method used to solve general convex constrained optimization has a convergence rate lower bounded by \(\Omega(1/\sqrt{T})\) \[5\]. However, assuming access to proximal mapping, Yu and Neely \[31\] proposed a new Lagrangian dual algorithm with an \(O(1/T)\) convergence rate by augmenting the Lagrange multiplier. Under smoothness assumption, the same \(O(1/T)\) convergence rate can be reached without accessing the proximal mapping \[30\]. Recently, Xu \[27\] pointed out that \(O(\exp(-T))\) convergence rate can be attained if the objective function possesses an additional strongly-convex property with a bounded number of constraints.

2 Preliminaries

2.1 Problem formulation

A discounted infinite-horizon CMDP model is a tuple \(M = (\mathcal{S}, \mathcal{A}, P, c_0, c_{1:m}, \rho, \gamma)\), where \(\mathcal{S}\) is the state space, \(\mathcal{A}\) is the action space, \(c_0 : \mathcal{S} \times \mathcal{A} \to [-1, 1]\) is the objective cost function, \(c_i : \mathcal{S} \times \mathcal{A} \to [-1, 1]\) is the \(i\)-th constraint cost function, \(\forall i \in [m] := \{1, \ldots, m\}\), \(P : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})\) is the transition kernel, \(\rho \in \Delta(\mathcal{S})\) is the starting state distribution over \(\mathcal{S}\), and \(\gamma \in [0, 1)\) is the discount factor. Above and throughout, \(\Delta(\mathcal{X})\) denotes the probability simplex over the set \(\mathcal{X}\) and \(|\mathcal{X}|\) denotes the cardinality of the set \(\mathcal{X}\). Given any stationary randomized policy \(\pi : \mathcal{S} \to \Delta(\mathcal{A})\) and any cost function \(c : \mathcal{S} \times \mathcal{A} \to [-1, 1]\), we define the state value function \(V^\pi_c : \mathcal{S} \to [-\frac{1}{1-\gamma}, \frac{1}{1-\gamma}]\) and the state-action value function \(Q^\pi_c : \mathcal{S} \times \mathcal{A} \to [-\frac{1}{1-\gamma}, \frac{1}{1-\gamma}]\) as

\[
V^\pi_c(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t c(s_t, a_t) \mid s_0 = s, \pi\right], \quad \forall s \in \mathcal{S},
\]

\[
Q^\pi_c(s, a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t c(s_t, a_t) \mid s_0 = s, a_0 = a, \pi\right], \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A},
\]

where \(\mathbb{E}\) is taken over the randomness of the trajectory of the Markov chain induced by policy \(\pi\) and transition kernel \(P\). With slight abuse of notation, denote \(V^\pi_c(\rho) := \mathbb{E}_{s \sim \rho}[V^\pi_c(s)]\).

Given the CMDP \(M = (\mathcal{S}, \mathcal{A}, P, c_0, c_{1:m}, \rho, \gamma)\), the CMDP problem is to solve the following constrained optimization problem

\[
\min_{\pi} V^\pi_{c_0}(\rho)
\]

subject to \(V^\pi_{c_i}(\rho) \leq 0, \forall i \in [m].\)

We make the following assumption to guarantee the feasibility of (1).

Assumption 2.1 (Existence of a feasible solution). There exists a policy \(\pi'\) such that \(V^\pi_{c_i}(\rho) \leq 0\) for every \(i \in [m]\).
Under this assumption, let \( \pi^* \) be the optimal policy of the CMDP problem in (1). It is well-known that in general the optimal policy \( \pi^* \) is randomized and the Bellman equation may not hold [3].

There are two standard approaches to solve a CMDP problem, namely, the linear programming (LP) approach and the Lagrangian approach [3]. Below, we give a brief description of each approach.

**Linear programming approach:** For any given policy \( \pi \), we define its discounted state-action visitation distribution as

\[
d^\pi_\rho(s,a) := \mathbb{E}_{s_0 \sim \rho} \left[ (1-\gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s, a_t = a | s_0) \right], \quad \forall (s,a) \in S \times A.
\]

It then follows that \( V^\pi_\rho(\rho) = \frac{1}{1-\gamma} \langle d^\pi_\rho, c \rangle \) by viewing \( d^\pi_\rho \) and \( c \) as \( |S||A| \)-dimensional vectors indexed by \( (s,a) \in S \times A \).

The constrained optimization problem in (1) can be reparameterized by viewing the discounted state-action visitation distribution as decision variables and rewritten as follows [3]:

\[
\begin{align*}
\min_{d \in D} & \quad \frac{1}{1-\gamma} \langle d, c_0 \rangle \\
\text{subject to} & \quad \frac{1}{1-\gamma} \langle d, c_i \rangle \leq 0, \quad \forall i \in [m],
\end{align*}
\]

where \( D \) is the domain of visitation distributions defined as

\[
D := \left\{ d \in \Delta(S \times A) : \gamma \sum_{s',a'} P(s'|s,a')d(s',a') + (1-\gamma)\rho(s) = \sum_a d(s,a), \forall s \in S \right\}.
\]

\( D \) is a compact convex set, and the linear programming (LP) formulation of the CMDP problem in (2) satisfies strong duality under Assumption 2.1, which can guarantee that the vector of optimal Lagrange multipliers (introduced in the next Lagrangian approach) is bounded, i.e., \( \|\lambda^*\| < \infty \).

LP approach can be computationally intractable for CMDPs with a large number of states and actions. Moreover, the LP approach requires explicit knowledge of the transition kernel \( P \), which makes it not amenable to gradient-based model-free RL algorithms.

**Lagrangian approach:** For the constrained optimization problem in (1), we define its Lagrangian as

\[
L(\pi, \lambda) := V^\pi_\rho(\rho) + \sum_{i=1}^{m} \lambda_i V^\pi_{c_i}(\rho),
\]

where \( \lambda_i \) is the Lagrange multiplier corresponding to the \( i \)-th constraint, for each \( i \in [m] \). Due to its equivalence to the LP formulation in (2) and the consequent strong duality [3], the optimal value of the CMDP equals to

\[
V^\pi_{c_0}(\rho) = \min_\pi \max_{\lambda \geq 0} L(\pi, \lambda) = \max_{\lambda \geq 0} \min_\pi L(\pi, \lambda).
\]

Notice that for any fixed vector \( \lambda \geq 0 \), the Lagrangian is actually the value function of the MDP with cost \( c_0 + \sum_{i=1}^{m} \lambda_i c_i \), i.e., \( L(\pi, \lambda) = V^\pi_{c_0 + \sum_{i=1}^{m} \lambda_i c_i}(\rho) \).

Algorithms for solving MDP problems can therefore be applied to tackle the problem \( \min_\pi L(\pi, \lambda) \) for any fixed \( \lambda \). The Langrange dual function, defined as \( D(\lambda) := \min_\pi L(\pi, \lambda) \), can then be used to update the dual variable. Denote \( \lambda^* := \arg \max_{\lambda \geq 0} D(\lambda) \) as the optimal dual variables (optimal Lagrange multipliers), which is bounded (i.e., \( \|\lambda^*\| < \infty \) based on Assumption 2.1) and the equivalence to the linear programming formulation. Then the optimal policy satisfies \( \pi^* \in \arg \min_\pi L(\pi, \lambda^*) \). In particular, this approach can be used to develop primal-dual gradient-based algorithms for solving the CMDP problem.
Primal-dual gradient-based approach: As mentioned in the introduction, our focus in this paper is to develop a gradient-based algorithm for solving the CMDP problem. Let \( \{ \pi_\theta | \theta \in \Theta \} \) be the class of parametric policies. The PG method updates the parameter \( \theta \) with learning rate \( \eta \) via \( \theta^{(t+1)} \leftarrow \theta^{(t)} + \eta \nabla_{\theta} V_{c_0 + \sum_{i=1}^m \lambda_i c_i} (\rho) \), while the NPG method adopts a pre-conditioned update

\[
\theta^{(t+1)} \leftarrow \theta^{(t)} + \eta F_\rho(\theta^{(t)})^\dagger \nabla_{\theta} V_{c_0 + \sum_{i=1}^m \lambda_i c_i} (\rho),
\]

where \( F_\rho(\theta)^\dagger \) is the Moore-Penrose inverse of the Fisher information matrix and is defined as \( F_\rho(\theta)^\dagger := \mathbb{E}_{s \sim \rho} \mathbb{E}_{a \sim \pi_\theta(s)} \left[ \nabla_{\theta} \log \pi_\theta(a | s) (\nabla_{\theta} \log \pi_\theta(a | s))^\top \right]^\dagger. \)

In particular, we focus on policies with the widely used softmax parameterization. In softmax parameterization (4), the NPG with the learning rate \( \eta \) takes the form

\[
\pi^{(t+1)}(a | s) = \frac{\exp(\theta_{s,a})}{\sum_{a' \in A} \exp(\theta_{s,a'})}, \quad \forall (s, a) \in S \times A.
\]

The above policy class is differentiable and complete in the sense that it covers almost any stochastic policy and its closure contains all stationary policies.

Under the softmax parameterization (4), the NPG with the learning rate \( \eta \) takes the form

\[
\pi^{(t+1)}(a | s) = \pi^{(t)}(a | s) \frac{\exp \left( \eta Q^{(t)}_{c_0 + \sum_{i=1}^m \lambda_i c_i} (s, a) \right)}{Z_t(s)},
\]

where \( Z_t(s) = \sum_{a \in A} \pi^{(t)}(a | s) \exp \left( \eta Q^{(t)}_{c_0 + \sum_{i=1}^m \lambda_i c_i} (s, a) \right) \). It was shown that (5) is equivalent to the mirror descent

\[
\pi^{(t+1)}(\cdot | s) = \arg \min_{\pi} \left\{ \langle Q^{(t)}_{c_0 + \sum_{i=1}^m \lambda_i c_i} (s, \cdot), \pi(\cdot | s) \rangle - \frac{1}{\eta} D(\pi(\cdot | s) || \pi^{(t)}(\cdot | s)) \right\}.
\]

Correspondingly, the conventional dual update with the learning rate \( \eta' \) will be \( \lambda^{(t+1)}_i = \max \{ \lambda^{(t)}_i + \eta' V^{(t+1)}_{c_i} (\rho), 0 \}, \forall i \in [m] \). Ding et al. [9] used the above NPG primal-dual (PD) approach, where the convergence rate is limited to \( O(1/\sqrt{T}) \). This is not surprising since the Lagrangian dual function \( D(\lambda) \) is piecewise linear and concave. Thus the negative Lagrangian dual function \( -D(\lambda) \) is neither smooth nor strongly convex. In general, the convergence rate of gradient-based methods for solving a non-smooth and non-strongly-convex function is at most \( \Theta(1/\sqrt{T}) \) [5]. It therefore seems impossible to achieve a faster rate, since even with direct access to \( \pi^*_\lambda \in \arg \min_\pi L(\pi, \lambda) \), using the gradient-based PD approach can not have a convergence rate faster than \( O(1/\sqrt{T}) \) due to the structure of \( D(\lambda) \). Yet it will be clear in Section 3 that our proposed approach can achieve a faster convergence rate \( O(\log(T)/T) \) through a well-designed Lagrange multiplier.

### 2.2 Auxiliary notation and lemmas

When it is clear from the context, with slight abuse of notation, we also denote the discounted state visitation distribution with respect to (w.r.t.) initial state distribution \( \rho \) and policy \( \pi \) by

\[
d''_\rho(s) := \mathbb{E}_{s_0 \sim \rho} \left[ (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s \mid s_0) \right], \quad \forall s \in S,
\]

which can be viewed as a marginal distribution of the state-action visitation \( d''_\rho(s, a) \), i.e., \( d''_\rho(s) = \sum_{a \in A} d''_\rho(s, a) \).
For any two policies \( \pi, \pi' \), the KL-divergence from \( \pi'\cdot|s\) to \( \pi\cdot|s\) is defined as \( D(\pi\cdot|s)||\pi'\cdot|s\) := \sum_{a \in A} \pi(a|s) \ln \frac{\pi'(a|s)}{\pi(a|s)} \), and for any state visitation distribution define
\[
D_d(\pi||\pi') := \sum_{s \in S} d(s) D(\pi\cdot|s)||\pi'\cdot|s) .
\] (6)

**Lemma 2.1.** Let \( d^\pi_\rho, d^{\pi'}_\rho \) be two discounted state-action visitation distributions corresponding to policies \( \pi' \) and \( \pi \), then
\[
\|d^\pi_\rho - d^{\pi'}_\rho \|_1 \leq \frac{\gamma \sqrt{2}}{1 - \gamma} \sqrt{\min \left( D_{d^{\pi'}_\rho}(\pi'||\pi), D_{d^{\pi'}_\rho}(\pi||\pi'), D_{d^{\pi'}_\rho}(\pi||\pi), D_{d^\pi_\rho}(\pi||\pi') \right)} .
\]

**Proof.** See Appendix [B] \[ \square \]

Define the pseudo KL-divergence between two discounted state-action visitation distributions \( d^\pi_\rho \) and \( d^{\pi'}_\rho \) as
\[
\tilde{D}(d^\pi_\rho||d^{\pi'}_\rho) := \sum_{(s,a) \in S \times A} d^\pi_\rho(s,a) \log \frac{d^\pi_\rho(s,a)}{d^{\pi'}_\rho(s,a)} .
\]

It is easy to verify that
\[
D_{d^{\pi'}_\rho}(\pi||\pi') = \sum_{s \in S} d^\pi_\rho(s) \sum_{a \in A} \pi(a|s) \log \left( \frac{\pi(a|s)}{\pi'(a|s)} \right)
= \sum_{s \in S} d^\pi_\rho(s) \sum_{a \in A} \frac{d^\pi_\rho(s,a)}{d^\pi_\rho(s)} \log \left( \frac{d^\pi_\rho(s,a)}{d^\pi_\rho(s,a)} \right)
= \sum_{(s,a) \in S \times A} d^\pi_\rho(s,a) \log \left( \frac{d^\pi_\rho(s,a)}{d^{\pi'}_\rho(s,a)} \right) = \tilde{D}(d^\pi_\rho||d^{\pi'}_\rho) .
\] (7)

The following lemma will play an important role when applying the pushback property (Lemma [A.1]) in the analysis of mirror descent.

**Lemma 2.2.** \( \tilde{D}(d^\pi_\rho||d^{\pi'}_\rho) \) is a Bregman divergence generated by the convex function
\[
\phi(d^\pi_\rho) = \sum_{(s,a) \in S \times A} d^\pi_\rho(s,a) \log d^\pi_\rho(s,a) - \sum_{s \in S} d^\pi_\rho(s) \log d^\pi_\rho(s).
\]

**Proof.** See Appendix [B] \[ \square \]

### 3 Policy mirror descent-primal dual algorithm and main results

In this section, we propose the policy mirror descent-primal dual (PMD-PD) approach (Algorithm [1]) for solving the CMDP problem in ([1]) with an \( O(\log(T)/T) \) convergence rate for both the optimality gap and the constraint violation. The PMD-PD algorithm updates Lagrange multipliers in the outer loop and executes the entropy-regularized NPG method under the softmax parameterization in the inner loop. Unlike the standard entropy-regularized NPG ([6]) that converges to the regularized optimal policy, which is suboptimal with respect to the unregularized problem, the learned policies of the PMD-PD approach can converge to the optimal policy of the unregularized problem.
Algorithm 1: Policy Mirror Descent-Primal Dual (PMD-PD)

Input: $\rho, K, \alpha, \eta, t_{0:K-1}$;

Initialization: Let $\pi_0$ take a random action with a uniform distribution in every state, and $\lambda_{0,i} = \max\{0, -V_{c_i}^{\pi_0}(\rho)\}$, $\forall i \in [m]$;

for $k = 0, 1, \ldots, K - 1$ do

(Policy update)
Take $\pi_{k}^{(0)} = \pi_{k}$ as the initialized policy (for optimizing $V_{k,\alpha}$ in (11) via NPG);

for $t = 0, 1, \ldots, t_k - 1$ do

Update the policy according to the NPG updating formula
$$
\pi_{k}^{(t+1)}(a|s) = \frac{1}{Z^{(t)}(s)}(\pi_{k}^{(t)}(a|s))^{1 - \frac{\eta}{1 - \gamma}} \exp \left( \eta Q_{k,\alpha}^{(t)}(s, a) \right), \forall (s, a) \in S \times A,
$$

where $Z^{(t)}(s) = \sum_{a' \in A} (\pi_{k}^{(t)}(a'|s))^{1 - \frac{\eta}{1 - \gamma}} \exp \left( \eta Q_{k,\alpha}^{(t)}(s, a') \right)$;

end

$$
\pi_{k+1}(a|s) = \pi_{k}^{(t_k)}(a|s), \forall (s, a) \in S \times A;
$$

(Dual update)
$$
\lambda_{k+1,i} = \max\{-V_{c_i}^{\pi_{k+1}}(\rho), \lambda_{k,i} + V_{c_i}^{\pi_{k+1}}(\rho)\} \text{ for each } i = 1, 2, \ldots, m;
$$

end

Output: $\bar{\pi} = \frac{1}{K} \sum_{k=0}^{K-1} \pi_k$.

Outer loop (dual update). For any macro step $k$, compared with the traditional dual update with step size 1, i.e., $\lambda_{k+1,i} = \max\{0, \lambda_{k,i} + V_{c_i}^{\pi_{k+1}}(\rho)\}$, we update the Lagrange multiplier to take a maximum with $-V_{c_i}^{\pi_{k+1}}(\rho)$ rather than 0. This technique, introduced in [31], is helpful to subsequently analyze the drift. The properties of the Lagrange multipliers are as follows.

**Lemma 3.1. Based on the Lagrange multipliers’ update in Algorithm 1**

1. For any macro step $k$, $\lambda_{k,i} \geq 0$, $\forall i \in [m]$.
2. For any macro step $k$, $\lambda_{k,i} + V_{c_i}^{\pi_k}(\rho) \geq 0$, $\forall i \in [m]$.
3. For macro step 0, $\|\lambda_{0,i}\|^2 \leq \|V_{c_i}^{\pi_0}(\rho)\|^2$; for any macro step $k > 0$, $\|\lambda_{k,i}\|^2 \geq \|V_{c_i}^{\pi_k}(\rho)\|^2$, $\forall i \in [m]$.

**Proof.** Similar properties were shown in [31] Lemma 3 for a convex constrained optimization problem. For completeness we include the proof in Appendix C.

The first property guarantees the feasibility of the Lagrange multipliers; the second property ensures that the Lagrangian in the inner loop can indeed minimize constraint costs (discussed in the inner loop (policy update) below); and the third property is a key supporting step for the convergence analysis.

Inner loop (policy update). We first define certain value functions. Compared with the conventional Lagrangian cost function via the Lagrange multiplier $\lambda_{k,i}$, we use $\lambda_{k,i} + V_{c_i}^{\pi_k}(\rho)$ as the modified Lagrange multiplier and denote
$$
\tilde{c}_k(s, a) := c_0(s, a) + \sum_{i=1}^{m} (\lambda_{k,i} + V_{c_i}^{\pi_k}(\rho))c_i(s, a), \forall (s, a) \in S \times A,
$$
as the Lagrangian cost function, where $|\tilde{c}_k(s, a)| \leq 1 + \sum_{i=1}^m \lambda_{k,i} + \frac{m}{1-\gamma}$. Due to the second property of Lemma 3.1, $\lambda_{k,i} + V_{\pi_i}^\pi(\rho)$ is positive. Otherwise the inner loop may significantly enlarge the constraint costs. Define the corresponding Lagrangian state value function and state-action value function as

$$
\tilde{V}_k^\pi(s) := \mathbb{E} \left[ \sum_{t \geq 0} \gamma^t \tilde{c}_k(s_t, a_t) \left| s_0 = s, \pi \right. \right], \quad \forall s \in S,
$$

(9)

$$
\tilde{Q}_k^\pi(s, a) := \mathbb{E} \left[ \sum_{t \geq 0} \gamma^t \tilde{c}_k(s_t, a_t) \left| s_0 = s, a_0 = a, \pi \right. \right], \quad \forall (s, a) \in S \times A.
$$

(10)

It follows that $|\tilde{V}_k^\pi(s)| \leq \frac{1+\sum_{i=1}^m \lambda_{k,i}}{1-\gamma} + \frac{m}{(1-\gamma)^2}$ and $|\tilde{Q}_k^\pi(s, a)| \leq \frac{1+\sum_{i=1}^m \lambda_{k,i}}{1-\gamma} + \frac{m}{(1-\gamma)^2}$. Define the KL-regularized value functions

$$
\tilde{V}_{k,\alpha}(s) := \mathbb{E} \left[ \sum_{t \geq 0} \gamma^t \left( \tilde{c}_k(s_t, a_t) + \alpha \log \frac{\pi(a_t|s_t)}{\pi^*(a_t|s_t)} \right) \left| s_0 = s, \pi \right. \right]
$$

$$
= \tilde{V}_k^\pi(s) + \frac{\alpha}{1-\gamma} D_{\text{KL}}(\pi||\pi_k), \quad \forall s \in S,
$$

(11)

$$
\tilde{Q}_{k,\alpha}(s, a) := \tilde{c}_k(s, a) + \alpha \log \frac{1}{\pi_k(a|s)} + \gamma \sum_{s' \in S} P(s'|s,a) \tilde{V}_{k,\alpha}(s'), \quad \forall (s, a) \in S \times A.
$$

(12)

With slight abuse of notation, denote

$$
\tilde{V}_{k,\alpha}(\rho) := \mathbb{E}_{s_0 \sim \rho} [\tilde{V}_{k,\alpha}(s)] \quad \text{and} \quad \tilde{V}_k(\rho) := \mathbb{E}_{s_0 \sim \rho} [\tilde{V}_k(s)].
$$

The inner loop of macro step $k$ essentially optimize $\tilde{V}_{k,\alpha}(\rho)$ with the newly designed Lagrange multiplier rather than the traditional Lagrangian via NPG. Notice that we can write

$$
\tilde{V}_{k,\alpha}(s) = \mathbb{E} \left[ \sum_{t \geq 0} \gamma^t \left( \tilde{c}_k(s_t, a_t) + \alpha \log \frac{1}{\pi_k(a_t|s_t)} \right) \right] + \alpha \mathbb{E} \left[ \sum_{t \geq 0} \gamma^t \log \pi(a_t|s_t) \right].
$$

Thus $\tilde{V}_{k,\alpha}(s)$ can be interpreted as a (negative) entropy-regularized value function with cost $\tilde{c}_k(s, a) + \alpha \log \frac{1}{\pi_k(a|s)}$. Indeed the updating formula (8) is the entropy-regularized NPG update under the softmax parameterization with the learning rate $\eta$ as shown in (6).

**Theorem 3.2.** Take $\alpha := \frac{2\gamma^2 m}{(1-\gamma)^2}$, $\eta := \frac{1-\gamma}{\alpha}$, and $t_k := \frac{1}{\eta\alpha} \log(3K C_k \gamma)$ with $C_k := 2\gamma(\frac{1+\sum_{i=1}^m \lambda_{k,i}}{1-\gamma} + \frac{m}{(1-\gamma)^2})$ in Algorithm 1 for any $K \geq 1$, we have the optimality gap and the constraint violation:

$$
\frac{1}{K} \sum_{k=1}^K \left( V_{c_0}^\pi(\rho) - V_{c_0}^\pi(\rho) \right) \leq \left( \frac{\alpha}{1-\gamma} \log(|A|) + \frac{2}{3\gamma} + 1 \right) \frac{1}{K}
$$

(13)

$$
\max_{i \in [m]} \left\{ \left( \frac{1}{K} \sum_{k=1}^K V_{c_i}^\pi(\rho) \right) \right\} \leq \left( \|\lambda^*\|^2 + \sqrt{\|\lambda^*\|^2 + \frac{2\alpha}{1-\gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1-\gamma)^2}} \right) \frac{1}{K},
$$

(14)

where $(a)_+ = \max\{a, 0\}$, $\lambda^*$ is the vector of optimal dual variables, and $\|\lambda^*\| := \|\lambda^*\|_2$ if there is no specific subscript.

**Proof.** The proof is sketched in Section 4 with details presented in Appendix C. \qed
4 Convergence analysis of the PMD-PD algorithm

We outline the proofs of bounds for the optimality gap in (13) and the constraint violation in (14), with the help of some key supporting lemmas.

4.1 Optimality gap analysis

Inner loop analysis. The goal of the inner loop in macro step $k$ is to approximately solve the MDP with value $\tilde{V}_k^{\pi}(\rho)$. Let $\pi_k^* := \arg\min_\pi \tilde{V}_k^{\pi}(\rho)$ be an optimal policy. We then have

$$\tilde{V}_k^{\pi_k^*}(s) \leq \tilde{V}_k^{\pi_k}(s) = \tilde{V}_k^\pi(s), \quad (15)$$

which implies $|\tilde{V}_k^{\pi_k^*}(s)|$ and $|\tilde{V}_k^{\pi_k}(s)|$ are both upper bounded by $\frac{1+\sum_{i=1}^{m} \lambda_{k,i}}{1-\gamma} + \frac{m}{(1-\gamma)^2}$. The optimal policy $\pi_k^*$ enjoys the pushback property presented in the following lemma.

Lemma 4.1. For any $k = 0, 1, \ldots, K-1$, and any policy $\pi$,

$$\tilde{V}_k^{\pi_k^*}(\rho) + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^{\pi_k^*}(\pi_k^* \| \pi_k) \leq \tilde{V}_k^\pi(\rho) + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^\pi(\pi \| \pi_k) - \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^\pi(\pi \| \pi_k^*).$$

Proof. Notice that $\tilde{V}_k^\pi(\rho)$ is a convex (linear) function with respect to the discounted state-action visitation distribution $d^k_{\rho}$ as shown in Equation (2) and $D(d^k_{\rho} || d^k_{\pi}) = D_{d^k_{\rho}}(\pi \| \pi_k)$ is a Bregman divergence with respect to $d^k_{\rho}$ according to Equation (7) and Lemma 2.2. We then conclude the proof by the pushback property (Lemma A.3) of the mirror descent for a convex optimization problem.

The policy $\pi_k^*$ is approximated by $\pi_k^{(t_k)}$ (i.e., $\pi_{k+1}$) via NPG, and it enjoys an almost similar pushback property with an additive approximation error term as in the following lemma.

Lemma 4.2. Let $\alpha, \eta, t_{0:K-1}$ be the same values as in Theorem 3.2. Then for any $k = 0, 1, \ldots, K-1$, and any policy $\pi$,

$$\tilde{V}_k^{\pi_{k+1}}(\rho) + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^{\pi_{k+1}}(\pi_{k+1} \| \pi_k) \leq \tilde{V}_k^\pi(\rho) + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^\pi(\pi \| \pi_k) - \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^\pi(\pi \| \pi_{k+1}) + \frac{1+2/(3\gamma)}{K}.$$

Proof. The proof utilizes the pushback property in Lemma 4.1 and also relies on the convergence of the entropy-regularized NPG given in [4]. See Appendix C for detail.

Comparing Lemma 4.2 with Lemma 4.1, there is an extra additive term $(1+2/(3\gamma))/K$. Taking $\pi = \pi^*$ in Lemma 4.2, since $\lambda_{k,i} + V_c^{\pi_k}(\rho) \geq 0$ by the second property in Lemma 3.1 and $V^{\pi^*}_{c_i}(\rho) \leq 0$ for any $i \in [m]$, we have

$$V_{c_{i+1}}^{\pi_k}(\rho) + \langle \lambda_k + V_{c_{i+1}}^{\pi_k}(\rho) \rangle + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^{\pi_{k+1}}(\pi_{k+1} \| \pi_k) \leq V_{c_{i+1}}^{\pi^*}(\rho) + \frac{\alpha}{1-\gamma} D_{d^k_{\rho}}^{\pi^*}(\pi^* \| \pi_{k+1}) + \frac{1+2/(3\gamma)}{K}. \quad (16)$$
Outer loop analysis The main objective of the analysis of the outer loop is to study the inner product term $\langle \lambda_k + V_{c_{1:m}}^{\pi k}, V_{c_{1:m}}^{\pi k+1}(\rho) \rangle$ in the Lagrangian by leveraging the update rule of dual variable in each macro step.

Lemma 4.3. For any $k = 0, 1, \ldots, K - 1$,

$$\langle \lambda_k, V_{c_{1:m}}^{\pi k+1}(\rho) \rangle \geq \frac{1}{2} \| \lambda_{k+1} \|^2 - \frac{1}{2} \| \lambda_k \|^2 - \| V_{c_{1:m}}^{\pi k+1}(\rho) \|^2.$$  

Proof. Recall $\lambda_{k+1,i} = \max \{ -V_{c_{1:m}}^{\pi k+1}(\rho), \lambda_{k,i} + V_{c_{1:m}}^{\pi k+1}(\rho) \}$ for each $i \in [m]$. If $\lambda_{k+1,i} = -V_{c_{1:m}}^{\pi k+1}(\rho)$, then $\lambda_{k,i}^{\pi k+1}(\rho) \geq -\frac{\lambda_{k,i}}{2} - \frac{1}{2} (V_{c_{1:m}}^{\pi k+1}(\rho))^2 = \frac{1}{2} \lambda_{k,i}^2 - \frac{1}{2} (V_{c_{1:m}}^{\pi k+1}(\rho))^2$. If $\lambda_{k+1,i} = \lambda_{k,i} + V_{c_{1:m}}^{\pi k+1}(\rho)$, then $\lambda_{k,i}^{\pi k+1}(\rho) = \frac{1}{2} \lambda_{k,i}^2 - \frac{1}{2} \frac{1}{2} \lambda_{k,i}^2 = \frac{1}{2} \lambda_{k,i}^2 - \frac{1}{2} (V_{c_{1:m}}^{\pi k+1}(\rho))^2$.  

Further notice that

$$\langle V_{c_{1:m}}^{\pi k}(\rho), V_{c_{1:m}}^{\pi k+1}(\rho) \rangle = \frac{1}{2} \| V_{c_{1:m}}^{\pi k}(\rho) \|^2 + \frac{1}{2} \| V_{c_{1:m}}^{\pi k+1}(\rho) \|^2 - \frac{1}{2} \| V_{c_{1:m}}^{\pi k}(\rho) - V_{c_{1:m}}^{\pi k+1}(\rho) \|^2,$$

the last term of which can be bounded by the following lemma.

Lemma 4.4. For any $k = 0, 1, \ldots, K - 1$, and any policy $\pi$ and $\pi'$,

$$\frac{1}{2} \| V_{c_{1:m}}^{\pi k}(\rho) - V_{c_{1:m}}^{\pi k+1}(\rho) \|^2 \leq \frac{\gamma^2 m}{(1 - \gamma)^4} D_{\rho'}^{\pi'}(\pi'||\pi).$$

Proof. For any $i = 1, 2, \ldots, m$, we know

$$\left| V_{c_{1:m}}^{\pi k}(\rho) - V_{c_{1:m}}^{\pi k+1}(\rho) \right| = \frac{1}{1 - \gamma} \left| \sum_{(s,a) \in S \times A} c_i(s,a) (d_{\rho}^{\pi k}(s,a) - d_{\rho}^{\pi k+1}(s,a)) \right| \leq \frac{1}{1 - \gamma} \left| d_{\rho}^{\pi k} - d_{\rho}^{\pi k+1} \right|_1 \leq \frac{\gamma \sqrt{2 \frac{2}{(1 - \gamma)^2}} \sqrt{D_{\rho'}^{\pi'}(\pi'\|\pi)},$$

where the last inequality is due to Lemma 2.1. The lemma is then proved since the inequality above is true for any $i \in [m]$.

By the two lemmas above, it then follows that

$$\langle \lambda_k + V_{c_{1:m}}^{\pi k}(\rho), V_{c_{1:m}}^{\pi k+1}(\rho) \rangle \geq \frac{1}{2} \left( \| \lambda_{k+1} \|^2 - \| \lambda_k \|^2 + \| V_{c_{1:m}}^{\pi k}(\rho) \|^2 - \| V_{c_{1:m}}^{\pi k+1}(\rho) \|^2 \right) - \frac{\gamma^2 m}{(1 - \gamma)^4} D_{\rho'}^{\pi k+1}(\pi_k || \pi_k).$$

Optimality gap bound With the supporting results in the inner loop and outer loop analysis, we are ready to upper bound the optimality gap. Plugging the lower bound of inner product $\langle \lambda_k + V_{c_{1:m}}^{\pi k}(\rho), V_{c_{1:m}}^{\pi k+1}(\rho) \rangle$ in (19) into (16) leads to

$$V_{c_{1:m}}^{\pi k+1}(\rho) + \frac{1}{2} \left( \| \lambda_{k+1} \|^2 - \| \lambda_k \|^2 + \| V_{c_{1:m}}^{\pi k}(\rho) \|^2 - \| V_{c_{1:m}}^{\pi k+1}(\rho) \|^2 \right) + \left( \frac{\alpha}{1 - \gamma} - \frac{\gamma^2 m}{(1 - \gamma)^4} \right) D_{\rho'}^{\pi k+1}(\pi_k || \pi_k)$$

$$\leq V_{c_{1:m}}^{\pi k+1}(\rho) + \frac{\alpha}{1 - \gamma} D_{\rho}^{\pi k+1}(\pi_k || \pi_k) + \frac{1 + 2/(3\gamma)}{k}.$$
is achieved by the optimal policy \( \pi \) \( \pi \)

\[ \text{Lagrangian with optimal dual variable } \]

To analyze the constraint violation, it therefore suffices to bound the dual variables. Consider the gap in Theorem 3.2 is then proved by dividing \( K \) on both sides.

\[ \text{Lemma 3.1. This implies } \]

\[ \sum_{k=1}^{K} V_{\pi_k}^{\pi_k}(\rho) \leq \lambda_{K,i} - \lambda_{0,i} \leq \lambda_{K,i} \leq \| \lambda_K \|. \]

To analyze the constraint violation, it therefore suffices to bound the dual variables. Consider the Lagrangian with optimal dual variable \( L(\pi, \lambda^*) = V_{\pi_0}^* (\rho) + \sum_{i=1}^m \lambda^*_i V_{\pi_i}^* (\rho) \), whose minimum value \( V_{\pi_0}^* (\rho) \) is achieved by the optimal policy \( \pi^* \). We know

\[ KV_{\pi_0}^* (\rho) = KL(\pi^*, \lambda^*) \leq \sum_{k=1}^{K} L(\pi_k, \lambda^*) = \sum_{k=1}^{K} V_{\pi_0}^* (\rho) + \sum_{i=1}^m \lambda^*_i \sum_{k=1}^{K} V_{\pi_k}^* (\rho) \leq \sum_{k=1}^{K} V_{\pi_k}^* (\rho) + \sum_{i=1}^m \lambda^*_i \lambda_{K,i} \]

\[ \leq KV_{\pi_0}^* (\rho) + \frac{\alpha}{1-\gamma} \left( D_{\pi_0}^* (\| \pi_0 \|) - D_{\pi_0}^* (\| \pi_K \|) \right) + 1 + \frac{2}{3\gamma} + \frac{1}{2}\| V_{\pi_1}^* (\rho) \| - \frac{\| \lambda_K \|^2}{2} + \sum_{i=1}^m \lambda^*_i \lambda_{K,i}. \]

(a) holds due to complementary slackness and (b) holds because Equation (20) and the third property of Lemma 3.1. This implies

\[ \frac{\alpha}{1-\gamma} \log(|A|) + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \leq \frac{2}{2-2\gamma m} \left( |V_{\pi_1}^* (\rho) - V_{\pi_1}^* (\rho) \|^2 + 1 + \frac{2}{3\gamma} + \frac{1}{2} \left( |V_{\pi_1}^* (\rho) - V_{\pi_1}^* (\rho) | + V_{\pi_1}^* (\rho) \right)^2 \right) \]

\[ \leq \frac{\alpha}{1-\gamma} \log(|A|) + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \leq \frac{2}{2-2\gamma m} \left( |V_{\pi_1}^* (\rho) - V_{\pi_1}^* (\rho) | + V_{\pi_1}^* (\rho) \right)^2, \]

where (c) holds due to Lemma 4.4 with \( \pi = \pi_K \), \( \pi^* = \pi^* \) and \( D_{\pi_0}^* (\| \pi_0 \|) \leq \log(|A|) \). For any \( a \neq \frac{1}{2} \) and vectors \( x, y \in \mathbb{R}^m \), we have

\[ -a \| x \|^2 + \frac{1}{2} \| x+y \|^2 = \left( \frac{1}{2} - a \right) \| x \|^2 + \frac{1}{2} \| x+y \|^2 + (x, y) \]

\[ = \left( \frac{1}{2} - a \right) \| x \|^2 + \frac{1}{2} \left( \frac{1}{2} - a \right) \| y \|^2 + \left( \frac{1}{2} - \frac{1}{1-2a} \right) \| y \|^2, \]

and assigning \( x = V_{\pi_1}^* (\rho) - V_{\pi_1}^* (\rho) \), \( y = V_{\pi_1}^* (\rho) \) and \( a = \frac{(1-\gamma)^{3\alpha}}{2\gamma m} \) leads to the equality (d). When \( a = \frac{(1-\gamma)^{3\alpha}}{2\gamma m} \), \( \gamma^{m-(1-\gamma)^{3\alpha}} \leq 0 \) and \( \frac{1}{2} - \frac{(1-\gamma)^{3\alpha}}{2\gamma m} = 1 \). It then follows that

\[ \frac{1}{2} \| \lambda^* - \lambda_K \|^2 = \frac{1}{2} \| \lambda^* \|^2 + \frac{1}{2} \| \lambda_K \|^2 - \sum_{i=1}^m \lambda^*_i \lambda_{K,i}, \]
\[ \leq \frac{1}{2} \| \lambda^* \|^2 + \frac{\alpha}{1 - \gamma} \log(|A|) + 1 + \frac{2}{3\gamma} + \| V_{c_0}^\pi (\rho) \|^2 \leq \frac{1}{2} \| \lambda^* \|^2 + \frac{\alpha}{1 - \gamma} \log(|A|) + 1 + \frac{2}{3\gamma} + \frac{m}{(1 - \gamma)^2}. \]

The constraint violation bound in Theorem 3.2 is thus follows from

\[ \sum_{k=1}^{K} V_{c_k}^\pi (\rho) \leq \| \lambda_K \| \leq \| \lambda^* \| + \| \lambda_K - \lambda^* \| \leq \| \lambda^* \| + \sqrt{\| \lambda^* \|^2 + \frac{2\alpha}{1 - \gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1 - \gamma)^2}}. \]

(24)

5 Zero constraint violation

Following recent work on zero constraint violation [18], we generalize the idea of “pessimism” to our setting to achieve zero constraint violation after some \( T \). To guarantee zero constraint violation, we require an additional assumption about Slater’s condition, which is mild and common in CMDP literature [9, 8, 11, 18].

Assumption 5.1 (Slater’s condition). There exists \( \xi > 0 \) and \( \pi \) such that \( V_{c_i}^\pi (\rho) \leq -\xi, \forall i \in [m] \).

Lemma 5.1. Under Assumption 5.1, the optimal dual variables \( \lambda^* \) satisfies

\[ \| \lambda^* \| \leq \| \lambda^* \|_1 \leq \frac{2}{\xi(1 - \gamma)}. \]

Proof. See Appendix D.

We introduce an additive pessimistic term \( \delta \in (0, \xi) \) in the original optimization problem (1), i.e.,

\[ \min_\pi V_{c_i}^\pi (\rho) \]

subject to \( V_{c_i}^\pi (\rho) \leq -\delta, \forall i \in [m] \).

Theorem 5.2. If we adopt Algorithm 1 to solve the new CMDP problem (25) (call the algorithm PMD-PD-Zero) with \( \alpha = \frac{2\gamma}{(1 - \gamma)}\frac{m}{1 - \gamma}, \eta = \frac{1 - \gamma}{\alpha}, t_k = \frac{1}{\eta} \log(3KC_k \gamma) \) with \( C_k = 2\gamma \left( \gamma \sum_{i=1}^{m} \frac{1}{1 - \gamma} + \frac{m}{(1 - \gamma)^2} \right) \), and

\[ \delta := \left( \frac{2}{\xi(1 - \gamma)} + \sqrt{\frac{4}{\xi^2(1 - \gamma)^2} + \frac{2\alpha}{1 - \gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1 - \gamma)^2}} \right) \frac{1}{K}, \]

then \( \forall K \geq \delta \), we have the optimality gap and the constraint violation:

\[ \frac{1}{K} \sum_{k=1}^{K} \left( V_{c_0}^\pi (\rho) - V_{c_0}^\pi (\rho) \right) \leq \left( \frac{\alpha}{1 - \gamma} \log(|A|) + 1 + \frac{2}{3\gamma} \right) \frac{1}{K} + \frac{2\delta}{\xi(1 - \gamma)} \]

\[ \max_{i \in [m]} \left\{ \left( \frac{1}{K} \sum_{k=1}^{K} V_{c_i}^\pi (\rho) \right) \right\} = 0. \]

Proof. This proof is based on Theorem 3.2 with detailed proof provided in Appendix D.
Corollary 5.2.1. The total number of iterations is \( T = \sum_{k=0}^{K-1} t_k = \frac{K}{1-\gamma} \log(3KC\gamma) \). According to Theorem 5.2, \( \forall T \geq \frac{c}{\xi(1-\gamma)} \log \left( \frac{3C\gamma}{\xi} \right) \), we have the optimality gap and the constraint violation:

\[
\frac{1}{K} \sum_{k=1}^{K} \left( V^{\pi_{\rho}}_{c_0}(\rho) - V^{\pi^*}_{c_0}(\rho) \right) \leq c_3 \frac{m \log(|A|) \log(T)}{(1-\gamma)^5 T} = O \left( \frac{\log(T)}{T} \right)
\]

\[
\max_{i \in [m]} \left\{ \left( \frac{1}{K} \sum_{k=1}^{K} V^{\pi_{\rho}}_{c_i}(\rho) \right) \right\} = 0,
\]

where \( c_3 \) is a universal constant.

6 Conclusion

We present an NPG-based algorithm that enjoys an \( O(\log(T)/T) \) global convergence rate for both the optimality gap and the constraint violation, which is nearly dimension-free. With the help of Slater’s condition, zero constraint violation can be further guaranteed after some \( T \) by employing an additional pessimistic term into safety constraints, while keeping the same order convergence rate for the optimality gap. A possible future direction for exploration is a faster convergence rate of the optimality gap.

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A  Supporting lemmas

**Lemma A.1** (Pushback property of Bregman divergences [25, Lemma 14]). Let $B : \Delta \times \Delta^o \to \mathbb{R}$ be a Bregman divergence function, where $\Delta$ is the probability simplex in $\mathbb{R}^d$ and $\Delta^o$ is the interior of $\Delta$. Let $f : \Delta \to \mathbb{R}$ be a convex function. Suppose $x^* = \arg\min_{x \in \Delta} f(x) + \alpha B(x, y)$ for a fixed $y \in \Delta^o$ and $\alpha > 0$, then, for any $z \in \Delta$,

$$f(x^*) + \alpha B(x^*, y) \leq f(z) + \alpha B(z, y) - \alpha B(z, x^*) .$$

**Lemma A.2** (Linear convergence of an exact entropy-regularized NPG [6, Theorem 1]). For any learning rate $0 < \eta \leq (1 - \gamma)/\alpha$ and any $k = 0, 1, \ldots, K - 1$, the entropy-regularized NPG updates satisfy

$$\left\| \bar{Q}^k_{k, \alpha} - Q^k_{k, \alpha} \right\|_{\infty} \leq C_k \gamma (1 - \eta \alpha)^t ,$$
$$\left\| \log \pi^*_{k} - \log \pi^{k+1}_{k} \right\|_{\infty} \leq 2C_k \alpha^{-1} (1 - \eta \alpha)^t ,$$
$$\left\| \bar{V}^k_{k, \alpha} - \bar{V}^{k+1}_{k, \alpha} \right\|_{\infty} \leq 3C_k \gamma (1 - \eta \alpha)^t ,$$

for all $t \geq 0$, where

$$C_k := \left\| \bar{Q}^0_{k, \alpha} - Q^0_{k, \alpha} \right\|_{\infty} + 2\alpha \left( 1 - \frac{\eta \alpha}{1 - \gamma} \right) \left\| \log \pi^*_{k} - \log \pi^{0}_{k} \right\|_{\infty} .$$

**Lemma A.3** (Pinsker’s inequality [7]). If $P$ and $Q$ are two probability distributions on a measurable space $(X, \Sigma)$, then

$$D_{TV}(P\|Q) \leq \sqrt{\frac{1}{2} D(P\|Q)} ,$$

where $D_{TV}(P\|Q) := \sup\{|P(A) - Q(A)| \mid A \in \Sigma \text{ is a measurable event}\}$ is the total variation between $P$ and $Q$.

B  Proofs of auxiliary lemmas

**Lemma B.1** (Restatement of Lemma [24]). Let $d^\pi'_{\rho}, d^\pi_{\rho}$ be two discounted state-action visitation distributions corresponding to policies $\pi'$ and $\pi$. Then

$$\left\| d^\pi'_{\rho} - d^\pi_{\rho} \right\|_1 \leq \frac{\gamma \sqrt{2}}{1 - \gamma} \sqrt{\min \left( D_{d^\pi'}(\pi'\|\pi), D_{d^\pi'}(\pi\|\pi'), D_{d^\pi}(\pi\|\pi), D_{d^\pi}(\pi\|\pi') \right)} .$$

**Proof of Lemma B.1** Let $d^\pi_{\rho,h} (\cdot, \cdot)$ be the state-action visitation distribution at step $h$, which implies

$$\frac{1}{1 - \gamma} d^\pi_{\rho} = \sum_{h \geq 0} \gamma^h d^\pi_{\rho,h} .$$

Let policy $\pi_h := \pi \times h + \pi' \times \infty$ denote the policy that implements policy $\pi$ for first $h$ steps and then commits to policy $\pi'$ thereafter and denote its corresponding discounted state-action visitation distribution by $d^\pi_{\rho,h}$. It follows that

$$\frac{1}{1 - \gamma} \left\| d^\pi'_{\rho} - d^\pi_{\rho} \right\|_1 \leq \frac{1}{1 - \gamma} \left\| \sum_{h=0}^{\infty} (d_{\rho,h}^\pi - d_{\rho,h+1}^\pi) \right\|_1 \leq \frac{1}{1 - \gamma} \sum_{h=0}^{\infty} \left\| d_{\rho,h}^\pi - d_{\rho,h+1}^\pi \right\|_1 .$$

$$\leq \sum_{h=0}^{\infty} \sum_{t \geq h+1} \gamma^t \left\| d_{\rho,h}^\pi - d_{\rho,h+1}^\pi \right\|_1 \leq \sum_{h=0}^{\infty} \sum_{t \geq h+1} \gamma^t \left\| d_{\rho,h}^\pi - d_{\rho,h+1}^\pi \right\|_1 .$$
\[
= \frac{\gamma}{1-\gamma} \sum_{h=0}^{\infty} \gamma^h E_{s \sim d^\pi_{\rho,h}} \|\pi(\cdot|s) - \pi'(\cdot|s)\|_1
\]
\[
\leq \frac{\gamma}{1-\gamma} \left( \sum_{h \geq 0} \gamma^h \right) \left( \sum_{h=0}^{\infty} \gamma^h E_{s \sim d^\pi_{\rho,h}} \|\pi(\cdot|s) - \pi'(\cdot|s)\|_1^2 \right)^{1/2}
\]
\[
= \frac{\gamma}{(1-\gamma)^2} \sqrt{E_{s \sim d^\pi_{\rho}} \|\pi(\cdot|s) - \pi'(\cdot|s)\|_1^2}.
\]

(a) holds by telescoping, (b) and (c) hold due to the triangle inequality of \(\ell_1\)-norm, (d) holds owing to the data processing inequality, and (e) holds due to the Cauchy-Schwarz inequality. Due to the symmetry between \(\pi\) and \(\pi'\), we can similarly derive
\[
\|d^\pi_{\rho} - d^\pi_{\rho}'\|_1 \leq \frac{\gamma}{1-\gamma} \sqrt{E_{s \sim d^\pi_{\rho}} \|\pi(\cdot|s) - \pi'(\cdot|s)\|_1^2}.
\]

We can conclude the proof by further applying Pinsker’s inequality (Lemma A.3).

\[\square\]

**Lemma B.2** (Restatement of Lemma 2.2). \(\tilde{D}(d^\pi_{\rho}||d^\pi_{\rho}')\) is a Bregman divergence generated by the convex function
\[
\phi(d^\pi_{\rho}) = \sum_{(s,a) \in S \times A} d^\pi_{\rho}(s,a) \log d^\pi_{\rho}(s,a) - \sum_{s \in S} d^\pi_{\rho}(s) \log d^\pi_{\rho}(s).
\]

**Proof of Lemma B.2** It is straightforward to verify that
\[
\tilde{D}(d^\pi_{\rho}||d^\pi_{\rho}') = \phi(d^\pi_{\rho}) - \phi(d^\pi_{\rho}') - \langle \nabla \phi(d^\pi_{\rho}), d^\pi_{\rho} - d^\pi_{\rho}' \rangle.
\]

Hence we only need to show that \(\phi(d^\pi_{\rho})\) is convex. The Hessian matrix of function \(\phi(d^\pi_{\rho})\) can be calculated as \(\text{diag}(H_1, H_2, \ldots, H_{|S|})\), where \(H_s = \frac{1}{d^\pi_{\rho}(s)} (\text{diag}(d^\pi_{\rho}(s)/d^\pi_{\rho}(s, \cdot)) - 11^T)\) is an \(|A| \times |A|\) matrix corresponding to state \(s\). For each \(H_s\), we know for any \(x_{1:|A|} \in \mathbb{R}^{|A|}\),
\[
x^TH_sx = \frac{1}{d^\pi_{\rho}(s)} \left( \sum_{a \in A} \frac{d^\pi_{\rho}(s)}{d^\pi_{\rho}(s,a)} x_a^2 - \left( \sum_{a \in A} x_a \right)^2 \right)
\]
\[
= \frac{1}{d^\pi_{\rho}(s)} \left( \sum_{a \in A} \frac{d^\pi_{\rho}(s,a)}{d^\pi_{\rho}(s)} \right) \left( \sum_{a \in A} \frac{d^\pi_{\rho}(s)}{d^\pi_{\rho}(s,a)} x_a^2 - \left( \sum_{a \in A} x_a \right)^2 \right)
\]
\[
\geq \frac{1}{d^\pi_{\rho}(s)} \left( \sum_{a \in A} |x_a| \right)^2 - \left( \sum_{a \in A} x_a \right)^2 \geq 0,
\]

where (a) is due to the Cauchy-Schwarz inequality. Thus the Hessian matrix of \(\phi(d^\pi_{\rho})\) is positive semi-definite, which implies that \(\phi(d^\pi_{\rho})\) is convex. \[\square\]

### C Details of analysis for PMD-PD

**Lemma C.1** (Restatement of Lemma 3.1). Based on the definition of Lagrange multipliers in Algorithm 4 we have
1. For any macro step \(k\), \(\lambda_{k,i} \geq 0\), \(\forall i \in [m]\).
2. For any macro step \(k\), \(\lambda_{k,i} + V^\pi_{c,i}(\rho) \geq 0\), \(\forall i \in [m]\).
3. For macro step 0, \(\|\lambda_0,i\|^2 \leq \|V^{\pi_0}(\rho)\|^2\); for any macro step \(k > 0\), \(\|\lambda_{k,i}\|^2 \geq \|V^{\pi_k}(\rho)\|^2\), \(\forall i \in [m]\).

**Proof of Lemma 3.1**

1. Fix \(i \in [m]\). Note that \(\lambda_{0,i} = \max\{0, -V^{\pi_0}(\rho)\} \geq 0\). Assume \(\lambda_{k,i} \geq 0\). If \(V^{\pi_{k+1}}(\rho) \geq 0\), then \(\lambda_{k+1,i} = \max\{-V^{\pi_{k+1}}(\rho), \lambda_{k,i} + V^{\pi_{k+1}}(\rho)\} \geq \lambda_{k,i} + V^{\pi_{k+1}}(\rho) \geq 0\). If \(V^{\pi_{k+1}}(\rho) < 0\), then \(\lambda_{k+1,i} = \max\{-V^{\pi_{k+1}}(\rho), \lambda_{k,i} + V^{\pi_{k+1}}(\rho)\} \geq -V^{\pi_{k+1}}(\rho) \geq 0\). Thus, \(\lambda_{k+1,i} \geq 0\). The result follows by induction.

2. Fix \(i \in [m]\). Note that \(\lambda_{0,i} + V^{\pi_0}(\rho) = \max\{0, V^{\pi_0}(\rho)\} \geq 0\). For \(k \geq 0\), we have \(\lambda_{k+1,i} = \max\{-V^{\pi_{k+1}}(\rho), \lambda_{k,i} + V^{\pi_{k+1}}(\rho)\} \geq -V^{\pi_{k+1}}(\rho)\).

3. Fix \(i \in [m]\). If \(V^{\pi_0}(\rho) \geq 0\), then \(\lambda_{0,i} = 0\), thus \(\|\lambda_{0,i}\| \leq \|V^{\pi_0}(\rho)\|\). If \(V^{\pi_0}(\rho) < 0\), then \(\lambda_{0,i} = -V^{\pi_0}(\rho)\), thus \(\|\lambda_{0,i}\| \leq \|V^{\pi_0}(\rho)\|\). For \(k \geq 0\), if \(V^{\pi_{k+1}}(\rho) \geq 0\), then \(\lambda_{k+1,i} = \max\{-V^{\pi_{k+1}}(\rho), \lambda_{k,i} + V^{\pi_{k+1}}(\rho)\} \geq \lambda_{k,i} + V^{\pi_{k+1}}(\rho) \geq V^{\pi_{k+1}}(\rho)\). If \(V^{\pi_{k+1}}(\rho) < 0\), then \(\lambda_{k+1,i} = \max\{-V^{\pi_{k+1}}(\rho), \lambda_{k,i} + V^{\pi_{k+1}}(\rho)\} \geq -V^{\pi_{k+1}}(\rho)\). Thus, \(\|\lambda_{k+1,i}\| \geq \|V^{\pi_{k+1}}(\rho)\|\).

\(\blacksquare\)

**Selection of \(t_k\) and \(C_k\).** According to Lemma [A.2] \(\|\tilde{V}_{k,\alpha} - \tilde{V}_{k,\alpha}^{*}\|_{\infty} \leq 3C_k\gamma(1 - \eta_0)^t_k\). If \(3C_k\gamma(1 - \eta_0)^t_k \leq \epsilon,\) we can guarantee \(|\tilde{V}_{k,\alpha} - \tilde{V}_{k,\alpha}^{*}| \leq \epsilon\). Let \(\epsilon := 1/K\) and choose \(t_k = 1/\eta_0 \log(3C_kK\gamma)\) noting that \(\log(1 - \eta_0) \leq -\eta_0\). Since \(\forall(s,a) \in S \times A,\)

\[
\left| \tilde{Q}_{k,\alpha}^{\pi_k}(s, a) - \tilde{Q}_{k,\alpha}^{\pi_k}(s, a) \right| = \gamma \sum_{s' \in S} P(s'|s, a) \left| \tilde{V}_{k,\alpha}^{\pi_k}(s) - \tilde{V}_{k,\alpha}^{\pi_k}(s) \right|
\leq \gamma \sum_{s' \in S} P(s'|s, a) \left| \tilde{V}_{k,\alpha}^{\pi_k}(s) - \tilde{V}_{k,\alpha}^{\pi_k}(s) \right|
\leq \gamma \| \tilde{V}_{k,\alpha}^{\pi_k} - \tilde{V}_{k,\alpha}^{\pi_k} \|_{\infty}
\leq 2\gamma \left( \frac{1 + \sum_{i=1}^{m} \lambda_{k,i}}{1 - \gamma} + \frac{m}{(1 - \gamma)^2} \right),
\]

we have

\[
\left\| \tilde{Q}_{k,\alpha}^{\pi_k} - \tilde{Q}_{k,\alpha}^{\pi_k} \right\|_{\infty} \leq 2\gamma \left( \frac{1 + \sum_{i=1}^{m} \lambda_{k,i}}{1 - \gamma} + \frac{m}{(1 - \gamma)^2} \right).
\]

Above, (a) holds due to Equation (15). Choosing \(C_k = 2\gamma \left( \frac{1 + \sum_{i=1}^{m} \lambda_{k,i}}{1 - \gamma} + \frac{m}{(1 - \gamma)^2} \right)\) can satisfy the conclusion of Lemma [A.2] since \(\eta = (1 - \gamma)/\alpha\).

**Lemma C.2** (Restatement of Lemma 1.2). Let \(\alpha, \eta, t_0, K-1\) be the same value as in Theorem 3.2. Then, for any \(k = 0, 1, \ldots, K - 1\) and any policy \(\pi,\)

\[
\tilde{V}^{\pi_{k+1}}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_{k+1}}(\pi_{k+1}\|\pi_k) \leq \tilde{V}^{\pi_k}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_k}(\pi_{k+1}\|\pi_k) + \frac{1 + 2/(3\gamma)}{K}.
\]

**Proof of Lemma C.2** After \(t_k\) iterations in macro step \(k\) with \(t_k = 1/\eta_0 \log(3C_kK\gamma)\), by Lemma [A.2], we have \(\tilde{V}_{k,\alpha}^{\pi_k}(\rho) \leq \tilde{V}_{k,\alpha}^{\pi_k}(\rho) + 1/\gamma\) and \(\log \pi_k - \log \pi_{k+1} \leq 2(1 - \gamma)2/(3\alpha_3K\gamma).\) It follows that

\[
\tilde{V}^{\pi_{k+1}}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_{k+1}}(\pi_{k+1}\|\pi_k) \leq \tilde{V}^{\pi_k}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_k}(\pi_{k+1}\|\pi_k) + 1/\gamma
\]

\[
\leq \tilde{V}_{k,\alpha}^{\pi_k}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_k}(\pi_{k+1}\|\pi_k) - \frac{\alpha}{1 - \gamma} D^{\pi_k}(\pi_{k}\|\pi_k) + 1/\gamma
\]

\[
= \tilde{V}_{k,\alpha}^{\pi_k}(\rho) + \frac{\alpha}{1 - \gamma} D^{\pi_k}(\pi_{k+1}\|\pi_k) + 1/\gamma + \frac{\alpha}{1 - \gamma} \sum_{(s,a) \in S \times A} d^{\pi_k}(s, a) \log \pi_{k+1}(a|s) / \pi_{k+1}(a|s)
\]

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\[
\begin{align*}
\leq & \tilde{V}_k^\pi(\rho) + \frac{\alpha}{1 - \gamma} D_{d\rho}(\pi_k\|\pi_{k+1}) - \frac{\alpha}{1 - \gamma} D_{d\rho}(\pi\|\pi_{k+1}) + \frac{1}{K} + \frac{\alpha}{1 - \gamma} \|\log \pi_k^\ast - \log \pi_{k+1}\|_\infty \\
\leq & \tilde{V}_k^\pi(\rho) + \frac{\alpha}{1 - \gamma} D_{d\rho}(\pi_k\|\pi_{k+1}) - \frac{\alpha}{1 - \gamma} D_{d\rho}(\pi\|\pi_{k+1}) + \frac{1 + 2/(3\gamma)}{K}.
\end{align*}
\]

Above, (a) holds due to Lemma 4.1.

## D Details of analysis of zero constraint violation

**Lemma D.1** (Restatement of Lemma 5.1). Under Assumption 5.1, the optimal dual variables \( \lambda^\ast \) satisfies

\[
\|\lambda^\ast\| \leq \|\lambda^\ast\|_1 \leq \frac{2}{\xi(1 - \gamma)}.
\]

**Proof of Lemma 5.1** Let \( \pi^\ast \) and \( \lambda^\ast \) achieve the minimax solution of the Lagrangian \( L(\pi, \lambda) \). If \( V_{c_0}^{\pi^\ast}(\rho) < 0 \) for some \( i \in [m] \), it follows that \( \lambda^*_i = 0 \). Due to Assumption 5.1,

\[
V_{c_0}^{\pi^\ast}(\rho) = V_{c_0}^{\pi^\ast}(\rho) + \sum_{i=1}^m \lambda^*_i V_{c_i}^{\pi^\ast}(\rho) \leq V_{c_0}^{\pi}(\rho) + \sum_{i=1}^m \lambda^*_i V_{c_i}^{\pi}(\rho) \leq V_{c_0}^{\pi}(\rho) - \xi \sum_{i=1}^m \lambda^*_i,
\]

which implies that

\[
\xi \|\lambda^\ast\| \leq \xi \|\lambda^\ast\|_1 = \xi \sum_{i=1}^m \lambda^*_i \leq V_{c_0}^{\pi}(\rho) - V_{c_0}^{\pi^\ast}(\rho).
\]

Hence,

\[
\|\lambda^\ast\| \leq \|\lambda^\ast\|_1 \leq \frac{V_{c_0}^{\pi}(\rho) - V_{c_0}^{\pi^\ast}(\rho)}{\xi} \leq \frac{2}{\xi(1 - \gamma)}.
\]

**Theorem D.2** (Restatement of Theorem 5.2). If we adopt Algorithm 2 to solve the new CMDP problem (call the algorithm PMD-PD-Zero) with \( \alpha = \frac{2\gamma^2m}{(1 - \gamma)^4} \), \( \eta = \frac{1 - \gamma}{\alpha} \), \( t_k = \frac{1}{\eta} \log(3KC_k\gamma) \) with \( C_k = 2\gamma \left( 1 + \frac{\sum_{i=1}^m \lambda^*_{i,k}}{1 - \gamma} + \frac{m}{(1 - \gamma)^2} \right) \), and

\[
\delta := \left( \frac{2}{\xi(1 - \gamma)} + \sqrt{\frac{4}{\xi(1 - \gamma)^2} + \frac{2\alpha}{1 - \gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1 - \gamma)^2} \right) \frac{1}{K},
\]

then \( \forall K \geq \frac{1}{\xi} \), we have the optimality gap and the constraint violation:

\[
\frac{1}{K} \sum_{k=1}^K \left( V_{c_0}^{\pi_k}(\rho) - V_{c_0}^{\pi^\ast}(\rho) \right) \leq \left( \frac{\alpha}{1 - \gamma} \log(|A|) + 1 + \frac{2}{3\gamma} \right) \frac{1}{K} + \frac{2\delta}{\xi(1 - \gamma)}
\]

\[
\max_{i \in [m]} \left\{ \left( \frac{1}{K} \sum_{k=1}^K V_{c_{i,k}}^{\pi_k}(\rho) \right) \right\} = 0.
\]
Proof of Theorem 5.2. Since $\pi(a|s) = d^\pi(s, a)/\sum_{a' \in A} d^\pi(s, a')$, we can define a mixed state-action visitation distribution $d^\pi(\delta)$ as
\[
d^\pi(\delta)(s, a) = \frac{\xi - \delta}{\xi} d^\pi^*(s, a) + \frac{\delta}{\xi} \bar{d}^\pi(s, a), \ \forall (s, a) \in S \times A.
\]
It is easy to verify that $\pi(\delta)$ is a feasible solution to the new CMDP formulation (25) since $\forall i \in [m],$
\[
V_{\pi_i}(\delta)(\rho) = (c_i, d^\pi_{\rho}(\delta))
= \frac{\delta - \xi}{\xi} V_i^*(\rho) + \frac{\delta}{\xi} V_{\pi_i}(\delta)(\rho)
\leq 0 + \frac{\delta}{\xi} (-\xi) = -\delta.
\]
Let $\pi^*(\delta)$ be the optimal policy of the new CMDP problem (25). It implies
\[
V_{\pi^*(\delta)}(\rho) - V_{\pi_0}(\rho) \leq \frac{2\delta}{\xi(1 - \gamma)}. \tag{26}
\]
Therefore,
\[
\frac{1}{K} \sum_{k=1}^{K} (V_{\pi_0}(\rho) - V_{\pi_0}(\rho)) = \frac{1}{K} \sum_{k=1}^{K} \left[ (V_{\pi_0}(\rho) - V_{\pi_0}(\delta)(\rho)) + (V_{\pi_0}(\delta)(\rho) - V_{\pi_0}(\rho)) \right]
\leq (\frac{\alpha}{1 - \gamma} \log(|A|) + 1 + \frac{2}{3\gamma}) \frac{1}{K} + \frac{2\delta}{\xi(1 - \gamma)}.
\]
(a) holds due to Equations (13) and (26). Define $\xi(\delta) = \xi + \delta$, we have
\[
\left( \frac{1}{K} \sum_{k=1}^{K} V_{\pi_i}^k(\rho) \right) + \left( \frac{1}{K} \sum_{k=1}^{K} (V_{\pi_i}^k(\rho) + \delta) - \delta \right)
\leq \left( \left\| \lambda_\delta^* \right\| + \sqrt{\left\| \lambda_\delta^* \right\|^2 + \frac{2\alpha}{1 - \gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1 - \gamma)^2}} \frac{1}{K} - \delta \right) +
\]
where $\left\| \lambda_\delta^* \right\| \leq 2/(\xi(\delta)(1 - \gamma)) = 2/(\xi + \delta)(1 - \gamma)$ according to Lemma 5.1 and (b) holds because of Equation (14). Choosing $\delta = \left( \frac{2}{\xi(1 - \gamma)} + \sqrt{\frac{4}{\xi^2(1 - \gamma)^2} + \frac{2\alpha}{1 - \gamma} \log(|A|) + 2(1 + \frac{2}{3\gamma}) + \frac{2m}{(1 - \gamma)^2}} \frac{1}{K} \right)$ concludes the proof. \qed