Hierarchical renormalization-group study on the planar bond-percolation problem

Seung Ki Baek and Petter Minnhagen

Abstract

For certain hierarchical structures, one can study the percolation problem using the renormalization-group method in a very precise way. We show that the idea can also be applied to two-dimensional planar lattices by regarding them as hierarchical structures. Either a lower bound or an exact critical probability can be obtained using this method and the correlation-length critical exponent is approximately estimated as $\nu \approx 1$.

PACS numbers: 64.60.ah, 64.60.ae, 05.10.Ln

The percolation problem is a question on how a global connection can be made possible by randomly filling local components by a certain probability $p$. While it can be explained in purely geometric terms without any interaction, when a global connection actually appears, the macroscopic behavior of the system exhibits all the characteristic features of a continuous phase transition with a diverging correlation length, just as we observe in other interacting spin systems such as the two-dimensional (2D) Ising model [1]. This analogy is given a precise meaning by the Fortuin–Kasteleyn representation of the $q$-state Potts model [2], where the percolation turns out to be equivalent to the limit of $q \to 1$. Since the percolation transition at a critical probability $p_c$ has a diverging correlation length, every microscopic length scale becomes irrelevant with respect to the critical phenomena, and the system behaves as if it does not have any specific length scale. This is a qualitative explanation of the reason why a percolating cluster connecting two opposite sides of a 2D plane has a fractal dimension at $p = p_c$. The lack of a specific length scale implies that the system remains statistically invariant even if we zoom the system up or down, and this scale invariance readily lends itself to a renormalization-group (RG) study of the percolation problem [3–5].

In certain cases, where the underlying structure itself is fractal, it is possible to carry out the RG calculation to a good approximation or exactly, exploiting this fractal property [6, 7]. Such fractal structures usually contain groups of bonds that connect longer and longer distances in a regular fashion. For this reason, one can sometimes arrange the groups of bonds in a hierarchical way according to their connection lengths. Figure 1(a) is an example of a hierarchical structure called the enhanced binary tree, which is obtained by adding horizontal bonds to the simple binary tree. It is hierarchical in the sense that filling a horizontal bond is comparable to a very long connection along the bottom layer and the connection length is dependent on the level of the horizontal bond [8]. That is, a horizontal bond in the highest level can connect two points at distance 7 along the bottom layer at maximum. For a horizontal bond at the next highest level, this maximum connection distance is only as large as three lattice spacings. An RG scheme for the enhanced binary tree is described in [8] as shown in figure 1(b): we calculate the probability for any of the leftmost points to connect to any of the rightmost points within the cell as a function of the bare coupling $p$ and a coarse-grained effective coupling $z_n$, and then replace this probability by a new effective bond with strength $z_{n+1}$. The resulting expression for $z_{n+1}$ is written as

$$z_{n+1} = p + (1 - p) \left[ (1 - p)^2 z_n^3 + 2p(1 - p)z_n^2 + p^2 z_n \right].$$

By asking when $z_n = z_{n+1} = z_\infty$ becomes 1, we obtained a lower bound of the percolation threshold as $p_c \geq 1/2$ [8], which is consistent with the conclusion in [9] that $p_c = 1/2$. Note that we get a lower bound since in iterating $z_n$ to $z_{n+1}$, there is a small chance to regard a layer as percolated when it is actually not (see, e.g. figure 1(c)), whereas the opposite is not possible.

Although the above RG scheme is devised to investigate a hierarchical structure, we show in this work that it can be applied to non-hierarchical planar lattices as well. In figure 2(a), we present a variation of the RG scheme shown above. The similarity is obvious: we have taken away only one bond out of those in figure 1(b), and this is meant to describe the triangular lattice. It leads us to the
coarse-grained into a single bond filled by probability $z_{n+1}$. (c) This layer is not connected from left to right even though the cells inside it appear as filled according to the recursion scheme in (b). The solid and dotted lines represent filled and empty bonds, respectively.

The following recursion:

$$z_{n+1} = p + (1-p) [ p + (1-p) z_n ]^2.$$  \hspace{1cm} (1)

Again, the bond connection in lower levels, composed of $p$ and $z_n$, is converted to a single bond with $z_{n+1}$ at a higher level. This distinction of levels might look arbitrary since the bonds in the plane do not have any hierarchy. However, the important point is that the argument above to find a lower bound still remains legitimate with regard to this viewpoint.

Solving (1) for $z_{n+1} = z_n = z_\infty$, we find that

$$z_\infty \equiv \frac{p(1 + p - p^2)}{(1-p)^3}$$

and consequently, $z_\infty = 1$ at $p^* = 1 - 1/\sqrt{2} \approx 0.293$.

Comparing this with the exact bond-percolation threshold in the triangular lattice, $p_c^T \approx 0.347$ [10], we see that our method indeed yields a lower bound. We now extend the cells to be renormalized by adding one more level. That is, let us denote the width of the cell as $w$ and consider the case of $w = 2$.

For the triangular lattice, the shape of such a larger cell is given in figure 2(b). By enumerating all the possible cases, the recursion relation is obtained as

$$z_{n+1} = 3p^6 z_n^3 - 25p^5 z_n^2 + 90p^4 z_n - 182p^3 z_n^2 + 224p^2 z_n^3 - 168p z_n^4 + 70p^3 z_n^3 - 10p^2 z_n^3 - 3p z_n^3 - 7p z_n^2 + 53p^2 z_n^2 - 171p z_n^3 + 303p^2 z_n^2 - 315p z_n^2 + 187p z_n^2 - 53p^2 z_n^2 + 2p z_n^3 + 5p z_n - 33p z_n + 89p z_n + 121p^2 z_n + 79p z_n - 11p z_n + 13z_n^2 + 5p z_n - 9p + 5p^2 - 8p + 12p^2 - 8p^4 - 4p^3 + 4p^2 + p.$$  

and we find its limiting value as

$$z_\infty \equiv F_1(p) - \sqrt{F_2(p)}$$

with

$$F_1(p) = 4p^6 - 16p^5 + 21p^4 - 6p^3 - 6p^2 + 2p + 1, \hspace{1cm} F_2(p) = 4p^5 - 40p^4 + 176p^3 - 400p^2 + 400p + 653p^5 - 508p^4 + 48p^5 + 236p^4 - 126p^3 - 32p^2 + 8p + 1$$

and $F_3(p) = 6p^6 - 32p^5 + 66p^4 - 64p^3 + 26p^2 - 2$. The solution of $z_\infty = 1$ is found at $p^* \approx 0.300$, which is an improved lower bound compared to the previous one, $p = 1 - 1/\sqrt{2} \approx 0.293$, even though the convergence turns out to be rather slow.

If we also regard the honeycomb lattice as hierarchical, we can consider an RG scheme as depicted in figure 2(c). By calculating the probability for any of the leftmost points to connect to any of the rightmost points within the cell, we find

$$z_{n+1} = (p + (1-p) z_n) z_n.$$

Using a little algebra as above, we find $z_\infty = p^2 / (1-p)^2$, which becomes one at $p^* = 1/2$. Again, this is lower than the exact value $p_c^H \approx 0.653$ [10]. One may expect an improved estimate by considering a larger cell shown in figure 2(d), which leads to

$$z_{n+1} = -4p^7 z_n^3 + 15p^6 z_n^3 - 62p^5 z_n^3 + 15p^4 z_n^3 - p^3 z_n^3 - 2p z_n^3 + 3p^2 z_n^2 - 35p^2 z_n^2 + 40p z_n^2 - 12p z_n^2 - 4p^3 z_n^2 + 2p z_n^2 + 2p z_n - 8p^2 z_n - 21p z_n + 12p z_n - 12 p z_n^2 + 6 p^2 z_n^2 + 4 p z_n^2 + 8 p z_n + 1.$$  

and

$$z_\infty \equiv G_1(p) - \sqrt{G_2(p)}$$

The limiting solution is

$$z_\infty \equiv \frac{G_1(p) - \sqrt{G_2(p)}}{G_3(p)},$$

where $G_1(p) = 4p^6 - 16p^5 + 61p^4 - 6p^3 - 6p^2 + 2p + 1$, $G_2(p) = 4p^5 - 40p^4 + 176p^3 - 400p^2 + 400p + 653p^5 - 508p^4 + 48p^5 + 236p^4 - 126p^3 - 32p^2 + 8p + 1$, and $G_3(p) = 6p^6 - 32p^5 + 66p^4 - 64p^3 + 26p^2 - 2$. The solution of $z_\infty = 1$ is found at $p^* \approx 0.293$, which is an improved lower bound compared to the previous one, $p = 1 - 1/\sqrt{2} \approx 0.293$, even though the convergence turns out to be rather slow.

If we also regard the honeycomb lattice as hierarchical, we can consider an RG scheme as depicted in figure 2(c). By calculating the probability for any of the leftmost points to connect to any of the rightmost points within the cell, we find

$$z_{n+1} = (p + (1-p) z_n) z_n.$$  

Using a little algebra as above, we find $z_\infty = p^2 / (1-p)^2$, which becomes one at $p^* = 1/2$. Again, this is lower than the exact value $p_c^H \approx 0.653$ [10]. One may expect an improved estimate by considering a larger cell shown in figure 2(d), which leads to

$$z_{n+1} = -4p^7 z_n^3 + 15p^6 z_n^3 - 62p^5 z_n^3 + 15p^4 z_n^3 - p^3 z_n^3 - 2p z_n^3 + 3p^2 z_n^2 - 35p^2 z_n^2 + 40p z_n^2 - 12p z_n^2 - 4p^3 z_n^2 + 2p z_n^2 + 2p z_n - 8p^2 z_n - 21p z_n + 12p z_n - 12 p z_n^2 + 6 p^2 z_n^2 + 4 p z_n^2 + 8 p z_n + 1.$$  

and

$$z_\infty \equiv G_1(p) - \sqrt{G_2(p)}$$

The limiting solution is

$$z_\infty \equiv \frac{G_1(p) - \sqrt{G_2(p)}}{G_3(p)},$$

where $G_1(p) = 4p^6 - 16p^5 + 61p^4 - 6p^3 - 6p^2 + 2p + 1$, $G_2(p) = 4p^5 - 40p^4 + 176p^3 - 400p^2 + 400p + 653p^5 - 508p^4 + 48p^5 + 236p^4 - 126p^3 - 32p^2 + 8p + 1$, and $G_3(p) = 6p^6 - 32p^5 + 66p^4 - 64p^3 + 26p^2 - 2$. The solution of $z_\infty = 1$ is found at $p^* \approx 0.293$, which is an improved lower bound compared to the previous one, $p = 1 - 1/\sqrt{2} \approx 0.293$, even though the convergence turns out to be rather slow.
where $G_1(p) \equiv -6p^2 + 6p^3 + 4p^4 + p^5 - 2p - 1$, $G_2(p) \equiv 4p^10 + 16p^9 + 24p^8 - 12p^7 + 12p^6 - 20p^5 - 11p^4 + 4p^3 + 10p^2 + 6p^1 + 1$ and $G_3(p) \equiv 8p^5 - 18p^4 + 8p^3 + 4p^2 - 2$. We find that $z_\infty = 1$ at $p^* \approx 0.557$. Using the duality relation $p^*_1 + p^*_2 = 1$ \cite{10}, we may turn this result to an upper bound of the bond-percolation threshold for the triangular lattice. That is, our method gives a possible region of the threshold as $0.300 \leq p^*_2 \leq 0.463$, or equivalently, $0.573 \leq p^*_3 \leq 0.700$.

A more interesting case is found by considering the horizontal bonds in figures 2(a) and (b) as fictitious (figures 2(e) and (f)). This corresponds to the square lattice, and the interaction in the horizontal direction will appear only as an effective one mediated by shorter bonds. Then we can simplify (1) as

$$z_{n+1} = \left[ p + (1 - p)z_n \right]^2,$$

which happens to be the same as (2). Therefore, we find $p^* = 1/2$ once again, but this value is identical to the exact value for the bond-percolation problem in the square lattice \cite{11}. Since this method is supposed to give a lower bound, it should not be possible to improve this result further, and so it will be worth checking whether this value really remains unchanged for a larger cell. From a larger cell depicted in figure 2(f), we obtain a recursion

$$z_{n+1} = -3p^6z_n^3 + 14p^5z_n^3 + 25p^4z_n^3 + 20p^3z_n^3 + 5p^2z_n^3 - 5p^3z_n^3 - 2pz_n^3 + 3p^4z_n^3 - 28p^3z_n^3 - 40p^2z_n^3 - 22p^3z_n^3 + 7p^2z_n^3 + 2p^4z_n^3 + 5p^5z_n^3 + 16p^4z_n^3 - 14p^3z_n^3 - 4p^5z_n^3 + 3p^4z_n^3 + 5p^6 - 2p^5 - 2p^4 - 2p^3 + p^2,$$

and find its limiting value as

$$z_\infty = \frac{H_1(p) - \sqrt{H_2(p)}}{H_3(p)},$$

with $H_1(p) \equiv 4p^4 - 6p^3 - p^2 + 2 + 1$, $H_2(p) \equiv 4p^8 - 16p^7 + 27p^6 - 12p^5 - 15p^4 + 6p^3 + 4p^2 + 1$ and $H_3(p) \equiv 6p^8 - 16p^7 + 12p^6 - 2$. The critical value making $z_\infty = 1$ is also $p^* = 1/2$, as expected. The fact that $p^*$ does not change with $w$ could be an evidence that the bond-percolation threshold is located exactly at $p = 1/2$ for the square lattice.

In addition, we can argue that the connection probability over distance $l$ would be roughly determined by $(z_\infty)^l = e^{l\log z_\infty}$ near the critical point. In other words, the correlation length would be written as $\xi = -1/\log z_\infty$. The slope of $z_\infty$ around $p = p_c$ does not vanish in every case considered above, and so it generally behaves as $z_\infty \sim a(p - p_c) + 1$ where $a \equiv \partial z_\infty / \partial p|_{p=p_c} \sim O(1)$ at $p = p_c + \epsilon$ with positive $\epsilon \ll 1$. Therefore, we see that

$$\xi \sim -\frac{1}{\log z_\infty} \sim -\frac{1}{\log[a(p - p_c) + 1]} \approx (p_c - p)^{-1},$$

by using $\log(1 - a\epsilon) \approx -a\epsilon$. Since the correlation length is assumed to diverge as $\xi \sim [p - p_c]^{-\nu}$, this argument gives us an approximate estimate of the critical exponent as $\nu \approx 1$, which is an underestimate compared to the exact value, $\nu = 4/3$ \cite{12}. It is worth noting that this RG scheme does not make use of any explicit scaling transformation: we do not zoom up or zoom down the system at criticality as usually found in RG studies \cite{3-5}. In arguing the value of $\nu$, therefore, we evaluate it directly in units of the given lattice spacing instead of any zooming ratio. By setting $z_n = z_{n+1}$, in a sense, it is the translational invariance that we are actually exploiting in this study.

In summary, we have shown that the RG scheme devised for a hierarchical structure can also be applied to the 2D lattices even though they are not hierarchical. It generally yields a lower bound, but correctly predicts the bond-percolation threshold for a square lattice. We have also approximately estimated $\nu \approx 1$. This method is more related to the translational invariance rather than to the scaling invariance at criticality.

Acknowledgment

We are grateful for support from the Swedish Research Council with grant no. 621-2008-4449.

References

[1] Stauffer D and Aharony A 2003 Introduction to Percolation Theory 2nd (London: Taylor and Francis)
[2] Fortuin C M and Kasteleyn P W 1972 On the random-cluster model: I. Introduction and relation to other models Physica 57 536
[3] Niemeijer Th and van Leeuwen J M 1973 Wilson theory for spin systems on a triangular lattice Phys. Rev. Lett. 31 1411
[4] Reynolds P J, Klein W and Stanley H E 1977 A real-space renormalization group for site and bond percolation J. Phys. C: Solid State Phys. 10 L167
[5] Reynolds P J, Stanley H E and Klein W 1980 Large-cell Monte Carlo renormalization group for percolation Phys. Rev. B 21 1223
[6] Rozenfeld H D and ben Avraham D 2007 Percolation in hierarchical scale-free nets Phys. Rev. E 75 061102
[7] Boettcher S, Cook J L and Ziff R M 2009 Patch percolation on a hierarchical network with small-world bonds Phys. Rev. E 80 041115
[8] Baek S K and Minnhagen P 2011 Bounds of percolation thresholds in the enhanced binary tree Physica A 390 1447
[9] Minnhagen P and Baek S K 2010 Analytic results for the percolation transitions of the enhanced binary tree Phys. Rev. E 82 011113
[10] Sykes M F and Essam J W 1963 Some exact critical percolation probabilities for bond and site problems in two dimensions Phys. Rev. Lett. 10 3
[11] Kesten H 1980 The critical probability of bond percolation on the square lattice equals 1/2 Commun. Math. Phys. 74 41
[12] den Nijs M P M 1979 A relation between the temperature exponents of the eight-vertex and $q$-state Potts model J. Phys. A: Math. Gen. 12 1857