On the problem of completeness of QM: von Neumann against Einstein, Podolsky, and Rosen

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Abstract

We performed a comparative analysis of the arguments of Einstein, Podolsky and Rosen – EPR, 1935: [1] (against the completeness of QM) and the theoretical formalism of QM (due to von Neumann, 1932: [2]). We found that the EPR considerations do not match at all with the von Neumann’s theory. Thus EPR did not criticize the real theoretical model of QM. The root of EPR’s paradoxical conclusion on incompleteness of QM is the misuse of von Neumann’s projection postulate. EPR applied this postulate to observables with degenerate spectra (which is totally forbidden by the axiomatics of QM).

1 Introduction

During last 70 years completeness of QM and ”quantum nonlocality” have been the most intriguing problems in quantum foundations. Since recently ideas on impossibility to provide a deterministic description of reality (to introduce ”hidden variables”) and on ”quantum nonlocality” diffused outside of physics, e.g., to philosophy, cognitive
science, genetics, psychology and even parapsychology, these problems became of the really multi-disciplinary character.

To understand correctly such fundamental problems, it is extremely important to read carefully original sources. And I would like to point out that the situation for mentioned problems is astonishing. Although the original paper of Einstein, Podolsky and Rosen – EPR, 1935: [1] is widely cited, it seems that not so many people read it carefully (if at all!).

1.1 Misuse of the von Neumann’s projection postulate in EPR’s argument

In the present article I perform a careful analysis of the EPR argument on the problem of completeness of QM. The conclusion of such analysis is that EPR simply made a mistake in consideration of the process of reduction of the wave function. The root of EPR’s paradoxical conclusion on incompleteness of QM is the misuse of von Neumann’s projection postulate. EPR applied this postulate to observables with degenerate spectra (which is totally forbidden by the axiomatics of QM, von Neumann, 1932: [2]).

I think that understanding of the real root of the EPR-paradox is extremely important for quantum foundations. I hope that the present paper would essentially clarify this problem.

1.2 Copenhagen and Växjö interpretations of QM

After publication of this preprint I was accused by some my colleagues that I ”changed the camp” and I took the side of the orthodox Copenhagen community, e.g. ”By reading your previous papers one had an impression that you believed that QT should be completed by some microscopic field theory. It seems quite strange that you are using now the axiomatic approach of von Neumann, who incorrectly claimed to prove the completeness of QT, in order to prove the incorrectness of EPR arguments.” Therefore I should explain from the very beginning the aim of this publication and my own position.

My own position is the same as before, see e.g. [3]. I do not think that the Copenhagen interpretation is the correct interpretation of QM. I recall the main distinguishing features of the Copenhagen
CH1: Any state of an individual physical system is described by a wave function $\psi$;

CH1: The state of a system after measurement is determined by the projection postulate.

I think that the correct interpretation is so-called statistical interpretation. Recently it also becomes known as the Växjö interpretation, see papers in [4]–[6].

I recall the main distinguishing features of the Växjö interpretation:

VXU1: A wave function $\psi$ is not an attribute of a single physical system (e.g., electron). A wave function $\psi$ (as well as a density matrix $\rho$) describes an ensemble of identically prepared physical systems.

VXU2: The projection postulate determines not the state of a system (after the corresponding measurement), but the probability distribution of an ensemble of (output-)systems.

This interpretation was supported by Einstein. In fact, article [1] was written to support this interpretation via proving inconsistency of the Copenhagen interpretation.

I am definitely on Einstein’s side regarding the interpretation of QM. However, I think that arguments used to criticize opponents should be perfectly rigorous. Otherwise such arguments might induce even more misunderstanding. The aim of my paper is to show that, in spite of good wish of EPR, their arguments were not rigorous. They misused the projection postulate. As a consequence, the EPR paper became the source of

a) **naive realism** – an attempt to ignore the role of measurement devices and assign values of e.g. two incompatible observables to the same system;

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1. The terminology "statistical interpretation" which was elaborated and advocated by L. Ballentine [8], [7] is sometimes misleading, because some people using the Copenhagen interpretation are also sure that they use "statistical interpretation", since they use Born’s rule. It became evident for me in a series of discussions with Slava Belavkin who definitely uses the Copenhagen interpretation, but at the same time he is sure that it is "statistical interpretation."

2. Thus, opposite to the Copenhagen interpretation, by the Växjö interpretation there is no difference between "pure" and "mixed" quantum states. Both types of states describes "subquantum mixtures".
b) quantum nonlocality.

At the first sight, the b) is surprising. EPR considered it as an absurd alternative to a). Nevertheless, quantum majority took this idea seriously. And we shall see that it was motivated by the very structure of the EPR-arguments.

Thus my reply to supporters of the Växjö interpretation is that even the orthodox Copenhagen interpretation is better than naive realism.

In this paper I shall show that one might work in the orthodox Copenhagen framework without quantum nonlocality! To proceed in this way, one should apply the projection postulate as it was proposed by von Neumann.

Thus the main aim of this paper is to liberate the orthodox Copenhagen interpretation from the monster of quantum nonlocality. It would be much easier to find common points between supporters of the local Copenhagen interpretation and the Växjö interpretation.

Concerning the critique of my colleagues from the Växjö side. I agree that if one starts from the very beginning with the statistical interpretation (the Växjö interpretation), one can easily resolve the EPR paradox, see e.g. the excellent paper of Kupczynski [9]. But it was not the aim of EPR! They used their arguments for another purpose – to destroy the Copenhagen interpretation.

1.3 Von Neumann’s postulate and Lüders postulate

The main point of this paper is that EPR applied the projection postulate to operators with degenerate spectrum. Even if one takes for a single system an operator with nondegenerate spectrum $A$, e.g., spin, then by considering a pair of particles one should realize this operator in the tensor product as $A \otimes I$. So, the latter has degenerate spectrum. Von Neumann’s [2] projection postulate is unaplicable in such a case. The postulate which was used by EPR became later formalized by Lüders, see [10] for discussion.

My colleagues became angry again. This time I was attacked from both sides, both from the Copenhagen and anti-Copenhagen. Surprisingly both groups have the same viewpoint to the projection postulate.

Copenhagen: "Whether or not it follows from von Neumanns’ axiomatization is irrelevant. There argument does follow from the ax-
iomatization adopted by all working physicists, still today. And I suppose the argument had been used before EPR, they did not invent it. When you have a composite system and you measure one part of it, the joint state is projected into the subspace obtained by taking the tensor product of the eigenspace of the observable you have measured on one of the components, with the whole of the second space. Are you saying that all books on quantum information should be thrown away because this axiom was not written down by von Neumann? Read any book on quantum information eg Nielsen and Chuang.”

Anti-Copenhagen: ”The thousands of physicists reading the EPR paper did not object the reduction argument because they used it in the same way. Note that presently nearly all people working in the field of quantum information are using the projection postulate similarly as it was used by EPR.”

First, I reply to the supporter of Copenhagen. Well, physicists ignores von Neumann’s distinction between operators with degenerate and nondegenerate spectra in application of the projection postulate. But they pay for this by QUANTUM NONLOCALTY. I think that it is too high price for ignorance.

But, even by using the Växjö interpretation one should be careful with the use of the projection postulate. In fact, VXU2 also might be interpreted in two ways: von Neumann’s like and Lüders-like. But, since this paper is solely based on the Copenhagen interpretation, we do not want to go into details.

Other people (experts in theory of so called ”quantum instruments”) pointed to me that they are well aware about different forms of the projection postulate, see e.g. [11]–[14]. And it is nothing new for them. However, they either proceed in purely mathematical framework or even simply ignore the principle physical difference between von Neumann’s and Lüders’ versions of the projection postulate. In the latter case they even speak about von Neumann-Lüders’ postulate by considering Lüders’ postulate as just a natural generalization of von Neumann’s one. Typically von Neumann’s postulate is considered as a ”primitive” one which was ”improved” by Lüders.
2 The role of the projection postulate in the EPR argument

The role of the projection postulate in the EPR-considerations is practically unknown (except of a few experts in quantum foundations). The main problem is that not so many people have read the original EPR-paper [1]. Even if one did this, it was not careful reading - since it was easier to understand the EPR-arguments from later books on QM. However the projection postulate is the basis of the EPR-definition of an element of reality. Hence, its use (in fact, misuse) is the main source of dilemma: either incompleteness or nonlocality. We shall see that the right (von Neumann) application of the projection postulate would not generate such a dilemma. In particular, so called "quantum nonlocality" would not at all appear in discussion on completeness of QM (its Copenhagen interpretation).

What was wrong in the EPR-considerations? The crucial point was misuse of reduction of wave function in QM. By speaking about QM one should pay attention both to its mathematical formalism and its interpretation. The EPR consideration was not consistent neither with the mathematical formulation (due to von Neumann [2]) nor interpretation (due to Bohr [16]).

We now present the EPR-arguments in detail, since otherwise it would be really impossible to criticize them: details are extremely important. We remind the EPR viewpoint on elements of reality:

"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity then there exists an element of physical reality corresponding to this physical quantity."

We emphasize that the main part of the EPR paper [1] consists

\[\text{From the very beginning we emphasize that the EPR-arguments were against QM as a theoretical model (including interpretational part). Thus the EPR story was not about "physical elements of reality", but about their theoretical counterparts in the formalism of QM. We recall that axiomatization of QM was performed by Dirac [15] and von Neumann [2]. Measurement theory was completely formalized in [2]. EPR's arguments are in fact about measurement theory. To be rigorous, they should speak about theoretical counterparts of "elements of reality" in von Neumann's axiomatic model. Unfortunately, EPR did not do this precisely (as we shall see). Instead of speaking about von Neumann's axiomatics, they criticized a QM model which was not rigorously formalized. I think that this absence of rigor was the main root of the "EPR-paradox."} \]
of considerations on description of reduction of the wave function in
QM. Their aim was to associate elements of reality with elements of
the theoretical model of QM. We recall that the *EPR critique was
against this model* (and not at all against some real experimental
designs). We shall see that EPR associated their elements of reality
with eigenfunctions of corresponding self-adjoint operators. We now
present their considerations on reduction.

If \( \psi \) is an eigenfunction of the operator \( \hat{A} \),
\[
\psi' \equiv \hat{A}\psi = a\psi, \tag{1}
\]
where \( a \) is a number, and so the physical quantity \( A \) has with certainty
the value \( a \) whenever the particle is in the state \( \psi \). By the criterion
of reality, for a particle in the state given by \( \psi \) for which (1) holds
there is an element of physical reality corresponding to the physical
quantity \( A \). For example,
\[
\psi = e^{i(p_0/\hbar)x}, \tag{2}
\]
where \( p_0 \) is some constant number, and \( x \) the independent variable.
Since the operator corresponding to the momentum of the particle is
\[
\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}, \tag{3}
\]
we obtain
\[
\psi' = \hat{p}\psi = \frac{\hbar}{i} \frac{\partial}{\partial x} \psi = p_0\psi. \tag{4}
\]
Thus in the state given by (2) the momentum has certainly the value
\( p_0 \). It thus has meaning to say that the momentum of the particle in
the state given by (2) is real.

On the other hand, if (1) does not hold we can no longer speak
of the physical quantity \( A \) having a particular value. This is the
case, for example, with the coordinate of the particle. The operator
corresponding to it, say \( \hat{q} \), is the operator of multiplication by the
independent variable. Thus
\[
\hat{q}\psi = x\psi \neq a\psi. \tag{5}
\]
In accordance with quantum mechanics we can only say that the relative
probability that a measurement of the coordinate will give a result
lying between \( a \) and \( b \) is
\[
P_{\psi}([a, b]) = \int_{a}^{b} \psi\bar{\psi}dx = \int_{a}^{b} dx = b - a. \tag{6}
\]
Since this probability depends upon the difference $b - a$, we see that all values of the coordinate are equally probable.

More generally, if the operators corresponding to two physical quantities, say $A$ and $B$, do not commute, that is, if $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this it follows that: either

a) the quantum mechanical description of reality given by the wave function is not complete;

or

b) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.

For if both of them had simultaneous reality— and thus definite values—their values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would be predictable.

By the Copenhagen interpretation of quantum mechanics it is assumed that the wave function does contain a complete description of the physical reality of the system in the state to which it corresponds.

Let us suppose that we have two systems $S_1$ and $S_2$ which we permit to interact from the time $t = 0$ to $t = T$, after which time we suppose that there is no longer any interaction between the two parts. We further suppose that the states of the two systems before $t = 0$ were known. We can then calculate, with the help of the Schrödinger equation, the state of the combined system $S_1 + S_2$ at any subsequent time; in particular, for any $t > T$.

Let us designate the corresponding wave function (calculated with the aid of the Schrödinger equation) by $\Psi$. This is the function of the two variables $x_1$ and $x_2$ corresponding to the systems $S_1$ and $S_2$ respectively, $\Psi = \Psi(x_1, x_2)$. We cannot, however, calculate the state in which either one of the two systems is left after the interaction. This, according to quantum mechanics, can be done with the help of the further measurements by a process known as the reduction of the wave function. Let us consider the essentials of this process.

Let $a_1, a_2, a_3, ...$ be the eigenvalues of an operator $\hat{A}$ corresponding to some physical quantity $A$ pertaining to the system $S_1$ and $u_1(x_1), u_2(x_1), u_3(x_1), ...$ the corresponding eigenfunctions. Then $\Psi$,
considered as a function of $x_1$, can be expressed as

$$
\Psi(x_1, x_2) = \sum_{n=1}^{\infty} u_n(x_1) \psi_n(x_2) \quad (7)
$$

Here the $\psi_n(x_2)$ are to be regarded merely as the coefficients of the expansion of $\Psi(x_1, x_2)$ into a series of orthogonal functions $u_n(x_1)$. Suppose now that the quantity $A$ is measured and is found to have the value $a_k$. It is then concluded that after the measurement the first system is left in the state given by the wave function $u_k(x_1)$, and the second system is left in the state given by the wave function $\psi_k(x_2)$. This is the process of reduction of the wave function; the wave function given by the infinite series (7) is reduced to a single term $u_k(x_1)\psi_k(x_2)$.

The set of functions $u_n(x_1)$ is determined by the choice of the physical quantity $A$. If, instead of this, we had chosen another quantity, say $B$, with the operator $\hat{B}$ having the eigenvalues $b_1, b_2, b_3, \ldots$ and eigenfunctions $v_1(x_1), v_2(x_1), v_3(x_1), \ldots$ we should have obtained, instead of (7), the expansion

$$
\Psi(x_1, x_2) = \sum_{s=1}^{\infty} v_s(x_1) \phi_s(x_2), \quad (8)
$$

where $\phi_s$ are the new coefficients. If the quantity $B$ is now measured and is found to have the value $b_r$, we conclude that after the measurement the system $S_2$ is left in the state given by $\phi_r(x_2)$.

Let us now go back to the consideration of the quantum state $\Psi$. As we have seen, as a consequence of two different measurements performed upon the first system $S_1$ (for the quantities $A$ and $B$) the second system may be left in states with two different wave functions -- $\psi_k(x_2)$ and $\phi_r(x_2)$. On the other hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system as a consequence of anything that may be done to the first system. This is, of course, merely a statement of what is meant by the absence of an interaction between the two systems. Thus it is possible to assign two different wave functions (in our example $\psi_k$ and $\phi_r$) to the same reality (the second system after the interaction with the first).

Now, it may happen that the two wave functions $\psi_k$ and $\phi_r$ are eigenfunctions of two non-commuting operators corresponding to some physical quantities $P$ and $Q$, respectively. That this may actually be the case can best be shown by an example, see [1].
3 On the logical scheme of the EPR argument

1). EPR provided their own definition of "an element of reality." We point out that it does not belong to the theoretical model of QM. Hence they should map "elements of reality" onto some conventional objects of the QM-model. EPR understood well that one could not criticize one theoretical model by using notions from a different model.

2). To perform such a task, EPR used the following consequence of the projection postulate. Let $A$ be a (self-adjoint) operator representing quantum observable. Let $\psi$ be its eigenvector. So, (1) holds. Then the value $A = a$ can be predicted with certainty. It justifies association of EPR’s elements of reality with eigenvectors. Thus (at least some) elements of reality can be represented by eigenvectors in the QM-model. It is important that any eigenvector represents an element of reality.

3). By using the QM-model EPR proved that one can assign to the same system eigenfunctions corresponding to noncommuting operators.

We shall criticize the last step of EPR’s considerations.

4 The von Neumann projection postulate

In von Neumann’s book $[2]$ the cases of observables with nondegenerate and degenerate spectra were sharply distinguished. The post-measurement state is well defined (and given by the corresponding eigenvector) only for observables with nondegenerate spectra. Only in this case EPR might say that one could assign the wave function with the physical system (after the measurement). However, if spectrum is degenerate, then by the von Neumann axiomatics of QM the post-measurement state is not determined.

Thus one could not assign the definite wave function with the physical system (after measurement).

It is amazing that EPR did not pay attention to this crucial point. I could not exclude that they even did not read von Neumann’s book. In their paper the projection postulate is applied for observables with degenerate spectra, but in such a way as if they were observables with
nondegenerate spectra.

By considering partial measurements on subsystems of composite systems one immediately moves to the domain of degenerate measurements. Those operators $A$ and $B$ considered by EPR have degenerate spectra. Therefore by measuring e.g. $A$ one would not determine the state of a composite system $S_1 + S_2$. Hence, the state of $S_2$ is not determined by $A$-measurement on $S_1$. The wave function $\psi_k(x_2)$ could not be assigned with $S_2$. It is impossible to proceed as EPR did at the very end of their general considerations on measurements on composite systems. Since even one wave function, $\psi_k(x_2)$, could not be assigned with $S_2$, it is totally meaningless to write about assigning of two different wave functions to the same reality.

**Conclusion.** EPR did not prove that QM is incomplete. They did mistake by assuming that by measurement of observable $A$ (respectively, $B$) on $S_1$ the linear combination (7) (respectively, (8)) is reduced to a single summand.

## 5 EPR is about precise correlations

My correspondence with readers of preprint [10] demonstrated that considerations of EPR on reduction of the wave function (which were presented in section 2) have never been discussed seriously. This part of EPR’s paper (two of totally four pages) is practically ignored. Instead of this, people have always been concentrated on the last page of the paper containing the discussion on precise correlations for the position and momentum. As e.g. Elena Loubentz and Joachim Kupsch pointed out in E-mails to me, the EPR paper is not about the projection postulate, but about measurements for states with precise correlations. We remark that mentioned ”presentation of the EPR without appealing to reduction of wave function” can be found in the book of Ballentine [8], p.583-584. He really believes that he simplified the EPR arguments and the he escaped using the notion of reduction. We come back to the original EPR argument.

The essence of the EPR conclusions is presented in short on page 780:

"Returning now to the general case contemplated in Eqs. (7) and

\[ \text{\footnote{Hans de Raedt pointed out (in Email to me) to Ballentine’s presentation of the EPR views in [8].}} \]
(8), we assume that $\psi_k$ and $\phi_r$ are indeed eigenfunctions of some non-commuting operators $P$ and $Q$, corresponding to the eigenvalues $p_k$ and $q_r$, respectively. Thus by measuring either $A$ or $B$ we are in a position to predict with certainty, and without in any way disturbing the second system, whether the value of the quantity $P$ (that is $p_k$) or the value of the quantity $Q$ (that is $q_r$). In accordance with our criterion of reality, in the first case we must consider the quantity $P$ as being an element of reality, in the second case the quantity $Q$ is an element of reality."

As I understood, the last sentence has always been considered as the very end of the story. However, (by some reason) EPR continued:

"But, as we have seen, both wave functions $\psi_k$ and $\phi_r$, belong to the same reality."

Opposite to the majority of readers of their paper or (and it was more common) some texts about their paper, EPR were not able to get the complete satisfaction via producing elements of reality for the second particle via $A$ and $B$ measurements on the first one. They had to come back to their rather long story (pages 788-789) on reduction of the wave function.

I think that this EPR’s comeback to reduction is the crucial point of their argument. Why did they need do this? I think that by the following reason. It is impossible to associate simultaneously two "experimental elements of reality" with $S_2$ on the basis of measurement on $S_1$, since (as everybody understood well) either $A$ or $B$ measurement could be performed on $S_1$ (but not both $A$ and $B$). Therefore EPR were able to associate with $S_2$ only "theoretical elements of reality" represented by the wave functions $\psi_k(x_2)$ and $\phi_r(x_2)$ - eigenfunctions of the two non-commuting operators $P$ and $Q$ (for the second particle).

And it was enough for their purpose, since they wanted to prove incompleteness of QM as a theoretical model, see section 3. Thus, although I have the great respect to the contribution of Ballentine to quantum foundations, I do not think that his viewpoint is correct. EPR were clever enough to restrict their argument to Ballentine’s type considerations, p.583-584. They did not do this just because they were not able to approach their aim in this way.

**Conclusion.** *EPR were not able to proceed without appealing to the projection postulate (with all consequences of its misuse).*
6 Refinement measurements

However, according to von Neumann by obtaining a fixed value, say $A = \alpha$, for measurement on $S_1$, one does not determine the state of $S_1 + S_2$ (and, hence, neither the state of $S_2$).

To determine the state of $S_1 + S_2$, one should perform some refinement measurement. In QM it is represented by an operator commuting with $A \otimes I$ and eliminating degeneration. Since any operator of the form $I \otimes C$ commutes with $A \otimes I$, it is natural to consider refinement observable corresponding to measurement on $S_2$. The position Q and momentum P operators considered by EPR give examples of von Neumann’s refinement measurements. Each of them determine the state of $S_1 + S_2$ (and hence $S_2$) uniquely.

Moreover, for any operator with degenerate spectrum its measurement is ambiguous. Thus in the EPR case measurement of $A$ could not at all be considered as measurement on $S_1 + S_2$. It is just measurement on $S_1$.

However, for EPR the story about so called EPR-states was not simply the standard story about von Neumann’s refinement measurements.

7 The EPR paper as the source of the idea about quantum nonlocality

At the very end of their paper EPR discussed a problem which later became known as the problem of quantum nonlocality:

"One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted. On this point of view, since either one or the other, but not both simultaneously, of the quantities $P$ and $Q$ can be predicted, they are not simultaneously real. This makes the reality of $P$ and $Q$ depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this."

5Here $A : L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3), A \otimes I : L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$. 

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Later nonlocality was coupled to the von Neumann projection postulate in the following way. To escape incompleteness of QM, one should not assign the wave function \( \psi_k(x_2) \) with \( S_2 \) before the \( A \)-measurement on \( S_1 \). One might say that the \( A \)-measurement on \( S_1 \) produces instantaneous action on \( S_2 \) and its state is collapsed into \( \psi_k(x_2) \). For example, one can find an example of such a reasoning in the paper of Alain Aspect [17].

This form of reasoning has nothing to do with QM. By the same von Neumann’s projection postulate the state of \( S_2 \) is NOT determined by measurement on \( S_1 \). There is no even trace of action at the distance!

**Conclusion.** "Quantum nonlocality" appeared as a consequence of misuse of the projection postulate. We also emphasize that EPR considered quantum nonlocality as a totally absurd alternative to their arguments in favor of incompleteness of QM.

8 Nonlocality of the experiment design as opposed to EPR state nonlocality

8.1 Quantum theory and joint measurements of compatible observables

We have already discussed that from the QM-viewpoint (based on von Neumann’s axiomatics) the whole EPR story is about refinement measurements for operators with degenerate spectra. It would be useful to analyse (by using the conventional QM-framework) the procedure of joint measurement of two compatible observables, say \( A \) and \( Q : [A, Q] = 0 \).

The crucial point is that by von Neumann, to design joint measurement of \( A \) and \( Q \), one should design measurement of third observable, say \( C \), such that \( A = f(C) \) and \( Q = g(C) \), where \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are some functions. In the EPR case we want to have \( C \) with nondegenerate spectrum and \( A \) is observable on \( S_1 \) and \( Q \) on \( S_2 \).

Since \( A \) and \( Q \) are measured in different domains of spacetime, the design of measurement of \( C \) should be nonlocal. It is an extremely important point.

What does it mean "nonlocal design"?
In particular, it means that one should perform the time synchronization between results of measurement of $A$ and $Q$. It is important to be totally sure that clicks of the $A$-detector (giving the result of measurement on $S_1$) and the $Q$-detector (giving the result of measurement on $S_2$) match each other. We emphasize that in the real experimental setup for the EPR-Bohm experiment for photon polarization, see e.g., [18], [19], such a time synchronization is really realized via the nonlocal experimental design - via using the time window. The time window constraint

$$|t^A_i - t^Q_i| < \Delta$$

is evidently nonlocal. We also point out to the synchronization of space frames. Orientations of polarization beam splitters are chosen in one fixed space frame (in the complete accordance with Bohr’s ideology [10]).

### 8.2 The EPR state nonlocality

If one proceeds with so called quantum nonlocality induced by the misuse of the projection postulate, then he should take such a nonlocality very seriously. It would be real physical nonlocality of states. We again recall that EPR considered such a nonlocality as totally absurd.

**Conclusion.** The correct application of the projection postulate implies the nonlocal experimental design of the EPR-type experiments; in particular, the time synchronization (e.g., via the time window) as well as the choice of the fixed space frame. This experimental design nonlocality has nothing to do with so called "quantum nonlocality".

### 9 Bohr’s reply to Einstein

It is typically emphasized that Bohr’s reply [16] is very difficult for understanding. I totally agree with such a common viewpoint. I was able to understand Bohr only on the basis of previous considerations on the role of the projection postulate in the EPR considerations. Unfortunately, in Bohr’s reply there was no even trace of von Neumann’s axiomatization of QM[6]. Consequently Bohr did not pay any attention

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[6] I strongly suspect that neither Einstein nor Bohr had read von Neumann’s book at that time.
to the role of the projection postulate in the EPR considerations. He missed the EPR-trick with assigning to $S_2$ two wave functions, $\psi_k(x_2)$ and $\phi_r(x_2)$, which are eigenfunctions of two noncommutative observables, say $P$ and $Q$. It is very important in the EPR considerations that these wave functions and not measurements by themselves represent "elements of reality" in QM (as a theoretical model). Thus, instead of analyzing this tricky point in the EPR paper, Bohr proceeded in the purely experimental framework. He simply recalled his ideas on complementarity of various measurement setups in relation to the EPR-considerations. In short his message was that since one could not combine two measurement setups for $S_1$ related to incompatible quantities, it is impossible to assign two corresponding elements of reality to $S_2$. Bohr concluded that the EPR notion of an element of reality was ambiguous.

The problem was that EPR "proved" that QM is incomplete as a theoretical model, but Bohr replied by supporting his old thesis that QM is complete as an experimental methodology. It seems that the resulting common opinion was not in favor of Bohr’s reply. And it is clear why. If EPR really were able to prove that the formalism of QM implies assigning to $S_2$ of two wave functions, $\psi_k(x_2)$ and $\phi_r(x_2)$, corresponding to two noncommuting operators $Q$ and $P$, I would (and I was!) on their side. The point (presented in this paper) is that they were not able to do this by using the QM formalism in the proper way.

**Conclusion.** Bohr’s reply in spite correctness of his arguments, did not contain the analysis of the real roots of the "EPR paradox". It induced a rather common impression that EPR’s argument is not trivially reduced to the old problem of complementarity. It was commonly accepted that the only possibility to escape assigning "elements of reality" corresponding to incompatible observables to the same particle is to accept quantum nonlocality.

**10 Concluding remarks**

It seems that the "EPR-paradox" was finally resolved in this paper. I hope that it would stimulate people to look for various ways beyond QM. By von Neumann’s axiomatics of QM \cite{2} the notion of measurement of observable $A$ with degenerate spectrum is ambiguous. It is well defined only via refinement measurement given by observable $C$ with nongenerate spectrum such that $A = f(C)$. Since any observ-
Able A on the subsystem $S_1$ of a composite system $S = S_1 + S_2$ has degenerate spectrum in the tensor Hilbert space of $S$-states, it is totally meaningless to discuss (as EPR did) its measurement without fixing a refinement measurement on $S_2$. If such a refinement is not fixed from the very beginning, then $A$-measurement has nothing to do with measurements on the composite systems $S$. It could not change the $S$-state and, hence, the $S_2$-state. Bohr’s reply [16] to Einstein could be interpreted in the same way. Thus the EPR-attack against QM was not justified. Unfortunately, this attack was the source of naive Einsteinian realism (assigning to the same system $S_2$ of two wave functions $\psi_k(x_2)$ and $\phi_r(x_2)$ corresponding to noncommutative operators) and quantum nonlocality. We also point out to practically unknown fact that so called EPR states were studied in detail by von Neumann [2], pp. 434-435. But he was able to proceed without assigning two wave functions (corresponding to noncommuting operators) to the same system. Consequently, no traces of incompleteness of QM or its nonlocality could be found in [2].

Finally, we remark that recently Bell-type inequalities for tests of compatibility of nonlocal realistic models with quantum mechanics were derived, see Leggett [20]. They were generalized and tested experimentally by Gröblacher et al. [21]. The conclusion of these theoretical and experimental studies is that the condition of nonlocality which was considered by Bell (of course, under the influence of EPR) plays a subsidiary role. It was proven that naive EPR-realism is incompatible with experimental data (and this fact has no relation to the EPR-Bell idea of nonlocality). It is an experimental confirmation that the analysis of the EPR-arguments performed in the present paper is correct. These arguments were wrong from the very beginning.

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