On Chebyshev type Inequalities using Generalized k-Fractional Integral Operator

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Abstract

In this paper, using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function), we establish new results on generalized k-fractional integral inequalities by considering the extended Chebyshev functional in case of synchronous function and some other inequalities.

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1 Introduction

In recent years, many authors have worked on fractional integral inequalities by using different fractional integral operator such as Riemann-Liouville, Hadamard, Saigo and Erdelyi-Kober, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15]. In [12] S. Kilinc and H. Yildirim establish new generalized k-fractional integral inequalities involving Gauss hypergeometric function related to Chebyshev functional. In [5, 10] authors gave the following fractional integral inequalities, using the Hadamard and Riemann-Liouville fractional integral for extended Chebyshev functional.

Theorem 1.1 Let $f$ and $g$ be two synchronous function on $[0, \infty]$, and $r, p, q : [0, \infty) \rightarrow [0, \infty)$. Then for all $t > 0$, $\alpha > 0$, we have

$$2 H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} (qfg)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (pfg)(t) \right] +$$

$$2 H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (rfg)(t) \geq$$

$$H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (pg)(t) \right] +$$

$$H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (rg)(t) \right] +$$

$$H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (pg)(t) + H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (rg)(t) \right]$$

(1.1)
Theorem 1.2 Let $f$ and $g$ be two synchronous function on $[0, \infty]$, and $r, p, q : [0, \infty) \rightarrow [0, \infty)$. Then for all $t > 0, \alpha > 0$, we have:

$$
H D_{1,t}^{-\alpha}r(t) ×
\left[
H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\beta}(pf g)(t) + 2H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\beta}(qf g)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha}(pf g)(t)
\right]
+ \left[
H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} q(t) + H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} q(t)
\right] H D_{1,t}^{-\alpha}(rfg)(t) \geq
H D_{1,t}^{-\alpha}r(t) \left[
H D_{1,t}^{-\alpha}(pf)(t) H D_{1,t}^{-\beta}(qf g)(t) + H D_{1,t}^{-\alpha}(qf g)(t) H D_{1,t}^{-\alpha}(pg)(t)
\right] +
H D_{1,t}^{-\alpha} p(t) \left[
H D_{1,t}^{-\alpha} r(f)(t) H D_{1,t}^{-\beta}(qf)(t) + H D_{1,t}^{-\alpha}(qf)(t) H D_{1,t}^{-\alpha}(r f)(t)
\right] +
H D_{1,t}^{-\alpha} q(t) \left[
H D_{1,t}^{-\alpha} r(f)(t) H D_{1,t}^{-\beta}(r f)(t) + H D_{1,t}^{-\alpha}(r f)(t) H D_{1,t}^{-\alpha}(r g)(t)
\right].
$$

(1.2)

The main objective of this paper is to establish some Chebyshev type inequalities and some other inequalities using generalized k-fractional integral operator. The paper has been organized as follows. In Section 2, we define basic definitions related to generalized k-fractional integral operator. In section 3, we obtain Chebyshev type inequalities using generalized k-fractional. In Section 4, we prove some inequalities for positive continuous functions.

2 Preliminaries

In this section, we present some definitions which will be used later discussion.

Definition 2.1 Two function $f$ and $g$ are said to synchronous (asynchronous) on $[a, b]$, if

$$(f(u) - f(v))(g(u) - g(v)) \geq (\leq) 0,
$$

(2.1)

for all $u, v \in [0, \infty)$.

Definition 2.2 [12, 15] The function $f(x), \text{ for all } x > 0$ is said to be in the $L_{p,k}[0, \infty)$, if

$$
L_{p,k}[0, \infty) = \left\{ f : \|f\|_{L_{p,k}[0,\infty)} = \left( \int_0^{\infty} |f(x)|^p x^k \, dx \right)^{1/p} < \infty \ 1 \leq p < \infty \ k \geq 0 \right\},
$$

(2.2)
Definition 2.3 \[12, 14, 15\] Let \( f \in L^1,0,\infty \). The generalized Riemann-Liouville fractional integral \( I^{\alpha,k}f(x) \) of order \( \alpha, k \geq 0 \) is defined by

\[
I^{\alpha,k}f(x) = \frac{(k + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (t^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt. \tag{2.3}
\]

Definition 2.4 \[12, 15\] Let \( k \geq 0, \alpha > 0 \mu > -1 \) and \( \beta, \eta \in R \). The generalized k-fractional integral \( I^{\alpha,\beta,\eta,\mu}_{t,k} \) (in terms of the Gauss hypergeometric function) of order \( \alpha \) for real-valued continuous function \( f(t) \) is defined by

\[
I^{\alpha,\beta,\eta,\mu}_{t,k}[f(t)] = \frac{(k + 1)^{\mu+\beta+1}t^{(k+1)}(-\alpha-\beta-2\mu)}{\Gamma(\alpha)} \int_0^t \tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - \left(\frac{\tau}{t}\right)^{k+1})t^k f(\tau) d\tau.
\]

where, the function \( 2F_1(-) \) in the right-hand side of (2.4) is the Gaussian hypergeometric function defined by

\[
2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n t, \tag{2.5}
\]

and \((a)_n\) is the Pochhammer symbol

\[
(a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.
\]

Consider the function

\[
F(t, \tau) = \frac{(k + 1)^{\mu+\beta+1}t^{(k+1)}(-\alpha-\beta-2\mu)}{\Gamma(\alpha)} \tau^{(k+1)\mu} (t^{k+1} - \tau^{k+1})^{\alpha-1} f(t) dt.
\]

It is clear that \( F(t, \tau) \) is positive because for all \( \tau \in (0, t) \), \((t > 0)\) since each term of the (2.6) is positive.
3 Fractional Integral Inequalities for Extended Chebyshev Functional

In this section, we establish some Chebyshev type fractional integral inequalities by using the generalized k-fractional integral (in terms of the Gauss hypergeometric function) operator. The following lemma is used for our main result.

Lemma 3.1 Let \( f \) and \( g \) be two synchronous function on \([0, \infty[, \) and \( x, y : [0, \infty) \rightarrow [0, \infty) \) be two nonnegative functions. Then for all \( k \geq 0, \ t > 0, \) \( \alpha > \max\{0, -\beta - \mu\}, \) \( \beta < 1, \) \( \mu > -1, \) \( \beta - 1 < \eta < 0, \) we have,

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(xf(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(yg(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(yf(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(xg(t)) \geq I_{t,k}^{\alpha,\beta,\eta,\mu}(xf(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(yg(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(yf(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(xg(t)).
\] (3.1)

Proof: Since \( f \) and \( g \) are synchronous on \([0, \infty[ \) for all \( \tau \geq 0, \) \( \rho \geq 0, \) we have

\[
(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0.
\] (3.2)

From (3.2),

\[
f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau).
\] (3.3)

Now, multiplying both side of (3.3) by \( \tau^k x(\tau)F(\tau, \tau), \) \( \tau \in (0, t), \) \( t > 0. \) Then the integrating resulting identity with respect to \( \tau \) from 0 to \( t, \) we obtain by definition (2.4)

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(xf(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(yf(t)) + f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg(t)).
\] (3.4)

Now, multiplying both side of (3.4) by \( \rho^k y(\rho)F(t, \rho), \) \( \rho \in (0, t), \) \( t > 0. \) Then the integrating resulting identity with respect to \( \rho \) from 0 to \( t, \) we obtain by definition (2.4)

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(y(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(xf(t)) + f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg(t)).
\] (3.5)

This complete the proof of (3.1)

Now, we gave our main result here.

Theorem 3.2 Let \( f \) and \( g \) be two synchronous function on \([0, \infty[, \) and \( r, p, q : [0, \infty) \rightarrow [0, \infty). \) Then for all \( k \geq 0, \ t > 0, \) \( \alpha > \max\{0, -\beta - \mu\}, \) \( \beta < 1, \) \( \mu > -1, \) \( \beta - 1 < \eta < 0, \) we have,

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(y(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(xf(t)) + f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg(t)).
\] (3.6)
\( \beta < 1, \mu > -1, \beta - 1 < \eta < 0, \) we have,

\[
2I_{t,k}^{\alpha,\beta,\eta,\mu}(p(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t)) + \\
2I_{t,k}^{\alpha,\beta,\eta,\mu}(p(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(pg)(t)) \geq \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t) + \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(pg)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t).
\]

**Proof:** To prove above theorem, putting \( x = p, y = q, \) and using lemma 3.1, we get

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(p(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t)) \geq \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(pg)(t).
\]

Now, multiplying both side by \((3.7) I_{t,k}^{\alpha,\beta,\eta,\mu}(r(t))\), we have

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(r(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t),
\]

putting \( x = r, y = q, \) and using lemma 3.1, we get

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(r(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t),
\]

multiplying both side by \((3.9) I_{t,k}^{\alpha,\beta,\eta,\mu}(p(t))\), we have

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(p(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(r(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(qf)(t) + I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t),
\]

With the same arguments as before, we can write

\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(q(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t) \geq \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(pf)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(rg)(t).
\]

(3.6)
Adding the inequalities (3.8), (3.10) and (3.11), we get required inequality (3.6).

Here, we give the lemma which is useful to prove our second main result.

Lemma 3.3 Let $f$ and $g$ be two synchronous function on $[0, \infty]$. and $x, y : [0, \infty[ \to [0, \infty]$. Then for all $k \geq 0$, $\alpha > 0$, $\gamma > max\{0, -\beta - \mu\}, \gamma > max\{0, -\delta - \upsilon\}$, $\beta, \delta < 1$, $\upsilon, \mu > -1$, $\beta - 1 < \eta < 0$, $\mu - 1 < \zeta < 0$, we have,

$$
I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(yf(t)) + I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))(t) \geq \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t)) + I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t))I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))(t).
$$

(3.12)

Proof: Now multiplying both side of (3.4) by

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)
$$

(3.13)

which remains positive in view of the condition stated in (3.12), $\rho \in (0, t)$, $t > 0$, we obtain

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)
$$

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)g(\rho)
$$

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)g(\rho)
$$

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)g(\rho)
$$

$$
(k + 1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}\Gamma(\gamma)\rho^{(k+1)v}y(\rho)g(\rho)
$$

then integrating (3.14) over $(0,t)$, we obtain

$$
I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t))(t) \geq \\
I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x(t))I_{t,k}^{\gamma,\delta,\xi,\upsilon}(y(t))(t),
$$

(3.15)

this ends the proof of inequality (3.12).
Theorem 3.4 Let $f$ and $g$ be two synchronous function on $[0, \infty]$, and $r, p, q : [0, \infty) \to [0, \infty)$. Then for all $t > 0$, $\alpha > 0$, we have:

$$I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \times$$

$$\left[ I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (pqf)(t) + 2I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qfg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} q(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pf)(t) \right]$$

$$+ \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} q(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} p(t) I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \right] I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \geq$$

$$I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (p(t)) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right] +$$

$$I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (r(f))(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right] +$$

$$I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (r(f))(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (pq)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].$$

(3.16)

**Proof:** To prove above theorem, putting $x = p$, $y = q$, and using lemma 3.3 we get

$$I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (p(t)) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right] \geq$$

$$I_{t,k}^{\alpha,\beta,\eta,\mu} (p(t)) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t).$$

(3.17)

Now, multiplying both side by (3.17) $I_{t,k}^{\alpha,\beta,\eta,\mu} r(t)$, we have

$$I_{t,k}^{\alpha,\beta,\eta,\mu} r(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (p(t)) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (pg)(t) \right],$$

(3.18)

putting $x = r$, $y = q$, and using lemma 3.3, we get

$$I_{t,k}^{\alpha,\beta,\eta,\mu} (r(f))(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t),$$

(3.19)

multiplying both side by (3.19) $I_{t,k}^{\alpha,\beta,\eta,\mu} p(t)$, we have

$$I_{t,k}^{\alpha,\beta,\eta,\mu} p(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (r(f))(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (qg)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu} (qf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].$$

(3.20)

With the same argument as before, we obtain

$$I_{t,k}^{\alpha,\beta,\eta,\mu} q(t) \left[ I_{t,k}^{\alpha,\beta,\eta,\mu} (r(f))(t) I_{t,k}^{\gamma,\delta,\zeta,\nu} (pq)(t) + (pf)(t) I_{t,k}^{\alpha,\beta,\eta,\mu} (rg)(t) \right].$$

(3.21)
Adding the inequalities (3.18), (3.20) and (3.21), we follow the inequality (3.16).

**Remark 3.1** If $f, g, r, p$ and $q$ satisfies the following condition,

1. The function $f$ and $g$ is asynchronous on $[0, \infty)$.
2. The function $r, p, q$ are negative on $[0, \infty)$.
3. Two of the function $r, p, q$ are positive and the third is negative on $[0, \infty)$.

then the inequality 3.6 and 3.16 are reversed.

### 4 Other fractional integral inequalities

In this section, we proved some fractional integral inequalities for positive and continuous functions which as follows:

**Theorem 4.1** Suppose that $f$, $g$, and $h$ be three positive and continuous functions on $[0, \infty]$, such that

$$
(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0; \ \tau, \rho \in (0, t) \ t > 0, \hspace{1cm} (4.1)
$$

and $x$ be a nonnegative function on $[0, \infty)$. Then for all $k \geq 0, \ t > 0, \ \alpha > \max\{0, -\beta - \mu\}, \gamma > \max\{0, -\delta - \nu\} \ \beta, \delta < 1, \ \nu, \mu > -1, \ \beta - 1 < \eta < 0, \ \delta - 1 < \zeta < 0$, we have,

$$
I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(x(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfgh)(t) + I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfg)(t)
+ I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xh)(t) + I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfg)(t)
\geq I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(x(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfgh)(t) + I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfg)(t)
+ I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfh)(t) + I_{t,k}^{\alpha,\beta,\gamma,\delta,\eta,\mu}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(xfg)(t). \hspace{1cm} (4.2)
$$

**Proof:** Since $f$, $g$, and $h$ be three positive and continuous functions on $[0, \infty]$, we can write

$$
f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau)
\geq f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho). \hspace{1cm} (4.3)
$$

Now, multiplying both side of (4.3) by $\tau^kx(\tau)F(t, \tau), \ \tau \in (0, t), \ t > 0$. Then the integrating resulting identity with respect to $\tau$ from 0 to $t$, we obtain by
definition (2.4)
\[ I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t) + f(\rho)g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f)(t) + g(\tau)h(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f)(t)) \]
\[ + f(\rho)g(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x h(t))(t) \geq g(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t) + g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f)(t) \]
\[ + f(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x g h(t))(t) + f(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x g)(t). \]  
(4.4)

Now multiplying both side of (4.4) by
\[ \frac{(k + 1)^{\nu+\delta+1}t^{(k+1)(-\delta-\gamma-2\nu)}}{\Gamma(\gamma)} \rho^{(k+1)^{\nu}} x(\rho) \]  
(4.5)
which remains positive in view of the condition stated in (4.2), \( \rho \in (0, t) \), \( t > 0 \) and integrating resulting identity with respective \( \rho \) from 0 to t, we obtain

\[ I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t) + I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t) \]
\[ + I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t) \]
\[ \geq I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t) \]
\[ + I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t)I_{t,k}^{\alpha,\beta,\gamma,\delta}(x f h(t))(t). \]  
(4.6)

which implies the proof inequality 4.2.

Here, we give another inequality which is as follows.

**Theorem 4.2** Let \( f, g \) and \( h \) be three positive and continuous functions on \([0, \infty[, \) which satisfying the condition (4.1) and \( x \) and \( y \) be two nonnegative functions on \([0, \infty). \) Then for all \( k \geq 0, \) \( t > 0, \) \( \alpha \geq \max\{0, -\beta - \mu\}, \gamma \geq \max\{0, -\delta - \nu\} \) \( \beta, \delta < 1, \) \( \nu, \mu > -1, \) \( \beta - 1 < \eta < 0, \) \( \delta - 1 < \zeta < 0, \) we have

\[ I_{t,k}^{\alpha,\beta,\eta,\mu}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f g h(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x h(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f g(t)) \]
\[ + I_{t,k}^{\alpha,\beta,\eta,\mu}(x f g(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y h(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x f g(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f h(t)) \]
\[ \geq I_{t,k}^{\alpha,\beta,\eta,\mu}(x f g(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y g h(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x f g(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f h(t)) \]
\[ + I_{t,k}^{\alpha,\beta,\eta,\mu}(x f g(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f h(t)) + I_{t,k}^{\alpha,\beta,\eta,\mu}(x f h(t))I_{t,k}^{\gamma,\delta,\zeta,\nu}(y f g(t)) \]  
(4.7)

**Proof:** Multiplying both side of (4.3) by \( \tau^{k}x(\tau)F(t, \tau), \) \( \tau \in (0, t), \) \( t > 0, \)
where \( F(t, \tau) \) defined by (2.6). Then the integrating resulting identity with
respect to \( \tau \) from 0 to \( t \), we obtain by definition (2.4)
\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(xfg)(t) + f(\rho)g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t)
\]
\[
+ f(\rho)g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) \geq g(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xfh)(t) + g(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xf)(t)
\]
\[
+ f(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xgh)(t) + f(\rho)h(\rho)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\]
(4.8)

Now multiplying both side of (4.8) by
\[
\frac{(k+1)^{v+\delta+1}t^{(k+1)(-\delta-\gamma-2\nu)}}{\Gamma(\gamma)}\rho^{(k+1)v}g(\rho)
\]
(4.9)

which remains positive in view of the condition stated in (4.7), \( \rho \in (0,t) \),
\( t > 0 \) and integrating resulting identity with respective \( \rho \) from 0 to \( t \), we obtain
\[
I_{t,k}^{\alpha,\beta,\eta,\mu}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,\nu}(y)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu}(yfg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(x)(t)
\]
\[
+ I_{t,k}^{\gamma,\delta,\zeta,\nu}(ygh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t) \geq I_{t,k}^{\gamma,\delta,\zeta,\nu}(ygh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t)
\]
\[
+ I_{t,k}^{\gamma,\delta,\zeta,\nu}(yfg)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xh)(t) + I_{t,k}^{\gamma,\delta,\zeta,\nu}(ygh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(x)(t)
\]
\[
+ I_{t,k}^{\gamma,\delta,\zeta,\nu}(yfh)(t)I_{t,k}^{\alpha,\beta,\eta,\mu}(xg)(t).
\]
(4.10)

which implies the proof inequality 4.7.

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