$c = 5/2$ Free Fermion Model of $WB_2$ Algebra

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We investigate the explicit construction of the $WB_2$ algebra, which is closed and associative for all values of the central charge $c$, using the Jacobi identity and show the agreement with the results studied previously. Then we illustrate a realization of $c = 5/2$ free fermion model, which is $m \to \infty$ limit of unitary minimal series, $c(WB_2) = \frac{5}{2}(1 - \frac{12}{(m+3)(m+4)})$ based on the cosets $\hat{B}_2 \oplus \hat{B}_2, \hat{B}_2$ at level $(1, m)$. We confirm by explicit computations that the bosonic currents in the $WB_2$ algebra are indeed given by the Casimir operators of $\hat{B}_2$.

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1 Introduction

Recently much progress has been made in our understanding the structure of two-dimensional rational conformal field theories (RCFTs) for which the spectrum contains a finite number of irreducible representations of chiral algebra, the operator product algebra (OPA) of all the holomorphic fields in the theory. Cardy \[1\] was able to show that the sum over irreducible representations of the conformal algebra must be infinite for the central charge \( c \geq 1 \). Therefore in order to study RCFTs with \( c \geq 1 \), extended conformal algebras which consist of the Virasoro algebra and additional higher spin currents, play an important role. Large classes of extended Virasoro algebras are known: affine Kac-Moody algebra \[2\], superconformal algebra \( (WB_1) \) \[3\], and a class of \( W \) algebras. Some examples of \( W \) algebras have been established: \( W_3 \) algebra \[4\], \( W_n \) algebra \[5\], and supersymmetric \( W \) algebra \[6\].

Fateev and Lukyanov \[7\] have considered an infinite dimensional, associative quantum algebra \( WB_n \), corresponding to the Lie algebras \( B_n \), consisting of \((n+1)\) fields of spin \( 4, 6, \cdots, 2n \), and a fermion field of spin \( (n + 1/2) \) in addition to the spin 2 energy momentum tensor. The addition of fermion field was necessary for the quantization of the Hamiltonian structures associated with Lie algebra whose simple roots have a different length. They conjectured that these \((n+1)\) fields form a closed OPA. These constructions produce representations of the algebras over a range of values of \( c \). For \( n = 1 \), the \( WB_1 \) algebra coincides with the \( N = 1 \) super Virasoro algebra.

On the other hand, Watts \[8\] has shown that \( W \) structures may be constructed in coset models of the form \((\hat{g}_n \oplus \hat{g}_n, \hat{g}_n)\) at level \((1, m)\) with \( g_n \) one of the finite Lie algebras ABDE and for \( m \) sufficiently large. An explicit expression for a fermion field which commutes with the diagonal subalgebra was given in terms of currents of \( \hat{B}_n \). The discrete series of \( c \) \[4\] coincide with values obtained from coset models based on \( \hat{B}_n \) algebra. It is expected, but has not been established that his construction are related to those in the free field construction \[7\]. Recently, he proved that the symmetry of Lie superalgebras of \( B(0, n) \) Toda theory was given
by the classical Poisson bracket analogue of the $WB_3$ algebra [9].

It wasn’t proven that algebra $WB_2$ are associative for all values of $c$ until Figueroa-O’Farrill et al. showed that the existence of $WB_2$ explicitly using the perturbative conformal bootstrap [10]. They have been able to show the equivalence with the findings conjectured by Fateev and Lukyanov [7] using Coulomb gas realization. We make an explicit construction of the Casimir of $WB_2$ from a fermion proposed by Watts in [8].

This paper is organized as follows. In section 2, we look at the explicit construction of $WB_2$ algebra using the associativity of Laurent expansion operators, which we investigate by considering a graded Jacobi identity for Laurent expansion modes. In section 3, we would like to understand how $WB_2$ currents emerge from fields of $\hat{B}_2$ algebra and prove a realization of $c = 5/2$ representation of section 2. Finally section 4 contains a few concluding remarks.

## 2 $WB_2$ Algebra

As our starting point, we follow the analysis in [7]. The presence of $W_2$ current of dimension 2 and $W_4$ current of dimension 4 can be understood from the operator product expansion (OPE) of $d$ current of dimension 5/2 with itself:

$$\frac{30}{(2c + 25)}d(z)d(w) = \frac{1}{(z-w)^5}c \frac{2}{5} + \frac{1}{(z-w)^3}2W_2(w) + \frac{1}{(z-w)^2}\partial W_2(w)$$

$$+ \frac{1}{(z-w)}\left[\frac{60}{(2c+25)}W_4(w) + \frac{1}{2}\partial^2 W_2(w)\right] + \cdots. \quad (2.1)$$

Notice that $W_4$ is not a primary field under the $W_2(z) = T(z)$ energy momentum tensor which generates the Virasoro algebra with a central charge $c$. The above OPE can be rewritten as the more familiar form in the following way:

$$U(z)U(w) = \frac{1}{(z-w)^5}c \frac{2}{5} + \frac{1}{(z-w)^3}2T(w) + \frac{1}{(z-w)^2}\partial T(w)$$

$$+ \frac{1}{(z-w)}\left[\frac{3}{10}\partial^2 T(w) + \frac{27}{(5c+22)}\Lambda(w) + C_{\frac{1}{12}}^4V(w)\right] + \cdots, \quad (2.2)$$
where
\[
\sqrt{\frac{30}{2c+25}} d(z) = U(z), \; \Lambda(z) = T^2(z) - \frac{3}{10} \partial^2 T(z),
\]
and
\[
W_4(z) = -\frac{(2c+25)^2}{120(5c+22)} \partial^2 T(z) + \frac{9(2c+25)}{20(5c+22)} T^2(z) + \frac{(2c+25)}{60} C^4_{\frac{5}{2}} V(z). \tag{2.4}
\]

It is convenient to use \( V(z) \), which is a Virasoro primary field of dimension 4, rather than \( W_4(z) \) for the future developments of \( WB_2 \) algebra. \( C^4_{\frac{5}{2}} \) is a coupling constant and to be determined later. Extended conformal algebra by a single primary field of \( 5/2 \) conformal dimension was found in [1]. Here enlarging the algebra by additional spin 4 field leads to the appearance of \( V(z) \) term of eq. (2.2). We would expect that \( C^4_{\frac{5}{2}} \) should vanish for \( c = -13/14 \). On the other hand, it has been proven that one extra spin 4 current algebra was determined by the closure of Jacobi identity [12, 11]. We are dealing with two-dimensional RCFTs, in which there exist conserved currents \( U(z) \) of spin \( 5/2 \) and \( V(z) \) of spin 4, and would expect that \( U(z) \) field dependence should appear in the OPE \( V(z)V(w) \). Recall that \( U^2(z) \) can be written as the combinations of \( \partial^3 T(z), \partial \Lambda(z) \) and \( \partial V(z) \) from eq. (2.2). We can see that the nontrivial \( U(z) \) field dependence has the form of \( \partial UU(z) \) ( or \( U \partial U(z) \) ). The next step is to work out the OPE \( V(z)V(w) \). Through an analysis of [11] with the symmetry under the interchange of \( z \) and \( w \), it leads to the following results:

\[
V(z)V(w) = \frac{1}{(z-w)^8} \left[ c + A\frac{1}{(z-w)^2} 2T(w) + \frac{1}{(z-w)^5} \partial T(w) \right. \\
- \left. \frac{1}{(z-w)^3} \frac{1}{12} \partial^3 T(w) + \frac{1}{(z-w)^6} \partial T(w) \right] + B\left[ \frac{1}{(z-w)^7} 2\partial^2 T(w) \right. \\
+ \left. \frac{1}{(z-w)^5} \partial^3 T(w) - \frac{1}{(z-w)^6} \frac{1}{12} \partial^5 T(w) \right] + C\left[ \frac{1}{(z-w)^8} 2\partial^4 T(w) \right. \\
+ \left. \frac{1}{(z-w)^6} \partial^5 T(w) \right] + D\left[ \frac{1}{(z-w)^7} 2\Lambda(w) \right. \\
- \left. \frac{1}{(z-w)^5} \frac{1}{12} \partial^3 \Lambda(w) \right] + E\left[ \frac{1}{(z-w)^8} 2\partial^2 \Lambda(w) \right. \\
\left. + \frac{1}{(z-w)^6} \partial^3 \Lambda(w) \right] \right] \tag{4}
\]
\[ + F \left[ \frac{1}{(z-w)^2} \partial \Xi(w) + \frac{1}{(z-w)^2} \partial^2 \Xi(w) \right] + G \left[ \frac{1}{(z-w)^2} \partial \Delta(w) \right] + H \left[ \frac{1}{(z-w)^4} 2 \partial V(w) + \frac{1}{(z-w)^3} \partial^3 V(w) \right] \\
- \frac{1}{(z-w)^2} \left[ \frac{1}{12} \partial^3 V(w) \right] + I \left[ \frac{1}{(z-w)^2} 2 \partial^2 V(w) + \frac{1}{(z-w)} \partial^3 V(w) \right] \\
+ J \left[ \frac{1}{(z-w)^2} 2 \partial \Omega(w) + \frac{1}{(z-w)} \partial \Omega(w) \right] \\
+ K \left[ \frac{1}{(z-w)^2} 2 \Gamma(w) + \frac{1}{(z-w)} \partial \Gamma(w) \right] + \cdots, \tag{2.5} \]

with

\[ \Xi(z) = \partial T \partial T(z) - \frac{3}{70} \partial^4 T(z) - \frac{31}{7(5c+22)} \partial^2 \Lambda(z), \]
\[ \Delta(z) = T \Lambda(z) - \frac{3}{10} \partial^2 \Lambda(z), \quad \Omega(z) = TV(z), \]
\[ \Gamma(z) = \partial U U(z) - \frac{5}{18} \partial T \partial T(z). \tag{2.6} \]

Note that the operator \( \Gamma(z) \) defined as above has no central term which makes the calculation easier. To guarantee that the complete OPA be associative, \( K \) terms are inevitable. In order to fix the coefficients in the OPE we consider the condition of associativity. We will use the Jacobi identities for Laurent expansion modes. The commutation relation \([V_m, V_n]\) can be obtained by contour integral, as usual. For the commutators of newly defined operators of (2.6) with Virasoro operators \( L_m \), see the reference [11]. Also we need to know \([L_m, \Gamma_n] = f_c \int_{C_0} (dw/2\pi i) w^{m+5} \int_{C_w} (dz/2\pi i) z^{n+1} T(z) \Gamma(w) : \]

\[ [L_m, \Gamma_n] = \frac{m(m^2 - 1)}{51180} \left[ 2020(m-2)(m-3) - 100(-10c+27)(m-2) \right. \]
\[ \times (m+n+2) + 2020(m+n+2)(m+n+3)] L_{m+n} \]
\[ + \frac{m}{318(5c+22)} \left[ 3159(m^2 - 1) - 3(m+1)(-100c+775)(m+n+4) \right] \Lambda_{m+n} \]
\[ + \frac{m}{12} C_4 \left[ 13(m^2 - 1) - 15(m+1)(m+n+4) \right] V_{m+n} + (5m-n) \Gamma_{m+n}. \tag{2.7} \]
After a straightforward computation of the following Jacobi identity
\[
[L_m, [V_n, V_p]] + [V_p, [L_m, V_n]] + [V_n, [V_p, L_m]] = 0,
\] (2.8)
we can have the intermediate results:
\[
A = 1, \quad B = \frac{3}{20}, \quad C = \frac{1}{168}, \quad D = \frac{21}{(5c + 22)},
\]
\[
E = \frac{22}{7(5c + 22)} - \frac{101}{36(5c + 22)} K,
\]
\[
F = -\frac{3(19c - 524)}{20(2c - 1)(7c + 68)} + \frac{7(50c^2 + 507c - 968)}{90(2c - 1)(7c + 68)} K,
\]
\[
G = \frac{12(72c + 13)}{(2c - 1)(7c + 68)(5c + 22)} - \frac{(734c + 49)}{(2c - 1)(7c + 68)(5c + 22)} K,
\]
\[
H = \frac{3(c + 24)}{28} J + \frac{19}{28} C_{\frac{4}{2} \frac{3}{2}} K, \quad I = \frac{5(c + 64)}{336} J - \frac{5}{112} C_{\frac{4}{2} \frac{3}{2}} K. \quad (2.9)
\]

We have still now three free parameters, \( C_{\frac{4}{2} \frac{3}{2}}, J, \) and \( K. \) As you can see, the above results reduce to ones in \([11, 12]\) when \( K \) goes to zero. We can check that the other Jacobi identity involving \( L_m, U_n, \) and \( U_p \) doesn’t give any further restriction on these free parameters.

Now we investigate the OPE \( V(z)U(w) \) in order to obtain the complete algebra. Then some local operators appearing in the r.h.s. of \( V(z)U(w) \) are made of the products \( T(w), U(w) \) and their derivatives. The most general form of the OPE of \( V(z)U(w) \) is
\[
V(z)U(w) = \frac{a}{(z-w)^4} U(w) + \frac{b}{(z-w)^3} \partial U(w)
\]
\[
+ \frac{1}{(z-w)^2} [d\partial^2 U(w) + eTU(w)]
\]
\[
+ \frac{1}{(z-w)} [f\partial^3 U(w) + g\partial TU(w) + h\partial(TU)(w)] + \cdots. \quad (2.10)
\]
In a similar manner, the unknown structure constants are determined by the following Jacobi identity :
\[
[L_m, [V_n, V_p]] + [U_p, [L_m, V_n]] + [V_n, [U_p, L_m]] = 0. \quad (2.11)
\]
Therefore it leads to the following results,

\[
C^4_{22}a = \frac{15(14c + 13)}{4(5c + 22)}, \quad C^4_{22}b = \frac{3(14c + 13)}{(5c + 22)}
\]

\[
C^4_{22}d = \frac{5(c + 8)(14c + 13)}{2(5c + 22)(2c + 25)}, \quad C^4_{22}e = \frac{45(14c + 13)}{(5c + 22)(2c + 25)}
\]

\[
C^4_{22}f = \frac{(2c - 5)}{(2c + 25)}, \quad C^4_{22}g = -\frac{162c + 2025}{(5c + 22)(2c + 25)}
\]

\[
C^4_{22}h = \frac{6(82c + 215)}{(2c + 25)(5c + 22)}
\]

In order to get \(C^4_{22}, J,\) and \(K\) completely, we should use explicit construction of \([V_m, \Lambda_n], \{U_m, (TU)_n\},\) and \(\{U_m, (\partial TU)_n\}\) to satisfy the following Jacobi identity:

\[
[V_m, \{U_n, U_p]\} - \{U_p, [V_m, U_n]\} - \{U_n, [V_m, U_p]\} = 0.
\]

But we refrain from giving the explicit expressions for the above (anti) commutators, which are rather complicated and not illuminating. Finally, with

\[
C^4_{22} = \sqrt{\frac{6(14c + 13)}{5c + 22}}, \quad J = \frac{4\sqrt{6}(7c - 115)}{(2c + 25)\sqrt{(5c + 22)(14c + 13)}}
\]

\[
K = \frac{30(5c + 22)}{(2c + 25)(14c + 13)},
\]

putting all together, all the coefficients can be determined in terms of only the Virasoro central charge \(c:\)

\[
a = \frac{15\sqrt{14c + 13}}{4\sqrt{6(5c + 22)}}, \quad b = \frac{3\sqrt{14c + 13}}{\sqrt{6(5c + 22)}}
\]

\[
d = \frac{5(c + 8)\sqrt{14c + 13}}{2(2c + 25)\sqrt{6(5c + 22)}}, \quad e = \frac{45\sqrt{14c + 13}}{(2c + 25)\sqrt{6(5c + 22)}}
\]

\[
f = \frac{(2c - 5)\sqrt{5c + 22}}{(2c + 25)\sqrt{6(14c + 13)}}, \quad g = -\frac{162c + 2025}{(2c + 25)\sqrt{6(5c + 22)(14c + 13)}}
\]
\[ h = \frac{6(82c + 215)}{(2c + 25)\sqrt{6(5c + 22)(14c + 13)}}, \]
\[ E = \frac{3696c^2 + 31957c - 34870}{42(2c + 25)(5c + 22)(14c + 13)}, \quad F = \frac{2158c + 21305}{60(2c + 25)(14c + 13)} \]
\[ G = \frac{54(32c - 5)}{(2c + 25)(5c + 22)(14c + 13)}, \quad H = \frac{3\sqrt{6}(2c^2 + 83c - 490)}{(2c + 25)\sqrt{(5c + 22)(14c + 13)}} \]
\[ I = \frac{\sqrt{6}(10c^2 - 197c - 2810)}{24(2c + 25)\sqrt{(5c + 22)(14c + 13)}}. \] (2.15)

Note that \( WB_2 \) algebra is a subalgebra of Zamolodchikov’s spin \( 5/2 \) algebra \([4]\) for \( c = -13/14 \) because \( C_{-3/2} \) vanishes for this value of \( c \). These results are in agreement with the findings of \([10]\).† The one thing which we would like to stress is the fact that all the remaining Jacobi identities, \([U_m, [V_n, V_p]] + \text{cycl.} = 0, [U_m, \{U_n, U_p]\} + \text{cycl.} = 0, \) and \([V_n, [V_n, V_p]] + \text{cycl.} = 0, \) are consistent with the above results after a long calculation with \( \text{MathematicaTM} \) [13].

### 3 The Five Free Fermion Model

The coset models \([8]\) are defined in terms of the currents \( E^{ab}_{(1)}(z), \) and \( E^{ab}_{(2)}(z), \) of level 1 and \( m, \) respectively, which generate the algebra \( g = \hat{B}_2 \oplus \hat{B}_2. \) The generator of the diagonal subalgebra \( g' = \hat{B}_2, \) which has level \( m' = 1 + m, \) is given by
\[ E^{\prime ab}(z) = E^{ab}_{(1)}(z) + E^{ab}_{(2)}(z). \] (3.1)

The coset Virasoro algebra is generated by the difference \( T_{(1)}(z) + T_{(2)}(z) - T'(z), \) where \( T_{(1)}(z) \) and \( T_{(2)}(z) \) are Sugawara stress energy tensors:
\[
\hat{T}(z) = -\frac{1}{16}E^{ab}_{(1)}E^{ab}_{(1)}(z) - \frac{1}{4(m + 3)}E^{ab}_{(2)}E^{ab}_{(2)}(z) + \frac{1}{4(m + 4)}E^{\prime ab}E^{\prime ab}(z). \] (3.2)

† In this way we discovered that there is a misprint of eq.(13) of ref.\([10]\). The factor 5 in the denominator should be in the numerator.
Of course, $\tilde{T}(z)$ commutes with $E^{ab}(z)$. The coset central charge of the unitary minimal models for $WB_2$ is

$$\tilde{c} = c(WB_2) = \frac{5}{2} + \frac{10m}{m+3} - \frac{10(m+1)}{m+4} = \frac{5}{2}(1 - \frac{12}{(m+3)(m+4)}) \quad (3.3)$$

where $m = 1, 2, \cdots$.

In this section we focus on the limit $m \to \infty$, which gives us a model of $c = 5/2$ that is invariant under the affine Lie algebra $\hat{B}_2$ at level 1. This model can be represented by 5 free fermions $\psi^a$ of dimension $1/2$, where the index $a$ takes values in the adjoint representation of $B_2$ and $a = 1, \cdots, 5$. We will show how the currents of $WB_2$ can be constructed from these free fermion fields or basic fields $E^{ab}$ of $\hat{B}_2$ and consider their OPA.

The defining OPE of the basic fermion fields is given by, as usual,

$$\psi^a(z)\psi^b(w) = \frac{1}{(z-w)}\delta^{ab} + \cdots \quad (3.4)$$

We can define dimension 1 currents $E^{ab}(z)$ as composites of the free fermions

$$E^{ab}(z) = \psi^a\psi^b(z) \quad (3.5)$$

which satisfy, at level 1, the usual $\hat{B}_2$ OPE

$$E^{ab}(z)E^{cd}(w) = \frac{1}{(z-w)^2}(\delta^{bc}\delta^{ad} - \delta^{bd}\delta^{ac})$$

$$+ \frac{1}{(z-w)}[\delta^{bc}E^{ad}(w) + \delta^{ad}E^{bc}(w) - \delta^{ac}E^{bd}(w) - \delta^{bd}E^{ac}(w)] + \cdots \quad (3.6)$$

Watts [8] has pointed out that $U(z)$ of dimension $5/2$, which is invariant under the horizontal subalgebra, can be expressed as follows using $B_2$ invariant $\epsilon^{abcde}$ tensor.

$$U(z) = \frac{1}{120}\epsilon^{abcde}\psi^aE^{bc}\psi^e(z). \quad \dagger$$

$\dagger$Multiple composite operators are always regularized from the right to left, unless otherwise stated. The normalization of $U(z)$ is chosen such that $\epsilon^{abcde}\epsilon^{abcde} = 120$. 9
After a tedious calculation, using the rearrangement lemmas [14], we arrive at the following result for the OPE of \( U(z) \) with \( U(w) \):

\[
U(z)U(w) = \frac{1}{(z-w)^5} - \frac{1}{(z-w)^3} \psi^a \partial \psi^a(w) - \frac{1}{(z-w)^2} \frac{1}{2} \psi^a \partial^2 \psi^a(w) + \frac{1}{(z-w)} \left[ \frac{1}{2} \psi^a \partial \psi^a \psi^b \partial \psi^b(w) - \frac{1}{6} \psi^a \partial^3 \psi^a(w) \right] + \cdots \quad (3.8)
\]

During this calculation, we used the fact that \( \epsilon_{abcdef} \epsilon_{efghi} ((E^bc E^{de})(E^{gh} E^{ij}))(z) = 864 \psi^a \partial \psi^a \psi^b \partial \psi^b(z) - 384 \psi^a \partial^3 \psi^a(z) \). Comparing this with eq. (2.2), one can readily see that

\[
V(z) = \frac{1}{8 \sqrt{69}} \left[ 7 \psi^a \partial \psi^a \psi^b \partial \psi^b(z) - 6 \partial \psi^a \partial^2 \psi^a(z) + \frac{2}{3} \psi^a \partial^3 \psi^a(z) \right]
\]

\[
= \sqrt{\frac{23}{192}} \left[ \frac{28}{23} T^2(z) + \frac{33}{23} \partial^2 T(z) + \psi^a \partial^3 \psi^a(z) \right]
\]

\[
= -\frac{1}{40 \sqrt{69}} E^{ab} E^{cd} E^{ac} E^{bd}(z) \quad (3.9)
\]

which is the unique (up to a normalization) dimension 4 primary field under the energy momentum tensor, \( T(z) = -\frac{1}{2} \psi^a \partial \psi^a(z) = -\frac{1}{16} E^{ab} E^{ab}(z) \) which is the form of the second order Casimir. For \( B_2 \) algebra, the number of independent Casimirs equals the rank of \( B_2 \) (=2). Therefore we have in addition to the second Casimir only a fourth order Casimir given by (3.9). The fact that the fourth order Casimir operator is generated in the OPE \( U(z)U(w) \) confirms Casimir algebras consisting of \( T(z) \) and \( V(z) \) are not the usual spin-4 algebras [11, 12].

In order to find the complete structure of \( WB_2 \), one has to take the OPE \( U(z)V(w) \):

\[
\sqrt{\frac{192}{23}} U(z)V(w) = \frac{1}{(z-w)} \frac{120}{23} \frac{1}{120} \epsilon^{abcdef} \psi^a E^{bc} E^{de}(w) + \frac{1}{(z-w)^3} \frac{24}{23} \frac{1}{120} \epsilon^{abcdef} \partial(\psi^a E^{bc} E^{de})(w)
\]

\(^{§}\)A product of two of \( \epsilon \) tensors can be expressed as a determinant in which the entries are \( \delta \)'s.
\[\begin{align*}
&+ \frac{1}{(z-w)^2} \frac{1}{120} \frac{48}{23} 15 \epsilon^{abcde} \psi^a \psi^b \psi^c \psi^d \partial^2 \psi^e (w) - \frac{8}{23} \epsilon^{abcde} \partial^2 (\psi^a E^{bc} E^{de}) (w) \\
&+ \frac{1}{(z-w)^2} \frac{1}{120} \left[ \frac{8}{23} 15 4 \epsilon^{abcde} \partial (\psi^a \psi^b \psi^c \psi^d \partial^2 \psi^e) (w) \\
&+ \frac{54}{23} \left( \frac{2}{3} \epsilon^{abcde} \partial^3 (\psi^a E^{bc} E^{de})(w) + 5 \epsilon^{abcde} \partial^3 \psi^a E^{bc} E^{de}(w) \\
&- \frac{5}{3} \epsilon^{abcde} \partial^3 (E^{bc} E^{de}) (w) + 20 \epsilon^{abcde} \psi^a \psi^b \psi^c \partial \psi^d \partial^2 \psi^e (w) \right) \right] \\
&= \frac{1}{(z-w)^4} \frac{120}{23} U(w) + \frac{1}{(z-w)^3} \frac{24}{23} \partial U(w) + \frac{1}{(z-w)^2} \frac{48}{23} \partial^2 U(w) \\
&- \frac{8}{23} \partial^2 U(w)] + \frac{1}{(z-w)} \left[ - \frac{8}{23} \partial (TU)(w) + \frac{54}{23} \partial TU(w) \right] + \cdots. \quad (3.10)
\end{align*}\]

Basically, it agrees with the expressions given in (2.10) and (2.15) for \( c = \frac{5}{2} \). We are ready to consider the OPE of \( V(z)V(w) \). We explicitly computed \( V(z)V(w) \), obtained from eq. (3.9), which is given by

\[\begin{align*}
V(z)V(w) &= \frac{1}{(z-w)^8} \frac{5}{8} + \frac{1}{(z-w)^6} 2T(w) + \frac{1}{(z-w)^5} \partial T(w) \\
&+ \frac{1}{(z-w)^4} \frac{2}{20} \partial^2 T(w) + \frac{14}{23} \Lambda(w) - \frac{27}{4 \sqrt{69}} V(w) \]

&+ \frac{1}{(z-w)^3} \left[ \frac{1}{15} \partial^3 T(w) + \frac{14}{23} \partial \Lambda(w) - \frac{27}{4 \sqrt{69}} \partial V(w) \right]

&+ \frac{1}{(z-w)^2} \left[ \frac{1}{168} \partial^4 T(w) + \frac{9083}{278208} \partial^2 \Lambda(w) + \frac{15}{184} \Delta(w) \right]

&+ \frac{89}{288} \Xi(w) - \frac{9}{4 \sqrt{69}} \partial^2 V(w) - \frac{13}{2 \sqrt{69}} \Omega(w) + \frac{23}{32} \Gamma(w) \]

&+ \frac{1}{(z-w)} \left[ \frac{1}{560} \partial^5 T(w) - \frac{5029}{278208} \partial^3 \Lambda(w) + \frac{15}{184} \partial \Delta(w) \right]

&+ \frac{89}{288} \partial \Xi(w) - \frac{27}{16 \sqrt{69}} \partial^2 V(w) - \frac{13}{2 \sqrt{69}} \partial \Omega(w) + \frac{23}{32} \partial \Gamma(w) \] \quad (3.11)

where \( \Lambda(z), \Delta(z), \Xi(z), \Omega(z) \) and \( \Gamma(z) \) which are expressed in terms of \( \psi^a(z) \)'s
according to

\[
\Lambda(z) = \frac{1}{4} \psi^a \partial \psi^a \psi^b \partial \psi^b(z) + \frac{21}{40} \partial \psi^a \partial^2 \psi^a(z) - \frac{7}{120} \psi^a \partial^3 \psi^a(z)
\]

\[
\Delta(z) = -\frac{1}{8} \psi^a \partial \psi^a \psi^b \partial \psi^b(z) - \frac{3}{20} \psi^a \partial^2 \psi^a \partial^2 \psi^b(z)
\]

\[
-\frac{63}{80} \psi^a \partial \psi^a \partial^2 \psi^b(z) + \frac{7}{80} \psi^a \partial \psi^a \psi^b \partial^2 \psi^b(z) - \frac{161}{400} \partial^2 \psi^a \partial^3 \psi^a(z)
\]

\[
+ \frac{49}{1600} \partial \psi^a \partial^4 \psi^a(z) + \frac{7}{1600} \psi^a \partial^5 \psi^a(z)
\]

\[
\Xi(z) = -\frac{31}{483} \psi^a \partial \psi^a \partial \psi^b \partial^2 \psi^b(z) - \frac{31}{483} \psi^a \partial \psi^a \psi^b \partial^3 \psi^b(z)
\]

\[
+ \frac{359}{1932} \psi^a \partial^2 \psi^a \psi^b \partial^2 \psi^b(z) + \frac{1084}{7245} \partial^2 \psi^a \partial^3 \psi^a(z)
\]

\[
- \frac{32}{7245} \psi^a \partial^5 \psi^a(z) + \frac{86}{7245} \partial \psi^a \partial^4 \psi^a(z)
\]

\[
\Omega(z) = \sqrt{\frac{23}{192}} \left[ -\frac{7}{46} \psi^a \partial \psi^a \psi^b \partial \psi^b \psi^c \partial \psi^c(z) - \frac{15}{46} \psi^a \partial \psi^a \partial \psi^b \partial^2 \psi^b(z) \right]
\]

\[
+ \frac{11}{46} \psi^a \partial \psi^a \psi^b \partial^3 \psi^b(z) + \frac{3}{23} \partial^2 \psi^a \partial^3 \psi^a(z)
\]

\[
- \frac{7}{92} \partial \psi^a \partial^4 \psi^a(z) + \frac{3}{460} \psi^a \partial^5 \psi^a(z) \right]
\]

\[
\Gamma(z) = -\frac{1}{6} \psi^a \partial \psi^a \psi^b \partial \psi^b \psi^c \partial \psi^c(z) + \frac{1}{6} \psi^a \partial \psi^a \psi^b \partial^3 \psi^b(z)
\]

\[
+ \frac{1}{18} \psi^a \partial^2 \psi^a \psi^b \partial^2 \psi^b(z) + \frac{1}{1080} \psi^a \partial^5 \psi^a(z)
\]

\[
- \frac{5}{108} \partial^2 \psi^a \partial^3 \psi^a(z),
\]

which agree with the formulas (2.6).

Crucial point to arrive at this result was to reexpress \( \partial UU(z) \) appearing in \( 1/(z - w)^2 \) of \( V(z)V(w) \) in terms of 7 independent fields, consisting of composites of \( \psi^a \) and their derivatives, and recombine with \( \partial^2 V(z) \), \( TV(z) \), and \( T(z) \) descendants. The OPE of two Virasoro primary fields can be represented as the sum
of Virasoro conformal families, i.e., Virasoro descendants and Virasoro primary fields \[13\]. Then we can identify unique Virasoro primary spin 6 field \[10\] with
\[
\Phi(z) = \frac{1}{576} (5\psi^a \partial \psi^b \psi^c \partial \psi^d(z) - \frac{17}{4} \psi^a \partial \psi^a \partial \psi^b \partial^2 \psi^c(z) - \frac{1}{5} \psi^a \partial^2 \psi^a \partial \psi^b \partial^2 \psi^c(z) + \frac{13}{20} \psi^a \partial \psi^a \partial \psi^b \partial^3 \psi^c(z) + \frac{1}{3} \partial^2 \psi^a \partial^3 \psi^c(z)
\]
(3.13)

Of course, \(\Phi(z)\) is a descendant w.r.t. the full \(WB_2\) algebra. Eq. (3.11) and (3.12) agree with the expression for the \(WB_2\) algebra as given in eq. (2.5) and (2.15) for \(c = 5/2\). The results obtained so far can be summarized as follows. In \(c = 5/2\) free fermion model, a consistent OPA can be made out of energy momentum tensor \(T(z)\) and additional currents \(U(z)\) of dimension 5/2 and \(V(z)\) of dimension 4, corresponding to a fourth order Casimir of \(B_2\).

## 4 Conclusion

The remaining problem is to construct the \(WB_2\) algebra in the coset models based on \((\hat{B}_2 \oplus \hat{B}_2, \hat{B}_2)\) at level \((1, m)\), which can be viewed as perturbations of the \(m \to \infty\) model discussed before. Then the dimension 5/2 coset field \(\tilde{U}(z)\) was given in \[8\] where \(\tilde{c}\) is as in eq. (3.3). We can do calculate the OPE \(\tilde{U}(z)\tilde{U}(w)\). Then the dimension 4 coset field \(\tilde{V}(z)\) can be obtained from the singular part of OPE \(\tilde{U}(z)\tilde{U}(w)\). We would like to show explicitly that the algebras, consisting of \(\tilde{T}(z), \tilde{U}(z)\) and \(\tilde{V}(z)\), closes in the coset model. Then, this construction will lead to an explicit realization of the \(c < 5/2\) unitary representations of \(WB_2\) algebra. We leave it further investigation \[16\].

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