Surface links with free abelian groups

By Inasa Nakamura

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Abstract. It is known that if a classical link group is a free abelian group, then its rank is at most two. It is also known that a \(k\)-component 2-link group (\(k > 1\)) is not free abelian. In this paper, we give examples of \(T^2\)-links each of whose link groups is a free abelian group of rank three or four. Concerning the \(T^2\)-links of rank three, we determine the triple point numbers and we see that their link types are infinitely many.

Introduction.

A classical link is the image of a smooth embedding of a disjoint union of circles into the Euclidean 3-space \(\mathbb{R}^3\). The link group is the fundamental group of the link exterior. It is known [13, Theorem 6.3.1] that if a classical link group is a free abelian group, then its rank is at most two. A surface link is the image of a smooth embedding of a closed surface into the Euclidean 4-space \(\mathbb{R}^4\). A 2-link (resp. \(T^2\)-link) is a surface link whose components are homeomorphic to 2-spheres (resp. tori). It is known [7, Chapter 3, Corollary 2] that a \(k\)-component 2-link group for \(k > 1\) is not a free abelian group.

The aim of this paper is to give concrete examples of \(T^2\)-links whose link groups are free abelian. It is known (see Remark 2.1) that a \(T^2\)-link called a “Hopf 2-link” [5] has a free abelian group of rank two. We give \(T^2\)-links with a free abelian group of rank three (Theorem 2.2). We also give a \(T^2\)-link with a free abelian group of rank four (Theorem 2.3). These \(T^2\)-links are “torus-covering \(T^2\)-links”, which are \(T^2\)-links in the form of unbranched coverings over the standard torus.

Further we study the \(T^2\)-links given in Theorem 2.2 i.e. \(T^2\)-links each of whose link groups is a free abelian group of rank three. We determine the triple point number of each \(T^2\)-link (Theorem 3.1), by which we can see that their link types are infinitely many. The triple point number of each \(T^2\)-link is a multiple of four, and it is realized by a surface diagram in the form of a covering over the torus. For other examples of surface links (not necessarily orientable) which realize large triple point numbers, see [6], [9], [12], [16], [17], [19].

The paper is organized as follows. In Section 1, we review the definition of a torus-covering \(T^2\)-link, and we review a formula how to calculate its link group. In Section 2, we show Theorems 2.2 and 2.3. In Section 3, we show Theorem 3.1.

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1. A torus-covering $T^2$-link and its link group.

In this section, we give the definition of a torus-covering $T^2$-link $S_m(a, b)$, which is determined from a pair of commuting $m$-braids $a$ and $b$ called basis braids. For the definition of a torus-covering link whose component might be of genus more than one, see [15]. We can compute the link group of $S_m(a, b)$ by using Artin’s automorphism associated with $a$ or $b$ [15].

1.1. Let $T$ be the standard torus in $\mathbb{R}^4$, i.e. the boundary of an unknotted solid torus in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Let $N(T)$ be a tubular neighborhood of $T$ in $\mathbb{R}^4$.

**Definition 1.1.** A torus-covering $T^2$-link is a surface link $F$ in $\mathbb{R}^4$ such that $F$ is embedded in $N(T)$ and $p|_F : F \to T$ is an unbranched covering map, where $p : N(T) \to T$ is the natural projection.

Let us consider a torus-covering $T^2$-link $F$. Let us fix a point $x_0$ of $T$, and take a meridian $m$ and a longitude $l$ of $T$ with the base point $x_0$. A meridian is an oriented simple closed curve on $T$ which bounds a 2-disk in the solid torus whose boundary is $T$ and which is not null-homologous in $T$. A longitude is an oriented simple closed curve on $T$ which is null-homologous in the complement of the solid torus in the three space $\mathbb{R}^3 \times \{0\}$ and which is not null-homologous in $T$. The intersections $F \cap p^{-1}(m)$ and $F \cap p^{-1}(l)$ are closures of classical braids. Cutting open the solid tori at the 2-disk $p^{-1}(x_0)$, we obtain a pair of classical braids. We call them basis braids [15]. The basis braids of a torus-covering $T^2$-link are commutative, and for any commutative braids $a$ and $b$, there exists a unique torus-covering $T^2$-link with basis braids $a$ and $b$ [15, Lemma 2.8]. For commutative $m$-braids $a$ and $b$, we denote by $S_m(a, b)$ the torus-covering $T^2$-link with basis $m$-braids $a$ and $b$.

1.2. We can compute the link group of a torus-covering $T^2$-link $S_m(a, b)$ [15]. As preliminaries, we will give the definition of Artin’s automorphism (see [11]). Let $c$ be an $m$-braid in a cylinder $D^2 \times [0, 1]$, and let $Q_m$ be the starting point set of $c$. Let $\{h_u\}_{u \in [0, 1]}$ be an isotopy of $D^2$ rel $\partial D^2$ such that $\cup_{u \in [0, 1]} h_u(Q_m) \times \{u\} = c$. Let $A_c : (D^2, Q_m) \to (D^2, Q_m)$ be the terminal map $h_1$, and consider the induced map $A^c : \pi_1(D^2 - Q_m) \to \pi_1(D^2 - Q_m)$. It is known [1] that $A^c$ is uniquely determined from $c$. We call $A^c$ Artin’s automorphism associated with $c$. Note that $\pi_1(D^2 - Q_m)$ is naturally isomorphic to the free group $F_m$ generated by the standard generators $x_1, x_2, \ldots, x_m$ of $\pi_1(D^2 - Q_m)$. By $A^c$, the braid group $B_m$ acts on $\pi_1(D^2 - Q_m)$. It is presented by

$$A^c_i(x_j) = \begin{cases} x_j x_{j+1} x_j^{-1} & \text{if } j = i \\ x_{j+1} & \text{if } j = i + 1 \\ x_j & \text{otherwise} \end{cases}$$
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and

\[ A_s^{-1}(x_j) = \begin{cases} x_{j+1} & \text{if } j = i \\ x_{j-1}^{-1}x_{j-1}x_j & \text{if } j = i + 1 \\ x_j & \text{otherwise} \end{cases} \]

where \( i = 1, 2, \ldots, m - 1 \) and \( j = 1, 2, \ldots, m \).

It is known [15, Proposition 3.1] that the link group of \( S_m(a, b) \) is presented by

\[ \pi_1(\mathbb{R}^4 - S_m(a, b)) = \langle x_1, \ldots, x_m \mid x_j = A_s^a(x_j) = A_s^b(x_j), \text{ for } j = 1, 2, \ldots, m \rangle. \]

2. \( T^2 \)-links whose link groups are free abelian.

In this section we show Theorems 2.2 and 2.3: There are torus-covering \( T^2 \)-links with a free abelian group of rank three (Theorem 2.2) or four (Theorem 2.3).

Remark 2.1. A Hopf 2-link [5] is a \( T^2 \)-link which is the product of a classical Hopf link in \( B^3 \) with \( S^1 \), embedded into \( \mathbb{R}^4 \) via an embedding of \( B^3 \times S^1 \) into \( \mathbb{R}^4 \), where \( B^3 \) is a 3-ball and \( S^1 \) is a circle. There are two link types according to the embedding of \( B^3 \times S^1 \), called a standard Hopf 2-link and a twisted Hopf 2-link [5]. A standard (resp. twisted) Hopf 2-link is the spun \( T^2 \)-link (resp. the turned spun \( T^2 \)-link) of a classical link \( L \). It is known [14], [2], [3] that the link group of the spun \( T^2 \)-link or the turned spun \( T^2 \)-link of a classical link \( L \) is isomorphic to the classical link group of \( L \). Thus we can see that a Hopf 2-link has a free abelian link group of rank two.

Let \( \sigma_1, \sigma_2, \ldots, \sigma_{m-1} \) be the standard generators of \( B_m \).

Theorem 2.2. The link group of \( S_3(\sigma_1\sigma_2^n, \Delta) \) is a free abelian group of rank three, where \( n \) is an integer and \( \Delta \) is a full twist of a bundle of three parallel strings.

Proof. Put \( S_n = S_3(\sigma_1\sigma_2^n, \Delta) \). Let us compute the link group \( G_n = \pi_1(\mathbb{R}^4 - S_n) \) by applying [15, Proposition 3.1]. Let \( x_1, x_2 \) and \( x_3 \) be the generators. Then the relations concerning the basis braid \( \sigma_1\sigma_2^n \) are

\[ x_1x_2 = x_2x_1, \]

\[ (x_2x_3)^n = (x_3x_2)^n. \]

The other relations concerning the other basis braid \( \Delta \) are

\[ x_1 = (x_1x_2x_3)x_1(x_1x_2x_3)^{-1}, \]

\[ x_2 = (x_1x_2x_3)x_2(x_1x_2x_3)^{-1}, \]

\[ x_3 = (x_1x_2x_3)x_3(x_1x_2x_3)^{-1}, \]

which are
\[ x_1 x_2 x_3 = x_2 x_3 x_1, \quad (2.3) \]
\[ x_2 (x_1 x_2 x_3) = (x_1 x_2 x_3) x_2, \quad (2.4) \]
\[ x_3 x_1 x_2 = x_1 x_2 x_3. \quad (2.5) \]

By (2.1), (2.3) is deformed to
\[ x_1 x_3 = x_3 x_1; \quad (2.6) \]

Similarly, by (2.4) and (2.1),
\[ x_2 x_3 = x_3 x_2. \quad (2.7) \]

We can see that all the relations are generated by the three relations (2.1), (2.6) and (2.7). Thus we have
\[ G_n = \langle x_1, x_2, x_3 | x_1 x_2 = x_2 x_1, x_2 x_3 = x_3 x_2, x_3 x_1 = x_1 x_3 \rangle, \]
which is a free abelian group of rank three.

**Theorem 2.3.** The link group of \( S_4(\sigma_1^2 \sigma_2^2 \sigma_3^2, \Delta) \) is a free abelian group of rank four, where \( \Delta \) is a full twist of a bundle of 4 parallel strings.

**Proof.** Similarly to the proof of Theorem 2.2, by [15, Proposition 3.1], for generators \( x_1, x_2, x_3 \) and \( x_4 \), we have the following relations:
\[ x_i x_{i+1} = x_{i+1} x_i, \quad (2.8) \]
where \( i = 1, 2, 3, \) and
\[ x_i = (x_1 x_2 x_3 x_4) x_i (x_1 x_2 x_3 x_4)^{-1}, \quad (2.9) \]
where \( i = 1, 2, 3, 4 \). Using \( x_1 x_2 = x_2 x_1 \) and \( x_3 x_4 = x_4 x_3 \) of (2.8), the latter four relations (2.9) are deformed as follows:
\[ x_1 x_3 x_4 = x_3 x_4 x_1, \quad (2.10) \]
\[ x_2 x_3 x_4 = x_3 x_4 x_2, \quad (2.11) \]
\[ x_3 x_1 x_2 = x_1 x_2 x_3, \quad (2.12) \]
\[ x_4 x_1 x_2 = x_1 x_2 x_4. \quad (2.13) \]

By \( x_2 x_3 = x_3 x_2 \) of (2.8), (2.11) is deformed to \( x_3 x_2 x_4 = x_3 x_4 x_2 \); hence
\[ x_2 x_4 = x_4 x_2. \quad (2.14) \]
Similarly, by $x_2 x_3 = x_3 x_2$ of (2.8) and (2.12),

$$x_3 x_1 = x_1 x_3, \tag{2.15}$$

and by (2.14) and (2.13),

$$x_4 x_1 = x_1 x_4. \tag{2.16}$$

We can see that all the relations are generated by the relations (2.8), (2.14), (2.15) and (2.16). Thus the link group is a free abelian group of rank four. $\square$

3. The triple point numbers of the $T^2$-links with a free abelian group of rank three.

The triple point number of a surface link $F$ is the minimal number of triple points among all the surface diagrams of $F$. In this section we study the $T^2$-links given in Theorem 2.2 i.e. $T^2$-links each of whose link group is a free abelian group of rank three.

**Theorem 3.1.** The triple point number of $S_n = S_3(\sigma_1^2 \sigma_2^{2n}, \Delta)$ given in Theorem 2.2 is $4n$ for $n > 0$ and $4(1 - n)$ for $n \leq 0$. Further it is realized by a surface diagram in the form of a covering over $T$, in other words, by a 3-chart on $T$ which presents $S_n$. Thus $T^2$-links with a free abelian group of rank three are infinitely many.

Here, a 3-chart [11] is a finite graph with certain additional data, which we review in Section 3.1.

This section is organized as follows. In Section 3.1, we review a surface diagram and an $m$-chart on $T$ which presents a torus-covering $T^2$-link (see [15], [11]). In Section 3.2, we review the result of [16] which gives lower bounds of triple point numbers. In Section 3.3, we prove Theorem 3.1.

3.1. Surface diagrams and $m$-charts presenting torus-covering $T^2$-links.

The notion of an $m$-chart on a 2-disk was introduced by Kamada [8] (see also [11]) to present a surface braid i.e. a 2-dimensional braid in a bi-disk (see [18], [11]). An $m$-chart on a disk is obtained from the singularity set of a surface diagram of a surface braid. By a minor modification, we can define an $m$-chart on $T$ presenting a torus-covering link [15].

For a torus-covering $T^2$-link $F$, we consider a surface diagram in the form of a covering over the torus, as in Section 3.1.1. Given $F$, we obtain such a surface diagram $D$, and from $D$ we obtain a graph called an $m$-chart on $T$ (without black vertices). Conversely, an $m$-chart on $T$ without black vertices presents such a surface diagram and hence a torus-covering $T^2$-link.

3.1.1. Surface diagrams.

We review a surface diagram of a surface link $F$ (see [4]). For a projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, the closure of the self-intersection set of $\pi(F)$ is called the singularity set. Let $\pi$ be a generic projection, i.e. the singularity set of the image $\pi(F)$ consists of double
points, isolated triple points, and isolated branch points; see Figure 3.1. The closure of the singularity set forms a union of immersed arcs and loops, which we call double point curves. Triple points (resp. branch points) form the intersection points (resp. the end points) of the double point curves. A surface diagram of \( F \) is the image \( \pi(F) \) equipped with over/under information along each double point curve with respect to the projection direction.

Throughout this paper, we consider the surface diagram of a torus-covering \( T^2 \)-link \( F \) by the projection which projects \( N(T) = I \times I \times T \) to \( I \times T \) for an interval \( I \), where we identify \( N(T) \) with \( I \times I \times T \) in such a way as follows. Since \( T \) is the boundary of the standard solid torus in \( \mathbb{R}^3 \times \{0\} \), the normal bundle of \( T \) in \( \mathbb{R}^3 \times \{0\} \) is a trivial bundle. We identify it with \( I \times T \). Then we identify \( N(T) \) with \( I \times I \times T \), where the second \( I \) is an interval in the fourth axis of \( \mathbb{R}^4 \). Perturbing \( F \) if necessary, we can assume that this projection is generic. We call this surface diagram the surface diagram of \( F \) in the form of a covering over the torus.

3.1.2. From surface diagrams to \( m \)-charts on \( T \).

Given a torus-covering \( T^2 \)-link \( F \), we obtain a graph on \( T \) from the surface diagram in the form of a covering over the torus, as follows. Now we have \( \text{Sing}(\pi(F)) \) in \( I \times T \). By the definition of a torus-covering \( T^2 \)-link, \( \text{Sing}(\pi(F)) \) consists of double point curves and triple points, and no branch points. We can assume that the singular set of the image of \( \text{Sing}(\pi(F)) \) by the projection to \( T \) consists of a finite number of double points such that the preimages belong to double point curves of \( \text{Sing}(\pi(F)) \). Thus the image of \( \text{Sing}(\pi(F)) \) by the projection to \( T \) forms a finite graph \( \Gamma \) on \( T \) such that the degree of its vertex is either 4 or 6. An edge of \( \Gamma \) corresponds to a double point curve, and a vertex of degree 6 corresponds to a triple point.

For such a graph \( \Gamma \) obtained from the surface diagram, we give orientations and labels to the edges of \( \Gamma \), as follows. Let us consider a path \( l \) in \( T \) such that \( l \cap \Gamma \) is a point \( P \) of an edge \( e \) of \( \Gamma \). Then \( F \cap p^{-1}(l) \) is a classical \( m \)-braid with one crossing in \( p^{-1}(l) \) such that \( P \) corresponds to the crossing of the \( m \)-braid. Let \( \sigma_i^\epsilon \) (\( i \in \{1, 2, \ldots, m - 1\} \), \( \epsilon \in \{+1, -1\} \)) be the presentation of \( F \cap p^{-1}(l) \). Then label the edge \( e \) by \( i \), and moreover give \( e \) an orientation such that the normal vector of \( l \) corresponds (resp. does not correspond) to the orientation of \( e \) if \( \epsilon = +1 \) (resp. \( -1 \)). We call such an oriented and labeled graph an \( m \)-chart of \( F \) (without black vertices).

In general, we define an \( m \)-chart on \( T \) as follows.
Definition 3.2. Let $m$ be a positive integer, and let $\Gamma$ be a finite graph on $T$. Then $\Gamma$ is called an $m$-chart on $T$ if it satisfies the following conditions:

(i) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.
(ii) Every vertex has degree 1, 4, or 6.
(iii) The adjacent edges around each vertex are oriented and labeled as shown in Figure 3.2, where we depict a vertex of degree 1 (resp. 6) by a black vertex (resp. white vertex).

Figure 3.2. Vertices in an $m$-chart.

A black vertex presents a branch point; see [11]. When an $m$-chart on $T$ without black vertices is given, we can reconstruct a torus-covering $T^2$-link [15] (see also [11]).

Two $m$-charts on $T$ are $C$-move equivalent [15] (see also [8], [10], [11]) if they are related by a finite sequence of ambient isotopies of $T$ and CI, CII, CIII-moves. We show several examples of CI-moves in Figure 3.3; see [11] for the complete set of CI-moves and CII, CIII-moves. For two $m$-charts on $T$, their presenting torus-covering links are equivalent if the $m$-charts are $C$-move equivalent [15] (see also [8], [10], [11]).

Figure 3.3. CI-moves. We give only several examples.

3.2. Triple point numbers.

For a surface link $F$, we denote by $t(F)$ the triple point number of $F$. It is shown [16] that for a pure $m$-braid $b$ ($m \geq 3$) and an integer $n$, a lower bound of $t(S_m(b, \Delta^n))$ is given by using the linking numbers of $\hat{b}$, and for a particular $b$, we can determine the triple point number. Here $\hat{b}$ denotes the closure of $b$.

For a pure 3-braid $b$, it follows from [16] that we can give a lower bound of $t(S_3(b, \Delta))$ as follows. We define the $i$th component of $\hat{b}$ by the component constructed by the $i$th string of $\hat{b}$ ($i = 1, 2, 3$). For positive integers $i$ and $j$ with $i \neq j$, the linking number of the $i$th and $j$th components of a classical link $L$, denoted by $\text{Lk}_{i,j}(L)$, is the total number of positive crossings minus the total number of negative crossings.
of a diagram of $L$ such that the under-arc (resp. over-arc) is from the $i$th (resp. $j$th) component. Put $\mu = \sum_{i<j} |L_{i,j}(\hat{b})|$, and put $\nu = \nu_{1,2,3} + \nu_{2,3,1} + \nu_{3,1,2}$, where $\nu_{i,j,k} = \min_{i,j,k} \{|L_{i,j}(\hat{b})|, |L_{j,k}(\hat{b})|\}$ if $L_{i,j}(\hat{b})L_{j,k}(\hat{b}) > 0$ and otherwise zero. Then, by [16],

$$t(S_3(b, \Delta)) \geq 4(\mu - \nu).$$

In particular, let $b$ be a 3-braid presented by a braid word which is an element of a monoid generated by $\sigma_1^2$ and $\sigma_2^{-2}$; note that $b$ is a pure braid. Then

$$t(S_3(b, \Delta)) = 4\mu,$$

and the triple point number is realized by a surface diagram in the form of a covering over the torus [16].

3.3. Proof of Theorem 3.1.

Put $b = \sigma_1^2 \sigma_2^{2n}$. We use the notations given in Section 3.2. Since $L_{i,j}(\hat{b}) = 1$ (resp. $n$) if $\{i, j\} = \{1, 2\}$ (resp. $\{2, 3\}$) and otherwise zero, we can see that $\mu = 1 + |n|$. Let us consider the case for $n \leq 0$. Since $b$ has the presentation which is an element of a monoid generated by $\sigma_1^2$ and $\sigma_2^{-2}$, $t(S_n) = 4\mu$ by [16]; thus $t(S_n) = 4(1 - n)$ ($n \leq 0$), and the triple point number is realized by a surface diagram in the form of a covering over the torus by [16].

Let us consider the case for $n > 0$. Since $L_{i,j}(\hat{b}) = 1$ (resp. $n$) if $\{i, j\} = \{1, 2\}$ (resp. $\{2, 3\}$) and otherwise zero, we can see that $\nu_{i,j,k} = 1$ if $(i, j, k) = (1, 2, 3)$ and zero if $(i, j, k) = (2, 3, 1)$ or $(3, 1, 2)$; thus $\nu = 1$, and hence $t(S_n) \geq 4(\mu - \nu) = 4n$ by [16]. It remains to show that there is a surface diagram of $S_n$ ($n > 0$) with 4$n$ triple points. It suffices to draw a 3-chart $\Gamma$ on $T$ which presents $S_n$ such that $\Gamma$ has exactly 4$n$ white vertices. We draw $\Gamma$ which presents $S_n$, and deform it to a 3-chart with 4$n$ white vertices by C-moves, as follows. First we draw $\Gamma$ as a 3-chart which consists of 2$n$ + 2 parts as follows, where we assume that a full twist $\Delta$ has the presentation $\Delta = (\sigma_1\sigma_2\sigma_1)^2$.

- (i) The part of $\Gamma$ with basis braids $\sigma_1$ and $\Delta$. We have two copies.
- (ii) The part of $\Gamma$ with basis braids $\sigma_2$ and $\Delta$. We have 2$n$ copies.

We draw the part (i) as in Figure 3.4 and we denote the white vertices by $t_{i_1}$ and $t_{i_2}$ as in Figure 3.4 for the $i$th copy ($i = 1, 2$). We draw the part (ii) as in Figure 3.5 and we denote the white vertices by $t_{i_1}$ and $t_{i_2}$ as in Figure 3.5 for the $(i - 2)$th copy.

![Figure 3.4](image-url)
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Figure 3.5. White vertices $t_{i1}$ and $t_{i2}$ \((i = 3, 4, \ldots, 2n + 2)\), for \(n > 0\).

\((i = 3, 4, \ldots, 2n + 2)\). There are \(4n + 4\) white vertices in \(\Gamma\). Let us apply a CI-move as in Figure 3.3 (3) to the pair \(\{t_{21}, t_{31}\}\) of white vertices in \(\Gamma\), and then to the pair \(\{t_{(2n+2)2}, t_{12}\}\); then we can eliminate the four white vertices, and the resulting 3-chart has \(4n\) white vertices. Hence \(t(S_n) = 4n\ (n > 0)\), and the triple point number is realized by this 3-chart on \(T\).

\[\square\]

Remark 3.3. There is an oriented \(T^2\)-link as in Figure 3.6 with a free abelian group of rank three and with the triple point number zero. It is a ribbon \(T^2\)-link (see [4] for the definition of a ribbon surface link). We briefly show that the link group is free abelian, as follows. In the surface diagram, there are six broken sheets (see [4]), consisting of three pairs of a sheet attached with \(x_i\) and a small disk \(D_i\) such that each pair forms the \(i\)th component of the \(T^2\)-link \((i = 1, 2, 3)\). Let us attach \(y_i\) to each \(D_i\). The link group has the presentation with generators \(x_i\) and \(y_i\ (i = 1, 2, 3)\) and the relations which are given around each double point curve (see [4], [11]). The singularity set consists of double point curves which form six circles. Around each circle in the \(i\)th component which does not bound \(D_i\ (i = 1, 2, 3)\), there are three broken sheets such that one is an over-sheet with \(x_i\) and the other two are under-sheets with the same generator \(x_{i+1}\), where \(x_4 = x_1\); together with the orientation, the relation is \(x_i = x_{i+1}x_ix_{i+1}^{-1}\), see [4], [11]. Around each circle \(\partial D_i\ (i = 1, 2, 3)\), there are three broken sheets such that one is an over-sheet with \(x_{i+1}\) and the other two are under-sheets with \(x_i\) and \(y_i\) respectively, where \(x_4 = x_1\); together with the orientation, the relation is \(y_i = x_{i+1}x_ix_{i+1}^{-1}\), see [4], [11]. Thus the link group is a free abelian group of rank three.

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Figure 3.6. A ribbon \(T^2\)-link with a free abelian group of rank three.
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Inasa Nakamura
Institute for Biology and Mathematics of Dynamical Cell Processes (iBMath)
Interdisciplinary Center for Mathematical Sciences
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba
Tokyo 153-8914, Japan
E-mail: inasa@ms.u-tokyo.ac.jp