Sensitivity analysis for expected utility maximization in incomplete Brownian market models

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Received: 17 February 2017 / Accepted: 14 December 2017 / Published online: 8 February 2018
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Abstract We examine the issue of sensitivity with respect to model parameters for the problem of utility maximization from final wealth in an incomplete Samuelson model and mainly, but not exclusively, for utility functions of positive-power type. The method consists in moving the parameters through change of measure, which we call a weak perturbation, decoupling the usual wealth equation from the varying parameters. By rewriting the maximization problem in terms of a convex-analytical support function of a weakly-compact set, crucially leveraging on the work (Backhoff and Fontbona in SIAM J Financ Math 7(1):70–103, 2016), the previous formulation let us prove the Hadamard directional differentiability of the value function with respect to the market price of risk, the drift and interest rate parameters, as well as for volatility matrices under a stability condition on their Kernel, and derive explicit expressions for the directional derivatives. We contrast our proposed weak perturbations against what we call strong perturbations, where the wealth equation is directly influenced by the changing parameters. Contrary to conventional wisdom, we find that both points of view generally yield different sensitivities unless e.g. if initial parameters and their perturbations are deterministic.

Julio Backhoff Veraguas is most grateful for partial support by the Austrian Science Fund (FWF) under grant Y782-N25 and the European Research Council (ERC) under grant FA506041, as well as to Humboldt-Universität zu Berlin and the funding by the Berlin Mathematical School. Francisco J. Silva acknowledges partial support by the Gaspar Monge Program for Optimization and Operation Research (PGMO).

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Introduction

The problem of continuous-time utility maximization in financial market models has a long and rich history going back to Merton [21,22], himself inspired in the work of Mirrlees and Samuelson in discrete times. The research on this topic continued in the eighties through the works of Pliska [30], Karatzas et al. (see e.g. [12,13]), Cox and Huang [7] and then probably culminated in the nineties with the general treatment of Kramkov and Schachermayer [16]. Naturally a comprehensive list would have to cover the works of many other people, but we do not intend to be exhaustive here and instead convey the interested reader to the books [14,29] for details. What all these works have in common, is that they provide an insight into the decision making problem of how to best select a portfolio from a given continuous-time, stochastic market model under the optimality criterion provided by the expected utility paradigm of von Neumann–Morgenstern.

It goes without a saying that in modelling the decision-making in such way, several parameters have to be chosen and therefore both the optimal portfolio rule and the optimal expected utility derived from it will be a function of these. Yet only recently the behaviour of the expected utility maximization problem in terms of its parameter-dependence has gained attention. In [16], for the case of general semimartingale models and an agent optimizing expected utility from final wealth only and no random endowment, the first-order sensitivity of the problem's value function (i.e. the optimal value) with respect to the initial wealth of the agent is studied, extending earlier results in [30]. More recently and in a similar setting, a second-order analysis of the value function is performed in [17] and even the first-order sensitivity of the optimizing wealth is carried out. A different trait in the literature has been the study of the stability (i.e. continuity) of the value function with respect to the so-called market price of risk or Sharpe ratio, which is a dynamic and stochastic parameter, heuristically measuring how much a given price model is away of a risk-neutral one (given by its martingale component). This analysis was performed in [20] initially (see also e.g. [23] for recent developments), and then extended in [15] for the case when a random endowment is present. The last article goes beyond that and actually proves stability of utility-based prices and admits misspecification of the utility functions themselves (see also [18] and the references given, for more on this subject). The previous articles focus on equivalent perturbations of a reference probability measure or a reference price process; recently [34] has showed that for non-equivalent perturbations the problem may be unstable/discontinuous.

In this article we focus on the first-order sensitivity analysis of the optimal value of the expected utility maximization problem with respect to the market price of risk and the drift and volatility coefficients of the model. We work in the classical setting where the utility function is defined on the positive half line, in the absence of consumption and random endowments, and we restrict ourselves to a Brownian filtration and the so-called Samuelson price model (e.g. geometric Brownian motion), which can be incomplete. In this framework, it is to be expected from the general stochastic maximum principle of [5] (specifically Sect. 2 therein) and recent results in [3], that the desired differentiability can be computed with
the help of the adjoint states appearing in the stochastic maximum principle. There are however several delicate points for this roadmap to work, the main one being that market prices of risk are multiplied by the decision variable (portfolio weights) in the controlled wealth equation, and so standard convex analysis arguments for convex perturbations are not applicable. Alternative arguments based in abstract optimization theory (see [4, Chapter 4] for the general theory and [3] for its application to stochastic control) seem difficult to apply since they require a normed vector space setting which is a priori absent in our problem. As a matter of fact, decision variables are a priori only almost surely square integrable with respect to the time variable. For these reasons, we choose in this article a different approach still allowing for a direct treatment of the first-order sensitivity question.

Let us be precise as to how we interpret parameter uncertainty/misspecification in this article. We take the widespread point of view of robust or worst-case stochastic optimization, in which one encodes uncertain parameters in uncertain probability measures under which the stochastic optimization problems are to be defined. See e.g. [32] or [6] in the context of model-misspecification and Knightean uncertainty in economics, or [15] for the question of stability in utility maximization/utility-based prices. Accordingly, we postulate that full knowledge of the parameters of a problem amounts to, in our case, a complete description of the price process, meaning concretely the drift, interest rate and volatility coefficients (and hence the market price of risk). Parameter uncertainty means for us that the actual possible trajectories of the price process, written in terms of the market price of risk, may have different “probability weights” than those specified by the law on the path space induced by the price process under the exact, “real” parameters. Consequently, the expected utility maximization problem under a perturbation of a “real” parameter consists for us in perturbing the reference probability measure away from the law induced by the “real” price process (that is, the one given the “real” parameters) yet otherwise leaving such “real” price process fixed. Naturally, the perturbation of the probability measure is defined with the help of Girsanov’s theorem and the optimal value of the new problem is referred to as the weakly perturbed value function. In this work, we shall study the differentiability and compute the directional derivatives of this weakly perturbed value function with respect to the market price of risk, the drift, and volatility coefficients, computed in a neighbourhood of the “real” parameters. As for all the articles around the topic of stability and sensitivity of the expected utility maximization problem already cited, only [15] takes this point of view. The others consider strong perturbations of the problem, meaning that the reference probability measure is kept fixed and the equations are perturbed. In this line, during the revision of our work we were made aware by a referee of the very recent work [25], where the authors study sensitivities with respect to strong perturbations via duality theory under great generality.

We remark that concurrently and independently from us, a related question has been posed and analyzed in [19] in the context of power utility functions with negative exponents and a semimartingale market model. We shall, on the contrary, focus our analysis on power utility functions with positive exponents, and more generally on utility functions dominated from above by such positive-power functions (see Theorem 2.1 and comments thereafter). In [19] the authors essentially study the dual problem and its associated dual value function, and from this they obtain the desired sensitivities of the primal, original problem. More substantially, type of exponent-regime (which is something that could certainly be amended both in [19] and here), the main difference with respect to our work is that in the cited article the sensitivity is studied in the strong sense (see our discussion after introducing Assumption (H) in Sect. 2) with respect to the market price of risk parameter. One of the advantages of the Brownian market model we consider is that it allows us to bring to light some delicate issues relating market incompleteness and the type of perturbations we are
able to handle. Indeed, in the weak formulation we are forced to consider a restricted space for perturbations of the volatility parameter, namely those which preserve the Kernel (see Remark 2.2). We believe that this discussion is essential and it seems absent in the literature. An additional nice feature of our Brownian framework is that it allows us to compare the sensitivity analysis in the strong and weak senses in a most transparent way. A detailed discussion about the differences between these approaches is provided in Sects. 2 and 5. As we will see, the sensitivities of the value function obtained from strong or weak perturbations need not coincide, and we provide examples in Sect. 2.2 for this situation. This is at odds with the implicit conventional wisdom that "it makes no difference how one perturbs parameters". We shall also show in an example that the weak sensitivity can behave in a counterintuitive fashion. Both phenomena occur when the nominal parameters are non-deterministic, so the lesson is that one should be cautious when applying weak (i.e. Girsanov-type) perturbations in such a situation. Although we do not provide a sensitivity analysis for strong perturbations, we can guess how the associated sensitivities would look like (consistently with [19]), and compare them to our weak sensitivities. Using Bismut’s integration by parts formula we find out exactly how these differ; see Eq. (5.2) in Sect. 5. It is also worth noticing that if both the nominal and the perturbed market parameters are deterministic functions, the directional sensitivities do coincide under our hypotheses, as we show in Proposition 2.1.

When performing the differentiability analysis of the weakly perturbed problem, we greatly rely on recent results having their origin in [1,2]. Indeed, the crucial fact is that we may interpret the expected utility maximization problem as the computation of a convex-analytical support function of a weakly-compact convex set in an explicit Banach space. The usefulness of working with weak perturbations and the weakly perturbed value function is that its differentiability and directional derivatives can then be computed by adapting Danskin’s Theorem for support functions and using the chain rule for directional derivatives. For this, the Fréchet directional differentiability of the Girsanov transform as an operator between essentially bounded integrands and elements in the pre-dual of the aforementioned Banach space has to be established. This issue poses most of the challenges in the present article. Our choice of dealing directly with the primal problem, via this support-function interpretation, is a second major distinction from [19].

In a nutshell our work has two original contributions. The first one is to provide new sensitivity results for weakly perturbed problems and fairly precise expressions for the directional derivatives. The main tool here is, as discussed in the previous paragraph, a hidden compactness property of the feasible set in a natural topological space. The second contribution is the detailed discussion on the type of perturbations allowed as well as on the difference between weak and strong perturbations and their associated sensitivities.

The paper is structured as follows. In Sect. 2 we present our Samuelson model, define the strong/weak perturbations and strongly/weakly perturbed value functions and describe our main result regarding differentiability of the value function under weak perturbations; Theorem 2.1. Of equal importance, we also prove that in the case of deterministic parameters and perturbations the strongly and weakly perturbed value functions do coincide, whereas we also provide two simple examples showing that in the general case the strong and weak sensitivities can differ. In Sect. 3 we provide for convenience of the reader a summary of the results in [2] needed for our proofs. Section 4 is the backbone of the article, where we prove the main sensitivity result. Then in Sect. 5 we present a discussion on how the strong and weak sensitivities are connected. Finally, in the “Appendix”, we briefly study support functions and prove a needed adaptation of the classical Danskin’s Theorem.
2 Problem statement and main results

We first fix some notations. In the entire article $\mathbb{R}_+$ ($\mathbb{R}_{++}$ respectively) will denote the set of non-negative (respectively strictly positive) real numbers. Given $T \in \mathbb{R}_{++}$, we work on a fixed filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{P} = \{\mathcal{F}_t\}_{t \leq T}, \mathbb{P})$, where $\mathbb{F}$ is the completed filtration of a Brownian motion $W$ in $\mathbb{R}^q$ (see e.g. [31]). Only in Sect. 3, in which we survey some of the findings in [2], we will lift this Brownian-type assumption on $\mathbb{F}$. We will denote by $L^0$ (resp. $L^0_\infty$) the set of all $\mathcal{F}_T$-measurable functions (resp. non-negative ones), and by $L^\infty_\mathbb{F}$ the set of essentially bounded real-valued progressively measurable processes endowed with the norm $\| \cdot \|_\infty$ defined as the least essential upper bound. Integration with respect to a measure $\mathbb{Q}$ shall be denoted $\int_{\cdot}^{\mathbb{Q}}$ except for $\mathbb{Q} = \mathbb{P}$, for which we reserve the notation $\int_{\cdot}^{\mathbb{P}}$. Given a local continuous martingale $M : \Omega \times [0, T] \to \mathbb{R}$, we denote by $L^2_{\text{loc}}(M)$ the set of all progressively measurable processes $H : \Omega \times [0, T] \to \mathbb{R}$ such that $\mathbb{P} \left( \int_0^T H_s^2 \, d\langle M \rangle_s < + \infty \right) = 1$, where $\langle M \rangle_t$ denotes the quadratic variation process associated to $M$. Finally, given a continuous semimartingale $Y$, we denote by $\mathcal{E}(Y)$, the Doléans-Dade stochastic exponential, defined as the solution of $Z_t = 1 + \int_0^T Z_s \, dY_s$, for $t \in [0, T]$.

We consider a general Samuelson’s price model, where discounted prices evolve continuously as geometric Brownian motions with progressively measurable drift and volatility coefficients. Specifically, suppose that the market consists of $d$ assets $S^1, \ldots, S^d$ whose prices (denoted likewise) evolve under $\mathbb{P}$ as

\[
\text{d}S^i_t = \text{diag}(S^i_t) \bar{\mu}^i_t \, dt + \text{diag}(S^i_t) \bar{\sigma}^i_t \, dW_t \quad \text{for } t \in [0, T],
\]

\[
S_0 = s_0 \in \mathbb{R}^d,
\]

where $S := (S^1, \ldots, S^d)$ and $W$ is the aforementioned $\mathbb{P}$-Brownian motion in $\mathbb{R}^n$ ($n \geq d$). The precise properties on the processes $\bar{\mu} \in (L^\infty_{\mathbb{F}})^d$ and $\bar{\sigma} \in (L^\infty_{\mathbb{F}})^{d \times n}$ shall be given shortly and will imply that the financial market is viable and moreover standard (see e.g. [14, Chapter 1] or [29, Chapter 7.2.4] for these concepts and the modelling details).

Given an initial wealth $x \in \mathbb{R}_{++}$ and a self-financing portfolio $\pi$ measured in units of wealth such that $\pi^i \in L^2_{\text{loc}}(W^k)$ ($i = 1, \ldots, d$ and $k = 1, \ldots, n$), which we denote $\pi \in \Pi$, the associated wealth process $X$ is defined through the equation

\[
\text{d}X^\pi_t = \pi^T_t \bar{\mu}^i_t \, dt + \pi^T_t \bar{\sigma}^i_t \, dW_t \quad \text{for } t \in [0, T],
\]

\[
X^\pi_0 = x.
\]

Let $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave utility function, whose properties will be specified in Sect. 3, but for the time being we suppose that $U(x) = -\infty$ if $x < 0$ and the restriction of $U$ to $\mathbb{R}_+$ takes values in $\mathbb{R}_+$ and is invertible. In this work, we consider the following utility maximization problem

\[
u(\bar{\mu}, \bar{\sigma}) := \sup_{\pi \in \Pi} \left\{ \mathbb{E} \left( U(X^\pi_T) \right) \; ; \; \pi \in \Pi \text{ and } X^\pi_t \geq 0 \; \forall \; t \in [0, T], \; \mathbb{P} - \text{a.s.} \right\}
\]

\[
\sup_{\pi \in \Pi} \mathbb{E} \left( U(X^\pi_T) \right) .
\]

The coefficients $(\bar{\mu}, \bar{\sigma})$ are our reference/nominal parameters, and the goal of this work is to study the behaviour of $u(\cdot)$ for infinitesimal perturbations of these. We thus consider perturbed coefficients $(\mu^\tau, \sigma^\tau)$ indexed by a “small size factor” $\tau > 0$. We will work under the following standing assumption:

\[
\text{[Standing Assumption]}
\]
The matrix $\bar{\sigma}$ has full rank and $(\bar{\sigma}\bar{\sigma}^\top)^{-1}$ is uniformly bounded in $(t, \omega)$. Moreover, the perturbations $\sigma^\tau$ of $\bar{\sigma}$ satisfy $\text{Ker}(\bar{\sigma}) = \text{Ker}(\sigma^\tau)$ (equivalently $\text{Im}(\bar{\sigma}^\top) = \text{Im}([\sigma^\tau]^\top)$) and $(\sigma^\tau(\sigma^\tau)^\top)^{-1}$ is uniformly bounded in $(t, \omega)$.

Remark 2.1 The assumptions for $\sigma^\tau$ in (H) are satisfied for $\sigma^\tau = \bar{\sigma} + A^\tau (\bar{\sigma}\bar{\sigma}^\top)^{-1}\bar{\sigma}$ where $A^\tau \in (L^\infty, \infty)^{d \times d}$ has small enough norm. This holds in particular for $\sigma^\tau = \bar{\sigma} + \tau A(\bar{\sigma}\bar{\sigma}^\top)^{-1}\bar{\sigma}$ with $A \in (L^\infty, \infty)^{d \times d}$ arbitrary and $\tau$ a small enough real number.

The crucial role of this assumption shall be discussed at length in Sect. 2.1 below. As customary in mathematical finance, we rewrite our problem in terms of the market price of risk/Sharpe ratio

$$\tilde{\lambda} := \bar{\sigma}^\top (\bar{\sigma}\bar{\sigma}^\top)^{-1}\bar{\mu} \in (L^\infty, \infty)^d,$$

and its associated return process

$$R^\tau_t := \int_0^t \tilde{\lambda}_s \, ds + W_t,$$

via

$$u(\tilde{\mu}, \tilde{\sigma}) = u(\tilde{\lambda}) := \sup \left\{ \mathbb{E} \left[ U \left( x + \int_0^T \xi_t^\top dR^\tau_t \right) \right] : \xi \in \bar{\sigma}^\top \Pi \right\},$$

with a harmless overload of notation for $u(\cdot)$. Here, $\tilde{\sigma}^\top \Pi$ is meant to denote the set of processes of the form $\tilde{\sigma}^\top \pi$ for some $\pi \in \Pi$. A first hint at the importance of Assumption (H), is that it implies

$$\bar{\sigma}^\top \Pi = [\sigma^\tau]^\top \Pi \quad \forall \tau > 0. \quad (2.5)$$

In order to perform a sensitivity analysis with respect to the new parameters $(\mu^\tau, \sigma^\tau)$ there are at least two modelling options. One, which we call the strongly perturbed formulation, is to consider a new process $S^\tau$ with dynamics like that of $S$ but under the new parameters, so that the perturbed wealth processes have the form

$$dX^\tau_{t, \pi} = \pi_t^\top \mu^\tau_t \, dt + \pi_t^\top \sigma^\tau_t \, dW_t \quad \text{for } t \in [0, T],$$

$$X^\tau_{0, \pi} = x,$$

and the new market price of risk and return processes are accordingly defined by

$$\lambda^\tau := (\sigma^\tau)^\top (\sigma^\tau(\sigma^\tau)^\top)^{-1}\mu^\tau. \quad R^\tau_t := \int_0^t \lambda^\tau_s \, ds + W_t.$$

The perturbed problem becomes (we use the $s$ to denote strongly perturbed)

$$u^s(\mu^\tau, \sigma^\tau) := \sup_{\pi \in \Pi} \mathbb{E} \left[ U \left( X^\tau_{T, \pi} \right) \right] = \sup \left\{ \mathbb{E} \left[ U \left( x + \int_0^T \xi_t^\top dR^\tau_t \right) \right] : \xi \in \tilde{\sigma}^\top \Pi \right\},$$

again with some harmless overload of notation, and where we used (2.5) crucially in order to keep the sets $[\sigma^\tau]^\top \Pi$ constant and equal to $\tilde{\sigma}^\top \Pi$.

Following [15], instead of fixing the reference probability measure $\mathbb{P}$ and considering perturbations directly affecting the dynamics of the $X$’s or of the $R$’s, it is reasonable to fix
the process \( f_0^t \tilde{\lambda} \, dt + W(\cdot) \) and to translate the perturbation of the price parameters into a perturbation of the law of \( f_0^t \tilde{\lambda} \, dt + W(\cdot) \). For this matter, it is natural to define

\[
d\mathbb{P}^\tau := \mathcal{E} \left( \int (\lambda^\tau - \tilde{\lambda})^\top dW \right)_T \, d\mathbb{P}.
\]

Note that Novikov’s condition implies that \( \mathbb{P}^\tau \) is a probability measure, equivalent to \( \mathbb{P} \). As explained in \([15, \text{Sect. 2.2}]\), if \((\mu^\tau, \sigma^\tau)\) converges to \((\tilde{\mu}, \tilde{\sigma})\) strongly in \((L_\infty^\infty)^d \times (L_\infty^\infty)^{d \times n}\), then \( \mathbb{P}^\tau \) converges to \( \mathbb{P} \) in the total variation norm. Therefore, taking this point of view, we define

\[
u_u^w(\mu^\tau, \sigma^\tau) := \sup_{\pi \in \Pi} \mathbb{E}^\mathbb{P}^\tau \left[ U(X_\tau^\tau) \right] = \sup_{\pi \in \Pi} \mathbb{E}^\mathbb{P} \left[ \mathcal{E} \left( \int (\lambda^\tau - \tilde{\lambda})^\top dW \right)_T U(X_\tau^\tau) \right]
\]

and we call \( \nu_u^w \) the weakly perturbed formulation of \( u(\cdot) \). Of course, for the nominal parameters we have

\[
u_u^w(\mu^\tau, \sigma^\tau) = u_u^w(\mu^\tau, \sigma^\tau) = \nu_u^w(\tilde{\lambda}) = u(\tilde{\lambda}).
\]

We will see in Sect. 2.2, that the values of \( \nu_u^w \) and \( u_u^w \), as well as their sensitivities, generally differ. On the other hand, the next result shows that if \( \tilde{\lambda}, \sigma \) and their perturbations \( \mu^\tau, \sigma^\tau \) are deterministic, then \( \nu_u^w \) and \( u_u^w \) (and so their sensitivities) do coincide. The proof is deferred to Sect. 2.1.

**Proposition 2.1** Assume that \( \mu^\tau, \sigma^\tau, \tilde{\lambda}, \sigma \) are deterministic, and that (H) holds. Then the weak and strong value functions coincide; \( \nu_u^w(\mu^\tau, \sigma^\tau) = u_u^w(\mu^\tau, \sigma^\tau) \).

As commented in the introduction, the continuity of \( u_u^w \) (in a broader context) as a function of \( \lambda \) was analysed in \([15]\). We move now towards the first-order analysis of \( u_u^w \), which is the object of study of our work, and present our main results. Consider the set

\[
\mathcal{M}^*(S) = \left\{ \mathbb{P}^\pi \sim \mathbb{P} : S \text{ is a } \mathbb{P}^\pi\text{-local martingale} \right\}.
\]

By \([29, \text{Proposition 7.2.1}]\) we have that \( \mathcal{M}^*(S) \) is characterized by the set of random variables \( Y^\nu_i(Z) \), where for \( \nu_i \in L^2_{\text{loc}}(W^j) \) \((j = 1, \ldots, m)\) and \( \nu \in \text{Ker}(\tilde{\sigma}) \) almost everywhere, and where the process \( Y^\nu_i \) is the exponential martingale \( Y^\nu_i := \mathcal{E} \left( - \int [\tilde{\lambda} + \nu]^\top dW \right)_t \). Given \( Z \in L^0 \), let us define

\[
J(Z) := \sup_{\mathbb{P} \in \mathcal{M}^*(S)} \mathbb{E}^\mathbb{M} \left[ U^{-1}(\{Z\}) \right].
\]

Note that \( \sup_{\pi \in \Pi} \mathbb{E}^\mathbb{P}^\tau \left[ U(X_\tau^\tau) \right] = \sup_{L \in C(Z)} \mathbb{E}^\mathbb{P}^\tau \left[ U(L) \right] \), where

\[
C(Z) := \left\{ L \in L^0_+ : \exists \, \pi \in \Pi, \, L \leq X_\tau^\tau \, \text{a.s.} \right\} = \left\{ L \in L^0_+ : \sup_{\mathbb{P} \in \mathcal{M}^*(S)} \mathbb{E}^\mathbb{M} (Z) \leq x \right\},
\]

the last equality being a consequence of \([29, \text{Corollary 7.2.1}]\). Letting \( Z = U(X_T) \) we can further rewrite problem (2.9) as:

\[
u_u^w(\lambda^\tau) = \sup \left\{ \mathbb{E} \left[ \mathcal{E} \left( \int (\lambda^\tau - \tilde{\lambda})^\top dW \right)_T Z \right] : J(Z) \leq x, \, Z \in L^0_+ \right\}.
\]

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Expression (2.12) opens the way to interpreting the sensitivity analysis of \( u^w \) as the study of a convex-theoretic support function of a convex and weakly-compact set. This will be made precise at the beginning of Sect. 4.

Under appropriate assumptions, we ultimately prove in Theorem 2.1 and Corollary 2.1 the following sensitivity results with respect to \( \lambda \) and \( (\mu, \sigma) \), respectively. We refer the reader to Definition 3.1 for the meaning of \( U \) being a utility function satisfying INADA conditions, and to the “Appendix” for the definition of Hadamard differentiability:

**Theorem 2.1** Suppose \( U \) is a utility function satisfying INADA conditions and such that \( U(0+) = 0 \) as well as the bound for some \( p \in (1, \infty) \):

\[
U(x) \leq Cx^{1/p}, \quad \text{for all} \quad x \geq 0.
\]

If \( \Delta \lambda \in (L^\infty_{\mathbb{F}})^n \) and we consider perturbations of the form \( \lambda^\tau := \tilde{\lambda} + \tau \Delta \lambda \), then the directional derivative

\[
D_\lambda u^w(\tilde{\lambda}) \Delta \lambda := \lim_{\tau \searrow 0} \frac{u^w(\lambda^\tau) - u^w(\tilde{\lambda})}{\tau},
\]

exists and equals

\[
\mathbb{E} \left[ U(\tilde{X}(T)) \int_0^T \Delta \lambda^\top dW \right],
\]

where \( \tilde{X}(T) \) is the unique optimal terminal wealth attaining \( u(\tilde{\lambda}) \). Furthermore, the function \( \lambda \mapsto u^w(\lambda) \) is Hadamard differentiable at \( \tilde{\lambda} \).

**Corollary 2.1** Under the same assumptions on \( U \) as in Theorem 2.1, consider a perturbation \((\Delta \mu, \Delta \sigma) \in (L^\infty_{\mathbb{F}})^d \times (L^\infty_{\mathbb{F}})^d \times \mathbb{R} \) and suppose that \((H)\) is satisfied for \((\mu^\tau, \sigma^\tau) := (\tilde{\mu} + \tau \Delta \mu, \tilde{\sigma} + \tau \Delta \sigma)\) and small enough \( \tau \). Then, the directional derivative \( D\mu u^w(\tilde{\mu}, \tilde{\sigma})(\Delta \mu, \Delta \sigma) \) exists and is given by

\[
D_\mu u^w(\tilde{\mu}, \tilde{\sigma}) \Delta \mu = \mathbb{E} \left[ U(\tilde{X}(T)) \int_0^T [\tilde{\sigma}^\top [\tilde{\sigma} \tilde{\sigma}^\top]^{-1} \Delta \mu]^\top dW \right].
\]

\[
D_\sigma u^w(\tilde{\mu}, \tilde{\sigma}) \Delta \sigma = \mathbb{E} \left[ U(\tilde{X}(T)) \int_0^T [\Delta \sigma^\top [\tilde{\sigma} \tilde{\sigma}^\top]^{-1} \tilde{\mu} - \tilde{\sigma}^\top [\tilde{\sigma} \tilde{\sigma}^\top]^{-1} [\tilde{\sigma} \tilde{\delta} \Delta \sigma^\top + \Delta \sigma \tilde{\delta} \Delta \sigma^\top + \tilde{\sigma} \tilde{\delta}^\top [\tilde{\sigma} \tilde{\delta}^\top]^{-1} \tilde{\mu}]^\top dW \right],
\]

where \( \tilde{X}(T) \) is the unique optimal terminal wealth attaining \( u(\tilde{\mu}, \tilde{\sigma}) \). Moreover, the application \((\tilde{\mu}, A) \in (L^\infty_{\mathbb{F}})^d \times (L^\infty_{\mathbb{F}})^d \times \mathbb{R} \mapsto u^w(\tilde{\mu}, A[\tilde{\sigma} \tilde{\sigma}^\top]^{-1} \tilde{\sigma}) \in \mathbb{R} \) is Hadamard differentiable at \((\tilde{\mu}, \tilde{\sigma} \tilde{\sigma}^{-1})\).

An example of \( U \) satisfying the assumptions in Theorem 2.1 is \( U(x) = x^{1/p} \) with \( p \in (1, \infty) \), the so-called positive power case. A further example is given e.g. by the inverse function of \( y \in [0, \infty) \mapsto R(y) := e^y - y - 1 \). Indeed, \( R^{-1} \) is non-negative, strictly concave and increasing, with \( R^{-1}(0) = 0 \). It is also differentiable in \((0, \infty)\) and from \( [R^{-1}](y) = 1/(R' \circ R^{-1})(x) \) we find that \([R^{-1}](0) = +\infty\) and \([R^{-1}](+\infty) = 0\). Finally, we easily see that \( R^{-1}(x) \leq \sqrt{2}x^{1/2} \), or equivalently \( y^2 \leq 2[e^y - y - 1] \), by Taylor expansion. On the other hand, our result does not cover the case of negative powers.

We finally remark that if the market defined by \((\tilde{\mu}, \tilde{\sigma})\) is complete, then \( n = d \) and \( \tilde{\sigma} \) is invertible (see e.g. [14, Theorem 6.6, Chapter 1]). In this case, \( u^w \) is Hadamard differentiable at \((\tilde{\mu}, \tilde{\sigma})\) and
\[ Du^w(\tilde{\mu}, \tilde{\sigma})(\Delta \mu, \Delta \sigma) = \mathbb{E} \left[ U(\tilde{X}(T)) \int_0^T [\tilde{\sigma}^{-1} \Delta \mu - \tilde{\sigma}^{-1} \Delta \sigma \tilde{\mu}]^T dW_t \right]. \quad (2.13) \]

2.1 On assumption (H)

Without Assumption (H), Eq. (2.5) need not hold. As a consequence the analysis of strong perturbations becomes much harder, since in (2.8) also the set upon which maximization is being performed, would be moving with the perturbation parameter. Actually, the work [34] even shows that the problem may not be continuous if \( \tilde{\sigma} \) is perturbed. As for weak perturbations, the role of Assumption (H) is already considered in [15, Remark 2.5] in the context of complete markets \( (n = d \text{ and } \tilde{\sigma} \text{ non-singular}) \). We, on the other hand, consider incomplete markets throughout.

Remark 2.2 Under (H) the strongly perturbed value function \( u^\pi(\mu^\top, \sigma^\top) \) coincides with the one presented in [19]. In fact, noting that (H) implies that

\[
\left\{ \int_0^T \tilde{\pi}_t^\top [\lambda^\top dt + dW_t] ; \ \tilde{\pi} = [\sigma^\top]^\top \pi, \ \pi \in \Pi \right\} = \left\{ \int_0^T \pi_t^\top [\tilde{\sigma} \lambda^\top dt + \tilde{\sigma} dW_t] ; \ \pi \in \Pi \right\},
\]

setting \( \tilde{\lambda}^\top := (\tilde{\sigma} \sigma^\top)^{-1} \tilde{\sigma} \lambda^\top \) we get

\[
u^\pi(\mu^\top, \sigma^\top) = \sup \left\{ \mathbb{E} \left[ U( x + \int_0^T \pi_t^\top [\lambda^\top dt + dW_t]) \right] ; \ \tilde{\pi} = [\sigma^\top]^\top \pi \text{ for some } \pi \in \Pi \right\}
= \sup \left\{ \mathbb{E} \left[ U( x + \int_0^T \pi_t^\top [\tilde{\sigma} \lambda^\top dt + \tilde{\sigma} dW_t]) \right] ; \ \pi \in \Pi \right\}
= \sup \left\{ \mathbb{E} \left[ U( x + \int_0^T \pi_t^\top [d(\tilde{\lambda}^\top + \tilde{\sigma} M) + dM_t]) \right] ; \ \pi \in \Pi \right\} =: \tilde{u}(\tilde{\lambda}^\top),
\]

We now provide the postponed proof of Proposition 2.1, which uses (H).

Proof of Proposition 2.1 Define \( B_t := W_t - \int_0^t [\lambda^\top - \tilde{\lambda}_s] ds \), so by Girsanov Theorem \( B \) is \( \mathbb{P}^\pi \)-Brownian motion. Notice that \( \mathcal{F}^B = \mathcal{F}^W \). Taking \( \pi \) feasible for the perturbed problem we have

\[
\begin{align*}
\mathbb{E}^{\mathbb{P}^\pi} \left[ U \left( x + \int_0^T \pi_t^\top (W)^\top \sigma^\top_l [\lambda^\top dt + dW_t] \right) \right] \\
= \mathbb{E}^{\mathbb{P}^\pi} \left[ U \left( x + \int_0^T \pi_t^\top (B)^\top \sigma^\top_l [\lambda^\top dt + dB_t] \right) \right] \\
= \mathbb{E}^{\mathbb{P}^\pi} \left[ U \left( x + \int_0^T \pi_t^\top (B)^\top \sigma^\top_l [\tilde{\lambda}_s dt + dW_t] \right) \right] \\
= \mathbb{E}^{\mathbb{P}^\pi} \left[ U \left( x + \int_0^T \tilde{\pi}_t^\top (W)^\top \sigma^\top_l [\tilde{\lambda}_s dt + dW_t] \right) \right] \\
= \mathbb{E}^{\mathbb{P}^\pi} \left[ U \left( x + \int_0^T \tilde{\pi}_t^\top \tilde{\lambda}_s [\tilde{\lambda}_s dt + dW_t] \right) \right] \\
\leq u^w(\mu^\top, \sigma^\top),
\end{align*}
\]

where we first used that \( B \) is \( \mathbb{P}^\pi \)-BM, then the definition of \( B \), then we built \( \tilde{\pi} \) by equality of filtrations, and finally the assumption (H) on the image of the matrices \( \tilde{\sigma} \) and \( [\sigma^\top]^\top \). Having begun with a feasible element for the unperturbed problem and reasoning as above, yields the opposite inequality.
Remark 2.3 Note that if $\mu^\tau, \sigma^\tau, \bar{\mu}, \bar{\sigma}$ are random, then the previous proof does not work. Indeed, following the lines of the proof, we would have that $B_t := W_t - \int_0^t \lambda^\tau(W) - \bar{\lambda}(W) \, ds$ is a $\mathbb{P}^\tau$-Brownian motion and so, following (2.14), we would get
\[
\mathbb{E}^{\mathbb{P}^\tau}\left[U\left(x + \int_0^T \pi_t(W)^\top \sigma_t(W) \lambda_t(W) \, dt + dW_t\right)\right] = \mathbb{E}^{\mathbb{P}^\tau}\left[U\left(x + \int_0^T \pi_t(B)^\top \sigma_t(B) \lambda_t(W) \, dt + dB_t\right)\right],
\]
whose right hand side generally differs from
\[
\mathbb{E}^{\mathbb{P}^\tau}\left[U\left(x + \int_0^T \pi_t(B)^\top \sigma_t(B) \bar{\lambda}(W) \, dt + dW_t\right)\right].
\]

Let us now proceed to the counterexamples promised before Proposition 2.1 and in the introduction.

2.2 Counterexamples

We illustrate how, even in the one-dimensional case, $u^t$ and $u^w$ (as well as their directional derivatives) generally defer. For this to be the case, it is important that the reference market price of risk $\bar{\lambda}$ be random.

Example 1 Let us take $U(x) = \log(x)$ if $x > 0$ and $U(x) = -\infty$ if $x \leq 0$. Although this utility function does not fulfil our assumption, we use it to illustrate the phenomenon we are discussing. It is well known (see e.g. [29, Chapter 7.3.5]) that for a market model $dS_t = \lambda d\langle M \rangle_t + dM_t$ for $M$ a martingale and $\lambda$ say essentially bounded, the optimal utility is $\log(x) + \frac{1}{2} \mathbb{E} \left[\int_0^T \lambda_i^\top d\langle M \rangle_i \lambda_i\right]$.

We thus conclude in our Brownian setting and for $\lambda^\tau = \bar{\lambda} + \tau \Delta$ that:
\[
u^t(\lambda^\tau) = \log(x) + \frac{1}{2} \mathbb{E} \left[\int_0^T |\bar{\lambda} + \tau \Delta|^2 dt\right],
\]
\[
= \log(x) + \frac{1}{2} \mathbb{E} \left[\int_0^T |\bar{\lambda}|^2 dt\right] + \tau \mathbb{E} \left[\int_0^T \lambda^\top \Delta dt\right] + \frac{\tau^2}{2} \mathbb{E} \left[\int_0^T |\Delta|^2 dt\right].
\]

On the other hand, denoting $d\mathbb{P}^\tau = \mathcal{E} (\tau \int \Delta dW) \, d\mathbb{P}$ so $W^\tau = W - \tau \int \Delta dt$ is a $\mathbb{P}^\tau$-Brownian motion by Girsanov’s theorem, and taking $\Delta$ deterministic so that the filtration $\mathcal{F}^{W^\tau}$ generated by $W^\tau$ coincides with $\mathcal{F}$, we get
\[
u^w(\lambda^\tau) = \log(x) + \frac{1}{2} \mathbb{E}^{\mathbb{P}^\tau} \left[\int_0^T |\bar{\lambda} + \tau \Delta|^2 dt\right],
\]
\[
= \log(x) + \frac{1}{2} \mathbb{E}^{\mathbb{P}^\tau} \left[\int_0^T |\bar{\lambda}|^2 dt\right] + \tau \mathbb{E}^{\mathbb{P}^\tau} \left[\int_0^T \lambda^\top \Delta dt\right] + \frac{\tau^2}{2} \mathbb{E}^{\mathbb{P}^\tau} \left[\int_0^T |\Delta|^2 dt\right].
\]
This already shows that the two value functions may easily differ, unless e.g. \( \tilde{\lambda} \) were further deterministic. Moreover, one can easily compute the first order sensitivities:

\[
\frac{du^s(\lambda^\tau)}{d\tau} \bigg|_{\tau=0} = \mathbb{E} \left[ \int_0^T \tilde{\lambda} \Delta d\tau \right],
\]

\[
\frac{du^w(\lambda^\tau)}{d\tau} \bigg|_{\tau=0} = \mathbb{E} \left[ \int_0^T \tilde{\lambda} \Delta d\tau \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T |\tilde{\lambda}|^2 d\tau \int_0^T \Delta dW \right],
\]

\[
= \mathbb{E} \left[ \int_0^T \tilde{\lambda} \Delta d\tau \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left\{ \int_0^t \Delta \epsilon dW_s \right\} |\tilde{\lambda}|^2 d\tau \right].
\]

We conclude that the sensitivities generally differ, unless again if e.g. \( \tilde{\lambda} \) was deterministic. To exemplify this point, the reader may take any bounded deterministic function \( \Delta \) and define \( \tilde{\lambda}(t, \omega) \) to be e.g. of euclidean norm 1 if \( \int_0^t \Delta \epsilon dW_s \) is positive and 0 otherwise.

This example also shows that the weak value function can behave in a counter-intuitive way in the presence of random parameters. For instance, taking \( \tilde{\lambda}_t := 1_{W_t<0} \) and \( \Delta \equiv 1 \) it is elementary to see that

\[
\frac{du^s(\tilde{\lambda} + \tau)}{d\tau} \bigg|_{\tau=0} = \frac{T}{2} \quad \text{and} \quad \frac{du^w(\tilde{\lambda} + \tau)}{d\tau} \bigg|_{\tau=0} = \frac{T}{2} - \frac{T^3/2}{3\sqrt{2\pi}},
\]

so as intuition suggest utility increases in the strong formulation whereas (for \( T \) large enough) it decreases in the weak one.

**Example 2** We now present an example that does fulfil our assumptions on the utility function. Let us take \( U(x) = 2\sqrt{x} \) if \( x \geq 0 \) and \( -\infty \) otherwise. We take \( x = 1 \) for simplicity. By e.g. [29, Chapter 7.3.5] we know, in the one-asset case, that the optimal utility for a market model \( dS = \lambda d\langle M \rangle + dM \) will be

\[
2\sqrt{\mathbb{E} \left[ \exp \left\{ \int_0^T \lambda d\langle M \rangle + \frac{1}{2} \int_0^T \lambda^2 d\langle M \rangle \right\} \right]}.
\]

Thus, in a one-dimensional Brownian setting and for \( \lambda^\tau = \tilde{\lambda} + \tau \Delta \) it holds:

\[
u^s(\lambda^\tau) = 2\sqrt{\mathbb{E} \left[ \exp \left\{ \int_0^T [\tilde{\lambda} + \tau \Delta] dW + \frac{1}{2} \int_0^T [\tilde{\lambda} + \tau \Delta]^2 d\tau \right\} \right]},
\]

and by Girsanov’s theorem and assuming \( \Delta \) deterministic:

\[
u^w(\lambda^\tau) = 2\sqrt{\mathbb{E} \left[ \exp \left\{ \int_0^T [\tilde{\lambda} + \tau \Delta] (dW - \tau d\tau) + \frac{1}{2} \int_0^T [\tilde{\lambda} + \tau \Delta]^2 d\tau \right\} \right]},
\]

where \( d\mathbb{P}^\tau = \mathcal{E} \left( \tau \int \Delta dW \right)_T d\mathbb{P} \). We thus obtain the following first order sensitivities:

\[
\frac{du^s(\lambda^\tau)}{d\tau} \bigg|_{\tau=0} = \mathbb{E} \left[ e^{\int_0^T \tilde{\lambda} dW + \frac{1}{2} \int_0^T \tilde{\lambda}^2 d\tau} \left| \int_0^T \Delta dW + \int_0^T \Delta d\tau \right| \right],
\]

\[
\frac{du^w(\lambda^\tau)}{d\tau} \bigg|_{\tau=0} = \mathbb{E} \left[ e^{\int_0^T \tilde{\lambda} dW + \frac{1}{2} \int_0^T \tilde{\lambda}^2 d\tau} \left\{ \int_0^T \Delta dW + \int_0^T \Delta d\tau \right\} \right].
\]
From this, we see that
\[ \frac{du^r(\lambda^T)}{d\tau} \bigg|_{\tau=0} = \frac{du^w(\lambda^T)}{d\tau} \bigg|_{\tau=0} \iff \mathbb{E} \left[ e^{\int_0^T \tilde{\lambda} dW + \frac{1}{2} \int_0^T \tilde{\lambda}^2 dt} \left( \int_0^T \Delta dW - \int_0^T \Delta \tilde{\lambda} dt \right) \right] = 0. \]
This shows that the sensitivities generally differ, unless if further e.g. \( \tilde{\lambda} \) is deterministic. To exemplify, with Girsanov theorem and the product formula, the expectation in the r.h.s above becomes
\[ \tilde{\mathbb{E}} \left[ \int_0^T \left( \int_0^t \Delta_s dW_s - \int_0^t \Delta_s \tilde{\lambda}_s ds \right) e^{\int_0^t \tilde{\lambda}_s^2 ds} \tilde{\lambda}_s^2 dt \right], \]
where \( \tilde{\mathbb{E}} \) denotes expectation under \( d\bar{\mathbb{P}} := \mathbb{E} \left( \int \tilde{\lambda} dW \right)_T d\mathbb{P}. \)

The reader may take any negative, bounded function \( \Delta \) and define \( \tilde{\lambda}(t, \omega) \) to be e.g. equal to 1 if \( \int_0^t \Delta_s dW_s \) is positive and 0 otherwise. Then \( \left( \int_0^t \Delta_s dW_s - \int_0^t \Delta_s \tilde{\lambda}_s ds \right) \tilde{\lambda}_s^2 \) is non-negative a.e. and can be seen to be strictly positive in a non-evanescent set. Thus the sensitivities differ in this case, and a fortiori also the value functions themselves.

3 The utility maximization problem as a support function of a weakly compact set

In this section we survey some of the results in [2], where the setting, similar to that of [16], is more general than ours as described so far. These results are crucial in the proof of Theorem 2.1.

Let there be \( d \) stocks and a bond, normalized to one for simplicity. Let \( S = (S^i)_{1 \leq i \leq d} \) be the price process of these stocks, and \( T < \infty \) a finite deterministic investment horizon. The process \( S \) is assumed to be a continuous semimartingale in a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})\), where \( \mathbb{P} \) will always stand for the reference measure. The expectation with respect to \( \mathbb{P} \) will be denoted by \( \mathbb{E} \) as before.

A (self-financing) portfolio \( \pi \) is defined as a couple \((X_0, H)\), where \( X_0 \geq 0 \) denotes the (constant) initial value associated to it and \( H = (H^i)_{i=1}^d \) is a predictable and \( d \)-integrable process which represents the number of shares of each type under possession. The wealth \( X = (X_t)_{t \leq T} \) associated to a portfolio \( \pi \) is defined as
\[ X_t = X_0 + \int_0^t H_u dS_u \quad \text{for all } t \in [0, T], \quad (3.1) \]
and the set of attainable wealths from \( x \) is defined as
\[ \mathcal{X}(x) = \{X \geq 0 \text{ a.s. in } \Omega \times [0, T] \mid X \text{as in (3.1) with } X_0 \leq x \}. \quad (3.2) \]

We assume in the sequel that the market is arbitrage-free, in the sense of NFLVR (see e.g. [10]), which implies that \( \mathcal{M}^\pi(S) \) (defined as in (2.10)) is not empty. As usual the market model is coined complete if \( \mathcal{M}^\pi(S) \) is reduced to a singleton, i.e. \( \mathcal{M}^\pi(S) = \{\mathbb{P}^\pi\} \), and incomplete otherwise. The following set, introduced in [16], plays a central role in portfolio optimization in incomplete markets
\[ \mathcal{Y}_\mathbb{P}(y) := \{Y \geq 0 | Y_0 = y, XY \text{ is } \mathbb{P} - \text{supermartingale } \forall X \in \mathcal{X}(1)\}. \]

The set \( \mathcal{Y}_\mathbb{P}(y) \) generalizes the set of density processes (with respect to \( \mathbb{P} \)) of risk neutral measures equivalent to it.

Now, we consider the following notion of utility function.
Definition 3.1 A function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is called a utility function if $U(x) = -\infty$ if $x \in (-\infty, 0)$ and on $[0, \infty)$ we have that $U$ is strictly increasing, strictly concave and continuously differentiable. We say that $U$ satisfies the INADA conditions ([11]) if

$$U'(0+) := \lim_{x \downarrow 0} U'(x) = \infty$$

and $U'(+\infty) = 0$.

As in e.g. [16], we will make use of the Fenchel conjugate of $-U(\cdot)$, namely:

$$V(y) := \sup_{x > 0} [U(x) - xy], \quad \forall \ y > 0.$$

Remark 3.1 The above assumption implies that $V \geq 0$ and the existence of and inverse $U^{-1} : (0, \infty) \to (0, \infty)$. Of course, by a translation argument we can assume that $U(0+) = 0$. In [2], on whose results we rely, it is assumed for simplicity that $U$ is unbounded from above, but this can be easily dispensed with from their work.

The usual way to dealing with the issue of existence of an element $\hat{X} \in \mathcal{X}(x)$ satisfying

$$\mathbb{E}[U(\hat{X}_T)] \geq \mathbb{E}[U(X_T)] \quad \text{for all } X \in \mathcal{X}(x),$$

uses crucially a result usually referred to as Komlos Theorem. This result states that, from a sequence of non-negative random variables which is bounded in probability, one can extract a subsequence of convex combinations convergent in probability to a finite limit. To apply this, one also needs growth conditions on $U$ and $U'$ (see e.g. [16] or [29, Theorem 7.3.4]). However, as a corollary of the analysis in [2] the authors show in [2, Proposition 5.22] that a shorter if more involved compactness argument can be applied; the same idea will allow us to prove the sensitivity results for $u^w$ in the next section.

The desired compactness property mentioned above holds in a suitably designed space. In order to motivate it, we start by observing that for $X \in \mathcal{X}(x)$:

$$\sup_{Y \in \mathcal{Y}} \mathbb{E}[YU^{-1} \circ U(X)] \leq x,$$

where $\mathcal{Y} := \mathcal{Y}_p(1)$, and we (now and often hereafter) write $Y$ for $Y_T$ and $X$ for $X_T$, as long as the context is unequivocal. We then see that setting

$$J(\cdot) := \sup_{Y \in \mathcal{Y}} \mathbb{E}[YU^{-1}(\cdot)] \tag{3.3}$$

for every $X \in \mathcal{X}(x)$ we have that $J(U(X)) \leq x$. We remark that (2.11) and (3.3) coincide by [16, Proposition 3.1], so notation is consistent. Therefore we may conjecture that if $J$ was connected to a norm (or say, grew stronger than it) and if the space defined by such a norm, which we shall soon call $L_J$, was a strong dual one, then we would get the weak* relative compactness of the set $\{U(X) : X \in \mathcal{X}(x)\}$ immediately from Banach-Aloulgu’s Theorem.

Let us now summarize the main topological results in [2, Section 5] for future reference. Consider $J$ as above and define $I : L^0 \to \mathbb{R} \cup \{+\infty\}$ as

$$I(Z) := \inf_{Y \in \mathcal{Y}} \mathbb{E}[|Z| \mathbb{V}(Y/|Z|)].$$

Lemma 3.1 Under Assumption (A1), the functions $I$ and $J$ are convex.
Proof See [2, Lemma 5.1].

We consider the spaces

\[ L_I := \{ Z \in L^0 : I(\alpha Z) < \infty \text{ for some } \alpha > 0 \}, \]
\[ E_I := \{ Z \in L^0 : I(\alpha Z) < \infty \text{ for every } \alpha > 0 \}, \]
\[ L_J := \{ Z \in L^0 : J(\alpha Z) < \infty \text{ for some } \alpha > 0 \}, \]
\[ E_J := \{ Z \in L^0 : J(\alpha Z) < \infty \text{ for every } \alpha > 0 \}, \]

and for \( F \) denoting \( I \) or \( J \), we set the equivalent norms (see [26, Theorem 1.10]):

\[ \| s \|_{F, \ell} := \inf \{ \beta > 0 : F(s/\beta) \leq 1 \} \quad \| s \|_{F, a} := \inf \left\{ \frac{1}{k} + \frac{E(k s)}{k} : k > 0 \right\}. \] (3.4)

**Lemma 3.2** Under Assumption (A1) and after identifying almost equal elements, for \( \gamma = \ell, a \) we have that \( (E_F, \| \cdot \|_{F, \gamma}), (L_F, \| \cdot \|_{F, \gamma}) \) are normed linear spaces. Moreover, \( E_F \) is a closed subspace of \( L_F \) and both \( E_I \) and \( L_J \) are Banach spaces.

Now, let us define \( \mathcal{Y}^\ast := \{ Y \in \mathcal{Y} : Y > 0 \text{ and } \forall \beta > 0, \mathbb{E}[V(\beta Y)] < \infty \} \) and suppose

\[ (A2) \quad \mathcal{Y}^\ast \neq \emptyset, \quad I(Z) = \inf_{Y \in \mathcal{Y}^\ast} \mathbb{E}[|Z|V(Y/Z)], \quad \text{and } J(X) = \sup_{Y \in \mathcal{Y}^\ast} \mathbb{E}[YU^{-1}(|X|)]. \]

**Remark 3.2** Condition (A2) is satisfied for instance if the price process \( S \) satisfies that \( dS = \lambda d(M) + dM \) for a continuous martingale \( M, \lambda \in L_{loc}^2(M) \), the market model is viable and \( \mathbb{E}[V(\beta \varepsilon(\lambda \cdot M)_T)] < \infty \) for all \( \beta > 0 \). See [2, Lemma 5.7] for a proof of this fact.

The next result, proved in [2, Proposition 5.10], establishes that \( L_J \) is a strong dual space.

**Theorem 3.1** Suppose that Assumptions (A1)–(A2) hold true. Then, the dual of \( (E_I, \| \cdot \|_{I, a}) \) is isometrically isomorphic to \( (L_J, \| \cdot \|_{J, \ell}) \).

To wrap up, and in light of the expression (2.12) for \( u^w \), we have given in this section conditions under which this weakly perturbed value function can indeed be viewed as a support function of a weakly compact set, namely \( \{ Z : J(Z) \leq x \} \). We proceed in the next section to take advantage of this fact, in the context outlined in Sect. 2, in order to perform the sensitivity analysis of our problem under weak perturbations.

**Remark 3.3** In the case of a complete market \( L_J \) and \( L_I \) are Orlicz spaces. In the general case they are modular spaces, introduced by Nakano (see [26,27]), which are generalizations of Orlicz spaces. By e.g. Hölder inequality for modular spaces (see [2, Proposition 5.9]) we have that \( u^w \), given by (2.12), is finite. Moreover, under our assumptions, [2, Proposition 5.22] shows that the supremum therein is attained. Finally, it is easy to see that this optimizer is unique, as it must lie in the image set of \( U \), which is a strictly concave function.

## 4 Stability and sensitivity

Let us go back to the weakly perturbed problem, defined in (2.12), and consider a fixed nominal parameter \( \bar{\lambda} = \bar{\sigma}^T(\bar{\sigma} \bar{\sigma}^T)^{-1}\bar{\mu} \in (L_{\infty}^\infty)^n \). In order to proceed, we initially suppose that

\[ U(x) = \begin{cases} px^{1/p} & \text{if } x \geq 0, \\ -\infty & \text{otherwise}, \end{cases} \] (4.1)
for some $p \in (1, +\infty)$. Notice that, in the more general context of the previous section, we clearly have that (4.1) implies (A1) and, thanks to Remark 3.2, assumption (A2) also holds true.

We denote $q := p/(p - 1) \in (1, +\infty)$ the conjugate exponent of $p$. Using (4.1) we have that

$$ L_I = \left\{ Z \in L^0 : \inf_{v \in K(\tilde{\sigma})} \mathbb{E} \left[ \mathcal{E} \left( -\int_{\tilde{\lambda}} + v \right)^T dW \right]^{1-q} |Z|^q \right\} < \infty, $$

where

$$ K(\tilde{\sigma}) := \left\{ v \in L^2_{loc}(W) : v(t, \omega) \in \text{Ker}(\tilde{\sigma}(t, \omega)) \text{ a.e.} \right\}, $$

as easily follows from [20, Proposition 3.2] and the fact that we are working on the Brownian filtration. In this context, we have that $L_I = E_I$ and, for some constant $C(p) \in \mathbb{R}_{++}$,

$$ \|Z\|_I := \|Z\|_{I,t} = C(p) \left( \inf_{v \in K(\tilde{\sigma})} \mathbb{E} \left[ \mathcal{E} \left( -\int_{\tilde{\lambda}} + v \right)^T dW \right]^{1-q} |Z|^q \right)^{\frac{1}{q}}. \quad (4.2) $$

Analogously,

$$ L_J = \left\{ X \in L^0 : \sup_{v \in K(\tilde{\sigma})} \mathbb{E} \left[ \mathcal{E} \left( -\int_{\tilde{\lambda}} + v \right)^T dW \right] |X|^p \right\} < \infty, $$

we have that $L_J = E_J$ and there exists a constant $c(p) \in \mathbb{R}_{++}$ such that

$$ \|X\|_J := \|X\|_{J,a} = c(p) \left( \sup_{v \in K(\tilde{\sigma})} \mathbb{E} \left[ \mathcal{E} \left( -\int_{\tilde{\lambda}} + v \right)^T dW \right] |X|^p \right)^{\frac{1}{p}}. $$

Since $c(p)$ and $C(p)$ play no role here, we shall ignore them. For a perturbation $\lambda^T$ of $\tilde{\lambda}$, we can rephrase problem (2.12) as:

$$ u^w(\lambda^T) = \sup \left\{ \langle g(\lambda^T), Z \rangle_{1,J} : Z \in \mathcal{K} \right\}, \quad (4.3) $$

where

$$ \mathcal{K} := \left\{ Z \in L^0_+ : J(Z) \leq x \right\} \subseteq L_J, \quad \text{and} \quad g(\lambda) := \mathcal{E} \left( \int [\lambda - \tilde{\lambda}]^T dW \right)_T \quad \forall \lambda \in (L^\infty_\mathbb{F})^n, \quad (4.4) $$

and $\langle \cdot, \cdot \rangle_{1,J}$ denotes the duality pairing between $L_I$ and $L_J$. Lemma 4.1(ii) below implies that the function $g : (L^\infty_\mathbb{F})^n \to L_I$ is well-defined. Theorem 3.1 and the Banach–Alaoglu theorem imply that $\mathcal{K}$ is a convex weak* compact subset of $L_J$. This compactness property plays an important role in the proof of differentiability of $u^w$, which is based on the fact that $u^w(\lambda^T)$ is the support function of $\mathcal{K}$ evaluated at $g(\lambda^T)$, and a suitable version of Danskin’s theorem (see Lemma A.1 in the “Appendix”).

We state now a simple lemma that we shall often invoke:

**Lemma 4.1** The following assertions hold true:

(i) Let $\rho \geq 2$, $A \in (L^\infty_\mathbb{F})^n$, $B$ progressive, $n$-dimensional, such that $\mathbb{E} \left[ \int_0^T |B_t|^\rho dt \right] < \infty$.
and $Z$ defined as the real-valued process solving $dZ = (ZA + B)\mathbb{T}dW$. Then, there exists a constant $c = c(\rho, T) > 0$ such that

$$\mathbb{E} \left[ \sup_{s \in [0,T]} |Z_s|^p \right] \leq c \left[ |Z_0|^p + \mathbb{E} \left[ \int_0^T |B_t|^p dt \right] \right] \exp \left\{ cT\|A\|_{\infty, \infty}^\rho \right\}.$$ 

(ii) For every $\Gamma \in [L^\infty_\mathbb{F}]^n$ we have $\mathcal{E} \left( \int \Gamma^\mathbb{T}dW \right)_{T} \in L_1$.

**Proof** The proof of the first assertion is a standard application of Gronwall’s Lemma (see e.g. [35, Chapter 6, Section 4]). For the second point, using that $\bar{\lambda}$ and $\Gamma$ are essentially bounded, we observe that

$$\alpha := \mathbb{E} \left[ \mathcal{E} \left( \int \Gamma^\mathbb{T}dW \right)^q_T \exp \left\{ \int_0^T (q - 1)\bar{\lambda}^\mathbb{T}dW + \frac{q - 1}{2} \int_0^T |\bar{\lambda}|^2 dt \right\} \right],$$

satisfies

$$\alpha \leq c\mathbb{E} \left[ \mathcal{E} \left( \int (q\Gamma + (q - 1)\bar{\lambda})^\mathbb{T}dW \right)_T \right] = c,$$

for some constant $c > 0$. Since $\alpha$ dominates $\|\mathcal{E} \left( \int \Gamma^\mathbb{T}dW \right)_T \|^q_2$, the result follows.

We prove now the Fréchet differentiability of $g$, defined in (4.4):

**Lemma 4.2** The map $g$ is locally Lipschitz and Fréchet differentiable. Moreover, for all $\Delta \lambda \in (L^\infty_\mathbb{F})^n$ we have that

$$Dg(\lambda)\Delta \lambda = \mathcal{E} \left( \int [\lambda - \bar{\lambda}]^\mathbb{T}dW \right)_T \left\{ \int_0^T \Delta \lambda_i^\mathbb{T}dW_t - \int_0^T (\lambda_i - \bar{\lambda}_i) \cdot \Delta \lambda_\ell dt \right\}.$$ 

**Proof** Let $\lambda_1, \lambda_2 \in (L^\infty_\mathbb{F})^n$. We have that, omitting the dependence on $t$ and denoting by $\| \cdot \|_2$ the $L^2$-norm with respect to $\mathbb{P}$,

$$\| g(\lambda_1) - g(\lambda_2) \|^q_2 \leq \| e^{q} - 1 \| \int_0^T (q - 1)\bar{\lambda}^\mathbb{T}dW + \frac{q - 1}{2} \int_0^T |\bar{\lambda}|^2 dt \|_2 \| g(\lambda_1) - g(\lambda_2) \|_2.$$ 

(4.6) Note that $\Delta g := g(\lambda_1) - g(\lambda_2)$ solves

$$d\Delta g = \left[ \Delta g(\lambda_1 - \bar{\lambda}) + g(\lambda_2)(\lambda_1 - \lambda_2) \right]^\mathbb{T}dW_t, \quad t \in [0, T], \quad \Delta g_0 = 0,$$

and so the local Lipschitz property follows from Lemma 4.1 and (4.6). Let us prove that $g$ is Gâteaux differentiable. Take $\lambda$ and call $\bar{\lambda} = \lambda - \bar{\lambda}$ and $\lambda^\varepsilon := \bar{\lambda} + \varepsilon \Delta \lambda$. We see that

$$\mathcal{E} \left( \int [\lambda^\varepsilon]^\mathbb{T}dW \right) = \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right) \exp \left\{ \varepsilon \int \Delta \lambda^\mathbb{T}dW - \varepsilon \int \Delta \lambda \cdot \bar{\lambda} dt - \frac{\varepsilon^2}{2} \int |\Delta \lambda|^2 dt \right\}.$$

Using that $e^x = 1 + x + x \int_0^1 [e^{tx} - 1]da$ and calling $x_\varepsilon$ the term inside $\exp\{\ldots\}$ in the expression above, we obtain

$$\frac{\mathcal{E} \left( \int [\lambda^\varepsilon]^\mathbb{T}dW \right) - \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right)}{\varepsilon} = \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right) \left[ \int \Delta \lambda^\mathbb{T}dW - \int \Delta \lambda \cdot \bar{\lambda} dt - \frac{\varepsilon}{2} \int |\Delta \lambda|^2 dt \right]$$

$$+ \varepsilon^{-1} x_\varepsilon \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right) \int_0^1 [e^{t\varepsilon x_\varepsilon} - 1]da.$$ 

In order to show (4.5), it suffices to prove that $\| \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right) \int |\Delta \lambda|^2 dt \|_2 < \infty$ and

$$\varepsilon^{-1} \left\| x_\varepsilon \mathcal{E} \left( \int \bar{\lambda}^\mathbb{T}dW \right) \int_0^1 [e^{t\varepsilon x_\varepsilon} - 1]da \right\|_L \to 0 \quad \text{as} \varepsilon \to 0.$$
The first claim is trivial, as $\Delta \lambda \in (L^\infty_{\mathbb{P}})^n$ and $g(\lambda) \in L_1$. For the second one, letting $v = 0$ in (4.2), it suffices to estimate

$$
\mathbb{E} \left[ e^{f(q-1)\tilde{\lambda}^T dW + \frac{q^2}{2} \int |\lambda|^2 d\tau} \mathcal{E} \left( \int \tilde{\Lambda}^T dW \right)^q \left( \frac{\lambda}{\epsilon} \right)^q \left( \int_0^1 [e^{a_{\lambda x} - 1}] d\tau \right)^q \right],
$$

which we may bound from above by the product of

$$
\sqrt{\mathbb{E} \left[ \mathcal{E} \left( \int \tilde{\Lambda}^T dW \right)^{2q} \left[ \int \Delta \lambda^T dW - \int \Delta \lambda \cdot \tilde{\lambda} d\tau - \frac{\epsilon}{2} \int |\Delta \lambda|^2 d\tau \right]^{2q} \right]},
$$

and

$$
\sqrt{\mathbb{E} \left[ e^{2(q-1)\int \tilde{\Lambda}^T dW + q(q-1) \int |\tilde{\lambda}|^2 d\tau} \left( \int_0^1 [e^{a_{\lambda x} - 1}] d\tau \right)^{2q} \right]},
$$

Using the Cauchy–Schwarz and the Burkholder–Davis–Gundy (BDG) inequalities we have that the first term is finite. As for the second one, in order to prove that it converges to zero it suffices to show that $\mathbb{E} \left[ \int_0^1 |e^{a_{\lambda x} - 1}|^{4q} d\tau \right] \to 0$. The term within the integral converges a.e. to zero as $\epsilon \to 0$. On the other hand, for some $c > 0$,

$$
|e^{a_{\lambda x} - 1}|^{4q} \leq c \left[ 1 + e^{4q} \int \tilde{\lambda} \cdot \Delta \lambda d\tau + 4q e \int \Delta \lambda^T dW \right],
$$

and $e^{4q} \int \Delta \lambda^T dW \leq e^{4q} \int \Delta \lambda^T dW + 1$, which is integrable. Thus, by dominated convergence, we have that (4.5) holds true.

In order to prove Fréchet differentiability it suffices to show the continuity of the application $\lambda \in (L^\infty_{\mathbb{P}})^n \mapsto Dg(\lambda)(\cdot) \in \mathcal{L}((L^\infty_{\mathbb{P}})^n, L_1)$, where $\mathcal{L}((L^\infty_{\mathbb{P}})^n, L_1)$ denotes the space of linear bounded operators from $(L^\infty_{\mathbb{P}})^n$ to $L_1$. Let $\Gamma, \lambda \in (L^\infty_{\mathbb{P}})^n$ and $\Delta \lambda \in (L^\infty_{\mathbb{P}})^n$ such that $\|\Delta \lambda\|_{\mathcal{L}^\infty, \mathcal{L}^1} \leq 1$. The triangle inequality yields

$$
\|Dg(\lambda) - Dg(\Gamma)\|_I \leq \left\| \mathcal{E} \left( \int (\lambda - \tilde{\lambda})^T dW \right) - \mathcal{E} \left( \int (\Gamma - \tilde{\lambda})^T dW \right) \right\|_I \\
+ \left\| \mathcal{E} \left( \int (\lambda - \tilde{\lambda})^T dW \right) \int \Delta \lambda \cdot (\lambda - \Gamma) d\tau \right\|_I \\
+ \left\| \mathcal{E} \left( \int (\lambda - \tilde{\lambda})^T dW \right) - \mathcal{E} \left( \int (\Gamma - \tilde{\lambda})^T dW \right) \right\|_I \int \Delta \lambda \cdot (\Gamma - \tilde{\lambda}) d\tau \\
.$$  

(4.7)

Up to taking $q$-root, the first and the third r.h.s terms can be bounded above, through repeated Cauchy-Schwarz, by

$$
\sqrt{\mathbb{E} \left[ e^{4(q-1)\int \tilde{\lambda}^T dW + (2q-1) \int |\tilde{\lambda}|^2 d\tau} \right] 4 \mathbb{E} \left[ \left| \int \Delta \lambda^T dW \right|^{4q} \right] 4 \mathbb{E} \left[ |Z_{\Gamma} - Z_{\lambda}|^{4q} \right]},
$$

and

$$
\sqrt{\mathbb{E} \left[ e^{4(q-1)\int \tilde{\lambda}^T dW + (2q-1) \int |\tilde{\lambda}|^2 d\tau} \right] 4 \mathbb{E} \left[ \left| \int \Delta \lambda \cdot [\Gamma - \tilde{\lambda}] d\tau \right|^{4q} \right] 4 \mathbb{E} \left[ |Z_{\Gamma} - Z_{\lambda}|^{4q} \right]},
$$

where $Z_{\Gamma} := \mathcal{E} \left( \int (\Gamma - \tilde{\lambda})^T dW \right)_T$ and $Z_{\lambda} := \mathcal{E} \left( \int (\lambda - \tilde{\lambda})^T dW \right)_T$. As in the proof of the local Lipschitzianity of $g$, we get that the last term in both expressions above tends to zero.
Therefore, the BDG inequality implies that the first and third terms in (4.7) tend to zero uniformly w.r.t. $\Delta \lambda$ satisfying that $\|\Delta \lambda\|_{\infty, \infty} \leq 1$.

Finally,
\[ \left\| E \left( \int (\lambda - \tilde{\lambda})^T dW \right) \int \Delta \lambda \cdot (\lambda - \Gamma) dt \right\|_{L} \leq T \|\lambda - \Gamma\|_{\infty, \infty} \left\| E \left( \int (\lambda - \tilde{\lambda})^T dW \right) \right\|_{L}. \]

The result follows.

Using the above fact we prove the stability (continuity) and the Hadamard differentiability of
\[ \lambda \in (L^\infty_{\mathbb{F}}) \mapsto u^w(\lambda) = \sup \left\{ \langle g(\lambda), Z \rangle_{L, I} \mid Z \in \mathcal{K} \right\} \in \mathbb{R}, \quad (4.8) \]
as a function of the market price of risk $\lambda$ assuming that $U$ is given by (4.1). The reader is referred to the “Appendix” for the definition of Hadamard directionally differentiable maps. Some parts of the following proof are independent of the choice of utility function, pointing out that we may in the future extend our approach:

**Proposition 4.1** Suppose that $U$ is given (4.1). Then, the function $u^w$ defined in (4.8) is continuous, Gâteaux and Hadamard directionally differentiable. Denoting by $X[\lambda]_{T}$ the optimal final wealth associated to $u^w(\lambda)$, which is unique, for all $\Delta \lambda \in (L^\infty_{\mathbb{F}}) \mapsto u^w(\lambda)$ the directional derivative is given by
\[ Du^w(\lambda) \Delta \lambda = E \left[ E \left( \int [\lambda - \tilde{\lambda}]^T dW \right) \right] U(X[\lambda]_{T}) \left\{ \int_0^T \Delta \lambda^T dW - \int_0^T (\lambda - \tilde{\lambda}) \cdot \Delta \lambda dt \right\}. \quad (4.9) \]

**Proof** Define $L_{I} \ni Y \mapsto F(Y) := \sup \left\{ E[YZ] : Z \in \mathcal{K} \right\} \in \mathbb{R}$, so that $u^w = F \circ g$. Lemma A.1(ii) in the “Appendix” implies that $F$ is Hadamard directionally differentiable. So Lemma 4.2 and the chain rule in [28, Theorem 2.28] imply that $u^w$ is Hadamard directionally differentiable.

Its directional derivative is given by
\[ Du^w(\lambda) \Delta \lambda = E \left[ Z(\lambda) E \left( \int [\lambda - \tilde{\lambda}]^T dW \right) \right] \left\{ \int_0^T \Delta \lambda^T dW - \int_0^T (\lambda - \tilde{\lambda}) \cdot \Delta \lambda dt \right\}, \]
with $Z(\lambda) = U(X[\lambda]_{T})$. Using Hölder’s inequality in [2, Proposition 5.9] we bound
\[ |Du^w(\lambda) \Delta \lambda| \leq \|Z(\lambda)\|_1 \left\| E \left( \int [\lambda - \tilde{\lambda}]^T dW \right) \right\| \left\{ \int_0^T \Delta \lambda^T dW - \int_0^T (\lambda - \tilde{\lambda}) \cdot \Delta \lambda dt \right\}. \]

Taking $k = 1$ in (3.4) and using that $J(Z(\lambda)) \leq x$, we obtain that $\|Z(\lambda)\|_1 \leq 1 + x$. The second term in the expression above is uniformly bounded whenever $\Delta \lambda$ is taken in a bounded set (as in the proof in Lemma 4.2). Thus, $Du^w(\lambda)(\cdot)$ is linear and continuous and so $u^w$ is Gâteaux differentiable.

We now we have all the elements to prove our main result Theorem 2.1:

**Proof of Theorem 2.1** We let $\bar{U}(x) = Cx^{1/p}$ and $\bar{V}$ its conjugate. Then for some other constant $c$ we have $V(y) \leq \bar{V}(y) = cy^{1/(1-p)}$ and so $zV(y/z) \leq cy^{1/(1-p)}z^{p/(p-1)}$. Writing $L_{I}$ for the modular space associated with $zV(y/z)$ and $L_{I}$ for the one associated with $cy^{1/(1-p)}z^{p/(p-1)}$ (as it has been described throughout most of this section) we conclude that
\[ L_I \subset L_I \] with continuous injection. Let \( \iota : L_I \to L_I \) be the identity map, which is then linear continuous and thus Fréchet differentiable with \( D\iota = \iota \). In particular \( G : (L^\infty_F)^n \to L_I \) given by \( G(\lambda) := \mathcal{E} \left( \int [\lambda - \tilde{\lambda}]^T dW \right)_F \) is well defined, and we have \( G = \iota \circ g \) with \( g \) as before. By Lemma 4.2 we conclude that \( G \) is locally Lipschitz and Fréchet differentiable with the same derivative as in (4.5). One can then repeat the proof of Proposition 4.1 to conclude.

We now provide a one-sided second order bound for the first order approximation error.

**Proposition 4.2** For any \( \delta > 0 \) there exists \( C(\delta) \geq 0 \) such that for all \(|\epsilon| \leq \delta\)

\[ u^w(\tilde{\lambda} + \epsilon \Delta \lambda) - u^w(\tilde{\lambda}) - \epsilon Du^w(\tilde{\lambda}) \Delta \lambda \geq -C(\delta)\epsilon^2. \]  

**(Proof)** Denoting \( \tilde{Z} \) the optimizer for \( u^w(\tilde{\lambda}) \), we have by Hölder’s inequality

\[ u^w(\tilde{\lambda} + \epsilon \Delta \lambda) - u^w(\tilde{\lambda}) - \epsilon Du^w(\tilde{\lambda}) \Delta \lambda \geq \mathbb{E} \left[ \tilde{Z} \left\{ g(\tilde{\lambda} + \epsilon \Delta \lambda) - 1 - \epsilon \int_0^T \Delta \lambda^T dW \right\} \right] \]

\[ \geq -\| \tilde{Z} \|_F \left\| g(\tilde{\lambda} + \epsilon \Delta \lambda) - 1 - \epsilon \int_0^T \Delta \lambda^T dW \right\|_I. \]

Defining \( Y_t := g(\tilde{\lambda} + \epsilon \Delta \lambda)_t - 1 - \epsilon \int_0^t \Delta \lambda^T dW \), we argue as in the proof of Lemma 4.2 that

\[ dY_t = \left[ Y_t \epsilon \Delta \lambda_t^T + \epsilon^2 \Delta \lambda_t \int_0^t \Delta \lambda_s^T dW_s \right] dW_t, \]

so by the SDE estimate in Lemma 4.1, we find

\[ \mathbb{E} \left[ Y_T^q \right] \leq ce^{k\epsilon \|\Delta \lambda\|_F \epsilon \epsilon} \mathbb{E} \left[ \int_0^T \left\{ \epsilon^2 \|\Delta \lambda_t\|_F \left( \int_0^t \Delta \lambda_s^T dW_s \right)^q \right\} dt \right], \]

thus \( \|Y_T\|_q^q \leq \tilde{C}(\delta) \epsilon^2 q \) and we conclude.

**Remark 4.1** It seems that a full second-order expansion or better, a sensitivity analysis of the optimal wealth, is beyond what we can reach by only looking at the primal problem. See [19] for such results via the duality method and for strong perturbations in the negative-power utility case. Such questions are out of the scope of the present paper, but it could be interesting to take advantage of the duality theory developed in [2] for the functional framework in Sect. 3 in order to tackle those problems.

We can now prove Corollary 2.1. We underline the fact that in this result assumption (H) is crucial for a meaningful analysis (see Sect. 2.1).

**Proof of Corollary 2.1** By (H) we have that \( (\tilde{\sigma} \tilde{\sigma}^T)^{-1} \) is essentially bounded. Thus, defining

\[ \mathcal{P} := \left\{ (\mu, \sigma) \in (L^\infty_F)^d \times (L^\infty_F)^d \times \mathbb{R}^{d \times n} : \sigma \sigma^T \text{ is a.e. invertible and} \right. \]

\[ \left. \text{ess sup}_{t, \omega} |(\sigma \sigma^T)^{-1} | < \infty \right\}, \]  

the mapping

\[ (\mu, \sigma) \in \mathcal{P} \mapsto \lambda(\mu, \sigma) := \sigma^T (\sigma \sigma^T)^{-1} \mu, \]
is Fréchet differentiable at \((\tilde{\mu}, \tilde{\sigma})\) and its directional derivative is given by

\[
D\lambda(\mu, \sigma)(\Delta\mu, \Delta\sigma) = \sigma^T [\sigma \sigma^T]^{-1} \Delta\mu + \Delta\sigma^T [\sigma \sigma^T]^{-1} \mu \\
- \sigma^T [\sigma \sigma^T]^{-1} \left\{ \sigma \Delta\sigma^T + \Delta\sigma \sigma^T \right\} [\sigma \sigma^T]^{-1} \mu.
\]

The result easily follows from Proposition 4.1 and the chain rule in [28, Theorem 2.28].

To conclude this section, we show how the results in Theorem 2.1 extend to the case of non trivial interest rate. More precisely, suppose now that the market comprises the previous \(d\) risky assets \(S^1, \ldots, S^d\) and also a riskless asset \(S^0\), satisfying that \(dS^0_t = \bar{r}S^0_t dt, S^0_0 = s^0_0 \in \mathbb{R}_{++}\), with \(r \in L_{\mathbb{F}}^{\infty, \infty}\). In this case the wealth process satisfies the SDE

\[
dX^\pi_T = \left[ r(t) X^\pi_T + \pi^T_t (\mu_t - \bar{r} \mathbf{1}) \right] dt + \pi^T_t \sigma_t dW_t, \quad t \in [0, T],
\]

where \(\mathbf{1}\) denotes the vector of ones in \(\mathbb{R}^d\). Let us fix \((\bar{r}, \tilde{\mu}, \tilde{\sigma}) \in L_{\mathbb{F}}^{\infty, \infty} \times \mathcal{P}\) (with \(\mathcal{P}\) given by (4.11)) and for any \((\bar{r}^T, \mu^T, \sigma^T) \in L_{\mathbb{F}}^{\infty, \infty} \times \mathcal{P}\) denote by \(u^s(\bar{r}^T, \mu^T, \sigma^T)\) the value of the strongly perturbed problem. Then, by a simple change of variable, for a \(p\)-power utility function \((p \in (1, \infty))\) we find that

\[
u^s(\bar{r}^T, \mu^T, \sigma^T) = \sup_{\pi \in \Pi} \mathbb{E}_\mathbb{P}^{\mathbb{F}} \left( \frac{1}{p} \int_0^T \bar{r}_t^T dU \left( \hat{X}^\pi_T \right) \right),
\]

where \(\hat{X}^\pi_T\) solves

\[
d\hat{X}^\pi_T = \pi^T_t (\mu_t^T - \bar{r}_t \mathbf{1}) dt + \pi^T_t \sigma_t^T dW_t, \quad t \in [0, T],
\]

\[
\hat{X}^\pi_0 = x.
\]

Assuming that \(\tilde{\sigma}\) and \(\sigma^T\) satisfy (H), we then define the weakly perturbed value function as

\[
u^w(\bar{r}^T, \mu^T, \sigma^T) = \sup_{\pi \in \Pi} \mathbb{E}_\mathbb{P}^{\mathbb{F}} \left( \frac{1}{p} \int_0^T \bar{r}_t^T dU \left( \hat{X}^\pi_T \right) \right),
\]

with \(\hat{X}^\pi_T\) solving

\[
d\hat{X}^\pi_T = \pi^T_t (\tilde{\mu}_t - \bar{r}_t \mathbf{1}) dt + \pi^T_t \tilde{\sigma}_t dW_t, \quad t \in [0, T],
\]

\[
\hat{X}^\pi_0 = x,
\]

and \(d\mathbb{P}^{\mathbb{F}} = \mathbb{E} \left[ \int (\lambda^r_T - \bar{\lambda}_r) dW \right]_T d\mathbb{P}\), with

\[
\lambda^r_T := (\sigma^T)^T [\sigma^T (\sigma^T)^T]^{-1} (\mu^T - \bar{r}^T \mathbf{1}), \quad \text{and} \quad \bar{\lambda}_r := \tilde{\sigma}^T [\tilde{\sigma} \tilde{\sigma}^T]^{-1} (\tilde{\mu} - \bar{r} \mathbf{1}).
\]

Thus, arguing exactly as before we obtain the following sensitivities; for every \((\Delta r, \Delta \mu, \Delta \sigma) \in L_{\mathbb{F}}^{\infty, \infty} \times (L_{\mathbb{F}}^{\infty, \infty})^d \times (L_{\mathbb{F}}^{\infty, \infty})^{d \times n}\) such that \(\sigma^T := \tilde{\sigma} + \tau \Delta \sigma\) satisfies (H) for \(\tau > 0\) small enough, we have

\[
\heartsuit \text{ Springer}
\]
\[ D_t u^w(\bar{\mu}, \bar{\sigma}, \Delta r) = \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T r_t dt} U(\hat{X}_T^\Delta) \left\{ \frac{1}{T} \int_0^T \Delta r_t dt - \int_0^T [\tilde{\sigma}^T \tilde{\sigma}^{-1} \Delta r^T] \mathbb{d} W \right\} \right], \]

\[ D_\mu u^w(\bar{\mu}, \bar{\sigma}, \Delta \mu) = \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T r_t dt} U(\hat{X}_T^\Delta) \int_0^T [\tilde{\sigma}^T \tilde{\sigma}^{-1} \Delta \mu^T] \mathbb{d} W \right], \]

\[ D_\sigma u^w(\bar{\mu}, \bar{\sigma}, \Delta \sigma) = \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T r_t dt} U(\hat{X}_T^\Delta) \int_0^T [\tilde{\sigma}^T \tilde{\sigma}^{-1} \Delta \sigma^T + \Delta \sigma \tilde{\sigma}^{-1} \Delta \mu] \mathbb{d} W \right] \]

- \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T r_t dt} U(\hat{X}_T^\Delta) \int_0^T [\tilde{\sigma}^T \tilde{\sigma}^{-1} \Delta \sigma^T + \Delta \sigma \tilde{\sigma}^{-1} (\tilde{\mu} - \bar{\mu} 1)^T] \mathbb{d} W \right].

5 A final discussion

As we have seen in Sect. 2.2 the sensitivities in the weak and strong formulations may differ. Proposition 2.1 and Remark 2.3, on the other hand, give a hint as to why this happens. We close the article by providing an expression, which we derive heuristically, connecting the sensitivities of the weakly and strongly perturbed problems. For simplicity, we restrict the analysis to varying market prices of risk only (and fixed volatilities, so only the drift is being perturbed). We work in canonical continuous-paths space.

Let us denote \( \theta^\epsilon (\omega) = \omega + \epsilon \int \delta \lambda ds \) a shift in canonical space and \( X^* \) the optimal wealth (\( \pi^* \) the optimal portfolio) under reference parameters. Then

\[
\mathbb{E} \left[ U(X^*(T) \circ \theta^\epsilon) \right] - \mathbb{E} \left[ U(X^*(T)) \right] \\
= \mathbb{E} \left[ U \left( x + \int_0^T [\pi^* \cdot \tilde{\lambda}] \circ \theta^\epsilon ds + \int_0^T \pi^* \circ \theta^\epsilon \cdot dW + \epsilon \int_0^T [\pi^* \circ \theta^\epsilon] \cdot \delta \lambda ds \right) \right] \\
- \mathbb{E} \left[ U(X^*(T)) \right] .
\]

From this we conclude that, if the corresponding directional derivatives in path-space are well-defined,

\[
\frac{d}{d\epsilon} \mathbb{E} \left[ U(X^*(T) \circ \theta^\epsilon) \right] \bigg|_{\epsilon=0} = \mathbb{E} \left[ U'(X^*(T)) \left\{ \int_0^T D[\pi^*_s \cdot \tilde{\lambda}_s](\omega, \delta \lambda) ds + \int_0^T D\pi^*_s(\omega, \delta \lambda) dW_s + \int_0^T \pi^* \cdot \delta \lambda ds \right\} \right].
\]

Now, by Bismut’s integration by parts formula (see e.g. [33, Chapter IV, Section 41] and the assumptions therein), under given conditions this implies:

\[
\mathbb{E} \left[ U(X^*(T)) \int_0^T \delta \lambda^T dW \right] \\
= \mathbb{E} \left[ U'(X^*(T)) \left\{ \int_0^T D[\pi^*_s \cdot \tilde{\lambda}_s](\omega, \delta \lambda) ds + \int_0^T D\pi^*_s(\omega, \delta \lambda) dW_s + \int_0^T \pi^* \cdot \delta \lambda ds \right\} \right].
\] (5.1)

We can reasonably conjecture, if anything like the “envelope” or “Danskin Theorem” is to hold for it, as well as a directional chain rule, that

\[
Du^w(\tilde{\lambda}) \delta \lambda = \mathbb{E} \left[ U'(X^*(T)) \int_0^T \pi^* \cdot \delta \lambda df \right].
\]
in accordance to [19] for the case of negative power utility, and so the l.h.s. in (5.1) is the
sensitivity associated to weak perturbations (see (4.9) in Proposition 4.1, evaluated at \( \hat{\lambda} \))
whereas the sensitivity for strong perturbations is contained in the r.h.s. Thus, we obtain the
sought after relationship between sensitivities:

\[
Du^w(\hat{\lambda})\delta\lambda - Du^s(\hat{\lambda})\delta\lambda = \mathbb{E} \left[ U'(X^*(T)) \left\{ \int_0^T D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right\} \right].
\]

It seems to us that a rigorous derivation of (5.2) is an interesting, and challenging, open prob-
lem. To overcome it, it would be needed to provide a precise analysis of the differentiability
of optimal portfolios, and of the sensitivity of utility maximization in the strong formulation.

We now make use of (5.2) to recover the result in Proposition 2.1. Let us assume that
\( \bar{\lambda} \) is deterministic and see what this can imply. Call

\[
R_t := \int_0^t D[\pi^s_x](\omega, \delta\lambda) \cdot \bar{\lambda}d\lambda + \int_0^t D\pi^s_x(\omega, \delta\lambda)dW_s = \int_0^t D[\pi^s_x](\omega, \delta\lambda) \cdot \{\bar{\lambda}d\lambda + dW_s\}.
\]

By duality and [20, Corollary 3.3] we know that there is a scalar \( a \) (making sure that
\( X^*(T) \) satisfies the budget constraint) such that

\[
U'(X^*(T)) = a\mathcal{E}(\mathbb{E} \left[ (\bar{\lambda} + \nu)R_T \right], \text{for some } \nu \in K(\hat{\sigma}); \text{ see Sect. 4.}
\]

We then see by the product formula that, upon defining \( dZ_t = -Z_t(\bar{\lambda} + \nu)dW_t \), we get:

\[
\mathbb{E}\left[ U'(X^*(T)) \left\{ \int_0^T D[\pi^s_x(\omega, \delta\lambda)]d\lambda + \int_0^T D\pi^s_x(\omega, \delta\lambda)dW_s \right\} \right] = \mathbb{E}\left[ Z_T R_T \right] = a\mathbb{E}\left[ \int_0^T R_t dZ_t \right] + a\mathbb{E}\left[ \int_0^T Z_t D[\pi^s_x(\omega, \delta\lambda)]d\lambda + dW_t \right] - a\mathbb{E}\left[ \int_0^T Z_t(\bar{\lambda}_t + \nu_t) \cdot D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right] = -a\mathbb{E}\left[ \int_0^T Z_t(\bar{\lambda}_t + \nu_t) \cdot D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right] + a\mathbb{E}\left[ \int_0^T Z_t(\bar{\lambda}_t + \nu_t) \cdot D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right].
\]

Under enough integrability conditions so that the Brownian integrals are martingales, we conclude

\[
\mathbb{E}\left[ U'(X^*(T)) \left\{ \int_0^T D[\pi^s_x(\omega, \delta\lambda)]d\lambda + \int_0^T D\pi^s_x(\omega, \delta\lambda)dW_s \right\} \right] = -a\mathbb{E}\left[ \int_0^T Z_t(\bar{\lambda}_t + \nu_t) \cdot D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right] + a\mathbb{E}\left[ \int_0^T Z_t(\bar{\lambda}_t + \nu_t) \cdot D[\pi^s_x(\omega, \delta\lambda)]d\lambda \right].
\]

and recalling that an optimal \( n \)-dimensional \( \pi^* \) corresponds to a \( \hat{\sigma}^\top \pi \) in the original \( d \)-assets,
we see that if \( \hat{\sigma} \) is deterministic then the r.h.s. also vanishes. All in all, we obtain

\[
Du^w(\hat{\lambda})\delta\lambda = Du^s(\hat{\lambda})\delta\lambda,
\]

which is in tandem with our Proposition 2.1, as well as [9, Lemma 9.2] and [24, Theorem 3.1] for instance.

**Appendix A**

We provide the proof of a version of the envelope or Danskin’s theorem (see [8]), adapted
to our purposes. First, we recall the notion of Hadamard differentiability. Given two Banach
spaces \((\mathcal{X}, \| \cdot \|_\mathcal{X})\) and \((\mathcal{Z}, \| \cdot \|_\mathcal{Z})\) a map \( f : \mathcal{X} \rightarrow \mathcal{Z} \) is directionally differentiable at \( x \) if
for all \( h \in \mathcal{X} \) the limit in \( \mathcal{Z} \)

\[
Df(x, h) := \lim_{\tau \downarrow 0} \frac{f(x + \tau h) - f(x)}{\tau},
\]
exists. If in addition, for all $h \in \mathcal{X}$ the following equality in $Z$ holds

$$Df(x, h) = \lim_{\tau \downarrow 0, h' \to h} \frac{f(x + \tau h') - f(x)}{\tau},$$

then we say that $f$ is directionally differentiable at $x$ in the Hadamard sense. An important property of Hadamard differentiable functions is the chain rule. More precisely, if $(\mathcal{V}, \| \cdot \|_\mathcal{V})$ is another Banach space, $g : \mathcal{V} \to \mathcal{X}$ is directionally differentiable at $v$ and $f$ is directionally differentiable at $g(v)$ in the Hadamard sense, then the composition $f \circ g$ is directionally differentiable at $v$ (see e.g. [4, Proposition 2.47]) and $D(g \circ f)(v, v') = Df(g(v), Dg(v, v'))$ for all $v' \in \mathcal{V}$. If in addition, $g$ is also Hadamard directionally differentiable at $v$, then $f \circ g$ is directionally differentiable at $v$ in the Hadamard sense.

Now, suppose that $K \subseteq \mathcal{X}$ is a weakly compact set. Let us consider the problem:

$$\sup_{Z \in K} \langle d, Z \rangle \quad \text{s.t.} \quad Z \in K,$$

(A.1)

where $d \in \mathcal{X}^*$ and $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing between $\mathcal{X}$ and $\mathcal{X}^*$. Let us define $v : \mathcal{X}^* \to \mathbb{R}$ as the optimal value of problem (A.1) and $S(d)$ the set of optimal solutions of (A.1), i.e.

$$v(d) := \sup_{Z \in K} \langle d, Z \rangle, \quad S(d) := \{ Z \in K \mid v(d) = \langle d, Z \rangle \}.$$

Note that $v$ is well defined, it is a Lipschitz function and $S(d) \neq \emptyset$. In fact,

$$|v(d_1) - v(d_2)| \leq \|d_1 - d_2\|_{\mathcal{X}^*} \sup_{Z \in K} \|Z\|_{\mathcal{X}}. \quad (A.2)$$

The proof of the following result is a simple modification of the proof in [4, Theorem 4.13].

**Lemma A.1** For any $\bar{d} \in \mathcal{X}^*$, the following assertions hold true

(i) The set $S(\bar{d})$ is weakly compact.

(ii) The function $v$ is directionally differentiable in the Hadamard sense and its directional derivative is

$$Dv(\bar{d}, \Delta d) = \sup_{Z \in S(\bar{d})} \langle \Delta d, Z \rangle \quad \text{for all} \Delta d \in \mathcal{X}^*.$$  

(A.3)

**Proof** The first assertion follows directly from the weak-continuity of $\langle \bar{d}, \cdot \rangle$, which implies the weak closedness of $S(\bar{d})$. Now, in view of [4, Proposition 2.49] and (A.1) it suffices to show that $v$ is directionally differentiable. Let $\bar{Z} \in S(\bar{d})$ be such that $\langle \Delta d, \bar{Z} \rangle = \sup_{Z \in S(\bar{d})} \langle \Delta d, Z \rangle$ and for $\tau > 0$ set $d_\tau := \bar{d} + \tau \Delta d$. By definition

$$v(d_\tau) - v(\bar{d}) \geq \langle d_\tau - \bar{d}, \bar{Z} \rangle = \tau \langle \Delta d, \bar{Z} \rangle,$$

which implies that

$$\liminf_{\tau \to 0} \frac{v(d_\tau) - v(\bar{d})}{\tau} \geq \langle \Delta d, \bar{Z} \rangle = \sup_{Z \in S(\bar{d})} \langle \Delta d, Z \rangle. \quad (A.4)$$

Analogously, let $Z_\tau \in S(d_\tau)$. Then

$$v(\bar{d}) - v(d_\tau) \geq -\langle d_\tau - \bar{d}, Z_\tau \rangle = -\tau \langle \Delta d, Z_\tau \rangle.$$  

(A.5)
On the other hand, using (A.1) we get that \( v(d_\tau) \rightarrow v(\bar{d}) \) as \( \tau \downarrow 0 \), which implies, since \( d_\tau \rightarrow \bar{d} \) strongly in \( \mathcal{X}^* \), that any weak limit point of \( Z_\tau \) belongs to \( S(\bar{d}) \). Thus, (A.4) yields

\[
\limsup_{\tau \rightarrow 0} \frac{v(d_\tau) - v(\bar{d})}{\tau} \leq \limsup_{\tau \rightarrow 0} \langle \Delta d, Z_\tau \rangle \leq \sup_{Z \in S(\bar{d})} \langle \Delta d, Z \rangle.
\]

(A.5)

Therefore, (A.2) is a consequence of (A.3) and (A.5).

References

1. Backhoff, J.: Functional analytic approaches to some stochastic optimization problems. PhD Thesis, Humboldt-Universität zu Berlin (2015)
2. Backhoff, J., Fontbona, J.: Robust utility maximization without model compactness. SIAM J. Financ. Math. 7(1), 70–103 (2016)
3. Backhoff, J., Silva, F.J.: Sensitivity results in stochastic optimal control: a Lagrangian perspective. ESAIM Control Optim. Calc. Var. 23(1), 39–70 (2017)
4. Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, New York (2000)
5. Cadenillas, A., Karatzas, I.: The stochastic maximum principle for linear convex optimal control with random coefficients. SIAM J. Control Optim. 33, 590–624 (1995)
6. Chen, Z., Epstein, L.: Ambiguity, risk, and asset returns in continuous time. Econometrica 70(4), 1403–1443 (2002)
7. Cox, J., Huang, C.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. J. Econom. Theory 49(1), 33–83 (1989)
8. Danskin, J.M.: The theory of max–min and its application to weapons allocation problems. In: Econometrics and Operations Research, vol. V. Springer, New York (1967)
9. Davis, M.: Optimal hedging with basis risk. In: From Stochastic Calculus to Mathematical Finance, pp. 169–187. Springer, Berlin (2006)
10. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. 300(3), 463–520 (1994)
11. Inada, K.: On a two-sector model of economic growth: comments and a generalization. Rev. Econ. Stud. 30(2), 119–127 (1963)
12. Karatzas, I., Lehoczky, J., Shreve, S.: Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. SIAM J. Control Optim. 25(6), 1557–1586 (1987)
13. Karatzas, I., Lehoczky, J., Shreve, S., Xu, G.L.: Martingale and duality methods for utility maximisation in an incomplete market. SIAM J. Control Optim. 29, 702–730 (1991)
14. Karatzas, I., Shreve, S.: Methods of mathematical finance. In: Applications of Mathematics, vol. 39. Springer, New York (1998)
15. Kardaras, C., Žitković, G.: Stability of the utility maximization problem with random endowment in incomplete markets. Math. Finance 21(2), 313–333 (2011)
16. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. 9(3), 904–950 (1999)
17. Kramkov, D., Sirbu, M.: On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. 16(3), 1352–1384 (2006)
18. Larsen, K.: Continuity of utility-maximization with respect to preferences. Math. Finance 19(2), 237–250 (2009)
19. Larsen, K., Mostovyi, O., Zitkovic, G.: An expansion in the model space in the context of utility maximization. arXiv:1410.0946v1 [q-fin.PM] (2014)
20. Larsen, K., Zitkovic, G.: Stability of utility-maximization in incomplete markets. Stoch. Process. Appl. 117(11), 1642–1662 (2007)
21. Merton, R.: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econom. Stat. 51, 247–257 (1971)
22. Merton, R.: Optimum consumption and portfolio rules in a continuous-time model. J. Econom. Theory 3(4), 373–413 (1971)
23. Mocha, M., Westray, N.: The stability of the constrained utility maximization problem: a BSDE approach. SIAM J. Financ. Math. 4(1), 117–150 (2013)
24. Monoyios, M.: Malliavin calculus method for asymptotic expansion of dual control problems. SIAM J. Financ. Math. 4(1), 884–915 (2013)
25. Mostovyi, O., Sirbu, M.: Sensitivity analysis of the utility maximization problem with respect to model perturbations. arXiv:1705.08291 (2017)
26. Musielak, J.: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
27. Nakano, H.: Generalized modular spaces. Stud. Math. 31, 439–449 (1968)
28. Penot, J.P.: Calculus Without Derivatives. Graduate Texts in Mathematics, vol. 266. Springer, New York (2013)
29. Pham H (2009) Continuous-time stochastic control and optimization with financial applications. In: Stochastic Modelling and Applied Probability, vol. 61. Springer, Berlin
30. Pliska, S.: A stochastic calculus model of continuous trading: optimal portfolios. Math. Oper. Res. 11(2), 370–382 (1986)
31. Protter, P.: Stochastic Integration and Differential Equations, 2nd edn. Springer, Heidelberg (2005)
32. Quenez, M.C.: Optimal portfolio in a multiple-priors model. In: Seminar on Stochastic Analysis, Random Fields and Applications IV, vol. 58, pp. 291–321. Birkhäuser, Basel (2004)
33. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and Martingales, Cambridge Mathematical Library, vol 2. Cambridge University Press, Cambridge (2000) (Itô calculus, Reprint of the second (1994) edition)
34. Weston, K.: Stability of utility maximization in nonequivalent markets. In: Forthcoming in Finance and Stochastics. arXiv:1410.0915v2 (2015)
35. Yong, J., Zhou, X.Y.: Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer, New York (2000)