C, P, and T of Braid Excitations in Quantum Gravity

Song He*

School of Physics, Peking University, Beijing, 100871, China,

Yidun Wan†

Perimeter Institute for Theoretical Physics,
31 Caroline st. N., Waterloo, Ontario N2L 2Y5, Canada, and
Department of Physics, University of Waterloo,
Waterloo, Ontario N2J 2W9, Canada

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Abstract

We study the discrete transformations of four-valent braid excitations of framed spin networks embedded in a topological three-manifold. We show that four-valent braids allow seven and only seven discrete transformations. These transformations can be uniquely mapped to C, P, T, and their products. Each CPT multiplet of actively-interacting braids is found to be uniquely characterized by a non-negative integer. Finally, braid interactions turn out to be invariant under C, P, and T.

*Email address: hesong@pku.edu.cn
†Email address: ywan@perimeterinstitute.ca
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1 Introduction

Since a ribbonized preon model\cite{1} was coded into local braided ribbon excitations\cite{2} there has been a large amount of research effort towards a quantum theory of gravity with matter as topological invariants\cite{3, 5, 6, 7, 9, 8, 11}. However, a serious limitation of the results of \cite{2} was realized soon that the conservation laws which preserve the braid excitations are exact. In other words, there is no possibility of dynamics of these exactly conserved excitations, e.g. creation and annihilation\cite{3}. Consequently, interpreting braid excitations found in \cite{2} as particles - in particular the Standard Model particles - is not going to work out unless interactions are successfully introduced to that model.

Meanwhile, a new model has also been put forward, which solves the problem of interaction and opens new interesting directions worth of investigation\cite{5, 6}. The new model involves framed four-valent rather than three-valent spin networks, embedded in a topological three-manifold, which also gives rise to local braid excitations, each of which is a 3-strand braid formed by the three common edges of two adjacent nodes of the network, and bases the dynamics on the dual Pachner moves naturally associated with four-valent graphs. The two main reasons of this extension are: that four-valent graphs and the corresponding dual Pachner moves naturally occur in spin foam models\cite{4}, and that vertices of four-valent spin networks have true correspondence to three-dimensional space.

The stable three-strand braids, under certain stability condition, are local excitations\cite{6, 11}. Among all stable braids, there is a small class of braids which are able to propagate on the spin network. The propagation of these braids are chiral, in the sense that some braids can only propagate to their left with respect to the local subgraph containing the braids, while some only propagate to their right and some do both\cite{5, 6}. There is another small class of braids, the actively-interacting braids; each is two-way propagating and is able to merge with its neighboring braid when the interaction condition is met\cite{6}. Braids that are not propagating are christened stationary braids.

\cite{5, 6} are based on a graphic calculus developed therein. However, although the graphic calculus has its own advantages - in particular in describing, e.g. the full procedure of the propagation a braid, it is not very convenient for finding conserved quantities of a braid which are useful to characterize the braid as a matter-like local excitation. In view of this, \cite{7, 8} proposed an algebraic notation of our braids and derived conserved quantities by means of the new notation.

One of our goals is to see whether some braid excitations of embedded 4-valent spin networks can eventually correspond to the standard model particles or are more fundamental matter degrees of freedom. Because CPT is a symmetry of quantum field theories, in this paper we investigate the discrete transformations of 3-strand braids of embedded 4-valent spin networks and map them to C, P, and T transformations and their products. We will see that the interaction of braids defined in \cite{6} respects CPT.

In fact, as a follow-up work of \cite{2}, in \cite{9} a similar study of CPT-symmetry is being taken for three-valent spin networks. However, in the 3-valent case there is no dynam-
ics, it is then unlikely to define what C, P, and T are meant to be physically. Besides, the largest discrete symmetry in the 3-valent case is $S_3 \times \mathbb{Z}_2$, giving more than C, P, T, and their products. On the contrary, in the 4-valent case we have dynamics which surprisingly and strongly constraints the number of possible discrete transformations to be exactly seven, excluding the identity, which are allowed on 3-strand braids. Moreover, the algebra developed and conserved quantities of braids found in our companion paper[8] also helps finding and mapping discrete transformations of braids to C, P, and T. This will become clear soon in the sequel. Let us summarize the main results of this paper as follows:

1. Discrete transformations C, P, T, and their products are found on 3-strand braids of embedded framed 4-valent spin networks.

2. Reversing the momentum direction of a braid is understood to be unambiguously associated with the flipping of the propagation chirality of the braid.

3. The "electric" charge of a braid is naturally represented by the effective twist number of the braid. A braid’s spin is argued to be related to the spin network labels on the braid.

4. Each CPT multiplet of actively-interacting braids is uniquely characterized by a non-negative integer.

5. Interactions of braids are found to be invariant under C, P, and T separately, and is thus invariant under CPT.

6. Possible future developments by means of tensor categories are pointed out.

2 Notation

It is worth of re-emphasizing an essential point. A 3-strand braid is a local sub network of the whole framed spin network embedded in a topological 3-manifold; however, many embeddings are diffeomorphic to each other, which gives rise to diffeomorphic (also called equivalent in our approach) braids. We study a braid through its 2-dimensional projection, called a braid diagram. We therefore will not distinguish braids from braid diagrams unless an ambiguity arises. A generic example of such a braid diagram is depicted in Fig. 1(a). Equivalent braid diagrams form an equivalence class. To choose an efficient representative of an equivalence class of braids is important; in [6] where we studied propagation and interactions of braids our choice was to represent an equivalence class by the representative which is a braid diagram which has zero external twists, which simplifies the interaction condition and the graphic calculus developed in [5–6]. Each class has one and only one such representative. Thus a braid represented this way is said to be in its unique representation.
An algebraic notation and the corresponding calculational method of braids were put forward in [7], which applies only to actively-interacting braids. This notation is extended to account for propagating braids and even stationary braids [8]. Nevertheless, in this paper we first analyze the discrete transformations of braids which will appear to be more transparent and lucid in terms of the graphic calculus in some cases. On the other hand, to identify the discrete transformations with C, P, T and their products and to sort out the conserved quantities the algebraic notation is more efficient. Therefore, we will use both the algebraic notation and the graphic notation.

Figure 1: (a) is a generic 3-strand braid diagram formed by the three common edges of two end-nodes. $S_l$ and $S_r$ are the states of the left and right end-nodes respectively, taking values in $+\text{ or } -$. $X$ represents a sequence of crossings, from left to right, formed by the three strands between the two nodes. $(T_a, T_b, T_c)$ is the triple of the internal twists respectively on the three strands from top to bottom, on the left of $X$. $T_l$ and $T_r$, called external twists, are respectively on the two external edges $e_l$ and $e_r$. All twists are valued in $\mathbb{Z}$ in units of $\pi/3$ [5]. (b) shows the four generators of $X$, which are also generators of the braid group $B_3$.

Let us briefly recall the algebraic notation of a braid introduced in [8]. A generic braid shown in Fig. 1 is characterized by an 8-tuple: \{$T_l, S_l, T_a, T_b, T_c, X, S_r, T_r$\}. The crossing sequence $X$ satisfies the definition of an ordinary 3-strand braid, an element of the braid group $B_3$; hence it is generated by the four generators shown in Fig. 1(b). The generators are assigned integral values according to their handedness, namely $u = d = 1$ and $u^{-1} = d^{-1} = -1$. Therefore, crossings in the $X$ of a braid can also be summed over to obtain an integer, the so-called crossing number: \[\sum_{i=1}^{\mid X\mid} x_i\] of the braid, where $\mid X\mid$ is the number of crossings forming $X$ and $x_i$ is a crossing in $X$.

The $X$ of a braid induces a permutation $\sigma_X$, which is an element in the permutation group $S_3$, of the three strands of the braid. The triple of internal twists on the left of $X$ and the one of the right of $X$ are thus related by $(T_a, T_b, T_c)\sigma_X = (T_{a'}, T_{b'}, T_{c'})$ and $(T_a, T_b, T_c) = \sigma_X^{-1}(T_{a'}, T_{b'}, T_{c'})$. That is, $\sigma_X$ is a left-acting function of the triple of internal...
twists, while its inverse, \( \sigma_X^{-1} \) is a right-acting function. The inverse relation between \( \sigma_X \) and \( \sigma_X^{-1} \) is understood as:

\[
\sigma_X^{-1} ((T_a, T_b, T_c) \sigma_X) = (\sigma_X^{-1} (T_a, T_b, T_c)) \sigma_X \equiv (T_a, T_b, T_c)
\]  

(1)

Note that the twists such as \( T_a \) and \( T_a' \) are abstract and have no meaning until their values and positions in a triple are fixed. Thus, \((T_a, T_b, T_c) = (T'_a, T'_b, T'_c)\) means \( T_a = T'_a \), etc, and \(- (T_a, T_b, T_c) = (-T_a, -T_b, -T_c)\). A generic braid diagram can now be denoted concisely by

\[
S_L[(T_a, T_b, T_c) \sigma_X]_{S_r}
\]

or by

\[
S_L[\sigma_X^{-1} (T_a', T_b', T_c')]_{S_r'}
\]

Since an active braid is always equivalent to trivial braid diagrams with external twists\^[5], and is usually represented by one such trivial braid\^[7, 8], it can be denoted as

\[
S_L[(T_a, T_b, T_c)]_{S_r}
\]

where both end-nodes are in the same state. This is called a trivial representation. However, a non actively-interacting braid is more conveniently to be represented by its unique representative with zero external twist, namely

\[
S_L[(T_a, T_b, T_c) \sigma_X]_{S_r}
\]

Because a main purpose of this paper is to find discrete transformations on all braids, we therefore will write an arbitrary braid in its generic form in most of the paper. The unique representation and trivial representation of braids will also be used when it is good to do so.

3 Discrete transformations

Though not separately, as a theorem the combined action of the three discrete transformations C, P and T, namely CPT, is a symmetry in any Lorentz invariant, local field theory. Being a concrete model of QFT, the Standard Model respects the CPT-symmetry too. The 3-strand braids of embedded 4-valent spin networks are local excitations, continuous transformations such as the equivalence moves of which have been analyzed in \^[5, 7, 8], it is then natural to look for the possible discrete transformations of these local excitations and check their correspondence with C, P, and T transformations. If the braids in our model would eventually be mapped to the Standard Model particles, or even if they are more fundamental entities on their own, which do not directly correspond to the Standard Model particles, they should be characterized by quantum numbers which have certain properties under the transformations of C, P, and T. In fact, investigating the action of
discrete transformations on our braid excitations can help us to construct quantum numbers of a braid, such as spin, charge, etc, out of the characterizing 8-tuple of the braid. If the 8-tuple, which only contains topological information of the embedding and framing, is not sufficient to produce all necessary quantum numbers, we may have to take spin network labels into account. One will see that this is indeed the case.

Due to the dynamics of the braids of embedded 4-valent spin networks, namely the propagation and interactions, there exist very natural constraints on the discrete transformations one can apply. The reason is that we should not allow that, for example, a discrete transformation turns an actively interacting braid into one which is not because there is no such a transformation in QFT which magically changes a particle to something else, and vice versa. Similarly, we can obtain other necessary rules. As a guideline, all rules are listed below as a condition.

**Condition 1.** A legal discrete transformation $D$ on an arbitrary braid $B$ must meet:

1. If $B$ is actively-interacting, then $D(B)$ also actively interacts.

2. If $B$ is not actively-interacting, $D(B)$ must remain so.

3. If $B$ is one-way (two-way) propagating, $D(B)$ must still be one-way (two-way) propagating; however, the propagation chirality of $B$ may be reversed in the one-way case.

4. If $B$ is stationary, $D(B)$ is stationary as well.

### 3.1 The group of discrete transformations

It is more convenient to write all discrete transformations in a compact, algebraic form. This can be achieved by introducing the so-called **atomic discrete operations** acting on the crossing sequence, the end-nodes, the triples of internal twists, and the pair of external twists separately. Each atomic transformation is not qualified as a legal discrete transformation on its own due to the violation of Condition 1. However, a legal discrete transformation can be written as a unique combination of the atomic ones. All atomic transformations are defined and listed with sufficient details in Appendix I, we thus in the rest of the main text will directly use them without further explanation but only a reference to the definition of each of them upon its first appearance.

In view of the parity transformation in QFT, the first kind of discrete transformations one may come up with is the mirror imaging of a braid. However, there are two ways of mirror reflecting a braid: one is to have the mirror perpendicular to the plane on which the braid is projected, the other is to arrange the mirror parallel to and behind the plane. Let us study them in order.

Fig. 2 illustrates the former case. Although only two generators of the crossing sequence $X$ of an arbitrary braid is shown in this figure, it is not hard to see that the order of the crossings of the original $X$ on the left of the mirror must be reversed by the mirror,
Figure 2: Shows how the crossing generators, end-nodes, twists, and the propagation chirality (indicated by the two thick arrows) of a braid (on the left) are mapped to their mirror images (on the right) via a mirror perpendicular to the plane on which the braid is projected.

resulting an $\mathcal{R}(X)$ (Def. 7) of the mirror image of the original braid. Besides, every crossing in $X$ is inverted by the mirror, giving rise to an $\mathcal{I}_X$ (Def. 5). As a result, the mirror imaging takes the $X$ to $X^{-1}$. In Fig. 2 uses only one edge between the two end-nodes of a braid to demonstrate the sign change of the twist of the edge by the mirror. This is sufficient to show that all twists of a braid should have a sign change via mirror imaging because the sign of a twist is unambiguously defined everywhere of an embedded spin network[5]. This means that the atomic operation $\mathcal{I}_T$ (Def. 6) must be part of this mirror image transformation. One can also find from Fig. 2 that due to the exchange of the two end-nodes, which indicates the atomic operations $\mathcal{E}_S$ (Def. 12) and $\mathcal{E}_T_e$ (Def. 10), and the existence of an $\mathcal{R}$, the left triple of internal twists should be exchanged with the right triple of internal twists, i.e. an $\mathcal{E}_T$ (Def. 9) is involved. These observations provide us an explicit definition of this mirror imaging as follows.

**Definition 1.** The **perpendicular mirror imaging** is such a discrete transformation, denoted by $\mathcal{M}_\perp$, that

$$\mathcal{M}_\perp = \mathcal{E}_S \mathcal{E}_{T_e} \mathcal{E}_T \mathcal{I}_T \mathcal{I}_X \mathcal{R},$$

and that for a generic braid $B = \frac{S_i}{T_i}[(T_a, T_b, T_c) \sigma_X]^{S_f}_{T_f}$, with $(T_a, T_b, T_c) \sigma_X = (T_a', T_b', T_c')$

$$\mathcal{M}_\perp(B) = \frac{S_f}{T_f}[-(T_a', T_b', T_c') \sigma_X^{-1}]^{S_i}_{T_i},$$

(2)

with $-(T_a', T_b', T_c') \sigma_X^{-1} = -(T_a, T_b, T_c)$.

A important point to address is that $\mathcal{M}_\perp$ flips the propagation chirality. That is, if a braid $\tilde{B}$ is (left-) right-propagating, then $\mathcal{M}_\perp(\tilde{B})$ is (right-) left-propagating, which is readily seen from Fig. 2 because a braid propagating towards the mirror from the left is mirrored to a braid towards the mirror from the right. This surely has no impact on a
two-way propagating braid. Since \( M_{\perp} \) is simply a mirror image, an actively-interacting braid stays so under this discrete transformation. Therefore, \( M_{\perp} \) fulfills Condition and hence is indeed a legal discrete transformation of braids. Concrete graphic examples are shown in Appendix II.

**Figure 3:** The crossing generators, end-nodes, twists, and the propagation chirality of a braid (above the mirror) are mapped to their mirror images (below the mirror) via a mirror parallel to the plane on which the braid is projected.

Fig. 3 presents the second type of mirror imaging of a braid, where the mirror is parallel to and beneath the plane on which the braid is projected. In contrast to the mirror imaging of the first kind, the second kind does not reverse the order of the crossings and does not exchange the two end-nodes, which leads to no exchange of triples of internal twists. However, from Fig. 3 it is clear that all crossings, twists, and the two end-node states are inverted, resulting in three atomic operations: \( I_X, I_T, \) and \( I_S \). As implied by the two thick arrows respectively above and below the mirror in Fig. 3 the propagation chirality of a braid should not be reversed under this mirror imaging. This implies that a stationary braid remains stationary under this transformation. It is not hard to see that this type of mirror imaging of an actively-interacting braid must still be active. Therefore, we have another legal discrete transformation of braids, as defined below.

**Definition 2.** The parallel mirror imaging, \( M_{\square} \), is a discrete transformation in the form

\[
M_{\square} = I_X I_T I_S,
\]

such that for a generic braid, \( B = \{S_{T_i}(T_a, T_b, T_c)\}_{i=1}^{S_{T_i}} \), with \( (T_a, T_b, T_c) \sigma_X = (T_a', T_b', T_c') \)

\[
M_{\square}(B) = \{S_{T_i}[-(T_a, T_b, T_c)]_{i=1}^{S_{T_i}} \}_{i=1}^{S_{T_i}} \sigma_{T_i},
\]

with \(- (T_a, T_b, T_c) \sigma_X = -(T_a', T_b', T_c').\)

After talking about reflections, it is now the turn to study other possibilities. The first is called a **vertical flip**, is depicted in Fig. 4 in which, rather than showing the flip of a whole generic braid with respect to the axis (the thick grey horizontal line in the figure), we illustrate how the generators of a crossing sequence and a trivial braid
diagram without crossings are transformed under such a flip, by which one can easily determine the corresponding transformation of an arbitrary braid. Note that this flip is not an equivalence rotational move, which is continuous, defined in [5] but rather a discrete operation, taking a braid, e.g. the one in the upper part of Fig. 4 directly to the one in the lower part of the same figure, without any continuous intermediate steps; hence, no extra twists or crossings are created or annihilated.

According to Fig. 4 the vertical flip neither reverses the order of crossings nor exchange the two end-nodes of a braid; however, it turns an upper crossing into a lower one and a lower one to an upper one, with their handedness unchanged, which gives rise to an $S_c$ (Def. 8), the chain shift of the crossing sequence $X$. From the figure an $I_S$ is also obtained. An interesting property of the vertical flip is that it swaps the top and bottom internal twists of a braid, as seen in Fig. 4 leading to an $S_T$ (Def. 11). The last atomic operation involved in this rotation is seen from the figure to be an $I_S$.

A braid’s propagation chirality is left intact under the vertical flip because its reducibility is unchanged due to the fact that, although each of it’s end-node states is flipped, the crossing next to each end-node is shifted to its counterpart of the same handedness, which ensures a reducible end-node being again reducible after the transformation. A stationary braid is thus still stationary under this transformation. By the same argument, a braid, which is actively-interacting, remains so too after the transformation. Therefore, the vertical flip exhibited in Fig. 4 is indeed a legal discrete transformation. We now present its explicit definition.

**Definition 3.** The **vertical flip**, $F_V$, is a discrete transformation in the form

$$F_V = I_S S_T S_c,$$
which for a generic braid \( B = \frac{S}{T_l}[(T_a, T_b, T_c)\sigma_X]_l^{S_T} \), with \((T_a, T_b, T_c)\sigma_X = (T'_{a'}, T'_{b'}, T'_{c'})\), satisfies

\[
\mathcal{F}_V(B) = \frac{S}{T_l}[(T_c, T_b, T_a)\sigma_{S_i(X)}]^{S_T}_{l},
\]

with \((T_c, T_b, T_a)\sigma_{S_i(X)} = (T'_{c'}, T'_{b'}, T'_{a'})\).

One may try to find out all other legal discrete transformations in a similar way. Nevertheless, our study shows that fortunately the aforementioned three discrete transformations and their products, seven altogether are the only allowable ones. In other words, \( \mathcal{M}_\perp, \mathcal{M}_\Box, \) and \( \mathcal{F}_V \) generate the largest group of legal discrete transformations which, denoted by \( G_D \), contains eight elements, including the identity transformation. This group of discrete transformations and their action on a generic braid is recorded in Table 1.

| Discrete Transformation | Algebraic Form | Action on \( B = \frac{S}{T_l}[(T_a, T_b, T_c)\sigma_X]_l^{S_T} \) | Prop-Chirality |
|------------------------|---------------|--------------------------------------------------|----------------|
| \( \mathcal{I} \)      | \( \mathcal{I} \)                             | \( \frac{S}{T_l}[(T_a, T_b, T_c)\sigma_X]_l^{S_T} \) | +              |
| \( \mathcal{M}_\perp \) | \( \mathcal{E}_S\mathcal{E}_T\mathcal{E}_T\mathcal{R}\mathcal{T}_X\mathcal{I}_T \) | \( \frac{S}{T_l}[-(T'_{a'}, T'_{b'}, T'_{c'})\sigma_{X^{-1}}]_{T^{-1}_l}^{S_T} \) | -              |
| \( \mathcal{M}_\Box \)  | \( \mathcal{I}_S\mathcal{I}_X\mathcal{I}_T \)     | \( \frac{S}{T_l}[-(T_a, T_b, T_c)\sigma_{T_X(X)}]_{T^{-1}_l}^{S_T} \) | +              |
| \( \mathcal{F}_V \)    | \( \mathcal{I}_S\mathcal{S}_c\mathcal{S}_T \)     | \( \frac{S}{T_l}[(T_c, T_b, T_a)\sigma_{S_i(X)}]^{S_T}_{l} \) | +              |
| \( \mathcal{F}_H = \mathcal{M}_\perp\mathcal{M}_\Box \) | \( \mathcal{I}_S\mathcal{E}_S\mathcal{E}_T\mathcal{E}_T\mathcal{R} \) | \( \frac{S}{T_l}[(T'_{a'}, T'_{b'}, T'_{c'})\sigma_{\mathcal{R}(X)}]_{T^{-1}_l}^{S_T} \) | -              |
| \( \mathcal{M}_\perp\mathcal{F}_V \) | \( \mathcal{I}_S\mathcal{E}_S\mathcal{E}_T\mathcal{E}_T\mathcal{S}_T\mathcal{S}_c\mathcal{I}_X \) | \( \frac{S}{T_l}[-(T'_{c'}, T'_{b'}, T'_{a'})\sigma_{T_XS_i(X)}]_{T^{-1}_l}^{S_T} \) | -              |
| \( \mathcal{M}_\Box\mathcal{F}_V \) | \( \mathcal{I}_T\mathcal{S}_c\mathcal{S}_T\mathcal{I}_X \) | \( \frac{S}{T_l}[-(T_c, T_b, T_a)\sigma_{T_XS_i(X)}]_{T^{-1}_l}^{S_T} \) | +              |
| \( \mathcal{F}_V\mathcal{F}_H \) | \( \mathcal{E}_S\mathcal{S}_T\mathcal{E}_T\mathcal{E}_T\mathcal{S}_c\mathcal{R} \) | \( \frac{S}{T_l}[(T'_{c'}, T'_{b'}, T'_{a'})\sigma_{S_i(R(X))}]_{T^{-1}_l}^{S_T} \) | -              |

Table 1: The group \( G_D \) and its action on a generic braid diagram. The last column shows whether the propagation chirality of a braid changes under the corresponding transformations in the first column; \( a^- \) means flipped and \( a^+ \) means unaffected.

One can easily check that \( G_D \), consisting of the transformations shown in the first column of Table 1 is indeed a group. That \( G_D \) is the largest group of legal discrete transformations of 3-strand braids is a result of the fact that \( G_D \) exhausts all possible combinations of the atomic operations defined in Appendix I, which meet Condition \( l \) and that there exists no new atomic discrete operations which can be constructed and combined with the current ones without violating Condition \( l \).
3.2 Conserved quantities

It is seen from Table 1 that each discrete transformation takes a braid to a new braid which is not equivalent to the original though they may be the same in some special cases. Focusing on the third column of Table 1, one readily finds that some characterizing quantities are invariant under some discrete transformations but are not under others. There are also composite quantities constructed from the characterizing 8-tuple, which are conserved under certain and changed under other transformations. It is then necessary and helpful to explicitly demonstrate these quantities, which is achieved by Table 2, not only for the purpose of mapping the discrete transformations to C, P, and T but also for the task to sort out the physically meaningful quantum numbers of a braid. The reason to consider the quantities listed in Table 2 apart from their properties under discrete transformations, is their conservation under interactions. The quantity \( \Theta \), defined in [8] as the effective twist number, of a single braid is conserved under both equivalence moves and evolution moves [5, 6]. It is also an additive conserved quantity under interactions of two braids, in the sense that the \( \Theta \)-value of the resulted braid of the interaction of two braids is equal to the sum of the \( \Theta \)-values of the two braids before the interaction [6]. This conservation law, though obtained in representing a braid by its unique representative, is independent of the choice of the representative of the braid.

In [7, 8], however, we also studied representing a braid by an extremum of it, i.e. an equivalent braid diagram with the least number of crossings (defined in [5]). An actively-interacting braid has infinite number of extrema, namely the trivial braid diagrams with

| \( G_D \) | \( (S_l, S_r) \) | \( (T_l, T_r) \) | \( [T_a, T_b, T_c] \) | \( \sum_{i=a}^{c} T_i \) | \( \sum_{i=1}^{|X|} x_i \) | \( \Theta' \) | \( T_l + T_r \) | \( \Theta \) |
|---|---|---|---|---|---|---|---|---|
| \( M_\perp \) | \( (S_r, S_l) \) | \( (T_r, T_l) \) | \( -[T_a', T_b', T_c'] \) | \( - \) | \( - \) | \( - \) | \( - \) | \( - \) |
| \( M_\Box \) | \( (S_l, S_r) \) | \( (T_l, T_r) \) | \( -[T_a, T_b, T_c] \) | \( - \) | \( - \) | \( - \) | \( - \) | \( - \) |
| \( \mathcal{F}_V \) | \( (S_r, S_l) \) | \( (T_r, T_l) \) | \( [T_c, T_b, T_a] \) | \( + \) | \( + \) | \( + \) | \( + \) | \( + \) |
| \( \mathcal{F}_H \) | \( (S_r, S_l) \) | \( (T_r, T_l) \) | \( [T_a', T_b', T_c'] \) | \( + \) | \( + \) | \( + \) | \( + \) | \( + \) |
| \( M_\perp \mathcal{F}_V \) | \( (S_r, S_l) \) | \( (T_r, T_l) \) | \( -[T_c', T_b', T_a'] \) | \( - \) | \( - \) | \( - \) | \( - \) | \( - \) |
| \( M_\Box \mathcal{F}_V \) | \( (S_l, S_r) \) | \( (T_l, T_r) \) | \( -[T_c, T_b, T_a] \) | \( - \) | \( - \) | \( - \) | \( - \) | \( - \) |
| \( \mathcal{F}_V \mathcal{F}_H \) | \( (S_r, S_l) \) | \( (T_r, T_l) \) | \( [T_c', T_b', T_a'] \) | \( + \) | \( + \) | \( + \) | \( + \) | \( + \) |

Table 2: Conserved quantities of a generic braid diagram under discrete transformations in \( G_D \). \( |X| \) is the number of crossings. \( \Theta = \sum_{i=a}^{c} T_i + T_l + T_r - 2 \sum_{i=1}^{|X|} x_i \) is the effective twist. \( \Theta' = \Theta - (T_l + T_r) \) is called the internal effective twist.
external twists, which are the very trivial representative aforementioned. Fortunately, it is shown in [7] that all extrema of an actively-interacting braid share the same value of the sum of the two external twists, i.e. $T_l + T_r$. Likewise, a propagating braid also has infinite number of extrema; however, as conjectured in [8], all extrema of a propagating braid have the same $T_l + T_r$, and $\sum_{i=1}^{c} T_i$, and $\sum_{i=1}^{c} x_i$ as well. So we may define another quantity, the internal effective twist, $\Theta' = \Theta - (T_l + T_r)$, which is the same for all extrema of a braid and is equal to $\Theta$ when the equivalence class of braid diagrams is represented by the unique representative free of external twists.

Importantly, [7, 8] showed that both $T_l + T_r$ and $\Theta'$, and hence $\Theta$ are additive conserved quantities under interactions. More precisely, for two braids, $B_1$ with $\Theta'_1$ and $B_2$ with $\Theta'_2$, the resulted braid $B_1 + B_2$ has $\Theta' = \Theta'_1 + \Theta'_2$.

In addition, all discrete transformations affect the end-nodes of a braid. However, the only meaningful quantity, made of the end-node states of a braid, is the so-called effective state, $\chi = S_l S_r (\text{tr} X)|X|$, which is a conserved quantity of braids under equivalence moves and also is a multiplicative conserved value under interactions of braids [8]. Table 2 reads that none of the discrete transformations changes the number of crossings of a braid; therefore, all discrete transformations preserve the $\chi$ of a braid.

4 C, P, and T

Since a braid is a local excitation, regardless of whether it corresponds to a Standard Model particle or not one can at least make an analogy between it and a one-particle state. This is the main task of this section.

4.1 Finding C, P, and T

To map the discrete transformations found in the last section to C, P, T, and their products, it is helpful to recall how the latter ones act on single particle states in the context of quantum field theory. In Table 3, we chose to denote C, P, and T transformations in the Hilbert space by calligraphic letters $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$. For this reason we have already used calligraphic letters for the legal discrete transformations of braids as well because braids are local excitations of embedded spin networks which are the states in the Hilbert space describing the fundamental space-time.

We emphasize here three things. Firstly, so far we have not incorporated spin network labels, which are normally representations of gauge groups, and that our scheme in this section is to obtain the map between two groups of transformations mentioned above by trying to utilize topological characterizing quantities of a braid as much as possible, without involving spin network labels. Secondly, for now we do not take into account the phase and sign factors in Table 3. Finally, all the transformations are restricted to
local braid states, rather than a full evolution picture. By doing so, surprisingly, there is a unique such map. Nevertheless, these two issues will be discussed in the next section.

According to Table 3, the four transformations $P$, $T$, $CP$, and $CT$ reverse the three momentum of a one-particle state. But then what do we mean by the momentum of a braid? In the case of Loop Quantum Gravity, there has not been a well-defined Hamiltonian yet but a Hamiltonian constraint which does not assign a well-defined energy and hence neither a momentum to a local excitation. In fact, the issue is more fundamental in the sense that what do we mean by a direction in space when there is no a notion of fundamental space-time but only superposed spin networks which might lead to a (semi-) classical space-time under some continuous limit? In the case of spin networks as a concept of quantum geometry in general [10], this problem of direction has not been solved either.

Nonetheless, we do not need an explicitly defined 3-momentum of a braid to pick out the discrete transformations which can flip the braid’s momentum. Each braid has a propagation chirality, namely it is either left-propagating or right-propagating or both. Propagation chirality is a locally defined property of a braid with respect to its neighboring subgraph which can be projected horizontally on the plane one is looking at. Actually the propagation chirality is an intrinsic property of a braid and is not the same as the propagation direction of the braid; the latter should be viewed with respect to the whole spin network embedded in a topological 3-manifold. Consequently a braid can actually propagate in any “direction” with respect to its spin network or to an observer, regardless its propagation chirality and how the semiclassical geometry is obtained. One may imagine looking at a braid which moves on a spin network along a “circle” and comes back to its original location.

However, locally, i.e. within a sufficiently small subgraph containing a braid, the braid’s propagation chirality coincides with its propagating direction. An immediate result of this is that if the local propagation chirality of a braid is flipped by a discrete trans-

| $|p, \sigma, n\rangle$ |
|-----------------|
| $C \propto |p, \sigma, n^c\rangle$ |
| $P \propto |-p, \sigma, n\rangle$ |
| $T \propto (-)^{J-\sigma}|-p, -\sigma, n\rangle$ |
| $CP \propto |-p, \sigma, n^c\rangle$ |
| $CT \propto (-)^{J-\sigma}|-p, -\sigma, n^c\rangle$ |
| $PT \propto (-)^{J-\sigma}|p, -\sigma, n\rangle$ |
| $CPT \propto (-)^{J-\sigma}|p, -\sigma, n^c\rangle$ |

Table 3: The action of $C$, $P$, $T$, and their products on a one-particle state, where $p$ is the 3-momentum, $\sigma$ is the third component of the particle spin $J$, and $n$ stands for the charge.
formation, so is its propagation direction. The direction of the 3-momentum of braid, by any means it is defined, is associated with the propagation direction. Therefore, the discrete transformations reversing the 3-momentum of a braid are exactly those flipping the propagation chirality of the braid, which are, according to Table 1, $\mathcal{M}_\perp$, $\mathcal{F}_H$, $\mathcal{M}_\perp \mathcal{F}_V$, and $\mathcal{F}_H \mathcal{F}_V$.

In other words, $\mathcal{M}_\perp$, $\mathcal{F}_H$, $\mathcal{M}_\perp \mathcal{F}_V$, and $\mathcal{F}_H \mathcal{F}_V$ are the only legal discrete transformations which can possibly be identified with $\mathcal{P}$, $\mathcal{T}$, $\mathcal{CP}$, and $\mathcal{CT}$, and our task is to find precisely which is which. We know that $\mathcal{P}$ is the transformation which does not change any quantum number but the 3-momentum of a particle. Hence the discrete transformation of a braid which reasonably corresponds to a $\mathcal{P}$ must have the fewest effects on the braid. From Tables 1 and 2 one can see that transformations $\mathcal{F}_H$ and $\mathcal{F}_V \mathcal{F}_H$ are the two candidates because they both reverse the 3-momentum without negating the twists, crossing values, and hence effective twists. Furthermore, $\mathcal{F}_H$ exchanges the left and the right triples of internal twists, but on top of this, $\mathcal{F}_V \mathcal{F}_H$ swaps the first and the third twists in the triple of internal twists of a braid. Therefore, $\mathcal{F}_H$ is the only candidate of a $\mathcal{P}$.

We need two more correspondences to pin down the complete mapping. For this we should find quantum numbers of a braid which are, or analogous to, charge and spin. Among all the conserved quantities of a braid, composed of characterizing topological quantities of the braid, only the total effective twist number $\Theta$ is independent of the representative of the equivalence class of the braid. Other conserved quantities, e.g. $\Theta'$, are not. We know that charges, e.g. electric and color, are unambiguous quantum numbers a particle. Consequently, representative-dependent conserved quantities of a braid, though maybe useful in other ways, should not be considered as charges. This means only $\Theta$ can be a candidate of determining certain charges of a braid.

Moreover, there are actually two more reasons, which are more heuristic and physical. As we know, the electric charge of a particle is quantized to be multiples of $1/3$. Now the interesting thing is that all our twists and hence the effective twists happen to be integers in units of $1/3$ too; the $1/3$ arises naturally rather than being put in by hand[5]. This is also an advantage of the 4-valent case because in the 3-valent case, in contrast, a factor of $1/3$ must be set by hand[9]. On the other hand, the framing of our spin networks which takes an edge to a tube is in fact a $U(1)$ framing; a tube coming from the framing of an edge is essentially an isomorphism from $U(1)$ to $U(1)$. If a tube is twist free, it simply means an identity map, whereas a twisted tube represents a non-trivial isomorphism. That is to say, a twist can be thought as characterizing the isomorphisms on $U(1)$ spaces. An interesting fact is that the electric charge is due to a $U(1)$ gauge symmetry. These suggest that $\Theta$ or an appropriate function of it can be viewed as the “electric charge” of a braid, which might serve as an explanation of why electric charge is quantized so.

Bearing this in mind, Table 2 presents four discrete transformations, $\mathcal{M}_\perp$, $\mathcal{M}_\Box$, $\mathcal{M}_\perp \mathcal{F}_V$, and $\mathcal{M}_\Box \mathcal{F}_V$, which negate the $\Theta$ value of a braid and hence correspond to $\mathcal{CP}$, $\mathcal{C}$, $\mathcal{CT}$, and $\mathcal{T}$ in certain manner. Our strategy is to find $\mathcal{C}$ first. Since a $\mathcal{C}$ does not flip the momentum,
Table 4: The map between legal discrete transformations of braids and $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$, and their products.

| $\mathcal{C}$ | $\mathcal{P}$ | $\mathcal{T}$ | $\mathcal{CP}$ | $\mathcal{CT}$ | $\mathcal{PT}$ | $\mathcal{CPT}$ |
|-------|------|------|-------|-------|-------|-------|
| $\mathcal{M}_\square$ | $\mathcal{F}_H$ | $\mathcal{F}_V \mathcal{F}_H$ | $\mathcal{M}_\perp$ | $\mathcal{M}_\perp \mathcal{F}_V$ | $\mathcal{F}_V$ | $\mathcal{M}_\square \mathcal{F}_V$ |

so does a $\mathcal{CPT}$, the transformations $\mathcal{M}_\square$ and $\mathcal{M}_\square \mathcal{F}_V$ are candidates of $\mathcal{C}$ and $\mathcal{CPT}$ for that they preserve the momentum, whereas $\mathcal{M}_\perp$ and $\mathcal{M}_\perp \mathcal{F}_V$ are possibly $\mathcal{CP}$ and $\mathcal{CT}$.

On a single particle state, a $\mathcal{CPT}$ has one more effect than a $\mathcal{C}$ because it also turns $\sigma$, the $z$-component spin, to $-\sigma$. We notice that a $\mathcal{M}_\square \mathcal{F}_V$ affects a braids more than a $\mathcal{M}_\square$ does; it swaps the first and the third elements in the triple of internal twists of a braid besides adding a negative sign to $\Theta$. As a result, although we do not know what of a braid behaves like the $\sigma$, we can now consider the transformation $\mathcal{M}_\square$ as $\mathcal{C}$ and the transformation $\mathcal{M}_\square \mathcal{F}_V$ to be $\mathcal{CPT}$.

Given the three correspondences we now have, it is easy to track down all the rest. As a summary, we list the map between $G_D$ and the group generated by $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ in Table 4.

### 4.2 CPT multiplets of braids

With the $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ we have found, one can see that certain diffeomorphism-inequivalent braids may not be totally different from each other, in the sense that they can belong to the same CPT multiplet. It would be very interesting to see if a CPT multiplet of braids has any characteristic property. It turns out that only actively-interacting braids which belong to a CPT multiplet have a clear and unique common topological property. We would like to formulate this claim as the theorem below.

**Theorem 1.** Each CPT multiplet of actively-interacting braids is uniquely characterized by a non-negative integer, $k$ - the number of crossings of all braids in the multiplet, when they are represented in the unique representation.

*Proof.* This theorem has a two-fold meaning: that all actively-interacting braids with the same number of crossings belong to the same CPT multiplet, and that all actively-interacting braids in a CPT multiplet must have the same number of crossings. We prove the former first. As demonstrated in [6, 5], a braid can interact actively if and only if it is completely reducible from both ends, and its end-nodes are in opposite states for odd number of crossings, and in the same state for even number of crossings. Bearing this in mind and by [5], we can straightforwardly work out the forms of all actively-interacting
braids in the unique representation with \( k \) crossings for \( k \) even

\[
B_1 = +[(T_{1a}, T_{1b}, T_{1c})\sigma_{(ud)k/2}]^+ \\
B_2 = +[(T_{2a}, T_{2b}, T_{2c})\sigma_{(ud)k/2}]^+ \\
B_3 = -(T_{3a}, T_{3b}, T_{3c})\sigma_{(du)k/2}]^- \\
B_4 = -(T_{4a}, T_{4b}, T_{4c})\sigma_{(du)k/2}]^- ,
\]

and for \( k \) odd,

\[
B'_1 = +[(T'_{1a}, T'_{1b}, T'_{1c})\sigma_{d(ud)(k-1)/2}]^- \\
B'_2 = +[(T'_{2a}, T'_{2b}, T'_{2c})\sigma_{d(ud)-(k+1)/2}]^- \\
B'_3 = -(T'_{3a}, T'_{3b}, T'_{3c})\sigma_{u(du)(k-1)/2}]^+ \\
B'_4 = -(T'_{4a}, T'_{4b}, T'_{4c})\sigma_{u(du)-(k+1)/2}]^+ ,
\]

where we have omitted all the external twists for that they are zero in the unique representation. The triple of internal twists of each actively-interacting braid in the unique representation is uniquely determined by the crossing sequence and end-node states of the braid for it to interact actively\[^6\]. In addition, if an exponent in Eq. 6 or 7 is positive, it means, for example, \((ud)^2 = udud\). By adopting from Appendix I the definition of \( X^{-1} \) with respect to \( X \), the meaning of the negative exponents in Eqs. 6 and 7 is clear: for instance, \((ud)^{-2} = d^{-1}u^{-1}d^{-1}u^{-1}\).

It is straightforward to see that if we apply \( M_C \), \( F_V \), and \( M_F \), or according to Table 5 \( C, PT \) and \( CPT \) on \( B_1 \) for even \( k \)'s and \( B'_1 \) for odd \( k \)'s, we get

\[
C(B_1) = -[(T_{1a}, T_{1b}, T_{1c})\sigma_{(du)k/2}]^- \\
PT(B_1) = -[(T_{1c}, T_{1b}, T_{1a})\sigma_{(du)k/2}]^- \\
CPT(B_1) = +[-(T_{1c}, T_{1b}, T_{1a})\sigma_{(ud)k/2}]^+
\]

for even \( k \)'s, and

\[
C(B'_1) = -[(T'_{1a}, T'_{1b}, T'_{1c})\sigma_{u(du)(k-1)/2}]^+ \\
PT(B'_1) = -[(T'_{1c}, T'_{1b}, T'_{1a})\sigma_{u(du)(k-1)/2}]^+ \\
CPT(B'_1) = +[-(T'_{1c}, T'_{1b}, T'_{1a})\sigma_{d(ud)-(k+1)/2}]^-
\]

for odd \( k \)'s. Comparing Eq. 6 with Eq. 8 and Eq. 7 with Eq. 9 one can see that the crossing sequence and end node states of \( C(B_1), PT(B_1) \) and \( CPT(B_1) \) are exactly the same as that of \( B_1, B_3 \) and \( B_2 \) respectively, for even \( k \)'s; similar observation holds for odd \( k \)'s as well. As for the internal twists, since they are uniquely determined by crossing sequence and end node states, one must have

\[
(T_{2a}, T_{2b}, T_{2c}) = -(T_{1c}, T_{1b}, T_{1a}) \\
(T_{3a}, T_{3b}, T_{3c}) = (T_{1c}, T_{1b}, T_{1a}) \\
(T_{4a}, T_{4b}, T_{4c}) = -(T_{1a}, T_{1b}, T_{1c})
\]
for braids with even crossings, and similar relations for braids with odd crossings. Therefore, in the unique representation, all actively-interacting braids with \( k \) crossings are related to each other by discrete transformations as following,

\[
\begin{align*}
C(B_1) &= B_4 \\
PT(B_1) &= B_3 \\
CPT(B_1) &= B_2
\end{align*}
\]

for even \( k \)'s, and

\[
\begin{align*}
C(B'_1) &= B'_4 \\
PT(B'_1) &= B'_3 \\
CPT(B'_1) &= B'_2
\end{align*}
\]

for odd \( k \)'s.

Pointed out in the last section, none of the discrete transformations on a braid diagram is able to change the number of crossings of the braid diagram; hence, all braid diagrams in a CPT multiplet must have the same number of crossings. This exhibits the latter meaning of the theorem.

Since in the unique representation, for each number of crossings we have only four actively-interacting braids, which have been shown being related only by three discrete transformations, namely \( C, PT \) and \( CPT \), applying the remaining four discrete transformations, viz \( P, T, CP \) and \( CT \) can not generate new braids with the same number of crossings, which means that their actions must be equivalent to, in certain order, those of \( C, PT, CPT \), and the identity \( 1 \) on actively-interacting braids.

As for braids that do not interact actively, we do not have a similar theorem. In fact, in the unique representation, for non-actively interacting braids with \( m \) crossings \((m > 1)\), we can always find those not related to each other by any discrete transformation. Here is an example with \( m = 2 \): for the braid, \( s[(T_a, T_b, T_c)\sigma_{ud-1}]s' \), and the braid, \( s[(T'_a, T'_b, T'_c)\sigma_{ud}]s' \), whatever their internal twists and end-node states are, it is straightforward to see that they can never be transformed into each other by the discrete transformations.

Nevertheless, it is still true for non actively-interacting braids that all braids in a CPT multiplet have the same number of crossings when they are represented in the same type of representation. This is so simply because discrete transformations do not change the representation type and the number of crossings of a braid.

### 4.3 Interactions under C, P, and T

We have seen the effects of \( C, P, \) and \( T \) on single braid excitations, it is then natural and important to discuss the action of these discrete transformations on braid interactions, defined in \([6]\) graphically, and in \([8]\) algebraically. Braid interactions turn out to be invariant under CPT, and more precisely, under \( C, P, \) and \( T \) separately.
This type of interaction always involve two braids, one of which must be actively-interacting. As pointed out in [8], in dealing with an interaction it is convenient to represent the actively-interacting braid, say $B$, by one of its trivial representatives, and represent the other braid, say $B'$, by its unique representative. Although [8] shows that the right-interaction of $B$ on $B'$, namely $B + B'$, and the left-interaction $B' + B$ are not equal in general, for the purpose here we need only to consider either of the two cases because the other case follows similarly; let us take $B + B'$ to study.

[8] has proven that the interaction $B + B'$ is independent of the trivial braid diagram representing $B$, and suggests to choose the one with zero right external twist to represent $B$. Thus we let $B = \frac{S_1}{T_1}[T_a, T_b, T_c]^S_0$, and $B' = \frac{S_1}{T_1}((T_a', T_b', T_c')\sigma X)_0^{S_r}$. Given this, we can adopt from [8] the algebraic form of $B + B'$ as follows

$$B'' = B + B' = \frac{S_1}{T_1}[(T_a + T_a', T_b + T_b', T_c + T_c')\sigma X]_0^{S_r},$$

where $S = S_1$, such that the so-called interaction condition is satisfied [6, 8]. There is a subtlety, the braid $B''$ in Eq. 11 is not the standard result which has zero external twist and should be obtained from this $B''$ by a rotation on its left end-node to remove the external twist, $T_l$. We do so to reduce the complexity of this proof, which has no harm because rotations, as equivalence moves, obviously commute with discrete transformations.

We now show that this interaction is invariant under a charge conjugation, i.e. to show $C(B) + C(B') = C(B + B')$. By Tables 4 and 1, we readily obtain

$$C(B) = C\left(\frac{S_1}{T_1}[T_a, T_b, T_c]^S_0\right) = -\frac{S_1}{T_1}[-T_a, -T_b, -T_c]^S_0$$

$$C(B') = C\left(\frac{S_1}{T_1}((T_a', T_b', T_c')\sigma X)_0^{S_r}\right) = -\frac{S_1}{T_1}[-(T_a', T_b', T_c')\sigma X_{(X)}]_0^{S_r}.$$ (12)

Hence, by the same token as how this kind of interaction is carried out in Eq. 11 we get

$$C(B) + C(B') = -\frac{S_1}{T_1}[-T_a, -T_b, -T_c]^S_0 + -\frac{S_1}{T_1}[-(T_a', T_b', T_c')\sigma X_{(X)}]_0^{S_r}$$

$$= -\frac{S_1}{T_1}[-(T_a + T_a', T_b + T_b', T_c + T_c')\sigma X_{(X)}]_0^{S_r}.$$ (13)

Now we directly apply a C-transformation on the braid $B''$ in Eq. 11 which leads to

$$C(B'') = C\left(\frac{S_1}{T_1}[(T_a + T_a', T_b + T_b', T_c + T_c')\sigma X]_0^{S_r}\right) = -\frac{S_1}{T_1}[-(T_a + T_a', T_b + T_b', T_c + T_c')\sigma X_{(X)}]_0^{S_r},$$ (14)

which is exactly the same as the RHS of Eq. 13 Therefore, due to the generality of $B$ and $B'$ all interactions are invariant under charge conjugation.

We now move on to the case of parity transformation. It is important to note that a discrete transformation acts on the whole process of an interaction - in particular the
complete states before and after the interaction. As a result, in view of the fact from Table I that a P-transformation, i.e. the $\mathcal{F}_H$, exchanges the two end-nodes, reverses the crossing sequence, and exchanges the twists on the left and on the right of the crossing sequence of a braid, it should also switch the positions of the two braids involved in an interaction before the interaction happens. That is to say, to show the invariance of an interaction, say $B + B' = B''$ (the same braids as above), under $P$, one needs to prove that the left-interaction, $\mathcal{P}(B') + \mathcal{P}(B) = \mathcal{P}(B'')$, holds.

Applying a P-transformation on $B$ and $B'$ respectively brings us

$$\mathcal{P}(B) = \mathcal{P}\left(\frac{S}{T_i}[T_a, T_b, T_c]_0^S\right) = -\frac{S}{0}[T_a, T_b, T_c]_{T_i}^{-S}$$

$$\mathcal{P}(B') = \mathcal{P}\left(\frac{S}{0}[(T'_a, T'_b, T'_c)\sigma_X]_0^{S_i}\right) = -\frac{S}{0}[(T'_a, T'_b, T'_c)\sigma_{R(X)}]_{T_i}^{-S_i}, \quad (15)$$

where $(T'_a, T'_b, T'_c)\sigma_X = (T'_a, T'_b, T'_c)$. Then according to [8], the left interaction of $\mathcal{P}(B)$ and $\mathcal{P}(B')$ reads

$$\mathcal{P}(B') + \mathcal{P}(B) = -\frac{S}{0}[(T'_a, T'_b, T'_c)\sigma_{R(X)}]_{T_i}^{-S_i} + -\frac{S}{0}[T_a, T_b, T_c]_{T_i}^{-S}$$

$$= -\frac{S}{0}[(T'_a, T'_b, T'_c) + \sigma_{-1}(T_a, T_b, T_c)]_{T_i}^{-S_i}$$

$$= -\frac{S}{0}[(T'_a + T_a, T'_b + T_b, T'_c + T_c)\sigma_X \sigma_{R(X)}]_{T_i}^{-S_i} \quad \text{Eq. (19)}$$

$$= -\frac{S}{0}[(T'_a + T_a, T'_b + T_b, T'_c + T_c)\sigma_X \sigma_{R(X)}]_{T_i}^{-S_i} \quad (16)$$

Note that in the last line of the equation above the $\sigma_X$ and $\sigma_{R(X)}$ should not be contracted by Eq. (17) and we did not do so, because the $\sigma_{R(X)}$ is present there not only for denoting the permutation but also for recording the crossing sequence, namely $R(X)$, of the resulted braid.

If we directly apply a P-transformation on $B''$ in Eq. (11), we attain

$$\mathcal{P}(B'') = \mathcal{P}\left(\frac{S}{T_i}[(T_a + T'_a, T_b + T'_b, T_c + T'_c)\sigma_X]_{0}^{S_i}\right)$$

$$= -\frac{S}{0}[(T'_a + T_a, T'_b + T_b, T'_c + T_c)\sigma_X \sigma_{R(X)}]_{T_i}^{-S_i},$$

for that the triple $(T'_a + T_a, T'_b + T_b, T'_c + T_c)\sigma_X$ is already the original triple of internal twists on the right of $X$, which should become the left one of $\mathcal{P}(B'')$ due to the effect of parity transformation. This equation is precisely the one on the RHS of Eq. (16). Therefore, the invariance of braid interaction under parity transformation is established.

Regarding interactions under time reversal there is a subtlety. The time reversal we have found is with respect to a single braid excitation; however, an interaction involves

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Footnote: The interaction condition is automatically satisfied in this way because the neighboring nodes of $B'$ and $B$ again are in the same state and have no twist on their common edge, after switching their positions. Please check [8] for details on the interaction condition.
the time evolution of a spin network under dynamical moves. To apply our T-transformation to an interaction, one should reverse all the dynamical moves, e.g. a $1 \rightarrow 4$ move taken in the interaction becomes a $4 \rightarrow 1$ move under the time reversal, at the same time. In this sense, to show the invariance of the interaction, $B + B' \rightarrow B''$, it suffices to show that $T(B'') \rightarrow T(B') + T(B)$, where the positions of $B$ and $B'$ are swapped as in the case of parity transformation. It is straightforward to prove the invariance of interactions under time reversal; the procedure is similar to that of the previous two cases, and is thus not explicitly presented here.

Consequently, all braid interactions of the type defined in \[6, 8\] are invariant under C, P, T, and any combination of them. This is not the case of the Standard Model of particles because of the absence of CP-violation. A possible reason is that all our current interactions are deterministic. However, it is suggested in \[8\] that one may consider superpositions of braids by possibly taking spin network labels into account. This is to be discussed in the next section.

It is however necessary to remark that the current study of discrete transformations of braids would not be impact by just adding spin network labels in a straightforward way in to our scheme. A reason is that the discrete transformations found in this paper do not change the spin network label of each existing edge of the network. One may try to construct discrete transformation of braids which change the spin network labels, but there could be many arbitrary ways of doing this and no a priori reason of making a special choice.

## 5 Discussion and conclusion

The results we have obtained so far are purely based on the algebra of the action of discrete transformations on topological quantities characterizing braids. Although our braids live on embedded spin networks, spin network labels have yet not been incorporated along this research line; spin networks are treated as framed graphs with merely topological properties. However, for reasons which will be clear soon, there is a necessity for spin network labels to be included.

The mapping between the discrete transformations of braids and the C, P, and T were determined without referring to the definition of the spin of a braid. This is because our analysis indicates that spin cannot be constructed out of the conserved topological quantities we have in hand of braids. The reasons are as follows.

In our language crossings of a braid and twists on the strands of the braid are on an equal footing due to equivalence moves which can trade crossings with twists or vice versa, which leads to the effective twist number $\Theta$ of a braid, a conserved quantity independent of the representative of the braid. For the aforementioned reason we may identify $\Theta$ or certain appropriate function of it as the charge of a braid. There is no a consistent way to directly associate $\Theta$ with spin as well. Furthermore, since in the context of particle physics, charge is a result of $U(1)$ gauge symmetry (and our twists are also related...
to $U(1)$ as previously explained), while spin results from Poincaré symmetry, a space-time symmetry, charge and spin thus could not be unified by superficially manipulating $\Theta$.

Spin network labels are usually representations of a gauge group or of a quantum group. In both of the original spin network proposal[10] and the Loop Quantum Gravity, spin network labels are $SU(2)$ or $SO(3)$ representations. In the framed case, one uses, for example, the quantum $SU(2)$, namely $SU_q(2)$. Consequently, it is more reasonable to associate the spin of a braid with the spin network labels on the strands and/or external edges, and the intertwiners at the end-nodes of the braid. If one decorate the 4-valent spin networks by representations of gauge groups other than $SU(2)$ and $SO(3)$, say $SU(3)$, it may be possible to define color charges of braids as well.

If we agree on this, then Table 2 gives us a hint how we would find spin for braids. The time reversal is identified with the discrete transformation $FVF$. Since a T-transformation flips the $z$-component of the spin of a single-particle state, and because external twists and internal twists of a braid are not separately conserved quantities independent of representatives of the braid, by the last line of Table 2, the only effect of $FVF$ on a braid, which possibly corresponds to the change of spin due to time reversal, is then the exchange of the top and the bottom strands of the braid, which is implied by the exchange of the top and the bottom internal twists explicitly shown in the table. This implies that the spin network labels respectively on the top and the bottom strands are also subject to exchange under a $FVF$, or simply a T-transformation. In other words, we would like to have a way of combining labels and/or intertwiners of a braid, which changes sign when the label on the top strand and the one on the bottom strand of the braid are exchanged. The sign factor of the action of any discrete transformation involving a time reversal, and other phase factors, in Table 3 will appear accordingly. Unfortunately, the precise form of this construction is unavailable for the moment; its discovery may require a complete and systematic approach which takes spin network labels into account.

Our observation that all interactions of braids are invariant under C, P, and T separately seems to indicate an issue that the interactions of braids we have studied so far are deterministic, in the sense that an interaction of two braids produces a definite new braid. Nevertheless, this may not be a problem at all because it could be due to the simple fact that we have only worked with definite vertices of interactions. In terms of vertices we have definite result for an interaction as to the case in QFT; this is similar to what have been done in spin foam models or group field theories. Besides, one can certainly argue that if our braids are more fundamental entities, the CP-violation in particle physics does not necessarily exist at this level. Putting this CP-violation problem aside; however, a fully quantum mechanical picture should be probabilistic³.

If we tend to take this as an issue, we may try to work with superpositions of braids and interactions of braids resulting in superposition of braids. A possible way out is to consider braids with the same topological structure but different sets of spin network labels as physically different. One may adopt from spin foam models the methods which

³One should note that a few theoretical physicists may not agree on this.
can assign amplitudes to the dynamical moves, namely the dual Pachner moves restricted by the stability condition\cite{6,11}, of the embedded 4-valent spin networks. A dynamical move such as $2 \rightarrow 3$ and $1 \rightarrow 4$ may then give rise to outcomes with the same topological configuration but different spin network labels; each outcome has a certain probability amplitude. As a result, an interaction of two braids may give rise to superposed braids, each of which has a certain probability to be observed, with the same topological content but different set of spin network labels. With this, CP-violating interactions may arise.

There seems to be a more elegant and unified way to resolve all the aforementioned issues once for all, which is the so-called tensor category, or more specifically the braided tensor category with twists. As previously explained, a twist of a strand of a braid can be interpreted as characterizing a non-trivial isomorphism from $U(1)$ to $U(1)$. However, the concept of twist can be generalized to vector spaces other than representation spaces of $U(1)$. This is the way how twists are defined in the language of tensor categories. In this manner, we may view spin network labels as they represent generalized framing of spin networks other than the $U(1)$ framing we have just studied, such that generalized twists can arise. The consequence is our twists and spin network labels, and hence gauge symmetries and space-time symmetries may be unified in this way.

Tensor categories naturally use isomorphisms between tensor products of vector spaces to account for braiding. This can be understood, for example, from the solutions of (Quantum) Yang-Baxter Equations. However, it is important to note that our braids are special because each of them has two 4-valent end-nodes and two external edges. All these will exert further constraints on the possible tensor categories we can use, or motivate new types of tensor categories. Tensor-categorized 4-valent braids and evolution moves may be evaluated by the relevant techniques already defined in theories of tensor category or new techniques adapted to our case.

There are also other works on unification of gravity and matter or on emergent matter degrees of freedom, which indicate that tensor category might be a correct underlying mathematical language towards this goal. The string network condensation by Wen et al\cite{12} is such an example.

In conclusion, we have found seven discrete transformations of 3-strand braids and mapped them to C, P, T, and their products. Along with this, the effective twist number of a braid has been demonstrated to be responsible for the electric charge of the braid. It is very interesting that in the braid representation without external twist, all actively-interacting braids of the same number of crossings form a CPT multiplet, whereas there are non actively-interacting braids of the same number of crossings which cannot be transformed into each other by any legal discrete transformation of our braids. This will help us to find a deeper correspondence between our braids and matter particles. Furthermore, braid interactions have been proven invariant under C, P, and T separately.

We have explained the necessity to incorporate spin network labels into this approach. This allows us to argue that the spin of a braid is related to the spin network labels of the braid. In addition, probability amplitudes of braid propagation and interaction may also be constructed with the help of spin network labels. A possible future direction regarding
a more generalized approach, in terms of tensor categories, along this research line is pointed out.

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Appendix I: Atomic discrete operations on braids

It is useful to represent a legal discrete transformation in an algebraic form. However, a discrete transformation of a braid normally acts on all elements in the characterizing 8-tuple of the braid. Consequently, one can split an action of a discrete transformation to minimal sub-operations on components of the 8-tuple. These minimal sub-transformations are named atomic discrete transformations, or atomic operations for short. An atomic operation can only act on one and only one type of component in the characterizing 8-tuple of a braid because otherwise it can be divided again and hence is not atomic.

Let us be precise. The characterizing 8-tuple of a braid, say \( \{ T_l, S_l, T_a, T_b, T_c, X, S_r, T_r \} \), consists of four types of components-to wit \((T_l, T_r)\), the pair of external twists, \((T_a, T_b, T_c)\), the triple of internal twists, the crossing sequence \(X\), and the pair of end-node states, \((S_l, S_r)\). In some cases, the external twists and internal twists can be considered together due to the fact that they are transformed simultaneously in the same manner. An atomic operation is only allowed to act on one of these four types or on the set of all twists. In addition, if an atomic operation acts on an element in the 8-tuple it must also acts on all other elements of the same type in a similar way because otherwise Condition 1 would be violated. This will be clarified case by case. We now try to sort out all legal atomic operations.

Since twists, crossings and end-node states can take both positive and negative values in our framework, it is natural to have discrete operations which flip their values. There are three such atomic operations, called inversions.

Definition 4. The inversion of the end-node states of a braid, denoted by \( \mathcal{I}_S \), is an atomic operation flipping the signs of both end-node states of the braid. That is,

\[
\mathcal{I}_S : (S_l, S_r) \mapsto (\bar{S}_l, \bar{S}_r).
\]
**Definition 5.** The inversion of the crossing sequence $X$ of a braid, denoted by $\mathcal{I}_X$, is an atomic operation taking each crossing in $X$ to its inverse, namely

\[
\mathcal{I}_X : \begin{align*}
    u &\mapsto u^{-1} \\
    d &\mapsto d^{-1}
\end{align*}
\]

\[X = x_1x_2\cdots x_n \mapsto x_1^{-1}x_2^{-1}\cdots x_n^{-1}.
\]

The integral value of each crossing is negated by this operation, so is the crossing number of the braid. In addition, we clearly have $\sigma_X = \sigma_{\mathcal{I}_X}$. The two atomic operations above must act on both end-nodes and on all crossings of $X$ respectively. Otherwise, one cannot combine them to make a discrete transformation which is legal for arbitrary braids.

**Definition 6.** The inversion of the twists of a braid, denoted by $\mathcal{I}_T$, is an atomic operation which multiplies a $-1$ to every twist of the braid, i.e.

\[
\mathcal{I}_T : \{T_l, T_a, T_b, T_c, T_r\} \mapsto \{-T_l, -T_a, -T_b, -T_c, -T_r\}.
\]

That this atomic discrete transformation acts on all the twists, internal and external, of a braid is largely due to Condition 1. The reason is that if not all but one or several of its twists are flipped, there is no way to combine such an operation with other atomic operations, albeit all other operations are legal, to keep any actively-interacting braids active, since twists play a key role in the activity of a braid. There are two more atomic operations acting on $X$.

**Definition 7.** The reversion is an atomic operation, denoted by $\mathcal{R}$, which reverses the order of the crossings in a crossing sequence $X$:

\[
\mathcal{R} : X = x_1x_2\cdots x_n \mapsto x_nx_{n-1}\cdots x_1.
\]

It is also useful for the sake of calculation to define $X^{-1}$ to be the combined result of $\mathcal{I}_X$ and $\mathcal{R}$ on $X$, i.e. $X^{-1} = \mathcal{I}_X\mathcal{R}(X)$. Note that for the permutation induced by $X$, $\sigma_X^{-1} \neq \sigma_{X^{-1}}$ in general. However, it is quite clear that

\[
\sigma_X\sigma_{\mathcal{R}(X)} = \sigma_{\mathcal{R}(X)}\sigma_X^{-1} \equiv 1
\]

(17)

The meaning of this equation must be understood from its action on triples of internal twists. That is,

\[(T_a, T_b, T_c)\sigma_X\sigma_{\mathcal{R}(X)} = (T_a, T_b, T_c).
\]

Keeping in mind that $\sigma_X^{-1}$ is a left-acting function, if we apply a $\sigma_X^{-1}$ to the left of both sides of the above equation, we get

\[
\sigma_X^{-1}(T_a, T_b, T_c)\sigma_X\sigma_{\mathcal{R}(X)} = \sigma_X^{-1}(T_a, T_b, T_c)
\]

\[
\Rightarrow (T_a, T_b, T_c)\sigma_{\mathcal{R}(X)} = \sigma_X^{-1}(T_a, T_b, T_c).
\]

(18)
Similary, what follows is also true:

$$\sigma_{R(X)}^{-1}(T_a, T_b, T_c) = (T_a, T_b, T_c)\sigma_X.$$  \hspace{1cm} (19)

**Definition 8.** A *chain shift* is an atomic operation on $X$, denoted by $S_c$, shifting every upper crossing in $X$ to a lower one and a lower one to an upper one, with however, the crossing’s chirality intact. That is,

$$S_c : \forall x_i \in X, x_i \mapsto \begin{cases} d, & \text{if } x_i = u \\ u, & \text{if } x_i = d \end{cases}, \quad i = 1 \ldots n.$$ 

The above two atomic operations on $X$ must apply to every crossing in $X$ simultaneously because otherwise an alternating braid could be transformed into a non-alternating braid and vice versa, which certainly causes the violation of Condition 1 if they are part of a discrete transformation of a braid.

There is actually a hidden triple in the characterizing 8-tuple of a braid, i.e. the triple of internal twists on the right of the crossing sequence $X$, $(T'_a, T'_b, T'_c)$. It is not explicitly included in the 8-tuple because is is related to the triple $(T_a, T_b, T_c)$ by the induced permutation $\sigma_X$, as aforementioned. However, one can have a transformation which exchanges these two triples.

**Definition 9.** The *exchange of triples of internal twists* of a braid is an atomic operation, denoted by $E_T$, doing the following:

$$E_T : (T_a, T_b, T_c) \mapsto (T'_a, T'_b, T'_c)$$ 

and vice versa, where $(T_a, T_b, T_c)\sigma_X = (T'_a, T'_b, T'_c)$.

An exchange of triples of internal twists is usually accompanied by an exchange of the two external twists of a braid.

**Definition 10.** The *exchange of external twists* of a braid, $E_{T_e}$, is an atomic operation, such that

$$E_{T_e} : (T_l, T_r) \mapsto (T_r, T_l).$$

The last possible atomic discrete transformation on the twists is defined as follows.

**Definition 11.** The *twist swap* is an atomic operation, denoted by $S_T$, which swaps the top and bottom internal twists of a braid:

$$S_T : (T_a, T_b, T_c) \mapsto (T_c, T_b, T_a).$$

Finally, one can exchanges the two end-node states of a braid.

**Definition 12.** The *exchange of end-node states*, $E_S$, is an atomic operation, such that

$$E_S : (S_l, S_r) \mapsto (S_r, S_l).$$

An important remark is that all above atomic operations in fact act on braids. For simplicity nevertheless, we only show here their definitions their effects on the relevant characterizing quantities of a braid. Another remark is that all atomic operations commute with each other and hence their relative positions in a combination as a discrete transformation do not matter.
Appendix II: Examples of braids under C, P, and T

The figure above illustrates two examples of braids under the action of $G_D$. The first row shows the two original braid diagrams: the left one is a right propagating braid in zero external twist representation, while the right one is an actively-interacting braid represented by a trivial braid diagram. The thick arrow on the lower left or right corner of a braid diagram indicates the propagation chirality of the braid.

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