SHARP RIESZ-FEJÉR INEQUALITY FOR HARMONIC HARDY SPACES

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Abstract. We prove sharp version of Riesz-Fejér inequality for functions in harmonic Hardy space $h^p(D)$ on the unit disk $D$, for $p > 1$, thus extending the result from [9] and resolving the posed conjecture.

1. Introduction

Let $D$ denote the unit disk in the complex plane. For holomorphic or harmonic function $f$ with $M_p(r, f)$ we denote the integral means:

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

The space of all holomorphic functions for which $M_p(r, f)$ is bounded for $0 < r < 1$ is the Hardy space $H^p(D)$, while the analogous space of harmonic functions is the harmonic Hardy space $h^p(D)$. Theory of Hardy spaces is a very well developed; for further background about these spaces, we refer reader, for instance, to the books [10] and [13].

One of the interesting results in this theory is the following inequality of Riesz and Fejér from [3]:

$$\int_{-1}^{1} |f(r)|^p dr \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta,$$

that holds for a function $f \in H^p(D)$ for every $0 < p < \infty$, where the values $f(e^{i\theta})$ denote the radial limits of the function $f$.

This inequality was generalized in several directions. Let us mention Beckenbach’s results: the same inequality holds where in place of $|f|^p$ we have a positive logarithmically subharmonic function. Some of generalizations can be found in [1], [2] and [7].

A recent significant result is an analog of this inequality for harmonic Hardy spaces, proved by Kayumov et al. Namely, they proved the next version of Riesz-Fejér inequality:

$$\int_{-1}^{1} |f(re^{is})|^p dr \leq K_p \int_0^{2\pi} |f(e^{i\theta})|^p d\theta,$$

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for all $s \in [0,2\pi]$ with $K_p = \frac{1}{2 \cos \frac{p \pi}{2}}$ for $1 < p < 2$ and $K_p = 1$ for $p \geq 2$.

The inequality is sharp for $p \in (1,2]$ and the authors made a conjecture that the inequality holds with $K_p = \frac{1}{2 \cos \frac{p \pi}{2}}$ for all $1 < p < \infty$. They also proved $K_p \geq \frac{1}{2 \cos \frac{p \pi}{2}}$ for these $p$, so the inequality with this $K_p$ would be the optimal one. The inequality for $1 < p < 2$ depends on an inequality of Kalaj, proved in [8] and Lozinski’s inequality from [11]. The proof of the first of these inequalities uses the plurisubharmonic method invented in [5]; recent update on this method can be found in [12]. The proof of Riesz-Fejér inequality for $p > 2$ uses a result of Frazer from [4].

The purpose of this paper is to prove the sharp version of Riesz-Fejér inequality for harmonic Hardy spaces for every $1 < p < \infty$ using Schur test for Poisson extension operator. Namely, we get the following theorem:

**Theorem 1.1.** For all $1 < p < \infty$ and $f \in h^p(\mathbb{D})$, we have:

$$\int_{-1}^{1} |f(re^{i\alpha})|^p |dr| \leq \frac{1}{2 \cos \frac{s \pi}{2}} \int_{0}^{2\pi} |f(e^{i\theta})|^p |d\theta|,$$

with $s \in [0, 2\pi]$.

Because of the rotational invariance of norm of functions in $h^p(\mathbb{D})$, we can consider only the case of $s = 0$, without any loss of generality.

2. Proof of the main theorem

We will prove Theorem 1.1 using the following version of Schur test as can be found in [6]:

**Lemma 2.1.** Let $X$ and $Y$ be measure spaces equipped with nonnegative, $\sigma$–finite measures and let $T$ be an operator from $L^p(Y)$ to $L^p(X)$ that can be expressed as

$$Tf(x) = \int_Y K(x,y)f(y)dy$$

for some nonnegative function $K(x,y)$. The adjoint operator $T^*$ is now given by

$$T^*f(y) = \int_X K(x,y)f(x)dx.$$

If we can find a measurable $h$ finite almost everywhere, such that:

$$T^*((Th)^{p-1}) \leq C_p h^{p-1}, \quad a.e. \ on \ Y$$

then for all $f \in L^p(Y)$, we have:

$$\int_X |T(f)|^p |dx| \leq C_p \int_Y |f|^p |dy|.$$

We apply the Schur test in the following setting. For spaces $X$ and $Y$ we set $X = [-1,1]$ with Lebesgue measure and $Y = \mathbb{T} = \partial \mathbb{D}$ with normalised arclength measure. Starting from a harmonic $f \in h^p(\mathbb{D})$, we first get the appropriate $f^*(e^{i\theta}) \in L^p(\mathbb{T})$, defined by its radial limits. Now, by acting with the operator $T$ of Poisson harmonic extension, we get:

$$Tf^*(r) = \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos \theta + r^2} f^*(e^{i\theta}) d\theta \frac{d\theta}{2\pi},$$
which is equal to \( f(r) \), because of harmonicity of \( f \). Hence, we easily see that the optimal constant in Riesz-Fejér inequality is equal to the \( p \)-th power of the operator norm of such \( T \). Since we consider \( T \) with normalised measure we have to find an \( h \) such that the constant \( C_p \) is equal to \( \frac{n}{\cos^{p} \theta} \). Also, \( T \) has positive kernel 

\[
K(r,\theta) = \frac{1-r^{2}}{1-2r \cos \theta + r^{2}},
\]

and therefore, it follows that 

\[
T^{*} f(e^{i\theta}) = \int_{-1}^{1} \frac{1-r^{2}}{1-2r \cos \theta + r^{2}} f(r) dr.
\]

We will work with \( h(z) = \Re(1-z^{2})^{-\frac{1}{p}} \). It is easy to find its values on the unit circle so that 

\[
\Re(1-e^{2i\theta})^{-\frac{1}{p}} = \Re(2 \sin \theta e^{i(\theta - \frac{p}{2})})^{-\frac{1}{p}} = 2^{-\frac{1}{p}} \sin \theta \cos \left( \frac{\pi}{2p} - \frac{\theta}{p} \right),
\]

for \( 0 \leq \theta \leq \pi \), 

\[
\Re(1-e^{2i\theta})^{-\frac{1}{p}} = \Re(2 \sin \theta e^{i(\theta - \frac{p}{2})})^{-\frac{1}{p}} = 2^{-\frac{1}{p}} |\sin \theta|^{-\frac{1}{p}} \cos \left( \frac{\pi}{2p} - \frac{\theta - \pi}{p} \right),
\]

for \( \pi \leq \theta \leq 2\pi \), while on the real line we have 

\[
\Re(1-z^{2})^{-\frac{1}{p}} = (1-r^{2})^{-\frac{1}{p}}.
\]

From the fact that 

\[
T h(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-r^{2}}{1-2r \cos \theta + r^{2}} \Re(1-e^{2i\theta})^{-\frac{1}{p}} d\theta = (1-r^{2})^{-\frac{1}{p}},
\]

for \( -1 < r < 1 \), we find that 

\[
T^{*}((Th)^{p-1}) = \int_{-1}^{1} \frac{1-r^{2}}{1-2r \cos \theta + r^{2}} (1-r^{2})^{-\frac{p-1}{p}} dr
\]

\[
= \int_{-1}^{1} \frac{(1-r^{2})^{\frac{1}{p}}}{1-2r \cos \theta + r^{2}} dr.
\]

Since \( \int_{-1}^{1} \frac{(1-r^{2})^{\frac{1}{p}}}{1-2r \cos \theta + r^{2}} dr = \int_{-1}^{1} \frac{(1-r^{2})^{\frac{1}{p}}}{1+2r \cos \theta + r^{2}} dr \), and substituting \( r \) with \(-r\) in the last integral, we easily see that it is enough to prove 

\[
\int_{-1}^{1} \frac{(1-r^{2})^{\frac{1}{p}}}{1-2r \cos \theta + r^{2}} dr \leq \frac{\pi}{\cos^{p} \frac{2\theta + 1}{p}} 2^{-\frac{p-1}{p}} \sin \left( \frac{\pi}{2p} - \frac{\theta}{p} \right),
\]

for \( 0 < \theta < \pi \), i.e. \( T^{*}((Th)^{p-1}) \leq C_{p}h^{p-1} \) almost everywhere on \( 0 < \theta < \pi \) and consequently on the whole domain.

Introducing a change of variables \( \frac{1-r^{2}}{1+r^{2}} = y \cot \frac{\theta}{2} \) in the integral, we have 

\[
\int_{-1}^{1} \frac{(1-r^{2})^{\frac{1}{p}}}{1-2r \cos \theta + r^{2}} dr = \int_{0}^{\infty} \frac{1}{1-2 \cos \theta} \frac{[1-\left( y \cot \frac{\theta}{2} \right)^{-1}]^{\frac{1}{p}}}{y \cot \frac{\theta}{2} + 1 + \left( y \cot \frac{\theta}{2} \right)^{2}} dy
\]

\[
= \int_{0}^{\infty} \frac{2 \cot \frac{\theta}{2} y^{\frac{\theta}{2} - 1}}{1 - 2 \cos \theta \cot \frac{\theta}{2} y^{\frac{\theta}{2} + 1} + (y \cot \frac{\theta}{2})^{2}} dy
\]
which is positive, since the integrand is positive for all \( x \in [0, \pi] \). Differentiating twice with respect to \( \theta \), we get:

\[
F''(\theta) = \frac{1}{2p} \int_0^{\frac{\pi}{2}} \Phi(x, \theta) dx,
\]

which is positive, since the integrand

\[
\Phi(x, \theta) = \frac{\sin^{\frac{1}{p}} x \cos^{\frac{1}{p}} x}{\left( (\sin(x + \frac{\theta}{2}))^{\frac{2}{2+p}} \right)^{\frac{1}{p}}} \left[ (1 + \frac{2}{p}) \cos^2 \left( x + \frac{\theta}{2} \right) + \sin^2 \left( x + \frac{\theta}{2} \right) \right]
\]

is positive for all \( x \in [0, \frac{\pi}{2}] \) and \( \theta \in [0, \pi] \). Thus, \( F(\theta) \) is convex on \([0, \pi] \).
By (2.1) and change of variable $x = \frac{\pi}{2} - t$, we get:

$$F(0) = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{p}} x \cos^{\frac{1}{p}} x \, dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{p}} t \cos^{-\frac{1}{p}} t \, dt = F(\pi).$$

Also, from the formula for Beta function we have: $F(0) = \frac{1}{2} B\left(\frac{1}{2} - \frac{1}{2p}, \frac{1}{2} + \frac{1}{2p}\right) = \frac{1}{2 \cos \frac{\pi}{2p}}$. □

Using Lemma 2.2, we easily finish the proof of the main inequality. Since $F(\theta)$ is convex, it attains its maximum at the end of the interval $[0, \pi]$, and by the same lemma its values at 0 and $\pi$ are both equal to $\frac{1}{2 \cos \frac{\pi}{2p}}$, hence $F(\theta) \leq \frac{1}{2 \cos \frac{\pi}{2p}}$.

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