Phase diagram of hot magnetized two-flavor color superconducting quark matter

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A two-flavor color superconducting (2SC) Nambu–Jona-Lasinio (NJL) model is introduced at finite temperature $T$, chemical potential $\mu$ and in the presence of a constant magnetic field $\tilde{e}B$. The effect of $(T, \mu, \tilde{e}B)$ on the formation of chiral and color symmetry breaking condensates is studied. The complete phase portrait of the model in $T - \mu$, $\mu - \tilde{e}B$, and $T - \tilde{e}B$ phase spaces for various fixed $\tilde{e}B$, $T$, and $\mu$ is explored. A threshold magnetic field $\tilde{e}B_t \simeq 0.5$ GeV$^2$ is found above which the dynamics of the system is solely dominated by the lowest Landau level (LLL) and the effects of $T$ and $\mu$ are partly compensated by $\tilde{e}B$.

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I. INTRODUCTION

Recently, the study of the properties of quark matter in the presence of strong uniform magnetic fields has attracted much attention. Possible effects caused by strong magnetic fields include magnetic catalysis [1, 2], modification of the nature of electroweak phase transition [3], spontaneous creation of axial currents [4], formation of $\pi_0$ domain walls [5] and chiral density waves [6], chiral magnetic effect [7, 8], and last but not least the influence on possible color-superconducting phases [9–16]. In this paper, we will focus on the magnetic catalysis and its possible effects on the phase diagram of a magnetized two-flavor color superconducting NJL model at finite temperature and chemical potential.\footnote{Magnetic catalysis has various application in cosmology [17], condensed matter physics [18], and particle physics [19–21].} In particular, the dependence of the included meson and diquark masses on thermodynamic parameters, and possible interplay between these parameters on the formation of meson and diquark condensates and on the nature of phase transition will be scrutinized.

At zero temperature, it is known that strong magnetic fields enhance the production of chiral and diquark condensates, albeit through different mechanisms, as it is described in [11]. Whereas magnetic catalysis of chiral symmetry breaking [1] is mainly responsible for dynamical mass generation and enhances the production of chiral condensates by increasing the particle-antiparticle interaction strength, a certain modification in the density of states of charged quarks near the Fermi surface, depending on...
the external magnetic field, reinforces the pairing of charged quarks and is made responsible for the enhancement of diquark production by a penetrating strong magnetic field \[11\]. In other words, in contrast to the effect of magnetic catalysis of chiral symmetry breaking, which is essentially based on a dimensional reduction of the dynamics of fermions from \( D = 3 + 1 \) to \( D = 1 + 1 \) dimensions to the regime of LLL dominance \[1\], the pairing mechanism by color superconductivity, involving the charged quarks near the Fermi surface, is already \( D = 1 + 1 \) dimensional, and therefore the external magnetic field does not lead to any further dimensional reduction \[11\]. It is the goal of the present paper to explore possible effects of finite \( T \) and \( \mu \) on the above mechanisms of chiral and diquark production in the presence of strong magnetic fields. We will show that in our setup a certain threshold magnetic field exists, above which the dynamics of the system is solely dominated by the lowest Landau level and the effects of \( T \) and \( \mu \) are partly compensated by very large \( \tilde{e}B \). The largest observed magnetic field in nature is \( \sim 10^{12} - 10^{13} \) Gauss in pulsars and up to \( \sim 10^{14} - 10^{15} \) Gauss on the surface of some magnetars, where the inner field is estimated to be of order \( \sim 10^{18} - 10^{20} \) Gauss \[22\]. In the early universe, magnetic fields of order \( \sim 10^{47} \) Gauss may be produced at the beginning of inflation \[23\]. Superconductive cosmic strings, if they exist, may have magnetic fields up to \( \sim 10^{47} - 10^{48} \) Gauss in their vicinities \[24\]. In \[25\], it is shown that “the maximum value of magnetic field that delimits the range of values admitted without revising QED” is of order \( \sim 10^{42} \) Gauss. There are also evidences for strong magnetic field creation in non-central heavy ion experiments \[26\]. The early estimate of the magnetic field for the RHIC energy was made in \[27\], where it was shown that the magnitude of the magnetic field for an earlier stage of noncentral Au-Au collision at energy \( \sqrt{s_{NN}} = 200 \) GeV and the impact parameter \( \sim 4 \) fm is about \( eB \simeq 1.3 \, m_{\pi}^2 \simeq 0.025 \) GeV\(^2\), that corresponds to \( B \simeq 4.3 \times 10^{18} \) Gauss.\(^2\) Using a microscopic transport model, the authors in \[28\] estimate the lowest bound of the maximal magnetic field strength at the LHC energy \( \sqrt{s_{NN}} = 4.5 \) TeV with the same impact parameter, to be \( eB \simeq 15 \, m_{\pi}^2 \simeq 0.3 \) GeV\(^2\), which is equivalent to \( B \simeq 5 \times 10^{19} \) Gauss. Numerically, it is not \textit{a priori} clear whether these amounts of magnetic fields are large enough to justify the LLL approximation, as it is done in \[1\] to demonstrate the magnetic catalysis of chiral symmetry breaking and in \[11\] to demonstrate the effect of strong magnetic field on color superconductivity.

We have tried to answer to this question, among others, in \[16\], where a color neutral and dense 2SC-NJL model has been introduced in a (rotated) constant magnetic field at zero temperature. To do this, first the dependence of chiral and color symmetry breaking condensates on the chemical potential \( \mu \) and the rotated magnetic field \( \tilde{e}B \) is determined analytically in a lowest Landau level (LLL) approximation. Then, the meson and diquark masses are computed numerically for arbitrary

\(^2\) Here, \( m_{\pi} = 140 \) MeV. Moreover, \( eB = 1 \) GeV\(^2\) corresponds to \( B = 1.69 \times 10^{20} \) Gauss.
magnetic fields. Comparing these analytical and numerical data, we have found a certain threshold magnetic field $\tilde{e}B_t \simeq 0.45 - 0.5 \text{ GeV}^2$, corresponding to $B \simeq 8 - 8.5 \times 10^{19} \text{ Gauss}$, above which the system turns out to be dominated by LLL. Below $\tilde{e}B_t$, the chiral and diquark mass gaps oscillate with the external magnetic field. These oscillations are the results of the well-known van Alfven-de Haas effect \cite{29}, that occurs whenever Landau levels pass the quark Fermi surface. They are also observed in \cite{21,30}, where the dependence of chiral symmetry breaking mass gaps on constant magnetic fields is explored. Similar oscillations are also perceived in \cite{14}, where diquark mass gap and the magnetization corresponding to the superconducting magnetized color-flavor locked (CFL) phase are determined as a function of external magnetic fields. For $\tilde{e}B > \tilde{e}B_t$, the system enters a “linear regime”, where the mass gaps and the magnetization depend linearly on the external magnetic field. In this regime, where the system is believed to be solely affected by the dynamics of the fermions in the LLL, the numerical data coincide with the analytical results (see \cite{16} for more detail). What concerns the phase diagram of the model introduced in \cite{16}, it is shown that a first order phase transition occurs between the chiral symmetry breaking ($\chi$SB) and the color symmetry breaking (CSC) phases at $T = 0$ and $\mu \sim 350 - 450 \text{ MeV}$. This transition is then followed by a second order phase transition between the CSC phase into the normal quark matter.\footnote{As it is known from \cite{31}, in the regime of low temperature and large chemical potential, the 2SC phase goes over into the three-flavor CFL phase. In a two-flavor model, however, where no CFL phase can be built, only a simple transition from the 2SC to the normal phase is assumed to exist. Same assumption is also made in \cite{16}. As we will note in Sec. III, our numerical results for low temperature and large chemical potential will be only of theoretical nature.}

In the present paper, the same magnetized 2SC model will be considered at finite temperature. We are in particular interested on the additional effects of finite temperature, and, will focus on possible interplay between $T, \mu$ and $\tilde{e}B$ on the formation of chiral and color symmetry breaking condensates and on the nature of phase transitions. Our results may be relevant for the physics of heavy ion collisions, where, recently the question of accessibility of color superconducting quark matter phases is pointed out \cite{32}. The authors in \cite{32} use a Polyakov-NJL (PNJL) model at finite temperature and density and present the corresponding QCD phase diagram including, among others, mixed phase regions of first order transition of 2SC-CFL quark matter and second order 2SC-normal phase transition. From this phase diagram, they conclude that color superconductor phase is already accessible at the present nuclotron-M energies $4 < E < 8 \text{ AGeV}$, and, that possible transition from 2SC to normal quark matter becomes attainable in the planned FAIR-CBM and NICA-MPD experiments at $2 < E < 40 \text{ AGeV}$. In this paper, the additional effect of constant magnetic field, that is believed to be created in non-central heavy ion collisions will be explored from purely theoretical point of view. A complete phenomenological answer to the question of accessibility of color superconducting phases in heavy ion
collisions is out of the scope of this paper, and will be postponed to future publications.

The organization of the paper is as follows: In Sec. II, we introduce our model for the hot magnetized two-flavor superconducting quark matter and determine the corresponding thermodynamic potential. In Sec. III, our numerical results on the dependence of the meson and diquark gaps on chemical potential, magnetic field, and temperature will be presented in Secs. III.A.1 – III.A.3. In Sec. III.B, the complete $T - \mu$, $T - \tilde{e}B$ and $\mu - \tilde{e}B$ phase portraits of the system at various fixed $\tilde{e}B, \mu$ and $T$ will be illustrated. The above mentioned threshold magnetic field, $\tilde{e}B_t$, will be determined by comparing the analytical and the numerical results, corresponding to the second order critical lines of the transition from the $\chi$SB and CSC phases to the normal phase, in a $T - \mu$ plane for $\tilde{e}B = 0.5, 0.7$ GeV$^2$ (Sec. III.B.1) and in a $T - \tilde{e}B$ plane for $\mu = 0$ MeV (Sec. III.B.2). The details of the analytical computations, that lead to second order critical surfaces of these transitions in a $(T, \mu, \tilde{e}B)$ phase space, will be presented in App. A. Section IV is devoted to concluding remarks.

II. MAGNETIZED 2SC QUARK MATTER AT FINITE $T$ AND $\mu$

In Sec. II.A, we briefly review our results from $[16]$ and introduce a two-flavor NJL model including the meson and diquark condensates at finite temperature and density and in the presence of a constant and uniform (rotated) magnetic field. In Sec. II.B, the corresponding one-loop thermodynamic potential will be determined in a mean field approximation.

A. The Model

The Lagrangian density of a two-flavor gauged NJL model is given by

$$
\mathcal{L}_f = \bar{\psi}(x)[i\gamma^\mu(\partial_\mu - ieQA_\mu - igT^8G_\mu^8) + \mu\gamma^0]\psi(x)
+ G_S[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)i\gamma_5\tau\psi(x))^2] + G_D[(i\bar{\psi}_C(x)\varepsilon_f\epsilon_c^3\gamma_5\psi(x))(i\bar{\psi}(x)\varepsilon_f\epsilon_c^3\gamma_5\psi_C(x))].
$$

(II.1)

Here, $\psi^C = C\bar{\psi}^T$ and $\psi^C = \psi^TC$ are charge-conjugate spinors, and $C = i\gamma^2\gamma^0$ is charge-conjugation matrix, $\tau = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices. Moreover, $(\varepsilon_f)_{ij}$ and $(\epsilon_c)^{ab} \equiv (\epsilon_c)^{ab3}$ are antisymmetric matrices in color and flavor spaces, respectively. For a theory with two quark flavors and three color degrees of freedom, the flavor indices $i, j = (1, 2) = (u, d)$, and the color indices $a, b = (1, 2, 3) = (r, g, b)$. The quarks are taken to be massless $m_u = m_d = 0$. The quark chemical potential which is responsible for nonzero baryonic density of quark matter is denoted by $\mu$. Here, $T^8 = \frac{\lambda_8}{2}$, where $\lambda_8 = \frac{1}{\sqrt{3}}$ diag(1, 1, -2) the $8^{th}$ Gell-Mann $\lambda$-matrix. The scalar and diquark couplings are denoted by $G_S$ and $G_D$, respectively. The charge matrix $Q \equiv Q_f \otimes 1_c$, where $Q_f \equiv \text{diag}(2/3, -1/3)$ is the
fermionic charge matrix coupled to $U(1)$ gauge field $A_{\mu}$. The same setup with an additional color chemical potential $\mu_8$, imposing the color neutrality of the theory, is also used in [16] to study the effect of magnetic field on quark matter under extreme conditions. Following the same steps as in [16] to determine the Lagrangian density containing the chiral and diquark condensates in an appropriate Nambu-Gorkov form, we define first the mesonic fields

$$\sigma = -2G_S(\bar{\psi}\psi), \quad \text{and} \quad \pi = -2G_S(\bar{\psi}\gamma^5\tau\psi), \quad (\text{II.2})$$

as well as the diquarks fields

$$\Delta = -2G_D(\bar{\psi}\varepsilon_f\epsilon_8^3\gamma_5\psi), \quad \text{and} \quad \Delta^* = -2G_D(\bar{\psi}\varepsilon_f\epsilon_8^3\gamma_5\psi^C). \quad (\text{II.3})$$

Combining then the gauge fields $A_{\mu}$ and $G_{\mu}^8$, using the “rotated” charge operator $\bar{Q} = Q_f \otimes 1_c - 1_f \otimes \left(\frac{\lambda_8}{2\sqrt{3}}\right)_c$, the rotated massless $U_{em}(1)$ field, $\tilde{A}_{\mu} = A_{\mu}\cos \theta - G_{\mu}^8\sin \theta$, as well as the massive in-medium $8^{th}$ gluon field, $\tilde{G}_{\mu}^8 = A_{\mu}\sin \theta + G^8_{\mu}\cos \theta$ can be derived (see [16] for more details). Replacing $\tilde{A}_{\mu}$ with an external gauge field $\tilde{A}_{\mu}^{\text{ext}} = (0,0,Bx,0)$ in the Landau gauge, a constant rotated background $U(1)$ magnetic field directed in the third direction $\tilde{B} = Be_3$ is induced. Neglecting then the massive gauge boson $\tilde{G}_{\mu}^8$, the total modified bosonized Lagrangian density, $\tilde{L} = \tilde{L}_k + \tilde{L}_f$, in the presence of a uniform magnetic field arises. It consists of a kinetic term

$$\tilde{L}_k = -\left(\frac{\sigma^2}{4G_S} + \frac{|\Delta|^2}{4G_D} + \frac{B^2}{2}\right), \quad (\text{II.4})$$

and an interaction term

$$\tilde{L}_f = \bar{\psi}(x)[i\gamma^\mu(\partial_\mu - ie\tilde{A}_{\mu}^{\text{ext}}) + \mu\gamma^0 - \sigma]\psi(x) - \frac{1}{2}[\Delta^*(i\bar{\psi}^C(x)\varepsilon_f\epsilon_8^3\gamma_5\psi(x)) + \Delta(i\bar{\psi}(x)\varepsilon_f\epsilon_8^3\gamma_5\psi^C(x))]. \quad (\text{II.5})$$

Assuming that the vacuum of the system is characterized by $\langle \sigma \rangle \neq 0$ and $\langle \bar{\pi} \rangle = 0$, we have neglected the $\bar{\pi}$ mesons. Moreover, using the definition of the rotated charge operator $\bar{Q}$ in a 6-dimensional flavor-color representation $(u_r,u_g,u_b,d_r,d_g,d_b)$, the rotated $\tilde{q}$ charges of different quarks, in units of $\hat{e} = e\cos \theta$, are given by

\[
\begin{array}{c|cccccc}
\text{quarks} & u_r & u_g & u_b & d_r & d_g & d_b \\
\hline
\tilde{q} & +\frac{1}{2} & +\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0
\end{array}
\]

To bring the above Lagrangian density $\tilde{L}_f$ in a more appropriate Nambu-Gorkov form, we introduce at this stage the rotated charge projectors $\Omega_{\tilde{q}}$

$$\Omega_0 = \text{diag}(0,0,0,0,0,1), \quad \Omega_1 = \text{diag}(0,0,1,0,0,0),$$

$$\Omega_{+\frac{1}{2}} = \text{diag}(1,1,0,0,0,0), \quad \Omega_{-\frac{1}{2}} = \text{diag}(0,0,0,1,1,0), \quad (\text{II.6})$$
that satisfy $\tilde{Q}\Omega_{\tilde{q}} = \tilde{q}\Omega_{\tilde{q}}$. The Nambu-Gorkov bispinors are then defined by
\[
\Psi_{\tilde{q}} = \begin{pmatrix} \psi_{\tilde{q}} \\ \psi_{\tilde{q}}^C \end{pmatrix},
\]
where $\psi_{\tilde{q}}(x) \equiv \Omega_{\tilde{q}}\psi(x)$. In terms of $\bar{\Psi}_{\tilde{q}}$ and $\Psi_{\tilde{q}}$, the Lagrangian density $\tilde{L}_f$ from (II.5) in the Nambu-Gorkov form reads (for more details see [16])
\[
\tilde{L}_f = \frac{1}{2} \sum_{\tilde{q} \in \{0,1,\pm \frac{1}{2}\}} \bar{\Psi}_{\tilde{q}}(x)S_{\tilde{q}}\Psi_{\tilde{q}}(x).
\]
For $\tilde{q} \in \{0,1\}$, $S_{\tilde{q}}$ is given by
\[
S_{\tilde{q} \in \{0,1\}} \equiv \begin{pmatrix} [G_{\tilde{q}}^+]^{-1} & 0 \\ 0 & [G_{\tilde{q}}^-]^{-1} \end{pmatrix},
\]
and for $\tilde{q} \in \{-\frac{1}{2}, +\frac{1}{2}\}$, it reads
\[
S_{\tilde{q} \in \{-\frac{1}{2}, +\frac{1}{2}\}} \equiv \begin{pmatrix} [G_{\tilde{q}}^+]^{-1} & -\kappa\Omega_{-\tilde{q}} \\ -\kappa^\prime\Omega_{\tilde{q}} & [G_{\tilde{q}}^-]^{-1} \end{pmatrix}.
\]
Here,
\[
[G_{\tilde{q}}^\pm]^{-1} \equiv \gamma^\mu(i\partial_\mu + \tilde{e}\tilde{q}\tilde{A}_\mu - \sigma \pm \mu\delta_\mu 0),
\]
and $\kappa^{ij,ab}_{\alpha\beta} \equiv i\Delta\tau^i_2\lambda_2^{ab}\gamma^5_{\alpha\beta}$ as well as $\kappa' \equiv \gamma_0\kappa^\dag\gamma_0 = i\Delta^\star\tau^2_2\lambda^5_2$. In the next section, the Lagrangian density $\tilde{\mathcal{L}}$ with $\tilde{\mathcal{L}}_k$ in (II.4) and $\tilde{\mathcal{L}}_f$ in (II.7)-(II.10) will be used to determine the thermodynamic potential of this model in the mean field approximation.

**B. Thermodynamic potential**

The quantum effective action of the theory, $\Gamma_{\text{eff}}$, is defined by integrating out the fermionic degrees of freedom using the path integral
\[
e^{i\Gamma_{\text{eff}}[\sigma,\Delta,\Delta^\star]} = \int D\psi D\bar{\psi} \exp \left( i \int d^4x \tilde{\mathcal{L}} \right).
\]

At one loop level, it consists of two parts: the tree-level and the one-loop effective action, $\Gamma_{\text{eff}}^{(0)}$ and $\Gamma_{\text{eff}}^{(1)}$. In the mean field approximation, where the order parameter $\sigma \equiv \langle \sigma(x) \rangle$, $\Delta \equiv \langle \Delta(x) \rangle$ and $\Delta^\star \equiv \langle \Delta^\star(x) \rangle$ are constant, the tree-level part of $\Gamma_{\text{eff}}$ is given by
\[
\Gamma_{\text{eff}}^{(0)}[\sigma,\Delta,\Delta^\star;B] = -\chi \left( \frac{\sigma^2}{4G_S} + \frac{\langle |\Delta|^2 \rangle}{4G_D} + \frac{B^2}{2} \right).
\]
Here, \( \mathcal{V} \) is a four-dimensional space-time volume and \( |\Delta|^2 = \Delta \Delta^* \). The one-loop effective action is given by

\[
\Gamma^{(1)}_{\text{eff}}[\sigma, \Delta, \Delta^*; B] = -\frac{i}{2} \sum_{\tilde{q}} \text{Tr}_{\text{NGFSX}} \ln[\mathcal{S}_{\tilde{q}}^{-1}],
\]

(II.13)

where \( \mathcal{S}_{\tilde{q}} \) is defined in (II.8)-(II.10). In (II.13) the trace operation (Tr) includes apart from a two-dimensional trace in the Nambu-Gorkov space (NG), a trace over color \((c)\), flavor \((f)\), and spinor \((s)\) degrees of freedom as well as a trace over a four-dimensional space-time coordinate \((x)\). After performing the trace operation over the NG,\(c,f\), and \(s\) using the method described in [16], we arrive at

\[
\tilde{\Gamma}^{(1)}_{\text{eff}}(\tilde{p}) = \sum_{\kappa \in \{r,g,b\}} \tilde{\Gamma}^{(1)}_{\text{eff}}(\tilde{p})^\kappa(n),
\]

(II.14)

that includes the contribution of the blue \((b)\), red \((r)\), and green \((g)\) quarks

\[
\tilde{\Gamma}^{(1)}_{\text{eff}}(\tilde{p})^b = -i \sum_{\tilde{q} \in \{0,1\}} \ln \det_x [(E_{\tilde{q}} + \mu)^2 - p_0^2] \{(E_{\tilde{q}} - \mu)^2 - p_0^2\}],
\]

(II.15)

\[
\tilde{\Gamma}^{(1)}_{\text{eff}}(\tilde{p})^c = -2i \sum_{\tilde{q} \in \{1, \frac{1}{2}, -\frac{1}{2}\}} \ln \det_x [(E_{\tilde{q}}^{(1)})^2 - p_0^2](E_{\tilde{q}}^{(-1)})^2 - p_0^2\] \]

Here, \( \tilde{p} \) is a modified four-momentum defined by

\[
\tilde{p}^\mu_{\tilde{q} \neq 0} = (p_0, 0, \frac{\tilde{q}}{|\tilde{q}|} \sqrt{2|\tilde{q}B|n}, p_3), \quad \text{for} \quad \tilde{q} = 1, \pm \frac{1}{2},
\]

(II.16)

\[
\tilde{p}^\mu_{\tilde{q} = 0} = (p_0, p), \quad \text{for} \quad \tilde{q} = 0,
\]

with \( p \equiv (p_1, p_2, p_3) \). In (II.15), \( E_{\tilde{q}} \) are given by the dispersion relations corresponding to the neutral and charged particles [16]

\[
E_{\tilde{q}} = \sqrt{2|\tilde{q}B|n + p_3^2 + \sigma^2}, \quad \text{for} \quad \tilde{q} = 1, \pm \frac{1}{2},
\]

(II.17)

\[
E_0 = \sqrt{p^2 + \sigma^2}, \quad \text{for} \quad \tilde{q} = 0.
\]

Moreover, we have \( E_{\tilde{q}}^{(1)} = \sqrt{(E_{\tilde{q}} \pm \mu) + |\Delta|^2} \). Performing the remaining determinant in the coordinate space, a space-time volume factor \( \mathcal{V} \) arises. Combining then the resulting expression with the tree-level part of the effective action from (II.12), the effective action of the theory can be given in terms of the effective thermodynamic (mean field) potential \( \Omega_{\text{eff}} \) as \( \Gamma_{\text{eff}} = -\mathcal{V} \Omega_{\text{eff}} \). Introducing now discrete Matsubara frequencies by replacing \( p_0 \) with \( p_0 = i\omega_\ell \), where \( \omega_\ell = \frac{\pi}{\beta}(2\ell + 1) \) and \( \beta \equiv T^{-1} \), the one-loop effective potential \( \tilde{\Omega}^{(1)}_{\text{eff}} \equiv -\frac{1}{V\beta} \tilde{\Gamma}^{(1)}_{\text{eff}} \) is first given by

\[
\tilde{\Omega}^{(1)}_{\text{eff}}(\tilde{B}, T, \mu) = -\frac{1}{V\beta} \sum_{\tilde{p}} \sum_{\ell = -\infty}^{\infty} \sum_{\kappa = \pm 1} \left\{ \sum_{\tilde{q} \in \{0,1\}} \ln \left[ \omega_\ell^2 + (E_{\tilde{q}} + \kappa \mu)^2 \right] + 2 \sum_{\tilde{q} \in \{1, \frac{1}{2}, -\frac{1}{2}\}} \ln \left[ \omega_\ell^2 + E_{\tilde{q}}^{(1)2} \right] \right\}.
\]

(II.18)

\footnote{In imaginary time formulation, the four-dimensional space-time volume \( \mathcal{V} \) is replaced by \( \mathcal{V} \rightarrow \mathcal{V}\beta \), where \( \mathcal{V} \) is the three-dimensional space volume and \( \beta = 1/T \) the compactification radius of imaginary time coordinate.}
Converting the logarithms into proper-time integrals over the dimensionful variable $s$, and using the Poisson resummation formula

$$\sum_{\ell=-\infty}^{\infty} e^{-s\left(\frac{2\pi\ell}{\beta} + x\right)^2} = \frac{\beta}{2\sqrt{\pi}s} \left[1 + 2 \sum_{\ell=1}^{\infty} \cos(x\beta\ell) \, e^{-\frac{\beta^2\ell^2}{4s}}\right], \quad (II.19)$$

to separate the resulting expression into a temperature dependent and a temperature independent part, we arrive at the one-loop effective potential of our model

$$\Omega_{\text{eff}}^{(1)}(\tilde{e} B, T, \mu) = \frac{1}{2} \sqrt{\frac{\pi V}{2}} \sum_{\mathbf{p}} \sum_{\kappa = \pm 1} \left\{ \sum_{\tilde{q} \in \{0,1\}} \int_{0}^{\infty} \frac{ds}{s^2} e^{-s(E_\tilde{q} + \kappa\mu)^2} + 2 \sum_{\tilde{q} \in \{\pm \frac{1}{2}, -\frac{1}{2}\}} \int_{0}^{\infty} \frac{ds}{s^2} e^{-sE_\tilde{q}^{(\kappa=2)}} \right\}$$

$$\times \left[1 + 2 \sum_{\ell=1}^{\infty} (-1)^\ell \, e^{-\frac{\beta^2\ell^2}{4s}}\right], \quad (II.20)$$

where $E_{\tilde{q}}$ are defined in (II.17). Replacing at this stage the discrete sum over momenta with continuous integrations over momenta by making use of

$$\frac{1}{V} \sum_{\mathbf{p}} f(\mathbf{p}_{\tilde{q}=0}) \rightarrow \int \frac{d^3 p}{(2\pi)^3} \, f(\mathbf{p}), \quad (II.21)$$

for neutral, and

$$\frac{1}{V} \sum_{\mathbf{p}} f(\mathbf{p}_{\tilde{q} \neq 0}) \rightarrow |\tilde{q}eB| \sum_{n=0}^{+\infty} \alpha_n \int_{-\infty}^{+\infty} \frac{dp_3}{8\pi^2} \, f(n,p_3), \quad (II.22)$$

for charged particles,\(^5\) and eventually adding the tree level part of the effective potential to the resulting expression, the full mean field effective potential at one-loop level is given by\(^6\)

$$\Omega_{\text{eff}}(\tilde{e} B, T, \mu) = \frac{\sigma^2}{4G_S} + \frac{|\Delta|^2}{4G_D} + \frac{B^2}{2} + \frac{1}{2} \sqrt{\frac{\pi}{s}} \sum_{\kappa = \pm 1} \left\{ \int \frac{d^3 p}{(2\pi)^3} \int_{0}^{\infty} \frac{ds}{s^2} e^{-s(E_0 + \kappa\mu)^2} \right\}$$

$$+ \tilde{e} B \sum_{n=0}^{\infty} \alpha_n \int_{0}^{\infty} ds \int_{0}^{\infty} dp_3 e^{-s(E_{n+1/2} + \kappa\mu)^2} + 2 e^{-sE_{\tilde{q}^{(\kappa=2)}}} \right\} \left[1 + 2 \sum_{\ell=1}^{\infty} (-1)^\ell \, e^{-\frac{\beta^2\ell^2}{4s}}\right]. \quad (II.23)$$

Note that in (II.22) as well as (II.23), $n$ denotes the discrete Landau levels and $\alpha_n = 2 - \delta_{n0}$ is introduced to consider the fact that Landau levels with $n > 0$ are doubly degenerate\(^12,14\). Moreover, we have used $E_{n+1/2} = E_{-1/2}$. In the next section, we will use (II.23) to determine numerically the chiral and diquark gaps and to present the complete phase structure of the magnetized two-flavor superconducting NJL model at finite $T$ and $\mu$.

\(^5\) Only charged particles interact with the external magnetic field.

\(^6\) In [16], the same effective potential (II.23) was determined using a different method (see (3.19)-(3.22) in [16]).
III. NUMERICAL RESULTS

In the previous section, we have determined the effective potential \( \Omega_{\text{eff}} \) of two-flavor NJL model including meson and diquark condensates at finite temperature, chemical potential and in the presence of constant magnetic fields in the mean field approximation at one-loop level. It is the purpose of this paper to have a complete understanding on the effect of these external parameters on the quark matter in the 2SC phase. This will complete our analysis in [16], where only the effect of \( \mu, \tilde{e}B \) was considered at \( T = 0 \). We start this section with presenting the numerical results on the \( \mu, T \), and \( \tilde{e}B \) dependence of the chiral and diquark condensates. We then continue with exploring the \( T - \mu, T - \tilde{e}B, \) and \( \mu - \tilde{e}B \) phase diagrams for fixed values of \( \tilde{e}B, \mu \) and \( T \), respectively. Before presenting our results, we will fix, in the subsequent paragraphs, our notations and describe our numerical method.

To determine the chiral and diquark gaps, the thermodynamic potential \( \Omega_{\text{eff}} \) from (II.23) is to be minimized. To solve the corresponding gap equations

\[
\frac{\partial \Omega_{\text{eff}}(\sigma, \Delta; T, \mu, \tilde{e}B)}{\partial \sigma}\bigg|_{\sigma_B, \Delta_B} = 0, \quad \text{and} \quad \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta; T, \mu, \tilde{e}B)}{\partial \Delta}\bigg|_{\sigma_B, \Delta_B} = 0, \tag{III.1}
\]

numerically, we have to fix the free parameters of the model. Our specific choice of parameters is

\[
\Lambda = 0.6533 \text{ GeV}, \; G_S = 5.0163 \text{ GeV}^{-2}, \quad \text{and} \quad G_D = \frac{3}{4} G_S, \tag{III.2}
\]

where \( \Lambda \) is the momentum cutoff and \( G_S \) as well as \( G_D \) are the chiral and diquark couplings. Using this special set of parameters, we have shown in [16], that at \( T = 0 \) no mixed phase characterized, with \((\sigma_B \neq 0, \Delta_B \neq 0)\), will appear. The same feature persists at finite \( T \). Moreover, for vanishing magnetic field and at zero temperature, the parameters in (III.2) yield the meson mass \( \sigma_0 \approx 323.8 \text{ MeV} \) at \( \mu = 250 \text{ MeV} \), and the diquark mass \( \Delta_0 \approx 126 \text{ MeV} \) at \( \mu = 460 \text{ MeV} \) [16].

To perform the momentum integration over \( p \) and \( p_3 \) in (II.23) numerically, we have introduced, as in [16], smooth cutoff functions (form factors)

\[
f_\Lambda = \frac{1}{1 + \exp \left( \frac{|p| - A}{\Lambda} \right)}, \quad \text{and} \quad f_{\Lambda, B}^n = \frac{1}{1 + \exp \left( \frac{\sqrt{p_3^2 + 2|p| \tilde{e}B n_A} - A}{\Lambda} \right)}, \tag{III.3}
\]

that correspond to neutral and charged particles, respectively. In (III.3), \( A \) is a free parameter and is chosen to be \( A = 0.05 \Lambda \). Similar smooth cutoff function (form factor) is also used in [13]. Here, as in

---

7 Although our free parameters \( \Lambda, G_D, \) and \( G_S \) coincide with the parameters used in [33], the numerical value of \( \sigma_0 \) is different from what is reported in [33]. The reason for this difference is apparently in the choice of the cutoff function. Whereas in [33] a sharp momentum cutoff is used, we have used smooth cutoff function (III.3) to perform the momentum integrations numerically.

8 In (II.23), the integrals proportional to \( \tilde{e}B \) and including a summation over Landau levels \( n \) arise from charged quarks with charges \( \tilde{q} = \pm \frac{1}{2}, +1 \).
the free parameter $A$ determines the sharpness of the cutoff scheme. At this stage, let us notice that the solutions of (III.1) are in general “local” minima of the theory. Keeping $(\sigma, \Delta) \neq (0, 0)$ and looking for global minima of the system described by complete $\Omega_{\text{eff}}(\sigma, \Delta; T, \mu, \bar{e}B)$ from (II.23), it turns out that in the regime $\mu \in [0, 800]$ MeV, $T \in [0, 250]$ MeV and $\bar{e}B \in [0, 0.8]$ GeV$^2$, the system exhibits two “global” minima. They are given by $(\sigma_B \neq 0, \Delta_B = 0)$ and $(\sigma_B = 0, \Delta_B \neq 0)$. We will denote the regime characterized by these two global minima by the $\chi_{SB}$ and the CSC phases, respectively. According to the above descriptions, in order to determine the chiral and diquark condensates, we will use instead of the gap equations (III.1),

$$\left. \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta_B = 0; T, \mu, \bar{e}B)}{\partial \sigma} \right|_{\sigma_B} = 0, \quad (\text{III.4})$$

in the $\chi_{SB}$ phase and

$$\left. \frac{\partial \Omega_{\text{eff}}(\sigma_B = 0, \Delta; T, \mu, \bar{e}B)}{\partial \Delta} \right|_{\Delta_B} = 0, \quad (\text{III.5})$$

in the CSC phase to simplify our computations. As it turns out, apart from the $\chi_{SB}$ and the CSC phases, there is also a normal phase characterized by $(\sigma_B = 0, \Delta_B = 0)$. Following the method presented in [20], we will determine the critical lines of first order transition between the $\chi_{SB}$ phase and the normal phase using

$$\left. \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta_B = 0)}{\partial \sigma} \right|_{\sigma_B} = 0, \quad \text{and} \quad \Omega_{\text{eff}}(\sigma, \Delta_B = 0) = \Omega_{\text{eff}}(\sigma_B = 0, \Delta_B = 0). \quad (\text{III.6})$$

Similarly, the first order phase transition between $\chi_{SB}$ and the CSC phases is determined by solving

$$\left. \frac{\partial \Omega_{\text{eff}}(\sigma_B = 0, \Delta_B = 0)}{\partial \sigma} \right|_{\sigma_B} = 0, \quad \left. \frac{\partial \Omega_{\text{eff}}(\sigma_B, \Delta)}{\partial \Delta} \right|_{\Delta_B} = 0, \quad (\text{III.7})$$

and

$$\Omega_{\text{eff}}(\sigma_B = 0, \Delta_B) = \Omega_{\text{eff}}(\sigma_B, \Delta_B = 0), \quad (\text{III.8})$$

simultaneously [20]. The second order critical lines between the $\chi_{SB}$ as well as CSC phase and the normal phase will be also determined by the method described in [20] (see Eq. (2.35) in [20]): To determine the second line between the $\chi_{SB}$ and the normal phase, we solve

$$\lim_{\sigma^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta = 0)}{\partial \sigma^2} = 0. \quad (\text{III.9})$$

The second line between the CSC and the normal phase is then determined by solving

$$\lim_{\Delta^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma = 0, \Delta)}{\partial \Delta^2} = 0. \quad (\text{III.10})$$
The same method is also used in [21] to find the second order phase transition between the $\chi_{SB}$ and the normal phase (see page 9 in [21]). To make sure that after the second order phase transition the global minima of the effective potential are shifted to $\sigma = 0$ in (III.9) and to $\Delta = 0$ in (III.10), and in order to avoid instabilities, an analysis similar to [9] is also performed. Same method is also used in [16] to determine second order critical line between the CSC and the normal phase.

### A. The $(\mu, T, \tilde{e}B)$ dependence of chiral and diquark condensates

#### 1. The $\mu$-dependence of $\sigma_B$ and $\Delta_B$

The $\mu$-dependence of both gaps at different temperatures, $T = 0, 20, 70, 150$ MeV, and magnetic fields, $\tilde{e}B = 0, 0.3, 0.5$ GeV$^2$, are demonstrated in Figs. 1 panels (a)-(c), respectively. The green dashed and solid lines denote the $\sigma_B$ mass gaps. The red solid lines determine the normal phase with $\sigma_B = \Delta_B = 0$. The diquark gaps $\Delta_B$ are demonstrated with blue solid lines. Dashed (solid) lines denote the first (second) order phase transitions.

![Graphs showing $\mu$-dependence of gaps](image)

**FIG. 1:** The $\mu$-dependence of $\sigma_B$ and $\Delta_B$ is demonstrated for $T = 0$ MeV and $\tilde{e}B = 0, 0.3, 0.5$ GeV$^2$ in (a), (b) and (c), respectively. The transitions from $\chi_{SB}$ to the CSC phase is of first order, the transitions from the CSC to the normal phase is second order.

The plots in Figs. 1(a)-1(c) show that at $T = 0$ MeV, the magnetic field enhances the formation of chiral condensate $\sigma_B$. The value of $\sigma_B$ is constant in $\mu$, with $\mu < \mu_c$ and $\mu_c \approx 320 - 350$ MeV, and increases with $\tilde{e}B$. On the other hand, for small value of $\tilde{e}B < 0.5$ GeV$^2$, $\Delta_B$ increases with $\mu$ in the regime $\mu_c < \mu < 600$ MeV [Figs. 1(a) and 1(b)]. Similar observation is also made in [16], where the same model as in the present paper is studied, and additionally the color chemical potential $\mu_8$ is assumed to be nonzero. Comparing the diagrams of Fig. 1 with the corresponding diagrams in Fig. 7 of [16], it turns out that $\mu_8$ has no significant effect on the $\mu$-dependence of $\sigma_B$ and $\Delta_B$ at $T = 0$ MeV.\footnote{In [3], in $\tilde{e}B = 0$ case, the color chemical potential $\mu_8$ is shown to be small and its effect is therefore neglected. See} For $\tilde{e}B = 0$ GeV$^2$, the $\mu$-dependence of diquark mass gap in Fig. 1(a) can be compared with
the analytical result

\[ \Delta_0^2 = C_2(\Lambda^2 - \mu^2) \exp \left( -\frac{\Lambda^2}{\mu^2} \left( \frac{1}{g_d} - 1 \right) \right), \]  

(III.11)

from \cite{16}. In (III.11), \( C_2 = 4e^{-3} \approx 0.2 \) and \( g_d \equiv \frac{4G_D\Lambda^2}{\pi} \). Plugging the numerical value of \( \Lambda \) and \( G_D \) from (III.2) in (III.11) and plotting the resulting expression in a \( \Delta_0 \) vs. \( \mu \) diagram, the result is in close agreement with the \( \mu \)-dependence of diquark mass gap in Fig. II(a) in the regime \( 320 < \mu < 600 \) MeV. In the same regime of \( \mu \), the \( \mu \)-dependence of \( \Delta_0 \) from (III.11) agrees also with the well-known result from \cite{34}. In contrast, as it can be seen in Fig. II(c), for \( \eB = 0.5 \) GeV\(^2\), \( \Delta_B \) decreases with \( \mu \in [350, 600] \) MeV. This behavior is expected from the analytical expression

\[ \Delta_B^2 = 4(\Lambda_B^2 - \mu^2) \exp \left( -\frac{\Lambda_B^2}{\mu} \left( \frac{1}{g_B} \right) \right), \]  

(III.12)

that is also computed in \cite{16} using an appropriate LLL approximation.\(^{10}\) In (III.12), \( \Lambda_B \equiv \sqrt{\eB} \) and \( g_d \equiv \frac{G_D\Lambda^2}{\pi} \). Plugging the numerical values of \( \Lambda \) and \( G_D \) from (III.2) in (III.12), and plotting the resulting expression in a \( \Delta_B \) vs. \( \mu \) diagram for \( \eB = 0.5 \) GeV\(^2\), it turns out that \( \Delta_B \) decreases with \( \mu \) in the regime \( 350 < \mu < 600 \) MeV, as it is shown in Fig. II(c). Similar expression as (III.12) is also derived in \cite{11} for the diquark mass gap in a three-flavor CFL model in the presence of strong magnetic field, using the LLL approximation.

In Figs. II(a)-II(c) the \( \mu \)-dependence of \( \sigma_B \) and \( \Delta_B \) is plotted at \( T = 20 \) MeV and for \( \eB = 0, 0.3, 0.5 \) GeV\(^2\). Comparing with the results from Figs. II(a)-II(c), it turns out that increasing temperature up to \( T = 20 \) MeV, has no significant effects on the results of \( T = 0 \) MeV. This is in contrast with the situation at \( T = 70 \) MeV. In Figs. III(a)-III(c), the \( \mu \)-dependence of \( \sigma_B \) and \( \Delta_B \) are plotted at \( T = 70 \)

\(^{10}\) Later we will see that \( \eB \approx 0.5 \) GeV\(^2\) is strong enough to justify LLL approximation.
MeV and for $\bar{e}B = 0, 0.3, 0.5 \text{GeV}^2$. As it turns out, in the regime $\mu < \mu_c$, the chiral condensate $\sigma_B$ increases with the magnetic field. The diquark condensate appears in the regime $\mu \in [480, 600] \text{ MeV}$ for $\bar{e}B = 0, 0.3 \text{ GeV}^2$ [Figs. 3(a) and 3(b)]. For $\bar{e}B = 0.5 \text{ GeV}^2$ in Fig. 3(c), however, no diquark condensate appears in the relevant regime $\mu < 600 \text{ MeV}$. Moreover, comparing Figs. 2 and 3, we notice that at $T = 70 \text{ MeV}$, in Fig. 3 in contrast to the situation at $T = 20 \text{ MeV}$, in Fig. 2, the first order transition from the $\chi_{SB}$ to the normal phase does not occur over the CSC phase; in Fig. 3 for $\bar{e}B < 0.5 \text{ GeV}^2$, there is a first order phase transition from the $\chi_{SB}$ to the normal phase, then a second order phase transition occurs from the normal to the CSC phase. For $\bar{e}B = 0.5 \text{ GeV}^2$, the phase transition from the $\chi_{SB}$ to the normal phase is of first order but no CSC phase appears in the regime $\mu < 600 \text{ MeV}$.

At higher temperature, as it is demonstrated for $T = 150 \text{ MeV}$ in Fig. 4, no diquark condensate appears at all in the relevant regime $\mu < 600 \text{ MeV}$ [Figs. 4(a)-4(c)]. Moreover, whereas for $\bar{e}B = 0, 0.3 \text{ GeV}^2$, the transitions from the $\chi_{SB}$ to the normal phase are of second order, for $\bar{e}B = 0.5 \text{ GeV}^2$, the $\chi_{SB}$-normal phase transition turns out to be of first order. The above observations suggest that while the magnetic field catalyzes the formation of the chiral condensate $\sigma_B$, the diquark condensate $\Delta_B$ is suppressed in the presence of a constant magnetic field. However, as it turns out, the latter feature depends on the strength of the magnetic field. This will be shown explicitly in the next section, where the effect of arbitrary magnetic fields on the formation of $\sigma_B$ and $\Delta_B$ for a wide range of $\bar{e}B \in [0, 0.8] \text{ GeV}^2$ will be explored.

FIG. 3: The $\mu$-dependence of $\sigma_B$ and $\Delta_B$ is demonstrated for $T = 70 \text{ MeV}$ and $\bar{e}B = 0, 0.3 \text{ GeV}^2$ in (a) and (b), respectively. While the transitions from $\chi_{SB}$ to the normal phase (solid red line) is of first order, the transitions from the normal to the CSC phase is of second order. c) The $\mu$-dependence of $\sigma_B$ is demonstrated for $T = 70 \text{ MeV}$ and $\bar{e}B = 0.5 \text{ GeV}^2$. The transition from the $\chi_{SB}$ to the normal phase is of first order and no CSC phase appears at $\mu > 330 \text{ MeV}$.
FIG. 4: The $\mu$-dependence of $\sigma_B$ is demonstrated for $T = 150$ MeV and $\tilde{e}B = 0, 0.3, 0.5$ GeV$^2$ in (a)-(c). While for $\tilde{e}B = 0, 0.3$ GeV$^2$ the transition from $\chi_S B$ to the normal phase is of second order, for $\tilde{e}B = 0.5$ GeV$^2$ this transition is of first order. No CSC phase appears at $T = 150$ MeV. The solid red line denotes the normal phase with $(\sigma_B = 0, \Delta_B = 0)$.

2. The $\tilde{e}B$-dependence of $\sigma_B$ and $\Delta_B$

The effect of external magnetic field on the formation of chiral condensates is studied intensively in the literature (see e.g. [1, 17–19]). As it is shown in [1], in the presence of strong magnetic field, the dynamics of the system is fully described by its dynamics in the LLL. However, there are, to the best of our knowledge, no evidences in the literature that fix quantitatively the strength of magnetic field which is enough to justify a LLL approximation. In [16], we have answered to this question numerically. First, we have determined the $\tilde{e}B$ dependence of the gaps in a LLL approximation, and then, compared the analytical results with the numerical ones including the effect of all Landau levels. It turned out that by increasing the magnetic field from $\tilde{e}B = 0$ GeV$^2$ to $\tilde{e}B = 0.8$ GeV$^2$, first in the regime $\tilde{e}B < 0.45$ GeV$^2$, the mass gaps underly small van Alfen-de Haas oscillations. Above a certain threshold magnetic field $\tilde{e}B_t \simeq 0.45 - 0.50$ GeV$^2$, the dependence of the condensates of $\tilde{e}B$ was linear. In this linear regime, our numerical results were comparable with the analytical results arising from the solution of the corresponding gap equations in the LLL approximation.

In Fig. 5, the $\tilde{e}B$ dependence of meson and diquark mass gaps, $\sigma_B$ and $\Delta_B$, are plotted at $T = 0$ and $\mu = 250$ MeV [Fig. 5(a)], and $\mu = 460$ MeV [Fig. 5(b)]. A comparison with similar plots from Fig. 2 of [16] shows that color neutrality condition has no significant effect on the above mentioned threshold magnetic field $\tilde{e}B_t \simeq 0.5$ GeV$^2$, and on the behavior of $\sigma_B$ and $\Delta_B$ below and above $\tilde{e}B_t$, consisting of van Alfen-de Haas oscillations in $\tilde{e}B < \tilde{e}B_t$ and the linear rise in $\tilde{e}B > \tilde{e}B_t$ regimes.\footnote{See Footnote 9.}

Similar strong van Alphen-de Haas oscillation of 2SC mass gap $\Delta_B$ in the regime $\tilde{e}B \in [0.4, 0.6]$ GeV$^2$ in Fig. 5(b) is also observed in [13, 14] in the three-flavor CFL phase in the presence of magnetic...
The dependence of the diquark gap $\Delta_B$ on $\tilde{e}B$ for fixed $\mu = 250$ MeV, at $T = 0$ MeV. (b) The dependence of diquark gap $\Delta_B$ on $\tilde{e}B$ for fixed $\mu = 460$ MeV, at $T = 0$ MeV.

fields, albeit in another regime of $\tilde{e}B/\mu^2$.

The dependence of the chiral gap $\sigma_B$ on $\tilde{e}B$ for fixed $\mu = 250$ MeV, at various temperatures. (b) The dependence of diquark gap $\Delta_B$ on $\tilde{e}B$ for fixed $\mu = 460$ MeV, at various temperatures.

In Fig. 6(a), the $\tilde{e}B$ dependence of $\sigma_B$ is demonstrated for fixed $\mu = 250$ MeV and at various fixed temperatures $T = 20, 60, 70$ MeV; after small oscillations in the regime below the threshold magnetic field $\tilde{e}B_t$, the system enters the linear regime, where only the contribution of the LLL affects the dynamics of the system. For $\tilde{e}B < \tilde{e}B_t$, $\sigma_B$ decreases by increasing the temperature. In the linear regime $\tilde{e}B > \tilde{e}B_t$, however, the effect of temperature is minimized. In Fig. 6(b), the dependence of the diquark gap $\Delta_B$ is demonstrated for fixed $\mu = 460$ MeV and at various temperatures $T = 20, 60, 70$ MeV. At $T = 20$ MeV, small oscillations occur in $\tilde{e}B < \tilde{e}B_t$. In contrast, $\Delta_B$ monotonically increases with $\tilde{e}B$ in the linear regime $\tilde{e}B > \tilde{e}B_t$. The behavior of $\Delta_B$ at $T = 20$ MeV is not too much different than its behavior at $T = 0$ MeV [see Fig. 5(b)]. This confirms our previous observation from the
comparison of Figs. 1 and 2. By increasing the temperature to \( T = 60 \text{ MeV} \), \( \Delta_B \) decreases. It vanishes in the range \( \tilde{e}B \in [0.4, 0.6] \text{ GeV}^2 \). Comparing to the \( T = 0 \text{ MeV} \) plot from Fig. 5(b), it turns out that the value of \( \Delta_B \) at \( T = 0 \text{ MeV} \) in this regime of \( \tilde{e}B \in [0.4, 0.6] \text{ GeV}^2 \) is in the same order of magnitude as 60 MeV. This is related with the well-known BCS ratio of critical temperature to zero-temperature gap in zero magnetic field case \[37\] [see the discussion below]. For \( \tilde{e}B > 0.6 \text{ GeV}^2 \), \( \Delta_B \) increases again with \( \tilde{e}B \). The situation demonstrated in Fig. 6(b) at \( T = 60 \text{ MeV} \) can be interpreted as a second order phase transition from the CSC to the normal phase at \( T = 60 \text{ MeV} \), \( \mu = 460 \text{ MeV} \) and \( \tilde{e}B \approx 0.4 \text{ GeV}^2 \). We believe that this CSC-Normal-CSC phase transition are caused by strong van Alfven-de Haas oscillations in the regime \( \tilde{e}B \in [0.4, 0.6] \text{ GeV}^2 \). A comparison between the \( T = 60 \text{ MeV} \) plot of Fig. 6(b) and the \( \mu = 460 \text{ MeV} \) plot of Fig. 14(i) confirms these statements.\[12\] Let us notice that van Alfven-de Haas oscillations, which are also observed in metals \[29\], are the consequence of oscillatory structure in the density of states of quarks and occur whenever the Landau level pass the quark Fermi surface \[14, 21\].

In Fig. 6(b), at \( T = 70 \text{ MeV} \) and in the regime for \( \tilde{e}B < \tilde{e}B_t \), the diquark condensate is not built at all, i.e. \( \Delta_B = 0 \). This result indicates that a normal phase exists at \( T \geq 70 \text{ MeV} \), \( \mu = 460 \text{ MeV} \) and in the regime \( \tilde{e}B \lesssim 0.65 \text{ GeV}^2 \), and that a second order phase transition from the normal to the CSC phase occurs at \((T, \mu) \sim (70, 460) \text{ MeV} \) and critical magnetic field \( \tilde{e}B \simeq 0.65 \text{ GeV}^2 \). These results coincide with the observations in the complete \( T - \tilde{e}B \) phase diagram from Fig. 14(i) for \( \mu = 460 \text{ MeV} \). Indeed, our numerical computations show that for \( \mu = 460 \text{ MeV} \), the critical temperatures in the whole range of \( \tilde{e}B \in [0, 0.6] \text{ GeV}^2 \) are smaller than 70 MeV. In \( \tilde{e}B = 0 \text{ GeV}^2 \) and \( \mu = 460 \text{ MeV} \), it is \( T_c \approx 68.5 \text{ MeV} \) and decreases with \( \tilde{e}B \lesssim 0.5 \text{ GeV}^2 \) [see also Fig. 14(i)]. Using this value of \( T_c \) in zero magnetic field and the numerical value of the diquark gap at zero temperature and zero magnetic field from Fig. 3(b), i.e. \( \Delta_B(T = 0) \approx 127.5 \text{ MeV} \), the above mentioned BCS ratio can be computed.

It is given by

\[
\frac{T_c}{\Delta_B(T = 0)} \approx \frac{68.5}{127.5} \approx 0.54,
\]

which is in good agreement with the BCS ratio \( \frac{T_c}{\Delta_B(T=0)} = e^{\gamma \pi^{-1}} + \mathcal{O}(g) \approx 0.567 + \mathcal{O}(g) \) of critical temperature to the zero-temperature 2SC mass gap of QCD in zero magnetic field \[37\]. Here, \( \gamma \approx 0.577 \) is the Euler-Mascheroni constant. As for the regime of strong magnetic fields, \( \tilde{e}B > 0.6 \text{ GeV}^2 \) at fixed \( T = 70 \text{ MeV} \), \( \Delta_B \) increases monotonically with \( \tilde{e}B \), as expected from (III.12) for \( \mu = 460 \text{ MeV} \), and \( \Lambda \) as well as \( G_D \) from (III.2).

\[12\] See also the explanation in the paragraph following Fig. 14 where we have compared Fig. 14(i) with Fig. 6(b).
In Fig. 7 the dependence of chiral and diquark gaps on the external magnetic field for constant $T$ are demonstrated for various $\mu$. As it is shown in Fig. 7(a), the chiral mass gap $\sigma_B$ decreases by increasing $\mu$. Small van Alfven-de Haas oscillations occur in the regime below the threshold magnetic field $\tilde{e} B_t \simeq 0.5 \text{ GeV}^2$ and die out in the linear regime above $\tilde{e} B_t$. In this regime $\sigma_B$ monotonically rises with $\tilde{e} B$, and the effect of $\mu$ is minimized. In the regime below $\tilde{e} B_t$, for $\mu = 300 \text{ MeV}$, the chiral mass gap $\sigma_B$ vanishes. This indicates a first order phase transition from the normal into the $\chi$SB phase at fixed $T = 100 \text{ MeV}$, $\mu = 300 \text{ MeV}$ and for $\tilde{e} B \simeq 0.5 \text{ GeV}^2$. This result agrees with our findings from the complete $T - \tilde{e} B$ phase diagram plotted in Fig. 14(c) for fixed $\mu = 300 \text{ MeV}$. As it can be checked in Fig. 14(c), at fixed $T = 100 \text{ MeV}$, the $\chi$SB phase appears first for $\tilde{e} B > 0.5 \text{ GeV}^2$. What concerns the $\tilde{e} B$-dependence of the diquark mass gap in Fig. 7(b), it increases with $\mu$ in the regime below the threshold magnetic field $\tilde{e} B_t$. In the regime above $\tilde{e} B_t$, however, $\Delta_B$ decreases by increasing $\mu$. Same phenomenon was also demonstrated in Fig. 2(c) at low temperature $T = 20 \text{ MeV}$ and for $\tilde{e} B = 0.5$ near $\tilde{e} B_t$, and was shown to be in full agreement with the analytical result [12, 16]. Note that strong van Alfven-de Haas oscillations are responsible for vanishing $\Delta_B$ in the regime $\tilde{e} B \in [0.4, 0.6] \text{ GeV}^2$. They induce CSC-Normal-CSC second order phase transitions, as it can be seen in the same regime of $\tilde{e} B$ in Fig. 14(a). The plots in Fig. 7(b) show also that the critical magnetic fields for the transition from the normal into the CSC phase increase by increasing the chemical potential [see Fig. 7(b)]$^{13}$.

$^{13}$ Detailed analysis on the dependence of critical $\tilde{e} B$ on $\mu$ and vice versa will be performed in Sec. III.B.
In Fig. the temperature dependence of the chiral condensate is presented for fixed $\mu = 250$ MeV (panel a), $\mu = 300$ MeV (panel b), and various magnetic fields $\vec{e}B = 0, 0.3, 0.5$ GeV$^2$. Whereas the transition from the $\chi$SB into the normal phase for $\mu = 250$ MeV is of second order [continuous decreasing of $\sigma_B$ to $\sigma_B = 0$ in Fig. a], it is of first order for $\mu = 300$ MeV [discontinuous transition from $\sigma_B \neq 0$ to $\sigma_B = 0$ in Fig. b]. As it is also expected from our previous results, the magnetic field enhances the production of $\sigma_B$, that increases by increasing the value of the magnetic field.

The temperature dependence of $\Delta_B$ is presented in Fig. for fixed $\mu = 480$ MeV and $\vec{e}B = 0, 0.4, 0.5$ GeV$^2$ (panel a), below the threshold magnetic field $\vec{e}B_t \simeq 0.45-0.50$ GeV$^2$, as well as for fixed $\mu = 480$ MeV and $\vec{e}B = 0.6, 0.7, 0.8$ GeV$^2$, above the threshold magnetic field. As it turns out magnetic fields below the threshold magnetic field, suppress the production of $\Delta_B$, whereas magnetic fields stronger than the threshold magnetic field enhances its production [see also Fig. b]. Similarly, for $\vec{e}B < \vec{e}B_t$, the value of critical temperature decreases by increasing the magnetic field, whereas for $\vec{e}B > \vec{e}B_t$, the critical temperatures increase by increasing the magnetic field.\footnote{The $\vec{e}B$ dependence of critical temperatures will be discussed in detail in III.B.}

The same effect is also observed in Fig. In Fig. a), the $T$-dependence of $\sigma_B$ is plotted for constant $\vec{e}B = 0.3$ GeV$^3$ and various $\mu = 100, 200, 300$ MeV. At low temperature $T < 50$ MeV, $\sigma_B$ remains constant, and decreases by increasing the temperature. Increasing the chemical potential only accelerate this decrease, i.e. the critical temperature decreases by increasing $\mu$. Whereas the transition at $\mu = 100, 200$ MeV is of second order (blue dots), a first order phase transition occurs at
that this result is also in full agreement to our result from Fig. 14(i).

\[ \mu = 300 \text{ MeV} \text{ (red dots). The } T\text{-dependence of diquark condensate } \Delta_B \text{ is demonstrated for } \tilde{e}B = 0.3 \text{ GeV}^2 \text{ below the threshold magnetic field [Fig. 10(b)], and for } \tilde{e}B = 0.7 \text{ GeV}^2 \text{ above the threshold magnetic field [Fig. 10(c)]. As it turns out, whereas the diquark condensate increases by increasing } \mu \text{ for fixed } T \text{ and } \tilde{e}B < \tilde{e}B_t \text{ [Fig. 10(b)], it decreases by increasing } \mu \text{ for } \tilde{e}B > \tilde{e}B_t \text{ [Fig. 10(c)]. This result coincides with our results from Fig. 7(b). Moreover, as we can see in Figs. 10(b) and 8(c), the diquark condensates decreases by increasing the temperature. Similar to what is observed in Fig. 9 for } \tilde{e}B = 0.3 \text{ GeV}^2, \text{ below } \tilde{e}B_t, \text{ the critical temperature increases by increasing } \mu \text{ [Fig. 10(b)], while for } \tilde{e}B = 0.7 \text{ GeV}^2 \text{ above } \tilde{e}B_t, \text{ the critical temperature decreases by increasing } \mu. \text{ Later, we will see that this result is also in full agreement to our result from Fig. 14(i).}

\[ \Delta_B = \begin{cases} 0 & \text{if } T < T_c \\
\text{monotonic increase} & \text{if } T > T_c 
\end{cases} \]

FIG. 9: The \( T\)-dependence of \( \Delta_B \) for fixed \( \mu = 480 \text{ MeV} \) for various \( \tilde{e}B = 0, 0.3, 0.5 \text{ GeV}^2 \) below the threshold magnetic field (panel a) and for \( \tilde{e}B = 0.6, 0.7, 0.8 \text{ GeV}^2 \) above the threshold magnetic field (panel b). Below (above) the threshold magnetic field the production of diquark is suppressed (enhanced).

\[ \sigma_B = \begin{cases} \text{monotonic increase} & \text{for } \mu = 500 \text{ MeV} \\
\text{monotonic decrease} & \text{for } \mu = 300 \text{ MeV} 
\end{cases} \]

FIG. 10: \( T\)-dependence of \( \sigma_B \) and \( \Delta_B \) for different \( \mu \) for fixed \( \tilde{e}B \).
B. Phase diagrams of hot magnetized 2SC quark matter

To complete our study on the effect of \((T, \mu, \tilde{e}B)\) on the quark matter including mesons and diquarks in the presence of constant magnetic field, we will present in this section the phase structure of the model in a \(T - \mu\) plane for various fixed \(\tilde{e}B\) in Fig. 11. The \(T - \tilde{e}B\) phase diagram of the model for various fixed \(\mu\) is presented in Fig. 14 and finally in Figs. 16 and 17 the \(\mu - \tilde{e}B\) phase diagram is explored for various fixed \(T\). As in the previous section, green dashed lines denote the first order phase transitions and the blue solid lines the second order transitions.

1. \(T - \mu\) phase diagram for various fixed \(\tilde{e}B\)

In Fig. 11 the \(T - \mu\) phase diagram of the model is demonstrated for various fixed \(\tilde{e}B\).

![Phase Diagrams](image)

**FIG. 11:** The \(T - \mu\) phase diagram of hot magnetized 2SC quark matter is presented for various \(\tilde{e}B\). Blue solid lines denote the second order phase transitions and the green dashed lines the first order transitions. The critical and tricritical points are denoted by \(C\) and \(T\), respectively.

As we have argued at the beginning of this section, three different phases appear in this system; the \(\chi_{SB}\) phase, the CSC phase and the normal phase. To study the effects of constant magnetic fields on the phase transition in the \(T - \mu\) plane, let us compare the plots from Fig. 11 with the plots from Figs. 2-4. As we have observed in Figs. 2(a)-2(c), at a fixed and low temperature \(T < T_1\), first order transitions occur between the \(\chi_{SB}\) and the CSC phase. By increasing the temperature to \(T_1 < T < T_2\) after a first order phase transition between the \(\chi_{SB}\) to the normal phase, the normal phase goes over


| $\tilde{e}B$ | ($\mu_{cr}, T_{cr}$) | ($\mu_{tr}, T_{tr}$) |
|-----------|----------------|----------------|
| 0         | (292, 76)      | (316, 45)      |
| 0.01      | (280, 84)      | (309, 43)      |
| 0.05      | (281, 87)      | (315, 44)      |
| 0.10      | (273, 94)      | (307, 41)      |
| 0.20      | (267, 87)      | (290, 28)      |
| 0.30      | (275, 109)     | (305, 41)      |
| 0.40      | (274, 141)     | (320, 46)      |
| 0.50      | (278, 156)     | (331, 52)      |
| 0.60      | (314, 189)     | (377, 68)      |
| 0.70      | (347, 245)     | (429, 87)      |
| 0.80      | (398, 271)     | (479, 104)     |
| 0.90      | (444, 304)     | (566, 111)     |
| 1.00      | (485, 345)     | (581, 132)     |

TABLE I: Critical and tricritical points in $T - \mu$ phase diagrams of Fig. 11, denoted by ($\mu_{cr}, T_{cr}$) and ($\mu_{tr}, T_{tr}$), respectively. Here, $\tilde{e}B$ is in GeV$^2$, $T$ and $\mu$ are in MeV.

into the CSC phase in a second order phase transition [see 2(a)-2(b)]. At higher temperature $T > T_2$, no CSC phase exists. Here, depending on the strength of the external magnetic fields, only second or first order transitions occur between the $\chi_{SB}$ and the normal phase [Figs. 4(a)-4(c)]. As it turns out, the values of $T_1$ and $T_2$ depend on the external magnetic field, as it can be seen also in the plots from Fig. 11.

The phase transition from $\chi_{SB}$ to CSC phase and from CSC to the normal phase are always of first and second order, respectively. The order of phase transition between the $\chi_{SB}$ and the normal phase depends, however, on the value of the external magnetic field $\tilde{e}B$. To describe this effect, let us consider the blue solid lines in Fig. 11 that demonstrate the second order phase transitions between the $\chi_{SB}$ and the normal phases. Starting from high temperature and zero chemical potential, they all end at the critical points denoted by $C$ (black bullets). They are then followed by first order critical lines (green dashed lines) between the $\chi_{SB}$ and the normal phase. The latter start at the critical points $C$ and end up at the tricritical points $T$ (red bullets), where three phases coexist. In Table I the values of the temperature and chemical potentials corresponding to the critical points ($T_{cr}, \mu_{cr}$) and the tricritical points ($T_{tr}, \mu_{tr}$) are presented.

As we can see in Fig. 11, by increasing the external magnetic field, the black bullets are shifted more and more to higher values of temperature and chemical potential (see also second column in
Table[1], so that the distance between the black and red bullets increases by increasing the strength of external magnetic field. This is another effect of the external magnetic field; increasing the magnetic field strength changes the type of the $\chi_{SB}$ to normal phase transition from second to first order. This can also be observed in Figs. [I(a)-I(c)], where at high enough temperature, the $\chi_{SB}$ phase transition to the normal phase is first of second order and then, for larger magnetic fields, changes to a first order phase transition.

Increasing the magnetic field strength leads also to an increase in the values of ($\mu_{tr}, T_{tr}$) corresponding to the tricritical point (see the third column of Table [I]). This confirms the conclusion at the end of Sec. III.A.3, where it was stated that the magnetic field above a certain threshold magnetic field enhances the production of the diquark condensate $\Delta_B$, and the CSC phase can therefore exists up to $T \simeq 100$ MeV. In what follows, we will show that above certain threshold magnetic field, $\tilde{e}B_t$, the analytical results arising in a LLL approximation are comparable with the numerical results including the contributions of all Landau levels. To do this, we will compare the analytical expression for the second order critical lines of a transition between the $\chi_{SB}$ and the normal phase with the corresponding numerical data to determine the threshold magnetic field for the LLL approximation. Similar comparison will be then performed for the second order critical line of the transition between the CSC and the normal phase.

![FIG. 12: The second order critical lines of transitions between the $\chi_{SB}$ and the normal phase. The red dots denote the analytical data that arise in a LLL approximation and the blue solid line the numerical data including the contributions of all Landau levels. For $\tilde{e}B = 0.7$ GeV$^2$ stronger than $\tilde{e}B_t$, the analytical and numerical data exactly coincide.](image)

The second order critical surface of the transition between the $\chi_{SB}$ and the normal phase is determined explicitly in App. [A1]. Here, we will present only the final results. Following the method presented in [20], in the phase space of the thermodynamical parameters $(T, \mu, \tilde{e}B)$, the second order
critical surface is determined by solving \[20, 21\]

\[
\lim_{\sigma^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta = 0)}{\partial \sigma^2} = 0. \tag{III.13}
\]

Setting \(n = 0\) in \(\Omega_{\text{eff}}\) from (II.23) and plugging the resulting expression in (III.13), leads to the second order critical surface in the phase space of these parameters [see (A.10) in App. A1]

\[
\frac{1}{4 G_S} = \frac{1}{4 \pi^2} \int_0^\lambda dz \left(2z + \frac{3 \tilde{e} B}{z}\right) F[z; T, \mu] = 0, \tag{III.14}
\]

with

\[
F[z; T, \mu] = \frac{\sinh(\beta z)}{\cosh(\beta z) + \cosh(\beta \mu)}. \tag{III.15}
\]

Expanding the integral in (III.14) in the orders of \(\kappa \equiv \frac{\mu}{\Lambda}\) and keeping only terms of the order \(O(\kappa^3)\), we arrive after a straightforward computation at [see also (A.17) in App. A1]

\[
\mu^2(T, \tilde{e} B; \Lambda) \approx \frac{1}{\alpha} \left\{ -\frac{\pi^2}{G_S} + 4T^2 \left[ \lambda^2 - \text{Li}_2(-e^{-2\lambda}) + 2\lambda \ln \left(1 + e^{-2\lambda}\right) - \frac{\pi^2}{12}\right] + 3\tilde{e} B \int_0^\lambda \frac{dz}{z^2} \tanh z \right\}, \tag{III.16}
\]

where \(\lambda \equiv \frac{\Lambda}{2T}\), and \(\alpha\) is defined by

\[
\alpha(T, \tilde{e} B; \Lambda) \equiv (\tanh \lambda + \lambda \tanh^2 \lambda - \lambda) + \frac{3 \tilde{e} B \tanh^2 \lambda}{8T^2} \frac{\lambda}{\lambda} + \frac{3 \tilde{e} B}{8T^2} \int_0^\lambda dz \frac{\tanh^2 z}{z^2}. \tag{III.17}
\]

Moreover, \(\text{Li}_2(z)\) in (III.16) is the dilogarithm function defined by

\[
\text{Li}_2(z) \equiv -\int_0^z \frac{\ln(1 - z)}{z} \, dz. \tag{III.18}
\]

To determine the second order critical line between the \(\chi_{SB}\) and the normal phase in the \(T - \mu\) plane, we have to fix \(\tilde{e} B\). The analytical results arising from the LLL approximation are demonstrated in Fig. 12 by red dots. Blue solid lines denote numerical results for second order critical lines including the contributions of all Landau levels. In Fig. 12, the analytical and the numerical results for \(\tilde{e} B = 0.5\) GeV\(^2\), approximately at the threshold, and for \(\tilde{e} B = 0.7\) GeV\(^2\), above the threshold magnetic field, are compared. The qualitative behavior of two curves coincides above the threshold magnetic field \(\tilde{e} B_t \simeq 0.5\) GeV\(^2\), where the system is in the regime of LLL dominance.

Similarly, the second order critical surface between the CSC and the normal phase is determined from \[20, 21\]

\[
\lim_{\Delta^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma = 0, \Delta)}{\partial \Delta^2} = 0. \tag{III.19}
\]

Setting \(n = 0\) in \(\Omega_{\text{eff}}\) from (II.23) and plugging the resulting expression in (III.19), we arrive at [see (A.21) in App. A2]

\[
\tilde{e} B^{-1}(T, \mu; \Lambda) = \frac{G_D}{\pi^2} \left[ H \left(\frac{\Lambda + \mu}{2T}\right) + H \left(\frac{\Lambda - \mu}{2T}\right) \right], \tag{III.20}
\]

23
where $H(z)$ is

$$H(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n(2^{2n} - 1)z^{2n-1}}{(2n-1)(2n)!} B_n.$$  \hfill (III.21)

Here, $B_n$ are the Bernoulli’s numbers. Equation (III.20) leads to a relation between the phase space parameters $(T, \mu, \tilde{e}B)$. In Fig. 13 we have fixed $\tilde{e}B$ to be $\tilde{e}B = 0.5, 0.7$ GeV$^2$, and compared the analytical data of the second order critical line (red dots), arising from (III.20)-(III.21) with the numerical data (blue solid line). For $\tilde{e}B > \tilde{e}B_t \approx 0.5$ GeV$^2$, the analytical and numerical data exactly coincide (see the plot of $\tilde{e}B = 0.7$ GeV$^2$ in Fig. 13). We conclude therefore that for $\tilde{e}B > \tilde{e}B_t$ the higher Landau level are decoupled and the properties of the phase transitions are essentially affected by the contribution from the LLL.

![Figure 13](image-url)

**FIG. 13:** The second order critical lines of transition between the CSC and the normal phase. The red dots denote the analytical data that arise in a LLL approximation and the Blue solid line the numerical data including the contributions of all Landau levels. For $\tilde{e}B = 0.7$ GeV$^2$ stronger than $\tilde{e}B_t$, the analytical and numerical data exactly coincide.

2. $T \sim \tilde{e}B$ phase diagram for various fixed $\mu$

In Fig. 14 the $T \sim \tilde{e}B$ phase diagram of hot 2SC quark matter is presented for various $\mu$. Blue solid lines denote the second order phase transitions and the green dashed lines the first order transitions. The critical points are denoted by $C$ and the tricritical points by $T$. The exact numerical values of the critical and tricritical points are presented in Table II.

In Fig. 14(a) and 14(b), the critical lines are plotted for $\mu = 0, 200, 250$ MeV and $\mu = 270, 280$ MeV, respectively. For relatively small values of $\mu \lesssim 270$ MeV the transition between the $\chi$SB and the normal phase is of second order for the whole range of magnetic field $\tilde{e}B \in [0, 0.8]$ GeV$^2$. When we increase the chemical potential to $\mu = 270, 280$ MeV in Fig. 14(b), the type of the phase transition
FIG. 14: The $T - \tilde{e}B$ phase diagram of hot magnetized 2SC quark matter is presented for various $\mu$. Blue solid lines denote the second order phase transitions and the green dashed lines the first order transitions. The critical points are denoted by $C$ and the tricritical points by $T$.

changes from second to first order in the regime $\tilde{e}B \in [0.1, 0.5]$ GeV$^2$, below the threshold magnetic field. By increasing the magnetic field and entering the regime of LLL dominance, the first order phase transition goes over into a second order transition [see in Table III for the exact values of the critical points]. The same effect has been previously observed in [21] (see Fig. 4(b) in [21] and compare it with Fig. 14(a) and 14(b) of the present paper). In Fig. 14(c) the chemical potential is increased to $\mu = 300$ MeV. Here, a small region of CSC phase appears in the regime $\tilde{e}B \in [0.1, 0.3]$ GeV$^2$, between two regions of the $\chi$SB phase. The critical points appearing in Fig. 14(a) are shifted to higher $(T, \tilde{e}B)$ [see also Table III]. The small CSC “island” enlarges by increasing the chemical potential to $\mu = 310$ MeV in Fig. 14(d), and remove the $\chi$SB phase appearing at $\tilde{e}B \lesssim 0.15$ GeV$^2$ totally at $\mu = 325$ MeV [Fig. 14(e)]. This is expected from Figs. 2(a), 2(c), where the critical chemical potential from the transition of $\chi$SB to CSC phase is approximately $\mu \simeq 325$ MeV. The $\chi$SB phase appearing only for
strong magnetic field survives, but it is pushed away to the regime of LLL dominance by increasing the chemical potential to \( \mu = 340 \) and \( \mu = 370 \) MeV in Figs. 14(f) and 14(g). We notice that the transition from \( \chi_{SB} \) to the CSC phase is always of first order, whereas the type of the phase transition from \( \chi_{SB} \) to the normal phase depends on the external magnetic field. This confirm our conclusion from the previous section (See Fig. 11 and the discussion in Sec. III.B.1). Increasing the chemical potential to \( \mu \geq 400 \) MeV, the \( \chi_{SB} \) region is completely removed from the interval \( \tilde{e}B \in [0, 0.8] \) GeV\(^2\), and a second order phase transition appears between the CSC and the normal phase in Fig. 14(h). In Fig. 14(i), the second order phase transition is plotted for \( \mu = 460, 500 \) MeV.\(^{15} \) Above the threshold magnetic field, the critical temperature increases by increasing the magnetic field. This may open the possibility to observe the 2SC phase in future heavy ion experiments.

Let us also emphasize that the plots in Fig. 14 are in very good agreement with plots from previous sections. As an example, let us consider Fig. 6(b), the \( \tilde{e}B \) dependence of \( \Delta_B \) at fixed \( \mu = 460 \) MeV, and compare it with the curve \( \mu = 460 \) MeV in Fig. 14(i). Let us then focus on the range of \( \tilde{e}B \in [0.4, 0.6] \) GeV\(^2\) in both figures. As it turns out from Fig. 6(b), at low temperature \( T = 20 \) MeV, \( \Delta_B \simeq 70 - 80 \) MeV in the interval \( \tilde{e}B \in [0.4, 0.6] \) GeV\(^2\), while at higher temperature \( T = 60, 70 \) MeV, \( \Delta_B \) vanishes and apparently a normal phase appears in this regime of \( \tilde{e}B \). The appearance of a normal phase can be checked in Fig. 14(i). For fixed \( T = 20 \) MeV, there is a CSC phase in the regime \( \tilde{e}B \in [0.4, 0.6] \) GeV\(^2\), while at higher temperature \( T = 60 \) MeV and \( T = 70 \) MeV, a normal phase

\(^{15} \) In Fig. 14(i), we have changed the scale of the plot from \( T \in [0, 360] \) MeV to \( T \in [35, 125] \) MeV, in order to magnify the van Alfven-de Haas oscillations that appear in the regime below the threshold magnetic field \( \tilde{e}B_t \simeq 0.5 \) GeV\(^2\).
appears in the same regime of $\tilde{e}B$. This phenomenon is because of van Alphen-de Haas oscillations that occur in this regime of $\tilde{e}B \in [0.4, 0.6]$ GeV$^2$ in Fig. 4 as well as in the phase diagram of Fig. 14(i). Moreover, the fact that $\Delta_B$ increases for $\tilde{e}B$ stronger than a certain threshold magnetic field $\tilde{e}B_t$ shows the important interplay between the effect of temperature and external magnetic field on the formation of bound states, which could be potentially relevant in connection with the question of accessibility of 2SC phase in experiments. Whereas below the threshold magnetic field, by increasing the temperature, the diquark condensate is destroyed and a normal phase is build up, the production of $\Delta_B$ is enhanced by magnetic fields stronger than $\tilde{e}B_t$ in a corresponding CSC phase. According to our previous studies in [16], we believe that above $\tilde{e}B_t$ the system enters the LLL dominant regime, where the effect of magnetic catalysis enhances the formation of mass gaps.

To show how the numerical results including the contributions of all Landau levels coincide with the analytical results consisting only of the contribution of the LLL above the above mentioned threshold magnetic field, we consider, as in the previous section, equation (III.14), expressing the second order critical surface between the $\chi_{SB}$ and the normal phase. Setting for instance $\mu = 0$, we get, as it is shown in App. A.1, the second order critical line in the $T - \tilde{e}B$ phase [see (A.12)]

$$\tilde{e}B(T, \mu = 0; \Lambda) = \frac{4\pi^2}{3H(\lambda)} \left\{ \frac{1}{4G_S} - \frac{\Lambda^2}{4\pi^2} + \frac{T^2}{12} + \frac{T^2}{\pi^2} \left[ \text{Li}_2(-e^{-2\lambda}) - 2\lambda \ln(1 + e^{-2\lambda}) \right] \right\}, \quad (\text{III.22})$$

where $\lambda \equiv \Lambda/2T$. Moreover, the dilogarithm function $\text{Li}_2(z)$ and $H(z)$ are defined in (III.18) and (III.21), respectively. In Fig. 15 the analytical and numerical data are compared. Red dots denote the analytical data and the blue solid line the numerical data. Whereas at the threshold magnetic field $\tilde{e}B_t \simeq 0.5$ GeV$^2$ the qualitative behavior of both data are similar, for $\tilde{e}B > \tilde{e}B_t$ the analytical and numerical data exactly coincides.

### 3. $\mu - \tilde{e}B$ phase diagram for various fixed $T$

In Figs. 16 and 17 the $\mu - \tilde{e}B$ phase diagram of hot magnetized 2SC quark matter is illustrated for various fixed temperature $T = 20, \cdots 200$ MeV. Green dashed lines denote the first order phase transitions and blue solid lines the second order phase transitions. The critical points are denoted by $C$ (black bullets) and the tricritical points by $T$ (red bullets).

Let us consider the first plot in Fig. 16 the $\mu - \tilde{e}B$ phase diagram at $T = 20$ MeV. This plot is similar to the phase diagram, which was found in [16] at $T = 0$. It includes a first order phase

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16 Similar van Alphen-de Haas oscillations occur in the “gap vs. $eB/\mu^2$ graphs” of a magnetized CFL model studied in [13] and [14]. In [16], where the effect of magnetic fields on the 2SC gap is studied at $T = 0$, same oscillations occur in the same regime of $\tilde{e}B$ for $\mu = 460$ MeV (see Fig. 5a in [10]).

17 See the explanation in Sec. I, for different mechanisms being responsible for the production enhancement of meson and diquark mass gaps by strong magnetic fields.
FIG. 15: The second order critical lines of transition between the $\chi_{SB}$ and the normal phase. Red dots denote the analytical data that arise in a LLL approximation and the blue solid line denotes the numerical data including the contributions of all Landau levels. The threshold magnetic field is at $\tilde{e}B \simeq 0.5$ GeV$^2$. For $\tilde{e}B > \tilde{e}B_t$ the analytical and numerical data exactly coincides.

transition between the $\chi_{SB}$ and the CSC phase. The latter goes over into the normal phase in a second order phase transition.\textsuperscript{18} This plot confirms our findings in Figs. 2(a)-2(c) and Fig. 11. At $T \simeq 50$ MeV, the CSC phase is broken into two separated islands by the normal phase, and a tricritical point (red bullet) appears at $(\tilde{e}B, \mu) = (0.490 \text{ GeV}^2, 333 \text{ MeV})$. The normal phase between two CSC islands appears at $\tilde{e}B \in [0.4, 0.6] \text{ GeV}^2$. This confirms our results from (b), where in the same regime of $\tilde{e}B$, the CSC condensate $\Delta_B$ vanishes and a normal phase occurs. By increasing the temperature both CSC islands shrink, so that at $T = 70$ MeV only three separated CSC islands are remained in the regime below the threshold magnetic field $\tilde{e}B \leq 0.5 \text{ GeV}^2$. They are then totally destroyed at $T \geq 100$ MeV and in the relevant interval $\tilde{e}B \in [0, 0.8] \text{ GeV}^2$. Moreover, by increasing the temperature from $T \simeq 50$ MeV to $T \simeq 100$ MeV, the CSC island and the corresponding tricritical point that exist in the regime above the threshold magnetic field are shifted away to higher values of $\mu$ and $\tilde{e}B$ (for the exact values of the tricritical points see Table III).

At $T \geq 100$ MeV, the $\mu - \tilde{e}B$ phase space consists only of the $\chi_{SB}$ phase and the normal phase. The critical points that appear at $T = 100$ MeV at $(0.14 \text{ GeV}^2, 267.3 \text{ MeV})$ is shifted to higher values of $\tilde{e}B$ and $\mu$ as it can be seen in Fig. 17 and Table III. High temperature suppresses the production of the chiral condensate in the regime of small magnetic fields (compare the curves in Fig. 17 in the regime $\tilde{e}B \in [0, 0.4] \text{ GeV}^2$). This effect is compensated by increasing the magnetic field to above the

\textsuperscript{18} As we have explained in Footnote 2, in a three-flavor NJL model at low temperature and $\mu > 500$ MeV, the CSC phase goes over into a CFL color-superconducting phase. In our model, where no CFL phase can be built, the second order transition between the CSC and the normal phase at low temperature and $\mu > 500$ MeV is only assumed to exist. The same assumption is also done in [16].
FIG. 16: The $\mu - \tilde{e}B$ phase diagram of hot magnetized 2SC quark matter is presented for various $T$. Blue solid lines denote second order phase transitions and the green dashed lines first order transitions. The critical points are denoted by $C$ and the tricritical points by $T$. The brown regions denote the CSC phases. A small CSC phase appears at $T = 100$ MeV on the right side of the last plot. The tricritical point (red bullet) is shifted to the regime $\tilde{e}B > 0.8$ GeV$^2$ (see Table III). Black bullet denotes the critical point.

FIG. 17: $\mu - \tilde{e}B$ phase diagram for $T = 120, 150, 180, 200$ MeV (from top to bottom).
| $T$  | $(\tilde{e}B_{cr},\mu_{cr})$ | $(\tilde{e}B_{tr},\mu_{tr})$ |
|------|------------------|------------------|
| $\lesssim 20$ | $-$ | $-$ |
| 50   | $-$ | $(0.490, 333)$ |
| 60   | $-$ | $(0.554, 356)$ |
| 65   | $-$ | $(0.580, 365)$ |
| 70   | $-$ | $(0.610, 387)$ |
| 100  | $(0.140, 267)$ | $(0.774, 467)$ |
| 120  | $(0.240, 245)$ | $-$ |
| 150  | $(0.458, 269)$ | $-$ |
| 180  | $(0.520, 273)$ | $-$ |
| 200  | $(0.610, 314)$ | $-$ |

TABLE III: Critical and tricritical points in $\tilde{e}B - \mu$ phase diagrams of Figs. 16 and 17, denoted by $(\tilde{e}B_{cr},\mu_{cr})$ and $(\tilde{e}B_{tr},\mu_{tr})$, respectively. Here, $\tilde{e}B$ is in GeV$^2$, $T$ and $\mu$ are in MeV.

IV. CONCLUDING REMARKS

In this paper, we have introduced a two-flavor superconducting NJL type model including mesons and diquarks at finite temperature $T$, chemical potential $\mu$, and in the presence of a constant (rotated) magnetic field $\tilde{e}B$. We were in particular interested on the effect of $(T, \mu, \tilde{e}B)$ on the formation of chiral and color symmetry breaking bound states, and on the nature of phase transitions. One of the most important effects of the magnetic field is its competition with $T$ and $\mu$ in the formation of bound states. This effect is demonstrated in Figs. 6 and 7. As it is illustrated in Fig. 6, although it is expected that increasing $T$ suppresses the formation of chiral and color condensates, it turns out that this specific effect of $T$ is minimized in the regime of strong magnetic field, for $\tilde{e}B$ larger than a certain threshold magnetic field $\tilde{e}B_t \approx 0.5$ GeV$^2$. However, whereas the chiral condensate $\sigma_B$ increases by increasing $\tilde{e}B$ in the whole range of $\tilde{e}B \in [0, 0.8]$ GeV$^2$ [Fig. 6(a)], there are regions below $\tilde{e}B_t$, where the color condensate $\Delta_B$ vanishes [Fig. 6(b)]. Same phenomenon is also observed in Fig. 7, where, at fixed temperature and various chemical potentials, the dependence of $\sigma_B$ and $\Delta_B$ on the magnetic field is plotted. Here, as in the previous case, the effect of $\mu$ in suppressing the production of the chiral and (2SC) color condensates is compensated by $\tilde{e}B$ in the regime above the threshold magnetic field.

According to our results from [16], we believe that in the regime above $\tilde{e}B_t$, the dynamics of the system is solely dominated by the lowest Landau level. In [16], a magnetized 2SC model at zero temperature is considered, and, the numerical value of the threshold magnetic field is determined by comparing the analytical $\tilde{e}B$ dependence of $\sigma_B$ and $\Delta_B$, arising in a LLL approximation, with
the corresponding numerical data including the contribution of all Landau levels. According to this comparison, the threshold magnetic field at $T = 0$ turned out to be $\tilde{e}B_t \simeq 0.5 \text{ GeV}^2$. In the present paper, the same model is considered at finite temperature. Comparing the analytical expression of the second order critical line of the transitions from $\chi_{SB}$ to the normal phase, arising from a LLL approximation, with the corresponding numerical data including the contribution of all Landau levels in Fig. 15, we arrive at the same threshold magnetic field as in $T = 0$ case. Note that in [16], the color neutrality condition was imposed on the magnetized 2SC quark matter. A comparison between the presented results in this paper, where no color neutrality is imposed, and the results from [16], shows that color neutrality, being very small [35], has no significant effect on the value of the above mentioned threshold magnetic field.\textsuperscript{19}

The nature of the phase transition is also affected by the external magnetic field. The most significant effect in this regard is illustrated in Fig. 11, where the $T - \mu$ phase space is plotted for various $\tilde{e}B$. As it is demonstrated in this figure, the phase space of the system includes three different phases. We observe that the distance between the black and red bullets, denoting the critical and the tricritical points, increases by increasing the strength of the external magnetic field. This implies that by increasing $\tilde{e}B$ and keeping $(T, \mu)$ fixed, the second order transition from the $\chi_{SB}$ phase to the normal quark matter changes into a first order transition. This can also be observed e.g. in Fig 17; on each isothermal critical curve, a second order phase transition changes into a first order one by increasing the strength of the magnetic field. Similar effect occurs in the electroweak phase transition by implying an external hypermagnetic field [3] and in type I QED superconductivity. Whereas the type of the phase transition from $\chi_{SB}$ to the normal phase changes by increasing the strength of $\tilde{e}B$, the latter has no effect on the nature of the phase transition between the $\chi_{SB}$ and the CSC phase (first order) and between the CSC and the normal phase (second order) [see the phase diagrams in Figs. 11 13 16]. The only crucial effect of $\tilde{e}B$ concerning the CSC phase, are the observed CSC-Normal-CSC second order phase transition induced by strong van Alften-de Haas oscillations in the regime $\tilde{e}B \in [0.4, 0.6] \text{ GeV}^2$ [see e.g. Figs. 6(b) and 14(i) at fixed $T = 60 \text{ MeV}$], and an increase of the tricritical temperature and chemical potential by increasing the magnetic field. The latter can be observed in Table II for $\tilde{e}B > \tilde{e}B_t \simeq 0.5 \text{ GeV}^2$. This could be relevant in relation with the question addressing the accessibility of 2SC phase in present or future heavy ion experiments, which is recently posed in [32]. We have extended the setup of the model used in [32] by considering the effect of constant magnetic fields, which are supposed to be created in the non-central heavy ion collisions [28] and are estimated to be of order $\tilde{e}B \simeq 0.03 \text{ GeV}^2$ at RHIC energies, and $\tilde{e}B \simeq 0.3 \text{ GeV}^2$ at LHC

\textsuperscript{19} See Footnote 9.
energies. These amounts of magnetic fields are, according to our observation in this extended model, far below the range of magnetic fields which could have significant effects on the CSC phase transition by partly compensating the effects of \((T, \mu)\), as described above. However, we believe, that more realistic models are to be examined to find a satisfactory answer to this interesting question.

The model which is used in this paper can be extended in many ways, e.g, by considering the 2SC-CFL phase including color neutrality. Similar computation can also be performed within a color neutral PNJL model. It is also intriguing to explore the effect of the axial anomaly in the same context.

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Appendix A: Second order critical lines in the LLL approximation

In this appendix, we will determine the main equations leading to the second order critical lines corresponding to the transition between the \(\chi_{SB}\) and the normal phase, as well as the transition between the \(\chi_{SB}\) and the CSC phase, in the limit of very strong magnetic field in the LLL approximation.

1. Transition between the \(\chi_{SB}\) and the normal phase

In the phase space spanned by the intensive thermodynamical parameters \((T, \mu, \tilde{e}B)\), the second order critical surface between the \(\chi_{SB}\) and the normal phase is determined by [20, 21]

\[
\lim_{\sigma^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma, \Delta = 0)}{\partial \sigma^2} = 0. \tag{A.1}
\]

Using \(\Omega_{\text{eff}}\) from [11,23] and setting \(n = 0\), to consider only the contribution of the lowest Landau level, we get first

\[
\frac{1}{4G_S} - \frac{1}{2\sqrt{\pi}} \sum_{\kappa = \pm 1} \left\{ \int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{ds}{\sqrt{s}} p + \kappa \mu e^{-s(p+\kappa \mu)^2} + 3\tilde{e}B \int_0^\infty \frac{ds}{\sqrt{s}} \int_0^\infty \frac{dp}{4\pi^2} p^3 + \kappa \mu e^{-s(p^3+\kappa \mu)^2} \right\} \\
\times \left[ 1 + 2 \sum_{\ell=1}^{\infty} (-1)^\ell e^{-\frac{\beta^2 \ell^2}{4s}} \right] = 0. \tag{A.2}
\]

Denoting the temperature independent part of (A.2) with \(I_{T=0}\) with \(I_{T=0}\), and using

\[
\frac{1}{2\sqrt{\pi}} \sum_{\kappa = \pm 1} \int_0^\infty \frac{ds}{\sqrt{s}} \frac{x - \kappa \mu}{x} e^{-s(x-\kappa \mu)^2} = \frac{\theta(x - \mu)}{x}, \tag{A.3}
\]
from \[38\], we get

\[
I_{T=0} = \frac{1}{4G_S} - \int_0^\infty \frac{dp}{2\pi^2} \theta(p - \mu) - 3\tilde{e}B \int_0^\infty \frac{dp_3}{4\pi^2} \frac{\theta(p_3 - \mu)}{p_3}
\]

\[
= \frac{1}{4G_S} - \frac{(\Lambda^2 - \mu^2)}{4\pi^2} - \frac{3\tilde{e}B}{4\pi^2} \ln \left(\frac{\Lambda}{\mu}\right).
\]  

(A.4)

To evaluate the momentum integration, we have introduced the ultraviolet cutoff \(\Lambda\). Note that (A.4) is comparable with our results from [16]. The temperature dependent part of (A.2), which is denoted by \(I_{T\neq0}\), can be evaluated using

\[
\sum_{\kappa = \pm 1} \int_0^\infty \frac{ds}{\sqrt{s}} \frac{z - \kappa \mu}{z} e^{-s(z - \kappa \mu)^2} e^{-\frac{z^2}{4s}} = \frac{\sqrt{\pi}}{z} \left( e^{-\beta \ell z - \mu} \text{sgn}(z - \mu) + e^{-\beta \ell (z + \mu)} \right),
\]  

(A.5)

and is given by

\[
I_{T\neq0} = \sum_{\ell=1}^\infty (-1)^\ell \left\{ \int_0^\mu \frac{dz}{2\pi^2} e^{-\ell \beta \mu} \sinh(\ell \beta z) \left( \frac{3\tilde{e}B}{z} + 2z \right) - \int_\mu^{\Lambda} \frac{dz}{2\pi^2} e^{-\ell \beta \mu} \cosh(\ell \beta \mu) \left( \frac{3\tilde{e}B}{z} + 2z \right) \right\},
\]  

(A.6)

where \(z\) is a generic integration variable replacing \(p = |p|\) or \(p_3\) in the one dimensional integrations of (A.2). To perform the summation over \(\ell\), we use

\[
\sum_{\ell=1}^\infty (-1)^\ell \sinh(\ell \beta z) e^{-\ell \beta \mu} = \frac{1}{2} F[z; T, \mu],
\]

\[
\sum_{\ell=1}^\infty (-1)^\ell \cosh(\ell \beta \mu) e^{-\ell \beta z} = \frac{1}{2} (F[z; T, \mu] - 1),
\]  

(A.7)

where

\[
F[z; T, \mu] = \frac{\sinh(\beta z)}{\cosh(\beta z) + \cosh(\beta \mu)}.
\]  

(A.8)

Plugging (A.7) in (A.6) and performing the integration over \(z\) as far as possible, the temperature dependent part \(I_{T\neq0}\) is given by

\[
I_{T\neq0} = -\frac{1}{4\pi^2} \int_0^{\Lambda} \frac{dz}{2\pi^2} \left( 2z + \frac{3\tilde{e}B}{z} \right) F[z; T, \mu] + \left( \frac{\Lambda^2 - \mu^2}{4\pi^2} \right) + \frac{3\tilde{e}B}{4\pi^2} \ln \left(\frac{\Lambda}{\mu}\right).
\]  

(A.9)

The second order critical surface of the transition between the \(\chi_{SB}\) and the normal phase as a function of the phase space variables \((T, \mu, \tilde{e}B)\) is then given by adding (A.4) and (A.9) and reads

\[
\frac{1}{4G_S} - \frac{1}{4\pi^2} \int_0^{\Lambda} \frac{dz}{2\pi^2} \left( 2z + \frac{3\tilde{e}B}{z} \right) F[z; T, \mu] = 0,
\]  

(A.10)

where \(F[z; T, \mu]\) is defined in (A.8). Note that (A.10) is only valid in the limit of strong magnetic field, where a LLL approximation is justified. Choosing \(\mu = 0\), (A.10) reads

\[
\frac{1}{4G_S} = \frac{1}{4\pi^2} \int_0^{\Lambda} \frac{dz}{2\pi^2} \left( 2z + \frac{3\tilde{e}B}{z} \right) \tanh \frac{\beta z}{2}.
\]  

(A.11)
This leads to the following relation between $T$ and $\tilde{e}B$

$$\tilde{e}B(T, \mu = 0; \Lambda) = \frac{4\pi^2}{3H(\lambda)} \left\{ \frac{1}{4G_S} - \frac{\Lambda^2}{4\pi^2} + \frac{T^2}{12} + \frac{T^2}{\pi^2} \left[ \text{Li}_2(-e^{-2\lambda}) - 2\lambda \ln(1 + e^{-2\lambda}) \right] \right\}, \quad (A.12)$$

where $\lambda \equiv \Lambda/2T$ and

$$H(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^n - 1)z^{2n-1}}{(2n-1)(2n)!} B_n, \quad (A.13)$$

and $\text{Li}_2(z)$ is the dilogarithm function defined by

$$\text{Li}_2(z) \equiv -\int_0^z dy \frac{\ln(1 - y)}{y}, \quad (A.14)$$

and $B_n$ are the Bernoulli’s numbers. This result will be used in Sec. III.B to determine the threshold magnetic field for the LLL approximation at $\mu = 0$. For arbitrary values of the chemical potential, we expand (A.10) in the orders of $\frac{\mu}{\Lambda}$ and keep terms up to $O((\frac{\mu}{\Lambda})^3)$, and arrive at the explicit ($T, \mu$) dependence of $\tilde{e}B$

$$\tilde{e}B(T, \mu; \Lambda) \approx \frac{1}{3\gamma} \left\{ -\frac{\pi^2}{G_S} + 4T^2 \left[ \lambda^2 - \text{Li}_2(-e^{-2\lambda}) + 2\lambda \ln \left(1 + e^{-2\lambda}\right) - \frac{\pi^2}{12} \right] - \mu^2 \left( \tanh \lambda + \lambda \tanh^2 \lambda - \lambda \right) \right\}, \quad (A.15)$$

where $\gamma$ is defined by

$$\gamma(T, \mu; \Lambda) \equiv \frac{\mu^2 \tanh^2 \lambda}{8T^2} - \int_0^\lambda dz \left( \frac{\tanh z}{z} - \frac{\mu^2 \tanh^2 z}{8T^2} \right). \quad (A.16)$$

Similarly, the ($T, \tilde{e}B$) dependence of $\mu$ is given by

$$\mu^2(T, \tilde{e}B; \Lambda) \approx \frac{1}{\alpha} \left\{ -\frac{\pi^2}{G_S} + 4T^2 \left[ \lambda^2 - \text{Li}_2(-e^{-2\lambda}) + 2\lambda \ln \left(1 + e^{-2\lambda}\right) - \frac{\pi^2}{12} \right] + 3\tilde{e}B \int_\lambda^\lambda dz \frac{\tanh z}{z} \right\}, \quad (A.17)$$

where $\alpha$ is defined by

$$\alpha(T, \tilde{e}B; \Lambda) \equiv \left( \tanh \lambda + \lambda \tanh^2 \lambda - \lambda \right) + \frac{3\tilde{e}B \tanh^2 \lambda}{8T^2} + \frac{3\tilde{e}B}{8T^2} \int_0^\lambda dz \frac{\tanh^2 z}{z^2}. \quad (A.18)$$

Relations (A.17) will be evaluated numerically in Sec. III.B to determine the threshold magnetic field for the LLL approximation.

2. Transition between the CSC and the normal phase

For the transition between the CSC and the normal phase, we use the equation [20, 21]

$$\lim_{\Delta^2 \to 0} \frac{\partial \Omega_{\text{eff}}(\sigma = 0, \Delta)}{\partial \Delta^2} = 0. \quad (A.19)$$
Setting $n = 0$ in $\Omega_{\text{eff}}$ from (II.23) and plugging the resulting expression in (A.19), we get

$$
\frac{1}{4G_D} - \tilde{e}B \sum_{n=\pm 1} \int_0^\infty \frac{ds}{\sqrt{s}} \int_0^\infty \frac{dp_3}{4\pi^2} e^{-s(p_3+\kappa\mu)^2} \left[ 1 + 2 \sum_{\ell=1}^\infty (-1)^\ell e^{-\frac{2\beta^2}{4s}} \right] = 0. \quad (A.20)
$$

Evaluating the $T$-dependent and independent parts of the expression in the r.h.s. of (A.20) using (A.3) and (A.5), respectively, we arrive after some straightforward manipulations at the relation between the phase space parameters $(T, \mu, \tilde{e}B)$,

$$
\tilde{e}B^{-1}(T, \mu; \Lambda) = \frac{G_D}{\pi^2} \left[ H \left( \Lambda + \frac{\mu}{2T} \right) + H \left( \frac{\Lambda - \mu}{2T} \right) \right], \quad (A.21)
$$

where $H(z)$ is defined in (A.13). Fixing one of these parameters in (A.21), the second order critical lines for the transition from CSC to the normal phase arises in the phase space of two other parameters.

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