STACKY GKM GRAPHS AND ORBIFOLD GROMOV-WITTEN THEORY

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ABSTRACT. Following Zong [63], we define an algebraic GKM orbifold \( \mathcal{X} \) is to be a smooth Deligne-Mumford stack equipped with an action of an algebraic torus \( T \), with only finitely many zero-dimensional and one-dimensional orbits. The 1-skeleton of \( \mathcal{X} \) is the union of its zero-dimensional and one-dimensional \( T \)-orbits; its formal neighborhood \( \hat{\mathcal{X}} \) in \( \mathcal{X} \) determines a decorated graph, called the stacky GKM graph of \( \mathcal{X} \). The \( T \)-equivariant orbifold Gromov-Witten (GW) invariants of \( \mathcal{X} \) can be computed by localization and depend only on the stacky GKM graph of \( \mathcal{X} \) with the \( T \)-action.

We also introduce abstract stacky GKM graphs and define their formal equivariant orbifold GW invariants. Formal equivariant orbifold GW invariants of the stacky GKM graph of an algebraic GKM orbifold \( \mathcal{X} \) are refinements of \( T \)-equivariant orbifold GW invariants of \( \mathcal{X} \).

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1. INTRODUCTION

In this paper, algebraic varieties and algebraic stacks are defined over the complex field \( \mathbb{C} \).

An algebraic GKM manifold, named after Goresky, Kottwitz, and MacPherson, is a non-singular algebraic variety equipped with an action of an algebraic torus \( T = (\mathbb{C}^*)^m \) such that there are finitely many zero-dimensional and one-dimensional orbits. In the seminar work [26], Goresky-Kottwitz-MacPherson showed that the \( T \)-equivariant cohomology of an equivariantly formal GKM manifold can be expressed in terms of a decorated graph known as the GKM graph. As an abstract graph, the vertices and edges of the GKM graph are in one-to-one correspondence with the zero-dimensional and one-dimensional orbits of the \( T \)-action on the GKM manifold. The additional decorations provide enough information to reconstruct the \( T \)-equivariant formal neighborhood of the 1-skeleton (the union of zero-dimensional and one-dimensional orbits of the torus action) in the GKM manifold. Toric manifolds are examples of GKM manifolds.
The quantum cohomology of a projective manifold \( X \) is equal to the rational cohomology as a vector space over \( \mathbb{Q} \), equipped with the quantum product which is a family of product parametrized by Novikov variables such that the classical cup product is recovered by setting all the Novikov variables to be zero. The quantum product is determined by genus zero Gromov-Witten invariants which are virtual counts of rational curves in \( X \). More generally, genus \( g \) Gromov-Witten invariants are virtual counts of genus \( g \) stable maps to \( X \). When \( X \) is a GKM manifold, Gromov-Witten invariants (which are rational numbers) admit an equivariant lifting (which takes values in \( H^*(BT;\mathbb{Q}) \)). Equivariant Gromov-Witten invariants can be computed by virtual localization on moduli of stable maps to \( X \) \([5, 24, 25]\) and depend only on the GKM graph \([44]\). The formula also makes sense for non-compact GKM manifolds if one works over the fractional field of \( H^*(BT;\mathbb{Q}) \). In particular, equivariant quantum cohomology is also an invariant of the GKM graph.

The GKM theory and GKM graphs have been generalized to orbifolds. To our knowledge, GKM graphs have been defined for smooth orbifolds having presentation as a global quotient of a smooth manifold by action of a torus \([28]\). Following Zong \([63]\), in the present paper we consider the most general possible definition that we can think of in the algebraic setting: an algebraic GKM orbifold is defined to be a smooth Deligne-Mumford (DM) stack equipped with an action by an algebraic torus, with only finitely many zero-dimensional and one-dimensional orbits. (We also make the mild assumption that any one-dimensional orbit contains at least one torus fixed point.) In Section \([5]\) we define the stacky GKM graph of an algebraic GKM orbifold. As an abstract graph, the vertices and edges are in one-to-one correspondence with the zero-dimensional and one-dimensional orbits of the torus action, respectively. Recall that in the manifold case, a zero-dimensional orbit is a (scheme) point, and a one-dimensional orbit is isomorphic to the complex projective line \( \mathbb{P}^1 \) or a complex affine line \( \mathbb{C} \). In the case of global quotients by torus action (such as smooth toric DM stacks), a zero-dimensional orbit is a stacky point \( BG = [\text{point}/G] \) where \( G \) is a finite abelian group, and a one-dimensional orbit is a one-dimensional smooth toric DM stack. In the general case studied in the present paper, a zero-dimensional orbifold is of the form \( BG = [\text{point}/G] \) where \( G \) is a (possibly non-abelian) finite group, and a one-dimensional orbit is a spherical DM curve in the sense of Behrend-Noohi \([8]\). Our definition of stacky GKM graphs relies on Behrend-Noohi’s presentation of a spherical DM curve as a global quotient of the form \( [(\mathbb{C}^2 - \{0\})/E] \), where \( E \) is a central extension of the fundamental group of the spherical DM curve by \( \mathbb{C}^* \) \([8]\). In the toric case the stacky GKM graph is determined by the stacky fan defining the smooth toric DM stack. The stacky GKM graph contains enough information to reconstruct the equivariant formal neighborhood of the 1-skeleton as a formal smooth DM stack equipped with a torus action.

In Section \([4]\) we axiomize the definition of a stacky GKM graph and introduce abstract stacky GKM graphs which are more general than stacky GKM graphs of honest algebraic GKM orbifolds. From an abstract stacky GKM graph we construct a formal GKM orbifold. This generalizes formal toric Calabi-Yau graphs and formal toric Calabi-Yau threefolds introduced in \([43]\).

Gromov-Witten invariants (and more generally \( \epsilon \)-stable quasimap invariants) of orbifold GIT quotients (including smooth toric DM stacks) are studied in \([13]\).
In the last two sections (Section 5 and Section 6) of this paper, we study equivariant Gromov-Witten invariants of algebraic GKM orbifolds. We compute torus equivariant orbifold Gromov-Witten invariants of any algebraic GKM orbifold $X$ by virtual localization on moduli of twisted stable maps to $X$. All the ingredients in the resulting localization formula can be extracted from the stacky GKM graph. In particular, this generalizes the localization computations in orbifold GW theory of smooth toric DM stacks [42, Section 9]. Moreover, the localization formula naturally generalizes to any abstract stacky GKM graph and defines formal equivariant orbifold Gromov-Witten invariants of the corresponding formal GKM orbifold. This generalizes formal Gromov-Witten invariants of formal toric Calabi-Yau threefolds defined in [43], and refines equivariant orbifold Gromov-Witten invariants when the GKM graph comes from an honest algebraic GKM orbifolds.

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2. Smooth Deligne-Mumford Stacks

We work over $\mathbb{C}$. Let $X$ be a smooth Deligne-Mumford (DM) stack. Let $\pi : X \to X$ be the natural projection to the coarse moduli space $X$.

2.1. The inertia stack and its rigidification. The inertia stack $\mathcal{I}X$ associated to $X$ is a smooth DM stack such that the following diagram is Cartesian:

$$
\begin{array}{ccc}
\mathcal{I}X & \longrightarrow & X \\
\downarrow & & \downarrow \Delta \\
X & \stackrel{\Delta}{\longrightarrow} & X \times X
\end{array}
$$

where $\Delta : X \to X \times X$ is the diagonal map. An object in the category $\mathcal{I}X$ is a pair $(x,g)$, where $x$ is an object in the category $X$ and $g \in \text{Aut}_X(x)$:

$$\text{Ob}(\mathcal{I}X) = \{ (x,g) \mid x \in \text{Ob}(X), g \in \text{Aut}_X(x) \}.$$
The morphisms between two objects in the category \( \mathcal{I} \mathcal{X} \) are:

\[
\text{Hom}_{\mathcal{I} \mathcal{X}}((x_1, g_1), (x_2, g_2)) = \{ h \in \text{Hom}_{\mathcal{X}}(x_1, x_2) \mid h \circ g_1 = g_2 \circ h \}.
\]

In particular,

\[
\text{Aut}_{\mathcal{I} \mathcal{X}}(x, g) = \{ h \in \text{Aut}_{\mathcal{X}}(x) \mid h \circ g = g \circ h \}.
\]

The rigidified inertia stack \( \mathcal{I} \mathcal{X} \) satisfies

\[
\text{Ob}(\mathcal{I} \mathcal{X}) = \text{Ob}(\mathcal{I} \mathcal{X}), \quad \text{Aut}_{\mathcal{I} \mathcal{X}}(x, g) = \text{Aut}_{\mathcal{I} \mathcal{X}}(x, g) / \langle g \rangle,
\]

where \( \langle g \rangle \) is the subgroup of \( \text{Aut}_{\mathcal{I} \mathcal{X}}(x, g) \) generated by \( g \).

There is a natural projection \( q : \mathcal{I} \mathcal{X} \to \mathcal{X} \) which sends \((x, g)\) to \( x \). There is a natural involution \( \iota : \mathcal{I} \mathcal{X} \to \mathcal{I} \mathcal{X} \) which sends \((x, g)\) to \((x, g^{-1})\). We assume that \( \mathcal{X} \) is connected. Let

\[
\mathcal{I} \mathcal{X} = \bigsqcup_{i \in I} \mathcal{X}_i
\]

be disjoint union of connected components. There is a distinguished connected component \( \mathcal{X}_0 \) whose objects are \((x, \text{id}_x)\), where \( x \in \text{Ob}(\mathcal{X}) \), and \( \text{id}_x \in \text{Aut}(x) \) is the identity element; note that \( \mathcal{X}_0 \cong \mathcal{X} \). The involution \( \iota \) restricts to an isomorphism \( \iota_i : \mathcal{X}_i \to \mathcal{X}_{\iota(i)} \). In particular, \( \iota_0 : \mathcal{X}_0 \to \mathcal{X}_0 \) is the identity functor.

**Example 2.1 (classifying space).** Let \( G \) be a finite group. The stack \( B G = \text{[point}/G \text{]} \) is a category which consists of one object \( x \), and \( \text{Hom}(x, x) = G \). The objects of its inertia stack \( IBG \) are

\[
\text{Ob}(IBG) = \{ (x, g) \mid g \in G \}.
\]

The morphisms between two objects are

\[
\text{Hom}( (x, g_1), (x, g_2) ) = \{ g \in G \mid g_2 g = g g_1 \} = \{ g \in G \mid g_2 = g g_1 g^{-1} \}.
\]

Therefore

\[
IBG \cong [G/G]
\]

where \( G \) acts on \( G \) by conjugation. We have

\[
IBG = \bigsqcup_{c \in \text{Conj}(G)} (BG)_c
\]

where \( \text{Conj}(G) \) is the set of conjugacy classes in \( G \), and \( (BG)_c \) is the connected component associated to the conjugacy class \( c \in \text{Conj}(G) \). We have

\[
(BG)_c = [c/G] \cong [\{h\}/C(h)] \cong B(C(h)).
\]

for any element \( h \) in the conjugacy class \( c \), where \( C(h) = \{ a \in G : ah = ha \} \) is the centralizer of \( h \).

In particular, when \( G \) is abelian, we have \( \text{Conj}(G) = G \), and

\[
IBG = \bigsqcup_{h \in G} (BG)_h
\]

where \( (BG)_h = [\{h\}/G] \cong BG \).
2.2. **Age.** Given a positive integer $s$, let $\mu_s$ denote the group of $s$-th roots of unity. It is a cyclic subgroup of $\mathbb{C}^*$ of order $s$, generated by

$$\zeta_s := e^{2\pi \sqrt{-1}/s}.$$ 

Given any object $(x, g) \in \mathcal{I}X$, $g : T_xX \to T_xX$ is a linear isomorphism such that $g^s = \text{id}$, where $s$ is the order of $g$. The eigenvalues of $g : T_xX \to T_xX$ are $\zeta_i^{l_i}, \ldots, \zeta_i^{l_r}$, where $l_i \in \{0, 1, \ldots, s-1\}$ and $r = \dim_{\mathbb{C}} X$. Define

$$\text{age}(x, g) := \frac{l_1 + \cdots + l_r}{s}.$$ 

Then $\text{age} : \mathcal{I}X \to \mathbb{Q}$ is constant on each connected component $X_i$ of $\mathcal{I}X$. Define

$$\text{age}(X_i) = \text{age}(x, g)$$

where $(x, g)$ is any object in the groupoid $X_i$. Note that

$$\text{age}(X_i) + \text{age}(X_{i(i)}) = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} X_i.$$

2.3. **The Chen-Ruan orbifold cohomology group.** In [11], W. Chen and Y. Ruan introduced the orbifold cohomology group of a complex orbifold. See [1, Section 4.4] for a more algebraic version.

As a graded $\mathbb{Q}$ vector space, the Chen-Ruan orbifold cohomology group of $X$ is defined to be

$$H^*_{\text{CR}}(X; \mathbb{Q}) := \bigoplus_{a \in \mathbb{Q} \geq 0} H^a_{\text{CR}}(X; \mathbb{Q})$$

where

$$H^a_{\text{CR}}(X; \mathbb{Q}) = \bigoplus_{i \in I} H^{a - 2\text{age}(X_i)}(X_{i}; \mathbb{Q}).$$

Suppose that $X$ is proper. Then we have the following proper pushforward to a point:

$$\int_X : H^*(X; \mathbb{Q}) \to H^*(\text{point}; \mathbb{Q}) = \mathbb{Q}.$$ 

The orbifold Poincaré pairing is defined by

$$(\alpha, \beta) := \begin{cases} 
\int_X \alpha \cup i^*_i \beta, & j = i(i), \\
0, & j \neq i(i), 
\end{cases}$$

where $\alpha \in H^*(X_i; \mathbb{Q}), \beta \in H^*(X_j; \mathbb{Q}).$

3. **Algebraic GKM Orbifolds and Stacky GKM Graphs**

In this section, we review the geometry of algebraic GKM orbifolds, and introduce the stacky GKM graph associated to an algebraic GKM orbifold. In the algebraic setting, algebraic GKM orbifolds are more general than the GKM orbifolds in Guillemin-Zara [27, 28] in the following ways:

1. Guillemin-Zara consider compact GKM manifolds or orbifolds, whereas we consider algebraic GKM orbifolds which are not necessarily compact.
2. By orbifolds Guillemin-Zara mean orbifolds having a presentation of the form $X/K$, $K$ being a torus and $X$ being a manifold on which $K$ acts in a faithful, locally free fashion [28, Section 1.2]. In particular, the inertia group of a point is a finite abelian group. Our algebraic GKM orbifolds do not have such a presentation in general, and the inertia group of a point is a possibly non-abelian finite group.
We do not assume the torus action on the algebraic GKM orbifold is faithful.

On the other hand, Guillemin-Zara work in the $C^\infty$ category and consider $C^\infty$-action by a compact torus $U(1)^m$, while we restrict ourselves to smooth Deligne-Mumford stacks and algebraic action by an algebraic torus $(\mathbb{C}^*)^m$ (which restricts to a $U(1)^m$-action).

3.1. Algebraic GKM orbifolds. The following definition of algebraic GKM orbifolds is the same as that in [63].

**Definition 3.1** (algebraic GKM orbifolds). Let $X$ be a smooth DM stack. We say $X$ is an algebraic GKM orbifold if it is equipped with an action of an algebraic torus $T = (\mathbb{C}^*)^m$ with only finitely many zero-dimensional and one-dimensional orbits.

The notion of a group action on a stack is discussed in [51].

Let $N = \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^m$ be the lattice of 1-parameter subgroups of $T$, and let $M = \text{Hom}(T, \mathbb{C}^*)$ be the character lattice of $T$. Then $M = \text{Hom}(N, \mathbb{Z})$ is the dual lattice of $N$. We introduce

$$N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}, \quad M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}, \quad N_\mathbb{Q} := N \otimes \mathbb{Z} \mathbb{Q}, \quad M_\mathbb{Q} := M \otimes \mathbb{Z} \mathbb{Q}.$$  

Then $M_\mathbb{Q}$ can be canonically identified with $H^2_T(\text{point}; \mathbb{Q}) = H^2(BT; \mathbb{Q})$, where $BT$ is the classifying space of $T$.

We make the following assumption on $X$.

**Assumption 3.2.**

1. The set $X^T$ of $T$ fixed points in $X$ is non-empty.
2. The coarse moduli space of the closure of a one-dimensional orbit is either a complex projective line $\mathbb{P}^1$ or a complex affine line $\mathbb{C}$.

Note that (1) and (2) hold when $X$ is proper. Indeed, if $X$ is proper then the coarse moduli space of the closure of any one-dimensional orbit is $\mathbb{P}^1$.

**Example 3.3.** If $X$ is a smooth toric DM stack defined by a finite fan [9, 20], then $X$ is an algebraic GKM orbifold.

**Example 3.4.** An algebraic GKM manifold (in the sense of [44]) is an algebraic GKM orbifold.

**Definition 3.5.** Let $X$ be an algebraic GKM orbifold. The 0-skeleton of $X$ is defined to be $X^0 := X^T$ which is the union of zero-dimensional orbits of the $T$-action on $X$. The 1-skeleton $X^1$ of $X$ is defined to be the union of zero-dimensional and one-dimensional orbits of the $T$-action on $X$.

3.2. A zero dimensional orbit and its normal bundle. A zero-dimensional orbit is a fixed (possibly stacky) point $p = BG$ under the $T$-action on $X$, where $G$ is a finite group. The normal bundle of $p$ in $X$ is the tangent space $T_p X$ to $X$ at $p$, which is a rank $r$ vector bundle over $BG$, or equivalently, a representation $\rho : G \to GL(r, \mathbb{C})$. The $T$-action on $X$ induces a $T$-action on $T_p X = [C'/G]$, which can be viewed as a $T$-equivariant vector bundle of rank $r$ over $BG$. The GKM assumption implies that $T_p X$ is a direct sum of $T$-equivariant line bundles $L_1, \ldots, L_r$ over $BG$, so that

$$\rho = \bigoplus_{i=1}^r \rho_i$$
is the direct sum of \( r \) one-dimensional representations of \( G \). We may choose coordinates on \( C' \) such that \( L_i \) corresponds to the \( i \)-th coordinate axis. Let \( \phi_i : G \to \mathbb{C}^* \) be the character of the one-dimensional representation \( \rho_i \). Then the \( G \)-action on \( C' \) is given by

\[
g \cdot (z_1, \ldots, z_r) = (\phi_1(g)z_1, \ldots, \phi_r(g)z_r)
\]

where \( g \in G \) and \( (z_1, \ldots, z_r) \in C' \). Let

\[
w_i := c_i^T(L_i) \in H^2(BG; \mathbb{Q}) \cong H^2(\text{point}; \mathbb{Q}) = M_Q.
\]

The GKM condition implies that \( w_i \) and \( w_j \) are linearly independent if \( i \neq j \). The tangent space \( T_pX = [C'/G] \), together with the \( T \)-action, is an affine GKM orbifold, characterized by the finite group \( G \), characters \( \phi_1, \ldots, \phi_r \in \mathbb{C}^* := \text{Hom}(G, \mathbb{C}^*) \), and the weights \( w_1, \ldots, w_r \in M_Q \).

We define the stacky GKM graph of \([C'/G]\) as follows. The underlying graph has a single vertex \( \sigma \) and \( r \) rays \( e_1, \ldots, e_r \) emanating from \( \sigma \). The vertex is decorated by the group \( G \); the edge \( e_i \) is decorated by the group \( G_i \); the flag \((e_i, \sigma)\) is decorated by \( \phi_i \) \( w_i \), and the injective group homomorphism \( G_i \hookrightarrow G \).

The image of \( \phi_i : G \to \mathbb{C}^* \) is a finite cyclic group \( \mu_{r_i} \) of order \( r_i > 0 \). Let \( G_i \) be the kernel of \( \phi_i \). For each \( i \), we have a short exact sequence of finite groups:

\[
1 \to G_i \to G \xrightarrow{\phi_i} \mu_{r_i} \to 1.
\]

The coarse moduli

\[
C'/G = \text{Spec} \left( \mathbb{C}[z_1, \ldots, z_r]^G \right)
\]

is an affine \( T \) scheme. Let \( x_i := z_i^{|i|} \). The \( i \)-th axis

\[
\ell_i = \{ (z_1, \ldots, z_r) \in C' : z_j = 0 \text{ for } j \neq i \} / G = \text{Spec} \left( \mathbb{C}[z_i]^G \right) = \text{Spec} \mathbb{C}[x_i]
\]

is a 1-dimension affine \( T \) subscheme of \( C'/G \). The \( T \)-action on \( C' \) restricts to a \( T \)-action on \( \ell_i \cong \mathbb{C} \) with weight \( r_i w_i \), so

\[
r_i w_i \in M.
\]

### 3.3. A proper one-dimensional orbit

Let \( l \subset X \) be a proper one-dimensional \( T \) orbit in \( X \). Then \( l \) contains exactly two zero-dimensional \( T \)-orbits \( x \) and \( y \) with inertia groups \( G_x \) and \( G_y \), respectively. The representation of \( G_x \) on \( (\text{res} G_y) \) on the tangent line \( T_x l \) (resp. \( T_y l \)) to the DM curve \( l \) at \( x \) (resp. \( y \)) determines a character \( \phi_x : G_x \to \mathbb{C}^* \) (resp. \( \phi_y : G_y \to \mathbb{C}^* \)) with image \( \mu_{r_x} \) (resp. \( \mu_{r_y} \)), where \( r_x \) and \( r_y \) are positive integers. Then \( l \) is a \( G \)-gerbe over its rigidification \( l^{\operatorname{rig}} \), where \( G \cong \operatorname{Ker}(\phi_x) \cong \operatorname{Ker}(\phi_y) \) and \( l^{\operatorname{rig}} \) is an orbifold DM curve isomorphic to the football \( F(r_x, r_y) \); here an orbifold DM curve is a 1-dimensional smooth DM stack with a trivial generic inertia group, and \( F(r_x, r_y) \) is an orbifold DM curve whose coarse moduli space is the projective line \( \mathbb{P}^1 \) and has exactly two orbifold points of orders \( r_x \) and \( r_y \). Let \( \tilde{x} \) and \( \tilde{y} \) be the images of \( x \) and \( y \) under the morphism \( l \to l^{\operatorname{rig}} \). The inertia groups of \( \tilde{x} \) and \( \tilde{y} \) are \( \mu_{r_x} \) and \( \mu_{r_y} \), respectively. The coarse moduli space \( \ell \) of \( l \) and \( l^{\operatorname{rig}} \) is isomorphic to the projective line \( \mathbb{P}^1 \).

The DM curve \( l \) is spherical in the sense of \[8\]. In the rest of this subsection, we recall some relevant facts from \[8\]. We thank Behrend and Noohi for explaining results in \[8\] to us.
(1) The open embeddings
\[ \iota_x : U_x := \mathfrak{R} \setminus \{ y \} \cong [C/\mu_x] \hookrightarrow \mathfrak{R}, \quad \iota_y : U_y := \mathfrak{R} \setminus \{ x \} \cong [C/\mu_y] \hookrightarrow \mathfrak{R} \]
induce surjective group homomorphisms
\[ \iota_{x*} : \pi_1(U_x) \cong \mu_x \rightarrow \pi_1(\mathfrak{R}) \cong \mu_a, \quad \iota_{y*} : \pi_1(U_y) \cong \mu_y \rightarrow \pi_1(\mathfrak{R}) \cong \mu_a, \]
where \( a = \text{g.c.d.}(r_x, r_y) \).

(2) The open embeddings
\[ \iota_x : U_x := \mathfrak{R} \setminus \{ y \} \cong [C/G_x] \hookrightarrow \mathfrak{R}, \quad \iota_y : U_y := \mathfrak{R} \setminus \{ x \} \cong [C/G_y] \hookrightarrow \mathfrak{R} \]
induce surjective group homomorphisms
\[ \iota_{x*} : \pi_1(U_x) \cong G_x \rightarrow \pi_1(\mathfrak{R}) \cong G \rightarrow \pi_1(\mathfrak{R}). \]

(3) \( \iota_{x*} \) restricts to a group homomorphism \( G \rightarrow \pi_1(\mathfrak{R}) \), whose kernel is a cyclic group \( \mu_b \) contained in the center \( Z(G) \) of \( G \), and whose cokernel is \( \pi_1(\mathfrak{R}) \cong \mu_a \). In other words, we have the following exact sequence of finite groups:
\[ 1 \rightarrow \mu_d \rightarrow G \rightarrow \pi_1(\mathfrak{R}) \rightarrow \mu_a \rightarrow 1. \]

(4) We have a commutative diagram
\[
\begin{array}{ccc}
1 & \rightarrow & G \\
\downarrow & & \downarrow \iota_{x*} \\
G_x & \rightarrow & \mu_x \\
\downarrow \text{id}_G & & \downarrow q_x \\
G & \rightarrow & \pi_1(\mathfrak{R}) \\
\downarrow \text{id}_G & & \downarrow q_y \\
1 & \rightarrow & G_y \\
\downarrow & & \downarrow \iota_{y*} \\
1 & \rightarrow & \mu_y \\
\end{array}
\]
where \( \text{id}_G : G \rightarrow G \) is the identity map, the rows are exact, and \( q_x, q_y \) are surjective. The maps \( \mu_x \rightarrow \text{Out}(G) \) and \( \mu_y \rightarrow \text{Out}(G) \) factor through \( \mu_a \rightarrow \text{Out}(G) \).

(5) We have \((r_x, r_y) = (ap, aq)\), where \( p, q \in \mathbb{Z}_{>0} \) are relatively prime. The universal cover of \( \mathfrak{R} \) is \( \mathcal{F}(p, q) = \mathcal{P}(p, q) \); the universal cover of \( \mathfrak{R} \) is the weighted projective line \( \mathcal{P}(dp, dq) \).
Recall that
- \( \mathcal{F}(m, n) = \mathcal{P}(m, n) \) if and only of \( m, n \) are relatively prime, and
- \( \mathcal{P}(m, n) \) is simply connected for any positive integers \( m, n \).

(6) There exist
- a central extension \( E \) of the finite group \( \pi_1(\mathfrak{R}) \) by \( C^* \), so that we have a short exact sequence of groups
\[ 1 \rightarrow C^* \rightarrow E \rightarrow \pi_1(\mathfrak{R}) \rightarrow 1 \]
where \( C^* \) is contained in the center \( Z(E) \) of \( E \), and
- a representation \( \rho : E \rightarrow GL(2, C) \),
such that \( \rho \circ (\iota(1)) = (t^{dp}, t^{dq}) \) and
\[ (3.1) \quad \mathfrak{R} \cong [(C^2 - \{0\})/E]. \]
The inclusion $i : C^* \hookrightarrow E$ induces a surjective morphism
\[ \pi : \tilde{\mathcal{I}} := \mathbb{P}(dp, dq) = [(C^2 - \{0\})/C^*] \longrightarrow \mathcal{I} = [(C^2 - \{0\})/E] \]
which is the universal covering map. Taking the rigidification yields
\[ \pi^{\text{rig}} : \tilde{\mathcal{I}}^{\text{rig}} = \mathbb{P}(p, q) = \mathcal{F}(p, q) \longrightarrow \mathcal{I}^{\text{rig}} = \mathcal{F}(r_x, r_y) = \mathcal{F}(ap, aq) \]
which is also the universal covering map.

The GKM condition implies the image of $\rho$ in (6) lies in (up to conjugation) the subgroup $GL(1, \mathbb{C}) \times GL(1, \mathbb{C})$ of diagonal matrices, i.e. $\rho = \rho_x \oplus \rho_y$ is the direct sum of two 1-dimensional representations of $E$.

Under the isomorphism (3.1) we have the following identifications:
\[ G_x = \text{Ker}(\rho_x), \quad G_y = \text{Ker}(\rho_y), \quad x = [1, 0], \quad y = [0, 1], \quad G = \text{Ker}(\rho), \]
\[ \rho_y(G_x) = \mu_x, \quad \rho_x(G_y) = \mu_y. \]

### 3.4. Normal bundle of a proper one-dimensional orbit.

Let
\[ \mathcal{I} = [(C^2 - \{0\})/E] \]
be as above. We have
\[ \text{Pic}(\mathcal{I}) = \text{Hom}(E, C^*). \]

The normal bundle of $\mathcal{I}$ in $\mathcal{X}$ is a direct sum of $(r - 1)$-line bundles over $\mathcal{I}$:
\[ N_{i/\mathcal{X}} = L_1 \oplus \cdots \oplus L_{r-1}. \]

For $i = 1, \ldots, r - 1$, let $\rho_i \in \text{Hom}(E, C^*)$ be the character associated to $L_i$ under the isomorphism (3.2). Then the total space of $L_i$ is the quotient stack
\[ L_i = [((C^2 - \{0\}) \times C)/E] \]
where the action of $E$ is given by
\[ g \cdot (X, Y, Z) = (\rho_x(g)X, \rho_y(g)Y, \rho_i(g)Z). \]

If $t \in C^* \subset E$ then
\[ t \cdot (X, Y, Z) = (t^{d_1}X, t^{d_2}Y, t^{d_i}Z) \]
for some $d_i \in \mathbb{Z}$. Recall that for any positive integers $m, n$,
\[ \text{Pic}(\mathbb{P}(m, n)) \cong \mathbb{Z} \]
is generated by
\[ O_{\mathbb{P}(m,n)}(1) = [((C^2 - \{0\}) \times C)/C^*] \]
where $t \in C^*$ acts by $t \cdot (X, Y, Z) = (t^mX, t^nY, tZ)$. We have
\[ \langle O_{\mathbb{P}(m,n)}(1), [\mathbb{P}(m,n)^{\text{rig}}] \rangle = \frac{1}{l.c.m.(m,n)} \]
where $l.c.m.(m,n)$ is the least common multiple of $m, n$.

\[ \pi^* L_i = O_{\mathbb{P}(dp,dq)}(d_i) \]
where $\pi : \mathbb{P}(dp,dq) \to \mathcal{I}$ is the universal cover. Define
\[ a_i := \langle c_1(L_i), [\mathcal{I}^{\text{rig}}] \rangle = \frac{d_i}{adpq}. \]
There is a map \( \pi^{rig} : \mathbb{P}(dp,dq) \to \mathbb{P}(p,q) \) from the universal cover \( \mathbb{P}(p,q) \) of \( \mathcal{L} \) to the universal cover of \( \mathbb{P}(p,q) \) of \( \mathcal{L}^{rig} \); this map can be identified with the map to rigidification, and is of degree \( 1/d \). We have

\[
(\pi^{rig})^*\mathcal{O}(p,q)(1) = \mathcal{O}(dp,dq)(d).
\]

The map from \( \mathcal{L} \) to \( \mathcal{L}^{rig} \) is of degree \( 1/|G| \). The universal covering maps \( \mathbb{P}(dp,dq) \to \mathcal{L} \) and \( \mathbb{P}(p,q) \to \mathcal{L}^{rig} = \mathcal{F}(ap,aq) \) is of degrees \( a|G|/d \) and \( a \), respectively.

We have

- The action of \( G_x \) on \( T_x \mathcal{L} \) and \( (L_i)_x \) is given by the \( G_x \)-characters \( \rho_y|_{G_x} \) and \( \rho_1|_{G_x} \), respectively;
- The action of \( G_y \) on \( T_y \mathcal{L} \) and \( (L_i)_y \) is given by the \( G_y \)-characters \( \rho_x|_{G_y} \) and \( \rho_1|_{G_y} \), respectively.

For \( i = 1, \ldots, r - 1 \), let \( w_{x,i} \) and \( w_{y,i} \) be the weights of \( T \)-action on \( (L_i)_x \) and \( (L_i)_y \), respectively; let \( w_{x,r} \) and \( x_{y,r} \) be the weights of \( T \)-action on \( T_y \mathcal{L} \) and \( T_y \mathcal{L} \), respectively. Then 

\[
r_x w_{x,r} + r_y w_{y,r} = 0.
\]

For \( i = 1, \ldots, r, \)

\[
w_{y,i} = w_{x,i} - a_i r_x w_{x,r} = w_{x,i} + a_i r_y w_{y,r}.
\]

In particular, \( a_r = \frac{1}{r_x} + \frac{1}{r_y} \).

The total space of \( N_{\mathcal{L}^{rig}/\mathcal{X}} \) is the quotient stack

\[
\frac{\left( (\mathbb{C}^2 - \{0\}) \times C^{r-1} \right)}{/E}
\]

where \( E \) acts on \( (\mathbb{C}^2 - \{0\}) \times C^{r-1} \) linearly by \( \rho_x \oplus \rho_y \oplus \rho_1 \oplus \cdots \oplus \rho_{r-1} \).

**Remark 3.6.** If \( \mathcal{X} \) is a smooth toric DM stack then \( N_{\mathcal{L}^{rig}/\mathcal{X}} \) is a smooth toric DM substack, and the above presentation as a quotient stack can be constructed by the stacky fan, where \( E \) is abelian.

We define the GKM graph of \( N_{\mathcal{L}^{rig}/\mathcal{X}} \) as follows.

- The underlying abstract graph is a tree with two \( r \)-valent vertices \( \sigma_x, \sigma_y \) connected by a compact edge \( e \). There are \( r - 1 \) rays \( e_1, \ldots, e_{r-1} \) emanating from the vertex \( \sigma_x \) and \( r - 1 \) rays \( e'_1, \ldots, e'_{r-1} \) emanating from the vertex \( \sigma_y \).
- The vertices \( \sigma_x \) and \( \sigma_y \) are decorated by finite groups \( G_x \) and \( G_y \), respectively.
- The edge \( e_i \) is decorated by the kernel \( G_i \) of \( \phi_{x,i} := \rho_i|_{G_x} \). The edge \( e'_i \) is decorated by the kernel \( G'_i \) of \( \phi_{y,i} := \rho_i|_{G_y} \). The edge \( e \) is decorated by the group \( G \).
- The flag \( (\sigma_x, e_i) \) is decorated by \( w_{x,i} \in M_{Q_x} \), the \( G_x \) character \( \phi_{x,i} \) and the injection \( G_i \hookrightarrow G_x \). The flag \( (\sigma_y, e'_i) \) is decorated by \( w_{y,i} \in M_{Q_y} \), the \( G_y \) character \( \phi_{y,i} \) and the injection \( G_i \hookrightarrow G_y \). The flag \( (\sigma_x, e) \) is decorated by \( w_y, \rho_y|_{G_y} \), and the injection \( G \hookrightarrow G_y \). The flat \( (\sigma_y, e) \) is decorated by \( w_x, \rho_x|_{G_x} \), and teh injection \( G \hookrightarrow G_x \).
- The unique compact edge \( e \) is decorated by the central extension \( E_E \) of \( \pi_1(\mathcal{L}) \) by \( C^* \) and \( \rho_x, \rho_y, \rho_1, \ldots, \rho_{r-1} \in \text{Hom}(E,C^*) \) with isomorphisms \( G_x \cong \text{Ker}(\rho_x), G_y \cong \text{Ker}(\rho_y), G \cong \text{Ker}(\rho_x) \cap \text{Ker}(\rho_y) \).
3.5. The stacky GKM graph of an algebraic GKM orbifold. Let $\mathcal{X}$ be an algebraic GKM orbifold of dimension $r$, so that $T = (\mathbb{C}^*)^m$ acts algebraically on $\mathcal{X}$. Similar to Guillemin-Zara [27, 28], we define the stacky GKM graph to $\mathcal{X}$.

The stacky GKM graph of an algebraic GKM orbifold.

Let $V(Y)$ (resp. $E(Y)$) denote the set of vertices (resp. edges) in $Y$.

(1) (Vertices) We assign a vertex $\sigma$ to each torus fixed point $p_\sigma$ in $\mathcal{X}$. Let $p_\sigma$ be the corresponding torus fixed point in the coarse moduli space $X$.

(2) (Edges) We assign an edge $\epsilon$ to each one-dimensional orbit $o_\epsilon$ in $X$, and choose a point $p_\epsilon$ on $o_\epsilon$. Let $l_\epsilon$ be the closure of $o_\epsilon$, and let $\ell_\epsilon$ be the coarse moduli space of $l_\epsilon$. Let $E(Y)_\epsilon := \{ \epsilon \in E(Y) : \ell_\epsilon \cong \mathbb{P}^1 \}$ be the set of compact edges in $Y$. (Note that $E(Y)_\epsilon = E(Y)$ if $\mathcal{X}$ is proper.)

(3) (Flags) The set of flags in the graph $Y$ is given by

$$F(Y) = \{(e, \sigma) \in E(Y) \times V(Y) : \sigma \in \epsilon \} = \{(e, \sigma) \in E(Y) \times V(Y) : p_\sigma \in \ell_\epsilon \}.$$ 

(4) (Inertia) For each $\sigma \in V(Y)$, we assign a finite group $G_\sigma$ which is the inertia group of $p_\sigma$, so that $p_\sigma = BG_\sigma$. For each $\epsilon \in E(Y)$, we assign a finite group $G_\epsilon$ which is the inertia group of $p_\epsilon$ in item (2) above.

(5) For every flag $(e, \sigma) \in F(Y)$, we choose a path from $p_\epsilon$ to $p_\sigma$, which determines an injective group homomorphism $j_{(e, \sigma)} : G_\epsilon \rightarrow G_\sigma$. Let $\phi_{(e, \sigma)} \in G_\sigma^*$ be the irreducible character which corresponds to the 1-dimensional $G_\sigma$ representation $T_{p_\epsilon,l_\epsilon}$. The image of $\phi_{(e, \sigma)} : G_\sigma \rightarrow \mathbb{C}^*$ is a finite cyclic group; let $r(e, \sigma)$ be the cardinality of this finite cyclic group. We have the following short exact sequence of finite groups:

$$1 \rightarrow G_\epsilon \xrightarrow{j_{(e, \sigma)}} G_\sigma \xrightarrow{\phi_{(e, \sigma)}} \mu_{r(e, \sigma)} \rightarrow 1.$$

So

$$r(e, \sigma) = \frac{|G_\sigma|}{|G_\epsilon|}.$$ 

(6) (fundamental groups) For each compact edge $\epsilon \in E(Y)_\epsilon$, there is a group homomorphism $G_\epsilon \rightarrow \pi_1(l_\epsilon)$ whose kernel is a cyclic subgroup of $\tilde{Z}(G_\epsilon)$, the center of $G_\epsilon$. Let $d_\epsilon$ be the cardinality of this cyclic subgroup. Then we have an exact sequence of finite groups:

$$1 \rightarrow \mu_{d_\epsilon} \rightarrow G_\epsilon \rightarrow \pi_1(l_\epsilon) \rightarrow \pi_1(l_\epsilon^{\text{rig}}) \rightarrow 1.$$

(7) (central extension of fundamental groups) For each compact edge $\epsilon \in E(Y)_\epsilon$, let $\sigma_x, \sigma_y \in V(Y)$ be the two ends, and let

$$x = p_{\sigma_x}, \quad y = p_{\sigma_y}, \quad r_x = r(e, \sigma_x), \quad r_y = r(e, \sigma_y), \quad a_\epsilon = \text{g.c.d.}(r_x, r_y).$$

Then $l_\epsilon^{\text{rig}}$ is the football $F(r_x, r_y)$, so that $\pi_1(l_\epsilon^{\text{rig}}) = \mu_{a_\epsilon}$. There is a triple $(t_\epsilon, E_\epsilon, \rho_\epsilon)$, where $t_\epsilon : \mathbb{C}^* \rightarrow E_\epsilon$ is a central injection and $E_\epsilon / \mathbb{C}^* \cong \pi_1(l_\epsilon)$, $\rho_\epsilon = (\rho_x, \rho_y) : E_\epsilon \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ is a group homomorphism and

$$\rho_\epsilon \circ t_\epsilon = (t_\epsilon^{r_x/a_\epsilon}, t_\epsilon^{r_y/a_\epsilon}).$$

We have an isomorphism $[(\mathbb{C}^2 - \{(0,0)\}) / E_\epsilon] \cong l_\epsilon$, and under this isomorphism

$$G_{\sigma_x} = \text{Ker}(\rho_x), \quad G_{\sigma_y} = \text{Ker}(\rho_y), \quad x = [1,0], \quad y = [0,1], \quad G_\epsilon = \text{Ker}(\rho).$$
Assumption 3.7. (connection) Let $e \in E(Y)_c$, and let $\sigma_x$ and $\sigma_y$ be as above, and let $E_x$ and $E_y$ be the set of edges emanating from $\sigma_x$ and $\sigma_y$, respectively. The normal bundle $N_{l_e/X}$ of $l_e$ in $X$ is a direct sum of line bundles

$$N_{l_e/X} = L_1 \oplus \cdots \oplus L_{r-1}. $$

For $i = 1, \ldots, r - 1$ there exist $e_i \in E_x$ and $e'_i \in E_y$ such that $(L_i)_x = T_x l_{e_i}$ and $(L_i)_y = T_y l_{e'_i}$. Then

$$E_x = \{e_1, \ldots, e_{r-1}, e\}, \quad E_y = \{e'_1, \ldots, e'_{r-1}, e\}. $$

Define a bijection $\theta_{(e,\sigma_x)} : E_x \rightarrow E_y$ by sending $e_i$ to $e'_i$ and sending $e$ to $e$; let $\theta_{(e,\sigma_y)} : E_y \rightarrow E_x$ be the inverse map. We say $\{e_i, e'_i\}$ is a pair of edges related by the parallel transport along the compact edge $e$. There exists $\rho_i \in \text{Hom}(E_e, C^*)$ such that $L_i = [(\mathbb{C}^2 - \{(0, 0)\}) \times C]/E$ where $E$ acts by $\rho_x \oplus \rho_y \oplus \rho_i$. Then $\rho_i \circ l_e : C^* \rightarrow C^*$ is given by $t \mapsto t^{d_i}$ for some $d_i \in \mathbb{Z}$ and

$$a_i := \langle c_1(L_i), [t^{d_i}] \rangle = \frac{d_i}{\text{l.c.m.}(r_x, r_y)} = \frac{d_i d_e}{r_x r_y d_e} \in \mathbb{Q}. $$

(Note that, if $l_e$ is the projective line $\mathbb{P}^1$ then $r_x = r_y = a_e = d_e = 1$, so $a_i = d_i \in \mathbb{Z}$. Let $\Delta_e$ be the set of pairs of edges related by the parallel transport along the compact edge $e$. For each pair $\delta \in \Delta_e$ we get a character $\rho_\delta \in \text{Hom}(E_e, C^*)$ which corresponds to a line bundle over $l_e$ which is a summand of $N_{l_e/X}$.

(9) (compatibility along compact edges)

$$\rho_x|_{C_{\sigma_x}} = \phi_{(e_i, \sigma_x)}, \quad \rho_x|_{C_{\sigma_y}} = \phi_{(e'_i, \sigma_y)}, \quad \rho_y|_{C_{\sigma_x}} = \phi_{(e_i, \sigma_x)}, \quad \rho_y|_{C_{\sigma_y}} = \phi_{(e_i, \sigma_y)}. $$

Assumption 3.2 can be rephrased in terms of the graph $Y$ as follows.

Assumption 3.7. (1) $V(Y)$ is non-empty.

(2) Each edge in $E(Y)$ contains at least one vertex.

Given a vertex $\sigma \in V(Y)$, we denote by $E_\sigma$ the set of edges containing $\sigma$, i.e.

$$E_\sigma := \{e \in E : (e, \sigma) \in F(Y)\}. $$

Then $|E_\sigma| = r$ for all $\sigma \in V(Y)$, so $Y$ is an $r$-valent graph.

Given a flag $(e, \sigma) \in F(Y)$, let $w(e, \sigma) \in M_Q$ be the weight of $T$-action on $T_{p_\sigma} l_e$, the tangent line to $l_e$ at the fixed point $p_\sigma = B G_{\sigma}$, namely

$$w(e, \sigma) := c_1(T_{p_\sigma} l_e) \in H^2_{\mathbb{R}}(p_\sigma, \mathbb{Q}) \cong M_Q. $$

This gives rise to a map $w : F(Y) \rightarrow M_Q$ satisfying the following properties.

(1) (GKM hypothesis) Given any $\sigma \in V(Y)$, and any two distinct edges $e, e' \in E_\sigma$, $w(e, \sigma)$ and $w(e', \sigma)$ are linearly independent in $M_Q$.

(2) (integrality) For any flag $(e, \sigma) \in F(Y)$, $\overline{w}(e, \sigma) := r(e, \sigma) w(e, \sigma) \in M$.

(3) Suppose that $e \in E(Y)_c$ is a compact edge and $\sigma_x, \sigma_y \in V(Y)$ are its two ends.

(a) $r(e, \sigma_x) w(e, \sigma_x) + r(e, \sigma_y) w(e, \sigma_y) = 0$, i.e. $\overline{w}(e, \sigma_x) + \overline{w}(e, \sigma_y) = 0$.

(b) Let $E_{\sigma_x} = \{e_1, \ldots, e_r\}$, where $e = e$, and let $e'_i := \theta_{(e_i, \sigma)}(e) \in E_{\sigma_y}$. Then

$$w(e'_i, \sigma_y) = w(e_i, \sigma_x) - a_i r(e, \sigma_x) w(e, \sigma_x) = w(e_i, \sigma_x) + a_i r(e, \sigma_y) w(e, \sigma_y). $$
Let $e$ be as in (2). The normal bundle of the 1-dimensional smooth DM stack $I_e$ in $\mathcal{X}$ is given by

$$N_{I_e/\mathcal{X}} \cong L_1 + \cdots + L_{r-1}$$

where $L_i$ is a degree $a_i$ $T$-equivariant line bundle over $I_e$ such that the weights of the $T$-action on the fibers $(L_i)_z$ and $(L_i)_y$ are $w(e_i, \sigma_x)$ and $w(e_i, \sigma_y)$, respectively. The map $w : F(Y) \to M_Q$ is called the axial function.

We call $\mathcal{Y}_i$, which is the abstract graph $Y$ together with the above decorations and constraints, the stacky GKM graph of the algebraic GKM orbifold $\mathcal{X}$ with the $T$-action.

If $\rho : T' \to T$ is a homomorphism between complex algebraic tori, then $T'$ acts on $X$ by $t' \cdot x = \rho(t') \cdot x$, where $t' \in T'$, $\rho(t') \in T$, $x \in X$. If the zero-dimensional and one-dimensional orbits of this $T'$-action coincide with those of the $T$-action, then the GKM graph with this $T'$-action is obtained by replacing $w(e, \sigma) \in M_Q$ by $\rho^* w(e, \sigma) \in M'_Q$, where

$$\rho^* : M_Q = H^2(BT; \mathbb{Q}) \to M_Q' := H^2(BT'; \mathbb{Q}).$$

### 3.6. Equivariant Chen-Ruan orbifold cohomology group

Let $\mathcal{X}$ be an algebraic GKM orbifold. The $T$-action on $\mathcal{X}$ induces a $T$-action on its inertia stack $\mathcal{I} \mathcal{X} = \bigsqcup_{i \in I} \mathcal{X}_i$ and on each $\mathcal{X}_i$.

Let $R_T := H^*_T(\text{point}; \mathbb{Q}) = H^* (BT; \mathbb{Q}) = \mathbb{Q}[u_1, \ldots, u_m]$ where $\deg(u) = 2$; let $Q_T = \mathbb{Q}(u_1, \ldots, u_m)$ be its fractional field.

As a graded $\mathbb{Q}$-vector space, $T$-equivariant Chen-Ruan orbifold cohomology group of an algebraic GKM orbifold is defined to be

$$H^a_{\text{CR}, T}(\mathcal{X}; \mathbb{Q}) := \bigoplus_{a \in \mathbb{Q}_{\geq 0}} H^a_{\text{CR}, T}(\mathcal{X}; \mathbb{Q})$$

where

$$H^a_{\text{CR}, T}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{i \in I} H^a_{T}(\mathcal{X}_i; \mathbb{Q}).$$

Suppose that $\mathcal{X}$ is proper. For each $i \in I$, we have the following proper push-forward to a point:

$$\int_{\mathcal{X}_i} : H^*_T(\mathcal{X}_i; \mathbb{Q}) \to H^*_T(\text{point}; \mathbb{Q}) = R_T$$

which is $R_T$-linear. The $T$-equivariant orbifold Poincaré pairing is defined by

$$(\alpha, \beta)_T := \begin{cases} \int_{\mathcal{X}_i} \alpha \cup t_j^* \beta, & j = i(i), \\ 0, & j \neq i(i), \end{cases}$$

where $\alpha \in H^*_T(\mathcal{X}_i; \mathbb{Q})$, $\beta \in H^*_T(\mathcal{X}_i; \mathbb{Q})$.

When $\mathcal{X}$ is not proper, we define a $T$-equivariant Poincaré pairing on

$$H^*_\text{CR,T}(\mathcal{X}; Q_T) = H^*_\text{CR,T}(\mathcal{X}; \mathbb{Q}) \otimes_{R_T} Q_T$$

as follows:

$$(\alpha, \beta)_T := \begin{cases} \int_{\mathcal{X}_i} \frac{(\alpha \cup t_j^* \beta)_{|\mathcal{X}_i}}{\mathcal{X}_i^*}, & j = i(i), \\ 0, & j \neq i(i), \end{cases}$$

where
where \( X^T_i \subseteq X_i \) is the \( T \) fixed substack, and \( e_T(N_{X^T_i / X_i}) \) is the \( T \)-equivariant Euler class of the normal bundle \( N_{X^T_i / X_i} \) of \( X^T_i \) in \( X_i \). Each \( X^T_i \) is a disjoint union of finitely many (stacky) points.

**Example 3.8 (affine GKM orbifold).** Let \( \mathcal{X} = [C^r / G] \) be an affine GKM orbifold. Let \( \rho_i : G \to C^* \), \( w_i \in M_Q \), and \( r_i \) be defined as in Section 3.2. Given \( h \in G \), let \( c_i(h) \) be the unique element in

\[
\left\{ 0, \frac{1}{r_i}, \ldots, \frac{r_i - 1}{r_i} \right\}
\]

such that

\[
e^{2\pi \sqrt{-1} c_i(h)} = \rho_i(h).
\]

Then

\[
\mathcal{X} = \bigoplus_{c \in \text{Conj}(G)} \mathcal{X}_c,
\]

where

\[
\mathcal{X}_c \cong [(C^r)^h / C(h)]
\]

for any \( h \in c \). We have

\[
\text{age}(\mathcal{X}_c) = \sum_{i=1}^{r} c_i(h)
\]

where \( h \) is any element in the conjugacy class \( c \).

Let \( 1_c \) denote the identity element of \( H^T_*(\mathcal{X}_c; \mathbb{Q}) \). Then

\[
H^*(\mathcal{X}_c; \mathbb{Q}) = \mathbb{Q} 1_c, \quad H^*_T(\mathcal{X}_c; \mathbb{C}) = R_T 1_c.
\]

So

\[
H^{CR}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{c \in \text{Conj}(G)} \mathbb{Q} 1_c
\]

as a \( \mathbb{Q} \) vector space, and

\[
H^{CR,T}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{c \in \text{Conj}(G)} R_T 1_c
\]

as an \( R_T \)-module.

Given \( c \in \text{Conj}(G) \), define

\[
e_c := e_T(T_{[0 / G]}(C^r)^h) = \prod_{i=1}^{r} w_i^{\delta_{c_i(h),0}}
\]

where \( h \) is any element in \( c \). Note that the right hand side of the above equation does not depend on the choice of \( h \in c \).

Given \( h \in G \), let \( [h] = \{ aha^{-1} : a \in G \} \) be the conjugacy class of \( h \).

The \( T \)-equivariant Poincaré pairing on

\[
H^{CR,T}(\mathcal{X}; Q_T) = \bigoplus_{c \in \text{Conj}(G)} Q_T 1_c
\]

is given by

\[
(1_{[h]}, 1_{[\nu]})_T = \frac{1}{|C(h)|} \delta_{[h^{-1} \nu, [h]}} e_{[h]}.
\]
3.7. **Cup product.** In this section, we describe the cup product on
\[ H^*_{CR,T}(\mathcal{X}; Q_T), \]
first for affine GKM orbifolds, and then for any algebraic GKM orbifolds.

Given \( c, c' \in \text{Conj}(G) \), define
\[ c_i(c, c') := c_i(h) + c_i(h') - c_i(hh') \in \{0, 1\}. \]
where \( h \in c \) and \( h' \in c' \); note that the right hand side of the above equation does not depend on choice of \( h \in c \) and \( h' \in c' \).

- Let \( \mathcal{X} = [C'/G] \) be an affine GKM orbifold as in Example 3.8. The cup product on \( H^*_{CR,T}(\mathcal{X}; Q_T) \) is given by
  \[ 1_c \star 1_{c'} = \prod_{i=1}^{r} \mathbb{Q} \frac{1}{|G|} \sum_{h \in c, h' \in c'} |C(hh')| \mathbb{1}_{[hh']}. \]

- Let \( \mathcal{X} \) be an algebraic GKM orbifold, and let \( \mathcal{Y} \) be the stacky GKM graph of \( \mathcal{X} \). Then we have an isomorphism of \( Q_T \)-algebras
  \[ H^*_{CR,T}(\mathcal{X}; Q_T) \cong \bigoplus_{\sigma \in V(\mathcal{Y})} H^*_{CR,T}(T_{p_\sigma} \mathcal{X}; Q_T) \]
  which preserves the \( T \)-equivariant Poincaré pairing; the isomorphism (3.5) is an isomorphism of Frobenius algebras over the field \( Q_T \).

4. **Abstract stacky GKM graphs and formal GKM orbifolds**

Given an algebraic GKM orbifold \( \mathcal{X} \) with a \( T \)-action, the formal completion \( \hat{\mathcal{X}} \) of \( \mathcal{X} \) along its 1-skeleton \( \mathcal{X}^1 \), together with the \( T \)-action inherited from \( \mathcal{X} \), can be reconstructed from the stacky GKM graph of \( \mathcal{X} \). In this section, we will define abstract stacky GKM graphs which are generalization of stacky GKM graphs of algebraic GKM orbifolds. Given an abstract stacky GKM graph, we will construct a formal GKM orbifold, which is a formal smooth DM stack together with an action by an algebraic torus \( T = C^m \). The construction of a formal GKM orbifold from an abstract stacky GKM graph can be viewed as generalization of the construction of \( \hat{\mathcal{X}} \) from the stacky GKM graph of \( \mathcal{X} \), and is inspired by the construction of a formal toric Calabi-Yau (FTCY) threefold from an FTCY graph in \([43, \text{Section 3}]\).

4.1. **Abstract stacky GKM graphs.** We fix \( T = (C^*)^m \) and a positive integer \( r \). An abstract stacky GKM graph is a decorated graph consisting of the following data.

1. (graph) The underlying graph \( \Gamma \) is a connected \( r \)-valent graph \( \Gamma \) with finitely many vertices and edges. Let \( V(Y) \) (resp. \( E(Y) \)) denote the set of vertices (resp. edges) in \( \Gamma \). Each edge in \( E(Y) \) is either a compact edge connecting two vertices or a ray emanating from one vertex. Let \( E(Y)_c \subset E(Y) \) be the set of compact edges. Let
  \[ F(Y) = \{(e, \sigma) \in E(Y) \times V(Y) : \sigma \in e\} \]
be the set of flags in \( \Gamma \). Given a vertex \( \sigma \in V(Y) \), let
  \[ E_\sigma := \{e \in E(Y) : (e, \sigma) \in F(Y)\} \]
be the set of edges emanating from the vertex \( v \). By the \( r \)-valent condition, \(|E_\sigma| = r \) for all \( r \in V(Y) \).
(2) (inertia and tangent characters) Each vertex $\sigma \in V(Y)$ (resp. edge $e \in E(Y)$) is decorated by a finite group $G_\sigma$ (resp. $G_e$). Each flag $(e, \sigma) \in F(Y)$ is decorated by
- an injective group homomorphism $j_{(e, \sigma)} : G_e \hookrightarrow G_\sigma$, and
- a character $\varphi_{(e, \sigma)} : G_\sigma \to \mathbb{C}^*$, such that $\text{Im}(j_{(e, \sigma)}) = \text{Ker}(\varphi_{(e, \sigma)})$.

Note that the image of $\varphi_{(e, \sigma)}$ is a finite cyclic group; let $r(e, \sigma)$ be the cardinality of this finite cyclic group. Then we have a short exact sequence of finite groups:

$$1 \to G_e \xrightarrow{j_{(e, \sigma)}} G_\sigma \xrightarrow{\varphi_{(e, \sigma)}} \mu_{r(e, \sigma)} \to 1.$$ 

(3) (fundamental groups and central extensions) Let $e \in E(Y)_c$ be a compact edge, and let $\sigma_x, \sigma_y \in V(Y)$ be the two ends of $e$. Let $a_e = \gcd(r(e, \sigma_x), r(e, \sigma_y))$. In addition to $G_e$, $e$ is decorated by:
- Another finite group $\pi_e$ together with a group homomorphism $G_e \to \pi_e$ such that we have the following exact sequence of finite groups:

$$1 \to \mu_{a_e} \to G_e \to \pi_e \to \mu_{a_e} \to 1$$

where $\mu_{a_e}$ is contained in the center of $G_e$.

- A central extension $1 \to \mathbb{C}^* \xrightarrow{i_{\sigma}} E_\sigma \to \pi_e \to 1$ of $\pi_e$ by $\mathbb{C}^*$, and a group homomorphism $\rho_e = (\rho_x, \rho_y) : E_\sigma \to \mathbb{C}^* \times \mathbb{C}^*$.

(4) (connection) Let $e \in E_c(Y)$ be a compact edge, and let $\sigma_x, \sigma_y \in V(Y)$ be the two ends of $e$. There are bijections $\theta_{(e, x)} : E_{\sigma_x} \to E_{\sigma_y}$ and $\theta_{(e, y)} : E_{\sigma_y} \to E_{\sigma_x}$ which are inverses of each other and send $e$ to $e$.

(5) (normal characters) Suppose that $e \in E_c(Y)$ is a compact edge, $\sigma_x, \sigma_y \in V(Y)$ are two ends of $e$, and $\delta = \{e_x, e_y\}$ is a pair of edges such that $e_x \in E_{\sigma_x} - \{e\}$ and $e_y = \theta_{(e, e_x)}(e_x) \in E_{\sigma_y} - \{e\}$. Such a pair is decorated by a character $\rho_{|\delta} \in \text{Hom}(E_\delta, \mathbb{C}^*)$.

(6) (compatibility along compact edges) In the notation of (3), (4), (5) above, $\text{Ker}(\rho_x) = G_{\sigma_x}, \text{Ker}(\rho_y) = G_{\sigma_y}, \text{Ker}(\rho_e) = G_e$.

$$\rho_y|_{G_{\sigma_x}} = \rho_{(e, x)}|_{\sigma_x}, \quad \rho_x|_{G_{\sigma_y}} = \rho_{(e, y)}|_{\sigma_y}, \quad \rho_e|_{G_{\sigma_x}} = \rho_{(e, x)}|_{\sigma_x}, \quad \rho_e|_{G_{\sigma_y}} = \rho_{(e, y)}|_{\sigma_y}.$$ 

(7) (axial function) There is a map $w : F(Y) \to M_Q$ satisfying the following properties.

(a) (GKM hypothesis) Given any $\sigma \in V(Y)$ and any two distinct edges $e, e' \in E_\sigma$, $w(e, \sigma)$ and $w(e', \sigma)$ are linearly independent vectors in $M_Q$.

(b) (integrality) For any $(e, \sigma) \in F(Y), w(e, \sigma) := r(e, \sigma)w(e, \sigma) \in M$.

(c) For any compact edge $e \in E(Y)_c$, let $\sigma_x, \sigma_y \in V(Y)$ be its two ends. Then the following properties hold.

(i) $r(e, \sigma_x)w(e, \sigma_x) + r(e, \sigma_y)w(e, \sigma_y) = 0$, i.e., $\mathbb{w}(e, \sigma_x) + \mathbb{w}(e, \sigma_y) = 0$.

(ii) Suppose that $E_{\sigma} = \{e_1, \ldots, e_r\}$, where $e_r = e$. Let $e'_i := \theta_{(e, \sigma)}(e_i) \in E_{\sigma'}$, so that $E_{\sigma'} = \{e'_1, \ldots, e'_r\}$. Let

$$a_i = \frac{d_{\sigma e}}{r(e, \sigma) r(e, \sigma')} d_e.$$
where \( d_i \in \mathbb{Z} \) is determined by by \( \rho_{(e, \sigma')} \circ t_c(t) = t^{d_i} \) for \( t \in \mathbb{C} \). Then
\[
w(e, \sigma) = w(e, \sigma) - a_r(e, \sigma_x)w(e, \sigma_x) = w(e, \sigma_i) + a_i r(e, \sigma_y)w(e, \sigma_y),
\]
or equivalently,
\[
w(e, \sigma) = w(e, \sigma) - a_\ell \mathcal{W}(e, \sigma) = w(e, \sigma_i) + a_i \mathcal{W}(e, \sigma_y),
\]
In particular, \( e'_r = e_r = e \) and \( a_r = \frac{1}{r(e, \sigma_x)} + \frac{1}{r(e, \sigma_y)} \).

Let \( \tilde{Y} \) denote the underlying abstract graph \( Y \) together with all the above decorations.

**Remark 4.1.** We may also define abstract GKM graphs by the following specialization.

- All the finite groups \( G_\sigma, G_r, \pi_\sigma \) are trivial, and we always have \( E_e = \mathbb{C}^* \) and \( \rho_x, \rho_y : \mathbb{C}^* \to \mathbb{C}^* \) are identity maps. So we do not need (2), (3), (6) above.
- In (7), the axial function \( w \) takes value in \( \mathbb{M} \) instead of \( \mathbb{M}_Q \), and
  \[
r(e, \sigma) = r(e, \sigma') = 1, \quad a_e = d_e = 1, \quad a_i = d_i \in \mathbb{Z}.
\]
- The normal characters in (5) are determined by the axial function.

Abstract GKM graphs are generalization of GKM graphs of algebraic GKM manifolds [44, Section 2.2].

### 4.2. Formal GKM orbifolds.

Given an abstract stacky GKM graph \( \tilde{Y} \) defined as in the previous section, we will construct a formal smooth DM stack \( \tilde{X}_Y \) of dimension \( r \) equipped with an action of \( T = (\mathbb{C}^*)^m \).

For any flag \((e, \sigma) \in F(Y)\), define a “stacky” affine line
\[
D_{(e, \sigma)} := \text{Spec} \mathbb{C}[z_{(e, \sigma)}]/G_{\sigma} \cong \mathbb{A}^1/G_{\sigma}
\]
where \( G_{\sigma} \) acts on \( \mathbb{A}^1 \) via the character \( \phi_{(e, \sigma)} : G_{\sigma} \to \mathbb{C}^* \). The coarse moduli space of \( D_{(e, \sigma)} \) is
\[
\text{Spec}(\mathbb{C}[z_{(e, \sigma)}]^{G_{\sigma}}) = \text{Spec}(\mathbb{C}[z_{(e, \sigma)}]) = \text{Spec}(\mathbb{C}[x_{(e, \sigma)}]) \cong \mathbb{A}^1
\]
where \( x_{(e, \sigma)} = z_{(e, \sigma)}^{r(e, \sigma)} \).

For any vertex \( \sigma \in V(Y) \), define an \( r \)-dimensional affine GKM orbifold
\[
X_\sigma = \text{Spec} \mathbb{C}[z_{(e, \sigma)}] : e \in E_{\sigma}/G_{\sigma} = \mathbb{A}^{E_{\sigma}}/G_{\sigma}.
\]
The \( T \)-action on \( z_{(e, \sigma)} \) is determined by \( w(e, \sigma) \in M_Q \). Let \( \tilde{X}_\sigma \) be the formal completion of \( X_\sigma \) along its 1-skeleton.

For any compact edge \( e \in E(Y) \), define
\[
I_e := [(\mathbb{C}^2 - \{0\})/E_{\sigma}]
\]
where the action of \( E_{\sigma} \) is given by the group homomorphism \( \rho_{\sigma} : E_{\sigma} \to \mathbb{C}^* \times \mathbb{C}^* \).

Let \( \sigma_x, \sigma_y \in V(Y) \) be its two ends. Suppose that \( E_{\sigma_x} = \{e_1, \ldots, e_{r-1}, e\} \), and let \( e'_1 = \theta_{(e, \sigma_x)}(e_i) \in E_{\sigma_y} - \{e\} \). Let \( \rho_i = \rho_{(e, \sigma')} \in \text{Hom}(E_{\sigma}, \mathbb{C}^*) \). Let \( L_i \) be the line bundle over the smooth DM curve \( I_e \) defined by
\[
L_i = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}]/E_{\sigma}
\]
where the action on the last factor \( \mathbb{C} \) is given by the character \( \rho_i \). Let \( \tilde{X}_e \) be the total space of \( L_1 \oplus \cdots \oplus L_{r-1} \), which is an algebraic GKM orbifold. Let \( \tilde{X}_{\sigma} \) be the formal
completion of $\mathcal{X}_\epsilon$ along $\mathcal{X}_1^1$. There are $T$-equivariant open embeddings of formal smooth DM stacks:

$$i_{(\epsilon_\sigma, \epsilon)} : \hat{\mathcal{X}}_{\epsilon_\sigma} \hookrightarrow \hat{\mathcal{X}}_\epsilon, \quad i_{(\epsilon_\eta, \epsilon)} : \hat{\mathcal{X}}_{\epsilon_\eta} \hookrightarrow \hat{\mathcal{X}}_\epsilon.$$ 

The formal DM stack $\hat{\mathcal{X}}_{\tilde{\Upsilon}}$ is the fiber product of the map

$$\bigcup_{\sigma \in V(Y)} \hat{\mathcal{X}}_{\epsilon_\sigma} \longrightarrow \bigcup_{\epsilon \in E(Y)_c} \hat{\mathcal{X}}_{\epsilon},$$

where the restrict of $i$ to $\hat{\mathcal{X}}_{\epsilon_\sigma}$ is given by

$$\prod_{\epsilon \in E_{\sigma} \cap E(Y)_c} i_{(\epsilon_\sigma, \epsilon)} : \hat{\mathcal{X}}_{\epsilon_\sigma} \longrightarrow \bigcup_{\epsilon \in E(Y)_c} \hat{\mathcal{X}}_{\epsilon}.$$ 

If $\tilde{\Upsilon}$ is the stacky GKM graph of an algebraic GKM orbifold $\mathcal{X}$ then $\hat{\mathcal{X}}_{\tilde{\Upsilon}}$ is the formal completion of $\mathcal{X}$ along its 1-skeleton $\mathcal{X}_1^1$.

### 4.3. Equivariant Chen-Ruan orbifold cohomology of an abstract stacky GKM graph

Given a stacky GKM graph $\tilde{\Upsilon}$, we define

$$\mathcal{H}_{\tilde{\Upsilon}} := \bigoplus_{\sigma \in V(Y)} H^*_{CR}(\mathcal{X}_\sigma; \mathbb{Q}_T)$$

as a Frobenius algebraic over the field $\mathbb{Q}_T$. If $\tilde{\Upsilon}$ is the stacky GKM graph of an algebraic GKM orbifold $\mathcal{X}$ then

$$\mathcal{H}_{\tilde{\Upsilon}} = H^*_{CR}(\mathcal{X}; \mathbb{Q}_T).$$

### 5. Orbifold Gromov-Witten theory

In [12], Chen-Ruan developed Gromov-Witten theory for symplectic orbifolds. The algebraic counterpart, the Gromov-Witten theory for smooth DM stacks, was developed by Abramovich-Graber-Vistoli [1, 2]. In this section, we give a brief review of algebraic orbifold Gromov-Witten theory, following [2].

#### 5.1. Twisted curves and their moduli

An $n$-pointed, genus $g$ twisted curve is a connected proper one-dimensional DM stack $\mathcal{C}$ together with $n$ disjoint closed substacks $\mathcal{C}_{x_1}, \ldots, \mathcal{C}_{x_n}$ of $\mathcal{C}$, such that

1. $\mathcal{C}$ is étale locally a nodal curve;
2. formally locally near a node, $\mathcal{C}$ is isomorphic to the quotient stack $\text{Spec}(\mathbb{C}[x,y]/(xy))/\mu_r$,

where the action of $\zeta \in \mu_r$ is given by $\zeta \cdot (x,y) = (\zeta x, \zeta^{-1} y)$;
3. each $\mathcal{C}_{x_i}$ is contained in the smooth locus of $\mathcal{C}$;
4. each stack $\mathcal{C}_{x_i}$ is an étale gerbe over $\text{Spec} \mathbb{C}$ with a section (hence trivialization);
5. $\mathcal{C}$ is a scheme outside the twisted points $\mathcal{C}_{x_1}, \ldots, \mathcal{C}_{x_n}$ and the singular locus;
6. the coarse moduli space $\mathcal{C}$ is a nodal curve of arithmetic genus $g$.

Let $\pi : \mathcal{C} \to \mathcal{C}$ be the projection to the coarse moduli space, and let $x_i = \pi(\mathcal{C}_{x_i})$. Then $x_1, \ldots, x_n$ are distinct smooth points of $\mathcal{C}$, and $(\mathcal{C}, x_1, \ldots, x_n)$ is an $n$-pointed, genus $g$ prestable curve.

Let $\mathcal{M}_{g,n}^{tw}$ be the moduli of $n$-pointed, genus $g$ twisted curves. Then $\mathcal{M}_{g,n}^{tw}$ is a smooth algebraic stack, locally of finite type [59].
5.2. **Riemann-Roch theorem for twisted curves.** Let \((\mathcal{C}, x_1, \ldots, x_n)\) be an \(n\)-pointed, genus \(g\) twisted curve, and let \((\mathcal{C}, x_1, \ldots, x_n)\) be the coarse curve, which is an \(n\)-pointed, genus \(g\) prestable curve. Let \(E \to \mathcal{X}\) be a vector bundle over \(\mathcal{X}\). Then \(r_i \equiv B\mu_{r_i}\), and \(\zeta_{r_i}\) acts on \(E|_{r_i}\) with eigenvalues \(\frac{r_i}{l_i}, \ldots, \frac{r_i}{l_i}\), where \(l_i \in \{0, 1, \ldots, r_i - 1\}\) and \(N = \text{rank} \, E\). Define

\[
\text{age}_{\beta_i}(\mathcal{E}) := \frac{l_1 + \cdots + l_N}{r_i} \in \mathbb{Q}.
\]

The Riemann-Roch theorem for twisted curves says

\[
\chi(\mathcal{E}) = \int_{\mathcal{C}} c_1(\mathcal{E}) + \text{rank}(\mathcal{E})(1 - g) - \sum_{i=1}^{n} \text{age}_{\beta_i}(\mathcal{E}).
\]

5.3. **Moduli of twisted stable maps.** Let \(\mathcal{X}\) be a smooth DM stack with a quasi-projective coarse moduli space \(\mathcal{X}\), and let \(\beta \in H_2(\mathcal{X}; \mathbb{Z})\) be an effective curve class in \(\mathcal{X}\). An \(n\)-pointed, genus \(g\), degree \(\beta\) twisted stable map to \(\mathcal{X}\) is a representable morphism \(f : \mathcal{C} \to \mathcal{X}\), where the domain \(\mathcal{C}\) is an \(n\)-pointed, genus \(g\) twisted curve, and the induced morphism \(\mathcal{C} \to X\) between the coarse moduli spaces is an \(n\)-pointed, genus \(g\), degree \(\beta\) stable map to \(X\).

Let \(\overline{M}_{g,n}(\mathcal{X}, \beta)\) be the moduli stack of \(n\)-pointed, genus \(g\), degree \(\beta\) twisted stable maps to \(\mathcal{X}\). Then \(\overline{M}_{g,n}(\mathcal{X}, \beta)\) is a DM stack; it is proper if \(X\) is projective.

For \(j = 1, \ldots, n\), there are evaluation maps \(\text{ev}_j : \overline{M}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}\). Given \(\vec{i} = (i_1, \ldots, i_n)\), where \(i_j \in I\), define

\[
\overline{M}_{g,n}(\mathcal{X}, \beta) := \bigcap_{j=1}^{n} \text{ev}_j^{-1}(\mathcal{X}_{ij}).
\]

Then \(\overline{M}_{g,n}(\mathcal{X}, \beta)\) is a union of connected components of \(\overline{M}_{g,n}(\mathcal{X}, \beta)\), and

\[
\overline{M}_{g,n}(\mathcal{X}, \beta) = \bigcup_{\vec{i} \in I^n} \overline{M}_{g,n}(\mathcal{X}, \beta).
\]

**Remark 5.1.** In the definition of twisted curves in Section 5.1 if we replace (4) by

(4)’ each stack \(x_i\) is an étale gerbes over \(\text{Spec}\, \mathbb{C}\);

i.e. without a section, then the resulting moduli space is \(\mathcal{K}_{g,n}(\mathcal{X}, \beta)\) in [2], and the evaluation maps take values in the rigidified inertial stack \(\mathcal{I}\mathcal{X}\) instead of the initial stack \(\mathcal{I}\mathcal{X}\).

5.4. **Obstruction theory and virtual fundamental classes.** The tangent space \(T^1\) and the obstruction space \(T^2\) at a moduli point \([f : (\mathcal{C}, x_1, \ldots, x_n) \to \mathcal{X}] \in \overline{M}_{g,n}(\mathcal{X}, \beta)\) fit in the tangent-obstruction exact sequence:

\[
0 \to \text{Ext}^0_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(x_1 + \cdots + x_n), \mathcal{O}_{\mathcal{C}}) \to H^0(\mathcal{C}, f^*T_{\mathcal{X}}) \to T^1
\]

\[
\to \text{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(x_1 + \cdots + x_n), \mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C}, f^*T_{\mathcal{X}}) \to T^2 \to 0
\]

where

- \(\text{Ext}^0_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(x_1 + \cdots + x_n), \mathcal{O}_{\mathcal{C}})\) is the space of infinitesimal automorphisms of the domain \((\mathcal{C}, x_1, \ldots, x_n)\),
- \(\text{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(x_1 + \cdots + x_n), \mathcal{O}_{\mathcal{C}})\) is the space of infinitesimal deformations of the domain \((\mathcal{C}, x_1, \ldots, x_n)\).
• $H^0(C, f^*T_X)$ is the space of infinitesimal deformations of the morphism $f$
for a fixed domain, and
• $H^1(C, f^*T_X)$ is the space of obstructions to deforming the morphism $f$ for
a fixed domain.

$T^1$ and $T^2$ form sheaves $T^1$ and $T^2$ on the moduli space $\mathcal{M}_{g,d}(X', \beta)$. This defines
a perfect obstruction theory of virtual dimension $d_{g,i,\beta}^{vir}$ on $\mathcal{M}_{g,d}(X', \beta)$, where

\begin{equation}
(5.3) \quad d_{g,i,\beta}^{vir} = \int_{\beta} c_1(T_X) + (\dim X' - 3)(1 - g) + n - \sum_{j=1}^{n} \text{age}(X_j).
\end{equation}

There is a virtual fundamental class

$$\mathcal{M}_{g,i}(X', \beta)^{vir} \in \mathcal{M}_{g,i}(X, \beta) \in \mathbb{Q}.$$  

The weighted virtual fundamental class is defined by

$$[\mathcal{M}_{g,i}(X', \beta)]^{vir} := \left( \prod_{j=1}^{n} \mathcal{M}_{g,i}(X', \beta)^{vir} \right).$$

5.5. Moduli of twisted stable maps to a formal GKM orbifold. Let $\hat{X}_{\hat{Y}}$ be the
formal GKM orbifold defined by an abstract GKM graph $\hat{Y}$, and let $\tilde{X}_{\hat{Y}}$ be its coarse
moduli space. Then

$$H_2(\tilde{X}_{\hat{Y}}; \mathbb{Z}) = \bigoplus_{e \in E(\hat{Y})_c} \mathbb{Z}[\ell_e].$$

Let

$$\hat{\beta} = \sum_{e \in E(\hat{Y})_c} d_e [\ell_e] \in H_2(\tilde{X}_{\hat{Y}}; \mathbb{Z})$$

where $d_e \geq 0$, so that $\hat{\beta}$ is effective. Let $\mathcal{M}_{g,n}(\hat{X}_{\hat{Y}}, \hat{\beta})$ be the moduli of genus $g$, $n$-
pointed, degree $\hat{\beta}$ twisted stable maps to $\hat{X}_{\hat{Y}}$, which is the analogue of $\mathcal{M}_{g,n}(X, \beta)$
defined in Section 5.3.

Let

$$\mathcal{M}_{g,n}(\hat{X}_{\hat{Y}}, \hat{\beta}) = \bigcup_{i \in I_{\hat{Y}}} (\hat{X}_{\hat{Y}})_i$$

be disjoint union of connected components. Let $\mathcal{M}_{g,i}(\hat{X}_{\hat{Y}}, \hat{\beta})$ be the analogue of $\mathcal{M}_{g,d}(X', \beta)$. Then we have a disjoint union

$$\mathcal{M}_{g,n}(\hat{X}_{\hat{Y}}, \hat{\beta}) = \bigcup_{i \in I_{\hat{Y}}} \mathcal{M}_{g,i}(\hat{X}_{\hat{Y}}, \hat{\beta}).$$

Each $\mathcal{M}_{g,i}(\hat{X}_{\hat{Y}}, \hat{\beta})$ is equipped with a $T$-action together with a $T$-equivariant perfect
obstruction theory of virtual dimension $d_{g,i,\beta}^{vir}$, where

\begin{equation}
(5.3) \quad d_{g,i,\beta}^{vir} = \int_{\beta} c_1(T_{\tilde{X}_{\hat{Y}}}) + (\dim \tilde{X}_{\hat{Y}} - 3)(1 - g) + n - \sum_{j=1}^{n} \text{age}(X_j).
\end{equation}

If $\hat{Y}$ is the stacky GKM graph of an algebraic GKM orbifold $X'$ then we have a
surjective map $I_{\hat{Y}} \rightarrow I$. 

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5.6. Hurwitz-Hodge integrals. By Example 2.1 when $X = BG$ we have

$$\mathcal{I}BG = \bigsqcup_{c \in \text{Conj}(G)} (BG)_{\beta}$$

where $\text{Conj}(G)$ is the set of conjugacy classes of $G$. Give $\bar{c} = (c_1, \ldots, c_n) \in \text{Conj}(G)^n$, let $\mathcal{M}_{g,\bar{c}}(BG) = \mathcal{M}_{g,c}(BG, \beta = 0)$. Then $\mathcal{M}_{g,\bar{c}}(BG)$ is a union of connected components of $\mathcal{M}_{g,n}(BG) := \mathcal{M}_{g,n}(BG, 0)$, and

$$\mathcal{M}_{g,n}(BG) = \bigsqcup_{\bar{c} \in \text{Conj}(G)^n} \mathcal{M}_{g,\bar{c}}(BG).$$

We now fix a genus $g$ and $n$ conjugacy classes $\bar{c} = (c_1, \ldots, c_n) \in \text{Conj}(G)^n$. Let $\pi : U \to \mathcal{M}_{g,\bar{c}}(BG)$ be the universal curve, and let $f : U \to BG$ be the universal map. Let $\rho : G \to GL(V)$ be an irreducible representation of $G$, where $V$ is a finite dimensional vector space over $\mathbb{C}$. Then $E_\rho := [V/G]$ is a vector bundle over $BG = [\text{point}/G]$. We have

$$\pi_* f^* E_\rho = \begin{cases} \mathcal{O}_{\mathcal{M}_{g,\bar{c}}(BG)} & \text{if } \rho : G \to GL(1, \mathbb{C}) \text{ is the trivial representation,} \\ 0 & \text{otherwise.} \end{cases}$$

The $\rho$-twisted Hurwitz-Hodge bundle $E_\rho$ can be defined as the dual of the vector bundle $R^1 \pi_* f^* E_\rho$. If $\rho = 1$ is the trivial representation, then $E_1 = e^* \mathcal{E}$, where $e : \mathcal{M}_{g,\bar{c}}(BG) \to \mathcal{M}_{g,n}$, and $\mathcal{E} \to \mathcal{M}_{g,n}$ is the Hodge bundle of $\mathcal{M}_{g,n}$. So $\text{rank} E_1 = g$. If $\rho$ is a nontrivial irreducible representation, it follows from the Riemann-Roch theorem for twisted curves (see Section 5.2) that

$$\text{rank} E_\rho = \text{rank}(E_\rho) (g - 1) + \sum_{j=1}^{n} \text{age}_{c_j}(E_\rho),$$

where $\text{age}_{c_j}(E_\rho)$ is given as follows. Choose $\rho \in c_j$. Let $s > 0$ be the order of $g$ in $G$, let $N = \text{rank} E_\rho = \dim V$. If the eigenvalues of $\rho(g) \in GL(V) = GL(N, \mathbb{C})$ are $\zeta_{l_1} s, \ldots, \zeta_{l_N} s$, where $l_1, \ldots, l_N \in \{0, 1, \ldots, s - 1\}$, then

$$\text{age}_{c_j}(E_\rho) = \frac{l_1 + \cdots + l_N}{s}.$$
Descendant classes. There is a map \( \epsilon: \overline{\mathcal{M}}_{g,c}(BG) \to \overline{\mathcal{M}}_{g,n} \). Define
\[
\tilde{\psi}_j = \epsilon^* \psi_j \in H^2(\overline{\mathcal{M}}_{g,c}(BG)), \quad j = 1, \ldots, n.
\]

Hurwitz-Hodge integrals are top intersection numbers of Hodge classes \( \lambda_i^0 \) and descendant classes \( \tilde{\psi}_j \):
\[
\int_{\overline{\mathcal{M}}_{g,c}(BG)} \tilde{\psi}_1^{d_1} \cdots \tilde{\psi}_n^{d_n} (\lambda_1^{\ell_1})^{k_1} \cdots (\lambda_g^{\ell_g})^{k_g}.
\]

In [62], J. Zhou described an algorithm of computing Hurwitz-Hodge integrals, as follows. By Tseng’s orbifold quantum Riemann-Roch theorem [58], Hurwitz-Hodge integrals can be reconstructed from descendant integrals on \( \overline{\mathcal{M}}_{g,c}(BG) \):
\[
\int_{\overline{\mathcal{M}}_{g,c}(BG)} \tilde{\psi}_1^{d_1} \cdots \tilde{\psi}_n^{d_n}.
\]

Jarvis-Kimura relate the descendant integrals on \( \overline{\mathcal{M}}_{g,c}(BG) \) to those on \( \overline{\mathcal{M}}_{g,n} \) [29]. We now state their result. Given \( g \in \mathbb{Z}_{\geq 0} \) and \( \bar{c} = (c_1, \ldots, c_n) \in \text{Conj}(G)^n \), let
\[
V^G_{g,c} := \{(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n) \in G^{2g+n} | \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^n e_j, e_j \in c_j \}.
\]

Then \( \overline{\mathcal{M}}_{g,c}(BG) \) is nonempty iff \( V^G_{g,c} \) is nonempty.

**Theorem 5.2** (Jarvis-Kimura [29] Proposition 3.4). Suppose that \( 2g - 2 + n > 0 \) and \( V^G_{g,c} \) is nonempty. Then
\[
\int_{\overline{\mathcal{M}}_{g,c}(BG)} \tilde{\psi}_1^{d_1} \cdots \tilde{\psi}_n^{d_n} = \frac{|V^G_{g,c}|}{|G|} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.
\]

5.7. Orbifold GW invariants. There is a morphism \( \epsilon: \overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta) \to \overline{\mathcal{M}}_{g,n}(\mathcal{X}; \beta) \). Define \( \tilde{\psi}_j = \epsilon^* \psi_j \).

Suppose that the coarse moduli space \( \mathcal{X} \) is projective. Then \( \overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta) \) is proper. Let
\[
\gamma_j \in H^{d_j}(\mathcal{X}_i; \mathbb{Q}) \subset H_{CR}^{d_j + 2\text{age}(\mathcal{X}_i)}(\mathcal{X}; \mathbb{Q}),
\]

Define orbifold Gromov-Witten invariants
\[
\langle \tilde{e}_{a_1} \gamma_1, \ldots, \tilde{e}_{a_n} \gamma_n \rangle^{\mathcal{X}}_{g,\beta} := \int_{\overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta)} \prod_{j=1}^n \text{ev}_1^* \gamma_1 \cup \tilde{\psi}_j^{d_1} \cup \cdots \cup \text{ev}_n^* \gamma_n \cup \tilde{\psi}_n^{d_n}
\]
which is zero unless
\[
\sum_{j=1}^n (d_j + 2\text{age}(\mathcal{X}_i) + 2a_j) = 2 \left( \int_{\beta} c_1(T_\mathcal{X}) + (1 - g)(\dim \mathcal{X} - 3) + n \right).
\]

5.8. Equivariant orbifold GW invariants. Suppose that \( \mathcal{X} \) is equipped with a \( T \)-action, which induces a \( T \)-action on \( \overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta) \) and on the perfect obstruction theory. Then there is a \( T \)-equivariant virtual fundamental class
\[
[\overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta)]^\text{vir, T} \in H^{2\dim \mathcal{X}}_{g,i,\beta}(\overline{\mathcal{M}}_{g,i}(\mathcal{X}; \beta); \mathbb{Q}).
\]
The weighted $T$-equivariant virtual fundamental class is defined by
\[
\left[\mathcal{M}_{g,j}(\mathcal{X}, \beta)\right]^{vir,T} = \left(\prod_{j=1}^{n} r_j\right) \left[\mathcal{M}_{g,j}(\mathcal{X}, \beta)\right]^{vir}.
\]

Suppose that $\mathcal{M}_{g,j}(\mathcal{X}, \beta)$ is proper. (If the coarse moduli space $X$ is projective then $\mathcal{M}_{g,j}(\mathcal{X}, \beta)$ is proper for any $g, j, d$.) Given $\gamma_i^T \in H^j_T(\mathcal{X}_i; \mathbb{Q}) \subset H^{d_j + 2\text{age}(\mathcal{X}_i)}_{\mathcal{CR}, T}(\mathcal{X}; \mathbb{Q})$ and $a_i \in \mathbb{Z}_{\geq 0}$, we define $T$-equivariant orbifold Gromov-Witten invariants
\[
\langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\check{g}, \beta} = \int_{[\mathcal{M}_{g,j}(\mathcal{X}, \beta)]^{vir,T}} e_T^*(ev_1^*\gamma_1^T \cup (\psi_1^T)^{a_1} \cup \ldots \cup ev_n^*\gamma_n^T \cup (\psi_n^T)^{a_n})
\]
\[
\in \mathbb{Q}[u_1, \ldots, u_m] \left(\sum_{j=1}^{n} (d_j + 2a_j) - 2d_i^{vir}\right).
\]

where $\mathbb{Q}[u_1, \ldots, u_m](2k)$ is the space of degree $k$ homogeneous polynomials in $u_1, \ldots, u_t$ with rational coefficients, and $\mathbb{Q}[u_1, \ldots, u_m](2k + 1) = 0$. In particular,
\[
\langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\check{g}, \beta} = \begin{cases} 0, & \sum_{j=1}^{n} (d_j + 2a_j) < 2d_i^{vir}, \\
\left(\langle \bar{e}_{a_1}(\gamma_1), \ldots, \bar{e}_{a_n}(\gamma_n) \rangle_{\check{g}, \beta} \right)^{X_T} \in \mathbb{Q}, & \sum_{j=1}^{n} (d_j + 2a_j) = 2d_i^{vir},
\end{cases}
\]

where $\gamma_i \in H^{d_j}(\mathcal{X}_i; \mathbb{Q})$ is the image of $\gamma_i^T$ under the map $H^j_T(\mathcal{X}_i; \mathbb{Q}) \to H^{d_j}(\mathcal{X}_i; \mathbb{Q})$.

Let $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T \subset \overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)$ be the substack of $T$ fixed points, and let $i : \overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T \to \overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)$ be the inclusion. Let $N^{vir}$ be the virtual normal bundle of substack $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T$ in $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)$; in general, $N^{vir}$ has different ranks on different connected components of $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T$. By virtual localization,
\[
\int_{\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T} e_T^*(ev_1^*\gamma_1^T \cup (\psi_1^T)^{a_1} \cup \ldots \cup ev_n^*\gamma_n^T \cup (\psi_n^T)^{a_n})
\]
\[
= \int_{\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T} e_T^*(N^{vir})
\]
(5.9)

Suppose that $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)$ is not proper, but $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T$ is proper. (If $\mathcal{X}$ is an algebraic GKM orbifold then $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T$ is proper for any $g, j, \beta$.) We define
\[
\langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\check{g}, \beta} = \int_{\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T} e_T^*(ev_1^*\gamma_1^T \cup (\psi_1^T)^{a_1} \cup \ldots \cup ev_n^*\gamma_n^T \cup (\psi_n^T)^{a_n})
\]
\[
= \int_{\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)^T} e_T^*(N^{vir})
\]
(5.10)

When $\overline{\mathcal{M}}_{g,j}(\mathcal{X}, \beta)$ is not proper, the right hand side of (5.10) is a rational function (instead of a polynomial) in $u_1, \ldots, u_m$. It can be nonzero when $\sum_{j=1}^{n} (d_j + 2a_j) < 2d_i^{vir}$, and does not have a nonequivariant limit $(u_i \to 0)$ in general.

5.9. **Formal equivariant GW invariants.** Let $\check{X}_\gamma$ be the formal GKM orbifold defined by an abstract GKM graph $\check{Y}$. Then there is a $T$-equivariant virtual fundamental class
\[
\left[\overline{\mathcal{M}}_{g,j}(\check{X}_\gamma, \check{\beta})\right]^{vir,T} \in H^2_{\mathcal{CR}, T}(\overline{\mathcal{M}}_{g,j}(\check{X}_\gamma, \check{\beta}); \mathbb{Q}).
\]
The weighted $T$-equivariant virtual fundamental class is defined by

$$[\mathcal{M}_{g,i}(\mathcal{X}, \beta)]_{vs,T} = \left( \prod_{j=1}^{n} r_j \right) [\mathcal{M}_{g,i}(\mathcal{X}, \beta)]_{vir,T}.$$  

Define

$$[\mathcal{M}_{g,n}(\mathcal{X}, \beta)]_{vs,T} = \sum_{\bar{\gamma} \in (\mathcal{L}_g)^n} [\mathcal{M}_{g,i}(\mathcal{X}, \beta)]_{vs,T}.$$  

Let $\mathcal{M}_{g,i}(\mathcal{X}, \beta)^T \subset \mathcal{M}_{g,i}(\mathcal{X}, \beta)$ be the substack of $T$ fixed points. Then $\mathcal{M}_{g,i}(\mathcal{X}, \beta)^T$ is proper. Given $\gamma_1^T, \ldots, \gamma_n^T \in H^*_{CR,T}(\mathcal{X}, \mathbb{Q}) \cong H_{\mathcal{Y}}$, we define

$$(5.11) \quad \langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\mathcal{X}, \beta}^{\mathcal{Y}} = \int_{[\mathcal{M}_{g,n}(\mathcal{X}, \beta)^T]_{vs,T}} \frac{i^* \prod_{j=1}^{n} \left( \text{ev}^*_{\gamma_j} \cup (\bar{q}_{\gamma_j})^T \right)}{e^! (N_{\text{vir}})}.$$  

If $\mathcal{Y}$ is the stacky GKM graph of an algebraic GKM orbifold $\mathcal{X}$ then we have a surjection

$$\phi : \bigoplus_{c \in E(Y)} \mathbb{Z}[\zeta_c] \to H_2(X; \mathbb{Z})$$

where $X$ is the coarse moduli of $\mathcal{X}$. Given $\gamma_1^T, \ldots, \gamma_n^T \in H^*_{CR,T}(\mathcal{X}, \mathbb{Q}) \cong H_{\mathcal{Y}}$, we have

$$\langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\mathcal{X}, \beta}^{\mathcal{Y}} = \sum_{\phi(\beta) = \beta} \langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\mathcal{X}, \beta}^{\mathcal{Y}}.$$  

In Section 6 we will compute

$$\langle \bar{e}_{a_1}(\gamma_1^T), \ldots, \bar{e}_{a_n}(\gamma_n^T) \rangle_{\mathcal{X}, \beta}^{\mathcal{Y}}$$

by localization.

6. Virtual localization

6.1. Morphisms from a football to a one-dimensional GKM orbifold. Let $l$ be a 1-dimensional GKM orbifold, let $l^{BG}$ be its rigidification, and let $\mathcal{X} \cong \mathbb{P}^1$ be the coarse moduli space of $l$ and $l^{BG}$. Let $C = \mathcal{F}(r_u, r_v)$ be a football, and let $C \cong \mathbb{P}^1$ be the coarse moduli space for $C$. In this subsection, we study morphisms $f : C \to l$ such that

- $f : C \to l$ of a representable map between smooth DM stacks;
- The induced morphism $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ between coarse moduli spaces is given by $[u, v] \mapsto [u^{a'}, v^{a'}]$ in terms of homogeneous coordinates on $\mathbb{P}^1$.

Let $a' = g \cdot c.d.(r_u, r_v)$. Then $\pi_1(C) = \mu_{a'}$. Let $p' = r_u/a', q' = r_v/a'$. We have a short exact sequence of abelian groups

$$1 \to C^* \to E' \to \mu_{a'} \to 1$$

and an injective group homomorphism $\rho' : E' \to C^* \times C^*$ with image $\{(t_1, t_2) \in C^* \times C^* : t_1^{a'} = t_2^{a'} \}$, and $\rho' \circ \iota' : C^* \to C^* \times C^*$ is given by $t \mapsto (t^{a'}, t^{a'})$. The map $\pi' : C = [(C^2 - \{0\})/E'] \to C = [(C^2 - \{0\})/C^*]$. 


where \( E' \) acts by \( \rho' \) and \( C^* \) acts diagonally, is given by \([\tilde{u}, \tilde{v}] \mapsto [\tilde{u}^\rho', \tilde{v}^\rho']\). The stabilizer \( G_u \) of \([1, 0]\) is generated by \( \tilde{\xi}_u = (1, e^{2\pi\sqrt{-1}/r_u}) \); the stabilizer \( G_v \) of \([0, 1]\) is generated by \( \tilde{\xi}_v = (e^{2\pi\sqrt{-1}/r_v}, 1) \).

Recall that there is a triple \((i, E, \rho)\) such that

\[
1 \to C^* \to E \to \pi_1(I) \to 1
\]

is a central extension of the fundamental group \( \pi_1(I) \) (a finite group) by \( C^* \), and a group homomorphism \( \rho = (\rho_x, \rho_y) : E \to C^* \times C^* \) such that

\[
i = [(C^2 - \{0\})/E], \quad \mathfrak{rig} = [(C^2 - \{0\})/(E/G)]
\]

where \( G = \text{Ker}(\rho) \). Let \( G_x = \text{Ker}(\rho_x) \) and \( G_y = \text{Ker}(\rho_y) \). Then \( G_x, G_y \) are finite subgroups of \( E \) and \( G_x = G_y = G \). Define

\[
r_x = |G_x|/|G|, \quad r_y = |G_y|/|G|, \quad a = g.c.d.(r_x, r_y), \quad p = r_x/a, \quad q = r_y/b.
\]

Then \( \mathfrak{rig} \cong \mathcal{F}(r_x, r_y) \) and \( \pi_1(\mathfrak{rig}) \cong \mu_a \). There is an exact sequence of groups

\[
1 \to \mu_a \to G \to \pi_1(I) \to \mu_a \to 1
\]

where \( \mu_a \) is contained in the center of \( G \). The map

\[
\pi : i = [(C^2 - \{0\})/E] \to \ell = [(C^2 - \{0\})/C^*]
\]

where \( C^* \) acts diagonally, is given by \([\tilde{x}, \tilde{g}] \mapsto [\tilde{x}^{r_x}, \tilde{g}^{r_y}]\). The map \( f : C \to i \) restricts to

\[
C - \{[0, 1]\} = [C/\mu_{ra}] \to i - \{[0, 1]\} = [C/G_x], \quad z \to z^{ra_d/ra}.
\]

Let \( g_x \) and \( g_y \) be the image of \( \tilde{\xi}_u \) and \( \tilde{\xi}_v \) under \( \mu_{ra} \to G_x \) and \( \mu_{ra} \to G_y \), respectively. The representability of \( f \) implies \( \text{ord}(g_x) = ra, \text{ord}(g_y) = r_v \).

\[
C - \{[1, 0]\} = [C/\mu_{ra}] \to i - \{[1, 0]\} = [C/G_y], \quad z \to z^{ra_d/r_y}
\]

where \( ra_d/r_y \in \mathbb{Z} \). Then \( \rho_y(g_x) = e^{2\pi\sqrt{-1}d/ra}, \rho_x(g_y) = e^{2\pi\sqrt{-1}d/ra} \).

The map \( f \) is determined by \( \tilde{f} \) and \( g_x \in G_x, g_y \in G_y \) such that \( \rho_y(g_x) = e^{2\pi\sqrt{-1}d/ra}, \rho_x(g_y) = e^{2\pi\sqrt{-1}d/ra} \); the domain \( \mathcal{F}(r_u, r_v) \) is determined by \( r_u = \text{ord}(g_x) \) and \( r_v = \text{ord}(g_y) \).

### 6.2. Torus fixed points and graph notation.

In this subsection, we describe the \( T \)-fixed points in \( \overline{\mathcal{M}}_{g,j}(X, \hat{\beta}) \). Given a twisted stable map \( f : (C, \Gamma_1, \ldots, \Gamma_n) \to X \) such that

\[
[f : (C, \Gamma_1, \ldots, \Gamma_n) \to X] \in \overline{\mathcal{M}}_{g,j}(X, \hat{\beta})^T,
\]

we will associate a decorated graph \( \hat{\Gamma} \). We first give a formal definition.

**Definition 6.1.** Let \( \hat{\beta} = \sum_{e \in E(Y)} \hat{\beta}_e [e] \), where \( \hat{\beta}_e \in \mathbb{Z}_{\geq 0} \). A decorated graph \( \hat{\Gamma} = (\Gamma, \hat{f}, \hat{d}, \hat{g}, \hat{s}, \hat{k}) \) for \( n \)-pointed, genus \( g \), degree \( \hat{\beta} \) twisted stable maps to \( X \) consists of the following data.

1. \( \Gamma \) is a compact, connected 1-dimensional CW complex. We denote the set of vertices (resp. edges) in \( \Gamma \) by \( V(\Gamma) \) (resp. \( E(\Gamma) \)). The set of flags of \( \Gamma \) is defined to be

\[
F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) \mid v \in e\}.
\]
(2) The label map $\tilde{f} : V(\Gamma) \cup E(\Gamma) \to V(Y) \cup E(Y)_{c}$ sends a vertex $v \in V(\Gamma)$ to a vertex $\sigma_v \in V(Y)$, and sends an edge $e \in E(\Gamma)$ to an edge $e_{c} \in E(Y)_{c}$. Moreover, $\tilde{f}$ defines a map from the graph $\Gamma$ to the graph $Y$: if $(e,v) \in F(\Gamma)$ then $(e_{c},\sigma_v) \in F(Y)$.

(3) The degree map $\tilde{d} : E(\Gamma) \to \mathbb{Z}_{>0}$ sends an edge $e \in E(\Gamma)$ to a positive integer $d_e$.

(4) The genus map $\tilde{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ sends a vertex $v \in V(\Gamma)$ to a nonnegative integer $g_v$.

(5) The marking map $\tilde{s} : \{1,2,\ldots,n\} \to V(\Gamma)$ is defined if $n > 0$.

(6) The twisting map $\tilde{k}$ sends a flag $(e,v)$ to an element $k_{(e,v)} \in G_v := G_{c_{(e)}}$ a marking $j \in \{1,\ldots,n\}$ to an element $k_j \in G_v$ if $\tilde{t}(j) = v$.

The above maps satisfy the following two constraints:

(i) (topology of the domain) $\sum_{v \in V(\Gamma)} g_v + |E(\Gamma)| - |V(\Gamma)| + 1 = g$.

(ii) (topology of the map) For any $e \in E(Y)$, $\sum_{\tilde{f}(e) = e} d_e = \beta_e$.

(iii) (compatibility at a flag) $\phi_{(e,\sigma_v)}(k_{(e,v)}) = e^{2\pi d_e/r(e,\sigma_v)}$.

(iv) (compatibility at a vertex) Given $v \in V(\Gamma)$, let $E_v$ and $S_v$ be defined as follows:

$E_v = \{e \in E(\Gamma) \mid (e,v) \in F(\Gamma)\}$

$S_v = \{i \in \{1,\ldots,n\} : x_i \in C_v\}$.

Then

$$\prod_{e \in E_v} k_{(e,v)}^{-1} \prod_{j \in S_v} k_j \in \begin{cases} \{1\} , & g = 0 ; \\ [G_v, G_v] , & g > 0 , \end{cases}$$

where 1 is the identity of $G_v$, and $[G_v, G_v]$ is the commutator subgroup of $G_v$. In particular, if $(e,v) \in F(\Gamma)$ and $v \in V(\Gamma)$ then $k_{(e,v)} = 1 \in G_v$.

(v) (compatibility with $\tilde{i} = (i_1,\ldots,i_n)$) Given $j \in \{1,\ldots,n\}$, if $\tilde{s}(j) = v$, then the pair $(p_{c_{(e,v)}}, k_j)$ represent a point in $X_{i_j}$, the connected component of $\mathcal{M}$ labelled by $i_j$.

Let $G_{g,\beta}(X,\beta)$ be the set of all decorated graphs $\tilde{\Gamma} = (\Gamma, \tilde{f}, \tilde{d}, \tilde{g}, \tilde{s}, \tilde{k})$ satisfying the above constraints.

Let $f : (C,x_1,\ldots,x_n) \to X$ be a twisted stable map which represents a $T$ fixed point in $\mathcal{M}_{g,\beta}(X,\beta)$. Let $f : (C,x_1,\ldots,x_n) \to X$ be the corresponding stable map between coarse moduli spaces. Then $\tilde{f} : (C,x_1,\ldots,x_n) \to X$ represents a $T$ fixed point in $\mathcal{M}_{g,\beta}(X,\beta)$, so we may define, as in Section 6.2, $\Gamma, \tilde{f}, \tilde{d}, \tilde{g}, \tilde{s}, \tilde{k}$ for each vertex $v \in V(\Gamma)$, and $C_v$ for each edge $e \in E(\Gamma)$. It remains to define the twisting map $\tilde{k}$. Let $C_v$ (resp. $C_{e}$) be the preimage of $C_v$ (resp. $C_e$) under the projection $C \to \mathcal{C}$.

By Section 6.1, given an edge $e \in E(\Gamma)$, the map $f_e := f|_{C_e} : C_e \to l_v$ is determined by the degree $d_{e}$ of the map $\tilde{f}_{e} := \tilde{f}|_{C_e} : C_e = \mathbb{P}^{1} \to \ell_{v} = \mathbb{P}^{1}$ and $k_{(e,v)} \in G_{v}, k_{(e,v')} \in G_{v'}$, where $v,v' \in V(\Gamma)$ are the two ends of $e$. 


Given \((e, v) \in F(\Gamma)\), let \(\eta(e, v) = C_{e} \cap C_{v}\). Define \(\tilde{k}(e, v) = k_{(e, v)} \in G_{v}\) to be the image of the generator of the stabilizer of the stacky point \(\eta(e, v)\) in the orbicurve \(C_{v}\).

- Under the evaluation map \(ev_{j}\), the \(j\)-th marked point \(p_{j}\) is mapped to \((p_{v}, k)\) in the inertial stack \(IX\), where \(\sigma \in V(\Sigma)\) and \(k \in G_{\sigma}\). Then \(\tilde{f} \circ \tilde{s}(j) = \sigma\).

Define \(r_{(e, v)} = |(k_{(e, v)})|\).

where \(\langle k_{(e, v)} \rangle\) is the subgroup of \(G_{v}\) generated by \(k_{(e, v)}\). Suppose that \(v, v' \in V(\Gamma)\) are the two end points of the edge \(e \in E(\Gamma)\). Then

\[
\mathcal{I}_{e}^{\text{rig}} \equiv C_{r(e,v)\sigma}, r(e,v)^{\prime}\), \quad C_{e} \equiv C_{r(e,v)^{\prime}}.
\]

To summarize, we have a map from \(\bar{\mathcal{M}}_{g, j}(\mathcal{X}, \beta)^{T}\) to the discrete set \(G_{g, j}(\mathcal{X}, \beta)\). Let \(\mathcal{F}_{\bar{\Gamma}} \subset \bar{\mathcal{M}}_{g, j}(\mathcal{X}, \beta)^{T}\) denote the preimage of \(\bar{\Gamma}\). Then

\[
\bar{\mathcal{M}}_{g, j}(\mathcal{X}, \beta)^{T} = \bigsqcup_{\Gamma \in G_{g, j}(\mathcal{X}, \beta)} \mathcal{F}_{\bar{\Gamma}}
\]

where the right hand side is a disjoint union of connected components.

We now describe the fixed locus \(\mathcal{F}_{\bar{\Gamma}}\) associated to each decorated graph \(\bar{\Gamma} \in G_{g, j}(\mathcal{X}, \beta)\). Given an edge \(e \in E(\Gamma)\), let \(v, v' \in V(\Gamma)\) be its two ends. The map \(f_{e} : C_{e} \to \mathcal{I}_{e}\), where \(e = \tilde{f}(e)\), is determined by \(d_{e}, k_{(e, v)}\) and \(k_{(e, v')}\) up to isomorphism. The automorphism group \(\text{Aut}(f_{e})\) of \(f_{e}\) is a finite group. The moduli space of \(f_{e}\) is

\[
\mathcal{M}_{e}^{\text{rig}} = B(\text{Aut}(f_{e})).
\]

Given a stable vertex \(v \in V^{*}(\Gamma)\), the map \(f_{v} := \tilde{f}|_{C_{v}} : C_{v} \to \mathcal{I}_{v} := BG_{v}\), where \(\sigma = \tilde{f}(v)\), represents a point in \(\bar{\mathcal{M}}_{g, j}(\mathcal{X}, \beta)\), where \(\mathcal{E}_{v}\) and \(\mathcal{S}_{v}\) are defined as in Definition 6.1. For each \(e \in E_{v} \subset E(\Gamma)\), there is an evaluation map

\[
ev_{(e,v)} : \bar{\mathcal{M}}_{g, j}(\mathcal{X}, \beta) \to \mathcal{I}_{E_{v}}.
\]

We have

\[
\mathcal{I}_{E_{v}} \cong \mathcal{I}BG_{v} = \bigcup_{k \in G_{v}} (BG_{v})_{k},
\]

where \((BG_{v})_{k}\) are connected components of \(\mathcal{I}BG_{v}\) (see Example 2.1). The moduli space of \(f_{v}\) is

\[
\bar{\mathcal{M}}_{g, j}(BG_{v}) := \bigcap_{e \in E_{v}} \ev_{(e,v)}^{-1}((BG_{v})_{k_{(e,v)}}) \cap \bigcap_{j \in S_{v}} \ev_{j}^{-1}((BG_{v})_{k_{j}}).
\]

To obtain a \(T\) fixed point \([f : (\mathcal{C}_{1}, \mathcal{X}_{1}, \ldots, \mathcal{X}_{h}) \to \mathcal{X}]\), we glue the the above maps \(f_{e}\) and \(f_{v}\) along the nodes. Let \(V^{*}(\Gamma)\) and \(F^{*}(\Gamma)\) be defined as in Definition 6.1. The nodes of \(\mathcal{C}\) are

\[
\{\eta_{(e,v)} = C_{e} \cap C_{v} \mid (e, v) \in F^{*}(\Gamma)\} \cup \{\eta_{v} = C_{v} \mid v \in V^{*}(\Gamma), E_{v} = \{e_{1}, e_{2}\}\}.
\]
We define $\tilde{\mathcal{M}}_F$ by the following 2-cartesian diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{M}}_F & \xrightarrow{f_E} & \prod_{e \in E(\Gamma)} \mathcal{M}_e \\
\downarrow f_V & & \downarrow \text{ev}_F \\
\prod_{v \in V^2(\Gamma)} \mathcal{M}_{\tilde{G}_v, \tilde{T}}(BG_v) & \xrightarrow{\text{ev}_V} & \prod_{(\varepsilon, v) \in F^2(\Gamma)} \mathcal{T}BG_v \times \prod_{v \in V^2(\Gamma)} \mathcal{T}BG_v
\end{array}
$$

where $\text{ev}_V$ and $\text{ev}_F$ are given by evaluation at nodes, and $\mathcal{T}BG_v$ is the rigidified inertia stack. More precisely:

- For every stable flag $(\varepsilon, v) \in F^2(\Gamma)$, let $\text{ev}_{(\varepsilon, v)}$ be the evaluation map at the node $\tilde{\eta}_{(\varepsilon, v)}$, and let $\overline{\text{ev}}_{(\varepsilon, v)} = \iota \circ \text{ev}_{(\varepsilon, v)}$ where $\iota$ is the involution on $\mathcal{T}BG_v$.
- For each $v \in V^2(\Gamma)$, let $E_v = \{e_1, e_2\}$ (we pick some ordering of the two edges in $E_v$), let $\text{ev}_{(\varepsilon, v)}$ be the evaluation map at the node $\tilde{\eta}_v$, and let $\text{ev}_{(2, v)} = \iota \circ \text{ev}_{(2, v)}$.

Define

$$
\text{ev}_V = \prod_{(\varepsilon, v) \in F^2(\Gamma)} \text{ev}_{(\varepsilon, v)}
$$

and

$$
\text{ev}_F = \prod_{(\varepsilon, v) \in F^2(\Gamma)} \overline{\text{ev}}_{(\varepsilon, v)} \times \prod_{v \in V^2(\Gamma)} \text{ev}_{(\varepsilon, v)}.
$$

The fixed locus associated to the decorated graph $\tilde{\Gamma}$ is

$$
\mathcal{F}_\Gamma = \tilde{\mathcal{M}}_F / \text{Aut}(\tilde{\Gamma}).
$$

From the above definitions, up to some finite morphism, $\mathcal{F}_\Gamma$ can be identified with

$$
\mathcal{M}_\Gamma := \prod_{v \in V^2(\Gamma)} \mathcal{M}_{\tilde{G}_v, \tilde{T}}(BG_v),
$$

and

$$
[\mathcal{F}_\Gamma] = c_\Gamma[\mathcal{M}_\Gamma] \in A_*(\mathcal{M}_\Gamma)
$$

where

$$
(6.3) \quad c_\Gamma = \frac{1}{|\text{Aut}(\tilde{\Gamma})| \prod_{e \in E(\Gamma)} |\text{Aut}(f_e)|} \cdot \prod_{(\varepsilon, v) \in F^2(\Gamma)} \frac{|G_v|}{r_{(\varepsilon, v)}} \cdot \prod_{v \in V^2(\Gamma)} \frac{|G_v|}{r_v}.
$$

In the above equation:

- $\frac{|G_v|}{r_{(\varepsilon, v)}} = |G_v / \langle k_{(\varepsilon, v)} \rangle|$, where $G_v / \langle k_{(\varepsilon, v)} \rangle$ is the automorphism group of $k_{(\varepsilon, v)}$ in the rigidified inertia stack $\mathcal{T}BG_v$.
- If $v \in V^2(\Gamma)$ and $E_v = \{e_1, e_2\}$, we define $r_v = r(e_1, v) = r(e_2, v)$.

6.3. **Virtual tangent and normal bundles.** Given a decorated graph $\tilde{\Gamma} \in \tilde{G}_{\tilde{g}, \tilde{t}}(X, \beta)$ and a twisted stable map $f : (C, f_1, \ldots, f_n) \to X$ which represents a point in the fixed locus $\mathcal{F}_\Gamma$ associated to $\tilde{\Gamma}$, let

$$
\begin{align*}
B_1 &= \text{Hom}(\Omega_C(f_1 + \cdots + f_n), \mathcal{O}_C), \quad B_2 = H^0(C, f^* T X) \\
B_4 &= \text{Ext}^1(\Omega_C(f_1 + \cdots + f_n), \mathcal{O}_C), \quad B_5 = H^1(C, f^* T X)
\end{align*}
$$
Deformations of the domain.

6.3.2. Automorphisms of the domain.

\[ B^m_1 = \bigoplus_{v \in V^S(\Gamma)} \operatorname{Hom}(\Omega_{C_v}(\eta(e, v) + \eta(e, v'))), \mathcal{O}_{C_v}) \]

\[ = \bigoplus_{v \in V^1(\Gamma), (e, v') \in F(\Gamma)} H^0(C_{\sigma_v^e}), T\mathcal{C}_e(-\eta(e, v) - \eta(e, v')) \]

\[ B^m_1 = \bigoplus_{v \in V^1(\Gamma), (e, v') \in F(\Gamma)} T\eta(e, v)C_{\sigma_v^e} \]

We define

\[ w_{(e, v')} := e^T(T\eta(e, v)C_{\sigma_v^e}) = \frac{r(e_v, \sigma_v)\omega(e_v, \sigma_v)}{r(e_v)\alpha_v} \in H^2_T(\eta(e, v)) = M_{\Omega}. \]

6.3.2. Deformations of the domain. Given any \( v \in V^S(\Gamma) \), define a divisor \( x_v \) of \( C_v \) by

\[ x_v = \sum_{i \in S_v} y_i + \sum_{e \in E_v} \eta(e, v). \]

Then

\[ B^m_1 = \bigoplus_{v \in V^S(\Gamma)} \operatorname{Ext}^1(\Omega_{C_v}(x_v), \mathcal{O}_{\mathcal{C}}) = \bigoplus_{v \in V^S(\Gamma)} T\mathcal{M}_{\nu_{\mathcal{C}}(BG_v)} \]

\[ = \bigoplus_{v \in V^1(\Gamma), E_v = (e, v')} T\eta_{\nu_v}(C_{\sigma_v^e}), T\eta_{(e, v')}C_{\sigma_v^e} \]

\[ \otimes_{(e, v') \in F(\Gamma)} T\eta(e, v)C_{\sigma_v^e} \]

where

\[ e^T(T\eta_{(e, v)}C_{\sigma_v^e} \otimes T\eta_{(e, v')}C_{\sigma_v^e}) = w_{(e, v)} + w_{(e, v')}, \quad v \in V^2(\Gamma) \]

\[ e^T(T\eta(e, v)C_{\sigma_v^e} \otimes T\eta(e, v)C_{\sigma_v^e}) = w_{(e, v)} - \frac{\tilde{Q}(e, v)}{r(e, v)}, \quad v \in V^S(\Gamma) \]
6.3.3. Unifying stable and unstable vertices. From the discussion in Section 6.3.1 and Section 6.3.2

\begin{equation}
\frac{e^T(B^m)}{e^T(B^*_m)} = \prod_{v \in V^1(\Gamma), (v, e) \in F(\Gamma)} w_v \prod_{v \in V^2(\Gamma), E_v = \{v, e\}} \frac{1}{w_v} - \frac{1}{w_v}
= \prod_{v \in V^3(\Gamma)} \prod_{E_v} \left( e^{(\psi_v)} - \frac{1}{r_v} \right).
\end{equation}

(6.6)

Recall that

\[ \mathcal{M}_F = \prod_{v \in V^3(\Gamma)} \mathcal{M}_{\bar{G}, \bar{r}_e}(BG_v). \]

\[ c_F = \frac{1}{|Aut(\bar{\Gamma}) \prod_{e \in E(\bar{\Gamma})} |Aut(f_e)|} \prod_{(e, v) \in F(\bar{\Gamma})} \prod_{v \in V^2(\bar{\Gamma})} \frac{|G_v|}{r_v} \frac{|G_v|}{r_v}. \]

To unify the stable and unstable vertices, we use the following convention for the empty sets $\mathcal{M}_{0, (1)}(BG)$ and $\mathcal{M}_{0, (e, c^{-1})}(BG)$, where $1 \in G$ is the identity element, and $c \in G$. Let $G$ be a finite abelian group. Let $w_1, w_2$ be formal variables.

- $\mathcal{M}_{0, (1)}(BG)$ is a $-2$ dimensional space, and

\begin{equation}
\int_{\mathcal{M}_{0, (1)}(BG)} \frac{1}{w_1 - \psi_1} = \frac{w_1}{|G|}
\end{equation}

(6.7)

- $\mathcal{M}_{0, (e, c^{-1})}(BG)$ is a $-1$ dimensional space, and

\begin{equation}
\int_{\mathcal{M}_{0, (e, c^{-1})}(BG)} \frac{1}{(w_1 - \psi_1)(w_2 - \psi_2)} = \frac{1}{(w_1 + w_2) \cdot |G|}
\end{equation}

(6.8)

\begin{equation}
\int_{\mathcal{M}_{0, (e, c^{-1})}(BG)} \frac{1}{w_1 - \psi_1} = \frac{1}{|G|}
\end{equation}

(6.9)

From (6.7), (6.8), (6.9), we obtain the following identities for non-stable vertices:

(i) If $v \in V^1(\Gamma)$ and $(v, e) \in F(\Gamma)$, then $r_v = 1$, and

\[ |G_v| \int_{\mathcal{M}_{0, (1)}(BG_v)} \frac{1}{w_v - \psi_v} = w_v. \]

(ii) If $v \in V^2(\Gamma)$ and $E_v = \{e, e'\}$, let $c = \rho(e, v) = \rho(v, e')^{-1} \in G_v$, then

\[ \frac{|G_v|}{r_v} \cdot \frac{|G_v|}{r_v} \cdot \int_{\mathcal{M}_{0, (e, c^{-1})}(BG_v)} \frac{1}{(w_v - \psi_v)(w_{e', v} - \psi_{e', v})} = \frac{1}{w_v + w_{e', v}}. \]

(iii) If $v \in V^1(\Gamma)$ and $(v, e) \in F(\Gamma)$, then

\[ \frac{|G_v|}{r_v} \cdot \frac{1}{w_v - \psi_v} = 1. \]
We then redefine $M_{\Gamma}$ and $c_{\Gamma}$ as follows:

\[(6.10) \quad M_{\Gamma} = \prod_{v \in V(G)} M_{g_v} \cdot \prod_{(e,v) \in F(G)} G_v, \quad \mathcal{F}_{\Gamma} = c_{\Gamma} [M_{\Gamma}], \]

and

\[(6.11) \quad c_{\Gamma} = \frac{1}{|\text{Aut}(\mathcal{T})|} \prod_{e \in E(G)} |\text{Aut}(f_e)| \prod_{(e,v) \in F(G)} |G_v|. \]

With the above conventions (6.7)–(6.11), we may rewrite (6.6) as

\[(6.12) \quad e^{T(B^m)} = \prod_{v \in V(G)} \left( \frac{1}{w_v - \bar{\psi}_v} \right). \]

The following lemma shows that the conventions (6.7), (6.8), and (6.9) are consistent with the stable case $M_{0,(c_1,\ldots,c_n)}(BG)$, $n \geq 3$.

**Lemma 6.2.** Let $G$ be a finite group. Let $\vec{c} = (c_1,\ldots,c_n) \in \text{Conj}(G)^n$. Let $w_1,\ldots,w_n$ be formal variables. Then

\[(a) \quad \int_{M_{0,(c_1,\ldots,c_n)}(BG)} 1 = \frac{|V_{0,c}|}{|G|} \cdot \left( \frac{1}{w_1} \cdot \ldots \cdot \frac{1}{w_n} \right)^{n-3}. \]

\[(b) \quad \int_{M_{0,(c_1,\ldots,c_n)}(BG)} \left( \frac{1}{w_1 - \psi_1} \right) = \frac{|V_{0,c}|}{|G|} \cdot w_2^{-n}. \]

**Proof.** The unstable cases $n = 1$ and $n = 2$ follow from the definitions (6.7) and (6.8), respectively. The stable case ($n \geq 3$) follows from Theorem 5.2 and the well known identity below.

\[\int_{M_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \frac{(n-3)!}{a_1! \cdots a_n!}. \]

\[\square\]

6.3.4. **Deformation of the map.** We first introduce some notation. Given $\sigma \in \Sigma(r)$ and $k \in G_{\sigma}$, let $(T_{p,\sigma})^k$ denote the subspace which is invariant under the action of $k$ on $T_{p,\sigma}$. Then

\[(T_{p,\sigma})^k = (T_{p,\sigma})^{k-1}. \]

Consider the normalization sequence

\[(6.13) \quad 0 \to \mathcal{O}_C \to \bigoplus_{v \in V(G)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(G)} \mathcal{O}_{C_e} \to \bigoplus_{v \in V(G)} \mathcal{O}_{\theta_v} \oplus \bigoplus_{(e,v) \in F(G)} \mathcal{O}_{\theta(e,v)} \to 0. \]
We twist the above short exact sequence of sheaves by $f^*TX$. The resulting short exact sequence gives rise to a long exact sequence of cohomology groups

$$0 \rightarrow B_2 \rightarrow \bigoplus_{v \in V^S(\Gamma)} H^0(C_v) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e) \rightarrow \bigoplus_{v \in V^S(\Gamma)} (T_{f(\eta_v)}X)^{k(e,v)} \oplus \bigoplus_{(e,v) \in F(\Gamma)} (T_{f(\eta(e,v))}X)^{k(e,v)} \rightarrow B_5 \rightarrow \bigoplus_{v \in V^S(\Gamma)} H^1(C_v) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e) \rightarrow 0.$$

where

$$H^i(C_v) = H^i(C_{e,v}, f_{e,v}^*TX), \quad H^i(C_e) = H^i(C_{e,v}, f_{e,v}^*TX)$$

for $i = 0, 1$.

The map $B_1 \rightarrow B_2$ sends $H^0(C_{e,v}, T_{f(\eta(e,v))}(-\eta(e,v) - \eta(e',v)))$ isomorphically to $H^0(C_{e,v}, f_{e,v}^*TL_{e,v})^f$, the fixed part of $H^0(C_{e,v}, f_{e,v}^*TL_{e,v})$.

It remains to compute

$$h(e,v) := e^T (T_{\eta(e,v)}X)^{k(e,v)} = \prod_{(e,v) \in F(\Gamma), (e',v) \subset C_e} w(e,v).$$

The map $B_1 \rightarrow B_2$ sends $H^0(C_{e,v}, T_{f(\eta(e,v))}(-\eta(e,v) - \eta(e',v)))$ isomorphically to $H^0(C_{e,v}, f_{e,v}^*TL_{e,v})^f$, the fixed part of $H^0(C_{e,v}, f_{e,v}^*TL_{e,v})$.

We first introduce some notation.

- If $v \in V^S(\Gamma)$, then there is a cartesian diagram

$$\begin{align*}
\tilde{C}_v \xrightarrow{\tilde{f}_0} & \quad \text{pt} \\
\downarrow & \\
C_v \xrightarrow{f_0} & \quad B G_v.
\end{align*}$$

Let $\tilde{G}_v$ denote the subgroup of $G_v$ generated by the monodromies of the $G_v$-cover $\tilde{C}_v \rightarrow C_v$. Then the number of connected components of $\tilde{C}_v$ is $|G_v/\tilde{G}_v|$, and each connected component is a $\tilde{G}_v$-cover of $C_v$.

- Given $(e,v) \in F(\Sigma)$, let $\phi(e,v) \in G_v^*$ be the irreducible character which corresponds to the 1-dimensional $G_v$-representation $T_{\eta(e,v)}$.

- Given an irreducible character $\phi$ of $G_v$, let $C_{\phi}$ denote the 1-dimensional $G_v$-representation associated to $\phi$. Define

$$\Lambda_\phi^\psi(u) = \sum_{i=0}^{\text{rank} E_{\phi}} (-1)^i \lambda_i^\phi u \text{rank} E_{\phi} - i,$$

where $\lambda_i^\phi \in A^i(M_{G_v^*}^0(BG_v))$ are Hurwitz-Hodge classes associated to $\phi \in G_v^*$. Here rank $E_{\phi}$ is the rank of $E_{\phi} \rightarrow M_{G_v^*}^0(BG_v)$. The rank of a Hurwitz-Hodge bundle $E_{\rho} \rightarrow M_{G_v^*}^0(BG_v)$, where $G$ is any finite group and $\rho \in G^*$, is given in Section 5.6.
Lemma 6.3. Suppose that \( v \in V^G(\Gamma) \) and \( \tilde{f}(v) = \sigma \in \Sigma(r) \). Then

\[
(6.15) \quad h(v) = \prod_{(\epsilon,\sigma) \in E(\Gamma)} \frac{\Lambda_{\phi(\epsilon,\sigma)}(w(\epsilon,\sigma))}{w(\epsilon,\sigma)}
\]

Proof. We have

\[
H^i(C_v, f_v^*TX) = \left( H^i(\tilde{C}_v, O_{\tilde{C}_v}) \otimes T_\sigma X \right)^{G_v} \cong \bigoplus_{(\epsilon,\sigma) \in F(\Gamma)} \left( H^i(\tilde{C}_v, O_{\tilde{C}_v}) \otimes C_{\phi(\epsilon,\sigma)} \right)^{G_v}.
\]

The group homomorphism \( G_v \to G_v/\tilde{G}_v \) induces an inclusion \( (G_v/\tilde{G}_v)^* \to G_v^* \) of sets of irreducible characters, so \( (G_v/\tilde{G}_v)^* \) can be viewed as a subset of \( G_v^* \). \( H^0(\tilde{C}_v, O_{\tilde{C}_v}) \) is the regular representation of \( G_v/\tilde{G}_v \), so

\[
H^0(\tilde{C}_v, O_{\tilde{C}_v}) = \bigoplus_{\phi \in (G_v/\tilde{G}_v)^*} C_{\phi}.
\]

\( \phi(\epsilon,\sigma) \in (G_v/\tilde{G}_v)^* \) iff \( \tilde{G}_v \subset G_{\epsilon,\sigma} \), so

\[
e_T\left( \left( H^0(\tilde{C}_v, O_{\tilde{C}_v}) \otimes C_{\phi(\epsilon,\sigma)} \right)^{G_v} \right) = \begin{cases} w(\epsilon,\sigma), & \tilde{G}_v \subset G_{\epsilon,\sigma} \\ 1, & \tilde{G}_v \not\subset G_{\epsilon,\sigma}. \end{cases}
\]

Therefore,

\[
(6.16) \quad e_T(H^0(\tilde{C}_v, f_v^*TX)^m) = e_T(H^0(\tilde{C}_v, f_v^*TX)) = \prod_{(\epsilon,\sigma) \in F(\Gamma), \tilde{G}_v \subset G_{\epsilon,\sigma}} w(\epsilon,\sigma)
\]

\[
\left( H^1(\tilde{C}_v, O_{\tilde{C}_v}) \otimes C_{\phi(\epsilon,\sigma)} \right)^{G_v} = E_{\phi(\epsilon,\sigma)}^V,
\]

so

\[
(6.17) \quad e_T(H^1(\tilde{C}_v, f_v^*TX)^m) = e_T(H^1(\tilde{C}_v, f_v^*TX)) = \prod_{(\epsilon,\sigma) \in F(\Gamma)} \Lambda_{\phi(\epsilon,\sigma)}(w(\epsilon,\sigma)).
\]

Equation (6.16) follows from (6.16) and (6.17). \( \square \)

Lemma 6.4. Suppose that \( v \in E(\Gamma) \). Let \( d = d_v \in \mathbb{Z}_{>0} \), and let \( \epsilon = \tilde{f}(v) \in E(Y)_e \). Define \( \sigma, \sigma', \epsilon, \epsilon_v, a_i \) as in Section 5.3. Suppose that \( (e,\sigma), (e,\sigma') \in F(\Gamma), \tilde{f}(v) = \sigma, \tilde{f}(v') = \sigma' \). Then \( k_{(e,\sigma)} \in G_v \) acts on \( T_{p,e} \) by multiplication by \( e^{2\pi\sqrt{-1}r(e,\sigma)} \), and acts on \( T_{p,e} \) by \( e^{2\pi\sqrt{-1}r(e,\sigma)} \), where

\[
\{ \frac{d}{r(e,\sigma)'}, \epsilon_1, \ldots, \epsilon_{r-1} \in \{0, \frac{1}{r(e,\sigma)'}, \ldots, \frac{r(e,\sigma) - 1}{r(e,\sigma)} \} \}.
\]

Define

\[
u = r(e,\sigma)w(e,\sigma) = -r(e,\sigma')w(e,\sigma').
\]
Then

\[ h(\epsilon) = \left( \frac{d}{u} \right)^{r(\epsilon, \sigma)} \left( -\frac{d}{u} \right)^{r(\epsilon, \sigma')} \prod_{i=1}^{r-1} b_i \]

where

\[ b_i = \begin{cases} \prod_{j=0}^{[d_i - \bar{e}_i]} (w(\epsilon_i, \sigma) - (j + \bar{e}_i)u) \frac{d}{r(\epsilon, \sigma)}, & a_i \geq 0, \\ \prod_{j=1}^{[\bar{e}_i - d_i - 1]} (w(\epsilon_i, \sigma) + (j - \bar{e}_i)u) \frac{d}{r(\epsilon, \sigma')}, & a_i < 0. \end{cases} \]

Proof. Let

\[ w_i = w(\epsilon_i, \sigma), \quad i = 1, \ldots, r - 1. \]

We have

\[ N_{f_c} = L_1 \oplus \cdots \oplus L_{r-1}. \]

- The weights of \( T \)-actions on \( (L_i)_{p_c} \) and \( (L_i)_{p_c'} \) are \( w_i \) and \( w_i - a_iu \), respectively.
- The weights of \( T \)-action on \( T_{p_c}L_i \) and \( T_{p_c'}L_i \) are \( u \frac{d}{r(\epsilon, \sigma)} \) and \( -u \frac{d}{r(\epsilon, \sigma')} \), respectively.
- Let \( p_c = f_c^{-1}(p_c) \), \( p_c' = f_c^{-1}(p_c') \) be the two torus fixed points in \( C_c \). Then the weights of \( T \)-action on \( T_{p_c}C_c \) and \( T_{p_c'}C_c \) are \( \frac{u}{d(\epsilon, \sigma)} \) and \( -\frac{u}{d(\epsilon, \sigma')} \), respectively.

By \[42\] Example 8.5],

\[ \text{ch}_T \left( H^1(C_c, f_c^* L_i) - H^0(C_c, f_c^* L_i) \right) = \begin{cases} -\sum_{j=0}^{[d_i - \bar{e}_i]} e^{w_i - (j + \bar{e}_i)u} \frac{d}{r(\epsilon, \sigma)}, & a_i \geq 0, \\ \sum_{j=1}^{[\bar{e}_i - d_i - 1]} e^{w_i + (j - \bar{e}_i)u} \frac{d}{r(\epsilon, \sigma')}, & a_i < 0. \end{cases} \]

Note that \( w_i - (j + \bar{e}_i)u \) and \( w_i + (j - \bar{e}_i)u \) are nonzero for any \( j \in \mathbb{Z} \) since \( w_i \) and \( u \) are linearly independent for \( i = 1, \ldots, r - 1 \). So

\[ \frac{e^T \left( H^1(C_c, f_c^* L_i)^m \right)}{e^T \left( H^0(C_c, f_c^* L_i)^m \right)} = \frac{e^T \left( H^1(C_c, f_c^* L_i) \right)}{e^T \left( H^0(C_c, f_c^* L_i) \right)} = b_i \]

where \( b_i \) is defined by \[6.19\].

By \[42\] Example 8.5] again,

\[ \text{ch}_T \left( H^1(C_c, f_c^* T L_i) - H^0(C_c, f_c^* T L_i) \right) \]

\[ = \sum_{j \in \mathbb{Z}, \frac{j}{r(\epsilon, \sigma)} \leq \frac{d}{r(\epsilon, \sigma')} \leq \frac{j}{r(\epsilon, \sigma')}} e^{\frac{u}{r(\epsilon, \sigma')} - (j + \frac{d}{r(\epsilon, \sigma')})u} \]

\[ = 1 + \sum_{j=1}^{[\frac{d}{r(\epsilon, \sigma')}]} e^{\frac{u}{r(\epsilon, \sigma')}} + \sum_{j=1}^{[\frac{d}{r(\epsilon, \sigma')}]} e^{-\frac{u}{r(\epsilon, \sigma')}}. \]
So
\[
e^T(H^1(C, f^* T\mathcal{X}))^{m} \quad e^T(H^0(C, f^* T\mathcal{X}))^{m} = \prod_{j=1}^{\lfloor \frac{d}{r(e, x)} \rfloor} \frac{1}{j!} \prod_{j=1}^{\lceil \frac{d}{r(e, x)} \rceil} \frac{1}{j!}
\]
\[
= \frac{\left( \frac{d}{u} \right)_{\lceil \frac{d}{r(e, x)} \rceil}}{\left( \frac{d}{r(e, x)} \right)_{\lceil \frac{d}{r(e, x)} \rceil}} \frac{\left( -\frac{d}{u} \right)_{\lfloor \frac{d}{r(e, x)} \rfloor}}{\left( \frac{d}{r(e, x)} \right)_{\lfloor \frac{d}{r(e, x)} \rfloor}}
\]

Therefore,
\[
e^T(H^1(C, f^* T\mathcal{X}))^{m} \quad e^T(H^0(C, f^* T\mathcal{X}))^{m} = e^T(H^1(C, f^* T\mathcal{X}))^{m} \cdot \prod_{i=1}^{r-1} e^T(H^1(C, f^* T\mathcal{X}))^{m} e^T(H^0(C, f^* T\mathcal{X}))^{m}
\]
\[
= \frac{\left( \frac{d}{u} \right)_{\lceil \frac{d}{r(e, x)} \rceil}}{\left( \frac{d}{r(e, x)} \right)_{\lceil \frac{d}{r(e, x)} \rceil}} \frac{\left( -\frac{d}{u} \right)_{\lfloor \frac{d}{r(e, x)} \rfloor}}{\left( \frac{d}{r(e, x)} \right)_{\lfloor \frac{d}{r(e, x)} \rfloor}} \prod_{i=1}^{r-1} b_i
\]

From the above discussion, we conclude that
\[
e^T(B^u) e^T(B^u) = \prod_{v \in V(\Gamma), E_v = \{e^r\}} h(e, v) \cdot \prod_{(e, v) \in F(\Gamma)} h(e, v) \cdot \prod_{v \in V^2(\Gamma)} h(v) \cdot \prod_{e \in E(\Gamma)} h(e)
\]

where \( h(e, v), h(v), \) and \( h(e) \) are defined by (6.14), (6.15), (6.18), respectively. To unify the stable and unstable vertices, we define
\[
h(v) := \begin{cases} 
\frac{1}{h(e, v)}, & v \in V^1(\Gamma) \cup V^1(\Gamma), \ E_v = \{e\}, \\
\frac{1}{h(e, v)} = 1, & v \in V^2(\Gamma), \ E_v = \{e, e^r\}.
\end{cases}
\]

Then
\[
e^T(B^u) e^T(B^u) = \prod_{v \in V(\Gamma)} h(v) \cdot \prod_{(e, v) \in F(\Gamma)} h(e, v) \cdot \prod_{e \in E(\Gamma)} h(e).
\]

6.4. Contribution from each graph.

6.4.1. Virtual tangent bundle. We have \( B_1^f = B_2^f, B_3^f = 0 \). So
\[
T^{1,f} = B_4^f = \bigoplus_{v \in V^3(\Gamma)} \mathcal{M}_{g_0, \mathcal{X}^f}^{-1}(BG_v), \quad T^{2,f} = 0.
\]

We conclude that
\[
[ \prod_{v \in V^3(\Gamma)} \mathcal{M}_{g_0, \mathcal{X}^f}(BG_v) ]^{\text{vir}} = \prod_{v \in V^3(\Gamma)} [ \mathcal{M}_{g_0, \mathcal{X}^f}(BG_v) ].
\]
6.4.2. Virtual normal bundle. Let $N^\text{vir}_\mathcal{F}$ be the virtual bundle on $\mathcal{M}_\mathcal{F}$ which corresponds to the virtual normal bundle of $\mathcal{F}_\mathcal{M}$ in $\overline{\mathcal{M}}_{g,i}(\mathcal{X}, \beta)$. Then

$$
\frac{1}{e_T(N^\text{vir}_\mathcal{F})} = e^T(B^n) e^T(B^n)

= \prod_{v \in V(\Gamma)} \frac{h(v)}{\prod_{e \in E_v} (w(e,v) - \phi(e,v)/r(e,v))} \prod_{(e,v) \in E(\Gamma)} h(e,v) \cdot \prod_{e \in E(\Gamma)} h(e)
$$

6.4.3. Integrand. Given $\sigma \in \Sigma(r)$, let

$$
i_r \colon A^\mathcal{F}_r(\mathcal{X}) \to A^\mathcal{F}_r(\mathcal{P}_r) = \mathbb{Q}[u_1, \ldots, u_r]
$$

be induced by the inclusion $i_r : \mathcal{P}_r \to \mathcal{X}$. Given $\mathcal{F} \in \mathcal{C}_{g,i}(\mathcal{X}, \beta)$, let

$$i_r \colon A^\mathcal{F}(\overline{\mathcal{M}}_{g,i}(\mathcal{X}, \beta)) \to A^\mathcal{F}(\mathcal{F}_\mathcal{M}) \cong A^\mathcal{F}(\mathcal{M}_\mathcal{F})
$$

be induced by the inclusion $i_r : \mathcal{F}_\mathcal{M} \to \overline{\mathcal{M}}_{g,i}(\mathcal{X}, \beta)$. Then

$$
i_r^n \prod_{i=1}^n \left( ev_i^* \gamma^\mathcal{F}_r \cup (\psi^\mathcal{F}_r)^{a_i} \right)

= \prod_{v \in V^1(\Gamma)} i_{e_v} \gamma^\mathcal{F}_r (-w(e,v))^{a_i} \cdot \prod_{v \in V^S(\Gamma)} \left( \prod_{i \in S_v} i_{e_v} \gamma^\mathcal{F}_r \prod_{e \in E_v} \psi^\mathcal{F}_r(e,v) \right)
$$

To unify the stable vertices in $V^S(\Gamma)$ and the unstable vertices in $V^{S,1}(\Gamma)$, we use the following convention: for $a \in \mathbb{Z}_{\geq 0}$,

$$
\int_{X_{0,(c_r, \ldots, c_n)}} \frac{\psi^a_2}{w_1 - \psi_1} = \frac{(-w_1)^a}{|G|}
$$

In particular, (6.9) is obtained by setting $a = 0$. With the convention (6.21), we may rewrite (6.20) as

$$
i_r^n \prod_{i=1}^n \left( ev_i^* \gamma^\mathcal{F}_r \cup (\psi^\mathcal{F}_r)^{a_i} \right) = \prod_{v \in V(\Gamma)} \left( \prod_{i \in S_v} i_{e_v} \gamma^\mathcal{F}_r \prod_{e \in E_v} \psi^\mathcal{F}_r(e,v) \right).
$$

The following lemma shows that the convention (6.21) is consistent with the stable case $\overline{\mathcal{M}}_{0,1,\ldots,c_n}(BG), n \geq 3$.

**Lemma 6.5.** Let $n, a$ be integers, $n \geq 2, a \geq 0$. Let $\bar{c} = (c_1, \ldots, c_n) \in \text{Conj}(G)^n$, where $c_1 \cdots c_n = 1$. Then

$$
\int_{\mathcal{M}_{0,\bar{c}}(BG)} \frac{\psi^a_2}{w_1 - \psi_1} = \begin{cases}
\prod_{i=1}^{n-1} (n-3-i) \frac{w_1^{a+2-n}|V^G_0|}{a!|G|}, & n = 2 \text{ or } 0 \leq a \leq n-3, \\
0, & \text{otherwise}.
\end{cases}
$$

**Proof.** The case $n = 2$ follows from (6.21). For $n \geq 3$,

$$
\int_{\mathcal{M}_{0,c}(BG)} \frac{\psi^a_2}{w_1 - \psi_1} = \frac{1}{w_1} \int_{\mathcal{M}_{0,c}(BG)} \frac{\psi^a_2}{1 - \frac{\psi_1}{w_1}} = w_1^{a+2-n} \int_{\mathcal{M}_{0,c}(BG)} \psi_1^{n-3-n} \psi^a_2

= w_1^{a+2-n} |V^G_0| \cdot \frac{1}{|G|} \cdot \frac{(n-3)!}{(n-3-a)! a!} \cdot \prod_{i=1}^{n-1} (n-3-i) \frac{w_1^{a+2-n}|V^G_0|}{a!|G|}.
$$

$\square$
6.4.4. Integral. Let

\[ i^* : A^+_T(\mathcal{M}_{g,i}^r(\mathcal{X}, \beta)) \to A^+_T(\mathcal{M}_{g,i}^r(\mathcal{X}, \beta)^T) \]

be induced by the inclusion \( i : \mathcal{M}_{g,i}^r(\mathcal{X}, \beta) \to \mathcal{M}_{g,i}^r(\mathcal{X}, \beta)^T \). The contribution of

\[
\int_{[\mathcal{M}_{g,i}^r(\mathcal{X}, \beta)^T]^\nu} i^* \prod_{i=1}^n (e v_i^* \gamma_i^T \cup (\psi_i^T)^{d_i}) e^T(N^{\nu_T})
\]

from the fixed locus \( F^T_{\Gamma} \) is given by

\[
c_p \prod_{e \in E(\Gamma)} h(e) \prod_{(v,v) \in F(\Gamma)} h(e,v) \prod_{v \in V(\Gamma)} \left( \prod_{i \in S_v} i_{e_i}^* \gamma_i^T \right) \cdot \prod_{v \in V(\Gamma)} \int_{\mathcal{M}_{g,0,v}^r(BG_v)} h(v) \prod_{i \in E_v} \tilde{\psi}_i^{d_i} \prod_{v \in E_v} (w_{(e,v)} - \tilde{\psi}_{(e,v)}/r_{(e,v)})
\]

where \( c_p \in Q \) is defined by (6.11).

6.5. Sum over graphs. Summing over the contribution from each graph \( \Gamma \) given in Section 6.4.4 above, we obtain the following formula.

**Theorem 6.6.**

\[
(\varepsilon_{a_1}(\gamma_1^T) \cdots \varepsilon_{a_n}(\gamma_n^T))_{\mathcal{X}_T}^\nu = \sum_{\Gamma \in G_{g,i}^r(\mathcal{X}, \beta)} c_{\Gamma} \prod_{e \in E(\Gamma)} h(e) \prod_{(v,v) \in F(\Gamma)} h(e,v) \prod_{v \in V(\Gamma)} \left( \prod_{i \in S_v} i_{e_i}^* \gamma_i^T \right) \cdot \prod_{v \in V(\Gamma)} \int_{\mathcal{M}_{g,0,v}^r(BG_v)} h(v) \prod_{i \in E_v} \tilde{\psi}_i^{d_i} \prod_{v \in E_v} (w_{(e,v)} - \tilde{\psi}_{(e,v)}/r_{(e,v)})
\]

(6.23)

where \( h(e), h(e,v), h(v) \) are given by (6.13), (6.14), (6.15), respectively, and we have the following formulae for the \( v \notin V^S(\Gamma) \):

\[
\int_{\mathcal{M}_{g,0}^r(BG)} \frac{1}{w_1 - \psi_2} = \frac{w_1}{|G|}, \quad \int_{\mathcal{M}_{g,0}^r(BG)} \frac{1}{(w_1 - \psi_1)(w_2 - \psi_2)} = \frac{1}{|G| \cdot (w_1 + w_2)},
\]

\[
\int_{\mathcal{M}_{g,0}^r(BG)} \frac{\psi_2^a}{w_1 - \psi_1} = \frac{(-w_1)^a}{|G|}, \quad a \in \mathbb{Z}_{\geq 0}.
\]

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