Double quiver gauge theory
and nearly Kähler flux compactifications

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Abstract

We consider $G$-equivariant dimensional reduction of Yang-Mills theory with torsion on manifolds of the form $M \times G/H$ where $M$ is a smooth manifold, and $G/H$ is a compact six-dimensional homogeneous space provided with a never integrable almost complex structure and a family of SU(3)-structures which includes a nearly Kähler structure. We establish an equivalence between $G$-equivariant pseudo-holomorphic vector bundles on $M \times G/H$ and new quiver bundles on $M$ associated to the double of a quiver $Q$, determined by the SU(3)-structure, with relations ensuring the absence of oriented cycles in $Q$. When $M = \mathbb{R}^2$, we describe an equivalence between $G$-invariant solutions of Spin(7)-instanton equations on $M \times G/H$ and solutions of new quiver vortex equations on $M$. It is shown that generic invariant Spin(7)-instanton configurations correspond to quivers $Q$ that contain non-trivial oriented cycles.
1 Introduction and summary

Realistic scenarios in string theory require compactification from ten-dimensional spacetime to four dimensions along a compact six-dimensional internal space $\mathbb{X}^6$. Initially, the cases where $\mathbb{X}^6$ is a coset space $G/H$ were considered, usually with Kähler structure. Subsequently, in the phenomenologically more interesting theories of heterotic strings, Calabi-Yau spaces $\mathbb{X}^6$ were utilized.

However, it has been realized in recent years that Calabi-Yau compactifications suffer from the presence of many massless moduli fields in the resulting four-dimensional field theories; Kähler cosets also lead to unrealistic effective field theories. This problem can be cured at least partially by allowing for $p$-form fluxes on $\mathbb{X}^6$. String vacua with $p$-form fields along the extra dimensions are called “flux compactifications” and have been intensively studied in recent years [1]. In particular, fluxes in heterotic string theory were analysed in [2, 3, 4, 5]. The allowed internal manifolds $\mathbb{X}^6$ in this case are not Kähler, but include quasi-Kähler and nearly Kähler manifolds [5], and more general almost hermitian manifolds with SU(3)-structure.

The analysis of Yang-Mills theory on product manifolds of the form $M \times \mathbb{X}^6$, where $\mathbb{X}^6$ is a quasi-Kähler six-manifold, is of great interest in this context. After constructing a vacuum solution in heterotic string theory, one should consider Kaluza-Klein dimensional reduction of heterotic supergravity over $\mathbb{X}^6$ by expansion around this background, and in particular describe the massless sector of induced Yang-Mills-Higgs fields on $M$. The $G$-equivariant dimensional reduction of Yang-Mills theory on Kähler homogeneous spaces $G/H$, for a compact Lie group $G$ with a closed subgroup $H$, was considered in [6, 7, 8]; the induced Yang-Mills-Higgs theory on $M$ in this case is a quiver gauge theory. This formalism was further developed and applied in a variety of contexts in [9, 10, 11, 12, 13].

In this paper we will extend the formalism of $G$-equivariant dimensional reduction from the Kähler case to the quasi-Kähler case, with the coset space $G/H$ endowed with a never integrable almost complex structure. We shall describe in detail the new quivers with relations associated to $G$-equivariant pseudo-holomorphic vector bundles on $G/H$, and establish a categorical equivalence between the corresponding quiver bundles on $M$ and invariant pseudo-holomorphic bundles on $M \times G/H$; this generalizes results of [7] from the Kähler case and the example of nearly Kähler dimensional reduction considered in [15]. We will spell out explicitly the properties of these new types of quivers, quiver gauge theories, and quiver vortex equations. These quiver gauge theories can have applications in flux compactifications of heterotic string theory, though these are left for future work.

For many explicit calculations we work on the six-dimensional complete flag manifold $G/H = SU(3)/U(1) \times U(1) =: \mathbb{F}_3$ with its associated SU(3)-equivariant pseudo-holomorphic homogeneous vector bundles. The requisite geometry and representation theory are best understood for this case, and detailed results can be obtained by following the formalism developed in [11]. Moreover, in [16] it is shown that, out of the four known compact nearly Kähler spaces in six dimensions, only $\mathbb{F}_3$ produces non-trivial heterotic string vacua. We shall demonstrate how the SU(3)-equivariant gauge theory derived in [11] changes when one considers dimensional reduction over $\mathbb{F}_3$ with a quasi-Kähler structure rather than a Kähler structure. For instance, quiver vortex equations associated to the nearly Kähler flag manifold $\mathbb{F}_3$ exhibit many qualitative differences compared to those associated with the Kähler geometry of $\mathbb{F}_3$ [11], as first pointed out in [15].

To the nearly Kähler manifold $\mathbb{F}_3$ (as well as to the other three compact nearly Kähler manifolds in six dimensions) one can associate quivers and quiver gauge theories which differ from those in the Kähler case. In particular, one can introduce two distinct almost complex structures with

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1 Another way in which quiver gauge theories arise as low-energy effective field theories in string theory is through considering orbifolds $\mathbb{X}^6$ and placing D-branes at the orbifold singularities [14].
corresponding fundamental two-forms $\omega$. The Yang-Mills lagrangian after equivariant dimensional reduction does not depend on the choice of (integrable or non-integrable) almost complex structure; in a real frame, the lagrangian and gauge field solutions of the equivariance conditions retain the same form. We obtain in this way two distinct quiver gauge theories by introducing an almost complex structure (and pulling it back to the gauge bundles), and writing all fields in a complex frame. Then the lagrangian can be rewritten as a “Bogomol’ny square” in two distinct ways using either a Kähler or a nearly Kähler two-form $\omega$, leading to two different sets of first-order hermitian Yang-Mills equations for the vacuum states of the quiver gauge theory. Moreover, one can consider equivariant dimensional reduction of Yang-Mills theory on homogeneous eight-manifolds $X^8 = \mathbb{R}^2 \times G/H$, with and without SU(4)-structure, to a quiver gauge theory on $\mathbb{R}^2$; the corresponding reductions of the BPS Yang-Mills equations have sharply different forms in the two cases. Solutions of these equations in the former case can be used to construct explicit vacua for heterotic supergravity and Yang-Mills theory in flux compactifications on nearly Kähler (and other almost hermitian) manifolds.

The organisation of the remainder of this paper is as follows. In section 2 we collect various geometrical facts concerning coset spaces and SU(3)-structure manifolds in six dimensions. In section 3 we work out these geometrical structures explicitly in the case of the six-manifold $\mathbb{P}_3$. In section 4 we recall the construction of quivers with relations and quiver gauge theories associated with holomorphic homogeneous vector bundles over complex flag varieties $G/H$, and discuss how they are modified in the pseudo-holomorphic case corresponding to a never integrable almost complex structure on $G/H$. In section 5 we describe in detail the new quivers and relations from a purely algebraic perspective. In section 6 we describe the corresponding quiver bundles and connections. In section 7 we construct natural corresponding quiver gauge theories and analyse their vacuum field equations. In section 8 we consider BPS-type gauge field equations on eight-manifolds $\mathbb{R}^2 \times G/H$ with SU(4)-structure, and their equivariant dimensional reduction to quiver vortex equations in two dimensions, comparing with the analogous reductions of the usual Kähler cases.

2 Homogeneous spaces and SU($n$)-structure manifolds

Coset space geometry. In this paper we will study dimensional reduction of gauge theories over certain coset spaces which can be used in flux compactifications of string theory. Let $G/H$ be a reductive homogeneous manifold of dimension $h = \dim G - \dim H$, where $H$ is a closed subgroup of a compact semisimple Lie group $G$ which contains a maximal abelian subgroup of $G$. Then the Lie algebra $\mathfrak{g}$ of $G$ admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \ ,$$

where $\mathfrak{h}$ is the Lie algebra of $H$, $\mathfrak{m}$ is an $H$-invariant subspace of $\mathfrak{g}$, i.e. $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and we choose $\mathfrak{m}$ so that it is orthogonal to $\mathfrak{h}$ with respect to the Cartan-Killing form. With a slight abuse of notation, we identify $\mathfrak{m}$ throughout with its dual $\mathfrak{m}^*$, i.e. the cotangent space $T^*_0(G/H)$ at the identity.

We choose a basis set $\{I_A\}$ for $\mathfrak{g}$ with $A = 1, \ldots, \dim G$ in such a way that $I_a$ for $a = 1, \ldots, h$ form a basis for the subspace $\mathfrak{m} \subset \mathfrak{g}$ and $I_i$ for $i = h + 1, \ldots, \dim G$ yield a basis for the holonomy subalgebra $\mathfrak{h}$. The structure constants $f^C_{AB}$ are defined by the Lie brackets

$$[I_A, I_B] = f^C_{AB} I_C \quad \text{with} \quad g_{AB} := f^D_{AC} f^C_{DB} = \delta_{AB} \ ,$$

where we have further chosen the basis so that it is orthonormal with respect to the Cartan-Killing form on $\mathfrak{g}$. Then $f_{ABC} := f^D_{AB} \delta^C_{DC}$ is totally antisymmetric in $A, B, C$. For a reductive homogeneous space, the relations (2.2) can be rewritten as

$$[I_i, I_j] = f^k_{ij} I_k \ , \quad [I_i, I_a] = f^b_{ia} I_b \quad \text{and} \quad [I_a, I_b] = f^c_{ab} I_c + f^d_{ab} I_i \ .$$
The invariant Cartan-Killing metric $g_{AB}$ is given by

$$g_{ab} = 2f_{ad}^i f_{db}^i + f_{cd}^i f_{eb}^i = \delta_{ab}, \quad (2.4)$$

$$g_{ij} = f_{id}^b f_{aj}^b + f_{jd}^b f_{ki}^b = \delta_{ij} \quad \text{and} \quad g_{ia} = 0. \quad (2.5)$$

The basis elements $I_a, I_i$ induce invariant one-forms $e^a, e^i$ on $G/H$, where $\{e^a\}$ form an invariant local orthonormal frame for the cotangent bundle $T^\ast(G/H) \cong G \times_H \mathfrak{m}$, while $\{e^i\}$ define the canonical connection $\omega^0 = e^i I_i$ which is the unique $G$-invariant connection on the principal $H$-bundle $G \to G/H$. They obey the Maurer-Cartan equations

$$de^a = -f_{ib}^a e^i \wedge e^b - \frac{1}{2} f_{bc}^a e^b \wedge e^c \quad \text{and} \quad de^i = -\frac{1}{2} f_{bc}^i e^b \wedge e^c - \frac{1}{2} f_{jk}^i e^j \wedge e^k, \quad (2.6)$$

and the $G$-invariant metric on $G/H$, lifted from the metric (2.4) on $\mathfrak{m}$, is given by

$$g = \delta_{ab} e^a \otimes e^b. \quad (2.7)$$

The first equation in (2.6) can be interpreted as a Cartan structure equation

$$de^a + \Gamma_b^a \wedge e^b = T^a = \frac{1}{2} T_{bc}^a e^b \wedge e^c, \quad (2.8)$$

where the metric connection $\Gamma_b^a = e^i f_{ib}^a$ is the canonical connection $\omega^0$ acting on one-forms via the adjoint action of $\mathfrak{h}$ on $\mathfrak{m}$, and $T_{bc}^a = -f_{bc}^a$ is its torsion. This connection has structure group $H$. In the sequel we will also consider more general metric connections with torsion tensor components $\varkappa f_{bc}^a$ for $\varkappa \in \mathbb{R}$.

**Quasi-Kähler manifolds.** An $H$-structure on a smooth orientable manifold $\mathbb{X}^m$ of dimension $m$ is a reduction of the structure group $GL(m, \mathbb{R})$ of the tangent bundle $TX^m$ to a closed subgroup $H$. Choosing an orientation and riemannian metric $g$ defines an $SO(m)$-structure on $\mathbb{X}^m$. We assume that $m = 2n$ is even and that $(\mathbb{X}^{2n}, g)$ is an almost hermitian manifold. Then there exists an almost complex structure $J \in \text{End}(T\mathbb{X}^{2n})$, with $J^2 = -I_{T\mathbb{X}^{2n}}$, which is compatible with the metric $g$, i.e. $g(JW, JZ) = g(W, Z)$ for all $W, Z \in T\mathbb{X}^{2n}$. This defines a $U(n)$-structure on $\mathbb{X}^{2n}$, and one introduces the fundamental two-form $\omega$ with

$$\omega(W, Z) := g(JW, Z) \quad (2.9)$$

for $W, Z \in T\mathbb{X}^{2n}$, i.e. $\omega$ is an almost Kähler form of type $(1,1)$ with respect to $J$.

The three-form $d\omega$ generally has $(3,0)+(0,3)$ and $(2,1)+(1,2)$ components with respect to $J$. The almost hermitian manifold $\mathbb{X}^{2n}$ is called quasi-Kähler or $(1,2)$-symplectic if only the $(3,0)+(0,3)$ components of $d\omega$ are non-vanishing [17, 18]. It is called nearly Kähler if in addition the Nijenhuis tensor $\mathfrak{N}$ of the canonical hermitian connection on $T\mathbb{X}^{2n}$ is totally antisymmetric; in this case $\mathfrak{N}$ is the real part of a $(3,0)$-form proportional to $d\omega$. Thus in the quasi-Kähler case there exists a global complex three-form $\Omega$ on $\mathbb{X}^{2n}$. In particular, for $n = 3$ the $(3,0)$-form $\Omega$ trivializes the canonical bundle of $\mathbb{X}^6$ and reduces the U(3) holonomy group to SU(3).

**SU(n)-structures.** An almost hermitian manifold $(\mathbb{X}^{2n}, g, J)$ with topologically trivial canonical bundle allows an $SU(n)$-structure. An $SU(n)$-structure on an oriented riemannian manifold $(\mathbb{X}^{2n}, g)$ is determined by a pair $(\omega, \Omega)$, where $\omega$ is a non-degenerate real two-form (an almost symplectic structure) and $\Omega$ is a decomposable complex $n$-form such that

$$\omega \wedge \Omega = 0 \quad \text{and} \quad \Omega \wedge \overline{\Omega} = \frac{(2i)^n}{n!} \omega^{\wedge n}. \quad (2.10)$$

The complex $n$-form

$$\Omega = \Theta^1 \wedge \cdots \wedge \Theta^n \quad (2.11)$$
determines an almost complex structure $\mathcal{J}$ on $\mathbb{X}^{2n}$ such that

$$\mathcal{J} \Theta^{\alpha} = i \Theta^{\alpha} \quad \text{for} \quad \alpha = 1, \ldots, n,$$

i.e. the forms $\Theta^{\alpha}$ span the space of forms of type $(1,0)$ with respect to $\mathcal{J}$. The $(n,0)$-form $\Omega$ is a global section of the (trivial) canonical bundle of $\mathbb{X}^{2n}$, so that $c_{1}(\mathbb{X}^{2n}) = 0$.

The fundamental two-form $\omega$ is of type $(1,1)$ with respect to $\mathcal{J}$ by virtue of (2.10), and $g = \omega \circ \mathcal{J}$ is an almost hermitian metric. We choose

$$g = \sum_{\alpha=1}^{n} \Theta^{\alpha} \otimes \bar{\Theta}^{\alpha} \quad \text{and} \quad \omega = \frac{i}{2} \sum_{\alpha=1}^{n} \Theta^{\alpha} \wedge \bar{\Theta}^{\alpha},$$

where $\Theta^{\bar{\alpha}} := \bar{\Theta}^{\alpha}$. Their non-vanishing components are therefore given by

$$g_{\alpha \bar{\beta}} = \frac{1}{2} \delta_{\alpha \bar{\beta}}, \quad g^{\alpha \bar{\beta}} = 2 \delta^{\alpha \bar{\beta}} \quad \text{and} \quad \omega_{\alpha \bar{\beta}} = \frac{1}{2} \delta_{\alpha \bar{\beta}}, \quad \omega^{\alpha \bar{\beta}} = -2i \delta^{\alpha \bar{\beta}}$$

with respect to this basis. For $n > 3$, these manifolds include Calabi-Yau torsion spaces for which the Nijenhuis tensor is the real part of a $(2,1)$-form.

**Nearly Kähler structures on six-manifolds.** A six-manifold $(\mathbb{X}^{6}, g)$ with SU(3)-structure $(\omega, \Omega)$ is nearly Kähler if the two-form $\omega$ and three-form $\Omega$ satisfy

$$d\omega = \frac{3}{2} W_{1} \text{Im} \Omega \quad \text{and} \quad d\Omega = W_{1} \omega \wedge \omega,$$

with a constant $W_{1} \in \mathbb{R}$ [19, 17, 20]. These backgrounds solve the Einstein equations with positive cosmological constant, and they yield a metric-compatible connection with totally antisymmetric (intrinsic) torsion and SU(3)-holonomy which admits covariantly constant spinor fields without coupling to other gauge fields. There are only four known examples of compact nearly Kähler six-manifolds, and they are all coset spaces

$$\text{SU}(3)/\text{U}(1) \times \text{U}(1) = \mathbb{F}_{3}, \quad \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) = \mathbb{C}P^{3},$$

$$G_{2}/\text{SU}(3) = S^{6} \quad \text{and} \quad \text{SU}(2)^{3}/\text{SU}(2) = S^{3} \times S^{3}.$$  \hspace{1cm} (2.16)

Here $\text{Sp}(1) \times \text{U}(1)$ is chosen to be a non-maximal subgroup of $\text{Sp}(2)$ [17], while in the final quotient $\text{SU}(2)$ is the diagonal subgroup of $\text{SU}(2)^{3}$. The cosets $G/H$ in (2.16) are all reductive, and in each case the subgroup $H$ of $G$ can be embedded in $\text{SU}(3)$.

The SU(3)-structure on a coset space in (2.16) is determined entirely from the Lie algebra $\mathfrak{g}$. The subgroup $H$ is the fixed point set of an automorphism of $G$ of order three, which induces a 3-symmetry $s_{s} : \mathfrak{g} \to \mathfrak{g}$ such that $s_{s}^{3} = 1_{\mathfrak{m}}$ on the tangent space $\mathfrak{m}$ and $s_{s} = 1_{\mathfrak{h}}$ on the Lie subalgebra $\mathfrak{h}$ [17] (see also [21]). The eigenspace decomposition of $s_{s}$ gives a complexification of the splitting (2.1),

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus (\mathfrak{m}^{+} \oplus \mathfrak{m}^{-})$$

which satisfies

$$[\mathfrak{m}^{\pm}, \mathfrak{m}^{\mp}] \subset \mathfrak{m}^{\mp}, \quad [\mathfrak{h}, \mathfrak{m}^{\pm}] \subset \mathfrak{m}^{\pm} \quad \text{and} \quad [\mathfrak{m}^{+}, \mathfrak{m}^{-}] \subset \mathfrak{h}^{\mathbb{C}}.$$  \hspace{1cm} (2.17)

We define an $H$-invariant almost complex structure $\mathcal{J}$ on $G/H$ by identifying $\mathfrak{m}^{+} = \Lambda^{1,0} T_{0}^{3}(G/H)$ and $\mathfrak{m}^{-} = \Lambda^{0,1} T_{0}^{3}(G/H)$. Note that $\mathcal{J}$ is a never integrable almost complex structure due to the first property of (2.18).

We choose a basis of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$, given by

$$-i I_{i}, \quad I_{\alpha} = \frac{1}{2} (I_{2\alpha - 1} - i I_{2\alpha}) \quad \text{and} \quad I_{\alpha}^{+} = \frac{1}{2} (I_{2\alpha - 1} + i I_{2\alpha})$$

\hspace{1cm} (2.19)
with \( i = 7, \ldots, \dim G \) and \( \alpha = 1, 2, 3 \), such that \( m^\pm = \text{span}_C \{ f^\pm_{\alpha\bar{\alpha}} \} \). Choosing an invariant local orthonormal basis \( \{ e^a \} \) of the cotangent bundle \( T^*(G/H) \) with \( a = 1, \ldots, 6 \) as above, the forms
\[
\Theta^\alpha = e^{2\alpha-1} + i e^{2\alpha}
\] (2.20)
are of type \((1,0)\) with respect to \( \mathcal{J} \). The metric \( g \) in (2.13) coincides in this case with the natural \( G \)-invariant metric (2.7) on the coset, while the fundamental two-form \( \omega \) in (2.13) reads
\[
\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 .
\] (2.21)
Furthermore, the three-form \( \Omega = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \) in (2.11) reads
\[
\Omega = \text{Re} \, \Omega + i \text{Im} \, \Omega = (e^{135} + e^{425} + e^{416} + e^{326}) + i (e^{136} + e^{426} + e^{145} + e^{235}) ,
\] (2.22)
where \( e^{a_1 \cdots a_k} := e^{a_1} \wedge \cdots \wedge e^{a_k} \). Then the pair \((\omega, \Omega)\) defines an invariant SU(3)-structure on the coset spaces \( G/H \) given in (2.16).

In what follows we will lower all algebra indices \( a, b, \ldots \) and \( i, j, \ldots \) with the help of the Cartan-Killing metric from (2.4)–(2.5). For all four nearly Kähler coset spaces in (2.16) one can choose the non-vanishing structure constants \( \{ f_{abc} \} \) such that
\[
f_{135} = f_{425} = f_{416} = f_{326} = -\frac{1}{2\sqrt{3}} .
\] (2.23)
This choice for the generators \( \{ I_a \} \) is correlated with (2.22) and the Maurer-Cartan equations. In particular,
\[
\text{Re} \, \Omega = \frac{1}{\sqrt{3}} (\text{Re} \, \Omega)_{abc} e^{abc} \quad \text{with} \quad (\text{Re} \, \Omega)_{abc} = -2\sqrt{3} f_{abc} ,
\] (2.24)
which coincides with the statement that
\[
T_{abc} = -f_{abc}
\] (2.25)
is a totally antisymmetric torsion and \( W_1 = \frac{1}{\sqrt{3}} \) [21]. The remaining structure constants are constrained by the identities [21]
\[
f_{aci} f_{bcj} = f_{acd} f_{bcd} = \frac{1}{3} \delta_{ab} ,
\] (2.26)
\[
\omega_{cd} f_{adi} = \omega_{ad} f_{cdi} \quad \text{and} \quad \omega_{ab} f_{abi} = 0 ,
\] (2.27)
where the first equality in (2.27) expresses invariance of the Lie bracket of \( \mathfrak{g} \) under the 3-symmetry \( s_* \), while the last identity expresses the embedding \( H \subseteq \text{SU(3)} \).

Using the \((1,0)\)-forms \( \Theta^\alpha \), we rewrite the Maurer-Cartan equations (2.6) as
\[
d\Theta^\alpha = -i C^{\alpha}_{ij\bar{\beta}} e^i \wedge \Theta^\beta - i C^{\alpha}_{i\bar{j}\bar{\beta}} e^\bar{i} \wedge \Theta^\beta - \frac{1}{2} C^{\alpha}_{i\bar{j} \bar{k}} \Theta^\beta \wedge \Theta^\gamma - C^{\alpha}_{i\bar{j} \bar{k}} \Theta^\beta \wedge \Theta^\gamma - \frac{1}{2} C^{\alpha}_{i\bar{j} \bar{k}} \Theta^\beta \wedge \Theta^\gamma , \]
\[
i d e^i = -i C^{\bar{i}}_{i\bar{j} \bar{\gamma}} \Theta^\beta \wedge \Theta^\gamma - C^{\bar{i}}_{i\bar{j} \bar{\gamma}} \Theta^\beta \wedge \Theta^\gamma - \frac{1}{2} C^{\bar{i}}_{i\bar{j} \bar{\gamma}} \Theta^\beta \wedge \Theta^\gamma - \frac{1}{2} C^{\bar{i}}_{i\bar{j} \bar{\gamma}} \Theta^\beta \wedge \Theta^\gamma ,
\] (2.28)
where we denote by \( C^{\alpha}_{i\bar{j} \bar{\gamma}} \), \( C^{i}_{i\bar{j} \bar{\gamma}} \), etc. the structure constants of \( \mathfrak{g}^C \) in the basis (2.19). From (2.23) we obtain
\[
C^{1}_{23} = C^{2}_{31} = C^{3}_{12} = C^{1}_{23} = C^{2}_{31} = C^{3}_{12} = -\frac{1}{2\sqrt{3}} .
\] (2.29)
Recalling that \( g_{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta} \), from (2.29) it follows that
\[
C^{\alpha}_{\beta\gamma} = g_{\alpha\bar{\alpha}} C^{\bar{\alpha}}_{\beta\gamma} = -\frac{1}{4\sqrt{3}} \varepsilon_{\alpha\beta\gamma} \quad \text{and} \quad C^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = g_{\alpha\bar{\alpha}} C^{\alpha}_{\beta\gamma} = -\frac{1}{4\sqrt{3}} \varepsilon_{\alpha\beta\gamma} .
\] (2.30)
The first equation of (2.28) can again be interpreted as structure equations
\[ d\Theta^\alpha + \Gamma^\alpha_\beta \wedge \Theta^\beta = T^\alpha \quad \text{and} \quad d\Theta^\alpha + \Gamma^\alpha_\beta \wedge \Theta^\beta = T^\bar{\alpha}, \]
where \( \Gamma = (\Gamma^\alpha_\beta) \) is the (torsionful) connection with holonomy \( \mathfrak{h} \) and the last term defines the Nijenhuis tensor (torsion) with components \( T^\alpha_\beta \) and their complex conjugates,
\[ T^\alpha = \frac{1}{2} T^\alpha_\beta \Theta^\beta \wedge \Theta^\gamma \quad \text{and} \quad T^\bar{\alpha} = \frac{1}{2} T^\bar{\alpha}_\beta \Theta^\beta \wedge \Theta^\gamma. \]
Using (2.30) one has
\[ T_{\alpha\beta\gamma} = \frac{1}{4\sqrt{3}} \varepsilon_{\alpha\beta\gamma} \quad \text{and} \quad T^{\bar{\alpha}\beta\gamma} = \frac{1}{4\sqrt{3}} \varepsilon^{\bar{\alpha}\beta\gamma}. \]

**A three-parameter family of SU(3)-structures.** Nearly Kähler six-manifolds form a special subclass of six-manifolds with SU(3)-structure. The generic case is classified by intrinsic torsion [19] which can be characterized by five irreducible SU(3)-modules with torsion classes \( W_1, \ldots, W_5 \).

Here \( W_1 \) is a complex function, \( W_2 \) is a (1,1)-form with \( \omega \wedge W_2 = 0 \), \( W_3 \) is the real part of a (2,1)-form with \( \omega \wedge W_3 = 0 \), and \( W_4 \) and \( W_5 \) are real one-forms. The symbol \( \wedge \) denotes contraction which is defined in terms of the Hodge duality operator \(*\) by \( u \wedge v = * (u \wedge * v) \). The five torsion classes are determined by the decompositions
\[ d\omega = \frac{3i}{4} (W_1 \overline{\Omega} - \overline{W_1} \Omega) + W_3 + \omega \wedge W_4, \]
\[ d\Omega = W_1 \omega \wedge \omega + \omega \wedge W_2 + \Omega \wedge W_5. \]

From (2.15) one sees that for nearly Kähler manifolds only \( W_1 \neq 0 \). We obtain from them more general almost hermitian SU(3)-manifolds with also \( W_2 \neq 0 \) by rescaling the one-forms \( \Theta^\alpha \) by constants \( \varsigma_\alpha \in \mathbb{R} \) as
\[ \Theta^\alpha \mapsto \tilde{\Theta}^\alpha = \frac{1}{2\sqrt{3}} \varsigma_\alpha^{-1} \Theta^\alpha \]
for \( \alpha = 1, 2, 3 \). Such manifolds are quasi-Kähler [17, 18].

After the rescaling (2.35), the structure constants (2.29) are rescaled as
\[ \tilde{C}^\alpha_\beta_\gamma = 2\sqrt{3} \frac{\varsigma_\alpha}{\varsigma_\beta} C^\alpha_\beta_\gamma = -\frac{\varsigma_\alpha}{\varsigma_\beta} \varepsilon^\alpha_\beta_\gamma, \quad \tilde{C}^i_j_k = C^i_j_k, \]
\[ \tilde{C}^\alpha_i_j = \frac{\varsigma_\alpha}{\varsigma_\alpha} C^\alpha_i_j, \quad \tilde{C}^\alpha_i_j_\beta = \frac{\varsigma_\beta}{\varsigma_\rho} C^\alpha_i_j_\beta, \quad \tilde{C}^\beta_i_\gamma_j = 12\varsigma_\beta \varsigma_\gamma C^\beta_i_\gamma_j, \quad \tilde{C}^i_\beta_\gamma = 12\varsigma_\beta \varsigma_\gamma C^i_\beta_\gamma, \]
plus their complex conjugates. The metric and the fundamental two-form become
\[ \tilde{g} = \tilde{\Theta}^1 \otimes \tilde{\Theta}^1 + \tilde{\Theta}^2 \otimes \tilde{\Theta}^2 + \tilde{\Theta}^3 \otimes \tilde{\Theta}^3 \quad \text{and} \quad \tilde{\omega} = \frac{i}{2} \left( \tilde{\Theta}^1 \wedge \tilde{\Theta}^1 + \tilde{\Theta}^2 \wedge \tilde{\Theta}^2 + \tilde{\Theta}^3 \wedge \tilde{\Theta}^3 \right). \]
The Maurer-Cartan equations for the \((1,0)\)-forms \( \tilde{\Theta}^\alpha \) are given by (2.28) but with the rescaled structure constants (2.36).

### 3 SU(3)-structures on \( \mathbb{F}_3 \)

**Homogeneous spaces of SU(3).** In this section we study in detail the first coset space in the list (2.16), whereby the geometry described in section 2 can be made very explicit. The projective plane \( \mathbb{C}P^2 \) and the complete flag manifold \( \mathbb{F}_3 \) on \( \mathbb{C}^3 \) are related through the fibrations
\[ \begin{array}{ccc}
\pi_3 & \longrightarrow & \mathbb{F}_3 \\
\pi_1 & \longrightarrow & \mathbb{C}P^2 \\
\pi_2 & \longrightarrow & \\
\end{array} \]

\(^2\)Here \( W_1 \) is complex, but one can always choose it to be real for nearly Kähler six-manifolds. Calabi-Yau manifolds correspond to the vanishing of all five intrinsic torsion classes.
with fibres $U(1) \times SU(2)$, $SU(2)/U(1)$ and $U(1) \times U(1)$ for the bundle projections $\pi_1, \pi_2$ and $\pi_3$, respectively. A representative element of the coset space $\mathbb{C}P^2 = SU(3)/U(1) \times SU(2)$ is a local section of the principal fibre bundle $\pi_1$ given by the $3 \times 3$ matrix
\[ V = \gamma^{-1} \begin{pmatrix} 1 & -T^\dagger \\ T & W \end{pmatrix} \in SU(3), \]  
where
\[ T := \begin{pmatrix} \tilde{y}^2 \\ y_1 \end{pmatrix}, \quad W := \gamma \begin{pmatrix} 1 & \frac{1}{\gamma + 1} T T^\dagger \end{pmatrix} \quad \text{and} \quad \gamma = \sqrt{1 + T T^\dagger} = \sqrt{1 + y^\alpha \tilde{y}^\alpha} \]  
(3.3)

and therefore $V^\dagger V = V V^\dagger = 1_3$. Here $y^1, y^2$ are local complex coordinates on $\mathbb{C}P^2$.

A representative element of the coset space $\mathbb{C}P^1 \cong SU(2)/U(1) \cong S^2$ is a local section of the Hopf fibration $S^3 \to S^2$ given by the matrix
\[ h = \frac{1}{\sqrt{1 + \zeta \xi}} \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} \in SU(2) \cong S^3, \]  
(3.5)

where $\zeta$ is a local complex coordinate on $\mathbb{C}P^1$. Then a representative element for the coset space $\mathbb{F}_3 = SU(3)/U(1) \times U(1)$ is a local section of the principal torus bundle $\pi_3$ given by the $3 \times 3$ matrix [22]
\[ \hat{V} = V \hat{h} = \gamma^{-1} \begin{pmatrix} 1 & -T^\dagger \\ T & W \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{h} \end{pmatrix} \in SU(3), \]  
(3.6)

with $V$ given in (3.2). The sphere bundle $\pi_2$ describes $\mathbb{F}_3$ as the twistor space of $\mathbb{C}P^2$.

**Monopole and instanton fields on $\mathbb{C}P^2$.** Let us introduce a flat connection on the trivial vector bundle $\mathbb{C}P^2 \times \mathbb{C}^3$ given by the $u(3)$-valued one-form
\[ A = V^{-1} dV =: \begin{pmatrix} 2b & -\frac{1}{\Lambda} \theta^1 \\ \frac{1}{\Lambda} \theta \end{pmatrix}, \]  
(3.7)

where the real parameter $\Lambda$ characterizes the “size” of the coset $\mathbb{C}P^2$ and from (3.2) one obtains
\[ b = \frac{1}{4 \gamma^2} (T^\dagger dT - dT^\dagger T) \quad \text{and} \quad B = \frac{1}{\gamma^2} (W dW + T dT^\dagger - \frac{1}{2} dT^\dagger T - \frac{1}{2} T^\dagger dT), \]  
(3.8)

and
\[ \theta = \frac{2\Lambda}{\gamma^2} W dT = \left( \frac{\theta^2}{\theta^1} \right) = \frac{2\Lambda}{\gamma} \left( \frac{dy^2}{dy^1} \right) - \frac{2\Lambda}{\gamma^2 (\gamma + 1)} \left( \tilde{y}^1 \right) (\tilde{y}^2 dy^1 + y^2 dy^2). \]  
(3.9)

Here $\theta^1$ and $\theta^2$ form a local $SU(3)$-equivariant orthonormal basis of $(1,0)$-forms on $\mathbb{C}P^2$. The flatness condition, $dA + A \wedge A = 0$, leads to the component equations
\[ f^- := db = \frac{1}{8\Lambda^2} \theta^1 \wedge \theta = -\frac{1}{8\Lambda^2} (\theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2) \]  
(3.10)

and
\[ F^+ = dB^+ + B^+ \wedge B^+ = \frac{1}{8\Lambda^2} \left( \theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2 \right) \left( \frac{2\theta^1 \wedge \theta^2}{-\theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2} \right), \]  
(3.11)

where
\[ B^+ = \begin{pmatrix} a_+ & -\bar{b}_+ \\ b_+ & -a_+ \end{pmatrix} = B + b 1_2 \]  
(3.12)
and
\[ F = dB + B \wedge B = \frac{1}{4\Lambda^2} \theta \wedge \theta^\dagger = -\frac{1}{4\Lambda^2} \begin{pmatrix} \theta^2 \wedge \theta^2 & \theta^1 \wedge \theta^2 \\ -\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^1 \end{pmatrix} =: F^+ - f^- 1_2. \] (3.13)

From (3.10) and (3.11) it follows that \( * f^- = -f^- \) and \( * F^+ = F^+ \), where * is the Hodge duality operator on \( CP^2 \) with respect to the Fubini-Study metric \( g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 \) for \( \theta^a \) given in (3.9), i.e. \( b \) is an anti-self-dual \( u(1) \)-connection (connection monopole potential) on a complex line bundle over \( CP^2 \) and \( B^+ \) is a self-dual \( su(2) \)-connection (instanton potential) on a complex vector bundle of rank two over \( CP^2 \). The gauge potential \( B \) is the (canonical) \( u(2) \)-valued Levi-Civita connection on the tangent bundle of the coset space \( CP^2 \).

**Generalized monopole and instanton fields on \( F_3 \).** Consider the trivial complex vector bundle \( F_3 \times C^3 \) endowed with a flat connection
\[ \hat{A} = V^{-1} dV = \hat{h} ^1 A \hat{h} + \hat{h} ^1 d\hat{h} =: \begin{pmatrix} \frac{2}{1+\zeta} \hat{\theta} & -\frac{1}{2\Lambda} \hat{\theta}^\dagger B \end{pmatrix}, \] (3.14)
where
\[ \hat{\theta} = h^1 \theta = \frac{1}{\sqrt{1+\zeta \zeta}} \begin{pmatrix} \theta^2 + \zeta \theta^1 \\ \theta^1 - \zeta \theta^2 \end{pmatrix} =: \begin{pmatrix} \hat{\theta}^2 \\ \hat{\theta}^1 \end{pmatrix}, \quad \hat{\theta}^\dagger = \theta^\dagger h = (\hat{\theta}^2 \hat{\theta}^1) \] (3.15)
and
\[ \hat{B} = h^1 B h + h^1 dh = \hat{B}^+ - b 1_2 =: \begin{pmatrix} \frac{1}{2\Lambda} \hat{\theta}^3 - \frac{1}{2\Lambda} \hat{\theta}^\dagger \hat{\theta}^\dagger \\ -\hat{a}_+ \end{pmatrix} - b 1_2, \] (3.16)
with
\[ \hat{a}_+ = \frac{1}{1+\zeta \zeta} \begin{pmatrix} (1-\zeta \zeta) a_+ + \bar{\zeta} b_+ - \zeta \bar{\zeta} - \frac{1}{2} (\zeta d\zeta - \bar{\zeta} d\bar{\zeta}) \end{pmatrix} \] (3.17)
and
\[ \hat{\theta}^3 = \frac{2R}{1+\zeta \zeta} (d\zeta + b + 2\zeta a_+ + \zeta^2 b_+) \] (3.18)
Here \( b, a_+ \) and \( b_+ \) are given in (3.12), while \( R \) is the radius of the fibre two-sphere \( S^2 \cong CP^1 \). The curvature of \( \hat{A} \) is given by
\[ \hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} = \begin{pmatrix} 2 db - \frac{1}{4 \Lambda} \hat{\theta}^\dagger \wedge \hat{\theta} - \frac{1}{2\Lambda} (d\hat{\theta}^\dagger + \hat{\theta}^\dagger \hat{B} - 2\hat{\theta}^\dagger \wedge b) \\ \frac{1}{2\Lambda} (d\hat{\theta} + \hat{B} \wedge \hat{B} - 2 b \wedge \hat{\theta}) \end{pmatrix}, \] (3.19)
and therefore from the flatness condition \( \hat{F} = 0 \) we obtain
\[ \hat{f^-} = f^- = db = -\frac{1}{8\Lambda^2} (\hat{\theta}^\dagger \wedge \hat{\theta}^\dagger - \hat{\theta}^2 \wedge \hat{\theta}^2) = -\frac{1}{8\Lambda^2} (\theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2) \] (3.20)
and
\[ \hat{F}^+ = dB^\dagger + B^\dagger \wedge B^+ = -\frac{1}{8\Lambda^2} \begin{pmatrix} \hat{\theta}^1 \wedge \hat{\theta}^\dagger \wedge \hat{\theta}^\dagger \wedge \hat{\theta}^2 \\ -2 \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^\dagger \wedge \hat{\theta} \end{pmatrix}, \] (3.21)
along with
\[ d\hat{\theta} + (\hat{B} - 2b 1_2) \wedge \hat{\theta} = 0. \] (3.22)
The gauge fields \( \hat{f^-} := \pi_2^* f^- \) and \( \hat{F}^+ := \pi_2^* F^+ \) are pull-backs of the monopole and instanton gauge fields \( f^- \) and \( F^+ \) on \( CP^2 \) to the flag manifold \( F_3 \) by the twistor fibration \( \pi_2 \) from (3.1).
In particular, the abelian gauge field $\hat{f}$ satisfies the hermitian Yang-Mills equations on $\mathbb{F}_3$ for both the Kähler and nearly Kähler geometries described below [22]. In the nearly Kähler case the abelian gauge field $\hat{d}a_\pm$ also satisfies the hermitian Yang-Mills equations. These monopole-type fields will be used later on in our constructions of quiver gauge theories.

**Kähler geometry of $\mathbb{F}_3$.** The metric and an almost Kähler structure on $\mathbb{F}_3$ read

$$\hat{g} = \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2 + \hat{\theta}^3 \otimes \hat{\theta}^3 \quad \text{and} \quad \hat{\omega} = \frac{i}{2} \left( \hat{\theta}^1 \wedge \hat{\theta}^1 + \hat{\theta}^2 \wedge \hat{\theta}^2 + \hat{\theta}^3 \wedge \hat{\theta}^3 \right), \quad (3.23)$$

where $\hat{\theta}^\alpha$ with $\alpha = 1, 2, 3$ are given in (3.15) and (3.18). The SU(3)-invariant one-forms $\hat{\theta}^\alpha$ define an integrable almost complex structure $\mathcal{J}_+$ on $\mathbb{F}_3$ such that

$$\mathcal{J}_+ \hat{\theta}^\alpha = i \hat{\theta}^\alpha, \quad (3.24)$$

i.e. $\hat{\theta}^\alpha$ are $(1,0)$-forms with respect to $\mathcal{J}_+$. From (3.19)–(3.22) we obtain the structure equations

$$d \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_+ - 3b & 0 & -\frac{1}{2\pi} \hat{\theta}^2 \\ 0 & -\hat{\alpha}_+ + 3b & -\frac{1}{2\pi} \hat{\theta}^1 \\ \frac{R}{4\Lambda^2} \hat{\theta}^2 & \frac{R}{4\Lambda^2} \hat{\theta}^1 & -2\hat{\alpha}_+ \end{pmatrix} = 0, \quad (3.25)$$

which define the Levi-Civita connection $\hat{\Gamma} = (\hat{\Gamma}^\alpha_\beta)$ on the tangent bundle of $\mathbb{F}_3$ by the formula

$$d\hat{\theta}^\alpha + \hat{\Gamma}^\alpha_\beta \wedge \hat{\theta}^\beta = 0. \quad (3.26)$$

From (3.25) it follows that $\hat{\omega}$ is Kähler, i.e. $d\hat{\omega} = 0$, if and only if

$$R^2 = 2\Lambda^2. \quad (3.27)$$

Then the connection matrix $\hat{\Gamma}$ in (3.25) takes values in the Lie algebra $\mathfrak{u}(3)$, i.e. the holonomy group is $\text{U}(3)$. The non-vanishing structure constants $\hat{C}^B_C$ of the Lie algebra $\mathfrak{su}(3)$ for the complex basis of one-forms $\hat{\theta}^\alpha$ adapted to the Kähler structure on $\mathbb{F}_3$ and the structure equations (3.25) are given by

$$\hat{C}^1_{23} = \hat{C}^2_{13} = -\frac{1}{2\sqrt{6}}, \quad \hat{C}^3_{12} = -\frac{1}{\sqrt{6}}, \quad (3.28)$$

$$\hat{C}^7_{11} = \hat{C}^7_{22} = \hat{C}^7_{33} = -\frac{1}{4\sqrt{3}}, \quad \hat{C}^8_{11} = \frac{1}{4} \quad \text{and} \quad \hat{C}^8_{22} = -\frac{1}{4}, \quad (3.29)$$

and their complex conjugates, plus

$$\hat{C}^1_{71} = \hat{C}^2_{72} = \frac{1}{2\sqrt{3}}, \quad \hat{C}^3_{73} = -\frac{1}{\sqrt{3}}, \quad \hat{C}^1_{81} = -\frac{1}{2}, \quad \hat{C}^2_{82} = \frac{1}{2}, \quad \hat{C}^8_{8a} = -\hat{C}^8_{8a}$$

for $\alpha = 1, 2, 3$. Here we have chosen $R^2 = 2\Lambda^2 = 6$.

**Nearly Kähler geometry of $\mathbb{F}_3$.** We have introduced above an integrable almost complex structure $\mathcal{J}_+$ and a Kähler structure on the flag manifold $\mathbb{F}_3$, defined via the $(1,0)$-forms $\hat{\theta}^\alpha$. Let us now introduce the forms

$$\Theta^1 := \hat{\theta}^1, \quad \Theta^2 := \hat{\theta}^2 \quad \text{and} \quad \Theta^3 := \hat{\theta}^3, \quad (3.30)$$

which are of type $(1,0)$ with respect to an almost complex structure $\mathcal{J} = \mathcal{J}_- [18]$, $\mathcal{J}_- \Theta^\alpha = i \Theta^\alpha$, defined in (2.12). The almost complex structure $\mathcal{J}_-$ is obtained from $\mathcal{J}_+$ by changing its sign along the $\mathbb{C}P^1$-fibres of the twistor bundle $\pi_2$, i.e. $\mathcal{J}_\pm \Theta^{1,2} = i \Theta^{1,2}$, $\mathcal{J}_\pm \Theta^3 = \mp i \Theta^3$. It is never integrable.
Using the redefinition (3.30), we obtain from (3.25) the structure equations
\[
\begin{align*}
d\begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \begin{pmatrix} -\tilde{a}_+ - 3b & 0 & 0 \\ 0 & -\tilde{a}_+ + 3b & 0 \\ 0 & 0 & 2\tilde{a}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} &= \frac{1}{2R} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ \Theta^1 \wedge \Theta^2 \end{pmatrix}.
\end{align*}
\] (3.31)

The first term here defines the (torsionful) connection \( \Gamma = (\Gamma^\alpha_\beta) \), with \( u(1) \oplus u(1) \) holonomy, whose components are obtained by comparing (3.31) with (2.31), and the last term defines the Nijenhuis tensor (torsion) with components \( T^\alpha_{\beta\gamma} \) and their complex conjugates. We also have
\[
\begin{align*}
db &= -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^1 - \Theta^2 \wedge \Theta^2) \\
d\tilde{a}_+ &= -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2) + \frac{1}{4R^2} \Theta^3 \wedge \Theta^3.
\end{align*}
\] (3.32) (3.33)

for the abelian gauge fields on \( F_3 \).

The pair of forms \( (\omega, \Omega) \) defined by (2.11) and (2.13) for \( n = 3 \) defines a one-parameter family of invariant SU(3)-structures on \( F_3 \), parametrized by the ratio \( \frac{R^2}{\Lambda^2} \). From (3.31) it follows that the conditions (2.15) for the coset space \( F_3 \) to be nearly Kähler yield
\[
R^2 = \Lambda^2.
\] (3.34)

We fix the scales of \( CP^1 \) and \( CP^2 \) in \( F_3 \) so that
\[
R = \Lambda = \sqrt{3}.
\] (3.35)

Then the connection \( \Gamma \) coincides with the canonical connection which was used throughout section 2. Notice that the Kähler and nearly Kähler structures correspond not only to different choices of almost complex structures \( J_+ \) and \( J_- \) on \( F_3 \), but also to metrics
\[
\tilde{g} = \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + 2\Theta^3 \otimes \Theta^3 \quad \text{and} \quad g = \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3
\] (3.36)

which differ by a factor of 2 along the fibre direction \( CP^1 \hookrightarrow F_3 \). Both \( \tilde{g} \) and \( g \) are Einstein metrics.

**Basis for SU(3)-generators.** The non-vanishing structure constants of \( su(3) \) which conform with the nearly Kähler structure (2.23)–(2.27) are given by
\[
f_{135} = f_{425} = f_{416} = f_{326} = -\frac{1}{2\sqrt{3}},
\]
\[
f_{127} = f_{347} = \frac{1}{2\sqrt{3}}, \quad f_{128} = -f_{348} = -\frac{1}{\sqrt{3}} \quad \text{and} \quad f_{567} = -\frac{1}{\sqrt{3}}.
\] (3.37)

Correspondingly, we choose the basis for \( 3 \times 3 \) matrices of the antifundamental representation of \( su(3) \) given by
\[
I_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad I_3 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
I_4 = \frac{1}{2\sqrt{3}} \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_5 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad I_6 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},
\]
Then the flat connection (3.14) can be written as
\[
\widehat{A} = e^i I_i + e^a I_a ,
\]}
(3.39)
where \(e^a\) from (2.20)–(2.22) form an orthonormal frame for the cotangent bundle \(T^\ast \mathbb{F}_3\), and \(e^i = \{e^7, e^8\}\) are two \(u(1)\)-valued gauge potentials on two line bundles of degree one over \(\mathbb{F}_3\) whose curvatures generate the cohomology group \(H^2(\mathbb{F}_3; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). Written in the form (3.39), the flatness condition \(\widehat{F} = d\widehat{A} + \widehat{A} \wedge \widehat{A} = 0\) is equivalent to the Maurer-Cartan equations (2.6). In the case at hand, the group \(H = U(1) \times U(1)\) is abelian and therefore \(f^i_{jk} = 0\).

Equivalently, we can write (3.39) as
\[
\widehat{A} = ie^i (-i I_i) + \Theta^a I_a^- + \Theta^a I_a^+ = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & i e^8 & -\Theta^2 & \sqrt{2} \Theta i e^8 \\ -\Theta^2 & -i e^7 & \sqrt{2} \Theta i e^8 & \Theta^3 \\ \sqrt{2} \Theta i e^8 & \Theta^3 & i e^7 & \sqrt{2} \Theta i e^8 \end{pmatrix},
\]}
(3.40)
where the matrices
\[
\begin{align*}
I_1^- & := \frac{1}{2} (I_1 - i I_2) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & I_1^+ & := \frac{1}{2} (I_1 + i I_2) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
I_2^- & := \frac{1}{2} (I_3 - i I_4) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & I_2^+ & := \frac{1}{2} (I_3 + i I_4) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
I_3^- & := \frac{1}{2} (I_5 - i I_6) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, & I_3^+ & := \frac{1}{2} (I_5 + i I_6) = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
-I_7 & := \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad -i I_8 = \frac{1}{6} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{align*}
\]}
(3.41)
form a basis for the complexified Lie algebra \(\mathfrak{sl}(3, \mathbb{C})\) in the antifundamental representation. Here complex conjugation acts by interchanging barred and unbarred indices. From (3.14) and (3.40) it follows that
\[
e^7 = 2\sqrt{3} i \hat{a}_+ \quad \text{and} \quad e^8 = -6 i b .
\]}
(3.42)
Explicit expressions for \(e^a\) in terms of \(SU(3)\)-invariant gauge potentials and for \(\Theta^a\) in terms of local coordinates on \(\mathbb{F}_3\) can also be easily extracted from (3.6)–(3.18).

Comparing (2.28) with (3.31)–(3.33), the non-vanishing structure constants of \(\mathfrak{sl}(3, \mathbb{C})\) in the basis (3.41) are given by
\[
\begin{align*}
C^1_{23} & = C^2_{31} = C^3_{12} = C^1_{23} = C^2_{31} = C^3_{12} = -\frac{1}{2\sqrt{3}}, \\
C^1_{71} & = C^2_{72} = -C^1_{71} = -C^2_{72} = \frac{1}{2\sqrt{3}}, \quad C^3_{73} = -C^3_{73} = -\frac{1}{\sqrt{3}}, \\
C^1_{81} & = -C^1_{81} = \frac{1}{2}, \quad C^2_{82} = -C^2_{82} = \frac{1}{2}, \\
C^7_{11} & = C^7_{22} = -\frac{1}{4\sqrt{3}}, \quad C^7_{33} = \frac{1}{2\sqrt{3}}, \quad C^8_{11} = \frac{1}{4} \quad \text{and} \quad C^8_{22} = -\frac{1}{4}.
\end{align*}
\]}
(3.43)
After the rescaling (2.35), the structure constants (3.43) are rescaled as

\[ \tilde{C}_{\beta\gamma} = 2\sqrt{3} \frac{s_{\lambda}}{s_{\alpha}} c_{\beta\gamma} = -\frac{s_{\lambda}}{s_{\alpha}} \tilde{c}_{\beta\gamma}, \]

\[ \tilde{C}_1^1 = C_1^1 = \frac{1}{2\sqrt{3}}, \quad \tilde{C}_2^2 = C_2^2 = \frac{1}{2\sqrt{3}}, \quad \tilde{C}_3^3 = C_3^3 = \frac{1}{\sqrt{3}}, \]

\[ \tilde{C}_1^2 = C_1^2 = \frac{1}{2}, \quad \tilde{C}_2^1 = C_2^1 = -\frac{1}{2}, \quad \tilde{C}_2^3 = C_2^3 = \frac{1}{2}, \quad \tilde{C}_3^2 = C_3^2 = \frac{1}{2}, \]

\[ \tilde{C}_1^3 = -\frac{2}{\sqrt{3}} C_1^3 = -\frac{2}{\sqrt{3}}, \quad \tilde{C}_2^1 = -\frac{2}{\sqrt{3}} C_2^1 = -\frac{2}{\sqrt{3}}, \quad \tilde{C}_2^3 = -\frac{2}{\sqrt{3}} C_2^3 = -\frac{2}{\sqrt{3}}, \quad \tilde{C}_3^2 = -\frac{2}{\sqrt{3}} C_3^2 = -\frac{2}{\sqrt{3}}, \]

plus their complex conjugates. In particular, by setting \( \varsigma_1 = \varsigma_2 = \varsigma_3 = 2\sqrt{3} R^{-1} \) we can restore the \( \mathbb{C}P^1 \) radius \( R \), and for \( \varsigma_1 = \varsigma_2 = 2\sqrt{3} \Lambda^{-1}, \varsigma_3 = 2\sqrt{3} R^{-1} \) we can restore both of our original size parameters \( \Lambda \) and \( R \).

4 Pseudo-holomorphic equivariant vector bundles

**Equivariant bundles.** In this paper we are interested in Yang-Mills theory with torsion and \( G \)-equivariant gauge fields on manifolds of the form

\[ \mathbb{X}^{d+h} = M \times G/H, \] \hspace{1cm} (4.1)

where \( M \) is a smooth manifold of real dimension \( d \) and \( G/H \) is a reductive homogeneous manifold of real dimension \( h = \dim G - \dim H \). The group \( G \) acts trivially on \( M \) and in the standard way by isometries of the coset space \( G/H \). By standard induction and reduction, there is an equivalence between smooth \( G \)-equivariant vector bundles \( \mathcal{E} \) over \( \mathbb{X}^{d+h} \) and smooth \( H \)-equivariant vector bundles \( E \) over \( M \), where \( H \) acts trivially on \( M \); a smooth \( H \)-equivariant bundle \( E \to M \) induces a smooth \( G \)-equivariant bundle \( \mathcal{E} \to \mathbb{X}^{d+h} \) by the fibred product

\[ \mathcal{E} = G \times_H E. \] \hspace{1cm} (4.2)

For each \( p \in M \), the restriction \( \mathcal{E}_p \) of \( \mathcal{E} \) to the coset \( G/H \cong \{ p \} \times G/H \to \mathbb{X}^{d+h} \) is a homogeneous vector bundle on \( G/H \) which is in correspondence, via (4.2), with the representation \( E_p \) of \( H \) on the fibres of the complex vector bundle \( E \to M \). Let \( T \) be a maximal torus of \( G \) such that \( T \subseteq H \), and let \( t \) be its Lie algebra. If \( V_{\lambda} \) is the irreducible representation of \( H \) with weight vector \( \lambda \in t^* \), then the fibred product

\[ V_{\lambda} := G \times_H V_{\lambda}, \] \hspace{1cm} (4.3)

is the induced irreducible smooth homogeneous vector bundle on \( G/H \) whose associated principal bundle has structure group \( H \). Then every smooth \( G \)-equivariant complex vector bundle \( \mathcal{E} \to \mathbb{X}^{d+h} \) can be equivariantly decomposed as \cite{7}

\[ \mathcal{E} = \bigoplus_{\lambda \in W} E_{\lambda} \otimes V_{\lambda}, \] \hspace{1cm} (4.4)

where \( E_{\lambda} \to M \) are smooth complex vector bundles with trivial \( H \)-action, and \( W \subseteq t^* \) is the finite set of eigenvalues for the action of \( H \) on \( E \).

**Holomorphic bundles.** On any coset space \( G/H \) which is a flag manifold, one can introduce an integrable almost complex structure and (almost) Kähler structure. Then the action of \( G \) on \( G/H \) is symplectic and \( G/H \) is diffeomorphic to the projective variety \( G^C/P \) with the canonical complex structure, where \( P \) is a parabolic subgroup of the complexification \( G^C \) of \( G \). We assume henceforth that \( M \) is a complex manifold (in particular, its dimension \( d \) is even), and denote by
$H^C$ the universal complexification of $H$. Then there is a one-to-one correspondence between $G^C$-invariant holomorphic structures on the $G$-equivariant vector bundle (4.2), and $H^C$-equivariant holomorphic structures on the $H$-equivariant vector bundle $E$ together with extensions of the $H^C$-action to a holomorphic $P$-action on $E$. At the level of (complex) Lie algebras, the extension of the reductive Levi subgroup $H^C$ to $P$ is described by a decomposition $\mathfrak{p} = \mathfrak{h}^C \oplus \mathfrak{u}$, where the nilpotent radical $\mathfrak{u} = \bigwedge^{0,1} T^*_0(G/H)$ for the canonical complex structure is an $H$-invariant subalgebra of $\mathfrak{g}^C$, i.e. $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}$ and $[\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{u}$.

By restriction and Corollary 1.13 of [7], in this case there is a functorial equivalence between the category of holomorphic homogeneous vector bundles on $G/H$ and the category of finite-dimensional representations of a certain bounded quiver $Q$ satisfying a set of relations $R$. The set of vertices $Q_0$ of $Q$ is the set of weights $W$ for the corresponding $H^C$-action and the set of arrows $Q_1$ is given by the actions of the generators of the nilpotent radical $\mathfrak{u}$ on the weight vectors, while the set of relations $R$ express commutativity of the quiver diagram through the Lie algebra relations of $\mathfrak{u}$ represented on weight vectors. The relations $R$ in this geometric framework appear as the condition for holomorphicity of the homogeneous bundles $\mathcal{V}$ over $G/H$, i.e. as the condition $\mathcal{F}^{0,2} = \mathcal{F}_A \wedge \mathcal{F}_A = 0$ for integrability of the Dolbeault operator $\mathcal{F}_A$ on $\mathcal{V}$ [7] with respect to a $G$-invariant connection $\mathcal{A}$ on $\mathcal{V}$ and the canonical complex structure. In this case, the relations $R$ are in one-to-one correspondence with integrability of the canonical $(0,1)$-distribution on $G/H$ and with a proper subalgebra $\mathfrak{u}$ of $\mathfrak{g}^C$.

In the example $G/H = SU(3)/U(1) \times U(1) = \mathbb{F}_3$, an integrable almost complex structure and (almost) Kähler structure is described by (3.23)-(3.27). In this case $P$ is a Borel subgroup of $SL(3, \mathbb{C})$ and the Levi decomposition is the usual root space decomposition. In the basis (3.41), the generators of $\mathfrak{u}$ are $I_1^+, I_2^-$ and $I_3^-$ which (after rescaling) close to the three-dimensional Heisenberg algebra with central element $I_1^+$. Thus the arrows $Q_1$ translate weight vectors by the set of positive roots of $\mathfrak{sl}(3, \mathbb{C})$, while the relations $R$ are induced by the Heisenberg commutation relations which express commutativity of the corresponding quiver diagrams [7, 11].

**Pseudo-holomorphic bundles.** In the following we will describe how these quivers and their relations are modified when on $G/H$ we consider instead a family of $SU(3)$-structures, as described in section 2. In this case the canonical complex structure on $G/H$ is replaced with a never integrable almost complex structure, and the Kähler structure is replaced by a quasi-Kähler structure. Instead of homogeneous holomorphic bundles $\mathcal{V}$ over $G/H$, we now consider pseudo-holomorphic bundles. Pseudo-holomorphicity of the homogeneous bundle is again defined by a connection $\mathcal{A}$ on $\mathcal{V}$ whose curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is of type $(1,1)$ with respect to the chosen almost complex structure, i.e. $\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}$ [20]. In sections 7 and 8 we will demonstrate that these equations lead to the appropriate set of relations on the pertinent quiver. We will establish a functorial equivalence between the category of pseudo-holomorphic homogeneous vector bundles on $G/H$ and a category of certain finite-dimensional “unitary” representations of a new quiver $\overline{Q}$ with new relations $\overline{R}$. Using (4.4), the extension from homogeneous pseudo-holomorphic bundles $\mathcal{V} \to G/H$ to $G$-equivariant pseudo-holomorphic bundles $\mathcal{E} \to \mathbb{X}^{d+1}$ is a straightforward technical task. In that case we obtain a functorial equivalence between the category of $G$-equivariant pseudo-holomorphic bundles over $\mathbb{X}^{d+1} = M \times G/H$ and a category of “hermitian” $(\overline{Q}, \overline{R})$-bundles on $M$. We shall show in later sections that this correspondence follows from using equivariant dimensional reduction of Yang-Mills theory on $\mathbb{X}^{d+1}$ to derive a quiver gauge theory on $M$ with relations arising from hermitian Yang-Mills equations on $\mathbb{X}^{d+1}$.

In section 5 we describe in detail the basic properties of the new quivers $\overline{Q}$ and relations $\overline{R}$ from a purely algebraic perspective. Since the $G$-equivariance conditions do not depend on an (either integrable or non-integrable) almost complex structure on $G/H$, the fundamental isotopical decomposition (4.4) always holds. Thus the set of vertices of the quiver $\overline{Q}$ is the same as that of the
holomorphic setting and coincides with the set of weights \( \lambda \in \mathfrak{t}^* \) occurring in the decomposition of the given \( H \)-module into irreducible representations \( V_{\lambda} \). Now, however, the vertices are connected together via the generators \( I_{\alpha} \) of (2.19) which do not close a subalgebra of \( \mathfrak{g} \) due to (2.18). With the given choice of almost complex structure, this just reflects the non-integrability of the \((0,1)\)-distribution on \( G/H \) in this case. This means that in the pseudo-holomorphic case one should consider, in a certain sense, representations of the full path algebra generated by the unoriented weight diagram \( W \). The purpose of section 5 is to make this statement precise.

5 Double quiver representations

Reduction of \( G \)-modules. We are interested in \( G \)-equivariant complex vector bundles \( \mathcal{V} \) over a reductive homogeneous manifold \( G/H \) induced by \( H \)-modules which descend from some finite-dimensional irreducible representation of \( G \) on \( \mathcal{V} \cong \mathbb{C}^q \). After restriction to \( H \subset G \) this representation decomposes into irreducible representations of \( H \) such that

\[
\mathcal{V} = \bigoplus_{q_r} V_{q_r} \quad \text{with} \quad \sum_{q_r} q_r = q \quad \text{and} \quad \mathcal{I}_i = \begin{pmatrix} I_i^{q_1} & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & I_i^{q_n} \end{pmatrix}, \tag{5.1}
\]

where \( I_i^{q_r} \) with \( i = h + 1, \ldots, \dim G \) are the generators of \( q_r \times q_r \) irreducible representations \( V_{q_r} \) of \( H \). Correspondingly, since we assume that \( H \) contains a maximal abelian subgroup of \( G \), the remaining generators \( \mathcal{I}_a \) of \( G \) in this representation have the off-diagonal form

\[
\mathcal{I}_a = \begin{pmatrix} 0 & I_a^{q_{12}} & \ldots & I_a^{q_{1n}} \\ I_a^{q_{21}} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_a^{q_{n-1,n}} \\ I_a^{q_{n1}} & \ldots & I_a^{q_{nn-1}} & 0 \end{pmatrix}, \tag{5.2}
\]

where \( I_a^{q_{rs}} \) with \( a = 1, \ldots, h \) are \( q_r \times q_s \) matrices. Depending on the representation of \( G \), some \( I_a^{q_{rs}} \) can be zero matrices.

From the commutation relations (2.3) one finds

\[
I_i^{q_r} I_a^{q_{rs}} - I_a^{q_{rs}} I_i^{q_r} = f_{ia}^{b} I_b^{q_{rs}}, \tag{5.3}
\]

\[
\sum_{r \neq s} (I_a^{q_{sr}} I_b^{q_{rs}} - I_b^{q_{sr}} I_a^{q_{rs}}) = f_{ab}^{c} I_c^{q_{sr}}, \tag{5.4}
\]

\[
\sum_{r \neq s,l} (I_a^{q_{sr}} I_b^{q_{rl}} - I_b^{q_{sr}} I_a^{q_{rl}}) = f_{ab}^{c} I_c^{q_{sl}}, \tag{5.5}
\]

for \( r, s, l = 1, \ldots, n \). (There are no sums over \( r \) and \( s \) in (5.3).) Additional constraints follow from the definition of the Lie algebra of \( G \). For example, in the case of unitary groups \( G \) one has

\[
(I_a^{q_{rs}})^\dagger = -I_a^{q_{sr}}. \tag{5.6}
\]

For the coset space \( \mathbb{F}_3 = SU(3)/U(1) \times U(1) \) with its never integrable almost complex structure, we can make this decomposition very explicit. For each fixed pair of non-negative integers \((k,l)\) there is an irreducible representation \( \mathcal{V}^{k,l} \) of \( SU(3) \) of dimension

\[
q^{k,l} = \frac{1}{2} (k + 1) (l + 1) (k + l + 2). \tag{5.7}
\]
The integer \( k \) is the number of fundamental representations \( \hat{V}^{1,0} \) and \( l \) the number of conjugate representations \( \hat{V}^{0,1} \) appearing in the usual tensor product construction of \( \hat{V}^{k,l} \), i.e. the Young diagram of \( \hat{V}^{k,l} \) contains \( k + l \) boxes in its first row and \( l \) boxes in its second row.

In this case \( n^{k,l} = q^{k,l} \) since all irreducible \( H \)-modules are one-dimensional, and the collection of weight vectors of \( H = U(1) \times U(1) \) in SU(3) label points in the weight diagram \( W^{k,l} \) for \( \hat{V}^{k,l} \). We denote them by \( (q,m)_n \), where \( q \) and \( m \) are respectively isospin and hypercharge eigenvalues, and the label by the total isospin integer \( n \) is used to keep track of multiplicities of states in the weight diagram. They may be conveniently parameterized by a pair of independent SU(2) spins \( j_+ \), with \( 2j_+ = 0, 1, \ldots, k \) and \( 2j_- = 0, 1, \ldots, l \), and the corresponding component spins \( m_+ \in \{-j_+, -j_+ + 1, \ldots, j_+ - 1, j_+\} \) as

\[
q = 2(m_+ + m_-), \quad m = 6(j_+ - j_-) - 2(k - l) \quad \text{and} \quad n = 2(j_+ + j_-). \quad (5.8)
\]

The SU(2) spin \( j_+ \) (resp. \( j_- \)) is the value of the isospin contributed by the upper (resp. lower) indices of the SU(3) tensor corresponding to the irreducible module \( \hat{V}^{k,l} \). The integers \( q, m \) all have the same even/odd parity.

To explicitly represent the coset generators in (5.1)–(5.2) in this case, we will use the Biedenharn basis for the irreducible representation \( \hat{V}^{k,l} \) of SU(3) [11]. The generators of \( H^C = (\mathbb{C}^\times)^2 \) for the irreducible module corresponding to the weight vector \( (q,m)_n \) in this basis are given by

\[
-i I_7^{(q,m)} = \frac{1}{4 \sqrt{3}} (q - m) \quad \text{and} \quad -i I_8^{(q,m)} = \frac{1}{12} (q + 3m),
\]

while the non-vanishing off-diagonal matrix elements of the remaining generators of SL(3, C) are

\[
I_1^{(q-1,m-3),n+1}(q,m)_n = \sqrt{\frac{n+q+1+1}{24(n+1+1)}} \lambda^+_k(n+1,m-3),
\]

\[
I_2^{(q+2,m)_n,q,m)_n} = \sqrt{\frac{n-q(n+q+2)}{48}},
\]

\[
I_3^{(q-1,m+3),n+1}(q,m)_n = \sqrt{\frac{n+q+1+1}{24(n+1)}} \lambda^-_k(n,m),
\]

where

\[
\lambda^+_k(n,m) = \frac{1}{\sqrt{n+2}} \sqrt{\left(\frac{k+2l}{3} + \frac{n}{2} + \frac{m}{6} + 1\right) \left(\frac{k-l}{3} + \frac{n}{2} + \frac{m}{6} + 1\right) \left(\frac{2k+l}{3} - \frac{n}{2} - \frac{m}{6}\right)},
\]

\[
\lambda^-_k(n,m) = \frac{1}{\sqrt{n}} \sqrt{\left(\frac{k+2l}{3} - \frac{n}{2} + \frac{m}{6} + 1\right) \left(\frac{l-k}{3} + \frac{n}{2} - \frac{m}{6} + 1\right) \left(\frac{2k+l}{3} + \frac{n}{2} - \frac{m}{6}\right)}.
\]

The last latter constants are defined for \( n > 0 \) and we set \( \lambda^-_k(0,m) := 0 \). The analogous relations for \( I^+_a \) can be derived by hermitian conjugation of (5.10) using the property (5.6).

**Quivers and path algebras.** The unoriented graph \( W \) associated to an irreducible \( G \)-module \( \hat{V} \) is composed of \( n \) vertices, one associated to each of the \( H \)-modules \( V_q \) appearing in the decomposition (5.1). After switching to a suitable basis (2.19) for the Lie algebra of the complexified group \( G^C \), an edge between two vertices \( v_r \) and \( v_s \) of the graph is associated to each pair of non-zero matrices \( I^{(q),rs}_a, I^{(q),rs}_a \). By giving the graph \( W \) an orientation, we turn it into a quiver \( Q = (Q_0, Q_1) \) with \( Q_0 \) the set of vertices \( v_r \) of \( W \), and \( Q_1 \) the set of arrows \( a_{rs} : v_r \rightarrow v_s \) associated with the non-zero generators \( I^{(q),rs}_a \). The quiver comes equipped with head and tail maps \( h, t : Q_1 \rightarrow Q_0 \), which for an arrow \( a : v \rightarrow v' \) are defined by \( h(a) = v' \) and \( t(a) = v \). A path in \( Q \) of length \( \ell \) is a sequence of \( \ell \) arrows in \( Q_1 \) which compose. If \( h(a) = t(a') \) for \( a, a' \in Q_1 \), then we may produce a path \( a' \) defined by \( \bullet \xrightarrow{a} \bullet \xrightarrow{a'} \bullet \), and so on. Each arrow \( a \in Q_1 \) itself is a path of length one. To each vertex \( v \in Q_0 \) we associate the trivial path \( 1_v \) of length zero with \( h(1_v) = t(1_v) = v \). More
generally, an oriented \( \ell \)-cycle in \( Q \) is a path \( p \) of length \( \ell \) with \( h(p) = t(p) \). Throughout this section we use various facts from the representation theory of quivers; see e.g. [23] for details.

The combinatorial data encoded by the quiver \( Q \) can be studied algebraically by introducing the path algebra \( \mathbb{C}Q \), the vector space over \( \mathbb{C} \) spanned by all paths together with multiplication given by concatenation of open oriented paths. If two paths do not compose then their product is defined to be 0. The trivial paths \( 1_v \) for \( v \in Q_0 \) are idempotents in this algebra and thereby define a system of mutually orthogonal projectors on the associative \( \mathbb{C} \)-algebra \( \mathbb{C}Q \), i.e. \( 1_v^2 = 1_v \) and \( 1_v 1_{v'} = 0 \) for \( v \neq v' \). For any arrow \( a : v \to v' \) in \( Q_1 \), one has \( a 1_v = a \) and \( 1_{v'} a = a \). Since \( Q \) is a finite quiver, and every path starts and ends at some vertex in \( Q_0 \), it follows that the algebra \( \mathbb{C}Q \) is unital with identity element

\[
1_{\mathbb{C}Q} = \sum_{v \in Q_0} 1_v .
\]  

The quiver \( Q \) and its path algebra \( \mathbb{C}Q \) are the basic building blocks of our ensuing constructions. However, for the reasons explained in section 4, we must instead work with a certain “completion” of this quiver in a sense that we explain precisely below.

The quiver \( Q^{k,l} \). Let us look at the quiver \( Q^{k,l} = (Q_0^{k,l}, Q_1^{k,l}) \) associated to an irreducible \( SU(3) \)-representation \( \hat{V}^{k,l} \). In this case the graph \( W^{k,l} \) is simply the weight diagram of \( \hat{V}^{k,l} \). The set of vertices \( Q_0^{k,l} \) consists of weights \( (q,m) \) constructed according to the formula (5.8). The arrows \( Q_1^{k,l} \) are built according to the non-vanishing matrix elements (5.10), which give the action of the off-diagonal coset generators on the weight vectors as

\[
\begin{align*}
I_1^- : (q,m)_n & \mapsto (q-1,m-3)_{n \pm 1} , \\
I_2^- : (q,m)_n & \mapsto (q+2,m)_n , \\
I_3^- : (q,m)_n & \mapsto (q-1,m+3)_{n \pm 1} .
\end{align*}
\]  

(5.13)

Compared to the quivers which arise in the holomorphic case [11], the directions of arrows associated to \( I_1^- \) are reversed, corresponding to the change in sign of the almost complex structure along the \( \mathbb{C}P^1 \)-fibre direction of \( F_3 \). Note that there can be multiple arrows emanating between two vertices due to degenerate weight vectors \( (q,m)_n \) and \( (q,m)_{n'} \) with \( n \neq n' \).

Let us consider some explicit constructions. For the three-dimensional fundamental representation \( \hat{V}^{1,0} \), this prescription gives the quiver

\[
Q^{1,0} : \quad (-1,1)_1 \quad \quad a_2 \quad \quad (1,1)_1
\]

\[
(0,-2)_0
\]

while the conjugate representation \( \hat{V}^{0,1} \) yields

\[
Q^{0,1} : \quad (0,2)_0 \quad \quad a_3 \quad \quad (-1,-1)_1
\]

\[
(0,-2)_0
\]

From the tensor product construction of \( \hat{V}^{k,l} \), it follows that arbitrary quivers \( Q^{k,l} \) can be built by gluing the fundamental triangles of the quivers (5.14)–(5.15) together in appropriate ways such that
coinciding edges have the same orientation. For instance, for the six-dimensional representation \( \hat{V}^{2,0} \) this prescription gives

\[
Q^{2,0} : \quad (-2, 2)_2 \quad \xrightarrow{c_2} \quad (0, 2)_2 \quad \xrightarrow{b_2} \quad (2, 2)_2
\]

while for the eight-dimensional adjoint representation \( \hat{V}^{1,1} \) we obtain

\[
Q^{1,1} : \quad (-1, 3)_1 \quad \xrightarrow{c_2} \quad (1, 3)_1
\]

The double quiver \( \overline{Q} \). Double quivers often arise as a means of translating combinatorial problems in graph theory into the algebraic framework of quivers; once an orientation is chosen on a graph, the “doubling” is inevitably necessary to retain the generic data of the original unoriented graph. A classic example is the McKay quiver which is a double quiver of an affine Dynkin diagram of type ADE describing the relationship between finite subgroups of SU(2) and kleinian singularities. In our case, the graph of interest is the weight diagram \( W \) of type ADE describing the relationship between finite subgroups of SU(2) and kleinian singularities.}

\[
(0, 2) \quad \xrightarrow{a_2} \quad (1, -1)_1
\]

while the double of the quiver \( (5.16) \) is

\[
\overline{Q}^{2,0} : \quad (-2, 2)_2 \quad \xrightarrow{c_2} \quad (0, 2)_2 \quad \xrightarrow{b_2} \quad (2, 2)_2
\]

Since the graph \( W \) does not contain any edge-loops (edges joining a vertex to itself), the path algebra \( \mathbb{C} \overline{Q} \) of the double quiver is canonically isomorphic to the path algebra \( \mathbb{C} W \), which is precisely the property we were after. The path algebra \( \mathbb{C} \overline{Q} \) is unital with the same identity
element (5.12). It has the natural structure of an involutive algebra with conjugate-linear anti-algebra involution \( \iota : \mathbb{C} \overrightarrow{Q} \to \mathbb{C} \overrightarrow{Q} \) defined by \( \iota(a) = a^* \) and \( \iota(a^*) = a \) for \( a \in Q_1 \). The inclusion of quivers \( Q \subset \overrightarrow{Q} \) makes \( \mathbb{C}Q \) a subalgebra of \( \mathbb{C} \overrightarrow{Q} \), whereas mapping each arrow \( a^* \in \overrightarrow{Q}_1 \setminus Q_1 \) to 0 gives a surjective algebra homomorphism \( \mathbb{C} \overrightarrow{Q} \to \mathbb{C}Q \).

**Representations of \( \overrightarrow{Q} \).** A representation of the quiver \( Q \) in a category \( C \) is given by a collection \( V = (V_v)_{v \in Q_0} \) of objects of \( C \) associated to each vertex together with a collection of morphisms \( \phi = (\phi_a : V_{t(a)} \to V_{h(a)})_{a \in Q_1} \) for each arrow. A morphism \( f : (V, \phi) \to (V', \phi') \) between two representations is a family of morphisms \( f = (f_v : V_v \to V'_v)_{v \in Q_0} \) such that \( f_a \circ \phi_a = \phi'_a \circ f_{t(a)} \) for all \( a \in Q_1 \). In this way, the representations of \( Q \) form a category. Any path \( p = a_1 \cdots a_\ell \) induces a morphism \( \phi(p) = \phi_{a_1} \cdots \phi_{a_\ell} : V_{t(p)} \to V_{h(p)} \). The morphism induced by the trivial path \( 1_v \) at \( v \in Q_0 \) is \( \phi(1_v) = 1_{V_v} : V_v \to V_v \). Similarly, one defines representations of the double quiver \( \overrightarrow{Q} \) in \( C \).

The fundamental case is when \( C \) is the abelian category of complex vector spaces. In this case a representation of \( Q \) is called a linear representation or a \( Q \)-module. The category of linear representations of \( Q \) is equivalent to the category of left modules over its path algebra \( \mathbb{C}Q \) [23]. Hence a \( Q \)-module \( (V, \phi) \) can be simply described as a \( \mathbb{C}Q \)-module \( V \), with the canonical identifications \( V_v = 1_vV \) for \( v \in Q_0 \). Under this equivalence, the representations of the trivial paths \( \phi(1_v) \) are orthogonal projections \( \Pi_v \) from the \( \mathbb{C}Q \)-module

\[
V = \bigoplus_{v \in Q_0} V_v \quad (5.20)
\]

onto the subspace \( V_v \) for \( v \in Q_0 \).

Any linear representation of the double quiver \( \overrightarrow{Q} \) is also a linear representation of \( Q \) by restricting the corresponding module over \( \mathbb{C} \overrightarrow{Q} \) to the subalgebra \( \mathbb{C}Q \subset \mathbb{C} \overrightarrow{Q} \). Conversely, given a \( Q \)-module \( (V, \phi) \), one naturally induces a \( \overrightarrow{Q} \)-module \( (\overrightarrow{V}, \overrightarrow{\phi}) \) by choosing hermitian inner products on each of the complex vector spaces \( V_v \) for \( v \in Q_0 \), and setting \( \overrightarrow{\phi}_a = \phi_a : V_{t(a)} \to V_{h(a)} \) and \( \overrightarrow{\phi}_a^* = \overrightarrow{\phi}_a^\dagger : V_{h(a)} \to V_{t(a)} \) the hermitian conjugate of \( \phi_a \) for \( a \in Q_1 \). Such a module will be referred to as a *unitary representation* of the double quiver \( \overrightarrow{Q} \); it defines an involutive representation of the path algebra \( \mathbb{C} \overrightarrow{Q} \) with the involution \( \iota \).

The \( \mathbb{C} \)-vector space of linear representations of the quiver \( Q \) with fixed \( V = (V_v)_{v \in Q_0} \) is

\[
\mathcal{R}(Q, V) = \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}) \quad (5.21)
\]

Upon choosing bases for \( V_v \cong \mathbb{C}^{q_v} \) for each \( v \in Q_0 \), this space may be identified with the affine variety \( \prod_{a \in Q_1} \text{Hom}(\mathbb{C}^{q_{t(a)}}, \mathbb{C}^{q_{h(a)}}) \cong \mathbb{C}^r \) where \( r = \sum_{a \in Q_1} q_{t(a)} q_{h(a)} \). The complex gauge group

\[
\mathcal{G}(V) = \left( \prod_{v \in Q_0} \text{GL}(V_v) \right) / \mathbb{C}^\times \quad (5.22)
\]

acts naturally by conjugating elements of (5.21) as bifundamental fields, i.e. \( \phi_a \mapsto g_{h(a)} \phi_a g_{t(a)}^{-1} \) for each \( a \in Q_1 \) and \( g = (g_v)_{v \in Q_0} \in \mathcal{G}(V) \). The corresponding gauge orbits are precisely the isomorphism classes of \( Q \)-modules with dimension vector \( \overrightarrow{q} = (q_v)_{v \in Q_0} \).

The vector space \( \mathcal{R}(\overrightarrow{Q}, V) \) of double quiver representations in \( V \) may be naturally identified with the *cotangent bundle* on \( \mathcal{R}(Q, V) \) through the trace pairing

\[
\mathcal{R}(\overrightarrow{Q}, V) = \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}^*, V_{h(a)}^*) = \mathcal{R}(Q, V) \oplus \mathcal{R}(Q, V)^* = T^* \mathcal{R}(Q, V) \quad (5.23)
\]

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In particular, it has a canonical $\mathcal{G}(V)$-invariant symplectic structure defined by the two-form

$$\omega_{\overline{Q}}(\overline{\phi}, \overline{\psi}) = \sum_{a \in Q_1} \text{tr}(\phi_a \psi_{a^*} - \phi_{a^*} \psi_a)$$  \hspace{1cm} (5.24)

for $\overline{\phi} = (\phi_a, \phi_{a^*})_{a \in Q_1}, \overline{\psi} = (\psi_a, \psi_{a^*})_{a \in Q_1} \in \mathcal{R}(\overline{Q}, V)$.\(^3\) Let $g(V) = (\bigoplus_{v \in Q_0} \mathfrak{g}(V_v)) \oplus \mathbb{C}$. Then the linear $\mathcal{G}(V)$-action on $\mathcal{R}(\overline{Q}, V)$ is hamiltonian and the corresponding $\mathcal{G}(V)$-equivariant moment map $\mu_V = (\mu_{V,v})_{v \in Q_0} : \mathcal{R}(\overline{Q}, V) \to g(V)^*$ is given by

$$\mu_{V,v}(\overline{\phi}) = \sum_{a \in h^{-1}(v)} \phi_a \phi_{a^*} - \sum_{a \in t^{-1}(v)} \phi_{a^*} \phi_a$$  \hspace{1cm} (5.25)

for $\overline{\phi} \in \mathcal{R}(\overline{Q}, V)$.\(^4\) A choice of hermitian metric on $V$ naturally makes $\mathcal{R}(\overline{Q}, V)$ into a flat hyper-Kähler space. For further details, see e.g. [24].

For any collection of complex numbers $\lambda = (\lambda_v)_{v \in Q_0}$, the points of $\mu_V^{-1}\left((\lambda_v 1_{V_v})_{v \in Q_0}\right)$ in $\mathcal{R}(\overline{Q}, V)$ can be identified in a gauge invariant way with modules of dimension vector $\overline{q}$ over the deformed preprojective algebra [25]

$$\mathcal{P}_\lambda = \mathbb{C} \overline{Q} / \left\langle \sum_{a \in Q_1} [a, a^*] - \sum_{v \in Q_0} \lambda_v \ 1_v \right\rangle .$$  \hspace{1cm} (5.26)

The corresponding Marsden-Weinstein symplectic quotient $\mu_V^{-1}\left((\lambda_v 1_{V_v})_{v \in Q_0}\right) // \mathcal{G}(V)$ is called an affine quiver variety. These gauge orbits classify the isomorphism classes of semi-simple representations of $\mathcal{P}_\lambda$ of dimension $\overline{q}$. The preprojective algebras $\mathcal{P}_0$ are of fundamental importance in diverse areas of representation theory, geometry, and quantum groups (see e.g. [26] for an overview). Their deformations commonly occur in problems of noncommutative geometry; for the example of the McKay quivers, they are related to algebras of functions on noncommutative deformations of kleinian singularities [25]. However, since our quivers $Q$ satisfy the hypotheses of Proposition 8.2.2 of [27], the center of the preprojective algebra $\mathcal{P}_0$ is trivial in this case, $Z(\mathcal{P}_0) = \mathbb{C} 1_{\mathcal{Q}}$.

Let us look at some explicit examples of this hamiltonian reduction associated to unitary $\overline{Q}^{k,l}$ modules induced by representations $(V^{k,l}, \phi)$ of the quiver $Q^{k,l}$ associated to an irreducible SU(3)-representation as above. For the antifundamental quiver (5.15), the moments maps are

$$\mu_{V^{0,1},(-1,-1)} = \phi_1 \phi_1^\dagger - \phi_2 \phi_2^\dagger ,$$
$$\mu_{V^{0,1},(1,-1)} = \phi_2 \phi_2^\dagger - \phi_3 \phi_3^\dagger ,$$
$$\mu_{V^{0,1},(0,2)} = \phi_3 \phi_3^\dagger - \phi_1 \phi_1^\dagger ,$$  \hspace{1cm} (5.27, 5.28, 5.29)

while for the quiver (5.16) one finds

$$\mu_{V^{2,0},(-2,2)} = \xi_3 \xi_3^\dagger - \xi_2 \xi_2^\dagger ,$$
$$\mu_{V^{2,0},(0,2)} = \xi_2 \xi_2^\dagger - \xi_1 \xi_1^\dagger + \psi_3 \psi_3^\dagger - \psi_2 \psi_2^\dagger ,$$
$$\mu_{V^{2,0},(2,2)} = \psi_2 \psi_2^\dagger - \psi_1 \psi_1^\dagger ,$$
$$\mu_{V^{2,0},(-1,-1)} = \xi_1 \xi_1^\dagger - \xi_3 \xi_3^\dagger + \phi_3 \phi_3^\dagger - \phi_2 \phi_2^\dagger ,$$
$$\mu_{V^{2,0},(1,-1)} = \psi_1 \psi_1^\dagger - \psi_3 \psi_3^\dagger + \phi_2 \phi_2^\dagger - \phi_1 \phi_1^\dagger ,$$
$$\mu_{V^{2,0},(0,-4)} = \phi_1 \phi_1^\dagger - \phi_3 \phi_3^\dagger .$$  \hspace{1cm} (5.30, 5.31, 5.32, 5.33, 5.34, 5.35)

\(^3\)Here we identify $\mathcal{R}(\overline{Q}, V)$ with its tangent space at any point.

\(^4\)Here we use the trace pairing to identify the dual space of the Lie algebra $g(V)$ with the traceless endomorphisms in $g(V)$.  

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Here we have abbreviated $\phi_\alpha := \phi_{a\alpha}$, $\psi_\alpha := \phi_{b\alpha}$, and $\xi_\alpha := \phi_{c\alpha}$ for $\alpha = 1, 2, 3$.

For a generic quiver $Q^{k,l}$, this construction can be straightforwardly generalized by using the structure of the corresponding weight diagram. Denoting the quiver module morphisms representing the adjoints of the arrows $(5.13)$ respectively by $\phi^{1,3}_{(q,m)_n}$, $\phi^{2}_{(q,m)_n}$, the associated moment maps are given by

$$
\mu_{V^{k,l},(q,m)_n} = \sum_\pm \left( \phi^{1}_{(q+1,m+3)_{n+1}} \phi^{1}_{(q+1,m+3)_{n+1}} - \phi^{1}_{(q,m)_n} \phi^{1}_{(q,m)_n} \right) \\
+ \left( \phi^{2}_{(q-2,m)_n} \phi^{2}_{(q-2,m)_n} - \phi^{2}_{(q,m)_n} \phi^{2}_{(q,m)_n} \right) \\
+ \sum_\pm \left( \phi^{3}_{(q+1,m-3)_{n+1}} \phi^{3}_{(q+1,m-3)_{n+1}} - \phi^{3}_{(q,m)_n} \phi^{3}_{(q,m)_n} \right),
$$

with the morphisms understood to be zero whenever the corresponding vertex labels map outside the range dictated by $(5.8)$. The associated modules over the preprojective algebra are then described by the moment map equations $\mu_{V^{k,l},(q,m)_n} = \lambda_{(q,m)_n} V_{(q,m)_n}$.

In section 6 we will construct, via equivariant dimensional reduction, natural representations of the quiver $Q$ in the case when $C$ is the category of complex vector bundles over a smooth manifold $M$. Such a representation is called a quiver bundle or $Q$-bundle. A quiver bundle on $M$ can be regarded as a family of quiver modules (the fibres of the quiver bundle), parametrized by points of $M$; hence many of the above constructions and results concerning quiver modules extend to this more general setting. In particular, a quiver bundle can be regarded from a more algebraic perspective as a locally free sheaf of modules over the path algebra bundle of the quiver $[8, 28]$.

By equipping a $Q$-bundle with a hermitian structure, one naturally induces a unitary representation of the double quiver $\overline{Q}$ in the category $C$. We call such a quiver bundle a hermitian $\overline{Q}$-bundle. In this case, the gauge group $(5.22)$ reduces to its unitary subgroup $\mathbb{U}(V)$. In section 8 we will see that the natural gauge theory equations associated with a hermitian $\overline{Q}$-bundle through dimensional reduction have an analogous moment map interpretation by combining the moment map $(5.25)$ (understood now in the category of complex vector bundles) with the moment map associated to the canonical $\mathbb{U}(V)$-invariant symplectic form on the space of unitary connections on a hermitian vector bundle over a Kähler manifold $M$ [8]. In particular, the equations naturally distinguish between the quasi-Kähler, nearly Kähler and Kähler reductions over the coset space $G/H$, corresponding to unitary representations of different preprojective algebras $(5.26)$.

The relations $\overline{R}$. The quivers associated to holomorphic homogeneous vector bundles over $G/H$ have the further crucial property that they contain no oriented $\ell$-cycles of length $\ell > 0$; equivalently, the corresponding path algebra is finite-dimensional. The absence of such cycles follows from parabolicity of the subgroup $P \subset G^C$ and it is of fundamental importance for obtaining nicely behaved quiver gauge theory moduli spaces. The crux of this condition is that it enables one to introduce total orderings on the vertex sets $Q_0$. This allows one to formulate and study the appropriate notion of stable quiver bundles and to relate them to solutions of quiver vortex equations, as the ordering can be exploited to construct natural $G$-equivariant filtrations of holomorphic homogeneous vector bundles by holomorphic sub-bundles over $G/H$ [7]. It also implies that all quiver representations with the same dimension vector are gauge equivalent [10].

In contrast, the quivers $Q$ associated to pseudo-holomorphic homogeneous vector bundles over $G/H$ contain non-trivial oriented cycles; for instance, the quivers $Q^{k,l}$ all contain $\ell$-cycles with $\ell \geq 3$. This will be the generic situation we encounter later on for the quiver bundles with connections that arise through equivariant dimensional reduction of Spin(7)-instanton equations; the Spin(7)-instanton moduli space thus has a very complicated stack structure that must be dealt with using
appropriate stability conditions, as in [8]. We will now define a generating set of relations on the corresponding double quiver $\overline{Q}$ which eliminates all oriented cycles in the quiver diagram of $\overline{Q}$, and hence of $Q$. The corresponding quiver bundles come from hermitian Yang-Mills equations and have better behaved moduli schemes, as in the holomorphic case.

A finite set of relations $R$ of the quiver $Q$ corresponds to a two-sided ideal of the path algebra $\mathbb{C}Q$, denoted $\langle R \rangle$, i.e. a collection of $\mathbb{C}$-linear combinations $r = \sum_i c_i p_i$ of paths $p_i$. In this paper we deal only with $admissible$ relations, i.e. the paths $p_i$ all have the same head and tail vertices. The bounded quiver with relations $(Q,R)$ is described algebraically by the factor algebra $\mathbb{C}Q/\langle R \rangle$. For the quivers associated to the $G$-module decompositions (5.1)–(5.2), the collection of relations comes from the commutation relations (5.3)–(5.5) among the block generators, written in an appropriate complex basis (2.19). In the pseudo-holomorphic case, these relations involve both $I^+_\alpha q\nu$ and $I^-_{\alpha q\nu} = (I^-_{\alpha q\nu})^\dagger$, and therefore must be understood as relations $\overline{R}$ on the double quiver $\overline{Q}$. Let us demonstrate this procedure explicitly on the quivers $Q^{k,l}$ above.

We start with the antifundamental double quiver (5.18), and consider the set of relations $\overline{R}^{0,1} = \{r_1, r_2, r_3\}$ given by

\begin{equation}
 r_1 = a_1 - a_2^* a_3^* , \quad r_2 = a_2 - a_3^* a_1^* \quad \text{and} \quad r_3 = a_3 - a_1^* a_2^* .
\end{equation}

In the corresponding factor algebra $\mathbb{C} \overline{Q}^{0,1}/\langle \overline{R}^{0,1} \rangle$, we have $r_\alpha = 0$ for $\alpha = 1, 2, 3$. By demanding that the involutive algebra structure of $\mathbb{C} \overline{Q}^{0,1}$ descend to the factor algebra under the canonical projection $\mathbb{C} \overline{Q}^{0,1} \to \mathbb{C} \overline{Q}^{0,1}/\langle \overline{R}^{0,1} \rangle$, we also have the conjugate relations $r^*_\alpha = \iota(r_\alpha)$ given by

\begin{equation}
 r^*_1 = a_1^* - a_3 a_2 , \quad r^*_2 = a_2^* - a_1 a_3 \quad \text{and} \quad r^*_3 = a_3^* - a_2 a_1 .
\end{equation}

The relations (5.37)–(5.38) express commutativity of the quiver diagram (5.18) along the primitive paths of $\overline{Q}^{0,1}$.

Let us look at the implications of the relations $r_\alpha = 0 = r^*_\alpha$ in the factor algebra. From (5.37)–(5.38) we obtain the relations

\begin{equation}
 (1_{(-1,-1)_1} - p_1) a_1 = 0 , \quad (1_{(1,-1)_1} - p_2) a_2 = 0 \quad \text{and} \quad (1_{(0,-2)_0} - p_3) a_3 = 0 ,
\end{equation}

with

\begin{equation}
 p_1 := a_1 a_3 a_2 , \quad p_2 := a_2 a_1 a_3 \quad \text{and} \quad p_3 := a_3 a_2 a_1 ,
\end{equation}

as well as relations

\begin{equation}
 a_1^* a_1 = p_3 = a_3 a_3^* , \quad a_2^* a_2 = p_1 = a_1 a_1^* \quad \text{and} \quad a_3^* a_3 = p_2 = a_2 a_2^* ,
\end{equation}

and

\begin{equation}
 a_1 (1_{(0,-2)_0} - p_3) = 0 , \quad a_2 (1_{(-1,-1)_1} - p_1) = 0 \quad \text{and} \quad a_3 (1_{(1,-1)_1} - p_2) = 0 .
\end{equation}

From (5.39) and (5.42) it follows that the oriented three-cycles (5.40) of the original quiver (5.15) are equivalent to trivial paths in the factor algebra,

\begin{equation}
 p_1 = 1_{(-1,-1)_1} , \quad p_2 = 1_{(1,-1)_1} \quad \text{and} \quad p_3 = 1_{(0,-2)_0} .
\end{equation}

Furthermore, from (5.41) it follows that the oriented two-cycles $a_\alpha^* a_\alpha$ and $a_\alpha a_\alpha^*$ for $\alpha = 1, 2, 3$ in the double quiver (5.18) are also all equivalent to trivial paths in $\mathbb{C} \overline{Q}^{0,1}/\langle \overline{R}^{0,1} \rangle$.

It is readily checked that the basic relations (5.37) (and all those implied by them above) are satisfied by the fundamental representation basis (3.41) for $\mathfrak{sl}(3, \mathbb{C})$. Similarly, one writes down relations analogous to (5.37) for the fundamental double quiver of (5.14), but with the
signs in (5.37) reversed due to the opposite orientations between the quiver diagrams (5.14)–(5.15). Arbitrary quivers $\overline{Q}^{k,l}$ are constructed by gluing fundamental triangles together. The corresponding relations $\overline{R}^{0,1}$ and $\overline{R}^{1,0}$ combine to give primitive commutativity conditions on the double quiver diagram. From an algebraic perspective, the relations around elementary parallelograms in the quiver diagram mimick the $\mathfrak{sl}(3, \mathbb{C})$ Lie algebra relations in higher irreducible representations $\hat{V}^{k,l}$ for the basis given by (5.9)–(5.10).

Let us consider a particular subdiagram of an arbitrary double quiver $\overline{Q}^{k,l}$ with the generic structure

\[
\begin{align*}
(q - 1, m + 3)_{n\pm 1} & \quad c_2^* \quad (q + 1, m + 3)_{n\pm 1} \\
(q, m)_{n} & \quad c_1^* \quad (q + 2, m)_{n} \\
(q + 1, m - 3)_{n\pm 1} & \quad a_2^* \quad (q + 2, m)_{n}
\end{align*}
\]

For clarity we do not indicate multiple arrows from the original quiver $Q^{k,l}$ involving degenerate weight vectors $(q', m')_{n\pm 1}$; they contribute identical relations to those given below. The primitive relations of $\overline{R}^{k,l}$ expressing equivalences of paths between opposite vertices connected by diagonal arrows in the two parallelograms of (5.44) are given by

\[
s_1 = c_1 + c_3^* c_2^* - a_2^* b_3^* \quad \text{and} \quad s_2 = a_2 + a_1^* a_3^* - b_3^* c_1^* ,
\]

plus their conjugates $s_1^* := \iota(s_1)$ and $s_2^* := \iota(s_2)$. In general, the subset of relations $\overline{R}^{k,l}$ in any elementary plaquette of $\overline{Q}^{k,l}$ are always of the generic form (5.45), with arrows set to 0 whenever the corresponding head or tail vertices lie outside the range (5.8). In particular, arrows on the boundary of the original quiver $Q^{k,l}$ are always subject to triangular relations such as those of (5.37)–(5.38). Note that commutativity is expressed only for paths between vertices that are joined by diagonal arrows of $Q^{k,l}$; in fact, one easily checks that other commutativity relations are inconsistent with the relations of (5.37)–(5.38) and (5.45). Thus the double quiver $\overline{Q}$ is not described by a commutative quiver diagram after imposing the relations $\overline{R}$. For example, the complete set of relations $\overline{R}^{2,0}$ for the double quiver (5.19) comprise the three-term relations (5.45) plus an additional one involving the $b_3$ arrow, and six triangular relations of the form (5.37) involving the arrows $a_1$, $a_3$, $b_1$, $b_2$, $c_1$, and $c_3$, together with their conjugates obtained as images under the involution $\iota$ of $\mathbb{C} \overline{Q}^{2,0}$.

Generally, one again establishes exactly as before that all oriented $\ell$-cycles with $\ell \geq 2$ in $\overline{Q}^{k,l}$ (and hence with $\ell \geq 3$ in $Q^{k,l}$) are equivalent to trivial cycles in the corresponding factor algebra $\mathbb{C} \overline{Q}^{k,l}/(\overline{R}^{k,l})$ with the induced $*$-involution $\iota$. Below we show that the relations $\overline{R}$ for the double quiver $\overline{Q}$ constructed in this way are in fact equivalent to the minimal requirement for triviality of all oriented cycles on the original quiver $Q$. This means that the generating set of relations $\overline{R}$ is complete.

A representation $(V, \phi)$ of $Q$ is a representation of the quiver with relations $(Q, R)$ if the collection of morphisms $\phi$ satisfy all relations $r = \sum_i c_i p_i$ in the ideal $\langle R \rangle \subset \mathbb{C} Q$ of the path algebra, i.e. $\phi(r) = \sum_i c_i \phi(p_i) = 0$. The category of linear representations of $(Q, R)$ is equivalent to the category of left modules over the factor algebra $\mathbb{C} Q/\langle R \rangle$. As before, we do not distinguish between a $(Q, R)$-module and the corresponding $\mathbb{C} Q/\langle R \rangle$-module. Since $R$ is a set of admissible relations for the quiver $Q$, it defines an affine subvariety $\mathcal{R}(Q, R, V) \subset \mathcal{R}(Q, V)$ of the representation variety (5.21) cut out by the polynomial equations $\phi(r) = 0$ coming from $R$. In subsequent sections, we will consider unitary representations of the double quiver with relations $(\overline{Q}, \overline{R})$, which thus have to satisfy all relations spelled out above.
The path groupoid $\mathcal{P}Q$. We complete our description of the relations $\mathcal{R}$ by demonstrating that the minimal requirement for absence of non-trivial cycles in the original quiver $Q$ implies the generating set of relations $\mathcal{R}$ constructed above for the double quiver $\mathcal{Q}$. For this, we need an alternative way to study equivalence relations on the paths of our quivers. We will use a categorical approach.

Let $\mathcal{P}Q$ be the additive path category of the quiver $Q$; its objects are the vertices $Q_0$ and its morphisms are paths in $Q$. Let $R$ be a generating set of admissible relations on $Q$ which renders all oriented cycles equivalent to trivial paths in the factor algebra $\mathbb{C}Q/\langle R \rangle$. Let $(R)$ be the minimal equivalence relation on $\mathcal{P}Q$ containing $R$ such that if $r \in (R)$, then $prp' \in (R)$ for any pair of morphisms $p,p'$ of $\mathcal{P}Q$ with $t(p) = h(r)$ and $h(p') = t(r)$. Let $\mathcal{P}Q/(R)$ be the corresponding factor category.

On the path category $\mathcal{P}\mathcal{Q}$ of the double quiver of $Q$, we introduce an equivalence relation by identifying $aa^*$ and $a^*a$ with the trivial paths $1_{(a)}$ and $1_{h(a)}$, respectively, for any arrow $a \in Q_1$. The corresponding factor category is a groupoid $\mathcal{P}Q$ over the base $Q_0$, called the path groupoid of the original quiver $Q$, with source and target maps $t,h : \mathcal{P}Q \rightrightarrows Q_0$; the equivalence classes $[1_v]$ for $v \in Q_0$ are partial identities for the concatenation product, and the inverse of a class $[p]$ of paths $\mathcal{P}$ on $\mathcal{Q}$ is $[p]^{-1} := [t(p)]$. The covariant functor $i : \mathcal{P}Q \to \mathcal{P}\mathcal{Q}$ defined by $p \mapsto [p]$ is an embedding which is universal: If $G$ is a groupoid and $\nu : \mathcal{P}Q \to G$ is a covariant functor, then $\nu$ can be extended to $\mathcal{P}\mathcal{Q}$ and hence to a unique functorial morphism of groupoids $\bar{\nu} : \mathcal{P}Q \to G$ such that $\nu = \bar{\nu} \circ i$.

Using the embedding $i$, we define $\langle \langle R \rangle \rangle$ to be the smallest normal subgroupoid of $\mathcal{P}Q$ containing $i(R)$, i.e. for any $x \in \mathcal{P}Q(v,v')$ with $v,v' \in Q_0$ and $r \in \langle \langle R \rangle \rangle(v)$, the element $x \cdot r \cdot x^{-1}$ belongs to $\langle \langle R \rangle \rangle(v')$. One can then construct the quotient groupoid $\mathcal{P}Q/\langle \langle R \rangle \rangle$ in the usual way, i.e. elements $x,y \in \mathcal{P}Q(v,v')$ are equivalent if and only if $x^{-1}y$ belongs to the subgroup $\langle \langle R \rangle \rangle(v) \subset \mathcal{P}Q(v,v)$. There is a natural functorial equivalence of factor categories

$$\mathcal{P}Q/\langle \langle R \rangle \rangle \xrightarrow{\simeq} \mathcal{P}\mathcal{Q}/\langle \mathcal{R} \rangle,$$

(5.46)

extending the injective map $\mathcal{P}Q/(R) \to \mathcal{P}Q/\langle \langle R \rangle \rangle$.

For example, the generating set of relations $R_{0,1}$ for the antifundamental quiver (5.15) are given by (5.43). By taking inverses in the path groupoid $\mathcal{P}Q_{0,1}$, one can reverse the steps in (5.37)-(5.43) to arrive at the generating set of relations $\mathcal{R}_{0,1}$ given by (5.37)-(5.38). For the general quivers $Q_{k,l}$, the relations $R_{k,l}$ are obtained by demanding that the arrows in $Q_{1}^{k,l}$ around each of the three elementary triangles in (5.44) all compose to trivial paths.

6 Equivariant dimensional reduction and quiver bundles

$G$-equivariant gauge fields. In this section we consider the dimensional reduction of invariant connections on equivariant vector bundles over the product manifold (4.1), and show that it naturally defines quiver bundles of the quiver $Q$ and its double $\mathcal{Q}$ described in section 5. The isotopical decomposition (4.4) associated to a $G$-module decomposition (5.1)-(5.6) is specified by $n$ complex vector bundles $E_1, \ldots, E_n$, with trivial $H$-action, over the smooth manifold $M$ of dimension $d$. We choose an $H$-invariant hermitian structure on each bundle $E_r \to M$, and let $A^r$ be a unitary connection on $E_r$. This data defines a hermitian vector bundle $\mathcal{E}$ over $\mathbb{X}^{d+h}$ of rank

$$N = \sum_{r=1}^{n} N_r q_r,$$

(6.1)

where $N_r$ is the rank of $E_r$ and the integers $q_r$ are introduced in (5.1). Without loss of generality [13] we assume $c_1(\mathcal{E}) = 0$, and that the structure group of $\mathcal{E}$ is $\text{SU}(N)$. 

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The matrices
\[
\bar{I}_i := \begin{pmatrix}
I_{N_i} \otimes I_{q_i} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & I_{N_n} \otimes I_{q_n}
\end{pmatrix}
\] (6.2)
for \(i = h + 1, \ldots, \dim G\) are generators of a reducible unitary representation of the group \(H\) on a complex vector space \(\bar{V} \cong \mathbb{C}^N\). Introduce a gauge connection
\[
A := \begin{pmatrix}
A^1 \otimes I_{q_1} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & A^n \otimes I_{q_n}
\end{pmatrix}
\] (6.3)
on the bundle \(\bigoplus_i E_i\) over \(M\), which by construction satisfies the identity
\[
[A, \bar{I}_i] = 0.
\] (6.4)
Its gauge group
\[
\mathcal{U}(\bar{V}) = \left( \prod_{r=1}^{n} U(N_r) \right) / U(1)
\] (6.5)
is the centraliser of the image of the homomorphism \(H \to SU(N)\) generated by \(\bar{I}_i\). Then any \(G\)-equivariant connection \(\mathcal{A}\) on the bundle \(\mathcal{E}\) over \(\mathbb{X}^{d+h} = M \times G/H\) in (4.4) is a natural extension of (6.3) given by
\[
\mathcal{A} = A + \bar{I}_i e^i + X_a e^a,
\] (6.6)
where \(X_a = \varrho(I_a) \in \mathfrak{su}(N)\) for \(a = 1, \ldots, h\) are matrices which depend only on the coordinates of \(M\) and locally form the components of a linear map \(\varrho : \mathfrak{m} \to \mathfrak{su}(N)\). The gauge potential (6.6) involves the canonical connection \(\bar{a}^0 = e^i \bar{I}_i\), which will contribute background \(H\)-fluxes proportional to \(\bar{I}_i\) in the expressions for the invariant gauge fields below.

For the curvature two-form \(\mathcal{F}\), we use the Maurer-Cartan equations (2.6) to obtain
\[
\mathcal{F} := dA + A \wedge A
\]
\[
= F + \left( dX_a + [A, X_a] \right) \wedge e^a
\]
\[-\frac{1}{2} \left( f_{ab}^i \bar{I}_i + f_{ab}^c X_c - [X_a, X_b] \right) e^a \wedge e^b + \left( [\bar{I}_i, X_a] - f_{ia}^b X_b \right) e^i \wedge e^a,
\] (6.7)
where \(F = dA + A \wedge A\). The last term involving the one-forms \(e^i = e^i_a e^a\) in (6.7) spoils \(G\)-invariance and so one should impose the conditions
\[
[\bar{I}_i, X_a] = f_{ia}^b X_b,
\] (6.8)
which ensure that the linear map \(\varrho : \mathfrak{m} \to \mathfrak{su}(N)\) is equivariant, i.e. it commutes with the action of \(H\). For \(\bar{I}_i\) given in (6.2), the general solution of (6.8) has the form
\[
X_a = \begin{pmatrix}
0 & \phi_{N_{12}} \otimes I_{q_{12}}^2 & \ldots & \phi_{N_{1n}} \otimes I_{q_{1n}}^n \\
\phi_{N_{21}} \otimes I_{q_{21}}^2 & 0 & \ddots & \\
\vdots & \ddots & \ddots & \phi_{N_{n-1,n}} \otimes I_{q_{n-1,n}}^n \\
\phi_{N_{n1}} \otimes I_{q_{n1}}^2 & \ldots & \phi_{N_{n,n-1}} \otimes I_{q_{n,n-1}}^n & 0
\end{pmatrix},
\] (6.9)
where $\phi_{Nrs}$ are any complex $N_r \times N_s$ matrices which depend only on the coordinates of $M$ and can be identified with sections (Higgs fields) of the bundles $E_r \otimes E_s^\vee$ over $M$.

This solution generically defines a $\overline{Q}$-bundle on $M$. The bifundamental scalar fields $\phi_{Nrs}$ in $\text{Hom}(E_s, E_r)$ are situated on the arrows $\overline{Q}_1$ of the double of the quiver $Q$. The bundles $E_r$ are sub-bundles of the $H$-equivariant complex vector bundle $E$ on $M$ with isotopical decomposition

$$E = \bigoplus_{r=1}^n E_r \otimes V_q^r,$$  

(6.10)

where $V_q^r$ is the $H$-equivariant vector bundle $M \times V_q$; it defines a locally free module over the bundle of path algebras $\mathbb{C}\overline{Q}$ of the quiver $\overline{Q}$. For anti-hermitian matrices $X_a \in \mathfrak{su}(N)$ we use (5.6) to infer

$$\phi^a_{Nrs} = \phi_{Nsr},$$  

(6.11)

and the quiver bundle thus induces a hermitian $\overline{Q}$-bundle of the double quiver of $Q$.

Thus for the $G$-equivariant gauge field $\mathcal{F}$ we have

$$\mathcal{F} = F + DX_a \wedge e^a - \frac{1}{2} \left( f^{ib}_{ac} \bar{I}_i + f^{ib}_{ac} X_c - [X_a, X_b] \right) e^a \wedge e^b,$$  

(6.12)

where

$$DX_a := dX_a + [A, X_a]$$  

(6.13)

is the gauge covariant derivative of $X_a$ on $M$. In a complex basis (2.19) for $\mathfrak{g}^C$ with redefinitions (2.20), one has

$$\mathcal{F} = F + DY_a \wedge \Theta^a + DY_\bar{a} \wedge \Theta^\bar{a} - \frac{1}{2} \left( -i C^i_{\alpha\beta} \bar{I}_i + C^\gamma_{\alpha\beta} Y_\gamma + C^\gamma_{\alpha\beta} Y_\bar{\gamma} - [Y_\alpha, Y_\beta] \right) \Theta^a \wedge \Theta^\beta$$

$$- \left( -i C^j_{\alpha\beta} \bar{I}_j + C^\gamma_{\alpha\beta} Y_\gamma + C^\gamma_{\alpha\beta} Y_\bar{\gamma} - [Y_\alpha, Y_\beta] \right) \Theta^\bar{a} \wedge \Theta^\bar{\beta}$$

$$- \frac{1}{2} \left( -i C^i_{\alpha\beta} \bar{I}_i + C^\gamma_{\alpha\beta} Y_\gamma + C^\gamma_{\alpha\beta} Y_\bar{\gamma} - [Y_\alpha, Y_\beta] \right) \Theta^\alpha \wedge \Theta^\beta,$$  

(6.14)

where

$$Y_a := \frac{1}{2} (X_{2a} - i X_{2a}) \quad \text{and} \quad Y_\bar{a} = -Y_\bar{a}^\dag.$$  

(6.15)

In the following we will also use a rescaled basis $\hat{\Theta}^a, \hat{\Theta}^\bar{a}$ (as in (2.35)–(2.36)); in this case one should substitute $Y_a \to \hat{Y}_a, Y_\bar{a} \to \hat{Y}_\bar{a}$ and $C^A_{BC} \to \hat{C}^A_{BC}$ in (6.14).

**Comparison with coset space dimensional reduction.** Suppose that the structure group of the bundle $\mathcal{E}$ over $\mathbb{R}^{d+h}$ is not the unitary group $\text{SU}(N)$ but an arbitrary compact Lie group $K$ with generators $L^A$ and structure constants $t^A_{BC}$. In the coset space dimensional reduction scheme [29, 30, 31], one assumes that the group $H$ appearing in the coset $G/H$ is embedded in $K$ and may be represented by generators (6.2). The matrices $X_a$ in (6.6) must then belong to the Lie algebra $\mathfrak{k}$ of $K$, $X_a = X_a^A L^A$, and the equations (6.8) yield the constraint equations

$$l^A_{iB} X_a^B = f^b_{ia} X_b^A$$  

(6.16)

on $X_a$ which ensure that they are intertwining operators connecting the induced representations of $H$ in $K$ and in $G$ [31]. It is difficult to solve the equations (6.16) in terms of unconstrained scalar fields, except in some special cases including the case where $G \subseteq K$ [31]. In contrast, the formula (6.9) for the case $K = \text{SU}(N)$ gives an explicit solution $X_a$ which does not arise in the coset space dimensional reduction scheme (except for some special cases) and relates it to a quiver bundle on $M$. One can extend all these considerations to arbitrary gauge groups $K$.  

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\(Q^{0.1}\)-bundles. For any concrete choice of quiver, gauge potential (6.3), and matrices (6.2) and (6.9), one can write the equivariant gauge fields explicitly in terms of \(A^r\) and \(\phi_{N_s}\). For example, by choosing the matrices (5.1)–(5.2) as generators of the antifundamental \(3 \times 3\) representation \(\tilde{V}^{0.1}\) of SU(3), we obtain from (6.6) and (3.40) an SU(3)-invariant gauge connection \(A\) as [15]

\[
A = \begin{pmatrix}
A^1 \otimes 1 + 1_{N_1} \otimes 2b & \varsigma_2 \phi_2^\dagger \otimes \Theta^2 & -\varsigma_1 \phi_1 \otimes \Theta^1 \\
-\varsigma_2 \phi_2 \otimes \Theta^2 & A^2 \otimes 1 + 1_{N_2} \otimes (\bar{a}_+ - b) & \varsigma_3 \phi_3 \otimes \Theta^3 \\
\varsigma_1 \phi_1 \otimes \Theta^1 & -\varsigma_3 \phi_3 \otimes \Theta^3 & A^3 \otimes 1 - 1_{N_3} \otimes (\bar{a}_+ + b)
\end{pmatrix},
\]

(6.17)

where \(A^1, A^2\) and \(A^3\) are \(u(N_1)\)-, \(u(N_2)\)- and \(u(N_3)\)-valued gauge potentials on hermitian vector bundles \(E_1, E_2\) and \(E_3\) over \(M\) with ranks \(N_1, N_2\) and \(N_3\), respectively, such that

\[
N_1 + N_2 + N_3 = N = \text{rank}(\mathcal{E}),
\]

(6.18)

while the bundle morphisms \(\phi_1 \in \text{Hom}(E_3, E_1), \phi_2 \in \text{Hom}(E_1, E_2)\) and \(\phi_3 \in \text{Hom}(E_2, E_3)\) are bifundamental scalar fields on \(M\). The bundles \(E_r\) are sub-bundles of the quiver bundle

\[
E^{0.1} = E_1 \otimes \mathbb{C} \oplus E_2 \otimes \mathbb{C} \oplus E_3 \otimes \mathbb{C}
\]

(6.19)

over \(M\) for the quiver \(\overline{Q}^{0.1}\) in (5.18), where the factors \(\mathbb{C}\) denote trivial \(H\)-equivariant line bundles over \(M\) arising from the decomposition of the representation \(\tilde{V}^{0.1} \cong \mathbb{C}^3\) into irreducible representations of \(H = U(1) \times U(1)\). In (5.18) the bundle \(E_1\) is assigned to the vertex \((-1, -1)_1\), \(E_2\) to \((1, -1)_1\), \(E_3\) to \((0, 0)_2\).

For the curvature \(F = dA + A \wedge A = (F^r)\) of the invariant connection (6.17) we obtain

\[
F^{11} = F^1 - \varsigma_1^2 \left(1_{N_1} - \phi_1 \phi_1^\dagger\right) \Theta^1 \wedge \Theta^1 + \varsigma_2^2 \left(1_{N_1} - \phi_2 \phi_2^\dagger\right) \Theta^2 \wedge \Theta^2,
\]

(6.20)

\[
F^{22} = F^2 - \varsigma_2^2 \left(1_{N_2} - \phi_2 \phi_2^\dagger\right) \Theta^2 \wedge \Theta^2 + \varsigma_3^2 \left(1_{N_2} - \phi_3 \phi_3^\dagger\right) \Theta^3 \wedge \Theta^3,
\]

(6.21)

\[
F^{33} = F^3 - \varsigma_3^3 \left(1_{N_3} - \phi_3 \phi_3^\dagger\right) \Theta^3 \wedge \Theta^3 + \varsigma_1^2 \left(1_{N_3} - \phi_1 \phi_1^\dagger\right) \Theta^1 \wedge \Theta^1,
\]

(6.22)

\[
F^{13} = -\varsigma_1 \left(\phi_1^\dagger A^1 - \phi_1 A^1\right) \Theta^1 \wedge \Theta^1 - \varsigma_2 \varsigma_3 \left(\phi_2 - \phi_2^\dagger \phi_3^\dagger\right) \Theta^2 \wedge \Theta^2,
\]

(6.23)

\[
F^{21} = -\varsigma_2 \left(\phi_2^\dagger A^2 - \phi_2 A^2\right) \Theta^2 \wedge \Theta^2 - \varsigma_1 \varsigma_3 \left(\phi_3 - \phi_3^\dagger \phi_1^\dagger\right) \Theta^1 \wedge \Theta^1,
\]

(6.24)

\[
F^{32} = -\varsigma_3 \left(\phi_3^\dagger A^3 - \phi_3 A^3\right) \Theta^3 \wedge \Theta^3 - \varsigma_1 \varsigma_2 \left(\phi_1 - \phi_1^\dagger \phi_3^\dagger\right) \Theta^1 \wedge \Theta^1,
\]

(6.25)

plus their hermitian conjugates \(F^{sr} = -(F^{sr})^\dagger\) for \(r \neq s\). In (6.20)–(6.25) the superscripts \(r, s\) label \(N_r \times N_s\) blocks in \(F\), and we have suppressed tensor products in order to simplify notation. Here \(F^r = dA^r + A^r \wedge A^r\) is the curvature of the connection \(A^r\) on the complex vector bundle \(E_r \rightarrow M\).

\(Q^{1.0}\)-bundles. Let us now describe how the analogous constructions for the fundamental quiver (5.14) are related to those of the \(Q^{0.1}\)-bundles above. The three-dimensional representations \(\tilde{V}^{1.0}\) and \(\tilde{V}^{0.1}\) of SU(3) are related as \(\tilde{V}^{1.0} = -\tilde{V}^{0.1}^\dagger\). The constructions of section 3 are all based on the representation \(\tilde{V}^{0.1}\) and the quiver \(Q^{0.1}\). To describe the quiver \(Q^{1.0}\), one reverses the orientation of \(F_3\); this amounts to interchanging coordinates on \(F_3\) in all formulas of section 3 with their complex conjugates, the forms \(\Theta^a\) with \(\Theta^a\), and the matrices \(I_{a\alpha}^\dagger\) with \(I_{a\alpha}\) in (3.41) (but keeping the same signs in (2.13) and (3.23)). Then via the replacement \(\mathcal{A} \rightarrow -\mathcal{A}^\dagger\) one gets the analogue of the flat connection (3.40) for the irreducible representation \(\tilde{V}^{1.0}\) and the quiver \(Q^{1.0}\).

In this way the SU(3)-invariant gauge connection (6.17) is modified to

\[
A' = \begin{pmatrix}
A^1 \otimes 1 - 1_{N_1} \otimes 2b & \varsigma_2 \psi_2 \otimes \Theta^2 & -\varsigma_1 \psi_1^\dagger \otimes \Theta^1 \\
-\varsigma_2 \psi_2 \otimes \Theta^2 & A^2 \otimes 1 - 1_{N_2} \otimes (\bar{a}_+ - b) & \varsigma_3 \psi_3 \otimes \Theta^3 \\
\varsigma_1 \psi_1 \otimes \Theta^1 & -\varsigma_3 \psi_3 \otimes \Theta^3 & A^3 \otimes 1 + 1_{N_3} \otimes (\bar{a}_+ + b)
\end{pmatrix},
\]

(6.26)
where \( \psi_1 \in \text{Hom}(E_1, E_3) \), \( \psi_2 \in \text{Hom}(E_2, E_1) \) and \( \psi_3 \in \text{Hom}(E_3, E_2) \). The vector bundle \( E_1 \) is now associated with the vertex \((1,1)\) of the quiver \((5.14)\), \( E_2 \) with \((-1,1)\), and \( E_3 \) with \((0,-2)\). The corresponding curvature two-form \( \mathcal{F}' = dA' + A' \wedge A' \) is given by

\[
\begin{align*}
\mathcal{F}'_{11} &= F^1 + \xi_2^2 \left( \Pi N_1 - \psi_1^\dagger \psi_1 \right) \Theta^1 \wedge \Theta^1 - \xi_2^2 \left( \Pi N_1 - \psi_2 \psi_2^\dagger \right) \Theta^2 \wedge \Theta^2, \\
\mathcal{F}'_{22} &= F^2 + \xi_2^2 \left( \Pi N_2 - \psi_2 \psi_2^\dagger \right) \Theta^2 \wedge \Theta^2 - \xi_2^2 \left( \Pi N_2 - \psi_3 \psi_3^\dagger \right) \Theta^3 \wedge \Theta^3, \\
\mathcal{F}'_{33} &= F^3 + \xi_2^2 \left( \Pi N_3 - \psi_3 \psi_3^\dagger \right) \Theta^3 \wedge \Theta^3 - \xi_2^2 \left( \Pi N_3 - \psi_1 \psi_1^\dagger \right) \Theta^1 \wedge \Theta^1, \\
\mathcal{F}'_{12} &= \xi_2 \left( \Pi N_1 + A_3 \psi_1 - \psi_1 A_1 \right) \Theta^1 \wedge \Theta^2 + \xi_2 \xi_3 \left( \psi_2 \psi_3 \psi_3^\dagger \right) \Theta^2 \wedge \Theta^3, \\
\mathcal{F}'_{13} &= \xi_2 \left( \Pi N_1 + A_2 \psi_3 - \psi_3 A_3 \right) \Theta^1 \wedge \Theta^2 + \xi_2 \xi_3 \left( \psi_2 \psi_3 \psi_3^\dagger \right) \Theta^2 \wedge \Theta^3.
\end{align*}
\]

\( \mathcal{Q}^{k,l} \)-bundles. The previous examples can be generalised to the quivers \( Q^{k,l} \) associated to higher irreducible representations \( \hat{V}^{k,l} \) of \( SU(3) \) given by \((5.9)-(5.11)\). Let \( E(q,m)_n \to M \) for \((q,m)_n \in W^{k,l} \) be hermitian vector bundles of rank \( N(q,m)_n \), with \( \sum (q,m)_n \in W^{k,l} N(q,m)_n = N \). Let \( A(q,m)_n \) be a unitary connection on \( E(q,m)_n \), and choose bifundamental scalar fields

\[
\phi_1^{(\pm)}(q,m)_n \in \text{Hom}(E(q-1,m-3)_n, E(q,m)_n), \quad \phi_2^{(q,m)} \in \text{Hom}(E(q+2,m)_n, E(q,m)_n)
\]

and

\[
\phi_3^{(\pm)}(q,m)_n \in \text{Hom}(E(q-1,m+3)_n, E(q,m)_n).
\]

Let \( \hat{\Pi}(q,m)_n \) be the hermitian projection of the \( H \)-restriction of \( \hat{V}^{k,l} \) onto the one-dimensional representation of \( H = U(1) \times U(1) \) with weight vector \((q,m)_n \in W^{k,l} \), and let \( \Pi(q,m)_n = \phi(1(q,m)_n) \) be the hermitian projection onto the sub-bundle \( E(q,m)_n \) of the quiver bundle

\[
E^{k,l} = \bigoplus_{(q,m)_n \in W^{k,l}} E(q,m)_n \otimes \mathbb{C}
\]

over \( M \) for the quiver \( \mathcal{Q}^{k,l} \).

Then an \( SU(3) \)-equivariant gauge connection \( A \) on the corresponding bundle \((4.4)\) over \( \mathbb{X}^{d+6} \) is given by

\[
A = \sum_{(q,m)_n \in W^{k,l}} \left[ A^{(q,m)}_n \otimes \hat{\Pi}(q,m)_n + \Pi(q,m)_n \otimes \left( 2qb - \frac{1}{2}(q-m) \left( \hat{a}_+ + b \right) \right) \hat{\Pi}(q,m)_n \right.
\]

\[
+ \xi_2 \left( \phi_1^{(\pm)}(q,m)_n \otimes \hat{\Pi}(q-1,m-3)_n \right) I^{-1}_1 \hat{\Pi}(q,m)_n \Theta^1 - \phi_1^{(\pm)}(q,m)_n \otimes \hat{\Pi}(q-1,m-3)_n \Theta^1 \\
\]

\[
+ \xi_2 \left( \phi_2^{(q,m)} \otimes \hat{\Pi}(q+2,m)_n \right) I^{-2}_2 \hat{\Pi}(q,m)_n \Theta^2 - \phi_2^{(q,m)} \otimes \hat{\Pi}(q,m)_n I^{-2}_2 \hat{\Pi}(q+2,m)_n \Theta^2 \\
\]

\[
+ \xi_2 \xi_3 \left( \phi_3^{(\pm)}(q,m)_n \otimes \hat{\Pi}(q-1,m+3)_n \right) I^{-3}_3 \hat{\Pi}(q,m)_n \Theta^3 - \phi_3^{(\pm)}(q,m)_n \otimes \hat{\Pi}(q-1,m+3)_n \Theta^3 \right].
\]
The diagonal curvature matrix elements of (6.14) at each vertex \((q,m)_n \in W^{k,l}\) of the weight diagram for \(\bar{V}^{k,l}\) are given by

\[
\mathcal{F}^{(q,m)}_{n} = \mathcal{F}^{(q,m)}_{n} \nonumber
\]

\[
+ \frac{s_2}{24} \Theta^1 \wedge \Theta^1 \sum_{\pm} \left[ \frac{n \pm q \pm 1 \mp 1}{n \mp 1} \lambda_{k,l}^{\pm}(n \pm 1, m + 3)^2 \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q,m)}_{n+1} \phi^{(q,m)}_{n+1} \right) \right) \right. 
\]

\[
- \frac{n \pm q + 1 \pm 1}{n \mp 1} \lambda_{k,l}^{\mp}(n, m)^2 \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q+1,m+3)}_{n+1} \right) \right) 
\]

\[
+ \frac{s_2}{48} \Theta^2 \wedge \Theta^3 \left[ (n - q) (n + q + 2) \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q,m)}_{n+1} \phi^{(q,m)}_{n+1} \right) \right) \right. 
\]

\[
- (n + q) (n + q + 2) \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q-2,m+3)}_{n+1} \right) \right) 
\]

\[
+ \frac{s_2}{24} \Theta^1 \wedge \Theta^3 \sum_{\pm} \left[ \frac{n \pm q \pm 1 \mp 1}{n \mp 1} \lambda_{k,l}^{\pm}(n, m)^2 \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q+1,m-3)}_{n+1} \right) \right) \right. 
\]

\[
- \frac{n \pm q + 1 \pm 1}{n \mp 1} \lambda_{k,l}^{\pm}(n \mp 1, m - 3)^2 \left( \Omega_n^{(q,m)} - \frac{1}{n+1} \left( \phi^{(q+1,m-3)}_{n+1} \right) \right) \]
and
\[
\mathcal{F}(q+1,m-3)_{n=1} (q+1,m+3)_{n=1} = \frac{\lambda^{\pm}_{k,l}(n,m)}{24} \lambda^{\pm}_{k,l}(n+1,m-3) \tilde{\Theta}^1 \wedge \tilde{\Theta}^3
\]
\[
\times \left( \phi^1_{(q+1,m-3)_{n=1}}(\mp) \phi^1_{(q+1,m+3)_{n=1}}(\mp) - \phi^1_{(q+2,m)_{n=1}}(\mp) \phi^1_{(q+2,m)_{n=1}}(\mp) \right),
\]
(6.40)
\[
\mathcal{F}(q-2,m)_{n=1} (q+1,m+3)_{n} = \frac{\lambda^{\pm}_{k,l}(n,m)}{24} \lambda^{\pm}_{k,l}(n+1,m-3) \tilde{\Theta}^2 \wedge \tilde{\Theta}^3
\]
\[
\times \left( \phi^2_{(q-2,m)_{n=1}}(\pm) \phi^2_{(q+1,m+3)_{n=1}}(\pm) - \phi^2_{(q-1,m+3)_{n=1}}(\pm) \phi^2_{(q-1,m+3)_{n=1}}(\pm) \right),
\]
(6.41)
\[
\mathcal{F}(q+2,m)_{n} (q-1,m+3)_{n=1} = \frac{\lambda^{\pm}_{k,l}(n,m)}{24} \lambda^{\pm}_{k,l}(n+1,m-3) \tilde{\Theta}^2 \wedge \tilde{\Theta}^3
\]
\[
\times \left( \phi^3_{(q+2,m)_{n}}(\pm) \phi^3_{(q-1,m+3)_{n=1}}(\pm) - \phi^3_{(q,m)_{n}}(\pm) \phi^3_{(q,m)_{n}}(\pm) \right),
\]
(6.42)
plus their hermitian conjugates \(\mathcal{F}(q',m')_{n'} (q,m)_n = -\left(\mathcal{F}(q,m)_n (q',m')_{n'}\right)^\dagger\) for \((q',m')_{n'} \neq (q,m)_n\).

Here
\[
D\phi^1_{(q,m)_n} = d\phi^1_{(q,m)_n} + A^{(q,m)_n} \phi^1_{(q,m)_n} = \phi^1_{(q,m)_n} A^{(q-1,m-3)_{n=1}},
\]
(6.43)
\[
D\phi^2_{(q,m)_n} = d\phi^2_{(q,m)_n} + A^{(q,m)_n} \phi^2_{(q,m)_n} = \phi^2_{(q,m)_n} A^{(q+2,m)_n},
\]
(6.44)
\[
D\phi^3_{(q,m)_n} = d\phi^3_{(q,m)_n} + A^{(q,m)_n} \phi^3_{(q,m)_n} = \phi^3_{(q,m)_n} A^{(q-1,m+3)_{n=1}},
\]
(6.45)
are bifundamental covariant derivatives of the Higgs fields on \(M\).

### 7 Quiver gauge theory

**Generalized instanton equations.** In this section we construct natural gauge theories on the quiver bundles of section 6. Fix a riemannian structure on the manifold \(M\), and let \(\Sigma\) be a differential form of degree \(d + h - 4\) on the manifold \(\mathbb{X}^{d+h} = M \times G/H\). When the coset space \(G/H\) is a nearly Kähler six-manifold, a natural choice for such a form on \(\mathbb{X}^{d+6}\) is given by
\[
\Sigma = \frac{1}{2} \kappa \ vol_d \wedge \omega,
\]
(7.1)
where \(\omega\) is the fundamental two-form on \(G/H\), \(\kappa \in \mathbb{R}\) is a constant and \(vol_d\) is the volume form on \(M\).

Let \(E\) be a complex vector bundle over \(\mathbb{X}^{d+h}\) endowed with a connection \(A\). The \(\Sigma\)-anti-self-dual Yang-Mills equations are defined as the first order equations [32]
\[
\star \mathcal{F} = -\Sigma \wedge \mathcal{F},
\]
(7.2)
on the connection \(A\) with curvature \(\mathcal{F} = dA + A \wedge A\). Here \(\star\) is the Hodge duality operator on \(\mathbb{X}^{d+h}\).

Taking the exterior derivative of (7.2) and using the Bianchi identity, we obtain
\[
d \star \mathcal{F} + A \wedge \star \mathcal{F} - \star \mathcal{F} \wedge A + \star \mathcal{H} \wedge \mathcal{F} = 0,
\]
(7.3)
where the three-form \(\mathcal{H}\) is defined by the formula
\[
\star \mathcal{H} := d\Sigma.
\]
(7.4)
The second order equations (7.3) differ from the standard Yang-Mills equations by the last term involving a three-form \(\mathcal{H}\) which can be identified with a totally antisymmetric torsion on \(\mathbb{X}^{d+h}\). This
torsion term naturally appears in string theory compactifications with fluxes where it is identified with a supergravity three-form field [2, 1]. When \( \mathcal{F} \) solves the first order equations (7.2), the torsion term in the Yang-Mills equations (7.3) can vanish in certain instances. For example, on nearly Kähler six-manifolds \( \mathbb{X}^6 \) the three-form \( d\Sigma \) from (7.1) (with \( d = 0 \)) is a sum of \((3,0)\)- and \((0,3)\)-forms, and \( \mathcal{F} \) is a \((1,1)\)-form, hence their exterior product vanishes.

The equations (7.2) are BPS-type instanton equations in \( D > 4 \) dimensions. For various classes of closed forms \( \Sigma \) (the integrable case) they were introduced and studied in [33, 34, 35, 36, 37]. Many of these equations, such as the hermitian Yang-Mills equations [34], naturally appear in superstring theory as the conditions for survival of at least one unbroken supersymmetry in the low-energy effective field theory in \( D \leq 4 \) dimensions. Some solutions of the first order gauge field equations (7.2) were described e.g. in [38, 39, 21, 40, 41].

**Reduction of the Yang-Mills action functional.** If \( M \) is a closed manifold, the Yang-Mills equations with torsion (7.3) are variational equations for the action \([40, 41]\)

\[
S = -\frac{1}{4} \int_{\mathbb{R}^{d+1}} \text{tr}(\mathcal{F} \wedge \star \mathcal{F} + \mathcal{F} \wedge \mathcal{F} \wedge \Sigma),
\]

where \( \text{tr} \) denotes the trace in the fundamental representation of the gauge group. Then \( S = 0 \) whenever the gauge field \( \mathcal{F} \) satisfies the \( \Sigma \)-anti-self-duality equation (7.2). Integrating the second term in (7.5) by parts using (7.4), we can write the action in the form

\[
S = -\frac{1}{4} \int_{\mathbb{R}^{d+1}} \text{tr}(\mathcal{F} \wedge \star \mathcal{F} + (\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \wedge \star \mathcal{H})
\]

\[
-\frac{1}{4} \int_{\mathbb{R}^{d+1}} d(\text{tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \wedge \Sigma),
\]

and the second integral in (7.6) vanishes by our assumption that \( \partial M = \emptyset \).

We will explicitly work out the dimensional reduction of (7.5) over \( \mathbb{X}^{d+6} = M \times G/H \) to a quiver gauge theory action on a \( \mathcal{Q} \)-bundle over \( M \), in the case where \( G/H \) is a six-dimensional nearly Kähler coset space with \( \Sigma \) given in (7.1). The \( G \)-equivariant gauge field \( \mathcal{F} \) from (6.12) has components

\[
\mathcal{F}_{\mu\nu} = F_{\mu\nu} \quad \text{and} \quad \mathcal{F}_{a\mu} = D_{\mu} X_a
\]

(7.7)

where the indices \( \mu, \nu, \ldots = 1, \ldots, d \) label components along the manifold \( M \) and \( a, b, \ldots = 1, \ldots, 6 \) label components along the internal coset space \( G/H \). Substituting (7.7) into the term involving the Yang-Mills form in (7.6), and using the structure constant identities (2.26) together with the equivariance conditions (6.8), we obtain

\[
S_{YM} := -\frac{1}{4} \int_{\mathbb{R}^{d+6}} \text{tr}(\mathcal{F} \wedge \star \mathcal{F})
\]

\[
= -\frac{1}{8} \int_{\mathbb{R}^{d+6}} \text{vol}_{d+6} \text{tr}(F_{\mu\nu} F^{\mu\nu} + 2 F_{\mu a} F^{\mu a} + F_{ab} F^{ab})
\]

(7.8)

\[
= -\frac{1}{8} \text{Vol}(G/H) \int_M \text{vol}_d \text{tr}(F_{\mu\nu} F^{\mu\nu} + 2 D_\mu X_a D^\mu X_a + f_{iab} f_{jcd} \mathcal{I}_i \mathcal{I}_j - \frac{1}{3} X_a X_a - 2 f_{abc} X_a [X_b, X_c] + [X_b, X_c] [X_b, X_c] [X_b, X_c]),
\]

where \( \text{vol}_{d+6} = \text{vol}_d \wedge e^{123456} \) and \( \text{Vol}(G/H) = \int_{G/H} e^{123456} \). To reduce the term involving the Chern-Simons three-form in (7.6), we use (6.6) to get

\[
A_\mu = A_\mu \quad \text{and} \quad A_a = e_a^i \mathcal{I}_i + X_a,
\]

(7.9)

\[\text{For recent discussions of heterotic string theory with torsion see e.g. [3, 4, 5] and references therein.}\]
and from the relation \( \star d\omega = \frac{1}{2} f_{abc} e^{abc} \) we obtain

\[
S_{\Sigma - \text{CS}} := -\frac{1}{4} \int_{X^{d+6}} \text{tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \wedge \star H
\]

\[
= -\frac{1}{24} \kappa \text{Vol}(G/H) \int_M \text{vol}_d \ f_{abc} \text{tr} (A_a F_{bc} - \frac{1}{3} A_a [A_b, A_c])
\]

\[
= -\frac{1}{24} \kappa \text{Vol}(G/H) \int_M \text{vol}_d \left( X_a \, X_a - 2 f_{abc} \, X_a [X_b, X_c] - f_{iab} f_{jab} \tilde{I}_i \tilde{I}_j \right).
\]

Thus the reduction of the total action \( S = S_{\text{YM}} + S_{\Sigma - \text{CS}} \) is given by

\[
S = -\frac{1}{8} \kappa \text{Vol}(G/H) \int_M \text{vol}_d \left( F_{\mu \nu} F^{\mu \nu} + 2 D_{\mu} X_a D^{\mu} X_a - \frac{1}{3} (\kappa - 3) f_{abc} f_{iab} f_{jab} \widetilde{I}_i \widetilde{I}_j \right) + \frac{1}{3} (\kappa - 1) X_a \, X_a - \frac{2}{3} (\kappa + 3) f_{abc} [X_b, X_c] + [X_b, [X_b, X_c]]
\]

Note that the value of \( \kappa \in \mathbb{R} \) is not arbitrary but defined by the torsion three-form components \( \kappa f_{abc} \) entering into the structure equations on \( G/H \). For the nearly Kähler case, from (2.25) one has \( \kappa = -1 \). This explains the choice of multiplier \( \frac{1}{6} \) in (7.1). Below we also consider more general situations with \( \kappa \neq -1 \), corresponding to non-canonical metric connections on the cotangent bundle of \( G/H \) with holonomy group larger than \( H \). In particular, the choice \( \kappa = 3 \) will be singled out by BPS conditions.

**Quiver matrix models.** Dimensional reduction of the action (7.11) further to a point truncates the quiver gauge theory to a quiver matrix model in \( d = 0 \) dimensions with action \( S = \frac{1}{8} \kappa \text{Vol}(G/H) S_{\kappa} \), which is naturally associated with a homogeneous vector bundle \( \tilde{V} = G \times_H \tilde{V} \) over \( G/H \) corresponding to a \( G \)-module decomposition (5.1)–(5.6). Upon substituting the equivariant decomposition (6.9) for \( X_a \), the action of the matrix model can be regarded as a Higgs potential

\[
\tilde{S}_{\kappa}(\phi) = -\text{tr} \left( [X_a, X_b]^2 - \frac{2}{3} (\kappa + 3) f_{abc} X_a [X_b, X_c] + \frac{1}{3} (\kappa - 1) X_a^2 - \frac{1}{3} (\kappa - 3) f_{iab} f_{jab} \tilde{I}_i \tilde{I}_j \right)
\]

in the original quiver gauge theory for the collection of bifundamental scalar fields \( \phi = (\phi_{N_r, s} \in \text{Hom}(E_s, E_r)) \). The critical points of this potential are determined by solutions of the matrix equations

\[
\frac{1}{3} (\kappa - 1) X_a - \frac{1}{3} (\kappa + 3) f_{abc} [X_b, X_c] - [X_b, [X_b, X_c]] = 0.
\]

The first term of (7.12) is the standard Yang-Mills matrix model action from reduction of Yang-Mills gauge theory in flat space. The Chern-Simons and mass deformations owe to the curvature of the original homogeneous manifold \( G/H \). The last term is a constant \( H \)-flux which is due to the topologically non-trivial gauge fields on \( G/H \); its appearance ensures e.g. that there is non-trivial dynamical mass generation from the vacuum state of the matrix model [13]. To compute it explicitly, we normalise the traces over the carrier spaces \( V_{q_r} \) using the quadratic Dynkin indices \( \chi_r \) of the representations so that

\[
\text{tr}_{V_{q_r}} (I_{i}^{q_r} I_{j}^{q_r}) = -\chi_r \delta_{ij},
\]

for \( r = 1, \ldots, n \) and \( i, j = 7, \ldots, \dim G \). Then using (2.26) the constant \( H \)-flux term in the action (7.12) evaluates to

\[
\frac{1}{3} (\kappa - 3) f_{iab} f_{jab} \text{tr}_{\tilde{V}} (\tilde{I}_i \tilde{I}_j) = -\frac{1}{3} (\kappa - 3) f_{iab} f_{jab} \sum_{r=1}^{n} N_r \chi_r = -\frac{2}{3} (\kappa - 3) \sum_{r=1}^{n} N_r \chi_r.
\]
For the special value $\kappa = 3$ (so that $\Sigma = \omega$ in (7.1) and the constant term (7.15) vanishes), the action (7.12) is non-negative and can be written in the complex parametrization (6.15), with structure constants (2.29)–(2.30), as a sum of squares

$$\tilde{S}_{\kappa=3}(\phi) = \text{tr} \left[ \varepsilon_{\delta\alpha\beta} \left(-iC^i_{\alpha\beta} \overline{I}_i + C^\gamma_{\alpha\beta} Y_\gamma + C^\sigma_{\alpha\beta} Y_\sigma - [Y_\alpha, Y_\beta] \right) \right]^2 + \text{tr} \left[ \delta^{\alpha\bar{\alpha}} \left(-iC^i_{\bar{\alpha}\bar{\beta}} \overline{I}_i + C^\bar{\gamma}_{\bar{\alpha}\bar{\beta}} Y_{\bar{\gamma}} + C^\bar{\sigma}_{\bar{\alpha}\bar{\beta}} Y_{\bar{\sigma}} - [Y_{\bar{\alpha}}, Y_{\bar{\beta}}] \right) \right]^2,$$

(7.16)

where we use the matrix notation $[Y]^2 := \frac{1}{2} (Y^\dagger Y + Y Y^\dagger)$. The vacuum solutions of the quiver matrix model in this case are thus determined by the equations

$$\phi(\overline{r}_\delta) := \varepsilon_{\delta\alpha\beta} \left([Y_\alpha, Y_\beta] - C^\gamma_{\alpha\beta} Y_\gamma - C^\sigma_{\alpha\beta} Y_\sigma + i C^i_{\alpha\beta} \overline{I}_i \right) = 0,$$

(7.17)

$$\mu_{\hat{\nu}} := \sum_{\alpha=1}^3 \left([Y_\alpha, Y_\alpha] - C^\beta_{\alpha\alpha} Y_\beta - C^\bar{\beta}_{\alpha\alpha} Y_{\bar{\beta}} + i C^i_{\alpha\alpha} \overline{I}_i \right) = 0.$$

(7.18)

This feature follows from the more general fact that the $\Sigma$-anti-self-duality equations (7.2) on a nearly Kähler six-manifold with $\Sigma = \omega$ are equivalent to the hermitian Yang-Mills equations [34, 32, 21]

$$\mathcal{F}^{0,2} = 0 \quad \text{and} \quad \omega_{\wedge} \mathcal{F} = 0,$$

(7.19)

whose solutions saturate the absolute minimum value $S = 0$ of the action functional (7.5) in this case. In the present instance, the curvature two-form $\mathcal{F}$ is given by (6.14) with $F = D Y_\alpha = 0$.

The system of equations (7.17) expresses the generating set of relations $\overline{R} = \{\overline{r}_\delta\}$ (plus their $*$-conjugates $\phi(\overline{r}_\delta^\dagger) : = \phi(\overline{r}_\delta)\dagger$) in the $\overline{Q}$-module $(\overline{V}, \phi)$ which is equivalent to pseudo-holomorphicity of the homogeneous vector bundle $\overline{V} \to G/H$, while (7.18) is the moment map equation for modules in $\mathcal{R}(\overline{Q}, \overline{V})$ over the deformed preprojective algebra (5.26) with deformation vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ fixed by the background fluxes $\overline{I}_i$ and the structure constants $C^i_{\alpha\beta}$. The presence of constant terms in these expressions only reflects the non-holonomic nature of the frame $\{\Theta^\alpha, \Theta^\beta\}$, i.e. they vanish in a holonomic frame. Moreover, the constant term in (7.17) vanishes for the quasi-Kähler case and the constant term in (7.18) vanishes for the nearly Kähler case.

**Gauge theory on $\overline{Q}^{0,1}$-bundles.** For an SU(3)-invariant connection $A$ on a hermitian quiver bundle (6.19) over the flag manifold $F_3$, we substitute into (7.7)–(7.11) the equivariant decompositions (6.17) and (6.20)–(6.25) in the nearly Kähler limit $\varsigma_1 = \varsigma_2 = \varsigma_3 =: \varsigma$, with the complex parametrization (6.15) and structure constants (3.44). After a bit of calculation, we arrive at the quiver gauge theory action

$$S^{0,1} = -\frac{1}{2} \text{Vol}(F_3) \int_M \text{vol}_d \left( \frac{1}{4} \sum_{s=1}^3 \text{tr}_N \left( (F^s)^\dagger_{\mu\nu} (F^s)^{\mu\nu} + \varsigma^2 (D_{\mu} \phi_s) (D^\mu \phi_s)^\dagger \right) + \varsigma^2 (D_{\mu} \phi_{s+1})^\dagger (D^\mu \phi_{s+1}) + \tilde{S}^{0,1}_{\varsigma,\kappa}(\phi_1, \phi_2, \phi_3) \right),$$

(7.20)

where we identify $\phi_s : = \phi_{s \text{mod} 3} \in \text{Hom}(E_{s+2}, E_s)$, and

$$D \phi_s = d \phi_s + A^{s} \phi_s - \phi_s A^{s+2}$$

(7.21)

are bifundamental covariant derivatives of the Higgs fields. The Higgs potential is given by

$$\tilde{S}^{0,1}_{\varsigma,\kappa}(\phi_1, \phi_2, \phi_3) = -\varsigma^4 \sum_{s=1}^3 \text{tr}_N \left( \frac{1}{4} (1 - \phi_s \phi_s^\dagger)^2 + \frac{1}{4} (1 - \phi^\dagger_{s+1} \phi_{s+1})^2 - 4 \varsigma \phi^\dagger_{s+1} \phi_{s+1} - 4 \varsigma \phi^\dagger \phi^\dagger_{s+1} \phi_{s+1} - \frac{1}{4} \phi^\dagger_{s+1} \phi^\dagger \phi^\dagger_{s+1} \phi^\dagger + \frac{1}{8} \varsigma^4 (N_1 + N_2 + N_3) \right).$$

(7.22)
In the reduction to a quiver matrix model in $d = 0$ dimensions at the special torsion value $\kappa = 3$, the action (7.22) reads

$$\mathcal{S}_{\kappa, \kappa = 3}^{0,1}(\phi_1, \phi_2, \phi_3) = \xi^4 \sum_{s=1}^{3} \text{tr} N_s \left( 4|\phi_s \phi_s^\dagger - \phi_s^\dagger \phi_{s+1}|^2 + |\phi_{s+1} - \phi_{s+2}^\dagger \phi_s|^2 \right).$$

(7.23)

The corresponding vacuum equations are

$$\phi(r_s) := \phi_s - \phi_{s+1}^\dagger \phi_{s+2}^\dagger = 0 \quad \text{and} \quad \mu_s := \phi_s \phi_s^\dagger - \phi_{s+1}^\dagger \phi_{s+1} = 0$$

for $s = 1, 2, 3$, which coincide respectively with the relations (5.37) (plus their $*$-conjugates (5.38)) and the moment map equations (5.27)–(5.29). The quiver modules in this case are thus identified with representations of the (undeformed) preprojective algebra $\mathcal{P}^{0,1}_0$. As we discuss in section 8, this is in marked contrast to the quiver representations associated to the Kähler geometry of $\mathbb{F}_3$ that generically correspond to deformations determined by the background (monopole) fluxes, which arise from contraction of $\omega$ and $\text{d}\tilde{a}_+$. In the nearly Kähler case this contraction vanishes because both monopole fields on $\mathbb{F}_3$ satisfy the hermitian Yang-Mills equations.

Gauge theory on $\mathcal{Q}^{k,l}$-bundles. For any given pair of non-negative integers $(k, l)$, the $Q^{k,l}$ quiver gauge theory can be described using the formulas (6.35)–(6.45). The general equations are rather lengthy and complicated, and not very informative. We will therefore satisfy ourselves by briefly commenting on the structure of the vacuum equations for the corresponding quiver matrix model, with $\kappa = 3$ and $\xi_1 = \xi_2 = \xi_3 =: \xi$, which follow from (7.19), (6.37)–(6.39), and (6.36) with constant Higgs fields and vanishing gauge potentials. One then finds a representation of the set of relations $\mathcal{F}^{k,l}$ given by

$$\phi(r_{(q,m)_n}^{1(\pm)}) = \sqrt{((n + 1 \pm 1)^2 - q^2)} (n \pm q + 1 \pm 1) \left( \phi_{(q,m)_n}^{1(\pm)} - \phi_{(q+1,m-3)_{n+1}}^{3(\mp)} \phi_{(q-1,m-3)_{n+1}}^{2(\mp)} \right. \left. - \sqrt{(n + q) (n - q + 2) (n \pm q + 1 \pm 1)} \left( \phi_{(q,m)_n}^{1(\pm)} - \phi_{(q-2,m)_n}^{2(\mp)} \phi_{(q-1,m-3)_{n+1}}^{3(\mp)} \right) \right),$$

(7.25)

$$\phi(r_{(q,m)_n}^{2(\mp)}) = \sum_{\pm} \left[ \frac{n \pm q + 1 \mp 1}{n+1} \lambda_{k,l}(n, m)^2 \phi_{(q,m)_n}^{2(\pm)} - \phi_{(q+1,m+3)_{n+1}}^{3(\mp)} \phi_{(q+2,m)_n}^{2(\mp)} \right] \phi_{(q,m)_n}^{1(\pm)} \right),$$

(7.26)

$$\phi(r_{(q,m)_n}^{3(\pm)}) = \sqrt{(n + q) (n - q + 2) (n \mp q + 1 \mp 1)} \left( \phi_{(q,m)_n}^{3(\pm)} - \phi_{(q-2,m)_n}^{2(\mp)} \phi_{(q-1,m+3)_{n+1}}^{3(\mp)} \right. \left. - \sqrt{(n + 1 \pm 1)^2 - q^2)} (n \mp q + 1 \mp 1) \left( \phi_{(q,m)_n}^{3(\pm)} - \phi_{(q+1,m+3)_{n+1}}^{1(\mp)} \phi_{(q-1,m-3)_{n+1}}^{2(\mp)} \right) \right),$$

(7.27)

and the moment map

$$\mu_{(q,m)_n} = \sum_{\pm} \left[ \frac{n \pm q + 1 \mp 1}{n+1} \lambda_{k,l}(n, m)^2 \phi_{(q,m)_n}^{1(\pm)} \phi_{(q,m)_n}^{1(\pm)} \right. \left. - \frac{n \pm q + 1 \mp 1}{n+1} \lambda_{k,l}(n, m)^2 \phi_{(q+1,m+3)_{n+1}}^{1(\pm)} \phi_{(q+2,m)_n}^{1(\pm)} \right] \phi_{(q,m)_n}^{1(\pm)} \right),$$

(7.28)

at each vertex $(q, m)_n \in W^{k,l}$ of the weight diagram for the SU(3)-module $\widehat{V}^{k,l}$. Note that the remaining field strength components (6.40)–(6.42) play no role in the vacuum sector of the quiver gauge theory.
8 Double quiver vortex equations

Spin(7)-instanton equations. In this section we consider the Σ-anti-self-dual Yang-Mills equations (7.2) on eight-dimensional manifolds and their reduction to quiver vortex equations in two dimensions. Consider an oriented riemannian manifold \((X^8, g)\) of dimension eight endowed with a four-form \(\Sigma\) which defines an almost Spin(7)-structure on \(X^8\). A Spin(7)-instanton is defined to be a connection \(A\) on a complex vector bundle \(E\) over \(X^8\) whose curvature \(F\) satisfies the Σ-anti-self-dual gauge field equations (7.2). For more details see e.g. \([32, 36, 37]\). As previously we assume that \(E\) has degree zero, i.e. \(c_1(E) = \frac{1}{2\pi} \text{tr}(F) = 0\).

Suppose that an (almost) Spin(7)-manifold \(X^8\) allows an almost complex structure \(\tilde{\mathcal{J}}\) and an SU(4)-structure, i.e. \(c_1(X^8) = 0\). Then there exists a non-degenerate \((4, 0)\)-form \(\tilde{\Omega}\) and a \((1, 1)\)-form \(\tilde{\omega}\) on \(X^8\) such that

\[
\tilde{\Omega} = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4 \quad \text{and} \quad \tilde{\omega} = \frac{1}{2} \left( \Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 + \Theta^3 \wedge \Theta^3 + \Theta^4 \wedge \Theta^4 \right). \tag{8.1}
\]

The one-forms \(\Theta^\alpha\) are defined by

\[
\tilde{\mathcal{J}} \Theta^\alpha = i \Theta^\alpha \quad \text{for} \quad \alpha = 1, 2, 3, 4, \tag{8.2}
\]

i.e. they are \((1, 0)\)-forms with respect to \(\tilde{\mathcal{J}}\).\(^7\)

Given any SU(4)-structure \((\tilde{\omega}, \tilde{\Omega})\) on an eight-manifold \(X^8\), there is a compatible Spin(7)-structure determined by

\[
\Sigma = \frac{1}{2} \tilde{\omega} \wedge \tilde{\omega} - \text{Re} \tilde{\Omega}. \tag{8.3}
\]

When the Spin(7)-structure is defined via the four-form (8.3), the Spin(7)-instanton equations (7.2) can be reduced via the inclusion \(\text{SU}(4) \subset \text{Spin}(7)\) to the equations

\[
\tilde{\omega} \lrcorner F = 0 \quad \text{and} \quad F^0,2_+ = 0. \tag{8.4}
\]

Here we have used the fact that the complex conjugate of \(\tilde{\Omega}\) induces an anti-linear involution \(*_{\tilde{\Omega}} : \wedge^0,2 T^*X^8 \to \wedge^0,2 T^*X^8\) on \(X^8\), so that one can introduce the corresponding self-dual part

\[
F^0,2_+ := \frac{1}{2} \left( F^{0,2} + *_{\tilde{\Omega}} F^{0,2} \right) \tag{8.5}
\]

of \(F^{0,2}\) in the +1 eigenspace of \(*_{\tilde{\Omega}}\) \([37]\). In the basis of \((1, 0)\)-forms \(\Theta^\alpha\) and \((0, 1)\)-forms \(\Theta^\dot{\alpha} := \overline{\Theta^\alpha}\), the equations (8.4) read

\[
\delta^{\dot{\alpha} \alpha} F_{\dot{\alpha} \alpha} = 0 \quad \text{and} \quad F_{\dot{\alpha} \dot{\beta}} = -\frac{1}{2} \varepsilon_{\dot{\alpha} \dot{\beta} \dot{\gamma} \dot{\rho}} F_{\dot{\gamma} \dot{\rho}}. \tag{8.6}
\]

The Spin(7)-instanton equations (8.4) are weaker than the hermitian Yang-Mills equations

\[
\tilde{\omega} \lrcorner F = 0 \quad \text{and} \quad F^{0,2} = 0. \tag{8.7}
\]

Any solution of the hermitian Yang-Mills equations is automatically a solution of the Spin(7)-instanton equations, but not conversely. Recall that \([20]\) any connection \(A\) on the bundle \(E\) which satisfies (8.7) defines a pseudo-holomorphic structure \(\tilde{\mathcal{J}}_A\) on \(E\). In the case of an integrable almost complex structure \(\tilde{\mathcal{J}}\), such hermitian Yang-Mills connections \(A\) define (semi)stable holomorphic vector bundles \(E \to X^8\) \([34]\).

\(^6\)One can omit the word ‘almost’ if \(\Sigma\) is closed \([36, 37]\).

\(^7\)For an integrable almost complex structure \(\tilde{\mathcal{J}}\) this defines a Calabi-Yau four-fold.
**SU(4)-structures on** \( G/H \times \mathbb{R}^2 \). Consider the manifold \( G/H \times \mathbb{R}^2 \), where the reductive six-dimensional coset \( G/H \) is endowed with an SU(3)-structure \((\omega, \Omega)\), an almost hermitian metric \( g \), and a compatible never integrable almost complex structure \( J \). The ensuing considerations hold not only on the plane \( M = \mathbb{R}^2 \) but also on the cylinder \( \mathbb{R} \times S^1 \) and on the torus \( T^2 = S^1 \times S^1 \). On the eight-manifold

\[
X^8 = G/H \times \mathbb{R}^2 
\]

we introduce an almost complex structure \( \tilde{J} = (J, i) \), where \( j \) is the canonical (integrable) almost complex structure on the plane \( \mathbb{R}^2 \) with coordinates \( x^7, x^8 \), i.e. \( j \) is defined so that

\[
\Theta^4 := dz^4 = dx^7 + i dx^8
\]

is a \((1,0)\)-form on \( \mathbb{R}^2 \cong \mathbb{C} \).

On \( (8.8) \) we introduce non-degenerate forms

\[
\tilde{\Omega} = \Omega \wedge \Theta^4 = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4, \quad (8.10)
\]

\[
\tilde{\omega} = \omega + \frac{i}{2} \Theta^4 \wedge \Theta^4 = \frac{i}{2} (\Theta^1 \wedge \Theta^4 + \Theta^2 \wedge \Theta^2 + \Theta^3 \wedge \Theta^3 + \Theta^4 \wedge \Theta^4),
\]

and the metric

\[
\tilde{g} = g + \Theta^4 \otimes \Theta^4 = \Theta^4 \otimes \Theta^4 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3 + \Theta^4 \otimes \Theta^4.
\]

This makes \( G/H \times \mathbb{R}^2 \) into an eight-dimensional riemannian manifold with an SU(4)-structure. Hence on the manifold \( (8.8) \) one can introduce both the Spin(7)-instanton equations \((8.4)\) as well as the hermitian Yang-Mills equations \((8.7)\).

**Quiver vortex equations on** \( \mathbb{R}^2 \). Let \( E \to G/H \times \mathbb{R}^2 \) be a \( G \)-equivariant complex vector bundle of rank \( N \) and degree zero, and let \( A \) be an \( su(N) \)-valued \( G \)-equivariant connection on \( E \) with curvature \( F \), described by \((6.6)\) and \((6.12)\). After substitution of the components from \((6.14)\) into the Spin(7)-instanton equations \((8.6)\), we obtain non-abelian vortex-type equations

\[
F_{\bar{z}z} = \left( -i \bar{C}_{\alpha}^i \bar{I}_i + \bar{C}_{\alpha}^\gamma \bar{Y}_\gamma + \bar{C}_{\alpha}^\eta \bar{Y}_\eta - \left[ \bar{Y}_\alpha , \bar{Y}_\alpha \right] \right) \delta^{\alpha \bar{\alpha}},
\]

\[
\partial_z \bar{Y}_\alpha + [A_z , \bar{Y}_\alpha] = -\frac{1}{2} \varepsilon_{\alpha \beta \gamma} \left( -i \bar{C}_{\beta}^i \bar{I}_i + \bar{C}_{\beta}^\gamma \bar{Y}_\gamma + \bar{C}_{\beta}^\eta \bar{Y}_\eta - \left[ \bar{Y}_\beta , \bar{Y}_\gamma \right] \right),
\]

where we use a rescaled basis \( \tilde{\Theta}^\alpha \) and the structure constants \((2.36)\). Here \( z := z^4 = x^7 + i x^8 \). Note that the right-hand side of \((8.13)\) is just the moment map of \((7.18)\) after rescaling, while the right-hand side of \((8.14)\) coincides with the relations in \((7.17)\).

In the example \( G/H = \mathbb{F}_3 = SU(3)/U(1) \times U(1) \), we substitute the explicit values \((3.44)\) of \( \bar{C}_{\alpha}^A_{BC} \) into \((8.13)\)–\((8.14)\) to get

\[
F_{\bar{z}z} = \frac{1}{\sqrt{3}} \left( \varsigma_1 + \varsigma_2 - 2 \varsigma_3 \right) \bar{I}_7 - \frac{i}{4} \left( \varsigma_1^2 - \varsigma_2^2 \right) \bar{I}_8 - \delta^{\alpha \bar{\alpha}} \left[ \bar{Y}_\alpha , \bar{Y}_\alpha \right],
\]

\[
\partial_z \bar{Y}_\alpha + [A_z , \bar{Y}_\alpha] = -\frac{1}{2} \varepsilon_{\alpha \beta \gamma} \left( \frac{2}{\sqrt{3}} \varepsilon_{\beta \gamma \delta} \varsigma_\delta \bar{Y}_\delta + \left[ \bar{Y}_\beta , \bar{Y}_\gamma \right] \right).
\]

In the nearly Kähler limit \( \varsigma_1 = \varsigma_2 = \varsigma_3 =: \varsigma \), these equations reduce to

\[
F_{\bar{z}z} = -\delta^{\alpha \bar{\alpha}} [Y_\alpha, Y_\alpha],
\]

\[
\partial_z Y_\alpha + [A_z, Y_\alpha] = -\frac{1}{2} \varepsilon_{\alpha \beta \gamma} \left( \frac{2}{\sqrt{3}} \varepsilon_{\beta \gamma \delta} Y_\delta + [Y_\beta, Y_\gamma] \right).
\]
It is instructive to compare these equations with those which arise when using a Kähler structure on $\mathbb{F}_3$; in that case one should instead substitute the structure constants (3.28)–(3.29) into (8.13)–(8.14) and use unscaled Higgs fields $Y_\alpha$. Then for the vortex equations in the Kähler case we obtain

$$F_{z\bar{z}} = \frac{i\sqrt{3}}{4} \mathcal{I}_7 - \delta^{\alpha\bar{\alpha}} [Y_\alpha, Y_{\bar{\alpha}}] ,$$  \hspace{1cm} (8.19)

$$\partial_z Y_\alpha + [A_z, Y_\alpha] = 0 ,$$  \hspace{1cm} (8.20)

$$\frac{1}{\sqrt{6}} Y_3 + [Y_1, Y_2] = 0 .$$  \hspace{1cm} (8.21)

Here we have taken into account that for the Kähler case there is no SU(4)-structure on the manifold $\mathbb{F}_3 \times \mathbb{R}^2$, and therefore one can impose only the hermitian Yang-Mills equations (8.7) on the gauge fields. This results in separate holomorphicity equations (8.20) and holomorphic relations (8.21).

For any concrete choice of quiver, gauge potential (6.3), and matrices (6.2) and (6.9), one can write all of these equations explicitly in terms of $A^r$ and $\phi_{N r}$.

\textbf{$\mathcal{Q}^{0,1}$-vortex equations.} For the special case (6.17)–(6.25), the non-abelian coupled vortex equations (8.13)–(8.14) are

$$F_{z\bar{z}}^1 = (\varsigma_1^2 - \varsigma_2^2) 1_{N_1} - \varsigma_1^2 \phi_1 \phi_1^\dagger + \varsigma_2^2 \phi_2 \phi_2^\dagger ,$$  \hspace{1cm} (8.22)

$$F_{z\bar{z}}^2 = (\varsigma_2^2 - \varsigma_3^2) 1_{N_2} - \varsigma_2^2 \phi_2 \phi_2^\dagger + \varsigma_3^2 \phi_3 \phi_3^\dagger ,$$  \hspace{1cm} (8.23)

$$F_{z\bar{z}}^3 = (\varsigma_3^2 - \varsigma_1^2) 1_{N_3} - \varsigma_3^2 \phi_3 \phi_3^\dagger + \varsigma_1^2 \phi_1 \phi_1^\dagger ,$$  \hspace{1cm} (8.24)

$$\partial_z \phi_1 + A_z^1 \phi_1 - \phi_1 A_z^1 = \frac{\varsigma_1 \varsigma_2}{\varsigma_1} (\phi_1 - \phi_2 \phi_2^\dagger ) ,$$  \hspace{1cm} (8.25)

$$\partial_z \phi_2 + A_z^2 \phi_2 - \phi_2 A_z^2 = \frac{\varsigma_2 \varsigma_3}{\varsigma_2} (\phi_2 - \phi_3 \phi_3^\dagger ) ,$$  \hspace{1cm} (8.26)

$$\partial_z \phi_3 + A_z^3 \phi_3 - \phi_3 A_z^3 = \frac{\varsigma_3 \varsigma_1}{\varsigma_3} (\phi_3 - \phi_1 \phi_1^\dagger ) .$$  \hspace{1cm} (8.27)

Let us denote by $\mathcal{D}_A$ the Dolbeault operator of the vector bundle $E^{0,1}$ over $M = \mathbb{R}^2$, acting on $\phi_r$ in (8.25)–(8.27). From these equations we see that the Higgs fields $\phi_r$, defining homomorphisms between the bundles $E_r \to M$ for $r = 1, 2, 3$, are not holomorphic. In fact, $\mathcal{D}_A \phi_s$ is proportional to the quiver relation $\phi(r_s)$ defined in (7.24), i.e. one has

$$\mathcal{D}_A \phi_s = \phi(r_s) \kappa_s ,$$  \hspace{1cm} (8.28)

with $\kappa_1 = \varsigma_1^{-1} \varsigma_2 \varsigma_3 \, d\bar{z}$, $\kappa_2 = \varsigma_1 \varsigma_2^{-1} \varsigma_3 \, d\bar{z}$, and $\kappa_3 = \varsigma_1 \varsigma_2 \varsigma_3^{-1} \, d\bar{z}$. This reflects the fact that the almost complex structure $\mathcal{J}$ on $\mathbb{X}^8 = \mathbb{F}_3 \times \mathbb{R}^2$ is not integrable, the bundle $\mathcal{E} \to \mathbb{X}^8$ is not holomorphic, and therefore $\mathcal{F}^{0,2}$ does not vanish. Of course, for the SU(3)-equivariant gauge connection $A$ on $\mathcal{E}$ one can impose the hermitian Yang-Mills equations (8.7) which are stronger than the Spin(7)-instanton equations. Then one obtains the quiver vortex equations with relations

$$\mathcal{D}_A \phi_s = 0 \quad \text{and} \quad \phi(r_s) = 0$$  \hspace{1cm} (8.29)

plus the equations (8.22)–(8.24). However, there exist solutions to (8.22)–(8.27) which do not reduce to solutions of (8.22)–(8.24) and (8.29) [40].
In the nearly Kähler case we have \( \zeta_1 = \zeta_2 = \zeta_3 =: \zeta \) and the equations (8.22)–(8.27) become

\[
F^1_{zz} = -\zeta^2 (\phi_1 \phi_1^\dagger + \phi_2 \phi_2^\dagger) , 
\]

\[
F^2_{zz} = -\zeta^2 (\phi_2 \phi_2^\dagger + \phi_3 \phi_3^\dagger) , 
\]

\[
F^3_{zz} = -\zeta^2 (\phi_3 \phi_3^\dagger - \phi_1 \phi_1^\dagger) , 
\]

\[
\partial_z \phi_1 + A^1_z \phi_1 - \phi_1 A^3_z = \zeta (\phi_1 - \phi_2^\dagger \phi_3^\dagger) , 
\]

\[
\partial_z \phi_2 + A^2_z \phi_2 - \phi_2 A^1_z = \zeta (\phi_2 - \phi_3^\dagger \phi_1^\dagger) , 
\]

\[
\partial_z \phi_3 + A^3_z \phi_3 - \phi_3 A^2_z = \zeta (\phi_3 - \phi_1^\dagger \phi_2^\dagger) . 
\]

For comparison, we note that in the Kähler case the vortex equations determined by the structure constants (3.28) and the hermitian Yang-Mills equations are given by

\[
F^1_{zz} = -\phi_1 \phi_1^\dagger + \phi_2 \phi_2^\dagger , 
\]

\[
F^2_{zz} = -\frac{1}{8} \mathbf{1}_{N_2} - \phi_2 \phi_2^\dagger + \phi_3 \phi_3^\dagger , 
\]

\[
F^3_{zz} = \frac{1}{8} \mathbf{1}_{N_3} - \phi_3 \phi_3^\dagger + \phi_1 \phi_1^\dagger , 
\]

\[
\partial_z \phi_1 + A^1_z \phi_1 - \phi_1 A^3_z = 0 , 
\]

\[
\partial_z \phi_2 + A^2_z \phi_2 - \phi_2 A^1_z = 0 , 
\]

\[
\partial_z \phi_3 + A^3_z \phi_3 - \phi_3 A^2_z = 0 , 
\]

\[
\phi_3 - \sqrt{6} \phi_2 \phi_1 = 0 . 
\]

This system, called a holomorphic triangle in [11], is the basic building block for all quiver vortices associated to the Kähler geometry of the flag variety \( F_3 \).

\( \mathcal{Q}^{k,l} \)-vortex equations. In the nearly Kähler limit of a general \( \mathcal{Q}^{k,l} \)-bundle over \( M = \mathbb{R}^2 \), the vortex equations are obtained by combining the moment map (7.28) with the natural moment map on the invariant symplectic space of unitary connections on the vector bundle \( E_{(q,m)_n} \rightarrow M \), and also the anti-holomorphic components of the bifundamental covariant derivatives (6.43)–(6.45) with the relations (7.25)–(7.27). Using the \( \text{Spin}(7) \)-instanton equations (8.6) and the field strength components (6.36)–(6.39), one thus finds

\[
F^1_{zz}^{(q,m)_n} = \frac{\zeta^2}{24} \mu_{(q,m)_n} , 
\]

\[
D_\zeta \phi^1_{(q,m)_n} = \frac{\zeta}{2^{24} (n+1)^3} \sqrt{\frac{24 (n+1)}{n+1}} \phi (r_{(q,m)_n}^1) , 
\]

\[
D_\zeta \phi^2_{(q,m)_n} = \frac{\zeta}{\sqrt{2} (n+1)^3} \phi (r_{(q,m)_n}^2) , 
\]

\[
D_\zeta \phi^3_{(q,m)_n} = \frac{\zeta}{2^{24} (n+1)^3} \sqrt{\frac{24 (n+1)}{n+1}} \phi (r_{(q,m)_n}^3) , 
\]

at each vertex \((q,m)_n \in W^{k,l}\). As in (7.25)–(7.27), the right-hand sides of the equations (8.44)–(8.46) all vanish in the case that the hermitian Yang-Mills equations (8.7) are satisfied. Then the double quiver relations \( \mathcal{R}^{k,l} \) are automatically fulfilled, and all Higgs fields are holomorphic. These equations can be compared with the non-abelian coupled quiver vortex equations corresponding to the Kähler geometry of \( F_3 \) [11, eqs. (4.19)–(4.23)].
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