Triangulordinary Selmer Groups

Jonathan Pottharst

May 16, 2008

Abstract

Let $p$ be a prime number, and let $K$ be a $p$-adic local field. We study a class of semistable $p$-adic Galois representations of $K$, which we call triangulordinary because it includes the ordinary ones yet allows non-étale behavior in the associated $(\varphi, \Gamma_K)$-modules over the Robba ring. Our main result provides a description of the Bloch–Kato local condition of such representations. We also propose a program, using variational techniques, that would give a definition of the Selmer group along the eigencurve of Coleman–Mazur, including notably its nonordinary locus.

Contents

1 Introduction 2
2 Review of $(\varphi, \Gamma_K)$-modules 4
   2.1 Definitions of many rings 4
   2.2 $\varphi$- and $(\varphi, \Gamma_K)$-modules over the Robba ring 6
   2.3 Galois cohomology 7
   2.4 de Rham theory 8
   2.5 $p$-adic monodromy 9
3 Local theory 12
   3.1 Triangulordinary $(\varphi, \Gamma_K)$-modules and Selmer groups 12
   3.2 Galois descent 13
   3.3 Irrelevance of $\varphi$-structure 17
   3.4 Cohomology of triangulordinary $(\varphi, \Gamma_K)$-modules 17
   3.5 Comparison with Bloch–Kato, ordinary and trianguline 22
   3.6 Examples of triangulordinary representations 24
4 Variational program 29
   4.1 Interpolation of $(\varphi, \Gamma_K)$-modules 29
   4.2 Interpolation of the triangulordinary theory 30
   4.3 Selmer groups via variation 32
   4.4 Review of the eigencurve 34
   4.5 Expectations for the eigencurve 35
1 Introduction

In his seminal work [12], Greenberg laid out a conjectural Iwasawa theory for a motive $M$ at an ordinary prime $p$. His ordinary hypothesis had the effect of drastically simplifying the $p$-adic Hodge theory of $M$, while on the other hand being expected to hold for a dense set of primes $p$.

Although our knowledge began to improve immediately after the time of Greenberg’s work, we learned that Iwasawa theory is, in comparison, very complicated at nonordinary primes. For example, Bloch–Kato found in [5] the right definition of the general Selmer group, and in [18] Perrin-Riou $p$-adically interpolated the Bloch–Kato dual exponential map, providing a close link between Euler systems (which bound Selmer groups) and $p$-adic $L$-functions. While these developments require no ordinary hypothesis, they rely heavily on difficult crystalline techniques, and do not lead to a convenient statement of the main conjecture punctually, let alone variationally.

Much more recently, there has been a major shift in the methods underlying $p$-adic Hodge theory. The work of many people has shown that, essentially, all the important information attached to a $p$-adic representation $V$ of the absolute Galois group $G_K$ of a $p$-adic local field $K$ can be read rather easily from its $(\varphi, \Gamma_K)$-module, an invariant originally associated to $V$ by Fontaine in [11], and subsequently refined by several authors. (See §2 for numerous details and references.) Notably, the $(\varphi, \Gamma_K)$-module of $V$ over the Robba ring may be dissected into subquotients in ways that are not readily visible on the level of the $p$-adic representation $V$ itself. This was first harnessed by Colmez, who called $V$ trianguline if its $(\varphi, \Gamma_K)$-module is a successive extension of 1-dimensional objects, and the latter notion has played a crucial role in our burgeoning understanding of the $p$-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

In this paper we use $(\varphi, \Gamma_K)$-modules to give a natural weakening of Greenberg’s ordinary hypothesis. We identify those representations whose $(\varphi, \Gamma_K)$-module is the same as that of an ordinary representation (except possibly as regards the $\varphi$-slopes of its ordinary filtration), and call them triangulordinary. We show how the $\varphi$-slopes are only rarely ever used when analyzing the $p$-adic Hodge theory of such $V$.

We present two pieces of evidence that our hypothesis is natural and timely. First, as our main result we show that the natural analogue of Greenberg’s Selmer groups coincide with those defined by Bloch–Kato. This generalizes a result of Flach (see [10, Lemma 2]), which was proved using Poitou–Tate local duality and Euler–Poincaré characteristic computations, to the case of arbitrary perfect residue field. Second, we propose a variational program to extend our theory to define the Selmer module of the universal finite-slope eigenform over a dense open subset of the Coleman–Mazur eigencurve (and the eigensurface obtained from it by cyclotomic twisting); such a definition has hitherto been unknown. Our program would encompass results of Kisin, which provide Selmer groups for all the individual overconvergent eigenforms in the family.

After the writing of this article, we found that many of our technical results appear in [1]. The works have slightly different aims, so let us briefly note how they differ. First, throughout [11] one has $K = \mathbb{Q}_p$, so that, in particular, $\varphi$ is a linear operator with well-defined eigenvalues; our theory does not even assume that $K/\mathbb{Q}_p$ is finite, which is necessary for all crystalline representations to be trianguline. As concerns Selmer groups, they only explicitly treat those associated to adjoint representations, by measuring when trianguline deformations are crystalline. Our work is valid even when there is no deformation-theoretic interpretation available. In any case, their
methods can easily show that $H^1_{\text{ord}} \subseteq H_f^1$; we explain when equality holds.

This work would not even have been attempted, were it not for the influence of many people. We owe particular thanks to Laurent Berger and Kiran Kedlaya for introducing us to this subject, and for their patience in explaining its ideas to us. Similarly, we would like to thank the organizers of the 2005 “Atelier sur les Représentations $p$-adiques” at CRM in Montreal, as well as the 2006 “Special Semester on Eigenvarieties” at Harvard—our experiences there incited us to take up a serious study of the ideas required to write this article. We thank Barry Mazur for his enthusiasm and encouragement throughout this project. We are indebted to Ruochuan Liu and Gaëtan Chenevier for extremely helpful conversations. Jan Nekovář arranged for our stay in Paris, during which time much of this work was hammered out. Finally, we heartily thank the NSF for its support through the MSGRFP, under which all this work was completed, and l’Institut de Mathématiques de Jussieu for its hospitality.

Let us conclude by describing the contents of the paper. In the following section, we gather in one place the facts about $(\varphi, \Gamma_K)$-modules, Galois cohomology, and $p$-adic Hodge theory that will be required in the sequel. Our aim is to provide a precise resumé and guide to the literature. In §3 we present our results concerning individual Galois representations. Here the reader will find the definition of triangulordinary representations and proofs of their basic properties, including the comparison of Selmer local conditions. The section concludes by describing the relationship to the notions of ordinary and trianguline, and discussing examples arising in nature, including abelian varieties and modular forms. In the §4 we propose a program to define Selmer modules for general variations of $p$-adic Galois representations, and show how this would apply to the eigencurve and overconvergent $p$-adic modular forms.
2 Review of \((\varphi, \Gamma_K)\)-modules

For this entire section, we fix a complete, discretely valued field \(K\) of characteristic 0, supposed to have a residue field \(k\) that is perfect of characteristic \(p > 0\). Choose once and for all an algebraic closure \(\overline{K}\) of \(K\) and set \(G_K = \text{Gal}(\overline{K}/K)\). Our goal in this section is to review the relevant theory of \((\varphi, \Gamma_K)\)-modules, which provide a means of describing continuous \(p\)-adic representations of \(G_K\) and their associated invariants.

2.1 Definitions of many rings

In terms of our fixed \(K\), we define a dizzying list of objects. Our notation most closely follows that of Colmez; in particular, our \(r\) varies inversely with Berger’s. For any field \(E\), write \(E_n = E(\mu_{p^n})\) for \(n \leq \infty\). If \(E\) carries a valuation, write \(O_E\) for its ring of integers.

**Fields.** Let \(F = \text{Frac} W(k)\) be the fraction field of the Witt vectors of \(k\). Then \(F\) embeds canonically into \(K\) as its maximal absolutely unramified subfield, and \(K/F\) is a finite, totally ramified extension. If \(k'\) denotes the residue field of \(K_{\infty}\), which is finite over \(k\), then we define \(F' = \text{Frac} W(k')\). Then \(F'\) is the maximal unramified extension of \(F\) in \(K_{\infty}\), and \(K'/F'\) is the maximal unramified extension of \(K\) in \(K_{\infty}\), so that \(K_{\infty}/K'\) is totally ramified. Observe that, since \(K' \subset K_{\infty}\), for \(n \gg 0\) one has \(K' \subset K_n\) and hence \(K'_n = K_n\) and \(K'_{\infty} = K_{\infty}\), and therefore \((K')' = K'\).

We set \(H_K = \text{Gal}(\overline{K}/K_\infty)\) and \(\Gamma_K = \text{Gal}(K_\infty/K)\). The latter group is rather simple. If \(E\) is any field of characteristic not equal to \(p\), then the action of \(\text{Gal}(\overline{E}/E)\) on \(\mu_{p^n}(E)\) is described by a uniquely determined character \(\chi_{\text{cycl}} : \text{Gal}(\overline{E}/E) \to \mathbb{Z}^\times_p\), called the cyclotomic character. The fundamental theorem of Galois theory in this case says that \(\chi_{\text{cycl}}\) identifies \(\text{Gal}(E_\infty/E)\) with a closed subgroup of \(\mathbb{Z}^\times_p\). In the case at hand, using the fact that \(K\) is discretely valued, one finds that \(K_{\infty}/K\) is infinite, so that \(\chi_{\text{cycl}}\) identifies \(\Gamma_K\) with an open subgroup of \(\mathbb{Z}^\times_p\), which by force must be procyclic or \(\{\pm 1\} \times \text{(procyclic)}\). One has \(H_K' = H_K\) by our earlier remarks, and \(\Gamma_K'\) has finite index in \(\Gamma_K\). Moreover, one has

\[
\Gamma_K/\Gamma_K' = \text{Gal}(K'/K) = \text{Gal}(F'/F) = \text{Gal}(k'/k).
\]

We have the following diagram, where (for readability) \(\star = \Gamma_K/\Gamma_K'\), “ur” means

\[1\text{One can have } F \subsetneq F' \text{ and } F' \not\subset K! \text{ Take, for example, } p = 3 \text{ and } K = \mathbb{Q}_3(\sqrt{3}).\]
unramified, and “tr” means totally ramified.

I would like to point out to the novice that dividing up $G_K$ into $H_K$ and $\Gamma_K$ is not traditional. Classically, one divides up $G_K$ into $I_K$ and $G_k$, where $I_K \subseteq G_K$ is the inertia subgroup and $G_k = G_K/I_k$ is the absolute Galois group of $k$. (Note that we have a canonical algebraic closure of $k$, namely the residue field $\overline{k}$ of $K$.) In fact, it ends up not being very hard to uncover traditional un/ramification information when using $H_K$ and $\Gamma_K$ instead, so this method is much more powerful, at least in the setting of $p$-adic representations.

Robba rings. There are three main variants of $(\varphi, \Gamma_K)$-modules, but we will only need the variety that live over the Robba ring, so we now make a beeline for these. All we really need is that the “field of norms” construction allows one to make a certain choice of an indeterminate $\pi_K$, and associates to $K$ a constant $e_K (= \text{ord}_{\overline{E}_+}(\pi_K)) > 0$. When $K = F$, there is a canonical uniformizer which is written $\pi$, and one can calculate that $e_F = p/(p - 1)$.

Berger’s Robba ring $B_{\text{rig}, K}^\dagger$ is defined to be the union of the rings $B_{\text{rig}, K}^{\dagger, r}$ for $r > 0$. The latter are defined by

$$B_{\text{rig}, K}^{\dagger, r} = \left\{ f(\pi_K) = \sum_{n \in \mathbb{Z}} a_n \pi_K^n \bigg| f(X) \text{ convergent for } 0 < \text{ord}_p(X) < r/e_K \right\}.$$

Although all these rings are non-Noetherian, they are not too unpleasant. For example, the rings $B_{\text{rig}, K}^{\dagger, r}$ are Bézout domains: they admit a theory of principal divisors, and they have a reasonable theory of finite free modules. See [3, §4.2] for details.

If $L$ is another CDVF with perfect residue field, with $K$ continuously embedded into it, there is a canonical embedding $B_{\text{rig}, K}^{\dagger, r} \hookrightarrow B_{\text{rig}, L}^{\dagger, r}$ for $r$ sufficiently small. More specifically, one can arrange for $\pi_L$ to satisfy an Eisenstein polynomial over a subring of $B_{\text{rig}, K}^{\dagger} \otimes_{F'} F_L'$ with respect to a suitable $\pi_K$-adic valuation. (The term $F_L'$ is the maximal absolutely unramified subfield of $L_{\infty}$, analogous to $F'$.) The constants $e_K$ and $e_L$ are normalized so that the growth conditions on power series coincide. When $L/K$ is finite, we see that the $B_{\text{rig}, K}^{\dagger, r} \subseteq B_{\text{rig}, L}^{\dagger, r}$ (for $r$ sufficiently small), and hence also $B_{\text{rig}, K}^{\dagger} \subseteq B_{\text{rig}, L}^{\dagger}$, are finite ring extensions.

A more delicate construction of these rings (as in [3]) endows them with natural, commuting ring-endomorphism actions of $\Gamma_K$ and an operator $\varphi$. One knows that
\( \varphi \) acts by Witt functoriality on \( a_n \in F' \), and \( \Gamma_K \) acts on \( a_n \) through its quotient \( \Gamma_K/\Gamma_K' = \text{Gal}(F'/F) \). The action on \( \pi_K \) is generally not explicitly given (especially since there is some choice in \( \pi_K \)), except when \( K = F \), in which case \( \varphi(\pi) = (1 + \pi)^p - 1 \) and \( \gamma \in \Gamma_K \) obeys \( \gamma(\pi) = (1 + \pi)^{\chi_{\gamma}} - 1 \). The embeddings \( B_{\text{rig},K}^{\dagger} \subseteq B_{\text{rig},L}^{\dagger} \) are \( \varphi \)- and \( \Gamma_L \)-equivariant (considering \( \Gamma_L \xhookrightarrow{} \Gamma_K \)).

Finally, we point out that the series \( \log(1 + \pi) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \pi^n \) converges in \( B_{\text{rig},\mathbb{Q}_p}^{\dagger} \) for every \( r > 0 \), and we call its limit \( t \). By means of the above embedding process, \( t \) is an element of every \( B_{\text{rig},K}^{\dagger} \). One has \( \varphi(t) = pt \) and \( \gamma(t) = \chi_{\gamma} t \) for all \( \gamma \in \Gamma_K \).

### 2.2 \( \varphi \)- and \( (\varphi, \Gamma_K) \)-modules over the Robba ring

Since many important facts about \( (\varphi, \Gamma_K) \)-modules arise from their underlying \( \varphi \)-modules, we first recall general properties of \( \varphi \)-modules over the Robba ring.

Suppose \( B \) is a ring equipped with a ring endomorphism \( \varphi \). A \( \varphi \)-module over \( B \) is a free, finite rank \( B \)-module \( D \) equipped with a semilinear action of \( \varphi \), satisfying the nondegeneracy condition that \( \varphi(D) \) span \( D \) over \( B \). The adjective “semilinear” indicates that one has \( \varphi(bd) = \varphi(b)\varphi(d) \) for \( b \in B \) and \( d \in D \), rather than \( \varphi(bd) = b\varphi(d) \). We write \( \mathcal{M}(\varphi)/B \) for the category of \( \varphi \)-modules. Unless otherwise specified, we understand that \( \varphi = B_{\text{rig},K}^{\dagger} \).

It is worth noting that, in general, \( \varphi(B_{\text{rig},K}^{\dagger}) \not\subseteq B_{\text{rig},K}^{\dagger} \), but instead \( \varphi(B_{\text{rig},K}^{\dagger}) \subseteq B_{\text{rig},K}^{\dagger} \) (the latter of which contains \( B_{\text{rig},K}^{\dagger} \)). With this in mind, it does not make sense to define a \( \varphi \)-module over \( B_{\text{rig},K}^{\dagger} \). The best we can (and will) ask for is a basis of \( D \) with respect to which the matrix for \( \varphi \) lies in \( B_{\text{rig},K}^{\dagger} \). In this regard, there is the following crucial lemma of Cherbonnier.

**Lemma 2.1 (2 Théorème I.3.3).** For any \( \varphi \)-module \( D \) and any \( r \) sufficiently small, say \( r < r(D) \), there exits a unique \( B_{\text{rig},K}^{\dagger} \)-lattice \( D^r \subset D \) such that \( B_{\text{rig},K}^{\dagger} \cdot \varphi(D^r) \) contains a basis of \( D^r \). One has

\[
B_{\text{rig},K}^{\dagger} \otimes_{B_{\text{rig},K}^{\dagger}} D^r \isomorph B_{\text{rig},K}^{\dagger} \cdot D^r = D^s \subset D
\]

for \( 0 < s \leq r < r(D) \).

We will make heavy use of the slope theory for \( \varphi \)-modules. Currently the best reference for this material is [14], whose proof easily specializes to give the classical Dieudonné–Manin theorem.

**Theorem 2.2 ([14]).** There is an over-ring \( \tilde{B}_{\text{rig}}^{\dagger} \) of \( B_{\text{rig},K}^{\dagger} \otimes F' \tilde{\mathbb{F}}_{\text{unr}} \), with an extension of the operator \( \varphi \) to it, such that the following claims hold.

1. For every \( \varphi \)-module \( D \) over \( \tilde{B}_{\text{rig}}^{\dagger} \) there is a finite extension \( L \) of \( \tilde{\mathbb{F}}_{\text{unr}} \) such that \( D \otimes_{\tilde{\mathbb{F}}_{\text{unr}}} L \) admits a basis of \( \varphi \)-eigenvectors, with eigenvalues in \( L \). The valuations of the eigenvalues, with multiplicity, are uniquely determined by \( D \). (Call them the slopes of \( D \).)

2. There exists a unique filtration \( \text{Fil}^s \subseteq D \) obeying the conditions that:
   - each of the \( \text{Gr}^s \) has only one slope \( s_n \) (but perhaps with multiplicity), and
• $s_1 < s_2 < \cdots < s_t$.

3. If $D$ descends to $B_{\text{rig}, K}^\dagger$, then $\text{Fil}^*$ descends with it uniquely.

One calls a $\phi$-module étale if its only slope is 0, and denotes by $M^{\text{et}}(\phi)/B_{\text{rig}, K}^1 \subset M(\phi)/B_{\text{rig}, K}^1$ the full subcategory of these.

A $(\phi, \Gamma_K)$-module (over $B_{\text{rig}, K}^\dagger$) is a $\phi$-module $D$ equipped with a semilinear action of $\Gamma_K$ that is continuous for varying $\gamma \in \Gamma_K$, and commutes with $\phi$. The category of these is denoted by $M(\phi, \Gamma_K)/B_{\text{rig}, K}^\dagger$. As a consequence of the uniqueness statements in the above theorems about $\phi$-modules, one finds that $\Gamma_K$ stabilizes the lattices $D^r$ and the slope filtration. A $(\phi, \Gamma_K)$-module is called étale if its underlying $\phi$-module is; the category of these is written $M^{\text{et}}(\phi, \Gamma_K)/B_{\text{rig}, K}^\dagger$.

The following theorem, which combines work of many people, is the reason that $(\phi, \Gamma_K)$-modules are important in the study of $p$-adic Galois representations. Let $\text{Rep}_{\mathbb{Q}_p}(G_K)$ denote the category of finite-dimensional $\mathbb{Q}_p$-vector spaces equipped with a continuous, linear action of $G_K$.

**Theorem 2.3.** There is a canonical fully faithful embedding

$$D_{\text{rig}} \colon \text{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow M(\phi, \Gamma_K)/B_{\text{rig}, K}^\dagger,$$

whose essential image is $M^{\text{et}}(\phi, \Gamma_K)/B_{\text{rig}, K}^\dagger$.

**Proof.** The equivalence of Galois representations with étale $(\phi, \Gamma_K)$-modules over the fraction field $B_K$ of the Cohen ring of the field of norms is proved in [11]. The overconvergence of such an object, which means that it can be uniquely defined over $B_{\text{rig}}^1 \subset B_K$, is proved in [6]. The equivalence of an object over $B_K^\dagger$ being étale over $B_K$ and étale over $B_{\text{rig}, K}^\dagger$, as well as the unique descent of an étale object over $B_{\text{rig}, K}^\dagger$ to $B_K^\dagger$, follows from the slope theory of [14].

Let $L$ be another CDVF with perfect residue field, with $K$ continuously embedded into it, and let $D$ be a $B_{\text{rig}, K}^\dagger$-module. We use the shorthand

$$D_L := D \otimes_{B_{\text{rig}, K}^\dagger} B_{\text{rig}, L}^\dagger$$

throughout this article. If $D$ has a $\phi$-action, then so does $D_L$. If $D$ has a $\Gamma_K$-action, then $D_L$ has a $\Gamma_L$-action. Suppose $\overline{L}$ is an algebraic closure of $L$ containing $\overline{K}$, and $G_L = \text{Gal}(\overline{L}/L)$. Then for all $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$, one has $D_{\text{rig}}(V|G_L) = D_{\text{rig}}(V)_L$.

Can one recover the invariants of $V$ from $D_{\text{rig}}(V)$? Indeed, and quite simply, as we recall in the next three sections.

### 2.3 Galois cohomology

Here we recall some results of Liu in [16], which are variants of those of Herr, that allow one to recover the Galois cohomology of $V$ from $D = D_{\text{rig}}(V)$.

Recall that $\Gamma_K \hookrightarrow \mathbb{Z}_p^\times$, and hence is procyclic, except when $p = 2$ and $-1 \in \text{img} \ \Gamma_K$, and in this case $\Gamma_K/\{\pm 1\}$ is procyclic. If $\Gamma_K$ is procyclic, we set $\Delta_K = \{1\}$, and otherwise we set $\Delta_K = \{\pm 1\}$. We let $\gamma \in \Gamma_K/\Delta_K$ denote a topological generator.
In his thesis, Herr associated to $D$ the complex

$$C^\bullet(D) = C^\bullet_{\varphi,\gamma}(D): 0 \to D^{\Delta_K} \xrightarrow{(\varphi - 1, \gamma - 1)} D^{\Delta_K} \oplus D^{\Delta_K} \xrightarrow{(1 - \gamma) \otimes (\varphi - 1)} D^{\Delta_K} \to 0,$$

concentrated in degrees $[0, 2]$. One can easily show that the cohomology groups

$$H^i(D) = H^i(C^\bullet_{\varphi,\gamma}(D))$$

are independent of $\gamma$, and moreover that they are canonically identified with the Yoneda groups:

$$H^i(D) = \text{Ext}^i_{\mathcal{M}(\varphi,\Gamma_K)_{/B_{\text{rig},K}^+}}(1, D),$$

where $1$ denotes the unit object (i.e. $B_{\text{rig},K}^+$ itself as a $(\varphi, \Gamma_K)$-module).

In the case where $D = D_{\text{rig},V}^1$ is étale, we recover continuous Galois cohomology. Namely, by [16, Theorem 2.6] there is a canonical isomorphism of $H^\ast(G_K, V) \cong H^\ast(D_{\text{rig},V}^1)$. In degree $i = 1$, this says that Galois cohomology classes of $V$ are in a natural bijection with extension classes of the $(\varphi, \Gamma_K)$-module $1$ by $D$. It is these extension classes that we will be measuring later in this article.

Similar (yet simpler) statements can be made for $\Gamma_K$-modules without $\varphi$-action: define $C^\bullet(D): 0 \to D^{\Delta_K} \xrightarrow{\gamma - 1} D^{\Delta_K} \to 0$, concentrated in degrees $[0, 1]$. Then the cohomology of this complex is independent of $\gamma$ and computes the Yoneda groups $\text{Ext}^i_{\mathcal{M}(\Gamma_K)_{/B_{\text{rig},K}^+}}(1, D)$, as well as the continuous group cohomology $H^\ast(\Gamma_K, D)$.

### 2.4 de Rham theory

We now explain the link between $(\varphi, \Gamma_K)$-modules and $p$-adic Hodge theory, first exploited by Cherbonnier–Colmez, and later extended to the Robba ring by Berger.

In his thesis, Berger constructed maps $\iota_n: B_{\text{rig},K}^1[p^{-n}] \to K_n'[t]$ for all $n \geq n(K)$. (In particular, one assumes $n$ is large enough so that $K_n'[t] = K_n[t]$.)

There are two ways to describe $\iota_n$. On the one hand, one first proves that $f \in B_{\text{rig},K}^1$ (as in the slope filtration theorem) converges in Fontaine’s ring $B_{\text{ir},K}^+$ if and only if $f \in B_{\text{rig},K}^1$. Write $\iota(f) = \text{img}(f) \in B_{\text{ir},K}^+$. One next shows that $\varphi^{-n}(\tilde{B}_{\text{rig},K}^1[p^{-n}]) \subseteq \tilde{B}_{\text{rig},K}^1$, and the image of $f \in B_{\text{rig},K}^1$ under $\iota_n = \iota \circ \varphi^{-n}$ lies in $K_n'[t]$.

On the other hand, there is the following geometric picture when $K = F$. An element of $B_{\text{rig},F}^1$ is a rigid analytic function on an annulus around $\pi = 0$. One has $r \geq 1$ if and only if $\epsilon^{(1)} - 1$ lies in this annulus. Since $t = t(\pi)$ vanishes to order 1 at every $\pi = \epsilon^{(n)} - 1$, it serves as a uniformizing parameter there. The map $\iota$ corresponds to the operation of taking the formal germ at $\pi = \epsilon^{(1)} - 1$. For $n \geq 0$, the operator $\varphi^{-n}$ stretches the domains of functions towards the center of the disk. After hitting a function by $\varphi^{-1}$ enough times (i.e., if $f \in B_{\text{rig},F}^1$ and we ensure $r \rho^n \geq 1$), the point $\epsilon^{(1)} - 1$ lies in its domain and we may localize there. Another way of saying this is that $\iota_n$ performs completion at $\epsilon^{(n)} - 1$.

Hopefully, the above description motivates the following formulas. When $K = F$, given $f = \sum_{k \in \mathbb{Z}} a_k \pi^k$ with $a_k \in F$, we may calculate $\iota_n(f)$ by means of the following rules:

$$\iota_n(a_k) = \varphi^{-n}(a_k) \quad \text{and} \quad \iota_n(\pi) = \epsilon^{(n)} \exp(t/p^n) - 1.$$
When \( K \) does not necessarily equal \( F \), a general \( f \) has the form \( \sum a_k \pi_K^k \) with \( a_k \in F' \). Then one still has \( \iota_n(a_k) = \varphi^{-n}(a_k) \), but \( \iota_n(\pi_K) \) is not generally explicit.

The \( \iota_n \) are \( \Gamma_K \)-equivariant, and for varying \( n \) they fit into the following diagram.

\[
\begin{array}{ccc}
B_{\text{rig},K}^{\dagger,r} & \xrightarrow{\iota_n} & K_n[t] \\
\varphi \downarrow & & \downarrow \\
B_{\text{rig},K}^{\dagger,r/p} & \xrightarrow{\iota_{n+1}} & K_{n+1}[t]
\end{array}
\]

Now we come to a fundamental construction. Given a \( \varphi \)-module \( D \) over \( B_{\text{rig},K}^{\dagger} \), we associate to it \( D_{\text{diff}}^{+} = D^r \otimes B_{\text{rig},K^{+},n}^{\dagger}\Gamma_K \lbrack t \rbrack \), and \( D_{\text{diff}} = D_{\text{diff}}^{+}[t^-1] \). Using the \( \varphi \)-structure in an essential way, one shows that this definition is independent of the choices of \( r \) and \( n \) satisfying \( r < r(V) \), \( n \geq n(K) \), and \( p^nr \geq 1 \).

If \( D \) is actually a \( (\varphi, \Gamma_K) \)-module, then \( D_{\text{diff}}^{+} \) and \( D_{\text{diff}} \) admit \( \Gamma_K \)-actions. Thus, we are able to define \( D_{\text{diff}}^{(+)} = (D_{\text{diff}}^{+})^{\Gamma_K} \) and \( D_{\text{DR}} = (D_{\text{diff}})_{\Gamma_K} \). These are \( K \)-vector spaces of dimension \( \leq \text{rank } D \), and they carry a decreasing, separated, and exhaustive filtration induced by the \( t \)-adic filtration on \( K_{\infty} \lbrack t \rbrack \). One says that \( D \) is de Rham (resp. +de Rham) if \( \text{dim}_K D_{\text{DR}} = \text{rank } D \) (resp. \( \text{dim}_K D_{\text{DR}}^{+} = \text{rank } D \)), and denotes by \( M_{\text{DR}}(\varphi, \Gamma_K)/B_{\text{rig},K}^{\dagger} \subset M(\varphi, \Gamma_K)/B_{\text{rig},K}^{\dagger} \) the full subcategory of de Rham objects.

**Theorem 2.4** ([3, §5.4]). There exist functorial identifications respecting filtrations, \( D_{\text{dR}}^{+}(V) = (D_{\text{rig}}^{+}(V))_{\text{dR}}^{+} \) and \( D_{\text{dR}}(V) = (D_{\text{rig}}(V))_{\text{dR}} \).

**Corollary 2.5.** A representation \( V \) is de Rham (resp. +de Rham) if and only if its \( (\varphi, \Gamma_K) \)-module is trivialized as semilinear \( \Gamma_K \)-module upon base change to \( K_{\infty} \lbrack t \rbrack \) (resp. \( K_{\infty}[t] \)).

**Remark 2.6.** Therefore, morally, only the existence of a \( \varphi \)-structure is needed in order to construct \( D_{\text{diff}}^{(+)} \), and the property of \( D \) being (+)de Rham is predominantly a condition on the \( \Gamma_K \)-action on \( D \). This observation is the basis of our entire method; see Proposition 5.3 below for a precise statement.

### 2.5 \( p \)-adic monodromy

We will require the following results at a crucial point of the proof of our main theorem, as well as to gain a more down-to-earth picture of its content.

Given a \( (\varphi, \Gamma_K) \)-module \( D \), we write for brevity

\[
D[t^-1] = D \otimes B_{\text{rig},K}^{\dagger} B_{\text{rig},K}[t^-1] \quad \text{and} \quad D[\log(\pi), t^-1] = D \otimes B_{\text{rig},K}^{\dagger} B_{\text{rig},K}[\log(\pi), t^-1],
\]

where \( t \) is the element of \( B_{\text{rig},K}^{\dagger} \) defined at the conclusion of [2.7] and where the element \( \log(\pi) \) is a free variable over \( B_{\text{rig},K}^{\dagger} \) equipped with

\[
\varphi(\log(\pi)) = p \log(\pi) + \log(\varphi(\pi)/\pi^p) \quad \text{and} \quad \gamma(\log(\pi)) = \log(\pi) + \log(\gamma(\pi)/\pi),
\]

\(^3\)Comparing [3, Proposition 5.7 and its proof] with [loc. cit., Corollaire 5.8] makes clear a slight typo in the statement of the proposition; it is the corrected form of the proposition that we use here.
the series $\log(\varphi(\pi)/\pi^p)$ and $\log(\gamma(\pi)/\pi)$ being convergent in $B_{\text{rig},Q_p}^\dagger$. We associate to $D$ the modules

$$D_{\text{crys}} = D[t^{-1}]_{\Gamma K} \quad \text{and} \quad D_{\text{st}} = D[\log(\pi), t^{-1}]_{\Gamma K}.$$  

These two modules are semilinear $\varphi$-modules over $F$, of $F$-dimension $\leq \text{rank } D$. They are related via the so-called monodromy operator $N$. Namely, consider the unique $B_{\text{rig},K}^\dagger$-derivation $N: B_{\text{rig},K}^\dagger[\log(\pi)] \to B_{\text{rig},K}^\dagger[\log(\pi)]$ satisfying $N(\log(\pi)) = -\frac{p}{p-1}$. It satisfies $N\varphi = p\varphi N$ and commutes with $\Gamma_K$, and thus gives rise to an operator $N$ on $D_{\text{st}}$ with the property that $D_{\text{crys}} = D_{\text{st}}^{N=0}$.

We say that $D$ is crystalline (resp. semistable) if $D_{\text{crys}}$ (resp. $D_{\text{st}}$) has the maximal $F$-dimension, namely $\dim_F D_{\text{crys}} = \text{rank } D$ (resp. $\dim_F D_{\text{st}} = \text{rank } D$). In particular, $D$ is crystalline if and only if it is semistable and $N = 0$ on $D_{\text{st}}$. One can show that $D_{\text{st}} \otimes_F K \hookrightarrow D_{\text{dR}}$, so that crystalline implies semistable and semistable implies de Rham. We call $D$ potentially crystalline (resp. potentially semistable) if there exists a finite extension $L/K$ such that $D_L$ is crystalline (resp. semistable), when considered as a $(\varphi, \Gamma_L)$-module. The following statement is known as Berger’s $p$-adic local monodromy theorem.

**Theorem 2.7 ([3]).** Every de Rham $(\varphi, \Gamma_K)$-module is potentially semistable.

The upshot of this theorem is that, whereas in the last section $D_{\text{dR}}$ was merely a filtered $K$-vector space, now we may equip it with much more structure. Given a de Rham $D$, let $L/K$ be finite Galois such that $D_L$ is semistable. Then $(D_L)_{\text{st}}$ is a $(\varphi, N)$-module over the maximal absolutely unramified subfield $F_L$ of $L$, and $(D_L)_{\text{st}} \otimes_{F_L} L = (D_L)_{\text{dR}}$ is a filtered $L$-vector space. Essentially because these data arise via restriction from $K$, they are naturally equipped with a semilinear action of $\text{Gal}(L/K)$ that commutes with $\varphi$ and $N$ and preserves the filtration. Such an object is called a filtered $(\varphi, N, \text{Gal}(L/K))$-module over $K$.

Given two extensions $L_1$ and filtered $(\varphi, N, \text{Gal}(L_1/K))$-modules $D_i$ (for $i = 1, 2$), we consider them equivalent if there exists an extension $L$ containing the $L_i$ such that the $D_i$ tensored up to $L$ are isomorphic. When we consider objects only up to this equivalence, we call them filtered $(\varphi, N, G_K)$-modules, thinking of the $G_K$-action as being through an unspecified finite quotient that determines the field of definition of the underlying vector spaces. We point out that if $D$ becomes semistable over both $L_1$ and $L_2$, then $(D_L)_{\text{st}}$ and $(D_{L_2})_{\text{st}}$ are equivalent. We call this equivalence class $D_{\text{pst}}$.

(To avoid set-theoretic issues, one simply deals with filtered $(\varphi, N, G_K)$-modules whose underlying $\varphi$-module is a vector space over $F^\text{unr}$, and underlying filtered vector space has coefficients in $\overline{K}$, with the assumption that $G_K$ acts discretely.) The category of filtered $(\varphi, N, G_K)$-modules is denoted $\text{MF}(\varphi, N, G_K)$.

Summarizing the $p$-adic monodromy theorem in terms of the above language, if $D$ is de Rham then $D_{\text{pst}}$ determines a filtered $(\varphi, N, G_K)$-module over $K$. Following Fontaine, a filtered $(\varphi, N, G_K)$-module $D_{\text{pst}}$ is called (weakly) admissible if, roughly, its Newton and Hodge polygons have the same endpoints, and all its $(\varphi, N)$-stable submodules satisfy “Newton on or above Hodge”. (See [2, §1.1] for details.)

**Theorem 2.8 ([3][2]).** The functor $D \mapsto D_{\text{pst}}$ is an equivalence of categories:

$$\text{M}^{dR}((\varphi, \Gamma_K))_{/B_{\text{rig},K}^\dagger} \sim \text{MF}(\varphi, N, G_K).$$
A de Rham $(\varphi, \Gamma_K)$-module $D$ is étale if and only if $D_{pst}$ is (weakly) admissible. It is potentially crystalline if and only if $N = 0$ on $D_{pst}$. It is semistable if and only if $D_{pst}$ can be realized as a filtered $(\varphi, N, \text{Gal}(K/K))$-module.

The two equivalent categories above are not abelian categories. In the first category, the coimage of a map $D \to D'$ is the set-theoretic image, while the image is the $t$-saturation of this set (i.e. the elements $x \in D'$ such that some $t^n x$ lies in the set-theoretic image). In the second category, the coimage and image have the same underlying $(\varphi, N, G_K)$-module, but different filtrations. The filtration on the coimage is induced by the surjection from $D$, and the filtration on the image is induced from the inclusion into $D'$. Thus, $t$-saturated $(\varphi, \Gamma_K)$-stable $B_{\text{rig}, K}^\dagger$-submodules of $D$ are in a natural correspondence with subspaces of $D_{pst}$ that are stable under the $(\varphi, N, G_K)$-actions (considered as being equipped with the filtration induced from $D_{pst}$). Furthermore, one can show that a $t$-saturated $B_{\text{rig}, K}^\dagger$-submodule is actually a direct summand, provided that it is $(\varphi, \Gamma_K)$-stable.

Moreover, the proof in [2] explains the following facts. For simplicity, assume that $D$ is crystalline. Consider $D$ and $D_{\text{crys}} \otimes_F B_{\text{rig}, K}^\dagger$ as $B_{\text{rig}, K}^\dagger$-lattices in $D[t^{-1}]$. As one passes from the first to the second, one invokes multiples by various $t^n$ (with $n \in \mathbb{Z}$) in order to trivialize the $\Gamma_K$-action on some basis. But multiplying by $t^n$ shifts $\varphi$-slopes upwards by $n$, and thus, as we change lattices, the $\varphi$-slopes get dragged to new values. But the powers of $t$ involved, which determine the amount of dragging, more directly determine the weights of the Hodge filtration on $D_{\text{crys}}$. So, there is a close (but complicated!) connection between the Hodge–Tate weights on $D_{\text{crys}}$ and the difference between the $\varphi$-slopes on $D$ and the $\varphi$-slopes on $D_{\text{crys}}$. In the case where $D$ is a trivial $\Gamma_K$-module, i.e. $D \approx (B_{\text{rig}, K}^\dagger)^{\oplus d}$ as a $\Gamma_K$-module, one clearly sees that $D$ is crystalline and that there is no change of lattice, so the $\varphi$-slopes on $D$ coincide with the $\varphi$-slopes on $D_{\text{crys}}$. 
3 Local theory

In this section we define triangulordinary representations, and prove that they have many amenable properties. We go on to define Selmer groups of representations that are triangulordinary at \( p \), and give examples.

We continue with the notations set forth in \( \S 2 \).

3.1 Triangulordinary \((\varphi, \Gamma_K)\)-modules and Selmer groups

When in doubt, \( D \) refers to an object in \( \text{M}(\varphi, \Gamma_K)/B_{\text{rig}, K}^1 \). In this subsection we set \( L = \widehat{K}^{\text{unr}} \), and remind the reader of the meaning of \( D_L \): the compositum \( \mathcal{L} := \overline{K}.L \) is an algebraic closure of \( L \), and \( G_L = \text{Gal}(\mathcal{L}/L) \cong I_K \subset G_K \), the inertia subgroup. Thus, for \( V \in \text{Rep}_{\mathbb{Q}_p}(G_K) \), one has \( D_{\text{rig}}^1(V)_L = D_{\text{rig}}^1(V|_{I_K}) \).

We say that \( D \) is triangulordinary if there exists a decreasing, separated and exhaustive, \((\varphi, \Gamma_K)\)-stable filtration \( F^* \subseteq D \) by \( B_{\text{rig}, K}^1 \)-direct summands, such that each \((\text{Gr}^n_F)_L \) is \( \Gamma_L \)-isomorphic to \((t^n B_{\text{rig}, L}^1)^{\oplus d_n} \). The \( d_n \) are called the multiplicities of the weights \( n \) in \( D \); for example, \( D \) has weights \( \geq 1 \) if and only if \( D = F^1 \). For an equivalent definitions see Corollary 3.7 and \( \S 3.5 \) and for examples see \( \S 3.6 \). We do not give these immediately, because discussing them rigorously requires several tools.

Given \( D \) and a triangulordinary filtration \( F^* \), we put

\[
H^1_{\text{ord}}(D) = H^1_{\text{ord}}(D; F^*) = \ker \left[ H^1(D) \to H^1((D/F^1)_L) \right],
\]

and call it the \textit{triangulordinary local condition}. It is possible that \( D \) be triangulordinary with respect to more than one filtration (see \( \S 3.6 \)), hence the need for the “\( F^* \)” in the notation. However, we will usually have a fixed filtration in mind, and therefore we will usually drop it from sight.

Here is the main theorem of this section.

\textbf{Theorem 3.1.} Let \( D \) be triangulordinary with filtration \( F^* \). Then the following claims hold.

1. \( D \) is de Rham (and moreover +de Rham if and only if \( F^1 = 0 \)), and even semistable.

2. Suppose for all \( n \leq 0 \) that \( n - 1 \) is not a \( \varphi \)-slope on \( \text{Gr}_F^n \). Then \( H^1_{\text{ord}}(D; F^*) \) coincides with \( H^1_{\text{rig}}(D) \), defined in \( \S 3.2 \).

We prove this theorem in \( \S 3.3 \); the intervening sections involve preparatory material.

For the remainder of this subsection, let us break with the running notation. Let \( K/\mathbb{Q} \) be a finite extension, and let \( S \) be a finite set of places of \( K \) containing all primes above \( p \) and \( \infty \). Write \( K_S \) for a maximal extension of \( K \) unramified outside \( S \), and \( G_{K,S} = \text{Gal}(K_S/K) \). Choose algebraic closures \( \overline{K}_v \) of \( K_v \), and write \( G_v = \text{Gal}(\overline{K}_v/K) \), for each finite place \( v \in S \); choose embeddings \( K_S \hookrightarrow \overline{K}_v \), which amount to maps \( G_v \to G_{K,S} \). Also, for \( v \in S \) with \( v \nmid p\infty \), denote by \( I_v \subset G_v \) the inertia subgroup, and write \( G_v = G_v/I_v \).

Let \( V \) be a finite-dimensional \( \mathbb{Q}_p \)-vector space equipped with a continuous, linear action of \( G_{K,S} \). Assume that, for each place \( v \) of \( K \) with \( v \mid p \), \( D_{\text{rig}}^1(V|_{G_v}) \) is equipped with a triangulordinary filtration \( F^*_v \). We define the \textit{triangulordinary local conditions}
as above: they are the respective subgroups $H^1_{\text{ord}}(K_v, V)$ corresponding, under the identifications $H^1(K_v, V) \cong H^1(D_{\text{rig}}^+(V|_{K^\text{unr}}), K_v)$, to the subgroups $H^1_{\text{ord}}(D_{\text{rig}}^+(V|_{K^\text{unr}}))$. Then, following the customary pattern, we define the triangulordinary Selmer group to be

$$H^1_{\text{ord}}(K, V) = \ker \left[ H^1(G_{K, S}, V) \to \bigoplus_{v \in S, v \nmid \infty} H^1(K_v, V) \bigoplus \bigoplus_{v \in S, v \mid p} H^1_{\text{ord}}(K_v, V) \right].$$ (2)

After proving Theorem 3.1, we will see how this definition generalizes Selmer groups defined by Greenberg, and agrees with those defined by Bloch–Kato.

### 3.2 Galois descent

In this section and the next we show that the de Rham property is rather flexible: it is easy to equate the validity of this property for one $(\varphi, \Gamma_K)$-module to its validity for another one. The instance in this section concerns the ability to discern that $D$ is de Rham (resp. crystalline), given that the restriction $D_L$ of $D$ to some possibly large overfield $L \supseteq K$ has the same property. I suspect that these facts are known to the experts, but I give precise statements and proofs because they do not appear in the literature in the generality of possibly non-étale $(\varphi, \Gamma_K)$-modules.

By a complete unramified extension $L$ of $K$, we mean the $p$-adic completion of an unramified (but possibly infinite) algebraic extension of $K$. Using an appropriate variant of the Witt vectors formalism, such fields $L$ lie in a natural bijection with algebraic extensions of $k$, and hence to closed subgroups $H$ of $G_K$. When $H$ is normal in $G_K$, we call $L$ normal and set $\text{Gal}(L/K) = G_k$. For example, the maximal complete unramified extension of $K$ is $L = \hat{K}^\text{unr}$, with $\text{Gal}(L/K) = G_k$.

By a complete discretely valued extension $L$ of $K$, we mean a finite extension of a complete unramified extension. These are the same as CDVF’s into which $K$ embeds continuously, such that the embedding induces an algebraic extension of residue fields. The complete discretely valued extensions $L$ of $K$ are in a natural bijection with closed subgroups $H$ of $G_K$ for which $H \cap I_K$ has finite index in $I_K$. If $H$ is normal in $G_K$, we call $L$ normal and we set $\text{Gal}(L/K) = G_K/H$. The class of complete discretely valued extensions is closed under finite composita, but does not admit a maximal element.

**Proposition 3.2.** Let $D \in M(\varphi, \Gamma_K)/B_{\text{rig}}^+(K)$, and let $L$ be a complete discretely valued extension of $K$. Then $\dim_K D_{dR}^{(+)} = \dim_L (D_L)_{dR}^{(+)}$.

**Remark 3.3.** In the down-to-earth terms of a Galois representation $V$, the proposition says that the de Rham periods of $V$ essentially only depend on the action upon $V$ of an arbitrarily small finite index subgroup of the inertia group $I_K \subseteq G_K$.

**Proof.** It is clear that $D_{dR}^{(+) \otimes_K L} \subseteq (D_L)_{dR}^{(+) \otimes_K L}$, which shows the inequality $\leq$. This proof consists of showing the reverse inequality. Notice that, for any particular $L$, and any complete discretely valued extension $L'/L$, we know the inequality $\leq$ for $L'/L$, and if we know $\geq$ for the extension $L'/K$, then we know as a result the inequality $\geq$ for $L/K$ and $L'/L$. Therefore, it suffices to prove the proposition with $L'$ in place of $L$, i.e. it
never hurts to enlarge the $L$ in question. In particular, by passing to the (completed) normal closure, we may assume that $L/K$ is normal.

We first establish the proposition in the case when $L/K$ is finite. The idea is to harness the ability to enlarge $L$ in order to really only treat two cases: when $L$ and $K_\infty$ are linearly disjoint over $K$, and when $L \subseteq K_\infty$. Let $L_0 = L \cap K_\infty$, so that $L$ and $K_\infty$ are linearly disjoint over $L_0$ (because $L$ and $K_\infty$ are both normal over $K$).

We treat the extension $L/L_0$ first. We have

$$(D_L)^{(+)}_{\text{diff}} = (D_{L_0})^{(+)}_{\text{diff}} \otimes_{L_0, \infty}[t] L_\infty[t].$$

Notice that $\text{Gal}(L/L_0)$ acts only on the right hand factor of this expression, and it commutes with the $\Gamma_L$-action (since $L$ and $K_\infty$ are linearly disjoint over $L_0$). Thus, the $\text{Gal}(L/L_0)$-invariants of $(D_L)^{(+)}_{\text{diff}}$ are the $\Gamma_L = \Gamma_{L_0}$-module $(D_{L_0})^{(+)}_{\text{diff}}$. In other words, $(D_{L_0})^{(+)}_{\text{dr}} = ((D_{L_0})^{(+)}_{\text{dr}})^{\text{Gal}(L/L_0)}$. But $(D_L)^{(+)}_{\text{dr}}$ is an $L$-vector space of dimension rank $D$ equipped with a continuous, semilinear action of $\text{Gal}(L/L_0)$, and Hilbert’s Theorem 90 for finite, Galois field extensions implies that all finite-dimensional, semilinear $\text{Gal}(L/L_0)$-modules over $L$ are trivial. Thus, $(D_L)^{(+)}_{\text{dr}}$ admits a basis of elements fixed under $\text{Gal}(L/L_0)$, which shows that $(D_{L_0})^{(+)}_{\text{dr}}$ has the same dimension as dimension $(D_L)^{(+)}_{\text{dr}}$, and we have $\geq$ in this case.

We may now assume that $L = L_0$, so that $L$ is contained in $K_\infty$. But now we have $(D_L)^{(+)}_{\text{diff}} = (D_{L_0})^{(+)}_{\text{diff}}$ as $K_\infty[t]$-modules, and the $\Gamma_L$-action on the left hand side obtained by restricting the $\Gamma_K$-action on the right hand side. Since $\Gamma_K$ is abelian, its action on $(D_L)^{(+)}_{\text{diff}}$ commutes with the $\Gamma_L$-action and induces a semilinear $\Gamma_K/\Gamma_L = \text{Gal}(L/K)$-action over $L$ on $(D_L)^{(+)}_{\text{diff}}$. We again apply Hilbert’s Theorem 90 to deduce the desired inequality.

Now we turn to the infinite case, and make use of the proposition in the finite case to simplify things. Suppose we are given a tower $K \subseteq L_0 \subseteq L$, with $L_0/K$ complete unramified and $L/L_0$ finite. Applying the finite case to $L/L_0$, we see that we have equality for $L_0/K$ if and only if we have equality for $L/K$. Thus we are reduced to proving the proposition when $L = L_0$, so that $L/K$ is complete unramified.

Next consider the extension $K'/K$. Since it is finite unramified, the extension $(K'/L)/L$ is also finite unramified. By the finite case of the proposition, we have equality for the extension $(K'/L)/L$, so we are reduced to the case where $L$ contains $K'$. Considering the tower $K \subseteq K' \subseteq L$, we have equality for $L/K$ if and only if we have equality for $L/K'$. Thus we can assume that $K = K'$ and $L/K$ is complete unramified.

Note that $K_\infty/K$ is now totally ramified, and hence also linearly disjoint with (any algebraic subextension of) $L/K$. This implies the following facts. First, $\Gamma_L = \Gamma_K$. Moreover, the actions of $\Gamma_L$ and $\text{Gal}(L/K)$ on $L_\infty$ commute with one another. Finally, we infer that $L_\infty/L$ is totally ramified, or, equivalently, that $L' = L$ (the left hand side being the maximal unramified extension of $L$ in $L_\infty$).

Consider the module $(D_L)^{(+)}_{\text{diff}}$, which can be written as $(D_{L_0})^{(+)}_{\text{diff}} \otimes_{K_\infty}[t] L_\infty[t]$. Notice that $\text{Gal}(L/K)$ acts only on the right hand factor of this expression for $(D_L)^{(+)}_{\text{diff}}$, and it commutes with the $\Gamma_L$-action. Thus, the $\text{Gal}(L/K)$-invariants of $(D_L)^{(+)}_{\text{diff}}$ are the $\Gamma_L = \Gamma_{L_0}$-module $(D_{L_0})^{(+)}_{\text{diff}}$. In other words, $D_{L_0}^{(+)} = ((D_{L_0})^{(+)}_{\text{dr}})^{\text{Gal}(L/K)}$. So, $(D_L)^{(+)}_{\text{dr}}$ is an $L$-vector space with a continuous, semilinear action of $\text{Gal}(L/K)$. Invoking Hilbert’s
Our main use of the proposition is in the case of extension classes. Let us describe how this occurs. Because the cohomology groups $H^*(D)$ defined in 2.3 coincide with Yoneda groups, to every $c \in H^1(D)$ there corresponds a class of extensions

$$0 \to D \to E_c \to 1 \to 0.$$ 

Since the functor $D \mapsto D_{\text{dif}}^{(+)}$ corresponds to changing the base rings of finite free objects (over Bezout domains), it preserves short exact sequences. Therefore, we can hit the exact sequence above with this functor to obtain an exact sequence of $\Gamma_K$-modules over $K_\infty[[t]]$ or $K_\infty((t))$. The we obtain thus a map $H^1(D) \to H^1(\Gamma_K, D_{\text{dif}}^{(+)}(1))$ by $[E_c] \mapsto [(E_c)_{\text{dif}}^{(+)}]$.

Also, if $L/K$ is an extension as in the proposition above, then $[E_c] \mapsto [(E_c)_L]$ defines a map denoted $\alpha_L: H^1(D) \to H^1(D_L)$.

The Bloch–Kato “$g(\pm)$” local condition is the subgroup of $H^1(D)$ determined by

$$H^1_g(\pm)(D) = \ker \left[ H^1(D) \to H^1(\Gamma_K, D_{\text{dif}}^{(+)}(1)) \right].$$

Proposition 3.2 now gives the following descent result for these subgroups.

**Corollary 3.4.** Let $D$ be a $(\varphi, \Gamma_K)$-module, and let $L/K$ be an extension, as in the preceding theorem. Then $D$ is $(+)$ de Rham if and only if $D_L$ is $(+)$ de Rham (the latter considered as a $(\varphi, \Gamma_L)$-module).

**Proof.** By the theorem, $\dim_K D_{\text{dif}}^{(+)} = \dim D$ if and only if $\dim_L(D_L)_{\text{dif}}^{(+)} = \dim D$.

Our main use of the proposition is in the case of extension classes. Let us describe how this occurs. Because the cohomology groups $H^*(D)$ defined in 2.3 coincide with Yoneda groups, to every $c \in H^1(D)$ there corresponds a class of extensions

$$0 \to D \to E_c \to 1 \to 0.$$ 

Since the functor $D \mapsto D_{\text{dif}}^{(+)}$ corresponds to changing the base rings of finite free objects (over Bezout domains), it preserves short exact sequences. Therefore, we can hit the exact sequence above with this functor to obtain an exact sequence of $\Gamma_K$-modules over $K_\infty[[t]]$ or $K_\infty((t))$. The we obtain thus a map $H^1(D) \to H^1(\Gamma_K, D_{\text{dif}}^{(+)}(1))$ by $[E_c] \mapsto [(E_c)_{\text{dif}}^{(+)}]$.

Also, if $L/K$ is an extension as in the proposition above, then $[E_c] \mapsto [(E_c)_L]$ defines a map denoted $\alpha_L: H^1(D) \to H^1(D_L)$.

The Bloch–Kato “$g(\pm)$” local condition is the subgroup of $H^1(D)$ determined by

$$H^1_{g(\pm)}(D) = \ker \left[ H^1(D) \to H^1(\Gamma_K, D_{\text{dif}}^{(+)}(1)) \right].$$

Proposition 3.2 now gives the following descent result for these subgroups.

**Corollary 3.5.** For any $D$ and $L/K$ as in the statement of Proposition 3.2 one has $H^1_{g(\pm)}(D) = \alpha_L^{-1} H^1_{g(\pm)}(D_L)$, where $\alpha_L$ is defined above.

In particular, taking $L = K_{\text{unr}}$, we see that $H^1_{g(\pm)}(D) := \ker \alpha_{K_{\text{unr}}} \subseteq H^1_{g(\pm)}(D)$.

**Proof.** A $\Gamma_K$-fixed vector of $(E_c)_{\text{dif}}^{(+)}$ not belonging to the subspace $D_{\text{dif}}^{(+)}$ is the same thing as a $\Gamma_K$-equivariant splitting of the map $(E_c)_{\text{dif}}^{(+)} \to 1_{\text{dif}}^{(+)}$. Thus, $(E_c)_{\text{dif}}^{(+)}$ is $\Gamma_K$-split if and only if $\dim_K (E_c)_{\text{dif}}^{(+)} = \dim_K D_{\text{dif}}^{(+)} + 1$. By the theorem, this holds if and only if the corresponding claim holds with $K$, $D$, and $E_c$ replaced by $L$, $D_L$, and $(E_c)_L$, respectively. Thus $(E_c)_{\text{dif}}^{(+)}$ is $\Gamma_K$-split if and only if $((E_c)_L)_{\text{dif}}^{(+)}$ is $\Gamma_L$-split. In other words, we have $c \in H^1_{g(\pm)}(D)$ if and only if $\alpha_L(c) \in H^1_{g(\pm)}(D_L)$, as was desired.

Suppose that $D$ is $(+)$ de Rham, and $c \in H^1(D)$. We see from the above proof that $(E_c)_{\text{dif}}^{(+)}$ is $\Gamma_K$-split if and only if

$$\dim_K (E_c)_{\text{dif}}^{(+)} = \dim_K D_{\text{dif}}^{(+)} + 1 = \dim D + 1 = \dim E,$$

which occurs if and only if $E_c$ is itself $(+)$ de Rham. Thus, in this case, the Bloch–Kato “$g(\pm)$” local condition can be interpreted as the subgroup of $H^1(D)$ determined by

$$H^1_{g(\pm)}(D) = \{ c \in H^1(D) \mid E_c \text{ is } (+) \text{ de Rham} \}.$$
We also prove a descent result for the crystalline property. Notice that it includes the case of finite unramified extensions $L/K$. For a complete discretely valued extension $L/K$, we write $F_L$ for the maximal absolutely unramified subfield of $L$, i.e. the field that would be called $F$ if we replaced $K$ with $L$.

**Proposition 3.6.** Let $D \in \mathcal{M}(\varphi, \Gamma_K)_{\B_{\text{rig}} \otimes K}$, and let $L/K$ be a complete unramified extension. Then $\dim_F D_{\text{crys}} = \dim_{F_L} (D_L)_{\text{crys}}$. In particular, $D$ is crystalline if and only if $D_L$ is.

**Proof.** Since $D_{\text{crys}} \otimes_F F_L \rightarrow (D_L)_{\text{crys}}$, we show the inequality $\geq$.

We may make reductions just as in the proof of Proposition 3.2, namely, it suffices to assume that $L$ is Galois over $K$ and contains $K'$, and to just treat independently the cases where $K = K'$ and $L = K'$.

Assume $L = K'$. Then since $(K')' = K'$ (see [2, 1]), one has $D_{K'} = D$ as sets, and we see that

$$(D_{K'})_{\text{crys}} = D_{K'}[t^{-1}]\Gamma_{K'} = D[t^{-1}]\Gamma_{K'}$$

is a semilinear $\Gamma_K/\Gamma_{K'} = \text{Gal}(F'/F)$-module over $F'$ satisfying $(D_{K'})_{\text{crys}}/\Gamma_{K'} = D_{\text{crys}}$. It must be trivial, by Hilbert’s Theorem 90, and hence $D_{\text{crys}}$ has the desired $F$-dimension.

Now assume $K = K'$. Since $K_{\text{Gal}}/K$ is totally ramified, and $L/K$ is unramified, the two extensions are linearly disjoint. In particular, $\Gamma_L = \Gamma_K$ and $L' = L$. We have

$$D_L[t^{-1}] = D[t^{-1}] \otimes_{\B_{\text{rig}} \otimes K} \B_{\text{rig}, L}$$

and by linear disjointness the $\text{Gal}(L/K)$-action on the right hand factor commutes with the $\Gamma_K$-action on the tensor product. This, combined with the fact that $\Gamma_L = \Gamma_K$, shows that $(D_L)_{\text{crys}} = D$. We know that $(D_L)_{\text{crys}}$ is a semilinear $\text{Gal}(L/K) = \text{Gal}(F_L/F)$-module over $F_L$. Applying Hilbert’s Theorem 90, it admits a basis of invariants, and hence $D_{\text{crys}}$ has the desired rank.

**Corollary 3.7.** Write $L = \hat{K}_{\text{unr}}$. For a $(\varphi, \Gamma_K)$-module $D$, the following claims are equivalent:

- $D$ is $\Gamma_K$-isomorphic to $(t^n B_{\text{rig}, K})^\oplus d$.
- $D_L$ is $\Gamma_L$-isomorphic to $(t^n B_{\text{rig}, L})^\oplus d$.
- $D$ is crystalline, and all its Hodge–Tate weights equal $n$.

**Proof.** The first condition clearly implies the second.

Assuming the second condition, $D_L$ is crystalline (since $(D_L)_{\text{crys}} = (t^n D_L)^{\Gamma_L}$ is clearly large enough), whence Proposition 3.6 shows that $D$ is crystalline. And $D_{\text{dif}}(D) = (t^n F_{\infty}[t])^\oplus d$, showing that the Hodge–Tate weights are all $n$.

Now assume that the third condition holds. The fact that $D$ is $\Gamma_K$-isomorphic to $(t^n B_{\text{rig}, K})^\oplus d$ results directly from a check of the construction $D_{\text{pstr}} \mapsto D$ given in [2, §III].

This corollary allows us to restate the condition that a filtration $F^* \subseteq D$ be triangular ordinary. The hypothesis becomes: each $\text{Gr}^E_D$ is crystalline, with all Hodge–Tate weights equal to $n$. 

16
3.3 Irrelevance of $\varphi$-structure

Next we state and prove a precise version of Remark 2.6.

**Proposition 3.8.** The $\Gamma_K$-isomorphism class of $D_{\text{d}i\text{f}}^+$ does not depend on which $\varphi$-structure $D$ is equipped with (although the existence of $\varphi$ is necessary to define $D_{\text{d}i\text{f}}^+$). The same claim holds for $D_{\text{d}i\text{f}}$.

**Proof.** The construction of $D^r$ in the proof of [2, Théorème I.3.3] shows that, for any $B_{\text{rig},K}^+$-basis $e = \{e_1, \ldots, e_d\}$, there exists $0 < r(e) < r(D)$ such that the $\Gamma_K$-action on $e$ is defined over $B_{\text{rig},K}^{1,r(e)}$ and if $0 < r \leq r(e)$ then $D^r$ is $B_{\text{rig},K}^{1,r}$-spanned by $e$. Moreover, all the modules $D^r \otimes_{B_{\text{rig},K}^{1,r}} K_\infty[t]$ for $0 < r < r(D)$, $n \geq n(K)$, and $rp^n \geq 1$ are isomorphic as $\Gamma_K$-modules. Thus, if we consider $D_{\text{d}i\text{f}}^+$ as the $K_\infty[t]$-span of $e$ with $\Gamma_K$-action piped through $\iota_n$, then the resulting isomorphism class stabilizes for $n \gg 0$. We simply take $n$ large enough to ensure this. (In other words, $\varphi$ only guarantees that the $\Gamma_K$-structures are equivalent and chooses the equivalence, but they are all equivalent no matter which $\varphi$ is used to show it.)

The claim for $D_{\text{d}i\text{f}}$ follows from the claim for $D_{\text{d}i\text{f}}^+$ after inverting $t$. \hfill \qed

**Corollary 3.9.** If $D$ and $D'$ are two $(\varphi,\Gamma_K)$-modules, and $D \cong D'$ as $\Gamma_K$-modules, then $D$ is $(+)$de Rham if and only if $D'$ is $(+)$de Rham.

We conclude this section by applying the Proposition 3.8 to $(\varphi,\Gamma_K)$-modules $D$ having the property that, as $\Gamma_K$-modules, they are isomorphic to $(t^n B_{\text{rig},K}^+) \otimes_d$. Such $D$ is crystalline, with $D_{\text{crys}} = (t^{-n}D)^{\Gamma_K}$ a $\varphi$-module over $F$ from which we can recover $D$ completely: $D = t^n B_{\text{rig},K}^+ \otimes_F D_{\text{crys}}$.

Although we know being crystalline implies being de Rham, one can also see that $D$ is de Rham by way of the proposition: $D_{\text{d}i\text{f}}^+$ is $\Gamma_K$-isomorphic to $(t^n K_\infty[t]) \otimes_d$, and therefore $D_{\text{d}i\text{f}}$ is $\Gamma_K$-isomorphic to $K_\infty((t)) \otimes_d$, which clearly has enough $\Gamma_K$-invariants. Moreover, one finds that

$$\dim_K H^q(\Gamma_K, D_{\text{d}i\text{f}}^+) = d \cdot \dim_K H^q(\Gamma_K, t^n K_\infty[t]) = \begin{cases} d & \text{if } n \leq 0 \\ 0 & \text{if } n \geq 1 \end{cases} \quad (3)$$

and

$$\dim_K H^q(\Gamma_K, D_{\text{d}i\text{f}}) = d \cdot \dim_K H^q(\Gamma_K, t^n K_\infty((t))) = d, \quad (4)$$

for both $q = 0, 1$.

3.4 Cohomology of triangulordinary $(\varphi,\Gamma_K)$-modules

As in [3.1], we set $L = \overline{K^\text{unr}}$. In this subsection we prove Theorem 3.1.

**Proof of Theorem 3.1.** We first apply the techniques of Galois descent.

We assume the theorem holds for $K = L$. Given a triangulordinary $D$, the theorem over $L$ shows that $D_L$ is $(+)$de Rham, and therefore by Corollary 3.4 we know that $D$ is $(+)$de Rham. Moreover, since the $(\text{Gr}_F^n)_L$ are unconditionally crystalline by the comments concluding [3.3] we can apply Proposition 3.6 to deduce that the $\text{Gr}_F^n$
themselves are crystalline, and hence semistable. Since $D$ is simultaneously de Rham and a successive extension of semistable pieces, [3 Théorème 6.2] asserts that $D$ is semistable. (Note that its proof applies without change to the non-étale case.) Thus we have proved: if $D_L$ is (+)de Rham, then $D$ is (+)de Rham and even semistable, in all cases.

By Corollary 3.5 one has

$$H^1_{g(+)}(D) = \alpha_L^{-1} H^1_{g(+)}(D_L).$$

On the other hand, by its very definition,

$$H^1_{\text{vord}}(D; F^*) = \alpha_L^{-1} H^1_{\text{vord}}(D_L; F^*_L).$$

Therefore, if $H^1_{\text{vord}}(D_L; F^*_L) = H^1_{g(+)}(D_L)$, then $H^1_{\text{vord}}(D; F^*) = H^1_{g(+)}(D)$.

The upshot is that we only need to prove (2) and the (+)de Rham claim of (1), assuming that $K = L = \hat{K}_{\text{unr}}$, i.e. $k$ is algebraically closed.

Under this assumption, we develop a number of properties of triangulordinary $(\varphi, \Gamma_K)$-modules, organized under the following lemma. (In particular, part (3) of the lemma finishes part (1) of the theorem.)

**Lemma 3.10.** Let $D$ be triangulordinary. Then the following claims hold.

1. If $F^1 = D$, then $H^q(\Gamma, D^+_{\text{df}}) = 0$ for $q = 0, 1$, and

$$H^1_{g+}(D) = H^1(D) = H^1_{\text{vord}}(D).$$

2. One has a decomposition $D^+_{\text{df}} = \bigoplus_n (\text{Gr}_F^n)^+_d$ as $\Gamma_K$-modules.

3. $D$ is de Rham, and it is +de Rham if and only if $F^1 = 0$.

4. The natural map $H^1(\Gamma_K, D^+_{\text{df}}) \to H^1(\Gamma_K, (D/F^1)^+_{\text{df}})$ is an isomorphism.

**Proof.** (1) To prove the first claim, we proceed by dévissage and induction on the length of $F^*$, and are immediately reduced to the case where $D = \text{Gr}_F^n$. But in this situation, Equations 3.3-4 provide exactly what we desire. As for the second claim, the first quality follows from the first claim, while the second equality follows from the fact that $F^1 = D$.

(2) We induct on the length of $F^*$, the case of length 1 being trivial. By twisting, we can assume that $F^0 = D$ and $F^1 \neq D$, and we must show that the extension class

$$0 \to (F^1)^+_{\text{df}} \to D^+_{\text{df}} \to (D/F^1)^+_{\text{df}} \to 0$$

is split. By Equation 3.3 $(D/F^1)^+_{\text{df}} \cong K_\infty[[t]]^{\oplus d_0}$. Therefore, as an extension class, we have

$$[D^+_{\text{df}}] \in \text{Ext}^1_K(1^{\oplus d_0}, (F^1)^+_{\text{df}}) = \text{Ext}^1_K(1, (F^1)^+_{\text{df}})^{\oplus d_0} = H^1(\Gamma_K, (F^1)^+_{\text{df}})^{\oplus d_0} = 0,$$

by part (1), since $F^1$ is triangulordinary of weights $\geq 1$. Hence, the desired extension class is split.
Invoking the decomposition in (2), we have
\[ D_{\text{dif}} = D_{\text{dif}}^{+}[t^{-1}] = \bigoplus_n (t^n K_\infty [t])^{\oplus d_n}[t^{-1}] = K_\infty (t)^{\oplus \text{rank } D}. \]

Therefore, \( \dim_K D_{\text{dR}} = \text{rank } D \), and \( D \) is de Rham. And, applying \( \Gamma_K \)-invariants directly to the decomposition in (2), Equation 3 shows that \( D \) is +de Rham if and only if \( d_n = 0 \) for all \( n \geq 1 \), i.e. \( F^1 = 0 \).

(4) This follows by applying \( H^1(\Gamma_L, \cdot) \) to the decomposition of (2), and noting Equation 3.

The rest of this section thus aimed at proving claim (2) of the theorem. Note that it is precisely at this point that we must work with the “g+” condition and not the “g” condition.

Consider the commutative diagram
\[
\begin{array}{ccc}
H^1(D) & \xrightarrow{\beta} & H^1(\Gamma_K, D^{+}_{\text{dif}}) \\
\downarrow & & \downarrow \\
H^1(D/F^1) & \xrightarrow{?} & H^1(\Gamma_K, (D/F^1)^{+}_{\text{dif}})
\end{array}
\]

where the isomorphism is by (4) of Lemma 3.10. The kernel of the top row is \( H^1_{g+}(D) \), so that \( H^1_{g+}(D) = \beta^{-1} H^1_{g+}(D/F^1) \). Also, directly from the definition, \( H^1_{\text{ord}}(D) = \beta^{-1} H^1_{\text{ord}}(D/F^1) \). Therefore, we have reduced to the case where \( F^1 = 0 \), i.e. \( D \) only has weights \( \leq 0 \). In this case (recall \( K = \hat{K}^{\text{unr}} \)), by definition \( H^1_{\text{ord}}(D) = 0 \), and so we must show that \( H^1_{g+}(D) = 0 \) as well, i.e. that \( H^1(D) \hookrightarrow H^1(\Gamma_K, D^{+}_{\text{dif}}) \).

By part (1) of the theorem, assuming that \( F^1 = 0 \) means that \( D \) is +de Rham. Therefore, \( H^1_{g+}(D) \) has an interpretation in terms of extension classes that are also +de Rham. We will harness this interpretation.

Considering the commutative diagram with exact rows
\[
\begin{array}{ccc}
H^1(F^j) & \xrightarrow{\beta} & H^1(D) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\text{onto}} & H^1(\Gamma_K, (F^j)^{+}_{\text{dif}}) \\
\downarrow & & \downarrow \\
H^1(\Gamma_K, (F^j)^{+}_{\text{dif}}) & \xrightarrow{\beta^{-1}} & H^1(\Gamma_K, (D/F^j)^{+}_{\text{dif}}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\text{onto}} & 0
\end{array}
\]

an easy diagram chase shows that if the outer vertical arrows are injective, then so is the middle. This allows us to induct on the length of the filtration \( F^* \), and reduce to the case where \( F^n = D \) and \( F^{n+1} = 0 \). In other words, we may assume that, as a semilinear \( \Gamma_K \)-module, we have \( D \cong (t^n B_{\text{rig},K}^\dagger)^{\oplus d} \). As mentioned in the concluding comments of §3.3, such objects are easy to classify: they are of the form \( D = t^n B_{\text{rig},K}^\dagger \boxtimes F D_{\text{crys}} \), where
\[
D_{\text{crys}} = (D[t^{-1}])^{\Gamma_K} = D^{\text{rig} \phi} = (t^{-n} D)^{\Gamma_K}
\]
(remember \( n \leq 0 \)) is a semilinear \( \phi \)-module over \( F \).

We may further simplify, by reducing the case of general \( n \) to the case when \( n = 0 \) by a descending induction on \( n \). So, we assume the claim holds for \( n+1 \leq 0 \), and show
it holds for \( n \). We consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
H^0(D/tD) & \longrightarrow & H^1(tD) & \longrightarrow & H^1(D) & \longrightarrow & H^1(D/tD) \\
\downarrow & & \downarrow a & & \downarrow b & & \downarrow \\
H^0(\Gamma_K, D_{\text{dif}}^+/tD_{\text{dif}}^+) & \longrightarrow & H^1(\Gamma_K, tD_{\text{dif}}^+) & \longrightarrow & H^1(\Gamma_K, D_{\text{dif}}^+) & \longrightarrow & H^1(\Gamma_K, D_{\text{dif}}^+/tD_{\text{dif}}^+) 
\end{array}
\]

In the bottom row, the first and last groups are respectively isomorphic to

\[ H^q(\Gamma_K, K^n t^n)^{\oplus d} \]  

for \( q = 0, 1 \),

and thus classically seen to be trivial; hence, the bottom middle arrow is an isomorphism. Using [16] Lemma 3.2(1–2), we have

\[
D/tD \cong (t^n B_{\text{rig},K}^+/t^{n+1} B_{\text{rig},K}^+)^{\oplus d} = \lim_{r} \left( t^n B_{\text{rig},K}^+/t^{n+1} B_{\text{rig},K}^+ \right)^{\oplus d} = \lim_{r} \prod_{m \geq n(r)} (K_m t^n)^{\oplus d}. \tag{5}
\]

Examining the Herr complex

\[
(D/tD)^{\Delta_K} \xrightarrow{(\varphi-1,\gamma-1)} (D/tD)^{\Delta_K} \oplus (D/tD)^{\Delta_K} \xrightarrow{(1-\gamma)\oplus(\varphi-1)} (D/tD)^{\Delta_K},
\]

we immediately deduce from Equation 5 that \( H^0(D/tD) \) vanishes, because \( n \neq 0 \). On the other hand, one easily uses Equation 5 and [16] Lemma 3.2(3–4) to calculate that \( H^1(D/tD) \) is isomorphic to \( \lim_m \left( ((K_m')^{\Delta_K} t^n)/(\gamma - 1) \right)^{\oplus d} \), and each term of this limit is zero, since \( n \neq 0 \). Hence, the top middle arrow in our commutative diagram is an isomorphism too. Notice that \( n - 1 \) is not a \( \varphi \)-slope on \( D \) if and only if \( n \) is not a \( \varphi \)-slope on \( tD \) (which is triangulordinary of all weights \( n + 1 \)). Therefore, the inductive hypothesis applies to \( tD \), and \( b \) is injective; we conclude that \( a \) is also injective. Thus it suffices to treat the case where \( n = 0 \).

Recall that we want to show that \( H^1_{g_+}(D) = 0 \). In other words, given a class \( c \in H^1(D) \), represented by an extension \( E_c \), we want to show that if \( E_c \) is \( + \)-de Rham then it must be split. If it is \( + \)-de Rham then, invoking [3] Théorème 6.2 again, it must be semistable. We will show that every semistable extension \( E_c \) is crystalline, under our hypothesis on the \( \varphi \)-slopes of \( D \). Then, we will show that every crystalline extension is split.

So let \( E = E_c \) be given, and assumed to be semistable. We write \( E_{\text{st}} = D_{\text{st}}(E) \) for the associated filtered \((\varphi, N)\)-module; showing that \( E \) is crystalline is tantamount to showing that \( N = 0 \) on \( E_{\text{st}} \). In fact, since \( D \) is crystalline, one has \( N = 0 \) on the corank 1 subspace \( D_{\text{crys}} \subset E_{\text{st}} \). To treat the remainder of \( E_{\text{st}} \), consider the \( \varphi \)-modules over \( F \) that underlie the the exact sequence

\[
0 \to D_{\text{crys}} \to E_{\text{st}} \to 1_{\text{crys}} \to 0.
\]

By the Dieudonné–Manin theorem, the category of \( \varphi \)-modules over \( F \) is semisimple (recall that \( k \) is algebraically closed), so we may split this extension of \( \varphi \)-modules, i.e. choose a \( \varphi \)-fixed vector \( e \in E_{\text{st}} \) that spans the complement of \( D_{\text{crys}} \). We will know that \( N = 0 \) on \( E_{\text{st}} \) as soon as we know that \( N(e) = 0 \). To see this, remember that \( D \) is \( \Gamma_K \)-isomorphic to \( (B_{\text{rig},K}^\dagger)^{\oplus d} \), and so the \( \varphi \)-slopes (à la Dieudonné–Manin) on \( D_{\text{crys}} \).
are equal to the \( \varphi \)-slopes (á la Kedlaya) on \( D \), which by hypothesis are not equal to \(-1\), while the \( \varphi \)-slope of \( e \) is 0. Denoting \( E^{(\lambda)}_{st} \) for the slope-\( \lambda \) part of \( E_{st} \), recall that 
\[
N\varphi = p\varphi N,
\]
and hence \( N(E^{(\lambda)}_{st}) \subseteq E^{(\lambda-1)}_{st} \). On the other hand, we have 
\[
N(e) \in \mathcal{N}(E_{st}^{(0)}) \subseteq E_{st}^{(-1)} = 0.
\]
Therefore, \( E \) is crystalline.

Given an extension \( E \) of \( D \) that is crystalline, we show that it is trivial. In fact, applying Dieudonné–Manin just as above, we find that \( E_{st} = E_{crys} \) is split as a \( \varphi \)-module:
\[
E_{crys} = D_{crys} \oplus 1_{crys}.
\]
But we may recover \( E \) as a \((\varphi, \Gamma_K)\)-module from \( E_{crys} \). In fact,
\[
E = B^{\dagger}_{rig,K} \otimes_F E_{crys} = B^{\dagger}_{rig,K} \otimes_F (D_{crys} \oplus 1_{crys}) = (B^{\dagger}_{rig,K} \otimes_F D_{crys}) \oplus (B^{\dagger}_{rig,K} \otimes_F 1_{crys}) = D \oplus 1,
\]
with \( \varphi \) acting diagonally and \( \Gamma_K \) acting on the left \( \otimes \)-factors. This shows that \( E = E_c \) is split as an extension of \((\varphi, \Gamma_K)\)-modules, and its corresponding class \( c \in H^1(D) \) is trivial. This completes the proof. \( \square \)

**Remark 3.11.** A curious byproduct of the final step of the argument is that, for \( D \) in the special form treated there, if an extension \( E \) of \( D \) as a \( \Gamma_K \)-module admits any \( \varphi \)-structure, then it admits only one \( \varphi \)-structure.

**Remark 3.12.** The hypothesis on the \( \varphi \)-slopes of \( D \) really is necessary when working with general \((\varphi, \Gamma_K)\)-modules, as the following example shows. The trouble seems to be that when we have left the category of Galois representations, i.e. étale \((\varphi, \Gamma_K)\)-modules and admissible filtered \((\varphi, N, G_K)\)-modules, there are simply too many objects to be handled via ordinary-theoretic techniques. Cf. the failure of the above proof to apply to the “g” local condition.

**Example 3.13.** Consider the filtered \((\varphi, N)\)-module \( E_{st} = \text{span}(e_0, e_{-1}) \), with \( \text{Fil}^0 = E_{st} \), \( \text{Fil}^1 = 0 \), and \( \varphi \) and \( N \) given by 
\[
\varphi(e_0) = e_0, \quad \varphi(e_{-1}) = p^{-1}e_{-1}, \quad \text{and} \quad N(e_0) = e_{-1}, \quad N(e_{-1}) = 0.
\]
Then \( E_{st} \) corresponds to a \((\varphi, \Gamma_K)\)-module \( E \) of rank 2 which is not étale, is semistable of Hodge–Tate weights both 0, and is not crystalline. Such an object is unheard of in the classical setting.

Notice that \( D_{st} = \text{span}(e_{-1}) \) corresponds to a subobject \( D \) of \( E \), with quotient object isomorphic to \( 1_{crys} \) (taking \( e_0 \) to the standard basis element). Thus, \( E \) represents a nontrivial extension class in \( H^1(D) \), which is actually +de Rham because its Hodge–Tate weights are all 0. On the other hand, \( D \) is only triangulordinary with respect to \( F^0 = D, F^1 = 0 \), and so if (for example) \( K = \overline{K} \) then \( H^1_{\text{ord}}(D) = 0 \), and \( H^1_{\text{ord}}(D) \subseteq H^1_{g+}(D) \).

The manner of the reduction steps in the proof of the theorem show that all counterexamples to its conclusion, outside the context of the slope hypothesis, arise by some manipulation (twisting, extensions, descent) from the above example.
We will see in §3.6 that the above counterexample really does occur as a graded piece within étale \((\varphi, \Gamma_K)\)-modules arising in nature, namely in the setting of a modular form with good reduction at \(p\) and slopes \(1, k - 2\) (where \(k\) is the weight of \(f\)). It is unclear to the author whether its obstruction can be worked around, even in explicit examples.

### 3.5 Comparison with Bloch–Kato, ordinary and triangulordinary

Since the reader might be wondering what “triangulordinary” means, we explain how triangulordinary representations and Selmer groups relate to common notions due to Bloch–Kato, Greenberg, and Colmez.

All the examples that motivate this work take place when \(D\) is étale, so that \(D = D_{\text{rig}}(V)\) for a bona fide \(p\)-adic representation \(V\) of \(G_K\). In order to place Theorem 3.1 into context, we recall the following facts, which are essentially due to Bloch–Kato [5].

**Proposition 3.14 ([5]).** Let \(V\) be de Rham. Then the following claims hold.

1. One always has \(H^1_g(V) = H^1_f(V)\), these two items being defined in §3.2.
2. If \(V\) is semistable and \(D_{\text{crys}}(V)^{p-1} = 0\), then \(H^1_g(V) = H^1_f(V)\).

In the second item, the local condition \(H^1_f(V)\) consists of those extension classes that are split after tensoring with \(B_{\text{crys}}\), and provides the correct Selmer group with which to state Bloch–Kato’s conjectural analytic class number formula.

Thus, in the triangulordinary setting, the local condition \(H^1_{\text{ord}}(V; F)\) usually measures \(H^1_f(V)\), and hence the triangulordinary Selmer group computes the Bloch–Kato Selmer group, which is of intrinsic interest.

We find it helpful to have access to the following equivalent formulation.

**Alternate definition 3.15.** We remind the reader that by Theorem 3.1(1), every triangulordinary \((\varphi, \Gamma_K)\)-module is semistable. Thus, there is no loss of generality in assuming this is the case from the outset.

Given a semistable \((\varphi, \Gamma_K)\)-module \(D\), the discussion at the conclusion of §2.5 relates semistable subobjects of \(D\) to subobjects of \(D_{\text{st}}\). We find that the triangulordinary filtrations \(F^* \subseteq D\) are in a natural correspondence with filtrations \(F^* \subseteq D_{\text{st}}\) by \((\varphi, N)\)-stable subvector spaces (each equipped with its Hodge filtration induced by \(D_{\text{st}}\), such that each \(\text{Gr}_n F D_{\text{st}}\) has all its induced Hodge–Tate weights equal to \(n\) (i.e., its induced Hodge filtration is concentrated in degree \(-n\)).

Given a corresponding pair \(F^* \subseteq D\) and \(F^* \subseteq D_{\text{st}}\), the gradeds \(\text{Gr}^n_F D\) and \(\text{Gr}^n_F D_{\text{st}}\) are linked by the formula

\[
\text{Gr}^n_F D \cong t^n D_{\text{rig}} \otimes_F \text{Gr}^n_F D_{\text{st}}
\]
as \((\varphi, \Gamma_K)\)-modules. Therefore, the \(\varphi\)-slopes on \(\text{Gr}^n_F D\) are \(n + \varphi\)-slopes on \(\text{Gr}^n_F D_{\text{st}}\).

As concerns Theorem 3.1(2), the requirement that for all \(n \leq 0, n - 1\) not be a \(\varphi\)-slope on \(\text{Gr}^n_F D\) becomes the condition that, for all such \(n\), the graded \(\text{Gr}^n_F D_{\text{st}}\) does not contain the \(\varphi\)-slope \(-1\).
Example 3.16 (Relation to ordinary representations). Let us see how ordinary representations, defined by Greenberg in [12], fit into our context. We are given a Galois representation $V$, so that $D = D_{\text{rig}}(V)$ is étale. The ordinary hypothesis is that $V$ admits a decreasing filtration $F^* \subseteq V$ by $G_K$-stable subspaces, such that, for each $n$, the representation $\chi_{n}^{-1} \otimes \text{Gr}_F^n V$ is unramified. Applying $D_{\text{rig}}$, we obtain a decreasing, $(\varphi, \Gamma_K)$-stable filtration $F^* \subseteq D$ by $B_{\text{rig}} \otimes_{\mathbb{Q}} \mathbb{Z}_p$-direct summands, such that each $\text{Gr}_F^n D_L$ is étale and $\Gamma_L$-isomorphic to $(t^nB_{\text{rig}} L)^{\otimes d_{\text{ad}}}$. Conversely, given such a filtration on $D$, Theorem 2.3 produces an ordinary filtration on $V$. Thus, given an étale $D$, the ordinary hypothesis is a strengthening of the triangulordinary hypothesis to require all the graded pieces to be étale.

In the language of filtered $(\varphi, N)$-modules, a triangulordinary filtration $F^* \subseteq D_{\text{st}}$ corresponds to an ordinary filtration precisely when all the $\text{Gr}_F^n D_{\text{st}}$ are admissibly filtered, which means here that each $\text{Gr}_F^n D_{\text{st}}$ is of pure $\varphi$-slope $-n$.

Moreover, Theorem 3.1(2) always applies to ordinary representations. Namely, for all $n \leq 0$, the number $n-1$ (which is $\leq -1$) never occurs as a $\varphi$-slope on $\text{Gr}_F^n D$ because the latter is étale. Thus, this theorem provides a generalization of Flach’s result [10, Lemma 2] from the case where $K = \mathbb{Q}_p$ to the case of arbitrary perfect residue field.

We alert the reader to the fact that, although $V$ admits at most one ordinary filtration, it may admit many different triangulordinary filtrations, as we will see below.

Before discussing trianguline representations, we point out that our entire theory works perfectly well with the $E$-coefficients replacing the $\mathbb{Q}_p$-coefficients of Galois representations, for any finite extension $E/\mathbb{Q}_p$.

Example 3.17 (Relation to trianguline representations). We now determine when a triangulordinary $(\varphi, \Gamma_K)$-module is trianguline, and give some comments about the converse. (More precisely, we discuss when one can modify a triangulordinary filtration into a trianguline one, and vice versa.) Recall that, following Colmez [8, §0.3], a $(\varphi, \Gamma_K)$-module $D$ is trianguline if it is a successive extension of rank 1 objects, i.e. if there exists a decreasing, separated and exhaustive filtration $F^* \subseteq D$ by $(\varphi, \Gamma_K)$-stable $B_{\text{rig}} \otimes_{\mathbb{Q}} \mathbb{Z}_p$-direct summands, with each graded of rank 1. We call the latter a trianguline filtration. When $D$ is semistable, these correspond precisely to refinements in the sense of Mazur: complete flags in $D_{\text{st}}$ by $(\varphi, N)$-stable subspaces.

A triangulordinary $D$ is trianguline precisely when the gradeds $\text{Gr}_F^s$ of the triangulordinary filtration $F^*$ are themselves trianguline. Sufficiency is clear. For necessity, note that $D$ is semistable, so assume given $D_{\text{st}}$, and think of triangulordinary filtrations as being $(\varphi, N)$-stable ones on $D_{\text{st}}$. Given our triangulordinary filtration and any (other) trianguline filtration on $D$, taking the intersections of the two filtrations gives a refinement of the triangulordinary filtration with rank one graded. (One could avoid using filtered $(\varphi, N)$-modules by converting this last step into the language of Bézout domains.)

In any case, with $D$ triangulordinary, each the $\text{Gr}_F^s$ is crystalline (by Corollary 3.7), so $D$ is trianguline if and only if the $\text{Gr}_F^s_{\text{crys}}$ admit refinements. Clearly, this is the case precisely when $D_{\text{crys}}$ is an extension of one-dimensional $\varphi$-stable subspaces. If the residue field $k$ is finite, then one can always replace the coefficient field $E$ by a finite extension in order to achieve this.

As for the converse, since triangulordinary $D$ are always semistable, we ask when a semistable trianguline $D$ is triangulordinary. It turns out that not all such $D$ are
triangulordinary. For a semistable trianguline $D$ that is not triangulordinary, we consider $E$ as in Example 3.13. Being constructed out of $E_{st}$, it is semistable; being an extension of $D$ by $1$ it is trianguline. Both its Hodge–Tate weights are 0, so a putative triangulordinary filtration $F^* \subseteq E$ would have $E = \text{Gr}_F^0$; by Corollary 3.7, in order for $E$ to be triangulordinary it must be crystalline. But $E$ is not crystalline.

The above example shows that, roughly, having a nonzero monodromy operator acting within fixed a Hodge–Tate weight part is an obstruction to being triangulordinary. Let us assume that this is not the case for $D$, and suppose we are given a trianguline filtration $F^* \subseteq D$. In order for $D$ to be triangulordinary, we must be able to arrange that the Hodge–Tate weights of the $\text{Gr}_F^*$ are nondecreasing, because then, weakening $F^*$ so that each $\text{Gr}_F^n$ has all Hodge–Tate weights equal to $n$, the resulting graded modules must also be crystalline (by our rough assumption), hence $\Gamma_L$-isomorphic to $(t^nB_{\text{rig},L}^{\hat{\varphi}})^{\otimes d_n}$ by Corollary 3.7. In order to rearrange $F^*$ to have Hodge–Tate weights in nondecreasing order, we must be able to break up any extension between adjacent gradeds that are in the wrong order. Given the intermediate extension of filtered $(\varphi,N)$-modules

$$0 \to (\text{Gr}_F^{n+1})_{\text{crys}} \to (F^n/F^{n+2})_{st} \to (\text{Gr}_F^n)_{\text{crys}} \to 0$$

whose Hodge–Tate weights are in decreasing order, one easily checks that any $(\varphi,N)$-equivariant splitting will do. But $(\varphi,N)$-equivariant splittings, in turn, might not exist. The $\varphi$-structure itself could be nonsemisimple; by Dieudonné–Manin, this would require the crystalline $\varphi$-slopes on the adjacent gradeds to be equal, and the $\varphi$-extension would only necessarily split upon restriction to $\hat{\mathbb{K}}_{\text{unr}}$. Assuming otherwise, that one can find a $\varphi$-eigenvector $v \in (F^n/F^{n+2})_{st}$ mapping onto a basis for $(\text{Gr}_F^n)_{\text{crys}}$, the extension is split as $(\varphi,N)$-module if and only if $N(v) = 0$, which might or might not hold.

In summary, the trianguline condition is roughly more general than the triangulordinary condition. Triangulordinary representations are semistable, and are trianguline when their filtrations may be further subdivided to have gradeds of rank 1; the latter always happens after an extension of coefficients when $k$ is finite.

Trianguline $D$ may be highly nonsemistable due to continuous variation of Sen weights. When they are semistable, they may fail to be triangulordinary, if they have nontrivial extensions with the wrong ordering of Hodge–Tate weights, or if they have extensions of common Hodge–Tate weight that are semistable but not crystalline.

### 3.6 Examples of triangulordinary representations

In this section we explain when abelian varieties and modular forms are triangulordinary. In passing, we gather for easy reference descriptions of the invariants of the cyclotomic character and modular forms. Since many different normalizations are used in the literature, we have made an effort to organize them systematically.

Let us begin with some discussion of normalizations. The general rules are summarized in the following table. The initial column says what kind of motive we are dealing with: one cut out of homology or cohomology. The first property is which Frobenius operator on $\ell$-adic realizations has $\ell$-adic integer eigenvalues. Next is which power of crystalline Frobenius, $\varphi$ or $\varphi^{-1}$, has $p$-adic integer eigenvalues. Then come the degrees in which we expect to see jumps in the Hodge filtration. Finally, we see which powers of the cyclotomic character tend to appear in the action of $\Gamma_{\mathbb{Q}_p}$ on basis elements of
the $D_{\text{dif}}^+$ (which is a rough indication of the $\Gamma_{Q_p}$-action on $D_{\text{rig}}^+$); these are the jumps we expect to see in a triangulordinary filtration.

|                | $\ell$-adic Frob | $\varphi$ | $\varphi^{-1}$ |
|----------------|-------------------|-----------|----------------|
| $\text{crystalline } \varphi$ | arithmetic        | $\varphi$ | $\varphi^{-1}$ |
| Hodge jumps    | nonpositive        | nonnegative| nonpositive    |
| $\Gamma_{Q_p}$ on $D_{\text{dif}}^+$ | nonnegative       | nonpositive| nonnegative |
| $\text{Vord jumps}$ | nonnegative       | nonpositive| nonpositive |

We give three reminders: the $\varphi$ on $D_{\text{rig}}^+$ is always étale, the cyclotomic character and Tate modules of abelian varieties are homological objects, and this table is invariant under the choice of sign of the Hodge–Tate weight of the cyclotomic character. (In this text, the cyclotomic character has Hodge–Tate weight +1.)

**Example 3.18** (The cyclotomic character). We consider the $p$-adic cyclotomic character $\chi_{\text{cycl}}$ as a 1-dimensional $Q_p$-vector space $Q_p \cdot e_{\text{cycl}}$, equipped with a $Q_p$-linear $G_K$-action via $g(e_{\text{cycl}}) = \chi_{\text{cycl}}(g) e_{\text{cycl}}$. One has $D_{\text{rig}}^+(\chi_{\text{cycl}}^n) = B_{\text{rig},K}^n \cdot (1 \otimes e_{\chi_{\text{cycl}}}^o)$ and $D_{\text{cris}}(\chi_{\text{cycl}}) = F \cdot (t^{-n} \otimes e_{\chi_{\text{cycl}}}^o)$. From these, we derive the following table, giving actions on the basis vectors just mentioned.

| $\ell$-adic Frob$_{\text{arith}}$ | $\varphi$ on $D_{\text{cris}}$ | $\varphi^{-1}$ | $\varphi$ on $D_{\text{rig}}^+$ | $\Gamma_{Q_p}$ on $D_{\text{rig}}^+$ |
|-------------------------------|----------------|--------------|----------------|----------------------------------|
| $p^n$                         | $p^{-n}$       | $-n$         | 1             | $\chi_{\text{cycl}}^n$          |

Finally, we point out that the powers of the cyclotomic character are all ordinary, hence triangulordinary. Since they are one-dimensional, the ordinary filtration is the only choice of triangulordinary filtration.

**Example 3.19** (Abelian varieties). Take a semistable abelian variety $B$ over $K$ of dimension $d \geq 1$, and consider $D_{\text{st}} = D_{\text{st}}(V)$, with $V = T_p B \otimes \mathbb{Q}$ the $p$-adic Tate module up to isogeny. Thus, when dealing with abelian varieties, we are primarily concerned with homology. It is well-known that the Hodge–Tate weights of $B$ are 0 and 1, each with multiplicity $d$, and this tells us that the Hodge filtration $H^* \subseteq D_{\text{dR}}$ satisfies $\dim_K \text{Gr}_H^d = \dim_K \text{Gr}_H^{-1} = d$, and our triangulordinary filtration $F^* \subseteq D_{\text{st}}$ must satisfy $\text{rank} \text{Gr}_F^d = \text{rank} \text{Gr}_F^{-1} = d$, and all other gradeds are trivial. So, $F^*$ consists of the single datum of a $(\varphi, N)$-stable $F$-subspace $F^1 \subseteq D_{\text{st}}$ of dimension $d$.

Weak admissibility, here, means: nonzero slopes do not meet the Hodge filtration $H$. Ordinary means that half these slopes are 0 (can lie anywhere), and half are $-1$ (cannot lie in $H$: span a weakly admissible submodule). Triangulordinary means, one can find half of these slopes, $\varphi$-stably, not contained in $H$. This means that corresponding subspace $F^1$ has induced Hodge filtration concentrated in degree $-1$, which automatically forces $\text{Gr}_F^d = D_{\text{st}}/F^1$ to have induced Hodge filtration concentrated in degree 0.

Thus, in short, a triangulordinary filtration for $V$ consists of a $d$-dimensional $(\varphi, N)$-stable subspace $F^1 \subseteq D_{\text{st}}$ such that $F^1 \otimes F K$ is complementary to the Hodge filtration $H^0 \subseteq D_{\text{dR}}$.

We stress that, because we are in a homological situation, the $\varphi$-slopes on $D_{\text{st}}(V)$ are nonpositive. In order for Theorem 3.1(2) to apply, all we need is that $-1$ does not
occur as a \( \varphi \)-slope on the quotient \( \text{Gr}^p D_{\text{crys}} = D_{\text{crys}}/F^1 \), or, equivalently, that every instance of slope \(-1\) occurs within \( F^1 \). (This hypothesis is a variant of a “noncritical slope” condition.)

Let us illustrate the above with some examples, assuming, for simplicity, that \( B \) has good reduction and our coefficients are \( E = \mathbb{Q}_p \):

Suppose \( B \) has slopes \(-1, -2/3, -1/3, 0\). Then \( B \) is nonordinary, and always triangulordinary: for the triangulordinary filtration, one can take either any of the two spaces with slopes \((-1, -2/3), (-1, 1/3)\). When the 0-slope is not in \( H \), one gets two more options. The theorem applies to the first two of these, but not to the possible latter two.

If \( B \) has slopes \(-1, -1/2, -1/2, 0\), then it is nonordinary, and always triangulordinary: its filtrations include the two slope \((-1, -1/2)\) spaces, and, if the slope 0 space is not in \( H \), then the two slope \((-1/2, 0)\) spaces are also valid. (In particular, having slopes equal to \(-1/2\) is not necessarily an obstruction.) The theorem applies to the first filtrations, and not to the second ones.

Let \( B \) have slopes \(-1, -1, -1, -1/2, -1/2, 0, 0, 0\), and assume \( \varphi \) acts irreducibly on its pure-slope spaces. Then \( B \) is not triangulordinary, simply because there are no \( \varphi \)-stable subspaces with half the total dimension.

We leave it to the reader to examine, when the residue field \( k \) is finite, what additional possibilities occur after enlarging the coefficient field \( E \) to break the pure-slope spaces into extensions of one-dimensional \( \varphi \)-stable spaces.

In the following example, fix a coefficient field \( E \).

**Example 3.20** (Elliptic modular eigenforms). Let \( f \in S_k(\Gamma_1(M), \psi; E) \) be a normalized elliptic modular cuspidal eigenform such that \( k \geq 2 \), having \( q \)-expansion \( \sum a_n q^n \).

Deligne has associated to \( f \) a 2-dimensional \( E \)-valued representation of the absolute Galois group of \( \mathbb{Q} \), which is unramified away from \( Mp_{\infty} \) and de Rham at \( p \). It is absolutely irreducible, so it is characterized (up to a scalar multiple) by its characteristic polynomials; by Chebotarev, it is enough to know the polynomials of the Frobenius elements \( \text{Frob}_\ell \) at primes \( \ell \nmid Mp \). For such \( \ell \), one has

\[
\text{trace}(\text{Frob}_\ell) = a_\ell \quad \text{and} \quad \text{det}(\text{Frob}_\ell) = \ell^{k-1} \psi(\ell).
\]

One vagueness that typically makes these matters confusing is whether \( \text{Frob}_\ell \) refers to the arithmetic or the geometric Frobenius. In the case where the above equations describe the arithmetic Frobenius, we say that the representation is the *homological* normalization, and denote it \( V_f^{\text{hom}} \). When they apply to the geometric Frobenius, we say that the representation is the *cohomological* variant, and we denote it \( V_f^{\text{coh}} \). These names originate in whether \( V_f^2 \) is found within the étale homology or cohomology of Kuga–Sato varieties, respectively. In either case, we only consider the restriction of the \( G_\mathbb{Q} \)-action to a decomposition group \( G_{\mathbb{Q}_p} \).

In what follows, we make the following hypothesis: \( f \) is semistable at \( p \), and the operator \( \varphi \) on \( \text{D}_{\text{st}}(V_f^{\text{coh}}) \) has *distinct* eigenvalues \( \lambda, \mu \) lying in \( E \). (This is equivalent to requiring the same on \( \text{D}_{\text{st}}(V_f^{\text{hom}}) \), with the the roots \( \lambda^{-1}, \mu^{-1} \in E \).) We order the roots so that \( \text{ord}_p \lambda \leq \text{ord}_p \mu \); one has \( \text{ord}_p \lambda + \text{ord}_p \mu = k - 1 \), with \( \text{ord}_p \lambda = 0 \) if and only if \( f \) is ordinary at \( p \).

The cohomological normalization has

\[
\text{D}_{\text{st}}(V_f^{\text{coh}}) = E \cdot e_\lambda \oplus E \cdot e_\mu \quad \text{with} \quad \left\{ \begin{array}{l}
\varphi(e_\nu) = \nu e_\nu, \\
\text{D} = F^0 \supseteq F^1 = \cdots = F^{k-1} \supseteq F^k = 0.
\end{array} \right.
\]
The “weak admissibility” condition means that $e_\lambda \notin F^1$, and $e_\mu \notin F^1$ unless possibly if ord$_p$ $\mu$ = $k$ – 1 (in which case $f$ is split ordinary at $p$). The monodromy operator $N$ is only nonzero when $p | M$; if $N \neq 0$ then ord$_p$ $\mu$ = ord$_p$ $\lambda$ + 1, and $N$ is determined by $N(e_\mu) = e_\lambda$ and $N(e_\lambda) = 0$ (up to rescaling $e_\lambda$).

Since the nonzero gradings for the Hodge filtration are $0, k – 1$, each with one-dimensional graded, the nonzero triangulordinary gradings are $1 – k, 0$, each with one-dimensional graded. Thus we consider the rank-one $\varphi$-stable subspaces of $D_{st}(V_f^{\text{coh}})$, and their corresponding $(\varphi, \Gamma_{Q_p})$-modules. Let $\nu$ be one of $\lambda$ or $\mu$ and let $\nu'$ be the other; let $D_\nu = E \cdot e_\nu \subset D_{st}$ and $D_{\nu}' = D_{st}/D_\nu$. Then, for comparison to the cyclotomic character, one has the following table. The parenthetical values are used precisely when $f$ is split ordinary at $p$ and $\nu = \mu$.

| $\varphi$ on $D_\nu$ | $\text{Gr}^1 \neq 0$ | $\varphi$ on $(D_\nu)^{T}_{\text{rig}}$ | $\Gamma_{Q_p}$ on $(D_\nu)^{T}_{\text{rig}}$ |
|---|---|---|---|
| $\nu$ | 0 ($k – 1$) | $\nu$ ($\nu'^{-1}$) | 1 ($\chi_{\text{cycl}}^{-k}$) |

The triangulordinary hypothesis on $F^\ast$ requires that $F^0$ not meet the Hodge filtration $H^1 = H^{k-1} \subset D_{st} \otimes_F K$, and, if this holds, then one obtains for free that $\text{Gr}^{1-k} D_{st} = D_{st}/F^0$ has induced Hodge–Tate weight $k – 1$, as is required. Examining the above table, we see that $e_\nu$ always defines a trianguline filtration, and that $e_\nu$ defines a triangulordinary filtration except in the parenthetical (split ordinary) case. In the split ordinary case, taking $\nu = \lambda$ still gives a triangulordinary filtration. Also, theorem 3.1(2) always applies, because the only nonzero $\text{Gr}^n$ with $n \leq 0$ is with $n = 0$, and the only $\varphi$-slope occurring there is nonnegative.

We obtain descriptions of the homological normalization by taking $E$-linear duals of everything above. Namely, $D_{st}$ has

$$D_{st}(V_f^{\text{hom}}) = E \cdot e_{\lambda -1} \oplus E \cdot e_{\mu -1} \text{ with } \begin{cases} \varphi(e_{\nu^{-1}}) = \nu^{-1}e_{\nu^{-1}}, \\ D = F^{1-k} \supset F^{2-k} = \cdots = F^0 \supset F^1 = 0. \end{cases}$$

The “weak admissibility” condition means that $e_{\mu -1} \notin F^1$, and $e_{\lambda -1} \notin F^1$ unless possibly if ord$_p$ $\lambda$ = 0 (in which case $f$ is split ordinary at $p$). The monodromy operator $N$ is only nonzero when $p | M$; if $N \neq 0$ then ord$_p$ $\mu$ = ord$_p$ $\lambda$ + 1, and $N$ is determined by $N(e_{\lambda -1}) = e_{\mu -1}$ and $N(e_{\mu -1}) = 0$ (after perhaps rescaling $e_{\mu -1}$). Note the role reversal between $\mu$ and $\lambda$; this is only because ord$_p$ $\mu^{-1}$ $\leq$ ord$_p$ $\lambda^{-1}$.

Our triangulordinary filtration must have one-dimensional nonzero gradings in degrees 0, 1, and so we consider the rank-one $\varphi$-stable subspaces of $D_{st}(V_f^{\text{hom}})$, and their corresponding $(\varphi, \Gamma_{Q_p})$-modules. Let $\nu$ be one of $\lambda$ or $\mu$ and let $\nu'$ be the other; let $D_{\nu^{-1}} = E \cdot e_{\nu^{-1}} \subset D_{st}$ and $D_{\nu'^{-1}} = D_{st}/D_{\nu^{-1}}$. Again, we have the following table. The parenthetical values are used precisely when $f$ is split ordinary at $p$ and $\nu^{-1} = \lambda^{-1}$.

| $\varphi$ on $D_{\nu^{-1}}$ | $\text{Gr}^1 \neq 0$ | $\varphi$ on $(D_{\nu^{-1}})^{T}_{\text{rig}}$ | $\Gamma_{Q_p}$ on $(D_{\nu^{-1}})^{T}_{\text{rig}}$ |
|---|---|---|---|
| $\nu^{-1}$ | 1 ($k – 1$) | $\nu'$ ($\nu'^{-1}$) | $\chi_{\text{cycl}}$ ($\lambda_{\text{cycl}}^{-1}$) |

| $\varphi$ on $D_{\nu'^{-1}}$ | $\text{Gr}^1 \neq 0$ | $\varphi$ on $(D_{\nu'^{-1}})^{T}_{\text{rig}}$ | $\Gamma_{Q_p}$ on $(D_{\nu'^{-1}})^{T}_{\text{rig}}$ |
|---|---|---|---|
| $\nu'^{-1}$ | 0 (1 – $k$) | $\nu'^{-1}$ ($\nu$) | 1 ($\chi_{\text{cycl}}^{-k}$) |

In particular, we see that $e_{\nu^{-1}}$ always defines a trianguline filtration, and that $e_{\nu^{-1}}$
defines a triangulordinary filtration except in the parenthetical (split ordinary) case. In the split ordinary case, taking $\nu = \mu^{-1}$ still gives a triangulordinary filtration. As concerns Theorem 3.1, $\text{Gr}^{1-k}$ always has nonnegative slope, and hence presents no obstruction. But $\text{Gr}^0$ has slope $\text{ord}_p \nu + 1 - k$, which is equal to $-1$ when $\text{ord}_p \nu = k - 2$. Thus, the theorem does not apply to modular forms with triangulordinary filtration determined by a $\varphi$-eigenvalue of slope $k - 2$.

We invite the reader to check that the above conclusions for $V^{\text{hom}}_f$ agree with the conclusions made, in the case $k = 2$, for $T_p B_f \otimes \mathbb{Q}$, where $B_f$ is the corresponding modular abelian variety.

Remark 3.21. The example above should generalize readily to Hilbert modular forms.

Remark 3.22. It is extremely unusual to naturally encounter a theorem that applies only to modular forms with a $U_p$-slope $\neq k - 2$, as does Theorem 3.1 for $V^{\text{hom}}_f$. The only special slopes are usually 0, $k - 1$, and $\frac{k-1}{2}$. 

28
4 Variational program

In this section we describe a conjectural program for obtaining triangulordinary filtrations, and hence Selmer groups, for families of Galois representations. Our primary guide here is Greenberg’s variational viewpoint, described in [13].

As a main example, we consider the eigencurve of Coleman–Mazur. We show how our program would recover results of Kisin (see [15]), and interpolate his Selmer groups for overconvergent modular forms of finite slope into a Selmer module over the entire eigencurve. Note that we use the homological normalization to maintain consistency with Greenberg on the ordinary locus, and with the statements of Kisin.

We retain the notations and conventions of the preceding sections, with the additional assumption that our local fields $K$ have finite residue fields (since this is required by Berger–Colmez in [4]). A careful reading of Berger–Colmez might allow this restriction to be removed.

4.1 Interpolation of $(\varphi, \Gamma_K)$-modules

We construct families of $(\varphi, \Gamma_K)$-modules over rigid analytic spaces corresponding to families of $p$-adic representations of $G_K$, using the theory of Berger–Colmez. In their work, the base of the family is a $p$-adic Banach space $S$. By an $S$-representation of $G_K$, we mean a finite free $S$-module $V$ equipped with a continuous, $S$-linear $G_K$-action. In order to get a $p$-adic Hodge theory for $V$, we must assume the mild condition that $S$ is a coefficient algebra as in [4, §2.1].

We require some terminology from $p$-adic functional analysis. Given a $p$-adic Banach algebra $S$ with norm $\| \cdot \|_S$ and a Fréchet space $T$ with norms $\{\| \cdot \|_i\}_{i \in I}$, we define norms $\{\| \cdot \|_{S,i}\}_{i \in I}$ on $S \otimes_{\mathbb{Q}_p} T$ by

$$\| x \|_{S,i} = \inf_{x = \sum_{k=1}^{\infty} s_k \otimes t_k} \left( \max_k \| s_k \|_S \cdot \| t_k \|_i \right).$$

This makes $S \otimes_{\mathbb{Q}_p} T$ into a pre-Fréchet space, and we declare $S \otimes T$ to be its Fréchet completion, consisting of equivalence classes of sequences that are simultaneously Cauchy with respect to all the norms. If $T$ is instead the direct limit of the Fréchet spaces $\{T_j\}_{j \in J}$ (henceforth, we say $T$ is LF), we define $S \otimes T$ to be the direct limit of the $S \otimes T_j$, each of the latter terms being defined above.

In particular, the above definitions apply to $T = B^{\dagger,r}_{\text{rig}, K}$, which is Fréchet: there are norms $\| \cdot \|_s$ on $B^{\dagger,r}_{\text{rig}, K}$, for $0 < s \leq r$, corresponding to the sup norms on the annuli $\text{ord}_p(X) = s/e_K$, which can be described easily in terms of the expansion of $f$ in $\pi_K$. The definitions also apply to $T = B^{\dagger}_{\text{rig}, K}$, which is the direct limit of the $B^{\dagger,r}_{\text{rig}, K}$ for $r > 0$, and hence is LF.

We write $\text{Spm} S$ for the collection of maximal ideals of $S$, and when we have a label $x$ for an element $m_x \in \text{Spm} S$, we abusively refer to $m_x$ by $x$. If $M$ is an $S$-module, we write $M_x$ for $M \otimes_S S/m_x$ throughout this section.

Applying $\otimes_{\text{Spm} B^\dagger_{\text{rig}, K}} (S \otimes B^\dagger_{\text{rig}, K})$ to [4 Théorème A], as in §6.2 of loc. cit., we see that one can canonically associate to $V$ a locally free $S \otimes B^\dagger_{\text{rig}, K}$-module $D^{\dagger}_{\text{rig}}(V)$ of rank equal to $\text{rank}_S V$, equipped with commuting, continuous, semilinear actions of $\varphi$ and $\Gamma_K$, with the property that $D^{\dagger}_{\text{rig}}(V)_x$ is canonically isomorphic to $D^{\dagger}_{\text{rig}}(V_x)$ in $M(\varphi, \Gamma_K)/B^{\dagger}_{\text{rig}, K}$ for all $x \in \text{Spm} S$. 


Let us globalize this result. Recall that our coefficient field for Galois representations is $E$, a finite extension of $\mathbb{Q}_p$. Let $\mathcal{X}/E$ be a reduced, separated rigid analytic space with structure sheaf $\mathcal{O}_{\mathcal{X}}$. The very notion of a rigid analytic space is that $\mathcal{X}$ is built from its admissible affinoid subdomains $\mathcal{U} = \text{Spm} \, S$, so that a sheaf is determined by its restriction to an admissible covering by admissible affinoid opens (for brevity, we will call this a *good cover*), and a quasi-coherent sheaf is determined by its values on such an open cover. Note that an affinoid algebra is naturally a $p$-adic Banach algebra; in fact, a reduced affinoid algebra is a coefficient algebra. A quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules is said to be locally free of finite rank (resp. locally free Banach, locally free Fréchet, locally free LF) if $\mathcal{X}$ admits a good cover by opens $\mathcal{U}$ for which $\Gamma(\mathcal{U}, M) \cong \Gamma(\mathcal{U}, \mathcal{O}) \hat{\otimes} T_\mathcal{U}$, where $T_\mathcal{U}$ is a finite-dimensional $\mathbb{Q}_p$-vector space (resp. a Banach space, a Fréchet space, an LF space).

For a commutative $\mathbb{Q}_p$-algebra $R$ that is finite-dimensional (resp. Banach, Fréchet, LF) as a $\mathbb{Q}_p$-module, we define $\mathcal{O}_{\mathcal{X}} \hat{\otimes} R$ to be the locally free sheaf of finite-dimensional (resp. Banach, Fréchet, LF) $\mathcal{O}_{\mathcal{X}}$-algebras with $T_\mathcal{U} = R$, as above, for every affinoid subdomain $\mathcal{U} \subseteq \mathcal{X}$. A locally free $\mathcal{O}_{\mathcal{X}} \hat{\otimes} R$-module of finite rank is a quasicoherent sheaf $M$ of $\mathcal{O}_{\mathcal{X}} \hat{\otimes} R$-modules on $\mathcal{X}$ such that, for $\mathcal{U}$ ranging over some good cover of $\mathcal{X}$, each $\Gamma(\mathcal{U}, M)$ is free of finite rank over $\Gamma(\mathcal{U}, \mathcal{O}) \hat{\otimes} R$.

In particular, we have defined the sheaf of rings $\mathcal{O}_{\mathcal{X}} \hat{\otimes} B_{\text{rig}, K}^\dagger$, and we have a notion of locally free $\mathcal{O}_{\mathcal{X}} \hat{\otimes} B_{\text{rig}, K}^\dagger$-module of finite rank.

Suppose we are given $\mathcal{X}/E$ as above, and a locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{V}$ of finite rank $d$ equipped with a continuous, linear action $G_K$ of $\mathcal{O}_{\mathcal{X}}$. If $\mathcal{U} = \text{Spm} \, S \subseteq \mathcal{X}$ is any admissible affinoid neighborhood over which $\mathcal{V}$ is free, then we can apply the theory of Berger–Colmez to obtain $D_{\text{rig}}^\dagger(\mathcal{V}|_\mathcal{U})$. One can check that the constructions of Berger–Colmez are compatible with localization to admissible affinoid subdomains of $\mathcal{U}$, so that the rule $\mathcal{U} \mapsto D_{\text{rig}}^\dagger(\mathcal{V}|_\mathcal{U})$ on admissible affinoid subdomains such that $\mathcal{V}|_\mathcal{U}$ is free determines a sheaf of locally free $\mathcal{O}_{\mathcal{X}} \hat{\otimes} B_{\text{rig}, K}^\dagger$-modules of rank $d$ on $\mathcal{X}$, which we call $\mathcal{D}$. It is equipped with commuting, continuous, semilinear actions of $\varphi$ and $\Gamma_K$, and satisfies $\mathcal{D}_x \cong D_{\text{rig}}^\dagger(\mathcal{V}_x)$ for all points $x \in \mathcal{X}$. (When writing "$x \in \mathcal{X}$", we always mean $x$ is a physical point of $\mathcal{X}$, in the sense of Tate’s rigid analytic spaces. The residue field $E(x)$ at $x$ is always finite over $\mathbb{Q}_p$, since we have assumed $E/\mathbb{Q}_p$ finite.)

### 4.2 Interpolation of the triangulordinary theory

It is our desire, in the future, to prove that a family consisting of triangulordinary representations admits a corresponding family of triangulordinary filtrations. We state our goal in a preliminary form, as the following conjecture.

Let $\mathcal{X}/E$ be a reduced, separated rigid analytic space, and let $\mathcal{D}$ be a locally free sheaf of $(\varphi, \Gamma_K)$-modules over $\mathcal{O}_{\mathcal{X}} \hat{\otimes} B_{\text{rig}, K}^\dagger$, of rank $d$. Let $0 < c < d$ be an integer. Consider the functor that associates to an $\mathcal{X}$-space $f : \mathcal{U} \to \mathcal{X}$ the collection of $(\varphi, \Gamma_K)$-stable $\mathcal{O}_{\mathcal{U}} \hat{\otimes} B_{\text{rig}, K}^\dagger$-local direct summands of $f^* \mathcal{D}$ rank $c$.

**Conjecture 4.1.** The functor described above is representable by a locally finite type morphism $p_c : \mathcal{X}(c) \to \mathcal{X}$. For each $x \in \mathcal{X}$, the fiber $\mathcal{X}(c)_x$ is a finite union of quasiprojective flag varieties over the residue field $E(x)$.

**Remark 4.2.** This conjecture is inspired by results of Kisin, notably [15, Proposition 5.4]. The statements proved there involve a number of technical hypotheses; thus, the
above conjecture may require some slight changes. Under Kisin’s hypotheses, we expect that his methods may be translated into the \((\varphi, \Gamma_K)\)-module language to establish the case where \(c = d - 1\). In any case, to prove the conjecture, it would suffice to work locally: to assume \(\mathcal{X}\) is affinoid, and to construct \(\mathcal{X}(c)\) with the desired property for maps \(\mathcal{U} \to \mathcal{X}\) with \(\mathcal{U}\) affinoid.

The underlying flag of a filtration \(F\) is simply the poset of its constituents \(F^i\), forgetting the indices. By a shape \(\sigma\) (of rank \(d\), and \(r\) constituents), we mean a finite sequence of dimensions \(\{d = d_0 > d_1 > \cdots > d_{r+1} = 0\}\). We say that a flag \(F\) of an object \(D\) has shape \(\sigma\) if \(\text{rank } D = d\) and its constituents have dimensions given precisely by the dimensions \(d_i\) of \(\sigma\); a filtration has shape \(\sigma\) if its underlying flag does. We can consider the integer \(c\) from above as the shape with one constituent given by \(\{d > c > 0\}\). If \(\sigma\) is an arbitrary shape of rank \(d\) then, by inducting on the number of constituents, Conjecture 4.1 implies the existence of a locally finite type morphism \(p_\sigma: \mathcal{X}(\sigma) \to \mathcal{X}\) classifying \((\varphi, \Gamma_K)\)-stable flags in \(\mathcal{D}\) with shape \(\sigma\). We write \(\mathcal{D}(\sigma) := p_\sigma^* \mathcal{D}\), and denote by \(F(\sigma)\) the corresponding universal flag in \(\mathcal{D}(\sigma)\) of shape \(\sigma\).

**Remark** 4.3. In the case \(\sigma\) is the shape of a complete flag, Bellaïche–Chenevier give in [1, Proposition 2.5.7] an affirmative answer to the Conjecture 4.1, at least infinitesimally locally: they prove the representability of the related deformation problem. They go on to undertake a considerably detailed study of what amounts to the formal completion of \(\mathcal{X}(\sigma)\) at a crystalline point.

We go on to explain how Conjecture 4.1 in the more general form just explained, should lead to triangulordinary filtrations on the level of families. First, we make precise what the latter means.

We call \(\mathcal{D}\) pretriangulordinary of shape \(\sigma\) if there is a Zariski dense subset \(\mathcal{D}^\text{alg} \subset \mathcal{X}\), all of whose points \(x \in \mathcal{D}^\text{alg}\) satisfy the following property: the \((\varphi, \Gamma_K)\)-module \(\mathcal{D}_x\) is triangulordinary, with some (hence every) triangulordinary filtration of shape \(\sigma\). We call \(\mathcal{D}\) triangular ordinary with respect to \(F\), where \(F^* \subset \mathcal{D}\) is a decreasing, separated and exhaustive filtration by \((\varphi, \Gamma_K)\)-stable \(O_\mathcal{X} \otimes B^{\text{rig}, K}\)-local direct summands, if there is a Zariski dense subset \(\mathcal{D}^\text{alg} \subset \mathcal{X}\) with the following property: for all points \(x \in \mathcal{D}^\text{alg}\), the image \(F^*_x\) has underlying flag equal to the underlying flag of some triangulordinary filtration of \(\mathcal{D}_x\). (For such \(x\), the choices of indices making \(F^*_x\) into the triangulordinary filtration are then uniquely determined by the Hodge–Tate weights.) Clearly, if \(\mathcal{D}\) is triangulordinary with respect to \(F\), and \(F\) has shape \(\sigma\), then \(\mathcal{D}\) is pretriangulordinary of shape \(\sigma\).

When \(\mathcal{D} = D^\text{rig}_\text{alg}(\mathcal{V})\), we say that \(\mathcal{V}\) is pretriangulordinary of shape \(\sigma\) (resp. triangulordinary with respect to \(F\)) if the said condition holds for \(\mathcal{D}\).

Suppose \(\mathcal{D}\) is pretriangulordinary of shape \(\sigma\), and let \(p_\sigma: \mathcal{X}(\sigma) \to \mathcal{X}\) classify \((\varphi, \Gamma_K)\)-stable flags of shape \(\sigma\), as above. We let \(\mathcal{X}^\text{alg} = \mathcal{X}^\text{alg}(\sigma)\) be the set of \(x \in p_\sigma^{-1} \mathcal{D}^\text{alg}\) such that \(F(\sigma)_x\) is the underlying flag of a triangulordinary filtration on \(\mathcal{D}_x\), and we let \(\mathcal{X}^\text{ord}\) be the Zariski closure of \(\mathcal{X}^\text{alg}\) inside \(\mathcal{X}\). We write \(\mathcal{D}_\text{ord}\) (resp. \(F_\text{ord}\)) for the restriction of \(\mathcal{D}(\sigma)\) (resp. \(F(\sigma)\)) to \(\mathcal{X}^\text{ord}\). By construction, \(\mathcal{D}_\text{ord}\) is a triangulordinary family over \(\mathcal{X}^\text{ord}\) of shape \(\sigma\) with respect to any choice of indices making the flag \(F_\text{ord}\) into a filtration. Moreover, by the hypothesis that \(\mathcal{X}\) is pretriangulordinary, the restriction of \(p_\sigma\) is a surjection \(\mathcal{D}^\text{ord} = \mathcal{D}^\text{alg}\), so that \(\mathcal{X}^\text{ord}\) is rather substantial in comparison with \(\mathcal{X}\).
Remark 4.4. It is not clear from the above discussion whether the construction singles out a choice of indices for the triangulordinary flag $F_{\nabla_{\text{ord}}}$ on $\mathcal{D}_{\nabla_{\text{ord}}}$. We would hope for the “most appropriate” choice of indexing to have the following property: for all $x \in \mathcal{X}_{\text{alg}}$, the constituent $F^1$ has image in $\mathcal{D}_x$ equal to the $F^1$ of some triangulordinary filtration on $\mathcal{D}_x$ satisfying the hypotheses of Theorem 3.1(2). Thus, the choice of indexing does affect which Selmer group is obtained from the definition given in the next section.

Whether a “most appropriate” indexing exists, and (if it exists) which indexing it is, are sensitive to the specification of $X_{\text{alg}} \subset X$. See Remark 4.10 for an example. (Although we have not stressed this, the construction of $X_{\nabla_{\text{ord}}}$ itself depends on $X_{\text{alg}}$.)

Example 4.5. Assume the notation at the end of §3.1, with $K = \mathbb{Q}$ and $p > 2$. Fix a 2-dimensional, irreducible, odd representation $\rho$ of $G_{\mathbb{Q},S}$ with values in the residue field $k_E$ of $E$. We take for $X$ the generic fiber of $\text{Spf} \, R_{S,\text{univ}}^\text{univ} (\rho)$, where $R_{S,\text{univ}}^\text{univ} (\rho)$ is the universal $\mathcal{O}_E$-valued deformation ring with “unramified” local conditions away from $S$, and no conditions at $S$. We take for $\mathcal{V}$ the universal representation on this space, and $\mathcal{D} = D_{\text{rig}}^\dagger (\mathcal{V}|_{G_p})$. Since the set $\mathcal{X}_{\text{alg}}$ of points $x \in \mathcal{X}$ for which $\mathcal{V}|_{G_p}$ is semistable with distinct Frobenius eigenvalues is Zariski dense, one can deduce that $\mathcal{D}$ is pretriangulordinary of shape $\sigma = \{2 > 1 > 0\}$.

Granting Conjecture 4.1, we expect that $\mathcal{X}_{\nabla_{\text{ord}}}(\sigma)$ is none other than the eigensurface of Coleman–Mazur (discussed towards the end of §4.5). Its restriction to the subspace of $\mathcal{X}$ having a vanishing Sen weight is expected to be the eigencurve, obtained as the resolution of the infinite fern of Gouvèa–Mazur [17] at its double points. Thus, our setup ought to give a clean realization of Kisin’s hope of constructing general eigenvarieties purely Galois-theoretically.

4.3 Selmer groups via variation

In order to define Selmer groups of families of Galois representations, we need to give a meaning to the Galois cohomology of a family. Let $G$ be a profinite group acting continuously on a locally free module $\mathcal{V}$ of finite rank over a rigid analytic space $\mathcal{X}$. We let

$$H^i(G, M) := \text{Ext}^i_{\mathcal{O}_{\mathcal{X}}(G)}(1, M),$$

the Yoneda group in the category of locally free $\mathcal{O}_{\mathcal{X}}$-modules with continuous $G$-actions. As is customary, when $G = G_K$ is the absolute Galois group of a field $K$, we write $H^i(K, M)$ for $H^i(G_K, M)$.

We now resume the notation at the end of §3.1. Namely, $K/\mathbb{Q}$ is a finite extension, $S$ is a finite set of places, and we have algebraic closures $\overline{K}_v$ containing $K_S$ and maps $G_v \to G_{K,S}$, for $v \in S$.

We let $\mathcal{X}/E$ be as in the preceding section, and let $\mathcal{V}$ be a locally free sheaf on $\mathcal{X}$ of finite rank, equipped with a continuous, $\mathcal{O}_{\mathcal{X}}$-linear $G_{K,S}$-action. We assume, for each place $v$ of $K$ with $v \mid p$, that $\mathcal{V}|_{G_v}$ is triangulordinary with respect to some filtration $F^*_v \subseteq \mathcal{D}_v := D_{\text{rig}}^\dagger (\mathcal{V}|_{G_v})$, in the sense described in §4.2. (Whether or not Conjecture 4.1 holds, we assume here that we are simply given the $F^*_v$.) For such $v$ we
define the local condition at $v$ to be

$$H^1_{\text{ord}}(K_v, \mathcal{V}) = \ker \left[ H^1(K_v, \mathcal{V}) = \text{Ext}^1_{\mathcal{O}_x[G_v]}(1, \mathcal{V}) \right]$$

Thus, it is reasonable to define the Selmer group of such points this agrees with the "g" local condition, and at most such points this agrees specialization-by-specialization, everywhere we are able to.

"g+" Bloch–Kato local conditions at the points of $\mathcal{X}$--algebra. An analogue of Mazur’s control theorem holds: when can we bound the kernel and is a natural specialization map $V$ of the filtration on $V$ over $S$ define the local condition at $v$ to be

$$H^1_{\text{ord}}(K_v, \mathcal{V}) = \ker \left[ H^1(K_v, \mathcal{V}) = \text{Ext}^1_{\mathcal{O}_x[G_v]}(1, \mathcal{V}) \right]$$

Assuming that, for $x \in \mathcal{X}$--alg, the specialization $(F^1_x)_x$ is the $F^1$ of a triangulordinary filtration on $\mathcal{Y}_x$, the above definition provides an interpolation over all of $\mathcal{X}$ of the "g+" Bloch–Kato local conditions at the points of $\mathcal{X}$--alg. By Proposition 3.14, at all such points this agrees with the "g" local condition, and at most such points this agrees with the "f" condition. Thus, it is reasonable to define the Selmer group of $\mathcal{V}$ over $\mathcal{X}$ to be

$$H^1_{\text{ord}}(K, \mathcal{V}) = \ker \left[ H^1(G_{K,S}, \mathcal{V}) \right]$$

Remark 4.6. The map labeled $\ast$ in Equation 6 is not known to be an isomorphism. In fact, in contrast to the situation of Theorem 2.3, the map $D^1_{\text{rig}}$ from families of Galois representations to étale families of $(\varphi, \Gamma_K)$-modules is not an equivalence of categories. Chenevier has given the following counterexample. Denote by $K(\mathcal{T})$ the Tate algebra over $K$ in the variables $\mathcal{T}$. Let $D = \mathbb{Q}_p(T, T^{-1}) \otimes_{\mathbb{Q}_p} B^1_{\text{rig}, K} \cdot e$, so that $D$ has rank 1 with basis element $e$, with actions given by $\Gamma_K \cdot e = e$ and $\varphi(e) = Te$. There is no Galois representation $V$ over $\mathbb{Q}_p(T, T^{-1})$ for which $D^1_{\text{rig}}(V) = D$.

We ask whether it is still the case that $\ast$ is an isomorphism: if two $(\varphi, \Gamma_K)$-modules over $S \otimes_{\text{rig}, K} B^1_{\text{rig}, K}$ come from $S$-representations of $G_K$, does every extension between them come from an $S$-representation? In any case, for every $x \in \mathcal{X}$--alg the diagram

$$
\begin{array}{ccc}
H^1(K_v, \mathcal{V}) & \rightarrow & \text{Ext}^1(1, \mathcal{D}) \\
\downarrow & & \downarrow \\
H^1(K_v, \mathcal{V}_x) & \sim & \text{Ext}^1(1, \mathcal{D}_x)
\end{array}
$$

commutes, so we at least know that we are imposing the correct local condition, specialization-by-specialization, everywhere we are able to.

We also obtain notions of Selmer groups $H^1_{\text{ord}}(K, \mathcal{V}_x)$ for all specializations $\mathcal{V}_x$ of $\mathcal{V}$ with $x \in \mathcal{X}$: namely, we define $F^1_{x,v}$ to be $(F^1_{x,v})_x$, and add subscripts $x$ everywhere in Equations 6 and 7. It follows from the commutativity of Diagram 8 above that there is a natural specialization map

$$H^1_{\text{ord}}(K, \mathcal{V})_x \rightarrow H^1_{\text{ord}}(K, \mathcal{V}_x)$$

for each $x \in \mathcal{X}$. Perhaps the most important open question in our program is whether an analogue of Mazur’s control theorem holds: when can we bound the kernel and
cokernel of the above map? Can this bounding be achieved, uniformly for \( x \) varying through a substantial subset of \( \mathcal{H} \)? Although we strongly desire to check this in a concrete setting, at present we cannot handle any particular nonordinary case.

We go on now to discuss in detail our model example: the eigencurve of Coleman–Mazur.

### 4.4 Review of the eigencurve

We continue with the notations of the end of \[ \text{3.1} \] We assume \( K = \mathbb{Q} \), and we fix positive integer \( N \) not divisible by \( p \), which we call the tame level. We take for \( S \) the set of primes dividing \( p \) and \( N \), together with the place \( \infty \).

By the weight space \( \mathcal{W} \) we mean the rigid analytic space over \( \mathbb{Q}_p \) arising as the generic fiber of \( \text{Spf} \mathbb{Z}_p[\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times] \). Its points \( \mathcal{W}(R) \) correspond to continuous characters of the form \( \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \to R^\times \). By class field theory, one can consider \( \mathcal{W} \) as being equipped with a free rank 1 bundle \( \mathcal{T} \) on which \( G_{\mathbb{Q},S} \) acts through its universal character. We let \( \mathcal{W}^{\text{alg}} \) consist of those points \( w \in \mathcal{W} \) corresponding to characters having the form \( a \mapsto a^{k_w} \) on some open subgroup of \( \mathbb{Z}_p^\times \subseteq \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \), with \( k_w \) an integer. Clearly, \( \mathcal{W} \) is triangulordinary of shape \( \{ 1 > 0 \} \).

The eigencurve \( \mathcal{E} = \mathcal{E}_{p,N} \), defined in \[ \text{7} \] in the case \( p > 2 \) and \( N = 1 \), and extended to general \( p \) and \( N \) by a variety of authors, is the following object. It is a rigid analytic space over \( \mathbb{Q}_p \), locally-on-the-base finite over \( \mathcal{W} \times (\mathcal{B}_1(0) \setminus \{ 0 \}) \), and locally-in-the-domain finite flat over \( \mathcal{W} \). Here, \( \mathcal{B}_1(0) = \text{Spm} \mathbb{Q}_p(U_p) \). The map to \( \mathcal{W} \) is called the weight (or, more precisely, weight-nebentypus), the map to \( \mathcal{B}_1(0) \) is called the \( U_p \)-eigenvalue, and the latter’s composite with the valuation map is called the slope. Finally, \( \mathcal{E} \) parameterizes a universal rigid analytic family of pairs \( (f, \alpha) \) with \( f \) a \( p \)-adic overconvergent elliptic modular eigenform of tame level \( N \) and \( \alpha \) a nonzero (“finite slope”) \( U_p \)-eigenvalue of \( f \). We let \( \mathcal{E}^{\text{alg}} \) be the collection of points \( x \in \mathcal{E} \) corresponding to pairs \( (f_x, \alpha_x) \) with \( f_x \) classical of weight \( k_x \geq 2 \), and with \( f_x \) at worst semistable at \( p \).

We remove two types of bad points on \( \mathcal{E} \). Namely, we say that \( x \in \mathcal{E} \) has critical slope if its weight \( w \) lies in \( \mathcal{W}^{\text{alg}} \) with \( k_w \) as above, \( k_w \geq 2 \), and \( k_w - 1 \) is the slope of \( x \). (This agrees with the terminology of \[ \text{3.6} \] except that it also includes some nonclassical \( x \).) We say that \( x \in \mathcal{E}^{\text{alg}} \) has does not have distinct eigenvalues if \( \alpha_x \) is a double root of the \( p \)-Hecke polynomial of (the newform associated to) \( f_x \). Write \( \mathcal{E}^0 \subseteq \mathcal{E} \) for the complement of the critical-slope and not-distinct-eigenvalue loci, and \( \mathcal{E}^{0,\text{alg}} = \mathcal{E}^0 \cap \mathcal{E}^{\text{alg}} \). (One can do slightly better as regards critical slope, and instead look at the complement of the points in the image of the \( \theta^{k-1} \)-map for each \( k \geq 2 \).)

The constructions of \( \mathcal{E} \) give rise to a locally free rank 2 bundle \( \mathcal{V}^0 \) over \( \mathcal{E}^0 \), equipped with a continuous, \( \mathcal{O}_{\mathcal{E}^0} \)-linear action of \( G_{\mathbb{Q},S} \). This representation has the property that, for any \( x \in \mathcal{E}^{0,\text{alg}} \), the fiber \( \mathcal{V}^0_x \) is isomorphic to the Galois representation \( V_{f_x}^{\text{hom}} \) associated to \( f_x \) by Deligne.

**Remark 4.7.** The reader will note that the Galois representation \( V_{f_x}^{\text{hom}} \) is defined for every \( x \in \mathcal{E} \) with \( f_x \) classical of weight \( k_x \geq 2 \). Our restriction to semistable points is because they would require modifying the triangulordinary theory to handle representations that become semistable over an abelian extension. We exclude \( \mathcal{E}^{\text{alg}} \setminus \mathcal{E}^{0,\text{alg}} \) from our consideration only because Kisin does so in \[ \text{15} \]; we have not tested whether our theory should make sense at these points.
We write $\mathcal{D}^0 = D_{\text{rig}}^\dagger(V_x^0|G_p)$, so that for $x \in \mathcal{E}^\text{alg}$ one has

$$\mathcal{D}^0_x \cong D_{\text{rig}}^\dagger(V_x^0|G_p) \cong D_{\text{rig}}^\dagger(V_x^\text{hom}|G_p).$$

For every $x \in \mathcal{E}^0,\text{alg}$ the representation $V_x^\text{hom}|G_p$ is semistable, and $\alpha_x$ is a $\varphi^{-1}$-eigenvalue of $\nu$ on $D_{\text{st}}(V_x^\text{hom}|G_p)$. Since we have removed the not-distinct-eigenvalue locus from $\mathcal{E}^0$, $D_{\text{st}}(V_x^\text{hom}|G_p)$ has distinct $\varphi$-eigenvalues, so the $\nu$-eigenspace gives rise to a canonical triangulordinary filtration

$$D_{\text{rig}}^\dagger(V_x^\text{hom}|G_p) = F^0_x \supseteq F^1_x = F^{k_w - 1}_x \supseteq F^{k_w}_x = 0, \quad (9)$$

as in [3.20] where $w$ is the weight of $x$. Therefore, $\mathcal{D}^0$ is pretriangulordinary of shape $\{2 > 1 > 0\}$. From now on, we denote this particular shape by $\sigma$.

### 4.5 Expectations for the eigencurve

We expect that $\mathcal{D}^0$ is triangulordinary of shape $\sigma$ in the following precise sense:

**Conjecture 4.8.** There exists a unique filtration

$$\mathcal{D}^0 \supseteq F^1 \supseteq 0$$

by a $(\varphi, \Gamma_{Q_p})$-stable locally $O_{\mathcal{E}^0} \otimes B_{\text{rig}, Q_p}$-direct summand $F^1$ of rank 1 with the property that, for each $x \in \mathcal{E}^0,\text{alg}$, $(F^1)_x = F^1_x$ under the identification of $\mathcal{D}^0_x \cong D_{\text{rig}}^\dagger(V_x^0|G_p) \cong D_{\text{rig}}^\dagger(V_x^\text{hom}|G_p)$.

An equivalent way of formulating the conjecture is as follows. Let $\mathcal{F} \subset \mathcal{D}^0$ be the subsheaf defined by

$$\mathcal{F} := \bigcap_{x \in \mathcal{E}^0,\text{alg}} \ker \left[ \mathcal{D}^0 \to \mathcal{D}^0_x / F^1_x \right].$$

Then we may phrase Conjecture 4.8 as asserting the existence of a unique $(\varphi, \Gamma_{Q_p})$-stable locally $O_{\mathcal{E}^0} \otimes B_{\text{rig}, Q_p}$-direct summand $F^1$ of rank 1 contained in $\mathcal{F}$. If so, then the specialization maps $\mathcal{D}^0_x \cong B_{\text{rig}}^\dagger(V_x^0|G_p)$ automatically identify $(F^1)_x \cong F^1_x$ for all $x \in \mathcal{E}^0,\text{alg}$.

Suppose that Conjecture 4.4 holds with $d = 2$ and $c = 1$, so that the morphism $p_{\sigma} : \mathcal{E}^0(\sigma) \to \mathcal{E}^0$ exists. Then Conjecture 4.8 says that the assignment

$$\mathcal{E}^0,\text{alg} \rightarrow \mathcal{E}^0,\text{ord} \subset \mathcal{E}^0(\sigma), \quad x \mapsto (x, F^1_x)$$

extends uniquely to a section $\mathcal{E}^0 \rightarrow \mathcal{E}^0(\sigma)$ of $p_{\sigma}$. In fact, any such section must have image in $\mathcal{E}^0,\text{ord}$, and we expect that this is the only section of $p_{\sigma}|_{\mathcal{E}^0,\text{ord}}$. We envision that $\mathcal{E}^0,\text{ord}$ can be divided into two parts: a component mapping isomorphically onto $\mathcal{E}^0$, and a disjoint union of points, one lying over each $x \in \mathcal{E}^0,\text{alg}$ with $f_x$ crystalline at $p$, corresponding to the “evil twin” of $(f_x, \alpha_x)$ (as in [17]).

The reader is advised to take note of the difference between the above picture and Example 4.5.

Conjecture 4.8 gives us a definition of the Selmer group over the eigencurve, as well as Selmer groups for all finite-slope overconvergent modular eigenforms, as in Equation 7.
Remark 4.9. An arbitrary (especially, nonclassical) \( x \in \mathcal{E}^0 \) is known to be trianguline by work of Kisin and Colmez. Let \( \mathbb{Q}_p(x) \) denote the residue field of \( \mathcal{E}^0 \) at \( x \), and \( (f_x, \alpha_x) \) the corresponding \( \mathbb{Q}_p(x) \)-valued overconvergent eigenform and \( U_p \)-eigenvalue. Then [13, Theorem 6.3] shows the existence of a nonzero, \( G_p \)-equivariant map \( V_{f_x}^{\text{hom}} \rightarrow (B_{\text{cry}}^+ \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(x))^\varphi=\alpha_x \). This is equivalent to a nonzero vector in \( D_{\text{crys}}(V_{f_x}^{\text{coh}}|_{G_p})^\varphi=\alpha_x \).

Then [8, Proposition 5.3] implies that \( V_{f_x}^{\text{hom}} \) is trianguline at \( p \), and so is \( V_{f_x}^{\text{hom}} \).

The trianguline subspace \( F \) inside \( D_{\text{rig}}(V_{f_x}^{\text{hom}}|_{G_p}) \) ought to coincide with the putative triangulordinary filtration \( F_x^1 \) described above when Conjecture 4.8 holds. In any case, using \( F \) in place of \( F_x^1 \), we obtain a definition of a local condition and Selmer group without assuming any conjecture.

We remind the reader that, although the work of Kisin and Colmez gives us trianguline filtrations at every point, they do not directly give us a filtration on the family.

A related example is the cyclotomic deformation of \( \mathcal{E}^0 \). This is the bundle \( \tilde{\mathcal{E}}^0 \) over the eigensurface \( \mathcal{E}^0 = \mathcal{E}^0 \times \mathcal{W} \) determined by \( p_1^1 \mathcal{E}^0 \otimes_{\tilde{\mathcal{E}}} p_2^1 \mathcal{F} \), where the \( p_i \) are the projections of \( \mathcal{E}^0 \) onto the respective factors. Letting \( \mathcal{E}_{\text{alg}}^0 = \mathcal{E}^0 \otimes_{\tilde{\mathcal{E}}} \mathcal{W} \), we see that \( \tilde{\mathcal{E}}^0 \) is pretriangulordinary of shape \( \sigma = \{ 2 > 1 > 0 \} \). Assuming Conjecture 4.8 and setting \( \tilde{F}^1 = p_1^1 F^1 \otimes_{\tilde{\mathcal{E}}} p_2^1 \mathcal{F} \), we see that \( \tilde{\mathcal{E}}^0 \) is also triangulordinary of shape \( \sigma \).

Remark 4.10. In the case of the eigencurve \( \mathcal{E}^0 \), every \( \mathcal{E}_{\mathcal{E}_{\text{alg}}}^0 \) with \( x \in \mathcal{E}^0 \otimes_{\tilde{\mathcal{E}}} \mathcal{W} \) has Hodge–Tate weights 0 and \( k - 1 > 0 \). Therefore, as seen in Equation 9, the triangulordinary filtrations all have \( F_x^1 \) equal to their rank-1 constituent. When assigning indices to the putative triangulordinary flag \( F \) on \( \mathcal{D}^0 \) given by Conjecture 4.8, this fact forces us to take \( F^1 \) to be the rank-1 constituent. In other words, the Galois theory provided us with a natural choice of filtration indexing, and, by consequence, a natural choice of Selmer group.

Consider the universal character \( \mathcal{F} \) of \( G_{\mathbb{Q}, S} \) over weight space \( \mathcal{W} \). The unique Hodge–Tate weight of \( w \in \mathcal{W} \) is \( k_w \), and these integers vary without bound. Thus there is “most appropriate” index at which to situate the jump in the triangulordinary filtration, compatibly over all of \( \mathcal{W} \) as defined in (17). Another viewpoint is that \( \mathcal{F} \) is the cyclotomic deformation of the trivial character \( \chi_{\text{triv}} \), and hence its triangulordinary filtration should be chosen to deform the natural one for \( \chi_{\text{triv}} \). Since \( \chi_{\text{triv}} \) has Hodge–Tate weight 0, this means taking \( \Gamma^0 \neq 0 \), and, in particular, \( F^1 = 0 \). Another way of achieving this would be to reduce \( \mathcal{W} \) to its subset consisting of those \( w \) with \( k_w = 0 \) (which is still Zariski dense). A third option is to note that \( \mathcal{F} \) is also the cyclotomic deformation of the cyclotomic character \( \chi_{\text{cycl}} \), which corresponds to replacing \( \mathcal{W} \) with the subset defined by \( k_w = 1 \), and which suggests taking \( F^1 = \mathcal{F} \). Thus, depending on the choice of \( \mathcal{W} \) as the most appropriate indexing of the triangulordinary flag either does not exist, has \( F^1 = 0 \), or has \( F^1 = \mathcal{F} \). The latter two possibilities give two different Selmer local conditions at \( p \) (respectively, they are the unramified and empty conditions).

The ambiguity described above passes on to \( \tilde{\mathcal{E}}^0 \): at a point \( (x, w) \), where \( x \) has weight \( k_x \) and \( w \) has weight \( k_w \), the Hodge–Tate weights of \( \tilde{\mathcal{E}}^0_{(x, w)} \) are \( k_w \) and \( k_w + k_x - 1 > k_w \), which vary roughly independently; thus \( \tilde{\mathcal{E}}_{\text{alg}}^0 \) does not admit a most appropriate choice of indices. Since we view the eigensurface as a cyclotomic deformation of the eigencurve, we expect that considering \( \mathcal{F} \) as the cyclotomic deformation of the trivial character is most appropriate (in this particular setting). This means reducing \( \tilde{\mathcal{E}}_{\text{alg}}^0 \) to the subset defined by \( k_w = 0 \), and taking for \( \tilde{F}^1 \) the rank-1 constituent of the
triangulordinary flag, which is given by $p_1^*F^1 \otimes_{\mathcal{O}} p_2^*\mathcal{T}$.

Since the reader is likely to be aware of the goals of Iwasawa theory, we conclude by saying that we expect the Selmer group $H^1_{\text{Tor}}(\mathbb{Q}, \mathcal{E}^0)$ (resp. $H^1_{\text{Tor}}(\mathbb{Q}, \tilde{\mathcal{E}}^0)$) to be related to the analytic standard $p$-adic $L$-function varying along the eigencurve (resp. eigensurface). But, the Selmer groups being highly non-integral (and likely non-torsion), and the $p$-adic $L$-functions being unbounded, the precise means by which these ought to be related related is far from clear.
References

[1] Joël Bellaïche and Gaëtan Chenevier. $p$-adic families of galois representations and higher rank selmer groups. arXiv:math/0602340v2.

[2] Laurent Berger. Équations différentielles $p$-adiques et $(\varphi, N)$-modules filtrés. Preprint (November 2006), available online at http://www.umpa.ens-lyon.fr/~lberger/publications.html.

[3] Laurent Berger. Représentations $p$-adiques et équations différentielles. Invent. Math., 148(2):219–284, 2002.

[4] Laurent Berger and Pierre Colmez. Familles de représentations de de rham et monodromie $p$-adique. Preprint (February 2007), available online at http://www.umpa.ens-lyon.fr/~lberger/publications.html.

[5] Spencer Bloch and Kazuya Kato. $L$-functions and Tamagawa numbers of motives. In The Grothendieck Festschrift, Vol. I, volume 86 of Progr. Math., pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[6] F. Cherbonnier and P. Colmez. Représentations $p$-adiques surconvergentes. Invent. Math., 133(3):581–611, 1998.

[7] R. Coleman and B. Mazur. The eigencurve. In Galois representations in arithmetic algebraic geometry (Durham, 1996), volume 254 of London Math. Soc. Lecture Note Ser., pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[8] Pierre Colmez. Série principale unitaire pour GL$_2$($\mathbb{Q}_p$) et représentations triangulines de dimension 2. Preprint (2005), available online at http://www.math.jussieu.fr/~colmez/triangulines.pdf.

[9] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations $p$-adiques semi-stables. Invent. Math., 140(1):1–43, 2000.

[10] Matthias Flach. A generalisation of the Cassels-Tate pairing. J. Reine Angew. Math., 412:113–127, 1990.

[11] Jean-Marc Fontaine. Représentations $p$-adiques des corps locaux. I. In The Grothendieck Festschrift, Vol. II, volume 87 of Progr. Math., pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

[12] Ralph Greenberg. Iwasawa theory for $p$-adic representations. In Algebraic number theory, volume 17 of Adv. Stud. Pure Math., pages 97–137. Academic Press, Boston, MA, 1989.

[13] Ralph Greenberg. Iwasawa theory and $p$-adic deformations of motives. In Motives (Seattle, WA, 1991), volume 55 of Proc. Sympos. Pure Math., pages 193–223. Amer. Math. Soc., Providence, RI, 1994.

[14] Kiran S. Kedlaya. Slope filtrations for relative frobenius. arXiv:math/0609272v2.

[15] Mark Kisin. Overconvergent modular forms and the Fontaine-Mazur conjecture. Invent. Math., 153(2):373–454, 2003.

[16] Ruochuan Liu. Cohomology and duality for $(\varphi, \Gamma)$-modules over the Robba ring. To appear in IMRN. arXiv:0711.4346v1.
[17] B. Mazur. An “infinite fern” in the universal deformation space of Galois representations. *Collect. Math.*, 48(1-2):155–193, 1997. Journées Arithmétiques (Barcelona, 1995).

[18] Bernadette Perrin-Riou. Théorie d’Iwasawa des représentations $p$-adiques sur un corps local. *Invent. Math.*, 115(1):81–161, 1994. With an appendix by Jean-Marc Fontaine.