A test of the Kugo-Ojima confinement criterion by lattice Landau gauge QCD simulations

Hideo Nakajima*
Department of Information science, Utsunomiya University
Sadakata Furui †
School of Science and Engineering, Teikyo University

The first test of the Kugo-Ojima colour confinement criterion by the lattice Landau gauge QCD simulation is performed. The parameter $u$ which is expected to be $-1\delta_{ab}$ in the continuum theory was found to be $-0.7\delta_{ab}$ in the strong coupling region. The data is analysed in connection with the theory of Zwanziger. In the weak coupling region, the expectation value of the horizon function is negative or consistent to 0.

1. Introduction

There are various manifestation of colour confinement in QCD. One is the linear potential between quarks, which appears in the quenched lattice QCD simulation. A mechanism for the appearance of the linear potential was proposed as well by Gribov [1] about 20 years ago. He showed that the Landau gauge fixing or the Coulomb gauge fixing does not fix the gauge field $A_\mu$ uniquely and that the restriction of $A_\mu$ to a physical space which is called the Gribov region enhances the singularity in the ghost propagator and induces the linear potential.

In the field theory, the confinement implies absence of free single coloured particle state in the asymptotic Hilbert space. The physical Hilbert space should satisfy symmetry specified by the Lagrangian, and the QCD Lagrangian is invariant under the BRS(Becchi-Rouet-Stora) transformation. Kugo and Ojima[2] proposed in 1978, a criterion for the colour confinement in the Landau gauge, based on the BRS symmetry, which consists of a two-point function produced by the ghost, the antighost and the gauge field becomes $-\delta_{ab}$, where $a$ and $b$ specify the colour in the adjoint representation. Analytical calculation of this value is extremely difficult and so far no verification was performed.

In the lattice QCD, the Gribov region still does not define the gauge field uniquely but there is a unique minimum $\| A \|$ in the fundamental modular region[3]. He argued that the restriction to the fundamental modular region implies the regularity of the horizon tensor, whose transverse projection is identical to the $-\delta_{ab}$ of the two-point function of Kugo-Ojima.

Both the Gribov-Zwanziger’s theory and the Kugo-Ojima’s theory suggest that in the lattice Landau gauge, the gluon propagator is infrared finite, which is confirmed by the lattice QCD simulation[4-6]. In this paper we simulate the two-point function specified

* e-mail nakajima@is.utsunomiya-u.ac.jp
† e-mail furui@dream.ics.teikyo-u.ac.jp
by the Kugo-Ojima in the lattice Landau gauge.

2. The Kugo-Ojima confinement criterion and the Gribov-Zwanziger’s theory

2.1. Kugo-Ojima’s theory

A sufficient condition of the colour confinement given by Kugo and Ojima[2] is that \( u^a_b \) defined by the two-point function of the FP (Faddeev-Popov) ghost fields, \( c(x) \) and \( \bar{c}(y) \), and \( A_\nu(y) \),

\[
\int e^{ip(x-y)} \langle 0 | TD_\mu c^a(x) g(A_\nu \times \bar{c})_b(y) | 0 \rangle dx = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) u^a_b(p^2)
\]

satisfies \( u^a_b(0) = -\delta^a_b \).

Essential points in their argument is based on the BRS invariance of the QCD Langrangian accompanied by the gauge fixing and FP terms. 1) The Ward-Takahashi identities implies that the gauge fields \( A^a_\mu(x) \), the auxiliary field \( B^a \), the covariant derivative of the ghost field \( D_\mu c^a \) and the antighost field \( \bar{c}^a(x) \) necessarily have massless asymptotic fields which forms the BRS-quartet. The Hilbert space is decomposed into the FP ghost number eigenstates and in this bases, the BRS-quartet space have the zero-norm. 2) The BRS charge \( Q_B \) is conserved, and the Noether current corresponding to the conservation of the colour symmetry is

\[
g J^a_\mu = \partial^\nu F^a_\mu + \{Q_B, D_\mu \bar{c}\},
\]

where its ambiguity by divergence of antisymmetric tensor should be understood, and this ambiguity is utilised so that massless contribution may be eliminated for the charge, \( Q^a \), to be well defined. The massless component in the current \( \{Q_B, D_\mu \bar{c}\} \) is absent if \( 1 + u = 0 \).

2.2. Gribov-Zwanziger’s theory

The Landau gauge of the QCD is specified by \( \partial_\mu A_\mu = 0 \). The Gribov region \( \Omega \) is specified by ensemble of local minimum points of gauge orbits under the variation with respect to \( g = e^\epsilon \) as follows.

\[
\Delta \|A^g\|^2 = -2(\partial A |\epsilon\rangle + \langle \epsilon | - \partial D |\epsilon\rangle) + \cdots
\]

\[
\Omega = \{ A | - \partial D \geq 0 , \partial A = 0 \} .
\]

The physical space of the gauge field is characterized by the condition that the FP determinant is positive. In the Coulomb gauge, the singulatity of the ghost propagator yields enhancement of the infrared singularity of the Coulomb potential[3].

In the lattice simulation, the unique gauge field configuration can be attained by the restriction to the fundamental modular region \( \Lambda[3] \), which is specified by the absolute minimum along the gauge orbits.

\[
\Lambda = \{ A | \|A\|^2 = \text{Min}_g \|A^g\|^2 \}, \quad \Lambda \subset \Omega .
\]

Let the gauge configuration be in the fundamental modular region obtained by the optimising function \( I(U^g) = \sum_{x,\mu} (1 - \frac{1}{n} \text{Retr} U^g_{x,\mu}) \), and let it be a global minimum even
under the gauge transformation of larger period (the region is called the core region), and
the two point tensor be defined as \( G_{\mu\nu} = \langle \text{tr} \left( \lambda^a D_\mu \frac{1}{\partial D} (-D_\nu) \lambda^b \right) \rangle_{xy} \). Then, in the
Zwanziger’s theory\[^3,10\], the horizon function \( H(U) \) defined as
\[
\langle H(U) \rangle \equiv (N^2 - 1) \left[ \lim_{p \to 0} G_{\mu\nu}(p) - e \right]
\]
is negative or 0 in finite volume, and 0 in the infinite volume limit. Here \( N \) is the number of
colour, \( V \) is the lattice volume, and \( e = \langle \sum_l \frac{1}{N} \text{Retr} U_l \rangle / V \).

Note that in a \( d \) dimensional lattice \( e/d = 1 \) if all links \( U_l = 1 \), and that the value of
\( 1 - e/d \) has the meaning of the distance from this vacuum.

3. The Lattice simulation

We define the gauge field\[^3\] on links as an element of \( SU(3) \) Lie algebra as, and perform
the gauge transformation as \( e^{A_{x,\mu}} = g^A_{\mu} e^{A_{x,\mu}} g^\dagger \).

The Landau gauge is realised by minimising \( \| A \|^2 \) via a gauge transformation \( g^\dagger U g \),
where \( g = e^\epsilon \). \( \epsilon \) is obtained by solving the equation with a suitable parameter \( \eta \)
\[
\mathcal{M} \epsilon = -\partial_\mu D_\mu (A) \epsilon = \eta \partial A
\]
(7)

The obtained norm \( \| A \| \) is close to that obtained after the smeared gauge fixing\[^7\]
within 1%.

The inverse FP operator, \( \mathcal{M}^{-1} U = (M_0 - M_1[U])^{-1} \), is calculated perturbatively by
using the Green function of the Poisson equation\[^3\].

In use of colour source \( |\lambda^a x, x_0 \rangle \) normalised as \( \langle \text{tr} |\lambda^a x \rangle |\lambda^b x_0 \rangle = \delta^{ab} \delta_{x, x_0} \),
the ghost propagator is given by
\[
G^{ab}(x, y) = \langle \text{tr} |\lambda^a x \rangle (\mathcal{M}[U])^{-1} |\lambda^b y \rangle \rangle
\]
(8)
where the outmost \( \langle \rangle \) specifies average over samples \( U \).

The ghost propagator is infrared divergent and its singularity can be parametrised as
\[
\tilde{Z}_3 p^{-2-\alpha} = Z_3 (4 \sin^2 \frac{\pi}{2})^{-1-\alpha/2}
\]
where \( \beta = 5.5 \), \( 8^3 \times 16 \) lattice indicates that the singularity is approximately \( p^{-2.2} \), while the data of \( 16^4 \) lattice indicates that it is \( p^{-2.8} \).

\( \tilde{Z}_3 \) and \( Z_3 \) are given by
\[
\tilde{Z}_3 = \sum_{\lambda} |\lambda^a x_0 \rangle = \sum_{\lambda} |\lambda^b x_0 \rangle - \delta_{x, x_0}
\]
(9)
(10)
These qualitative features are in agreement with the analysis of the Dyson-Schwinger equation\[^8\].

Using the above inverse FP operator, we obtained that \( u_0^a(0) \) is consistent to \( -c\delta_0^a \), \( c = 0.7 \) in \( SU(3) \) quenched simulation, \( \beta = 5.5 \), on \( 8^4, 12^4 \) and \( 16^4 \). Fig.2 shows the value of
\( u_0^a \).

In terms of the Kugo-Ojima parameter \( c \), the function \( G_{\mu\nu}(0) \) can be written as \( 1 + 3c \).

In the \( 16^4 \) lattice, \( e/d = 0.78 \) and 0.86 for \( \beta = 5.5 \) and \( \beta = 6 \), respectively. They are numerically close to the inverse of the renormalization factor of the ghost propagator \( 1/\tilde{Z}_3 \). The Slavnov-Taylor relation \( Z_1 = Z_3 / \tilde{Z}_3 = 1 \) implies \( e/d \approx 1/\tilde{Z}_3 \).

Our data of \( 16^4 \) lattice suggest that when \( \beta \) becomes larger, \( G_{\mu\nu}(0) \) becomes smaller,
while the Zwanziger’s parameter \( e \) has the opposite tendency.

It is to be remarked that in the Zwanziger’s theory, Kugo-Ojima criterion \( c = 1 \) does
not hold in view of \( e/d \neq 1 \). In the case of \( \beta = 5.5 \), our data of \( G_{\mu\nu}(0) \) agrees with \( e \).
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Figure 1. The ghost propagator as function of the lattice momentum. The data are $\beta = 5.5, 16^4$. The fitted curve is $1.287/p^{2.779}$.

Figure 2. The Kugo-Ojima parameter $|u^a|$ as the function of the spatial extent of the lattice $aL(fm)$. The data are $\beta = 6, 8^4, 12^4, 16^4$ and $\beta = 5.5, 8^4, 12^4, 16^4$ from left to right, respectively.

| $\beta$ | $c$      | $G_{\mu\nu}(0)$ | $e$     | $e/d$   | $S$    | $1/Z_3$  |
|---------|----------|------------------|---------|---------|--------|----------|
| 5.5     | 0.712(18)| 3.14(5)          | 3.13(1) | 0.783   | 0.657  | 0.777    |
| 6.0     | 0.628(56)| 2.88(17)         | 3.45(1) | 0.863   | 0.694  | 0.860    |

However, when the optimising function $I(A^g) = \|A\|^2 = \sum_{x,\mu} tr A^g A^g$ is used instead of $I(U^g)$, the function $e/d$ is to be replaced by the link average $\langle S(A) \rangle = \langle \frac{A/2}{\tanh A/2} \rangle$, where $A = \text{adj}A$, and the value is reduced by about 20%. The positive horizon function for $\beta = 5.5$ implies our configurations are not in the core region. In the weak coupling region ($\beta = 6.0$), the horizon function is negative or consistent to 0.

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