Derived equivalence for stratified Mukai flop on $G(2, 4)$

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1 Introduction

We consider a conjecture on the derived equivalence for $K$-equivalent varieties in the case of a certain flop between 9-dimensional varieties. The following is a very general version of this conjecture on the fully faithful derived embedding for log $K$-related varieties ([11] Conjecture 2.2):

**Conjecture 1.1.** Let $(X, B)$ and $(Y, C)$ be pairs of quasi-projective varieties with $\mathbb{Q}$-divisors such that there exist quasi-finite and surjective morphisms $\pi : U \to X$ and $\sigma : V \to Y$ from smooth varieties, which may be reducible, such that $\pi^*(K_X + B) = K_U$ and $\sigma^*(K_Y + C) = K_V$. Let $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$ be natural morphisms from the associated Deligne-Mumford stacks. Assume that there are proper birational morphisms $\mu : W \to X$ and $\nu : W \to Y$ from a third variety $W$ such that $\mu^*(K_X + B) \leq \nu^*(K_Y + C)$. Then there exists a fully faithful exact functor $D^b(Coh(X)) \to D^b(Coh(Y))$.

The conjecture is proved to be true in some cases ([2], [3], [4], [9], [6], [13], [10], [11], and [1]).

On the other hand, Namikawa [14] proved that a naturally defined functor between the derived categories for the stratified Mukai flop on $G(2, 4)$ is not an equivalence. This is a flop between 9-dimensional varieties which is the total space of a 3-parameter degeneration of standard 3-dimensional flops of $(-1, -1)$-curves and the fiber over the most degenerate point is isomorphic to $G(2, 4)$.

So it is worthwhile to check the conjecture in this special case. We shall prove that there is nevertheless another functor between the same categories which is an equivalence.
2 Stratified Mukai flop

We recall the construction of stratified Mukai flops due to Markman [12] (see also [14]).

Let $G = G(r, n)$ be the Grassmann variety of $r$-dimensional subspaces in an $n$-dimensional vector space $V$. We assume that $2r \leq n$. Let $S$ (resp. $Q$) be the tautological subbundle (resp. quotient) bundle on $G$:

$$0 \rightarrow S \rightarrow V \otimes \mathcal{O}_G \rightarrow Q \rightarrow 0.$$ 

The polarization of $G$ is given by

$$\det Q = \mathcal{O}_G(1).$$

We have $\Omega^1_G \cong \mathcal{H}om(Q, S)$. Since

$$\text{Ext}^1(\mathcal{O}_G, \Omega^1_G) \cong \text{Ext}^1(Q, S) \cong \mathbb{C},$$

there exists a non-trivial extension of vector bundles on $G$ induced by the natural homomorphism $\mathcal{O}_G \rightarrow \mathcal{H}om(S, S)$:

$$0 \rightarrow \Omega^1_G \rightarrow \tilde{\Omega}^1_G \rightarrow \mathcal{O}_G \rightarrow 0.$$ 

$$0 \rightarrow \mathcal{H}om(Q, S) \rightarrow \mathcal{H}om(V \otimes \mathcal{O}_G, S) \rightarrow \mathcal{H}om(S, S) \rightarrow 0.$$ 

Let $X_0 = T^*G$ and $X$ be the total spaces of $\Omega^1_G$ and $\tilde{\Omega}^1_G$, respectively, and $\pi : X \rightarrow G$ the projection. We write $\pi^*S = S_X$ and $\pi^*\mathcal{O}_G(1) = \mathcal{O}_X(1)$. A point of $X$ is given by a pair consisting of a point $p \in G$ and a homomorphism $A : V \rightarrow S_p \subset V$ which induces a homothety $t : S_p \rightarrow S_p$ ($t \in \mathbb{C}$). We have $(p, A) \in X_0$ if and only if $t = 0$. Let

$$Z = \{A \in \text{End}(V) \mid \exists t \in \mathbb{C} \text{ s.t. } A^2 = tA, \text{rank } A \leq r\}$$

with a morphism $t : Z \rightarrow \mathbb{C}$, and set $Z_0 = t^{-1}(0)$. Then there is a natural morphism $\phi : X \rightarrow Z$ given by $\phi(p, A) = A$, which induces a morphism $\phi_0 : X_0 \rightarrow Z_0$. We know that $\phi_0$ and $\phi$ are projective birational morphisms. We write

$$W^{(k)} = \{(p, A) \in X \mid \text{rank } A \leq k\}.$$
Then the exceptional loci of both $\phi_0$ and $\phi$ are equal to $W = W^{(r-1)}$. We have $\dim G = r(n-r)$, $\dim X_0 = 2r(n-r)$, $\dim X = 2r(n-r) + 1$, and \(\dim W = 2(r-1)(n-r+1) + n - 2r + 1\).

We have a dual construction starting from the dual vector space $V^*$. Let $G^+$ be the Grassmann variety of $r$-dimensional subspaces in $V^*$, $S^+$ (resp. $Q^+$) the tautological subbundle (resp. quotient bundle) on $G^+$, $\det Q^+ = \mathcal{O}_{G^+}(1)$, $X^+_0$ (resp. $X^+$) the total space of $\Omega^1_{G^+}$ (resp. $\Omega^1_{G^+}$), $\pi^+ : X^+ \to G^+$ the projection, $\pi^+ S^+ = S^+_X$, and $\pi^+ \mathcal{O}_{G^+}(1) = \mathcal{O}_{X^+}(1)$. There is a natural projective birational morphism $\phi^+ : X^+ \to Z$ given by $\phi^+(q, B) = ^tB$ which induces a projective birational morphism $\phi_0^+ : X^+_0 \to Z_0$. We write

$$W^{+(k)} = \{(q, B) \in X^+ \mid \text{rank } B \leq k\}.$$ 

The exceptional locus of $\phi$ is $W^+ = W^{+(r-1)}$. The diagram

$$X \xrightarrow{\phi} Z \xleftarrow{\phi^+} X^+$$

thus obtained is called a stratified Mukai flop. By restricting to the subspaces defined by $t = 0$, we have a smaller diagram

$$X_0 \xrightarrow{\phi} Z_0 \xleftarrow{\phi^+} X^+_0$$

which is also called a stratified Mukai flop.

If $r = 1$, then the above diagrams are reduced to a standard flop and a Mukai flop.

The birational map $(\phi^+)^{-1} \circ \phi : X \to X^+$ is decomposed into blow-ups and downs $[12]$. We consider only the case $r = 2$ in the following. Let $f_i : X_i \to X$ be the blowing up along the center $G = W^{(0)}$, the 0-section of the projection $\pi$. Then the strict transform $W'$ of $W = W^{(1)}$ is smooth. Indeed, the projection $W' \to G$ is smooth and its fibers are isomorphic to the cone over $\mathbb{P}^1 \times \mathbb{P}^{n-3} \subset \mathbb{P}^{2n-5}$. Let $f_2 : X_2 \to X_1$ be the blowing up along the center $W'$. Let $E_i$ be the exceptional divisor of $f_i$ for $i = 1, 2$, and $E'_1 = f_2^{-1}E_1$, the strict transform of $E_1$. We set $f = f_2 \circ f_1$. If $r \geq 3$, then we have similar construction with more blow-ups $[12]$.

On the dual side, let $f_1^+ : X_1^+ \to X^+$ be the blowing up along the center $G^+ = W^{+(0)}$, the 0-section of the projection $\pi^+$. Then the strict transform $(W^+)'$ of $W^+ = W^{+(1)}$ is smooth. Let $f_2^+ : X_2^+ \to X_1^+$ be the blowing up along the center $(W^+)'$. Let $E_i^+$ be the exceptional divisor of $f_i^+$ for $i = 1, 2$, and $(E_i^+)' = (f_2^+)^{-1}E_1^+$. We set $f^+ = f_2^+ \circ f_1^+$. 

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The birational map \((\phi^+)^{-1} \circ \phi : X^- \to X^+\) induces an isomorphism \(X_2 \to X_2^+\) \([12]\). We write \(Y = X_2 = X_2^+\) under this identification. Then we have \(E_1' = (E_1^+)', E_2 = E_2^+\), and a commutative diagram

\[
\begin{array}{rcl}
A \in Z & \xrightarrow{\phi} & X \\
\| & & \| \\
A^+ \in Z^+ & \xleftarrow{\phi^+} & X^+ \\
\| & & \| \\
& \xleftarrow{f_1} & X_1 \xleftarrow{f_2} X_2 \\
\end{array}
\]

Since \(\dim X = 4n - 7\), \(\dim G = 2n - 4\) and \(\dim W = 3n - 5\), we have \(K_Y = f^*K_X + (2n - 4)E_1' + (n - 3)E_2\).

The isomorphism \(X_2 \to X_2^+\) can be described set theoretically in the following way.

1. For a point \((p, A) \in X \setminus W\), we have \(\dim A = 2\) and \(S_p = \text{Im} A\). The corresponding point \((q, \iota A) \in X^+ \setminus W^+\) is given by \(S_q^+ = \text{Im} \iota A = (\text{Ker} A)^{1}\).

2. For a point \(\lim_{\epsilon \to 0}(p, \epsilon A_1) \in E_1 \setminus W'\), with rank \(A_1 = 2\) and \(S_p = \text{Im} A_1\), the corresponding point is \(\lim_{\epsilon \to 0}(q, \epsilon A_1) \in E_1^+ \setminus (W^+)', \text{where} S_q^+ = \text{Im} \iota A_1\).

3. For a point \((p, A) \in W \setminus G\), we have \(\dim A = 1\) and \(S_p \supset \text{Im} A\). We take \(A_2\) such that \(S_p = \text{Im} (A + A_2)\). Then a point \(\lim_{\epsilon \to 0}(p, \epsilon A_2) \in E_2 \setminus E_1'\) corresponds to a point \(\lim_{\epsilon \to 0}(q, \epsilon (A + \epsilon A_2)) \in E_2^+ \setminus (E_1^+)',\text{which is over a point} q \in G^+\text{given by} S_q^+ = \lim_{\epsilon \to 0} \text{Im} \iota (A + \epsilon A_2)\).

4. For a point \(\lim_{\epsilon \to 0}(p, \epsilon A_1) \in E_1 \cap W'\) with rank \(A_1 = 1\) and \(S_p \supset \text{Im} A_1\), we take \(A_2\) such that \(S_p = \text{Im}(A_1 + A_2)\). Then a point \(\lim_{\epsilon \to 0}(p, \epsilon A_1 + \epsilon^2 A_2) \in E_1' \cap E_2\) corresponds to a point \(\lim_{\epsilon \to 0}(q, \epsilon (A_1 + \epsilon^2 A_2)) \in (E_1^+)' \cap E_2^+\text{which is over a point} q \in G^+\text{given by} S_q^+ = \lim_{\epsilon \to 0} \text{Im} \iota (A_1 + \epsilon A_2)\).

The above description shows that the birational map \(X_2^- \to X_2^+\) induced by \((\phi^+)^{-1} \circ \phi\) is an isomorphism by the Zariski main theorem. As a consequence, we have the following lemma:

**Lemma 2.1.**

\[S_X^*|_{X \setminus W} = S_X^+|_{X^+ \setminus W^+}.\]

Moreover, \(f^*S_X^*\) is a locally free subsheaf of \(f^{++}S_X^+\), such that a local section of the latter belongs to the former if and only if its value in \(S_q^+\) at any point in \((f^+)^{-1}(x^+)\) for \(x^+ = (q, B)\) is contained in \(\text{Im} B\).

**Proof.** For \((p, A) \in X \setminus W\), we have \(\text{Ker} \iota A = (\text{Im} A)^{1}\) from (1) above. Therefore, \(\iota A : V^* \to V^*\) induces an isomorphism \(S_p^* = V^*/\text{Ker} \iota A \to S_q^+\), hence the first assertion.
Any element $v \in V^*$ determines a global section of $f^*S_X^+$, and the sheaf $f^*S_X^+$ is generated by such sections. The value at the point $(q, ^tA) \in X^+ \setminus W^+$ of the corresponding section of $S_X^+$ is given by $^tAv \subset S_q^+$, hence the second assertion.

By taking the determinants, we obtain:

**Corollary 2.2.**

$$f^*O_X(1) = f^{++}O_X(1) \otimes O_Y(-2E_1 - E_2).$$

Let $P$ be a fiber of $f_1$ above a point in $G$, $P_0 = P \cap f_1^{-1}X_0$ and $P_1 = P \cap W'$. Then the sequence $P_1 \subset P_0 \subset P$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-3} \subset \mathbb{P}^{2n-5} \subset \mathbb{P}^{2n-4}$, where the first inclusion is the Segre embedding and the second is linear. Let $P' = f_2^{-1}P$, $P'_0 = f_2^{-1}P_0$, and $P'_1 = f_2^{-1}(P_1)$. Let $l_1$ be a line on $P_0$ which meets $W'$ at 2 points, $l'_1 = f_2^{-1}l_1$, and $l_2$ a fiber of $f_2$ over a point in $W'$.

**Lemma 2.3.** The cone of curves $\overline{NE}(Y/X)$ is generated by the classes of $l'_1$ and $l_2$. The intersection numbers are given by the following table:

$$(E'_1 \cdot l'_1) = -1, (E'_1 \cdot l_2) = 0, (E_2 \cdot l'_1) = 2, (E_2 \cdot l_2) = -1.$$  

**Proof.** $P_1$ is a determinantal variety defined by quadratic equations. Therefore we obtain the formula for $\overline{NE}(Y/X)$. The intersection numbers are obvious.

\section{Derived equivalence}

Let $X \rightarrow Z \leftarrow X^+$ and $X_0 \rightarrow Z_0 \leftarrow X_0^+$ be stratified flops defined in the previous section. We shall compare the bounded derived categories of coherent sheaves $D^b(\text{Coh}(X))$ with $D^b(\text{Coh}(X^+))$ and $D^b(\text{Coh}(X_0))$ with $D^b(\text{Coh}(X_0^+))$. By [10] Lemma 5.6 and Corollary 5.7, the derived equivalence for the former pair implies the latter.

We begin with an easy case $r = 1$, i.e., the standard flop and the usual Mukai flop.
Proposition 3.1. Let \( l \) be an arbitrary integer. Then a functor
\[
\Phi_l : \text{D}^b(\text{Coh}(X)) \to \text{D}^b(\text{Coh}(X^+))
\]
defined by
\[
\Phi_l(a) = f^+_s(f^*a \otimes \mathcal{O}_Y(lE))
\]
is an equivalence.

Proof. The adjoint functor \( \Psi_l : \text{D}^b(\text{Coh}(X^+)) \to \text{D}^b(\text{Coh}(X)) \) is given by
\[
\Psi_l(b) = f^*_s(f^{++}_s a \otimes \mathcal{O}_Y((n-1-l)E)).
\]
The category \( \text{D}^b(\text{Coh}(X)) \) is spanned by the set of sheaves \( \mathcal{O}_X(k) \) for \( l-n+1 \leq k \leq l \). By the Kodaira vanishing theorem, we have
\[
\Phi_l : \mathcal{O}_X(k) \mapsto \mathcal{O}_Y(k,0)(lE) = \mathcal{O}_Y(0,-k)((l-k)E) \mapsto \mathcal{O}_{X^+}(-k)
\]
and
\[
\Psi_l : \mathcal{O}_{X^+}(-k) \mapsto \mathcal{O}_Y(0,-k)((n-1-l)E) = \mathcal{O}_Y(k,0)((n-1-l+k)E) \mapsto \mathcal{O}_X(k).
\]
There is an adjunction morphism of functors \( F : \text{Id}_{\text{D}^b(\text{Coh}(X))} \to \Psi_l\Phi_l \), which is reduced to the identity when restricted to the open subset \( X \setminus W \). By the above argument, we have isomorphisms \( \omega \cong \Psi\Phi(\omega) \) for a spanning class \( \Omega = \{ \omega \} \). Since the \( \omega \) are invertible sheaves, it follows that the morphisms \( F(\omega) \) are isomorphisms. Therefore, the natural homomorphisms
\[
\Phi : \text{Hom}^p(\omega_1, \omega^2) \to \text{Hom}^p(\omega_1, \Psi\Phi(\omega^2)) \to \text{Hom}^p(\Phi(\omega_1), \Phi(\omega^2))
\]
for any \( \omega_1, \omega_2 \in \Omega \) and \( p \in \mathbb{Z} \) are isomorphisms. Then \( \Phi \) is an equivalence by [2] and [3].

We assume that \( r = 2 \) in the rest of the paper. We consider exact functors between bounded derived categories
\[
\Phi : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y))
\]
\[
\Psi : D^b(\text{Coh}(Y)) \to D^b(\text{Coh}(X))
\]
(3.1)
defined by
\[
\Phi(a) = Rf^+_s(Lf^*_s(a) \otimes \mathcal{O}_Y((2n-5)E'_1 + (n-3)E_2))
\]
\[
\Psi(b) = Rf^*_s(Lf^{++}_s(b) \otimes \mathcal{O}_Y(E'_1)).
\]
They are adjoints each other because \( K_Y = f^*K_X + (2n-4)E'_1 + (n-3)E_2 \).
Lemma 3.2. $D^b(Coh(X))$ is spanned by the set of the following locally free sheaves:

$$Sym^iS^*_X \otimes O_X(j)$$

for $0 \leq i$, $0 \leq j$ and $i + j \leq n - 2$.

Proof. By [8], any point sheaf $O_p$ for $p \in G$ has a finite locally free resolution whose terms are direct sums of the sheaves

$$Sym^iS^* \otimes O_G(j)$$

for $0 \leq i$, $0 \leq j$ and $i + j \leq n - 2$. Hence $O_{\pi^{-1}(p)}$ is resolved by our set. Then so is any point sheaf $O_x$ for $x \in X$ because $\pi^{-1}(p)$ is an affine space. \qed

Let $\mathcal{E}^+$ be a subsheaf of $S^+_{X^+}$ such that a local section of the latter belongs to the former if and only if its value in $S^+_q$ at a point $(q, B)$ is contained in $\text{Im } B$. We denote by $\mathcal{E}^+_i$ for $i > 0$ the image of the natural homomorphism $\text{Sym}^{i-1}S^+_{X^+} \otimes \mathcal{E}^+ \rightarrow \text{Sym}^iS^+_{X^+}$.

Lemma 3.3. (1) For $0 \leq j \leq n - 3$,

$$\Phi(O_X(j)) = O_{X^+}(-j)$$
$$\Psi(O_{X^+}(-j)) = O_X(j).$$

(2) Let $I_{W^+}$ be the ideal sheaf of $W^+ \subset X^+$. Then

$$\Phi(O_X(n - 2)) = I_{W^+} \otimes O_{X^+}(-n + 2).$$

(3) For $0 < i$, $0 \leq j$ and $i + j \leq n - 3$,

$$\Phi(\text{Sym}^iS^*_X \otimes O_X(j)) = \text{Sym}^iS^+_{X^+} \otimes O_{X^+}(-j)$$
$$\Psi(\text{Sym}^iS^+_{X^+} \otimes O_{X^+}(-j)) = \text{Sym}^iS^*_X \otimes O_X(j).$$

(4) For $0 < i \leq n - 2$,

$$\Phi(\text{Sym}^iS^*_X \otimes O_X(n - 2 - i)) = \mathcal{E}^+_i \otimes O_{X^+}(-n + 2 + i).$$

Proof. (1) By Corollary 2.2

$$f^*(O_X(j)) \otimes O_Y((2n - 5)E'_1 + (n - 3)E_2)$$
$$= f^{**}O_{X^+}(-j) \otimes O_Y((2n - 5 - 2j)E'_1 + (n - 3 - j)E_2).$$

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Since $K_Y = f^*K_X + (2n - 4)E'_1 + (n - 3)E_2$, the higher direct images for $f^+$ vanish because $-(2j + 1)E'_1 - jE_2$ is nef. Since $(2n - 5 - 2j)E'_1 + (n - 3 - j)E_2$ is effective, we have the first formula.

Similarly, we have

$$f^+ ((O_X + (-j)) \otimes O_Y(E'_1) = f^*O_X(j) \otimes O_Y((2j + 1)E'_1 + jE_2).$$

The higher direct images for $f$ vanish because $(2j - 2n + 5)E'_1 - jE_2$ is nef. $(2j + 1)E'_1 + jE_2$ is effective, hence the second formula.

(2) We already proved in (1) that the higher direct images vanish. We obtain our formula from

$$f^+ O_Y(E'_1 - E_2) = I_{W^+}.$$ 

(3) We have an exact sequence

$$0 \to O_{E_2}(-E'_1) \to f^+ S^+_{X+} \otimes O_{E_2}(-E'_1) \to O_{E_2}(-E'_1) \otimes f^{++} O_{X+}(-1) \to 0,$$

where the first term is the image of $f^*S^*_X$. We define a decreasing filtration of the sheaf $f^*\text{Sym}^i S^*_X$ by locally free subsheaves $\mathcal{F}^{k,l}$ for $0 \leq l \leq k \leq i$ by

$$\mathcal{F}^{k,l} = f^*\text{Sym}^i S^*_X \cap f^{++}\text{Sym}^i S^+_{X+}(-iE'_1 - kE_2) \cap (f^*\text{Sym}^i S^*_X(-lE_2) + f^{++}\text{Sym}^i S^+_{X+}(-iE'_1 - (k + 1)E_2)).$$

We have

$$f^*\text{Sym}^i S^*_X = F^{0,0} \supset F^{1,0} \supset F^{1,1} \supset F^{2,0} \supset \cdots \supset F^{i-1,i-1} \supset F^{i,0} = f^{++}\text{Sym}^i S^+_{X+}(-iE'_1 - iE_2),$$

and

$$\mathcal{F}^{k,l} / \mathcal{F}^{k,l+1} \cong O_{E_2}(-iE'_1 - kE_2) \otimes f^{++} O_{X+}(-k + l)$$

for $0 \leq l \leq k < i$, where we put $\mathcal{F}^{k,k+1} = \mathcal{F}^{k+1,0}$. Since

$$R(f^+_2)_* O_{E_2}(tE_2) = 0$$

for $0 < t \leq n - 3$, we have

$$R(f^+_2)_* (\mathcal{F}^{k,l} / \mathcal{F}^{k,l+1} \otimes O_Y((2n - 5 - 2j)E'_1 + (n - 3 - j)E_2)) = 0$$

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because $i + j \leq n - 3$. Therefore,
\[
\Phi(\text{Sym}^i S_X^+ \otimes \mathcal{O}_X(j))
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X + (-j)
\otimes Rf^*_+ \mathcal{O}_Y((2n - 5 - i - 2j)E_1^i + (n - 3 - i - j)E_2)
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X + (-j) \otimes Rf^*_+ \mathcal{O}_{X_i}^+((2n - 5 - i - 2j)E_1^i)
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X + (-j).
\]

For the inverse direction, we climb back along the filtration $\mathcal{F}^{k,l}$. Since
\[
Rf_{2*}(\mathcal{F}^{k,l}/\mathcal{F}^{k,l+1} \otimes \mathcal{O}_Y((i + 2j + 1)E_1^i + (i + j)E_2)) = 0
\]
for $0 \leq l \leq k < i$, we have
\[
\Psi(\text{Sym}^i S_X^+ \otimes \mathcal{O}_X + (-j))
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X(j) \otimes Rf_* \mathcal{O}_Y((i + 2j + 1)E_1^i + (i + j)E_2)
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X(j) \otimes Rf_* \mathcal{O}_{X_1}((i + 2j + 1)E_1)
= \text{Sym}^i S_X^+ \otimes \mathcal{O}_X(j).
\]

(4) We already proved in (3) that
\[
Rf^*_+(f^*(\text{Sym}^i S_X^+ \otimes \mathcal{O}_X(n - 2 - i)) \otimes \mathcal{O}_Y((2n - 5)E_1^i + (n - 3)E_2))
\cong Rf^*_+(F^{i-1,0} \otimes f^+ \mathcal{O}_X(-n + 2 + i) \otimes \mathcal{O}_Y((2i - 1)E_1^i + (i - 1)E_2)).
\]

Moreover
\[
R^pf^*_+(f^+(\text{Sym}^i S_X^+ \otimes \mathcal{O}_X + (-n + 2 + i)) \otimes \mathcal{O}_Y((i - 1)E_1^i - E_2))
\to R^pf^*_+(F^{i-1,0} \otimes f^+ \mathcal{O}_X(-n + 2 + i) \otimes \mathcal{O}_Y((2i - 1)E_1^i + (i - 1)E_2))
\]
is surjective for $p > 0$. Therefore, there are no higher direct images for $f^+$. Since
\[
0 \to F^{i-1,0} \to f^+ \text{Sym}^i S_X^+ \otimes \mathcal{O}_Y(-iE_1^i - (i - 1)E_2)
\to \mathcal{O}_{E_2}(-iE_1^i - (i - 1)E_2) \otimes f^+ \mathcal{O}_X + (-i) \to 0
\]
we have
\[
0 \to f^*_+(F^{i-1,0} \otimes f^+ \mathcal{O}_X(-n + 2 + i) \otimes \mathcal{O}_Y((2i - 1)E_1^i + (i - 1)E_2))
\to \text{Sym}^i S_X^+ \otimes \mathcal{O}_X(-n + 2 + i) \to \mathcal{O}_{\mathcal{W}^+} \otimes \mathcal{O}_X + (-n + 2) \to 0.
\]
On the other hand, we have
\[ f^+_*(F^{i-1,0} \otimes \mathcal{O}_X(-n + 2 + i) \otimes \mathcal{O}_Y((2i - 1)E'_1 + (i - 1)E_2))|_{X^+ \setminus G^+} \]
\[ \cong \mathcal{E}^+_i \otimes \mathcal{O}_{X^+}(-(n + 2 + i)|_{X^+ \setminus G^+}. \]

By taking the direct image sheaves of both sides under the inclusion morphism \( X^+ \setminus G^+ \to X^+ \), we obtain
\[ f^+_*(F^{i-1,0} \otimes \mathcal{O}_X(-n + 2 + i) \otimes \mathcal{O}_Y((2i - 1)E'_1 + (i - 1)E_2)) \]
\[ \cong \mathcal{E}^+_i \otimes \mathcal{O}_{X^+}(-(n + 2 + i). \]

We assume that \( n = 4 \) besides \( r = 2 \) in the following. Then \( \dim G = 4 \), \( \dim X_0 = 8 \), \( \dim X = 9 \), and \( \dim W = 7 \). In particular, \( W \) is locally complete intersection. We have \( K_Y = f^*K_X + 4E'_1 + E_2 \).

In this case, Namikawa [14] proved that the functors
\[ \Phi': D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y)), \Psi': D^b(\text{Coh}(Y)) \to D^b(\text{Coh}(X)) \]
defined by
\[ \Phi'(a) = Rf^+_*(Lf^*(a) \otimes \mathcal{O}_Y(4E'_1 + E_2)), \Psi'(b) = Rf_*(Lf^{**}(b) \otimes \mathcal{O}_Y). \]
are not equivalences.

**Lemma 3.4.** Assume that \( n = 4 \). Then
1. \( \Psi(I_{W^+} \otimes \mathcal{O}_{X^+}(-2)) = \mathcal{O}_X(2) \).
2. \( \Psi(\mathcal{E}^+_i \otimes \mathcal{O}_{X^+}(i - 2)) = \text{Sym}^i S^* \otimes \mathcal{O}_Y(2 - i) \) for \( 0 < i \leq 2 \).

**Proof.** (1) \( W^+ \) is a divisor on \( X^+_0 \) and \( \phi^+ \) is the contraction of \((-2)\)-curves along the generic points of \( \phi(W^+) \). Hence \( \mathcal{O}_{X^+_0}(-W^+) = \mathcal{O}_{X^+_0}(2) \). We have an exact sequence
\[ 0 \to \mathcal{O}_{X^+}(-2 - X^+_0) \to I_{W^+} \otimes \mathcal{O}_{X^+}(-2) \to \mathcal{O}_{X^+_0} \to 0. \]
Since \( X^+_0 \) is a Cartier divisor, we have \( L_k f^{**}\mathcal{O}_{X^+_0} = 0 \) for \( k > 0 \). Hence
\[ 0 \to f^{**}\mathcal{O}_{X^+}(-2 - X^+_0) \to f^{**}(I_{W^+} \otimes \mathcal{O}_{X^+}(-2)) \to f^{**}\mathcal{O}_{X^+_0} \to 0. \]
On the other hand,
\[ 0 \to f^{**}\mathcal{O}_{X^+}(-2 - X^+_0) \to I_{W^+} \mathcal{O}_Y \otimes f^{**}\mathcal{O}_{X^+}(-2) \to \mathcal{O}_{(X^+_0)^\vee} \to 0. \]
where \((X_0^+)'' = (f^+_*)^{-1}X_0^+\). Thus
\[
0 \rightarrow \mathcal{O}_{(E_1^+)''}(-X_0^+)'' \rightarrow f^{++}(I_{W^+} \otimes \mathcal{O}_{X^+}(-2))
\rightarrow I_{W^+}\mathcal{O}_{Y} \otimes f^{++}\mathcal{O}_{X^+}(-2) \rightarrow 0.
\]
We have
\[
\mathcal{O}_{(E_1^+)''}(-X_0^+)'' \cong \mathcal{O}_{E_1' + E_2'}(E_1' + E_2).
\]
Since \(Rf_{2*}\mathcal{O}_{E_2}(E_2) = 0\) and \(Rf_{1*}\mathcal{O}_{E_1}(2E_1) = 0\), we have
\[
Rf_*\mathcal{O}_{E_1' + E_2'}(2E_1' + E_2) \cong Rf_{1*}\mathcal{O}_{E_1}(2E_1) \cong 0.
\]
Therefore, we have
\[
Rf_*(f^{++}(I_{W^+} \otimes \mathcal{O}_{X^+}(-2)) \otimes \mathcal{O}_{Y}(E_1')) \cong Rf_*(I_{W^+}\mathcal{O}_{Y} \otimes f^{++}\mathcal{O}_{X^+}(-2) \otimes \mathcal{O}_{Y}(E_1')).
\]
Let \(F^+ = (X_0^+)'' \cap (E_1^+)'' \subset Y\). The projection \(f^+ : F^+ \rightarrow G^+\) is a \(\mathbb{P}^3\)-bundle whose fiber \((P_0^+)''\) is contracted by \(f\). We have
\[
I_{W^+}\mathcal{O}_{Y} = I_{F^+}(-E_1' - E_2)
\]
Thus
\[
I_{W^+}\mathcal{O}_{Y} \otimes f^{++}\mathcal{O}_{X^+}(-2) \otimes \mathcal{O}_{Y}(E_1') = I_{F^+}f^*\mathcal{O}_{X}(2) \otimes \mathcal{O}_{Y}(4E_1' + E_2).
\]
On the other hand, there is an exact sequence we have \(\mathcal{O}_{(P_0^+)''}(E_1') \cong \mathcal{O}_{\mathbb{P}^3}(-1)\) and \(\mathcal{O}_{(P_0^+)''}(E_2) \cong \mathcal{O}_{\mathbb{P}^3}(2)\). Hence \(\mathcal{O}_{(P_0^+)''}(4E_1' + E_2) \cong \mathcal{O}_{\mathbb{P}^3}(-2)\). It follows that \(Rf_*\mathcal{O}_{F^+}(4E_1' + E_2) = 0\). Therefore,
\[
Rf_*(I_{F^+}\mathcal{O}_{Y} \otimes f^{++}\mathcal{O}_{X^+}(-2) \otimes \mathcal{O}_{Y}(E_1')) \cong \mathcal{O}_{X}(2).
\]
(2) We have an exact sequence
\[
0 \rightarrow \mathcal{E}_i^+ \otimes \mathcal{O}_{X^+}(i - 2) \rightarrow \text{Sym}^1 S_{X^+}^+ \otimes \mathcal{O}_{X^+}(i - 2) \rightarrow \mathcal{O}_{W^+} \otimes \mathcal{O}_{X^+}(-2) \rightarrow 0.
\]
Thus the torsion part of \(f^{++}\mathcal{E}_i^+ \otimes \mathcal{O}_{X^+}(i - 2)\) is isomorphic to that of \(f^{++}I_{W^+} \otimes \mathcal{O}_{X^+}(-2)\). Therefore, by the argument in (1), we have
\[
\Psi(\mathcal{E}_i^+ \otimes \mathcal{O}_{X^+}(i - 2)) \cong Rf_*(f^{++}\mathcal{E}_i^+/\text{torsion} \otimes f^{++}\mathcal{O}_{X^+}(i - 2) \otimes \mathcal{O}_{Y}(E_1')).
\]
We have an exact sequence

\[ 0 \to f^{++} \mathcal{E}^+_i / \text{torsion} \otimes f^{++} \mathcal{O}_{X^+}(i - 2) \otimes \mathcal{O}_Y(E'_1) \]
\[ \to f^{++}(\text{Sym}^i S_{X^+}^+ \otimes \mathcal{O}_{X^+}(i - 2)) \otimes \mathcal{O}_Y(E'_1) \]
\[ \to f^{++}(\mathcal{O}_{W^+} \otimes \mathcal{O}_{X^+}(-2)) \otimes \mathcal{O}_Y(E'_1) \to 0. \]

Therefore, we have an exact sequence

\[ 0 \to f^{++} \mathcal{E}^+_i / \text{torsion} \otimes f^{++} \mathcal{O}_{X^+}(i - 2) \otimes \mathcal{O}_Y(E'_1) \]
\[ \to F^{i-1,0} \otimes f^{++} \mathcal{O}_{X^+}(i - 2) \otimes \mathcal{O}_Y((i + 1)E'_1 + (i - 1)E_2) \to Q \to 0 \]

where the cokernel \( Q \) has a decomposition as follows

\[ 0 \to \mathcal{O}_{F^+}(-E_2) \otimes f^{++} \mathcal{O}_{X^+}(-2) \to Q \to \mathcal{O}_{E'_1}(E'_1 - E_2) \otimes f^{++} \mathcal{O}_{X^+}(-2) \to 0. \]

As in (1), \( F^+ = F = X'_0 \cap E'_1 \) is a \( \mathbb{P}^3 \)-bundle over \( G \) with fibers \( P'_0 \) such that \( \mathcal{O}_{P'_0}(E'_1) \cong \mathcal{O}_{\mathbb{P}^3}(-1) \) and \( \mathcal{O}_{P'_0}(E_2) = \mathcal{O}_{\mathbb{P}^3}(2) \). Thus

\[ \mathcal{O}_{F^+}(-E_2) \otimes f^{++} \mathcal{O}_{X^+}(-2) \otimes \mathcal{O}_{P'_0} \cong \mathcal{O}_{P'_0}(4E'_1 + E_2) \cong \mathcal{O}_{P'_0}(-2) \]

and we have \( Rf_* \mathcal{O}_{F^+}(-E_2) = 0 \). We have also

\[ Rf_* \mathcal{O}_{E'_1}(E'_1 - E_2) \otimes f^{++} \mathcal{O}_{X^+}(-2) \]
\[ \cong Rf_* \mathcal{O}_{E'_1}(3E'_1 + E_2) \otimes \mathcal{O}_X(2) \]
\[ \cong Rf_1* \mathcal{O}_{E'_1}(3E_1) \otimes \mathcal{O}_X(2) \cong 0. \]

Therefore

\[ \Psi(\mathcal{E}^+_i \otimes \mathcal{O}_{X^+}(i - 2)) \]
\[ \cong Rf_* (F^{i-1,0} \otimes f^{++} \mathcal{O}_{X^+}(i - 2) \otimes \mathcal{O}_Y((i + 1)E'_1 + (i - 1)E_2)) \]
\[ \cong Rf_* (F^{i-1,0} \otimes \mathcal{O}_Y((5 - i)E'_1 + E_2)) \otimes \mathcal{O}_X(2 - i). \]

If \( i = 1 \), then \( F^{0,0} = f^* \text{Sym}^i S^* \) and \( Rf_* \mathcal{O}_Y(4E'_1 + E_2) = \mathcal{O}_X \), hence the result.

If \( i = 2 \), then \( \mathcal{F}^{0,0}/\mathcal{F}^{1,0} \cong \mathcal{O}_E(-2E'_1) \), hence

\[ Rf_* (\mathcal{F}^{0,0}/\mathcal{F}^{1,0} \otimes \mathcal{O}_Y(3E'_1 + E_2)) = 0 \]

and we complete the proof. \( \square \)
Theorem 3.5. If \( n = 4 \), then \( \Phi \) and \( \Psi \) in (3.1) are equivalences.

Proof. There is an adjunction morphism of functors \( F : \text{Id}_{D^b(\text{Coh}(X))} \to \Psi \Phi \), which is reduced to the identity when restricted to the open subset \( X \setminus W \). By Lemmas 3.2, 3.3 and 3.4, we have isomorphisms \( \omega \cong \Psi \Phi(\omega) \) for the spanning class \( \Omega = \{ \omega \} \) given by Lemma 3.2. Since the \( \omega \) are locally free sheaves, it follows that the morphisms \( F(\omega) \) are isomorphisms. Therefore, the natural homomorphisms

\[
\Phi : \text{Hom}^p(\omega_1, \omega_2) \to \text{Hom}^p(\omega_1, \Psi \Phi(\omega_2)) \to \text{Hom}^p(\Phi(\omega_1), \Phi(\omega_2))
\]

for any \( \omega_1, \omega_2 \in \Omega \) and \( p \in \mathbb{Z} \) are isomorphisms. Then \( \Phi \) is an equivalence by [2] and [3]. \[\square\]

Remark 3.6. (1) Let \( F = \bigoplus \omega \) be the sum of the spanning sheaves given in Lemma 3.2. Then \( F \) is not an almost exceptional object in the sense of [1] in our case \( n = 4 \) and \( r = 2 \), because \( R^1\phi_* \mathcal{O}_X(-2) \neq 0 \) implies \( \text{Hom}^1(F, F) \neq 0 \).

(2) In order to extend our argument to the case \( n \geq 5 \), the Eagon-Northcott resolution on \( X_0^+ \) would be useful (7):

\[
0 \to \pi_0^+(\text{Sym}^{n-2}S^2)^* \otimes \bigwedge^2 V^* \to \pi_0^+(\text{Sym}^{k}S^2)^* \otimes \bigwedge^k V^* \to \pi_0^+(\text{Sym}^{k-1}S^2)^* \otimes \bigwedge^{k+1} V^* \to \pi_0^+(\text{Sym}^{k-2}S^2)^* \otimes \bigwedge^{k+2} V^* \to \cdots
\]

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