Effective Hamiltonian for Scalar Theories in the Gaussian Approximation

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Abstract

We use a Gaussian wave functional for the ground state to reorder the Hamiltonian into a free part with a variationally determined mass and the rest. Once spontaneous symmetry breaking is taken into account, the residual Hamiltonian can, in principle, be treated perturbatively. In this scheme we analyze the $O(1)$ and $O(2)$ scalar models. For the $O(2)$-theory we first explicitly calculate the massless Goldstone excitation and then show that the one-loop corrections of the effective Hamiltonian do not generate a mass.
1 Introduction

Hamiltonian field theory has been the starting point of modern quantum field theory, but since then has been less used than Lagrangian field theory or action oriented approaches. The advantage of a Hamiltonian formulation \[1\] is the appearance of "wave functions", which are functionals of the fields and allow an intuitive understanding of the ground state. Recently, for the analysis of gauge theories \[2\] and light–cone theory \[3\] a Hamiltonian treatment has been revived. The concept of an effective Hamiltonian, as exploited in this paper, has actually been developed recently in the context of one–dimensional light–cone theories \[4\]. There exists a substantial literature \[5, 6\] on the use of variational treatments in Hamiltonian field theory. One even finds quantitative estimates of the Higgs–mass, based on this technique \[5\]. The variational technique is inherently nonperturbative, so one hopes to capture features which go beyond the standard loop expansion. In many–body theory the Gaussian wave functional with an effective mass corresponds to the self–consistent mean field theory. In fact, also more sophisticated methods of many–body theory like the cluster expansion \[8\] or the RPA–approximation can be used to improve the mean field result.

In this paper we treat \(O(N)\)–models for \(N = 1, 2\) to analyse two main features, (i) the bounds on the mass \(m(1)\) of the massive scalar after renormalization, and (ii) the mass \(m(2)\) of the Goldstone particle. In the recent literature \[7\] a Higgs–mass of 2 TeV has been "predicted" for the experimentally given vacuum expectation value of the Higgs field in the Standard Model. We use a variational treatment of the scalar sector with a finite cutoff and present results different from the previous estimates in Hamiltonian field theory: \(M_{\text{Higgs}} \leq 1.7\) TeV.

In previous Hamiltonian based work on the \(O(2)\)–model a finite Goldstone boson mass \(m(2) \approx m(1)/2.06\) \[1\] is obtained, in contradiction to Goldstone’s theorem. We will show that a symmetric Gaussian ansatz and a further rediagonalization of the effective Hamiltonian exactly gives a zero mass Goldstone excitation. This result forms the basis for any further exploration of the Higgs model or more complicated gauge theories like QCD. Possible problems due to the violation of local gauge invariance, which in general appear in approximative treatments, will be avoided by starting with "gauge fixed Hamiltonians" from which the redundant degrees of freedom have been eliminated \[3\].

There has been an extensive discussion of the Gaussian Effective Potential (GEP) \[3\]. The emphasis of that work has been to calculate the energy of the vacuum as a function of the symmetry breaking zero mode of the field. In general the mass gap or the spectrum of particles, however, is the more interesting phenomenological quantity. Therefore one has to address the problem of calculating the energy and the dispersion relation of excited states. From our point of view the Gaussian wave functional presents an efficient way of reordering the Hamiltonian into quadratic and higher polynomial parts. Our approach is time–independent; recently, also time–dependent variational equations have been investigated in \(\phi^4\) field theory \[10\].

In \(0 + 1\) dimensions, i.e. quantum mechanics, the anharmonic oscillator is an elucidating example. It shows the problems we are facing in the case of spontaneous symmetry breaking (SSB) in the \(O(2)\)–model. The ground state solution of \(H\)

\[
H = \frac{1}{2} \left( p^2 + m^2 x^2 \right) + \lambda x^4,
\]

is approximated by the Gaussian wave function

\[
\phi_G(x) = \left( \frac{\Omega}{\pi} \right)^{\frac{1}{4}} \exp \left[ -\frac{1}{2} \Omega (x - x_0)^2 \right].
\]

Minimization of the expectation value of the Hamiltonian operator,

\[
V_G(x_0, \Omega) = \langle \phi_G | H | \phi_G \rangle = \frac{1}{4} \Omega + \frac{m^2}{4 \Omega} + \frac{1}{2} m^2 x_0^2 + \lambda \left[ \frac{3}{\Omega} x_0^2 + \frac{3}{4 \Omega^2} + x_0^4 \right]
\]

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\]
with respect to $x_0$ and $\Omega$ gives the equations

$$x_0 \left( m^2 + \frac{6\lambda}{\Omega} + 4\lambda x_0^2 \right) = 0,$$

$$\Omega^2 - m^2 - 12\lambda x_0^2 - \frac{6\lambda}{\Omega} = 0. \quad (4)$$

For the case $x_0 \neq 0$ one explicitly finds

$$\Omega^2 = 8\lambda x_0^2.$$

Equivalently, one defines the trial ground state via creation and annihilation operators $a_\Omega^+$ and $a_\Omega$,

$$a_\Omega |\phi_G\rangle = 0,$$

$$\langle \phi_G | \phi_G \rangle = 1,$$

$$\langle \phi_G | x | \phi_G \rangle = x_0. \quad (6)$$

Note that in this way excited states are also implicitly defined. Coordinate and momentum are correspondingly decomposed

$$p = -i\sqrt{\frac{\Omega}{2}} (a_\Omega - a_\Omega^+) \quad \text{and} \quad \bar{x} := x - x_o = \sqrt{\frac{1}{2\Omega}} \left( a_\Omega + a_\Omega^+ \right). \quad (7)$$

Normal ordering with respect to $a_\Omega$ and $a_\Omega^+$, which is denoted by $: :$, yields

$$: x^2 : = x^2 - \frac{1}{2\Omega},$$

$$: x^3 : = x^3 - \frac{3}{2\Omega} : x : ,$$

$$: x^4 : = x^4 - \frac{6}{2\Omega} : x^2 : - 3 \left( \frac{1}{2\Omega} \right)^2,$$

$$: p^2 : = p^2 - \frac{\Omega}{2}. \quad (8)$$

The normal ordered Hamiltonian reads

$$H = \frac{1}{2} \left( p^2 + m^2 x^2 \right) + \lambda x^4$$

$$+ \frac{\Omega}{4} + \frac{m^2}{4\Omega} + \frac{3\lambda}{\Omega} x^2 + \frac{3\lambda}{4\Omega^2} : :. \quad (9)$$

Herewith one easily verifies eq. (3). Now we consider the difference $H_R = H - V_G$ and using eqs. (4) we obtain

$$H_R = \frac{1}{2} \left( p^2 + \left( m^2 + \frac{6\lambda}{\Omega} \right) (x^2 - x_0^2) \right) + \lambda (x^4 - x_0^4) : :$$

$$= \frac{1}{2} \left( p^2 + (\Omega^2 - 12\lambda x_0^2) (x^2 - x_0^2) \right) + \lambda (x^4 - x_0^4) : :$$

$$= \frac{1}{2} \left( p^2 + \Omega^2 \bar{x}^2 \right) + \lambda \bar{x}^4 + 4\lambda x_0 \bar{x}^3 : :. \quad (10)$$

$H_R$ can be rewritten in terms of the creation and annihilation operators

$$H_R = H_0 + H_I, \quad (11)$$
with

\[ H_0 = \Omega a_\Omega a_\Omega, \quad (12) \]
\[ H_I = 4\lambda x_0 \left( \frac{1}{2\Omega} \right)^2 \left( a_\Omega^3 + 3a_\Omega^2 a_{\Omega^2} + 3a_{\Omega^2} a_\Omega + a_{\Omega^2}^3 \right) \]
\[ + \lambda \left( \frac{1}{2\Omega} \right)^2 \left( a_{\Omega^4}^4 + 4a_{\Omega^2}^3 a_{\Omega^2} + 6a_{\Omega^2}^2 a_{\Omega^2} + 4a_{\Omega^4}^3 a_{\Omega^2} + a_{\Omega^4}^4 \right), \quad (13) \]

where we separated the harmonic part \( H_0 \). The remaining part of the Hamiltonian, i.e. \( H_I \), will be treated in perturbation theory. We will see that the use of perturbation theory is well justified if \( \Omega \) is large. This is obvious for \( x_0 = 0 \), where only the quartic interaction is present. In the "broken" phase, however, there is a cubic interaction with an effective coupling constant containing \( x_0 \). An explicit (second order) calculation shows that for localized trial states the relevant gap between the first excited state \( E_1 \) and the ground state \( E_0 \) is dominated by the splitting of energy levels for the unperturbed Hamiltonian

\[ E_1 - E_0 = \Omega \left( 1 - \frac{6\lambda}{\Omega^3} \right). \quad (14) \]

In our system of units \( H, m \) and \( \Omega \) have units of energy, \([ H ] = [ m ] = [ \Omega ] = E\), \([ x ] = 1/\sqrt{E}\) and \([ \lambda ] = E^3\). One can see the intuitively obvious fact that the corrections are small for large \( \Omega \). To be more specific, \( \lambda/\Omega^3 \) has to be smaller than unity for perturbation theory to hold. In the opposite case the Gaussian approach fails and higher clusters play an important role.

For the two–dimensional problem in quantum mechanics, a similar variational treatment is possible. The Hamiltonian of this \( O(2) \)–model reads

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + \frac{1}{2} m^2 \left( x^2 + y^2 \right) + \lambda \left( x^2 + y^2 \right)^2. \quad (15) \]

Here, in quantum mechanics, the best approach is to rewrite the Hamiltonian in spherical variables and solve for the radial wave function. However, this cannot be easily generalized to quantum field theory and therefore we will stick to Cartesian coordinates. Our main new idea is to use a Gaussian wave function

\[ \phi_G(x, y) = \left( \frac{\Omega}{\pi} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \Omega (x - x_0)^2 - \frac{1}{2} \Omega y^2 \right], \quad (16) \]

with the same size parameter in longitudinal and transverse direction. This will be seen to be a good starting point for quantum field theory, because it preserves Goldstone’s theorem for higher dimensions of space–time. Thus the same parameter \( \Omega \) appears in the decomposition of coordinates and momenta in two dimensions

\[ p_x = -i \sqrt{\frac{\Omega}{2}} (a_\Omega - a_\Omega^\dagger), \quad \bar{x} = \sqrt{\frac{1}{2\Omega}} (a_\Omega + a_\Omega^\dagger), \]
\[ p_y = -i \sqrt{\frac{\Omega}{2}} (b_\Omega - b_\Omega^\dagger), \quad y = \sqrt{\frac{1}{2\Omega}} (b_\Omega + b_\Omega^\dagger). \]

The same variational treatment as before leads to the normal ordered form of the effective Hamiltonian, \( H_R = H - V_G = H_0 + H_I \),

\[ H_0 = : \frac{1}{2} p_x^2 + \frac{1}{2} (2\Omega^2)x^2 + \frac{1}{2} p_y^2 :; \]
\[ H_I = : 4\lambda x_0 \bar{x}^3 + \lambda \bar{x}^4 + 2\lambda y^2 \bar{x}^2 + 4\lambda x_0 \bar{x} y^2 + \lambda y^4 :. \quad (17) \]
Only the more interesting case $x_0 \neq 0$ is considered. Because of the appearance of $2\Omega^2 = 8\lambda x_0^2$, with $x_0 := [(m^2 - 8\lambda/\Omega)/4\lambda]^{1/2}$, in front of the quadratic term $\bar{x}^2$ and the vanishing of the term purely quadratic in $y$, the normal ordered form $H_0$ has not the usual representation in terms of the occupation number operators $a_\Omega^\dagger a_\Omega$ and $b_\Omega^\dagger b_\Omega$. It is therefore necessary to introduce two Bogoliubov transformations for the pair of operators $a_\Omega$, $a_\Omega^\dagger$ and $b_\Omega$, $b_\Omega^\dagger$ in order to diagonalize $H_0$. To preserve the commutation relations one has

$$\begin{align*}
u &= \cosh \alpha a_\Omega + \sinh \alpha a_\Omega^\dagger, \\
v &= \cosh \beta b_\Omega + \sinh \beta b_\Omega^\dagger, \quad (18)
\end{align*}$$

where $\alpha$ and $\beta$ are determined from commuting $u$ and $v$ with $H_0$:

$$\begin{align*}
[u, H_0] &= \omega_1 u, \\
[v, H_0] &= \omega_2 v. \quad (19)
\end{align*}$$

The eigenvalues of these commutation relations are indeed the physical energy eigenvalues. One finds \( \tanh \alpha = 3 - 2\sqrt{2}, \ \tanh \beta = -1 \) and

\begin{align*}
\omega_1 &= \Omega \sqrt{2}, \\
\omega_2 &= 0. \quad (20)
\end{align*}

The zero eigenvalue for the $y$–oscillator is related to the fact that the effective classical potential is flat after spontaneous symmetry breaking in $x$–direction. The Bogoliubov transformation, however, is formally not well defined, since we have $\beta = -\infty$. This reflects the fact that there is no unitary transformation between non–normalizable plane waves and normalizable oscillator wave functions. In section (4) we will regularize the theory such that the second frequency is non–zero and, consequently, the wave function is normalizable. This also renders the Bogoliubov transformation to be well behaved, i.e. unitary. Furthermore, the perturbative corrections due to $H_I$ will yield a finite frequency, even in the limit where the regulator is removed.

2 Effective Hamiltonian for $\phi^4$ scalar field theory

The methods described above can be applied to field theory in $\nu$ spatial dimensions. In this section we discuss the one–component theory, i.e. the $\phi^4$ model. The expectation value of the Hamiltonian with respect to a Gaussian trial state is calculated, minimized and subtracted from the Hamiltonian. This yields an effective Hamiltonian containing a renormalized mass and new interaction terms. The latter only appear if the original reflection symmetry, $\phi \to -\phi$, is spontaneously broken. In this method symmetry breaking effects are taken into account in a nonperturbative way. The resulting effective Hamiltonian may be treated in a perturbative way. In field theory the appearing integrals are ultraviolet divergent; in the following they are assumed to be regularized via a momentum cutoff or dimensional regularization. One may introduce renormalization constants for the mass, the coupling constant and the wave function. In dimensional regularization one can avoid an explicit mass renormalization. It is important to see that the effective Hamiltonian scheme contains a new physical mass. In this section we do not perform an explicit renormalization. We merely assume the theory to be regularized and that the resulting equations can be solved. Details of a full renormalization and explicit solutions will be discussed in section (5).

The Hamiltonian for $\phi^4$ theory in terms of ”bare” quantities is given by

$$H = \int d^\nu x \left[ \frac{1}{2} \pi_B^2 + \frac{1}{2} (\nabla \phi_B)^2 + \frac{1}{2} m_B^2 \phi_B^2 + \lambda_B \phi^4 \right]. \quad (21)$$
where \( \pi_B \) is the conjugate momentum of \( \phi_B \),

\[
[\pi_B(\vec{x}), \phi_B(\vec{y})] = -i \delta^\nu(\vec{x} - \vec{y}).
\]  

(22)

This commutator is to be understood as an equal time commutator. Since we work in a time–independent scheme, the time variable is suppressed. We explicitly introduce a field renormalization constant \( Z_\phi \),

\[
\phi_B = Z_\phi^{1/2} \phi,
\]

\[
\pi_B = Z_\phi^{-1/2} \pi.
\]

(23)

The canonical commutator, eq. (22) is preserved in this way. Inserting this into the Hamiltonian yields

\[
H = \int d^nx \left[ \frac{1}{2} Z_\phi^{-1} \pi^2 + \frac{1}{2} Z_\phi(\nabla \phi)^2 + \frac{1}{2} m_B^2 Z_\phi \phi^2 + \lambda_B Z_\phi^2 \phi^4 \right].
\]

(24)

The variational calculation starts with the introduction of a trial ground state, \( |0\rangle_{\Omega,\phi_0} \). It is defined via

\[
a_{\Omega}(\vec{k}) |0\rangle_{\Omega,\phi_0} = 0,
\]

(25)

with corresponding field (momenta) expansion

\[
\phi(\vec{x}) = \phi_0 + \int (dk)_{\Omega} \left[ a_{\Omega}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + a_{\Omega}^\dagger(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right],
\]

\[
\pi(\vec{x}) = -i Z_\phi^{1/2} \int (dk)_{\Omega} \omega(\vec{k}) \left[ a_{\Omega}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} - a_{\Omega}^\dagger(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right].
\]

(26)

The energies \( \omega(\vec{k}) = \sqrt{\vec{k}^2 + \Omega^2} \) also appear in the measure

\[
(dk)_{\Omega} = \frac{d^nk}{2(2\pi)^{\nu} \omega(\vec{k})},
\]

(27)

as well as in the canonical commutation relations of the creation and annihilation operators

\[
[a_{\Omega}(\vec{k}), a_{\Omega}^\dagger(\vec{k}')] = 2(2\pi)^\nu \omega(\vec{k}) \delta^\nu(\vec{k} - \vec{k}').
\]

(28)

Note that the \( \phi_0 \) dependences of \( a_{\Omega}(\vec{k}) \) and \( a_{\Omega}^\dagger(\vec{k}') \) are also suppressed. The state \( |0\rangle_{\Omega,\phi_0} \) is normalized,

\[
_{\Omega,\phi_0} \langle 0 | 0 \rangle_{\Omega,\phi_0} = 1.
\]

(29)

The quantities \( \Omega \) and \( \phi_0 \) are the variational parameters of the calculation. As a next step we again normal order with respect to the operators \( a_{\Omega} \) and \( a_{\Omega}^\dagger \); this normal ordering is denoted by \( : \cdot : \). One obtains

\[
: \phi^2 : = \phi^2 - Z_\phi^{-1} I_0(\Omega^2),
\]

\[
: \phi^4 : = \phi^4 - 6 Z_\phi^{-1} I_0(\Omega^2) : \phi^2 : - 3 Z_\phi^{-2} I_0(\Omega^2)^2,
\]

\[
: (\nabla \phi)^2 : = (\nabla \phi)^2 - Z_\phi^{-1} I_1(\Omega^2) + Z_\phi^{-2} \Omega^2 I_0(\Omega^2),
\]

\[
: \pi^2 : = \pi^2 - Z_\phi I_1(\Omega^2),
\]

(30)

where the integrals \( I_N \) are defined as

\[
I_N(\Omega^2) = \int (dk)_{\Omega} (\omega^2(\vec{k}))^N.
\]

(31)
For certain combinations of $\nu$ and $N$ these integrals are divergent. Here we assume them to be regularized in such a way that the "naive" relation

\[
\frac{dI_N(\Omega^2)}{d\Omega} = (2N - 1)\Omega I_{N-1}(\Omega^2), \tag{32}
\]

holds for the regularized equations. With eqs. (30) one can write the Hamiltonian in normal ordered form

\[
H = : \int d^nx \left[ \frac{1}{2}Z^{-1}_\phi \pi^2 + \frac{1}{2}Z_\phi(\nabla \phi)^2 + \frac{1}{2}(m_B^2 + 12\lambda_B I_0)Z_\phi \phi^2 + \lambda_B Z_\phi^2 \phi^4 + I_1 - \frac{1}{2}\Omega^2 I_0 + \frac{1}{2}m_B^2 I_0 + 3\lambda_B I_0^2 \right] :, \tag{33}
\]

where we suppressed the argument of the integrals: $I_N \equiv I_N(\Omega^2)$. The expectation value of the Hamiltonian density, $H$, with respect to the trial state is known as the Gaussian Effective Potential (GEP) [1].

\[
V_G(\phi_0, \Omega^2) = \langle 0 \mid H \mid 0 \rangle_{\Omega,\phi_0} = I_1 + \frac{1}{2}(m_B^2 - \Omega^2)I_0 + 3\lambda_B I_0^2 + \frac{1}{2}(m_B^2 + 12\lambda_B I_0)Z_\phi \phi_0^2 + \lambda_B Z_\phi^2 \phi_0^4. \tag{34}
\]

Because of the possibility of spontaneous symmetry breaking, normal ordered fields still can contribute to $V_G$, e.g.

\[
\langle 0 \mid : \phi^2 : \mid 0 \rangle_{\Omega,\phi_0} = \phi_0^2. \tag{35}
\]

The variational principle, which is expressed by the equations $\frac{\partial V_G}{\partial \phi} = 0$ and $\frac{\partial V_G}{\partial \phi_0} = 0$, yields the gap equation

\[
\Omega^2 = m_B^2 + 12\lambda_B \left[ I_0 + Z_\phi \phi_0^2 \right], \tag{36}
\]

and

\[
m_B^2 Z_\phi \phi_0 + 4\lambda_B Z_\phi^2 \phi_0^3 + 12\lambda_B I_0 Z_\phi \phi_0 = 0. \tag{37}
\]

It has to be verified that these equations do correspond to a minimum [1]. Here we proceed by assuming that these equations hold and that their solutions indeed minimize the trial vacuum energy. The vacuum expectation value is subtracted from the Hamiltonian:

\[
H_R = H - \int d^nx V_G =: \int d^nx \left[ \frac{1}{2}Z^{-1}_\phi \pi^2 + \frac{1}{2}Z_\phi(\nabla \phi)^2 + \frac{1}{2}(m_B^2 + 12\lambda_B I_0)Z_\phi(\phi^2 - \phi_0^2) + \lambda_B Z_\phi^2(\phi^4 - \phi_0^4) \right] :, \tag{38}
\]

The bare mass $m_B^2$ can be eliminated with the gap equation, eq. (33),

\[
H_R = : \int d^nx \left[ \frac{1}{2}Z^{-1}_\phi \pi^2 + \frac{1}{2}Z_\phi(\nabla \bar{\phi})^2 + \frac{1}{2}\Omega^2 Z_\phi(\phi^2 - \bar{\phi}_0^2) \right.
\]

\[
+ \lambda_B Z_\phi^2(\phi^4 - \bar{\phi}_0^4) - 6\lambda_B Z_\phi^2 \phi_0^2(\phi^2 - \bar{\phi}_0^2) \left. \right] :, \tag{39}
\]

and $H_R$ is rewritten in terms of the fluctuating field $\bar{\phi} := \phi - \phi_0$ ($\bar{\pi} = \pi$),

\[
H_R = : \int d^nx \left[ \frac{1}{2}Z^{-1}_\phi \pi^2 + \frac{1}{2}Z_\phi(\nabla \bar{\phi})^2 + \frac{1}{2}\Omega^2 Z_\phi \bar{\phi}^2 + \lambda_B Z_\phi^2 \bar{\phi}^4 + 4\lambda_B Z_\phi^2 \phi_0 \bar{\phi}^3 \right] :. \tag{40}
\]

This simple form, in particular the vanishing of the linear term in $\bar{\phi}$, follows from the variational equations [36, 37]. Redefining field and momentum as,

\[
\bar{\phi} = Z^{-\frac{1}{2}}_\phi \phi, \quad \bar{\pi} = Z^{-\frac{1}{2}}_\phi \pi, \tag{41}
\]
finally yields the effective Hamiltonian

\[ H_R = \int d^v x \left[ \frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\nabla \phi)^2 + \lambda B \tilde{\phi}^4 + 4 \lambda B Z^3 \phi_0 \tilde{\phi}^3 \right]. \] (42)

Obviously, the mass is given by the variational parameter \( \Omega^2 \). If the symmetry is spontaneously broken, i.e. \( \phi_0 \neq 0 \), the physical mass and the vacuum expectation value are related by (cf. eq. (43))

\[ \Omega^2 = 8 \lambda B Z \phi_0. \] (43)

A similar relation appears in the \( O(2) \)-model and has important consequences for phenomenology.

Standard perturbation theory with the unperturbed free Hamiltonian \( H_0 \)

\[ H_0 = \int d^v x \left[ \frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \Omega^2 \phi^2 \right]. \] (44)

and the interaction term \( H_I \)

\[ H_I = \int d^v x \left[ 4 \lambda B Z^3 \phi_0 \phi^3 + \lambda B \tilde{\phi}^4 \right]. \] (45)

yields the ground state energy

\[ \frac{1}{V} \langle 0 | H | 0 \rangle = V_c(\phi_0) + \frac{1}{V} \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \mathcal{T} \int dt_1 \cdots \int dt_n \langle 0 | H_I(0) H_I(t_1) \cdots H_I(t_n) | 0 \rangle, \] (46)

where \( H_I(t) \) is the interaction Hamiltonian in the interaction picture:

\[ H_I(t) := e^{iH_0 t} H_I e^{-iH_0 t}. \] (47)

In ref. [12] the expectation value \( \langle H \rangle / V \) is called generalized Gaussian Effective Potential. It consists of the standard GEP, obtained variationally, and perturbative corrections due to the effective interaction Hamiltonian. Note that the latter contains a new, cubic interaction term generated nonperturbatively by spontaneous symmetry breaking.

\section{Effective Hamiltonian for the \( O(2) \)-model and Goldstone bosons}

Two–component self–interacting scalar field theory is the paradigm for the Goldstone mechanism. A simple, classical treatment serves as textbook example for the occurrence of massless particles due to the spontaneous breaking of a continuous symmetry. It merely consists of a longitudinal shift of the field by a non–zero vacuum expectation value which is found by minimization of the classical potential. A vanishing mass term with respect to the transverse direction is interpreted to correspond to a massless excitation, i.e. the Goldstone boson.

Up to now, a satisfactory quantum mechanical analysis is lacking. Of course, quantum corrections could destroy the simple, classical picture and its consequences sketched above. In fact, this necessarily happens in one space dimension because of Coleman’s theorem [13]. Since exact, analytical solutions, even restricted to the ground state properties and masses, are beyond present possibilities, approximations have to be made. A variational approach, using Gaussian wave functionals with two mass parameters, indeed yields spontaneous breaking of the \( O(2) \) rotational symmetry. As mentioned in the introduction both excitations, however, are massive; in conflict with Goldstone’s theorem. The reason is that the symmetry is not only broken because of a non–vanishing vacuum expectation value of the field but also by the use of different masses in the wave functional. The variational ansatz was formulated in the ”Cartesian” basis; a formulation in
Again we introduce the renormalization constant $Z$. Note that only one wave function renormalization of renormalized fields and momenta, one–component theory we introduce a trial ground state where radial and angular "coordinates" may avoid this problem. Conceptual problems with the quantization have prohibited such an analysis until now.

The (Cartesian) Gaussian ansatz with different masses is also not compatible with an interpretation in terms of charged scalar particles. This happens in the Abelian Higgs model: the Cartesian components of the field are combined in a complex field, describing particle and anti–particle excitations. Of course, particles and anti–particles should have the same mass. The problems concerning the ansatz with two mass parameters \[9\] will be discussed in more detail in section (6).

In view of these arguments we do a variational calculation with one mass. The procedure retraces the steps outlined in the introduction. First, the $O(2)$–symmetry can only be spontaneously broken due to a non–vanishing expectation value of the field. The second step, of course absent in the symmetric phase, is to diagonalize the quadratic part of the effective Hamiltonian. In contrast to the one–component case this is, in principle, a non–trivial Bogoliubov transformation. It is necessary because of the use of only one mass parameter and, consequently, only one gap equation. This transformation finally leads to two different masses. In fact, in the broken phase we obtain zero mass excitations, i.e. the Goldstone bosons.

The Hamiltonian for the $O(2)$–model is given by

$$H = \int d^\nu x \left[ \frac{1}{2} (\pi^2_{1B} + \pi^2_{2B}) + \frac{1}{2} (\nabla \phi_{1B})^2 + (\nabla \phi_{2B})^2 + \frac{1}{2} m^2_B \phi^2_B + \lambda_B \phi^4_B \right],$$

where $\phi^2_B = \phi^2_{1B} + \phi^2_{2B}$. The equal time commutation relations are

$$[\pi_{kB}(\vec{x}), \phi_{lB}(\vec{y})] = -i \delta_{kl} \delta^\nu(\vec{x} - \vec{y}).$$

Again we introduce the renormalization constant $Z_\phi$ and rewrite the Hamiltonian in terms of renormalized fields and momenta,

$$H = \int d^\nu x \left[ \frac{1}{2} Z^{-1}_\phi (\pi^2_1 + \pi^2_2) + \frac{1}{2} Z_\phi ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) + \frac{1}{2} Z_\phi m^2_B \phi^2 + \lambda_B Z^2_\phi \phi^4 \right].$$

Note that only one wave function renormalization $Z_\phi$ is necessary. Analogous to the one–component theory we introduce a trial ground state

$$|0\rangle_{\Omega, \phi_0} = |0\rangle_{\Omega, \phi_{01}} \otimes |0\rangle_{\Omega, \phi_{02}},$$

$$\phi_1(\vec{x}) = \phi_{01} + Z^{-\frac{1}{2}}_\phi \int (dk)\Omega \left[ a_1(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + a^\dagger_1(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right],$$

$$\phi_2(\vec{x}) = \phi_{02} + Z^{-\frac{1}{2}}_\phi \int (dk)\Omega \left[ a_2(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + a^\dagger_2(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right],$$

$$\pi_1(\vec{x}) = -i Z^{\frac{1}{2}}_\phi \int (dk)\Omega \omega(\vec{k}) \left[ a_1(\vec{k}) e^{i\vec{k} \cdot \vec{x}} - a^\dagger_1(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right],$$

$$\pi_2(\vec{x}) = -i Z^{\frac{1}{2}}_\phi \int (dk)\Omega \omega(\vec{k}) \left[ a_2(\vec{k}) e^{i\vec{k} \cdot \vec{x}} - a^\dagger_2(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right],$$

where the index $\Omega$ of the creation and annihilation operators is omitted. We normal order with respect to the trial ground state state (cf. eqs. (51)):

$$: \phi^2_j : = \phi^2_j - Z^{-1}_\phi I_0(\Omega^2),$$

$$: \phi^4 : = \phi^4 - 8 Z^{-1}_\phi I_0(\Omega^2) : \phi^2 : - 8 Z^{-2}_\phi I_0(\Omega^2)^2,$$

$$: (\nabla \phi_j)^2 : = (\nabla \phi_j)^2 - Z^{-1}_\phi I_1(\Omega^2) + Z^{-1}_\phi \Omega^2 I_0(\Omega^2),$$

$$: \pi^2_j : = \pi^2_j - Z_\phi I_1(\Omega^2),$$

(53)
where \( j = 1, 2 \). In the following we again suppress the argument of the regularized integrals: \( I_N \equiv I_N(\Omega^2) \). The Hamiltonian in normal ordered form reads

\[
H = \int d^nx \left[ \frac{1}{2} Z^{-1}_\phi (\pi_1^2 + \pi_2^2) + \frac{1}{2} Z_\phi ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) \\
+ \frac{1}{2} (m_0^2 + 16\lambda B I_0) Z_\phi \phi^2 + \lambda_0 Z_\phi^2 \phi^4 + 2I_1 + (m_0^2 - \Omega^2) I_0 + 8\lambda_0 I_0^2 \right].
\]  

(54)

The expectation value of the Hamiltonian density in the trial ground state is readily calculated from this expression,

\[
V_G(\phi_0, \Omega^2) = \frac{1}{2} (m_0^2 + 16\lambda B I_0) Z_\phi \phi_0^2 + \lambda_0 Z_\phi^2 \phi_0^4 + 2I_1 + (m_0^2 - \Omega^2) I_0 + 8\lambda_0 I_0^2,
\]

(55)

with \( \phi_0^2 = \phi_{01}^2 + \phi_{02}^2 \). The variational principle yields

\[
\Omega^2 = m_0^2 + 8\lambda B \left[ 2I_1 + Z_\phi \phi_0^2 \right],
\]

(56)

and

\[
m_0^2 Z_\phi \phi_0 + 4\lambda B Z_\phi^2 \phi_0^3 + 16\lambda B I_0 Z_\phi \phi_0 = 0.
\]

(57)

For \( \phi_0^2 \neq 0 \) these equations imply \( \Omega^2 = 4\lambda B \phi_0^2 Z_\phi \). Note that only the magnitude \( \phi_0^2 \) follows from these equations. Because of the rotational symmetry the direction is arbitrary. Furthermore, these equations have to be supplemented by the condition that the extremum indeed corresponds to a minimum. As in the previous section we assume these equations to be fulfilled and continue by subtracting the vacuum expectation value from the Hamiltonian

\[
H_R = H - \int d^nx V_G = \int d^nx \left[ \frac{1}{2} Z^{-1}_\phi (\pi_1^2 + \pi_2^2) + \frac{1}{2} Z_\phi ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) \\
+ \frac{1}{2} (m_0^2 + 16\lambda B I_0) Z_\phi (\phi^2 - \phi_0^2) + \lambda_0 Z_\phi^2 (\phi^4 - \phi_0^4) \right].
\]

(58)

The bare mass parameter is eliminated via the variational principle, i.e. with eq. (56),

\[
H_R = \int d^nx \left[ \frac{1}{2} Z^{-1}_\phi (\pi_1^2 + \pi_2^2) + \frac{1}{2} Z_\phi ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) \\
+ \frac{1}{2} (\Omega^2 - 8\lambda B Z_\phi^2) Z_\phi (\phi^2 - \phi_0^2) + \lambda_0 Z_\phi^2 (\phi^4 - \phi_0^4) \right].
\]

(59)

Concomitantly, the divergent integrals have been removed. For \( \phi_0 = 0 \) one again finds (after rescaling) that the masses are both given by \( \Omega^2 \). The resulting effective Hamiltonian obviously is \( O(2) \) symmetric.

In the following we will consider the case \( \phi_0 \neq 0 \). The shifted fields \( \tilde{\phi} \) are defined as

\[
\tilde{\phi}_1 = \phi_1 - \phi_0 = \phi_1 - \phi_0 \cos \alpha, \\
\tilde{\phi}_2 = \phi_2 - \phi_0 = \phi_2 - \phi_0 \sin \alpha.
\]

(60)

Then one obtains

\[
\phi^2 - \phi_0^2 = \tilde{\phi}^2 + 2\phi_0 L(\tilde{\phi}),
\]

(61)

and

\[
\phi^4 - \phi_0^4 = \tilde{\phi}^4 + 4\phi_0 L(\tilde{\phi}) \tilde{\phi}^2 + 4\phi_0^2 L^2(\tilde{\phi}) + 2\phi_0^2 \phi_0^2 + 4\phi_0^3 L(\tilde{\phi}),
\]

(62)

where

\[
L(\tilde{\phi}) = \phi_1 \cos \alpha + \phi_2 \sin \alpha.
\]

(63)
Substituting this in the effective Hamiltonian and using the extremum conditions, eqs. (56, 57), gives

\[ H_R = : \int d^\nu x \left[ \frac{1}{2} Z_\phi^{-1} (\pi_1^2 + \pi_2^2) + \frac{1}{2} Z_\phi((\nabla \phi_1)^2 + (\nabla \phi_2)^2) \right. \]
\[ + \Omega^2 Z_\phi L^2(\phi) + \lambda B Z_\phi^2(\phi^4 + 4\phi_0 L(\phi) \phi^2) \left. \right] : . \]  

(64)

In terms of the rescaled fields \( \tilde{\phi}_j \) and momenta \( \tilde{\pi}_j \),

\[ \tilde{\phi}_j = Z_{\phi}^{-\frac{1}{2}} \phi_j; \]
\[ \tilde{\pi}_j = Z_{\phi}^{\frac{1}{2}} \pi_j, \]  

(65)

one finally gets the effective Hamiltonian for the vacuum with spontaneously broken symmetry. It is characterized by \( \phi_0 \) and \( \alpha \):

\[ H_R = : \int d^\nu x \left[ \frac{1}{2} (\tilde{\pi}_1^2 + \tilde{\pi}_2^2) + \frac{1}{2} ((\nabla \tilde{\phi}_1)^2 + (\nabla \tilde{\phi}_2)^2) \right. \]
\[ + \Omega^2 L^2(\tilde{\phi}) + \lambda B (\tilde{\phi}^4 + 4Z_{\phi}^2 \phi_0 L(\tilde{\phi}) \tilde{\phi}^2) \left. \right] : . \]  

(66)

We emphasize that spontaneous symmetry breaking, i.e. the non–trivial vacuum and its consequences, are taken into account in a nonperturbative way. Consider the quadratic part of the Hamiltonian, which is normal ordered with respect to the operators \( a_j \). It is non–diagonal and needs to be diagonalized by means of a Bogoliubov transformation. This can be done for an arbitrary angle \( \alpha \) as will be shown in Appendix A. In the following we will choose most conveniently \( \alpha = 0 \), for which the \( \phi_1 \) and \( \phi_2 \) modes decouple and the mass term simplifies. The quadratic part of \( H_R \) then has the following form:

\[ H_0 = : \int d^\nu x \left[ \frac{1}{2} (\tilde{\pi}_1^2 + \tilde{\pi}_2^2) + \frac{1}{2} ((\nabla \tilde{\phi}_1)^2 + (\nabla \tilde{\phi}_2)^2) + \frac{1}{2} (2\Omega^2) \tilde{\phi}^2 \right] :, \]  

(67)

and corresponds to eq. (13) for the quantum mechanical example. In terms of the annihilation and creation operators one explicitly has the non–diagonal form:

\[ H_0 = \int (dk) \left[ \omega(k) \left\{ a_1^\dagger(k) a_1(k) + a_2^\dagger(k) a_2(k) \right\} \right. \]
\[ \left. + \frac{\Omega^2}{4\omega(k)} \left[ a_1^\dagger(k) a_1^\dagger(-k) + 2a_1^\dagger(k) a_1(k) + a_1(k) a_1(-k) \right] \right. \]
\[ - \left. a_2(k) a_2(-k) + 2a_2^\dagger(k) a_2(k) + a_2^\dagger(k) a_2^\dagger(-k) \right\} \right]. \]  

(68)

For the following it is more convenient to change to a new set of creation and annihilation operators with \( \omega–\)independent commutation relations

\[ a_1(k) = \sqrt{2(2\pi)^\nu \omega(k)} a(k), \]
\[ a_2(k) = \sqrt{2(2\pi)^\nu \omega(k)} b(k). \]  

(69)

Analogous expressions hold for \( a_1^\dagger \) and \( a_2^\dagger \). Recall that \( \omega(k) = \sqrt{k^2 + \Omega^2} \). In order to diagonalize the quadratic part of the Hamiltonian we define the following Bogoliubov transformation:

\[ u(k) = \cosh \alpha(k) a(k) + \sinh \alpha(k) a^\dagger(-k) = U a(k) U^\dagger, \]
\[ v(k) = \cosh \beta(k) b(k) + \sinh \beta(k) b^\dagger(-k) = U b(k) U^\dagger, \]  

(70)
with

\[ U = \exp \left\{ - \int d^d k \left\{ \alpha(k) \left( a^\dagger(k) a^\dagger(-k) - a(k) a(-k) \right) + \beta(k) \left( b^\dagger(k) b^\dagger(-k) - b(k) b(-k) \right) \right\} \right\}. \]

In terms of the operators \( u \) and \( v \) the quadratic part of the Hamiltonian \( H_0 \) now reads

\[ H_0 = \int d^d k \left[ \omega_1(k) u^\dagger(k) u(k) + \omega_2(k) v^\dagger(k) v(k) \right] + C, \]

where \( C \) is an irrelevant \( c \)-number. This determines the functions \( \alpha(k) \) and \( \beta(k) \):

\[ \tanh \alpha(k) = \frac{\Omega^{-2} \left[ 2\omega(k)^2 + \Omega^2 - 2\omega(k) \sqrt{k^2 + 2\Omega^2} \right]}{1 + 2\omega(k) \sqrt{k^2 + 2\Omega^2}}, \]
\[ \tanh \beta(k) = \frac{\Omega^{-2} \left[ -2\omega(k)^2 + \Omega^2 + 2\omega(k) k \right]}{1 - 2\omega(k) \sqrt{k^2 + 2\Omega^2}}. \]

In the limit \( k \to 0 \) these "angles" agree with those of the quantum mechanical example. The energy eigenvalues are readily obtained by commuting \( H_0 \) with the quasiparticle operators \( u \) and \( v \) (cf. section (1)). This yields

\[ \omega_1(k) = \sqrt{k^2 + M_1^2}, \]
\[ \omega_2(k) = \sqrt{k^2 + M_2^2}, \]

where the physical masses are

\[ M_1 = \Omega \sqrt{2}, \]
\[ M_2 = 0. \]

The Goldstone boson is indeed massless. The ground state \( |0\rangle_{M_1,M_2} \) of the quadratic part of the Hamiltonian \( H_0 \) is given by

\[ |0\rangle_{M_1,M_2} = U |0\rangle_{\Omega,B_0}. \]

It contains a coherent superposition of correlated boson pairs with vanishing total momentum. Analogous states appear in the theory of superfluidity and in the BCS theory.

Finally, we note that the explicit renormalization for the one–component theory (cf. section (5)) can straightforwardly be extended to the \( O(2) \)–model. Due to our ansatz with only one mass parameter, the structure of the Gaussian Effective Potentials, eqs. (34) and (55), as well as the variational equations (cf. eqs. (36), (37), (56) and (57)) is identical. These expressions only differ by some numerical coefficients and we do not expect that the qualitative behaviour of the solutions changes.

### 4 Perturbation theory for the \( O(2) \)–model

The Bogoliubov transformation, cf. eqs. (70, 71), also modifies the non–quadratic terms of the effective Hamiltonian. As we will show below, new quadratic terms are generated and thus the Goldstone bosons could acquire a mass. These new terms, however, are one–loop contributions and, in order to work consistently, all one–loop contributions of the effective theory must be taken into account. Alternatively speaking, the possible mass corrections are of order \( \lambda_B \) in the effective theory and, again for consistency, one needs to include all contributions of order \( \lambda_B \). Note that, since \( \phi^2_0 \sim \frac{\Omega^2}{\lambda_B} \), \( \phi_0 \lambda_B \) is of order \( \sqrt{\lambda_B} \).

In this way, the equivalence of the loop- and coupling constant expansion is transparent to this order.

In this section we will explicitly show that the possible mass corrections cancel and, as a consequence, that the Goldstone bosons remain massless in field theory. In order to control infrared divergences in perturbation theory with massless particles we introduce an explicit symmetry breaking term in the Hamiltonian. This also slightly modifies the variational calculation. Finally, the infrared regulator is taken to be zero. Again we treat the quantum mechanical model first. It is interesting that, although the one–loop contributions are infrared finite, the zero frequency disappears in quantum mechanics.
4.1 Quantum mechanics

We start by adding an explicit symmetry breaking term to the Hamiltonian:

\[ H_\delta = -\delta^2 f x. \]  \hfill (77)

The same variational approach, with the new variational parameters \( \Omega_\delta \) and \( x_\delta \) requires the minimization of

\[ V_G^\delta = V_G - \delta^2 f x_\delta. \]  \hfill (78)

After doing this we take \( f = x_0 \), i.e. the expectation value for \( \delta = 0 \) (cf. section (1)), and we obtain

\[ \Omega_\delta^2 = 4\lambda x_0^2 + \delta^2 + O(\delta^4). \]  \hfill (79)

Furthermore, one can easily verify that

\[ \Omega_\delta^2 = \Omega^2 + O(\delta^2) \]
\[ x_\delta^2 = x_0^2 + O(\delta^2). \]  \hfill (80)

More explicit expressions, which are irrelevant at the moment, will be given for the field theoretical case. The physical frequencies correspond to the diagonal quadratic part of the Hamiltonian

\[ \omega_1^2 = 2\Omega_\delta^2 - \delta^2 + O(\delta^4), \]
\[ \omega_2^2 = \delta^2 - O(\delta^4), \]  \hfill (81)

i.e. the normal modes in the \( y \)-direction have a finite frequency. After the unitary Bogoliubov transformation the Hamiltonian reads

\[ H_R^\delta = H_0 + H_I, \]  \hfill (82)

with

\[ H_0 = \omega_1 u^\dagger u + \omega_2 v^\dagger v, \]  \hfill (83)

and

\[ H_I = : \lambda \left[ (\tilde{x}^2 + y^2)^2 + \left( \frac{3}{\omega_1} + \frac{1}{\omega_2} - \frac{4}{\Omega_\delta} \right) \tilde{x}^2 + \left( \frac{3}{\omega_2} + \frac{1}{\omega_1} - \frac{4}{\Omega_\delta} \right) y^2 \right] + 4x_\delta \lambda \left[ \tilde{x} y^2 + \left( \frac{3}{2\omega_1} + \frac{1}{2\omega_2} - \frac{2}{\Omega_\delta} \right) \tilde{x} + \tilde{x}^3 \right]:_u,v. \]  \hfill (84)

As denoted the Hamiltonian is reordered with respect to the \((u, v)\) ground state. Moreover, a constant is omitted.

We are interested in the energy shift of the first excited state of the \( y \)-oscillator. The energy difference between this state and the ground state is \( \delta \) for \( H_0 \), but will be changed by the perturbative part of the Hamiltonian. We want to calculate its correction in first order in \( \lambda \). The \( y^2 \) term, from now on denoted by \( V_0 \), gives a contribution of order \( \lambda \) and will be taken into account in first order perturbation theory. The terms proportional to \( x_\delta \) must be taken into account in second order perturbation theory since \( x_\delta^2 \lambda^2 \sim \lambda \). Let us define

\[ U = 4\lambda x_\delta (U_1 + U_2 + U_3), \]  \hfill (85)

where

\[ U_1 = : \tilde{x} y^2 :; \]
\[ U_2 = \left( \frac{3}{2\omega_1} + \frac{1}{2\omega_2} - \frac{2}{\Omega_\delta} \right) \tilde{x}; \]
\[ U_3 = : \tilde{x}^3 :. \]  \hfill (86)
The calculation of the remaining part is facilitated by noting that the possible contributing intermediate states, starting from the state $|0;1\rangle$, are $|1;1\rangle$, $|1;3\rangle$ via $U_1$, $|1;1\rangle$ via $U_2$, and $|3;1\rangle$ via $U_3$, respectively. Therefore, there is only one contributing "interference" term, i.e. $U_1U_2$ with the intermediate state $|1;1\rangle$. The $U_3^2$ and $U_3^2$ terms are obviously cancelled by the ground state subtractions, whose relevant states are $|1;2\rangle$, $|1;0\rangle$ and $|3;0\rangle$ for $U_1$, $U_2$ and $U_3$, respectively. As a consequence, also part of the $U_1^2$ contribution is cancelled. In this way we obtain

$$\Delta E_{v}^{(2)} = \frac{16\lambda^2 x_3^2}{2 \omega_2^2 \omega_1} \left[ \frac{16 \lambda^2 x_3^2}{\omega_2 \omega_1^2(2\omega_2 + \omega_1)} - \frac{16 \lambda^2 x_3^2}{\omega_2 \omega_1^2} \left( \frac{3}{2 \omega_1} + \frac{1}{2 \omega_2} - \frac{4}{\Omega_\delta} \right) \right],$$

(90)

where the first line is the $U_1^2$ contribution via the intermediate state $|1;1\rangle$. The second line is also a $U_1^2$ contribution, but from the intermediate state $|1;3\rangle$ (including the ground state subtraction). The third line is the contribution of the interference term $U_1U_2$. Using $16\lambda^2 x_3^2 = 2\lambda(\omega_1^2 - \omega_2^2) + O(\delta^4)$ gives

$$\Delta E_{v}^{(2)} = -\frac{\lambda}{\omega_2^2} + \frac{\lambda}{\omega_1^2} \frac{\omega_2^2 - \omega_1^2}{\omega_1 \omega_2^2(2\omega_2 + \omega_1)} + \frac{\lambda}{\omega_1^2} - \frac{\lambda}{\omega_2} \left( \frac{3}{\omega_1} + \frac{1}{\omega_2} - \frac{4}{\Omega_\delta} \right) + O(\delta).$$

(91)

Further expanding in $\delta$ and adding the terms finally yields

$$\Delta E_v = -\frac{\lambda}{\omega_1^2} + O(\delta).$$

(92)

Consequently, all the possible infrared divergences cancel and the limit $\delta \to 0$ is trivial. The (would be) zero frequency becomes finite in perturbation theory.

Naively, one may expect more severe infrared problems in the corrections to the frequency of the $x$–oscillator. An explicit calculation, however, shows that also in this case these infrared divergences cancel and the order $\lambda$ correction is finite in the limit $\delta \to 0$. Of course, this does not exclude infrared infinities in higher order.
4.2 Field Theory

The phenomenon we described above could generate a Goldstone mass in field theory. We will explicitly demonstrate, however, that due to a different infrared behaviour this does not happen. In contrast to quantum mechanics the zero Goldstone mass does not get changed in perturbation theory to this order.

We add an explicit symmetry breaking term to the Hamiltonian of the $O(2)$–model (cf. eq. (48)),

$$ H^\delta = H - f^2 \sqrt{Z_\phi} \int d^nx \phi_1(\vec{x}). $$

(93)

We repeat the variational calculation of section (3) but with the new parameters $\Omega^\delta$ and $\phi^\delta$. Consistent with the added symmetry breaking term we take $\alpha = 0$ from the beginning, i.e. $\phi_{01} = \phi_0$ and $\phi_{02} = 0$. The expectation value of the Hamiltonian density now reads

$$ V^\delta_G(\phi^\delta, \Omega^\delta) = V_G(\phi^\delta, \Omega^\delta) - f^2 \sqrt{Z_\phi} \phi^\delta, $$

(94)

with $V_G$ from the previous $O(2)$ calculation, eq. (55). Minimization yields the same gap equation (56), but in the new variables. The $\phi^\delta$ equation, however, is modified

$$ Z_\phi \phi^\delta (m^2_B + 4\lambda_B Z_\phi \phi^2 + 16\lambda_B I_0) = \sqrt{Z_\phi} f^2. $$

(95)

For convenience we fix $f$ in the usual way, i.e. $f = \sqrt{Z_\phi} \phi_0$, where $\phi_0$ is determined by eqs. (56) and (57). (These equations also yield $\Omega$). With this choice one obtains

$$ \begin{align*}
\Omega^2_\delta &= \Omega^2 + \delta^2 \frac{2}{1 - 8\lambda_B I_{-1}(\Omega^2)} + O(\delta^4), \\
\phi^2_\delta &= \phi^2_0 + \delta^2 \frac{1 + 8\lambda_B I_{-1}(\Omega^2)}{4\lambda_B Z_\phi} \Omega^2_\delta + O(\delta^4).
\end{align*} $$

(96)

In the following we proceed as in section (3). First, the vacuum expectation value of the Hamiltonian is subtracted and the bare mass is eliminated with the gap equation. Secondly, we shift the field $\phi_1, \bar{\phi}_1 = \phi_1 - \phi^\delta$ and insert eq. (95) in order to get rid of the linear terms. Finally, we rescale the fields and find

$$ H^\delta_R = : \int d^nx \left[ \frac{1}{2}(\bar{\pi}_1^2 + \bar{\pi}_2^2) + \frac{1}{2}((\nabla \bar{\phi}_1)^2 + (\nabla \bar{\phi}_2)^2) \\
+ \frac{1}{2} M_1^2 \bar{\phi}_1^2 + \frac{1}{2} M_2^2 \bar{\phi}_2^2 + \lambda_B (\bar{\phi}_4^2 + 4 Z_\phi \bar{\phi}_0 \bar{\phi}_1 \bar{\phi}_2) \right] :. $$

(97)

with

$$ \begin{align*}
M_1^2 &= 2\Omega^2_\delta - \delta^2 + O(\delta^4), \\
M_2^2 &= \delta^2 + O(\delta^4).
\end{align*} $$

(98)

The normal ordering is still with respect to $\Omega^\delta$. The Bogoliubov transformation which diagonalizes the quadratic part of the Hamiltonian corresponds to a reordering with respect to the corresponding operators $u$ and $v$. As before, this simplicity is due to the choice $\alpha = 0$. Apart from a constant, the result is

$$ H^\delta_R = H_0 + H_I, $$

(99)

with the free, diagonal Hamiltonian

$$ H_0 = : \int d^nx \left[ \frac{1}{2}(\pi_1^2 + \pi_2^2) + \frac{1}{2}((\nabla \phi_1)^2 + (\nabla \phi_2)^2) + \frac{1}{2} M_1^2 \phi_1^2 + \frac{1}{2} M_2^2 \phi_2^2 \right] :u,v $$

(100)
and the interaction terms

\[ H_I = : \int d^\nu x \lambda_B \left[ \phi^4 + 4\sqrt{Z_\phi} \phi_0 \phi_1 \phi^2 \right. \]
\[ + 4\sqrt{Z_\phi} \left( 3I_0(M_1^2) + I_0(M_2^2) - 4I_0(\Omega_3^2) \right) \phi_1 \]
\[ + \left( 6I_0(M_1^2) + 2I_0(M_2^2) - 8I_0(\Omega_3^2) \right) \phi_2 \]
\[ \left. + \left( 6I_0(M_2^2) + 2I_0(M_1^2) - 8I_0(\Omega_3^2) \right) \phi_3 \right] :_{u,v} . \]

\( H_0 \) describes free particles with energies \( E_1^{(0)}(\vec{p}) = \sqrt{M_1^2 + \vec{p}^2} \) and \( E_2^{(0)}(\vec{p}) = \sqrt{M_2^2 + \vec{p}^2} \), for given 3-momentum \( \vec{p} \). We are interested in possible mass corrections to the Goldstone boson, i.e. particle 2. It is convenient to take the limit \( \vec{p} \to 0 \) and eventually we want to study the limit \( \delta \downarrow 0 \) in order to see whether the Goldstone boson remains massless.

The perturbative calculation is completely analogous to the quantum mechanical example. The latter is actually contained in the field theoretical framework; it formally corresponds to zero space dimension (\( \nu = 0 \)) without the particle interpretation. The extension of the definitions of the potentials is straightforward and will be assumed from now on. Again the \( U_2 \) pieces are disconnected and, consequently, irrelevant. As in quantum mechanics, the first correction is trivially calculated.

\[ \Delta E_2^{(1)}(\vec{p} = 0) = \lim_{\vec{p} \to 0} \frac{(0; \vec{p}) | V_0 | (0; \vec{p})}{\langle 0; \vec{p} | 0; \vec{p} \rangle} = \frac{2\lambda_B}{M_2} \left[ I_0(M_1^2) + 3I_0(M_2^2) - 4I_0(\Omega_3^2) \right] . \]

The rest of the calculation is slightly more involved than in quantum mechanics since one has to integrate over the momenta of the particles in the intermediate states. After subtracting the disconnected pieces we obtain

\[ \Delta E_2^{(2)}(\vec{p} = 0) = \frac{16\lambda_B^2 Z_\phi \phi_3^2}{M_2} \int \frac{d^\nu q}{(2\pi)^3} \frac{1}{\sqrt{q^2 + M_1^2}} \frac{1}{\sqrt{q^2 + M_2^2}} \frac{1}{M_2 - \sqrt{q^2 + M_1^2} - \sqrt{q^2 + M_2^2}} \]
\[ - \frac{16\lambda_B^2 Z_\phi \phi_3^2}{M_2} \int \frac{d^\nu q}{(2\pi)^3} \frac{1}{\sqrt{q^2 + M_1^2}} \frac{1}{\sqrt{q^2 + M_2^2}} \frac{1}{M_2 + \sqrt{q^2 + M_1^2} + \sqrt{q^2 + M_2^2}} \]
\[ - \frac{16\lambda_B^2 Z_\phi \phi_3^2}{M_2} \frac{1}{M_1^2} \left[ 3I_0(M_1^2) + I_0(M_2^2) - 4I_0(\Omega_3^2) \right] . \]

Here the same order of presentation as in eq. (103) has been chosen. Recall that some ultraviolet regularization is assumed. The numerator of the common prefactor can be written as

\[ 16\lambda_B^2 Z_\phi \phi_3^2 = 4\lambda_B(\Omega_3^2 - \delta^2) + O(\delta^4) \]
\[ = 2\lambda_B(M_1^2 - M_2^2) + O(\delta^4) . \]

This enables us to rewrite eq. (103) as

\[ \Delta E_2^{(2)}(\vec{p} = 0) = \frac{2\lambda_B}{M_2} \left[ I_0(M_1^2) - I_0(M_2^2) \right. \]
\[ + I_0(M_1^2) - I_0(M_2^2) \]
\[ - 3I_0(M_2^2) - I_0(M_2^2) + 4I_0(\Omega_3^2) \right] + O(\delta) . \]

This equation is proved in Appendix B where also the quantum mechanical result will be rederived. Consequently,

\[ \Delta E_2(\vec{p} = 0) = \Delta E_2^{(1)}(\vec{p} = 0) + \Delta E_2^{(2)}(\vec{p} = 0) = O(\delta) , \]
which means that in the limit $\delta \downarrow 0$ the Goldstone boson indeed remains massless to this order. The $O(\delta)$ in the last two equations is only valid for space–dimension $\nu \geq 2$, which is indeed relevant for the Goldstone boson interpretation. In the one–dimensional case a logarithm appears, i.e. one finds $O(\delta \ln \delta)$. This also vanishes in the limit $\delta \downarrow 0$ and, consequently, does not reflect Coleman’s theorem \cite{13}. In quantum mechanics there are finite corrections: the frequency is non–zero even in the limit $\delta \downarrow 0$. Again the non–normalizable plane waves (with zero momentum) are avoided. Finally, just as in quantum mechanics, the analogous calculation in field theory yields that the one–loop corrections to the massive excitation are infrared finite.

5 Renormalization of the GEP at finite momentum cutoff

Much effort has been put into the renormalization of the Gaussian Effective Potential of scalar $\lambda \phi^4$ \cite{5,7,9}. Generally, it is believed that after the regulator is removed, $\lambda \phi^4$–theory becomes trivial (see e.g. \cite{14}). We do not consider here the so called ”autonomous” renormalization scheme, which has been developed in ref. \cite{9} and critically examined in ref. \cite{15}. The great success of the Standard Model gives special importance to a detailed understanding of spontaneous symmetry breaking and of the Higgs–Mechanism. One expects that the effective potential of the true underlying theory develops a nontrivial minimum for certain values of the parameters. The $\lambda \phi^4$ theory might not be a sufficiently good approximation to the underlying theory at energies much larger than some maximum value $\Lambda$, which may be given by the scale of Grand Unification or the breaking of supersymmetry. Therefore we explicitly keep a finite momentum cutoff, i.e. all modes with momentum larger than $\Lambda$ will be discarded and the regulator will not be removed after the reparametrization in terms of the new ”renormalized” parameters. All dimensionful parameters, variables and resulting expressions for the GEP will be expressed in units of the cutoff $\Lambda$:

\begin{align*}
x &:= \Omega^2/\Lambda^2, \\
\hat{m}^2 &:= m^2/\Lambda^2, \\
\hat{m}^2_R &:= m^2_R/\Lambda^2,
\end{align*}

where $\Omega^2_0$ is defined as the solution of the gap equation (36) for $\phi_0 = 0$. The effects of vacuum fluctuations become finite and the resulting $I_N$–integrals can be written in units of $\Lambda$ (see Appendix C).

In analogy to ref. \cite{14} we assert that the physical parameters do not exceed one half of the value $\Lambda$:

\begin{align*}
\sqrt{x} &\leq \frac{1}{2}, \\
\Phi_0 &\leq \frac{1}{2}.
\end{align*}

(108)

As we will see later, the phenomenological results of the calculation are rather sensitive to the exact value of this upper limit. In any case it should not exceed unity.

We impose the renormalization conditions:

\begin{align*}
\hat{m}^2_R &:= \frac{d^2 \mathcal{V}_C(\Phi_0)}{d\Phi_0^2}\bigg|_{\Phi_0^2=0}, \\
\lambda_R &:= \frac{1}{4!} \frac{d^4 \mathcal{V}_C(\Phi_0)}{d\Phi_0^4}\bigg|_{\Phi_0^2=0}.
\end{align*}

(109)

(110)

By inserting the expression for the GEP (eq. (34)) we calculate the flows of the bare–parameters $\hat{m}^2, \lambda_R$, i.e. their dependence on the renormalized parameters $\hat{m}^2_R, \lambda_R$. In the self–consistent Hartree–approximation we take $Z_\phi = 1$ (see the discussion in ref. \cite{15}). One has to distinguish between two cases, $\hat{m}^2 \geq 0$ and $\hat{m}^2 < 0$:
a) $\hat{m}_R^2 \geq 0$: The renormalization conditions together with the expression of the GEP lead to

$$\begin{align*}
\hat{m}_R^2 &= \hat{m}_B^2 + 12\lambda_B I_0(x_0), \\
\lambda_R &= \lambda_B \frac{1 - 12\lambda_B I_{-1}(x_0)}{1 + 6\lambda_B I_{-1}(x_0)}.
\end{align*}$$

(111)

(112)

Now a specific choice for $x_0$ has to be made. There are two possibilities for the extremum condition

$$0 = \frac{\partial V_G}{\partial \sqrt{x}} = \frac{1}{2} \sqrt{x} \frac{I_{-1}(x)}{I_{-1}(x)} \left( x - \hat{m}_B^2 - 12\lambda_B \left[ \Phi_0^2 + I_0(x) \right] \right),$$

(113)

to be satisfied. The first one is $x \equiv 0$, independent of $\Phi_0$ and the bare parameters. The other is given by the solution of the "optimization equation":

$$x = \hat{m}_B^2 + 12\lambda_B [\Phi_0^2 + I_0(x)].$$

(114)

In the present case ($\hat{m}_R^2 \geq 0$) only

$$x_0 = \hat{m}_B^2 + 12\lambda_B I_0(x_0) = \hat{m}_R^2$$

(115)

applies (cf. Appendix D), where we used eq. (111) for the second identity. By solving eq. (111) and eq. (112) one now obtains expressions for $\lambda_B, \hat{m}_B^2$ in terms of the renormalized parameters. For the explicit calculation of the renormalized GEP we use the parametrization

$$\begin{align*}
\hat{m}_B^2 &= \frac{\hat{m}_R^2}{I_{-1}(x_0)} - 12\lambda_B I_0(0), \\
\lambda_B &= \frac{\eta}{I_{-1}(x_0)},
\end{align*}$$

(116)

(117)

which allows us to write the following equations in a more convenient form. We will need a subtraction formula for the $I_N$-integrals,

$$I_0(x_0) - I_0(0) = -\frac{x}{2} I_{-1}(x_0) + 2f'(x),$$

(118)

where $x_0 = x(\Phi_0 = 0)$ has been chosen as a reference point. The function $f(x)$ and its derivative $f'(x)$ are defined and calculated in Appendix C. Moreover, it is more transparent to start from the derivative of the GEP with respect to $\Phi_0^2$, to insert the flow eqs. (116), (117) as well as the optimization equation (114) and to integrate the resulting expression:

$$\frac{dV_G}{d\Phi_0^2} = \frac{1}{2\Phi_0} \frac{dV_G}{d\Phi_0} = \frac{1}{2} \left( x - 8\lambda_B \Phi_0^2 \right)$$

$$= \frac{1}{(1 + 6\eta) I_{-1}} \left[ \hat{m}_R^2 + 2\lambda BI_{-1} + 2\eta(1 - 12\eta)\Phi_0^2 \right].$$

(119)

where

$$\begin{align*}
x &= \frac{\hat{m}_R^2}{I_{-1}} + \frac{12\eta}{I_{-1}} \left[ \Phi_0^2 - \frac{x}{2} I_{-1} + 2f'(x) \right] \\
&= \frac{1}{(1 + 6\eta) I_{-1}} \left[ \hat{m}_R^2 + 12\eta \left( \Phi_0^2 + 2f'(x) \right) \right].
\end{align*}$$

(120)
Starting with eq. (120) we omit the argument of \( I_{-1}(x_0) \): \( I_{-1} \equiv I_{-1}(x_0) \). Now expression (119) is transformed into a total derivative via

\[
\frac{dx}{d\Phi_0^2} = \frac{12\eta}{(1 + 6\eta)I_{-1}} \left[ 1 + 2 \frac{dx}{d\Phi_0^2} \hat{f}''(x) \right]
\]

\[
= \left[ \frac{(1 + 6\eta)I_{-1}}{12\eta} - 2 \hat{f}''(x) \right]^{-1}.
\] (121)

Inserting this into eq. (119) leads to

\[
\frac{d\mathcal{V}_G}{d\Phi_0^2} = \hat{m}^2 + 4\eta(1 - 12\eta)\Phi_0^2 + \left( 1 - \frac{24\eta\hat{f}''(x)}{(1 + 6\eta)I_{-1}} \right) \hat{f}'(x) \frac{dx}{d\Phi_0^2},
\] (122)

and after integration to

\[
\mathcal{V}_G(\Phi_0) = \int_0^{\Phi_0} \frac{d\mathcal{V}_G}{d\Phi_0^2} d\Phi_0^2 = \hat{f}(x) - \hat{f}(x_0) + \frac{\hat{m}^2\Phi_0^2 + 2\eta(1 - 12\eta)\Phi_0^4 - 24\eta (\hat{f}'(x)^2 - \hat{f}'(x_0)^2)}{2(1 + 6\eta)I_{-1}}
\]

\[
= \frac{1}{2} \hat{m}_R^2 \Phi_0^2 + \lambda_R \Phi_0^4 + \hat{f}(x) - \hat{f}(x_0)
\]

\[
- \frac{12\eta}{(1 + 6\eta)I_{-1}} \left( \Phi_0^2 \hat{f}'(x_0) + \hat{f}'(x)^2 - \hat{f}'(x_0)^2 \right).
\] (123)

This is the resulting (dimensionless) expression for the renormalized GEP which will be analyzed below with respect to the position of local or absolute minima in terms of the renormalized parameters. Fig. 1 (right) shows an example of the renormalized GEP for \( \hat{m}_R^2 > 0 \).

b) \( \hat{m}_R^2 < 0 \): In this case a minimization of the GEP at \( \Phi_0 = 0 \) is only possible through \( x_0 = 0 \) (cf. eq. (113)): By definition there is no solution \( x_0 < 0 \) to the optimization equation (114)). Thus eqs. (109), (110) become

\[
\hat{m}_R^2 = \hat{m}_B^2 + 12\lambda_B I_0(0),
\] (124)

\[
\lambda_R = \lambda_B,
\] (125)

and for the flows one obtains

\[
\hat{m}_B^2 = \hat{m}_R^2 - 12\lambda_R I_0(0),
\] (126)

\[
\lambda_B = \lambda_R.
\] (127)

With this flow the optimization equation (cf. eq. (114))

\[
x = \hat{m}_R^2 + 12\lambda_B \left[ \Phi_0^2 + I_0(x) - I_0(x_0) \right]
\] (128)

has no positive solution for \( x \) within the interval

\[
0 \leq \Phi_0^2 < \Phi_{0,\text{crit}} := -\frac{\hat{m}_R^2}{12\lambda_B}.
\] (129)

Thus the integration has to be split into two parts:

\[
\mathcal{V}_G(\Phi_0^2 < \Phi_{0,\text{crit}}) = \frac{1}{2} \hat{m}_R^2 \Phi_0^2 + \lambda_R \Phi_0^4,
\] (130)
Here we use \( v \) radial mode in the }\( U(1) \)-model, then one obtains as an upper limit to the Higgs–mass: }\( M_{\text{Higgs}} \leq 6.9 \cdot 246 \text{ GeV} \simeq 1.7 \text{ TeV}. \) (135)

Here we use }\( v = (\sqrt{2} G_F)^{-1/2} \simeq 246 \text{ GeV}. \) This mass in turn corresponds to a minimum cutoff of 3.4 TeV. In comparison, the customary upper limit to the Higgs–mass in the literature is }\( M_{\text{Higgs}} \leq 1.2 \text{ TeV}, \) obtained from unitarity arguments in }\( W^\pm \)-scattering with Higgs exchange. Of course, also our limit on }\( M_{\text{Higgs}} \) may change when we consider the complete Higgs model with vector fields.

On the basis of our upper bound we calculate the parameters which appear in our renormalization procedure for a cutoff value of 3.4 TeV, which is twice the limit on the Higgs–mass. Furthermore, we choose three different values for the Higgs–mass (200 GeV, 800 GeV and 1.2 TeV) and take the vacuum expectation value to be 246 GeV as above. The results are shown in Table 1. As can be seen, the light Higgs bosons are coupled more weakly than the heavy ones. The first two Higgs bosons correspond to case B) }\( \hat{m}_R^2 < 0, \) i.e. the renormalized GEP has only one minimum at }\( v = 246 \text{ GeV} \) and a local maximum at }\( \Phi_0 = 0. \) This also leads to }\( \lambda_R \) being equal to }\( \lambda_R. \) The heavy Higgs boson corresponds to case a) }\( \hat{m}_R^2 \geq 0, \) so the renormalized potential has two local minima, one at }\( \Phi_0 = 0 \) as well as an absolute minimum at a finite value of }\( \Phi_0. \) One can show that case a) can only become relevant for cutoff values larger than about 2.7 TeV. Figure 2 shows the bare potential as well as the renormalized GEP for Higgs–masses of 200 GeV (left) and 1.2 TeV (right). The bare potential has been rescaled by a factor 100.

The underlying }\( O(1) \)-symmetry may be realized in the ”symmetric” as well as in the spontaneously broken phase, characterized by an absolute minimum of the GEP at }\( \Phi_0^2 > 0, \) Fig. 3 (right) shows the phase diagram in terms of the renormalized parameters. The critical line in coupling constant space is determined by }\( V_G(v) = V_G(0). \) The range of dimensionless parameters plotted in Fig. 3 corresponds to the scaling condition, eq. (108), which the physical mass has to obey. The system is in the broken phase for all pairs of coupling constants which lie below the critical line. There is no symmetric phase for }\( \hat{m}_R^2 < 0 \) and no symmetry breaking for }\( \lambda_R > 0. \)

Since renormalization amounts to a finite reparametrization for a given cutoff, the phase diagram may be represented also in terms of bare parameters. This is shown in Fig.
| $M_{\text{Higgs}}$ | 200 GeV | 800 GeV | 1200 GeV |
|-------------------|---------|---------|---------|
| $\lambda_B$       | 0.083   | 1.322   | 2.974   |
| $\lambda_R$       | 0.083   | 1.322   | −1.700  |
| $m^2_B$            | −(405 GeV)$^2$ | −(1559 GeV)$^2$ | −(2261 GeV)$^2$ |
| $m^2_R$            | −(136 GeV)$^2$ | −(326 GeV)$^2$ | (220 GeV)$^2$ |
| $\Omega^2_0$      | 0       | 0       | (220 GeV)$^2$ |

Table 1: Bare and renormalized couplings for three different Higgs–masses for a cutoff of $\Lambda = 3.4$ TeV and a vacuum expectation value $v = 246$ GeV.

The symmetry is spontaneously broken for all values between the critical line and the $-\hat{m}_B^2$–axis. This is similar to the quasiclassical behaviour (tree–level). The relation of the bare–parameters $\lambda_B$ and $\hat{m}_B^2$ is almost linear, which results from the rather linear behaviour of the integrals $I_0(x)$ and $I_1(x)$ within the range $0 < x < \frac{1}{4}$ (cf. Appendix C).

### 6 Discussion and Conclusions

Although spontaneous symmetry breaking for the $O(N)$–model has been discussed before in the framework of the Gaussian Effective Potential \cite{9}, only massive excitations were found. This violation of Goldstone’s theorem was ascribed to symmetry breaking ”at the operator level”, i.e. current non–conservation because of the non–zero vacuum expectation value of the field. As will be demonstrated below, the earlier approach indeed explicitly breaks the $O(2)$–symmetry. However, the argument concerning the Noether current needs some refinement. The explicit symmetry breaking and, equivalently the non–conserved current, is due to a variational ansatz with two mass parameters. In contrast, our approach, using one mass parameter, does not have these problems and we recover Goldstone’s theorem. The two different masses emerge, via a Bogoliubov transformation, after shifting the field. This transformation modifies the trial ground state and, concomitantly, its energy. Therefore, the ansatz with one mass parameter can eventually be superior to the ansatz with two mass parameters.

Let us compare our method to the one of ref. \cite{9} in some more detail. For the Noether current of the $O(2)$–model one has

\begin{align}
\rho &= \phi_2 \pi_1 - \phi_1 \pi_2, \\
\vec{j} &= \phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1. \tag{136}
\end{align}

Since $\partial_0 \phi_j = \pi_j (j = 1, 2)$, one generally obtains

\begin{align}
\partial_0 \rho + \nabla \vec{j} &= \phi_2 (\partial_0 \pi_1 - \Delta \phi_1) - \phi_1 (\partial_0 \pi_2 - \Delta \phi_2). \tag{137}
\end{align}

It is easy to verify that Heisenberg’s equations of motion yield current conservation. The ansatz in ref. \cite{9} corresponds to the field expansion

\begin{align}
\phi_1(x) &= \phi_0 + Z^{\frac{1}{2}}_\phi \int (dk)_{\Omega_1} \left[a_{\Omega_1}(\vec{k})e^{i\vec{k} \cdot \vec{x}} + a^\dagger_{\Omega_1}(\vec{k})e^{-i\vec{k} \cdot \vec{x}}\right], \\
\phi_2(x) &= Z^{\frac{1}{2}}_\phi \int (dk)_{\Omega_2} \left[a_{\Omega_2}(\vec{k})e^{i\vec{k} \cdot \vec{x}} + a^\dagger_{\Omega_2}(\vec{k})e^{-i\vec{k} \cdot \vec{x}}\right], \\
\pi_1(x) &= -iZ^{\frac{1}{2}}_\phi \int (dk)_{\Omega_1} \omega(\vec{k}) \left[a_{\Omega_1}(\vec{k})e^{i\vec{k} \cdot \vec{x}} - a^\dagger_{\Omega_1}(\vec{k})e^{-i\vec{k} \cdot \vec{x}}\right], \\
\pi_2(x) &= -iZ^{\frac{1}{2}}_\phi \int (dk)_{\Omega_2} \omega(\vec{k}) \left[a_{\Omega_2}(\vec{k})e^{i\vec{k} \cdot \vec{x}} - a^\dagger_{\Omega_2}(\vec{k})e^{-i\vec{k} \cdot \vec{x}}\right]. \tag{138}
\end{align}
Figure 1: The renormalized GEP $V_c \times 10^4$ (thin line) compared to the bare potential $V_{\text{bare}} \times 10^2$ (thick line). On the left side a Higgs–mass $M_{\text{Higgs}} = 200$ GeV has been chosen, on the right side $M_{\text{Higgs}} = 1.2$ TeV. The bare potential has been rescaled by a factor 100.

Figure 2: The phase diagram in terms of the bare (left) and the renormalized parameters (right).

a) Bare parameters: the system is in the broken phase below the curve and above the $-\hat{m}_B^2$ axis. The end of the critical line corresponds to a maximum value $1/4$ for $x_v$.

b) Renormalized parameters: the system is in the broken phase below the critical line.
Comparing to our ansatz, cf. eq. (52), we encounter two different mass parameters $\Omega_1$ and $\Omega_2$. It is important that the variational principle then indeed yields two different values for the masses $\Omega_1$ and $\Omega_2$ in case of a non–zero vacuum expectation value of the scalar field. Inserting eqs. (138) into eq. (137) gives

$$\partial_0 \rho + \nabla \vec{j} = (\Omega_2^2 - \Omega_1^2) \phi_1 \phi_2.$$ (139)

We see that the Noether current associated with the $O(2)$–symmetry is non–conserved for $\phi_0 \neq 0$. If the symmetry is unbroken, i.e. $\phi_0 = 0$, one obtains $\Omega_1 = \Omega_2$ and the current is conserved. It is evident from eq. (139) that the ansatz with one mass parameter, eq. (52), does not violate current conservation, irrespective of the vacuum expectation value of the field. In this way one can combine spontaneous symmetry breaking and current conservation, which are both basic ingredients of the proof of Goldstone’s theorem.

It may also be instructive to compare the different approaches at the level of the effective Hamiltonian which is defined in this work. We claim that the $O(2)$–symmetry gets broken explicitly when we use the gap equations of [9]. This can be seen as follows. Normal ordering of the Hamiltonian with respect to the ground state defined with two mass parameters [9] and a subsequent subtraction of the vacuum expectation value give

$$H_R = H - \int d^d x V_G =$$

$$= \int d^d x \left[ \frac{1}{2} Z_\phi^{-1} (\pi_1^2 + \pi_2^2) + \frac{1}{2} Z_\phi ((\nabla \phi_1)^2 + (\nabla \phi_2)^2) + \frac{1}{2} (m_B^2 + 12 \lambda_B I_0^{(1)}) Z_\phi \phi_1^2 + \lambda_B Z_\phi^2 \phi_2^2 + \lambda_B Z_\phi^2 \phi_1 \phi_2 + C \right].$$ (140)

where $C = C(\Omega_1, \Omega_2, \phi_0)$ is an irrelevant constant and $I_0^{(j)} = I_0(\Omega_j^2), j = 1, 2$. Since only operator identities were inserted and a constant was subtracted, the Hamiltonian $H_R$ is $O(2)$–symmetric. The transformation properties of the normal ordered operators appearing in $H_R$ are remarkable. For example, an $O(2)$–rotation with an angle $\beta$ leads to

$$: \phi_1^2 : \rightarrow : (\phi_1 \cos \beta - \phi_2 \sin \beta)^2 + \sin^2 \beta (I_0^{(2)} - I_0^{(1)}) :,$$

$$: \phi_2^2 : \rightarrow : (\phi_1 \sin \beta - \phi_2 \cos \beta)^2 + \sin^2 \beta (I_0^{(1)} - I_0^{(2)}) :,$$ (141)

and consequently the bare mass term $W_0 := \frac{1}{2} m_0^2 \left[ : \phi_1^2 : + : \phi_2^2 : \right]$ is invariant. Moreover, from the invariance of the original $\phi^4$ term follows the transformation of the normal ordered quartic term

$$: \phi^4 : \rightarrow : \phi^4 - (6 I_0^{(1)} + 2 I_0^{(2)}) \left[ (\phi_1 \cos \beta - \phi_2 \sin \beta)^2 - \phi_1^2 + \sin^2 \beta (I_0^{(2)} - I_0^{(1)}) \right]$$

$$- (2 I_0^{(1)} + 6 I_0^{(2)}) \left[ (\phi_1 \sin \beta + \phi_2 \cos \beta)^2 - \phi_2^2 + \sin^2 \beta (I_0^{(1)} - I_0^{(2)}) \right] :.$$ (142)

Therefore, the term $W_1 := \lambda_B \left[ \phi^4 + (6 I_0^{(1)} + 2 I_0^{(2)}) \phi_1^2 + (2 I_0^{(1)} + 6 I_0^{(2)}) \phi_2^2 \right]$ is also invariant. Now consider the two gap equations (cf. eq. (56)), which read

$$\Omega_1^2 = m_B^2 + 4 \lambda_B \left[ 3 I_0^{(1)} + I_0^{(2)} + Z_\phi \phi_0^2 \right],$$

$$\Omega_2^2 = m_B^2 + 4 \lambda_B \left[ I_0^{(1)} + 3 I_0^{(2)} + Z_\phi \phi_2^2 \right].$$ (143)

They are supplemented by the $\phi_0$ equation (cf. eq. (57))

$$m_B^2 Z_\phi \phi_0 + 4 \lambda_B Z_\phi^2 \phi_0^3 + 4 \lambda_B Z_\phi \phi_0 (3 I_0^{(1)} + I_0^{(2)}) = 0.$$ (144)
These are the variational equations and symmetry properties in the two mass parameter case. In ref. [9] it is shown that the equations (143, 144) indeed support solutions $\phi_0 \neq 0, \Omega_1 \neq \Omega_2$ and for $\phi_0 = 0$ one obviously has $\Omega_1 = \Omega_2$. In the following, the broken phase is considered. The sum of the invariant terms,

$$W = W_0 + W_1,$$

appears in the Hamiltonian. As in section (3), the bare mass and the $I_0$ functions can be eliminated by means of the variational equations. This simplifies $W$:

$$W = \lambda_B \phi^4 + \left( \frac{1}{2} \Omega_1^2 - 6\lambda_B \phi_0^2 \right) \phi_1^2 + \left( \frac{1}{2} \Omega_2^2 - 2\lambda_B \phi_0^2 \right) \phi_2^2 :$$

(146)

Let us now, i.e. after this substitution, check the invariance of $W$. Obviously, the last term of $W$ is invariant. A rotation of the quartic term in eq. (146), however, is not cancelled by the corresponding one in the operator $\frac{1}{2} \Omega_2^2 \phi_2^2$. In other words, the symmetry is explicitly broken by the insertion of the variational equations of ref. [9]. On the other hand, one easily verifies that the procedure followed in our work, i.e. starting with one mass parameter, preserves the explicit $O(2)$–symmetry (cf. eq. (59)). Only after the shift the symmetry is no longer manifest (see eq. (66)). This explains why we obtain massless bosons, in accordance with Goldstone’s theorem.

In this paper, effective Hamiltonians for scalar field theories have been derived within a variational approach using Gaussian wave functionals as ansatz for the ground state. The regularized expectation value of the Hamiltonian has been minimized and, subsequently been subtracted from the Hamiltonian in order to arrive at an effective theory. This effective theory can be split into a free Hamiltonian, which already contains variationally determined parameters, and a residual part. The latter can be treated perturbatively. In contrast to perturbation theory from the onset, spontaneous symmetry breaking and its consequences, are –in principle– included in such a framework. Furthermore, the appearance of spontaneous symmetry breaking is demonstrated beyond the classical level, i.e. on the level of a self–consistent quantum theory.

The main new results of our work are obtained for the $O(2)$–model, the standard example for Goldstone’s theorem. The appearance of massless particles is conventionally derived on the classical level. One minimizes the classical potential and, after shifting the relevant component of the field, zero mass terms occur in the Lagrangian. Of course, this does not guarantee massless excitations on the quantum level. The variational calculation of this paper, including quantum effects via the Gaussian Effective Potential, indeed shows spontaneous symmetry breaking. In the ansatz with one mass parameter the symmetry can only be broken by a non–zero expectation value of the field. In this case, the quadratic part of the effective Hamiltonian is not diagonal and, in contrast to the one-component case, the mass parameter is not equal to the physical mass. Diagonalization of the quadratic Hamiltonian via a unitary Bogoliubov transformation yields one massive and one massless excitation, i.e. the Goldstone boson. Moreover, it has been explicitly shown that taking into account the one–loop corrections to the effective Hamiltonian, or equivalently including the remaining terms in lowest order perturbation theory, the Goldstone boson remains massless. The analogous calculation in quantum mechanics (zero space dimension), however, leads to a finite non–zero frequency.

The expectation value of the Hamiltonian is a priori ill–defined. Also the equations following from the variational principle contain infinities. Consequently, the theory has been regularized. On a formal level the results are independent of the chosen scheme. The only assumption is that the variational equations indeed are satisfied. Here we have chosen to work with a finite momentum cutoff and redefined (only) mass and coupling constant. In contrast to the case where the regulator is finally removed, spontaneous symmetry breaking occurs. Other possibilities including a wave function renormalization, e.g. the autonomous scheme, are not discussed. It should be emphasized that the formal developments of this work do not exclude such schemes.
Up to now, the Standard Model is in perfect agreement with experiment. The only missing link is the Higgs particle, which has not been detected yet. The simple reason may be that the Higgs is too heavy for the present energies. Phenomenologically, a bound on the Higgs–mass is therefore of crucial importance. Assuming the value of the symmetry breaking parameter, the condensate, and interpreting the variational mass of the one–component theory as the Higgs–mass, we obtain an upper bound for the Higgs–mass of 1.7 TeV. The two–component scalar theory probably improves this estimate but we believe that a complete treatment of the Higgs model is in order. Also from the theoretical side it is interesting to investigate the Higgs model along the lines given here. A pertinent question is, for example, whether it is possible to describe both the Higgs as well as the Coulomb phase within one gauge. At present, a study of the Abelian model in the Coulomb gauge is in progress.

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Appendix A: Diagonalization of the quadratic Hamiltonian for arbitrary angles

In order to diagonalize the Hamiltonian for arbitrary angles it is more convenient to use the complex field representation

\[
\Phi = \sqrt{\frac{1}{2}} (\phi_1 + i\phi_2),
\]

\[
\Phi^\dagger = \sqrt{\frac{1}{2}} (\phi_1 - i\phi_2),
\]

\[
\Pi = \sqrt{\frac{1}{2}} (\pi_1 - i\pi_2),
\]

\[
\Pi^\dagger = \sqrt{\frac{1}{2}} (\pi_1 + i\pi_2),
\]

(A.1)

and the corresponding relations for the creation and annihilation operators. The quadratic part of the Hamiltonian in terms of these fields reads

\[
H_0 = : \int d^\nu x \left[ \Pi^\dagger \Pi + (\nabla \Phi^\dagger)(\nabla \Phi) + \Omega^2 \Phi^\dagger \Phi \\
+ \frac{1}{2} \Omega^2 [e^{2i\alpha} \Phi \Phi + e^{-2i\alpha} \Phi^\dagger \Phi^\dagger] \right] :.
\]

(A.2)

Of course, one can also use the complex representation from the beginning; the result is the same. In momentum space one has

\[
H_0 = \int (dk) \left[ \omega(k) \left[ \tilde{a}^\dagger(k) \tilde{a}(k) + \tilde{b}^\dagger(k) \tilde{b}(k) \right] + \right. \\
+ \frac{\Omega^2}{2\omega(k)} \left( \frac{1}{4} e^{2i\alpha} \left[ \tilde{a}^\dagger(k) \tilde{a}^\dagger(-k) + 2\tilde{b}^\dagger(k) \tilde{b}(k) + \tilde{b}(k) \tilde{b}(-k) \right] \right]
\]

\[
+ \left. \frac{\Omega^2}{2\omega(k)} \left( \frac{1}{4} e^{-2i\alpha} \left[ \tilde{a}^\dagger(-k) \tilde{a}^\dagger(k) + 2\tilde{b}^\dagger(-k) \tilde{b}(k) + \tilde{b}(-k) \tilde{b}(k) \right] \right) \right].
\]
The measure in the field expansion also needs to be adjusted

\[ \frac{1}{2} e^{-2i\alpha} \left[ \tilde{a}(\vec{k}) \tilde{a}(-\vec{k}) + 2 \tilde{b}^\dagger(\vec{k}) \tilde{a}(\vec{k}) + \tilde{b}^\dagger(\vec{k}) \tilde{b}^\dagger(-\vec{k}) \right] \].

These creation and annihilation operators are linear combinations of those in the main text (cf. section (3)). Anticipating massless particles we first change the normalization

\[ \tilde{a}(\vec{k}) = \sqrt{2(2\pi)^\nu \omega(\vec{k})} a(\vec{k}), \]

and analogous expressions for \( a^\dagger, b \) and \( b^\dagger \). It should be emphasized that the \( a, a^\dagger \) and \( b, b^\dagger \) are not the operators of section (3) (with the same symbol) but depend linearly on them. (We thought it would be more confusing to introduce new symbols than dealing with this small inconvenience with respect to the notation.) The commutation relations are now independent of \( \omega \)

\[
\begin{align*}
[a(\vec{k}), a^\dagger(\vec{k}')] &= \delta(\vec{k} - \vec{k}'), \\
[b(\vec{k}), b^\dagger(\vec{k}')] &= \delta(\vec{k} - \vec{k}').
\end{align*}
\]

The measure in the field expansion also needs to be adjusted

\[ \Phi(\vec{x}) = \int \{dk\} \Omega \{b(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + a^\dagger(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \}, \]

with

\[ \{dk\} \Omega = \frac{d^\nu k}{\sqrt{2(2\pi)^\nu \omega(\vec{k})}}. \]

For the Hamiltonian one obtains

\[ H_0 = \int d^\nu k \left[ \omega(\vec{k}) \left[ a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) \right] \\
+ \frac{\Omega^2}{2\omega(\vec{k})} \left[ \frac{1}{2} e^{2i\alpha} \left[ a^\dagger(\vec{k}) a^\dagger(-\vec{k}) + 2 a^\dagger(\vec{k}) b(\vec{k}) + b(\vec{k}) b(-\vec{k}) \right] \\
+ \frac{1}{2} e^{-2i\alpha} \left[ a(\vec{k}) a(-\vec{k}) + 2 b^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b^\dagger(-\vec{k}) \right] \right] \].

In order to diagonalize we make the ansatz for the operators \( C \) and \( D \)

\[
\begin{align*}
C(\vec{k}) &= \alpha_1 a(\vec{k}) + \alpha_2 a^\dagger(-\vec{k}) + \beta_1 b(\vec{k}) + \beta_2 b^\dagger(-\vec{k}), \\
C^\dagger(\vec{k}) &= \alpha_1^* a^\dagger(\vec{k}) + \alpha_2^* a(-\vec{k}) + \beta_1^* b^\dagger(\vec{k}) + \beta_2^* b(-\vec{k}), \\
D(\vec{k}) &= \gamma_1 a(\vec{k}) + \gamma_2 a^\dagger(-\vec{k}) + \delta_1 b(\vec{k}) + \delta_2 b^\dagger(-\vec{k}), \\
D^\dagger(\vec{k}) &= \gamma_1^* a^\dagger(\vec{k}) + \gamma_2^* a(-\vec{k}) + \delta_1^* b^\dagger(\vec{k}) + \delta_2^* b(-\vec{k}),
\end{align*}
\]

where the \( \alpha_j, \beta_j, \gamma_j, \delta_j \) are complex c–number functions of \( k, \Omega \) and \( \alpha \). The new operators fulfill the standard commutation relations

\[
\begin{align*}
[C(\vec{k}), C^\dagger(\vec{k}')] &= \delta(\vec{k} - \vec{k}'), \\
[D(\vec{k}), D^\dagger(\vec{k}')] &= \delta(\vec{k} - \vec{k}').
\end{align*}
\]

Since the Hamiltonian should be diagonal in these operators, one has

\[
\begin{align*}
[C(\vec{k}), H_0] &= \epsilon_1(\vec{k}) C(\vec{k}), \\
[D(\vec{k}), H_0] &= \epsilon_2(\vec{k}) D(\vec{k}),
\end{align*}
\]
where \( \epsilon_{1,2}(\vec{k}) \) are the energy eigenvalues. Inserting eq. (A.9) yields an algebraic eigenvalue equation for each \( \vec{k} \). Its solution gives the energies

\[
\begin{align*}
\epsilon_1(\vec{k}) &= (\vec{k}^2 + 2\Omega^2)^{\frac{1}{2}} = \epsilon(k), \\
\epsilon_2(\vec{k}) &= (\vec{k}^2)^{\frac{1}{2}} = k,
\end{align*}
\]

and the complex functions

\[
\begin{align*}
\alpha_1 &= \frac{1}{4} \sqrt{2} e^{-i \alpha} \frac{\epsilon + \omega}{\sqrt{\epsilon \omega}}, \\
\alpha_2 &= \frac{1}{4} \sqrt{2} e^{i \alpha} \frac{\epsilon - \omega}{\sqrt{\epsilon \omega}}, \\
\beta_1 &= \frac{1}{4} \sqrt{2} e^{i \alpha} \frac{k + \omega}{\sqrt{k \omega}}, \\
\beta_2 &= \frac{1}{4} \sqrt{2} e^{-i \alpha} \frac{k - \omega}{\sqrt{k \omega}}, \\
\gamma_1 &= \frac{1}{4} \sqrt{2} e^{-i \alpha} \frac{k + \omega}{\sqrt{k \omega}}, \\
\gamma_2 &= -\frac{1}{4} \sqrt{2} e^{i \alpha} \frac{k - \omega}{\sqrt{k \omega}}, \\
\delta_1 &= -\frac{1}{4} \sqrt{2} e^{i \alpha} \frac{k + \omega}{\sqrt{k \omega}}, \\
\delta_2 &= \frac{1}{4} \sqrt{2} e^{-i \alpha} \frac{k - \omega}{\sqrt{k \omega}}.
\end{align*}
\]

These functions determine the transformation, eq. (A.9), which, as one can explicitly check, indeed diagonalizes the Hamiltonian, eq. (A.8).

\[
H_0 = \int d^\nu k \left[ \epsilon(k) C^\dagger(\vec{k}) C(\vec{k}) + k D^\dagger(\vec{k}) D(\vec{k}) \right],
\]

where we omitted an irrelevant constant. Moreover, with eqs. (A.3) and (A.9), the invariance of the canonical commutation relations, cf. eq. (A.10), follows. Note that the diagonalized Hamiltonian, eq. (A.11), is independent of the angle \( \alpha \). This a posteriori justifies fixing it to a convenient value as was done in the main text.

Finally, we want to derive the explicit operator form of the unitary transformation, eq. (A.9). In other words, the operator \( U \) is to be determined such that

\[
\begin{align*}
C(\vec{k}) &= U a(\vec{k}) U^\dagger, \\
C^\dagger(\vec{k}) &= U a^\dagger(\vec{k}) U^\dagger, \\
D(\vec{k}) &= U b(\vec{k}) U^\dagger, \\
D^\dagger(\vec{k}) &= U b^\dagger(\vec{k}) U^\dagger.
\end{align*}
\]

After defining the canonical operators \( A \) and \( B \) as

\[
\begin{align*}
A(\vec{k}) &= \frac{1}{2} \sqrt{2} \left( e^{-i \alpha} a(\vec{k}) + e^{i \alpha} b(\vec{k}) \right), \\
A^\dagger(\vec{k}) &= \frac{1}{2} \sqrt{2} \left( e^{i \alpha} a^\dagger(\vec{k}) + e^{-i \alpha} b^\dagger(\vec{k}) \right), \\
B(\vec{k}) &= \frac{1}{2} \sqrt{2} \left( e^{-i \alpha} a(\vec{k}) - e^{i \alpha} b(\vec{k}) \right), \\
B^\dagger(\vec{k}) &= \frac{1}{2} \sqrt{2} \left( e^{i \alpha} a^\dagger(\vec{k}) - e^{-i \alpha} b^\dagger(\vec{k}) \right),
\end{align*}
\]

the transformation reads

\[
\begin{align*}
C(\vec{k}) &= \frac{1}{2} \frac{\epsilon + \omega}{\sqrt{\epsilon \omega}} A(\vec{k}) + \frac{1}{2} \frac{\epsilon - \omega}{\sqrt{\epsilon \omega}} A^\dagger(-\vec{k}) =: A(\vec{k}) \cosh \rho + A^\dagger(-\vec{k}) \sinh \rho, \\
C^\dagger(\vec{k}) &= \frac{1}{2} \frac{\epsilon + \omega}{\sqrt{\epsilon \omega}} A^\dagger(\vec{k}) + \frac{1}{2} \frac{\epsilon - \omega}{\sqrt{\epsilon \omega}} A(-\vec{k}) =: A^\dagger(\vec{k}) \cosh \rho + A(-\vec{k}) \sinh \rho, \\
D(\vec{k}) &= \frac{1}{2} \frac{k + \omega}{\sqrt{k \omega}} B(\vec{k}) - \frac{1}{2} \frac{k - \omega}{\sqrt{k \omega}} B^\dagger(-\vec{k}) =: B(\vec{k}) \cosh \sigma + B^\dagger(-\vec{k}) \sinh \sigma, \\
D^\dagger(\vec{k}) &= \frac{1}{2} \frac{k + \omega}{\sqrt{k \omega}} B^\dagger(\vec{k}) - \frac{1}{2} \frac{k - \omega}{\sqrt{k \omega}} B(-\vec{k}) =: B^\dagger(\vec{k}) \cosh \sigma + B(-\vec{k}) \sinh \sigma,
\end{align*}
\]

(27)
with
\[ \tanh \rho = \frac{\epsilon - \omega}{\epsilon + \omega}, \quad (A.18) \]
and
\[ \tanh \sigma = \frac{\omega - k}{\omega + k}. \quad (A.19) \]
Consequently
\[
C(\vec{k}) = U_3 A(\vec{k}) U_3^\dagger, \\
C^\dagger(\vec{k}) = U_3 A^\dagger(\vec{k}) U_3^\dagger, \\
D(\vec{k}) = U_3 B(\vec{k}) U_3^\dagger, \\
D^\dagger(\vec{k}) = U_3 B^\dagger(\vec{k}) U_3^\dagger, \quad (A.20)
\]
with
\[
U_3 = \exp \left[ - \int d\nu \nu \left\{ \rho(k) \left( A(\vec{k}) A^\dagger(-\vec{k}) - A(\vec{k}) A(-\vec{k}) \right) + \sigma(k) \left( B^\dagger(\vec{k}) B(-\vec{k}) - B(\vec{k}) B(-\vec{k}) \right) \right\} \right]. \quad (A.21)
\]
The transformation given by eq. (A.16) is most conveniently written as the product of two unitary transformations. With
\[
\tilde{a}(\vec{k}) := e^{-i\alpha} a(\vec{k}) \quad \tilde{b}(\vec{k}) := e^{i\alpha} b(\vec{k}), \\
\tilde{a}^\dagger(\vec{k}) := e^{i\alpha} a^\dagger(\vec{k}) \quad \tilde{b}^\dagger(\vec{k}) := e^{-i\alpha} b^\dagger(\vec{k}), \quad (A.22)
\]
it consists of a rotation over an angle \( \varphi, \varphi = \frac{\pi}{4} \)
\[
A(\vec{k}) = \tilde{a}(\vec{k}) \cos \varphi + \tilde{b}(\vec{k}) \sin \varphi, \\
A^\dagger(\vec{k}) = \tilde{a}^\dagger(\vec{k}) \cos \varphi + \tilde{b}^\dagger(\vec{k}) \sin \varphi, \\
B^\dagger(\vec{k}) = \tilde{a}(\vec{k}) \cos \varphi - \tilde{b}(\vec{k}) \sin \varphi, \\
B^\dagger(\vec{k}) = \tilde{a}^\dagger(\vec{k}) \cos \varphi - \tilde{b}^\dagger(\vec{k}) \sin \varphi. \quad (A.23)
\]
The second unitary transformation therefore is
\[
U_2 = \exp \left[ \varphi \int d\nu \nu \left\{ \tilde{b}^\dagger(\vec{k}) \tilde{a}(\vec{k}) - \tilde{a}^\dagger(\vec{k}) \tilde{b}(\vec{k}) \right\} \right]. \quad (A.24)
\]
The first one only yields the phases, cf. eq. (A.22),
\[
U_1 = \exp \left[ i\alpha \int d\nu \nu \left\{ \tilde{b}^\dagger(\vec{k}) \tilde{a}(\vec{k}) - \tilde{a}^\dagger(\vec{k}) \tilde{b}(\vec{k}) \right\} \right]. \quad (A.25)
\]
In this way we finally have
\[
U = U_3 U_2 U_1. \quad (A.26)
\]

**Appendix B: Infrared Analysis and Comparison**

In order to prove eq. (105) and to compare to quantum mechanics a careful infrared analysis is in order. Of course, the dimensionality of space is of crucial importance. Consider, for instance, the infrared behaviour of the integrals \( I_0(\delta^2) \). For \( \delta \downarrow 0 \) one finds
\[
I_0(\delta^2) \sim \delta^{-1}, \quad \nu = 0, \\
I_0(\delta^2) \sim \log \delta, \quad \nu = 1, \\
I_0(\delta^2) \sim \delta, \quad \nu = 2, \\
I_0(\delta^2) \sim \delta^2, \quad \nu = 3. \quad (B.1)
\]
In quantum mechanics, i.e. for \( \nu = 0 \), we actually have

\[
I_0(M^2) = \frac{1}{2M}.
\] (B.2)

Let us now come back to eq. (103) for \( \nu \geq 1 \), i.e. field theory, and introduce the notation \( A = \sqrt{q^2 + M_1^2} \), \( B = \sqrt{q^2 + M_2^2} \).

Rewrite the first integrand as

\[
\frac{1}{AB} \left( \frac{1}{M_2 - A - B} \right) = \frac{1}{AB} \left( -\frac{1}{A + B} + \frac{M_2}{(M_2 - A - B)(A + B)} \right).
\] (B.3)

Herewith, the first contribution can be expressed in terms of the \( I_0 \) integrals and a remaining term \( R_1 \),

\[
\frac{16\lambda^2_2 Z_\phi \phi^2}{M_2} \int \frac{d^\nu q}{2(2\pi)^3} \frac{1}{AB M_2 - A - B} = \frac{2\lambda_B}{M_2} [I_0(M_1^2) - I_0(M_2^2)] + O(\delta \log \delta) + R_1. \] (B.4)

The last term explicitly reads

\[
R_1 = 2\lambda_B (M_1^2 + O(\delta^2)) \int \frac{d^\nu q}{2(2\pi)^3} \frac{1}{AB(M_2 - A - B)(A + B)}. \] (B.5)

In one space dimension this is logarithmically divergent. Similar manipulations for the second contribution lead to the same result as eq. (B.4) but with the remainder

\[
R_2 = 2\lambda_B (M_1^2 + O(\delta^2)) \int \frac{d^\nu q}{2(2\pi)^3} \frac{1}{AB(M_2 + A + B)(A + B)}. \] (B.6)

Adding \( R_1 \) and \( R_2 \) actually improves the infrared behaviour:

\[
R_1 + R_2 = 2\lambda_B (M_1^2 + O(\delta^2)) \int \frac{d^\nu q}{2(2\pi)^3} \frac{1}{AB(A + B)(M_2^2 - (A + B)^2)}. \] (B.7)

Thus we obtain an \( O(\delta \log \delta) \) contribution for \( \nu = 1 \) and an \( O(\delta) \) contribution for \( \nu \geq 2 \), respectively. With eqs. (B.4) and (B.7), the proof of eq. (105) is now trivially completed.

These infrared arguments cannot be taken over immediately to quantum mechanics, \( \nu = 0 \). The \( O(\delta \log \delta) \) (\( \nu = 1 \)) terms have to be replaced by \( O(1) \) and thus one cannot exclude finite corrections. Indeed, these were obtained in the explicit quantum mechanical calculation. However, one can also derive that result starting from eq. (B.3). Apart from eliminating the basic integral via eq. (B.2), one only needs evident identifications, e.g. \( M_1 \leftrightarrow \omega_u \).

**Appendix C: Integrals \( I_N \) at finite cutoff**

The integrals \( I_N \) with finite momentum cutoff read

\[
I_1(\Omega^2) = \int_0^\Lambda \frac{dk}{(2\pi)^2} k^2 \sqrt{k^2 + \Omega^2} = \frac{\Lambda^4}{16\pi^2} \left[ \sqrt{(1 + x)^3} - \frac{x}{2} \sqrt{1 + x} - \frac{x^2}{2} \ln \left( \frac{1 + \sqrt{1 + x}}{\sqrt{x}} \right) \right],
\]

\[
I_0(\Omega^2) = \int_0^\Lambda \frac{dk}{(2\pi)^2} \frac{k^2}{\sqrt{k^2 + \Omega^2}} = \frac{\Lambda^2}{8\pi^2} \left[ \sqrt{1 + x} - x \ln \left( \frac{1 + \sqrt{1 + x}}{\sqrt{x}} \right) \right],
\]
\[ I_{-1}(\Omega^2) = \int_0^\Lambda \frac{dk}{(2\pi)^2} \frac{k^2}{\sqrt{(k^2 + \Omega^2)^3}} = \frac{1}{4\pi^2} \left[ -\frac{1}{\sqrt{1 + x}} + \ln \left( \frac{1 + \sqrt{1 + x}}{\sqrt{x}} \right) \right] \] (C.1)

\( k := |\vec{k}| \). Fig. 3 shows the above integrals (\( I_N(x) := I_N(\Omega^2)/\Lambda^{2(N+1)} \)) as functions of \( x \) for all \( x \) which satisfy the scaling condition (103). The subtraction formula used in the text,

\[ I_o(\Omega^2) - I_o(0) = -\frac{\Omega^2}{2} I_{-1}(\Omega^2) + 2f'(\Omega^2), \] (C.2)

where \( \Omega_o = \Omega(\phi_o = 0) \) has been chosen as a reference point, defines the derivative \( \hat{f}'(x) = df(x)/dx \) of a function \( \hat{f} \):

\[ \hat{f}'(x) := \frac{f'(\Omega^2)}{\Lambda^2} = \frac{1}{16\pi^2} \left[ \sqrt{1 + x} - 1 - \frac{x}{\sqrt{1 + x_o}} \right] \]

\[ -x \ln \left( \frac{\sqrt{x_o}(1 + \sqrt{1 + x})}{\sqrt{x}(1 + \sqrt{1 + x_o})} \right), \] (C.3)

where \( x_o := \Omega^2_o/\Lambda^2 \). With the condition \( f(0) = 0 \) it follows that

\[ \hat{f}(x) := \frac{f(\Omega^2)}{\Lambda^4} = \frac{1}{16\pi^2} \left[ \left( 1 + \frac{x}{2} \right) \sqrt{1 + x} - (1 + x) \right] - \frac{x^2}{2\sqrt{1 + x_o}} \]

\[ -\frac{x^2}{2} \ln \left( \frac{\sqrt{x_o}(1 + \sqrt{1 + x})}{\sqrt{x}(1 + \sqrt{1 + x_o})} \right). \] (C.4)

The second derivative will be used as well:

\[ \hat{f}''(x) := f''(\Omega^2) = \frac{1}{16\pi^2} \left[ \frac{1}{\sqrt{1 + x}} - \frac{1}{\sqrt{1 + x_o}} - \ln \left( \frac{\sqrt{x_o}(1 + \sqrt{1 + x})}{\sqrt{x}(1 + \sqrt{1 + x_o})} \right) \right] \] (C.5)

\( \hat{f}''(x) = d^2\hat{f}(x)/dx^2, f''(\Omega^2) = d^2f(\Omega^2)/d\Omega^2)^2 \).

**Appendix D: The case \( \Omega = 0 \)**

The minimization of the GEP by \( \Omega = 0 \) at the nontrivial minimum does not correspond to a stable minimum of the potential with respect to \( \phi_o \) and \( \Omega \). The minimum in this case would be at \( \phi_o^2 = v_{cl}^2 := -m_R^2/4\lambda_R \), which is only possible for \( m_R^2 < 0 \). One can easily realize that the extremum is unstable by considering

\[ \left. \frac{\partial^2 V_G}{\partial \Omega^2} \right|_{\phi_o = v_{cl}, \Omega = 0} = m_R^2 I_{-1}(0) < 0, \]

\[ \det \text{ Hess } (V_G(\phi_o, \Omega)) = \begin{vmatrix} \frac{\partial^2 V_G}{\partial \phi_o \partial \Omega} & \frac{\partial^2 V_G}{\partial \phi_o \partial \phi_o} \\ \frac{\partial^2 V_G}{\partial \phi_o \partial \phi_o} & \frac{\partial^2 V_G}{\partial \phi_o^2} \end{vmatrix}_{\phi_o = v_{cl}, \Omega = 0} \]

\[ = -2m_R^4 I_{-1}(0) < 0. \] (D.1)

For \( m_R^2 \geq 0 \), the GEP which is minimized by \( \Omega = 0 \) has a minimum only at \( \phi_o = 0 \).
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Figure 3: The integrals $I_N$ as functions of $x = \Omega^2/\Lambda^2$, for all $x$ which satisfy the scaling condition (108).