A general formula by $LDL^T$ decomposition in the minimal seesaw model and conditions of $CP$ symmetry for Lagrangian parameters

Masaki J. S. Yang$^1$

$^1$Department of Physics, Saitama University, Shimo-okubo, Sakura-ku, Saitama, 338-8570, Japan

In this letter, using a general formula by $LDL^T$ decomposition in the minimal seesaw model, we obtain the basis-independent conditions of $CP$ symmetry for Lagrangian parameters of the neutrino mass matrix $m$. The conditions are found to be $\text{Re} (M_{22}a_i - M_{12}b_i) \text{Im} (M_{22}a_j - M_{12}b_j) = - \det M \text{Re} b_i \text{Im} b_j$ or $= - \det M \text{Im} b_i \text{Re} b_j$ for the Yukawa matrix $Y_{ij} = (a_j, b_j)$ and the right-handed neutrino mass matrix $M_{ij}$. In other words, the real or imaginary part of $b_i$ must be proportional to the real or imaginary part of the quantity $(M_{22}a_i - M_{12}b_i)$.

I. INTRODUCTION

Recently, CP violation (CPV) in the neutrino oscillation has been strongly suggested by T2K \cite{1} and NO$\nu$A \cite{2}. For this reason, CPV in the lepton sector has been widely studied using the seesaw mechanism \cite{3,4}. Conventionally, it is common to reduce parameters by diagonalization and/or phase redefinition for the analysis of seesaw relations \cite{5,6}. However, such a representation obscures symmetries and relationships of the original Lagrangian. Thus, in this letter, we analyze the conditions for $CP$ symmetry of the light neutrino mass matrix $m$ in the minimal seesaw model \cite{9,10} using a formula by $LDL^T$ decomposition \cite{24}. As a result, basis-independent conditions are obtained for the general Lagrangian parameters. Furthermore, we discuss relationships between the obtained solution and generalized $CP$ symmetry (GCP) \cite{25,26}. 

$^*$Electronic address: yang@krishna.th.phy.saitama-u.ac.jp
II. A GENERAL FORMULA IN THE MINIMAL SEESAW MODEL

In the beginning, we consider a general formula by $LDLT^T$ decomposition \cite{24} in the minimal seesaw models \cite{9-13}. By setting the vacuum expectation value of Higgs to one, the two mass matrices of neutrinos are defined as follows

$$Y = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \equiv \begin{pmatrix} Y_1^T \\ Y_2^T \\ Y_3^T \end{pmatrix}. \quad (1)$$

Here, two-dimensional complex vectors $(Y_i)_j \equiv (Y_{ij})$ and $(M_i)_j \equiv M_{ij}$ have mass dimension one. Let us consider a case where $M$ and its mass eigenvalues $M_i$ are hierarchical;

$$|M_{22}| \gg |M_{12}|, |M_{11}|, \quad M_2 \simeq M_{22} + \frac{M_{12}^2}{M_{22}}, \quad M_1 \simeq \frac{\det M}{M_2}. \quad (2)$$

As in the case of the type-I seesaw mechanism, we first perform an approximate spectral decomposition for $M^{-1}$;

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{12} & \frac{M_{12}^2}{M_{22}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/M_{22} \end{pmatrix} \quad (3)$$

$$= M^{(1)} + M^{(2)}. \quad (4)$$

The eigenvalues of the $M^{(1,2)}$ are $(M_{22}^2 + M_{12}^2)/M_{22} \det M$ and $M_2^{-1}$, and it corresponds to the first-order approximation of spectral decomposition.

The mass matrix of the light neutrinos $m$ is found to be

$$m = Y(M^{(1)} + M^{(2)})Y^T \equiv m^{(1)} + m^{(2)} \quad (5)$$

$$= \frac{M_{22}}{\det M} \begin{pmatrix} \tilde{a}_1^2 & \tilde{a}_1 \tilde{a}_2 & \tilde{a}_1 \tilde{a}_3 \\ \tilde{a}_1 \tilde{a}_2 & \tilde{a}_2^2 & \tilde{a}_2 \tilde{a}_3 \\ \tilde{a}_1 \tilde{a}_3 & \tilde{a}_2 \tilde{a}_3 & \tilde{a}_3^2 \end{pmatrix} + \frac{1}{M_{22}} \begin{pmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{pmatrix}, \quad (6)$$

where

$$\tilde{a}_i \equiv \frac{\det(Y_i, M_2)}{M_{22}} = a_i - b_i \frac{M_{12}}{M_{22}}. \quad (7)$$

This is equivalent to the general formula for the type-I seesaw mechanism by setting $M_{33} \to \infty$. Since no approximation is used obviously, this formula is valid in any basis. If the three-dimensional vectors $a \equiv (a_i)$ and $b \equiv (b_i)$ are linearly independent, the rank of this matrix $m$
is two and it has one zero eigenvalue. The eigenvector that belongs to the massless mode is proportional to the cross product $\tilde{a} \times b$.

A deformed Yukawa matrix $\tilde{Y}$ is defined by

$$
(\tilde{Y})_{ij} \equiv \begin{pmatrix} \tilde{a} & b \end{pmatrix} = (a_i - b_i \frac{M_{12}}{M_{22}}, b_i) = Y \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix} \equiv YL,
$$

where $L$ is a lower unitriangular matrix

$$
L = \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{M_{12}}{M_{22}} & 1 \end{pmatrix},
$$

that has all the diagonal entries equal to one. The mass matrix of heavy neutrinos $M$ is diagonalized by $L$ as

$$
m = \tilde{Y} \tilde{M}^{-1} \tilde{Y}^T, \quad \tilde{M}^{-1} = L^{-1} M^{-1} (L^{-1})^T = \begin{pmatrix} \frac{M_{22}}{\det M} & 0 \\ 0 & \frac{1}{M_{22}} \end{pmatrix}.
$$

This is called the $LDL^T$ (or generalized Cholesky) decomposition.

### III. BASIS-INDEPENDENT CONDITIONS FOR CP SYMMETRY

In this section, using the general formula (1), we investigate the basis-independent conditions for $CP$ symmetry in the minimal seesaw model. First, Eq. (8) is rewritten as

$$
m = \frac{M_{22}}{|M|} \tilde{a} \otimes \tilde{a}^T + \frac{1}{M_{22}} b \otimes b^T,
$$

where $|M| \equiv \det M$ and $\tilde{a}_i \equiv \tilde{a}_i, b_i \equiv b_i$ are three-dimensional vectors. By redefining phases of the right-handed neutrinos, we can choose a basis in which $M_{22}$ and $|M|$ are real-positive. It is easy to incorporate the effects of these overall phases into the final result by inverse redefinitions.

If the matrix $m$ satisfies the $CP$ symmetry, each imaginary part must cancel in Eq. (11);

$$
\text{Im} m_{ij} = \frac{M_{22}}{|M|} \text{Im} (\tilde{a}_i \tilde{a}_j) + \frac{1}{M_{22}} \text{Im} (b_i b_j) = 0.
$$

By separating the imaginary parts of products,

$$
\frac{M_{22}}{|M|} (\text{Re} \tilde{a}_i \text{Im} \tilde{a}_j + \text{Im} \tilde{a}_i \text{Re} \tilde{a}_j) + \frac{1}{M_{22}} (\text{Re} b_i \text{Im} b_j + \text{Im} b_i \text{Re} b_j) = 0.
$$

(13)
Furthermore, by considering cross products of $\text{Im} b_i$ and $\text{Im} b_j$ for $i$ and $j$ components, the term containing $1/M_{22}^2$ becomes zero:

$$
\frac{M_{22}^2}{|M|}[(\text{Im} b \times \text{Re} \tilde{a}) \otimes (\text{Im} \tilde{a} \times \text{Im} b)^T + (\text{Im} b \times \text{Im} \tilde{a}) \otimes (\text{Re} \tilde{a} \times \text{Im} b)^T] = 0. \quad (14)
$$

This is an antisymmetric condition for a matrix. However, since the only antisymmetric matrix with a rank less than one is the zero matrix, we obtain

$$
\text{Im} b \times \text{Re} \tilde{a} = 0 \quad \text{or} \quad \text{Im} \tilde{a} \times \text{Im} b = 0, \quad \text{Re} \tilde{a} \propto \text{Im} b \quad \text{or} \quad \text{Im} \tilde{a} \propto \text{Im} b. \quad (15)
$$

Similarly, we can show $\text{Re} \tilde{a} \propto \text{Re} b$ or $\text{Im} \tilde{a} \propto \text{Re} b$. As a result, the conditions that $m$ is $CP$-invariant are equivalent to

$$
M_{22}^2 \text{Re} \tilde{a}_i \text{Im} \tilde{a}_j = -|M| \text{Re} b_i \text{Im} b_j \quad \text{or} \quad -|M| \text{Im} b_i \text{Re} b_j. \quad (16)
$$

Thus, the real and imaginary parts of $\tilde{a}_i$ are proportional to those of $b_i$, and their coefficients are determined by Eq. (16). At first glance, these conditions appear to give nine constraints. However, since $A \equiv \text{Re} \tilde{a}_i \text{Im} \tilde{a}_j$ with rank one satisfies $A_{ii} A_{jj} = A_{ij} A_{ji}$, it is necessary and sufficient to give $A_{1i}$ and $A_{1j}$. Therefore, there are only five independent constraints.

By the conditions, we can express $\text{Re} \tilde{a}$ and $\text{Im} \tilde{a}$ for given $\text{Re} b$ and $\text{Im} b$;

$$
(M_{22} \text{Re} \tilde{a}, M_{22} \text{Im} \tilde{a}) = (\pm r \sqrt{|M|} \text{Re} b, \mp \frac{1}{r} \sqrt{|M|} \text{Im} b) \quad \text{or} \quad (\pm r' \sqrt{|M|} \text{Im} b, \mp \frac{1}{r'} \sqrt{|M|} \text{Re} b). \quad (17)
$$

Here, $r$ and $r'$ are real constants defined by the non-zero $\text{Im} \tilde{a}_i$ and $\text{Re} b_j$ or $\text{Im} b_j$,

$$
r \equiv \pm \frac{M_{22}}{\sqrt{|M|} \text{Re} b_j} \frac{\text{Re} \tilde{a}_j}{\text{Im} \tilde{a}_i}, \quad r' \equiv \pm \frac{M_{22}}{\sqrt{|M|} \text{Im} b_j} \frac{\text{Re} \tilde{a}_i}{\text{Im} \tilde{a}_j}. \quad (18)
$$

The sign $\pm$ is linked in each solution, and these four solutions correspond to $-\tilde{a}$ and $\pm i\tilde{a}^*$ for a given solution $\tilde{a}$.

$\text{Re} \tilde{a} = 0$ or $\text{Im} \tilde{a} = 0$ is a special case where all the denominators of $r^{(l)}$ or $1/r^{(l)}$ cannot be defined and the magnitudes of $\tilde{a}_i$ and $b_i$ are unconstrained. In such a case, the elements of $\tilde{Y}$ are real or pure imaginal;

$$
\tilde{Y} = (\tilde{a}_i, b_i) = \{(x_i, y_i), \ (ix_i, y_i), \ (x_i, iy_i), \ (ix_i, iy_i)\}, \quad (19)
$$

with real $x_i, y_i \in \mathbb{R}$. In each case, $\tilde{Y}$ has a (generalized) $CP$ symmetry;

$$
\tilde{Y}^* = \{\tilde{Y}, \ \tilde{Y} R, \ -\tilde{Y} R, \ -\tilde{Y}\}, \quad (20)
$$
where $R \equiv \text{diag}(-1, 1)$.

For example, if $\tilde{Y}$ has the canonical $CP$ and $\text{Im} \tilde{a}_i = \text{Im} b_i = 0$ holds, then $a_i = \tilde{a}_i + \frac{M_{12}}{M_{22}} b_i$ rewrites $\tilde{Y}$ to $Y$ as

\begin{align}
\text{Re} \, a_i &= \text{Re} \, \tilde{a}_i + \text{Re} \, \frac{M_{12}}{M_{22}} \text{Re} \, b_i, \\
\text{Im} \, a_i &= \text{Im} \, \frac{M_{12}}{M_{22}} \text{Re} \, b_i,
\end{align}

for any $i$. In other words, the imaginary part of $a$ and the real part of $b$ are required to be proportional. They can be regarded as a kind of alignment conditions [52]. For the remaining three cases, similar alignments holds between $\text{Re} \, a_i, \text{Im} \, a_i$ and $\text{Re} \, b_i, \text{Im} \, b_i$.

### A. Understanding from the original raw formula

We can also understand the result (16) without $LDL^T$ decomposition. By adjusting coefficients in Eq. (16) and rewriting $\tilde{a}$ to $a$,

\begin{equation}
\text{Re} \, (M_{22} a_i - M_{12} b_i) \text{Im} \, (M_{22} a_j - M_{12} b_j) = -|M| \text{Re} \, b_i \text{Im} \, b_j \quad \text{or} \quad -|M| \text{Im} \, b_i \text{Re} \, b_j.
\end{equation}

With some transformation, we obtain

\begin{equation}
\text{Im} [(a_i M_{22} - b_i M_{12}) a_j] = -\text{Im} [(b_i M_{11} - a_i M_{12}) b_j].
\end{equation}

This is equivalent to the condition $m = m^*$ in the original raw formula. From Eq. (1), the mass matrix $m$ is written as

\begin{equation}
m = Y M^{-1} Y^T = \frac{1}{|M|} \begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2 \\
a_3 & b_3
\end{pmatrix} \begin{pmatrix}
-(M_2 \times Y_1)_z & -(M_2 \times Y_2)_z & -(M_2 \times Y_3)_z \\
(M_1 \times Y_1)_z & (M_1 \times Y_2)_z & (M_1 \times Y_3)_z
\end{pmatrix},
\end{equation}

\begin{equation}
m_{ij} = \frac{1}{|M|} [-a_i(M_{21} b_j - M_{22} a_j) + b_i(M_{11} b_j - M_{12} a_j)],
\end{equation}

where $(M_i \times Y_j)_z \equiv M_{i1} Y_j - M_{i2} Y_{j1}$. Note that the resulting $m$ is a symmetric matrix although the representation is not symmetric at a glance. Since the $CP$-invariance requires cancellation of the imaginary parts between the two terms, Eq. (24) is obtained as conditions.
IV. UNDERSTANDING FROM GENERALIZED CP SYMMETRY

The solution (17) can also be understood from GCP. By defining \( X \equiv \bar{Y} \sqrt{M^{-1}} \), \( m \) is written only in \( X \);

\[
X = \left( \sqrt{\frac{M_{22}}{|M|}} \tilde{a} \sqrt{\frac{1}{M_{22}}} \right), \quad m = XX^T. \tag{27}
\]

In order for \( m \) to have CP symmetry, \( X = (u, v) \) must satisfy the following GCP with a complex orthogonal matrix \( O \);

\[
X^* O = (u^*, v^*) \begin{pmatrix} c_z & -s_z \\ s_z & c_z \end{pmatrix} = (c_x u^* + s_x v^*, -s_x u^* + c_x v^*) = (u, v) = X, \tag{28}
\]

where \( c_z \equiv \cos z, s_z \equiv \sin z \).

Since \( XO^* O = X^* O = X \) holds from a conjugation of Eq. (28), the matrix \( O \) satisfies \( O^* = O^{-1} = O^T \) and \( O = O^\dagger \). In other words, \( O \) is a Hermitian orthogonal matrix. The situation is divided into two cases with \( \det O = \pm 1 \) \([17, 22, 53]\);

\[
O = \begin{pmatrix} \cosh x & -i \sinh x \\ i \sinh x & \cosh x \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{pmatrix}, \tag{29}
\]

where \( x \) is a real parameter.

When \( O \) is real, the CP-invariant condition is

\[
X^* O = (u^*, v^*) \begin{pmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{pmatrix} = (c_x u^* + s_x v^*, s_x u^* - c_x v^*) = (u, v) = X. \tag{30}
\]

From this,

\[
u^* = \frac{1}{s_x} (v + c_x v^*), \quad u = \frac{c_x}{s_x} (v + c_x v^*) + s_x v^* = \frac{c_x}{s_x} v + \frac{1}{s_x} v^*. \tag{31}
\]

And since \( v, v^* = \text{Re} \ v \pm i \text{Im} \ v \) holds,

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Re} \tilde{a} = \text{Re} u = \frac{c_x + 1}{s_x} \text{Re} v = \frac{c_x + 1}{s_x} \sqrt{\frac{1}{M_{22}}} \text{Re} b, \tag{32}
\]

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Im} \tilde{a} = \text{Im} u = \frac{c_x - 1}{s_x} \text{Im} v = \frac{c_x - 1}{s_x} \sqrt{\frac{1}{M_{22}}} \text{Im} b. \tag{33}
\]

Obviously, these coefficients satisfy

\[
\frac{c_x + 1}{s_x} \times \frac{c_x - 1}{s_x} = -1, \tag{34}
\]
and it corresponds to the first solution of Eq. (17).

The other solution with \( \det O = 1 \) is

\[
X^* O = (u^*, v^*) \begin{pmatrix} \cosh x & -i \sinh x \\ i \sinh x & \cosh x \end{pmatrix} = (ch_x u^* + ish_x v^*, -ish_x u^* + ch_x v^*) = (u, v) = X,
\]

(35)

where \( ch_x \equiv \cosh x, sh_x \equiv \sinh x \). A similar calculation yields

\[
\sqrt{\frac{M_{22}}{|M|}} \Re \tilde{a} = \Re u = \frac{-ch_x - 1}{sh_x} \Im v = \frac{-ch_x - 1}{sh_x} \sqrt{\frac{1}{M_{22}}} \Im b, \tag{36}
\]

\[
\sqrt{\frac{M_{22}}{|M|}} \Im \tilde{a} = \Im u = \frac{ch_x - 1}{sh_x} \Re v = \frac{ch_x - 1}{sh_x} \sqrt{\frac{1}{M_{22}}} \Re b. \tag{37}
\]

Since the product of the two coefficients satisfies

\[
\frac{-ch_x - 1}{sh_x} \times \frac{ch_x - 1}{sh_x} = \frac{-ch_x^2 + 1}{sh_x^2} = -1, \tag{38}
\]

it corresponds to the second solution of Eq. (17).

By understanding the solution \( X^* O = X \) in a matrix form, the form of \( \tilde{Y} \) and \( Y \) can be specified. Let us consider the following solution.

\[
X_0^* = X_0, \quad X_1^* = X_1 T, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{39}
\]

where \( X_0 \) is a real solution and \( X_1 = (u, v) \) is a solution with \( u = v^* \). By defining \( X_{0,1}' \equiv X_{0,1} Q \) with a complex orthogonal matrix \( Q \), conjugation of \( X_{0,1}' \) becomes

\[
(X_0')^* = (X_0 Q)^* = X_0 Q^* = X_0 Q^T Q^*, \tag{40}
\]

\[
(X_1')^* = (X_1 Q)^* = X_1 T Q^* = X_1 Q^T T Q^*. \tag{41}
\]

Thus, if we define \( O_0 \equiv Q^T Q \) and \( O_1 \equiv Q^T T Q \), these \( O_{0,1} \) are Hermitian orthogonal matrices satisfying the GCP (28).

\[
(X_0')^* O_0 = X_0', \quad (X_1')^* O_1 = X_1'. \tag{42}
\]

Specifically,

\[
Q \equiv \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix} \Rightarrow O_0 = Q^T Q = \begin{pmatrix} \cos(w - w^*) & -\sin(w - w^*) \\ \sin(w - w^*) & \cos(w - w^*) \end{pmatrix}, \tag{43}
\]

\[
O_1 = Q^T T Q = \begin{pmatrix} \sin(w + w^*) & \cos(w + w^*) \\ \cos(w + w^*) & -\sin(w + w^*) \end{pmatrix}. \tag{44}
\]
Indeed $O_{0,1}$ is Hermitian, and they agree with Eq. \((29)\) by suitable redefinitions.

From $X = X_{0,1}Q$, the following holds for the Yukawa matrix;

\[
\tilde{Y} = X\sqrt{M} = X_{0,1}Q\sqrt{M}, \quad Y = \tilde{Y}L^{-1} = X_{0,1}Q\sqrt{ML}^{-1}.
\] (45)

Recalling that $L^{-1}M^{-1}(L^T)^{-1} = \tilde{M}^{-1}$ from Eq. \((10)\), we finally obtain

\[
m = X_0QQ^TX_0^T,
\] (46)

and $m$ is \(CP\)-invariant. In the end, in order to make $m$ \(CP\)-invariant, the Yukawa matrix $Y$ has the degrees of freedom $Q$ like the Casas-Ibarra parameterization \([6, 7]\). Although such results may be well known, the main result of this letter is Eqs. \((16)\) and \((17)\), the conditions of \(CP\) invariance of $m$ for basis-independent parameters of the Lagrangian.

V. SUMMARY

In this letter, using a general formula by $LDL^T$ decomposition in the minimal seesaw model, we obtain the basis-independent conditions of \(CP\)-invariance for Lagrangian parameters of the neutrino mass matrix $m$. The conditions are found to be \(\text{Re} (M_{22}a_i - M_{12}b_i) \text{Im} (M_{22}a_j - M_{12}b_j) = - \det M \text{Re} b_i \text{Im} b_j \) or $= - \det M \text{Im} b_i \text{Re} b_j$ for the Yukawa matrix $Y_{ij} = (a_j, b_j)$ and the right-handed neutrino mass matrix $M_{ij}$. In other words, the real or imaginary part of $b_i$ must be proportional to the real or imaginary part of the quantity $(M_{22}a_i - M_{12}b_i)$.

We also discussed the relevance of the generalized \(CP\) symmetry that exists in such a solution. As a result, Yukawa matrix $Y$ found to be restricted to the form $Y = X_{0,1}Q\sqrt{ML}^{-1}$ with \(CP\)-invariant Yukawa matrices $X_{0,1}$, a complex orthogonal matrix $Q$, the diagonal right-handed neutrino mass matrix $\tilde{M}$ by the $LDL^T$ decomposition, and the unitriangular matrix $L$ at that time. This result can be extended to other seesaw mechanisms and GCP.
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