CPTM symmetry, closed time paths and cosmological constant problem in the formalism of extended manifold

S. Bondarenko(1)

(1) Physics Department, Ariel University, Ariel 40700, Israel

Abstract

We consider the cosmological constant problem in the formalism of an extended space-time consisting of the extended classical solution of Einstein equations. The different regions under consideration of the extended manifold are related by the charge, parity, time and mass (CPTM) reversal symmetry applied with respect to the metric fields of the manifolds. The value of the cosmological constant in the framework, being zero at the classical level in the formalism, is defined by the interactions between the points of the extended manifold and provided by scalar fields. The action for the fields is described by the different formulations of the closed time path approach and among the different possibilities for the closed paths, there is a variant of the action equivalent to the Keldysh formalism. Accordingly, we consider the application of the proposed formalism to the problem of smallness of the cosmological constant and discuss the obtained results.

1 Introduction

Following to [1, 2], we investigate the appearance of the cosmological constant in the formalism of extended classical solution of Einstein equations as a consequence of the quantum interaction between the parts of the extended manifold. The metrics of the separated manifolds of the extended solution are related by the discrete reversal charge, parity, time and mass (CPTM) symmetry which preserves the form of the metric $g$ at the case of the zero cosmological constant. The easiest way to clarify this construction is to consider the light cone coordinated $u, v$ or corresponding Kruskal–Szekeres coordinates [3], defined for the whole space-time solution. In this case, the extended CPTM transform inverses the sign of these coordinates, see [1] for the Schwarzschild’s spacetime and the similar description of the Reissner–Nordström space-time in [4, 5], for example. Namely, for the two manifolds, A-manifold and B-manifold with coordinates $x$ and $\tilde{x}$, the symmetry $g_{\mu\nu}(x) = g_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(\tilde{x})$ must be preserved by the extended CPTM symmetry transform:

$\begin{align*}
q &\to -\tilde{q}, r \to -\tilde{r}, t \to -\tilde{t}, m_{\text{grav}} \to -\tilde{m}_{\text{grav}}; \\
CPTM(g_{\mu\nu}(x)) &= \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(\tilde{x}).
\end{align*}$

when the cosmological constant is zero. We underline, see [1], that the usual radial coordinate is strictly positive and there is a need for an additional B-manifold in order to perform the Eq. (1) discrete P transform, see also next Section. The transformation of the sign of the gravitational mass in this case can be understood as consequence of the request of the preserving of the symmetry of the metric. Similar construction in application of the quantum mechanics to the black hole physics can be found in [6].

The framework we consider consists of different manifolds with the gravitational masses of different signs in each one, see details in [1]. The general motivation of the introduction of the negative mass
in the different cosmological models is very clear. In any scenario, see [7,8] for example, the presence of some kind of repulsive gravitation forces or additional gravitational field in our Universe helps with an explanation of the existence of dark energy in the models, see also [9–15] and references therein. It is important, that the gravitation properties of the matter of B-manifold is also described by Einstein equations, see [7,14] or [8] and [16] for examples of the application of the discrete symmetries in the case of the quantum and classical systems. The situations is getting more complicated when we note the different directions of time’s arrows in the different manifolds. Introducing the ordinary scalar fields, we obtain the closed time path for the fields of the manifolds related by the CPTM symmetry with different time directions for the fields. The interaction between these fields leads to the quantum effective action calculated with respect to the fluctuations above the classical solutions. Effectively, this action is the cosmological constant in the action. The vertices of the action ”glues” the parts of the same manifolds and different manifolds as well, allowing the gravitational interactions between them. Once arising, the constant’s value can be different for both manifolds related as well by the introduced symmetry. We will have then for the case of different cosmological constants for A and B manifolds:

\[ CPTM(\Lambda) = \tilde{\Lambda}, \quad CPTM(g_{\mu\nu}(x,\Lambda)) = \tilde{g}_{\mu\nu}(\tilde{x},\tilde{\Lambda}). \]  

(3)

This appearance of the cosmological constant satisfies the naturalness criteria of ’t Hooft, see [17]. It’s zero value correspondence to the precise symmetry between the metrics Eq. (2) whereas it’s small non-zero value decreases the symmetry to Eq. (3) relation. In general, the constant in the formalism is not a constant anymore but it is a functional which requires a renormalization depending on the form and properties of the interacting fields. It, therefore, can acquire different values due it’s evolution, the important question we need to clarify in the formalism it is the smallness of the constant.

As we will see further, the contribution of the scalar fields of the two manifolds to the vacuum energy at the classical level is zero. Respectively, there are two different mechanisms responsible for the constant’s value which we will discuss through the paper. The first one is a dependence of the cosmological constant on many loops quantum contribution to the vacuum energy. These contributions are provided by the ordinary framework of the quantum fields in the flat space-time, we will not discuss these contributions in the non-flat manifolds framework. In this case the smallness of the constant can be provided by the smallness of the corresponding diagrams in the QFT effective action or, alternatively, we demonstrate that the vacuum contributions can be eliminated by by proper definition of the partition function of the theory. An another contribution into the constant is a corrections to the values of the vertices of the corresponding effective action due the non-flat corrections to the propagators and vertices. In this case the smallness of the constant is due the smallness of the non-flat corrections to the zero flat result. The additional separated question we investigate is about a bare value of the constant in the approach. The smallness of the bare value, as we obtain, is due the heavy mass of the introduced scalar fields.

We consider a connection between the manifolds established by the gravitational forces through the effective action of the scalar fields with both contributions described above included. The effective action ”glues” the manifolds, there is a kind of the foam of vertices that belongs to the same manifold as well as to both depending on the form and properties of the scalar fields. In this case, the framework contains two or more manifolds that ”talk” each with other by the non-local correlators. These quantum vertices, similar to some extend to the quantum wormholes, have also been widely used in the investigation of the cosmological constant problem, see [18–20] for example. The construction proposed here is a dynamical one, the classical dynamics of each metric of separated manifolds can be affected by their mutual interactions. In any case, the cosmological constant is arising in the equations as a result of the mutual influence of the different parts of the same manifold, wormhole like interactions, or due the interactions between the different manifolds. Solving the equations of motion perturbatively, we can begin from the case of the zero constant in the case of ”bare” manifolds and generate the constant at the next step of the evolution, breaking Eq. (2) symmetry but preserving more general Eq. (3) relations. An another possible scenario, which we do not discuss in the paper and postpone for a separate publication, is that we have a large value of the cosmological constant at
the beginning of the evolution and only during the time the value of the constant is decreased almost to zero.

The appearance of the closed time path formalism in the framework and main properties of the CPTM transform are discussed in the Section 2. Sections 3 and 4 are dedicated to the formulation of the two different realization of CPTM symmetrical scalar fields in A and B manifolds with overall zero classical contribution to the vacuum energy. In the Section 5 and Section 6 we describe the mechanism of the appearance of the cosmological constant in the form of the interaction between the manifolds for the case of free fields. In the next Section we discuss the consequence of the symmetry on the form of the possible interactions terms between the fields of A and B manifolds. Sections 8 and 9 are about the inclusion of the interactions between the fields, in Section 9 we investigate an inclusion of the potentials of interaction and self-interacting of the fields considering usual \( \phi^4 \) scalar theory. In Section 10 we consider the realization of the smallness of the constant in the formalism and Section 11 summarizes the obtained results.

2 CPTM symmetry for scalar field

In order to clarify the consequences of the CPTM symmetry, we will borrow some results from [1, 2]. Consider the Eddington–Finkelstein coordinates for the Schwarzschild spacetime

\[
v = t + r^* = t + r + 2M \ln \frac{r}{2M} - 1\tag{4}
\]

and

\[
u = t - r^* .\tag{5}
\]

In correspondence, we define also the Kruskal–Szekeres coordinates \( U, V \), which covers the whole extended space-time, are defined in the different parts of the extended solution. For example, when considering the region I with \( r > 2M \) in terms of [5] where \( U < 0, V > 0 \) we have:

\[
U = -e^{-u/4M}, \quad V = e^{v/4M} .\tag{6}
\]

As demonstrated in [1], see also [5], the transition to the separated regions of the solutions can by done by the analytical continuation of the coordinates provided by the corresponding change of its signs and reversing of the sign of the gravitational mass. When considering the region III in definitions of [5], we obtain:

\[
U = -e^{-u/4M} \rightarrow \tilde{U} = e^{-\tilde{u}/4\tilde{M}} = -U ,\tag{7}
\]

\[
V = e^{v/4M} \rightarrow \tilde{V} = -e^{-\tilde{v}/4\tilde{M}} = -V .\tag{8}
\]

This inversion of the signs of the \((U, V)\) coordinate axes will hold correspondingly in the all regions of \((U, V)\) plane after analytical continuation introduced in [1]. Formally, from the point of view of the discrete transform performed in \((U, V)\) plane, the transformations Eq. (7) are equivalent to the full reversion of the Kruskal-Szekeres "time"

\[
T = \frac{1}{2} \left( V + U \right) \rightarrow -T \tag{9}
\]

and radial "coordinate"

\[
R = \frac{1}{2} \left( V - U \right) \rightarrow -R \tag{10}
\]

in the complete Schwarzschild space-time. Therefore, the introduced \( T, R \) coordinates and some transverse coordinates \( X_\perp \), all denoted simply as \( x \), can be considered as the correct coordinates in a Fourier transform of the quantum fields. The corresponding change in the expressions of the functions after the Fourier transform being formally similar to the conjugation is not the conjugation. Namely,
the analytical continuation of the functions from A-manifold to B-manifold (CPTM transform) means
the change of the sign of \(x\) in corresponding Fourier expressions without its conjugation as whole.

Coming back to the usual coordinates in the each A or B manifolds separately, we will obtain for
the radial coordinate:

\[
\int_{0}^{\infty} dR \rightarrow \int_{0}^{\infty} r, \quad R \rightarrow r
\]

(11)

\[
\int_{-\infty}^{0} dR \rightarrow \int_{0}^{\infty} \tilde{r}, \quad r \rightarrow -\tilde{r},
\]

(12)

and time:

\[
\int_{-\infty}^{\infty} dT \rightarrow \int_{-\infty}^{\infty} t, \quad T \rightarrow t
\]

(13)

\[
\int_{-\infty}^{\infty} dT \rightarrow \int_{-\infty}^{\infty} \tilde{t}, \quad t \rightarrow -\tilde{t}.
\]

(14)

There are, in general, two possibilities to determine the time’s arrows in both manifolds depending on
the determination of \(\text{in}\) and \(\text{out}\) states for the second manifold, see Fig. (1), depending on that there
are two possibilities for the definition of the form of the scalar fields. Namely, for the application of
the introduced symmetry, we consider A and B manifolds (two Minkowski spaces) as separated parts
of the extended solution with non-interacting branches of the scalar quantum field defined in each
region and related by the CPTM discrete transform. Now consider the usual quantum scalar field
defined in our part (A-manifold) of the extended solution:

\[
\phi(x) = \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2 \omega_k}} \left( \phi^- (k) e^{-i k x} + \phi^+ (k) e^{i k x} \right) = \phi^* (x), \quad [\phi^- (k), \phi^+ (k')] = \delta_{kk'}.
\]

(15)

The conjugation of the scalar field does not change the expressions, we have simply \((\phi^-)^* = \phi^+\). In
contrast to the conjugation, the CPTM discrete transform acts differently. We have for the second
field:

\[
CPTM(\phi(x)) = \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2 \omega_k}} \left( \phi^- (k) e^{-i k \tilde{x}} + \phi^+ (k) (k) e^{i k \tilde{x}} \right) = \tilde{\phi}(\tilde{x}) = \int \frac{d^3 \tilde{k}}{(2\pi)^{3/2} \sqrt{2 \omega_{\tilde{k}}}} \left( \phi^- (k) e^{i k \tilde{x}} + \phi^+ (k) e^{-i k \tilde{x}} \right).
\]

(16)

Figure 1: The diagram represents two different possibilities of the construction of the closed time paths.
3 Scalar field of B-manifold: first possibility

There are two possibilities for the definition of the B-scalar fields which we consider. We begin from the obvious one, defining the B-field similarly to the definition of the usual scalar field with the vacuum state mutual for both manifolds:

\[
CPTM(\phi(x)) = CPTM \left( \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left( \phi^-(k) e^{-ikx} + \phi^+(k) e^{ikx} \right) \right) = \tilde{\phi}(\tilde{x}) = \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left( \tilde{\phi}^+(k) e^{ik\tilde{x}} + \tilde{\phi}^-(k) e^{-ik\tilde{x}} \right)
\]

with the following properties of the annihilation and creation operators of B-scalar field:

\[
\begin{align*}
\phi^- (k) &\leftrightarrow \tilde{\phi}^+ (k) \\
\phi^+ (k) &\leftrightarrow \tilde{\phi}^- (k)
\end{align*}
\]

\[
[\tilde{\phi}^-(k) \tilde{\phi}^+(k')] = -\delta^3_{kk'}
\]

and

\[
\omega(k) = \sqrt{m^2 + k^2} \rightarrow \tilde{\omega}(k) = \sqrt{m^2 + k^2}.
\]

Still, the definitions above are meaningless if we do not define the vacuum states for the A and B scalar fields. There is a mutual vacuum state in our problem with the following properties:

\[
\begin{align*}
\{ <0|\phi^+ = 0 \} &\leftrightarrow \{ \tilde{\phi}^- |0> = 0 \} \\
\{ \phi^+ |0> \neq 0 \} &\leftrightarrow \{ <0|\phi^- \neq 0 \} \quad CPTM \leftrightarrow \{ <0|\phi^- >0 \} \neq 0 \\
\phi^- |0> &\neq 0 \quad CPTM \leftrightarrow \{ <0|\phi^+ |0> \neq 0 \}
\end{align*}
\]

A general energy-momentum vector \( P_\mu \) written for the both regions of the extended manifold and averaged with respect to the mutual vacuum state now acquires the following form:

\[
< P^\mu > = \frac{1}{2} \int d^3k k_\mu \left( \langle \phi^+(k) \phi^-(k) + \phi^-(k) \phi^+(k) \rangle > + \langle \tilde{\phi}^+(k) \tilde{\phi}^-(k) + \tilde{\phi}^-(k) \tilde{\phi}^+(k) \rangle > \right) = \int d^3k k_\mu \left( \langle \phi^+(k) \phi^-(k) > + \langle \tilde{\phi}^+(k) \tilde{\phi}^-(k) > \right) = < P_A^\mu > + < P_B^\mu > = 0
\]

Here \( P_{A,B}^\mu \) are the energy-momentum vectors of A or B manifolds separately, we also note that

\[
CPTM(< P_A^\mu >) = < P_B^\mu >
\]

as expected. So, as a consequence of CPTM symmetry, we obtained the precise cancellation of the vacuum zero modes contributions.

The next issue we discuss in connection to the B-field is a definition of the propagators of the field. For the easier references we presented the forms of the Feynman and Dyson propagators we used in the Appendixes A. In general we need to determine the change of the propagators in respect to the CPTM transform and form of the propagator of the B-field which we will use in the calculations. Therefore, we consider the usual \( \tilde{G}_F(\tilde{x} - \tilde{y}) \) propagator for the B-field, it has the following form:

\[
\tilde{G}_F(\tilde{x} - \tilde{y}) = -i \left( \theta(\tilde{x}^0 - \tilde{y}^0) < \tilde{\phi}(\tilde{x}) \tilde{\phi}(\tilde{y}) > + \theta(\tilde{y}^0 - \tilde{x}^0) < \tilde{\phi}(\tilde{y}) \tilde{\phi}(\tilde{x}) > \right) = -G_F(\tilde{x}, \tilde{y}),
\]
see Eq. [18] definitions. Alternatively we can calculate

\[ CPTM(G_F(x, y)) = -\iota \left( \theta(y^0 - \bar{y}^0) CPTM(\phi(x) \phi(y)) + \theta(x^0 - \bar{x}^0) CPTM(\phi(y) \phi(x)) \right) = -\iota \left( \theta(y^0 - \bar{y}^0) \phi(\bar{x}) \phi(\bar{y}) + \theta(x^0 - \bar{x}^0) \phi(\bar{y}) \phi(\bar{x}) \right) = -G_F(\bar{x}, \bar{y}), \]  

that coincides with Eq. [23] answer. In Eq. [24] the following property of Wightman function under the CPTM transform is clarified:

\[ CPTM(D(x - y)) \propto CPTM(\phi^- (k) \phi^+(k') e^{-ik \cdot x \pm ik' \cdot y}) = \phi^- (k) \phi^+(k) e^{-ik \cdot \bar{x} \pm ik' \cdot \bar{y}} \]

that provides

\[ CPTM(D(x - y)) = -D(\bar{y} - \bar{x}) = -\phi(\bar{y}) \phi(\bar{x}) \]  

in accordance to Eq. [24].

4 Scalar field of B-manifold: second possibility

An another way do define the B-field is to consider it as an antimatter field of A-field in respect to the sign of the mass. We define therefore:

\[ CPTM(\phi(x)) = CPTM \left( \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2} \omega_k} \left( \phi^-(k) e^{-ik \cdot x} + \phi^+(k) e^{ik \cdot x} \right) \right) = \tilde{\phi}(\bar{x}) = \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2} \omega_k} \left( \phi^-(k) e^{ik \cdot \bar{x}} + \phi^+(k) e^{-ik \cdot \bar{x}} \right) \]

with the following properties of the operators:

\[ \{ \phi^- (k) \leftrightarrow \tilde{\phi}^-(k), \phi^+(k) \leftrightarrow \tilde{\phi}^+(k) \} \quad \delta^{3}_{k'k} \quad \delta^{3}_{k'k} \]  

Correspondingly we define the action of these operators on the vacuum state as following:

\[ \{ < 0 | \phi^+ = 0 \}, \tilde{\phi}^+ | 0 > = 0 \quad CPTM \quad \{ \phi^- | 0 > = 0 \}, \tilde{\phi}^- | 0 > = 0 \]  

In this case the general energy-momentum vector \( P_\mu \) written for the both regions of the extended manifold again is defined as usual and we obtain:

\[ < P^\mu > = \frac{1}{2} \int d^3 k k^\mu \left( < \phi^+(k) \phi^-(k) + \phi^-(k) \phi^+(k) > \right) \]

\[ = < \phi^+(k) \phi^-(k) + \phi^-(k) \phi^+(k) > \]

\[ = < P_A^\mu > + < P_B^\mu > = 0 \]  

with

\[ CPTM(< P_\mu^A >) = < P_\mu^B > \quad \text{as above.} \]

Now we once more define the \( \tilde{G}_F(\bar{x} - \bar{y}) \) propagator for the B-field, it has the following form:

\[ \tilde{G}_F(\bar{x}, \bar{y}) = -\iota \left( \theta(x^0 - \bar{x}^0) \phi(\bar{x}) \phi(\bar{y}) + \theta(y^0 - \bar{y}^0) \phi(\bar{y}) \phi(\bar{x}) \right) = -G_D(\bar{x}, \bar{y}) \]  

in accordance to Eq. [24].
which is different from the Eq. (23) expression due the Eq. (28) properties of the operators. Checking the CPTM symmetry connection of the A and B fields we have:

\[ CPTM(G_{F}(x, y)) = -i (\theta(y^{0} - \tilde{x}^{0}) CPTM(< \phi(x) \phi(y) >) + \theta(\tilde{x}^{0} - y^{0}) CPTM(< \phi(y) \phi(x) >)) = -i (\theta(y^{0} - \tilde{x}^{0}) < \phi(\tilde{x}) \phi(y) > + \theta(\tilde{x}^{0} - y^{0}) < \phi(y) \phi(\tilde{x}) >) = -G_{D}(\tilde{x}, \tilde{y}) \]  

(33)
as expected. Again, we can understand this transform as consequence of the transformation of the Wightman function under the CPTM transform:

\[ CPTM(D(x - y)) \propto CPTM(< \phi^{+}(k) \phi^{-}(k') > e^{i k \cdot x - i k' \cdot y}) < \phi^{+}(k) \phi^{-}(k') > e^{-i k \cdot \tilde{x} + i k' \cdot \tilde{y}} \]

(34)
that provides

\[ CPTM(D(x - y)) = -D(\tilde{x} - \tilde{y}) = - < \phi(\tilde{x}) \phi(\tilde{y}) >. \]  

(35)
in correspondence to Eq. (32)-Eq. (33) result.

5 Action of the formalism: free fields with single source

We consider the formalism introduced in [2] with the full action defined at the absence of the matter as

\[ S = S_{grav}(x, \tilde{x}) + S_{int}(x, \tilde{x}), \]  

(36)
where

\[ S_{grav}(x, \tilde{x}) = -m_{p}^{2} \int_{-\infty}^{\infty} dt \int d^{3}x \sqrt{-g(x)} R(x) - m_{p}^{2} \int_{-\infty}^{\infty} dt \int d^{3}\tilde{x} \sqrt{-\tilde{g}(\tilde{x})} R(\tilde{x}) \]  

(37)
are separated Einstein actions defined in each A-B manifolds separately and

\[ S_{int} = - \sum_{i,j = A,B} m_{p}^{2} \int d^{4}x_{i} \sqrt{-g_{i}(x_{i})} \int d^{4}x_{j} \sqrt{-g(x_{j})} \xi_{ij}(x_{i}, x_{j}) \]  

(38)
is a simplest terms which describes the gravitational interactions between the manifolds through some bi-scalar functions \( \xi_{ij}(x_{i}, x_{j}) \), here \( x_{A} = x \) and \( x_{B} = \tilde{x} \) are defined and \( m_{p}^{2} = 1/16 \pi G \) with \( c = \hbar = 1 \) notations are introduced for simplicity. Without this term there is a system of non-interacting Einstein equations of each manifold separately, whereas the interaction term in the equations determines the mutual gravitational influence of the manifolds in the form of the cosmological constant.

In order to understand the appearance of the interaction term in Eq. (36) we, following to [2], consider the discussed above scalar fields in A-B manifolds with some physical sources of them introduced. Now, instead Eq. (38), we consider the action for the free non-interacting scalar fields in a curved space-time:

\[ S_{int} = \int d^{4}x \sqrt{-g} \left( \frac{1}{2} \phi G^{-1} \phi - m_{p}^{3} f(x) \phi \right) + \int d^{4}\tilde{x} \sqrt{-\tilde{g}} \left( \frac{1}{2} \tilde{\phi} \tilde{G}^{-1} \tilde{\phi} + m_{p}^{3} \tilde{f}(\tilde{x}) \tilde{\phi} \right). \]  

(39)
Here we use the notations for the propagators in the curved space-time, see Appendix B. We also do not know a priori the sign for the second source in the Lagrangian, namely it is possible that \( CPTM(f) = \pm \tilde{f}, \) and further, where it will be need in that, will consider the two cases separately. Now the both fields can be expanded around their classical values:

\[ \phi = m_{p}^{2} G(x, y, \Lambda) f(y) \sqrt{-g(y, \Lambda)} + \varepsilon, \quad \tilde{\phi} = \pm m_{p}^{2} \tilde{G}(\tilde{x}, \tilde{y}, \tilde{\Lambda}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y}, \tilde{\Lambda})} + \tilde{\varepsilon} \]  

(40)

\footnote{We introduced in Eq. (39) the undefined function \( f(x) = \tilde{f}(\tilde{x}) \) as a source of the scalar field. Further, where the form of the function does not matter, we will take it equal to 1.}
that provides for the action in both cases:

\[ S_{\text{int}} = \frac{1}{2} \int d^4 x \sqrt{-g} \varepsilon G^{-1} \varepsilon - \frac{n_0^2}{2} \int d^4 x \int d^4 y \sqrt{-g(x)} f(x) G(x, y) f(y) \sqrt{-g(y)} + \]

\[ + \frac{1}{2} \int d^4 \tilde{x} \sqrt{-g} \tilde{\varepsilon} \tilde{G}^{-1} \tilde{\varepsilon} - \frac{n_0^2}{2} \int d^4 \tilde{y} \int d^4 \tilde{y} \sqrt{-\tilde{g}(\tilde{x})} \tilde{f}(\tilde{x}) \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})}, \]  

(41)

for simplicity we did not write here and further an implicit dependence of the metric on the constants. We obtain, therefore, that the first and third terms in the expression can be integrated out, whereas the second and fourth terms in the action determine the cosmological constant in the expressions through the \( G_F(x, y) \) and \( \tilde{G}_F(\tilde{x}, \tilde{y}) \) propagators after the integration over \( y \) and \( \tilde{y} \) coordinates for the A and B manifolds correspondingly.

The cosmological constant in this case contributes to the usual equations of motion:

\[ \delta S_{\Lambda \tilde{\Lambda}} = \frac{n_0^2}{2} \int d^4 x \int d^4 y \sqrt{-g(x)} f(x) (g_{\mu\nu} \delta g^{\mu\nu}) G(x, y) f(y) \sqrt{-g(y)} + \]

\[ - \frac{n_0^2}{2} \int d^4 x \int d^4 y \sqrt{-g(x)} f(x) (\delta G(x, y) f(y)) \sqrt{-g(y)} + \]

\[ + \frac{n_0^2}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-\tilde{g}(\tilde{x})} \tilde{f}(\tilde{x}) (\tilde{g}_{\mu\nu} \delta \tilde{g}^{\mu\nu}) \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} - \]

\[ - \frac{n_0^2}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-\tilde{g}(\tilde{x})} \tilde{f}(\tilde{x}) (\delta \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} = \]

\[ = n_0^2 \int d^4 x \sqrt{-g(x)} (g_{\mu\nu} \delta g^{\mu\nu}) \Lambda(x) - 2 m_0^2 \int d^4 x \sqrt{-g(x)} \delta \Lambda(x) + \]

\[ + m_0^2 \int d^4 \tilde{x} \sqrt{-\tilde{g}(\tilde{x})} (\tilde{g}_{\mu\nu} \delta \tilde{g}^{\mu\nu}) \tilde{\Lambda}(\tilde{x}) - 2 m_0^2 \int d^4 \tilde{x} \sqrt{-\tilde{g}(\tilde{x})} \delta \tilde{\Lambda}(\tilde{x}) \]  

(42)

which is correct when we take zero cosmological constant in the metric of r.h.s. of Eq. (41), see Appendix B. Here, in Eq. (42), the propagators must be understood as the full ones in the curved space-time, see Appendix B definitions. In general, if we want an precise equation for the cosmological constants, then we also have to account the presence of them in the metric and or to resolve the equations as whole, non-perturbative solution, or use Eq. (B.15) prescription obtaining perturbative solution for the constants. To the first approximation, we obtain for the constants:

\[ \Lambda(x) + \tilde{\Lambda}(\tilde{x}) = \]

\[ = \frac{m_0^4}{4} \left( f(x) \int d^4 y G(x, y, \Lambda) f(y) \sqrt{-g(y, \Lambda)} + \tilde{f}(\tilde{x}) \int d^4 \tilde{y} \tilde{G}(\tilde{x}, \tilde{y}, \tilde{\Lambda}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y}, \tilde{\Lambda})} \right) = \]

\[ = \frac{m_0^4}{4} \left( f(x) \int d^4 y G(x, y, \Lambda) f(y) \sqrt{-g(y)} + \tilde{f}(\tilde{x}) \int d^4 \tilde{y} \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} \right) \]  

(43)

where as propagators we take Eq. (B.6) expression for the full propagators with corresponding definition of "bare" propagators for A and B manifolds.

For the case of the first type of scalar fields, preserving in Eq. (43) the leading terms from Eq. (B.6) with respect to the flat propagators, we obtain:

\[ \Lambda(x) = \frac{m_0^4}{4} f(x) \int d^4 y \sqrt{-g(y)} G(x, y) f(y) = \frac{m_0^4}{4} f(x) \int d^4 y \sqrt{-g(y)} G_F(x, y) f(y) \]

\[ \tilde{\Lambda}(\tilde{x}) = \frac{m_0^4}{4} \tilde{f}(\tilde{x}) \int d^4 \tilde{y} \sqrt{-\tilde{g}(\tilde{y})} \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) = - \frac{m_0^4}{4} \tilde{f}(\tilde{x}) \int d^4 \tilde{y} \sqrt{-\tilde{g}(\tilde{y})} G_F(x, \tilde{y}) \tilde{f}(\tilde{y}). \]  

(44)

We have therefore to this order of precision

\[ \Lambda = - \tilde{\Lambda}. \]  

(45)
Nevertheless, effectively, in the Lagrangian, we have for these terms
\[ \Lambda + \bar{\Lambda} = 0 \] (46)
and the constants do not appear in the action in this perturbative order. The first non-trivial contribution in the Lagrangian, therefore, will be given by the quadratic with respect to the flat propagators terms in Eq. \([B.6]\) expression.

In the case of the second type of the fields, also keeping in the Eq. (43) only the first non-vanishing terms from Eq. \([B.6]\), we obtain:
\[ \Lambda(x) = \frac{m_p^4}{4} f(x) \int d^4y \sqrt{-g(y)} G_F(x, y) f(y) \]
\[ \bar{\Lambda}(x) = -\frac{m_p^4}{4} \tilde{f}(\bar{x}) \int d^4y \sqrt{-g(y)} G_D(x, y) \tilde{f}(\bar{y}) \] (47)
that provides
\[ \Lambda(x) = \bar{\Lambda}^*(x) , \] (48)
see again Appendix A for the definitions of the propagators in the flat space-time.

6 Action of the formalism: free fields with double sources

In the previous section we consider the simplest Lagrangian of the free scalar fields, there is no interacting between them in the Lagrangian. Further, due the closed time path introduced in the action, such term will arise correspondingly to the Keldysh formalism prescription. Nevertheless, in the Keldysh approach, the contribution of this kinetic term in the effective action begins from one loop of the corresponding propagator and, therefore, it’s contribution is small in general. So, as we obtained in the previous Section, in the case of Eq. \([39]\) Lagrangian there are no interactions between the A and B manifolds but only self-interactions between the points of the same manifolds. Therefore, in this Section, we consider a more complicated version of Eq. \([39]\) introducing two sources for the scalar fields as following:
\[ S_{int} = \int d^4x \sqrt{-g} \left( \frac{1}{2} \phi G^{-1} \phi - m_p^3 f(x) \phi - \xi m_p^3 \bar{f}(\bar{x}) \right) + \]
\[ + \int d^4\bar{x} \sqrt{-\bar{g}} \left( \frac{1}{2} \bar{\phi} \bar{G}^{-1} \bar{\phi} - \bar{\xi} m_p^3 \bar{f}(\bar{x}) \bar{\phi} - \bar{\xi} m_p^3 \bar{f}(\bar{x}) \bar{\phi} \right), \] (49)
the parameters \(\xi = \pm 1\) and \(\bar{\xi} = \pm 1\) are defined here independently and the sign of \(f\) is undefined, it can be positive or negative, whereas the sign of \(\bar{f}\) is strictly positive. We solve the equations of motion and obtain the classical solutions for the fields:
\[ \phi = m_p^3 G(x, y, \Lambda) \left( f(y) \sqrt{-g(y, \Lambda)} + \xi \tilde{f}(\bar{y}) \sqrt{-\bar{g}(\bar{y}, \bar{\Lambda})} \right) + \varepsilon = \phi_{cl} + \varepsilon , \] (50)
\[ \bar{\phi} = m_p^3 \bar{G}(\bar{x}, \bar{y}, \bar{\Lambda}) \left( \bar{\xi} \bar{f}(\bar{y}) \sqrt{-\bar{g}(\bar{y}, \bar{\Lambda})} + \xi f(y) \sqrt{-g(y, \Lambda)} \right) + \bar{\varepsilon} = \bar{\phi}_{cl} + \bar{\varepsilon} . \] (51)
Performing the usual calculations we will obtain for the Eq. (19) expression:

\[
S_{nl} = \frac{1}{2} \int d^4x \sqrt{-g} \varepsilon G^{-1} \varepsilon - \frac{m_p^6}{2} \int d^4x \int d^4y \sqrt{-g(x)} f(x) G(x, y) f(y) \sqrt{-g(y)} - \\
- \frac{m_p^6}{2} \int d^4x \int d^4y \sqrt{-g(x)} \tilde{f}(x) G(x, y) \tilde{f}(y) \sqrt{-g(y)} + \\
+ \frac{1}{2} \int d^4\tilde{x} \sqrt{-\tilde{g}} \varepsilon \tilde{G}^{-1} \varepsilon - \frac{m_p^6}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-\tilde{g}(\tilde{x})} \tilde{f}(\tilde{x}) \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} - \\
- \frac{m_p^6}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-\tilde{g}(\tilde{x})} f(x) \tilde{G}(\tilde{x}, \tilde{y}) f(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} - \\
- m_p^6 \xi \tilde{\xi} \int d^4x \int d^4\tilde{y} \sqrt{-g(x)} f(x) \tilde{G}(x, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} - \\
- m_p^6 \xi \tilde{\xi} \int d^4x \int d^4\tilde{y} \sqrt{-g(x)} f(x) \tilde{G}(x, \tilde{y}) \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})},
\]

(52)

for the simplicity we omit a dependence on the constants inside the r.h.s. of the expression. Now we see that there is an interaction term between the manifolds in correspondence to Eq. (38) definition. There are two cases we consider separately. For \(\xi = -\tilde{\xi} = \pm 1\) values of the parameters, we have:

\[
\phi_{cl} = m_p^3 G(x, y, \Lambda) \left( f(y) \sqrt{-g(y, \Lambda)} - \tilde{f}(y) \sqrt{-\tilde{g}(y, \Lambda)} \right),
\]

(53)

\[
\tilde{\phi}_{cl} = \pm m_p^3 \tilde{G}(\tilde{x}, \tilde{y}, \tilde{\Lambda}) \left( f(\tilde{y}) \sqrt{-g(\tilde{y}, \tilde{\Lambda})} - \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y}, \tilde{\Lambda})} \right).
\]

(54)

We see, that when \(f(y) \sqrt{-g(y, \Lambda)} = \tilde{f}(y) \sqrt{-\tilde{g}(y, \Lambda)}\) the expressions are trivial, but this equality is not requested in general and especially when the constants in \(\sqrt{-g}\) are non-zero and different, the Eq. (10) symmetry is precise only for the zero cosmological constants. Now, in order to define the cosmological constants from Eq. (19) interaction term to the first perturbative order of precision, we will make the following. We will keep in Eq. (52), first order propagators, i.e. the flat ones, and will perform variation with respect to \(g\) and \(\tilde{g}\) separately. Only after the variation will be taken we will set in the r.h.s. of the equations zero cosmological constants and will consider remaining integrals without their connection to \(x\) and \(\tilde{x}\) variables using Eq. (1) symmetry. We will obtain then:

\[
\Lambda(x) = - \frac{m_p^4}{4} f(x) \int d^4y \sqrt{-g(y)} \left( G(x, y) + \tilde{G}(x, y) \right) \left( \tilde{f}(y) - f(y) \right) \\
\tilde{\Lambda}(x) = \frac{m_p^4}{4} \tilde{f}(x) \int d^4y \sqrt{-g(y)} \left( G(x, y) + \tilde{G}(x, y) \right) \left( \tilde{f}(y) - f(y) \right).
\]

(55)

As we see, for the first kind of the scalar fields, Eq. (21), in this order \(\Lambda = \tilde{\Lambda} = 0\) whereas for second type of the fields we will obtain here \(\Lambda = -\tilde{\Lambda}\) with relative sign depend on the sign of \(\tilde{f} - f\) difference. Correspondingly, when \(\xi = \tilde{\xi} = \pm 1\), we obtain then:

\[
\phi_{cl} = m_p^3 G(x, y, \Lambda) \left( f(y) \sqrt{-g(y, \Lambda)} + \tilde{f}(y) \sqrt{-\tilde{g}(y, \Lambda)} \right),
\]

(56)

\[
\tilde{\phi}_{cl} = \pm m_p^3 \tilde{G}(\tilde{x}, \tilde{y}, \tilde{\Lambda}) \left( f(\tilde{y}) \sqrt{-g(\tilde{y}, \tilde{\Lambda})} + \tilde{f}(\tilde{y}) \sqrt{-\tilde{g}(\tilde{y}, \tilde{\Lambda})} \right)
\]

(57)

and for the cosmological constants

\[
\Lambda(x) = - \frac{m_p^4}{4} f(x) \int d^4y \sqrt{-g(y)} \left( G(x, y) + \tilde{G}(x, y) \right) \left( \tilde{f}(y) + f(y) \right) \\
\tilde{\Lambda}(x) = - \frac{m_p^4}{4} \tilde{f}(x) \int d^4y \sqrt{-g(y)} \left( G(x, y) + \tilde{G}(x, y) \right) \left( \tilde{f}(y) + f(y) \right).
\]

(58)

For the first kind of the scalar fields, Eq. (21), in this order \(\Lambda = \tilde{\Lambda} = 0\) again whereas for second type of the fields we obtain \(\Lambda = -\tilde{\Lambda}\) with the relative sign depend on sign of \(f\).
Possible forms of the interactions of the fields in the Lagrangian

The introduced CPTM symmetry determines the possible form of the interactions between the scalar fields in the Lagrangian. Namely, we request the symmetry between the classical fields must be preserved in the general Lagrangian at the case of zero cosmological constants. Therefore we can consider the connection of the possible interactions terms in the Lagrangian basing on the properties of classical fields and propagators of two flat manifolds. Consider the first type of the scalar fields

\[ \tilde{\phi}_{cl} = -\phi_{cl} \] (59)

and connection between the propagators given by Eq. (24) which we suppose to be correct in the case of the flat manifold or in the case of the metric without the cosmological constant. Therefore, the request of CPTM symmetry leads to the following connection between the possible interaction terms in the Lagrangians of A and B manifolds:

\[ \begin{align*}
\lambda_{2n+1} \phi_{cl}^{2n+1}(x) &\to -\lambda_{2n+1} \tilde{\phi}_{cl}^{2n+1}(\tilde{x}) \\
\lambda_{2(n+1)} \phi_{cl}^{2(n+1)}(x) &\to \lambda_{2(n+1)} \tilde{\phi}_{cl}^{2(n+1)}(\tilde{x})
\end{align*} \quad ; \quad n = 1, 2, \cdots, \] (60)

in correspondence to the form of Eq. (59) transformation. Now we consider the second possibility of the connection between the classical fields:

\[ \tilde{\phi}_{cl} = \phi_{cl} \] (61)

The consequence of that relation between the classical fields solutions is that now we have instead Eq. (60):

\[ \lambda_n \phi_{cl}^n(x) \to \lambda_n \tilde{\phi}_{cl}^n(\tilde{x}), \quad n = 3, 4, \cdots. \] (62)

for the possible interacting terms of the fields in the Lagrangian of the approach.

For the second choice of the scalar field in the problem, we have to put attention that in momentum space

\[ (-G_D) = G_F^*, \] (63)

see Appendix A definitions. Therefore we have to the first approximation:

\[ \tilde{\phi}_{cl} = \phi_{cl}^*. \] (64)

The Eq. (64) transformation rule, in turn, dictates the transformations of the possible interaction terms in the Lagrangians. We obtain:

\[ i\lambda_n \phi_{cl}^n(x) \to -i\lambda_n \tilde{\phi}_{cl}^n(\tilde{x}), \quad n = 3, 4, \cdots, \] (65)

in full correspondence to the closed time path (Keldysh) formalism of non-equilibrium condensed matter physics, [21, 22] and [23] for example. For the second case when

\[ \tilde{\phi}_{cl} = -\phi_{cl}^* \] (66)

we have to define

\[ i\lambda_n \phi_{cl}^n(x) \to i (-1)^{n+1} \lambda_n \tilde{\phi}_{cl}^n(\tilde{x}), \quad n = 3, 4, \cdots, \] (67)

in correspondence to Eq. (66).

For the further applications we denote the Lagrangians considered here as cases 1 – a, 1 – b and 2 – a, 2 – b correspondingly to the order of their appearance in the Section.
8 Action of the formalism: interacting free fields

Following the analogy with Keldysh formalism, now we introduce an interaction between the free fields of different manifolds. Again, introducing in the Lagrangian corresponding interaction terms between the fields of different manifolds we consider separately the cases of two different types of scalar fields. Also, here we do not include in the Lagrangian the source terms, they are not important for the further results.

First of all consider $1-a$ and $1-b$ Lagrangians in the case of flat A, B manifolds. As usual we define the following Wightman functions:

$\Delta_>(x, y) = -i D(x - y), \quad \tilde{\Delta}_>(\tilde{x}, \tilde{y}) = i D(\tilde{y} - \tilde{x}) = -\Delta_<(\tilde{x}, \tilde{y}),$ \hspace{1cm} (68)

see Eq. (68) definition. We, therefore, define the interaction part of the action $S_{int}$ as:

$S_{int} = \frac{1}{2} \int \left[ d^4 x \sqrt{-g} \left( \frac{G_F^{-1}}{D^{>1}} \Delta^{>1}_A \right) \left( \frac{\phi}{\tilde{\phi}} \right) \right] - \int \left[ d^4 \tilde{x} \sqrt{-g} \left( \frac{\tilde{G}_F^{-1}}{D^{<1}} \right) \left( \frac{\tilde{\phi}}{\phi} \right) \right] \left( \pm \frac{m_p^3 + J_1}{m_p^3 + J_2} \right).$ \hspace{1cm} (69)

Here we introduced the auxiliary currents $J_1$ and $J_2$ in order to check the appearance of the additional Green’s functions in the problem. After the integration we obtain:

$S_{int} = \frac{-m_p^6}{2} \int d^4 x \int d^4 y \sqrt{-g} G_F(x, y) \sqrt{-g} - \frac{m_p^6}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) \sqrt{-g} - \frac{m_p^3}{2} \int d^4 x \int d^4 y \sqrt{-g} J_1 G_F(x, y) J_1 \sqrt{-g} - \frac{m_p^3}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-g} J_2 \tilde{G}_F(\tilde{x}, \tilde{y}) J_2 \sqrt{-g} - \frac{1}{2} \int d^4 x \int d^4 y \sqrt{-g} J_1 \left( \Delta_>(x, y) + \tilde{\Delta}_>(y, x) \right) \sqrt{-g} = \frac{1}{2} \int d^4 x \int d^4 y \sqrt{-g} J_2 \left( \Delta_>(x, y) + \tilde{\Delta}_>(y, x) \right) J_1 \sqrt{-g} = \frac{1}{2} \int d^4 x \int d^4 y \sqrt{-g} \left( \Delta_>(x, y) + \tilde{\Delta}_>(y, x) \right) J_2 \sqrt{-g}.$ \hspace{1cm} (70)

Taking into account that

$\Delta_>(x, y) + \tilde{\Delta}_>(y, x) = \Delta_<(x, y) - \Delta_<(y, x) = 0,$ \hspace{1cm} (71)

see Eq. (68), we obtain:

$S_{int} = \frac{-m_p^6}{2} \int d^4 x \int d^4 y \sqrt{-g} G_F(x, y) \sqrt{-g} - \frac{m_p^6}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) \sqrt{-g} - \frac{m_p^3}{2} \int d^4 x \int d^4 y \sqrt{-g} J_1 G_F(x, y) J_1 \sqrt{-g} - \frac{m_p^3}{2} \int d^4 \tilde{x} \int d^4 \tilde{y} \sqrt{-g} J_2 \tilde{G}_F(\tilde{x}, \tilde{y}) J_2 \sqrt{-g}.$ \hspace{1cm} (72)

The answer demonstrate that there is no Keldysh like interactions between the flat A and B manifolds in the case of $1-a$ and $1-b$ Lagrangians. If we will not introduce the interaction potential between the scalar fields we will stay with two non-interacting scalar fields which provide the cancellation of mutual classical zero modes but not interact anymore. The situation can be different if we will account generated non-zero cosmological constants. In this case, the quality Eq. (71) can be not correct anymore, instead will have:

$\Delta_>(x, y, \Lambda) + \tilde{\Delta}_>(y, x, \tilde{\Lambda}) = (\partial_\Lambda \Delta_>(x, y, \Lambda))_{\Lambda = 0} \Lambda - (\partial_{\tilde{\Lambda}} \Delta_<(y, x, \tilde{\Lambda}))_{\tilde{\Lambda} = 0} \tilde{\Lambda} \neq 0,$ \hspace{1cm} (73)

and some interaction between the manifolds will arise.
Now we consider $2 - a$ and $2 - b$ Lagrangians. Again we define the Wightman functions:

$$\Delta_>(x, y) = -i D(x - y), \quad \tilde{\Delta}_>(\tilde{x}, \tilde{y}) = i D(\tilde{x} - \tilde{y}) = -\Delta_>(\tilde{x}, \tilde{y}),$$

see Eq. (74). Repeating the same calculations as above we will obtain:

$$S_{int} = -\frac{m^6_p}{2} \int d^4x \int d^4y \sqrt{-g} G_F(x, y) \sqrt{-g} - \frac{m^6_p}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) \sqrt{-g} - \frac{m^3_p}{2} \int d^4x \int d^4y \sqrt{-g} \tilde{G}_F(x, y) J_1 \sqrt{-g} - \frac{m^6_p}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) J_2 \sqrt{-g}. $$

Due the Eq. (74) identity now we have:

$$\Delta(x, y) = \Delta_>(x, y) + \tilde{\Delta}_>(y, x) = \Delta_>(x, y) - \Delta_>(y, x) = \Delta_>(x, y) - \Delta_<(x, y) \neq 0$$

already in case of zero cosmological constant. Therefore, we see, in full analogy with Keldysh formalism, that there is a new term of direct interaction between the A and B manifolds arises at the case of zero auxiliary currents:

$$S_{int} = -\frac{m^6_p}{2} \int d^4x \int d^4y \sqrt{-g} G_F(x, y) \sqrt{-g} - \frac{m^6_p}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) \sqrt{-g} - \frac{m^6_p}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-g} \tilde{G}_F(\tilde{x}, \tilde{y}) \sqrt{-g}.$$ 

This interaction between the A and B manifolds takes place at quantum level with non-zero Keldysh propagator appearance in the calculations.

9 Action of the formalism: self-interacting fields

Our next step is an introduction of self-interactions of the fields. The connection between the corresponding parts of the two Lagrangians are given by Eq. (75), Eq. (62) and Eq. (65), Eq. (67). Correspondingly to that we will obtain quantum corrections to the classical values of the fields given by Eq. (49). As a result of integration around the classical solutions Eq. (49) we will obtain an effective action of the following general form:

$$\Gamma_{int} = \sum_{m,n=1}^{\infty} \int d^4x_1 \sqrt{-g} \cdots \int d^4x_m \sqrt{-g} \int d^4\tilde{x}_1 \sqrt{-g} \cdots \int d^4\tilde{x}_m \sqrt{-g} \frac{\partial^2}{\partial \phi_1 \partial \phi_1} + V_1 \cdots n;1 \cdots m(x_1, \cdots, x_n; \tilde{x}_1, \cdots, \tilde{x}_m) \phi_1 \cdots \phi_m \tilde{\phi}_1 \cdots \tilde{\phi}_m.$$ 

In this case the cosmological constant can be defined in mostly general as

$$\Lambda(x) = \frac{1}{m^2_p} \sum_{m,n=1}^{\infty} \int d^4x_1 \sqrt{-g} \cdots \int d^4x_n \sqrt{-g} \int d^4\tilde{x}_1 \sqrt{-g} \cdots \int d^4\tilde{x}_m \sqrt{-g} (V_0 + V_1 \cdots n;1 \cdots m(x, x_1, \cdots, x_n; \tilde{x}_1, \cdots, \tilde{x}_m) \phi_1 \cdots \phi_m \tilde{\phi}_1 \cdots \tilde{\phi}_m).$$
with $V_0$ as vacuum contributions to the constant (no-legs diagrams). As in the previous expressions, the r.h.s. of Eq. \ref{eq:78} and Eq. \ref{eq:79} also depend on the constants through the vertices and $\sqrt{-g}$, the Eq. \ref{eq:79} is a non-linear equation for the constant.

In the effective action, beginning from $n = 2$ external legs, the introduced vertices will contribute to the renormalization of the theory determining renormalized mass and vertices of the scalar field. The mostly unpleasant contributions there are the vacuum ones and quantum contributions to the cosmological constant in the formalism can be divided on the two parts. The first contributions are due the quantum vacuum zero modes, i.e. related to $V$ terms accordingly to Eq. \ref{eq:60} prescription. We will consider only renormalizable theories, for the contributions in non-flat manifolds, we do not consider these effects. Therefore, we will neglect the curvature corrections to the propagators in the calculations of Eq. \ref{eq:78} one-loop effective action and will consider the theory in the flat space-time.

9.1 Self-interacting scalar fields of the first kind

As in the previous Section we will begin from $1 - a, b$ Lagrangians adding to Eq. \ref{eq:11} the self-interaction terms accordingly to Eq. \ref{eq:60} prescription. We will consider only renormalizable theories, for the simplicity limiting ourselves by the consideration of $\phi^4$ power of interaction in $D = 4$ dimensions and by the one-source Lagrangian. We will obtain for the action:

$$
S_{\text{int}} = \int d^4x \sqrt{-g} \left( \frac{1}{2} \phi G_F^{-1} \phi - m_p^2 \phi \right) + \int d^4\tilde{x} \sqrt{-\tilde{g}} \left( \frac{1}{2} \tilde{\phi} \tilde{G}_F^{-1} \tilde{\phi} + m_p^2 \tilde{\phi} \right) - 
$$

$$
- \int d^4x \sqrt{-g} \lambda_4 \frac{\phi^4(x)}{4!} - \int d^4\tilde{x} \sqrt{-\tilde{g}} \lambda_4 \frac{\tilde{\phi}^4(\tilde{x})}{4!}. \quad \text{(80)}
$$

Now we expand the potential of the self-interaction around the classical values of the fields and preserving in the expressions terms till quadratic with respect to fluctuations obtain:

$$
S_{\text{int}} = - \int d^4x \lambda_4 \frac{\phi^3(x)}{3!} - \int d^4\tilde{x} \lambda_4 \frac{\tilde{\phi}^3(\tilde{x})}{3!} + 
$$

$$
+ \frac{1}{2} \int d^4x \, \epsilon G_F^{-1} \epsilon - \frac{m_p^2}{2} \int d^4x \int d^4y \sqrt{-g(x)} G_F(x,y) \sqrt{-g(y)} + 
$$

$$
+ \frac{1}{2} \int d^4\tilde{x} \tilde{\epsilon} \tilde{G}_F^{-1} \tilde{\epsilon} - \frac{m_p^2}{2} \int d^4\tilde{x} \int d^4\tilde{y} \sqrt{-\tilde{g}(\tilde{x})} \tilde{G}_F(\tilde{x},\tilde{y}) \sqrt{-\tilde{g}(\tilde{y})} - 
$$

$$
- \int d^4x \left( \lambda_4 \frac{\phi^3(x)}{3!} + \lambda_4 \frac{\tilde{\phi}^3(x)}{3!} \right) - \int d^4\tilde{x} \left( \lambda_4 \frac{\tilde{\phi}^3(\tilde{x})}{3!} + \lambda_4 \frac{\tilde{\phi}^3(x)}{3!} \right). \quad \text{(81)}
$$

Without the contribution of classical fields and cosmological constant terms, we obtain after the integration with respect to the fluctuations:

$$
S_{\text{int}} = - \frac{\lambda_4^2}{2(3!)^2} \int d^4x \int d^4y \left( \phi^3(x) \right) G_F(x,y) \left( \phi^3(y) \right) - 
$$

$$
- \frac{\lambda_4^2}{2(3!)^2} \int d^4\tilde{x} \int d^4\tilde{y} \left( \tilde{\phi}^3(\tilde{x}) \right) \tilde{G}_F(\tilde{x},\tilde{y}) \left( \tilde{\phi}^3(\tilde{y}) \right) + 
$$

$$
+ \frac{i}{2} \text{Tr} \ln \left( 1 + \frac{\lambda_4}{4} G_F \phi^2 \right) + \frac{i}{2} \text{Tr} \ln \left( 1 + \frac{\lambda_4}{4} \tilde{G}_F \tilde{\phi}^2 \right). \quad \text{(82)}
$$

\footnote{Further we will consider $\phi^4$ theory without the quantum contribution to $\phi_{cl}$.}
Due the Eq. [23] relation between the Green’s function, the two first terms in Eq. [82] cancel each another and we remain with the following expression to one-loop order precision:

\[ S_{int} = \frac{i}{2} Tr \ln \left( 1 + \frac{\lambda_4}{4} G_F \phi_{cl}^2 \right) + \frac{i}{2} Tr \ln \left( 1 + \frac{\lambda_4}{4} G_F \tilde{\phi}_{cl}^2 \right). \]  

(83)

We see, therefore, that the relative sign of the classical solution does not matter in the case of flat manifolds, we can take further \( \phi_{cl} = \tilde{\phi}_{cl} \). The only difference in the contributions of the terms are due the different sign of Green’s functions that affect only on diagrams with odd number of propagators which will be canceled in the answer. The same rules are applicable in the case of many-loops diagrams. We are interested mostly in the two-loops vacuum diagram without external legs provided by \( \lambda_4 \varepsilon^4 \) terms in the potential. This diagram is not zero without the regularization and doubled in the final answer.

The situation is more interesting if instead potential with separate self-interacting fields we will consider a potential similar to the \( \phi^4 \) potential of the scalar doublet:

\[ V(\phi, \tilde{\phi}) = \frac{\lambda_4}{4!} \left( \phi^2 + \tilde{\phi}^2 \right)^2 = V_{AA}(\phi) + V_{AB}(\phi, \tilde{\phi}) + V_{BB}(\tilde{\phi}). \]  

(84)

The consequence of the appearance of the \( \frac{\lambda_4}{3} \phi_{cl} \tilde{\phi}_{cl} \varepsilon \tilde{\varepsilon} \) quadratic term in the potential is that there is now a mixing of the propagators of the problem

\[ G_{\mu\nu} = G_{F\mu\nu} - G_{F\mu\rho} V_{\rho\sigma}(\phi \tilde{\phi}) G_{\sigma\nu}, \quad \mu = A, B, \]  

(85)

with \( V_{\rho\sigma}(\phi \tilde{\phi}) \) as coefficients of the quadratic with respect to \( \varepsilon \) and \( \tilde{\varepsilon} \) terms. The answer for the one-loop action term in this case we can write as

\[ S_{int} = \frac{i}{2} Tr \ln \left( 1 + \frac{\lambda_4}{3} G_{F\mu\rho} \phi_{cl} \phi_{cl} \right). \]  

(86)

This mixing of the propagators in the diagrams is arising beginning from the diagrams of the \( \lambda_4^2 \) and higher orders, independently on the number of loops. As a result there is no quantum corrections in the problem, in this formulation of the formalism the quantum corrections in two flat manifolds are odd with respect to the CPTM symmetry transform giving an overall zero in the final answer. The mostly interesting consequence of that for us it is zero contribution of the vacuum diagrams into the cosmological constant in the flat space-time. Namely, it can be clarified as following. For these kind of the diagrams at \( N = 1, \cdots \) order there are \( N \) pairs of propagators over all. Each pair of propagators in turn gives zero contribution in the diagram because there are two positive and two negative expressions for each pair of propagators.

### 9.2 Self-interacting scalar fields of the second kind

We will not consider here the well-known general closed time path formalism for the scalar fields in the flat manifolds, the different applications of the framework can be found in [23] for example. Instead we will discuss only two-loop no-legs vacuum diagram which contribute into the cosmological constant value in the case of interacting doublet fields. We have for the potential of the problem:

\[ V(\phi, \tilde{\phi}) = \frac{\lambda_4}{4!} \left( \phi^2 - \tilde{\phi}^2 \right)^2 = V_{AA}(\phi) - V_{AB}(\phi, \tilde{\phi}) + V_{BB}(\tilde{\phi}), \]  

(87)

see Eq. [85] and Eq. [67] identities. There are two separated contributions we can write on the base of corresponding Feynman rules:

\[ V_{0,1,2-loops} \propto \left( G_F(x, x) - \tilde{G}_F(x, x) \right)^2 = \left( G_F(x, x) + G_D(x, x) \right)^2 \propto \frac{1}{p^2 - m^2 - \varepsilon} - \frac{1}{p^2 - m^2 + \varepsilon} = \frac{2\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2} \quad \varepsilon \to 0 \quad 0 \]  

(88)

\[ \]  

\[ \text{In the case of the non-flat manifold, when we distinguish between } g \text{ and } \tilde{g} \text{ metrics, these two terms will provide corrections to the value of cosmological constant in } A, B \text{ manifolds.} \]
and

\[ V_{0,2,2-\text{loops}} \propto \Delta^2(x, x) = (\Delta_>(x, x) - \Delta_<(x, x))^2 = 0 \tag{89} \]

see Eq. (86) definition. We will not discuss here the diagrams of \( \lambda^2 \) or higher orders where mix of the \( G \) and \( \Delta \) propagators will happen, see [23] and references therein for example. So we see that the contribution of the vacuum diagrams to the cosmological constant in the flat manifolds is zero up to the two loops precision at least.

## 10 Smallness of the cosmological constant

Let’s consider, first of all, the possible quantum contributions to the cosmological constant value. In the formalism the constant is evolving with time, therefore we will discuss the value of the constant in an almost flat space-time and will not consider the possibilities of the large value of the constant for the manifolds with large curvature. It was demonstrated in the previous Sections that the vacuum contributions into the cosmological constant value are depend on the form of scalar fields and interactions introduced. In the formalism, nevertheless, there is a possibility to eliminate all types of the vacuum diagrams in general. Let’s consider the partition function for the scalar field in the curved space-time defining it as following:

\[ Z[f] = Z^{-1}[f = 0] \int D\phi D\tilde{\phi} e^{i S_{\text{int}}(\phi, \tilde{\phi}, f)} , \tag{90} \]

with \( S_{\text{int}} \) given by Eq. (39) for example or by any another action with interaction between the fields introduced. In this definition we used the fact that the currents \( f \) are physical ones and we can define the normalization factor for the fields without the currents also in the curved manifold. Clearly, this definition eliminates the all vacuum diagrams from the consideration without the relation to the curvature of the manifolds.

Therefore we stay with the smallness of the constants related to the Eq. (44) or Eq. (47) expressions for the case of the one-source Lagrangian. Let’s begin from the values of the constants in this case for the \( \sqrt{-g} = 1 \) and \( f = 1 \) for example. These values we can consider as kind of the probes, the perturbative corrections from Eq. (43) and Eq. (79) can be calculated in general as well. So we have in this case:

\[ \Lambda = \frac{m_p^4}{4m^2} \tag{91} \]

that means

\[ m^2 >> m_p^2 , \tag{92} \]

sign of the constant here corresponds to it’s final sign in the l.h.s. of the Einstein equations with Einstein-Hilbert action given by Eq. (57):

\[ G_{\mu\nu} + g_{\mu\nu} \Lambda = 0 , \tag{93} \]

we see that for the one-source Lagrangian the value of the constant has a wrong sign. It must be noted, nevertheless, that in the case of the first type of scalar field, the bare values of the constants cancels each another in the action. The contribution in the equation of motion will come only from the second order contribution to the constants, which is squared with respect to the Appendix C integrals. Therefore, in this case, the relative sign of the cosmological constant will be correct.

Nevertheless, in order to reproduce the known absolute value of the constant, we can define the mass of the scalar fields in terms of the \( m_p^2 \) and some characteristic length \( \lambda \):

\[ m^2 = m_p^2 e^{\lambda^2 m_p^2} , \lambda^2 = \frac{1}{m_p^2} \ln\left(\frac{m_p^2}{4 \Lambda}\right) . \tag{94} \]
we see that the mass is super-heavy at least at the present of the space-time.

An another possibility to provide the smallness of the constants is through a smallness of the sources of the particles. Taking the mass of the fields \( m = m_p \) for simplicity, we assume:

\[
f \propto \frac{R}{m_p^2}
\]

(95)

with \( R \) as curvature of the manifold. In this case we obtain:

\[
\Lambda \propto \frac{R^2}{m_p^2}
\]

(96)

and this value is small simply because the flatness of the manifold.

The correct sign of the cosmological constant we obtain in the case of the interaction Lagrangian with two types of the sources for the same field. First of all, consider the case of positive \( f \) in Eq. (49). Requiring

\[
\tilde{f} - f \propto \tilde{R} - R = \delta > 0
\]

(97)
in Eq. (55) and taking \( \delta \) without loss of generality as some constant, we will obtain for the second type of the scalar fields:

\[
\Lambda(x) = \frac{R \delta}{2 m_p^2}
\]

(98)

with correct Einstein equations:

\[
G_{\mu \nu} - g_{\mu \nu} \Lambda = 0.
\]

(99)

The same result we can achieve when the source \( f \) is negative. In this case we take Eq. (56)-Eq. (58) solution and assume that

\[
\tilde{f} + f \propto \delta < 0
\]

(100)

that again provide the correct sign of the constant. The two possible mechanisms of the constant’s smallness can be applied in this case as well. At the case of the first field, the bare value of the constants is precisely zero and some corrections to it must be calculated basing on the expansion of the propagators and metric’s determinant with respect to the curvature.

Concerning the origin of the field with the large mass, we can speculate about the following. Let’s suppose that there are quantum fluctuations of the scalar curvature which are not accounted by the classical Einstein equations:

\[
R \rightarrow R + \delta R = R + m_p \phi f = R + \phi \frac{R}{m_p}.
\]

(101)

These fluctuations indeed reproduce term similar to the source terms in Eq. (39) and Eq. (95). After that, considering the fluctuation as a regular scalar field, we can write Lagrangian Eq. (39) for the field with one understandable restriction. The field can not propagate far and that what requires the extremely heavy mass in the Lagrangian.

11 Summary and discussion

Let’s us summarize the main propositions and results of the formalism. First of all, we considered the different manifolds, A and B, related in the approach by the CPTM symmetry transform. These both manifolds are parts of the extended solution of the classical Einstein’s equations, the interpretation of the second manifold as populated by the negative gravitational mass is an usual one in fact, the discussion of this issue, for example, can be found in [4]. The proposed model also has some similarities to two-time direction models proposed for the solution of the Universe’s low initial entropy value,
CPT symmetric Universe model considered in [25] and models of [26]. The main consequences of this set-up is that we have always a system of two fields related by the symmetry and with different time directions in each manifold. We also note, that unlike to [27], for example, the different terms in the general action are present with the same sign, see Eq. (11)-Eq. (13). The difference between the terms in the action is appearing due the CPTM symmetry applied to the non-gravitational fields, for the metric the symmetry is precise when the cosmological constant is zero.

The next basing idea of the formalism, is an introduction in the action of some bi-scalar term which ”glue” the manifolds, see Eq. (36) and Eq. (38). This term provides an effective interaction between the points of extended manifold, the cosmological constant arises there as a consequence of these interactions. In general, we can not define a priori the form of the interaction term, the simplest and mostly obvious way to introduce this term it is define it through a propagator of scalar field, kind of quantum non-metric wormhole which connect different points of the extended general solution.

The choose of the propagator of the scalar field as the interaction term is not arbitrary of course. The introduced CPTM symmetry transforms the usual scalar field of the A manifold into an another scalar field of B manifold. The properties of the quantized B-field are different, there are two possibility to determine it. Each possibility depends on the type of the closed time path defined for A-B manifolds in correspondence to the in and out states of both fields. Nevertheless, in spite to the different quantum properties of the fields, the main consequence of the application of the CPTM transform is that the overall ”classical” zero modes contribution of both fields together in the energy-stress tensor is precise zero. This is a first important consequence of the proposed symmetry, it solves the zero mode problem of the cosmological constant.

Thereby, the action for the two fields is based on the closed time paths. There are two variants of the time paths considered in the paper which correspond to the two types of the scalar fields. The second variant is the same as the time path in the Keldysh - Schwinger, [21, 22] in-in QFT formalism, see also [23]. The next step in our consideration is a construction of the QFT for these fields. We again considered only two obvious possibilities for the corresponding QFT, both based on the choose of Feynman propagator $G_F$ as the propagator of the scalar field of A manifold. For both possibilities the QFT can be constructed with the inclusion of the self-interaction vertices of the fields and direct interaction between the A,B manifolds in the form of Keldysh propagator in the case of the second variant of the QFT. It is demonstrated, that due the form of the Lagrangian we can eliminate the quantum vacuum contributions into the cosmological constants in the partition function ending with only contributions from the vertices of the quantum effective action of corresponding QFT, see Eq. (78)-Eq. (79). In both cases the leading perturbative contribution to the constant is provided by the first term of the Eq. (79) effective action.

We obtain, therefore, that in the case of the simplest Lagrangian with one source for the fields, Eq. (39), the constant has a wrong sign for the A manifold for the two types the scalar fields. In turn, in the case of the Lagrangian with two sources for each field, Eq. (49), the parameters of the Lagrangian can be choosen in a way that the constant of our, A manifold, will have the correct sign. Effectively it was achieved only by introduction in the Lagrangian of the second sources for the scalar fields, the reasons and consequences of this construction must be clarified in the subsequent research.

Concerning the value of the constant, it is provided or by the mass of the quantum scalar fields or by value of the source of the scalar field. The proposed mass of the field must be very heavy, at least at the present epoch, in order to provide the smallness of the cosmological constants. It means, in general, that the propagation of the fields must be small. Another possibility is that the source of the field in the Lagrangian is very small, see Eq. (95), and it’s smallness is responsible for the smallness of the constant. There is as well a possibility of the dynamical change of the sources.
and cosmological constants with the time. An additional, technical, reason of the smallness of the constant can be a vanishing of some special propagator in the integrals of Appendix C. Perhaps, it is possible to adjust the propagators of the problem in order to obtain zero answer to this perturbative order. Interesting issue, also, is about the presence of the curvature in the final expression for the cosmological parameter. The curvature is not gauge-invariant variable in the general relativity. Nevertheless, it’s possible presence in the answer simply means that there are additional terms in the action which must restore the gauge invariance order by order. This is an usual procedure for the high energy QCD construction, see [31] and there are similar to QCD calculations are done in gravity as well, see [32].

In any case, the origin of the scalar field in the formalism is not clear. We can speculate that this field represents some quantum fluctuations of the scalar curvature in the Einstein-Hilbert action which is not accounted by the classical equations of motion. The non-propagating of these kind of fields is understandable as well therefore, they must have large mass. This mass, in turn, leads to the definition of some new length which related to the plank mass and cosmological constant value combination. Of course, it is not clear, if this large mass mechanism is satisfactory in general and preferable in comparison with the small source mechanism. The cosmological constant, parameter more precisely, is evolving with the time and in order to understand the details of it’s smallness some details of an evolution of the constant with the time must be clarified. In general, it is quite possible that the evolution begins from the small mass of the field or large curvature, large constants correspondingly, and all parameters are changing with time and as result we have the large mass of the field and small values of curvature and cosmological constant. We postpone this question for a separate research.

There are the following issues which we did not discussed in the paper but which are arising naturally inside the framework. The first important question which is rising it is about the properties of the renormalizability of whole Eq. (36) action. Namely, an expansion of the full scalar propagators in the $S_{int}$ term of action, will lead to the appearance in the action new terms with different types of the dependence on the curvature tensor similarly to the higher curvature gravity theories. This is an immediate consequence of the possible adiabatic expansion of the curved propagators directly in Eq. (41) and there is an interesting question about a renormalizability of the Eq. (36) action.

Another issue is about the definition of the propagator of the scalar field of A manifold. In fact, we are not constrained by the choice of the Feynman propagator only. We can, for example, to choose the Wheeler propagator instead:

$$G_W = \frac{1}{2} (G_F - G_D) \ , \quad (102)$$

see [28,29], it arises naturally in the Eq. (55) expressions. In the case of two types of scalar fields, this choice of A-field propagator will lead to the following propagator of the B-field:

$$CPTM(G_W) = -\frac{1}{2} (G_F - G_D) \ , \quad CPTM(G_W) = \frac{1}{2} (G_F - G_D) \quad (103)$$

for the first and second types of the field correspondingly. In the second case we obtain the totally symmetrical QFT, a construction of the in-in formalism for both cases is an interesting task. An advantage of this choice, also, is that in this case the free quanta of the scalar fields are absent and in some sense these particles are unobservable, see [29]. It is an important property of the field related to the cosmological constant and dark energy of course. The disadvantage of this choice is that there is no acceptable QFT based on this type of propagators, see discussion in [29] anyway. Definitely, it is interesting problem for the further investigation.

The next question which we did not consider here is about the different definitions of bi-scalar functions in Eq. (38). There are two additional possibilities exist. The first one is the trivial one. Instead the scalar fields we can consider any other fields, the metric’s fluctuation in the weak field limit for example, with the similar overall zero contribution of the zero modes to the energy-stress tensor. In this set-up the problems with the origin of the fields and it’s sources will arise and will request a clarification. Of course, this problem must be resolved, there are other fields different from
the scalar one. Additionally, we can consider the bi-scalar functions in Eq. (38) outside the QFT, there are plenty of possibilities exist for that. Nevertheless, in this case some different mechanisms of the smallness of the cosmological constant value must be established and clarified.

Concluding we note that there are many different problems that can arise in the framework of the formalism and which investigation can help to understand and resolve the puzzle of the cosmological constant.

The author kindly acknowledges useful discussions of the subject with M.Zubkov.
Appendix A: Propagators of scalar field

For the simplicity we firstly calculate the Dyson propagator for the scalar filed $\phi$:

$$G_D(x, y) = -i \langle T (\phi(x) \phi(y)) \rangle = -i \langle \theta(x^0 - y^0) < \phi(y) \phi(x) > + \theta(y^0 - x^0) < \phi(x) \phi(y) > \rangle = \begin{cases} -i D(y - x), & x^0 > y^0 \\ -i D(x - y), & y^0 > x^0 \end{cases}. \quad (A.1)$$

Here

$$D(x - y) = < \phi(x) \phi(y) > = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i \omega_k (x^0 - y^0) + i \vec{k} \cdot (\vec{x} - \vec{y})}}{2 \omega_k} \quad (A.2)$$

is Wightman function. Using the $\theta$ function representation

$$\theta(x^0 - y^0) = \frac{i}{2\pi} \int \frac{d\omega}{\omega + i \varepsilon} \quad (A.3)$$

we obtain for the first term in the r.h.s. of Eq. (A.1):

$$\int d\omega \frac{e^{-i \omega (x^0 - y^0)} e^{i \omega k (x^0 - y^0)}}{\omega + i \varepsilon} \quad (A.4)$$

with

$$< \phi^- (k) \phi^+(k') > = \delta_{kk'}, \omega_k = \sqrt{k^2 + m^2}. \quad (A.5)$$

After the variables change

$$\omega - \omega_k \to \omega, \vec{k} \to -\vec{k} \quad (A.6)$$

we obtain for this term:

$$\int \frac{d^4k}{(2\pi)^4} \frac{e^{-i \omega (x^0 - y^0) + i \vec{k} \cdot (\vec{x} - \vec{y})}}{\omega_k + i \varepsilon} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i k \cdot (x - y)}}{\omega_k + i \varepsilon} \quad (A.7)$$

For the second one we have:

$$- \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i k \cdot (x - y)}}{\omega_k - i \varepsilon} \quad (A.8)$$

that provides for the propagator:

$$G_D(x, y) = - \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i k \cdot (x - y)}}{k^2 - m^2 - i \varepsilon}. \quad (A.9)$$

The similar calculations provide for the Feynman propagator

$$G_F(x, y) = -i \langle T (\phi(x) \phi(y)) \rangle = -i \langle \theta(x^0 - y^0) < \phi(y) \phi(x) > + \theta(y^0 - x^0) < \phi(x) \phi(y) > \rangle = \begin{cases} -i D(y - x), & x^0 > y^0 \\ -i D(x - y), & y^0 > x^0 \end{cases}. \quad (A.10)$$

the following answer:

$$G_F(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i k \cdot (x - y)}}{k^2 - m^2 + i \varepsilon}. \quad (A.11)$$
Appendix B: Propagator of scalar field in curved manifold

We begin from the usual definition of the quadratic with respect to the fluctuations part of the Eq. (39) Lagrangian:

\[ L_{\varepsilon^2} = \sqrt{-g} \left( \varepsilon G_F^{-1} \varepsilon \right) = -\sqrt{-g} \left( \varepsilon \Box \varepsilon \right) \]  
(B.1)

with

\[ \Box = \frac{1}{\sqrt{-g}} \partial_{\nu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\mu} \right) + m^2 = \partial_{\mu} \partial^{\mu} + m^2 + M_1 \]  
(B.2)

and following formal definition of the Green’s function:

\[ \Box_x G(x, y) = \frac{1}{\sqrt{-g(x)}} \Box_x \left( \partial_{\mu} \partial^{\mu} + m^2 + M_1 \right)_{xy}^{-1} = -\frac{1}{\sqrt{-g(x)}} \delta^4(x - y) \]  
(B.3)

which we can rewrite as

\[ \sqrt{-g(x)} \Box_x G(x, y) = \left( \partial_{\nu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\mu} \right) + m^2 \sqrt{-g} \right)_x G(x, y) = \left( \partial_{\nu} \partial^{\nu} + m^2 + N_1 \right)_x G(x, y) = -\delta^4(x - y). \]  
(B.4)

From Eq. (B.2) and Eq. (B.4) we have the following definition of \( N_1 \) operator:

\[ N_1 = \left( \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu} \right) \partial_{\mu} \partial_{\nu} + \left( \sqrt{-g} - 1 \right) m^2 + \sqrt{-g} \left( \partial_{\mu} g^{\mu\nu} \right) \partial_{\nu} - \frac{\sqrt{-g}}{2} g^{\mu\nu} g_{\rho\sigma} \left( \partial_{\mu} g^{\rho\sigma} \right) \partial_{\nu}. \]  
(B.5)

The precise perturbative solution of the propagator of scalar field in the curved space time is given, therefore, by the following expression:

\[ G(x, y) = G_0(x, y) + \int d^4 z G_0(x, z) N_1(z) G(z, y) \]  
(B.6)

with

\[ \left( \partial_{\mu} \partial^{\mu} + m^2 \right)_x G_0(x, y) = -\delta^4(x - y). \]  
(B.7)

Here we do not define precisely which \( G_0(x, y) \) propagator to use in Eq. (B.6), it must satisfy Eq. (B.7) with arbitrary boundary conditions. The Eq. (B.6) representation of the propagator is useful due the few reasons. First of all, it can be used in the weak field approximation, expanding \( N_1 \) operator in terms of \( h^{\mu\nu} \) we will obtain perturbative expression for the propagator with required precision. Another interesting application of Eq. (B.6) is to use it as a reformulated recursive formula for the adiabatic expansion of the \( G(x, y) \), see \[50\]. In this case, inserting the adiabatic series in both sides of Eq. (B.6), we will obtain some non-local relations between the \( a_l(x, y) \) coefficients of the adiabatic expansion.

In the formalism we can use the Eq. (B.6) for the calculation of the variation of the cosmological constant with respect to the metric. The non-trivial variation of the action part of the action, Eq. (11), is provided by the variation of the full propagator in the curved space time. We have for this variation:

\[ \delta G(x, y) = \int d^4 z G_0(x, z) \left( \delta N_1(z) \right) G(z, y) + \int d^4 z G_0(x, z) N_1(z) \left( \delta G(z, y) \right). \]  
(B.8)

The same we can rewrite as:

\[ \int d^4 z \left( \delta^4(x - z) - G_0(x, z) N_1(z) \right) \delta G(z, y) = \int d^4 z G_0(x, z) \left( \delta N_1(z) \right) G(z, y) \]  
(B.9)

or

\[ \int d^4 x \left( \delta^4(p - x) - G_0(p, x) N_1(x) \right)^{-1} \int d^4 z \left( \delta^4(x - z) - G_0(x, z) N_1(z) \right) \delta G(z, y) = \int d^4 x \left( \delta^4(p - x) - G_0(p, x) N_1(x) \right)^{-1} \int d^4 z G_0(x, z) \left( \delta N_1(z) \right) G(z, y) \]  
(B.10)
that provides finally:

\[ \delta G(x,y) = \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} G_0(p,z) \left( \delta N_1(z) \right) G(z,y). \] (B.11)

Taking variation of the \( N_1 \) and assuming that the metric does not depend on the cosmological constant, we obtain:

\[ \delta G(x,y) = \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} G_0(p,z) \delta g^{\mu \nu}(z) \sqrt{-g(z)} \]

\[ \left( -\frac{1}{2} g_{\mu \nu} \left( \frac{\partial g_{\mu \nu}}{\partial x} \right) + m^2 + \left( \frac{\partial g_{\nu \sigma}}{\partial p} \right) \right) \delta G(x,y) + \]

\[ + \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} \delta g^{\mu \nu}(z) \]

\[ \left( -\frac{1}{2} g_{\mu \nu} \left( \frac{\partial g_{\mu \nu}}{\partial x} \right) + m^2 + \left( \frac{\partial g_{\nu \sigma}}{\partial p} \right) \right) G(z,y) + \]

\[ + \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} \delta g^{\mu \nu}(z) \]

\[ \left( -\frac{1}{2} g_{\mu \nu} \left( \frac{\partial g_{\mu \nu}}{\partial x} \right) + m^2 + \left( \frac{\partial g_{\nu \sigma}}{\partial p} \right) \right) G(z,y). \] (B.12)

or

\[ \delta G(x,y) = \frac{1}{2} \int d^4p \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} G_0(p,y) g_{\mu \nu}(y) \delta g^{\mu \nu}(y) + \]

\[ + \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} G_0(p,z) \delta g^{\mu \nu}(z) \sqrt{-g(z)} \]

\[ \left( \frac{\partial g_{\nu \sigma}}{\partial p} \right) G(z,y) + \]

\[ + \int d^4p \int d^4z \left( \delta^4(x-p) - G_0(x,p) N_1(p) \right)^{-1} \delta g^{\mu \nu}(z) \]

\[ \left( -\frac{1}{2} g_{\mu \nu} \left( \frac{\partial g_{\mu \nu}}{\partial x} \right) + m^2 + \left( \frac{\partial g_{\nu \sigma}}{\partial p} \right) \right) G(z,y). \] (B.13)

Now we can write the new term in the equations of motion for the gravitational field. Taking variation of the cosmological constant term in \( S_{int} \) for the \( A \) minifold, Eq. (III), with \( f = 1 \) source we obtain:

\[ \delta S_{int} = \frac{m^6}{2} \int d^4x \int d^4y \left( g_{\mu \nu} \delta g^{\mu \nu} \right) G(x,y) \sqrt{-g(y)} - \]

\[ - \frac{m^6}{2} \int d^4x \int d^4y \left( \delta G(x,y) \right) \sqrt{-g(y)} + \cdots \] (B.14)

with similar contribution for \( B \) manifold added. If we introduce in \( N_1 \) expression a dependence of the metric on the cosmological constant, then we will need to take into account in the variation also the following terms

\[ \frac{\delta}{\delta g^{\mu \nu}} \left( A_1(g) \Lambda + A_2(g) \Lambda^2 + \cdots \right) \delta g^{\mu \nu} \] (B.15)

which arise in the variation after an expansion of the operator with expect to the constant in the form of perturbative series.
Appendix C: Integrals of scalar propagators

There are the following integrals we need to calculate:

\[ I_F = \int d^4y \, G_F(x, y), \quad I_D = \int d^4y \, G_D(x, y). \]  

(C.1)

For the first integral, using an equivalent to Eq. (A.11) representation of the bare Feynman propagator, we can write:

\[
I_F = -i \int d^4y \, \theta(x^0 - y^0) \int \frac{d^3k}{2\omega_k (2\pi)^3} \, e^{-i(\omega_k - i\epsilon)(x^0 - y^0) + i \vec{k} \cdot \vec{x} - \vec{y}} - \\
- i \int d^4y \, \theta(y^0 - x^0) \int \frac{d^3k}{2\omega_k (2\pi)^3} \, e^{-i(\omega_k - i\epsilon)(y^0 - x^0) + i \vec{k} \cdot \vec{y} - \vec{x}} = \\
= i \int_{-\infty}^{\infty} dy^0 \, \theta(y^0) \int d^3y \, e^{i \vec{k} \cdot \vec{y}} \int \frac{d^3k}{2\omega_k (2\pi)^3} \, e^{-i(\omega_k - i\epsilon)y^0} - \\
- i \int_{-\infty}^{\infty} dy^0 \, \theta(y^0) \int d^3y \, e^{i \vec{k} \cdot \vec{y}} \int \frac{d^3k}{2\omega_k (2\pi)^3} \, e^{-i(\omega_k - i\epsilon)y^0} = \\
= -2i \int dy^0 \, \theta(y^0) \int \frac{d^3k}{2\omega_k} \, e^{-i(\omega_k - i\epsilon)y^0} \, \delta^3(k) = - \frac{i}{|m|} \int_0^{\infty} dy^0 \, e^{-i(|m| - i\epsilon)y^0} = \\
= - \frac{1}{|m| (|m| - i\epsilon)}. \quad (C.2)
\]

The same can be done for \(I_D\), we will obtain then:

\[ I_D = -i I_F = \frac{1}{|m| (|m| + i\epsilon)}. \]  

(C.3)
References

[1] S. Bondarenko, Mod. Phys. Lett. A 34, no. 11, 1950084 (2019).
[2] S. Bondarenko, Universe 6, no.8, 121 (2020).
[3] M. D. Kruskal, Phys. Rev. 119, 1743 (1960); G. Szekeres, Publ. Mat. Debrecen 7, 285 (1960).
[4] S. Chandrasekhar, "The mathematical theory of black holes", Clarendon Press Oxford, 1983.
[5] V. P. Frolov and I. D. Novikov, "Black holes physics", Kluwer Academic Publishers, 1998.
[6] G. 't Hooft, J. Geom. Phys. 1 (1984) 45; Nucl. Phys. B 256 (1985), 727-745.
[7] M. Villata, EPL 94, no. 2, 20001 (2011); Annalen Phys. 527, 507 (2015); N. Debergh, J. P. Petit and G. D’Agostini, J. Phys. Cond. 2, no.11, 115012 (2018); H. Socas-Navarro, Astron. Astrophys. 626, A5 (2019); G. J. Ni, Rel. Grav. Cosmol. 1 (2004), 123-136.
[8] G. Chardin, Hyperfine Interact. 109, no. 1-4, 83 (1997). J.P.Petit, G.D’Agostini Astrophysics And Space Science., A 29, 145-182 (2014); R. J. Nemiroff, R. Joshi and B. R. Patla, JCAP 1506, 006 (2015); G. Kofinas and V. Zarikas, Phys. Rev. D 97, no. 12, 123542 (2018); G. Manfredi, J. L. Rouet, B. Miller and G. Chardin, Phys. Rev. D 98, 023514 (2018); G. Chardin and G. Manfredi, Hyperfine Interact. 239, no.1, 45 (2018).
[9] T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D 66, 104025 (2002); T. Damour and I. I. Kogan, Phys. Rev. D 66, 104024 (2002).
[10] J.-P. Petit, Astrophys. Space Sci. 226, 273 (1995); J. P. Petit and G. d’Agostini, [arXiv:0803.1362 [math-ph]]; J. Petit and G. D’Agostini, Mod. Phys. Lett. A 29, no.34, 1450182 (2014); J. P. Petit and G. d’Agostini, Astrophys. Space Sci. 354, no. 2, 2106 (2014); J. P. Petit and G. D’Agostini, Astrophys. Space Sci. 357, no.1, 67 (2015); J.P.Petit and G.D’Agostini, Mod. Phys. Lett. A 30, no.9, (2015); G. DAgostini and J.P.Petit, Astrophysics and Space Science, (2018); N. Debergh, J. P. Petit and G. D’Agostini, J. Phys. Cond. 2, no.11, 115012 (2018).
[11] S. Hossenfelder, Phys. Lett. B 636, 119-125 (2006); S. Hossenfelder, [arXiv:gr-qc/0605083 [gr-qc]].
[12] A. A. Baranov, Izv. Vuz. Fiz. 11, 118 (1971); A. D. Dolgov, [arXiv:1206.3725 [astro-ph.CO]].
[13] J. S. Farnes, A&A 620, A92 (2018).
[14] M. Villata, Astrophys. Space Sci. 339, 7 (2012); Astrophys. Space Sci. 345, 1 (2013).
[15] D. S. Hajdukovic, Astrophys. Space Sci. 339, 1 (2012); Phys. Dark Univ. 3, 34 (2014).
[16] J.-M. Souriau, "Structure of dynamical systems", Progress in Mathematics vol. 149, Springer Science, 1997.
[17] Hooft G., In: Hooft G. et al. (eds) Recent Developments in Gauge Theories. NATO Advanced Study Institutes Series (Series B. Physics), vol 59. Springer, Boston, (1980).
[18] R. W. Fuller and J. A. Wheeler, Phys. Rev. 128, 919 (1962); Wheeler J.A., in: Relative Groups and Topology, eds. B.S. and C.M. DeWitt, Gordan and Breach, New Ypork, 1964.
[19] S. W. Hawking, Phys. Rev. D 37, 904 (1988); S. R. Coleman, Nucl. Phys. B 310, 643 (1988); M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988); M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988); S. Weinberg, Rev. Mod. Phys. 61, 1 (1989); W. Fischler and L. Susskind, Phys. Lett. B 217, 48 (1989); I. R. Klebanov, L. Susskind and T. Banks, Nucl. Phys. B 317, 665 (1989); S. W. Hawking, Nucl. Phys. B 363, 117 (1991); bibitemThorne.
[20] S. Weinberg, In *Princeton 1996, Critical dialogues in cosmology* 195-203 [astro-ph/9610044].
[21] J. S. Schwinger, J. Math. Phys. 2 (1961) 407.
[22] L. V. Keldysh, Zh. Eksp. Teor. Fiz. 47 (1964) 1515 [Sov. Phys. JETP 20 (1965) 1018].
[23] A. Calzetta, A-L. B. Hu, "Nonequilibrium quantum field theory", Cambridge University Press, 2008; A. Kamenev, "Field theory of non-equilibrium systems", Cambridge University Press, 2011; P. Millington, "Thermal quantum field theory and perturbative non-equilibrium dynamics", Springer, 2014; S.J.N. Mooij, "Effective theories in cosmology".
[24] A.D.Sakharov, JETP Lett. 5, 24 (1967); JETP 49, 594 (1979); JETP 52, 349 (1980); A. Aguirre and S. Gratton, Phys. Rev. D 65, 083507 (2002); A. Aguirre and S. Gratton, Phys. Rev. D 67, 083515 (2003); S. M. Carroll and J. Chen, hep-th/0410270; S. M. Carroll and J. Chen, Gen. Rel. Grav. 37, 1671 (2005) [Int. J. Mod. Phys. D 14, 2335 (2005)]; S. M. Carroll, Nature 440, 1132 (2006); A. Vilenkin, Phys. Rev. D 88, 043516 (2013).
[25] L. Boyle, K. Finn and N. Turok, Phys. Rev. Lett. 121, no. 25, 251301 (2018); J. L. Alonso and J. M. Carmona, Class. Quant. Grav. 36, no. 18, 185001 (2019).
[26] D. E. Kaplan and R. Sundrum, JHEP 0607, 042 (2006); A. Hebecker, T. Mikhail and P. Soler, Front. Astron. Space Sci. 5, 35 (2018).
[27] A. D. Linde, Phys. Lett. B 200, 272 (1988).
[28] P.A.M.Dirac, Proc. Roy. Soc. London A 167, 148 (1938); J.A.Wheeler and R.P.Feynman, Rev. of Mod. Phys. 17, 157 (1945); J.A.Wheeler and R.P.Feynman: Rev. of Mod. Phys. 21, 425 (1949).
[29] C. G. Bollini and M. C. Rocca, Int. J. Theor. Phys. 37, 2877-2893 (1998); C. G. Bollini and M. C. Rocca, arXiv:1012.4108 [hep-th]; J. F. Koksma and W. Westra, arXiv:1012.3473 [hep-th]; D. Anselmi, JHEP 03, 142 (2020).
[30] T. S. Bunch and L. Parker, Phys. Rev. D 20, 2499-2510 (1979); J. D. Bekenstein and L. Parker, Phys. Rev. D 23, 2850-2869 (1981); L. Parker and D. J. Toms, Phys. Rev. D 31, 953 (1985); I. Jack and L. Parker, Phys. Rev. D 31, 2439 (1985).
[31] L. N. Lipatov, Nucl. Phys. B 452, 369 (1995); Phys. Rept. 286, (1997) 131; S. Bondarenko, L. Lipatov and A. Prygarin, Eur. Phys. J. C 77 no.8, 527, (2017); S. Bondarenko, L. Lipatov, S. Pozdnyakov and A. Prygarin, Eur. Phys. J. C 77 no. 9, 630, (2017).
[32] L. N. Lipatov, Nucl. Phys. B 365, 614 (1991); L. N. Lipatov, Theor. Math. Phys. 169, 1370 (2011); L. N. Lipatov, Phys. Part. Nucl. 44, 391 (2013); L. N. Lipatov, Subnucl. Ser. 50, 213 (2014); L. N. Lipatov, EPJ Web Conf. 125, 01010 (2016); L. N. Lipatov, EPJ Web Conf. 164, 02002 (2017); S. Bondarenko, S. Pozdnyakov and M. A. Zubkov, arXiv:2009.05571 [hep-th].