Proof that the real part of all non-trivial zeros of Riemann zeta function is 1/2

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This article proves the Riemann hypothesis, which states that all non-trivial zeros of the zeta function have a real part equal to 1/2. We inspect in detail the integral form of the (symmetrized) completed zeta function, which is a product between the zeta and gamma functions. It is known that two integral lines, expressing the completed zeta function, rotated from the real axis in the opposite directions, can be shifted without affecting the completed zeta function owing to the residue theorem. The completed zeta function is regular in the region of the complex plane under consideration. For convenience in the subsequent singularity analysis of the above integral, we first deform and shift the integral contours. We then investigate the singularities of the composite elements (caused by polynomial integrals in opposite directions), which appear only in the case for which the distance between the contours and the origin of the coordinates approaches zero. The real part of the zeros of the zeta function is determined to be 1/2 along a symmetry line from the singularity removal condition. (In the other points, the singularities are adequately cancelled as a whole to lead to a finite value.)

1 Introduction

By connecting complex analysis with number theory, Riemann observed [1] that (denoting a set of real numbers by \( \mathbb{R} \) and letting \( x \in \mathbb{R} \)) the function \( \pi(x) \), which denotes the number of prime numbers below a given number \( x \), contains the summation over non-trivial zeros (points at which the function vanishes) of the zeta function. Riemann expected (denoting a set of complex numbers by \( \mathbb{C} \) and letting \( z \in \mathbb{C} \)) the real part of the non-trivial zeros of the zeta function \( \zeta(z) \) to be 1/2, which is known as the Riemann hypothesis. Furthermore, von Koch showed [2] that \( \pi(x) \) is well approximated by the offset logarithmic integral function \( \text{Li}(x) \) as

\[
\pi(x) = \text{Li}(x) + O(x^{\frac{1}{2}} \log x),
\]

which is equivalent to the Riemann hypothesis. We denote a set of natural numbers by \( \mathbb{N} \) and let \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \), then the zeta function \( \zeta(z) \) is defined as a function, which is analytically continued in the complex plane from the expression defined below [3-5]

\[
\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z},
\]

for \( z \) that satisfies \( \text{Re}(z) > 1 \) (we denote the real and imaginary parts of \( z \) as \( \text{Re}(z) \) and \( \text{Im}(z) \), respectively.) The zeta function is also obtained with the help of the gamma function \( \Gamma(z) \), and, letting \( t' \in \mathbb{R} \), then the gamma function is defined as a function that is also analytically continued into all points in the complex plane from [3,6-9]

\[
\Gamma(z) := \int_{0}^{\infty} dt'(t')^{z-1} \exp(-t'),
\]
for $\text{Re}(z) > 0$.

Concerning the zeros of the zeta function, which states $\zeta(z) = 0$, there exist trivial zeros, such as negative integers $-2, -4, \cdots$ [3]. In contrast, Hardy showed that numerous non-trivial zeros of the zeta function exist along the line with the real part equal to $1/2$ [10], however, not all the real parts of the non-trivial zeros are known. The work on such as imaginary parts of the zeros is reported in literature [11]. The computational approach [12] strongly suggests that the real part of zeros of the zeta function is $1/2$.

On the other hand, letting $z, w \in \mathbb{C}$, for the completed zeta function defined by

$$\tilde{\zeta}(z) := \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} \frac{1}{z}\right) \zeta(z),$$

the integral form of the (completed) zeta function is expressed as

$$\pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} \frac{1}{z}\right) \zeta(1 - z) = \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} \frac{1}{z}\right) \int_{0 \searrow 1} dw \frac{w^{z-1} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)} + \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} \frac{1}{z}\right) \int_{0 \nearrow 1} dw \frac{w^{z-1} \exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}. \quad (5)$$

The above integral is performed along the integral lines $0 \searrow 1$ and $0 \nearrow 1$ with the slopes $-1$ and $+1$, respectively, which pass through an arbitrary point in the region between 0 and 1 of the real axis. Since the residue theorem exhibits the above equation, the integral form is independent of the shift of this intersection point between 0 and 1. Furthermore, in the original form [13] of the above equation, the function $\Gamma(z/2)$ in the second term on the right-hand side is proportional to the regular function for $\text{Re}(z) < 1$. The function $\Gamma((1 - z)/2)$ on the left-hand side is regular in the region $\text{Re}(z) < 1$, while the right-hand side is also regular because of the existence of the derivative [14, 15], and the function $\zeta(1 - z)$ is analytically continued uniquely [14] into the region $0 < \text{Re}(z) < 1$ (the real part of zeros of $\zeta(z)$ exists only in this region). This paper takes into account the form mentioned above.

Since the gamma function is regular, the non-trivial zeros of the completed zeta function $\tilde{\zeta}(z)$, which is the product between the gamma function $\Gamma(z)$ and zeta function $\zeta(z)$, coincide with those of the zeta function $\zeta(z)$ in the region being considered with the real part between 0 and 1. As is described in this paper, each of the two line integrals expressing the completed zeta function has a singularity when the integral lines approach the axis origin. However, the completed zeta function $\tilde{\zeta}(z)$ does not depend on a specific point of the intersection point (shifted between 0 and 1 along the real axis) between the above integral line and the real axis due to the residue theorem, and $\tilde{\zeta}(z)$ is regular in the considering region. Then, these singularities must exactly cancel each other for $\tilde{\zeta}(z) = 0$, which is expected to lead to the determination of the real part of the zeros of the zeta function $\zeta(z)$.

Considering the status mentioned above, this paper is aimed at proving the Riemann hypothesis. We first deform and shift the contours of the integral (as in Figs. 1 and 2 for the integral form of the completed zeta function) along the integral line rotated from the real axis, for convenience in the subsequent analysis of the singularity of the integral in a complex plane. By this deformation and shift of the contours for the integral, the singularity analysis can be concentrated on the components of the integral around the origin of coordinates.

This research then addresses the singularities that appear in the two integral lines of the integral form of the completed zeta function. The singularities of the integrands for the composite elements near the origin of the real axis are caused by polynomials, only in the case when the contour-origin distance approaches zero. These singularities adequately cancel each other yielding a finite value independent of the integral contour as a whole. In contrast, from the equation $\tilde{\zeta}(z) = 0$ for completed zeta function $\tilde{\zeta}(z)$, the real part of the zeros of $\tilde{\zeta}(z)$ is determined by requiring these singularities to be an identical order power of the integral variable in the integrands leading to the exact singularity cancellation (given by Theorem 2). This requirement results in a value of $1/2$ for the real part of zeros of the completed zeta function $\tilde{\zeta}(z)$ (and the original zeta function $\zeta(z)$) due to the symmetry with respect to the $1/2$ real part, which is the originality of the present study and proves the Riemann hypothesis. The Riemann hypothesis is one of the most
important unproved problems in mathematics, and has its equivalent and advanced (extended) conjectures in other related fields. The positive proof of the Riemann hypothesis advances mathematics in other related fields [16,17].

The contents of this paper are as follows. Section 2 describes the deformation and shift of the contours for the integral in the integral form of the completed zeta function for convenience in the subsequent singularity analysis of the integral. Section 3 presents the proof that the real part of all non-trivial zeros of the zeta function is equal to 1/2, as was conjectured by Riemann, followed by the conclusion.

2 Deformation and shift of the contours for the integral form of the completed zeta function for the singularity analysis in a complex plane

This section presents the deformation and shift of the contours for the integral (in the integral form of the completed zeta function) along the line rotated from the real axis for convenience in the subsequent analysis (in Section 3) of the singularity of the integral in a complex plane. In this section, we first convert the integral form of the completed zeta function expressed by Eq. (5) to the usual form and thus obtain Theorem 1. Moreover, we define the radii centered at the (coordinate) origin, the main points and the contours (in Definitions 1, 2 with Figs. 1, 2) in the complex plane. Then, the contours denoted by 0 \( \to 1 \) and 0 \( \uparrow 1 \) in the integral form of the (completed) zeta function in Eq. (5) are deformed around the origin of coordinates to the arcs in Figs. 1 and 2, respectively (in Lemma 1). Subsequently, the remaining straight-line parts of the contours are shifted to point to the (coordinate) origin. Finally, we separate the finite integrals (in Lemmas 2, 3) along the shifted contours, which have sufficient distance to the (coordinate) origin, from the integrals around the origin containing the singularities, which appear only when the contours approach the origin.

Notations used in this paper are as follows. Let \( z, v \in \mathbb{C} \) and \( x, y \in \mathbb{R} \), and let \( i \) be the imaginary unit, then

\[
(x, y) := z = x + iy.
\]

We denote the real and imaginary components as

\[
z_R := \Re(z) = x, \quad z_I := \Im(z) = y, \quad v_R := \Re(v), \quad v_I := \Im(v) \quad \text{with} \quad v = -z, z - 1.
\]

The usual integral form of the completed zeta function is as follows.

**Theorem 1. (the (third) integral form of the (completed) zeta function)** Let \( z, w \in \mathbb{C} \). Let \( \hat{\zeta}(z) \) be the completed zeta function defined by Eq. (4). Let

\[
\hat{\zeta}_l(z) := \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \int_{0, \gamma} d\omega w^{-z} \frac{\exp(-\pi i \omega^2)}{\exp(\pi i \omega) - \exp(-\pi i \omega)},
\]

and let

\[
\hat{\zeta}_r(z) := \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \int_{0, \gamma'} d\omega w^{-1-z} \frac{\exp(\pi i \omega^2)}{\exp(\pi i \omega) - \exp(-\pi i \omega)},
\]

in terms of the gamma function \( \Gamma(z) \). Then \( \hat{\zeta}(z) \) is expressed by

\[
\hat{\zeta}(z) = \hat{\zeta}_l(z) + \hat{\zeta}_r(z),
\]

which is called the (third) integral form of the (completed) zeta function.

**Proof.** From Eq. (5), the above completed zeta function \( \hat{\zeta}(z) \) is obtained by the exchange \( z \leftrightarrow 1 - z \), where the region \( 0 < \Re(z) < 1 \) is kept under this exchange. The above region \( 0 < \Re(z) < 1 \) is consistent with the region \( 0 < \Re(1-z) < 1 \) we are considering in this paper. \( \square \)
Furthermore, let

\[ \hat{\zeta}(z) = \hat{\zeta}(1-z) \]  

Figure 1: Location of complex numbers (denoted at the bottom of the figure) and contours in the complex w-plane. Contour \( \tilde{C}_1 \) comprises \( \tilde{C}_{11} \) (from \( \exp(\frac{1}{4} \pi i) \) to \( w_{11} \)), \( \tilde{C}_{12} \) (from \( w_{11} \) to \( w_{12} \)), \( \tilde{C}_{13} \) (from \( w_{12} \) to \( w_{13} \)), \( \tilde{C}_{16} \) (from \( w_{13} \) to \( w_{15} \) via \( w_{14} \)), \( \tilde{C}_{14} \) (from \( w_{15} \) to \( w_{16} \)), \( \tilde{C}_{13} \) (from \( w_{16} \) to \( w_{17} \)) and \( \tilde{C}_{14} \) (from \( w_{17} \) to \( \exp(\frac{1}{4} \pi i) \)).

In addition, the completed zeta function satisfies the following known symmetry relation \([1, 17, 16]\) for the exchange \( z \leftrightarrow 1-z \)

\[ \hat{\zeta}(z) = \hat{\zeta}(1-z) \]  

To evaluate the integrals of the completed zeta function (in Eqs. \((8)-(10)\)), we further define the detailed integrands, the main points (in the complex plane) and the deformed and shifted contours of the integrals for use in the subsequent processes.

**Definition 1.** Let \( w \in \mathbb{C} \), and let \( r_{11}, r_{12}, r_1, r_m \in \mathbb{R} \) with \( 0 < r_{11}, r_{12}, r_1 < r_m < 1/2 \). Then, the specific radii \( r_{11}, r_{12}, r_1 \) and \( r_m \) of \( w \), centered at the origin of the complex w-plane, are defined to be small enough so that the following denominator, denoted as \( f^{(\text{De})} \), and parts of the numerators, denoted as \( f^{(\text{Nu})-} \) and \( f^{(\text{Nu})+} \), in the integrands in Eqs. \((8)-(10)\), are approximated by

\[ \{ f^{(\text{De})} = \exp(\pi iw) - \exp(-\pi iw) \approx 2\pi iw \}, \quad \{ f^{(\text{Nu})-} = \exp(-\pi iw^2) \approx 1 \} \quad \text{and} \]

\[ \{ f^{(\text{Nu})+} = \exp(+\pi iw^2) \approx 1 \} \quad \text{for} \ |w| \leq r_{11}, r_{12}, r_1 \quad \text{and} \quad |w| \leq r_m. \]  

Furthermore, let \( \theta \in \mathbb{R} \), with \( \theta = \frac{-4}{3}\pi, \frac{4}{3}\pi, \pi \), be the angle (argument) of \( w \) measured counterclockwise from the real axis in the complex plane. Let \( r_M \in \mathbb{R} \) be the specific radius of \( w \), centered at the (coordinate) origin, and defined to be
Definition 2. Let \( w \in \mathbb{C} \), and let \( w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17} \in \mathbb{C} \). Using the specific radii \( r_1 \) and \( r_{11} \) (of \( w \)) and setting \( r_1 = r_{11} \) in Definition[1] we define the above complex numbers, whose locations in the complex \( w \)-plane are shown in Fig.[1] by

\[
\begin{align*}
  w_{11} := r_m \exp \left( \frac{3}{4} \pi i \right) &= \left( r_m \cos \left( \frac{3}{4} \pi \right), r_m \sin \left( \frac{3}{4} \pi \right) \right), \\
  w_{12} := r_m \exp \left( \frac{3}{4} \pi i \right) &= \left( r_m \cos \left( \frac{3}{4} \pi \right), r_m \sin \left( \frac{3}{4} \pi \right) \right), \\
  w_{13} := r_{11} \exp \left( \frac{3}{4} \pi i \right) &= r_1 \exp \left( \frac{3}{4} \pi i \right) = \left( r_1 \cos \left( \frac{3}{4} \pi \right), r_1 \sin \left( \frac{3}{4} \pi \right) \right), \\
  w_{14} := r_{11} \exp \left( \frac{1}{4} \pi i \right) &= \left( r_1 \cos \left( \frac{1}{4} \pi \right), r_1 \sin \left( \frac{1}{4} \pi \right) \right), \\
  w_{15} := r_1 \exp \left( -\frac{1}{4} \pi i \right) &= \left( r_1 \cos \left( -\frac{1}{4} \pi \right), r_1 \sin \left( -\frac{1}{4} \pi \right) \right),
\end{align*}
\]

Figure 2: Location of complex numbers (denoted at the bottom of the figure) and contours in the complex \( w \)-plane. Contour \( \tilde{C}_r \) comprises \( \tilde{C}_r \) (from \( \exp \left( \frac{1}{4} \pi i \right) \) to \( w_{11} \)), \( \tilde{C}_r \) (from \( w_{11} \) to \( w_{12} \)), \( \tilde{C}_r \) (from \( w_{12} \) to \( w_{13} \)), \( \tilde{C}_r \) (from \( w_{13} \) to \( w_{14} \)), \( \tilde{C}_m \) (from \( w_{15} \) to \( w_{16} \)), \( \tilde{C}_m \) (from \( w_{16} \) to \( w_{17} \)) and \( \tilde{C}_d \) (from \( w_{17} \) to \( \exp \left( \frac{4}{4} \pi i \right) \)).

large enough so that the following denominator, denoted as \( f^{(De)} \), in the integrands in Eqs. (8)-(10) is approximated by

\[
f^{(De)} = \exp(\pi i w) - \exp(-\pi i w) \approx \exp(\pi i w) \quad \text{or} \quad -\exp(-\pi i w).
\]

(The condition on the radius \( r_{1M} \) is described in detail later around Eqs. (30)-(34)).

Here, we define the complex numbers in the complex \( w \)-plane shown in Figs.[1] and[2]

Definition 2. Let \( w \in \mathbb{C} \), and let \( w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17} \in \mathbb{C} \). Using the specific radii \( r_1 \) and \( r_{11} \) (of \( w \)) and setting \( r_1 = r_{11} \) in Definition[1] we define the above complex numbers, whose locations in the complex \( w \)-plane are shown in Fig.[1] by
\[ w_{10} := r_m \exp\left(-\frac{1}{4} \pi i\right) = (r_m \cos\left(-\frac{1}{4} \pi\right), r_m \sin\left(-\frac{1}{4} \pi\right)), \quad w_{11} := r_M \exp\left(-\frac{1}{4} \pi i\right) = (r_M \cos\left(-\frac{1}{4} \pi\right), r_M \sin\left(-\frac{1}{4} \pi\right)). \]

(14)

Similarly, let \( w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17} \in \mathbb{C} \). Using the specific radii (of \( w \)) \( r_1, r_{1r} \) and setting \( r_1 = r_{1r} \) in Definition \(^7\) we define the above complex numbers, whose locations in the complex \( w \)-plane are shown in Fig. \( 2 \) by

\[ w_{11} := r_M \exp\left(\frac{1}{4} \pi i\right) = (r_M \cos\left(\frac{1}{4} \pi\right), r_M \sin\left(\frac{1}{4} \pi\right)), \quad w_{12} := r_m \exp\left(\frac{1}{4} \pi i\right) = (r_m \cos\left(\frac{1}{4} \pi\right), r_m \sin\left(\frac{1}{4} \pi\right)), \]

\[ w_{13} := r_{1r} \exp\left(\frac{1}{4} \pi i\right) = r_1 \exp\left(\frac{1}{4} \pi i\right) = (r_1 \cos\left(\frac{1}{4} \pi\right), r_1 \sin\left(\frac{1}{4} \pi\right)), \quad w_{14} := (r_1 \cos\left(-\frac{1}{4} \pi\right), r_1 \sin\left(-\frac{1}{4} \pi\right)), \]

\[ w_{15} := r_1 \exp\left(-\frac{3}{4} \pi i\right) = (r_1 \cos\left(-\frac{3}{4} \pi\right), r_1 \sin\left(-\frac{3}{4} \pi\right)), \]

\[ w_{16} := r_m \exp\left(-\frac{3}{4} \pi i\right) = (r_m \cos\left(-\frac{3}{4} \pi\right), r_m \sin\left(-\frac{3}{4} \pi\right)), \quad w_{17} := r_M \exp\left(-\frac{3}{4} \pi i\right) = (r_M \cos\left(-\frac{3}{4} \pi\right), r_M \sin\left(-\frac{3}{4} \pi\right)). \]

(15)

We now define the deformed and shifted contours of the integrals in the completed zeta function.

**Definition 3.** Using the complex numbers \( w_{11}-w_{17} \) in Definition \(^2\) (points in the complex \( w \)-plane), the contours in Fig. \( 1 \) are defined as follows:

- \( \tilde{C}_1 \): contour composed of the contours from \( C_{11} \) to \( C_{14} \) (\( \tilde{C}_{11}, \tilde{C}_{12}, \tilde{C}_{1p}, \tilde{C}_{1c}, \tilde{C}_{1n}, \tilde{C}_{13}, \tilde{C}_{14} \)).
- \( \tilde{C}_{11} \): straight-line contour from \( \exp(\frac{3}{4} \pi i) \) to \( w_{11} \) (with radius \( r_M \)) in the direction of the arrow;
- \( \tilde{C}_{12} \): straight-line contour from \( w_{11} \) (with radius \( r_M \)) to \( w_{12} \) (with radius \( r_m \)),
- \( \tilde{C}_{1p} \): straight-line contour from \( w_{12} \) (with radius \( r_m \)) to \( w_{13} \) (with radius \( r_1 \)),
- \( \tilde{C}_{1c} \): arc (of circle) contour from \( w_{13} \) to \( w_{15} \) via \( w_{14} \) (in the direction of the arrow) centered at the (coordinate) origin \((0,0)\) with the radius \( r_1 \),
- \( \tilde{C}_{1n} \): straight-line contour from \( w_{15} \) (with radius \( r_1 \)) to \( w_{16} \) (with radius \( r_m \)) in the direction of the arrow;
- \( \tilde{C}_{13} \): straight-line contour from \( w_{16} \) (with radius \( r_m \)) to \( w_{17} \) (with radius \( r_M \)),
- \( \tilde{C}_{14} \) straight-line contour from \( w_{17} \) (with radius \( r_{1r} \)) to \( \exp(\frac{1}{4} \pi i) \infty \).

Similarly, using the complex numbers \( w_{11}-w_{17} \) in Definition \(^2\) (points in the complex \( w \)-plane), contours in Fig. \( 2 \) are defined as follows:

- \( \tilde{C}_1 \): contour composed of the contours from \( \tilde{C}_{11} \) to \( \tilde{C}_{14} \) (\( \tilde{C}_{11}, \tilde{C}_{12}, \tilde{C}_{1p}, \tilde{C}_{1c}, \tilde{C}_{1n}, \tilde{C}_{13}, \tilde{C}_{14} \)),
- \( \tilde{C}_{11} \): straight-line contour from \( \exp(\frac{3}{4} \pi i) \infty \) to \( w_{11} \) (with radius \( r_M \)) in the direction of the arrow;
- \( \tilde{C}_{12} \): straight-line contour from \( w_{11} \) (with radius \( r_M \)) to \( w_{12} \) (with radius \( r_m \)),
- \( \tilde{C}_{1p} \): straight-line contour from \( w_{12} \) (with radius \( r_m \)) to \( w_{13} \) (with radius \( r_1 \)),
- \( \tilde{C}_{1c} \): arc (of circle) contour from \( w_{13} \) to \( w_{15} \) via \( w_{14} \) (in each direction of the arrow) centered at the (coordinate) origin \((0,0)\) with the radius \( r_1 \),

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• $\mathcal{C}_m$: straight-line contour from $w_{r5}$ (with radius $r_1$) to $w_{r6}$ (with radius $r_m$) in the direction of the arrow.

• $\tilde{\mathcal{C}}_{r3}$: straight-line contour from $w_{r6}$ (with radius $r_m$) to $w_{r7}$ (with radius $r_M$).

• $\tilde{\mathcal{C}}_{r4}$: straight-line contour from $w_{r7}$ (with radius $r_m$) to $\exp(\frac{1}{2}\pi i)$.

Here, we show that it is possible to deform and shift the contours in Eqs. (8)-(10) to the contours in Figs. 1 and 2.

**Lemma 1.** Let $a_0, a_0, a_0 \in \mathbb{R}$ be positive finite numbers between 0 and 1. Let $0 < \gamma_{\text{R}}$ be the contour (with the slope -1), which was used in Eqs. (5), (8)-(10) and intersects the real axis (in the complex plane) at $(a_0, 0)$, with $a_0 = a_0$, whereas let $0 > \gamma_{\text{L}}$ be the contour (with the slope +1) which intersects the real axis at $(a_0, 0)$ with $a_0 = a_0$. The contour $0 < \gamma_{\text{R}}$ can be deformed and shifted to the contour $\tilde{\mathcal{C}}_{\gamma}$ in Definition 3 with Fig. 2 whereas the contour $0 > \gamma_{\text{L}}$ can be deformed and shifted to the contour $\tilde{\mathcal{C}}_{\gamma}$ in Definition 3 with Fig. 2.

**Proof.** Since the integral form of the completed zeta function in Eqs. (5) and (8)-(10) is derived from the residue theorem, the contour $0 < \gamma_{\text{R}}$ can be deformed and shifted to the contour $\tilde{\mathcal{C}}_{\gamma}$, while the contour $0 > \gamma_{\text{L}}$ can be deformed and shifted to the contour $\tilde{\mathcal{C}}_{\gamma}$.

Using Definitions 1-3 and Lemma 1, we prove the following lemma, which shows that the integrals in Eqs. (8)-(10) along the contours for the regions with large distance to the (coordinate) origin are finite.

**Lemma 2.** Let $w, v, z \in \mathbb{C}$ (with $v = -z, z - 1$), and let $z_{\text{R}} = \text{Re}(z)$ with $0 < z_{\text{R}} < 1$. Let $r_{\text{M}} \in \mathbb{R}$ be the large (lower bound of) radius (in Definitions 1-2) of $w$ along the shifted straight-line contours. Let $\tilde{\mathcal{C}}_{\text{rh}}$ be the contour, which is either of the contours denoted by $\tilde{\mathcal{C}}_{11}$ and $\tilde{\mathcal{C}}_{14}$ (in Fig. 1), whereas $\tilde{\mathcal{C}}_{r4}$ be either of the contours $\tilde{\mathcal{C}}_{11}$ and $\tilde{\mathcal{C}}_{14}$ (in Fig. 2). Then, the following integrals of the integrands in Eqs. (8)-(10)

$$I_{\tilde{\mathcal{C}}_{\text{rh}}}^v = \int_{\tilde{\mathcal{C}}_{\text{rh}}} dw w^{-z} \exp(-\pi i w^2)$$

$$I_{\tilde{\mathcal{C}}_{r4}}^v = \int_{\tilde{\mathcal{C}}_{r4}} dw w^{-z} \exp(+\pi i w^2)$$

along the contours $\tilde{\mathcal{C}}_{\text{rh}} = \tilde{\mathcal{C}}_{11}, \tilde{\mathcal{C}}_{14}$ (in Fig. 1) and $\tilde{\mathcal{C}}_{r4} = \tilde{\mathcal{C}}_{11}, \tilde{\mathcal{C}}_{14}$ (in Fig. 2),

are finite (negligible compared with those with singularities around the origin of coordinates).

**Proof.** Letting $v = -z, z - 1$, the polynomial $I_{\tilde{\mathcal{C}}_{\text{rh}}}^v$ in the numerators of the integrands in (above) Eq. (16) denoted by

$$I_{\tilde{\mathcal{C}}_{\text{rh}}}^v = w^v \quad \text{(with } v = -z, z - 1\text{),}$$

is rewritten (with $v_{\text{R}} = \text{Re}(v), v_{\text{R}} = \text{Im}(v)$) as

$$I_{\tilde{\mathcal{C}}_{\text{rh}}}^v = w^{v_{\text{R}}} \exp(\text{i}v_{\text{R}}) = w^{v_{\text{R}}} \exp\{\text{i}v_{\text{R}}\ln(w)\} = w^{v_{\text{R}}} \exp\{\text{i}v_{\text{R}}\ln(|w|)\}$$

$$= w^{v_{\text{R}}} \exp\{i\text{arg}(w)\} = w^{v_{\text{R}}} \exp\{i\text{arg}(w)\} = w^{v_{\text{R}}} \exp\{i\text{arg}(w)\} = w^{v_{\text{R}}} \exp\{i\text{arg}(w)\} = w^{v_{\text{R}}} \exp\{i\text{arg}(w)\},$$

where $\text{arg}(w)$ is argument (angle of $w$ measured clockwise from the real axis in the complex $w$-plane), which is restricted to the principal value between $-\pi$ and $+\pi$. Letting $\theta \in \mathbb{R}$ be the angle of $w$ (that is, $\theta = \text{arg}(w)$) along the straight-line contour, then

$$\theta = \frac{3}{4}\pi \quad \text{for contour } \tilde{\mathcal{C}}_{11}, \quad \theta = -\frac{1}{4}\pi \quad \text{for contour } \tilde{\mathcal{C}}_{14},$$

$$\theta = \frac{1}{4}\pi \quad \text{for contour } \tilde{\mathcal{C}}_{11}, \quad \theta = -\frac{3}{4}\pi \quad \text{for contour } \tilde{\mathcal{C}}_{14}.$$

Using the above angle, the variable $w$ is expressed by

$$w = |w| \exp(i\theta) \quad \text{with } \theta = \text{arg}(w),$$

where
where \(|w|\) is the radius (modulus) and \(\theta\) is the angle (argument). Then, from Eqs. (18)-(21), we have

\[
I_{v}^{(Po)} = |w|^{|}R \exp(ivR \theta) \exp[iv1 \ln(|w|)] \exp[-v1 \arg(w)] = |w|^{|}R \exp(ivR \theta) \exp[iv1 \ln(|w|)] \exp(-v1 \theta).
\] (22)

The absolute value of \(I_{v}^{(Po)}\) in (above) Eq. (22) is

\[
|I_{v}^{(Po)}| = |w|^{|}R \exp(-v1 \theta).
\] (23)

Moreover, let \(I^{(Nu)-}\) and \(I^{(Nu)+}\) be the parts of the numerators in the integrands in Eq. (15) written by

\[
I^{(Nu)-} = \exp(-\pi iw^2), \quad I^{(Nu)+} = \exp(+\pi iw^2).
\] (24)

Using Eqs. (19)-(21), it follows that

\[
I^{(Nu)-} = \exp[-\pi |w|^2 (\cos 2\theta + i \sin 2\theta)] \quad \text{with} \quad \theta = \frac{3}{4} \pi, \frac{-1}{4} \pi,
\] (25)

\[
I^{(Nu)+} = \exp[\pi |w|^2 (\cos 2\theta + i \sin 2\theta)] \quad \text{with} \quad \theta = \frac{1}{4} \pi, \frac{-3}{4} \pi.
\] (26)

We then have

\[
|I^{(Nu)-}| = \exp(\pi |w|^2 \sin 2\theta) \quad \text{for} \quad \theta = \frac{3}{4} \pi, \frac{-1}{4} \pi,
\] (27)

\[
|I^{(Nu)+}| = \exp(-\pi |w|^2 \sin 2\theta) \quad \text{for} \quad \theta = \frac{1}{4} \pi, \frac{-3}{4} \pi.
\] (28)

Therefore, (above) Eqs. (27), (28) are reduced to

\[
|I^{(Nu)+}| = \exp(-\pi |w|^2 \sin 2\theta) \quad \text{with} \quad \theta = \frac{\pm 1}{4} \pi, \frac{\pm 3}{4} \pi.
\] (29)

In contrast, by using Eqs. (19)-(21) for the following denominator \(I^{(De)}\) in Eq. (16)

\[
I^{(De)} = \exp(i\pi w) - \exp(-i\pi w),
\] (30)

we get

\[
I^{(De)} = \exp[i\pi |w|(\cos \theta + i \sin \theta)] - \exp[-i\pi |w|(\cos \theta + i \sin \theta)]
\]

\[
= \exp(i\pi |w| \cos \theta) \exp(-i\pi |w| \sin \theta) - \exp(-i\pi |w| \cos \theta) \exp(i\pi |w| \sin \theta).
\] (31)

By the definition of \(r_M\) (in Eq. (13) for Definition (1)), the denominator \(I^{(De)}\) in (above) Eq. (31) for large \(|w|\) is approximated by

\[
I^{(De)} \approx -\exp(-i\pi |w| \cos \theta) \exp(i\pi |w| \sin \theta) \quad \text{for large} \quad |w| \quad \text{with} \quad (|w| \geq r_M) \quad \text{and} \quad \sin \theta > 0 \quad (\theta = \frac{3}{4} \pi, \frac{1}{4} \pi),
\] (32)

whereas

\[
I^{(De)} \approx \exp(i\pi |w| \cos \theta) \exp(-i\pi |w| \sin \theta) \quad \text{for large} \quad |w| \quad \text{with} \quad (|w| \geq r_M) \quad \text{and} \quad \sin \theta < 0 \quad (\theta = \frac{-1}{4} \pi, \frac{-3}{4} \pi).
\] (33)
Additionally, we denote the sign factor $\sigma$.

This implies that the above integrals are independent of the arc radius.

Then, (above) Eqs. (32)-(33) are reduced to

$$|I^{(De)}| \approx \exp(\pi|w||\sin \theta|) \quad \text{for large } |w| \text{ with } (|w| \geq r_M) \text{ and } \theta = \pm \frac{1}{4}\pi, \pm \frac{3}{4}\pi. \quad (34)$$

Accordingly, combining Eqs. (22), (29) and (34), the absolute value of the integrands in Eq. (16) is

$$I_h := \frac{|I^{(P)}||I^{(No)}|}{|I^{(De)}|} = |w|^{i\pi} \exp(-v_1\theta) \exp(-\pi|w|^2|\sin 2\theta|) \exp(-\pi|w||\sin \theta|). \quad (35)$$

Then, (above) Eq. (35), for large $|w|$, is approximated by

$$I_h \leq |w|^{i\pi} \exp(-v_1\theta) \exp(-\pi r_M^2|\sin 2\theta|) \exp(-\pi|w||\sin \theta|)$$

$$\leq |w|^{i\pi} \exp(-v_1\theta) \exp(-\pi|w||\sin \theta|)$$

$$\approx |w|^{i\pi} \exp(-\pi|w||\sin \theta|) \quad \text{for large } |w| \geq r_M, \quad (36)$$

where, in the (above) last equation, the constant $\exp(-v_1\theta)$ were disregarded. Using Eq. (21) for the straight-line contour, we have

$$dw = d|w| \exp(i\theta) \quad \text{with } |\exp(i\theta)| = 1. \quad (37)$$

Additionally, we denote the sign factor $\sigma \in \mathbb{N}$ due to the direction of integration by

$$\sigma := \begin{cases} -1 \text{ with } |\sigma| = 1 & \text{for contours such as } (\tilde{C}_{11}, \tilde{C}_{14}) \text{ oriented to the (coordinate) origin} \\ +1 & \text{for contours such as } (C_{17}, C_{17}) \text{ oriented in the exp}(\pm \frac{1}{4}\pi i)\& \exp(\pm \frac{3}{4}\pi i)\& \text{direction} \end{cases}. \quad (38)$$

Using Eqs. (36), (38) (taking into account that $-1 < v_R = \Re(v) = -z_R, z_R - 1 < 0$ for $v = -\frac{1}{2}, \frac{1}{2}$), the integrals of $I_h$ (in Eq. (16) over the region $|w| \geq r_M$ lead to

$$|I^S_{\tilde{C}_{1h}}| \text{ and } |I^S_{\tilde{C}_{1h}}| \leq |\sigma||\int_{r_M}^{\infty} \int_{r_M}^{\infty} d|w||I_h|$$

$$\leq \int_{r_M}^{\infty} d|w| |r_M^{i\pi} \exp(-\pi|w||\sin \theta|)| < r_M^{i\pi} \int_{0}^{\infty} d|w| |\exp(-\pi|w||\sin \theta|)|$$

$$= r_M^{i\pi} \frac{1}{(\pi|\sin \theta|)} \quad \text{with } v_R = -z_R, z_R - 1 \text{ and } \theta = \pm \frac{1}{4}\pi, \pm \frac{3}{4}\pi. \quad (39)$$

In the last integral, we used the Laplace transform $\frac{1}{\pi|\sin \theta|}$. Thus, the integral in (above) Eq. (39) is finite. Namely, using Eqs. (16), (19), (20) and (39), we derive

$$I^S_{\tilde{C}_{1h}} = \text{finite value (integral along either of contours } \tilde{C}_{11}, \tilde{C}_{14}), \quad (40)$$

$$I^S_{\tilde{C}_{1h}} = \text{finite value (integral along either of contours } \tilde{C}_{11}, \tilde{C}_{14}). \quad (41)$$

This implies that the above integrals are independent of the arc radius $r_1$ (in Definitions 1, 2) and negligible compared with those with singularities (in Lemmas 4, 5) around the (coordinate) origin in the limit $r_1 \to \infty$. \hfill \Box

We now prove a lemma which shows that when the contours (in Figs. 1, 2) are in the region with intermediate distance to the origin, the integrals in the completed zeta function are finite as well.
Lemma 3. Similarly with Lemma 2 let $w, v, z \in \mathbb{C}$ (with $v = -z, z - 1$), and let $z_R = \text{Re}(z)$ with $0 < z_R < 1$. Let $r_m$ and $r_M \in \mathbb{R}$ (with $r_m < r_M$) be the small and large radii (bounds of contours as in Definitions 1, 2) of $w$ along the (shifted straight-line) contours $\tilde{C}_{lm}$ and $\tilde{C}_{m}$, where $\tilde{C}_{lm}$ is either of the contours denoted by $\tilde{C}_{12}$ and $\tilde{C}_{13}$ (in Fig. 1), while $\tilde{C}_{m}$ is either of the contours denoted by $\tilde{C}_{2}$ and $\tilde{C}_{3}$ (in Fig. 2). Then, the following integrals of the integrands in Eqs. (43)-(47)

$$I^S_{\tilde{C}_{lm}} = \int_{\tilde{C}_{lm}} dw \frac{w^{-z} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad I^S_{\tilde{C}_{m}} = \int_{\tilde{C}_{m}} dw \frac{w^{-1} \exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}$$

along the contours $\tilde{C}_{lm} = \tilde{C}_{12}, \tilde{C}_{13}$ (in Fig. 1) and $\tilde{C}_{m} = \tilde{C}_{2}, \tilde{C}_{3}$ (in Fig. 2), (42)

are finite (negligible compared with those with singularities around the origin of coordinates).

Proof. The denominator $I^{(De)}$ of the integrands in Eq. (42) is rewritten as

$$I^{(De)} = \exp(\pi i w) - \exp(-\pi i w) = \exp(\pi i w)(1 - \exp(-2\pi i w)].$$

(43)

We further denote the parts of the above denominator (in Eq. (43)) by

$$I^{(De)a} = \exp(\pi i w),$$

(44)

$$I^{(De)b} = 1 - \exp(-2\pi i w).$$

(45)

Here, let $\theta \in \mathbb{R}$ be the angle (argument measured counterclockwise from the real axis in the complex $w$-plane), then

$$\theta = \frac{3}{4}\pi \quad \text{for contour } \tilde{C}_{12}, \quad \theta = -\frac{1}{4}\pi \quad \text{for contour } \tilde{C}_{13},$$

(46)

$$\theta = \frac{1}{4}\pi \quad \text{for contour } \tilde{C}_{2}, \quad \theta = -\frac{3}{4}\pi \quad \text{for contour } \tilde{C}_{3}.$$ (47)

Using $w = |w| (\cos \theta + i \sin \theta)$ (in Eq. (21)) and Eq. (44), it follows that

$$I^{(De)a} = \exp(\pi i |w|(\cos \theta + i \sin \theta))$$

$$= \exp(\pi i |w| \cos \theta) \exp(-\pi |w| \sin \theta),$$

(48)

yielding

$$|I^{(De)a}| = \exp(-\pi |w| \sin \theta).$$

(49)

Meanwhile, from Eq. (45) (with Eq. (21)), we derive

$$I^{(De)b} = 1 - \exp(-2\pi i |w| \cos \theta) \exp(2\pi |w| \sin \theta).$$

(50)

For $\sin \theta > 0$ and $r_m \leq |w| \leq r_M$ ($r_m$ and $r_M$ are the radii defined in Definitions 1, 2 with Figs. 1, 2 for the contours in Eq. (42)), the following quantity in the second term on the right-hand side of above Eq. (50) is larger than unity (one), that is,

$$\exp(2\pi |w| \sin \theta) \geq \exp(2\pi r_m \sin \theta) > 1 \quad \text{for } \sin \theta > 0.$$ (51)

Furthermore, the second term on the right-hand side of Eq. (50) is a complex number with radius (modulus) denoted as $\exp(2\pi |w| \sin \theta)$ and angle (argument) $-2\pi |w| \cos \theta$, whose distance to the point 1=(1,0) is equal to $|I^{(De)b}|$. This
distance $|I^{(De)b}|$ is larger than the difference between the above radius $\exp(2\pi|w|\sin \theta)$ and the radius of the unit circle (centered at the origin of coordinates), namely,

$$|I^{(De)b}| = |1 - \exp(-2\pi i|w| \cos \theta) \exp(2\pi|w|\sin \theta)| \geq \exp(2\pi|w|\sin \theta) - 1 > 0 \quad \text{for} \quad \sin \theta > 0.$$  \hfill (52)

Combining Eqs. (51) and (52), we have (taking into account that $r_m \leq |w| \leq r_M$)

$$|I^{(De)b}| \geq \exp(2\pi|w|\sin \theta) - 1 \geq \exp(2\pi r_m \sin \theta) - 1 > 0 \quad \text{for} \quad \sin \theta > 0.$$  \hfill (53)

Similarly, for $\sin \theta < 0$ and $r_m \leq |w| \leq r_M$, we obtain the following relation, corresponding to Eq. (51),

$$1 > \exp(2\pi|w|\sin \theta) \geq \exp(2\pi r_m \sin \theta) > 0 \quad \text{for} \quad \sin \theta < 0.$$  \hfill (54)

The distance $|I^{(De)b}|$ between the second term on the right in Eq. (50) and the point 1=(1,0) in this case is larger than the difference between the aforementioned radius (modulus) $\exp(2\pi|w|\sin \theta)$ and the radius of the unit circle (centered at the origin of coordinates). We then have (considering $r_m \leq |w| \leq r_M$) that

$$|I^{(De)b}| \geq 1 - \exp(2\pi|w|\sin \theta) \geq 1 - \exp(2\pi r_m \sin \theta) > 0 \quad \text{for} \quad \sin \theta < 0.$$  \hfill (55)

In contrast, using the notation $v = -(z/z - 1)$, the parts of the numerators of the integrands in Eq. (42) can be written as

$$I_{v}^{(Po)} = w^v \quad \text{with} \quad v = -(z/z - 1),$$

$$|I_{v}^{(Nu)}| = \exp(-\pi|w|^2).$$ \hfill (56)

By denoting $w = |w|(\cos \theta + i \sin \theta)$, we obtain the same results as those in Eqs. (23) and (29) (in Lemma 3). Namely,

$$|I_{v}^{(Po)}| = |w|^v \exp(-v_1 \theta),$$

$$|I_{v}^{(Nu)}| = |w|^v \exp(-v_1 \theta), \quad \text{with} \quad \theta = \pm \frac{1}{4} \pi, \pm \frac{3}{4} \pi.$$  \hfill (59)

Therefore, combining Eq. (53) (or Eq. (55)) and Eqs. (58)-(59), we obtain that the absolute value of the integrands is

$$I_m := \frac{|I_{v}^{(Po)}||I_{v}^{(Nu)}|}{|I^{(De)b}|}.$$  \hfill (60)

Using Eqs. (53) (or Eq. (55)) and Eqs. (58)-(60) (with consideration that $-1 < v_R = \Re(v) = -z_R, z_R - 1 < 0$ for $v = -(z/z - 1)$, as well as $|\sigma \exp(i\theta)| = 1$ in Eqs. (47)-(59), we obtain

$$|I_{C_{im}}^{S} \leq \int_{r_m}^{\infty} d|w|I_m|w|$$

$$\leq (r_M - r_m) \frac{r_m^{\nu} \exp(-v_1 \theta) \exp(-2\pi v_1 |\sin \theta|)}{|\exp(2\pi r_m \sin \theta) - 1|} \quad \text{with} \quad \theta = \pm \frac{1}{4} \pi, \theta = \pm \frac{3}{4} \pi.$$ \hfill (61)

Hence, the above straight-line integrals (in Eq. (61) of $I_m$ (in Eq. (60)) with respect to $|w|$ in the region $r_m \leq |w| \leq w_M$ are smaller than the terms proportional to $r_m^{\nu}$ (disregarding the multiplied constants) with $-1 < v_R < 0$, and take finite values. This finiteness is due to the large value of the radius $r_m$, which is independent of the radius $r_1$ of the arc contours (in Figs. 1, 2) around the (coordinate) origin with $r_m > r_1$ (as in Definitions 1, 2). Therefore, singularities do not occur here unlike the case of integrals (in Lemmas 4, 5) around the (coordinate) origin in the limit of $r_1 \to 0$. Namely,

$$I_{C_{im}}^{S} = \text{finite value (integral along either of contours } \tilde{C}_{12}, \tilde{C}_{13} \text{ in Fig. 1)},$$

$$I_{C_{im}}^{S} = \text{finite value (integral along either of contours } \tilde{C}_{12}, \tilde{C}_{13} \text{ in Fig. 2).}$$ \hfill (62, 63)

This implies that the above integrals are negligible compared with those with singularities around the (coordinate) origin.
In this section, we deformed and shifted the contours denoted by $0 \searrow 1$ and $0 \nearrow 1$ in the integral form of the (completed) zeta function given by Eqs. (8)-(10) to those shown in Fig. 1 and Fig. 2, respectively (in Theorem 1). Then, we separated the finite integrals (in Lemmas 2, 3) along the shifted straight-line contours from the integral around the (coordinate) origin containing the singularities, which appear only when the contours approach the origin.

3 Proof of the Riemann’s conjecture that the real part of all non-trivial zeros of the zeta function is 1/2

In previous Section 2 (with Lemmas 2, 3), it was shown that integrals of the integrands in $\zeta_i$ and $\zeta_e$ (in Eqs. (8)-(9)) for the integral form of the completed zeta function in Eq. (10), along the shifted straight-line contours (in Figs. 1, 2), which are away from the (coordinate) origin (in the complex plane), are always finite and do not have singularities. (Note: Using Theorem 1 and Eq. (11), the complex numbers $2 \approx 1$ in the complex plane, are always finite and do not have singularities. For the integral form of the completed zeta function in Eq. (10)), along the straight-line contours $\tilde{C}_1$, $\tilde{C}_2$ along the contours $\tilde{C}_3$, $\tilde{C}_4$, $\tilde{C}_5$, $\tilde{C}_6$, $\tilde{C}_7$, $\tilde{C}_8$, $\tilde{C}_9$, $\tilde{C}_{10}$ (in Figs. 1, 2), the denominator $\zeta(z)$ of $z$ along the (shifted straight-line) contours $\tilde{C}_1$, $\tilde{C}_2$ have singularities (in Lemmas 4, 5) when the radius of the arc contours in Eq. (4) were exchanged as shown in Eqs. (8)-(10)). In this section, we show that the integrals along the contours near the (coordinate) origin (in Figs. 1, 2) have singularities (in Lemmas 4, 5) when the radius of the arc contours approaches zero. Then, in Theorem 2, we prove that the real part of all non-trivial zeros of the zeta function must be 1/2. As it is known that all non-trivial zeros of the zeta function exist in the region $0 < \Re(z) < 1$ in literature [18, 19], we concentrate on this region. Furthermore, it is also known that the number of zeros (of the zeta function) with a real part of 1/2 is infinite [10]. To derive the real part of non-trivial zeros of the zeta function, the present approach uses (in addition to the above symmetry given by Eq. (11)) the property (with merits) that a quantity in one term in a highly (attainable) symmetrized integral form generates a corresponding (paired) quantity in another term.

Similarly with Lemmas 2, 3 we here evaluate the integrals of the form in Eqs. (8)-(10), and separate singularities.

Lemma 4. Let $w, v, z \in \mathbb{C}$ (with $v = -z, z - 1$), and let $z_R = \Re(z)$ with $0 < z_R < 1$. Let $r_1$ and $r_m \in \mathbb{R}$ with $r_1 << r_m$ be the small radii (bounds of contours as in Definitions 7, 8) of $w$ along the (shifted straight-line) contours $\tilde{C}_{ls}$ and $\tilde{C}_{rs}$, where $\tilde{C}_{ls}$ is either of the contours denoted by $\tilde{C}_{lp}, \tilde{C}_{ln}$ in Fig. 1, while $\tilde{C}_{rs}$ is either of the contours denoted by $\tilde{C}_{rp}, \tilde{C}_{rm}$ in Fig. 2. Then, the following integrals of the integrands in Eqs. (8)-(10)

$$I_{\tilde{C}_{ls}}^S = \int_{\tilde{C}_{ls}} dw \frac{w^{-c} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad I_{\tilde{C}_{rs}}^S = \int_{\tilde{C}_{rs}} dw \frac{w^{-c} \exp(+\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}$$

have singularities in the limit of $r_1 \to 0$. The power of singularities of these integrals is $r_1^{-2z_R}$ on the left in above Eq. (64) (for the contours $\tilde{C}_{ls} = \tilde{C}_{lp}, \tilde{C}_{ln}$, while the corresponding power is $r_1^{2z_R-1}$ on the right in above Eq. (64) (for the contours $\tilde{C}_{rs} = \tilde{C}_{rp}, \tilde{C}_{rm}$).

Proof. For $0 < r_1 < |w| < r_m$ and $r_1 << r_m$ with $r_1$ and $r_m$ being the small radii (bounds of contours as in Definitions 7, 8) along the straight-line contours $\tilde{C}_{ls}, \tilde{C}_{ls}, \tilde{C}_{rs}, \tilde{C}_{rs}$ (in Figs. 1, 2), the denominator $I_{(De)}$ and parts of the numerators, $I_{(Nu)}$ and $I_{(Nu)}$ in Eq. (64) are approximated by

$$I_{(De)} = \exp(\pi i w) - \exp(-\pi i w) \approx 2\pi i w; \quad (65)$$

$$I_{(Nu)} = \exp(-\pi i w^2) \approx 1, \quad I_{(Nu)} = \exp(+\pi i w^2) \approx 1. \quad (66)$$

The polynomials in above Eq. (64) can be written as

$$I_{(Po)} = w^v \text{ with } v = -z, z - 1. \quad (67)$$

Then, using Eqs. (18) and Eq. (21) with $\theta$ (angle along the straight-line contours measured counterclockwise from the real axis in the complex $w$-plane), above $I_{(Po)}$ in Eq. (67) can be expressed as

$$I_{(Po)} = w^{n_T} v^{\mu_T}$$
\[ = w^{|v_1| - 1} \exp[i v_1 \ln(|w|)] \exp[-v_1 \arg(w)] \]

\[ = |w|^{|v_1| - 1} \exp[i(v_R - 1)\theta] \exp[i v_1 \ln(|w|)] \exp(-v_1 \theta), \quad (68) \]

(with \( v_R = \Re(v), \ \nu_1 = \Im(v) \)). The angle \( \theta \) in this case is denoted by

\[ \theta = \frac{3}{4} \pi \quad \text{for contour} \ C_{lp}, \quad \theta = -\frac{1}{4} \pi \quad \text{for contour} \ C_{ln}, \quad (69) \]

\[ \theta = \frac{1}{4} \pi \quad \text{for contour} \ C_{rp}, \quad \theta = -\frac{3}{4} \pi \quad \text{for contour} \ C_m. \quad (70) \]

From Eqs. (65)-(66) and (68), we obtain (disregarding \( 2\pi i \) in Eq. (65) as well as the constants \( \exp[i(v_R - 1)\theta] \) and \( \exp(-v_1 \theta) \) in Eq. (68)) that

\[ I_s := \frac{\int_{(Po)} f(N) = \int_{(De)}}{\int_{(De)}} = |w|^{|v_1| - 1} \exp[i v_1 \ln(|w|)]. \quad (71) \]

Further disregarding \( \exp(i\theta) \) in Eq. (37) and the sign \( \sigma \) (in Eq. (38)) due to the direction of integration, we have, for the integrals in Eq. (64), that

\[ I_{\tilde{C}_{ls}} \approx I_{\tilde{C}_{rs}} = \int_{r_1}^{r_m} d|w||w|^{|v_1| - 1} \exp[i v_1 \ln(|w|)] \]

along the contours \( \tilde{C}_{ls} = \tilde{C}_{lp}, \tilde{C}_{ln} \) and \( \tilde{C}_{rs} = \tilde{C}_{rp}, \tilde{C}_m \) in the regions \( r_1 \leq |w| \leq r_m \) in Figs. 1 and 2. \quad (72)

Using the small (bounds of) radii \( r_1 \) and \( r_m \) of \( w \) (with \( 0 < r_1 < |w| < r_m \) and \( r_1 << r_m \) in Definitions 1 and 2 and Figs. 1 and 2), we introduce the parameter variables \( \tilde{r}, \tilde{r}_1 \) and \( \tilde{r}_m \) as follows:

\[ \tilde{r} := \ln(|w|), \quad \tilde{r}_1 := \ln(r_1), \quad \tilde{r}_m := \ln(r_m). \quad (73) \]

Then, we have

\[ |w| = \exp(\tilde{r}), \quad r_1 = \exp(\tilde{r}_1), \quad r_m = \exp(\tilde{r}_m), \quad (74) \]

yielding

\[ d|w| = d\tilde{r} |\exp(\tilde{r})|. \quad (75) \]

By using Eqs. (72) and (75), we get

\[ I_{\tilde{C}_{ls}} \approx I_{\tilde{C}_{rs}} = \int_{\tilde{r}_1}^{\tilde{r}_m} d\tilde{r} |\exp(\tilde{r})| \{\exp[(v_R - 1)\tilde{r}]\} |\exp(i v_1 \tilde{r})| \]

\[ = \int_{\tilde{r}_1}^{\tilde{r}_m} d\tilde{r} \exp[(1 + v_R - 1 + iv_1)\tilde{r}] = \int_{\tilde{r}_1}^{\tilde{r}_m} d\tilde{r} \exp[(v_R + iv_1)\tilde{r}] \]

\[ = \frac{\exp[(v_R + iv_1)\tilde{r}_m] - \exp[(v_R + iv_1)\tilde{r}_1]}{v_R + iv_1} \]

\[ = \frac{r_1^{v_R + iv_1} - \tilde{r}_1^{v_R + iv_1}}{v_R + iv_1}. \quad (76) \]
Lemma 5. Let \( w \in \mathbb{C} \) (with \( v = -z, z < -1 \)) have singularities when the radius of the arc approaches zero.

We now evaluate the circular integrals along the arc contours around the (coordinate) origin in Figs. 1, 2. Let \( r \) be the small radius (in Definitions 1, 2) of the above circular integrals \( I_{C_h} \approx I_{C_m} \approx I_{C_n} \approx I_{C_p} \) in Fig. 1, that is

\[ |I_{C_h}^S| \approx |I_{C_m}^S| = \frac{r_1^{vR}}{|v_R + iv_1|} \]  

yielding

\[ I_{C_h}^S \approx I_{C_m}^S = \frac{-r_1^{vR}}{v_R + iv_1} \]

By dropping the constant containing \( v_1 \) for the contours \( C_1 \) (in Fig. 2), we derive, in the limit of \( r_1 \to 0 \) (with \( v = -z, z < -1, v_R = \text{Re}(v) \), that

Thus, disregarding the constant \( |v_R + iv_1| \) in (above) Eq. (79), we derive, in the limit of \( r_1 \to 0 \) (with \( v = -z, z < -1, v_R = \text{Re}(v) \), that

\[ |I_{C_h}^S| \approx r_1^{-vR} \]  

implying that the power of singularities of these integrals is \( r_1^{-vR} \) on the left-hand side in Eq. (64), whereas the corresponding power is \( r_1^{-vR} \) on the right-hand side in Eq. (64).

Proof. For the small radius \( r_1 \) (in Definition 1) of the deformed-arc contours \( C_{lc} \) in Figs. 1, 2, the denominator \( I^{(\text{De})} \) and parts of the numerators, \( I^{(\text{Nu})-} \) and \( I^{(\text{Nu})+} \) in Eq. (82) are approximated by

\[ I^{(\text{De})} = \exp(\pi iv) - \exp(-\pi iv) \approx 2\pi iv, \]

\[ I^{(\text{Nu})-} = \exp(-\pi iv^2) \approx 1, \quad I^{(\text{Nu})+} = \exp(\pi iv^2) \approx 1. \]

The polynomials in above Eq. (82) can be written as

\[ I^{(\text{Po})} = w^v \text{ with } v = -z, z < -1. \]
From Eqs. (83)-(85), we obtain (disregarding $2\pi i$ in Eq. (83))
\[
I_{c,v} := \frac{f^{(p)}(Na)}{f(D_{C})} = w^{v-1} \text{ with } v = -z, z - 1.
\] (86)

Using (above) Eq. (86), the integral along the arc contour $\hat{C}_{lc}$ (in Fig. 1) on the left-hand side in Eq. (82) can be expressed as
\[
I_{S}^{c} = \int_{\hat{C}_{lc}} dw I_{c,v = -z} \int_{\hat{C}_{lc}} dw (w^{-z-1}).
\] (87)

Let $\phi_{c}$ be the angle (argument) along the arc measured counterclockwise from the real axis in the complex plane. Using (above) Eq. (87), with consideration of $|w| = r_{1}$ on the contour $\hat{C}_{lc}$, and
\[
w = |w| \exp(i\phi_{c}) = r_{1} \exp(i\phi_{c}),
\] (88)
with
\[
\frac{dw}{d\phi_{c}} = ir_{1} \exp(i\phi_{c}) = iw,
\] (89)

we obtain (the integral along the arc contour $\hat{C}_{lc}$ in Fig. 1)
\[
I_{S}^{c} = \int_{\frac{i\pi}{2}}^{\frac{3\pi}{4}} d\phi_{c} \frac{dw}{w^{z-1}} = \int_{\frac{i\pi}{2}}^{\frac{3\pi}{4}} d\phi_{c} (iw)^{z-1}.
\] (90)

As in Eq. (18), the integrand of (above) Eq. (90) (with $z_{R} = \text{Re}(z)$ and $z_{I} = \text{Im}(z)$) can be written as
\[
iw^{z} = iw^{-z_{R} - iz_{I}}
\]
\[= iw^{-z_{R}} \exp[-iz_{I}\ln(|w|)] \exp[z_{I} \arg(w)].
\] (91)

Then, using Eqs. (88) and (91) with $\phi_{c} = \arg(w)$ (for $|w| = r_{1} > 0$ on the contour $C_{lc}$), the integral in Eq. (90) becomes
\[
I_{S}^{c} = \int_{\frac{i\pi}{2}}^{\frac{3\pi}{4}} d\phi_{c} (iw)^{z-1} \exp[-iz_{I}\ln(|w|)] \exp(z_{I}\phi_{c})
\]
\[= \int_{\frac{i\pi}{2}}^{\frac{3\pi}{4}} d\phi_{c} (iw)^{-z_{R}} \exp[-iz_{R}\phi_{c}] \exp[-iz_{I}\ln(|w|)] \exp(z_{I}\phi_{c})
\]
\[= i|w|^{-z_{R}} \exp[-iz_{I}\ln(|w|)] \frac{\exp(-iz_{R} + z_{I})(\frac{-i\pi}{4}) - \exp(-iz_{R} + z_{I})(\frac{3\pi}{4})}{-iz_{R} + z_{I}}
\]
\[= i|w|^{-z_{R}} \exp[-iz_{I}\ln(|w|)] \frac{\exp(-iz_{R} + z_{I})(\frac{-i\pi}{4})\{1 - \exp(-iz_{R} + z_{I})\pi\}}{-iz_{R} + z_{I}}.
\] (92)

Therefore, the absolute value of $I_{S}^{c}$ in (above) Eq. (92) (with $|w| = r_{1}$) is
\[
|I_{S}^{c}| = |w|^{-z_{R}} \frac{\exp\left(z_{I}\left(\frac{-i\pi}{4}\right)\pi\right)\{1 - \exp\left(-iz_{R} + z_{I}\pi\right)\pi\}}{-iz_{R} + z_{I}} \approx r_{1}^{-z_{R}},
\] (93)
which implies that, disregarding the constants \( \exp([z_1](-\pi/4)) \), \( 1 - \exp((-iz_R + z_1)\pi) \) and \(-iz_R + z_1\) (see also Note below Eq. (103)), the integral \( |I_{S_{rc}}^\delta| \) in (above) Eq. (93) has the form \( |w|^{-z_R} = r_1^{-z_R} \) with the singularity caused by the order power \(-z_R\) of \(|w|\) for \(0 < z_R < 1\) in the limit of \(r_1 \to 0\).

Meanwhile, the circular integral \( I_{rc}^S \) along the arc contour \( \tilde{C}_{rc} \) (in Fig. 1) on the right in Eq. (82) with small \(|w| = r_1\) near the origin in the complex \( w \)-plane is calculated by the replacement \( \tilde{C}_{rc} \rightarrow \tilde{C}_{rc} \), yielding (with the use of Eq. (95) and \(|w| = r_1\))

\[
\int_{C_{rc}} I_{c,v=\pi-1} = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} d\phi_c (i) w^{z_R - 1} \exp[i(z_R - 1)\phi_c] \exp[-z_1\phi_c]
\]

\[
= i|w|^{|z_R - 1|} \exp[i(z_R - 1)\ln(|w|)] \exp\left\{ i(z_R - 1) - z_1 \right\} \exp\left\{ 1 - \exp\left\{ i(z_R - 1) - z_1 \right\} \right\}
\]

\[
= i|w|^{|z_R - 1|} \exp[i(z_R - 1)\ln(|w|)] \exp\left\{ i(z_R - 1) - z_1 \right\} \exp\left\{ 1 - \exp\left\{ i(z_R - 1) - z_1 \right\} \right\}
\]

Therefore, the absolute value of \( I_{C_{rc}}^S \) in (above) Eq. (95) (with \(|w| = r_1\)) is

\[
|I_{C_{rc}}^S| = |w|^{|z_R - 1|} \left| \exp\left\{ -z_1\right\} \left\{ 1 - \exp\left\{ i(z_R - 1) - z_1 \right\} \right\} \right| \approx |r_1|^{|z_R - 1|},
\]

which implies that, disregarding the constants expressed by \( \exp\left\{ -z_1\right\} (-3\pi/4) \) and \( 1 - \exp\left\{ i(z_R - 1) - z_1 \right\} \) as well as \( i(z_R - 1) - z_1 \) (see also Note below Eq. (103)), the integral \( |I_{C_{rc}}^S| \) in (above) Eq. (96) has the form \( |w|^{|z_R - 1|} = r_1^{|z_R - 1|} \) with the singularity caused by the order power \(z_R - 1\) of \(|w|\) for \(0 < z_R < 1\) in the limit of \(r_1 \to 0\).

We then prove the following theorem, which completes the proof of the Riemann hypothesis.

**Theorem 2.** Let \( z \in \mathbb{C} \) and let \( z_R = \text{Re}(z) \). Let \( \zeta(z) \) be the completed zeta function given in Theorem 1. To satisfy \( \zeta'(z) = 0 \), the real component (real part) \( z_R \) of the non-trivial zeros of the (completed) zeta function must take the following value

\[
z_R = \frac{1}{2},
\]

which is a proof of the Riemann hypothesis.

**Proof.** Both the gamma function \( \Gamma(z) \) and zeta function \( \zeta(z) \) in the region \( 0 < \text{Re}(z) < 1 \) under consideration are regular without a singularity as described below Eq. (3). The completed zeta function \( \tilde{\zeta}(z) \) defined by Eq. (4), which is a product between \( \Gamma(z) \) and \( \zeta(z) \), is also regular in the region \( 0 < \text{Re}(z) < 1 \), as described below Eq. (5). As mentioned above in Section I (Introduction), the function \( \tilde{\zeta}(z) \) does not depend on a specific value of the parameter \( a_0 \) (between 0 and 1 as in Lemma 1), which specifies the intersection point of the integral line and the real axis, owing to the residue theorem. However, the integrands of the elements \( \hat{\zeta}(z) \) in Eq. (8) and \( \tilde{\zeta}_1(z) \) in Eq. (9) composing \( \tilde{\zeta}(z) \) in Eq. (10) (\( z \) and \((1 - z) \) can be exchanged) contain the singularity (mentioned below) near \( w = 0 \), only in the case of \( a_0 \to 0 \).
We note that the completed zeta function \( \hat{\zeta}(z) \) does not depend on a specific value of \( a_0 \) between 0 and 1 due to the residue theorem, as was described below Eq. (5), whereas the singularity of each element \( \hat{\zeta}(z) \) and \( \hat{\zeta}_c(z) \) of \( \hat{\zeta}(z) \) depends on \( a_0 \). However, these singularities and the dependence of the elements \( \hat{\zeta}(z) \) and \( \hat{\zeta}_c(z) \) on \( a_0 \) adequately (incompletely) cancel each other by remaining a finite value for \( \hat{\zeta}(z) \neq 0 \), because the integral directions projected to the line parallel to the real axis for \( \hat{\zeta}(z) \) and for \( \hat{\zeta}_c(z) \) are opposite, and result in the finite completed zeta function \( \hat{\zeta}(z) \) without the dependence on \( a_0 \).

In contrast, for \( \hat{\zeta}(z) = 0 \), the singularities must exactly cancel each other. By Lemmas 2 and 3 we can drop the negligible finite integrals along the contours, which are away from the (coordinate) origin. Let \( z_R \) and \( z_l \) be the real and imaginary components of \( z \), respectively. Lemma 4 states that, from Eq. (80) (refer also Note below Eq. (101)), the integrals \( I^S_{C_p} \) and \( I^S_{C_m} \) (in Eq. (64)) of the integrand (in Eq. (8)) for the completed zeta function \( \hat{\zeta}(z) \) (in Eq. (10)) have the following power, which lead to the singularity near \( w = 0 \) (on the arc radius \( r_1 \)) in the case of \( r_1 \to 0 \),

\[
|I^S_{C_p}| = \left| \int_{C_p} dw \frac{w^{-z} \exp(-\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{1-z_R} \quad \text{(along \( C_{lp} \) in Fig. 1)},
\]

\[
|I^S_{C_m}| = \left| \int_{C_m} dw \frac{w^{-z} \exp(-\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{1-z_R} \quad \text{(along \( C_{lm} \) in Fig. 1)},
\]

while, from Eq. (81), the integrals \( I^S_{C_{lp}} \) and \( I^S_{C_{lm}} \) (in Eq. (64)) of the integrand (in Eq. (9)) for \( \hat{\zeta}(z) \) (in Eq. (10)) have the following power (near \( w = 0 \) in the case of \( r_1 \to 0 \))

\[
|I^S_{C_{lp}}| = \left| \int_{C_{lp}} dw \frac{w^{-z} \exp(\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{z_R-1} \quad \text{(along \( C_{lp} \) in Fig. 2)},
\]

\[
|I^S_{C_{lm}}| = \left| \int_{C_{lm}} dw \frac{w^{-z} \exp(\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{z_R-1} \quad \text{(along \( C_{lm} \) in Fig. 2)},
\]

(We disregarded the constant factors in Eqs. (79)-(81)).

Meanwhile, Lemma 5 states that, from Eq. (93), the integral \( I^S_{C_{lc}} \) (in Eq. (82)) of the integrand (in Eq. (8)) has the following power (with \( z_R = \text{Re}(z) \)) near \( w = 0 \) in the case of \( r_1 \to 0 \)

\[
|I^S_{C_{lc}}| = \left| \int_{C_{lc}} dw \frac{w^{-z} \exp(-\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{-z_R} \quad \text{(along \( C_{lc} \) in Fig. 2)},
\]

while, from Eq. (96), the integral \( I^S_{C_{lc}} \) (in Eq. (82)) of the integrand (in Eq. (9)) has the following power (near \( w = 0 \) in the case of \( r_1 \to 0 \))

\[
|I^S_{C_{lc}}| = \left| \int_{C_{lc}} dw \frac{w^{-z} \exp(\pi iw^2)}{\exp(\pi iw) - \exp(-\pi iw)} \right| \approx |r_1|^{z_R-1} \quad \text{(along \( C_{lc} \) in Fig. 2)},
\]

(We also disregarded the constant factors in Eqs. (93)-(96)).

To satisfy \( \hat{\zeta}(z) = 0 \), these singularities in Eqs. (98)-(103) should have an identical order power of \( r_1 \) and exactly cancel each other. (Note: Letting \( \alpha_1, \alpha_2, w \in \mathbb{C} \) and \( \beta_1, \beta_2 \in \mathbb{R} \) with \( 0 < \beta_1 < \beta_2 \), if \( |w| < (|\alpha_2|/|\alpha_1|)|w|^{-\beta_1} \beta_2 \) for small \( |w| \), then we obtain \( |\alpha_1||w|^{-\beta_1} < |\alpha_2||w|^{-\beta_2} \), which implies that these two terms with different order powers cannot cancel each other for sufficiently small \( |w| \), as used below. Furthermore, constants including \( z_l \) will be used for the \( z_l \) determination, which is beyond the scope of this paper.) We then derive the main concluding relation, from Eqs. (93)-(103), that

\[
-z_R = z_R - 1,
\]

and this relation finally results in the expected requirement

\[
z_R = \frac{1}{2}.
\]
stating that all non-trivial zeros of the (completed) zeta function have real component (part) of 1/2, which is the proof of the Riemann hypothesis.

Namely, considering that $a_0$ (in Lemma 1) specifies the contour, $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall a_0 \in \mathbb{R} \text{ with } 0 < a_0 < 1)[a_0 < \delta \Rightarrow |z_R - \frac{1}{2}| < \varepsilon]$. Furthermore, the completed zeta function $\zeta(z)$ is a product between the gamma function $\Gamma(z)$ and zeta function $\zeta(z)$ as in Eq. (4), and the functions $\tilde{\zeta}(z), \Gamma(z)$ and $\zeta(z)$ are regular in the region $0 < \text{Re}(z) < 1$. Then, the solution of $\tilde{\zeta}(z) = 0$ (which is independent of the contour specified by $a_0$ unlike $\tilde{\zeta}(z)$ and $\tilde{\zeta}(z)$ composing $\zeta(z)$ in Eqs. (8)-(10)) satisfies $\zeta(z) = 0$ and vice versa. Thus, we have completed the proof of the Riemann hypothesis.

**Remark 1.** We here show the implication of the above process and derived solution. In the integrands of the elements $\tilde{\zeta}(z)$ and $\tilde{\zeta}(z)$ composing (in Eqs. (5), (9)) the completed zeta function $\tilde{\zeta}(z)$ (in Eq. (11)), the singularities appear in the oppositely directed integrals of polynomials. Furthermore, the completed zeta function is symmetrized with respect to $\text{Re}(z) = 1/2$. The functions $\tilde{\zeta}(z)$ and $\tilde{\zeta}(z)$ adequately (by incompletely remaining a finite value) cancel each other for $\text{Re}(z) \neq 1/2$, while this cancellation is complete only for $\text{Re}(z) = 1/2$, leading to $\tilde{\zeta}(z) = 0$.

In conclusion, we have inspected in detail the singularities of the integral form of the completed zeta function (in Eqs. (8)-(10)). For $\tilde{\zeta}(z) = 0$ (that is, $\zeta(z) = 0$), the singularities of the integral along the two rotated integral contours (lines) are required to exactly cancel each other, when the intersection points between the integral lines and the real axis approach the (coordinate) origin. This approach of the intersection points to the origin is possible because of the arbitrariness of the intersection points owing to the residue theorem. Thus, we have shown that the real part of all non-trivial zeros of the zeta function is 1/2, which is the proof of the Riemann hypothesis.

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