We propose and verify a wave-vector-space version of generalized extended self similarity \[\text{(ESS)}\] and broaden its applicability to uncover intriguing, universal scaling in the far dissipation range by computing high-order (\(\leq 20\)) structure functions numerically for: (1) the three-dimensional, incompressible Navier Stokes equation (with and without hyperviscosity); and (2) the GOY shell model for turbulence. Also, in case (2), with Taylor-microscale Reynolds numbers \(4 \times 10^4 \leq Re_\lambda \leq 3 \times 10^6\), we find that the inertial-range exponents \((\zeta_p)\) of the order - \(p\) structure functions do not approach their Kolmogorov value \(p/3\) as \(Re_\lambda\) increases.

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The central concern of studies of homogeneous, isotropic turbulence is the scaling of order-\(p\) velocity structure functions, e.g., \(S_p(r) \approx \langle |v_i(x + r) - v_i(x)|^p \rangle\), where \(i = 1, 2,\) or 3 is the Cartesian component of the velocity \(v(x)\) at point \(x\), and the angular brackets imply, in principle, a spatiotemporal average. Kolmogorov (K41)\(^2\) predicted that, at high Reynolds numbers \(Re_\lambda\) and for the inertial range \(20\eta_d \lesssim r \ll L\) (\(\eta_d\) and \(L\) are, respectively, dissipation and forcing scales and \(\lambda\) is the Taylor microscale), \(S_p(r) \sim r^{\zeta_p}\) with \(\zeta_p = p/3\). Subsequent experimental and theoretical studies \[\text{(ESS)}\] have argued for: (1) multiscale, i.e., \(\zeta_p = p/3 - \delta\zeta_p\), with \(\delta\zeta_p > 0\) but \(\zeta_p\) a nonlinear, monotonically increasing function of \(p\); and (2) extended self similarity (ESS) \[\text{(ESS)}\], in which \(\zeta_p\) is obtained from \(S_p \sim S_p^{3p}\), since this extends the apparent inertial range down to \(r \approx 5\eta_d\). A recent generalization \[\text{(ESS)}\] uses \(G_p(r) \equiv S_p(r)/[S_3(r)]^{p/3}\) and suggests that a log-log plot of \(G_p\) versus \(\delta\zeta_p = k/\eta_d\) is a straight line with slope \(p\delta\zeta_p = \left[\zeta_p - p\zeta_3/3\right]/\left[\zeta_q - q\zeta_3/3\right]\) for the lowest resolvable values of \(r\). This generalized extended self similarity (GESS) has been tested \[\text{(ESS)}\] to some extent \((p, q \leq 6)\).

Here we show how GESS is modified at sufficiently small \(r\) by computing wave-vector-space \((k\text{-space})\) analogs of high-order \((\leq 20)\) structure functions for: (1) the three-dimensional, incompressible Navier Stokes equation \((3d\,\text{NS})\), with and without hyperviscosity, and (2) the GOY shell model for turbulence \[\text{(ESS)}\] \((\text{GOY})\), where we attain both large \(Re_\lambda\) and \(k \gg k_d \equiv \eta_d/L\). We further propose a \(k\text{-space GESS}\), show that it holds for \(L^{-1} \ll k \lesssim 1.5k_d\), but then crosses over to another form in the far dissipation range. To study this we postulate \(k\text{-space ESS}\) \((\text{ESS})\) for real-space structure functions we use the symbols \(S\) and \(G\) and for their \(k\text{-space analogs (not}\) Fourier transforms) the symbols \(S\) and \(G\): \[S_p \equiv \langle |v(k)|^p \rangle \approx A_{Ip}(S_3)^{\zeta_p},\quad L^{-1} \ll k \lesssim 1.5k_d,\]

\[S_p \equiv \langle |v(k)|^p \rangle \approx A_{Dp}(S_3)^{\zeta_p},\quad 1.5k_d \lesssim k \ll \Lambda,\]

where \(A_{Ip}\) and \(A_{Dp}\) are, respectively, nonuniversal amplitudes for inertial and dissipation ranges and \(\Lambda^{-1}\) the (molecular) length at which hydrodynamics fails (see \[\text{(ESS)}\] for real-space analogs). Our study shows (Figs. 1-2) that Eq. (1) holds with two different exponents \(\alpha_p\) and \(\zeta_p\). In the GOY model \(\zeta_p = \zeta_p\), but we find explicitly \([\text{insert(b)}, \text{Fig}1]\) that, for the 3d NS case, \(\zeta_p = 2(\zeta_p + 3p^2)/11\) (i.e., \(S_p(k) \sim k^{-3(p^2+3)/2}\)) in the inertial range \[\text{(ESS)}\]; the difference between the two arises because of phase-space factors. Both \(\zeta_p\) and \(\alpha_p\) (Fig.2) seem universal \((\text{the same for all GOY and 3d NS runs (Table I))})\] \[\text{(ESS)}\]. \(\zeta_p\) agrees fairly with the She-Leveque (SL) \[\text{(ESS)}\] formula \(\zeta_p^{SL} = p/9 + 2[1 - (2/3)p^2]/3\) for the ranges of \(p\) and \(Re_\lambda\) in Fig 2 and \(\alpha_p\) is close to, but \textit{systematically less than}, \(p/3\).

The \(k\) dependences of the inertial- and dissipation-range asymptotic behaviors follow now from the dependence of \(S_3\) on \(k\): We find

\[S_3 \approx B_1 k^{-\zeta_3 - 9/2},\quad L^{-1} \ll k \lesssim 1.5k_d,\]

\[S_3 \approx B_2 k^\delta \exp(-ck/k_d),\quad 1.5k_d \lesssim k \ll \Lambda,\]

where \(B_1\) and \(B_2\) are, respectively, nonuniversal amplitudes (Eq. (2) holds \[\text{(ESS)}\] for 3d NS; for GOY the factor \(9/2\) is absent). Thus, in the far dissipation range, all \(S_p \sim k^\delta \exp(-\theta_p k/k_d)\) for \(1.5k_d \lesssim k \ll \Lambda\), with \(\theta_p = \alpha_p\delta\), \(\delta\) a form not easy to verify numerically for large \(p\), given the rapid decay at large \(k\), and suggested hitherto \[\text{(ESS)}\] only for \(S_2\). In Eq. (3), \(\delta, c, k_d\) are not universal, but we extract the universal part of the crossover via our \(k\)-space GESS: Define \(G_p \equiv S_p/(S_3)^{3/p}\); log-log plots of \(G_p\) versus \(\zeta_p\) now yield curves (Figs. 3a and 3b) with asymptotes which have \textit{universal}, but \textit{different}, slopes in inertial and dissipation ranges. The inertial-range asymptote has a slope \([\rho(p, q)]\) (as in real-space GESS \[\text{(ESS)}\]) which follows from the formulae above; the resulting \(\zeta_p\) are in fair agreement with the SL formula \[\text{(ESS)}\]. The dissipation-range asymptote has a slope \([\omega(p, q)]\) \((\text{see Eq.}(1)\) and the definition of \(G_p\). \(\omega(p, q)\) are universal, but the point at which the curve veers off from the inertial-range asymptote depends on the model (GOY, NS, etc.). However, a simple transformation yields a \textit{universal crossover scaling function} (different for each \(p, q\) pair because of multiscaling): Define

\[\text{universal crossover scaling function}\]
\log(H_{pq}) \equiv D_{pq} \log(G_p) \text{ and } \log(H_{qp}) \equiv D_{qp} \log(G_q);
the scale factors \(D_{pq} = D_{qp}\) are nonuniversal, but plots of \(\log(H_{pq})\) versus \(\log(H_{qp})\) show data from all GOY and 3d NS runs collapsing onto one universal curve within all our error bars (Fig.1 for \(p = 6\) and \(q = 9\)) for all \(k\) and \(R_{\lambda}\).
Both ESS (Fig.1) and GESS (Fig.3) remove the exponential controlling factor [2] from the leading asymptotic behavior of \(S_p\) in the far dissipation range and expose the remaining power-law dependence on \(k\). Also, it is easy to see analytically that GESS plots (Fig.3) amplify slope differences between inertial- and dissipation-range asymptotes relative to ESS plots (Fig.1).

How robust is the fair agreement of \(\zeta_p\) (Fig.2) with the SL formula? Some studies [17–19] suggest that, as \(R_{\lambda} \to \infty\), \(\delta\zeta_p \equiv (p/3 - \zeta_p) \to 0\). Numerical solutions of the 3d NS equation can at best achieve \([11, 21]\) \(R_{\lambda} \lesssim 220\), too small, by far, to resolve this issue, so we address it for the GOY model, by studying the range \(4 \times 10^4 \lesssim R_{\lambda} \lesssim 3 \times 10^6\). We find (Fig.3) that \(\delta\zeta_p\) does not vanish with increasing \(R_{\lambda}\) if anything, it rises marginally [2]. Systematic experimental studies at high \(R_{\lambda}\) are perhaps the best way to check if the trends of Fig.3 obtain in the 3d NS case.

We remark that, if we assume the hierarchy \([G_{p+1}/G_p] = [G_p/G_{p-1}]^{\alpha} \times \lim_{p \to \infty} G_{p+1}/G_p^{1-\gamma}\) with \(\nu^3 = 2/3\) (whose real-space analog is equivalent [1] to the SL moment hierarchy for the energy dissipation [3]) and use \([22]\) \(G_p(k) \approx C_p k^{\beta_p}\), we get a difference equation for \(\beta_p\) identical to the SL one (our \(\beta_p\) is their \(-\tau_p/3\)). This, when solved with the boundary conditions \(\beta_0 = \beta_1 = 0\) and \(\lim_{p \to \infty} (\beta_{p+1} - \beta_p) = 2/9\), yields the SL formula (via \(\zeta_p = -\beta_p + p \zeta_3/3\)). However, our GESS yields \([G_{p+1}/G_p] \approx C_p [G_p/G_{p-1}]^{\gamma_p}\) with \(\gamma_p = (\zeta_p - \zeta_p - 1/3)/(\zeta_p - \zeta_p - 1/3)\). Superficially, this might seem to violate the hierarchy assumed above, but it turns out to be consistent with our GESS form, if \(\zeta_p = \zeta_{p}^\text{SL}\). Superficially, this might seem to violate the hierarchy assumed above, but it turns out to be consistent with our GESS form, if \(\zeta_p = \zeta_{p}^\text{SL}\). Superficially, this might seem to violate the hierarchy assumed above, but it turns out to be consistent with our GESS form, if \(\zeta_p = \zeta_{p}^\text{SL}\).
TABLE I. Parameters \( \nu \) (viscosity), \( \nu_H \) (hyperviscosity), \( Re_\lambda \) (Taylor-microscale Reynolds number), \( \tau_e \) (box-size eddy-turnover time), \( \tau_{av} \) (averaging time), \( \tau_t \) (transient time) and \( k_d \) (dissipation-scale wavenumber) for our 3d NS runs NS1-4 \((k_{max} = 64)\) and GOY-model runs G1-8 \((k_{max} = 2^{12}k_0)\). The step size\((\delta t)\) used is 0.02 for NS1-4, \(10^{-4}\) for G1-4, and \(2 \cdot 10^{-5}\) for G5-8.

| Run   | \( \nu \) | \( \nu_H \) | \( Re_\lambda \) | \( \tau_e/\delta t \) | \( \tau_t/\tau_e \) | \( \tau_{av}/\tau_e \) | \( k_{max}/k_d \) |
|-------|------------|-------------|------------------|-------------------|-------------------|-------------------|---------------|
| NS1   | \( 5 \cdot 10^{-4} \) | 0           | \( \approx 3.5 \) | \( \approx 3 \cdot 10^2 \) | \( \approx 1 \)   | \( 2 \)           | \( \approx 4 \) |
| NS2   | \( 2 \cdot 10^{-4} \) | 0           | \( \approx 8 \)   | \( \approx 3 \cdot 10^2 \) | \( \approx 1 \)   | \( \approx 2.5 \)  | \( \approx 2.3 \) |
| NS3   | \( 5 \cdot 10^{-4} \) | \( 5 \cdot 10^{-6} \) | \( \approx 3.5 \) | \( \approx 3 \cdot 10^2 \) | \( \approx 1 \)   | \( \approx 1 \)    | \( \approx 6.5 \) |
| NS4   | \( 5 \cdot 10^{-4} \) | \( 10^{-6} \) | \( \approx 22 \)   | \( \approx 3 \cdot 10^3 \) | \( \approx 10 \)   | \( \approx 7 \)    | \( \approx 2 \)   |
| G1-4  | \( 5 \cdot 10^{-6} - 10^{-7} \) | 0           | \( 4 \cdot 10^4 - 3 \cdot 10^5 \) | \( (1.5 - 2.0)10^4 \) | \( \approx 500 \)  | \( \approx 2500 \) | \( \approx 2^5 - 2^3 \) |
| G5-8  | \( 5 \cdot 10^{-8} - 10^{-9} \) | 0           | \( 3.5 \cdot 10^5 - 3 \cdot 10^6 \) | \( (0.7 - 1)10^5 \) | \( \approx 500 \)  | \( \approx 2500 \) | \( \approx 2^3 - 1 \) |

FIG. 3. Log-log (base 10) plots of \( G_6 \) versus (a) \( G_{15} \) and (b) \( G_9 \) illustrating our \( k \)-space GESS; (c) \( H_{5.9} \) versus \( H_{6.6} \) showing the universal inertial- to dissipation-range crossover (see text). The line shows the SL, inertial-range prediction.

NS runs). Fortunately, this hardly affects our exponents: any attendant systematic errors in Fig. 2 are certainly less than the random errors indicated. Also, the agreement between our GOY and NS runs confirms our results. Our GOY-model data are, of course, of much better quality. Here Fourier components of the velocity are labeled by a discrete set of wave vectors \( k_n = k_0q \). The dynamical variables are the complex, scalar velocities \( v_n \) for each shell \( n \); \( v_n \) is affected directly only by the velocities in nearest and next-nearest shells. In spite of its simplicity, this model yields scaling properties akin to experimental ones. The GOY-model equations are:

\[
dv[n]{v}_n = iC_n - \nu k_n^2 v_n + f_n, \tag{4}
\]

where \( \nu \) is the kinematic viscosity, \( f_n \) the external force on shell \( n \), \( C_n = (a k_n v_{n+1} v_{n+2} + b k_n^{-1} v_{n-1} v_{n+1} + c k_n^{-2} v_{n-1} v_{n-2}) \), and \( a \), \( b \), and \( c \) can be fixed up to a constant by demanding \( [4] \) for \( \nu, f_n = 0 \), that: \( v_n \approx k_n^{-1/3} \) be a stationary solution of Eq.(4); and the GOY-model kinetic energy and helicity be conserved. We adopt the conventional parameters \( k_0 = 2^{-4}, q = 2, a = 1, b = c = -1/2 \) and use \( f_n = 5 \cdot 10^{-5} (1 + i) \delta n_{11}, \) i.e., we force the first shell \( [3] \). The GOY-model structure functions are \( S_{n,p} \equiv \langle |v_n|^p \rangle \sim k_n^{-\zeta_p} [4][5] \); reliable values of \( \zeta_p \) obtain \( [3] \) if we use \( \Sigma_{n,p} = \langle |v_n v_{n+1} v_{n+2} + v_{n-1} v_n v_{n+1}/4|^{p/2} \rangle \) since this removes an underlying 3-cycle. We have used \( \Sigma_{n,p} \) to obtain Fig. 4 \([24] \), but \( S_{n,p} \) in Figs. \([3],[6],[7] \) for consistency with 3d NS. We use an Adams-Bashforth scheme \([10] \) (step size \( \delta t \)) to integrate Eq. (4). The average of the time scale associated with the smallest wavenumber, \( (|v| / k_1)^{-1} \), gives the “box-size” eddy turnover time. Table \([1]\) lists other parameters for our 8 GOY-model runs G1-8, for which we use (cf., \([10] \)) \( E(k) = S_{n,2}/k_n \), \( \lambda = 2n/k_0[\Sigma_n S_{n,2}/\Sigma_n k_n S_{n,2}]^{1/2} \), and \( v_{rms} = [k_0 \Sigma_n S_{n,2}/\pi]^{1/2} \). This yields \( Re_\lambda \sim \nu^{-0.5} \), as expected \([25] \) at large \( Re_\lambda \).

Experimental evidence for the slope change in the dissipation range in real-space analogs of Fig. 4 was given by Stolovitzky and Sreenivasan \([3] \), who postulated \( S_p \sim S_0^{\alpha_p} \) in the dissipation range and suggested \( \alpha_p^{\prime} \approx (\zeta_{3p/2} + p/2)/2(\zeta_{3/2} + 3/2) \). We have not been able to obtain a simple relation between our \( \alpha_p \) and their \( \alpha_p^{\prime} \) (unlike \([3] \) that between \( \zeta_p \) and \( \zeta_p^{\prime} \)) since \( S_p \) does not have a power-law dependence on \( k \) in the dissipation range. It would be very interesting to extend such experimental studies to test the universality of dissipation-range asymptotics (e.g., in different flows) and the crossover suggested here. The universal multiscaling in the dissipation range that we have elucidated is a manifestation of strongly intermittent (multifractal) dissipation which is believed to occur \([13] \) even at low \( Re_\lambda \). We believe that this multiscaling should extend far enough into the dissipation range before corrections set in because of
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