SELF-INTERSECTION OF DUALIZING SHEAVES OF ARITHMETIC SURFACES WITH REDUCIBLE FIBERS

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Abstract. Let $K$ be an algebraic number field and $O_K$ the ring of integers of $K$. Let $f : X \to \text{Spec}(O_K)$ be a stable arithmetic surface over $O_K$ of genus $g \geq 2$. In this short note, we will prove that if $f$ has a reducible geometric fiber, then $(\omega_{X/O_K}^A \cdot \omega_{X/O_K}^A) \geq \log 2/6(g - 1)$.

Let $K$ be a number field and $O_K$ the ring of integers of $K$. Let $f : X \to \text{Spec}(O_K)$ be a stable arithmetic surface of genus $g \geq 2$. One of interesting problems on arithmetic surfaces is a question whether $(\omega_{X/O_K}^A \cdot \omega_{X/O_K}^A) > 0$. This question is closely related to Bogomolov conjecture, which claims that, for a curve $C$ over a number field and an embedding $j : C \to \text{Jac}(C)$ into the Jacobian $\text{Jac}(C)$, the image $j(C)$ in $\text{Jac}(C)$ is discrete in terms of the Néron-Tate height of $\text{Jac}(C)$ (cf. [Sz] and [Zh2]). Currently, the positivity of $(\omega_{X/O_K}^A \cdot \omega_{X/O_K}^A)$ is known in the following cases:

1. $f$ has a bad reduction. ([Zh1] and [Zh2]).
2. $f$ has a reducible geometric fiber, or $\text{Jac}(X_K)$ has a complex multiplication. ([Bu])
3. $\text{End}(\text{Jac}(X_K))_\mathbb{R}$ is not isomorphic to $\mathbb{R}, \mathbb{C}$, or the quaternion division algebra $\mathbb{D}$. ([Zh3])

In this short note, we would like to give an effective lower bound of $(\omega_{X/O_K}^A \cdot \omega_{X/O_K}^A)$ under the assumption that $f$ has a reducible geometric fiber, namely,

**Theorem 1.** Let $K$ be an algebraic number field and $O_K$ the ring of integers of $K$. Let $f : X \to \text{Spec}(O_K)$ be a stable arithmetic surface over $O_K$ of genus $g \geq 2$. Assume that, for $P_1, \ldots, P_n \in \text{Spec}(O_K)$, geometric fibers $X_{P_1}, \ldots, X_{P_n}$ of $X$ at $P_1, \ldots, P_n$ are reducible. Then, we have

$$(\omega_{X/O_K}^A \cdot \omega_{X/O_K}^A) \geq \sum_{i=1}^{n} \frac{\log \#(O_K/P_i)}{6(g - 1)}.$$
In particular, \((\omega^A_{X/O_K} \cdot \omega^A_{X/O_K}) \geq \log 2/6(g - 1)\) if \(f\) has a reducible geometric fiber.

As compared with methods used in [Bu], [Zh1], [Zh2] and [Zh3], our method is very elementary. Besides it, we can give the exact positive lower bound \(\log 2/6(g - 1)\).

Let \(h : Y \to \text{Spec}(O_K)\) be a regular semistable arithmetic surface over \(O_K\) of genus \(g \geq 2\). Then, \(\omega_{Y/O_K}\) is \(h\)-nef, i.e., \(\deg(\omega_{Y/O_K} |_C) \geq 0\) for all vertical curves \(C\) on \(Y\). Let \(P \in \text{Spec}(O_K)\) and \(Y_P\) the fiber of \(Y\) at \(P\). To distinguish Arakelov intersection and geometric intersection in the fiber \(Y_P\), we will use a symbol \((\ ,\ )\) for Arakelov intersection and \(\langle\ ,\ \rangle\) for geometric intersection. \((\ ,\ )\) and \(\langle\ ,\ \rangle\) are related by a formula

\[ (\ ,\ ) = \langle\ ,\ \rangle \log \#(O_K/P). \]

Let \(Y_P = Y_P \otimes (O_K/P)\) be the geometric fiber at \(P\) and \(C\) an irreducible component of \(Y_P\). We say \(C\) is an **essential irreducible component** if \(\langle\omega_{Y/O_K} \cdot C\rangle > 0\). Here we assume that \(Y_P\) is reducible. We set

\[ \beta(P) = \min \left\{ -\frac{\langle\omega_{Y/O_K} \cdot C\rangle^2}{\langle C \cdot C\rangle} \middle| C \text{ is an essential irreducible component of } Y_P, \right\}. \]

**Lemma 2.** \(\beta(P) \geq 1/3\).

**Proof.** Let \(C\) be an irreducible component of \(Y_P\) with \(\langle\omega_{Y/O_K} \cdot C\rangle > 0\). We need to prove

\[ \frac{\langle\omega_{Y/O_K} \cdot C\rangle^2}{\langle C \cdot C\rangle} \leq -\frac{1}{3}. \]

By adjunction formula, if \(q = h^1(O_C)\) is an arithmetic genus of \(C\), then

\[ \langle\omega_{Y/O_K} \cdot C\rangle + \langle C \cdot C\rangle = 2q - 2. \]

Thus,

\[ \frac{\langle\omega_{Y/O_K} \cdot C\rangle^2}{\langle C \cdot C\rangle} = \langle C \cdot C\rangle - 4(q - 1) + \frac{4(q - 1)^2}{\langle C \cdot C\rangle}. \]

Therefore, since \(\langle C \cdot C\rangle \leq -1\), if \(q \geq 1\), our assertion is trivial.

Here we assume \(q = 0\). Then, the previous formula is

\[ \frac{\langle\omega_{Y/O_K} \cdot C\rangle^2}{\langle C \cdot C\rangle} = \langle C \cdot C\rangle + 4 + \frac{4}{\langle C \cdot C\rangle}. \]

Since \(\langle\omega_{Y/O_K} \cdot C\rangle > 0\), we have \(\langle C \cdot C\rangle \leq -3\). Therefore, we get our lemma. \(\square\)

Let us consider the following conditions for \(Y_P\):

(C.1) Every irreducible component of \(Y_P\) is geometrically irreducible and every node of the geometric fiber \(Y_P\) is defined over \(O_K/P\).

(C.2) For any point closed \(x\) of the fiber \(Y_P\), there is an irreducible component \(C\) of \(Y_P\) such that \(\langle\omega_{Y/O_K} \cdot C\rangle > 0\) and \(x \notin C\).

Note that (C.2) holds if (C.1) holds and number of essential irreducible components is greater than or equal to 3.
Lemma 3. Let $\Gamma$ be a horizontal prime divisor on $Y$, and $\rho : \tilde{\Gamma} = \text{Spec}(O_{K'}) \rightarrow \Gamma$ a normalization of $\Gamma$. Let $Y' \rightarrow Y \times_{\text{Spec}(O_K)} \tilde{\Gamma}$ be a minimal resolution of $Y \times_{\text{Spec}(O_K)} \tilde{\Gamma}$. We set induced morphisms as follows:

$$
\begin{array}{ccc}
Y & \xleftarrow{\rho'} & Y' \\
\downarrow h & & \downarrow h' \\
\text{Spec}(O_K) & \xleftarrow{\rho} & \tilde{\Gamma} = \text{Spec}(O_{K'})
\end{array}
$$

Let $\Gamma'$ be a section of $g'$ with $\rho'(\Gamma') = \Gamma$. Under the assumption (C.1) and (C.2) for $Y_P$, there is a vertical $\mathbb{Q}$-divisor $E_P$ on $Y'$ with following properties:

3.1. $\rho'(E_P) \subseteq Y_P$.
3.2. $\left( (\omega_{Y'/O_{K'}}^A - (2g-2)\Gamma' - E_P) \cdot E_P \right) = 0$.
3.3. $(E_P \cdot E_P) \leq -\beta(P) \log \#(O_K/P)[K' : K]$.

Proof. First of all, it is well known that $\omega_{Y'/O_{K'}} = \rho'^*(\omega_{Y/O_K})$. We set $PO_{K'} = Q_1^{e_1} \cdots Q_t^{e_t}$.

For each $i$ ($1 \leq i \leq t$), we would like to find $\mathbb{Q}$-divisors $E_i$ on $Y$ with the following properties:

1. $E_i$ is a scalar of an essential irreducible component $C$ of $Y_P$ by a rational number, i.e., $E_i = aC$ for some $a \in \mathbb{Q}$.
2. $\langle E_i \cdot E_i \rangle \leq -\beta(P)$.
3. If we set $E'_i = \rho'^*(E_i)|_{Y_{Q_i}}$, then $\left( (\omega_{Y'/O_{K'}}^A - (2g-2)\Gamma' - E'_i) \cdot E'_i \right) = 0$.

Assuming the existence of $E_i$'s as above, we proceed to our argument. We set $E_P = E'_1 + \cdots + E'_t$.

Then, 3.1 and 3.2 are obvious by (1) and (3). Let us try to evaluate $\langle E'_i \cdot E'_i \rangle$. Let $C'$ be an irreducible component of $Y'_Q$, such that $\rho'(C') = C$. Then,

$$E'_i = e_i aC' + (\text{exceptional divisors}).$$

Thus, if we set $\rho'_*(C') = fC$, then

$$\langle E'_i \cdot E'_i \rangle = e_i a \langle \rho'^*(E_i) \mid_{Y_{Q_i}} \cdot C' \rangle = e_i a \langle E_i \cdot \rho'_*(C') \rangle = e_i af \langle E_i \cdot C \rangle = e_i f \langle E_i \cdot E_i \rangle \leq -\beta(P)e_i.$$
Therefore,
\[
(E_P \cdot E_P) = \sum_{i=1}^{t} (E_i' \cdot E_i') = \sum_{i=1}^{t} \langle E_i' \cdot E_i' \rangle \log #(O_{K'}/Q_i)
\]
\[
\leq \sum_{i=1}^{t} -\beta(P)e_i \log #(O_{K'}/Q_i)
\]
\[
= -\beta(P) \log #(O_K/P) \sum_{i=1}^{t} e_i [O_{K'}/Q_i : O_K/P]
\]
\[
= -\beta(P) \log #(O_K/P)[K' : K].
\]

Thus \(E_P\) satisfies 3.3.

We need to construct \(E_i\) satisfying (1), (2) and (3). By our assumptions (C.1) and (C.2), there is an essential irreducible component \(C\) of \(Y_P\) such that \(\rho'(Y_{Q_i} \cap \Gamma') \notin C\). We set
\[
E_i = \frac{\langle \omega_{Y/O_{K'}/C} \cdot C \rangle}{\langle C \cdot C \rangle} C.
\]

Then, by the definition of \(\beta(P)\), \(\langle E_i \cdot E_i \rangle \leq -\beta(P)\). Let \(C'\) be an irreducible component of \(Y_{Q_i}'\) such that \(\rho'(C') = C\). Then, we can write
\[
E_i' = \rho'^*(E_i) \mid_{Y_{Q_i}'} = bC' + F,
\]
where \(b\) is a rational number and \(F\) is an exceptional divisor of \(\rho'\). By our choice of \(C\), \(C' \cap \Gamma' = \emptyset\) and \(F \cap \Gamma' = \emptyset\). Therefore, if we set \(\rho'(C') = fC\), then
\[
\langle (\omega_{Y'/O_{K'}/C} - (2g-2)\Gamma' - E_i') \cdot E_i' \rangle = \langle (\omega_{Y'/O_{K'}/C} - (2g-2)\Gamma' - E_i') \cdot bC' \rangle
\]
\[
= \langle (\omega_{Y'/O_{K'}/C} - E_i') \cdot bC' \rangle
\]
\[
= b \langle \omega_{Y/O_{K}} \cdot \rho_*(C') \rangle - b \langle E_i \cdot \rho_*(C') \rangle
\]
\[
= bf \langle \omega_{Y/O_{K}} \cdot C \rangle - \langle E_i \cdot C \rangle = 0.
\]

which completes the proof of Lemma 3. \(\square\)

Using Lemma 3, we get

**Lemma 4.** Let \(P_1, \ldots, P_n \in \text{Spec}(O_K)\) such that geometric fibers \(Y_{P_1}, \ldots, Y_{P_n}\) at \(P_1, \ldots, P_n\) are reducible. If \(Y_{P_1}, \ldots, Y_{P_n}\) satisfy the conditions (C.1) and (C.2), then, for any prime divisor \(\Gamma\) (including an infinite fiber) on \(Y\), we have
\[
\left[ \left( \omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar} \right) + \sum_{i=1}^{n} \beta(P_i) \log #(O_K/P_i) \right] \deg(\Gamma_K) \leq 2g(2g-2) \left( \omega_{Y/O_K}^{Ar} \cdot \Gamma \right).
\]
Proof. Since $\omega_{Y/O_K}$ is $h$-nef, if $\Gamma$ is an irreducible component of a fiber, or an infinite fiber, our assertion is trivial. So we assume that $\Gamma$ is horizontal. We use the same notation as in Lemma 3. By virtue of Lemma 3, for each $P_i$, there is a vertical $Q$-divisor $E_{P_i}$ on $Y'$ with the following properties:

(a) $\rho'(E_{P_i}) \subseteq Y_P$.
(b) $\left(\omega_{Y'/O_{K'}}^{Ar} - (2g - 2)\Gamma' - E_{P_i}\right) \cdot E_{P_i} = 0$.
(c) $(E_{P_i} \cdot E_{P_i}) \leq -\beta(P_i) \log #(O_K/P_i)[K' : K]$.

We set $E = E_{P_1} + \cdots + E_{P_n}$. Then, by (a), (b), (c) and Hodge index theorem ([Fa, Theorem 4]),

\[
\left(\omega_{Y'/O_{K'}}^{Ar} - (2g - 2)\Gamma'\right)^2 = \left(\omega_{Y'/O_{K'}}^{Ar} - (2g - 2)\Gamma' - E + E\right)^2 \\
= \left(\omega_{Y'/O_{K'}}^{Ar} - (2g - 2)\Gamma' - E\right)^2 + (E \cdot E) \\
\leq \sum_{i=1}^{n} (E_{P_i} \cdot E_{P_i}) \\
\leq \sum_{i=1}^{n} -\beta(P_i) \log #(O_K/P_i)[K' : K].
\]

It follows from adjunction formula that

\[
(\omega_{Y'/O_{K'}}^{Ar} \cdot \omega_{Y'/O_{K'}}^{Ar}) - 2g(2g - 2)(\omega_{Y'/O_{K'}}^{Ar} \cdot \Gamma') \leq -\sum_{i=1}^{n} \beta(P_i) \log #(O_K/P_i)[K' : K].
\]

Hence, we obtain

\[
\left[ (\omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar}) + \sum_{i=1}^{n} \beta(P_i) \log #(O_K/P_i) \right] \deg(\Gamma_K) \leq 2g(2g - 2) \left(\omega_{Y/O_K}^{Ar} \cdot \Gamma\right).
\]

□

We need the following technical fact of algebra.

**Lemma 5.** Let $K$ be an algebraic number field and $O_K$ the ring of integers of $K$. Let $P_1, \ldots, P_r$ be non-zero prime ideals of $O_K$ and $n$ a positive integer. Then, there is a finite extension field $K'$ of $K$ with the following properties:

1. $[K' : K] = n$.
2. $Q_i = P_iO_{K'}$ is a prime ideal for every $i$.
3. $[O_{K'}/Q_i : O_K/P_i] = n$ for every $i$. 

We need the following technical fact of algebra.
**Proof.** Let \((O_K)_{P_i}\) be the localization of \(O_K\) at \(P_i\) and \(F_i \in (O_K)_{P_i}[X]\) such that \(F_i\) is monic and of degree \(n\), and the class \(\bar{F}_i\) in \((O_K)_{P_i}/P_i(O_K)_{P_i}[X] (= O_K/P_i[X])\) is irreducible. We set
\[
F_i = X^n + a_{i1}X^{n-1} + \cdots + a_{in}.
\]
By approximation theorem (cf. [La, XII, Theorem 1.2]), for every \(1 \leq j \leq n\), there is \(a_j \in K\) such that
\[
a_j - a_{ij} \in P_i(O_K)_{P_i}
\]
for all \(1 \leq i \leq r\). We set
\[
F = X^n + a_1X^{n-1} + \cdots + a_n.
\]
Then, \(F \in (O_K)_{P_i}[X]\) and \(F \equiv F_i \pmod{P_i(O_K)_{P_i}[X]}\) for all \(1 \leq i \leq r\). Since \(\bar{F} = \bar{F}_i\) in \((O_K)_{P_i}/P_i(O_K)_{P_i}[X]\) is irreducible, \(F\) is irreducible over \(K\) by Gauss Lemma (cf. [La, IV, Corollary 2.2]). Let \(\alpha\) be a root of \(F\) and \(K' = K(\alpha)\). Then, \([K' : K] = n\). Let \(Q_i\) be a prime ideal of \(O_{K'}\), lying over \(P_i\). Since \(\alpha\) is integral over \((O_K)_{P_i}\), \(\alpha \in (O_{K'})_{Q_i}\). Let \(\bar{\alpha}\) be the class of \(\alpha\) in \(O_{K'}/Q_i = (O_{K'})_{Q_i}/Q_i(O_{K'})_{Q_i}\). Then, since \(\bar{\alpha}\) is a root of \(\bar{F}_i\) and \(\bar{F}_i\) is irreducible, \([O_{K'}/Q_i : O_K/P_i] \geq n\). On the other hand, by [La, XII, Corollary 6.3], \([O_{K'}/Q_i : O_K/P_i] \leq n\) and equality holds if and only if \(Q_i = P_iO_{K'}\). Thus, we get our lemma. \(\square\)

Let us start the proof of Theorem 1. Actually, we will prove the following. Indeed, Theorem 6 implies Theorem 1 if we consider a minimal resolution \(Y \to X\).

**Theorem 6.** Let \(h : Y \to \Spec(O_K)\) be a regular semistable arithmetic surface of genus \(g \geq 2\). Let \(P_1, \ldots, P_n \in \Spec(O_K)\) such that, for every \(1 \leq i \leq n\), a geometric fiber \(Y_{P_i}\) at \(P_i\) has at least two essential irreducible components. Then, we have the following:

(A) For all effective Arakelov divisors \(D\) on \(Y\),
\[
\left(\omega^A_{Y/O_K} \cdot \omega^A_{Y/O_K}\right) + \sum_{i=1}^{n} \frac{\log \#(O_K/P_i)}{6} \deg(D_K) \leq 2g(2g - 2) \left(\omega^A_{Y/O_K} \cdot D\right).
\]

(B) \(\left(\omega^A_{Y/O_K} \cdot \omega^A_{Y/O_K}\right) \geq \sum_{i=1}^{n} \frac{\log \#(O_K/P_i)}{6(g - 1)}\).

**Proof.** First, let us consider (A). Here we claim:

**Claim 7.** We may assume that \(Y_{P_i}\) satisfies (C.1) for every \(1 \leq i \leq n\).

Let \(k_i\) be a finite extension field of \(O_K/P_i\) such that every irreducible component of \(Y_{P_i} \otimes k_i\) is geometrically irreducible and every node of \(Y_{P_i} \otimes k_i\) is a \(k_i\)-rational point. Let \(l\) be a positive integer such that \([k_i : O_K/P_i] \leq l\) for all \(1 \leq i \leq n\). Let \(P_{n+1}, \ldots, P_r \in \Spec(O_K)\) be other critical values of \(h\). Then, by Lemma 5, there is a finite extension
field $E$ of $K$ such that $[E : K] = l$, $R_i = P_iO_E$ is a prime ideal and $[O_E/R_i : O_K/P_i] = l$ for every $1 \leq i \leq r$. Let us consider $Z = Y \times_{\text{Spec}(O_K)} \text{Spec}(O_E)$. Then, $Z$ is regular and semistable, and for $1 \leq i \leq n$, a fiber $Z_{R_i}$ at $R_i$ satisfies (C.1). Thus, we get

$$\left[ (\omega_{Z/O_E}^{Ar} \cdot \omega_{Z/O_E}^{Ar}) + \sum_{i=1}^{n} \log \#(O_E/R_i) \right] \deg(\rho^*(D)_E) \leq 2g(2g - 2) \left( \omega_{Z/O_E}^{Ar} \cdot \rho^*(D) \right)$$

for all effective Arakelov divisors $D$ on $Y$, where $\rho : Z \to Y$ is the induced morphism. Here, $\omega_{Z/O_E}^{Ar} = \rho^*(\omega_{Y/O_K}^{Ar})$, and $\log \#(O_E/R_i) = \log \#(O_K/P_i)[E : K]$. Thus, we obtain our claim.

Moreover, in the same way as in Claim 7, we may assume that if we take a blowing-up at a node of $Y_{P_i}$, then new nodes are also $(O_K/P_i)$-rational points.

Let $t_{P_i}$ be a local parameter of $P_i(O_K)_{P_i}$. We set $K' = K(\sqrt[1^n]{t_{P_i}})$. Then, there is a unique prime ideal $Q_i$ of $O_{K'}$ such that $Q_i \cap O_K = P_i$ and the ramification index between $t_{P_i}$ and $Q_i$ is equal to 2. Let $Y_1 = Y \times_{\text{Spec}(O_K)} \text{Spec}(O_{K'})$ and $Y' \to Y_1$ a minimal resolution of $Y_1$. If number of essential irreducible components of $Y_{P_i}$ is greater than or equal to 3, then $Y_{P_i}$ satisfies the condition (C.2). So does $Y'_{Q_i}$. We assume that number of essential irreducible components of $Y_{P_i}$ is equal to 2. Let $C_1$ and $C_2$ be essential irreducible components of $Y_{P_i}$. Let $C_1'$ and $C_2'$ be irreducible components of a fiber $(Y_{1})_{Q_i}$ corresponding to $C_1$ and $C_2$. Then, $Y_{1}$ has $A_1$ type singularities at $C_1' \cap C_2'$. Therefore, corresponding irreducible components $C_1''$ and $C_2''$ in $Y'_{Q_i}$ has no common points. (Note that every node of $Y'_{Q_i}$ is defined over $O_{K'}/Q_i = O_K/P_i$.) Thus, $Y'_{Q_i}$ satisfies the condition (C.2). Hence we can apply Lemma 4 for $Y'$. Namely if $\mu : Y' \to Y$ is the induced morphism, then,

$$\left[ (\omega_{Y'/O_{K'}}^{Ar} \cdot \omega_{Y'/O_{K'}}^{Ar}) + \sum_{i=1}^{n} \beta(Q_i) \log \#(O_{K}/Q_i) \right] \deg(\mu^*(D)_{K'})$$

$$\leq 2g(2g - 2) \left( \omega_{Y'/O_{K'}}^{Ar} \cdot \mu^*(D) \right).$$

Thus, by Lemma 2, we get (A) because $\omega_{Y'/O_{K'}}^{Ar} = \mu^*(\omega_{Y/O_K}^{Ar})$ and $[K' : K] = 2$.

Next let us consider (B). Let $F$ be an infinite fiber and $r$ a real number with

$$\omega_{Y/O_K}^{Ar} - rF \cdot \omega_{Y/O_K}^{Ar} > 0,$$

i.e. $r < \frac{(\omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar})}{2(\omega_{Y/O_K}^{Ar} \cdot F)}$. Then, by [Fa, Corollary of Theorem 3], there is an effective Arakelov divisor $E$ on $Y$ such that $E$ is arithmetically linearly equivalent to $n(\omega_{Y/O_K}^{Ar} - rF)$ for sufficiently large $n$. Thus, by (A), we have

$$\left( \omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar} \right) + \sum_{i=1}^{n} \log \#(O_K/P_i) \leq 2g \left( \omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar} - rF \right).$$

Taking a limit $r \to \frac{(\omega_{Y/O_K}^{Ar} \cdot \omega_{Y/O_K}^{Ar})}{2(\omega_{Y/O_K}^{Ar} \cdot F)}$, we get (B). \qed
References

[Bu] J.-F. Burnol, Weierstrass points on arithmetic surfaces, Invent. Math. 107 (1992), 421–432.
[Fa] G. Faltings, Calculus on arithmetic surfaces, Ann. of Math. 119 (1984), 387–424.
[La] S. Lang, Algebra, (3nd Edition), Addison-Wesley, 1993.
[Sz] L. Szpiro, Sur les propriétés numériques du dualisant-relatif d’une surface arithméthique, The Grothendieck Festschrift, vol. 3, Birkhauser, 1990, pp. 229–246.
[Zh1] S. Zhang, Positive line bundles on arithmetic surfaces, Thesis at Columbia University (1991).
[Zh2] ——, Admissible pairing on a curve, Invent. Math. 112 (1993), 171–193.
[Zh3] ——, Small points and adelic metrics, J. Algebraic Geometry (to appear).

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