Double Potts chain and exact results for some two-dimensional models

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Abstract

A closed-form exact analytical solution for the $q$-state Potts model on a ladder $2 \times \infty$ with arbitrary two-, three-, and four-site interactions in a unit cell is presented. Using the obtained solution it is shown that the finite-size internal energy equation [6] yields an accurate value of the critical temperature for the triangular Potts lattice with three-site interactions in alternate triangular faces. It is argued that the above equation is exact at least for self-dual models on isotropic lattices.

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1 Introduction

The methods allowing to extract information about a multi-dimensional system from solutions of their lower dimensional counterparts play an important role in statistical physics. One of the most well-known examples of this kind is the finite-size scaling approach [1, 2].

There are cases which evoke particular interest when the critical properties of a system experiencing a phase transition can be determined exactly from data only for its subsystems. For instance, for the Ising strips the intersection point of the locus of partition function zeros in a complex temperature plane with the real positive axis yields the exact value of the critical temperature for the two-dimensional Ising model [3]. Exact critical temperatures for the $S = 1/2$ Ising models on square, triangular, honeycomb, and centered square (Union Jack) anisotropic lattices are obtained by using strip clusters when an effective field is applied to one side of a strip only [3]. Another exotic way of estimating the critical point of the square-lattice Ising model was proposed in [5]. The authors of this paper showed that, in the quasi-diagonal form of a transfer matrix of finite width strip, all coefficients of the characteristic equation for the sub-block containing the largest eigenvalue have extremum located precisely at the exact value of phase transition temperature of infinite lattice.

In the present paper we concentrate our attention on the method to calculate the critical temperature, proposed by Wosiek [6] (see also [7, 8, 9, 10, 11, 12]). The author [6] introduced a maximum criterion for the ratio of moments of the transfer matrix, and obtained the following equation for determining the position of critical point in a $d$-dimensional system:

$$u_1(K_c) = u_2(K_c).$$

(1)

Here $u_1$ and $u_2$ are the internal energies of $(d - 1)$-dimensional and two coupled $(d - 1)$-dimensional subsystems, respectively, and $K_c$ is the critical coupling (normalized inverse critical temperature) of the $d$-dimensional system.

It is remarkable that at $d = 2$ Eq. (1), shown in [6], yields the exact value of $K_c$ for the isotropic square and triangular Ising lattices, as well as for the three-site Potts model on the square lattice also with isotropic interactions. Later on some other models were added to the list. Now it includes another isotropic Baxter model (two square Ising lattices coupled by four-particle interactions), the Baxter-Wu model (triangular lattice with three-site interactions of Ising spins) [10], and $q$-state Potts model on the isotropic square lattice with arbitrary value of $q$ [12]. Physical nature of Eq. (1) may be elucidated when it yields either (a) an exact solution, or (b) approximate estimation, or (c) does not give any solution at all for a given model. The more such examples are found, the better we understand its sense.

For a two-dimensional system, Eq. (1) connects the internal energies of infinitely long linear and double chains. Therefore, to obtain a possibility to test...
Eq. (1) rigorously, it is necessary to have analytical solutions for such subsystems.

In the second section of the present paper, we give an exact analytical solution for the two-chain Potts strip with a large number of independent parameters. As a special case, it contains a solution for the linear Potts chain.

Our solution for the double Potts chain enables us to cover all earlier revealed cases in which Eq. (1) reproduces exactly the critical temperatures for the two-dimensional models of Ising, Baxter-Wu, and Potts. Besides, we discover (section 3) a new model for which Eq. (1) yields the exact result. This is the $q$-state Potts model on the triangular lattice with pure three-site interactions in half of the triangular faces [13].

In the fourth section of this paper a discussion of results is presented. In particular, it is shown that duality is a sufficient condition for the validity of Eq. (1) for the isotropic spin lattices.

Finally, in section 5, we summarize the results obtained in the work.

2 Solution of double $q$-state Potts chain with $S_q$ symmetry

Consider a two-chain (ladder) lattice with spin variables $\sigma_i^l$ attached to its sites ($i = 1, 2$ is the chain index, and $l = 1, 2, 3, \ldots$ labels the sites in the longitudinal direction of the ladder); the spin variables take the values $1, 2, \ldots, q$.

Let us write the Hamiltonian of a system in the form

$$ H = -\sum_{l} H(\sigma_1^l, \sigma_2^l; \sigma_{1+1}^l, \sigma_{2+1}^l). $$

(2)

The locality of interactions in the above Hamiltonian allows one to introduce the transfer matrix $V$ with elements

$$ \langle \sigma_1, \sigma_2 | V | \sigma_1', \sigma_2' \rangle = \exp[H(\sigma_1, \sigma_2; \sigma_1', \sigma_2')/k_B T] $$

(3)

($T$ is the temperature and $k_B$ is Boltzmann’s constant) and reduce the problem of calculating the free energy density $f$ of infinitely long strip to a search for the largest eigenvalue $\lambda_1$ of the matrix $V$:

$$ f = \frac{1}{2} \ln \lambda_1. $$

(4)

The transfer matrix (3) has the size $q^2 \times q^2$. It is real and, moreover, all its elements are positive, but the matrix is, generally speaking, non-symmetrical ($V_{ij} \neq V_{ji}$).

To solve the eigenvalue problem for the transfer matrix, we shall use a group-theoretical approach (in this connection see, e.g., Ref. [14] where such an approach was applied to a quasidiagonalization of a transfer matrix of Ising model.
on parallelepipeds \( L \times L \times \infty \). In order to obtain a solution for the two-leg spin ladder (in which we are particularly interested) in a most general form, we will go in reverse order. Namely, we first select a symmetry group in the space \(|\sigma_1, \sigma_2\rangle\), that would enable us to quasidiagonalize the transfer matrix up to sub-blocks secular equations which can be solved analytically; only then we expand the Hamiltonian density \(H\) into a series on invariants of the symmetry group.

Let us take a model which is invariant, for example, under transformations of the symmetric group \(S_q\) of degree \(q\). For the Potts model this means, as a matter of fact, that we are dealing with a system in the zero external field. But fortunately the field is not required to test Eq. (1).

It is known (see, for example, [15]) that the largest eigenvalue of a transfer matrix is located in the sub-block of the identity irreducible representation (IR). In accordance with group theory, the basis vectors \(\psi_i\) of the identity IR can be obtained acting by the permutation operators of a group \(S_q\) successively on the orths \(|1,1\rangle, |1,2\rangle, \ldots, |q,q\rangle\). Acting by elements of symmetric group first on the orth \(|1,1\rangle\) and then on the orth \(|1,2\rangle\), we find that the two linear combinations obtained involve all orths. The normalized basis vectors are given by

\[
\psi_1 = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i, i\rangle, \quad \psi_2 = \frac{1}{\sqrt{q(q-1)}} \sum_{i,j=1}^{q} |i, j\rangle \tag{5}
\]

(the prime at the second sum indicates that the terms with \(i = j\) are omitted). Hence, the sub-block of identity IR has sizes 2 by 2 and, therefore, its eigenvalues (one of which is \(\lambda_1\)) can easily be obtained by solving an algebraic equation only of second degree. Notice that if one takes the group \(S_q \times C_s\) (\(C_s\) is the group of mirror reflections in the plane going between the chains of the two-leg ladder), the sub-block corresponding to the identity IR will again have the size 2 \times 2 and therefore this symmetry, which only reduces the number of independent parameters in a Hamiltonian, does not justify itself in the given case.

We now represent the Hamiltonian (2) in the form of a sum of terms which are invariant under transformations of the group \(S_q\):

\[
\mathcal{H} = -\sum_l \left[ J_1 \delta_{\sigma_l^1, \sigma_{l+1}^1} + J_2 \delta_{\sigma_l^2, \sigma_{l+1}^2} + J_0 \delta_{\sigma_l^1, \sigma_l^2} + J' \delta_{\sigma_l^1, \sigma_{l+1}^2} + J'' \delta_{\sigma_l^2, \sigma_{l+1}^1} 
+ J_3 \delta_{\sigma_l^1, \sigma_{l+1}^1} + J_4 \delta_{\sigma_l^2, \sigma_{l+1}^2} + J_5 \delta_{\sigma_l^1, \sigma_l^2} + J_6 \delta_{\sigma_l^1, \sigma_{l+1}^2} + J_7 \delta_{\sigma_l^2, \sigma_{l+1}^1} \right]. \tag{6}
\]

The Kronecker symbols entering here are defined as follows:

\[
\delta_{\sigma_1 \cdots \sigma_k} = \begin{cases} 
1, & \text{if } \sigma_1 = \ldots = \sigma_k \\
0, & \text{otherwise}
\end{cases} \tag{7}
\]
The structure of two-site couplings in the Hamiltonian \((\text{H})\) is shown in Fig. 1. Matrix elements of the original transfer matrix are written as

\[
\langle \sigma_1, \sigma_2 | V | \sigma_1', \sigma_2' \rangle = \exp[K_1 \delta_{\sigma_1, \sigma_1'} + K_2 \delta_{\sigma_2, \sigma_2'} + \frac{1}{2} K_0 (\delta_{\sigma_1, \sigma_2} + \delta_{\sigma_1', \sigma_2'}) + K' \delta_{\sigma_1, \sigma_1'} + K'' \delta_{\sigma_2, \sigma_2'} + K_3 \delta_{\sigma_1, \sigma_1'} + K_3' \delta_{\sigma_1', \sigma_1'} + K_4 \delta_{\sigma_1, \sigma_2} + K_4' \delta_{\sigma_1, \sigma_2'} + K_5 \delta_{\sigma_1', \sigma_2} + K_5' \delta_{\sigma_1', \sigma_2'}],
\]

where \(K_0 = J_0/k_B T\), \(K_1 = J_1/k_B T\), \(K_2 = J_2/k_B T\), \(K' = J'/k_B T\), \(K'' = J''/k_B T\), \(K_3 = J_3/k_B T\), \(K_3' = J_3'/k_B T\), \(K_3'' = J_3''/k_B T\), and \(K_4 = J_4/k_B T\).

Using expressions \((\text{6})\) and \((\text{8})\), we calculate the matrix elements \(Q_{ij} = \psi_i^\dagger V \psi_j\) of the sub-block corresponding to the identity IR:

\[
\begin{align*}
Q_{11} &= |q-1 + \exp(K_1 + K_2 + K' + K'' + K_3 + K_3' + K_3'')| e^{K_0}, \\
Q_{12} &= (q-1)^{1/2} |q-2 + \exp(K_1 + K'' + K_3)| e^{K_0}, \\
Q_{21} &= (q-1)^{1/2} |q-2 + \exp(K_1 + K' + K_3)| e^{K_0}, \\
Q_{22} &= (q-2)(q-3 + e^{K_1} + e^{K_2} + e^{K'} + e^{K''})| e^{K_0}.
\end{align*}
\]

As a result, we find that the largest eigenvalue of the transfer matrix of the double \(q\)-state Potts chain with the Hamiltonian \((\text{H})\) reads

\[
\lambda_1^{(2)} = \frac{1}{2} (Q_{11} + Q_{22}) + \frac{1}{4} (Q_{11} - Q_{22})^2 + (q-1) A e^{K_0}]^{1/2},
\]

where

\[
A = [q-2 + \exp(K_1 + K'' + K_3)|q-2 + \exp(K_1 + K' + K_3)| e^{K_0}.
\]

Those variants of the double Potts chains solved earlier \([6, 12, 13, 17, 18]\) correspond to a particular choice of the interaction constants. Setting \(J_0 = J' = J\) with all other interaction constants vanishing, we arrive at the solution for the linear Potts chain \([19]\):

\[
\lambda_1^{(1)}(K) = e^K + q - 1.
\]

### 3 Triangular Potts lattice with three-site interactions on alternate triangle faces

A large number of independent parameters in the model, solved in the previous section, enables us to test Eq. \((\text{H})\) for a wide class of two-dimensional spin systems.
In addition to the cases listed in the Introduction, in which Eq. (1) is satisfied exactly, let us consider the Potts model on a triangular lattice with three-site interactions in each up-triangle (Fig. 2). The position of the critical point in this model was found with both three- and two-site interactions [13]. However, it is known [12] that for the triangle lattice with pair couplings, Eq. (1) yields the exact result only for the Ising case \((q = 2)\). Therefore we discuss the model with purely three-site interactions. For this case,

\[ K_c = \ln(1 + q). \]  

(13)

Let us show that this value satisfies Eq. (1) by subsystems in the shape of strips with a periodic boundary condition in the transverse direction.

The internal energy of one-dimensional subsystem is

\[ u_1(K) \equiv \frac{\partial f_1}{\partial K} = \left((q - 1)e^{-K} + 1\right)^{-1}. \]  

(14)

Substituting \(K_3 = \tilde{K}_3\) with all other interaction constants vanishing, from (4), (10), and (11) we obtain an expression for the free energy density of double Potts chain:

\[ f_2(K) = \frac{1}{2} \ln \left[ \frac{1}{2} \left( e^{2K} + q^2 - 1 \right) + \left( \frac{1}{4} e^{2K} - (q - 1)^2 \right)^2 + q(q - 1)(2e^K + q - 2) \right]^{1/2}. \]  

(15)

The internal energy \(u_2(K) = \partial f_2/\partial K\). Differentiating Eq. (15) with respect to \(K\) one finds an expression for \(u_2(K)\).

An analysis shows that the dependences \(u_1(K)\) and \(u_2(K)\) have a crossing point which lies exactly at \(K = K_c = \ln(1 + q)\) both for integer and non-integer \(q\). The internal energy of a system at critical point is equal to

\[ u_\infty(K_c) = u_1(K_c) = u_2(K_c) = \frac{1}{2}(1 + q^{-1}). \]  

(16)

Thus, using solutions only for the linear and double Potts chains, Eq. (1) has enabled us to extract the exact value of \(K_c\) for the bulk two-dimensional Potts model on a triangular lattice alternating faces which interact by three-site forces.

4 Discussion

In the paper [8] Eq. (1) was extended up to the form

\[ u_L(K_c) = u_{L'}(K_c) \quad (L, L' = 1, 2, 3, \ldots). \]  

(17)

Here \(u_L\) is the internal energy per site of \(L\) coupled \((d - 1)\)-dimensional subsystems. In the two-dimensional case \(L\) denotes the width of a strip.
The validity of condition (17) for arbitrary \( L \) and \( L' \) means the absence of a “singular” (i.e. depending on \( L \)) part of the internal energy density at critical point:

\[
u_L(K_c) = \text{const. upon } L.
\]

In other words, the amplitudes of all finite-size corrections to the critical internal energy of a system \( u_\infty(K_c) \) are equal to zero.

For the square isotropic Ising lattice, the derivative of the inverse correlation length \( \kappa_L(K) \) with respect to a temperature-like variable \( K \) has a similar property \cite{20, 21}:

\[
\left. \frac{\partial \kappa_L}{\partial K} \right|_{K=K_c} = \left. \frac{\partial \kappa_{L'}}{\partial K} \right|_{K=K_c},
\]

i.e. \( \partial \kappa_L / \partial K |_{c} \) does not depend on \( L \). This property has enabled us to determine exactly the value of the thermal critical exponent \( \eta_1 = 1 \) for this model using only the finite-size data \cite{20, 21}.

Equations (1) and (17) are valid for the ferromagnetic isotropic square Potts lattices. These models are self-dual and the critical coupling for them (in the anisotropic case) is determined from the condition

\[
(e^{K_x} - 1)(e^{K_y} - 1) = q.
\]

For the antiferromagnetic square-lattice Potts model the criticality condition is \cite{22}

\[
(e^{K_x} + 1)(e^{K_y} + 1) = 4 - q,
\]

where both \( K_x < 0 \) and \( K_y < 0 \). We performed a verification and found that in the antiferromagnetic case the curves \( u_1(K) \) and \( u_2(K) \) do not have any self-crossing point and therefore Eq. (1) does not lead to the exact value which follows from Eq. (21) or even to any approximate estimate for the critical point.

It is not difficult to show that if a model is self-dual and by this the dual point coincides with the original one, then Eqs. (1) and (17) are valid.

Indeed, consider for instance the Ising model on the isotropic square lattice \( L \times N \) with toroidal boundary conditions. The partition function of such a system has a fundamental property: it is invariant (up to multiplicative factor exponentially depending on \( L \)) under the duality transformation (see \cite{23}):

\[
Z_{L,N}(K^*) = (\sinh 2K)^{-LM} Z_{L,M}(K),
\]

where \( K \) and \( K^* \) are related by the condition

\[
\tanh K^* = e^{-2K}.
\]

(We here used another normalization of the exchanged constant in the Ising model, namely \( J_{\text{Potts}} = 2J_{\text{Ising}} \).) In the limit of an infinitely long strip \( (N \to \infty) \), Eq. (23) transforms to the duality condition for the largest eigenvalue:

\[
\lambda_1^{(L)}(K^*) = (\sinh 2K)^{-L} \lambda_1^{(L)}(K).
\]
It follows from here that the values of the normalized internal energy in dually conjugated points \((K \text{ and } K^*)\) are connected by the relation

\[ u_L(K^*) \frac{\partial K^*}{\partial K} = u_L(K) - 2u_0(K) \]  

in which the additive term \(u_0 (= \coth 2K)\) does not depend on \(L\). Another important feature, this time related to isotropy of a lattice, is that the dually conjugated points \(K\) and \(K^*\) confluent into one point at criticality:

\[ K^* = K = K_c. \]  

Using condition (23) we find that at the critical point \(K_c = (1/2) \ln(1 + \sqrt{2})\) the derivative \(\partial K^*/\partial K|_c = -1\). Consequently

\[ u_L(K_c) = u_0(K_c) = \sqrt{2}. \]  

Thus, the critical internal energy per site \(u_L(K_c)\) of an Ising cylinder with isotropic square cells obeys the condition (18) for all \(L = 1, 2, \ldots\). That, in turn, leads to the validity of Eqs. (1) and (17).

In an analogous way, Eqs. (1) and (17) can be derived for other isotropic spin models partition functions which satisfy a functional equation like

\[ Z_L(K^*) = [g(K)]^L Z_L(K). \]  

In those cases when the model is self-dual but the critical manifold is a line or a surface (as, e.g., for anisotropic lattices), Eqs. (1) and (17) no longer hold.

This is not difficult to prove if we consider again a two-dimensional Ising model. For the anisotropic square lattice the duality condition now looks as

\[ \lambda_1^L(K^*_x, K^*_y) = (\sinh 2K_x \sinh 2K_y)^{-L/2} \lambda_1^L(K_x, K_y) \]  

with

\[ \tanh K^*_x = e^{-2K_y} \quad \text{and} \quad \tanh K^*_y = e^{-2K_x}. \]  

From this it follows that on the critical line

\[ \sinh 2K_x \sinh 2K_y = 1 \]  

a condition (23) relates the values of the free energy in distinct (dually conjugated) points \((K_x, K_y)\) and \((K_y, K_x)\):

\[ f_L(K_x, K_y) = f_L(K_y, K_x) + \frac{1}{2} \ln(\sinh 2K_x \sinh 2K_y). \]  

This circumstance prevents fulfillment of the validity of Eqs. (1) and (17) which identify the internal energies at the same point.
The critical internal energy density of a strip $L \times \infty$ cut out from anisotropic lattice depends on the size $L$. This is easy to verify if, using the results of the section 2, one calculates the values $u_1(K_c)$ and $u_2(K_c)$ for the anisotropic Ising and Potts lattices.

On the other hand, we can establish the same property if we take the Onsager solution [24] for the two-dimensional Ising model. The dominant eigenvalue of a transfer matrix of cylinder $L \times \infty$ with spatially anisotropic interactions is equal to

$$
\lambda^{(L)}(K; \alpha) = (2 \sinh 2K)^{L/2} \exp[(\gamma_1 + \gamma_3 + \ldots + \gamma_{2L-1})/2],
$$

where $\alpha = J_y/J_x$ is the lattice anisotropy parameter and $\gamma_r$ are positive solutions of the equations

$$
cosh \gamma_r = \cosh 2\alpha K \coth 2K - \frac{\sinh 2\alpha K}{\sinh 2K} \cos \left( \frac{\pi r L}{L} \right).
$$

From this, for the internal energy per site, we obtain

$$
u_L(K; \alpha) = \coth 2K + \frac{1}{2L} \left( \frac{\partial \gamma_1}{\partial K} + \frac{\partial \gamma_3}{\partial K} + \ldots + \frac{\partial \gamma_{2L-1}}{\partial K} \right).
$$

The dependences $\gamma_r(K)$ have a smooth extremum (minimum) which lies in the isotropic case ($\alpha = 1$) exactly at $K = K_c$ and therefore

$$
\frac{\partial \gamma_r}{\partial K} \bigg|_{K=K_c} = 0 \quad (r \neq 0).
$$

As a result, the second term in Eq. (35) disappears and the critical internal energy ceases to depend on $L$. When $\alpha \neq 1$ then Eq. (36) does not have a place and $u_L(K; \alpha)$ depends on the strip width in a complicated way. This explains the failures of exact calculations of $K_c$ from Eq. (1) in the anisotropic Ising lattice [10].

Ending the section we note that, in spite of Eqs. (1) and (17), Eq. (19) cannot be deduced from a dual invariance of system.

5 Conclusions

Using a group-theoretical approach we obtained the exact analytical solution for the double Potts chain with the Hamiltonian (6). The solution gives a possibility to examine Eq. (1) for a large number of models both with Ising ($q = 2$) and arbitrary Potts spins (including non-integer $q$). The validity of Eq. (1) for the triangular Potts lattice with pure three-site interactions in alternate triangular faces was established.
In this paper it was also shown that Eqs. (1) and (17) are a consequence of a dual symmetry of models for which the critical point coincides with its dual image.

As far as the author knows, the inverse theorem has not been proved. Duality plus isotropy or, more correctly, self-conjugation of a critical point are not necessary conditions for Eq. (1). Therefore, generally speaking, there can exist systems which are not invariant under dual transformation or a combination of dual and star-triangular ones, but for which all amplitudes of finite-size corrections to the critical internal energy (or to some other quantity) are equal to zero.

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Figure captions

Fig. 1. Geometry of two-site couplings in the double $q$-state Potts chain with $S_q$ symmetry.

Fig. 2. Fragment of Potts lattice with three-site interactions in alternate triangular faces (shaded).