Groupoid Models for the C*-Algebra of Labelled Spaces

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Abstract
We define a groupoid from a labelled space and show that it is isomorphic to the tight groupoid arising from an inverse semigroup associated with the labelled space. We then define a local homeomorphism on the tight spectrum that is a generalization of the shift map for graphs, and show that the defined groupoid is isomorphic to the Renault-Deaconu groupoid for this local homeomorphism. Finally, we show that the C*-algebra of this groupoid is isomorphic to the C*-algebra of the labelled space as introduced by Bates and Pask.

Keywords C*-algebra · Labelled space · Groupoid

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1 Introduction

The definition of a C*-algebra associated with a labelled graph, or more precisely a labelled space, was first given by Bates and Pask (2007). Among the examples given by them were the classes of graph C*-algebras (Kumjian et al. 1997; Raeburn 2005), ultragraph C*-algebras (Tomforde 2003) and Carlsen-Matsumoto algebras (Carlsen and Matsumoto 2004; Carlsen 2008). These encompass other important classes of C*-algebras such as Cuntz algebras (Cuntz 1977), Cuntz-Krieger algebras (Cuntz and Krieger 1980) and Exel-Laca algebras (Exel and Laca 1999). The original definition
was later revised in Bates and Pask (2009) and then in Bates et al. (2017). We point out that the authors in Boava et al. (2017b) independently found the revised definition given in Bates et al. (2017).

One powerful tool when studying C*-algebras is to describe them as groupoid C*-algebras as pioneered by Renault (1980). Several of the classes mentioned above were shown to be isomorphic to a groupoid C*-algebra (Renault 1980; Kumjian et al. 1997; Renault 2000; Paterson 2002; Marrero and Muhly 2008; Thomsen 2010).

The main goal of this paper is to prove that the C*-algebra associated with a labelled space is also isomorphic to a groupoid C*-algebra. This was partially done for the countable case in a paper by (Carlsen et al. 2017, Theorem 8.3 and Example 11.1) where they work in the setting of Boolean dynamical systems, under an additional assumption on the labelled space. However, certain C*-algebras, such as those associated with shift spaces, when seen from the point of view of labelled graphs, deal with uncountable sets (in the case of the shift, the vertices are the points of the shift). Their approach was to use a universal property of groupoid C*-algebras described by Exel in Exel (2008) in the case of second countable étale groupoids. In a footnote of Exel (2008), Exel observes that it might be possible to circumvent the second countability hypothesis by working out the results therein using Baire σ-algebras instead of Borel σ-algebras, and it is thus possible that Carlsen, Ortega and Pardo’s results could be generalized to the uncountable case. We follow a different path, more in the lines of what was done by Kumjian et al. (1997).

Our approach is also heavily based in the framework developed by Exel (2008), working with inverse semigroups, their semilattices of idempotents and tight spectra. The interplay between inverse semigroups, groupoids and C*-algebras already appears in Renault’s monograph Renault (1980) and in Paterson’s book Paterson (1999). Paterson associated a universal groupoid with an inverse semigroup, however in several cases this was not the “correct” groupoid, in a sense; it was also necessary to restrict this groupoid by looking at the object used to define the inverse semigroup. One of Exel’s main motivations was to find the correct groupoid solely with the inverse semigroup.

The original purpose of Boava et al. (2017a, b) was to lay down the framework for the results now presented here. In Boava et al. (2017a), the authors defined an inverse semigroup associated with a labelled space by mimicking the product of elements inside the C*-algebras associated with this labelled space; contrary to what is done by Carlsen et al. (2017), however, the elements of the inverse semigroup are not seen inside the C*-algebra and, as it turns out, they need not even be the same (Boava et al. 2017b, Example 7.3). We then gave a description of the tight spectrum associated with the defined inverse semigroup that is somewhat similar to the boundary path space of a graph Webster (2014) and, depending on the labelled space, they actually coincide (Boava et al. 2017a, Proposition 6.9).

The boundary path space of a graph is the unit space of the groupoid that gives the graph C*-algebra; it is also the spectrum of a commutative C*-subalgebra, called the diagonal subalgebra. The authors extended this result to the case of labelled spaces (Boava et al. 2017b) and, in doing so, the revised definition for labelled space C*-algebras mentioned in the first paragraph was found.
After reviewing the necessary tools, we begin our paper by describing, from a labelled space, a groupoid isomorphic to the tight groupoid defined by Exel (2008), but concretely instead of as a quotient. First we work in a purely algebraic setting and then in a topological setting. By using some cutting and gluing maps defined in Boava et al. (2017b), we build a local homeomorphism and show that our groupoid can be seen as a Renault-Deaconu groupoid (Renault 1980; Deaconu 1995; Renault 2000). We then prove that the C*-algebra of this groupoid is isomorphic to the C*-algebra associated with the labelled space.

2 Preliminaries

2.1 Labelled Spaces

A (directed) graph \( E = (E^0, E^1, r, s) \) consists of non-empty sets \( E^0 \) (of vertices), \( E^1 \) (of edges), and range and source functions \( r, s : E^1 \to E^0 \). A vertex \( v \) such that \( s^{-1}(v) = \emptyset \) is called a sink, and the set of all sinks is denoted by \( E^0_{\text{sink}} \).

A path of length \( n \) on a graph \( E \) is a sequence \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_n \) of edges such that \( r(\lambda_i) = s(\lambda_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). We write \( |\lambda| = n \) for the length of \( \lambda \) and regard vertices as paths of length 0. \( E^n \) stands for the set of all paths of length \( n \) and \( E^* = \cup_{n \geq 0} E^n \). We extend the range and source maps to \( E^* \) by defining \( s(\lambda) = s(\lambda_1) \) and \( r(\lambda) = r(\lambda_n) \) if \( n \geq 2 \) and \( s(v) = v = r(v) \) for \( n = 0 \). Similarly, we define a path of infinite length (or an infinite path) as an infinite sequence \( \lambda = \lambda_1 \lambda_2 \cdots \) of edges such that \( r(\lambda_i) = s(\lambda_{i+1}) \) for all \( i \geq 1 \); for such a path, we write \( |\lambda| = \infty \); \( E^\infty \) denotes the set of all infinite paths.

A labelled graph consists of a graph \( E \) together with a surjective labelling map \( L : E^1 \to A \), where \( A \) is a fixed non-empty set, called an alphabet, and whose elements are called letters. \( A^* \) stands for the set of all finite words over \( A \), together with the empty word \( \omega \), and \( A^\infty \) is the set of all infinite words over \( A \). A labelled graph is said to be left-resolving if for each \( v \in E^0 \) the restriction of \( L \) to \( r^{-1}(v) \) is injective.

The labelling map \( L \) extends in the obvious way to \( L : E^n \to A^* \) and \( L : E^\infty \to A^\infty \). \( L^n = L(E^n) \) is the set of labelled paths of length \( n \), and \( L^\infty = \{ \alpha \in A^\infty \mid \alpha_{1,n} \in L^n, \forall n \} \) is the set of infinite labelled paths.\(^1\) The length of a labelled path \( \alpha \) is denoted by \( |\alpha| \). We consider \( \omega \) as a labelled path with \( |\omega| = 0 \), and set \( L^{\geq 1} = \cup_{n \geq 1} L^n \), \( L^* = \{ \omega \} \cup L^{\geq 1} \), and \( L^{\leq \infty} = L^* \cup L^\infty \).

For \( \alpha \in L^* \) and \( A \in \mathcal{P}(E^0) \) (the power set of \( E^0 \)), the relative range of \( \alpha \) with respect to \( A \) is the set

\[
r(A, \alpha) = \{ r(\lambda) \mid \lambda \in E^n, L(\lambda) = \alpha, s(\lambda) \in A \}
\]

if \( \alpha \in L^{\geq 1} \) and \( r(A, \omega) = A \) if \( \alpha = \omega \). The range of \( \alpha \), denoted by \( r(\alpha) \), is the set

\(^1\) This is different from Boava et al. (2017a). The authors realized that the original description of \( L^\infty \) was incorrect—for instance, (Boava et al. 2017a, Proposition 4.18) did not hold with \( L^\infty \) as originally described. With this change, all results involving \( L^\infty \) hold.
\[ r(\alpha) = r(\mathcal{E}^0, \alpha), \]

so that \( r(\omega) = \mathcal{E}^0 \) and, if \( \alpha \in \mathcal{L}^{\geq 1} \), then \( r(\alpha) = \{ r(\lambda) \in \mathcal{E}^0 \mid \mathcal{L}(\lambda) = \alpha \} \).

We also define
\[
\mathcal{L}(A \mathcal{E}^1) = \{ \mathcal{L}(e) \mid e \in \mathcal{E}^1 \text{ and } s(e) \in A \} = \{ a \in A \mid r(A, a) \neq \emptyset \}.
\]

A labelled path \( \alpha \) is a beginning of a labelled path \( \beta \) if \( \beta = \alpha \beta' \) for some labelled path \( \beta' \); also, \( \alpha \) and \( \beta \) are comparable if either one is a beginning of the other. If \( 1 \leq i \leq j \leq |\alpha| \), let \( \alpha_{i,j} = \alpha_i \alpha_{i+1} \cdots \alpha_j \) if \( j < \infty \) and \( \alpha_{i,j} = \alpha_i \alpha_{i+1} \cdots \) if \( j = \infty \). If \( j < i \) set \( \alpha_{i,j} = \omega \).

A labelled space is a triple \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) where \((\mathcal{E}, \mathcal{L})\) is a labelled graph and \( \mathcal{B} \) is a family of subsets of \( \mathcal{E}^0 \) that is closed under finite intersections and finite unions, contains all \( r(\alpha) \) for \( \alpha \in \mathcal{L}^{\geq 1} \), and is closed under relative ranges, that is, \( r(A, \alpha) \in \mathcal{B} \) for all \( A \in \mathcal{B} \) and all \( \alpha \in \mathcal{L}^* \). If in addition \( \mathcal{B} \) is closed under relative complements, we say the labelled space is normal. Finally, a labelled space \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) is weakly left-resolving if for all \( A, B \in \mathcal{B} \) and all \( \alpha \in \mathcal{L}^{\geq 1} \) we have \( r(A \cap B, \alpha) = r(A, \alpha) \cap r(B, \alpha) \).

For a given \( \alpha \in \mathcal{L}^* \), let
\[
\mathcal{B}_\alpha = \mathcal{B} \cap \mathcal{P}(r(\alpha)).
\]

If the labelled space is weakly left-resolving and normal, then the set \( \mathcal{B}_\alpha \) is a Boolean algebra for each \( \alpha \in \mathcal{L}^{\geq 1} \), and \( \mathcal{B}_\omega = \mathcal{B} \) is a generalized Boolean algebra as in Stone (1935). By Stone duality there is a topological space associated with each \( \mathcal{B}_\alpha \) with \( \alpha \in \mathcal{L}^* \), which we denote by \( X_\alpha \), consisting of the set of ultrafilters in \( \mathcal{B}_\alpha \).

### 2.2 The Inverse Semigroup of a Labelled Space

For a given labelled space \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) that is weakly left-resolving, consider the set
\[
S = \{ (\alpha, A, \beta) \mid \alpha, \beta \in \mathcal{L}^* \text{ and } A \in \mathcal{B}_\alpha \cap \mathcal{B}_\beta \text{ with } A \neq \emptyset \} \cup \{ 0 \}.
\]

A binary operation on \( S \) is defined as follows: \( s \cdot 0 = 0 \cdot s = 0 \) for all \( s \in S \) and, given \( s = (\alpha, A, \beta) \) and \( t = (\gamma, B, \delta) \) in \( S \),
\[
s \cdot t = \begin{cases} (\alpha \gamma', r(A, \gamma') \cap B, \delta), & \text{if } \gamma = \beta \gamma' \text{ and } r(A, \gamma') \cap B \neq \emptyset, \\ (\alpha, A \cap r(B, \beta'), \delta \beta'), & \text{if } \beta = \gamma \beta' \text{ and } A \cap r(B, \beta') \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]

If for a given \( s = (\alpha, A, \beta) \in S \) we define \( s^* = (\beta, A, \alpha) \), then the set \( S \), endowed with the operation above, is an inverse semigroup with zero element 0 (Boava et al. 2017a, Proposition 3.4), whose semilattice of idempotents is
\[
E(S) = \{ (\alpha, A, \alpha) \mid \alpha \in \mathcal{L}^* \text{ and } A \in \mathcal{B}_\alpha \} \cup \{ 0 \}.
\]

The natural order in the semilattice \( E(S) \) is described in the next proposition.
**Proposition 2.1** (Boava et al. 2017a, Proposition 4.1) Let \( p = (\alpha, A, \alpha) \) and \( q = (\beta, B, \beta) \) be non-zero elements in \( E(S) \). Then \( p \leq q \) if and only if \( \alpha = \beta \alpha' \) and \( A \subseteq r(B, \alpha') \).

### 2.3 Filters in \( E(S) \)

For \( E \) a (meet) semilattice with 0, there is a bijection between the set of filters in \( E \) (upper sets that are closed under meets and that do not contain 0) and the set \( \hat{E}_0 \) of characters of \( E \) (zero and meet-preserving nonzero maps from \( E \) to the Boolean algebra \( \{0, 1\} \)). Under the topology of pointwise convergence in \( \hat{E}_0 \), the closure of the subset \( \hat{E}_\infty \) of characters that correspond to ultrafilters in \( E \) is denoted by \( \hat{E}_{\text{tight}} \), and is called the **tight spectrum** of \( E \); its elements are the **tight characters** of \( E \), and their corresponding filters are **tight filters**. See (Exel 2008, Section 12) for details.

For a given labelled space \((E, \mathcal{L}, \mathcal{B})\) that is weakly left-resolving and normal, we now review a description of the tight spectrum of \( E(S) \) given in Boava et al. (2017a):

- Let \( \alpha \in \mathcal{L}_{\leq \infty} \) and \( \{F_n\}_{0 \leq n \leq |\alpha|} \) (understanding that \( 0 \leq n \leq |\alpha| \) means \( 0 \leq n < \infty \) when \( \alpha \in \mathcal{L}_{\infty} \)) be a family such that \( F_n \) is a filter in \( \mathcal{B}_1 \alpha_n \) for every \( n > 0 \) and \( F_0 \) is either a filter in \( \mathcal{B} \) or \( F_0 = \emptyset \). The family \( \{F_n\}_{0 \leq n \leq |\alpha|} \) is said to be **complete for** \( \alpha \) if

\[
F_n = \{ A \in \mathcal{B}_1 \alpha_n | r(A, \alpha_{n+1}) \in F_{n+1} \}
\]

for all \( n \geq 0 \).

It is worth pointing out that in the case of a labelled space that is weakly left-resolving and normal, if a filter in a complete family is an ultrafilter, then all filters in the family coming before it are also ultrafilters (Boava et al. 2017a, Proposition 5.7).

**Theorem 2.2** (Boava et al. 2017a, Theorem 4.13) Let \((E, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving labelled space and \( S \) be its associated inverse semigroup. Then there is a bijective correspondence between filters in \( E(S) \) and pairs \((\alpha, \{F_n\}_{0 \leq n \leq |\alpha|})\), where \( \alpha \in \mathcal{L}_{\leq \infty} \) and \( \{F_n\}_{0 \leq n \leq |\alpha|} \) is a complete family for \( \alpha \).

Filters are of **finite type** if they are associated with pairs \((\alpha, \{F_n\}_{0 \leq n \leq |\alpha|})\) for which \( |\alpha| < \infty \), and of **infinite type** otherwise.

A filter \( \xi \) in \( E(S) \) with associated labelled path \( \alpha \in \mathcal{L}_{\leq \infty} \) is sometimes denoted by \( \xi^{\alpha} \) to stress the word \( \alpha \); in addition, the filters in the complete family associated with \( \xi^{\alpha} \) will be denoted by \( \xi_n^{\alpha} \) (or simply \( \xi_n \)). Specifically,

\[
\xi_n^{\alpha} = \{ A \in \mathcal{B} | (\alpha_{1,n}, A, \alpha_{1,n}) \in \xi^{\alpha} \}.
\]

**Remark 2.3** It follows from (Boava et al. 2017a, Propositions 4.4 and 4.8) that for a filter \( \xi^{\alpha} \) in \( E(S) \) and an element \((\beta, A, \beta) \in E(S)\) we have \((\beta, A, \beta) \in \xi^{\alpha}\) if and only if \( \beta \) is a beginning of \( \alpha \) and \( A \in \xi_{|\beta|}^{\alpha} \).
Theorem 2.4  (Boava et al. 2017a, Theorems 5.10 and 6.7) Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving, normal labelled space and \(S\) be its associated inverse semigroup. Then the tight filters in \(E(S)\) are:

(i) The filters of infinite type for which the non-empty elements of their associated complete families are ultrafilters.
(ii) The filters of finite type \(\xi^\alpha\) such that \(\xi^\alpha|\alpha\) is an ultrafilter in \(\mathcal{B}_\alpha\) and for each \(A \in \xi^\alpha|\alpha\) at least one of the following conditions hold:

(a) \(\mathcal{L}(A\mathcal{E}^1)\) is infinite.
(b) There exists \(B \in \mathcal{B}_\alpha\) such that \(\emptyset \neq B \subseteq A \cap \mathcal{E}_0^\text{sink}^1\).

The set \(T\) of tight filters in \(E(S)\) is endowed with the topology induced from the topology of pointwise convergence of characters of \(E(S)\), and is thus treated as being (homeomorphic to) the tight spectrum of \(E(S)\). For \(\alpha \in \mathcal{L}^*\), we denote by \(T_\alpha\) the set of all tight filters in \(E(S)\) for which the associated word is \(\alpha\). Also, in what follows \(F\) denotes the set of all filters in \(E(S)\).

2.4 Filter Surgery in \(E(S)\)

From now on, \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) stands for a weakly left-resolving normal labelled space unless stated otherwise, and all sets \(X_\alpha\), as presented at the end of Sect. 2.1, are considered as topological spaces, with the topology given by convergence of filters (that is, the pointwise convergence of the corresponding characters). For all omitted details and proofs in this subsection, see (Boava et al. 2017b, Section 4).

Given \(\alpha, \beta \in \mathcal{L}^\geq 1\) such that \(\alpha\beta \in \mathcal{L}^\geq 1\), the relative range map \(r(\cdot, \beta) : \mathcal{B}_\alpha \rightarrow \mathcal{B}_\alpha\beta\) is a morphism of Boolean algebras and, therefore, we have its dual morphism

\[ f_{\alpha[\beta]} : X_{\alpha\beta} \rightarrow X_\alpha \]

given by \(f_{\alpha[\beta]}(\mathcal{F}) = \{A \in \mathcal{B}_\alpha \mid r(A, \beta) \in \mathcal{F}\}\) (think of this as the map on ultrafilters induced by cutting \(\beta\) from the end of the labelled path \(\alpha\beta\)).

When \(\alpha = \omega\), if \(\mathcal{F} \in \mathcal{B}_\beta\) then \(\{A \in \mathcal{B} \mid r(A, \beta) \in \mathcal{F}\}\) is either an ultrafilter in \(\mathcal{B} = \mathcal{B}_\omega\) or is the empty set, and we can therefore consider \(f_{\omega[\beta]} : X_{\beta} \rightarrow X_\omega \cup \{\emptyset\}\). These functions \(f_{\alpha[\beta]}\) are continuous, and \(f_{\alpha[\beta|\gamma]} = f_{\alpha[\beta]} \circ f_{\alpha[\beta|\gamma]}\).

We now review functions described in Boava et al. (2017b) that generalize two operations that can easily be done with paths on a graph \(\mathcal{E}\): gluing paths, that is, given \(\mu\) and \(\nu\) paths on \(\mathcal{E}\) such that \(r(\mu) = s(\nu)\), it is easy to see that \(\mu\nu\) is a new path on \(\mathcal{E}\); and cutting paths, that is, given a path \(\mu\nu\) on \(\mathcal{E}\) then \(\nu\) is also a path on the graph.

In the context of labelled spaces, we have an extra layer of complexity because filters in \(E(S)\) are described not only by a labelled path but also by a complete family of filters associated with it, by Theorem 2.2. When we cut or glue labelled paths, the Boolean algebras where the filters lie change because they depend on the labelled path. We also note that, since we are only interested in tight filters in \(E(S)\), it is enough to consider families consisting only of ultrafilters, by Theorem 2.4.
Let us begin with the gluing: for composable labelled paths \( \alpha \in \mathcal{L}_{\geq 1} \) and \( \beta \in \mathcal{L}^* \) (that is, such that \( \alpha \beta \in \mathcal{L}_{\geq 1} \)), consider the subspace \( X_{(\alpha)\beta} \) of \( X_\beta \) given by

\[
X_{(\alpha)\beta} = \{ F \in X_\beta \mid r(\alpha \beta) \in F \}.
\]

There is then a continuous map

\[
g_{(\alpha)\beta} : X_{(\alpha)\beta} \rightarrow X_{\alpha \beta}
\]

on ultrafilters induced by gluing \( \alpha \) at the beginning of the labelled path \( \beta \) given by

\[
g_{(\alpha)\beta}(F) = \{ C \cap r(\alpha \beta) \mid C \in F \}.
\]

(2.2)

The following simple result will be needed later on.

**Lemma 2.5** Suppose that \( A \in \mathcal{B}_\gamma \) and \( F \in X_{(\alpha)\gamma} \). Then, \( A \in F \) if and only if \( A \cap r(\alpha \gamma) \in g_{(\alpha)\gamma}(F) \).

**Proof** If \( A \in F \) then \( A \cap r(\alpha \gamma) \in g_{(\alpha)\gamma}(F) \) from the definition of \( g_{(\alpha)\gamma}(F) \). For the converse, suppose that \( A \cap r(\alpha \gamma) \in g_{(\alpha)\gamma}(F) \). Since \( g_{(\alpha)\gamma}(F) \subseteq F \), \( A \in \mathcal{B}_\gamma \) and \( F \) is a filter in \( \mathcal{B}_\gamma \), it follows that \( A \in F \). \( \square \)

Now for composable labelled paths \( \alpha \in \mathcal{L}_{\geq 1} \) and \( \beta \in \mathcal{L}_{\leq \infty} \), let \( T_{(\alpha)\beta} \) be the subspace of \( T_\beta \) given by

\[
T_{(\alpha)\beta} = \{ \xi \in T_\beta \mid \xi_0 \in X_{(\alpha)\omega} \}.
\]

We can then define a gluing map

\[
G_{(\alpha)\beta} : T_{(\alpha)\beta} \rightarrow T_{\alpha \beta}
\]

taking a tight filter \( \xi \in T_{(\alpha)\beta} \) to the tight filter \( \eta \in T_{\alpha \beta} \), whose complete family of (ultra)filters is obtained by gluing and cutting labelled paths appropriately, as follows:

- If \( \beta = \omega \),
  \[
  \eta_{|\alpha|} = g_{(\alpha)\omega}(\xi_0) = \{ C \cap r(\alpha) \mid C \in \xi_0 \}
  \]
  and, for for \( 0 \leq i < |\alpha| \),
  \[
  \eta_i = f_{[a_{i+1},|\alpha|]}(\eta_{|\alpha|}) = \{ D \in \mathcal{B}_{[a_{i+1},|\alpha|]} \mid r(D, \alpha_{i+1},|\alpha|) \in \eta_{|\alpha|} \};
  \]
- If \( \beta \neq \omega \), for \( 1 \leq n \leq |\beta| \) (or \( n < |\beta| \) if \( \beta \) is infinite)
  \[
  \eta_{|\alpha|+n} = g_{(\alpha)\beta_{1,n}}(\xi_n) = \{ C \cap r(\alpha \beta_{1,n}) \mid C \in \xi_n \}
  \]
  and, for \( 0 \leq i \leq |\alpha| \),
\[ \eta_i = f_{\alpha_{1,i} [\alpha_{i+1}, \beta_1]}(\eta_{\alpha_{i+1}}) = \{ D \in \mathcal{B}_{\alpha_{1,i}} \mid r(D, \alpha_{i+1}, \beta_1) \in \eta_{\alpha_{i+1}} \}. \]

Finally, for \( \alpha = \omega \) set \( T_{(\omega)\beta} = T_\beta \) and let \( G_{(\omega)\beta} \) be the identity function on \( T_\beta \).

Next, we describe the cutting: for composable labelled paths \( \alpha \in \mathcal{L}^{\geq 1} \) and \( \beta \in \mathcal{L}^* \), there is a continuous map

\[ h_{[\alpha]\beta} : X_{\alpha\beta} \rightarrow X_{(\alpha)\beta} \]

induced on ultrafilters by cutting \( \alpha \) from the beginning of \( \alpha\beta \) given by

\[ h_{[\alpha]\beta}(\mathcal{F}) = \uparrow_{\mathcal{B}_\beta} \mathcal{F} = \{ C \in \mathcal{B}_\beta \mid D \leq C \text{ for some } D \in \mathcal{F} \}. \quad (2.3) \]

**Lemma 2.6** Suppose that \( A \in \mathcal{B}_{a\gamma} \) and \( \mathcal{F} \in X_{a\gamma} \). Then, \( A \in \mathcal{F} \) if and only if \( A \in h_{[\alpha]\gamma}(\mathcal{F}) \).

**Proof** If \( A \in \mathcal{F} \) then \( A \in h_{[\alpha]\gamma}(\mathcal{F}) \) from the definition of \( h_{[\alpha]\gamma}(\mathcal{F}) \). Now, if \( A \in h_{[\alpha]\gamma}(\mathcal{F}) \), using that \( \mathcal{F} = \mathcal{g}_{(a)\gamma}(h_{[\alpha]\gamma}(\mathcal{F})) = \{ B \cap r(a\gamma) \mid B \in h_{[\alpha]\gamma}(\mathcal{F}) \} \), since \( A \subseteq r(a\gamma) \), we conclude that \( A \in \mathcal{F} \). \[ \square \]

For composable labelled paths \( \alpha \in \mathcal{L}^{\geq 1} \) and \( \beta \in \mathcal{L}_X \), these give rise to a cutting map

\[ H_{[\alpha]\beta} : T_{\alpha\beta} \rightarrow T_{(\alpha)\beta} \]

that takes a tight filter \( \xi \in T_{\alpha\beta} \) to the tight filter \( \eta \in T_{(\alpha)\beta} \) such that, for all \( n \) with \( 0 \leq n \leq |\beta| \),

\[ \eta_n = h_{[\alpha]\beta_{1,n}}(\xi_{n+|\alpha|}). \]

For \( \alpha = \omega \) define \( H_{[\alpha]\beta} \) to be the identity function over \( T_\beta \).

**Theorem 2.7** (Boava et al. 2017b, Theorem 4.17) Suppose the labelled space \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) is weakly left-resolving and normal, and let \( \alpha \in \mathcal{L}^{\geq 1} \) and \( \beta \in \mathcal{L}_X \) be such that \( \alpha\beta \in \mathcal{L}^{\leq \infty} \). Then \( H_{[\alpha]\beta} = (G_{(\alpha)\beta})^{-1} \).

**Theorem 2.8** (Boava et al. 2017b, Lemmas 4.13 and 4.16) Suppose the labelled space \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) is weakly left-resolving and normal, and let \( \alpha \), \( \beta \in \mathcal{L}^{\geq 1} \) and \( \gamma \in \mathcal{L}_X \) be such that \( \alpha\beta\gamma \in \mathcal{L}^{\leq \infty} \). Then \( G_{(\alpha\beta)\gamma} = G_{(\alpha)\beta\gamma} \circ G_{(\beta)\gamma} \) and \( H_{(\beta)\gamma} \circ H_{[\alpha]\beta\gamma} = H_{[\alpha\beta]\gamma} \).

### 2.5 The C*-Algebra of a Labelled Space

Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving, normal labelled space. The C*-algebra associated with \((\mathcal{E}, \mathcal{L}, \mathcal{B})\), denoted by \( C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \), is the universal C*-algebra generated by projections \( \{ p_A \mid A \in \mathcal{B} \} \) and partial isometries \( \{ s_a \mid a \in \mathcal{A} \} \) subject to the relations

1. \( p_{\alpha \cap \beta} = p_{\alpha \beta} + p_{\alpha \cup \beta} = p_{\alpha} + p_{\beta} - p_{\alpha \cap \beta} \) and \( p_{\emptyset} = 0 \), for every \( A, B \in \mathcal{B} \);
2. \( p_{A s_a} = s_a p_{f(A,a)} \), for every \( A \in \mathcal{B} \) and \( a \in \mathcal{A} \);
(iii) \( s_a^*s_a = p_{r(a)} \) and \( s_b^*s_a = 0 \) if \( b \neq a \), for every \( a, b \in A \);
(iv) For every \( A \in \mathcal{B} \) for which \( 0 < \#L(AE^1) < \infty \) and there does not exist \( B \in \mathcal{B} \) such that \( \emptyset \neq B \subseteq A \cap \mathcal{E}^0_{\text{sink}} \),
\[
p_A = \sum_{a \in L(AE^1)} s_{a} p_{r(A,a)} s_{a}^*.
\]

For each word \( \alpha = a_1a_2 \cdots a_n \), define \( s_\alpha = s_{a_1}s_{a_2} \cdots s_{a_n} \); we also set \( s_\omega = 1 \), where \( \omega \) is the empty word. We observe that \( s_\omega \) does not belong to \( C^*(E, L, \mathcal{B}) \) unless it is unital—we work with \( s_\omega \) to simplify our statements. For example, \( s_\omega p_A s_\omega^* \) means \( p_A \). We never use \( s_\omega \) alone.

**Proposition 2.9** Let \((E, L, \mathcal{B})\) be a weakly left-resolving normal labelled space. Then
\[
C^*(E, L, \mathcal{B}) = \overline{\text{span}\{s_{\alpha} p_A s_{\beta}^* | \alpha, \beta \in L^* \text{ and } A \in \mathcal{B}_\alpha \cap \mathcal{B}_\beta\}}.
\]

For details, see Bates et al. (2017) and Boava et al. (2017b).

**3 The Groupoid Associated With a Labelled Space**

In (Exel 2008, Section 4) a certain action of inverse semigroups on their tight spectra is constructed, from which one can associate a groupoid of germs. In this section we give an isomorphism between this groupoid of germs and a kind of boundary path groupoid (such as in Farthing et al. 2005 and Yeend 2007). This description will facilitate the study of the groupoid C*-algebra in Sect. 5.

We begin by constructing, in the present context, the tight groupoid \( \mathcal{G}_{\text{tight}} \) as in Exel (2008). Let \((E, L, \mathcal{B})\) be a labelled space, with associated inverse semigroup \( S \). For each idempotent \( e \in E(S) \), define \( D_e = \{ \phi \in \hat{E}_{\text{tight}} | \phi(e) = 1 \} \) and \( \Omega = \{(s, \phi) \in S \times \hat{E}_{\text{tight}} | \phi \in D_s \} \). The action \( \theta \) of \( S \) on \( \hat{E}_{\text{tight}} \) is given by
\[
\theta_s(\phi)(e) = \phi(s^*es).
\]

The following is an equivalence relation on \( \Omega \): \((s, \phi) \sim (t, \psi)\) if and only if \( \phi = \psi \) and there exists \( e \in E(S) \) such that \( \phi \in D_e \) and \( se = te \). Let \( \mathcal{G}_{\text{tight}} = \Omega/ \sim \) and denote the class of \((s, \phi)\) by \([s, \phi]\). Set
\[
\mathcal{G}_{\text{tight}}^{(2)} = \{ ([s, \phi], [t, \psi]) \in \mathcal{G}_{\text{tight}} \times \mathcal{G}_{\text{tight}} | \phi = \theta_t(\psi) \}
\]
and for \(([s, \phi], [t, \psi]) \in \mathcal{G}_{\text{tight}}^{(2)} \) define
\[
[s, \phi] \cdot [t, \psi] = [st, \psi].
\]

Also, for \([s, \phi] \in \mathcal{G}_{\text{tight}}\) let
\[
[s, \phi]^{-1} = [s^*, \theta_s(\phi)].
\]
Then $G_{\text{tight}}$ is a groupoid with operations defined as above.

**Remark 3.1** Given a non zero element $s = (\mu, A, v) \in S$ and $\phi \in \hat{E}_{\text{tight}}$, let us characterize when $\phi(s^*s) = 1$. Since $s^*s = (v, A, v)$, if $\xi^\alpha$ is the filter on $E(S)$ associated with $\phi$ then $\phi(s^*s) = 1$ if and only if $(v, A, v) \in \xi^\alpha$; from (2.1), this happens if and only if $v$ is a beginning of $\alpha$ and $A \in \xi^\alpha_{[\nu]}$.

**Proposition 3.2** Let $s = (\mu, A, v), t = (\beta, B, \gamma)$ be two non zero elements in $S$ and $\phi$ be a tight character associated with a filter $\xi^\alpha$ on $E(S)$. Suppose that $(s, \phi), (t, \phi) \in \Omega$ and that $v$ is a beginning of $\gamma$ with $\gamma = v\gamma'$. Then $(s, \phi) \sim (t, \phi)$ if and only if $\beta = \mu\gamma'$.

**Proof** Suppose that $(s, \phi) \sim (t, \phi)$, then there exists $e = (\delta, N, \delta) \in E(S)$ such that $\phi \in D_e$ and $se = te$. Since $\phi \in D_e$, we have $\phi(e) = 1$; therefore, Remark 3.1 ensures $\delta$ is a beginning of $\alpha$ and $N \in \xi^\alpha_{[\nu]}$. Additionally, it follows from $(t, \phi) \in \Omega$ that $\phi \in D_{r\gamma}$. Remark 3.1 then says $\gamma$ (and thus also $v$, since $\gamma = v\gamma'$) is a beginning of $\alpha$ and hence $\delta$ is comparable with $\gamma$ (and $v$). There are a few cases to consider.

First consider that $|\delta| \leq |v| \leq |\gamma|$. Writing $v = \delta v'$ and $\gamma = \delta \gamma''$ for $\mu', \gamma'' \in \mathcal{L}^*$, we have

$$se = (\mu, A \cap r(N, v'), v)$$
$$te = (\beta, B \cap r(N, \gamma''), \gamma).$$

It follows from $se = te$ that $\gamma = v$, that is, $\gamma' = \omega$, hence $\beta = \mu = \mu\gamma'$ as desired.

Now suppose that $|v| \leq |\gamma| \leq |\delta|$. In this case $\delta = \gamma\delta' = v\gamma'\delta'$ for $\delta' \in \mathcal{L}^*$, and

$$se = (\mu\gamma'\delta', r(A, \gamma'\delta') \cap N, \delta)$$
$$te = (\beta\delta', r(B, \delta') \cap N, \delta),$$

whence $\beta\delta' = \mu\gamma'\delta'$ and therefore $\beta = \mu\gamma'$.

Finally, if $|v| < |\delta| < |\gamma|$, from the previous cases the third coordinate of $se$ is $\delta$ and the third coordinate of $te$ is $\gamma$; however, since $|\delta| < |\gamma|$, we have $\delta \neq \gamma$ and $se \neq te$, which is a contradiction.

For the converse, suppose that $\beta = \mu\gamma'$ and define $e = (\gamma, r(A, \gamma') \cap B, \gamma)$. Let us first check that $e \in E(S) \setminus \{0\}$. From $(s, \phi), (t, \phi) \in \Omega$, Remark 3.1 gives $A \in \xi^\alpha_{[\nu]}$ and $B \in \xi^\alpha_{[\gamma]}$. Using that $|\gamma| = |v| + |\gamma'|$, $A \in \xi^\alpha_{[\nu]}$ and the completeness of the family $\{\xi^\alpha_n\}$ we obtain $r(A, \gamma') \in \xi^\alpha_{|\gamma|}$. In particular $B \cap r(A, \gamma') \in \xi^\alpha_{[\gamma]}$ since $\xi^\alpha_{[\gamma]}$ is a filter and so $\emptyset \neq r(A, \gamma') \cap B \in \mathbb{B}_\gamma$, hence $e \in E(S) \setminus \{0\}$. Now

$$se = (\mu\gamma', r(A, \gamma') \cap B, \gamma)$$
$$te = (\beta, r(A, \gamma') \cap B, \gamma),$$

so that $se = te$ and $(s, \phi) \sim (t, \phi)$. \hfill \Box

**Remark 3.3** Let $s = (\mu, A, v), t = (\beta, B, \gamma)$ be such that $(s, \phi), (t, \phi) \in \Omega$ for some $\phi \in \hat{E}_{\text{tight}}$. If $\xi^\alpha$ is the filter associated with $\phi$ then $v$ and $\gamma$ are both beginnings of $\circledast S
α so that they are comparable. It follows from Proposition 3.2 that \((s, \phi) \sim (t, \phi)\) if and only if \(\mu\) and \(v\) are beginnings of \(\beta\) and \(\gamma\) respectively with the same ending, or the other way around.

We now focus our attention in defining an analogue to the boundary path groupoid of a graph (Farthing et al. 2005; Yeend 2007) in the setting of labelled spaces. Let \(s = (\mu, A, v) \in S\) and \(\phi \in \hat{\mathcal{E}}_{\text{tight}}\) be such that \((s, \phi) \in \Omega\). If \(\xi^\alpha\) is the filter corresponding to \(\phi\) then by Remark 3.1, \(v\) is a beginning of \(\alpha\), that is \(\alpha = v\alpha'\) for some \(\alpha' \in \mathcal{L}_{\leq \infty}\).

As with graphs, a candidate to be associated with the class \([s, \phi]\) would be the triple \((\mu \alpha', |\mu| - |v|, v\alpha')\), but doing so would ignore the information contained in the family \(\{\xi^\alpha_n\}_n\) of ultrafilters. In order not to lose this information we cut and glue tight filters adequately, using filter surgery as described in Sect. 2.4, to build a new tight filter \(\eta^\mu \alpha^\nu\) from \(\xi^\alpha\) by cutting off \(v\) from the beginning of \(\alpha = v\alpha'\) and gluing \(\mu\) in its place.

**Proposition 3.4** Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving, normal labelled space and define

\[
\Gamma = \{(\xi^{\alpha \gamma}, |\alpha| - |\beta|, \eta^{\beta \gamma}) \in T \times \mathbb{Z} \times T \mid H_{[\alpha] \gamma}(\xi^{\alpha \gamma}) = H_{[\beta] \gamma}(\eta^{\beta \gamma})\}.
\]

Then \(\Gamma\) is a groupoid with product given by

\[
(\xi, m, \eta)(\eta, n, \rho) = (\xi, m + n, \rho)
\]

and an inverse given by

\[
(\xi, m, \eta)^{-1} = (\eta, -m, \xi).
\]

**Proof** The only difficulty lies in proving that the product \((\xi, m + n, \rho)\) is an element of \(\Gamma\). On the one hand, from \((\xi, n, \eta) \in \Gamma\) we can write \(\xi = \xi^{\alpha \gamma}\) and \(\eta = \eta^{\beta \gamma}\) with \(|\alpha| - |\beta| = m\); on the other hand, since \((\eta, n, \rho) \in \Gamma\) we have \(\eta = \eta^{\beta' \gamma'}\) and \(\rho = \rho^{\delta \gamma'}\) with \(|\beta'| - |\delta| = n\). Now \(\eta = \eta^{\beta' \gamma'} = \eta^{\delta \gamma'}\), so \(\beta' = \beta' \gamma''\); this ensures that either \(\beta\) is a beginning of \(\beta'\) or \(\beta'\) is a beginning of \(\beta\).

In the case that \(\beta\) is a beginning of \(\beta'\), say \(\beta' = \beta' \gamma''\),

\[
\beta' \gamma'' = \beta' \gamma' = \beta \gamma'\gamma',
\]

so that \(\gamma' = \gamma'' \gamma'\). Therefore \(\xi = \xi^{\alpha \gamma} = \xi^{(\alpha \gamma') \gamma'}, \rho = \rho^{\delta \gamma'},\) and

\[
m + n = (|\alpha| - |\beta|) + (|\beta'| - |\delta|)
\]

\[
= |\alpha| - |\beta| + |\beta'\gamma''| - |\delta|
\]

\[
= |\alpha \gamma''| - |\delta|.
\]
Additionally, using that $H_{[\alpha']}(\xi) = H_{[\beta']}(\eta)$ and $H_{[\beta']}(\eta) =$ $H_{[\beta']}(\rho)$,

\[
H_{[\alpha''\gamma']}(\xi) = (H_{[\gamma']}(\alpha') \circ H_{[\alpha']}(\xi)) = (H_{[\gamma']}(\alpha') \circ H_{[\alpha']}(\xi))
\]

\[= (H_{[\gamma']}(\alpha') \circ H_{[\beta']}(\eta)) = (H_{[\gamma']}(\alpha') \circ H_{[\beta']}(\eta))
\]

\[= H_{[\beta']}(\eta) = H_{[\beta']}(\eta)
\]

\[= H_{[\delta]}(\rho)
\]

whence $(\xi, m + n, \rho) \in \Gamma$. The case that $\beta'$ is a beginning of $\beta$ is analogous. \(\square\)

**Lemma 3.5** Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving, normal labelled space. Suppose that $\alpha, \beta \in \mathcal{L}^*$ are such that $\alpha\beta \in \mathcal{L}^*$, and that $\mathcal{F}$ is a filter on $\mathcal{B}_{\alpha\beta}$. If $A \in \mathcal{F}$ and $C \in \mathcal{T}_\beta$, then $A \cap C \in \mathcal{F}$.

**Proof** Notice that $A \cap C \in \mathcal{B}_{\alpha\beta}$ since $\mathcal{B}$ is closed under intersections and $A \cap C \subseteq A \subseteq r(\alpha\beta)$. Also, there exists $X \in \mathcal{F}$ such that $X \subseteq C$. Since $A \cap X \subseteq A \cap C$ and $A \cap X \in \mathcal{F}$, we have that $A \cap C \in \mathcal{F}$. \(\square\)

**Lemma 3.6** Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving, normal labelled space. Let $(t, \phi)$ be an element of $\Omega$ with $t = (\beta, A, \gamma)$ and $\phi \in \mathcal{E}_{\text{tight}}$ be a character associated with $\xi^\alpha$ so that $\alpha = \gamma\alpha'$ for some $\alpha' \in \mathcal{L}^{\geq \infty}$. Then $\eta = (G(\beta)\alpha' \circ H_{[\gamma]})^{(\xi^\alpha)}$, where $\alpha = \gamma\alpha'$, is well defined, and $\eta$ is the character associated with $\theta_t(\phi)$ given by (3.1).

**Proof** Let $(\delta, D, \delta) \in E(\mathcal{S})$. Then, using Remark 3.1,

\[
\theta_t(\phi)(\delta, D, \delta) = \phi(\gamma, A, \beta)(\delta, D, \delta)(\beta, A, \gamma)
\]

\[= \begin{cases}
\phi(\gamma, \delta', r(A, \delta') \cap D, \gamma \delta') & \text{if } \delta = \beta \delta' \\
\phi(\gamma, A \cap r(D, \beta'), \gamma) & \text{if } \beta = \delta \beta' \\
0 & \text{otherwise}
\end{cases}
\]

\[= \begin{cases}
[\delta' \text{ is a beginning of } \alpha' \text{ and } r(A, \delta') \cap D \in \xi^\alpha_{[\gamma]}] & \text{if } \delta = \beta \delta' \\
A \cap r(D, \beta') \in \xi^\alpha_{[\gamma]} & \text{if } \beta = \delta \beta' \\
0 & \text{otherwise},
\end{cases}
\]

where \([\ ]\) represents the boolean function that returns 0 if the argument is false and 1 if it is true. Note that for $\theta_t(\phi)(\delta, D, \delta) = 1$ to hold, it is necessary for $\delta$ to be a beginning of $\beta\alpha'$. There is also a condition on $D$ that depends on whether $\delta = \beta \delta'$ or $\beta = \delta \beta'$.

To see that $\eta = (G(\beta)\alpha' \circ H_{[\gamma]})^{(\xi^\alpha)}$ is well defined first notice that $(\beta, A, \gamma)$ is an element of $\mathcal{S}(E, \mathcal{L}, \mathcal{B})$ so that $\emptyset \neq A \subseteq r(\beta) \cap r(\gamma)$. If $\beta = \omega$, then the domain of $G(\beta)\alpha'$ is $\mathcal{T}_\alpha$, which contains the range of $H_{[\gamma]}\alpha'$. If $\beta \neq \omega$ then by Remark 3.1 $A \in \xi^\alpha_{[\gamma]}$, and since $A \subseteq r(\beta)$, this implies that $r(\beta) \in H_{[\gamma]}\alpha'(\xi^\alpha_0)$ by the definition of $H$. Finally, from the definition of $\mathcal{T}(\beta)\alpha'$ we conclude that $H_{[\gamma]}\alpha'(\xi^\alpha_0) \in \mathcal{T}(\beta)\alpha'$, which is the domain of $G(\beta)\alpha'$.

Let $\psi$ be the character associated with $\eta = (G(\beta)\alpha' \circ H_{[\gamma]})^{(\xi^\alpha)}$, then

\[
\psi(\delta, D, \delta) = [\delta \text{ is a beginning of } \beta\alpha' \text{ and } D \in \eta_{[\delta]}].
\]
If \( \delta = \beta \delta' \) then, by (2.2) and (2.3),
\[
\eta_{|\delta|} = (g(\beta)_{\delta'} \circ h_{|\gamma\delta'|})(\xi_{|\gamma\delta'|})
\]
\[
= g(\beta)_{\delta'} \left( \uparrow_{\mathcal{B}_{\delta'}} \xi_{|\gamma\delta'|} \right)
\]
\[
= \left\{ C \cap r(\beta \delta') \mid C \in \uparrow_{\mathcal{B}_{\delta'}} \xi_{|\gamma\delta'|} \right\}.
\]

We claim that \( D \in \eta_{|\delta|} \) if and only if \( D \cap r(A, \delta') \in \xi_{|\gamma\delta'|} \). On the one hand, if \( D \in \eta_{|\delta|} \), then there exists \( C \in \uparrow_{\mathcal{B}_{\delta'}} \xi_{|\gamma\delta'|} \) such that \( D = C \cap r(\beta \delta') \). Since \( (\beta, A, \gamma) \in S \) we have \( A \in \mathcal{B}_\beta \cap \mathcal{B}_\gamma \) and so \( r(A, \delta') \subseteq r(\beta \delta') \). Now, \( D \cap r(A, \delta') = C \cap r(\beta \delta') \subseteq r(\gamma \delta') \), which is an element of \( \xi_{|\gamma\delta'|} \) by Lemma 3.5.

On the other hand, if \( D \cap r(A, \delta') \in \xi_{|\gamma\delta'|} \), since \( D \in \mathcal{B}_\delta \subseteq \mathcal{B}_{\delta'} \) and \( r(A, \delta') \cap D \subseteq D \), then \( D \subseteq \uparrow_{\mathcal{B}_{\delta'}} \xi_{|\gamma\delta'|} \) and \( D = D \cap r(\beta \delta') \in \eta_{|\delta|} \).

Now suppose that \( \beta = \delta \beta' \). In this case by the definitions of \( G \) and \( H \) given in Sect. 2.4,
\[
\eta_{|\delta|} = (f_{\delta|\beta'|} \circ g_{\beta|\omega}) h_{|\gamma\omega|}(\xi_{\gamma|\omega|})
\]
\[
= (f_{\delta|\beta'|} \circ g_{\beta|\omega}) \left( \uparrow_{\mathcal{B}_{\omega}} \xi_{\gamma|\omega|} \right)
\]
\[
= f_{\delta|\beta'|} \left( \left\{ C \cap r(\beta) \mid C \in \uparrow_{\mathcal{B}} \xi_{\gamma|\omega|} \right\} \right)
\]
\[
= \left\{ F \in \mathcal{B}_\delta \mid r(F, \beta') \in \left\{ C \cap r(\beta) \mid C \in \uparrow_{\mathcal{B}} \xi_{\gamma|\omega|} \right\} \right\}.
\]

We claim that \( D \in \eta_{|\delta|} \) if and only if \( A \cap r(D, \beta') \in \xi_{\gamma|\omega|} \). In fact, if \( D \in \eta_{|\delta|} \) then there exists \( C \in \uparrow_{\mathcal{B}} \xi_{\gamma|\omega|} \) such that \( r(D, \beta') = C \cap r(\beta) \). Since \( A \subseteq r(\beta) \) and \( A \in \xi_{\gamma|\omega|} \), we have that \( A \cap r(D, \beta') = A \cap C \cap r(\beta) = A \cap C \) which is an element of \( \xi_{\gamma|\omega|} \) by Lemma 3.5.

On the other hand, if \( A \cap r(D, \beta') \in \xi_{\gamma|\omega|} \), since \( A \in \xi_{\gamma|\omega|} \), by choosing \( C = r(D, \beta') \) in the expression for \( \eta_{|\delta|} \), we see that \( D \in \eta_{|\delta|} \).

By comparing the formulas for \( \psi \) and \( \theta_1(\phi) \) we conclude that they are equal, that is, \( \eta \) is the filter associated with \( \theta_1(\phi) \). \( \square \)

**Theorem 3.7** Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving, normal labelled space. Then
\[
\Phi : \mathcal{G}_{tight} \rightarrow \Gamma
\]
\[
[t, \phi] \mapsto ((G_{(\beta)\omega} \circ H_{|\gamma\omega|})(\xi_{\omega})), |\beta| - |\gamma|, \xi_{\omega})
\]
is a well defined isomorphism of groupoids where \( \xi_{\omega} \) is the filter associated with \( \phi \) and \( t = (\beta, A, \gamma) \) is such that \( (\gamma, A, \gamma) \in \xi_{\omega} \), so that \( \alpha = \gamma \alpha' \) for some \( \alpha' \in \mathcal{L}_{\leq \infty} \).

**Proof** We divide the proof in several steps.

- \( \Phi \) is well defined:
For $t$ and $\phi$ as in the statement, $(G(\beta)\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha))$ is well defined by Lemma 3.6. Also, by Theorem 2.7,

$$(H_{[\beta]\alpha'} \circ G(\beta)\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha)) = H_{[\gamma]\alpha'}(\xi^\alpha)$$

so that $((G(\beta)\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha)), |\beta| - |\gamma|, \xi^\alpha) \in \Gamma$.

Now, let $s = (\mu, B, v) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be such that $\phi \in D_{s+t}$ and $[s, \phi] = [t, \phi]$. By Remark 3.3, it is sufficient to suppose that $\gamma = v\gamma'$ and $\beta = \mu\gamma'$ for some $\gamma' \in \mathcal{L}^*$. Then

$$|\beta| - |\gamma| = |\mu\gamma'| - |v\gamma'| = |\mu| - |v|$$

and, by Theorems 2.7 and 2.8,

$$(G(\beta)\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha)) = (G(\mu)\gamma'\alpha' \circ G(\gamma)\alpha' \circ H_{[\gamma]\alpha'} \circ H_{[\gamma]\alpha'}(\xi^\alpha))$$
$$= (G(\mu)\gamma'\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha))$$

so that $\Phi$ does not depend on the representative.

- $\Phi$ is injective:

Suppose that $\Phi[t, \phi] = \Phi[s, \psi]$, where $t = (\beta, A, \gamma)$ and $s = (\mu, B, v)$. Since the third entries of $\Phi[t, \phi]$ and $\Phi[s, \psi]$ are the filters associated with $\phi$ and $\psi$ respectively, for the equality $\Phi[t, \phi] = \Phi[s, \psi]$ to be true it is necessary that $\phi = \psi$. Let $\xi^\alpha$ be the filter associated with $\phi$. By Remark 3.1, $\gamma$ and $v$ are beginnings of $\alpha$ so that they are comparable. Without loss of generality, suppose that $\gamma = v\gamma'$ for some $\gamma' \in \mathcal{L}^*$.

By the definitions of $G$ and $H$, the first entries of $\Phi[t, \phi]$ and $\Phi[s, \psi]$ are filters with associated words $\beta\alpha'$ and $\mu\gamma'\alpha'$ respectively. From $\Phi[t, \phi] = \Phi[s, \psi]$, it follows that $\beta\alpha' = \mu\gamma'\alpha'$ and hence $\beta = \mu\gamma'$. By Proposition 3.2, $[t, \phi] = [s, \phi]$ and $\Phi$ is injective.

- $\Phi$ is surjective:

If $(\eta^\alpha, m, \xi^\alpha) \in \Gamma$ then there exists $\beta, \gamma \in \mathcal{L}^*$ and $\alpha' \in \mathcal{L}^{\leq \infty}$ such that $\tilde{\alpha} = \beta\alpha'$, $\alpha = \gamma\alpha'$, $m = |\beta| - |\gamma|$ and $H_{[\gamma]\alpha'}(\xi^\alpha) = H_{[\beta]\alpha'}(\eta^\alpha)$. Let $\phi$ be the filter associated with $\xi^\alpha$. Suppose that $\gamma \neq \omega$ and define $t = (\beta, r(\beta) \cap r(\gamma), \gamma)$. Since $H_{[\gamma]\alpha'}(\xi^\alpha) = H_{[\beta]\alpha'}(\eta^\alpha)$ (Boava et al. 2017b, Proposition 4.6) we have that $r(\beta) \cap r(\gamma) \neq \emptyset$. It follows from Theorem 2.7 that

$$\Phi[t, \phi] = ((G(\beta)\alpha' \circ H_{[\gamma]\alpha'}(\xi^\alpha)), |\beta| - |\gamma|, \xi^\alpha)$$
$$= ((G(\beta)\alpha' \circ H_{[\beta]\alpha'}(\eta^\alpha)), |\beta| - |\gamma|, \xi^\alpha)$$
$$= (\eta^\alpha, m, \xi^\alpha).$$

If $\gamma = \omega$ and $\alpha' \neq \omega$, we can use Theorem 2.8 to define $\gamma' = \alpha'_1$, $\beta' = \beta\alpha'_1$ and $\alpha'' = \alpha'_{2,|\alpha'|}$, and repeat the above argument. If $\gamma = \alpha' = \omega$, then there exists $A \in \xi^\alpha$ and we then take $t = (\beta, r(\beta) \cap A, \gamma)$ and again show that $\Phi[t, \phi] = (\eta^\alpha, m, \xi^\alpha)$. 

\begin{thebibliography}{9}
\end{thebibliography}
\( (\Phi \times \Phi) \left( \mathcal{G}^{(2)}_{\text{tight}} \right) \subseteq \Gamma^{(2)} \) and \( \Phi \) preserves multiplication:

Let \( ([s, \theta_t(\phi)], [t, \phi]) \in \mathcal{G}^{(2)}_{\text{tight}} \) with \( s = (\mu, B, v) \), \( t = (\beta, A, \gamma) \) and \( \xi^\alpha \) be the filter associated with \( \phi \). Also, let \( \eta^{\beta \alpha'} \) be the filter associated with \( \theta_t(\phi) \) as in Lemma 3.6. It follows from the definition of \( \Omega \) that \( st \neq 0 \), and so from the definition of the product given in Sect. 2.2 we have two cases to consider.

**Case 1:** \( \beta = v \beta' \) for some \( \beta' \in \mathcal{L}^* \). In this case \( \beta \alpha' = v \beta' \alpha' \) and

\[
st = (\mu \beta', r(B, \beta') \cap A, \gamma).
\]

On the one hand

\[
\Phi[st, \phi] = ((G(\mu \beta') \circ H_{[\gamma] \alpha'})(\xi^\alpha), |\mu \beta'| - |\gamma|, \xi^\alpha).
\]

On the other hand

\[
\Phi[s, \theta_t(\phi)] = ((G(\mu) \beta' \circ H_{[v] \beta'})(\eta^{\beta \alpha'}), |\mu| - |v|, \eta^{\beta \alpha'}
\]

and, by Lemma 3.6,

\[
\Phi[t, \phi] = ((G(\beta) \alpha' \circ H_{[\gamma] \alpha'})(\xi^\alpha), |\beta| - |\gamma|, \xi^\alpha) = (\eta^{\beta \alpha'}, |\beta| - |\gamma|, \xi^\alpha),
\]

which implies that \( (\Phi[s, \theta_t(\phi)], \Phi[t, \phi]) \in \Gamma^{(2)} \).

Multiplying, we obtain

\[
\Phi[s, \theta_t(\phi)] \Phi[t, \phi] = ((G(\mu) \beta' \circ H_{[v] \beta'})(\eta^{\beta \alpha'}), |\mu| - |v| + |\beta| - |\gamma|, \xi^\alpha).
\]

Since \( \beta = v \beta' \),

\[
|\mu| - |v| + |\beta| - |\gamma| = |\mu| - |v| + |v \beta'| - |\gamma| = |\mu \beta'| - |\gamma|.
\]

Also, from Theorems 2.7 and 2.8,

\[
(G(\mu) \beta' \circ H_{[v] \beta'})(\eta^{\beta \alpha'}) = (G(\mu) \beta' \circ H_{[v]} \beta' \circ G(\beta) \alpha' \circ H_{[\gamma] \alpha'})(\xi^\alpha)
\]

\[
= (G(\mu) \beta' \circ H_{[v]} \beta' \circ G(\beta) \alpha' \circ G(\beta') \alpha' \circ H_{[\gamma] \alpha'})(\xi^\alpha)
\]

\[
= (G(\mu) \beta' \circ G(\beta') \alpha' \circ H_{[\gamma] \alpha'})(\xi^\alpha)
\]

\[
= (G(\mu \beta') \alpha' \circ H_{[\gamma] \alpha'})(\xi^\alpha).
\]

It follows that

\[
\Phi[st, \phi] = \Phi[s, \theta_t(\phi)] \Phi[t, \phi].
\]

**Case 2:** \( v = \beta \nu' \) for some \( \nu' \in \mathcal{L}^* \). Since \( (s, \theta_t(\phi)) \in \Omega \), \( v \) is a beginning of \( \beta \alpha' \) and in this case \( \alpha' = \nu' \alpha'' \) for some \( \alpha'' \in \mathcal{L}^{\leq \infty} \). Also, we have that

\[
st = (\mu, B \cap r(A, \nu'), \gamma \nu').
\]
Now,
\[ \Phi[st, \phi] = ((G(\mu)\alpha'' \circ H[\gamma\nu]\alpha'')(\xi^\alpha), |\mu| - |\gamma\nu|, \xi^\alpha). \]

On the other hand
\[ \Phi[s, \theta_t(\phi)] = ((G(\mu)\alpha'' \circ H[\nu]\alpha'')(\eta^\beta\alpha'), |\mu| - |\nu|, \eta^\beta\alpha') \]
and
\[ \Phi[t, \phi] = ((G(\nu)\alpha' \circ H[\gamma]\alpha')(\xi^\alpha), |\beta| - |\gamma|, \xi^\alpha) = (\eta^\beta\alpha', |\beta| - |\gamma|, \xi^\alpha), \]
which again implies that \((\Phi[s, \theta_t(\phi)], \Phi[t, \phi]) \in \Gamma^{(2)}\).

Multiplying, we obtain
\[ \Phi[s, \theta_t(\phi)] \Phi[t, \phi] = ((G(\mu)\alpha'' \circ H[\nu]\alpha'')(\eta^\beta\alpha'), |\mu| - |\nu| + |\beta| - |\gamma|, \xi^\alpha). \]

Since \(\nu = \beta\nu'\),
\[ |\mu| - |\nu| + |\beta| - |\gamma| = |\mu| - |\beta\nu'| + |\beta| - |\gamma| = |\mu| - |\gamma\nu'|. \]

From Theorems 2.7 and 2.8,
\[
(G(\mu)\alpha'' \circ H[\nu]\alpha'')(\eta^\beta\alpha') = (G(\mu)\alpha'' \circ H[\nu]\alpha'' \circ G(\beta)\alpha' \circ H[\gamma]\alpha')(\xi^\alpha) \\
= (G(\mu)\alpha'' \circ H[\nu']\alpha'' \circ H[\beta]\alpha' \circ G(\beta)\alpha' \circ H[\gamma]\alpha')(\xi^\alpha) \\
= (G(\mu)\alpha'' \circ H[\gamma\nu']\alpha'' \circ H[\gamma]\alpha')(\xi^\alpha) \\
= (G(\mu)\alpha'' \circ H[\gamma\nu']\alpha')(\xi^\alpha).
\]

It follows that
\[ \Phi[st, \phi] = \Phi[s, \theta_t(\phi)] \Phi[t, \phi]. \]

\[ \Box \]

4 Topological Considerations

Our goal in this section is to describe the topology on \(\Gamma\) induced by the isomorphism given in Theorem 3.7 from the topology of \(\mathcal{G}_{tight}\) given in Exel (2008). First we recall a basis of compact-open sets for \(F\) given in Lawson (2012). For \(e \in E(S)\) define
\[ U_e = \{ \xi \in F \mid e \in \xi \} \]
and, if we are also given a finite (possibly empty) set \(\{e_1, \ldots, e_n\}\), define
\[ U_{e:e_1,\ldots,e_n} = U_e \cap U_{e_1}^c \cap \cdots \cap U_{e_n}^c. \]
As observed in Lawson (2012),
\[ U_{e; e_1, \ldots, e_n} = U_e \cap U^C_{e_1} \cap \cdots \cap U^C_{e_n} \]
so that we may assume that \( e_i \leq e \) for all \( i \in \{1, \ldots, n\} \).

**Proposition 4.1** (Lawson 2012, Lemmas 2.22 and 2.23) The sets \( U_{e; e_1, \ldots, e_n} \) form a basis of compact-open sets for \( F \) such that the resulting topology is Hausdorff.

This is the topology induced by the topology of pointwise convergence on characters, given the bijection between filters and characters.

**Remark 4.2** Since \( T \) is a closed subset of \( F \), the sets
\[ V_{e; e_1, \ldots, e_n} = U_{e; e_1, \ldots, e_n} \cap T \]
as above form a basis of compact-open sets for the relative topology on \( T \). As a particular case, we denote \( V_e = U_e \cap T \).

Now we recall the topology on \( G^\text{tight} \) given in Exel (2008). Given \( s \in S \) and an open set \( U \subseteq D_{s^*s} \), let
\[ \Theta(s, U) = \{ [s, \phi] \in G^\text{tight} \mid \phi \in U \} . \]

Then, the collections of all \( \Theta(s, U) \) is basis for a topology on \( G^\text{tight} \) making it a topological étale groupoid.

Finally we define a collection of subsets on \( \Gamma \) as follows. Given \( s = (\mu, A, \nu) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B}) \) and \( e_1, \ldots, e_n \in E(S) \), let
\[ Z_{s; e_1, \ldots, e_n} = \{ (\eta^{\mu \gamma}, |\mu| - |\nu|, \xi^{\nu \gamma}) \in \Gamma \mid \xi \in V_{e; e_1, \ldots, e_n} \text{ and } H_{[\mu]y}(\eta) = H_{[\nu]y}(\xi) \} . \]

**Remark 4.3** In the above definition, if \( e = s^*s \), we denote it by \( Z_{s; e_1, \ldots, e_n} \). We also allow \( n \) to be zero and in this case we denote it simply by \( Z_{s, e} \) or \( Z_s \) when \( e = s^*s \).

**Proposition 4.4** The sets \( Z_{s; e_1, \ldots, e_n} \) defined above form a basis of compact-open sets for the topology on \( \Gamma \) induced by the map \( \Phi \) of Theorem 3.7 from the topology on \( G^\text{tight} \) given by the sets \( \Theta(s, U) \).

**Proof** Given an open set \( U \subseteq D_{s^*s} \) for some \( s \in S(\mathcal{E}, \mathcal{L}, \mathcal{B}) \), the corresponding set of filters \( V \) is an open set of \( T \). Let \( [s, \phi] \in \Theta(s, U) \) with \( \phi \in U \) and associated filter \( \xi \in V \) be given. Then there exists \( e, e_1, \ldots, e_n \) such that \( \xi \in V_{e; e_1, \ldots, e_n} \subseteq V \). Let \( U_{e; e_1, \ldots, e_n} \) be the corresponding open set on \( D_{s^*s} \) so that \( \phi \in U_{e; e_1, \ldots, e_n} \subseteq U \) and \( [s, \phi] \in \Theta(s, U_{e; e_1, \ldots, e_n}) \subseteq \Theta(s, U) \).

It follows that the family of all sets \( \Theta(s, U_{e; e_1, \ldots, e_n}) \) is also a basis for \( G^\text{tight} \) where \( s \in S(\mathcal{E}, \mathcal{L}, \mathcal{B}) \) and \( U_{e; e_1, \ldots, e_n} \) is the corresponding set of characters associated with the elements of \( V_{e; e_1, \ldots, e_n} \).
By Theorem 2.7, $H_{[v]}(\gamma)(\eta) = H_{[v]}(\xi)$ if and only if $\eta = (G_{(\mu)} \circ H_{[v]})(\xi)$. By the definition of the isomorphism $\Phi$ of Theorem 3.7, we have that

$$\Phi(\Theta(s, U_{e:e_1,...,e_n})) = Z_{s,e:e_1,...,e_n}$$

and then it is easy to see that the sets $Z_{s,e:e_1,...,e_n}$ form a basis for the induced topology by $\Phi$ on $\Gamma$.

To see that they are compact, we use Proposition 4.15 of Exel (2008). We have the homeomorphisms

$$Z_{s,e:e_1,...,e_n} \cong \Theta(s, U_{e:e_1,...,e_n}) \cong U_{e:e_1,...,e_n} \cong V_e:e_1,...,e_n,$$

where the last is compact by Remark 4.2.

Let us now check that, with the above topology, $\Gamma$ is Hausdorff. This follows from an algebraic property of the semigroup as studied in Exel (2008).

**Definition 4.5** An inverse semigroup $S$ with zero is said to be $E^*$-unitary if for every $s \in S$ such that $e = se$ for some $e \in E(S) \setminus \{0\}$, we have that $s \in E(S)$.

**Proposition 4.6** Let $(E, L, B)$ be a weakly left-resolving labelled space, then $S(E, L, B)$ is a $E^*$-unitary inverse semigroup.

**Proof** Let $s = (\mu, B, v)$ and $e = (\delta, D, \delta) \in E(S) \setminus \{0\}$. Suppose that $e = se$, that is

$$\delta = \nu \delta'$$

and $\delta = \mu \delta'$ for some $\delta' \in L^*$. If that is the case then $\mu = v$, that is, $s = (\mu, B, \mu) \in E(S)$.

**Corollary 4.7** $\Gamma$ with the above topology is Hausdorff.

**Proof** This follows immediately from Corollary 10.9 of Exel (2008).

Next, we show that $\Gamma$ can be seen as a Renault-Deaconu groupoid (Renault 1980; Deaconu 1995; Renault 2000). For that we define, for each $n \in \mathbb{N}$ with $n \geq 1$,

$$\mathcal{T}^{(n)} = \{\xi^\alpha \in \mathcal{T} | \alpha \in L_{\leq n}, |\alpha| \geq n\}$$

and $\sigma : \mathcal{T}^{(1)} \rightarrow \mathcal{T}$ by $\sigma(\xi^\alpha) = H_{[\alpha]}(\xi)$ if $\alpha = \gamma \alpha'$. 

**Proposition 4.8** Let $\mathcal{T}^{(1)}$ and $\sigma$ be as above. Then:

(i) $\mathcal{T}^{(1)}$ is open;

(ii) $\sigma$ is a local homeomorphism.

**Proof** To prove (i), observe that if $\xi = \xi^\alpha$ with $|\alpha| \geq 1$, then $(\alpha_1, r(\alpha_1), \alpha_1) \in \xi$. Hence

$$\mathcal{T}^{(1)} = \bigcup_{a \in \mathcal{A}} V_{(a,r(a),a)},$$

which is an open set.
For (ii) we prove that, when restricted to $V_{(a,r(a),a)}$, $\sigma$ is a homeomorphism between $V_{(a,r(a),a)}$ and $V_{(\omega,r(\omega),\omega)}$. Denote by $\sigma_a$ the restriction of $\sigma$ to $V_{(a,r(a),a)}$, with $V_{(\omega,r(\omega),\omega)}$ as codomain. We use Theorem 2.7 in what follows. To see that $\sigma_a$ is injective, suppose that $\sigma_a(\xi) = \xi' = \sigma_a(\eta)$. In this case, $\xi = \xi'' = \eta''$ and $H_{[a]1}(\xi) = \sigma_a(\eta) = H_{[a]1}(\eta)$. Since $H_{[a]1}$ is injective, $\xi = \eta$. For the surjectivity, let $\zeta = \xi'' = \sigma_a(\eta)$. We show that $a$ and $\gamma$ are composable. This is trivial if $\gamma = \omega$. Supposing that $|\gamma| \geq 1$, let $n \in \mathbb{N}$ be such that $1 \leq n \leq |\gamma| (n < |\gamma|)$, if $\xi$ is infinite), then $r(a \gamma_{1,n}) = r(a) \gamma_{1,n}) \in \zeta_n$. Since $\xi_n$ is a filter, $r(a \gamma_{1,n}) \neq \emptyset$, which implies that $a \gamma_{1,n} \in \mathcal{L}^*$ and therefore $a$ and $\gamma$ are composable. It follows that $\zeta \in T_{(a)1}$. Define $\xi = G_{(a)1}(\xi)$, so that $\xi \in V_{(a,r(a),a)}$ and $\sigma_a(\xi) = H_{[a]1}(\xi) = \zeta$.

To show that $\sigma_a$ is a homeomorphism, one can use Lemma 2.6 to conclude that

$$
\sigma_a(V_{(a',a)\gamma}) = V_{(a',a)\gamma}
$$

for an arbitrary basic open set $V_{(a',a)\gamma}$ of $V_{(a,r(a),a)}$, and Lemma 2.5 to see that

$$
\sigma_a^{-1}(V_{(a',a)\gamma}) = V_{(a',a)\gamma}
$$

for an arbitrary basic open set $V_{(a',a)\gamma}$ of $V_{(a,r(a),a)}$. Hence $\sigma_a$ and $\sigma_a^{-1}$ are continuous. \hfill \Box

Notice that for $\xi^a \in T(1)$, the length of the word associated with $\sigma(\xi)$ is $|a| - 1$. This implies that for $n \in \mathbb{N}$ with $n \geq 1$, $\text{dom}^n = T(n)$. Also if $\xi = \xi^{|a|}$ then by Theorem 2.8, $\sigma^{|a|}(\xi) = H_{|a|}(\xi)$. This implies that

$$
\Gamma = \{(\eta, m - n, \xi) \mid m, n \in \mathbb{N}, \eta \in \text{dom}^m, \xi \in \text{dom}^n, \sigma^m(\eta) = \sigma^n(\xi)\},
$$

that is, $\Gamma$ is the Renault-Deaconu groupoid associated with $\sigma$ (Renault 2000, Definition 2.5).

Now, let us prove that the topology given in Proposition 4.4 is the same as the one given in Renault (2000), which is the topology defined by the basic open sets

$$
V(X, Y, m, n) = \{(\eta, m - n, \xi) \mid (\eta, \xi) \in X \times Y, \sigma^m(\eta) = \sigma^n(\xi)\} \quad (4.1)
$$

where $X$ (resp. $Y$) is an open subset of $\text{dom}^m$ (resp. $\text{dom}^n$) for which $\sigma^m|_X$ (resp. $\sigma^n|_Y$) is injective.

**Lemma 4.9** If $(a, A, B, \beta) \in S(\mathcal{E}, \mathcal{L}, \mathbb{B})$ and $e_1 = (\beta \delta_1, B_1, \beta \delta_1), \ldots, e_n = (\beta \delta_n, B_n, \beta \delta_n) \in \mathcal{E}(S)$ are such that $B_i \subseteq r(A, \delta_i)$, then $\sigma^{|a|}$ (resp. $\sigma^{\beta}$) is injective restricted to $V_{s^{|a|}f_1,...,f_n}$ (resp. $V_{s^{|a|}e_1,...,e_n}$) and $Z_{s^{|a|}e_1,...,e_n} = V_{s^{|a|}f_1,...,f_n}$, $V_{s^{|a|}e_1,...,e_n}, \{a, b|\}$, where $f_1 = (\alpha \delta_1, B_1, \alpha \delta_1), \ldots, f_n = (\alpha \delta_n, B_n, \alpha \delta_n)$.

**Proof** For injectivity, it is sufficient to show that $\sigma^{|a|}$ is injective in $V_{s^{|a|}}$. This is indeed the case, for, if $\sigma^{|a|}(\eta) = \sigma^{|a|}(\xi) = \xi'$ then $\eta = G_{(a)}(\xi') = \xi'$ by Theorem 2.7.
Now, by the definition of $S(E, L, B)$, $A \subseteq r(\alpha) \cap r(\beta)$. Since the labelled space is weakly left-resolving, this implies that for each $i = 1, \ldots, n$,

$$B_i \subseteq r(A, \delta_i) \subseteq r(r(\alpha) \cap r(\beta), \delta_i) = r(\alpha \delta_i) \cap r(\beta \delta_i).$$

From this, it follows that for $(\eta^{\alpha \nu}, |\alpha| - |\beta|, \xi^{\beta \nu}) \in \Gamma, \eta \in V_{S^{\ast}:f_1, \ldots, f_n}$ if and only if $\xi \in V_{S^{\ast}:e_1, \ldots, e_n}$.

The equality between the sets in the statement is then a consequence of their respective definitions.

**Proposition 4.10** The topologies on $\Gamma$ given by Proposition 4.4 and the basic open sets given by (4.1) coincide.

**Proof** By Lemma 4.9, it is sufficient to show that for each $(\eta, m - n, \xi) \in V(X, Y, m, n)$, there exists $s \in S(E, L, B)$ and $e_1, \ldots, e_n \in E(S)$ such that $(\eta, m - n, \xi) \in Z_{s^{e_1} \ldots e_n} \subseteq V(X, Y, m, n)$. Given such $(\eta, m - n, \xi)$, there exist labelled paths $\alpha, \beta, \gamma$ such that $|\alpha| = m, |\beta| = n, \eta = \eta^{\alpha \nu}, \xi = \xi^{\beta \nu}$, and $H_{[\alpha]^{\nu}}(\eta) = H_{[\beta]^{\nu}}(\xi)$. Since $\xi \in Y$ and $Y$ is open there exists $e, e_1, \ldots, e_n \in E(S)$ such that $\xi \in V_{e^{e_1} \ldots e_n} \subseteq Y$. This implies, by Remark 2.3, that $e$ is of the form $(\beta \delta, A, \beta \delta)$ for some beginning $\delta$ of $\gamma$. Due to the construction of $H$, $A' = A \cap r(\alpha \delta) \neq \emptyset$. Define $s = (\alpha \delta, A', \beta \delta)$, so that $Z_{s^{e_1} \ldots e_n}$ is the desired set.

**Corollary 4.11** The groupoid $\Gamma$ is amenable, so that the reduced and full C*-algebras coincide.

**Proof** This is an immediate consequence of Proposition 2.9 from Renault (2000).

5 The C*-Algebra of the Labelled Graph as a Groupoid C*-Algebra

We begin this section recalling from Renault (1980) that, since $\Gamma$ is an étale groupoid, the corresponding Haar system is given by counting measures and, for $f, g \in C_c(\Gamma)$, their product is given by

$$(f \ast g)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} f(\eta, m, \zeta)g(\zeta, n - m, \xi). \quad (5.1)$$

**Proposition 5.1** Consider the following functions in $C_c(\Gamma)$ viewed as elements of $C^*(E, L, B)$:

$$P_A = \chi_{Z_{(\omega, A, \omega)}}$$

$$S_a = \chi_{Z_{(\omega, r(a), \omega)}}$$

where $A \in B, a \in A$ and $\chi$ represents the characteristic function of the given set. Then the families $\{P_A\}_{A \in B}$ and $\{S_a\}_{a \in A}$ satisfy the relations defining $C^*(E, L, B)$. 
Proof Fix an arbitrary element \((\eta, n, \xi) \in \Gamma\). By the definition of \(Z_{(\omega, A, \omega)}\), \(P_A(\eta, n, \xi) = 1\) if and only if \(n = 0\), \(\eta = \xi\), and \(A \in \xi_0\). Similarly, by the definition of \(Z_{(a, r(a), \omega)}\), \(S_a(\eta, n, \xi) = 1\) if and only if \(n = 1\), \(\eta = \eta^{a^\gamma}\) and \(H_{[a]_{\gamma}}(\eta) = \xi\). For the remainder of the proof, let \(A, B \in \mathcal{B}\) and \(a, b \in A\) be given. When we use (5.1) below, all the sums are either zero or only have one term which is not zero, in which case the sum will be equal to 1.

Since \(\xi_0\) is a filter in \(\mathcal{B}\), \(\emptyset \notin \xi_0\) and hence \(P_\emptyset = 0\).

Now,

\[
(P_A * P_B)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} P_A(\eta, m, \zeta) P_B(\xi, n - m, \xi).
\]

This sum is not zero precisely when \(m = n = 0\), \(\eta = \xi\), \(A \in \xi_0\) and \(B \in \xi_0\). Using that \(x_0\) is a filter in \(\mathcal{B}\), \(A \in \xi_0\) and \(B \in \xi_0\) if and only if \(A \cap B \in \xi_0\), and therefore \(P_A * P_B = P_{A \cap B}\).

The equality \(P_{A \cup B} = P_A + P_B - P_{A \cap B}\) follows from the fact that \(\xi_0\) is an ultrafilter and therefore a prime filter, so that \(A \cup B \in \xi_0\) if and only if \(A \in \xi_0\) or \(B \in \xi_0\).

We compute both \(P_A * S_a\) and \(S_a * P_{r(A,a)}\) at \((\eta, n, \xi)\) and analyse when they are not zero. On the one hand

\[
(P_A * S_a)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} P_A(\eta, m, \zeta) S_a(\xi, n - m, \xi),
\]

which is not zero if and only if \(m = 0\), \(n = 1\), \(\zeta = \eta = \eta^{a^\gamma}\), \(A \in \eta_0\), \(H_{[a]_{\gamma}}(\eta) = \xi\) and \(r(A) \in \xi_0\). On the other hand

\[
(S_a * P_{r(A,a)})(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} S_a(\eta, m, \zeta) P_{r(A,a)}(\xi, n - m, \xi),
\]

which is not zero if and only if \(m = n = 1\), \(\eta = \eta^{a^\gamma}\), \(H_{[a]_{\gamma}}(\eta) = \xi = \xi\) and \(r(A, a) \in \xi_0\). In the condition that \(H_{[a]_{\gamma}}(\eta) = \xi\) we have that \(\xi_0 \subseteq \eta_1\) and, in this case, \(r(A, a) \in \xi_0\) if and only if \(r(a) \in \xi_0\) (since \(r(A, a) \subseteq r(a)\) and \(\xi_0\) is a filter) and \(A \in \eta_0\) (see the definition of complete family in Sect. 2.3). The equality \(P_A * S_a = S_a * P_{r(A,a)}\) follows.

We have that

\[
(S_a * S_b)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} S_a(\eta, m, \zeta) S_b(\xi, n - m, \xi)
= \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} S_a(\xi, -m, \eta) S_b(\xi, n - m, \xi).
\]

A necessary condition for the above sum not to be zero is that the labelled path associated with \(\xi\) must start with \(a\) and \(b\) and, in particular, it is necessary that \(a = b\). Hence, \(S_a * S_b = 0\) if \(a \neq b\). When \(a = b\), the above sum is not zero if and only if \(\zeta = \xi^{a^{\gamma}}, \eta = H_{[a]_{\gamma}}(\xi) = \xi, m = -1, n = 0\) and \(r(a) \in \xi_0\). This implies that \(S_a * S_a = P_{r(a)}\).
Finally, for the last relation, let \( A \in \mathcal{B} \) be such that \( 0 < \# \mathcal{L}(A \mathcal{E}^1) < \infty \), and such that there is no \( C \in \mathcal{B} \) with \( \emptyset \neq C \subseteq A \cap \mathcal{E}^0_{\text{sink}} \). We need to verify that

\[
P_A = \sum_{a \in \mathcal{L}(A \mathcal{E}^1)} S_a * P_{r(A,a)} * S_a^* = P_A * \sum_{a \in \mathcal{L}(A \mathcal{E}^1)} S_a * S_a^*. \tag{5.2}
\]

Let us first apply \( S_a * S_a^* \) to \((\eta, n, \xi)\),

\[
(S_a * S_a^*)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} S_a(\eta, m, \zeta) S_a^*(\eta, n-m, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} S_a(\eta, m, \zeta) S_a(\eta, m-n, \zeta),
\]

which is not zero if and only if \( m = 1, n = 0, \zeta = \zeta', \xi = \xi^{a\gamma}, \eta = \eta^{a\gamma} \) and \( H_{[a]\gamma}(\eta) = \zeta = H_{[a]\gamma}(\xi) \). Now

\[
P_A * (S_a * S_a^*)(\eta, n, \xi) = \sum_{\zeta, m: (\eta, m, \zeta) \in \Gamma} P_A(\eta, m, \xi)(S_a * S_a^*)(\eta, n-m, \xi),
\]

and using the above calculations, we see that the sum is not zero if and only if \( m = n = 0, A \in \eta_0, \zeta = \eta = \eta^{a\gamma}, \xi = \xi^{a\gamma} \) and \( H_{[a]\gamma}(\eta) = H_{[a]\gamma}(\xi) \). Applying \( G_{(a)\gamma} \) to the last equality we conclude that in this case \( \gamma = \omega \). On the other hand, as seen above, \( P_A(\eta, n, \xi) \neq 0 \) if and only if \( n = 0, A \in \eta_0 \) and \( \xi = \eta \). By Theorem 2.4 and the assumptions on \( A \), if \( \eta = \eta' \), then \( A \in \eta_0 \) implies that \( \gamma \neq \omega \), and in this case \( \gamma = a\gamma' \) for some \( a \) such that \( a \in \mathcal{L}(A \mathcal{E}^1) \). Hence, Eq. (5.2) holds. \( \square \)

Using the universal property of \( C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \), there is a \(*\)-homomorphism

\[
\pi : C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \to C^*(\Gamma) \tag{5.3}
\]

such that \( \pi(p_A) = P_A \) for all \( A \in \mathcal{B} \) and \( \pi(s_a) = S_a \) for all \( a \in \mathcal{A} \). Our next goal is to show that \( \pi \) is an isomorphism. To prove that \( \pi \) is injective we use the gauge-invariance uniqueness theorem.

**Theorem 5.2** (Bates et al. 2017, Corollary 3.10) Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a weakly left-resolving normal labelled space. Let \( \{p_A, s_a | A \in \mathcal{B}, a \in \mathcal{A}\} \) be the universal representation of \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) that generates \( C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \), let \( \{q_A, t_a | A \in \mathcal{B}, a \in \mathcal{A}\} \) be a representation of \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) in a \( C^*\)-algebra \( \mathcal{X} \) and let \( \varphi \) be the unique \(*\)-homomorphism from \( C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \) to \( \mathcal{X} \) that maps each \( p_A \) to \( q_A \) and each \( s_a \) to \( t_a \). Then \( \varphi \) is injective if and only if \( q_A \) is non-zero whenever \( A \neq \emptyset \), and for each \( z \in \mathbb{T} \) there exists a \(*\)-homomorphism \( \rho_z : C^*(\{q_A, t_a | A \in \mathcal{B}, a \in \mathcal{A}\}) \to C^*(\{q_A, t_a | A \in \mathcal{B}, a \in \mathcal{A}\}) \) such that \( \rho_z(q_A) = q_A \) and \( \rho_z(t_a) = z t_a \) for \( A \in \mathcal{B} \) and \( a \in \mathcal{A} \).

**Corollary 5.3** The homomorphism \( \pi \) in (5.3) is injective.

**Proof** To find the action \( \rho \) of \( \mathbb{T} \) on \( C^*(\Gamma) \), we consider the one-cocycle \( c : \Gamma \to \mathbb{R} \) given by \( c(\eta, k, \xi) = k \) analogous to what is done in Deaconu (1995). \( \square \)
To prove that \( \pi \) is surjective we follow the ideas of Kumjian et al. (1997), but first we need a few lemmas.

**Lemma 5.4** Let \((\alpha, A, \beta), (\mu, B, \nu) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})\), then

\[
Z_{(\alpha, A, \beta)} \cap Z_{(\mu, B, \nu)} = \begin{cases} \ Z_{(\mu, r(A, \delta) \cap B, \nu)} & \text{if } \mu = \alpha \delta, \ \nu = \beta \delta \text{ and } (r(A, \delta) \cap B) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}
\]

**Proof** Let \((\eta, k, \xi) \in \Gamma\) be given. From the definition of \(Z_{(\alpha, A, \beta)}\), \((\eta, k, \xi) \subseteq Z_{(\alpha, A, \beta)}\) if and only if \(\eta = \eta^{\alpha, \gamma}, \xi = \xi^{\beta, \gamma}, (\beta, A, \beta) \subseteq \xi\) and \(H_{[\alpha]_\gamma}(\eta) = H_{[\beta]_\gamma}(\xi)\). Similarly, \((\eta, k, \xi) \subseteq Z_{(\nu, A, \mu)}\) if and only if \(\eta = \eta^{\mu, e}, \xi = \xi^{\nu, e}, (\nu, B, \nu) \subseteq \xi\) and \(H_{[\mu]_e}(\eta) = H_{[\nu]_e}(\xi)\).

If \((\eta, k, \xi) \subseteq Z_{(\alpha, A, \beta)} \cap Z_{(\mu, B, \nu)}\) then one of the first two conditions of (5.4) must hold. To show the equality is true in the first two cases, simply use the definition of complete family given in Sect. 2.3 to conclude that \(A \subseteq \xi_{[\alpha]}\) if and only if \(r(A, \delta) \subseteq \xi_{[\mu]} = \xi_{[\nu]|+\delta}\).

**Lemma 5.5** Let \((\alpha_1, A_1, \beta_1), \ldots, (\alpha_n, A_n, \beta_n)\) be elements of \(S(\mathcal{E}, \mathcal{L}, \mathcal{B})\). Then, there exist \(B_1, \ldots, B_n \subseteq \mathcal{B} \setminus \{\emptyset\}\) such that \(B_i \subseteq A_i\) for all \(i = 1, \ldots, n\) and

\[
\bigcup_{i=1}^n Z_{(\alpha_i, B_i, \beta_i)} = \bigcup_{i=1}^n Z_{(\alpha_i, A_i, \beta_i)}
\]

where \(\cup\) represents a disjoint union.

**Proof** By induction on \(n\). We start with the case \(n = 2\). Due to Lemma 5.4, we can suppose without loss of generality that \((\alpha_1, A_1, \beta_1) = (\alpha, A, \beta)\) and \((\alpha_2, A_2, \beta_2) = (\alpha \delta, B, \beta \delta)\) in such a way that \(r(A, \delta) \cap B \neq \emptyset\). We claim that

\[
Z_{(\alpha, A, \beta)} \cup Z_{(\alpha \delta, B, \beta \delta)} = Z_{(\alpha, A, \beta)} \cup Z_{(\alpha \delta, B, \beta \delta)}.
\]

The union on the left hand side is indeed disjoint by Lemma 5.4 and, by the definition of these basic open sets, it is contained in the right hand side. To prove the other inclusion, let \((\eta, n, \xi) \subseteq Z_{(\alpha \delta, B, \beta \delta)}\) be given. Since \(\xi_{[\beta \delta]}\) is an ultrafilter, either \(r(A, \delta) \cap B\) or \(B \setminus r(A, \delta)\) belongs to \(\xi_{[\beta \delta]}\). In the first case \((\eta, n, \xi) \subseteq Z_{(\alpha, A, \beta)}\), and in the second case \((\eta, n \xi) \subseteq Z_{(\alpha \delta, B, \beta \delta)}\).

Now suppose we are given a disjoint union \(\bigcup_{i=1}^n Z_{(\alpha_i, A_i, \beta_i)}\) (by using the induction hypothesis) and a basic open set \(Z_{(\alpha, A, \beta)}\). We use the case \(n = 2\) with \(Z_{(\alpha, A, \beta)}\) and \(Z_{(\alpha_1, A_1, \beta_1)}\) to find \(C_1 \subseteq A\) and \(B_1 \subseteq A_1\) such that \(Z_{(\alpha, A, \beta)} \cup Z_{(\alpha_1, A_1, \beta_1)} = Z_{(\alpha, C_1, \beta)} \cup Z_{(\alpha_1, B_1, \beta_1)}\). Now, use the case \(n = 2\) with \(Z_{(\alpha, C_1, \beta)}\) and \(Z_{(\alpha_2, A_2, \beta_2)}\) and repeat the process to find sets \(B_i\) and \(C_i\) such that \(B_i \subseteq A_i, C_i \subseteq C_{i-1}\) and \(Z_{(\alpha, C_{i-1}, \beta)} \cup Z_{(\alpha_i, A_i, \beta_i)} = Z_{(\alpha, C_i, \beta)} \cup Z_{(\alpha_i, B_i, \beta_i)}\). Defining \(B = C_n\), it follows that

\[
Z_{(\alpha, A, \beta)} \cup \bigcup_{i=1}^n Z_{(\alpha_i, A_i, \beta_i)} = Z_{(\alpha, B, \beta)} \cup \bigcup_{i=1}^n Z_{(\alpha_i, B_i, \beta_i)}.
\]
As in the case of \( C^* (\mathcal{E}, \mathcal{L}, \mathcal{B}) \), for \( \alpha = \alpha_1 \ldots \alpha_n \in \mathcal{L}^{\geq 1} \), we will denote by \( S_\alpha \) the product \( S_{\alpha_1} \ast \cdots \ast S_{\alpha_n} \) in \( C^* (\Gamma) \).

**Lemma 5.6** For all \((\alpha, A, \beta) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})\), \( S_\alpha \ast P_A \ast S^*_\beta = \chi_{Z(\alpha, A, \beta)} \).

**Proof** Following arguments similar to the ones used in the proof of Proposition 5.1, one can show that \((S_\alpha \ast P_A \ast S^*_\beta)(\eta, A, \xi) \) is non-zero if and only if \((\alpha, A, \alpha) \in \eta, (\beta, A, \beta) \in \xi, \eta = \eta^{\alpha \gamma}, \xi = \xi^{\beta \gamma} \) and \( H_{[\alpha] \gamma}(\eta) = H_{[\beta] \gamma}(\xi) \), but this happens if and only if \((\eta, A, \xi) \in Z(\alpha, A, \beta) \). \( \square \)

**Proposition 5.7** The map \( \pi \) in (5.3) is surjective.

**Proof** As mentioned above, we follow the ideas of Kumjian et al. (1997). The argument goes as follows. By Proposition 2.9, it suffices to show that \( S := \text{span}(S_\alpha \ast P_A \ast S^*_\beta | (\alpha, A, \beta) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})) = \text{span}(\chi_{Z(\alpha, A, \beta)} | (\alpha, A, \beta) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})) \) is dense in \( C^* (\Gamma) \), where the equality is due to Lemma 5.6.

Since \( C_\mathcal{C}(\Gamma) \) is dense in \( C^* (\Gamma) \) it is sufficient to approximate an element \( f \in C_\mathcal{C}(\Gamma) \). Using Lemma 5.5, and the fact that the family of compact-open sets \( \{Z(\alpha, A, \beta) | (\alpha, A, \beta) \in S(\mathcal{E}, \mathcal{L}, \mathcal{B})\} \) covers \( \Gamma \), we can suppose that \( f \in C(Z(\alpha, A, \beta)) \) for some \((\alpha, B, \beta) \). The sup norm dominates the I-norm, which dominates the norm in \( C^* (\Gamma) \) (see Renault 1980). The result will follow from the Stone-Weierstrass theorem by proving that \( S \cap C(Z(\alpha, A, \beta)) \) separates points.

Due to Lemma 5.4, elements of \( S \cap C(Z(\alpha, A, \beta)) \) must be a linear combination of elements of the form \( \chi_{Z(\alpha, B, \beta \delta)} \) for some \( \delta \) and \( B \) such that \( r(\alpha, \delta) \in B \neq \emptyset \). Now fix two elements \((\eta^{\alpha \delta}, n, \xi^{\beta \delta}), (\chi^{\alpha \gamma}, n, \rho^{\beta \gamma}) \) of \( Z(\alpha, A, \beta) \) (here \( n = |\alpha| - |\beta| \), and \( \gamma \) and \( \delta \) may be infinite paths). If there is a labelled path \( \varepsilon \) that is a beginning of \( \gamma \) and not \( \delta \) (or vice-versa), then we can consider the characteristic function of the basic open set \( Z(\alpha, r(A, \varepsilon), B \delta) \) to separate the points. Otherwise, \( \delta = \gamma \). If this labelled path is infinite, then \( \xi \) and \( \rho \) are ultrafilters in \( E(S) \) (by Theorem 2.4) that must be equal since they have a common element, namely \((\beta, A, \beta) \). If the labelled path is finite, then it is also the case that \( \xi = \rho \) because \( \xi^{\beta \delta} \) and \( \rho^{\beta \delta} \) are ultrafilters in \( B \beta \delta \) that contain the element \( r(A, \delta) \). This implies that \( \xi = \rho \) (by the definition of complete family given in Sect. 2.3). So, in both cases \( \xi = \rho \), and since \( H_{[\alpha] \delta} \) is injective, by the definition of the groupoid it follows that \( \eta = \zeta \), and we do not have two different points to separate. \( \square \)

Putting everything together we have the following.

**Theorem 5.8** Let \((\mathcal{E}, \mathcal{L}, \mathcal{B})\) be a normal weakly left-resolving labelled space, and \( \mathcal{G}_{\text{tight}} := \mathcal{G}_{\text{tight}}(S) \) be the tight groupoid of the inverse semigroup described in Sect. 2.2. Then \( C^* (\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^* (\mathcal{G}_{\text{tight}}) \).

We now give two examples, describing all the steps in order to characterize the \( C^* \)-algebra of the labelled space.

**Example 5.9** Consider the labelled space on the alphabet \( \mathcal{A} = \{0, 1\} \) whose labelled graph is given by

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with $L(e_0) = 1$, $L(e_1) = 0$, $L(e_2) = 0$, and with $B = \mathcal{P}(E^0)$. We have that $r(0) = \{v_0, v_1\}$ and $r(1) = \{v_0\}$. More generally, writing $1 \in \alpha$ if 1 is a letter of the word $\alpha \in \mathcal{L}^*$, and $1 \notin \alpha$ if it is not, we have that

$$r(\alpha) = \begin{cases} \{v_0\} & \text{if } 1 \in \alpha \text{ and } \alpha \text{ ends with an even number of 0's}, \\ \{v_1\} & \text{if } 1 \in \alpha \text{ and } \alpha \text{ ends with an odd number of 0's}, \\ \{v_0, v_1\} & \text{if } 1 \notin \alpha. \end{cases} \quad (5.5)$$

With this we can describe all possible ultrafilters in $B_\alpha$. In the first case the only possible ultrafilter is $\{\{v_0\}\}$, in the second case the only possible ultrafilter is $\{\{v_1\}\}$, and in the third case the ultrafilters are $G_0 := \{\{v_0\}, \{v_0, v_1\}\}$ and $G_1 := \{\{v_1\}, \{v_0, v_1\}\}$.

The non-zero elements of $E(S)$ are triples $(\alpha, A, \beta)$ satisfying one of the following conditions:

- $1 \in \alpha$, $1 \in \beta$ and the number of zeroes at the end of $\alpha$ and of $\beta$ have the same parity. In this case $A = \{v_0\}$ or $A = \{v_1\}$ depending on the parity as per the formula of $r(\alpha)$ above.

- $1 \in \alpha$ and $1 \notin \beta$, or $1 \notin \alpha$ and $1 \in \beta$. In this case $A = \{v_0\}$ or $A = \{v_1\}$ depending on the parity of the labelled path containing 1.

- $1 \notin \alpha$ and $1 \notin \beta$. In this case $A$ can be any non-empty subset of $E^0$.

In order to describe $T$, we first notice that since the graph has no sinks nor infinite emitters, then we only have tight filters of infinite type. For any labelled path containing 1, the complete family is completely determined because of Eq. (5.5) and the discussion following it. For the labelled path $0^\infty = 0000\ldots$, we notice that $f_{0[0]}(G_0) = G_1$ and $f_{0[1]}(G_1) = G_0$, so we have two complete families for $0^\infty$, namely $\{G_0, G_1, G_0,\ldots\}$ and $\{G_1, G_0, G_1,\ldots\}$. In fact, we have that $T \cong \partial E$ as proven in (Boava et al. 2017a, Proposition 6.9). This can be described explicitly as follows: to each tight filter whose associated labelled path contains 1, we associate the only path on the graph that produces this labelled path; for the tight filters whose associated labelled path is $0^\infty$, if the family is $\{G_0, G_1, G_0,\ldots\}$, we take the path that starts at $v_0$ and goes through the edges with label 0, and analogously for the other family where we take the path starting at $v_1$.

We now describe the maps $h_{[\alpha]\beta}$ of Sect. 2.4. If $1 \in \beta$, then there is only one filter both in $B_{\alpha\beta}$ and $B_\alpha$, and the map $h$ sends one to the other. If $1 \notin \beta$, but $1 \in \alpha$, then there is only one filter $B_{\alpha\beta}$, say $\{\{v_i\}\}$, that is sent to the filter $\{\{v_i\}, \{v_0, v_1\}\}$. Finally $1 \notin \alpha\beta$, then we have two possible filters, $G_0, G_1$ which are kept fixed by $h$ if $|\beta|$ is even, and are sent to one another if $|\beta|$ is odd. The description of $g_{(\alpha)\beta}$ is analogous.

With the description of the maps $g$ and $h$, as well as the homeomorphism between $T$ and $\partial E$, we can see that $E_{tight}$ is the same as the groupoid of the graph given...
in Kumjian et al. (1997), $\mathcal{G}(\mathcal{E}) = \{(\mu\rho, |\mu| - |\nu|, \nu\rho) \in \partial\mathcal{E} \times \mathbb{Z} \times \partial\mathcal{E}\}$. One can easily check the topologies of the groupoids are the same, which then implies that $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^*(\mathcal{G}_{\text{right}}) \cong C^*(\mathcal{G}(\mathcal{E})) \cong C^*(\mathcal{E})$, where the latter is the C*-algebra associated to the underlying graph. We can also give a concrete isomorphism as follows: if $p_{v_0}, p_{v_1}, s_{e_0}, s_{e_1}, s_{e_2}$ are the generators of $C^*(\mathcal{E})$, we can define a map from $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$ to $C^*(\mathcal{E})$ by $p_0 \mapsto 0$, $p_{v_0} \mapsto p_{v_0}$, $p_{v_1} \mapsto p_{v_1}$, $p_{v_0 + v_0} \mapsto p_{v_0}$, $s_0 \mapsto s_{e_1} + s_{e_2}$ and $s_1 \mapsto s_{e_0}$. The inverse is given by $p_{v_0} \mapsto p_{v_0}$, $p_{v_1} \mapsto p_{v_1}$, $s_{e_0} \mapsto s_{e_1}$, $s_{e_1} \mapsto p_{v_1}s_{e_0}$ and $s_{e_2} \mapsto p_{v_0}s_{e_0}$.

**Example 5.10** Consider the labelled space on the alphabet $A = \{0, 1\}$ whose underlying labelled graph has a single vertex $v$ and a countably infinite number of edges, such that at least one edge has label 0, and at least one edge has label 1. Let $\mathcal{B} = \mathcal{P}(\mathcal{E}^0)$.

Since $\{v\}$ is the only non-empty subset of $\mathcal{E}^0$, the non-zero elements of $E(S)$ are of the form $(\alpha, \{v\}, \beta)$, where $\alpha, \beta$ are finite words over the alphabet $A$. Arguing as in the previous example we see that $T \cong \{0, 1\}^\mathbb{N}$, and that $\mathcal{G}_{\text{right}}$ is isomorphic to the groupoid

$$O_2 := \{(\mu\rho, |\mu| - |\nu|, \nu\rho) \in \{0, 1\}^\mathbb{N} \times \mathbb{Z} \times \{0, 1\}^\mathbb{N}\}$$

given in (Renault 1980, III.2). It then follows that $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^*(\mathcal{G}_{\text{right}}) \cong C^*(O_2) \cong O_2$, where the latter is the Cuntz algebra $\mathcal{O}_n$ with $n = 2$. On the other hand, it is well known that $C^*(\mathcal{E})$, the C*-algebra of the underlying graph, is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$. Therefore, $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^*(\mathcal{E})$ in this example.

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