AN EXTENDED REICH FIXED POINT THEOREM

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Abstract. We obtain an extended Reich fixed point theorem for the setting of generalized cone rectangular metric spaces. The main contractivity condition is given in terms of a weak distance altering function and the underlying cone is not assumed to be normal. Our work is a generalization of the main result in [3] and [14].

1. Introduction

A well-known fixed point theorem for metric spaces is the Banach contraction mapping theorem, which states that if $(X, d)$ is a complete metric space and the map $T : X \to X$ is a contraction, i.e. $d(Tx, Ty) \leq \lambda d(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$, then there exists a unique fixed point for the map $T$. In [15], Kannan considered a self-map $T$ on a complete metric space $(X, d)$ that satisfies

$$d(Tx, Ty) \leq \alpha d(x, Tx) + d(y, Ty) \}$$

for all $x, y \in X$. He proved that there exists a unique fixed point for the map $T$, if $\alpha \in [0, 1/2)$. Several successful attempts have been made to improve the Banach and Kannan fixed point theorems, mainly along three different directions: (a) by finding better contractivity conditions on the map $T$, (b) by replacing the underlying metric space with a more general space, for example - a partial metric space, a generalized metric space, an ordered metric space etc., and (c) by relaxing the completeness assumption. A small sample of such results can be found in [2] - [11], [19]. One such attempt is due to Reich, who in [20] proved that if $T$ is a self-map on a complete metric space $(X, d)$ that satisfies

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \mu d(x, y),$$

where $\alpha, \beta, \mu \geq 0$ and $\alpha + \beta + \mu < 1$, then $T$ has a unique fixed point.

In [12], Huang and Zhang introduced the notion of a (normal) cone metric, which is more general than a metric, and proved the Banach contraction mapping theorem for that setting. This initiated a series of articles generalizing the Banach, Kannan and other fixed point theorems to (normal)

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cone metric spaces. Hambarani and Rezapur Sh., in [21], further generalized the results of [12] by dropping the normality assumption of the cone. Along the lines of [7], Azam et al. in [3], introduced the notion of a cone rectangular metric, which is obtained by replacing the triangle inequality in the definition of a cone metric by a rectangular inequality, i.e. an inequality that involves four points and proved the Banach fixed point theorem for cone rectangular metric spaces. Jleli and Samet in [14] proved the Kannan fixed point theorem for such spaces. Further generalization to the setting of TVS-cone rectangular metric spaces was done by Abdeljawad et al. in [4]. In this article, we obtain an extended Reich fixed point theorem for the setting of generalized cone rectangular metric spaces. Being inspired by and using the techniques in [21], we also do away with the normality of the underlying cone. This work generalizes the results found in [3] and [14].

2. Preliminaries and The Main Result

Let $E$ be a real Banach space. A non-empty closed subset $P$ of $E$ is said to be a cone if

(a) $P + P \subseteq P$
(b) $\alpha P \subseteq P$ for all $\alpha \in [0, \infty)$
(c) $P \cap (-P) = \{0\}$.

The cone $P$ is said to be solid if the interior of $P$, which we will denote by $\text{int} P$, is non-empty.

Examples of Solid Cones:

(1) Let $E = \mathbb{R}$ and $P = [0, \infty)$.
(2) Let $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$.

A cone $P$ in a real Banach space $E$, induces the following partial order $\leq$ on $E$. For $x, y \in E$,

$$x \leq y \iff y - x \in P.$$ 

In the case of a solid cone $P$, we will use the notation $x \ll y$ to denote $y - x \in \text{int} P$.

A cone $P$ is said to be normal if for all $x, y \in P$ such that $x \leq y$, there exists a constant $\kappa \geq 1$ such that $\|x\| \leq \kappa \|y\|$. The examples (1) and (2) above are normal cones with $\kappa = 1$. An example of a cone that is not normal is the following (See [21]). Let $E$ be the real Banach space $C'[0, 1]$ with the norm defined as $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$ and $P$ be the cone $\{f : f \geq 0\}$.

A function $\psi : P \to P$ is said to be a weak distance altering function if

(i) $\psi$ is subadditive, i.e. for $x, y \in P$, $\psi(x + y) \leq \psi(x) + \psi(y)$.
(ii) $\psi$ is monotonically increasing, i.e. for $x, y \in P$, $\psi(x) \leq \psi(y)$ if and only if $x \leq y$.
(iii) For $0 \leq \eta \leq 1$, $x \in P$, $\psi(\eta x) \geq \eta \psi(x)$.
(iv) If for a sequence \((a_n)\) in \(P\), for every \(c \gg 0\), there exists \(N \in \mathbb{N}\) such that \(a_n \ll c\) for all \(n \geq N\), then for every \(d \gg 0\), there exists \(M \in \mathbb{N}\) such that \(\psi(a_n) \ll d\) for all \(n \geq M\).

**Remark 1:** The above definition is adapted from [17]. Note that condition (i) in the above definition implies that for \(n \in \mathbb{N}\), \(\psi(na) \leq n\psi(a)\) for all \(a \in P\) and condition (ii) implies that \(\psi\) is injective. It is indeed true that if \(P\) is normal, then condition (iii) in the above definition is equivalent to the statement \(a_n \to 0\) implies \(\psi(a_n) \to 0\), i.e. the continuity of \(\psi\) at 0.

**Examples:**

1. \(E\) is a Real Banach space with cone \(P\) and \(\psi(a) = \alpha a\), \(\alpha \geq 0\).
2. Let \(E = \mathbb{R}\), \(P = [0, \infty)\) and \(\psi(a) = a^\alpha\), \(0 \leq \alpha \leq 1\).

Let \(X\) be a nonempty set, \(E\) be a real Banach space and \(P \subset E\) be a solid cone. A map \(d : X \times X \to E\) is said to be a generalized cone rectangular metric, if there exists an integer \(s \geq 1\) such that for all \(x, y \in X\) and for all distinct elements \(u, v \in X \setminus \{x, y\}\),

(a) \(d(x, y) \geq 0\), i.e. \(d(x, y) \in P\).
(b) \(d(x, y) = 0\) if and only if \(y = x\).
(c) \(d(x, y) \leq d(x, u) + sd(u, v) + d(v, w)\).

The pair \((X, d)\) is called a generalized cone rectangular metric space. The quantity \(s\) is called a weight of \((X, d)\).

**Example:** Let \(X = \{1, 2, 3, 4\}\), \(E\) and \(P\) be as in Example (2) above. Define \(d : X \times X \to E\) by

\[
d(1, 2) = d(2, 1) = (4, 6); \\
d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = d(3, 4) = (1, 1); \\
d(k, k) = 0\ and \ d(i, j) = d(j, i); \ i,j,k \in X.
\]

\((X, d)\) is a generalized cone rectangular metric space with \(s = 4\).

**Remark 2:** Letting \(s = 1\) in the definition of the generalized cone rectangular metric \(d\) yields the definition of a cone rectangular metric, which was introduced by Azam et al. in [3]. Thus the collection of all generalized cone rectangular metric spaces includes all the cone rectangular metric spaces. Moreover, the inclusion is strict. Observe that \(d(1, 2) \not\leq d(1, 3) + d(3, 4) + d(4, 2)\).

The following is our main result, which concerns a weakly complete generalized cone rectangular metric space. i.e. a space where every Cauchy sequence \(\{x_n\}\) converges to some \(x\) in the space, in a weak sense namely, given a neighborhood \(U\) of \(x\) of a certain type, there exists a natural number
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N such that \( x_n \in U \) for all \( n \geq N \). Please see Section 4 for the actual definition of weak convergence.

**Theorem 1.** Let \((X,d)\) be a weakly complete generalized cone rectangular metric space with weight \( s \geq 1 \), \( T : X \rightarrow X \) be a map and \( \psi \) be a weak distance altering function. If for every \( x, y \in X \),

\[
\psi(d(Tx,Ty)) \leq \mu \psi(d(x,y)) + \alpha \psi(d(x,Tx)) + \beta \psi(d(y,Tx)),
\]

where \( \mu, \alpha, \beta, \gamma \geq 0 \), \( 0 \leq \mu + \alpha + \beta < 1 \), \( \mu < \frac{1}{s} \) and \( \mu + \gamma \in [0,1) \), then \( T \) has a unique fixed point.

The proof of the above Theorem is given in Section 5. Letting \( \psi \) to be the identity function and \( s = 1 \), the main result of [3] can be recovered by taking \( \alpha = \beta = \gamma = 0 \), and the main result of [14] can be recovered by taking \( \alpha = \beta \) and \( \mu = \gamma = 0 \). Also, here we do not assume the solid cone \( P \) to be normal, as was the case in [14]. The absence of the normality assumption on the cone is handled by methods given in [21].

3. SOME PROPERTIES OF A CONE IN A REAL BANACH SPACE

Recall the definition of a cone in a real Banach space given in Section 2. The following are some well-known properties of a cone \( P \) in a real Banach space \( E \). A few more of them can be found in [12], [13].

**Lemma 1.** Let \( E \) be a real Banach space and \( P \) be a cone in \( E \).

(i) \( P + \text{int} P \subset \text{int} P \)

(ii) \( \alpha (\text{int} P) \subset \text{int} P \) for all \( \alpha \in [0,\infty) \).

(iii) If \( u \in P \) and \( u \leq ku \) for some \( k \in [0,1) \), then \( u = 0 \).

**Lemma 2.** Let \( E \) be a real Banach space, \( P \) be a solid cone in \( E \) and \( u, v, w \in E \).

(i) If \( 0 \ll v \leq u \), then \( 0 \ll u \). i.e. \( u \in \text{int} P \).

(ii) If \( u \leq v \) and \( v \ll w \), then \( u \ll w \).

(iii) If \( 0 \leq u \ll c \) for all \( c \gg 0 \), then \( u = 0 \).

(iv) If \( (x_n) \) is a sequence in \( E \) such that \( x_n \geq 0 \) and \( x_n \rightarrow 0 \), then given \( c \gg 0 \), there exists \( N \in \mathbb{N} \) such that \( x_n \ll c \) for all \( n \geq N \).

**Remark 3:** Setting \( a_n = 0 \in P \), in the definition of \( \psi \), an application of part (iii) of Lemma 2 yields \( \psi(0) = 0 \). By injectivity of \( \psi \) it follows that \( \psi(a) = 0 \) if and only if \( a = 0 \).

4. MORE ON GENERALIZED CONE RECTANGULAR METRIC SPACES

Recall the definition of a generalized cone rectangular metric space given in Section 2. For \( x \in X \) and \( c \gg 0 \), define \( B(x,c) = \{ y : d(x,y) \ll c \} \subset X \). The collection \( \mathcal{B} = \{ B(x,c) : x \in X, c \gg 0 \} \) being a subbasis generates a topology on \( X \), say \( \Gamma \). Note that \( \Gamma \) consists of all unions of finite intersections of elements of \( \mathcal{B} \). In particular \( \mathcal{B} \subset \Gamma \). We will henceforth view \((X,\Gamma)\) as a
topological space. That whether the topological space \((X, \Gamma)\) is Hausdorff, remains to be seen. The following definitions are adapted from [3], [12], [14].

A sequence \((x_n)\) in a generalized cone rectangular metric space \(X\) is said to be \textit{Cauchy}, if given \(c \gg 0\), there exists \(N \in \mathbb{N}\), which is independent of \(k\), such that \(d(x_n, x_{n+k}) \ll c\) whenever \(n \geq N\).

A sequence \((x_n)\) in a generalized cone rectangular metric space \(X\) is said to \textit{converge weakly} to \(x \in X\), if given \(c \gg 0\), there exists an \(N \in \mathbb{N}\) such that \(d(x_n, x) \ll c\) for all \(n \geq N\). i.e. \(x_n \in B(x, c)\) for all \(n \geq N\). We will denote \((x_n)\) converging weakly to \(x\) by \(x_n \rightarrow x\).

A generalized cone rectangular metric space \(X\) is said to be \textit{weakly complete} if every Cauchy sequence in the space converges weakly.

**Example:** Let \((X, d)\) be the generalized cone rectangular metric space given in the Example in Section 2. It can easily be verified that the only Cauchy sequences in \(X\) are the eventually constant sequences, which of course converge weakly in \(X\). Thus \((X, d)\) is a complete generalized cone rectangular metric space.

**Remark 4:** For a general topological space \(Y\), one says that a sequence \((x_n)\) in \(Y\) converges to \(x\) if and only if given any open set \(U\) containing \(x\), there exists an \(N \in \mathbb{N}\) such that \(x_n \in U\), for all \(n \geq N\). Observe that for our purposes, we only consider a weaker form of convergence. The type of convergence defined above is weaker, because we demand the existence of an \(N \in \mathbb{N}\) not for all, but only for certain open sets containing \(x\), namely, sets of the form \(B(x, c)\).

The following lemma is a minor variant of Lemma 1.10 from [14].

**Lemma 3.** Let \((x_n)\) be a Cauchy sequence of distinct points in a generalized cone rectangular metric space \(X\). If \(x, y \not\in \{x_n : n \in \mathbb{N}\}\) and \((x_n)\) converges weakly to both \(x\) and \(y\), then \(x = y\).

**Proof.** Given \(c \gg 0\), by the hypotheses, there exists \(N \in \mathbb{N}\) such that \(d(x, x_n) \ll \frac{c}{16}, d(x_n, x_{n+1}) \ll \frac{c}{32}\) and \(d(x, y) \ll \frac{c}{32}\), for all \(m, n, \ell \geq N\). Using part (i) of Lemma 1 and part (ii) of Lemma 2 it follows that

\[d(x, y) \leq d(x, x_N) + sd(x_N, x_{N+1}) + d(x_{N+1}, y) \ll c.\]

An application of part (iii) of Lemma 2 completes the proof. \(\square\)

5. **Proof of The Main Result**

This section contains a proof of our main result stated in Section 2. A part of it is adapted from Theorem 2.1 in [14] and Theorem 1 of [18].
Proof of Theorem 1. First we prove uniqueness of the fixed point. Suppose that \( x, y \in X \) are fixed points of \( T \). From inequality (\( \Pi \)), it follows that
\[
\psi(d(x, y)) \leq \mu \psi(d(x, y)) + \alpha \psi(d(x, Tx)) + \beta \psi(d(y, Ty)) + \gamma \psi(d(y, Tx))
\]
\[
= (\mu + \gamma) \psi(d(y, x)).
\]
By part (iii) of Lemma 1, it follows that \( \psi(d(x, y)) = 0 \). Remark 3 implies that \( x = y \).

Let \( \delta = \frac{\mu + \alpha}{1 - \beta} \). Note that \( \delta \in (0, 1) \). Fix \( x \in X \). For \( n \in \mathbb{N} \), using inequality (\( \Pi \)), we get
\[
\psi(d(T^n x, T^{n+1} x)) \leq (\mu + \alpha) \psi(d(T^{n-1} x, T^n x)) + \beta \psi(d(T^n x, T^{n+1} x)).
\]
Thus for each \( n \in \mathbb{N} \),
\[
\psi(d(T^n x, T^{n+1} x)) \leq \delta \psi(d(T^{n-1} x, T^n x)).
\]
Iterating we get,
\[
(2) \quad \psi(d(T^n x, T^{n+1} x)) \leq \delta^n \psi(d(x, Tx)).
\]

Without loss of generality, one can assume that the sequence \( (T^n x) \) consists of distinct elements. This is because if \( T^m x = T^n x \) for some \( m > n \), then
\[
\psi(d(T^n x, T^{n+1} x)) = \psi(d(T^{m-n} T^n x, T^{m-n+1} T^n x)) \leq \delta^{m-n} \psi(d(T^n x, T^{n+1} x)).
\]
From part (iii) of Lemma 1 and Remark 3, it follows that \( d(T^n x, T^{n+1} x) = 0 \), i.e. \( T^n x \) is a fixed point of \( T \). Henceforth, we will assume \( T^n x \neq T^m x \) for any \( n \neq m \).

Next we prove that the sequence \( (T^n x) \) is a Cauchy sequence in \( X \). For \( k \in \mathbb{N} \), consider \( d(T^n x, T^{n+k} x) \).

Suppose that \( k \) is odd. We have,
\[
d(T^n x, T^{n+k} x) \leq \{ d(T^n x, T^{n+1} x) + s d(T^{n+1} x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) \\
+ s d(T^{n+3} x, T^{n+4} x) + \cdots + d(T^{n+k-1} x, T^{n+k} x) \}
\leq s \{ d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) \\
+ d(T^{n+3} x, T^{n+4} x) + \cdots + d(T^{n+k-1} x, T^{n+k} x) \}
\]
The monotonicity and sub-additivity of \( \psi \) yield,
\[
\psi(d(T^n x, T^{n+k} x)) \leq s \{ \psi(d(T^n x, T^{n+1} x)) + \psi(d(T^{n+1} x, T^{n+2} x)) \\
+ \psi(d(T^{n+2} x, T^{n+3} x)) + \cdots + \psi(d(T^{n+k-1} x, T^{n+k} x)) \}
\leq s (\delta^n + \delta^{n+1} + \cdots + \delta^{n+k-1}) \psi(d(x, Tx))
\leq \frac{s \delta^n}{1 - \delta} \psi(d(x, Tx)).
\]
(3)
Suppose that \( k = 2 \). Using inequality (1), the facts \( 0 \leq \alpha, \beta, \gamma \leq 1 \) and \( 0 \leq \mu < \frac{1}{s} \) and the monotonicity and sub-additivity of \( \psi \), we get
\[
\psi(d(T^n x, T^{n+2} x)) \leq \{ \mu \psi(d(T^{n-1} x, T^{n+1} x)) + \alpha \psi(d(T^{n-1} x, T^n x)) \\
+ \beta \psi(d(T^{n+1} x, T^{n+2} x)) + \gamma \psi(d(T^{n+1} x, T^n x)) \}
\leq \{ \mu \psi(d(T^{n-1} x, T^n x)) + s \psi(d(T^n x, T^{n+2} x)) \\
+ \psi(d(T^{n+2} x, T^{n+1} x)) + \alpha \psi(d(T^{n-1} x, T^n x)) \\
+ \beta \psi(d(T^{n+1} x, T^{n+2} x)) + \gamma \psi(d(T^{n+1} x, T^n x)) \}
\leq \{ (\mu + \alpha) \psi(d(T^{n-1} x, T^{n+1} x)) + (\mu + \beta) \psi(d(T^{n+1} x, T^n x)) \\
+ (\mu + \gamma) \psi(d(T^{n+1} x, T^n x)) + \mu s \psi(d(T^n x, T^{n+2} x)) \}
\leq \{ (\mu + \alpha) \delta^{n-1} + (\mu + \beta) \delta^{n+1} + \gamma \delta^n \} \psi(d(x, T x))
\leq 5 \delta^{n-1} \psi(d(x, T x)) + \mu s \psi(d(T^n x, T^{n+2} x)).
\]
Thus,
\[
(4) \quad \psi(d(T^n x, T^{n+2} x)) \leq \frac{5 \delta^{n-1}}{1 - \mu s} \psi(d(x, T x)).
\]
Suppose that \( k \) is even. We have
\[
d(T^n x, T^{n+k} x) \leq \{ d(T^n x, T^{n+2} x) + s d(T^{n+2} x, T^{n+3} x) + d(T^{n+3} x, T^{n+4} x) \\
+ s d(T^{n+4} x, T^{n+5} x) + \cdots + d(T^{n+k-1} x, T^{n+k} x) \}
\leq d(T^n x, T^{n+2} x) + s \{ d(T^{n+2} x, T^{n+3} x) + d(T^{n+3} x, T^{n+4} x) \\
+ d(T^{n+4} x, T^{n+5} x) + \cdots + d(T^{n+k-1} x, T^{n+k} x) \}
\]
The monotonicity and sub-additivity of \( \psi \) and inequality (1) together imply,
\[
\psi(d(T^n x, T^{n+k} x)) \leq \psi(d(T^n x, T^{n+2} x)) + s \{ \psi(d(T^{n+2} x, T^{n+3} x)) \\
+ \psi(d(T^{n+3} x, T^{n+4} x)) + \cdots + \psi(d(T^{n+k-1} x, T^{n+k} x)) \}
\leq \frac{5 \delta^{n-1}}{1 - \mu s} \psi(d(x, T x)) + s \{ \delta^{n+2} + \cdots + \delta^{n+k-1} \} \psi(d(x, T x))
\leq \left( \frac{5 \delta^{n-1}}{1 - \mu s} + \frac{s \delta^n}{1 - \delta} \right) \psi(d(x, T x)).
\]
It follows from inequalities (3), (4) and (5) that
\[
(6) \quad \psi(d(T^n x, T^{n+k} x)) \leq \left( \frac{5 \delta^{n-1}}{1 - \mu s} + \frac{s \delta^n}{1 - \delta} \right) \psi(d(x, T x)) \quad \text{for all } n, k \in \mathbb{N}.
\]
Let \( M \in \mathbb{N} \) be such that \( 0 \leq \frac{5 \delta^{n-1}}{1 - \mu s} + \frac{s \delta^n}{1 - \delta} \leq 1 \) for all \( n \geq M \). For each \( k \in \mathbb{N} \), it follows from property (iii) in the definition of \( \psi \) and inequality (5) that
\[
\psi(d(T^n x, T^{n+k} x)) \leq \psi(y_n), \quad \text{for all } n \geq M.
\]
Thus, using inequality (6) together with part (iv) of Lemma 2, it follows that for each $k \in \mathbb{N}$, it follows that
\[
d(T^n x, T^{n+k} x) \leq y_n, \quad \text{for all } n \geq M. \tag{7}
\]

Since the sequence $(y_n)$ in $P$ converges to 0, for a given $c \gg 0$, using part (iv) of Lemma 2 one can choose a natural number $K \geq M$ such that $y_n \ll c$ for all $n \geq K$. Thus for each $k \in \mathbb{N}$ and $n \geq K$, we get
\[
d(x_n, x_{n+k}) \leq y_n \ll c. \tag{8}
\]

An application of part (ii) of Lemma 2 implies that the sequence $(T^n x)$ is Cauchy. Since $X$ is weakly complete, there exists $u \in X$ such that $T^n x \to u$. Note that the uniqueness of the limit $u$ of the sequence $(T^n x)$ is guaranteed by Lemma 3. Our next claim is that $u$ is a fixed point of $T$.

Recall that $(T^n x)$ is a sequence in $X$ of distinct points. Suppose that $u = T^m x$ for some $m \in \mathbb{N}$. Taking $a_n = d(u, T^n x)$ in property (iv) of $\psi$ and using inequality (5) together with part (iv) of Lemma 2 it follows that for every $c \gg 0$, there exists $N \geq m + 2$ such that
\[
\psi(d(u, T^n x)) \ll \frac{(1 - \beta)c}{6} \quad \text{and} \quad \psi(d(T^{\ell} x, T^{\ell+1} x)) \ll \frac{(1 - \beta)c}{6s},
\]
for all $n, \ell \geq N$.

Since $d(u, Tu) \leq d(u, T^N x) + s d(T^N x, T^{N+1} x) + d(T^{N+1} x, Tu)$, the monotonicity and sub-additivity of $\psi$ together with inequality (11) yield,
\[
\psi(d(u, Tu)) \leq \psi(d(u, T^N x)) + s \psi(d(T^N x, T^{N+1} x)) + \psi(d(T^{N+1} x, Tu)) \\
\leq \{ \psi(d(u, T^N x)) + s \psi(d(T^N x, T^{N+1} x)) + \mu \psi(d(T^N x, u)) \\
+ \alpha \psi(d(T^N x, T^{N+1} x)) + \beta \psi(d(u, Tu)) + \gamma \psi(d(u, T^{N+1} x)) \} \\
\leq \{ 2 \psi(d(u, T^N x)) + 2 s \psi(d(T^N x, T^{N+1} x)) + \beta \psi(d(u, Tu)) \\
+ 2 \psi(d(u, T^{N+1} x)) \}. \tag{9}
\]

Thus,
\[
\psi(d(u, Tu)) \leq \frac{2}{1 - \beta} \{ \psi(d(u, T^N x)) + \psi(d(u, T^{N+1} x)) \} \\
+ \frac{2s}{1 - \beta} \psi(d(T^N x, T^{N+1} x)) \\
\ll c.
\]

By part (iii) of Lemma 2 it follows that $\psi(d(u, Tu)) = 0$ and hence by Remark 3, $d(u, Tu) = 0$. i.e. $T^m u = T^{m+1} u$. This is a contradiction to the fact that the sequence $(T^n x)$ in $X$ consists of distinct elements. Thus $u \neq T^n x$ for any $n \in \mathbb{N}$.

Suppose that $Tu = T^m x$ for some $m \in \mathbb{N}$. Let $N \in \mathbb{N}$ be as above. Using the fact that $u \neq T^n x$ for any $n \in \mathbb{N}$, it follows from an argument similar to
that \( d(u, Tu) = 0 \), i.e. \( u = Tu = T^m x \). This contradicts the fact proved above namely, \( T^n x \neq u \) for any \( n \in \mathbb{N} \).

Henceforth, we will assume that \( (T^n x) \) is in fact a sequence of distinct terms in \( X \setminus \{u, Tu\} \). Suppose that \( u \neq Tu \). Let \( N \in \mathbb{N} \) be such that inequalities in (\ref{ineq}) hold for all \( n, \ell \geq N \). An argument as in (\ref{ineq}) and an application of part (iii) of Lemma 2 plus Remark 3 imply
\[
d(u, Tu) = 0,
\]
a contradiction to our assumption that \( u \) is not a fixed point of \( T \). Thus indeed, \( u = Tu \), and this completes the proof.

**Corollary 1.** Let \( E = \mathbb{R}, P = [0, \infty) \). If \((X, d)\) is a generalized cone rectangular metric space with weight \( s \geq 1 \) and \( T : X \to X \) satisfies \( (d(Tx, Ty))^t \leq \frac{1}{t} \{ (d(Tx, x))^t + (d(Ty, y))^t \} \), where \( 0 \leq t \leq 1 \), then \( T \) has a unique fixed point.

**Proof.** Taking \( \mu = \gamma = 0, \alpha = \beta = \frac{1}{2} \) and \( \psi(a) = a^t \) in Theorem 1 yields the desired result.

**Corollary 2.** Let \( E = \mathbb{R}, P = [0, \infty) \). If \((X, d)\) is a generalized cone rectangular metric space with weight \( s \geq 1 \) and \( T : X \to X \) satisfies \( (d(Tx, Ty))^t \leq \mu (d(x, y))^t \), where \( 0 \leq \mu < 1 \) and \( 0 \leq t \leq 1 \), then \( T \) has a unique fixed point.

**Proof.** Taking \( \gamma = \alpha = \beta = 0 \) and \( \psi(a) = a^t \) in Theorem 1 yields the desired result.

**Corollary 3.** Let \( E = \mathbb{R}, P = [0, \infty) \). If \((X, d)\) is a cone rectangular metric space and \( T : X \to X \) satisfies \( (d(Tx, Ty))^t \leq \alpha (d(Tx, x))^t + \beta (d(Ty, y))^t + \mu (d(x, y))^t \), where \( 0 \leq t \leq 1 \) and \( \alpha, \beta, \mu \geq 0 \) with \( \alpha + \beta + \mu < 1 \), then \( T \) has a unique fixed point.

**Proof.** Taking \( s = 1, \gamma = 0 \) and \( \psi(a) = a^t \) in Theorem 1 yields the desired result.

**Remark 5:** Indeed Theorem 4 can be made to work for arbitrary \( s \geq 1 \), by working with the integer \( \lfloor s \rfloor + 1 \) in place of \( s \), where \( \lfloor s \rfloor \) is the integral part of \( s \).

**Remark 6:** If in the definition of \( \psi \), condition (i) is dropped, condition (ii) is replaced by the condition \( \psi(a) = 0 \) if and only if \( a = 0 \) and condition (iii) is replaced by the condition \( \psi(x) \geq x \), then we get a variant of the distance altering function considered in [17]. We point out that, a (simpler) proof of Theorem 1 for these \( \psi \) can be obtained along the lines of the given proof with suitable (minor) modifications.

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