THE b-CHROMATIC NUMBER OF HELM GRAPH

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Abstract: The b-chromatic number \( \varphi(G) \) of a graph \( G \) is the greatest integer \( k \) such that \( G \) admits a proper \( k \)-coloring in which every color class has a vertex realizing color 1 that is proficient to correspond with all the others color classes. The paper estimates the b-chromatic number of helm graph, central graph of helm graph and middle graph of helm graph which is denoted by \( H_n \), \( C(H_n) \) and \( M(H_n) \) respectively.

Keywords: b-chromatic number; Helm graph; Central graph; Middle graph

1. INTRODUCTION AND PRELIMINARIES

All graphs in this note are finite and simple. For notation not defined here we refer to Harary [5]. A vertex coloring of a graph \( G \) is a function that maps vertices of graph to a set of positive integer. If the adjacent vertices receive the different colors then minimum number of colors \( k \) is the chromatic number of graph \( G \) and it is known as proper \( k \)-coloring. The area of graph coloring is quite old and still very active. Many problems can be formulated as a graph coloring problem including channel assignment in radio stations, time tabling, clustering in data mining, scheduling, automatic reading system, micro-economics and register allocation etc. There are many problems in graph coloring, one such problem is b-coloring problem. We refer the color class as a community. The b-coloring of graph \( G \) is a proper \( k \)-coloring in which every communities contains a vertex that is adjacent to at least one vertex in each of other communities. Such a vertex is called a color-dominating vertex. We denote by \( \varphi(G) \) the maximum number \( k \) for which there exists a b-coloring of graph \( G \) using \( k \) colors. This parameter of b-coloring was introduced by Irving and Manlove [12], and is called the b-chromatic number of graph \( G \). In that paper they prove that determining the b-chromatic number is NP-hard in general and polynomial for trees. Let \( m(G) \) be the largest integer \( k \) such that \( G \) has \( k \) vertices of degree at least \( k - 1 \), and let \( \Delta(G) \) be the maximum degree in \( G \). For a given graph \( G \), it may be easily remarked that \( b(G) \leq m(G) \leq \Delta(G) + 1 \). This parameter has received exceptional consideration by many authors. Kratochvil et al. [11] showed that determining \( \varphi(G) \) is NP-hard even for bipartite graphs. Corteel et al. [4] proved that there is no constant \( \epsilon > 0 \) for which this problem can be approximated within a factor of \( 120/113 - \epsilon \) in polynomial time, unless \( P = NP \). Hoang and Kouider [6] characterize all bipartite graphs \( G \) and all P4-sparse graphs \( G \) such that each induced subgraph \( H \) of \( G \) satisfies \( b(H) = \chi(H) \), where \( \chi(H) \) is the chromatic number of \( H \). The b-chromatic number of the Cartesian product of general graphs was studied by Kouider and Zaker [9] and Kouider and Maheo [10]. Exact value of the b-chromatic number of central graph, middle graph, total graph and line graph of star graphs has been premeditated by Vijayalaksmi et al. [13]. Vivin and Venkatachalam [14] investigated the b-chromatic number of corona graph of any graph with path, cycle and complete graph. For some special families of graphs, some authors have obtained upper or lower bound for \( \varphi(G) \). The bounds for the b-chromatic number have been studied by Kouider and Maheo [7] in general and for some graph classes, see Balakrishnan et al. [2], and Chaouche and Berrachedi [3]. We also recall the following result of Kouider [8]. Let \( G \) be a graph of girth at least 3 then \( \varphi(G) > \min \{ \delta, \frac{n}{2} \} \) for minimum degree \( \delta \) and diameter \( D \).

Let \( G \) be a graph with vertex set \( V(G) \) and the edge set \( E(G) \). The central graph [1] of \( G \), denoted \( C(G) \), is obtained by subdividing each edge of \( G \) exactly once and joining all the non-adjacent vertices of \( G \). The central graph of a graph \( G \) is denoted by \( C(G) \) and the maximum degree in \( C(G) \) is \( \max \{ \Delta(G) \} \).

The middle graph [15] of \( G \) denoted by \( M(G) \), is defined with the vertex set \( V(G) \cup E(G) \) where two vertices are adjacent if they are either adjacent edges of \( G \) or one is a vertex and the other is an edge incident with it.

In this paper, we study the b-coloring of helm graph \( H_n \), central graph of helm graph \( C(H_n) \), and middle graph of helm graph \( M(H_n) \) and we obtain the b-chromatic number for these graphs.
2. HELM GRAPH

A Helm graph $H_n$ of order $n$ which contains a cycle of order $n - 1$, for which every graph vertex in the cycle is connected to hub and each vertex of the cycle adjoin by a pendant edge.

**Theorem 2.1.**

If $n \geq 6$, $\varphi(H_n) = 5$, $n$ is the number of vertices in $H_n$.

**Proof.**

Let $H_n$ be a helm graph. Let $V(H_n) = \{v_0\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq n\}$ (taken in order clockwise); where $v_0$ is the hub, $v_i$ for $1 \leq i \leq n$ are the vertex set of cycle and $u_i$ for $1 \leq i \leq n$ are the pendant vertices. Now let $E(H_n) = \{e_i; 1 \leq i \leq n\} \cup \{e_i; 1 \leq i \leq n - 1\} \cup \{e_i; 1 \leq i \leq n\}$; where $e_i$ is the edge $v_iu_i$ for $1 \leq i \leq n$, $e_i$ is the edge $v_iu_{i+1}$ (for $1 \leq i \leq n - 1$), $e_i$ is the edge $v_iu_i$ (for $1 \leq i \leq n$) and $e_n$ is edge $v_nv_1$.

By definition of central graph each edge of a graph is subdivided by new vertex. Let us assume that each edge $e_i$, $e_i$, $e_i$ is subdivided by $v_i', v_i$ and $u_i$ for $1 \leq i \leq n$ respectively and joining all non adjacent vertices. So that $V(C(H_n)) = V(H_n) \cup \{v_i'; 1 \leq i \leq n\}$ and $V(H_n) \cup \{v_i'; 1 \leq i \leq n\} \cup \{v_i'; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq n\}$.

A procedure to obtain b-chromatic number of $C(H_n)$ is as follows:

First we find $t = \left[\frac{n}{2}\right]$, where $n$ is number of vertices in $H_n$.

Assign the color $i + 1$ to $v_i$ for $1 \leq i \leq n - 1$, $i$ to $v_n$, and $u_n$, $i + 1$ to $v_i$ and $u_i$ for $1 \leq i \leq n$, $n + 1 + t$ to $v_n$ and $u_n$ for $1 \leq i \leq n - 1$. If $n$ is odd then assign $n + 1 + t$ to $v_n$. Now assign $i$ to $v_{2i-1}$ and $v_{2i}$ for $1 \leq i \leq t$.

If $\varphi(C(H_n)) = n + \left[\frac{n}{2}\right] + 2$ for $n \geq 3$ then by upper bound given Irving and Manlove [12], there must be at least $n + \left[\frac{n}{2}\right] + 2$ vertices of degree $n + \left[\frac{n}{2}\right] + 1$ in $C(H_n)$, all with distinct colors and each neighbours to vertices of the other color. E.g. if we assign distinct colors $a$ and $b$ to $v_1$ and $v_2$ respectively then pendant vertices $u_1$ and $u_2$ can’t have adjacent vertices of color $a$ and $b$ respectively, so that any recipe of colors fails to lift up new color. Therefore $n + \left[\frac{n}{2}\right] + 2$ coloring is not possible. Thus $\varphi(C(H_n)) \leq n + \left[\frac{n}{2}\right] + 1$. Hence $\varphi(C(H_n)) = n + \left[\frac{n}{2}\right] + 1$, for $n \geq 3$ (see Fig. 2). \(\square\)

**Theorem 2.2.**

If $n \geq 3$, $\varphi(C(H_n)) = n + 1 + \left[\frac{n}{2}\right]$, $n$ is the number of vertices in $H_n$.

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**Figure 1** Helm graph $H_n$.

**Figure 2** Central graph of Helm graph $C(H_n)$. 
Theorem 2.3.

If \( n \geq 8 \), then \( \phi(M(H_n)) = n + 1 \), where \( n \) is the number of vertices in \( H_n \).

Proof.

Let \( H_n \) be a helm graph of order \( n \). Let \( V(H_n) = \{v_0\} \cup \{v_1; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq 4\} \) (taken in order clockwise), where \( v_0 \) is the hub, \( v_i \) for \( 1 \leq i \leq n \) are the vertex set of cycle and \( u_i \) for \( 1 \leq i \leq n \) are the pendant vertices. Now let \( E(H_n) = \{e_i; 1 \leq i \leq n\} \cup \{e_i; 1 \leq i \leq n - 1\} \cup \{e_i; 1 \leq i \leq n\} \cup \{e_i\} \), where \( e_i \) is the edge \( v_0v_i \) (for \( 1 \leq i \leq n \)), \( e_i \) is the edge \( v_iu_{i+1} \) (for \( 1 \leq i \leq n - 1 \)), \( e_i \) is the edge \( v_iu_i \) (for \( 1 \leq i \leq n \)) and \( e_0 \) is edge \( v_0v_1 \).

By definition of middle graph each edge of helm graph is subdivided by new vertex therefore assume that each edge \( e_i \), \( e_i \) and \( e_i \) is subdivided by \( v_i, v_i \) and \( u_i \) for \( 1 \leq i \leq n \) respectively. So that \( V(C(M(H_n))) = V(Y_0) \cup E(Y_0) = \{v_0\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq n\} \). Note that the vertices \( v_i, 1 \leq i \leq n \) with \( v_0 \) induces a clique of order \( n + 1 \) in \( M(H_n) \), i.e., \( \phi(M(H_n)) \geq n + 1 \) (where \( n \) is the number of vertices in \( H_n \)). Assign proper coloring to the vertices of \( M(H_n) \) as follows:

Assign \( i + 2 \) to \( v_i \) for \( 1 \leq i \leq 5 \), \( 2 \) to \( v_i \) for \( 6 \leq i \leq n \), \( 7 \) to \( v_i \), \( 8 \) to \( v_i \), \( 9 \) to \( v_i \), \( 10 \) to \( v_i \), \( 11 \) to \( v_i \), \( 12 \) to \( v_i \). Assign \( 5 \) to \( u_{2k-1} \) for \( \geq 8 \) and \( 4 \leq k \leq \frac{n}{2} \) and \( 4 \) to \( v_2k \) for \( \geq 8 \) and \( 4 \leq k \leq \frac{n}{2} \).

Now assign \( 8 \) to \( u_{i+1}, 6 \) to \( u_i, 3 \) to \( u_i \) for \( 3 \leq i \leq n \), \( 3 \) to \( u_i \) for \( 1 \leq i \leq n \). Now assign \( i \) to \( v_i \) for \( 1 \leq i \leq n \) and \( n + 1 \) to \( v_0 \). We get a b-coloring with b-vertices \( v_i \) (for \( 1 \leq i \leq n \)) and \( v_0 \) for the color classes \( 1, 2, 3, \ldots, n \) and \( n + 1 \) respectively. It is maximal since \( \phi(M(H_n)) = m(G) \). Hence \( \phi(M(H_n)) = n + 1 \), \( n \geq 8 \) (see Fig. 3). \( \square \)

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