Generalized Chacon polynomial constructions

Vladislav Slyusarev

November 9, 2018

1 Introduction

The present work is devoted to Chacon’s automorphism, which was first described by Friedman in [1]. This rank-one automorphism is well-known for a number of unusual ergodic and spectral properties, hence it is used in the construction of examples in the theory of dynamical systems. The classical version of the automorphism if built in the ‘cutting-and-stacking’ procedure.

We build the transformation inductively. As a base, we have a Rokhlin tower of height $h_0 := 1$, which we will call Tower 0. At the $n$-th step, Tower $n$ (of height $h_{n-1}$) is divided into 3 equal sub-columns and a spacer is inserted above the middle column. Then we stack them together to get Tower $n$, of height $h_n = 3h_{n-1} + 1$. We further denote this transformation, acting on the Borel probability space $(X, A, \mu)$, as $T$.

Chacon’s automorphism $T$ is known as an example of a weakly mixing but not strongly mixing transformation (proved by Chacon in [2] for a historical version of the automorphism, constructed with division into two sub-columns instead of three, though but the arguments also apply in the above-described case). It is proved by Del Junco in [3] that $T$ has a trivial centralizer. This result was improved by Del Junco, Rahe and Swanson in [4]: they have shown that the transformation has minimal self-joinings. Prikhod’ko and Ryzhikov proved in [5] that the convolution powers of its maximal spectral type are pairwise mutually singular. They considered the Koopman operator $\hat{T}$ and the weak closure of its powers. The connection between Chacon’s automorphism and an infinite family of polynomials over the rationals was first shown in this paper, where the polynomials as functions of $\hat{T}$ are identified to prove the statement.

For a given $\alpha \in [0, 1]$, an automorphism $S$ is said to be $\alpha$-weakly mixing (for some $0 \leq \alpha \leq 1$) if there exists an integer sequence $(m_j)$ such that $S^{m_j}$ converges weakly to $\alpha \Theta + (1 - \alpha)\text{Id}$, where $\Theta$ is the ortho-projector to constants. The disjointness of the convolution powers follows from $\alpha$-weakly mixing property when $0 < \alpha < 1$ (see [6] and [7]). This property is widely used in ergodic theory to build counterexamples ([8]). The question of $\alpha$-weak mixing for Chacon’s automorphism can be considered as a special case of the general problem of description of the weak limits of powers of $T$. A similar method with application of the Koopman operator and the weak limits of its powers is used in [9]. For examples of transformations with non-trivial explicit weak closure of powers, see [10].

E. Janvresse, A. A. Prikhod’ko, T. de la Rue, and V. V. Ryzhikov have shown in [11] that the weak closure of the powers of $T$ is reduced to $\Theta$ and an explicit family of polynomials $P_n(T)$:

$$L = \Theta \cup \{ P_{m_1}(\hat{T}), \ldots, P_{m_r}(\hat{T})\hat{T}^m, r \geq 0, 1 \leq m_1 \leq \ldots \leq m_r, n \in \mathbb{Z} \}.$$

These polynomials are obtained from the representation of $T$ as an integral automorphism over the 3-adic odometer (this representation was also used earlier in [5]). Several properties and the recurrence formulae for these polynomials are also described in [11].

The purpose of the present paper is to generalize the results of [11] connected to the family of polynomials. We consider a parametric set of transformations similar to the Chacon’s automorphism and infer the properties and the recurrence equations for the polynomials generated by these transformations. The constructions are parametrized by an integer $p \geq 3$, and the classic case of Chacon’s automorphism corresponds to the parameter value $p = 3$. In Section 2, we give a definition of the functional $\phi$, the polynomials $P_n^\phi(t)$ and the generalized automorphism $T$. We provide inductive formulas for these polynomials in Section 4. The proofs of several important properties of $P_n^\phi(t)$, such as the palindromic property and the sequence of degrees, are provided in Section 3 and Section 5 respectively.
2 Basic definitions

2.1 The polynomials $P^n_m$ in the p-adic group

Let $p \geq 3$ be an arbitrary integer. Consider the compact group of $p$-adic numbers

$$\Gamma := \{x = (x_0, x_1, x_2, \ldots), x_k \in \{0, 1, \ldots, p - 1\}\}$$

We will also use the set

$$\Gamma' := \Gamma \setminus \{(p - 1, p - 1, \ldots)\},$$

where each element has only a finite number of leading elements equal to $(p - 1)$.

Let $\lambda$ be the Haar measure on $\Gamma$. Under $\lambda$, the coordinates $x_k$ are i.i.d., uniformly distributed in \{0, 1, \ldots, p - 1\}.

We define two measure-preserving (in terms of $\lambda$) transformations on $\Gamma'$:

- The shift-map $\sigma: x = (x_0, x_1, x_2, \ldots) \mapsto \sigma x = (x_1, x_2, \ldots)$
- The adding map $S: x \mapsto x + 1$, where $1 := (1, 0, 0, \ldots) \in \Gamma'$. (In general, each integer $j$ is identified with an element of $\Gamma'$, so that $S^j x = x + j$ for all $j \in \mathbb{Z}$ and all $x \in \Gamma'$.)

Let $\phi: \Gamma' \to \mathbb{Z}$ be the «first not $(p - 1)$» functional:

$$\phi(x) := x_i \text{ if } x = (p - 1, \ldots, p - 1, x_i, x_{i+1}, \ldots)$$

$$\phi^{(0)}(x) := 0; \quad \phi^{(m)}(x) := \phi(x) + \phi(Sx) + \ldots + \phi(S^{m-1} x)$$

Let us define $\pi_m$ as the probability distribution of $\phi^{(m)}$ on $\mathbb{Z}$:

$$\pi_m(j) = \lambda(\phi^{(m)} = j)$$

We consider the sequences of polynomials $P^n_m$ produced by $\pi_m$ for fixed $p$ and $m$:

$$P^n_m(t) := E_{\lambda} \left[t^{\phi^{(m)}(x)}\right] = \sum_{j=0}^{m} \pi_m(j) t^j$$

2.2 Integral automorphisms over the p-adic odometer

Fix an arbitrary $p \geq 3$. Let \{h_n\}_{n \geq 0} be the sequence of heights: $h_0 := 1$, $h_n = ph_{n-1} + 1$.

For each $n \geq 0$, we define

$$X_n := \{(x, i) : x \in \Gamma', \ 0 \leq i \leq h_{n-1} + \phi(x)\}$$

We consider the transformation $T_n$ of $X_n$, defined by

$$T_n(x, i) := \begin{cases} (x, i + 1), & \text{if } i + 1 \leq h_n - 1 + \phi(x) \\ (Sx, 0), & \text{if } i = h_n - 1 + \phi(x). \end{cases}$$

The bijective map $\psi_n: X_n \mapsto X_n + 1$, defined by

$$\psi_n(x, i) := (\sigma x, x_0h_n + i + \mathbb{I}\{x_0 = p - 1\}).$$

Observe that $\psi_n$, it conjugates the transformations $T_n$ and $T_{n+1}$. We introduce the probability measure $\mu_n$ on $X_n$: for a fixed $i$ and a subset $A \subset \{(x, i), x \in \Gamma'\}$,

$$\mu_n(A) := \frac{1}{h_n + 1/2} \lambda\{(x \in \Gamma', (x, i) \in A)\}.$$
is a Rokhlin tower of height $h_n$ for $T_n$. Yet, for any $n \geq 0$, and any $0 \leq i \leq h_n - 1$,

$$\psi_n(En,i) = E_{n+1,i} \sqcup E_{n+1,h_n+i} \sqcup E_{n+1,2h_n+i+1}.$$ 

Fix $n_0$. A composition of the isomorphisms ($\psi_n$), lets us observe all these Rokhlin towers inside $X_{n_0}$. It follows from the formula above that the towers are embedded in the same way as the towers of Chacon’s automorphism described in the introduction. In case $p = 3$, it literally defines the Chacon’s automorphism: $(X_{n_0}, \mu_{n_0}, T_{n_0})$ is isomorphic to $(X, \mu, T)$. For an arbitrary $p > 3$, it generates a similar dynamical system which we call the generalized Chacon automorphism. In this paper we discuss the properties of the generalized automorphism and infer how the known properties of the classical Chacon’s automorphism change in the general case.

## 3 Palindromic property

In this section, we are going to prove that, just like in the classical case, the polynomials $P_m^n$ are palindromic for any parameter $p$. Moreover, we generalize the definition of $\phi$ functional and describe the class of Chacon-like functionals which produce palindromic polynomials.

Let $p \geq 3$ be an arbitrary integer.

We can generalize the definition of $\phi$ functional:

**Definition 1.**

$$\phi(x) = \begin{cases} 
\omega(x_0), & 0 \leq x_0 \leq p - 2 \\
\phi(\sigma x), & x_0 = p - 1
\end{cases}$$

The classic case is hence described by $\omega(j) = j$. This is the only non-trivial $\omega$ for $p = 3$.

**Definition 2.** Let $\phi_* = \min_x \phi(x), \phi^* = \max_x \phi(x), \delta = \phi^* - \phi_*$. We say that a functional $\phi$ has the palindromic property iff

$$\forall j, 0 \leq j \leq m\delta : \pi_m(m\phi_* + j) = \pi_m(m\phi^* - j)$$

Let us find the sufficient conditions for $\phi$ to have the palindromic property.

**Definition 3.** We call a function $\omega$ antipalindromic iff

$$\forall j, 0 \leq j \leq p - 2 : \omega(j) = \omega(p - 2 - j)$$

**Notation.** We further use $[M, N]$ instead of $\{M, M + 1, \ldots, N\}$.

Note that given $\omega$ such that $\text{Ran } \omega = [0, \zeta]$, we have $\text{Ran } \phi = [0, \zeta]$ and $\text{Ran } \phi^{(m)} = [0, m\zeta]$

**Example.** The usual $\omega(j) = j$ is antipalindromic for $p = 3$. Indeed,

$$\omega(0) = 0 = 1 - \omega(1) = 1 - \omega(3 - 2 - 0)$$

$$\omega(1) = 1 = 1 - \omega(0) = 1 - \omega(3 - 2 - 1)$$

In the same way it is shown that $\omega(j) = j$ is antipalindromic for any $p \geq 3$.

**Theorem 3.1.** If $\omega$ is antipalindromic, the corresponding $\phi$ has the palindromic property.

We will need the following lemma to prove the theorem.

**Lemma 3.1.** If $\omega$ is antipalindromic, the probability distributions of the random sequences $\{\phi(S^j x)\}_{j \geq 0}$ and $\{\zeta - \phi(S^{-j} x)\}_{j \geq 0}$ are the same.

**Proof.** Let $x \in \Gamma \setminus \{(p - 1, p - 1, \ldots)\}$. We say that order($x$) $= k \geq 0$ if $x_0 = x_1 = \ldots = x_{k-1} = p - 1$ and $x_k \neq p - 1$. Since the first digit in the sequence $(\ldots, x - 1, x, x + 1)$ follows a periodic pattern $(\ldots, 0, 1, 2, \ldots, p - 2, p - 1, 0, 1, 2, \ldots, p - 2, p - 1, \ldots)$, the contribution of points of order $0$ in the sequence $\{\phi(S^{j} x)\}_{j \geq 0}$ provides a sequence of blocks $\omega(0), \omega(1), \ldots, \omega(p - 2)$ separated by one symbol given by a point of higher order. To fill in the missing symbols corresponding to positions $j$ such that order($x + j$) $\geq 1$, we observe that, if $x$ starts with a $p - 1$, then for all $j \geq 0$, $\phi(x + pj) = \phi(\sigma x + j)$.
Hence the missing symbols are given by the symbols \( \{ \phi(S^j x) \}_{j \geq 0} \).

Let us observe an example for \( p = 7 \) produced by the Legendre symbol-like \( \omega(x) = ((\frac{x+1}{7}) + 1)/2 \) (which is antipalindromic due to the Quadratic reciprocity):

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 0 & 0 & . & 1 & 1
\end{array}
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 0 & 0 & . & 1 & 1
\end{array}
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}
\begin{array}{cccccccc}
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}
\]

\( \iff \) contribution of order 0

\( \iff \) contribution of order 1

\( \iff \) the whole sequence

We may build \( \{ \zeta - \phi(S^{-j} x) \}_{j \geq 0} \) from \( \{ \phi(S^j x) \}_{j \geq 0} \) with a composition of two measure-preserving transformations:

1. The 'reverse' transformation \( \{ a_j \}_{j \geq 0} \mapsto \{ a_{-j} \}_{j \geq 0} \)

2. The 'flip' transformation which substitutes each element \( k \in \{ b_j \}_{j \geq 0} \) with \( \zeta - k \), turning the sequence into \( \{ \zeta - b_j \}_{j \geq 0} \).

By the definition of antipalindromic functions, it is clear that this procedure works as identity transformation when applied to \( \{ \phi(S^j x) \}_{j \geq 0} \).

\[ \square \]

**Theorem 3.2.** If \( \omega \) is antipalindromic, the corresponding \( \phi \) has the palindromic property.

**Proof.** In Definition 1, we may substitute \( \phi_* = 0, \phi^* = \delta = \zeta \). Hence it is enough to show that

\[ \forall j, 0 \leq j \leq m \zeta : \pi_m(j) = \pi_m(m \zeta - j). \]

\[ \pi_m(m \zeta - j) = \lambda(\phi^{(m)}(x) = m \zeta - j) = \sum_{(\phi_1, \ldots, \phi_m)} \lambda(\phi(x) = \phi_1, \phi^{(2)}(x) = \phi_2, \ldots, \phi^{(m)}(x) = \phi_m) I(\phi_1 + \ldots + \phi_k = m \zeta - j) = \sum_{(\phi_1, \ldots, \phi_m)} \lambda(\phi(x) = \phi_1, \phi^{(2)}(x) = \phi_2, \ldots, \phi^{(m)}(x) = \phi_m) I((\zeta - \phi_1) + \ldots + (\zeta - \phi_m) = j) \]

Using Lemma 3.1, this equals to:

\[ \sum_{(\phi_1, \ldots, \phi_m)} \lambda(\phi(x) = \zeta - \phi_m, \phi^{(2)}(x) = \zeta - \phi_{m-1}, \ldots, \phi^{(m)}(x) = \zeta - \phi_1) I(\sum_k (\zeta - \phi_k) = j) = \langle \psi : \zeta - \phi_k = j \rangle \]

\[ \sum_{(\phi_1, \ldots, \phi_m)} \lambda(\phi(x) = \psi_1, \phi^{(2)}(x) = \psi_2, \ldots, \phi^{(m)}(x) = \psi_m) I(\psi_1 + \ldots + \psi_m) = j) = \lambda(\phi^{(m)}(x) = j) = \pi_m(j) \]

\[ \square \]

We may describe a larger set of functionals \( \phi \) having the palindromic property with the use of following lemma.

**Lemma 3.2** (On the affine transformations). Let \( \omega : [0, p - 2] \to [0, \zeta] \) be antipalindromic, then for any \( a > 0, b \geq 0 \):

\[ \phi'(x) = \begin{cases} a \omega(x), & 0 \leq x \leq p - 2 \\ \phi'(x), & x = p - 1 \end{cases} \]

is antipalindromic.

**Proof.** Let \( \pi'_m(j) = \lambda(\phi^{(m)}(x) = j) \). In terms of Definition 1, \( \phi_* = b, \phi^* = a \zeta + b, \delta = a \zeta \). Let us prove that \( \pi'_m(mb + j) = \pi'_m(m(a \zeta + b) + j) \).

First, we perform the division with remainder: \( j = qa + r \). It follows from the construction of \( \phi' \) that \( \pi'_m(mb + qa + r) = 0 \) if \( r \neq 0 \). Yet, \( m(a \zeta + b) - j = m(a \zeta + b) - qa - r = mb + (m \zeta - q)a - r \) and hence \( \pi'_m(mb + j) = 0 \) if \( r \neq 0 \).

Thus, it remains to prove that \( \pi'_m(mb + qa) = \pi'_m(m(a \zeta + b) - qa) \). Note that we may restore the values of \( \phi \) produced by \( \omega \) from the values of \( \phi' \). Indeed, consider the bijection \( i : \{ b, a+b, 2a+b, \ldots, a \zeta + b \} \to [0, \zeta] \) such that \( i(j) = \frac{mb}{m} \). It’s easy to see that \( i(\phi'(x)) = \phi(x) \). Subsequently, we may define \( i^{(m)}(j) = \frac{mb}{m} \) and conclude \( i^{(m)}(\phi^{(m)}(x)) = \phi(x) \).

Now let us prove \( \pi'_m(mb + qa) = i^{(m)}(m(a \zeta + b) - qa) \) using the fact that \( i^{(m)} \) is bijective.

\[ \pi'_m(mb + qa) = \lambda(\phi^{(m)}(x) = mb + qa) = \lambda(i^{(m)}(\phi^{(m)}(x))) = i^{(m)}(mb + qa) = \lambda(\phi^{(m)}(x) = mb + qa - mb) = \lambda(\phi^{(m)}(x) = q) = \pi_m(q) \]

Similarly, \( \pi'_m(m(a \zeta + b) - qa) = \pi_m(m \zeta - q) \). Since \( \omega \) is antipalindromic, it follows from Theorem 2 that \( \pi_m(q) = \pi_m(m \zeta - q) \) and then \( \pi'_m(mb + qa) = \pi'_m(m(a \zeta + b) - qa) \).
Proposition 3.1 (On inheritance of palindromic property). Let \( \phi \) have the palindromic property, and let there be \( \phi' \) such that for any \( m \in \mathbb{N} \) there exists bijection \( i^{(m)} : \text{Ran} \phi^{(m)} \to \text{Ran} \phi^{(m)} \). Then \( \phi' \) has the palindromic property.

The proof of this is the same as for Lemma 3.2.

Theorem 3.3. The polynomials \( P^p_m \) produced by the generalized Chacon automorphism have the palindromic property for any \( p \geq 3 \).

Proof. We have already shown that \( \omega(j) = j \) is antipalindromic. Hence this theorem is a direct corollary of Theorem 3.2. \( \square \)

4 Recurrence formulae for \( P^p_m(t) \)

Notation. We denote \( n \)th triangle number \( \frac{n(n+1)}{2} \) as \( \Delta_n \)

Lemma 4.1. \( P^p_{pm}(t) = t^m \Delta_{p-2} P^p_m(t) \)

Proof. Let \( x \in \Gamma' \). Recalling the structure of \( \{ \phi(S^jx) \}_{j \geq 0} \), the value of \( \phi^{(pm)}(x) \) is the sum of:

- the order-0 points. There are exactly \( (p-1)m \) points following the repeating pattern \( \ldots, (p-2), 0, 1, 2, \ldots, (p-2), 0, 1, \ldots \). Their contribution is \( m \) times the sum of integers \( 0, 1, \ldots, (p-2) \) which is \( \frac{(p-1)(p-2)}{2} m = m \Delta_{p-2} \)

- the higher-order points. Their contribution is \( \phi^{(m)}(\sigma x) \).

Hence \( \phi^{pm}(x) = m \Delta_{p-2} + \phi^{(m)}(\sigma x) \). By the definition of polynomials \( P^p_m(t) \) it implies \( P^p_m(t) = \mathbb{E}_{\lambda} \left[ \phi^{(pm)(\lambda)}(x) \right] = \mathbb{E}_{\lambda} \left[ m \Delta_{p-2} + \phi^{(m)}(\sigma x) \right] = t^m \Delta_{p-2} P^p_m(t) \) \( \square \)

Notation. Similarly to \( \phi^{(m)} \), we denote \( \omega^{(m)}(j) = \omega(j) + \omega(j+1) + \ldots + \omega(j+m-1) \) if \( j \leq m < p-1 \).

Proposition 4.1. \( \omega^{(k)}(x_0) \) is the sum of an arithmetic progression \( x_0, x_0 + 1, \ldots, x_0 + k - 1 \). \( \square \)

Lemma 4.2. Let \( 0 < k < p \). Consider \( x = (x_0, x_1, \ldots) \in \Gamma' \).

\[
\phi^{(pm+k)}(x) = \begin{cases} \omega^{(k)}(x_0) + m \Delta_{p-2} + \phi^{(m)}(\sigma x), & x_0 < p-k \\ \omega^{(p-1-x_0)}(x_0) + \Delta_{x_0+k-p-1} + m \Delta_{p-2} + \phi^{(m+1)}(\sigma x), & x_0 \geq p-k \end{cases}
\]

Proof. First, by the definition of \( \phi^{(k)}(x) \) we state

\[
\phi^{(pm+k)}(x) = \phi(x) + \phi(Sx) + \ldots + \phi(S^{k-1}x) + \phi^{(pm)}(S^k x) = \phi^{(k)}(x) + \phi^{(pm)}(S^k x).
\]

With the use of the previous lemma holds the equality \( \phi^{(pm+k)}(x) = \phi^{(k)}(x) + m \Delta_{p-2} + \phi^{(m)}(\sigma S^k x) \). By the definition of \( \phi(x) \) there are two cases in the computation of \( \phi^{(pm+k)}(x) \):

- The regular case \( x_0 < p-k \). In this case each term in \( \phi^{(k)}(x) \) is computed directly: \( \phi^{(k)}(x) = \omega^{(k)}(x) \). Yet \( \sigma S^k x = \sigma x \); since \( S^k \) affects only the first digit of \( x \), its effect if erased from \( \sigma S^k x \). We may compute \( \phi^{(pm+k)}(x) = \omega^{(k)}(x_0) + m \Delta_{p-2} + \phi^{(m)}(\sigma x) \).

- In another case, if \( x_0 \geq p-k \), some term in \( \phi^{(k)}(x) \) evaluates with recursion: there exists \( j \in [0, k-1] \) such that \( S^j x \) begins with \( p-1 \) and hence \( \phi(S^j x) = \phi(\sigma x) \). Therefore we may not compute \( \phi^{(k)}(x) \) directly. Instead, we divide it into \( \phi^{(p-1-x_0)}(x) = \omega^{(p-1-x_0)}(x) \), which is computed directly, and the rest of the terms. The latter form the sum \( Z := \phi(S^j x) + \phi(S^{j+1} x) + \ldots + \phi(S^{k-1}) \). Consider this sum. As stated before, \( S^j x \) begins with \( p-1 \), so the first digits of \( S^{j+1} x, S^{j+2} x, \ldots, S^{k-1} x \) are \( 0, 1, \ldots, x_0 + k - p - 1 \). We know \( \phi(S^j x) = \phi(\sigma x) \), and, knowing the first digits of the following terms, we may evaluate \( Z = \phi(\sigma x) + 0 + 1 + \ldots + (x_0 + k - p - 1) = \omega(\sigma x) + \Delta_{x_0+k-p-1} \). Now we aggregate \( \phi^{(k)}(x) = \omega^{(p-1-x_0)}(x) + \Delta_{x_0+k-p-1} + \phi(\sigma x) \). Using the equality \( \phi^{(m)}(\sigma S^k x) = \phi^{(m)}(\sigma x) + m \Delta_{p-2} + \phi^{(m+1)}(\sigma x) - \phi(x) \), we get

\[
\phi^{(pm+k)}(x) = \phi^{(k)}(x) + m \Delta_{p-2} + \phi^{(m)}(\sigma S^k x) = \omega^{(p-1-x_0)}(x_0) + \Delta_{x_0+k-p-1} + \phi(\sigma x) + m \Delta_{p-2} + \phi^{(m+1)}(\sigma x) - \phi(x) = \omega^{(p-1-x_0)}(x_0) + \Delta_{x_0+k-p-1} + m \Delta_{p-2} + \phi^{(m+1)}(\sigma x).
\]
Lemma 4.3. Let $0 < k < p$.

$$P_{pm+k}(t) = \frac{1}{p} t^{m \Delta_{p-2} + \Delta_{k-1}} \sum_{j=0}^{p-k-1} t^{jk} P_{m}^p(t) + \frac{1}{p} t^{m \Delta_{p-2} + \Delta_{k-2}} \sum_{j=0}^{k-1} t^{j(p-k)} P_{m+1}^p(t)$$

Proof. This lemma is shown using the previous result and the law of total expectation.

$$P_{pm+k}(t) = E_{\lambda} \left[ t^{\phi(pm+k)(x)} \right] =$$

$$= E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 < p - k \right] \lambda(x_0 < p - k) + E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 \geq p - k \right] \lambda(x_0 \geq p - k)$$

$$= E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 < p - k \right] \frac{p - k}{p} + E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 \geq p - k \right] \frac{k}{p} \quad (1)$$

Now we calculate each conditional expectation according to Lemma 4.2. Note that the digits of $x$ are mutually independent, hence $x_0$ $(x_1, x_2, \ldots)$ and then $x_0 \perp \sigma x$, which is used in the calculation.

$$E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 < p - k \right] = E_{\lambda} \left[ t^{\omega(k)(x_0) + m \Delta_{p-2} + \Delta_{k-1}} | x_0 < p - k \right] =$$

$$= t^{m \Delta_{p-2}} E_{\lambda} \left[ t^{\omega(k)(x_0)} | x_0 < p - k \right] E_{\lambda} \left[ t^{\phi(m)(x_0)} \right] =$$

The distributions of $\sigma x$ and $x$ are the same, which implies $E_{\lambda} \left[ t^{\phi(m)(x_0)} \right] = E_{\lambda} \left[ t^{\phi(m)(x)} \right] = P_m^p(t)$. It remains to evaluate $E_{\lambda} \left[ t^{\omega(k)(x_0) | x_0 < p - k} \right]$.

$$E_{\lambda} \left[ t^{\omega(k)(x_0) | x_0 < p - k} \right] = \frac{1}{p - k} \sum_{j=0}^{p-k-1} t^{\omega(k)(j)} = \frac{1}{p - k} \sum_{j=0}^{p-k-1} t^{kj} + \Delta_{k-1} = \frac{1}{p - k} t^{k} \sum_{j=0}^{p-k-1} t^{kj}$$

Thus,

$$E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 < p - k \right] = \frac{1}{p - k} t^{m \Delta_{p-2} + \Delta_{k-1}} \sum_{j=0}^{p-k-1} t^{jk} P_{m}^p(t) \quad (2)$$

Similarly, we evaluate the second conditional expectation.

$$E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 \geq p - k \right] = E_{\lambda} \left[ t^{\omega(p-1-x_0)(x_0) + \Delta_{x_0+k-p-1} + m \Delta_{p-2} + \phi(m+1)(x_0)} | x_0 \geq p - k \right] =$$

$$= t^{m \Delta_{p-2}} E_{\lambda} \left[ t^{\omega(p-1-x_0)(x_0) + \Delta_{x_0+k-p-1}} | x_0 \geq p - k \right] E_{\lambda} \left[ t^{\phi(m+1)(x_0)} \right] =$$

$$= t^{m \Delta_{p-2}} P_{m+1}^p(t) E_{\lambda} \left[ t^{\omega(p-1-x_0)(x_0) + \Delta_{x_0+k-p-1}} | x_0 \geq p - k \right] =$$

$$= \frac{1}{p} t^{m \Delta_{p-2}} P_{m+1}^p(t) \sum_{j=p-k}^{p-1} t^{(p-1-j) + \Delta_{p-2-j} + \Delta_{j+k-p-1}} =$$

We substitute the index $j$ with $q = p - 1 - j$ and simplify the exponent:

$$(p - 1 - j) + \Delta_{p-2-j} + \Delta_{j+k-p-1} = q(p - 1 - q) + \Delta_{q-1} + \Delta_{k-2-q} = q(p - k) + \Delta_{k-2}$$

Now we collect the result:

$$E_{\lambda} \left[ t^{\phi(pm+k)(x)} | x_0 \geq p - k \right] = \frac{1}{k} t^{m \Delta_{p-2} + \Delta_{k-2}} P_{m+1}^p(t) \sum_{q=0}^{k-1} t^{q(p-k)} \quad (3)$$

It remains to substitute (2) and (3) into (1) to obtain the recurrence equation:

$$P_{pm+k}(t) = \frac{1}{p} t^{m \Delta_{p-2} + \Delta_{k-1}} \sum_{j=0}^{p-k-1} t^{jk} P_{m}^p(t) + \frac{1}{p} t^{m \Delta_{p-2} + \Delta_{k-2}} \sum_{j=0}^{k-1} t^{j(p-k)} P_{m+1}^p(t)$$
Bringing together the results of Lemma 4.1 and Lemma 4.3, we obtain the general recurrence law for \( P_n^p(t) \)

**Theorem 4.1** (Recurrence formulae on \( P_n^p(t) \)).

\[
P_{pm}^p(t) = t^{m\Delta_{p-2}} P_{m}^p(t)
\]

\[
P_{pm+k}^p(t) = \frac{1}{p} t^{m\Delta_{p-2}+\Delta_{k-1}} \sum_{j=0}^{p-k-1} t^{jk} P_{m}^p(t) + \frac{1}{p} t^{m\Delta_{p-2}+\Delta_{k-2}} \sum_{j=0}^{k-1} t^{(p-k)j} P_{m+1}^p(t), \quad 0 < k < p
\]

### 5 Degrees of polynomials

In this section, our goal is to describe a handy symmetric form of the polynomials \( P_n^p(t) \). This requires a more detailed consideration of the degrees of these polynomials.

#### 5.1 Sequence \( \hat{S}_j \)

Here we define the substitution-produced sequence \( \{\hat{S}_j\}_{j \geq 0} \). It will be useful to evaluate the degrees of \( P_n^p(t) \). We build this sequence recursively. First, fix the parameter value \( p \geq 3 \).

**Definition 4** (Zero-tier construction). Let \( \hat{S}_0^{(0)} = p - 2 \), \( \hat{S}_j^{(0)} = p - 2 - (j - 1) \) for \( j = 1, p - 1 \).

Literally, \( \{\hat{S}_j^{(0)}\} \) is \( p - 2, p - 2, p - 3, \ldots, 2, 1, 0 \).

**Definition 5** (Number substitution rule). We substitute

- \( 0 \) with \( p - 2, p - 3, \ldots, 2, 1, 0, 0 \);
- \( 1 \) with \( p - 2, p - 3, \ldots, 2, 1, 1, 0 \);
- \( 2 \) with \( p - 2, p - 3, \ldots, 2, 2, 1, 0 \);
- \( \ldots \)
- \( p - 3 \) with \( p - 2, p - 3, p - 3, \ldots, 2, 1, 0 \);
- \( p - 2 \) with \( p - 2, p - 2, p - 3, \ldots, 2, 1, 0 \);
- \( \ldots \)
- \( \text{etc.} \)

Each number \( q = \overline{0, p - 2} \) becomes the descendent sequence of numbers from \( p - 2 \) to \( 0 \) with \( q \) taken twice in a row.

**Definition 6** (First-tier construction). We have built this sequence \( \{\hat{S}_j^{(0)}\} \) of \( p \) integers. To define \( \hat{S}_j^{(1)} \), we substitute each number in \( \{\hat{S}_j^{(0)}\} \) according to the rule. As a result, we have the sequence \( \{\hat{S}_j^{(1)}\} \) of \( p^2 \) integers.

We continue the procedure ad infinitum, substituting each number in \( \{\hat{S}_j^{(l)}\} \) to define \( \{\hat{S}_j^{(l+1)}\} \). It is easy to see that

- on each step the length of the sequence multiplies by \( p \) and
- for each \( l \geq 0 \), \( \{\hat{S}_j^{(l)}\} \) is a prefix of \( \{\hat{S}_j^{(l+1)}\} \).

There exists a countably infinite sequence \( \{\hat{S}_j^{(\infty)}\} \) which contains each of \( \{\hat{S}_j^{(l)}\} \) as a prefix.

**Definition 7.** Let \( \{\hat{S}_j\}_{j \geq 0} := \{\hat{S}_j^{(\infty)}\}_{j \geq 0} \).

We could as well define the substitution rule first and then generate the sequence from initial datum \( s_0 = p - 2 \), i.e. the sequence is is determined in a unique way by the rule and the initial element. This notion will be useful further.
Example. Let \( p = 4 \).

\[
\begin{array}{c|cccc|c|cccc|c|cccc|c}
\hat{S}_0 & 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 \\
\hat{S}_1 & 2210 & 2210 & 2110 & 2100 & 2210 & 2210 & 2110 & 2100 & 2210 & 2110 & 2110 & 2100 & 2210 & 2110 & 2100 & 2100
\end{array}
\]

Proposition 5.1 (Characteristic property of \( \hat{S}_j \)). For each \( m > 0 \), \( 0 \leq j \leq p - 1 \): \( \hat{S}_{pm+j} = \hat{S}_{p(p-1-S_m)+j} \)

Proof. First, we find the number of the first tier where the \((pm+j)\)-th element appears: \( l = \log_p m \).

This \( l \)-th tier consists of \( p \) blocks, each having \( p^l \) elements, and each block represents an element from tier \( l-1 \). The element \( \hat{S}_{pm+j} \) belongs to the \( m \)-th block of tier \( l \), which is the expansion of the \( m \)-th element in tier \( l-1 \). Hence, \( \hat{S}_{pm+j} \) is the \( j \)-th element in the substitution of \( \hat{S}_m \).

Note that all the substitutions are listed in tier 1, which consists of the elements from \( \hat{S}_0 \) to \( \hat{S}_{p^2-1} \). Rigorously, the substitution of \( q \) if represented with the items from \( \hat{S}_{p^2-p-(q-p-1)} \) to \( \hat{S}_{p^2-(p-2)-q-p-1} \).

Example. Let \( p = 4 \). Our goal is to find \( \hat{S}_{141} \).

\[
\hat{S}_{141} = \hat{S}_{4\cdot35+1} = \hat{S}_{4(3-\hat{S}_m)+1}
\]

\[
\hat{S}_{35} = \hat{S}_{4\cdot8+3} = \hat{S}_{4(3-S_3)+3}
\]

\[
\hat{S}_8 = 2 \Rightarrow \hat{S}_{35} - \hat{S}_{4(3-2)+3} = \hat{S}_7 = 0
\]

\[
\hat{S}_{141} = \hat{S}_{4(3-S_8)+1} = \hat{S}_{4(3-0)+1} = \hat{S}_{13} = 1
\]

The answer is 1.

With the use of this property, we may make a more handy definition of \( \hat{S} \). Indeed, as shown in the example, for any \( j \geq p^2 \) we may find \( \hat{S}_j \) applying the equation from Proposition 5.1 sufficient number of times. Hence it is enough to define only the tier 1 (i.e. the values of \( \hat{S}_0, \ldots, \hat{S}_{p^2-1} \) to construct the whole sequence.

Definition 8. Define \( \{\hat{S}_j^{(1)}\} \) as in Definition 6. Let

\[
\hat{S}_{pm+j} = \begin{cases} 
\hat{S}_0, & m < p \\
\hat{S}_{p(p-1-S_m)+j}, & m \geq p 
\end{cases}
\]

We will use this definition further.

5.2 Formulae for degrees

Definition 9. Let \( m \geq 0 \). We define \( D_m^p := \deg P_m^p \).

Proposition 5.2. For any \( m \geq 0 \), \( 0 \leq k < p \):

\[
D_{pm}^p = m \Delta_{p-2} + D_m^p
\]

\[
D_{pm+k}^p = m \Delta_{p-2} + (p-k)(k-1) + \Delta_{k-2} + D_m^p + \max\{p-k-1, D_{m+1}^p - D_m^p\}
\]

Proof. Directly from the (2) and (3) we obtain the equations for degrees:

\[
D_{pm}^p = m \Delta_{p-2} + D_m^p
\]

\[
D_{pm+k}^p = \max\{m \Delta_{p-2} + \Delta_{k-1} + k(p-k-1) + D_m^p, m \Delta_{p-2} + \Delta_{k-2} + (p-k)(k-1) + D_{m+1}^p\}
\]

To show this proposition, it is enough to take the common terms out of the maximum operator.

The following holds for arbitrary fixed \( p \geq 3 \). Keeping this in mind, we use \( D_j \) instead of \( D_j^p \) for short.
Proposition 5.3.

\[
S_{pm+k} = p - k - 1 - \mathbb{1}\{S_m < p - 1 - k\}
\]  

(6)

Proof. We will show this using Proposition 5.2 and Definition 10. If we try to evaluate \(S_j\), it collapses into two cases:

- Let \(k < p - 1\).

\[
S_{pm+k} = D_{pm+k+1} - D_{pm+k} = m\Delta_{p-2} + (p-k-1)k + \Delta_{k-1} + D_m + \max\{p - k - 1, D_{m+1} - D_m\} - m\Delta_{p-2} - (p-k)(k-1) - \Delta_{k-2} - D_m - \max\{p - k, D_{m+1} - D_m\} = p - k - 1 + \max\{p - k - 1, D_{m+1} - D_m\} - \max\{p - k, D_{m+1} - D_m\};
\]

We substitute \(D_{m+1} - D_m = S_m\) and observe that

\[
\max\{p - k - 1, D_{m+1} - D_m\} - \max\{p - k, D_{m+1} - D_m\} = \mathbb{1}\{S_m < 1 - k\}.
\]

This substitution leads to exactly the equation in the proposition.

- \(S_{pm+k-1} = D_{pm+1} - D_{pm+3} = (m+1)\Delta_{p-2} + D_{m+1} - m\Delta_{p-2} - (p-1-1) - \Delta_{p-2} - D_m - \max\{0, D_{m+1} - D_m\} = \Delta_{p-2} + (D_{m+1} - D_m) - (p-2) - \Delta_{p-3} - (D_{m+1} - D_m) = 0\)

A straightforward substitution shows that the second case is governed by the same equation.

Lemma 5.1. For \(j = 0, 1, \ldots, p^2 - 1\): \(S_j = S_j^{(1)}\)

Proof. We compute the initial \(p^2\) terms of this sequence. Recall that \(P_0^p(t) = 1\), \(P_1^p(t) = \frac{1}{t} \sum_{j=0}^{p-2} t^j\), hence \(D_0 = 0\), \(D_1 = p - 2\) and \(S_0 = p - 2\). We will use this as the initial condition in the recurrence equation from (6). The right part consists of a periodic term \(p - k - 1\) with period \(p\) and the step function \(-\mathbb{1}\{S_m < p - 1 - k\}\) which is determined by the number of period. Thus we evaluate the sequence \(\{S_j\}\) period-wise.

In the first period, \(\{S_{kp+k}\}\), we have already shown \(S_0 = p - 2\). The step function is \(-\mathbb{1}\{k < 1\}\), so it affects none of the terms \(S_1, S_2, \ldots, S_{p-1}\). They form the descending progression \(p - 2, p - 3, \ldots, 1\). Now we have the first \(p\) terms of \(\{S_j\}\) which are to be used to obtain the subsequent terms.

In the second period, the step function is \(-\mathbb{1}\{S_1 < p - 1 - k\}\). We know that \(S_1 = 2\), thus the second period repeats the first. In the third period we have \(-\mathbb{1}\{S_1 < p - 1 - k\} = -\mathbb{1}\{p - 3 < p - 1 - k\} = -\mathbb{1}\{k < 2\}\) which affects first two terms. The third period is \(p - 2, p - 3, p - 4, \ldots, 1\).

Similarly, we use all of the terms in the first period to evaluate each of \(S_j\) for \(j < p^2\). It is shown with ease that they the same as in \(S_j^{(1)}\).

Proposition 5.4. \(\forall j \geq 0\): \(S_j \in \{0, 1, \ldots, p - 2\}\)

Proof. The explicit evaluation of \(S_0, S_1, \ldots, S_{p-1}\) shows

\[
\{0, 1, \ldots, p - 2\} \subset \text{Ran } \{S_j\}
\]

(7)

Obviously, \(S_j \geq 0\). We prove by induction that \(\forall m \geq p, 0 < k < p - 1 : S_{pm+k} < p - 1\). The order \(\log_p m := a\) is the parameter of induction. The explicit evaluation of \(S_p, S_{p+1}, \ldots, S_{p^2-1}\) shows the base of induction. Let the statement be true for an arbitrary \(u\). When computing the values of \(S_j\) for order \(u+1\) from (6), we refer to the values of \(S_m\) of order \(u\). In (6), \(p-k-1 \leq p-1\) and \(-\mathbb{1}\{S_m < p - 1 - k\} \leq 0\), thus \(S_j < p - 1\). Let us show that the equality is never reached. Indeed, \(p - k - 1 = p - 1\) if true only for \(k = 0\). When we substitute \(k = 0\) into the step function, we get \(-\mathbb{1}\{S_m < p - 1\}\). For \(S_m\) of order \(u\), the statement in brackets is true due to the induction hypothesis. So, in order \(u + 1\) the statement \(S_j < p - 1\) holds. The inductive step is now shown.

Now we may assert \(\text{Ran } \{S_j\} \subset \{0, 1, \ldots, p - 2\}\). Bringing together this and (7), we infer the proposition.
Lemma 5.2. For each \( m \geq p, 0 < k < p - 1 \): 
\[ S_{pm+k} = S_{p(p-1-S_m)+k} \]

Proof. First, we observe that for any \( j \geq 0 \), \( S_j = S_{p-1-S_j} \). Using the Proposition 5.4, it is enough to show the statement for \( S_j = 0, 1, \ldots, p - 2 \). For example, let \( S_j = 0 \). We make sure that \( 0 = S_{p-1} \) which we know from the direct computation. Similarly, we do this for any \( S_j = 1, \ldots, p - 2 \).

To prove the lemma, we will this equation. Let \( m \geq p, 0 < k < p - 1 \). From (6) we obtain \( S_{pm+k} = p - k - 1 \cdot \mathbb{1}\{S_m < p - 1 - k\} \) and \( S_{p(p-1-S_m)+k} = p - k - 1 - \mathbb{1}\{S_{p-1-S_m} < p - 1 - k\} \). One may easily observe that the equality \( S_m = p - 1 - S_m \) turns this into an identity.

\[ \Box \]

Theorem 5.1. For each \( j \geq 0 \): \( S_j = \bar{S}_j \).

Proof. The results of Lemma 5.1 and Lemma 5.2 show that Definition 8 is true for \( \{S_j\}_{j \geq 0} \).

\[ \Box \]

Corollary 5.1. \( D_m = \sum_{j=0}^{m-1} \bar{S}_j \)

5.3 Formulae for lower degrees

Definition 11. Let \( m \geq 0 \). The lower degree \( d^p_m \) of the polynomial \( P^p_m(t) \) is the smallest degree of \( t \) among the terms of \( P^p_m(t) \).

In other words, \( d^p_m \) is such an integer that \( P^p_m(t) = t^{d^p_m}Q(t) \), where \( Q(t) \) is a polynomial having a constant term.

In this subsection we will show that the lower degree \( d^p_m \) follows a substitution pattern similar to the degree \( D^p_m \).

Proposition 5.5. For any \( m \geq 0, 0 < k < p \):
\[ d^p_{pm} = m\Delta_{p-2} + d^p_m \]
\[ d^p_{pm+k} = m\Delta_{p-2} + \Delta_{k-2} + d^p_m + \min\{k-1, d^p_{m+1} - d^p_m\} \]

Proof. This is shown directly from the equations (4) and (5).

\[ \Box \]

Definition 12. For any \( m \geq 0 \) and a fixed \( p \), we define \( s_m := s^p_{m+1} - s^p_m \).

Similarly to \( S_m \) from the previous subsection, \( s_m \) is a substitution sequence. This fact is shown in complete analogy with Theorem 5.1, so we omit the details of the proof and just describe the substitution rule.

Theorem 5.2. The sequence \( s_m \) is generated by substitutions:

- \( s_0 = 0 \)
- each \( s_j \) is substituted with \( 0, 1, \ldots, s_j-1, s_j, s_j+1, \ldots, p-2 \) (which is the increasing sequence of integers from \( 0 \) to \( p-2 \) with \( s_j \) taken twice).

Example. We evaluate the first elements of \( \{s_j\}_{j \geq 0} \) for \( p = 4 \) with the substitution rule: \( 0 \mapsto 0012 \mapsto 001201120122 \mapsto [0012001201120122] [0012011201120122] [0012011201120122] [0012011201120122] \mapsto \ldots \). Writing the last expression in a row, we get the first 64 elements of \( \{s_j\} \).

5.4 Symmetric forms of polynomials

Here we use the results of Theorem 5.1 and Theorem 5.2 to evaluate a reduced form of the recurrence equations (4) and (5).

Lemma 5.3. For each \( j \geq 0 \), \( S_j + s_j = p - 2 = \text{const} \)

Proof. We prove this by induction. As a base, we recall that \( S_j \) is generated from \( S_0 = p - 2 \) and \( s_j \) is generated from \( s_0 = 0 \). For index 0, the statement holds: \( S_0 + s_0 = p - 2 \).

Now we prove the induction step. Let \( j \) be such that \( S_j + s_j = p - 2 \). We state that when we substitute \( S_j \) and \( S_j \) with the proper sequences of length \( p \), the statement holds for each pair of elements in these subsequences. This follows directly from the substitution rules.

\[ \Box \]
Definition 13. The mid-degree $\delta_m^p$ of a polynomial $P_m^p(t)$ is the arithmetic mean of its degree and its lower degree: $\delta_m^p = \frac{D_m^p + d_m^p}{2}$

Corollary 5.2. $\delta_m^p = \frac{(p-2)m}{2}$

Proof. For a fixed $p$, $\delta_m^p = \frac{D_m^p + d_m^p}{2} = \frac{\sum_{j=0}^{m-1} S_j + \sum_{j=0}^{m-1} s_j}{2} = \frac{\sum_{j=0}^{m-1} (S_j + s_j)}{2} = \frac{m(p-2)}{2}$

Definition 14. We define the symmetric form of polynomials $P_m^p(t)$:

$$o P_m^p(t) = t^{\delta_m^p} P_m^p(t)$$

These are polynomial-like functions of $t$. The terms in the symmetric form are monomials of $t$ in real degrees. If the number of terms in $P_m^p(t)$ is odd, then the degrees of $t$ in $o P_m^p(t)$ are integers, otherwise they are half-integers.

The degrees of $t$ in $o P_m^p(t)$ take from $-\frac{D_m^p}{2}$ to $\frac{D_m^p}{2}$, half positive and half negative (and a term of degree 0, i.e. constant, if the number of terms is odd).

Theorem 5.3. (Recurrence equations on the symmetric forms of $P_m^p$)

$$o P_m^p(t) = o P_m^p(t)$$

$$o P_{m+k}^p(t) = \frac{1}{p} \left( \sum_{j=-\frac{k}{2}}^{-\frac{k-1}{2}} t^j \right) o P_m^p(t) + \frac{1}{p} \left( \sum_{j=-\frac{k}{2}}^{\frac{k-1}{2}} t^j \right) o P_{m+1}^p(t)$$

In these sums, indexes may not be integers, which is not absolutely strict though compact. It is implied that the sum index $j$ increases by one, anyway. For example, given $p = 5$ and $k = 1$, the expression $\left( \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} t^j \right)$ means $t^{-\frac{1}{2}} + t^{\frac{1}{2}} + t^1$. 

Proof. Substituting $P_m^p(t) = t^{\delta_m^p} o P_m^p(t)$, we deduce these equations directly from (4) and (5).
References

[1] Nathaniel A. Friedman. *Introduction to ergodic theory*. Van Nostrand Reinhold Mathematical Studies, 1970.

[2] R. V. Chacon. Weakly mixing transformations which are not strongly mixing. *Proceedings of the American Mathematical Society*, 1969.

[3] Andrés del Junco. A simple measure-preserving transformation with trivial centralizer. *Pacific J. Math.*, 79(2):357–362, 1978.

[4] A. del Junco, M. Rahe, and L. Swanson. Chacon’s automorphism has minimal self-joinings. *Journal d’Analyse Mathematique*, 1980.

[5] A. A. Prikhod’ko and V. V. Ryzhikov. Disjointness of the convolutions for chacon’s automorphism. *Colloquium Mathematicum*, 85:67–74, 2000.

[6] A.B. Katok. *Combinatorial Constructions in Ergodic Theory and Dynamics*. Universi Series. American Mathematical Society, 2003.

[7] Anatolii M Stepin. Spectral properties of generic dynamical systems. *Mathematics of the USSR-Izvestiya*, 29(1):159, 1987.

[8] Andrés Del Junco and Mariusz Lemańczyk. Generic spectral properties of measure-preserving maps and applications. *Proceedings of the American Mathematical Society*, 115(3):725–736, 1992.

[9] El Abdalaoui Abdalaoui, Mariusz Lemanczyk, and Thierry de la Rue. On spectral disjointness of powers for rank-one transformations and mains orthogonality. *Journal of Functional Analysis*, 266, 01 2013.

[10] Valery Ryzhikov. Minimal self-joinings, bounded constructions, and weak closure of ergodic actions. arXiv:1212.2602, 2012.

[11] E. Janvresse, A. A. Prikhod’ko, T. de la Rue, and V. V. Ryzhikov. Weak limits of powers of chacon’s automorphism. *Ergodic Theory and Dynamical Systems*, 35(1):128–141, 2015.