On Spherically Symmetric Non-Static Space-Times Admitting Homothetic Motions

Ragab M. Gad
Mathematics Department, Faculty of Science, Minia University, 61915 El-Minia, EGYPT.

Abstract

Spherically symmetric solutions admitting a homothetic Killing vector field (HKVF) either orthogonal, $\eta_\perp$, or parallel, $\eta_\parallel$, to the 4-velocity vector field, $u^a$, are studied. New self-similar solution of Einstein’s field equation is found in the case when HKVF is in a general form. Some physical properties of the obtained solution are examined.

PACS: 04.20.-q-Classical general relativity.
PACS: 04.20.-Jb- Exact solutions.

1 Introduction

Recently, symmetries in general relativity have attracted much attention, not only because of their classical physical significance, but also because they simplify Einstein field equations. Many survey articles are given to discuss the concept of these symmetries from the mathematical and physical viewpoints (see for example [1]).

One of the most important symmetries is the self-similarity. Self-similar solutions of the Einstein field equations are of great interest in physics because they are often found to play an important role in describing the asymptotic properties of more general solutions [2]. These solutions have relevance in astrophysics and critical phenomena in gravitational collapse (see for example [3] - [7] and references therein).

In a recent paper [8], Gad and Hassan studied a non-static spherically symmetric solutions. They assumed that these space-times admit a homothetic vector field orthogonal to the 4-velocity vector, $u^a$, and obtained an exact solution. This solution has non-vanishing expansion, acceleration and shear. They derived another solution by assuming, in additional to space-like homothetic motion, the matter in this fluid is represented by perfect fluid. This solution has zero expansion.

Email Address: ragab2gad@hotmail.com
Many exact solutions has been derived by imposing the condition of the existence of a conformal Killing vector orthogonal to the 4-velocity (see for example [10], [9]). In the present paper we study the cases when space-time admitting HKVF either orthogonal or parallel to 4-velocity vector. Several authors have studied the solutions admitting the first symmetry. Most of them have restricted their intention to the solutions discovered by Gutman-Bosal’ke [16], which are given in another form by Wesson [20]. These solutions are denoted by (GBW). Collins and Land [14] have studied these solutions as well as a stiff equation of state. Sussman [19] investigated the properties of them, and obtained interesting results.

The second aim of this paper is to obtain an exact self-similar solution and explore some of its physical properties.

The paper has been organized as follows: In the next section, we shall comment on the singularities inherent the solutions obtained in [8] and we examine when these singularities could be possible naked. We find the form of HKVF when it is either orthogonal or parallel to $u^a$. We shall derive an exact new self-similar solution. In section 3, we shall discuss the physical properties of the obtained solution. Finally, in section 4, we shall conclude the results.

## 2 Homothetic Motion

A global vector field $\eta$ on a space-time $M$ is called homothetic if either one of the following conditions holds on a local chart:

$$\mathcal{L}_\eta g_{ab} = \eta_{a;b} + \eta_{b;a} = 2\Phi g_{ab}, \quad H_{a;b} = \Phi g_{ab} + F_{ab},$$

(2.1)

where $\Phi$ is a constant on $M$, $\mathcal{L}$ stands for the Lie derivative operator, a semicolon denotes a covariant derivative with respect to the metric connection, and $F_{ab} = -F_{ba}$ is the so-called homothetic bivector. If $\Phi \neq 0$, $\eta$ is called proper homothetic and if $\Phi = 0$, $\eta$ is called Killing vector field on $M$.

For a geometrical interpretation of (2.1) we refer the reader to [15], [16], and for a physical properties we refer for example to [3].

For the study of non-static spherically symmetric motion, we used the model given by [21]

$$ds^2 = \alpha d\nu^2 + 2\beta d\nu dr - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(2.2)

where $\alpha$ and $\beta$ are positive function of $\nu$ and $r$. 

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Gad and Hassan [8] assumed an additional symmetry to the spherically symmetric, space-like homothetic motion, and they obtained

$$\alpha = r^2 h(\nu), \quad \beta = rf(\nu), \quad (2.3)$$

where $h(\nu)$ and $f(\nu)$ are arbitrary positive functions. In addition to the above symmetry, they assumed that the matter is represented by a perfect fluid and found the relation between $f(\nu)$ and $h(\nu)$ as follows

$$h(\nu) = \frac{1}{2} f^2(\nu). \quad (2.4)$$

This solution is scalar-polynomial singular along $r = 0$ [8].

In the following we examine when this singularity could be possible naked. To do this, we consider the transverse radial null geodesics. The equations governing these geodesics are

$$f(\nu) \ddot{\nu} + (f'(\nu) - h(\nu)) \dot{\nu} = 0, \quad (2.5)$$

$$r^2 h(\nu) \dot{\nu}^2 + 2rf(\nu) \dot{\nu} \dot{r} = 0. \quad (2.6)$$

It is clear from equation (2.6) that the ingoing null geodesics are the line $\nu = constant$. The outgoing geodesics obey the equation

$$\frac{dr}{d\nu} = -\frac{rh(\nu)}{2f(\nu)}. \quad (2.7)$$

By integrating this equation, we get

$$r = c_1 \exp \left( -\int \frac{h(\nu)}{2f(\nu)} d\nu \right), \quad c_1 \neq 0, \quad (2.8)$$

we can see the structure of the space-time by examining equation (2.8). For example, if there exists a solution of equation (2.8) which starts from the singularity and ends at the future null infinity, the singularity is globally naked. Unfortunately, we cannot solve equation (2.8) unless the special choices of the functions $f(\nu)$ and $h(\nu)$ are given. Now, we have two cases are depending on the value of integrand $\int \frac{h(\nu)}{f(\nu)} d\nu$, inside equation (2.8).

1. If the integrand has negative values, then the geodesics will never meet the singular point $r = 0$.

2. If the integrand has positive values, then the geodesics are meeting the singular point $r = 0$.
Proposition 2.1 All non-static spherically symmetric solutions described by metric (2.2) admit a homothetic vector field orthogonal to the 4-velocity vector in the form $\eta_\perp = \Phi r \partial_r$.

**proof:** Consider the homothetic Killing equation (2.1) and $\eta$ is a homothetic Killing vector field having the general form

$$\eta = A(\nu, r) \partial_\nu + \Gamma(\nu, r) \partial_r.$$  \hfill (2.9)

For the metric (2.2), we have

$$u^a = \frac{1}{\sqrt{\alpha(\nu, r)}}.$$  

If $\eta_\perp$ is everywhere orthogonal to $u^a$, then

$$\eta_\perp^a = \Gamma(\nu, r) \delta_r^a.$$  

By straightforward calculations, using (2.3) and the Christoffel symbols of second kind (see Appendix), we get that this vector satisfies the condition (2.1) if $\Gamma(\nu, r) = \Phi r$.

According to the above proposition and using (2.4), the following result has been established

**Proposition 2.2** All perfect fluid solutions described by the metric (2.2) admit a homothetic vector field orthogonal to the 4-velocity vector in the form $\eta_\perp = \Phi r \partial_r$.

Now, we study the case when the homothetic vector field is parallel to the four-velocity vector field.

**Proposition 2.3** All non-static spherically symmetric solutions described by metric (2.2) admit a homothetic vector field parallel to the 4-velocity vector in the form $\eta_\parallel = \Phi \nu \partial_\nu$.

**proof:** Consider the general form of HVF (2.9) and using the relation, since HVF is parallel to $u^a$,

$$\eta_\parallel^a = \text{const.} u^a,$$

then

$$\eta_\parallel^a = A(\nu, r) \delta_\nu^a.$$
By straightforward calculations, using (2.3) and the Christoffel symbols of second kind (see Appendix), we get that this vector satisfies the condition (2.1) if $A(\nu, r) = \Phi \nu$.

By the same manner, see proposition (2.2), we can prove that if the fluid is a perfect fluid, then it admits HVF parallel to $u^a$ in the form $\eta_{\parallel} = \Phi \nu \partial_{\nu}$.

According to the above propositions, the HVF $\eta$ takes the following form

$$\eta = \phi r \partial_r + \phi \nu \partial_{\nu}$$  \hspace{1cm} (2.10)

This vector satisfies the conditions (2.1).

Now we assume that the line element (2.2) admits HVF (2.10), then the (non-trivial) equations arising from (2.2), are

$$\nu \beta \nu + r \beta_r = 0,$$

$$r \alpha_r + \nu \alpha_{\nu} = 0.$$

Using equations (2.3) and (2.4), the solutions of the above equations are

$$\alpha = \frac{1}{2} \left( \frac{r}{\nu} \right)^2,$$

$$\beta = \left( \frac{r}{\nu} \right).$$

According to the above results, the line element (2.2) can be written in the following form

$$ds^2 = \frac{1}{2} \left( \frac{r}{\nu} \right)^2 dv^2 + 2 \left( \frac{r}{\nu} \right) dv dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (2.11)

In the following section, we shall discuss some of the physical properties of the obtained solution given by (2.11).

3 Physical Properties

3.1 Kinematic of the Velocity Field

For a given space-time the kinematics properties (acceleration, expansion scalar, rotation, shear and scalar shear) are respectively defined as below [16]:

The acceleration is defined by

$$\dot{u}_a = u_{a;\mu} u^\mu.$$
The expansion scalar, which determines the volume behavior of the fluid, is defined by

\[ \Theta = u^a_{;a}. \]

The rotation is given by

\[ \omega_{ab} = u_{[a;b]} + \dot{u}_{(a}u_{b]} . \]

The shear tensor, which provides the distortion arising in a fluid flow leaving the volume invariant, can be found by

\[ \sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}u_{b)} - \frac{1}{3}\Theta h_{ab}, \]

where \( h_{ab} = g_{ab} + u_a u_b \).

The shear invariant is given by

\[ \sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}. \]

For the solution given by (2.11)

The acceleration is

\[ \dot{u}_a = -\frac{1}{2r} \delta^1_a. \]

For the expansion scalar, we find

\[ \Theta = 0. \]

The only non-vanishing components of rotation is given by

\[ \omega_{41} = \frac{1}{\sqrt{2}\nu}. \]

The only non-zero component of the shear tensor is

\[ \sigma_{11} = -\frac{2\sqrt{2}}{r}, \]

and the shear scalar, is given by

\[ \sigma^2 = \frac{1}{r^2} \]
\section{Pressure and Density}

In addition to self-similarity, we assume that the matter is represented by a perfect fluid, that is, the Einstein field equations, \( G_{ab} = -\kappa T_{ab} \), are satisfied with the energy momentum tensor

\[ T_{ab} = (\rho + p)u_a u_b - pg_{ab}. \]

For the line element (2.11), the Einstein field equations reduce to the following equations

\[
\frac{1}{r^2} = \kappa(\rho + p),
\]

\[
\frac{1}{2r^2} = \kappa p.
\]

From the above equations, we obtain the expression for the pressure and density in the form

\[ p = \rho = \frac{1}{2\kappa r^2}. \]

\section{Tidal Forces}

The components of the Riemann curvature tensor \( R^a_{bcd} \), which describe tidal forces (relative acceleration) between two particles in free fall, are the components \( R^i_{i0j0} \), \((i, j = 1, 2, 3)\). [17].

For the line element (2.11), we obtain

\[ R^1_{010} = 0, \]

and the only non-vanishing relevant components are

\[ R^2_{020} = R^3_{030} = \frac{1}{4\nu^2}. \]

Then the equations of geodesic deviation (Jacobi equations), which connected the behavior of nearby particles and curvature, are reduce to the following equations

\[
\frac{D^2 \zeta^r}{d\tau^2} = 0, \tag{3.12}
\]

\[
\frac{D^2 \zeta^\theta}{d\tau^2} = -\frac{1}{2r^2} \zeta^\theta, \tag{3.13}
\]

\[
\frac{D^2 \zeta^\phi}{d\tau^2} = -\frac{1}{2r^2} \zeta^\phi, \tag{3.14}
\]
where $\zeta^r$, $\zeta^\theta$, $\zeta^\phi$ are the components of Jacobi vector field.

Hence, equation (3.12) indicates tidal forces in radial direction will not stretch an observer falling in this fluid. The equations (3.13) and (3.14) are indicate a pressure or compression in the transverse directions, that is, the tidal forces will not squeeze the observer in the transverse directions.

4 Conclusion

In the theory of general relativity, there are different types of self-similarity. To distinguish between them we refer the reader to the topical review by Carr and Coley [3]. In this paper we have restricted our intention to the first type of self-similarity, which characterized by the existence of a homothetic Killing vector field. We have obtained the form of homothetic Killing vector field when it is either orthogonal or parallel to the 4-velocity vector field. In the case when HKVF takes a general form, we have derived self-similar solution. This solution has zero expansion, non-vanishing acceleration and non-vanishing shear and satisfies the equation of state $\rho = p$. Furthermore, we have shown that the tidal forces in radial direction will not stretch an observer falling in this fluid and they not squeeze him in transverse directions.

Appendix

We use $(x^0, x^1, x^2, x^3) = (\nu, r, \theta, \phi)$ so that the non-vanishing Christoffel symbols of the second kind of the line element (2.2) are

$$
\Gamma^1_{11} = \frac{\beta_r}{\beta}, \\
\Gamma^1_{22} = -\frac{r\alpha}{\beta^2}, \\
\Gamma^1_{01} = \frac{\alpha_r}{2\beta}, \\
\Gamma^1_{00} = -\frac{\alpha(\beta_\nu - \frac{1}{2}\alpha_r)}{\beta^2} + \frac{\alpha_\nu}{2\beta}, \\
\Gamma^0_{00} = \frac{\beta_\nu - \frac{1}{2}\alpha_r}{\beta}.
$$

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