COUPLING OF GAUSSIAN FREE FIELD WITH GENERAL SLIT SLE

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Abstract. We consider a coupling of the Gaussian free field with slit holomorphic stochastic flows, called $(\delta, \sigma)$-SLE, which contains known SLE processes (chordal, radial, and dipolar) as particular cases. In physical terms, we study a free boundary conformal field theory with one scalar bosonic field, where Green’s function is assumed to have some general regular harmonic part. We establish which of these models allow coupling with $(\delta, \sigma)$-SLE, or equivalently, when the correlation functions induce local $(\delta, \sigma)$-SLE martingales (martingale observables).

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2010 Mathematics Subject Classification. 30C35, 34M99, 60D05, 60G57, 60J67.
Key words and phrases. Löwner equation, stochastic flows, general Löwner theory, SLE, conformal field theory, boundary conformal field theory, Gaussian free field, coupling.

The authors were supported by EU FP7 IRSES program STREVCOMS, grant no. PIRSES-GA-2013-612669, and by the grants of the Norwegian Research Council #239033/F20.
1. Introduction

1.1. Simple example of coupling. The paper is focused on the coupling of the Gaussian free field (GFF) with the \((\delta, \sigma)\)-SLE, and studies in detail some special cases. Let us explain briefly the case of the chordal SLE\(_4\) and its coupling with the Gaussian free field subject to the Dirichlet boundary conditions, see [18] and [26] for details. We review some known generalizations, and then, explain the main results of the paper.

Let \(G_t : \mathbb{H} \setminus K_t \to \mathbb{H}\) be a conformal map from a subset \(\mathbb{H} \setminus K_t\) of the upper half-plane \(\mathbb{H} := \{ z \in \mathbb{C} : \text{Im} z > 0\}\) onto \(\mathbb{H}\) defined by the initial value problem for the stochastic differential equation

\[
\frac{d^{\text{Itô}} G_t(z)}{dt} = \frac{2}{G_t(z)} \frac{d}{dt} G_t(z) - \sqrt{\kappa} \frac{d^{\text{Itô}} B_t}{dG_t(z)} , \quad G_0(z) = z, \quad t \geq 0, \quad \kappa > 0,
\]

where \(d^{\text{Itô}}\) denotes the Itô differential and \(B_t\) stands for the standard Brownian motion (we avoid writing \(\circ\) for the Stratonovich integral because of possible confusion with superposition used at the same time, so we explicitly write \(d^S\) or \(d^{\text{Itô}}\)). This initial value problem is known as the chordal SLE whose solution \(G_t\) is a random conformal map which defines a strictly growing random set \(K_t := \mathbb{H} \setminus G_t^{-1}(\mathbb{H})\), called the SLE hull, \((K_t \subset K_s, s > t)\) bounded almost surely. In particular, it is a random simple curve for \(\kappa \leq 4\) that generically tends to infinity as \(t \to \infty\). It turns out that the random law defined this way is related to various problems of mathematical physics.

It is straightforward to check that the following random processes are local martingales for \(\kappa = 4\)

\[
M_{1t}(z) := \arg G_t(z), \quad M_{2t}(z, w) := -\log \left| \frac{G_t(z) - G_t(w)}{G_t(z) - G_t(w)} \right| + \arg G_t(z) \arg G_t(w),
\]

\[
M_{3t}(z, w, u) := -\log \left| \frac{G_t(z) - G_t(w)}{G_t(z) - G_t(w)} \right| \arg G_t(u) + \arg G_t(z) \arg G_t(w) \arg G_t(u) + \{ z \leftrightarrow w \leftrightarrow u \},
\]

for \(z, w, u \in \mathbb{H}\) until a stopping time \(t_z\), which is geometrically defined by the fact that \(K_t\) touches \(z\) for the first time, i.e. \(G_t(z)\) is no longer defined for \(t > t_z\). The Wick pairing structure can be recognized in this collection of martingales. It appears in higher moments of multivariant Gaussian distributions as well as in the correlation functions in the free quantum field theory. We consider the Gaussian free field (GFF) \(\Phi\) which is a random real-valued functional over smooth functions in \(\mathbb{H}\). Let us use the heuristic notation ‘\(\Phi(z)\)’ \((z \in \mathbb{H})\) until a rigorous definition is given in Section 2.7.

For some special choice of GFF the moments are

\[
S_1(z) := \mathbb{E} [\Phi(z)] = \arg z,
\]

\[
S_2(z, w) := \mathbb{E} [\Phi(z)\Phi(w)] = -\log \left| \frac{z - w}{z - \bar{w}} \right| + \arg z \arg w,
\]

\[
S_3(z, w, u) := \mathbb{E} [\Phi(z)\Phi(w)\Phi(u)] = -\log \left| \frac{z - w}{z - \bar{w}} \right| \arg u + \arg z \arg w \arg u + \{ z \leftrightarrow w \leftrightarrow u \},
\]

Roughly speaking, \(\Phi(z)\) is a Gaussian random variable for each \(z\) with the expectation \(S_1(z)\) proportional to \(\arg z\), and with the covariance \(-\log \left| \frac{z - w}{z - \bar{w}} \right|\). It can be shown that the
stochastic processes
\begin{equation}
M_{nt} := S_n(G_t(z_1), G_t(z_2), \ldots G_t(z_n))
\end{equation}
are martingales for all \(n = 1, 2, \ldots\).

This fact is closely related to the observation that the random variable \(\Phi(G_t(z))\) agrees in law with \(\Phi(z)\), where \(\Phi\) and \(G_t\) are sampled independently. It was also obtained, see [23], that the properly defined zero-level line of the random distribution \(\Phi(z)\) starting at the origin in the vertical direction agrees in law with SLE\(_4\).

This connection, i.e., coupling, can be generalized for an arbitrary \(\kappa > 0\), see [29]. In order to explain its geometric meaning let us associate another field \(J(z)\) of unit vectors \(|J(z)| = 1\) with \(\Phi(z)\) by
\[
J(z) := e^{i\Phi(z)/\chi}
\]
for some \(\chi > 0\). A unit vector field \(J\) transforms from a domain \(D\) to \(D \setminus K\) according to the rule
\[
J(z) \rightarrow \tilde{J}(\tilde{z}) = \frac{G'(\tilde{z})}{G'(\tilde{z})}J(G(\tilde{z}))
\]
where \(G\) is the conformal map \(G : D \setminus K \rightarrow D\) (the coordinate transformation). Thereby, \(\Phi(z)\) transforms as
\begin{equation}
\Phi(z) \rightarrow \tilde{\Phi}(\tilde{z}) = \Phi(G(\tilde{z})) - \chi \arg G'(\tilde{z}).
\end{equation}
The coupling for \(\kappa > 0\) is the agreement in law of \(\Phi(G_t(z)) - \chi \arg G'(\tilde{z})\) with \(\Phi(z)\), where
\begin{equation}
\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.
\end{equation}
Besides, the flow line of \(J(z)\) starting at the origin agrees in law with the SLE\(_\kappa\) curve.

The statement above can be extended from the chordal equation (1.1) to the radial equation, see [2, 19], or to the dipolar one, see [5, 20, 17], see also [16] for the SLE\(_{(\kappa, \rho)}\) case.

In [13, 15], we considered slit holomorphic stochastic flows, called in this paper \((\delta, \sigma)\)-SLE, which contain all the above mentioned SLEs as special cases except SLE\(_{(\kappa, \rho)}\). In the present paper, we consider a general case of the coupling of \((\delta, \sigma)\)-SLE with GFF. In particular, we study the known cases of the couplings in a systematic way as well as consider some new ones.

There is also another type of coupling described in [29] for the chordal case. The usual forward Löwner equation is replaced by its reverse version with the opposite sign at the drift term proportional to \(dt\) in (1.1). The corresponding solution is a conformal map \(G_t : \mathbb{H} \rightarrow \mathbb{H} \setminus K_t\), and \(\Phi(z)\) agrees in law with \(\Phi(G_t(z)) + \gamma \log |G_t'(z)|\) for some real \(\kappa\)-dependent constant \(\gamma\).

In addition to the coupling, there are also interesting connections to other aspects of conformal field theory such as the highest weight representations of the Virasoro algebra [3], and the vertex algebra [18]. On the other hand, the crossing probabilities, such as touching the real line by an SLE curve, are connected with the CFT stress tensor correlation functions, see [3, 11]. Both of these directions, as well as the reverse coupling, exceed the scope of this paper.

1.2. \((\delta, \sigma)\)-SLE overview. The \((\delta, \sigma)\)-SLE is a unification of the well-known chordal, radial, and dipolar stochastic Löwner equations described first in [13]. We will give a rigorous definition in Section 2 and will use a simplified version in Introduction. Let
$D \subset \mathbb{C}$ is a simply connected hyperbolic domain. A $(\delta, \sigma)$-SLE or a slit holomorphic stochastic flow is the solution to the stochastic differential equation

\begin{equation}
(1.5) \quad d^S G_t(z) = \delta(G_t(z)) dt + \sigma(G_t(z)) d^S B_t, \quad G_0(z) = z, \quad z \in D, \quad t \geq 0,
\end{equation}

where $B_t$ is the standard Brownian motion, $\sigma: D \to \mathbb{C}$ and $\delta: D \to \mathbb{C}$ are some fixed holomorphic vector fields defined in such a way that the solution $G_t$ is always a conformal map $G_t: D \setminus K_t \to D$ for some random curve-generated subset $K_t$. The differential $d^S$ is the Stratonovich differential, which is more convenient in our setup than the more frequently used Itô differential $d^{I}$. The same equation in the Itô form is

\begin{equation}
(1.6) \quad d^{I} G_t(z) = \left(\delta(G_t(z)) + \frac{1}{2} \sigma'(G_t(z)) \sigma(G_t(z))\right) dt + \sigma(G_t(z)) d^{I} B_t, \quad G_0(z) = z, \quad t \geq 0,
\end{equation}

see Appendix B for details. Equation (1.6) can be easily reformulated for any other domain $\tilde{D} \subset \mathbb{C}$ using the transition function $\tau: \tilde{D} \to D$. Define $\tilde{G}_t := \tau^{-1} \circ G_t \circ \tau$. The equation for $\tilde{G}_t$ is of the same form as (1.5), but with new fields $\tilde{\delta}$ and $\tilde{\sigma}$ defined as

$$
\tilde{\delta}(\tilde{z}) = \frac{1}{\tau'(\tilde{z})} (\delta(\tau(\tilde{z}))), \quad \tilde{\sigma}(\tilde{z}) = \frac{1}{\tau'(\tilde{z})} \sigma(\tau(\tilde{z})).
$$

The general form of $\delta(z)$ and $\sigma(z)$ for $D = \mathbb{H}$ (the upper half-plane) is

\begin{align}
(1.7) \quad & \delta^\mathbb{H}(z) = \frac{\delta_{-2}}{z} + \delta_{-1} + \delta_{0} z + \delta_{1} z^2, \\
& \sigma^\mathbb{H}(z) = \sigma_{-1} + \sigma_{0} z + \sigma_{1} z^2,
\end{align}

$\delta_{-1}$, $\delta_{0}$, $\delta_{1}$, $\sigma_{-1}$, $\sigma_{0}$, $\sigma_{1} \in \mathbb{R}$, $\sigma_{-1} \neq 0$, $\delta_{-2} > 0$.

The values of $\delta$ and $\sigma$ for the classical SLEs (in the upper half-plane) are summarized in the following table.

| SLE type      | $\delta^\mathbb{H}(z)$ | $\sigma^\mathbb{H}(z)$ |
|--------------|-------------------------|-------------------------|
| Chordal      | $2/z$                   | $-\sqrt{\kappa}$       |
| Radial       | $1/(2z) + z/2$          | $-\sqrt{\kappa(1 + z^2)/2}$ |
| Dipolar      | $1/(2z) - z/2$          | $-\sqrt{\kappa(1 - z^2)/2}$ |
| ABP SLE, see [14] | $1/(2z)$               | $-\sqrt{\kappa(1 + z^2)/2}$ |

We use the half-plane formulation in this table just for simplicity. In order to obtain the commonly used form of the radial equation in the unit disk one applies the transition map

\begin{equation}
(1.8) \quad \tau_{\mathbb{H},\mathbb{D}}(z) := i \frac{1 - z}{1 + z}, \quad \tau_{\mathbb{H},\mathbb{D}}: \mathbb{D} \to \mathbb{H},
\end{equation}

for the unit disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$. For more details, see Example 2 and Section 4.3.

The same procedure with the transition map

\begin{equation}
(1.9) \quad \tau_{\mathbb{H},\mathbb{S}}(z) = th \frac{z}{2}, \quad \phi_{\mathbb{H},\mathbb{S}}: \mathbb{S} \to \mathbb{H},
\end{equation}

for the strip $\mathbb{S} := \{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}$, gives the common form for the dipolar SLE, see also Section 4.2.

It was shown in [13], that the choice of $\delta$ and $\sigma$ given by (1.7) guarantees that the random set $K_t$ is curve-generated similarly to the standard known SLE. Moreover, $K_t$ has the same local behaviour (the fractal dimension, dilute phases, and etc.).

The main difference between the classical SLEs and the case of the general $(\delta, \sigma)$-SLE is the absence of fixed normalization points, in general, e.g., in the classical cases the
solution \( G_t \) to the radial SLE equation always fixes an interior point (0 in the unit disk formulation), the dipolar SLE preserves two boundary points \((-\infty, +\infty\) in the strip coordinates \( S \)), the chordal equation is normalized at the boundary point at infinity in the half-plane formulation. The chordal case can be considered as the limiting case of the dipolar one as both fixed boundary points collide, or the limiting case of the radial equation when the fixed interior point approaches the boundary. In the general case of \((\delta, \sigma)\)-SLE, there is no such a simple normalization. Numerical experiments show that the curve (or the curve-generated hull for \(\kappa \geq 4\)) \( K_t \) tends to a random point inside the disk as \( t \to \infty \). A version of the domain Markov property still holds due to an autonomous form of equation (1.6).

1.3. Overview and purpose of this paper. We consider the general form of 2-dimensional GFF \( \Phi \) with some expectation 
\[
\eta(z) := \mathbb{E}[\Phi(z)],
\]
and the covariance
\[
\Gamma(z, w) := \mathbb{E}[\Phi(z)\Phi(w)] - \mathbb{E}[\Phi(z)]\mathbb{E}[\Phi(w)].
\]
We postulate that \( \Gamma(z, w) \) is symmetric fundamental solution to the Laplace equation,
\[
\triangle \Gamma(z, w) = 2\pi \delta(z, w), \quad \Gamma(z, w) = \Gamma(w, z),
\]
but imposing no boundary conditions. In other words, \( \Gamma(z, w) \) has the form
\[
\Gamma(z, w) = -\frac{1}{2} \log |z - w| + \text{symmetric harmonic function of } z \text{ and } w.
\]
We address the following question: which of so defined GFF can be coupled with some \((\delta, \sigma)\)-SLE in the sense that \( \Phi(z) \) and \( \Phi(G_t(z)) \) agree in law? In particular, we show (Theorem 1) that this is possible if and only if the following system of partial differential equation is satisfied
\[
(1.10)
\]
\[
\left( L_\delta + \frac{1}{2} L_\sigma^2 \right) \eta(z) = 0, \\
L_\delta \Gamma(z, w) + L_\sigma \eta(z) L_\sigma \eta(w) = 0, \\
L_\sigma \Gamma(z, w) = 0,
\]
where, following [18], we understand under \( L \) a generalized version of the Lie derivatives defined by
\[
L_v \eta(z) := v(z) \partial_z + \overline{v(z)} \partial_{\overline{z}} + \chi \text{Im } v'(z), \\
L_v \Gamma(z, w) := v(z) \partial_z + \overline{v(z)} \partial_{\overline{z}} + v(w) \partial_w + v(w) \partial_{\overline{w}}.
\]
Here, \( v = \delta \) or \( v = \sigma \) and \( \chi \) refers to the SLE parameter \( \kappa \) by (1.4). These equations generalize the corresponding relation from [29], see the table on page 45, and they are versions of (M-cond') and (C-cond') from [16]. The second equation in (1.10) is known as a version of Hadamard’s variation formula, and the third states that the covariance \( \Gamma \) must be invariant with respect to the Möbius automorphisms generated by the vector field \( \sigma \).

The paper is a continuation and extension of [15] and is organized as follows. The rest of Introduction provides some remarks about the relations between equations (1.10) and the BPZ equation considered first in [6]. In Section 2 we give all necessary definitions for \((\delta, \sigma)\)-SLE as well as a very basic definitions and properties of GFF that we will need in what follows. Further, in Section 3 we prove the coupling Theorem 1. It states that for any \((\delta, \sigma)\)-SLE a proper pushforward of \( S_n \) denoted by \( G_t^{-1} S_n \) (a generalisation of (1.2) with an additional \( \chi \)-term similar to (1.3)) is a local martingale if and only if the system (1.10) is satisfied for given \( \delta, \sigma, \Gamma, \) and \( \eta \). The same theorem also states that both: the local martingale property and the system of equations are equivalent to a local coupling
which is a weaker version of the coupling from \[29\] discussed above. We expect, but do not consider in this paper, that the local coupling leads to the same property of the flow lines of $e^{\Phi(z)/x}$ to agree in law with the $(\delta, \sigma)$-SLE curves.

The general solution to the system \[(1.10)\] gives all possible ways to couple $(\delta, \sigma)$-SLEs with the GFF at least in the frameworks of our assumptions of the pre-pre-Schwarzian behaviour of $\eta$ and of the scalar behaviour of $\Gamma$. In Section 4 we assume the simplest choice of the Dirichlet boundary conditions for $\Gamma$ and study all $(\delta, \sigma)$-SLEs that can be coupled. It turns out that only the classical SLEs with drift are allowed plus some exceptional cases, see Sections 4.5 and 4.6, that define the same measure as for the chordal SLE up to time reparametrization. Observe however, that the coupling with the classical SLEs with drift has not been considered in the literature so far.

Further, in Sections 5 and 6, we assume less trivial choices of $\Gamma$ and obtain that among all $(\delta, \sigma)$-SLEs only the dipolar and radial SLEs with $\kappa = 4$ can be coupled. The first case was considered in $[17]$. The second one requires a construction of a specific ramified GFF $\Phi$ twisted, that changes its sign to the opposite while being turned once around some interior point in $D$. This construction was considered before and it is called the ‘twisted CFT’ as we were informed by Nam-Gyu Kang, see $[21]$.

1.4. Nature of coupling. One of the approaches to conformal field theory (CFT) boils down to consideration of a probability measure in a space of function in $D$. The simplest choice is GFF. The chordal SLE/CFT correspondence is revealed in, for example, $[8]$ and $[18]$. Here, we extend this treatments to a $(\delta, \sigma)$-SLE/CFT correspondence. This section is less important for the general objective of the paper and is dedicated mostly to a reader with physical background.

We consider a boundary CFT (BCFT) defined on a domain $D \subset \mathbb{C}$ with one free scalar bosonic field $\Phi$. One of the standard approach to define a quantum field theory is the heuristic functional integral formulation with the classical action $I[\Phi]$, see for instance, $[7]$. The following triple definition of the Schwinger functions $S(z_1, z_2, \ldots z_n)$ manifests the relation between the probabilistic notations, functional integral formulation, and the operator approach.

\begin{equation}
S_n(z_1, z_2, \ldots, z_n) := \mathbb{E}[\Phi(z_1)\Phi(z_2)\ldots\Phi(z_n)] = \\
e^{-\frac{1}{2} \int \Phi(z_1)\Phi(z_2)\ldots\Phi(z_n)e^{-I[\Phi]}d\Phi} = \\
= \langle \{T[\hat{\Psi}(a)\hat{\Phi}(z_1)\hat{\Phi}(z_2)\ldots\hat{\Phi}(z_n)] \} \rangle, \\
z_i \in D, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots \tag{1.11}
\end{equation}

Here in the second term, ‘$\langle \rangle$’ is the vacuum state, $\hat{\Phi}(z)$ the primary operator field, $\hat{\Psi}$ a certain operator field taken at a boundary point $a \in D$, and $T[\ldots]$ is the time-ordering, which is often dropped in the physical literature, we refer to $[28]$ for details. The second string in (1.11) contains a heuristic integral over some space of functions $\Phi(z)$ on $D$, which corresponds to the operator $\hat{\Phi}(z)$.

The first string in (1.11) is a mathematically precise formulation of the second one. The expectation $\mathbb{E}[\ldots]$ can be understood as the Lebesgue integral over the space $\mathcal{D}'(D)$ of linear functionals over smooth functions in $D$ with compact support. The expression $e^{-I[\Phi]}d\Phi$ can be in its turn understood as the differential w.r.t. the measure.

We emphasize here that the correlation functions are not completely defined by the action $I[\Phi]$ because one has to specify also the space of functions the integral is taken

\[\int \mathcal{D}\Phi e^{-I[\Phi]} \]
over and the measure. For instance, the Euclidean free field action
\[ I[\Phi] = \frac{1}{2} \int_D \partial \Phi(z) \bar{\partial} \Phi(z) d^2 z \]
defines only the singular part of the 2-point correlation function, which is \((-2\pi)^{-1} \log|z|\). To illustrate the statement above let us consider the following example. Let the expectation of \( \Phi \) be identically zero,
\[ S_1(z) = \int \Phi(z) e^{-I[\Phi]} \mathcal{D}\Phi = 0, \quad z \in D. \]
Depending on the choice of the space of functions \( \mathcal{H}' \ni \Phi \) and of the measure on it, the 2-point correlation function
\[ \Gamma(z_1, z_2) = \int \Phi(z_1) \Phi(z_2) e^{-I[\Phi]} \mathcal{D}\Phi \]
vary. For example in \([18, 19, 20, 23]\), \( \Gamma \) was assumed to possess the Dirichlet boundary conditions, see \((2.37)\). But in \([17]\), \( \Gamma \) vanishes only on a part of the boundary and on the other part the boundary conditions are Neumann’s. In \([16]\), even more general boundary conditions were considered.

In fact, under CFT one can understand a probability measure on the space of functions \( \mathcal{H}' \) in \( D \), which is just a version of \( \mathcal{D}'(D) \). In the case of the free field, the Schwinger functions (equivalently, correlation functions or moments) \( S_n \) are of the form
\[
S_1[z_1] = \eta[z_1],
S_2[z_1, f_2] = \Gamma[f_1, f_2] + \eta[f_1] \eta[f_2],
S_3[z_1, f_2, f_3] = \Gamma[f_1, f_2] \eta[f_3] + \Gamma[z_2, z_3] \eta[f_2] + \Gamma[z_2, z_3] \eta[f_1] + \eta[f_1] \eta[z_2] \eta[f_3],
S_4[f_1, f_2, f_3, f_4] = \Gamma[z_1, z_2] \Gamma[z_3, z_4] + \Gamma[f_1, f_2] \Gamma[f_3, f_4] + \Gamma[f_1, f_3] \Gamma[f_2, f_4] + \Gamma[f_1, f_4] \Gamma[z_2, z_3] + \Gamma[z_1, z_2] \eta[z_3] \eta[z_4] + \Gamma[z_1, z_3] \eta[z_2] \eta[z_4] + \Gamma[z_1, z_4] \eta[f_2] \eta[z_3] + \eta[z_1] \eta[z_2] \eta[z_3] \eta[z_4],
\]
\[
\ldots
\]
In this case, the measure is characterized by the space \( \mathcal{H}' \) equipped with certain topology, by the field expectation \( \eta(z) = S_1(z) \), and by Green’s function (covariance) \( \Gamma(z, w) = S_2(z, w) - S_1(z) S_1(w) \).

Assume now that
\[
(1.12) \quad \mathcal{L} S_n(z_1, z_2, \ldots, z_n) = 0, \quad n = 1, 2, \ldots
\]
for some first order differential operator \( \mathcal{L} \). Then, in the language of quantum field theory, the functions \( S_n \) are called ‘correlation functions’ and the relations \( (1.12) \) can be called the Ward identities. This identities usually correspond to invariance of \( S_n \) with respect to some Lie group. For example, if
\[
(1.13) \quad \mathcal{L} := \sum_{i=1}^{\infty} \sigma(z_i) \partial_{z_i} + \overline{\sigma(z_i)} \partial_{z_i}
\]
for a vector field \( \sigma \) of the form \( (1.17) \), then \( (1.12) \) is equivalent to
\[
S_n(z_1, z_2, \ldots, z_n) = S_n(H_s[\sigma](z_1), H_s[\sigma](z_2), \ldots, H_s[\sigma](z_n)), \quad s \in \mathbb{R},
\]
where \( \{H_t[\sigma]\}_{t \in \mathbb{R}} \) is a one parametric Lie group of Möbius automorphisms \( H_t[\sigma] : D \to D \) induced by \( \sigma \):
\[
(1.14) \quad dH_t[\sigma](z) = \sigma(H_t[\sigma](z)) dt, \quad z \in D, \quad t \in \mathbb{R}, \quad H_0[\sigma](z) = z,
\]
because

\[ (1.15) \quad dS_n(H_s[\sigma](z_1), H_s[\sigma](z_2), \ldots, H_s[\sigma](z_n))|_{s=0} = \mathcal{L} S_n(z_1, z_2, \ldots, z_n) ds. \]

The relations (1.12) with (1.13) are satisfied if and only if

\[ (\sigma(z) \partial_z + \overline{\sigma(z)} \partial_{\overline{z}}) \eta(z) = 0, \]

\[ (\sigma(z) \partial_z + \overline{\sigma(z)} \partial_{\overline{z}} + \sigma(w) \partial_w + \overline{\sigma(w)} \partial_{\overline{w}}) \Gamma(z, w) = 0. \]

Consider now the vector field \( \delta \) in place of \( \sigma \). It is still holomorphic but generates a semigroup \( \{H_t[\sigma]\}_{t \in (-\infty, 0]} \) of endomorphisms \( G_t : D \to D \setminus \gamma_t \) (\( \gamma_t \) is some growing curve in \( D \)) instead of the group of automorphisms of \( D \). Then the second identity in (1.16) does not hold because

\[ (\delta(z) \partial_z + \overline{\delta(z)} \partial_{\overline{z}} + \delta(w) \partial_w + \overline{\delta(w)} \partial_{\overline{w}}) \Gamma(z, w) \neq 0. \]

Geometrically this can be explained by the fact that \( H_t[\delta] \) is not just a change of coordinates but also it necessarily shrinks the domain \( D \).

Let now \( G_t \) satisfy (1.15), which is just a stochastic version of (1.14), where \( v(z)dt \) is replaced by \( \delta(z)dt + \sigma(z) dB_t \). The vector field \( \delta \) induces endomorphisms, \( \sigma \) induces automorphisms, and the multiple of \( dB_t \) can be understood as the white noise. Such a stochastic change of coordinates \( G_t \) in the infinitesimal form, according to the Itô formula, leads to the second order differential operator \( \delta(z) \partial_z + \frac{1}{2} (\sigma(z) \partial_z)^2 \) instead of the first order Lie derivative \( v(z) \partial_z \).

The first two Ward identities become

\[ (\delta(z) \partial_z + \frac{1}{2} (\sigma(z) \partial_z)^2 + \text{complex conjugate}) \eta(z) = 0, \]

\[ (\delta(z) \partial_z + \delta(w) \partial_w + \frac{1}{2} (\sigma(z) \partial_z + \sigma(w) \partial_w)^2 + \text{complex conjugate}) \times \Gamma(z, w) + \eta(z) \eta(w) = 0, \]

which is equivalent to (1.10) if \( \chi = 0 \). Due to a version of the Itô formula we have

\[ \left. d^{\text{Itô}} S_n(G_s(z_1), G_s(z_2), \ldots, G_s(z_n)) \right|_{s=0} = \left( \mathcal{L}_\sigma + \frac{1}{2} \mathcal{L}_\sigma^2 \right) S_n(z_1, z_2, \ldots, z_n) dt + \mathcal{L}_\sigma S_n(z_1, z_2, \ldots, z_n) dB_t, \]

which is an analog to the relation (1.15). In other words, the system (1.10) represents the local martingale conditions.

For the case

\[ (1.18) \quad \delta(z) = \frac{2}{z}, \quad \sigma(z) = -\sqrt{\kappa}, \quad \kappa > 0, \quad z \in D = \mathbb{H}, \]

the identities (1.17) is an analog to the BPZ equation (5.17) in [9]. The choice of (1.18) corresponds to the chordal SLE (see Example 1) and it was considered first in [8], and later in [18]. In this paper, we assume that the field \( \delta \) is holomorphic, has a simple pole of positive residue at a boundary point \( a \in \partial D \), and tangent at the rest of the boundary.

2. Preliminaries

Each version of the Löwner equations and holomorphic flows are usually associated with a certain canonical domain \( D \subset \mathbb{C} \) in the complex plane specifying fixed interior or boundary points, for example, in the case of the upper half-plane, the unit disk, etc. It is always possible to map these domains one to another if necessary. In this paper, we focus on conformally invariant properties, i.e., those which are not related to a specific
choice of the canonical domain. This can be achieved by considering a generic hyperbolic simply connected domain and conformally invariant structures from the very beginning. It could be also natural to go further and work with a simply connected hyperbolic Riemann surface $\mathcal{D}$ (with the boundary $\partial \mathcal{D}$). In what follows, $\mathcal{D}$ is thought of as a generic domain with a well-defined boundary or a Riemann surface.

We use mostly global chart maps $\psi : \mathcal{D} \to \mathbb{D} \subset \mathbb{C}$ from $\mathcal{D}$ to a domain of the complex plane, writing $\psi$ for a chart $(\mathcal{D}, \psi)$ for simplicity. The charts $\psi \mathbb{H} : \mathcal{D} \to \mathbb{H}$ corresponding to the upper half-plane or $\psi \mathbb{D} : \mathcal{D} \to \mathbb{D}$ to the unit disk are related by \cite{128}. Another example is a multivalued map $\psi_L : \mathcal{D} \to \mathbb{D}$ to the upper half-plane or $\psi$ plane, writing thereby (2.2)

We consider a conformal homeomorphism $G : \mathcal{D} \setminus \mathcal{K} \to \mathcal{D}$ (inverse endomorphism) that in a given chart $\psi$ is given as $G^\psi := \psi \circ G \circ \psi^{-1} : \mathcal{D}^\psi \setminus \mathcal{K}^\psi \to \mathcal{D}$. A single-valued branch of $\psi_L : \mathcal{D} \to \mathbb{D}$ is not a global chart map.

2.1. **Vector fields and coordinate transform.** Consider a holomorphic vector field $v$ on $\mathcal{D}$, which can be defined as a holomorphic section of the tangent bundle. We also can define it as a map $\psi \mapsto v^\psi$ from the set of all possible global chart maps $\psi : \mathcal{D} \to \mathcal{D}^\psi$ to the set of holomorphic functions $v^\psi : \mathcal{D}^\psi \to \mathbb{C}$ defined in the image of these maps $\mathcal{D}^\psi := \psi(\mathcal{D})$. For the vector fields, the following coordinate change holds. For any $\tilde{\psi} : \mathcal{D} \to \tilde{\mathcal{D}}$, we have

\begin{equation}
\tilde{v}^\tilde{\psi}(\tilde{z}) = \frac{1}{(\tau'(\tilde{z}))}v^\psi(\tau(\tilde{z})), \quad \tau := \psi \circ \tilde{\psi}^{-1}, \quad \tilde{z} \in \tilde{\mathcal{D}}.
\end{equation}

We consider a conformal homeomorphism $G : \mathcal{D} \setminus \mathcal{K} \to \mathcal{D}$ (inverse endomorphism) that in a given chart $\psi$ is given as $G^\psi := \psi \circ G \circ \psi^{-1} : \mathcal{D}^\psi \setminus \mathcal{K}^\psi \to \mathcal{D}$. A vector field in other words is a $(-1, 0)$-differential and the pushforward of a vector field $G_{s}^{-1} : v^\psi \mapsto \tilde{v}^\psi$ is defined by the rule

\begin{equation}
G_{s}^{-1}v^\psi(z) := v^{G \circ \psi^{-1} \circ G^{-1}}(z) = v^{G \circ \psi^{-1} \circ G^{-1}}(z) = v^{G \circ \psi^{-1} \circ G^{-1}}(z) =
\end{equation}

thereby $G^\psi$ plays the role of $\tau$.

It is easy to see that for any two given maps $G$ and $\tilde{G}$ as above

$\tilde{G}_{s}^{-1}G_{s}^{-1} = (\tilde{G}^{-1} \circ G^{-1})_{s}$,

which motivates the upper index $-1$ in the definition of $G_{s}$, because in this case we are working with a left module.

The pushforward map $G_{s}^{-1}$ also can be obtained as a map from the tangent space to $\mathcal{D}$ induced by $G$. We follow the way above because it can be generalized then to sections of tangent and cotangent spaces and their tensor products, see Section 2.3.

Let $v_t$ and $\tilde{v}_t$ be two holomorphic vector fields depending on time continuously such as the following differential equations has continuously differentiable solutions $F_t$ and $\tilde{F}_t$ in some time interval

$$
\dot{F}_t = v_t \circ F_t,
$$

$$
\dot{\tilde{F}}_t = \tilde{v}_t \circ \tilde{F}_t.
$$

Then, we can conclude that

$$
\frac{\partial}{\partial t}(F_t \circ \tilde{F}_t) = (v_t + F_t \tilde{v}_t \tilde{F}_t) \circ F_t \circ \tilde{F}_t
$$

$$
\tilde{F}_t^{-1} = -\left(F_t^{-1}v_t\right) \circ F_t^{-1}.
$$
in the same $t$-interval and in the region of $D$ where $F_t$ and $\tilde{F}_t$ are defined. The latter relation can be reformulated in a fixed chart $\psi$ as

$$F_t^{-1}\psi(z) = -\left(\left(F_t^{-1}\right)'(z)\right)\psi'(z).$$

2.2. $(\delta, \sigma)$-SLE basics. Here we repeat briefly some necessary material about the slit holomorphic stochastic flows $\text{SLE}(\delta, \sigma)$ considered first in the paper [13] that we advice to follow for more details.

A holomorphic vector field $\sigma$ on $D$ is called complete if the solution $H_s[\sigma]^\psi(z)$ of the initial valued problem

$$(2.3) \quad \dot{H}_s[\sigma]^\psi(z) = \sigma^\psi(H_s[\sigma]^\psi(z)), \quad H_0[\sigma]^\psi(z) = z, \quad z \in D^{\psi}$$

is defined for $s \in (-\infty, \infty)$. The solution $H_s[\sigma]^\psi : D^{\psi} \rightarrow D^{\psi}$ is a conformal automorphism of $D^{\psi} = \psi(D)$. Here and further on, we denote the partial derivative with respect to $s$ as $\dot{H}_s := \frac{\partial}{\partial s} H_s$. It is straightforward to see that the differential equation is of the same form in any chart $\psi$. So it is reasonable to drop the index $\psi$ and variable $z$ as follows

$$(2.4) \quad \dot{H}_s[\sigma] = \sigma \circ H_s[\sigma], \quad H_0[\sigma] = \text{id}. $$

Let $\psi_{\text{H}}$ be a chart map onto the upper half-plane $D^{\psi} = \mathbb{H}$. We denote $X^{\text{H}} := X^{\psi_{\text{H}}}$ if $X$ is a vector field, a conformal map, or a pre-pre-Schwarzian form (defined below) on $D$.

A complete vector field $\sigma$ in the half-plane chart admits the form

$$(2.5) \quad \sigma^{\text{H}}(z) = \sigma_{-1} + \sigma_0 z + \sigma_1 z^2, \quad z \in \mathbb{H}, \quad \sigma_{-1}, \sigma_0, \sigma_1 \in \mathbb{R}. $$

A vector field $\delta$ is called antisemicompllicate if the initial valued problem

$$\dot{H}_s[\delta] = \delta \circ H_s[\delta], \quad H_0[\delta] = \text{id},$$

has a solution $H_t[\sigma]$, which is a conformal map $H_t[\sigma] : D \setminus \mathcal{K}_t \rightarrow D$ for all $t \in [0, +\infty)$ for some family of subsets $\mathcal{K}_t \subset D$.

The linear space of all antisemicomplete fields is essentially bigger and infinite-dimensional, but we restrict ourselves to the from

$$(2.6) \quad \delta^{\text{H}}(z) = \frac{\delta_{-2}}{z} + \delta_{-1} + \delta_0 z + \delta_1 z^2, \quad z \in \mathbb{H}, \quad \delta_{-1}, \delta_0, \delta_1 \in \mathbb{R}, \quad \delta_{-2} > 0,$$

which guarantees that the set $\mathcal{K}_t$ is curve-generated, see [13]. The first term gives just a simple pole in the boundary at the origin with a positive residue. The sum of the last three terms is just a complete field.

Let us consider first a continuously differentiable driving function function $u_t : t \mapsto \mathbb{R}$ for motivation. Then the solution $G_t$ of the initial value problem

$$(2.7) \quad \dot{G}_t = \delta \circ G_t + \dot{u}_t \sigma \circ G_t, \quad G_0 = \text{id}, \quad t \geq 0,$$

is a family of conformal maps $G_t : D \setminus \mathcal{K}_t \rightarrow D$, where the family of subsets $\{\mathcal{K}_t\}_{t \geq 0}$ depends on the driving function $u$. To avoid the requirement of continuous differentiability we use the following method.

Define a conformal map

$$g_t := H_{u_t}[\sigma]^{-1} \circ G_t, \quad t \geq 0.$$ 

It satisfies the equation

$$(2.8) \quad \dot{g}_t = (H_{u_t}[\sigma]^{-1} \delta) \circ g_t, \quad g_0 = \text{id},$$

where $H_{u_t}[\sigma]^{-1} \delta$ is defined in (2.2) for $G = H_{u_t}[\sigma]$. Reciprocally, (2.7) can be obtained from (2.3), although (2.3) is defined for a continuous function $u_t$, not necessary continuously differentiable. This motivates the following definition.
Definition 1. Let \( \sigma \) and \( \delta \) be a complete and a semicomplete vector fields as in (2.5) and (2.6), and let \( u_t \) be a continuous function \( u : [0, \infty) \to \mathbb{R} \). Then the solution \( g_t \) to the initial value problem (2.8) is called the (forward) general slit Löwner chain. Respectively, the map \( G_t \) is defined by
\[
G_t := H_{u_t}[\sigma] \circ g_t, \quad t \geq 0.
\]

The stochastic version of equation (2.8) can be set up by introducing the Brownian measure on the set of driving functions, or equivalently, as follows.

Definition 2. Let \( \sigma \) and \( \delta \) be as in (2.5) and (2.6), and let \( B_t \) be the standard Brownian motion (the Wiener process). Then the solution to the stochastic differential equation (2.9)
\[
d^S G_t = \delta \circ G_t dt + \sigma \circ G_t d^S B_t, \quad G_0 = \text{id}, \quad t \geq 0
\]
(where \( d^S \) is the Stratanovich differential) is called a slit holomorphic stochastic flow or \((\delta, \sigma)\)-SLE.

In order to formulate (2.9) in the Itô form we have to chose some chart \( \psi \):
\[
d^\text{Itô} G_t(z) = \left( \delta^\psi + \frac{1}{2} \sigma^\psi \left( \sigma^\psi \right) \right) \circ G_t(z) dt + \sigma^\psi \circ G_t(z) d^\text{Itô} B_t, \quad G_0^\psi(z) = z,
\]
see, for example, [9, Section 4.3] for the definition of the Stratanovich and Itô’s differentials and for the relations between them. A disadvantage of Itô’s form is that the coefficient at \( dt \) transforms from chart to chart in a complicated way.

Example 1. Chordal Löwner equation.
Let us show how the construction (2.7–2.10) works in the case of the chordal Löwner equation. Define
\[
\delta^\mathbb{H}(z) = \frac{2}{z}, \quad \sigma^\mathbb{H}(z) = -1,
\]
in the half-plane chart. Then \( H[\sigma] \) can be found from equation (2.4) as
\[
\dot{H}_s[\sigma]^\mathbb{H}(z) = -1, \quad H_0[\sigma]^\mathbb{H}(z) = z, \quad z \in \mathbb{H},
\]
\[
H_s[\sigma]^\mathbb{H}(z) = z - s \quad z \in \mathbb{H}.
\]
The equation for \( g_t \) becomes
\[
\dot{g}_t^\mathbb{H} = \frac{2}{g_t^\mathbb{H} - u_t},
\]
and it is known as the chordal Löwner equation.

For a differentiable \( u_t \) we can write
\[
G_t^\mathbb{H}(z) = \frac{2}{G_t^\mathbb{H}(z)} - \dot{u}_t
\]
The stochastic version in the Stratanovich form can be obtained by substituting \( u_t = \sqrt{\kappa} B_t \):
\[
d^S G_t^\mathbb{H}(z) = \frac{2}{G_t^\mathbb{H}(z)} dt - \sqrt{\kappa} d^S B_t,
\]
and it is of the same form in the Itô case in the half-plane chart because \( \sigma^\mathbb{H}(z) \equiv 0 \),
\[
d^\text{Itô} G_t^\mathbb{H}(z) = \frac{2}{G_t^\mathbb{H}(z)} dt - \sqrt{\kappa} d^\text{Itô} B_t.
\]
In other charts \((\sigma^X)'(z) \neq 0\), and the Stratonovich and Itô forms differ. The space \(G[\delta, \sigma]\) consists of endomorphisms that in the half-plane chart are normalized as

\[
G^H(z) = z + O\left(\frac{1}{z}\right).
\]

**Example 2. Radial Leowner equation.**

Let \(\psi_D : D \to \mathbb{D}\) be the chart map defined by \((1.8)\), and let \(\tau_{H,D} = \psi_H \circ \psi_D^{-1}\). The transition function \(\tau_{H,D}\) maps the point 1 in the unit disk chart to the point 0 in the half-plane chart, the point \(-1\) to a point at infinity, and the point 0 to \(+i\). Similarly to what we have done in the half-plane case, we define \(X_D := X_H^D\) and call it the unit disk chart.

Let

\[
\delta^D(z) = -z\frac{z + 1}{z - 1}, \quad \sigma^D(z) = -iz,
\]

that corresponds to

\[
\delta^H(z) = \frac{1}{2}\left(\frac{1}{z} + z\right), \quad \sigma^H(z) = -\frac{1}{2}(1 + z^2)
\]

in the half-plane chart.

The equation for \(H_s[\sigma]\) in the unit disk chart becomes

\[
\dot{H}_s[\sigma] = -iH_s[\sigma] + \mathbf{1}, \quad H_0[\sigma] = z, \quad z \in \mathbb{D}.
\]

The solution takes the form

\[
H_s[\sigma] = e^{-is}z,
\]

and

\[
\dot{G}_t^D(z) = -G_t^D(z)\frac{G_t^D(z) + 1}{G_t^D(z) - 1} - iG_t^D(z)\dot{u}_t.
\]

Thus, the equation for \(g_t\) becomes

\[
\dot{g}_t^D(z) = \left(\frac{1}{H_{u_t}[\sigma]'}\left(-H_{u_t}[\sigma]H_{u_t}[\sigma] + 1\right)\right)\circ \dot{g}_t^D(z) = \frac{1}{e^{-iu_t}} \left(e^{-iu_t}g_t^D(z) + \frac{1}{e^{-iu_t}g_t^D(z) - 1}\right)
\]

or

\[
\dot{g}_t^D(z) = g_t^D(z)\left(e^{iu_t} + g_t^D(z)\right) - g_t^D(z).
\]

Its stochastic version in the Stratonovich form is

\[
d^S G_t^D(z) = G_t^D(z)\frac{G_t^D(z) + 1}{G_t^D(z) - 1} dt - i\sqrt{\kappa}G_t^D(z)d^SB_t.
\]

or in the half-plane chart

\[
d^S G_t^H(z) = \frac{1}{2} \left(\frac{1}{G_t^H(z)} + G_t^H(z)\right) dt - \frac{\sqrt{\kappa}}{2} \left(1 + \left(G_t^H(z)^2\right)^2\right) d^SB_t.
\]

The collection \(G[\delta, \sigma]\) consists of maps that in the unit disk chart are normalized by

\[
G^D(0) = 0.
\]
It turns out that some different combinations of $\delta$ and $\sigma$ induce measures that can be transformed one to another in a simple way. For example, let $m : D \to D$ be a Möbius automorphism fixing a point $a \in \partial D$ where $\delta$ has a pole. Then

$$\delta \to m_* \delta, \quad \sigma \to m_* \sigma$$

are also vector fields of the form (2.15) and (2.16). For instance, let $\delta$ and $\sigma$ be as in Example 2 and let $m^c : D \to D$ map the center point $0$ of the unit disk to another point inside it. Then we come to an equation defined by $m_* \delta$ and $m_* \sigma$, which is still in fact, the radial equation written in different coordinates.

Another example of such a transformation preserving the form (2.15) and (2.16) can be constructed as follows.

$$\delta_{D}(z) \to c^2 \delta_{D}(z), \quad \sigma_{D}(z) \to c \sigma_{D}(z), \quad t \to c^{-2} t, \quad c > 0.$$  

The solution is not changed as a random law, because $cB_{t/2}$ and $B_t$ agree in law.

It is important to know which of the equations defined by the parameters $\delta_{-2}$, $\delta_{-1}$, $\delta_0$, $\delta_1$, $\sigma_0$, and $\sigma_0$ are ‘essentially different’. A systematic analysis of this question was presented in [13]. Without lost of generality we can restrict ourselves to the from

$$\delta_{D}(z) = \frac{2}{z} + \delta_{-1} + \delta_0 z + \delta_1 z^2, \quad \sigma_{D}(z) = -\sqrt{\kappa}(1 + \sigma_0 z + \sigma_1 z^2), \quad \kappa > 0.$$  

Transformations (2.16) and

$$\delta \to r_c \delta, \quad \sigma \to r_c \sigma, \quad r^D_c(z) := \frac{z}{1 - cz}, \quad c \in \mathbb{R}$$

preserve (2.17) and keep $\kappa$ unchanged. Thus, we can fix some 2 of 6 parameters in (2.17). Besides, the transform

$$\delta \to \delta + \frac{\nu}{\sqrt{\kappa}} \sigma, \quad \nu \in \mathbb{R},$$

can be interpreted as an insertion of a drift to the Brownian measure. Thus, all 6 parameters can be fixed by $\kappa$ (responsible for the fractal dimension of the slit), $\nu$ (the drift), and some 2 parameters that set the equation type (for example, chordal, radial, and dipolar).

Due to the autonomous form of the equation (2.7) the solution $G_t$ possesses the following property.

**Proposition 1** ([13]). Let $\tilde{u}_{t-s} := u_{t} - u_{s}$ for some fixed $s : t \geq s \geq 0$, and let $\tilde{G}_t$ be defined by (2.7) with the driving function $\tilde{u}_i$. Then

$$G_t = \tilde{G}_{t-s} \circ G_s.$$  

In the stochastic case the process $\{G_t\}_{t \geq 0}$, taking values in the space of inverse endomorphisms of $D$, is a continuous homogeneous Markov process. In particular,

**Proposition 2** ([13]). If $G_t$ and $\tilde{G}_s$ are two independently sampled $(\delta, \sigma)$-SLE maps, then $G_t \circ \tilde{G}_s$ has the same law as $G_{t+s}$.

2.3. Pre-pre-Schwarzian. A collection of maps $\eta^\psi : \psi(D) \to \mathbb{C}$, each of which is given in a global chart map $\psi : D \to \psi(D) \subset \mathbb{C}$, is called a pre-pre-Schwarzian form of order $\mu, \mu^* \in \mathbb{C}$ if for any chart map $\psi$

$$\eta^{\psi}(\tilde{z}) = \eta^\psi(\tau(\tilde{z})) + \mu \log \tau'(\tilde{z}) + \mu^* \log \tilde{\tau}'(\tilde{z}), \quad \tau = \psi \circ \tilde{\psi}^{-1}, \quad \tilde{z} \in \tilde{D}, \quad \forall \psi, \tilde{\psi},$$

for any chart map $\tilde{\psi}$. If $\eta$ is defined for one chart map, then it is automatically defined for all chart maps. We borrowed the term ‘pre-pre-Schwarzian’ from [18]. In [29], an analogous object is called ‘AC surface’.
Analogously to vector fields in Section 2.1 we define the pushforward of a pre-pre-Schwarzian by
\[
G^{-1}_* \eta^\psi(z) := \eta^{v*G}(z) = \eta^\psi(G^\psi(z)) + \mu \log \left( G^\psi(z) \right)'(z) + \mu^* \log \left( G^\psi(z) \right)(z)
\]
We are interested in two special cases. The first one corresponds to \( \mu = \mu^* = \gamma/2 \in \mathbb{R} \), and
\[
G^{-1}_* \eta^\psi(z) = \eta^\psi(G^\psi(z)) + \gamma \log |G^\psi(z)|.
\]
The second one is \( \mu = i\chi/2, \mu^* = -i\chi/2, \chi \in \mathbb{R} \), and
\[
G^{-1}_* \eta^\psi(z) = \eta^\psi(G^\psi(z)) - \chi \arg G^\psi(z).
\]
In both cases \( \eta \) can be chosen real in all charts. Moreover, if the pre-pre-Schwarzian is represented by a real-valued function it is one of two above forms in all charts.

A \((\mu, \mu^*)\)-pre-pre-Schwarzian can be obtained from a vector field \( v \) by the relation
\[
\eta^\psi(z) := -\mu \log v^\psi(z) - \mu^* \log \overline{v^\psi(z)}.
\]
For two special cases above we have
\[
\eta = -\gamma \log |v|\]
and
\[
\eta = \chi \arg v,
\]
where we drop the upper index \( \psi \).

In Section 2.5 we obtain the transformation rules (2.21) by taking logarithm of a \((1, 1)\)-differential. The second type of the real pre-pre-Schwarzian is connected to a sort of an imaginary analog of the metric.

We can define the Lie derivative of \( X \) as
\[
\mathcal{L}_v X^\psi(z) := \frac{\partial}{\partial \alpha} H^{-1}_* [v] X^\psi(z) \bigg|_{\alpha=0},
\]
where \( X \) can be a pre-pre-Schwarzian, a vector field, or even an object with a more general transformation rule, see (2.23) for any two holomorphic vector fields \( v \). If \( X \) is a holomorphic vector field \( w \), then
\[
\mathcal{L}_w v^\psi = [v, w]^\psi(z) := v^\psi(z)w^{\psi'}(z) - w^\psi(z)v^{\psi'}(z),
\]
see (2.22).

If \( X \) is a pre-pre-Schwarzian, then
\[
\mathcal{L}_v \eta^\psi(z) = v^\psi(z) \partial_z \eta^\psi(z) + \overline{v^\psi(z)} \partial_z \eta^\psi(z) + \mu v^\psi(z) + \mu^* \overline{v^\psi(z)}.
\]
Here and further on, we use notations
\[
\partial_z := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial \overline{z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\]
and \( f'(z) := \partial_z f(z) \) for a holomorphic function \( f \).

If \( \mu = \mu^* = 0 \), then \( \eta \) is called a scalar. It is remarkable that if \( \eta \) is a pre-pre-Schwarzian, then \( \mathcal{L}_v \eta \) is a scalar anyway, which is stated in the following lemma.

Lemma 1. Let \( \eta \) be a pre-pre-Schwarzian of order \( \mu, \mu^* \), let \( v \) be a holomorphic vector field, and let \( G \) be a conformal self-map. Then
\[
G^{-1}_* (\mathcal{L}_v \eta)^\psi(z) = (\mathcal{L}_v \eta)^\psi \circ G^\psi(z)
\]
or in the infinitesimal form,
\[
\mathcal{L}_w (\mathcal{L}_v \eta)^\psi(z) = w^\psi(z) \partial_z (\mathcal{L}_v \eta)^\psi(z) + \overline{w^\psi(z)} \partial_z (\mathcal{L}_v \eta)^\psi(z).
\]
Proof. The straightforward calculations imply
\[
G^{-1}_s(\mathcal{L}_v \eta)\psi(z) = G^{-1}_s \left( v^\psi(z) \partial_z \eta^\psi(z) + v^\psi(z) \partial_z \eta^\psi(z) + \mu v^\psi(z) + \mu^* v^\psi(z) \right) = \\
= \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} \partial_z \left( \eta^\psi \circ G^\psi(z) + \mu \log G^\psi(z) + \mu^* \log G^\psi(z) \right) + \\
+ \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} \partial_z \left( \eta^\psi \circ G^\psi(z) + \mu \log G^\psi(z) + \mu^* \log G^\psi(z) \right) + \\
+ \mu \partial_z \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} + \mu^* \partial_z \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} = \\
= v^\psi \circ G^\psi(z) (\partial \eta^\psi) \circ G^\psi(z) + \mu \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} \partial_z \log G^\psi(z) + 0+ \\
+ \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} (\partial \eta^\psi) \circ G^\psi(z) + 0 + \mu^* \frac{v^\psi \circ G^\psi(z)}{G^\psi(z)} \partial_z \log G^\psi(z) + \\
+ \mu v^\psi \circ G^\psi(z) - \mu \frac{v^\psi \circ G^\psi(z) G^\psi(z)}{G^\psi(z)^2} + \mu^* v^\psi \circ G^\psi(z) - \mu \frac{v^\psi \circ G^\psi(z) G^\psi(z)}{G^\psi(z)^2} = \\
= (v^\psi \partial \eta^\psi) \circ G^\psi(z) + \mu v^\psi \circ G^\psi(z) + \\
+ (v^\psi \partial \eta^\psi) \circ G^\psi(z) + \mu^* v^\psi \circ G^\psi(z) = \\
= (v^\psi \partial \eta^\psi + v^\psi \partial \eta^\psi + \mu v^\psi \circ G^\psi(z) + \mu^* v^\psi \circ G^\psi(z) \circ G^\psi(z) = \\
= (\mathcal{L}_v \eta)\psi \circ G^\psi(z)
\]
\[
\square
\]

2.4. Test functions. We will define the Schwinger functions $S_n$ and the Gaussian free field $\Phi$ in terms of linear functionals over some space of smooth test functions defined in what follows.

Let $\mathcal{H}_s$ be a linear space of real-valued smooth functions $f: D^\psi \rightarrow \mathbb{R}$ in the domain $D^\psi := \psi(D) \subset \mathbb{C}$ with compact support equipped with the topology of homogeneous convergence of all derivatives on the corresponding compact, namely, the topology is generated by following collection of neighborhoods of the zero function

$$U_K := \bigcap_{n,m=0,1,2,...} \{ f(z) \in C^\infty(D) : \text{supp} f \subseteq K \land |\partial^n \partial^m f^\psi(z)| < \varepsilon_{n,m}, \quad z \in K \},$$

$$\varepsilon_{n,m} > 0, \quad n, m = 0, 1, 2, \ldots ,$$

where $K \subset D^\psi$ is any compact subset of $D^\psi$.

We call $f^\psi \in \mathcal{H}_s$ the test functions and assume that they are $(1,1)$-differentials

$$f^\psi(\tilde{z}) = \tau'(\tilde{z}) \tau'(\tilde{z}) f^\psi(\tau(\tilde{z})), \quad \tau := \psi \circ \tilde{\psi}^{-1},$$

It is straightforward to check that any transition map $\tau$ induces a homeomorphism between $\mathcal{H}_s$ and $\mathcal{H}_s^\psi$. Thereby, we will drop the index $\psi$ at $\mathcal{H}_s$, and consider the space $\mathcal{H}_s$ as a topological space of smooth $(1,1)$-differentials with compact support.

The space $\mathcal{H}_s$ does not match all cases of coupling. For the couplings with radial SLE we use spaces $\mathcal{H}_{s,b}$ and $\mathcal{H}_{s,b}^\psi$ defined in corresponding Sections 4.3 and 6. Henceforth, we denote by $\mathcal{H}$ any of those nuclear spaces $\mathcal{H}_s$, $\mathcal{H}_{s,b}$, or $\mathcal{H}_{s,b}^\psi$ for shortness. An important property of $\mathcal{H}$ is nuclearity, see [10][12][24] which is necessary and sufficient to admit the uniform Gaussian measure on the dual space $\mathcal{H}'$ (the GFF).
Constructing such a uniform Gaussian measure on a finite dimensional linear space is a trivial problem, however, it is not possible on an infinite-dimensional Hilbert space. On the other hand, if a space $H$ is nuclear as $H_s$, then the dual space $H'$ admits a uniform Gaussian measure. A general recipe holds not only for Gaussian measures and is given by the following theorem.

**Proposition 3. (Bochner-Minols [11, 12, 11])**

Let $H$ be a nuclear space, and let $\hat{\mu}: H \to \mathbb{C}$ be a functional (non-linear). Then the following 3 conditions

1. $\hat{\mu}$ is positive definite
   \[
   \forall \{z_1, z_2, \ldots, z_n\} \in \mathbb{C}^n, \forall \{f_1, f_2, \ldots, f_n\} \in H^n \Rightarrow \sum_{1 \leq k, l \leq n} z_k \bar{z}_l \hat{\mu}[f_k - f_l] \geq 0;
   \]

2. $\hat{\mu}(0) = 1$;

3. $\hat{\mu}$ is continuous

are satisfied if and only if there exists a unique probability measure $P_\Phi$ on $(\Omega_\Phi, \mathcal{F}_\Phi, P_\Phi)$ for $\Omega_\Phi = H'$, with $\hat{\mu}$ as a characteristic function

\[
\hat{\mu}[f] := \int_{\Phi \in H'} e^{i\Phi[f]} P_\Phi(d\Phi), \quad \forall f \in H.
\]

The corresponding $\sigma$-algebra $\mathcal{F}_\Phi$ is generated by the cylinder sets

\[
\{F \in H': F[f] \in B\}, \quad \forall f \in H, \quad \forall \text{Borel sets } B \text{ of } \mathbb{R}.
\]

The random law on $H'$ is called uniform with respect to a bilinear functional $B: H \times H \to \mathbb{R}$ if the characteristic function $\hat{\mu}$ is of the form

\[
\hat{\mu}[f] = e^{-\frac{1}{2}B[f,f]}, \quad f \in H.
\]

We consider the class of bilinear functionals we work with in Section 2.6. First, we study the linear and bilinear functionals over $H_s$ and their transformation properties.

### 2.5. Linear functionals and change of coordinates.

In this section, we consider linear functionals over $H_s$ and $H$ that transform as pre-pre-Schwarzians.

Let $\eta^\psi \in H^\psi_s$ be a linear functional over $H^\psi_s$ for a given chart $\psi$. The functional is called regular if there exists a locally integrable function $\eta^\psi(z)$ such that

\[
\eta^\psi[f] := \int_{D^\psi} \eta^\psi(z) f^\psi(z) l(dz),
\]

where $l$ is the Lebesgue measure on $\mathbb{C}$. We use the brackets $[\cdot]$ for functionals and the parentheses $(\cdot)$ for corresponding functions (kernels).

We assume that $f$ transforms according to (2.27). If $\eta^\psi(z)$ is a scalar, then the number $\eta^\psi[f] \in \mathbb{R}$ does not depend on the choice of the chart $\psi$. Indeed, for any choice of another chart $\tilde{\psi}$, we have

\[
\eta^{\tilde{\psi}}[f] := \int_{D^{\tilde{\psi}}} \eta^{\tilde{\psi}}(\tilde{z}) f^{\tilde{\psi}}(\tilde{z}) l(d\tilde{z}) = \int_{D^\psi} \eta^\psi(\tau(\tilde{z})) f^\psi(\tau(\tilde{z})) |\tau'(\tilde{z})|^2 l(d\tilde{z}) = \int_{D^\psi} \eta^\psi(z) f^\psi(z) l(dz) = \eta^\psi[f],
\]
If \( \eta^\psi(z) \) is a pre-pre-Schwarzian, then

\[
\eta^\psi[f] = \int_{D^\psi} \eta^\psi(\hat{z}) f^\psi(\hat{z}) l(d\hat{z}) = \\
= \int_{D^\psi} \left( \eta^\psi(\tau(\hat{z})) + \mu \log \tau'(\hat{z}) + \mu^* \log \tau'(\hat{z}) \right) f^\psi(\tau(\hat{z})) |\tau'(\hat{z})|^2 l(d\hat{z}) = \\
(2.29) = \int_{D^\psi} \left( \eta^\psi(z) - \mu \log \tau^{-1}(z) - \mu^* \log \tau^{-1}(z) \right) f^\psi(z) l(dz) = \\
= \eta^\psi[f] - \int_{D^\psi} \left( \mu \log \tau^{-1}(z) + \mu^* \log \tau^{-1}(z) \right) f^\psi(z) l(dz)
\]

according to (2.20).

If \( \eta^\psi \) is not a regular pre-pre-Schwarzian but just a functional from \( \mathcal{H}' \) we can consider the last line of (2.29) as a definition of the transformation rule for \( \eta[f] \) from a chart \( \psi \) to a chart \( \psi'. \)

Let us denote by \( \mathcal{H}' \) the linear space of pre-pre-Schwarzians as above. Consider now the pushforward operation \( G^{-1}_s \) on \((1,1)\)-differentials \( f \) defined by

\[
G^{-1}_s f^\psi(z) := \left| G^\psi'(z) \right|^2 f^\psi \left( G^\psi(z) \right).
\]

The right-hand side is well-defined only for \( \psi(D \setminus K) \). Here we define \( F_s \) only on a subset of \( \mathcal{H}_s \) of test functions that are supported in \( D \setminus K = F^{-1}(D) \).

Define the pushforward operation by

\[
G^{-1}_s \eta^\psi[f] = \eta^\psi G[f] = \\
(2.30) = \eta^\psi[G_s f] + \int_{\text{supp } f^\psi} \left( \mu \log G^\psi(z) + \mu^* \log G^\psi(z) \right) f^\psi(z) l(dz), \\
f \in \mathcal{H}_s; \quad \text{supp } f \subset G^{-1}(D).
\]

It can be understood as a pushforward \( F_s : \mathcal{H}' \to \mathcal{H}' \) in the dual space.

Functionals over the space \( \mathcal{H}_s \) are differentiable infinitely many times. According to (2.25) the Lie derivative is defined by

\[
\mathcal{L}_s \eta[f] = \frac{\partial}{\partial s} H^{-1}_s[v_s \eta^\psi[f]] \bigg|_{s=0} = \\
= - \eta^\psi[\mathcal{L}_s f] + \int_{\text{supp } f^\psi} \left( \mu v^\psi(z) + \mu^* v^\psi z(z) \right) f^\psi(z) l(dz),
\]

where

\[
\mathcal{L}_s f^\psi(z) = \frac{\partial}{\partial s} H^{-1}_s[v_s f^\psi] \bigg|_{s=0} = \\
=v^\psi(z) \partial_z f^\psi(z) + \overline{v^\psi(z)} \partial_{\bar{z}} f^\psi(z) + v^\psi(z) f^\psi(z) + \overline{v^\psi(z)} f^\psi(z).
\]

2.6. Fundamental solution to the Laplace-Beltrami equation. In this section, we consider linear continuous functionals with respect to each argument in \( \mathcal{H}_s \). An important example is the Dirac functional

\[
(2.31) \delta_{\lambda}[f, g] := \int_{\psi(D)} f^\psi(z) g^\psi(z) \frac{1}{\lambda^\psi(z)} l(dz), \quad f, g \in \mathcal{H}_s,
\]
where $\lambda(z)l(dz)$ is some measure on $D^\psi$, which is absolutely continuous with respect to the Lebesgue measure $l(dz)$. The Radon-Nikodym derivative $\lambda^\psi(z)$ transforms as a $(1,1)$-differential:

$$
\lambda^\psi(\tilde{z}) = \tau'(\tilde{z})\tau^((\tilde{z})) \lambda^\psi(\tau(\tilde{z})), \quad \tau := \psi \circ \tilde{\psi}^{-1}.
$$

It is easy to see that the right-hand side of (2.31) does not depend on the choice of $\psi$.

We call the functional regular if there exists a function $B^\psi(z, w)$ on $\psi^D \times \psi^D$ such that

$$
(2.32) \quad B^\psi[f, g] := \int_{\psi^D} \int_{\psi^D} B^\psi(z, w)f^\psi(z)g^\psi(w)l(dz)l(dw), \quad f, g \in \mathcal{H}_s.
$$

Let us use the same convention about the brackets and parentheses as for the linear functionals. We consider only scalar regular bilinear functionals and require the transformation rules

$$
B^\psi(\tilde{z}, \tilde{w}) = B^\psi(\tau(\tilde{z}), \tau(\tilde{w})), \quad \tau := \psi \circ \tilde{\psi}^{-1}, \quad z, w \in \tilde{\psi}(D).
$$

Thus, the right-hand side of (2.32) does not depend on the choice of the chart $\psi$ and we can drop the index $\psi$ in the left-hand side.

The pushforward is defined by

$$
(2.33) \quad F_\psi B^\psi(z, w) = B^\psi_{\circ F}(z, w) := B^\psi((F^\psi)^{-1}(z), (F^\psi)^{-1}(w)), \quad z, w \in \text{Im}(F),
$$

which becomes

$$
F_\psi B^\psi[f, g] = B^\psi_{\circ F}[f, g] := B^\psi[f, f] - B^\psi[f, g] + B^\psi[g, f], \quad f, g \in \mathcal{H}_s; \supp f \subset \text{Im}(F),
$$

for an arbitrary functional $F$. The same remarks remain true in this case as in the previous section for $\eta$.

Define now the Lie derivative in the same way as before

$$
(2.34) \quad \mathcal{L}_v B^\psi(z, w) := \frac{\partial}{\partial s} \mathcal{H}_s[v]^s B^\psi(z, w) \bigg|_{s=0} = v^\psi(z) \partial_z B^\psi(z, w) + v^\psi(w) \partial_w B^\psi(z, w) + v^\psi(w) \partial_w B^\psi(z, w).
$$

We remark that $\mathcal{L}_v B$ is also scalar in two variables. Functionals $\delta_\lambda$ and $B$ are both scalar and continuous with respect to each variable.

Define the Laplace-Beltrami operator $\Delta_\lambda$ as

$$
\Delta_{\lambda_1} B^\psi(z, w) := -\frac{4}{\lambda^\psi(z)} \partial_z \partial_{\tilde{z}} B^\psi(z, w),
$$

where the lower index ‘1’ means that the operator acts only with respect to the first argument.

Let a regular bilinear functional $\Gamma_\lambda$ be a solution to the equation

$$
(2.35) \quad \Delta_{\lambda_1} \Gamma_\lambda[f, g] = 2\pi \delta_\lambda[f, g], \quad \Gamma_\lambda[f, g] = \Gamma_\lambda[g, f], \quad f, g \in \mathcal{H}_s.
$$

The boundary conditions will be fixed later. This equation is conformally invariant in the sense that if $\Gamma_\lambda^\psi(z, w)$ is a solution on a chart $\psi$, then

$$
\Gamma_\lambda^\psi(\tau(\tilde{z}), \tau(\tilde{w})) = \tau^{-1} \circ \psi(\tilde{z}, \tilde{w}).
$$

is a solution in the chart $\tau^{-1} \circ \psi$.

The solution $\Gamma_\lambda^\psi(z, w)$ is a collection of smooth and harmonic functions on $\psi(D) \times \psi(D) \setminus \{ z \times w: z = w \}$ of general form

$$
(2.36) \quad \Gamma_\lambda^\psi(z, w) = -\frac{1}{2} \log(z - w)(\tilde{z} - \tilde{w}) + H^\psi(z, w),
$$

where $H^\psi(z, w)$ is a collection of smooth and harmonic functions on $\psi(D) \times \psi(D)$. 

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where $H^\psi(z, w)$ is an arbitrary symmetric harmonic function with respect to each variable that is defined by the boundary conditions and will be specified in what follows.

It is straightforward to verify that the function $\Gamma^\psi_\lambda(z, w)$ does not depend on the choice of $\lambda$ because the identity (2.35) in the integral form becomes

$$
\int_{\psi(D)} \int_{\psi(D)} -\frac{4}{\lambda^\psi(z)} \partial_z \partial_{\bar{z}} \Gamma^\psi_\lambda(z, w)f^\psi(z)g^\psi(w)l(dz)l(dw) =
\int_{\psi(D)} f^\psi(z)g^\psi(z) \frac{1}{\lambda^\psi(z)}l(dz).
$$

The change $\lambda \rightarrow \tilde{\lambda}$ is equivalent to the change $f^\psi(z) \rightarrow \frac{\lambda^\psi(z)}{\lambda^\psi(z)}f^\psi(z)$. We will drop the lower index $\lambda$ in $\Gamma_\lambda$ in what follows. The fundamental solutions to the Laplace equation are also known as Green’s functions (for the free field).

**Example 3. Dirichlet boundary conditions.** Let us denote by $\Gamma_D$ the solution $\Gamma$ to (2.35) satisfying the zero boundary conditions, namely,

$$\left. \Gamma^H_D(z, w) \right|_{z \in \mathbb{R}} = 0, \quad \lim_{z \to \infty} \Gamma^H_D(z, w) = 0, \quad w \in \mathbb{H}.$$

Then, $\Gamma_D$ admits the form

$$\Gamma^H_D(z, w) := -\frac{1}{2} \log \frac{(z-w)(\bar{z}-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})},$$

and possesses the property of symmetry with respect to all Möbius automorphisms $H : D \rightarrow D$,

$$H \cdot \Gamma_D = \Gamma_D$$

or

$$\mathcal{L}_\sigma \Gamma_D(z, w) = 0, \quad \forall \text{ complete vector field } \sigma.$$

**Example 4. Combined Dirichlet-Neumann boundary conditions.** Let $\Gamma_{DN}$ denote the solution to (2.35) satisfying the following boundary conditions in the strip chart

$$\left. \Gamma^S_{DN}(z, w) \right|_{z \in \mathbb{R}} = 0, \quad \partial_y \Gamma^S_{DN}(x + iy, w) \bigg|_{y = \pi} = 0, \quad x \in \mathbb{R}, \quad \lim_{z \to \infty \wedge \text{Re } z > 0} \Gamma^S_{DN}(z, w) = 0, \quad \lim_{z \to \infty \wedge \text{Re } z < 0} \Gamma^S_{DN}(z, w) = 0, \quad w \in \mathbb{H}.$$

We consider this case in Section 5 and the exact form of $\Gamma_{DN}$ is given by (5.1). It is not invariant with respect to all Möbius automorphisms but it is invariant if the automorphism preserves the points of change of the boundary conditions, which are $\pm \infty$ in the strip chart.

We will consider another example ($\Gamma_{tw,b}$) in Section 6.

### 2.7. Gaussian free field.

**Definition 3.** For some nuclear space of smooth functions $\mathcal{H}$, let the linear functional $\eta$ and some Green’s functional $\Gamma$ be given. Assume in Theorem 3

$$\hat{\mu}[f] := \exp \left( -\frac{1}{2} \Gamma[f, f] + i\eta[f] \right), \quad f \in \mathcal{H}.$$

Then the $\mathcal{H}$-valued random variable $\Phi$ is called the Gaussian free field (GFF). We will denote it by $\Phi(\mathcal{H}, \Gamma, \eta)$. 
For convenience, we change the definition of the characteristic function from (2.28) to
\[ \hat{\phi}[f] := \int_{\Phi \in \mathcal{H}'} e^{\Phi[f] P\Phi(d\Phi)}, \quad \forall f \in \mathcal{H}, \]
and (2.39) changes to
\[ (2.40) \quad \hat{\phi}[f] = e^{\left( \frac{1}{2} \Gamma[f,f] + \eta[f] \right)}, \]
which is possible for the Gaussian measures.

The expectation of a random variable \( X[\Phi] \) (\( X : \mathcal{H}' \to \mathbb{C} \)) is defined as
\[ \mathbb{E} [X] := \int_{\Phi \in \mathcal{H}'} X[\Phi] P\Phi(d\Phi). \]

An alternative and equivalent (see, for example [12]) definition of GFF can be formulated as follows:

**Definition 4.** The Gaussian free field \( \Phi \) is a \( \mathcal{H}' \)-valued random variable, that is a map \( \Phi : \mathcal{H} \times \Omega \to \mathbb{R} \) (measurable on \( \Omega \) and continuous linear on the nuclear space \( \mathcal{H} \)), or a measurable map \( \Phi : \Omega \to \mathcal{H}' \), such that \( \text{Law} [\Phi[f]] = N \left( \eta[f], \Gamma[f,f] \right) \), \( f \in \mathcal{H} \), i.e., it possesses the properties
\[ \mathbb{E} [\Phi[f]] = \eta[f], \quad \forall f \in \mathcal{H}, \]
\[ \mathbb{E} [\Phi[f] \Phi[f]] = \Gamma[f,f] + \eta[f] \eta[f], \quad \forall f \in \mathcal{H} \]
for Green’s bilinear positively defined functional \( \Gamma \), and for a linear functional \( \eta \).

The random variable \( \Phi \) introduced this way transforms from one chart to another according to the pre-pre-Schwarzian rule
\[ (2.41) \quad \Phi^\psi[f] = \Phi[f] - \int_{\psi(D)} \left( \mu \log \tau^{-1}(z) + \mu^* \log \tau^{-1}(z) \right) f^\psi(z) l(dz), \quad \tau := \psi \circ \tilde{\psi}^{-1}, \]
due to the corresponding property (2.29) of \( \eta \).

The pushforward can also be defined by
\[ G_s^{-1} \Phi^\psi[f] = \Phi^\psi[G_s f] + \int_{\text{supp} f^\psi} \left( \mu \log G^{\psi'}(z) + \mu^* \log G^{\psi'}(z) \right) f^\psi(z) l(dz), \quad f \in \mathcal{H}_s: \text{ supp } f \subset G^{-1}(D) \]
as well as the Lie derivative becomes
\[ \mathcal{L}_v \Phi[f] = \frac{\partial}{\partial s} H_s^{-1}[v] \Phi^\psi[f] \bigg|_{s=0} = - \Phi^\psi[\mathcal{L}_v f] + \int_{\text{supp} f^\psi} \left( \mu v^{\psi'}(z) + \mu^* \psi^{\psi'}(z) \right) f^\psi(z) l(dz), \]

**Example 5.** Let \( \mathcal{H} := \mathcal{H}_s, \Gamma := \Gamma_D \) (as in Example 2.37), and let \( \eta^\psi(z) := 0 \) in all charts \( \psi \) (\( \mu = \mu^* = 0 \)). Then we call \( \Phi(\mathcal{H}_s, \Gamma_D, 0) \) the Gaussian free field with zero boundary condition.

**Example 6.** Relax the previous example. Let \( \eta^\psi \) be a harmonic function in \( D^\psi \) continuously extendable to the boundary \( \partial D^\psi \) if the chart map \( \psi \) can be extended to \( \partial D \). Then we call \( \Phi \) the Gaussian free field with the Dirichlet boundary condition.
We can define the Laplace-Beltrami operator $\Delta_\lambda$ over $\Phi$ as well as the Lie derivative by

$$(\Delta_\lambda \Phi)[g] := \Phi[\Delta_\lambda g], \quad g \in \mathcal{H},$$

where $\Delta_\lambda$ on a $(1,1)$-differential is defined by

$$\Delta_\lambda g^\psi(z) := -4 \partial_z \partial_{\bar{z}} \frac{g^\psi(z)}{\lambda^\psi(z)}$$

in any chart $\psi$. If $\eta$ is harmonic the identity

$$\mathbb{E} [(\Delta_\lambda \Phi)[g] \Phi[f_1] \Phi[f_2] \ldots \Phi[f_n]] = \sum_{i=1,2,\ldots,n} \delta_\lambda [g, f_i] \mathbb{E} [\Phi[f_1] \Phi[f_2] \ldots \Phi[f_{i-1}] \Phi[f_{i+1}] \ldots \Phi[f_n]]$$

is satisfied. Thereby, one can write heuristically

$$\Delta_\lambda \Phi(z) = 0, \quad z \not\in \text{supp } f_1 \cup \text{supp } f_2 \cup \ldots \cup \text{supp } f_n.$$

It turns out that the characteristic functional $\hat{\phi}$ is also a derivation functional for the correlation functions. Define the variational derivative over some functional $\nu$ as a map $\frac{\delta}{\delta f}: \nu \mapsto \frac{\delta}{\delta f} \nu$ to the set of functionals by

$$\left( \frac{\delta}{\delta f} \nu \right)[g] := \left. \frac{\partial}{\partial \alpha} \nu[g + \alpha f] \right|_{\alpha=0}, \quad \forall f, g \in \mathcal{H}.$$

If $\nu$ is such that $\nu[g + \alpha f]$ is an analytic function with respect to $\alpha$ for each $f$ and $g$, like $\hat{\phi}$, it is straightforward to see that for each $g, f_1, f_2, \ldots \in \mathcal{H}$,

$$\nu[g] = \nu[0], \quad g \in \mathcal{H} \quad\iff\quad \left( \frac{\delta}{\delta f_1} \frac{\delta}{\delta f_2} \ldots \frac{\delta}{\delta f_n} \nu \right)[0] = 0, \quad n = 1, 2, \ldots.$$

Define the Schwinger functionals as

$$S_n[f_1, f_2, \ldots, f_n] := \mathbb{E} [\Phi[f_1] \Phi[f_2] \ldots \Phi[f_n]] = \left( \frac{\delta}{\delta f_1} \frac{\delta}{\delta f_2} \ldots \frac{\delta}{\delta f_n} \hat{\phi} \right)[0]$$

where $\hat{\phi}: \mathcal{H} \to \mathbb{R}$ is defined in (2.40).

The identity (2.42) can be reformulated as

$$\mathbb{E} [(\Delta_\lambda \Phi)[g] e^{\Phi[f]}] = (\delta_\lambda [g, f] + \eta[\Delta_\lambda f]) \hat{\phi}[f], \quad f, g \in \mathcal{H}.$$

### 2.8. The Schwinger functionals.

In this section, we consider the Schwinger functionals defined by (2.43) and their derivation functional $\hat{\phi}$ in detail.

For any finite collection $\{f_1, f_2, \ldots, f_n\}$ of functions from $\mathcal{H}_s$ or $\mathcal{H}_r$, the collection of random variables $\{\Phi[f_1], \Phi[f_2], \ldots, \Phi[f_n]\}$ has the multivariate normal distribution. Thus, we have

$$\mathbb{E} [\Phi[f_1] \Phi[f_2] \ldots \Phi[f_n]] = \sum_{\text{partitions } k} \prod_{i} \Gamma[f_{i_k}, f_{j_k}],$$

for $\eta(z) = 0$, where the sum is taken over all partitions of the set $\{1, 2, \ldots, n\}$ into disjoint pairs $\{i_k, j_k\}$. In particular, the expectation of the product of an odd number of fields is identically zero. For the general case ($\eta \neq 0$) the Schwinger functionals are

$$S[f_1, f_2, \ldots, f_n] := \mathbb{E} [\Phi[f_1] \Phi[f_2] \ldots \Phi[f_n]] = \sum_{\text{partitions } k} \prod_{i} \Gamma[f_{i_k}, f_{j_k}] \prod_{l} \eta[f_{l_k}],$$
where the sum is taken over all partitions of the set \{1, 2, \ldots, n\} into disjoint non-ordered pairs \{i_k, j_k\}, and non-ordered single elements \{\eta_i\}. In particular,

\[
S_1[f_1] = \eta[f_1], \\
S_2[f_1, f_2] = \Gamma[f_1, f_2] + \eta[f_1]\eta[f_2], \\
S_3[f_1, f_2, f_3] = \Gamma[f_1, f_2]\eta[f_3] + \Gamma[f_3, f_1]\eta[f_2] + \Gamma[f_2, f_3]\eta[f_1] + \eta[f_1]\eta[f_2]\eta[z_2]\eta[f_3], \\
S_4[f_1, f_2, f_3, f_4] = \Gamma[f_1, f_2]\Gamma[f_3, f_4] + \Gamma[f_1, f_3]\Gamma[f_2, f_4] + \Gamma[f_1, f_4]\Gamma[f_2, f_3] + \\
+ \Gamma[f_1, f_2]\eta[f_3]\eta[f_4] + \Gamma[f_1, f_3]\eta[f_2]\eta[f_4] + \Gamma[f_1, f_4]\eta[f_2]\eta[f_3] + \\
+ \eta[f_1]\eta[f_2]\eta[f_3]\eta[f_4].
\]

Such correlation functionals are called the *Schwinger functionals*. Their kernels

\[
S_n(z_1, z_2, \ldots, z_n)
\]

are known as Schwinger functions or \(n\)-point functions. For regular functionals \(\Gamma\) and \(\eta\), the Schwinger functions are also regular but it is still reasonable to understand \(S_n\) as a functional because the derivatives are not regular. For example,

\[
\Delta_{\lambda_1}S_2^\psi(z, w) = 2\pi\delta_\lambda(z - w).
\]

The transformation rules for \(S_n\) (the behaviour under the action of \(G_s\)) are quite complex. We present here only the infinitesimal ones

\[
\mathcal{L}_v S_n^\psi[f_1, f_2, \ldots] = -\sum_{1 \leq k \leq n} S_n^\psi[f_1, f_2, \ldots, f_k, \ldots, f_n] - \\
-\sum_{1 \leq k \leq n} S_{n-1}^\psi[f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{n-1}] \int_{\psi(D)} (\mu \psi^\mu(z) + \mu^* \psi^\nu^\nu(z)) f_k^\psi(z) l(dz)
\]

We prefer to work with the characteristic functional \(\hat{\phi}\), rather than with \(S_n\). For instance, for any inverse endomorphism \(G\colon D \setminus K \to D\), we can define the pushforward \(G^{-1}\colon \hat{\phi}(\Gamma, \eta) \mapsto \hat{\phi}(F, \Gamma, F_s\eta)\) that maps the functionals on \(D\) to functionals on \(D \setminus K\).

Equivalently,

\[
(G^{-1}_s\hat{\phi}(\Gamma, \eta)) [\hat{f}] := \hat{\phi}(G^{-1}_s\Gamma, G^{-1}_s\eta)[\hat{f}], \quad \hat{f} \in \mathcal{H}_s[D]
\]

(we need to mark the dependence on the functionals \(\Gamma\) and on \(\eta\) here).

The Lie derivative \(\mathcal{L}_v\) over an arbitrary nonlinear functional \(\rho\colon \mathcal{H}_s \to \mathbb{C}\) can be also defined as

\[
\mathcal{L}_v \rho[f] := (\mathcal{L}_v \rho)[f] = \frac{\partial}{\partial t} (H_\alpha[v, s^{-1}_s \rho])[f] \big|_{t=0}
\]

(if the partial derivative w.r.t. \(\alpha\) is well-defined).

For example,

\[
\mathcal{L}_v \exp(\rho[f]) = (\mathcal{L}_v \rho)[f] \exp(\rho[f]), \\
\mathcal{L}_v^2 \exp(\rho[f]) = \left(\mathcal{L}_v^2 \rho[f] + (\mathcal{L}_v \rho[f])^2\right) \exp(\rho[f]).
\]

In our case \(\rho[f] = \hat{\phi}[f] = \exp\left(\frac{1}{2}\Gamma[f, f] + \eta[f]\right)\). We remind that the Lie derivative of \(\eta\) and \(\Gamma\) are defined in (2.25) and (2.31) respectively.

The operations \(G^{-1}_s\) and \(\frac{\delta}{\delta \hat{f}}\) or \(\mathcal{L}\) and \(\frac{\delta}{\delta \hat{f}}\) commute. Thus, for example, we have

\[
\mathcal{L}_v S_n[f_1, f_2, \ldots, f_n] = \left(\frac{\delta}{\delta f_1} \frac{\delta}{\delta f_2} \cdots \frac{\delta}{\delta f_n} \mathcal{L}_v \hat{\phi}\right) [0].
\]

We use this to deduce the martingale properties of \(G^{-1}_t S_n\) and of all their variational derivatives from the martingale property of \(G^{-1}_t \hat{\phi}\), which will be discussed in the next section.
3. Coupling between SLE and GFF

Let \((\Omega^\Phi, \mathcal{F}^\Phi, P^\Phi)\) be the probability space for GFF \(\Phi\) and let \((\Omega^B, \mathcal{F}^B, P^B)\) be the independent probability space for the Brownian motion \(\{B_t\}_{t \in [0, +\infty)}\), which governs some \((\delta, \sigma)\)-SLE \(\{G_t\}_{t \in [0, +\infty)}\). In this section, we consider a coupling between these random laws.

The pushforward \(G_t^{-1} \Phi[f]\) of the GFF \(\Phi[f]\) is well-defined if \(f \in \text{image}[G_t^{-1}]\). In order to handle this, we introduce a stopping time \(T[f]\), for which the hull \(K_t\) of \((\delta, \sigma)\)-SLE touches some small neighborhood \(U\) of \(\text{image}[G_t^{-1}]\) of the support of \(f\) for the first time:

\[
T[f] := \sup\{t > 0; K_t \cap U(\text{image}[G_t^{-1}]) = \emptyset\}, \quad f \in \mathcal{H}.
\]

The neighborhood \(U(\text{image}[G_t^{-1}])\) can be defined, for example, as the set of points from \(\text{image}[G_t^{-1}]\) with the Poincare distance less than some \(\varepsilon > 0\). Thus, \(T[f] > 0\) a.s. We consider a stopped process \(\{G_{t \wedge T[f]}\}_{t \in [0, +\infty)}\). This approach was also used in \([16]\). The most important property of the process \(\{G_{t \wedge T[f]}^{-1} \Phi[f]\}_{t \in [0, +\infty)}\) is that it is a local martingale. A stopped local martingale is also a local martingale. That is why a stopping of \(\{G_t\}_{t \in [0, +\infty)}\) does not change our results. However, we lose some information, which makes the proposition of coupling less substantial than one possibly expects.

We present here two definitions of the coupling. The first one is similar to \([29, 16]\). The second one is a weaker statement that we shall use in this paper.

**Definition 5.** A GFF \(\Phi(\mathcal{H}, \Gamma, \eta)\) is called coupled to the forward or reverse \((\delta, \sigma)\)-SLE, driven by \(\{B_t\}_{t \in [0, +\infty)}\), if the random variable \(G_{t \wedge T[f]}^{-1} \Phi[f]\) obtained by independent sampling of \(\Phi\) and \(G_t\) has the same law as \(\Phi^\psi[f]\) for any test function \(f \in \mathcal{H}\), chart map \(\psi\), and \(t \in [0, +\infty)\).

If the coupling holds for a fixed chart map \(\psi\) and for any \(f \in \mathcal{H}\), then it also holds for any chart map \(\psi\), due to \((2.41)\). We also give a weaker version of the coupling statement that we plan to use here. To this end, we have to consider a stopped versions of the stochastic process \(\{G_{t \wedge T[f]}^{-1} \Phi[f]\}_{t \in [0, +\infty)}\).

A collection of stopping times \(\{T_n\}_{n=1, 2, \ldots}\) is called a fundamental sequence if \(0 \leq T_n \leq T_{n+1} \leq +\infty\), \(n = 1, 2, \ldots\) a.s., and \(\lim_{n \to \infty} T_n = \infty\) a.s.

A stochastic process \(\{x_t\}_{t \in [0, +\infty)}\) is called a local martingale if there exists a fundamental sequence of stopping times \(\{T_n\}_{n=1, 2, \ldots}\), such that the stopped process \(\{x_{t \wedge T_n}\}_{t \in [0, +\infty)}\) is a martingale for each \(n = 1, 2, \ldots\).

Let now the statement of coupling above be valid only for the process \(\{G_{t \wedge T[f]}^{-1} \Phi[f]\}_{t \in [0, +\infty)}\) stopped by \(T_n\) for each \(n = 1, 2, \ldots\). Namely, \(G_{t \wedge T[f] \wedge T_n}^{-1} \Phi[f]\) has the same law as \(\Phi^\psi[f]\) for each \(n = 1, 2, \ldots\).

We are ready now to define the local coupling.

**Definition 6.** A GFF \(\Phi(\mathcal{H}, \Gamma, \eta)\) is called locally coupled to \((\delta, \sigma)\)-SLE, driven by \(\{B_t\}_{t \in [0, +\infty)}\), if there exists a fundamental sequence \(\{T_n[f, \psi]\}_{n=1, 2, \ldots}\), such that the random variable \(G_{t \wedge T[f]}^{-1} \Phi[f]\) obtained by independent sampling of \(\Phi\) and \(G_t\) has the same law as \(\Phi^\psi[f]\) until the stopping time \(T_n[f, \psi]\) for each \(n = 1, 2, \ldots\), for any test function \(f \in \mathcal{H}\), and a chart map \(\psi\).

**Remark 1.** If \(T[f, \psi] = +\infty\) a.s. for each \(f \in \mathcal{H}\), then the coupling is not local.

The following theorem generalizes the result of \([29]\).

**Theorem 1.** The following three statements are equivalent:

1. GFF \(\Phi(\mathcal{H}, \Gamma, \eta)\) is locally coupled to \((\delta, \sigma)\)-SLE;
2. \(G_{t \wedge T[f]}^{-1} \Phi[f]\) is a local martingale for \(f \in \mathcal{H}\) in any chart \(\psi\);
3. The system of the equations

\begin{align}
\mathcal{L}_\delta \eta[f] + \frac{1}{2} \mathcal{L}_\sigma^2 \eta[f] &= 0, \quad f \in \mathcal{H}, \\
\mathcal{L}_\delta \Gamma[f, g] + \mathcal{L}_\sigma \eta[f] \mathcal{L}_\alpha \eta[g] &= 0, \quad f, g \in \mathcal{H},
\end{align}

and

\begin{equation}
\mathcal{L}_\sigma \Gamma[f, g] = 0, \quad f, g \in \mathcal{H}.
\end{equation}

is satisfied.

We start the proof after some remarks. Just for clarity (but not for applications) we reformulate the system (3.2–3.4) directly in terms of partial derivatives using (2.25), (2.26), (2.33), and (A.1) as

\begin{align}
\delta(z) \partial_z \eta(z) + \delta(z) \partial_z \eta(z) + \mu \delta'(z) + w \delta'(z) + \\
\frac{1}{2} \sigma(z) \partial_z \eta(z) + \frac{1}{2} \sigma(z) \partial_z \eta(z) + \sigma(z) \sigma(z) \partial_z \eta(z) + \\
\frac{1}{2} \sigma(z) \sigma'(z) \partial_z \eta(z) + \frac{1}{2} \sigma(z) \sigma'(z) \partial_z \eta(z) + \mu \sigma(z) \sigma''(z) + \mu \sigma(z) \sigma''(z) = 0;
\end{align}

\begin{align}
\delta(z) \partial_z \Gamma(w, w) + \delta(w) \partial_w \Gamma(w, w) + \delta(z) \partial_z \Gamma(w, w) + \\
\Gamma \sigma'(z) \partial_z \eta(z) + \mu \sigma'(z) + \mu \sigma'(z) + \\
\sigma(z) \partial_z \Gamma(w, w) + \sigma(w) \partial_w \Gamma(w, w) + \sigma(z) \partial_z \Gamma(w, w) + \\
\sigma(w) \partial_w \Gamma(w, w) = 0,
\end{align}

where we drop the upper index \( \psi \) for shortness.

The first equation (3.2) is just a local martingale condition for \( \eta \). The second one (3.3) is a special case of Hadamard’s variation formula, where the variation is concentrated at one point at the boundary. The third equation means that \( \Gamma \) should be invariant under the one-parametric family of Möbius automorphisms generated by \( \sigma \).

**Proof of Theorem 1** Let us start with showing how the statement 1 about the coupling implies the statement 2 about the local martingality.

1. \( \leftrightarrow \) 2. Let \( G_{t \wedge T_f \wedge T_n[f, \psi]} \) be a stopped process \( G_{t \wedge T_f} \) by the stopping times \( T_n[f, \psi] \) forming some fundamental sequence. The coupling statement can be reformulated as an equality of characteristic functions for the random variables \( G_{t \wedge \tilde{T}_n[f, \psi]}^{-1} \Phi[f] \) and \( \Phi[f] \) for all test functions \( f \). Namely, the following expectations must be equal

\begin{equation}
\mathbb{E}_B \left[ \mathbb{E}_\Phi \left[ e^{G_{t \wedge T_f \wedge T_n[f, \psi]}^{-1} \Phi[f]} \right] \right] = \mathbb{E}_\Phi \left[ e^{\Phi[f]} \right], \quad f \in \mathcal{H}, \quad t \in [0, +\infty), \quad n = 1, 2, \ldots,
\end{equation}

which in particular, means the integrability of \( e^{G_{t \wedge \tilde{T}_n[f, \psi]}^{-1} \Phi[f]} \) with respect to \( \Omega_B \) and \( \Phi \).

We used \( \mathbb{E}_B [\cdot] \) for the expectation with respect to the random law of \( \{B_t\}_{t \in [0, +\infty)} \) (or \( \{G_t\}_{t \in [0, +\infty)} \)) and \( \mathbb{E}_\Phi [\cdot] \) for the expectation with respect to \( \Phi \). Let us use (2.40) and (2.41) to simplify this identity to

\begin{equation}
\mathbb{E}_B \left[ G_{t \wedge T_f \wedge T_n[f, \psi]}^{-1} \tilde{\Phi}^\psi[f] \right] = \tilde{\Phi}^\psi[f], \quad f \in \mathcal{H}, \quad t \in [0, +\infty), \quad n = 1, 2, \ldots.
\end{equation}

After substituting \( f \to \tilde{G}_{s \wedge T_f} f \) for some independently sampled \( \tilde{G}_s \) and \( s \in [0, +\infty) \), we obtain

\begin{equation}
\mathbb{E}_B \left[ G_{t \wedge T_f \wedge T_n[f, \psi]}^{-1} \tilde{G}_{s \wedge T_f} f \right] = \tilde{\Phi}^\psi[\tilde{G}_{s \wedge T_f} f], \quad f \in \mathcal{H}, \quad t \in [0, +\infty), \quad n = 1, 2, \ldots.
\end{equation}
We can conclude that
\[ \delta, \sigma \]
and by making use of \((2.30)\) and \((2.40)\), we conclude that
\begin{equation}
\mathbb{E}_B \left[ \tilde{G}_{s T(f)}^{-1} \circ G_{t T(G_{t T(f)}^{-1} \circ F_{s T(f)})}^{-1} \phi_x[f] \right] = \tilde{G}_{s T(f)}^{-1} \phi_x[f],
\end{equation}
\[ f \in \mathcal{H}, \ t \in [0, +\infty), \ n = 1, 2, \ldots. \]
Defined now the process
\[ \tilde{G}_{t+s} := G_t \circ \tilde{G}_s, \ s, t \in [0, +\infty), \]
which has the law of \((\delta, \sigma)\)-SLE. Its stopped version possesses the identity
\[ G_{t+s T(f)} = G_{t T(G_{s T(f)}^{-1} \circ F_{s T(f)})} \circ \tilde{G}_{s T(f)}^{-1} \phi_x[f], \ s, t \in [0, +\infty), \ f \in \mathcal{H} \]
The left-hand side of \((3.5)\) is equal to
\begin{align*}
&= \mathbb{E}_B \left[ \left( \tilde{G}_{t+s T(f)}^{-1} \circ G_{t T(G_{s T(f)}^{-1} \circ F_{s T(f)})}^{-1} \phi_x[f] \right) \right] = \\
&= \mathbb{E}_B \left[ \left( \tilde{G}_{t+s T(f)}^{-1} \circ G_{t T(G_{s T(f)}^{-1} \circ F_{s T(f)})}^{-1} \phi_x[f] \right) \right]
\end{align*}
We use now the Markov property of \((\delta, \sigma)\)-SLE and conclude that \(T_n^*[f, \psi] := T_n[\tilde{G}_{s T(f)}^{-1} \circ F_{s T(f)}] + s\) is a fundamental sequence for the pair of \(f, \psi\). Thus, \((3.5)\) simplifies to
\begin{equation}
\mathbb{E}_B \left[ G_{t+s T(f)}^{-1} \circ F_{s T(f)}^{-1} \phi_x[f] \right] = G_{t+s T(f)}^{-1} \phi_x[f],
\end{equation}
hence, \(\{G_{t+s T(f)}^{-1} \phi_x[f]\}_{t \in [0, +\infty)}\) is a local martingale.
The inverse statement can be obtained by the same method in the reverse order.

2.\(\iff\)3. According to Proposition 8, Appendix A, the drift term, i.e., the coefficient at \(dt\), vanishes identically when
\begin{equation}
\mathcal{A} W[f] + \frac{1}{2} (\mathcal{L}_x W[f])^2 = 0, \quad f \in \mathcal{H}.
\end{equation}
The left-hand side is a functional polynomial of degree 4. We use the fact that a regular symmetric functional \(P[f] := \sum_{k=1}^{n} p_k[f, f, \ldots, f]\) of degree \(n\) over such spaces as \(\mathcal{H}_s, \mathcal{H}_{s,b}, \mathcal{H}_{s,b}^\pm\) is identically zero if and only if
\[ p_k[f_1, f_2, \ldots, f_n] = 0, \quad k = 1, 2, \ldots n, \quad f \in \mathcal{H}. \]
Thus, each of the following functions must be identically zero:
\begin{align*}
\mathcal{A} \eta[f] & = 0, \quad \frac{1}{2} \mathcal{A} \Gamma[f, g] + \frac{1}{2} \mathcal{L}_x \eta[f] \mathcal{L}_x \eta[g] = 0, \\
\mathcal{L}_x \eta[f] \mathcal{L}_x \Gamma[g, h] + \text{symmetric terms} & = 0, \\
\mathcal{L}_x \Gamma[f, g] \mathcal{L}_x \Gamma[h, l] + \text{symmetric terms} & = 0,
\end{align*}
\(f, g, h, l \in \mathcal{H}\).
We can conclude that \(\mathcal{L}_x \Gamma[f, g] = 0, \mathcal{A} \Gamma[f, g] = \mathcal{L}_x \Gamma[f, g]\) for any \(f, g \in \mathcal{H}\), and this system is equivalent to the system \((3.2) - (3.4)\). For the case \(\mathcal{H} = \mathcal{H}_s\) we can write \((3.7)\) in...
terms of functions on \( \psi(D) \):

\[
\mathcal{A} \eta(z) = 0, \quad \frac{1}{2} \mathcal{A} \Gamma(z, w) + \frac{1}{2} \mathcal{L}_\sigma \eta(z) \mathcal{L}_\sigma \eta(w) = 0,
\]

\[
\mathcal{L}_\sigma \eta(z) \mathcal{L}_\sigma \Gamma(w, u) + \text{symmetric terms} = 0,
\]

\[
\mathcal{L}_\sigma \Gamma(z, w) \mathcal{L}_\sigma \Gamma(u, v) + \text{symmetric terms} = 0,
\]

\[z, w, u, v \in \psi(D), \quad z \neq w, \ u \neq v, \ldots .\]

**Remark 2.** Fix a chart \( \psi \). The coupling and the martingales are not local if in addition to the proposition 3 in Theorem 2 the relation

\[
\mathbb{E}_B \left[ \int_0^t \exp \left( G^{-1}_{\tau \wedge T[f]} W^{\psi}[f] \right) G^{-1}_{\tau \wedge T[f]} \mathcal{L}_\sigma W^{\psi}[f] \, d^h_0 B_\tau \right] < \infty, \quad t \geq 0,
\]

holds. This is the condition that the diffusion term at \( d^h_0 B_\tau \) in (A.12) is in \( L_1(\Omega^B) \). However, this may not be true, in general, in another chart \( \psi \). Meanwhile, if the local martingale property of \( G^{-1}_{\tau \wedge T[f]} \hat{\psi}[f] \) is satisfied in one chart \( \psi \) for any \( f \in \mathcal{H} \), then it is also true in any chart due to the invariance of the condition (3.6) in the proof.

The study of the general solution to (3.2-3.4) is an interesting and complicated problem. Take the Lie derivative \( \mathcal{L}_\sigma \) in the second equation, the Lie derivative \( \mathcal{L}_\delta \) in the third equation, and consider the difference of the resulting equations. It is an algebraically independent equation

\[
\mathcal{L}_{[\delta, \sigma]} \Gamma[f, g] = - \mathcal{L}_\delta^2 \eta[f] \mathcal{L}_\sigma \eta[g] + \mathcal{L}_\delta^2 \eta[f] \mathcal{L}_\sigma \eta[g].
\]

Continuing by induction we obtain an infinite system of a priori algebraically independent equations because the Lie algebra induced by the vector fields \( \delta \) and \( \sigma \) is infinite-dimensional. Thereby, the existence of the solution to the system (3.2,3.4) is a special event that is strongly related to the properties of this algebra.

Before studying special solutions to the system (3.2,3.4), let us consider some of its general properties. We also reformulate it in terms of the analytic functions \( \eta^+, \Gamma^+ \) and \( \Gamma^+ \), which is technically more convenient.

**Lemma 2.** Let \( \delta, \sigma, \eta, \) and \( \Gamma \) be such that the system (3.2,3.4) is satisfied, let \( \Gamma \) be a fundamental solution to the Laplace equation (see (2.30)), which transforms as a scalar, see (2.33), and let \( \eta \) be a pre-pre-Schwarzian. Then,

1. \( \eta \) is a \((i\chi/2, -i\chi/2)\)-pre-pre-Schwarzian (2.22) given by a harmonic function in any chart with \( \chi \) given by (1.4).
2. The boundary value of \( \eta \) undergoes a jump \( 2\pi/\sqrt{\kappa} \) at the source point \( a \), namely, its local behaviour in the half-plane chart is given by (3.21) up to a sign;
3. The system (3.2,3.4) is equivalent to the system (3.5, (3.13), (3.14), (3.17), and (3.18).

**Proof.** The system (3.2,3.4) defines \( \eta \) only up to an additive constant \( C \) that we keep writing in the formulas for \( \eta \) below. The condition for the pre-pre-Schwarzian \( \eta \) to be real leads to only two possibilities:

1. \( \mu = -\mu^* \) and is pure imaginary as in (2.22);
2. \( \mu = \mu^* \) and is real as in (2.21).

The equation (3.3) shows that the functional \( \mathcal{L}_\sigma \eta \) has to be given by a harmonic function as well as \( \mathcal{L}_\delta \eta \) in any chart. On the other hand, (3.2) implies that \( \mathcal{L}_\delta \eta \) is also harmonic. The vector fields \( \delta \) and \( \sigma \) are transversal almost everywhere. We conclude that \( \eta \) is harmonic. We used also the fact that the additional \( \mu \)-terms in (2.25) are harmonic.
The harmonic function $\eta^\psi(z)$ can be represented as a sum of an analytic function $\eta^{+\psi}(z)$ and its complex conjugate in any chart $\psi$
\begin{equation}
\eta^\psi(z) = \eta^{+\psi}(z) + \eta^{-\psi}(z).
\end{equation}

Below in this proof, we drop the chart index $\psi$, which can be chosen arbitrarily.

Let us define $\eta^+$ and $\eta^-$ to be pre-pre-Schwarzians of orders $(\mu, 0)$ and $(0, \mu^*)$ respectively by (2.25). Thus, $\eta^+$ is defined up to a complex constant $C^+$. We denote
\begin{equation}
j^+ := L_\sigma \eta^+.
\end{equation}
and
\begin{equation}
j := L_\sigma \eta = L_\sigma \eta^+ + L_\sigma \eta^-.
\end{equation}
The reciprocal formula is
\begin{equation}
\eta^+(z) := \int \frac{j^+(z) - \mu \sigma'(z)}{\sigma(z)} dz.
\end{equation}
This integral can be a ramified function if $\sigma(z)$ has a zero inside of $\mathcal{D}$ (the elliptic case). We consider how to handle this technical difficulty in Section 4.3.

Let us reformulate now (3.2) in terms of $j^+$. Using the fact that
\begin{equation}
L_\sigma^2 (\eta^+ + \eta^-) = L_\sigma^2 \eta^+ + L_\sigma^2 \eta^-,
\end{equation}
we arrive at
\begin{equation}
L_\delta \eta^+ + \frac{1}{2} L_\sigma^2 \eta^+ = C^+.
\end{equation}
Here $C^+ = i \beta$ for some $\beta \in \mathbb{R}$ for the forward case. For the reverse case, $C^+ = -\beta + i \beta'$ for some $\beta, \beta' \in \mathbb{R}$ because (3.12) is an identity in sense of functionals over $\mathcal{H}_s^*$.

The relation (3.12) is equivalent to
\begin{align}
\frac{\delta}{\sigma} L_\sigma \eta^+ + \frac{\sigma L_\delta \eta^+ - \delta L_\sigma \eta^+}{\sigma} + \frac{1}{2} L_\sigma^2 \eta^+ = C^+ & \iff \\
\frac{\delta}{\sigma} j^+ + \frac{\sigma \delta \partial \eta^+ + \mu \sigma' - \delta \sigma \partial \eta^+ - \mu \delta \sigma'}{\sigma} + \frac{1}{2} L_\sigma j^+ = C^+ & \iff \\
(3.13)
\end{align}
by (2.25) and (2.24).

Consider now the function $\Gamma^H(z, w)$. It is harmonic with respect to both variables with the only logarithmic singularity. Hence, it can be split as a sum of four terms
\begin{equation}
\Gamma^H(z, w) := \Gamma^{++H}(z, w) + \Gamma^{++-H}(z, w) - \Gamma^{+-H}(z, \bar{w}) - \Gamma^{-+H}(z, w),
\end{equation}
where $\Gamma^{++H}(z, w)$ and $\Gamma^{+-H}(z, w)$ are analytic with respect to both variables except the diagonal $z = w$ for $\Gamma^{++H}(z, w)$.

So, e.g., $\Gamma^{+-H}(z, \bar{w})$ is anti-analytic with respect to $z$ and analytic with respect to $w$. We can assume that both $\Gamma^{++}(z, w)$ and $\Gamma^{+\ast}(z, w)$ transform as scalars represented by analytic functions in all charts and symmetric with respect to $z \leftrightarrow w$. Observe that these functions are defined at least up to the transform
\begin{align}
\Gamma^{++H}(z, w) \rightarrow \Gamma^{++H}(z, w) + \epsilon^H(z) + \epsilon^H(w), \\
\Gamma^{+-H}(z, w) \rightarrow \Gamma^{+-H}(z, w) + \epsilon^H(z) + \epsilon^H(w)
\end{align}
for any analytic function $\epsilon^H(z)$ such that
\begin{equation}
\epsilon^H(z) = \epsilon^H(\bar{z}).
\end{equation}
These additional terms are cancelled due to the choice of minus in the pairs ‘∓’ in (3.14). In the reverse case, the contribution of these functions is equivalent to zero bilinear functional over $H^s$.

Consider the equation (3.14). It leads to
\begin{equation}
\mathcal{L}_\sigma \Gamma^{++H}(z,w) = \beta_2^H(z) + \beta_2^H(w), \quad \mathcal{L}_\sigma \Gamma^{+-H}(z,w) = \beta_2^H(z) + \beta_2^H(w)
\end{equation}
for any analytic function $\beta_2^H(z)$ such that $\overline{\beta_2^H(z)} = \beta_2^H(\overline{z})$. One can fix this freedom, i.e., the function $\beta_2^H$, by the conditions
\begin{equation}
\mathcal{L}_\sigma \Gamma^{++H}(z,w) = 0, \quad \mathcal{L}_\sigma \Gamma^{+-H}(z,w) = 0.
\end{equation}
Thus, $\Gamma^{++H}(z,w)$ and $\Gamma^{+-H}(z,w)$ are fixed up to a non-essential constant.

The second equation (3.3) can be reformulated now as
\begin{equation}
\mathcal{L}_\delta \Gamma^{++H}(z,w) + \mathcal{L}_\sigma \eta^{H}(z) \mathcal{L}_\sigma \eta^{+H}(w) = \beta_1^H(z) + \beta_1^H(w),
\end{equation}
for any analytic function $\beta_1^H(z)$ such that $\overline{\beta_1^H(z)} = \beta_1^H(\overline{z})$ analogously to (3.15). We can conclude now that the system (3.2–3.4) is equivalent to the system (3.8), (3.13), (3.14), (3.17), and (3.16).

Use now the fact
\begin{equation}
\Gamma^{++}(z,w) = -\frac{1}{2} \log(z - w) + \text{analytic terms}
\end{equation}
to obtain a singularity of $j^{+H}$ about the origin in the half-plane chart. Relation (2.17) yields
\begin{equation}
2 \frac{z}{z} \partial_z \left(-\frac{1}{2} \log(z - w)\right) + 2 \frac{w}{w} \partial_w \left(-\frac{1}{2} \log(z - w)\right) = \frac{1}{zw},
\end{equation}
hence,
\begin{equation}
j^{+H}(z) = -\frac{i}{z} + \text{holomorphic part}.
\end{equation}
The choice of the sign of $j^{+H}(z)$ is irrelevant. We made the choice above to be consistent with [29]. The analytic terms in (3.18) can give a term with the sum of simple poles at $z$ and $w$ but in the form of the product $1/(zw)$.

From (3.11) we conclude that the singular part of $\eta^{H}(z)$ is proportional to the logarithm of $z$:
\begin{equation}
\eta^{H}(z) = \frac{i}{\sqrt{\kappa}} \log z + \text{holomorphic part}.
\end{equation}
Thus, we have
\begin{equation}
\eta^{H}(z) = -\frac{2}{\sqrt{\kappa}} \text{arg } z + \text{non-singular harmonic part}.
\end{equation}
We can chose the additive constant such that, in the half-plane chart, we have
\begin{equation}
\eta^{H}(+0) = -\eta^{H}(-0) = \frac{\pi}{\sqrt{\kappa}}
\end{equation}
in the forward case. This provides the jump $2\pi/\sqrt{\kappa}$ of the value of $\eta$ at the boundary near the origin, which is exactly the same behaviour of $\eta$ needed for the flow line construction in [23] and [29]. However, the form (3.22) is not chart independent, and only the jump $2\pi/\sqrt{\kappa} = \eta^{H}(+0) - \eta^{H}(-0)$ does not change its value if the boundary of $\psi(D)$ is not singular in the neighbourhood of the source $\psi(a)$. 
Substitute now \((3.20)\) in \((3.13)\) in the half-plane chart, use \((2.17)\), and consider the corresponding Laurent series. We are interested in the coefficient near the first term \(\frac{1}{z^2}\):

\[
\frac{2}{z} \frac{1}{\sqrt{\kappa}} \frac{-i}{z} + \mu \frac{-2}{z^2} + \frac{1}{2} \left( -\sqrt{\kappa} \right) \frac{i}{z^2} + o \left( \frac{1}{z^2} \right) = C^+.
\]

We can conclude that

\[
(3.23) \quad \mu = \frac{i(4 - \kappa)}{4\sqrt{\kappa}}.
\]

Thus, the pre-pre-Schwarzians \((2.22)\) with \(\chi\) given by \((1.4)\) is only one that can be realized.

4. **Coupling of GFF with the Dirichlet boundary conditions**

In this section, we consider a special solution to the system \((3.2–3.4)\) with the help of Lemma 2. We assume the Dirichlet boundary condition for \(\Gamma\) considered in Example 3 and find the general solution in this case. In other words, we systematically study which of \((\delta, \sigma)\)-SLE can be coupled with GFF if \(\Gamma = \Gamma_D\).

Let us formulate the following general theorem, and then consider each of the allowed cases of \((\delta, \sigma)\)-SLE individually.

**Theorem 2.** Let a \((\delta, \sigma)\)-SLE be coupled to the GFF with \(\mathcal{H} = \mathcal{H}_s\), \(\Gamma = \Gamma_D\), and let \(\eta\) be the pre-pre-Schwarzian \((2.22)\) of order \(\chi\). Then the only special combinations of \(\delta\) and \(\sigma\) for \(\kappa \neq 6\) and \(\nu \neq 0\) summarized in Table 1, and their arbitrary combinations when \(\kappa = 6\) and \(\nu = 0\) are possible.

Table 1 consists of 6 cases, each of which is a one-parameter family of \((\delta, \sigma)\)-SLEs parametrized by the drift \(\nu \in \mathbb{R}\), and by a parameter \(\xi \in \mathbb{R}\). These cases may overlap for vanishing values of \(\nu\) or \(\xi\).

In other words, different combinations of \(\delta\) and \(\sigma\) can correspond essentially to the same process in \(D\) but written in different coordinates. We give one example of such choices in each case of \(\delta\) and \(\sigma\).

Some particular cases of CFTs studied here were considered earlier in the literature. The chordal SLE without drift (case 1 from the table with \(\nu = 0\)) was considered in [18], the radial SLE without drift (case 3 from the table with \(\nu = 0\)) in [19], and the dipolar SLE without drift (case 4 from the table with \(\nu = 0\)) appeared in [20]. The case 2 actually corresponds to the same measure defined by the chordal SLE but stopped at the time \(t = 1/4\xi\) (see Section 4.5). The cases 5 and 6 are mirror images of each other. They are discussed in Section 4.6.

**Remark 3.** An alternative approach to the relation between CFT and SLE based on the highest weight representation of the Virasoro algebra was considered in [4] and [8]. We remark that such a representation can not be constructed for non-zero drift \((\nu \neq 0)\).

**Proof of Theorem 2** Let us use Theorem 1 and assume the Dirichlet boundary conditions for \(\Gamma = \Gamma_D\).

\[
\Gamma^{++}(z,w) = -\frac{1}{2} \log(z - w), \quad \Gamma^{+-}(z,\bar{w}) = -\frac{1}{2} \log(z - \bar{w})
\]

in Theorem 2. The condition \((3.15)\) is satisfied for any complete vector field \(\sigma\) and some \(\sigma\)-dependent \(\beta_2\) which is irrelevant.
In order to obtain \( j^+ \) we remark first that due to the Möbius invariance (2.38) we can ignore the polynomial part of \( \delta^H(z) \)

\[
\mathcal{L}_\delta \Gamma^H(z, w) = \left( \frac{2}{z} \partial_z + \frac{2}{\bar{z}} \partial_{\bar{z}} + \frac{2}{w} \partial_w + \frac{2}{\bar{w}} \partial_{\bar{w}} \right) \Gamma^H(z, w).
\]

Using (3.17), (2.37), and (3.19) we obtain that

\[
(4.1) \quad j^+ = \frac{-i}{z} + i\alpha, \quad \alpha \in \mathbb{C},
\]

with

\[
\beta_1(z) = \frac{\alpha}{z} - \frac{\alpha^2}{2}.
\]

In order to satisfy all conditions formulated in Lemma 2 we need to check (3.13). Substituting (4.1) to (3.13) gives

\[
\delta j^+ + \mu \left[ \sigma, \delta \right] + \frac{1}{2} \mathcal{L}_\sigma j^+ = i\beta \quad \Leftrightarrow
\]

\[
(4.2) \quad \delta j^+ + \mu \left[ \sigma, \delta \right] + \frac{1}{2} \mathcal{L}_\sigma j^+ - i\beta \sigma = 0 \quad \Leftrightarrow
\]

\[
\delta^H(z) \left( \frac{-i}{z} + i\alpha \right) + \mu \left[ \sigma, \delta \right]^H(z) + \frac{1}{2} \left( \sigma^H(z) \right)^2 i \left( \frac{-i}{z} + i\alpha \right) - i\beta \sigma^H(z) = 0.
\]

In what follows, we will use the half-plane chart in the proof. With the help of (2.16) and (2.18) we can assume without loss of generality that \( \sigma^H \) is one of three possible forms:

1. \( \sigma^H(z) = -\sqrt{\kappa} \),
2. \( \sigma^H(z) = -\sqrt{\kappa} (1 - z^2) \),
3. \( \sigma^H(z) = -\sqrt{\kappa} (1 + z^2) \).

| N | Name | \( \delta^H(z) \) | \( \sigma^H(z) \) | \( \alpha \) | \( \beta \) |
|---|------|-----------------|-----------------|---------|---------|
| 1 | Chordal with drift | \( \frac{1}{z} - \nu \) | \( -\sqrt{\kappa} \) | \( -\frac{\nu}{2} \) | \( \frac{\nu}{2\sqrt{\kappa}} \) |
| 2 | Chordal with fixed time change | \( \frac{2}{z} + 2\xi z \) | \( -\sqrt{\kappa} \) | 0 | \( \frac{\xi (8 - \kappa)}{2\sqrt{\kappa}} \) |
| 3 | Dipolar with drift | \( 2 \left( \frac{1}{z} - z \right) - \nu (1 - z^2) \) | \( -\sqrt{\kappa} (1 - z^2) \) | \( -\frac{\nu}{2} \) | \( \frac{4 - \nu^2}{2\sqrt{\kappa}} \) |
| 4 | One right fixed boundary point | \( \frac{2}{z} + \kappa - 6 + 2(3 - \kappa + \xi) z + (\kappa - 2 - 2\xi) z^2 \) | \( -\sqrt{\kappa} (1 - z^2) \) | \( \frac{1}{2} (\kappa - 6) \) | \( \frac{\xi (8 - \kappa)}{2\sqrt{\kappa}} \) |
| 5 | One left fixed boundary point | \( \frac{2}{z} - (\kappa - 6) + 2(3 - \kappa + \xi) z + (\kappa - 2 - 2\xi) z^2 \) | \( -\sqrt{\kappa} (1 - z^2) \) | \( -\frac{1}{2} (\kappa - 6) \) | \( \frac{\xi (8 - \kappa)}{2\sqrt{\kappa}} \) |
| 6 | Radial with drift | \( \frac{1}{z} + 2z \) | \( -\sqrt{\kappa} (1 + z^2) \) | \( -\frac{\nu}{2} \) | \( \frac{4 - \nu^2}{2\sqrt{\kappa}} \) |

Table 1. \((\delta, \sigma)\)-SLE types that can be coupled with CFT with the Dirichlet boundary conditions \((\Gamma = \Gamma_0)\). Each of the pairs of \( \delta \) and \( \sigma \) is given in the half-plane chart for simplicity. For the same purpose we use the normalization \((2.17)\). The details are in Sections 4.1 and 4.3. See also the comments after Theorem 2.
Let us consider these cases turn by turn.

1. $\sigma(z) = -\sqrt{\kappa}$.

Inserting (2.17) to relation (4.2) reduces to

$$\frac{-2 + \frac{\kappa}{2} - 2i\sqrt{\kappa}\mu}{z^2} + \frac{2\alpha - \delta_{-1}}{z} + \left(\beta\sqrt{\kappa} + \alpha\delta_{-1} - \delta_0 + i\sqrt{\kappa}\mu\delta_0\right) +$$

$$+ z \left(\alpha\delta_0 - \delta_1 + 2i\sqrt{\kappa}\mu\delta_1\right) + z^2 \alpha\delta_1 \equiv 0 \iff (3.23), 2\alpha - \delta_{-1} = 0, \beta\sqrt{\kappa} + \alpha\delta_{-1} - \delta_0 + i\sqrt{\kappa}\mu\delta_0 = 0, \alpha\delta_0 - \delta_1 + 2i\sqrt{\kappa}\mu\delta_1 = 0, \alpha\delta_1 = 0.$$

There are three possible cases:

(1) $\delta_{-1} = 2\alpha, \quad \delta_0 = 0, \quad \delta_1 = 0, \quad \kappa > 0, \quad \beta = -\frac{2\alpha^2}{\sqrt{\kappa}}.$

It is convenient to use the drift parameter $\nu$. Thus,

$$\nu = -2\alpha,$$

which is related to the drift in the chordal equation. This case is presented in the first line of Table 1.

(2) $\delta_{-1} = 0, \quad \delta_0 = -\frac{4\beta\sqrt{\kappa}}{\kappa - 8}, \quad \delta_1 = 0, \quad \kappa > 0, \quad \alpha = 0.$

This case is presented in the second line of Table (\(\xi \in \mathbb{R}\)) and discussed in details in Section 4.5.

(3) $\delta_{-1} = 0, \quad \delta_0 = 2\sqrt{5}\beta, \quad \delta_1 \in \mathbb{R}, \quad \kappa = 6, \quad \alpha = 0.$

This is a general case of $\delta$ with $\kappa = 6$ and $\nu = 0$.

2. $\sigma^H(z) = -\sqrt{\kappa}(1 - z^2)$.

Relation (4.2) reduces to

$$\frac{-2 + \frac{\kappa}{2} + 2i\sqrt{\kappa}\mu}{z^2} + \frac{2\alpha - \delta_{-1}}{z} + \left(\beta\sqrt{\kappa} - \kappa + 6i\sqrt{\kappa}\mu + \alpha\delta_{-1} - \delta_0 + i\sqrt{\kappa}\mu\delta_0\right) +$$

$$+ z \left(2i\sqrt{\kappa}\mu\delta_{-1} + \alpha\delta_0 - \delta_1 + 2i\sqrt{\kappa}\mu\delta_1\right) z^2 \left(-\beta\sqrt{\kappa} + \kappa \mu + i\sqrt{\kappa}\mu\delta_0 + \alpha\delta_1\right) = 0 \iff (3.23), 2\alpha - \delta_{-1} = 0, \beta\sqrt{\kappa} - \kappa + 6i\sqrt{\kappa}\mu + \alpha\delta_{-1} - \delta_0 + i\sqrt{\kappa}\mu\delta_0 = 0, \quad 2i\sqrt{\kappa}\mu\delta_{-1} + \alpha\delta_0 - \delta_1 + 2i\sqrt{\kappa}\mu\delta_1 = 0, \quad -\beta\sqrt{\kappa} + \kappa \mu + i\sqrt{\kappa}\mu\delta_0 + \alpha\delta_1 = 0.$$

There are four solutions each of which is a two-parameter family. The first one corresponds to the dipolar SLE with the drift $\nu$, line 3 in Table 1. The second and the third equations are ‘mirror images’ of each other, as it can be seen from the lines 4 and 5 in the table. They are parametrized by $\xi := \frac{2\beta\sqrt{\kappa}}{8 - \kappa}$ and discussed in details in Section 4.6. The fourth case is given by putting

$$\delta_{-1} = 0, \quad \delta_0 = 2(\sqrt{6}\beta - 3), \quad \delta_1 \in \mathbb{R}, \quad \kappa = 6, \quad \alpha = 0.$$

This is a general form of $\delta$ with $\kappa = 6$ and $\nu = 0.$
3. \( \sigma^H(z) = -\sqrt{\kappa}(1 + z^2) \).

Relation (4.2) reduces to
\[
-2 + \frac{\kappa}{z^2} - 2i\sqrt{\kappa\mu} + \frac{2\alpha - \delta_{-1}}{z} + \\
+ \left( \beta\sqrt{\kappa} - \kappa + 6i\sqrt{\kappa\mu} + \alpha\delta_{-1} - \delta_0 + i\sqrt{\kappa\mu}\delta_0 \right) + \\
z \left( 2i\kappa\mu\delta_{-1} + \alpha\delta_0 - \delta_1 + 2i\sqrt{\kappa\mu}\delta_1 \right) + \\
z^2 \left( -\beta\sqrt{\kappa} + \frac{\kappa}{2} + i\sqrt{\kappa\mu}\delta_0 + \alpha\delta_1 \right) = 0 \\
\iff
\delta_{-1} = 0, \quad \delta_0 = 2(\sqrt{6\beta} - 3), \quad \delta_1 \in \mathbb{R}, \quad \kappa = 6, \quad \alpha = 0.
\]

The first solution is presented in the line 6 of Table 1 where it is again convenient to introduce the parameter \( \nu \) related to the drift in the radial equation. The second solution is
\[
\delta_{-1} = 0, \quad \delta_0 = 2(\sqrt{6\beta} - 3), \quad \delta_1 \in \mathbb{R}, \quad \kappa = 6, \quad \alpha = 0.
\]

This is a general form of \( \delta \) with \( \kappa = 6 \) and \( \nu = 0 \). \( \square \)

4.1. **Chrodal SLE with drift.** It is natural to study this case in the half-plane chart, where
\[
(4.3) \quad \delta^H_c(z) := \frac{2}{z} - \nu, \quad \sigma^H_c(z) := -\sqrt{\kappa}, \quad \nu \in \mathbb{R}.
\]

The form of \( \eta^+ \) can be found from (3.9) by substituting (4.1) as
\[
-\kappa^{\frac{1}{2}} \partial_z \eta^+^H(z) + \mu \cdot 0 = \frac{-i}{z} + i\alpha.
\]

Then
\[
(4.4) \quad \eta^+^H(z) = \frac{i}{\sqrt{\kappa}} \log z - \frac{i\alpha z}{\sqrt{\kappa}} + C^+,
\]

and taking into account that \( \alpha = -\frac{\kappa}{2} \) we obtain
\[
(4.5) \quad \eta^H(z) = \frac{-2}{\sqrt{\kappa}} \arg z - \frac{\nu}{\sqrt{\kappa}} \Im z + C.
\]

Let us present here an explicit form of the evolution of the one-point function \( S_1(z) = \eta(z) \)
\[
M_t^H(z) = (G_t^{-1}, \eta)^H(z) = \frac{-2}{\sqrt{\kappa}} \arg G_t^H(z) - \frac{\nu}{\sqrt{\kappa}} \Im G_t^H(z) + \frac{1}{2\sqrt{\kappa}} \arg G_t^H(z) + C
\]

This expression with \( \nu = 0 \) coincides (up to a constant) with the analogous one from [18, Section 8.5].

Now we need to work with a concrete form of the space \( \mathcal{H}_s \) discussed in Section 2.4. It is convenient to define it in the half-plane chart. In other charts it can be obtained with the rule (2.27). We choose the subspace \( C^\infty_0 \mathcal{H} \) of \( C^\infty \)-smooth functions with compact support in the half-plane chart. The function \( \phi \) defining metric can be, for example, zero in the half-plane chart, \( \phi^\mathcal{H}(z) \equiv 0 \). This choice guarantees that the integrals in (A.9) and (A.11) are well-defined with \( \eta \) as above and \( \Gamma = \Gamma_D \).
4.2. Dipolar SLE with drift. The dipolar SLE equation is usually defined in the strip chart, see (1.9), as

\[(4.6)\]

\[d^{\text{Itô}} G_t^S(z) = \frac{\text{cth} G_t(z)}{2} dt - \sqrt{\tau} d^{\text{Itô}} B_t - \frac{\nu}{2} dt,\]

where we add the drift term \(\frac{\nu}{2} dt\) in the Itô differentials with the same form in terms of Stratonovich

\[(4.7)\]

\[d^S G_t^S(z) = \frac{\text{cth} G_t(z)}{2} dt - \sqrt{\tau} d^S B_t - \frac{\nu}{2} dt\]

due to \(\sigma^S(z)\) is constant. The vector fields \(\delta\) and \(\sigma\) in the strip chart, see (1.9), are

\[\delta^S(z) = 4 \text{cth} \frac{z}{2} - \frac{\nu}{2}, \quad \sigma^S(z) = -\sqrt{\tau}, \quad \nu \in \mathbb{R},\]

The form of \(\delta\) and \(\sigma\) in the half-plane can be obtained by (2.1) as

\[(4.8)\]

\[\delta^H_d(z) := 2 \left( \frac{1}{z} - z \right) - \nu (1 - z^2), \quad \sigma^H_d(z) := -\sqrt{\tau} (1 - z^2), \quad \nu \in \mathbb{R}\]

that possess normalization (2.17) used in Table 1.

Let us first find \(\eta^+, \eta^H, \text{and } M_{c_t}\) in the half-plane chart. The same way as in the previous subsection we calculate

\[-\sqrt{\tau}(1 - z^2) \partial_z \eta^H(z) + \mu \left( -\sqrt{\tau}(1 - z^2) \right)' = \frac{-i}{z} + i\alpha.\]

Taking into account (3.23) and \(\alpha = -\nu/2\) we obtain

\[\eta^H(z) = \frac{i}{\sqrt{\tau}} \log z + \frac{i(k - 6)}{4\sqrt{\tau}} \log(1 - z^2) + \frac{i\nu}{2\sqrt{\tau}} \text{arcth} z + C^+,\]

\[\eta^H(z) = -\frac{2}{\sqrt{\tau}} \arg z - \frac{(k - 6)}{2\sqrt{\tau}} \arg(1 - z^2) - \frac{\nu}{\sqrt{\tau}} \text{Im arcth} z + C,\]

\[M_{c_t}^H(z) = (G_t^{-1})^H(z) = \]

\[-\frac{2}{\sqrt{\tau}} \arg G_t^H(z) - \frac{(k - 6)}{2\sqrt{\tau}} \arg(1 - G_t^H(z)^2) - \frac{\nu}{\sqrt{\tau}} \text{Im arcth} G_t^H(z) + \frac{4}{2\sqrt{\tau}} \arg G_t^H(z) + C\]

The corresponding relations in the strip chart are

\[\delta^S_d(z) = 4 \text{cth} \frac{z}{2} - 2^*, \quad \sigma^S_d(z) = -2\sqrt{\tau}, \quad * \in \mathbb{R},\]

obtained with the help of (2.19). Then

\[\eta^S(z) = \eta^H(\tau_{H,S}(z)) + \frac{k - 4}{2\sqrt{\tau}} \text{Im } \varphi_{H,S}(z),\]
where we used (1.4) and the expression for \( \tau_{H,S}(z) = \tau_{S,H}^{-1}(z) = \psi_H \circ \psi_S^{-1}(z) \) that defines the strip chart (1.9). Alternatively \( \eta^S(z) \) can be found as the solution to (3.9) in the strip chart

\[
\eta^S(z) = -\frac{2}{\sqrt{\kappa}} \arg \text{sh} \frac{z}{2} - \frac{\nu}{2\sqrt{\kappa}} \text{Im} z + C,
\]

\[
M_{t\eta}^S(z) = (G_t^{-1} \eta)^S(z) = -\frac{2}{\sqrt{\kappa}} \arg \text{sh} \frac{G_t^S(z)}{2} - \frac{\nu}{2\sqrt{\kappa}} \text{Im} G_t^S(z) + \frac{\nu - 4}{2\sqrt{\kappa}} \arg G_t^S(z) + C.
\]

The expression under the square root vanishes only at the same points as \( \tau \) where we used (1.4) and the expression for \( \eta^S(z) \).

As for now, we just remark that from the heuristic point of view this is not an essential normalization and define

\[
\phi(t) := \frac{r}{\sqrt{\kappa}} \text{Im} \arg z + C,
\]

The expression for \( \eta^S \) in the strip chart becomes

\[
\Gamma_D^S(z,w) = \Gamma^H_D (\tau_{H,S}(z), \tau_{H,S}(w)) = -\frac{1}{2} \log \frac{\text{sh}(\frac{z-w}{2})\text{sh}(\frac{z-w}{2})}{\text{sh}(\frac{z-w}{2})\text{sh}(\frac{z-w}{2})}.
\]

We remark that \( \eta \) can be presented in a chart-independent form as a function of \( \delta_d \) and \( \sigma_d \) using (2.23) as

\[
\eta = -\frac{2}{\sqrt{\kappa}} \arg \frac{\sigma_d}{\sqrt{\sigma_d^2 - \frac{\kappa}{4} (\delta_d - \frac{\nu}{\sqrt{\kappa}} \sigma_d)^2}} + \frac{\nu}{\sqrt{\kappa}} \text{Im} \text{arcth} \left( \frac{2}{\sqrt{\kappa}} \frac{\alpha_d}{\text{Im} \text{sh} \left( \frac{\alpha_d}{\sqrt{\kappa}} \right)} \right) + C.
\]

The expression under the square root vanishes only at the same points as \( \sigma_d \). As before, the choice of the branch is irrelevant because \( \sigma \) can vanish at infinity only at the boundary, and \( \eta \) is defined up to a constant \( C \).

Now we work with the concrete form of the space \( \mathcal{H}_* \) discussed in Section 2.4. It is convenient to define it in the strip chart. We choose the subspace \( C^\infty_0 \) of \( C^\infty \)-smooth functions with compact support in the strip chart. The function \( \phi \) defining the metric can be, for example, zero in the strip chart, \( \phi^S(z) \equiv 0 \), which guarantees that the integrals in (A.9) and (A.11) are well-defined with \( \eta \) as above and \( \Gamma = \Gamma_D \).

4.3. Radial SLE with drift. The radial SLE equation (2.13) is usually formulated in the unit disk chart. It can be defined with the vector fields (2.12), which admit the form (2.13) in the half-plane chart. By the same reasons as for the dipolar SLE, we can change normalization and define

\[
\sigma^H_r(z) := 2 \left( \frac{1}{z} + z \right) - \nu (1 + z^2), \quad \sigma^H_r(z) := -\sqrt{\kappa} (1 + z^2), \quad \nu \in \mathbb{R}
\]

that coincides with the expressions in Table 1.

Let us give here the expressions for \( \delta, \sigma, \Gamma_D, \eta \) and \( M_{1|1} \) in three different charts: half-plane, logarithmic (see below for the details), and the unit disk using the same method as before. The calculations are similar to the dipolar case. In fact, it is enough to change some signs and replace the hyperbolic functions by trigonometric. In contrast to the dipolar case, \( \eta \) is multiply defined. We discuss this difficulty at the end of this subsection. As for now, we just remark that from the heuristic point of view this is not an essential problem. In any chart \( \eta^\psi(z) \) just changes its value only up to an irrelevant constant after the harmonic continuation about the fixed point of the radial equation.

In the half-plane chart, we have

\[
-\sqrt{\kappa} (1 + z^2) \partial_z \eta^H(z) + \mu \left( -\sqrt{\kappa} (1 + z^2) \right)' = -\frac{i}{z} + i\alpha,
\]

\[
\eta^H(z) = -\frac{2}{\sqrt{\kappa}} \arg z - \frac{(\kappa - 6)}{2\sqrt{\kappa}} \arg (1 + z^2) - \frac{\nu}{\sqrt{\kappa}} \text{Im} \text{arctg} z + C,
\]

as above and \( \delta_D \).
\[ M_t^H(z) = (G_t^{-1} \eta)^H(z) = \]
\[ = \frac{-2}{\sqrt{\kappa}} \arg G_t^H(z) - \frac{(\kappa - 6)}{2\sqrt{\kappa}} \arg(1 + G_t^H(z)^2) - \]
\[ - \frac{\nu}{\sqrt{\kappa}} \Im \arctg z + \frac{\nu}{2\sqrt{\kappa}} \arg G_t^H(z) + C \]

analogously to (4.1).

The unit disk chart is defined in (1.8), and
\[ \eta^D(z) = \frac{-2}{\sqrt{\kappa}} \arg(1 - z) - \frac{\kappa - 6}{2\sqrt{\kappa}} \arg z + \frac{\nu}{2\sqrt{\kappa}} \log|z| + C, \]
\[ M_t^D(z) = (G_t^{-1} \eta)^D(z) = \]
\[ = \frac{-2}{\sqrt{\kappa}} \arg(1 - G_t^D(z)) - \frac{\kappa - 6}{2\sqrt{\kappa}} \arg G_t^D(z) + \frac{\nu}{2\sqrt{\kappa}} \log|G_t^D(z)| + \frac{\kappa - 4}{2\sqrt{\kappa}} \arg G_t^D(z) + C, \]
\[ \Gamma^D(z, w) = \frac{1}{2} \log (z - w)(\bar{z} - \bar{w}). \]

The third chart is called logarithmic, and it is defined by the transition map
\[ (4.11) \quad \tau_{L, L}(z) := e^{iz} : \mathbb{H} \to \mathbb{D}, \quad \tau_{L, D}(z) = \tau_{D, L}(z) = -i \log z. \]

Therefore,
\[ \tau_{H, L}(z) = \tau_{H, D} \circ \tau_{D, L}(z) = \frac{z}{2}, \quad \tau_{H, D}(z) = \frac{1}{2} \arctg z, \]
and \( \psi^L \) is not a global chart map as we used before because there is a point (the origin in the unit-disc chart) which is mapped to infinity. Besides, the function \( \log \) is multivalued and the upper half-plane contains infinite number of identical copies of the radial SLE slit \((\tau_{L, L}(z + 2\pi) = \tau_{L, H}(z))\). The advantage of this chart is that the automorphisms \( H_t[\sigma_r]^L \) induced by \( \sigma_r \) (see [23]) are horizontal translations because \( \sigma_r(z) \) is a real constant (see below). The corresponding relations for the radial SLE in the logarithmic chart can be easily obtained from the dipolar SLE in the strip chart just by replacing the hyperbolic functions by their trigonometric analogs as
\[ \delta^L_r(z) = 4 \tg \frac{2}{z} - 2^\ast, \quad \alpha^L_t(z) = -2\sqrt{2}^\ast, \quad \ast \in \mathbb{R}. \]
\[ \eta^L(z) = \frac{-2}{\sqrt{\kappa}} \arg \sin \frac{z}{2} - \frac{\ast}{2\sqrt{\kappa}} \Im z + C, \]
\[ M_t^L(z) = (G_t^{-1} \eta)^L(z) = \]
\[ = \frac{-2}{\sqrt{\kappa}} \arg \sin \frac{G_t^L(z)}{2} - \frac{\nu}{2\sqrt{\kappa}} \Im G_t^L(z) + \frac{\kappa - 4}{2\sqrt{\kappa}} \arg G_t^L(z) + C. \]
\[ \Gamma^L_D(z, w) = \frac{1}{2} \log \frac{\sin(z - w)}{\sin(\bar{z} - \bar{w})} \sin(\frac{z - w}{2}) \sin(\frac{\bar{z} - \bar{w}}{2}). \]

This relations above coincide up to a constant with the analogous ones established in [19] [21].

We remark again that \( \eta \) can be represented in a chart-independent form as a function of \( \delta_r \) and \( \sigma_r \) with the help of (2.23) by the relation
\[ \eta = \left( \frac{2}{\sqrt{\kappa}} \arg \frac{\sigma_r^\ast}{\sqrt{\sigma_r^2 + \frac{\kappa}{4} (\delta_r - \frac{\nu}{\sqrt{\kappa}} \sigma_r)^2}} \right) + \frac{\nu}{\sqrt{\kappa}} \Im \arctg \frac{2}{\sqrt{\kappa}} \frac{\alpha_r}{f_{\theta r} - \frac{\nu}{\sqrt{\kappa}} \alpha_r} + C. \]
In order to define GFF for radial SLE carefully we need to generalize slightly the above approach. Let \( b \in D \) be a zero point of \( \delta, \) and \( \sigma \) simultaneously inside \( D: \delta^\psi(b) = \sigma^\psi(b) = 0 \) (for any \( \psi \)). We have \( \psi^\psi(b) = 0 \) in the unit disk chart, \( \psi^\dD(b) = i \) in the half-plane chart, and \( \psi^\dD(b) = \infty \) in the logarithmic chart.

Let \( \hat D_b \) be the universal cover of \( D \setminus \{b\} \). Then the logarithmic chart map \( \psi^\dD: \hat D \to \mathbb{H} \) defines a global chart map of \( \hat D_b \). The space \( \mathcal{H}_d[\hat D_b] \) in the logarithmic chart is defined as in Section 2.4 with \( \phi^\dD \equiv 0 \), and we require in addition, that the support is bounded. The last condition guarantees the finiteness of functionals such as \( s \).

The compatibility condition of \( H \) \( \Löwner \) equation to have a fixed point \( b \) defines a global chart map of \( \hat D \) as in Section 2.4 with \( \phi, \psi \) half-plane chart, and \( \hat \kappa \) chordal case and other cases considered in the next two subsections.

Consideration of \( \hat D_b \) instead of \( D \) is possible thanks to a special property of radial \( \Löwner \) equation to have a fixed point \( b \in D \). The branch point \( b \) is in fact eliminated from the domain of definition and the pre-pre-Schwarzian \( \eta \) is well-defined on \( \hat D_b \).

4.4. General remarks. Here we are aimed at explaining why all three cases of \( \eta \) above has the same form for \( \kappa = 6 \) and \( \nu = 0 \). Besides, we explain the relations between the chordal case and other cases considered in the next two subsections.

Let \( G_t \) be a \((\delta, \sigma)-SLE\) driven by \( B_t \), and let and \( \hat G_t \) be a \((\tilde \delta, \tilde \sigma)\)-SLE driven by \( \tilde B_t \) with the same parameter \( \kappa \). Then there exists a stopping time \( \tilde \tau > 0 \), a family of random \( \text{Möbius} \) automorphisms \( M_t: D \to D, \tilde t \in [0, \tilde \tau) \), and a random time reparametrization \( \lambda: [0, \tilde \tau) \to [0, \tau) \) \((\tau := \lambda(\tilde \tau))\), such that

\[
\hat G_t = M_t \circ G_{\lambda(t)}, \quad \tilde t \in [0, \tilde \tau)
\]

and

\[
d\tilde B_t = a_t d\tilde t + \left(\tilde \lambda(\tilde t)\right)^{-\frac{\kappa}{2}} dB_{\lambda(t)}, \quad \tilde t \in [0, \tilde \tau)
\]

for some continuous \( a_t \). In particular, this means that the laws of \( \mathcal{K}_t \) and \( \hat \mathcal{K}_t \) induced by the \((\delta, \sigma)\)-SLE and \((\tilde \delta, \tilde \sigma)\)-SLE correspondingly are absolutely continuous with respect to each other until some stopping time. We proved this fact in [13]. However, it is possible to show a bit more: if \( \nu = 0 \) for both \((\delta, \sigma)\)-SLE and \((\tilde \delta, \tilde \sigma)\)-SLE, then the coefficient \( a_t \) is proportional to \( \kappa - 6 \). Here the drift parameter \( \nu \) is defined by

\[
\nu := \delta_{-1} + 3\sigma_0,
\]

see (1.7). This definition agrees with (1.3), (1.8) and (1.9) and is invariant with respect to (2.15). Since \( \tilde \lambda_{\tilde t}^\frac{\kappa}{2} B_{\lambda(t)} \) agrees in law with \( B_t \), the random laws of \( \mathcal{K}_t \) and \( \hat \mathcal{K}_t \) are identical, not just absolutely continuous as above, at least until some stopping time.

It can be observed that \( \eta \) for the chordal (1.3), dipolar (1.5), and radial (1.10) cases are identical for \( \kappa = 6 \) and \( \nu = 0 \). This is a consequence of the above fact. Special cases of chordal and radial SLEs were considered in [27].

Besides, there are two special situations when \( a_t \) is identically zero for all values of \( \kappa > 0 \), not only for \( \kappa = 6 \) as above. In order to study them, let us consider the chordal SLE \( G_t \), see [21], and a differentiable time reparametrization \( \lambda \), which possesses property (B.8).

Set

\[
\tilde G_t := s_{c_t} \circ G_{\lambda(t)},
\]

where \( s_c: D \to D \) is the scaling flow \((s_c(z) = e^{-c}z, \ c \in \mathbb{R})\). In the half-plane chart we have

\[
(4.12) \quad \tilde G^H_t(z) = e^{-ct}G^H_{\lambda(t)}(z).
\]
The Stratonivich differential of $\tilde{G}^H_t(z)$ is

$$d^S \tilde{G}^H_t(z) = (d^S e^{-\xi t}) G^H_{\lambda t}(z) + e^{-\xi t} d^S \tilde{G}^H_t(z) =$$

$$= (d^S e^{-\xi t}) G^H_{\lambda t}(z) + e^{-\xi t} \lambda_t \left( \frac{2}{G^H_{\lambda t}(z)} d\tilde{t} - \sqrt{\kappa} d^S B_{\lambda t} \right)$$

Due to \((B.7)\), we have to assume that

$$e^{-\xi t} \lambda_t \equiv \lambda_t^\frac{1}{2},$$

in order to have an autonomous equation. So

$$e^{-\xi t} = \lambda_t^{-\frac{1}{2}},$$

and, consequently,

$$d^S e^{-\xi t} = -\frac{1}{2} e^{-3\xi t} a_t d\tilde{t} - \frac{1}{2} e^{-3\xi t} b_t d^S \tilde{B}$$

where we used \((B.6)\). Eventually, we conclude that

\[(4.13)\]

$$d^S \tilde{G}^H_t(z) = \left( -\frac{1}{2} e^{-3\xi t} a_t d\tilde{t} - \frac{1}{2} e^{-3\xi t} b_t d^S \tilde{B} \right) e^{\xi t} \tilde{G}^H_t(z) + \frac{2}{G^H_{\lambda t}(z)} d\tilde{t} - \sqrt{\kappa} d^S \tilde{B} + \frac{1}{4} \sqrt{\kappa} e^{-2\xi t} b_t d\tilde{t}. $$

In order to have time independent coefficients we assume that $a_t$ and $b_t$ are proportional to $e^{2\xi_t}$. Hence, define $\xi \in \mathbb{R}$ by

$$a_t = -4\xi e^{2\xi t}. $$

Without lost of generality, we can assume that $b_t$ is of the following three possible forms

1. $b_t = 0,$
2. $b_t = 4 \sqrt{\kappa} e^{2\xi t},$
3. $b_t = -4 \sqrt{\kappa} e^{2\xi t},$

because all other choices can be reduced to these three by \((2.13)\). The first case is considered in Section \[4.5\]. Other two cases are discussed in Section \[4.6\].

4.5. Chordal SLE with fixed time reparametrization. Let $\xi \in (\infty, -\infty) \setminus \{0\}$, and let $G_t$ be a chordal stochastic flow, i.e., the chordal SLE \((4.3)\). Define

$$\tilde{G}^H_t(z) = e^{2\xi t} G^H_{\lambda t}(z)$$

in the half-plane chart and assume that

$$\lambda(\tilde{t}) := \frac{1 - e^{-4\xi \tilde{t}}}{4 \xi};$$

$$\lambda : [0, +\infty) \rightarrow [0, (4\xi)^{-1}], \quad \xi > 0;$$

$$\lambda : [0, +\infty) \rightarrow [0, +\infty), \quad \xi < 0.$$ 

This choice of $\lambda$ corresponds to $c_{\tilde{t}} = -2\xi_{\tilde{t}}$, in the previous subsection. We remark, that the time reparametrization here is not random.

The flow $\tilde{G}_t$ satisfies the autonomous equation \((2.9)\) with

\[(4.14)\]

$$\delta^H(z) = \frac{2}{z} + 2\xi z, \quad \sigma^H(z) = -\sqrt{\kappa},$$

which are the vector fields from the second string of Table \[4\] and a special case of \(1.13\) with $a_t = -4\xi e^{2\xi t}$ and $b_t = 0$.

There is a common zero of $\delta$ and $\sigma$ at infinity in the half-plane chart, so infinity is a stable point $\tilde{G}^H_t : \infty \rightarrow \infty$. But in contrast to the chordal case the coefficient at $z^{-1}$ in the Laurent series is not 1 but $e^{2\xi t}$. The vector field $\delta$ is of radial type if $\xi > 0$, and of
dipolar type if $\xi < 0$. It is remarkable that if $\xi < 0$, then the equation induces exactly the same measure as the chordal stochastic flow but with a different time parametrization. If $\xi > 0$ the measures also coincide when the chordal stochastic flow is stopped at the time $t = (4\xi)^{-1}$.

By the reasons described above it is natural to expect that the GFF coupled with such kind of $(\delta, \sigma)$-SLE is the same as in the chordal case, because it is supposed to induce the same random law of the flow lines. Indeed, $\sigma$ from (4.14) coincides with that from the chordal case, hence, $\eta$ defined by (3.10), with $\alpha = 0$ (see the table) also coincides with (4.5) with $\nu = 0$. Thus, the martingales are the same as in the chordal case.

4.6. SLE with one fixed boundary point. Let the vector fields $\delta$ and $\sigma$ be defined by the 5th and the 6th strings of Table 1. There are two ‘mirror’ cases. The $(\delta, \sigma)$-SLE denoted here by $G_t$ is characterized by the stable point at $z = 1$ (the 5th case) or $z = -1$ (the 6th case) in the half-plane chart. We will consider only the first (the 5th string) case, the second (the 6th string) is similar.

We will show below that this $(\delta, \sigma)$-SLE coincides with the chordal SLE up to a random time reparametrization for all values of $\kappa > 0$. Let us apply a Möbius transform $r_c: \mathcal{D} \rightarrow \mathcal{D}$ defined in (2.18) with $c = -1$

$$r_{-1}^H(z) = \frac{z}{1 + z}.$$ 

In the half-plane chart, it maps the stable point $z = 1$ to infinity keeping the origin and the normalization (2.17) unchanged. It results in

$$\tilde{G}_t := r_{-1} \circ G_t \circ r_{-1}^{-1},$$

$$r_{-1} \cdot \delta^H(z) = \tilde{\delta}^H(z) = \frac{2}{z} + \kappa + 2\xi z,$$

$$r_{-1} \cdot \sigma^H(z) = \tilde{\sigma}^H(z) = -\sqrt{\kappa}(1 + 2z),$$

and the equation for $\tilde{G}_t$ becomes

$$(4.15) \quad d^S \tilde{G}_t^H(z) = \left( \frac{2}{G_t^H(z)} + \kappa + 2\xi \tilde{G}_t^H(z) \right) dt - \sqrt{\kappa} \left( 1 + 2\tilde{G}_t^H(z) \right) d^S B_t,$$

which is a special case of (4.13) with $a_i = -4\xi \bar{l}$ and $b_i = 4\sqrt{\kappa} e^{\bar{i}}$. In other words, the relation (4.15) can be obtained from (4.12) with $c_i = -2\xi \bar{l} + 2\sqrt{\kappa} \bar{B}_t$ under the random time reparametrization $\lambda_t = e^{4\xi \bar{l} - 4\sqrt{\kappa} \bar{B}_t}$.

It is remarkable that the subsurface $\tilde{I} \subset \mathcal{D}$ defined in the half-plane chart as

$$\psi^H(\tilde{I}) = \{ z \in \mathbb{H} : \text{Re}(z) > \frac{1}{2} \}$$

is invariant $(G_t^{-1}(\mathcal{D}) \subset \mathcal{D})$ if and only if $\xi \geq \kappa$. In order to see this, it is enough to calculate the real parts of

$$\tilde{\delta}^H(z) = \frac{2}{z} + \kappa + 2\xi z,$$

$$\tilde{\sigma}^H(z) = -\sqrt{\kappa}(1 + 2z),$$
which are actually the horizontal components of the vector fields at the boundary of \( \psi^H(I) \) in \( \mathbb{H} \), \( \{ z \in \mathbb{H} : \text{Re}(z) = -\frac{1}{2} \} \),

\[
\text{Re} \left( f_1^H \left( -\frac{1}{2} + ih \right) \right) = \text{Re} \left( \frac{2}{-\frac{1}{2} + ih} + \gamma + 2 \left( -\frac{1}{2} + ih \right) \right) = -\frac{1}{h^2 + \frac{1}{4}} + \kappa - \xi,
\]

\[
\text{Re} \left( \alpha_1^H \left( -\frac{1}{2} + ih \right) \right) = \text{Re} \left( \sqrt{\tau} \left( 1 + 2 \left( -\frac{1}{2} + ih \right) \right) \right) = 0, \quad h > 0.
\]

The first number is negative for all values of \( h \) if and only if \( \xi \geq \kappa \).

We remark that the \( r^{-1} \)-transform has the invariant subsurface \( I := r^{-1}(I) \subset \mathcal{D} \) for the \( (\delta, \sigma) \)-SLE above, which is an upper half of the unit disk

\[
\psi^H(I) = \{ z \in \mathbb{H} : |z| < 1 \}
\]

Similarly to the previous subsection it is reasonable to expect that the GFF coupled with this \( (\tilde{\delta}, \tilde{\sigma}) \)-SLE is the same as in the chordal case, because it is supposed to induce the same random law of the flow lines. Indeed, the solution to \( (3.9) \), with \( \sigma \) and \( \alpha \) as in the 5th string of the table, is

\[
\eta^H(z) = \frac{i}{\sqrt{\kappa}} \log z + \frac{\kappa - 6}{2\sqrt{\kappa}} \text{arg}(1 - z) + C^+.
\]

Thus,

\[
\eta^H(z) = \frac{-2}{\sqrt{\kappa}} \text{arg} z - \frac{\kappa - 6}{\sqrt{\kappa}} \text{arg}(1 - z) + C.
\]

After the \( r^{-1} \)-transform for \( \tilde{\delta} \) and \( \tilde{\sigma} \), we have

\[
\eta^H(z) = \frac{-2}{\sqrt{\kappa}} \text{arg} z + C.
\]

The last relation coincides with \( (4.5) \) with \( \nu = 0 \). We remind that \( \Gamma_0 \) is invariant under Möbius transforms, in particular, under \( r^{-1} \).

### 5. Coupling of GFF with Dirichlet-Neumann boundary conditions

We assume in this chapter that \( \Gamma = \Gamma_{DN} \), see Example 4 which becomes

\[
\Gamma_{DN}^S(z, w) = -\frac{1}{2} \log \frac{\text{th} \frac{z-w}{4} + \text{th} \frac{z+w}{4}}{\text{th} \frac{z-w}{4} + \text{th} \frac{z+w}{4}}, \quad z, w \in S := \{ z : 0 < \text{Im} z < \beta \}.
\]

in the strip chart \( (1.9) \). It is exactly Green’s function used in \( [17] \) (it is also a special case of \( [16] \)).

The function \( \Gamma_{DN}^S \) satisfies the boundary conditions

\[
\Gamma_{DN}^S(x, w) \big|_{x \in \mathbb{R}} = 0, \quad \partial_y \Gamma_{DN}^S(x + iy, w) \big|_{x \in \mathbb{R}, y = \pi} = 0,
\]

the symmetry property, and

\[
\mathcal{L}_\sigma \Gamma_{DN}(z, w) = 0.
\]

The coupling of GFF with this \( \Gamma \) to the dipolar SLE is geometrically motivated. We also require that both zeros of \( \delta \) and \( \sigma \) are at the same boundary points where \( \Gamma_{DN} \) changes the boundary conditions from Dirichlet to Neumann. In the strip chart these points are \( \pm \infty \).

**Proposition 4.** Let the vector fields \( \delta \) and \( \sigma \) be as in \( (4.8) \), let \( \Gamma = \Gamma_{DN} \), and let \( \eta \) be a pre-pre-Schwarzian. Then the coupling is possible only for \( \kappa = 4 \) and \( \nu = 0 \).
Proof. We use Lemma 2 in the strip chart. From (6.2) we obtain that
\[ \Gamma^+_{DN}(z, w) = -\frac{1}{2} \log \frac{z-w}{4}, \quad \Gamma^+_{DN}(z, \bar{w}) = -\frac{1}{2} \log \frac{z-\bar{w}}{4}, \]
and relations (3.16) hold. From (3.17) we find that
\[ j^+(z) = -\frac{i}{\sinh \frac{z}{2}} + i\alpha, \quad \alpha \in \mathbb{R}. \]
Substituting (5.2) in (3.13) gives
\[ -i(\beta \sqrt{\kappa - \nu} \sinh \frac{z}{2} + \nu \cosh \frac{z}{2} - 2i \sqrt{\kappa - \nu} \sinh \frac{z}{2} \left(4 \sinh^2 \frac{z}{2} + (\nu - 4)\right)) = 0, \]
which is possible only for \( \kappa = 4, \nu = 0, \beta = 0 \) and \( \mu = 0 \), where the latter agrees with (3.23). □

From (3.9) we obtain that
\[ \eta^+(z) = \frac{i}{2} \log \frac{z}{4} + C, \]
and
\[ \eta^-(z) = -2 \arctan \frac{z}{4} + C. \]

We also present here the relations in the half-plane chart
\[ \eta^H(z) = -2 \arg \frac{z}{1 + \sqrt{1 - z^2}} + C, \]
\[ \Gamma^H_{DN}(z) = -\frac{1}{2} \log \left(\frac{z-w}{\bar{z}-\bar{w}}(1-\bar{w})(1-w)\right) \left(1-\bar{z}w + \sqrt{1-\bar{w}^2}\right) \left(1-z\bar{w} + \sqrt{1-w^2}\right) \left(1-z\bar{w} + \sqrt{1-z^2}\right) \left(1-\bar{z}\bar{w} + \sqrt{1-\bar{w}^2}\right). \]

The pre-pre-Schwarzian \( \eta \) is scalar in this case, and in its chart-independent form is
\[ \eta = -\arg \frac{\sqrt{\kappa}(\delta - \nu \sigma) + \sqrt{\nu}(\delta - \nu \sigma)^2 - \sigma^2}{\sigma}. \]

6. COUPLING OF TWISTED GFF

This model is similar to the previous one. As it will be shown below, it is enough at the algebraic level to replace formally all hyperbolic functions in the dipolar case in the strip chart \( S \) by the corresponding trigonometric functions in order to obtain the relations for the radial SLE in the logarithmic chart \( L \). But at the analytic level, we have to consider the correlation functions which are doubly defined on \( D \) and change their sign after the analytic continuation about the center point. This construction was considered before as we were informed by Num-Gyu Kang. [21]

We have to generalize slightly the general approach similarly to Section 4.3 considering the double cover \( D^\pm_b \) instead of the infinitely ramified cover of \( D \setminus \{b\} \). Let us define the space \( H_s[D^\pm_b] \) of test functions \( f: D^\pm_b \rightarrow \mathbb{R} \) as in Section 2.4 with \( \phi^\pm(z) \equiv 0 \) and with an extra condition \( f(z_1) = -f(z_2) \), where \( z_1 \) and \( z_2 \) are two points of \( D^\pm_b \) corresponding to the same point of \( D \setminus \{b\} \). Thus, in the logarithmic chart, we have
\[ f_L(z) = f^L(z + 4\pi k) = -f^L(z + 2\pi k), \quad k \in \mathbb{Z}, \quad z \in \mathbb{H}. \]
Such functions are \( 4\pi \)-periodic and \( 2\pi \)-antiperiodic. In particular, \( f_L \) is not of compact support, but we require in addition that
\[ \sup \text{Im} \{z \in \mathbb{H} : f^L(z) \neq 0\} < \infty \]
in order to maintain compatibility. In some sense, the ‘value’ of $\Phi_{tw}$ changes its sign after horizontal translation by $\pi$.

The twisted Gaussian free field $\Phi_{tw}$ is defined similarly to the usual one but taking values in $D_b^\pm$.

In this section, we define $\Gamma$ by

$$\Gamma_{tw}^L(z, w) = -\frac{1}{2} \log \frac{\tg \frac{z-w}{4} \tg \frac{\bar{z}-\bar{w}}{4}}{\tg \frac{\bar{z}-w}{4} \tg \frac{\bar{z}-w}{4}}, \quad z, w \in \mathbb{H}. \quad (6.2)$$

in the logarithmic chart. Observe that

$$\Gamma_{tw}^L(z, w) = \Gamma_{tw}^L(z + 4\pi k, w) = -\Gamma_{tw}^L(z + 2\pi k, w), \quad k \in \mathbb{Z}. \quad$$

In the unit disk chart the covariance $\Gamma_{tw}^D$ admits the form

$$\Gamma_{tw}^D(z, w) = -\frac{1}{2} \log \frac{(\sqrt{z} - \sqrt{w})(\sqrt{\bar{z}} - \sqrt{\bar{w}})(\sqrt{\bar{z}} + \sqrt{\bar{w}})(\sqrt{z} + \sqrt{w})}{(\sqrt{z} + \sqrt{w})(\sqrt{\bar{z}} + \sqrt{\bar{w}})(\sqrt{\bar{z}} - \sqrt{\bar{w}})(\sqrt{z} - \sqrt{w})},$$

or in the half-plane chart,

$$\Gamma_{tw}^H(z) = -\frac{1}{2} \log \left( \frac{(z - w)(\bar{z} - \bar{w})(1 + \bar{z}w + \sqrt{1 + \bar{z}^2}(\sqrt{1 + w^2})}{(\bar{z} - w)(z - w)(1 + zw + \sqrt{1 + z^2}(\sqrt{1 + \bar{w}^2}))}(1 + \bar{z}w + \sqrt{1 + z^2}(\sqrt{1 + \bar{w}^2}) (1 + zw + \sqrt{1 + \bar{z}^2}(\sqrt{1 + w^2}) \right).$$

It is doubly defined because of the square root, and the analytic continuation about the center changes its sign.

The covariance $\Gamma_{tw}^L$ satisfies the Dirichlet boundary conditions and tends to zero as one of the variables tends to the center point $b$ (or $\infty$ in the $\mathbb{L}$ chart)

$$\Gamma_{tw}^L(x, w) \big|_{x \in \mathbb{R}} = 0, \quad \lim_{y \to +\infty} \Gamma_{tw}^L(x + iy, w) = 0, \quad x \in \mathbb{R}, \quad w \in \mathbb{H};$$

$$\Gamma_{tw}^D(z, w) \big|_{|z| = 1} = 0, \quad \lim_{z \to 0} \Gamma_{tw}^D(z, w) = 0, \quad w \in \mathbb{D}.$$ 

The $\sigma$-symmetry property

$$\mathcal{L}_\sigma, \Gamma_{tw}(z, w) = 0$$

holds.

As we will see below, $\eta$ also possesses property (6.1). Thus, the construction of the level (flow) lines can be performed for both layers simultaneously and the lines will be identical. In particular, this means that the line can turn around the central point and appears in the second layer but can not intersect itself. This agrees with the property of the SLE slit which evoids self-intersections.

Similarly to the dipolar case in the previous section the following proposition can be proved.

**Proposition 5.** Let the vector fields $\delta$ and $\sigma$ be as in (4.3), let $\Gamma = \Gamma_{tw}$, and let $\eta$ be a pre-pre-Schwarzian. Then the coupling is possible only for $\kappa = 4$ and $\nu = 0$.

The proof in the logarithmic chart actually repeats the proof of Proposition 4.

We give here the expressions for $\eta$ in the logarithmic, unit-disk and half-plane charts:

$$\eta^L(z) = -2 \arg \tg \frac{4}{z} + C.$$ 

$$\eta^D(z) = -2 \arg \frac{1 - \sqrt{z}}{1 + \sqrt{z}} + C = 4 \Im \arctgh \sqrt{z} + C.$$ 

$$\eta^H(z) = -2 \arg \frac{z}{1 + \sqrt{1 + z^2}} + C.$$ 

From this relation it is clear that $\eta$ is antiperiodic.
The pre-pre-Schwarzian \( \eta \) is scalar in this case and its chart-independent form becomes

\[
\eta = -\frac{2}{\sqrt{\kappa}} \arg \frac{\sqrt{\kappa} (\delta - \frac{\nu}{\kappa} \sigma) + \sqrt{\frac{\nu}{4} (\delta - \frac{\nu}{\kappa} \sigma)^2 + \sigma^2}}{\sigma}.
\]

We defined the linear functional \( \Phi_{tw} \) on the space of antiperiodic functions before, however, such functional can be also defined on the space of functions with bounded support in the logarithmic chart. Thus, we can use the same space \( \mathcal{H}_4 \) as in Section 4.3.

**Perspectives**

1. The coupling with the reverse \((\delta, \sigma)\)-SLE can also be established using the same classification as in Table 1.

2. We did not prove in this paper but our experience shows that \( \Gamma \) transforms as a scalar, see (2.33), and that \( \eta \) is a pre-pre-Schwarzian. It would be useful to prove this.

3. The pre-pre-Schwarzian rule (2.20) is motivated by the local geometry of SLE curves [29]. In principle, one can consider alternative rules. Moreover, the scalar behaviour of \( \Gamma \) can also be relaxed because the harmonic part \( H^\psi(z, w) \) can transform in many ways. Such more general coupling is intrinsic and can be thought of as a generalization of the coupling in Sections 3 and 6 for arbitrary \( \kappa \).

4. We considered only the simplest case of one Gaussian free field. It would be interesting to examine tuples of \( \Phi_i \), \( i = 1, 2, \ldots n \) which transform into non-trivial combinations \( \Phi_i = G_{i1}[\Phi_1, \Phi_2, \ldots \Phi_n] \) under conformal transforms \( G \).

5. The Bochner-Minols Theorem [3] suggests to consider not only free fields, but for example, some polynomial combinations in the exponential of (2.39). In particular, the quartic functional corresponds to conformal field theories related to 2-to-2 scattering of particles in dimension two.

**Acknowledgement.** The authors gratefully acknowledge many useful and inspiring conversations with Nam-Gyu Kang and Georgy Ivanov.

**APPENDIX A. TECHNICAL REMARKS**

In this appendix section, we prove some technical propositions needed in the proof of Theorem 1.

Consider an Itô process \( \{X_t\}_{t \in [0, +\infty)} \) such that

\[
d^{10} X_t = a_t dt + b_t \, d^{10} B_t, \quad t \in [0, +\infty),
\]

for some continuous processes \( \{a_t\}_{t \in [0, +\infty)} \) and \( \{b_t\}_{t \in [0, +\infty)} \). We denote by \( \{X_{t\wedge T}\}_{t \in [0, +\infty)} \) the stopped process by a stopping time \( T \). It satisfies the following SDE:

\[
d^{10} X_{t\wedge T} = \theta(T - t)a_t dt + \theta(T - t)b_t \, d^{10} B_t, \quad t \in [0, +\infty),
\]

where

\[
\theta(t) := \begin{cases} 
0 & \text{if } t \leq 0 \\
1 & \text{if } t > 0.
\end{cases}
\]

If \( \{X_t\}_{t \in [0, +\infty)} \) is a local martingale (\( a_t = 0 \), \( t \in [0, +\infty) \)), then \( \{X_{t\wedge T}\}_{t \in [0, +\infty)} \) is also a local martingale.

We consider below the stopped processes \( Y(G_{t\wedge T})\) instead of \( \{Y(G_t)\}_{t \in [0, +\infty)} \) for some functions \( Y : \mathcal{G} \to \mathbb{R} \), and the corresponding Itô SDE. In order to make the relations less cluttered, we usually drop the terms ‘\( \ldots \wedge T[f] \)’ and \( \theta(T - t) \). However, in the places where it is essential to remember them, e.g., the proof of Theorem 1, we specify the stopping times explicitly.
Define the diffusion operator
\begin{equation}
\mathcal{A} := \mathcal{L}_\delta + \frac{1}{2} \mathcal{L}_\sigma^2.
\end{equation}
and consider how a regular pre-pre-Schwarzian \( \eta \) changes under the random evolution \( G_t \).

We also define the stopping time \( \mathbb{T} \).
Then, Itô's differential of the process is given by
\begin{equation}
\mathcal{L}_\sigma \eta \text{ and } \mathcal{L}_\delta \eta \text{ are well-defined. Then}
\end{equation}

\begin{equation}
\frac{d}{dt} G_t^{-1} \eta^G(z) = G_t^{-1} \left( \mathcal{A} \eta^G(z) \ dt + \mathcal{L}_\sigma \eta^G(z) \ dt \right).
\end{equation}

Proof of Proposition 6. Let \( \{G_t\} \) be a \((\delta, \sigma)\)-SLE.

1. Let \( \eta \) be a regular pre-pre-Schwarzian such that the Lie derivatives \( \mathcal{L}_\sigma \eta \) are valid not only for pre-pre-Schwarzians but, for instance, for vector fields, and even more generally, for assignments whose transformation rules contain an arbitrary finite number of derivatives at a finite number of points. To this end let us prove the following lemma.

Lemma 3. Let \( X^i(t) (i = 1, 2, \ldots, n) \) be a finite collection of stochastic processes defined by the following system of equations in the Stratonovich form
\begin{equation}
d\mathbb{S} X^i_t = \alpha^i(X_t) dt + \beta^i(X_t) d\mathbb{S} B_t,
\end{equation}
for some fixed functions \( \alpha, \beta: \mathbb{R}^n \to \mathbb{R}^n \). Let us define \( Y^i_s, Z^i_s \) as the solution to the initial value problem
\begin{equation}
Y^i_s = \alpha^i(Y_s), \quad Y^i_0 = 0,
\end{equation}
\begin{equation}
Z^i_s = \beta^i(Z_s), \quad Z^i_0 = 0,
\end{equation}
in some neighbourhood of \( s = 0 \). Let also \( F: \mathbb{R}^n \to \mathbb{C} \) be a twice-differentiable function. Then, Itô’s differential of \( F(X_t) \) can be written in the following form
\begin{equation}
\frac{d}{dt} F(X_t) = \frac{\partial}{\partial s} F(X_t + Y_s) dt + \frac{\partial}{\partial s} F(X_t + Z_s) dt + \frac{1}{2} \frac{\partial^2}{\partial s^2} F(X_t + Z_s) dt \bigg|_{s=0}.
\end{equation}

Proof. The direct calculation of the right-hand side of (A.6) gives
\begin{equation}
F_t'(X_t) \left( \alpha^i(X_t) + \frac{1}{2} \beta^i(X_t) \right) dt + F_t'(X_t) \beta^i(X_t) d\mathbb{S} B_t + \frac{1}{2} F_t''(X_t) \beta^i(X_t) \beta^j(X_t) dt,
\end{equation}
which is indeed Itô’s differential of \( F(X_t) \). We employed summation over repeated indices and used the notation \( F_t'(X) := \frac{\partial}{\partial X^i} F(X) \).

Proof of Proposition 6. We use the lemma above. Let \( n = 4 \), and let us define a vector valued linear map \( \{x\} \) for an analytic function \( x(z) \) as
\begin{equation}
x(z) := \{ \text{Re } x(z), \text{Im } x(z), \text{Re } x'(z), \text{Im } x'(z) \}.
\end{equation}
For example,
\begin{align*}
X_t &:= \{ G_1^\psi(t) \} = \{ \text{Re } G_i(z), \text{Im } G_i(z), \text{Re } G'_i(z), \text{Im } G'_i(z) \}.
\end{align*}
Then a similar observation for other terms in (A.8) implies that Proposition 7.

Under the conditions of Proposition 6 the following holds:

proposition above, the classical Itô formula, and the stochastic Fubini theorem.

propositions are special cases required for this paper. They are consequences of the

the authors are not aware of similar results for nonlinear functionals. The following

\[ H_t \] over \( t \in [0, \infty) \).

But (A.8)

From (2.9) we have

\[ \partial \frac{\partial F(X_s + Y_s)}{\partial s} \bigg|_{s=0, t=0} = \frac{\partial}{\partial s} F\{ z + H_s[\delta]\psi(z) - z \} \bigg|_{s=0} = \frac{\partial}{\partial s} \{ H_s[\delta]^{-1} \psi(z) \} \bigg|_{s=0} = \mathcal{L}_\delta \psi(z). \]

A similar observation for other terms in (A.8) implies that

\[
d\hat{G}_t^{-1} \eta\psi(z)_{t=0} = \mathcal{L}_\delta \eta\psi(z) dt + \mathcal{L}_\sigma \eta\psi(z) d\hat{B}_t + \frac{1}{2} \mathcal{L}_\sigma^2 \eta\psi(z) dt = A \eta\psi(z) dt + \mathcal{L}_\sigma \eta\psi(z) d\hat{B}_t.
\]

For \( t > 0 \) we conclude that

\[
d\hat{G}_t^{-1} \eta\psi(z) = d\hat{G}_t^{-1} \left( \hat{G}_{t_0} \circ G_{t_0} \right)^{-1} \eta\psi(z) = d\hat{G}_t^{-1} \left( \hat{G}_{t_0} \circ \hat{G}_{t_0} \right)^{-1} \eta\psi(z)_{t_0=t} = G_t^{-1} \eta \psi(z)_{s=0} = \mathcal{L}_\sigma \eta\psi(z) (A \eta\psi(z) dt + \mathcal{L}_\sigma \eta\psi(z) d\hat{B}_t).
\]

The proof of (A.3) is analogous. The only difference is that we do not have the pre-pre-Schwarzian terms with the derivatives but there are two points \( z \) and \( w \). We can assume

\[ \{x\} := \{ \text{Re } x(z), \text{Im } x(z), \text{Re } x(w), \text{Im } x(w) \} \]

instead of (A.7) and the remaining part of the proof is the same. \( \square \)

We will obtain below the Itô differential of \( G_t^{-1} \eta[f] \) and \( G_t^{-1} \Gamma[f, g] \) for \( (\delta, \sigma) \)-SLE \( \{ G_t \}_{t \in [0, \infty)} \) and \( f, g \in \mathcal{H} \). To this end we need the Itô formula for nonlinear functionals over \( \mathcal{H} \). For linear functionals on the Schwartz space this has been shown in [22]. However, the authors are not aware of similar results for nonlinear functionals. The following propositions are special cases required for this paper. They are consequences of the proposition above, the classical Itô formula, and the stochastic Fubini theorem.

**Proposition 7.** Under the conditions of Proposition 6 the following holds:
1. The Itô differential is interchangeable with the integration over $\mathcal{D}$. Namely,

$$
\int_{\psi(\text{supp } f)} G_t^{-1} \eta^\psi(z) f^\psi(z) l(dz) = \int_{\psi(\text{supp } f)} G_t^{-1} A \eta^\psi(z) f^\psi(z) l(dz) dt + \int_{\psi(\text{supp } f)} G_t^{-1} L_\sigma \eta^\psi(z) f^\psi(z) l(dz) d\text{Itô } B_t.
$$

(A.9)

An equivalent shorter formulation is

$$
\text{d}\text{Itô } G_t^{-1} \eta[f] = G_t^{-1} A \eta[f] dt + G_t^{-1} L_\sigma \eta[f] d\text{Itô } B_t.
$$

(A.10)

2. The Itô differential is interchangeable with the double integration over $\mathcal{D}$, namely,

$$
\int_{\psi(\text{supp } f)} \int_{\psi(\text{supp } f)} G_t^{-1} \Gamma(x, y) f^\psi(x) f^\psi(y) l(dx) l(dy) = \int_{\psi(\text{supp } f)} \int_{\psi(\text{supp } f)} G_t^{-1} A \Gamma(x, y) f^\psi(x) f^\psi(y) l(dx) l(dy) dt + \int_{\psi(\text{supp } f)} \int_{\psi(\text{supp } f)} G_t^{-1} L_\sigma \Gamma(x, y) f^\psi(x) f^\psi(y) l(dx) l(dy) d\text{Itô } B_t.
$$

(A.11)

An equivalent shorter formulation is

$$
\text{d}\text{Itô } G_t^{-1} \Gamma[f, g] = G_t^{-1} A \Gamma[f, g] dt + G_t^{-1} L_\sigma \Gamma[f, g] d\text{Itô } B_t.
$$

Proof. The relation (A.9) in the integral form becomes

$$
\int_{\psi(\text{supp } f)} G_t^{-1} \eta^\psi(z) f^\psi(z) l(dz) = \eta[f] + \int_0^t \int_{\psi(\text{supp } f)} G_t^{-1} A \eta^\psi(z) f^\psi(z) l(dz) d\tau + \int_0^t \int_{\psi(\text{supp } f)} G_t^{-1} L_\sigma \eta^\psi(z) f^\psi(z) l(dz) d\text{Itô } B_\tau.
$$

The order of the Itô and the Lebesgue integrals can be changed using the stochastic Fubini theorem, see, for example [25]. It is enough now to use (A.2) to obtain (A.9).

The proof of (A.10) is analogous.

\[ \square \]

Proposition 8. Let

$$
\hat{\phi}[f] = \exp (W[f]), \quad W[f] := \frac{1}{2} \Gamma[f, f] + \eta[f].
$$

Then $G_t^{-1} \hat{\phi}[f]$ is an Itô process defined by the integral

(A.12)

$$
G_t^{-1} \hat{\phi}[f] = \int_0^t \exp \left( G_{\tau}^{-1} W[f] \right) \left( G_{\tau}^{-1} A W[f] d\tau + G_{\tau}^{-1} L_\sigma W[f] d\text{Itô } B_\tau + \frac{1}{2} \left( G_{\tau}^{-1} L_\sigma W[f] \right)^2 d\tau \right).
$$
Proof. The stochastic process $G_t^{-1} \ast W^t[f]$ has the integral form

$$G_t^{-1} \ast W^t[f] = \frac{1}{2} G_t^{-1} \ast \Gamma^t[f, f] + G_t^{-1} \ast \eta^t[f] =$$

$$= \int_{\psi(supp \, f)} G_t^{-1} \ast \Gamma(z, w) f^t(z) f^t(w) l(dz) l(dw) + \int_{\psi(supp \, f)} G_t^{-1} \ast \eta^t(z) f^t(z) l(dw).$$

due to Proposition 7. In terms of the Itô differentials it becomes

$$d^{Itô} G_t^{-1} \ast W^t[f] = G_t^{-1} \ast A W^t[f] dt + G_t^{-1} \ast \mathcal{L}_\sigma W^t[f] d^{Itô} B_t.$$ 

In order to obtain the exponential function we can just use Itô’s lemma

$$d^{Itô} G_t^{-1} \ast \exp \left( W^t[f] \right) = d^{Itô} \exp \left( G_t^{-1} \ast W^t[f] \right) =$$

$$= \exp \left( G_t^{-1} \ast W^t[f] \right) \left( G_t^{-1} \ast A W^t[f] dt + G_t^{-1} \ast \mathcal{L}_\sigma W^t[f] d^{Itô} B_t + \frac{1}{2} \left( G_t^{-1} \ast \mathcal{L}_\sigma W^t[f] \right)^2 dt \right).$$

\[\square\]

**Appendix B. Some formulas from stochastic calculus**

We refer to [9], [30], and [25] for the definitions and properties of the Itô and Stratonovich calculus and use the following relation between the Itô and Stratonovich integrals

$$\int_0^T F(x_t, t) d^S B_t = \int_0^T F(x_t, t) d^{Itô} B_t + \frac{1}{2} \int_0^T b_t \partial_1 F(x_t, t) dt.$$ 

The latter item can also be expressed in terms of the covariance

$$\int_0^T b_t \partial_1 F(x_t, t) dt = \langle F(x_T), B_t \rangle.$$ 

In order to obtain (2.10) from (2.9), let us assume

$$x_t := G_t(z), \quad b_t := \sigma(G_t(z)), \quad F(x_t, t) := \sigma(x_t) = \sigma(G_t(z)).$$ 

Then

$$\partial_1 F(x_t, t) = \sigma'(G_t(z)),$$ 

and

$$\int_0^T \sigma(G_t(z)) d^S B_t = \int_0^T \sigma(G_t(z)) d^{Itô} B_t + \frac{1}{2} \int_0^T \sigma(G_t(z)) \sigma'(G_t(z)) dt.$$ 

It is enough now to add $\int_0^T \delta(G_t(z)) dt$ to both parts to obtain the right-hand sides of the integral forms of (2.10) and (2.9).

We also use in this paper that

$$\tilde{B}_t := \int_0^T \lambda^t \, d^{Itô} B_{\lambda_t} = \int_0^{\tau_t} \lambda^t \, d^{Itô} B_t$$

has the same law as $B_t$ for any monotone and continuously differentiable function $\lambda: [0, \tilde{T}] \to [0, T]$. In differential form this relation becomes

$$d^{Itô} \tilde{B}_t = \lambda^t d^{Itô} B_t.$$ 

We need to reformulate relation (B.5) in the Stratonovich form. Let now $\lambda$ satisfy

$$d^S \lambda_t = a_t d\tilde{t} + b_t d^S \tilde{B}_t,$$

and

$$d^{Itô} \lambda_t = \lambda_t d^{Itô} B_t.$$
\[
\int_0^T d^{S} \tilde{B}_t = \int_0^T d^{t_0} \tilde{B}_t = \int_0^{\tilde{t}} \lambda_t^{\frac{1}{2}} d^{t_0} B_t = \int_0^{\tilde{t}} \lambda_t^{\frac{1}{2}} d^S B_t - \frac{1}{2} \langle \lambda_t^{\frac{1}{2}}, B_{\lambda_t} \rangle = \\
= \int_0^{\tilde{t}} \lambda_t^{\frac{1}{2}} d^S B_{\lambda_t} - \frac{1}{2} \langle \lambda_t^{\frac{1}{2}}, \int_0^T \lambda_t^{\frac{1}{2}} d\tilde{B}_t \rangle = \int_0^{\tilde{t}} \lambda_t^{\frac{1}{2}} d^S B_{\lambda_t} - \frac{1}{2} \int_0^T \frac{1}{2} \lambda_t^{-\frac{1}{2}} b_t \lambda_t^{-\frac{1}{2}} dt = \\
= \int_0^{\tilde{t}} \lambda_t^{\frac{1}{2}} d^S B_{\lambda_t} - \frac{1}{4} \int_0^T b_t d\tilde{t}.
\]

We conclude that

\[(B.7) \quad d^S \tilde{B}_t = \lambda_t^{\frac{1}{2}} d^S B_{\lambda_t} - \frac{1}{4} \frac{b_t}{\lambda_t} d\tilde{t}.
\]

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