A limiting form of the \( q \)-Dixon \( 4\varphi_3 \) summation
and related partition identities

by

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Abstract

By considering a limiting form of the \( q \)-Dixon \( 4\varphi_3 \) summation, we prove a weighted partition theorem involving odd parts differing by \( \geq 4 \). A two parameter refinement of this theorem is then deduced from a quartic reformulation of Göllnitz’s (Big) theorem due to Alladi, and this leads to a two parameter extension of Jacobi’s triple product identity for theta functions. Finally, refinements of certain modular identities of Alladi connected to the Göllnitz-Gordon series are shown to follow from a limiting form of the \( q \)-Dixon \( 4\varphi_3 \) summation.

§1: Introduction

The \( q \)-hypergeometric function \( r+1\varphi_r \) in \( r+1 \) numerator parameters \( a_1, a_2, \ldots, a_{r+1} \), and \( r \) denominator parameters \( b_1, b_2, \ldots, b_r \), with base \( q \) and variable \( t \), is defined as

\[
r+1\varphi_r \left( \frac{a_1, a_2, a_3, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r}; q, t \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_{r+1}; q)_k t^k}{(b_1; q)_k (b_2; q)_k \cdots (b_r; q)_k (q; q)_k}.
\]

(1.1)

When \( r = 3 \), for certain special choices of parameters \( a_i, b_j \), and variable \( t \), it is possible to evaluate the sum on the right in (1.1) to be a product. More precisely, the \( 4\varphi_3 \) \( q \)-Dixon summation ([4], (II. 13), p. 237) is

\[
4\varphi_3 \left( \frac{a, -q\sqrt{a}, b, c}{-\sqrt{a}, a, q\sqrt{a}, bc}; q, q\sqrt{a} \right) = \frac{(aq; q)_\infty (a\sqrt{a}; q)_\infty (\frac{aq}{\sqrt{a}}; q)_\infty (\frac{a^2}{bc}; q)_\infty}{(a; q)_\infty (q^2; q)_\infty (aq^2; q)_\infty (aq^2; q)_\infty (aq^2; q)_\infty}.
\]

(1.2)

Here and in what follows we have made use of the standard notation

\[
(a; q)_n = (a)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - aq^j), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \end{cases}
\]

(1.3)

for any complex number \( a \) and a non-negative integer \( n \), and

\[
(a)_\infty = (a; q)_\infty = \lim_{n \to \infty} (a; q)_n = \prod_{j=0}^{\infty} (1 - aq^j), \text{ for } |q| < 1.
\]

(1.4)

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Sometimes, as in (1.3) and (1.4), when the base is \( q \), we might suppress it, but when the base is anything other than \( q \), it will be made explicit.

Our first goal is to prove Theorem 1 is §2 which is a weighted identity connecting partitions into odd parts differing by \( \geq 4 \) and partitions into distinct parts \( \not\equiv 2 \) (mod 4). We achieve this by showing that the analytic representation of Theorem 1 is

\[
\sum_{k=0}^{\infty} z^k q^{4T_k - 1} q^{3k} (z^2 q^2; q^2)_k (1 + z q^{2k+1}) (q^2; q^2)_k = (-z q^2)_\infty (z^2 q^4; q^4)_\infty
\]

and establish (1.5) by utilizing a limiting form of the \( q \)-Dixon summation formula (1.2). In (1.5), \( T_k = \frac{k(k+1)}{2} \) is the \( k \)-th triangular number.

It is possible to obtain a two parameter refinement of Theorem 1 by splitting the odd integers into residue classes 1 and 3 (mod 4) and keeping track of the number of parts in each of these residue classes. This result, which is stated as Theorem 2 in §3, is a special case of a weighted reformulation of Göllnitz’s (Big) theorem due to Alladi (Theorem 6 of [2]). In §3 we also state an analytic identity (see (3.3)) in two free parameters \( a \) and \( b \) that is equivalent to Theorem 2, and note that (1.5) follows from this as the special case \( a = b = z \). Identity (3.3) can be viewed as a two parameter generalization of Jacobi’s celebrated triple product identity for theta functions (see (3.5) in §3).

Identity (3.3) is itself a special case of key identity in three free parameters \( a, b, \) and \( c \), due to Alladi and Andrews ([3], eqn. 3.14), for Göllnitz’s (Big) Theorem. The proof of this key identity of Alladi and Andrews in [3] utilizes Jackson’s \( q \)-analog of Dougall’s summation for \( {}_6 \phi_5 \). Note that the left hand side of (3.3) is a double summation. On the other hand, the left side of (1.5) is just a single summation, and its proof requires only a limiting form of the \( q \)-Dixon summation for \( {}_4 \phi_3 \). Owing to the choice \( a = b = z \), the double sum in (3.3) reduces to a single summation in (4.6) resembling (1.5), and this process is described in §4. Finally, certain modular identities for Göllnitz-Gordon functions due to Alladi [2] are refined in §5 using a limiting case of the \( q \)-Dixon summation (1.2).

We conclude this section by mentioning some notation pertaining to partitions. For a partition \( \pi \) we let

\[
\sigma(\pi) = \text{the sum of all parts of } \pi,
\nu(\pi) = \text{the number of parts of } \pi,
\nu(\pi; r, m) = \text{the number of parts of } \pi \text{ which are } \equiv r \text{ (mod } m),
\nu_d(m) = \text{the number of different parts of } \pi,
\nu_d,\ell(m) = \text{the number of different parts of } \pi \text{ which are } \geq \ell, \text{ and}
\lambda(\pi) = \text{the least part of } \pi.
\]
§2: Combinatorial interpretation and proof of (1.5)

Let $O_4$ denote the set of partitions into odd parts differing by $\geq 4$. Given $\pi \in O_4$, a chain $\chi$ in $\pi$ is defined to be a maximal string of consecutive parts differing by exactly 4. Let $N_\chi(\pi)$ denote the number of chains in $\pi$ with least part $\geq \lambda$.

Next let $D_{2,4}$ denote the set of partitions into distinct parts $\not\equiv 2(\text{mod } 4)$. We then have

**Theorem 1**: For all integers $n \geq 0$

$$\sum_{\pi \in O_4} z^{\nu(\pi)} (1 - z^2)^N_\chi(\pi) = \sum_{\pi \in D_{2,4}} z^{\nu(\pi;1,2)} (-z^2)^{\nu(\pi;0,2)} \sigma(\pi) = n$$

So, for partitions $\pi \in O_4$, we attach the weight $z$ to each part, and the weight $(1 - z^2)$ to each chain having least part $\geq 5$. The weight of $\pi$ is then defined multiplicatively. Similarly, for partitions $\pi \in D_{2,4}$, each odd part is assigned weight $z$, and each even part is assigned the weight $-z^2$, where all these even parts are actually multiples of 4. For example, when $n = 10$, the partitions in $O_4$ are $9 + 1$ and $7 + 3$, with weights $z^2(1 - z^2)$ and $z^2$ respectively. These weights add up to yield $2z^2 - z^4$. The partitions of 10 in $D_{2,4}$ are $9 + 1$, $7 + 3$, and $5 + 4 + 1$, with weights $z^2, z^2$, and $z^2(-z^2)$ respectively. These weights also add up to $2z^2 - z^4$, verifying Theorem 1 for $n = 10$.

We will now show that Theorem 1 is the combinational interpretation of (1.5).

It is clear that the product

$$\prod_{m=1}^{\infty} (1 + zq^{2m-1})(1 - z^2q^{4m})$$

on the right in (1.5) is the generating function of partitions $\pi \in D_{2,4}$, with weights as in Theorem 1. So we need to show that the series on the left in (1.5) is the generating function of partitions $\pi \in O_4$ with weights as specified in Theorem 1. For this we consider two cases.

**Case 1**: $\lambda(\pi) \neq 1$.

If $\pi$ is non-empty, then $\lambda(\pi) \geq 3$.

Since the parts of $\pi$ differ by $\geq 4$, we may subtract 0 from the smallest part, 4 from the second smallest part, 8 from the third smallest, ..., $4k - 4$ from the largest part of $\pi$, assuming $\nu(\pi) = k$. We call this procedure the *Euler subtraction*. After the Euler subtraction is performed on $\pi$, we are left with a partition $\pi'$ into $k$ odd parts such that the number of different parts of $\pi'$ is precisely the number of chains in $\pi$. If we denote by $G_{3,k}(q, z)$ the generating function of partitions $\pi \in O_4$ with $\lambda(\pi) \neq 1$, $\nu(\pi) = k$, and counted with weight $z^{\nu(\pi)} (1 - z^2)^{N_\chi(\pi)}$, then the Euler subtraction process yields

$$G_{3,k}(q, z) = z^k q^{4k-1} g_{3,k}(q, z),$$

(2.2)
where $g_{3,k}(q, z)$ is the generating function of partitions $\pi'$ into $k$ odd parts each $\geq 3$ and counted with weight $(1 - z^2)^{v_{d,s}(\pi')}$.  

At this stage we make the observation that if a set of positive integers $J$ is given, then

$$\prod_{j \in J} \left(1 - \frac{twq^j}{1 - t q^j}\right) = \prod_{j \in J} (1 + (1 - w) \{tq^j + t^2q^{2j} + t^3q^{3j} + \ldots\})$$  \hspace{1cm} (2.3)$$

is the generating function of partitions $\pi'$ into parts belonging to $J$ and counted with weight $t^{\nu(\pi')} (1 - w)^{v_{d,s}(\pi')}$. So from the principles underlying (2.3) it follows that

$$\sum_{k=0}^{\infty} g_{3,k}(q, z)t^k = \frac{1}{(1 - tq^3)} \prod_{j=0}^{\infty} \left(1 - tz^2q^{2j+5}\right) = \frac{(tz^2q^5; q^2)^{\infty}}{(tq^3; q^2)^{\infty}}.$$  \hspace{1cm} (2.4)$$

Using Cauchy's identity

$$\frac{(at)_{\infty}}{(t)_{\infty}} = \sum_{k=0}^{\infty} \frac{(a)_kt^k}{(q)_k},$$  \hspace{1cm} (2.5)$$

we can expand the product on the right in (2.4) as

$$\frac{(tz^2q^5; q^2)^{\infty}}{(tq^3; q^2)^{\infty}} = \sum_{k=0}^{\infty} \frac{t^k q^{3k} (z^2q^2; q^2)_k}{(q^2; q^2)_k}.$$  \hspace{1cm} (2.6)$$

So by comparing the coefficients of $t^k$ in (2.4) and (2.6) we get

$$g_{3,k}(q, z) = \frac{q^{3k} (z^2q^2; q^2)_k}{(q^2; q^2)_k}.$$  \hspace{1cm} (2.7)$$

Thus (2.7) and (2.2) yield

$$G_{3,k}(q, z) = \frac{z^k q^{4T_k-1} (z^2q^2; q^2)_k}{(q^2; q^2)_k}, \text{ for } k \geq 0.$$  \hspace{1cm} (2.8)$$

Case 2: $\lambda(\tilde{\pi}) = 1$.

Here for $k > 0$ we denote by $G_{1,k}^*(q, z)$ the generating function of partitions $\tilde{\pi} \in \mathcal{O}_4$ having $\lambda(\tilde{\pi}) = 1$, $\nu(\tilde{\pi}) = k$, and counted with weight $z^{\nu(\tilde{\pi})} (1 - z^2)^{N_k(\tilde{\pi})}$. The Euler subtraction process yields

$$G_{1,k}^*(q, z) = z^k q^{4T_k-1} g_{1,k}^*(q, z),$$  \hspace{1cm} (2.9)$$

where $g_{1,k}^*(q, z)$ is the generating function of partitions $\pi'$ into $k$ odd parts counted with weight $(1 - z^2)^{v_{d,s}(\pi')}$. The principles underlying (2.3) show that

$$\sum_{j=1}^{\infty} g_{1}^*(q, z)t^k = \frac{tq}{1 - tq} \prod_{j=1}^{\infty} \left(1 - \frac{tz^2q^{2j+1}}{1 - tq^{2j+1}}\right) = \frac{tq(tz^2q^3; q^2)^{\infty}}{(tq; q^2)^{\infty}} = \sum_{k=0}^{\infty} t^{k+1} q^{k+1} (z^2q^2; q^2)_k$$  \hspace{1cm} (2.10)$$

by Cauchy's identity (2.5). Thus by comparing the coefficients of $t^k$ at the extreme ends of (2.10), we get

$$g_{1,k}^*(q, z) = \frac{q^k (z^2q^2; q^2)_{k-1}}{(q^2; q^2)_{k-1}}.$$  \hspace{1cm} (2.11)$$
This when combined with (2.9) yields
\[ G_{1,k}(q, z) = \frac{z^k q^k q^{4T_k-1}(z^2 q^2: q^2)_{k-1}}{(q^2: q^2)_{k-1}}. \tag{2.12} \]

Finally, it is clear that
\[ \sum_{k=0}^{\infty} G_{3,k}(q, z) + \sum_{k=1}^{\infty} G_{1,k}(q, z) \]

is the generating function of partitions \( \tilde{\pi} \in \mathcal{O}_4 \) counted with weight as specified in Theorem 1.

From (2.8) and (2.12), the sum in (2.13) can be seen to be
\[ \sum_{k=0}^{\infty} z^k q^{3k} q^{4T_k-1}(z^2 q^2: q^2)_k + \sum_{k=1}^{\infty} z^k q^k q^{4T_k-1}(z^2 q^2: q^2)_{k-1} \]
\[ = \sum_{k=0}^{\infty} z^k q^{3k} q^{4T_k-1}(z^2 q^2: q^2)_k + \sum_{k=0}^{\infty} z^{k+1} q^{k+1} q^{4T_k}(z^2 q^2: q^2)_k \]
\[ = \sum_{k=0}^{\infty} z^k q^{3k} q^{4T_k-1}(z^2 q^2: q^2)_k(1 + zq^{2k+1}) \tag{2.14} \]

which is the series on the left in (1.5).

From (2.14) and (2.1) it follows that Theorem 1 is the combinatorial interpretation of (1.5).

Thus to prove Theorem 1 it suffices to establish (1.5) and this is what we do next.

From the definition of the \( q \)-hypergeometric function in (1.1) we see that
\[ (1+zq)_{4\varphi_3}\left(\frac{z^2 q^2, -zq^3, \rho, \rho}{-q^3, \frac{z^2 q^2}{\rho}, \frac{z^2 q^2}{\rho}}: \frac{q^2}{\rho^2}\right) \]
\[ = (1 + zq) \sum_{k=0}^{\infty} \frac{(z^2 q^2: q^2)_k}{(q^2: q^2)_k} \frac{(-zq^3)_k}{(-q^3)_k} (\rho q^2)_k (\rho: q^2)_k \frac{1}{(\rho q^3: q^2)_k} \frac{1}{(z^2 q^4: q^2)_k} \frac{1}{\rho^{2k}} \tag{2.15} \]

Next observe that
\[ \lim_{\rho \to \infty} \frac{(\rho q^3)^k}{\rho^k} = \lim_{\rho \to \infty} \frac{(1 - \rho)(1 - \rho q^2) \ldots (1 - \rho q^{2k-2})}{\rho^k} = (-1)^k q^{2T_k-1} \tag{2.16} \]

and
\[ \lim_{\rho \to \infty} \left(\frac{z^2 q^4}{\rho}: q^2\right)_k = 1. \tag{2.17} \]

Thus (2.15), (2.16), and (2.17) imply that
\[ (1 + zq) \lim_{\rho \to \infty} \frac{(\rho q^3)^k}{\rho^k} = \sum_{k=0}^{\infty} z^k q^{3k} q^{4T_k-1}(z^2 q^2: q^2)_k(1 + zq^{2k+1}) \frac{1}{(q^2: q^2)_k} \tag{2.18} \]

which is the series on the left in (1.5).
At this stage we observe that the hypergeometric function $_4\phi_3$ on the left in (2.18) is precisely the one in the $q$–Dixon summation (1.2) with the replacements

$$ q \mapsto q^2, \ a \mapsto z^2 q^2, \ b \mapsto \rho, \ c \mapsto \rho. \quad (2.19) $$

Thus with substitutions (2.19) in (1.2) we deduce that

$$ (1 + z q) \lim_{\rho \to \infty} \sum_{\tau \in \mathcal{O}_4^\nu} \sigma(\tau) = (1 + z q) \lim_{\rho \to \infty} \frac{(z^2 q^4; q^2)_\infty (z^4 q^2; \rho q^2)_\infty (z^4 q^2; \rho^2 q^2)_\infty}{(z q^2; \rho^2 q^2)_\infty} = (1 + z q)^2 \frac{(z^2 q^4; q^2)_\infty (z^4 q^2; \rho^2 q^2)_\infty}{(z q^2; \rho^2 q^2)_\infty}. \quad (2.20) $$

Thus (1.5) follows from (2.18) and (2.20) and this completes the proof of Theorem 1.

### §3: Two parameter refinement

When a partition $\tilde{\pi} \in \mathcal{O}_4$ is decomposed into chains, the parts in a given chain all belong to the same residue class mod 4. This suggests that there ought to be a two parameter refinement of Theorem 1 in which we can keep track of parts in residue classes 1 and 3 (mod 4) separately. Theorem 2 stated below is such a refinement. Actually Theorem 2 is a special case of a refinement and reformulation of a deep theorem of Göllnitz [5] in three parameters $a, b, c$ due to Alladi ([2], Theorem 6) by setting one of the parameters $c = -ab$.

**Theorem 2:** For all integers $n \geq 0$ and complex numbers $a$ and $b$ we have

$$ \sum_{\pi \in \mathcal{O}_4, \sigma(\tilde{\pi}) = n} a^{\nu(\tilde{\pi}; 1,4)} b^{\nu(\tilde{\pi}; 3,4)} (1 - ab)^{N_5(\tilde{\pi})} = \sum_{\pi \in \mathcal{D}_{2,4}, \sigma(\pi) = n} a^{\nu(\pi; 1,4)} b^{\nu(\pi; 3,4)} (-ab)^{\nu(\pi; 0,4)}. $$

Since Theorem 2 is a two parameter refinement of Theorem 1 which has the analytic representation (1.5), it is natural to ask for an analytic identity in two free parameters that reduces to (1.5). We will now obtain such a two parameter identity, namely, (3.3) below. Instead of deriving (3.3) combinatorially from Theorem 2 by following the method in §2, we will now illustrate a different approach which involves a certain cubic reformulation of Göllnitz’s (Big) theorem due to Alladi [1], and its key identity in three free parameters $a, b, c$ due to Alladi and Andrews [3]. More precisely, we will show that (3.3) is the analytic representation of Theorem 2 after a
Then we have
decompose it into chains, where a chain here is a maximal string of parts differing by exactly 3.
of partitions into parts differing by \( \geq D \) is replaced by the modulus 4 in Theorem 2. More precisely, we may view
corresponding weights are listed below:

in \([3]\) and the specialization \( c \) makes the weights equal to 0 if the partition has a multiple of 3 in it. Thus from the analysis

ab



Weights

Partitions

9, \( 8+1 \), \( 7+2 \), \( 6+3 \), \( 6+2+1 \), \( 5+4 \), \( 5+3+1 \), \( 4+3+2 \)

\(-ab\), \( ab\), \( (-ab)^2\), \( ab(-ab)\), \( ab\), \( ab(-ab)\), \( ab(-ab)\)

The above weights when added also yield \( 2ab(1-ab) \) thereby verifying Theorem 3 for \( n = 9 \).

Theorems 3 and 2 are really the same because the role of the modulus 3 in Theorem 3

is replaced by the modulus 4 in Theorem 2. More precisely, we may view \( \mathcal{D}_3 \) and \( \mathcal{O}_4 \) as

sets of partitions into parts differing by \( \geq k \) and containing only parts in the residue classes

\( \pm 1(m o d \ k) \), for \( k = 3, 4 \). Similarly, we may think of \( \mathcal{D} \) and \( \mathcal{D}_{2,4} \) as sets of partitions into parts

\( \equiv 0, \pm 1(m o d \ k) \), for \( k = 3, 4 \). Note that the functions \( \nu(\pi; r, 3) \) and \( \nu(\tilde{\pi}; r, 3) \) in Theorem 3 are

replaced \( \nu(\pi; r, 4) \) and \( \nu(\tilde{\pi}; r, 4) \) in Theorem 2. Pursuing this line of correspondence, \( N_3(\tilde{\pi}) \) in

Theorem 3 should be replaced by \( N_4(\tilde{\pi}) \) in Theorem 2, but this is the same as having \( N_5(\tilde{\pi}) \) in

Theorem 2 because \( \tilde{\pi} \in \mathcal{O}_4 \).
Having observed the correspondence between Theorems 2 and 3, we deduce that the analytic representation of Theorem 2 (in the form of a two parameter $q$–hypergeometric identity) is obtained from (3.1) by the substitutions

$$ q \mapsto q^{4/3}, a \mapsto aq^{-1/3}, b \mapsto bq^{1/3}, ab \mapsto ab, $$

which yields

$$\sum_{i,j \geq 0} \frac{a^i q^{2i^2 - i(ab; q^4)_i}}{(q^4; q^4)_i} \cdot \frac{b^i q^{4ij^2 + j(ab; q^4)_j}}{(q^4; q^4)_j} \left( \frac{1 - abq^{4(i+j)}}{1 - ab} \right) = (-aq; q^4)_\infty (-bq^3; q^4)_\infty (ab; q^4)_\infty. \quad (3.3)$$

Identities (3.1) and (3.3) are interesting for another reason. They can be considered as two parameter extensions of Jacobi’s celebrated triple product identity for theta functions. More precisely, if we put $ab = 1$ in (3.1) and (3.3), then on the left hand side of each of these identities, only the terms having either $i = 0$ or $j = 0$ survive, and so the identities reduce to

$$\sum_{i=-\infty}^{\infty} a^i q^{3i^2 - i} = (-aq; q^3)_\infty (-a^{-1}q^2; q^3)_\infty (q^3; q^3)_\infty \quad (3.4)$$

and

$$\sum_{i=-\infty}^{\infty} a^i q^{3i^2 - i} = (-aq; q^4)_\infty (-a^{-1}q^3; q^4)_\infty (q^4; q^4)_\infty \quad (3.5)$$

which are equivalent to Jacobi’s triple product identity.

Identities (3.4) and (3.5) can be deduced combinatorially from Theorems 3 and 2 respectively, by setting $ab = 1$. This is because $ab = 1$ forces $N_3(\tilde{\pi}) = 0$ (resp. $N_5(\tilde{\pi}) = 0$) in Theorem 3 (resp. Theorem 2) and this brings about a drastic reduction in the type of partitions to be enumerated in $D_3$ (resp. $O_4$). We refer the reader to Alladi [1], [2] for these combinatorial arguments.

§4: Reduction to a single summation

The left hand side of identity (3.3) is a double summation. It turns out that if we set

$$a = b = z,$$

then the left side of (3.3) reduces to a single infinite sum. It is quite instructive to see how this happens, and so we describe it now.

First observe that

$$2i^2 - i + 2j^2 + j + 4ij = 2(i+j)^2 - (i+j) + 2j. \quad (4.1)$$
Thus if we set \( a = b = z \) and reassemble the terms in (3.3) with \( k = i + j \), then (4.1) shows that (3.3) becomes

\[
\sum_{k=0}^{\infty} z^k q^{2k^2-k} \frac{1 - z^2 q^{4k}}{(1 - z^2)^{2}} \left\{ \sum_{i+j=k} \frac{q^{2j}(z^2 q^4)^i (z^2 q^4)^j}{(q^4 q^4)^i (q^4 q^4)^j} \right\} = \left(-zq q^2 \right) \infty (z^2 q^4 q^4)^\infty. \tag{4.2}
\]

At this point we take the product in Cauchy’s identity (2.5) and decompose it as

\[
\frac{(at)_{\infty}}{(t)_{\infty}} = \frac{(at; q^2)_{\infty}}{(t; q^2)_{\infty}} \cdot \frac{(atq; q^2)_{\infty}}{(tq; q^2)_{\infty}}. \tag{4.3}
\]

If we now substitute the expansion in (2.5) for each of the products in (4.3), we get

\[
\sum_{k=0}^{\infty} \frac{(a)_k t^k}{(q)_k} = \left( \sum_{i=0}^{\infty} \frac{(a; q^2)_i t^i}{(q^2 q^2)_i} \right) \left( \sum_{j=0}^{\infty} \frac{(a; q^2)_j t^j q^j}{(q^2 q^2)_j} \right). \tag{4.4}
\]

By comparing the coefficients of \( t^k \) on both sides of (4.4) we obtain

\[
\frac{(a)_k}{(q)_k} = \sum_{i+j=k} \frac{(a; q^2)_i (a; q^2)_j q^j}{(q^2 q^2)_i (q^2 q^2)_j}. \tag{4.5}
\]

If we replace \( q \mapsto q^2 \) and \( a \mapsto z^2 \), we see that the sum in (4.5) becomes the expression within the parenthesis (namely the inner sum) on the left in (4.2). Thus with these replacements (4.5) implies that (4.2) can be written as the single summation identity

\[
1 + \sum_{k=1}^{\infty} z^k q^{2k^2-k} \frac{(z^2 q^2 q^2)_{k-1}}{(q^2 q^2)_k} (1 - z^2 q^{4k}) \left\{ \sum_{i+j=k} \frac{q^{2i}(z^2 q^4)^i (z^2 q^4)^j}{(q^4 q^4)^i (q^4 q^4)^j} \right\} = \left(-zq q^2 \right) \infty (z^2 q^4 q^4)^\infty. \tag{4.6}
\]

Identity (4.6) is an analytic representation of Theorem 1. Note however that the series in (4.6) is different from the series in (1.5). The explanation of this is as follows.

If we add \( G_{3,k}(q, z) \) and \( G_{1,k}^*(q, z) \) for each \( k \geq 1 \), we get from (2.8) and (2.12)

\[
G_{3,k}(q, z) + G_{1,k}^*(q, z) = \frac{z^k q^{4k} q^{4T_k} (z^2 q^2 q^2)_{k-1} (1 - q^{2k} + q^{2k} (1 - z^2 q^{4k}))}{(q^2 q^2)_k}
\]

which is the \( k \)-th summand in (4.6). The starting term 1 in (4.6) is to be interpreted as \( G_{3,0}(q, z) \). On the other hand in (2.14) we are considering

\[
G_{3,k}(q, z) + G_{1,k+1}^*(q, z), \quad \text{for } k \geq 0,
\]

and this leads to the series in (1.5) which is different from (4.6).

The reason we preferred (1.5) to (4.6) is because (1.5) could be proved using only a limiting form of the \( q-Dixon \) summation of 4\( \varphi_3 \), whereas (4.6) would have required a limiting form of Jackson’s 6\( \varphi_5 \) summation ([4], (II.21), p. 238).
§5: Modular identities for the refined Göllnitz-Gordon functions.

The well known Göllnitz-Gordon identities are

\[ G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_\infty(q^4; q^8)_\infty(q^3; q^8)_\infty} \]  \hspace{1cm} (5.1)

and

\[ H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q^3; q^8)_\infty(q^4; q^8)_\infty(q^5; q^8)_\infty}. \]  \hspace{1cm} (5.2)

Identities (5.1) and (5.2) are actually (36) and (34) on Slater’s list [7], but it was Göllnitz [5] and Gordon [6] who realized their partition significance and their relationship with a continued fraction. More precisely, the Göllnitz-Gordon partition theorem is: For \( i=1,2 \), the number of partitions of an integer \( n \) into parts differing by \( \geq 2 \), with strict inequality if a part is even, and least part \( \geq 2i-1 \), equals the number of partitions of \( n \) into parts \( \equiv 4, \pm(2i-1) \pmod{8} \).

In view of the form of the series-product identities (5.1) and (5.2), their partition interpretation given above, and their relationship with a certain continued fraction, the Göllnitz-Gordon identities are considered as the perfect analogues for the modulus 8 for what the celebrated Rogers-Ramanujan identities are for the modulus 5.

In [2] Alladi established four reformulations of Göllnitz’s (Big) partition theorem using four quartic transformations. One of the reformulations yielded Theorem 2 of §3. Using one of the other reformulations, Alladi [2] deduced the modular identity

\[ G(-q^2) + qH(-q^2) = (-q; q^4)_\infty(q^2; q^4)_\infty(-q^3; q^4)_\infty \]  \hspace{1cm} (5.3)

combinatorially. Alladi [2] then defined the twisted Göllnitz-Gordon functions

\[ G_t(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q; q^2)_n}{(q^2; q^2)_n}, \] \hspace{1cm} (5.4)

and

\[ H_t(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(q; q^2)_n}{(q^2; q^2)_n}, \] \hspace{1cm} (5.5)

and deduced the modular identity

\[ G_t(q^2) + qH_t(q^2) = (-q; q^4)_\infty(-q^2; q^4)_\infty(q^3; q^4)_\infty \] \hspace{1cm} (5.6)

combinatorially from the same reformulation of Göllnitz’s (Big) theorem.

The twisted Göllnitz-Gordon functions do not have the product representations of the type \( G(q) \) and \( H(q) \) possess. But the modular identity (5.6) implies that

\[ G_t(q^2) = \frac{(-q; q^4)_\infty(-q^2; q^4)_\infty(q^3; q^4)_\infty + (q; q^4)_\infty(-q^2; q^4)_\infty(-q^3; q^4)_\infty}{2} \] \hspace{1cm} (5.7)
and
\[ H_t(q^2) = \frac{(-q; q^4)_\infty (-q^2; q^4)_\infty (q^2; q^4)_\infty - (q; q^4)_\infty (-q^2; q^4)_\infty (q^3; q^4)_\infty}{2q}. \] (5.8)

In the absence of product representations, (5.7) and (5.8) show that \( G_t(q^2) \) and \( G_t(q^2) \) are arithmetic means of interesting products. From (5.3) it follows that \( G(-q^2) \) and \( H(-q^2) \) have representations similar to (5.7) and (5.8).

By utilizing a limiting form of the \( q \)-Dixon summation (1.2) for \( a \neq 3 \), we will now establish a more general modular identity (see (5.13) below) that contains both (5.3) and (5.6). To this end let \( c = \delta \sqrt{aq} \) with \( \delta = \pm 1 \) in (1.2), and multiply both sides by \( 1 + \sqrt{a} \). This way we find
\[ \sum_{k=0}^{\infty} \frac{\delta^k b^{-k} q^{k/2} (1 + \sqrt{aq} k)}{(q)_k} \frac{(a)_k}{(a)_{2k}} (q) \frac{(\sqrt{a} q)_{2k}}{(\sqrt{a} q)_k} = \frac{(aq)_{2k}}{(aq)_{k}} \frac{(\sqrt{aq})_3}{(\sqrt{aq})_2} \frac{(\sqrt{aq})_2}{(\sqrt{aq})_1}. \] (5.9)

Analogous to (2.16) we now have
\[ \lim_{b \to \infty} \frac{(b)_k}{b^k} = (-1)^k q_k. \] (5.10)

Thus (5.9) and (5.10) imply that by going to the limit \( b \to \infty \) we get
\[ \sum_{k=0}^{\infty} \frac{(-\delta)^k q^{k/2} (1 + \sqrt{aq} k)}{(q)_k} \frac{(a)_k}{(a)_{2k}} (q) \frac{(\sqrt{aq})_{2k}}{(\sqrt{aq})_k} = \frac{(aq)_{2k}}{(aq)_{k}} \frac{(\sqrt{aq})_3}{(\sqrt{aq})_2} \frac{(\sqrt{aq})_2}{(\sqrt{aq})_1} \text{ for } \delta = \pm 1. \] (5.11)

In (5.11) if we replace \( q \to q^4 \) and \( a \to a^2 q^2 \), we obtain
\[ \sum_{k=0}^{\infty} \frac{(-\delta)^k q^{k^2} (a^2 q^2; q^4)_k}{(a^2 q^2; q^4)_k} (1 + aq^{k+1}) = \frac{(a^2 q^2; q^4)_\infty}{(aq; q^4)_\infty} \frac{(\delta q^2; q^4)_\infty}{(\delta a q^3; q^4)_\infty} = \frac{(-aq; q^4)_\infty}{(aq; q^4)_\infty} \frac{(\delta a q^3; q^4)_\infty}{(-\delta a q^3; q^4)_\infty} \] (5.12)

Note that the special case \( \delta = 1, a = 1 \) in (5.12) yields (5.3), whereas \( \delta = -1, a = 1 \) in (5.12) is (5.6). We may write (5.12) in the form of the modular identity
\[ G_{a^2, \delta}(-q^2) + aq H_{a^2, \delta}(-q^2) = (-aq; q^4)_\infty (\delta q^2; q^4)_\infty (-\delta a q^3; q^4)_\infty, \text{ for } \delta = \pm 1 \] (5.13)
for the refined Göllnitz-Gordon functions which we define as
\[ G_{a, \delta}(q) = \sum_{k=0}^{\infty} \frac{\delta^k q^{k^2}}{(q^2; q^2)_k} (-aq; q^2)_k \] (5.14)
and
\[ H_{a, \delta}(q) = \sum_{k=0}^{\infty} \frac{\delta^k q^{k^2 + 2k}}{(q^2; q^2)_k} (-aq; q^2)_k. \] (5.15)

Here \( G_{a, \delta}(q) \) is the generating function of partitions into parts differing by \( \geq 2 \), with strict inequality if a part is even, where each odd part is assigned weight \( \delta \), and each even part given weight \( \delta a \). The weight of the partition under consideration is the product of the weights of its
parts. The function $H_{\alpha,\delta}(q)$ has a similar interpretation, except for the added restriction that the least part in $\geq 3$ for the partitions enumerated.

We note that the combinatorial arguments in [2] which yielded (5.3) and (5.6) could be used to derive the more general modular relation (5.13).

References

1. K. Alladi, A combinatorial correspondence related to Göllnitz’s (Big) partition theorem and applications, Trans. Amer. Math Soc. 349 (1997), 2721-35.

2. K. Alladi, On a partition theorem of Göllnitz and quartic transformations (with an appendix by B. Gordon), J. Num. Th. 69 (1998), 153-180.

3. K. Alladi and G. E. Andrews, A new key identity for Göllnitz’s (Big) partition theorem, Contemp. Math. 210 (1998), 229-241.

4. G. Gasper and M. Rahman, Basic hyper-geometric series, Encyclopedia of Mathematics and its Applications, Vol.10, Cambridge (1990).

5. H. Göllnitz, Partitionen mit Differenzenbedingungen, J. Reine Angew Math. 225 (1967), 154-190.

6. B. Gordon, Some continued fractions of the Rogers-Ramanujan type, Duke Math. J. 32 (1965), 741-748.

7. L. J. Slater, Further identities of Rogers-Ramanujan type, Proc. London Math. Soc. (2) 54 (1952), 147-167.

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