SOME EXISTENCE AND REGULARITY RESULTS
FOR POROUS MEDIA AND FAST DIFFUSION EQUATIONS
WITH A GRADIENT TERM

BOUMEDIENE ABDELLAOUI, IRENEO PERAL, AND MAGDALENA WALIAS

To the memory of Juan Antonio Aguilar, our dearest friend

Abstract. In this article we consider the problem

\[
\begin{aligned}
(P) \quad & u_t - \Delta u^m = |\nabla u|^q + f(x,t), \quad u \geq 0 \quad \text{in } \Omega_T \equiv \Omega \times (0,T), \\
& u(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T), \\
& u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded regular domain, \( N \geq 1 \), \( 1 < q \leq 2 \), and \( f \geq 0 \), \( u_0 \geq 0 \) are in a suitable class of measurable functions.

We obtain some results for the so-called elliptic-parabolic problems with measure data related to problem \((P)\) that we use to study the existence of solutions to problem \((P)\) according with the values of the parameters \( q \) and \( m \).

1. Introduction

In this work we will study the problem

\[
\begin{aligned}
(1) \quad & u_t - \Delta u^m = |\nabla u|^q + f(x,t), \quad u \geq 0 \quad \text{in } \Omega_T \equiv \Omega \times (0,T), \\
& u(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T), \\
& u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, \( N \geq 1 \), \( m > 0 \), \( 1 < q \leq 2 \), and \( f \geq 0 \), \( u_0 \geq 0 \) are in a suitable class of measurable functions. If \( m > 1 \), problem (1) is a model of growth in a porous medium; see for instance [6].

We refer to the fundamental monograph by J. L. Vázquez, [26], and the references therein for the basic results about Porous Media Equations (PME) and Fast Diffusion Equation (FDE) without gradient term. An optimal existence result for the homogeneous Cauchy problem (without source term) can also be found in [8].

Here we are interested in the existence and regularity of solutions to problem (1) related to the parameters \( q \) and \( m \). If \( q = 2 \), some results were obtained in [18] and [20] for bounded data.

One of the main new features of this paper is to find a general class of data according to the values of the parameters in the problems, in order to obtain the existence of nonnegative solutions to problem (1). This study is in some way motivated by reference [2], where the stationary problem was analyzed.

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A pertinent formal remark is the following: The equation $u_t - \Delta u^m = \mu$, after the change $v = u^m$, becomes

$$b(v)_t - \Delta v = \mu \text{ with } b(s) = s^{\frac{1}{m}}.$$  

The last formulation usually is known in the literature as an elliptic-parabolic equation. References for problems related to these equations are [4], [9], [10], [12], [15], [16] and [25] among others. We will study the elliptic-parabolic problems with $\mu$ a bounded Radon measure, which is the natural class of data in the application to the analysis of problem (1). The strategy that we follow in this work can be summarized in the following points:

1. We consider approximated problems that kill the degeneration, or the singularity, in the principal part ((PME), or (FDE), respectively), and we truncate the first order term in the right hand side. With respect to these approximated problems, the existence of a solution follows using the well known results obtained in [13]. Here the natural setting is to find a weak solution in the sense of Definition 2.2 (where it is formulated for the corresponding elliptic-parabolic equation).

2. We obtain uniform estimates of the solution of the approximated problems in such a way that the first order part in the second member is uniformly bounded in $L^1(\Omega_T)$.

3. The previous step motivates the study of a problem with measure data. To have more flexibility in the calculation we formulate the problem as an elliptic-parabolic equation and look for a reachable solution in the sense of Definition 2.4. One of the new features in this work is the proof of the almost everywhere convergence of the gradients of the solutions of the approximated problems.

4. The final step is to use the uniform estimates and the a.e. convergence of the gradients to prove that, up to a subsequence, the second members of the approximated problems of (1) converge strongly in $L^1_{\text{loc}}(\Omega_T)$. That is, we find a distributional solution.

The organization of the paper is as follows. In Section 2, we prove the existence of reachable solutions for a class of elliptic-parabolic problems with measure data, which include both cases, namely, (PME) and (FDE). To obtain the existence of reachable solutions, we show suitable a priori estimates for the solutions of the truncated problems and the pointwise convergence of the gradients that allow us to conclude. In the last part of Section 2, as an application, we find the existence of a solution for the porous media and fast diffusion equations with a Radon measure data. A key point is the proof of a.e. convergence of the gradients, which will be used in the following sections. It is worthy to point out that these results improve in some way the ones obtained in [24] for $m > 1$, and give a proof for the (FDE) with measure data.

Section 3 deals with the (PME) with a gradient term. We follow the strategy describe above. In the first part of the section we prove the existence results for the interval $1 < m \leq 2$. The main contribution in this case is Theorem 3.1. In the last part of this section we consider the complementary interval of $m$, that is, $m > 2$. In this case we are able to prove the existence of a solution with $L^1(\Omega_T)$ data. This result is the content of Theorem 3.3. In the last part of Section 3, we point out the particular behavior of the case $p = 2$, $m = 2$. Indeed, if the source
term \( f = 0 \), via a change of variable, we show that problem (11) is equivalent to a homogeneous (PME) equation with \( m = \frac{5}{3} \). As a consequence, we obtain in this case the finite speed of propagation property, and also a self-similar solution with compact support in each fixed positive time.

Finally, in Section 4 we analyze the fast diffusion equation, i.e. \( 0 < m < 1 \), with a convenient hypothesis of integrability of the source term. The main result of this section is Theorem (4.1). We also prove Theorem (4.3) that gives us the finite extinction time property of a regular solution (see Definition (4.2)) if \( 0 < m < 1 \) and \( q = 2 \). In a very recent paper, [3], we obtain some results of nonuniqueness for \( 0 < m \leq 2 \) and \( q = 2 \), that is, quadratic growth in the gradient.

We have tried to write the paper in an almost self-contained form; moreover, we give precise references for all the points that are not detailed in the work.

2. SOME RESULTS FOR AN ELLIPTIC-PARABOLIC PROBLEM WITH MEASURE DATA

We will consider the general problem

\[
\begin{cases}
(b(v))_t - \Delta v = \mu & \text{in } \Omega \times (0, T), \\
v(x, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
b(v(x, 0)) = b(v_0(x)) & \text{in } \Omega,
\end{cases}
\]

where \( b : \mathbb{R} \to \mathbb{R} \) is a continuous strictly increasing function such that \( b(0) = 0 \), \( b(v_0) \in L^1(\Omega) \) and \( \mu \) is a Radon measure whose total variation is finite in \( \Omega_T \). We will assume the following hypotheses on \( b \):

\[
\begin{cases}
(B1) \text{There exists } a_1 > 0 \text{ such that } b(s) \geq Cs^{a_1} \text{ for } s \gg 1. \\
(B2) \text{There exists } a_2 < 1 \text{ such that } |b'(s)| \leq \frac{1}{s^{a_2}} \text{ for } s \ll 1. \\
(B3) \text{Either } b' \in C([0, \infty)) \text{ and } |b'(s)|b^{2a_1-1}(s) \leq s^{\frac{N+2a_1}{N}} - \varepsilon \text{ as } s \to \infty \text{ or } |b'(s)| \leq b^{2-2a_3-\varepsilon}(s) \text{ as } s \to \infty.
\end{cases}
\]

Remark 2.1. The following examples of \( b \) will be considered in this work:

(1) \( b(s) = s^\sigma \) if \( s \geq 0 \), for some \( \sigma > \frac{(N - 2)_+}{N} \).

(2) \( b(s) = \frac{1}{m} \int_0^s (\Lambda^{-1}(\sigma))^{\frac{1}{m} - 1} d\sigma = \frac{1}{m} \int_0^{\Lambda^{-1}(s)} \sigma^{\frac{1}{m} - 1} \Lambda'(\sigma) d\sigma \), where

\[
\Lambda(s) = \frac{4}{5} s^5 \text{ if } m = 2, \quad \Lambda(s) = \int_0^s \frac{\tilde{e}}{m (2-m)} dt \text{ if } 0 < m < 2.
\]

(3) \( b_n(s) := b(s + b(\frac{1}{n})) - b(\frac{1}{n}), b(s) = s^\sigma \) and \( \sigma > \frac{(N - 2)_+}{N} \). In this case we are able to show that the estimates obtained for approximated problems are uniform in \( n \geq 1 \).

A similar observation can be done for the truncation of \( \Lambda \).

For \( \mu \in L^\infty(\Omega_T) \) and \( b(v_0) \in L^\infty(\Omega) \), we have the following definition of the weak solution.
Definition 2.2. Assume that $\mu \in L^\infty(\Omega_T)$ and $b(v_0) \in L^\infty(\Omega)$. We say that $v$ is a weak solution to (2) if

1. $v \in L^2((0,T);W^{1,2}_0(\Omega)) \cap L^\infty(\Omega_T)$,
2. the function $b(v) \in C((0,T);L^q(\Omega))$ for all $q < \infty$,
3. $(b(v))_t \in L^2((0,T);W^{-1,2}(\Omega))$.

Also, for every $\phi \in L^2((0,T);W^{1,2}_0(\Omega))$ the following identity holds:

$$\int_0^T \langle b(v)_t, \phi \rangle + \iint_{\Omega_T} \nabla v \cdot \nabla \phi = \iint_{\Omega_T} \mu \phi.$$  

The following result is well known.

Theorem 2.3. Assuming $\mu \in L^\infty(\Omega_T)$ and $b(v_0) \in L^\infty(\Omega)$, there exists a unique weak solution to problem (2) in the sense of Definition 2.2.

The proof of Theorem 2.3 can be found in [4] and [16].

2.1. Reachable solutions. Since we are considering problems with general data, we need to be precise in the sense in which the solution is defined. For elliptic equations the notion of reachable solutions was introduced in [19]. We refer to [17] for the parabolic equation. See also [1] for some particular cases. If $\mu \in L^1(\Omega_T)$, the renormalized solution is studied in [12]. In our case, we have the following definition of a reachable solution.

Definition 2.4. Assume that $\mu$ is a Radon measure whose total variation is finite in $\Omega_T$ and $b(v_0) \in L^1(\Omega)$.

We say that $v$ is a reachable solution to (2) if

1. $T_k(v) \in L^2((0,T);W^{1,2}_0(\Omega))$ for all $k > 0$.
2. For all $t > 0$ there exist both one-side limits $\lim_{\tau \to t} b(v(\cdot, \tau))$ weakly-* in the sense of measures.
3. $b(v(\cdot, t)) \rightharpoonup b(v_0(\cdot))$ weakly-* in the sense of measures as $t \to 0$.
4. There exist three sequences, $\{v_n\}_n$ in $L^2((0,T);W^{1,2}_0(\Omega))$, $\{h_n\}_n$ in $L^\infty(\Omega_T)$ and $\{g_n\}_n$ in $L^\infty(\Omega)$, such that if $v_n$ is the weak solution to problem

$$\begin{cases}
(b(v_n))_t - \Delta v_n &= h_n \quad \text{in } \Omega \times (0,T), \\
v_n(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T), \\
v_n(x,0) &= b^{-1}(g_n(x)) \quad \text{in } \Omega,
\end{cases}$$

then

(a) $g_n \rightharpoonup b(v_0)$ in $L^1(\Omega)$.
(b) $h_n \rightharpoonup \mu$ as measures.
(c) $\nabla v_n \rightharpoonup \nabla v$ strongly in $L^\sigma(\Omega_T)$ for $1 \leq \sigma < \frac{N + 2a_1}{N + a_1}$.
(d) The sequence $\{b(v_n)\}_n$ is bounded in $L^\infty((0,T);L^1(\Omega))$ and $b(v_n) \rightharpoonup b(v)$ strongly in $L^1(\Omega_T)$.

One of the main goals of our analysis is to prove that, under condition (B), we obtain the regularity of the spatial gradients that extends the regularity in the linear case.

To prove the existence of a reachable solution, we need the following lemma whose proof can be obtained by approximation.
Lemma 2.5. Let \( v \in L^2(0,T;W^{1,2}_0(\Omega)) \) satisfy \( b(v)_t \in L^2(0,T;W^{-1,2'}(\Omega)) \). Assume that \( \phi(s) : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz–continuous function such that \( \phi(0) = 0 \). Then, if

\[
\Phi(s) = \int_0^s b'(\sigma) \phi(\sigma) \, d\sigma, 
\]

the following integration by parts formula holds:

\[
\int_{t_1}^{t_2} \langle b(u)_t, \phi(u) \rangle_{W^{-1,2'}(\Omega),W^{1,2}_0(\Omega)} \, dt = \int_{\Omega} \Phi(u(x,t_2)) \, dx - \int_{\Omega} \Phi(u(x,t_1)) \, dx, 
\]

for every \( 0 \leq t_1 < t_2 \).

We will now prove the existence of a reachable solution to problem \( (\text{I}) \).

2.2. Some a priori estimates. Let us consider the following approximating problems:

\[
\begin{aligned}
\{ \begin{array}{l}
\frac{\partial v_n}{\partial t} - \Delta v_n = h_n, \\
v_n(x,t) = 0, \\
v_n(x,0) = b^{-1}(g_n(x)),
\end{array} \quad (x,t) \in \Omega_T, \\
\end{aligned}
\]

where \( g_n \to b(v_0) \) strongly in \( L^1(\Omega) \) and \( h_n \to \mu \) in the weak-* sense in \( \Omega_T \). The existence of weak solutions to these problems follows from Theorem 2.3.

Let us begin by proving the next proposition.

Proposition 2.6. Let \( \{v_n\}_n \) be a sequence of solutions of the approximate problems \( (\text{I}) \). Then

1. The sequence \( \{(|v_n| + 1)^{\beta} - 1\}_n \) is bounded in \( L^2(0,T;W^{1,2}_0(\Omega)) \), for each \( 0 < \beta < \frac{1}{2} \).
2. The sequence \( \{b(v_n)\}_n \) is bounded in the space \( L^\infty(0,T;L^4(\Omega)) \) and the sequence \( \{(b(v_n))_t\}_n \) is bounded in \( L^1(\Omega_T) + L^\sigma(0,T;W^{-1,\sigma}(\Omega)) \), for some \( \sigma > 1 \).

Moreover,

\[
\int \int_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq C \quad \text{for all } \alpha > 0.
\]

Furthermore, the sequence \( \{\nabla v_n\}_n \) is bounded in the Marcinkiewicz space \( M^q(\Omega_T) \) where \( q = \frac{N+2a_1}{N+a_1} \) and \( \{v_n\}_n \) is bounded in the space \( M^\sigma(\Omega_T) \) where \( \sigma = \frac{N+2a_1}{N} \).

Proof. Take \( \phi(v_n) \equiv T_k(v_n) \) in Lemma 2.5 with \( k > 0 \), as a test function in the weak formulation of \( (\text{I}) \). Then

\[
\Phi_k(v_n(x,t)) \, dx - \int_{\Omega} \Phi_k(b^{-1}(g_n(x))) \, dx + \int_{\Omega_T} |\nabla T_k(v_n)|^2 \leq k|\mu|(\Omega_T),
\]

where \( \Phi_k(s) = \int_0^s T_k(\sigma) b'(\sigma) \, d\sigma \) which is a nonnegative function. Since \( \Phi_k(s) \leq k|b(s)| \), it follows from \( (\text{I}) \) that

\[
\int_{\Omega} |\Phi_k(v_n(x,t))| \, dx + \int_{\Omega_T} |\nabla T_k(v_n)|^2 \leq ck \int_{\Omega_T} (1 + |b(v_n)|) + ck.
\]

Now, dropping a nonnegative term, dividing by \( k \) and taking the limit for \( k \) going to \( 0 \), it yields

\[
\int_{\Omega} |b(v_n(x,t))| \, dx \leq c \int_{\Omega_T} |b(v_n)| + c.
\]
Thus, Gronwall’s Lemma implies that

\[(11) \sup_{t \in [0,T]} \int_\Omega |b(v_n(x,t))| \, dx \leq C,\]

and going back to (10) we get

\[(12) \iint_{\Omega_T} |\nabla T_k(v_n)|^2 \leq C k.\]

In order to prove the estimate (5), we define \(\theta(s) = (1 - (1 + |s|)^{-\alpha})\), with \(0 < \alpha\), and we take \(\theta(v_n)\) as a test function in the approximating problems. There results

\[(13) \int_\Omega \Theta(v_n(T)) \, dx - \int_\Omega \Theta(b^{-1}(g_n)) \, dx + \alpha \int_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq ||\mu||,\]

where \(\Theta(s) = \int_0^s b'(\alpha) \theta(\sigma) \, d\sigma\). Hence

\[
\int_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq C \text{ for all } \alpha > 0,
\]

and then \(\{(v_n| + 1)^{\beta} - 1\}_n\) is bounded in \(L^2(0,T; W^{1,2}_0(\Omega))\) for all \(0 < \beta < \frac{1}{2}\).

To get the estimates on the Marcinkiewicz spaces, we follow closely the arguments in [7]; see also [5]. Since \((B1)\) holds, we obtain that \(b(s) \geq C_1 s^{a_1} - C\) for all \(s \geq 0\). By again using (11) there exists a positive constant \(C\) such that

\[(14) |\{x \in \Omega : |v_n(x,t)| > k\}| \leq \frac{C}{k^{a_1}} \text{ for almost all } t \in [0,T], \text{ all } k > 0 \text{ and all } n \in \mathbb{N}.\]

Then, by Sobolev’s inequality and (12),

\[(15) \int_0^T \left(\{x \in \Omega : |v_n(x,t)| \geq k\} \right)^{2/2^*} dt \leq \int_0^T \left(\frac{\|T_k(v_n(x,t))\|_{2^*}^{2^*}}{k^{2^*}}\right) dt \leq C \int_0^T \frac{\|\nabla T_k(v_n(x,t))\|_2^2}{k^2} dt \leq \frac{C}{k} \text{ for all } k > 0 \text{ and all } n \in \mathbb{N}.\]

Therefore, combining (15) and (14) we obtain, for all \(k > 0\) and all \(n \in \mathbb{N},\)

\[(16) \left|\{(x,t) \in \Omega_T : |v_n(x,t)| \geq k\}\right| \leq \left|\{(x,t) \in \Omega_T : |\nabla v_n(x,t)| \geq h\}\right| + \left|\{(x,t) \in \Omega_T : |\nabla T_k(v_n(x,t))| \geq h\}\right| \leq \frac{C}{k^{N+2a_1}} + \int_{\Omega_T} \frac{\|\nabla T_k(v_n)\|^2}{h^2} \leq \frac{C}{k^{N+2a_1}} + \frac{Ck}{h^2}.\]

A similar estimate for the gradients is now easy to obtain. Indeed, for every \(h, k > 0\), we have

\[
|\{(x,t) \in \Omega_T : |\nabla v_n(x,t)| \geq h\}| \leq |\{(x,t) \in \Omega_T : |v_n(x,t)| \geq k\}| + |\{(x,t) \in \Omega_T : |\nabla T_k(v_n(x,t))| \geq h\}| \leq \frac{C}{k^{N+2a_1}} + \int_{\Omega_T} \frac{\|\nabla T_k(v_n)\|^2}{h^2} \leq \frac{C}{k^{N+2a_1}} + \frac{Ck}{h^2}.\]
Minimizing in $k$, we obtain that for $k = h^{\frac{N}{N+a_1}}$,
\begin{equation}
|\{(x,t) \in \Omega_T : |\nabla v_n(x,t)| \geq h\}| \leq \frac{C}{h^{\frac{N+2a_1}{N+a_1}}}
\end{equation}
for all $h > 0$ and all $n \in \mathbb{N}$.

Hence,
\begin{equation}
\iint_{\Omega_T} |v_n|^\rho \leq C \quad \text{for all } 0 < \rho < \frac{N+2a_1}{N},
\end{equation}
\begin{equation}
\iint_{\Omega_T} |\nabla v_n|^r \leq C \quad \text{for all } 0 < r < \frac{N+2a_1}{N+a_1}.
\end{equation}

From (7) we obtain that $\{(b(v_n))\}_n$ is bounded in $L^1(\Omega_T) + L^\sigma(0,T;W^{-1,\sigma}(\Omega))$ for some $\sigma \geq \frac{N+2a_1}{a_1}$.

Consider $\varrho$ such that
\begin{equation}
-\Delta \varrho = 1 \quad \text{in } \Omega, \quad \varrho \in W^{1,2}_0(\Omega).
\end{equation}
We claim that for all $0 < \delta < \min\{1, \frac{1}{a_2}\}$,
\begin{equation}
\int_0^T \int_\Omega \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho dx \leq C \text{ uniformly in } n.
\end{equation}

To prove the claim we define $K(s) = \int_s^0 b'(\sigma) \frac{1}{(\sigma + \frac{1}{n})^\delta} d\sigma$. By (B2) we get easily that $K(v_n)$ is well defined, $K(0) = 0$ and
\begin{equation}
\sup_{t \in [0,T]} \int_\Omega K(v_n(x,t)) \varrho dx \leq C.
\end{equation}

Taking $\frac{\varrho}{(v_n + \frac{1}{n})^\delta}$ as a test function in (7), it follows that
\begin{align*}
\int_\Omega K(v_n(x,T)) \varrho dx + \frac{1}{1-\delta} \int_\Omega \left[\left(v_n + \frac{1}{n}\right)^{1-\delta} - \left(\frac{1}{n}\right)^{1-\delta}\right] (-\Delta \varrho) \\
= \int_\Omega \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho + \int_\Omega \frac{h_n}{(v_n + \frac{1}{n})^{1+\delta}} \varrho + \int_\Omega K(v_n(x,0)) \varrho dx.
\end{align*}

Since $0 < \delta < 1$, and then using (18), we get
\begin{align*}
\frac{1}{1-\delta} \int_\Omega \left[\left(v_n + \frac{1}{n}\right)^{1-\delta} - \left(\frac{1}{n}\right)^{1-\delta}\right] \leq C \text{ uniformly in } n.
\end{align*}
As a consequence, and by (22),
\begin{equation}
\iint_{\Omega_T} \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho \leq C_1 \text{ uniformly in } n.
\end{equation}
In particular the claim follows.

In the same way, using $1 - \frac{1}{(b(v_n) + 1)^\delta}$, where $0 < \delta$, as a test function in (7), we reach that
\begin{equation}
\sup_{t \in [0,T]} \int_\Omega b(v_n(x,t)) bx + \int_0^{T_1} \int_\Omega \frac{b'(v_n)|\nabla v_n|^2}{(b(v_n) + 1)^{1+\delta}} \leq C \text{ uniformly in } n.
\end{equation}
To finish we have just to prove that
\begin{equation}
b(v_n) \to b(v) \text{ strongly in } L^1(\Omega_T).
\end{equation}
Notice that \( \|b(v_n)\|_{L^1(\Omega_T)} \leq C \) and \( b(v_n) \to b(v) \) a.e. in \( \Omega_T \).

If \( b(s) \leq C s^{ \frac{N+2\alpha_1}{N} - \varepsilon } \), for some \( \varepsilon > 0 \), as \( s \to \infty \), then using (18) and by Vitali’s lemma we reach the strong convergence in (25).

Assume that condition (B3) holds. Let \( w_n = (b(v_n) + 1) \beta \), where \( \beta \leq 1 \) is to be chosen later; then \( \|w_n\|_{L^\infty(0,T;L^1(\Omega))} \leq C \) and \( \nabla w_n = \beta b'(v_n)(b(v_n) + 1)^{\beta - 1} \nabla v_n \). Thus, for \( \delta > 0 \), we have

\[
\int_\Omega |\nabla w_n| = \beta \int_\Omega b'(v_n)(b(v_n) + 1)^{\beta - 1} |\nabla v_n| \\
= \beta \int_\Omega b'(v_n)(b(v_n) + 1)^{2\beta + \delta - 1} \frac{|\nabla v_n|}{(b(v_n) + 1)^{\frac{\delta}{2}}} \\
\leq \frac{1}{2} \beta \int_\Omega \frac{b'(v_n)|\nabla v_n|^2}{(b(v_n) + 1)^{\delta+1}} + \frac{1}{2} \beta \int_\Omega b'(v_n)(b(v_n) + 1)^{2\beta + \delta - 1}.
\]

From (24), we obtain that

\[
\int_\Omega \frac{b'(v_n)|\nabla v_n|^2}{(b(v_n) + 1)^{\delta+1}} \leq C.
\]

Since condition (B3) holds, then by choosing \( \delta \) small enough and \( \beta = a_3 \), we find that

\[
b'(s)(b(s) + 1)^{2\beta + \delta - 1} \leq C_1 s^{ \frac{N+2\alpha_1}{N} - \varepsilon } \quad \text{as } s \to \infty \quad \text{for some } \varepsilon > 0.
\]

Thus

\[
\frac{\beta}{2} \int_\Omega b'(v_n)(b(v_n) + 1)^{2\beta + \delta - 1} \leq C \quad \text{uniformly in } n.
\]

If the second condition in (B3) holds, with the same \( \beta \) and for \( \delta \) small,

\[
b'(s)(b(s) + 1)^{2\beta + \delta - 1} \leq b^{1-\varepsilon}(s) \quad \text{as } s \to \infty,
\]

and hence

\[
\frac{\beta}{2} \int_\Omega b'(v_n)(b(v_n) + 1)^{2\beta + \delta - 1} \leq C \quad \text{uniformly in } n.
\]

Therefore, we conclude that \( \|w_n\|_{L^1(0,T;W^{1,1}(\Omega))} \leq C \) for all \( n \). Hence, using the Gagliardo-Nirenberg inequality we conclude that \( \|w_n\|_{L^\infty(\Omega_T)} \leq C \).

Since \( \beta \frac{N}{N-1} > 1 \), then \( \{b(v_n)\}_n \) is bounded in \( L^{1+\varepsilon}(\Omega_T) \) for some \( \varepsilon > 0 \). Hence, using Vitali’s lemma, we obtain that the sequence \( \{b(v_n)\}_n \) is compact in \( L^1(\Omega_T) \) and so we may extract a subsequence (also labelled by \( n \)) such that \( b(v_n) \to b(v) \) strongly in \( L^1(\Omega_T) \).

2.3. Pointwise convergence of the gradients. In this subsection we prove that up to a subsequence

\[
\nabla v_n \to \nabla v, \quad \text{a.e. in } \Omega_T \text{ as } n \to \infty.
\]

Hence we will obtain that the sequence \( \{v_n\}_n \) satisfies condition (4) (c) in Definition 2.4.

Proposition 2.7. Consider \( \{v_n\}_n \), the solution of the approximated problems (7). Then, up to subsequence,

\[
(26) \quad \nabla T_k(v_n) \to \nabla T_k(v) \quad \text{almost everywhere in } \Omega_T.
\]

As a consequence, \( \nabla v_n \to \nabla v \) almost everywhere in \( \Omega_T \).
Proof. We recall the time–regularization of functions due to Landes and Mustonen (see [22], [23]). Consider $w$ such that $T_k(w) \in L^2(0, T: W^{1,2}_0(\Omega)) \cap C([0, T] : L^2(\Omega))$. For every $\nu \in \mathbb{N}$, we define $(T_k w)_\nu$ as the solution of the Cauchy problem

$$
\begin{cases}
\frac{1}{\nu} [(T_k w)_\nu, t] + (T_k w)_\nu &= T_k w; \\
(T_k w)_\nu(0) &= T_k(w_0).
\end{cases}
$$

(27)

Then, one has $(T_k w)_\nu \in L^2(0, T; W^{1,2}_0(\Omega))$, $((T_k w)_\nu, t) \in L^2(0, T; W^{-1,2}_0(\Omega))$

\[ ||(T_k w)_\nu||_{L^\infty(\Omega_T)} \leq ||T_k w||_{L^\infty(\Omega_T)} \leq k, \]

and if $\nu \to \infty$, $(T_k w)_\nu \to T_k w$ strongly in $L^2(0, T; W^{1,2}_0(\Omega))$ as $\nu \to \infty$.

From now on, we will denote by $\omega(n, \nu, \varepsilon)$ any quantity such that

\[ \lim_{\varepsilon \to 0^+} \limsup_{\nu \to \infty} \limsup_{n \to \infty} |\omega(n, \nu, \varepsilon)| = 0. \]

Taking $T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu))$ as a test function in (17), we obtain

\[ \int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt + \int_{\Omega_T} \nabla v_n \cdot \nabla T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \]

\[ = \int_{\Omega_T} h_n T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu). \]

We will analyze the integrals which appear in the previous identity. For simplicity of typing we set $w_n = b(v_n)$, $w = b(v)$ and $\psi = b^{-1}$.

It is clear that

\[ \int_{\Omega_T} h_n T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \leq C \varepsilon. \]

Notice that

\[ \int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt = \int_0^T \langle ((w_n)_t, T_\varepsilon(w_n - (T_k(w)))_\nu) \rangle dt. \]

Then using the same arguments as in [14] (see also (3.37) in [1]) we reach that

$$
\begin{align*}
\int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt &\geq \omega(n, \nu, \varepsilon).
\end{align*}
$$

(28)

We set

\[ I = \int_{\Omega_T} \nabla v_n \cdot \nabla T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu). \]

Claim. It holds that

$$
I \geq \int_{\{|w_n| \leq k\}} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(T_k(w_n) - (T_k w)_\nu) + \omega^\varepsilon(n, \nu).
$$

(29)
Indeed,

\[ I = \int \int_{\{ w_n \leq k \}} \psi'(w_n) \nabla T_k(w_n) \cdot \nabla T_\varepsilon (T_k(w_n) - (T_k w)_\nu) \]

\[ + \int \int_{\{ w_n > k \}} \psi'(w_n) \nabla w_n \cdot \nabla T_\varepsilon (w_n - (T_k w)_\nu) \]

\[ \geq \int \int_{\{ w_n \leq k \}} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon (T_k(w_n) - (T_k w)_\nu) \]

\[ - \int \int_{\{ w_n > k, |w_n - (T_k w)_\nu| \leq \varepsilon \}} \psi'(w_n) \nabla w_n \cdot \nabla (T_k w)_\nu. \]

Since \( \|(T_k w)_\nu\|_\infty \leq k \), the last integrand is different from zero only in the set where \( |w_n| \leq k + \varepsilon \); therefore the last integrand is bounded by

\[ c_1(k) \left[ \int \int_{\Omega_T} \left| \nabla T_{k+\varepsilon} w_n \right|^2 \right]^{\frac{1}{2}} \left[ \int \int_{\Omega_T} \left| \nabla (T_k w)_\nu \right|^2 \chi_{\{ w_n > k \}} \right]^{\frac{1}{2}} \]

\[ \leq c_2(k, \varepsilon) \left[ \int \int_{\Omega_T} \left| \nabla T_k w \right|^2 \chi_{\{ w > k \}} \right]^{\frac{1}{2}} + \omega^\varepsilon(n, \nu) = \omega^\varepsilon(n, \nu). \]

Here we have used the a.e. convergence of \( \chi_{\{ w_n > k \}} \) to \( \chi_{\{ w > k \}} \) as \( n \to +\infty \), which holds for all \( k \) (see for instance [14]).

Therefore, putting together the above estimates, we reach that

\[ \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon (T_k w n_n - (T_k w)_\nu) \]

\[ = \int \int_{\{ w_n \leq k \}} \psi'(w_n) \nabla T_k(w_n) \cdot \nabla T_\varepsilon (w_n - (T_k w)_\nu) \leq \omega(n, \nu, \varepsilon). \]

Thus

\[ \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w n_n - (T_k w)_\nu) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} \leq \omega(n, \nu, \varepsilon). \]

Hence it follows that

\[ \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w n_n - T_k w) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} \]

\[ = \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w n_n - T_k w) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} \]

\[ + \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla ((T_k w)_\nu - T_k w) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} \]

\[ = \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w n_n - (T_k w)_\nu) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} + \omega(n, \nu, \varepsilon). \]

Therefore, by (31) we obtain

\[ \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k w \cdot \nabla (T_k w n_n - T_k w) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} \leq \omega(n, \nu, \varepsilon). \]

On the other hand, we have that

\[ \int \int_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k w \cdot \nabla (T_k w n_n - T_k w) \chi_{\{ T_k(w_n) - (T_k w)_\nu \leq \varepsilon \}} = \omega^{\nu, \varepsilon}(n). \]
Indeed, $\psi'(T_k(w_n))\nabla T_k w \to \psi'(T_k(w))\nabla T_k w$ strongly in $L^2(\Omega_T)$ and $\nabla T_k w_n \rightharpoonup \nabla T_k w$ weakly in $L^2(\Omega_T)$. Hence, we deduce from (32) that

$$
\int_{\{|T_k(w_n)-(T_k w)_\nu|\leq \varepsilon\}} \psi'(T_k(w_n))|\nabla T_k(w_n) - \nabla T_k w|^2 \leq \omega(n, \nu, \varepsilon).
$$

Denoting $\Psi_{n,k} = \psi'(T_k(w_n))|\nabla T_k(w_n) - \nabla T_k w|^2$, then

$$
\int_{\{|T_k(w_n)-(T_k w)_\nu|\leq \varepsilon\}} \Psi_{n,k} \leq \omega(n, \nu, \varepsilon),
$$

since

$$
\chi(\{|T_k(w_n)-(T_k w)_\nu|> \varepsilon\}) \rightharpoonup \chi(\{|T_k(w)-(T_k w)_\nu|> \varepsilon\}) \quad \text{strongly in } L^p(\Omega_T), \forall p \geq 1.
$$

Then using Hölder inequality and the fact that $\Psi_{n,k}$ is bounded in $L^1(\Omega_T)$, we deduce that $\int_{\Omega_T} \Psi_{n,k} = \omega(n, \nu, \varepsilon)$ for all $\theta < 1$. Since $\psi'(T_k(s)) \geq c(k)(\psi'(T_k(s)))^2$, it follows that

$$
\lim_{n \to \infty} \int_{\Omega_T} \left[ (\psi'(T_k(w_n)))^2 |\nabla T_k(w_n) - \nabla T_k w|^2 \right]^{\theta} = 0.
$$

Hence we get $\psi'(T_k(w_n))\nabla T_k(w_n) \to \psi'(T_k(w))\nabla T_k(w)$ a.e. in $\Omega$, and then

$$
\nabla \psi(T_k(w_n)) \to \nabla \psi(T_k(w)) \quad \text{a.e. in } \Omega.
$$

Taking into consideration that the function $\psi$ is a strictly monotone function we reach

$$
\nabla T_k(w_n) \to \nabla T_k(w) \quad \text{a.e. in } \Omega
$$

and then $\nabla T_k(v_n) \to \nabla T_k(v)$ a.e. in $\Omega$.

We summarize the previous result in the following theorem.

**Theorem 2.8.** Let $b$ be a function verifying the hypotheses (B). Let $\mu$ be a finite Radon measure, and consider $\{v_n\}_n$ to be a sequence of solutions to (7). Then there exists a measurable function $v$ which is a reachable solution to the problem (2), namely,

1. $\nabla v_n \to \nabla v$ strongly in $L^\sigma(\Omega_T)$ for all $1 \leq \sigma < \frac{N+2\alpha_1}{N+\alpha_1}$.
2. From estimate (36), we reach

$$
\nabla T_k(v_n) \to \nabla T_k(v) \quad \text{strongly in } L^\sigma(\Omega_T) \text{ for all } \sigma < 2.
$$

3. For every $\Phi \in C^\infty(\overline{\Omega_T})$, such that $\Phi(\cdot, t) \in C_0(\Omega)$ for all $t \in (0, T)$ and $\Phi(x, T) = 0$ for all $x \in \Omega$, the following identity holds:

$$
- \int_{\Omega_T} b(v_0(x))\Phi(x, 0) \, dx - \int_{\Omega_T} b(v)\Phi_t \, dx + \int_{\Omega_T} \nabla v \cdot \nabla \Phi = \int_{\Omega_T} \Phi \, d\mu.
$$

2.4. **Application to the porous media and fast diffusion equations with a Radon measure.** The results in the above subsection allow us to consider the problem

$$
\begin{cases}
    u_t - \Delta u^m = \mu & \text{in } \Omega \times (0, T), \\
    u(x, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
    u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$
with \( m > \frac{(N-2)_+}{N} \), \( u_0 \in L^1(\Omega) \) and \( \mu \) a Radon measure whose total variation is finite in \( \Omega_T \).

We can directly use the result of Theorem 2.8 to get the existence of a reachable solution to problem (39). However we will use an equivalent approach that will be useful in the following sections to analyze the truncated problems in (PME) and (FDE) with a gradient term.

In the case of the porous media equation, i.e., \( m > 1 \), the existence results are obtained in [24] by using some result in [21]: however our approach is different and follows using the elliptic-parabolic equation. Here we extend the results to the interval \( \frac{(N-2)_+}{N} < m < 1 \), by proving, moreover, the a.e. convergence of the gradients of the truncated problems to the gradient of the solution of problem (39). Our approach is different and we use the elliptic-parabolic framework discussed above.

We will consider the approximated form

\[
\begin{aligned}
  u_{nt} - \text{div}(m(u_n + \frac{1}{n})^{m-1}\nabla u_n) &= h_n & \text{in } \Omega \times (0, T), \\
  u_n(x, t) &= 0 & \text{on } \partial \Omega \times (0, T), \\
  u_n(x, 0) &= T_n(u_0) & \text{in } \Omega.
\end{aligned}
\]

The main goal of this subsection is to show compactness results for the sequences \( \{\nabla u_n\}_n \) and \( \{T_k(u_n)\}_n \) including the case \( \frac{(N-2)_+}{N} < m < 1 \).

Define \( v_n \equiv (u_n + \frac{1}{n})^m - (\frac{1}{n})^m \); then \( v_n \) solves

\[
\begin{aligned}
  (b_n(v_n))_t - \Delta v_n &= h_n, & (x, t) &\in \Omega_T, \\
  v_n(x, t) &= 0, & (x, t) &\in \partial \Omega \times (0, T), \\
  v_n(x, 0) &= b_n^{-1}(T_n(u_0(x))), & x &\in \Omega,
\end{aligned}
\]

where \( b_n(s) = (s + (\frac{1}{n})^m)^{\frac{1}{m}} - \frac{1}{n}, n \geq 1 \).

**Theorem 2.9.** Consider \( v_n \) and \( u_n = b(v_n) \), the solutions to (41) and (40) respectively. Then, there exists a measurable function \( u \) such that \( u^n \in L^r(0, T; W_0^{1, r}(\Omega)) \) for all \( r < 1 + \frac{1}{Nm+1} \), and, up to a subsequence,

1. \( \nabla v_n \rightarrow \nabla u \) a.e. in \( \Omega_T \) and then \( \nabla v_n \rightarrow \nabla v \) a.e. in \( \Omega_T \) where \( v = u^m \).
2. \( T_k(v_n) \rightarrow T_k(v) \) strongly in \( L^\sigma(0, T; W_0^{1, \sigma}(\Omega)) \) for all \( k > 0 \) and for all \( \sigma < 2 \).

**Proof.** It is easy to check that by the same computation as in Theorem 2.8 we obtain

1. \( \nabla v_n \rightarrow \nabla v \) strongly in the space \( L^\sigma((0, T); L^\sigma(\Omega)) \) for all \( 1 \leq \sigma < 1 + \frac{1}{Nm+1} \) and
2. \( \nabla T_k(v_n) \rightarrow \nabla T_k(v) \) strongly in \( L^\sigma(\Omega_T) \) for all \( \sigma < 2 \).

Let \( \psi \in C_0^\infty(\Omega_T) \) be such that \( \psi \geq 0 \) in \( \Omega_T \); we claim that

\[
\left\{ \left| \nabla T_k(u_n) \right| \left( u_n + \frac{1}{n} \right)^\theta \psi \right\}_n \text{ is uniformly bounded in } L^1(\Omega_T) \text{ for some } \theta > 0.
\]

To prove the claim we will consider separately the cases \( 0 < m < 1 \) and \( m > 1 \).
Let us begin with the case $0 < m < 1$. Using $T_k(u_n)$ as a test function in (39), we get
\[ \iint_{\Omega_T} (u_n + \frac{1}{n})^{m-1} |\nabla T_k(u_n)|^2 \leq C. \]
Thus the claim follows with $\theta = 1 - m > 0$.

We deal now with the case $m > 1$. Using $\psi(v_n + \frac{1}{n})^m$ where $\delta < \frac{1}{m}$, as a test function in (41), we obtain that
\[ \iint_{\Omega_T} |\nabla v_n|^2 (v_n + \frac{1}{n})^{-\delta} \leq C. \]
Notice that
\[ \frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^\theta} = \frac{|\nabla T_k(v_n)|}{(v_n + \frac{1}{n})^{\theta - \frac{1}{m}} \psi}. \]
Now using (43) we conclude that \( \{ \frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^\theta} \psi \}_n \) is bounded in $L^1(\Omega_T)$ for all $\theta < 1$. Then, the claim is proved.

By similar computation as above, taking $\psi(u_n + \frac{1}{n})^m$ as a test function in (40) with $\delta < 1$, we reach that
\[ \delta \iint_{\Omega_T} |\nabla u_n|^2 (u_n + \frac{1}{n})^{\delta + 1 - m} \psi + \iint_{\Omega_T} h_n (u_n + \frac{1}{n})^\delta \psi \leq C \text{ for all } n. \]
We will prove point (1) in the theorem, that is, $\nabla u_n \to \nabla u$ a.e. in $\Omega_T$ where $u = v^{\frac{1}{m}}$

To prove this assertion it is sufficient to show that, for some $s \in (0, 1)$,
\[ \iint_{\Omega_T} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi \to 0 \text{ as } n \to \infty. \]
Consider $A \equiv \{(x, t) \in \Omega_T : u(x, t) = 0\} \equiv \{(x, t) \in \Omega_T : v(x, t) = 0\}$. Notice that $\nabla u_n \to \nabla v$ in $\Omega_T \setminus A$. Since $\nabla T_k(u) = \nabla T_k(v) = 0$ in $A$, then
\[ \iint_{\Omega_T} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi = \iint_{\{u = 0\}} |\nabla T_k(u_n)|^s \psi \]
Using the fact that $s < 1$ and $\nabla u_n^m \to \nabla v^m$ strongly in the space $L^\sigma(\Omega_T)$ for all $1 \leq \sigma < 1 + \frac{1}{Nm+1}$, it follows that
\[ \iint_{\Omega_T \setminus A} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi \to 0 \text{ as } n \to \infty. \]
To conclude the proof, it is sufficient to show that $\int_A |\nabla T_k(u_n)|^s \psi \to 0$ as $n \to \infty$.

Notice that $T_k(u_n) \to T_k(u)$ strongly in the space $L^\sigma(\Omega_T)$ for all $\sigma > 1$. Then by Egorov’s Lemma, for every $\epsilon > 0$, there exists a measurable set $E_\epsilon$ such that
\[ |E_\varepsilon| \leq \varepsilon \text{ and } T_k(u_n) \rightarrow T_k(u) \text{ uniformly in } \Omega_T \setminus E_\varepsilon. \]

Then

\[
\int \int_{\{T_k(u) = 0\}} |\nabla T_k(u_n)|^s \psi = \int \int_{\{T_k(u) = 0\} \cap E_\varepsilon} |\nabla T_k(u_n)|^s \\
+ \int \int_{\{T_k(u) = 0\} \cap \Omega \setminus E_\varepsilon} |\nabla T_k(u_n)|^s \psi \\
= I_1 + I_2.
\]

Using the fact that \(|\nabla T_k(u_n)|\}_n is uniformly bounded in \(L^1(\Omega_T)\), and choosing \(s < 1\), we find that

\[ I_1 \leq C |E_\varepsilon|^{1-s}. \]

By the uniform convergence of \(T_k(u_n)\) in \(\Omega_T \setminus E_\varepsilon\), we obtain

\[
I_2 \leq \int \int_{\{T_k(u_n) \leq M\} \cap \Omega \setminus E_\varepsilon} |\nabla T_k(u_n)|^s \psi \\
\leq (M + \frac{1}{n})^a \int \int_{\{T_k(u_n) \leq M\} \cap \Omega \setminus E_\varepsilon} \left(\frac{|\nabla T_k(u_n)|}{u_n + \frac{1}{n}}\right)^s \psi,
\]

where \(a > 0\). Since estimate (42) holds, it is sufficient to pick up \(s < 1\) and \(a > 0\) such that

\[
\int \int_{\{T_k(u_n) \leq M\} \cap \Omega \setminus E_\varepsilon} \left(\frac{|\nabla T_k(u_n)|}{u_n + \frac{1}{n}}\right)^s \psi \leq C.
\]

Taking limits for \(M \rightarrow 0\) in (47), the result follows. \(\square\)

**Remark 2.10.** If \(0 < m \leq \frac{(N - 2)_+}{N}\), the difficulty is to show the strong convergence of the sequence \(\{u_n\}_n\) in \(L^1(\Omega_T)\). This can be proved by assuming additional hypotheses on the measure \(\mu\); see Section 4. Once proving this strong convergence, the result of Theorem 2.9 holds with the same conclusions.

### 3. The Porous Medium Equation with Gradient Term

The main goal of this section is to prove the existence of a solution to problem (\textcircled{1}). We start by obtaining *a priori estimates* for the truncated problems, in order to be able to apply the results for an associated *elliptic-parabolic* problem.

More precisely, the proof of the existence result is a consequence of the following steps:

1. We prove some a priori estimates that allow us to show that the right hand side of the truncated problems converge weak-* to a Radon measure.
2. We transform in a natural way the problem to an *elliptic-parabolic* problem.
3. By using the results of Theorem 2.9 and some compactness arguments, we identify the measure limit as the second member of problem (\textcircled{1}).

We divide the section in two parts according to the values of \(m\).

#### 3.1. The case \(1 < m \leq 2\).

In this subsection we will consider the problem

\[
\begin{cases}
  u_t - \Delta u^m = |\nabla u|^q + f(x,t) & \text{in } \Omega_T \equiv \Omega \times (0, T), \\
  u(x, t) \geq 0 & \text{in } \Omega_T, \\
  u(x, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) = u_0(x) & \text{if } x \in \Omega,
\end{cases}
\]

(48)
where \( m > 1, \, q \leq 2, \, \Omega \subset \mathbb{R}^N \) is a bounded domain, and \( f \) and \( u_0 \) are nonnegative functions under suitable hypotheses given below.

We will use as a starting point the results in [13] for bounded data, \( f \in L^\infty(\Omega_T) \) and \( u_0 \in L^\infty(\Omega) \). Since \( 1 < m \leq 2 \) and \( 1 \leq q \leq 2 \) we will be able to obtain \textit{a priori} estimates in the framework of [24]. Notice that in [21], these estimates are used to analyze the behavior of a \textit{viscosity supersolution} to the porous medium equation. See [21] and [24] for more details concerning this framework.

More precisely we have the next theorem.

**Theorem 3.1.** Assume that \( 1 < m \leq 2 \) and \( q \leq 2 \). Then

1. If \( q'(m-1) > 2 \), \( u_0 \in L^{1+\theta}(\Omega) \) and \( f \in L^{1+\frac{2q}{m}}(0,T;L^{\frac{(\theta+m)N}{mN+2q}}(\Omega)) \) where \( \theta \geq 2 - m \), then problem (48) has a distributional solution.
2. If \( q'(m-1) \leq 2 \)
   a. If \( q < m \), problem (48) has a solution for all \( f, u_0 \) as in the first case.
   b. If \( m \leq q \leq 2 \), then problem (48) has a solution if \( e^{\alpha u_0} \in L^{1}(\Omega) \) for some \( \alpha > 0 \) and \( f \in L^{r}(0,T;L^{s}(\Omega)) \) where \( 1 < r < \infty, s > \frac{N}{2} \) and \( \frac{1}{r} + \frac{N}{2s} = 1 \).

**Proof.**

\textbf{Step 1. A priori estimates.} We prove each case separately.

\textbf{(I)} \( q'(m-1) > 2 \). Assume that \( q'(m-1) > 2 \) and fixed \( \theta > 2 - m \), since \( m > 1 \), and then \( q < 2 \). Let \( u_0 \in L^{1+\theta}(\Omega) \) and \( f \in L^{1+\frac{2q}{m}}(0,T;L^{\frac{(\theta+m)N}{mN+2q}}(\Omega)) \). Then there exist sequences \( \{f_n\}, \{u_{0n}\} \) such that \( f_n \in L^{\infty}(\Omega_T), u_{0n} \in L^{\infty}(\Omega), u_n \uparrow u_0 \) in \( L^{1+\theta}(\Omega) \) and \( f_n \uparrow f \) in \( L^{1+\frac{2q}{m}}(0,T;L^{\frac{(\theta+m)N}{mN+2q}}(\Omega)) \).

Define \( u_n \) to be the bounded solution of the approximated problem

\begin{equation}
\begin{cases}
  u_{nt} - \text{div}(m(u_n + \frac{1}{n})^{m-1}\nabla u_n) = \frac{\nabla u_n|^q}{|\nabla u_n|^q + \frac{1}{n}} + f_n & \text{in } \Omega_T, \\
  u_n(x,t) = 0 & \text{on } \partial \Omega \times (0,T), \\
  u_n(x,0) = u_{0n}(x) & \text{if } x \in \Omega.
\end{cases}
\end{equation}

Notice that the existence and the boundedness of \( u_n \) follow using the result of [13].

Taking \( (G_k(u_n))^\theta \) as a test function in (49), with \( \theta > 2 - m \), it follows that

\begin{equation}
\frac{d}{dt} \frac{1}{\theta + 1} \int_\Omega (G_k(u_n))^{\theta+1} dx + m \int_\Omega u_n^{m-1} (G_k(u_n))^{\theta-1} |\nabla u_n|^2 dx \\
\leq \int_\Omega (G_k(u_n))^{\theta} |\nabla u_n|^q dx + \int_\Omega f_n (G_k(u_n))^\theta dx.
\end{equation}

Using the Hölder inequality,

\begin{equation}
\int_\Omega (G_k(u_n))^{\theta} |\nabla u_n|^q dx \leq \varepsilon \int_\Omega (G_k(u_n))^{m+\theta-2} |\nabla u_n|^2 dx + c(\varepsilon) \int_\Omega (G_k(u_n))^{\theta + \frac{2(2-m)}{2-q}} dx.
\end{equation}

Since \( q'(m-1) > 2 \), then \( \theta + \frac{q(2-m)}{2-q} < \theta + 1 \). Thus

\begin{equation}
\int_\Omega (G_k(u_n))^{\theta} |\nabla u_n|^q dx \leq \varepsilon \int_\Omega (G_k(u_n))^{m+\theta-2} |\nabla u_n|^2 dx + c(\varepsilon) \int_\Omega (G_k(u_n))^{\theta+1} dx + C(\Omega).
\end{equation}
We deal with the last term in (50). Using Hölder, Young and Sobolev inequalities we reach that
\[
\int_\Omega f_n(G_k(u_n))^{\theta} dx \leq \epsilon \left( \int (G_k(u_n))^{\frac{(m+\theta)N}{N-2}} dx \right)^{\frac{N-2}{N}} + C(\epsilon) \left( \int \frac{f^{(m+\theta)N}}{N+m} dx \right)^{\frac{mN+2\theta}{N-m}}
\leq \frac{\epsilon}{S} \int_\Omega |\nabla(G_k(u))|^{\frac{m+\theta}{2}} dx + C(\epsilon) \left( \int \frac{f^{(m+\theta)N}}{N+m} dx \right)^{\frac{mN+2\theta}{N-m}}.
\]
Choosing \(\epsilon\) small enough, it follows that
\[
\frac{d}{dt} \frac{1}{\theta+1} \int_\Omega (G_k(u_n))^{\theta+1} dx + c \int_\Omega (G_k(u_n))^{\theta+2} |\nabla G_k(u_n)|^2 dx 
\leq c(\epsilon) \int_\Omega (G_k(u_n))^{\theta+1} dx + C(\epsilon) \left( \int \frac{f^{(m+\theta)N}}{N+m} dx \right)^{\frac{mN+2\theta}{N-m}} + C(\Omega).
\]
Integrating in time and using Gronwall’s Lemma there results that
\[
C \frac{1}{\theta+1} \int_\Omega (G_k(u_n))^{\theta+1} dx + c \int_{\Omega_T} (G_k(u_n))^{\theta+2} |\nabla G_k(u_n)|^2 dx 
\leq \frac{1}{\theta+1} \int_\Omega u_0^{\theta+1} dx + \int_0^T \left( \int \frac{f^{(m+\theta)N}}{N+m} dx \right)^{\frac{mN+2\theta}{N-m}} + C(\Omega, T).
\]
Now, taking \(T_k(u_n)\) as a test function in the problem of \(u_n\), we reach that
\[
\int_\Omega \Theta_k(u_n) dx + m \int_{\Omega_T} (u_n + \frac{1}{n})^{m-1} |\nabla T_k(u_n)|^2 
\leq \int_{\Omega_T} T_k(u_n) |\nabla u_n|^q dx + k \int_{\Omega_T} f 
\leq \int_{\{u_n \leq \sigma\}} T_k(u_n) |\nabla u_n|^q dx + k \int_{\{u_n \geq \sigma\}} |\nabla u_n|^q dx + k \int f,
\]
where \(\Theta_k(s) = \int_0^s T_k(\sigma) d\sigma\). From (50) we obtain that \(\int_{\{u_n \geq \sigma\}} |\nabla u_n|^q dx \leq C(\sigma, f)\); thus
\[
\int_\Omega \Theta_k(u_n) dx + m \int_{\Omega_T} (T_k(u_n))^{m-1} |\nabla T_k(u_n)|^2 
\leq \int_{\{u_n \leq \sigma\}} T_k(u_n) |\nabla u_n|^q dx + C(k, a, f).
\]
Notice that, for \(\sigma \ll k\) small, there exists \(C(k) \gg 1\) such that \(T_k^{m-1}(s) \geq C(k)T_k(s), 0 \leq s \leq \sigma\). Hence, using Young’s inequality (if \(q < 2\)), there results that
\[
\int_\Omega \Theta_k(u_n) dx + c \int_{\Omega_T} T_k^{m-1}(u_n) |\nabla T_k(u_n)|^2 \leq C(\Omega, T, k).
\]
Therefore, combining (53) and (55) we conclude that
(1) \{u_n^{1+\theta}\}_n is bounded in \( L^\infty(0,T; L^1(\Omega)) \).
(2) \{(G_k(u))_{\frac{n^m}{2}}\}_n is bounded in \( L^2(0,T; W^{1,2}_0(\Omega)) \), and then using Poincaré
inequality, it follows that \{u_n^{m+\theta}\}_n is bounded in \( L^1(\Omega_T) \).

We claim that \{\|\nabla T_k(u_n)\|\}_n is bounded in \( L^2(\Omega_T) \) if \( m < 2 \), while, \{\|\nabla T_k(u_n)\|\}_n is bounded in \( L^2(\Omega_T, \delta(x)) \) if \( m = 2 \), where \( \delta(x) \equiv \text{dist}(x, \partial \Omega) \).

If \( m < 2 \), using \( w_n \equiv e^{\frac{c}{m(2-m)}}(T_k(u_n) + \frac{1}{n})^{2-m} - e^{\frac{c}{m(2-m)}}(\frac{1}{n})^{2-m} \) as a test function in (49), we obtain
\[
\int_\Omega L_n(u_n)dx + c \int_\Omega \int_\Omega e^{\frac{c}{m(2-m)}}(T_k(u_n) + \frac{1}{n})^{2-m} |\nabla T_k(u_n)|^2 dx dt \\
\leq \int_\Omega L_n(u_0)dx + \int_\Omega \int_\Omega e^{\frac{c}{m(2-m)}}(T_k(u_n) + \frac{1}{n})^{2-m} |\nabla u_n|^q dx dt + C(k) \int_\Omega \int_\Omega \phi dx,
\]
where \( L_n(s) = \int_0^s \left( e^{\frac{c}{m(2-m)}}(T_k(s) + \frac{1}{n})^{2-m} - e^{\frac{c}{m(2-m)}}(\frac{1}{n})^{2-m} \right) ds \). Notice that \( L_n(s) \leq C(k)s \).

Thus choosing \( c \gg 1 \) and using estimate (53) on \( G_k(u_n) \) there results that
\[
\int_\Omega \int_\Omega |\nabla T_k(u_n)|^2 dx dt \leq C(k).
\]

Therefore the claim follows in this case.

Assume that \( m = 2 \) and consider \( \phi \), defined in (20). For fixed \( 0 < \alpha < 1 \), to be chosen later, and using \( \frac{\theta}{(u_n + \frac{1}{n})^\alpha} \) as a test function in (49), we get
\[
\frac{1}{1-\alpha} \int_\Omega \left( u_n + \frac{1}{n} \right)^{1-\alpha} \phi dx + \frac{1}{2-\alpha} \int_\Omega \left( (u_n + \frac{1}{n})^{2-\alpha} - (\frac{1}{n})^{2-\alpha} \right) \phi dx \\
= \alpha \int_\Omega \int_\Omega \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\alpha} \phi dx + \int_\Omega \int_\Omega \frac{\phi}{(u_n + \frac{1}{n})^\alpha} |\nabla u_n|^q dx \\
+ \int_\Omega \int_\Omega \frac{f_n \phi}{(u_n + \frac{1}{n})^\alpha} dx + \frac{1}{1-\alpha} \int_\Omega (u_n + \frac{1}{n})^{1-\alpha} \phi dx.
\]
Choosing \( \alpha \) such that \( 1 - \theta < \alpha < 1 \), then from (53) it follows that the first term in the above identity is uniformly bounded in \( n \). Thus
\[
\alpha \int_\Omega \int_\Omega \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\alpha} \phi dx \leq C
\]
and then
\[
\int_\Omega \int_\Omega |\nabla T_k(u_n)|^2 \phi \leq C(k).
\]

Since \( \phi \preceq \delta(x) \), the claim follows in this case.

Combining the result of the claim and estimates (53) and (55), we get easily that
(1) \{\|u_n^q\|\}_n is bounded in \( L^1(\Omega_T) \) if \( m < 2 \),
(2) \{\|u_n^q\|\}_n is bounded in \( L^1_{\text{loc}}(\Omega_T) \) if \( m = 2 \).

Notice that in both cases we can prove that \{\|u_n^{\theta+m-2}\|\}_n is bounded in \( L^1(\Omega_T) \).
(II) $q'(m-1) \leq 2$ and $q < m$. We deal now with the case (2)(a). Assume that $q'(m-1) \leq 2$ and $q < m$. Then the result follows using the same kind of computations as in the first case, with the main difficulty being to estimate the second term in (51), where we will use the Poincaré inequality. This is possible since $\theta + \frac{n(2-m)}{2-q} < m + \theta$. More precisely, we have, using the Hölder and Poincaré inequalities,

$$\int_\Omega (G_k(u_n))^{\theta + \frac{n(2-m)}{2-q}} dx \leq \varepsilon \int_\Omega (G_k(u_n))^{m+\theta} dx + C(\varepsilon, \Omega) \leq \frac{\varepsilon}{\lambda_1} \int_\Omega |\nabla (G_k(u_n))|^{\frac{m+\theta}{\theta}} dx + C(\varepsilon, \Omega).$$

Choosing $\varepsilon$ small enough and going back to estimate (53), it follows that (57)

$$C \frac{1}{\theta + 1} \int_\Omega (G_k(u_n))^{\theta+1} dx + c \int_\Omega (G_k(u_n))^{\theta+2-\theta} |\nabla G_k(u_n)|^2 dx \leq C(u_0, f, T, \Omega).$$

(III) $q'(m-1) \leq 2$ and $m \leq q$. Consider now the case where $q'(m-1) \leq 2$ and $m \leq q$. The existence result in this case follows using the same arguments as in [18]; for the reader's convenience we include some details.

Taking $e^{\alpha(G_k(u_n))} - 1$, with $\alpha > 0$, as a test function in the approximated problem for $u_n$, and calling $H_k(s) = \int_0^s (e^{\alpha(G_k(s))} - 1) d\sigma$, we reach that

$$\frac{d}{dt} \int_\Omega H_k(u_n) dx + m\alpha \int_\Omega u_n^{m-1} e^{\alpha(G_k(u_n))} |\nabla G_k(u_n)|^2 dx = \int_\Omega (e^{\alpha(G_k(u_n))} - 1)|\nabla u_n|^q dx + \int_\Omega f_n (e^{\alpha(G_k(u_n))} - 1) dx.$$ 

Without loss of generality we can assume that $k \geq 1$.

a) If $q = 2$, choosing $m\alpha > 1$, it follows that

$$\frac{d}{dt} \int_\Omega H_k(u_n) dx + m\alpha \int_\Omega (u_n^{m-1} - 1)e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + \int_\Omega |\nabla G_k(u_n)|^2 dx \leq \int_\Omega f_n (e^{\alpha(G_k(u_n))} - 1) dx.$$ 

b) If $q < 2$, by the Young inequality we obtain that

$$\int_\Omega (e^{\alpha(G_k(u_n))} - 1)|\nabla u_n|^q dx \leq \varepsilon \int_\Omega e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + C(\varepsilon) \int_\Omega (e^{\alpha(G_k(u_n))} - 1) dx.$$ 

Hence

$$\frac{d}{dt} \int_\Omega H_k(u_n) dx + m\alpha \int_\Omega (u_n^{m-1} - \varepsilon)e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + \int_\Omega |\nabla G_k(u_n)|^2 dx \leq C(\varepsilon) \int_\Omega (e^{\alpha(G_k(u_n))} - 1) dx + \int_\Omega f_n (e^{\alpha(G_k(u_n))} - 1) dx.$$
For fixed $k \geq 2$, then integrating in $[0, \tau]$ and taking the maximum on $\tau$, it follows that
\begin{equation}
\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau))dx + C \iint_{\Omega\tau} u_n^{m-1}e^{\alpha(G_k(u_n))}|\nabla u_n|^2
\leq \int_{\Omega} H_k(u_0(x))dx + C \iint_{\Omega\tau} (e^{\alpha(G_k(u_n))} - 1)
+ \iint_{\Omega\tau} f_n(e^{\alpha(G_k(u_n))} - 1) \text{ if } q < 2.
\end{equation}

Moreover,
\begin{equation}
\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau))dx + C \iint_{\Omega\tau} u_n^{m-1}e^{\alpha(G_k(u_n))}|\nabla u_n|^2
\leq \int_{\Omega} H_k(u_0(x))dx + \iint_{\Omega\tau} f_n(e^{\alpha(G_k(u_n))} - 1) \text{ if } q = 2.
\end{equation}

Let us analyze the term $\iint_{\Omega\tau} f_n(e^{\alpha(G_k(u_n))} - 1)$:
\[\iint_{\Omega\tau} f_n(e^{\alpha(G_k(u_n))} - 1) \leq \iint_{\Omega\tau} f(e^{2\alpha(G_k(u_n))} - 1)^2 + C \leq \|f\|_{r,s'}(e^{2\alpha(G_k(u_n))} - 1)^2 + C.\]

For simplicity of notation we set $w_n = e^{2\alpha(G_k(u_n))} - 1$. By the Gagliardo-Nirenberg inequalities there results that
\[\|w_n\|_{r',s'}^2 \leq C\|w_n\|_{\infty,2}^2 \left( \iint_{\Omega\tau} |\nabla w_n|^2 \right)^{\frac{1}{2}} \leq C \left( \sup_{\tau \in [0, T]} \int_{\Omega} w_n^2dx \right)^{\frac{1}{2}} \left( \iint_{\Omega\tau} |\nabla w_n|^2 \right)^{\frac{1}{2}}.\]

Hence using Young’s inequality we reach that
\[\iint_{\Omega\tau} f_n(e^{\alpha(G_k(u_n))} - 1)dx \leq C(\varepsilon)\|f\|_{r,s'} \iint_{\Omega\tau} |\nabla w_n|^2 + \varepsilon(\sup_{\tau \in [0, T]} \int_{\Omega} w_n^2dx) + C.\]

Notice that $H_k(u_n) \geq c_1w_n^2 - c_2$; then choosing $\varepsilon$ small it follows that
\[\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau))dx + C(m\alpha - \varepsilon) \iint_{\Omega\tau} u_n^{m-1}|\nabla w_n|^2dx \leq C(\varepsilon)\|f\|_{r,s'} \iint_{\Omega\tau} |\nabla w_n|^2 + C(T).\]

If $\|f\|_{r,s'}$ is sufficiently small we get
\[\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau))dx + C(m\alpha - \varepsilon) \iint_{\Omega\tau} u_n^{m-1}|\nabla w_n|^2 \leq C.\]

If not, then we can choose $t_1 < T$ such that $\|f\|_{L^r((0,t_1),L^s(\Omega))}$ is sufficiently small, and repeating the same computation as above in the set $(0,t_1) \times \Omega$,.
it follows that
\[
\sup_{\tau \in [0,t_1]} \int_{\Omega} H_k(u_n(x,\tau)) dx + c(m\alpha - \varepsilon) \int_{0}^{t_1} \int_{\Omega} u_n^{m-1} |\nabla w_n|^2 dx \leq C.
\]

Then the general result follows by iteration. Hence we conclude that
\[
(61) \sup_{\tau \in [0,T]} \int_{\Omega} H_k(u_n(x,\tau)) dx + c\int_{0}^{T} \int_{\Omega} u_n^{m-1} e^{\alpha G_k(u_n)} |\nabla u_n|^2 dx \leq C(\Omega, T).
\]

Now, taking \(T_k(u_n)\) as a test function in the problem satisfied by \(u_n\) and using the previous estimate, it follows that
\[
(62) \int_{\Omega} \Theta_k(u_n) dx + c\int_{\Omega} T_k^{m-1}(u_n) |\nabla T_k(u_n)|^2 \leq k\int_{\Omega} f + C(\Omega, T, k).
\]

Thus \(|\nabla u_n|^q + f_n\) is bounded in \(L^1(\Omega_T)\).

**Step 2. Passage to the limit.** To get the existence of the solution we need to prove that
\[
|\nabla u_n|^q + f_n \to |\nabla u|^q + f \text{ strongly in } L^1_{loc}(\Omega_T).
\]

**Claim.** The following inequality holds:
\[
(64) \int_{\{u_n \leq M\}} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n}|\nabla u_n|)^q(u_n + \frac{1}{n})^s} \rho \leq C
\]
for all \(s < 1\), where \(\rho\) is defined in (20).

To prove the claim, consider \(\rho(u_n + \frac{1}{n})^s\), with \(s < 1\), as a test function in (49). Therefore
\[
\int_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n}|\nabla u_n|)^q(u_n + \frac{1}{n})^s} \rho \leq \int_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n}|\nabla u_n|)^q(u_n + \frac{1}{n})^s} \rho
\]
and then
\[
\int_{\Omega_T} (1 + \frac{1}{n}|\nabla u_n|)^q(u_n + \frac{1}{n})^s \rho 
\leq \int_{\Omega_T} \frac{1}{1-s}(u_n + \frac{1}{n})^{1-s} \rho dx + \frac{1}{m-s} \int_{\Omega_T} (u_n + \frac{1}{n})^{m-s} dx.
\]

Since \(s < 1\) we obtain
\[
(65) \int_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n}|\nabla u_n|)^q(u_n + \frac{1}{n})^s} \rho \leq C,
\]
and the claim follows.

By using the *a priori* estimates of the first step, there results that
\[
\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} + f_n \equiv h_n \text{ is bounded in } L^1(\Omega_T) \text{ (resp. in } L^1_{loc}(\Omega_T)) \text{ if } m < 2 \text{ (resp. if } m = 2\).
Now define
\[ v_n = \left( u_n + \frac{1}{n} \right)^m - \left( \frac{1}{n} \right)^m, \]
then \( v_n \) solves the problem
\[
\begin{cases}
  b(v_n) - \Delta v_n = h_n, & (x,t) \in \Omega_T, \\
  v_n(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\
  v_n(x,0) = b_n^{-1}(T_n(u_0(x))), & x \in \Omega,
\end{cases}
\]
where \( b(s) = (s + (\frac{1}{n})^m)^\frac{1}{m} - \frac{1}{n} \) satisfies the hypotheses \((B)\) uniformly in \( n \). Therefore by Theorem 2.9 we have that
\[
\nabla v_n \to \nabla v \text{ strongly in } L^\sigma(\Omega_T) \text{ for all } 1 \leq \sigma < 1 + \frac{1}{1 + Nm}.
\]
To conclude we only need to prove (63). The case \( q = 2 \) is treated in [18] by using some kind of exponential change of variables. We deal with the case \( q < 2 \).

By Theorem 2.9 we have
\[ \nabla u_n \to \nabla u \text{ a.e. in } \Omega_T. \]
Since \( \{ |\nabla T_k(u_n)| \} \) is bounded in \( L^2(\Omega_T) \), then
\[ \nabla T_k(u_n) \to \nabla T_k(u_n) \text{ strongly in } L^q(\Omega_T). \]
We will use Vitali’s lemma. Consider \( E \subset \Omega_T, \) a measurable set; then we have
\[
\int\int_E |\nabla u_n|^q \, dx \, dt = \int\int_{E \cap \{ u_n < k \}} |\nabla T_k(u_n)|^q \, dx \, dt + \int\int_{E \cap \{ u_n \geq k \}} |\nabla u_n|^q \, dx \, dt.
\]
By using the strong convergence of the truncations, we have
\[
\int\int_E |\nabla T_k u_n|^q \, dx \, dt \to \int\int_E |\nabla T_k u|^q \, dx \, dt.
\]
We deal with the term \( \int\int_{E \cap \{ u_n \geq k \}} |\nabla u_n|^q \). By (53), (57) and (61) we reach that, in all the cases,
\[
\int\int_{\{ u_n \geq k \}} |\nabla u_n|^2 \leq C.
\]
Then
\[
\int\int_{E \cap \{ u_n \geq k \}} |\nabla u_n|^q \, dx \, dt \leq C \left( \int\int_{\{ u_n \geq k \}} |\nabla u_n|^2 \right)^{\frac{q}{2}} \left( \int\int_{\{ u_n \geq k \}} 1 \right)^{\frac{1}{2}} \leq C \left( \int\int_{\{ u_n \geq k \}} 1 \right)^{\frac{1}{2}}
\]
It is clear that \( |\{ u_n \geq k \}| \to 0 \) as \( k \to \infty \) uniformly in \( n \). Hence the result follows using Vitali’s lemma.

If \( m = 2 \), we can repeat the same arguments above to handle the term \( |\nabla u_n|^q \psi \), for \( \psi \in C_0^\infty(\Omega_T) \).

Therefore in both cases we reach
\[ \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \to |\nabla u|^q \text{ strongly in } L^1_{loc}(\Omega_T) \]
and the existence result follows. \( \square \)

**Remark 3.2.** Notice that \( 1 + \theta, 1 + \frac{2\theta}{mN}, \frac{(\theta + m)N}{mN + 2\theta} \to 1 \) as \( \theta \to 0 \). Then fixed \( q < 2 \), for all \( \varepsilon > 0 \), there exists \( 1 < m(\varepsilon) < 2 \) such that if \( m > m(\varepsilon) \), then problem (48) has a nonnegative solution for all \( u_0 \in L^{1+\varepsilon}(\Omega) \) and \( f \in L^{1+\varepsilon}(\Omega_T) \). This motivates the existence result studied in the next subsection.
3.2. The case \( m > 2 \): \( L^1 \) data. In the elliptic case if \( q(\frac{1}{m} - 1) < -1 \), then the existence result holds for all \( L^1 \) data, without any restriction on its size; see [2].

Notice that, since \( q \leq 2 \), the above condition implies that \( m > 2 \); however, in this section we assume that \( m > 2 \) and \( q \leq 2 \), without any other restriction. In particular, our result can be seen as a slight improvement of the result obtained in the elliptic case. The key is to prove some \( a \) \( priori \) estimates that allow us to show that the problem [38] has a distributional solution for all \( f \in L^1(\Omega_T) \) and \( u_0 \in L^1(\Omega) \). The main existence result in this section is the following.

**Theorem 3.3.** Let \( f, u_0 \) be such that \( f \in L^1(\Omega_T) \) and \( u_0 \in L^1(\Omega) \). Assume \( 1 < q \leq 2 \) and \( m > 2 \); then problem [38] has a distributional solution \( u \) such that \( |\nabla u|^m \in L^\sigma_{loc}(\Omega_T) \) for all \( 1 \leq \sigma < 1 + \frac{1}{N + 1} \).

**Proof.** We will consider separately the cases \( q < 2 \) and \( q = 2 \).

Let us begin with the case \( q < 2 \). Define \( u_n \) to be a solution to the approximated problem

\[
\begin{aligned}
\left\{
\begin{aligned}
u_{nt} - \text{div}(m(u_n + \frac{1}{n})^{m-1}\nabla u_n) &= \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} + T_n(f) & \text{in } \Omega_T, \\
u_n(x, t) &= 0 & \text{on } \partial \Omega \times (0, T), \\
u_n(x, 0) &= T_n(u_0(x)) & \text{if } x \in \Omega.
\end{aligned}
\right.
\]

Using \( e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho \), where \( \varrho \) is defined in [20], as a test function in [68], it follows that

\[
\int_\Omega D_n(u_n) \varrho dx + c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \int_\Omega K_n(u_n) dx dt \\
\leq \int_\Omega D_n(u_n_0) \varrho dx + c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt \\
+ c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho T_n(f) dx dt
\]

with

\[
D_n(s) = \int_0^s e^{-\frac{c}{(m-2)(\sigma + \frac{1}{n})^{m-2}}} d\sigma
\]

and

\[
K_n(s) = \int_0^s m(\sigma + \frac{1}{n})^{m-1} e^{-\frac{c}{(m-2)(\sigma + \frac{1}{n})^{m-2}}} d\sigma.
\]

Since \( e^{-\frac{c}{(m-2)s^{m-2}}} \leq C \), for \( s \geq 0 \), then \( c_1 s - c_2 \leq D_n(s) \leq s \) and \( K_n(s) \geq c_1 s^m - c_2 \). Hence using Young’s inequality,

\[
\int_\Omega D_n(u_n) \varrho dx + c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \int_\Omega K_n(u_n) dx dt \\
\leq \int_\Omega u_n_0 \varrho dx + c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt \\
+ c \int_\Omega \int_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho dx dt + \int_\Omega f dx dt.
\]
Choosing $\varepsilon$ small it follows that
\[
\int_\Omega D_n(u_n) \varphi dx + c \int_\Omega \int_\Omega e^{-\frac{\varepsilon}{(u_n+\frac{1}{n})^\alpha}} |\nabla u_n|^2 \varphi dx dt + \int_\Omega K_n(u_n) dx dt \leq C.
\]

As a consequence, $\{u_n\}_n$ is bounded in $L^\infty(0,T;L^1_{loc}(\Omega))$, $\{u^m_n\}_n$ is bounded in $L^1(\Omega_T)$ and $\{G_k(u_n)\}_n$ is bounded in $L^2(0,T;W^{1,2}_{loc}(\Omega))$.

Taking $T_k(u_n)\varphi$ as a test function in (68) we reach that the sequence $\{\Pi(u_n)\}_n$ is bounded in the space $L^2(0,T;W^{1,2}_{loc}(\Omega))$ where $\Pi(s) = \int_0^s (T_k(\sigma))^{\frac{m-1}{2}} d\sigma$.

In the same way we easily get that
\[
\int_\Omega \left( \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} + T_n(f) \right) \varphi \leq C \text{ for all } n.
\]

For fixed $s < \min\{1, m-2\}$, and using $\frac{\varrho}{(u_n + \frac{1}{n})^s}$ as a test function in (68), there results that
\[
\int_\Omega (u_n + \frac{1}{n})^{m-2-\alpha} |\nabla u_n|^2 \varphi dx dt + \int_\Omega \frac{T_n(f) \varrho}{(u_n + \frac{1}{n})^\alpha} dx dt \leq C \text{ for all } n
\]
and
\[
\int_\Omega (u_n + \frac{1}{n})^{m-2} |\nabla u_n|^2 \varphi dx dt \leq C \text{ for all } n.
\]

Thus $\{u^m_n - |\nabla u|^2\}_n$ is bounded in $L^1_{loc}(\Omega)$.

Notice that, if $2 < m < 3$, we get
\[
\int_\Omega |\nabla T_k(u_n)|^2 \varphi \leq C \text{ for all } n.
\]

We claim that $G_k(u_n) \to G_k(u)$ strongly in $L^2(0,T;W^{1,\alpha}_{loc}(\Omega))$ for all $\alpha < 2$ and for all $k > 0$.

By Theorem 2.9 we have that $\nabla u_n \to \nabla u$ a.e. in $\Omega_T$; in particular, $\nabla G_k(u_n) \to \nabla G_k(u)$ a.e. in $\Omega_T$. Hence to get the desired result we will use Vitali’s lemma. Let $M > k$ and $\psi \in C_0^\infty(\Omega_T)$; then for a measurable set $E \subset \Omega_T$ we have
\[
\int_E |\nabla G_k(u_n)|^\alpha \psi = \int_{E \cap \{u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi + \int_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^\alpha \psi
\]
\[
= \int_{E \cap \{k \leq u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi + \int_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^\alpha \psi.
\]

Since
\[
T_k(u^m_n) \to T_k(u^m) \text{ strongly in } L^2(0,T;W^{1,\alpha}_{loc}(\Omega)) \text{ for all } \alpha < 2,
\]
then
\[
\int_{E \cap \{k < u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi \leq \frac{\varepsilon}{2} \text{ if } |E| \leq \delta \varepsilon.
\]
We deal now with the second term. Using (70) we obtain
\[ \int \int_{E \cap \{u_n \geq M\}} |\nabla u_n|^\alpha |\psi dx| \leq C \left( \int \int_{\{u_n \geq M\}} u_n^{m-2-s} |\nabla u_n|^{\alpha (m-2-s)} \psi dx dt \right) \]
where \( s > 0 \) is chosen such that \( 0 < s < m - 2 \). Taking \( M \) large enough, we reach
\[ \int \int_{E \cap \{u_n \geq M\}} |\nabla u_n|^\alpha |\psi dx| \leq \frac{\varepsilon}{2}. \]
Therefore the strong convergence of \( \{\nabla G_k(u_n)\}_n \) follows and the claim is proved.

As in the previous subsection, to get the existence result we just have to show that
\[ |\nabla u_n|^{q \left( 1 + \frac{1}{n} |\nabla u_n|^q \right)} \rightarrow |\nabla u|^q \text{ strongly in } L^1_{loc}(\Omega_T). \]

From Theorem 2.9 and by the above estimate there results that
\[ \frac{|\nabla u_n|^{q |\nabla u_n|^q}}{1 + \frac{1}{n} |\nabla u_n|^q} \rightarrow |\nabla u|^q \text{ a.e. in } \Omega_T. \]
Using (71), we can prove that
\[ \limsup_{M \to 0} \int \int_{\{u_n \leq M\}} \frac{|\nabla u_n|^{q |\nabla u_n|^q}}{1 + \frac{1}{n} |\nabla u_n|^q} \psi dx = 0 \text{ uniformly in } n. \]
Using the result of the last claim and (72) we obtain that
\[ \frac{|\nabla u_n|^{q |\nabla u_n|^q}}{1 + \frac{1}{n} |\nabla u_n|^q} \rightarrow |\nabla u|^q \text{ strongly in } L^1_{loc}(\Omega_T \cap \{u > 0\}). \]
If \( \{|u = 0|\} = 0 \), then (73) follows. Assume that \( \{|u = 0|\} > 0 \). Since \( |\nabla u|^q \in L^1_{loc}(\Omega_T) \), then we conclude that \( |\nabla u|^q = 0 \) on the set \( \{u = 0\} \). Hence to finish the proof we just have to prove that
\[ \frac{|\nabla u_n|^{q |\nabla u_n|^q}}{1 + \frac{1}{n} |\nabla u_n|^q} \rightarrow 0 \text{ strongly in } L^1_{loc}(\Omega_T \cap \{u = 0\}). \]
This follows using (74) and Egorov’s Lemma. Thus the existence result follows in this case.

We deal now with the case \( q = 2 \). From the result of [20], there exists a bounded solution \( u_n \) to the problem
\[ \begin{aligned}
  u_{nt} - \text{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= |\nabla u_n|^2 + T_n(f) \quad \text{in } \Omega_T, \\
  u_n(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
  u_n(x, 0) &= T_n(u_0(x)) \quad \text{if } x \in \Omega.
\end{aligned} \]
Using the same argument as in the case \( q < 2 \), we find the same estimates for the sequence \( \{u_n\}_n \); moreover, for all \( \psi \in C_0^\infty(\Omega_T) \), \( \psi \geq 0 \),
\[ \int \int_{\Omega_T} |\nabla u_n|^2 \psi < C \text{ for all } n \]
and
\[ \int \int_{\Omega_T} \frac{|\nabla u_n|^2 \psi}{(u_n + \frac{1}{n})^s} \leq C \text{ for all } n. \]
Again using Theorem 2.9, \( \nabla u_n \to \nabla u \) a.e. in \( \Omega_T \), and then, as above,
\[
\tag{78} T_k(u^m_n) \to T_k(u^m) \quad \text{strongly in } L^2(0, T; W^{1,\alpha}_{loc}(\Omega)) \quad \text{for all } \alpha < 2.
\]

Now from (76), we conclude that
\[
u_n \to \nu \quad \text{strongly in } L^2(0, T; W^{1,\alpha}_{loc}(\Omega)) \quad \text{for all } \alpha < 2.
\]

Consider \( w_n \equiv T_{2k}(u_n - T_h(u_n) + T_k(u_n) - (T_k(u))_\nu) \), where \( (T_k(u))_\nu \) is defined as in (27) and \( \gamma(s) = \frac{1}{m(2-m)}(s + \frac{1}{n})^{2-m} \). Let \( h > 2k > 0 \) to be chosen later. It is clear that \( \nabla w_n \equiv 0 \) for \( u_n > M \equiv 4k + h \).

Taking \( w_n e^{\gamma(u_n)} \) as a test function in (75), it follows that
\[
\int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \rangle dt + m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nu T_M(u_n) \nabla w_n + m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} w_n \nabla T_M(u_n) \nabla \psi \leq m \int_{\Omega_T} T_n(f) e^{\gamma(u_n)} w_n \psi.
\]

It is clear that
\[
\int_{\Omega_T} T_n(f) e^{\gamma(u_n)} w_n \psi \leq \omega(n, \nu)
\]
and
\[
|\int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} w_n \nabla T_M(u_n) \nabla \psi| \leq \omega(n, \nu).
\]

From the results in [18], we deduce that
\[
\int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \rangle dt \geq \omega(n) + \omega(\nu).
\]

Let us analyze the term \( m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \psi \):
\[
\int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \psi
= \int_{\{u_n \leq k\}} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla w_n \psi
+ \int_{\{u_n > k\}} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n
\geq \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu) + \psi
- \int_{\{u_n > k\}} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| |\nabla (T_k(u))_\nu| \psi.
\]

Define
\[
\Gamma_n(s) = \begin{cases} e^{\gamma(s)}(s + \frac{1}{n})^{m-1} & \text{if } s \leq k, \\ 0 & \text{if } s \geq k; \end{cases}
\]
then
\[ \int\int_{\Omega T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu) + \psi \]
\[ = \int\int_{\Omega T} \Gamma_n(u_n) \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu) + \psi \]
\[ = \int\int_{\Omega T} \Gamma_n(u_n) |\nabla (T_k(u_n) - (T_k(u))_\nu)|^2 \psi + \omega(n, \nu). \]

On the other hand, for $M$ fixed,
\[ \int\int_{\{u_n > k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| \nabla (T_k(u))_\nu \psi = \omega(n, \nu). \]

Hence
\[ \int\int_{\Omega T} \Gamma_n(u_n) |\nabla (T_k(u_n) - (T_k(u))_\nu)|^2 \psi \]
\[ \leq \int\int_{\Omega T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla \nu \psi \]
\[ + \int\int_{\{u_n > k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| \nabla (T_k(u))_\nu \psi + \omega(n, \nu). \]

Thus
\[ \int\int_{\Omega T} \Gamma_n(u_n) |\nabla (T_k(u_n) - (T_k(u))_\nu)|^2 \psi \leq \omega(n, \nu). \]

In the same way we reach that
\[ \int\int_{\Omega T} \Gamma_n(u_n) |\nabla (T_k(u_n) - (T_k(u))_\nu)|^2 \psi \leq \omega(n, \nu). \]

Since $\Gamma_n(s) \geq C_1$ if $s > C_2 > 0$, uniformly in $n$, then
(79) \[ |\nabla u_n| \chi_{\{c_1 < u_n < c_2\}} \rightarrow |\nabla u| \chi_{\{c_1 < u < c_2\}} \text{ strongly in } L^2_{loc}(\Omega_T). \]

We claim that
(80) \[ |\nabla G_k(u_n)| \rightarrow |\nabla G_k(u)| \text{ strongly in } L^2_{loc}(\Omega_T). \]

We again use Vitali’s lemma. Let $\psi \in C^\infty_0(\Omega_T)$ be such that $\psi \geq 0$. Consider $E \subset \Omega_T$, a measurable set; then,
\[ \int\int_E |\nabla G_k(u_n)|^2 \psi = \int\int_{E \cap \{u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi + \int\int_{E \cap \{u_n > M\}} |\nabla G_k(u_n)|^2 \psi \]
\[ = \int\int_{E \cap \{k \leq u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi + \int\int_{E \cap \{u_n > M\}} |\nabla G_k(u_n)|^2 \psi. \]

From (79), given $\epsilon > 0$ there exists $\delta > 0$ such that
\[ \limsup_{n \to \infty} \int\int_{E \cap \{k \leq u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi \leq \frac{\epsilon}{2} \text{ if } |E| \leq \delta. \]

We deal now with the term $\int_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi$. 

Using (70), we get, for some $\alpha < m - 2$,
\[
\int\int_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi = \int\int_{E \cap \{u_n \geq M\}} \frac{\left(u_n + \frac{1}{n}\right)^{m-2-\alpha}}{(M + \frac{1}{n})^{m-2-\alpha}} |\nabla u_n|^2 \psi \leq \frac{C}{(M + \frac{1}{n})^{m-2-\alpha}}.
\]

Thus
\[
\limsup_{n \to \infty} \int\int_{E} |\nabla G_k(u_n)|^2 \psi \leq \frac{\varepsilon}{2} + \frac{C}{(M + \frac{1}{n})^{m-2-\alpha}}.
\]

Letting $M \to \infty$, we reach the claim.

In the same way and by using estimates (77), we can prove that
\[
\int\int_{\Omega_T} |\nabla (T_k(u_n) - T_k(u))|^2 \psi \to 0 \text{ as } n \to \infty.
\]

Hence
\[
|\nabla u_n|^2 \to |\nabla u|^2 \text{ strongly in } L^1_{loc}(\Omega_T),
\]

and then the existence result follows.

\[\square\]

**Remark 3.4.** If $m = q = 2$ and $f = 0$, we can prove that the solution to problem (48) has the **finite speed propagation property**. This follows by setting $w = \frac{2}{3} \left(\frac{4}{5}\right)^{\frac{2}{5}} u^\frac{5}{2}$; then $w$ solves
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
  w_t - \frac{4}{5} \left(\frac{3}{2}\right)^{\frac{2}{5}} \Delta w^{\frac{2}{5}} &= 0 & \text{in } \Omega_T, \\
  w(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\
  w(x, 0) &= \frac{2}{3} \left(\frac{4}{5}\right)^{\frac{2}{5}} u_0^{\frac{5}{2}}(x) & \text{in } \Omega.
\end{array}
\right.
\end{align*}
\]

If $u_0 \in L^\infty(\Omega)$ has compact support, by using a convenient Barenblatt self-similar supersolution (see [26], for instance) we obtain the **finite speed of propagation property**. The inverse change of variable allows us to conclude the same result for problem
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
  u_t - \Delta u^2 &= |\nabla u|^2 & \text{in } \Omega_T, \\
  u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\
  u(x, 0) &= u_0(x) & \text{in } \Omega.
\end{array}
\right.
\end{align*}
\]

4. **The fast diffusion equation**

In this section we consider the case $0 < m < 1$, usually called **fast diffusion equation** in the literature. We will prove the following existence result.

**Theorem 4.1.** Assume that $0 < m < 1$, $q \leq 2$ and

1. $f \in L^r(0, T; L^s(\Omega))$, where $1 < r < \infty, s > \frac{N}{2}$ with $\frac{1}{r} + \frac{N}{2s} = 1$,
2. $e^{\alpha u_0^{\frac{2}{m}}} \in L^1(\Omega)$, where
   - (a) either $\alpha > 0$ is any positive constant if $q < 2$,
   - (b) or $\alpha m (2-m) > 2$ if $q = 2$.

Then problem (48) has a distributional solution.
Proof. Let \( \{f_n\}, \{u_{0n}\} \) be sequences of bounded nonnegative functions such that \( u_{0n} \uparrow u_0 \) and \( f_n \uparrow f \).

Let \( u_n \) be the bounded solution of

\[
\left\{ \begin{array}{l}
    u_{nt} - \text{div}(n(u_n + \frac{1}{n})^{m-1}\nabla u_n) = \frac{u_n}{1 + \frac{1}{n} u_n^q} + f_n & \text{in } \Omega_T, \\
    u_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\
    u_n(x,0) = u_{0n}(x) & \text{if } x \in \Omega,
\end{array} \right.
\]

with data \((f_n, u_{0n})\). Notice that the existence and the boundedness of \( u_n \) follow using the result of [13].

Taking \( e^{\alpha u_n^{2-m}} - 1, \alpha > 0 \), as a test function in (83), we find that

\[
\frac{d}{dt} \int_{\Omega} H(u_n)dx + m\alpha(2-m) \int_{\Omega} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 \, dx
\]

\[
\leq \int_{\Omega} (e^{\alpha u_n^{2-m}} - 1) |\nabla u_n|^q \, dx + \int_{\Omega} f_n(e^{\alpha u_n^{2-m}} - 1) \, dx,
\]

where \( H(s) = \int_{0}^{s} (e^{\alpha s^{2-m}} - 1) \, d\sigma \).

Using Young’s inequality, integrating in \([0, \tau]\) and taking the maximum on \( \tau \), \( q < 2 \)

\[
\sup_{\tau \in [0,T]} \int_{\Omega} H(u_n(x, \tau)) \, dx + (\alpha m(2-m) - \varepsilon) \int_{\Omega_T} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 \, dx
\]

\[
\leq C \int_{\Omega_T} e^{\alpha u_n^{2-m}} - 1 + \int_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1) \, dx + \int_{\Omega} H(u_0(x)) \, dx
\]

and \( q = 2 \)

\[
\sup_{\tau \in [0,T]} \int_{\Omega} H(u_n(x, \tau)) \, dx + (\alpha m(2-m) - 1) \int_{\Omega_T} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 \, dx
\]

\[
\leq C \int_{\Omega_T} e^{\alpha u_n^{2-m}} - 1 + \int_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1) \, dx + \int_{\Omega} H(u_0(x)) \, dx
\]

Let us analyze the last term in (84):

\[
\int_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1) \, dx \leq \int_{\Omega_T} f(e^{\frac{\alpha}{2} u_n^{2-m}} - 1)^2 + C
\]

\[
\leq \|f\|_{r,s} (e^{\frac{\alpha}{2} u_n^{2-m}} - 1)_{r,s}^2 + C.
\]

We set \( w_n = e^{\frac{\alpha}{2} u_n^{2-m}} - 1 \); then using the Gagliardo-Nirenberg inequality we obtain that

\[
\|w_n\|_{r,s}^2 \leq C\|w_n\|_{\infty,2}^\frac{2}{r} \left( \int_{0}^{T} \int_{\Omega} |\nabla w_n|^2 \right)^\frac{1}{r'}
\]

\[
\leq C \left( \sup_{\tau \in [0,T]} \int_{\Omega} w_n^2 \, dx \right)^\frac{1}{2} \left( \int_{0}^{T} \int_{\Omega} |\nabla w_n|^2 \right)^\frac{1}{2}.
\]
Thus
\[
\int_{\Omega} f_n(e^{\alpha u_n^{2-m}} - 1)dx \leq C(\varepsilon)||f||_{r,s} \int_0^\tau \int_{\Omega} |\nabla w_n|^2 + \varepsilon(\sup_{\tau \in [0,T]} \int_{\Omega} w_n^2dx) + C.
\]

Using the fact that \(H(u_n) \geq c_1 w_n^2 - c_2\) and then choosing \(\varepsilon\) small, it follows that
\[
\sup_{\tau \in [0,T]} \int_{\Omega} H(u_n(x,\tau))dx + c(\alpha m(2 - m) - \varepsilon) \int_{\Omega} |\nabla w_n|^2dx \\
\leq C(\varepsilon)||f||_{r,s} \int_0^\tau \int_{\Omega} |\nabla w_n|^2 + C(\Omega, T).
\]

If \(||f||_{r,s}\) is sufficiently small we get
\[
\sup_{\tau \in [0,T]} \int_{\Omega} H(u_n(x,\tau))dx + c(\alpha m(2 - m)\varepsilon) \int_{\Omega} |\nabla w_n|^2dx \leq C.
\]

If not, then we can choose \(t_1 < T\) such that \(||f||_{L^r((0,t_1)\times(0,\Omega))}\) is sufficiently small. Then, repeating the above computation in the set \((0,t_1) \times (0,\Omega)\), we obtain
\[
\sup_{\tau \in [0,t_1]} \int_{\Omega} H(u_n(x,\tau))dx + c(\alpha m(2 - m) - \varepsilon) \int_0^{t_1} \int_{\Omega} |\nabla w_n|^2 \leq C.
\]

Therefore, the general result follows by iteration. Hence we conclude that
(85) \[
\sup_{\tau \in [0,T]} \int_{\Omega} H(u_n(x,\tau))dx + c \int_{\Omega} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 dx \leq C(\Omega, T),
\]

and as a consequence \(|\nabla u_n|^q + f_n\) is uniformly bounded in \(L^1(\Omega_T)\).

As a consequence of the above estimate, \(\{u_n\}_n\) is bounded in the spaces \(L^2(0,T;W^{1,2}_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))\) and \(\{u_{nt}\}_n\) is bounded in the spaces \(L^2(0,T;W^{-1,2}_0(\Omega)) + L^1(\Omega_T)\). Then, there exists a measurable function \(u \in L^2(0,T;W^{1,2}_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))\) such that, up to a subsequence, \(u_n \to u\) strongly in \(C(0,T;L^2(\Omega))\). It is clear that, by Theorem 2.3, we have \(\nabla u_n \to \nabla u\) a.e. in \(\Omega_T\). Thus if \(q < 2\), then by the previous estimate we obtain that \(|\nabla u_n|^q \to |\nabla u|^q\) strongly in \(L^1(\Omega_T)\) and then the result follows.

We deal now with the case \(q = 2\). Consider \(\psi \in C^\infty_0(\Omega_T), \psi \geq 0\). By using \(\psi(u_n + \frac{1}{n})^\delta\) as a test function in (83), where \(\delta < m\), there results that
(86) \[
\int_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{2+\delta-m}} \psi \leq C \quad \text{for all } n \text{ and for all } n.
\]

Notice that from (86), it follows that
(87) \[
\int_{\{u_n \leq M\}} |\nabla u_n|^2 \psi \leq C(M + \frac{1}{n})^{2+\delta-m} \quad \text{uniformly in } n,
\]

and
(88) \[
\int_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{2+\delta-m}} \psi \leq C \quad \text{for all } n.
\]

Let us prove now that
(89) \[
|\nabla T_k(u_n)| \to |\nabla T_k(u)| \quad \text{strongly in } L^2_{loc}(\Omega_T).
\]
Consider \( \varphi \) a real differentiable function such that \( \varphi(0) = 0 \) and \( (2k)^{m-1} \varphi' - |\varphi| \geq C > 0 \). Consider \( w_n \equiv e^{-\gamma(T_k(u_n))} \varphi(T_k(u_n) - (T_k(u))_+) \), where \( (T_k(u))_+ \) is defined as in \([27]\) and
\[
\gamma(s) = \frac{1}{m(2-m)} \left( s + \frac{1}{n} \right)^{2-m} - \left( \frac{1}{n} \right)^{2-m}.
\]

Taking \( w_ne^{\gamma(u_n)} \psi \) as a test function in \([33]\), it follows that
\[
\int_0^T \langle (u_n)_t, e^{\gamma(u_n)}w_n \psi \rangle dt + m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla u_n \nabla w_n \psi
\]
\[
+ m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} w_n \nabla u_n \nabla \psi \leq \int_{\Omega_T} fe^{\gamma(u_n)}w_n \psi.
\]
Notice that
\[
\int_0^T \langle (u_n)_t, e^{\gamma(u_n)}w_n \psi \rangle dt \geq \omega(n) + \omega(\nu)
\]
(see for instance \([18]\)). Then by the hypothesis on \( f \) and by \([33]\) it follows that
\[
\int_{\Omega_T} fe^{\gamma(u_n)}w_n \psi \leq \omega(n, \nu).
\]
On the other hand,
\[
| \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} w_n \nabla u_n \nabla \psi |
\]
\[
\leq \left( \int_{\text{Supp}(\psi)} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{2(m-1)}|\nabla u_n|^2 \right)^{\frac{1}{2}} \left( \int_{\text{Supp}(\psi)} e^{\gamma(u_n)}w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{\text{Supp}(\psi)} e^{\gamma(u_n)}w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}}.
\]
Since
\[
e^{\gamma(u_n)}w_n^2 |\nabla \psi|^2 \to 0 \quad \text{a.e. in } \Omega_T \quad \text{and } e^{\gamma(u_n)}w_n^2 |\nabla \psi|^2 \leq Ce^{\gamma(u_n)},
\]
therefore by the Sobolev inequality and the estimates \([33]\) and \([36]\), we get
\[
\left( \int_{\text{Supp}(\psi)} e^{\gamma(u_n)}w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}} = \omega(n).
\]
By a direct computation we obtain that
\[
m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla u_n \nabla w_n \psi \leq \omega(n) + \omega(\nu).
\]
Notice that
\[
m \int_{\Omega_T} e^{\gamma(u_n)}(u_n + \frac{1}{n})^{m-1} \nabla u_n \nabla w_n \psi
\]
\[
= m \int_{\Omega_T} e^{\gamma((u_n)) - \gamma(T_k(u_n))}(u_n + \frac{1}{n})^{m-1}
\]
\[
\cdot \nabla u_n \nabla (T_k(u_n) - T_k(u) + \varphi'((T_k(u_n) - T_k(u))_+) \psi
\]
\[
- m \int_{\{u_n \leq k\}} |\nabla u_n|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi.
\]
Let us analyze each term in the previous identity:
\[ m \int \int_{\Omega_T} e^{\gamma((u_n)) - \gamma(T_k(u_n))}(u_n + \frac{1}{n})^{m-1} \nabla u_n \nabla (T_k(u_n) - T_k(u))_+ \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi \]
\[ = m \int \int_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} |\nabla (T_k(u_n) - (T_k(u))_+|^2 \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi \]
\[ + m \int \int_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi \]
\[ - m \int \int_{\{u_n \geq k\}} (u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi. \]

It is clear that
\[ | \int \int_{\{u_n \geq k\}} (u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ + \varphi'((T_k(u_n) - T_k(u))_+) \psi | \]
\[ \leq \omega(n, \nu). \]

Now,
\[ \int \int_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi \]
\[ = \int \int_{\{u_n \leq k\} \cap \{T_k(u_n) \geq (T_k(u))_+ \}} \psi(u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ \cdot \varphi'((T_k(u_n) - T_k(u))_+) \psi. \]

Since \( m < 1 \), by using the dominated convergence theorem and by estimate \( \text{[SS]} \),
\[ \psi(u_n + \frac{1}{n})^{m-1} |\nabla ((T_k(u))_+|X(T_k(u_n) \geq (T_k(u))_+ \rightarrow u^{m-1} |\nabla T_k(u)| \psi \]
\[ \text{strongly in } L^2(\Omega_T) \text{ as } n, \nu \rightarrow \infty. \]

Thus using a duality argument, we get
\[ m \int \int_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \nabla ((T_k(u))_+ \nabla (T_k(u_n) - (T_k(u))_+ + \varphi'((T_k(u_n) - T_k(u))_+) \psi \]
\[ = \omega(n, \nu). \]

It is clear that
\[ -m \int \int_{\{u_n \leq k\}} |\nabla u_n|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi \]
\[ \geq -m \int \int_{\{u_n \leq k\}} |\nabla (T_k(u_n) - (T_k(u))_+|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi + \omega(n) + \omega(\nu). \]

Therefore combining the above estimates, there results that
\[ m \int \int_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \varphi' - \varphi |\nabla (T_k(u_n) - (T_k(u))_+|^2 \leq \omega(n) + \omega(\nu). \]
Using the properties of $\varphi$, 
\[
\int_\Omega |\nabla(T_k(u_n) - T_k(u))_+|^2 \psi \to 0 \text{ as } n \to \infty.
\]
In a similar way, we obtain 
\[
\int_\Omega |\nabla(T_k(u_n) - T_k(u))_-|^2 \psi \to 0 \text{ as } n \to \infty.
\]
Hence 
\[
|\nabla T_k(u_n)| \to |\nabla T_k(u)| \text{ strongly in } L^2_{\text{loc}}(\Omega_T).
\]
Therefore, by (86), (89) and Vitali’s lemma, we can prove that 
\[
|\nabla u_n| \to |\nabla u| \text{ strongly in } L^2_{\text{loc}}(\Omega_T).
\]
Hence the existence result follows.

4.1. Finite time extinction. Assume that $f \equiv 0 \text{ and } q = 2$. Consider the problem 
\[
\begin{cases}
    u_t - \Delta u^m = |\nabla u|^2 & \text{in } \Omega_T, \\
    u(x,t) = 0 & \text{on } \partial \Omega \times (0,T), \\
    u(x,0) = u_0(x) & \text{if } x \in \Omega,
\end{cases}
\]
where $0 < m < 1$.

We will prove that the regular solutions of (90), in a suitable sense, become zero in finite time, provided the initial datum $u_0 \in L^{1+\theta_0}(\Omega)$ for some $\theta_0 > 0$. We begin by making precise the meaning of regular solutions to (90).

Definition 4.2. Let $\Lambda(s) = \int_0^s \frac{e^{2-m}}{s(m-1)} dt$ and define 
\[
\beta(s) = \frac{1}{m} \int_0^s (\Lambda^{-1}(\sigma))^{\frac{1}{m}-1} d\sigma.
\]
We say that $u$ is a regular solution to problem (90) in $\Omega_T$ if $v \equiv \Lambda(u^m) \in L^2((0,T); W^{1,2}_0(\Omega)) \cap C([0,T]; L^2(\Omega)), \beta(v)_t \in L^2((0,T); W^{-1,2}(\Omega)),$ 
and for all $\phi \in L^2((0,T); W^{1,2}_0(\Omega))$ we have 
\[
\int_0^T \langle (\beta(v))_t, \phi \rangle + \int_0^T \int_\Omega \nabla v \cdot \nabla \phi = 0.
\]
It is clear that the existence of a regular solution follows using Theorem 4.1 for $q = 2, f = 0$ and the regularity of the initial datum $u_0$. Now we are able to state the next result.

Theorem 4.3. Assume that $0 < m < 1$. If $u$ is the regular solution of problem (90) in the sense of Definition 4.2, then there exists a positive, finite time $t_0$, depending on $N$, and $u_0$ such that $u(x,t) \equiv 0$ for $t > t_0$.

Proof. To get the desired result we have just to show that $v(x,t) \equiv 0$ for $t > t_0$. It is clear that $v$ solves 
\[
\begin{cases}
    (\beta(v))_t - \Delta v = 0 & \text{in } \Omega \times (0,T), \\
    v(x,t) = 0 & \text{on } \partial \Omega \times (0,T), \\
    v(x,0) = v_0(x) & \text{in } \Omega,
\end{cases}
\]
with \( v_0 \in L^2(\Omega) \). Using \( v^\theta \), where \( \theta > 0 \) is to be chosen later, as a test function in (93), there results that

\[
\frac{d}{dt} \int_\Omega \Psi(v(x,t)) \, dx + c(\theta) \int_\Omega \left| \nabla v^{\theta+1} \right|^2 \, dx = 0,
\]

where

\[
\Psi(s) = \int_0^s s^\theta (\Lambda^{-1}(\sigma))^{\frac{1}{m}-1} \, d\sigma.
\]

Since

\[
\lim_{s \to \infty} \frac{\Lambda^{-1}(s)}{s^\varepsilon} = 0 \quad \text{for all } \varepsilon > 0
\]

it follows that

\[
\Psi(s) \leq c(\varepsilon) s^{\theta+1+\varepsilon(\frac{1}{m}-1)} \quad \text{for every } s \geq 0.
\]

For fixed \( \theta \) such that \( \theta + 1 + \varepsilon(\frac{1}{m} - 1) = 1 + \theta_0 \), then using Sobolev’s and Hölder’s inequalities,

\[
\int_\Omega \left| \nabla v^{\theta+1} \right|^2 \, dx \geq c_1(N,\theta) \left[ \int_\Omega \left( v^{\theta+1} \right)^{2^*} \, dx \right]^{2/2^*} \geq c_2(N,\theta,|\Omega|) \left[ \int_\Omega (v^{a(\theta+1)}) \, dx \right]^{1/a},
\]

where \( 1 < a < \frac{2^*}{2} \) is chosen such that \( a(\theta + 1) = \theta + 1 + \varepsilon(\frac{1}{m} - 1) \). Hence it follows that

\[
\int_\Omega \left| \nabla v^{\theta+1} \right|^2 \, dx \geq c(N,\theta,|\Omega|) \left[ \int_\Omega \Psi(v(x,t)) \, dx \right]^{1/a}.
\]

Define

\[
\xi(t) = \int_\Omega \Psi(v(x,t)) \, dx;
\]

then

\[
\frac{\xi'(t)}{\xi(t)^{1/a}} \leq -c_4 < 0.
\]

Notice that by the assumption on \( v_0 \) we reach that \( \xi(0) < \infty \). Integrating in \( t \), one obtains

\[
\frac{a}{a-1} \left( \xi(t)^{\frac{a-1}{a}} - \xi(0)^{\frac{a-1}{a}} \right) \leq -c_4 t.
\]

Thus, as long as \( \xi(t) > 0 \), one has

\[
\xi(t)^{\frac{a-1}{a}} \leq \xi(0)^{\frac{a-1}{a}} - c_4 \frac{a-1}{a} t.
\]

Therefore, \( \xi(t) \equiv 0 \) for \( t \) large enough. \( \square \)

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DÉPARTEMENT DE MATHEMÁTIQUES, UNIVERSITÉ ABOUBEKR BELKAÏD, TLEMÇEN, TLEMÇEN 13000, ALGERIA

E-mail address: boumediene.abdellaoui@uam.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN

E-mail address: ireneo.peral@uam.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN

E-mail address: magdalena.walias@uam.es