Observable estimation of entanglement for arbitrary finite-dimensional mixed states

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We present observable upper bounds of squared concurrence, which are the dual inequalities of the observable lower bounds introduced in [F. Mintert and A. Buchleitner, Phys. Rev. Lett. 98, 140505 (2007)] and [L. Aolita, A. Buchleitner and F. Mintert, Phys. Rev. A 78, 022308 (2008)]. These bounds can be used to estimate entanglement for arbitrary experimental unknown finite-dimensional states by few experimental measurements on a twofold copy $\rho \otimes \rho$ of the mixed states. Furthermore, the degree of mixing for a mixed state and some properties of the linear entropy also have certain relations with its upper and lower bounds of squared concurrence.

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I. INTRODUCTION

Entanglement is not only one of the most fascinating features of quantum theory that has puzzled generations of physicists, but also an essential resource in quantum information [1, 2, 3, 4]. Thus, the detection [5, 6, 7, 8, 9] of physicists, but also an essential resource in quantum information science. A number of measures have been proposed to quantify entanglement, such as concurrence [10, 11], negativity [12] and tangle [13]. However, for arbitrary experimental unknown mixed states, the observable upper bound is non-trivial since it also provides an estimation of entanglement as well as the lower bound in experiments.

In this paper, we present observable upper bounds of squared concurrence which, together with the observable lower bounds introduced by Mintert et al. [20, 21], can estimate entanglement for arbitrary experimental unknown states. These bounds can be easily obtained by few experimental measurements on a twofold copy $\rho \otimes \rho$ of the mixed states. Actually, the upper bounds are the dual one of the lower bounds in Refs. [20, 21]. Furthermore, the degree of mixing for a mixed state and some properties of the linear entropy also have certain relations with its upper and lower bounds of squared concurrence.

The paper is organized as follows. In Sec. II we propose an observable upper bound of squared concurrence for bipartite states and multipartite states. The relations with properties of the linear entropy is shown in Sec. III. In Sec. IV we discuss a tighter upper bound of squared concurrence for two-qubit states, and give a brief conclusion of our results.

II. OBSERVABLE UPPER BOUND FOR ARBITRARY MIXED STATES

Bipartite mixed states. The $I$ concurrence of a bipartite pure state is defined as [12, 19]

$$ C(\ket{\psi}) \equiv \sqrt{2(1 - \text{Tr}\rho_A^2)} = \sqrt{\bra{\psi} \otimes \bra{\psi} A |\psi\rangle \otimes |\psi\rangle}, $$

(1)

where the reduced density matrix $\rho_A$ is obtained by tracing over the subsystem B and $A = 4P^{(1)}_i \otimes P^{(2)}_i$. $P^{(i)}_i$ is the projector on the antisymmetric subspace $\mathcal{H}_i \wedge \mathcal{H}_i$ (symmetric subspace $\mathcal{H}_i \otimes \mathcal{H}_i$) of the two copies of the $i$th subsystem $\mathcal{H}_i \otimes \mathcal{H}_i$, which has been defined as follows [19]

$$ P^{(i)}_i = \frac{1}{4} \sum_{j,k} \{ |\alpha_j \alpha_k \rangle \pm |\alpha_k \alpha_j \rangle \} \langle |\alpha_j \alpha_k \rangle \mp \langle |\alpha_k \alpha_j \rangle \}, $$

(2)

where $\{ |\alpha_j \rangle \}$ is an arbitrary complete set of orthogonal bases of $\mathcal{H}_i$. The definition of $I$ concurrence can
be extended to mixed states \( \rho \) by the convex roof,
\[
C(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,
\]
for all possible decomposition into pure states, where \( p_i \geq 0 \)
and \( \sum_i p_i = 1 \). Ref. \[21\] introduced lower bounds of
squared concurrence for arbitrary finite-dimensional bi-partite states,
\[
[C(\rho)]^2 \geq \text{Tr}(\rho \otimes \rho V_i),
\]
with \( V_1 = 4(\rho^{(1)}_+ - \rho^{(1)}_-) \otimes I^{(2)}_+ \) and \( V_2 = 4(\rho^{(2)}_- - \rho^{(2)}_+) \). We conjecture its dual inequality as follows
\[
[C(\rho)]^2 \leq \text{Tr}(\rho \otimes \rho K_i),
\]
with \( K_1 = 4\rho^{(1)}_+ \otimes I^{(2)}_+ \) and \( K_2 = 4(I^{(1)}_+ \otimes \rho^{(2)}_-) \). The proof of this inequality is shown in the following.

\[
[C(\rho)]^2 = \left[ \inf_i \sum \rho_i C(|\psi_i\rangle) \right]^2
\]
\[
\leq \inf_i \left( \sqrt{\rho_i} C(|\psi_i\rangle) \right)^2 \cdot \left( \sqrt{\rho_i} \right)^2
\]
\[
= \inf_i \left( 2 \rho_i (1 - \text{Tr}(\rho_i^2)) \right)
\]
\[
= 2(1 - \text{Tr}(\rho K_i)),
\]
where \( \rho_i^A = \text{Tr}_B|\psi_i\rangle\langle\psi_i| \). The first inequality holds by
applying the Cauchy-Schwarz inequality \[32\], the second one, which has also been proved in Ref. \[8\], holds due to
the convex property of \( \text{Tr} \rho_i^2 \), and the last equality can
be proved directly using the definition of \( P^{(i)}_\pm \). Similarly,
one can also obtain the inequality \( [C(\rho)]^2 \leq \text{Tr}(\rho \otimes \rho K_2) \).

Similar to the lower bounds, inequality \[4\] implies some interesting consequences:

1. The upper bounds can be expressed in terms of the purities of \( \rho_A \) and \( \rho_B \), i.e.,
\[
\text{Tr}(\rho \otimes \rho K_1) = 2(1 - \text{Tr}(\rho_A^2)),
\]
\[
\text{Tr}(\rho \otimes \rho K_2) = 2(1 - \text{Tr}(\rho_B^2)),
\]
which coincide with Eq. \[1\] for pure state concurrence.
Notice that Ref. \[20\] has introduced similar equations for lower bounds \( V_1 \) and \( V_2 \),
\[
\text{Tr}(\rho \otimes \rho V_1) = 2(2 \text{Tr}(\rho^2) - \text{Tr}(\rho_A^2)),
\]
\[
\text{Tr}(\rho \otimes \rho V_2) = 2(2 \text{Tr}(\rho^2) - \text{Tr}(\rho_B^2)).
\]

2. The upper bounds can be directly measured, since
it is given in terms of expectation values of \( P_- \). It is
a little different from the experimental measurement of
pure state concurrence \( 4\rho^{(1)}_+ \otimes \rho^{(2)}_+ \). Notice that \( 4\rho^{(1)}_+ \otimes I^{(2)}_+ \)
\( = 4\rho^{(1)}_+ \otimes I^{(2)}_+ + 4\rho^{(1)}_+ \otimes I^{(2)}_+ \). For pure state \(|\psi\rangle\),
\(|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes \rho^{(1)}_+ \otimes I^{(2)}_+ = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes \rho^{(1)}_+ \otimes I^{(2)}_+ \). Actually, Refs. \[28\], \[29\] and \[31\]
measured their concurrence via \( 4\rho^{(1)}_+ \otimes I^{(2)}_+ \) instead of \( 4\rho^{(1)}_+ \otimes I^{(2)}_+ \).

In this sense, they obtained an upper bound rather than concurrence itself.

3. Interestingly, it is worth noting that
\[
\text{Tr}(\rho \otimes \rho K_1) - \text{Tr}(\rho \otimes \rho V_1) = 2(1 - \text{Tr}(\rho^2)),
\]
i.e., the degree of mixing can be easily calculated out
based on the upper and lower bounds.

Let us simulate the observable upper bound on mixed
random states of \( 3 \times 3 \)-dimensional systems. Mixed random
states with different degrees of mixing were obtained
via the generalized depolarizing channel \[33\], as Ref. \[21\]
did. The observable upper bound versus lower bound is
shown in Fig. 1. Interestingly, the upper bounds in Fig. 1
are always in parallel with the lower bounds, which
actually coincides with Eq. \[7\]. For weakly mixed states,
the bounds provide an excellent estimation of concurrence;
for strongly mixed states, they also provide a region for concurrence.

**Multipartite mixed states.** The generalized concurrence for
multipartite pure state is not unique. For instance, Ref. \[19\]
introduced several inequivalent alternatives. In this section, we choose the multipartite concurrence introduced in \[24\], \[34\]:
\[
C_N(\Psi) \equiv 2^{1-N/2} \sqrt{\left(2^N - 2 \right) - \sum_i \text{Tr}(\rho_i^2)},
\]
where \( i \) labels all \( 2^N - 2 \) different reduced density matrices. The definition can also be expressed as
\[
C_N(\Psi) = \sqrt{\langle \Psi | \otimes \langle 
\Psi | A | \Psi \rangle \otimes | \Psi \rangle} \text{ with } A = \left( P_+ - P^{(1)}_+ \otimes \right)
\]

![Fig. 1: (Color online). Measurable upper bound \( \text{Tr}(\rho \otimes \rho K_1) - \text{Tr}(\rho \otimes \rho V_1) = 2(1 - \text{Tr}(\rho^2)) \) for squared mixed-state concurrence \( [C(\rho)]^2 \), versus its observable lower bound \( \text{Tr}(\rho \otimes \rho (V_1 + V_2)/2) \) for \( 3 \times 3 \)-dimensional random states with different degrees of mixing: a shows weakly mixed states \( \text{Tr}(\rho^2) = 0.98 \), b displays intermediate mixing \( \text{Tr}(\rho^2) = 0.88 \), c corresponds to strongly mixed states \( \text{Tr}(\rho^2) = 0.78 \). The dashed lines denote the lower bound.](image-url)
\( \cdots \otimes P_1^{(N)} \). \( P_+ \) (\( P_- \)) is the projector onto the globally symmetric (antisymmetric) space \(^{[22]}\). For mixed states, it is also given by the convex roof, \( C_N(\rho) = \inf \{ p_i | \Psi_i \} \sum \rho_i C_N(\Psi_i) \), for all possible decomposition into pure states, where \( p_i \geq 0 \) and \( \sum_i p_i = 1 \). Ref. \(^{[21]}\) introduced lower bounds of squared concurrence for arbitrary multipartite states,

\[
[C_N(\rho)]^2 \geq \text{Tr}(\rho \otimes \rho V),
\]

with \( V = 4(P_+ - P_+^{(1)} \otimes \cdots \otimes P_+^{(N)} - (1 - 2^{1-N})P_-) \). We introduce an observable \( K \) such that

\[
[C_N(\rho)]^2 \leq \text{Tr}(\rho \otimes \rho K),
\]

with \( K = 4(P_+ - P_+^{(1)} \otimes \cdots \otimes P_+^{(N)} + (1 - 2^{1-N})P_-) \). The proof of this inequality is shown in the following.

\[
[C_N(\rho)]^2 \leq \inf \sum_i p_i [C_N(\Psi_i)]^2
= \inf \sum_i p_i 2^{-N} \sum_k c_k^2(\Psi_i)
\leq 2^{2-N} \sum_k (2 - \text{Tr}p_k - \text{Tr}p_k^2)
= 2^{2-N} \sum_k \text{Tr}(\rho \otimes \rho 2(P_-^{(k)} \otimes 1 + 1 \otimes P_-^{(k)}))
= \text{Tr}(\rho \otimes \rho K),
\]

where \( \sum_k \) is taken over all the bipartite concurrence \( c_k \) corresponding to each subdivision of the entire system into two subsystems \(^{[21]}\), \( k \) denotes one subsystem and \( \overline{k} \) denotes the other one. We have used that \( \sum_k P_-^{(k)} \otimes P_-^{(\overline{k})} = 2^{N-2}(P_+ - P_+^{(1)} \otimes \cdots \otimes P_+^{(N)}) + \sum_k (P_-^{(k)} \otimes P_+^{(k)} \otimes P_+^{(\overline{k})} = (2^{N-1} - 1)P_- \).

Inequality (10) also implies some interesting consequences: (1) The upper bound can also be expressed in terms of the purities of reduced density matrices, i.e., \( \text{Tr}(\rho \otimes \rho K) = 2^{2-N}(2^N - 2 - \sum \text{Tr}p_k^2) \), which coincides with Eq. (8) for pure state concurrence. (2) The upper bound can be directly measured, since it is given in terms of expectation values of symmetric and antisymmetric projectors. It is a little different from the lower bound \( \text{Tr}(\rho \otimes \rho V) \). (3) Interestingly, it is worth noting that

\[
\text{Tr}(\rho \otimes \rho K) - \text{Tr}(\rho \otimes \rho V) = 4(1 - 2^{1-N})(1 - \text{Tr}\rho^2),
\]

i.e., the degree of mixing can be easily calculated out based on the upper and lower bounds.

We also simulate the observable upper bound on mixed random states of \( 2 \times 2 \times 2 \)-dimensional systems with different degrees of mixing obtained via the generalized depolarizing channel \(^{[33]}\). The observable upper bound versus lower bound is shown in Fig. 2. The upper bounds in Fig. 2 are always in parallel with the lower bounds as well, which actually coincides with Eq. (11). For weakly mixed states, the bounds provide an excellent estimation of concurrence; for strongly mixed states, they also provide a region for concurrence.

\[\text{FIG. 2: (Color online). Measurable upper bound } \text{Tr}(\rho \otimes \rho K) \text{ versus its lower bound } \text{Tr}(\rho \otimes \rho V) \text{ for } 2 \times 2 \times 2 \text{-dimensional mixed random states. Degrees of mixing are the same as Fig. 1.} \]

\section*{III. RELATIONS WITH PROPERTIES OF THE LINEAR ENTROPY}

Interestingly, the upper and lower bounds of squared concurrence have certain relations with some properties of the linear entropy, such as the triangle inequality. The linear entropy is defined as follows \(^{[35]}\)

\[
E(\rho) = 1 - \text{Tr} \rho^2.
\]

It can be regarded as a kind of linearized von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \log_2 \rho) \), and has several same properties as \( S(\rho) \). In the following, we will give simple proofs of the triangle inequality and subadditivity of the linear entropy.

The triangle inequality can be proved directly using the upper and lower bounds. Notice that the following inequalities hold for arbitrary bipartite states:

\[
\text{Tr}(\rho \otimes \rho K_1) \geq [C(\rho)]^2 \geq \text{Tr}(\rho \otimes \rho V_2),
\]

\[
\text{Tr}(\rho \otimes \rho K_2) \geq [C(\rho)]^2 \geq \text{Tr}(\rho \otimes \rho V_1).
\]

Since Eqs. (10) and (11) hold, we can obtain some new inequalities,

\[
1 - \text{Tr}\rho_A^2 \geq \text{Tr}\rho_B^2 - \text{Tr}\rho_B^2,
1 - \text{Tr}\rho_B^2 \geq \text{Tr}\rho_B^2 - \text{Tr}\rho_A^2;
1 - \text{Tr}\rho^2 \geq (1 - \text{Tr}\rho_B^2) - (1 - \text{Tr}\rho_A^2),
1 - \text{Tr}\rho^2 \geq (1 - \text{Tr}\rho_A^2) - (1 - \text{Tr}\rho_B^2).
\]

Obviously, inequalities (15) can be directly calculated out from inequalities (14), and they are actually triangle inequalities of the linear entropy,

\[
E(\rho_{AB}) \geq E(\rho_B) - E(\rho_A),
E(\rho_{AB}) \geq E(\rho_A) - E(\rho_B).
\]
Before embark on proving the subadditivity of the linear entropy, let us review the universal state inverter $\tilde{\rho}$ introduced in Ref. \[32\]

\[
\tilde{\rho} \equiv \text{Tr}(\rho)\mathbb{1} \otimes (1 - \rho^\dagger A \otimes (1 - \rho^\dagger B + \rho^\dagger)
= \sum_\alpha \sigma_\alpha \otimes \sigma_\alpha (Q_\alpha \rho Q_\alpha)^* \sigma_\alpha \otimes \sigma_\alpha,
\]
where $Q_\alpha = P^{(\alpha)}_A \otimes P^{(\alpha)}_B$, $P^{(\alpha)}_A = |i\rangle A (|i\rangle + |i\rangle A (i\rangle$ and $P^{(\alpha)}_B = |j\rangle B (|j\rangle + |j\rangle B (j\rangle$. The universal state inverter $\tilde{\rho}$ is a semi-positive definite operator, since each term in the sum $\sigma_\alpha \otimes \sigma_\alpha (Q_\alpha \rho Q_\alpha)* \sigma_\alpha \otimes \sigma_\alpha$ is semi-positive definite. Therefore, $\sqrt{\rho \tilde{\rho}} \rho$ has the semi-positive definite property as well, and we can obtain the following inequality,

\[
1 + \text{Tr}\rho^2 - \text{Tr}\rho^2_A - \text{Tr}\rho^2_B = \text{Tr}\sqrt{\rho \tilde{\rho}} \rho \geq 0, \tag{17}
\]

where we have used $\text{Tr}\sqrt{\rho \tilde{\rho}} \rho = \text{Tr}\rho = \text{Tr}[\rho (1 - \rho A \otimes (1 - \rho B + \rho)] = 1 + \text{Tr}\rho^2 - \text{Tr}\rho^2_A - \text{Tr}\rho^2_B$. Thus, $1 + \text{Tr}\rho^2 - \text{Tr}\rho^2_A - \text{Tr}\rho^2_B \geq 0$ holds, i.e. the subadditivity of the linear entropy

\[
E(\rho_A) + E(\rho_B) \geq E(\rho_{AB}) \tag{18}
\]

holds \[33\] in fact, it is not the first time to prove the subadditivity of the linear entropy. For instance, Ref. \[37\] has proved the subadditivity of the linear entropy. Furthermore, the triangle inequality of the linear entropy can be proved from the subadditivity \[38\]. Compared with this earlier proof, roughly speaking, our proof is a little simpler. The main purpose of this section is that these properties of the linear entropy are the natural results from the positive semidefiniteness of the universal state inverter and the upper and lower bounds, and they also indicate the validity of these bounds.

IV. DISCUSSIONS AND CONCLUSIONS

Actually, for two-qubit states, $\text{Tr}(\rho \otimes \rho \cdot 4P^{(1)}_\perp \otimes P^{(2)}_\perp)$ is a tighter upper bound of squared concurrence than $\text{Tr}(\rho \otimes \rho K)$. Because the equation

\[
\text{Tr}(\rho \otimes \rho \cdot 4P^{(1)}_\perp \otimes P^{(2)}_\perp) = \text{Tr}\rho \tilde{\rho} \tag{19}
\]

holds for arbitrary two-qubit states, where $\tilde{\rho} = \sigma_A \otimes \sigma_B \rho^* \sigma_A \otimes \sigma_B$. Eq. \[19\] has also been proved in \[39\]. Furthermore, notice that $C = \max \{\lambda_1 - \lambda_2, -\lambda_3 - \lambda_4\}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are squared roots of eigenvalues of $\rho \tilde{\rho}$ in the decreasing order. Therefore, it is easily concluded that $\text{Tr}(\rho \otimes \rho \cdot 4P^{(1)}_\perp \otimes P^{(2)}_\perp) = \text{Tr}\rho \tilde{\rho} = \sum_\lambda \lambda^2 \geq (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)^2 \geq C^2$. However, the new upper bound $\text{Tr}(\rho \otimes \rho \cdot 4P^{(1)}_\perp \otimes P^{(2)}_\perp)$ is hard to generalize to arbitrary finite-dimensional bipartite states.

We give a brief discussion on the experimental measurement of our upper bound. As only the projector $P_-$ on one of the subsystems, rather than a complete set of observables, is required, our upper bound could be easily measured. In particular, for two-dimensional systems, $P_-$ is simply the projector onto the singlet state $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. Let us take the photonic system for example. The simplest way to project two photons onto the singlet state is using a Hong-Ou-Mandel interferometer \[40\]. This method has been widely used since the teleportation \[41\] experiment. Another method, employed in \[28\], \[29\], is distinguishing the Bell states with a controlled-NOT gate, which can transform the Bell states to separable states \[42\].

In conclusion, we present observable upper bounds of squared concurrence, which are the dual bound of the observable lower bounds introduced by Mintert et al. These bounds can estimate entanglement for arbitrary finite-dimensional experimental unknown states by few experimental measurements on a twofold copy $\rho \otimes \rho$ of the mixed states. Furthermore, the degree of mixing for a mixed state and some properties of the linear entropy also have certain relations with its upper and lower bounds of squared concurrence. Last but not least, we discuss a tighter upper bound for two-qubit states only, and it remains an open question to generalize it to arbitrary finite-dimensional bipartite systems.

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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] A. Einstein, B Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[3] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) 416, 608 (2002).
[4] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[5] O. Rudolph, e-print arXiv:quant-ph/0202121 K. Chen and L.-A. Wu, Quantum Inf. Comput. 3, 193 (2003); H. Fan, e-print arXiv:quant-ph/0210168.
[6] P. Aniello and C. Lupi, J. Phys. A: Math. Theor. 41, 355303 (2008); C. Lupi, P. Aniello, and A. Scardicchio, ibid. 41, 415301 (2008).
[7] J.I. de Vicente, Quantum Inf. Comput. 7, 624 (2007); M. Li, S.-M. Fei, and Z.-X. Wang, J. Phys. A: Math. Theor. 41, 202002 (2008); O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. 99, 130504 (2007).
[8] J.I. de Vicente, J. Phys. A 41, 065309 (2008).
[9] M. Seevinck and J. Uffink, Phys. Rev. A 76, 042105.
P. Rungta, V. Buzek, C. M. Caves, M. Hillery and G. J. Milburn, Phys. Rev. A 64, 042315 (2001).

H. Fan, K. Matsumoto and H. Imai, J. Phys. A 36, 4151 (2003); H. Fan, V. Korepin, and V. Roychowdhury, Phys. Rev. Lett. 93, 227203 (2004).

Y.-C. Ou and H. Fan, Phys. Rev. A 75, 062308 (2007); Y.-C. Ou, ibid., 75, 034305 (2007); H. Fan, Y.-C. Ou, and V. Roychowdhury, e-print arXiv:0707.1578.

F. Mintert, M. Kuś and A. Buchleitner, Phys. Rev. Lett. 98, 052302 (2007); F. A. Brandão, K. Audenaert, and J. Eisert, New J. Phys. 9, 46 (2007).

O. Gühne, M. Reimpell, and R. F. Werner, Phys. Rev. Lett. 98, 110502 (2007); ibid., Phys. Rev. A 77, 052317 (2008).

C.-S. Yu and H.-S. Song, Phys. Rev. A 76, 022324 (2007); C.-S. Yu, C. Li, and H.-S. Song, ibid. 77, 012305 (2008).

X.-J. Ren, Z.-W. Zhou, X.-X. Zhou, and G.-C. Guo, Phys. Rev. A 77, 054302 (2008).

S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Nature 440, 1022 (2006).

S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Phys. Rev. A 75, 032338 (2007).

F.-W. Sun, J.-M. Cai, J.-S. Xu, G. Chen, B.-H. Liu, C.-F. Li, Z.-W. Zhou, and G.-C. Guo, Phys. Rev. A 76, 052303 (2007).

B. Terhal, Phys. Lett. A 271, 319 (2000); G. Tóth and O. Gühne, Phys. Rev. Lett. 94, 060501 (2005); F. A. Bovino, G. Castagnoli, A. Ekert, P. Horodecki, C. M. Alves and A. V. Sergienko, Phys. Rev. Lett. 95, 240407 (2005); O. Gühne and N. Litkenhaus, ibid. 96, 170502 (2006); F. Mintert, Phys. Rev. A 75, 052302 (2007); R. Augusiak, M. Demianowicz, P. Horodecki, ibid. 77, 030301(R) (2008).

The first inequality has been proved in T. J. Osborne, Phys. Rev. A 72, 022309 (2005).

M. L. Aolita, I. García-Mata, and M. Saraceno, Phys. Rev. A 70, 062301 (2004).

A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 93, 230501 (2004).

E. Santos and M. Ferrero, Phys. Rev. A 62, 024101 (2000).

Actually, it is also held that \( \text{Tr}(\rho \otimes \rho \cdot 4P_{-}^{(1)} \otimes P_{-}^{(2)}) = \sqrt{\text{Tr}\rho \cdot \sqrt{\text{Tr}\rho}} \), since we have \( \text{Tr}(\rho \otimes \rho \cdot 4P_{-}^{(1)} \otimes P_{-}^{(2)}) = \text{Tr}[\rho \otimes \rho(V_1 + K_2)/2] = \text{Tr}[\rho \otimes \rho(V_2 + K_1)/2] = 1 + \text{Tr}^2 - \text{Tr}^2 \). Therefore, the proof of the subadditivity above has a little relations with the upper and lower bounds. Eq. (41) indicates that the average of the upper and lower bounds \( \text{Tr}[\rho \otimes \rho(V_1 + K_2)/2] \) is nonnegative.

J.-M. Cai, Z.-W. Zhou, S. Zhang, and G.-C. Guo, Phys. Rev. A 75, 052324 (2007).

J. Preskill, Lecture Notes on Physics 229: Quantum Information and Computation, available at http://www.theory.caltech.edu/people/preskill/ph229/. Chapter 5, Exercise 5.2 (g). Although this exercise is for the von Neumann entropy, the same hint can be used for the linear entropy.

J.-M. Cai and W. Song, e-print arXiv:0804.2246.

C.-K. Hong, Z.-Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044 (1987).

D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature 390, 575 (1997).

A. Barenco, D. Deutsch, A. Ekert, and R. Jozsa, Phys. Rev. Lett. 74, 4083 (1995).