Learning pythagorean theorem from ancient China: A preliminary study

A D Fachrudin¹, R Ekawati², A W Kohar³, S Widada⁴, I B Kusumawati¹ and R Setianingsih²

¹Mathematics Education Department, STKIP PGRI Sidoarjo, Indonesia
²Faculty of Mathematics and Natural science, Universitas Negeri Surabaya, Indonesia

E-mail: dh4nyy@gmail.com

Abstract. In this article, we explore how History of the Pythagorean theorem, especially from ancient China, as a source of inspiration for learning design. Based on historical examples and problem solving from the history of mathematics, we try to arrange instructional tasks for seventh-grade secondary education students and how its role in supporting students' understanding of Pythagorean theorem. We focus on explaining the preliminary studies or the first phase of the three main phases in design research. The basic idea of instructional design is to introduce historical geometric diagrams and solve the right triangle problem in the manner of Liu Hui (3rd century CE). The purpose of introducing geometric diagrams is to connect students’ symbolic algebraic thinking with visual thinking about geometric shapes especially in the case of right triangles. In this study, we also conducted interviews and debrief sessions with teachers and tested the student’s history-based tasks to several students for initial investigation in the matter of developing the Hypothetical Learning Trajectory. Results of this study show that teachers have a positive response toward the history-based instructional task developed. On the other hand, we find the resolution of Right Triangles problems with diagrams by the students involving three steps: translation, transformation, and diagrammatic reasoning.

1. Introduction
In recent decades, integrating the history of mathematics (HoM) in mathematics education has become an important study in mathematics education worldwide. As an example, Fauvel et al [1] explained and mentioned in detail hundreds of published works of the relation between HoM and mathematics pedagogy that written in eight different languages. Moreover, topic namely “The role of the use history of mathematics in mathematics education” become one of Topic Study Group (TGS) in International Congress on Mathematical Education (ICME), the biggest international congress in mathematics education, regularly. Furthermore, in some countries, the history of mathematics has been integrated into the curriculum of mathematics education. researchers have also suggested integrating HoM in classroom (i.e. [2] [3] [4] [5] and [6]) Thus, this makes the history of mathematics an interesting topic to be studied deeply, especially how HoM can be integrated in classrooms for certain topic.

In Indonesia, the study of the relation between the History of Mathematics and pedagogy were addressed by some researchers. Fachrudin, Putri, Kohar, & Widada [7] developed a design of learning of quadratic equation that inspired by the Babylonian approach. Ekawati, Fiangga, & Siswono [8] studied about how historical perspectives in Indonesian Mathematics Textbook support students understanding about the mathematics concept. Fachrudin & Putri [9] addressed about how the ancient Geometric method helps students’ understanding of the concept of quadratic equation, especially on
how student bring the geometric solution into algebraic form. Revealing a positive impact on improving student understanding, all these studies suggest the use of HoM in mathematics learning.

Swetz [10] pointed out that the history of mathematics contains a worth of material (i.e. historical problems and problem-solving situations) that can be used in today classroom. Furthermore, Tzanakis & Archavi [11] argued that using the history of mathematics in the classroom can support the learning process in many ways, including developing a teaching approach inspired by history. Therefore, by designing history-based activities, which involve problem-solving, students will get the mathematicians’ experience about a certain topic which they are studying has been developed also the alternative ways of thinking about mathematical reasoning. Regarding the usefulness of using HoM in mathematics pedagogy, Siu [12] addressed the use of Chinese problem solving from ancient Chinese mathematics, from Chiu Chang Suan Shu (Nine Chapters on Mathematical Art), can support student visual reasoning. In the last chapter of “Nine Chapter”, presents about geometric manipulation method in solving the various problem especially related to right triangle.

By means of various problems and ancient Chinese geometric diagram, we try to design a set of instructional tasks for learning Pythagorean Theorem. The idea is by concerning the ancient China geometric manipulation, students can associate their symbolic algebraic thinking with visual thinking about geometric shapes to build their understanding related to the Pythagorean Theorem. In particular, the aim of this study was to describe our preliminary stage in constructing the Hypothetical Learning Trajectory in learning Pythagoras by using ancient china history-based problem.

1.1. History-based Problem from Ancient China “Nine Chapter”

The old Chinese work that provides excellent work in mathematics was Chiu Chang Suan Shu or known as “Nine Chapter”. It consists of nine different part, the first three-part about surveying and engineering formulas, next three chapters about taxation and bureaucratic administration, and last three chapters about specific computational techniques. The last chapter or ninth chapter consists of twenty-four problems concerning about right triangle’s properties. In this section, we choose and discuss several problems in the ninth chapter, namely chapter Kou Ku, as an initial problem or as a problem to assist students in performing geometric manipulations to improve students’ visual reasoning.

In a right triangle, the sort side adjacent to the right angle is called kou. The longer side adjacent to the right angle is called ku. the side opposite the right angle is called hsien. Hsien is longer than ku, and ku is longer than Kou [13]. This term of definition will assist the understanding of the solution method. Here, we will show rule and problem presented in Kou Ku we choose as history-based problem-solving in compiling the design of the instructional task.

The Rule: Add the square of kou and ku. The square root of the sum is equal to Hsien.

![Image](image-source)

**Figure 1.** Relation among Hsien, Ku, and Kou (image source: Swetz and Kao, 1977)
Based on the above picture, it is known that the Pythagorean rule was known by ancient Chinese. The pairs of triangles 3, 4, 5 are used as examples and have been widely known by many ancient societies.

Furthermore, we will take an example of Liu Hui’s explanation of the solution to the sixth problem of Kou Ku in Nine Chapter.

In the center of a square pond whose side is 10 Chi (feet), grows a reed whose top reaches 1 Chi above the water level. If we pull the reed toward the bank, its top is even with the water's surface. What is the depth of the pond and the length of the plant?

1 Chi = 1 Feet

**Figure 2. Problem 6 of Kou Ku in Nine Chapter**

To solve the type of problem using algebraic symbols, indeed students need prior knowledge about product formula of algebraic sum or difference. However, rather than using calculation of algebraic symbols, we may imagine the right triangle devised by Liu Hui as that described by Swetz and Kao [13], Dauben [14], Chemla and Gou [15], Guevara-Casanova & Burgués-Flamarich [16], although Liu Hui did not explain any figures in his commentary.

**Figure 3. Geometric illustration of the problem 6**

Based on the picture, the length of the reed as Hsien and the depth of the pond as Ku. It is known that the sum of the areas of the squares rooted on the Kou and Ku of the triangle should equal to the area of square rooted on the Hsien (see fig. 3). Then, by performing geometric manipulation, placing the Ku in Hsien, we know that the rest of the area must be the same with the area of Kou. Then by changing the gnomon into a rectangle, it will be easily obtained that Ku must be 12 Chi (see fig.4)
Based on the fig.4, then it will be easy to determine the value of Hsien, 13 Chi. That is one example of a problem and how to use geometric manipulation to supports students’ geometric reasoning in determining solutions to problems related to the Pythagorean theorem.

2. Method
Design research was chosen as a method used in this research. The core of this type of research is developing instructional sequences and local instruction theories that underpin them [18]. Design Research typically consists of cycles of three main phases, namely Preliminary Design, Teaching Experiment, and Retrospective Analysis. However, in this paper, we restricted the discussion just in the preliminary phase. By conducting a literature review and interviews with teachers, we compiled a teaching sequence in order to support students’ understanding in learning Pythagoras by exploring their visual reasoning before perfecting the teaching sequence into a complete Hypothetical Learning Trajectory (HLT). Simon [18] defined the hypothetical learning trajectory contain of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process or the conjecture of student thinking. we tested the teaching sequence on four students. We did this in the preliminary phase in order to support the development of HLT and predict the various of a conjecture of student thinking. Moreover, we conducted interviews and debrief sessions with teachers to get information about their experiences and attitude toward using HoM in the classroom.

3. Result and Discussion
3.1. Teacher Interviews
In this part, we shall present the result of debriefing season with the teacher and the teaching sequence that we have developed in order to support student understanding about Pythagoras through learning instruction involving history-based problem. By introducing the historical geometric diagrams and solving the right triangle problem in the manner of Liu Hui will help students connecting their symbolic algebraic thinking with visual thinking about geometric shapes especially in the case of right triangles.

Based on the interview session we got some information about the initial knowledge and attitude of the teachers towards the history of mathematics and its use in learning. Teachers complained that many students experience difficulties, especially in contextual problems. However, one of the teachers believed that it caused by students’ weakness in memorizing the triple Pythagoras. this shows that the
teacher still considers that success in learning mathematics is related to only memorizing the formulas, and he has not yet realized the importance of understanding the basic concepts of a material.

On the other hand, from deep interviews, we get the information that teachers did not get enough information about HoM just from a snippet of historical figures in student’s book while the other did not even understand about History relating to Pythagoras. It can be inferred that the teacher still does not have sufficient access to learn more about HoM and its role in mathematics pedagogy formally. However, the teacher has a positive response when we ask about their opinions toward using the history of mathematics in the classroom.

3.2. History-based Instructional Tasks

After conducting teacher interviews and literature studies on the history of Pythagoras in China, especially from Nine Chapters, we compiled a series of learning activities with the aim of supporting students’ understanding of Pythagoras by exploring students’ visual reasoning abilities. The first activities, learning started by introducing the rule (sum of Kou² and Ku² must be Hsien²), from the first problem in Kou Ku, of Chinese Pythagorean. Students were asked to prove that the sum of Kou² and Ku² equal with Hsien² using the geometric method. In second activities, based on their prior geometric manipulation knowledge, students are asked to solve a problem related to Pythagorean theorem material. In the last activities, we presented a history-based problem from Nine Chapter, Problem 6 and 7 in Kou Ku, and ask them to solve it using geometric approach. Why using geometric approach? Our assumption, the problem posed in Kou Ku is more difficult, because the data given, consisting of one side of the triangle and the difference or the sum of the other two. Therefore, when the problem is expressed to an algebraic equation, some ability in determining the algebraic square product is required to solve this problem.

Here, we try to give a brief explanation and analysis of our set of instructional tasks developed. Furthermore, we also analyzed the student work to be used as initial information, especially for constructing the conjectures of student thinking in developing HLT.

a. Activity I: proof of Chinese Pythagorean Rule using diagrams/ geometric manipulation

In this part, students were asked to prove the Chinese Pythagorean rule below, problem 1 in Kou Ku, using diagrams or geometric approach.

The rule: **Add the square of kou and ku. The square root of the sum is equal to Hsien**, (Swetz & Kao, 1977)

By proving this rule, students will understand the various models of geometric interpretation to show that square of hsien is equal to the sum of square of kou and ku. Based on figure 5 below, we know that there are two types of proof posed by the students. First, let the kou square and cut out ku square then obtained the sum of the area is equal with hsien and vice versa.

![Figure 5. student’s work in activity 1](image)

b. Activity II: using Chinese Pythagorean rule to solve right triangle problem

In the second activity, students are asked to solve a right triangle contextual problem using Pythagoras. This to help students achieve a better understanding of solving various contextual problems related to Pythagoras. Indeed, all student tend to use an algebraic method rather than geometric approach because
the problems algebraic form of the problems faced are simple and do not require complicated algebraic computation to solve it.

c. Activity III: solving right triangle problem in the manner of Liu Hui.
In this part, students are presented with several contextual right triangle problems from Kou Ku in Nine Chapters. The situation in the tasks is more difficult since the information given involving the difference or the sum of two sides of the right triangle. Guevara-Casanova & Burgués-Flamarich [16] stated that difficulty can be overcome by introducing geometric approach or diagrams, on the basis of which students are able to argue visually and follow the procedure for calculating the solution to the problem by manipulating geometric shapes that are introduced after understanding the problem.

The first assignment for this activity was the problem 6 of Kou Ku, the problem about reed that grows on the pond, that we have presented in the previous section. the following is the example of student answers.

Figure 6. student’s visual reasoning in solving problem 6 in Kou Ku

The second problem given in this part is problem 7 in Kou Ku that has a similar type with the previous problem. The difference is only in the context and situation used.

A rope is tied to the top of a pole. The rope is 3 Chi longer than the pole. if we pull the rope (taut), the end will just touch the ground 8 chi from the (base of the) pole. what is the length of the rope? [13]

Guevara-Casanova & Burgués-Flamarich [16] state to solve this type of problem using diagram students will take several steps namely: constructing, processing, interpretation, and reading, or simplified into three procedures: translation, transformation and diagrammatic reasoning. The following is the student answer in solving problem 7.

Figure 7. the process of solving Rope Problem

When referring to the procedures of solving with diagrams stated by Guevara-Casanova & Burgués-Flamarich, namely translation, transformation and diagrammatic reasoning, the translation process occurs at an early stage when students change contextual problems into geometric problems, in the
picture above students determine that what is known from the problem is pole height or $ku = a$ feet and rope length or $Hsien = a + 3$ feet while Kou is 8 feet. Then students use the relationship between the area of square $kou$, $ku$, and $hsien$, and did the geometric manipulation. The step resulted that the gnomon form (shaded area) of that has the same area with the $kou$ square ($64$ ft$^2$). We call that step as a \textit{transformation} procedure. The last process, \textit{diagrammatic reasoning}, by analyzing the relation between the area of the gnomon ($64$ ft$^2$) and the side known, they found that $a = \frac{27.5}{3}$ or they wrote $a \approx 9,16$ (actually the most appropriate answer is $a \approx 9,167$) and the length of the rope is 12,16.

4. Conclusion and implications
In the preliminary design, we conducted a literature review of Chinese Pythagoras, interviews with teachers and implement the instructional task to get the various conjecture of student thinking. We use the data to revise and improve the learning design. Based on the teacher's interview, we addressed that the teachers had a positive attitude towards the use of history in mathematics learning. However, professional teacher development or similar programs in Indonesia still does not support teachers to improve their knowledge of HoM and its various benefit and use in mathematics pedagogy. Tzanakis et al [19] addressed that there was a wide range of possible ways of implementing HoM in the classroom, e.g. as historical snippets, primary sources, worksheet, also problem-solving from the HoM.

From the result regarding the implementation of instructional task developed, in general, we conclude that problem-solving from the history of mathematics can support students’ understanding of the concept of Pythagoras. The visual argument in solving the right triangle problem help student improve their algebraic understanding through three main procedures of geometric manipulation: \textit{translation}, \textit{transformation}, and \textit{diagrammatic reasoning}. Furthermore, through the history-based problem as context students get a chance to learn Pythagorean theorem meaningfully. This is in line with Freudenthal [20] statements that learning will happen when meaningful for students.

In general, this study revealed that HoM, in this term we use ancient China Pythagoras, have a plentiful material and didactical source to improve the quality of mathematics pedagogy and design of mathematics instruction. Therefore, we recommend teachers to integrate the history of mathematics in the classroom and researcher to conduct other similar research to enrich teachers’ references in using HoM in mathematics pedagogy.

Acknowledgments
This research was supported and funded by Kemenristekdikti.

References
[1] Fauvel J \textit{et al} 2002 Bibliography for further work in the area \textit{History in Mathematics Education} (Springer Dordrecht) pp 371-418
[2] Fauvel J and Van Maanen J 2002 \textit{The History in mathematics education} (Dordrecht Netherlands: Kluwer Academic Publishers)
[3] Katz V 2000 Using History To Teach Mathematics: An International Perspective (USA: The Mathematical Association of America)
[4] Panasuk R M and Horton L B 2012 Integrating History of Mathematics into Curriculum: What are the Chances and Constraints? \textit{International Electronic Journal of Mathematics Education} 7 3-20
[5] Radford L 1996 The roles of geometry and arithmetic in the development of algebra: Historical remarks from a didactic perspective \textit{Approaches to algebra} (Dordrecht: Springer) pp 39-53
[6] Radford L and Guerette G 2000 Second degree equations in the classroom: A babylonian approach \textit{Using History To Teach Mathematics. An International Perspective} (Washington: The Mathematical Association of America) pp. 69-75
[7] Fachrudin A D, Putri R I I, Kohar A W and Widayah S 2018 Developing a local instruction theory for learning the concept of solving quadratic equation using babylonian approach \textit{Journal of Physics: Conference Series} (IOP Publishing) \textbf{1108} 012069
[8] Ekawat R, Fiangga S and Siswono T Y E 2018 Historical aspect of mathematics on Indonesian mathematics textbook IOP Conference Series: Materials Science and Engineering (IOP Publishing) 434 012001

[9] Fachrudin A D and Putri R I I 2014 building students' understanding of quadratic equation concept using naïve geometry Indonesian Mathematical Society Journal on Mathematics Education 5 192-202

[10] Swetz F 2000 Problem solving from the history of mathematics Using History To Teach Mathematics: An International Perspective (Washington: The Mathematical Association of America)

[11] Tzanakis C and Archavi A 2000 Integrating history of mathematics in the classroom: an analytic survey History in mathematics education: the ICMI study (New York: Kluwer Academic Publishers) pp 201-40

[12] Siu M K 2000 An excursion in ancient Chinese mathematics Using history to teach mathematics. An international perspective (Washington, DC: The Mathematical Association of America) pp 159–66

[13] Swetz F and Kao T I 1977 Was Pythagoras Chinese?: an examination of right triangle theory in ancient China (No. 40) (Pennsylvania: Penn State Press)

[14] Dauben J W 2007 Chinese mathematics The mathematics of Egypt, Mesopotamia, China, India and Islam. A sourcebook (New Jersey, NJ: Princeton University Press) pp 187-384

[15] Chemla K and Guo S (Eds.) 2005 Les Neuf Chapitres: Le Classique Mathématique De La Chine Ancienne Et Ses Commentaires (París: Dunod)

[16] Guevara-Casanova I and Burgués-Flamarich C 2018 Geometry and Visual Reasoning. Mathematics, Education and History (Springer, Cham) pp 165-92

[17] Gravemeijer K 2004 Local instruction theories as means of support for teachers in reform mathematics education Math. Think. Learn 6 105–28

[18] Simon M A 1995 Reconstructing mathematics pedagogy from a constructivist perspective Journal for Research in Mathematics Education 114-45

[19] Tzanakis C et al 2002 Integrating history of mathematics in the classroom: an analytic survey History in mathematics education (Springer: Dordrecht) pp 201-40

[20] Freudenthal 1991 Revisiting Mathematics Education: China Lectures (Dordrecht, the Netherlands: Kluwer Academic Publishers)