On commuting probabilities in finite groups and rings

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Abstract.
We show that the set of all commuting probabilities in finite rings is a subset of the set of all commuting probabilities in finite nilpotent groups of class \( \leq 2 \). We believe that these two sets are equal; we prove they are equal, when restricted to groups and rings with odd number of elements.

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1 Introduction and preliminaries

In 1940, Philip Hall [17] introduced the notion of the commuting probability in groups. Feit and Fine [12], derived a combinatorial formula and a generating function for commuting probability in matrix rings over finite fields. In the second half of 1960’s, the series of papers [8], [9], [10], [11] by Erdős and Turán, gave birth to the statistical group theory. In the fourth paper, among other results, the authors derived a lower bound for commuting probability in a finite group of order \( n \), and showed that the commuting probability in the symmetric group \( S_n \) is asymptotically equal to \( \frac{1}{n} \).

A number of research and expository papers on commuting probability in groups appeared during late sixties and the seventies: Joseph [19], [20], Galagher
Gustafson [16], Machale [22], and Rusin [27], to name a few. Rusin [27], characterized all finite groups with commuting probability > $\frac{11}{32}$. In the nineties, Lescot [21], re-derived classification of groups with commuting probability > $\frac{1}{2}$, using the notion of isoclinism in groups introduced by Hall [17].

There has also been interest in the study of commuting probability of other algebraic structures; MacHale [23], investigated the notion of commuting probability in rings. Commuting probability in semigroups has been studied in [14], [24], [26] and [29].

The dawn of the twenty-first century has seen a renewed interest in the study of the commuting probability in groups and rings, and other types of probabilities in rings, such as anticommuting and annihilating probability. Papers [28], [7], [15], [6] and [18] deal with commuting probability in finite groups. Buckley et. al., [3] and [1], classified all finite rings with commuting probability $\geq \frac{11}{32}$ and anticommuting probability $\geq \frac{15}{32}$, respectively.

Throughout this paper, $|A|$ denotes cardinality of the set $A$. $Z(G)$ denotes its center of a group $G$. For $a, b \in G$, $[a, b] = a^{-1}b^{-1}ab$ denotes the commutator of $a$ and $b$, and $[G, G]$ denotes the derived subgroup of $G$ generated by all commutators in $G$. Recall that $G$ is nilpotent of class $n$, if its lower central series (of normal subgroups) terminate in the trivial subgroup after $n$ steps, i.e.

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e_G\},$$

where $G_i = [G_{i-1}, G]$ for $i = 1, 2, \ldots, n$, and $G_{n-1} \neq \{e_G\}$. Commuting probability in a group $G$ is defined to be the number

$$Pr_c(G) = \frac{|\{(a, b) \in G \times G : ab = ba\}|}{|G|^2}.$$ 

For class $\mathcal{G}$ of finite groups, the set $\mathfrak{S}_c(\mathcal{G}) = \{Pr_c(G) : G \in \mathcal{G}\}$ is called the commuting spectrum of $\mathcal{G}$.

Rings are not assumed to be associative or unitary. By $R(+)$ we denote the additive group of $R$.

Recall that a ring $R$ is called antisymmetric if for all $a, b \in R$, $ab = -ba$. $R$ is called strongly antisymmetric if the dinipotent condition, $a^2 = 0$, is satisfied for all $a \in R$. Strong antisymmetry implies antisymmetry. A ring $R$ is said to be nilpotent class $\leq n$ if the product of any $n$ elements with any correct distribution of brackets is zero. For a prime $p$, $R$ is called a $p$-ring if $|R| = p^n$ for some positive integer $n$.

The symbol $[,]$ denotes the commutator in both a group $G$ and a ring $R$; whenever needed, we will write $[,]_G$ and $[,]_R$ to distinguish between the two cases.

Buckley [22], introduced the following generalization of the notion of commuting probability in rings. Let $f(X, Y) = aXY + bYX$ be a formal ”non-commutative polynomial” with integer coefficients. For any ring $R$ define a

\[\text{1. Dixon [7], provides an extensive list of publications on statistical group theory in the references, up to the year 2002.}\]
\[\text{2. Some publications use the term commuting degree in place of the commuting probability.}\]
function \( f^R : R \times R \to R, (x, y) = axy + byx \). Let

\[
\Pr_f(R) = \frac{|\{(x, y) \in R \times R : f^R(x, y) = 0\}|}{|R|^2}
\]

For class \( \mathcal{R} \) of finite rings, the set \( \mathcal{S}_f(\mathcal{R}) = \{\Pr_f(R) : R \in \mathcal{R}\} \) is called the \( f \)-spectrum of \( \mathcal{R} \).

Here, we are going to be mostly concerned with the commuting spectrum, \( \mathcal{S}_c(\mathcal{R}) \) and the annihilating spectrum, \( \mathcal{S}_{\text{ann}}(\mathcal{R}) \), with the associated formal "non-commutative polynomials" \( f(X, Y) = XY - YX \) and \( f(X, Y) = XY \), respectively. The commuting probability and the annihilating probability in a ring \( R \) are denoted by \( \Pr_c(R) \) and \( \Pr_{\text{ann}}(R) \), respectively.

We will use the following classes of groups and rings:

- \( \mathcal{G} \) the class of finite groups;
- \( \mathcal{G}_{\text{nil}} \) the class of finite nilpotent groups;
- \( \mathcal{G}_{\text{nil}}^{(2)} \) the class of finite nilpotent groups of class \( \leq 2 \);
- \( \mathcal{R} \) the class of finite rings;
- \( \mathcal{R}_{\text{nil}}^{(2)} \) the class of finite nilpotent rings of class \( \leq 3 \);
- \( \mathcal{R}_{\text{sa}} \) the class of finite strongly antisymmetric rings;
- \( \mathcal{R}_p \) the class of \( p \)-rings;
- for class \( \mathcal{C} \) of finite sets, denote \( \text{ODD}(\mathcal{C}) = \{A \in \mathcal{C} : |A| \text{ is odd}\} \).

Recall the following well know construction. For a given ring \( R \), we construct the ring \( N(R) \) in the following way: the additive group of \( N(R) \) is \((R \times R, +)\) with multiplication \((a, x)(b, y) = (0, ab)\). The following Lemma is immediate.

**Lemma 1.1.** Let \( R \) be a ring. Then \( N(R) \) is a nilpotent ring of class at most 3. Furthermore, if \( f(X, Y) = aXY + bYX \) is a formal non-commutative polynomial with integer coefficients and \( R \) is finite, then

\[
\Pr_f(R) = \Pr_f(N(R)).
\]

In particular, the Lemma implies

\[
\mathcal{S}_f(\mathcal{R}) = \mathcal{S}_f(\mathcal{R}_{\text{nil}}^{(2)}). \tag{1}
\]

Ever since it was discovered that there are no finite groups with commuting probability in the open interval \((1, \frac{5}{8})\), there has been an interest to understand the structure of the commuting spectrum of groups, and later, the structure of the commuting spectrum of rings and semigroups. The commuting spectrum for semigroups turned out to be the simplest to understand. Given [14] showed that the commuting spectrum for semigroups is dense in the interval \([0, 1]\). Later Ponomarenko and Selinski [26] proved that for any rational number in \( r \in (0, 1] \), there is a finite semigroup \( S \) such that the commuting probability in \( S \) is equal to \( r \). Soule [29] found a single family of semigroups that has this property.

Contrastingly, for groups, Hegarty [18] showed that for any limit point \( l \in (\frac{2}{3}, 1) \) of \( \mathcal{S}_c(\mathcal{G}) \), there is no increasing sequence of numbers \( \{a_n\} \subset \mathcal{S}_c(\mathcal{G}) \), such that \( l = \lim_{n \to \infty} a_n \).
Recently, Buckley and MacHale investigated relations between the commuting spectra of finite groups and rings. Comparing the structure of these two spectra for large probabilities, the authors formulated two conjectures, [4], page 9:

Conjecture 1. \( S_c(R) \subset S_c(G) \).

Conjecture 2. \( S_c(R) = S_c(G_{nil}) \) or \( S_c(G) = S_c(G_{nil}^{(2)}) \).

In this paper, we positively resolve the first conjecture and partially resolve the second.

2 Main results

Theorem 2.1. \( S_c(R) \subseteq S_c(G_{nil}^{(2)}) \subseteq S_{ann}(Rsa \cap R_{nil}^{(2)}) \).

In [5], the authors determined all values in \( S_c(R) \) that are \( \geq \frac{11}{32} \). These are

\[ \frac{1}{16}, \frac{7}{27}, \frac{11}{64}, \frac{25}{32}, \frac{11}{27}, \frac{25}{32}, \text{ and } \frac{22k + 1}{2^{2k+1}} \text{ for } k = 1, 2, 3, \ldots. \]

Thus, \( \frac{1}{2} \not\in S_c(R) \). But, \( \frac{1}{2} \in S_c(G) \), (27), page 246, and so \( S_c(R) \neq S_c(G) \). In particular, \( Pr_e(S_3) = \frac{1}{2} \) (see [20]); \( S_3 \) denotes the symmetric group of order 3. This, together with the first inclusion of Theorem 2.1, positively resolves Conjecture 1. As for Conjecture 2, the Theorem states \( S_c(R) \subseteq S_c(G_{nil}^{(2)}) \). Now that we know \( S_c(R) \) is a subset of the potentially smaller one of the two sets, \( S_c(G_{nil}^{(2)}) \) and \( S_c(G_{nil}) \) (it is unknown whether or not \( S_c(G_{nil}^{(2)}) = S_c(G_{nil}) \)), we ask the following question: Does

\[ S_c(R) = S_c(G_{nil}^{(2)}) \] (2)

hold true? We don’t know. But, Equation (2) does hold true, when restricted to finite groups and finite rings with odd number of elements. In fact, we prove the following:

Theorem 2.2.

\[ S_c(\text{ODD}(R)) = S_c(\text{ODD}(G_{nil}^{(2)})) = S_{ann}(\text{ODD}(Rsa \cap R_{nil}^{(2)})). \]

Next, we would like to formulate a condition, purely in terms of probabilities in rings, that would imply Equation (2). Using Theorem 2.1, one obvious choice could be \( S_{ann}(Rsa \cap R_{nil}^{(2)}) \subseteq S_c(R) \). We can do slightly better. Because things are working smoothly when restricted to rings with odd number of elements, it is sufficient to focus on the "trouble makers" which are the 2-rings.

Proposition 2.3. If \( S_{ann}(R_{sa} \cap R_{nil}^{(2)} \cap R_{2}) \subseteq S_c(R) \), then Equation (2) holds true.

\textsuperscript{3}The authors would like to thank Victor Bovdi for his interest in this paper.
The condition of Proposition 2.3 implies a stronger statement: If \( \mathcal{S}_{ann}(\mathcal{R}_{sa} \cap \mathcal{R}^{(2)}_{nil} \cap \mathcal{R}_2) \subseteq \mathcal{S}_c(\mathcal{R}) \), then both inclusions in Theorem 2.1 can be replaced by equal signs. Note that if there is a counterexample to the condition above, i.e. if there exists a ring \( R \) such that \( R \in \mathcal{R}_{sa} \cap \mathcal{R}^{(2)}_{nil} \cap \mathcal{R}_2 \) and \( \Pr_{ann}(R) \notin \mathcal{S}_c(\mathcal{R}) \), then \( \Pr_{ann}(R) < \frac{1}{2} \). We conjecture that \( \mathcal{S}_c(\mathcal{R}) = \mathcal{S}_{ann}(\mathcal{R}_{sa}) \).

3 Proofs

Let \( N \) be an associative nilpotent ring of class \( n \). Then \( N \), endowed with "circular multiplication", \( a \circ b = a + b + ab \), is a group which we will denote by \( G_N \). 0 is the unit element in \( G_N \) and \( a^{-1} = -a + a^2 - a^3 + \cdots + (-1)^{n-1} a^{n-1} \) is the inverse of \( a \) in \( G_N \), \( a \circ a^{-1} = a^{-1} \circ a = 0 \). Since, \( ab = ba \) if and only if \( a \circ b = b \circ a \), then, if \( N \) is finite,

\[
\Pr_c(N) = \Pr_c(G_N),
\]

**Lemma 3.1.** Let \( N \) be a nilpotent ring of class at most 3 (hence, also an associative ring). Let \( a, b, c \in N \). Then

(i) \( [a, b]_{G_N} = [a, b]_N \),

(ii) \( [a, b]_{G_N} \circ [c, d]_{G_N} = [a, b]_N + [c, d]_N \),

(iii) \( G_N \) is a nilpotent group of class \( \leq 2 \).

**Proof.** (i) follows by direct computation.

(ii). By (i),

\[
[a, b]_{G_N} \circ [c, d]_{G_N} = [a, b]_N \circ [c, d]_N
\]

\[
= [a, b]_N + [c, d]_N + [a, b]_N [c, d]_N = [a, b]_N + [c, d]_N.
\]

(iii). By (i), \( [[a, b]_{G_N}, c]_{G_N} = [[a, b]_N, c]_N = 0 \). \( \blacksquare \)

Let \( G \) be a nilpotent group of class \( \leq 2 \) and let \( Z = Z(G) \) be the center of \( G \). Then \( G/Z \) is abelian. By \( R_G \), denote the ring with the additive group \( G/Z \oplus Z \), and the multiplication defined by

\[
(aZ, x) \cdot (bZ, y) = (Z, [a, b]),
\]

where \( [a, b] = a^{-1}b^{-1}ab \) is the commutator in \( G \). Explicitly, the addition in \( R_G \) is given by

\[
(aZ, x) + (bZ, y) = (abZ, xy).
\]

\( (Z, e_G) \) is the zero element and \( (a^{-1}Z, x^{-1}) \) is the additive inverse of \( (aZ, x) \).

\[\text{footnote}{\text{Another way to associate a group to a ring such that their commuting probabilities equate can be obtained by modifying a construction of Mal’cev [25]. For an arbitrary ring } R, \text{ define a binary operation on } R \times R \text{ by } (a, b) \cdot (c, d) = (a + c, ac + b + d). \text{ This operation is associative, has unit } (0, 0) \text{ and } (a, b)^{-1} = (-a, a^2 - b). \text{ } G = (R \times R, \cdot) \text{ is a nilpotent group of class at most } 2 \text{ and } \Pr_{c}(R) = \Pr_{c}(G). \text{ Note that, unlike the construction of } G_N, \text{ the ring } R \text{ is not required to be nilpotent or associative!}}\]
To verify that \( R_G \) is indeed a ring, the distributive laws have to be satisfied. Let \( a, b, c \in G \) and \( x, y, z \in Z \). We have
\[
(cZ, z) \cdot ((aZ, x) + (bZ, y)) = (cZ, z) \cdot (abZ, xy) = (Z, [c, ab]).
\]
On the other hand,
\[
(cZ, z) \cdot (aZ, x) + (cZ, z) \cdot (bZ, y) = (Z, [c, a]) + (Z, [c, b]) = (Z, [c, a][c, b]).
\]
Using \([G, G] \subseteq Z\), we deduce
\[
[c, a][c, b] = c^{-1}a^{-1}cab^{-1} = c^{-1}a^{-1}b^{-1}cb = c^{-1}b^{-1}a^{-1}cb = c^{-1}b^{-1}a^{-1}cab = c^{-1}(ab)^{-1}c(ab) = [c, ab].
\]

Hence, the left distributive law is satisfied. The proof of the right distributive law is similar.

**Lemma 3.2.** Let \( G \) be a nilpotent group of class at most 2. Then \( R_G \) is a strongly antisymmetric nilpotent ring of class at most 3. If \( G \) is finite, then \(|R_G| = |G| \) and
\[
Pr_c(G) = Pr_{\text{ann}}(R_G).
\]

**Proof.** \(|R_G| = |G/Z||Z| = |G|\). \( R_G^2 = 0 \) and strong antisymmetry of \( R_G \) follows immediately from the multiplication formula (4) and the fact that \( G \) is a nilpotent group of class \( \leq 2 \). To prove (5), it suffices to note that \((aZ, x) \cdot (bZ, y) = (Z, [a, b]) = (Z, e_G) \) if and only if \([a, b] = e_G\). But, this is exactly when \( ab = ba\).

**Proof of Theorem 2.1**
We first show that \( \mathcal{G}_c(\mathcal{R}) \subseteq \mathcal{G}_c(\mathcal{G}^{(2)}_{\text{nil}}) \). Let \( r \in \mathcal{G}_c(\mathcal{R}) \) By Lemma 1.1 there is a nilpotent ring \( N \) of class at most 3 such that \( r = Pr_c(N) \). By Lemma 3.2 (iii), \( G_N \) is a nilpotent group of class at most 2 and by Equation (3), \( Pr_c(G_N) = Pr_c(N) \). We conclude that \( r \in \mathcal{G}_c(\mathcal{G}^{(2)}_{\text{nil}}) \).

To prove the second inclusion, consider \( G \in \mathcal{G}^{(2)}_{\text{nil}} \). By Lemma 3.2 \( R_G \in \mathcal{R}_{\text{sa}} \cap \mathcal{R}^{(2)}_{\text{nil}} \) and \( Pr_{\text{ann}}(R_G) = Pr_c(G) \).

**Lemma 3.3.** Let \( R \) be a finite antisymmetric ring and with odd number of elements. Then
\[
Pr_c(R) = Pr_{\text{ann}}(R).
\]

**Proof.** In an antisymmetric ring, \( ab = -ba \). Hence, \( ab = ba \) iff \( 2ab = 0 \). Since \(|R|\) is odd, \( 2ab = 0 \) iff \( ab = 0 \).

**Proof of Theorem 2.2**
To prove \( \mathcal{G}_c(\text{ODD}(\mathcal{R})) \subseteq \mathcal{G}_c(\text{ODD}(\mathcal{G}^{(2)}_{\text{nil}})) \subseteq \mathcal{G}_{\text{ann}}(\text{ODD}(\mathcal{R}_{\text{sa}} \cap \mathcal{R}^{(2)}_{\text{nil}})) \), we follow the proof of Theorem 2.1 and note that \(|N| = |G_N| \) and \(|G| = |R_G| \).

\[\text{Proposition 3 [3], states that the condition } [c, a][c, b] = [c, ab] \text{ for all } a, b, c \in G \text{ is equivalent to } G \text{ being nilpotent of class } \leq 2.\]
To conclude the proof of Theorem 2.2, it suffices to show
\[ S_{\text{ann}}(\text{ODD}(\mathcal{R}_{\text{sa}} \cap \mathcal{R}_{\text{nil}}^{(2)})) \subseteq S_{c}(\text{ODD}(\mathcal{R})). \]

Let \( r \in S_{\text{ann}}(\text{ODD}(\mathcal{R}_{\text{sa}} \cap \mathcal{R}_{\text{nil}}^{(2)})) \) and let \( R \in \text{ODD}(\mathcal{R}_{\text{sa}} \cap \mathcal{R}_{\text{nil}}^{(2)}) \) such that \( r = \text{Pr}_{\text{ann}}(R) \). By Lemma 3.3, \( r = \text{Pr}_{\text{ann}}(R) = \text{Pr}_{c}(R) \in S_{c}(\text{ODD}(\mathcal{R})). \)

In a ring, the additive order of \( ab \) divides the additive orders of both \( a \) and \( b \). In particular, if the additive orders of \( a \) and \( b \) are relatively prime, then \( ab = 0 \).

As a consequence of this fact and the GH fundamental theorem of finite abelian groups, GH a finite ring is a product of \( p \)-rings. In turn, this implies that
\[ \text{Pr}_{\text{ann}}(R_1 \times R_2) = \text{Pr}_{\text{ann}}(R_1) \text{Pr}_{\text{ann}}(R_2), \]
for any two rings \( R_1, R_2 \).

We say that a class \( \mathcal{R} \) of finite rings is \textit{hereditary}, if any subring of a ring in \( \mathcal{R} \) is also in \( \mathcal{R} \).

Let \( p \) be a prime number and assume that a class \( \mathcal{C} \) of finite rings is hereditary. Then \( \mathcal{C}_p = \mathcal{C} \cap \mathcal{R}_p \) also is hereditary. Furthermore, \( S_{\text{ann}}(\mathcal{C}) \) and \( S_{\text{ann}}(\mathcal{C}_p) \), are monoids and values in \( S_{\text{ann}}(\mathcal{C}) \) are finite products of values taken from the set \( \bigcup_p S_{\text{ann}}(\mathcal{C}_p) \), where \( p \) runs all prime number.

**Proof of Proposition 2.3**

It is easy to see that the class \( \mathcal{C} = \mathcal{R}_{\text{sa}} \cap \mathcal{R}_{\text{nil}}^{(2)} \) is hereditary. Assume \( S_{\text{ann}}(\mathcal{C} \cap \mathcal{R}_p) \subseteq S_{c}(\mathcal{R}) \). If \( p \neq 2 \) is a prime, by Theorem 2.2, \( S_{\text{ann}}(\text{ODD}(\mathcal{C})) = S_{c}(\text{ODD}(\mathcal{R})) \) and so \( S_{\text{ann}}(\mathcal{C} \cap \mathcal{R}_p) \subseteq S_{c}(\mathcal{R}) \). Hence, for all primes \( p \), the monoids \( S_{\text{ann}}(\mathcal{C} \cap \mathcal{R}_p) \subseteq S_{c}(\mathcal{R}) \). Using the considerations above, we conclude \( S_{\text{ann}}(\mathcal{C}) \subseteq S_{c}(\mathcal{R}) \). By Theorem 2.1, the reverse inclusion is satisfied, and so the proposition follows.

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\footnote{Similar arguments show that \( \text{Pr}_f(R_1 \times R_2) = \text{Pr}_f(R_1) \text{Pr}_f(R_2) \), for any noncommutative formal polynomial \( f(X, Y) = axY + byX \).}
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