Estimation of surgical tool-tip tracking error distribution in coordinate reference frame involving pivot calibration uncertainty

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Accurate understanding of surgical tool-tip tracking error is important for decision making in image-guided surgery. In this Letter, the authors present a novel method to estimate/model surgical tool-tip tracking error in which they take pivot calibration uncertainty into consideration. First, a new type of error that is referred to as total target registration error (TTRE) is formally defined in a single-rigid registration. Target localisation error (TLE) in two spaces to be registered is considered in proposed TTRE formulation. With first-order approximation in fiducial localisation error (FLE) or TLE magnitude, TTRE statistics (mean, covariance matrix and root-mean-square (RMS)) are then derived. Second, surgical tool-tip tracking error in optical tracking system (OTS) frame is formulated using TTRE when pivot calibration uncertainty is considered. Finally, TTRE statistics of tool-tip in OTS frame are then propagated relative to a coordinate reference frame (CRF) rigid-body. Monte Carlo simulations are conducted to validate the proposed error model. The percentage passing statistical tests that there is no difference between simulated and theoretical mean and covariance matrix of tool-tip tracking error in CRF space is more than 90% in all test cases. The RMS percentage difference between simulated and theoretical tool-tip tracking error in CRF space is within 5% in all test cases.

1. Introduction: Surgical tool-tip tracking is an essential technique in image-guided surgery (IGS) [1, 2]. On the one hand, it can be used to provide the real-time surgical tool-tip position in tracking system frame during surgery. On the other hand, it can also be adopted to acquire fiducials’ positions in patient space for an image-to-patient registration [3]. Statistics of surgical tool-tip tracking error can provide real-time feedback to help surgeons make correct decisions (e.g. avoiding potentially dangerous tool movements) during surgery [4]. Among various tracking systems, optical tracking system (OTS) is most commonly adopted for its easy implementation. The purpose of this Letter is to describe and validate a closed-form solution to surgical tool-tip tracking error model problem in CRF while pivot calibration uncertainty is considered. To do this, we first define and develop a new type of error metric called total target registration error (TTRE) in a single rigid registration. Target localisation error (TLE) in two spaces to be registered is considered in the formulation of TTRE. Tool-tip tracking error in OTS frame is represented by TTRE where TLE in TRF space is caused by pivot calibration. Then TTRE model is extended to the case where an optically tracked tool’s pose is measured relative to a CRF. A closed-form formulation of statistics (i.e. mean, covariance matrix, RMS
(root-mean-square) of surgical tool-tip tracking error in CRF is then derived. Simulation results show that the proposed model can (i) predict the mean and covariance matrix of tool-tip tracking error in CRF well (at least 90% of test cases accepting the null-hypothesis of hypothesis tests); and (ii) predict RMS value of tool-tip tracking error in CRF well (RMS percentage difference between predicted and simulated data is within 5% for all test cases). We summarise our contributions as follows: (i) A new tool-tip tracking error in CRF well (RMS percentage difference can (i) predict the mean and covariance matrix of tool-tip tracking error in CRF well (at least 90% of test cases accepting the null-

**2. Method:** The coordinate frames and transformation matrices involved in this Letter are first defined for clarity:

- OTS – optical tracking system;
- TRF – tool reference frame;
- CRF – coordinate reference frame;
- $T'$ – measured transformation matrix relating frames $A$ and $B$;
- $T'$ or $R'$ – true transformation or rotation matrix;
- $d_p$ – true value of vector $p$ in frame $A$;
- $d^p$ – measured value of vector $p$ in frame $A$.

**2.1. PPRR problem:** The PPRR problem is to determine the rigid transformation $T \in SE(3)$ composed of a rotation matrix $R \in SO(3)$ and a translation vector $t \in \mathbb{R}^3$ that minimises the following term [10]:

$$ \text{FRE}^2 = \sum_{i=1}^{N} |W_i(R(x_i + \Delta x_i) + t - (y_i + \Delta y_i))|^2 $$

where FRE is the weighted fiducial registration error, $N \geq 3$ is the number of fiducials, $x_i = (x_{i1}, \ldots, x_{iN}) \in \mathbb{R}^{N \times 1}$ and $y_i = (y_{i1}, \ldots, y_{iN}) \in \mathbb{R}^{N \times 1}$ represent corresponding fiducials’ position sets in $X$ (e.g. TRF) and $Y$ (e.g. OTS frame) spaces to be registered, $\{\Delta x_1, \ldots, \Delta x_N\} \in \mathbb{R}^{3 \times N}$ and $\{\Delta y_1, \ldots, \Delta y_N\} \in \mathbb{R}^{3 \times N}$ represent FLE vector sets in $X$ and $Y$ spaces, $W_i \in \mathbb{R}^{3 \times 1}$ is a non-singular weighting matrix of the $i$th fiducial. Without loss of generality, $\Delta x_i$ and $\Delta y_i$ are modelled as independent zero-mean random variables (only reasonable for passive OTS [5]) satisfying $\Delta x_i \sim \mathcal{N}(0, \text{cov}[x_i])$ and $\Delta y_i \sim \mathcal{N}(0, \text{cov}[y_i])$, where $\text{cov}[v]$ denotes the covariance matrix of one random variable $v$ with itself.

**2.2. Total target registration error:** A new type of error metric that is referred to as total target registration error at a given ‘nominal’ target point $r$ is proposed and defined as follows:

$$ \text{TRE}(r) = R(r + \Delta r) + t + \Delta y - (Rr + t') $$

$$ = Rr + t - (R'r + t') + R\Delta r + \Delta y $$

$$ \approx \text{TRE}(0) + \Delta RR'r + t - t' + R\Delta r + \Delta y $$

where $R' \in SO(3)$ and $t' \in \mathbb{R}^3$ are the ‘true’ rotation matrix and translation vector relating $X$ and $Y$ spaces, $r \in \mathbb{R}^3$ is the ‘true’ target location in $X$ space, $\Delta r \in \mathbb{R}^3 \sim \mathcal{N}(0, \text{cov}[\Delta r])$ and $\Delta y \in \mathbb{R}^3 \sim \mathcal{N}(0, \text{cov}[\Delta y])$ are independent TLE vectors in $X$ and $Y$ spaces. TLE = $(R\Delta r + \Delta y) - \mathcal{N}(0, \text{cov}[\Delta r] R' + \text{cov}[\Delta y])$ is the ‘two-space’ TLE vector. The concept of TLE is illustrated in Fig. 1. Two assumptions are now made to simplify (2): (a) both FLE and TLE magnitudes are small; (b) approximation to first-order in FLE or TLE magnitude is utilised. It was proved in [10] that $\Delta R = R(R'^{-1})' - I_{3 \times 3}$ is of first-order in FLE or TLE magnitude. With the two assumptions, we can see TLE equals the following:

$$ \text{TRE}(r) = \text{TRE}(0) + \Delta RR' \Delta r + R' \Delta r + \Delta y $$

$$ \approx \text{TRE}(0) + R' \Delta r + \Delta y $$

where the term $\Delta RR' \Delta r$ disappears in last line of (3) as it is of second order in FLE.

**2.2.1 Mean of TLE:** The mean of TLE is calculated by taking the expectation of TLE vector in (3)

$$ \text{TRE}(r) = \langle \text{TRE}(r) \rangle + \langle R' \Delta r \rangle + \langle \Delta y \rangle $$

where we have adopted the property that $R'$ is a constant matrix in going from the first to second line of (4).

**2.2.2 Covariance matrix of TLE:** The covariance matrix of TLE is calculated using the expected value of the outer product of TLE
vector with itself
\[ \text{cov}[\text{TTRE}(r)] = \langle \text{TTRE}(r) \cdot (\text{TTRE}(r))^T \rangle \] (5)

Substitute (3) into (5), together with (4), the following holds:
\[ \text{cov}[\text{TTRE}(r)] = \text{cov}[(\text{TTRE}(r) + \text{cov}[\text{R'} \cdot \text{DR}_r] + \text{cov}[\text{DR}_r]) \cdot (\text{R'})^T + \text{cov}[\text{DR}_r]] \]
(6)

where we have utilised the property that terms TRE, DR_r and DR_r are co-independent and thus uncorrelated with each other. The expression of \( \text{cov}[\text{TTRE}(r)] \) was developed in [10].

2.2.3 RMS of TTRE: The TTRE RMS value is acquired by calculating the trace of TTRE covariance matrix
\[ \langle (\text{TTRE}(r))^2 \rangle = \text{trace} (\text{cov}[\text{TTRE}(r)]). \] (7)

2.3. Surgical tool-tip tracking: Two paired-point rigid registrations are involved in determining the tool-tip position in CRF space (denoted by \( \text{crf} \)): (i) TRF-attached fiducials’ measured positions in OTS frame are registered to corresponding fiducials’ calibrated positions in TRF and \( \text{ots} \) TRF is acquired; (ii) CRF-attached fiducials’ measured positions in OTS frame are registered to corresponding fiducials’ calibrated positions in CRF and \( \text{ots} \) CRF is acquired. After the two registrations, \( \text{crf} \) can be calculated as:
\[ \text{crf} = \text{ots} \cdot \text{TRF} \quad \text{or} \quad \text{crf} = \text{ots} \cdot \text{TRF} \quad \text{or} \quad \text{crf} \]

The above two registrations are denoted as ‘to’ and ‘oc’ hereafter, respectively. It is worth mentioning that we still assume that both TRF-attached and CRF-attached fiducials are in their own respective local coordinate frames (i.e. TRF and CRF) are well calibrated. Mathematically, let \( \{x_i^{\text{crf}}\}_{i=1}^N \) be the CRF-attached fiducials’ calibrated positions in CRF, we assume \( \{x_i^{\text{ots}}\}_{i=1}^N \) be the TRF-attached fiducials’ calibrated positions in OTS.

2.3.1 Surgical tool-tip tracking error in OTS frame: In surgical tool-tip tracking, tool-tip tracking error in OTS frame is actually an adapted version of TTRE in (2)
\[ \text{TTRE}_{\text{o},\text{ots}}(r) = \text{ots} \cdot (\text{RT} + \text{DR}_r) + \text{ots} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \]
(8)

where TRF is X space and OTS frame is Y space in (8), target (tool-tip) localisation error \( \text{DR}_r \) in TRF space is caused by pivot calibration. Notice \( \text{DR}_r \) disappears in (8) as the tracking system does not make any direct localisation of the tool-tip in OTS frame.

2.3.2 Surgical tool-tip tracking error in CRF space: We shift back to use term \( \text{TTRE}_{\text{r},\text{crf}}(r) \) to represent surgical tool-tip tracking error vector in CRF
\[ \text{TTRE}_{\text{crf}}(r) = \text{crf} \cdot (\text{RT} + \text{DR}_r) \\
\text{TTRE}_{\text{crf}}(r) = \text{crf} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \]
(9)

where
\[ \text{TTRE}_{\text{crf}}(r) = \text{crf} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \]
(10)

Notice since \( \text{ots} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \) represents a different vector and is not a spatial position, the transformation \( \text{ots} \) can be reduced to the rotation matrix \( \text{ots} \cdot \text{R} \) in the last line of (9) [15].

2.3.3 Mean, covariance matrix and RMS of tool-tip tracking error in CRF space: The mean of TRE in CRF space is a zero vector
\[ \langle \text{TRE}_{\text{comb}}(r) \rangle = 0_{3 \times 1} \] (12)
The covariance matrix of TRE in CRF space is the following:
\[ \text{cov}[\text{TRE}_{\text{comb}}(r)] = \langle \text{TRE}_{\text{comb}}(r) \cdot (\text{TRE}_{\text{comb}}(r))^T \rangle \] (13)

Substitute the last expression of (9) into (13), with some expansions, we can obtain
\[ \text{cov}[\text{TRE}_{\text{comb}}(r)] = \text{crf} \cdot \text{TTRE}_{\text{crf}}(r) \cdot \text{crf} \cdot \text{TTRE}_{\text{crf}}(r)^T \]
(14)

Due to the two registrations, respectively, denoted by ‘oc’ and ‘to’ are independent, the two random variables \( \text{crf} \cdot \text{TTRE}_{\text{crf}}(r) \) and \( \text{ots} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \) are uncorrelated. Thus, the last term in (14) disappears and together with (4), we obtain a more concise expression of \( \text{cov}[\text{TRE}_{\text{comb}}(r)] \):
\[ \text{cov}[\text{TRE}_{\text{comb}}(r)] = \text{crf} \cdot \text{TTRE}_{\text{crf}}(r) \cdot \text{crf} \cdot \text{TTRE}_{\text{crf}}(r)^T \]
(15)

where \( \text{crf} \cdot \text{TTRE}_{\text{crf}}(r) \) can be computed using the expression developed in [10], \( \text{ots} \cdot \text{TTRE}_{\text{r},\text{crf}}(r) \) is calculated using (9). The RMS value of surgical tool-tip tracking error in CRF space is further calculated as the following:
\[ \text{RMS}_{\text{comb}}(r) = \sqrt{\langle \text{RMS}_{\text{o},\text{oc}}(r)^2 \rangle + \langle \text{RMS}_{\text{meo},\text{to}}(r)^2 \rangle} \] (16)

where \( \text{RMS}_{\text{o},\text{oc}}(r)^2 \) and \( \text{RMS}_{\text{meo},\text{to}}(r)^2 \) can be computed using (7).

3. Experiments: We conducted extensive simulations using two different surgical tool configurations. The two surgical tool configurations are shown clearly in Figs. 2a and b. In all simulations, the number of TRF-attached or CRF-attached fiducials \( N \) is 4. More specifically, for the first kind of surgical tool, the coordinates of fiducials in TRF, \( \{x_i^{\text{crf}}\}_{i=1}^4 \), are: [35.5, 27, 0]', [−35.5, 27, 0]', [35.5, −27, 0]', [−35.5, −27, 0]' mm [19]. For the second kind of surgical tool, the coordinates of fiducials in TRF, \( \{x_i^{\text{crf}}\}_{i=1}^4 \), are: [0, 50, 0]', [−50, 0, 0]', [50, 0, 0]', [0, 50, 0]' mm [21]. As it is shown in Fig. 2c, the CRF rigid-body is a square centred at \( O_t \) with side length \( l \) being 32 or 64 mm. The coordinates of CRF-attached fiducials in CRF, \( \{x_i^{\text{crf}}\}_{i=1}^4 \), are: [1/2, 1/2, 0]', [1/2, 1/2, 0]', [−1/2, −1/2, 0]', [−1/2, −1/2, 0]' mm. The distance \( d \) between CRF origin \( O_t \) and pivot point or tool-tip position \( P \) was set to be 100, 200, 300 or 400 mm. For the first kind of tool, the distance \( \rho \) between tool tip \( P \) and marker centroid \( O_t \) was 85 mm; for the second kind of tool, \( \rho \) equals 200 mm. The FLE covariance matrix \( \Sigma_{\text{FLE}} \) in OTS frame was set to be identical for all TRF-attached and CRF-attached fiducials: \( \Sigma_{\text{FLE}} = \text{diag}([0.02^2, 0.02^2, 0.2^2]) \) [20]. The pivot calibration...
uncertainty covariance matrix $\Sigma_{\text{pivot}}$ was set to be a matrix whose eigenvalues' square roots were [0.31, 0.40, 0.91]$^T$ [4]. For all simulated cases, the rotation matrix between CRF and OTS stays the same and is denoted as $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$. For each simulated case with certain values of $l$ and $d$, $M = 100$ random orientations of surgical tool $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]_{j=1}^{M}$ were generated while the tool-tip was fixed at the pivot point $P$. For the $j$th tool orientation $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$, $N_f = 2000$ samples of $2N$ FLE vectors and pivot calibration uncertainty vectors $\Delta_R$ are generated independently according to $\Sigma_{\text{Sim}}$ and $\Sigma_{\text{pivot}}$, respectively. Let $k = 1, \ldots, N_f$ denote the index of $N_f$ error samples. For the $k$th sample, the generated $2N + 1$ vectors were added to ‘true’ transformed fiducials’ positions in OTS frame $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and the ‘real’ tip position in TRF $r_{\text{def}}$. In this way, the $k$th measured TRF-attached and CRF-attached positions in OTS frame $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ were acquired. Then $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ were registered to their corresponding calibrated ones in TRF and CRF spaces $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$, respectively. The registration algorithm introduced in [22] was used in the above two registrations with the weighting matrix $W_f$ being $(\Sigma_{\text{Sim}})^{-1/2}$ for all the fiducials. After the two registrations $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ were acquired. So for $j$th tool orientation, $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ and $\left[ \begin{array}{c} \cos \theta^T \sin \theta \\ \sin \theta \cos \theta \end{array} \right]$ were calculated in all. For each tool orientation, TRE statistics of tool-tip in CRF space were calculated using above simulated data and (8), (9). At the same time, for each tool orientation, the predicted TRE statistics in CRF were computed using (5)–(7), (12), (15) and (16).

Two Wishart distribution hypothesis tests ($\alpha = 0.05$) similar to those in [6] were conducted for each simulation case (with the null hypothesis stating that there was no difference between simulated and theoretical TRE covariance matrix (or mean and covariance matrices)). For each simulated case, the percentage passing the $M$ Wishart distribution hypothesis tests was also calculated for each tool orientation [%diff = 100(RMS$_{\text{sim}}$ – RMS$_{\text{true}}$)/RMS$_{\text{true}}$]. Statistics (mean, standard deviation, maximum and minimum) of RMS percentage difference were further calculated for each simulated case.

### Results and discussion: Simulation results are summarised in Tables 1 and 2. The worst cases in each column are emphasised using black bold texts. For the first kind of tool, at least 93 and 94% accept the null hypothesis of first and second hypothesis test, respectively. The RMS percentage difference is within ±2.78% (95% confidence interval (CI)) with maximum and minimum values being 4.41 and –4.23%. For the second kind of tool, at least 93 and 95% accept the null hypothesis of first and second hypothesis tests, respectively. The RMS percentage difference is within ±2.60% (95% CI) with maximum and minimum values being 4.33 and –3.71%. Thus, we can conclude proposed error model can well predict the simulated/measured tool-tip tracking error magnitudes. With FLE RMS and pivot calibration uncertainty vector RMS being 0.20 and 1.27 mm,

| Case | Ref. size | Working distance | Accepted | RMS percent difference summary statistics |
|------|-----------|-----------------|----------|------------------------------------------|
|      | $l$, mm   | $d$, mm         | 1, %     | 2, %          | Mean, % | Std. dev, % | Max, % | Min, % |
| 1    | 32        | 100             | 93.00    | 97.00    | 0.04   | 1.28      | 3.10   | −2.94 |
| 2    | 32        | 200             | 95.00    | 99.00    | −0.06  | 1.39      | 3.75   | −4.23 |
| 3    | 32        | 300             | 95.00    | 95.00    | −0.01  | 1.16      | 3.41   | −3.36 |
| 4    | 32        | 400             | 97.00    | 94.00    | 0.07   | 1.23      | 3.09   | −2.67 |
| 5    | 64        | 100             | 97.00    | 100.00   | 0.14   | 1.11      | 3.43   | −2.05 |
| 6    | 64        | 200             | 97.00    | 99.00    | 0.02   | 1.01      | 2.13   | −3.24 |
| 7    | 64        | 300             | 95.00    | 96.00    | 0.08   | 1.26      | 4.41   | −3.02 |
| 8    | 64        | 400             | 98.00    | 100.00   | 0.06   | 1.22      | 3.38   | −2.68 |
The 95% CI boundary of predicted (red) and simulated (green) tool-tip tracking error covariance matrix (95% CI boundary) in CRF for one simulation case using first kind of surgical tool; (Right) similar statistics are visualised for one simulation case using the second kind of surgical tool.

**Table 2** Monte Carlo simulation results for second kind of surgical tool with various reference tool size \( l \) and working distance \( d \). Null hypothesis for test 1 is \( \Sigma_{\text{sim}} = \Sigma_{\text{pre}} \) and test 2 is \( H_0: \Sigma_{\text{sim}} = \Sigma_{\text{pre}} \).

| Case | Ref size \( l \), mm | Working distance \( d \), mm | Accepted 1, % | Accepted 2, % | Mean, % | Std. dev, % | Max, % | Min, % |
|------|----------------------|-----------------------------|----------------|----------------|---------|-------------|--------|--------|
| 1    | 32                   | 100                         | 94.00          | 96.00          | -0.12   | 1.30        | 2.74   | -3.71  |
| 2    | 32                   | 200                         | 95.00          | 99.00          | -0.01   | 1.16        | 3.28   | -3.13  |
| 3    | 32                   | 300                         | 93.00          | 98.00          | 0.04    | 1.06        | 3.36   | -2.28  |
| 4    | 32                   | 400                         | 95.00          | 95.00          | -0.03   | 1.21        | 3.18   | -2.98  |
| 5    | 64                   | 100                         | 96.00          | 100.00         | 0.10    | 1.15        | 3.06   | -2.46  |
| 6    | 64                   | 200                         | 97.00          | 100.00         | 0.09    | 1.12        | 3.23   | -2.48  |
| 7    | 64                   | 300                         | 95.00          | 99.00          | 0.01    | 1.16        | 3.04   | -2.97  |
| 8    | 64                   | 400                         | 96.00          | 100.00         | 0.07    | 1.21        | 4.33   | -3.58  |

**Fig. 3** (Left) Predicted (red) and simulated (green) tool-tip tracking error covariance matrix (95% CI boundary) in CRF for one simulation case using first kind of surgical tool; (Right) similar statistics are visualised for one simulation case using the second kind of surgical tool.

respectively, the model’s performance varies little with respect to different side lengths \( l \) of CRF and working distances \( d \).

The 95% CI boundary of predicted (red) and simulated (green) covariance matrices are visualised in Fig. 3. The three ellipses in each plot represent the three principal directions of tool-tip tracking error covariance matrices in CRF. As it is shown in the plots of Fig. 3, predicted covariance matrices agree very well with the simulated ones. It is worth mentioning the tool-tip tracking error distribution is anisotropic in CRF. More specifically, we are more uncertain of tool-tip position in the direction with larger ellipse.

One issue in applying the error model to real surgical tool tracking scenario is that the ‘true’ rotation matrix \( \otimes_{\text{true}} R \) in (6), (15) is not known. In real implementations, \( \otimes_{\text{true}} R \) can be approximated using measured rotation matrix \( \otimes_{\text{meas}} R \). Another one is the choice of visualisation methods in order to better convey the information of tool-tip tracking error statistics in a precise way for all test cases. More specifically, the magnitude (RMS), position (mean) and shape (covariance matrix) of surgical tool-tip tracking error are very well modelled for two kinds of surgical tools.

Future extensions include incorporating the proposed error model into a commercial surgical navigation system to provide useful feedback for surgeon during surgery. The proposed model will also be extended to the case where a multi-camera tracking system is adopted to eliminate the occlusion problem of existing stereo-camera tracking system. In a multi-camera tracking system, FLEs of TRF-attached and CRF-attached fiducials should be considered to be inhomogeneous and anisotropic. The inhomogeneity of FLE is partly caused by different number of cameras seeing each fiducial.

**5. Conclusions:** In this Letter, we have presented a closed-form formulation of surgical tool-tip tracking error distribution in CRF. Pivot calibration uncertainty is included in the proposed error model. Results show that the proposed model can predict tool-tip tracking error statistics in a precise way for all test cases. More specifically, the magnitude (RMS), position (mean) and shape (covariance matrix) of surgical tool-tip tracking error are very well modelled for two kinds of surgical tools.

Future extensions include incorporating the proposed error model into a commercial surgical navigation system to provide useful feedback for surgeon during surgery. The proposed model will also be extended to the case where a multi-camera tracking system is adopted to eliminate the occlusion problem of existing stereo-camera tracking system. In a multi-camera tracking system, FLEs of TRF-attached and CRF-attached fiducials should be considered to be inhomogeneous and anisotropic. The inhomogeneity of FLE is partly caused by different number of cameras seeing each fiducial.

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