THE ACTION OF CONTACT TRANSFORMATIONS
PSEUDOGROUP ON THE SECOND ORDER ODES WHICH
ARE CUBIC IN SECOND DERIVATIVE

Vadim V. Shurygin, jr
Kazan Federal University, Russia

Abstract
In the present paper we consider the problem of local equivalence of second order
ODEs which are cubic in second derivative under the action of the pseudogroup of
contact transformations. We concentrate on the case when the associated equation
is linearizable and reduce the problem to the equivalence problem of 2-parametric
families of curves in $\mathbb{R}^3$ under the action of the group $SL_3$.

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1 Introduction
The problem of equivalence of second order ODEs of the form
$$y'' = f(x, y, y')$$
under the action of some transformations pseudogroup was the subject of many papers.
S. Lie proved that any two such equations are equivalent under the action of the pseudo-
group of contact transformations on the space $\mathbb{R}^3(x, y, p)$. A. Tresse [12] found the complete
set of differential invariants of the ODE (1.1) under the action of the pseudogroup of point
transformations. B. Kruglikov [4] completely described the algebra of differential invariants
and solved the problem of point equivalence of such ODEs.

We consider the problem of equivalence of the ODEs of the form
$$y''^n + A_1(x, y, y')y''^{n-1} + \ldots + A_{n-1}(x, y, y')y'' + A_n(x, y, y') = 0$$
under the action of the pseudogroup of contact transformations. We suppose that this
equation has $n$ distinct roots
$$y'' = \lambda_i(x, y, y'), \quad i = 1, \ldots, n.$$  

The case $n = 2$ was solved in [13]. The equivalence problem of such ODEs reduces to the problem of point equivalence of some ODEs of the form (1.1), associated with the original equations. In the present paper we show how the case $n = 3$ may be reduced to the case $n = 2$. Moreover, we consider the most interesting situation for $n = 3$, namely, the case when the associated equation for the quadratic equation $(y'' - \lambda_1)(y'' - \lambda_2) = 0$, is linearizable. This case reduces to the equivalence problem of 2-parametric families of integral curves of Cartan distribution in $\mathbb{R}^3$ under the action of projective group $SL_3$ admitted by the associated equation. This problem can be reformulated as a classification problem of ODEs of the form (1.1) under the action of $SL_3$. We also solve the more general problem of $SL_3$-equivalence of arbitrary 2-parametric families of curves in $\mathbb{R}^3$. 

1
2 Basic definitions and constructions

Let $\pi$ be a vector bundle and $G$ be a pseudogroup of diffeomorphisms acting on $\pi$. The action of $G$ naturally lifts to the action on the space $J^k\pi$ of $k$-jets of sections of $\pi$.

By a smooth function $f \in C^\infty(J^\infty\pi)$ we mean the smooth function on the finite-dimensional jet space $J^k\pi$ for some $k \geq 1$.

Definition. The function $I \in C^\infty(J^k\pi)$ is called the (absolute) scalar differential invariant of $k$th order, if it is constant along the orbits of the lifted action of $G$ on $J^k\pi$.

The space $I$ of all invariants forms an algebra with respect to algebraic operations of linear combinations over $R$, multiplication and the composition $I_1, \ldots, I_m \rightarrow I = F(I_1, \ldots, I_m)$ for any smooth function $F \in C^\infty(R^m) \rightarrow R$, $m \geq 1$.

Let $G$ be a connected Lie group and let $g$ be its Lie algebra. For $X \in g$ denote by $\hat{X}$ the lift of $X$ to $J^\infty\pi$ (see [2]). Then $I$ is an absolute differential invariant if and only if $L_{\hat{X}}(I) = 0$ for all $X \in g$. For more information on differential invariants see, e.g., [2, 3, 8].

Definition. The function $F \in C^\infty(J^k\pi)$ is called the relative scalar differential invariant of $k$th order, if for all $g \in G$ there holds $g^*F = \mu(g) \cdot F$, where $\mu : G \rightarrow C^\infty(J^\infty\pi)$ is a smooth function satisfying the condition $\mu(g \cdot h) = h^*\mu(g) \cdot \mu(h), \quad \mu(e) = 1$.

In other words, the equation $F = 0$ is invariant under the action of $G$. The function $\mu$ is called the weight function.

The infinitesimal analogue of this definition is that $F$ is a relative invariant if and only if $L_{\hat{X}}(F) = \mu_X \cdot F$, where the map $\mu : g \rightarrow C^\infty(J^\infty\pi)$ satisfies the condition

$$\mu_{[X,Y]} = L_{\hat{X}}(\mu_Y) - L_{\hat{Y}}(\mu_X), \quad \forall X, Y \in g.$$ 

Let $(x_1, \ldots, x_n)$ be the coordinates in the base manifold of the bundle $\pi$.

Definition. An invariant derivation is a linear combination of total derivatives

$$\nabla = \sum_{i=1}^n A_i \frac{d}{dx^i}, \quad A_i \in C^\infty(J^\infty\pi), \quad i = 1, \ldots, n,$$

which commutes with all prolongations of vector fields $X \in g$. For each differential invariant $I$ the function $\nabla(I)$ also is a differential invariant (usually, of order, higher than the order of $I$). This fact allows one to obtain new invariants out of known ones using invariant derivations. For the algebra $I$ of differential invariants the Lie-Tresse theorem is valid. It claims that there exists a finite number of basic differential invariants and basic invariant derivations, such that any other differential invariant can be expressed in terms of these basic invariants and its derivations (see [3, 5, 8]).

The following result of V. Lychagin [7] allows one to find invariant derivations. Let $G$ be a Lie group acting on $R^n$. Let us identify the elements of its Lie algebra $g$ with contact vector fields $X_f$ on $J^1(R^n)$ (here $f$ is the generating function of $X_f$).
Proposition 1. Let \( x_1, \ldots, x_n \) be coordinates in \( \mathbb{R}^n \), and let \((x_1, \ldots, x_n, u, p_1, \ldots, p_n)\) denote the corresponding canonical coordinates in \( J^1(\mathbb{R}^n) \). Then the derivation

\[
\nabla = \sum_{i=1}^{n} A_i \frac{d}{dx^i}
\]

is \( \mathfrak{g} \)-invariant if and only if functions \( A_i \in C^\infty(J^\infty(\mathbb{R}^n)) \), \( i = 1, \ldots, n \), satisfy the following system of PDEs

\[
X_f(A_i) + \sum_{j=1}^{n} \frac{d}{dx^j} \left( \frac{\partial f}{\partial p_i} \right) A_j = 0
\]

for all \( j = 1, \ldots, n \), \( X_f \in \mathfrak{g} \).

Let \( \mathcal{C} \) denote the standard contact distribution (Cartan distribution) on \( \mathbb{R}^3(x, y, p) \) and let \( \omega = dy - pdx \) be the contact form. Geometrically each of the equations \( y'' = \lambda_i(x, y, y') \) determines the 1-dimensional distribution \( \mathcal{F}_i \) in the 3-dimensional contact manifold \( \mathbb{R}^3(x, y, p) \). This distribution is given by a vector field

\[
X_i = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + \lambda_i(x, y, p) \frac{\partial}{\partial p}.
\]

(2.1)

and lies in \( \mathcal{C} \).

In the paper [13] we solved the problem of contact equivalence for the equation (1.2) when \( n = 2 \):

\[
\mathcal{E} = \{(y'' - \lambda_1(x, y, y'))(y'' - \lambda_2(x, y, y')) = 0\}. \tag{2.2}
\]

Let \( X_1 \) and \( X_2 \) be the vector fields (2.1) corresponding to the equation (2.2) and let \( a_1, b_1 \) be any two independent integrals of \( X_1 \), and \( a_2, b_2 \) be independent integrals of \( X_2 \). Any three of these four functions are independent and the fourth one may be expressed in terms of these three. Let, for example, \( b_2 = h(a_1, b_1, a_2) \). Let us introduce the new coordinate system

\[
a = a_1, \quad b = b_1, \quad c = -\frac{h_{a_1}}{h_{b_1}}
\]

(the lower indices \( a_i, b_i \) denote the partial derivatives with respect to \( a_i, b_i \)). In these coordinates the fields \( X_1, X_2 \) have the following form (up to a multiple)

\[
X_1 = \frac{\partial}{\partial c}, \quad X_2 = \frac{\partial}{\partial a} + c \frac{\partial}{\partial b} + G_2(a, b, c) \frac{\partial}{\partial c},
\]

where

\[
G_2(a, b, c) = -\frac{h_{a_1} h_{b_1} - 2h_{a_1} h_{b_1} h_{a_1 b_1} + h_{b_1} h_{a_1}^2}{h_{b_1}^3}.
\]

The exterior 1-form \( \theta = db - c da \) determines the contact structure on \( \mathbb{R}^3 \).

We will say that the ODE

\[
\mathcal{E}_2^a = \{y'' = G_2(x, y, y')\},
\]

...
determined by $X_2$, is associated with $\mathcal{E}$. The choice of another pairs of integrals of $X_1$ and $X_2$ leads to another associated equation which is point equivalent to the previous one. Moreover, if we interchange $X_1$ and $X_2$, then we obtain another equation $\mathcal{E}_1^a = \{y'' = G_1(x, y, y')\}$ associated with $\mathcal{E}$. We will say that these two equations are dual. Two dual equations are not necessary point equivalent (see [13]). Thus, it is correct to speak about the two equivalence classes $([\mathcal{E}_1^a], [\mathcal{E}_2^a])$ associated with $\mathcal{E}$.

**Theorem 1. [13]** Let

$$\mathcal{E} = \{(y'' - \lambda_1)(y'' - \lambda_2) = 0\} \quad \text{and} \quad \tilde{\mathcal{E}} = \{(y'' - \tilde{\lambda}_1)(y'' - \tilde{\lambda}_2) = 0\}$$

be two ODEs of the form (2.2) and let $([\mathcal{E}_1^a], [\mathcal{E}_2^a])$ and $([\tilde{\mathcal{E}}_1^a], [\tilde{\mathcal{E}}_2^a])$ be the pairs of equivalence classes of their associated equations. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent under the action of the pseudogroup of contact transformations if and only if one of the classes $([\mathcal{E}_1^a], [\mathcal{E}_2^a])$ coincides with one of the classes $([\tilde{\mathcal{E}}_1^a], [\tilde{\mathcal{E}}_2^a])$.

It follows that the equation $y'' = \lambda(x, y, y')$ is equivalent to the 2-parametric family of curves

$$S = \{a(x, y, p) = \text{const}, \quad b(x, y, p) = \text{const}\}$$

which are integral curves of Cartan distribution in $\mathbb{R}^3(x, y, p)$. The functions $a$ and $b$ satisfy the conditions

$$da \wedge db \wedge \omega = 0, \quad da \wedge db \neq 0,$$

where $\omega$ is the Cartan form. These functions are essential up to diffeomorphisms

$$(a, b) \rightarrow (\varphi(a, b), \psi(a, b)).$$

**3 From ODEs to families of curves**

Let now the equation

$$y''' + A(x, y, y')y'' + B(x, y, y')y' + C(x, y, y') = 0$$

be given and let

$$y'' = \lambda_i(x, y, y'), \quad i = 1, 2, 3,$$

be its roots. Choose any two of them, say $\lambda_1$ and $\lambda_2$, and let

$$y'' = G(x, y, y')$$

be the equation associated with $(y'' - \lambda_1)(y'' - \lambda_2) = 0$. In the paper [13] we showed that the action of contact transformations pseudogroup on the ODE (2.2) induces the action of point transformations pseudogroup on the associated ODEs. It follows that the group $\mathcal{G}$ of point transformations, admitted by the equation (3.2) acts on the set of integrals of the third equation $y'' = \lambda_3(x, y, y')$. It is well-known that such a group may have dimension 0, 1, 2, 3 or 8. Thus, the problem of contact equivalence of ODEs (3.1) reduces to the problem of equivalence of 2-parametric families of curves (2.3) in $\mathbb{R}^3(x, y, p)$ under the
action of the group $G$. In the present paper we restrict ourselves to the most interesting case of the projective group $SL_3$ of dimension 8. The ODEs that admit this group, are linearizable, that is, point equivalent to the ODE $y'' = 0$ (see, e.g., [1, 6]).

In what follows we assume that the coordinate system $(x, y, p)$ in $\mathbb{R}^3$ is chosen in such a way that the associated equation (3.2) has the form $y'' = 0$. The Lie algebra $sl_3$ of the group $SL_3$ consists of the vector fields

$$(a_0 + a_{11}x + a_{12}y) \frac{\partial}{\partial x} + (b_0 + a_{21}x + a_{22}y) \frac{\partial}{\partial y} + (c_1x + c_2y) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),$$

(3.3)
on $\mathbb{R}^2(x, y)$, where $a_0$, $b_0$, $a_{ij}$, $c_i$ are constants. We prolong these fields to $R^3(x, y, p)$ using the standard formulas [1, 2, 9]

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \mapsto \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial p}, \quad \zeta = \eta x + p(\eta_y - \xi_x) - p^2 \xi_y.$$

Let $S$ be a family of curves (2.3). Such a family is a section of a 2-dimensional bundle

$$\mathbb{R}^5 \to \mathbb{R}^3, \quad (x, y, p, a, b) \to (x, y, p).$$

(3.4)
Since the functions $a$ and $b$ are significant up to fiberwise diffeomorphisms (2.4), we pass to another bundle

$$\pi : \mathbb{R}^5 \to \mathbb{R}^3, \quad (x, y, p, f, g) \to (x, y, p).$$

The fibers of $\pi$ consist of the values of the functions

$$f := \frac{a_x b_y - a_y b_x}{a_y b_p - a_p b_y}, \quad g := \frac{a_p b_x - a_x b_p}{a_y b_p - a_p b_y}.$$ 

(3.5)

The group of fiberwise diffeomorphisms of the bundle (3.4) acts of $\pi$ trivially. The functions $a$ and $b$ are not arbitrary, they are integrals of the contact vector field

$$X = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + G(x, y, p) \frac{\partial}{\partial p}.$$ 

It follows that the functions (3.5) satisfy the relations

$$f(x, y, p) = G(x, y, p), \quad g(x, y, p) = p.$$ 

(3.6)

The conditions (3.6) determine the 1-dimensional subbundle $\tilde{\pi}$ of $\pi$. To solve the equivalence problem of families (2.3) under the action of $SL_3$ we need to find the algebra of differential invariants of this action on $\tilde{\pi}$. This will be done in the section 5. In the section 4 we solve the problem of $SL_3$-equivalence of 2-parametric families of curves for the case when $a, b \in C^\infty(\mathbb{R}^3(x, y, p))$ are arbitrary functions. The computations of differential invariants were performed using the Maple packages DifferentialGeometry and JetCalculus by I.M. Anderson.
4 Projective classification of 2-parametric families of curves

At first, we will find the algebra of differential invariants of the action of $SL_3$ on the bundle $\pi$. The lift of the field

$$X = \xi(x, y, p)\frac{\partial}{\partial x} + \eta(x, y, p)\frac{\partial}{\partial y} + \zeta(x, y, p)\frac{\partial}{\partial p}$$

(4.1)

to $\pi$ has the form

$$\hat{X}^{(0)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial p} + \left(\zeta f + \zeta y + \zeta x - \eta f + \eta g + \zeta_x f\right)\frac{\partial}{\partial f} +
\left(\eta f + \eta y + \eta x - \zeta f + \zeta y g + \zeta_x g\right)\frac{\partial}{\partial g}. \quad (4.2)$$

Proposition 2. The action of $SL_3$ on $\pi$ has the following relative differential invariants of orders 0, 1, 2.

Relative invariant of order 0

$$I_0 = p - g.$$

Relative invariants of order 1

$$I_1 = g_p,$$
$$L_1 = g_x + gg_y + fg_p,$$
$$L_2 = ((-2 + g_p)f + (p - g)f_p - pg_y - g_x),$$
$$L_3 = (-f^2 + (pg_y + g_x)f + (pf_y + f_x)(p - g)),$$
$$L_4 = (-2 + g_p)f^2 + (g_x + gg_y)f + (f_x + g f_y + f f_p)(p - g).$$

Relative invariants of order 2

$$L_5 = (p - g)(f_{pp}(p - g) + f g_{pp} + 2f_p(g_p - 1)) + 2f(p - 1)^2,$$
$$L_6 = (p - g)(g_{xp} + gg_{yp} + fg_{pp} + (4f_p - 2g_y)g_p) + 6f g_p(g_p - 1),$$
$$L_7 = g_{pp}(p - g) + 2g^2 + 2g_p,$$
$$L_8 = p(p - g)g_{yp} + (p - g)g_{xp} + 2g_p(pg_y + g_x + f),$$
$$L_9 = (p - g)\left(3g_{xx} + 2gg_{xy} + g^2 g_{yy} + f(2g_{xp} + 2gg_{yp} + f g_{pp}) + 4f_p(g_x + gg_y + f g_p)\right) +
\left(3f g_{pp}(3f(p - 1) + 4g_x + 4g g_y) - 2(g_x + gg_y)(4f - g_x - gg_y).\right)$$
$$L_{10} = (p - g)\left(g_{xx} + (p + g)g_{xy} + pg g_{yy} + f(g_{xp} + pg_{yp}) + g_p(p f_y + f_x + 3f g_y)\right) + \left(3f g_{pp}(g_x + gg_y) + (g_x + pg_y)(3g_x + (p + 2g)y).\right)$$
The corresponding weights are:

\[
\begin{align*}
\mu(I_0) &= -a_{11} + a_{22} - a_{12}(p + g) - c_1 x + c_2(y - x(p + g)), \\
\mu(I_1) &= 2(c_2 x + a_{12})(p - g), \\
\mu(L_1) &= -2a_{11} + a_{22} - 3a_{12}g - 3c_1 x - 3c_2 g x, \\
\mu(L_2) &= -2a_{11} + a_{22} - a_{12}(p + 2g) - 3c_1 x - c_2(p + 2g) x, \\
\mu(L_3) &= -4a_{11} + 2a_{22} - 2a_{12}(2p + g) - 6c_1 x - 2c_2(2p + g) x, \\
\mu(L_4) &= -4a_{11} + 2a_{22} - 3a_{12}(p + g) - 6c_1 x - 3c_2(p + g) x, \\
\mu(L_5) &= -2a_{11} + a_{22} - 3a_{12}g - 3c_1 x - 3c_2 g x = \mu(L_1), \\
\mu(L_6) &= -2a_{11} + a_{22} + a_{12}(p - 4g) - 3c_1 x + c_2(p - 4g) x, \\
\mu(L_7) &= 3(c_2 x + a_{12})(p - g) = 3\mu(I_1)/2, \\
\mu(L_8) &= -2a_{11} + a_{22} - 3a_{12}g - 3c_1 x - 3c_2 g x = \mu(L_1), \\
\mu(L_9) &= -4a_{11} + 2a_{22} - a_{12}(p + 5g) - c_2(p + 5g) x, \\
\mu(L_{10}) &= -4a_{11} + 2a_{22} - 2a_{12}(p + 2g) - 6c_1 x - 2c_2(p + 2g) x.
\end{align*}
\]

The four weights \(\mu(I_0), \mu(I_1), \mu(L_1), \mu(L_3)\) are generators over \(\mathbb{Q}\).

**Definition.** By an (absolute) differential invariant of the action of \(SL_3\) on \(\pi\) we understand a function \(I \in J^\infty(\pi)\) which is constant along the orbits of the action of \(SL_3\) and is polynomial in derivatives of second order and higher and in the functions \(L_1^{-1}\) and \(|I_1|^{-1/2}\).

**Theorem 2.** The action of \(SL_3\) on \(\pi\) has the following absolute differential invariants of orders 1 and 2.

**Invariants of order 1:**

\[
J_1 = \frac{|I_1|^{1/2} \cdot L_2}{L_1}, \quad J_2 = \frac{I_1^2 L_3}{L_1^2}, \quad J_3 = \frac{|I_1|^{3/2} \cdot L_4}{L_1^2}.
\]

**Invariants of order 2:**

\[
K_1 = \frac{L_5}{L_1}, \quad K_2 = \frac{L_6}{|I_1|^{1/2} \cdot L_1}, \quad K_3 = \frac{L_7}{|I_1|^{3/2}}, \quad K_4 = \frac{L_8}{L_1}, \quad K_5 = \frac{|I_1|^{1/2} \cdot L_9}{L_1^2}, \quad K_6 = \frac{I_1 L_{10}}{L_1^2}.
\]

Let \(\frac{d}{dx}, \frac{d}{dy}\) and \(\frac{d}{dp}\) denote the total derivatives.

**Definition.** By an \(\mathfrak{sl}_3\)-invariant derivation we mean the linear combination of total derivatives

\[
\nabla = A \frac{d}{dx} + B \frac{d}{dy} + C \frac{d}{dp}, \quad A, B, C \in C^\infty(J^\infty(\pi)),
\]

invariant under the prolonged action of \(SL_3\). The functions \(A, B, C\) are supposed to be polynomial in derivatives of second order and higher and in the functions \(L_1^{-1}\) and \(|I_1|^{-1/2}\).

Using the Proposition 1, we find three basic \(\mathfrak{sl}_3\)-invariant derivations:

\[
\nabla_1 = C_1 \frac{d}{dp}, \quad \nabla_2 = A_2 \frac{d}{dx} + B_2 \frac{d}{dy}, \quad \nabla_3 = A_3 \frac{d}{dx} + B_3 \frac{d}{dy} + C_3 \frac{d}{dp}, \quad (4.3)
\]
where
\[ C_1 = \frac{p - g}{\sqrt{|g_p|}} = \frac{I_0}{|I_1|^{1/2}}, \]
\[ A_2 = \frac{(p - g)g_p}{g_x + gg_p + fg_p} = \frac{I_0I_1}{L_1}, \quad B_2 = pA_2, \]
\[ A_3 = \frac{(p - g)\sqrt{|g_p|}}{g_x + gg_p + fg_p} = \frac{I_0 \cdot |I_1|^{1/2}}{L_1}, \quad B_3 = gA_3, \quad C_3 = fA_3. \]

These derivations obey the following commutation relations:
\[
[\nabla_1, \nabla_2] = \frac{1}{2} K_4 \cdot \nabla_1 + (K_3 - K_2 + 3J_1) \cdot \nabla_2 - \nabla_3,
\]
\[
[\nabla_1, \nabla_3] = \frac{1}{2} (K_2 - 2J_1) \cdot \nabla_1 + \nabla_2 + \frac{1}{2} (K_3 - 2K_2 + 6J_1) \cdot \nabla_3,
\]
\[
[\nabla_2, \nabla_3] = J_2 \cdot \nabla_1 + (K_5 - K_2 + J_3) \cdot \nabla_2 + \frac{1}{2} (K_4 - 2K_6) \cdot \nabla_3.
\]

Note that invariants \( L_5, L_6, L_7 \) are linear in second derivatives \( f_{xx}, \ldots, g_{pp} \), and that \( C_1, A_2, B_2, A_3, B_3, C_3 \) are functions on \( J_1(\pi) \). It follows that the functions \( K_j \) and \( \nabla_i J_k \) are linear in second derivatives and the functions \( \nabla_i K_j \) are linear in third derivatives \((i,j,k = 1,2,3)\). Consider the following 12 invariants of second order:
\[
K_1, \quad K_2, \quad K_3, \quad \nabla_i J_k, \quad i,k = 1,2,3. \quad (4.4)
\]

Let \( U_{12} \) be the \((12 \times 12)\)-matrix of coefficients in \( f_{xx}, \ldots, g_{pp} \) in the functions \((4.4)\). Direct computations show that
\[
\det U_{12} = \frac{2I_0^{23}I_1^{14}(L_1L_3 + L_2L_4)}{L_1^{23}} = \frac{2I_0^{23}I_1^{12}(J_2 + J_1 J_3)}{L_1^{20}}. \quad (4.5)
\]

If \( \det U_{12} \neq 0 \) then the invariants \((4.4)\) have linear independent symbols.

Let
\[
U = \begin{pmatrix}
\frac{dJ_1}{dx} & \frac{dJ_2}{dx} & \frac{dJ_3}{dx} \\
\frac{dJ_1}{dy} & \frac{dJ_2}{dy} & \frac{dJ_3}{dy} \\
\frac{dJ_1}{dz} & \frac{dJ_2}{dz} & \frac{dJ_3}{dz}
\end{pmatrix}, \quad W = \begin{pmatrix}
\nabla_1 J_1 & \nabla_1 J_2 & \nabla_1 J_3 \\
\nabla_2 J_1 & \nabla_2 J_2 & \nabla_2 J_3 \\
\nabla_3 J_1 & \nabla_3 J_2 & \nabla_3 J_3
\end{pmatrix}.
\]

By virtue of \((1.3)\), the determinants of \( U \) and \( W \) satisfy
\[
\det W = C_1(A_2B_3 - A_3B_2) \cdot \det U,
\]
whence
\[
\det U = -\frac{L_1^2}{I_0^4I_1} \det W.
\]
**Definition.** We will say that a point $x_k \in J^k(\pi)$ is regular, if the inequality

$$I_0 I_1 L_1 (L_1 L_3 + L_2 L_4) \det W \neq 0$$

(4.6)

holds at this point.

This is equivalent to the fact that the $SL_3$-orbit passing through the regular point has dimension equal to 8. In what follows we consider only the orbits of regular points.

**Theorem 3.** Algebra of differential invariants of the action of $SL_3$ on $\pi$ is generated by invariants $J_1, J_2, J_3, K_1, K_2, K_3$ and invariant derivations $\nabla_1, \nabla_2, \nabla_3$. This algebra separates regular orbits of $SL_3$.

**Proof.** Let $O_n$ be the orbit of action of $SL_3$ in $J^n(\pi)$. Then its projection $O_{n-1} = \pi_{n,n-1}(O_n) \subset J^{n-1}(\pi)$ also is an orbit. The codimension of regular orbit $O_1 \subset J^1(\pi)$ equals $\dim J^1(\pi) - \dim SL_3 = 3$. At the same time, there exist three independent invariants $J_1, J_2, J_3$ of order 1. It can be checked by direct computations that at a regular point $x_1 \in \dim J^1(\pi)$ one has $h \cdot x_1 = x_1, h \in SL_3$ if and only if $h = e$. Consequently, invariants $J_1, J_2, J_3$ generate the space of invariants of order 1 and separate regular orbits.

The fibers $F_2$ of the bundle $J^2(\pi) \to J^1(\pi)$ are 12-dimensional. Since the second order invariants $\{J_i\}$ are linearly independent at the fibers of this bundle, one can choose them as coordinates on these fibers. The intersection of the regular orbit and the fiber $O_2 \cap F_2$ has zero dimension, so the functions $\{J_i\}$ generate the space of invariants of second order and separate regular orbits.

The situation in the order 3 and higher is similar. Differential invariants obtained by invariant derivations of invariants of smaller order, are linear in fiber coordinates. The fiber $F_n$ of the bundle $J^n(\pi) \to J^{n-1}(\pi)$ has dimension $(n+1)(n+2)$. Let $N_1, \ldots, N_{n(n+1)}$ be the generators in the space of invariants of order $n - 1$. One can check that among the invariants $\nabla_i N_k, i = 1, 2, 3, k = 1, \ldots, n(n + 1)$, there exist $(n + 1)(n + 2)$ linearly independent ones. It follows that they generate the algebra of invariants of order $n$ and separate regular orbits. □

Consider the space $\mathbb{R}^3$ with coordinates $(x, y, p)$ and the space $\mathbb{R}^{15}$ with coordinates $(j_1, j_2, j_3, k_1, k_2, k_3, j_{11}, j_{12}, j_{13}, j_{21}, j_{22}, j_{23}, j_{31}, j_{32}, j_{33})$. For every pair of functions $\alpha = (f, g)$ define the map

$$\sigma_\alpha : \mathbb{R}^3 \to \mathbb{R}^{15}$$

by

$$j_i = J_i^\alpha, \quad k_i = K_i^\alpha, \quad j_{ik} = (\nabla_k J_i)^\alpha,$$

where $i, k = 1, 2, 3$, and the upper index $\alpha$ means that the differential invariants are evaluated at $\alpha = (f, g)$.

Let $\Phi \in SL_3$. From the definition of the invariant it follows that

$$\sigma_\alpha \circ \Phi = \sigma_{\Phi^\ast(\alpha)}.$$

Hence, the image

$$\Sigma_\alpha = \text{im}(\sigma_\alpha) \subset \mathbb{R}^{15}$$
depends only on equivalence class of $\alpha$ under the action of $SL_3$.

**Definition.** We say that the germ of $\alpha = (f, g)$ is regular if 3-jets of $\alpha$ lie in regular orbits.

**Theorem 4.** Two regular germs of pairs of functions $\alpha$ and $\overline{\alpha}$ are locally $SL_3$-equivalent if and only if

$$\Sigma_\alpha = \Sigma_{\overline{\alpha}}. \quad (4.7)$$

**Proof.** The necessity is obvious.

Assume that (4.7) holds. Let us show that $\alpha$ and $\overline{\alpha}$ are equivalent.

Since det $W \neq 0$ at a regular point, the germs of functions $j_1, j_2, j_3$ are coordinates on $\Sigma_\alpha$. Let

$$K^\alpha_i = k^\alpha_i(J_1, J_2, J_3), \quad J^\alpha_{im} = j^\alpha_{im}(J_1, J_2, J_3) \quad (4.8)$$
on the submanifold $\Sigma_\alpha$ and

$$K^{\overline{\alpha}}_i = k^{\overline{\alpha}}_i(J_1, J_2, J_3), \quad J^{\overline{\alpha}}_{im} = j^{\overline{\alpha}}_{im}(J_1, J_2, J_3)$$
on $\Sigma_{\overline{\alpha}}$. The condition (4.7) means that

$$K^\alpha_i = K^{\overline{\alpha}}_i, \quad J^\alpha_{im} = J^{\overline{\alpha}}_{im}. \quad (4.9)$$

The formulas (4.8) determine the system of 12 PDEs of second order on the functions $f$ and $g$. This system is of finite type (see [10]). It represents invariants $K_i$ and invariant derivations $\nabla_i$ in coordinates $(J_1, J_2, J_3)$. The pairs $\alpha$ and $\overline{\alpha}$ are solutions of this system. It follows from (4.9) that invariants of all orders for $\alpha$ and $\overline{\alpha}$ are equal. Since this system is of finite type, any solution is determined by its projection to $J^1(\pi)$. The action of $SL_3$ on the fiber of the bundle $J^1(\pi) \to \mathbb{R}^3$ is transitive. Consequently, $\alpha$ and $\overline{\alpha}$ lie in the same orbit, hence are $SL_3$-equivalent. □

5 Projective classification of families of integral curves of Cartan distribution

Now we pass to the case when the curves (2.3) are integral curves of Cartan distribution. At first, we find the algebra of differential invariants of the action of $SL_3$ on the subbundle $\tilde{\pi}$. This problem cannot be reduced to the previous one, since all the invariants, mentioned in the Theorem 2, are constant along $\tilde{\pi}$.

The action of $SL_3$ has the following relative invariants: one relative invariant of order 0, one of order 1

$$I_0 = f, \quad I_1 = pf_yf_p - 3f f_y + f_x f_p$$
and five of order 2

\[ H_1 = 3f_{pp}f^2 - 2f_f^2, \]
\[ H_2 = 3f(f_{xx} + 2pf_{xy} + p^2f_{yy}) - 4(pf_y + f_x)^2; \]
\[ H_3 = 3f(f_{pp}f_{xp} + 3f(pf_f - f_f)f_{yp} + 3f(pf_y + f_x)f_{yp} + f_p(9f_fy - 5f_p(pf_f + f_x)), \]
\[ H_4 = 3f(f_{pp}f_{xx} + (2pf_p - 3f)f_{xy} + (pf_y + f_x)(f_{xp} + pf_{yp}) + p(pf_p - 3f)f_{yp}) - \]
\[ -(pf_y + f_x)(7f_p(pf_y + f_x) - 12f_fy), \]
\[ H_5 = 3f(f_{pp}f_{xx} + 2f_p(pf_p - 3f)f_{xy} + (pf_p - 3f^2f_{yy}) + 2(2pf_yf_p + 2fxf_p - 3ff_f)f_{xp} + \]
\[ + 2(2p^2f_yf_p + 2pf_xf_p - 6pf_yf_x - 3ff_x)f_{yp} + (pf_y + f_x)^2f_{pp}) - \]
\[ - 18f_p^2(pf_y + f_x)^2 + 60f_yf_p(pf_y + f_x) - 36f^2f_y^2. \]

The corresponding weights are

\[ \mu(I_0) = -2a_{11} + a_{22} - 3a_{12}p - 3c_1x - 3c_2xp, \]
\[ \mu(I_1) = -4a_{11} + a_{22} - 5a_{12}p - 7c_1x - c_2(5xp + 2y), \]
\[ \mu(H_1) = -4a_{11} + a_{22} - 5a_{12}p - 7c_1x - c_2(5xp + 2y) = \mu(I_1), \]
\[ \mu(H_2) = 2(-3a_{11} + a_{22} - 4a_{12}p - 5c_1x - c_2(4xp + y)), \]
\[ \mu(H_3) = -5a_{11} + a_{22} - 6a_{12}p - 9c_1x - 3c_2(2xp + y), \]
\[ \mu(H_4) = -7a_{11} + 2a_{22} - 9a_{12}p - 12c_1x - 3c_2(1xp + y), \]
\[ \mu(H_5) = 2(-4a_{11} + a_{22} - 5a_{12}p - 7c_1x - c_2(5xp + 2y)) = 2\mu(I_1). \]

Two weights \( \mu(I_0) \) and \( \mu(I_1) \) are generators over \( \mathbb{Q} \).

**Definition.** By an (absolute) differential invariant of the action of \( SL_3 \) on \( \tilde{\pi} \) we understand a function \( I \in J^\infty(\tilde{\pi}) \) which is constant along the orbits of the action of \( SL_3 \) and is polynomial in derivatives of second order and higher and in the functions \( |I_0|^{-1/2} \) and \( |I_1|^{-1/2} \). We impose the same condition on the coefficients of invariant derivations.

Since \( \dim J^1(\tilde{\pi}) = 7 \) and \( \mu(I_0) \) is not proportional to \( \mu(I_1) \), there are no differential invariants of order less than 2.

**Theorem 5.** The action of \( SL_3 \) on \( \tilde{\pi} \) has the following absolute differential invariants of order 2:

\[ M_1 = \frac{H_1}{I_1}, \quad M_2 = \frac{H_2}{I_0I_1}, \quad M_3 = \frac{|I_0|^{1/2} \cdot H_3}{|I_1|^{3/2}}, \quad M_4 = \frac{H_4}{|I_0|^{1/2} \cdot |I_1|^{3/2}}, \quad M_5 = \frac{H_5}{I_1^2} \]

and the following invariant derivations

\[ \nabla_1 = C_1 \frac{d}{dp}, \quad \nabla_2 = A_2 \frac{d}{dx} + B_2 \frac{d}{dy}, \quad \nabla_3 = A_3 \frac{d}{dx} + B_3 \frac{d}{dy} + C_3 \frac{d}{dp}, \]
where

\[
C_1 = \frac{f^{3/2}}{(p_y f_p - 3f_y f_x + f_x f_p)^{1/2}} = \frac{|I_0|^{3/2}}{|I_1|^{1/2}},
\]

\[
A_2 = \frac{f^{1/2}}{(p_y f_p - 3f_y f_x + f_x f_p)^{1/2}} = \frac{|I_0|^{1/2}}{|I_1|^{1/2}},
\]

\[
B_2 = p A_2,
\]

\[
A_3 = \frac{f f_p}{p y f_p - 3 f y f_x + f_x f_p} = \frac{f p I_0}{I_1},
\]

\[
B_3 = \frac{f (p f_p - 3 f)}{p y f_p - 3 f y f_x + f_x f_p} = \frac{(p f_p - 3 f) I_0}{I_1},
\]

\[
C_3 = \frac{f (p f_y + f_x)}{p y f_p - 3 f y f_x + f_x f_p} = \frac{(p f_y + f_x) I_0}{I_1}.
\]

The following commutation relations hold

\[
[\nabla_1, \nabla_2] = \frac{1}{6} M_4 \cdot \nabla_1 - \frac{1}{6} M_3 \cdot \nabla_2 - \frac{1}{3} \cdot \nabla_3,
\]

\[
[\nabla_1, \nabla_3] = \frac{1}{6} (M_5 - 4) \cdot \nabla_1 + \frac{1}{3} M_1 \cdot \nabla_2 - \frac{1}{3} M_3 \cdot \nabla_3,
\]

\[
[\nabla_2, \nabla_3] = \frac{1}{3} M_2 \cdot \nabla_1 + \frac{1}{6} (M_5 + 4) \cdot \nabla_2 - \frac{1}{3} M_4 \cdot \nabla_3.
\]

We call the point \( x_k \in J^k(\tilde{\pi}) \) regular, if \( I_0 I_1 \neq 0 \) at this point. As above, this is equivalent to the fact the \( SL_3 \)-orbit passing through it has dimension 8. We will consider the orbits of regular points only.

**Theorem 6.** Algebra of differential invariants of the action of \( SL_3 \) on \( \tilde{\pi} \) is generated by five invariants \( M_1, M_2, M_3, M_4, M_5 \) and three invariant derivations \( \nabla_1, \nabla_2, \nabla_3 \). This algebra separates regular orbits.

**Proof.** There are no differential invariants of order less than 2. The codimension of a regular orbit \( O_2 \subset J^2(\tilde{\pi}) \) equals 5. At the same time, there exist five independent invariants of order 2. It can be checked by direct computations that at a regular point \( x_2 \in \dim J^2(\tilde{\pi}) \) one has \( h \cdot x_2 = x_2, h \in SL_3 \) if and only if \( h = e \). Consequently, invariants \( M_1, M_2, M_3, M_4, M_5 \) generate the space if invariants of second order and separate regular orbits.

The fiber \( F_3 \) of the bundle \( J^3(\tilde{\pi}) \rightarrow J^2(\tilde{\pi}) \) has dimension 10. We denote \( M_{ik} := \nabla_i M_k, i = 1, 2, 3, k = 1, \ldots, 5 \). The following 10 invariants of third order have independent symbols

\[
M_{11}, M_{12}, M_{13}, M_{14}, M_{15}, M_{21}, M_{22}, M_{24}, M_{25}, M_{35}.
\]

The determinant of the \((10 \times 10)\)-matrix \( U_{10} \) of coefficients in \( f_{xxx}, \ldots, f_{ppp} \) in the functions \([5.2]\) is

\[
\det U_{10} = -\frac{3^{20} I_{30}^{30}}{I_1^{25}}.
\]

It does not vanish at regular points. As in the Theorem 3, the invariants \( M_1, \ldots, M_5 \) are linear in second order derivatives and the invariants \( M_{ik} \) are linear in third order
derivatives. Hence, the functions (5.2) are independent on the fibers of the bundle $J^3(\tilde{\pi}) \to J^2(\tilde{\pi})$.

Invariants $M_k$ and $M_{ik}$, $i = 1, 2, 3$, $k = 1, \ldots, 5$, satisfy five syzygy relations

\begin{align*}
6M_{34} - 6M_{25} - M_4M_5 + 12M_4 + 6M_2M_3 + 12M_2 &= 0, \\
12M_{33} - 12M_{15} + 24M_{21} + 12M_1M_4 - 2M_3M_5 - 24M_3 &= 0, \\
3M_{23} - 3M_{14} + M_5 &= 0, \\
6M_{31} - 6M_{13} + 2M_1M_5 - 3M_3^2 - 6M_1 &= 0, \\
6M_{32} - 6M_{24} + 2M_2M_5 - 3M_1^2 + 6M_2 &= 0. \quad (5.3)
\end{align*}

The first three of them follow from the commutation relations (5.1) and the Jacobi identity for derivations $\nabla_1, \nabla_2, \nabla_3$. The last two can be checked directly.

As in the proof of Theorem 3, differential invariants obtained by invariant derivations of invariants of orders 2 and higher, are linear in fiber coordinates. The fiber $F_n$ of the bundle $J^n(\tilde{\pi}) \to J^{n-1}(\tilde{\pi})$ has dimension $\frac{1}{2}(n+1)(n+2)$. Let $N_1, \ldots, N_{n(n+1)/2}$ be generators in the space of invariants of order $n - 1$. Among the symbols of invariants $\nabla_i N_k$, $i = 1, 2, 3$, $k = 1, \ldots, \frac{1}{2}n(n+1)$, one can find $\frac{1}{2}(n+1)(n+2)$ independent ones. The corresponding invariants generate the space of invariants of order $n$ and separate regular orbits. □

Consider the space $\mathbb{R}^3$ with coordinates $(x, y, p)$ and the space $\mathbb{R}^{10}$ with coordinates $(m_1, m_2, m_3, m_4, m_5, m_{11}, m_{12}, m_{13}, m_{21}, m_{22})$. For every function $f$ we define the map

$$
\sigma_f : \mathbb{R}^3 \to \mathbb{R}^{10}
$$

by

$$
m_i = M_i^f, \quad m_{ik} = (\nabla_i M_k)^f,
$$

where $i, k = 1, 2, 3$, and the upper index $f$ means that the invariants are evaluated at $f$. The image

$$
\Sigma_f = \text{im}(\sigma_f) \subset \mathbb{R}^{10}
$$

depends only on equivalence class of $f$.

**Definition.** We say that the germ of $f$ is regular at a point $a \in \mathbb{R}^3$, if

i) 3-jets of $f$ belong to regular orbits;

ii) the germ $\sigma_f(D)$ is a germ of smooth 3-dimensional manifold in $\mathbb{R}^{10}$ for a domain $D \subset \mathbb{R}^3$, containing $a$;

iii) the germs of three functions of five $m_1$, $m_2$, $m_3$, $m_4$, $m_5$ are coordinates on $\Sigma_f$.

The proof of the following theorem is similar to that of Theorem 4.

**Theorem 7.** Two regular germs of functions $f$ and $\overline{f}$ are locally $SL_3$-equivalent if and only if

$$
\Sigma_f = \Sigma_{\overline{f}}.
$$

**Remark.** Note that from the formulas (3.6) it follows that the results of Section 5 can be transferred to the classification of ODEs $y'' = f(x, y, y')$ under the action of $SL_3$ as
a subgroup of pseudogroup of point transformations. Thus, Theorems 6 and 7 also solve this classification problem.

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Institute of Mathematics and Mechanics,
Kazan (Volga Region) Federal University,
Kazan, RUSSIA

E-mail: vadimjr@ksu.ru, vshjr@yandex.ru