A network communication through McGee graph and Antimagic labeling

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Abstract
Let $G$ be a simple graph with $p$ nodes and $q$ links. A one to one correspondence between the set of links and the set of integers $\{1, 2, \ldots, q\}$ is called the Antimagic labeling if the sum of the link labels incident with a node is different for all nodes. If the Antimagic labeling is assignable on a graph, it is termed as an Antimagic graph. In this paper, on the McGee graph, the Antimagic, the Even Antimagic and the Odd Antimagic graphs can be allotted is proved. Through the McGee graph and Antimagic labeling, an application for Network Communication is presented.

Keywords
McGee graph, Antimagic labeling, Even Antimagic labeling, Odd Antimagic labeling, Bus topology and Star topology.

AMS Subject Classification
05C78.

1. Introduction
The Authors of this paper were inspired by the Dynamic Survey of Graph Labelings by J.A.Gallian [1] to take up their Research work on labelings. Having presented a coding technique with a combination of McGee Graph and a Prime Cordial labeling [2], they were looking for another labeling. They were motivated to choose the Antimagic labeling from the result, “a non-bipartite, regular graph of at least degree three is an Antimagic graph” [3]. As the McGee Graph is non-bipartite and a regular cubic graph, they worked on McGee Graph [4], Antimagic labeling and provided a solution for a network problem using Bus topology and Star topology [5].
A graph with Odd Antimagic labeling is called Odd Antimagic graph.

**Definition 2.5 (Bus Topology).** Bus topology is used to LAN (Local Area Network). All the computers are connected to a single communication line. So all the computers can receive the information. If the communication line fails, the entire network is lost.

**Definition 2.6 (Star Topology).** All the stations in a star topology are connected to a central unit called hub. The hub refers to a common connection for all the stations on the network. Each station has its own direct cable connection to the hub.

### 3. Results

**Theorem 3.1.** The McGee graph is an Antimagic graph.

*Proof.* Let \( \{v_1, v_2, \ldots, v_{24}\} \) be the vertices and \( \{e_1, e_2, \ldots, e_{36}\} \) be the edges of the McGee graph \( G \). Define the labeling function \( f \) as follows:

\[
f : E(G) \to \{1, 2, \ldots, q\}
\]

\[
f(e_i) = \begin{cases} 
  i & \text{for } i = 1 - 12, 15, 16, 19, 20; \\
  23 - 28; 31 - 36 & \\
  i + 1 & \text{for } i = 13, 17, 21, 29 \\
  i - 1 & \text{for } i = 14, 18, 22, 30
\end{cases}
\]

**Verification of vertex labeling:**

The above edge labeling pattern gives different label sum at each vertex and it is verified here. \( S(v_i) \) is the total sum of the labelings of the edges with \( v_i \) as an end vertex for \( i = 1 - 24 \).

1. \( e_1, e_{24}, e_{25} \) are incident at \( v_1 \). Therefore
   \[
   S(v_1) = f(e_1) + f(e_{24}) + f(e_{25}) = 1 + 24 + 25 = 50
   \]

2. At \( v_{14} \), the edges \( e_{13}, e_4, e_{35} \) are incident and they have different rules for the labeling. 

   For \( e_{13} 'i' + 1' \); for \( e_4 'i' - 1' \) and for \( e_{35} 'i' \). Therefore
   \[
   S(v_{14}) = f(e_{13}) + f(e_4) + f(e_{35}) = (13 + 1) + (14 - 1) + 35 = 62
   \]

It is easily seen that \( S(v_i) \neq S(v_j) \) for \( i \neq j \) where \( i, j \in \{1, 2, \ldots, 24\} \).

Hence the McGee graph is found to admit the Antimagic labeling. So, the McGee graph is an Antimagic graph is proved. \( \square \)

**Theorem 3.2.** The McGee graph is an Even Antimagic Graph.

*Proof.* Let \( \{v_1, v_2, \ldots, v_{24}\} \) be the vertices and \( \{e_1, e_2, \ldots, e_{36}\} \) be the edges of the McGee graph \( G \). Define the labeling function \( f \) as follows:

\[
f : E(G) \to \{2, 4, \ldots, 2q\}
\]

\[
f(e_i) = \begin{cases} 
  2i & \text{for } i = 1 - 12, 15, 16, 19, 20; \\
  23 - 28; 31 - 36 & \\
  2i + 2 & \text{for } i = 13, 17, 21, 29 \\
  2i - 2 & \text{for } i = 14, 18, 22, 30
\end{cases}
\]

**Verification of vertex labeling:**

The above edge labeling pattern gives different label sum at each vertex and it is verified here. \( S(v_i) \) is the total sum of the labelings of the edges with \( v_i \) as an end vertex for \( i = 1 - 24 \).

1. At \( v_2 \), the edges \( e_1, e_2, e_{26} \) are incident. Therefore
   \[
   S(v_2) = f(e_1) + f(e_2) + f(e_{26}) = 1 + 3 + 51 = 55
   \]

2. At \( v_{18} \), the edges \( e_{17}, e_{18} \) and \( e_{34} \) are incident and they have different rules for the labeling.

\[
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{McGee graph with Even Antimagic labeling}
\end{figure}
For \( e_{17} \cdot 2i + 1 \); for \( e_{18} \cdot 2i - 3 \) and for \( e_{34} \cdot 2i - 1 \). Therefore 
\[
S(v_{18}) = f(e_{17}) + f(e_{18}) + f(e_{34}) = (34 + 1) + (36 - 3) + (68 - 1) = 135
\]
and so on.

From the graph \( S(v_i) \neq S(v_j) \) for \( i \neq j \) where \( i, j \in \{1, 2, \ldots, 24\} \). Hence the McGee graph is found to admit the Odd Antimagic labeling. So, the McGee graph is an Odd Antimagic Graph is proved.

![Figure 2. McGee graph with Odd Antimagic labeling](image1)

4. Applications

The following is a problem posed and solved using McGee graph, Antimagic labelling in providing the network connection.

NATIVI, the most famous builders of the city is in the completion of Sha-She Gardens consisting of 24 massive buildings, each with 100 apartments. The builders face a problem in providing a network connection which has to satisfy the following conditions.

- All buildings must be connected.
- Every building must be connected to only three other buildings directly.
- The number of connections between two buildings varies from a single connection, 2 connections and so on with a maximum number of 36 connections.
- No two buildings should have the same number of connections and the total number of connections to any building should not exceed 100.

The Builders entrusts the work to Alb-Sam, the network consultant. The network consultant visualises the problem.

The 24 buildings, each connected to 3 buildings imply that 72 connections are required. But connection between \( B_i \) to \( B_j \) is not different from \( B_j \) to \( B_i \). So, the 72 connections get reduced to 36 connections.

The number of buildings 24 and the number of connections 36 make the person at solving, strike at the McGee graph, realising the role of the Antimagic labeling and provide the solution for connection as shown below.

![Figure 3. McGee graph with Antimagic labeling](image2)

Here, the Bus topology and Star topology are made use of

- A single connection to 24 buildings through a Bus topology.

![Figure 4. Bus Topology between the buildings](image3)

- A Star topology with the hub at each building connected to three buildings as mentioned above.

- Another Star topology between the apartments in two different buildings.

Any 24 apartments of \( B_{24} \) to be connected to an apartment of \( B_1 \). This can be considered as a star topology with \( B_1(A_i) \) as the hub, where \( i, i_1, i_2, \ldots, i_{24} \) take any value between 1 to 100. As required by the Builders the network connection is given.
5. Conclusion

The problem given by the Builders looks complicated. The Network Connection problem is simplified with a graph and a suitable graph labeling. Here a heterogeneous connectivity problem in Networks is given a solution with the knowledge of graph theory.

It is expected that more and more graphs with a variety of graph labelings may provide solutions for much complicated network problems.

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