Chang’s conjecture and semiproperness of nonreasonable posets

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Abstract Let $\mathcal{Q}$ denote the poset which adds a Cohen real then shoots a club through the complement of $\left(\left[\omega_2\right]^\omega\right)^V$ with countable conditions. We prove that the version of Strong Chang’s conjecture from Todorčević and Torres-Pérez (MLQ Math Log Q 58(4–5):342–347, 2012) implies semiproperness of $\mathcal{Q}$, and that semiproperness of $\mathcal{Q}$—in fact semiproperness of any poset which is sufficiently nonreasonable in the sense of Foreman and Magidor (Ann Pure Appl Log 76(1):47–97, 1995)—implies the version of strong Chang’s conjecture from Woodin (The axiom of determinacy, forcing axioms, and the nonstationary ideal, de Gruyter Series in Logic and its Applications, 2nd ed., vol 1, Walter de Gruyter GmbH & Co. KG, Berlin, 2010) and Todorčević (Conjectures of Rado and Chang and cardinal arithmetic, Kluwer Academic, Dordrecht, 1993). In particular, semiproperness of $\mathcal{Q}$ has large cardinal strength, which answers a question of Friedman and Krueger (Trans. Am. Math. Soc. 359(5):2407–2420, 2007). One corollary of our work is that the version of Strong Chang’s Conjecture from Todorčević and Torres-Pérez (MLQ Math Log Q 58(4–5):342–347, 2012) does not imply the existence of a precipitous ideal on $\omega_1$.

Keywords Semiproper forcing · Chang’s conjecture · Reasonable forcing · Forcing axioms · Martin’s maximum · Stationary reflection
1 Introduction

Foreman et al. [6] proved the consistency of Martin’s Maximum (MM), and isolated an interesting consequence:

† : Every poset which preserves stationary subsets of \( \omega_1 \) is semiproper.

In fact they showed that MM implies generalized stationary set reflection, which in turn implied \( \dagger \). They proved that \( \dagger \) implies precipitousness of the nonstationary ideal on \( \omega_1 \); thus \( \dagger \) has large cardinal strength. They also proved that generalized stationary set reflection implies presaturation of the nonstationary ideal on \( \omega_1 \); recently Usuba [21] reduced the assumption to “\( \dagger \) holds for posets of size \( \leq 2^{\omega_1} \)” . He also proved that this bounded dagger principle implies a version of Chang’s Conjecture.

The \( \dagger \) principle is also interesting for particular posets definable in ZFC. For example, it is a theorem of ZFC that Namba forcing preserves stationary subsets of \( \omega_1 \) (and even stronger properties, by [16]). Moreover:

**Theorem 1** (Shelah [16]; see also Sect. 3 of Doebler [2]) Semiproperness of Namba forcing is equivalent to a certain version of Strong Chang’s Conjecture (the version we call SCC\textsuperscript{cof} in Sect. 2).

This paper is about the \( \dagger \) principle for the poset which adds a Cohen real, then shoots a continuous \( \subset \)-chain of length \( \omega_1 \) through \( [\omega_2]^{\omega_1} - V \) using countable conditions, which we’ll denote by

\[
\text{Add}(\omega) \ast \dot{\mathcal{C}}(\mathcal{C}(\omega_2) - V).
\]

This poset has appeared in several applications in the literature, such as separating internal unboundedness from internal stationarity (Krueger [12]) and for applications involving thin stationary sets and disjoint club sequences Friedman and Krueger [8]. It always preserves stationary subsets of \( \omega_1 \), which follows from the following very useful fact:

**Fact 2** (Abraham and Shelah [1], Gitik [9], Velickovic [23]) If \( \sigma \) is Add(\( \omega \))-generic over \( V \), then \( V[\sigma] \) believes that \( [\omega_2]^{\omega_1} - V \) is projective stationary;\(^2\)

In fact Friedman and Krueger [8] proved that the poset from (1) always satisfies a stronger (and RCS-iterable) condition of Shelah which is intermediate between “preserves stationary subsets of \( \omega_1 \)” and “semiproper”. They asked:

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1. Shelah [16] Chapter XII proves that semiproperness of Namba forcing implies SCC\textsuperscript{cof}; and a minor variation in the proof of Sect. 3 of Doebler [2] proves that SCC\textsuperscript{cof} implies semiproperness of Namba forcing.

2. That is, for every stationary \( S \subseteq \omega_1 \) there are stationarily many \( z \in [\omega_2]^{\omega_1} - V \) such that \( z \cap \omega_1 \in S \). The Gitik and Velickovic arguments actually prove something much more general: if \( W \) is an outer model of \( V \) and \( W \) has some real that is not in \( V \), then for every \( W \)-regular \( \kappa \geq \omega_2 \), \( W \) believes that \( [\kappa]^{\omega_1} - V \) is projective stationary.
Question 3 (Question 1 of Friedman and Krueger [8]) Assuming Martin’s Maximum, the poset $\text{Add}(\omega) \ast \check{C}([\omega_2]^\omega - V)$ is semiproper. Is this poset semiproper in general?

We give a strong negative answer to Question 3, which we now describe. Foreman and Magidor [5] defined a poset to be reasonable iff it preserves the stationarity of $([\theta]^\omega)^V$ for all $\theta \geq \omega_1$; this is a weak version of proper forcings (which are required to preserve all stationary subsets of $[\theta]^\omega$). Intuitively, a nonreasonable poset is as non-proper as possible while (possibly) preserving $\omega_1$; it kills the stationarity of the former club $[\theta]^\omega$ for some $\theta$. In the following results it will be useful to stratify the notion of reasonableness: let us say that a poset is reasonable at $[\theta]^\omega$ if it preserves the stationarity of $([\theta]^\omega)^V$, and nonreasonable at $[\theta]^\omega$ otherwise. So in particular, the poset from Question 3 is nonreasonable at $[\omega_2]^\omega$. Notice that any $\omega_1$-preserving poset is reasonable at $[\omega_1]^\omega$; so for $\omega_1$-preserving posets, the strongest possible degree of nonreasonableness is to be nonreasonable at $[\omega_2]^\omega$. Namba forcing is always nonreasonable at $[\omega_2]^\omega$, and so is the poset (1); however the latter preserves all uncountable cofinalities because $\check{C}([\omega_2]^\omega - V)$ is forced by $\text{Add}(\omega)$ to be $\sigma$-distributive.

We prove that semiproperness of the poset from the Friedman and Krueger question, and semiproperness of nonreasonable posets in general, are closely related to strong versions of Chang’s Conjecture. The definitions of SCC, SCC^cof, and SCC^gap are given in Sect. 2.

Theorem 4 If there exists a semiproper poset which is nonreasonable at $[\omega_2]^\omega$, then Strong Chang’s Conjecture (SCC) holds.

Theorem 5 The principle $\text{SCC}^{\text{cof}}_\text{gap}$ implies that

$$\text{Add}(\omega) \ast \check{C}([\omega_2]^\omega - V)$$

is semiproper.

We also prove the following theorem, which is a minor modification of an argument of Sakai [14]:

Theorem 6 (After [14]) Assume there exists a normal ideal $\mathcal{J}$ on $\omega_2$ such that $\varphi(\omega_2)/\mathcal{J}$ is a proper forcing. Then $\text{SCC}^{\text{cof}}_\text{gap}$ holds.

Now by Jech et al. [10], an ideal satisfying the hypothesis of Theorem 6 can be forced from a measurable cardinal. Moreover SCC implies Chang’s Conjecture (CC) which is equiconsistent with an $\omega_1$-Erdős cardinal. So the results above have the following corollary:

In fact one can arrange that the quotient is forcing equivalent to a $\sigma$-closed poset; and moreover the ideal can consistently be the nonstationary ideal on $\omega_2$ restricted to ordinals of uncountable cofinality.
Corollary 7

\[ \text{CON}(ZFC + \text{there is a measurable cardinal}) \]

\[ \implies \text{CON}(ZFC + \text{the poset } \text{Add}(\omega) \ast \dot{\mathcal{C}}([\omega_2]^\omega - V) \text{ is semiproper}) \]

\[ \implies \text{CON}(ZFC + \text{there is an } \omega_1\text{-Erdős cardinal}). \]

We can also draw another corollary from Theorem 6 and core model theory. By Foreman et al. [6], the † principle implies that NS_{\omega_1} is precipitous; and † implies semiproperness of Namba forcing which in turn (by Shelah’s Theorem 1) is equivalent to SCC^{cof}. In light of these facts, a natural question is whether SCC^{cof} implies precipitousness of NS_{\omega_1}. It does not; not even the stronger SCC^{cof}_{gap} implies there is a precipitous ideal on \omega_1:

Corollary 8 The principle SCC^{cof}_{gap} (the strongest of the Chang’s Conjecture variations considered in this paper) does not imply that there is a precipitous ideal on \omega_1.

Section 2 provides the relevant background. Section 3 examines the relationship between Martin’s Maximum, †, and the principle SCC^{cof}_{gap}. Section 4 proves Theorem 4; in fact a stronger theorem is proved there. Section 5 proves Theorems 5 and 6. Section 6 proves Corollary 8. Section 7 provides some concluding remarks about the relationship between the various strong Chang’s Conjectures and special Aronszajn trees on \omega_2, and the relationship between bounded dagger principles and semiproperness of the poset (1).

2 Preliminaries

If M and N are sets which have transitive intersection with \omega_1, we write \(M \subseteq N\) to mean that \(M \subseteq N\) and \(M \cap \omega_1 = N \cap \omega_1\). A poset \(Q\) is semiproper iff for all sufficiently large \(\theta\) and club-many (equivalently, every) countable \(M \prec (H_\theta, \in, Q)\) and every \(q \in M \cap Q\) there is a \(q' \leq q\) such that

\(q' \Vdash M \subseteq \dot{M}[\dot{G}]\).

We frequently use the following fact (see e.g. Larson and Shelah [13]):

Fact 9 If \(\theta\) is regular uncountable, \(\mathfrak{A}\) is a structure on \(H_\theta\) in a countable language which has definable Skolem functions, \(M \prec \mathfrak{A}\), and \(Y\) is a subset of some \(\eta \in M\), then

\[\text{Sk}^{\mathfrak{A}}(M \cup Y) = \{f(y) \mid y \in [Y]^{<\omega} \text{ and } f \in M \cap [\eta]^{<\omega}H_\theta\}\].

The classic Chang’s Conjecture, which we will abbreviate by CC, has many equivalent formulations.\(^4\) One version states: for every \(\theta \geq \omega_2\) and every algebra \(\mathfrak{A}\) on \(H_\theta\), there is an \(X \prec \mathfrak{A}\) with \(|X \cap \omega_2| \geq \omega_1\) and \(X \cap \omega_1 \in \omega_1\).

\(^4\) CC is often expressed by \((\omega_2, \omega_1) \rightarrow (\omega_1, \omega)\).
We will refer to several strengthenings of Chang’s Conjecture. We caution the reader that the notation for various strengthenings of CC is very inconsistent across the literature. For example:

- The notation $\text{CC}^*$ is used in the literature to refer to at least four distinct concepts (which are not known to be equivalent, as far as the author is aware). The $\text{CC}^*$ from Todorčević and Torres-Pérez [19] is what we are calling $\text{SCC}^{\text{cof}}_{\text{gap}}$, whereas the apparently weaker $\text{CC}^*$ from Usuba [21] and Torres-Pérez and Wu [20] is what we are calling $\text{SCC}^{\text{cof}}$. The $\text{CC}^*$ from Todorčević [18] is what we are calling SCC. The $\text{CC}^*$ from Doebler and Schindler [3] is yet another version which is much stronger and will not be considered here.\(^5\)

- “Strong Chang’s Conjecture” from Woodin [24] is not the same as “Strong Chang’s Conjecture” from Sharpe and Welch [15] (Woodin’s is what we call SCC, and Sharpe and Welch’s is what we call $\text{SCC}^{\text{cof}}$). A similar discrepancy appears in the use of the notation $\text{CC}^+$ in [15,24], though we will not deal with either of these versions. The “Strong Chang’s Conjecture” of Foreman et al. [6] is apparently weaker than the “Strong Chang’s Conjecture” of Woodin [24].

Table 1 provides a translation for the various uses in the literature.

**Definition 10** Strong Chang’s Conjecture (SCC) is the statement: for all sufficiently large regular $\theta$, all wellorders $\Delta$ of $H_\theta$, and all countable $M \prec (H_\theta, \in, \Delta)$, there exists a $\tilde{M}$ such that

- $\tilde{M} \prec (H_\theta, \in, \Delta)$;
- $M \subseteq \tilde{M}$; and
- $\tilde{M} \cap [\sup(M \cap \omega_2), \omega_2) \neq \emptyset$.

SCC implies CC. In fact, SCC is equivalent to saying that club-many $M \in [H_\theta]^{\omega_1}$ can be $\subseteq$-extended to a model whose intersection with $\omega_2$ is uncountable; whereas CC is equivalent to this holding for just stationarily many $M \in [H_\theta]^{\omega_1}$. SCC is strictly stronger than CC because SCC implies $2^{\omega_1} \leq \omega_2$, whereas CC places no bound on the continuum (see Section 2 of Todorčević [18]).

We will use even further strengthenings of SCC. The following requires that one can not only obtain proper end-extensions (as SCC requires), but that an end-extension with arbitrarily large supremum below $\omega_2$ can be found:

**Definition 11** $\text{SCC}^{\text{cof}}$ is the statement: for all sufficiently large regular $\theta$ and every wellorder $\Delta$ of $H_\theta$ and every countable $N \prec (H_\theta, \in, \Delta)$, there are cofinally many $\alpha \in \omega_2$ such that there exists an $N' \prec (H_\theta, \in, \Delta)$ where:

1. $N \subseteq N'$;
2. $N' \cap [\alpha, \omega_2) \neq \emptyset$.

Finally, the strongest version we will encounter is the following, which requires arbitrarily large gaps above the model to be $\subseteq$-extended:

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\(^5\) Doebler and Schindler [3] proved that their version implies $\dagger$, which by Usuba [21] implies presaturation of $\text{NS}_{\omega_1}$. Thus by Steel [17] and Jensen and Steel [11], the Doebler and Schindler version of $\text{CC}^*$ has consistency strength at least a Woodin cardinal; whereas all the versions of Chang’s Conjecture considered in this paper can be forced from a measurable cardinal.
Elsewhere in the literature | Corresponds to our
--- | --- | --- | ---
Source | Their notation | SCC | SCC^cof | SCC^cof^\text{gap}
--- | --- | --- | --- | ---
Todorčević [18] | CC^* | ✓ | | |
Todorčević and Torres-Pérez [19] | CC^* | ✓ | | |
Usuba [21] | CC^* | ✓ | | |
Usuba [21] | CC^** | ✓ | | |
Torres-Pérez and Wu [20] | CC^* | ✓ | | |
Doebler [2] | CC^* | ✓ | | |
Shelah [16] | Version in XII Theorem 2.5 | ✓ | | |
Sharpe and Welch [15] | SCC | ✓ | | |
Woodin [24] | SCC (Def 9.101 part 2) | ✓ | | |

**Definition 12** SCC\text{cof}^\text{gap} is defined exactly the same as SCC\text{cof}, except the following additional requirement is placed on the \( N' \) from Definition 11:

\[
N' \cap [\sup(N \cap \omega_2), \alpha) = \emptyset.
\]

The additional requirement for SCC\text{cof}^\text{gap} will be important in the proof of Theorem 5. The following implications are straightforward:

\[
\text{SCC} \implies \text{SCC}^\text{cof} \implies \text{SCC}^\text{cof}^\text{gap} \implies \text{CC}
\]

The following lemma is standard and streamlines arguments involving variants of SCC, by allowing one to replace “every” by “club-many”, but without having to strengthen the algebra in which the end extensions are required to be elementary.

**Lemma 13** The following are equivalent:

1. SCC\text{cof}^\text{gap} (as in Definition 11);
2. There are club-many \( N \in [H_{\omega_3}]^{\omega_3} \) such that for cofinally many \( \alpha \in \omega_2 \), there exists an \( N' \prec (H_{\omega_3}, \in) \) where:
   (a) \( N \subseteq N' \);
   (b) \( N' \cap [\alpha, \omega_2) \neq \emptyset \); and
   (c) \( N' \cap [\sup(N \cap \omega_2), \alpha) = \emptyset \).

Lemma 13 is similar to Lemma 9.103 of [24]; however since there are some confusing typos in the “3 implies 1” direction of the latter, we provide a short proof.\(^6\)

**Proof** That 1 implies 2 is trivial. For the other direction, fix a regular \( \theta > |H_{\omega_3}| \) and a wellorder \( \Delta \) on \( H_\theta \). Fix a countable \( \vec{N} \prec (H_\theta, \in, \Delta) \). The assumptions and the elementarity of \( \vec{N} \) imply there is some algebra \( \mathcal{A} \) on \( H_{\omega_3} \) such that \( \mathcal{A} \in \vec{N} \) and \( \mathcal{A} \) has

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\(^6\) Lemma 9.103 of [24] is the version for SCC; Lemma 13 above is the version for SCC\text{cof}^\text{gap}.
the properties listed in 2. In particular since \( \mathfrak{A} \in \tilde{N} \) then \( N := \tilde{N} \cap H_{\omega_3} \prec \mathfrak{A} \); so there are cofinally many \( \alpha < \omega_2 \) with the properties listed in 2. Fix such an \( \alpha \) and an \( N' \supseteq N \) such that \( N' \prec (H_{\omega_3}, \in) \) and \( N' \) has the other properties listed in 2. Define
\[
\tilde{N}' := \{ f(y) \mid f \in \tilde{N} \cap \omega_2 H_\theta \text{ and } y \in N' \cap \omega_2 \}.
\]
Since \( \tilde{N} \prec (H_\theta, \in, \Delta) \) and \( (H_\theta, \in, \Delta) \) has definable Skolem functions, then Fact 9 implies that \( \tilde{N}' \prec (H_\theta, \in, \Delta) \). Clearly \( \tilde{N} \subset \tilde{N}' \). Furthermore if \( f(y) < \omega_2 \) where \( f \in \tilde{N} \) and \( y \in N' \) then without loss of generality \( f : \omega_2 \to \omega_2 \); so \( f \in \tilde{N} \cap H_{\omega_3} = N \subset N' \). Since \( y \) and \( f \) are both in \( N' \) then \( f(y) \in N' \). This shows that \( \tilde{N}' \) has the desired properties with respect to \( \tilde{N} \).

The following lemma is very similar to Lemma 13, so we omit the proof:

**Lemma 14** The following are equivalent:

1. Strong Chang’s Conjecture (Definition 10)
2. There are club-many \( N \in [H_{\omega_3}]^{\omega} \) such that there exists an \( N' \prec (H_{\omega_3}, \in) \), where:
   a. \( N \subset N' \); and
   b. \( \text{Sup}(N \cap \omega_2), \omega_2) \neq \emptyset \).

The following lemma basically says that if \( M \sqsubseteq N \) and they have access to the same wellorder of \( H_{\omega_2} \), then \( N \cap \omega_2 \) is an end extension of \( M \cap \omega_2 \).

**Lemma 15** Suppose \( w \) is a wellorder on \( H_{\omega_2} \) and \( M \) is a countable elementary substructure of \( (H_{\omega_2}, \in, w) \). Suppose \( N \) is another countable model, perhaps in some outer model of \( V \), such that \( M \sqsubseteq N \) and \( N \cap H_{\omega_2} \prec (H_{\omega_2}^V, \in, w) \). Then \( N \cap \omega_2^V \) is an end-extension of \( M \cap \omega_2^V \); i.e.
\[
N \cap \sup(M \cap \omega_2^V) = M \cap \omega_2^V.
\]

**Proof** One direction is trivial, since \( M \subset N \) by assumption. For the other direction, let \( \zeta \in N \cap \sup(M \cap \omega_2^V) \). Since \( \sup(M \cap \omega_2^V) \) is a limit ordinal there is some \( \beta \in M \cap \omega_2^V \) such that \( \zeta < \beta \). Let \( f \) be the \( w \)-least bijection from \( \omega_1 \to \beta \). Since \( \beta \in M \subset \tilde{N} \), \( M \prec (H_{\omega_2^V}, \in, w) \), and \( N \cap H_{\omega_2} \prec (H_{\omega_2}^V, \in, w) \), then \( f \in M \cap N \). Then
\[
\zeta \in N \cap \beta = f[N \cap \omega_1] = f[M \cap \omega_1] = M \cap \beta.
\]

Finally we recall a standard fact:

**Fact 16** If \( \mathbb{P} \) is a proper poset, \( S \) is a stationary subset of \( [H_\theta]^\omega \) for some \( \theta \geq 2|\mathbb{P}| \), and \( G \) is generic for \( \mathbb{P} \), then \( V[G] \) believes that there are stationarily many \( N \in S \) such that \( G \) includes a master condition for \( N \).

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7 Lemma 15 is the reason that our Definition 10 of SCC is equivalent to part 2 of Definition 9.101 of [24].
Proof If not then there is some condition \( p \) and some name \( \hat{A} \) for an algebra on \( H^V_\theta \) such that

\[
p \models \forall N \in S \ N < \hat{A} \implies \hat{G} \text{ does not include a master condition for } N.
\] (2)

The stationarity of \( S \) ensures that there is some \( N \in S \) such that \( p \in N \) and \( N = \tilde{N} \cap H_\theta \) for some

\[
\tilde{N} < (H|_{H_\theta}^+, \in, \hat{A}).
\]

Since \( \mathbb{P} \) is proper and \( p \in \tilde{N} \) then there is a \( p' \leq p \) which is a master condition for \( \tilde{N} \). Since \( \hat{A} \in \tilde{N} \) and \( p' \) is a master condition for \( \tilde{N} \) then

\[
p' \models \tilde{N} \cap H^V_\theta = N < \hat{A}, \ N \in S, \text{ and } \hat{G} \text{ includes a master condition for } N.
\]

Since \( p' \leq p \), this contradicts (2). \( \square \)

3 Martin’s maximum, †, and SCC_{gap}^{cof}

Recall from the introduction that Martin’s Maximum (MM) implies †, which in turn implies SCC_{gap}^{cof}. In this section we show that MM implies the principle SCC_{gap}^{cof} introduced in Sect. 2, while † does not.

If \( \Gamma \) is a subclass of \( \{W : |W| = \omega_1 \subset W\} \), RP_{\Gamma} abbreviates the statement: For every regular \( \theta \geq \omega_2 \) and every stationary \( S \subseteq [H_\theta]^{\omega_1} \), there exists a \( W \in \Gamma \cap [H_\theta]^{\omega_1} \) such that \( S \cap [W]^{\omega_1} \) is stationary. The following is a standard fact:

Fact 17 If RP_{\Gamma} holds, then for every \( \theta \geq \omega_2 \) and every stationary \( S \subseteq [H_\theta]^{\omega_1} \) there are in fact stationarily many \( W \in [H_\theta]^{\omega_1} \cap \Gamma \) such that \( S \cap [W]^{\omega_1} \) is stationary.

Proof If not, there is a function \( F : [H_\theta]^{<\omega_1} \rightarrow H_\theta \) and a stationary \( S \subseteq [H_\theta]^{\omega_1} \) such that \( S \cap [W]^{\omega_1} \) is nonstationary for every \( W \in \Gamma \) that is closed under \( F \). Let \( S' := \{M \in S : M \text{ is closed under } F\} \). Then \( S' \) is stationary, so by assumption there is a \( W \in \Gamma \) such that \( S' \cap [W]^{\omega_1} \) is stationary (and hence \( S \cap [W]^{\omega_1} \) is also stationary). If \( p \in [H_\theta]^{<\omega_1} \cap W \) there is some \( M \in S' \cap [W]^{\omega_1} \) such that \( p \in M \), and since \( M \in S' \) we have \( F(p) \in M \subset W \). So \( W \) is closed under \( F \), a contradiction. \( \square \)

Two particular subclasses of \( \{W : |W| = \omega_1 \subset W\} \) are relevant in what follows. IA denotes the class of \( W \) such that \( \omega_1 \subset W \) and there is some \( \subseteq \)-increasing, continuous sequence \( \langle N_\xi : \xi < \omega_1 \rangle \) of countable sets such that \( W = \bigcup_{\xi < \omega_1} N_\xi \) and \( N_\xi \cap N_\zeta = \emptyset \) for every \( \xi < \omega_1 \) (the IA stands for “internally approachable”). IC denotes the class of \( W \) such that \( |W| = \omega_1 \subset W \) and \( W \cap [W]^{\omega_1} \) contains a club in \( [W]^{\omega_1} \) (the IC stands for “internally club”, as introduced in Foreman and Todorcevic [7]).

Lemma 18 RP_{IC} implies SCC_{gap}^{cof}.
Proof Suppose not. By Lemma 13 there is a stationary $S \subseteq [H_{\omega_3}]^{< \omega_3}$ such that for every $M \in S$, there is a $\beta_M < \omega_2$ such that for all $\beta \in [\beta_M, \omega_2)$, there is no countable $N \prec (H_{\omega_3}, \in)$ such that $M \subseteq N$, $N \cap \sup(M \cap \omega_2)^+$ = $\emptyset$, and $N \cap [\beta, \omega_2) \neq \emptyset$. By Fact 17 there exists a $W \in \mathcal{IC} \cap [H_{\omega_3}]^{< \omega_3}$ such that $S \cap [W]^{< \omega_3}$ is stationary, and $W \prec (H_{\omega_3}, \in, S, P, \Delta)$ where $\Delta$ is a wellordering of $H_{\omega_3}$ and $P$ is the predicate $(\{M, \beta_M\} \ M \in S)$. Since $W \in \mathcal{IC}$, $S \cap W \cap [W]^{< \omega_3}$ is stationary. It follows by normality and $\sigma$-completeness of the nonstationary ideal that there is some $M \in S \cap W \cap [W]^{< \omega_3}$ such that $N \cap W = M$, where $N := \text{Sk}^{(H_{\omega_3}, \in, \Delta)}(M \cup [W \cap \omega_2])$.

Then $M \subseteq N$, $N \cap \sup(M \cap \omega_2)^+ = \emptyset$, and $W \cap \omega_2 \in N$; it follows that $\beta_M \geq W \cap \omega_2$. On the other hand, since $M \in W$ and $W$ is elementary with respect to the predicate $P$, $\beta_M < W \cap \omega_2$. Contradiction.

Corollary 19 Martin’s Maximum implies $\text{SCC}^\text{cof}_{\text{gap}}$.

Proof MM implies $\text{RP}_{IA}$ (see [4, 6]). Clearly $\text{IA} \subseteq \text{IC}$, and so $\text{RP}_{IA} \implies \text{RP}_{IC}$. The corollary then follows from Lemma 18.

Since $\dag \implies \text{SCC}^\text{cof}$ by Shelah’s Theorem 1, it is natural to ask if $\dag$ also implies $\text{SCC}^\text{cof}_{\text{gap}}$. It does not. To see this we use a result of Usuba [22]. For $m < n$ let $S^n_m$ denote the set $\omega_n \cap \text{cof}(\omega_m)$. A sequence $\vec{d} = (d_\alpha: \alpha \in S^n_3)$ is called a nonreflecting ladder system for $S^n_0$ iff each $d_\alpha$ is a cofinal subset of $\alpha$ of ordertype $\omega$, and for every $\gamma \in S^n_1$ there exists a club $D \subseteq \gamma$ and an injective function $f: D \rightarrow \text{ORD}$ such that $f(\alpha) \in d_\alpha$ for all $\alpha \in D$.

Lemma 20 Suppose there is a nonreflecting ladder system for $S^n_0$. Then $\text{SCC}^\text{cof}_{\text{gap}}$ fails.

Proof Let $\Delta$ be a wellordering of $H_{\omega_3}$, and let $\vec{d}$ be the $\Delta$-least nonreflecting ladder system for $S^n_0$. Let $S$ be the set of $M \in [H_{\omega_3}]^{< \omega_3}$ such that $M < 2^\omega := (H_{\omega_3}, \in, \Delta)$ and $M \subseteq d_{\sup(M \cap \omega_3)}$; $S$ is easily seen to be stationary.\footnote{In fact it is stationary and costationary, as shown in Usuba [22].} Fix $M \in S$. We prove that if $N$ is any countable elementary substructure of $2^\omega$ such that $M \subseteq N$ and $N \cap \sup(M \cap \omega_2)^+ \neq \emptyset$, then $\sup(M \cap \omega_2) \in N$; since $S$ is stationary this will imply that $\text{SCC}^\text{cof}_{\text{gap}}$ fails.\footnote{The “cofinal” requirement of $\text{SCC}^\text{cof}_{\text{gap}}$ isn’t used here, just the “gap” requirement. That is, the proof actually shows that if there is a nonreflecting ladder system for $S^n_0$, then there are stationarily many models $M$ for which there is no $\beta \in \sup(M \cap \omega_2)$ such that $\text{Sk}^{2^\omega}(M \cup \{\beta\}) \cap \beta = M \cap \omega_2$.} So fix such an $N$, and let $\gamma$ be the least member of $N \cap \sup(M \cap \omega_2)^+$. Suppose toward a contradiction that $\gamma > \sup(M \cap \omega_2)$; then $\gamma$ must have cofinality $\omega_1$. Since $\vec{d}$ is nonreflecting and $N < 2^\omega$, in $N$ there is a club $D \subseteq \gamma$ and an injective $f: D \rightarrow \text{ORD}$ such that $f(\alpha) \in d_\alpha$ for every $\alpha \in D$. By minimality of $\gamma$ and the facts that $D \in N$
and $D$ is unbounded in $\gamma$, $\sup(M \cap \omega_2)$ is a limit point of $D$, and hence an element of $D$ because $D$ is closed. Now $f(\sup(M \cap \omega_2)) \in d_{\sup(M \cap \omega_2)} \subset M$ because $M \in S$, and hence $f(\sup(M \cap \omega_2)) \in N$. But then the injectivity of $f$, and the fact that $f \in N$, ensure that $\sup(M \cap \omega_2) \in N$, a contradiction. \qed

Section 6 of Usuba [22] produces a model where $\dagger$ holds\(^{10}\) and there exists a nonreflecting ladder system for $S_0^2$. Together with Lemma 20 this yields a model witnessing the following corollary.

**Corollary 21** The $\dagger$ principle does not imply $\text{SCC}^{\omega}$.  

### 4 Proof of Theorem 4

If $H \supseteq \omega_1$ and $S \subseteq [H]^{\omega}$, we say that $S$ is *semistationary* iff

$$\{ N \in [H]^{\omega} | \exists M \in S \ M \subseteq N \}$$

is stationary.

Clearly every stationary set is semistationary, but the converse is false. Just as properness is equivalent to preservation of stationary sets, Shelah [16] shows that semiproperness of a poset $Q$ is equivalent to: every semistationary set in $V$ remains semistationary in $V^Q$. This, in turn, is easily equivalent to saying that every stationary set in $V$ remains at least semistationary in $V^Q$. We prove the following theorem, which is slightly more general than Theorem 4 because it deals with arbitrary semistationary preserving outer models, rather than just forcing extensions.

**Theorem 22** Assume $V \subset W$ are models of ZFC, every stationary set of countable models in $V$ remains semistationary in $W$, and $(\omega_2^V)$ is nonstationary in $W$. Then

$$V \not\models \text{SCC}.$$  

**Proof** First note that the assumptions ensure that $V$ and $W$ have the same $\omega_1$; otherwise the stationary set $(\omega_1^V)$ in $V$ would fail to be semistationary in $W$.

Working in $V$, fix a regular $\theta \geq \omega_3$ and a wellorder $\Delta$ of $H_\theta$. Let

$$\mathcal{A} = (H_\theta, \in, \Delta).$$

By Lemmas 14 and 15 it suffices to prove that there are club-many $M \in [H_\theta]^{\omega}$ which can be $\sqsubseteq$-extended to an elementary substructure of $\mathcal{A}$ which includes some ordinal in $\omega_2 - M$. Suppose toward a contradiction that this fails; let $S$ denote the stationary collection of counterexamples, and without loss of generality assume $M \prec \mathcal{A}$ for every $M \in S$.

The hypotheses of the theorem ensure that

$$W \models S \text{ is semistationary in } [H_\theta^V]^{\omega}. \tag{3}$$

\(^{10}\) He shows that the model satisfies “Semistationary set reflection”, which is equivalent to $\dagger$.  

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Work in $W$. Let $F : [\omega_2^V]^2 \rightarrow \omega_2^V$ witness that $([\omega_2]^\omega)^V$ is nonstationary in $W$; so

$$W \models \forall z \in [\omega_2^V]^\omega \ z \text{ closed under } F \implies z \notin H_0^V.$$  

(Here we use $H_0^V$, which is an element of $W$, because we are not necessarily assuming that $V$ is definable in $W$). Let $\Omega > |H_0|$ be regular and define

$$\mathcal{B} := (H_\Omega^W, \in, \{\mathfrak{A}, F\}).$$

By (3) and standard facts about liftings of stationary sets, in $W$ there is some countable $N < \mathcal{B}$ such that $N \ni M$ for some $M \in S$. Since $F \in N$ then $N \cap \omega_2^V \notin V$; together with the facts that $M \in V$ and $M \subseteq N$ this implies

$$M \cap \omega_2^V \subsetneq N \cap \omega_2^V.$$  

Pick some $\zeta \in \omega_2^V \cap (N - M)$ and consider the following set, which is an element of $V$ (note that $\mathfrak{A}$ has definable Skolem functions):

$$M' := \text{Sk}_{\mathfrak{A}}(M \cup \{\zeta\}).$$

Since $N < \mathcal{B}$ (so $\mathfrak{A} \in N$) and $M \cup \{\zeta\} \subseteq N$, then $M' \subseteq N$. So in summary we have that $M' \prec \mathfrak{A}$, $M \subseteq M' \subseteq N$ and $M \nsubseteq N$. It follows that $M \nsubseteq M' \prec \mathfrak{A}$. This contradicts that $M \in S$. \qed

**Remark 23** Recall that Shelah [16] proved that semiproperness of Namba forcing implies $\text{SCC}^{\text{cof}}$. It is tempting to try to modify the proof of Theorem 22 above to achieve $\text{SCC}^{\text{cof}}$, rather than just $\text{SCC}$, as follows. Instead of working with $M$, work instead with some maximal $\subseteq$-extension of $M$ which is elementary in $\mathfrak{A}$. The problem is that Theorem 22 implies that such a $\subseteq$-maximal extension will be uncountable, and thus apparently irrelevant to the preservation of semistationary sets of countable models.

**5 Proofs of Theorems 5 and 6**

Before proceeding to the proofs of Theorems 5 and 6, note that Sakai [14] proved that if there is a normal ideal $\mathcal{J}$ on $\omega_2$ such that $P(\omega_2)/\mathcal{J}$ is a semiproper poset, then $\text{SCC}^{\text{cof}}$ holds. However it is not clear if $\text{SCC}^{\text{cof}}$ would suffice to prove Theorem 5; i.e. we seem to need $\text{SCC}^{\text{cof}}_{\text{gap}}$, not just $\text{SCC}^{\text{cof}}$, to prove semiproperness of the poset

$$\text{Add}(\omega) * \dot{\bigcap}([\omega_2]^\omega - V).$$
5.1 Proof of Theorem 5

Let $\theta > |H_{\omega_2}|$. Fix some $w \in H_{\theta}$ which is a wellorder of $H_{\omega_2}$. Fix any $N < (H_{\theta}, \in)$ such that $w \in N$. Since $w \in N$ then Lemma 15 implies:

$$N \subseteq Q \sqsubseteq R \quad \text{and} \quad Q, R < (H_{\theta}, \in) \quad \implies \quad R \cap \sup(Q \cap \omega_2) = Q \cap \omega_2. \quad (4)$$

Let $(p, \dot{f})$ be a condition in $N \cap \text{Add}(\omega) \star \dot{\mathbb{C}}([\omega_2]^\omega - V)$; we want to find a semigeneric condition for $N$ below it.

Let $\sigma$ be $(V, \text{Add}(\omega))$-generic with $p \in \sigma$ and $f := \dot{f}_\sigma$.

**Claim 24** There is some $M < (H_{\theta}[\sigma], \in)$ such that

- $f \in M$;
- $M \cap \omega_2 \notin V$; and
- $N[\sigma] \sqsubseteq M$.

Note that this claim will finish the proof of Theorem 5, because there will then be some $f' \leq f$ which is a totally generic condition for $(M, \mathbb{C}([\omega_2]^\omega - V))$. To construct such an $f'$ (assuming the claim holds), first define a descending chain $\langle f_n : n \in \omega \rangle$ with $f_0 = f$ such that the upward closure of $\{f_n : n \in \omega\}$ is an $(M, \mathbb{C}([\omega_2]^\omega - V))$-generic filter. By Fact 2, $[\omega_2]^\omega - V$ is projective stationary and in particular unbounded in $([\omega_2]^\omega)^V[\sigma]$, so an easy density argument ensures that $\bigcup_{n \in \omega} \text{range}(f_n) = M \cap \omega_2$ and $\sup_{n \in \omega} \text{dom}(f_n) = M \cap \omega_1$. Then

$$\bigcup_{n \in \omega} f_n \cup \{M \cap \omega_1 \mapsto M \cap \omega_2\}$$

satisfies the continuity requirement, and is a condition because $M \cap \omega_2 \notin V$. Since $N[\sigma] \sqsubseteq M$ and $f'$ is a generic condition for $(M, \mathbb{C}([\omega_2]^\omega - V))$, $f'$ is a semigeneric condition for $(N[\sigma], \mathbb{C}([\omega_2]^\omega - V))$; and since $p$ is an $(N, \text{Add}(\omega))$ master condition then $(p, \dot{f}')$ will be the semigeneric condition we seek (where $\dot{f}'$ is a name for $f'$).

**Proof** (of Claim 24) The following coding argument in some ways resembles arguments from Gitik [9] and Velickovic [23]. In $V[\sigma]$ we recursively define three sequences of elementary substructures of $(H_{\theta}^V, \in)$:

$$\langle Q_n^M \mid n < \omega \rangle$$

$$\langle Q_n^Y \mid n < \omega \rangle$$

$$\langle Q_n^N \mid n < \omega \rangle.$$  

Intuitively, the “$M$” (for Move) sequence will tell us when to move to the next decimal place; the “$Y$” (for Yes) sequence will indicate where to put a 1; and the “$N$” (for No) sequence will indicate where to put a 0.

Define a function

$$\text{Active} : \omega \to \{\mathcal{M, Y, N}\}$$
as follows. If \( n \) is even, say \( n = 2k \), then \( \text{Active}(n) = \mathcal{Y} \) if the \( k \)-th bit of \( \sigma \) is 1, \( \text{Active}(n) = \mathcal{N} \) if the \( k \)-th bit of \( \sigma \) is 0. If \( n \) is odd, then \( \text{Active}(n) \) is always \( \mathcal{M} \). For \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \) and \( n < \omega \) we say that \( \mathcal{X} \) is active at stage \( n \) if \( \text{Active}(n) = \mathcal{X} \); otherwise \( \mathcal{X} \) is passive at stage \( n \).

Set \( Q_0^\mathcal{X} := \mathcal{N} \) for all \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \). Assume \( Q_n^\mathcal{X} \) is defined for each \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \). Set \( s_n^\mathcal{X} := \sup(Q_n^\mathcal{X} \cap \omega_2) \) for each \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \), and \( s_n := \max(s^\mathcal{M}_n, s^\mathcal{Y}_n, s^\mathcal{N}_n) \). We then define the \( n + 1 \)-st models as follows:

- If \( \mathcal{X} \) is passive at stage \( n \) then \( Q_{n+1}^\mathcal{X} := Q_n^\mathcal{X} \).
- If \( \mathcal{X} \) is active at stage \( n \), then we use that SCC_{gap} holds in \( V \) to find some \( Q_{n+1}^\mathcal{X} \in V \cap [H^{\mathcal{V}}_\theta]^\omega \) such that:
  - \( Q_{n+1}^\mathcal{X} \prec (H^{\mathcal{V}}_\theta, \in) \);
  - \( Q_n^\mathcal{X} \subseteq Q_{n+1}^\mathcal{X} \);
  - \( Q_{n+1}^\mathcal{X} \cap [s_n^\mathcal{X}, s_n) = \emptyset \); and
  - \( Q_{n+1}^\mathcal{X} \cap [s_n, \omega_2) \neq \emptyset \).

This completes the recursive definition of the three sequences of models. Note that (4) and the construction of the models implies that for all \( n < \omega \) and each \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \):

\[
Q_{n+1}^\mathcal{X} \cap s_n = Q_n^\mathcal{X} \cap \omega_2.
\] (5)

Let

\[
Q_\omega^\mathcal{X} := \bigcup_{n<\omega} Q_n^\mathcal{X}
\]

and set \( z^\mathcal{X} := Q_\omega^\mathcal{X} \cap \omega_2 \) for each \( \mathcal{X} \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \). Also define a sequence \( \vec{\alpha} \) by setting \( \alpha_0 := \sup(N \cap \omega_2) \), and for each \( n \in \omega \), define \( \alpha_{n+1} \) to be the least element of \( Q_{n+1}^{\text{Active}(n)} \cap [s_n, \omega_2) \).

Notice that the construction of the sequences of models ensures:

1. If \( H \) is a transitive \( ZF^- \) model and \( z^\mathcal{M}, z^\mathcal{Y}, \) and \( z^\mathcal{N} \) are all elements of \( H \), then \( \vec{\alpha} \in H \), via the following algorithm. Clearly \( \alpha_0 \in H \) by transitivity. Given \( \alpha_n \), there is a unique \( \mathcal{X}_n \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \) such that \( \alpha_n \in z^\mathcal{X}_n \);\(^{11}\) then \( \alpha_{n+1} \) is the smallest ordinal \( > \alpha_n \) which is missing from \( z^\mathcal{X}_n \) but is in \( \bigcup(z^\mathcal{M}, z^\mathcal{Y}, z^\mathcal{N}) \). (This makes use of (5))

2. \( \sigma \) can be decoded from the parameters \( \vec{\alpha}, z^\mathcal{Y}, \) and \( z^\mathcal{N} \) as follows: for \( k \in \omega \), \( \sigma(k) = 1 \) if \( \alpha_{2k} \in z^\mathcal{Y} \), and \( \sigma(k) = 0 \) if \( \alpha_{2k} \in z^\mathcal{N} \).

Thus, since \( \sigma \notin V \), it follows that there is at least one \( \mathcal{X}^* \in \{ \mathcal{M}, \mathcal{Y}, \mathcal{N} \} \) such that \( z^{\mathcal{X}^*} \notin V \). Note that since \( \text{Add}(\omega) \) is ccc, then in particular \( \sigma \) automatically includes a master condition (namely \( 0 \)) for \( Q_n^{\mathcal{X}^*} \), for every \( n < \omega \). It follows that \( Q_n^{\mathcal{X}^*}[\sigma] \cap ORD = Q_n^{\mathcal{X}^*} \cap ORD \) for every \( n \in \omega \). Also \( (Q_n^{\mathcal{X}^*}[\sigma] \mid n \in \omega) \) is a \( \prec \)-chain of elementary submodels of \( (H_\theta[\sigma], \in) \). Notice by construction that all these models have the same intersection with \( \omega_1 \) also; namely \( N \cap \omega_1 \).

\(^{11}\) Namely \( \mathcal{X}_n = \text{Active}(n) \), though the function \( \text{Active} \) is not assumed to be available to \( H \).
\[ M := \bigcup_{n < \omega} Q_n^{X^*}[\sigma]. \]

Then \( N[\sigma] \subseteq M, \ M \prec (H_\theta[\sigma], \in), \) and
\[ M \cap \omega_2 = \bigcup_{n < \omega} \left( Q_n^{X^*}[\sigma] \cap \omega_2 \right) = \bigcup_{n < \omega} \left( Q_n^{X^*} \cap \omega_2 \right) = z^{X^*} \notin V. \]

This completes the proof of the claim. \( \square \)

### 5.2 Proof of Theorem 6

Let \( J \) be a normal ideal on \( \omega_2 \) such that \( \wp(\omega_2)/J \) is a proper forcing. Fix a sufficiently large regular \( \theta \) and a wellorder \( \Delta \) on \( H_\theta; \) let
\[ \mathcal{A} := (H_\theta, \in, \Delta, \{ J \}). \]

Since \( J \)-positive sets are unbounded in \( \omega_2 \), then to prove SCC\(_{\text{gap}}^\text{Cof} \) it suffices (by Lemma 13) to find club-many \( N \in [H_\theta]^\omega \) such that:
\[ \left\{ \alpha < \omega_2 \mid \text{Sk}^{\mathcal{A}}(N \cup \{ \alpha \}) \cap \alpha = N \cap \alpha \right\} \in J^+. \]

By Fact 9, for any \( N \prec \mathcal{A} \):
\[ \text{Sk}^{\mathcal{A}}(N \cup \{ \alpha \}) = \{ f(\alpha) \mid f \in N \text{ and } f \text{ is a function } \}. \]

Thus it suffices to find club-many \( N \in [H_\theta]^\omega \) such that
\[ A_N := \{ \alpha < \kappa \mid \forall f \in N \ f(\alpha) < \alpha \implies f(\alpha) \in N \} \in J^+. \]

Suppose toward a contradiction that
\[ S := \{ N \prec \mathcal{A} \mid A_N \in J \} \text{ is stationary in } [H_\theta]^\omega. \]

Let \( U \) be generic for \( P(\omega_2)/J \) and let \( j : V \rightarrow_U M_U \) be the generic ultrapower embedding. By Fact 16, there is some \( N \in S \) (in fact stationarily many) such that \( U \) includes a master condition for \( N \). Fix such an \( N \) for the remainder of the proof.

Set \( \kappa := \omega_2^V \). Since \( N \in S \) then \( A_N \in J \), which implies that \( A_N \notin U \) and thus \( \kappa \notin j(A_N) \). Also since \( |N|^V < \text{crit}(j) \) then \( j(N) = j[N] \), so
\[ j(A_N) = \{ \alpha < j(\kappa) \mid \forall f \in j(N) = j[N] \ f(\alpha) < \alpha \implies f(\alpha) \in j(N) = j[N] \}. \]

Since \( \kappa \notin j(A_N) \) there is some \( f' \in j[N] \) such that \( f'(\kappa) < \kappa \) but \( f'(\kappa) \notin j[N] \); say \( f' = j(f) \) where \( f \in N \). Also note that \( j[N] \cap \kappa = N \cap \kappa \). In summary, we have found an \( f \) such that:

\( \diamond \) Springer
\( f \in N, \ j(f)(\kappa) < \kappa, \ \text{and} \ j(f)(\kappa) \notin N. \quad (6) \)

Set \( \beta := j(f)(\kappa); \) then \( \kappa \in j\left( f^{-1}[\{\beta]\} \right) \) and so \( f^{-1}[\{\beta\}] \in U. \) Since \( U \) is a filter on \( P^V(\kappa) \) then

\[ V[U] \models \beta \ \text{is the unique ordinal such that} \ f^{-1}[\{\beta\}] \in U. \]

Back in \( V, \) let \( \hat{\beta}_f \) be the name which denotes the unique value for which \( f \) is constant on a \( \hat{U} \)-measure one set, if such a thing exists. Since \( f \in N \) then we can assume \( \hat{\beta}_f \in N. \) But then

\[ j(f)(\kappa) = \beta = (\hat{\beta}_f)_U \in N \]

where the last relation is due to the fact that \( U \) includes a master condition for \( N. \) This contradicts (6).

\[ \square \]

6 Proof of Corollary 8

In this brief section we use Theorem 6 to produce a model where \( \text{SCC}^{\text{cof}}_{\text{gap}} \) holds, but there is no precipitous ideal on \( \omega_1. \)

Assume \( 0 \)-pistol does not exist, and let \( K \) be the core model (see Chapter 7 of [25]). Work in \( K. \) Assume \( \kappa \) is a measurable cardinal and let \( G \) be \( (K, \text{Col}(\omega_1, < \kappa)) \)-generic. By Jech et al. [10], in \( K[G] \) there is a normal ideal \( J \) on \( \omega_2 = \kappa \) such that \( P(\kappa)/J \) is forcing equivalent to a \( \sigma \)-closed poset. By Theorem 6,

\[ K[G] \models \text{SCC}^{\text{cof}}_{\text{gap}}. \]

Now \( K \) is absolute for set forcing; so \( K \) is the core model from the point of view of \( K[G] \) ([25], Theorem 7.4.11). If \( K[G] \) had a precipitous ideal on \( \omega_1^{K[G]} \), then \( \omega_1^{K[G]} \) would be measurable in \( K \) ([25], Theorem 7.4.8). But since \( \text{Col}(\omega_1, < \kappa) \) preserves \( \omega_1 \) then

\[ \omega_1^{K[G]} = \omega_1^K \]

so it is impossible for \( \omega_1^{K[G]} \) to be measurable in \( K. \) So \( K[G] \) satisfies \( \text{SCC}^{\text{cof}}_{\text{gap}} \) but has no precipitous ideal on \( \omega_1. \)

7 Some remarks about strong Chang’s conjecture, special Aronszajn trees on \( \omega_2, \) and bounded dagger principles

We call attention to the following two theorems:

\textbf{Theorem 25} (Todorčević and Torres-Pérez [19], Theorem 2.2) \textit{If CH fails and} \( \text{SCC}^{\text{cof}}_{\text{gap}} \) \textit{holds, then there are no special Aronszajn trees on} \( \omega_2. \)}
Theorem 26 (Torres-Pérez and Wu [20], Theorem 3.1) If CH fails and SCC\textsuperscript{cof} holds, then there are no Aronszajn trees on $\omega_2$ (i.e. the Tree Property holds at $\omega_2$).

Theorem 26 strengthens Theorem 25 by weakening the hypothesis and strengthening the conclusion. We observe that the hypothesis of Theorem 25 can in fact be weakened all the way to SCC, and the proof actually follows via a circuitous route from several older theorems:

**Theorem 27** If CH fails and SCC holds, then there are no special Aronszajn trees on $\omega_2$.

Theorem 27 follows immediately from the following three facts:

1. Todorčević (Lemma 6 of [18]) proved that SCC implies WRP([$\omega_2$]$^\omega$).
2. WRP([$\omega_2$]$^\omega$) implies—in fact is equivalent to—the non-existence of a costationary, local club subset of [$\omega_2$]$^\omega$;\textsuperscript{12} and
3. If there is a special Aronszajn tree on $\omega_2$ then there is a thin local club subset $T$ of [$\omega_2$]$^\omega$ (Theorem 2.3 of Friedman and Krueger [8]);\textsuperscript{13} and if CH fails then this $T$ must be co-stationary in [$\omega_2$]$^\omega$, by a result of Baumgartner and Taylor (see Theorem 2.7 of [8]).

Note that while Theorem 26 subsumes Theorem 25 in a strong way, it does not subsume Theorem 27 because it uses SCC\textsuperscript{cof} instead of just SCC. In fact the proof of Theorem 26 heavily uses the “cofinal” requirement in the definition of SCC\textsuperscript{cof}.

Finally, we remark on a theorem of Usuba. Let $\dagger_{\omega_2}$ abbreviate the statement: every poset of size $\leq \omega_2$ which preserves stationary subsets of $\omega_1$ is semiproper. Let

$$Q := \text{Add}(\omega) \ast \check{\text{C}}([\omega_2]^\omega - V).$$

We observe:

$$\dagger_{\omega_2} \implies Q \text{ is semiproper. (7)}$$

To see this, first observe that $Q$ always has the following properties:

- it preserves stationary subsets of $\omega_1$ (see [8]); and
- it has cardinality $\max\{\omega_2, 2^\omega\}$.

Now Usuba (Theorem 1.7 of [21]) proved that $\dagger_{\omega_2}$ implies $2^\omega \leq \omega_2$.\textsuperscript{14} So if $\dagger_{\omega_2}$ holds then in particular

$$|Q| = \max\{\omega_2, 2^\omega\} = \omega_2$$

and since $Q$ preserves stationary subsets of $\omega_1$ then $\dagger_{\omega_2}$ applies to it. Thus $Q$ is semiproper.

\textsuperscript{12} A set $T \subseteq [\omega_2]^\omega$ is called local club iff $T \cap [\beta]^\omega$ contains a club for every $\beta < \omega_2$.

\textsuperscript{13} $T$ is thin if for every $\beta < \omega_2$: $|{a \cap \beta \mid a \in T}| \leq \omega_1$.

\textsuperscript{14} And SCC, though we won’t use that.
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