LOCAL MINIMIZERS IN SPACES OF SYMMETRIC
FUNCTIONS AND APPLICATIONS

LEONELO ITURRIAGA, EDERSON MOREIRA DOS SANTOS, AND PEDRO UBILLA

Abstract. We study $H^1$ versus $C^1$ local minimizers for functionals defined
on spaces of symmetric functions, namely functions that are invariant by
the action of some subgroups of $O(N)$. These functionals, in many cases,
are associated with some elliptic partial differential equations that may have
supercritical growth. So we also prove some results on classical regularity for
symmetric weak solutions for a general class of semilinear elliptic equations
with possibly supercritical growth. We then apply these results to prove the
existence of a large number of classical positive symmetric solutions to some
concave-convex elliptic equations of Hénon type.

1. Introduction

We study $H^1$ versus $C^1$ local minimizers for functionals defined in spaces of
symmetric functions, namely functions that are invariant by the action of some
subgroups of $O(N)$. The functionals considered in this paper may not be defined
in the whole space $H^1_0(B)$, but on some proper subspaces of symmetric functions.
Throughout in this paper $B$ stands for the open unit ball centered at zero in $\mathbb{R}^N$,
$N \geq 1$. In order to prove the equivalence in the $C^1$-topology and $H^1$-topology
of local symmetric minimizers, it is essential to have the classical regularity for
symmetric weak solutions of the Euler-Lagrange equations associated with these
functionals. Problems with supercritical growth in the classical sense are involved
and so classical regularity results, as in Brezis and Kato [7] based on the Moser’s
iteration technique [36], cannot be directly applied. By the same reason, the
principle of symmetric criticality of Palais [39] does not apply. Hence we prove
some regularity results, namely Theorems 2.2 and 2.5 which cover a large class of
elliptic partial differential equations and extend and simplify the proofs of some
results in [28, Sections 5.1 and 5.2].

We then apply these results to prove the existence of a large number of positive
solutions to some classes of elliptic partial differential equations of concave-convex
type. We prove the existence of at least three solutions and, if $N \geq 3$, up to
$\left\lceil \frac{N}{2} \right\rceil + 2$ solutions, each of them exhibiting certain symmetry. In comparison with
the pioneering work of Brezis and Nirenberg [39] and Ambrosetti et al. [3], our approach allows us to obtain the existence of more solutions and to treat problems that are critical or supercritical in the classical sense.

We consider elliptic equations of the type

\[
-\Delta u = f(x, u) \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B,
\]

where \( f \) satisfies some suitable hypotheses regarding symmetry with respect to the first variable and growth that may even be supercritical in the classical sense. We also assume that \( f \) is Carathéodory, that is, for each \( u \in \mathbb{R}, \ x \mapsto f(x, u) \) is measurable and, \( u \mapsto f(x, u) \) is continuous for almost every \( x \in B \).

As we will describe next, many interesting problems involving partial differential equations are invariant by the action of certain groups of symmetries and there are two major lines of research on this type of problems: the symmetry that solutions inherit from the problem, and the existence of solutions exhibiting the problem’s symmetry.

On the first direction we mention the seminal work of Gidas et al. [23] in which, assuming quite sharp conditions on \( f \), radial symmetry for any positive solution of (1.1) is proved. Bearing on this subject and related to the problems treated in this paper we also mention the results on symmetry breaking for least energy solutions of the Hénon equation [29], i.e. in case \( f(x, u) = |x|^a |u|^{p-1} u \) with \( \alpha > 0 \) and \( p > 1 \), proved in [13 11 13] and the results about the Schwarz foliated symmetry for least energy solutions proved in [42 38].

On the second direction, within which this paper contributes, the search of symmetric solutions naturally induces the study of spaces of symmetric functions. Here we mention the work of Strauss [45] on solitary waves; the work of Ni [37] on symmetric solutions; the work of de Figueiredo et al. [28] about embeddings of Sobolev spaces; the work of de Figueiredo et al. [28] about embeddings of Sobolev spaces of symmetric functions in weighted \( L^p \)-spaces.

To state the results about \( H^1 \) versus \( C^1 \) local minimizers in spaces of symmetric functions, we introduce some notations: \( F(x, u) := \int_0^u f(x, s)ds, \alpha \geq 0 \) and

\[
2^\ast = \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2, \end{cases}
\]

\[
2^\ast_\alpha = \begin{cases} 2(N + \alpha)/(N-2) & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2, \end{cases}
\]

which are, in the case of \( N \geq 3 \), the critical exponents for the embeddings \( H^1_0(B) \hookrightarrow L^p(B) \) and \( H^1_{0,rad}(B) \hookrightarrow L^p(B, |x|^\alpha) \), respectively.

**Theorem 1.1 (\( H^1 \) versus \( C^1 \) local minimizers: space of radially symmetric functions).** Assume the symmetry and growth conditions

\[
(1.2) \quad \begin{cases} f(x, u) = f(|x|, u), \quad \forall u \in \mathbb{R}, \ \forall x \in B, \\ |f(x, u)| \leq C|x|^\alpha (1 + |u|^q), \quad \forall x \in B, \quad \forall u \in \mathbb{R}, \quad C > 0 \text{ is a constant,} \\ \alpha \geq 0, \ q = 2^\ast_\alpha - 1 \quad \text{in case} \ N \geq 3 \quad \text{and any} \ 1 < q \quad \text{in case} \ N = 1, 2, \end{cases}
\]

and set

\[
H^1_{0,rad}(B) = \{ u \in H^1_0(B); u = u \circ O, \ \forall O \in \mathcal{O}(N) \} \quad \text{and} \\
C^1_{0,rad}(B) = \{ u \in C^1(B); u = u \circ O, \ \forall O \in \mathcal{O}(N) \quad \text{and} \ u = 0 \quad \text{on} \ \partial B \}. 
\]
Associated with (1.1) we consider the integral functional
\[
\Phi_{\text{rad}}(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B F(x, u) dx, \quad u \in H^1_{0, \text{rad}}(B).
\]
Let \( u_0 \in H^1_{0, \text{rad}}(B) \) be a local minimum of \( \Phi_{\text{rad}} \) for the \( C^1_{\text{rad}} \)-topology, that is, there exists \( r > 0 \) such that
\[
(1.3) \quad \Phi_{\text{rad}}(u) \leq \Phi_{\text{rad}}(u_0 + v), \quad \forall \; v \in C^1_{0, \text{rad}}(B) \quad \text{with} \quad \|v\|_{C^1} \leq r.
\]
Then \( u_0 \) is also a local minimum of \( \Phi_{\text{rad}} \) for the \( H^1_{\text{rad}} \)-topology, that is, there exists \( \delta > 0 \) such that
\[
(1.4) \quad \Phi_{\text{rad}}(u) \leq \Phi_{\text{rad}}(u_0 + v), \quad \forall \; v \in H^1_{0, \text{rad}}(B) \quad \text{with} \quad \|\nabla v\|_2 \leq \delta.
\]

**Theorem 1.2** (\( H^1 \) versus \( C^1 \) local minimizers: spaces of partially symmetric functions). Let \( N \geq 4, \; l \in \mathbb{N} \) such that \( 2 \leq N - l \leq l \). Set \( 2l = \frac{2(l+1)}{l+2} \).
\[
H^1_l(B) = \{ u \in H^1_0(B); u = u \circ O, \; \forall \; O \in \mathcal{O}(l) \times \mathcal{O}(N-l) \} \quad \text{and} \quad C^1_{0,l}(B) = \{ u \in C^1(B); u = u \circ O, \; \forall \; O \in \mathcal{O}(l) \times \mathcal{O}(N-l) \quad \text{and} \quad u = 0 \; \text{on} \; \partial B \}.
\]
Assume the symmetry and growth conditions
\[
(1.5) \begin{cases}
  f(y, z, u) = f(|y|, |z|, u), \; \forall \; x = (y, z) \in B, \; \forall \; u \in \mathbb{R}, \\
  |f(x, u)| \leq C|x|^\alpha(1 + |u|^p), \; \forall \; x \in B, \; \forall \; u \in \mathbb{R}, \; C > 0 \; \text{is a constant}, \\
  \alpha \geq \alpha_0(N, l, p) > 0, \; \quad 1 < p < 2l - 1 = \frac{N+3}{N+1},
\end{cases}
\]
where \( \alpha_0(N, l, p) \) is given as in Theorem 2.5 below. Associated with (1.1) we consider the integral functional
\[
\Phi_l(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B F(x, u) dx, \quad u \in H^1_l(B).
\]
Let \( u_0 \in H^1_l \) be a local minimum of \( \Phi_l \) for the \( C^1_l \)-topology, that is, there exists \( r > 0 \) such that
\[
\Phi_l(u_0) \leq \Phi_l(u_0 + v), \quad \forall \; v \in C^1_{0,l}(B) \quad \text{with} \quad \|v\|_{C^1} \leq r.
\]
Then \( u_0 \) is also a local minimum of \( \Phi_l \) for the \( H^1 \)-topology, that is, there exists \( \delta > 0 \) such that
\[
\Phi_l(u_0) \leq \Phi_l(u_0 + v), \quad \forall \; v \in H^1_l(B) \quad \text{with} \quad \|\nabla u\|_2 \leq \delta.
\]

A model problem to which the above results apply is
\[
(1.6) \begin{cases}
  -\Delta w = \lambda |x|^\alpha(w + 1)^p \quad \text{in} \; B, \\
  w > 0 \quad \text{in} \; B, \quad w = 0 \quad \text{on} \; \partial B,
\end{cases}
\]
with \( \alpha > 0, \; p > 1 \) and a parameter \( \lambda > 0 \), which is related to the Hénon equation
\[-\Delta w = |x|^\alpha w^p \quad \text{in} \; B, \quad w > 0 \quad \text{in} \; B, \quad w = 0 \quad \text{on} \; \partial B, \]
and to the equation
\begin{equation}
-\Delta w = \lambda (w + 1)^p \quad \text{in } B, \quad w > 0 \quad \text{in } B, \quad w = 0 \quad \text{on } \partial B,
\end{equation}
for which we cite the works \[32, 31, 15, 8, 22\].

The equation (1.7) has been extensively studied due to its application to physical models, in particular as the steady-state problem corresponding to a nonlinear reaction-diffusion equation; cf. \[30, 31\]. By adding to (1.7) the weight \(|x|^\alpha\), with \(\alpha > 0\), which turns out to be the equation (1.6), we mean that the medium \(B\) has some intrinsic properties that interfere in the reaction rate. Moreover, the partial differential equation (1.6), literally the identity (1.6), says that such intrinsic properties of \(B\) hinder diffusion close to its center, because \(|x|^\alpha\) vanishes at zero and it is very small close to \(x = 0\). Therefore, the existence of solutions that concentrate on the boundary, as \(\alpha \rightarrow +\infty\), is somehow expected; cf. \[11, 13, 35\].

In view of the published literature on the non-weighted problem (1.7), e.g. \[32, 15, 8\], the next theorem is quite conventional as it describes the range of the parameter \(\lambda\) in which (1.6) has a solution.

**Theorem 1.3 (existence of a solution).** Suppose \(N \geq 1\), \(p > 1\) and \(\alpha > 0\). There exists \(\lambda_* = \lambda_*(N, \alpha, p) > 0\) such that:

(i) There is no classical solution of (1.6) if \(\lambda > \lambda_*\);
(ii) There exists at least one radial classical solution of (1.6) if \(0 < \lambda < \lambda_*\);
(iii) If \(N \geq 3\) also assume \(p \leq 2^* - 1\). If \(\lambda = \lambda_*\), then (1.6) has at least one radial classical solution.

We then combine Theorems 1.1, 1.2 and 1.3 to guarantee the existence of a large number solutions to (1.6) and, in addition, we classify their symmetry.

**Theorem 1.4 (multiple symmetric solutions).** Suppose \(N \geq 1\) and \(\alpha > 0\). Let \(\lambda_* > 0\) be as above.

(I) Let \(1 < p\) and if \(N \geq 3\) also assume \(p \leq 2^* - 1\). If \(0 < \lambda < \lambda_*\), then (1.6) has at least two radial classical solutions.

(II) Let \(1 < p\) and \(N = 1, 2\). There exists \(\alpha_0 = \alpha_0(N, p) > 0\) such that the problem (1.6) has at least three non rotational equivalent classical solutions, if \(\alpha > \alpha_0\) and \(0 < \lambda < \lambda_0(N, p, \alpha)\). Two of them are radially symmetric. If \(N = 1\), the third solution is not even. If \(N = 2\), the third solution is not radially symmetric but Schwarz foliated symmetric.

(III) Let \(N \geq 3\) and \(1 < p < 2^* - 1\). There exists \(\alpha_0 = \alpha_0(N, p) > 0\) such that the problem (1.6) has at least \(\left\lceil \frac{N}{2} \right\rceil + 2\) non rotational equivalent classical solutions, if \(\alpha > \alpha_0\) and \(0 < \lambda < \lambda_0(N, p, \alpha)\). Two of them are radially symmetric. The third solution is not radially symmetric but Schwarz foliated symmetric. Each of the others \(\left\lceil \frac{N}{2} \right\rceil - 1\) solutions has a \(O(l) \times O(N - l)\) symmetry for some \(l \in \mathbb{N}\) such that \(2 \leq N - l \leq l\).

**Remark 1.5.** In the case of \(N \geq 4\) we have more information about the existence of multiple solutions to (1.6). Indeed, we prove the existence of more than two solutions even with \(p + 1 \geq 2^*\); cf. Remark 7.7. The main ingredients in the proof
are the results about $H^1$ versus $C^1$ minimizers applied to the space $H^1_{0,\text{rad}}(B)$ combined with a careful asymptotic analysis, as $\lambda \to 0^+$, of the associated symmetric mountain pass levels.

In contrast, in the non-weighted case, with $1 < p$ if $N = 1, 2$, and $1 < p \leq 2^* - 1$ if $N \geq 3$, it is proved in [30] that (1.7) has precisely two solutions for every $0 < \lambda < \lambda_*$. Our results also apply to the Hénon equation with inhomogeneous boundary condition, namely to

\begin{equation}
-\Delta u = |x|^\alpha u^p \quad \text{in} \quad B, \quad u > 0 \quad \text{in} \quad B, \quad \text{with} \quad u = a \quad \text{on} \quad \partial B.
\end{equation}

Indeed, if we write $u = a(w + 1)$, then (1.8) is equivalent to (1.6) with $a^{p-1} = \lambda$. So, Theorems 1.3 and 1.4 can be restated in the context of (1.8).

Since the works of Hénon [29] and Ni [37], the problem (1.8) with $a = 0$ has been extensively studied; cf. [13, 35, 11, 5, 11, 28, 12, 17, 26] and references therein. Nevertheless, less attention has been devoted to the study of (1.8) with $a > 0$ and, as far as we know, [40] is the only published paper on this subject.

One of the first applications of the result of Brezis and Nirenberg [9] about $C^1$ versus $H^1$ local minimizers appears in concave-convex problems as in Ambrosetti et al. [3], whose weighted version reads

\begin{equation}
\begin{cases}
-\Delta u = \lambda |x|^\beta u^q + |x|^\alpha u^p & \text{in} \quad B, \\
u > 0 & \text{in} \quad B, \\
u = 0 & \text{on} \quad \partial B,
\end{cases}
\end{equation}

with $\beta \geq 0$, $\alpha \geq 0$ and $0 < q < 1 < p \leq 2^* - 1$; cf. [14] where (1.9) is considered in the case of $q = 1$. We stress that Theorems 1.1 and 1.2 can be applied to study equation (1.9) and to obtain results in the sense of Theorems 1.3 and 1.4 above.

Our results also apply to some weighted problems posed in exterior domains, as to

\begin{equation}
\begin{cases}
-\Delta U = \frac{U^p}{|x|^\beta} & \text{in} \quad \mathbb{R}^N \setminus B, \\
U > 0 & \text{in} \quad \mathbb{R}^N \setminus B, \\
U = a & \text{on} \quad \partial B, \\
U \to 0 & \text{as} \quad |x| \to \infty,
\end{cases}
\end{equation}

with $N \geq 3$, $a \geq 0$, $\beta \in \mathbb{R}$ and $p > 0$; cf. Section 6. In this direction our results slightly complement the results in [16, Theorem 2].

This manuscript is organized as follows. In Section 2 we prove some regularity results for symmetric weak solutions of (1.1), namely Theorems 2.2 and 2.5. In Sections 3, 4 and 5 we prove Theorems 1.1, 1.3 and 1.4 respectively. Then, in Section 6 we describe how our procedure can be used to prove results on the existence and the multiplicity of solutions to the equation (1.10).

### 2. Regularity Results

In this section we prove classical regularity for symmetric weak solutions of elliptic equations of the type (1.1). We point out that problems with supercritical growth in the classical sense are involved and so classical regularity results, as in Brezis and Kato [7] based on the Moser’s iteration technique [36], cannot be
directly applied. In addition, since the functionals may not be defined on \( H^1_0(B) \), the principle of symmetric criticality \([39]\) cannot be applied as well.

2.1. **Radial solutions.** We recall that from \([37]\), eq. (4)), for \( N \geq 3 \) and any \( u \in H^1_{0, \text{rad}}(B) \) we have

\[
(2.1) \quad |u(x)| \leq \frac{1}{\sqrt{\omega_N(N - 2)}} \frac{\|\nabla u\|_2}{|x|^{\frac{N}{2} - 1}}, \quad \forall x \in B \setminus \{0\},
\]

where \( \omega_N \) stands for the surface area of \( S^{N-1} \subset \mathbb{R}^N \).

Then, (1.2), (2.1) for \( N \geq 3 \) and the classical Sobolev embeddings of \( H^1_0(B) \) for \( N = 1, 2 \) guarantee that

\[
\Phi_{\text{rad}}(u) = \frac{1}{2} \int_B |\nabla u|^2 \, dx - \int_B F(x, u) \, dx, \quad u \in H^1_{0, \text{rad}}(B),
\]

is a \( C^1(H^1_{0, \text{rad}}(B), \mathbb{R}) \)-functional.

**Definition 2.1.** We say that \( u \in H^1_{0, \text{rad}}(B) \) is a weak radial solution of (1.1) if \( \Phi'_{\text{rad}}(u) = 0 \), that is, if

\[
\int_B \nabla u \nabla v \, dx = \int_B f(x, u) v \, dx, \quad \forall v \in H^1_{0, \text{rad}}(B).
\]

**Theorem 2.2.** Assume (1.2). If \( u \in H^1_{0, \text{rad}}(B) \) is a weak radial solution of (1.1), then \( u \in W^{2, t}(B) \cap W^{1, \frac{1}{t}}(B) \) for every \( t \geq 1 \) and it strongly solves (1.1). Consequently, by classical embeddings of Sobolev spaces, \( u \in C^{1, \theta}(B) \) for every \( 0 < \theta < 1 \).

**Proof.** Let \( u \in H^1_{0, \text{rad}}(B) \) be a weak solution of (1.1). Then, by (1.2), \( f(x, u) \in L^t(B) \) for all \( t \geq 1 \) in case \( N = 1, 2 \). On the other hand, in case \( N \geq 3 \), from (1.2) we infer that

\[
(2.2) \quad |f(x, u)|^\frac{2N}{N + 2} \leq (C|x|^\alpha (1 + |u|^{\frac{N + 2 + 2\alpha}{N}}))^\frac{2N}{N + 2} \leq C(|x|^{\frac{2N}{N + 2}} + |u|^{\frac{2N}{N + 2}} |x|^{\frac{\alpha N}{N + 2}} |u|^{\frac{\alpha N}{N + 2}}),
\]

and, by (2.1), \( |x|^\frac{2N}{N + 2} |u|^{\frac{\alpha N}{N + 2}} \in L^{\infty}(B) \). Then \( f(x, u) \in L^{\frac{2N}{N + 2}}(B) \) in case \( N \geq 3 \).

Let \( w \) be the strong solution of

\[
-\Delta w = f(x, u) \quad \text{in} \quad B, \quad \text{with} \quad w = 0 \quad \text{on} \quad \partial B.
\]

Then, by the standard elliptic regularity for second order elliptic operators as in \([1]\), \( w \in W^{2, t}(B) \cap W^{1, \frac{1}{t}}(B) \) for all \( t \geq 1 \) in case \( N = 1, 2 \) and \( w \in W^{2, \frac{2N}{N + 2}}(B) \cap W^{1, \frac{1}{\frac{2N}{N + 2}}}(B) \) in case \( N \geq 3 \). In addition, \( w \) is radially symmetric. Then, for every \( \varphi \in C^\infty_{0, \text{rad}}(B) \),

\[
(2.3) \quad \int_B w(-\Delta \varphi) \, dx = \int_B (-\Delta w) \varphi \, dx = \int_B f(x, u) \varphi \, dx = \int_B \nabla u \nabla \varphi = \int_B u(-\Delta \varphi) \, dx.
\]
Let $\psi \in C^\infty_{c,\text{rad}}(B)$ be an arbitrary function and $\varphi$ be the solution of
\[-\Delta \varphi = \psi \quad \text{in} \quad B, \quad \text{with} \quad \varphi = 0 \quad \text{on} \quad \partial B.

Then $\varphi \in C^\infty_{0,\text{rad}}(B)$ and so, by means of (2.3),
\[\int_B (w - u)\psi dx = 0 \quad \forall \psi \in C^\infty_{c,\text{rad}}(B),\]
which implies
\[\int_0^1 [w(r) - u(r)]\psi(r)r^{N-1}dr = 0 \quad \forall \psi \in C^\infty((0,1)).\]

Therefore, $w = u$ a.e. in $B$.

At this point we have proved that $u$ strongly solves (1.1) and that $u \in W^{2,t}(B) \cap W^{1,1}_0(B)$ for all $t \geq 1$ in case $N = 1,2$ and $u \in W^{2,\frac{2N}{N+2}}(B) \cap W^{1,\frac{2N}{N+2}}_0(B)$ in case $N \geq 3$.

In case $N \geq 3$, since $W^{2,\frac{2N}{N+2}}(B) \hookrightarrow H^1_0(B)$ and since $u \in W^{2,\frac{2N}{N+2}}(B) \cap W^{1,\frac{2N}{N+2}}_0(B)$ strongly solves (1.1) we also obtain that $u$ weakly solves (1.1) in the sense of $H^1_0(B)$. In addition, by (1.2) we write
\[(2.4) \quad |f(x,u)| \leq C|x|^\alpha (1 + |u|^\frac{N+2+2\alpha}{N-2}) = a(x)(1 + |u|), \quad \forall x \in B,
\]
with
\[a(x) = C|x|^\alpha \frac{1 + |u|^\frac{N+2+2\alpha}{N-2}}{1 + |u|}.
\]

Then
\[\left( a(x) \right)^\frac{N}{N-2} \leq \left( C|x|^\alpha (1 + |u|^\frac{2(N+2)}{N-2}) \right)^\frac{N}{N-2} \leq C_1 \left( |x|^\frac{2N}{N-2} + |x|^\frac{N}{N-2} |u|^\frac{(2+\alpha)N}{(N-2)2} \right) = C_1 \left( |x|^\frac{2N}{N-2} + |x|^\frac{N}{N-2} |u|^\frac{2N}{N-2} \right)\]
and by Ni’s pointwise estimate (2.1), we infer that $|x|^\frac{2N}{N-2} |u|^\frac{2N}{N-2} \in L^\infty(B)$. Therefore, $a(x) \in L^{\frac{N}{N-2}}(B)$. Then, by [7, Theorem 2.3], see also [10, B.3 Lemma], and [25, Lemma 9.17], it follows that $u \in W^{2,t}(B) \cap W^{1,1}_0(B)$ for all $t \geq 1$ and strongly solves (1.1).

**Remark 2.3.** If $f(x,u)$ satisfies (1.2) and is also suitably regular, that is the case of $f(x,u) = \lambda|x|^\alpha |u + 1|^{p-1}(u + 1)$ with $\lambda \in \mathbb{R}$, $\alpha \geq 0$ and $p > 1$, then we combine Theorem 2.2 and the classical Schauder’s estimates to obtain that any weak radial solution of (1.1) lies in $C^{2,\gamma}(\overline{B})$, for a certain $0 < \gamma < 1$, and classically solves (1.1).
2.2. Partially symmetric solutions. In this part

\[ N \geq 4, \quad l \in \mathbb{N} \quad \text{s.t.} \quad 2 \leq N - l \leq l, \quad \alpha > \max \left\{ \frac{(N + 2)^2}{2(N - 2)}, \frac{(N - 2)^2 + 5}{N - 3} \right\}. \]

Set \( 2l = \frac{2(l+1)}{l} \) and

\[ H_l(B) = \left\{ u \in H^1_l(B); u = u \circ O \quad \text{for every} \quad O \in \mathcal{O}(l) \times \mathcal{O}(N - l) \right\} = \left\{ u \in H^1_l(B); u(y, z) = u(|y|, |z|), \quad x = (y, z) \in B \right\}, \]

where \( y = (y_1, \ldots, y_l), \quad z = (z_1, \ldots, z_{N-l}) \). Then, as proved in [5, Theorem 2.1], we have \( H_l(B) \hookrightarrow L^{2l}(B, |x|^{\alpha}) \) continuously. Observe that instead of \( \alpha \geq N + 2 \), the inequality \( \alpha > \max \left\{ \frac{(N+2)^2}{2(N-2)}, \frac{(N-2)^2 + 5}{N-3} \right\} \) is the correct assumption for the proof of [5, Theorem 2.1].

Here we assume (2.5) and the symmetry and growth conditions (1.5). Under these hypotheses

\[ \Phi_l(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B F(x, u) dx, \quad u \in H_l(B), \]

is a \( C^1(H_l(B), \mathbb{R}) \)-functional.

**Definition 2.4.** We say that \( u \in H_l(B) \) is a weak \((l, N-l)\)-symmetric solution of \((1.1)\) if \( \Phi'_l(u) = 0 \), that is, if

\[ \int_B \nabla u \nabla v dx = \int_B f(x, u)v dx, \quad \forall v \in H_l(B). \]

**Theorem 2.5.** Assume \( N \geq 4, \quad l \in \mathbb{N}, \quad 2 \leq N - l \leq l \) and that (1.5) holds. Then there exists \( \alpha_0 = \alpha_0(N, l, p) > 0 \) with the following property: if \( \alpha \geq \alpha_0(N, l, p) \) and \( u \in H_l(B) \) is a weak \((l, N-l)\)-symmetric solution of (1.1), then \( u \in W^{2,1}_l(B) \cap W^{2,1}_0(B) \) for every \( t \geq 1 \) and it strongly solves (1.1). Consequently, by classical embeddings of Sobolev spaces, \( u \in C^{1,\gamma} (\overline{B}) \) for every \( 0 < \gamma < 1 \).

**Proof.** By (1.5) and the imbedding \( H_l(B) \hookrightarrow L^{2l}(B, |x|^\alpha) \), it follows that \( f(x, u) \in L^{2l}(B) \). Let \( w \) be the strong solution of

\[ -\Delta w = f(x, u) \quad \text{in} \quad B, \quad \text{with} \quad w = 0 \quad \text{on} \quad \partial B. \]

Then, by the standard elliptic regularity for second order elliptic operators as in [1], \( w \in W^{2,2}_l(B) \cap W^{1,2}_0(B) \). Then, for every \( \varphi \in C^\infty_{0,l}(\overline{B}) \),

\[ \int_B w(-\Delta \varphi)dx = \int_B (-\Delta w)\varphi dx = \int_B f(x, u)\varphi dx = \int_B \nabla u \nabla \varphi = \int_B u(-\Delta \varphi)dx. \]

Given \( \psi \in C^\infty_{c,l}(B) \) let \( \varphi \) be the solution of

\[ -\Delta \varphi = \psi \quad \text{in} \quad B, \quad \text{with} \quad \varphi = 0 \quad \text{on} \quad \partial B. \]

Then \( \varphi \in C^\infty_{0,l}(\overline{B}) \) and so, by means of (2.6),

\[ \int_B (w - u)\psi dx = 0 \quad \forall \psi \in C^\infty_{c,l}(B), \]
which implies
\[
\int_0^1 \int_0^1 \sqrt{t} [w(s, t) - u(s, t)] \psi(s, t) s^{t-1} t^{N-1-\ell} ds dt = 0 \quad \forall \psi \in C_c^\infty(Q),
\]
where \(Q = \{(s, t) \in \mathbb{R}^2; s, t > 0 \text{ and } 0 < s^2 + t^2 < 1\}\). Therefore, \(w = u\) a.e. in \(B\).

At this point we have proved that \(u \in W^{2, \frac{2l}{p}}(B) \cap W^{1, \frac{2l}{p}}_0(B)\) and strongly solves \((1.1)\).

If \(p(l - 1) - 4 \leq 0\) and \(\alpha\) is large enough, then \(u \in W^{2, \theta}(B)\) for all \(\theta \geq 1\) as a consequence of [28, Corollary 1.5] with \(m = 2\) and the standard regularity for second order elliptic operators.

In case \(p(l - 1) - 4 > 0\), [28 Corollary 1.5] is the key in the bootstrap argument. To show that \(u \in W^{2, \theta}(B)\) for all \(\theta \geq 1\) when \(\alpha\) is large enough, we combine [28 Corollary 1.5] with \(m = 2\) and the standard regularity for second order elliptic operators to prove the statement:

\[
\text{if } u \in W^{2, q}_l(B) \text{ for some } q \geq \frac{2l}{p}, \text{ then } u \in W^{2, \theta}(B) \text{ with } \frac{\theta}{q} \geq \frac{l - 1}{p(l - 1) - 4}.
\]

And the bootstrap argument works because \(\frac{l - 1}{p(l - 1) - 4} > 1\) is sharply guaranteed by the hypothesis \(p < 2l - 1\). \(\square\)

**Remark 2.6.** If \(f(x, u)\) satisfies (1.5) and is also suitably regular, that is the case of \(f(x, u) = \lambda|x|^\alpha|u + 1|^{p-1}(u + 1)\) with \(\lambda \in \mathbb{R}, \alpha \geq 0\) and \(p > 1\), then we combine Theorem 2.3 and the classical Schauder’s estimates to obtain that any weak \((l, N-l)\)-symmetric solution of \((1.1)\) lies in \(C^{2, \gamma}(\overline{B})\), for a certain \(0 < \gamma < 1\), and classically solves \((1.1)\).

### 3. \(H^1\) Versus \(C^1\) Local Minimizers: Radially Symmetric Functions

In this part we consider the functional
\[
\Phi_{\text{rad}}(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B F(x, u) dx, \quad u \in H^{1, \text{rad}}(B).
\]

Here we suppose that \(f\) verifies the growth and symmetry conditions \((1.2)\) and we prove Theorem 1.1. We also mention [10 Proposition 3.9] and [10 Lemma 2.2] from where we borrow some ideas.

**Proof of Theorem 1.1**

Step 1. We infer that \(u_0 \in W^{2, t}(B) \cap W^{1, t}_0(B)\) for every \(t \geq 1\) and strongly solves \((1.1)\). In particular, \(u_0 \in C^{1, \gamma}(\overline{B})\) for every \(0 < \gamma < 1\).

Let \((\rho_n)\) a regularizing sequence formed by radial functions, for example,
\[
\rho(x) = \begin{cases} 
\exp \left( \frac{1}{|x|^{\gamma} - 1} \right), & |x| < 1, \\
0, & |x| \geq 1,
\end{cases}
\]
and \(\rho_n(x) = \left( \int_{\mathbb{R}^N} \rho(x) dx \right)^n \rho(nx), n \in \mathbb{N}.
\]

Then observe that \(\rho_n * w\) is radially symmetric if \(w\) is radially symmetric. Then \(C^{\infty, \text{rad}}_{c}(B)\), and consequently \(C^{1, \text{rad}}_{0}(B)\), is dense in \(H^{1, \text{rad}}_{0}(B)\). As in (2.2), we have...
that $f(x, u_0) \in L^{\frac{2N}{N+2}}(B)$ in case $N \geq 3$, and $f(x, u_0) \in L^t(B)$ for all $t \geq 1$ in case $N = 1, 2$.

From (1.2), we have

$$\int_B \nabla u_0 \nabla vdx - \int_B f(x, u_0) vdx = 0 \quad \forall v \in C^1_{0, \text{rad}}(B),$$

and since $C^1_{0, \text{rad}}(B)$ is dense in $H^1_{0, \text{rad}}(B)$ and the above integrability of $f(x, u_0)$,

$$\int_B \nabla u_0 \nabla vdx - \int_B f(x, u_0) vdx = 0 \quad \forall v \in H^1_{0, \text{rad}}(B),$$

that is, $u_0$ is a critical point of $\Phi_{\text{rad}}$. Therefore, by Theorem 2, $u_0 \in W^{2, t}(B) \cap W^{1, t}_0(B)$ for every $t \geq 1$ and strongly solves (1.1). In particular, $u_0 \in C^{1, \gamma}(\overline{B})$ for every $0 < \gamma < 1$.

**Step 2.** We infer that $u_0$ satisfies (1.4).

For each $j \in \mathbb{N}$ consider the truncated functional

$$\Phi_j(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B F_j(x, u) dx, \quad u \in H^1_{0, \text{rad}}(B),$$

where $F_j(x, u) = \int_0^u f_j(x, s) ds$ and

$$f_j(x, s) = \begin{cases} f(x, -j), & \text{if } s < -j, \\ f(x, s), & \text{if } -j \leq s \leq j, \\ f(x, j), & \text{if } s > j. \end{cases}$$

Then from (1.2), there exists $C > 0$, that does not depend on $j$, such that

$$|F_j(x, u)| \leq C|x|^{\alpha}(1 + |u|^{q+1}), \quad \forall u \in \mathbb{R}, \quad \forall x \in B,$$

for certain $q > 1$ in case $N = 1, 2$ and $q = \frac{N+2(1+\alpha)}{N-2}$ in case $N \geq 3$. Also observe that, by the dominated convergence theorem, $\Phi_j(u) \to \Phi(u)$, as $j \to \infty$, for every $u \in H^1_{0, \text{rad}}(B)$.

Suppose that (1.4) does not hold. Then, by the continuous embedding $H^1_{0, \text{rad}}(B) \hookrightarrow L^{q+1}(B, |x|^{\alpha})$, for each $n \in \mathbb{N}$ there exists $v_n \in H^1_{0, \text{rad}}(B)$ and $j_n \in \mathbb{N}$ sufficiently large such that $|v_n - u_0|_{q+1, \alpha} \leq \frac{1}{n}$, $\Phi_{\text{rad}}(u_0) = \Phi_{j_n}(u_0)$, $\Phi_{\text{rad}}(v_n) < \Phi_{\text{rad}}(u_0)$ and $\Phi_{j_n}(v_n) < \Phi_{j_n}(u_0)$; recall that $u_0 \in L^\infty(B)$. In addition, we take $(j_n)$ increasing such that $j_n \to \infty$ as $n \to \infty$. Then

$$\Phi_{j_n}(u) \geq \frac{1}{2} \int_B |\nabla u|^2 dx - C \int_B |x|^\alpha |u|^{q+1} dx - C, \quad \forall u \in H^1_{0, \text{rad}}(B), \quad \forall n \in \mathbb{N},$$

and, from the subcritical growth of $F_{j_n}$, there exists $u_n \in H^1_{0, \text{rad}}(B)$ such that $|u_n - u_0|_{q+1, \alpha} \leq \frac{1}{n}$ and

$$\Phi_{j_n}(u_n) = \min_{u \in H^1_{0, \text{rad}}(B), |u - u_0|_{q+1, \alpha} \leq \frac{1}{n}} \Phi_{j_n}(u).$$

In particular,

$$\Phi_{j_n}(u_n) \leq \Phi_{j_n}(v_n) < \Phi_{\text{rad}}(u_0) = \Phi_{j_n}(u_0),$$

(3.1)
and there exists $\mu_n \in \mathbb{R}$ such that

$$-\Delta u_n = f_{j_n}(x, u_n) + \mu_n |x|^{\alpha}|u_n - u_0|^{q-1}(u_n - u_0) \text{ in } B, \quad u_n = 0 \text{ on } \partial B.$$  \hfill (3.2)

Observe that $\mu_n$ is the Lagrange multiplier associated to the minimization of $\Phi_{j_n}$ on

$$C_n = \{ u \in H^1_0, \text{rad}(B); |u - u_0|^{q+1}_{q+1, \alpha} \leq n^{-(q+1)} \}.$$  

Indeed, observe that $|u_n - u_0|^{q+1}_{q+1, \alpha} = \frac{1}{n}$ or $u_n$ is a local minimum for $\Phi_{j_n} : H^1_0, \text{rad}(B) \rightarrow \mathbb{R}$.

Now, since $u_n + t(u_0 - u_n) \in C_n$ for every $t \in [0, 1]$, we obtain

$$0 \leq \Phi'_{j_n}(u_n)(u_0 - u_n) = -\mu_n \int_B |x|^{\alpha}|u_n - u_0|^{q+1} dx$$

and, therefore, $\mu_n \leq 0$. In addition, arguing as in Step 1, for every $n \in \mathbb{N}$, it follows that $u_n$ strongly solves (3.2) and $u_n \in C^{1, \gamma}(\overline{B})$ for every $0 < \gamma < 1$.

We prove below that

$$u_n \rightarrow u_0 \quad \text{in } C^1_{0, \text{rad}}(\overline{B}) \quad \text{as } n \rightarrow \infty.$$  \hfill (3.3)

For this moment assume (3.3), which in particular means that $(u_n)$ is bounded in $L^\infty(B)$. Then, from (3.1), for every $n$ sufficiently large

$$\Phi(u_n) = \Phi_{j_n}(u_n) < \Phi(u_0)$$

which contradicts (1.3).

Now we prove (3.3). We distinguish two cases according to the behavior of $(\mu_n)$ as $n \rightarrow \infty$, namely,

(i) $(\mu_n)$ is bounded;

(ii) there exists a subsequence of $(\mu_n)$ that converges to $-\infty$.

Case (i). In this case, from (3.2), $u_n$ strongly solves

$$-\Delta u_n = g_n(x, u_n) \quad \text{in } B, \quad u_n = 0 \quad \text{on } \partial B,$$  \hfill (3.4)

and also in the sense of $H^1_0(B)$. Here $g_n(x, u_n) = f_{j_n}(x, u_n) + \mu_n |x|^{\alpha}|u_n - u_0|^{q-1}(u_n - u_0)$. In particular, $(u_n)$ is bounded in $H^1_0(B)$.

Again we split the proof into two cases: $N = 1, 2$ and $N \geq 3$.

In case $N = 1, 2$, $H^1_0(B) \hookrightarrow L^t(B)$ continuously for every $t \geq 1$. Then, since $(u_n)$ is bounded in $H^1_0(B)$, for each $t \geq 1$, there exists $C_t > 0$ such that $|g_n(x, u_n)| \leq C_t$. Therefore, by the classical elliptic regularity as in [25] Lemma 9.17, for each $t \geq 1$ there exists $C_t > 0$ such that $\|u_n\|_{W^{2, t}} \leq C_t$ for all $n \in \mathbb{N}$. Therefore, (3.3) follows from the classical Sobolev embeddings.

In case $N \geq 3$, since $H^1_{0, \text{rad}}(B) \hookrightarrow L^{2N}(B, |x|^\alpha)$ continuously and $(u_n)$ is bounded in $H^1_{0, \text{rad}}(B)$, there exists $a(x) \in L^{\frac{N}{2}}(B)$, see (2.4), such that

$$|g_n(x, u_n(x))| = |f_{j_n}(x, u_n) + \mu_n |x|^{\alpha}|u_n - u_0|^{q-2}(u_n - u_0)|$$

$$\leq a(x)(1 + |u_n(x)|), \quad \forall n \in \mathbb{N}, \quad \forall x \in B.$$  

Then, arguing as in [27, Proposition 1.2] and by [25, Lemma 9.17], for each $t \geq 1$ there exist $C_t > 0$ such that $\|u_{n}\|_{W^{2,t}} \leq C_t$ for all $n \in \mathbb{N}$. Therefore, once more, (3.3) follows from the classical Sobolev embeddings.

Case (ii). To simplify notation, we also denote by $(\mu_n)$ the subsequence of $(\mu_n)$ that converges to $-\infty$. Observe that (1.2) and the fact that $u_0$ is bounded, guarantees the existence $n_0 \in \mathbb{N}$ and $M > 0$, that do not depend on $n$, such that

$$g_n(x, s) = f_n(x, s) + \mu_n |x|^\alpha |s - u_0|^{q-1} (s - u_0) \begin{cases} < 0 & \text{if } s \geq M, \\
 > 0 & \text{if } s \leq -M, \end{cases}$$

for all $x \in B \setminus \{0\}$ and all $n \geq n_0$. Set $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Then $(u_n - M)^+, (u_n + M)^- \in H^1_0(B)$ and taking $(u_n - M)^+, (u_n + M)^-$ as test functions in (3.3) we obtain that $|u_n(x)| \leq M$ for every $n \geq n_0$ and for every $x \in B$.

Let $s > 1$. Then, from (1.1) and (3.4), with $|u_n - u_0|^{q-1}(u_n - u_0)$ as a test function we obtain

$$0 \leq \int_{B} |u_n - u_0|^{q-1} |\nabla u_n - u_0|^2 \, dx$$

$$= \int_{B} (f_n(x, u_n) - f(x, u_0)) |u_n - u_0|^{q-1}(u_n - u_0) \, dx + \mu_n \int_{B} |x|^\alpha |u_n - u_0|^{q+s} \, dx.$$

Now, since $(u_n)$ remains bounded in $L^\infty$, from Hölder’s inequality with $p = \frac{2+s}{q-s}$ applied to the above inequality we obtain

$$-\mu_n \int_{B} |u_n - u_0|^{q+s} |x|^\alpha \, dx \leq C \int_{B} |u_n - u_0|^{q} |x|^\alpha \, dx$$

$$\leq C \left( \int_{B} |x|^\alpha \, dx \right)^{\frac{q}{p+s}} \left( \int_{B} |u_n - u_0|^{q+s} |x|^\alpha \, dx \right)^{\frac{q}{q+s}}$$

and therefore

$$(-\mu_n)^{\frac{1}{s}} \left( \int_{B} |u_n - u_0|^{q+s} |x|^\alpha \, dx \right)^{\frac{1}{q+s}} \leq C \left( \int_{B} |x|^\alpha \, dx \right)^{\frac{s}{q+s}}.$$

Finally, taking the limit as $s \to \infty$ in the above inequality we obtain that $(-\mu_n)\|u_n - u_0\|_{W^{2,q}}$ is bounded and hence we argue as in Case (i).

Remark 3.1 (H^1 versus C^1 local minimizers: partially symmetric functions). The proof Theorem 1.2 follows similarly as the proof of Theorem 1.1a basically replacing Theorem 2.2 by Theorem 2.7 hence omitted here.

4. Existence results and local minimum solutions
In this part we start our study on the boundary value problem (1.6). Set

$$\lambda_{1,\alpha} = \inf_{u \in H^1_0(B) \setminus \{0\}} \frac{\int_{B} |\nabla u|^2 \, dx}{\int_{B} |x|^\alpha|u|^2 \, dx}.$$
Then $\lambda_{1,\alpha}$ is the first eigenvalue for the weighted eigenvalue problem

$$-\Delta u = \lambda |x|^\alpha u \text{ in } B, \quad \text{with } u = 0 \text{ on } \partial B,$$

which is simple and has a positive eigenfunction $\varphi_{1,\alpha}$. Consequently, $\varphi_{1,\alpha}$ is radially symmetric and strictly radially decreasing. Here we consider $\varphi_{1,\alpha}$ with $\int |x|^\alpha \varphi_{1,\alpha}^2 \, dx = 1$.

**Lemma 4.1.** If $\lambda \geq \lambda_{1,\alpha}$, then (1.6) has no solution.

*Proof.* If $w$ is a solution of (1.6), then

$$\lambda_{1,\alpha} \int_B |x|^\alpha \varphi_1 w \, dx = \lambda \int_B |x|^\alpha (w+1)^p \varphi_1 \, dx > \lambda^p \int_B |x|^\alpha \varphi_1 w \, dx,$$

which implies $\lambda^p < \lambda_{1,\alpha}$. \qed

Let $e_\alpha$ and $e_0$ be the solutions of

$$-\Delta e_\alpha = |x|^\alpha, \quad -\Delta e_0 = 1 \text{ in } B, \quad e_\alpha = e_0 = 0 \text{ on } \partial B.$$

Then, by the strong maximum principle, $0 < e_\alpha < e_0$ in $B$ for all $\alpha > 0$.

**Lemma 4.2.** If $0 < \lambda \leq \frac{(p-1)^{p-1}}{p^p |e_\alpha|_\infty}$, then (1.6) has a solution.

*Proof.* First observe that, for every $\lambda > 0$, $w = \lambda e_\alpha$ is a lower solution of (1.6).

Now we search for an upper solution $w$ of the form $w = Me_\alpha$. Then $w = Me_\alpha$ is an upper solution for (1.6) if, and only if,

$$\left(1 + M |e_\alpha|_\infty^p \right) \leq \lambda.$$

Let

$$h(t) = \left(1 + t |e_\alpha|_\infty^p \right), \quad t > 0.$$

Observe that $h$ attains its minimum value $\frac{p^p |e_\alpha|_\infty^p}{(p-1)^{p-1}}$ at $t_\ast = \frac{1}{(p-1)|e_\alpha|_\infty}$. Therefore, if $0 < \lambda \leq \frac{(p-1)^{p-1}}{p^p |e_\alpha|_\infty}$, then $\overline{\omega} = Me_\alpha$ with $M = t_\ast$ is an upper solution for (1.6) such that $w \leq \overline{\omega}$. Then we can apply the monotone iteration technique, as in [2], to prove the existence of a solution to (1.6). \qed

**Lemma 4.3.** There exists $\lambda_0 = \lambda_0(N,p) \in (0,1)$ such that for every $\lambda \in (0,\lambda_0)$, every integer $k \geq 2$, then $\overline{\omega} = \lambda^{1/k} e_\alpha$ is an upper solution for (1.6) such that $w = \lambda e_\alpha < \lambda^{1/k} e_\alpha = \overline{\omega}$ in $B$.

*Proof.* It is a straightforward consequence of (4.1). Observe that $|e_\alpha|_\infty \leq |e_0|_\infty$ for every $\alpha \geq 0$ and this allows us to get $\lambda_0(N,p)$ independent of $\alpha$. \qed

We say that a solution $w_0$ of

(4.2) $-\Delta w = f(x,w) \text{ in } \Omega, \quad w > 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega,$

is a minimal solution if

$$w_0(x) \leq w(x) \quad \forall x \in B.$$
for any other solution \( w \) of (1.2).

Let

\[
\lambda_* = \sup\{ \lambda > 0; \ (1.6) \text{ has a solution}\}.
\]

To treat (1.6) we will also consider an equivalent problem, namely, if we write

\[ v = aw, \]

then (1.6) is equivalent to

\[
-\Delta v = |x|^\alpha (v + a)^p \quad \text{in } B, \quad v > 0 \text{ in } B, \quad v = 0 \text{ on } \partial B, \quad \text{with } a^{p-1} = \lambda.
\]

**Proposition 4.4.** Let \( \lambda_* \) be as above. Then \( 0 < \lambda_* < \infty \) and:

(i) There is no solution of (1.6) if \( \lambda > \lambda_* \);

(ii) If \( 0 < \lambda < \lambda_* \), then (1.6) has a minimal solution \( w_{\alpha, \lambda} \). Let \( v_{\alpha, \lambda} \) and \( w_{\alpha, \lambda} \) be the corresponding minimal solutions of (4.3) and (1.6), respectively. Then \( w_{\alpha, \lambda} \to 0 \) uniformly w.r.t. \( x \) and \( \alpha \) as \( \lambda \to 0 \).

(iii) \( v_{\alpha, \lambda}, w_{\alpha, \lambda} \) are positive, radially symmetric and radially decreasing.

**Proof.** Regarding the existence on minimal solution, we apply the monotone iteration technique, as in [2]. Indeed the minimal solution \( w_{\alpha, \lambda} \) is obtained by the lower and upper solution method with a monotone iteration starting from the lower solution \( \tilde{w} = \lambda e_\alpha \) and therefore it is radially symmetric and radially decreasing.

Fix an integer \( k \geq 2 \). Then, by Lemma 4.3 and the strong maximum principle, we conclude that

\[
(4.4) \quad w_{\alpha, \lambda}(x) < \lambda^{1/k} e_\alpha(x) < \lambda^{1/k} ||e_0||_\infty \quad \forall x \in B \quad \text{and} \quad \lambda \in (0, \lambda_0(N, p)).
\]

Items (i) and (iii) are quite evident. \( \square \)

**Proposition 4.5.** Let \( v_{\alpha, \lambda} \) and \( w_{\alpha, \lambda} \) be the corresponding minimal solutions of (4.3) and (1.6), respectively. Then

\[
\lim_{\lambda \to 0} \frac{v_{\alpha, \lambda}}{\lambda e_\alpha} = 1, \quad \lim_{\lambda \to 0} \frac{w_{\alpha, \lambda}}{\lambda^{p/(p-1)} e_\alpha} = 1, \quad \text{uniformly w.r.t. } x \text{ and } \alpha.
\]

**Proof.** By the strong maximum principle we have that \( \lambda e_\alpha < w_{\alpha, \lambda} \) in \( B \). On the other hand, given \( \epsilon > 0 \), we have by (1.6) that there exists \( \lambda_\epsilon > 0 \) such that

\[
|w_{\alpha, \lambda}|_\infty < \epsilon, \quad \forall 0 < \lambda < \lambda_\epsilon, \forall \alpha > 0.
\]

Then

\[
-\Delta w_{\alpha, \lambda} = \lambda |x|^\alpha (1 + w_{\alpha, \lambda})^p < \lambda |x|^\alpha (1 + \epsilon)^p = -(1 + \epsilon)^p \Delta (\lambda e_\alpha),
\]

and, by strong maximum principle, we obtain

\[
\lambda e_\alpha < w_{\alpha, \lambda} < (1 + \epsilon)^p \lambda e_\alpha, \quad \forall 0 < \lambda < \lambda_\epsilon, \forall \alpha > 0.
\]

For the second limit observe that \( v_{\alpha, \lambda} = aw_{\alpha, \lambda} \) with \( a^{p-1} = \lambda \). \( \square \)
4.1. On radial local minimum solutions and the proof Theorem 1.3. Let $1 \leq p$ and $0 < \lambda < \lambda_*$ in case $N \geq 3$ also assume $p \leq 2^*_a - 1$. Then, see Remark 2.3, we know that the (positive) critical points of the $C^1(H^1_{0,\text{rad}}(B), \mathbb{R})$-functional

$$J_{\lambda,\text{rad}}(v) = \frac{1}{2} \int_B |\nabla v|^2 dx - \frac{1}{p+1} \int_B |x|^\alpha |v + a|^{p+1} dx,$$

are precisely the classical radial solutions of (4.3).

**Proposition 4.6.** Let $1 < p$ and in case $N \geq 3$ also assume $p \leq 2^*_a - 1$. If $0 < \lambda < \lambda_*$, then there exists $\bar{v}_{\alpha,\text{rad},\lambda} > 0$ in $B$ such that $J_{\lambda,\text{rad}}$ has a local minimum at $\bar{v}_{\alpha,\text{rad},\lambda}$.

$$J_{\lambda,\text{rad}}(\bar{v}_{\alpha,\text{rad},\lambda}) < 0 \quad \text{and} \quad \lim_{\lambda \to 0} \frac{\bar{v}_{\alpha,\text{rad},\lambda}}{\lambda^{p/(p-1)} e_\alpha} = 1, \quad \text{uniformly w.r.t. } x \text{ and } \alpha.$$

**Proof.** Given $0 < \lambda < \lambda_*$ fix any $\lambda' \in (\lambda, \lambda_*)$. Then observe that $0$ and $v_{\alpha,\lambda'}$ are, respectively, strict lower and upper solutions of (4.3). Consider

$$[0, v_{\alpha,\lambda}] = \{ v \in C^1_{0,\text{rad}}(B); 0 \leq v \leq v_{\alpha,\lambda'} \text{ in } B \}.$$

Then arguing as [16] Theorem 2.4, there exists $\bar{v}_{\alpha,\text{rad},\lambda} \in [0, v_{\alpha,\lambda}]$ such that

$$J_{\lambda,\text{rad}}(\bar{v}_{\alpha,\text{rad},\lambda}) = \min_{v \in [0, v_{\alpha,\lambda}]} J_{\lambda,\text{rad}}(v).$$

Then, by Theorems 1.1 and 2.2, $\bar{v}_{\alpha,\text{rad},\lambda}$ is a classical solution of (4.3) and, by the strong maximum principle, $0 < \bar{v}_{\alpha,\text{rad},\lambda} < v_{\alpha,\lambda'}$ in $B$. Then $\bar{v}_{\alpha,\text{rad},\lambda}$ is a local minimum of $J_{\lambda,\text{rad}}$ in the $H^1_{0,\text{rad}}(B)$ topology and, since $J_{\lambda,\text{rad}}(0) < 0$, it follows that $J_{\lambda,\text{rad}}(\bar{v}_{\alpha,\text{rad},\lambda}) < 0$. Finally, from $v_{\alpha,\lambda} \leq \bar{v}_{\alpha,\text{rad},\lambda} < v_{\alpha,\lambda'}$ in $B$, we obtain (4.6) from Proposition 4.5.

**Remark 4.7.** We do not know whether or not $\bar{v}_{\alpha,\text{rad},\lambda} = v_{\alpha,\lambda}$. See [32] and also eq. (2.76) for a similar problem where it is proved that the minimal positive solution is the local minimum solution.

**Proposition 4.8.** Let $N \geq 1, \alpha > 0, \lambda \in (0, \lambda_*], 1 < p$ and in case $N \geq 3$ also assume $p \leq 2^*_a - 1$. Fix $0 < \gamma < \min\{\alpha, 1\}$. Then there exists a positive constant $C$ such that any positive radial solution $v$ of (4.3) with $J_{\lambda,\text{rad}}(v) < 0$ satisfies

$$\|v\|_{0,\gamma} < C.$$

As a consequence, (4.6) has a solution with $\lambda = \lambda_*$.

**Proof.** We recall that $a^{p-1} = \lambda$. We have that

$$0 > J_{\lambda,\text{rad}}(v) = \frac{1}{2} \int_B |\nabla v|^2 dx - \frac{1}{p+1} \int_B |x|^\alpha |v + a|^{p+1} dx = \frac{1}{2(p+1)} \int_B |x|^\alpha ((p-1)(v + a)^{p+1} - (v + a)^{p+2}) dx,$$

and then,

$$\int_B |x|^\alpha |v + a|^{p+1} dx \leq C,$$
with $C$ independent of $\alpha$. Once more using that $J_{\text{rad}}(v) < 0$, we get

$$\|v\| < C \quad \forall 0 < \lambda < \lambda_*,$$

again with $C$ independent of $\alpha$. Then, arguing as in the proof of Theorem 1.1 based on [27, Proposition 1.2], we get $\|v\|_\infty \leq C$ for all $0 < \lambda < \lambda^*$.

Now, from Proposition 4.6, we have that $J_{\lambda,l}(\tilde{v}_{\alpha,l},\lambda) < 0$. Then using that $0 < v_{\alpha,\lambda} \leq \tilde{v}_{\alpha,l,\lambda}$ for all $0 < \lambda < \lambda_*$ and the above a priori bounds, we obtain the existence of a solution to (4.3) with $\lambda = \lambda_*$ as a limit of $v_{\alpha,\lambda}$ as $\lambda \uparrow \lambda_*$. \(\square\)

**Proof of Theorem 1.3.** It follows directly from Proposition 4.4 and 4.8. \(\square\)

4.2. On partially symmetric local minimum solutions. Let $1 < p$, $0 < \lambda < \lambda_*$ and assume all the hypotheses of Theorem 2.5. Then from Remark 2.6, we know that the (positive) critical points of the $C^1(H_0^1(B),\mathbb{R})$-functional

$$J_{\lambda,l}(v) = \frac{1}{2} \int_B |\nabla v|^2 dx - \frac{1}{p+1} \int_B |x| \alpha |v + a|^{p+1} dx, \quad v \in H_0^1(B), \quad a^{p-1} = \lambda,$$

are precisely the classical $(l,N-l)$-symmetric solutions of (4.3).

**Proposition 4.9.** Let $1 < p$ and assume all the hypotheses of Theorem 2.5. If $0 < \lambda < \lambda_*$, then exists $\tilde{v}_{\alpha,l,\lambda} > 0$ in $B$ such that $J_{\lambda,l}$ has a local minimum at $\tilde{v}_{\alpha,l,\lambda}$ and

$$\lim_{\lambda \to 0} \frac{\tilde{v}_{\alpha,l,\lambda}}{\lambda^{p/(p-1)} e_\alpha} = 1, \quad \text{uniformly w.r.t. } x \text{ and } \alpha.$$

**Proof.** It is very similar to the proof of Proposition 4.6 at this time with the aid of Theorem 1.2. \(\square\)

4.3. On local minimum solutions without symmetry. Let $1 < p$, $N \geq 1$, $0 < \lambda < \lambda_*$ and assume $1 < p < 2^* - 1$. Then we know that classical solutions of (4.3) are precisely the (positive) critical points of the $C^1(H_0^1(B),\mathbb{R})$-functional

$$J_\lambda(v) = \frac{1}{2} \int_B |\nabla v|^2 dx - \frac{1}{p+1} \int_B |x|^\alpha |v + a|^{p+1} dx, \quad a^{p-1} = \lambda.$$

**Proposition 4.10.** Let $1 < p$, $N \geq 1$ and assume $1 < p < 2^* - 1$. If $0 < \lambda < \lambda_*$, then exists $\tilde{v}_{\alpha,\lambda} > 0$ in $B$ such that $J_\lambda$ has a local minimum at $\tilde{v}_{\alpha,\lambda}$ and

$$\lim_{\lambda \to 0} \frac{\tilde{v}_{\alpha,\lambda}}{\lambda^{p/(p-1)} e_\alpha} = 1, \quad \text{uniformly w.r.t. } x \text{ and } \alpha.$$

**Proof.** It is very similar to the proof of Proposition 4.6 at this time with the aid of [9]. \(\square\)

**Remark 4.11.** We do not know whether or not $\tilde{v}_{\alpha,\text{rad},\lambda} = v_{\alpha,\lambda} = \tilde{v}_{\alpha,l,\lambda} = \tilde{v}_{\alpha,\lambda}$. 
5. Multiple positive solutions: proof of Theorem 1.4

5.1. Proof of Theorem 1.4 (I): under the extra assumption $1 < p < 2^*_\alpha - 1$ in case $N \geq 3$. In this case $J_{\lambda,\text{rad}}$, as defined by (1.5), satisfies the (PS) condition. In addition, $J_{\lambda,\text{rad}}$ has a local minimum at $\tilde{v}_{\alpha,\text{rad},\lambda}$, with $\tilde{v}_{\alpha,\text{rad},\lambda}$ given by Proposition 4.6. We recall that $-\Delta \tilde{v}_{\alpha,\text{rad},\lambda} > 0$ in $B$, $\tilde{v}_{\alpha,\text{rad},\lambda} > 0$ in $B$ and that $J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}) < 0$. Let $r_\lambda > 0$ such that

\[ J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}) \leq J_{\lambda,\text{rad}}(v) \quad \forall v \in H^1_{0,\text{rad}}(B) \text{ s.t. } \|v - \tilde{v}_{\alpha,\text{rad},\lambda}\| < r_\lambda. \]

We have to consider two cases.

**Case 1:** There exist $\epsilon_\lambda > 0$ and $r \in (0, r_\lambda)$ such that

\[ J_{\lambda,\text{rad}}(v) > J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}) + \epsilon_\lambda, \quad \forall v \in H^1_{0,\text{rad}}(B), \|v - \tilde{v}_{\alpha,\text{rad},\lambda}\| = r. \]

Choose $v_0 \in H^1_{0,\text{rad}}(B)$ such that $v_0 \geq 0$ in $B$, $\|v_0 - \tilde{v}_{\alpha,\text{rad},\lambda}\| > r_\lambda$ and $J_{\lambda,\text{rad}}(v_0) < J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda})$. Set

\[ \Gamma_{\alpha,\lambda,\text{rad}} = \{ \gamma \in C([0, 1], H^1_{0,\text{rad}}(B)); \gamma(0) = v_\alpha, \lambda \text{ and } \gamma(1) = v_0 \} \text{ and } m_{\alpha,\lambda,\text{rad}} = \inf_{\gamma \in \Gamma_{\alpha,\lambda,\text{rad}}} \max_{t \in [0, 1]} J_{\lambda,\text{rad}}(\gamma(t)). \]

Now observe that $J_{\lambda,\text{rad}}(|v|) \leq J_{\lambda,\text{rad}}(v)$ for every $v \in H^1_{0,\text{rad}}(B)$. Then, from [49, Theorem 2.8], any mountain pass solution of $J_{\lambda,\text{rad}}$ associated to the mountain pass level $m_{\alpha,\lambda,\text{rad}}$ is positive in $B$. Take $V_{\alpha,\lambda,\text{rad}}$ a mountain pass critical point of $J_{\lambda,\text{rad}}$ associated to the mountain pass level $m_{\alpha,\lambda,\text{rad}}$. Then, $V_{\alpha,\lambda,\text{rad}}$ is a classical solution of (4.3) such that $J_{\lambda,\text{rad}}(V_{\alpha,\lambda,\text{rad}}) > J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda})$. Consequently, $V_{\alpha,\lambda,\text{rad}}$ has at least two radial solutions.

**Case 2:** Suppose that Case 1 does not hold. In this case, for every $r \in (0, r_\lambda)$,

\[ \inf\{ J_{\lambda,\text{rad}}(v); \|v - \tilde{v}_{\alpha,\text{rad},\lambda}\| = r \} = J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}). \]

Therefore, by [17, Theorem 5.10], for every $r \in (0, r_\lambda)$ there exists $v_{0,r} \in H^1_{0,\text{rad}}(B)$ such that

\[ \|v_{0,r} - \tilde{v}_{\alpha,\text{rad},\lambda}\| = r \quad \text{and} \quad J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}) = J_{\lambda,\text{rad}}(v_{0,r}). \]

Then we have that each $v_{0,r}$ is also a local minimum of $J_{\lambda,\text{rad}}$ and therefore a classical solution of

\[ -\Delta v = |x|^\alpha|v + a|^{p-1}(v + a) \text{ in } B, \quad v = 0 \text{ on } \partial B, \quad \text{with } a^{p-1} = \lambda. \]

We claim that each $v_{0,r} \geq 0$ in $B$ and therefore (by the strong maximum principle) another radial solution of (4.3) as desired. By contradiction suppose that "$v_0 \geq 0$ in $B$" is not satisfied. Then observe that

\[
\begin{align*}
\int_B |\nabla v_0 - \nabla \tilde{v}_{\alpha,\text{rad},\lambda}|^2 dx &= \int_B |\nabla v_0 - \nabla \tilde{v}_{\alpha,\text{rad},\lambda}|^2 dx - 2 \int_B \nabla (|v_0| - v_0) \nabla \tilde{v}_{\alpha,\text{rad},\lambda} dx \\
&= \int_B |\nabla v_0 - \nabla \tilde{v}_{\alpha,\text{rad},\lambda}|^2 dx - 2 \int_B (|v_0| - v_0)(-\Delta \tilde{v}_{\alpha,\text{rad},\lambda}) dx \leq \int_B |\nabla v_0 - \nabla \tilde{v}_{\alpha,\text{rad},\lambda}|^2 dx.
\end{align*}
\]
that is
\[ \|v_0 - \tilde{v}_{\alpha,\text{rad},\lambda}\| \leq \|v_0 - \tilde{v}_{\alpha,\text{rad},\lambda}\| < r \]
and
\[ J_{\lambda,\text{rad}}(\|v_0\|) < J_{\lambda,\text{rad}}(\|v_0\|) = J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}), \]
which contradicts (5.1). Therefore \( v_0 \geq 0 \) in \( B \) and, by the strong maximum principle, we infer that \( v_0 > 0 \) in \( B \). Consequently, (1.6) has at least two radial solutions.

5.2. **Proof of Theorem 1.4 (I):** with \( N \geq 3 \) and \( p = 2^*_\alpha - 1 \). In this case \( J_{\lambda,\text{rad}} \) does not satisfies the \((PS)\) condition at all levels. To overcome this difficulty we adopt an approach different from that in Section 5.1 and close to the proof of [18, Theorem 2.5].

For that, it is essential to study
\[ -\Delta u = |x|^\alpha u^{2^*_\alpha - 1} \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N, \]
and in this direction we cite the pioneering work [20]. Let.
\[ D_{\text{rad}}^{1,2}(\mathbb{R}^N) = \{ u \in D^{1,2}(\mathbb{R}^N), u \text{ is radially symmetric} \}. \]
Arguing as in [37], [28], see also [45], we know that every element \( u \in D_{\text{rad}}^{1,2}(\mathbb{R}^N) \) has a representative \( U \), also denoted by \( u \), which is continuous in \( \mathbb{R}^N\setminus\{0\} \) and satisfies
\[ |u(x)| \leq \frac{1}{[(N-2)\omega_N]^{1/2}} \left( \int_{\mathbb{R}^N} |\nabla u| dx \right)^{1/2} |x|^{\frac{2-N\alpha}{2}}, \quad \forall x \in \mathbb{R}^N\setminus\{0\}. \]

At this time we use the identity \( u(x) = -\int_{|x|}^\infty u'(t)dt \). Then using the embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N) \) and arguing as in [28, p. 3742] we conclude that \( D_{\text{rad}}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N, |x|^\alpha) \).

Now consider
\[ S_\alpha(\mathbb{R}^N) = \inf_{u \in D_{\text{rad}}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*_\alpha} |x|^\alpha dx \right)^{2/2^*_\alpha}}. \]

We know that any solution of \( S_\alpha(\mathbb{R}^N) \) induces, up to multiplication by a constant, a \( C^2(\mathbb{R}^N) \) radial solution of (5.2). On the other hand, it is proved in [24 eq. (A.9)] that all \( C^2(\mathbb{R}^N) \) radial solutions of (5.2) are explicitly given by
\[ u_{\alpha,\theta}(x) = \sqrt{\frac{\theta(N - 2)(N + \alpha)}{\theta + |x|^{\alpha + 2}}} \left[ \frac{x}{|x|} \right]^{N-2 \over \alpha + 2}, \quad x \in \mathbb{R}^N, \quad \theta > 0. \]

**Lemma 5.1.** Given \( R > 0 \) let \( B_R = \{ x \in \mathbb{R}^N; |x| < R \} \). Set
\[ S_\alpha(B_R) = \inf_{u \in H_{\alpha,\text{rad}}(B_R)\setminus\{0\}} \frac{\int_{B_R} |\nabla u|^2 dx}{\left( \int_{B_R} |u|^{2^*_\alpha} |x|^\alpha dx \right)^{2/2^*_\alpha}}. \]
Then \( S_\alpha(B_R) = S_\alpha(\mathbb{R}^N) \).

**Proof.** Extending by zero an element in \( H^1_{0, \text{rad}}(B_R) \) we obtain an element in \( D^{1,2}_\text{rad}(\mathbb{R}^N) \). Then \( S^\alpha(\mathbb{R}^N) \leq S_\alpha(B_R) \).

Now, let \((w_n) \subset D_{\text{rad}}(\mathbb{R}^N)\) be a minimizing sequence for \( S_\alpha(\mathbb{R}^N) \) such that, for each \( n \), \( \text{supp}(w_n) \subset B_{r_n}(0) \). Then \( u_n(x) = \left( \frac{r_n}{R} \right)^{\frac{N-2}{2}} w \left( \frac{r_n}{R} x \right) \in H^1_{0, \text{rad}}(B) \) and

\[
\frac{\int_{B_R} |\nabla u_n|^2 \, dx}{\left( \int_{B_R} |u_n|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}} = \frac{\int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx}{\left( \int_{\mathbb{R}^N} |w_n|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}}.
\]

Therefore, \( S_\alpha(B_R) = S_\alpha(\mathbb{R}^N) \). \( \square \)

In view of Lemma 5.1 we will denote \( S_\alpha(\mathbb{R}^N) \) and \( S_\alpha(B_R) \) by \( S_\alpha \).

Fix \( \lambda \) with \( 0 < \lambda < \lambda_* \). We recall that, with \( a^{p-1} = \lambda \),

\[
J_{\lambda, \text{rad}}(v) = \frac{1}{2} \int_B |\nabla v|^2 \, dx - \frac{1}{p+1} \int_B |x|^\alpha |v|^{p+1} \, dx, \quad v \in H^1_{0, \text{rad}}(B),
\]

has a local minimum at \( \bar{v}_{\lambda, \text{rad}}, J_{\lambda, \text{rad}}(\bar{v}_{\lambda, \text{rad}}) < 0, -\Delta \bar{v}_{\lambda, \text{rad}} > 0 \) in \( B \) and \( \bar{v}_{\lambda, \text{rad}} > 0 \) in \( B \); see Proposition 4.6. It is enough to prove the existence of a nontrivial solution \( w \) of

\[
\begin{cases}
-\Delta w = |x|^\alpha g(x, w) & \text{in } B, \\
w = 0 & \text{on } \partial B,
\end{cases}
\]

where \( g(x, s) = (\bar{v}_{\lambda, \text{rad}} + s^+ + a)^{2^*_\alpha-1} - (\bar{v}_{\lambda, \text{rad}} + a)^{2^*_\alpha-1} \), since \( w + \bar{v}_{\lambda, \text{rad}} \) will correspond to a second radial solution of (5.3). Observe that any nontrivial solution \( w \) of (5.3) satisfies \( w > 0 \) in \( B \).

The functional associated to (5.3) is

\[
J(w) = \frac{1}{2} \int_B |\nabla w|^2 - \int_B |x|^\alpha G(x, w) \, dx, \quad w \in H^1_{0, \text{rad}}(B),
\]

where

\[
G(x, s) := \int_0^s g(x, t) \, dt = \frac{(\bar{v}_{\lambda, \text{rad}} + s^+ + a)^{2^*_\alpha}}{2^*_\alpha} - \frac{(\bar{v}_{\lambda, \text{rad}} + a)^{2^*_\alpha}}{2^*_\alpha} - (\bar{v}_{\lambda, \text{rad}} + a)^{2^*_\alpha-1} s^+.
\]

Then 0 is a local minimum of \( J \) on \( H^1_{0, \text{rad}}(B) \) (see [4] Lemma 4.2) for a similar result) and we are going to prove the existence of a nonzero critical point for \( J \).

By contradiction, assume that 0 is the only critical point of \( J \). Then for some ball \( B(0, r) \) in \( H^1_{0, \text{rad}}(B) \)

\[
J(0) < J(w)
\]

for all \( w \in B(0, r) \setminus \{0\} \). The following lemma, which is similar to [4] Lemma 4.3 (ii), will be proved below.
Lemma 5.2. Assume that 0 is the only critical point of $J$. Then $J$ satisfies the (PS)$_c$ condition for all levels $c$ with

$$c < c_0 := S_\alpha \frac{N + \alpha}{N + 2} \left( \frac{\alpha + 2}{2(N + \alpha)} \right).$$

From the preceding lemma together with [17, Theorem 5.10], which just requires the (PS)$_c$ condition to hold at the level of the local minimum (here the level $J(0) = 0 < c_0$), one deduces from (5.4) that

$$J(0) < \inf \{ J(w) : w \in H^1_{0,\text{rad}}(B) \text{ and } \|w\| = \rho \},$$

holds for all $\rho$ in $(0, r)$. We intend to apply the mountain pass theorem. For this purpose we will show the existence of $w_1 \in H^1_{0,\text{rad}}(B)$ such that $\|w_1\| > \rho$, $J(w_1) < 0$ and that the infmax value of $J$ over the family of all continuous paths from 0 to $w_1$ is smaller than $c_0$. Once this is done, the usual mountain pass theorem yields the existence of a nonzero critical point for $J$, a contradiction which will complete the proof of this part of the theorem (Theorem 1.4 (I)).

To construct a $w_1$ as above, we consider as in [3] functions of the form $tu_{\alpha, \epsilon}$ with $t > 0$ and

$$u_{\alpha, \epsilon}(x) := \frac{\varphi(x) \epsilon^{\frac{N-2}{2(\alpha+2)}}}{(\epsilon + |x|^{\alpha+2})^\frac{N-2}{2}}, \quad x \in \mathbb{R}^N,$$

where $\epsilon > 0$, $\varphi$ is a fixed $C^\infty_{c,\text{rad}}(B)$-function such that $\varphi \equiv 1$ in $B(0,1/2)$ and $0 \leq \varphi \leq 1$. Note that

$$U_\alpha(x) = \frac{1}{(1 + |x|^{\alpha+2})^\frac{N-2}{2}}, \quad x \in \mathbb{R}^N,$$

is a function where $S_\alpha$ is attained. Set

$$K_1 = \int_{\mathbb{R}^N} |\nabla U_\alpha|^2 dy = (N - 2)^2 \int_{\mathbb{R}^N} \frac{|y|^{2(\alpha+1)}}{(1 + |y|^{\alpha+2})^\frac{N-2}{2}} dy$$

and

$$K_2 = |U_\alpha|_{2^*_\alpha}^2 = \left( \int_{\mathbb{R}^N} \frac{|y|^{\alpha}}{(1 + |y|^{\alpha+2})^\frac{N-2}{2}} dy \right)^\frac{N-2}{N+\alpha}.$$

Then observe that $S_\alpha = K_1/K_2$.

The following lemma implies that for $\epsilon$ sufficiently small, the infmax value of $J$ over the family of all continuous paths from 0 to $w_1 = t_{\epsilon}u_{\alpha, \epsilon}$ is indeed smaller than $c_0$.

Lemma 5.3. We have that

$$\sup_{t > 0} J(tu_{\alpha, \epsilon}) < c_0$$

for $\epsilon > 0$ sufficiently small.

The above two lemmas, that are proved below, and the fact that

$$\lim_{t \to \infty} J(tu_{\alpha, \epsilon}) = -\infty$$

complete the proof of Theorem 1.4 (I).
Proof of Lemma 5.2  Let \( w_n \) be a \((PS)_c\) sequence with \( c < c_0 \), i.e.

\[
(5.6) \quad \frac{1}{2} \| w_n \|^2 - \int_B |x|^\alpha G(x, w_n) \, dx \to c
\]

\[
(5.7) \quad \left| \int_B \nabla w_n \nabla \phi \, dx - \int_B |x|^\alpha g(x, w_n) \phi \, dx \right| \leq \varepsilon_n \| \phi \|, \quad \forall \phi \in H^1_{0,\text{rad}}(B),
\]

where \( \varepsilon_n \to 0 \). We first observe that \( w_n \) remains bounded in \( H^1_{0,\text{rad}}(B) \). This follows by multiplying (5.7) with \( \phi = \bar{v}_{\alpha,\text{rad},\lambda} + w_n \) by \( 1/2^n \alpha \) and subtracting from (5.6): the terms of power \( 2\alpha \) cancel and the remaining dominating term is \( \| w_n \|^2 \), which yields the boundedness of \( w_n \). So, for a subsequence, \( w_n \to w_0 \) in \( H^1_{0,\text{rad}}(B) \) and \( w_n \to w_0 \) in \( L^r(B, |x|^{\alpha}) \) for any \( 1 \leq r < 2\alpha \). Then \( w_0 \) solves

\[
\begin{cases}
-\Delta w_0 = |x|^\alpha g(x, w_0) & \text{in } B \\
w_0 = 0 & \text{on } \partial B,
\end{cases}
\]

and consequently, by the assumption of this lemma, \( w_0 = 0 \). We now go back to (5.7) with \( \phi = w_0 + w_n \), multiply again by \( 1/2^n \alpha \) and subtract from (5.6) to get

\[
(5.8) \quad \lim_{n \to \infty} \| w_n \|^2 = 2c \left( \frac{N + \alpha}{\alpha + 2} \right).
\]

There are two possibilities: either \( c = 0 \) or \( c \neq 0 \). If \( c = 0 \) then \( w_n \to w_0 \) in \( H^1_{0,\text{rad}}(B) \) by (5.8) and we are done. We will now see that \( c \neq 0 \) leads to a contradiction. For that purpose we deduce from (5.7) with \( \phi = w_n \) that

\[
\lim_{n \to \infty} \| w_n \|^2 = \lim_{n \to \infty} \int_B |x|^\alpha g(x, w_n) \, dx = \lim_{n \to \infty} \int_B |x|^\alpha (w_n^+)^{2\alpha} \, dx.
\]

By definition of \( S_\alpha \), we have that

\[
(5.9) \quad \| w_n \|^2 \geq S_\alpha \left( \int_B |x|^\alpha |w_n|^{2\alpha} \, dx \right)^{2/2^n \alpha} \geq S_\alpha \left( \int_B |x|^\alpha (w_n^+)^{2\alpha} \, dx \right)^{2/2^n \alpha}.
\]

It follows from (5.8)–(5.9) that

\[
2c \left( \frac{N + \alpha}{\alpha + 2} \right) \geq S_\alpha \left( 2c \left( \frac{N + \alpha}{\alpha + 2} \right) \right)^{2/2^n \alpha},
\]

and then \( c \geq c_0 \). This contradicts (5.9) and completes the proof of Lemma 5.2. \( \square \)

Proof of Lemma 5.8  We have that

\[
(5.10) \quad g(x, s) = (\bar{v}_{\alpha,\text{rad},\lambda} + s^+ + a)^{2\alpha - 1} - (\bar{v}_{\alpha,\text{rad},\lambda} + a)^{2\alpha - 1} \geq (s^+)^{2\alpha - 1} + (2\alpha - 1)(\bar{v}_{\alpha,\text{rad},\lambda} + a)^{2\alpha - 2} s^+.
\]

Consequently,

\[
J(tu_{\alpha,c}) \leq \frac{t^2}{2} \| u_{\alpha,c} \|^2 - \frac{t^{2\alpha}}{2\alpha} \int_B |x|^\alpha u_{\alpha,c}^{2\alpha} - \frac{t^2}{2} (2\alpha - 1) \int_B |x|^\alpha (\bar{v}_{\alpha,\text{rad},\lambda} + a)^{2\alpha - 2} u_{\alpha,c}^2.
\]
Since $\bar{u}_{\alpha, \lambda} \geq b_0 > 0$ on the support of $u_{\alpha, \epsilon}^2$, we deduce

$$J(tu_{\alpha, \epsilon}) \leq \frac{t^2}{2} \frac{||u_{\alpha, \epsilon}||^2}{2\alpha} - \frac{t^{2\alpha}}{2\alpha} \int_B |x|^\alpha u_{\alpha, \epsilon}^2 - \frac{t^2}{2} (2^\alpha - 1) (b_0 + a)^{2\alpha - 2} \int_B |x|^\alpha u_{\alpha, \epsilon}^2.$$

By definition we have

$$\nabla u_{\alpha, \epsilon}(x) = \epsilon^{\frac{N-2}{N+2}} \left[ \frac{\nabla \varphi(x)}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} - (N-2) \frac{\varphi(x)|x|^\alpha}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \right].$$

Then, since $N \geq 3$ and $\varphi \equiv 1$ in $B(0,1/2)$, we get

$$\int_B |\nabla u_{\alpha, \epsilon}|^2 dx = \epsilon^{\frac{N-2}{N+2}} \left[ O(1) + (N-2)^2 \int_{\mathbb{R}^N} \frac{|x|^{2(\alpha+1)}}{(\epsilon + |x|^{\alpha+2})^{\frac{2N+2}{N+2}}} dx \right]$$

and so

$$\int_B |\nabla u_{\alpha, \epsilon}|^2 dx = \epsilon^{\frac{N-2}{N+2}} \left[ O(1) + K_1 \epsilon^{\frac{N-2}{N+2}} \right] = O(\epsilon^{\frac{N-2}{N+2}}) + K_1.$$

Now, since $\varphi \equiv 1$ in $B(0,1/2)$, we have

$$\int_B |u_{\alpha, \epsilon}|^{2\alpha} |x|^{2\alpha} dx = \epsilon^{\frac{N+2}{N+2}} \int_B \frac{\varphi^{2\alpha}(x)|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{2N+2}{N+2}}} dx$$

$$= \epsilon^{\frac{N+2}{N+2}} O(1) + \epsilon^{\frac{N+2}{N+2}} \int_{\mathbb{R}^N} \frac{|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{2N+2}{N+2}}} dx$$

$$= O(\epsilon^{\frac{N+2}{N+2}}) + K_2'$$

where $K'_2 = K_2^{2\alpha'/2}$. Then observe that $\left( O(\epsilon^{\frac{N+2}{N+2}}) + K_2 \right)^{2\alpha} = K_2 + O(\epsilon^{\frac{N+2}{N+2}})$.

Now we estimate $\int_B |u_{\alpha, \epsilon}|^{2\alpha} |x|^{2\alpha} dx$. We prove that

$$\int_B |u_{\alpha, \epsilon}|^{2\alpha} |x|^{2\alpha} dx = \begin{cases} K_3 \epsilon + O(\epsilon^{\frac{N-2}{N+2}}), & 0 < \alpha < N - 4, \\ K_3 \epsilon \log \epsilon + O(\epsilon), & 0 < \alpha = N - 4, \\ K_3 \epsilon^{\frac{N-2}{N+2}} + o(\epsilon^{\frac{N-2}{N+2}}), & N - 4 < \alpha, \end{cases}$$

with, respectively,

$$K_3 = \int_{\mathbb{R}^N} \frac{|x|^{\alpha}}{(1 + |x|^{\alpha+2})^{\frac{2N+2}{N+2}}} dx, \quad K_3 = \frac{\omega_N}{N-2}, \quad K_3 = \int_B \frac{\varphi^{2}(x)dx}{|x|^{2(N-2)-\alpha}}.$$

Indeed, since $\varphi \equiv 1$ in $B(0,1/2)$ we have

$$\int_B |u_{\alpha, \epsilon}|^{2\alpha} |x|^{2\alpha} dx = \epsilon^{\frac{N+2}{N+2}} \int_B \frac{\varphi^{2}(x)|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{2N+2}{N+2}}} dx$$

$$= \epsilon^{\frac{N+2}{N+2}} \left[ O(1) + \int_B \frac{|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{2(N+2)}{N+2}}} dx \right].$$
Case $0 \leq \alpha < N - 4$. In this case, from (5.13), we have
\[
\int_B |u_{\alpha, \epsilon}|^2 |x|^{\alpha} \, dx = O(\epsilon^{\frac{N-2}{N+2}}) + \epsilon \int_{\mathbb{R}^N} \frac{|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx
\]

$$= O(\epsilon^{\frac{N-2}{N+2}}) + \epsilon K_3,$$

with $K_3 = \int_{\mathbb{R}^N} \frac{|x|^{\alpha}}{(1 + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx$. Then, from (5.11) and (5.12), we obtain
\[
\sup_{t \geq 0} J(tu_{\alpha, \epsilon}) \leq \left( \frac{\alpha + 2}{2(N + \alpha)} \right) (S_{\alpha}^{\frac{N+\alpha}{N}} + O(\epsilon^{\frac{N-2}{N+2}}) - K_4 \epsilon),
\]
for some positive constant $K_4$, and we are done since $0 \leq \alpha < N - 4$.

Case $0 \leq \alpha = N - 4$. First, we have
\[
\int_B \frac{|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx = \omega_N \int_{\mathbb{R}^N} \frac{r^{\alpha+N-1}}{(\epsilon + r^{\alpha+2})^{\frac{N-2}{N+2}}} \, dr
\]

\[
= \frac{\omega_N}{\alpha + 2} \int_0^{1+\epsilon} \left( \frac{z - \epsilon}{z^2} \right)^{\frac{N-2}{N+2}} \, dz.
\]

From (5.14) and using that $\alpha = N - 4$ we get
\[
\int_B \frac{|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx = \frac{\omega_N}{\alpha + 2} \int_{\epsilon}^{1+\epsilon} \left( \frac{z - \epsilon}{z^2} \right)^{\frac{N-2}{N+2}} \, dz = O(1) + \frac{\omega_N}{N - 2} |\log \epsilon|.
\]

Then,
\[
\int_B |u_{\alpha, \epsilon}|^2 |x|^{\alpha} \, dx = O(\epsilon) + \frac{\omega_N}{N - 2} |\log \epsilon|,
\]
and at this time we have
\[
\sup_{t \geq 0} J(tu_{\alpha, \epsilon}) \leq \left( \frac{\alpha + 2}{2(N + \alpha)} \right) (S_{\alpha}^{\frac{N+\alpha}{N}} + O(\epsilon) - K_4 |\log \epsilon|),
\]
for some positive constant $K_4$ and we are done.

Case $N - 4 < \alpha$. In this case we can apply the dominated convergence theorem to obtain that
\[
\int_B \frac{\phi^2(x)|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx = \int_B \frac{\phi^2(x)dx}{|x|^{2(N-\alpha)}} + o(1)
\]

and then that
\[
\int_B |u_{\alpha, \epsilon}|^2 |x|^{\alpha} \, dx = \epsilon^{\frac{N-2}{N+2}} \int_B \frac{\phi^2(x)|x|^{\alpha}}{(\epsilon + |x|^{\alpha+2})^{\frac{N-2}{N+2}}} \, dx
\]

\[
= o(\epsilon^{\frac{N-2}{N+2}}) + \epsilon^{\frac{N-2}{N+2}} \int_B \frac{\phi^2(x)dx}{|x|^{2(N-\alpha)}}.
\]

In this case the inequality (5.10) is not suitable for our purposes. We emphasize that the inequality $N < \alpha + 4$ corresponds to critical dimensions associated to
problem (1.6) compare with the critical dimension $N = 3$ ($\alpha = 0$) in the paper of Brezis and Nirenberg [8]. Instead of (5.10) we use that

$$g(x, s) = (\tilde{v}_{\alpha, \text{rad}}, \lambda + s^+ + a)^{2^*_\alpha} - (\tilde{v}_{\alpha, \text{rad}}, \lambda + a)^{2^*_\alpha}$$

$$\geq (s^+)^{2^*_\alpha} - 1(\tilde{v}_{\alpha, \text{rad}}, \lambda + a)(s^+)^{2^*_\alpha - 2}.$$ Consequently,

$$J(tu_{\alpha, \eps}) \leq \frac{t^2}{2} \parallel u_{\alpha, \eps} \parallel^2_B - \frac{t^2}{2} \int_B \parallel x \parallel^\alpha u_{\alpha, \eps}^{2^*_\alpha} - t^2 \int_B \parallel x \parallel^\alpha (\tilde{v}_{\alpha, \text{rad}}, \lambda + a)u_{\alpha, \eps}^{2^*_\alpha}$$

and observe that $2^*_\alpha - 2 < 2^*_\alpha - 1 < 2^*_\alpha; \text{ compare with } [8, \text{ eq. (0.6)}].$ Since $\tilde{v}_{\alpha, \text{rad}, \lambda} \geq b_0 > 0$ on the support of $u_{\alpha, \eps}^{2^*_\alpha}$ we deduce that

$$J(tu_{\alpha, \eps}) \leq \frac{t^2}{2} \parallel u_{\alpha, \eps} \parallel^2_B - \frac{t^2}{2} (2^*_\alpha - 1)(b_0 + a) \int_B \parallel x \parallel^\alpha u_{\alpha, \eps}^{2^*_\alpha - 1}.$$ Now, since $\varphi \equiv 1$ in $B(0, 1/2)$, we have

$$\int_B \parallel u_{\alpha, \eps} \parallel^{2^*_\alpha - 1} \parallel x \parallel^\alpha \, dx = \epsilon \int_{\mathbb{R}^N} \varphi^{2^*_\alpha - 1}(x) \parallel x \parallel^\alpha \frac{N + 2(1 + \alpha)}{\alpha + 2} \, dx$$

$$= \epsilon \int_{\mathbb{R}^N} \frac{N + 2(1 + \alpha)}{\alpha + 2} \left[ O(1) + \int_{\mathbb{R}^N} \frac{\parallel x \parallel^\alpha}{(\epsilon + \parallel x \parallel^{\alpha + 2})^{N + 2(1 + \alpha)}} \, dx \right]$$

$$= \epsilon \int_{\mathbb{R}^N} \frac{N + 2(1 + \alpha)}{\alpha + 2} \left[ O(1) + \epsilon^{-1} \int_{\mathbb{R}^N} \frac{\parallel x \parallel^\alpha}{(1 + \parallel x \parallel^{\alpha + 2})^{N + 2(1 + \alpha)}} \, dx \right] = O(\epsilon^{N + 2(1 + \alpha)\alpha + 2}) = O(\epsilon^{N + 2(1 + \alpha)\alpha + 2}) + K_5 \epsilon^{N - 2}. $$

with $K_5 = \int_{\mathbb{R}^N} \frac{\parallel x \parallel^\alpha}{(1 + \parallel x \parallel^{\alpha + 2})^{N + 2(1 + \alpha)}} \, dx.$ Then from (5.11) and (5.12) we infer that

$$\sup_{t \geq 0} J(tu_{\alpha, \eps}) \leq \left( \frac{\alpha + 2}{2(N + \alpha)} \right) (S_{\alpha}^{N + p \alpha + 2} + O(\epsilon^{N - 2}) - K_4 \epsilon^{N - 2}),$$

for some positive constant $K_4$ and we are done. \hfill \Box

**Remark 5.4.** In the proof of Lemma 5.3 we could have used (5.15) in all the cases $0 < \alpha < N - 4,$ $\alpha = N - 4$ and $\alpha > N - 4.$ However, we decided also to use (5.10) to emphasize the critical dimensions $N \in \{3, \alpha + 4\}$ associated to the equation (1.0).

5.3. On the existence of a radial mountain pass solution to (1.0). In Section 5.2 we proved that (1.0) has two radial solutions for every $0 < \lambda < \lambda_1$ and for every $1 < p \leq 2^*_\alpha - 1$. Here we guarantee the existence of a mountain pass solution in the case that $\lambda > 0$ is sufficiently small.

**Proposition 5.5.** Let $1 < p$ and in case $N \geq 3$ also assume $p < 2^*_\alpha - 1.$ Then there exists $\lambda_0 = \lambda_0(N, p) \in (0, \lambda_1),$ such that for every $0 < \lambda < \lambda_0$ the functional $J_{\lambda, \text{rad}}$ has a mountain pass solution associated to its local minimum $\tilde{v}_{\alpha, \text{rad}, \lambda}$ with $\tilde{v}_{\alpha, \text{rad}, \lambda}$ as in Proposition 4.4. We emphasize that $\lambda_0(N, p)$ does not depend on $\alpha.$
Proof. First we recall that
\[ \epsilon_\alpha < \epsilon_0 \quad \text{in} \quad B \quad \forall \alpha > 0. \]

From Proposition 4.6, more precisely from (4.6) (cf. Lemma 4.3), we infer that
\[ \lambda \lesssim 2^{p/(p-1)} \epsilon_0 \quad \forall \lambda \in (0, \lambda_0). \]

Then, taking into account that \( a = \lambda^{1/(p-1)} \), we get that
\[
\int_B |\nabla \hat{v}_{\alpha, \text{rad}, \lambda}|^2 \, dx = \int_B |x|^\alpha (\nabla \hat{v}_{\alpha, \text{rad}, \lambda} + a)^p \hat{v}_{\alpha, \text{rad}, \lambda} \, dx \\
\leq \int_B \hat{v}_{\alpha, \text{rad}, \lambda} + a | \hat{v}_{\alpha, \text{rad}, \lambda} |^{p+1} \, dx \leq C \lambda^{(p+1)/(p-1)} \quad \forall \lambda \in (0, \lambda_0).
\]

Now observe that
\[
|J_{\lambda, \text{rad}}(v) - J_{0, \text{rad}}(v)| = \frac{1}{p+1} \left| \int_B |x|^\alpha v + a |x|^\alpha + |x|^\alpha v^{p+1} \, dx \right| \\
\leq 2^{p-1} a \int_B |x|^\alpha (|v|^{p+1} + a^p) \, dx \leq 2^{p-1} \lambda^{1/(p-1)} (C |v|^p + \lambda^{p/(p-1)} |B|)
\]
with \( C = C(N, p) \) is such that
\[
\int_B |u|^p |x|^\alpha \, dx \leq C(N, p) \left( \int_B |\nabla u|^2 \, dx \right)^{p/2}, \quad \forall u \in H_{0, \text{rad}}^1(B).
\]

Observe that \( C(N, p) \) may be taken independently of \( \alpha \). Indeed, for every \( u \in H_{0, \text{rad}}^1(B) \), we have from Ni’s pointwise estimate (2.1) that
\[
\left( \int_B |u|^2 |x|^\alpha \, dx \right)^{1/2} \leq \left( \int_B |u|^2 |x|^{2N/(N-2)} |x|^\alpha \right)^{1/2} \\
\leq \left( \left\| \nabla u \right\|_{2^{\alpha}}^{2^{\alpha}} \right)^{1/2} \left( \frac{1}{(\omega_{N-1}(N-2))^{\frac{N-2}{N}}} \int_B |u|^2 \, dx \right)^{1/2} \\
\leq \left( \left\| \nabla u \right\|_{2^{\alpha}}^{2^{\alpha}} \right)^{1/2} \frac{S_N^{\frac{N}{(N-2)}}}{((N-2)\omega_{N-1})^{\frac{N}{(N-2)^2}}} \left( \int_B |\nabla u|^2 \, dx \right)^{1/2}.
\]

Then take into account that the constants for the embeddings \( L^{2^{\alpha}}(B, |x|^\alpha) \hookrightarrow L^p(B, |x|^\alpha) \) can be bounded from above uniformly with respect to \( \alpha \).

Then, we recall that
\[ J_{0, \text{rad}}(v) = \frac{1}{2} \int_B |\nabla v|^2 \, dx - \frac{1}{p+1} \int_B |x|^\alpha |v|^{p+1} \, dx \\
\] has a strict minimum at \( v = 0 \). Moreover, using again (5.18), now with \( p + 1 \) in place of \( p \), there exists \( \epsilon(N, p) > 0 \) and \( r = r(N, p) > 0 \) such that
\[ J_{0, \text{rad}}(v) \geq \epsilon \quad \forall v \in H_{0, \text{rad}}^1(B) \quad \text{s.t.} \quad \|v\| = r.\]
Hence, combining the last inequality with (5.16) and (5.17), there exists $0 < \tilde{\lambda}_0 \leq \lambda_0$, with $\tilde{\lambda}_0 = \tilde{\lambda}_0(N, p)$, such that for every $0 < \lambda < \tilde{\lambda}_0$:

$$
\|\tilde{v}_{\alpha,\text{rad},\lambda}\| < \frac{r}{2}, \quad J_{\lambda,\text{rad}}(\tilde{v}_{\alpha,\text{rad},\lambda}) < 0, \quad \text{and} \quad J_{\lambda,\text{rad}}(v) > \frac{c}{2} \quad \forall \ v \in H^1_{0,\text{rad}}(B) \ \text{s.t.} \ \|v\| = r.
$$

It is also clear that $J_{\lambda,\text{rad}}(R\tilde{v}_{\alpha,\text{rad},\lambda}) < 0$ for $R > 0$ sufficiently large. Therefore, since $J_{\lambda,\text{rad}}$ satisfies the $(PS)_c$ condition at every level $c$, we apply the standard version of the mountain pass lemma [4].

5.4. Partially symmetric mountain pass solutions. Assume all the hypotheses from Proposition 4.9. So,

$$
J_{\lambda,l}(v) = \int_B |\nabla u|^2 \, dx - \frac{1}{p + 1} \int_B |x|^\alpha |v + a|^{p+1} \, dx, \quad v \in H^1_l(B), \quad a^{p-1} = \lambda,
$$

has a local minimum at $\tilde{v}_{\alpha,l,\lambda}$ and a mountain pass solution in the case that $0 < \lambda < \lambda_0(N, p)$; cf. Section 5.3. At this time, to see that $\lambda_0(N, p)$ can be taken independent of $\alpha$ and $l$ we refer to [3, Corollary 2.3]. As before, we prove that any mountain pass solution of $J_{\lambda,l}$ associated to the mountain pass level

$$
m_{\alpha,\lambda,l} = \inf_{\gamma \in \Gamma_{\alpha,\lambda,l}} \max_{t \in [0,1]} J_l(\gamma(t)),
$$

$$
\Gamma_{\alpha,\lambda,l} = \{ \gamma \in C([0,1], H^1_l(B)); \gamma(0) = \tilde{v}_{\alpha,l,\lambda} \ \text{and} \ \gamma(1) = v_0 \},
$$

is positive in $B$. Take $V_{\alpha,\lambda,l}$ a mountain pass critical point of $J_l$ associated to the mountain pass level $m_{\alpha,\lambda,l}$.

5.5. Solutions in the space $H^1_0(B)$. Here we suppose $1 < p < 2^* - 1$. So, by Proposition 4.9

$$
J_{\lambda}(v) = \int_B |\nabla u|^2 \, dx - \frac{1}{p + 1} \int_B |x|^\alpha |v + a|^{p+1} \, dx, \quad v \in H^1_0(B), \quad a^{p-1} = \lambda,
$$

has a local minimum at $\tilde{v}_{\alpha,\lambda}$ and a mountain pass solution in the case that $0 < \lambda < \lambda_0(N, p)$; cf. Section 5.3. At this time, to see that $\lambda_0(N, p)$ can be taken independent of $\alpha$ we can use that

$$
\int_B |v|^{p+1} |x|^\alpha \, dx \leq \int_B |v|^{p+1} \, dx
$$

and we use the classical Sobolev embeddings. As before, we prove that any mountain pass solution of $J_{\lambda}$ associated to the mountain pass level

$$
m_{\alpha,\lambda} = \inf_{\gamma \in \Gamma_{\alpha,\lambda}} \max_{t \in [0,1]} J(\gamma(t)),
$$

$$
\Gamma_{\alpha,\lambda} = \{ \gamma \in C([0,1], H^1_0(B)); \gamma(0) = \tilde{v}_{\alpha,\lambda} \ \text{and} \ \gamma(1) = v_0 \},
$$

is positive in $B$. Take $V_{\alpha,\lambda}$ a mountain pass critical point of $J$ associated to the mountain pass level $m_{\alpha,\lambda}$. 
In case \( N \geq 2 \), arguing as in [44, Proposition 3.1], we can prove that for each closed half-space \( H \) in \( \mathbb{R}^N \), the polarized function
\[
V_{\alpha,\lambda}^H = \begin{cases} 
\max\{u, u \circ \sigma_H\} & \text{on } H \cap B, \\
\min\{u, u \circ \sigma_H\} & \text{on } (\mathbb{R}^N \setminus H) \cap B,
\end{cases}
\]
is also a solution of (4.3) associated to the critical level \( m_{\alpha,\lambda} \). Then we argue as in [6, Lemmas 16 and 17] to prove that \( V_{\alpha,\lambda} \) is Schwarz foliated symmetric indeed.

5.6. Proof of Theorem 1.4 (II) and (III).

Proposition 5.6. Let \( N \geq 1 \), \( \alpha > 0 \) and \( 0 < \gamma < \min\{1, \alpha\} \).

(i) If \( 1 < p < 2^*_\alpha - 1 \), then
\[
\lim_{\lambda \to 0^+} V_{\alpha,\lambda,\text{rad}} = V_{\alpha,\text{rad}} \text{ in } C^{2,\gamma}(B).
\]

(ii) If \( 1 < p < 2^* - 1 \), then
\[
\lim_{\lambda \to 0^+} V_{\alpha,\lambda,l} = V_{\alpha,l} \text{ and } \lim_{\lambda \to 0^+} V_{\alpha,\lambda} = V_{\alpha} \text{ in } C^{2,\gamma}(B).
\]

Here \( V_{\alpha,\text{rad}} \), \( V_{\alpha,l} \) and \( V_{\alpha} \) are mountain pass critical points of \( J_{0,\text{rad}} : H^1_{0,\text{rad}}(B) \to \mathbb{R} \) and \( J_0 : H^1_0(B) \to \mathbb{R} \) respectively, associated to the problem
\[
-\Delta Z = |x|^\alpha |Z|^{p-1}Z \text{ in } B, \quad Z = 0 \text{ on } \partial B.
\]

We also have the respective convergence of the mountain pass levels.

Proof. Here we can closely follow the proof of [21, Theorem 2] and [22, Theorem 6], which are based on a priori estimates for positive solution of (4.3) for \( 0 \leq \lambda < \lambda_* \). Since such arguments are indeed very similar to those in the proof of [21, Theorem 2] and [22, Theorem 6], we omit them here.

We stress that, in the case (i), the a priori estimates for radial positive solutions of (4.3) with \( 1 < p < 2^*_\alpha - 1 \) follows from [40, p. 2529, Case 2]; observe that \( P_k = 0 \), for every \( k \), for radial solutions. The a priori estimates in the case (ii) (for all positive solutions of (4.3)) is presented in [40, Theorem 1.3]. Moreover, these a priori estimates depends on \( N, p \) and \( \alpha \). Since these a priori estimates depend on \( \alpha \) we see from (5.17) that the convergence of the mountain pass levels (as \( \lambda \to 0^+ \)) also depends on \( \alpha \).

Proof of Theorem 1.4 (II) and (III) completed. For each \( N \geq 1 \), the mountain pass levels of \( V_{\alpha,\text{rad}} \), \( V_{\alpha,l} \) and \( V_{\alpha} \) are different, provided \( \alpha > a_0(N,p) \); see [43, 11, 5]. Then, for every \( \alpha > a_0(N,p) \), we obtain from Propositions 5.5 and 5.6 that there exists \( \lambda_0 = \lambda_0(N,p,\alpha) \) such that for every \( 0 < \lambda \leq \lambda_0 \) the solutions \( V_{\alpha,\lambda,\text{rad}} \), \( V_{\alpha,\lambda,l} \) and \( V_{\alpha,\lambda} \) are non rotational equivalent, because they have different positive energy levels. Observe that \( \lambda_0 \) depends on \( \alpha \) because, as we explained in the proof of Proposition 5.5, the convergence of the mountain pass levels may depend on \( \alpha \). Then, for every \( N \geq 1 \), \( V_{\alpha,\lambda,\text{rad}} \), \( V_{\alpha,l} \) and \( \tilde{v}_{\alpha,\lambda,\text{rad}} \), with \( \tilde{v}_{\alpha,\lambda,\text{rad}} \) from Proposition 4.6 produce three non rotational equivalent solutions of (1.6). In addition, \( V_{\alpha,\lambda} \) is not radially symmetric, and in case \( N \geq 2 \), \( V_{\alpha,\lambda} \) is Schwarz foliated symmetric.
Using the same arguments, in case $N \geq 4$, $V_{\alpha, \lambda, l}$ and $V_{\alpha, \lambda, j}$ are non rotational equivalent if $j \neq l$; see [3, 33] for the limit problem with $\lambda = 0$. Then we get the existence of at least $\lceil N/2 \rceil + 2$ non rotational equivalent solutions for (1.6), since we have $\lceil N/2 \rceil - 1$ choices of $l \in \mathbb{Z}$ such that $2 \leq N - l \leq l$.

**Remark 5.7.** Assume $N \geq 4$, $l \in \mathbb{N}$, $2 \leq N - l \leq l$ and $2 < p + 1 < \frac{2(l+1)}{l-1}$. Then, as we argued in the proof of Theorem 1.4 (III), there exist $\alpha_0(N, p) > 0$ and $\lambda_0 = \lambda_0(N, p, \alpha)$ such that for all $\alpha > \alpha_0(N, p)$ and $0 < \lambda < \lambda_0$ the solutions $V_{\alpha, \lambda, l}$ have distinct positive critical levels and $\bar{V}_{\alpha, \lambda, l}$ has a negative critical level. Therefore we get the existence of at least three solutions to (1.6). Observe that an estimate like (II.4), involving $J_{\alpha, \lambda}$ and $J_{0, l}$, can be used to prove uniformly bound of the mountain pass level, and so of the mountain pass solutions, as $\alpha \to 0^+$. Finally, observe that the condition $2 < p + 1 < \frac{2(l+1)}{l-1}$ includes cases with $p + 1 \geq 2^*$.

6. A Weighted Problem Posed in an Exterior Domain

Here we consider $N \geq 3$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $p > 0$ and the problem

$$\begin{align*}
-\Delta u &= \frac{U^p}{|x|^\beta} \quad \text{in} \quad \mathbb{R}^N \setminus B, \quad U > 0 \quad \text{in} \quad \mathbb{R}^N \setminus B, \\
u &= a \quad \text{on} \quad \partial B, \quad U \to 0 \quad \text{as} \quad |x| \to \infty.
\end{align*}$$

(6.1)

If $U : \mathbb{R}^N \setminus B \to \mathbb{R}$, then the Kelvin transform $u(x) = U \left(\frac{x}{|x|^2}\right) |x|^{2-N}$ is such that $u : \partial B \setminus \{0\} \to \mathbb{R}$ and

$$\Delta u(x) = |x|^{-N-2} \Delta U \left(\frac{x}{|x|^2}\right).$$

On the other hand, if $u : \partial B \to \mathbb{R}$ is a $C^2(B) \cap C^2(\mathbb{R}^N \setminus B)$, then $U(x) = u \left(\frac{|x|}{|x|^2}\right) |x|^{2-N}$ is such that, $U : \mathbb{R}^N \setminus B \to \mathbb{R}$ is continuous, $U \in C^2(\mathbb{R}^N \setminus B)$ and $\lim_{|x| \to \infty} U(x)|x|^{N-2} = u(0)$.

So, if we search for a solution of (6.1), let $u(x) = U \left(\frac{|x|}{|x|^2}\right) |x|^{2-N}$. Then we are led to study the following problem

$$\begin{align*}
-\Delta u &= |x|^{-N-2+\beta+p(N-2)} u^p \quad \text{in} \quad B, \\
u &= a \quad \text{on} \quad \partial B, \quad u > 0 \quad \text{in} \quad B,
\end{align*}$$

which is similar to problem (1.3).

The case with $a = 0$. As a consequence of [27, 33, 11]:

(i) If $\beta \leq 0$, then there exists at least one positive radial $U$ for (6.1) for all $p > \frac{N+2-2\beta}{N-2}$.

(ii) Consider $0 < \beta \leq \frac{N+2}{N-2}$. Then there exists at least one positive radial $U$ for (6.1) for all $p \geq \frac{N+2-\beta}{N-2}$ and $p \neq 1$.

(iii) Consider $\beta > \frac{N+2}{2}$. Then there exists at least one positive radial solution $U$ for (6.1) for all $p > 0$ and $p \neq 1$. 

(iv) Consider $N = 1, 2$, $p > 1$ and $\beta > 0$ large. Then (6.1) has at least two non rotational equivalent solutions.

(v) Consider $N \geq 3$, $1 < p < 2^* - 1$ and $\beta > 0$ large. Then (6.1) has at least $\left\lceil \frac{N}{2} \right\rceil + 1$ non rotational equivalent solutions.

The case with $a > 0$.

(i) In case $p > 1$ and $\beta \geq N + 2 - p(N - 2)$. Then (6.1) has a solution if, and only if, $a$ is suitably small; see Theorem 1.3.

(ii) In case $\beta \leq 0$ and $p > \frac{N + 2 - 2\beta}{N - 2}$, then (6.1) has at least two radial solutions in case $a > 0$ is suitably small; see Theorem 1.4 (I).

(iii) In case $\beta > 0$ is large, $a > 0$ is suitably small and $1 < p < 2^* - 1$, then Theorem 1.4 (II) and (III) apply to prove the existence of multiple positive solutions to (6.1).

Acknowledgments

Leonelo Iturriaga has been partially supported by Programa Basal PFB 03, CMM, U. de Chile; Fondecyt grant 1120842 and USM grant No. 12.12.11. Ederson Moreira dos Santos has been partially supported by CNPq #309291/2012-7 grant and FAPESP #10/19320-7 grant. Pedro Ubilla has been partially supported Fondecyt grant 1120524.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math., 12:623–727, 1959.

[2] H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev., 18(4):620–709, 1976.

[3] A. Ambrosetti, H. Brezis, and G. Cerami. Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal., 122(2):519–543, 1994.

[4] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. J. Functional Analysis, 14:349–381, 1973.

[5] M. Badiale and E. Serra. Multiplicity results for the supercritical H"{e}non equation. Adv. Nonlinear Stud., 4(4):453–467, 2004.

[6] E. Berchio, F. Gazzola, and T. Weth. Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems. J. Reine Angew. Math., 620:165–183, 2008.

[7] H. Brézis and T. Kato. Remarks on the Schrödinger operator with singular complex potentials. J. Math. Pures Appl. (9), 58(2):137–151, 1979.

[8] H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math., 36(4):437–477, 1983.

[9] H. Brezis and L. Nirenberg. $H^1$ versus $C^1$ local minimizers. C. R. Acad. Sci. Paris Sér. I Math., 317(5):465–472, 1993.

[10] F. Brock, L. Iturriaga, and P. Ubilla. A multiplicity result for the $p$-Laplacian involving a parameter. Ann. Henri Poincaré, 9(7):1371–1386, 2008.

[11] J. Byeon and Z.-Q. Wang. On the Hénon equation: asymptotic profile of ground states. I. Ann. Inst. H. Poincaré Anal. Non Linéaire, 23(6):803–828, 2006.

[12] D. Cao and S. Peng. Asymptotic behavior for elliptic problems with singular coefficient and nearly critical Sobolev growth. Ann. Mat. Pura Appl. (4), 185(2):189–205, 2006.
[13] D. Cao, S. Peng, and S. Yan. Asymptotic behaviour of ground state solutions for the Hénon equation. *IMA J. Appl. Math.*, 74(3):468–480, 2009.

[14] P. Clément, D. G. de Figueiredo, and E. Mitidieri. Quasilinear elliptic equations with critical exponents. *Topol. Methods Nonlinear Anal.*, 7(1):133–170, 1996.

[15] M. G. Crandall and P. H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.*, 58(3):207–218, 1975.

[16] J. Dávila, M. del Pino, and M. Musso. The supercritical Lane-Emden-Fowler equation in exterior domains. *Comm. Partial Differential Equations*, 32(7-9):1225–1243, 2007.

[17] D. G. de Figueiredo. *Lectures on the Ekeland variational principle with applications and detours*, volume 81 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1989.

[18] D. G. de Figueiredo, J.-P. Gossez, and P. Ubilla. Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity. *J. Eur. Math. Soc. (JEMS)*, 8(2):269–286, 2006.

[19] D. G. de Figueiredo, J.-P. Gossez, and P. Ubilla. Local “superlinearity” and “sublinearity” for the \( p \)-Laplacian. *J. Funct. Anal.*, 257(3):721–752, 2009.

[20] R. Fowler. Further studies of Emden’s and similar differential equations. *Q. J. Math., Oxf. Ser.*, 2:259–288, 1931.

[21] F. Gazzola. Critical growth quasilinear elliptic problems with shifting subcritical perturbation. *Differential Integral Equations*, 14(5):513–528, 2001.

[22] F. Gazzola and A. Malchiodi. Some remarks on the equation \(-\Delta u = \lambda (1 + u)^p\) for varying \(\lambda\), \(p\) and varying domains. *Comm. Partial Differential Equations*, 27(3-4):809–845, 2002.

[23] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[24] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.

[25] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[26] F. Gladiali, M. Grossi, and S. L. N. Neves. Nonradial solutions for the Hénon equation in \(\mathbb{R}^N\). *Adv. Math.*, 249:1–36, 2013.

[27] M. Guedda and L. Véron. Quasilinear elliptic equations involving critical Sobolev exponents. *Nonlinear Anal.*, 13(8):879–902, 1989.

[28] D. Guedes de Figueiredo, E. M. dos Santos, and O. H. Miyagaki. Sobolev spaces of symmetric functions and applications. *J. Funct. Anal.*, 261(12):3735–3770, 2011.

[29] M. Hénon. Numerical experiments on the stability of spherical stellar systems. *Astronomy and Astrophysics*, 24:229–238, 1973.

[30] D. D. Joseph and T. S. Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.*, 49:241–269, 1972/73.

[31] D. D. Joseph and E. M. Sparrow. Nonlinear diffusion induced by nonlinear sources. *Quart. Appl. Math.*, 28:327–342, 1970.

[32] J. P. Keener and H. B. Keller. Positive solutions of convex nonlinear eigenvalue problems. *J. Differential Equations*, 16:103–125, 1974.

[33] Y. Y. Li. Existence of many positive solutions of semilinear elliptic equations on annulus. *J. Differential Equations*, 83(2):348–367, 1990.

[34] P.-L. Lions. Symétrie et compacité dans les espaces de Sobolev. *J. Funct. Anal.*, 49(3):315–334, 1982.

[35] E. Moreira dos Santos and F. Pacella. Hénon type equations and concentration on spheres. To appear in *Indiana Univ. Math. J.* [arXiv:1407.6581].

[36] J. Moser. A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.
[37] W. M. Ni. A nonlinear Dirichlet problem on the unit ball and its applications. *Indiana Univ. Math. J.*, 31(6):801–807, 1982.

[38] F. Pacella. Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities. *J. Funct. Anal.*, 192(1):271–282, 2002.

[39] R. S. Palais. The principle of symmetric criticality. *Comm. Math. Phys.*, 69(1):19–30, 1979.

[40] Q. H. Phan and P. Souplet. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations. *J. Differential Equations*, 252(3):2544–2562, 2012.

[41] A. Pistoia and E. Serra. Multi-peak solutions for the Hénon equation with slightly subcritical growth. *Math. Z.*, 256(1):75–97, 2007.

[42] D. Smets and M. Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations*, 18(1):57–75, 2003.

[43] D. Smets, M. Willem, and J. Su. Non-radial ground states for the Hénon equation. *Commun. Contemp. Math.*, 4(3):467–480, 2002.

[44] M. Squassina and J. Van Schaftingen. Finding critical points whose polarization is also a critical point. *Topol. Methods Nonlinear Anal.*, 40(2):371–379, 2012.

[45] W. A. Strauss. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.*, 55(2):149–162, 1977.

[46] M. Struwe. *Variational methods*, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.

[47] J. Wei and S. Yan. Infinitely many nonradial solutions for the Hénon equation with critical growth. *Rev. Mat. Iberoam.*, 29(3):997–1020, 2013.

[48] M. Willem. *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

**Leonelo Iturriaga**

Departamento de Matemática - Universidad Técnica Federico Santa María

Av. España 1680, Casilla 110-V, Valparaíso - Chile

leonelo.iturriaga@usm.cl

**Ederson Moreira dos Santos**

Instituto de Ciências Matemáticas e de Computação - Universidade de São Paulo

C.P. 668, CEP 13560-970 - São Carlos - SP - Brazil

ederson@icmc.usp.br

**Pedro Ubilla**

Departamento de Matemáticas e C. C. - Universidad de Santiago de Chile

Casilla 307, Correo 2, Santiago - Chile

pedro.ubilla@usach.cl