Circular orbit spacecraft control at the L4 point using Lyapunov functions

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Abstract—The objective of this work is to demonstrate the utility of Lyapunov functions in control synthesis for the purpose of maintaining and stabilizing a spacecraft in a circular orbit at the L4 point in the circular restricted three body problem (CR3BP). Incorporating the requirements of a fixed radius orbit and a desired angular momentum, a Lyapunov function is constructed and the requisite analysis is performed to obtain a controller. Asymptotic stability is proved in a defined region around the L4 point with the help of LaSalle’s principle.

I. INTRODUCTION

The restricted three body problem is defined as follows: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of this third body [10].

Libration points are the natural equilibrium solutions of the restricted three-body problem (R3BP). In the last few years, the interest concerning the libration points for space applications has risen within the scientific community [1]. This is because the libration points offer the unique possibility to have a fixed configuration with respect to two primaries. Therefore, a libration point mission could fulfill a lot of mission constraints that are not achievable with the classical Keplerian two-body orbits. Moreover, exploiting the stable and unstable part of the dynamics related to these equilibria, low-energy station keeping missions of practical interest can be obtained.

Considerable work has been done on the CR3BP with linearized dynamics [4] [7] [9]. Work on existence of formation flight trajectories near the triangular libration point in the CR3BP using the linearized equations of motion has been presented in [2]. Feedback linearization techniques have been applied to various formation flights near the vicinity of the libration points [11] [8]. An extensive bibliographical survey of problems in this context can be found in [5].

In [3], a Lyapunov based control is derived to achieve transfer between elliptic Keplerian orbits. Taking inspiration from [3], we attempt to exploit the theory of Lyapunov functions and related stability notions to derive a feedback control law to keep the spacecraft in a circular orbit around the libration point. We use Newton’s universal law of gravitation to express the spacecraft dynamics. This approach of using the original nonlinear dynamics of the system to derive the feedback controller has not been adopted in the area of control related to CR3BP.

The document is organized as follows. In the second section we introduce the notations used in the text and derive the dynamics of the spacecraft in the CR3BP. In the third section we define the characteristics of the desired orbit and propose a candidate Lyapunov function. We carry out further computation and analysis to construct a suitable controller so as to render the desired orbit stable. In the fourth section we use LaSalle’s invariance principle to prove asymptotic stability of the desired orbit. Finally, in the fifth section, we present the numerical simulation results with respect to the Earth-Moon system.

II. NOTATION AND SYSTEM MODELLING

We first present the notation used in the document and follow this up with a dynamic model.

• \(m_1, m_2, m_s\) - two primaries and spacecraft
• \(m_1, m_2, L^4\) form a plane of rotation - synodic frame
• \(r_{m_j}\) - distance of \(m_j\) from centre of mass (COM) of \(m_1 m_2\)
• \(r_{c}\) - position vector of L4 point from the COM
• \(r_{s}\) - position vector of spacecraft from L4
• \(r_{js}\) - position vector of spacecraft from \(m_j\)
• \(\phi\) - angle of rotation of the synodic frame with respect to the sidereal frame
• subscript \(b\) - vectors in synodic frame
• subscript \(i\) - vectors in inertial frame
• \(x_i, y_i\) are along the inertial frame
• \(x_s, y_s\) are along the synodic frame
• \(\omega\) - angular velocity vector of the two primaries about the COM
• \(\hat{\omega}\) - skew symmetric matrix of \(\omega\) given by

\[
\hat{\omega} = \phi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

From Newton’s universal law of gravitation, we have the dynamics of the satellite expressed as

\[
m_3 \hat{\vec{r}}_{s_i} = \frac{k m_1 m_3}{||\vec{r}_{1s_i}||^2} (\vec{r}_{1s_i}) + \frac{k m_2 m_3}{||\vec{r}_{2s_i}||^2} (\vec{r}_{2s_i})
\]

where \(k\) is the gravitational constant, \(\vec{r}_{1s_i}\) and \(\vec{r}_{2s_i}\) are the position vectors of the spacecraft in the inertial frame from masses \(m_1\) and \(m_2\) respectively. The sidereal (inertial) and synodic coordinate representations of the position vectors are related by

\[\]
which yields

\[ (\mathbf{1}) \text{ now reads as} \]

We proceed as follows. We substitute (3) in (1). Also, our objective is to express (1) in the body frame coordinates.

\[ \text{where} \quad \phi \text{ denotes the angle of rotation. Note that we are concerned with planar rotations. Differentiating (2) twice, we have} \]

\[ \ddot{r}_{s_i} = \ddot{R}r_{sb} + 2\ddot{R}r_{sb} + \dddot{R}r_{sb} \] (3)

Our objective is to express (1) in the body frame coordinates. We proceed as follows. We substitute (3) in (1). Also, \[ \mathbf{r}_{1s_i} = R\mathbf{r}_{sb} + R(\mathbf{r}_{1cb} + \mathbf{r}_{csb}) \]

which yields

\[ \frac{km_1m_3}{||\mathbf{r}_{1s_i}||^3} (\mathbf{r}_{1s_i}) + \frac{km_2m_3}{||\mathbf{r}_{2s_i}||^3} (\mathbf{r}_{2s_i}) \]

\[ = -Rkm_1m_3 \frac{\mathbf{r}_{1cb} + \mathbf{r}_{csb}}{||\mathbf{r}_{1s_i}||^3} + \frac{-Rkm_2m_3}{||\mathbf{r}_{2s_i}||^3} (\mathbf{r}_{2cb} + \mathbf{r}_{csb}) \]

\[ \text{(1) now reads as} \]

\[ m_3(\dddot{R}r_{sb} + 2\dddot{R}r_{sb} + \dddot{R}r_{sb}) = -Rkm_1m_3 \frac{\mathbf{r}_{1cb} + \mathbf{r}_{csb}}{||\mathbf{r}_{1s_i}||^3} + \frac{-Rkm_2m_3}{||\mathbf{r}_{2s_i}||^3} (\mathbf{r}_{2cb} + \mathbf{r}_{csb}) \]

Regrouping terms

\[ \dddot{r}_{sb} = -R^{-1}\dddot{R}r_{sb} - 2R^{-1}\dddot{R}r_{sb} \]

\[ -k\left( \frac{m_1(\mathbf{r}_{1cb} + \mathbf{r}_{csb})}{||\mathbf{r}_{1s_i}||^3} + \frac{m_2(\mathbf{r}_{2cb} + \mathbf{r}_{csb})}{||\mathbf{r}_{2s_i}||^3} \right) \]

and using the expression

\[ \dddot{R} = R\dddot{\omega}, \quad \dddot{R} = R\dddot{\omega} = R\dddot{\omega} \]

we have

\[ \dddot{r}_{csb} = -\dddot{R}r_{csb} + \dddot{R}r_{csb} - 2\dddot{R}r_{csb} \]

\[ = \frac{km_1}{||\mathbf{r}_{1s_i}||^3} (\mathbf{r}_{1cb} + \mathbf{r}_{csb}) - \frac{km_2}{||\mathbf{r}_{2s_i}||^3} (\mathbf{r}_{2cb} + \mathbf{r}_{csb}) \]

(4)

Now (4) represents the natural dynamics of the spacecraft near the L4 point. Note that for the circular restricted three body problem, \( \phi \) is assumed to be a constant and is given by

\[ \dot{\phi} = \sqrt{\frac{F_{12}}{m_1r_{m_1}}} \Rightarrow \dddot{R} = -\dddot{R}r_{csb} \]

where \( F_{12} \) is the gravitational force exerted by \( m_2 \) on \( m_1 \). Also \[ r_{sb} = r_{cb} + r_{csb} \] with \( \dddot{r}_{sb} = \dddot{r}_{csb} \) since \( \dddot{r}_{sb} = 0 \) as it is the L4 point and is stationary in the sidereal frame.

Now we introduce the control input as follows:

\[ \dddot{r}_{csb} = f(r_{csb}, \dddot{r}_{csb}) + \dddot{u}_{sb} \]

(5)

where \( f(r_{csb}, \dddot{r}_{csb}) \) is the whole right hand side of (4), \( \dddot{u}_{sb} \in \mathbb{R}^3 \) denotes the three independent control inputs. Note that in practice control is achieved through force generated by firing thrusters. The control law here has the units of acceleration. The force applied will be acceleration times the mass of spacecraft.

### III. Lyapunov Function and Analysis

For further analysis, we characterize the orbits around the L4 point by a two-tuple \( (d, L_d) \), where \( d \) stands for the desired magnitude of the radius of the orbit and \( L_d \) is the desired angular momentum. For a circular orbit of a pre-specified radius around the triangular libration point the candidate Lyapunov function must hence incorporate three objectives:

- The velocity vector is perpendicular to the position vector,
  \[ r_{csb} \cdot \dot{r}_{csb} = 0 \]

- The angular momentum is constant,
  \[ r_{csb} \times \dot{r}_{csb} = 0 \]

- The orbital radius is of a specified magnitude,
  \[ ||r_{csb}|| = d \]

Based on these requirements, the following candidate Lyapunov function is selected

\[ V(r_{csb}, \dot{r}_{csb}) = \frac{1}{2}(||r_{csb} \cdot \dot{r}_{csb}||^2 + ||r_{csb} \times \dot{r}_{csb} - L_d||^2) \]

\[ + \frac{a}{2}(||r_{csb}|| - d)^2 \]

(6)
where $L_d$ is the desired angular momentum and $a > 0$ is a tuning parameter for control design purpose. Now

$$\frac{dV}{dt} = (r_{csb} \cdot \dot{r}_{csb})(||\dot{r}_{csb}||^2 + r_{csb} \cdot \dot{r}_{csb})$$
$$+ (r_{csb} \times \dot{r}_{csb} - L_d) \cdot (r_{csb} \times \dot{r}_{csb})$$

$$+ a(||r_{csb}|| - d)\frac{r_{csb} \cdot \dot{r}_{csb}}{||r_{csb}||}$$

Using the identity: $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$

$$(r_{csb} \times \dot{r}_{csb}) \cdot (r_{csb} \times \dot{r}_{csb}) = ||r_{csb}||^2(\dot{r}_{csb} \cdot \dot{r}_{csb})$$
$$- (r_{csb} \cdot \dot{r}_{csb})(r_{csb} \cdot \dot{r}_{csb})$$

and

$$\frac{dV}{dt} = (r_{csb} \cdot \dot{r}_{csb})||\dot{r}_{csb}||^2 + (r_{csb} \cdot \dot{r}_{csb})(r_{csb} \cdot \dot{r}_{csb})$$
$$+ ||r_{csb}||^2(\dot{r}_{csb} \cdot \dot{r}_{csb}) - (r_{csb} \cdot \dot{r}_{csb})(r_{csb} \cdot \dot{r}_{csb})$$
$$- L_d \cdot (r_{csb} \times \dot{r}_{csb}) + a(||r_{csb}|| - d)\frac{r_{csb} \cdot \dot{r}_{csb}}{||r_{csb}||}$$

(7)

Now we substitute (3) in the above equation and evaluate each term of the right hand side separately to obtain

$$\dot{r}_{csb} \cdot \dot{r}_{csb} = r_{csb} \cdot r_{csb} (\dot{r}_{c1b}^2 - \frac{k_m}{||r_{1sb}||^3} - \frac{k_m}{||r_{2sb}||^3})$$
$$+ \dot{r}_{csb} \cdot (\dot{r}_{c1b}^2 - \frac{k_m}{||r_{1sb}||^3})(r_{1sb})$$
$$- \frac{k_m}{||r_{2sb}||^3}(r_{2sb}) + \dot{r}_{csb} \cdot \bar{u}_b$$

(8)

$$r_{csb} \times \dot{r}_{csb} = r_{csb} \times (\dot{r}_{c1b}^2 r_{c1b} - 2\dot{r}_{csb} - \frac{k_m}{||r_{1sb}||^3}(r_{1sb}))$$
$$- \frac{k_m}{||r_{2sb}||^3}(r_{2sb}) + r_{csb} \times \bar{u}_b$$

(9)

Evaluating (4) at L4 we get the following identity

$$\dot{r}_{c1b}^2 - \frac{k_m}{||r_{1sb}||^3}(r_{1sb}) - \frac{k_m}{||r_{2sb}||^3}(r_{2sb}) = 0$$

(10)

To simplify further analysis, we now make the following assumption:

**Assumption 1:** The radius of the orbit of the spacecraft is small compared to the distance of the libration point from the two primaries such that $||r_{1sb}|| \approx ||r_{1cb}||(1 + \epsilon_1)$ and $||r_{2sb}|| \approx ||r_{2cb}||(1 + \epsilon_2)$ where $|\epsilon_1| < < 1$.

With the above assumption and using a binomial identity, (5) and (10) reduce to

$$\dot{r}_{csb} \cdot \dot{r}_{csb} = r_{csb} \cdot r_{csb} (\dot{r}_{c1b}^2 - \frac{k_m}{||r_{1sb}||^3} - \frac{k_m}{||r_{2sb}||^3})$$
$$+ \dot{r}_{csb} \cdot (\dot{r}_{c1b}^2 - \frac{k_m}{||r_{1sb}||^3})(r_{1sb})$$
$$+ \dot{r}_{csb} \cdot \bar{u}_b$$

$$r_{csb} \times \dot{r}_{csb} = r_{csb} \times (\dot{r}_{c1b}^2 r_{c1b} - 2\dot{r}_{csb} - \frac{k_m}{||r_{1sb}||^3}(r_{1sb}))$$
$$- \frac{k_m}{||r_{2sb}||^3}(r_{2sb}) + r_{csb} \times \bar{u}_b$$

(11)

Thus (7) reduces to

$$\frac{dV}{dt} = ||r_{csb}||^2(r_{csb} \cdot \dot{r}_{csb})(\dot{r}_{c1b}^2 + \frac{k_m}{||r_{1sb}||^3} - \frac{k_m}{||r_{2sb}||^3})$$
$$- \frac{k_m}{||r_{2sb}||^3}||r_{2sb}||^3 - \frac{k_m}{||r_{2sb}||^3}||r_{2sb}||^3$$
$$- L_d \cdot (r_{csb} \times \dot{r}_{csb}) + a(||r_{csb}|| - d)\frac{r_{csb} \cdot \dot{r}_{csb}}{||r_{csb}||}$$

Using the identity: $a \cdot (b \times c) = c \cdot (a \times b)$

$$\frac{dV}{dt} = ||r_{csb}||^2(r_{csb} \cdot \dot{r}_{csb})(\dot{r}_{c1b}^2 + \frac{k_m}{||r_{1sb}||^3} - \frac{k_m}{||r_{2sb}||^3})$$

(12)

Based on the need to render the time-derivative of the Lyapunov function to be negative semi-definite, we select the controller as

$$\bar{u}_b = -\beta (r_{csb} \cdot r_{csb})^2 (L_d \times r_{csb}) + \dot{r}_{csb} \times \eta$$
$$+ \frac{q \dot{r}_{csb} - a(||r_{csb}|| - d) \frac{r_{csb} \cdot \dot{r}_{csb}}{||r_{csb}||}}{||r_{csb}||^3}$$

(13)

where $\beta > 0$, $\eta \in \mathbb{R}^3$ and $q \in \mathbb{R}$ are to be chosen on further analysis. Note that the controller is such that the first term renders the fourth term of (12) as negative. However the other terms do remain. Hence the additional two degrees of freedom $\eta$ and $q$ appear in the proposed form.

Define

$$p \triangleq \dot{r}_{csb}^2 + \frac{||r_{csb}||^2}{||r_{1sb}||^3} - \frac{k_m}{||r_{1sb}||^3}$$

and rewrite

$$\frac{dV}{dt} = -\beta ||(r_{csb} \cdot r_{csb})^2 - (L_d \times r_{csb})||^2$$
$$+ ||r_{csb}||^2(r_{csb} \cdot \dot{r}_{csb})(p + q)$$
$$- (2\dot{r}_{csb} + \dot{r}_{csb} \times \eta) \cdot (L_d \times r_{csb})$$
$$+ (\dot{r}_{csb} \times \eta) \cdot (r_{csb} \cdot r_{csb})^2 - (q \dot{r}_{csb}) \cdot (L_d \times r_{csb})$$
Using the assumption made earlier (13) with constant angular momentum, the acceleration above should be equal to the centripetal acceleration

\[
\ddot{r}_{c_{sb}} = -\frac{||\dot{r}_{c_{sb}}||^2}{||r_{c_{sb}}||^3} r_{c_{sb}}
\]

Equations (19) and (22) imply that the orbits lying in the limit set E are circular with constant angular momentum. Further, the acceleration term (19) can be reduced as follows:

\[
\ddot{r}_{c_{sb}} = \phi^2 r_{c_{sb}} - \frac{km_1}{||r_{1_{cb}}||^3} r_{1_{cb}} - \frac{km_2}{||r_{2_{cb}}||^3} r_{2_{cb}} - \frac{||\dot{r}_{c_{sb}}||^2}{||r_{c_{sb}}||^2} r_{c_{sb}} - \beta (||\dot{r}_{c_{sb}}||^2 - L_d \times r_{c_{sb}}) - \frac{a(||r_{c_{sb}}|| - d)}{||r_{c_{sb}}||^3} r_{c_{sb}}
\]

Using the assumption made earlier (19) with \(c_1 << 1\) and equation (10) we have

\[
\ddot{r}_{c_{sb}} = -\beta (||\dot{r}_{c_{sb}}||^2 - L_d \times r_{c_{sb}}) - \frac{||\dot{r}_{c_{sb}}||^2}{||r_{c_{sb}}||^2} r_{c_{sb}} - \frac{a(||r_{c_{sb}}|| - d)}{||r_{c_{sb}}||^3} r_{c_{sb}}
\]

When the system approaches steady state the first term in above equation renders zero. So we have following

\[
\ddot{r}_{c_{sb}} = -\frac{||\dot{r}_{c_{sb}}||^2}{||r_{c_{sb}}||^2} r_{c_{sb}} - \frac{a(||r_{c_{sb}}|| - d)}{||r_{c_{sb}}||^3} r_{c_{sb}}
\]

Since the steady state orbit is a circular orbit with constant speed and angular momentum, the acceleration above should be zero, which results in

\[
||r_{c_{sb}}|| = d
\]

Therefore the steady state orbit is a circular orbit with constant angular momentum (22) and desired radius (25).  

**Remark 1:** The control law (13) is observed to have a feedback linearization structure. It can be written as

\[
\bar{u}_b = -\beta e_1 - \bar{a} e_2 - f(r_{c_{sb}}, \dot{r}_{c_{sb}}) - \frac{||\dot{r}_{c_{sb}}||^2}{||r_{c_{sb}}||^2} r_{c_{sb}}
\]

where \(e_1 = \dot{r}_{c_{sb}} - L_d \times r_{c_{sb}}\) and \(e_2 = \frac{||r_{c_{sb}}|| - d}{||r_{c_{sb}}||} r_{c_{sb}}\). The first term represents error in the current state and the second term, \(f(r_{c_{sb}}, \dot{r}_{c_{sb}})\) is the natural dynamics of the spacecraft given by the right hand side of (4) and the fourth term is the desired dynamics.

**V. NUMERICAL SIMULATIONS**

Simulations were performed for a spacecraft initially located near the Earth-Moon L4 point using a numerical integration scheme. For computing the control law it is required to know the distance of the spacecraft, \(||r_{1_{cb}}||\) and \(||r_{2_{cb}}||\), from both the primaries. To keep the control...
magnitude under bounds, a saturation was introduced and the control law implemented was

$$u = \begin{cases} \bar{u}_b & \text{if } ||\bar{u}_b|| \leq u_{\max} \\ u_{\max} & \text{if } ||\bar{u}_b|| > u_{\max} \end{cases}$$  \hspace{1cm} (26)$$

where $\bar{u}_b$ is given by (13). Two representative cases are presented here. The first one corresponds to the initial position in the plane of rotation of the two primaries. The second one corresponds to an arbitrary initial position in the 3D space. Parameters and conditions corresponding to both the cases are given below.

| Parameter | Case 1 (2D) | Case 2 (3D) | Units |
|-----------|-------------|-------------|-------|
| $r_{c_{sb}}$ at $t = 0$ | $[100000 \ 0 \ 0]$ | $[75000 \ 75000 \ 1000]$ | m |
| $\dot{r}_{c_{sb}}$ at $t = 0$ | $[0 \ 8000 \ 0]$ | $[100 \ 7500 \ 10]$ | m s$^{-1}$ |
| $L_d$ | $[0 \ 0 \ 800000000]$ | $[0 \ 0 \ 1000000]$ | m$^2$ s$^{-1}$ |
| $d$ | 100000 | 100000 | m |
| $u_{\max}$ | 500 | 1e-11 | m s$^{-2}$ |
| $\alpha$ | 1e-11 | m$^{-1}$ s$^{-1}$ |
| $\omega$ | $[0 \ 0 \ 2.66e-06]$ | rad s$^{-1}$ |
| $k$ | 6.633e-11 | N m$^2$ kg$^{-2}$ |
| $m_1$ | 5.972e24 | kg |
| $m_2$ | 7.3476e22 | kg |

**TABLE I**

PARAMETERS AND CONDITIONS FOR THE TWO CASES

Figure 2 shows the trajectory of the spacecraft in the first case. It can be seen that it takes about 13.5 hours to converge to the desired orbit. In the initial few seconds the control law shoots to a maximum and then becomes almost zero (fig 3). It remains close to zero till the spacecraft reaches the desired orbit. Near the desired orbit there is a spike in the control input. The spacecraft distance from the L4 point oscillates about the desired distance and then finally settles down to a circular orbit with radius within 0.5% of the desired radius. This shows that the initial control input provides the spacecraft a momentum such that it goes into a spiral orbit. The sudden spike in the control near desired orbit is similar to orbit insertion burns performed by rocket thrusters in spacecraft to enter an orbit around a planetary body.

The results for the first case of 2D motion are similar to the second case of 3D motion. The trajectory followed by the spacecraft depends on the initial error as it will contribute to the momentum imparted initially. The larger the error, larger is the momentum imparted and hence can lead to larger oscillations about the desired orbit before it stabilizes. Further, in the stable state a continuous control input is required to maintain the spacecraft in the orbit. This is the centripetal force and depends on the desired angular momentum.

**VI. CONCLUSION AND FUTURE WORK**

In this work, using Lyapunov theory we have shown that asymptotically stable orbits around the triangular libration point can be achieved with low thrust solutions. The initial thrust is required for a short time to transfer from an arbitrary initial position to a transfer orbit. Subsequent stationkeeping requires negligible thrust. This concept can be extended to transfer from one circular orbit to another.
This analysis did not include the perturbation effects by solar gravity and other planets. Further work could include these factors and modify the control law accordingly. Also the assumption about $\epsilon_i$ requires that we determine a region of stability around the L4 point. This can be done by observing the unstable manifolds of the collinear libration points. This can then lay the ground work for spacecraft formations.

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