ON THE FOURIER TRANSFORM OF SCHWARTZ 
FUNCTIONS ON RIEMANNIAN SYMMETRIC SPACES.

BY NILS BYRIAL ANDERSEN.

ABSTRACT. Consider the (Helgason-) Fourier transform on a Riemannian symmetric space $G/K$. We 
give a simple proof of the $L^p$-Schwartz space isomorphism theorem ($0 < p \leq 2$) for $K$-finite functions. 
The proof is a generalization of J.-Ph. Anker’s proof for $K$-invariant functions.

1. Introduction.

Let $G/K$ be a Riemannian symmetric space, where $G$ is a connected, non-compact semisimple 
Lie group with finite center, and $K$ is the maximal compact subgroup fixed by a Cartan involution. 
Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, the Lie algebra of $G$, and let $\mathfrak{a}$ be a maximal abelian 
subspace of $\mathfrak{p}$. Let $M = Z_K(\mathfrak{a})$, then $K/M$ is a symmetric space.

Let as usual $\rho$ denote half the sum of the positive roots, and let $W$ denote the Weyl group. Let 
$\varepsilon \geq 0$. Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$, let $C^p$ be the convex hull of the set $W \cdot \varepsilon \rho$ in $\mathfrak{a}^*$, and let 
$\mathfrak{a}_c^* = \mathfrak{a}^* + iC^p$ be the tube with basis $C^p$ in the complex dual $\mathfrak{a}_c^*$.

Let $\mathcal{H}$ denote the (Helgason-) Fourier transform on $G/K$, and consider the $L^p$-Schwartz spaces 
$\mathcal{S}^p(G/K)$, $0 < p \leq 2$, and the (semi-classical) Schwartz spaces $\mathcal{S}(\mathfrak{a}_c^* \times K/M)$. The Schwartz space 
isomorphism theorem ([Eg, Theorem 4.1.1]) states that:

**Theorem 1.** Let $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$. The Fourier transform $\mathcal{H}$ is a topological isomorphism 
between $\mathcal{S}^p(G/K)$ and $\mathcal{S}(\mathfrak{a}_c^* \times K/M)$, with the usual inverse.

J.-Ph. Anker gave in [An] a simple and beautiful proof of Theorem 1 when restricted to $K$-invariant 
functions, in which case the Fourier transform reduces to the spherical transform. In Section 2 of 
this note we extend Anker’s proof to $K$-finite functions for the real hyperbolic spaces (the rank 1 
case), and in Section 3 we sketch how to prove the general case.

**Remark:** This note consists of two slightly reworked Chapters from my "Progress Report": "On the Fourier transform on real Hyperbolic Spaces", Aarhus University, 1995. Recently, two articles 
on the same subject were published:

J. Jana and P. Sarkar, *On the Schwartz space isomorphism theorem for rank one symmetric space*, 
Proc. Indian Acad. Sci. Math. Sci., 117 (2007), no. 3, 333–348, and

J. Jana, *On the Schwartz space isomorphism theorem for the Riemannian symmetric spaces*, 
arXiv:1002.4855.

Further material may also be found in Jana’s Thesis: "Isomorphism of Schwartz spaces under 
Fourier transform", Indian Statistical Institute, Kolkata, July 2008.

The proofs by Jana and Sarkar use a reduction to the $K$-invariant result by Anker; indeed they 
show that the various spaces of a fixed $K$-type are isomorphic to the similar space of trivial $K$-type. 
The proof in the present note is a more straightforward generalization of Anker’s proof, with the
use of a (generalized) Abel (or Radon) transform and a cut-off function to show that the inverse transform is continuous from the Paley–Wiener space to the space of smooth functions with compact support on $G/K$. Unfortunately, I also conclude that the proof cannot be generalized to the general case without restriction to $K$-finite functions. However, I hope, and think, that the present notes could form a nice supplement to the papers and Thesis mentioned above.

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**2. The rank one case.**

In the following $c$ will denote (possibly different) positive constants.

**2.1. Notation and preliminaries.**

Let $G$ be a connected semisimple Lie group with finite center, and let $\theta$ be a Cartan involution of $G$. Then the fixed point group $K = G^\theta$ is a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the respective Lie algebras, we then have a Cartan decomposition of $\mathfrak{g}$ given by: $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. The Killing form on $\mathfrak{g}$ induces an $AdK$-invariant scalar product on $\mathfrak{p}$, and hence a $G$-invariant Riemannian metric on $X = G/K$. With this structure, $X = G/K$ becomes a Riemannian globally symmetric space of the noncompact type.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. This induces the Iwasawa decomposition $G = KAN$, where $A$ and $N$ are the Lie groups corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively. In this Section we will assume $G$ to be of rank one, i.e., dim $\mathfrak{a} = 1$. In the next Section we will sketch how to remove this condition. We identify $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ with $\mathbb{R}$ (for the unique positive root, $\alpha$ in $\mathfrak{a}^*$, we make the identification $\alpha = 1$, for the unique element $H$ in $\mathfrak{a}$ such that $\alpha(H) = 1$, we put $H = 1$. Then we define $a_t = \exp tH$. Thus every $g \in G$ can be written as $ka_t\mathfrak{n}$, where $k \in K$, $t \in \mathbb{R}$ and $\mathfrak{n} \in N$ are unique. We will denote the Iwasawa projections on the $K$-part and $A$-part by $\kappa(g) = \kappa(ka_t\mathfrak{n}) = k$ and $H(g) = H(ka_t\mathfrak{n}) = t$. Furthermore we will consider the ”reverse” Iwasawa decomposition, namely: $G = NAK$, where we will denote the projection onto the $A$-part by: $A(g) = A(na_tk) = t$. Remark that $a_{-t} = a_t^{-1}$ and $H(g) = -A(g^{-1}).$

Define $A_+ = \{a_t \in A | t > 0\}$ and $\mathbb{A}_+ = \{a_t \in A | t \geq 0\}$, corresponding to the open and closed positive Weyl chambers, then we have the Cartan (or Polar) decomposition of $G$ given by: $G = K\mathbb{A}_+K$, that is, every element $g \in G$ can be written as $k_1a_tk_2$, where $t \in \mathbb{R}_+$ is unique and $k_1, k_2 \in K$. We will define $|g| = |k_1a_tk_2| = t$. For the rank one case we have the basic estimate: $|H(g)| \leq |g|$ (Given a finite-dimensional irreducible representation of $G$, $H(g)$ and $|g|$ can be found using a normalized highest weight vector). In the Cartan decomposition the Haar measure is given by:

$$\int_G f(x) dx = c \int_K \int_{\mathbb{R}_+} \int_K f(k_1a_tk_2) \sinh^n t dk_1 dt dk_2,$$

where $n = \dim \mathfrak{n}$. In the hyperbolic case, $G = SO(p, 1)$, $K = SO(p)$, we get: $n = p - 1$. In $\mathfrak{a}^* \cong \mathbb{R}$ we define the element $\rho(H) = \frac{1}{2} \text{tr}(adH_{\mathfrak{n}})$, $H \in \mathfrak{a}$, or under the identification with $\mathbb{R}$: $\rho = \frac{1}{2} n$. We have the obvious estimate: $0 \leq \sinh^n t \leq ce^{2\rho t}$.

Define $M = Z_K(\mathfrak{a})$, then $B = K/M$ is a symmetric space. We will sometimes identify functions on $X = G/K$ ($B = K/M$) as right-$K$-invariant (right-$M$-invariant) functions on $G$ ($K$). Then the invariant measure on $X$ is given by:

$$\int_X f(x) dx = c \int_K \int_{\mathbb{R}_+} f(k_1 k) \sinh^n t dk dt,$$

for some constant. This is the Cartan decomposition of the measure on $X$. Define $A(x, b) = A(gK, kM) = A(k^{-1}g)$. 


Definition 2.1.1 (The Fourier transform on $G/K$). For $f \in C_c^\infty(G/K)$, we define the Fourier transform by:

$$\mathcal{H} f(\nu, b) = \hat{f}(\nu, b) = \int_X f(x) e^{-(\nu + \rho)(A(x, b))} dx,$$

for all $\nu \in \mathbb{C}$, $b \in B$.

Remarks: Let $f \in C_c^\infty(K \backslash G/K)$. Then: (for $k \in K$)

$$\int_X f(x) e^{-(\nu + \rho)(A(x, b))} dx = \int_X f(k \cdot x) e^{-(\nu + \rho)(A(x, b))} dx = \int_X f(x) e^{-(\nu + \rho)(A(x, k \cdot b))} dx.$$

Since $K$ acts transitively on $B$, we see that $\hat{f}$ is independent of $b \in B$. Integrating over $K$ we get:

$$= \int_K \int_X f(x) e^{-(\nu + \rho)(A(x, kM))} dx dk = \int_X f(x) \int_K e^{-(\nu + \rho)(A(x, kM))} dk dx.$$

Let $x = gK$, we then recognize the spherical function $\varphi_{-\nu}$, $\nu \in \mathbb{C}$ on $G$:

$$\varphi_{-\nu}(g) = \int_K e^{-(\nu + \rho)(A(kg))} dk,$$

see [He1, Theorem 4.3]. For $f \in C_c^\infty(K \backslash G/K)$, the Fourier transform thus reduces to the spherical transform:

$$\mathcal{H} f(\nu) = \int_X f(x) \varphi_{-\nu}(x) dx.$$

treated in [An], [He1], [GV], etc.

From [He2, Chapter III, §1, §5], we get Theorems 2.1.2, 2.1.3 and 2.1.5, where $c(\cdot)$ is the Harish-Chandra $c$-function:

Theorem 2.1.2 (The inversion formula). Let $f \in C_c^\infty(X)$, $x \in X$. Then:

$$f(x) = c \int_{\mathbb{R}^+ \times B} e^{(i
u + \rho)(A(x, b))} \hat{f}(i\nu, b) |c(i\nu)|^{-2} d\nu db.$$

Theorem 2.1.3 (The Plancherel formula). Let $f_1, f_2 \in C_c^\infty(X)$, then:

$$\int_X f_1(x) \overline{f_2(x)} dx = c \int_{\mathbb{R}^+ \times B} \hat{f_1}(i\nu, b) \overline{\hat{f_2}(i\nu, b)} |c(i\nu)|^{-2} d\nu db.$$

The Fourier transform extends to an isometry of $L^2(X)$ onto $L^2(i\mathbb{R}^+ \times B, c|c(i\nu)|^{-2} d\nu db)$.

A $C^\infty$-function $\psi(z, b)$ on $\mathbb{C} \times B$, holomorphic in $z$, is called a holomorphic function of uniform exponential type $R$, if there exists a constant $R \geq 0$ such that for each $N \in \mathbb{N}$ we have:

$$\sup_{z \in \mathbb{C}, b \in B} e^{-R|\text{Re}z|}(1 + |z|)^N |\psi(z, b)| < \infty.$$
Definition 2.1.4. The space of holomorphic functions of uniform exponential type \( R \) will be denoted \( H^R(C \times B) \). Furthermore denote by \( H(\mathbb{C} \times B) \) their union over all \( R > 0 \). Let \( H_c(\mathbb{C} \times B) \) denote the space of functions \( \psi \in H(\mathbb{C} \times B) \) satisfying the symmetry condition \((SC1)\):

\[
\int_B e^{(-\nu+\rho)(A(x,b))} \psi(-\nu,b)db = \int_B e^{(\nu+\rho)(A(x,b))} \psi(\nu,b)db, \quad \nu \in \mathbb{C}, \ x \in X.
\]

We usually call \( H(\mathbb{C} \times B) \) the Paley-Wiener space. For \( \psi \in H_c(\mathbb{C} \times B) \) independent of \( b \), \((SC1)\) reduces to the symmetry condition: \( \varphi_{-\nu}(x)\psi(-\nu) = \varphi_{\nu}(x)\psi(\nu) \), where \( \varphi_{\nu} \) is the spherical function indexed by \( \nu \). Since \( \varphi_{-\nu} = \varphi_{\nu} \) this again reduces to \( \psi \) being an even function of \( \nu \), that is, the usual symmetry condition for the spherical transform (Weyl group invariance), see [He1, p. 450].

Consider \( f \in C^\infty_c(X) \). By the Cartan (polar) decomposition, the polar distance from the point \( x = gK = ka \) to the origin \( x_a = eK \) is \( |x| = |t| \). Let \( R > 0 \). We say that \( \text{supp}(f) \subset \tilde{B}(0,R) \) if and only if the function \( f \) has support inside \( K \times \tilde{B}(0,R) \) or if and only if \( f(x) = 0 \) for \( |x| > R \).

The Paley-Wiener Theorem 2.1.5. The Fourier transform is a bijection of the space \( C^\infty_c(X) \) onto the space \( H_c(\mathbb{C} \times B) \), the inverse transform being given by Theorem 2.1.2. Moreover \( \text{supp}(f) \subset \tilde{B}(0,R) \) if and only if \( f \in H^\infty_c(\mathbb{C} \times B) \).

We decompose the Fourier transform into two transforms. Let \( f \in C^\infty_c(G/K) \), then the Radon transform is defined as follows:

\[
\mathcal{R}f(t,k) = e^{i\rho} \int_N f(ka_t n)dn.
\]

Proposition 2.1.6. The Radon transform \( \mathcal{R} \) maps \( C^\infty_c(G/K) \) into \( C^\infty_c(\mathbb{R} \times B) \). If \( \text{supp}f \subset \tilde{B}(0,R) \), then \( \text{supp}\mathcal{R}f \subset \tilde{B}(0,R) \times B \).

Proof. Let \( k, k_1, k_2 \in K, n \in N, t \in \mathbb{R} \), we then get: \( |ka_t n| \geq |t| = |k_1 a_t k_2| \), hence \( \text{supp}\mathcal{R}f \subset \text{supp}f \times B \). \( \square \)

Remark: For \( f \in C^\infty_c(K \setminus G/K) \), the Radon transform reduces to the Abel transform, see [An].

Let \( \phi \in C^\infty_c(\mathbb{R} \times B) \), then the "classical" Fourier transform on \( \mathbb{R} \times B \) is defined as:

\[
\mathcal{F}\phi(\nu, b) = \int_\mathbb{R} \phi(t,b)e^{-i\nu t}dt, \quad \nu \in i\mathbb{R}, \ b \in B.
\]

Let \( \psi \) be a nice function on \( i\mathbb{R} \times B \), then the "classical" inverse Fourier transform is defined by:

\[
\mathcal{F}^{-1}\psi(t,b) = \frac{1}{2\pi} \int_\mathbb{R} \psi(i\nu,b)e^{i\nu t}d\nu, \ t \in \mathbb{R}, \ b \in B.
\]

We then have: \( (b = kM) \)

\[
\hat{f}(\nu, kM) = \int_\mathbb{R} e^{-i\nu t} \left\{ e^{i\rho} \int_N f(ka_t n)dn \right\} dt,
\]

i.e., we have the following commutative diagram:

\[
\begin{array}{ccc}
H_c(\mathbb{C} \times B) & \xrightarrow{\mathcal{H}} & C^\infty_c(G/K) \\
\mathcal{R} & \xrightarrow{\mathcal{F}} & \mathcal{R}C^\infty_c(G/K) \subset C^\infty_c(\mathbb{R} \times B)
\end{array}
\]

The commutativity is an easy consequence of the definitions of the various transforms. Furthermore we get:
Lemma 2.1.9. The spherical functions are all bi-

Proof. The Paley-Wiener Theorems above and below.

Remark: We have indirectly introduced a symmetry condition for functions in $C_c^\infty(\mathbb{R} \times B)$, namely that the Fourier transformed function should be in $H_c(\mathbb{R} \times B)$. For functions of a specific $K$-type ($K$ acting on the $B$-variable), this symmetry condition becomes somewhat easier to describe, see later. For the trivial $K$-type, or for $B$-invariant functions, it reduces to: $g(-t) = g(t), g \in C_c^\infty(\mathbb{R})$, or $g$ even, that is, as in [An], where the Abel transform maps $C_c^\infty(K \backslash G/K)$ onto even functions in $C_c^\infty(\mathbb{R})$.

Theorem 2.1.8 (A Paley-Wiener Theorem on $\mathbb{R} \times B$). Let $\phi \in C_c^\infty(\mathbb{R} \times B)$ have support in $B(0, R) \times B$, and let $f(\nu, b) = \int_{\mathbb{R}} \phi(t, b)e^{-\nu t}dt$, $\nu \in i\mathbb{R}, b \in B$. Then $f \in C_c(\mathbb{C} \times B)$, and $f(\cdot, b)$ is an entire function for fixed $b$, such that for all $N \in \mathbb{N}$, we have:

$$\sup_{z \in \mathbb{C}, b \in B} e^{-R|\text{Re}z|}(1 + |z|)^N|f(z, b)| < \infty.$$  

Conversely, let $f \in C_c(\mathbb{C} \times B)$ satisfy the above. Then there exists a function $\phi \in C_c^\infty(\mathbb{R} \times B)$, with support in $B(0, R) \times B$, such that $f = \mathcal{F}\phi$.

Proof. An easy generalization of [Ru, Theorem 7.22].

We will in the following sections need some estimates on the spherical functions $\varphi_\nu$ introduced before and on the Harish-Chandra c-function.

Lemma 2.1.9. The spherical functions are all bi-$K$-invariant, $\varphi_{-\nu} = \varphi_\nu$, and:

i) For $t \geq 0$, $\exists c > 0$ such that: $e^{-\rho t} < \varphi_\nu(a_t) < c(1 + t)e^{-\rho t}$.

ii) Let $\epsilon \geq 0$, $|\text{Re} \nu| \leq \epsilon \rho$ and $t \geq 0$. Then: $|\varphi_\nu(a_t)| \leq c(1 + t)e^{(\epsilon - 1)\rho t}$.

For the $c$-function we have:

iii) Let $\text{Re} \nu \geq 0$. $\exists \gamma \geq 0$ such that: $|c(\nu)|^{-1} \leq \gamma(1 + |\nu|)^{\frac{3}{2}}$.

Proof. i) [He1, Chap IV, Ex.B1; GV, Sect.4.6].

ii) Assume $\nu \geq 0$: Then:

$$\varphi_\nu(a_t) \leq e^{\rho t} \varphi_\nu(a_t) \leq ce^{\rho t}(1 + t)e^{-\rho t} = c(1 + t)e^{(\epsilon - 1)\rho t}.$$  

iii) Properties of the $\Gamma$-function, see [He1, Chapter IV, Proposition 7.2].

2.2. Schwartz spaces and dense subspaces.

We have come to the definition of the Schwartz spaces. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. The elements of $U(\mathfrak{g})$ act on $C_c(\mathbb{R})$ as differential operators on both sides. We shall write $f(D; g; E)$ for the action of $(D, E) \in U(\mathfrak{g}) \times U(\mathfrak{g})$ on $f \in C_c(\mathbb{R})$ at $g \in G$, more explicitly we have:

$$f(D; g; E) = \left(\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_d} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_e}\right)_{s_1=\cdots=s_d=t_1=\cdots=t_e=0} \times f((\exp s_1X_1) \cdots (\exp s_dX_d)g(\exp t_1Y_1) \cdots (\exp t_eY_e)).$$

if $D = X_1 \cdots X_d, E = Y_1 \cdots Y_e \in \mathfrak{g}$.
Definition 2.2.1. Let $0 < p \leq 2$. The $L^p$-Schwartz space $S^p(G/K)$ is the space of all functions $f \in C^\infty(G/K)$, such that

$$\sup_{g \in G} (1 + |g|)^N \varphi_\sigma(g)^{-\frac{2}{p}} |f(D; g; E)| < \infty,$$

for any $D, E \in U(g)$, and any nonnegative integer $N$. Here $\varphi_\sigma$ is the spherical function with $\nu = 0$.

The topology of $S^p(G/K)$ is defined by the seminorms:

$$\sigma^p_{D,E,N}(f) = \sup_{g \in G} (1 + |g|)^N \varphi_\sigma(g)^{-\frac{2}{p}} |f(D; g; E)|.$$

Remarks

1) Consider the Cartan decomposition of the integral of $X$, then:

$$\int_X f(x)dx = \int_{K \times \mathbb{R}^+} f(ka_\epsilon) \sinh^n t \,dkdt.$$

From above we see that the factor $\varphi_\sigma(g)^{-\frac{2}{p}}$ will control the factor $\sinh^n t$, and hence the definition resembles the natural definition of a Schwartz space on $K \times \mathbb{R}^+$.

2) In fact we see that: $S^p(G/K) \subset L^q(G/K)$ for $0 < p \leq q \leq 2$, while $S^p(G/K) \not\subset L^q(G/K)$ for $0 < q < p \leq 2$. For $0 < p \leq q < 2$ the inclusion above is continuous.

3) Obviously $C^\infty_c(G/K) \subset S^p(G/K)$ for $0 < p \leq 2$, and the inclusion is continuous.

Lemma 2.2.2. Let $0 < p \leq q \leq 2$. Then:

(i) $S^p(G/K)$ is a Fréchet space.

(ii) $C^\infty_c(G/K)$ is a dense subspace of $S^p(G/K)$.

(iii) $S^p(G/K)$ is a dense subspace of $S^q(G/K)$.

(iv) $S^p(G/K)$ is a dense subspace of $L^q(G/K)$.

(v) $S^p(K\backslash G/K)$ is a closed subspace of $S^q(G/K)$ in the Fréchet topology.

Proof. See [GV, sect. 6.1, 7.8].

For $E \in U(\mathfrak{t})$ define: $Ef(k) = f(k; E)$. Let $\Omega_K$ denote the Casimir element of $U(\mathfrak{t})$, then for the Laplace-Beltrami operator $\Delta_B$ on $B$ we have (modulo a constant): $\Delta_B f = f(\cdot; \Omega_K) = f(\Omega_K; \cdot)$ ($\Omega_K \in \mathfrak{t}(\mathfrak{t})$). Furthermore $(P \left( \frac{\partial}{\partial \nu} \right), E)$ will denote differentiation with $P \left( \frac{\partial}{\partial \nu} \right)$ on the $\nu$-variable and with $E$ on the $k$-variable.

Definition 2.2.3. Fix $\varepsilon \geq 0$. Let $i\mathbb{R}_\varepsilon = i\mathbb{R} + [-\varepsilon, \varepsilon]$. The Schwartz space $S(i\mathbb{R}_\varepsilon \times B)$ consists of all complex valued functions $f \in C^\infty(i\mathbb{R}_\varepsilon \times B)$ such that:

(i) For fixed $b \in B$, $f(\cdot, b)$ is holomorphic in the interior of $i\mathbb{R}_\varepsilon$.

(ii) $f$ and all its derivatives extend continuously to $i\mathbb{R}_\varepsilon \times B$.

(iii) For any polynomial $P$, any $E \in U(\mathfrak{t})$ and nonnegative integer $N$, we have the estimate:

$$\sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} (1 + |\nu|)^N \left| P \left( \frac{\partial}{\partial \nu} \right), E \right| f(\nu, k) < \infty.$$

The topology of $S(i\mathbb{R}_\varepsilon \times B)$ is defined by the seminorms:

$$\tau^p_{P,E,N}(f) = \sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} (1 + |\nu|)^N \left| P \left( \frac{\partial}{\partial \nu} \right), E \right| f(\nu, k) < \infty.$$
Remarks: For $\varepsilon = 0$ condition i) is empty. For $\varepsilon > 0$ ii) and iii) are equivalent to:

(iv) For any polynomial $P$, any $E \in U(\mathfrak{t})$ and any nonnegative integer $N$ we have the estimate:

$$
\sup_{\nu \in i\mathbb{R}^n, k \in K} (1 + |\nu|)^N \left| \left( P \left( \frac{\partial}{\partial \nu} \right), E \right) f(\nu, k) \right| < \infty.
$$

To see this, observe that for fixed $k \in K$, $(P \left( \frac{\partial}{\partial \nu} \right), E) f(\nu, k)$ is bounded on $i\mathbb{R}^n$ for all polynomials $P$. $i\mathbb{R}^n$ is convex, hence by the Mean Value Theorem we get:

$$
|f(x_1, k) - f(x_2, k)| \leq \left\{ \sup_{\nu \in i\mathbb{R}^n} |\nabla f(\nu, k)| \right\} |x_1 - x_2|, \quad x_1, x_2 \in i\mathbb{R}^n,
$$

where $\nabla$ is the operator $\nabla = \frac{dt}{db}$, which shows that $f(\cdot, k)$ is uniformly continuous on $i\mathbb{R}^n$, hence by density $f(\cdot, k)$ extends to a uniformly continuous function on $i\mathbb{R}^n$. This can also be done for all derivatives of $f$. We also note that the space defined above is homeomorphic to the space of functions $f$ in $C^\infty(i\mathbb{R}_+ \times \mathbb{B})$ satisfying i), ii) and:

(v) For any polynomial $P$ and any nonnegative integers $M$ and $N$ we have the estimate:

$$
\sup_{\nu, k \in K} (1 + |\nu|)^N \left| \left( P \left( \frac{\partial}{\partial \nu} \right), \Omega_K^M \right) f(\nu, k) \right| < \infty.
$$

Here we define the topology by the seminorms:

$$
\tau_{P,M,N}^\varepsilon(f) = \sup_{\nu, k \in K} (1 + |\nu|)^N \left| \left( P \left( \frac{\partial}{\partial \nu} \right), \Omega_K^M \right) f(\nu, k) \right| < \infty,
$$

that is, restriction to powers of the Casimir element does not alter the space or topology, see e.g. [Eg, p.193].

Let $\mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$ denote the space of functions $\psi \in \mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$ satisfying the symmetry condition (SC2):

$$
\int_B e^{(-\nu+\rho)(A(x,b))} \psi(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} \psi(\nu, b) db, \quad \nu \in i\mathbb{R}_+, x \in X.
$$

Lemma 2.2.4. Let $\varepsilon \geq 0$.

(i) $\mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$ is a Fréchet space.

(ii) $\mathcal{H}(\mathbb{C} \times \mathbb{B})$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$.

Proof. (i) Clearly $\mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$ is a closed subspace of the Fréchet space $\mathcal{S}(i\mathbb{R}_+ \times \mathbb{B})$, hence it is a Fréchet space in the inherited topology.

(ii) Let $\mathcal{S}(\mathbb{R} \times \mathbb{B})$ be the "classical" Schwartz space on $\mathbb{R} \times \mathbb{B}$, i.e. the space of functions $f \in C^\infty(\mathbb{R} \times \mathbb{B})$ such that:

$$
\sup_{t \in \mathbb{R}, k \in K} \left| (1 + |t|)^N \left( P \left( \frac{\partial}{\partial t} \right), E \right) f(t, k) \right| < \infty,
$$

for any polynomial $P, E \in U(\mathfrak{t})$ and $N \in \mathbb{N}$. We consider the function $t \mapsto \cosh(\varepsilon \rho t)$. Let $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times \mathbb{B})$ be the space of functions $f \in C^\infty(\mathbb{R} \times \mathbb{B})$ such that:

$$
\sup_{t \in \mathbb{R}, k \in K} \left| (1 + |t|)^N \cosh(\varepsilon \rho t) \left( P \left( \frac{\partial}{\partial t} \right), E \right) f(t, k) \right| < \infty.
$$
for any polynomial $P, E \in U(t)$ and $N \in \mathbb{N}$. Then $\mathcal{S}_{\rho}(\mathbb{R} \times B)$ is a Fréchet space for the obvious topology. We now have two topological isomorphisms:

1) the map $f \mapsto \cosh(\epsilon \rho) f$ between $\mathcal{S}_{\rho}(\mathbb{R} \times B)$ and the classical Schwartz space $\mathcal{S}(\mathbb{R} \times B)$. (obvious)

2) the "classical" Fourier transform $\mathcal{F}$ between $\mathcal{S}_{\rho}(\mathbb{R} \times B)$ and $\mathcal{S}(i\mathbb{R}_c \times B)$.

Consider:

$$\hat{f}(\nu, b) = \int_{\mathbb{R}} f(t, b)e^{-\nu t} dt.$$ 

The factor $e^{-\nu t}$ is bounded by $2\cosh(\epsilon \rho t)$ for $|\text{Re} \nu| \leq \epsilon \rho$, hence we can write the above as:

$$\hat{f}(\nu, b) = \int_{\mathbb{R}} 2 \cosh(\epsilon \rho t) f(t, b) \frac{1}{2 \cosh(\epsilon \rho t)} e^{-\nu t} dt.$$ 

For fixed $k \in K$, we see by Morera’s Theorem that $\hat{f}(\cdot, k)$ is holomorphic in $i\mathbb{R}_c$. Now consider:

$$\left| (1 + \nu)^N \left( P \left( \frac{\partial}{\partial \nu} \right), E \right) \hat{f}(\nu, k) \right|$$

$$= \left| \int_{\mathbb{R}} \left( \left( 1 + \left( \frac{d}{dt} \right) \right)^N P(-t)Ef(t, k) \right) e^{-\nu t} dt \right|$$

$$\leq c \sup_{\nu \in \mathbb{R}, b \in B} \cosh(\epsilon \rho t)(1 + |t|)^2 \left( 1 + \left( \frac{d}{dt} \right) \right)^N \left| P(-t)Ef(t, k) \right|$$

$$\leq c \sup_{\nu \in \mathbb{R}, b \in B} \cosh(\epsilon \rho t)(1 + |t|)^M \left( \hat{\mu} \left( \frac{\partial}{\partial t} \right), E \right) f(t, k) < \infty,$$

for some polynomial $\hat{\mu}$ and $M \in \mathbb{N}$, i.e., $\hat{f} \in \mathcal{S}(i\mathbb{R}_c \times B)$ and the Fourier transform is continuous as an operator from $\mathcal{S}_{\rho}(\mathbb{R} \times B)$ to $\mathcal{S}(i\mathbb{R}_c \times B)$. Now consider the inverse Fourier transform:

$$\tilde{f}(t, k) = \frac{1}{2\pi} \int_{\mathbb{R}} f(i\nu, k)e^{i\nu t} d\nu.$$ 

Let $P$ and $Q$ be polynomials, then by Cauchy’s Theorem, shifting integral from $i\nu$ to $i\nu + \epsilon \rho$, we get:

$$P(t) \cosh(\epsilon \rho t) \left( Q \left( \frac{\partial}{\partial \nu} \right), E \right) \tilde{f}(t, k) =$$

$$\frac{1}{4\pi} \int_{\mathbb{R}} P \left( i \frac{\partial}{\partial \nu} \right) \{Q(i\nu - \epsilon \rho)Ef(i\nu - \epsilon \rho, k) + Q(i\nu + \epsilon \rho)Ef(i\nu + \epsilon \rho, k)\} e^{i\nu t} d\nu,$$

i.e.,

$$\sup_{t \in \mathbb{R}} \left| P(t) \cosh(\epsilon \rho t) \left( Q \left( \frac{\partial}{\partial \nu} \right), E \right) \tilde{f}(t, k) \right|$$

$$\leq c \sup_{\nu \in i\mathbb{R}_c} \left| (1 + |\nu|^2) P \left( -\frac{\partial}{\partial \nu} \right) Q(\nu) Ef(\nu, k) \right| < \infty,$$

and we see that the inverse Fourier transform also is continuous.

Since $C_c^\infty(\mathbb{R} \times B)$ is a dense subspace of $\mathcal{S}(\mathbb{R} \times B)$, we get from above that $C_c^\infty(\mathbb{R} \times B)$ is a dense subspace of $\mathcal{S}_{\rho}(\mathbb{R} \times B)$, and hence from the Paley-Wiener Theorem on $\mathbb{R} \times B$ we conclude ii).
Let in the following $\varepsilon \geq 0$. We want to show that $\mathcal{H}_e(C \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$, that is, we have to look at functions in each space satisfying the symmetry conditions (SC1-2). We will make this problem a little easier by looking at $K$-types for the left-regular representation of $K$ (l) on the Fréchet space $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ (acting on the second variable). It is straightforward to check that this representation is a smooth Fréchet representation of $K$. Let $\hat{K}$ denote the set of $\mathbb{R}$-valued unitary irreducible representations $(\delta, V_\delta)$ of $K$. Define $V_\delta^M = \{v \in V_\delta | \delta(m)v = v, m \in M\}$ and $\hat{K}_M = \{\delta \in \hat{K} | V_\delta^M \neq 0\}$. For $\delta \in \hat{K}$, let $d(\delta)$ and $\chi_\delta$ denote the dimension and character of $\delta$. Let $S(i\mathbb{R}_\varepsilon \times B)_\delta$ be the closed subspace of $S(i\mathbb{R}_\varepsilon \times B)$ consisting of functions of $K$-type $\delta$. The continuous projection of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ onto $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$ is given by:

$$P_\delta f(\cdot, b) = f_\delta(\cdot, b) = d(\delta) \int_K \chi_\delta(k^{-1}) f(\cdot, k^{-1} \cdot b) dk.$$ [He1, Chapter IV, Lemma 1.7]. The space of $K$-finite functions in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ is given as:

$$\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K \equiv \{ f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B) | \dim \text{span}(l(K)f) < \infty \}.$$ 

Then $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K \iff f = \sum_{\delta \in \hat{K}_M} f_\delta$, where the sum is finite. For $\delta \in \hat{K}$ we will also consider the contragredient representation $\delta^\vee \in \hat{K}$. $\delta(k)$ can be identified as the operator in $\text{Hom}(V^\vee_\delta, V^\vee_\delta)$ defined by $\delta(k) = \delta(k^{-1})^t$, where $V^\vee_\delta$ is the dual space of $V_\delta$ and $t$ denotes transpose. We remark that $\delta \in \hat{K}_M \iff \delta^\vee \in \hat{K}_M$. We have the following important result, see [He1, Chapter V, Theorem 3.1]:

**Theorem 2.2.5.** $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$. Furthermore we have the expansion $f = \sum_{\delta \in \hat{K}_M} f_\delta = \sum_{\delta \in \hat{K}_M} f_{\delta^\vee}$, where the sums are absolutely convergent.

We easily see that $\mathcal{H}(C \times B)$ is invariant under $P_\delta$, hence we will consider the subset $\mathcal{H}(C \times B)_{\delta} = \mathcal{H}(C \times B) \cap \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta}$. Since $P_\delta$ is continuous we see that $\mathcal{H}(C \times B)_{\delta}$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta}$. Denote by $\mathcal{H}(C \times B)_{\delta,e}$, respectively by $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$, the set of functions in $\mathcal{H}(C \times B)_{\delta}$ respectively in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta}$ satisfying (SC2) (a function in $\mathcal{H}(C \times B)_{\delta}$ that satisfies (SC2) will automatically by holomorphicity satisfy (SC1)). We want to show that $\mathcal{H}(C \times B)_{\delta,e}$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$. For $\delta \in \hat{K}_M$, and then by Theorem 2.2.5 conclude that $\mathcal{H}_e(C \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{e}$.

Let $\delta \in \hat{K}_M$ act on $V_\delta$. Looking at matrix entries, we can define the spaces $\mathcal{H}(C \times B, \text{Hom}(V_\delta, V_\delta))$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta))$ of Paley-Wiener functions on $C \times B$, respectively Schwartz functions on $i\mathbb{R}_\varepsilon \times B$, taking values in $\text{Hom}(V_\delta, V_\delta)$. We equip $\mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta))$ with the obvious topology. Define $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta = \{ F \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta)) : F(\cdot, k \cdot b) = \delta(k) F(\cdot, b) \}$, and define $\mathcal{H}(C \times B)^\delta = \mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta \cap \mathcal{H}(C \times B, \text{Hom}(V_\delta, V_\delta)).$ For $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)$, we define:

$$P^\delta f(\cdot, b) = f^\delta(\cdot, b) = d(\delta) \int_K \delta(k) f(\cdot, k^{-1} \cdot b) dk.$$ 

We easily see that $P^\delta$ takes $\mathcal{H}(C \times B)$ into $\mathcal{H}(C \times B)^\delta$, and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ into $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$. Denote by $\mathcal{H}(C \times B)_{e,\delta}$, respectively by $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{e,\delta}$, the set of functions $F$ in $\mathcal{H}(C \times B)^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$ satisfying:

$$\int_B e^{(-\nu + \rho)(A(x,b))} F(-\nu, b) db = \int_B e^{(\nu + \rho)(A(x,b))} F(\nu, b) db, \quad \nu \in i\mathbb{R}_\varepsilon, x \in X.$$
Proposition 2.2.6. Let $\delta \in \hat{K}_M$.

(i) The map:

$$Q : F(\nu, b) \to \text{Tr}(F(\nu, b))$$

is a homeomorphism of $S(i\mathbb{R}_e \times B)^{\delta}$ onto $S(i\mathbb{R}_e \times B)^{\delta}_e$, with inverse $f \to P^{\delta}f = f^{\delta}$. Furthermore $Q$ takes $\mathcal{H}(\mathbb{C} \times B)^{\delta}$ onto $\mathcal{H}(\mathbb{C} \times B)^{\delta}_e$. Here $\text{Tr}$ denote trace in $\text{Hom}(V_{\delta}, V_{\delta})$.

(ii) The image of $\mathcal{H}(\mathbb{C} \times B)^{\delta}_e$, respectively of $S(i\mathbb{R}_e \times B)^{\delta}_e$, under $Q$ is $\mathcal{H}(\mathbb{C} \times B)^{\delta}_e$, respectively $S(i\mathbb{R}_e \times B)^{\delta}_e$.

(iii) The maps:

$$P_\delta : S(i\mathbb{R}_e \times B) \to S(i\mathbb{R}_e \times B)^{\delta}_e \text{ and } P^{\delta} : S(i\mathbb{R}_e \times B) \to S(i\mathbb{R}_e \times B)^{\delta}_e$$

are continuous open surjections, and the images are closed in $S(i\mathbb{R}_e \times B)$ and $S(i\mathbb{R}_e \times B, \text{Hom}(V_{\delta}, V_{\delta}))$ respectively.

Proof. (i), (iii) As in [He1,p.395f].

(ii) Consider $f \in S(i\mathbb{R}_e \times B)^{\delta}_e$. In (SC2) replace $x$ by $k^{-1} \cdot x$, use $A(k^{-1} \cdot x, b) = A(x, k \cdot b)$, make the substitution $b \mapsto k^{-1} \cdot b$, multiply by $\delta(k)$ and integrate over $K$. The result is:

$$\int_{B} e^{(-\nu+\rho)(A(x,b))}f^{\delta}(-\nu, b)db = \int_{B} e^{(\nu+\rho)(A(x,b))}f^{\delta}(\nu, b)db, \quad \nu \in i\mathbb{R}_e, \ x \in X,$$

and thus $P^{\delta}$ takes $S(i\mathbb{R}_e \times B)^{\delta}_e$ into $S(i\mathbb{R}_e \times B)^{\delta}_e$. Taking trace we see that $Q$ maps $S(i\mathbb{R}_e \times B)^{\delta}_e$ into $S(i\mathbb{R}_e \times B)^{\delta}_e$. \hfill \Box

From Proposition 2.2.6i), we see that $\mathcal{H}(\mathbb{C} \times B)^{\delta}$ is a dense subspace of $S(i\mathbb{R}_e \times B)^{\delta}$. Furthermore, given $F \in S(i\mathbb{R}_e \times B)^{\delta}_e$, we can find $f \in S(i\mathbb{R}_e \times B)^{\delta}_e$ such that $F = f^{\delta}$. We will use this identification in the following. Consider $\delta \in \hat{K}_M$. Define the evaluation map:

$$(evf)(\nu) = f^{\delta}(\nu) \equiv f^{\delta}(\nu, eM).$$

We see that $\delta(m)f^{\delta}(\nu) = f^{\delta}(\nu)$, hence $f^{\delta}(\nu) \in \text{Hom}(V_{\delta}, V_{\delta}^M)$. The evaluation map is a homeomorphism between $S(i\mathbb{R}_e \times B)^{\delta}_e$ and $S(i\mathbb{R}_e, \text{Hom}(V_{\delta}, V_{\delta}^M)) \equiv S(i\mathbb{R}_e)^{\delta}_e$, and between $\mathcal{H}(\mathbb{C} \times B)^{\delta}$ and $\mathcal{H}(\mathbb{C}, \text{Hom}(V_{\delta}, V_{\delta}^M)) \equiv \mathcal{H}(\mathbb{C})^{\delta}$. Let’s consider $f \in S(i\mathbb{R}_e \times B)^{\delta}_e$. Then:

$$\int_{B} e^{(-\nu+\rho)(A(x,b))}f^{\delta}(-\nu, b)db = \int_{B} e^{(\nu+\rho)(A(x,b))}f^{\delta}(\nu, b)db \Leftrightarrow$$

$$\int_{K} e^{(-\nu+\rho)(A(x,kM))}\delta(k)dkf^{\delta}(-\nu) = \int_{K} e^{(\nu+\rho)(A(x,kM))}\delta(k)dkf^{\delta}(\nu) \Leftrightarrow$$

$$\Phi_{-\nu,\delta}(x)f^{\delta}(-\nu) = \Phi_{\nu,\delta}(x)f^{\delta}(\nu),$$

where $\Phi_{\nu,\delta}(x) = \int_{K} e^{(\nu+\rho)(A(x,kM))}\delta(k)dk$ is a generalized spherical function (or Eisenstein integral). We will consider the spaces $\mathcal{H}(\mathbb{C})^{\delta}$ in $\mathcal{H}(\mathbb{C})^{\delta}_e$ and $S(i\mathbb{R}_e)^{\delta}_e$ in $S(i\mathbb{R}_e)^{\delta}_e$ of functions satisfying:

$$\Phi_{-\nu,\delta}(x)f^{\delta}(-\nu) = \Phi_{\nu,\delta}(x)f^{\delta}(\nu), \quad \nu \in i\mathbb{R}_e, \ x \in X,$$

and from above we see that we have homeomorphism between $\mathcal{H}(\mathbb{C} \times B)^{\delta}_e$ and $\mathcal{H}(\mathbb{C})^{\delta}_e$ and between $S(i\mathbb{R}_e \times B)^{\delta}_e$ and $S(i\mathbb{R}_e)^{\delta}_e$. Since $G/K$ is a symmetric space of rank one, we see from [He2, Chapter II, Corollary 6.8] and [He1, Chapter V, Theorem 3.5] that $\dim V_{\delta}^M = 1$. Let $v$ span $V_{\delta}^M$ and let $v_1, v_2, \ldots, v_{d(\delta)}$ with $v_1 = v$ be an orthonormal basis of $V_{\delta}$.
Lemma 2.2.7. Let
\[ \varphi_{\nu,\delta}(x) = \langle \Phi_{\nu,\delta}(x)v, v \rangle = \int_K e^{(\nu+\rho)(A(x,kM))}(\delta(k)v, v)dk, \]
and let
\[ \varphi_{\nu,\delta}^j(x) = \langle \Phi_{\nu,\delta}(x)v, v_j \rangle = \int_K e^{(\nu+\rho)(A(x,kM))}(\delta(k)v, v_j)dk, 1 \leq j \leq d(\delta). \]
Then \( \varphi_{\nu,\delta}^j(ka \cdot x_o) = (\delta(k)v, v_j)\varphi_{\nu,\delta}(a \cdot x_o), \) where \( x_o = \epsilon K, k \in K, a \in A. \)

Proof. Define \( F : X \to V_{\delta} \) by:
\[ F(x) = \int_K e^{(\nu+\rho)(A(x,kM))}\delta(k)vdk = \Phi_{\nu,\delta}(x)v. \]
Then \( \varphi_{\nu,\delta}^j(x) = \langle F(x), v_j \rangle \) and since \( F(k \cdot x) = \delta(k)F(x) \) we have \( F(a \cdot x_o) \in V_{\delta}^M (\delta(m)F(a \cdot x_o) = F(\epsilon m \cdot x_o) = F(a \cdot x_o), m \in M = Z_K(A)). \) Since \( \dim V_{\delta}^M = 1 \) we deduce: \( F(a \cdot x_o) = \varphi_{\nu,\delta}(a \cdot x_o)v. \)
Then:
\[ \varphi_{\nu,\delta}^j(ka \cdot x_o) = \langle F(ka \cdot x_o), v_j \rangle = (\delta(k)F(a \cdot x_o), v_j) = (\delta(k)v, v_j)\varphi_{\nu,\delta}(a \cdot x_o). \]

Since \( f^\delta(\nu) \in \text{Hom}(V_{\delta}, V_{\delta}^M) \) we see from Lemma 2.2.7 that (1) is equivalent to:
\[ \varphi_{-\nu,\delta}(a \cdot x_o)f^\delta(-\nu) = \varphi_{\nu,\delta}(a \cdot x_o)f^\delta(\nu), \quad \nu \in i\mathbb{R}_e. \]

(2)

We can determine \( \varphi_{\nu,\delta} \) quite explicitly in terms of the hypergeometric functions (See [He2, Chapter III, Theorem 11.2]). As a corollary we get:

Lemma 2.2.8. The functions \( \varphi_{\nu,\delta} \) satisfies the symmetry condition:
\[ \varphi_{\nu,\delta}(a \cdot x_o) = \varphi_{-\nu,\delta}(a \cdot x_o) \frac{p_\delta(\nu)}{p_\delta(-\nu)}, \]
where \( p_\delta(\nu) \) is a polynomial of the form:
\[ p_\delta(\nu) = (\nu + \rho + s - 1) \cdots (\nu + \rho), \quad s \in \mathbb{N} \quad \text{or} \quad p_\delta(\nu) \equiv 1. \]

Proof. [He2, Chapter III, Corollary 11.3].

(3)

So (2) is equivalent to:
\[ p_\delta(-\nu)f^\delta(-\nu) = p_\delta(\nu)f^\delta(\nu), \quad \nu \in i\mathbb{R}_e. \]
Consider the set of even functions in \( \mathcal{H}(\mathbb{C})^\delta, \mathcal{H}(\mathbb{C})^\delta_1 \) and the set of even functions in \( S(i\mathbb{R}_e)^\delta, S(i\mathbb{R}_e)^\delta_1. \)
Then we have the following lemma:

Lemma 2.2.9. The map:
\[ G(\nu) \to F(\nu) = p_\delta(-\nu)G(\nu) \]
is a homeomorphism of \( S(i\mathbb{R}_e)^\delta_1 \) onto \( S(i\mathbb{R}_e)^\delta \) taking \( \mathcal{H}(\mathbb{C})^\delta_1 \) to \( \mathcal{H}(\mathbb{C})^\delta. \)
Proof. The map clearly is continuous, and it takes $\mathcal{H}(\mathbb{C})^\delta_e$ into $\mathcal{H}(\mathbb{C})^\delta_e$ and $S(i\mathbb{R}_e)^\delta$ into $S(i\mathbb{R}_e)^\delta_e$. Since $p_\delta$ is a nonzero polynomial we see that the map is injective. Let $F \in S(i\mathbb{R}_e)^\delta_e$, and consider the function:

$$G(\nu) = \frac{F(\nu)}{p_\delta(-\nu)},$$

$G$ is even:

$$G(-\nu) = \frac{F(-\nu)}{p_m(-\nu)} = \frac{F(\nu)}{p_m(\nu)} = \frac{F(\nu)}{p_m(-\nu)} = G(\nu).$$

Since $\nu$ and $-\nu$ are not both roots for $p_\delta$ we see that one of the expressions $G(\nu) = \frac{F(\nu)}{p_m(-\nu)} = \frac{F(-\nu)}{p_m(\nu)}$ always will be welldefined, and then $G$ will satisfy the same differentiability and growth conditions as $F$. Hence the map is surjective, and by the closed graph theorem the map is a homeomorphism with the required properties.

Via the classical Fourier transform it is easy to verify that $\mathcal{H}(\mathbb{C})^\delta_e$ is dense in $S(i\mathbb{R}_e)^\delta_e$, which yields that $\mathcal{H}(\mathbb{C})^\delta_e$ is dense in $S(i\mathbb{R}_e)^\delta_e$, and thus we get the desired result:

**Theorem 2.2.10.** Let $\varepsilon \geq 0$ and let $\delta \in \hat{K}_M$, then:

(i) $\mathcal{H}(\mathbb{C})^\delta_e$ is dense in $S(i\mathbb{R}_e)^\delta_e$.

(ii) $\mathcal{H}(\mathbb{C} \times B)^{\delta,e}$ is a dense subspace of $S(i\mathbb{R}_e \times B)^{\delta,e}$.

(iii) $\mathcal{H}_e(\mathbb{C} \times B)$ is a dense subspace of $S(i\mathbb{R}_e \times B)^{\delta,e}$.

Proof. (ii) Since $\mathcal{H}(\mathbb{C} \times B)^{\delta,e}$ is dense in $S(i\mathbb{R}_e \times B)^{\delta,e}$ for all $\delta \in \hat{K}_M$.

(iii) $S(i\mathbb{R}_e \times B)^{\delta,e}$ is closed in $S(i\mathbb{R}_e \times B)$.

$\Rightarrow$: In (SC2) replace $x$ by $k^{-1} \cdot x$, use $A(k^{-1} \cdot x, b) = A(x, k \cdot b)$, make the substitution $b \mapsto k^{-1} \cdot b$, multiply by $\chi_\delta(k^{-1})$ and integrate over $K$. The result is:

$$\int_B e^{-(\nu + \rho)(A(x, b))} f_\delta(-\nu, b) db = \int_B e^{(\nu + \rho)(A(x, b))} f_\delta(\nu, b) db.$$ 

which exactly means that $f_\delta \in S(i\mathbb{R}_e \times B)^{\delta,e}$.

By Theorem 2.2.10ii), every finite sum of the type $\sum_{\delta \in \hat{K}_M} f_\delta$, $f_\delta \in S(i\mathbb{R}_e \times B)^{\delta,e}$, can be approximated by a finite sum $\sum_{\delta \in \hat{K}_M} g_\delta$, $g_\delta \in \mathcal{H}(\mathbb{C} \times B)^{\delta,e}$, and the theorem then follows by Theorem 2.2.5.

**Remark:** As the primary goal was to show Theorem 1 via Anker’s method, Theorem 1.2.10iii) is very much an important result. But since we can only handle the $K$-finite case, the important result is actually Theorem 2.10i).

Later we will need some estimates of the matrix coefficients of the generalized spherical functions (as for spherical functions in [An]). Let $P = MAN$ be the minimal parabolic subgroup of $G$. Consider the characters $\{ma_0 n \mapsto e^{\nu t}, \nu \in \mathbb{C}\}$ of $P$. The spherical principal series representations $\pi_\nu$ on $G$ are then the induced representations coming from these characters. They are realized on $L^2(K/M)$ via the formula:

$$\{\pi_\nu(g)f\}(k) = e^{-(\nu + \rho)\mathcal{H}(g^{-1}k)} f(\kappa(g^{-1}k)), \quad f \in L^2(K/M).$$

The restriction of $\pi_\nu$ to $K$ is: $\{\pi_\nu(k_1)f\}(k_2) = f(k_1^{-1}k_2)$.
Proposition 2.2.11. Let $\phi \in C^\infty(K/M)$, and consider the function:

$$
\psi_\nu(g) = \int_K \phi(k) e^{-\nu \rho(H(\nu^{-1}k))} dk.
$$

Given $D, E \in U(g)$, there is a constant $c$ and elements $D_l \in U(\mathfrak{g})$, $1 \leq l \leq M$ such that:

$$
|\psi_\nu(D; g; E)| < c(1 + |\nu|)^{\text{deg} D + \text{deg} E} \varphi_{\text{Re}(\nu)}(g) \sup_{k \in K} \sum_{l=1}^M |\phi(k; D_l)|.
$$

Proof. We can write $|\psi_\nu(D; g; E)|$ as:

$$
|\int_K \phi(k) \pi_\nu(D; g) E1_{K/M} dk| = |\int_K \phi(k) \pi_\nu(D) \pi_\nu(g) \pi_\nu(E) 1_{K/M} dk|

= |(\pi_\nu(g) \pi_\nu(E) 1_{K/M}, \pi_\nu(D^*) \phi(k))_2|

\leq \|\pi_\nu(g) \pi_\nu(E) 1_{K/M}\|_1 \|\pi_\nu(D^*) \phi\|_\infty,
$$

by the Hölder inequality. $\pi_\nu(E) 1_{K/M}$ can be considered as a function on $K/M : \xi$, hence we have:

$$
[\pi_\nu(g) \xi](k) = e^{-\nu \rho H(\nu^{-1}k)} \xi(g(k^{-1}k)).
$$

Again using Hölder this gives us:

$$
\|\pi_\nu(g) \pi_\nu(E) 1_{K/M}\|_1 \leq \|\xi(k)\|_\infty \|\pi_\nu(g) 1_{K/M}\|_1 = \|\pi_\nu(E) 1_{K/M}\|_\infty \varphi_{\text{Re}(\nu)}(g).
$$

We are left by estimating the two norms:

(i) $\|\pi_\nu(E) 1_{K/M}\|_\infty$
(ii) $\|\pi_\nu(D^*) \phi(k)\|_\infty$

From [An, p.336] we get an estimate on (i) $\|\pi_\nu(E) 1_{K/M}\|_\infty \leq c(1 + |\nu|)^{\text{deg} E}$. So consider (ii).

Introduce the auxiliary function $f_\nu(g) = e^{-\nu \rho H(g)}$, then by the Leibniz rule of differentiation:

$$
\pi_\nu(D^*) \phi(k) = \sum_{\text{deg} D = \text{deg} D' + \text{deg} D''} f_\nu(D'; k) \phi(\kappa(D''; k))

= \sum_{\text{deg} D = \text{deg} D' + \text{deg} D''} f_\nu(D'; k) \phi(\kappa(k; Ad(k^{-1}D''))).
$$

We see that $f_\nu(D'; k) = \{\pi_\nu(D'^*) 1_{K/M}\}(k)$. There exists functions $\xi_1, \ldots, \xi_m$ in $C^\infty(K)$ and elements $D_1, \ldots, D_m$ in $U(\mathfrak{g})$ of degree $\leq \text{deg} D$ such that $Ad(k^{-1}D'') = \sum_{i=1}^m \xi_i(k) D_i$. By the PBW-Theorem we can write:

$$
U(g) = U(\mathfrak{g}) \oplus U(\mathfrak{g})(a + n).
$$

Let $D'_l$ be the projection of $D_l$ on $U(\mathfrak{g})$. Then:

$$
|\phi(\kappa(k; Ad(k^{-1}D''))| = |\phi(\kappa(k; \sum_{l=1}^m \xi_l(k) D_l))| = \sum_{l=1}^m |\xi_l(k)\phi(\kappa(k; D_l))|

= \sum_{l=1}^m |\xi_l(k)\phi(\kappa(k; D'_l))| \leq \sum_{l=1}^m \sup_{k \in K} |\phi(k; D'_l)|.
$$
5) By the Cartan decomposition we have:

\[
\|\pi_\nu(D^*)\phi\|_\infty \leq c(1 + |\nu|)^{\deg D} \sum_{\deg D' \leq \deg D} \sum_{1 \leq i \leq m} \sup_{k \in K} |\phi(k; D'_i)|.
\]

\[\square\]

2.3. The isomorphism of the Fourier transform on \( K \)-finite elements in the \( L^p \)-Schwartz spaces.

In this section we show the generalization of [An] in the rank 1 case. Let \( f \in \mathcal{S}^p(G/K) \), \( 0 < p \leq 2 \). Since \( C^\infty_c(G/K) \subset \mathcal{S}^p(G/K) \subset L^2(G/K) \), we can define the Fourier transform \( \hat{f} \in L^2(i\mathbb{R}_+ \times B, e|c(i\nu)|^{\frac{2}{p}}) \). From the definition of \( \mathcal{S}^p(G/K) \), and the Cartan decomposition of the measure on \( X \), we see that the extension of the Fourier transform from \( C^\infty_c(G/K) \) to \( \mathcal{S}^p(G/K) \) is trivial, that is:

\[
\mathcal{H}f(\nu, b) = \hat{f}(\nu, b) = \int_X f(x)e^{(-\nu + \rho)(A(x, b))}dx,
\]

for all \( \nu \in i\mathbb{R}, b \in B \). Furthermore we see that the above integral is well defined for \( (\nu, b) \in i\mathbb{R}_+ \times B \). We actually have:

**Lemma 2.3.1.** Let \( f \in \mathcal{S}^p(G/K) \), and let \( \varepsilon = \frac{2}{p} - 1 \). Then \( \hat{f} \in C^\infty(i\mathbb{R}_+ \times B) \), and \( \hat{f}(-, b) \) is holomorphic in \( i\mathbb{R}_+^\circ \) for fixed \( b \in B \).

**Proof.** Let

\[
\hat{f}(\nu, b) = \int_X f(x)e^{(-\nu + \rho)(A(x, b))}dx.
\]

By the Cartan decomposition we have:

\[
\hat{f}(\nu, b) = \int_0^\infty \int_K f(ka_t)e^{(-\nu + \rho)(A(ka_t, x, b))}dk \sinh^n t dt.
\]

Consider the above integral over \( K \):

\[
\left| \int_K f(ka_t)e^{(-\nu + \rho)(A(ka_t, x, b))}dk \right| \leq \left\{ \sup_{k \in K} |f(ka_t)| \right\} \int_K e^{(-Re\nu + \rho)(A(ka_t, x, b))}dk
\]

\[
= \left\{ \sup_{k \in K} |f(ka_t)| \right\} \varphi_{Re\nu}(a_t).
\]

We then see:

1) \( e^{-2(\frac{1}{p} - 1)\rho t}(1 + t)^{-1}\varphi_{Re\nu}(a_t) \) is a bounded function for \( \nu \in i\mathbb{R}_+^\circ \) (by Lemma 2.1.9).

2) \( t \mapsto \{\sup_{k \in K} |f(ka_t)|\} \in C(\mathbb{R}_+) \).

3) \( \sup_{t > 0} (1 + t)^N \varphi_{\nu}(a_t)^{-\frac{2}{p}}\{\sup_{k \in K} |f(ka_t)|\} \leq \sup_{g \in G} (1 + |g|^N \varphi_{\nu}(g)^{-\frac{2}{p}}|f(g)| < \infty \).

4) \( t \mapsto e^{2(\frac{1}{p} - 1)\rho t}(1 + t)\{\sup_{k \in K} |f(ka_t)|\} \in L^1(\mathbb{R}_+, \sinh^n t dt) \) (Lemma 2.1.9).

We thus see that:

5) \( \hat{f}(\nu, b) \) is well defined for all \( \nu \in i\mathbb{R}_+^\circ, b \in B \).

6) By Morera’s Theorem we see that \( \hat{f}(-, b) \) is holomorphic in \( i\mathbb{R}_+^\circ \). Differentiability follows from Lebesgue’s Dominated Convergence Theorem. \[\square\]
Theorem 2.3.2. Let \( 0 < p \leq 2, \ v = \frac{2}{p} - 1 \). The Fourier transform \( H \) is an injective and continuous homomorphism from \( S^p(G/K) \) into \( \mathcal{S}(i\mathbb{R}_c \times B)_c \).

Proof. Let \( f \in S^p(G/K) \). By Lemma 2.3.1 we see that \( Hf(\cdot, b) \) is holomorphic in \( i\mathbb{R}_c^\circ \) for fixed \( b \in B \). Now we observe that \( \mathcal{S}(i\mathbb{R}_c \times B)_c \) and it’s topology also is determined by the set of seminorms:

\[
\tilde{\tau}_{P,M,N}(f) = \sup_{\nu \in \mathbb{R}_c^\circ, k \in K} \left| \left( P \left( \frac{\partial}{\partial \nu} \right), \Omega_K^M \right) \left\{ (\nu^2 - \rho^2 + d)^N f(\nu, k) \right\} \right|,
\]

where \( P, M \) and \( N \) are as before, and \( d \) is a constant such that \( \nu^2 - \rho^2 + d \neq 0 \). Fix \( \tilde{\tau}_{P,M,N} \). Then

\[
\left( P \left( \frac{\partial}{\partial \nu} \right), \Omega_K^M \right) \left( (\nu^2 - \rho^2 + d)^N Hf(\nu, k) \right) = \left( P \left( \frac{\partial}{\partial \nu} \right), \Omega_K^M \right) \left( (\nu^2 - \rho^2 + d)^N \int X f(x) e^{(-\nu + \rho)(A(x,kM))} dx \right) = \int X \left( \Omega_K^M (\Delta + d)^N f \right)(x) P \left( \frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x,kM))} dx,
\]

where \( \Omega_K^M f(x) = \Omega_K^M f(g) = f(\Omega_K^M g), x = gK \). Now:

\[
\left| P \left( \frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x,kM))} \right| = \left| P(-A(x,kM)) e^{(-\nu + \rho)(A(x,kM))} \right| \leq c(1 + |x|)^{degP} e^{(-2\nu + \rho)(A(x,kM))}.
\]

As before (Lemma 2.3.1), we use the Cartan decomposition and the estimate \( \sinh^n t \leq e^{2pt} \):

\[
\int X \left| \left( \Omega_K^M (\Delta + d)^N f \right)(x) P \left( \frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x,kM))} \right| dx \leq c \int_0^\infty \left\{ \sup_{k \in K} |(\Omega_K^M (\Delta + d)^N f)(k')| \right\} (1 + t)^{degP} \int K e^{(-2\nu + \rho)(A(k',k'M))} dk' e^{2pt} dt = c \int_0^\infty \left\{ \sup_{k \in K} |(\Omega_K^M (\Delta + d)^N f)(ka_t)| \right\} (1 + t)^{degP} e^{2pt} dt.
\]

By Lemma 2.1.9 ii) there exists a constant \( c \) such that \( |\varphi_{Re\nu}(a_t)| \leq c(1 + t)(\frac{2}{p} - 2) pt \), for \( t > 0 \). Let

\[
\sigma(f) = \sup_{g \in G} \left( 1 + |g| \right)^{M'} \varphi_o(g)^{-\frac{2}{p}} \left| (\Omega_K^M (\Delta + d)^N f)(g) \right|,
\]

where we choose \( M' \in \mathbb{N} \) s.t. \( M' > degP + 1 + \frac{2}{p} \). Then from Lemma 2.1.9 i) we get:

\[
\left| (\Omega_K^M (\Delta + d)^N f)(g) \right| \leq c\sigma(f) \left( 1 + |g|^M' \right)^{-1+\frac{2}{p}} e^{-\frac{2}{p} |g|}.
\]

Putting this together we get:

\[
\tilde{\tau}_{P,M,N}(Hf) \leq c\sigma(f)
\]

and \( \sigma \) is obviously bounded by a sum of seminorms for \( S^p(G/K) \), and we have shown continuity.

Injectivity follows from Theorem 2.1.3. \( \square \)

Note, that the restriction to powers of \( \Omega_K \) is not crucial, we could prove the theorem for the topology with \( \Omega_K \) replaced by \( E \in U(g) \) in the same fashion.
Now, of course, we would like to show surjectivity of $\mathcal{H}$. In order to generalize the proof from [An], we need a cut-off function, $\omega_j$, w.r.t. to the first variable of $H(\cdot, \cdot)$, in order to control the involved functions in the commutative diagram below:

\[
\begin{array}{ccc}
\mathcal{H} \ni h & \xrightarrow{\sim} & \mathcal{H}_e(\mathbb{C} \times B) \\
f \in C_c^\infty(G/K) & \xrightarrow{\sim} & H \in \mathcal{R}C_c^\infty(G/K) \\
\end{array}
\]

Unfortunately the symmetry conditions imposed on $h$ and $H$ are rather obscure, as we have already encountered, therefore looking at the function $H$ decomposed w.r.t. to the cut-off function,

\[
H(t, b) = \omega_j(t) H(t, b) + (1 - \omega_j(t)) H(t, b), \quad H_j(t, b) = (1 - \omega_j(t)) H(t, b),
\]

will not necessarily give us a function $H_j$ in $\mathcal{R}C_c^\infty(G/K)$. This means that we will be unable to define functions $h_j = FH_j \in \mathcal{H}_e(\mathbb{C} \times B)$ and $f_j = \mathcal{R}^{-1} H_j \in C_c^\infty(G/K)$, as e.g., $f_j$, though welldefined, doesn’t need to have compact support. A way to avoid the difficulties created by the symmetry conditions is to look at fixed $K$-types, as we did with the density argument.

So consider the left-regular representation of $K$ ($\iota$) on the Fréchet space $S^p(G/K)$. It is straightforward to check that this representation is a smooth Fréchet representation of $K$. With $\delta \in \hat{K}_M$ acting on $V_\delta$, consider the spaces $C_c^\infty(G/K, \text{Hom}(V_\delta, V_\delta))$ and $S^p(G/K, \text{Hom}(V_\delta, V_\delta))$ of compactly supported differentiable functions on $G/K$, respectively $L^p$-Schwartz functions on $G/K$, taking values in $\text{Hom}(V_\delta, V_\delta)$ (looking at the matrix-entries). Define $S^p(G/K) \delta = \{ F \in S^p(G/K, \text{Hom}(V_\delta, V_\delta)) : F(k \cdot \cdot x) = \delta(k) F(x) \}$, and define $C_c^\infty(G/K) \delta \subseteq S^p(G/K) \delta \cap C_c^\infty(G/K, \text{Hom}(V_\delta, V_\delta))$. For $f \in S^p(G/K)$ we define:

\[
P_\delta f(x) = f_\delta(x) = d(\delta) \int_K \delta(k) f(k^{-1} \cdot x) dk.
\]

We see that $P_\delta$ takes $C_c^\infty(G/K)$ into $C_c^\infty(G/K) \delta$, and $S^p(G/K)$ into $S^p(G/K) \delta$. Let furthermore $C_c^\infty(G/K) \delta$ and $S^p(G/K) \delta$ denote the space of $K$-finite functions in $C_c^\infty(G/K)$, respectively in $S^p(G/K)$, of type $\delta$. The continuous projection of $C_c^\infty(G/K)$, respectively of $S^p(G/K)$, onto $C_c^\infty(G/K) \delta$ and $S^p(G/K) \delta$ is given by:

\[
P_\delta f(x) = f_\delta(x) = d(\delta) \int_K \chi_\delta(k^{-1}) f(k^{-1} \cdot x) dk.
\]

**Proposition 2.3.3.** Let $\delta \in \hat{K}_M$.

(i) The map:

\[
Q : F(x) \to \text{Tr}(F(x))
\]

is a homeomorphism of $S^p(G/K) \delta$ onto $S^p(G/K) \delta$, with inverse $f \to P_\delta f = f_\delta$. Furthermore $Q$ takes $C_c^\infty(G/K) \delta$ onto $C_c^\infty(G/K) \delta$. Here $\text{Tr}$ denotes trace in $\text{Hom}(V_\delta, V_\delta)$.

(ii) The maps:

\[
P_\delta : S^p(G/K) \to S^p(G/K) \delta, \quad \text{and} \quad P_\delta : S^p(G/K) \to S^p(G/K) \delta,
\]

are continuous open surjections, and the images are closed in $S^p(G/K)$, respectively in $S^p(G/K, \text{Hom}(V_\delta, V_\delta))$.

**Proof.** As for proposition 2.2.6. \qed
Definition 2.3.4. Let $\delta \in \hat{K}_M$. For $f \in S^0(G/K)_\delta$ the $\delta$-spherical transform is defined by:

$$H^\delta f(\nu) = (ev \circ P^\delta H f)(\nu)$$

$$= d(\delta) \int_K \int_X f(x) e^{-(\nu + \rho)(A(x,kM))} dx \delta(k^{-1}) dk$$

$$= d(\delta) \int_X f(x) \int_K e^{-(\nu + \rho)(A(x,kM))} \delta(k^{-1}) dk dx$$

$$= d(\delta) \int_X f(x) \Phi_{-\nu,\delta}(x)^* dx,$$

where: $\Phi_{-\nu,\delta}(x)^* = \int_K e^{-(\nu + \rho)(A(x,kM))} \delta(k^{-1}) dk$ is the adjoint of the generalized spherical function $\Phi_{-\nu,\delta}$, $^*$ denoting the adjoint in Hom$(V_\delta, V_{\delta'})$, and ev is the evaluation map, ev$(f) = f(\nu, eM)$.

For the trivial representation we have $\Phi_{-\nu,1}(x)^* = \varphi_\nu(x)$, that is, the 1-spherical transform $H^1$ is the "classical" spherical transform $H$. For the $\delta$-spherical transform we have the following Paley-Wiener Theorem:

Theorem 2.3.5. The $\delta$-spherical transform $f \mapsto H^\delta f$ is a bijection of $C_\infty^c(G/K)_\delta$ onto $H(\mathbb{C})^\delta$.

Proof. It is evident from the definition that $H^\delta$ maps $C_\infty^c(G/K)_\delta$ into $H(\mathbb{C})^\delta$, and also that $H^\delta$ is injective since $H$ is an injective map. For surjectivity, let $\psi \in H(\mathbb{C})^\delta$. The function $\Psi(\nu, kM) \equiv Tr(\delta(k)\psi(\nu))$ is clearly a holomorphic function of uniform exponential type on $\mathbb{C} \times B$. By Proposition 2.2.6 and the discussion afterwards, we see that $\Psi \in H(\mathbb{C} \times B)_\delta$. By the Paley-Wiener Theorem 2.1.5, there exists a unique $F \in C_\infty^c(G/K)$ such that $\Psi = HF$. By Proposition 2.3.3 the function $P_\delta F = F_\delta$ belongs to $C_\infty^c(G/K)_\delta$, and we have:

$$H^\delta F_\delta(\nu) = \{ P^\delta H P_\delta F \}(\nu, eM) = \{ P^\delta P_\delta H F \}(\nu, eM) = \{ P^\delta P_\delta \Psi \}(\nu, eM)$$

$$= d^2(\delta) \int_K \int_K \Psi(\nu, u^{-1}kM) \chi_\delta(u) \delta(k^{-1}) du dk$$

$$= d^2(\delta) \int_K \int_K Tr(\delta(k)\psi(\nu)) \chi_\delta(u) \delta(k^{-1}u^{-1}) du dk$$

$$= d^2(\delta) \int_K \int_K Tr(\delta(k)\psi(\nu)) \chi_\delta(u) \delta(k^{-1}) \delta(u^{-1}) du dk$$

$$= d^2(\delta) \int_K \delta(k^{-1}) Tr(\delta(k)\psi(\nu)) dk \int_K \chi_\delta(u) \delta(u^{-1}) du$$

$$= d(\delta) \int_K \delta(k^{-1}) Tr(\delta(k)\psi(\nu)) dk$$

$$= P^\delta Tr(\delta(k)\psi(\nu))$$

$$= \psi(\nu),$$

since the orthogonality relations yields:

$$d(\delta) \int_K \delta_{i,j}(u) \delta(u) du = E_{i,j}.$$

We also have the following inversion Theorem:
Theorem 2.3.6. The $\delta$-spherical transform is inverted by:

$$f(x) = c \text{Tr} \left\{ \int_0^\infty \Phi_{iv,\delta}(x) (\mathcal{H}^\delta f(i\nu)) |c(i\nu)|^{-2} d\nu \right\}, \quad f \in C_c^\infty(G/K)_\delta,$$

where

$$\Phi_{iv,\delta}(x) = \int_K e^{(i\nu + \rho)(A(x,kM))} \delta(k) dk.$$

Proof. If $f \in C_c^\infty(G/K)_\delta$, then:

$$\mathcal{H} f(\nu, kM) = \mathcal{H}' P_\delta f(\nu, kM) = P_\delta' \mathcal{H} f(\nu, kM) = \text{Tr} P_\delta^* \mathcal{H} P_\delta f(\nu, kM) = \text{Tr} \mathcal{H}^\delta f(\nu, kM) = \text{Tr} (\delta(k) \mathcal{H}^\delta f(\nu)).$$

So by the "classical" inversion Theorem, Theorem 2.1.2, we get:

$$f(x) = c \int_{\mathbb{R}^+ \times K} e^{(i\nu + \rho)(A(x,kM))} \text{Tr} (\delta(k) \mathcal{H}^\delta f(i\nu)) |c(i\nu)|^{-2} d\nu$$

$$= c \text{Tr} \left\{ \int_{\mathbb{R}^+ \times K} e^{(i\nu + \rho)(A(x,kM))} \delta(k) \mathcal{H}^\delta f(i\nu) |c(i\nu)|^{-2} d\nu \right\}.$$

Let $v$ span $V_\delta^M$, and let $v_1, v_2, \ldots, v_{d(\delta)}$, with $v_1 = v$, be an orthonormal basis of $V_\delta$. Then the inverse $\delta$-spherical transform can be written as:

$$f(x) = c \int_0^\infty \text{Tr} (\Phi_{iv,\delta}(x) \mathcal{H}^\delta f(i\nu)) |c(i\nu)|^{-2} d\nu$$

$$= c \int_0^\infty \sum_{i=1}^{d(\delta)} \Phi_{iv,\delta}(x)_{1,i} \mathcal{H}^\delta f(i\nu)_{1,i} |c(i\nu)|^{-2} d\nu,$$

where $i, j$ denote matrix entries. We see that $\Phi_{iv,\delta}(x)_{1,1} = \phi_{iv,\delta}(x)$ (see Lemma 2.2.7).

Moreover we also have a Plancherel Theorem:

Theorem 2.3.7.

$$\int_X |f(x)|^2 dx = \int_0^\infty \text{Tr} \{ (\mathcal{H}^\delta f(i\nu) (\mathcal{H}^\delta f(i\nu))^* (i\nu)) |c(i\nu)|^{-2} d\nu$$

$$= c \int_0^\infty \| \{ (\mathcal{H}^\delta f(i\nu)) \} \|^2_{\mathcal{H}^\delta} |c(i\nu)|^{-2} d\nu.$$

Proof. As above. $\square$

Consider the classical Fourier transform $\mathcal{F}$ acting on vector valued functions, and define $C_c^\infty(\mathbb{R})^\delta_{\otimes} = \mathcal{F}^{-1} \mathcal{H} \mathcal{C}^\delta_{\otimes}$, that is, functions in $C_c^\infty(\mathbb{R}, \text{Hom}(V_\delta, V_\delta))$ satisfying certain symmetry conditions, then we have the following commutative diagram:

$$\begin{array}{ccc}
C_c^\infty(G/K)_\delta & \xrightarrow{\mathcal{H}^\delta} & \mathcal{H} \mathcal{C}^\delta_{\otimes} \\
\mathcal{R} & \downarrow & \uparrow \mathcal{F} \\
C_c^\infty(\mathbb{R} \times B)_{\otimes\otimes} & \xrightarrow{\text{ev} \circ P^\delta} & C_c^\infty(\mathbb{R})^\delta_{\otimes}
\end{array}$$
where:

\[ \mathcal{P}^\delta f(t, b) = d(\delta) \int_K \delta(k) f(t, k^{-1} \cdot b) dk, \]

and \( R \) is the Radon transform. We define a new transform \( T \) by:

\[ T = ev \circ \mathcal{P}^\delta R. \]

More exactly we have:

\[ T f(t) = e^{\rho t} \int_{K \times N} f(k, n) \delta(k^{-1}) dk, \]

which can be thought of as a generalized Abel transform.

**Theorem 2.3.8.** The transform \( T \) is an isomorphism between \( C^\infty_c(G/K) \) and \( C^\infty_c(R) \). Moreover \( \text{supp} f \subset \overline{B}(0, R) \) if and only if \( \text{supp} T f \subset \overline{B}(0, R) \).

**Proof.**

i) By definition.

ii) \( \Leftarrow \) As Proposition 2.1.6. \( \Rightarrow \) Let \( g \in C^\infty_c(R) \) such that \( \text{supp} g \subset \overline{B}(0, R) \). Then \( \mathcal{F} g \in \mathcal{H}(\mathbb{C}) \), with matrix entries \( \{ \mathcal{F} g \}_{i,j} \) in \( \mathcal{H}(\mathbb{C}) \). Now consider the function \( G(z, kM) = \text{Tr}(\delta(k) \{ \mathcal{F} g \}(z)) \) in \( \mathcal{H}_e^R(\mathbb{C} \times \mathbb{B}) \). The proof of Theorem 2.3.6 yields that:

\[ T^{-1} g(x) = \mathcal{H}^{-1} G(x), \]

and then by Proposition 2.1.7, we see that \( \text{supp} T^{-1} g \subset \overline{B}(0, R) \).

Now we arrive at the essential theorem:

**Theorem 2.3.9.** Let \( 0 < p \leq 2, \varepsilon = \frac{2}{p} - 1 \). Then the \( \delta \)-spherical transform \( \mathcal{H}^\delta \) is a topological isomorphism between \( \mathcal{S}^p(G/K) \) and \( \mathcal{S}(i\mathbb{R})^\delta \). The inverse transform is given by Theorem 2.3.6.

**Proof.**

a) We can write \( \mathcal{H}^\delta \) as a composition of continuous operators:

\[ \mathcal{H}^\delta = ev \circ P^\delta \mathcal{H}. \]

Hence \( \mathcal{H}^\delta \) is an injective continuous homomorphism into \( \mathcal{S}(i\mathbb{R})^\delta \).

b) We now want to show surjectivity. By density, Theorem 2.2.10i), it is enough to show that the inverse \( \delta \)-spherical transform is continuous as a map from \( \mathcal{H}(\mathbb{C})^\delta \) to \( C^\infty_c(G/K) \), with the topologies induced by \( \mathcal{S}(i\mathbb{R})^\delta \) and \( \mathcal{S}^p(G/K) \). So let \( f \in C^\infty_c(G/K) \), \( h = \mathcal{H}^\delta f \) and \( H = T f \) as in the commuting diagram below.

Let \( \sigma^p_{D,E,N} \) be a seminorm for \( \mathcal{S}^p(G/K) \):

\[ \sigma^p_{D,E,N}(f) = \sup_{g \in G} (1 + |g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(D; g; E)|. \]
We will consider the function: $F(g) = (1 + |g|)^{N} \varphi_{o}(g)^{-\frac{2}{N}} |f(D; g; E)|$. Our goal now is to estimate $F$ on the intervals $[j, j + 1]$. Remark that all positive constants appearing below may depend on $N$ and $p$, but not on $f$.

Step 1 ($[0, 2]$): By the inversion formula, Theorem 2.3.6, we get:

$$f(D; g; E) = c Tr \left\{ \int_{-\infty}^{\infty} \Phi_{i \nu, \delta}(D; g; E) h(i \nu) |c(i \nu)|^{-2} d\nu \right\}.$$ 

Using Lemma 2.1.9 iii) and Proposition 2.2.11, on the functions $\phi = \delta_{i, 1}$, we get:

$$|f(D; g; E)| \leq c \varphi_{o}(g) \int_{\mathbb{R}^+} (1 + |\nu|)^{R} \sum_{l=1}^{d(\delta)} |h_{1, l}(i \nu)| d\nu,$$

for some $R \in \mathbb{N}$. We thus get:

$$\sup_{|g| \in [0, 2]} F(x) \leq c \int_{\mathbb{R}^+} (1 + |\nu|)^{R} \sum_{l=1}^{d(\delta)} |h_{1, l}(i \nu)| d\nu$$

$$\leq \sup_{\nu \in \mathbb{R}^+} (1 + |\nu|)^{R+2} \sum_{l=1}^{d(\delta)} |h_{1, l}(i \nu)|$$

$$= c \sum_{l=1}^{d(\delta)} \tau_{1, R+2}^{l}(h_{1, l}),$$

using the compactness of $[0, 2]$.

Step 2: For $j \in \mathbb{N}$ we introduce an even auxiliary function $\omega_{j} \in C_{c}^{\infty}(\mathbb{R})$ defined such that:

$$\omega_{j}(t) = \begin{cases} 1, & t \in [0, j - 1] \\ 0, & t \in [j, \infty], \end{cases}$$

and $\omega_{j+1}(t) = \omega_{j}(t - 1)$. We write $H$ as:

$$H(t) = p_{\delta} \left( -\frac{\partial}{\partial t} \right) \{ (1 - \omega_{j})(t) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \}(t) \}$$

$$+ p_{\delta} \left( -\frac{\partial}{\partial t} \right) \{ \omega_{j}(t) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \}(t) \}.$$ 

Consider the functions:

$$H^j(t) = p_{\delta} \left( -\frac{\partial}{\partial t} \right) \{ (1 - \omega_{j})(t) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \}(t) \},$$

$$h^j(\nu) = \mathcal{F} H^j(\nu) = p_{\delta}(\nu) \{ \mathcal{F} \{ (1 - \omega_{j})(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \} \} \}(\nu).$$

We observe that $\{ (1 - \omega_{j})(t) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \}(t) \}$ is an even function. For $h^j$ to be in $\mathcal{H}(\mathbb{C})^\delta$, it has to satisfy: $p_{\delta}(\nu) h^j(\nu) = p_{\delta}(\nu) h^j(\nu)$. 

$$p_{\delta}(\nu) h^j(\nu) = p_{\delta}(\nu) p_{\delta}(\nu) \{ \mathcal{F} \{ (1 - \omega_{j})(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \} \} \}(\nu)$$

$$= p_{\delta}(\nu) p_{\delta}(\nu) \{ \mathcal{F} \{ (1 - \omega_{j})(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_{\delta}^{-1}(\cdot) \} \} \}(\nu)$$

$$= p_{\delta}(\nu) h^j(\nu),$$
and hence \( h^j \in \mathcal{H}(\mathbb{C})^\delta \) and \( H^j \in C^\infty_c(\mathbb{R})^\delta \). Let \( f^j \) be the corresponding element of \( C^\infty_c(G/K)^\delta \). Since \( \omega_j \) has support in \([0,j]\), Proposition 2.3.8 tells us that \( f \) may differ from \( f^j \) only inside \( K \times [0,j] \times K \).

Step 3 ((\( j, j + 1 \))): As in step 1 we get:

\[
|f^j(D; g; E)| \leq c_\varphi_o(g) \int_{\mathbb{R}^+} (1 + |\nu|)^R \sum_{l=1}^{d(\delta)} |h^j_{1,l}(i\nu)| d\nu
\leq c_\varphi_o(g) \sum_{l=1}^{d(\delta)} \tau_{1,R+2}^o(h^j_{1,l}).
\]

It follows that:

\[
\sup_{|g| \in [j,j+1]} |F(g)| \leq c j^N e^{\varepsilon j} \sum_{l=1}^{d(\delta)} \tau_{1,R+2}^o(h^j_{1,l}),
\]

by estimates on \( \varphi_\nu \) (Lemma 2.1.9).

Step 4: We now want to find a constant \( c \) and \( m, n \in \mathbb{N} \) s.t.

\[
j^N e^{\varepsilon j} \tau_{1,R+2}^o(h^j_{1,l}) \leq c \sum_{k=0}^{m} \sup_{\nu \in \mathbb{R}} (1 + |\nu|)^n |\nabla^k h^j_{1,l}(\nu)|,
\]

for all \( l \). In the following \( \nabla \) will denote either \( \nabla = \frac{d}{d\nu} \) or \( \nabla = \frac{d}{dt} \). The connection between \( h^j_{1,l} \) and \( H^j_{1,l} \) is:

\[
h^j_{1,l}(\nu) = \mathcal{F}H^j_{1,l}(\nu) = \int_{\mathbb{R}} H^j_{1,l}(t) e^{-\nu t} dt,
\]

and

\[
H^j_{1,l}(t) = \mathcal{F}^{-1} h^j_{1,l}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} h^j_{1,l}(\nu) e^{\nu t} d\nu.
\]

Thus:

\[
\tau_{1,R+2}^o(h^j_{1,l}) = \sup_{\nu \in \mathbb{R}} (1 + |\nu|)^R |h^j_{1,l}(\nu)|
= \sup_{\nu \in \mathbb{R}} (1 + |\nu|)^R |\mathcal{F}H^j_{1,l}(\nu)|
= \sup_{\nu \in \mathbb{R}} (1 + |\nu|)^R \int_{\mathbb{R}} H^j_{1,l}(t) e^{-\nu t} dt
\leq c \sum_{k=0}^{R+2} \int_{\mathbb{R}} |\nabla^k H^j_{1,l}(t)| dt
\leq c \sup_{t \geq 0} \sum_{k=0}^{R+2} (1 + t)^2 |\nabla^k H^j_{1,l}(t)|.
\]

We compute the derivatives of \( H^j_{1,l}(\cdot) \) by the Leibniz rule. \( 1 - \omega_j \) and its derivatives vanish on \([0,j-1] \), and are bounded on \([j-1, \infty[, \) uniformly in \( j \). Consequently:

\[
j^N e^{\varepsilon j} \tau_{1,R+2}^o(h^j_{1,l})
\leq c \sum_{k=0}^{R+2} \sup_{t \geq j-1} (1 + t)^{N+2} e^{\varepsilon j} \left| \nabla^k \omega_j \left( - \frac{\partial}{\partial t} \right) \mathcal{F}^{-1} \{h^j_{1,l}(\cdot)\} \right|(t)
\leq c \sum_{k=0}^{R+2} \sup_{t \geq 0} (1 + t)^{N+2} e^{\varepsilon j} \left| \nabla^k \omega_j \left( - \frac{\partial}{\partial t} \right) \mathcal{F}^{-1} \{h^j_{1,l}(\cdot)\} \right|(t).
\]
Let $P$ and $Q$ be polynomials, then by Cauchy’s Theorem, shifting integral from $i\nu$ to $i\nu + \rho \varepsilon$, we get:

$$P(t)e^{\rho \varepsilon t}Q\left(\frac{\partial}{\partial t}\right)\mathcal{F}^{-1}\{h_{1,t}(\cdot)p_{\delta}^{-1}(-\cdot))\}(t)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} P\left(\frac{\partial}{\partial \nu}\right)\{Q(i\nu - \rho \varepsilon)h_{1,t}(i\nu - \rho \varepsilon)p_{\delta}^{-1}(-i\nu + \rho \varepsilon)\}e^{i\nu t}d\nu.$$ 

Hence we get:

$$\sum_{k=0}^{R+2} \sup_{t \geq 0} (1 + t)^{N+2} e^{\varepsilon pt} \left| \nabla^k p_{\delta}(\frac{\partial}{\partial t}) \mathcal{F}^{-1}\{h_{1,t}(\cdot)p_{\delta}^{-1}(-\cdot))\}(t) \right|$$

$$\leq c \sum_{k=0}^{N+2} \int_{\mathbb{R}} (1 + |\nu|)^{\tilde{R} + 2} \left| (\nabla^k(h_{1,t}p_{\delta}^{-1}(-\cdot)))(i\nu - \rho \varepsilon) \right| d\nu,$$

where $\tilde{R} = \deg p_{\delta} + R$. Now the remaining problem is to estimate $h_{1,t}(\nu)p_{\delta}^{-1}(-\cdot)$ and derivatives on the boundary of the tube $i\mathbb{R}_\varepsilon$. Recall that the polynomials $p_{\delta}(\nu)$ are of the form:

$$p_{\delta}(\nu) = (\nu + \rho + s - 1) \cdots (\nu + \rho), \ s \in \mathbb{N} \ \text{or} \ \ p_{\delta}(\nu) \equiv 1.$$

So we see that $p_{\delta}^{-1}(-\cdot)$ has at most one singularity on the boundary. Assume that we a singularity $\nu_0$ on the boundary of $i\mathbb{R}_\varepsilon$. Since $h_{1,t}(\nu_0) = 0$, we can write $h_{1,t}$ as:

$$h_{1,t}(\nu) = (\nu - \nu_0)h_{1,t}^\#(\nu)$$

$$h_{1,t}^\#(\nu) = \int_0^1 h_{1,t}'(\nu_0 + t(\nu - \nu_0))dt.$$ 

We thus have: $h_{1,t}(\nu)p_{\delta}^{-1}(-\nu) = p_{\delta}^{-1}(-\nu)(\nu - \nu_0)h_{1,t}^\#(\nu)$, where the function $p_{\delta}^{-1}(-\nu)$ has no singularity on the boundary. Furthermore we see that the $N$ first derivatives of $h_{1,t}^\#$ can be estimated by the $(N + 1)$ first derivatives of $h_{1,t}$. Hence we get:

$$\sum_{k=0}^{N+2} \int_{\mathbb{R}} (1 + |\nu|)^{\tilde{R} + 2} \left| (\nabla^k(h_{1,t}p_{\delta}^{-1}(-\cdot)))(i\nu - \rho \varepsilon) \right| d\nu$$

$$\leq c \sum_{k=0}^{N+3} \int_{\mathbb{R}} (1 + |\nu|)^{\tilde{R} + 2} \left| (\nabla^k h_{1,t})\right| (i\nu - \rho \varepsilon) d\nu$$

$$\leq c \sum_{k=0}^{N+3} \sup_{\nu \in i\mathbb{R}_\varepsilon} (1 + |\nu|)^{\tilde{R} + 4} \left| (\nabla^k h_{1,t}(\nu)) \right|.$$ 

All in all this gives, for some positive constant $c$

$$\sigma_{D,E,N}^p(f) \leq c \sum_{l=1}^{d(\delta)} \sum_{k=0}^{N+3} \sup_{\nu \in i\mathbb{R}_\varepsilon} (1 + |\nu|)^{\tilde{R} + 4} \left| (\nabla^k h_{1,t}(\nu)) \right|.$$ 

This concludes the proof. \qed
FOURIER TRANSFORM OF SCHWARTZ FUNCTIONS

Corollary 2.3.10. Let \( 0 < p \leq 2 \). The Fourier transform \( \mathcal{H} \) is a topological isomorphism between \( S^p(G/K)_\delta \) and \( S(i\mathbb{R}_c \times B)_\delta \), taking \( C^\infty_c(G/K)_\delta \) to \( \mathcal{H}(C^\infty \times B)_\delta \). The inverse transform is given by Theorem 2.1.2.

Proof. Theorem 2.3.9 and the isomorphisms discussed in Proposition 2.2.6 and thereafter. Actually let \( h \in \mathcal{H}(C \times B)_\delta \), then we get the following estimate on \( f = \mathcal{H}^{-1}h \in C_c^\infty(G/K)_\delta \), for some positive constant \( c \)

\[
\sigma^p_{D,E,N}(f) \leq c d(\delta)^2 \sum_{l=1}^{d(\delta)N+3} \sup_{k \in K} \sup_{\nu \in i\mathbb{R}_c, k \in K} (1 + |\nu|)^{\tilde{N}+4} |\nabla^k h_{1,l}(\nu, kM)|.
\]

\( \square \)

Let \( S^p(G/K)_K \) and \( S(i\mathbb{R}_c \times B)_\delta \) denote the \( K \)-finite elements of \( S^p(G/K) \) and \( S(i\mathbb{R}_c \times B)_\epsilon \), then we finally get the \( K \)-finite version of Theorem 1:

Theorem 2.3.11. Let \( 0 < p \leq 2 \). The Fourier transform \( \mathcal{H} \) is a topological isomorphism between \( S^p(G/K)_K \) and \( S(i\mathbb{R}_c \times B)_\epsilon \). The inverse transform is given by Theorem 2.1.2.

Remark: It is not possible via a density argument to use the above theorem to prove Theorem 1 in general, since we don’t have a polynomial or uniform bound in the estimate above for the various \( K \)-types. In estimating the derivatives of \( H^1_{\mathcal{H}} \), we used boundedness of finite derivations of \( \omega_j \), but unfortunately we cannot have a uniform or polynomial bound on the derivatives of \( \omega_j \), as this would imply analycity of \( \omega_j \). Hence, as there is no bound on the degree of the polynomials \( p_B \), we cannot have a uniform or polynomial bound on the constants \( c \).

3. General rank.

We will in this Section sketch how to remove the \( \dim a = 1 \) condition in Section 2. So let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) be a Cartan decomposition of \( \mathfrak{g} \), and fix a maximal subspace \( \mathfrak{a} \in \mathfrak{p} \) (\( \dim \mathfrak{a} \geq 1 \)). Denote its real dual by \( \mathfrak{a}^* \) and its complex dual by \( \mathfrak{a}_C^* \). Let \( \Sigma \subset \mathfrak{a}^* \) be the root system of \( (\mathfrak{g}, \mathfrak{a}) \), and let \( W \) be the Weyl group associated to \( \Sigma \) \( (\cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})) \). Choose a set \( \Sigma^+ \) of positive roots, and let \( \mathfrak{a}_+ \subset \mathfrak{a} \) and \( \mathfrak{a}_+^* \subset \mathfrak{a}^* \) be the corresponding positive Weyl chambers. We will define \( \rho \) as in Section 2. Fix \( \varepsilon \geq 0 \), let \( C^{a^\varepsilon} \) be the convex hull of the set \( W \cdot \varepsilon \rho \) in \( \mathfrak{a}^* \) and let \( \mathfrak{a}_C^\varepsilon = \mathfrak{a}^* + i C^{a^\varepsilon} \) be the tube in \( \mathfrak{a}_C^* \) with basis \( C^{a^\varepsilon} \).

In the following, the variable \( \nu \in \mathbb{C} \) in Section 2 will correspond to the variable \( i\lambda \in \mathfrak{a}_C^* \) and differentiation with \( P \left( \frac{d}{d\lambda} \right) \) will correspond to \( P \left( \frac{d}{d\lambda} \right) \), where \( P \in S(\mathfrak{a}^* \times B) \). On nice functions on \( X = G/K \), we will define the Fourier transform as:

\[
\mathcal{H} f(\lambda, b) = f(\lambda, b) = \int_X f(x) e^{-(i\lambda + \rho)(A(x,b))} dx,
\]

for \( \lambda \in \mathfrak{a}_C, b \in B \), when welldefined, see [He2, Chapter III, §1]. Inversion formulas, Plancherel formulas and a Paley-Wiener Theorem can be found in [He2, Chapter III]. For \( \varepsilon \geq 0 \), let \( S(\mathfrak{a}_C^\varepsilon \times B) \) be the Schwartz space on \( a^\varepsilon \times B \) (replace \( \mathbb{R}^d \) with \( \mathfrak{a}_C^\varepsilon \) in definition 2.2.3), and let \( S(\mathfrak{a}_C^\varepsilon \times B)^W \) be the subspace of \( S(\mathfrak{a}_C^\varepsilon \times B) \) of functions satisfying the symmetry condition (SC):

\[
\int_B e^{i(\varepsilon l + \rho)(A(x,b))} \psi(s, \lambda, b) db = \int_B e^{i(\lambda + \rho)(A(x,b))} \psi(\lambda, b) db,
\]

for \( s \in W, \lambda \in \mathfrak{a}_C^\varepsilon \times X \). For \( 0 < p \leq 2 \), \( S^p(G/K) \) denotes the \( L^p \) Schwartz space on \( G/K \). The group \( K \) acts naturally on \( S^p(G/K) \) and \( S(\mathfrak{a}_C^\varepsilon \times B)^W \) (on the second variable, the action preserving the symmetry condition (SC)). Let \( S^p(G/K)_K \) and \( S(\mathfrak{a}_C^\varepsilon \times B)^W_K \) denote the \( K \)-finite elements of \( S^p(G/K) \) and \( S(\mathfrak{a}_C^\varepsilon \times B)^W \) respectively. In this setup, Theorem 2.3.2 and Theorem 2.3.11 become:
Theorem 3.1. Let $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$.

(i) The Fourier transform is an injective and continuous homomorphism from $S^p(G/K)$ into $S(a_\varepsilon^* \times B)^W$.

(ii) The Fourier transform $\mathcal{H}$ is a topological isomorphism between $S^p(G/K)_K$ and $S(a_\varepsilon^* \times B)_K^W$.

(iii) The inverse transform is given by: $(\psi \in S(a_\varepsilon^* \times B)_K^W)$

$$\mathcal{H}^{-1}(\psi)(x) = c \int_{a_\varepsilon^* \times B} e^{(i\lambda + \rho)(\lambda(x,b))} \hat{f}(\lambda, b)|c(\lambda)|^{-2} d\lambda db.$$

As in Section 1, the difficult part is to prove (ii) and (iii). Let $C^\infty_c(X)$ be as usual. Consider the Paley-Wiener space $\mathcal{H}(a_\varepsilon^* \times B)$ (replace $\mathbb{C}$ with $a_\varepsilon^*$ and $|Re \cdot|$ with $|Im \cdot|$), and denote by $\mathcal{H}(a_\varepsilon^* \times B)^W$ the subspace of $\mathcal{H}(a_\varepsilon^* \times B)$ of functions satisfying the symmetry condition (SC) for $\lambda \in a_\varepsilon^*$. Then we have:

Theorem 3.2. Let $0 < p \leq 2$ and $\varepsilon \geq 0$. Then:

(i) $C^\infty_c(X)$ is a dense subspace of $S^p(G/K)$.

(ii) $\mathcal{H}(a_\varepsilon^* \times B)$ is a dense subspace of $S(a_\varepsilon^* \times B)$.

(iii) $\mathcal{H}(a_\varepsilon^* \times B)^W$ is a dense subspace of $S(a_\varepsilon^* \times B)^W$.

Proof. (i) See Lemma 2.2.2.

(ii) See Lemma 2.2.4 and [An].

To prove (iii), we will consider the symmetry conditions (SC) for various $K$-types, and again they reduce to polynomial symmetry conditions. Recall the definitions of the various projections in Section 2, 2.2, and define the subspaces: $S(a_\varepsilon^* \times B)_\delta^W$, $S(a_\varepsilon^* \times B)^{\delta W}$, $\mathcal{H}(a_\varepsilon^* \times B)_\delta^W$ and $\mathcal{H}(a_\varepsilon^* \times B)^{\delta W}$ of functions of $K$-type $\delta$ satisfying (SC). Using the evaluation map, we get isomorphisms between $S(a_\varepsilon^* \times B)^{\delta W}$ and $S(a_\varepsilon^* \times B)_\delta^W$, and between $\mathcal{H}(a_\varepsilon^* \times B)^{\delta W}$ and $\mathcal{H}(a_\varepsilon^*)_\delta^W$, where:

$$\mathcal{H}(a_\varepsilon^*)_\delta^W = \{ F \in \mathcal{H}(a_\varepsilon^*) : (Q^\delta)^{-1}F \text{ is } W\text{-invariant} \}$$

$$S(a_\varepsilon^*)_\delta^W = \{ F \in S(a_\varepsilon^*) : (Q^\delta)^{-1}F \text{ is } W\text{-invariant} \}.$$

Here $Q^\delta(\lambda)$ is the Kostant $Q$-polynomial, see [He2, Chapter III, 2.3.5]. Fix an orthonormal basis $v_1, \cdots, v_{l(\delta)}$ of $V_\delta$ such that $v_1, \cdots, v_{l(\delta)}$ span $V_\delta^M$. Then the members of $\mathcal{H}(a_\varepsilon^*)_\delta^W$ resp. $S(a_\varepsilon^*)_\delta^W$ become matrix valued holomorphic functions on $a_\varepsilon^*$, respectively on $a_\varepsilon^*$, and $Q^\delta$ is an $l(\delta) \times l(\delta)$ matrix whose entries are polynomials on $a_\varepsilon^*$ ([He2, p.287]). Let $\mathcal{H}(a_\varepsilon^*)_\delta^W$ and $S(a_\varepsilon^*)_\delta^W$ denote the spaces of Weyl group invariants in $\mathcal{H}(a_\varepsilon^*, \text{Hom}(V_\delta, V_\delta^M))$, respectively in $S(a_\varepsilon^*, \text{Hom}(V_\delta, V_\delta^M))$ (corresponding to $Q^\delta = I$), then we get the crucial fact:

Fact: The mapping $\psi(\lambda) \mapsto Q^\delta(\lambda)\psi(\lambda)$ is a homeomorphism of $S(a_\varepsilon^*)_\delta^W$ onto $S(a_\varepsilon^*)_\delta^W$, taking $\mathcal{H}(a_\varepsilon^*)_\delta^W$ onto $\mathcal{H}(a_\varepsilon^*)_\delta^W$.

The map clearly is into, and from [He2, Chapter III, Lemma 5.12] we see that the map takes $\mathcal{H}(a_\varepsilon^*)_\delta^W$ onto $\mathcal{H}(a_\varepsilon^*)_\delta^W$. We can write:

$$Q^\delta(\lambda)^{-1} = Q_{\text{c}(\lambda)}(\text{det } Q^\delta(\lambda))^{-1},$$

where $Q_{\text{c}}$ is a matrix whose entries are polynomials on $a_\varepsilon^*$, and $\text{det } Q^\delta(\lambda)$ is a product of polynomials coming from the rank one reduction, see [He2, p.263 (50)] and [He2, Chapter III, Theorem 4.2]. From [He2, §11], we conclude that $\text{det } Q^\delta(\lambda)$ is non-zero in a neighbourhood of $a^* + i\mathfrak{a}^*_+, and considering only one Weyl chamber, using Weyl group invariance, we conclude the result. Using the matrix
valued classical Fourier transform (see below) we conclude that $\mathcal{H}(\mathfrak{a}_C^*)$ is dense in $\mathcal{S}(\mathfrak{a}_C^*)$, and hence $\mathcal{H}(\mathfrak{a}_C^*)$ is dense in $\mathcal{S}(\mathfrak{a}_C^*)$. Elaborating on these results, as in Section 2, we get iii). □

Let $\mathcal{R}$ be the Radon transform (replace $a_t \in A$ with $\exp(H)$, $H \in \mathfrak{a}$). Let $\phi \in C_c^\infty(\mathfrak{a} \times B)$, then the "classical" Fourier transform on $\mathfrak{a} \times B$ is defined as:

$$ F\phi(\lambda,b) = \int_a \phi(H,b)e^{-i\lambda(H)}dH, \lambda \in \mathfrak{a}_C^*, b \in B. $$

We have the following commutative diagram:

$$ \xymatrix{ \mathcal{H}(\mathfrak{a}_C^* \times B)^W \ar[rd]^F \ar[rr]^\mathcal{R} & & \mathcal{R} C_c^\infty(G/K) \subset C_c^\infty(\mathfrak{a} \times B) } $$

As in Section 2, all transforms preserves $K$-types. We define the $\delta$-spherical transform:

$$ \mathcal{H}\delta f(\lambda) = d(\delta) \int_X f(x)\Phi_{\lambda,\delta}(x)^*dx $$

where $\Phi_{\lambda,\delta}$ is the generalized spherical function:

$$ \Phi_{\lambda,\delta} = \int_K e^{i(\lambda + \rho)(A(x,kM) + \delta(k))}dk, x \in X, $$

see [He2, Chapter 3, §2, §5].

**Theorem 3.3.** The $\delta$-spherical transform $f \mapsto \mathcal{H}\delta f$ is a bijection of $C_c^\infty(G/K)_\delta$ onto $\mathcal{H}(\mathfrak{a}_C^*)^\delta_W$.

**Proof.** [He2, Chapter III, Theorem 5.11]. □

Defining an operator $\mathcal{T}$ as in Section 2, and considering the "classical" Fourier transform on matrix coefficients

$$ F\phi(\lambda,b) = \int_a \phi(H)e^{i\lambda(H)}dH, \lambda \in \mathfrak{a}_C, $$

we get the commuting diagram:

$$ \xymatrix{ \psi \in \mathcal{H}(\mathfrak{a}_C^*)^\delta_W \ar[rd]^F \ar[rr]^\mathcal{H}\delta & & \phi \in C_c^\infty(\mathfrak{a})^\delta_W } $$

The subscript $W$ on $C_c^\infty(\mathfrak{a})^\delta_W$ indicates some kind of symmetry condition. To show Theorem 3.1 ii)+iii), we are left to show that $(\mathcal{H}\delta)^{-1}$ is a continuous homomorphism from $\mathcal{H}(\mathfrak{a}_C^*)^\delta_W$ into $C_c^\infty(G/K)_\delta$ in the induced topologies. Again we want to use a cut-off function, as in Section 2 and [An]. To do so, we need to reformulate the Paley-Wiener Theorems for the various transforms. Define convex $W$-invariant sets in $\mathfrak{a}$ and $G$ by: $\mathfrak{a}_R = \{H \in \mathfrak{a}|\rho(w \cdot H) \leq R, \forall w \in W\}$, and $G_R = K(\exp \mathfrak{a}_R)K$. Furthermore define the gauge associated to $\mathfrak{a}_R$ by $q(\lambda) = \sup_{H \in \mathfrak{a}_R} \lambda(H)$. 
Theorem 3.4. The "classical" Fourier transform $\mathcal{F}$ is an isomorphism between the space of all functions $\phi \in C_c^\infty(a)$, such that $\text{supp} \phi \subset a_R$, and the space of all entire functions $\psi$ on $a^*_\mathbb{C}$, such that
\[
\sup_{\lambda \in a^*_\mathbb{C}} e^{-\eta \lvert \Im \lambda \rvert} (1 + \lvert \lambda \rvert)^N \lvert \psi(\lambda) \rvert < \infty,
\]
for all $N \in \mathbb{N}$.

Theorem 3.5. The $\delta$-spherical transform $\mathcal{H}_\delta$ is an isomorphism between the space of functions $f \in C_c^\infty(G/K)$, with $\text{supp} f \subset G_R$, and the space of functions $\psi \in \mathcal{H}(a^*_\mathbb{C})^\delta_W$, with matrix entries satisfying (5).

Proof. A slight modification of the proof of [He2, Chapter III, Theorem 5.11], using the ideas from [An, p. 341-2].

Corollary 3.6. The transform $\mathcal{T}$ is an isomorphism between the spaces $C_c^\infty(G/K)$ and $C_c^\infty(a)^\delta_W$. Moreover $\text{supp} f \subset G_R$ if and only if $\text{supp} \mathcal{T} f \subset a_R$.

Consider the commuting diagram (4). Given a seminorm $\sigma$ in $\mathcal{S}^p(G/K)$, we shall find a seminorm $\tau$ on $\mathcal{S}(a^*_\mathbb{C})^\delta_W$, such that $\sigma(f) \leq \tau(\psi)$, for all $f$ and $\psi$. Instead of looking at the intervals $[0, j]$ and $[j, j + 1]$, as we did in Section 2, we will consider the sets $G_j$ and $G_{j+1} \setminus G_j$. The crucial point in the proof is then to estimate $f(D; g; E)$ on $G_{j+1} \setminus G_j$. Let $\omega \in C_c^\infty(\mathbb{R})$, with $\omega \equiv 0$ on $]-\infty; 0]$, and $\omega \equiv 1$ on $[1; \infty[$. Introduce a $W$-invariant function in $C_c^\infty(a)$ by:
\[
\omega_j(H) = \prod_{\omega \in W} (\omega(j - \rho(w \cdot H))).
\]
We see that $\omega_j \equiv 1$ on $a_{j-1}$, and $\omega_j \equiv 0$ outside $a_j$. Moreover $\omega_j$ and all its derivatives are bounded in $j$. We decompose $\phi$ as:
\[
\phi(H) = Q^\delta \left( -i \frac{\partial}{\partial H} \right) \{ (1 - \omega_j)(H) \mathcal{F}^{-1} \{(Q^\delta)^{-1}(\cdot)(\psi(\cdot))(H) \} \\
+ Q^\delta \left( -i \frac{\partial}{\partial H} \right) \{ \omega_j(H) \mathcal{F}^{-1} \{(Q^\delta)^{-1}(\cdot)(\psi(\cdot))(H) \} \}.
\]

Consider the functions
\[
\phi_j(H) = Q^\delta \left( -i \frac{\partial}{\partial H} \right) \{ (1 - \omega_j)(H) \mathcal{F}^{-1} \{(Q^\delta)^{-1}(\cdot)(\psi(\cdot))(H) \}),
\]
\[
\psi_j(\lambda) = \mathcal{F} \phi_j(\lambda) = Q^\delta(\lambda) \{ \mathcal{F} \{(1 - \omega_j)(\cdot)\mathcal{F}^{-1} \{(Q^\delta)^{-1}(\cdot)(\psi(\cdot))\}(\cdot)\}\}(\lambda),
\]
\[
f_j(x) = ((\mathcal{H}_\delta)^{-1} \psi_j)(x) = (T^{-1} \phi_j)(x).
\]
We see that $\psi_j \in \mathcal{H}(a^*_\mathbb{C})^\delta_W$ and $\phi_j \in C_c^\infty(a)^\delta_W$. Since $\omega_j$ has support in $a_j$, Corollary 3.6 tells us that $f$ may differ from $f_j$ only inside $G_j$. Continuing as in [An] or Section 2, using elementary Fourier analysis, we see that the remaining problem is to estimate $(Q^\delta)^{-1}(\cdot)(\psi(\cdot))$ and its derivatives on the boundary of $a^*_\mathbb{C}$, with similar estimates on $\psi(\cdot)$. Following the argument of Theorem 2.3.9, using the knowledge from the proof of Theorem 3.2, we reach the result.

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Department of Mathematics, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark

E-mail address: byrial@imf.au.dk