Solving one-dimensional unconstrained global optimization problem using parameter free filled function method

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Abstract

It is generally known that almost all filled function methods for one-dimensional unconstrained global optimization problems have computational weaknesses. This paper introduces a relatively new parameter free filled function, which creates a non-ascending bridge from any local isolated minimizer to other first local isolated minimizer with lower or equal function value. The algorithm’s unprecedented function can be used to determine all extreme and inflection points between the two considered consecutive local isolated minimizers. The proposed method never fails to carry out its job. The results of the several testing examples have shown the capability and efficiency of this algorithm while at the same time, proving that the computational weaknesses of the filled function methods can be overcome.

Keywords: curvature, filled function, global optimization, local optima, Newton’s method

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1. Introduction

This article presents a new method for finding the global minimum of a non-convex function. A variety of fields, including engineering, operational research, finance and social sciences can be transformed as the optimization model where its objective function is non-convex. This non-convexity property makes the main reason a classical optimization method often fails to obtain a global minimizer (maximizer) [1-10]. The existing methods for finding a global minimizer provide the global descent [11, 12], the interval [13-17], the simulated annealing [18], the genetic algorithm [19], one-dimensional global optimization with Lipschitz conditions [20, 21], the filled function methods [4-6, 22-25] ideas and etc. Most of the filled function methods work quite different compared to the methods described in [11-22, 26-28]. This article will focus on the filled function methods and the comparison among them.

Suppose that \( x_k^* \) \((k = 0, ..., m)\) are \( m \) isolated minimizers of \( f \) Four different parametric filled functions at \( x_k^* \) are defined by

\[
F_0 = F_0(x, x_k^*, r, \rho) = \frac{1}{r + f(x)} \exp \left( -\frac{\|x-x_k^*\|^2}{\rho^2} \right),
\]

\[
F_2 = F_2(x, x_k^*, r, \mu) = f(x_k^*) - \min(f(x_k^*), f(x)) - \rho\|x-x_k^*\|^2 + \mu\left\{ \max(0, f(x) - f(x_k^*)) \right\}^2,
\]

\[
F_{LS} = F_{LS}(x, x_k^*, \tau, \rho) = \eta(0.5\|x-x_0\|^2) + \varphi(\tau[f(x) - f(x_k^*) + \rho]), \quad \text{and}
\]

\[
F_{\lambda} = F_{\lambda}(x, x_k^*, \lambda) = \frac{1}{\ln(1+f(x) - f(x_k^*))} - a\|x-x_k^*\|^2,
\]

are proposed in [4, 26, 29, 22] respectively where parameter \( a \) is defined by

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a = \frac{\xi |f'(x_0)|}{\sqrt{\frac{1}{|x_0 - x_i|} (1 + f(x_0)) (\ln(1 + f(x_0)) - f(x_0))}}. \tag{5}

Unfortunately, the existing filled function methods [23, 26-29] can not solve the global optimization problems since:

a. cannot assure the existence of a better local minimizer in a lower basin [29, 30];

b. require the assumption that \( f \) has only a finite number of local minimizer which have different function values, i.e., \( f(x_i) \neq f(x_j) \) if \( x_i \neq x_j \);

c. difficult to adjust an appropriate parameter to satisfy the conditions of filled function;

d. iteratively updated the parameter;

e. can only obtain one global optimizer, and

f. contain exponential or logarithmic expressions in their forms which make a large amount of computation.

For filled function [24], its two parameters, one of which relies on the diameter of a bounded closed domain which contains all global minimizers, and the other on Lipschitz constant of \( f \) respectively. The parameter free filled function (PFFF) was initially introduced in [30-35]. A PFFF proposed by Ma et al. [34] is

\[ F_p(x, x_k) = -\text{sign}(f(x) - f(x_k)) \arctan(||x - x_k||^2) \text{ where sign}(t) = \begin{cases} 1, & (t \geq 0) \\ -1, & (t < 0) \end{cases} \tag{6} \]

where this method also has weaknesses as the others.

Our new PFFF method (or simply IYRH’s method) is based on PFFF [30-35] for global optimization of \( f: D \subseteq R \to R \) where \( f \) satisfies the following seven assumptions:

A1. \( f \) is a trice continuously differentiable on \( D \) (or \( f \in C^3(D) \))

A2. \( f \) has only a finite number of extreme and inflection points in \( D \), and \( f^{(n)} x_j \neq 0 \) for \( n \geq 3 \) where \( x_j \) is an inflection point of \( f \).

A3. \( f^{(1)}, f^{(2)} \) and \( f^{(3)} \) of \( f \) are Lipschitz-continuous with computable constants.

A4. \( f(x) \to \infty \) as \( |x| \to \infty \).

A5. For \( x_k \) ([36]), \( f(x) - f(x_k) = 0 \) yields at most two nearest points \( x_k^- \) and \( x_k^+ \) located on the left and the right hand sides of \( x_k \), respectively such that \( f(x_k^-) = f(x_k) = f(x_k^+) \) and \( x \in [x_k^-, x_k^+] \).

A6. \( f(x) > f(x_k) \) for \( x \in (x_k^-, x_k^+) \) and \( f(x) = f(x_k) \) if \( x = x_k^- \) or \( x = x_k^+ \).

A7. There exists only one \( x \), between two consecutive minimizer and maximizer of \( f(x) \).

The reason why we need to solve one-dimensional multimodal function is described in many references cited in [20]. The needed is appeared in scientific and engineering applications especially in electrical engineering optimization problem. One of the important issues in global optimization is “the region of attraction” where its detailed explanation can be seen in [3].

This paper is organized as follows. Section 2 describes the IYRH’s function. Section 3 describes how to find all extreme and inflection points using the IYRH’s function. Section 4 discusses the relationship between \( f \) and IYRH’s function. In Section 5, the idea of curvature is described. Section 6 contains the convergence theorem. The numerical results of IYRH’s algorithm will be presented in Section 7. Comparison and discussion will be given in Section 8. Section 9 contains the conclusion and the brief explanation on how this one-dimensional case can be extended to n-dimensional case.

2. A Relatively New Parameter Free Filled Function

In this section, the IYRH’s will be derived. Definition 1 (One-Dimensional PFFF):

Suppose that \( f: [a, b] \subseteq R \to R \) satisfies A1–A7. A new \( F(x, x_k) \) \( x \in [x_k^-, x_k^+] \) called IYRH’s function of \( f \) at \( x_k \), is defined by:

\[ F(x, x_k) = \begin{cases} -\int_{x_k^-}^{x_k} (f(s) - f(x_k)) ds & (x_k^- \leq x \leq x_k) \\ -\int_{x_k^-}^{x_k^+} (f(s) - f(x_k)) ds & (x_k \leq x \leq x_k^+) \end{cases} \tag{7} \]

if \( F(x, x_k) \) satisfies the following 3 conditions. C1. \( x_k \) is a local isolated maximizer of \( F(x, x_k) \), C2. \( F(x, x_k) \) has no stationary point in the interval \( (x_k^-, x_k^+) \cup (x_k, x_k^+) \), and C3.
If \( x_k^\pm \) is not a global minimizer of \( f \), then \( x_k^- \) and \( x_k^+ \) are the minimizer or stationary points of \( F(x, x_k^\pm) \). It is enough to consider the second integration of (7) which can be rewritten as follows:

\[
F(x, x_k) = -\int_{x_k^-}^{x_k^+} (f(x) - f(x_k^\pm)) \, dx, \quad (x_k^- \leq x \leq x_k^+)
\]

by using A5. The following results can be proved. Theorem 1: 1) \( f \in C^3(a, b) \); 2) \( x_k^\pm \in [x_k^-, x_k^+] \subseteq [a, b] \) and; 3) \( F(x, x_k) \) is defined by (6), then \( x_k^\pm \) must be a local isolated maximizer of \( F(x, x_k) \). Theorem 2: If the hypotheses of Theorem 1 are valid, then \( F(x, x_k) \) does not have any stationary point in the interval \( I_k = \{ x : f'(x) > f'(x_k^\pm), x \in (x_k^- , x_k^+) \} \). By (8), for \( x_1, x_2 \in [x_k^-, x_k^+] \) with \( x_1 < x_2 \), \( F(x_1, x_k) - F(x_2, x_k) = \int_{x_1}^{x_2} (f(x) - f(x_k^\pm)) \, dx \geq 0 \). Thus, \( F(x_k, x_k) \) decreases over \([x_k^-, x_k^+]. By similar argument, \( F(x, x_k) \) increases over \([x_k^-, x_k^+]. Theorem 3: If the hypotheses of Theorem 1 are valid and \( x_k^\pm \) is not a global minimizer of \( f(x) \), then \( x_k^- \) and \( x_k^+ \) are the minimizer or stationary point of \( F(x, x_k) \).

The obtaining of all the extreme and inflection points in every \([x_k^-, x_k^+]\) \((k = 0, 1, 2, ..., t < \infty)\) is an indicator that this algorithm never fail to obtain the global one. That is why it makes this method explores along the entire domain which very much different to other methods [4-6, 11-21].

We are not aware with the method in [37], which quiet similar with our method. Fortunately, our method has been published first as mentioned in [30-35]. However, we did not know how the authors [37] compute their hypotheses of Theorem 1 are never been computed as had been done in [30-35]. Therefore, IYRH’s algorithm very much different compared with others.

3. Sequences of Extreme and Inflection Points

The IYRH’s function \( F(x, x_k) \) \((x \in [x_k^-, x_k^+])\) has the following properties:

P1. \( F(x, x_k) \) is concave downward at \( x_k^\pm \) and concave upward at both \( x_k^- \) and \( x_k^+ \).

P2. \( F(x, x_k^+) = F^1(x, x_k^+), F^2(x, x_k^+), \) and \( F^3(x, x_k^+) \) are continuous.

P3. \( F(x, x_k) < 0 \) for \([x_k^-, x_k^+] \) and \( F(x_k, x_k) = 0 \).

P4. \( F(x, x_k^+ \pm) \) are increasing and decreasing over \([x_k^-, x_k^+] \) and \([x_k^-, x_k^+] \) respectively.

P5. \( F^1(x, x_k) > 0 \) \((x \in [x_k^-, x_k^+])\) and \( F^1(x, x_k^+) < 0 \) \((x \in [x_k^-, x_k^+]) \) except at inflection points

P6. \( F(x, x_k) \) has isolated minimizer or stationary point at \( x_k^- \) or \( x_k^+ \).

As an example, the graph of our PFFF for \( \sin x + \sin(2x/3) \) can be plotted as in Figure 1. By P1, there exists at least two inflection points of \( F(x, x_k) \) each lies in \((x_k^-, x_k^+) \) and \((x_k^-, x_k^+) \). By P1-P6, Figure 1 and \([x_k^-, x_k^+] \) \((x_k^-, x_k^+) \), the IYRH’s function generates the sequence of:

\[
x_0, x_1, x_2, ..., x_k, ... (x_0, x_{k-1}^{-1}, x_{k-1}^{-2}, ..., x_{k-1}^{-k}, ...)
\]

using any suitable optimization tools except \( x_0 \), starting at:

\[
x_0, x_0^+, x_1^+, x_2^+, ..., x_k^+, ..., x_k^+, x_0^-, x_1^-, x_2^-, ..., x_k^-, ...
\]

respectively where as beginning \( x_0 \) is any given point for obtaining \( x_0 \). Therefore, we have 2 phases as follows:

Phase I: Starting at \( x_k \) \((k = 0, 1, 2, ...). Minimize \( f(x) \) to obtain isolated minimizer \( x_k^- \).

Phase II: \( F(x, x_k^-) \) is constructed to find \( x_k^+ \). Replace \( x_k \) with \( x_k^+ \). Restart Phase 1.

However, it is not easy to obtain (9) and (10) since \( F(x, x_k) \) contains at least 1 inflection point in \([x_k^-, x_k^+] \) and also in \([x_k^-, x_k^-]\). The analytical existence of inflection points is shown as follows.

a. \( F^1(x, x_k) = f(x) - f(x_k) \) \((x_k^- \leq x \leq x_k^+) \) and \( F^2(x, x_k) = -f^1(x) \) \((x_k^- \leq x \leq x_k^+) \). If \( F^2(x, x_k) = 0 \), then \( f^1(x) = 0 \). Therefore, the solution of \( F^2(x, x_k) = 0 \) becomes the critical point of \( f(x) \).

b. \( F^3(x, x_k) = -f^2(x) \) \((x_k^- \leq x \leq x_k^+) \), if \( F^3(x, x_k) = 0 \), then \( f^2(x) = 0 \). Therefore, by A2, the solution of \( F^3(x, x_k) = 0 \) \((the critical point of \( F^3(x, x_k) \)) \) becomes the inflection point of \( f(x) \) but this solution becomes the critical point (maximizer or minimizer) of \( F^2(x, x_k) \).
Therefore, the isolated extreme points i.e. $x_{M,k}^{(1)}$ (local maximizer) and $x_{m,k}^{(1)}$ (local minimizer) of $f(x)$ as shown in Figure 2, are inflection points of $F(x,x_k^*)$ where $(1)$ the superscript of $x_{M,k}^{(1)}$ and $x_{m,k}^{(1)}$, denotes the first inner iteration in the interval $[x_k^a, x_k^b]$.

Note that there might exist more than one inner iteration in the interval $[x_k^a, x_k^b]$ and this will happen when more than two inflection points occurred in $[x_k^a, x_k^b]$. Therefore for Phase II, we need to analyse the behaviour of $F(x,x_k^*)$. Since $x_k^*$ cannot be used blindly to minimize the filled function, then in the phase II, a $\delta > 0$ must be chosen such that $x_k^* + \delta$ can be safely utilized to minimize $F(x,x_k^*)$. For handling these difficulties, consider the relationship between $f$, $F$ and $F^{(2)}$ as illustrated in Figure 2. Since Newton’s method [38] sometimes fails to converge to $x_k^*$, we need IYRH’s function method to handle it. From Figure 2 and the above discussion, it is clear that all minimizers and maximizers of $F^{(2)}(x,x_k^*)$ become the inflection points of $f(x)$, and all the roots of $F^{(2)}(x,x_k^*)$ become minimizers or maximizers of $f(x)$. These special properties are only possessed by IYRH’s function.

4. Computation of the Inflection Points

Based on the discussion in Section 3, we have proved the following theorems: Theorem 4: If the hypotheses of Theorem 1 are valid, then the solution of $F^{(3)}(x,x_k^*) = 0$ becomes the critical point of $f(x)$. Theorem 5: If the hypotheses of Theorem 1 are valid, then the critical point of $F^{(2)}(x,x_k^*)$ becomes the inflection points of $f(x)$.

By A5 and Figure 2, $x_{l,k}^{(1)}$, $x_{M,k}^{(1)}$ and $x_{m,k}^{(1)}$ are the first isolated inflection, maximum and minimum points of $f(x)$ respectively, $x_{l,k}^{(2)}$ is the second inflection point of $f(x)$ found after $x_k^*$ where $x_{l,k}^{(1)} < x_{M,k}^{(1)} < x_{l,k}^{(2)} < x_{m,k}^{(1)}$, and it might continue with another sequence of extreme and inflection points until $x_k^{b+}$ such that $f(x_k^{a+}) = f(x_k^*$) and $f(x_k^{b+}) < f(x) \ (x \in (x_k^*,x_k^{b+}))$. In order to guarantee no extreme or inflection points of $f(x)$ missed during the computation, the outer and inner iterations are used over $[x_k^a,x_k^{b+}]$. In inner iteration, $F^{(3)}(x,x_k^*)$ and $F^{(2)}(x,x_k^*)$ are used to compute inflection and extreme points of $f(x)$ respectively whereas in outer iteration, $f(x)$ is minimized or solve $F^{(3)}(x,x_k^*) = 0$ to obtain $x_k^{b+}$. The following steps implement those both inner and outer iterations:

Outer Iteration
Step 1 : construct $F(x,x_k^*)$ at $x_k^*$.
Inner Iteration
Step 2 : Solve $F^{(3)}(x,x_k^*) = 0$ by Newton’s method for inflection point of $f(x)$ nearest to $x_k^*$.
Step 3 : Solve $F^{(2)}(x,x_k^*) = 0$ by Newton’s method for isolated maximizer of $f(x)$.
Step 4 : Solve $F^{(3)}(x,x_k^*) = 0$ by Newton’s method for next inflection point of $f(x)$.
Step 5 : Solve $F^{(2)}(x,x_k^*) = 0$ by Newton’s method for isolated minimizer of $f(x)$.
Step 6: If \( f(x) > f(x^*_i) \) and \( x < x^*_k \), then repeat Step 2. Step 5 else solve \( F^{(1)}(x, x^*_i) = 0 \) by Newton’s method for \( x^*_k \) such that \( f(x^*_k) = f(x^*_i) \).

Step 7: Use \( x^*_k \) to yield \( x^*_{k+1}, k := k + 1 \). Go to Step 1 if \( x^*_{k+1} < b \).

5. Convergence with Curvature

The curvature \([39]\) and radius of curvature are defined by:

\[
\kappa(x) = \frac{|d^2y|}{dx^2} = \frac{|d^2y(x)/dx^2|}{(1 + (dy/dx)^2)^{3/2}} \quad \text{and} \quad \rho(x) = \frac{1}{\kappa(x)}
\]

respectively. Basically, to make Newton’s method converges to \( x_\ast \), the solution of \( f(x) = 0 \), we need an initial estimation which closes enough to \( x_\ast \). Assign the radius of curvature of \( f(x) \) to \( \rho \), therefore \( x^*_k + \rho \) becomes the initial best estimator for Newton’s method to solve \( f(x) = 0 \). We will prove that \( \eta = |x^*_k + \rho - x_\ast| \) is the radius of the largest interval around \( x_\ast \) such that the Newton’s method converges to \( x_\ast \in (x_\ast - \eta, x_\ast + \eta) \). However, we will need the following definition.

Definition 2 \([40]\): The function \( f: D \subseteq R \rightarrow R \) is Lipschitz continuous function with constant \( \gamma \) in \( D \), written \( f \in \text{Lip}_\gamma(D) \), if for every \( x, y \in D \), \( |f(x) - f(y)| \leq \gamma |x - y| \).

For the convergence of Newton’s method, we need \( f^{(1)} \in \text{Lip}_\gamma(D) \) which had been shown in \([40]\).

Lemma 1 \([40]\): If 1) \( f: D \subseteq R \rightarrow R \) for an open interval \( D \); 2) \( f^{(1)} \in \text{Lip}_\gamma(D) \), then for any \( x, y \in D \), \( |f(y) - f(x) - f^{(1)}(x)(y - x)| \leq \gamma |y - x|^2/2 \).

For most problems, Newton’s method will converge \( q \)-quadratically to its root \([40]\).

Theorem 6 \([40]\): If 1) \( f: D \subseteq R \rightarrow R \) for an open interval \( D \); 2) \( f^{(1)} \in \text{Lip}_\gamma(D) \), 3) for some \( \beta > 0 \), \( |f^{(1)}(x)| \geq \beta \) \( \forall \in \( D \) \); 4) \( f(x) = 0 \) has a solution \( x_\ast \in D \), then there is some \( \eta > 0 \) such that if \( |x_0 - x_\ast| < \eta \), then \( \{x_n\} \) generated by \( x_{n+1} = x_n - (f(x)/f^{(1)}(x)) \) \( (n = 0, 1, 2, \ldots) \) and converges to \( x_\ast \). Furthermore, \( |x_{n+1} - x_\ast| \leq (\gamma / 2\beta) |x_n - x_\ast|^2 \) \( (n = 0, 1, 2, \ldots) \).

Now, we prove that \( \eta = |x^*_k + \rho - x_\ast| \), the radius of the largest interval around the solution of \( f^{(1)}(x) = 0 \) holds Theorem 6. The similarity proof is applied for \( F^2(x, x^*_k) \). Theorem 7: If 1) \( f: D \subseteq R \rightarrow R \) is an objective function; 2) \( x^*_1 \) is a local isolated minimizer of \( f(x) \); 3) \( f^{(1)}: D \subseteq R \rightarrow R \) and \( f^{(2)} \in \text{Lip}_\gamma(X) \); 4) for some \( \rho > 0 \), \( |f^{(2)}(x)| \geq \rho \) \( \forall \in \( D \) \); 5) \( f^{(1)}(x) = 0 \) has a solution \( x_\ast \in D \), then there is some \( \eta > 0 \) such that if \( |x_0 - x_\ast| < \eta \), then \( \{x_n\} \) generated by \( x_{n+1} = x_n - (f^{(1)}(x)/f^{(2)}(x)) \) \( (n = 0, 1, 2, \ldots) \) and converges to \( x_\ast \). Furthermore, \( |x_{n+1} - x_\ast| \leq (\gamma / 2\beta) |x_n - x_\ast|^2 \) \( (n = 0, 1, 2, \ldots) \).

6. Convergence of the IYRH’s Algorithm

By A2, IYRH’s algorithm actually generates (9) and (10) according to the following pattern:

\[
\begin{align*}
x^*_1 & \uparrow \quad x^*_2 & \uparrow \quad \ldots \uparrow \quad x^*_k & \uparrow \quad \ldots \uparrow \quad x^*_n \\
x_0 & \rightarrow \quad x_1 & \rightarrow \quad \ldots \rightarrow \quad x_{k-1} & \rightarrow \quad \ldots \rightarrow \quad x_{n-1} \\
Z^+ & \quad Z^+ & \quad \ldots \quad Z^+ & \quad \ldots \quad Z^+ \\
x_1 & \rightarrow \quad x_2 & \rightarrow \quad \ldots \rightarrow \quad x_{k-1} & \rightarrow \quad \ldots \rightarrow \quad x_{n-1} \\
x_0 & \rightarrow \quad x_1 & \rightarrow \quad \ldots \rightarrow \quad x_{k-1} & \rightarrow \quad \ldots \rightarrow \quad x_{n-1} \\
\end{align*}
\]

which satisfy \( f(x^*_1) \geq f(x^*_2) \geq \ldots \geq f(x^*_k) \geq \ldots \geq f(x^*_n) \) where \( x_0 \) is any given point in the considered interval. Therefore, IYRH’s algorithm generates a finite sequence \( \{x^*_1, x^*_2, \ldots, x^*_k, x^*_k, \ldots, x^*_n, x^*_n\} \). Thus, by A2, IYRH’s algorithm converges to \( x^*_k \) as a global minimizer. IYRH’s algorithm also automatically generates at least a set of finite sequence of inflection, local isolated maximizers and isolated minimizers, \( A_1 = \{x^{(1)}_{1,k}, x^{(2)}_{1,k}, \ldots, x^{(n)}_{1,k}\} \) in every subinterval \( [x^*_n, x^*_k] \) \( (k = 0, 1, \ldots, n) \) if exist where the superscript \( (1) \) on \( x^{(1)}_{1,k} \) denotes the first number of local maximizer and subscript of \( A \) denotes the number of local maximizer in \( [x^*_n, x^*_k] \). If it contains two local isolated maximizers, then it generates...
\[ A_2 = \{ x^{(1)}_{M,k}, x^{(2)}_{M,k}, x^{(1)}_{m,k}, x^{(2)}_{m,k} \} \] and so forth. However for \( A_2 \), the inflection point \( x^{(4)}_{i,k} \) is option. Thus, the following theorem is proved.

Theorem 8 (Convergence Theorem): If 1) all the hypothesis of Theorem 6 and Theorem 7 are valid for \( f, f^{(1)}, f^{(2)}, f^{(3)} \) and 2) \( F(x,x^*) \) is IYRH's function at \( x^* \), the local isolated minimizer of \( f \), then IYRH's algorithm converges to the right solution.

7. Numerical Results

The test examples are listed in Tables 1–3. In Table 1, where \( N, f(x), \psi^*_k, \) and \( m^*_k \) denote the number of function, expression of the objective function, the domain, global minimum value and global minimizer respectively. The numerical results will be presented to compare the capability of the IYRH's method with two-parameter filled function methods [4, 26, 29], one-parameter filled function methods [6, 22, 28], and the PFFF method [34]. We also present the results for observing the sensitivity of IYRH's method due to different initial points. Therefore, the presentation is arranging into four categories.

| \( N \) | \( f(x) \) | \( D \) | \( \psi^*_k \) | \( m^*_k \) |
|---|---|---|---|---|
| 1. \( \frac{1}{6}x^6 - \frac{52}{25}x^5 + \frac{39}{80}x^4 + \frac{71}{10}x^3 - \frac{79}{20}x^2 - x + \frac{1}{10} \) | \([-1.5,1.1]\) | \(-29763.233\) | 10 |
| 2. \( \sin x + \sin (10/3)x \) | \([-9.4,10]\) | \(-12.03124\) | 5.145735 |
| 3. \(-\sum_{k=1}^{5} k \sin((k+1)x+k) \) | \([-9.4,10]\) | \(-12.03124\) | 5.791785 |
| 4. \(-16x^2 - 24x + 5)e^{-x} \) | \([-10,10]\) | \(-0.824239\) | 0.679579 |
| 5. \(-(-3x+1.4) \sin(18x) \) | \([-10,10]\) | \(-1.48077\) | 0.96609 |
| 6. \(-(x+\sin(x))e^{-x} \) | \([-10,10]\) | \(-1.48077\) | 0.96609 |
| 7. \( \sin(x) + \sin(10/3x) + \ln x - 0.84x + 3 \) | \([-79,1.1]\) | \(-10.48077\) | 0.679579 |
| 8. \(-\sum_{k=1}^{5} k \cos((k+1)x+k) \) | \([-79,1.1]\) | \(-10.48077\) | 0.679579 |
| 9. \( \sin x + \sin (2/3)x \) | \([-1.57,6.28]\) | \(-3\) | 4.76837e - 009 |
| 10. \( -\sin x \) | \([-0.10]\) | \(-7.91673\) | 7.9787 |
| 11. \( -2 \cos x - \cos 2x \) | \([-1.57,6.28]\) | \(-3\) | 4.76837e - 009 |
| 12. \( \sin^3 x + \cos^3 x \) | \([-0.628]\) | \(-1\) | \( \pi \) |
| 13. \( -x^{2/3} - (1 - x^{1/3}) \) | \([0.001,0.99]\) | \(-1.5874\) | \( \sqrt[3]{2} \) |
| 14. \( -e^{-x} \sin 2nx \) | \([0.4]\) | \(-0.788685\) | 0.224885 |
| 15. \( (x^2 - 5x + 6) + (x^2 + 1) \) | \([-5.5]\) | \(-7.03553\) | \(-0.41422\) |
| 16. \( 2(x - 3)^2 + e^{-x/2} \) | \([-3.3]\) | \(0.011109\) | 3 |
| 17. \( x^2 - 15x^4 + 27x^2 + 250 \) | \([-4.4]\) | \(-3\) | \(-3\) |
| 18. \( (x - 2)^2 \leq 3 \) | \( (x - 2)^2 \leq 3 \) | \( 0 \) | 2 |
| 19. \( -\sin 3x + x + 1 \) | \([0.65]\) | \(0.467511\) | 0.41032 |
| 20. \( -(x + \sin x) e^{-x^2} \) | \([-10,10]\) | \(-0.634905\) | \(-1.19514\) |

For first category, the results of Table 4 shows that IYRH's algorithm can solve the global optimization problems listed in Table 1. In Table 4, \( k (k \geq 0) \) is the number of outer iteration, \( j (j \geq 0) \) is the number of inflection points \( x^{(i)}_{i,k} \) and \( x^{(i+1)}_{i,k} \) where \( i \) refers to the word “inflection", \( n \) is the number of local isolated maximizer \( x^{(n)}_{M,k} \) and minimizer \( x^{(n)}_{m,k} \) where \( M \) and \( m \) denote maximizer and minimizer respectively, \( x^{*}_i (k \geq 0) \) is isolated minimizer and \( x^{+}_k \) and \( x^{-}_k (k \geq 0) \) are points such that \( f(x^{+}_k) = f(x^{-}_k) = f(x^{*}_k) \).
In inner iteration, there are several cases that $x_{m,k}^{(n)}$ equals to $x_k^{2r}$ and $x_{k+1}^{2r}$ as shown in Table 4 for $k = 2, 3$ of example 3 ($N = 3$), $k = 3$ of example 8 ($N = 8$), $k = 1$ of example 12 ($N = 12$) and $k = 0$ of example 17 ($N = 17$). For second category, Tables 5–8 compare IYRH’s algorithm with New algorithm [42], the direct method [42] and Lagrange interpolation [43], using test functions in Table 1 [44] and 100 one-dimensional randomized test functions [45]. Table 5 shows the relative errors [42] of global minimum values and global minimizers of functions as shown in Table 1 obtained by IYRH’s algorithm is better than Lagrange interpolant on 81 Chebyshev nodes [43]. Table 6 shows that a “fortune effect” does not happened to IYRH’s algorithm when it is applied to the example given in Table 3 for $r = 67$ and $x'_r$ is chosen randomly and differently where its graph is shown in Figure 3.

| Table 2. 7 Test Functions for Comparison with Existed Filled Function Methods |
|--------------------------|--------------------------|
| $N$ | $f(x), D, x_0$ |
| --- | --- |
| 1 | $f(x) = \sin(x) + \sin(10x/3) + \ln(x) - 0.84x$ |
| 2 | $f(x) = -\sum_{i=1}^{5} \sin((i + 1)x + i)$ |
| 3 | $f(x) = \frac{\pi}{n} \left( k \sin^2(\pi y_i) + \sum_{i=1}^{n-1} [(y_i - A)^2(1 + k \sin^2(\pi y_{i+1})) + (y_i - A)^2] \right)$ |
| 4 | $f(x) = \frac{\pi}{n} \left( k \sin^2(\pi x_i) + \sum_{i=1}^{n-1} [(x_i - A)^2(1 + k \sin^2(\pi x_{i+1})) + (x_i - A)^2] \right)$ |
| 5 | $f(x) = k \sin^2(x_0) + k_1 \sum_{i=1}^{n-1} [(x_i - A)^2(1 + k \sin^2(\pi x_{i+1})) + (x_i - A)^2(1 - k_0 \sin^2(x_0))]$ |

where the constants in this equation have been fixed as follows: $k_0 = 1, k_1 = 0.1, A = 1, l_0 = 3$.

| Table 3. Test Functions for “Fortune Effect” of IYRH’s Function [44] |
|--------------------------|--------------------------|
| $r$ | $f_r$ | $D$ |
| 1, ..., 100 | $0.025(x - x'^2_r) + \sin^2((x - x'^2_r) + (x - x'^2_r) + \sin^2(x - x'^2_r)$ | $[-5,5]$ |

For third category, Table 7 compares the results of IYRH’s algorithm with Ma et al. ’s filled function and Lucidi and Piccially’s filled function for a set of 5 test examples in Table 2. For the last category, the results presented in Table 8 is used to observe the sensitivity of IYRH’s algorithm due to three initial points using example 3 from Table 2. It is clear that IYRH’s function can be used to solve the global optimization problems from any initial point.

![Figure 3. Graph of $f_{67}(x)$ one of the 100 one-dimensional randomized test functions](image-url)
Table 4. Numerical Results of a Set of 20 Test Functions by IYR’s Algorithm

| N  | k   | j/n  | x̂  | 𝑥̂(1) | 𝑥̂(2) | 𝑥̂(3) | 𝑥̂(4) | 𝑥̂(5) | 𝑥̂(6) | 𝑥̂(7) | 𝑥̂(8) | 𝑥̂(9) | 𝑥̂(10) | 𝑥̂(11) | 𝑥̂(12) | 𝑥̂(13) | 𝑥̂(14) | 𝑥̂(15) | 𝑥̂(16) | 𝑥̂(17) | 𝑥̂(18) | 𝑥̂(19) | 𝑥̂(20) | 𝑥̂(21) | 𝑥̂(22) | 𝑥̂(23) | 𝑥̂(24) | 𝑥̂(25) | 𝑥̂(26) | 𝑥̂(27) | 𝑥̂(28) | 𝑥̂(29) | 𝑥̂(30) | 𝑥̂(31) | 𝑥̂(32) | 𝑥̂(33) | 𝑥̂(34) | 𝑥̂(35) | 𝑥̂(36) | 𝑥̂(37) | 𝑥̂(38) | 𝑥̂(39) | 𝑥̂(40) | 𝑥̂(41) | 𝑥̂(42) | 𝑥̂(43) | 𝑥̂(44) | 𝑥̂(45) | 𝑥̂(46) | 𝑥̂(47) | 𝑥̂(48) | 𝑥̂(49) | 𝑥̂(50) | 𝑥̂(51) | 𝑥̂(52) | 𝑥̂(53) | 𝑥̂(54) | 𝑥̂(55) | 𝑥̂(56) | 𝑥̂(57) | 𝑥̂(58) | 𝑥̂(59) | 𝑥̂(60) | 𝑥̂(61) | 𝑥̂(62) | 𝑥̂(63) | 𝑥̂(64) | 𝑥̂(65) | 𝑥̂(66) | 𝑥̂(67) | 𝑥̂(68) | 𝑥̂(69) | 𝑥̂(70) | 𝑥̂(71) | 𝑥̂(72) | 𝑥̂(73) | 𝑥̂(74) | 𝑥̂(75) | 𝑥̂(76) | 𝑥̂(77) | 𝑥̂(78) | 𝑥̂(79) | 𝑥̂(80) | 𝑥̂(81) | 𝑥̂(82) | 𝑥̂(83) | 𝑥̂(84) | 𝑥̂(85) | 𝑥̂(86) | 𝑥̂(87) | 𝑥̂(88) | 𝑥̂(89) | 𝑥̂(90) | 𝑥̂(91) | 𝑥̂(92) | 𝑥̂(93) | 𝑥̂(94) | 𝑥̂(95) | 𝑥̂(96) | 𝑥̂(97) | 𝑥̂(98) | 𝑥̂(99) | 𝑥̂(100) | 𝑥̂(101) | 𝑥̂(102) | 𝑥̂(103) | 𝑥̂(104) | 𝑥̂(105) | 𝑥̂(106) | 𝑥̂(107) | 𝑥̂(108) | 𝑥̂(109) | 𝑥̂(110) | 𝑥̂(111) | 𝑥̂(112) | 𝑥̂(113) | 𝑥̂(114) | 𝑥̂(115) | 𝑥̂(116) | 𝑥̂(117) | 𝑥̂(118) | 𝑥̂(119) | 𝑥̂(120) | 𝑥̂(121) | 𝑥̂(122) | 𝑥̂(123) | 𝑥̂(124) | 𝑥̂(125) | 𝑥̂(126) | 𝑥̂(127) | 𝑥̂(128) | 𝑥̂(129) | 𝑥̂(130) | 𝑥̂(131) | 𝑥̂(132) | 𝑥̂(133) | 𝑥̂(134) | 𝑥̂(135) | 𝑥̂(136) | 𝑥̂(137) | 𝑥̂(138) | 𝑥̂(139) | 𝑥̂(140) | 𝑥̂(141) | 𝑥̂(142) | 𝑥̂(143) | 𝑥̂(144) | 𝑥̂(145) | 𝑥̂(146) | 𝑥̂(147) | 𝑥̂(148) | 𝑥̂(149) | 𝑥̂(150) | 𝑥̂(151) | 𝑥̂(152) | 𝑥̂(153) | 𝑥̂(154) | 𝑥̂(155) | 𝑥̂(156) | 𝑥̂(157) | 𝑥̂(158) | 𝑥̂(159) | 𝑥̂(160) | 𝑥̂(161) | 𝑥̂(162) | 𝑥̂(163) | 𝑥̂(164) | 𝑥̂(165) | 𝑥̂(166) | 𝑥̂(167) | 𝑥̂(168) | 𝑥̂(169) | 𝑥̂(170) | 𝑥̂(171) | 𝑥̂(172) | 𝑥̂(173) | 𝑥̂(174) | 𝑥̂(175) | 𝑥̂(176) | 𝑥̂(177) | 𝑥̂(178) | 𝑥̂(179) | 𝑥̂(180) | 𝑥̂(181) | 𝑥̂(182) | 𝑥̂(183) | 𝑥̂(184) | 𝑥̂(185) | 𝑥̂(186) | 𝑥̂(187) | 𝑥̂(188) | 𝑥̂(189) | 𝑥̂(190) | 𝑥̂(191) | 𝑥̂(192) | 𝑥̂(193) | 𝑥̂(194) | 𝑥̂(195) | 𝑥̂(196) | 𝑥̂(197) | 𝑥̂(198) | 𝑥̂(199) | 𝑥̂(200) | 𝑥̂(201) | 𝑥̂(202) | 𝑥̂(203) | 𝑥̂(204) | 𝑥̂(205) | 𝑥̂(206) | 𝑥̂(207) | 𝑥̂(208) | 𝑥̂(209) | 𝑥̂(210) | 𝑥̂(211) | 𝑥̂(212) | 𝑥̂(213) | 𝑥̂(214) | 𝑥̂(215) | 𝑥̂(216) | 𝑥̂(217) | 𝑥̂(218) | 𝑥̂(219) | 𝑥̂(220) | 𝑥̂(221) | 𝑥̂(222) | 𝑥̂(223) | 𝑥̂(224) | 𝑥̂(225) | 𝑥̂(226) | 𝑥̂(227) | 𝑥̂(228) | 𝑥̂(229) | 𝑥̂(230) | 𝑥̂(231) | 𝑥̂(232) | 𝑥̂(233) | 𝑥̂(234) | 𝑥̂(235) | 𝑥̂(236) | 𝑥̂(237) | 𝑥̂(238) | 𝑥̂(239) | 𝑥̂(240) | 𝑥̂(241) | 𝑥̂(242) | 𝑥̂(243) | 𝑥̂(244) | 𝑥̂(245) | 𝑥̂(246) | 𝑥̂(247) | 𝑥̂(248) | 𝑥̂(249) | 𝑥̂(250) | 𝑥̂(251) | 𝑥̂(252) | 𝑥̂(253) | 𝑥̂(254) | 𝑥̂(255) | COVID-19 pandemic, the conflict between people and those who are not vaccinated, and the economic and political situations around the world. These factors are likely to cause the pandemic to spread more widely and for longer periods of time than expected. Therefore, it is crucial to have robust and effective strategies for managing the pandemic, including vaccination, social distancing, and quarantine measures. Additionally, it is important to address the socio-economic impacts of the pandemic and to ensure that vulnerable populations are protected. The pandemic has highlighted the importance of preparedness and resilience in health systems, and this will need to be a key area of focus going forward. As the pandemic continues to evolve, it is essential to remain adaptable and responsive to new developments and challenges. COVID-19 has shown that public health measures are crucial in controlling the spread of the virus, and that these measures must be tailored to the specific context of each country and region.
Solving one-dimensional unconstrained global optimization problem... (Ismail Bin Mohd)
Theorem 6 and Theorem 7

The meaning of the abbreviations used in Table 8 is as follows:
- nFL, nFM and nFL are the number of function evaluations needed to yield the global minimum of IYRH’s, Ma et al.’s and Lucidi and Piciably’s algorithms respectively.
- nFI, nFM and nFL are the number of function evaluations needed to satisfy the stopping criterion of IYRH’s, Ma et al.’s and Lucidi and Piccially’s algorithms respectively.
- nFI, nFM and nFL are the number of filled function evaluations needed to obtain the global minimum of IYRH’s, Ma et al.’s and Lucidi and Piccially’s algorithms respectively.
- 5. “failed” means the method of Lucidi and Piccially fails to achieve the results.

Table 8. Numerical Results due to 3 Different Initial Points for Example 3 of Table 2

| x₀ | k | j/κ | x₃⁺ | x₄⁺ | x₅⁺ | x₆⁺ |
|----|---|-----|-----|-----|-----|-----|
| -12 | 0 | 1/1 | -10.8789 | -9.99355 | -9.10281 | -8.00645 | -7.87069 |
| 1 | 1/1 | -6.91954 | -5.99355 | -5.06151 | -4.00645 | -3.67996 |
| 2 | 1/1 | -2.95985 | -1.99355 | -1.02048 | -0.00645 | 0.594662 |
| 3/2 | 1/1 | 2.00645 | 3.02048 | 3.99355 | 4.95985 |
| 5/3 | 10.0065 | 11.1028 | 11.9935 |
| -9.6 | 0 | 1/1 | -10.8789 | -9.99355 | -9.10281 | -8.00645 | -7.87069 |
| 1 | 1/1 | -6.91954 | -5.99355 | -5.06151 | -4.00645 | -3.67996 |
| 2 | 1/1 | -2.95985 | -1.99355 | -1.02048 | -0.00645 | 0.594662 |
| 3 | 1/1 | 2.00645 | 3.02048 | 3.99355 | 4.95985 |
| 5/3 | 10.0065 | 11.1028 | 11.9935 |
| -5.6 | 0 | 1/1 | -6.91954 | -5.99355 | -5.06151 | -4.00645 | -3.67996 |
| 1 | 1/1 | -2.95985 | -1.99355 | -1.02048 | -0.00645 | 0.594662 |
| 2 | 1/1 | 2.00645 | 3.02048 | 3.99355 | 4.95985 |
| 3 | 1/1 | 2.00645 | 3.02048 | 3.99355 | 4.95985 |
| 5/3 | 10.0065 | 11.1028 | 11.9935 |

8. Comparison and Discussion
The graphical comparison between IYRH’s filled function method with other best current filled function methods (1)-(4) included tunneling and bridging methods, will be presented.

8.1. Comparison with the Tunneling Method [8]
The weakness of tunneling method [8] \( T(x, \Gamma) = (f(x) - f(x_i))/[(x - x_i)\Gamma(x - x_i)] \) appeared when Newton’s method is used since the non-convexity problem. Fortunately, IYRH’s filled function can be utilized (Theorem 6 and Theorem 7) using the radius of curvature applied to Newton’s method to find the root of non-convex problems. For example 7 [8], the tunneling method can only obtain the global minimizer, whereas IYRH’s algorithm can obtain the entire extreme and inflection points in considered domain.

8.2. Comparison with the Bridging Method [41]
The bridging function [41]:

Table 7. Comparison of the Numerical Results by IYRH’s Algorithm with Two Other Methods

| Example | nFI/nFM/nF'L | nFI/nFM/nF'L | nFI/nFM/nF'L | nFI/nFM/nF'L |
|---------|--------------|--------------|--------------|--------------|
| 1       | 27/1361/failed | nFI/nFM/nF'L | 55/1141/failed | 62/1211/failed |
| 2       | 16/170/failed  | nFI/nFM/nF'L | 16/72/failed  | 22/132/failed |
| 3       | 104/339/failed | nFI/nFM/nF'L | 35/236/failed | 57/296/failed |
| 4       | 8/418/failed   | nFI/nFM/nF'L | 9/309/failed  | 19/429/failed |
| 5       | 18/505/failed  | nFI/nFM/nF'L | 20/397/failed | 31/457/failed |

Comparison and Discussion

The graphical comparison between IYRH’s filled function method with other best current filled function methods (1)-(4) included tunneling and bridging methods, will be presented.
which is strongly dependent on the parameters $\delta_0$, $\delta_1$, $\varepsilon_0$, $\delta$ and $d$ where $\delta_0$, $\delta_1$, and $\varepsilon_0$ must be predetermined, where $r(x) = r(x, x_0, \delta) = f(x_0) - \delta(x - x_0)$, and $l(x) = l(x, x_0, \delta) = f(x_0) + \delta(x - x_0)$.

The authors suggested that $\delta_0$ should be chosen first and it should be big enough so that when $\delta \geq \delta_0$, the computer does not treat $-\delta$ as zero. They suggested $\delta_0 = 10^{-2}$ or $10^{-3}$, $\delta_1 = 0.1$ or 1, and $d$ is chosen such that $\delta_1/d^k = \delta_0$. We are lucky since IYRH’s algorithm did not face any situation like that.

8.3. Comparison with the Two-Parameter Filled Function

The graph of $f_e(x) = \cos(3x/5) \cos(2x) + \sin(x)$ (0.5 $\leq x \leq 12$) is given in Figure 4 (a) and the graph of $F_{IRH}(x, x^*_1) \ (x \in [1.34096, 273.151])$ of $f$ is given in Figure 4 (b) where $x^*_1 = 1.34096$. The graphs of the filled functions (1) with $\rho = 1$ and $r = 1 - f(x)$, (2) with $\rho = 5$ and $\mu = 1$, (3) with $\rho = 1$ and $r = 1 - f(x)$ and (4) with $x_0 = 2$, $r = 10$ are given in Figures 5 (a)-(d) respectively. Contrast to $F_{IRH}$, (a) $F_G$ has the infinity structure (flat) and strongly dependent on $\rho$ and $r$. Therefore $F_G$ becomes inefficient, (b) $F_Z$ discontinuous at a point $x' \in (2, 3)$. This condition makes the minimizing difficult, (c) according to (3), they actually have four parameters to be adjusted (see [29]) and (d) $a$ (see (5)) contain a parameter $\zeta > 0$, which is also difficult to be adjusted, and it is clear that $F_Z$ is not a one-parameter filled function.

![Graphs](attachment:image.png)

Figure 4. Graphs of (a) $f(x)$ and (b) $F_{IRH}$ at $x^*_1 = 1.34096$
8.4. Comparison to the Parameter Free Filled Function

Ma et al. [34] suggest a PFFF (6) at $x_k^*$ of $f$. When applied to Example 9 in Table 1 at $x_k^* = 5.36225$, it yields a graph as in Figure 6. It is clear that, $F_M$ is discontinuous at $x_k^*$, and almost flat for $x$ such that $f(x) < f(x_k^*)$ whereas $F_{IRH}$ is continuous as shown in Figure 7.

8.5. Conclusion

This article introduces a new IYRH's method which absolutely different from other filled function methods, in finding all extreme and inflection points of $f: D \subseteq R \rightarrow R$ According to the results listed in Tables 4-8, this method never fail compute all those points. Thus, this method is an efficient and reliable method for solving the global optimization problems numerically and analytically. Therefore, the IYRH's method is far more advanced and superior than other most of the filled function methods published in the literature.

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