STRONG METRIC (SUB)REGULARITY OF KKT MAPPINGS FOR PIECEWISE LINEAR-QUADRATIC CONVEX-COMPOSITE OPTIMIZATION

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ABSTRACT. This work concerns the local convergence theory of Newton and quasi-Newton methods for convex-composite optimization: minimize \( f(x) := h(c(x)) \), where \( h \) is an infinite-valued proper convex function and \( c \) is \( C^2 \)-smooth. We focus on the case where \( h \) is infinite-valued piecewise linear-quadratic and convex. Such problems include nonlinear programming, mini-max optimization, estimation of nonlinear dynamics with non-Gaussian noise as well as many modern approaches to large-scale data analysis and machine learning. Our approach embeds the optimality conditions for convex-composite optimization problems into a generalized equation. We establish conditions for strong metric subregularity and strong metric regularity of the corresponding set-valued mappings. This allows us to extend classical convergence of Newton and quasi-Newton methods to the broader class of non-finite valued piecewise linear-quadratic convex-composite optimization problems. In particular we establish local quadratic convergence of the Newton method under conditions that parallel those in nonlinear programming when \( h \) is non-finite valued piecewise linear.

1. INTRODUCTION

This work concerns local convergence theory of Newton and quasi-Newton methods for the solution of the convex-composite problem:

\[
\begin{align*}
\text{(P)} \quad & \min_{x \in \mathbb{R}^n} f(x) := h(c(x)), \\
\end{align*}
\]

where \( h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) is piecewise linear-quadratic (PLQ) and convex, and \( c : \mathbb{R}^n \to \mathbb{R}^m \) is \( C^2 \)-smooth. When \( h = \frac{1}{2} \|\cdot\|^2 \), \( P \) is the classical nonlinear least-squares problem. Numerous other problems fall within this class including nonlinear programming (NLP), mini-max optimization, estimation of nonlinear dynamics with non-Gaussian noise as well as many modern approaches to large-scale data analysis and machine learning [1,2,11]. Convex-composite optimization has a long history with investigations in the 1970s [29,30], 1980s [3,4,22,34,35,39,40], and 1990s [6,7,12,37], where much of the emphasis was on a calculus for compositions and its relationship to nonlinear programming (NLP) and exact penalization [19]. Recently, there has been a resurgence of interest in local [15,18] and global [9,10,15–17,24] algorithms for this class of problems especially with respect to establishing the iteration complexity of first-order methods for \( P \). Much of this work has focused on the case where the function \( h \) is finite-valued.

These, and almost all other methods for solving \( P \), use a direction-finding subproblem similar to

\[
\begin{align*}
\text{(P_k)} \quad & \min_{x \in \mathbb{R}^n} h(c(x^k) + \nabla c(x^k)[x - x^k]) + \frac{1}{2}[x - x^k]^\top H_k[x - x^k], \\
\end{align*}
\]

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where $H_k$ is the Hessian of a Lagrangian for $P$ [4]. When the Hessian $H_k$ is used in the subproblems, the method corresponds to a Newton method (5), and when $H_k$ is approximated by a matrix $B_k$, it corresponds to a quasi-Newton method (6). In either case, the subproblems $P_k$ may or may not be convex depending on whether $H_k, B_k \succeq 0$. In the context of the broader class of prox-regular $h$, Lewis and Wright [24] take $B_k = \mu_k I$ at each iteration, thereby guaranteeing existence and uniqueness of the "proximal step" and a global descent algorithm. Instead, our focus is on developing methods possessing fast local rates of convergence by taking advantage of second-order information together with the convex geometry of $\text{dom}(h)$ developed by Rockafellar [35].

When $h$ is assumed to be a finite-valued piecewise linear convex function, Womersley [38] established second-order rates of convergence for these algorithms under conditions comparable to those used in NLP, i.e., linear independence of the active constraint gradients, strict complementarity, and strong second-order sufficiency. Notwithstanding this correspondence to NLP, the method of proof differs significantly from the standard methodology to establishing such results in the NLP case developed by Robinson [31,32]. Notably, in the case of NLP, the function $h$ is piecewise linear but not finite-valued. In subsequent work, Robinson [33] introduced the revolutionary idea of generalized equations, whose variational properties can be used to establish local rates of convergence for Newton’s method for NLP. By employing the techniques of generalized equations, Cibulka et. al. [8] recently connected classical second-order necessary and sufficient conditions for a local minimizer of $P$ with strong metric subregularity (see Definition 5.1) of the underlying KKT mapping when $h$ is piecewise linear convex but not necessarily finite-valued. However, their analysis relies heavily on the fact that $h$ is piecewise linear. And so, the old question of what conditions imply local quadratic convergence when $h$ is not piecewise linear remains open. However, their technique created the possibility of an extension to the case where $h$ is a member of the PLQ class. This extension is our goal. It is hoped that the methods and techniques developed in this paper provide insight into how to extend these results beyond the PLQ class.

As noted above, we couch the analysis in the context Newton’s method for generalized equations. The first-order necessary conditions of a local minimum of $P$ are encoded through a generalized equation of the form $g(x, y) + G(x, y) \geq 0$, where $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is a $C^1$-smooth function, $G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ is a set-valued mapping, $(x, y)$ represents a primal-dual pair, and the function $\nabla g(x, y)$ is a KKT matrix for $P$ (see Definition 3.5). Newton’s method (5) for solving this generalized equation corresponds to solving the optimality conditions for $P_k$. The Newton iterate at $(x^k, y^k)$ is obtained by solving the following linearized generalized equation:

\[
(1) \quad \text{Find } (x^{k+1}, y^{k+1}) \text{ such that } g(x^k, y^k) + \nabla g(x^k, y^k) \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} + G(x^{k+1}, y^{k+1}) \geq 0.
\]

The details of this derivation appear in Section 3.

The goal of this paper is to establish local convergence rates for algorithms based on iteratively solving $P_k$ in the case where $h$ is a PLQ convex function. We do this by augmenting the strategy of Cibulka et. al. [8] with additional innovations by Lewis [23] and Rockafellar [35]. In particular, we are able to establish conditions under which these algorithms are locally quadratically convergent. The first phase of our analysis involves extensive application of the first- and second-order PLQ calculus [35,37] to establish conditions under which the underlying generalized equation is strongly metrically subregular. This allows us to establish sufficient conditions for the superlinear convergence of quasi-Newton methods for algorithms whose direction finding subproblems are based on $P_k$. The second phase of our analysis employs the technique of partly smooth functions in the sense of [20,23] to establish conditions under which a local approximation to the underlying generalized
equation is strongly metrically regular (see Definition 7.1). This allows us to give conditions for the local quadratic convergence of the Newton method based on $P_k$.

We also note that recent work by Drusvyatskiy and Lewis [15] considers similar types of results for convex-composite optimization problems of the form $\varphi(x) = h(c(x)) + g(x)$, where $h$ is finite-valued and $L-$Lipschitz, $\nabla c$ is $\beta$-Lipschitz, and $g$ is closed, proper, convex, but infinite-valued. One of their goals is to understand the convergence of prox-linear type methods through either the subregularity [15, Theorems 5.10 and 5.11] or strong regularity [15, Theorem 6.2] of $\partial \varphi$ at stable strong minima or sharp minima of $\varphi$ [15, Theorems 7.1 and 7.2].

When $h$ is only assumed to be finite-valued convex and $g$ is zero, the first result on the local quadratic convergence for convex-composite problems was that of Burke and Ferris [6]. In that work, the authors established a constraint qualification for the inclusion $c(\overline{x}) \in \arg \min h$ that ensures the local quadratic convergence of constrained Gauss-Newton methods. In [6], the authors assumed $\arg \min h$ was a set of weak sharp minima [5]. However, it was observed by Li and Wang [26] that the sharpness hypothesis was not required. Rather, a local quadratic growth condition [26, Theorem 2] was sufficient for the proof techniques in [6] to succeed. The authors continued research [25] in relaxations of the constraint qualification on $c(\overline{x}) \in \arg \min h$ and studied proximal methods [21] for their convergence.

Our focus on the PLQ class is motivated by the great variety of modern problems in data analysis, estimation of dynamical systems, inverse problems, and machine learning that are posed within this class. The key to the success of the convex-composite structure is that it separates the data associated to the problem, the function $c$, from the model within which we wish to explore the data, the function $h$. Consequently, the broader the class of functions $h$ available, the greater the variety of ways within which we can explore underlying extremal properties of the input function $c$, e.g., sparsity, robustness, network structure, dynamics, influence of hyperparameters, etc. Importantly, we have learned that features of the data can be more readily extracted by imposing nonsmoothness in the function $h$.

The roadmap of the paper is as follows. Section 2 collects tools from convex and variational analysis used throughout the paper. Section 3 formally presents the convex-composite problem class. We take advantage of the structure of the problem class to rewrite the general first-order optimality conditions for proper functions in the presence of various constraint qualifications used in this work. We also present the generalized equation (10) associated with the first-order optimality conditions for $P$. Section 4 discusses the convex geometry and differential theory of piecewise linear-quadratic functions collected in [37]. The second-order theory of [37] allows us to rewrite the general second-order necessary and sufficient conditions for a local minimum of $P$. We extract a crucial result from [37] that highlights natural candidates for manifolds of partial smoothness [23] inherent to the function $h$. Section 5 extends the result [8, Theorem 7.1] relating the strong metric subregularity of (10) to the second-order sufficient conditions of local minima and ends with a convergence study of quasi-Newton methods for $P$. Section 6 establishes conditions for the partly smooth structure of PLQ convex functions and sets the stage for Section 7, where we analyze the local quadratic convergence of Newton’s method as in [13].

2. Notation

These sections summarize the relevant notation and tools of convex and variational analysis used in this work. Unless otherwise stated, we follow the notation in [13,23,37].
2.1. Preliminaries. We work in \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) with the standard inner product \(\langle x, y \rangle = x^\top y = \sum_{i=1}^n x_i y_i\), and \(\|x\|^2 = x^\top x\). Throughout, we switch between the notations \(\langle x, y \rangle\) and \(x^\top y\) for clarity considerations. Let \(B := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}\) be the closed unit ball. For \(A \in \mathbb{R}^{m \times n}\), its \(range\), \(null\) \(space\), and \(transpose\) are \(\text{Ran}\ (A), \text{Null}\ (A), A^\top\) respectively, and for a finite collection of mappings \(\{A_k\}_{k \in J}\) with index set \(J\), let \(\text{diag}A_k\) denote the block diagonal matrix with \(k\)th block \(A_k\). Let \(e_j \in \mathbb{R}^n\) denote the standard unit coordinate vector.

2.2. Convex Analysis. A set \(C \subset \mathbb{R}^m\) is \emph{locally closed} at a point \(\tau\), not necessarily in \(C\), if there exists a closed neighborhood \(V\) of \(\tau\) such that \(C \cap V\) is closed. Any closed set is locally closed at all of its points, and the closure and interior of \(C\) is denoted by \(c\) \(C\) and \(int\) \(C\), respectively.

For a closed convex set \(C \subset \mathbb{R}^m\), let \(\text{aff} C\) denote the \emph{affine hull} of \(C\) and \(\text{par} (C)\) the \emph{subspace parallel} to \(C\). Then, for any \(c \in C\), \(\text{par} (C) := \text{aff} C - c = \mathbb{R}(C - C)\), where we employ Minkowski set algebra for addition of sets: for sets \(C_1, C_2 \subset \mathbb{R}^m\) and \(t \in \mathbb{R}\), define \(C + C' := \{c + c' \mid c \in C, c' \in C'\}\) and \(\Lambda C := \{\lambda c \mid \lambda \in \Lambda, c \in C\}\). When \(C = \{c\}\), we omit the set braces and write \(c + C'\). The \emph{relative interior} of \(C\) is given by \(\text{ri} (C) = \{x \in \text{aff} C \mid \exists (\epsilon > 0) (x + \epsilon B) \cap C \subset C\}\).

2.3. Variational Analysis. The functions in this paper take values in the extended reals \(\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}\). For \(f : \mathbb{R}^n \to \bar{\mathbb{R}}\), the \emph{domain} of \(f\) is \(\text{dom} (f) := \{x \in \mathbb{R}^n \mid f(x) < \infty\}\), and the \emph{epigraph} of \(f\) is \(\text{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}\).

We say \(f\) is \emph{closed} if \(\text{epi} f\) is a closed subset of \(\mathbb{R}^{n+1}\), \(f\) is \emph{proper} if \(\text{dom} (f) \neq \emptyset\) and \(f(x) > -\infty\) for all \(x \in \mathbb{R}^n\), and \(f\) is \emph{convex} if \(\text{epi} f\) is a convex subset of \(\mathbb{R}^{n+1}\).

Suppose \(f : \mathbb{R}^n \to \bar{\mathbb{R}}\) is finite at \(\bar{x}\) and \(w, v \in \mathbb{R}^n\). The \emph{subderivative} \(d f(\bar{x}) : \mathbb{R}^n \to \bar{\mathbb{R}}\) and \emph{one-sided directional derivative} \(f' (\bar{x}; w)\) at \(\bar{x}\) for \(w\) are

\[
d f(\bar{x})(w) := \lim_{t \searrow 0} \frac{f(\bar{x} + tw) - f(\bar{x})}{t}, \quad f'(\bar{x}; w) := \lim_{t \searrow 0} \frac{f(\bar{x} + tw) - f(\bar{x})}{t}.
\]

At points \(w \in \mathbb{R}^n\) such that \(f'(\bar{x}; w)\) exists and is finite, the \emph{one-sided second directional derivative} is

\[
f''(\bar{x}; w) := \lim_{t \searrow 0} \frac{f(\bar{x} + tw) - f(\bar{x}) - tf'(\bar{x}; w)}{t^2}.
\]

For any \(w, v \in \mathbb{R}^n\), the \emph{second subderivative} at \(\bar{x}\) for \(v\) and \(w \in \mathbb{R}^n\) is

\[
d^2 f(\bar{x}|v)(w) := \lim_{t \searrow 0} \Delta^2 f(\bar{x}|v)(w), \quad \Delta^2 f(\bar{x}|v)(w) := \frac{f(\bar{x} + tw') - f(\bar{x}) - t \langle v, w' \rangle}{t^2}.
\]

The structure of our problem class allows the classical one-sided first and second directional derivatives \(f'(\bar{x}; \cdot)\) and \(f''(\bar{x}; \cdot)\) to entirely capture the variational properties of their more general counterparts.

Suppose \(f : \mathbb{R}^n \to \bar{\mathbb{R}}\) is finite at \(\bar{x}\). Define the \emph{(Fréchet) regular subdifferential}

\[
\partial f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \right\},
\]

and the \emph{(limiting or Mordukhovich) subdifferential} by

\[
\partial f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \exists (x^n \to \bar{x}) \exists (v^n \to v) \forall (n \in \mathbb{N}) \: v^n \in \partial f(x^n) \right\}, \quad (2)
\]
where \( x^n \to \bar{x} \) denotes \textit{f-attentive convergence}, i.e., that \( x^n \to \bar{x} \), with \( f(x^n) \to f(\bar{x}) \). In the case of a closed, proper, convex function \( f \), the set \( \partial f(\bar{x}) \) is the usual subdifferential of convex analysis. The tools of first and second subderivative functions and subdifferential sets allow us to concisely write first-order necessary conditions and second-order necessary and sufficient conditions for local minima.

**Theorem 2.1** (First-order necessity, second-order necessity and sufficiency). [37, Theorems 10.1, 13.24] For a proper function \( f : \mathbb{R}^n \to \mathbb{R} \), consider the problem \( \min_x f(x) \).

(a) If \( f \) has a local minimum at \( \bar{x} \), then \( 0 \in \partial f(\bar{x}) \) and for all \( w \in \mathbb{R}^n \), \( df(\bar{x})(w) \geq 0 \) and \( d^2 f(\bar{x})(w) \geq 0 \).

(b) If \( 0 \in \partial f(\bar{x}) \) and \( d^2 f(\bar{x})(w) \geq 0 \) for \( w \neq 0 \), then \( \bar{x} \) is a local minimizer of \( f \).

(c) The statement \( 0 \in \partial f(\bar{x}) \) and \( d^2 f(\bar{x})(w) \geq 0 \) for \( w \neq 0 \) is equivalent to \( \bar{x} \) being a strong local minimizer of \( f \), i.e., there exists a neighborhood \( U \) of \( \bar{x} \) and a constant \( \gamma > 0 \) such that

\[
\gamma \| x - \bar{x} \|^2 \leq f(x) \leq f(\bar{x}) + \gamma \| x - \bar{x} \|^2 \quad \text{for all } x \in U \cap \text{dom } f.
\]

A set-valued mapping \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a mapping from \( \mathbb{R}^n \) into the power set of \( \mathbb{R}^m \), so for each \( x \in \mathbb{R}^n \), \( S(x) \subset \mathbb{R}^m \). The graph and domain of \( S \) are defined to be

\[
gph S := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in S(x) \right\}, \quad \text{and } \text{dom } S := \left\{ x \in \mathbb{R}^n \mid S(x) \neq \emptyset \right\},
\]

and \( S \) is \textit{graph-convex} whenever \( \text{gph } S \) is a convex subset of \( \mathbb{R}^n \times \mathbb{R}^m \). For a point \( (\bar{x}, \bar{y}) \in \text{gph } S \), and neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \), a \textit{graphical localization} of \( S \) at \( \bar{x} \) for \( \bar{y} \) is a set-valued mapping \( \tilde{S} \) defined by \( \text{gph } \tilde{S} = \text{gph } S \cap (U \times V) \). A \textit{single-valued localization} of \( S \) at \( \bar{x} \) for \( \bar{y} \) is a graphical localization that is also function. If the domain of \( \tilde{S} \) is a neighborhood of \( \bar{x} \), \( \tilde{S} \) is called a single-valued localization of \( S \) around \( \bar{x} \). The mapping \( \tilde{S} \) is \textit{outer semicontinuous} at \( \bar{x} \) relative to \( X \subset \mathbb{R}^n \) if

\[
\limsup_{x \to \bar{x}} S(x) := \left\{ u \mid \exists (x^n) \to (\bar{x}) \exists (u^n) \to u \forall (n \in \mathbb{N}) u^n \in S(x^n) \right\} \subset S(\bar{x}),
\]

and is \textit{inner semicontinuous} relative to \( X \subset \mathbb{R}^n \) if

\[
S(\bar{x}) \subset \liminf_{x \to \bar{x}} S(x) := \left\{ u \mid \forall (x^n) \to (\bar{x}) \exists (n \in \mathbb{N}, u^n \to u) \forall (n \geq N) u^n \in S(x^n) \right\},
\]

where \( x^n \to \bar{x} \Leftrightarrow x^n \to \bar{x} \) with \( x^n \in X \). Then, (2) is \( \partial f(\bar{x}) := \limsup_{x \to \bar{x}} \tilde{\partial} f(x) \). The last notion employed from variational analysis is that of normal and tangent vectors. Let \( C \subset \mathbb{R}^n \), and \( \bar{\sigma} \in C \). Define the \textit{normal cone} to \( C \) at \( \bar{\tau} \) as

\[
N(\bar{\tau} \mid C) := \limsup_{c \to \bar{\tau}} \tilde{N}(c \mid C), \quad \text{where } \tilde{N}(c \mid C) := \left\{ v \mid \forall (c') \in C \langle v, c' - c \rangle \leq o(\| c' - c \|) \right\},
\]

and the \textit{tangent cone} to \( C \) at \( \bar{\tau} \) as \( T(\bar{\tau} \mid C) := \limsup_{t \to 0} t^{-1}(C - \bar{\tau}) \). A set \( C \) is \textit{Clarke regular} at \( \bar{\tau} \in C \) if \( C \) is locally closed at \( \bar{\tau} \) and \( N(\bar{\tau} \mid C) = \tilde{N}(\bar{\tau} \mid C) \). A nonempty, closed, convex set \( C \) is Clarke regular at all \( \bar{\tau} \in C \), with \( N(\bar{\tau} \mid C) = \left\{ v \mid \langle v, c - \bar{\tau} \rangle \leq 0 \text{ for all } c \in C \right\} \), and \( T(\bar{\tau} \mid C) = \left\{ v \mid \langle v, w \rangle \leq 0 \text{ for all } w \in N(\bar{\tau} \mid C) \right\} = \text{cl } \{ \mathbb{R}^+ (C - \bar{\tau}) \} \), [37, Theorem 6.9]. We refer the reader to [37, Chapter 6] for a thorough exposition.

Suppose \( g : \mathbb{R}^n \to \mathbb{R}^m \) is \( C^1 \)-smooth, \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a set-valued mapping with closed graph and
\{B_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^{m \times n}$. Consider the generalized equation \(0 \in g(z) + G(z)\). The Newton method for \(g + G\) is the iteration
\[
(5) \quad \text{find } z^{k+1} \text{ such that } 0 \in g(z^k) + \nabla g(z^k)(z^{k+1} - z^k) + G(z^{k+1}), \text{ for } k \in \mathbb{N},
\]
and the quasi-Newton method for \(g + G\) is the iteration
\[
(6) \quad \text{find } z^{k+1} \text{ such that } 0 \in g(z^k) + B_k(z^{k+1} - z^k) + G(z^{k+1}), \text{ for } k \in \mathbb{N}.
\]

3. Convex-composite first- and second-order theory

We begin by recalling the basic ingredients of convex-composite optimization and the associated variational structures.

**Definition 3.1** (Convex-composite functions). Let \(h : \mathbb{R}^m \to \bar{\mathbb{R}}\) be a closed, proper, convex function and \(c : \mathbb{R}^n \to \mathbb{R}^m\) a \(C^2\)-smooth function. Define \(f : \mathbb{R}^n \to \bar{\mathbb{R}}\) by \(f(x) := h(c(x))\). We say the function \(f\) is convex-composite.

**Definition 3.2** (Convex-composite Lagrangian). \([4]\) For any \(y \in \mathbb{R}^m\), define the function \((yc) : \mathbb{R}^n \to \mathbb{R}\) by \((yc)(x) := \langle y, c(x) \rangle\). The Lagrangian for the convex-composite \(f\) is defined by \(L(x,y) := (yc)(x) - h^*(y)\), where \(h^* : \mathbb{R}^m \to \bar{\mathbb{R}}\) denotes the Fenchel conjugate of the convex function \(h\) defined by \(h^*(y) := \sup_{z \in \mathbb{R}^m} \langle z, y \rangle - h(z)\). The Hessian of \(L\) in its first variables is denoted
\[
(7) \quad \nabla^2_L(x,y) = \nabla^2(yc)(x) = \sum_{i=1}^m y_i \nabla^2 c_i(x).
\]

**Definition 3.3** (Convex-composite multiplier sets). Suppose \(f\) is convex-composite. Define the set of multipliers at \(\bar{x} \in \text{dom } f\) for \(v \in \mathbb{R}^n\) as in \([37, \text{Theorem 13.14}]\) by
\[
(8) \quad Y(\bar{x}, v) := \left\{ y \left| \begin{array}{c} v \\ 0 \end{array} \right. \in \left( \begin{array}{c} \frac{\partial x}{y} L(\bar{x}, y) \\ \frac{\partial y}{(-L)(\bar{x}, y)} \end{array} \right) \right\} = \left\{ y \in \partial h(c(\bar{x})) \left| \nabla c(\bar{x})^\top y = v \right. \right\},
\]
and define the set of multipliers at \(\bar{x}\) for 0 by
\[
(9) \quad M(\bar{x}) := Y(\bar{x}, 0) = \text{Null } \left( \nabla c(\bar{x})^\top \right) \cap \partial h(c(\bar{x})).
\]

A calculus for convex-composite functions at a point \(\bar{x} \in \text{dom } f\) requires various types of “constraint qualifications.” Stronger versions of the basic constraint qualification (BCQ) will be employed to ensure uniqueness of the multiplier and underlying strict complementarity properties in later sections.

**Definition 3.4** (Convex-composite constraint qualifications). Suppose \(f\) is convex-composite and \(\bar{x} \in \text{dom } f\). We say \(f\) satisfies the
- basic constraint qualification at \(\bar{x}\) if
  \[
  \text{(BCQ)} \quad \text{Null } \left( \nabla c(\bar{x})^\top \right) \cap N \left( c(\bar{x}) \left| \text{dom } h \right) \right) = \{0\},
  \]
- transversality condition at \(\bar{x}\) if
  \[
  \text{(TC)} \quad \text{Null } \left( \nabla c(\bar{x})^\top \right) \cap \text{par } \left( \partial h(c(\bar{x})) \right) = \{0\},
  \]
- strict criticality condition at \(\bar{x} \in \text{dom } f\) for \(\bar{y}\) if
  \[
  \text{(SC)} \quad \text{Null } \left( \nabla c(\bar{x})^\top \right) \cap \text{ri } \left( \partial h(c(\bar{x})) \right) = \{\bar{y}\}.
  \]
Remark 1. Following [37, Definition 10.23], one says that a convex-composite function \( f \) is strongly amenable at \( \overline{x} \in \text{dom} \ (f) \) if \( f \) satisfies (BCQ) at \( \overline{x} \). One says that \( f \) is fully amenable at \( \overline{x} \in \text{dom} \ (f) \) if \( f \) satisfies (BCQ) at \( \overline{x} \) and the function \( h \) is PLQ convex. Here, we make use of the underlying assumption that \( c \) is \( C^2 \)-smooth.

Notice the basic constraint qualification is a local property in the following sense. If \( f \) satisfies (BCQ) at \( \overline{x} \), then there exists a neighborhood \( U \) of \( \overline{x} \) such that \( f \) satisfies (BCQ) at all \( x \in [U \cap c^{-1}(\text{dom} \ (h))] \). Moreover, the basic constraint qualification ensures that the chain rule applies in the subdifferential calculus for convex-composite functions and establishes a foundation for the application of tools from variational analysis.

Theorem 3.1 (Convex-composite first order necessary conditions). Suppose \( f \) is convex-composite and \( \overline{x} \in \text{dom} \ (f) \) is such that \( f \) satisfies (BCQ) at \( \overline{x} \). Then, \( \partial f(\overline{x}) = \nabla c(\overline{x})^\top \partial h(c(\overline{x})) \), and for any \( d \in \mathbb{R}^n \), \( df(\overline{x})(d) = h'(c(\overline{x})); \nabla c(\overline{x})d \). Suppose, in addition, that \( \overline{x} \) is a local solution to \( P \). Then, \( M(\overline{x}) := \text{Null} \left( \nabla c(\overline{x})^\top \right) \cap \partial h(c(\overline{x})) \neq \emptyset \), or equivalently, \( 0 \in \partial f(\overline{x}) \), and for any \( d \in \mathbb{R}^n \), \( h'(c(\overline{x})); \nabla c(\overline{x})d \geq 0 \).

Proof. This follows from Theorem 2.1 and [37, Proposition 8.21, Exercise 10.26(b)].

We now establish a relationship between the various notions of a constraint qualification given in Definition 3.4.

Lemma 3.1. Suppose \( f \) is convex-composite, \( \overline{x} \in \text{dom} \ (f) \), and \( \overline{y} \in \mathbb{R}^m \). Then, the following implications hold:

\[ (\text{SC}) \Rightarrow (\text{TC}) \Rightarrow (\text{BCQ}) \]

\[ (M(\overline{x}) = \{\overline{y}\}) \]

Proof. [\((\text{TC}) \Rightarrow (\text{BCQ})\)] By [37, Proposition 8.12], at any point \( \overline{x} \in \text{dom} \ (\partial h) \), \( N (\overline{x} | \text{dom} \ (h)) \subset \text{par} (\partial h(\overline{x})) \). The implication follows.

\[ [(M(\overline{x}) = \{\overline{y}\}) \Rightarrow (\text{BCQ})] \]

Let \( M(\overline{x}) = \{\overline{y}\} \) and suppose there exists

\[ 0 \neq v \in \text{Null} \left( \nabla c(\overline{x})^\top \right) \cap N (c(\overline{x}) | \text{dom} \ (h)) \subset \text{Null} \left( \nabla c(\overline{x})^\top \right) \cap \text{par} (\partial h(c(\overline{x}))) \]

Then, by the subgradient inequality, \( v + \overline{y} \in \text{Null} \left( \nabla c(\overline{x})^\top \right) \cap \partial h(c(\overline{x})) = M(\overline{x}) \), which is a contradiction.

The rest of the proof appears in Lemma 9.1 in the appendix as general facts about closed convex sets \( C \) and linear maps \( A \).

Gauss-Newton methods for iteratively solving \( P \) are based on finding a search direction that approximates a solution to subproblems of the form

\[
(\hat{P}) \quad \text{minimize}_{d \in \mathbb{R}^n} \quad h(c(\hat{x})) + \nabla c(\hat{x})d + \frac{1}{2}d^\top \hat{H}d.
\]

Local rates of convergence for algorithms of this type, where the function \( h \) is assumed to be finite-valued and piecewise linear convex were developed by Womersley [38] based on tools developed for classical nonlinear programming. More recently, Cibulka et. al. [8] successfully applied a modern
approach through generalized equations to obtain similar and stronger results again in the piecewise linear convex case. Inspired by these results and the existence of a sophisticated first- and second-order subdifferential calculus for piecewise linear-quadratic convex functions [37], we develop a convergence theory in the piecewise linear-quadratic case from the generalized equations perspective. The basic notational objects for our development are given in the next definition.

**Definition 3.5** (Convex-composite generalized equations). Let \( f \) be convex-composite, and define the set-valued mapping \( g + G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m} \) by

\[
g(x, y) = \begin{pmatrix} \nabla c(x)\,^\top\! y \\ -c(x) \end{pmatrix}, \quad G(x, y) = \begin{pmatrix} \{0\}_n \\ \partial h^*(y) \end{pmatrix}.
\]

For a fixed \( (\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m \), define the linearization mapping

\[
G : (x, y) \mapsto g(\bar{x}, \bar{y}) + \nabla g(\bar{x}, \bar{y}) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + G(x, y),
\]

where \( \nabla g(\bar{x}, \bar{y}) = \begin{pmatrix} \nabla^2 (\bar{y} c)(\bar{x}) - \nabla c(\bar{x})\,^\top \\ 0 \end{pmatrix} \).

Observe that for any \( \bar{x} \in \text{dom} (f) \) where \( f \) satisfies (BCQ), \( \bar{x} \) satisfies the first-order necessary conditions of Theorem 2.1 for the problem \( P \) if and only if there exists \( \bar{y} \) such that \( (\bar{x}, \bar{y}) \) solves the generalized equation \( g + G \succeq 0 \). More precisely, we have

\[
0 \in g(\bar{x}, \bar{y}) + G(\bar{x}, \bar{y}) \iff \nabla c(\bar{x})\,^\top\! \bar{y} = 0 \quad \text{and} \quad \bar{y} \in \partial h(c(\bar{x})) \iff M(\bar{x}) \neq \emptyset.
\]

The relationship between the linearization of the generalized equation described in (11) and the subproblems \( \bar{P} \) is described in the following lemma. The proof follows from Theorem 3.1.

**Lemma 3.2.** Let \( f \) be convex-composite and \( (\tilde{x}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m \) be such that \( f \) satisfies (BCQ) at \( \tilde{x} \), and define \( \bar{H} := \nabla^2 (\tilde{y} c)(\tilde{x}) \). Then, \( (\bar{d}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m \) satisfy the optimality conditions for

**\( \bar{P} \)**

\[
\min_{d \in \mathbb{R}^n} \quad h(c(\tilde{x}) + \nabla c(\tilde{x})d) + \frac{1}{2} d\,^\top\! \bar{H} d
\]

if and only if \( (\tilde{x} + \bar{d}, \bar{y}) \) solves the Newton equations for \( g + G \): \( 0 \in g(\tilde{x}, \bar{y}) + \nabla g(\tilde{x}, \bar{y}) \begin{pmatrix} x - \tilde{x} \\ y - \bar{y} \end{pmatrix} + G(x, y) \).

4. **Geometry of PLQ Functions and Their Domains**

In this section, unless otherwise stated, we let \( f := h \circ c \) where \( h \) is piecewise linear-quadratic convex and \( c \) is \( C^2 \)-smooth.

**Definition 4.1** (piecewise linear-quadratic). A proper function \( h : \mathbb{R}^m \to \overline{\mathbb{R}} \) is called piecewise linear-quadratic (PLQ) if \( \text{dom} (h) \neq \emptyset \) and \( \text{dom} (h) \) can be represented as the union of \( K \geq 1 \) polyhedral sets of the form

\[
C_k = \left\{ c \mid \langle a_k, c \rangle \leq \beta_k, \text{ for all } j \in \{1, \ldots, s_k\} \right\}
\]

relative to each of which \( h(c) \) is given by an expression of the form \( \frac{1}{2} \langle c, Q_k c \rangle + \langle b_k, c \rangle + \beta_k \) for some scalar \( \beta_k \in \mathbb{R} \), vector \( b_k \in \mathbb{R}^n \), and symmetric matrix \( Q_k \).

** Remark 2.** The sets \( C_k \) do not necessarily form a partition of the set \( C \).
The following lemma is straightforward.

**Lemma 4.1.** Suppose \( h \) is piecewise linear-quadratic convex. Then, for any \( k \in \mathcal{K} \), the matrices \( Q_k \) satisfy \( \langle c, Q_k c \rangle \geq 0 \) for all \( c \in \text{par}(C_k) \).

For the sake of reference we recall the normal and tangent cone structure for polyhedral sets.

**Definition 4.2** (Active indices). For a piecewise linear-quadratic function \( h \) and a point \( \overline{c} \in \text{dom}(h) \), define the set \( \mathcal{K}(\overline{c}) := \{ k \in \mathcal{K} \mid \overline{c} \in C_k \} \), and write \( \overline{T} := |\mathcal{K}(\overline{c})| \), so that \( \mathcal{K}(\overline{c}) = \{ k_1, k_2, \ldots, k_T \} \).

**Theorem 4.1** (Normal and Tangent Cones to Polyhedra). \([37, \text{Theorem } 6.46]\) Suppose \( c \in C_k \) with \( C_k \) polyhedral as in (13). Let \( I_k(c) = \{ j \in \{ 1, \ldots, s_k \} \mid \langle c, a_{kj} \rangle = \alpha_k \} \), and let \( \ell_k = |I_k(c)| \). Then,

\[
\mathcal{N}(c | C_k) = \left\{ \sum_{j \in I_k(c)} \lambda_j a_{kj} \mid \lambda_j \geq 0, \ j \in I_k(c) \right\} \\
\mathcal{T}(c | C_k) = \left\{ v \mid \langle a_{kj}, v \rangle \leq 0, \ j \in I_k(c) \right\}.
\]

Our first- and second-order analysis in the PLQ case heavily depends on the following results from [37].

**Proposition 4.1.** \([37, \text{Propositions } 10.21, 13.9]\) If \( h: \mathbb{R}^m \to \mathbb{R} \) is piecewise linear-quadratic, then \( \text{dom}(h) \) is closed, \( h \) is continuous relative to \( \text{dom}(h) \). Consequently, \( h \) is closed. At any point \( \overline{c} \in \text{dom}(h) \), \( h'(\overline{c}, \cdot) = \partial h(\overline{c}) \), and \( h'(\overline{c}, \cdot) \) is piecewise linear with \( \text{dom}(h'(\overline{c}, \cdot)) \). In particular, for \( k \in \mathcal{K}(\overline{c}) \) and \( w \in \mathcal{T}(\overline{c} | C_k) \),

\[
h'({\overline{c}}, w) = \langle Q_k{\overline{c}} + b_k, w \rangle.
\]

If, in addition, \( h \) is convex, then \( \text{dom}(h) \) is polyhedral,

\[
\emptyset \neq \partial h(\overline{c}) = \bigcap_{k \in \mathcal{K}(\overline{c})} \left\{ y \mid y - Q_k{\overline{c}} - b_k \in \mathcal{N}(\overline{c} | C_k) \right\},
\]

\( h''(\overline{c}, \cdot) \) is piecewise linear-quadratic, but not necessarily convex, and for any \( w \in \mathbb{R}^m \),

\[
0 \leq h''(\overline{c}, w) = \begin{cases} \langle w, Q_k w \rangle & \text{when } w \in \mathcal{T}(\overline{c} | C_k), \\
\infty & \text{when } w \notin \mathcal{T}(\overline{c} | \text{dom}(h)).
\end{cases}
\]

For every \( y \in \partial h(\overline{c}) \), \( d^2h(\overline{c}|y) \) is piecewise linear-quadratic and convex. Let \( K(\overline{c}, y) := \{ w \mid h''(\overline{c}, w) = \langle y, w \rangle \} \). Then, \( K(\overline{c}, y) \) is a polyhedral cone, and

\[
d^2h(\overline{c}|y)(w) = \lim_{\tau \searrow 0} \Delta^2_{\tau} h(\overline{c}|y)(w) = \begin{cases} h''(\overline{c}, w) & w \in K(\overline{c}, y), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Moreover, there exists a neighborhood \( V \) of \( \overline{c} \) such that

\[
h(c) = h(\overline{c}) + h'(\overline{c}, c - \overline{c}) + \frac{1}{2} h''(\overline{c}, c - \overline{c}) \text{ for } c \in V \cap \text{dom}(h).
\]

**Theorem 4.2.** \([37, \text{Theorem } 13.14]\) Let \( f = h \circ c \) for a \( C^2 \) mapping \( c: \mathbb{R}^n \to \mathbb{R}^m \) and a piecewise linear-quadratic convex \( h: \mathbb{R}^m \to \mathbb{R} \). Let \( \overline{x} \in \text{dom}(f) \), and suppose \( f \) satisfies (BCQ) at \( \overline{x} \). Then, for any \( v \in \partial f(\overline{x}) \), the set \( Y(\overline{x}, v) \) given by (8) is compact as well as convex and nonempty, and for any \( w \in \mathbb{R}^n \)

\[
d^2f(\overline{x}|v)(w) = d^2\overline{f}(\overline{x}|v)(w) + \max \left\{ \langle w, \nabla^2 y(c)(\overline{x})w \rangle \mid y \in Y(\overline{x}, v) \right\},
\]

with \( \overline{f}(x) := h(c(\overline{x}) + \nabla c(\overline{x})(x - \overline{x})) \) piecewise linear-quadratic convex.
The standard development of first- and second-order optimality conditions requires the notion of directions of non-ascent.

**Definition 4.3.** Let the directions of non-ascent for any proper \( f : \mathbb{R}^n \to \mathbb{R} \) at \( x \in \text{dom}(f) \) be denoted by \( D(x) := \{ d \in \mathbb{R}^n \mid df(x)(d) \leq 0 \} \). By Theorem 3.1, if \( f \) is convex-composite and \( f \) satisfies (BCQ) at \( x \), then

\[
D(x) = \{ d \in \mathbb{R}^n \mid h'(c(x)); \nabla c(x)d \leq 0 \}
\]

In the PLQ convex case, (BCQ) ensures that we have the following convenient representation of the set \( D(\overline{x}) \).

**Lemma 4.2.** Let \( f \) be as in \( \mathcal{P} \), and let \( \overline{x} \in \mathbb{R}^n \) be such that \( f \) satisfies (BCQ) at \( \overline{x} \). Set \( \overline{c} := c(\overline{x}) \). Then, \( D(\overline{x}) \) is convex and the union of finitely many polyhedral closed convex sets with following the representation:

\[
D(\overline{x}) = \bigcup_{k \in K(\overline{x})} \left\{ d \mid \nabla c(\overline{x})d \in T(\overline{x} \mid C_k), \langle Q_k \overline{c} + b_k, \nabla c(\overline{x})d \rangle \leq 0 \right\}
\]

\[
= \bigcup_{k \in K(\overline{x})} \left\{ d \mid \left\{ \begin{array}{l} \langle Q_k \overline{c} + b_k, \nabla c(\overline{x})d \rangle \leq 0 \\ \langle a_{kj}, \nabla c(\overline{x})d \rangle \leq 0, j \in I_k(\overline{x}) \end{array} \right\} \right\}
\]

**Proof.** (\( \subseteq \)) Suppose \( d \in D(\overline{x}) \). By (21), \( \nabla c(\overline{x})d \in \text{dom}(h'(\overline{c}; \cdot)) \). In particular, by Proposition 4.1, \( \nabla c(\overline{x})d \in T(\overline{x} \mid C_k) \) for some \( k \in K(\overline{x}) \). By (15), we also have \( \langle Q_k \overline{c} + b_k, \nabla c(\overline{x})d \rangle = h'(c(\overline{x}); \nabla c(\overline{x})d) \leq 0 \).

(\( \supseteq \)) If \( d \in \bigcup_{k \in K(\overline{x})} \left\{ d \mid \nabla c(\overline{x})d \in T(\overline{x} \mid C_k), \langle Q_k \overline{c} + b_k, \nabla c(\overline{x})d \rangle \leq 0 \right\} \), then for some \( k \in K(\overline{x}), \nabla c(\overline{x})d \in T(\overline{x} \mid C_k) \). Then, again by Proposition 4.1, \( h'(c(\overline{x}); \nabla c(\overline{x})d) = \langle Q_k \overline{c} + b_k, \nabla c(\overline{x})d \rangle \leq 0 \), so \( d \in D(\overline{x}) \). \( \square \)

We now have the tools necessary to rewrite Theorem 2.1 in the context of piecewise linear-quadratic convex functions \( h \).

**Theorem 4.3** (PLQ second-order necessary and sufficient conditions). [37, Theorems 13.24(b), 13.14], [35, Theorem 3.4]. Let \( h : \mathbb{R}^m \to \mathbb{R} \) be piecewise linear-quadratic and convex with \( \overline{x} \in \text{dom}(f) \) such that \( f \) satisfies (BCQ) at \( \overline{x} \).

(a) If \( f \) has a local minimum at \( \overline{x} \), then \( 0 \in \nabla c(\overline{x})^\top \partial h(c(\overline{x})) \) and

\[
h''(c(\overline{x}); \nabla c(\overline{x})d) + \max \left\{ \langle d, \nabla^2(yc)(\overline{x})d \rangle \mid y \in M(\overline{x}) \right\} \geq 0
\]

for all \( d \in D(\overline{x}) \).

(b) If \( 0 \in \nabla c(\overline{x})^\top \partial h(c(\overline{x})) \) and

\[
h''(c(\overline{x}); \nabla c(\overline{x})d) + \max \left\{ \langle d, \nabla^2(yc)(\overline{x})d \rangle \mid y \in M(\overline{x}) \right\} > 0
\]

for all \( d \in D(\overline{x}) \setminus \{0\} \), then \( \overline{x} \) is a strong local minimizer (see (3)) of \( f \).
5. STRONG METRIC SUBREGULARITY OF THE KKT MAPPING

In this section we establish conditions under which the set-valued mapping Definition 3.5 satisfies strong metric subregularity.

**Definition 5.1** (Strong metric subregularity). A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in \text{gph} S$ and there exists $\kappa \geq 0$ and a neighborhood $U$ of $\bar{x}$ such that $\|x - \bar{x}\| \leq \kappa \text{dist} \left( \bar{y}, S(x) \right)$ for all $x \in U$.

Our discussion of strong metric subregularity only requires $f$ to satisfy (BCQ) at $\bar{x} \in \text{dom}(f)$.

**Lemma 5.1.** Consider the KKT mapping $g + G$ and the mapping $G$ given in Definition 3.5. Then, strong metric subregularity of $g + G$ at $(\bar{x}, \bar{y})$ for 0 is equivalent to the property that $(\bar{x}, \bar{y})$ is an isolated point of $G^{-1}(0)$.

**Proof.** By [13, Corollary 31.10], strong metric subregularity of $g + G$ at $(\bar{x}, \bar{y})$ for 0 is equivalent to strong metric subregularity of the linearization $G$ (11) at $(\bar{x}, \bar{y})$.

By [37, Theorem 11.14, Proposition 12.30] the mapping $G(x, y)$ is polyhedral; that is, $\text{gph} G$ is the union of finitely many polyhedral sets. Then [13, Corollary 31.11] establishes the equivalence of strong metric subregularity of $G$ at $(\bar{x}, \bar{y})$ for 0 and $(\bar{x}, \bar{y})$ being an isolated point of $G^{-1}(0)$. □

The main result of this section now follows.

**Theorem 5.1.** Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is piecewise linear-quadratic and convex with $\bar{x} \in \text{dom}(f)$ such that $f$ satisfies (BCQ) at $\bar{x}$. Then, the following are equivalent:

1. The set $M(\bar{x}) := \text{Null} \left( \nabla c(\bar{x})^T \right) \cap \partial h(c(\bar{x}))$ in (9) is a singleton and the second-order sufficient conditions of Theorem 4.3 are satisfied at $\bar{x}$;

2. The mapping $g + G$ is strongly metrically subregular at $(\bar{x}, \bar{y})$ for 0 and $\bar{x}$ is a strong local minimizer of $f$.

**Proof.** For a point $x \in \text{dom}(f)$, define $\Delta f(x; d) := h(c(x) + \nabla c(x)d) - h(c(x))$.

$(\Rightarrow)$ By Lemma 5.1 we argue strong metric subregularity of $g + G$ at $(\bar{x}, \bar{y})$ for 0 by showing that there is a neighborhood of $(\bar{x}, \bar{y})$ on which $(\bar{x}, \bar{y})$ is the unique solution to the generalized equation $G \ni 0$ (11). After the change of variables $d := x - \bar{x}$, we show that there is a neighborhood $U$ of $(0, \bar{y})$ such that $(d, y) = (0, \bar{y})$ is the unique solution to the generalized equation

\begin{align}
Hd + \nabla c(\bar{x})^Ty &= 0 \tag{23} \\
c(\bar{x}) + \nabla c(\bar{x})d &\in \partial h^*(y) (\iff y \in \partial h(c(\bar{x}) + \nabla c(\bar{x})d)), \tag{24}
\end{align}

where $H := \nabla^2_{xx} L(\bar{x}, \bar{y})$. Suppose there is no such neighborhood. Then, there exists a sequence of vectors $\{(d^i, y^i)\}_{i \in \mathbb{N}}$ converging to $(0, \bar{y})$ with $(d^i, y^i) \neq (0, \bar{y})$ that solve the generalized equation (23), (24). First assume $d^i \neq 0$ for all $i \in \mathbb{N}$. Define for each $i \in \mathbb{N}$, $t_i := \|d^i\|$, $v^i := d^i / \|d^i\|$, and assume without loss of generality that $v^i \to \bar{v}$ and that

\begin{align}
\Bigset{ c(\bar{x}) + \nabla c(\bar{x})d^i}_{i \in \mathbb{N}} \subset C_{k_0} \text{ for some } k_0 \in K(c(\bar{x}) + \nabla c(\bar{x})d^i) \subset K(\bar{x}), \tag{25}
\end{align}
since $d^i \to 0$. Taking the inner product on both sides of (23) with $d^i$, we obtain
\begin{equation}
0 = \left\langle d^i, H d^i \right\rangle + \left\langle d^i, \nabla c(\bar{x})^\top y^i \right\rangle \text{ for all } i \in \mathbb{N}.
\end{equation}
(26)

The subgradient inequality for $h$ at $c(\bar{x}) + \nabla c(\bar{x})d^i$ with subgradient $y_i$ gives
\begin{equation}
\Delta f(\bar{x}; d^i) \leq \left\langle d^i, \nabla c(\bar{x})^\top y_i \right\rangle = -\left\langle d^i, H d^i \right\rangle.
\end{equation}
(27)

Dividing through by $t_i > 0$ and letting $i \to \infty$, $df(\bar{x})/\bar{v} \leq \liminf_i \Delta f(\bar{x}; t_i, v_i)$, hence by (BCQ), Theorem 4.2 and (27), $h'(c(\bar{x}), \nabla c(\bar{x})\bar{v}) = df(\bar{x})/\bar{v} \leq \lim_i -\left\langle v^i, H d^i \right\rangle = 0$, and so $\bar{v} \in D(\bar{x}) \setminus \{0\}$.

By second-order sufficiency, $h''(c(\bar{x}), \nabla c(\bar{x})\bar{v}) + \bar{v}^\top H \bar{v} > 0$. We now show $\nabla c(\bar{x})\bar{v} \in T(\bar{v} | C_{k_0})$.

By (25) and the computation $\nabla c(\bar{x})\bar{v} = \bar{v}^\top \nabla c(\bar{x})^\top Q_{k_0} \nabla c(\bar{x})\bar{v}$, so that
\begin{equation}
\bar{v}^\top H \bar{v} + \bar{v}^\top \nabla c(\bar{x})^\top Q_{k_0} \nabla c(\bar{x})\bar{v} > 0.
\end{equation}

(28)

On the other hand, by (16),
\begin{equation*}
y^i \in \partial h(c(\bar{x}) + \nabla c(\bar{x})d^i) = \bigcap_{k \in K(c(\bar{x}) + \nabla c(\bar{x})d^i)} \left\{ y \mid y - Q_k(c(\bar{x}) + \nabla c(\bar{x})d^i) - b_k \in N \left( c(\bar{x}) + \nabla c(\bar{x})d^i \mid C_k \right) \right\},
\end{equation*}
and so $y^i - Q_{k_0}(c(\bar{x}) + \nabla c(\bar{x})d^i) - b_{k_0} \in N \left( c(\bar{x}) + \nabla c(\bar{x})d^i \mid C_{k_0} \right)$ for all $i \in \mathbb{N}$. Since $c(\bar{x}) \in C_{k_0}$, we have
\begin{align*}
0 &\geq \left\langle y^i - [Q_{k_0}(c(\bar{x}) + \nabla c(\bar{x})d^i) + b_{k_0}], c(\bar{x}) - [c(\bar{x}) + \nabla c(\bar{x})d^i] \right\rangle \\
&= \left\langle y^i - Q_{k_0}(c(\bar{x}) + \nabla c(\bar{x})d^i) - b_{k_0}, -\nabla c(\bar{x})d^i \right\rangle \\
&= -\left\langle d^i, \nabla c(\bar{x})^\top y^i \right\rangle + \left\langle Q_{k_0}(c(\bar{x}) + \nabla c(\bar{x})d^i) + b_{k_0}, \nabla c(\bar{x})d^i \right\rangle.
\end{align*}
Together with (26),
\begin{align*}
0 &\geq \left\langle d^i, H d^i \right\rangle + \left\langle Q_{k_0}(c(\bar{x}) + \nabla c(\bar{x})d^i) + b_{k_0}, \nabla c(\bar{x})d^i \right\rangle \\
&= \left\langle d^i, H d^i \right\rangle + \left\langle \nabla c(\bar{x})d^i, Q_{k_0} \nabla c(\bar{x})d^i \right\rangle + \left\langle Q_{k_0}c(\bar{x}) + b_{k_0}, \nabla c(\bar{x})d^i \right\rangle \\
&= \left\langle d^i, H d^i \right\rangle + \left\langle \nabla c(\bar{x})d^i, Q_{k_0} \nabla c(\bar{x})d^i \right\rangle + h'(c(\bar{x}); \nabla c(\bar{x})d^i) \text{ (by (15))} \\
&\geq \left\langle d^i, H d^i \right\rangle + \left\langle \nabla c(\bar{x})d^i, Q_{k_0} \nabla c(\bar{x})d^i \right\rangle,
\end{align*}
where the final inequality follows from Theorem 2.1, Theorem 4.3, and the observation that $\nabla c(\bar{x})d^i \in C_{k_0} - c(\bar{x}) \subset T(c(\bar{x}) \mid C_{k_0})$. Next, divide the inequality $0 \geq \left\langle d^i, H d^i \right\rangle + \left\langle \nabla c(\bar{x})d^i, Q_{k_0} \nabla c(\bar{x})d^i \right\rangle$ by $t_i^2$ and let $i \to \infty$ to yield the contradiction $0 \geq \bar{v}^\top H \bar{v} + \bar{v}^\top \nabla c(\bar{x})^\top Q_{k_0} \nabla c(\bar{x})\bar{v} > 0$.

Consequently, $d^i = 0$ for all $i$ sufficiently large, so without loss of generality, we now suppose $d^i = 0$ for all $i \in \mathbb{N}$. Hence by hypothesis, and $y^i \neq \bar{v}$ for all $i \in \mathbb{N}$. But then we contradict uniqueness of $M(\bar{x})$.

($\Leftarrow$) By Lemma 5.1, $(\bar{x}, \bar{v})$ is an isolated point of $G^{-1}(0)$. That is, there is a neighborhood $U$ of $(\bar{x}, \bar{v})$ on which $(\bar{x}, \bar{v})$ is the unique solution to the generalized equation
\begin{align*}
H(x - \bar{x}) + \nabla c(\bar{x})^\top y &= 0 \\
c(\bar{x}) + \nabla c(\bar{x})(x - \bar{x}) &\in \partial h^*(y).
\end{align*}
For $x = \bar{x}$, this implies there is a neighborhood $U_{\bar{y}}$ about $\bar{y}$ such that

\begin{equation}
U_{\bar{y}} \cap M(\bar{x}) = \{ \bar{y} \}. 
\end{equation}

Suppose there is $y \in (M(\bar{x})) \setminus U_{\bar{y}}$. Then $y_t = (1 - t)\bar{y} + ty \in M(\bar{x})$ for $t \in [0, 1]$. But for $t$ small, $y_t \in U_{\bar{y}} \cap M(\bar{x})$, which contradicts (29), so $M(\bar{x})$ is the singleton $\{ \bar{y} \}$. Therefore, it only remains to show that the second-order sufficient conditions of Theorem 4.3 are satisfied at $\bar{x}$.

Since $\bar{x}$ is local minimizer of $f$ at which $f$ satisfies (BCQ), Theorem 3.1 gives $0 \in \nabla c(\bar{x})^T \partial h(c(\bar{x}))$ and $h'(c(\bar{x})); \nabla c(\bar{x})d \geq 0$ for all $d \in \mathbb{R}^n$. Let $\bar{d} \in \mathbb{R}^n \setminus \{ 0 \}$ with $h'(c(\bar{x})); \nabla c(\bar{x})\bar{d} = 0$, or equivalently, $\bar{d} \in D(\bar{x})$. Without loss of generality, suppose $\| \bar{d} \| = 1$. In particular, by (22), there exists $k_0 \in K(\bar{x})$ such that

\begin{equation}
\nabla c(\bar{x})\bar{d} \in T(\bar{x} | C_{k_0}) \text{ and } \langle Q_{k_0}\bar{c} + b_{k_0}, \nabla c(\bar{x})\bar{d} \rangle = h'(c(\bar{x})); \nabla c(\bar{x})\bar{d} = 0 
\end{equation}

Since $h$ is PLQ convex, the second-order necessary conditions of Theorem 4.3 imply $h''(c(\bar{x})); \nabla c(\bar{x})\bar{d} + \bar{d}^T H\bar{d} \geq 0$.

We show this inequality is strict to complete the proof. Suppose to the contrary that

\begin{equation}
h''(c(\bar{x})); \nabla c(\bar{x})\bar{d} + \bar{d}^T H\bar{d} = 0. 
\end{equation}

Then, $\bar{d} \neq 0$ solves the program

\begin{equation}
\text{minimize } \quad h'(c(\bar{x})); \nabla c(\bar{x})d + \frac{1}{2} h''(c(\bar{x})); \nabla c(\bar{x})d + \frac{1}{2} d^T H d \\
\text{subject to } \quad d \in D(\bar{x}). 
\end{equation}

By (19) and continuity of $d \mapsto c(\bar{x}) + \nabla c(\bar{x})d$, there exists $\varepsilon > 0$ so that

\begin{equation}
\Delta f(\bar{x}; d) = h'(c(\bar{x})); \nabla c(\bar{x})d + \frac{1}{2} h''(c(\bar{x})); \nabla c(\bar{x})d \quad \text{for } \quad d \in \varepsilon B \cap \left\{ d \mid c(\bar{x}) + \nabla c(\bar{x})d \in \text{dom}(h) \right\}. 
\end{equation}

By (30) and polyhedrality, $c(\bar{x}) + t\nabla c(\bar{x})\bar{d} \in \text{dom}(h)$ for sufficiently small $t > 0$. It follows, after shrinking $\varepsilon > 0$ if necessary, that

\begin{equation}
\Delta f(\bar{x}; \bar{d}) + \frac{t^2}{2} \bar{d}^T H\bar{d} = 0 \quad \text{for all } \quad 0 \leq t < \varepsilon. 
\end{equation}

Since $0 \in \partial f(\bar{x})$ and $f$ satisfies (BCQ) at $\bar{x}$, (20) with $v = 0$, $y = \bar{y}$, and $w \in \mathbb{R}^n$ gives $d^2 f(\bar{x}|0)(w) = d^2 \bar{f}(\bar{x}|0)(w) + w^T Hw$, where $\bar{f}$ is also piecewise linear-quadratic by the discussion following (20).

Since $\bar{x}$ is a strong local minimizer,

\begin{equation}
d^2 f(\bar{x}|0)(w) = \lim_{\tau \searrow 0} \frac{f(\bar{x} + \tau w') - f(\bar{x})}{\tau^2} \geq \lim_{\tau \searrow 0} \gamma \|w'\|^2 = \gamma \|w\|^2 \quad \text{(see Theorem 2.1).} 
\end{equation}

Then, we have $d^2 f(\bar{x}|0)(w) = d^2 \bar{f}(\bar{x}|0)(w) + w^T Hw \geq \gamma \|w\|^2$. By (18) the limit inf defining $d^2 \bar{f}(\bar{x}|0)(w)$ is also expressed as a limit only in $\tau$ (because $\bar{f}$ is piecewise linear-quadratic), so

\begin{equation}
d^2 \bar{f}(\bar{x}|0)(w) \leq \lim_{\tau \searrow 0} \frac{\bar{f}(\bar{x} + \tau w') - \bar{f}(\bar{x})}{\tau^2} = \lim_{\tau \searrow 0} \frac{\Delta f(\bar{x}; \tau w)}{\tau^2}. 
\end{equation}

Putting the last two observations together, $d^2 f(\bar{x}|0)(w) = \lim_{\tau \searrow 0} \frac{\Delta f(\bar{x}; \tau w)}{\tau^2} + w^T Hw \geq \gamma \|w\|^2$. But, for $0 < \tau < \varepsilon$ and $w = \bar{d}$, (32) gives the contradiction $0 = \lim_{\tau \searrow 0} \left\{ \frac{\Delta f(\bar{x}; \tau w) + \bar{d}^T H\bar{d}}{\tau^2} \right\} = d^2 f(\bar{x}|0)(\bar{d}) \geq \gamma \|\bar{d}\|^2 = \gamma > 0$. \qed
5.1. Application: superlinear convergence of quasi-Newton methods. Let \( f \) and \( g + G \) be given by Definition 3.5 and consider the corresponding quasi-Newton method (6) initialized at \((x^0, y^0)\). In this section, we assume the \( B_k \) defined in (6) take the form

\[
B_k = \begin{pmatrix}
B_k & \nabla c(x^k) \\
-\nabla c(x^k) & 0
\end{pmatrix}.
\]

This choice allows us to relate the optimality conditions for the subproblems \( Q_k \) defined in Lemma 5.2 for solving \( P \) to the quasi-Newton method of (6). As in Section 3, the following is immediate:

**Lemma 5.2.** Let \( f \) be convex-composite, and let \((x^k, y^k) \in \mathbb{R}^n \times \mathbb{R}^m\) be such that \( f \) satisfies (BCQ) at \( x^k \), let \( B_k \in \mathbb{R}^{m \times n} \). Then, \((d^k, y^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^m\) satisfy the optimality conditions for

\[
\minimize d \in \mathbb{R}^n \quad h(c(x^k) + \nabla c(x^k)d) + \frac{1}{2} d^\top B_k d
\]

if and only if \((x^{k+1}, y^{k+1})\) satisfy the quasi-Newton update for \( g + G \) given by Definition 3.5, with the choice (33). Namely, \( 0 \in G(x^k, y^k) + B_k \left( x^{k+1} - x^k \right) \) and satisfies \( x^{k+1} := x^k + d^k \).

As a consequence of strong metric subregularity of the linearization \( G \) given by (11), we have the following convergence result:

**Theorem 5.2.** [13, Dennis-Moré Theorem for Generalized Equations] Let \((\overline{x}, \overline{y})\) be a solution of \( g + G \ni 0 \) given by Definition 3.5 and let \( U \) be a neighborhood of \((\overline{x}, \overline{y})\). For some starting point \((x^0, y^0) \in U\) consider a sequence \( \{(x^k, y^k)\}_{k \in \mathbb{N}} \) generated by (6) which remains in \( U \) for all \( k \in \mathbb{N} \) and satisfies \((x^k, y^k) \neq (\overline{x}, \overline{y})\) for all \( k \in \mathbb{N} \). Define \( E_k := B_k - \nabla G(\overline{x}, \overline{y}) \) and \( s^k := (x^{k+1} - x^k, y^{k+1} - y^k) \). If the linearization mapping \( G \) given by (11) is strongly metrically subregular at \((\overline{x}, \overline{y})\) for 0 and the sequence \( \{(x^k, y^k)\}_{k \in \mathbb{N}} \) satisfies \((x^k, y^k) \to (\overline{x}, \overline{y})\) and \( E_k s^k = o(||s^k||) \) then \((x^k, y^k) \to (\overline{x}, \overline{y})\) superlinearly.

**Remark 3.** Suppose the function \( g \) is \( C^1 \)-smooth and \((x^k, y^k) \to (\overline{x}, \overline{y})\). Then, \( E_k s^k = o(||s^k||) \iff [B_k - \nabla G(x^k, y^k)]s^k = o(||s^k||)\).

The following corollary is of algorithmic significance.

**Corollary 5.1.** Let \( f \) be as in \( P \). Suppose \( M(\overline{x}) = \{\overline{y}\} \) and the second-order sufficient conditions of Theorem 4.3 are satisfied at \( \overline{x} \). Then, \((\overline{x}, \overline{y})\) solves \( 0 \in G(\overline{x}, \overline{y}) + G(\overline{x}, \overline{y}) \). Moreover, there exists a neighborhood \( U \) of \((\overline{x}, \overline{y})\) such that if \((x^0, y^0) \in U\), the sequence \( \{(x^k, y^k)\}_{k \in \mathbb{N}} \) generated from the optimality conditions for \( Q_k \) remains in \( U \) with \((x^k, y^k) \neq (\overline{x}, \overline{y})\) for all \( k \in \mathbb{N} \), and

\[
(x^k, y^k) \to (\overline{x}, \overline{y}) \quad \text{and} \quad (B_k - \nabla^2 (y^k c)(x^k))[x^{k+1} - x^k] = o(||s^k||),
\]

then \((x^k, y^k) \to (\overline{x}, \overline{y})\) superlinearly.

**Remark 4.** Consequently, the sufficient conditions for superlinear convergence of quasi-Newton methods require us to choose \( B_k \) as an approximation to the Hessian of the Lagrangian \( \nabla^2_{xx} L(x^k, y^k) = \nabla^2 (y^k c)(x^k) \) in the update direction \( x^{k+1} - x^k \) at every iteration.

6. Partial Smoothness

The notion of partial smoothness, introduced by Lewis [23], generalizes classical notions of nondegeneracy, strict complementarity, and active constraint identification by illuminating the appropriate
underlying manifold geometry of optimization problems. This allows for a more thorough understanding of the convergence behavior of algorithms applied to nonsmooth optimization problems, where solutions lie on well-defined submanifolds of the parameter space on which the function behaves smoothly and off of which it behaves nonsmoothly. Partial smoothness in the context of P allows us in Section 7 to establish metric regularity properties of the solution mapping.

**Definition 6.1.** Define a set \( \mathcal{M} \subset \mathbb{R}^m \) to be a manifold of codimension \( \ell \) around \( \tau \in \mathbb{R}^m \) if \( \tau \in \mathcal{M} \), and there exists an open set \( V \subset \mathbb{R}^m \) containing \( \tau \) and a \( C^2 \)-smooth function \( F : V \to \mathbb{R}^\ell \) with surjective derivative throughout \( V \) such that \( \mathcal{M} \cap V = \{ c \in V : F(c) = 0 \} \). In which case (see [23]), the tangent space to \( \mathcal{M} \) at \( \tau \) is \( T(\tau | \mathcal{M}) = \text{Null}(\nabla F(\tau)) \), the normal space to \( \mathcal{M} \) at \( \tau \) is \( N(\tau | \mathcal{M}) = \text{Ran} \left( \nabla F(\tau)^\top \right) \), both independent of the choice of \( F \). Moreover, the set \( \mathcal{M} \) is Clarke regular at \( \tau \), and \( N(\tau | \mathcal{M}) \) equals the normal cone defined in (4).

**Definition 6.2** (Partial smoothness for closed, convex functions). Suppose \( h : \mathbb{R}^m \to \mathbb{R} \) is a closed, proper, convex function and that \( \tau \in \mathcal{M} \subset \mathbb{R}^m \). The function \( h \) is partly smooth at \( \tau \) relative to \( \mathcal{M} \) if \( \mathcal{M} \) is a manifold around \( \tau \) and the following four properties hold:

(a) (restricted smoothness) the restriction \( h|_{\mathcal{M}} \) is smooth around \( \tau \), in that there exists a neighborhood \( V \) of \( \tau \) and a \( C^2 \)-smooth function \( g \) defined on \( V \) such that \( h = g \) on \( V \cap \mathcal{M} \);

(b) (existence of subgradients) at every point \( c \in \mathcal{M} \) close to \( \tau \), \( \partial h(c) \neq \emptyset \);

(c) (normals and subgradients parallel) \( \text{par}(\partial h(\tau)) = N(\tau | \mathcal{M}) \);

(d) (subgradient inner semicontinuity) the subdifferential map \( \partial h \) is inner semicontinuous at \( \tau \) relative to \( \mathcal{M} \).

We say that \( h \) is partly smooth relative to \( \mathcal{M} \) if \( \mathcal{M} \) is a manifold and \( h \) is partly smooth at each point in \( \mathcal{M} \) relative to \( \mathcal{M} \).

**Remark 5.** By [23, Proposition 2.4], requiring (a)-(d) in the definition is equivalent to requiring (a), (b), (d), and normal sharpness:

\[
(34) \quad h'(\tau; -w) > -h'(\tau; w), \quad \forall w \in N(\tau | \mathcal{M}) \setminus \{0\},
\]

and is also equivalent to requiring (a), (b), (d), and linearity and tangent equality:

\[
(35) \quad \left\{ w \in \mathbb{R}^m \mid -h'(\tau; w) = h'(\tau; -w) \right\} =: \text{lin} h'(\tau ; \cdot) = T(\tau | \mathcal{M}).
\]

In the context of the PLQ functions given in Definition 4.1, a natural choice for the active manifold at a point \( \tau \in \text{dom}(h) \) for \( P \) is the set given by

\[
(36) \quad \mathcal{M}_{\tau} := \text{ri} \left( \bigcap_{k \in \mathcal{K}(\tau)} C_k \right),
\]

where \( \mathcal{K}(\tau) \) are the active indices at \( \tau \) (see Definition 4.2). The analysis of the manifold \( \mathcal{M}_{\tau} \) requires a more thorough understanding of the structure of \( \text{dom}(h) \), which we obtain from the following key result due to Rockafellar and Wets.

**Lemma 6.1.** [37, Lemma 2.50] Suppose \( C \) is a convex set which is the union of a finite collection of polyhedral sets \( C_k \). If the polyhedral sets \( \{C_k\}_{k=1}^K \) are represented in terms of a single family of non-constant affine functions \( l_i(x) = \langle a_i, x \rangle - \alpha_i \) indexed by \( i = 1, \ldots, s \), then for each \( k \) there is a subset \( I_k \) of \( \{1, \ldots, s\} \) such that \( C_k = \left\{ x \mid l_i(x) \leq 0 \text{ for all } i \in I_k \right\} \). Let \( I \) denote the set of indices \( i \in \{1, \ldots, s\} \) such that \( l_i \leq 0 \) for all
Then, 

\[ C = \left\{ x \mid I_i(x) \leq 0 \text{ for all } i \in I \right\}. \]

If \( \text{int} \ C \neq \emptyset \), then \( C \) can be written as the union of a finite collection of polyhedral sets \( \left\{ D_j \right\}_{j \in J} \) such that

(a) each set \( D_j \) is included in one of the sets \( C_k \),

(b) \( \text{int} D_j \neq \emptyset \), so \( D_j = \text{cl} \text{int} D_j \),

(c) \( \text{int} D_{j_1} \cap \text{int} D_{j_2} = \emptyset \) when \( j_1 \neq j_2 \).

This result implies that the domain of \( h \) has a finite stratification [14, Definition 3.1] for which \( h \) is a stratifiable function [14, Definition 3.2]. This stratification is central to our discussion of partial smoothness and is referred to as the Rockafellar-Wets PLQ Representation.

**Theorem 6.1** (Rockafellar-Wets PLQ Representation). Suppose \( h \) is piecewise linear-quadratic convex and \( \text{int} \text{dom} \ (h) \neq \emptyset \). Then, without loss of generality, we may assume the polyhedral sets \( \{C_k\}_{k=1}^K \) defining \( h \) are given in terms of a common set of \( s > 0 \) hyperplanes \( \mathcal{H} := \left\{ (a_j, \alpha_j) \right\}_{j=1}^s \subset (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R} \), so that for all \( k \in \{1, \ldots, K\} \),

\[
C_k = \left\{ c \mid \langle \omega_{kj} a_j, c \rangle \leq \omega_{kj} \alpha_j \text{ for all } j \in \{1, \ldots, s\} \right\},
\]

with \( \omega_{kj} \in \{\pm 1\} \),

\[
I_k(c) = \left\{ j \mid \langle \omega_{kj} a_j, c \rangle = \omega_{kj} \alpha_j \right\} = \left\{ j \mid \langle a_j, c \rangle = \alpha_j \right\} \subset \{1, \ldots, s\},
\]

and

(a) \( \emptyset \neq \text{int} C_k = \left\{ c \mid \langle \omega_{kj} a_j, c \rangle < \omega_{kj} \alpha_j \text{ for all } j \in \{1, \ldots, s_k\} \right\} \), for all \( k \in \{1, \ldots, K\} \),

(b) \( \text{int} C_{k_1} \cap \text{int} C_{k_2} = \emptyset \) when \( k_1 \neq k_2 \).

Condition (b) implies that if \( c \in C_{k_1} \cap C_{k_2} \), then \( c \in \text{bdry} C_{k_1} \cap \text{bdry} C_{k_2} \) when \( k_1 \neq k_2 \).

**Proof.** The proof of the previous lemma shows that for every polyhedron \( D_j \) and every \( i \in \{1, \ldots, s\} \), either \( I_i(x) \leq 0 \) for all \( x \in D_j \) or \( I_i(x) \geq 0 \) for all \( x \in D_j \). Therefore each affine function is used in the definition of \( D_j \), and \( D_j \) is contained entirely within one of the sets \( C_k \), relative to which \( h \) takes the form \( \frac{1}{2} \langle c, Q_k c \rangle + \langle b_k, c \rangle + \beta_k \).

The basic assumptions employed for the remainder of this section are listed below.

**Assumption 1.**

(a) The function \( h \) is PLQ convex with \( \text{dom} \ (h) \) given by the Rockafellar-Wets PLQ representation described in Theorem 6.1,

(b) \( \tau \in \text{dom} \ (h) \) satisfies \( \overline{k} := |\mathcal{K}(\tau)| \geq 2 \).

**Remark 6.** Whenever \( \mathcal{K}(\tau) = \{k_0\} \), \( h \) is continuously differentiable on \( \text{int} C_{k_0} \). Therefore, we assume that \( \overline{k} \geq 2 \) and delay the discussion of \( \overline{k} = 1 \) to Section 7.2.

The following lemma further supports the choice for the manifold \( \mathcal{M}_\tau \).

**Lemma 6.2.** Let \( \mathcal{M}_\tau \) be as in (36) and let Assumption 1 hold. Then, for any \( c \in \mathcal{M}_\tau \), \( \mathcal{K}(c) = \mathcal{K}(\tau) \), and so \( \mathcal{M}_c = \mathcal{M}_\tau \). Moreover, for any \( k \in \mathcal{K}(\tau) \), the active index sets \( I_k(c) \) satisfy \( I_k(c) = I_k(\tau) \).
Proof. Suppose $K(c) \neq K(\overline{c})$. Since the definition of $M_\sigma$ implies $K(\overline{c}) \subset K(c)$, there exists $j \in K(c') \setminus K(\overline{c})$. By (b) in Theorem 6.1, we necessarily have $c \in \text{bdry } C_j$.

We first argue the existence of $\varepsilon > 0$ such that that $(\sigma + \varepsilon B) \cap C_k = \emptyset$ for all $k \notin K(\overline{c})$. If no such $\varepsilon$ exists, since there are only finitely many $k \in K \setminus K(\overline{c})$, there would exist an index $k_0 \notin K(\overline{c})$ and an infinite sequence $c^n \to \sigma$ with $\{c^n\} \subset C_{k_0}$. By closedness of the set $C_{k_0}$, $\sigma \in C_{k_0}$, which is a contradiction.

Since $c, \sigma \in M_\sigma$, by [36, Theorem 6.4] there exists a $\mu > 1$ such that $\varepsilon := (1 - \mu)\sigma + \mu c \in \bigcap_{k \in K(\overline{c})} C_k$.

Since $c \in \text{bdry } C_j$, there exists a $z \in \text{int } C_j$ sufficiently close to $c$ so that the ray $R := \{\varepsilon + \lambda(z - \sigma) \mid 0 \leq \lambda\}$ meets $\sigma + B$. We consider two cases. To set the stage, for any two points $x, y \in \mathbb{R}^m$, denote the line segment connecting them by $[x, y] = \{(1 - \lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\}$.

Case 1. There is a point $x \in R \cap (\sigma + B) \cap C$. Then $z \in [\sigma, x] \subset C_k$ for some $k \in K(\overline{c})$. But then $z \in (\text{int } C_j) \cap C_k$, a contradiction.

Case 2. We have $R \cap (\sigma + B) \cap C = \emptyset$. Then there is a point $x \in (\sigma + B) \setminus C$ such that $z \in [\sigma, x]$. Since $x \notin C$, there is a first point, which we denote by $\tilde{z}$, in $C_j$ on this line segment as one moves from $x$ to $c$. Then the line segment $[\hat{z}, \sigma] \subset C$. The point $\hat{z}$ is not on the line segment $[\sigma, \tilde{z}]$ since then both $c'$ and $z$ would be on the same line segment $[\sigma, \tilde{z}]$ and so $\text{int } C_j \cap \text{bdry } C_k \neq \emptyset$ for some $k \in K(\overline{c})$, a contradiction. Consequently, the points $\tilde{z}, \sigma$ and $\hat{z}$ are not all collinear and hence form a triangle inside of $C$. Let $\hat{z}$ be on the boundary of $\sigma + B$ and on the line segment $[\tilde{z}, \sigma]$. Then the line segment $[\hat{z}, \sigma]$ passes through $\text{int } C_j$. This is again a contradiction.

Therefore, no such $c$ exists, and $K(c) = K(\overline{c})$ for all $c \in M_\sigma$.

For the second claim, suppose there exists $k \in K(\overline{c})$, $c \in M_\sigma$ and $j \in \{1, \ldots, s\}$ with
\[
\left\langle c, \omega_{kj} a_j \right\rangle < \omega_{kj} a_j \quad \text{and} \quad \left\langle \sigma, \omega_{kj} a_j \right\rangle = \omega_{kj} a_j.
\]

Again by [36, Theorem 6.4], we may choose $\mu > 1$ so that $\mu \sigma + (1 - \mu)c \in M_\sigma$. In particular, $\mu \sigma + (1 - \mu)c \in C_k$. But writing $\mu = 1 + \varepsilon$ with $\varepsilon > 0$ gives the contradiction
\[
\omega_{kj} a_j \geq \left\langle \mu \sigma + (1 - \mu)c, \omega_{kj} a_j \right\rangle
\]
\[
= (1 + \varepsilon) \left\langle \sigma, \omega_{kj} a_j \right\rangle - \varepsilon \left\langle c, \omega_{kj} a_j \right\rangle > \omega_{kj} a_j \text{ by (38)}.
\]

Therefore $I_k(\overline{c}) \subset I_k(c)$. Reversing the roles of $c$ and $\sigma$ in (38) gives the other inclusion. 

The previous lemma tells us distinct points $c, c' \in M_\sigma$ have the same active indices $K(c)$ and $K(c')$. Moreover, for any active polyhedron $C_k$, the active hyperplanes for that polyhedron, $I_k(c)$ and $I_k(c')$, at $c$ and $c'$ are the same. This observation offers a global description of $M_\sigma$ in terms of the active hyperplanes at $\sigma$ alone.

Lemma 6.3. Let $M_\sigma$ be as in (36), and let Assumption 1 hold. Then,
\[
M_\sigma = \left\{ c \left| \begin{array}{l}
\left\langle c, a_j \right\rangle = a_j \text{ for all } k \in K(\overline{c}), j \in I_k(\overline{c}) \\
\left\langle c, \omega_{kj} a_j \right\rangle < \omega_{kj} a_j \text{ for all } k \in K(\overline{c}), j \notin I_k(\overline{c})
\end{array} \right. \right\}.
\]

In particular, $I_{k_1}(c) = I_{k_2}(c)$ for all $c \in M_\sigma$ and $k_1, k_2 \in K(\overline{c})$. Moreover, for any $k \in K(\overline{c})$ and $c \in M_\sigma$, $T(c \mid M_\sigma) = \text{Null} \left( A_k(\overline{c})^\top \right)$, and $N(c \mid M_\sigma) = \text{Ran} \left( A_k(\overline{c}) \right)$, where $A_k(\overline{c})$ is the matrix whose columns are the gradients of the active constraints at $\sigma \in C_\sigma$ in some ordering.

Remark 7. By Lemma 6.2 and Lemma 6.3, for all $c \in M_\sigma$, $k \in K(\overline{c})$, and $j \in K(c)$, $\text{Ran} \left( A_k(\overline{c}) \right) = \text{Ran} \left( A_j(c) \right)$. This observation becomes important in a structural definition to follow.
Proof. Define

$$C_1 := \bigcap_{k \in \mathcal{K}(\bar{c})} C_k, \quad C_2 := \left\{ c \left| \begin{array}{l}
\langle c, \omega_{kj} a_j \rangle = \omega_{kj} a_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \in I_k(\bar{c}) \\
\langle c, \omega_{kj} a_j \rangle \leq \omega_{kj} a_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \notin I_k(\bar{c})
\end{array} \right. \right\}.$$ 

We aim to show $\text{ri } (C_1) \supset \text{ri } (C_2)$. For $k \in \mathcal{K}(\bar{c})$ and $j \in I_k(\bar{c})$ define $C_{kj} := \left\{ c \left| \begin{array}{l}
\langle c, \omega_{kj} a_j \rangle = \omega_{kj} a_j
\end{array} \right. \right\}$, and for $k \in \mathcal{K}(\bar{c})$ and $j \notin I_k(\bar{c})$, let $D_{kj} := \left\{ c \left| \begin{array}{l}
\langle c, \omega_{kj} a_j \rangle \leq \omega_{kj} a_j
\end{array} \right. \right\}$. Then by definition of $I_k(\bar{c})$,

$$\bar{c} \in \bigcap_{k \in \mathcal{K}(\bar{c})} \text{ri } (C_{kj}) \cap \bigcap_{k \notin \mathcal{K}(\bar{c})} \text{ri } (D_{kj}),$$

so [36, Theorem 6.5] gives

$$\text{ri } (C_2) = \left\{ c \left| \begin{array}{l}
\langle c, \omega_{kj} a_j \rangle = \omega_{kj} a_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \in I_k(\bar{c}) \\
\langle c, \omega_{kj} a_j \rangle < \omega_{kj} a_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \notin I_k(\bar{c})
\end{array} \right. \right\}.$$ 

Moreover, $C_1 \supset C_2$ with $C_2$ not entirely contained within the relative boundary of $C_1$ because $\bar{c} \in C_2 \cap \mathcal{M}_\tau$. By [36, Corollary 6.5.2], $\mathcal{M}_\tau := \text{ri } (C_1) \supset \text{ri } (C_2)$. Lemma 6.2 shows $\mathcal{M}_\tau := \text{ri } (C_1) \subset \text{ri } (C_2)$ because $I_k(c) = I_k(\bar{c})$ throughout $\mathcal{M}_\tau$.

For the second claim, the structure of $\mathcal{M}_\tau$ implies that if $\langle c, \omega_{kj} a_j \rangle = \omega_{kj} a_j$ for some $k_1 \in \mathcal{K}(\bar{c})$, then $\langle c, \omega_{k_2} a_j \rangle = \omega_{k_2} a_j$ for any other $k_2 \in \mathcal{K}(\bar{c})$ as $\omega_{kj} \in \{\pm 1\}$. Hence $I_{k_2}(c) \supset I_{k_1}(c)$, and this argument is symmetric in $k_1$ and $k_2$.

The tangent and normal cone formulas hold throughout $\mathcal{M}_\tau$ by Theorem 4.1. \hfill \Box

Based on Lemma 6.3 and Remark 7, we now establish the notational tools required for our analysis.

**Definition 6.3.** Let $\mathcal{M}_\tau$ be as in (36), and let Assumption 1 hold. Define $A_{k}(c)$ to be the matrix whose columns are the gradients of the active constraints at $c \in C_k$ in some ordering. By Theorem 6.1 and Lemma 6.3, without loss of generality, we can define $A := A_{k}(c)$ independent of the choice of $c \in \mathcal{M}_\tau$, and for any $j \in \{1, \ldots, k\}$, there exists a diagonal matrix $P_j$ with entries $\pm 1$ on the diagonal such that

$$A_{k}(c) = A_{k}(c) \text{ independent of } c \in \mathcal{M}_\tau.$$ 

We let $\ell$ be the common number of columns $\ell := |I_k(\bar{c})| = |I_k(\bar{c})|$ for all $k, k' \in \mathcal{K}(\bar{c})$, so that $A \in \mathbb{R}^{m \times \ell}, P_j \in \mathbb{R}^{\ell \times \ell}, P_\tau = I_\ell$, and define the following block matrices $Q := \text{diag}(Q_k), \hat{A} := \text{diag}A P_j$

$$A := \begin{pmatrix}
(1 - \bar{k}) A P_1 & A P_2 & \cdots & A  \\
A P_1 & (1 - \bar{k}) A P_2 & \cdots & A  \\
\vdots & \ddots & \ddots & \vdots  \\
A P_1 & A P_2 & \cdots & (1 - \bar{k}) A
\end{pmatrix}, \quad Q := \begin{pmatrix}
Q_{k_1}  \\
Q_{k_2}  \\
\vdots  \\
Q_{k_{\bar{m}}} 
\end{pmatrix}, \quad B := \begin{pmatrix}
b_{k_1}  \\
b_{k_2}  \\
\vdots  \\
I_m
\end{pmatrix}, \quad J := \begin{pmatrix}
I_m  \\
I_m  \\
\vdots  \\
I_m
\end{pmatrix}$$

and averaged quantities

$$\overline{Q} = (1/\bar{k}) J^T \hat{Q} J, \quad \overline{A} = (1/\bar{k}) J^T \hat{A}, \quad \overline{b} = (1/\bar{k}) J^T B, \quad \lambda_0(\bar{c}) = \overline{Q} \bar{c} + \overline{b}.$$
In a fashion similar to the structure functional approach of [27, 28, 38], we give a formula for the subdifferential in terms of the active manifold structure previously laid out.

**Lemma 6.4.** Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. For any $c \in \mathcal{M}_\tau$, $\partial h(c)$ can be given by two equivalent formulations:

\begin{equation}
\partial h(c) = \left\{ y \mid \exists \mu = (\mu_1^\top, \ldots, \mu_k^\top)^\top \geq 0 \text{ such that } J y = Q c + B + \bar{A} \mu \right\} = \lambda_0(c) + \bar{A} U(c),
\end{equation}

where

\begin{equation}
U(c) := \left\{ \mu \geq 0 \mid A \mu = \bar{k} \left[ Q c + B - J(\overline{Q} c + \overline{b}) \right] \right\}.
\end{equation}

**Proof.** By (16) and Lemma 6.2, $y \in \partial h(c)$ if and only if $y \in Q_{k_j} c + b_{k_j} + N\left(c \mid C_{k_j}\right)$ for all $j \in \{1, \ldots, \overline{k}\}$. In terms of the active indices at $c$ for the polyhedron $C_{k_j}$, (14) and (39) imply

\[
y = Q_{k_j} c + b_{k_j} + A P_j \mu_j, \quad \text{where } j \in \{1, \ldots, \overline{k}\}, \mu_j \geq 0.
\]

Hence $y \in \partial h(c)$ if and only if there exists $\mu = (\mu_1^\top, \ldots, \mu_k^\top)$ such that $(y, \mu)$ satisfies the system

\[
J y = Q c + B + \bar{A} \mu, \quad \mu = (\mu_1^\top, \ldots, \mu_k^\top)^\top \geq 0.
\]

Since $J^\top J = \bar{k} I_m$, multiplying both sides of the first equation in (41) by $(1/\bar{k}) J^\top$ gives $y = \overline{Q} c + \overline{b} + \bar{A} \mu$, where $\mu$ satisfies

\[
\overline{Q} c + \overline{b} + \bar{A} \mu = A P_j \mu_j + Q_{k_j} c + b_{k_j}, \quad \text{for all } j \in \{1, \ldots, \overline{k}\}, \mu \geq 0.
\]

The set of $\mu$ that satisfy the display defines membership in $U(c)$, so $\partial h(c) = \lambda_0(c) + \bar{A} U(c)$. \qed

The notion of nondegeneracy that we use imposes linear independence of the columns of $A$.

**Definition 6.4** (Nondegeneracy). Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. We say that $\mathcal{M}_\tau$ satisfies the nondegeneracy condition if $\text{Null}(A) = \{0\}$.

Nondegeneracy yields a uniqueness property of the multipliers $\mu \in U(c)$.

**Lemma 6.5.** Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Suppose $\mathcal{M}_\tau$ satisfies the nondegeneracy condition of Definition 6.4, $c \in \mathcal{M}_\tau$, and $y \in \partial h(c)$. Then, there is a unique $\mu \in U(c)$, given by $\mu(c, y)_j = P_j (A^\top A)^{-1} A^\top (y - (Q_{k_j} c + b_{k_j})), j \in \{1, \ldots, \overline{k}\}$ so that $y = \lambda_0(c) + \bar{A} \mu(c, y)$.

**Proof.** For any $j \in \{1, \ldots, \overline{k}\}$, Lemma 6.4 implies there exists $\mu_j \geq 0$ such that $y = Q_{k_j} c + b_{k_j} + A P_j \mu_j$. Nondegeneracy implies $\mu_j$ is given uniquely by the equation $\mu(c, y)_j = P_j (A^\top A)^{-1} A^\top (y - (Q_{k_j} c + b_{k_j}))$. \qed

A corresponding notion of strict complementarity is provided by the next lemma.

**Lemma 6.6.** Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Suppose $c \in \mathcal{M}_\tau$ and $\text{ri}(\partial h(c)) \neq \emptyset$. Then $y \in \text{ri}(\partial h(c))$ if and only if $\mu(c, y)_i > 0$ for all $i \in \{1, \ldots, \overline{k}\}$.
Proof. By [36, Theorem 6.4], $y \in \text{ri} (\partial h(c))$ if and only if for all $y' \in \partial h(c)$, there exists $t > 1$ so that $ty + (1 - t)y' \in \partial h(c)$. Choose a $y' \in \partial h(c)$ with $y' \neq y$.

$(\Rightarrow)$ If there exists $i_0 \in \{1, \ldots, \overline{c}\}$ and $j \in \{1, \ldots, \ell\}$, with $(\mu(c, y)_{i_0})_j = 0$, then, by (41),

$$
\partial h(c) \ni ty + (1 - t)y' = Q_0c + b_{i_0} + AP_{i_0}[\mu(c, y)_{i_0} + (1 - t)\mu(c, y')_{i_0}].
$$

By Lemma 6.5, $\mu(c, ty + (1 - t)y')_{i_0} = \mu(c, y)_{i_0} + (1 - t)\mu(c, y')_{i_0}$. By assumption, the right-hand side has its $j$th component is negative for all $t > 1$, a contradiction.

$(\Leftarrow)$ We must show there exists $\epsilon > 0$ such that if $t := 1 + \epsilon$ then $\mu(c, y)_{i_0} + (1 - t)\mu(c, y')_{i_0} > 0$. After rearranging, this is equivalent to finding $\epsilon > 0$ so that $\mu(c, y)_{i_0} + \epsilon[\mu(c, y)_{i_0} - \mu(c, y')_{i_0}] > 0$. If $\mu(c, y)_{i_0} - \mu(c, y')_{i_0} \geq 0$, the claim is immediate. Otherwise, we choose $\epsilon$ via

$$
0 < \epsilon < \min \left\{ \frac{(\mu(c, y)_{i_0})_j}{(\mu(c, y)_{i_0})_j - (\mu(c, y')_{i_0})_j} \mid (\mu(c, y)_{i_0})_j - (\mu(c, y')_{i_0})_j < 0, j \in \{1, \ldots, \ell\} \right\}.
$$

Then $y \in \text{ri} (\partial h(c))$. \hfill $\square$

However, a weaker notion of strict complementarity in conjunction with nondegeneracy suffices to show that $\text{ri} (\partial h(c)) \neq \emptyset$ throughout $\mathcal{M}_\tau$.

**Definition 6.5** ($k$-strict complementarity). Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. We say $k$-strict complementarity holds at $(c, y)$ for $\mu = (\mu_1^\top, \ldots, \mu_{\overline{c}}^\top)^\top$ if

(a) $c \in \mathcal{M}_\tau$, $y \in \partial h(c)$,

(b) There exists $k \in \mathcal{K}(\overline{c})$ with $\mu_k > 0$,

(c) Whenever there exists $j \in \mathcal{K}(c) \setminus \{k\}$ and $i \in \{1, \ldots, \ell\}$ with $(\mu_j)_i = 0$, then the scalars $(P'_{j})_{ii} = 1$

for all $j' \in \mathcal{K}(c)$,

(d) $(y, \mu)$ satisfies (41).

**Remark 8.** When $k$-strict complementarity holds at a pair $(c, y)$ and an index $j$ satisfies (c), the active polyhedra $\{c_k\}_{k \in \mathcal{K}(\overline{c})}$ are all within the same closed half-space of the corresponding hyperplane. Also observe that $y \in \text{ri} (\partial h(c))$ implies $k$-strict complementarity at $(c, y)$.

A requirement of partial smoothness is that the normal space to $\mathcal{M}_\tau$ and $\text{par} (\partial h(c))$ are equal. The nondegeneracy condition allows us to describe $\text{par} (\partial h(c))$ using the vectors in $\mathcal{U}(c)$ rather than the subgradients in $\partial h(c)$.

**Lemma 6.7.** Let $\mathcal{M}_\tau$ be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Suppose $\mathcal{M}_\tau$ satisfies the nondegeneracy condition. Then, for any $c \in \mathcal{M}_\tau$,

$$
\text{par} (\partial h(c)) = \text{Ran} (A) \iff \text{par} (\mathcal{U}(c)) = \text{Null} (A).
$$

**Proof.** By Lemma 6.3, $N (c \mid \mathcal{M}_\tau) = \text{Ran} (A)$, and by Lemma 6.4, $\partial h(c) = \lambda_0(c) + \overline{\mathcal{A}} \mathcal{U}(c)$. The system of linear equations (42) in $\mathcal{U}(c)$ has coefficient matrix $A$ defined in (40) which is block-circulant and can be block row-reduced to

$$
\begin{pmatrix}
AP_1 & 0 & 0 & \cdots & -A \\
0 & AP_2 & 0 & \cdots & -A \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & AP_{\overline{c} - 1} & -A \\
0 & \cdots & 0 & -A & 0
\end{pmatrix}.
$$

(44)
We now compute \( \text{Null} \left( \mathcal{A} \right) \). Suppose \( \mu = (\mu_1^\top, \ldots, \mu_k^\top)^\top \in \text{Null} \left( \mathcal{A} \right) \). Then (44) and nondegeneracy imply that \( \mu \in \text{Null} \left( \mathcal{A} \right) \) if and only if \( \mu_j = P_j \mu_k \) for all \( j \in \{1, \ldots, k-1\} \), i.e.,

\[
\text{Null} \left( \mathcal{A} \right) \subseteq \left\{ \begin{pmatrix} P_1 \mu_k \\ \vdots \\ P_{k-1} \mu_k \\ \mu_k^\top \\
 \end{pmatrix} \bigg| \mu_k \in \mathbb{R}^\ell \right\}, \text{with basis} \left\{ \begin{pmatrix} P_1 e_p \\ \vdots \\ P_{k-1} e_p \\ e_p^\top \\
 \end{pmatrix} \bigg| p \in \{1, \ldots, \ell\} \right\} =: \{\zeta_1, \ldots, \zeta_\ell\}.
\]

By (42),

\[
\text{par} \left( \mathcal{U}(c) \right) := \mathbb{R} \left( \mathcal{U}(c) - \mathcal{U}(\cdot) \right) \subseteq \text{Null} \left( \mathcal{A} \right),
\]

and since \( \mathcal{A} = \frac{1}{\kappa} \left[ A P_1 \cdots A P_{k-1} A \right] \), (41) implies

\[
\text{par} \left( \partial h(c) \right) = \text{par} \left( \mathcal{A} \mathcal{U}(c) \right) = \mathcal{A} \text{par} \left( \mathcal{U}(c) \right) \subseteq \mathcal{A} \text{Null} \left( \mathcal{A} \right) = \left\{ A \mu_k \big| \mu_k \in \mathbb{R}^\ell \right\} = \text{Ran} \left( \mathcal{A} \right),
\]

so \((\Leftarrow)\) in (43) is clear as “\( \subset \)” becomes an equation. For \((\Rightarrow)\), suppose strict containment: \( \text{par} \left( \mathcal{U}(c) \right) \subset \text{Null} \left( \mathcal{A} \right) \). Then there exists \( p \in \{1, \ldots, \ell\} \) such that \( \zeta_p \notin \text{par} \left( \mathcal{U}(c) \right) \). This implies that the \( p \)-th column of \( A \) is not in \( \text{par} \left( \partial h(c) \right) \) which we have assumed equal to \( \text{Ran} \left( \mathcal{A} \right) \). This contradiction establishes (43). \( \square \)

We now show that nondegeneracy and \( k \)-strict complementarity together imply that the normal space and subdifferential are parallel.

**Lemma 6.8.** Let \( \mathcal{M}_\tau \) be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Suppose \( \mathcal{M}_\tau \) satisfies the nondegeneracy condition, and the \( k \)-strict complementarity of Definition 6.5 holds at \((c, \nu)\) for \( \mu \). Then,

\[
\text{par} \left( \partial h(c) \right) = N \left( c \mid \mathcal{M}_\tau \right),
\]

where it is shown in Lemma 6.3 that \( N \left( c \mid \mathcal{M}_\tau \right) = \text{Ran} \left( \mathcal{A} \right) \). Moreover, (47) holds throughout \( \mathcal{M}_\tau \), and \( \partial h \) is inner semicontinuous relative to \( \mathcal{M}_\tau \).

**Proof.** We first show that a sufficient condition to guarantee the right-hand side of (43) is \((c, \nu)\) satisfying the \( k \)-strict complementarity condition of Definition 6.5 for \( \mu \in \mathcal{U}(c) \). To see this note that, by relabeling the active polyhedral sets if necessary, we can assume without loss of generality that the index \( k \) in \( k \)-strict complementarity is \( \overline{k} \). Let \( p \in \{1, \ldots, \ell\}, \ t \in \mathbb{R}, \) and consider the step given by \( \mu + t \zeta_p \), where \( \zeta_p \) is the \( p \)-th basis element of \( \text{Null} \left( \mathcal{A} \right) \) given in (45), i.e.,

\[
\mu + t \zeta_p := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{\overline{k}-1} \\ \mu_k^\top \\
 \end{pmatrix} + t \begin{pmatrix} P_1 e_p \\ \vdots \\ P_{\overline{k}-1} e_p \\ e_p^\top \\
 \end{pmatrix},
\]

We consider two cases. If, for all \( j \in \{1, \ldots, k\} \), \((\mu_j)_p > 0\), then for sufficiently small \( t \), \( \mu + t \zeta_p \geq 0 \), and \( A(\mu + t \zeta_p) = A\mu \). That is, both \( \mu \in \mathcal{U}(c) \) and \( \mu + t \zeta_p \in \mathcal{U}(c) \), which implies \( \zeta_p \notin \text{par} \left( \mathcal{U}(c) \right) \). Otherwise, there exists \( j \in \{1, \ldots, k\} \) with \((\mu_j)_p = 0 \). By part (c) of \( k \)-strict complementarity, the scalars \( P_j e_p = 1 \) for all \( j' \in \{1, \ldots, \overline{k}\} \), so repeating the previous argument with \( t > 0 \) gives \( \zeta_p \notin \text{par} \left( \mathcal{U}(c) \right) \). Since \( p \in \{1, \ldots, \ell\} \) was arbitrary, \( k \)-strict complementarity is a sufficient condition
guaranteeing par \((U(c)) = \text{Null } (A)\).

This argument shows, under nondegeneracy, that
\[(49)\]
k-strict complementarity at \((c, y)\) for \(\mu \mapsto \text{ri } (\partial h(c)) \neq \emptyset\),
because, given any \(\mu \in U(c)\), the fact that par \((U(c)) = \text{Null } (A)\) together with (41) implies there
exists a strictly positive \(\tilde{\mu} \in U(c)\) and a \(\tilde{y} \in \partial h(c)\) given by
\(\tilde{y} = \lambda_0(c) + \tilde{A}\tilde{\mu}\), with \(\mu(c, \tilde{y}) = \tilde{\mu}\). By Lemma 6.6, \(\tilde{y} \in \text{ri } (\partial h(c))\).

We now argue that if, for some \(c \in M_\tau\), \(y \in \partial h(c)\), k-strict complementarity holds at \((c, y)\) for \(\mu\), then \(\text{ri } (\partial h(c)) \neq \emptyset\) throughout \(M_\tau\). This will imply (47) holds throughout \(M_\tau\) as well. By (49),
suppose \(y \in \text{ri } (\partial h(c))\) so that \(\mu(c, y) > 0\) by Lemma 6.6.

Choose any other \(c' \in M_\tau\). Since \(M_\tau\) is relatively open, there exists \(c'' \in M_\tau\) and \(\lambda \in (0, 1)\) so that \(c' = \lambda c + (1 - \lambda)c''\). Let \(y'' \in \partial h(c'')\). By Lemma 6.5, there exists a unique vector \(\mu(c'', y'')\) associated with \((c'', y'')\). Since \(c, c'' \in M_\tau\) and \(\mu(c, y) > 0\), \(\lambda \mu(c', y') + (1 - \lambda)\mu(c, y) > 0\). It follows from (41)
that for all \(j \in \{1, \ldots, \ell\}\) and \(\lambda \in (0, 1)\),
\[(50)\]
\[
\lambda y + (1 - \lambda)y'' = Q_{k_j} c' + b_{k_j} + AP_j (\lambda \mu(c, y) + (1 - \lambda)\mu(c'', y''))
\]

Define \(y' := \lambda y + (1 - \lambda)y''\). Then (50) implies that the equations (41) defining member-
ship \(y' \in \partial h(c')\) are satisfied, with \(\mu(c', y') = \lambda \mu(c, y) + (1 - \lambda)\mu(c'', y'') > 0\), so \(y' \in \text{ri } (\partial h(c'))\) by Lemma 6.6. Since \(c' \in M_\tau\) was arbitrary, \(\text{ri } (\partial h(c')) \neq \emptyset\) for all \(M_\tau\).

We lastly establish \(\partial h(c)\) is inner semicontinuous relative to \(M_\tau\). The previous paragraph and (50)
showed \(\partial h|_{M_\tau}\) is graph-convex. By defining \(S(c) = \partial h(c)\) for \(c \in M_\tau\) and \(S(c) = \emptyset\)
or otherwise and noting the convex sets \(\{c\}\) and \(M_\tau\) cannot be separated, [37, Theorem 5.9(b)] gives inner semconti-
nuity of \(\partial h\) at all \(c \in M_\tau\) relative to \(M_\tau\).

\(\square\)

The main result of this section shows that partial smoothness follows from nondegeneracy and
k-strict complementarity.

**Theorem 6.2.** Let \(M_\tau\) be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Suppose
\(M_\tau\) satisfies the nondegeneracy condition, and \(c \in M_\tau\) and \(y \in \partial h(c)\) are such that \((c, y)\) satisfies the k-strict
complementarity condition of Definition 6.5. Then \(h\) is partly smooth relative to \(M_\tau\).

**Proof.** By definition of \(M_\tau\), for any \(k \in K(\tau)\) and any \(c \in M_\tau\),
\(h(c) = \frac{1}{2} \langle c, Q_k c \rangle + \langle b_k, c \rangle + \beta_k\), so \(h|_{M_\tau}\) is smooth. By Proposition 4.1, \(\text{dom } (\partial h) = \text{dom } (h) \supset M_\tau\), so existence of subgradients holds
throughout \(M_\tau\) as well. The normal cone and subdifferential being parallel along with subdifferen-
tial inner semicontinuity relative to \(M_\tau\) are the content of Lemma 6.8.

\(\square\)

**Remark 9.** Observe that if the hypotheses of Theorem 6.2 are satisfied, the assumption that \(f\) satisfies
(TC) at \(x\) is equivalent to requiring
\[(51)\]
\[
\text{Null } (\nabla c(x)^T) \cap \text{Ran } (A) = \{0\}.
\]

This condition and the nondegeneracy condition imply the \(n \times \ell\) matrix \(\nabla c(x)^T A\) has full rank
equal to \(\ell \leq n\), i.e., \(\text{Null } (\nabla c(x)^T A) = \{0\}\).

We now show the assumptions of Theorem 6.2 allow us to write the cone of non-ascent directions as a subspace at strictly critical points.

**Lemma 6.9 (Non-ascent directions).** Let \(M_\tau\) be as in (36), let Assumption 1 hold, and recall the nota-
tion of Definition 6.3. Suppose \(f\) satisfies (BCQ) at \(x\), \(\vec{y} \in M(\vec{x})\), and \(\vec{c} := c(\vec{x})\). Then,
\(D(\bar{x}) \supset \text{Null } (\bar{A}^T \nabla c(\bar{x}))\). If, in addition, \(f\) satisfies (SC) at \(x\) for \(\vec{y}\) and \(M_\tau\) satisfies the nondegeneracy condition,
then \( D(\bar{x}) \subset \text{Null} \left( A^\top \nabla c(\bar{x}) \right) \).

**Proof.** Since \( f \) satisfies (BCQ) at \( \bar{x} \), Theorem 3.1 gives
\[
D(\bar{x}) = \left\{ d \in \mathbb{R}^n \mid h'(c(\bar{x}); \nabla c(\bar{x})d) \leq 0 \right\}.
\]
(\( \supset \)) Since \( \bar{y} \in M(\bar{x}) \), by (41), there exists \( \bar{y} \in \mathcal{U}(\bar{x}) \) so that \( \bar{y} = Q\bar{x} + B + \hat{A}\bar{y} \). Then, for any \( j \in \{1, \ldots, k\} \),
\[
\begin{align*}
D(\bar{x}) &= \bigcup_{j=1}^{k} \left\{ d \left| \left\langle Q_k \bar{x} + b_k, \nabla c(\bar{x})d \right\rangle \leq 0 \right. \right\} \quad \text{by (22), Definition 6.3} \\
&= \bigcup_{j=1}^{k} \left\{ d \left| \frac{\langle \bar{y} - AP_j \bar{x}, \nabla c(\bar{x})d \rangle}{P_j A^\top \nabla c(\bar{x})d} \leq 0 \right. \right\} \quad \text{since } \bar{y} \in M(\bar{x})
\end{align*}
\]

The inclusion follows.

(\( \subset \)) Let \( 0 \neq d \in D(\bar{x}) \), and suppose to the contrary that \( d = d_1 + d_2 \), where \( d_1 \in \text{Null} \left( A^\top \nabla c(\bar{x}) \right) \) and \( d_2 = \nabla c(\bar{x})^\top Aw, w \neq 0 \). By Lemma 6.8, \( \text{Ran} (A) \subset \partial h(\bar{x}) \). Since \( \bar{y} \in \text{ri} (\partial h(\bar{x})) \), there exists \( \epsilon > 0 \) so that \( \bar{y} + \epsilon Aw \in \partial h(\bar{x}) \). Then,
\[
0 \geq h'(c(\bar{x}); \nabla c(\bar{x})d)
\]
\[
= \sup_{y \in \partial h(\bar{x})} \left\langle \nabla c(\bar{x})^\top y, d \right\rangle
\]
\[
\geq \left\langle \bar{y} + \epsilon Aw, \nabla c(\bar{x})(d_1 + \nabla c(\bar{x})^\top Aw) \right\rangle
\]
\[
\geq \left\langle \nabla c(\bar{x})^\top \bar{y}, d \right\rangle + \epsilon \left\| \nabla c(\bar{x})^\top Aw \right\|^2
\]
\[
= \epsilon \left\| \nabla c(\bar{x})^\top Aw \right\|^2,
\]
so \( w = 0 \) (see Remark 9). \( \square \)

By a continuity argument in \((x, y)\), we have the following result which is important for our discussion of the metric regularity of Newton’s iteration in the next section. It states that, in the presence of partial smoothness, (TC) and the curvature condition are local properties.

**Lemma 6.10.** Suppose (51) holds and that for all \( j \in \mathcal{K}(\bar{x}) \) and
\[
d^\top \nabla c(\bar{x})^\top Q_j \nabla c(\bar{x})d + d^\top \nabla^2 (\nabla c(\bar{x})^\top Q_j \nabla c(\bar{x})d) > 0, \quad \forall d \in \text{Null} \left( A^\top \nabla c(\bar{x}) \right) \setminus \{0\}.
\]

Then, there exists a neighborhood \( \mathcal{N} \) of \((\bar{x}, \bar{y})\) such that if \((x, y) \in \mathcal{N} \) then for all \( j \in \mathcal{K}(\bar{x}) \),
\[
d^\top \nabla c(x)^\top Q_j \nabla c(x)d + d^\top \nabla^2 (\nabla c(x)^\top Q_j \nabla c(x)d) > 0, \quad \forall d \in \text{Null} \left( A^\top \nabla c(x) \right) \setminus \{0\}.
\]
and \( \text{Null} (\nabla c(x)^\top) \cap \text{Ran} (A) = \{0\} \).

The following examples are inspired by the discussion in [23].
Example 1. In \( \mathbb{R}^2 \), let \( h_a(c) = \|c\|^2 \), so \( h \) is piecewise linear-quadratic convex. If \( \mathcal{M} := \{0\} \), then \( h_a \) is not partly smooth relative to \( \mathcal{M} \) because \( \partial h_a(0) = \{0\} \) while \( N(0 | \mathcal{M}) = \mathbb{R}^n \). On the other hand, if \( h_b(c) = \|c\| \) with the same domain representation, then \( \partial h(0) = B_{\infty} \), in which case \( h_b \) is partly smooth relative to \( \mathcal{M} \).

Suppose we represent the domain of \( h_a \) and \( h_b \) as the four quadrants in the plane, relative to each of which \( h_a, h_b \) are linear-quadratic. This representation meets the criteria of the Rockafellar-Wets PLQ representation of Theorem 6.1. For both \( h_a \) and \( h_b \), the nondegeneracy condition for \( \mathcal{M} \) holds since \( A \) can be taken to be \( I_2 \).

Example 2. In \( \mathbb{R}^2 \), the domain of \( h_a \) and \( h_b \) in the previous example can be presented in the following way. Take each of the four quadrants in the plane and split them along their respective diagonal. Define \( h_a \) as usual on each of the pieces. Then this presentation describes \( \text{dom}(h_a) \) using 4 hyperplanes and also meets the Rockafellar-Wets PLQ representation theorem. However, the nondegeneracy condition fails for \( \mathcal{M} \) in this representation.

On the manifold \( \mathcal{M} \) given by an “artificial” diagonal, the matrix \( A \) is comprised of a single column, with \( N(c | \mathcal{M}) = \text{Ran}(A) \) for any \( c \in \mathcal{M} \). However, \( h_a \) is smooth on \( \mathcal{M} \) with \( \text{par} \partial h(c) = \{0\} \).

We end this section with a relationship between partial smoothness and the convergence analysis of quasi-Newton methods studied in 5.1. The following result is a finite identification property for any algorithm solving \( \mathcal{P} \) in the presence of an active manifold at a solution.

Theorem 6.3. [24, Theorem 4.10] Suppose the closed, proper, convex function \( h : \mathbb{R}^m \to \overline{\mathbb{R}} \) is partly smooth at the point \( \overline{\tau} \in \mathbb{R}^n \) relative to a manifold \( \mathcal{M} \subset \mathbb{R}^n \). Consider a subgradient \( \overline{y} \in \text{ri} \partial h(\overline{\tau}) \). Suppose the sequence \( \{\overline{c}_k\} \subset \mathbb{R}^m \) satisfies \( \overline{c}_k \to \overline{\tau} \) and \( h(\overline{c}_k) \to h(\overline{\tau}) \). Then, \( \overline{c}_k \in \mathcal{M}_\overline{\tau} \) for all large \( k \) if and only if \( \text{dist} \left( \overline{y}, \partial h(\overline{c}_k) \right) \to 0 \).

Combining Corollary 5.1 and Theorem 6.3, we have the following relationship between the sufficient conditions for superlinear convergence of the quasi-Newton method \( Q_k \) and the finite identification of an active manifold at a solution.

Corollary 6.1. Let \( \mathcal{M}_\overline{\tau} \) be as in (36), let Assumption 1 hold, and recall the notation of Definition 6.3. Let \( \overline{\tau} \in \text{dom}(f) \) and \( \overline{\tau} := c(\overline{\tau}) \).

Suppose

(a) \( \mathcal{M}_\overline{\tau} \) satisfies the nondegeneracy condition,

(b) the \( k \)-strict complementarity condition of Definition 6.5 holds at \( (c, y) \in \mathbb{R}^n \times \mathbb{R}^m \),

(c) \( M(\overline{\tau}) = \{\overline{y}\} \), and

(d) the second-order sufficient conditions of Theorem 4.3 are satisfied at \( \overline{\tau} \).

Consider the neighborhood \( U \) of \( (\overline{\tau}, \overline{y}) \) of Corollary 5.1, and a starting point \( (x^0, y^0) \in U \). Suppose the sequence \( \{(x^k, y^k)\}_{k \in \mathbb{N}} \) is generated from the optimality conditions for \( Q_k \), remains in \( U \) for all \( k \in \mathbb{N} \), and satisfies \( (x^k, y^k) \neq (\overline{\tau}, \overline{y}) \) for all \( k \in \mathbb{N} \). Then, the sufficient conditions for superlinear convergence of Corollary 5.1 imply \( c(x^k) + \nabla c(x^k)(x^{k+1} - x^k) \in M(\overline{\tau}) \) for all large \( k \).

Proof. Since \( x^k \to \overline{\tau} \), \( d^k \to 0 \). By continuity, \( \overline{c}_k := c(x^k) + \nabla c(x^k)(x^{k+1} - x^k) \to \overline{\tau} \). The quasi-Newton method (6) with \( B_k \) given by (33) implies \( y^{k+1} \in \partial h(\overline{c}_k) \), so \( \{\overline{c}_k\} \subset \text{dom}(h) \). By Proposition 4.1, \( h(\overline{c}_k) \to h(\overline{\tau}) \). Since \( y^k \to \overline{y} \), \( \text{dist} \left( \overline{y}, \partial h(\overline{c}_k) \right) \leq \|\overline{y} - y^{k+1}\| \to 0 \). Then, by partial smoothness and Theorem 6.3, \( \overline{c}_k \in M(\overline{\tau}) \) for all large \( k \). \( \square \)
The point of this section is to marry the partial smoothness hypothesis to the hypotheses used to establish strong metric subregularity in Section 6 to establish strong metric regularity of a solution mapping that is an appropriately defined local version of \( g + G \) in (10). In addition, we establish the local quadratic convergence of the Newton method for \( g + G \).

**Definition 7.1** (Metric regularity). A set-valued mapping \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is metrically regular at \( \bar{x} \) for \( \bar{y} \) when \( \bar{y} \in S(\bar{x}) \), the graph of \( S \) is locally closed at \((\bar{x}, \bar{y})\), and there exists \( \kappa \geq 0 \) and neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that \( \text{dist} \left( x \mid S^{-1}(y) \right) \leq \kappa \text{dist} \left( y \mid S(x) \right) \) for all \((x, y) \in U \times V \). The infimum of \( \kappa \) over all \((\kappa, U, V)\) satisfying the display is called the metric regularity modulus of \( S \) at \( \bar{x} \) for \( \bar{y} \), and is denoted \( \text{reg}(S; \bar{x}; \bar{y}) \).

**Definition 7.2** (Strong metric regularity). A set-valued mapping \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is strongly metrically regular at \( \bar{x} \) for \( \bar{y} \) when it is metrically regular at \( \bar{x} \) for \( \bar{y} \) and \( S^{-1} \) has a single-valued localization at \( \bar{y} \) for \( \bar{x} \). Equivalently, when \( S^{-1} \) has a Lipschitz continuous single-valued localization around \( \bar{y} \) for \( \bar{x} \).

### 7.1. Partly Smooth Problems.

In this section, we make the following assumptions:

**Assumption 2.** Let \( f \) be as in \( P \), \((\bar{x}, \bar{y}) \in \text{dom}(f) \times \mathbb{R}^m \), \( \bar{c} := c(\bar{x}) \), \( \bar{\kappa} = |K(\bar{c})| \), where \( K(\bar{c}) \) are the active indices given in Definition 4.2. Let \( \mathcal{M}_\bar{x} \) be the active manifold defined in (36) and let \( \bar{p}_j \in \mathbb{R}^\ell \) for \( j \in \{1, \ldots, \bar{\kappa}\} \), where \( \ell = |I_k(\bar{c})| \) for any \( k \in K(\bar{c}) \) with \( I_k(\bar{c}) \) defined in (37). Recall that \( \ell \) is well-defined by Lemma 6.2. With these specifications, we assume that

1. \( \text{dom}(h) \) is given by the Rockafellar-Wets PLQ representation of Theorem 6.1,
2. \( c \) is \( C^3 \)-smooth,
3. \( \mathcal{M}_\bar{x} \) satisfies the nondegeneracy condition (in particular, \( \bar{\kappa} \geq 2 \)),
4. \( f \) satisfies (SC) at \( \bar{y} \); i.e., \( \text{Null} \left( \nabla c(\bar{x})^\top \right) \cap \text{ri} \left( \partial h(\bar{c}) \right) = \{\bar{y}\} \), so that in particular, as in (41), \( f(\bar{y}) = Q_c + B + A\bar{p} \), where \( \bar{p} = (\bar{p}_1^\top, \ldots, \bar{p}_{\bar{\kappa}}^\top)^\top > 0 \) by Lemma 6.6,
5. \( \bar{x} \) satisfies the second-order sufficient conditions of Theorem 4.3, i.e.,

\[
h''(c(\bar{x}); \nabla c(\bar{x})d) + \left\langle d, \nabla^2(c(\bar{x}))d \right\rangle > 0 \quad \forall d \in \text{Null} \left( A^\top \nabla c(\bar{x}) \right) \setminus \{0\},
\]

where, by Lemma 3.1, \( M(\bar{x}) = \{\bar{y}\} \), and by Lemma 6.9, \( D(\bar{x}) = \text{Null} \left( A^\top \nabla c(\bar{x}) \right) \).

The conditions (c) - (e) in Assumption 2 can be interpreted in terms of similar assumptions employed in classical NLP. Condition (c) corresponds to the linear independence of the active constraint gradients, (d) corresponds to strict complementary slackness, and (e) corresponds to the strong second-order sufficiency condition. The convergence results developed in this section subsume those known for NLP, since they follow from the case in which \( h \) is non finite-valued piecewise linear convex.

We begin with a key technical lemma important for establishing metric regularity.

**Lemma 7.1.** In the notation of Definition 6.3, for any \( i, j \in \{1, \ldots, \bar{\kappa}\} \), \( (Q_{k_i} - Q_{k_j}) \text{Null} \left( A^\top \right) \subset \text{Ran}(A) \).
Proof. Let \( w \in \text{Null} \left( A^\top \right) \). By polyhedrality, there exists \(|]\| > 0\) such that \( c_t := \tau + tw \in M_\tau \). By Proposition 4.1, \( \text{dom} \left( \partial h \right) = \text{dom} \left( h \right) \), so there exists \( v \in \partial h(c_t) \) and \( \overline{v} \in \partial h(\tau) \). By (41), \((v, \mu(c_t, v))\) and \((\tau, \overline{\mu})\) satisfy \( Jv = Qc_t + B + \tilde{A}\mu(c_t, v) \) and \( J\overline{v} = Q\tau + B + \tilde{A}\overline{\mu} \). Then for any \( i, j \in K(\tau) \),

\[
0 = (Q_{k_i} - Q_{k_j})c_t + A(P_i\mu(c_t, v)_i - P_j\mu(c_t, v)_j) + b_{k_i} - b_{k_j},
\]

\*

Subtracting the second equation from the first and rearranging gives

\[
(Q_{k_i} - Q_{k_j})w = t^{-1}A \left\{ P_i\mu(c_t, v)_j - P_j\mu(c_t, v)_i - P_i\mu(c_t, v)_i - P_j\mu(c_t, v)_j \right\}.
\]

We now define a family of local approximations to \( g + G \) for which strong metric regularity is established.

**Definition 7.3.** For a point \( \tau \in M_\tau \) and each \( j \in \{1, \ldots, k\} \), define \( g_j : \mathbb{R}^{n+m+\ell} \to \mathbb{R}^{n+m+\ell+\ell} \).

\[
g_j(x, y, \mu_j) := \begin{pmatrix} \nabla c(x)^\top y \\ y - Q_{k_j}c(x) - b_{k_j} - AP_j\mu_j \\ A^\top [c(x) - \tau] \\ -\mu_j \end{pmatrix}, \quad G_0 := \begin{pmatrix} \{0\}^n \\ \{0\}^m \\ \{0\}^\ell \\ \mathbb{R}^\ell \end{pmatrix}
\]

and set \( \overline{x}_j := (\overline{x}, \overline{y}, \overline{\mu}_j) \in \mathbb{R}^{n+m+\ell} \), where \( \overline{x}, \overline{y}, \overline{\mu}_j \) are as in Assumption 2. Then

\[
\nabla g_j(x, y, \mu_j) = \begin{pmatrix} \nabla^2 (yc)(x) & \nabla c(x)^\top & 0 \\ -Q_{k_j} & I & -AP_j \\ A^\top & 0 & 0 \\ 0 & 0 & -I_\ell \end{pmatrix}, \quad g_j(\overline{x}_j) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\overline{\mu}_j \end{pmatrix} \in -G_0 \text{ (see Assumption 2 (d)).}
\]

In parallel to the study in Section 5, we introduce the linearization of these mappings.

**Definition 7.4 (M_\tau-restricted KKT Mappings).** Let \( \tau \) and \( \overline{\tau} \) be given by Assumption 2, and \( g_j \) and \( G_0 \) be as in Definition 7.3. For all \( j \in \{1, \ldots, k\} \), define the linearization of \( g_j + G_0 \) at \( u = (\overline{x}, \overline{y}, \overline{\mu}_j) \)

\[
G_j^l(x) := g_j(u) + \nabla g_j(u)(x - u) + G_0, \text{ or equivalently,}
\]

\[
G_j^l(x, y, \mu_j) := g_j(\overline{x}, \overline{y}, \overline{\mu}_j) + \nabla g_j(\overline{x}, \overline{y}, \overline{\mu}_j) \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \\ \mu_j - \overline{\mu}_j \end{pmatrix} + G_0.
\]

For any \( u = (\overline{x}, \overline{y}, \overline{\mu}_j) \), define the function

\[
F_u(x, z) := g_j(u) + \nabla g_j(u)(x - u) - z = \begin{pmatrix} \nabla c(\overline{x})^\top y + \nabla^2 (yc)(\overline{x})[x - \overline{x}] - z_1 \\ y - Q_{k_j}[c(\overline{x})] + \nabla c(\overline{x})[x - \overline{x}] - b_{k_j} - AP_j\mu_j - z_2 \\ A^\top [c(\overline{x}) + \nabla c(\overline{x})[x - \overline{x}] - \tau] - z_3 \\ -\mu_j - z_4 \end{pmatrix}.
\]

Then,

\[
gph G_j^l = \left\{ (x, z) \mid F_u(x, z) \in -G_0 \right\}.
\]
with \( \text{dom} \left( G^i_u \right) = \mathbb{R}^{n+m+\ell} \). Explicitly,

\[
gph G^j_{(\bar{x}, \bar{y}, \bar{\mu})} = \left\{ (x, y, \mu_j, z_1, z_2, z_3, z_4) \mid \begin{align*}
  z_1 &= \nabla c(\bar{x})^\top y + \nabla^2(\bar{y} c)(\bar{x})[x - \bar{x}] \\
  z_2 &= y - Q_{kj}[c(\bar{x}) + \nabla c(\bar{x})[x - \bar{x}]] - b_{kj} - AP_j \mu_j \\
  z_3 &= A^\top[c(\bar{x}) + \nabla c(\bar{x})[x - \bar{x}] - \bar{c}] \\
  z_4 &\in -\mu_j + \mathbb{R}_+^\ell
\end{align*} \right\}.
\]

The next lemma shows that the error in the Newton iterates can be measured in terms of \( (x, y) \) alone, independent of the vectors \( \mu_j \).

**Lemma 7.2.** Let \( \bar{x}, \bar{y}, \bar{\mu}, \bar{c}, \bar{k} \), and \( Q \) be as in Assumption 2, and \( g_j \) and \( G_0 \) be as in Definition 7.3. For any \( j \in \{1, \ldots, \bar{k}\} \), define \( \eta_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m+\ell} \) by

\[
\eta_j(x, y) := \begin{pmatrix}
  \nabla c(x)^\top y \\
  Q_{kj}(\bar{c} - c(x)) \\
  A^\top(c(x) - \bar{c})
\end{pmatrix}.
\]

Observe that for any \( (x, y, \mu_j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \),

\[
g_j(x, y, \mu_j) = \left( \begin{pmatrix}
  \nabla c(x)^\top y \\
  Q_{kj}(\bar{c} - c(x)) \\
  A^\top(c(x) - \bar{c})
\end{pmatrix} \right) + \begin{pmatrix}
  0 \\
  y - \bar{y} + AP_j(\bar{\mu}_j - \mu_j) \\
  -\mu_j
\end{pmatrix}
\]

and \( \nabla g_j(x, y, \mu_j) = \begin{pmatrix}
  \nabla \eta_j(x, y) & 0 \\
  0 & 0
\end{pmatrix} + \begin{pmatrix}
  0 & 0 & 0 \\
  0 & I & -AP_j \\
  0 & 0 & -I
\end{pmatrix} \)

Set \( \bar{\nu}_j := (\bar{x}, \bar{y}, \bar{\mu}_j) \). Then, for any \( u := (\bar{x}, \bar{y}, \bar{\mu}_j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \),

\[
\| F_u(\bar{\nu}_j, g_j(\bar{\nu}_j)) \| = \| \eta_j(\bar{x}, \bar{y}) + \nabla \eta_j(\bar{x}, \bar{y}) \begin{pmatrix}
  \bar{c} - \bar{x} \\
  \bar{c} - \bar{y}
\end{pmatrix} - \eta_j(\bar{x}, \bar{y}) \|
\]

since \( \eta_j(\bar{x}, \bar{y}) = 0 \).

The following lemma uses the strict criticality assumption to show the normal cone to the graph of these linearization are captured by the range of \( \nabla F_{\bar{\nu}_j} \).

**Lemma 7.3.** Let \( \bar{x}, \bar{y}, \bar{\mu}, \bar{c}, \bar{k} \), and \( Q \) be as in Assumption 2 and set \( \bar{\nu}_j := (\bar{x}, \bar{y}, \bar{\mu}_j) \). Then, for all \( j \in \{1, \ldots, \bar{k}\} \), the mapping \( G_{\bar{\nu}_j}^j \) in (56) has \( N \left( (\bar{x}, 0) \mid gph G_{\bar{\nu}_j}^j \right) = \text{Ran} (W) \), where

\[
W := \begin{pmatrix}
  \nabla^2(\bar{y} c)(\bar{x}) & -\nabla c(\bar{x})^\top Q_{kj} & \nabla c(\bar{x})^\top A \\
  \nabla c(\bar{x}) & I_m & 0 \\
  0 & -P_j A^\top & 0 \\
  -I_n & 0 & 0 \\
  0 & -I_m & 0 \\
  0 & 0 & -I_\ell
\end{pmatrix}.
\]
Proof. The set $\text{gph} G^j_{\bar{x}_j} = \{ (x, z) \mid F_{\bar{x}_j}(x, z) \in -G \}$ defined in (56) is closed with $(\bar{x}_j, 0) \in \text{gph} G^j_{\bar{x}_j}$.

In addition, $\bar{\pi}_j > 0$, $N \left( F_{\bar{x}_j}(\bar{x}_j, 0) \mid -G_0 \right) = \mathbb{R}^{n+m+\ell} \times \{0\}^\ell$, and

$$
\nabla F_{\bar{x}_j}(\bar{x}_j, 0) = \begin{pmatrix}
\nabla^2(\bar{y}_c)(\bar{x}) & -\nabla c(\bar{x})^\top Q_{k_j} & \nabla c(\bar{x})^\top A & 0 \\
\n\nabla c(\bar{x}) & I_m & 0 & 0 \\
0 & -P_j A^\top & 0 & I_{\ell} \\
-I_{n} & 0 & 0 & 0 \\
0 & -I_m & 0 & 0 \\
0 & 0 & -I_{\ell} & 0 \\
0 & 0 & 0 & I_{\ell}
\end{pmatrix} = \begin{pmatrix} W \mid R \end{pmatrix},
$$

where the matrix $R$ is being defined by this expression. Combining the facts in the previous two sentences, the constraint qualification (81) in Theorem 9.1 (see appendix), for $N \left( (\bar{x}_j, 0) \mid \text{gph} G^j_{\bar{x}_j} \right)$ is the requirement that $\text{Null} (W) = \{0\}$. If we verify $\text{Null} (W) = \{0\}$, then $N \left( (\bar{x}_j, 0) \mid \text{gph} G^j_{\bar{x}_j} \right) = \text{Ran} (W)$ by Theorem 9.1. But the presence of the identity matrices in $W$ immediately give $\text{Null} (W) = \{0\}$. 

The metric regularity of the mappings $g_j + G_0$ follow from the second-order sufficient conditions of Theorem 4.3.

Lemma 7.4. Let $\bar{x}, \bar{y}, \bar{\pi}, \bar{\pi}, \bar{\pi}, j$ and $Q$ be as in Assumption 2, $W$ as in (60) and set $\bar{x}_j := (\bar{x}, \bar{y}, \bar{\pi}_j)$. For all $j \in \{1, \ldots, k\}$,

$$(0, -z) \in N \left( (\bar{x}_j, 0) \mid \text{gph} G^j_{\bar{x}_j} \right) \iff z = 0,$$

where $G^j_{\bar{x}_j}$ is given by (56). Then, $G^j_{\bar{x}_j}$ is metrically regular at $\bar{x}_j$ for $0$ and

$$
\begin{pmatrix}
\nabla^2(\bar{y}_c)(\bar{x}) & \nabla c(\bar{x})^\top & 0 \\
\n-\nabla c(\bar{x}) & I_m & -AP_j \\
A^\top \nabla c(\bar{x}) & 0 & 0
\end{pmatrix}
$$

is nonsingular.

Proof. By Lemma 7.3, $N \left( (\bar{x}_j, 0) \mid \text{gph} G^j_{\bar{x}_j} \right) = \text{Ran} (W)$, and so the statement

$$(0, -z) \in N \left( (\bar{x}_j, 0) \mid \text{gph} G^j_{\bar{x}_j} \right) \iff z = 0$$

is equivalent to

$$(d) = \begin{pmatrix} d \\ v \\ w \end{pmatrix} \text{ for some } \begin{pmatrix} d \\ v \\ w \end{pmatrix} \iff \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = 0.$$
Since \((\Leftarrow)\) is trivial, we only establish \((\Rightarrow)\). Define \(H := \nabla^2(\bar{y}c)(\bar{x})\). Then the left-hand side of (61) becomes

\[
0 = Hd - \nabla c(\bar{x})^\top Q_k v + \nabla c(\bar{x})^\top Aw,
\]

(62)

\[
0 = \nabla c(\bar{x})d + v,
\]

(63)

\[
z_1 = d, \quad z_2 = v, \quad z_3 = w, \quad z_4 = 0.
\]

Since \(z_4 = 0\), we need only show \(z_1 = z_2 = z_3 = 0\), which we establish by showing \(d = v = w = 0\). First suppose \(d \neq 0\). From (64) and Definition 6.3, \(v \in \text{Null} \left( A^\top \right)\). Then (63) and gives \(\nabla c(\bar{x})d = -v \in \text{Null} \left( A^\top \right)\). By Lemma 6.9, \(d \in D(\bar{x}) \setminus \{0\}\). Taking the inner product on both sides of (62) with \(d\) and using (63) gives

\[
d^\top Hd = d^\top \nabla c(\bar{x})^\top Q_k v = -d^\top \nabla c(\bar{x})^\top Q_k \nabla c(\bar{x})d,
\]

so

\[
d^\top \nabla c(\bar{x})^\top Q_k \nabla c(\bar{x})d + d^\top Hd = 0.
\]

But the second-order sufficient conditions of Theorem 4.3 imply that for any \(j \in \{1, \ldots, \bar{k}\}\),

\[
d^\top \nabla c(\bar{x})^\top Q_k \nabla c(\bar{x})d + d^\top Hd > 0.
\]

This contradiction implies \(d = 0\). But then \(v = 0\) by (63). Finally, (62) states that \(w\) must satisfy \(Aw \in \text{Null} \left( \nabla c(\bar{x})^\top \right) \cap \text{Ran} (A) = \{0\}\). By the nondegeneracy condition of Definition 6.4, \(w = 0\).

Equation (56) gives local closedness of \(G_{\bar{x}}^j\) at \((\bar{x}, 0)\), so the coderivative criterion for metric regularity [13, Theorem 4C.2] implies \(G_{\bar{x}}^j\) is metrically regular at \(\bar{x}\) for \(0\), as required.

The metric regularity of the mappings \(G_{\bar{x}}^j\) imply a parameterized uniform version of metric regularity, where we allow \(\bar{x}\) to move.

**Lemma 7.5.** Let \(\bar{x}, \bar{y}, \bar{u}, \bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{u}}\), and \(\bar{Q}\) be as in Assumption 2, set \(\bar{x}_j := (\bar{x}, \bar{y}, \bar{u}_j)\), and let \(G_{\bar{x}_j}^j\) be given by (56). For all \(j \in \{1, \ldots, \bar{k}\}\), there exists a neighborhood \(U_j \subset \mathbb{R}^n+m+\ell\) of \(\bar{x}_j\) and a neighborhood \(V_j \subset \mathbb{R}^n+m+\ell+\ell\) of \(0\) such that the mapping

\[
(u, z) \mapsto G_{u}^{-j}(z) := (G_{u}^j)^{-1}(z) \quad \text{for} \quad (u, z) \in U_j \times V_j
\]

is single-valued with \(G_{u}^{-j}(0) \in U_j\).

**Proof.** Fix \(j \in \{1, \ldots, \bar{k}\}\). By Lemma 7.4 and [13, Theorem 6D.1], for every \(\lambda > \text{reg}(G_{\bar{x}_j}^j; \bar{x}_j; 0)\) there exists \(a > 0\) and \(b > 0\) such that

\[
\text{dist} \left( x \left| G_{u}^{-j}(z) \right. \right) \leq \lambda \text{dist} \left( z \left| G_{u}^j(x) \right. \right), \quad \text{for every} \quad u, x \in \bar{x}_j + aB, z \in bB.
\]

By reducing \(a\), if necessary, we may assume the conclusion of Lemma 6.10 holds on \(\bar{x}_j + aB\). We follow the argument given in [13, Theorem 6D.2] by recalling (58) and choosing

\[
L > \text{lip}(\nabla \eta_j; (\bar{x}, \bar{y})) := \lim \sup (x,y),(x',y') \to (\bar{x},\bar{y})\ \lim\sup (x,y),(x',y') \neq (x',y') \frac{||\nabla \eta_j(x,y) - \nabla \eta_j(x',y')||}{||(x,y) - (x',y')||}, \quad \text{and} \quad \gamma > \frac{1}{2} \lambda L.
\]
Define $\overline{r} := \min \left\{ \frac{1}{t}, a \right\} > 0$, $U_j := \overline{x}_j + \overline{r}B$, and $V_j := bB$. We first establish nonemptiness of $G_u^{-1}(z)$. Fix $x = \overline{x}_j$, and choose any $(u, z) \in U_j \times V_j$, and consider two cases in (65). If $\operatorname{dist} \left( z \mid G_u'(\overline{x}_j) \right) = 0$, then by closedness of the set $G_u' \overline{x}_j$, it follows that $\overline{x}_j \in G_u^{-1}(z)$. On the other hand, if $0 < \operatorname{dist} \left( z \mid G_u'(\overline{x}_j) \right) < \infty$, where finiteness is guaranteed because $\operatorname{dom} \left( G_u' \right) = \mathbb{R}^{m+n+\ell}$. Then the implication

$$\operatorname{dist} \left( \overline{x}_j \mid G_u^{-1}(z) \right) \leq \lambda \operatorname{dist} \left( z \mid G_u'(\overline{x}_j) \right) \implies \operatorname{dist} \left( \overline{x}_j \mid G_u^{-1}(z) \right) < \infty$$

holds, so in both cases $G_u^{-1}(z) \neq \emptyset$.

We now show single-valuedness. For the same $j$, $u$, and $z$, write $u = (\tilde{x}, \tilde{y}, \tilde{\mu}, \tilde{\gamma})$, and suppose there are two points $x_1 = (x_1, y_1, \mu_1)$, $x_2 = (x_2, y_2, \mu_2)$ satisfying $x_1, x_2 \in G_u^{-1}(z)$. Then subtracting the equations in (57) gives

\begin{align*}
0 &= \nabla^2(\tilde{y}c)(\tilde{x})[x_2 - x_1] + \nabla c(\tilde{x})^\top (y_2 - y_1) \\
y_2 - y_1 &= Q_k \nabla c(\tilde{x})[x_2 - x_1] + AP(\mu_j - \mu_1) \\
0 &= A^\top \nabla c(\tilde{x})[x_2 - x_1].
\end{align*}

Then $\nabla c(\tilde{x})[x_2 - x_1] \in \null \left( A^\top \right)$. Suppose $x_2 \neq x_1$. Taking the inner product on both sides of (66) and using the choice of $\overline{r}$ in accordance with Lemma 6.10,

$$0 = [x_2 - x_1]^\top \nabla^2(\tilde{y}c)(\tilde{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\tilde{x})^\top (y_2 - y_1)$$

by (66)

$$= [x_2 - x_1]^\top \nabla^2(\tilde{y}c)(\tilde{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\tilde{x})^\top [Q_k \nabla c(\tilde{x})[x_2 - x_1] + AP(\mu_j - \mu_1)]$$

by (67)

$$= [x_2 - x_1]^\top \nabla^2(\tilde{y}c)(\tilde{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\tilde{x})^\top Q_k \nabla c(\tilde{x})[x_2 - x_1]$$

by (68)

$$> 0,$$

so $x_2 = x_1$. But then (66), (67), and Lemma 6.10 imply

$$y_2 - y_1 \in \null \left( \nabla c(\tilde{x})^\top \right) \cap \ran (A) = \{0\},$$

so $y_2 = y_1$. The nondegeneracy condition of Definition 6.4 and (67) together imply

$$0 = AP(\mu_j - \mu_1) \implies \mu_j = \mu_1,$$

so single-valuedness is established. We conclude the proof by following the proof given in [13, Theorem 6D.2] and write $(x, y, \mu) = \gamma = G_u^{-1}(0)$. Then the quadratic bound lemma and the choice of $\gamma$ gives

$$\left\| \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix} \right\| \leq \left\| x - \overline{x} \right\|$$

by (54) and $-g_j(\overline{x}_j) \in G_0$
\[
\begin{align*}
= \frac{2\gamma}{L} \| F_u(\overline{x}_j, \overline{g}_j(\overline{x}_j)) \| & \quad \text{by (55)} \\
= \frac{2\gamma}{L} \left\| \eta_j(\overline{x}, \overline{y}) + \nabla \eta_j(\overline{x}, \overline{y}) \left( \frac{\overline{x} - \overline{x}}{\overline{y} - \overline{y}} \right) - \eta_j(\overline{x}, \overline{y}) \right\| & \quad \text{by (59)} \\
\leq \gamma \left\| \frac{\overline{x} - \overline{x}}{\overline{y} - \overline{y}} \right\|^2 & \quad \text{by (59)} \\
\leq \gamma \left\| u - \overline{x} \right\|^2 & \quad \text{by (59)} \\
< \pi,
\end{align*}
\]
so \( x = G_u^i(0) \in U_j \).

Our work so far implies that Newton's method applied to the individual mappings \( G^i_{\overline{x}_j} \) exhibit local quadratic convergence.

**Theorem 7.1.** Let \( \overline{x}, \overline{y}, \overline{\pi}, \overline{\kappa}, \) and \( Q \) be as in Assumption 2, set \( \overline{x}_j := (\overline{x}, \overline{y}, \overline{\pi}_j) \), and let \( g^j_{\overline{x}_j} \) be given by (56). Then, the mappings \( \{ g^j_{\overline{x}_j} \}_{j=1}^{\overline{k}} \) are strongly metrically regular (see Definition 7.2) at \( \overline{x}_j \) for \( 0 \). Moreover, for all \( j \in \{1, \ldots, \overline{k}\} \), there exists a neighborhood \( U_j \) of \( \overline{x}_j \) such that, for every \( x^0 \in U_j \), there is a unique sequence \( x^k = (x^k, y^k, \mu^k) \subset U_j \) generated by Newton's method for \( g_j + G_0 \) (5). Both this sequence, and the sequence \( (x^k, y^k) \), converge at a quadratic rate to \( \overline{x}_j \) and \( (\overline{x}, \overline{y}) \) respectively.

**Proof.** The metric regularity at \( \overline{x}_j \) for \( 0 \) was established in Lemma 7.4. Lemma 7.5 with \( u = \overline{x}_j \) shows \( G_{\overline{x}_j}^i \) has a single-valued localization around \( 0 \) for \( \overline{x}_j \), so the strong metric regularity of \( G_{\overline{x}_j}^i \) at \( \overline{x}_j \) for \( 0 \) follows.

For the second claim, we again follow the proof in [13, Theorem 6D.2] by taking \( U_j \) as in Lemma 7.5, and choosing any \( x^0 \in U_j \). Following the proof of the final claim of Lemma 7.5, we find, for every \( k \geq 1 \), the existence and uniqueness of \( x^k \) given \( x^{k-1} \) satisfying

\[
0 \in G^i_{x^{k-1}}(x^k), \quad \left( \frac{x^k - \overline{x}}{y^k - \overline{y}} \right) \leq \| x^k - \overline{x} \| \leq \gamma \left( \frac{x^{k-1} - \overline{x}}{y^{k-1} - \overline{y}} \right) \leq \gamma \| x^{k-1} - \overline{x} \|, \quad \text{and} \quad x^k \in U_j.
\]

Moreover, since \( \theta := \gamma \| x^0 - \overline{x} \| = \gamma \overline{\pi} < 1 \), \( \| x^k - \overline{x} \| \leq \theta^{k-1} \| x^0 - \overline{x} \| \) for all \( k \geq 1 \), which completes the proof of quadratic convergence of both sequences. \( \square \)

We now move from an isolated analysis of the mappings \( G_u^i \) to how they behave as a whole. The goal is to guarantee the \( y \) obtained by solving \( 0 \in G_u^i(x) \) at some \( u = (\hat{x}, \hat{y}, \hat{\mu}_i) \) for \( x = (x, y, \mu) \) has \( y \in \partial c(\hat{x}) + \nabla c(\hat{x}) [x - \hat{x}] \).

**Theorem 7.2.** Let \( \overline{x}, \overline{y}, \overline{\pi}, \overline{k}, \) and \( Q \) be as in Assumption 2, set \( \overline{x}_j := (\overline{x}, \overline{y}, \overline{\pi}_j) \), and let \( G_{\overline{x}_j}^i \) be given by (56). Suppose \( i \neq j \) and \( i, j \in \{1, \ldots, \overline{k}\} \). There exists a neighborhood \( N \) of \( (\overline{x}, \overline{y}, \overline{\pi}_1, \ldots, \overline{\pi}_{\overline{k}}) =: (\overline{x}, \overline{y}, \overline{\pi}) \in \mathbb{R}^{n+m+\overline{k}} \) such that, if \( (\hat{x}, \hat{y}, \hat{\mu}_1, \ldots, \hat{\mu}_{\overline{k}}) \in N \) and \( u_j := (\hat{x}, \hat{y}, \hat{\mu}_j) \), \( u_i := (\hat{x}, \hat{y}, \hat{\mu}_i) \), with \( \hat{\mu}_i > 0 \) and \( \hat{\mu}_j > 0 \),
then
\[ x_j := G^{-1}_{u_j}(0) = \begin{pmatrix} x_j \\ y_j \\ \mu_j \end{pmatrix}, \quad x_i := G^{-1}_{u_i}(0) = \begin{pmatrix} x_i \\ y_i \\ \mu_i \end{pmatrix} \text{ satisfy } \begin{pmatrix} x_j \\ y_j \\ \mu_j \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \\ \mu_i \end{pmatrix} \text{ for all } i, j \in \{1, \ldots, k\}. \]

That is, there exists \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\) such that \((x, y) = (x_i, y_i)\) for all \(i \in \{1, \ldots, k\}\). Moreover,

(i) \(c(\tilde{x}) + \nabla c(x)[x - \tilde{x}] \in \mathcal{M}_\tau\),

(ii) \(\mu(c(\tilde{x}) + \nabla c(x)[x - \tilde{x}], y) > \mu_j > 0\) for all \(j \in \{1, \ldots, k\}\),

(iii) \(y \in \text{ri} (\partial h(c(\tilde{x}) + \nabla c(x)[x - \tilde{x}]))\),

where the mapping \(\mu(c, y)\) is defined in Lemma 6.5.

Proof. For \(j \in \{1, \ldots, k\}\), define \(\pi_j : \mathbb{R}^{n+m+\tilde{t}} \to \mathbb{R}^{n+m+\tilde{t}}\) by \(\pi_j(x, y, \mu_1, \ldots, \mu_j, \ldots, \mu_{\tilde{t}}) := (x, y, \mu_j)\).

We first show there exists a neighborhood \(\mathcal{N}\) of \((\tilde{x}, \tilde{y}, \tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{t}})\) such that, for all \(j \in \{1, \ldots, k\}\) and all \((\tilde{x}_j, \tilde{y}_j, \tilde{\mu}_j) = u_j \in \mathcal{N}_j := \pi_j(\mathcal{N})\),

(a) the mappings \(\{G^{-1}_{u_j}(0)\}_{j=1}^k\) are single-valued with \(G^{-1}_{u_j}(0) \in \mathcal{N}_j\),

(b) \(\mu_j\) associated to \(G^{-1}_{u_j}(0)\) has \(\mu_j > 0\),

(c) the condition (52) is satisfied at all \((x, y, \mu_j) \in \mathcal{N}_j\), and

(d) \(c(\tilde{x}_j) + \nabla c(\tilde{x}_j)[x_j - \tilde{x}_j] \in \mathcal{M}_\tau\), where \((x_j, y_j, \mu_j) = G^{-1}_{(\tilde{x}_j, \tilde{y}_j, \tilde{\mu}_j)}(0)\).

Parts (a), (b), and (c) are a consequence of Lemma 7.5. We now justify (d). For any \(j \in \{1, \ldots, k\}\), the definition of \((x_j, y_j, \mu_j) = G^{-1}_{(\tilde{x}_j, \tilde{y}_j, \tilde{\mu}_j)}(0)\) implies, in particular, \(A^+[c(\tilde{x}_j) + \nabla c(\tilde{x}_j)[x_j - \tilde{x}_j] - c(\tilde{x})] = 0\).

By the polyhedral structure of \(\mathcal{M}_\tau\), for any \(w \in \text{Null}(A^+) = T(\tilde{c} | \mathcal{M}_\tau)\), there exists \(\tau > 0\) such that \(\tau + tw \in \mathcal{M}_\tau\) for all \(|t| < \tau\). Lemma 7.5 argued that, for all sufficiently small \(\epsilon > 0\),

\[ G^{-1}_{u_j}(0) \in (\tilde{x}_j + \epsilon \mathbb{B}) \text{ for all } u \in (\tilde{x}_j + \epsilon \mathbb{B}) \text{ (see (a)).} \]

The continuity of \(c\) and (70) imply that for \(u_j\) sufficiently close to \(\tilde{x}_j\), \(c(\tilde{x}_j) + \nabla c(\tilde{x}_j)[x_j - \tilde{x}_j]\) can be made as close to \(c(\tilde{x})\) as desired. Then there exists a neighborhood of \((\tilde{x}, \tilde{y}, \tilde{\mu}_j)\) such (d) holds. The neighborhood \(\mathcal{N}_j\) also exists because there are only finitely many indices \(j\) in consideration.

Now let \(u_j := (\tilde{x}_j, \tilde{y}_j, \tilde{\mu}_j) \in \mathcal{N}_j, u_i := (\tilde{x}_i, \tilde{y}_i, \tilde{\mu}_i) \in \mathcal{N}_i,\) with \(\tilde{\mu}_i > 0\) and \(\tilde{\mu}_j > 0\), and denote

\[ G^{-1}_{u_j}(0) = \begin{pmatrix} x_j \\ y_j \\ \mu_j \end{pmatrix}, \quad G^{-1}_{u_i}(0) = \begin{pmatrix} x_i \\ y_i \\ \mu_i \end{pmatrix}. \]

By (57),

\[ 0 = \nabla^2(\tilde{y}c)(\tilde{x})[x_j - x_i] + \nabla c(\tilde{x})^T(y_j - y_i) \]

\[ y_i = Q_k(c(\tilde{x}) + \nabla c(\tilde{x})[x_j - \tilde{x}]) + AP_j \mu_i + b_{k_i} \]

\[ y_j = Q_k(c(\tilde{x}) + \nabla c(\tilde{x})[x_j - \tilde{x}]) + AP_j \mu_j + b_{k_j} \]

\[ 0 = A^T \nabla c(\tilde{x})[x_j - x_i] \]
Define \( \tilde{c}_i := c(\tilde{x}) + \nabla c(\tilde{x})[x_i - \tilde{x}] \in \mathcal{M}_\tau \) by (d). By Assumption 3, \( \overline{y} = Q_k \overline{c} + b_k + AP \overline{\mu}_i = Q_k \overline{c} + b_k + AP_j \overline{\mu}_j \), and in particular, 
\[
Q_k \overline{c} + b_k - b_j = Q_k \overline{c} + AP_j \overline{\mu}_j - AP_i \overline{\mu}_i.
\]
Then (53) with \( w := \tilde{c}_i - \overline{c} \in \text{Null} \left( A^\top \right) , \ i = 1 \), and any \( y \in \partial h(\tilde{c}_i) \) gives
\[
y_i = Q_k w + Q_k \overline{c} + b_k + AP_i \mu_i
\]
where the fourth equivalence follows from (75). This implies
\[
y_j - y_i - Q_k j \nabla c(\tilde{x})[x_j - x_i] \in \text{Ran} \left( A \right).
\]

Taking the inner product on both sides of (71) with \( x_j - x_i \) gives
\[
0 = [x_j - x_i]^\top \nabla^2 (\overline{y}c)(\tilde{x})[x_j - x_i] + [x_j - x_i]^\top \nabla c(\tilde{x})^\top (y_j - y_i)
\]
By Lemma 6.10 and (74), \( x_j = x_j \). Then (76), (71), and (c) imply \( y_j - y_i \in \text{Ran} \left( A \right) \cap \text{Null} \left( \nabla c(\tilde{x})^\top \right) = \{0\} \), which proves (69).

Since \( i \) and \( j \) were arbitrary, let \( x \) and \( y \) denote the common values of the first two components of \( \mathcal{G}_u^{-1}(0) \) for each \( j \in \{1, \ldots, \overline{k}\} \). Then \( f_j = Q(c(\tilde{x}) + \nabla c(\tilde{x})[x - \tilde{x}]) + B + \tilde{A} \mu_j \), with \( c(\tilde{x}) + \nabla c(\tilde{x})[x - \tilde{x}] \in \mathcal{M}_\tau \), and \( \mu_1, \ldots, \mu_{\overline{k}} > 0 \). By (41) and Lemma 6.6, \( \mu(c(\tilde{x}) + \nabla c(\tilde{x})[x - \tilde{x}], y_j) = \mu_j > 0 \), with \( y \in \text{ri} \left( \partial h(c(\tilde{x}) + \nabla c(\tilde{x})[x - \tilde{x}]) \right) \).

Our final theorem integrates the ideas from Section 6 and our work in this section to establish the local quadratic convergence of Newton’s method for \( P \).

**Theorem 7.3.** Let \( \mathcal{X}, \overline{y}, \overline{\mu}, \overline{x}, \overline{c}, \overline{\tau}, \text{ and } Q \) be as in Assumption 2, set \( \overline{x} := (\overline{x}, \overline{y}, \overline{\mu}) \), and let \( \mathcal{G}_u^{x_j, \overline{k}, \overline{c}, \overline{m}} \) be given by (56). There exists a neighborhood \( \mathcal{N} \) of \( (\overline{x}, \overline{y}, \overline{\mu}) \) on which the conclusions of Lemma 6.10 are satisfied such that if \( (x^0, y^0, \mu^0) \in \mathcal{N} \), then there exists a unique sequence \( \left\{ (x^k, y^k, \mu^k) \right\} \) \( k \in \mathbb{N} \) satisfying the optimality conditions of \( P_k \) for all \( k \in \mathbb{N} \), with

(a) \( c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}] \in \mathcal{M}_\tau \),

(b) \( \mu(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}], y^k)_j > 0 \) for all \( j \in \{1, \ldots, \overline{k}\} \),

(c) \( y_k^* \in \text{ri} \left( \partial h(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]) \right) \),

(d) \( H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0 \),

(e) \( x^k - x^{k-1} \) is a strong local minimizer of the model function \( \phi_{x^{k-1}, y^{k-1}} \), given by Definition 9.1.

Moreover, the sequence \( (x^k, y^k) \) converges to \( (\overline{x}, \overline{y}) \) at a quadratic rate.
Proof. All claims except (e) follow from Theorem 7.1 and Theorem 7.2. By Lemma 9.3, Lemma 9.4, and (d), claim (e) is equivalent to showing

\[ h''(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]; \nabla c(x^{k-1})\delta) + \delta^T H_{k-1}\delta > 0 \quad \forall \delta \in \text{Null}\left( A^T \nabla c(x^{k-1}) \right) \setminus \{0\}. \]

Using (17) and partial smoothness,

\[ h''(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]; \nabla c(x^{k-1})\delta) = \delta^T \nabla c(x^{k-1})^T Q_j \nabla c(x^{k-1})\delta, \quad \forall j \in \mathcal{K}(\overline{c}), \]

so (52) gives (77). \(\square\)

Remark 10. The fact that \(\{x^k - x^{k-1}\}\) is a strong local minimizer of \(\phi_{(x^{k-1}, y^{k-1})}\) does not mean that there are not other critical points for the model function outside the neighborhood of interest. It may be that at any iteration the problem \(\overline{P}\) does not have a finite optimal value, in particular, should there exist directions of negative curvature orthogonal to the manifold.

7.2. Smooth Problems. In this section, we make the following assumptions:

Assumption 3. Let \(f\) be as in \(\overline{P}\) and \((\overline{x}, \overline{y}) \in \text{dom}\ (f) \times \mathbb{R}^m\), \(\overline{y} := c(\overline{x})\), \(\overline{c} = |\mathcal{K}(\overline{c})|\), where \(\mathcal{K}(\overline{c})\) are the active indices given in Definition 4.2. Let \(\mathcal{M}_f\) be the active manifold defined in (36). We assume that

(a) \(\text{dom}\ (h)\) is given by the Rockafellar-Wets PLQ representation of Theorem 6.1,

(b) \(c\) is \(C^3\)-smooth,

(c) \(\mathcal{K}(\overline{c}) = \{k_0\}\),

(d) \(\overline{x}\) satisfies the second-order sufficient conditions of Theorem 4.3,

Remark 11. Since \(\overline{c} = 1\), we omit reference to the index \(k_0\) for the rest of this section.

Remark 12. By (a) and (c), \(c(\overline{x}) \in \text{int dom}\ (h)\) and \(\partial h(\overline{x}) = \{\overline{y}\}\). Then, (d) becomes

\[ \overline{y} = Q\overline{c} + b, \quad \nabla c(\overline{x})^T \overline{y} = 0, \quad d^T \nabla c(\overline{x})^T Q \nabla c(\overline{x})d + d^T \nabla^2(\overline{y}c)(\overline{x})d > 0 \quad \forall \ d \in \mathbb{R}^n \setminus \{0\}, \]

where \(D(\overline{x}) = \mathbb{R}^n\).

As in Lemma 6.10, we have the following stability result.

Lemma 7.6. Suppose \(d^T \nabla c(x)^T Q \nabla c(\overline{x})d + d^T \nabla^2(\overline{y}c)(\overline{x})d > 0\) for all \(d \in \mathbb{R}^n \setminus \{0\}\). Then, there exists a neighborhood \(\mathcal{N}\) of \((\overline{x}, \overline{y})\) such that if \((x, y) \in \mathcal{N}\) then,

\[ d^T \nabla c(x)^T Q \nabla c(x)d + d^T \nabla^2(yc)(x)d > 0, \quad \forall \ d \in \mathbb{R}^n \setminus \{0\}, \]

and \(c(x) \in \text{int dom}\ (h)\).

Our local analogue of the KKT mapping (10) is the following.

Definition 7.5. Define \(g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}\) by

\[ g(x, y) := \begin{pmatrix} \nabla c(x)^T y \\ y - Qc(x) - b \end{pmatrix}, \quad G := \{0\}^{n+m}, \]

and set \(\nabla g(x, y) = \begin{pmatrix} \nabla^2 (yc)(x) & \nabla c(x)^T \\ -Q \nabla c(x) & I_m \end{pmatrix}, \quad g(\overline{x}, \overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)
Assumption 3 (d) implies $\nabla g(\bar{x}, \bar{y})$ is nonsingular. Consequently, and the Newton method (5) corresponds to the classical Newton’s method for solving the equation $g(x, y) = 0$. Namely,

\begin{equation}
(79) \quad \text{Find } (x^{k+1}, y^{k+1}) \text{ such that } g(x^k, y^k) + \nabla g(x^k, y^k) \left( x^{k+1} - x^k, y^{k+1} - y^k \right) = 0.
\end{equation}

The local quadratic convergence of the iteration (79) near $(\bar{x}, \bar{y})$ with $\nabla g(\bar{x}, \bar{y})$ is nonsingular is well-known, with (79) corresponding to the optimality conditions for $P_k$. We conclude with the following theorem, which parallels Theorem 7.3.

**Theorem 7.4.** Let $\mathcal{N}$, $\mathcal{M}_\mathcal{N}$ be as in Assumption 3. Then, there exists a neighborhood $\mathcal{N}$ of $(\bar{x}, \bar{y})$ on which the conclusions of Lemma 7.6 are satisfied such that if $(x^0, y^0) \in \mathcal{N}$, then there exists a unique sequence $\left\{ (x^k, y^k) \right\}_{k \in \mathbb{N}}$ satisfying the optimality conditions of $P_k$ for all $k \in \mathbb{N}$, with

(a) $c(x^{k-1}) + \nabla c(x^{k-1}) [x^k - x^{k-1}] \in \mathcal{M}_\mathcal{N}$,

(b) $\partial h(c(x^{k-1}) + \nabla c(x^{k-1}) [x^k - x^{k-1}]) = \{y^k\}$,

(c) $H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0$,

(d) $x^k - x^{k-1}$ is a strong local minimizer of the model function $\phi_{(x^{k-1}, y^{k-1})}$, given by Definition 9.1.

Moreover, the sequence $(x^k, y^k)$ converges to $(\bar{x}, \bar{y})$ at a quadratic rate.

8. Acknowledgement

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9. Appendix

Lemma 9.1. Suppose $C \subseteq \mathbb{R}^m$ is a nonempty, closed, convex set and $A \in \mathbb{R}^{n \times m}$. Consider the following equations:

(a) $\text{Null}(A) \cap \text{ri}(C) = \{\overline{y}\}$,
(b) $\text{Null}(A) \cap \text{par}(C) = \{0\}$,
(c) $\text{Null}(A) \cap C = \{\overline{y}\}$.

Then (a) $\implies$ (b) $\implies$ (c).
Proof. [(a) $\Rightarrow$ (b)] Since $\overline{y} \in C$, there exists an integer $n \geq 1$ and points $\{y_1, \ldots, y_n\} \subset \text{aff } C$ that span $\{y_i - \overline{y}\}_{i=1}^n = \text{par } (C)$. By convexity and the assumption $\overline{y} \in \text{ri } (C)$, we can further assume $\{y_1, \ldots, y_n\} \subset \text{ri } (C)$. By [36, Theorem 6.4], there exists $\{z_1, \ldots, z_n\} \subset \text{ri } (C)$ and $t_i > 0$ such that, for all $i \in \{1, \ldots, n\}$, $y_i - \overline{y} = -t_i(z_i - \overline{y})$. Then, after relabeling, we may suppose $\{y_1, \ldots, y_n\} \subset \text{ri } (C)$ satisfies

\[
\text{par } (C) = \left\{ \sum_{i=1}^n \mu_i (y_i - \overline{y}) \mid \mu_i \geq 0, i \in \{1, \ldots, n\} \right\}
\]

Now suppose (b) does not hold. Then, there exists $0 \neq z \in \text{Null } (A) \cap \text{par } (C)$. By (80), $z = \sum_i \mu_i (y_i - \overline{y})$ with $\mu_i \geq 0$ and $\sum_i \mu_i \neq 0$. Define $t := \frac{1}{\sum_i \mu_i}$, and for $i \in \{1, \ldots, n\}$, define $\lambda_i := t \mu_i$. Then $\lambda_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$, with $\sum_{i=1}^n \lambda_i = 1$. Then by [36, Theorem 6.1] $\overline{y} + tz = \overline{y} + \sum_i \lambda_i (y_i - \overline{y}) = \sum_i \lambda_i y_i \in \text{ri } (C)$. But then $\overline{y}$ and $\overline{y} + tz$ are two points in $\text{Null } (A) \cap \text{ri } (C)$, so (b) must hold.

[(b) $\Rightarrow$ (c)] Suppose (b) and that there exists $y_1, y_2 \in \text{Null } (A) \cap C$. Then $y_1 - y_2 \in \text{Null } (A) \cap \text{par } (C) = \{0\}$, so $y_1 = y_2$. \qed

**Theorem 9.1** (Normals Cones to Sets with Constraint Structure). [37, Theorem 6.14] Let $C = \left\{ x \in X \mid F(x) \in Z \right\}$ for closed convex sets $X \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^m$ and a $C^1$-mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. Suppose $\overline{x} \in C$ satisfies the constraint qualification

\[
[y \in N (F(\overline{x}) \mid Z), -\nabla F(\overline{x})^\top y \in N (\overline{x} \mid X)] \iff y = (0, \ldots, 0).
\]

Then $N (\overline{x} \mid C) = \left\{ \nabla F(\overline{x})^\top y + v \mid y \in N (F(\overline{x}) \mid Z), v \in N (\overline{x} \mid X) \right\}$. \hfill (81)

**Definition 9.1** (The model function at $\hat{x}$). Let $f$ be as in $\mathcal{P}$ and $\hat{x} \in \text{dom } (f)$. Suppose $f$ satisfies (BCQ) at $\hat{x}$. Define $u := (\hat{x}, \hat{y})$, $\hat{H} := \nabla^2 (\hat{y}c)(\hat{x})$,

$$
\psi(v, w) := h(v) + w, \text{ and } \Phi_u(d) := \begin{pmatrix} c(\hat{x}) + \nabla c(\hat{x})d \\ \frac{1}{2} d^\top \hat{H}d \end{pmatrix}.
$$

Then, for any $(v, w) \in \text{dom } (h) \times \mathbb{R}$ and $(d, s) \in \mathbb{R}^n \times \mathbb{R}$,

$$
\nabla \Phi_u(d) = \begin{pmatrix} \nabla c(\hat{x}) \\ \frac{1}{2} d^\top \hat{H}d \end{pmatrix}, \quad \psi'((v, w); (d, s)) = h'(v; d) + s, \quad \psi''((v, w); (d, s)) = h''(v; d).
$$

Set $\phi_u(d) := \psi(\Phi_u(d)) = h(c(\hat{x}) + \nabla c(\hat{x})d) + \frac{1}{2} d^\top \hat{H}d$. By Theorem 4.2, $\phi_u$ is piecewise linear-quadratic, though not necessarily convex because $\hat{H}$ may not be positive semi-definite. However, $\phi_u$ is convex-composite with $\psi$ piecewise linear-quadratic convex.

The following lemma shows that if $f$ satisfies (BCQ) at $\hat{x}$, then the model function at $\hat{x}$ satisfies its (BCQ) throughout its domain.

**Lemma 9.2**. Let $f$ be as in $\mathcal{P}$, and suppose $f$ satisfies (BCQ) at $\hat{x}$. Then, $\phi_u$ given in Definition 9.1 satisfies (BCQ) at all points $\overline{d} \in \text{dom } (\phi_u) = \left\{ d \mid c(\hat{x}) + \nabla c(\hat{x})d \in \text{dom } (h) \right\}$.

**Proof.** Let $\overline{d} \in \left\{ d \mid c(\hat{x}) + \nabla c(\hat{x})d \in \text{dom } (h) \right\}$. By definition,

$$
\text{Null } \left( \nabla \Phi_u(\overline{d})^\top \right) = \text{Null } \left( \begin{pmatrix} \nabla c(\hat{x})^\top \\ \frac{1}{2} \overline{d}^\top \hat{H}\overline{d} \end{pmatrix} \right) \text{ and } N \left( \Phi_u(\overline{d}) \mid \text{dom } (\psi) \right) = N \left( c(\hat{x}) + \nabla c(\hat{x})\overline{d} \mid \text{dom } (h) \times \{0\} \right).
$$
Suppose \( v = (v_1, v_2) \in \text{Null} \left( \nabla \Phi(u(\bar{d})^\top) \cap N \left( \Phi(u(\bar{d}) \mid \text{dom}(\phi)) \right) \right) \). Then \( v_2 = 0 \), and
\[
v_1 \in \text{Null} \left( \nabla c(\bar{x})^\top \right) \cap N \left( c(\bar{x}) + \nabla c(\bar{x})\bar{d} \mid \text{dom}(h) \right) \subset \text{Null} \left( \nabla c(\bar{x})^\top \right) \cap N \left( c(\bar{x}) \mid \text{dom}(h) \right) = \{0\},
\]
where the inclusion follows since \( \left\langle v_1, \nabla c(\bar{x})\bar{d} \right\rangle = 0 \). 

\[\Box\]

**Lemma 9.3.** Let \( \Phi \) be as in Definition 9.1, and suppose \( f \) satisfies (BCQ) at \( \bar{x} \). Consider the problem
\[
(P_{\Phi},) \quad \min_{\bar{d}} \Phi(u(\bar{d}))
\]
Then, the cone of non-ascent directions \( D_{\Phi}(\bar{d}) \) at any \( \bar{d} \in \text{dom}(\Phi) \) is given by
\[
D_{\Phi}(\bar{d}) = \left\{ \delta : h'(c(\bar{x}) + \nabla c(\bar{x})\bar{d}; \nabla c(\bar{x})\delta) + \bar{d}^\top \bar{H} \delta \leq 0 \right\}.
\]
Moreover, the second-order necessary and sufficient conditions of Theorem 4.3 applied to \( \Phi \) are
\[\begin{align*}
(1) \quad & \text{If } \Phi \text{ has a local minimum at } \bar{d}, \text{ then } 0 \in \bar{H} \bar{d} + \nabla c(\bar{x})^\top \partial_h(c(\bar{x}) + \nabla c(\bar{x})\bar{d}) \\
& \quad \text{and } h''(c(\bar{x}) + \nabla c(\bar{x})\bar{d}; \nabla c(\bar{x})\delta) + \delta^\top \bar{H} \delta \geq 0,
\end{align*}\]
for all \( \delta \in D_{\Phi}(\bar{d}) \).
\[\begin{align*}
(2) \quad & \text{If } 0 \in \bar{H} \bar{d} + \nabla c(\bar{x})^\top \partial_h(c(\bar{x}) + \nabla c(\bar{x})\bar{d}) \text{ and } \\
& \quad h''(c(\bar{x}) + \nabla c(\bar{x})\bar{d}; \nabla c(\bar{x})\delta) + \delta^\top \bar{H} \delta > 0,
\end{align*}\]
for all \( \delta \in D_{\Phi}(\bar{d}) \setminus \{0\} \), then \( \bar{d} \) is a strong local minimizer of \( \Phi \).

**Proof.** Since (BCQ) is satisfied at all points \( \bar{d} \in \text{dom}(\Phi) \), the chain rule of Theorem 3.1 gives
\[
\partial \Phi(\bar{d}) = \bar{H} \bar{d} + \nabla c(\bar{x})^\top \partial_h(c(\bar{x}) + \nabla c(\bar{x})\bar{d}),
\]
which is (82). The set of Lagrange multipliers for \( \Phi \) becomes
\[
M_{\Phi}(d) := \text{Null} \left( \nabla \Phi(\bar{d})^\top \right) \cap \partial \Phi(\bar{d})
\]
(83)
\[
= \text{Null} \left( \left( \nabla c(\bar{x})^\top \bar{H} \bar{d} \right) \right) \cap \partial_h(c(\bar{x}) + \nabla c(\bar{x})\bar{d}) \times \{1\},
\]
so that \( (y_1, y_2) \in M_{\Phi}(d) \iff \left\{ \bar{H} \bar{d} + \nabla c(\bar{x})^\top \bar{y}_1, \bar{y}_1 \in \partial h(c(\bar{x}) + \nabla c(\bar{x})\bar{d}), \bar{y}_2 = 1 \right\} \). The Lagrangian in [4] is \( L(d,y) := \left\langle y, \Phi(u(\bar{d})) \right\rangle - \psi^*(y), \quad y = (y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}, \text{ with } \nabla^2(y\Phi(u))(d) = y_2 \bar{H} \). Then, from Theorem 4.3, for any \( \delta \in \mathbb{R}^n \),
\[
\psi''(\Phi(u(\bar{d})); \nabla \Phi(u(\bar{d})\delta) + \max \left\{ \left\langle \delta, \nabla^2(y\Phi(u))(d)\delta \right\rangle \mid y \in M_{\Phi}(d) \right\} = h''(c(\bar{x}) + \nabla c(\bar{x})\bar{d}; \nabla c(\bar{x})\delta) + \delta^\top \bar{H} \delta.
\]

The following lemma relates an active manifold at a solution to \( P \) to the directions of non-ascent for the model function Definition 9.1. It is an immediate consequence of Theorem 6.2, Lemma 6.9, and (82), and the proof is identical to Lemma 6.9.
Lemma 9.4 (Model non-ascent directions). Let $f$ be as in $P$, $\bar{x} \in \text{dom}(f)$, $\bar{\tau} := c(\bar{x})$, $\bar{k} = |\mathcal{K}(\bar{\tau})|$, where $K(\bar{\tau})$ are the active indices given in Definition 4.2. Let $(\hat{x}, \hat{y})$ and $\phi_u$ be as in Definition 9.1, and let the active manifold $\mathcal{M}_\tau$ be as in (36), with dom$(h)$ given by the Rockafellar-Wets PLQ representation theorem. Suppose $0 = \hat{H}d + \nabla c(\hat{x})^\top \hat{y}$, $c(\hat{x}) + \nabla c(\hat{x})d \in \mathcal{M}_\tau$, and $\hat{y} \in \text{ri} \left( \partial h(c(\hat{x}) + \nabla c(\hat{x})d) \right)$. Then, $\phi_u$ satisfies (SC) at $d$ for $(\hat{y}, 1)$, and

if $\bar{k} \geq 2$, then, in the notation of Definition 6.3, $D_{\phi_u}(\bar{d}) = \text{Null} \left( A^\top \nabla c(\hat{x}) \right)$.

if $\bar{k} = 1$, then, $D_{\phi_u}(\bar{d}) = \mathbb{R}^n$. 
