AN AVOIDANCE PRINCIPLE AND MARGULIS FUNCTIONS FOR EXPANDING TRANSLATES OF UNIPOTENT ORBITS

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Abstract. We prove an avoidance principle for expanding translates of unipotent orbits for some semisimple homogeneous spaces. In addition, we prove a quantitative isolation result of closed orbits and give an upper bound on the number of closed orbits of bounded volume. The proofs of our results rely on the construction of a Margulis function and the theory of finite dimensional representations of semisimple Lie groups.

1. Introduction

Avoidance principles — quantifying how much time trajectories avoid certain subsets of the ambient space — have been fruitful in the study of dynamical systems. An important example is the non-divergence of unipotent flows which goes back to Margulis [Mar71]. A quantitative version of non-divergence appears in Dani [Dan79] and were key in Ratner’s seminal theorems on unipotent flows [Rat90b, Rat90a, Rat91a, Rat91b].

Two successful strategies to prove such avoidance principles has been through the construction of Margulis functions which originated in the influential work of Eskin–Margulis–Mozes [EMM98] and the linearization technique of Dani–Margulis [DM93].

The flexibility offered by the construction of a Margulis function make them applicable to settings where unipotent dynamics are not available or poorly understood. For example, they appear in the important work of Benoist-Quint [BQ11, BQ13a, BQ13b] and the recent generalizations of Eskin–Lindenstrauss [EL18b, EL18a] on stationary measures of homogeneous spaces. Additionally, Margulis functions are utilized in Eskin–Mirzakhani–Mohammadi [EMM15] to prove an avoidance principle that was crucially used to show an analog of Ratner’s orbit closure theorem.

We highlight some other examples to indicate the breadth of Margulis functions, but we recommend the wonderful survey of Eskin–Mozes [EM22] for a more complete overview of the literature. Margulis functions appear in the setting of Teichmüller dynamics by Eskin–Masur [EM01] and Athreya [Ath06], in the space of lattices by Kadyrov–Kleinbock–Lindenstrauss–Margulis [KKLM17] and Kleinbock–Mirzadeh [KM20], for infinite homogeneous spaces by Mohammadi–Oh [MO20], and in the space of closed subgroups of a semisimple Lie group equipped with the Chabauty topology in the work of Gelander–Levit–Margulis [GLM21] and Fraczyk–Gelander [FG21].

We use Margulis functions and the theory of finite dimensional representations of semisimple Lie groups to prove an avoidance principle. Broadly speaking, our results rely on the

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hyperbolicity of diagonal actions and the fact that the perturbation by a foliation often places one in a general position where one expects expansion by the diagonal direction.

Throughout this paper, $G$ will be a semisimple algebraic Lie group without compact factors and $H$ will be a semisimple subgroup of $G$ without compact factors such that $C_G(H)$ is finite. We let $X := G/\Gamma$ where $\Gamma$ is a lattice.

We equip $\text{Lie}(G)$ with an inner product that induces a Riemannian metric on $G$. The notion of distances and volumes makes sense with respect to this Riemannian metric. Denote by $\text{inj}(x)$ the injectivity radius at point $x$. See the next section for formal descriptions of these notions.

**Definition 1.** For a pair of positive real numbers $(V, d)$, we say that a point $x \in X$ is $(V, d)$-Diophantine with respect to $H$ if the following holds: for any intermediate subgroup $H \subseteq S \subseteq G$ and any closed $S$-orbit $Y = Sx'$ with $\text{vol}(Y) \leq V$, the distance between $x$ and $Y$ is at least $d$; namely, $\text{dist}(x, Y) \geq d$.

For $r > 0$, if a point $x \in X$ is $(V, d)$-Diophantine with respect to $H$ and $\text{inj}(x) \geq r$, then we say that $x$ is $(V, d, r)$-Diophantine with respect to $H$.

We fix a one parameter subgroup of diagonalizable elements $\{a_t\} \subseteq H$ and let $U$ be the unstable horospherical subgroup with respect to $\{a_t\}$; $U = \{u \in G : a tua_{-t} \to e \text{ as } t \to -\infty\}$.

We work with the operators $$(A_{r,t}f)(x) = \frac{1}{m_U(B_r^U)} \int_{B_r^U} f(a_tux) \, dm_U(u),$$ where $B_r^U$ is the ball of radius $r$ in $U$ and $m_U$ is the Haar measure on the Lie subgroup $U$ normalized so that $B_1^U$ has measure 1. Here the implicit metric on $U$ comes from the identification of $\text{Lie}(U)$ with a Euclidean space. See the next section for details. When considering $A_{1,t}$, we use the notation $A_t$.

We use the operators $A_{r,t}$ to prove a result on the behavior of points of the form $a_tux$ for $u \in B_1^U$ and large $t > 0$. The following is our main theorem.

**Theorem 2** (Avoidance Principle). Let $G$ be a semisimple group without compact factors and $H$ be a semisimple subgroup without compact factors such that $C_G(H)$ is finite. Let $X = G/\Gamma$ where $\Gamma$ is a lattice. There exists absolute constants $D = D(\dim(G)) > 0$, $A = A(G/\Gamma, H) > 0$, and $C = C(G/\Gamma, H) > 0$ such that the following dichotomy holds: for any $x \in X$, there exists $T_x > 0$ such that for any pair of $T > T_x$ and $R > 2$, either:

1. $x$ is not $(R, 1/T)$-Diophantine or
2. for all $t \geq A \log T$,
   $$m_U(\{u \in B_1^U : a_tux \text{ is not } (R, R^{-D}, R^{-D})\text{-Diophantine with respect to } H\}) < CR^{-1}.$$

Moreover, if $K \subseteq X$ is compact, then, $T_x$ can be chosen to be uniform over all $x \in K$.

**Remark 3.** We note a generalization of our main result for solvable epimorphic subgroups. Recall, a subgroup $G'$ of a real algebraic group $G$ is called epimorphic in $G$ if any $G'$-fixed
vector is also $G$-fixed for any finite dimensional algebraic representation of $G$. Proposition 2.2. of Shah and Weiss [SW00] gives an analogous result to our Linear Algebra Lemma (Lemma 3) for solvable epimorphic groups. Hence, it is plausible that our result can be further generalized so that $B_1^U$ in condition (2) is replaced by $B_1^N$ where $N \subseteq U$ is an algebraic unipotent subgroup normalized by $\{a_t\}$, such that the subgroup generated by $\{a_t\}$ and $N$ is solvable and epimorphic in $G$.

The result is similar to Lindenstrauss–Margulis–Mohammadi–Shah [LMMS19] though they are interested in avoidance principles for unipotent flows and work in a more general setting. It is also similar to the work of Bénard–de Saxcé [BS21].

To prove our main result, we need results on the quantitative isolation of closed orbits which are interesting in their own right. The following theorem is analogous to Lemma 10.3.1 of Einsiedler–Margulis–Venkatesh [EMV09].

**Theorem 4** (Quantitative Isolation of closed orbits). There exists a global constant $D = D(\dim(G)) > 0$ such that the following holds: for all intermediate subgroup $H \subseteq S \subseteq G$ and closed $S$-orbits $Y = Sy$ and $Z = Sz$ of finite volume,

$$\text{dist}(Y \cap K, Z) \gg K \text{vol}(Y)^{-D} \text{vol}(Z)^{-D}$$

where $K$ is a compact subset of $X$.

We note that the proof of [EMV09] Lemma 10.3.1, relies on uniform spectral gap for periodic $S$-orbits ($H \subseteq S \subseteq G$) in congruence quotients. Our proof is arguably softer. In particular, it does not require $\Gamma$ to be arithmetic. The main idea of the proof is to estimate the size of the additive constant of a Margulis function, and goes back to Margulis’ unpublished notes (see also Theorem 1.1. of [MO20]).

Using Theorem 4 above, an upper bound can be obtained on the number of closed orbits of bounded volume. The theorem below is analogous to Corollary 10.7 of Mohammadi–Oh [MO20].

**Theorem 5** (Upper bound for the number of closed orbits of bounded volume). There exists a global constant $D = D(\dim(G)) \gg 1$ such that for any intermediate subgroup $H \subseteq S \subseteq G$,

$$\# \{Y : Y = Sy \text{ is a closed } S\text{-orbit and } \text{vol}(Y) \leq R\} \ll R^D.$$  

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2. **Preliminaries**

In this section we fix notation.

Equip $\text{Lie}(G)$ with the Killing form. This induces

(1) a norm $\| \cdot \|$ on $\text{Lie}(G)$.  

(2) a right-invariant Riemannian metric on $G$ that induces a right-invariant metric on $G$ denoted as $\text{dist}_G$.

(3) a metric on $X = G/\Gamma$ denoted as $\text{dist}$ so that the canonical projection $G \to X$ is a local isometry.

(4) a volume for a closed orbit $H$-orbit on $X$ induced from the Riemannian structure on $G$ which we denote with $\text{vol}$.

With respect to the norm $\| \cdot \|$ on $\text{Lie}(G)$, we can define the unit ball in $\text{Lie}(G)$ which we denote as $B_{1}^{\text{Lie}(G)}$.

We choose an inner product on $\text{Lie}(U)$ that comes from the identification of $\text{Lie}(U)$ to $\mathbb{R}^{d_U}$ where $d_U$ denotes the dimension of $\text{Lie}(U)$. For any $\eta > 0$, we can use the inner product on $\text{Lie}(U)$ to define a norm (resp. metric) on $\text{Lie}(U)$ (resp. $U$). This allows us to make sense of the unit ball in $\text{Lie}(U)$ which we denote as $B_{1}^{\text{Lie}(U)}$ (resp. in $U$ which we denote as $B_{1}^{U}$).

For each $x \in X$, we denote by $\text{inj}(x)$ the injectivity radius at point $x$; the supremum of all $\eta > 0$ for which the projection map $g \to gx$ from $G$ to $X = G/\Gamma$ is injective on $B_{\eta}^{G}$. In Section 6, we shall choose a specific $\varepsilon_X > 0$ and denote $X_{\varepsilon_X} := \{x \in X : \text{inj}(x) \geq \varepsilon_X\}$ as the compact part of $X$. Since the exponential map $\text{Lie}(G) \to G$ defines a local diffeomorphism, there exists an absolute constant $\sigma_0 > 1$ such that for all $w \in \text{Lie}(G)$ with $\|w\| \leq \varepsilon_X$ and $x \in X_{\varepsilon_X}$,

$$\sigma_0^{-1}\|w\| \leq \text{dist}(x, \exp(w)x) \leq \sigma_0\|w\|.$$  

By noting that the canonical projection $G \to X$ is a local isometry, we have a way of locally measuring distances in $X$ with the norm on $\text{Lie}(G)$.

For any intermediate subgroup $H \subseteq S \subseteq G$, we denote the dimension of $\text{Lie}(S)$ by $\dim(S)$ or simply, $d_S$.

We will denote the Haar measure on $G$ by $m_G$. For the horospherical subgroup $U$ of $G$, we denote the Haar measure on $U$ by $m_U$.

Let $T$ denote a maximal Cartan subgroup containing $(a_t)_{t \in \mathbb{R}}$. Let $\rho : G \to \text{GL}(V)$ be a finite dimensional representation. Let $\Phi$ denote the root system of $\text{Lie}(G)$ and decompose the vector space into weight spaces $V = \bigoplus_{\beta \in \Phi} V_\beta$ where

$$V_\beta = \{ v \in V : \rho(\tau)v = \exp(\beta(\log(\tau)))v, \forall \tau \in T \}$$

is the weight space with weight $\beta \in \Phi$. Choose a basis $(v_{\beta,i})_{i=1}^{\dim V_\beta}$ so that every $v \in V$ can be written in the form

$$v = \sum_{\beta \in \Phi} \sum_{i=1}^{\dim V_\beta} c_{\beta,i} v_{\beta,i}$$

for some scalars $c_{\beta,i}$.

Let $S$ be an intermediate subgroup with $H \subseteq S \subseteq G$ and consider the decomposition of $\text{Lie}(G)$ given by $\text{Lie}(G) = \text{Lie}(S) \oplus V_S$ where $V_S$ is $\text{Ad}(\text{Lie}(S))$-invariant, but not necessarily irreducible. If we decompose $V_S$ into $\text{Ad}(\text{Lie}(S))$-invariant subspaces, then each subspace will be non-trivial since $C_G(H)$ is finite. This will be an important fact that we use throughout the paper when working with the adjoint representation.

We end the section by introducing two results from Einsiedler–Margulis–Venkatesh [EMV09] on intermediate subgroups $H \subset G$. 

Lemma 6 (Lemma 3.4.1, [EMV09]). Suppose $H \subseteq G$ are semisimple Lie groups without compact factors such that $C_G(H)$ is finite. Then there are only finitely many intermediate subgroups $H \subseteq S \subseteq G$. Each such $S$ is semisimple and without compact factors.

Lemma 7 (Appendix A, [EMV09]). If $G$ is a semisimple Lie group without compact factors, then there exists a finite collection of semisimple subgroups $\mathcal{H}$ such that the following holds: for any semisimple Lie subgroup $H \subseteq G$ with no compact factors and $C_G(H)$ finite, there exists $H' \in \mathcal{H}$ and $g \in G$ such that $H = gH'g^{-1}$.

3. **Linear Algebra Lemma**

In this section we state some key technical lemmas related to the action of horospherical subgroups and diagonal subgroups from [Kat20] and [Sha96] and prove extensions of these results. The main result of this section applies to representations that are not necessarily irreducible.

Lemma 8 (Linear algebra lemma). Suppose $\rho : G \to GL(V)$ is a faithful finite dimensional representation of a semisimple Lie group $G$. Suppose $V$ decomposes into non-trivial and irreducible subspaces $V = \bigoplus_i V_i$. There exists an absolute constant $0 < \delta_0 = \delta_0(\dim(G)) \ll 1$ such that for all $0 < \delta < \delta_0$ and $0 < c < 1$, there exists $t_{\delta,c} = t_{\delta,c}(G,H) > 0$ with

$$\frac{1}{m_U(B^U_{2r})} \int_{B^U_{2r}} \frac{1}{\|\rho(u)v\|^6} \, dm_U(u) < c \|v\|^6$$

for every $v \in V$, $t \geq t_{\delta,c}$.

We now state some lemmas from Katz [Kat20]. In brief, these lemmas show that the action of the diagonal and horospherical subgroups on a vector space expand the norm. While we will follow the exposition of Katz [Kat20], we would like to draw the readers attention to Shah [Sha96], specifically Section 5.

The following lemmas are essentially Lemma 3.1 and Lemma 3.2 of [Kat20].

Lemma 9 (Lemma 3.1, [Kat20]). Let $u = \exp(u) \in B^U_r$. There exists polynomials $f_{\beta,j} : B^{Lie(U)}_r \to \mathbb{R}$ with

$$\rho(u)v = \sum_{\beta \in \Phi} \sum_{j=1}^{\dim V_\beta} f_{\beta,j}(u)v_{\beta,j}.$$

Proof. The proof of Lemma 3.1 in [Kat20] works for any $u \in B^U_r$ where $r > 0$.

Lemma 10 (Anchor Lemma, Lemma 3.2, [Kat20]). Let $\rho : G \to GL(V)$ be a finite dimensional irreducible representation of a semisimple Lie group $G$. Then for any $r > 0$ and non-zero $v \in V$, there is a positive root $\beta \in \Phi^+$ and $1 \leq j \leq \dim V_\beta$ such that

$$\sup_{u \in B^U_r} |f_{\beta,j}(u)| > 0.$$

Proof. The proof of Lemma 3.2 in [Kat20] works for any open ball $B^U_r$ with $r > 0$. 

□
By the Anchor lemma, the projection of the action of $U$ in the expanding direction is nonzero. Thus, the norm under the action of $a_t$ grows. By noting that Lemmas 3.1 and 3.2 of Katz [Kat20] hold for any open ball $B^U_\rho$, we have the following minor generalization of Lemma 2.3 of [Kat20].

**Lemma 11.** Suppose $\rho : G \to GL(V)$ is an irreducible finite dimensional representation of a semisimple Lie group $G$. There exists $0 < \delta_0 = \delta_0(\dim(G)) \ll 1$ such that for all $0 < \delta < \delta_0$ and $0 < c < 1$, there exists $t_c > 0$ with

$$\frac{1}{m_U(B^U_\rho)} \int_{B^U_\rho} \frac{1}{\|\rho(a_t u)v\|^{\delta}} dm_U(u) < \frac{c}{\|v\|^{\delta}}$$

for every $v \in V$ and $t \geq t_c$.

We conclude with a proof of the lemma stated at the beginning of the section.

**Proof of Lemma 11.** This essentially follows from the irreducible version of [Kat20]. We equip $V = \oplus V_\iota$ with the max norm. That is, for $v = (v_\iota)$, $\|v\| = \max \|v_\iota\|$. We also note that the inequality we aim to prove is independent of the choice of norm.

Let $0 < c < 1$ and $0 < \delta < \delta_0$. We will choose $\delta_0$ in the course of the proof.

Given $v = (v_\iota)$, let $i_0$ be the index with $\|v\| = \|v_{i_0}\|$. Then,

$$\|\rho(a_t u)v\| = \max_i \|\rho(a_t u)v_i\| \geq \|\rho(a_t u)v_{i_0}\|.$$ 

By the irreducible case (Lemma 11), we have the existence of $\delta_i \in (0,1)$ such that for every $\delta \in (0,\delta_i)$ contraction occurs for every $v_i \in V_\iota$ and $t$ sufficiently large. To finish the proof, take $\delta_0 := \min_i \delta_i$ and we have

$$\frac{1}{m_U(B^U_\rho)} \int_{B^U_\rho} \frac{1}{\|\rho(a_t u)v\|^{\delta}} dm_U(u) \leq \frac{1}{m_U(B^U_\rho)} \int_{B^U_\rho} \frac{1}{\|\rho(a_t u)v_{i_0}\|^{\delta}} dm_U(u)$$

$$\leq \frac{c}{\|v_{i_0}\|^{\delta}} = \frac{c}{\|v\|^{\delta}}$$

for every $v \in V$ and $t$ sufficiently large. \qed

Now we apply the above Linear algebra lemma to a specific representation which will be used to control the height function. First we give a definition. See also the end of section 2 of Eskin–Margulis [EM04].

**Definition 12 (Maximal Parabolic Subgroups).** When a lattice $\Gamma$ is non-uniform, we define a finite collection $\Delta$ of maximal parabolic subgroups of $G$ as follows. A parabolic subgroup $P$ of $G$ is called $\Gamma$-rational if $\Gamma \cap R_u(P)$ is a lattice in $R_u(P)$, where $R_u(P)$ is the unipotent radical of $P$. If $G$ is of real rank 1, then we let $\Delta = \{P_0\}$ where $P_0$ is a $\Gamma$-rational minimal parabolic subgroup of $G$. The existence of $P_0$ follows from Garland–Raghunathan [GR70]. If the real rank of $G$ is not less than 2, then by the Margulis Arithmeticity theorem, $\Gamma$ is arithmetic. Hence we let $\Delta = \{P_1, P_2, \ldots, P_r\}$ where $P_k$ are standard parabolic subgroups of $G$, with respect to its maximal $\mathbb{Q}$-split torus $A_0$. For every $1 \leq k \leq r$, there exists a finite-dimensional irreducible representation $\rho_k : G \to GL(W_k)$ and vectors $w_k \in W_k$ such that the stabilizer of $\mathbb{R}w_k$ is $P_k$. 


Remark 13 (Upper bound on the dimension of $W_k$). For later computational purposes (see Lemma 19 and Lemma 33), we take $W_k$ to have dimension no greater than $\dim(G)^2$. We can do so by choosing

$$W_k := \wedge^{\dim(R_u(P_k))} \text{Lie}(R_u(P_k)) \subseteq \wedge^{\dim(R_u(P_k))} \text{Lie}(G)$$

and $w_k$ to be a normalized diagonal element of $W_k$.

The following result is an analogue of Condition A of Eskin–Margulis [EM04]. We can deduce it by applying Lemma 8 to the representation above.

**Lemma 14 (Linear algebra lemma for Height functions).** If $\Gamma$ is a non-uniform lattice of $G$, then there exists $0 < \delta_1 = \delta_1(\dim(G)) \ll 1$ such that for all $0 < \delta < \delta_1$ and $0 < c < 1$, there exists $t_{\delta,c} = t_{\delta,c}(G, H) > 0$ such that for every $v \in Gw_k$ and $t \geq t_{\delta,c}$,

$$\frac{1}{m_U(B_U^2)} \int_{B_U^2} F(a_tux) \, dm_U(u) < \frac{c}{\|v\|^{\delta}}.$$ 

4. **Abstract Margulis inequality**

In this section we prove an abstract result that yields exponential decay for Margulis functions.

**Theorem 15.** Suppose $F : X \to (0, \infty)$ satisfies the following properties

- **(Log Continuity)** For any compact subset $K \subset G$, there exists $\sigma = \sigma_F(K) > 1$ such that for all $g \in K$ and $x \in X$,

  $$\sigma^{-1} F(x) \leq F(gx) \leq \sigma F(x)$$

- **(Margulis Inequality for $F$)** There exists constants $0 < c < 1$, $t = t_c \gg 1$, and $b > 0$ such that for any $x \in X$,

  $$(A_2(tF))(x) := \frac{1}{m_U(B_U^2)} \int_{B_U^2} F(a_tux) \, dm_U(u) < cF(x) + b.$$  

Then, there exist absolute constants $C > 0$ and $B > 0$ such that for any $t \geq t_c$,

$$(A_tF) \leq C \cdot e^{t/t_c} F + B.$$  

**Proof.** **Step 1:** We find an upper bound on $A_n tF$.

Recall,

$$(A_n tF)(x) := \frac{1}{m_U(B_U^2)} \int_{B_U^2} F(a_n tux) \, dm_U(u).$$

By iterating our operator, we have

$$(A_{2,t}^n F)(x) < c^n F(x) + B'$$

where $B' = O(b) = b \sum_{j=0}^n c^j$. On the other hand,

$$(A_{2,t}^n F)(x) = \frac{1}{m_U(B_U^2)} \int_{B_U^2} \left( \frac{1}{m_U(B_U^2)} \int_{B_U^2} F(a_n t \phi_n(u) u_1 x) \, dm_U(u_1) \right) \, d(m_U)^{n-1}(\bar{u})$$
Throughout this section, $\Gamma$ will be taken to be a non-uniform lattice of $G$. The height function in use is essentially the same as in Eskin–Margulis [EM04]. However, how high a point $x \in X$ is in the cusp. We prove that $h$ satisfies a Margulis inequality. The height function in use is essentially the same as in Eskin–Margulis [EM04]. However,
instead of taking the average over some random walk on $X$, we average over the expanding translates of $B_1^U$, the unit ball in the horospherical subgroup.

**Theorem 16.** For any $0 < \delta < \delta_1$ (\(\delta_1\) as in Lemma 14), there exists a height function $h = h_3 : X \to (0, \infty)$ such that the following holds: for any $0 < c < 1$, there exists $t_c > 0$ such that for any $t \geq t_c$, there exists absolute constant $B_t > 0$ such that

$$A_{2t}h < ch + B_t.$$

Note that Theorem 16 directly implies the (Margulis inequality) hypothesis in Theorem 15 for $h$. In Lemma 19 we shall see that $h$ is log-continuous also, and thus we get the following exponential decay property for $h$.

**Corollary 17.** For any $0 < \delta < \delta_1$, height function $h = h_3$ (the same height function as in Theorem 16) satisfies the following: there exists $t_h > 0$, $C_h > 0$ and $B_h > 0$ such that for all $t \geq t_h$,

$$A_t h \leq \frac{C_h}{2^{t/t_h}} : h + B_h.$$

**Proof.** The result directly follows from Theorem 15, Theorem 16 and Lemma 19. We note that $t_h$ is equal to $t_{1/2}$, defined as in Theorem 16.

\[\square\]

**Construction of the height function** $h$. Let $P_0$ denote a minimal $\Gamma$-rational parabolic subgroup of $G$. Then we have the Langlands decomposition $P_0 = M_0A_0N_0$ where $M_0$ is semisimple, $A_0$ is abelian, and $N_0$ is the unipotent radical of $P_0$. If $G$ has real rank greater than 2, then we let $A_0$ to be the fixed maximal $\mathbb{Q}$-split torus of $G$ in Definition 12. Let $\mathfrak{a}$ denote the Lie algebra of $A_0$. We shall identify $\mathfrak{a}$ with its dual via the Killing form. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ denote the roots which we view as elements of the dual of $\mathfrak{a}$. A **Siegel set** is a set $\mathcal{S} = KMAN$ where $K$ is the maximal compact subgroup of $G$, $M \subseteq M_0$ and $N \subseteq N_0$ are compact, and $A = \mathcal{P} \{ a \in A_0 : \alpha_k (\log a) < C \text{ for all } 1 \leq k \leq r \}$ for some positive constant $C$.

Note that for appropriate choices of $M$, $N$, and $C$, there exists a finite set $J \subseteq G$ such that for every $g \in G$, the intersection $\mathcal{S} \cap g\Gamma J$ is not empty. See Dani–Margulis [DM93] for details.

For $1 \leq k \leq r$, define $d_k(g) := \| \rho_k(g)w_k \|$ where $\rho_k : G \to GL(W_k)$ and $w_k \in W_k$ are defined as in Definition 12. By structure theory, there exists absolute constants $C_0$ and $c_1, c_2, \ldots, c_r$ such that for each $1 \leq k \leq r$,

$$d_k(g) = d_k(a) \text{ and } | \log(d_k(a)) - c_k \omega_k(\log a) | < C_0$$

for all $g \in G$ where $g = kman$ is the Langlands decomposition of $g$ with respect to $P_k$ and $\omega_k$ is the co-root corresponding to $\alpha_k$; i.e. $\omega_k(\alpha_k) = 1$ and $\omega_k(\alpha_j) = 0$ for all $j \neq k$. Let

$$\beta_k(g) = \max_{\gamma \in \Gamma} \frac{1}{d_k(g \gamma)^{1/c_k}}.$$

Also, for $x = g \Gamma \in X$, we shall define $\beta_k(x) := \beta_k(g)$.
Remark 18. There exists an absolute constant $C = C(G/\Gamma) > 1$ such that for any $g \in G$ and $g_1 \in \mathcal{G} \cap g\Gamma J$,

$$C^{-1}\beta_k(g_1) < \beta_k(g) < C\beta_k(g_1).$$

Lastly, we choose a sequence of positive real numbers \( \{q_k\}_{k=1}^r \) so that \( \sum_k q_k \omega_k \) belongs to the positive Weyl chamber of \( \mathfrak{a} \) and let

$$h_k(g) = \beta_k(g)^{1/q_k}.$$ 

For later use (see Proposition 22), we shall take \( \{q_k\}_{k=1}^r \) to be normalized so that

$$\min_{1 \leq k \leq r} \{c_k q_k\} = 1.$$

Our height function \( h \) will be defined to be

$$h := C_* \sum_{k=1}^r t_k^\delta_*$$

for some \( \delta_* \) and \( C_* \). The condition \( \delta_* \ll 1 \) will be later verified in the proof of Proposition 22 so that \( h \) satisfies the Margulis inequality, and the constant \( C_* \gg 1 \) will be later chosen in Remark 21 so that \( h \) is bounded away from 1.

Lemma 19 (Log continuity of height function). For any compact subset \( K \subset G \), there exists \( \sigma_h = \sigma_h(K) \geq 1 \) such that for all \( g \in K \) and \( x \in X \),

$$\sigma_h^{-1} \cdot h(x) \leq h(gx) \leq \sigma_h \cdot h(x).$$

Moreover, \( \sigma_h \) can be chosen to be a constant only depending on a compact set \( K \) and \( \dim(G) \), each of which will be independent of lattice \( \Gamma \).

Proof. For each \( 1 \leq k \leq r \), the map \( d_k = \|\rho_k(\cdot)w_k\| \) is log continuous. Thus, there exists \( \sigma_k = \sigma_k(K) \) such that for all \( g' \in K \) and \( g \in G \),

$$\sigma_k^{-1} \cdot d_k(g) \leq d_k(g'g) \leq \sigma_k \cdot d_k(g).$$

Note that \( \sigma_k \) only depends on compact set \( K \) and \( \dim(W_k) \). By Remark 13, \( \dim(W_k) \leq \dim(G)^2 \) so that \( \sigma := \max_{1 \leq k \leq r} \{\sigma_k\} \) is a constant only depending on \( K \) and \( \dim(G) \).

Now for each \( 1 \leq k \leq r \), let \( \sigma_k' := \sigma^{1/c_k} \), where \( c_k \) are the constants used to define \( \beta_k \). Then, for all \( g' \in K \) and \( g \in G \),

$$(\sigma_k')^{-1} \cdot \beta_k(g) \leq \beta_k(g'g) \leq \sigma_k' \cdot \beta_k(g).$$

If the maximum in \( \beta_k(g'g) \) is achieved by the same \( \gamma \in \Gamma \) as in \( \beta_k(g) \), then the result directly follows from log continuity of \( d_k \) and definition of \( \sigma_k' \). Suppose that the maximum is achieved by different choice of \( \gamma \);

$$\beta(k(g) = \frac{1}{d_k(g\gamma_0)^{1/c_k}} \quad \text{and} \quad \beta_k(g'g) = \frac{1}{d_k(g'g\gamma_1)^{1/c_k}}$$

for some \( \gamma_0 \neq \gamma_1 \in \Gamma \). Then,

$$\beta_k(g'g) \geq \frac{1}{d_k(g'g\gamma_0)^{1/c_k}} = d_k(g'g\gamma_0)^{-1/c_k} \geq (\sigma \cdot d_k(g\gamma_0))^{-1/c_k} = (\sigma_k')^{-1} \cdot \beta_k(g)$$

and

$$\beta_k(g'g) = d_k(g'g\gamma_1)^{-1/c_k} \leq (\sigma^{-1} \cdot d_k(g\gamma_1))^{-1/c_k} < \sigma_k' \cdot d_k(g\gamma_0))^{-1/c_k} = \sigma_k' \cdot \beta_k(g).$$
Lastly, by definition
\[ h := C_\star \sum_{k=1}^{r} h_k^{\delta_s} = C_\star \sum_{k=1}^{r} \beta_k^{\delta_s/q_k}, \]
so it follows that the height function \( h \) has log continuity with constant
\[ \sigma_h := \max_{1 \leq k \leq r} \{ (\sigma_k')^{\delta_s/q_k} \} = \sigma_{\max_{1 \leq k \leq r} \{ \delta_s/c_k q_k \}} = \sigma_{\delta_s}. \]
(Here we are using the fact that the \( \{ q_k \}_{k=1}^{r} \) are normalized to satisfy \( \min_{1 \leq k \leq r} \{ c_k q_k \} = 1 \).
Since \( \delta_s = \frac{1}{2} \delta_1 \) is a constant only dependent on \( \dim(G) \) (see Lemma 14), \( \sigma_h \) is only dependent on compact set \( K \) and \( \dim(G) \).

\[ \Box \]

The following lemma is a direct result of equation (33) from Eskin–Margulis [EM04].

**Lemma 20.** For any constant \( C > 1 \), there exists an absolute constant \( D_C = D_C(G/\Gamma) > 1 \) such that the following holds: if for some \( 1 \leq k \leq r \) and \( g \in G \) there exists \( g_1 \in \mathcal{S} \cap g \Gamma J \) and \( g_2(\neq g_1) \in g \Gamma J \) such that \( d_k(g_2) < C d_k(g_1) \), then

\[ h_k(g) \leq D_C \prod_{j \neq k} h_j^{\lambda_{j,k}}(g) \]
where
\[ \lambda_{j,k} := \frac{g_j|_{<a_j,a_k>}}{q_k|_{<a_j,a_k>}}. \]

In view of Remark 18 we can rewrite the above Lemma 20 as follows.

**Lemma 21** (Upper bound for \( h_k \)). For any constant \( C > 1 \), there exists an absolute constant \( D'_C = D'_C(G/\Gamma) > 1 \) such that the following holds. If \( \beta_k(g) = \frac{1}{d_k(g \gamma_0)^{1/c_k}} \) for some \( \gamma_0 \in \Gamma \) and there exists \( \gamma_1(\neq \gamma_0) \in \Gamma \) such that \( d_k(g \gamma_1) < C d_k(g \gamma_0) \), then

\[ h_k(g) \leq D'_C \prod_{j \neq k} h_j^{\lambda_{j,k}}(g). \]

Now, we replace Condition A of Eskin–Margulis [EM04] with Lemma 14 and prove an Margulis inequality for our averaging operator \( A_\star \).

**Proposition 22** (Upper bound for \( A_\star h_k^{\delta} \)). For any \( 0 < \delta < \delta_1 \) and \( 0 < c < 1 \), for all \( t \geq t_{\delta,c} \) (where \( t_{\delta,c} \) is as in Lemma 14), there exists an absolute constant \( D_t > 0 \) (depending only on \( t \)) such that for any \( 1 \leq k \leq r \),

\[(A_{2,t} h_k^{\delta})(g) := \frac{1}{m_U(B_{2,t} \backslash K_\Gamma)} \int_{B_{2,t} \backslash K_\Gamma} h_k^{\delta}(a_t u g) \ dm_U(u) \leq c h_k^{\delta}(g) + D_t \prod_{j \neq k} h_j^{\lambda_{j,k}}(g) \]
for any \( g \in G \).

**Proof.** If for every \( a_t u g \) (varying \( u \) over \( B_{2,t} \)) the maximum in \( \beta_k \) is achieved by the exact same \( \gamma \in \Gamma \) as in \( \beta_k(g) \), then we get \( (A_{2,t} h_k^{\delta})(g) \leq c h_k^{\delta}(g) \) directly from Lemma 14, since

\[ h_k(g) = \frac{1}{d_k(g)^{\delta/c_k q_k}} = \frac{1}{\| p_k(g) w_k \|^{|\delta/c_k q_k}|} \]
and $\delta/c_k q_k < \delta_1/c_k q_k \leq \delta_1$. (Here we are using the fact that the $\{q_k\}_{k=1}^r$ are normalized to satisfy $\min_{1 \leq k \leq r} \{c_k q_k\} = 1$.)

Suppose that the maximum in $\beta_k$ is achieved by a different $\gamma$ for some $u \in B_2^U$:

$$\beta_k(g) = \frac{1}{d_k(g\gamma_0)^{1/c_k}} \text{ and } \beta_k(g') = \frac{1}{d_k(g'\gamma_1)^{1/c_k}}$$

for some $g' \in a_t B_2^U$ and $\gamma_0 \neq \gamma_1 \in \Gamma$. By definition of $\beta_k$, compactness of $a_t B_2^U$, and log continuity of $d_k$, we have

$$d_k(g\gamma_1) < C d_k(g'\gamma_1) < C d_k(g'\gamma_0) < C^2 d_k(g\gamma_0),$$

where $C = C_{t,k} = \sigma_{d_k}(a_t B_2^U)$. By Lemma 21 and by log continuity of $h_k$,

$$\frac{1}{m_U(B_2^U)} \int_{B_2^U} h_k^\delta(a_t u g) dm_U(u) \leq \frac{1}{m_U(B_2^U)} \int_{B_2^U} C' h_k^\delta(g) dm_U(u) = C' h_k^\delta(g)$$

$$\leq C' D_{C'2} \prod_{j \neq k} h_j^{\delta\lambda_j,k}(g)$$

where $C' = C'_{t,k} = \sigma_{h_k}(a_t B_1^U)$, and $D_{C'2}$ is defined as in Lemma 21 for our choice of $C = C_{t,k}$. We note that $C = C_{t,k}$, $C' = C'_{t,k}$, and $D_{C'2} = D_{C'2,t,k}$ are all absolute constants, only depending on $t$ and $k$. Thus, by taking $D_i := \max_{1 \leq k \leq r} \{C'_{t,k} D_{C'2,t,k}\}$, we get the desired result.

**Theorem 23.** For any triple of $0 < \delta < \delta_1$, $0 < c < 1$, and $t \geq t_{\delta,c/2}$ ($t_{\delta,c/2}$ as in Lemma 14), there exists $0 < \varepsilon = \varepsilon_{t,c} \ll 1$ such that the height function $h = h_{t,c} := \sum_{k=1}^r (\varepsilon h_k)^\delta$ satisfies

$$A_{2,t} h \leq c h + 1.$$ 

**Proof.** By Proposition 22 we have

$$A_{2,t} h_k^\delta \leq \frac{c}{2} \cdot h_k^\delta + D_t \cdot \prod_{j \neq k} h_j^{\delta\lambda_j,k}. $$

Taking the sum over all $1 \leq k \leq r$ and multiplying $\varepsilon^\delta$ to both sides of the above equation yields,

$$A_{2,t} \left( \sum_{k=1}^r (\varepsilon h_k)^\delta \right) \leq \frac{c}{2} \cdot \sum_{k=1}^r (\varepsilon h_k)^\delta + D_t \cdot \varepsilon \sum_{k=1}^r \varepsilon h_k \prod_{j \neq k} (\varepsilon h_j)^{\delta\lambda_j,k}. $$
where \( \varepsilon_k = \varepsilon^\delta (1 - \sum_{j \neq k} \lambda_{j,k}) \) for each \( 1 \leq k \leq r \). Since \( \sum_k q_k \omega_k \) belongs to the positive Weyl chamber, we have that \( \sum_{j \neq k} \lambda_{j,k} < 1 \) for each \( 1 \leq k \leq r \). Hence, by Jensen’s inequality
\[
\prod_{j \neq k} (\varepsilon h_j)^{\delta \lambda_{j,k}} = \exp \left( \sum_{j \neq k} \delta \lambda_{j,k} \cdot \left( \log(\varepsilon h_j)^\delta \right) \right) \leq \sum_{j \neq k} \delta \lambda_{j,k} \cdot \left( \log(\varepsilon h_j)^\delta \right) + (1 - \sum_{j \neq k} \lambda_{j,k}) \cdot \exp(0)
\]
and thus,
\[
A_{2,t} \left( \sum_{k=1}^r (\varepsilon h_k)^\delta \right) \leq \frac{c}{2} \cdot \sum_{k=1}^r (\varepsilon h_k)^\delta + \left( D_t \sum_{k=1}^r \varepsilon_k \right) \cdot \left( \sum_{k=1}^r (\varepsilon h_k)^\delta + 1 \right)
\]
Since \( \varepsilon_k = \varepsilon^{\delta(1 - \sum_{j \neq k} \lambda_{j,k})} \) and \( \sum_{j \neq k} \lambda_{j,k} < 1 \) for each \( 1 \leq k \leq r \), we can choose \( \varepsilon = \varepsilon_{\delta,c} \ll 1 \) small enough so that \( (D_t \sum_{k=1}^r \varepsilon_k) < \frac{c}{2} \).

**Proof of Theorem 16** For fixed \( 0 < \delta < \delta_1 \), the class of height functions \( \{ h_{t,c} \} \) with \( 0 < c < 1 \) and \( t \geq t_{\delta,c}/2 \) defined as in Theorem 23 are linear. That is, if we let \( h = h_{\delta} = \sum_{k=1}^r h_k^\delta \), then
\[
h_{t,c} = \sum_{k=1}^r (\varepsilon h_k) = \sum_{k=1}^r \varepsilon h_k^\delta = \varepsilon_{t,c} \cdot h.
\]
By Theorem 23 for any pair of \( 0 < c < 1 \) and \( t \geq t_{\delta,c}/2 \), we have
\[
(A_{2,t} h_{t,c})(x) \leq c h_{t,c}(x) + 1.
\]
Multiply \( B_t := 1/\varepsilon_{t,c}^\delta \) to both sides of the inequality and we get
\[
(A_{2,t} h)(x) \leq c h(x) + B_t.
\]
Lastly, take \( t_c = t_{\delta,c}/2 \) and we are done. \( \square \)

**Remark 24** (Lower bound for \( h \)). For computational reasons that will become apparent later, we want our \( h \) to be large and bounded away from 1. Since \( \{ d_k(g) : g \in \mathcal{G} \} \) (\( \mathcal{G} \) is the Siegel set) is bounded away from zero, by Remark 18 we have that for each \( 1 \leq k \leq r \), \( h_k \) is bounded away from zero and therefore, \( h_{\delta} = \sum_{k=1}^r h_k^\delta \) is bounded away from zero. Thus, by multiplying some large \( C \gg 1 \), we can make our height function \( h := Ch_{\delta} \) to be no less than 2. Note that constant multiple does not effect Margulis inequalities (see proof of Theorem 16) except that it only makes the additive constant \( B_t \) bigger; our newly defined \( h = Ch_{\delta} \) also satisfies Theorem 16 and Corollary 17.

For the remainder of the paper, if \( \Gamma \) is a non-uniform lattice, then we fix \( \delta_* := \frac{1}{2} \delta_1 \) and let our height function to be \( h = C_h h_{\delta_*} \geq 2 \). However, note that the exact value of \( \delta_* \) is not important and all arguments in the later sections apply for any choice of \( 0 < \delta_* < \delta_1 \).
6. Return Lemma and Number of Nearby Sheets

Let $H \subseteq S \subseteq G$ be an intermediate orbit. For a closed $S$-orbit $Y = Sy$ and point $x \in X$, we shall define a window set $I_Y(x)$ which collects all the sheets of $Y$ that are nearby $x$. Roughly, the idea is to collect sheets of $Y$ within the injective ball $B_{\text{inj}(x)}^G(x)$, but the exact size of our windows will be much smaller, and will be given in terms of the height function $h$. A formal definition of $I_Y(x)$ will be given in Definition 31.

The aim of this section is to show that $\#I_Y(X)$, the number of nearby sheets, is bounded in terms of volume of $Y$.

**Proposition 25** (Number of nearby sheets). There exists a global constant $C_1 > 0$ such that for any intermediate subgroup $H \subseteq S \subseteq G$, closed $S$-orbit $Y = Sy$, and $x \in X$, we have

$$\#I_Y(x) < C_1 \text{vol}(Y).$$

6.1. Height function and Radius of Injectivity. First we compare $h(x)$, the value of the our height function at $x \in X$, with $\text{inj}(x)$, the injectivity radius at point $x$. We note that the following Proposition is as an analog of Lemma 6.3. of Benoist-Quint [BQ11].

**Proposition 26.** If $\Gamma$ is a non-uniform lattice of $G$, then there is absolute constants $C_2 > 0$ and $m > 0$ such that for all $x \in X$,

$$\text{inj}(x)^{-1} \leq C_2 h(x)^m.$$

**Proof.** Let $\text{dist}_G$ denote the left invariant Riemannian metric on $G$. Suppose that for some $g_1 \neq g_2 \in B^G_\epsilon(e)$ and $x = g\Gamma \in X$, $g_1x = g_2x \in X$. Then, for any $\gamma \in \Gamma$, $(\gamma g)^{-1}(g_1^{-1}g_2)(g\gamma)$ is in $\Gamma$. Moreover, for each point in $G$, there is a neighborhood on which the metric $\text{dist}_G$ is Lipschitz equivalent to the metric derived from matrix norm. Thus,

$$2\epsilon \geq \text{dist}_G(e, g_1^{-1}g_2) \geq \text{dist}_G(e, (\gamma g)^{-1}(g_1^{-1}g_2)g\gamma)||\text{Ad}(g\gamma)^{-1}||^{-1}.$$

Since $\Gamma$ is a lattice, $\inf_{\epsilon \neq \gamma \in \Gamma} \text{dist}_G(e, \gamma) > 0$ and thus, $\text{inj}(x) \gg \min_{\gamma \in \Gamma} ||\text{Ad}(\gamma)||^{-1}$. Since $\mathcal{S} \cap g\Gamma J$ is non-empty, we choose $g' \in \mathcal{S} \cap g\Gamma J$ and take its Langlands decomposition $g' = k'a'n'$ with respect to $P_0$; we get $||\text{Ad}(g')|| \asymp ||\text{Ad}(a')||$ and

$$||\text{Ad}(a')|| \ll \left(\frac{1}{\min_{1 \leq k \leq r \exp(\omega_k(\log(a')))}\right)^r.$$

By Remark 18, $\min_{\gamma \in \Gamma} ||\text{Ad}(\gamma)||^{-1}$ is comparable with $||\text{Ad}(g')||^{-1}$ and $h(x)$ is comparable with $h(g')$. Therefore,

$$\text{inj}(x)^{-1} \ll h(x)^m$$

where $m := \min_{1 \leq k \leq r} a_k = \frac{r}{\delta^*} \cdot \max_{1 \leq k \leq r} \{q_k\}$. (Here, $\delta^*$ is from the definition of height function $h$, as in Remark 21 and $\{q_k\}_{1 \leq k \leq r}$ are the positive real numbers used to define $h_k = \beta_k^{1/q_k}$.)

Later in the proof of Proposition 25 and also in the construction of Margulis function $F_Y$ (see Theorem 35), we shall see that Proposition 26 plays a key role, together with Theorem 16 and Corollary 17. That is, the key property of the height function $h$ is that it is a Margulis function that is comparable with the injectivity radius.
We now generalize our definition of height function \( h : X \to [2, \infty) \) to the case when \( \Gamma \) is cocompact, so that Theorem\(^{[10]}\) Corollary\(^{[17]}\) and Proposition\(^{[20]}\) all holds true also when \( \Gamma \) is cocompact.

**Definition 27.** If \( \Gamma \) is a cocompact lattice of \( G \), then we define \( h : X \to (0, \infty) \) to be the constant function,

\[
h \equiv 2.
\]

**Proof of Theorem\(^{[10]}\) and Corollary\(^{[17]}\) (for cocompact \( \Gamma \)).** All constant functions are Margulis functions. That is, for any \( c > 0, t > 0, \) and \( x \in X \), we have

\[
(A_{2,t}h)(x) := \frac{1}{m_U(B^U_2)} \int_{B^U_2} h(a_{t}ux)dm_U(u) = \frac{1}{m_U(B^U_2)} \int_{B^U_2} 2dm_U(u) = 2 < ch(x) + 2.
\]

Corollary\(^{[17]}\) can be proved in a similar way. Simply take \( t_n = 1 \); for any \( t \geq 2t_h \), we have

\[
(A_{t}h)(x) = 2 < \frac{1}{2t/t_h}h(x) + 2.
\]

\[\square\]

**Proof of\(^{[26]}\) (for cocompact \( \Gamma \)).** If \( \Gamma \) is a cocompact lattice of \( G \), then \( X = G/\Gamma \) is compact and \( \varepsilon_0 := \inf_{x \in X} \text{inj}(x) > 0 \). Therefore, for any \( x \in X \),

\[
\text{inj}(x)^{-1} \leq \frac{1}{\varepsilon_0} = \frac{1}{2\varepsilon_0}h(x).
\]

\[\square\]

For the remainder of the paper, the height function \( h \) will refer to either of the two; if \( \Gamma \) is non-uniform, then \( h \) will refer to the function defined in Section 4 and if \( \Gamma \) is cocompact, then \( h \) will refer to the constant function in Definition\(^{[27]}\) above.

6.2. **Return Lemma and Number of Sheets.** In Lemma\(^{[32]}\) we shall first give a weaker bound on \( \#I_{Y}(x) \), depending on both \( \text{vol}(Y) \) and \( h(x) \). This shows that \( \#I_{Y}(x) \) is uniformly bounded in terms of \( \text{vol}(Y) \) in the compact part of \( X \). In the case where \( \Gamma \) is non-uniform, we shall make use of height function \( h \) to analyze the cuspidal part of \( X \). We shall show that if the sheets of \( Y \) are very dense nearby a point \( x \) in the cuspidal part of \( X \), then for some \( a_{t}ux \) which lies in the compact part of \( X \) the sheets of \( Y \) must be very dense nearby \( a_{t}ux \) too. This gives us the desired result; even when \( x \) is high up in the cuspidal part of \( X \), there is a uniform bound for \( \#I_{Y}(x) \), only in terms of \( \text{vol}(Y) \). Results in this section are analogous to Section 8 of Mohammadi–Oh \[^{[MO20]}\] and Section 8 of Eskin–Mirzakhani–Mohammadi \[^{[EMM15]}\].

**Lemma 28** (Return lemma). There exists a global constant \( C_3 = C_3(\dim(G)) > 0 \) and \( Q = Q(G/\Gamma, H) \) such that the following holds: for every \( x \in X \), there exists \( u \in B^U_1 \) so that \( a_{t_{x}}ux \in X_{\text{cpx}} := \{ x \in X : h(x) \leq Q \} \) where \( t_{x} = C_3 \log(h(x)) \). (Constants \( t_h, C_h \) and \( B_h \) are as in Corollary\(^{[17]}\)).

**Proof.** By Lemma\(^{[7]}\) there exists \( H' \in \mathcal{H} \) and \( g \in G \) such that \( H = gH'g^{-1} \). Let \( \{a'_i\} \) denote the fixed one parameter subgroup of diagonalizable elements in \( H' \) and let \( U' \) be the unstable horospherical subgroup of \( G \) with respect to \( \{a'_i\} \). Note that

\[
\{a_i\} = g\{a'_i\}g^{-1} \text{ and } B^U_1 = gB^U_{1'}g^{-1}.
\]
Let $t'_h, C'_h$, and $B'_h$ be the constants from Corollary 17 with respect to $H' \subset G$. That is, $t'_h, C'_h$, and $B'_h$ are constants so that for all $t \geq t'_h$ and $x \in X$,

$$A'_h(x) := \int_{B'_h} h(a'_t u' x) \, dm_{U'}(u') \leq \frac{C'_h}{2^{t'_h} t'_h} \cdot h(x) + B'_h.$$ 

Take $C_3 := \frac{1}{\log 2} \max_{H' \in \mathcal{H}}(t'_h)$. Also, let $\sigma_g > 1$ be the absolute constant following from the log continuity of $h$ so that for any $x \in X$, $\sigma_g^{-1} h(x) \leq h(g^{\pm 1} x) \leq \sigma_g h(x)$. (Here, $g$ is an element of $G$ so that $H = gH'g^{-1}$, as defined above.) Then, for every $x \in X$,

$$(A'_h(x)(g^{-1}) = \int_{B'_h} h(a'_t u'(g^{-1} x)) \, dm_{U'}(u') \leq \frac{C'_h}{2^{t'_h} t'_h} \cdot h(g^{-1} x) + B'_h$$

and thus, there exists $u' \in B'_h$ such that $h(a'_t u' g^{-1} x) \leq C'_h \sigma_g + B'_h$.

Now, if we take $u := gu'g^{-1} (\in B'_h)$, then we have that

$$h(a'_t u x) = h(g a'_t u' g^{-1} \cdot gu' g^{-1} x) = h(g a'_t u' g^{-1} x) \leq \sigma_g h(a'_t u' g^{-1} x) = \sigma_g (C'_h \sigma_g + B'_h).$$

Take $Q := \sigma_g (C'_h \sigma_g + B'_h)$. All that is left is to show that $C_3$ is a constant that only depends on $\dim(G)$ (and hence is independent of $H$ and lattice $\Gamma$).

Recall that $t'_h$ (see the proof of Theorem 23 and also the proof of Corollary 17) is the constant $t'_h = t'_h(G, H')$ as in Lemma 14. From Lemma 14, we have that $\delta_1 = \frac{1}{2} \delta_1$ depends only on $\dim(G)$ and therefore, $t'_h$ is a constant that only depends on $G$ and $H'$ (see Lemma 14).

Since $\mathcal{H}$ is finite data that only depend on $\dim(G)$ (Lemma 7),

$$C_3 = \frac{1}{\log 2} \max_{H' \in \mathcal{H}}(t'_h)$$

is a constant that only depends on $\dim(G)$. The fact that $C_3$ only depends on $\dim(G)$ will be later used to show that constant $D$ in Theorem 2 only depends on $\dim(G)$.

\[ \square \]

Remark 29. Let $\sigma_U > 1$ be a global constant such that for all $v \in \text{Lie}(G)$ and $u \in B'_h$,

$$\sigma_U^{-1} ||v|| \leq ||u v|| \leq \sigma_U ||v||$$

where $u v$ denotes the adjoint representation of $u \in G$ on $v \in \text{Lie}(G)$. For later computational purposes (see the proof of Proposition 25), we shall assume that $Q > \sigma_U$. Even if we replace $Q = \sigma_g (C'_h \sigma_g + B'_h)$ by $Q = \max \{ \sigma_g (C'_h \sigma_g + B'_h), \sigma_U \}$, Lemma 28 still holds.

We shall also assume that $C_3 \geq 1$ (this will also be used in the proof of Proposition 25). Even if we replace $C_3 = \frac{1}{\log 2} \max_{H' \in \mathcal{H}}(t'_h)$ with $C_3 = \max \{ \frac{1}{\log 2} \max_{H' \in \mathcal{H}}(t'_h), 1 \}$, Lemma 28 still holds.

Remark 30. Note that by Proposition 20, $X_{\text{cpt}} \subseteq X_\varepsilon = \{ x \in X : \text{inj}(x) \geq \varepsilon \}$, where $\varepsilon = C_2^{-1} Q^{-m}$ (where $C_2$ and $m$ as in Proposition 20): $\text{inj}(a_t u x) \geq C_2^{-1} h(x)^{-m} \geq C_2^{-1} Q^{-m} = \varepsilon_X$. 

\[ \square \]
Recall, from the preliminaries that there exists an absolute constants \( \sigma_0 > 1 \) such that for all \( w \in \text{Lie}(G) \) with \( ||w|| \leq \epsilon_X \),
\[
\sigma_0^{-1}||w|| \leq \text{dist}(x, \exp(w)x) \leq \sigma_0||w||.
\]

**Definition 31.** Let \( S \) be a subgroup with \( H \subseteq S \subseteq G \) and consider the decomposition of \( \text{Lie}(G) \) given by \( \text{Lie}(G) = \text{Lie}(S) \oplus V_S \) where \( V_S \) is \( \text{Ad}(\text{Lie}(S)) \)-invariant, but not necessarily irreducible. For each closed \( S \)-orbit \( Y = Sy \) and \( x \in X \), we define the set
\[
I_Y(x) = \{ v \in V_S \setminus \{0\} : ||v|| \leq \varepsilon_h \cdot h(x)^{-\kappa}, \exp(v)x \in Y \}
\]
where global constants \( \kappa \gg 1 \) and \( \varepsilon_h \ll 1 \) are chosen to be; \( \kappa := \max\{m, 3C_3\} \) and \( \varepsilon_h = \min\{2^k\epsilon_X, \frac{1}{2}\sigma_0^{-1}C_2^{-1}\} \) \( (m \text{ and } C_2 \text{ as in Proposition } \text{26}, C_3 \text{ as in Lemma } \text{28} \text{ and } \epsilon_X \text{ as in Remark } \text{30}) \).

**Lemma 32.** Let \( Y = Sy \) be a closed \( S \)-orbit where \( H \subseteq S \subseteq G \). For all \( x \in X \), we have that
\[
\#I_Y(x) < C_4h(x)^{dsm}\text{vol}(Y)
\]
where \( d_S \) is the dimension of \( \text{Lie}(S) \) and \( C_4 = (4C_2)^{d_S} \).

**Proof.** For any \( x \in X \) and \( v \in I_Y(x) \),
\[
\text{dist}(x, \exp(v)x) \leq \sigma_0||v|| \leq \sigma_0\varepsilon_h h(x)^{-\kappa} \leq \frac{1}{2}C_2^{-1}h(x)^{-m} \leq \frac{1}{2}\text{inj}(x).
\]
(Since \( h \geq 2 \) and \( \varepsilon_h \leq 2^k\epsilon_X \), \( ||v|| \leq \varepsilon_h h(x)^{-\kappa} \leq \epsilon_X \) and thus \( \text{dist}(x, \exp(v)x) \leq \sigma_0||v|| \). Also, since \( h \geq 1 \), \( \kappa \leq m \), and \( \varepsilon_h \leq \frac{1}{2}\sigma_0^{-1}C_2^{-1} \), we have \( \sigma_0\varepsilon_h h(x)^{-\kappa} \leq \frac{1}{2}C_2^{-1}h(x)^{-m} \).

It follows that for each \( v \in I_Y(x) \), \( \text{inj}(\exp(v)x) \geq \frac{1}{2}\text{inj}(x) \), which means that the balls
\[
(B_Y(\exp(v)x, \text{inj}(x)/4))_{v \in I_Y(x)}
\]
are disjoint from each other. Hence,
\[
\#I_Y(x) \cdot \text{vol}(B_S(e, \text{inj}(x)/4)) = \text{vol}\{\bigcup (B_Y(\exp(v)x, \text{inj}(x)/4)) : v \in I_Y(x)\}
\leq \text{vol}(B_Y(x, \text{inj}(x))) \leq \text{vol}(Y).
\]
Therefore,
\[
\#I_Y(x) \leq \text{vol}(B_S(e, \text{inj}(x)/4))^{-1} \cdot \text{vol}(Y)
\leq 4^{ds}\text{inj}(x)^{-ds} \cdot \text{vol}(Y) \leq 4^{ds}C_2^{ds}h(x)^{dsm}\text{vol}(Y).
\]
\[
\]

**Proof of Proposition 26** If \( x \in X \) is so that \( h(x) \leq Q^2 \), then by Lemma 32,
\[
\#I_Y(x) < (4C_2)^{ds}Q^{2dsm}\text{vol}(Y).
\]
Suppose that \( h(x) > Q^2 \). By Lemma 28 there exists \( u \in B_U^1 \) and \( t_x = C_3\log(h(x)) \) such that \( h(t_xux) \leq Q^2 \).

We claim that if \( v \in I_Y(x) \), then \( a_{t_x}u.v \in I_Y(a_{t_x}ux) \) and moreover, the map \( a_{t_x}u : I_Y(x) \rightarrow I_Y(a_{t_x}ux) \) which sends \( v \mapsto a_{t_x}u.v \) is injective. If \( v \in I_Y(x) \), then
\[
||a_{t_x}u.v|| \leq e^{t_x}\sigma_U||v|| = \exp(C_3\log(h(x)))\sigma_U||v|| = \sigma_Uh(x)^{C_3}||v||
\leq \sigma_Uh(x)^{C_3}\varepsilon_h h(x)^{-\kappa} \leq \sigma_U\varepsilon_h h(x)^{-2k/3} \leq \sigma_U\varepsilon_h Q^2(-2k/3) = \sigma_U\varepsilon_h Q^{-4k/3} \leq \varepsilon_h Q^{-\kappa}
\]
since $\kappa \geq 3C_3 \geq 3$, $h(x) > Q^2$, and $Q \geq \sigma_U$. On the other hand, $h(a_tux)^{−\kappa} \geq Q^{−\kappa}$ and thus,

$$||a_t.u.v|| < \epsilon_1 h(a_tux)^{−\kappa}. $$

Moreover,

$$a_t \exp(v)x = \exp(a_tu.v)a_tux \in Y$$

since $Y$ is $S$-invariant. Therefore, $a_tu.v \in I_Y(a_tux)$ and the map $v \mapsto a_tu.v$ from $I_Y(x)$ to $I_Y(a_tux)$ is injective. Consequently,

$$\#I_Y(x) \leq \#I_Y(a_tux) < (4C_2)^dS^2d_{\sigma}^m\text{vol}(Y).$$

We complete the proposition by taking $C_1 = (4C_2)^dS^2d_{\sigma}^m$ where $d_G$ denotes the dimension of Lie($G$).

We end this section with the following lemma on $\kappa$, which will be later used to show that constant $D$ in Theorem 2 is an absolute constant that only depends on dim($G$).

**Lemma 33.** Constant $\kappa$ in Definition 31 has an upper bound as a function of dim($G$).

**Proof.** Since $\kappa := \max\{m, 3C_3\}$ and $C_3$ is a constant that only depends on dim($G$) (see Lemma 28), it is enough to show that $m$ from Proposition 26 has an upperbound as a function of dim($G$).

Recall that (see the proof of Proposition 26)

$$m = \frac{r}{\delta_s} \cdot \max_{1 \leq k \leq r} \{q_k\} = \frac{r}{\frac{r}{\delta_1}} \cdot \max_{1 \leq k \leq r} \{q_k\}$$

where $\delta_1$ is as in Lemma 14 and $q_k$ are as in definition of $h_k$. By Lemma 14, $\delta_1$ is a constant that only depends on dim($G$). From reduction theory, we have that $r \leq \text{dim}(G)$. Lastly, since the root system $\{\omega_k\}$ is a data that only depends on $G$, the value of $\max_{1 \leq k \leq r} \{q_k\}$ also has an upper bound in terms of dim($G$).

7. Margulis function: Construction and estimates

In this section we construct Margulis functions that we associate to the orbit of an intermediate subgroup. The main result of this section is a Margulis inequality for the functions we consider (Theorem 38).

**Definition 34 (Margulis function).** For an intermediate subgroup $H \subseteq S \subseteq G$ and closed $S$-orbit $Y = Sy$, define $f_Y := X \rightarrow (0, \infty)$ by

$$f_Y(x) := \begin{cases} \sum_{v \in I_Y(x)} \|v\|^{−\delta_F}, & \text{if } I_Y(x) \neq \emptyset \\ h(x), & \text{otherwise} \end{cases}$$

where $\delta_F := \min\{\delta_0/2, 1/\kappa\}$, $\delta_0$ as in Lemma 8 applied to the adjoint representation of $G$ on the Lie algebra Lie($G$) and $\kappa$ as in Definition 31.

For $\lambda \geq 1$, define $F_{\lambda,Y} : X \rightarrow (0, \infty)$ by

$$F_{\lambda,Y}(x) = f_Y(x) + \lambda\text{vol}(Y)h(x).$$
Thus, we have a bound of the form
\[ g \in v \implies \text{for every } I \]
where \( C \) is an absolute constant that only depends on \( \dim(G) \). Later, this will imply that constant \( D \) in Theorem 2 is only dependent on \( \dim(G) \).

**Remark 35.** Note that by Lemma 8 and Lemma 33, \( \delta_F \) can be thought of as a constant that only depends on \( \dim(G) \). Later, this will imply that constant \( D \) in Theorem 2 is only dependent on \( \dim(G) \).

**Proposition 36 (Log continuity of \( F_{\lambda,Y} \)).** Let \( K \) be a compact subset of \( G \). Then there exists an absolute constant \( \sigma = \sigma(K) \) (only depending on \( K \) and independent of the choice of \( Y \) and \( \lambda \)) such that for all Margulis function \( F_{\lambda,Y} \), point \( x \in X \), and \( g \in K \),
\[
\sigma^{-1} F_{\lambda,Y}(x) \leq F_{\lambda,Y}(gx) \leq \sigma F_{\lambda,Y}(x).
\]

**Proof.** Recall that we denote the adjoint representation of elements \( g \in G \) and \( v \in \text{Lie}(G) \) as \( g.v \). Since \( K \subset G \) is compact, there exists \( R_K \geq 1 \) so that
\[
R_K^{-1} \| v \| \leq \| g.v \| \leq R_K \| v \|
\]
for every \( g \in K \) and \( v \in \text{Lie}(G) \). Also, by the log continuity of \( h \), there exists \( \sigma_h = \sigma_h(K) \geq 1 \) so that
\[
\sigma_h^{-1} \cdot h(x) \leq h(gx) \leq \sigma_h \cdot h(x)
\]
for every \( g \in K \) and \( x \in X \).

If \( I_Y(gx) \) is empty, then
\[
f_Y(gx) = h(gx) \leq \sigma_h \cdot h(x)
\]
Now suppose that \( I_Y(gx) \) is not empty. Set \( \varepsilon = R_K^{-1} \varepsilon h(x)^{-\kappa} \). Note that we can write
\[
f_Y(gx) = \sum_{v \in I_Y(gx), \| v \| < \varepsilon} \| v \|^{-\delta_F} + \sum_{v \in I_Y(gx), \| v \| \geq \varepsilon} \| v \|^{-\delta_F}.
\]
By Proposition 23 since \( \# I_Y(gx) \leq C_1 \text{vol}(Y) \), then we have the following bound for the second term above,
\[
\sum_{v \in I_Y(gx), \| v \| \geq \varepsilon} \| v \|^{-\delta_F} \leq C_1 \text{vol}(Y) \varepsilon^{-\delta_F} = (C_1(R_K \varepsilon_h^{-1})^\delta_F \text{vol}(Y)) \cdot h(x)^{\delta_F} \leq C_5 \text{vol}(Y) h(x)
\]
where \( C_5 := C_1(R_K \varepsilon_h^{-1})^\delta_F \). (Here we are using that \( h \geq 1 \) and \( \delta_F \leq 1/\kappa \).) If there is no \( v \in I_Y(gx) \) with \( \| v \| < \varepsilon \), then this proves the claim. If there is \( v \in I_Y(gx) \) with \( \| v \| < \varepsilon \), then
\[
\| g^{-1} v \| < R_K \varepsilon = \varepsilon h(x)^{-\kappa}.
\]
Thus, \( g^{-1}. v \in I_Y(x) \). Setting \( v' = g^{-1} v \) yields that
\[
\sum_{v \in I_Y(gx), \| v \| < \varepsilon} \| v \|^{-\delta_F} \leq \sum_{v' \in I_Y(x)} \| g.v' \|^{-\delta_F} \leq R_K^\delta F \sum_{v' \in I_Y(x)} \| v' \|^{-\delta_F} = R_K^\delta F f_Y(x).
\]
In total, we have a bound of the form
\[
f_Y(gx) \leq R_K^\delta F f_Y(x) + (C_5 \text{vol}(Y) + \sigma_h) h(x).
\]
Thus,
\[
F_{\lambda,Y}(gx) = f_Y(gx) + \lambda \text{vol}(Y) h(gx)
\leq R_K^\delta F f_Y(x) + (C_5 \text{vol}(Y) + \sigma_h) h(x) + \lambda \text{vol}(Y) \cdot \sigma_h h(x).
\]
Note that \( \text{vol}(Y) \) is bounded away from zero. That is, there exists an absolute constant \( \tau > 0 \) such that for any intermediate subgroup \( H \subseteq S \subseteq G \) and closed \( S \)-orbit \( Y = Sy \), \( \text{vol}(Y) > \tau \). (See Lemma 37 below.) Since \( \lambda \geq 1 \), we have

\[
F_{\lambda,Y}(gx) \leq R^{{\delta_F}}_K f_Y(x) + (C_5 + \sigma_h \tau^{-1} + \sigma_h \lambda) \text{vol}(Y) h(x)
\]

\[
\leq R^{{\delta_F}}_K f_Y(x) + (C_5 + \sigma_h \tau^{-1} + \sigma_h \lambda) \text{vol}(Y) h(x).
\]

Put \( \sigma := \max\{R^{{\delta_F}}_K, C_5 + \sigma_h (\tau^{-1} + 1)\} \). We note that \( \sigma \) is independent of \( Y \) and \( \lambda \).

The remaining inequality

\[
\sigma^{-1} F_{\lambda,Y}(x) \leq F_{\lambda,Y}(gx)
\]

is proved in a similar fashion. \( \square \)

**Lemma 37.** There exists an absolute constant \( \tau > 0 \) such that for any intermediate subgroup \( H \subseteq S \subseteq G \) and closed \( S \)-orbit \( Y = Sy \), \( \text{vol}(Y) > \tau \).

**Proof.** This follows from the quantitative non-divergence of Dani–Margulis [DM91]. Let \( U' \) be a 1-parameter unipotent subgroup of \( H \). By the quantitative non-divergence of the action of \( U' \) on \( X \), there exists some \( \rho > 0 \) such that \( m_Y(X \setminus X_\rho) < 0.01 \) for every closed \( S \)-orbit \( Y = Sy \) \((H \subseteq S \subseteq G \text{ and } y \in Y)\), where \( m_Y \) is the probability Haar measure on \( Y \).

Note that we have the Lie algebra decomposition \( \text{Lie}(G) = \text{Lie}(S) \oplus V_S \). Let \( \eta \prec \rho \) be so that the map \( g \mapsto gx \) is injective for all \( x \in X_\rho \) and all

\[
g \in \text{Box}(\eta) := \exp(B_\eta^{\text{Lie}(S)}) \exp(B_\eta^{V_S}).
\]

For any connected component \( C \) of \( Y \cap \text{Box}(\eta)z \) with \( z \in X_\rho \), there exists some \( v \in V_S \) such that

\[
C = C_v := \exp(B_\eta^{\text{Lie}(S)}) \exp(v)z.
\]

Since \( m_Y(X \setminus X_\rho) < 0.01 \), \( Y \cap X_\rho \neq \emptyset \) and thus,

\[
\text{vol}(Y) \geq \eta^{d_S} \geq \eta^{d_H}.
\]

\( \square \)

**Theorem 38** (Margulis Inequality for \( F_Y \)). Let \( H \subseteq S \subseteq G \) be an intermediate subgroup and \( Y = Sy \) a closed \( S \)-orbit. Let \( F_{\lambda,Y} (\lambda \geq 1) \) denote the Margulis functions associated to \( Y \). For any \( 0 < c < 1 \), there exists \( t = t_{F,c} > 0 \) such that there exists global constants \( \lambda_1 \geq 1 \) and \( E_1 > 0 \) such that the following holds for any closed orbit \( Y = Sx \) \((H \subseteq S \subseteq G)\) and its corresponding Margulis function \( F_Y := F_{\lambda_1,Y} : A_{2,t}F_Y \leq cF_Y + E_1 \text{vol}(Y) \).

**Proof.** We have that \( \text{Lie}(G) = \text{Lie}(S) \oplus V_S \) where \( V_S \) is \( \text{Ad}(H) \)-invariant, but has no \( \text{Ad}(H) \)-invariant vectors. Since \( \delta_F \leq \delta_0/2 < \delta_0 \), by the Linear Algebra Lemma (Lemma 8) there exists \( t'_c > 0 \) so that

\[
\frac{1}{m_U(B_2^{V_S})} \int_{B_2^{U}} \frac{1}{\|a_t u.v\|^{\delta_F}} dm_U(u) < \frac{c}{\|v\|^{\delta_F}}
\]

for every \( v \in V_S \) and \( t \geq t'_c \).
On the other hand, by Theorem 16 there exists \( t'_c > 0 \) such that for all \( t \geq t'_c \), there exists absolute constant \( B_t \) (depending on \( t \)) so that

\[
A_{2,t}h \leq \frac{c}{2}h + B_t.
\]

Let \( t'_c \) and \( t''_c \) be as above and take \( t = t_{F,c} = \max\{t'_c, t''_c\} \).

We first find a bound on \( A_{t}f_Y \). Then we pick a particular value for \( \lambda \) and combine the first bound with the bound from Theorem 16 to reach the desired bound on \( F_Y \).

Fix compact set \( K_t = a_tB_t^U \) and let \( R_{K_t} \geq 1 \) be a constant so that

\[
R_{K_t}^{-1}\|v\| \leq \|g.v\| \leq R_{K_t}\|v\|
\]

for every \( g \in K_t \) and \( v \in V_\mathcal{S} \).

If \( I_Y(gx) \) is empty for every \( g \in K_t \), then \( F_Y \) is just a constant multiple of the height function \( h \) and thus by (1),

\[
A_{2,t}F_Y = A_{2,t}(\lambda \text{vol}(Y)h) \leq \lambda \text{vol}(Y) \left( \frac{c}{2}h + B_t \right) \\
\leq c(\lambda \text{vol}(Y)h + \text{vol}(Y)B_t) \leq cF_Y + \lambda B_t \text{vol}(Y).
\]

Suppose that \( I_Y(gx) \) is not empty. Set \( \epsilon = R_{K_t}^{-1}\epsilon_hh(x)^{-\kappa} \). Note that we can write

\[
f_Y(gx) = \sum_{v \in I_Y(gx), \|v\| < \epsilon} \|v\|^{-\delta_F} + \sum_{v \in I_Y(gx), \|v\| \geq \epsilon} \|v\|^{-\delta_F}.
\]

and by the calculation from Proposition 36, we obtain a bound of the form,

\[
f_Y(gx) \leq \sum_{v' \in I_Y(x)} \|g.v'\|^{-\delta_F} + (C_5 \text{vol}(Y) + \sigma_h)h(x)
\]

where \( C_5 = C_1(R_{K_t}\epsilon_h^{-1})^{-\delta_F} \) and \( \sigma_h = \sigma_h(K_t) \) is the constant from the log continuity property of \( h \). Integrating over \( B_2^U \) yields

\[
\frac{1}{m_U(B_2^U)} \int_{B_2^U} f_Y(a_tux) \, dm_U(u) \\
\leq \sum_{v' \in I_Y(x)} \frac{1}{m_U(B_2^U)} \int_{B_2^U} \|a_tu.v'\|^{-\delta_F} \, dm_U(u) + (C_5 \text{vol}(Y) + \sigma_h)h(x)
\]

and since \( t \geq t'_c \) by Lemma 3,

\[
\sum_{v' \in I_Y(x)} \frac{1}{m_U(B_2^U)} \int_{B_2^U} \|a_tu.v'\|^{-\delta_F} \, dm_U(u) < c \sum_{v' \in I_Y(x)} \|v\|^{-\delta_F}.
\]

To summarize, we get the bound

\[
(2) \quad \frac{1}{m_U(B_2^U)} \int_{B_2^U} f_Y(a_tux) \, dm_U(u) \leq cf_Y(x) + (C_5 + \sigma_h^{-1})\text{vol}(Y)h(x)
\]

where \( \tau \) is as in Lemma 37.
Combining equation (1) and equation (2) together we have
\[ A_{2,t}F_Y \leq c \cdot f_Y + (C_5 + \sigma h \tau^{-1}) \text{vol}(Y) \cdot h + \lambda \text{vol}(Y) \left( \frac{c}{2} \cdot h + B_t \right) \]
\[ = c \cdot f_Y + \left( C_5 + \sigma h \tau^{-1} + \frac{\lambda c}{2} \right) \text{vol}(Y) \cdot h + \lambda B_t \text{vol}(Y). \]

Now choose \( \lambda_1 := 2(C_5 + \sigma h \tau^{-1})/c \) so that \( \left( C_5 + \sigma h \tau^{-1} + \frac{\lambda_1 c}{2} \right) = \lambda_1 c \) and we get the desired result,
\[ A_{2,t}F_Y \leq cF_Y + E_1 \text{vol}(Y) \]
where \( E_1 = \lambda_1 B_t = 2(C_5 + \sigma h \tau^{-1})B_t/c \). Note that \( \lambda_1 \geq 1 \), since \( C_5 \) and \( \sigma h \) are constants larger than 1 and \( \tau \) and \( c \) are constants smaller than 1. Also note that both \( \lambda_1 \) and \( E_1 \) are constants independent of \( Y \).

For the remainder of the paper, \( F_Y \) will refer to the Margulis function \( F_{\lambda_1,Y} \) with fixed \( \lambda_1 = 2(C_5 + \sigma h \tau^{-1})/c \) as in Theorem 38.

**Corollary 39** (Exponential decay). There exists global constants \( C_F > 0 \) and \( E_2 > 0 \) such that for any closed orbit \( Y \) and for any \( t \geq t_F \) (where \( t_F := t_{F,1/2} \) defined as in Theorem 38),
\[ (A_t F_Y)(x) \leq \frac{C_F}{2^{t_F/t_F}} F_Y(x) + E_2 \text{vol}(Y) \]

**Proof.** The result follows from Theorem 15, Proposition 36, and Theorem 38. Especially, the fact that \( C_F \) and \( E_2 \) are global constants, independent of \( Y \) follows from the fact that the log continuity constants for \( F_Y \) depends only on the compact set and is independent on the choice of \( Y \) (see Proposition 36).

The following is following result is standard, see \[\text{MO20, Lemma 7.3}] or \[\text{EMM15, Lemma 11.1}].

**Proposition 40** (Margulis function on average). Let \( H \subseteq S \subseteq G \) denote an intermediate subgroup and \( Y = Sy \) be a closed \( S \)-orbit. Let \( F_Y \) denote the associated Margulis function from Theorem 38. Let \( \mu \) be an \( A \)-ergodic \( U \)-invariant measure with \( \mu(Y) = 0 \). Then \( F_Y \in L^1(\mu) \).

**Proof.** In this proof we will drop the subscript \( Y \) in \( F_Y \) for simplicity. For \( k \in \mathbb{N} \), let \( F_k := \min(F, k) \). Take \( t \) to be \( t_F \), the constant obtained from Corollary 39.

By Moore’s ergodicity theorem, we have that the action of \( A = \{a_t : t \in \mathbb{R}\} \) is ergodic \( X \). Then, by the Birkhoff ergodic theorem, for \( \mu \)-a.e. \( x \in X \) and \( k \in \mathbb{N} \),
\[ \lim \frac{1}{N} \sum_{n=1}^{N} F_k(a_{nt}x) = \int F_k \, d\mu. \]

There exists some \( x_0 \in X \) such that for \( m_U \)-a.e. \( u \in B_1^U \),
\[ \lim \frac{1}{N} \sum_{n=1}^{N} F_k(a_{nt}ux_0) = \int F_k \, d\mu. \]
Thus, by Egoroff’s theorem, for each $k \in \mathbb{N}$ there exists a subset $E_k \subseteq B_1^U$ with $m_U(E_k) > \frac{1}{2}$ and $N_k \in \mathbb{N}$ such that for every $N > N_k$ and $u \in E_k$,

$$\frac{1}{N} \sum_{n=1}^{N} F_k(a_n t u x_0) > \frac{1}{2} \int F_k \, d\mu.$$ 

Integrate this inequality over $B_1^U$ to obtain

$$\frac{1}{N} \sum_{n=1}^{N} \int_{B_1^U} F_k(a_n t u x_0) \, dm_U(u) > \frac{1}{2} \int F_k \, d\mu.$$ 

By Corollary 39 for all $n \in \mathbb{N}$,

$$\int_{B_1^U} F_k(a_n t u x_0) \, dm_U(u) \leq \int_{B_1^U} F(a_n t u x_0) \, dm_U(u) < \frac{C}{2^n} F(x_0) + b$$

where $C = C_F$ and $b = E_2 \text{vol}(Y)$.

Choose $n_0$ so that $\frac{1}{2^n} F(x_0) \leq 1$. Then for each $n \geq n_0$ and $N > \max(N_k, k n_0)$,

$$\frac{1}{2} \int F_k \, d\mu < \frac{1}{N} \sum_{n=1}^{N} \int_{B_1^U} F_k(a_n t u x_0) \, dm_U(u)$$

$$= \frac{1}{N} \sum_{n=1}^{n_0} \int_{B_1^U} F_k(a_n t u x_0) \, dm_U(u) + \frac{1}{N} \sum_{n=n_0+1}^{N} \int_{B_1^U} F_k(a_n t u x_0) \, dm_U(u)$$

$$\leq \frac{n_0 k}{N} + \frac{1}{N} \sum_{n=n_0+1}^{N} \left( \frac{C}{2^n} F(x_0) + b \right)$$

$$\leq 1 + \frac{1}{N} \sum_{n=n_0+1}^{N} (C + b) = C + b + 1.$$ 

Thus,

$$\int F_k \, d\mu \leq 2(C + b + 1).$$

Taking $k \to \infty$ and using the monotone convergence theorem, we have $F \in L^1(\mu)$. 

\[\square\]

8. Isolation of closed orbits

In this section, we prove Theorem 4 and Theorem 5. Results in this section are analogous to Section 10 of [MO20].

Proof of Theorem 4 We shall prove the following: for any two distinct closed $S$-orbits $Y = S y$ and $Z = S z$ ($H \subseteq S \subseteq G$) of finite volume,

$$\text{dist}(Y \cap K, Z) \gg_K \text{vol}(Y)^{-1/\delta_F} \text{vol}(Z)^{-1/\delta_F}$$

where $K$ is a compact subset of $X$ and $\delta_F$ is as in Definition \[34\]. Recall that $\delta_F$ is a global constant only depending on $G$ and $H$ (and thus, independent of the choice of $\Gamma$, see Remark \[35\]).

Let $m_Y$ denote the Haar probability measure on $Y$. Since $m_Y$ is an $A$-ergodic $S$-invariant probability measure, $m_Y(A_tF_Z) = m_Y(F_Z)$. Thus, by integrating the Margulis inequality

$$A_t(F_Z) < cF_Z + E_2\text{vol}(Z)$$

($c$ is some positive constant smaller than 1) from Corollary \[39\] over $Y$, we get

$$m_Y(F_Z) \leq \frac{E_2}{1-c}\text{vol}(Z).$$

Since $K$ is compact, by log continuity of $F_Z$, there exists $\sigma = \sigma_{F_Z}(K) > 1$ such that for any $x \in X$ and $g \in B^G_\epsilon$,

$$F_Z(x) \leq \sigma F_Z(gx).$$

Recall that the log continuity coefficients for $F_Z$ is independent of closed orbit $Z$ that only depends on the compact set that $g$ belongs to (Proposition \[36\]); $\sigma$ is a global constant that only depends on $K$.

Now since $K$ is compact, $\epsilon = \epsilon_K := \min_{x \in K} \text{inj}(x) > 0$. For any point $y \in Y \cap K$,

$$f_Z(y) \leq F_Z(y) \leq \frac{1}{m_Y(B^S_\epsilon(y))} \int_{g \in B^S_\epsilon(\epsilon)} \sigma F_Z(gy)dm_Y(gy)$$

$$\leq \frac{1}{m_Y(B^S_\epsilon(y))}\sigma m_Y(F_Z) \leq \frac{\sigma}{\epsilon^d \epsilon} \cdot \frac{E_2}{1-c}\text{vol}(Y)\text{vol}(Z).$$

Lastly, we observe that $\text{dist}(y, Z)^{-\delta_F} \ll f_Z(y)$. If $I_Z(y)$ is non-empty, then

$$f_Z(y) = \sum_{v \in I_Z(y)} \|v\|^{-\delta_F} \geq \text{dist}(y, Z)^{-\delta_F}.$$ 

If $I_Z(y)$ is empty, then $\text{dist}(y, Z) > \epsilon_h h(y)^{-\kappa}$ and so

$$f_Z(y) = h(y) \geq h(y)^{\kappa \delta_F} \geq \epsilon_h^{\delta_F} \text{dist}(y, Z)^{-\delta_F}.$$ 

(Here, we are using that $h \geq 1$ and $\delta_F := \min\{\delta_0/2, 1/\kappa\} \leq 1/\kappa.$) \hfill \QED

**Proof of Theorem \[5\]** We shall prove the following: there exists a global constant $C_6 > 0$ such that for any intermediate subgroup $H \subseteq S \subseteq G$,

$$\#\{Y : Y = Sy \text{ is a closed } S\text{-orbit and } \text{vol}(Y) \leq R\} < C_6 R^{d_G/\delta_F},$$

where $d_G$ is the dimension of $\text{Lie}(G)$ and $\delta_F$ is as in Definition \[34\].

We define constants $\rho > 0$ and $\eta > 0$ as in Lemma \[37\]. Let $\rho > 0$ be a constant such that $m_Y(X \setminus X_\rho) < 0.01$ for every closed $S$-orbit $Y = Sy$ ($H \subseteq S \subseteq G$ and $y \in Y$) and let $\eta \times \rho$ be a constant so that the map $g \mapsto gx$ is injective for all $x \in X_\eta$ and all

$$g \in \text{Box}(\eta) := \exp(B^\text{Lie}(S)_\eta) \exp(B^V_S),$$

(here, $\text{Lie}(G) = \text{Lie}(S) \oplus V_S$ is the Lie algebra decomposition). Then, for any connected component $C$ of $Y \cap \text{Box}(\eta)z$ with $z \in X_\rho$, there exists some $v \in V_S$ such that

$$C = C_v := \exp(B^\text{Lie}(S)_\eta)\exp(v)z.$$
For $R > 0$, let
\[ \mathcal{Y}(R) = \{ Y : Y = Sy \text{ is closed $S$-orbit and } R/2 < \text{Vol}(Sy) \leq R \} \]

By Theorem 4 for any distinct connected components $C_v$ and $C_{v'}$ in $\mathcal{Y}(2^k) \cap \text{Box}(\eta)z$, we have that
\[ \|v - v'\| \gg \rho 2^{-2k/\delta_F}. \]

The cardinality of any $2^{-2k/\delta_F}$-separated set in $B^V_S \eta$ is up to multiplicative constant,
\[ (2^{2k/\delta_F})^{(d_G - d_S)} \cdot (d_G - d_S). \]

Since $\text{vol}(\text{Box}(\eta)) = \eta d_G$, we can cover $X_\rho$ by $M = O(\eta^{-d_G})$ many sets of the form $\text{Box}(\eta)z$. Choose such a cover \{Box(\eta)z_j : j = 1, ..., M\}.

Then,
\[ \# \mathcal{Y}(2^k) \leq 2^{-k+1} \sum_{Y \in \mathcal{Y}(2^k)} \text{vol}(Y) \ll 2^{-k+1} \sum_{j=1}^M \sum_{C_v \in \text{Box}(\eta)z_j} \text{vol}(C_v). \]

Since (1) $\text{vol}(C_v) = \eta d_s \ll 1$ for each $C_v$, (2) $\# \{ C_v \in \text{Box}(\eta)z_j \} \ll (2^{2k/\delta_F})^{(d_G - d_S)}$ for each $\text{Box}(\eta)z_j$, and (3) $M = O(1)$, we have
\[ 2^{-k+1} \sum_{j=1}^M \sum_{C_v \in \text{Box}(\eta)z_j} \text{vol}(C_v) \ll 2^{2k(d_G - d_S)/\delta_F - k + 1}. \]

Recall that $\text{vol}(Sy) \geq \eta d_s$ since the volume of the orbit needs to contain at least one connected component $C_v$ in some $\text{Box}(\eta)z_j$. Let $n_0 = [d_s \log_2(\eta)]$ and $n_R = [\log_2(R)]$.

Since
\[ \{ Sy : Sy \text{ is closed and } \text{vol}(Sy) \leq R \} \subseteq \bigcup_{k=n_0}^{n_R} \mathcal{Y}(2^k), \]

we get
\[ \# \{ Y : Y = Sy \text{ is a closed $S$-orbit and } \text{vol}(Y) \leq R \} \leq \sum_{k=n_0}^{n_R} \# \mathcal{Y}(2^k) \ll R^{d_G/\delta_F}. \]

9. Proof of the Main Theorem

In this section we prove Theorem 2.

Proof of Theorem 2 For each point $x \in X$, our choice of $T_x$ will be $T_x := h(x)^{1/\delta_F}$. Note that $h(x)$ is bounded in the compact part of $X$ and thus, $T_x$ can be chosen uniformly within a compact subset of $X$.

We fix a point $x \in X$. Let $(T, R)$ be a pair of real numbers such that $T > T_x$ and $R > 2$ and suppose that $x$ is $(R, 1/T)$-Diophantine with respect to $H$. Our final goal will be to show
that there exist absolute constants $D = D(\dim(G))$, $A = A(G/\Gamma, H)$, and $C = C(G/\Gamma, H)$ independent of $x$, $R$, and $T$, such that condition (2) of Theorem 2 holds: for all $t \geq A \log T$,

$$m_U \left( \{ u \in B_1^U : a_tux \text{ is not} (R, R^{-D}, R^{-D})-\text{Diophantine} \} \right) < CR^{-1}.$$  

Recall that $x' \in X$ is $(R, R^{-D}, R^{-D})$-Diophantine with respect to $H$ if and only if

1. $\text{inj}(x) \geq R^{-D}$ and
2. for all intermediate subgroups $H \subseteq S \subseteq G$ and all closed $S$-orbit $Y = Sx'$ with $\text{vol}(Y) \leq R$, we have $\text{dist}(x, Y) \geq R^{-D}$.

**Step 1: Recurrence to the compact part.**

First, we show that there exists $D_1 = D_1(\dim(G)) > 0$ and $A_1 = A_1(G/\Gamma, H) > 0$ such that for all $D \geq D_1$ and $A \geq A_1$, the following is true: for all $t \geq A \log T$,

$$m_U(\{ u \in B_1^U : \text{inj}(a_tux) < R^{-D} \}) \ll R^{-1}.$$

Take $A_1 = \delta_F t_h / \log 2$. Then, for any $t \geq A \log T$,

$$t \geq \log T \geq A_1 \log T_x = A_1 \log(h(x)^{1/\delta_F}) \geq A_1/\delta_F \cdot \log 2 = t_h$$

and so by Corollary 17

$$(A_t h)(x) = \int_{B_1^U} h(a_tux) \, d\mu_U(u) \leq \frac{C_h}{2^{t_h}} h(x) + B_h$$

$$\leq \frac{C_h}{2^{A_1 \log T / t_h}} h(x) + B_h$$

$$\leq \frac{C_h}{2^{A_1 \log T_x / t_h}} h(x) + B_h$$

$$= \frac{C_h}{2^{(A_1/\delta_F t_h) \log(h(x))}} h(x) + B_h = C_h + B_h.$$  

For $x' \in X$, if $\text{inj}(x') \leq R^{-D}$, then by Proposition 26

$$h(x') \geq C_2^{-1/m} \cdot \text{inj}(x)^{-1/m} > C_2^{-1/m} \cdot R^{D/m}.$$  

Applying this observation to points $a_tux$ and using Chebyshev’s theorem we obtain,

$$m_U(\{ u \in B_1^U : \text{inj}(a_tux) \leq R^{-D} \})$$

$$\leq m_U(\{ u \in B_1^U : h(a_tux) > C_2^{-1/m} \cdot R^{D/m} \})$$

$$< C_2^{1/m} R^{-D/m} \cdot (A_t h)(x)$$

$$\leq C_7 R^{-D/m}$$

where $C_7 := C_2^{1/m}(C_h + B_h)$ is a global constant. Recall that $m$ is a constant that depends only on $\dim(G)$ (see proof of Lemma 35). Thus, by taking $D_1 = m$, we get the desired result.

**Step 2: Avoidance principle.** Let $H \subseteq S \subseteq G$ be an intermediate orbit. First we shall fix a single closed $S$-orbit $Y = Sy$ with volume less that $R$, and show that there exists
$D_2 = D_2(\dim(G)) > 0$ and $A_2 = A_2(G/T, H) > 0$ such that for all $D \geq D_2$ and $A \geq A_2$, the following is true: for all $t \geq A\log T$,

$$m_U\{u \in B^U_1 : \text{dist}(a_tux, Y) < R^{-D}\} \ll R^{-1}.$$  

Then, we will use Lemma 6 and Corollary 5 to piece together the results for different choices of $Y$: we show that there exists $D_3 = D_3(\dim(G)) > D_2$ such that for all $D \geq D_3$ and $A \geq A_2$, for all $t \geq A\log T$,

$$m_U\{u \in B^U_1 : \text{dist}(a_tux, Y) < R^{-D} \text{ for some } Y \in O_R\} \ll R^{-1}$$

(here, $O_R = \{Y = Sy : H \subseteq S \subseteq G, Y \text{ is closed, vol}(Y) < R\}$ is the set of all closed orbits of volume less than $R$).

**Step 2.1: Avoiding single closed orbit.**

Construct Margulis functions $f_Y$ and $F_Y$ with respect to $Y$ as in Section 6. If $I_Y(x)$ is empty, then

$$f_Y(x) = h(x) \leq T^\delta_F \leq T^\delta_F.$$

Otherwise, we have

$$f_Y(x) = \sum_{v \in I_Y(x)} \|v\|^{\delta_F} \leq \text{dist}(x, Y)^{-\delta_F} \#I_Y(x).$$

Since $x$ is $(R, 1/T)$–Diophantine with respect to $H$, $x$ is $(R, 1/T)$–Diophantine with respect to $Y$ and thus, $\text{dist}(x, Y) \geq 1/T$. Combining with Proposition 25, we have

$$\text{dist}(x, Y)^{-\delta_F} \#I_Y(x) \leq \text{dist}(x, Y)^{-\delta_F} \cdot C_1 \text{vol}(Y) \leq C_1 T^\delta_F R.$$

Thus, using that $h(x) \leq T^\delta_F$, we conclude

$$F_Y(x) = f_Y(x) + \lambda_1 \text{vol}(Y)h(x) \leq C_1 T^\delta_F R + \lambda_1 T^\delta_F R = C_8 T^\delta_F R$$

where $C_8 := C_1 + \lambda_1$ is a global constant independent of $Y$.

Now take $A_2 = \delta_F t_F / \log 2$ (here, $t_F$ is as in Theorem 38). If $t \geq A\log T$, then

$$t \geq A\log T \geq A_2 \log T \geq A_2 \log h(x)^{1/\delta_F} \geq A_2 / \delta_F \cdot \log 2 = t_F$$

and so by Corollary 39 we have

$$A_1(F_Y(x)) \leq \frac{C_F}{2^{t_F}} F_Y(x) + E_2 \text{vol}(Y) \leq \frac{C_F}{2^{A_2 \log T / t_F}} \cdot C_8 T^\delta_F R + E_2 R \leq \left(\frac{1}{2A_2 / t_F}\right)^{\log T} \cdot C_F C_8 T^\delta_F R + E_2 R = (C_F C_8 + E_2)R.$$

If $\text{dist}(x', Y) < R^{-D}$ for some $x' \in X$, then either there exists $v \in I_Y(x')$ with $\|v\| < R^{-D}$ or $\varepsilon_h h(x')^{-\kappa} < R^{-D}$. In either case, we have that

$$F_Y(x') = f_Y(x') + \lambda_1 \text{vol}(Y)h(x') > \min\{R^{D\delta_F}, \lambda_1 \varepsilon_h^{1/\kappa} R^{D/\kappa}\} \geq C_9 \cdot R^{D\delta_F}$$
where $C_9 := \lambda_1 r\varepsilon_1^{1/\kappa}$ is a global constant independent of $Y$. (Here, we are again using that $\delta_F := \min\{\delta_0/2, 1/\kappa\}$ and thus $\delta_F \leq 1/\kappa$.)

Applying this observation to points $a_tux$ and using Chebyshev’s theorem we obtain,

\[
m_U(\{u \in B^U_Y : \text{dist}(a_tux, Y) < R^{-D}\}) \\
\leq m_U(\{u \in B^U_Y : F_Y(a_tux) < C_9 \cdot R^D\delta_F\}) \\
< C_9^{-1} R^{-D\delta_F} \cdot A_t(F_Y(x)) \\
< C_{10} R^{-(D\delta_F - 1)}
\]

where $C_{10} := (C_F C_8 + D_2)/C_9$ is another global constant independent of $Y$. Recall that $\delta_F$ is a global constant that depends only on $G$ and $H$ (see Remark 35). Take $D_2 = 2/\delta_F$.

**Step 2.2: Avoiding all closed orbits of small volume.**

By Lemma 6, the number of intermediate subgroups $H \subseteq S \subseteq G$ is finite; we shall denote this number as $N(G, H)$. By Theorem 5 for each fixed $S$, the number of closed $S$-orbits $Y$ with $\text{vol}(Y) < R$ is bounded by $C_6 R^{d_G/\delta_F}$. Therefore, the cardinality of the set

\[
O_R = \{Y = S y : H \subseteq S \subseteq G, Y \text{ is closed}, \text{vol}(Y) < R\}
\]

is bounded above by $N(G, H) \cdot C_6 R^{d_G/\delta_F}$ and so

\[
m_U(\{u \in B^U_Y : \text{dist}(a_tux, Y) < R^{-D} \text{ for some } Y \in O_R\}) \\
\leq \sum_{Y \in O_R} m_U(\{u \in B^U_Y : \text{dist}(a_tux, Y) < R^{-D}\}) \\
\leq \sum_{Y \in O_R} C_9 R^{-(D\delta_F - 1)} \\
< N(G, H) \cdot C_6 R^{d_G/\delta_F} \cdot C_{10} R^{-(D\delta_F - 1)} \\
= N(G, H) C_6 C_{10} \cdot R^{-(D\delta_F + d_G/\delta_F + 1)}.
\]

Take $D_3 = D_3(\dim(G)) := (d_G/\delta_F + 2)/\delta_F$ (so that $-D\delta_F + d_G/\delta_F + 1 \geq -1$ for any $D \geq D_3$) and we get the desired avoidance principle: for all $D \geq D_3$ and $A \geq A_2$, for all $t \geq A \log T$,

\[
m_U(\{u \in B^U_Y : \text{dist}(a_tux, Y) < R^{-D} \text{ for some } Y \in O_R\}) < C_{11} R^{-1}
\]

(here, $C_{11} := N(G, H) C_6 C_{10}$ is another global constant).

Now combine the results of Step 1 and Step 2. Take $D = D(\dim(G)) := \max\{D_1, D_3\}$, $A = A(G/\Gamma, H) := \max\{A_1, A_2\}$, and $C = C(G/\Gamma, H) := C_7 + C_{11}$, and we get

\[
m_U(\{u \in B^U_Y : a_tux \text{ is not } (R, R^{-D}, R^{-D})\text{-Diophantine with respect to } H\}) \\
\leq m_U(\{u \in B^U_Y : \text{inj}(a_tux) < R^{-D}\}) \\
+ m_U(\{u \in B^U_Y : \text{dist}(a_tux, Y) < R^{-D} \text{ for some } Y \in O_R\}) \\
< C_7 R^{-1} + C_{11} R^{-1} = CR^{-1}.
\]
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