Semiunital Semimonoidal Categories
(Applications to Semirings and Semicorings)

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Abstract

The category $A\mathcal{S}_A$ of bisemimodules over a semialgebra $A$, with the so called Taka-hashi’s tensor product $- \boxtimes_A -$, is semimonoidal but not monoidal. Although not a unit in $A\mathcal{S}_A$, the base semialgebra $A$ has properties of a semiunit (in a sense which we clarify in this note). Motivated by this interesting example, we investigate semiunital semimonoidal categories $(\mathcal{V}, \bullet, I)$ as a framework for studying notions like semimonoids (semicomonoids) as well as a notion of monads (comonads) which we call $J$-monads ($J$-comonads) with respect to the endo-functor $J := I \bullet - \simeq - \bullet I : \mathcal{V} \to \mathcal{V}$. This motivated also introducing a more generalized notion of monads (comonads) in arbitrary categories with respect to arbitrary endo-functors. Applications to the semiunital semimonoidal variety $(A\mathcal{S}_A, I_A, A)$ provide us with examples of semiunital $A$-semirings (semicounital $A$-emicorings) and semiunitary semimodules (semicounitary semicomodules) which extend the classical notions of unital rings (counital corings) and unitary modules (counitary comodules).

1 Introduction

A semiring is, roughly speaking, a ring not necessarily with subtraction. The first natural example of a semiring is the set $\mathbb{N}_0$ of non-negative integers. Other examples include the set $\text{Ideal}(R)$ of (two-sided) ideals of any associative ring $R$ and distributive complete lattices. A semimodule is, roughly speaking, a module not necessarily with subtraction. The category of Abelian groups is nothing but the category of modules over $\mathbb{Z}$; similarly, the category of commutative monoids is nothing but the category of semimodules over $\mathbb{N}_0$.

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Semirings were studied by many algebraists beginning with Dedekind [Ded1894]. Since the sixties of the last century, they were shown to have significant applications in several areas as Automata Theory, Optimization Theory, Tropical Geometry and Idempotent Analysis (for more, see [Gol1999a]). Recently, Durov [Dur2007] demonstrated that semirings are in one-to-one correspondence with the algebraic additive monads on the category Set of sets. The theory of semimodules over semirings was developed by many authors including Takahashi, Patchkoria and Katsov (e.g. [Tak1981], [Tak1982a], [Pat2006], [Kat1997]).

A strong connection between corings [Swe1975] over a ring $A$ (coalgebras in the monoidal category $A \text{Mod}_A$ of bimodules over $A$) and their comodules on one side and comonads induced by the tensor product $- \otimes_A -$ and their comodules on the other side has been realized by several authors (e.g. [BW2003]). Moreover, the theory of monads and comonads in (autonomous) monoidal categories received increasing attention in the last decade and extensions to arbitrary categories were carried out in several recent papers (e.g. [BBW2009]).

Using the so called Takahashi’s tensor-like product $- \boxtimes_A -$ of semimodules over an associative semiring $A$ [Tak1982a], notions of semiunital semirings and semicounital semicorings were introduced by the author in 2008. However, these could not be realized as monoids (comonoids) in the category $A \text{S}_A$ of $(A, A)$-bisemimodules. This is mainly due to the fact that the category $(A \text{S}_A, \boxtimes_A, A)$ is not monoidal in general (an alternative tensor product $- \otimes_A -$ was recalled by Katsov in [Kat1997]; in fact $(A \text{S}_A, \otimes_A, A)$ is monoidal. For the relation between $- \otimes_A -$ and $- \boxtimes_A -$ see [Abu]). Motivated by the desire to fix this defect, we introduce and investigate a notion of semiunital semimonoidal categories with prototype $(A \text{S}_A, \boxtimes_A, A)$ and investigate semimonoids (semicomonoids) in such categories as well as their categories of semimodules (semicomodules). In particular, we realize our semiunital $A$-semirings (semicounital $A$-semicorings) as semimonoids (semicomonoids) in $(A \text{S}_A, \boxtimes_A, A)$. Moreover, we introduce and study $\mathbb{J}$-monads ($\mathbb{J}$-comonads) in any arbitrary category $\mathfrak{A}$, where $\mathbb{J} : \mathfrak{A} \rightarrow \mathfrak{A}$ is an endo-functor, and apply them to semiunital semimonoidal categories in general and to $A \text{S}_A$ in particular. Our results extend recent ones on monoids (comonoids) in monoidal categories as well as monads (comonads) in arbitrary categories to semimonoids (semicomonoids) in semiunital semimonoidal categories as well as $\mathbb{J}$-monads ($\mathbb{J}$-comonads) in arbitrary categories.

Throughout, $\mathbb{I}$ denotes the identity endo-functor on the category under consideration. The paper is organized as follows. After this introduction, we present in Section 2 our (generalized) notion of $\mathbb{J}$-monads and $\mathbb{J}$-comonads in arbitrary categories. In Section 3, we introduce and investigate semiunits in semimonoidal categories. In Section 4, we introduce semimonoids (semicomonoids) in semiunital semimonoidal categories as well as their categories of semimodules (semicomodules). Moreover, we prove two reconstruction results, namely Theorems 4.8 and 4.17. In Section 5, we consider the semiunital semimonoidal category (variety) of bisemimodules $A \text{S}_A$ over a semialgebra $A$ which provides us with a rich source of concrete examples for applying our results. As mentioned above, these concrete examples were the main motivation behind introducing all the abstract notions in this paper. Further investigations of $\mathbb{J}$-bimonads and Hopf $\mathbb{J}$-monads as well as bisemimonoids and Hopf semimonoids in semiunital semimonoidal categories will be the subject of a forthcoming paper.
2 Monads and Comonads

Recall first the so called Godement product of natural transformations between functors:

2.1. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be any categories. Any natural transformations \( \psi : F \to G \) and \( \phi : F' \to G' \) of functors \( \mathcal{A} \xrightarrow{F,G} \mathcal{B} \xrightarrow{F',G'} \mathcal{C} \) can be multiplied using the Godement product to yield a natural transformation \( \phi \psi : F'F \to G'G \), where

\[
\phi_{G(X)} \circ F'_{\psi X} = (\phi \psi)_X = G'_{\psi X} \circ \phi_{F(X)} \quad \text{for every } X \in \mathcal{A}.
\]

Moreover, if \( \mathcal{A} \xrightarrow{H,H'} \mathcal{B} \xrightarrow{H',K} \mathcal{C} \) are functors and \( \delta : G \to H, \theta : G' \to H' \) are natural transformations, then the following interchange law holds

\[
(\delta \circ \psi)(\theta \circ \phi) = (\theta \delta) \circ (\phi \psi).
\]

2.2. Let \( \mathcal{A} \) and \( \mathcal{B} \) be categories, \( L : \mathcal{A} \to \mathcal{B}, R : \mathcal{B} \to \mathcal{A} \) be functors and \( \mathcal{J} : \mathcal{A} \to \mathcal{A}, \mathcal{K} : \mathcal{B} \to \mathcal{B} \) be endo-functors such that \( R \mathcal{K} \simeq \mathcal{J}R \) and \( L \mathcal{J} \simeq \mathcal{K}L \). We say that \( (L,R) \) is a \((\mathcal{J}, \mathcal{K})\)-adjoint pair iff we have natural isomorphisms in \( X \in \mathcal{A} \) and \( Y \in \mathcal{B} \):

\[
\mathcal{B}(L(J(X)), K(Y)) \simeq \mathcal{A}(J(X), R \mathcal{K}(Y)).
\]

For the special case \( \mathcal{J} = \mathbb{I}_\mathcal{A} \) and \( \mathcal{K} = \mathbb{I}_\mathcal{B} \), we recover the classical notion of adjoint pairs.

Till the end of this section, \( \mathcal{A} \) is an arbitrary category.

2.3. Let \( T : \mathcal{A} \to \mathcal{A} \) be an endo-functor. An object \( X \in \mathrm{Obj}(\mathcal{A}) \) is said to have a \( T \)-action or to be a \( T \)-act iff there is a morphism \( \tau_X : T(X) \to X \) in \( \mathcal{A} \). For two objects \( X, X' \) with \( T \)-actions, we say that a morphism \( \varphi : X \to X' \) in \( \mathcal{A} \) is a morphism of \( T \)-acts iff the following diagram is commutative

\[
\begin{array}{ccc}
T(X) & \xrightarrow{e_X} & X \\
\downarrow{T(\varphi)} & & \downarrow{\varphi} \\
T(X') & \xrightarrow{e_{X'}} & X'
\end{array}
\]

The category of \( T \)-acts is denoted by \( \mathbf{Act}_T \). Dually, one can define the category \( \mathbf{Coact}_T \) of \( T \)-coacts.

Remark 2.4. The objects of \( \mathbf{Coact}_F \), where \( F : \mathbf{Set} \to \mathbf{Set} \) is any endo-functor, play an important role in logic and theoretical computer science. They are called \( F \)-systems (e.g. \cite{Rut2000}). Some references call these \( F \)-coalgebras (e.g. \cite{Gum1999}). For us, coalgebras are always coassociative and counital unless something else is explicitly specified.
\section{\texttt{J}-Monads}

\subsection{2.5.}
Let \( \texttt{J} : \texttt{A} \longrightarrow \texttt{A} \) be an endo-functor. With a \textit{\texttt{J}-monad} on \( \texttt{A} \) we mean a datum \((\texttt{M}, \mu, \omega, \nu; \texttt{J})\) consisting of an endo-functor \( \texttt{M} : \texttt{A} \longrightarrow \texttt{A} \) associated with natural transformations
\[
\mu : \texttt{MM} \longrightarrow \texttt{M}, \quad \omega : \mathbb{I} \longrightarrow \texttt{J} \quad \text{and} \quad \nu : \texttt{J} \longrightarrow \texttt{M}
\]
such that the following diagrams are commutative
\[
\begin{array}{ccc}
\texttt{MM} & \xrightarrow{\mu_{\texttt{M}}} & \texttt{MM} \\
\downarrow{\mu_{\texttt{M}}} & & \downarrow{\mu} \\
\texttt{M} & & \texttt{M}
\end{array}
\quad \quad 
\begin{array}{ccc}
\texttt{MM} & \xrightarrow{\mu} & \texttt{IM} \\
\downarrow{\nu_{\texttt{M}}} & & \downarrow{\omega_{\texttt{M}}} \\
\texttt{JM} & & \texttt{JM}
\end{array}
\quad \quad 
\begin{array}{ccc}
\texttt{MM} & \xrightarrow{\mu} & \texttt{MI} \\
\downarrow{\nu} & & \downarrow{\omega_{\texttt{M}}} \\
\texttt{MJ} & & \texttt{MJ}
\end{array}
\]
i.e. for every \( X \in \texttt{A} \) we have
\[
\mu_X \circ \texttt{M}(\mu_X) = \mu_X \circ \mu(\texttt{M}(X)) = \texttt{M}(\omega_X \circ \mu_X) = \texttt{I}_{\texttt{MM}(X)}.
\]

\subsection{2.6.}
With \( \texttt{JMonad}_\texttt{A} \) we denote the category whose objects are \( \texttt{J}\)-monads, where \( \texttt{J} \) runs over the class of endo-functors on \( \texttt{A} \). A morphism \((\phi; \xi) : (\texttt{M}, \mu, \omega, \nu; \texttt{J}) \longrightarrow (\texttt{M}', \mu', \omega', \nu'; \texttt{J}')\) in this category consists of natural transformations \( \phi : \texttt{M} \longrightarrow \texttt{M}' \) and \( \xi : \texttt{J} \longrightarrow \texttt{J}' \) such that the following diagrams are commutative
\[
\begin{array}{ccc}
\texttt{MM} & \xrightarrow{\mu} & \texttt{M} \\
\downarrow{\phi} & & \downarrow{\phi} \\
\texttt{MM}' & \xrightarrow{\mu'} & \texttt{M}'
\end{array}
\quad \quad 
\begin{array}{ccc}
\texttt{J} & \xrightarrow{\nu} & \texttt{M} \\
\downarrow{\xi} & & \downarrow{\xi} \\
\texttt{J}' & \xrightarrow{\nu'} & \texttt{M}'
\end{array}
\]
i.e. for every \( X \in \texttt{A} \) we have
\[
\phi_X \circ \mu_X = \mu'_X \circ \phi(\texttt{M}(X)) = \texttt{M}(\phi_X) \circ \omega_X \quad \text{and} \quad \phi_X \circ \nu_X = \nu'_X \circ \xi_X.
\]

For a fixed endo-functor \( \texttt{J} : \texttt{A} \longrightarrow \texttt{A} \), we denote by \( \texttt{J-Monad}_\texttt{A} \) the subcategory of \( \texttt{JMonad}_\texttt{A} \) of \( \texttt{J}\)-monads on \( \texttt{A} \) with \( \omega \) the identity natural transformation. In the special case \( \texttt{J} = \mathbb{I}_\texttt{A} \) and \( \omega \) is the identity natural transformation, we drop the prefix and recover the classical notion of \textit{monads} on \( \texttt{A} \).

\textit{Remark 2.7.} As we saw above, a \( \texttt{J}\)-monad \((\texttt{M}, \mu, \omega, \nu; \texttt{J})\) is a generalized notion of a monad. However, it can also be seen as just a monad \((\texttt{M}, \mu, \eta)\) whose unit \( \eta := \mathbb{I} \xrightarrow{\omega} \texttt{J} \xrightarrow{\nu} \texttt{M} \) factorizes through \( \texttt{J} \). Having this in mind, a morphism \((\phi; \xi) : (\texttt{M}, \mu, \omega, \nu; \texttt{J}) \longrightarrow (\texttt{M}', \mu', \omega', \nu'; \texttt{J}')\) in \( \texttt{JMonad}_\texttt{A} \) is just a morphism of monads which is compatible with the factorizations of the units through \( \texttt{J} \) and \( \texttt{J}' \).
2.8. Let \((\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \text{JMonad}_A\). An \((\mathbb{M}; \mathbb{J})\)-module is an object \(X \in \text{Obj}(\mathfrak{A})\) with a morphism \(\varrho_X : \mathbb{M}(X) \rightarrow X\) such that the following diagrams are commutative

\[
\begin{array}{c}
\mathbb{M}(X) \xrightarrow{\varrho_X} X \\
\downarrow \mu_X \quad \downarrow \varrho_X \\
\mathbb{M}(X) \xrightarrow{\mathbb{M}^{(\varrho_X)}} \mathbb{M}(X)
\end{array}
\quad
\begin{array}{c}
\mathbb{M}(X) \xrightarrow{\varrho_X} X \\
\downarrow \nu_X \quad \downarrow \omega_X \\
\mathbb{J}(X) \xrightarrow{\mathbb{J}(\varrho_X)} \mathbb{J}(X)
\end{array}
\]

The category of \((\mathbb{M}; \mathbb{J})\)-modules and morphisms those of \(\mathbb{M}\)-acts is denoted by \(\mathfrak{A}_{(\mathbb{M}; \mathbb{J})}\). In case \(\mathbb{J} \simeq \mathbb{I}_\mathfrak{A}\) and \(\omega\) is the identity natural transformation, we recover the category of \(\mathbb{M}\)-modules of the monad \(\mathbb{M}\).

2.9. Let \((\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \text{JMonad}_A\). For every \(X \in \text{Obj}(\mathfrak{A})\), \(\mathbb{M}(X)\) is an \((\mathbb{M}; \mathbb{J})\)-module through

\[\varrho_{\mathbb{M}(X)} : \mathbb{M}(\mathbb{M}(X)) \xrightarrow{\mu_X} \mathbb{M}(X)\]

Such modules are called free \((\mathbb{M}; \mathbb{J})\)-modules and we have the so called free functor

\[\mathcal{F}_{(\mathbb{M}; \mathbb{J})} : \mathfrak{A} \rightarrow \mathfrak{A}_{(\mathbb{M}; \mathbb{J})}, \ X \mapsto \mathbb{M}(X)\]

The full subcategory of free \((\mathbb{M}; \mathbb{J})\)-modules is called the Kleisli category and is denoted by \(\tilde{\mathfrak{A}}_{(\mathbb{M}; \mathbb{J})}\).

Remark 2.10. Let \((\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \text{JMonad}_A\) with \(\mathbb{M}\mathbb{J} \simeq \mathbb{J}\mathbb{M}\). If \(X\) is an \((\mathbb{M}; \mathbb{J})\)-module, then \(\mathbb{J}(X)\) is also an \((\mathbb{M}; \mathbb{J})\)-module through

\[\varrho_{\mathbb{J}(X)} : \mathbb{M}(\mathbb{J}(X)) \xrightarrow{\mathbb{J}(\mathbb{M}X)} \mathbb{J}(X)\]

Moreover, if \(Y = \mathbb{M}(X)\) is a free \((\mathbb{M}; \mathbb{J})\)-module, then \(\mathbb{J}(Y) = \mathbb{J}\mathbb{M}(X) \simeq \mathbb{M}\mathbb{J}(X)\) is also a free \((\mathbb{M}; \mathbb{J})\)-module. One can easily see that \(\mathbb{J}\) can be lifted to endo-functors \(\mathbb{J}' : \mathfrak{A}_{(\mathbb{M}; \mathbb{J})} \rightarrow \mathfrak{A}_{(\mathbb{M}; \mathbb{J})}\) and \(\tilde{\mathbb{J}} : \tilde{\mathfrak{A}}_{(\mathbb{M}; \mathbb{J})} \rightarrow \tilde{\mathfrak{A}}_{(\mathbb{M}; \mathbb{J})}\).

2.11. Let \((\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \text{JMonad}_A\) and assume that \(\mathbb{M}\mathbb{J} \simeq \mathbb{J}\mathbb{M}\). Then we have a natural isomorphism for every \(X \in \mathfrak{A}\) and \(Y \in \mathfrak{A}_{(\mathbb{M}; \mathbb{J})} : \mathfrak{A}_{(\mathbb{M}; \mathbb{J})}(\mathcal{F}_{(\mathbb{M}; \mathbb{J})}(X), \mathbb{J}(Y)) \simeq \mathfrak{A}(X, \mathbb{J}(Y)), \ f \mapsto f \circ (\nu \circ \omega)_X\)

with inverse \(g \mapsto \varrho_{\mathbb{J}(Y)} \circ \mathcal{F}_{(\mathbb{M}; \mathbb{J})}(g)\). Consider the forgetful functor \(U : \mathfrak{A}_{(\mathbb{M}; \mathbb{J})} \rightarrow \mathfrak{A}\) and the endo-functor \(\mathbb{J}' : \mathfrak{A}_{(\mathbb{M}; \mathbb{J})} \rightarrow \mathfrak{A}_{(\mathbb{M}; \mathbb{J})}\) (see Remark[2.10]). Then we have a natural isomorphism

\[\mathfrak{A}_{(\mathbb{M}; \mathbb{J})}((\mathcal{F}_{(\mathbb{M}; \mathbb{J})}(\mathbb{J}(X)), \mathbb{J}'(Y)) \simeq \mathfrak{A}(\mathbb{J}(X), U(\mathbb{J}'(Y)));
\]
i.e. \((\mathcal{F}_{(\mathbb{M}; \mathbb{J})}(-), U)\) is a \((\mathbb{J}, \mathbb{J}')\)-adjoint pair.
\(J\)-Comonads

2.12. Let \(J\) be an endo-functor on \(\mathfrak{A}\). With a \(J\)-\textit{comonad} on \(\mathfrak{A}\) we mean a datum \((C, \Delta, \omega, \theta)\) consisting of an endo-functor \(C : \mathfrak{A} \rightarrow \mathfrak{A}\) associated with natural transformations 
\[
\Delta : C \rightarrow CC, \quad \omega : I \rightarrow J \quad \text{and} \quad \theta : C \rightarrow J
\]
such that the following diagrams are commutative

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & CC \\
\downarrow & & \downarrow \\
CC & \xrightarrow{c\Delta} & CCC
\end{array}
\quad \begin{array}{ccc}
IC & \xrightarrow{\Delta} & CC \\
\downarrow & & \downarrow \\
JC & \xrightarrow{c\Delta} & JC
\end{array}
\quad \begin{array}{ccc}
II & \xrightarrow{\Delta} & CC \\
\downarrow & & \downarrow \\
JI & \xrightarrow{c\Delta} & JI
\end{array}
\]

i.e. for every \(X \in \mathfrak{A}\) we have
\[
\Delta_{C(X)} \circ \Delta_X = C(\Delta_X) \circ \Delta_X, \quad \theta_{C(X)} \circ \Delta_X = \omega_{C(X)} \quad \text{and} \quad C(\theta_X) \circ \Delta_X = C(\omega_X).
\]

2.13. By \(J\text{-Comonad}_\mathfrak{A}\) we denote the category whose objects are \(J\)-comonads, where \(J\) runs over the class of endo-functors on \(\mathfrak{A}\). A morphism \((\psi; \xi) : (C, \Delta, \omega, \theta; J) \rightarrow (C', \Delta', \omega', \theta'; J')\) in this category consists of natural transformations \(\psi : C \rightarrow C'\) and \(\xi : J \rightarrow J'\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & CC \\
\downarrow & & \downarrow \\
C' & \xrightarrow{\Delta'} & C'C'
\end{array}
\quad \begin{array}{ccc}
C & \xrightarrow{\theta} & J \\
\downarrow & & \downarrow \\
C' & \xrightarrow{\theta'} & J'
\end{array}
\]

i.e. for every \(X \in \mathfrak{A}\) we have
\[
\psi_{C(X)} \circ C(\psi_X) \circ \Delta_X = \Delta'_X \circ \psi_X \quad \text{and} \quad \xi_X \circ \theta_X = \theta'_X \circ \psi_X.
\]

For a fixed endo-functor \(J : \mathfrak{A} \rightarrow \mathfrak{A}\), we denote by \(J\text{-Comonad}_\mathfrak{A}\) the subcategory of \(J\)-comonads on \(\mathfrak{A}\) with \(\omega\) the identity transformation. When we drop the prefix, we have the special case \(J = I_\mathfrak{A}\) and \(\omega\) is the identity natural transformation and recover the notion of comonads on \(\mathfrak{A}\).

\textbf{Remark 2.14.} \(J\)-Comonads are not fully dual to \(J\)-monads. Recall from Remark 2.7 that a \(J\)-monad can be seen as a monad whose unit factorizes through \(J\). On the other hand, \(J\)-comonads cannot be seen as a special type of comonads. The lack of duality is because not all arrows are reversed; the arrow \(\omega : I \rightarrow J\) is assumed for both. Notice that keeping this arrow is suggested by the concrete example in Section 5.
2.15. Let \((\mathcal{C}, \Delta, \omega, \theta; \mathbb{J}) \in \textbf{JComonad}_{\mathbb{A}}\). A \((\mathcal{C}; \mathbb{J})\)-comodule is an object \(X \in \text{Obj}(\mathbb{A})\) along with a morphism \(\varrho^{X} : X \rightarrow \mathcal{C}(X)\) in \(\mathbb{A}\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\varrho^{X}} & \mathcal{C}(X) \\
\downarrow{\varrho^{X}} & & \downarrow{\mathcal{C}(\varrho^{X})} \\
\mathcal{C}(X) & \xrightarrow{\Delta_X} & \mathcal{C}(X)
\end{array}
\quad \quad
\begin{array}{ccc}
X & \xrightarrow{\varrho^{X}} & \mathcal{C}(X) \\
\downarrow{\varrho^{X}} & & \downarrow{\omega_{X}} \\
\mathcal{C}(X) & \xrightarrow{\Delta_X} & \mathcal{C}(X)
\end{array}
\]

The category of \((\mathcal{C}; \mathbb{J})\)-comodules and morphisms those of \(\mathcal{C}\)-coacts is denoted by \(\mathbb{A}^{(\mathcal{C}; \mathbb{J})}\). In case \(\mathbb{J} = \mathbb{I}_{\mathbb{A}}\) and \(\omega\) is the identity natural transformation, we recover the category of \(\mathcal{C}\)-comodules for the comonad \(\mathcal{C}\).

2.16. Let \((\mathcal{C}, \Delta, \epsilon; \mathbb{J}) \in \textbf{JComonad}_{\mathbb{A}}\). For every \(X \in \text{Obj}(\mathbb{A})\), \(\mathcal{C}(X)\) has a canonical structure of a \((\mathcal{C}; \mathbb{J})\)-comodule through

\[\varrho^{C(X)} : \mathcal{C}(X) \xrightarrow{\Delta_X} \mathcal{C}(X)\]

Such comodules are called cofree \((\mathcal{C}; \mathbb{J})\)-comodules and we have the so called cofree functor

\[\mathcal{F}^{\mathcal{C}} : \mathbb{A} \rightarrow \mathbb{A}^{(\mathcal{C}; \mathbb{J})}, \quad X \mapsto \mathcal{C}(X)\]

The full subcategory of cofree \((\mathcal{C}; \mathbb{J})\)-comodules is called the Kleisli category of \(\mathcal{C}\) and is denoted by \(\mathbb{A}^{(\mathcal{C}; \mathbb{J})}\).

Remark 2.17. Let \((\mathcal{C}, \Delta, \omega, \theta; \mathbb{J}) \in \textbf{JComonad}_{\mathbb{A}}\) with \(\mathbb{J}C \simeq \mathbb{C}\mathbb{J}\). If \(X\) is a \((\mathcal{C}; \mathbb{J})\)-comodule, then \(\mathbb{J}(X)\) is also a \((\mathcal{C}; \mathbb{J})\)-comodule through

\[\varrho^{\mathbb{J}(X)} : \mathbb{J}(X) \xrightarrow{\mathbb{J} \varrho^{X}} \mathbb{J}(\mathcal{C}(X)) \simeq \mathbb{C}(\mathbb{J}(X))\]

If \(Y = \mathbb{C}(X)\) is a cofree \((\mathcal{C}; \mathbb{J})\)-comodule, then \(\mathbb{J}(Y) = \mathbb{J}(\mathbb{C}(X)) \simeq \mathbb{C}(\mathbb{J}(X))\) is also a cofree \((\mathcal{C}, \mathbb{J})\)-comodule. One case easily see that \(\mathbb{J}\) lifts to endo-functors \(\mathbb{J}' : \mathbb{A}^{(\mathbb{C}; \mathbb{J})} \rightarrow \mathbb{A}^{(\mathbb{C}; \mathbb{J})}\) and \(\widehat{\mathbb{J}} : \mathbb{A}^{(\mathbb{C}; \mathbb{J})} \rightarrow \mathbb{A}^{(\mathbb{C}; \mathbb{J})}\).

2.18. Let \((\mathcal{C}, \Delta, \omega, \theta; \mathbb{J}) \in \textbf{JComonad}_{\mathbb{A}}\) with \(\mathbb{J}\) idempotent and \(\mathbb{J}C \simeq \mathbb{C}\mathbb{J}\). Consider the forgetful functor \(U : \mathbb{A}^{(\mathbb{C}; \mathbb{J})} \rightarrow \mathbb{A}\) and the endo-functor \(\mathcal{J}' : \mathbb{A}^{(\mathbb{C}; \mathbb{J})} \rightarrow \mathbb{A}^{(\mathbb{C}; \mathbb{J})}\). Then we have a natural isomorphism for \(X \in \mathbb{A}\) and \(Y \in \mathbb{A}^{(\mathbb{C}; \mathbb{J})}\):

\[\mathbb{A}^{(\mathbb{C}; \mathbb{J})}(\mathcal{J}'(Y), \mathcal{F}^{\mathcal{C}}(\mathbb{J}(X))) \simeq \mathbb{A}(U(\mathcal{J}'(Y)), \mathbb{J}(X)), \quad f \mapsto \theta_{\mathbb{J}(X)} \circ f\]

with inverse \(g \mapsto \mathcal{F}^{\mathcal{C}}(g) \circ \varrho^{\mathbb{J}(Y)}\); i.e. \((U, \mathcal{F}^{(\mathbb{C}; \mathbb{J})}(-))\) is a \((\mathcal{J}', \mathbb{J})\)-adjoint pair.
Proposition 2.19. Let $\mathcal{A}$ and $\mathcal{B}$ be categories, $L : \mathcal{A} \rightarrow \mathcal{B}$, $R : \mathcal{B} \rightarrow \mathcal{A}$ be functors and $\mathcal{J} : \mathcal{A} \rightarrow \mathcal{A}$, $\mathcal{K} : \mathcal{B} \rightarrow \mathcal{B}$ endo-functors such that $\mathcal{L} \mathcal{J} \simeq \mathcal{K} \mathcal{L}$, $\mathcal{J} \mathcal{R} \simeq \mathcal{R} \mathcal{K}$ and $(L, R)$ is a $(\mathcal{J}, \mathcal{K})$-adjoint pair.

1. $(\mathcal{L}, \mathcal{R})$ is an adjoint pair where $\mathcal{L} : \mathcal{J}(\mathcal{A}) \xrightarrow{\eta} \mathcal{K}(\mathcal{B})$ and $\mathcal{R} : \mathcal{K}(\mathcal{B}) \xrightarrow{\varepsilon} \mathcal{J}(\mathcal{A})$ with unit and counit of adjunction given by

   $$\eta : \mathcal{J} \rightarrow \mathcal{R} \mathcal{L} \quad \text{and} \quad \varepsilon : \mathcal{L} \mathcal{R} \mathcal{K} \rightarrow \mathcal{K}.$$

2. $\mathcal{R} \mathcal{L}$ is a monad on $\mathcal{J}(\mathcal{A})$ with

   $$\mu_{\mathcal{R} \mathcal{L}} : (\mathcal{R} \mathcal{L})(\mathcal{R} \mathcal{L}) \simeq \mathcal{R}(\mathcal{L} \mathcal{K}) \xrightarrow{\mathcal{R} \eta} \mathcal{K} \mathcal{L} \simeq (\mathcal{R} \mathcal{L}) \mathcal{J}$$

   and $\eta_{\mathcal{R} \mathcal{L}} := \eta$.

3. $\mathcal{L} \mathcal{R}$ is a comonad on $\mathcal{K}(\mathcal{B})$ with

   $$\Delta_{\mathcal{L} \mathcal{R}} : (\mathcal{L} \mathcal{R}) \mathcal{K} \simeq \mathcal{L} \mathcal{R} \mathcal{L} \xrightarrow{\mathcal{L} \eta} \mathcal{L} \mathcal{R} \mathcal{K} \simeq (\mathcal{L} \mathcal{R}) \mathcal{K}$$

   and $\varepsilon_{\mathcal{L} \mathcal{R}} := \varepsilon$.

4. $L$ is a monad on $\mathcal{J}(\mathcal{A})$ if and only if $R$ is a comonad on $\mathcal{K}(\mathcal{B})$. In this case, $\mathcal{J}(\mathcal{A})_L \simeq \mathcal{K}(\mathcal{B})^R$.

5. $L$ is a comonad on $\mathcal{J}(\mathcal{A})$ if and only if $R$ is a monad on $\mathcal{K}(\mathcal{B})$. In this case, $\mathcal{J}(\mathcal{A})^L \simeq \mathcal{K}(\mathcal{B})_R$. 

Proof. By assumption $\mathcal{L} \mathcal{J} \simeq \mathcal{K} \mathcal{L}$ whence $\mathcal{L}(\mathcal{J}(\mathcal{A})) := L \mathcal{J}(\mathcal{A}) = \mathcal{K} \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{B})$ and $\mathcal{J} \mathcal{R} \simeq \mathcal{R} \mathcal{K}$ whence $\mathcal{R}(\mathcal{K}(\mathcal{B})) := \mathcal{R}(\mathcal{K}(\mathcal{B})) = \mathcal{J}(\mathcal{B}) \subseteq \mathcal{J}(\mathcal{A})$. The assumptions imply that $(\mathcal{L}, \mathcal{R})$ is an adjoint pair. The result follows now from the classical result on right adjoint pairs (e.g. [EM1965] Proposition 3.1, [BBW2009, 2.5, 2.6]).

3 Semiunital Semimonoidal Categories

A semimonoidal category is roughly speaking a monoidal category not necessarily with a unit object. The reader might consult the literature for the precise definitions and for the notions of (op)-semimonoidal functors between such categories. In this section, we introduce a notion of semiunital semimonoidal categories and semiunital (op-)semimonoidal functors.

Semiunits

3.1. Let $(\mathcal{V}, \bullet)$ be a semimonoidal category. We say that $I \in \mathcal{V}$ is a semiunit iff

1. there is a natural transformation $\omega : I \rightarrow (I \bullet -)$;
2. there exists an isomorphisms of functors $I \bullet - \simeq - \circ I$, i.e. there is a natural isomorphism $I \bullet X \overset{\ell_X}{\sim} X \bullet I$ in $\mathcal{V}$ with inverse $\varphi_X$, for each object $X$ of $\mathcal{V}$, such that $\ell_I = \varphi_I$ and the following diagrams are commutative for all $X,Y \in \mathcal{V}$:

$$
\begin{array}{cccc}
(I \bullet X) \bullet Y & \overset{\gamma_{I,X,Y}}{\longrightarrow} & I \bullet (X \bullet Y) & \overset{\ell_{X \circ Y}}{\longrightarrow} & (X \bullet Y) \bullet I \\
\downarrow \ell_{X \circ Y} & & \downarrow & & \downarrow \\
(X \bullet I) \bullet Y & \overset{\gamma_{X \bullet I,Y}}{\longrightarrow} & X \bullet (I \bullet Y) & \overset{X \bullet \ell_Y}{\longrightarrow} & X \bullet (Y \bullet I) \\
\end{array}

\begin{array}{cccc}
(I \bullet X) \bullet Y & \overset{\omega_{I,Y}}{\longrightarrow} & X \bullet Y & \overset{X \bullet \omega_Y}{\longrightarrow} & X \bullet (I \bullet Y) \\
\downarrow \omega_{I,Y} & & \downarrow & & \downarrow \\
I \bullet (X \bullet Y) & \overset{\gamma_{X,Y}}{\longrightarrow} & I \bullet (X \bullet Y) & \overset{\lambda_X \bullet Y}{\longrightarrow} & X \bullet Y \\
\end{array}
$$

If $X \overset{\omega_X}{\simeq} I \bullet X (\overset{\ell_X}{\simeq} X \bullet I)$, then we say that $X$ is firm.

**Notation.** If $X$ is firm, then we set $\lambda_X := \omega_X^{-1} : I \bullet X \longrightarrow X$ and $\rho_X : X \bullet I \overset{\varphi_X}{\simeq} I \bullet X \overset{\lambda_X}{\longrightarrow} X$. With $\mathcal{V}^{\text{firm}}$ we denote the full subcategory of firm objects in $\mathcal{V}$.

**Remark 3.2.** If $I$ is firm (called also pseudo-idempotent) and $\omega_I^{-1} \bullet I = I \bullet \omega_I^{-1}$, then one says that $I$ is idempotent [Koc2008].

**Remark 3.3.** Let $(\mathcal{V}, \bullet)$ be a semimonoidal category. One says that $\mathcal{V}$ is monoidal [Mac1998] iff $\mathcal{V}$ has a unit (or an LR unit), i.e. a distinguished object $I \in \mathcal{V}$ with natural isomorphisms $I \bullet X \overset{\lambda_X}{\simeq} X$ and $X \bullet I \overset{\rho_X}{\simeq} X$ such that $X \bullet \lambda_Y = \rho_X \bullet Y$ for all $X,Y \in \mathcal{V}$ (equivalently, $\lambda_I = \rho_I$, $\lambda_X \bullet Y = \lambda_X \bullet Y$ and $\rho_{X \bullet Y} = X \bullet \rho_Y$ for all $X,Y \in \mathcal{V}$). Kock [Koc2008] called an object $I \in \mathcal{V}$ a Saavedra unit – called also a reduced unit – iff it is pseudo-idempotent and cancellable in the sense that the endo-functors $I \bullet -$ and $- \circ I$ are full and faithful (equivalently, $I$ is idempotent and the endo-functors $I \bullet -$ and $- \circ I$ are equivalences of categories). Moreover, he showed that $I$ is a unit if and only if $I$ is a Saavedra unit. Indeed, every unit is a semunit, whence our notion of semiunital semimonoidal categories generalizes the classical notion of monoidal categories.

**3.4.** Let $(\mathcal{V}, \bullet, I_\mathcal{V}; \omega_\mathcal{V})$ and $(\mathcal{W}, \otimes, I_\mathcal{W}; \omega_\mathcal{W})$ be semiunital semimonoidal categories. A semimonoidal functor $F : \mathcal{V} \longrightarrow \mathcal{W}$, with a natural transformation $\phi : F(-) \otimes F(-) \longrightarrow F(- \bullet -)$, is said to be semiunital semimonoidal iff there exists a coherence morphism...
\( \phi : I_W \rightarrow F(I_V) \) in \( W \) such that the following diagram is commutative

\[
\begin{array}{ccc}
I_W \otimes F(X) & \xrightarrow{\ell_{F(X)}} & F(X) \otimes I_W \\
\phi \otimes F(X) & & F(X) \otimes \phi \\
F(I_V) \otimes F(X) & \xrightarrow{\phi_{I_V,X}} & F(X) \otimes F(I_V) \\
\phi_{I_V,X} & & \phi_{X,I_V} \\
F(I_V \bullet X) & \xrightarrow{F(\ell_X)} & F(X \bullet I_V) \\
\end{array}
\]

Moreover, we say that \( F \) is a \textit{strong (strict) semiunital semimonoidal functor} iff \( F \) is strong (strict) as a semimonoidal functor and \( \phi \) is an isomorphism (identity). For two semimonoidal functors \( F, F' : \mathcal{V} \rightarrow \mathcal{W} \), we say that a semimonoidal natural transformation \( \xi : F \rightarrow F' \) is \textit{semiunital semimonoidal} iff the following diagram is commutative

\[
\begin{array}{ccc}
I_W & \xrightarrow{\phi} & I_W \\
\xi_{I_V} & & \xi_{F(I_V)} \\
F(I_V) & \xrightarrow{\xi_{I_V}} & F'(I_V) \\
\end{array}
\]

One can dually define \textit{semiunital (strong, strict) op-semimonoidal functors} and semiunital natural transformations between them.

\textit{Remarks 3.5.} Let \( (\mathcal{V}, \bullet, I; \omega) \) be a semiunital semimonoidal category and consider the functor

\[ J := I \bullet - : \mathcal{V} \rightarrow \mathcal{V}. \]

1. We have natural isomorphisms

\[ J(I) \bullet X \simeq J(J(X)) \simeq X \bullet J(I) \quad \text{and} \quad J(X) \bullet Y \simeq X \bullet J(Y) \]

for any \( X, Y \in \mathcal{V} \).

2. \( J \) is op-semimonoidal; the natural transformation \( \delta_{X,Y} : J(- \bullet -) \rightarrow J(-) \bullet J(-) \) is given by the collection of morphisms

\[ \delta_{X,Y} : I \bullet (X \bullet Y) \xrightarrow{\omega^{(X,Y)}} (I \bullet I) \bullet (X \bullet Y) \simeq (I \bullet X) \bullet (I \bullet Y) \]

for all \( X, Y \in \mathcal{V} \).

3. Assume that \( I \) is firm.
(a) $\mathcal{J}$ is strong semiunital semimonoidal with
\[
\phi_{X,Y} : (\mathbf{I} \otimes X \otimes \mathbf{I} \otimes Y) \simeq (\mathbf{I} \otimes I \otimes (X \otimes Y)) \omega^{-1}_{I \otimes (X \otimes Y)} \simeq \mathbf{I} \otimes (X \otimes Y) \quad \text{and} \quad \phi := \omega_{I} : \mathbf{I} \longrightarrow \mathbf{I} \mathbf{I}
\]
for all $X, Y \in \mathcal{V}$.

(b) $\mathcal{J}$ is strong semiunital op-semimonoidal with
\[
\delta := \omega_{I}^{-1} : \mathbf{I} \cdot \mathbf{I} \longrightarrow \mathbf{I}.
\]

(c) the full subcategory $(\mathcal{U} \mathcal{V}, \cdot, \mathbf{I})$ is monoidal.

(d) $(\mathcal{J} \mathcal{V}, \cdot, \mathbf{I})$ is a monoidal full subcategory of $(\mathcal{U} \mathcal{V}, \cdot, \mathbf{I})$ with
\[
\lambda_{\mathbf{I} \otimes X} : \mathbf{I} \cdot (\mathbf{I} \otimes X) \simeq \mathbf{I} \otimes \mathbf{I} \otimes X \quad \text{and} \quad \rho_{\mathbf{I} \otimes X} : (\mathbf{I} \otimes X) \cdot \mathbf{I} \simeq (\mathbf{I} \otimes \mathbf{I}) \otimes \mathbf{I} \cdot (\mathbf{I} \otimes X)
\]
for every $X \in \mathcal{V}$.

**Definition 3.6.** Let $(\mathcal{V}, \cdot, \mathbf{I}; \omega)$ be a semiunital semimonoidal category. We say that $V \in \mathcal{V}$ has a left dual iff there exists $V^\oplus \in \mathcal{V}$ along with morphisms $\upsilon : \mathbf{I} \longrightarrow \mathbf{I} \cdot V \cdot V^\oplus$ and $\varpi : \mathbf{I} \cdot V^\oplus \cdot \mathbf{V} \longrightarrow \mathbf{I}$ in $\mathcal{V}$ such that
\[
(V \cdot \varpi) \circ (\ell_V \cdot V^\oplus \cdot \mathbf{V}) \circ (\upsilon \cdot V) = \ell_V \quad \text{and} \quad (\varpi \cdot V^\oplus) \circ (\phi_{V^\oplus} \cdot V \cdot V^\oplus) \circ (V^\oplus \cdot \upsilon) = \phi_{V^\oplus}.
\]
A right dual $V^\otimes$ of $V$ is defined symmetrically. We say that $\mathcal{V}$ is left (right) autonomous, or left (right) rigid iff every object in $\mathcal{V}$ has a left (right) dual.

**Definition 3.7.** Let $(\mathcal{V}, \cdot, \mathbf{I}; \omega)$ be a semiunital semimonoidal category. We say that $\mathcal{V}$ is right (left) closed iff for every $V \in \mathcal{V}$, the functor $- \cdot V : \mathcal{J} \mathcal{V} \longrightarrow \mathcal{J} \mathcal{V}$ ($V \cdot - : \mathcal{J} \mathcal{V} \longrightarrow \mathcal{J} \mathcal{V}$) has a right-adjoint, i.e. there exists a functor $G : \mathcal{J} \mathcal{V} \longrightarrow \mathcal{J} \mathcal{V}$ and a natural isomorphism for every pair of objects $X, Y \in \mathcal{V}$:
\[
\mathcal{V}(X \cdot \mathbf{I} \cdot V, Y \cdot \mathbf{I}) \simeq \mathcal{V}(X \cdot \mathbf{I}, G(Y \cdot \mathbf{I})) \quad \text{(resp.} \quad \mathcal{V}(V \cdot \mathbf{I} \cdot X, Y \cdot \mathbf{I}) \simeq \mathcal{V}(X \cdot \mathbf{I}, G(Y \cdot \mathbf{I})).
\]

Moreover, $\mathcal{V}$ is said to be closed iff $\mathcal{V}$ is left and right closed.

**Lemma 3.8.** Let $(\mathcal{V}, \cdot, \mathbf{I}; \omega)$ be a semiunital semimonoidal category. If $V \in \mathcal{V}$ has a left (right) dual $V^\oplus$, then $(- \cdot V, - \cdot V^\oplus)$ is a $(\mathcal{J}, \mathcal{J})$-adjoint pair. In particular, if $\mathcal{V}$ is left (right) autonomous, then $\mathcal{V}$ is right (left) closed.

**Proof.** Assume that $V \in \mathcal{V}$ has a left dual $V^\oplus$. For any $X, Y \in \mathcal{V}$ we have a natural isomorphism
\[
\mathcal{V}(X \cdot \mathbf{I} \cdot V, Y \cdot \mathbf{I}) \simeq \mathcal{V}(X \cdot \mathbf{I}, Y \cdot \mathbf{I} \cdot V^\oplus), \quad \phi \longmapsto (f \cdot V^\oplus) \circ (X \cdot \upsilon)
\]
with inverse $g \longmapsto (Y \cdot \varpi) \circ (g \cdot V)$. \(\blacksquare\)
4 Semimonoids and Semicomonoids

In this section, we introduce notions of semimonoids and semicomonoids in semiunital semimonoidal categories.

Throughout, \((\mathcal{V}, \bullet, I; \omega)\) is a semiunital semimonoidal category, where \(I\) is a semiunit and \(\omega : I \rightarrow J\) is a natural transformation between the identity functor and \(J := I \bullet - \simeq - \bullet I : \mathcal{V} \rightarrow \mathcal{V}\) (we assume the existence of natural isomorphisms \(I \bullet X \simeq X \bullet I\) with inverse \(X \bullet I \overset{\varphi_X = \ell_X^1}{\longrightarrow} I \bullet X\) for every \(X \in \mathcal{V}\)).

Semimonoids

4.1. A \(\mathcal{V}\)-semimonoid consists of a datum \((A, \mu, \eta)\), where \(A \in \mathcal{V}\) and \(\mu : A \bullet A \rightarrow A\), \(\eta : I \rightarrow A\) are morphisms in \(\mathcal{V}\) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
A \bullet A \bullet A & \xrightarrow{\mu \bullet A} & A \bullet A \\
A \bullet A & \xrightarrow{\mu} & A \\
& & \\
A \bullet A & \xrightarrow{\eta \bullet A} & I \bullet A \\
& & \\
& & \\
& & A \bullet I
\end{array}
\]

If \(A \overset{\omega A}{\simeq} I \bullet A\), then we say that \(A\) is a unital \(\mathcal{V}\)-semimonoid. A morphism of \(\mathcal{V}\)-semimonoids \(f : (A, \mu, \eta) \rightarrow (A', \mu', \eta')\) is a morphism in \(\mathcal{V}\) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
A \bullet A & \xrightarrow{\mu} & A \\
& \xleftarrow{f} & \\
A' \bullet A' & \xrightarrow{\mu'} & A'
\end{array}
\]

The category of \(\mathcal{V}\)-semimonoids is denoted by \(\text{SMonoid}(\mathcal{V})\); the full subcategory of unital \(\mathcal{V}\)-semimonoids is denoted by \(\text{USMonoid}(\mathcal{V})\).

4.2. Let \((A, \mu, \eta)\) be a \(\mathcal{V}\)-semimonoid. A right \(A\)-semimodule is a datum \((M, \varrho_M)\) where \(M \in \mathcal{V}\) and \(\varrho_M : M \bullet A \rightarrow M\) is a morphism in \(\mathcal{V}\) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
M \bullet A \bullet A & \xrightarrow{\varrho_M \bullet A} & M \bullet A \\
M \bullet A & \xrightarrow{\varrho_M} & M \\
& & \\
M \bullet A & \xrightarrow{\phi_M} & M \\
& & \\
& & \\
& & M \bullet I \overset{\ell_M}{\longrightarrow} I \bullet M
\end{array}
\]
If \( M \cong_\omega I \cdot M \), then we say that \( M \) is a unitary right \( A \)-semimodule. A morphism of right \( A \)-semimodules is a morphism \( f: M \to M' \) in \( \mathcal{V} \) such that the following diagram is commutative

\[
\begin{array}{ccc}
M \cdot A & \xrightarrow{\varrho_M} & M \\
\downarrow_{f \cdot A} & & \downarrow_{f} \\
M' \cdot A & \xrightarrow{\varrho_M'} & M'
\end{array}
\]

The category of right \( A \)-semimodules is denoted by \( S_A \); the full subcategory of unitary right \( A \)-semimodules is denoted by \( US_A \). Analogously, one can define the category \( _A S \) of left \( A \)-semimodules and its full subcategory \( _A US \) of unitary left \( A \)-semimodules.

**Example 4.3.** If \( I \cong_\omega I \cdot I \), then \( I \) is a unital \( \mathcal{V} \)-semimonoid with

\[
\mu_I : I \cdot I \xrightarrow{\omega_I^{-1}} I \text{ and } \eta_I : I \xrightarrow{id} I.
\]

Moreover, every \( M \in \mathbb{U} \mathcal{V} \) is a unitary \((I, I)\)-bisemimodule through \( \varrho_M^r : M \cdot I \cong_\omega M \) and \( \varrho_M^l : I \cdot M \cong_\omega M \).

**4.4.** Let \( A \) be a \( \mathcal{V} \)-semimonoid and \( M \) a right \( A \)-semimodule. We have a functor

\[
- \cdot M : \mathcal{V} \to S_A,
\]

where for any \( X \in \mathcal{V} \) we have a structure of a right \( A \)-semimodule on \( X \cdot M \) given by

\[
\varrho_X: (X \cdot M) \cdot A \xrightarrow{\gamma_{X,M,A}} X \cdot (M \cdot A) \to X \cdot M.
\]

Similarly, if \( M \) is a left \( A \)-semimodule, then we have a functor \( M \cdot - : \mathcal{V} \to _A S \).

**4.5.** Let \( A \) and \( B \) be \( \mathcal{V} \)-semimonoids. Let \( M \) be a left \( B \)-semimodule as well as a right \( A \)-semimodule and consider \( B \cdot M \in S_A \) and \( M \cdot A \in B \mathcal{M} \). We say that \( M \) is a \((B, A)\)-bisemimodule iff \( \varrho_{(M; B)} : B \cdot M \to M \) is a morphism in \( S_A \), equivalently, \( \varrho_{(M; A)} : M \cdot A \to M \) is a morphism in \( B \mathcal{S} \). The category of (unitary) \((A, B)\)-bisemimodules with morphisms being in \( B \mathcal{S} \cap S_A \) is denoted by \( B \mathcal{S}_A (B \mathcal{U} \mathcal{S}_A) \). Indeed, every (unital) \( \mathcal{V} \)-semimonoid \( A \) is a (unitary) \((A, A)\)-bisemimodule in a canonical way.

**Proposition 4.6.** Every semiunital semimonoidal functor \( F : (\mathcal{V}, \cdot, I_\mathcal{V}) \to (\mathcal{W}, \otimes, I_\mathcal{W}) \) lifts to a functor

\[
\tilde{F} : S\text{Mon}(\mathcal{V}) \to S\text{Mon}(\mathcal{W}), \ A \mapsto F(A)
\]

that commutes with the forgetful functors \( U_\mathcal{V} : S\text{Mon}(\mathcal{V}) \to \mathcal{V} \) and \( U_\mathcal{W} : S\text{Mon}(\mathcal{W}) \to \mathcal{W} \).
Proof. Let \((A, \mu_A, \eta_A)\) be a semimonoid in \(\mathcal{V}\) and consider \(B := F(A)\). Define
\[
\mu_B : F(A) \otimes F(A) \xrightarrow{\phi_{A, A}} F(A \bullet A) \xrightarrow{F(\mu_A)} F(A);
\]
\[
\eta_B : I_W \xrightarrow{\phi} F(I_V) \xrightarrow{F(\eta_A)} F(A).
\]
One checks easily that \((B, \mu_B, \eta_B)\) is a semimonoid in \(\mathcal{W}\). If \(f : A \rightarrow A'\) is a morphism of \(\mathcal{V}\)-semimonoids, then examining the involved diagrams shows that \(F(f) : F(A) \rightarrow F(A')\) is a morphism of \(\mathcal{W}\)-semimonoids. Finally, it is clear that \(U_W \circ \tilde{F} = F \circ U_V\).

Proposition 4.7. Let \((A, \mu, \eta)\) be a \(\mathcal{V}\)-semimonoid.

1. We have \(\mathbb{J}\)-monads
   \[- \bullet A : \mathcal{V} \rightarrow \mathcal{V} \text{ and } A \bullet - : \mathcal{V} \rightarrow \mathcal{V}\]
   and isomorphisms of categories
   \(S_A \simeq \mathcal{V}_{(- \bullet A; \mathbb{J})}\) and \(A S \simeq \mathcal{V}_{(- \bullet A; \mathbb{J})}\).

2. If \(B\) is a \(\mathcal{V}\)-semimonoid, then we have \(\mathbb{J}\)-monads
   \[- \bullet A : B S \rightarrow B S \text{ and } B \bullet - : S A \rightarrow S A\]
   and isomorphisms of categories
   \((B S)_{(- \bullet A; \mathbb{J})} \simeq B S A \simeq (S A)_{(B \bullet -; \mathbb{J})}\).

Proof. Consider the natural transformations
\[
\mu : (- \bullet A) \bullet A \rightarrow - \bullet A, \quad \mu_X : (X \bullet A) \bullet A \xrightarrow{\gamma_{X, A, A}} X \bullet (A \bullet A) \xrightarrow{X \bullet \eta} X \bullet A,
\]
\[
\nu : \mathbb{J} \rightarrow - \bullet A, \quad \nu_X : I \bullet X \xrightarrow{f_X} X \bullet I \xrightarrow{X \bullet \eta} X \bullet A.
\]
One can easily check that \((- \bullet A, \mu, \omega, \nu)\) is a \(\mathbb{J}\)-monad. The isomorphism \(S M_A \simeq \mathcal{V}_{(- \bullet A; \mathbb{J})}\) follows immediately from comparing the corresponding diagrams. The other assertions can also be checked easily.

An object \(G\) in cocomplete category \(\mathfrak{A}\) is said to be a (regular) generator iff for every \(X \in \mathfrak{A}\), there exists a canonical (regular) epimorphism \(f_X : \bigsqcup_{f \in \mathfrak{A}(G, X)} G \rightarrow X\) (BW2005, p. 199) (see also [Kel2005, Ver]); recall that an arrow in \(\mathfrak{A}\) is said to be a regular epimorphism iff it is a coequalizer (of its kernel pair).

Theorem 4.8. Let \(\mathcal{V}\) be cocomplete, \(\mathbf{I}\) and \(A \in \mathcal{V}\) be firm and assume that \(\mathbf{I}\) is a regular generator in \(\mathcal{V}\) and that both \(A \bullet -\) and \(- \bullet A\) preserve colimits in \(\mathcal{V}\). There is a bijective correspondence between the semimonoid structures on \(A\), the \(\mathbb{J}\)-monad structures \((- \bullet A, \mu, \omega, \nu; \mathbb{J})\) and the \(\mathbb{J}\)-monad structures \((A \bullet -, \mu, \omega, \nu; \mathbb{J})\).
Proof. Assume that \((- \cdot A, \mu, \omega, \nu)\) is a \(J\)-monad and consider
\[
\begin{align*}
\mu : & A \cdot A \xrightarrow{\omega \cdot A} I \cdot A \xrightarrow{\mu} I \cdot A \xrightarrow{\lambda} A; \\
\eta : & I \xrightarrow{\omega I} I \cdot I \xrightarrow{\mu I} I \cdot A \xrightarrow{\lambda I} A.
\end{align*}
\]
Clearly, \((A, \mu, \eta)\) is a (unital) semimonoid. The converse follow by Proposition 4.7. The proof of the bijective correspondence is similar to that in the proof of [Ver, Theorem 3.9]. The statement corresponding to the endo-functor \(A \cdot -\) can be proved analogously. 

Semicomonoids

4.9. A \(V\)-semicomodule is a datum \((C, \Delta, \varepsilon)\) where \(C \in V\), \(\Delta : C \rightarrow C \cdot C\), \(\varepsilon : C \rightarrow I\) are morphisms in \(V\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
C \xrightarrow{\Delta} C \cdot C & \Delta \cdot C \\
\downarrow \Delta & \downarrow \Delta \\
C \cdot C \xrightarrow{\Delta} C \cdot C \cdot C & C \cdot C \xrightarrow{\Delta} C \cdot C
\end{array}
\]

\[
\begin{array}{ccc}
C \xrightarrow{\Delta} C \cdot C & C \xrightarrow{\Delta} C \cdot C & \omega_C \\
\downarrow \varepsilon \cdot C & \downarrow \Delta & \downarrow C \cdot \varepsilon \\
I \cdot C \xrightarrow{\varepsilon} I \cdot I & C \cdot C \xrightarrow{\Delta} C \cdot C
\end{array}
\]

If \(C \xrightarrow{\omega_C} I \cdot C\), then we say that \(C\) is a counital \(V\)-semicomodule. A morphism of \(V\)-semicomodons \(f : (C, \Delta, \varepsilon) \rightarrow (C', \Delta', \varepsilon')\) is a morphism in \(V\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
C \xrightarrow{\Delta} C \cdot C & C \xrightarrow{\varepsilon} I \\
\downarrow f & \downarrow f \\
C' \xrightarrow{\Delta'} C' \cdot C' & C' \xrightarrow{\varepsilon'} I
\end{array}
\]

The category of \(V\)-semicomodons is denoted by \(SComonoid(V)\); the full subcategory of counital \(V\)-semicomodons is denoted by \(USComonoid(V)\).

4.10. Let \((C, \Delta, \varepsilon)\) be a \(V\)-semicomodule. A right \(C\)-semicomodule is a datum \((M, \varrho^M)\) where \(M \in V\) and \(\varrho^M : M \rightarrow M \cdot C\) are morphisms in \(V\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
M \xrightarrow{\varrho^M} M \cdot C & M \xrightarrow{\varrho^M} M \cdot C \\
\downarrow \varrho^M & \downarrow \varrho^M \cdot C \\
M \cdot C \xrightarrow{\varrho^M \cdot C} M \cdot C \cdot C & I \cdot M \xrightarrow{\varrho_M} M \cdot I
\end{array}
\]

\[
\begin{array}{ccc}
M \xleftarrow{\varrho^M} M \cdot C & M \xleftarrow{\varrho^M} M \cdot C \\
\downarrow \varrho^M & \downarrow \varrho^M \cdot C \\
M \cdot C \xleftarrow{\varrho^M \cdot C} M \cdot C \cdot C & I \cdot M \xleftarrow{\varrho_M} M \cdot I
\end{array}
\]
A morphism of right $C$-semicomodules is a morphism $f : M \to M'$ in $\mathcal{V}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
M & \xrightarrow{\rho^M} & M \cdot C \\
\downarrow f & & \downarrow f \cdot C \\
M' & \xrightarrow{\rho^{M'}} & M' \cdot C \\
\end{array}
$$

The category of right $C$-semicomodules is denoted by $S^C$; the category of counitary right $C$-semicomodules is denoted by $CS^C$. Analogously, one can define the category $^CS^C$ of left $C$-semicomodules and its full subcategory $CS^C$ of counitary left $C$-semicomodules.

**Remark 4.11.** We prefer to use the terminology unital semimonoids (counital semicomonoids) over monoids (comonoids) which we reserve for monoidal categories. For example, the category of unital semimonoids in the monoidal category $\text{Set}$ of sets is the category $\text{Monoid}$ of usual monoids of the sense of Abstract Algebra. The same applies for unitary semimodules (counitary semicomodules). This is also consistent with the classical terminology of semirings and semimodules used in Section 5.

**4.12.** Let $C$ be a $\mathcal{V}$-semicomonoid and $M$ a semicounitary right $C$-semicomodule. Then we have a functor

$$- \cdot M : \mathcal{V} \to S^C,$$

where for any $X \in \mathcal{V}$, we have a structure of a right $C$-semicomodule on $X \cdot M$ given by

$$\rho^{X \cdot M} : X \cdot M \xrightarrow{X \cdot \rho^M} X \cdot (M \cdot C) \xrightarrow{\tau_{X,M,C}^X} (X \cdot M) \cdot C.$$

Similarly, if $M$ is a left $C$-semicomodule, then we have a functor $M \cdot - : \mathcal{V} \to ^CS^C$.

**4.13.** Let $C$ and $D$ be $\mathcal{V}$-semicomonoids. Let $M$ be a left $D$-semicomodule and a right $C$-semicomodule and consider $D \cdot M \in S^C$ and $M \cdot C \in ^DS$. We say that $M$ is a $(D, C)$-bisemicomodule iff $\varrho^{(M, D)} : M \to D \cdot M$ is a morphism in $S^C$; equivalently, $\varrho^{(M, C)} : M \to M \cdot C$ is a morphism in $^DS$. The category of $(D, C)$-bisemicomodules with morphisms in $^DS \cap S^C$ is denoted by $^DS^C$. The full subcategory of counitary $(D, C)$-bisemicomodules is denoted by $^DCS^C$. Indeed, every $\mathcal{V}$-semicomonoid $C$ is a $(C, C)$-bisemicomodule in a canonical way.

**Example 4.14.** $I$ is $\mathcal{V}$-semicomonoid with

$$\Delta_I : I \xrightarrow{\omega_I} I \cdot I \quad \text{and} \quad \varepsilon_I : I \xrightarrow{id} I.$$

Moreover, every (firm) $M \in \mathcal{V}$ is a (counitary) $(I, I)$-bisemicomodule with $\varrho^I_M : M \xrightarrow{\omega_M} I \cdot M$ and $\varrho^I_M : M \xrightarrow{\ell_M \cdot \omega_M} M \cdot I$. 

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Dual to Proposition 4.6, we have

**Proposition 4.15.** Every semiunital op-semimonoidal functor \( F : \mathcal{V} \to \mathcal{W} \) lifts to a functor

\[
\tilde{F} : \text{SCMonoid}(\mathcal{V}) \to \text{SCMonoid}(\mathcal{W}), \quad C \mapsto F(C)
\]

which commutes with the forgetful functors \( U_{\mathcal{V}} : \text{SCMon}(\mathcal{V}) \to \mathcal{V} \) and \( U_{\mathcal{W}} : \text{SCMon}(\mathcal{W}) \to \mathcal{W} \).

Dual to Proposition 4.7, we obtain

**Proposition 4.16.** Let \( (C, \Delta, \varepsilon) \) be a \( \mathcal{V} \)-semicomonoid.

1. We have \( \mathcal{J} \)-comonads

\[
- \circ C : \mathcal{V} \to \mathcal{V} \quad \text{and} \quad C \circ - : \mathcal{V} \to \mathcal{V}
\]

and isomorphisms of categories

\[
\mathcal{S}^C \simeq \mathcal{V}^{(- \circ C ; \mathcal{J})} \quad \text{and} \quad C^\mathcal{S} \simeq \mathcal{V}^{(- C \circ ; \mathcal{J})}.
\]

2. If \( D \) is a \( \mathcal{V} \)-semicomonoid, then we have \( \mathcal{J} \)-comonads

\[
- \circ C : D^\mathcal{S} \to D^\mathcal{S} \quad \text{and} \quad D \circ - : C^\mathcal{S} \to C^\mathcal{S}
\]

and isomorphisms of categories

\[
(D^\mathcal{M})^{(- \circ C ; \mathcal{J})} \simeq D^{C^\mathcal{S}} \simeq (S^C)^{(D \circ - ; \mathcal{J})}.
\]

Our second reconstruction result is obtained in a way similar to that of Theorem 4.8:

**Theorem 4.17.** Let \( \mathcal{V} \) be cocomplete, \( I \) and \( C \in \mathcal{V} \) be firm and assume that \( I \) is a regular generator and that both \( C \circ - \) and \( - \circ C \) respect colimits in \( \mathcal{V} \). There is a bijective correspondence between the semicomonoid structures on \( C \), the \( \mathcal{J} \)-comonad structures \( (- \circ C, \Delta, \omega, \varepsilon; \mathcal{J}) \) and the \( \mathcal{J} \)-comonad structures \( (C \circ -, \Delta, \omega, \varepsilon; \mathcal{J}) \).

**Proposition 4.18.** Let \( (C, \Delta, \varepsilon) \) be a semicomonoid and \( (A, \mu, \eta) \) a unital semimonoid. Then \( (\mathcal{V}(C,A), *, \epsilon) \) is a monoid in \( \text{Set} \) with multiplication and unit given by

\[
f \ast g := \mu \circ (f \circ g) \circ \Delta \quad \text{and} \quad \epsilon := \eta \circ \varepsilon.
\]

**Proposition 4.19.** Let \( \phi : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D) \) be a morphism of semicomonoids and \( \psi : (A, \mu_A, \eta_A) \to (B, \mu_B, \eta_B) \) be a morphism of unital semimonoids. Then

\[
\mathcal{V}(D,A) \xleftarrow{\phi_A} \mathcal{V}(C,A), \quad f \mapsto f \circ \phi \quad \text{and} \quad \mathcal{V}(C,B) \xrightarrow{\psi_C} \mathcal{V}(C,B), \quad g \mapsto \psi \circ g
\]

are morphisms of monoids in \( \text{Set} \). In particular, we have functors

\[
\mathcal{V}(-, A) : \text{SCMonoid}_\mathcal{V} \to \text{Monoid} \quad \text{and} \quad \mathcal{V}(C, -) : \text{SMonoid}_\mathcal{V} \to \text{Monoid}.
\]
5 A concrete example

In this section we give applications to the category of bisemimodules over a base semialgebra.

Semirings and Semimodules

For the convenience of the reader and to make the manuscript self-contained, we begin this section by recalling some basic definitions and results on semirings and their semimodules.

Definition 5.1. A semiring is an algebraic structure \((S, +, \cdot, 0, 1)\) consisting of a non-empty set \(S\) with two binary operations “+” (addition) and “\(\cdot\)” (multiplication) satisfying the following axioms:

1. \((S, +, 0)\) is a commutative monoid with neutral element 0;
2. \((S, \cdot, 1)\) is a monoid with neutral element 1;
3. \(x \cdot (y + z) = x \cdot y + x \cdot z\) and \((y + z) \cdot x = y \cdot x + z \cdot x\) for all \(x, y, z \in S\);
4. \(0 \cdot s = 0 = s \cdot 0\) for every \(s \in S\) (i.e. 0 is absorbing).

5.2. Let \(S, S'\) be semirings. A map \(f : S \to S'\) is said to be a morphism of semirings iff for all \(s_1, s_2 \in S:\)

\[ f(s_1 + s_2) = f(s_1) + f(s_2),\ f(s_1 s_2) = f(s_1) f(s_2),\ f(0_S) = 0_{S'}\text{ and } f(1_S) = 1_{S'}.

The category of semirings is denoted by \(\text{SRng}\).

5.3. Let \((S, +, \cdot)\) be a semiring. We say that \(S\) is cancellative iff the additive semigroup \((S, +)\) is cancellative, i.e. whenever \(s, s', s'' \in S\) we have

\[ s + s' = s + s'' \implies s' = s''.\]

commutative iff the multiplicative semigroup \((S, \cdot)\) is commutative;
semifield iff \((S \setminus \{0\}, \cdot, 1)\) is a commutative group.

Examples 5.4. Rings are indeed semirings. The first natural example of a (commutative) semiring which is not a ring is \((\mathbb{N}_0, +, \cdot)\), the set of non-negative integers. The semirings \((\mathbb{R}_0^+, +, \cdot)\) and \((\mathbb{Q}_0^+, +, \cdot)\) are indeed semifields. For any associative ring \(R\) we have a semiring structure \((\text{Ideal}(R), +, \cdot)\) on the set \(\text{Ideal}(R)\) of (two-sided) ideals of \(R\). Any distributive complete lattice \((\mathcal{L}, \land, \lor, 0, 1)\) is a semiring. For more examples, the reader may refer to \([Gol1999a]\). In the sequel, we always assume that \(0_S \neq 1_S\).
Definition 5.5. Let $S$ be a semiring. A right $S$-semimodule is an algebraic structure $(M, +, 0_M)$ consisting of a non-empty set $M$, a binary operation “+” along with a right $S$-action

$$M \times S \longrightarrow M, \ (m, s) \mapsto ms,$$

such that:

1. $(M, +, 0_M)$ is a commutative monoid with neutral element $0_M$;
2. $(ms)s' = m(ss'), (m + m')s = ms + m's$ and $m(s + s') = ms + ms'$ for all $s, s' \in S$ and $m, m' \in M$;
3. $m1_S = m$ and $m0_S = 0_M = 0_Ms$ for all $m \in M$ and $s \in S$.

5.6. Let $M, M'$ be right $S$-semimodules. A map $f : M \longrightarrow M'$ is said to be a morphism of $S$-semimodules (or $S$-linear) iff for all $m_1, m_2 \in M$ and $s \in S$:

$$f(m_1 + m_2) = f(m_1) + f(m_2) \text{ and } f(ms) = f(m)s.$$

The set $\text{Hom}_S(M, M')$ of $S$-linear maps from $M$ to $M'$ is clearly a commutative monoid under addition. The category of right $S$-semimodules is denoted by $\mathbb{S}_S$. Analogously, one can define the category $\mathbb{S}_S$ of left $S$-semimodules. A right (left) $S$-semimodule is said to be cancellative iff the semigroup $(M, +)$ is cancellative. With $\mathbb{CS}_S \subseteq \mathbb{S}_S$ (resp. $\mathbb{SC}_S \subseteq \mathbb{S}_S$) we denote the full subcategory of cancellative right (left) $S$-semimodules. For two semirings $S, T$, an $(S, T)$-bisemimodule $M$ has a structure of a left $S$-semimodule and a right $T$-semimodule such that $(sm)t = s(mt)$ for all $m \in M$, $s \in S$ and $t \in T$. The category of $(S, T)$-bisemimodules and $S$-linear $T$-linear maps is denoted by $\mathbb{S}_S T$; the full subcategory of cancellative $(S, T)$-bisemimodules is denoted by $\mathbb{SC}_S T$.

5.7. Let $M$ be a right $S$-semimodule. An $S$-congruence on $M$ is an equivalence relation $\equiv$ such that

$$m_1 \equiv m_2 \Rightarrow m_1s + m \equiv m_2s + m \text{ for all } m_1, m_2, m \in M \text{ and } s \in S.$$

In particular, we have an $S$-congruence relation $\equiv_{[0]}$ on $M$ defined by

$$m \equiv_{[0]} m' \iff m + m'' = m' + m'' \text{ for some } m'' \in M.$$

The quotient $S$-semimodule $M/ \equiv_{[0]}$ is indeed cancellative and we have a canonical surjection $c_M : M \longrightarrow c(M)$, where $c(M) := M/ \equiv_{[0]}$, with

$$\text{Ker}(c_M) = \{ m \in M \mid m + m'' = m'' \text{ for some } m'' \in M \}.$$

The class of cancellative right $S$-semimodules is a reflective subcategory of $\mathbb{S}_S$ in the sense that the functor $c : \mathbb{S}_S \longrightarrow \mathbb{CS}_S$ is left adjoint to the embedding functor $\mathbb{CS}_S \hookrightarrow \mathbb{S}_S$, i.e. for any $S$-semimodule $M$ and any cancellative $S$-semimodule $N$ we have a natural isomorphism of commutative monoids $\text{Hom}_S(c(M), N) \simeq \text{Hom}_S(M, N)$ [Tak1981, p.517].
Takahashi’s Tensor-like Product

5.8. ([Gol1999a] page 187) Let $M_S$ be a right $S$-semimodule, $sN$ a left $S$-semimodule and consider the Abelian monoid $U := S(M \times N) \times S(M \times N)$. Let $U' \subseteq S(M \times N) \times S(M \times N)$ be the symmetric $S$-subsemimodule generated by the set of elements of the form

$$(\delta_{(m_1+m_2,n)}, \delta_{(m_1,n)} + \delta_{(m_2,n)}), \quad (\delta_{(m_1,n)} + \delta_{(m_2,n)}, \delta_{(m_1+m_2,n)}),$$

$$(\delta_{(m,n_1+n_2)}, \delta_{(m,n_1)} + \delta_{(m,n_2)}), \quad (\delta_{(m,n_1)} + \delta_{(m,n_2)}, \delta_{(m,n_1+n_2)}),$$

$$(\delta_{(m,s,n)}), \quad (\delta_{(m,s,n)}),$$

where

$$\delta_{m,n}(m, n) = \begin{cases} 1_S, & m = n \\ 0, & m \neq n. \end{cases}$$

Let $\equiv$ be the $S$-congruence relation on $S(M \times N)$ defined by

$$f \equiv f' \iff f + g = f' + g' \text{ for some } (g, g') \in U'.$$

Takahashi’s tensor-like product of $M$ and $N$ is defined as $M \bowtie_S N := F/\equiv$. Notice that there is an $S$-balanced map

$$\tilde{\tau} : M \times N \longrightarrow M \mathcal{Z}_S N, \ (m, n) \mapsto m \mathcal{Z}_S n := (m, n)/\equiv$$

with the following universal property [Tak1982a]: for every commutative monoid $G$ and every $S$-bilinear $S$-balanced map $\beta : M \times N \longrightarrow G$ there exists a unique morphism of monoids $\gamma : M \mathcal{Z}_S N \longrightarrow \mathcal{C}(G)$ such that we have a commutative diagram

\[ \begin{array}{ccc} M \times N & \beta \longrightarrow & G \\ \downarrow \tau & & \downarrow \iota_G \\ M \mathcal{Z}_S N & \longrightarrow & \mathcal{C}(G) \end{array} \] (4)

The following result collects some properties of $- \mathcal{Z}_S -$ (compare with [Abu] and [Gol1999a Proposition 16.15, 16.16]):

**Proposition 5.9.** Let $M$ be a right $S$-semimodule and $N$ a left $S$-semimodule.

1. $M \mathcal{Z}_S N$ is a cancellative commutative monoid.
2. $M_S (sN)$ is cancellative if and only if $\mathcal{C}(M) \simeq M (\mathcal{C}(N) \simeq N)$. In this case, we
3. We have natural isomorphisms of functors

$$- \mathcal{Z}_S S \simeq \mathcal{C}(-) : S_S \longrightarrow S_S \text{ and } S \mathcal{Z}_S - \simeq \mathcal{C}(-) : sS \longrightarrow sS.$$  

Moreover, we have isomorphisms of functors

$$- \mathcal{Z}_S S \simeq \mathcal{C}(-) \simeq S \mathcal{Z}_S - : S_S \longrightarrow S_S.$$  

We set

$$M \mathcal{Z}_S S \overset{\vartheta_M}{\simeq} M \text{ and } S \mathcal{Z}_S N \overset{\varphi_M}{\simeq} N.$$
4. We have idempotent functors

\[ J : S \boxtimes_S - : S S \longrightarrow S S \text{ and } K := - \boxtimes_T T : S T \longrightarrow S T. \]  

In particular, \( c(c(M)) \simeq c(M) \) and \( c(c(N)) \simeq c(N) \).

5. We have natural isomorphisms of commutative monoids

\[ c(M) \boxtimes_S N \simeq c(M) \boxtimes_S c(N) \simeq M \boxtimes_S c(N) \simeq M \boxtimes_S N \simeq c(M \boxtimes_S N). \]

Proposition 5.10. Let \( S \) and \( T \) be semirings, \( M \) a right \( S \)-semimodule and \( N \) an \((S,T)\)-bisemimodule. Consider the functors

\[ - \boxtimes_S N : S_S \longrightarrow S_T, \ N \boxtimes_T - : T_S \longrightarrow S_S \]

and the endo-functors \( J \) and \( K \) in (5).

1. \((- S_N, \text{Hom}_{-T}(N, -))\) is a \((J, K)\)-adjoint pair.

2. \((N \boxtimes_T -, \text{Hom}_{S-}(N, -))\) is a \((K, J)\)-adjoint pair.

Proof. For every right \( T \)-semimodule \( G \) we have natural isomorphisms

\[
\begin{align*}
\text{Hom}_{-T}(J(M) \boxtimes_S N), \ K(G) & \simeq \text{Hom}_{-T}(c(M) \boxtimes_S N), \ c(G)) \\
& \simeq \text{Hom}_{-T}(M \boxtimes_S N, \ c(G)) \\
& \simeq \text{Hom}_{-S}(M, \text{Hom}_{-T}(N, \ c(G)))) \quad \text{[Gol1999a, 16.15]} \\
& \simeq \text{Hom}_{-S}(c(M), \text{Hom}_{-T}(N, \ c(G)))) \quad \text{[Tak1981, p. 517]} \\
& \simeq \text{Hom}_{-S}(J(M), \text{Hom}_{-T}(N, \ K(G)))).
\end{align*}
\]

The second statement can be proved symmetrically. ■

**Semiunital Semirings and Semicounitary Semimodules**

In what follows, \( S \) denotes a **commutative** semiring with \( 1_S \neq 0_S \), \( A \) is an \( S \)-semialgebra (i.e. a semiring with a morphism of semirings \( \iota_A : S \longrightarrow A \)), \( A S_A \) is the category of \((A, A)\)-bisemimodules and \( ACS_A \) is its **full** subcategory of cancellative \((A, A)\)-bisemimodules. Moreover, we fix the idempotent endo-functor

\[ J := c(-) \simeq A \boxtimes_A - \simeq - \boxtimes_A A : A S_A \longrightarrow A S_A. \]

Summarizing the observations above, we obtain

**Theorem 5.11.** 1. \((A S_A, \boxtimes, A)\) is a closed semiunital semimonoidal category.

2. \((A CS_A, \boxtimes, c(A))\) is a closed monoidal category.
5.12. By a **semiunital $A$-semiring** we mean an $(A,A)$-bisemimodule $A$ associated with $(A,A)$-bilinear maps $\mu_A : A \boxtimes_A A \rightarrow A$ and $\eta_A : A \rightarrow A$ such that the following diagrams are commutative

$$
\begin{array}{ccc}
A \boxtimes_A A & \xrightarrow{\mu_A} & A \\
A \boxtimes_A A & \xrightarrow{\mu_A} & A \\
A \boxtimes_A A & \xrightarrow{\mu_A} & A \\
\end{array}
$$

Let $A$ and $A'$ be semiunital $A$-semirings. An $(A,A)$-bilinear map $f : A \rightarrow A'$ is called a **morphism of semiunital $A$-semirings** iff

$$f \circ \mu_A = \mu_{A'} \circ (f \boxtimes_A f) \text{ and } f \circ \eta_A = \eta_{A'}.$$

The set of morphisms of semiunital $A$-semirings form $A$ to $A'$ is denoted by $\text{SSRng}_A(A,A')$. The category of semiunital $A$-semirings will be denoted by $\text{SSRng}_A$. Indeed, we have an isomorphism of categories $\text{SSRng}_A \cong \text{SMonoid}(A_S^A)$.

5.13. Let $A$ be a semiunital $A$-semiring. A **semiunitary right $A$-semimodule** is a right $A$-semimodule along with a right $A$-linear map $\vartheta_M : M \boxtimes_A A \rightarrow M$ such that the following diagrams are commutative

$$
\begin{array}{ccc}
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
\end{array}
$$

A **morphism of semiunitary right $A$-semimodules** ($A$-linear) is an $A$-linear map $f : M \rightarrow M'$ such that the following diagram is commutative

$$
\begin{array}{ccc}
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
M \boxtimes_A A & \xrightarrow{\vartheta_M} & M \\
\end{array}
$$

The category of semiunitary right $A$-semimodules and $A$-linear maps is denoted by $\text{SS}_A$. Analogously, one can define the category $\text{SS}_A^B$ of **semiunital left $A$-semimodules**. For two semiunital $A$-semirings $A$ and $B$, one can define the category $\text{SS}_A^B$ of $(B,A)$-bisemimodules in the obvious way. Considering semiunital $A$-semirings as semimonoids in $A_S^A$, we have indeed isomorphisms of categories

$$
\text{SS}_A \cong S_A, \quad \text{SS}_A^B \cong S_B, \quad \text{SS}_A^B \cong S_A, \quad \text{CS}_A \cong \text{US}_A, \quad \text{CS}_B \cong \text{US}_B, \quad \text{CS}_A \cong \text{US}_A. \quad (7)
$$
Remark 5.14. We use semiunital $A$-semirings to stress that such semimonoids are defined in the semiunital semimonoidal category $(\mathcal{A}\mathcal{S}_A, \otimes_A, A)$ and to avoid confusion with (unital) $A$-semirings which can be defined as monoids in the monoidal category $(\mathcal{A}\mathcal{S}_A, \otimes, A)$. The same applies for semicounitary $A$-semicorings below.

5.15. Being a variety, in the sense of Universal Algebra, the category $\mathcal{A}\mathcal{S}_A$ of $(A, A)$-bisemimodules is cocomplete. The class of regular epimorphism in $\mathcal{A}\mathcal{S}_A$ coincides with that of surjective $(A, A)$-bilinear maps. For every $(A, A)$-bisemimodule $M$, there is a surjective $(A, A)$-bilinear map from a free $(A, A)$-bisemimodule to $M$ (compare with [Gol1999a, Proposition 17.11]); whence, $A$ is a regular generator. Moreover, for any $(A, A)$-bisemimodule $X$, both $X \boxtimes_A -$, $- \boxtimes_A X : \mathcal{A}\mathcal{S}_A \rightarrow \mathcal{A}\mathcal{CS}_A$ respect colimits since they are left adjoints [Tak1982a, Corollary 4.5].

Applying Theorem 4.8 to $\mathcal{A}\mathcal{S}_A$, we obtain:

Corollary 5.16. Let $A$ be cancellative and $A$ a cancellative $(A, A)$-bisemimodule. There is a bijective correspondence between the structures of unital $A$-semirings on $A$, $c$-monads on $A \boxtimes_A -$ and $c$-monads $- \boxtimes_A A$.

Semicounital Semicorings and Semicounitary Semicomodules

5.17. A semicounital $A$-semicoring is an $(A, A)$-bisemimodule associated with $(A, A)$-bilinear maps $\Delta_C : C \rightarrow C \boxtimes_A C$ and $\varepsilon_C : C \rightarrow A$ such that the following diagrams are commutative

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \boxtimes_A C \\
\downarrow{\Delta_C} & & \downarrow{\varepsilon_C \otimes A \Delta_C} \\
C \boxtimes_A C & \xrightarrow{\Delta_C \boxtimes_A \varepsilon_C} & C \boxtimes_A C \boxtimes_A C \\
\end{array}
\]

\[
\begin{array}{ccc}
C \boxtimes_A C & \xrightarrow{\Delta_C} & C \boxtimes_A C \\
\varepsilon_C \otimes A \Delta_C & & \varepsilon_C \\
C \boxtimes_A C & \xrightarrow{\Delta_C \boxtimes_A \varepsilon_C} & C \boxtimes_A C \boxtimes_A C \\
\end{array}
\]

The map $\Delta_C (\varepsilon_C)$ is called the comultiplication (counity) of $C$. Using Sweedler-Heuneman’s notation, we have for every $c \in C$:

\[
\sum c_{11} \boxtimes_A c_{12} \boxtimes_A c_{2} = \sum c_1 \boxtimes_A c_{21} \boxtimes_A c_2; \\
\varepsilon(\sum c_1 \varepsilon(c_2)) = \varepsilon_M(c) = \varepsilon(\sum \varepsilon(c_1)c_2).
\]

Let $(C, \Delta, \varepsilon)$ and $(C', \Delta', \varepsilon')$ be semicounital $A$-semicorings. We call an $(A, A)$-bilinear map $f : C \rightarrow C'$ a morphism of $A$-semicorings iff

\[
(f \boxtimes_A f) \circ \Delta_C = \Delta_{C'} \circ f \text{ and } \varepsilon_{C'} \circ f = \varepsilon_C.
\]

The set of $A$-semicoring morphisms from $C$ to $C'$ is denoted by $\text{SSCog}_A(C, C')$. The category of semicounital $A$-semicorings is denoted by $\text{SSCrng}_A$. Indeed, we have an isomorphism of categories $\text{SSCrng}_A \simeq \text{SCMonoid}(\mathcal{A}\mathcal{S}_A)$. 

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5.18. Let \((C, \Delta, \varepsilon)\) be an \(A\)-semicoring. A **semicounitary right \(C\)-semicomodule** is a right \(A\)-semimodule \(M\) associated with an \(A\)-linear map
\[
\varrho^M : M \to M \otimes_A C, \quad m \mapsto \sum m_{<0>} \otimes_A m_{<1>},
\]
such that the following diagrams are commutative
\[
\begin{array}{ccc}
M & \xrightarrow{\varrho^M} & M \otimes_A C \\
\downarrow{\varrho_M} & & \downarrow{\varrho^M_{\otimes A \Delta_C}} \\
M \otimes_A C & \xrightarrow{\varrho_M \otimes A \varepsilon_C} & M \otimes_A C \otimes_A C \\
\end{array}
\]

Using Sweedler-Heyneman’s notation, we have for every \(m \in M\):
\[
\sum m_{<0>} \otimes_A m_{<1>} \otimes_A m_{<1>2} = \sum m_{<0><0>} \otimes_A m_{<1>} \otimes_A m_{<1>};
\]
\[
\varepsilon\left(\sum m_{<0>} \varepsilon_C(m_{<1>})\right) = \varepsilon_M(m).
\]

For semicounitary right \(C\)-comodules \(M, M'\), we call an \(A\)-linear map \(f : M \to M'\) a **morphism of semicounitary right \(C\)-semicomodules** (or \(C\)-colinear) if the following diagram is commutative
\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\varrho_M} & & \downarrow{\varrho_N} \\
M \otimes_A C & \xrightarrow{f \otimes A C} & N \otimes_A C \\
\end{array}
\]

The category of semicounitary right \(C\)-semicomodules and \(C\)-colinear maps is denoted by \(\mathcal{SS}^C\); the full subcategory of counitary right \(C\)-semicomodules is denoted by \(\mathcal{CS}^C\). Analogously, one can define the category \(\mathcal{DSS}^C\) of **semicounitary left \(C\)-semicomodules** and its full subcategory of **counitary left \(C\)-semicomodules**. For two semicounital \(A\)-semicorings \(C\) and \(D\) one can define the category \(\mathcal{PSS}^C\) of **semicounitary \((D,C)\)-bisemicomodules** and its full subcategory of **counitary \((D,C)\)-bisemicomodules** in the obvious way. Considering semicounital \(A\)-semicorings as semicomonoids in \(A \mathcal{S}_A\), we have indeed isomorphisms of categories
\[
\mathcal{SS}^C \simeq \mathcal{S}^C, \mathcal{DSS}^C \simeq \mathcal{DS}^C, \mathcal{CS}^C \simeq \mathcal{CS}^C, \mathcal{DSS}^C \simeq \mathcal{DS}^C, \mathcal{CS}^C \simeq \mathcal{CS}^C. \quad (9)
\]

Applying Theorem 4.17 to \(A \mathcal{S}_A\), we obtain:

**Corollary 5.19.** Let \(A\) be cancellative and \(C\) a cancellative \((A,A)\)-bisemimodule. There is a bijective correspondence between the structures of counitary \(A\)-semicorings on \(C\), \(\varepsilon\)-comonads on \(C \otimes_A \) and \(\varepsilon\)-comonads on \(- \otimes_A C\).

Almost all structures of corings over rings (e.g. [Abu], [BW2003]) can be transferred to obtain structures of semicorings over semirings.
Example 5.20. Let $\kappa : B \to A$ be an extension of $S$-semialgebras and consider $B$ as a $(B, B)$-bimodule in the canonical way. One can define Sweedler’s counital $A$-semiring $C := (A \boxtimes_B A, \Delta, \varepsilon)$ with

$$
\Delta : A \boxtimes_B A \to (A \boxtimes_B A) \boxtimes A (A \boxtimes_B A), \quad a \boxtimes_B \tilde{a} \mapsto (a \boxtimes_B 1_A) \boxtimes (1_A \boxtimes_B \tilde{a});
$$

$$
\varepsilon : A \boxtimes_B A \to A, \quad a \boxtimes_B \tilde{a} \mapsto a\tilde{a}.
$$

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