In this paper, we propose a novel probability distribution that asymptotically represents a power-law, $\psi(t) \sim t^{-\alpha-1}$, with $0 < \alpha < 2$. The main feature of the distribution is that it has a simple expression in the Laplace transform representation, making it suitable for performing calculations in stochastic processes, particularly non-Poissonian processes.

**Keywords:** stochastic processes, random walks, diffusion

**Contents**

1. Introduction  
2. Laplace transform of a power-law distribution  
3. Rate event function  
4. Master equation for a stochastic dichotomous process  
5. Conclusions
1. Introduction

Stochastic processes are extensively used to model processes across all scientific disciplines. In addition to physical sciences, stochastic processes are applied to a wide range of fields, from biology and chemistry to material science and social sciences. The main study objective of a stochastic process is the probability density distribution $P(x, t)$, where $x = (x_1, x_2, \ldots, x_n)$ are the stochastic variables and $t$ represents the time. For Markovian processes, i.e., processes with no memory, or more general for renewal processes, the associated equations frequently change into convoluted forms in space and time. A primary role is played by the continuous time random walk (CTRW), whereby an indefinite number of processes can be modelled (see references [1, 2] and reference [3] for an extensive review). An excellent example is provided by the probability density of the random walker position, initially located at $x = 0$ in the CTRW scheme. The expression of the probability density is

$$P(x, t) = \sum_{n=0}^{\infty} P_n(x) \int_0^t \psi_n(\tau) \Psi(t - \tau) d\tau,$$

where $\psi_n(\tau)$ is defined as

$$\psi_n(t) = \int_0^t \psi_{n-1}(\tau) \psi(t - \tau) d\tau, \quad \psi_0(t) = \delta(t), \quad \psi_1(t) \equiv \psi(t),$$

and $\psi_n(\tau)d\tau$ is the probability that the $n$'th step occurred at some time between $\tau$ and $\tau + d\tau$, moving the random walker toward $x$. In several processes $\psi(\tau)$ is also called the waiting-time distribution. The function,

$$\Psi(t) = \int_0^\infty \psi(\tau) d\tau,$$

is the survival probability that ensures that no additional step is taken between the times $\tau$ and $t > \tau$. Setting $\psi_0(t) = \delta(t)$ and because at $t = 0$ the initial condition is $P_0(x) = \delta(x)$, the Laplace transform of $P(x, t)$ is found directly as

$$\tilde{P}(x, s) = \frac{1 - \tilde{\psi}(s)}{s} \sum_{n=0}^{\infty} P_n(x) \tilde{\psi}^n(s), \quad \tilde{\psi}(s) \equiv \int_0^\infty \psi(t) \exp[-st] dt.$$

Setting conditions on $P_n(x)$, a further elaboration of equation (4) can be performed; however, we refer the reader to reference [3] for further elaboration of equation (4). For our purposes, we limit ourselves to stress the key role in the resulting expression of the Laplace transform for the waiting-time distribution.

The random processes leading to anomalous diffusion, using CTRW or the fractional Fokker–Planck equation (see references [4–6] and references therein), have been inten-
Distribution with a simple Laplace transform and its applications to non-Poissonian stochastic processes

sively studied using a power-law as the waiting-time distributions, $\psi(t) \sim t^{-\alpha-1}$, with $0 < \alpha < 1$. In this regard, the Mittag–Leffler function plays an important role in modeling a power-law distribution. An exhaustive study on this topic can be found in reference [7], where the authors presented, in detail, the key role of the Mittag–Leffler function in renewal processes that are relevant to the theories of anomalous diffusion, particularly in the CTRW approach and fractional Fokker–Planck equation. The Mittag–Leffler function is defined as [8, 9],

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}$$  \hspace{1cm} (5)

where the associated distribution, the Mittag–Leffler waiting-time density, is,

$$\psi(t) = -\frac{d}{dt}E_\alpha(-t^\alpha).$$  \hspace{1cm} (6)

The asymptotic behavior of the Mittag–Leffler waiting-time density is a power-law $\psi(t) \sim t^{-\alpha-1}$ with $0 < \alpha < 1$, i.e., the first moment of the distribution does not exist. Due to its simple Laplace transform expression, $\hat{\psi}(s) = 1/(s^\alpha + 1)$, the Mittag–Leffler waiting-time density has been extensively used (see reference [7] and references therein). Moreover, when $\alpha = 1$ the function describes a Poissonian process. Indeed, in this case, we have $\psi(t) = \exp[-t]$.

Another relevant quantity associated with several stochastic processes is the rate event function, which can be defined as

$$R(t) = \sum_{n=1}^{\infty} \psi_n(t),$$  \hspace{1cm} (7)

with $\psi_n(t)$ defined as in equation (2). This function plays a central role in several stochastic processes [10–17] and it is studied in detail in section 3 (for a detailed explanation see reference [18]). The expressions containing $R(t)$ as part of a more complicated expression can be found in subordination processes, such as in the Montroll–Weiss formalism [1, 19–21] or statistics of rare events in renewal theory [22].

From an analytical point of view, the range $1 < \alpha < 2$, i.e., a distribution with a finite first moment, is less known. As we may infer from the above discussion, having a simple expression for the Laplace transform of $\psi(t)$, $\hat{\psi}(s)$ is crucial to be able to invert the Laplace transform of the final expression in the Laplace representation. Particularly, in this paper, a novel probability distribution representing asymptotically a power-law is presented and we focus on non-Poissonian processes where $\psi(t)$ is a power-law asymptotically behaving as $t^{-\alpha-1}$ with $0 < \alpha < 2$. As largely assumed, what matters is the asymptotic expression of the waiting-time distribution rather than its exact expression. However, if the stochastic process can be reduced to a closed expression in the Laplace representation, as usually is the case, the knowledge of the Laplace transform is crucial. Thus, we need to focus on the expression of the Laplace transform $\hat{\psi}(s)$ and not on the details of the waiting-time distribution in the time representation, $\psi(t)$.

https://doi.org/10.1088/1742-5468/ab96b1

J. Stat. Mech. (2020) 073201
Distribution with a simple Laplace transform and its applications to non-Poissonian stochastic processes

2. Laplace transform of a power-law distribution

The inversion of the Laplace transform is, in general, a challenging task where exact results are difficult to find. Frequently, we turn to Tauberian theorems to find approximated expressions, in particular, asymptotic expressions. When we have a Laplace transform, let us say \( \hat{f}(s) \) and we are willing to go back to the \( t \)-representation, \( f(t) \), we may utilize the Tauberian theorem to obtain \( f(t) \) when \( t \to \infty \). This implies a Taylor expansion for \( s \to 0 \) of the transformed function, i.e., \( \hat{f}(s) = a_0 + a_1 s^\alpha + \ldots \). However, the conditions ensuring that the Taylor expansion has a correct correspondence with the asymptotic expression of \( f(t) \) are quite strict. Frequently, one inverts the expression for \( \hat{f}(s) \) without ensuring the conditions and checks for \textit{a posteriori} if the result is correct. When the conditions of the Tauberian theorem do not hold, an important question is where to stop the Taylor expansion. Whether more terms are added or not can produce different results (see the example in reference [23]). In stochastic processes, we frequently deal with the waiting-time distributions. For non-Poissonian processes, the waiting-time distribution is a power-law that admits a complicated expression for its Laplace transform. For example, in reference [24, 25] the authors considered a waiting-time distribution given by \( \psi(t) = \alpha (1 + t)^{-\alpha - 1} \). The expression in time representation is simple, whereas the Laplace transform, \( \hat{\psi}(s) \), has a complicated expression, making it difficult to perform the Laplace inversion for the related quantities. Typically, we are interested in the asymptotic behavior of the distributions in time representation. Therefore, the exact form of \( \psi(t) \) is not important, whereas having a manageable Laplace transform of \( \hat{\psi}(s) \) is crucial. As we have already seen, a notable example of a power-law distribution with a simple expression for its Laplace transform is the derivative of the Mittag–Leffler function, whose power series expression provides a useful expression,

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}.
\]  

(8)

Alternatively, it can be expressed through an integral representation [8]. In the interval \( t \in (0, \infty) \), the negative derivative of \( E_\alpha(-t^\alpha) \) for \( 0 < \alpha < 1 \) is positive definite and integrable. In other words, properly normalized, it represents a power-law distribution. Its asymptotic behavior is,

\[
-\frac{d}{dt}E_\alpha(-t^\alpha) \approx \frac{\sin(\pi\alpha)\Gamma(1 + \alpha)}{\pi} \frac{1}{t^{\alpha+1}}
\]  

(9)

where, for the sake of simplicity, we set the time scale parameter equal to the unit. Note that for \( \alpha = 1 \), the function ceases to be a power-law distribution since \( E_1(-t) = \exp[-t] \).

Starting from its definition, equation (8), finding the Laplace transform of \( E_\alpha(-t^\alpha) \) and its negative derivative are not difficult, both are written as a closed expression, i.e.,

\[
\mathcal{L}[E_\alpha(-t^\alpha)] = \frac{s^{\alpha-1}}{s^{\alpha} + 1}, \quad \mathcal{L}\left[-\frac{d}{dt}E_\alpha(-t^\alpha)\right] = \frac{1}{s^{\alpha} + 1}.
\]  

(10)

Due to its simple functional form in the Laplace representation, the above distribution has been used to evaluate the inverse Laplace transform of expressions such as the
one in reference [26], which will be thoroughly examined in section 4. As mentioned above, the validity region of the Mittag–Leffler distribution as a power-law distribution is limited to the power parameter ranging in the interval $0 < \alpha < 1$. To go beyond the above distribution and cross the critical value $\alpha = 1$, we require functions that have a simple Laplace transform and contain an asymptotic power-law behavior, for which good candidates are defined as follows [27, 28],

$$E^\alpha_t \equiv D^\alpha_t \exp[t] = \sum_{n=0}^{\infty} \frac{t^{n-\alpha}}{\Gamma(n + 1 - \alpha)},$$

(11)

where $D^\alpha_t$ is the Riemann–Liouville fractional derivative. The asymptotic behavior is,

$$E^\alpha_t \approx \exp[t] - \frac{\sin(\pi \alpha)\Gamma(1 + \alpha)}{\pi} \frac{1}{t^{\alpha+1}}, \text{ for } t \to \infty.$$  

(12)

Its Laplace transform is,

$$L[E^\alpha_t] = \frac{s^\alpha}{s - 1}.$$  

(13)

In principle, $E^\alpha_t$ has a Laplace transform defined for $\alpha < 1$; however, as we will see, this constraint can be bypassed. Note that the derivative of the Mittag–Leffler function and function $E^\alpha_t$ are related to each other. In the case of a rational index, it is straightforward to express the relationship explicitly. For example for $\alpha = 1/2$ we have,

$$L \left[ -\frac{d}{dt} E_{1/2}^{1/2}(-t^{1/2}) \right] = \frac{1}{s^{1/2} + 1} = \frac{s^{1/2} - 1}{s - 1}.$$  

(14)

which implies,

$$L^{-1} \left[ \frac{s^{1/2} - 1}{s - 1} \right] = E_{1/2}^t - \exp[t] = -\frac{d}{dt} E_{1/2}^{1/2}(-t^{1/2}).$$  

(15)

To determine a new distribution with similar characteristics of the distribution given in equation (10), we shall consider the associated functions with the imaginary argument,

$$\cos_\alpha t \equiv \frac{E^\alpha_t + E^{-\alpha}_{-it}}{2}, \quad \sin_\alpha t \equiv \frac{E^\alpha_t - E^{-\alpha}_{-it}}{2i},$$

(16)

or in terms of series

$$\cos_\alpha t = \sum_{n=0}^{\infty} \frac{t^{n-\alpha} \cos \left( \frac{\pi \alpha}{2} (n - \alpha) \right)}{\Gamma(n - \alpha + 1)}, \quad \sin_\alpha t = \sum_{n=0}^{\infty} \frac{t^{n-\alpha} \sin \left( \frac{\pi \alpha}{2} (n - \alpha) \right)}{\Gamma(n - \alpha + 1)}.$$  

(17)

Note that $\cos_\alpha t$ and $\sin_\alpha t$ are divergent at $t = 0$ as $t^{-\alpha}$, such that their Laplace transform is defined for $\alpha < 1$. We may overcome this difficulty by selecting an appropriate combination of those functions. We build the required distribution as

$$\psi(t) = \frac{\sin \left( \frac{\pi \alpha}{2} \right) \cos t + \cos \left( \frac{\pi \alpha}{2} \right) \sin t}{\cos \left( \frac{\pi \alpha}{2} \right)} - \frac{\sin \left( \frac{\pi \alpha}{2} \right) \cos_\alpha t + \cos \left( \frac{\pi \alpha}{2} \right) \sin_\alpha t}{\cos \left( \frac{\pi \alpha}{2} \right)}.$$  

(18)

https://doi.org/10.1088/1742-5468/ab96b1
Distribution with a simple Laplace transform and its applications to non-Poissonian stochastic processes

The cosine function in the denominator is a normalization factor, whereas the trigonometric coefficients in the numerator are selected to erase the first divergent terms of (17). The above expression can be rewritten as

$$\psi(t) = \frac{\sin \left( t + \frac{\pi \alpha}{2} \right) - D^\alpha_t \sin(t)}{\cos \left( \frac{\pi \alpha}{2} t \right)}.$$  \hfill (19)

Now, we need to show that the function defined in equation (19) is positive and integrable. Let us first show the positivity of $\psi(t)$. We use the following result (see references [27, 28]),

$$\int_0^\infty \frac{x^\alpha}{x^2 + 1} \exp(-tx) \, dx = \frac{\pi}{\sin \left( \frac{\pi \alpha}{2} \right)} \left[ \sin \left( t + \frac{\pi \alpha}{2} \right) - D^\alpha_t \sin(t) \right] = \frac{\pi}{2 \sin \left( \frac{\pi \alpha}{2} \right)} \psi(t).$$  \hfill (20)

As the integral in the left-hand side of equation (20) is a positive quantity for $\alpha \in (-1, \infty)$, the right-hand side of equation (20) should also be positive. Particularly, for $\alpha \in (0, 2)$ the factor $\sin \left( \frac{\pi \alpha}{2} \right)$ is a positive quantity and consequently $\psi(t)$ has to be positive, which concludes the demonstration.

Once we have ensured the positivity of the function, we study the integrability and the asymptotic behavior of $\psi(t)$. At the origin $\psi(t)$ behaves as

$$\psi(t) \approx \tan \left( \frac{\pi \alpha}{2} \right) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha) \cos \left( \frac{\pi \alpha}{2} \right)} + t + \cdots, \quad \text{for } t \to 0,$$  \hfill (21)

showing that $\psi(t)$ is integrable at the origin for $0 < \alpha < 2$ and $\alpha \neq 1$. For $t \to \infty$ we have,

$$\psi(t) \approx \frac{2 \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(\alpha + 1)}{\pi t^{\alpha+1}}, \quad \text{for } t \to \infty.$$  \hfill (22)

Equation (22) shows that $\psi(t)$ is integrable for $t \to \infty$ for $0 < \alpha < 2$ and it has the correct asymptotic behavior, namely $\psi(t) \sim t^{-\alpha-1}$. Two plots of $\psi(t)$ are shown in figures 1 and 2. Despite its complicated structure in time representation, its Laplace transform is,

$$\hat{\psi}(s) = \frac{1 + s \tan \left( \frac{\pi \alpha}{2} \right) - \sec \left( \frac{\pi \alpha}{2} \right) s^\alpha}{s^2 + 1}, \quad 0 < \alpha < 2, \quad \alpha \neq 1.$$  \hfill (23)

The case $\alpha = 1$ has to be evaluated in equation (23) by taking the limit for $\alpha \to 1$ of $\hat{\psi}(s)$. It is straightforward to obtain,

$$\hat{\psi}(s) = \lim_{\alpha \to 1} \frac{1 + s \tan \left( \frac{\pi \alpha}{2} \right) - \sec \left( \frac{\pi \alpha}{2} \right) s^\alpha}{s^2 + 1} = \frac{1 + \frac{2}{\pi} s \log s}{s^2 + 1}.$$  \hfill (24)

We have achieved our goal, i.e., to express the Laplace transform of $\psi(t)$ in an expression easy to manipulate, namely through a fraction of powers of the Laplace variable $s$. 

https://doi.org/10.1088/1742-5468/ab96b1 6
3. Rate event function

As an example, let us consider the following quantity, called rate event function,

\[ R(t) = \sum_{n=1}^{\infty} \psi_n(t), \quad (25) \]

with,

\[ \psi_n(t) = \int_{0}^{t} \psi_{n-1}(\tau)\psi(t-\tau)d\tau, \quad \psi_0(t) = \delta(t), \quad \psi_1(t) \equiv \psi(t). \quad (26) \]

and whose expression in the Laplace representation is,

\[ \hat{R}(s) = \frac{\hat{\psi}(s)}{1 - \hat{\psi}(s)}. \quad (27) \]

The function \( R(t) \) describes the number of events for a unit of time, and it is a relevant quantity present in several stochastic processes, including aging processes. Similar expressions can be found in subordination processes such as in the Montroll–Weiss process.
formalism [1, 19–21] through the typical kernel in the Laplace representation,
\[ \hat{\Phi}(s) = \frac{s\hat{\psi}(s)}{1 - \hat{\psi}(s)}. \] (28)

Going back to \( R(t) \) and using \( \hat{\psi}(s) \) given by equation (23), we obtain its Laplace transform, i.e.,
\[ \hat{R}(s) = \frac{-\sec\left(\frac{\pi\alpha}{2}\right) s^\alpha + s \tan\left(\frac{\pi\alpha}{2}\right) + 1}{\sec\left(\frac{\pi\alpha}{2}\right) s^\alpha + s^2 - s \tan\left(\frac{\pi\alpha}{2}\right)}. \] (29)

It can be observed that for rational values of \( \alpha \), an exact analytical expression can be found for \( R(t) \). For example, \( \alpha = 1/2 \) gives [29],
\[ R(t) = \exp[2t] \left[ 1 - \text{erf}\left(\sqrt{2t}\right) \right] \approx \frac{1}{\sqrt{2\pi t}} \] (30)
where \( \text{erf}(z) \) is an error function. Considering \( \alpha = 3/2 \), we have the exact result as in [29],
\[ R(t) = 1 + \sqrt{\frac{2}{\pi t}}. \] (31)

In general, we have that for \( 0 < \alpha < 1 \) it holds,
\[ \hat{R}(s) \approx \frac{1}{\sec\left(\frac{\pi\alpha}{2}\right) s^\alpha} \Rightarrow R(t) \propto \frac{1}{t^{1-\alpha}} = \frac{1}{t^{2-2\mu}}, \] (32)
while for \( 1 < \alpha < 2 \), we obtain,
\[ \hat{R}(s) \approx \frac{-\cos\left(\frac{\pi\alpha}{2}\right)}{s \sin\left(\frac{\pi\alpha}{2}\right) - s^\alpha} \Rightarrow R(t) \propto 1 + \frac{1}{t^{\alpha-1} \sin\left(\frac{\pi\alpha}{2}\right)} = 1 + \frac{1}{t^{\mu-2} \sin\left(\frac{\pi\alpha}{2}\right)}. \] (33)

4. Master equation for a stochastic dichotomous process

In this section, we use the distribution previously introduced to study the analytical expression of the distributions associated with a stochastic process. Particularly, we focus on the Lévy walks presented in reference [24] and also studied in references [30, 31]. We are also able to confirm that the trajectory approach, using walkers, and the density approach, using the Liouville equation, lead to the same result. The authors of reference [24] generated a stochastic trajectory, \( x(t) \), with a walker that starts moving spending time \( \tau_1 \) in a uniform motion with speed \( W \), and then the walker tosses a coin to decide whether to keep moving in the same direction or reverse it. Subsequently, the walker moves for a time \( \tau_2 \) with speed \( W \) and continues repeating the process. Without loss of generality, we may set \( W = 1 \). The adopted time distribution for \( \tau_i \) is,
\[ \psi_M(\tau) = \frac{\alpha T_M^\alpha}{(\tau + T_M)^{\alpha+1}}, \quad 1 < \alpha < 2. \] (34)
The authors demonstrated numerically that the corresponding probability distribution $P(x, t)$ for $t \to \infty$ is a Lévy distribution. However, to the best of our knowledge, this has not been shown analytically. In reference [26], the above process has been studied adopting the density point of view as a starting point (Liouville equation). The authors of reference [26] found an analytical expression in the double Laplace transform of the distribution $P(x, t)$ given by

$$
\hat{P}(s, u) = \frac{1}{4} \left[ \frac{\hat{\psi}(s) \left[ 1 - \hat{\psi}(u)^2 \right]}{u[1 - \hat{\psi}(s)\hat{\psi}(u)]} + \frac{1 - \hat{\psi}(s)^2}{s[1 - \hat{\psi}(s)\hat{\psi}(u)]} \right] + \frac{1}{4} \left[ \frac{1 - \hat{\psi}(u)}{u} + \frac{1 - \hat{\psi}(s)}{s} \right],
$$

(35)

where $s$ and $u$ correspond to $\frac{t+x}{2}$ and $\frac{t-x}{2}$, respectively, in the space-time representation. The inverse Laplace transform of $\hat{P}(s, u)$ has been studied in detail for power-law distribution $\psi(t) \propto t^{-\alpha-1}$ with $0 < \alpha < 1$ (reference [26]). The inverse Laplace transform of equation (35), for $1 < \alpha < 2$, has not been analytically studied due to the difficulty of inverting the double Laplace transform. Using the distribution introduced in equation (18), we may perform a series of exact calculations and, for rational power, the corresponding expression can be reduced to a sum of polynomial-like terms.

First, we need to find the relationship between the time scale $T$ of distribution (18) and the time scale $T_M$ of $\psi_M(t)$ of reference [24], i.e., equation (34). Using equation (22) explicitly written with respect to the time scale $T$ and equating it to the asymptotic expression of $\psi_M(t)$ given by equation (34), i.e.,

$$
\frac{\alpha T_M^\alpha}{t^{\alpha+1}} \approx T^\alpha \frac{2 \sin \left( \frac{\alpha \pi}{2} \right) \Gamma(\alpha+1)}{\pi t^{\alpha+1}}, \quad \text{for } t \to \infty,
$$

(36)

we may write the connection between the time scale $T_M$ and the time scale associated with distribution (18), $T$, as

$$
T_M = T \left[ \frac{2 \sin \left( \frac{\alpha \pi}{2} \right) \Gamma(\alpha+1)}{\pi \alpha} \right]^{1/\alpha}.
$$

(37)

We set $T = 1$ and focus on $\alpha = 3/2$. Due to the evident symmetry of $P(s, u)$, it is enough to consider the term,

$$
\hat{P}_1(s, u) = \frac{1}{4} \left[ 1 - \hat{\psi}(s)^2 \right] \hat{\psi}(u),
$$

(38)

while the last term in the square bracket on the right side of equation (35) represents the two ballistic peaks multiplied by the survival probability. Performing the calculation, we have,

$$
\hat{P}_1(s, u) = \frac{\hat{\psi}(u)}{4} \frac{1 + \hat{\psi}(s)}{\sqrt{2} \sqrt{s \epsilon(u) + s + \epsilon(u)}}, \quad \epsilon(u) \equiv 1 - \psi(u).
$$

(39)
We may rewrite the equation as
\[
\hat{P}_1(s, u) = \frac{\psi(u)}{4 \sqrt{2}} \frac{1 + \psi(s)}{\sqrt{2 - \epsilon(u)} \sqrt{\epsilon(u)}} \frac{2}{2} \left[ \frac{1}{\sqrt{s + \lambda(u)}} - \frac{1}{\sqrt{s + \lambda^*(u)}} \right]
\] (40)

where
\[
\lambda(u) = \frac{\epsilon(u)}{\sqrt{2}} - i \frac{\sqrt{2 - \epsilon(u)} \sqrt{\epsilon(u)}}{\sqrt{2}}
\] (41)

and \(\lambda^*(u)\) is the conjugate. Note that at this stage the inversion of Laplace transform with respect to the parameter \(s\) can be performed exactly. Using the result of equation (15) and the following equality [28],
\[
\int_0^\infty \frac{z^\beta}{z + a} \exp[-uz]du = \frac{\pi a^\beta}{\sin \pi \beta} \left( E_{\beta}^{wa} - E_{\beta}^{wa} \right).
\] (42)

we may rewrite equation (40), in the limit \(s \to 0\),
\[
\hat{P}_1(v, u) \approx \text{Im} \left[ \frac{1}{\sqrt{2} \sqrt{2 - \epsilon(u)} \sqrt{\epsilon(u)}} \frac{1 - \epsilon(u)}{\pi} \int_0^\infty \frac{\sqrt{z} \exp[-vz]}{z + \lambda(u)^2} \, dz \right].
\] (43)

Taking now the limit \(u \to 0\) we have,
\[
\lambda(u)^2 \approx -u + (1 - i)\sqrt{2}u^{3/2}, \quad \sqrt{2 - \epsilon(u)} \sqrt{\epsilon(u)} \approx \sqrt{2}\sqrt{u}.
\] (44)

Thus,
\[
\hat{P}_1(v, u) \approx \frac{1}{2\pi} \text{Im} \left[ \int_0^\infty \frac{\sqrt{z}}{(1 - i) \sqrt{2} \sqrt{u}} \left[ \frac{A_1(z)}{\sqrt{u - u_1(z)}} + \frac{A_2(z)}{\sqrt{u - u_2(z)}} + \frac{A_3(z)}{\sqrt{u - u_3(z)}} \right] \exp[-vz]dz \right],
\] (45)

with \(A_k(z)\) given by
\[
A_1 = -\frac{1}{(u_2 - u_1)(u_1 - u_3)}, \quad A_2 = -\frac{1}{(u_1 - u_2)(u_2 - u_3)}, \quad A_3 = -\frac{1}{(u_1 - u_3)(u_3 - u_2)}
\]

and \(u_k\) are the roots of the equation \(\sqrt{2}(1 - i)y^3 - y^2 + z = 0\). For the sake of compactness, we omitted the \(z\)-dependence of \(A_k\) and \(u_k\). Therefore, asymptotically we have,
\[
u_k \approx a + bz^{1/3}, \quad A_k \approx \frac{c}{z^{2/3}},
\] (46)

with \(a\), \(b\) and \(c\) complex constants. Using inverse Laplace transform,
\[
\mathcal{L}^{-1} \left[ \frac{1}{\sqrt{u}} \frac{1}{\sqrt{u - u_k}} \right] = e^{w^2} \text{erf} \left( u_k \sqrt{w} \right) + 1,
\] (47)
where erf(z) is the error function, we finally obtain,

\[ P_1(v, w) \approx \frac{1}{2\pi} \text{Im} \left[ \int_0^\infty dz \frac{\sqrt{z}}{(1-i)\sqrt{2}} \sum_{k=1}^{3} A_k \exp \left[ u_k^2 w \right] \left[ \text{erf} \left( u_k \sqrt{w} \right) + 1 \right] \exp[-vz] \right]. \] (48)

Moreover, the total distribution is

\[ P(x, t) \approx \frac{1}{2\pi} \text{Im} \left[ \int_0^\infty dz \frac{\sqrt{z}}{(1-i)\sqrt{2}} \left( \text{erf} \left[ u_k \sqrt{\frac{t+x}{2}} \right] + 1 \right) \exp \left[ -\frac{t-x}{2} \right] \sum_{k=1}^{3} A_k \exp \left[ u_k^2 \frac{t+x}{2} \right] \right. \]
\[ + \int_0^\infty dz \frac{\sqrt{z}}{(1-i)\sqrt{2}} \left( \text{erf} \left[ u_k \sqrt{\frac{t-x}{2}} \right] + 1 \right) \exp \left[ -\frac{t+x}{2} \right] \]
\[ \times \sum_{k=1}^{3} A_k \exp \left[ u_k^2 \frac{t-x}{2} \right] \] \[ \left. \theta(t-|x|) + \frac{1}{2} \Psi(t) \delta(t-|x|) \right], \] (49)

where \( \Psi(t) = \int_0^\infty \psi(\tau) d\tau \) is the survival probability. This expression has to be compared with the asymptotic theoretical \( P_L(x, t) \) represented by the Lévy distribution [24],

\[ P_L(x, t) = \frac{1}{\pi} \int_0^\infty \cos(kx) \exp \left[ -bk^{3/2}t \right] dk \] (50)

where \( b = \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(2-\alpha)T_M^{\alpha-1} \) and \( \alpha = 3/2 \). The good agreement between the two distributions is showed in figure 3, while in figure 4 shows the percent of error \( \Delta E = 100|P(x, t) - P_L(x, t)|/P_L(x, t) \).
Figure 4. Plot of the percent error $\Delta E = 100|P(x, t) - P_L(x, t)|/P_L(x, t)$ at $t = 440$ with $P(x, t)$ given by equation (49) and $P_L(x, t)$ given by equation (50). The maximum percent error is of the order of 2%.

5. Conclusions

In this paper, a new distribution, representing a power-law, $\psi(t) \sim t^{-\alpha-1}$ asymptotically with $0 < \alpha < 2$, has been presented. The main feature of the distribution is having a simple expression of the Laplace transform. Its application in several examples has been shown to produce exact results. We applied the new distribution, rigorously showing the conjecture that the Lévy walks generate a Lévy distribution, as numerically showed in reference [24] and analytically, via the decoupling approximation, in reference [31]. Simultaneously, we have also demonstrated that the trajectory and the density approach lead to the same result. Processes based on a power-law distribution in the region $0 < \alpha < 1$ can be analytically studied using the Mittag–Leffler distribution. The main contribution of this paper is that processes based on a power-law distribution in the range $1 < \alpha < 2$ can be analytically studied using the presented distribution.

Acknowledgments

The author acknowledges Phy.CA for providing logistical support.

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Distribution with a simple Laplace transform and its applications to non-Poissonian stochastic processes

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