EXTREMAL FUNCTIONS IN POINCARÉ–SOBOLEV INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATION

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Dedicated to Jean-Pierre Gossez, on the occasion of his 65th birthday

Abstract. If \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain and \( q \in (0, \frac{n}{n-1}) \) we consider the Poincaré–Sobolev inequality
\[
c \left( \int_{\Omega} |u|^\frac{nq}{n-1} \right)^{\frac{1}{n-1}} \leq \int_{\Omega} |Du|,
\]
for every \( u \in \text{BV}(\Omega) \) such that \( \int_{\Omega}|u|^{q-1}u = 0 \). We show that the sharp constant is achieved. We also consider the same inequality on an \( n \)–dimensional compact Riemannian manifold \( M \). When \( n \geq 3 \) and the scalar curvature is positive at some point, then the sharp constant is achieved. In the case \( n \geq 2 \), we need the maximal scalar curvature to satisfy some strict inequality.

1. Introduction

If \( \Omega \subset \mathbb{R}^n \) is smooth and has finite measure and if \( p \in (1, n) \), there exists \( c > 0 \) such that for every \( u \) in the Sobolev space \( W^{1,p}(\Omega) \) with \( \int_{\Omega} u = 0 \),
\[
c \left( \int_{\Omega} |u|^{\frac{n}{n-p}} \right)^{1-\frac{n}{n-p}} \leq \int_{\Omega} |\nabla u|^p.
\]
This inequality follows from the classical Sobolev inequality
\[
\left( \int_{\Omega} |u|^{\frac{n}{n-p}} \right)^{1-\frac{n}{n-p}} \leq C \left( \int_{\Omega} |\nabla u|^p + |u|^p \right)
\]
and a standard compactness argument (see for example E. Giusti [11, §3.6]).

We are interested in whether the sharp constant in (1), that is the value
\[
\inf \left\{ \int_{\Omega} |\nabla u|^p : u \in W^{1,p}(\Omega), \int_{\Omega} |u|^{\frac{n}{n-p}} = 1 \text{ and } \int_{\Omega} u = 0 \right\},
\]
is achieved. Since the embedding of \( W^{1,p}(\Omega) \) in \( L^{\frac{n}{n-p}}(\Omega) \) is not compact, the solution to this problem is not immediate.

In the case where \( \Omega = \mathbb{R}^n \) and the condition \( \int_{\Omega} u = 0 \) is dropped, this was solved by T. Aubin [1] and G. Talenti [19]. When the condition \( \int_{\Omega} u = 0 \) is replaced by \( u = 0 \) on \( \partial \Omega \), it is known that the constant is not achieved. However, if \( u \) is only required to vanish on a part \( \Gamma \) of the boundary and \( \Gamma \) has

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some good geometric properties, P.-L. Lions, F. Pacella and M. Tricarico have showed that the corresponding sharp constant is achieved for every \( p \in (1, \bar{p}) \) where \( \bar{p} \in (1, n] \) depends on \( \Omega \) and \( \Gamma \) \cite{17}. Returning to our problem P. Girão and T. Weth \cite{10} have showed that the sharp constant is achieved for \( p = 2 \).

A. V. Demyanov and A. I. Nazarov \cite{5, 18} have proved that there exists \( \delta > 0 \) depending on \( \Omega \) such that the sharp constant is achieved for \( p \in (1, n + \frac{1}{2} + \delta) \).

M. Leckband \cite{15} has given an alternative proof of this statement for a ball.

We are interested in the same question when \( p = 1 \). The counterpart of the Sobolev space \( W^{1,p}(\Omega) \) in this case is the space of functions of bounded variation \( BV(\Omega) \), and the inequality (1) becomes

\[
(2) \quad c \left( \int_{\Omega} |u|^{\frac{n}{n-1}} \right)^{1 - \frac{1}{n}} \leq \int_{\Omega} |Du|,
\]

where now \( |Du| \) is a measure. The sharp constant is then

\[
c_1^1 = \inf \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), \int_{\Omega} |u|^{\frac{n}{n-1}} = 1 \text{ and } \int_{\Omega} u = 0 \right\}.
\]

When \( \Omega \) is a ball, A. Cianchi \cite{3} has showed that the sharp constant is achieved. In the general case, Zhu M. \cite{22, Theorem 1.3} has showed that if one restricts the inequality to functions in \( BV(\Omega) \) that take two values, the sharp constant is achieved.

Our first result is

**Theorem 1.** Let \( n \geq 2 \). If \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^2 \) boundary, then \( c_1^1 \) is achieved.

When \( n = 2 \), this answers a question mentioned by H. Brezis and J. Van Schaftingen \cite{2, problem 3].

Instead of considering the inequality (1) under the constraint \( \int_{\Omega} u = 0 \), one can drop the condition and take the infimum over functions that only differ by a constant:

\[
(3) \quad c \inf_{\lambda \in \mathbb{R}} \left( \int_{\Omega} |u - \lambda|^\frac{np}{n-p} \right)^{1 - \frac{p}{n}} \leq \int_{\Omega} |\nabla u|^p;
\]

this is equivalent to (1) under the constraint \( \int_{\Omega} |u|^\frac{np}{n-p - 2} u = 0 \). In this setting, A. V. Demyanov and A. I. Nazarov \cite{5, theorem 7.4} have proved that when

\[
1 < p < \min \left( \frac{3n + 1 - \sqrt{5n^2 + 2n + 1}}{2}, \frac{n^2 + 3n + 1 + \sqrt{n^4 + 6n^3 - n^2 - 2n + 1}}{2(3n + 2)} \right),
\]

the optimal constant

\[
\inf \left\{ \int_{\Omega} |\nabla u|^p : u \in W^{1,p}(\Omega), \int_{\Omega} |u|^{\frac{np}{n-p}} = 1 \text{ and } \int_{\Omega} |u|^{\frac{np}{n-p} - 2} u = 0 \right\}
\]

is achieved.

We consider the corresponding problem of determining whether

\[
c_1^{\frac{1}{n-1}} = \inf \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), \int_{\Omega} |u|^{\frac{n}{n-1}} = 1 \text{ and } \int_{\Omega} |u|^{\frac{1}{n-1} - 1} u = 0 \right\}
\]
is achieved.

**Theorem 2.** Let \( n \geq 2 \). If \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^2 \) boundary, then \( c_{\Omega}^{1} \) is achieved.

More generally, we can consider the quantity

\[
c_{\Omega}^{q} = \inf \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), \int_{\Omega} |u|^q = 1 \text{ and } \int_{\Omega} |u|^{q-1}u = 0 \right\}
\]

for \( q \in (1, \frac{n}{n-1}) \). If \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^1 \) boundary, \( c_{\Omega}^{q} > 0 \). In this setting theorems 1 and 2 are particular cases of

**Theorem 3.** Let \( n \geq 2 \). If \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^2 \) boundary, then for every \( q \in (0, \frac{n}{n-1}) \), \( c_{\Omega}^{q} \) is achieved.

The inequality (1) is also valid on a compact manifold without boundary \( M \). In this setting, Zhu M. [20] has showed that the sharp constant is achieved when \( p \in (1, (1 + \sqrt{1 + 8n})/4) \) on the sphere. A. V. Demyanov and A. I. Nazarov have showed that if there exist a point of \( M \) at which the scalar curvature is positive, then there exists \( \delta > 0 \) such that the corresponding sharp constant is achieved for \( n \geq 3 \) and \( p \in (1, \frac{n+2}{3} + \delta) \) [5, theorem 5.1]. For the inequality (3), A. V. Demyanov and A. I. Nazarov [5, theorem 6.1] have proved that if the scalar curvature is positive at some point and

\[
1 < p < \max \left( 2n + 1 - \sqrt{3n^2 + 2n + 1}, \frac{n^2 + 6n + 2 + \sqrt{n^4 + 12n^3 - 8n + 4}}{2(5n + 4)} \right),
\]

then the sharp constant is achieved. Moreover, they have proved that for \( p \geq \frac{n+1}{2} \), the sharp constant is not achieved on the \( n \)-dimensional sphere.

For a compact \( C^1 \) Riemannian manifold \( M \) of dimension \( n \geq 2 \) we consider whether the quantity

\[
c_{M}^{q} = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in BV(\Omega), \int_{\Omega} |u|^\frac{n}{n-1} = 1 \text{ and } \int_{\Omega} |u|^{q-1}u = 0 \right\}
\]

is achieved, with \( q \in (0, \frac{n}{n-1}) \). In the case where \( n \geq 3 \) and the manifold has somewhere positive scalar curvature, one has the counterpart of theorem 3.

**Theorem 4.** Let \( n \geq 3 \) and \( M \) be an \( n \)-dimensional compact Riemannian \( C^2 \) manifold. If there exists a \( a \in M \) such that the scalar curvature \( S(a) \) at \( a \) is positive, then for every \( q \in (0, \frac{n}{n-1}) \), \( c_{M}^{q} \) is achieved.

In dimension 2, the same method only yields

\footnote{In some cases, the condition on the scalar curvature is reversed. Considers the quantity}

\[
\sup \left\{ \left( \int_{M} |u|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} - K_{2,n} \int_{M} |\nabla u|^{2} : \int_{M} |u|^{1} = 1 \right\}
\]

\footnote{This is in contradiction with a result of Zhu M. [20, theorem 3.1].}
Theorem 5. Let $M$ be a 2–dimensional compact Riemannian $C^2$ manifold. If there exists $a \in M$ such that the scalar curvature $S_a$ at $a$ is positive, then for every $q \in (0, 1)$, $c_M^q$ is achieved.

If we strengthen the condition on the curvature we obtain

Theorem 6. Let $M$ be a 2–dimensional compact Riemannian $C^2$ manifold. If there exists $a \in M$ such that the scalar curvature $S_a$ at $a$ satisfies

\[ S_a > \frac{8\pi}{\mathcal{H}^2(M)}, \]

then for every $q \in (0, 2)$, $c_M^q$ is achieved.

Here $\mathcal{H}^2(M)$ denotes the two-dimensional Hausdorff measure of the manifold $M$.

In particular, theorem 6 allows to solve completely the case of surfaces of Euler–Poincaré characteristic 2 of nonconstant gaussian curvature.

Theorem 7. Let $M$ be a 2–dimensional compact $C^2$ Riemannian manifold with nonconstant scalar curvature. If $\chi(M) = 2$, then for every $q \in (0, 2)$, $c_M^q$ is achieved.

While the sphere does not satisfy the hypotheses of the previous theorem, we have

Theorem 8. For every $q \in (0, 2)$, $c_{S^2}^q$ is achieved.

2. Preliminaries

Recall that for $\Omega \subseteq \mathbb{R}^n$ open, $\text{BV}(\Omega)$ denotes the space of functions $u \in L^1(\Omega)$ such that

\[ \sup \left\{ \int_\Omega u \text{div} \varphi : \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \text{ and } |\varphi| \leq 1 \right\} < \infty. \]

If $u \in \text{BV}(\Omega)$, then there exists a vector measure $Du$ such that for every $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$,

\[ \int_\Omega u \text{div} \varphi = -\int_\Omega \varphi \cdot Du. \]

In particular, one can consider the variation $|Du|$ of $Du$ which is a bounded measure on $\Omega$.

The optimal Sobolev inequality of H. Federer and W. H. Fleming [8] states that for every $u \in \text{BV}(\mathbb{R}^n)$,

\[ \frac{\pi^{\frac{2}{n}}}{\Gamma(\frac{n}{2} + 1)^{\frac{1}{n}}} \left( \int_{\mathbb{R}^n} |u|^\frac{n}{n-1} \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |Du|. \]

The proof also shows that the constant if optimal and that it is achieved by multiples of characteristic functions of balls (see also [4]). If $\mathbb{R}^n_+$ denotes the $n$–dimensional half-space, one deduces from [4] by a reflexion argument that for every $u \in \text{BV}(\mathbb{R}^n_+)$, one has

\[ \frac{\pi^{\frac{2}{n}}}{2\pi^{\frac{2}{n}} \Gamma(\frac{n}{2} + 1)^{\frac{1}{n}}} \left( \int_{\mathbb{R}^n_+} |u|^\frac{n}{n-1} \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n_+} |Du|. \]
One can show that the constant is achieved by characteristic functions of intersections of balls centered on the boundary of $\mathbb{R}^n_+$ with $\mathbb{R}^n$ itself.

A consequence that we shall use is

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a domain with a $C^1$ boundary. For every $a \in \partial \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in BV(\Omega)$ and $\text{supp } u \subset B(a, \delta)$, then

$$
\left( \frac{c_n^*}{2\pi} - \varepsilon \right) \left( \int_\Omega |u|^{n-1} \right)^{1-n} \leq \int_\Omega |Du|.
$$

### 3. Extremal functions on bounded domains

#### 3.1. Existence by concentration-compactness.

A first ingredient in our proof of theorem 3 is

**Proposition 3.1.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^2$ boundary. If $c_q^* \Omega < c_n^*$, then $c_q^* \Omega$ is achieved.

In the case of the sharp constants for embeddings of $W^{1,p}(\Omega)$, with $1 < p < n$, the counterpart has been proved has been proved by A. V. Demyanov and A. I. Nazarov [5, proposition 7.1]. An alternative argument has been provided by S. De Valeriola and M. Willem [6, Theorem 4.1].

Our main tool shall be

**Proposition 3.2.** Let $(u_m)_{m \in \mathbb{N}}$ in $BV(\Omega)$ converge weakly to some $u \in BV(\Omega)$. Assume that there exist two bounded measures $\mu$ and $\nu$ on $\bar{\Omega}$ such that $(|u_m|^{n-1})_{m \in \mathbb{N}}$ and $(|Du_m|)_{m \in \mathbb{N}}$ converge weakly in the sense of measures to $\mu$ and $\nu$ respectively. Then there exists some at most countable set $J$, distinct points $x_j \in \bar{\Omega}$ and real numbers $\nu_j > 0$ with $j \in J$ such that

$$
\nu = |u|^{n-1} + \sum_{j \in J} \nu_j \delta_{x_j},
$$

$$
\mu \geq |Du| + \frac{c_n^*}{2\pi} \sum_{j \in J} \nu_j^{1-\frac{1}{n}} \delta_{x_j}.
$$

This result is a variant of the corresponding result on $\mathbb{R}^n$ due to P.-L. Lions [16, Lemma 1.1]. P.-L. Lions, F. Pacella and M. Tricarico [17, lemma 2.2] have adapted it to functions vanishing on a part of the boundary.

**Proof.** We follow the proof of P.-L. Lions [16, lemma 1.1]. First assume that $u = 0$. Then, using Lemma 2.1 and Rellich’s compactness theorem, one shows that for every $a \in \bar{\Omega}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varphi \in C(\Omega)$, $\varphi \geq 0$ and $\text{supp } \varphi \subset B(a, \delta)$, then

$$
\left( \int_\Omega \varphi^{n-1} \mu \right)^{1-\frac{1}{n}} \leq \int_\Omega \varphi \nu.
$$

One deduces then the conclusion when $u = 0$ by the argument of [16, lemma 1.2].

The case $u \neq 0$ follows then by standard arguments. \qed
Proof of proposition 3.1. Let \((u_m)_{m \in \mathbb{N}}\) be a sequence in \(BV(\Omega)\) such that
\[
\int_\Omega |Du_m| \to c_{\Omega}^q, \quad \int_\Omega |u_m|^{\frac{n}{n-1}} = 1, \quad \int_\Omega |u_m|^{q-1} u_m = 0.
\]
Going if necessary to a subsequence, we can assume that the assumptions of lemma 3.2 are satisfied. Since \((u_m)_{m \in \mathbb{N}}\) converges weakly to \(u \in BV(\Omega)\) and \(q < \frac{n}{n-1}\), by Rellich’s compactness theorem,
\[
\int_\Omega |u|^{q-1} u = \lim_{m \to \infty} \int_\Omega |u_m|^{q-1} u_m = 0.
\]
Assume by contradiction that
\[
\int_\Omega |u|^{\frac{n}{n-1}} < 1.
\]
In view of proposition 3.2 we have
\[
\lim_{m \to \infty} \int_\Omega |u_m|^{\frac{n}{n-1}} = \int_\Omega \nu = \int_\Omega |u|^{\frac{n}{n-1}} + \sum_{j \in J} \nu_j,
\]
and thus \(J \neq \emptyset\). On the other hand,
\[
c_{\Omega}^q = \lim_{m \to \infty} \int_\Omega |Du_m| = \int_\Omega |Du| + \sum_{j \in J} \frac{c_n^*}{2\pi} \frac{1}{|j|^{\frac{n-1}{n}}}
\[
\geq c_{\Omega}^q \left( \int_\Omega |u|^{\frac{n}{n-1}} \right)^{1-\frac{1}{n}} + c_{\Omega}^q \sum_{j \in J} \nu_j^{1-\frac{1}{n}}
\[
\geq c_{\Omega}^q,
\]
which is a contradiction.

\[\square\]

3.2. Upper estimate on the sharp constant. We shall now prove that the condition of proposition 3.1 is indeed satisfied.

Proposition 3.3. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with \(C^2\) boundary. If \(q < \frac{n^2}{n^2-1}\), then
\[
c_{\Omega}^q < \frac{c_n^*}{2\pi}.
\]

Proof. Since \(\Omega\) is bounded, there exists \(a, b \in \partial \Omega\) such that \(|a-b| = \sup\{|x-y| : x, y \in \partial \Omega\}\). Since \(\partial \Omega\) is of class \(C^2\), its mean curvature \(H_a\) at \(a\) satisfies
\[
H_a \geq \frac{1}{|a-b|} > 0.
\]
For \(\varepsilon > 0\) such that \(\Omega \setminus B(a, \varepsilon) \neq \emptyset\), consider the function \(u_\varepsilon : \Omega \to \mathbb{R}\) defined by
\[
u = \lambda_{\Omega \cap B(a, \varepsilon)} - \beta_\varepsilon \lambda_{\Omega \setminus B(a, \varepsilon)},
\]
where
\[
\beta_\varepsilon = \left( \frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\Omega \cap B(a, \varepsilon))} - 1 \right)^{-\frac{1}{q}}.
\]
The quantity \(\mathcal{L}^n(\Omega \cap B(a, \varepsilon))\) can be expanded in terms of the mean curvature [13] equation (1) as
\[
\mathcal{L}^n(\Omega \cap B(a, \varepsilon)) = \frac{\pi^\frac{n}{2} \varepsilon^n}{2\Gamma\left(\frac{n}{2} + 1\right)} \left( 1 - \frac{nH_a \varepsilon}{(n+1)B\left(\frac{1}{2}, \frac{n-1}{2}\right)} + o(\varepsilon) \right),
\]
where $B$ denotes Euler’s beta function. In particular, one has
\[ \beta_\varepsilon = \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2} + 1)} \right)^{\frac{1}{n}} \varepsilon^\frac{n}{2} (1 + o(1)). \]

Since $q < \frac{n^2}{n^2 - 1}$, we have $\beta_\varepsilon^{-1} = o(\varepsilon^{n+1})$ and therefore
\[ \int \frac{|u_\varepsilon|^n}{\varepsilon^{n-1}} = \mathcal{L}^n (\Omega \cap B(a, \varepsilon)) - \beta_\varepsilon^{-1} \mathcal{L}^n (\Omega \setminus B(a, \varepsilon)) = \frac{\pi^{\frac{n}{2}} n}{2\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{n}{n+1} \frac{H_a \varepsilon}{B(\frac{1}{2}, \frac{n-1}{2})} + o(\varepsilon) \right). \]

Similarly, one computes
\[ \int |D u_\varepsilon| = (1 + \beta_\varepsilon) \int |D \chi_{\Omega \cap B(a, \varepsilon)}| = \varepsilon^{n-1} \frac{\pi^{\frac{n}{2}} n}{2\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{H_a \varepsilon}{B(\frac{1}{2}, \frac{n-1}{2})} + o(\varepsilon) \right). \]

One has finally, since $\beta_\varepsilon = o(\varepsilon)$,
\[ \frac{\int \mathcal{L}^n |D u_\varepsilon|}{\left( \int \varepsilon^{\frac{n}{n-1}} |u_\varepsilon|^{-\frac{n}{n-1}} \right)^{1 - \frac{n}{n}}} = \frac{\pi^{\frac{n}{2}} n}{2\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{2H_a \varepsilon}{(n+1)B(\frac{1}{2}, \frac{n-1}{2})} + o(\varepsilon) \right), \]

it follows then that for $\varepsilon > 0$ sufficiently small,
\[ \frac{\int \mathcal{L}^n |D u_\varepsilon|}{\left( \int \varepsilon^{\frac{n}{n-1}} |u_\varepsilon|^{-\frac{n}{n-1}} \right)^{1 - \frac{n}{n}}} < c_n^*, \]

which is the desired conclusion. \qed

In the previous proof, the existence of a point of the boundary with positive mean curvature is crucial. We would like to point out that in the problem of optimal functions for Sobolev–Hardy inequalities with a point singularity, one needs the boundary to have negative mean curvature at that point of the boundary [9].

**Proposition 3.4.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^1$ boundary. For every $q \in (0, \frac{n}{n-1})$,
\[ c_\Omega^q \leq c_\Omega^{\frac{1}{n-1}}. \]

**Proof.** Let $u \in \text{BV}(\Omega)$ be such that $\int_\Omega |u|^{\frac{1}{n-1}} - 1 = 0$. Consider $\lambda \in \mathbb{R}$ such that $\int_\Omega |u - \lambda|^{q-1}(u - \lambda) = 0$. One has then
\[ c_\Omega^q \left( \int |u|^{\frac{\frac{1}{n-1}}{q-1}} \right)^{1 - \frac{1}{n}} \leq c_\Omega^q \left( \int |u - \lambda|^{\frac{\frac{n}{n-1}}{q-1}} \right)^{1 - \frac{1}{n}} \leq \int |D(u - \lambda)| = \int |D u|, \]

and therefore $c_\Omega^q \leq c_\Omega^{\frac{1}{n-1}}$. \qed

By combining proposition 3.3 together with proposition 3.4 we obtain
Proposition 3.5. Let \( n \geq 2 \) and \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^2 \) boundary. For every \( q \in (0, \frac{n}{n-1}) \),
\[
 c^q_{\Omega} < \frac{c^*_{n}}{2^n}.
\]

Proof. Since \( n \geq 2 \), one has
\[
 \frac{1}{n-1} < \frac{n^2}{n^2 - 1}.
\]
Therefore, by proposition 3.3 \( c^{\frac{1}{n-1}}_{\Omega} < \frac{c^*_{n}}{2^n} \). Hence, by proposition 3.4,
\[
 c^q_{\Omega} \leq c^{\frac{1}{n-1}}_{\Omega} < \frac{c^*_{n}}{2^n}.
\]

We are now in position to prove theorem 3, which contains theorems 1 and 2 as particular cases.

Proof of theorem 3. By proposition 3.5 \( c^q_{\Omega} < \frac{c^*_{n}}{2^n} \). Proposition 3.1 is thus applicable and yields the conclusion.

□

4. Extremal functions on manifolds

4.1. Existence by concentration compactness. The counterpart of proposition 3.1 on manifolds is given by

Proposition 4.1. Let \( n \geq 2 \) and let \( M \) be an \( n \)-dimensional compact Riemannian \( C^1 \) manifold. If
\[
 c^q_{M} < c^*_{n},
\]
then \( c^q_{M} \) is achieved.

4.2. Upper estimate on the sharp constant. We now turn on to estimates on the sharp constant,

Proposition 4.2. Let \( n \geq 3 \) and \( M \) be an \( n \)-dimensional compact Riemannian \( C^2 \) manifold. If there exists \( a \in M \) such that the scalar curvature \( S_a \) at \( a \) is positive, then for every \( q \in \left(0, \frac{n^2}{n^2 + n - 2}\right) \),
\[
 c^q_{M} < c^*_{n}.
\]

Proof. For \( \varepsilon > 0 \) such that \( M \setminus \overline{B(a, \varepsilon)} \neq \emptyset \), where \( B(a, \varepsilon) \) is a geodesic ball of radius \( \varepsilon \) centered at \( a \), consider the function \( u_\varepsilon : M \to \mathbb{R} \) defined by
\[
 u_\varepsilon = \chi_{B(a, \varepsilon)} - \beta_\varepsilon \chi_{M \setminus B(a, \varepsilon)},
\]
where
\[
 \beta_\varepsilon = \left( \frac{\mathcal{H}^n(M)}{\mathcal{H}^n(B(a, \varepsilon))} - 1 \right)^{-\frac{1}{q}}.
\]
The measure of the ball can be extended as
\[
 \mathcal{H}^n(B(a, \varepsilon)) = \frac{\pi^\frac{2n}{n}}{\Gamma(\frac{n}{2} + 1)} \left(1 - \frac{S_a \varepsilon^2}{6(n + 2)} + o(\varepsilon^2)\right)
\]
(see for example [12, Theorem 3.1]). In particular, one has
\[
 \beta_\varepsilon = \left( \frac{\pi^\frac{2n}{n}}{\Gamma(\frac{n}{2} + 1) \mathcal{H}^n(M)} \right)^{\frac{1}{q}} \varepsilon \left(1 + o(1)\right).
\]
Since $q < \frac{n^{2}}{n^{2} + n - 2}$, we have $\beta_{\varepsilon} = o(\varepsilon^{n+2})$ and therefore

$$\int_{M} |u_{\varepsilon}|^{n-1} = \frac{\pi^{\frac{n}{2}} \varepsilon^{n}}{\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{S_{a} \varepsilon^{2}}{6(n + 2)} + o(\varepsilon^{2}) \right).$$

One also computes

$$\int_{M} |Du_{\varepsilon}| = (1 + \beta_{\varepsilon}) \int_{M} |D\chi_{B(a,\varepsilon)}|$$

$$= \frac{n \pi^{\frac{n}{2}} \varepsilon^{n-1}}{\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{S_{a} \varepsilon^{2}}{6n} + o(\varepsilon^{2}) \right).$$

The combination of the previous developments gives

$$\int_{M} |Du_{\varepsilon}| \left( \int_{M} |u_{\varepsilon}|^{\frac{n}{n-1}} \right)^{1-\frac{n}{2}} = \frac{n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \left( 1 - \frac{S_{a} \varepsilon^{2}}{2n(n + 2)} + o(\varepsilon^{2}) \right).$$

It follows then that for $\varepsilon > 0$ sufficiently small,

$$\int_{M} |Du_{\varepsilon}| \left( \int_{M} |u_{\varepsilon}|^{\frac{n}{n-1}} \right)^{1-\frac{n}{2}} < c^{*}_{n},$$

which is the desired conclusion. □

The counterpart of proposition 3.4 can be obtained straightforwardly.

**Proposition 4.3.** Let $n \geq 2$ and $M \subset \mathbb{R}^{n}$ be an $n$-dimensional compact Riemannian manifold. For every $q \in (0, \frac{n}{n-1})$,

$$c_{q}^{M} \leq c_{1}^{\frac{1}{n-1}}.$$

This allows us to obtain a counterpart of proposition 3.5.

**Proposition 4.4.** Let $n \geq 3$ and $M$ be an $n$-dimensional compact Riemannian $C^{2}$ manifold. If there exists $a \in M$ such that the scalar curvature $S_{a}$ at $a$ is positive, then for every $q \in (0, \frac{n}{n-1})$,

$$c_{q}^{M} \leq c_{n}^{*}.$$

**Proof.** One checks that if $n \geq 3$, $\frac{1}{n-1} < \frac{n^{2}}{n^{2} + n - 2}$. One can then apply proposition 4.2 with $q = \frac{1}{n-1}$ and then conclude with proposition 4.3. □

This allows us to prove theorem 4 on manifolds.

**Proof of theorem 4.** Since $n \geq 3$, this follows from proposition 4.4 and proposition 4.1. □

We can also prove theorem 5 on surfaces.

**Proof of theorem 5.** This follows from proposition 4.2 and proposition 4.1. □
4.3. **Refined upper estimates.** We now give a condition on the scalar curvature that gives a strict inequality in the critical case $q = \frac{n^2}{n^2 + n - 2}$. Although we only need the result for $n = 2$, we state it for all dimensions.

**Proposition 4.5.** Let $M$ be an $n$–dimensional compact Riemannian $C^2$ manifold. If there exists $a \in M$ such that the scalar curvature $S_a$ at $a$ satisfies

\begin{equation}
S_a > \frac{2(n + 2)}{n - 1} \left( \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right)}{H^n(M)} \right)^{\frac{2}{n}},
\end{equation}

then for $q = \frac{n^2}{n^2 + n - 2}$,

$$c_q^g M < c_q^*.$$ 

**Proof.** Since $q = \frac{n^2}{n^2 + n - 2}$, the computations of the proof of proposition 4.2 give instead of (5)

\begin{equation}
\int_M |u_\varepsilon|^\frac{n}{n - 1} = \frac{\pi^{\frac{n}{2}} \varepsilon^n}{\Gamma\left(\frac{n}{2} + 1\right)} \left( 1 + \left( \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\pi^{\frac{n}{2}} H^n(M)} \right)^{\frac{2}{n}} \varepsilon^2 - \frac{S_a}{6(n + 2)} \varepsilon^2 + o(\varepsilon^2) \right),
\end{equation}

and then, instead of (6),

\begin{equation}
\int_M |Du_\varepsilon| \left( \int_\Omega |u_\varepsilon|^\frac{n}{n - 1} \right)^{1 - \frac{1}{n}} = \frac{n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)^{\frac{1}{n}}} \left( 1 + \frac{n - 1}{n} \left( \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\pi^{\frac{n}{2}} H^n(M)} \right)^{\frac{2}{n}} \varepsilon^2 - \frac{S_a}{2n(n + 2)} \varepsilon^2 + o(\varepsilon) \right),
\end{equation}

and one checks that in view of (7), the inequality is satisfied by taking some small $\varepsilon > 0$. \hfill \Box

**Proof of theorem 6** One first notes that proposition 4.3 is applicable with $q = 1$. Therefore by proposition 4.3, for every $q \in (0, 2)$, $c_q^g M < c_q^*$. The conclusion comes from proposition 4.1. \hfill \Box

**Proof of theorem 7** Since $M$ does not have constant scalar curvature, there exists $a \in M$ such that

$$S_a > \frac{1}{H^2(M)} \int_M S.$$ 

Since by the Gauss–Bonnet formula

$$\int_M S = 4\pi \chi(M),$$

we have

$$S_a > \frac{8\pi}{H^2(M)},$$

The conclusion is then given by theorem 6. \hfill \Box
4.4. The case of the sphere.

Proof of theorem 8. By proposition 4.1, we can assume that \( c^q_{S^2} \geq c^*_{S^2} \). Let \( a \in S^2 \) and consider the function \( u : S^2 \to \mathbb{R} \) defined by
\[
u = \chi_{B(a, \frac{\pi}{2})} - \chi_{S^2 \setminus B(a, \frac{\pi}{2})},
\]
i.e., the difference between characteristic functions of opposite hemispheres. One checks that
\[
\int_{S^2} |u|^q - 2 u = 0
\]
and
\[
\frac{\int_{S^2} |Du|}{\left(\int_{S^2} |u|^2\right)^{\frac{1}{2}}} = 2\sqrt{\pi} = c^*_{S^2}.
\]
Since we have assumed that \( c^q_{S^2} \geq c^*_{S^2} \), this proves that the \( c^q_M \) is achieved. □

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