Abstract

We consider the model of auto-ignition (thermal explosion) of a free round reactive turbulent jet introduced in [11]. This model falls into the general class of Gelfand-type problems and constitutes a boundary value problem for a certain semi-linear elliptic equation that depends on two parameters: \( \alpha \) characterizing the flow rate and \( \lambda \) (Frank-Kametskii parameter) characterizing the strength of the reaction. Similarly to the classical Gelfand problem, this equation admits a solution when the Frank-Kametskii parameter \( \lambda \) does not exceed some critical value \( \lambda^*(\alpha) \) and admits no solutions for larger values of \( \lambda \). We obtain the sharp asymptotic behavior of the critical Frank-Kametskii parameter in the strong flow limit (\( \alpha \gg 1 \)). We also provide a detailed description of the extremal solution (i.e., the solution corresponding to \( \lambda^* \)) in this regime.

Keywords: Gelfand problem, strong advection, extremal solutions, stable solutions, thermal explosion.

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1 Introduction

In this paper we are concerned with the existence and quantitative properties of solutions of the following problem:

\[
\begin{aligned}
-\Delta u - \alpha r \varphi(r) \frac{\partial u}{\partial r} &= \lambda \psi(r) f(u) & \text{in } B, \\
u &> 0 & \text{in } B, \\
u &= 0 & \text{on } \partial B,
\end{aligned}
\]

where \( B \) is the unit disk in \( \mathbb{R}^2 \) centered at the origin, \( \lambda > 0, \alpha > 0 \) are parameters, \( f : [0, \infty) \to (0, \infty) \) is a \( C^1 \) convex non-decreasing function satisfying:

\[
\int_0^\infty \frac{ds}{f(s)} < \infty,
\]

\( \varphi(r), \psi(r) \) are non-negative, non-increasing Lipschitz continuous functions on \( [0,1] \) satisfying \( \varphi(0) = \psi(0) = 1 \). Moreover, we assume that \( \varphi(r) \) is positive in \( [0,1) \). In addition we assume that

\[
\int_0^1 M(s) ds < \infty,
\]

where

\[
M(s) := \max_{r \in [0,s]} \frac{\psi(r)}{\varphi(r)}.
\]

Equation (1.1) was recently introduced in [11] as a model of the autoignition event of free round reactive turbulent jets. In the context of this model, \( u, f, \varphi, \text{ and } \psi \) are respectively the (appropriately normalized) temperature, reaction rate, flow velocity profile, and product of concentrations of the oxidizing and reactive
components over cross-sections of the jet, while \( \lambda \) is the Frank-Kamenetskii parameter representing the ratio of the heat release of the reaction and the thermal conductivity and \( \alpha \) is the ratio of the injection velocity and the thermal conductivity.

The typical examples considered in the physical literature (e.g. \([1][16]\)) are \( f(u) = \exp(u) \), \( \varphi(r) = \exp(-4r^2) \) or \( \varphi(r) = (1 - r^2)^2 \), and \( \psi = \varphi^{2\alpha_c} \), where \( \alpha_c \) is a Schlichting number which for round turbulent jets is \( \alpha_c \approx 0.75 \), in which cases our assumptions \([12][13]\) are easily seen to be satisfied.

Problem \([11]\) falls into the general class of Gelfand-type problems. The classical Gelfand problem can be obtained from problem \([11]\) by removing the advection term, making the right hand side of the equation independent of \( r \), that is, by setting \( \alpha = 0 \) and \( \psi = 1 \), and replacing the unit disk \( B \) in \( \mathbb{R}^2 \) by a general bounded domain in \( \mathbb{R}^n \). That problem was introduced in 1938 by Frank-Kamenetskii as a model of thermal explosion in a combustion vessel with ideally thermally conducting walls (see \([3][17][19]\) for more details), but became known in the mathematical community due to the chapter written by Barenblatt in a famous review of Gelfand \([9]\). The general properties of solutions of the classical Gelfand problem were studied quite extensively in both mathematical and physical literature, see book \([6]\) for a review of results. The problem considered in this paper inherits many nice features of the classical Gelfand problem. The following proposition summarizes the properties of solutions of problem \([11]\) relevant to the present work.

**Proposition 1.1.** For fixed \( \alpha > 0 \), there exists an extremal value of the Frank-Kamenetskii parameter \( \lambda^* = \lambda^*(\alpha) \in (0, \infty) \) such that:

i) Problem \([11]\) admits a unique minimal (i.e. smallest) classical positive solution \( u_{\lambda,\alpha} \) for \( \lambda \in (0, \lambda^*) \);

ii) Problem \([11]\) admits a unique extremal solution \( u_{\alpha}^* \) defined as

\[
u_{\alpha}^*(x) := \lim_{\lambda \to \lambda^*} u_{\lambda,\alpha}(x),
\]

which is also classical.

iii) Minimal solutions of \([11]\) for \( \lambda \in (0, \lambda^*) \) are radially symmetric, strictly decreasing and satisfy the semi-stability condition

\[
\int_B |\nabla \eta|^2 \mu dx \geq \lambda \int_B \psi f'(u_{\lambda,\alpha}) \eta^2 \mu dx, \quad \forall \eta \in H^1_0(B),
\]

where

\[
\mu(r) = \exp(\alpha \int_0^r \varphi(s) ds);
\]

iv) There are no solutions for \([11]\) when \( \lambda > \lambda^* \).

The proposition above is quite standard. We will present a sketch of the proof in the next section for completeness.

In the context of the autoignition problem, the extremal value \( \lambda^*(\alpha) \) and the extremal solution \( u_{\alpha}^* \) play a very special role. Indeed, in the context of the theory developed in \([11]\), as any theory based on Frank-Kamenetskii approach, the existence of a solution for \([11]\) indicates autoignition failure. From physical standpoint, that means that the reactive component undergoes partial oxidation, which results in establishing a self-similar temperature profile given by the minimal solution of \([11]\). In contrast, the absence of a solution for \([11]\) indicates successful autoignition. Therefore, \( \lambda^*(\alpha) \) determines the boundary between the successful autoignition and the absence thereof. The extremal value of the Frank-Kamentskii parameter \( \lambda^*(\alpha) \) indicates the maximal reaction intensity for a given flow rate for which auto-ignition does not take place. The extremal solution determines the maximal possible self-similar profile.

In practical applications \( \alpha \gg 1 \) and hence one needs to understand the behavior of \( \lambda^*(\alpha) \) for large \( \alpha \). This observation raises the question of asymptotic behavior of \( \lambda^*(\alpha) \) as \( \alpha \to \infty \). Lower and upper bounds on \( \lambda^*(\alpha) \) as \( \alpha \to \infty \) were derived in \([11]\) Theorem 3.1 in the special case \( f(u) = \exp(u) \). In this paper we establish a sharp asymptotic of \( \lambda^*(\alpha) \) and give quite precise description of an extremal solution in this limit for a general class of nonlinearities \( f(u) \) and functions \( \varphi, \psi \) under very mild regularity assumptions. Our main results are given by the following theorems.

The first theorem gives sharp asymptotic for \( \lambda^*(\alpha) \) for large values of \( \alpha \):
Theorem 1.1. Let
\[ K := \int_0^\infty \frac{ds}{f(s)}. \] (1.8)

Then,
\[ \lim_{\alpha \to \infty} \lambda^*(\alpha) \left( \frac{2K\alpha}{\log \alpha} \right)^{-1} = 1. \] (1.9)

The second theorem provides details of the behavior of the extremal solution when \( \alpha \gg 1 \):

Theorem 1.2. Let \( u^*_\alpha \) be the extremal solution of (1.1). Then, as \( \alpha \to \infty \), we have

i) \( u^*_\alpha(x) \to 0 \quad \forall x \in B \setminus \{0\}, \quad u^*_\alpha(0) \to \infty, \) (1.10)

and

ii) \( \int_B u^*_\alpha(x) dx \to 0, \quad \int_B \psi(x)f(u^*_\alpha(x)) dx \to f(0) \int_B \psi(x) dx. \) (1.11)

While Proposition [1.1] ensures that extremal solutions of problem (1.1) are bounded, the establishment of a reasonable uniform upper bound appeared to be a difficult task. However, in case of sufficiently regular non-linearities or non-linearities with sufficiently fast growth at infinity we have the following result.

Theorem 1.3. Assume that there exist constants \( 0 < c_0 < 1, c_1 > 1 \) and \( t_0 > 0 \) such that
\[ f(t_2) \geq c_1 f(t_1), \quad t_2 > t_1 > t_0, \] (1.12)

implies
\[ (1 - c_0)f'(t_2) \geq f'(t_1). \] (1.13)

Then,
\[ u^*_\alpha(0) \leq cA, \] (1.14)

where \( A \) is the solution of
\[ f'(A) = c \log \alpha \] (1.15)

and \( c > 0 \) is some constant independent of \( \alpha \).

Moreover,
\[ \int_B (u^*_\alpha(x))^p dx \to 0 \quad \text{as} \quad \alpha \to \infty, \] (1.16)

for any \( 1 \leq p < \infty \).

Remark 1.1. It is easy to check that the assumptions of Theorem 1.3 are satisfied for most typical non-linearities such as \( f(u) = \exp(u) \) and \( f(u) = (1 + u)^p \) for \( p > 1 \). Moreover, they are also satisfied provided \( f \) is a \( C^2 \) function such that \( f'(s), f''(s) > 0 \) and \( f''(s) \) is strictly increasing on \( (0, \infty) \) (see Lemma 5.2).

Remark 1.2. In the case of exponential nonlinearity, we also have that
\[ \lambda^*(\alpha) = \frac{2\alpha}{\log \alpha} \left( 1 + O \left( \frac{1}{\sqrt{\log \alpha}} \right) \right) \] (1.17)

and
\[ u^*_\alpha(0) = O(\log(\log \alpha)), \] (1.18)

see Lemma [5.2] These results are consistent with the formal asymptotic extremal solution of (1.1) obtained in [11].

The paper is organized as follows. In section 2 we are setting up the stage by giving necessary definitions, providing standard results and introducing rescaling which makes the analysis more convenient. In sections 3, 4, 5 we give proofs of theorems 1.1, 1.2, and 1.3 respectively.
2 Preliminaries

In this section we outline a proof of Proposition 1.1 and introduce alternative forms of problem (1.1), which will be used in the later sections.

First we observe that problem (1.1) can be written in the divergence form. Indeed one can verify by direct computations that (2.1) can be rewritten as follows

\[
\begin{cases}
-\nabla \cdot (\mu(r)\nabla u) = \lambda \mu(r)\psi(r)f(u) & \text{in } B,
\quad u > 0 \quad \text{in } B,

\quad u = 0 \quad \text{on } \partial B,
\end{cases}
\]

(2.1)

where \(\mu\) is given by (1.7). We note that \(\mu \in C^{1,\omega} (0 < \omega < 1)\) as follows from definition and properties of \(\varphi\), while \(\psi\) is Lipshitz continuous. The results presented in this section deal with the situation when \(\alpha > 0\) is fixed. Therefore, we will omit subscript \(\alpha\) when referring to minimal and extremal solutions of (2.1).

Proposition 1.1 is basically a compilation of well known results (or their minimal adaptations) presented in [2, 5, 12, 14, 15]. It follows from the sequence of Lemmas 2.1, 2.2, 2.3, presented below.

A proof of Proposition 1.1 is based on the construction of sub and super-solutions for problem (2.1). Following [7, 15], we define a classical positive super-solution of (2.1) as a function \(\bar{u}\) classically. Moreover, since each \(u\) is radial, as follows from the uniqueness. We now define the minimal solution of (2.1) as

\[
\begin{cases}
-\nabla \cdot (\mu(r)\nabla \bar{u}) \geq \lambda \mu(r)\psi(r)f(\bar{u}) & \text{in } B, \\
\quad \bar{u} \geq 0 \quad \text{on } \partial B,
\end{cases}
\]

(2.2)

and classical non-negative sub-solution of (2.1) as a function \(\bar{u} \leq 0 \text{ in } B\) such that

\[
\begin{cases}
-\nabla \cdot (\mu(r)\nabla \bar{u}) \leq \lambda \mu(r)\psi(r)f(\bar{u}) & \text{in } B, \\
\quad \bar{u} = 0 \quad \text{on } \partial B,
\end{cases}
\]

(2.3)

We note that under the assumptions of this paper \(\bar{u} = 0\) is always a sub-solution.

Lemma 2.1. Assume (2.1) admits positive, classical super-solution \(\bar{u}\). Then (2.1) admits a unique minimal, positive classical solution \(u_{\lambda} \in C^{2,\omega}(\bar{B})\). This minimal solution is radially symmetric, strictly decreasing and bounded by \(\bar{u}\) from above.

Proof. The minimal solution \(u_{\lambda}\) of (2.1) is obtained by a construction using monotone iteration arguments. Namely, we consider a sequence of functions \(\{u_n\}_{n=0}^{\infty}\) with \(u_0 = 0\) and \(u_n\) defined as

\[
\begin{cases}
-\nabla \cdot (\mu(r)\nabla u_n) + \Omega u_n = \lambda \mu(r)\psi(r)f(u_{n-1}) + \Omega u_{n-1} & \text{in } B, \\
\quad u_n = 0 \quad \text{on } \partial B,
\end{cases}
\]

(2.4)

for \(n \geq 1\), where \(\Omega > 0\) is an arbitrary constant.

As follows from [10] Theorem 6.14 for each \(n\) problem (2.1) admits a unique solution \(u_n \in C^{2,\omega}(\bar{B})\). Each \(u_n\) is radial, as follows from the uniqueness. We now define the minimal solution of (2.1) as

\[
u_{\lambda}(x) := \lim_{n \to \infty} u_n(x).\]

(2.5)

By [18] Theorem 2.1 we have that \(u_{\lambda}\) defined by (2.5) satisfies \(\bar{u} \geq u_{\lambda} > 0\) in \(B\), belongs to \(C^{2,\omega}(\bar{B})\) and solves (2.1) classically. Moreover, since each \(u_n\) is radially symmetric we have that \(u_{\lambda}\) is also radially symmetric so that \(u_{\lambda}(x) = u_{\lambda}(|x|) = u_{\lambda}(r)\). Consequently, any minimal solution constructed above satisfies

\[
\begin{cases}
-\frac{d}{dr}(r \mu(r) \frac{d}{dr} u_{\lambda}) = \lambda r \mu(r)\psi(r)f(u_{\lambda}) & 0 < r < 1, \\
\frac{d}{dr} u_{\lambda}(0) = 0, \\
u_{\lambda}(1) = 0.
\end{cases}
\]

(2.6)

Integrating (2.6) we also have that for \(r \in (0,1]\),

\[
\frac{d}{dr} u_{\lambda}(r) = -\frac{\lambda}{r \mu(r)} \int_0^r \mu(s)\psi(s)f(u_{\lambda}(s))ds < 0,
\]

(2.7)

and hence \(u_{\lambda}\) is strictly decreasing. \(\square\)
The following lemma uses the notion of a weak solution. Similarly to [3], we define a weak solution of (2.1) as a non-negative function \( u \in L^1(B) \) such that \( \psi f(u) \) is \( L^1(B) \), where dist\(_{\partial B}(x)\) is the distance from \( x \) to the boundary of \( B \) and

\[
- \int_B u \nabla \cdot (\mu \nabla \xi) dx = \int_B \psi f(u) \xi dx,
\]

for all \( \xi \in C^2(B) \) with \( \xi = 0 \) on \( \partial B \).

**Lemma 2.2.** Problem (2.1) admits a minimal classical solution \( u_\lambda \) for \( 0 < \lambda < \lambda^* < \infty \). Moreover, the extremal solution \( u^* \) defined by (1.5) is a weak solution of (2.1).

**Proof.** First observe that Lemma 2.2. Problem (2.1) with \( \lambda \) is a super-solution for (2.1) provided \( \lambda \) is a non-decreasing function of \( \lambda \). This follows from the fact that \( u_\lambda \) is a super-solution for problem (2.1) with \( \lambda < \lambda' \). Hence, if (2.1) with \( \lambda = \lambda' \) admits a classical solution, then (2.1) admits a classical solution for \( \lambda \in (0, \lambda'] \).

Next, let \( \tau \) be a solution of

\[
\begin{cases}
- \nabla \cdot (\mu \nabla \tau) = 1 & \text{in } B, \\
\tau = 0 & \text{on } \partial B.
\end{cases}
\]

It is easy to see that \( \tau \) is a super-solution for (2.1) provided \( \lambda \leq (\mu(1)f(\tau(0)))^{-1} \). This establishes the existence of a minimal solution for small enough \( \lambda \).

Now let us show that \( \lambda^* < \infty \), which is done by a slight adaptation of [3] Lemma 5]. This adaptation is needed because \( \psi \) might be zero in some portion of \( B \). By convexity of \( f \) we have that there is \( \varepsilon > 0 \) such that \( f(s) > \varepsilon s \) for \( s \geq 0 \). Hence

\[
- \nabla \cdot (\mu \nabla u) \geq \varepsilon \lambda \mu u \quad \text{in } B.
\]

Let \( \kappa_1, \xi_1 \) be the principal eigenvalue and the corresponding eigenfunction of the generalized eigenvalue problem

\[
\begin{cases}
- \nabla \cdot (\mu \nabla \xi) = \kappa \mu \xi & \text{in } B, \\
\xi = 0 & \text{on } \partial B.
\end{cases}
\]

Variational characterization of \( \kappa_1 \) and arguments identical to these of [10] Theorem 8.38 show that \( \kappa_1 > 0 \) and \( \xi_1 > 0 \) in \( B \).

Multiplying (2.10) by \( \xi_1 \) and integrating by parts we obtain that

\[
\kappa_1 \int_B \psi \mu u \xi_1 dx \geq \varepsilon \lambda \int_B \psi \mu u \xi_1 dx.
\]

Thus, \( \varepsilon \lambda \leq \kappa_1 \) and hence \( \lambda^* \leq \kappa_1 / \varepsilon \).

Finally, proceeding as in the proof of [3] Lemma 5], with the only modification that \( -\Delta \) is replaced by \( -\nabla \cdot (\mu \nabla (\cdot)) \), we recover that \( u^* \) is a weak solution of (2.1).

**Lemma 2.3.** Problem (2.1) admits neither classical, nor weak solutions for \( \lambda > \lambda^* \).

**Proof.** The proof of this lemma is a line by line adaptation of [3] Theorem 3] with the only difference that \( -\Delta \) is replaced by \( -\nabla \cdot (\mu \nabla (\cdot)) \) and is omitted here.

As a next step we give a proof of the semi-stability condition (1.6). The proof is similar to the one of [18] Theorem 4.2], with a slight modification, which is required due to the restriction on regularity of the nonlinear term in (2.1).

**Lemma 2.4.** The semi-stability condition (1.6) holds for any minimal solution \( u_\lambda \) with \( \lambda \in (0, \lambda^*] \).

**Proof.** Let

\[
\mathcal{L} \eta = - \nabla \cdot (\mu \nabla \eta) - \lambda \mu \psi f'(u_\lambda) \eta, \quad \tilde{\mathcal{L}} \tilde{\eta} = - \nabla \cdot (\mu \nabla \tilde{\eta}) - \lambda \mu \psi f'_\varepsilon(u_\lambda) \tilde{\eta},
\]

where \( f'_\varepsilon \) is \( C^\omega \) function such that

\[
|f'(s) - f'_\varepsilon(s)| < \varepsilon \quad \text{for } s \in [0, u_\lambda(0)],
\]

5
for some $\varepsilon > 0$ and set $\lambda_1, \tilde{\lambda}_1$ to be the principal eigenvalues of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ respectively.

Assume first that $\lambda < \lambda^*$. We claim that $\lambda_1 \geq 0$. To show that, we note that the first eigenfunction of $\tilde{\mathcal{L}}$, $\tilde{\eta}_1$, is positive in $B$ and $\tilde{\eta}_1 \in C^{2,\infty}(B)$ as follows from [10] Theorem 6.15, Theorem 8.38]. Choosing the normalization in such a way that $||\tilde{\eta}_1||_{L^2(B)} = 1$, we observe that $\tilde{u} = u_\lambda - \varepsilon \tilde{\eta}_1$ is in $C^{2,\infty}(B)$ and positive in $B$, provided that $\varepsilon$ is sufficiently small. The latter is guaranteed by the fact that the normal derivative of $u_\lambda$ on the boundary is strictly positive as follows from Hopf’s lemma [15] Chapter 2, Theorem 7.

We next observe that

$$-\nabla \cdot (\mu \nabla \tilde{u}) - \lambda \mu \psi \tilde{u} = -\varepsilon \tilde{\lambda}_1 \tilde{\eta}_1 + \lambda \mu \psi R_\varepsilon,$$

with

$$R_\varepsilon = [f(u_\lambda) - \varepsilon f'(u_\lambda)\tilde{\eta}_1 - f(u_\lambda - \varepsilon \tilde{\eta}_1)] + \varepsilon [f'(u_\lambda) - f'(u_\lambda)] \tilde{\eta}_1.$$

Clearly $|R_\varepsilon| = o(\varepsilon \eta_1)$ as $\varepsilon \to 0$. Thus, if $\tilde{\lambda}_1 < 0$, then $\tilde{u}$ is a classical positive super-solution of (2.11) strictly below $u_\lambda$ in $B$, which contradicts the minimality of $u_\lambda$. Hence, $\lambda_1 \geq 0$.

Next observe that $\lambda_1, \tilde{\lambda}_1$ admit the variational characterization

$$\lambda_1 = \inf_{\eta \in \mathcal{A}} \int_B (|\nabla \eta|^2 - \lambda \psi f'(u_\lambda) \eta^2) \mu dx,$$

$$\tilde{\lambda}_1 = \inf_{\tilde{\eta} \in \mathcal{A}} \int_B (|\nabla \tilde{\eta}|^2 - \lambda \psi f'(u_\lambda) \tilde{\eta}^2) \mu dx,$$

where $\mathcal{A} := \{ \eta \in H^1_0(B) : ||\eta||_{L^2(B)} = 1 \}$. This immediately implies that $\lambda_1 \geq \tilde{\lambda}_1 - \lambda \mu(1)\varepsilon$. Since $\varepsilon$ is arbitrarily small, we have that $\lambda_1 \geq 0$ and hence (1.6) holds for $\lambda < \lambda^*$. Moreover, since $u^*$ is an increasing point-wise limit of $u_\lambda$, it also holds for $\lambda = \lambda^*$.

**Lemma 2.5.** An extremal solution of (2.1) is classical.

**Proof.** Taking $\tilde{f}(s) = f(s) - f(0)$ and setting $\eta = \tilde{f}(u_\lambda)$ in (1.6) we have that

$$\int_B |\nabla u_\lambda| (f'(u_\lambda))^2 \mu dx \geq \lambda \int_B f'(u_\lambda) (\tilde{f}(u_\lambda))^2 \mu dx.$$

Next taking $g(t) = \int_0^t (f'(s))^2 ds$, multiplying first equation in (2.1) by $g(u_\lambda)$ and integrating the result by parts we have

$$\int_B |\nabla u_\lambda|^2 (f'(u_\lambda))^2 \mu dx = \lambda \int_B f(u_\lambda) g(u_\lambda) \psi \mu dx.$$

Combining (2.18), (2.19), we have

$$\int_B f'(u_\lambda) (\tilde{f}(u_\lambda))^2 \psi \mu dx \leq \int_B f(u_\lambda) g(u_\lambda) \psi \mu dx.$$

Using this inequality instead of Eq.(4) in [14] and arguing exactly as in [14] Theorem 1] we conclude that the extremal solution is classical.

In this paper we are mostly concerned with minimal solutions of problem (1.1), which by Proposition [11] are classical and radially symmetric. Therefore, we will only consider solutions of (1.1) that are radially symmetric.

To study radially symmetric solutions, it is convenient to introduce the following rescaling

$$u(r) := v(\sqrt{\alpha r}) = v(y).$$

Substituting (2.21) into (1.1) we get

$$\begin{cases}
-\nu'' - \left(\frac{4}{y} + y \phi_\alpha(y)\right) v' = \frac{2}{\alpha} \psi_\alpha(y) f(v), & 0 < y < \sqrt{\alpha}, \\
v'(0) = 0, \\
v(\sqrt{\alpha}) = 0.
\end{cases}$$

(2.22)
Here and below, $(\cdot)' = \frac{d}{dy}(\cdot)$ and
\[
\varphi_\alpha(y) := \varphi\left(\frac{y}{\sqrt{\alpha}}\right), \quad \psi_\alpha(y) := \psi\left(\frac{y}{\sqrt{\alpha}}\right).
\]  
(2.23)

We now set
\[
G(v) := \int_v^\infty \frac{ds}{f(s)}
\]
(2.24)
and note that $G : [0, +\infty) \to (0, K]$ is a $C^2$ strictly monotone decreasing bijection and hence (2.24) implicitly defines the inverse $G^{-1} : (0, K] \to [0, +\infty)$, which is also strictly monotone decreasing. Differentiating (2.24), we obtain
\[
G'(v) = -\frac{v'}{f(v)}, \quad G''(v) = -\frac{v''}{f(v)} + \left(\frac{v'}{f(v)}\right)^2 f_v(v),
\]
(2.25)
where
\[
f_v(v) = \frac{df(v)}{dv}.
\]
(2.26)
Combining (2.22) and (2.25) yields
\[
\begin{aligned}
\{ & G'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right)G' = \frac{1}{\alpha} \psi_\alpha(y) + (G')^2 f_v(v), \quad 0 < y < \sqrt{\alpha}, \\
& G'(0) = 0,
\end{aligned}
\]
where $v = G^{-1}(y)$ is implicitly defined by (2.24).

In what follows we will work with both (2.22) and (2.27) as alternative versions of (1.1).

We next define super-solution for problem (2.22) and sub-solution for (2.27). A function $\bar{v} > 0$ on $[0, \sqrt{\alpha})$ belonging to $C^2((0, \sqrt{\alpha})) \cap C([0, \sqrt{\alpha}])$ is a positive super-solution of (2.22) provided it verifies
\[
\begin{aligned}
\{ & -\bar{v}'' - \left(\frac{1}{y} + y\varphi_\alpha(y)\right)\bar{v}' \geq \frac{1}{\alpha} \psi_\alpha(y) f(\bar{v}), \quad 0 < y < \sqrt{\alpha}, \\
& \bar{v}'(0) = 0,
\end{aligned}
\]
(2.28)
Similarly, a function $\underline{G} > 0$ on $[0, \sqrt{\alpha}]$ belonging to $C^2((0, \sqrt{\alpha})) \cap C([0, \sqrt{\alpha}])$ is a positive classical sub-solution of (2.27) if
\[
\begin{aligned}
\{ & G'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right)G' \geq \frac{1}{\alpha} \psi_\alpha(y) + (G')^2 f_v(v), \quad 0 < y < \sqrt{\alpha}, \\
& G'(0) = 0,
\end{aligned}
\]
(2.29)
The following results follow from Lemma 2.1.

**Corollary 2.1.** Assume (2.22) has a positive super-solution $\bar{v}$, then (2.22) has a minimal positive classical solution $v_\lambda$. Moreover, $v_\lambda(y) \leq \bar{v}(y)$ in $y \in [0, \sqrt{\alpha}]$.

**Corollary 2.2.** Assume that $\underline{G}$ is a positive sub-solution of (2.27). Then, the function $\bar{v}$ implicitly defined by (2.24) is a positive super-solution of (2.22).

### 3 Asymptotic behavior of $\lambda^*$.

In this section we establish an asymptotic behavior of $\lambda^*(\alpha)$ for sufficiently large $\alpha$ and give a proof of Theorem 1.1. In what follows we will work with (2.27), which is an alternative form of (1.1).

We start with the upper bound for $\lambda^*$, which is given by the following lemma.
Lemma 3.1. Assume \((1.1)\) has a solution. Then, for \(\alpha\) sufficiently large, \(\lambda^*(\alpha)\) obeys the following upper bound:

\[
\lambda^*(\alpha) \leq \frac{2K \alpha}{\log \alpha} \left(1 + \frac{c}{\log \alpha}\right),
\]

(3.1)

where \(c > 0\) is some constant independent of \(\alpha\).

Proof. First we observe that the first equation in \((2.22)\) can be rewritten in the divergence form

\[
\left( y \exp \left( \int_0^y s \varphi_\alpha(s) ds \right) G'(y) \right)' = \left[ \frac{\lambda}{\alpha} \varphi_\alpha(y) + (G'(y))^2 f_\alpha(v(y)) \right] y \exp \left( \int_0^y s \varphi_\alpha(s) ds \right).
\]

Therefore,

\[
\left( y \exp \left( \int_0^y s \varphi_\alpha(s) ds \right) G'(y) \right)' \geq \frac{\lambda}{\alpha} \varphi_\alpha(y) y \exp \left( \int_0^y s \varphi_\alpha(s) ds \right).
\]

(3.3)

Integrating \((3.3)\) from 0 to \(y\) and using the boundary condition at zero in \((2.27)\), we get

\[
G'(y) \geq \frac{\lambda}{\alpha} \frac{1}{y} \int_0^y z \varphi_\alpha(z) \exp \left(- \int_z^y s \varphi_\alpha(s) ds \right) dz.
\]

(3.4)

Next observe that due to the monotonicity of \(\psi\) and \(\varphi\) we have

\[
G'(y) \geq \frac{\lambda}{\alpha} \frac{1}{y} \int_0^y z \exp \left(- \int_z^y s \varphi_\alpha(s) ds \right) dz = \frac{\lambda}{\alpha} \frac{\varphi_\alpha(y)}{y} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right).
\]

(3.5)

Since \(\psi(0) = 1\) and \(\varphi\) is Lipshitz continuous, we have that

\[
\psi(x) \geq 1 - \frac{x}{h}
\]

(3.6)

for some constant \(0 < h \leq 1\) independent of \(\alpha\). Therefore,

\[
\psi_\alpha(y) \geq 1 - \frac{y}{h \sqrt{\alpha}}
\]

(3.7)

Consequently, \((3.5)\) and \((3.7)\) yield that for \(y \in [0, h \sqrt{\alpha}]\)

\[
G'(y) \geq \frac{\lambda}{\alpha} \frac{1}{y} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right) + \frac{\lambda}{\alpha} \frac{(\psi_\alpha(y) - 1)}{y} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right)
\]

\[
\geq \frac{\lambda}{\alpha} \frac{1}{y} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right) - \frac{\lambda}{h \alpha^{1/2}} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right).
\]

(3.8)

Integrating \((3.8)\) from 0 to \(h \sqrt{\alpha}\), we obtain

\[
G(h \sqrt{\alpha}) - G(0) \geq \frac{\lambda}{\alpha} \left( J_1 - J_2 \right),
\]

(3.9)

where

\[
J_1 = \int_0^{h \sqrt{\alpha}} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right) \frac{dy}{y}, \quad J_2 = \frac{1}{h \sqrt{\alpha}} \int_0^{h \sqrt{\alpha}} \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right) dy.
\]

(3.10)

Since \(G(h \sqrt{\alpha}) \leq K\) and \(G(0) > 0\) we have from \((5.9)\) that

\[
K \alpha \geq \lambda (J_1 - J_2).
\]

(3.11)

Next observe that the following estimates for \(J_1\) and \(J_2\) hold:

\[
J_1 = \int_1^{h \sqrt{\alpha}} \frac{dy}{y} - \int_1^{h \sqrt{\alpha}} \exp \left(- \frac{y^2}{2} \right) \frac{dy}{y} + \int_0^1 \left( 1 - \exp \left(- \frac{y^2}{2} \right) \right) \frac{dy}{y}
\]

\[
\geq \int_1^{h \sqrt{\alpha}} \frac{dy}{y} - \int_1^{\infty} \exp \left(- \frac{y^2}{2} \right) \frac{dy}{y} = \frac{\log \alpha}{2} + \log h - c,
\]

(3.12)
(here and below \( c \) stands for a positive constant independent of \( \alpha \)) and

\[
J_2 \leq \frac{1}{h\sqrt{\alpha}} \int_0^{h\sqrt{\alpha}} dy = 1.
\]

(3.13)

As a result of (3.12) and (3.13) we have that

\[
J_1 - J_2 \geq \frac{\log \alpha}{2} - c.
\]

(3.14)

Combining (3.11) and (3.14), we conclude that for (2.27) to admit a solution, we necessarily need

\[
\lambda \leq 2K\alpha \log \alpha - c \leq 2K\alpha \log \alpha \left( 1 + \frac{c \log \alpha}{\log \alpha} \right),
\]

(3.15)

for \( \alpha \) sufficiently large.

This observation immediately implies (3.1), which establishes an upper bound for \( \lambda^*(\alpha) \).

To obtain a lower bound for \( \lambda^*(\alpha) \) we need two technical lemmas, where the role of the parameter \( w \) will be clarified later.

**Lemma 3.2.** Let \( \tilde{G} \) be the solution of the following linear problem:

\[
\begin{align*}
\tilde{G}'' + \left( \frac{1}{y} + y\varphi_\alpha (y) \right) \tilde{G}' &= \beta_w \left( \psi_\alpha (y) + \frac{M^2(y)}{\alpha} \right), & 0 < y < \sqrt{\alpha} \\
\tilde{G}(0) &= 0, \\
\tilde{G}(\sqrt{\alpha}) &= K,
\end{align*}
\]

(3.16)

with sufficiently large \( \alpha \). Here

\[
M_\alpha(t) := M \left( \frac{t}{\sqrt{\alpha}} \right),
\]

(3.17)

with \( M \) given by (1.4), \( \varphi_\alpha, \psi_\alpha \) given by (2.23),

\[
\beta_w = \frac{2(K - \varepsilon_w)}{\log \alpha + c}, \quad \varepsilon_w = \int_w^{\infty} \frac{ds}{f(s)},
\]

(3.18)

where \( w > 0 \) is an arbitrary number and \( c \) is a sufficiently large constant independent of \( \alpha \).

Then, \( \tilde{G}(y) \) is an increasing function on \( y \in [0, \sqrt{\alpha}] \) and

\[
\tilde{G}(y) \geq \varepsilon_w > 0, \quad \text{on } [0, \sqrt{\alpha}].
\]

(3.19)

**Proof.** Using computations identical to (3.2), we rewrite (3.16) in the divergence form. Integrating the result from 0 to \( y \) and using the boundary condition at zero in (3.16), we get

\[
\tilde{G}'(y) = \frac{\beta_w}{y} \int_0^y z\varphi_\alpha(z) \exp \left( - \int_z^y s\varphi_\alpha(s)ds \right) dz + \frac{\beta_w}{\alpha y} \int_0^y zM^2_\alpha(z) \exp \left( - \int_z^y s\varphi_\alpha(s)ds \right) dz = J_3 + J_4.
\]

(3.20)

Clearly \( J_3, J_4 > 0 \) and hence \( \tilde{G} \) is strictly increasing. Therefore, we only need to prove the positivity of \( \tilde{G}(0) \).

As a first step, we establish upper bounds on \( J_3 \) and \( J_4 \). To do so, let

\[
F(t) := \int_0^t s\varphi_\alpha(s)ds,
\]

(3.21)

and observe that

\[
F(t) \leq \frac{t^2}{2},
\]

(3.22)

\[
\int_0^y z\varphi_\alpha(z) \exp \left( - \int_z^y s\varphi_\alpha(s)ds \right) dz = \int_0^y z\varphi_\alpha(z) \exp \left( F(z) - F(y) \right) dz = 1 - \exp \left( -F(y) \right) \leq 1 - \exp \left( -\frac{y^2}{2} \right),
\]

(3.23)
\[
\int_0^y M_\alpha(z)dz = \int_0^y M \left( \frac{z}{\sqrt{\alpha}} \right) dz = \sqrt{\alpha} \int_0^{\frac{y}{\sqrt{\alpha}}} M(s)ds \leq \sqrt{\alpha} \int_0^1 M(s)ds = m \sqrt{\alpha},
\]
where
\[
m := \int_0^1 M(s)ds.
\]

Using (3.22) – (3.26) we obtain the following upper bounds on \(J_3, J_4\):
\[
J_3(y) = \frac{\beta w}{y} \int_0^y \frac{\psi_\alpha(z)}{\varphi_\alpha(z)} \psi_\alpha(z) \exp \left( - \int_z^y s \varphi_\alpha(s) ds \right) dz
\leq \frac{\beta w}{y} M_\alpha(y) \int_0^y \psi_\alpha(z) \exp \left( - \int_z^y s \varphi_\alpha(s) ds \right) dz \leq \frac{\beta w}{y} M_\alpha(y) \left( 1 - \exp \left( - \frac{y^2}{2} \right) \right),
\]
and
\[
J_4 \leq \frac{\beta w}{\alpha y} \int_0^y \frac{z M_\alpha^2(z)dz}{\alpha} \leq \frac{\beta w}{\alpha} \int_0^y \frac{M_\alpha^2(z)dz}{\alpha} \leq \frac{\beta w}{\alpha} M_\alpha(y) \int_0^y M_\alpha(z)dz \leq m \frac{\beta w}{\sqrt{\alpha}} M_\alpha(y).
\]

Bounds (3.26), (3.27), (3.28) and the observations presented below allow one to estimate the difference
\[
\hat{G}(\sqrt{\alpha}) - \hat{G}(0) = \int_0^{\sqrt{\alpha}} \hat{G}'(y)dy = \int_0^{\sqrt{\alpha}} J_3(y)dy + \int_0^{\sqrt{\alpha}} J_4(y)dy.
\]
Indeed, we have
\[
\int_0^1 \frac{M_\alpha(y)}{y} \left( 1 - \exp \left( - \frac{y^2}{2} \right) \right) dy \leq M_\alpha(1) \int_0^1 \left( 1 - \exp \left( - \frac{y^2}{2} \right) \right) dy \leq \left( 1 + \frac{1}{\sqrt{\alpha}} \right) c \leq c,
\]
and
\[
\int_1^{\sqrt{\alpha}} (M_\alpha(y) - 1) \frac{dy}{y} = \int_1^{k \sqrt{\alpha}} (M_\alpha(y) - 1) \frac{dy}{y} + \int_{k \sqrt{\alpha}}^{\sqrt{\alpha}} (M_\alpha(y) - 1) \frac{dy}{y}
\leq \frac{l}{\sqrt{\alpha}} \int_1^{k \sqrt{\alpha}} dy + \frac{1}{k \sqrt{\alpha}} \int_{k \sqrt{\alpha}}^{\sqrt{\alpha}} M_\alpha(y)dy \leq lk + \frac{1}{k} \int_k^1 M(s)ds \leq lk + \frac{m}{k} = c.
\]

Thanks to estimates (3.30) and (3.31) we have
\[
\int_0^{\sqrt{\alpha}} J_3(y)dy \leq \beta w \int_0^1 \frac{M_\alpha(y)}{y} \left( 1 - \exp \left( - \frac{y^2}{2} \right) \right) dy + \beta w \int_1^{\sqrt{\alpha}} \frac{dy}{y} + \beta w \int_1^{\sqrt{\alpha}} (M_\alpha(y) - 1) \frac{dy}{y}
\leq \beta w \left( \frac{\log \alpha}{2} + c \right),
\]
and
\[
\int_0^{\sqrt{\alpha}} J_4(y)dy \leq m \frac{\beta w}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha}} M_\alpha(y)dy = m \beta w \int_0^1 M(s)ds = m^2 \beta w.
\]
Combining (3.29), (3.32) and (3.33) we obtain

\[ \tilde{G}(\sqrt{\alpha}) - \tilde{G}(0) \leq \beta_w \left( \frac{\log \alpha}{2} + c \right). \]  

(3.34)

That yields

\[ \tilde{G}(0) \geq K - \beta_w \left( \frac{\log \alpha}{2} + c \right). \]  

(3.35)

Consequently,

\[ \tilde{G}(0) \geq \epsilon_w. \]  

(3.36)

In view of the monotonicity of \( \tilde{G} \) we then have that \( \tilde{G} \geq \epsilon_w \) on \([0, \sqrt{\alpha}]\).

\[ \lambda \leq \frac{2K \alpha}{\log \alpha} \left( 1 - \frac{\epsilon_w}{K} - \frac{c f_v(w) + 1}{\log \alpha} \right), \]  

(3.37)

where \( \epsilon_w \) is as in Lemma 3.2. Then problem (1.1) admits a minimal positive strictly increasing solution satisfying \( G(0) \geq \epsilon_w \).

**Proof.** We claim that \( \tilde{G} \) constructed in Lemma 3.2 is a sub-solution for problem (2.27), provided \( \lambda \) satisfies (3.37). Indeed, using (3.20), (3.26) and (3.27) we observe that

\[ \left( \tilde{G}'(y) \right)^2 \leq c \beta_w^2 M_\alpha(w) \left( \frac{1}{\alpha} + \left[ 1 - \exp \left( \frac{-y^2}{\alpha} \right) \right]^2 \right). \]  

(3.38)

In particular, this observation and (3.28) imply that

\[ \left( \tilde{G}'(y) \right)^2 \leq c \beta_w^2 \leq c \beta_w^2 \left( \psi_\alpha(y) + \frac{M_\alpha(w)}{\alpha} \right) \quad \text{for} \quad y \in [0, \tilde{k} \sqrt{\alpha}], \]  

(3.39)

and

\[ \left( \tilde{G}'(y) \right)^2 \leq c \beta_w^2 M_\alpha(w) \left( \psi_\alpha(y) + \frac{M_\alpha(w)}{\alpha} \right) \quad \text{for} \quad y \in [\tilde{k} \sqrt{\alpha}, \sqrt{\alpha}], \]  

(3.40)

where \( \tilde{k} = \min[k, h/2] \) and \( k, h \) are as in Lemmas 3.2, 3.1 respectively. Therefore,

\[ \left( \tilde{G}'(y) \right)^2 \leq c \beta_w^2 \left( \psi_\alpha(y) + \frac{M_\alpha(w)}{\alpha} \right) \quad \text{for} \quad y \in [0, \sqrt{\alpha}]. \]  

(3.41)

Next we define implicitly \( \tilde{v}(y) \) by the following formula

\[ \tilde{G}(y) = \int_{\tilde{v}(y)}^{\infty} \frac{ds}{f(s)}, \]  

(3.42)

i.e. \( \tilde{v}(y) = G^{-1}(\tilde{G}(y)) \) where \( G^{-1} : (0, K] \to [0, \infty) \) is implicitly defined by (2.24). Since \( G^{-1} \) is decreasing and \( \tilde{G}(y) \) is an increasing function of \( y \) we have that \( \tilde{v}(y) \) is a decreasing function of \( y \). In view of this observation and an assumption that \( f \) is convex and increasing we have

\[ f_v(\tilde{v}(y)) \leq f_v(w) \quad \text{for} \quad y \in [0, \sqrt{\alpha}], \]  

(3.43)

Therefore, \( \tilde{G} \) will be a sub-solution for (2.27), provided that

\[ \tilde{G}'' + \left( \frac{1}{y} + y \varphi_\alpha(y) \right) \tilde{G}' \geq \frac{\lambda}{\alpha} \psi_\alpha(y) + \left( \tilde{G}' \right)^2 f_v(w), \quad 0 < y < \sqrt{\alpha}. \]  

(3.44)
Using (3.15) and (3.41), we observe that this condition is automatically satisfied if
\[
\beta_w \left( \psi_\alpha(y) + \frac{M_\alpha^2(y)}{\alpha} \right) \geq \frac{\lambda}{\alpha} \psi_\alpha(y) + c\beta_w f_v(w) \left( \psi_\alpha(y) + \frac{M_\alpha^2(y)}{\alpha} \right) \quad 0 < y < \sqrt{\alpha},
\]
which is in turn satisfied when
\[
\beta_w (1 - c\beta_w f_v(w)) \geq \frac{\lambda}{\alpha}.
\]
Straightforward computations show that (3.46) holds for all values of \( \lambda \) satisfying (3.37). Consequently, for \( \lambda \) satisfying (3.37), problem (2.27) admits a positive sub-solution and thus by Corollaries 2.1, 2.2 problem (2.22) and hence problem (1.1) admits a minimal solution.

An immediate consequence of this lemma is the following corollary.

**Corollary 3.1.**
\[
\lambda^*(\alpha) \geq \sup_{w \in (0, \infty)} \frac{2K}{\log \alpha} \left( 1 - \frac{\varepsilon_w}{K} - \frac{f_v(w) + 1}{\log \alpha} \right),
\]
where \( w \) and \( \varepsilon_w \) are as in Lemma 3.2.

We now can proceed to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** The statement of the theorem follows directly from Lemma 3.1 and Corollary 3.1. First we observe that by (3.1)
\[
\limsup_{\alpha \to \infty} \frac{\lambda^*(\alpha) \log \alpha}{\alpha} \leq 2K.
\]
On the other hand given \( \varepsilon > 0 \) arbitrary small, we can choose \( w \) sufficiently large so that \( \varepsilon_w/K < \varepsilon/2 \). This observation together with (3.47) gives
\[
\liminf_{\alpha \to \infty} \frac{\lambda^*(\alpha) \log \alpha}{\alpha} \geq 2(K - \varepsilon).
\]
In view of the fact that \( \varepsilon \) is arbitrarily small, (3.48) and (3.49) give (1.9).

4 Asymptotic behavior of the extremal solution as \( \alpha \to \infty \).

In this section we give a proof of Theorem 1.2. For convenience we split the section into two parts dealing with local and integral properties of the extremal solution respectively, that is, with parts one and two of Theorem 1.2. In this section we will use (2.22) as an alternative version of (1.1).

4.1 Local properties of the extremal solution

In this subsection we give a proof of the first part of Theorem 1.2. The proof is based on the following two lemmas.

**Lemma 4.1.** Assume that
\[
\vartheta < \frac{\lambda}{\lambda^*} \leq 1 \quad \text{with} \quad 0 < \vartheta \leq 1
\]
Then, any radial solution of (1.1) satisfies for any \( 1/\vartheta > \delta > 0 \) the following upper bound:
\[
u(x) \leq c \left( 1 + \frac{1/|x|}{\log \alpha} \right), \quad |x| \in \left[ \frac{1}{\alpha^{1/2^\delta - 2}}, 1 \right],
\]
where constant \( c = c(\vartheta, \delta) > 0 \) is independent of \( \alpha \).
Proof. First we observe that (2.22) can be rewritten in the divergence form
\[- \left( y \exp \left( \int_0^y s\varphi_\alpha(s)ds \right) v'(y) \right)' = \frac{\lambda}{\alpha} y \exp \left( \int_0^y s\varphi_\alpha(s)ds \right) \psi_\alpha(y) f(v(y)). \tag{4.3}\]
Integrating this equation and taking into account the boundary condition at zero in (2.22), we get
\[- yv'(y) = \frac{\lambda}{\alpha} I(y), \tag{4.4}\]
where
\[I(y) = \int_0^y f(v(z)) z\varphi_\alpha(z) \exp \left( - \int_z^y s\varphi_\alpha(s)ds \right) dz = \exp(-F(y)) \int_0^y f(v(z)) z\varphi_\alpha(z) \exp(F(z)) dz, \tag{4.5}\]
and $F(t)$ is defined in (3.21).

We claim that
\[I(y_2) \leq cM_\alpha(y_2) I(y_1) \quad \text{for} \quad y_2 > y_1, \quad y_1 \in [1, q\sqrt{\alpha}], \tag{4.6}\]
with some $0 < q \leq 1$ independent of $\alpha$.

To prove this claim we observe that
\[I(y_2) = \exp(F(y_1) - F(y_2)) I(y_1) + \exp(-F(y_2)) \int_{y_1}^{y_2} f(v(z)) \frac{\psi_\alpha(z)}{\varphi_\alpha(z)} z\varphi_\alpha(z) \exp(F(z)) dz. \tag{4.7}\]
Since $F(t)$ and $M_\alpha(t) \geq 1$ are increasing and $f(v(t))$ is decreasing on $[0, \sqrt{\alpha}]$, using calculations similar to (4.3), we obtain
\[I(y_2) \leq I(y_1) + \exp(-F(y_2)) f(v(y_1)) M_\alpha(y_2) \int_{y_1}^{y_2} z\varphi_\alpha(z) \exp(F(z)) dz \leq I(y_1) + f(v(y_1)) M_\alpha(y_2). \tag{4.8}\]
Moreover, since $\varphi, \psi$ are Lipshitz continuous and $\varphi(0) = \psi(0) = 1$ we can choose $q > 0$ independent of $\alpha$ such that
\[\frac{\psi_\alpha(y)}{\varphi_\alpha(y)} \geq \frac{1}{2}, \quad \varphi_\alpha(y) \geq \frac{1}{2} \quad \text{for} \quad y \leq q\sqrt{\alpha}. \tag{4.9}\]

Therefore, as follows from (4.9) and (4.3)
\[I(y_1) \geq f((v(y_1))) \exp(-F(y_1)) \int_0^{y_1} z\varphi_\alpha(z) \exp(F(z)) dz \geq \frac{1}{2} f((v(y_1))) \exp(-F(y_1)) \int_0^{y_1} z\varphi_\alpha(z) \exp(F(z)) dz = \frac{1}{2} f((v(y_1))) \left( 1 - \exp(-F(y_1)) \right) \geq cf(v(y_1)), \tag{4.10}\]
provided $y_1 < q\sqrt{\alpha}$. Combining (4.8) and (4.10), we get (4.6).

Now we fix $\delta > 0$ sufficiently small. We claim that for large enough $\alpha,$
\[v(\alpha^\delta) \leq c, \tag{4.11}\]
where $c$ depends only on $\vartheta$ and $\delta.$ To show that, we integrate (3.3) from zero to $\alpha^\delta$ and arguing as in Lemma 3.1 obtain
\[G(\alpha^\delta) \geq \frac{\lambda}{\alpha} (\delta \log \alpha - c). \tag{4.12}\]
Since for $\alpha$ large enough,
\[\lambda^* \geq \frac{K\alpha}{\log \alpha}, \tag{4.13}\]

we have that for \( \lambda \) satisfying (4.1)

\[
G(\alpha^\delta) \geq \frac{\phi \delta K}{2}.
\]  

(4.14)

Using definition of \( G \) (see (2.24)), we have

\[
\frac{1}{K} \int_{v(\alpha^\delta)}^{\infty} ds \frac{d}{df(s)} \geq \frac{\phi \delta}{2} > 0.
\]  

(4.15)

Since the integral in (4.15) is bounded from below away from zero independently of \( \alpha \), we conclude that the lower limit in this integral is bounded from above, which gives (4.11).

Next, taking \( y_1 = \alpha^2 \), we have from (4.6) that

\[
I(\alpha^2) \leq cM_\alpha(\alpha^2)I(y), \quad \alpha^2 \leq y \leq \alpha^2.
\]  

(4.16)

Since \( 1 \leq M_\alpha(\alpha^2) < c \), we have

\[
I(y) \geq cI(\alpha^2), \quad \alpha^2 \leq y \leq \alpha^2,
\]  

(4.17)

and hence by (4.4)

\[
-yv'(y) \geq -c\alpha^2v'(\alpha^2) = cD,
\]  

(4.18)

where

\[
D = -\alpha^2v'(\alpha^2).
\]  

(4.19)

Integrating inequality (4.18), we obtain

\[
v(\alpha^\delta) - v(\alpha^2) = -\int_{\alpha^\delta}^{\alpha^2} v'(y)dy \geq D \int_{\alpha^\delta}^{\alpha^2} dy = cD \log \alpha.
\]  

(4.20)

Therefore, from (4.11) and (4.20) we have

\[
D \leq \frac{c}{\log \alpha}.
\]  

(4.21)

On the other hand, from (4.6) with \( y_1 = \alpha^2 \) and \( y_2 = y \) we have

\[
I(y) \leq cM_\alpha(\alpha^2)I(\alpha^2), \quad y \geq \alpha^2,
\]  

(4.22)

and, therefore, by (4.4), (4.19), (4.21) and (4.22)

\[
-yv'(y) \leq cM_\alpha(y) \left(-\alpha^2v'(\alpha^2)\right) = cDM_\alpha(y) \leq \frac{c}{\log \alpha} M_\alpha(y).
\]  

(4.23)

Thus, for \( y \geq \alpha^2 \), we have

\[
-v'(y) \leq \frac{c}{\log \alpha} \frac{M_\alpha(y)}{y}.
\]  

(4.24)

Integrating (4.24), we then obtain

\[
v(y) = -\int_{\alpha^2}^{y} v'(y)dy \leq \frac{c}{\log \alpha} \int_{\alpha^2}^{y} M_\alpha(y) \frac{dy}{y}.
\]  

(4.25)

Now using (3.17), (3.24), (3.25) and (3.28) we estimate the right hand side of (4.25). First assume that \( y \geq k\sqrt{\alpha} \), where \( k \) is as in Lemma 3.2. Then

\[
\int_{y}^{\alpha^2} M_\alpha(y) \frac{dy}{y} \leq \int_{k\sqrt{\alpha}}^{\alpha^2} M_\alpha(y) \frac{dy}{y} \leq \frac{1}{k\sqrt{\alpha}} \int_{k\sqrt{\alpha}}^{\alpha^2} M_\alpha(y)dy \leq \frac{1}{k} \int_{k}^{1} M(t)dt \leq \frac{m}{k} < c.
\]  

(4.26)
Next, when \( y < k \sqrt{\alpha} \), we have
\[
\int_y^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} = \int_{k \sqrt{\alpha}}^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} + \int_y^{k \sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} \leq c + \int_y^{k \sqrt{\alpha}} \left( 1 + \frac{y}{\sqrt{\alpha}} \right) \frac{dy}{y} \leq
c + \int_y^{k \sqrt{\alpha}} \frac{dy}{y} + \frac{l}{\sqrt{\alpha}} \int_y^{k \sqrt{\alpha}} dy = c + \log(k \sqrt{\alpha}) - \log(y) + lk - \frac{y}{\sqrt{\alpha}} \leq c + \log \left( \frac{\sqrt{\alpha}}{y} \right). \tag{4.27}
\]
Combining (4.25), (4.26) and (4.27), we get
\[
v(y) \leq \frac{c}{\log \alpha} \left( 1 + \log \left( \frac{\sqrt{\alpha}}{y} \right) \right), \quad y \geq \alpha^{28}. \tag{4.28}
\]
The latter inequality in terms of the original (unscaled) variables (see (2.21)) gives (4.2), which completes the proof.

**Remark 4.1.** We note that the statement of the lemma above concerns not only extremal but all radial solutions of problem (1.1). That is, any radial solution of (1.1) with \( \lambda \) comparable with \( \lambda^* \) obeys (1.2). This fact, in particular, implies that any radial solution of (1.1) with \( \lambda \approx \lambda^* \) tends to zero outside of the origin as \( \alpha \to \infty \).

**Lemma 4.2.** The extremal solution \( u^*_\alpha \) of (1.1) satisfies
\[
u^*_\alpha(0) \to \infty \quad \text{as} \quad \alpha \to \infty. \tag{4.29}
\]

**Proof.** First we observe that due to the monotonicity of \( G' \), inequality (3.8), and arguments identical to those given in Lemma 3.1 we have
\[
G(\sqrt{\alpha}) - G(\tilde{c}) = \int_{\tilde{c}}^{\sqrt{\alpha}} G'(y)dy \geq \int_{\tilde{c}}^{k \sqrt{\alpha}} G'(y)dy \geq \frac{\lambda \log \alpha}{\alpha} \left( 1 - \frac{c}{\log \alpha} \right). \tag{4.30}
\]
where \( \tilde{c} \geq 0 \) is arbitrary constant independent of \( \alpha \). Also we observe that Theorem 1.1 implies
\[
\lambda^*(\alpha) = \frac{2K_\alpha}{\log \alpha} (1 - \sigma(\alpha)), \tag{4.31}
\]
with
\[
\sigma(\alpha) \to 0, \quad \text{as} \quad \alpha \to \infty. \tag{4.32}
\]

Since \( G(\sqrt{\alpha}) = K \) \tag{4.30} and \( \tilde{c} \) \tag{4.31} give that for \( \lambda = \lambda^* \) the following estimate holds:
\[
G^*_\alpha(\tilde{c}) \leq K - \frac{\lambda^*(\alpha) \log \alpha}{2\alpha} \left( 1 - \frac{c}{\log \alpha} \right) \leq c \left( \frac{1}{\log \alpha} + \sigma(\alpha) \right), \tag{4.33}
\]
where \( G^*_\alpha(y) = G(v^*_\alpha(y)) \).

Thus, \( G^*_\alpha(\tilde{c}) \to 0 \) as \( \alpha \to \infty \). Therefore,
\[
\int_{v^*_\alpha(\tilde{c})}^{\infty} \frac{ds}{f(s)} \to 0. \tag{4.34}
\]
Hence \( v^*_\alpha(\tilde{c}) \to \infty \) as \( \alpha \to \infty \) and, consequently, \( v^*_\alpha(0) \to \infty \) as \( \alpha \to \infty \).

Since \( u^*_\alpha(0) = v^*_\alpha(0) \) we conclude that \( u^*_\alpha(0) \to \infty \) as \( \alpha \to \infty \). \( \square \)

We now can give a proof of the first part of Theorem 1.2

**Proof of Theorem 1.2 part 1.** By Lemma 4.1 we have that
\[
u^*_\alpha(x) \leq c \frac{\log(\log \alpha)}{\log \alpha}, \quad |x| \geq \frac{1}{\log \alpha}. \tag{4.35}
\]
Taking a limit as \( \alpha \to \infty \) in (4.35), we obtain that \( u^*_\alpha(x) \to 0 \) for \( x \neq 0 \) as \( \alpha \to \infty \). The fact that \( u^*_\alpha(0) \to \infty \) as \( \alpha \to \infty \) follows directly from Lemma 1.2. \( \square \)

We now proceed to the proof of the second part of Theorem 1.2
4.2 Integral properties of the extremal solution

In this section we complete the proof of Theorem 1.2, which follows from the following two lemmas.

**Lemma 4.3.** Let \( \theta^*(\alpha) \) be the largest solution of the equation

\[
\frac{f(\theta)}{\theta} = c \log \alpha, \tag{4.36}
\]

where \( c > 0 \) is a fixed constant.

Then, for arbitrarily small \( \gamma > 0 \), we have

\[
\theta^*(\alpha) \leq \alpha^\gamma, \tag{4.37}
\]

provided \( \alpha \) is sufficiently large.

**Proof.** Let \( \theta > 1 \). Then, by the convexity of \( f \), we have

\[
f(s) \leq g(s) = f(\sqrt{\theta}) + \frac{f(\theta) - f(\sqrt{\theta})}{\theta - \sqrt{\theta}} (s - \sqrt{\theta}), \quad s \in [\sqrt{\theta}, \theta], \tag{4.38}
\]

and

\[
f(\theta) \leq tf(\theta) + (1 - t)f(0), \quad t \in [0, 1]. \tag{4.39}
\]

In particular, setting \( t = 1/\sqrt{\theta} \), the latter inequality gives

\[
f(\sqrt{\theta}) \leq f(\theta) \frac{\theta}{\sqrt{\theta}} + c. \tag{4.40}
\]

Next let

\[
\rho(\theta) = \int_0^\theta \frac{ds}{\sqrt{\pi} f(s)}. \tag{4.41}
\]

By (4.38) we have

\[
\rho(\theta) \geq \int_0^\theta \frac{ds}{\sqrt{\pi} g(s)} = \theta - \sqrt{\theta} \log \left( \frac{f(\theta)}{f(\sqrt{\theta})} \right). \tag{4.42}
\]

This observation together with (4.40) implies that for \( \theta \) sufficiently large we have

\[
\rho(\theta) > c \frac{\theta \log(\theta)}{f(\theta)}. \tag{4.43}
\]

By (4.2), \( \rho(\theta) \to 0 \) as \( \theta \to \infty \) and therefore,

\[
\frac{f(\theta)}{\theta} > \log(\theta) \chi(\theta), \tag{4.44}
\]

with some function \( \chi(\theta) \) having the property that \( \chi(\theta) \to \infty \) as \( \theta \to \infty \). In view of this observation we have

\[
\log(\theta^*) \chi(\theta^*) \leq c \log \alpha. \tag{4.45}
\]

The statement of the lemma then follows immediately.

**Lemma 4.4.** Let \( \delta, \gamma > 0 \) be arbitrary fixed small numbers such that \( \gamma + 4\delta < 1 \). If \( \alpha \) is large enough, then there exists a point \( a < \alpha^{2\delta} \) such that

\[
\int_0^a v^*_\alpha(y) dy \leq c \alpha^{\gamma + 4\delta}, \quad \int_0^a \psi_\alpha(y) f(v^*_\alpha(y)) dy \leq c \alpha^{\gamma + 4\delta} \log \alpha, \tag{4.46}
\]

and

\[
v^*_\alpha(y) \leq c \alpha^{\gamma}, \quad f(v^*_\alpha(y)) \leq c \alpha^{\gamma} \log \alpha \quad \text{when} \quad y \geq a, \tag{4.47}
\]

where \( c = c(\delta, \gamma) > 0 \) is a constant independent of \( \alpha \).
Proof. First we claim that
\[ v_\alpha^*(\alpha^2) > c. \] (4.48)
Indeed, arguing as in Lemma 4.1 (see Eqs. (4.3), (4.4), (4.5)), we have that
\[-v'(y) = \frac{\lambda}{\alpha} \exp(-F(y)) \int_0^y \frac{\psi_\alpha(z)}{z} \varphi_\alpha(z) f(v(z)) \exp(F(z)) dz, \] (4.49)
where \( F \) is defined by (3.21).
Next, using (4.9), (3.23), and the fact that \( f(v(y)) \geq f(0) > 0 \), we have that for \( \alpha \delta^2 \leq y \leq q \sqrt{\alpha} \)
\[-v'(y) \geq \frac{\lambda}{\alpha} \frac{1}{y} (1 - \exp(-F(y)) \geq \frac{\lambda}{\alpha} \frac{1}{y} (1 - \exp(-ca^{4\delta}) \geq \frac{\lambda}{\alpha} \frac{1}{y}. \] (4.50)
Integrating this expression from \( \alpha \delta^2 \) to \( q \sqrt{\alpha} \) we get
\[ v(\alpha^2) \geq v(q \sqrt{\alpha}) + \frac{\lambda}{\alpha} \frac{1}{2} - 2\delta)(\log \alpha - c). \] (4.51)
For \( \delta < \frac{1}{4} \), the latter inequality implies that for sufficiently large \( \alpha \)
\[ v(\alpha^2) \geq c \frac{\lambda}{\alpha} \log \alpha. \] (4.52)
In particular, we have
\[ v_\alpha^*(\alpha^2) \geq c \frac{\lambda^* \log \alpha}{\alpha} \geq c, \] (4.53)
which proves our claim.
Next let
\[ \Gamma(y) = y[v_\alpha^*(y)]' + \frac{\lambda}{\alpha} \varphi_\alpha(y)v_\alpha^*(y). \] (4.54)
We note that since \( \varphi \) is Lipshitz continuous \( \varphi' \) is defined almost everywhere and \( |\varphi'(y)| < c \). Direct computations give
\[ \Gamma'(y) = \left[ 2\varphi_\alpha(y) + \frac{\varphi'(y)}{\sqrt{\alpha}} \right] v_\alpha^*(y) - \frac{\lambda^*}{\alpha} \psi_\alpha(y)f(v_\alpha^*(y)) \] \[ \] (4.55)
for almost every \( y \).
Clearly, we have \( \Gamma(0) = \Gamma'(0) = 0 \). Assume first that \( v_\alpha^*(0) \geq v_0 \) where \( v_0 \) is the largest solution of
\[ 4v_0 = \frac{\lambda^*}{\alpha} f(v_0). \] (4.56)
Then \( \Gamma(y) \) is negative in some small neighborhood of \( y = 0 \). On the other hand \( \Gamma(\alpha^2) > 0 \) as follows from (4.24) and (4.53). Consequently, there exists a point \( y = a \in (0, \alpha^2) \) such that \( \Gamma(a) = 0 \). This in particular implies that there exists a point \( 0 < a_0 < a \) where \( \Gamma \) attains its minimum. At that point we have
\[ \left( 2\varphi_\alpha(a_0) + O \left( \frac{a_0}{\sqrt{\alpha}} \right) \right) v_\alpha^*(a_0) = \frac{\lambda^*}{\alpha} \psi_\alpha(a_0)f(v_\alpha^*(a_0)). \] (4.57)
This implies that
\[ cv_\alpha^*(a_0) = \frac{\lambda^*}{\alpha} f(v_\alpha^*(a_0)). \] (4.58)
Therefore, as follows from Lemma 4.3 and the monotonicity of $v^*_\alpha(y)$, for $y \geq a_0$ we have

$$v^*_\alpha(y) < c \alpha^\gamma, \quad f(v^*_\alpha(y)) \leq c \alpha^\gamma \log \alpha. \quad (4.59)$$

Taking into account that $a_0 < a$ and the monotonicity of $v^*_\alpha(y)$, for $y \geq a_0$ we have

$$v^*_\alpha(y) < c \alpha^\gamma, \quad f(v^*_\alpha(y)) \leq c \alpha^\gamma \log \alpha. \quad (4.59)$$

Next integrating (4.55) we have

$$\int_0^a \Gamma'(y)dy = \int_0^a \left[ 2\varphi_\alpha(y) + \frac{\varphi'(y)}{\sqrt{\alpha}} y \right] v^*_\alpha(y)dy = 0. \quad (4.60)$$

Therefore,

$$\int_0^a \left( 2\varphi_\alpha(y) + \frac{\varphi'(y)}{\sqrt{\alpha}} y \right) v^*_\alpha(y)dy = \int_0^a \frac{\lambda}{\alpha} \psi_\alpha(y)f(v^*_\alpha(y))dy. \quad (4.61)$$

This observation and Jensen inequality imply that

$$\langle v \rangle \geq c \frac{\lambda}{\alpha} f(\langle v \rangle), \quad (4.62)$$

where

$$\langle v \rangle = \frac{2}{a} \int_0^a v^*_\alpha(y)dy, \quad (4.63)$$

is an average of $v$ over $[0, a]$. Consequently, by Lemma 4.3 we have

$$\langle v \rangle \leq c \alpha^\gamma, \quad (4.64)$$

and thus

$$\int_0^a v^*_\alpha(y)dy < c \alpha^\gamma + 4\delta. \quad (4.65)$$

The latter inequality and (4.61) imply that

$$\int_0^a \psi_\alpha(y)f(v^*_\alpha(y))dy < c \alpha^\gamma + 4\delta \log \alpha. \quad (4.66)$$

This proves (4.46).

Assume now that $v^*_\alpha(0) \leq v_0$. In this case by Lemma 4.3

$$v^*_\alpha(y) \leq \alpha^\gamma, \quad f(v^*_\alpha(y)) \leq c \alpha^\gamma \log \alpha \quad \text{on} \quad [0, \sqrt{\alpha}], \quad (4.67)$$

which proves (4.47). Using these inequalities and taking $\alpha = \alpha^{2\delta}$ we also have (4.46) for this case.

We now can proceed to the proof of the second part of Theorem 1.2.

**Proof of Theorem 1.2 part 2.** We first observe that

$$\int_0^{\sqrt{\alpha}} v^*_\alpha(y)dy = \left\{ \int_0^a + \int_{a^{2\delta}}^{\sqrt{\alpha}} + \int_{\sqrt{\alpha}}^{\log \alpha} \right\} v^*_\alpha(y)dy = L_1 + L_2 + L_3 + L_4. \quad (4.68)$$

By (4.46), (4.47) (see Lemma 4.3) we have that

$$L_1, L_2 \leq c \alpha^{\gamma + 4\delta}. \quad (4.69)$$

Moreover by (4.28) (see Lemma 4.1) we also have that

$$L_3 \leq c \frac{\alpha}{(\log \alpha)^2}, \quad L_4 \leq c \frac{\log(\log \alpha)}{\log \alpha}. \quad (4.70)$$
Hence,
\[ \int_0^{\sqrt{\alpha}} v^*_{\alpha}(y)dy \leq c \frac{\log(\log \alpha)}{\log \alpha}. \] (4.71)

Observing that
\[ \int_B u^*_{\alpha}(x)dx = \frac{2\pi}{\alpha} \int_0^{\sqrt{\alpha}} v^*_{\alpha}(y)dy \leq c \frac{\log(\log \alpha)}{\log \alpha}. \] (4.72)

Taking a limit as \( \alpha \to \infty \) in the right hand side of (4.72) and using the positivity of \( u^*_{\alpha} \), we obtain the first part of (1.11).

Next, we perform a computation similar to those above,
\[ \int_0^{\sqrt{\alpha}} \psi_{\alpha}(y)f(v^*_{\alpha}(y))dy = \left\{ \int_0^{a} + \int_{a}^{a^{2\delta}} + \int_{a^{2\delta}}^{\sqrt{\alpha}} \right\} \psi_{\alpha}(y)f(v^*_{\alpha}(y))dy = P_1 + P_2 + P_3 + P_4. \] (4.73)

By (4.46), (4.47) (see Lemma 4.4) we have that
\[ P_1, P_2 \leq c \alpha^{\gamma+1/4} \log \alpha. \] (4.74)

By (4.11) we have
\[ P_3 \leq c \alpha (\log \alpha)^2, \] (4.75)

and by (4.28)
\[ P_4 \leq f \left( \frac{\log(\log \alpha)}{\log \alpha} \right) \int_0^{\sqrt{\alpha}} \psi_{\alpha}(y)dy, \] (4.76)

Arguing as above we then have
\[ \int_B \psi(x)f(u^*_{\alpha}(x))dx = \frac{2\pi}{\alpha} \int_0^{\sqrt{\alpha}} \psi_{\alpha}(y)f(v^*_{\alpha}(y))dy \leq \frac{c}{(\log \alpha)^2} \int_0^{\sqrt{\alpha}} \psi_{\alpha}(y)dy \]
\[ = \frac{c}{(\log \alpha)^2} + f \left( \frac{\log(\log \alpha)}{\log \alpha} \right) \int_{B \setminus B(a^{1/4\log \alpha})} \psi(x)dx. \] (4.77)

Therefore,
\[ \int_B \psi(x)f(u^*_{\alpha}(x))dx \leq f(0) \int_B \psi(x)dx + \tilde{\sigma}(\alpha), \] (4.78)

for some \( \tilde{\sigma}(\alpha) \) having the property that \( \tilde{\sigma}(\alpha) \to 0 \) as \( \alpha \to \infty \). In view of this observation and the fact that
\[ \int_B \psi(x)f(u^*_{\alpha}(x))dx > f(0) \int_B \psi(x)dx, \] (4.79)

which follows from the positivity of \( u^*_{\alpha} \), we have the second part of (1.11), which completes the proof.

We now turn to the proof of Theorem 1.3.
5 Proof of Theorem 1.3

The proof of Theorem 1.3 requires the following lemma. This lemma is based on a rescaled version of inequality (2.20) which was first introduced in [5].

Lemma 5.1. Let \( v_\alpha^* \) be an extremal solution of (2.22) and set
\[
\begin{align*}
f^*(t) &= f(v_\alpha^*(1) + t), \quad f^*_t(t) = f_t(v_\alpha^*(1) + t), \quad f^t(t) = f^*(t) - f^*(0), \quad g^t(t) = \int_0^t (f^t(s))^2 ds. \quad (5.1)
\end{align*}
\]
Assume that there exist constants \( 0 < \tilde{c}_0 < 1 \) and \( \tilde{c}_1 > 1 \) such that for sufficiently large \( \alpha \)
\[
f^*(t) \geq \tilde{c}_1 f^*(0), \quad t \geq 0, \quad (5.2)
\]
implies
\[
(1 - \tilde{c}_0) \tilde{f}^2(t) f^*_t(t) \geq g^t(t), \quad (5.3)
\]
Then, for sufficiently large \( \alpha \),
\[
v_\alpha^*(0) \leq v_\alpha^*(1) + c \frac{f(v_\alpha^*(1))}{\log \alpha}, \quad (5.4)
\]
where \( c > 0 \) is some constant independent of \( \alpha \).

Proof. As a first step we establish an inequality similar to (2.20).

Let
\[
\phi(y) := v_\alpha^*(y) - v_\alpha^*(1). \quad (5.5)
\]
By (2.22), this function verifies
\[
\begin{cases}
-(y^t(y)\phi')' + \frac{\lambda}{\alpha} y \psi_\alpha(y) \mu_\alpha(y) f^t(y), & 0 < y < 1, \\
\phi'(0) = 0, \quad \phi(1) = 0, 
\end{cases} \quad (5.6)
\]
where
\[
\mu_\alpha(y) = \exp \left( \int_0^y s \varphi_\alpha(s) ds \right). \quad (5.7)
\]
Multiplying the first equation in (5.6) by \( g^t(\phi) \), integrating by parts and taking into account the boundary conditions in (5.6), we get
\[
\int_0^1 (f^t_\phi(\phi))'^2 d\tilde{\nu} = \frac{\lambda^*}{\alpha} \int_0^1 f^t(\phi) g^t(\phi) d\nu, \quad (5.8)
\]
where
\[
d\tilde{\nu}(y) = y \mu_\alpha(y) dy, \quad d\nu(y) = \psi_\alpha(y) d\tilde{\nu}(y). \quad (5.9)
\]
We also note that the semi-stability condition (1.6) implies that
\[
\int_0^{\sqrt{\alpha}} (\eta')^2 d\tilde{\nu} \geq \frac{\lambda^*}{\alpha} \int_0^{\sqrt{\alpha}} f^t_\phi(\phi) \eta^2 d\nu, \quad \forall \eta \in H^1_0((0, \sqrt{\alpha})). \quad (5.10)
\]
Taking (in the spirit of the arguments in [14] and [6, Section 4.3])
\[
\eta(y) = \begin{cases} 
\tilde{f}^t(\phi(y)) & 0 \leq y \leq 1, \\
0 & 1 < y \leq \sqrt{\alpha},
\end{cases} \quad (5.11)
\]
and substituting this test function into (5.10), we obtain
\[
\int_0^1 (f^2_v(\phi)^2) \, d\nu \geq \frac{\lambda^*}{\alpha} \int_0^1 f^2_v(\phi)(\tilde{f}^2(\phi))^2 \, d\nu.
\] (5.12)

Combining (5.8) and (5.12), we obtain
\[
\int_0^1 f^2_v(\phi)(\tilde{f}^2(\phi))^2 \, d\nu \leq \int_0^1 f^2_v(\phi)g^2(\phi) \, d\nu.
\] (5.13)

Next let \( \tilde{c}_2 > \tilde{c}_1 \). Then,
\[
\int_0^1 f^2_v(\phi)(\tilde{f}^2(\phi))^2 \, d\nu = \left\{ \int_{X_1} + \int_{X_2} \right\} f^2_v(\phi)g^2(\phi) \, d\nu = I_1 + I_2,
\] (5.14)
where
\[
X_1 = \{ f^2(\phi) \leq \tilde{c}_2 f^2(0) \}, \quad X_2 = \{ f^2(\phi) > \tilde{c}_2 f^2(0) \}.
\] (5.15)

By the assumption of the lemma, we have
\[
I_2 \leq (1 - \tilde{c}_0) \int_{X_2} f^2_v(\phi)\tilde{f}^2(\phi) f^2_v(\phi) \, d\nu.
\] (5.16)

Moreover,
\[
f^2(\phi) \leq \frac{\tilde{c}_2}{\tilde{c}_2 - 1} \tilde{f}^2(\phi) \quad \text{on} \quad X_2.
\] (5.17)

Combining these two observations, we conclude that
\[
I_2 \leq \left( 1 - \frac{\tilde{c}_0}{2} \right) \int_{X_2} f^2_v(\phi)\left(\tilde{f}^2(\phi)\right)^2 \, d\nu
\] (5.18)
provided \( \tilde{c}_2 \) is sufficiently large.

Next we observe that (5.13), (5.14) and (5.18) imply that
\[
\frac{\tilde{c}_0}{2} \int_{X_2} f^2_v(\phi)\left(\tilde{f}^2(\phi)\right)^2 \, d\nu \leq I_1
\] (5.19)
Noting that \( f^2_v \) is non-decreasing, we have
\[
g^2(t) = \int_0^t f^2_v(s)f^2_v(s)ds \leq f^2_v(t) \int_0^t f^2_v(s)ds = f^2_v(t)\tilde{f}^2(t).
\] (5.20)

Consequently,
\[
I_1 \leq \int_{X_1} f^2_v(\phi)f^2_v(\phi)f^2(\phi) \, d\nu
\] (5.21)
Hence, from (5.19) and (5.21) we get
\[
\frac{\tilde{c}_0}{2} \int_{X_2} f^2_v(\phi)\left(\tilde{f}^2(\phi)\right)^2 \, d\nu \leq \int_{X_1} f^2_v(\phi)f^2_v(\phi)f^2(\phi) \, d\nu.
\] (5.22)

Since
\[
\min_{X_2} f^2_v(\phi) \geq \max_{X_1} f^2_v(\phi),
\] (5.23)
we have from (5.22) that
\[
\frac{\tilde{c}_0}{2} \int_{X_2} \left(\tilde{f}^2(\phi)\right)^2 \, d\nu \leq \int_{X_1} f^2_v(\phi)f^2(\phi) \, d\nu,
\] (5.24)
and thus
\[
\int_{X_2} (f^2(\phi))^2 \, dv \leq c(f^2(0))^2. \tag{5.25}
\]

On the other hand, as follows from the definition of \(X_1\), we have
\[
\int_{X_1} (f^2(\phi))^2 \, dv \leq c(f^2(0))^2. \tag{5.26}
\]

Consequently,
\[
\int_0^1 (f^2(\phi))^2 \, dv \leq c(f^2(0))^2. \tag{5.27}
\]

This estimate and the standard elliptic \(L^p\)-estimates \cite{[10]} Theorem 8.16 imply that
\[
\phi(0) \leq \frac{\lambda_t^T}{\alpha} f^2(0), \tag{5.28}
\]
and therefore
\[
v_\alpha^*(0) \leq v_\alpha^*(1) + c f(v_\alpha^*(1)) \log \alpha. \tag{5.29}
\]

We now turn to the proof of Theorem \ref{thm1.3}.

**Proof of Theorem \ref{thm1.3}**. First let us show that the assumptions of Lemma \ref{lem5.1} hold under the assumptions of Theorem \ref{thm1.3}. Observe that, as follows from the proof of Lemma \ref{lem4.2}, \(v_\alpha^*(1) \to \infty\) as \(\alpha \to \infty\). In view of this fact, the assumptions of Lemma \ref{lem5.1} can be rewritten in the form:
\[
f(t_2) > \bar{c}_1 f(t_1) \quad t_2 > t_1 > t_0, \tag{5.30}
\]

implies
\[
(1 - \bar{c}_0) f_c(t_2) (f(t_2) - f(t_1)) \geq \int_{t_1}^{t_2} (f_c(s))^2 \, ds. \tag{5.31}
\]

Assume that the assumptions of Theorem \ref{thm1.3} hold. Choose \(t_1 < \tilde{t} < t_2, \bar{c}_1 > c_1\) such that
\[
f(\tilde{t}) = \frac{1}{c_1} f(t_2). \tag{5.32}
\]

Then, by \ref{lem1.33}
\[
(1 - c_0) f_c(t_2) \geq f_c(\tilde{t}). \tag{5.33}
\]

Using the monotonicity of \(f_c, \ref{5.32}\) and \ref{5.33}, we obtain
\[
\int_{t_1}^{t_2} (f_c(s))^2 \, ds = \left\{ \int_{t_1}^{\tilde{t}} + \int_{\tilde{t}}^{t_2} \right\} (f_c(s))^2 \, ds \leq f_c(\tilde{t}) \left( f(\tilde{t}) - f(t_1) \right) + f_c(t_2) \left( f(t_2) - f(\tilde{t}) \right) \leq
\]
\[
f_c(t_2) (f(t_2) - f(t_1)) \left\{ \left( \frac{f(t_2) - f(\tilde{t})}{f(t_2) - f(t_1)} \right) + (1 - c_0) \left( \frac{f(\tilde{t}) - f(t_1)}{f(t_2) - f(t_1)} \right) \right\} = \tag{5.34}
\]
\[
f_c(t_2) (f(t_2) - f(t_1)) \left\{ 1 - c_0 \left( \frac{f(\tilde{t}) - f(t_1)}{f(t_2) - f(t_1)} \right) \right\} \leq f_c(t_2) (f(t_2) - f(t_1)) \left\{ 1 - \frac{c_0}{c_1} \left( \frac{f(t_2) - c_1 f(t_1)}{f(t_2)} \right) \right\}.
\]

Now taking \(\bar{c}_1 > 2c_1\) in \ref{5.30} we get
\[
\left( \frac{f(t_2) - c_1 f(t_1)}{f(t_2)} \right) \geq \frac{1}{2}. \tag{5.35}
\]
Hence, as follows from
\[
\int_{t_1}^{t_2} (f_v(s))^2 \, ds \leq \left(1 - \frac{c_0}{2c_1}\right) f_v(t_2)(f(t_2) - f(t_1)),
\]  
(5.36)
which gives (5.31) with \(\tilde{c}_0 = c_0/2c_1\).

We next turn to proof of (1.14). First note that taking an arbitrary smooth test function \(\eta\) with support on \([1/2, 1]\) in (5.10) and using the monotonicity of \(f_v\) and \(v^*_\alpha\) we have that
\[
f_v(v^*_\alpha(1)) \leq c \log \alpha.
\]  
(5.37)
Next, by convexity,
\[
f_v(t) \geq \frac{f(t) - f(0)}{t}, \quad t > 0.
\]  
(5.38)
Hence,
\[
v^*_\alpha(1)f_v(v^*_\alpha(1)) + c \geq f(v^*_\alpha(1)).
\]  
(5.39)
In view of (5.37) and (5.39) we have that for sufficiently large \(\alpha\),
\[
f_v(v^*_\alpha(1)) \log \alpha \leq cv^*_\alpha(1).
\]  
(5.40)
Combining this result with (5.4) we have
\[
v^*_\alpha(0) \leq cv^*_\alpha(1),
\]  
(5.41)
which implies the result.

Finally, let us prove (1.16). Observe that (5.37), (5.38) and Lemma 4.1 imply that
\[
v^*_\alpha(y) < \begin{cases} 
    c\alpha^\gamma & y \in [0, \alpha^{2\delta}], \\
    c \frac{\log \alpha}{\log \alpha} & y \in [\alpha^{2\delta}, \sqrt{\alpha}], \\
    c \frac{\alpha \log \alpha}{\log \alpha} & y \in [\sqrt{\alpha}, \alpha].
\end{cases}
\]  
(5.43)
where \(\delta > 0\) is arbitrarily small. Hence,
\[
\int_0^{\sqrt{\alpha}} (v^*_\alpha(y))^p \, dy \leq c \left\{ \frac{1}{\alpha^{1-p\gamma-4\delta}} + \frac{1}{(\log \alpha)^2} + \left(\frac{\log \alpha}{\log \alpha}\right)^p \right\} \alpha.
\]  
(5.44)
We thus have that for any \(1 \leq p < \infty\)
\[
\int_B (u^*_\alpha(x))^p \, dx = \frac{2\pi}{\alpha} \int_0^{\sqrt{\alpha}} (v^*_\alpha(y))^p \, dy \leq \left\{ \frac{1}{\alpha^{1-p\gamma-4\delta}} + \frac{1}{(\log \alpha)^2} + \left(\frac{\log \alpha}{\log \alpha}\right)^p \right\} \alpha.
\]  
(5.45)
In view that \(\delta\) and \(\gamma\) can be chosen arbitrarily small we conclude that the expression in the braces in (5.45) goes to zero as \(\alpha \to \infty\). This observation concludes the proof of the theorem.

The following lemma gives an example of rather general nonlinearity that satisfies assumptions of Theorem 1.3.

**Lemma 5.2.** Assume that \(f \in C^2, \frac{df(s)}{ds}, \frac{d^2f(s)}{ds^2} > 0,\) and \(\frac{d^2f(s)}{ds^2}\) is strictly increasing on \((0, \infty)\). Then, \(f\) satisfies the assumptions of Theorem 1.3.
Proof. Let us show that under the assumption of this lemma the assumptions of Theorem 1.3 are satisfied with $c_1 = 4$ and $c_0 = \frac{1}{3}$.

Take $t_2 > t_1$ such that

$$f(t_2) \geq 4f(t_1),$$

(5.46)

and consider two cases: case I in which $t_2 \leq 2t_1$, and case II in which $t_2 \geq 2t_1$.

We start with case I. Assume that

$$f_v(t_1) \geq \frac{2}{3}f_v(t_2).$$

(5.47)

Then, assumption (5.47) implies

$$\int_{t_2}^{t_1} \frac{d^2}{ds^2} f(s) = f_v(t_2) - f_v(t_1) \leq \frac{1}{3} f_v(t_2).$$

(5.48)

Let $t_3 = 2t_1 - t_2$ and note that under the assumption of case I we have that $0 \leq t_3 \leq t_1$. Since $t_1 - t_3 = t_2 - t_1$ and $\frac{df(s)}{ds}$ is strictly increasing, we have from (5.48) that

$$\int_{t_3}^{t_1} \frac{d^2}{ds^2} f(s) = f_v(t_1) - f_v(t_3) \leq \frac{1}{3} f_v(t_2).$$

(5.49)

Therefore,

$$f_v(t_3) \geq f_v(t_1) - \frac{1}{3} f_v(t_2),$$

(5.50)

and hence, by (5.47),

$$f_v(t_3) \geq \frac{1}{3} f_v(t_2).$$

(5.51)

Since $f_v$ is an increasing function we also have

$$f_v(s) \geq \frac{1}{3} f_v(t_2), \quad s \in [t_3, t_1].$$

(5.52)

Integrating (5.52) from $t_3$ to $t_1$ we have

$$f(t_1) - f(t_3) \geq \frac{1}{3} f_v(t_2)[t_2 - t_1].$$

(5.53)

By convexity,

$$f_v(t_2)[t_2 - t_1] \geq f(t_2) - f(t_1).$$

(5.54)

Thus, from (5.53), (5.54) we obtain

$$f(t_3) \leq f(t_1) - \frac{1}{3}(f(t_2) - f(t_1)) = \frac{1}{3}(4f(t_1) - f(t_2)).$$

(5.55)

Therefore, by (5.46) we have

$$f(t_3) \leq 0,$$

(5.56)

which contradicts the strict positivity of $f$. Hence, we must have

$$f_v(t_1) \leq \frac{2}{3} f_v(t_2),$$

(5.57)

which completes the proof in case I.
We now turn to case II. In this case by monotonicity of \( \frac{d^2 f(s)}{ds^2} \) we have
\[
f_v(t_2) - f_v(0) \geq \int_0^{t_2} \frac{d^2 f(s)}{ds^2} ds \geq \int_0^{2t_1} \frac{d^2 f(s)}{ds^2} ds \geq 2 \int_0^{t_1} \frac{d^2 f(s)}{ds^2} ds = 2(f_v(t_1) - f_v(0)). \tag{5.58}
\]
Thus,
\[
f_v(t_2) \geq 2f_v(t_1) - f_v(0). \tag{5.59}
\]
Choosing \( t_0 \) large enough, so that \( f_v(0) \leq \frac{1}{2} f_v(t_0) \) we obtain \((5.57)\) which completes the proof.

Finally, we summarize properties of extremal solutions for exponential nonlinearity.

**Lemma 5.3.** Let \( u_\alpha(x) \) be an extremal solution for problem \((1.1)\) with \( f(u) = e^u \). Then for sufficiently large \( \alpha \) we have
\[
\lambda^*(\alpha) = \frac{2\alpha}{\log \alpha} \left( 1 + O \left( \frac{1}{\sqrt{\log \alpha}} \right) \right), \tag{5.60}
\]
\[
u^*_\alpha(0) = O(\log \log \alpha), \tag{5.61}
\]
\[
\int_B (u^*_\alpha(x))^p dx \to 0, \quad 1 \leq p < \infty, \tag{5.62}
\]
\[
\int_B \psi(x) \exp(u^*_\alpha(x)) dx \to \int_B \psi(x) dx. \tag{5.63}
\]

*Proof.* We first prove \((5.60)\). We first observe that an estimate \((3.47)\) imply that for \( f(u) = \exp(u) \)
\[
\lambda^*(\alpha) \geq \frac{2\alpha}{\log \alpha} \left( 1 - \frac{c}{\log \alpha} - c\tilde{R}(w) \right), \quad \tilde{R}(w) = \exp(-w) + \frac{\exp(w)}{\log \alpha} \tag{5.64}
\]
with an arbitrary \( w \). It is easy to verify that \( \tilde{R}(w) \) attains its minimum at
\[
w = \frac{1}{2} \log(\log(\alpha)), \tag{5.65}
\]
hence
\[
\min_{w \in (0, \infty)} \tilde{R}(w) = \frac{2}{\sqrt{\log(\alpha)}}. \tag{5.66}
\]
Since \((3.47)\) holds for an arbitrary \( w \) we have
\[
\lambda^*(\alpha) \geq \frac{2\alpha}{\log(\alpha)} \left( 1 - \frac{c}{\log(\alpha)} \right). \tag{5.67}
\]
On the other hand in the case \( f(u) = \exp(u) \) an estimate \((3.1)\) takes a form
\[
\lambda^*(\alpha) \leq \frac{2\alpha}{\log(\alpha)} \left( 1 + \frac{c}{\log(\alpha)} \right), \tag{5.68}
\]
Combining two estimates above we have \((5.60)\).

Next let us show that \((5.61)\) holds. By \((5.37)\)
\[
\exp(u^*_\alpha(1)) \leq c \log \alpha, \tag{5.69}
\]
and hence
\[ v^*_\alpha(1) < c \log(\log \alpha). \] (5.70)

Using this observation (5.41) and the fact that \( u^*_\alpha(0) = v^*_\alpha(0) \), we have that
\[ u^*_\alpha(0) \leq c \log(\log \alpha). \] (5.71)

On the other hand we have from (5.67) and (4.33)
\[ \exp(-u^*_\alpha(0)) \leq c \frac{1}{\sqrt{\log \alpha}}. \] (5.72)

Hence,
\[ u^*_\alpha(0) \geq \frac{1}{2} \log(\log \alpha) - c. \] (5.73)

Combining, (5.71) and (5.73) we have (5.61).

Finally (5.62) and (5.63) follow from Theorem 1.3 and Theorem 1.2 respectively.

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