WHAT DO TOPOLOGISTS WANT FROM SEIBERG–WITTEN THEORY?

KEVIN IGA
Natural Science Division
Pepperdine University
24255 Pacific Coast Hwy.
Malibu, CA 90263, USA

Received (Day Month Year)
Revised (Day Month Year)

In 1983, Donaldson shocked the topology world by using instantons from physics to prove new theorems about four-dimensional manifolds, and he developed new topological invariants. In 1988, Witten showed how these invariants could be obtained by correlation functions for a twisted \( N = 2 \) SUSY gauge theory. In 1994, Seiberg and Witten discovered dualities for such theories, and in particular, developed a new way of looking at four-dimensional manifolds that turns out to be easier, and is conjectured to be equivalent to, Donaldson theory.

This review describes the development of this mathematical subject, and shows how the physics played a pivotal role in the current understanding of this area of topology.

Keywords: Seiberg–Witten; instantons; Donaldson theory; topology; four-dimensional manifolds.

When, in 1994, Nathan Seiberg and Edward Witten introduced Seiberg–Witten theory to the physics world, the study of supersymmetry was revolutionized. But surprisingly, mathematicians were also getting excited about Seiberg–Witten theory. They started asking physicists questions and seeking answers. They started making new discoveries in their mathematical fields, supposedly because of the physics. And many of these mathematicians were not mathematical physicists, but topologists. What did topologists want from Seiberg–Witten theory?

Stereotypically, a physicist supposedly comes up with a mathematical way of phrasing a physics problem, and a mathematician helps solve the problem. The result is answers for the physicist, and interesting areas of research for the mathematician. For Seiberg–Witten theory, however, the situation was reversed. Mathematicians were looking to the physicists for answers to mathematical questions.

There were actually many cases of this sort of interaction, especially in the last twenty years. Developments in high-energy physics have led to completely new techniques in topology, resulting in an impressive growth in our knowledge of four-dimensional manifolds, complex manifolds, knot theory, differential geometry, and symplectic geometry.
To understand what Seiberg–Witten theory has to do with topology, though, we need to delve into one of these earlier cases: in the early 1980s, when physics affected four-dimensional topology through the study of instantons. In 1983, Simon Donaldson showed how studying instantons led to new theorems and powerful new techniques in understanding the topology of four-dimensional manifolds. Both the instanton revolution of 1983 and the Seiberg–Witten revolution of 1994 are interesting cases of physics leading to new breakthroughs in mathematics.

The purpose of this review is to study these two breakthroughs and examine how questions from physics helped address fundamental questions in four-dimensional topology. We will also see the impact of physics on what was known in the subject, and how the physics led to the discovery of intriguing relationships to other areas of mathematics. As a mathematician, I hope to convey to a physics audience an appreciation what has happened, in language that I hope is as physics-friendly as possible.

This article will outline the development of the subject in roughly chronological order. In sections 1 and 2, I will describe the problem of classifying four-dimensional manifolds, and indicate what was known before physics got involved. In sections 3 through 9, I will describe S. Donaldson’s work in 1983 showing how finding \( SU(2) \) instantons on the manifold can help solve some of these questions, while introducing new topological invariants. Section 10 describes Witten’s 1988 derivation of these topological invariants using a supersymmetric topological quantum field theory. Sections 11 through 13 describe how notions of duality in supersymmetry discovered by Seiberg and Witten in 1994 give rise to a new topological quantum field theory that is easier to handle mathematically. Sections 15 through 17 describe some advances in four-dimensional topology using this dual theory. Section 18 has some philosophical conclusions.

1. Topology

One of the central problems of topology is to classify manifolds. Two manifolds are said to be the same if there is a diffeomorphism between them. To illustrate this problem, consider the classification of compact connected surfaces without boundary. This problem was solved by Poincaré in the early twentieth century, and it goes like this: some surfaces are orientable, and some are not. Here is a list of the compact connected surfaces without boundary that are orientable: the two-sphere \( S^2 \), the two-torus \( T^2 \), the double-torus (like a torus but with two handles), the triple-torus, and so on (see Figure 1). A good way to think of these is as a connected sum of tori. The connected sum of two connected surfaces \( X \) and \( Y \) is what you get when you remove a disk from \( X \) and a disk from \( Y \), then glue the result.

---

\[\text{This review is based primarily on a lecture I gave to the Stanford physics department in 1998, but more details taken from a talk I gave to the UCLA mathematics department in 2001, though the mathematics talk is translated into physics language for the purposes of this review, as far as I was able.}\]
What do Topologists want from Seiberg–Witten theory?

Fig. 1. Classification of compact connected surfaces without boundary: orientable surfaces are on top and non-orientable surfaces are on the bottom. The non-orientable ones cannot be embedded in $\mathbb{R}^3$ so those drawings are to be suggestive at best. The Euler characteristic $\chi$ of each is shown beneath each picture.

The Euler characteristic $\chi$, defined in a standard course in algebraic topology, is a number that is easy to assign to each surface. Once you know whether or not a surface is orientable, the Euler characteristic uniquely determines the surface. This is what is meant by a classification of compact connected surfaces without boundary. More generally, we would like to classify manifolds, the $n$-dimensional version of surfaces. Whether or not we insist on connectedness is not very important, since any disconnected manifold is just a union of connected manifolds. The criterion of compactness is more worthwhile, since any open subset of a manifold is also a manifold, and we don’t want to get bogged down in the classification of open subsets. There is more here than that, and much of it is very interesting, but as we will see, it is a hard enough question to classify compact manifolds, that it makes sense not to be too ambitious too quickly. For similar reasons we will focus on manifolds without boundary. From now on, when I mention classification of manifolds, I will mean the classification of compact manifolds without boundary.

If we were to pattern the project of classification of manifolds after the classification above for surfaces, then one way to describe the problem would be to say that we wish to assign some mathematical object (such as a number, a group, or anything just as easy to understand) to each manifold (hopefully in a way that is easy to compute) so that if two manifolds are diffeomorphic, they have the same
mathematical object (in which case the object is called a topological invariant), and so that if two manifolds are not diffeomorphic, then they are not assigned the same mathematical object (in which case the topological invariant is called a complete topological invariant).

In the case of surfaces, we had two important topological invariants: the Euler characteristic (a number assigned to each surface), and the orientability (a “yes” or a “no” assigned to each surface, answering the question of whether or not it was orientable). Neither alone is a complete topological invariant of compact connected surfaces without boundary; but the ordered pair is.

In general, we don’t hope to come up with a single object that is our complete topological invariant right away; we expect to come up with many topological invariants which together (we hope) classify manifolds completely.

The classification of two-dimensional manifolds is mentioned above (the classification of surfaces). The classification of one-dimensional (compact, connected, no boundary) manifolds is also easy. There is only one: the circle $S^1$. If we allowed for non-compact manifolds we could have the real line $\mathbb{R}$, and if we allowed for boundary we could have intervals like $[0, 1]$. Similarly for zero-dimensional compact connected manifolds: the only such manifold is a single point.

Now that we have the easy examples out of the way, we might ask about $n$-dimensional manifolds where $n \geq 3$. There is much that is known and much that is not known for such dimensions. Throughout the 1930s through the 1950s, the subject of algebraic topology developed. Algebraic topology defined many kinds of topological invariants that were defined for $n$-dimensional manifolds (in fact they were usually defined for arbitrary topological spaces). For instance if $X$ is a connected space, its fundamental group $\pi_1(X)$ is a group, and if two manifolds are diffeomorphic, then they have the same fundamental group. Therefore, $\pi_1$ is a topological invariant.

There are generalizations $\pi_2(X), \pi_3(X), \ldots,$ that are also topological invariants, which are actually abelian groups. There are other sequences of topological invariants that are groups: the homology of a manifold $X$ is a sequence of abelian groups $H_0(X), H_1(X), \ldots,$ and the cohomology $H^0(X), H^1(X), \ldots,$ and there are others. A brief account for physicists is found in Nash and Sen’s book Topology and Geometry for Physicists, and a more complete text on the subject is Elements of Algebraic Topology by J. Munkres.

For compact manifolds, these groups are all finitely generated, and the point is that inasmuch as finitely generated groups are understood (they are not) and inasmuch as finitely generated abelian groups are understood (they are), these invariants should make it easier to understand the problem of classification of manifolds. The problem is that it is not clear whether or not these form a set of complete invariants, and furthermore, which values of the invariants are possible.

Actually, it is possible to prove that in dimension 4 and higher, any group with finitely many generators and relations can be $\pi_1(X)$ for some manifold $X$. This can
be done explicitly enough that the classification of manifolds would also produce a
classification of groups with finitely many generators and relations. The bad news
is that the classification of groups with finitely many generators and relations has
been proven to be impossible,\footnote{The connected sum $X \# Y$ of $n$-dimensional manifolds $X$ and $Y$ is defined analogously to surfaces: remove a small ball (a copy of $B^n$) from $X$ and from $Y$, and glue along their boundary (a copy of $S^{n-1}$). For dimension three and higher, $X \# Y$ is simply connected if and only if $X$ and $Y$ are.} and therefore, the classification of manifolds must
be impossible, too.

This would seem to answer the main problem in a spectacularly negative fash-
on: if $n \geq 4$, then the classification of compact $n$-manifolds without boundary is
algorithmically impossible.

But this is not the end of the story. We could restrict our attention to simply
connected manifolds (those for which $\pi_1(X)$ is the trivial group), or manifolds with
$\pi_1(X)$ some group that is easy to understand (finite groups, cyclic groups, etc.). And
it is precisely for dimensions 4 and higher that we know of many, many manifolds
that are simply connected, so classifying simply connected manifolds (as before,
compact, connected, no boundary) is a very interesting question, and perhaps one
we can hope to answer.

For example, in dimension 4, we have the sphere $S^4$, we have $S^2 \times S^2$, we have
the complex projective plane $\mathbb{C}P^2$, there are connected sums of these, and there
are many more that arise naturally in algebraic geometry.

In dimensions 5 and higher, remarkably, the problem of classifying simply con-
nected compact manifolds without boundary is solved, whereas the analogous clas-
sification in dimensions 3 and 4 is still unsolved today. This strange circumstance,
suggesting that dimensions 5 and higher are easier than dimensions 3 and 4, comes
about because there are certain techniques that are very powerful, but require a
certain amount of room before you can use them. A readable account can be found
in Kosinski’s *Differential Manifolds* and Ranicki’s book on surgery theory (which
should be read in that order). This classification also extends to the classification
of manifolds whose $\pi_1(X)$ is understood sufficiently well.

This leaves the problem of classifying manifolds of dimensions 3 and 4. Again,
for dimension four, we would like to insist that the manifolds are simply connected,
or at least that $\pi_1(X)$ be sufficiently well understood. In dimension 3, it is not
clear whether or not we need to be concerned with $\pi_1$, and it is not known if there
are other simply connected compact three-dimensional manifolds without boundary
other than the three-sphere $S^3$. It is interesting that the classification problem is
not solved in dimensions 3 and 4, the two dimensions that have long been of interest
to physics (space and space-time) until the advent of string theory. I will focus on
dimension four, since that is where Seiberg–Witten theory has had its impact.
2. What was classically known in dimension four

Before the 1980s, there was not much known about simply connected four-dimensional manifolds. It was possible to compute homology and cohomology groups, and so on, but invariants like these from algebraic topology gave limited information, and it was not clear whether or not there was more to the classification story.

The homology groups look like this:

\[ H_0(X^4) \cong \mathbb{Z} \]
\[ H_1(X^4) \cong 0 \]
\[ H_2(X^4) \cong \mathbb{Z}^{b_2} \]
\[ H_3(X^4) \cong 0 \]
\[ H_4(X^4) \cong \mathbb{Z} \]

and the higher homology groups are all trivial. The vanishing of \( H_1 \) occurs because \( X^4 \) is simply connected (using the Hurewicz theorem); the vanishing of \( H_3 \) occurs because of Poincaré duality. So if you were to use only homology, the only topological invariant we could get was one number: the second Betti number \( b_2 \).

The cohomology groups can be calculated using the universal coefficient theorem, and in this case, the table for cohomology groups is identical to the one for homology groups above. But the cohomology groups have some extra information, because cohomology classes can be multiplied via the wedge product. In our case, the only case to consider is multiplying two elements of \( H_2(X^4) \), which gives rise to an element of \( H_4(X^4) \). We can view this as number by integrating over \( X^4 \):

\[ I(\omega_1, \omega_2) = \int_{X^4} \omega_1 \wedge \omega_2 \]  

where \( \omega_1 \) and \( \omega_2 \) are elements of \( H^2(X^4) \). This can be viewed as a bilinear form on \( H^2(X^4) \), taking two cohomology classes and returning a number.

If we use singular cohomology with integer coefficients instead of using differential forms, it would be more apparent that \( (1) \) is an integer, and in that language, the wedge product is called the cup product.

By Poincaré duality, we can interpret \( (1) \) in terms of homology instead of cohomology, and this is what happens: an element of \( H_2(X^4) \) can be viewed as a surface embedded in \( X^4 \), and if \( \Sigma_1 \) and \( \Sigma_2 \) are two such, they will generically intersect in a finite set of points. If these are counted with appropriate signs, the number of points in the intersection will be the integer that corresponds to \( (1) \). In this way, we can define \( I(\Sigma_1, \Sigma_2) \) as a bilinear form on \( H_2(X^4) \).

Whichever way you wish to think of it, there is a bilinear form on \( H_2(X^4) \) or equivalently on \( H^2(X^4) \) called the intersection form, and it is symmetric, integer-valued, and non-degenerate. If we choose a basis for \( H_2(X^4) \), this intersection form can be viewed as a square \( b_2 \times b_2 \) matrix of integers. This matrix is symmetric and its determinant is \( \pm 1 \).
What do Topologists want from Seiberg–Witten theory?

This intersection form is a topological invariant: every simply connected compact four-dimensional manifold without boundary gives rise to an integer-valued $b_2 \times b_2$ symmetric matrix with determinant $\pm 1$. This would be convenient (matrices are very convenient and easy to understand), except for one problem: the intersection form may be a bilinear form on $H_2(X^4)$; but identifying it as a matrix requires choosing a basis. Changing this basis with an invertible integer-valued matrix $S$ will change the matrix $A$ to $S^T AS$, and both $A$ and $S^T AS$ are descriptions of the same bilinear form but in different bases.

The classification of symmetric integer-valued bilinear forms with determinant $\pm 1$, up to integer change of basis, is a difficult subject in general. If we were to allow any real change of basis, the classification of these bilinear forms is just a matter of counting the number of positive eigenvalues and the number of negative eigenvalues (since the determinant is $\pm 1$, there are no zero eigenvalues). Let $b^+_2$ be the number of positive eigenvalues and $b^-_2$ the number of negative eigenvalues.

Since we are only allowed integer change of bases, the problem is more difficult than this. We still do have $b^+_2$ and $b^-_2$, but several matrices may have the same values for $b^+_2$ and $b^-_2$ but not be equivalent.

In particular, the classification of definite symmetric bilinear forms ($b^+_2 = 0$ or $b^-_2 = 0$) is not known at all, but it is known that the number of these forms of even moderate size is quite large. If the classification of simply connected four-dimensional manifolds depends on understanding this classification, we are in some trouble.\(^6\)

The classification of the indefinite case (neither $b^+_2$ nor $b^-_2$ is zero) is actually much better: we know the classification of these completely. When at least one diagonal element is odd, it is possible to change the basis so that the matrix is diagonal with only 1’s and −1’s on the diagonal; and when all diagonal elements are even, the basis can be chosen so that the matrix breaks up into $2 \times 2$ blocks and $8 \times 8$ blocks, where the $2 \times 2$ blocks are the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the $8 \times 8$ blocks are each the Cartan matrix for the Lie group $E_8$:

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

\(^6\)Not as much trouble as we were with $\pi_1(X)$. This classification is not algorithmically impossible; just poorly understood.
Kevin Iga

Indefinite

\[ m(1) \oplus n(-1) \]
\[ m, n \geq 1 \]

Definite

\[ \pm I, \ E_8 \oplus (1), \text{many more (unknown)} \]

|        | Indefinite                  | Definite                          |
|--------|-----------------------------|-----------------------------------|
| Odd    | \[ m(1) \oplus n(-1) \]    | \[ \pm I, \ E_8 \oplus (1), \text{many more (unknown)} \] |
|        | \[ m, n \geq 1 \]          |                                   |
| Even   | \[ mH \oplus nE_8 \]       | \[ nE_8, \ SO(32), \text{Leech lattice, many more (unknown)} \] |
|        | \[ m \geq 1, n \geq 0 \]   |                                   |

Fig. 2. Classification of symmetric, integer-valued bilinear forms of determinant \( \pm 1 \), up to change of basis. “Even” means all the diagonal elements are even, and “odd” means at least one diagonal element is odd.

| Manifold   | \( b_2 \) | \( I \) | \( b_2^+ \) | \( b_2^- \) |
|------------|----------|--------|------------|------------|
| \( S^4 \)  | 0        | ( )    | 0          | 0          |
| \( S^2 \times S^2 \) | 2 | \( H \) | 1          | 1          |
| \( CP^2 \) | 1        | (1)    | 1          | 0          |
| \( CP^2 \) | 1        | (-1)   | 0          | 1          |
| \( K3 \)   | 22       | \( 3H \oplus 2E_8 \) | 19 | 3          |
| \( (CP^2)^m \# (CP^2)^n \) | \( m + n \) | \( m(1) + n(-1) \) | \( m \) | \( n \) |
| \( K3 \# m \# (S^2 \times S^2)^n \) | \( 22m + 2n \) | \( (3m + n)H \oplus (2m)E_8 \) | \( 19m + n \) | \( 3m + n \) |

Fig. 3. Some intersection forms of simply connected four-dimensional manifolds

The matrix \( E_8 \) is definite, so there must be at least one \( H \), or we would be considering the definite case above, instead of the indefinite case.

Note that \( b_2 = b_2^+ + b_2^- \). As usual, \( \sigma(X^4) = b_2^+ - b_2^- \) is called the signature. If the orientation of the manifold is reversed, the matrix is replaced by its negative, and therefore the \( b_2^+ \) and \( b_2^- \) reverse roles. So \( b_2^+ \) and \( b_2^- \) are not really topological invariants, but \( |\sigma| \) is. Alternately, we can try to classify manifolds together with their orientations, and then we have \( b_2^+ \) and \( b_2^- \) as invariants of manifolds with orientation.

Now for some examples of intersection forms. The four-sphere \( S^4 \) has \( b_2 = 0 \), so the matrix is the empty zero-by-zero matrix. For \( S^2 \times S^2 \), \( b_2 = 2 \), and the intersection form is \( H \) above, and \( b_2^+ = 1 \), and \( b_2^- = 1 \), so \( \sigma = 0 \). For \( CP^2 \), we have \( b_2 = 1 \), and the intersection form is \( (1) \) (assuming the usual orientation on \( CP^2 \)). Then \( b_2^+ = 1 \) and \( b_2^- = 0 \), and \( \sigma = 1 \). We call \( CP^2 \) the manifold \( CP^2 \) with the reverse orientation, so that \( b_2^+ = 0 \), \( b_2^- = 1 \), and \( \sigma = -1 \).

We can also take the connected sum of two simply connected four-dimensional manifolds, resulting in a new simply connected four-dimensional manifold. The second Betti number of the resulting manifold is \( b_2(X \# Y) = b_2(X) + b_2(Y) \), and similarly \( b_2^+ \), \( b_2^- \), and \( \sigma \) are additive. Furthermore, the resulting intersection form
can be put into blocks

\[
\begin{pmatrix}
I_X & 0 \\
0 & I_Y
\end{pmatrix}
\]

where \(I_X\) is the matrix for the intersection form on \(X\) and \(I_Y\) is the matrix for the intersection form on \(Y\). So if we take connected sums of copies of \(CP^2\) and \(\overline{CP^2}\), we can form a manifold with intersection form with arbitrarily many 1’s and −1’s down the diagonal, with the rest of the matrix zero. If we only use \(CP^2\)’s, we get the identity matrix, and if we only use \(\overline{CP^2}\)’s, we get minus the identity matrix.

Note that \(S^2 \times S^2\) and \(CP^2 \# \overline{CP^2}\) have the same homology groups (\(b_2 = 2\) in both cases) but their intersection forms are different.

A remarkable manifold is the K3 surface (the four-dimensional equivalent to Calabi–Yau manifolds). The K3 surface can be described as the solutions to

\[
x^4 + y^4 + z^4 = 1
\]

where \(x, y,\) and \(z\) are complex numbers, and where we compactify the parts that go to infinity by considering \((x, y, z)\) as coordinates on \(CP^3\). The K3 surface is a smooth four-dimensional manifold, simply connected, and has \(b_2 = 22\), with \(b_+^2 = 19, b_-^2 = 3\), and \(\sigma = 16\). The intersection form is a \(22 \times 22\) matrix, which can be put into block diagonal form with three \(H\)’s and two \(E_8\) blocks.

Because of the block structure, we might suspect the K3 surface is a connected sum of various pieces, two that have \(E_8\) as their intersection form, and three that have \(H\) as their intersection form, and in fact this might have been supposed before the 1980s, but this turns out not to be true, as we now know from what is described below.

There was a little bit known beyond this before the 1980s, but not much. Basically nothing was known about which intersection forms were possible, and basically nothing was known about whether it was possible for two manifolds to have the same intersection form (implicit is that they would have the same \(b_2\), and in particular, the same homology and cohomology).

The reader may have noticed that we have spent some time with homology and cohomology groups, and a lot of time with the intersection form, but we have not discussed the higher homotopy groups. The higher homotopy groups are in general too difficult to calculate, but in the end turn out not to give new information anyway for simply connected four-dimensional manifolds.

For a more detailed account of this section, see Kirby’s book *The Topology of 4-Manifolds*.

---

\(\text{footnote}{^4}\) Actually, because of Rokhlin’s theorem, the two \(E_8\)’s cannot be separated, but we might have thought that was the only restriction.

\(\text{footnote}{^5}\) There was a little more than what I have mentioned here: a theorem by Rokhlin, and an invariant by Kirby and Siebenmann, and so on. These are summarized in Kirby’s book mentioned above.
3. The two breakthroughs in the 1980s

There were two breakthroughs in the 1980s that suddenly added remarkable clarity to what was going on for simply connected four-dimensional manifolds, and they happened at roughly the same time. On the one hand was the work of Michael Freedman that was completely topological, and on the other was the work of Simon Donaldson that used instantons. These two breakthroughs were complementary in the sense that they addressed two disjoint sides of the question.

Freedman’s work classified topological manifolds (where the coordinate charts need not patch together smoothly) up to homeomorphism (for two topological manifolds to be homeomorphic, all that is necessary is the existence of a continuous map from one to the other with a continuous inverse) as opposed to Donaldson’s work which described what happens to smooth manifolds (where the coordinate charts patch together differentiably) up to diffeomorphism (so that the map relating the two and its inverse must be differentiable). It turns out the stories for the smooth classification and for the non-smooth classification are very different.

Freedman’s work, published in 1982, showed that for simply connected compact four-dimensional manifolds without boundary, all intersection forms depicted in Figure 2 are possible, and with an additional $\mathbb{Z}_2$-valued invariant known as the Kirby–Siebenmann invariant, these data completely determine the manifold up to homeomorphism. Thus, the question of classifying simply connected compact topological four-dimensional manifolds without boundary up to homeomorphism was finally solved. The idea behind Freedman’s work is to show that a more sophisticated version of what works for dimensions five and higher actually works for dimension four. In dimensions five and higher, it is often necessary to “simplify” a description of a manifold by finding a complicated subset and showing it is really a ball. The same idea works in dimension four, except that sometimes the necessary subset is infinitely complicated, and Freedman was able to show that such a subset is homeomorphic (though perhaps not diffeomorphic) to a ball. Since this did not involve physics, we will not discuss this work further, but a good place to learn this is in the book by Freedman and Quinn.

On the other hand, Donaldson’s work, starting with the seminal publication in 1983, dealing with smooth manifolds up to diffeomorphism, did not result in as

\[ \text{The condition that the manifold be simply connected can be somewhat loosened, and the story is fairly similar. For the rest of this article we assume the manifold is simply connected, to make the notation clearer, and to avoid having lots of complicated restrictions on the statements of the results. Even though classification of four-dimensional manifolds is impossible, the techniques we describe are still useful in general.} \]

\[ \text{The non-smooth classification is sometimes referred to as the topological classification, since notions of continuity are required but not notions of differentiability. It is nevertheless common for mathematicians to use the word “topological” in the context of the smooth classification, in phrases like “topological invariant”, when there is no chance for confusion, and since beyond this section the primary consideration is with the smooth classification problem, we will sometimes use the word “topological” in this way.} \]

\[ \text{This does not include the fact that intersection forms are not classified, of course.} \]
What do Topologists want from Seiberg–Witten theory?

|    | Indefinite                                                                 | Definite       |
|----|-----------------------------------------------------------------------------|----------------|
| Odd| \( m(1) \oplus n(-1) \) \( m, n \geq 1 \)                                 | \( \pm I \)   |
| Even| \( mH \oplus nE_8 \) \( m, n \ ? \) see Figure 7                           | nothing        |

Fig. 4. Classification of intersection forms of smooth manifolds, due to Donaldson. Note that the unknown areas of Figure 2 have disappeared, and almost all entries are represented by manifolds given in Figure 3. The only situation unknown is the number of \( m \) and \( n \) possible for even intersection forms.

complete an answer, but what Donaldson discovered resulted in a simplification along a completely different direction. By considering a Yang–Mills \( SU(2) \) gauge field on the four-dimensional manifold, and studying instantons, Donaldson was able to prove that the intersection form must be either indefinite (in which case we know how to classify such intersection forms) or plus or minus the identity. In other words, the situation where we didn’t know how to classify intersection forms, the case where it was definite, is the situation where we this classification is unnecessary, since smooth manifolds can’t have them as intersection forms anyway, with the exception of the identity and minus the identity (see Figure 4 and compare to the earlier Figure 2).

This is known as Donaldson’s Theorem A, since there are other important theorems in that paper, all deriving from analyzing the equations for instantons. The original papers are Ref. [10] and Ref. [11]. A friendly introduction to the subject is Freed and Uhlenbeck’s book [12], and a detailed textbook is a book by Donaldson and Kronheimer [13].

Before giving a sense for how Donaldson’s Theorem A was proved, and before giving other important uses of this technique, let us recall a few things about instantons.

4. Instantons

Consider a pure \( SU(2) \) gauge field theory on flat \( \mathbb{R}^4 \), as described in standard textbooks like Peskin and Schroeder [17]. Let \( i\sigma^a \) be the standard Pauli basis for the Lie algebra of \( SU(2) \), where \( a = 1, \ldots, 3 \). Let \( A_\mu = A_\mu^a \sigma_a \) be an \( SU(2) \) connection, with \( \mu = 1, \ldots, 4 \) a spatial index, and \( F_\mu^a \) = \( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \) is its curvature tensor, so that

\[
F_\mu^a = A_{[\mu, \nu]} + [A_\mu, A_\nu].
\]

Consider the action

\[
S = \int_{\mathbb{R}^4} \| F \|^2 d^4x = \int_{\mathbb{R}^4} F_\mu^a F_{\mu}^a d^4x.
\]
If we replace the Lorentzian \((-+++)\) metric with the Euclidean \((++++)\) metric, we can obtain classical minima of the action above. These are called instantons, and are useful in calculating tunneling amplitudes \(^{18}\) (the rotation from time to imaginary time is what is involved in the WKB approximation).

We care not about \(\mathbb{R}^4\) but about arbitrary compact manifolds (\(\mathbb{R}^4\) is not compact). The question of finding instantons is basically unchanged, except when \(\mathbb{R}^4\) is replaced by a non-trivial manifold, we need to consider some topological considerations. Namely, the gauge field corresponds to a vector bundle \(E\) (in this case, a two dimensional complex vector bundle) on the manifold. The connection is locally defined on coordinate patches, and transforms as we go from one patch to another by gauge transformations.

For each such vector bundle \(E\) over our manifold \(X^4\) we can associate the second Chern class

\[
c_2(E) = -\frac{1}{8\pi^2} \int_{X^4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}
\]

which is an integer\(^1\). The second Chern class is defined above in terms of the connection \(A\), through its curvature \(F\), but in fact it is independent of the connection and only depends on the vector bundle \(E\). The first Chern class \(c_1(E)\), incidentally, is zero because the group is \(SU(2)\). In the \(U(1)\) gauge theory, the first Chern class in generally non-zero and measures the monopole charge for a Dirac monopole. There are higher Chern classes but they are all zero for \(SU(2)\).

It turns out that the second Chern class completely classifies the vector bundle topologically, so that there is a unique vector bundle up to topological vector bundle isomorphism for every integer value of \(c_2\). The trivial bundle has \(c_2 = 0\).

For each vector bundle \(E\), we can look for connections \(A\) that minimize the Yang–Mills action. One choice might be the trivial connection \(A = 0\), which gives rise to the action being equal to zero. This is clearly an absolute minimum, because the action in our case cannot be negative. But this trivial connection only exists in the trivial bundle. When we plug in this connection into the formula for the second Chern class, we get \(c_2 = 0\). More generally, any flat connection is a minimum, but also exists only in the trivial bundle with \(c_2 = 0\).

For other vector bundles, the minima are not as obvious. The trick to understanding these minima is to split the curvature \(F\) into the \(+1\) and \(-1\) eigenvalues of the duality operator \(*\), where \(*F = \tilde{F}\). We define \(F^+ = \frac{1}{2}(F + \tilde{F})\) and \(F^- = \frac{1}{2}(F - \tilde{F})\). Then \(F = F^+ + F^-\), where \(*F^+ = F^+\) and \(*F^- = -F^-\). Furthermore, \(F^+\) and \(F^-\) are orthogonal. The formula for \(c_2\) gives

\[
c_2(E) = -\frac{1}{8\pi^2} \int_{X^4} \left( (F^+ + F^-)_{\mu\nu} (* (F^+ + F^-))^{\mu\nu} \right) = -\frac{1}{8\pi^2} \int_{X^4} F^+_{\mu\nu} (* F^+)^{\mu\nu} + F^-_{\mu\nu} (* F^-)^{\mu\nu}
\]

\(^1\)More precisely, \(c_2(E)\) is the four-form in the integrand; it is an element of \(H^4(X^4) \cong \mathbb{Z}\). The isomorphism \(H^4(X^4) \cong \mathbb{Z}\) is realized by taking the integral.
What do Topologists want from Seiberg–Witten theory?

\[
= -\frac{1}{8\pi^2} \int_{X^4} F_{\mu\nu}^+ F^{+\mu\nu} - F_{\mu\nu}^- F^{-\mu\nu}
\]
\[
= \frac{1}{8\pi^2} \int_{X^4} -\|F^+\|^2 + \|F^-\|^2
\]

while the formula for the action is

\[
S = \int_{X^4} (F^+ + F^-)^{\mu\nu} (F^+ + F^-)^{\mu\nu}
\]
\[
= \int_{X^4} \|F^+\|^2 + \|F^-\|^2.
\]

Thus we see that when \(c_2(E) < 0\), the action is minimized when \(F^- = 0\), so that for instantons, \(*F = F\) (in which case we call \(F\) self-dual) and when \(c_2(E) > 0\), the action is minimized when \(F^+ = 0\), so that instantons have \(*F = -F\) (in which case we call \(F\) anti-self-dual, and we sometimes call such solutions anti-instantons). When \(c_2(E) = 0\), the action is minimized when \(F = 0\), which we observed before.

Suppose we have an instanton with \(c_2(E) = 1\). We view this as a minimum of the action. When we ask the question as to why this is the minimum when the connection \(A = 0\) clearly gives a lower value for the action, the answer is that \(A = 0\) does not exist in our bundle. To “decay” from our instanton to zero would require that we “tear” our bundle first to untwist it. This is what we mean when we say that the instanton cannot decay for topological reasons. The number \(c_2(E)\) (more conventionally, \(-c_2(E)\)) is called the instanton number of the solution, and we imagine that instantons with \(c_2(E) = 2\) are in some sense “non-linear” combinations of two instantons with \(c_2(E) = 1\). When we combine a solution with \(c_2 = -1\) (an instanton) with a solution with \(c_2 = 1\) (an anti-instanton), they can cancel and flow down to a flat connection.

Before I continue, let us consider a point about these solutions. The critical points of the action could have been found using very standard classical techniques using the calculus of variations, and it turns out that this gives us the equation \(D(*F) = 0\). Since the connection \(A\) is the dynamical field we are interested in, and \(F\) involves a derivative of \(A\), we see that \(D(*F) = 0\) is a second-order equation. Instead, we have just derived the equations \(*F = F\) or \(*F = -F\) which are first-order equations in \(A\). The difference is that these first-order equations hold for only the absolute minima, and do not hold for other relative (local) minima, nor do they hold for any sort of “saddle” points of the action. The Bianchi identities \(DF = 0\) can assure us that any solution to \(*F = \pm F\) will also satisfy \(D(*F) = 0\), but the reverse is not true. It turns out that there are many critical points (those that satisfy the Euler–Lagrange equation \(D(*F) = 0\)) that are not absolute minima (instantons, satisfying \(*F = \pm F\)). Now the physics reasons for studying instantons really does prefer absolute minima, anyway, so it could be argued that \(*F = \pm F\) is really what we want to solve. From the mathematical perspective, we get to choose whatever equations we happen to like, and it is solutions to \(*F = \pm F\), not \(D(*F) = 0\), that led to new developments in four-dimensional topology, so that is what we will focus on here.
Kevin Iga

Fig. 5. Schematically drawn here, the moduli space of instantons with $c_2(E) = -1$ on $S^4$ is a five-dimensional ball, with the original $S^4$ as its boundary.

We now consider instantons on $S^4$. Readers who are familiar with instantons on $\mathbb{R}^4$ will see many similarities. The reason is that the Yang–Mills action above has a conformal symmetry, and there is a conformal map from $\mathbb{R}^4$ to $S^4$ that covers everything except for one point. The work of Atiyah, Drinfeld, Hitchin, and Manin gives an explicit description of these instantons, and we will here describe their results for $c_2(E) = -1$.

In the case $c_2(E) = -1$, we are looking for self-dual connections on $E$, which involves solving the differential equation $*F = F$ for $A$. It turns out that the set of instantons on a bundle with $c_2(E) = -1$ on $S^4$, modulo gauge symmetry, is naturally a five-dimensional non-compact manifold. More specifically, it is a five-dimensional open ball. We call this set the moduli space. It turns out we can identify $S^4$ with the missing boundary of the ball in a sense I will describe in a moment.

But before doing that, we should first consider how it came to be that the set of minima is not unique. Usually, a function has a unique absolute minimum. It is possible to have functions that have many absolute minima, by arranging it so that many points take on the same minimum value of the function. But we usually regard this as an unusual phenomenon, and in the world of physics, where the formulas are given to us by nature rather than specifically dreamed up to have multiple minima, we should expect there to be only one absolute minimum. If we see more than one absolute minimum, this is a phenomenon to be explained.

There are, indeed, circumstances in physics that give multiple absolute minima, and even continuous families of absolute minima, but these are usually explained by the existence of a group of symmetries. Take, for example, the Higgs mechanism in a $\phi^4$ theory. The theory has a spherical symmetry, and so the set of minima might be a sphere, and small perturbations that preserve this symmetry will still have a spherical set of minima.

In the case of instantons on $S^4$, the existence of many minima can also be explained by symmetry. There is the gauge symmetry, but recall that we have
already quotiented out by this symmetry. But there are also conformal symmetries of $S^4$, and since the action is conformally invariant, these conformal symmetries will take instantons to other instantons. In fact, the conformal symmetries of $S^4$ are enough to explain the entire set of solutions in this case. Therefore, from one solution, we can use the conformal symmetries to explain the entire moduli space. The fact that there are no other solutions for $c_2(E) = -1$ was shown by Atiyah and Ward.\footnote{These limiting degenerate configurations are sometimes called small instantons, and while physicists are used to viewing them as instantons of a special kind, mathematicians tend not to view them as instantons, since $A_p$ is not even well-defined at the point $p$. But it is possible to define a “small instanton” and add these small instantons to the moduli space in a natural way. The result makes the moduli space compact and this process is called compactifying the moduli space.}

Taking this idea of using the conformal symmetry, we can take a conformal symmetry that flows all of $S^4$ concentrating more and more of it closer to any given point on $S^4$. The effect of this is to concentrate the instanton near a given point of $S^4$. This explains why the boundary of the set of solutions is $S^4$. The conformal symmetry that concentrates most of $S^4$ near a point $p \in S^4$ will also move instantons in the moduli space (recall it is a five-dimensional ball) near a corresponding point on its boundary. The limiting connection is degenerate, and in a sense that is reminiscent of a Dirac delta function, is flat everywhere on $S^4$ except at $p$, where it has infinite curvature. Thus we can add to our moduli space these extra limiting connections, thereby turning our non-compact ball to a compact ball with boundary.\footnote{When we say, “manifold”, we perhaps should say “orbifold” instead, since singularities due to quotients of group actions sometimes occur in moduli spaces, as we will see later. But referring to the moduli space as a manifold is entrenched in the literature and we imagine that it is a manifold, but perhaps with a few singularities. Besides this, we already saw in the case of $S^4$ that the moduli space is not necessarily compact, until we include the small instantons which may add a boundary.}

More generally, the moduli space of instantons on a four-dimensional manifold $X^4$, with $c_2(E) < 0$ given, is a manifold\footnote{When $c_2(E) > 0$, we are solving the anti-self-dual equation $\ast F = -F$, with the same gauge-fixing condition, and the Atiyah–Singer index theorem gives the dimension as

$$d = 8c_2(E) - 3(1 - b_1(X^4) + b_2^+(X^4)).$$

The dimension may be zero, in which case the moduli space would be a set of points, or the dimension may be negative, in which case the moduli space will be empty (so that there would be generically no instantons with that value of $c_2$).} of dimension

$$d = -8c_2(E) - 3(1 - b_1(X^4) + b_2^{-}(X^4)).$$

This formula is obtained by the Atiyah–Singer index theorem, by viewing the self-dual equations as zeros of a differential operator, together with a suitable gauge-fixing condition like $d^*(A - A_0) = 0$ once a fixed reference connection $A_0$ is identified. Similarly, when $c_2(E) > 0$, we are solving the anti-self-dual equation $\ast F = -F$, with the same gauge-fixing condition, and the Atiyah–Singer index theorem gives the dimension as

$$d = 8c_2(E) - 3(1 - b_1(X^4) + b_2^{+}(X^4)).$$
Kevin Iga

If this dimension $d$ is positive, we should in general have many absolute minima, and we might want to explain why this is the case. We no longer have the conformal symmetry of $S^4$ in general, so we have no reason to suspect multiple solutions. In fact, with many known cases, we see no obvious symmetry in the moduli space. This is an example of a situation where the dictum that multiple minima must come from a group is unfounded. The reason a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ typically has a unique absolute minimum, or at least zero-dimensional relative minima, is that the criterion $\nabla f = 0$ gives $n$ equations. A system of $n$ equations and $n$ variables typically results in a zero-dimensional set of solutions. If instead of $\nabla f = 0$ we had $n - 1$ equations, we would expect a one-dimensional set of solutions.

In our case, the action $S$ has an infinite dimensional domain (the set of connections), and the criterion of $\nabla S = 0$ results in infinitely many equations. But we can no longer make any sense of comparing the number of equations and number of variables. In fact, in general, infinite dimensional problems like this have many pathologies, and there is nothing more we can say. But in our particular case, because the differential equations are elliptic, we have many nice results. For instance, even though we have an infinite number of “variables” and an infinite number of “equations”, we have a well-defined notion of the difference between the two dimensions, and this number is called the index. This is the dimension of the set of solutions, and is what the Atiyah–Singer index theorem calculates. This is the number given above for the dimension of the moduli space.

In other words, for infinite-dimensional problems like this, we must discard our intuition that was based on finite-dimensional problems, and if we have an elliptic differential equation (as in our case) we can be thankful that the intuition need not be completely discarded, but only modified. An example that may be more familiar to you is the fact that the minima of

$$\int_{X^4} ||d\omega||^2 + ||d^*\omega||^2$$

where $\omega$ ranges over $k$-forms are those $k$-forms $\omega$ that are harmonic, that is, satisfy $\Delta \omega = 0$. The set of these is the $k$-th cohomology group $H^k(X^4; \mathbb{R})$, and this is a vector space of dimension $b_k$.

There is no group that guarantees a non-zero-dimensional family of solutions: the “correct” dimension of the set of minima is simply given by the Atiyah–Singer index theorem. If you plug in $S^4$ and $c_2(E) = -1$ in the dimension formula (2), (note that for $S^4$ we have $b_1 = 0$ and $b_2 = 0$) we get $d = 5$, which says that the five-dimensionality of the moduli space there is not really a consequence of the conformal symmetry group after all, in the sense that the moduli space would

---

1Some may point out that this has a symmetry, too, in that harmonic forms act on the set of solutions by addition. But this begs the point, since if there were not an infinite family of harmonic forms, there would be no group to act on the set. Anyway, the point is that analogous situations come up in more elementary settings, and the Atiyah–Singer index theorem predicts the correct dimension of the space.
5. Donaldson’s Theorem A

As mentioned above, Donaldson’s Theorem A states:

**Theorem 1:** Let $X^4$ be a simply connected compact four-dimensional manifold (no boundary) with definite intersection form. Then its intersection form, in some basis, is plus or minus the identity matrix.

A rough proof goes as follows: By changing the orientation on $X^4$ we can assume that the intersection form is positive-definite. Then $b_2^- = 0$. For simply connected manifolds, we saw above that $b_1 = 0$. Then if we are interested in the bundle $E$ over $X^4$ with $c_2(E) = -1$, we see that the formula for the dimension of the moduli space gives us that the moduli space of instantons will be a five-dimensional manifold.

Analogously to the case for $S^4$, where $S^4$ could be viewed as the boundary of the moduli space, we can similarly “compactify” the moduli space by including small instantons (the set of small instantons looks like a copy of $X^4$) so that the resulting moduli space is a five-dimensional manifold with boundary $X^4$. The procedure is

---

**Fig. 6.** The moduli space of instantons with $c_2(E) = -1$ on $X^4$ with $b_2^- = 0$. Note the boundary as $X^4$ itself, and the $m/2$ singularities. This drawing is intended to be schematic or suggestive: the four-dimensional manifold $X^4$ is drawn as a circle, and the five-dimensional moduli space is drawn as a surface.

continue to be five-dimensional even if we were to slightly perturb the metric on $S^4$ so that it no longer has conformal symmetry.

For a more detailed description of the moduli space of instantons on $S^4$, see Ref. [12] and Ref. [13].
analogous to the one mentioned above, and was developed by Karen Uhlenbeck and was developed by Karen Uhlenbeck.

Now I mentioned under my breath that the moduli space may not be quite a manifold, because it may have singularities. It turns out that in the situation we are describing, there are finitely many singularities, each isolated and locally isomorphic to a cone on $\mathbb{CP}^2$. They can be counted in the following way: let $m$ be the number of elements $v \in H_2(X^4)$ so that $v^T I v = 1$, where $I$ is the intersection form of $X^4$. Then there will be $m/2$ singularities.

These singularities come about from the fact that the gauge group does not always act freely. When the complex two-dimensional bundle $E$ can be split into two one-dimensional bundles $E = L_1 \oplus L_2$, in such a way that the connection $A$ turns out to be the product of connections on each of the one-dimensional bundle factors, so that the connection is actually a product of $U(1)$ connections, then a part of the gauge group will fix $A$. In particular, a constant $U(1)$ gauge transformation will leave this reducible connection $A$ invariant. Such connections are called reducible, and if this does not occur, we call it irreducible.

The result is that when $A$ is reducible, and we quotient by the global gauge group, there will be the kind of singularity mentioned above: a cone on $\mathbb{CP}^2$.

Studying the self-dual equations for connections of this special type shows that each splitting of $E$ into two factors contributes a unique reducible connection, and for this splitting to happen $c_1(L_1) + c_1(L_2) = 0$ and $c_1(L_1)^T I c_1(L_2) = c_2(E)$. So these correspond to elements $v = c_1(L_1) \in H_2(X^4)$ so that $v^T I v = 1$, and this is a one-to-one correspondence up to swapping the roles of $L_1$ and $L_2$. This explains the number of singularities.

These singularities are isolated and do not occur on the glued-in $X^4$.

Therefore, we can take our moduli space of instantons and modify it as follows: first, glue in the $X^4$ so that the moduli space becomes a compact manifold with boundary and with singularities. Then excise a small open ball around each of the $m/2$ singularities. What we now have is a five-dimensional manifold with $X^4$ as one boundary component, and $m/2$ other boundary components, each of which is a $\mathbb{CP}^2$.

Now, an old topological theorem is that when a union of four-dimensional manifolds is the boundary of a five-dimensional manifold, the sum of their signatures is zero. In our case, this means $\sigma(X^4) + \frac{m}{2} \sigma(\mathbb{CP}^2) = 0$. Since $\sigma(\mathbb{CP}^2) = -1$, this means $\sigma(X^4) = m/2$. Since the intersection form for $X^4$ is positive definite, $\sigma = b_2$, so there are $b_2$ singularities in the original moduli space. Therefore, there are $b_2$ solutions to $v^T I v = 1$.

A few words about the $v \in H_2(X^4)$ with $v^T I v = 1$. Note that if $v$ satisfies this, so does $-v$, which partly explains the naturality of dividing by 2. Furthermore, if $v$ and $w$ are two such, and $v \neq \pm w$, it turns out that $v^T I w = 0$.

\[\text{For this theorem to apply we need the five-dimensional manifold to be orientable. It turns out that these moduli spaces are always orientable.}\]
What do Topologists want from Seiberg–Witten theory?

Fig. 7. The possible number of $E_8$’s and $H$’s. Rokhlin’s theorem prevents us from having an odd number of $E_8$’s, and Donaldson’s work says that if we have at least two $E_8$’s, we must have at least three $H$’s. At the end of the review we describe Furuta’s work, using Seiberg-Witten theory, that says the number of $H$’s must be at least the number of $E_8$’s plus one. The work of Furuta, Kametani, and Matsue, also using Seiberg-Witten theory, excludes $4E_8 + 5H$. The filled dots indicate which combinations are known to exist. The question marks indicate the current unknown areas.

Choose one $v$ from each $\pm$ pair of solutions to $v^T Iv = 1$. The collection of these $v$ will then form a basis, and in this basis, $I$ will be the identity matrix.

This proves Donaldson’s Theorem A.

6. Other questions about existence

The question of classification of smooth manifolds might be split into two questions: For each intersection form, do there exist manifolds with that intersection form? And for each intersection form, how many manifolds have that same intersection form? The first question might be viewed as an “existence” question, and the second question might be viewed as a “uniqueness” question.

What Donaldson’s Theorem A does is eliminate a great many intersection forms from consideration, showing that manifolds do not exist that have those intersection forms. So it weighs in on the “existence” question. The extent to which it works is apparent when you realize that the only intersection forms left to consider are blockwise combinations of $H$’s and $E_8$’s, and diagonal matrices with $\pm 1$ on the diagonal (see Figure 7).

The diagonal matrices can be obtained by taking the connected sum of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$, so these are definitely possible. As above, $S^2 \times S^2$ has $H$, so it is possible to get arbitrarily many $H$’s. The K3 surface, as described above, has three $H$’s and two $E_8$’s. By connect summing K3 to another copy of K3, and iterating this procedure, it is possible to get $3kH \oplus 2kE_8$. By connect summing these by
S\(^2\) × S\(^2\)'s, we can get \(mH \oplus nE_8\), where \(n\) is even, and \(m \geq \frac{1}{2}n\). In terms of \(\sigma\) and \(b_2\), where we note that \(b_2 = 2m + 8n\) and \(\sigma = 8n\), we see that we can get any \(b_2\) and \(\sigma\) with \(b_2 \geq \frac{11}{8}\sigma\).

The question as to whether or not the other combinations of \(H\) and \(E_8\) are possible is a difficult one. A natural conjecture, called the *eleven-eighths conjecture*, is that it is impossible to have a combination of \(H\)'s and \(E_8\)'s with \(b_2 < \frac{11}{8}|\sigma|\). This has not yet been proven nor disproven. Donaldson’s techniques were able to make some progress (such as \(m \geq 3\) when \(n > 0\)). Seiberg–Witten theory allowed for even more progress, described in section 17.

7. Uniqueness: the Donaldson invariants

The question of discerning different manifolds that have the same intersection form comes down to finding new invariants. Here, too, instantons turn out to be useful. Using them, Donaldson defined what are now known as *Donaldson invariants*, or *Donaldson polynomials*. A thorough elaboration of this section can be found in Donaldson and Kronheimer’s book.

To get some idea of how these might be defined, consider a simply connected four-dimensional manifold \(X^4\). Suppose some choice of \(c_2(E) > 0\) makes the dimension of the moduli space \(d = 8c_2(E) - 3(1 - b_1(X^4) + b_2^+(X^4))\) equal to zero. For instance, if \(b_1 = 0\) (as is required for \(X^4\) to be simply connected) and \(b_2^+ = 7\) (as is the case with a connected sum of seven \(\mathbb{C}P^2\)'s), then for the bundle \(E\) over \(X^4\) with \(c_2(E) = 3\), the moduli space of instantons would have dimension zero, and so would be a collection of points. These points actually come with multiplicity and sign. The Donaldson invariant of the manifold \(X^4\) would be the count of how many points there are in the moduli space, counted with appropriate multiplicity and sign.

The main objection to this idea is that in order to define these invariants, we had to assume a metric on \(X^4\)—and if we had used a different metric, we would surely have different solutions to the relevant differential equations, and hence, a different moduli space.

What makes this invariant a topological invariant is that this count is independent of the metric. The reason is that if \(g_0\) and \(g_1\) are two metrics on \(X^4\), then since the set of metrics is connected, we can consider a path of metrics \(g_t\), \(0 \leq t \leq 1\) on \(X^4\). Then over \(X^4 \times [0, 1]\), with the metric \(g_t\) on the slice \(X^4 \times \{t\}\), the moduli space over each slice joins together to form a one-dimensional manifold. The result is a diagram as in Figure 8.

Though this looks like a Feynmann diagram, note that \(t\) does not really mean time; it is the parameter through which we are changing our metric. Time has already been made space-like, because we are looking for instantons. But the same kind of behavior appears: positive solutions and negative solutions may cancel, or
pairs of positive and negative solutions may appear. So if we count these solutions with appropriate multiplicity and sign, the number does not change.

Note, by the way, that these signs of $+$ and $-$ are not the same thing as instantons and anti-instantons. In our example, $c_2(E) = 3$, so each point on the moduli space is a 3-anti-instanton solution to the differential equation, and there are no “positive” instantons. The vertical axis does not represent space, either. It represents (schematically) the set of connections. So a point over $t = 0$ refers to a particular 3-anti-instanton solution on $X^4$ with metric $g_0$, and as $t$ increases, the particular solution changes gradually. When two points “annihilate”, what is really going on is two 3-anti-instanton solutions become more and more similar as the metric is varied, and at a certain metric, they become identical, and then the solution disappears completely. Fans of catastrophe theory may recognize this phenomenon.

There are some minor problems with this picture. One problem is that singularities may appear, and to avoid this requires that $b_2^+ > 1$. There is nothing we can do about this except to only define the Donaldson invariant when $b_2^+ > 1$. This restriction thus turns up in many results related to Donaldson theory.

The other problem is that the formula for the dimension of the moduli space does not usually give zero, and besides, we would like to get as many invariants in as many situations as possible. So we should define Donaldson invariants in the situation where the expected dimension of the moduli space is non-zero. This leads to the more general Donaldson invariants which we now define.

In general, we consider the set of all irreducible connections, modulo gauge transformation, as an infinite-dimensional manifold (call it $B^* = A^*/G$\[^{[1]}\]) and we look at the moduli space of solutions modulo gauge as a finite-dimensional subman-
ifold (call it $\mathcal{M}_g$). As the metric $g$ changes, the set of solutions $\mathcal{M}_g$ moves inside the space of connections $\mathcal{B}^*$.

If $g_0$ and $g_1$ are metrics on $X^4$, then $\mathcal{M}_{g_0}$ and $\mathcal{M}_{g_1}$ are related as follows: if $g_t$ is a path from $g_0$ to $g_1$ parameterized by $t$, then the moduli spaces $\mathcal{M}_{g_t}$ sweep out a manifold that has as boundary $\mathcal{M}_{g_0}$ and $\mathcal{M}_{g_1}$. Therefore $\mathcal{M}_{g_0}$ and $\mathcal{M}_{g_1}$ are cobordant. The study of submanifolds of $\mathcal{B}^*$ up to cobordism allows us to conclude that the homology element defined by $\mathcal{M}_{g_0}$ in $H_d(\mathcal{B}^*)$ (where $d$ is the dimension of the moduli space) is the same as the homology element defined by $\mathcal{M}_{g_1}$, or indeed, the homology element defined by any $\mathcal{M}_g$. Therefore the element $[\mathcal{M}]$ of $H_d(\mathcal{B}^*)$ is independent of the metric and is a topological invariant.

So this Donaldson invariant might be viewed as an element of $H_d(\mathcal{B}^*)$. It turns out that elements of $H_4(\mathcal{B}^*)$ can be expressible as functions on the set of formal polynomials in the homology of $X^4$, in a sense I will now explain. We recall from above that when $X^4$ is a simply connected compact four-dimensional manifold, the homology of $X^4$ (with real coefficients) is as follows:

$$
\begin{align*}
H_0(X; \mathbb{R}) &\cong \mathbb{R} \\
H_1(X; \mathbb{R}) &\cong 0 \\
H_2(X; \mathbb{R}) &\cong \mathbb{R}^{b_2} \\
H_3(X; \mathbb{R}) &\cong 0 \\
H_4(X; \mathbb{R}) &\cong \mathbb{R}
\end{align*}
$$

Let $x$ be a generator for $H_0(X^4; \mathbb{R})$, and let $y_1, \ldots, y_{b_2}$ be a basis for $H_2(X^4; \mathbb{R})$, and let $z$ be a generator for $H_1(X^4; \mathbb{R})$. An argument from algebraic topology using spectral sequences shows that there is a homomorphism $\mu : H_*(X^4; \mathbb{R}) \to H^4-*(\mathcal{B}^*; \mathbb{R})$ so that $\mu(z) = 1 \in H^4(\mathcal{B}^*; \mathbb{R})$, $\mu(y_1), \ldots, \mu(y_{b_2})$ form a basis for $H^2(\mathcal{B}^*)$, and $\mu(x) \in H^4(\mathcal{B}^*; \mathbb{R})$. Using wedge product we can generate all of $H^*(\mathcal{B}^*; \mathbb{R})$, and there are no relations other than $\mu(z) = 1$. In other words, any element in $H^*(\mathcal{B}^*; \mathbb{R})$ can be written as a polynomial in $\mu(x), \mu(y_1), \ldots, \mu(y_{b_2})$. We view $\mu(x)$ as degree 4, and $\mu(y_i)$ as degree 2. Note that we are only working with differential forms of even degree here, so that the wedge product is commutative.

If we have an element of the homology of $\mathcal{B}^*$, like $[\mathcal{M}] \in H_4(\mathcal{B}^*)$, then we can pair it on any homogeneous polynomial $p$ in $\mu(x), \mu(y_1), \ldots, \mu(y_{b_2})$ of degree $d$:

$$
\int_{\mathcal{M}} p(\mu(1), \mu(y_1) \wedge \ldots \wedge \mu(y_{b_2})).
$$

For polynomials of the wrong dimension, we define this number to be zero. It therefore defines a function from $\mathbb{R}[x, y_1, \ldots, y_{b_2}]$ to $\mathbb{R}$, where for the sake of conciseness we can omit writing $\mu$ since we are dealing with formal expressions anyway.

In this way, every $SU(2)$ bundle $E$ and metric $g$ on $X^4$, $\mathcal{M}_{g,E}$ gives rise to a linear function on polynomials

$$
Q_{X^4, E, g} : \mathbb{R}[x, y_1, \ldots, y_{b_2}] \to \mathbb{R}.
$$

It does not depend on the metric $g$ as long as $b_2^+ > 1$. This is called the Donaldson polynomial invariant for $(X^4, E)$. It is sometimes indexed not by $E$ but by the
expected dimension of the moduli space \( d \), given in (3), so we sometimes write \( Q_{X^4,d} \).

There are a number of technical difficulties, and a number of restrictions due to the non-compactness of the moduli space and the existence of singularities. But for the most part, these problems have been mostly solved or circumvented, as long as
\[ b_2^+ > 1. \]

15, 16

These Donaldson polynomial invariants have been calculated in a number of circumstances, and it has been shown that many manifolds that are indistinguishable using the “classical” (i.e. pre-1983) invariants (such as the intersection form) turn out not to be diffeomorphic to each other, by virtue of having different Donaldson polynomials. From this, and from the earlier work of Freedman mentioned above, it can be proven that there are pairs of compact four-dimensional manifolds that are homeomorphic to each other but not diffeomorphic to each other.

As might be expected, it is in general very difficult to calculate these invariants, and it is not even clear if there is a general method for calculating them, so cases where Donaldson polynomials helped distinguish four-dimensional manifolds were rare.

In 1993, Kronheimer and Mrowka showed how to put the various polynomials for each dimension \( d \) into a generating function, called a Donaldson series. This is a formal non-linear function on \( H_2(X^4) \):

\[
D(h) = \sum_d \frac{Q_{2d}(h^d)}{d!} + \frac{1}{2} \sum_d \frac{Q_{2d+4}(xh^d)}{d!}
\]

where the sum is taken over dimensions \( d \) where an \( SU(2) \) bundle exists such that the expected dimension of the moduli space is \( d \), and \( h \) is an element of \( H_2(X^4) \).

It is not clear that this series converges, but the convergence is not crucial since the manipulations involved are formal. Besides, in many (perhaps all) cases the series does converge, giving rise to an honest function \( D : H_2(X) \to \mathbb{R} \).

Kronheimer and Mrowka say that a manifold \( X^4 \) satisfies the simple type condition if \( Q_{|z|+8}(x^2 z) = 4Q_{|z|}(z) \) for all \( z \in \mathbb{R}[x, y_1, \ldots, y_b] \) and where \( |z| \) means the degree of \( z \). This essentially says that the four-dimensional class \( x \) is not independent in the series \( Q_d \), but satisfies \( x^2 = 4 \), so that these invariants depend only on the \( H_2(X^4) \) part.

When the simple type condition holds, then this series can be written as

\[
D(h) = e^{I(h,h)/2} \cdot \left( r_1 e^{K_1(h)} + \ldots + r_m e^{K_m(h)} \right) \tag{4}
\]

where \( I \) is the intersection form, viewed as bilinear function on \( H_2(X^4) \), \( r_1, \ldots, r_m \) are rational numbers, and \( K_1, \ldots, K_m \) are elements in \( H^2(X^4) \), thought of as linear functions on \( H_2(X^4) \).

They showed that for an impressive number of examples, the simple type condition actually holds, and it has been conjectured that all four-dimensional manifolds
with $b^+_2 > 1$ are of simple type. This appears likely, especially in light of more recent developments.

In many cases, these series helped in the calculation of Donaldson invariants, especially with examples from algebraic geometry.

8. Algebraic and Kähler geometry

When the manifold admits a Kähler metric, the anti-self-dual equations to find instantons are related to the notion of stable holomorphic bundles. Similarly, the self-dual equations related to stable anti-holomorphic bundles. Because mathematicians prefer to deal with holomorphic objects rather than anti-holomorphic objects, the orientation was chosen to deal with the anti-self-dual equations.

But beyond this, this meant that for Kähler manifolds, there were a number of ways of calculating the set of instantons without having to actually solve a differential equation, and this led to important advances in calculating Donaldson invariants, especially once Kronheimer and Mrowka gathered them into the Donaldson series. For example, the K3 surface has Donaldson series

$$\exp(I/2),$$

and more generally, an elliptic surface $E(\chi, m_1, \ldots, m_r)$ (which has elliptic curves as fibers and $S^2$ as base, where the holomorphic Euler characteristic of this fibration is $\chi$ and $m_1, \ldots, m_r$ are the multiplicities of the singular fibers) has Donaldson series

$$\exp(I/2) \left( \frac{\sinh F}{\prod \sinh(F/m_i)} \right)^{p_g-1+r},$$

where $F$ is the class generated by the fiber and $p_g$ is the geometric genus. Similarly, many such formulas were computed for complex algebraic surfaces of various kinds.

The main point here is that for complex algebraic surfaces, the Donaldson invariants could be calculated systematically, once the systematic framework of the Donaldson series of Kronheimer and Mrowka was in place. We will soon refer to the fact that the Donaldson invariants of K3 are non-zero.

9. Floer homology and topological quantum field theories

One idea to calculate Donaldson invariants for non-Kähler manifolds is to split the manifold into pieces, each of which might be easier to analyze, perhaps because

\footnote{There are examples, like $\mathbb{C}P^2$, that are not of simple type, but none of these have $b^+_2 > 1$.}

\footnote{Essentially, if we view manifolds as coming with a particular orientation (the fact that an orientation can be chosen at all follows from the simply connected criterion), then we can view solutions to the self-dual equations as solutions to the anti-self-dual equations with the reverse orientation. When the orientation is reversed, $b^+_2$ becomes $b^-_2$, the sign convention on $c_2(E)$ (as an integer) reverses, the intersection form becomes its negative, and self-dual solutions become anti-self-dual solutions, but nothing else changes. So we can emphasize the theory of anti-self-dual connections and discard the theory of self-dual connections, without losing any real mathematics, as long as we consider manifolds as coming with an orientation.}
they can be viewed as pieces of Kähler manifolds. This is easier said than done. For an analogy, consider a large molecule. Strictly speaking, to predict the structure of the molecule, you would need to solve a huge quantum $n$-body problem that is beyond the capabilities of even the best computers. But a more enlightening approach might be to solve the Schrödinger equation for individual atoms, then understand how orbitals combine to form molecular orbitals in particular bonds, and so on. Of course, you would run into a few surprises that happen on a more global scale, but for the most part, this idea works remarkably well.

For Donaldson theory, the idea would be to find a three-dimensional submanifold that splits the four-dimensional manifold into two parts. Since the Donaldson invariants are invariant under change of metric, we can imagine altering the metric on the manifold by moving the two parts further apart, which stretches a neighborhood of the three-dimensional manifold into a neck (see Figure 9).

Then instantons on the four-dimensional manifold can be viewed as instantons on each piece which are glued together along instantons in the cylindrical neck. When we adjust the metric on the four-dimensional manifold so that the neck becomes very long, the cylindrical neck can be viewed as $Y^3 \times \mathbb{R}$, where $Y^3$ is the three-dimensional manifold. The instanton solutions, from the perspective of each unstretched side, become solutions that converge to a constant solution on the $Y^3 \times \mathbb{R}$ cylindrical end, while on the neck itself, these solutions become solutions on the cylinder $Y^3 \times \mathbb{R}$.

A dimensional reduction can be done to turn the anti-self-dual equations on the cylindrical neck $Y^3 \times \mathbb{R}$ into a problem of connections $A$ on $Y^3$ (taking the temporal gauge $A_0 = 0$). Viewing $\mathbb{R}$ as time, the anti-self-dual equations become the following flow:

$$\frac{d}{dt} A = - * F.$$  \hspace{1cm} (5)

This equation is exactly the gradient flow equation in Morse theory. Morse theory considers a smooth potential function $V : X^n \to \mathbb{R}$ and its critical points (points $p \in X^n$ so that $\nabla V(p) = 0$). The second-derivative test tells us whether the critical point is a maximum, a minimum, or some kind of saddle in between. More precisely, the number of negative eigenvalues of the Hessian at $p$ is called the index...
of the critical point \( p \). Minima have index 0, and maxima have index \( n \).

Morse theory describes how the number of critical points of each index relates to the homology of the manifold \( X^n \). Essentially, each critical point of \( V \) of index \( k \) gives rise to a \( k \)-dimensional cell in a decomposition of \( X^n \) into cells, and these cells can be used to compute the homology of \( X^n \). The trick is to look at trajectories \( x(t) \) in \( X^n \) that satisfy

\[
\frac{dx}{dt} = -\nabla V(x(t)).
\]

If \( p \in X^n \) is a critical point of \( V \), then the set of points that lie on trajectories that come from \( p \) forms a cell whose dimension is the index of \( p \). These cells, in turn, can be used to calculate the homology of \( X^n \), and in particular, give rise to formulas like

\[
\sum_{i=0}^{n} (-1)^i b_i = \sum_{i=0}^{n} (-1)^i m_i
\]

and

\[
b_i \leq m_i
\]

where \( m_i \) is the number of critical points of index \( i \) and \( b_i \) is the \( i \)th Betti number, that is, the rank of \( H_i(X^n) \).

Given two critical points \( p \) and \( q \), there may be trajectories that start from \( p \) and end at \( q \). When the index of \( p \) is one greater than the index of \( q \), there will be only finitely many such trajectories, up to translation in \( t \). The number of these trajectories can be used to reconstruct the homology of \( X^n \), as long as the dimension \( n \) is finite.

Now replace \( X^n \) with the set \( \mathcal{A} \) of \( SU(2) \) connections on \( Y^3 \), up to gauge equivalence. This is an infinite-dimensional manifold. For our potential function \( V \) we take the Chern–Simons functional

\[
CS[A] = \frac{1}{2} \int_{Y^3} \text{Tr} \left( dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right)
\]

Then

\[
\frac{dA}{dt} = -\nabla CS[A(t)]
\]

becomes

\[
\frac{dA}{dt} = -* F
\]

which is equation (5). Analogously to the finite-dimensional example, we can define a sort of “homology” by counting flow lines between pairs of critical points of \( CS \) whose indices differ by one. Because of the infinite dimensionality, the result is not the homology of the configuration space, but it is interesting nonetheless. This is called the Floer homology of \( Y^3 \). There are a number of technical difficulties
in carrying this idea out, because the set of connections modulo gauge is infinite-dimensional and because of the sort of non-compactness associated with instantons, so that certain ideas of Morse theory are no longer available and other techniques need to be invented. But for three-dimensional manifolds $Y^3$ with $b_1(Y^3) = 0$, this idea was carried out by A. Floer and refined by others (see, for instance, Ref. 24).

The result is that the Donaldson invariants on a four-dimensional manifold with $Y^3$ as boundary can be viewed as an element of the Floer homology of $Y^3$, and when two four-dimensional manifolds have the same boundary and are glued together along the boundary, an inner product on the Floer homology pairs these invariants together to get actual numbers (which, put together, form the Donaldson series).

This idea was explained by Sir Michael Atiyah. He views it as a topological quantum field theory, where three-dimensional manifolds are viewed as a possibility for a space-like slice, and four-dimensional manifold that are cobordisms between these three-dimensional manifolds are viewed as world-sheets. The Hilbert space of states associated to a space-like slice is the Floer homology of the three-dimensional manifold, and the Donaldson invariant on the four-dimensional manifold is the operator for time-translation. This is called a topological field theory since there is no local dynamics (the Hamiltonian $H$ is zero). The dynamics can only happen because of topological changes in the space. For a clear axiomatic treatment of topological quantum field theories, see Ref. 26 by Atiyah.

In the case where $X^4$ is a connected sum of two manifolds $X_+$ and $X_-$, so that $X = X_+ \# X_-$, we are attaching the two pieces along an $S^3$. Now $S^3 \times \mathbb{R}$ is fairly easy to analyze, so in this case, it is possible to show that if $b_2^+(X_+)$ and $b_2^-(X_-)$ are greater than zero, then the Donaldson invariants for $X = X_+ \# X_-$ vanish.

Therefore, since the Donaldson invariants were known not to vanish for the K3 surface, we know that the K3 surface cannot be a connected sum unless one of the pieces has $b_2^+ = 0$. By Donaldson’s Theorem A, if this other piece has $E_8$’s, it must have at least one $H$, which would make $b_2^+ > 0$; so if the K3 surface is a connected sum, it must be with a piece that has the homology of a sphere.

Though there is a great deal of physics that underlies Donaldson theory, it was possible for a mathematician to work in the field with little or no knowledge of physics. The reason is that although the physics motivated the notion of gauge theory, it is possible to treat the Yang–Mills theory as simply a minimization problem, and the self-dual and anti-self-dual equations as partial differential equations. This is like learning, in an ordinary differential equations course, how to solve second-order linear differential equations with constant coefficients, without ever knowing about Hooke’s law about springs. In the case of finding instantons, there was a great deal of interesting analysis, geometry, and topology that can be done without knowing the physics, and once it was appreciated that on Kähler manifolds these instantons related to stable holomorphic bundles, it was possible to use algebraic geometry and complex geometry, too. Atiyah’s notion of a topological quantum field theory, also, could be appreciated without even knowing what an ordinary
quantum field theory was, simply by thinking about what happened to instantons when you stretch the neck.

Thus, many mathematicians went into the subject with little or no background in quantum field theory, and were able to make important contributions. The physics had its impact in posing a mathematical problem (the problem of finding instantons) and from then on, mathematicians could play with the problem without knowing the physics origins. But there was also a sense that if physics made an impact on mathematics in this unexpected way once, perhaps it might happen again, and so ignoring the physics would be a bad idea. Perhaps physics might have more to say about instantons and Donaldson invariants. It did, as we will see next.

10. Witten’s work on Donaldson invariants

Edward Witten found another way to understand Morse theory, relating the cohomology of $X^n$ to critical points of a potential function $V : X^n \to \mathbb{R}$. He used a Hamiltonian formulation of a certain $N = 2$ supersymmetric quantum mechanical system. He defined, for every $t \geq 0$,

$$d_t = e^{-tV} d e^{tV}, \quad d_t^* = e^{tV} d^* e^{-tV},$$

then defined

$$Q_{1t} = d_t + d_t^*, \quad Q_{2t} = i(d_t - d_t^*), \quad H_t = d_t d_t^* + d_t^* d_t,$$

and showed that for all $t$,

$$Q_{1t}^2 = Q_{2t}^2 = H_t, \quad \{Q_{1t}, Q_{2t}\} = 0.$$  

The number of zero-modes of $H_0$ is the space of harmonic $k$-forms, which has dimension $H^k(X^n)$. As $t \to \infty$, the zero-modes of $H_t$ concentrate at the critical points of $V$, and it is possible to derive the same kinds of relations between critical points of $V$ and $H^k$ that show up in Morse theory.

As we saw in the previous section, Donaldson invariants on a cylinder $Y^3 \times \mathbb{R}$ is formally Morse theory for the Chern–Simons functional. So applying Witten’s idea to Donaldson invariants on $Y^3 \times \mathbb{R}$ gives rise to an $N = 2$ supersymmetric quantum system, but since this Morse theory is on a set of gauge fields, we get a quantum field theory.

Witten generated this quantum field theory in 1988. Roughly, it involves gauge fields $A_i^a$ and anti-commuting fields $\psi_i^a$ and $\chi_i^a$ on $Y^3$. In the following discussion, we will use the term boson to refer to commuting fields and fermion to refer to anti-commuting fields, regardless of their spin. Thus, $A_i^a$ is a boson, and $\psi_i^a$ and $\chi_i^a$ are fermions, even though all three fields have spin 1. This particular theory is not relativistic, so the spin-statistics theorem does not apply.
The gauge group $SU(2)$ acts as usual on $A^a_i$ and on fermions $\psi^a_i$ and $\chi^a_i$ it acts via the adjoint action. The Hamiltonian of this system is

$$H = \int d^3x \left[ \frac{1}{2} \sum_{i,a} \left( -i \delta \frac{\delta}{\delta A^a_i(x)} \right)^2 + \frac{t^2}{2} \text{Tr} \tilde{F}^i \tilde{F}^i + t\epsilon_{ijk} \text{Tr} \psi^i D^j \chi^k \right].$$

The Latin index $i$ goes between 1 and 3, and represents space in $Y^3$.

Roughly speaking, for each $i,a$, and $x$, the $\psi^a_i(x)$ play the role of a basis of one-forms on the set of connections $A$, and the $\chi^a_i(x)$ are a dual basis of vector fields on $A$.

Zero modes of the Hamiltonian will then calculate instantons on the cylindrical manifold $Y^3 \times \mathbb{R}$. Since this theory was designed for $Y^3$ and describes dynamics on $Y^3 \times \mathbb{R}$, this cannot work for a general four-dimensional manifold. What was needed was an extension of this theory into a four-dimensional covariant theory.

Witten then (in the same paper) came up with such a relativistically covariant quantum field theory which was still supersymmetric, by generalizing the existing fields and adding new fields. In this setting, $A^a_i$ became the gauge field $A^a_\mu$, where $\mu = 1,2,3,4$ is a spatial 4-index; $\psi^a_\mu$ became $\psi^a_\mu(x)$, and $\chi^a_\mu (x)$ became a self-dual two-form $\chi^a_{\mu\nu}$ (so that $\tilde{\chi}_{\mu\nu} = \chi^a_{\mu\nu} = -\chi^a_{\nu\mu}$). As before, $A^a_\mu$ is bosonic and transforms as a connection under gauge transformations, and $\phi^a_\mu$ and $\chi^a_{\mu\nu}$ are fermionic and transform under the gauge group by the adjoint action. There is also a new fermionic scalar $\eta^a$. Roughly speaking, the $\psi_\mu(x)$, $\chi_{\mu\nu}(x)$ and $\eta(x)$ fields occur because the anti-self-dual equations, together with the gauge condition, to first order are

$$(d + *d)A = 0$$
$$d(*A) = 0$$

and we can think of this as finding the zeros of an operator $(d + *d, *d*)$ that sends one-forms to a pair of a self-dual two-form and a scalar. The $\psi^a_\mu(x)$, $\chi^a_{\mu\nu}(x)$, and $\eta^a(x)$ form a basis for these spaces of forms.

Witten then added two bosonic scalar fields $\phi^a$ and $\rho^a$ to balance the fermionic and bosonic degrees of freedom. These also transform via the adjoint action of the gauge group.

The commuting fields (bosons) are $(A_\mu, \phi, \rho)$ and the anti-commuting fields (fermions) are $(\psi_\mu, \chi_{\mu\nu}, \eta)$. Note that there is no relation between the spin and the statistics (all the fields have integer spin), but this is explained by the fact that some of these fields are ghosts.

The Lagrangian Witten discovered is

$$\mathcal{L} = \int_{X^4} d^4x \sqrt{g} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \phi D_\mu D^\mu \rho - i \eta D_\mu \psi^\mu + i D_\mu \psi_\nu \chi^{\nu\mu} \right]$$
$$- \frac{i}{8} \phi [\chi_{\mu\nu}, \chi^{\mu\nu}] - \frac{i}{2} \rho [\psi_\mu, \psi^\mu] - \frac{i}{2} \phi [\eta, \eta] - \frac{1}{8} [\phi, \rho]^2 + \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right]$$

The last term is a multiple of the second Chern class $c_2(E)$ and for infinitesimal variations is irrelevant; but for later calculations it is convenient. Note that in
Ref. Witten calls $\mathcal{L}$ the Lagrangian without this last term, and $\mathcal{L}'$ the Lagrangian with it included.

If $(A, \phi, \rho, \eta, \psi, \chi)$ have scaling dimensions $(1, 0, 2, 2, 1, 2)$, this Lagrangian is scale invariant, and there is another additive quantum number $U$ for which the fields have values $(0, 2, -2, -2, -1, 1, -1)$, which is preserved by the Lagrangian. This $U$ was a holdover from the Floer theory, where quantum-mechanical violations of $U$ measured the degree in Floer homology. In this new setting, quantum-mechanical $U$ violation will turn out to measure the dimension of the moduli space. Furthermore, there is a constant spinless supersymmetry $Q$:

$$\delta A_\mu = i \epsilon \psi_\mu, \quad \delta \phi = 0, \quad \delta \rho = 2i \epsilon \eta,$$

$$\delta \eta = \frac{1}{2} \epsilon [\phi, \rho], \quad \delta \psi_\mu = -\epsilon D_\mu \phi, \quad \delta \chi_{\mu \nu} = \epsilon (F_{\mu \nu} + \tilde{F}_{\mu \nu})$$

where $\epsilon$ is an antisymmetric scalar constant. We write $-i \epsilon \{Q, O\}$ for $\delta O$. Then $Q^2 = 0$ (though this fact uses the equations of motion in the case of $\chi$).

Again, the supersymmetry being a scalar is unexpected, but remember that the spin-statistics theorem need not apply here.

This supersymmetric field theory may have been motivated by the Morse theoretic view of instantons on $Y^3 \times \mathbb{R}$, but Witten points out that on flat $\mathbb{R}^4$, it is a “twisted” version of the usual $N = 2$ supersymmetric gauge theory.

The usual $N = 2$ SUSY theory with gauge fields and no matter fields has the following multiplet structure (the “$N = 2$ gauge multiplet”):

- $A_\mu^a$
- $(\lambda_\alpha^a, \bar{\lambda}_{\dot{\alpha}}^a)$
- $(\psi_\alpha^a, \bar{\psi}_{\dot{\alpha}}^a)$
- $\varphi^a$

The first row is the vector gauge field, the second row has two fermionic spinors, and the bottom row is a complex bosonic scalar field. There is a global internal symmetry $SU(2)_R \times U(1)_U$. The top and bottom rows are singlets for $SU(2)_R$, and the middle row is an $SU(2)_R$ doublet. $U(1)_U$ does not act on the gauge field, it has charge 1 on $\lambda$ and $\bar{\psi}$, and charge 2 on $\varphi$.

In Euclidean $(+ + ++)$ space, $\lambda_\alpha$ and $\bar{\lambda}_{\dot{\alpha}}$ are not complex conjugates and are therefore separate fields. So we really have four independent fermionic spinors. Also, $\varphi^a$ and $\varphi^{\dagger a}$ are two independent real fields that make up one complex bosonic scalar field.

We will write our particle content, then, using the notation $(n_-, n_+, n_R)^T$ where the numbers in the parentheses reflect the representation of $SU(2)_- \times SU(2)_+ \times SU(2)_R$, and the superscript is the representation of $U(1)_U$.

| field | spin | statistics |
|-------|------|------------|
| $A_\mu$ | 1 | boson $(1/2, 1/2, 0)^0$ |
| $(\lambda_\alpha, \psi_\alpha)$ | 1/2 | fermion $(1/2, 0, 1/2)^1$ |
| $(\bar{\lambda}_{\dot{\alpha}}, \bar{\psi}_{\dot{\alpha}})$ | 1/2 | fermion $(0, 1/2, 1/2)^{-1}$ |
| $\varphi$ | 0 | boson $(0, 0, 0)^2$ |
| $\varphi^{\dagger}$ | 0 | boson $(0, 0, 0)^{-2}$ |
Let $SU(2)'_+$ be the diagonal in $SU(2)_- \times SU(2)_R$. Then $SU(2)_- \times SU(2)'_+ \times U(1)_U$ is a symmetry. We now view $SU(2)_- \times SU(2)'_+$ as the spatial symmetry. This is the “twist” we referred to.

Under $SU(2)_- \times SU(2)'_+ \times U(1)_U$, the gauge fields $A_\mu$ will not be affected, since there was no $SU(2)_R$ symmetry to begin with. Its representation is $\left(1/2,1/2\right)^0$.

The spinor $SU(2)_R$ doublet $(\lambda_\alpha, \psi_\alpha)$ now acquires some $SU(2)'_+$, and becomes a vector $\psi_\mu$:

$\left(1/2,1/2\right)^1$.

The other spinor $SU(2)_R$ doublet $(\bar{\lambda}_\dot{\alpha}, \bar{\psi}_\dot{\alpha})$ splits into two representations: $\chi_{\mu\nu}$ and a boson $\eta$.

$\left(0,1\right)^{-1} \oplus \left(0,0\right)^{-1}$

The scalar bosons are unchanged, though we split $(\varphi, \varphi^1)$ into its real and imaginary parts, and call these $\phi$ and $\rho$.

$\left(0,0\right)^2 \oplus \left(0,0\right)^{-2}$.

This is summarized in the following table:

| field     | spin | statistics | scale dim. | $(SU(2)_- \times SU(2)'_+ \times U(1)_U)$ |
|-----------|------|------------|------------|------------------------------------------|
| $A_\mu$   | 1    | boson      | 1          | $\left(1/2,1/2\right)^0$               |
| $\psi_\mu$| 1    | fermion    | 1          | $\left(1/2,1/2\right)^1$               |
| $\chi_{\mu\nu}$ | 1    | fermion    | 2          | $\left(0,1\right)^{-1}$               |
| $\eta$    | 0    | fermion    | 2          | $\left(0,0\right)^{-1}$               |
| $\phi + i\rho$ | 0    | boson      | 0          | $\left(0,0\right)^2$               |
| $\phi - i\rho$ | 0    | boson      | 0          | $\left(0,0\right)^{-2}$               |

We note that these are precisely the particle fields in Witten’s Lagrangian above that is supposed to mimic Donaldson theory. This is what is meant when we say that Donaldson theory is a twisted $N = 2$ SUSY theory.

There are two supersymmetries in the standard $N = 2$ theory. Under the group $SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)_U$ the two supersymmetries transform in the representation $(1/2,0,1/2)^{-1}$ and $(0,1/2,1/2)^1$. Under our twisted action of $SU(2)_- \times SU(2)'_+ \times U(1)_U$, the first one becomes $(1/2,1/2)^{-1}$ and the second splits into $(0,1)^1 \oplus (0,0)^1$. The supersymmetry $Q$ we had above was the $(0,0)^1$ component.

Since the usual $N = 2$ theory is supersymmetric, so is Witten’s twisted theory, at least in $\mathbb{R}^4$ with the flat Euclidean metric. What is not clear is what happens when $\mathbb{R}^4$ is replaced by a more general manifold, and Witten checks this and notes that Riemann curvature considerations, which might normally appear, turn out not to appear at all in this case.
Witten then describes, through a formal, non-rigorous argument, how to compute the Donaldson invariants as expectation values (that is, correlation functions) of the form

$$\langle W \rangle = Z(W) = \int (DX) \exp(-L/e^2)W$$

where $DX$ indicates the integration over all the fields $(A, \phi, \rho, \eta, \psi, \chi)$, $e$ is a constant viewed as a “gauge coupling constant”, and $W$ is a polynomial in the fields $A, \phi, \rho, \eta, \psi, \chi$.

We will not take an arbitrary $W$, but only those for which $\langle W \rangle$ is invariant under changes in the metric, because otherwise we won’t have a topological invariant.

Before we proceed, we need a few basic facts. The supersymmetry of $L$ and $DX$ give rise to the equation $\langle \{Q,W\} \rangle = 0$. Infinitessimal changes in the metric produce a change in the Lagrangian by $\frac{1}{2} \int X^4 \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress-energy tensor. Witten calculates this and shows that it is of the form $T_{\mu\nu} = \{Q, \lambda_{\mu\nu}\}$ where $\lambda_{\mu\nu}$ is an expression involving the fields that Witten writes down explicitly.

Furthermore $\{Q, V\} = L$ for some expression $V$ in terms of the fields.

We will use these facts to find topological invariants. For example, we will now show that the partition function $Z = Z(1) = (1)$ is invariant under changes in the metric. To do this we perturb the metric $g^{\mu\nu}$, and get:

$$\delta Z = \int (DX)[\exp(-L/e^2)] \cdot -\frac{1}{e^2} \delta L$$

$$= \int (DX)[\exp(-L/e^2)] \cdot -\frac{1}{e^2} \frac{1}{2} \int X^4 \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu}$$

$$= \int (DX)[\exp(-L/e^2)] \cdot -\frac{1}{e^2} \frac{1}{2} \int X^4 \sqrt{g} \delta g^{\mu\nu} \{Q, \lambda_{\mu\nu}\}$$

$$= -\frac{1}{2e^2} \left\langle \left\{ Q, \int X^4 \sqrt{g} \delta g^{\mu\nu} \lambda_{\mu\nu} \right\} \right\rangle$$

$$= 0$$

so that $Z$ is a topological invariant.

The partition function $Z$ is also invariant under changes in $e$, as follows:

$$\delta Z = \int (DX) \exp(-L/e^2) \delta(-1/e^2) L$$

$$= \delta(-1/e^2) \int (DX) \exp(-L/e^2) \{Q, V\}$$

$$= \delta(-1/e^2) \langle \{Q, V\} \rangle = 0$$

Therefore to calculate $Z$ we can take the limit of very small $e$, so that the path integral is dominated by the classical minima. The term depending on the gauge field $A$ is at a minimum whenever $A$ is an instanton. We can write the linearization of the anti-self-dual equations, and the linearization of the gauge fixing condition,
What do Topologists want from Seiberg–Witten theory?

and compute the number of degrees of freedom. This is, of course, exactly the index calculation earlier that gave the expected dimension of the moduli space in (3):

\[ d = 8c_2(E) - 3(1 - h_1(X^4) + b^+_2(X^4)). \]

The fermion zero modes turn out to give linearized equations that are identical in form, but on \( \psi_\mu \) instead of \( A_\mu \). Hence the number of fermion zero modes equal the number of gauge zero modes. It turns out there are generically no zero modes for \( \eta \) and \( \chi \).

Because of this, \( Z \) vanishes, unless \( d = 0 \). This is precisely the case where in classical Donaldson theory, we would want to count points in the moduli space. But this is exactly what this path integral is doing. The only thing to check is that the use of signs agrees, which is delicate, but works.

Can we derive the other Donaldson invariants for \( d > 0 \)? Yes. What is needed is the other expectation values \( \langle W \rangle \).

We originally showed \( Z \) is a topological invariant by varying the metric. The more general \( Z(W) \) is a topological invariant when \( W \) does not depend explicitly on the metric \( g \) and when \( \{ Q, W \} = 0 \). So we need to find such expressions.

If \( W = \{ Q, \mathcal{O} \} \), then although it is true that \( \{ Q, W \} = 0 \), it is also true that \( Z(W) = \langle \{ Q, \mathcal{O} \} \rangle = 0 \), so this does not help us. Therefore, the set of expressions that we might use would be those \( W \) (independent of \( g \)) for which \( \{ Q, W \} = 0 \), modulo those \( W \) which are \( W = \{ Q, \mathcal{O} \} \). Thus, we are essentially looking for BRST singlet operators.

Upon examining the supersymmetry on the fields in equation (9), we see that \( \phi^a \) is invariant under \( Q \), so that \( \{ Q, \phi^a \} = 0 \), and \( \phi^a \) is also not in the image of \( Q \). This means \( \phi^a \) might be a good candidate for \( W \), except that \( \phi^a \) is not gauge invariant.

But \( \text{Tr} \phi^2 \) is gauge invariant, and of course is still BRST singlet. So \( W_0(P) = \text{Tr} \phi^2(P) \) (where \( P \) is a given point on \( X^4 \)) is the kind of functional we need, and \( \langle W_0(P) \rangle \) is the corresponding topological invariant. It turns out not to depend on \( P \), as can be verified by differentiating with respect to \( P \):

\[
\frac{\partial}{\partial x^\mu} W_0 = \frac{\partial}{\partial x^\mu} \left( \frac{1}{2} \text{Tr} \phi^2(P) \right) = \text{Tr} \phi D_\mu \phi = i \{ Q, \text{Tr} \phi \psi_\mu \}
\]

which has expectation value zero since it is of the form \( \{ Q, \cdot \} \).

Motivated by this, we define \( W_1 = \text{Tr}(\phi \psi_\mu) dx^\mu \) as an operator valued 1-form on \( X^4 \). Thus,

\[ 0 = i \{ Q, W_0 \}, \quad dW_0 = i \{ Q, W_1 \} \]

Similarly, we can define \( W_2, W_3, \ldots \) like this:

\[ dW_1 = i \{ Q, W_2 \}, \quad dW_2 = i \{ Q, W_3 \}, \quad dW_3 = i \{ Q, W_4 \}, \quad dW_4 = 0 \] (10)

The range of possible \( W \) is slightly more general but this is not important for our purposes.
and obtain the formulas

\[ W_2 = \text{Tr}(\frac{1}{2} \psi \wedge \psi + i \phi \wedge F), \quad W_3 = i \text{Tr}(\psi \wedge F), \quad W_4 = -\frac{1}{2} \text{Tr}(F \wedge F). \]  

(11)

Then \( W_k \) is an operator-valued \( k \)-form on \( X^4 \).

If \( \gamma \) is a \( k \)-dimensional homology cycle on \( X^4 \), consider \( I = \int_\gamma W_k \). We note that

\[ \{Q, I\} = \int_\gamma \{Q, W_k\} = -i \int_\gamma dW_{k-1} = 0, \]

so that \( \langle I \rangle \) is a topological invariant. If \( \gamma = \partial \beta \), then

\[ I = \int_\gamma W_k = \int_\beta dW_k = i \int_\beta \{Q, W_{k+1}\} = i \left\{ Q, \int_\beta W_{k+1} \right\}, \]

so that in that case, \( \langle I \rangle = 0 \). So \( I \) only depends on the homology class of \( \gamma \).

So if \( \gamma_1, \ldots, \gamma_r \) are homology classes in degree \( k_1, \ldots, k_r \), then

\[ \left\langle \int_{\gamma_1} W_{k_1} \cdots \int_{\gamma_r} W_{k_r} \right\rangle \]

is a topological invariant. This is zero unless

\[ \sum_{i=1}^r (4 - k_r) = d \]

where \( d \) is the expected dimension of the moduli space. Witten then adjusts \( e \) to be small, as before, and relates it to the notion of integrating over the moduli space \( \mathcal{M} \) differential forms that are canonically defined in \( B^* \). This leads differential forms are, essentially, \( \mu(\gamma_i) \), and so these correlation functions turn out to be, indeed, the Donaldson invariants \( Q(\gamma_1 \ldots \gamma_r) \).

There is much that is unusual about this theory. But one that is striking is that what appeared to be a classical problem of counting classical field theory solutions to the anti-self-dual equation (finding instantons) turns out to be expressible as correlation functions in a supersymmetric quantum field theory.

Although this formulation caught the attention of many, this approach did not lead to many mathematicians using this approach to prove theorems about Donaldson invariants and about four-dimensional manifolds, perhaps for three reasons: first, it takes time and effort for mathematicians trained in analysis and topology to learn the relevant physics; second, the mathematics needed to make the physics rigorous was not (and still is not) available, and the problem of making all the arguments rigorous seems daunting; and third, it was not clear whether or not these physical insights could lead to new theorems, or even lead to new explicit calculations. In 1994, Witten showed how to calculate Donaldson invariants for Kähler manifolds, but this was just already discovered by the work of Kronheimer and Mrowka. Using other techniques (Witten refers to these developments in his paper), and it was not clear whether or not the non-physics-related methods could do everything these physics-related methods could do.
But also in 1994, Witten’s supersymmetric theory proved itself useful again, this time in a far more dramatic way.

11. Seiberg–Witten theory and S duality

In 1988, Nathan Seiberg had discovered new techniques to show that certain supersymmetric theories had very explicit formulas for quantum corrections, obtained by considerations of holomorphicity and symmetry. The idea is that if the quantum theory has supersymmetry, this constrains the form of the effective Lagrangian so much that it is possible to describe to write down explicit formulas, at which point it so happens that the quantum corrections vanish after the one-loop stage to all orders in perturbation theory. Instantons give rise to non-perturbative effects, but the form of this is highly constrained, too, so that it is possible to derive explicit formulas for the effective Lagrangian.

In 1994, Nathan Seiberg and Edward Witten used these techniques to discover new dualities in supersymmetric theories, and was able, in a short amount of time, to illustrate many features of supersymmetric gauge field theories that had eluded physicists for decades for more general theories (like quark confinement). These dualities exchange electric and magnetic charges, while turning a weak coupling constant into a strong one by $g \leftrightarrow 1/g$.

Dualities of this sort were first conjectured for gauge theories by Olive and Montonen, and were verified for $N = 4$ by Olive and Witten. Seiberg and Witten were able to put this result in a broader framework of dualities for more general supersymmetric theories. In other supersymmetric theories, the duality related two different descriptions of the same theory. These ideas led to the current program of the unification of string theories into $M$-theory, and many other exciting developments in supersymmetric theories in various dimensions. A brief description of this idea was reviewed by Seiberg recently in this journal, and an introduction for beginners was written by Luiz Alvarez-Gaumé and S. F. Hassan.

The duality for pure $N = 2$ supersymmetric gauge theory was worked out in detail by Seiberg and Witten. We have just seen that the $SU(2)$ supersymmetric gauge theory can be twisted into a topological field theory, for which the Donaldson invariants are the correlation functions of BRST singlet operators. The dual theory also has $N = 2$ supersymmetry, and so the same twist can be applied and we will get a new, dual, topological field theory. We would then expect that the correlation functions of the BRST singlet operators in this dual field theory should also be the Donaldson invariants. This idea was first described in a 1994 talk by Witten. The ideas were made more precise and explicit by Moore and Witten.

In the remainder of this section, I will derive the dual topological field theory following the arguments of Seiberg and Witten.

First, let us review a few facts about $N = 2$ SUSY Yang–Mills theory. The gauge multiplet
Kevin Iga

\[ \begin{align*}
\left( \lambda^a, \bar{\lambda}^a \right) & \quad A^a_{\mu} \\
\left( \psi^a, \bar{\psi}^a \right) & \quad \varphi^a
\end{align*} \]

can be viewed in \( N = 1 \) superspace terms as a chiral superfield \( \Phi = (\varphi, \psi_\alpha) \) and a vector superfield \( V = (A_\mu, \lambda_\alpha) \); or in \( N = 2 \) superspace terms as a gauge superfield \( \Psi = (\varphi, \psi_\alpha, A_\mu, \lambda_\alpha) \). The renormalizable Lagrangian can be written as

\[ L = \frac{1}{4\pi} \text{Im} \text{Tr} \int d^2\theta d^2\bar{\theta} \frac{1}{2} \tau \Psi^2 \quad (12) \]

where \( \tau = \frac{\theta}{2\pi} + 4\pi i/e^2 \). When we wish to find the effective Lagrangian, if it still has \( N = 2 \) supersymmetry then we should expect the effective Lagrangian to have the same form, except that since we do not need this to be renormalizable we can replace \( \Psi^2 \) by any holomorphic function \( F(\Psi) \). This function \( F \) is called the prepotential.

\[ L = \frac{1}{4\pi} \text{Im} \text{Tr} \int d^2\theta d^2\bar{\theta} F(\Psi) \quad (13) \]

Now Seiberg calculates the prepotential \( F \) using basic facts about the symmetries, and obtains the form

\[ F(\Psi) = \frac{i}{2\pi} \Psi^2 \ln \frac{\Psi^2}{\Lambda^2} + \sum_{k=1}^{\infty} F_k \Lambda^{4k} \Psi^{2-4k} \quad (14) \]

where \( \Lambda \) is a fixed dynamically generated scale. The first term is the one-loop correction, and there are no other perturbative corrections. The sum is due to non-perturbative “instanton” corrections, and the coefficients \( F_k \) are only known for small values of \( k \) on \( \mathbb{R}^4 \).

In terms of the \( N = 1 \) fields \( \Phi \) and \( V \), we can write the Lagrangian as

\[ \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F}{\partial \Phi} + \int d^2\theta \frac{1}{2} \frac{\partial^2 F}{\partial \Phi^2} W_\alpha W^\alpha \right] \quad (15) \]

where \( W \) is the field strength of the vector superfield \( V \). Furthermore the metric can be written as \( \text{Im} \frac{\partial^2 F}{\partial \varphi^2} \), and the coupling constant is given by

\[ \tau = \frac{\partial^2 F}{\partial \varphi^2}. \]

One feature of the \( N = 2 \) theory is a Higgs-like classical vacuum. The bosonic terms in the Lagrangian, after eliminating auxiliary fields, are

\[ L_{\text{bason}} = \frac{1}{e^2} \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + (D_\mu \varphi)^4 D^\mu \varphi - \frac{1}{2} [\varphi^\dagger, \varphi]^2 \right). \quad (16) \]

There is a Higgs potential here \( [\varphi, \varphi^\dagger] \). We have a classical vacuum when \( \varphi \) is a covariantly constant scalar field, such that \( [\varphi^\dagger, \varphi] = 0 \). This happens when the real and imaginary parts of \( \varphi \) point in the same direction. After a gauge transformation to point the real part of \( \varphi \) in the direction \( \sigma_3 \), we end up with \( \varphi = \frac{a}{2} \sigma_3 \) for \( a \in \mathbb{C} \).
Classically, \( a \) can take any value. This turns out to be true also in the full quantum theory.

In order to describe the classical moduli space of vacua, we cannot use \( \varphi^a \) since that is not gauge-invariant, so we use \( W_0 = \text{Tr}(\varphi^2(P)) = \frac{1}{2}a^2 \in \mathbb{C} \) to parameterize the classical moduli space. The set of vacua here (the complex plane), as with classical instantons, is a family that is not generated by symmetries, and in fact the effective theory localized around each of these vacua is in general different.

For \( W_0 \neq 0 \), the \( SU(2) \) gauge group is broken to \( U(1) \). Localized around this point in the moduli space, we have classical monopole and dyon solutions. These satisfy the Bogomol’nyi–Prasad–Sommerfield mass relation

\[
M = a \sqrt{g^2 a^2 n_c^2 + \frac{a_D^2 n_m^2}{g^2}}
\]

where \( n_c \) and \( n_m \) are integers.

Seiberg takes the exact prepotential \( F \), and shows that the quantum moduli space of vacua is this same degenerate moduli space, and that \( u = \langle W_0(P) \rangle \in \mathbb{C} \) parametrizes this moduli space of quantum vacua. The situation where \( u \) is large is where coupling becomes weak, and the classical approximation in valid. So here, \( u \approx \frac{1}{2}a^2 \), and the theory is singular at \( u = \infty \). Now \( a \) cannot be a good parameter, or else the metric, given by the harmonic function

\[
\text{Im} \frac{\partial^2 F}{\partial a^2} da d\bar{a}
\]

would eventually be negative by the maximum principle of harmonic functions.

As we move \( u \) in a large circle where the theory is classical, we see that since \( u = \frac{1}{2}a^2 \), \( a \) will go to \(-a\). If we let

\[
a_D = \frac{\partial F}{\partial a}
\]

which by (14) is

\[
a_D = \frac{2ia}{\pi} \ln(a/\Lambda) + \frac{ia}{\pi} + \sum_{k=1}^{\infty} (2 - 4k) F_k \Lambda^{4k} a^{1-4k}
\]

then we see that \( a_D \) goes to \(-a_D + 2a\). This can be viewed as a monodromy

\[
\begin{pmatrix}
a_D \\
a
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & 2 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
a_D \\
a
\end{pmatrix}.
\]

If there is a monodromy like this at \( u = \infty \), there must be other singularities for finite \( u \), where the classical description is no longer valid. Furthermore, some monodromies must fail to commute with the monodromy above, or else \( a \) would be a good parameter, and the metric would fail to be positive. Therefore, there must be at least two singularities in the finite \( u \) plane.

There is a symmetry \( u \rightarrow -u \), so it makes sense to suppose there are precisely two singularities, which after rescaling the \( u \) plane, may be at \( \pm 1 \). The situation
$u = 0$ would classically give rise to a singularity, but this singularity no longer occurs quantum mechanically.

There are many strong indications that there are only two singular points in the finite $u$ plane, ranging from checking particular cases explicitly, to calculating $\mathcal{F}_1$ and comparing the results with those already known by explicit calculation, but a rigorous proof is still lacking. Seiberg and Witten show that the monodromies around $\pm 1$ multiply to the monodromy at $\infty$, giving further credibility to the idea that there are only two singularities in the finite $u$ plane. The fact that this duality gives rise to a theory mathematicians are interested in might be considered as further evidence, as I will describe later.

The monodromies generate $SL_2(\mathbb{Z})$, and in particular include

$$ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} $$

which sends $a$ to $a_D$ and $a_D$ to $-a$. It also sends the coupling constant to its reciprocal. This is the $S$ duality mentioned above. Also, $a$ and $a_D$ have the following interpretation: a dyon will have electric charge $an_e$ and magnetic charge $a_Dn_m$, where $n_e$ and $n_m$ are integers. They will satisfy the BPS bound, $M = \sqrt{g^2(a_n e)^2 + (a_D n_m)^2}/g^2$.

The only immediately recognizable source of the singularities in the $u$ plane is the possibility that the dyons become massless. This occurs when $a_D = 0$ and $n_e = 0$, so that these dyons are magnetic monopoles.

The theory around these points $u = \pm 1$, therefore, appears to involve massless magnetic monopoles, which are described as an $N = 2$ $U(1)$ gauge multiplet

$$ (\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}) $$

and an $N = 2$ $U(1)$ hypermultiplet

$$ (\psi_\alpha, \tilde{\psi}_{\dot{\alpha}}) \quad (\phi, \tilde{\phi}) $$

where $\xi$ and $\tilde{\xi}$ are spinors that are fixed by the $SU(2)_R$ symmetry, and $\Phi$ and $\tilde{\Phi}$ are scalars that fit into one $SU(2)_R$ doublet. Near $u = \pm 1$, the low energy theory is dominated by this behavior, with coupling constant $-1/\tau$ instead of $\tau$. At the actual singularity, $\tau$ goes to infinity, but in terms of the monopoles, the coupling constant goes to zero, and we can use a semiclassical approximation.

This is the dual theory to $N = 2$ supersymmetric gauge theory.

12. Seiberg–Witten duality on Donaldson theory

As we saw before, Donaldson theory is a twisted version of $N = 2$ SUSY pure gauge theory, so we might apply the above to Donaldson theory.

Recall the relationship between Donaldson invariants and $N = 2$ SUSY pure gauge theory. Starting from $N = 2$ SUSY pure gauge theory, and using the Eu-
clidean $++++$ metric, we “twist” the $\text{Spin}(4) = SU(2)_- \times SU(2)_+$ with the $SU(2)_R$ and obtain a topological quantum field theory where certain correlation coefficients do not depend on the metric or the coupling constant $e$.

Consider a one-parameter family of metrics $g_t = t^2 g_1$ for some fixed metric $g_1$. If $t \to 0$, we get the weak coupling limit $e \to 0$, and the theory is dominated by the $A_\mu$ fields and their minima, which are the classical instantons. In terms of the $u$ plane description, as the coupling goes to zero, the region in the $u$ plane where the theory is classical (near $u = \infty$) expands to include more and more of the $u$ plane, until in the limit, the contribution only comes from $u = 0$, where classically the full $SU(2)$ gauge theory is unbroken.

Since the correlation functions do not depend on the metric or the gauge coupling, we can compute the same things as $t \to \infty$. In this direction, we no longer have a classical theory, and it appears we should integrate over the $u$ plane. But it turns out that for most of the $u$ plane, there are no contributions, as long as $b_+^2 > 1$. The reason is roughly that there are too many fermionic zero-modes that force the contribution to be zero away from $u = \pm 1$.

So just as in a contour integral in complex analysis, the contributions only come from $u = \pm 1$. At this point, we can use the dual description of the theory in terms of massless monopoles, and in this description, the coupling constant is small.

We apply the Witten twist again, and this time we get not only the gauge multiplet (with $SU(2)$ broken to $U(1)$) but a hypermultiplet describing the monopole. The gauge multiplet twists in exactly the same way as in the Donaldson case above. The hypermultiplet goes from

| field | spin | statistics | $SU(2)_- \times SU(2)_+ \times SU(2)_R$ |
|-------|------|------------|
| $\xi_\alpha$ | $1/2$ | fermion | $(1/2,0,0)$ |
| $\Phi_\alpha, \tilde{\Phi}$ | $0$ | boson | $(0,0,1/2)$ |
| $\tilde{\xi}_\alpha$ | $1/2$ | fermion | $(0,1/2,0)$ |

and we see that we get only spinors, one of which is a boson. We take the limit as the “new” coupling constant goes to zero, and we get a theory dominated by $A_\mu$ as before but also the spinor $\Phi_\alpha$. The correlation functions are dominated by solutions to a certain classical equation of motion, which we describe in the next section.

This new classical theory is what mathematicians have called Seiberg–Witten theory. The use of this term led to some confusion when discussing these developments with physicists, who were used to using this term for the general approach to supersymmetry that these techniques inspired. Thus, mathematicians were in the unusual situation of using a term in a more restrictive sense than physicists.

These results are explored more fully in Moore and Witten’s paper and the precise relationship between Donaldson theory and this dual theory is also ex-
The relationship between Donaldson theory, counting instantons, on the bottom left, and the dual theory, described in detail in the next section, on the top right, counting monopoles.

Fig 10. The relationship between Donaldson theory, counting instantons, on the bottom left, and the dual theory, described in detail in the next section, on the top right, counting monopoles.
13. The Seiberg–Witten equations

Recall that for Donaldson invariants, we considered the moduli space of instantons, that is, classical $SU(2)$ gauge-fields that satisfy the self-dual or anti-self-dual equations, on a four-dimensional manifold with positive definite metric. The topology of the moduli space inside the set of all connections modulo gauge gives rise to the Donaldson invariants.

In this dual Seiberg–Witten theory, we are again considering the moduli space of classical solutions to a differential equation coming from a gauge theory, on a four-dimensional manifold with positive-definite metric. We again consider the topology of the moduli space inside the set of all possible fields modulo gauge, and we hope to define Seiberg–Witten invariants.

The gauge group $SU(2)$ is replaced by $U(1)$, though in addition to the connection $A_\mu$, we also have the bosonic Weyl spinor $\Phi_\alpha$. The vector bundle $E$ is now replaced by a complex line bundle $L$, in accordance with $SU(2)$ breaking to $U(1)$.

Let us consider what sort of object $\Phi_\alpha$ is. It arises from the Witten twist on two scalars $\Phi$ and $\tilde{\Phi}$. These scalars lie in representations of $U(1)$. Therefore, $\Phi_\alpha$ transforms under $\text{Spin}(4) \times U(1)$. Actually, under this action $(-1, -1) \in \text{Spin}(4) \times U(1)$ acts trivially, so the symmetry group of $\Phi_\alpha$ is $\text{Spin}^c(4) = (\text{Spin}(4) \times U(1))/\pm 1$ where the $\pm 1$ acts diagonally on both factors. By projecting onto the first factor, we have a homomorphism $\pi_1 : \text{Spin}^c(4) \to \text{Spin}(4)/\pm 1 = SO(4)$. One way to describe the configuration $\Phi$ is to view it as a section of a two-dimensional complex vector bundle $W$ over $X^4$ with a structure group $\text{Spin}^c(4)$ so that the projection to $SO(4)$ gives the same $SO(4)$ bundle as the tangent bundle. There is similarly a projection $\pi_2 : \text{Spin}^c(4) \to U(1)/\pm 1 \cong U(1)$ on the second factor. It is useful to think of the $\text{Spin}^c(4)$ bundle as having these two components: the first component is an $SO(4)$ bundle that is the tangent bundle, and the second component is a $U(1)$ bundle that corresponds to the gauge freedom in the theory.

The group $\text{Spin}^c(4)$ is the complex analogue of $\text{Spin}(4)$ in the sense that it comes from the complexified Clifford algebra in the same way as $\text{Spin}(4)$ comes from the Clifford algebra. There are many similarities between the two, including the existence of gamma matrices and (given a $U(1)$ connection for the $U(1)$ piece) a Dirac operator.

For $\text{Spin}(4)$, we know that the tangent bundle to a manifold $X^4$ has structure group $\text{Spin}(4)$ if and only if a certain Stiefel-Whitney class $w_2$ is equal to zero. For $\text{Spin}^c(4)$, it is possible to work out the analogous conditions, and in four dimensions, the tangent bundle always has a $\text{Spin}^c(4)$ bundle, without further conditions. There are typically many ways to make it a $\text{Spin}^c(4)$ bundle, parameterized by the $U(1)$ part. One way to extract the $U(1)$ information from $W$ is to construct a complex line bundle over $X^4$ so that the gauge transformations on the line bundle are given by the determinants of gauge transformations on $W$. This complex line bundle is called $\text{det}(W)$. Taking the first Chern class $c_1(\text{det}(W))$ gives a class in $H^2(X^4)$. It turns out that this class in $H^2(X^4)$ completely classifies the $\text{Spin}^c(4)$ bundle.
Then the Seiberg–Witten equations are

\[ F_{\mu\nu} + \ast F_{\mu\nu} = -i\frac{1}{2} \bar{\Phi} \Gamma_{\mu\nu\dot{a}\dot{b}} \Phi^\dot{b} \]  

(17)

\[ \Gamma_{\dot{a}\dot{b}}(D + A)_{\mu} \Phi^\dot{b} = 0. \]  

(18)

Here \( F_{\mu\nu} \) is the curvature of \( A_{\mu} \), and \( \bar{\Phi} \Gamma_{\mu\nu} \Phi \) lies naturally in \( \Lambda^2 + T^*X^4 \), which makes the first equation sensible. The second equation is, of course, the Dirac equation, which requires that we use the \( U(1) \) connection \( A_{\mu} \) together with the Levi-Civita Spin covariant derivative \( D \) to get the Spin\(^c\) (4) covariant derivative \( (D + A)_{\mu} \).

Also recall that \( A_{\mu} \) and \( \Phi_{\alpha} \) are both c-numbers, even though \( \Phi_{\alpha} \) is a spinor.

As in the instanton case, we need to find the set of classical solutions to equations (18), up to gauge equivalence. As before, it is possible to prove that the set of solutions forms a manifold of dimension

\[ d = \frac{1}{4} (c_1(\text{det}(W))^2 - (2 + 2b_1 + 4b_2^+ - 2b_2^-)) . \]  

(19)

So we have another theory that seems to be similar to Donaldson theory, at least in the basic features. Of course, this development would not have been a revolution in our understanding of four-dimensional manifolds unless it were a simplification of the anti-self-dual equations of Donaldson theory, and so it is. The fact that the gauge group has gone from \( SU(2) \) to what is essentially \( U(1) \) might be expected to make matters much simpler, but this is somewhat of a red herring. The biggest reason why abelian theories are easier than non-abelian ones is because the curvature is linear, rather than non-linear, operator on the set of connections. But the fact that we have gone from a non-abelian gauge theory to an abelian one is offset by the fact that we have two quadratic non-linearities that appear: \( \bar{\Phi} \Gamma \Phi \) in the first equation, and the interaction between \( A_{\mu} \) and \( \Phi_{\alpha} \) because of the second equation.

In other words, we may not have self-interactions in the gauge field, but we have interactions between the gauge field and the matter field in two different ways, and this amounts to the same kind of problems.

There is, however, an important and useful feature of these Seiberg–Witten equations that is absent in Donaldson theory: the set of solutions is bounded by the scalar curvature. The reason for this is we have a Weitzenböck formula

\[ (\Gamma(D + A))^* (\Gamma(D + A)) \Phi = \nabla_A^2 \Phi + \frac{R}{4} \Phi - \frac{1}{2} F^+ \Gamma \Phi \]

where \( R \) is scalar curvature, and \( \Gamma \) is assumed to carry the indices to contract completely with the curvature \( F \) or the covariant derivative \( D \), as the context suggests. Note that although \( R \) is not necessarily constant on \( X^4 \), we assume that \( X^4 \) is compact, so that the values of \( R \) are bounded on \( X^4 \).

Consider a configuration \((A_{\mu}, \Phi_{\alpha})\) that satisfies the Seiberg–Witten equations (18). At a point \( P \in X^4 \) where \( \Phi \) is at a maximum, we note that \( \Delta |\Phi|^2 \leq 0 \). We then use the Weitzenböck formula to rewrite the Laplacian in terms of the Dirac
operators. Using the equations of motion we can derive that at such a maximum point \( P \in X^4 \),
\[
-R(P) \frac{1}{2} |\Phi(P)|^2 - \frac{1}{2} |\Phi(P)|^4 \geq 0
\]
which shows that either \(|\Phi(P)|^2 = 0\) or \(|\Phi(P)|^2 = -R(P)\). Of course, at the point \( P \), \(|\Phi(P)|\) was assumed to be a maximum, so if \(|\Phi(P)|^2 = 0\), then \( \Phi \) is zero everywhere on \( X^4 \). Otherwise if the scalar curvature is bounded from below by some number \( k \) (which it must be if \( X^4 \) is compact), then \( R \geq k \), and \(|\Phi| \leq \sqrt{-k} \).

In case \( k > 0 \), so that the metric has positive scalar curvature everywhere, then the only solutions have \( \Phi = 0 \). Then the Seiberg–Witten equations become the anti-self-dual equations for \( A \), and usual techniques show that when the gauge group is \( U(1) \), \( A \) must be a flat connection, and when \( X^4 \) is simply connected, then up to gauge \( A \) must be the trivial connection. Therefore, for manifolds with positive scalar curvature, the only solution is the trivial solution \((A, \Phi) = (0, 0)\).

Furthermore, even when the manifold does not admit a metric of positive scalar curvature, we still have the bound \(|\Phi| \leq \max \{-R\}\). So when the manifold \( X^4 \) is compact, since the scalar curvature must be bounded, we have that when \((A, \Phi)\) is a solution of the Seiberg–Witten equations, \(|\Phi|\) must also be bounded. It is possible to prove that for solutions, the connection \( A \), modulo gauge, is also bounded. This shows that the moduli space of solutions is compact. The fact that the moduli space is compact is one of the most important reasons why the Seiberg–Witten equations are easier to work with than the anti-self-dual equations.

We can try to define Seiberg–Witten invariants in analogy to Donaldson invariants. To do this we need to be certain that the moduli space is always a compact manifold, and that when two different metrics are used, \( g_0 \) and \( g_1 \), a generic path \( g_t \) between them provides a cobordism between the two moduli spaces.

We’ve mentioned that these moduli spaces are compact, and as long as the gauge group acts freely, we can avoid singularities if \( b^+_2 > 1 \), as before.

When the expected dimension \([13]\) is zero, we can count the number of solutions (with appropriate sign and multiplicity) and we will get an invariant if \( b^+_2 > 1 \), for the same reason that this idea works for Donaldson invariants. When the expected dimension is not zero, we can consider the moduli space of solutions as it sits inside the space of fields modulo gauge, and compute what homology class it represents, as in the situation with Donaldson theory. In this way, we get Seiberg–Witten invariants \( SW(w) \) for each \( \text{Spin}^c(4) \) structure \( W \), and thus, for each cohomology class \( w = c_1(\det(W)) \in H^2(X^4) \).

It is conjectured that if \( b^+_2 > 1 \), the Seiberg–Witten invariants are only non-zero when the dimension \([13]\) is zero. This statement is true for the many known cases.

\[^1\text{It is sometimes useful to instead perturb the first equations slightly using a self-dual two-form, instead of perturbing the metric, since this would require redefining the spinors, and allows removing the trivial solution when it is not generic. This is a technical point and not crucial to our story.}\]
This condition, called Seiberg–Witten simple type, is supposed to be equivalent to
the simple type condition for Donaldson invariants.

14. Donaldson = Seiberg–Witten?

The duality given by Seiberg and Witten’s work is more detailed than saying the
theories are in some vague sense “equivalent”. The duality also predicts particular
formulas that relate Donaldson invariants to Seiberg–Witten invariants.

Recall that (assuming \( b_2^+ > 1 \) and simple type) the Donaldson invariants can be
written as a series which can be factored as

\[
D = e^{I/2} \left( r_1 e^{K_1} + \ldots + r_m e^{K_m} \right)
\]  (20)

where \( I \) is the intersection form, \( r_1, \ldots, r_m \) are rational numbers, and \( K_1, \ldots, K_m \)
are elements in \( H^2(X^4) \).

Recall that (assuming \( b_2^+ > 1 \) and Seiberg–Witten simple type) the Seiberg–
Witten invariants assign to each class \( w \in H^2(X^4) \) an integer \( \text{SW}(w) \in \mathbb{Z} \)
counting the number of solutions to the Seiberg–Witten equations with sign.

I can now present the relationship between Donaldson invariants and Seiberg–
Witten invariants. According to the (somewhat non-rigorous) argument by Seiberg
and Witten, the classes \( K_i \) that appear in (20) are the classes \( w \) for which
\( \text{SW}(w) \neq 0 \), and the rational coefficient \( r_i \) is equal to \( \text{SW}(w) \), up to a factor:

\[
r_i = 2^{1/4} (18 + 14b_1 + 18b_2 - 4b_2^-) \text{SW}(K_i)
\]  (21)

The power of two in front is a kind of renormalization factor, and although the
form of the power came out of the theory, the actual coefficients were discovered by
plugging in particular known examples. The formula also holds up in many other
known cases, so many people are confident that the formula is in general true.

A rigorous proof of this result is still lacking, however. Seiberg and Witten’s
work do not constitute a proof, since there are a number of non-rigorous argu-
ments, ranging from concluding that no other factors arise from integration in the
\( u \)-plane, to the whole notion of functional integration (which is still not founded
on rigorous mathematics, even today). The conjectured relationship (21) turns out
to work in the many cases where the Donaldson invariants and the Seiberg–Witten
invariants are both known. This “empirical” evidence may be convincing, but for
mathematicians concerned with calculating these invariants, the lack of a rigorous
proof is problematic.

In 1995, Victor Pidstrigach and Andrei Tyurin proposed a program to prove
the relationship (21) between the Donaldson invariants and the Seiberg–Witten
invariants. Their approach is to consider a theory that contains both the scalar
field \( \Phi \) and the non-abelian gauge group \( SO(3) \) (which is basically \( SU(2) \), except it
identifies \(+I\) with \(-I\)). The theory they examine is analogous to the Seiberg–Witten
equations (18), though slightly more complicated.
The moduli space of solutions to these equations behaves similarly to the moduli spaces for $SU(2)$ instantons in Donaldson theory, but the behavior of the singularities is more intricate. Pidstrigach and Tyurin claim that there are two kinds of singularities that can occur: those that appear because of solutions to the anti-self-dual equation $*F = -F$, and those that appear because of solutions to the Seiberg–Witten equations (18). The moduli space of $SO(3)$ monopoles, then, is a cobordism between the moduli space in Donaldson theory and the moduli space in Seiberg–Witten theory. This would be helpful in proving equation (21).

Carrying out this program involves a great deal of difficult mathematics, and this mathematics is being developed by Paul Feehan and Thomas Leness in a series of papers. The difficulties associated with working with the Pidstrigach–Tyurin theory are the difficulties with the Donaldson theory combined with the difficulties of the Seiberg–Witten equations, so these papers involve delicate functional analysis. Unlike the Seiberg–Witten equations, there is no compactness result, and the gauge group is non-abelian. The analytical details are still being developed by Feehan and Leness. Meanwhile, with what they have accomplished so far, Feehan and Leness have proved Witten’s conjectured relationship (21) between Donaldson invariants and Seiberg–Witten invariants for a large class of manifolds, up to a certain number of terms. Given the impressive work so far, it is reasonable to hope that this program will eventually prove the equivalence of the Donaldson invariants and the Seiberg–Witten invariants.

This Pidstrigach–Tyurin–Feehan–Leness program does not follow the Seiberg–Witten S duality approach. It might be instructive to investigate if there is a way of phrasing this program in terms of S duality. If so, this might open a new way of thinking about dualities in physics.

Even if this program does not illuminate S duality, physicists will still benefit from this Pidstrigach–Tyurin–Feehan–Leness program. The fact that the expected relationships between Donaldson invariants and Seiberg–Witten invariants do work out may be an indication that the results from Seiberg–Witten theory are true and dependable when applied to other theories.

15. Seiberg–Witten invariants, Kähler geometry, and Riemannian geometry

The Seiberg–Witten equations might be studied independently of whether or not they relate to the Donaldson invariants, since to a topologist, instantons were not an end anyway, but merely a means to an end. So it is possible to try to find ways to study the Seiberg–Witten invariants and see what it has to say about four-dimensional manifolds, even without linking Seiberg–Witten invariants to Donaldson invariants.

In the case where $X^4$ carries a Kähler metric, the Donaldson theory simplified considerably. As it turns out, the Seiberg–Witten theory simplifies even more dramatically. When a manifold carries a Kähler metric, there is a canonical class
\( K_X \in H^2(X) \). In this case, it is possible to rewrite the Seiberg–Witten equations as equations involving complex holomorphic sections, and explicitly derive the solutions to the Seiberg–Witten equations. When this is done, we see that the Seiberg–Witten invariants on \( K_X \) have the values \( SW(K_X) = 1 \), \( SW(-K_X) = \pm 1 \), and all other Seiberg–Witten invariants are zero.\footnote{There is a formula that determines whether \( SW(-K_X) \) is 1 or \(-1 \), but that would be distracting at this point.}

Among the first papers that used Seiberg–Witten theory was the proof by Kronheimer and Mrowka\footnote{Kronheimer and Mrowka proved the Thom conjecture, which states that if an embedded surface \( \Sigma^2 \subset \mathbb{C}P^2 \) represents a class in \( H_2(\mathbb{C}P^2) \cong \mathbb{Z} \), the genus of \( \Sigma^2 \) must be at least \( (d-1)(d-2)/2 \), where \( d \) is an integer labeling the classes in \( H_2(\mathbb{C}P^2) \). The formula \( (d-1)(d-2)/2 \) is interesting, because that is exactly the genus of \( \Sigma^2 \) in the case where \( \Sigma^2 \) is algebraic. More generally, surfaces inside Kähler manifolds that are algebraic have the least possible genus for their homology class. Besides answering an important question relating topology and algebraic geometry, what was particularly striking was how short the paper was compared to many papers that used Donaldson theory to prove various kinds of results. In other words, Seiberg–Witten theory was easier than Donaldson theory.}
of the Thom conjecture, which states that if an embedded surface \( \Sigma^2 \subset \mathbb{C}P^2 \) represents a class in \( H_2(\mathbb{C}P^2) \cong \mathbb{Z} \), the genus of \( \Sigma^2 \) must be at least \( (d-1)(d-2)/2 \), where \( d \) is an integer labeling the classes in \( H_2(\mathbb{C}P^2) \). The formula \( (d-1)(d-2)/2 \) is interesting, because that is exactly the genus of \( \Sigma^2 \) in the case where \( \Sigma^2 \) is algebraic. More generally, surfaces inside Kähler manifolds that are algebraic have the least possible genus for their homology class. Besides answering an important question relating topology and algebraic geometry, what was particularly striking was how short the paper was compared to many papers that used Donaldson theory to prove various kinds of results. In other words, Seiberg–Witten theory was easier than Donaldson theory.

Even when the manifold contains a symplectic form \( \omega \) that is not necessarily Kähler, Taubes showed that \( SW([\omega]) \) is non-zero\footnote{This has led to new ways of thinking about symplectic geometry, \( J \)-holomorphic curves, and its relations to contact geometry.} and even related it to counting \( J \)-holomorphic curves in \( X^4 \) (the Gromov–Witten invariants in symplectic geometry)\footnote{The applications to Riemannian geometry were also extensive and intriguing. As we saw above, when the manifold \( X^4 \) has a metric with positive scalar curvature, there is only one solution, and this can be perturbed away if \( b_2^+ > 1 \). In other words, when the \( X^4 \) admits a metric of positive scalar curvature and \( b_2^+ > 1 \), then \( SW(w) = 0 \) for all \( w \in H^2(X^4) \). Therefore, such manifolds cannot be Kähler or even symplectic. Claude LeBrun used arguments related to this observation to compute the Yamabe invariants for certain Kähler manifolds, and found a large class of four-dimensional manifolds that do not admit Einstein metrics.}

The idea of pulling apart four-dimensional manifolds along necks has been more successful for Seiberg–Witten theory than with Donaldson theory, partly because of compactness. It is possible to define a Floer-like homology (Seiberg–Witten–Floer homology) using solutions to these equations on \( Y^3 \times \mathbb{R} \), \( Y^3 \) has positive scalar curvature (for instance, \( Y^3 \) is a sphere \( S^3 \)), we get the same kind of results as in Donaldson theory, that is, the Seiberg–Witten invariants vanish on \( X^4 = X_+ \# X_- \) when \( b_2^+(X_+) \) and \( b_2^+(X_-) \) are both positive. Therefore, if \( X_1 \) and \( X_2 \) have \( b_2^+ > 1 \), then \( X_1 \# X_2 \) cannot be symplectic.

The relations of Seiberg–Witten solutions to \( J \)-holomorphic curves mentioned above allow interpretations that have implications to symplectic and contact topol-
ogy, and also suggest ways to compute Seiberg–Witten invariants (see the next section).

There are many other situations where it can be proved that the Seiberg–Witten invariants are zero, and whenever this happens and \( b_2^+ > 1 \), we can be assured that the manifold is not symplectic and therefore, not Kähler. One recent example is Scott Baldridge’s work, which shows that if a manifold \( X^4 \) has an effective \( S^1 \) action with a fixed point, and \( b_2^+ > 1 \), then all the Seiberg–Witten invariants vanish (and therefore cannot be symplectic). In particular, the subject of symplectic manifolds with \( S^1 \) action has been simplified dramatically.

16. Using Seiberg–Witten invariants to distinguish manifolds

The most direct use of a new invariant is to use it to distinguish manifolds. We now know of many examples of two manifolds that are homeomorphic (in particular have the same homology, cohomology, intersection form) that are not diffeomorphic, because their Seiberg–Witten invariants are different.

Along these lines, R. Fintushel and R. Stern gave an infinite collection of smooth manifolds homeomorphic to the K3 manifold but with different Seiberg–Witten invariants. These were obtained by taking a knot or link in \( S^3 \), and removing a small neighborhood of the knot or link from \( S^3 \), then taking the resulting manifold and forming the cartesian product with \( S^1 \). The resulting four-dimensional manifold has a \( T^3 \) boundary. We then take a K3 manifold and remove a neighborhood of a particular \( T^2 \), and this gives us a four-dimensional manifold with \( T^3 \) boundary. Then glue the two manifolds together along this boundary in a particular way.

It turns out that these have the same intersection matrix as K3, so by Freedman’s work, they are all homeomorphic. But the Seiberg–Witten invariants are essentially the coefficients of the Alexander polynomial of the original knot or link. Taking different knots or links gives different manifolds that are homeomorphic but have different Seiberg–Witten invariants, and so are not diffeomorphic.

More generally, there is a great deal of work that relates Seiberg–Witten Floer homology to the Alexander polynomial for links, and various kinds of topological torsion.

Computing Seiberg–Witten invariants, though less hopeless than computing Donaldson invariants, is still not necessarily easy, and it is not clear whether or not there will eventually be a general technique to compute them. Work in this direction is indicated by Peter Ozsvath and Zoltan Szabó, who have developed a kind of Seiberg–Witten-like invariant that they conjecture is equal to the Seiberg–Witten invariants, but is more combinatorial in nature and is easier to calculate.

17. The eleven-eighths conjecture

The eleven-eighths conjecture is about manifolds whose intersection matrix is broken into a certain number of \( H \)'s and a certain number of \( E_8 \)'s. The conjecture says
Kevin Iga

that the number of $H$'s must be at least $3/2$ the number of $E_8$'s, and when this is written in terms of $b_2$ and $\sigma$, the result is $b_2 \geq \frac{11}{8} |\sigma|$.

The most dramatic progress so far in proving the eleven-eighths conjecture is the work of M. Furuta, showing that as long as we have at least one $E_8$, the number of $H$'s must be at least one larger than the number of $E_8$'s. This statement can be expressed as $b_2 \geq \frac{10}{8} |\sigma| + 2$. See Figure 7.

This was proved by looking at the solutions to the Seiberg–Witten equations as zeros of a non-linear operator, and approximating the linear part of the operator by a finite-dimensional operator operating on the first several eigenspaces. By adding in the non-linear part, it is possible to construct a finite-dimensional approximation to the Seiberg–Witten operator. The overall constant gauge symmetries provide a symmetry in finite dimensions that give rise to equivariant maps of spheres. By classic $K$-theoretic results on equivariant maps of spheres, Furuta obtains his result.

Furuta, Kametani and Matsue furthermore prove that if there are four $E_8$'s, then there must be at least 6 $H$'s.

18. The Future

The Seiberg–Witten equations have been more than a way to distinguish four-dimensional manifolds. Apart from their relation to physics, there are intriguing relationships to symplectic geometry, scalar curvature, the Alexander polynomial for links, Reidemeister torsion, and so on. It is possible that Seiberg–Witten theory is a part of a bigger picture that unifies these concepts. Seiberg–Witten Floer homology for three-dimensional manifolds have similar relationships to these subjects, but in three-dimensional topology there is already the program of W. Thurston, that seeks to understand three-dimensional manifolds as combinations, along spheres and incompressible tori, of geometrically uniform three-dimensional manifolds (so that their geometry is one of eight geometries). The fact that Seiberg–Witten theory gives trivial invariants when we split along spheres and to some extent along tori of a certain type, and when the manifolds have positive scalar curvature, might suggest that the kind of decomposition of Thurston might be natural for Seiberg–Witten theory also, but no relationships have yet been found.

In other words, there are a number of intriguing relationships between the Seiberg–Witten equations and other ideas, and if these are more than coincidental, we can look forward to many fruitful synergies that result from understanding how these subjects are related.

The relationship to physics is perhaps the most direct one, since the Seiberg–Witten equations came directly from physics. It is gratifying to see more mathematicians and physicists working together because of these and other influences of physics on mathematics (mirror symmetry, Yang–Baxter, Monstrous Moonshine, the Penrose inequality, etc.). It is possible to suggest that this collaboration has much further to go, since mathematicians have still not found a way to understand much of quantum field theory in ways that have sound mathematical footing, and
much of the mathematical work with Seiberg–Witten theory uses the equations on their own terms, instead of looking to the original supersymmetry theories. Perhaps this is because the problems involved in making functional integrals (for instance) completely rigorous are considered too difficult, and certainly risky for those in a publish-or-perish environment. But if and when mathematicians find a solid mathematical foundation for the arguments involved in the work of Seiberg and Witten, there are bound to be many new developments in both mathematics and physics.

Witten seems to hope for this, when in his conclusion on his work on Supersymmetry and Morse theory he writes:

It is not at all clear whether supersymmetry plays a role in nature. But if it does, this is a field in which mathematical input may make a significant contribution to physics.

In section 2 of his work relating Donaldson theory to a certain SUSY theory, Witten gets a little more explicit:

In this section, we will see what can be obtained by formal manipulations of Feynman path integrals. Of course, a rigorous framework for four dimensional quantum gauge theory has not yet been developed to a sufficient extent to justify all of our considerations. Perhaps the connection we will uncover between quantum field theory and Donaldson theory may serve to broaden the interest in constructive field theory, or even stimulate the development of new approaches to that subject.

If Witten were the sort of person to say, “I told you so,” he would have strong justification for doing so, in light of the Seiberg–Witten equations. In one of his papers with Moore on the relationship between the Seiberg–Witten equations and Donaldson theory, Witten more modestly concludes:

In this paper, we have obtained a more comprehensive understanding of the relation between the Donaldson invariants and the physics of $N = 2$ supersymmetric Yang–Mills theory. In particular, we have explained the role of the $u$-plane in Donaldson theory more thoroughly than had been done before, both for $b_2^+ = 1$ and for hypothetical manifolds of $b_2^+ > 1$ that are not of simple type. We hope that in the process the power of the quantum field theory approach to Donaldson theory and the rationale for the role of modular functions in Donaldson theory have become clearer.

For mathematicians, the lesson is clear: never underestimate the importance of physics is solving mathematical problems, and perhaps effort invested in solving problems in physics will reap rewards in mathematics down the line.

For physicists, the application of duality to an area of mathematics may argue for the importance and validity of supersymmetry and duality. We might view four-dimensional topology as an experimental apparatus. Since mathematicians are
finding that the Seiberg–Witten invariants are really related to the Donaldson invariants in ways that Seiberg and Witten predicted, then this lends credence to the idea that duality, derived from mathematical manipulations that are not always rigorous, really does work, and gives one hope that at least supersymmetric gauge theories will one day be on firm ontological foundation, or at least be proved as mathematically consistent as the rest of mathematics.

Inasmuch as many important theories in physics have led to unexpected powerful developments in mathematics, perhaps we have some inductive evidence that the reverse is true: if a theory leads to unexpected powerful developments in mathematics, the theory may be an important one in physics. We have just seen that supersymmetric gauge theories lead to the kind of development in mathematics that one might associate with a good physical theory.

This may also be an indication that there is something deeper going on, where a more general mathematical theory explains what why we should expect supersymmetry to be relevant to four-dimensional topology. This more general theory may, in turn, lay the foundation for new physical theories.

Acknowledgements

Thanks to the many physicists, especially Michael Peskin, who, even though I am a mathematician, have spent time to make the physics they study clear and interesting to me. I hope I have made the mathematics I study clear and interesting to physicists in return.

References

1. C. Nash and S. Sen, Topology and Geometry for physicists, Academic Press, 1983.
2. J. Munkres, Elements of Algebraic Topology, Addison Wesley, 1984.
3. W. Boone, The Word problem, Annal. of Math. 70 (1959), pp. 207–265.
4. W. Boone and W. Haken and V. Poenaru, On recursively unsolvable problems in topology and their classification, Contributions to Math. Logic (Colloquium, Hanover, 1966), North–Holland, 1968, pp. 37–74.
5. A. Kosinski, Differential Manifolds, Academic Press, 1993.
6. A. Ranicki, Algebraic and Geometric Surgery, Oxford Univ. Press, 2002.
7. R. Kirby, The Topology of 4-manifolds, Springer, Lecture notes in Mathematics no. 1374, 1989.
8. M. Freedman, The Topology of four-dimensional manifolds, Journ. Differ. Geom., 17 (1982), pp. 357–453.
9. M. Freedman and F. Quinn, Topology of 4-manifolds, Princeton Univ. Press, Princeton Math. Series no. 39, 1990.
10. S. Donaldson, An application of gauge theory to the topology of four-manifolds, Jour. Differ. Geom., 18 (1983), pp. 269–316.
11. S. Donaldson, The orientation of Yang–Mills moduli spaces and 4-manifold topology, Jour. Differ. Geom., 26 (1987), pp. 397.
12. D. Freed and K. Uhlenbeck, Instantons and four manifolds, Springer 1984.
13. S. Donaldson and P. Kronheimer, The geometry of four-manifolds, Oxford Univ. Press, 1990.
What do Topologists want from Seiberg–Witten theory?

14. S. Donaldson, Polynomial invariants for smooth 4-manifolds, *Topology*, **29** (1990), pp. 257–315.
15. R. Friedman and J. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, 1994.
16. J. Morgan and T. Mrowka, A note on Donaldson’s polynomial invariants, *Int. Math. Research Notices*, **10** (1992), pp. 223–230.
17. M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, Addison Wesley, 1995.
18. S. Coleman, *Aspects of Symmetry*, Cambridge Univ. Press, 1985.
19. M. Atiyah and V. Drinfeld and N. Hitchin and Y. Manin, Construction of instantons, *Physics Letters*, **65A**, pp. 185–187.
20. M. Atiyah and R. Ward, Instantons and Algebraic geometry, *Comm. Math. Phys.*, **55** (1977), pp. 117–124.
21. K. Uhlenbeck, Removable singularities in Yang–Mills fields. *Communications in Mathematical Physics*, **83**, (1982), pp. 11–29.
22. P. Kronheimer and T. Mrowka, Embedded surfaces and the structure of Donaldson’s polynomial invariants *Jour. Differ. Geom.* **41** (1995), pp. 573–734.
23. R. Fintushel and R. Stern, Donaldson invariants of 4-manifolds with simple type, *Journal of Differ. Geom.*, **42** (1995), pp. 577–633.
24. A. Floer, *An instanton invariant for three manifolds*, *Comm. Math. Phys.*, **118** (1988), pp. 215–240.
25. M.F. Atiyah, New invariants of three and four dimensional manifolds, *Symposium on the mathematical heritage of Hermann Weyl*, R. Wells, et al., eds., Univ. of North Carolina, May 1987.
26. M.F. Atiyah, *The Geometry and Physics of Knots*, Cambridge Univ. Press, 1990.
27. E. Witten, Supersymmetry and Morse theory, *Jour. Differ. Geom.*, **17** (1982), pp. 661–692.
28. E. Witten, Topological Quantum Field Theory, *Comm. Math. Phys.*, **117** (1988), pp. 353–386.
29. L. Alvarez-Gaumé and S. F. Hassan, Introduction to S-Duality in \( N = 2 \) supersymmetric gauge theories, *Fortschritte Phys.* **45** (1997), pp. 159–236.
30. E. Witten, Supersymmetric Yang–Mills theory on a four-manifold, *Jour. Math. Phys.*, **35** (1994), pp. 5101–5135.
31. N. Seiberg, Supersymmetry and Non-perturbative Beta functions, *Physics Letters* **B206** (1988), pp. 75–80.
32. N. Seiberg, Exact results on the space of vacua of Four dimensional SUSY gauge theories, *Phys. Rev.* **D49** (1994), pp. 6857–6863, [hep-th/9402044](https://arxiv.org/abs/hep-th/9402044).
33. N. Seiberg, The power of duality—exact results in 4D SUSY field theory, *Int’l. Journal of Modern Physics A*, **16** (2001) pp. 4365–4376.
34. N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in \( N = 2 \) supersymmetric Yang–Mills theory, *Nucl. Phys.* **B426** (1994) pp. 19-52; Erratum-ibid. **B430** (1994), pp. 485-486, [hep-th/9407087](https://arxiv.org/abs/hep-th/9407087).
35. C. Montonen and D. Olive, Magnetic monopoles as gauge particles?, *Phys. Lett.* **B72** (1977), pp. 117.
36. E. Witten and D. Olive, Supersymmetry algebras that include topological charges, *Phys. Lett.* **B78** (1978), pp. 97–101.
37. N. Seiberg and E. Witten, Monopoles, Duality, and Chiral Symmetry Breaking in \( N = 2 \) supersymmetric QCD, *Nucl. Phys.*, **B431** (1994), pp. 484-550, [hep-th/9408099](https://arxiv.org/abs/hep-th/9408099).
38. E. Witten, Monopoles and Four-manifolds, *Math. Res. Letters*, **1** (1994), pp. 769–796.
39. G. Moore and E. Witten, Integration over the \( u \)-plane in Donaldson theory, *Adv.
Kevin Iga

Theor. Math. Phys., 1 (1997) no. 2, pp. 298–387., hep-th/9709193.
40. V. Pistriagich and A. Tyurin, Localisation of the Donaldson’s invariants along Seiberg–Witten classes, dg-ga/9507004.
41. P.M.N. Feehan and T.G. Leness, PU(2) monopoles. I: Regularity, compactness and transversality, J. Differential Geom. 49 (1998), pp. 265-410, arXiv:dg-ga/9710032.
42. P.M.N. Feehan and T.G. Leness, PU(2) monopoles and relations between four-manifold invariants, Topology Appl. 88 (1998), pp. 111-145, arXiv:dg-ga/9709022.
43. P.M.N. Feehan, Critical-exponent Sobolev norms and the slice theorem for the quotient space of connections, Pac. J. Math. 200 (2001), pp. 71-118, arXiv:dg-ga/9711004.
44. P.M.N. Feehan and T.G. Leness, PU(2) monopoles and links of top-level Seiberg-Witten moduli spaces, J. Reine Angew. Math. 538 (2001), to appear, arXiv:math.DG/0007190.
45. P.M.N. Feehan and T.G. Leness, PU(2) monopoles and links of top-level Seiberg-Witten moduli spaces, and Witten’s conjecture in low degrees, J. Reine Angew. Math. 538 (2001), to appear, arXiv:math.DG/0007190.
46. P.M.N. Feehan and T.G. Leness, SO(3) monopoles, level-one Seiberg-Witten moduli spaces, and Witten’s conjecture in low degrees, Topology Appl., to appear, arXiv:math.DG/0106298.
47. J.W. Morgan, The Seiberg–Witten equations and applications to the topology of smooth four-manifolds, Mathematical Notes, Princeton Univ. Press, 1996.
48. P.B. Kronheimer and T.S. Mrowka, The Genus of Embedded Surfaces in the Projective Plane, Math. Res. Letters, 1 (1994), pp. 797–808.
49. J. Morgan and Z. Szabó and C. Taubes, A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture, Journal of Differ. Geom., 44 (1996), pp. 706–788.
50. C.H. Taubes, The Seiberg–Witten invariants and symplectic forms, Math. Res. Letters, 1 (1994), pp. 809–822.
51. C.H. Taubes, The Seiberg–Witten and Gromov invariants, Math. Res. Letters, 2 (1995), pp. 221–238.
52. C. LeBrun, Four-manifolds without Einstein metrics, Math. Res. Letters, 3 (1996), pp. 133–147.
53. M. Marcolli, Equivariant Seiberg–Witten–Floer homology, dg-ga 9606003.
54. B.L. Wang, Seiberg–Witten–Floer theory for homology three-spheres, unpublished.
55. M. Marcolli and B.L. Wang, Equivariant Seiberg-Witten-Floer homology, Comm. Anal. Geom. 9 (2001), no. 3, pp. 451–639.
56. K. Frøyshov, The Seiberg–Witten equations and four-manifolds with boundary, Math. Res. Letters, bf 3 (1996), pp. 373–390.
57. T. Mrowka and P. Ozsvath and B. Yu, Seiberg–Witten monopoles on Seifert fibered spaces, Comm. Anal. and Geom., 4 (1997), pp. 685–791.
58. K. Iga, Stanford Univ. Ph.D. Thesis, 1998.
59. L. Nicolaescu, Notes on Seiberg–Witten Theory, Amer. Math. Soc., Grad. Studies in Math. 28, 2000. theory.
60. S. Baldridge, Seiberg–Witten vanishing theorem for $S^1$-manifolds with fixed points, math.GT/0201034.
61. R. Fintushel and R. Stern, Knots, links, and 4-manifolds, Inventiones Mathematicae, 134 (1998), pp. 363–400.
62. J.W. Morgan and Z. Szabó, Homotopy $K^3$ surfaces and mod 2 Seiberg–Witten In-
variants, *Math. Res. Letters* 4 (1997), pp. 17–21.
64. G. Meng and C. Taubes, SW=Milnor Torsion, *Math. Res. Lett.*, 3 (1996), pp. 661–674.
65. M. Hutchings, *Reidemeister Torsion in generalized Morse theory*, Harvard Univ. Ph.D. Thesis, 1998.
66. M. Hutchings and Y. Lee, Circle-valued Morse theory and Reidemeister torsion, *Topology*, 38 (1999), pp. 861–888.
67. P. Ozsvath and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, preprint.
68. P. Ozsvath and Z. Szabó, *Holomorphic disks and topological invariants for rational homology three-spheres*, preprint.
69. P. Ozsvath and Z. Szabó, *Holomorphic triangles and invariants for smooth four-manifolds*, preprint.
70. M. Furuta, The Monopole Equations and the 11/8 conjecture, *Math. Res. Letters*, 8 (2001), pp. 279–291.
71. M. Furuta and Y. Kametani and H. Matsue, Spin 4-manifolds with signature=−32, *Math. Res. Letters*, 8 (2001), pp. 293–301.