1D Log Gases and the Renormalized Energy : Crystallization at Vanishing Temperature

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Abstract

We study the statistical mechanics of a one-dimensional log gas with general potential and arbitrary \( \beta \), the inverse of temperature, according to the method we introduced for two-dimensional Coulomb gases in [SS2]. Such ensembles correspond to random matrix models in some particular cases. The formal limit \( \beta = \infty \) corresponds to “weighted Fekete sets” and is also treated.

We introduce a one-dimensional version of the “renormalized energy” of [SS1], measuring the total logarithmic interaction of an infinite set of points on the real line in a uniform neutralizing background. We show that this energy is minimized when the points are on a lattice.

By a suitable splitting of the Hamiltonian we connect the full statistical mechanics problem to this renormalized energy \( W \), and this allows us to obtain new results on the distribution of the points at the microscopic scale: in particular we show that configurations whose \( W \) is above a certain threshold (which tends to min \( W \) as \( \beta \rightarrow \infty \)) have exponentially small probability. This shows that the configurations have increasing order and crystallize as the temperature goes to zero.

1 Introduction

In [SS2] we studied the statistical mechanics of a 2D classical Coulomb gas (or two-dimensional plasma) via the tool of the “renormalized energy” \( W \) introduced in [SS1], a particular case of which is the Ginibre ensemble in random matrix theory.

In this paper we are interested in doing the analogue in one dimension, i.e. first defining a “renormalized energy” for points on the real line, and applying this tool to the study of the classical log gases i.e. to probability laws of the form

\[
d\mathbb{P}_n^\beta(x_1, \ldots, x_n) = \frac{1}{Z_n^\beta} e^{-\frac{2}{\beta} w_n(x_1, \ldots, x_n)} dx_1 \cdots dx_n
\]

where \( Z_n^\beta \) is the associated partition function, i.e. a normalizing factor such that \( \mathbb{P}_n^\beta \) is a probability, and

\[
w_n(x_1, \ldots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^n V(x_i).
\]

Here the \( x_i \)'s belong to \( \mathbb{R} \), \( \beta > 0 \) is a parameter corresponding to (the inverse of) a temperature, and \( V \) is a relatively arbitrary potential, satisfying some growth conditions. For
a general presentation, we refer to the textbook \cite{For}. Minimizers of \( w_n \) are also called “weighted Fekete sets” and arise in interpolation, cf. \cite{SaTo}.

There is an abundant literature on the random matrix aspects of this problem (the connection was first pointed out by Wigner and Dyson \cite{Wi, Dy}), which is the main motivation for studying log gases. Indeed, for the quadratic potential \( V(x) = x^2/2 \), particular cases of \( \beta \) correspond to the most famous random matrix ensembles: for \( \beta = 1 \) the law \( P_\beta^n \) is the law of eigenvalues of matrices of the Gaussian Orthogonal Ensemble (GOE), while for \( \beta = 2 \) it corresponds to the Gaussian Unitary Ensemble (GUE), for general reference see \cite{For, AGZ, Me}. For \( V(x) \) still quadratic, general \( \beta \)'s have been shown to correspond to tri-diagonal random matrix ensembles, cf. \cite{DE, ABF}. This observation allowed Valkó and Virág \cite{VV} to derive the sine-\( \beta \) processes as the local spacing distributions of these ensembles. When \( \beta = 2 \) and \( V(x) \) is more general, the model corresponds (up to minor modification) to other determinantal processes called orthogonal polynomial ensembles (see e.g. \cite{Ko} for a review).

The study of \( P_\beta^n \) via the random matrix aspect is generally based on explicit formulas for correlation functions and local statistics, obtained via orthogonal polynomials, as pioneered by Gaudin, Mehta, Dyson, cf. \cite{Me, D, DG}. We emphasize that for general \( \beta \) and \( V \), which is the setting of the present work, there is no random matrix model associated and no explicit formulas available, and fewer results. Notable exceptions are the work of Ben Arous and Guionnet \cite{BG} also valid for every \( \beta \) and the recent work of Bourgade, Erdős and Yau \cite{BEY1, BEY2}, which are in the same setting as ours. We will discuss them more below. The global and local statistics (e.g. spacings) of eigenvalues of large random matrices are expected to be independent of the fact that the entries are specifically Gaussian, this is referred to as “universality”, cf. \cite{TV, EPRSY} for latest results. Here we are rather interested in questions of universality with respect to the potential \( V \), as in \cite{BEY1, BEY2}.

The results here are counterparts of the results we obtained in \cite{SS2} for \( x_1, \ldots, x_n \) belonging to \( \mathbb{R}^2 \) (this corresponds for \( V \) quadratic and \( \beta = 2 \) to the Ginibre ensemble of non-Hermitian Gaussian random matrices), or in other words the two-dimensional Coulomb gas. The study in \cite{SS2} relied on relating the Hamiltonian \( w_n \) to a Coulomb “renormalized energy” \( W \) introduced in \cite{SS1} in the context of Ginzburg-Landau vortices. This relied crucially on the fact that the logarithm is the Coulomb kernel in two dimensions, or in other words the fundamental solution to the Laplacian. When looking at the situation in one dimension, i.e. the present situation of the 1D log-gas, the logarithmic kernel is no longer the Coulomb kernel, and it is not a priori clear that anything similar to the study in two dimensions can work. Note that the 1D Coulomb gas, corresponding to \( P_\beta^n \) where the logarithmic interaction is replaced by the 1D Coulomb kernel \(|x|\), has been studied, notably by Lenard \cite{Le1, Le2}, Brascamp-Lieb \cite{BL}, Aizenman-Martin \cite{AM}. The situation there is rendered again more accessible by the Coulomb nature of the interaction and its less singular character. In particular \cite{BL} prove cristallization (i.e. that the points tend to arrange along a regular lattice) in the limit of a small temperature, we will get a similar result for the log-gas.

The starting point of our study is that even though the logarithmic kernel is not Coulombic in dimension 1, we can view the particles on the real line as embedded into the two-dimensional plane and interacting as Coulomb charges there. This provides a way of defining an analogue of the “renormalized energy” of \cite{SS1} in the one-dimensional setting, still called \( W \), which goes “via” the two-dimensional plane, see Section \ref{sec:setup} below.

Once this is accomplished, we connect in the same manner as \cite{SS2} the Hamiltonian \( w_n \) to
the renormalized energy $W$ via a “splitting formula” (cf. Lemma 1.7 below), and we obtain the counterparts results to [SS2]:

- a next-order expansion of the partition function in terms of $n$ and $\beta$, cf. Theorem 3.
  For $V$ quadratic, exact formulas were already known via Selberg integrals.

- the proof that ground states of $w_n$, or “weighted Fekete sets”, converge to minimizers of $W$, cf. Theorem 4.

- the proof that the minimum of $W$ is achieved by the one-dimensional regular lattice $\mathbb{Z}$, called the “clock distribution” in the context of orthogonal polynomial ensembles [Si]. This is in contrast with the dimension 2 where the identification of minimizers of $W$ is still open (but conjectured to be “Abrikosov” triangular lattices.)

- A large deviations type result which shows that events with high $W$ become less and less likely as $\beta \to \infty$, proving in particular the crystallization as the temperature tends to 0, in view of the result of the previous item.

As far as we know, this is the first proof of crystallization for one-dimensional log gases. Our renormalized energy $W$, which serves to prove the crystallization also appears (like its two-dimensional version) to be a measurement of “order” of a configuration at the microscopic scale $1/n$. What we show here is that there is more and more order (or rigidity) in the log gas, as the temperature gets small. Of course, as already mentioned, it is known that eigenvalues of random matrices, even of general Wigner matrices, should be regularly spaced. In fact the local statistics of the random process are completely known from [VV], and [BEY1, BEY2] showed that this could be extended to general $V$’s. Our results approach this question sort of orthogonally, by exhibiting a unique number which measures the average rigidity, without using explicit formulas for local statistics relying on determinantal forms. Note that in [BS2] the second author and Borodin used $W$ as a way of quantifying the order of random point processes, in particular those arising as local limits in random matrix theory.

Our study here differs technically from the two-dimensional one in two ways: the first one is in the definition of $W$ by embedding the problem into the plane, as already mentioned. The second one is more subtle: in both settings a crucial ingredient in the analysis is to reduce the evaluation of the interactions to an extensive quantity (instead of sums of pairwise Coulomb interactions); that quantity is essentially the $L^2$ norm of the “electric field” generated by the Coulomb charges. Test-configurations can be built and their energy evaluated by “copying and pasting”, provided a cut-off procedure is devised: it consists essentially in taking a given electric field and making it vanish on the boundary of a given box while not changing its energy too much. In physical terms, this corresponds to screening the field. The point is that screening is much easier in two dimensions than in one dimension, because in two dimensions there is more geometric flexibility to move charges around. The screening result of [SS2] does not apply to the 1-dimensional setting. We found that in fact, in dimension 1, not all configurations with finite energy can be effectively screened. However, we also found that generic “good” configurations can be, and this suffices for our purposes. The screening construction, which is different from the two-dimensional one, is one of the main difficulties here, and forms a large part of the paper.

The rest of the introduction is organized as follows: in Section 1.1, we present the embedding of the problem into two-dimensions and the definition of the one-dimensional renormalized energy $W$. In Section 1.2 we introduce notation and well-known facts on the equilibrium
measure (i.e. the minimizer of the mean-field limiting Hamiltonian), as well as the “splitting formula” that connects \( w_n \) to \( W \). Finally, in Section 1.3 we state our main results.

1.1 Definition of the renormalized energy in 1 dimension

As mentioned above, the renormalized energy between points in 1D is obtained by “embedding” the real line in the plane and computing the renormalized energy in the plane, as defined in \cite{SS1}. More specifically, we introduce the following definitions:

\( \mathbb{R} \) will denote the set of real numbers, but also the real line of the plane \( \mathbb{R}^2 \). For the sake of clarity, we will denote points in \( \mathbb{R} \) by the letter \( x \) and points in the plane by \( z = (x, y) \).

For a function \( \chi \) on \( \mathbb{R} \), we define its natural extension \( \bar{\chi} \) to a function on \( \mathbb{R}^2 \) by \( \bar{\chi}(x, y) := \chi(x) \). \( I_R \) will denote the intervals \([−R/2, R/2]\) in \( \mathbb{R} \) and \( \bar{I}_R \) the strips \( I_R \times \mathbb{R} \) in \( \mathbb{R}^2 \). \( \delta_{\mathbb{R}} \) denotes the measure of length on \( \mathbb{R} \) seen as embedded in \( \mathbb{R}^2 \), that is

\[
\int_{\mathbb{R}^2} \varphi \delta_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(x, 0) \, dx
\]

for any smooth compactly supported test function \( \varphi \) in \( \mathbb{R}^2 \). This measure can be multiplied by bounded functions on the real-line. For any measurable set \( U \), \( |U| \) denotes its Lebesgue measure, and if \( U \) is a finite set \( \#U \) denotes its cardinal. \( \delta_p \) denotes the Dirac mass at \( p \).

The renormalized energy \( W \) is an energy of an infinite configuration of points on the real line, screened by a uniform background charge, again on the real line. It will later be obtained as a limit as \( n \to \infty \) of logarithmic interactions of systems of \( n \) particles/points.

We first introduce the classes \( \mathcal{A}_m \) corresponding to infinite configurations on the real line with density \( m \).

**Definition 1.1.** Let \( m \) be a nonnegative number. Let \( E \) be a vector field in \( \mathbb{R}^2 \). We say \( E \) belongs to the admissible class \( \mathcal{A}_m \) if

\[
\text{div} \, E = 2\pi(\nu - m\delta_{\mathbb{R}}), \quad \text{curl} \, E = 0 \quad \text{in} \quad \mathbb{R}^2
\]

where \( \nu \) has the form

\[
\nu = \sum_{p \in \Lambda} \delta_p \quad \text{for some discrete set} \ \Lambda \subset \mathbb{R} \subset \mathbb{R}^2,
\]

and

\[
\frac{\nu(I_R)}{R} \quad \text{is bounded by a constant independent of} \quad R > 1.
\]

**Definition 1.2.** Let \( m \) be a nonnegative number. For any bounded function \( \chi \) and any \( E \in \mathcal{A}_m \) satisfying (1.3) we let

\[
W(E, \chi) = \lim_{\eta \to 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \eta)} |E|^2 + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right)
\]

where \( \bar{\chi} \) is the natural extension of \( \chi \).
Here we thus look at any configuration of points in the plane through an “electric field” $E$ that it generates in the whole plane $\mathbb{R}^2$. Observe that if $E$ satisfies (1.3), there exists a function $H$ such that $E = -\nabla H$, we will call this function the potential generated by the point charges. We also note that the distribution of charges on the real line is compensated by a background charge $m\delta_0$ which is also concentrated on the real line. We will use the notation $\chi_R$ for positive cutoff functions over $\mathbb{R}$ satisfying, for some constant $C$ independent of $R$,

$$|\nabla \chi_R| \leq C, \quad \text{Supp}(\chi_R) \subset I_R, \quad \chi_R(x) = 1 \text{ if } |x| < R/2 - 1. \quad (1.7)$$

**Definition 1.3.** The renormalized energy $W$ is defined, for $E \in \mathcal{A}_m$, by

$$W(E) = \limsup_{R \to \infty} \frac{W(E, \chi_R)}{R}, \quad (1.8)$$

where $\{\chi_R\}_{R>0}$ satisfies (1.7).

Returning to writing $E = -\nabla H$, we see that while $W$ in 2D can be viewed as a “renormalized” way of computing $\|H\|_{H^1(\mathbb{R}^2)}$, in 1D it amounts rather to a renormalized computation of $\|H\|_{H^{1/2}(\mathbb{R})}$ (where $H^s$ denote the fractional Sobolev spaces). In other words, because the logarithmic kernel is not Coulombic in one-dimension, the associated energy is non-local (and the associated operator is the fractional Laplacian $\Delta^{1/2}$). Augmenting the dimension by 1 allows to make it local and Coulombic again. This well-known extension idea is also now commonly used in the study of general fractional Laplacians [CS].

As in the two dimensional case, we have the following properties:

- The value of $W$ does not depend on $\{\chi_R\}_R$ as long as it satisfies (1.7).
- $W$ is insensitive to compact perturbations of the configuration.
- Scaling: it is easy to check that if $E$ belongs to $\mathcal{A}_m$ then $E' := \frac{1}{m}E(\cdot/m)$ belongs to $\mathcal{A}_1$ and

$$W(E) = m \left( W(E') - \pi \log m \right). \quad (1.9)$$

- If $E \in \mathcal{A}_m$ then in the neighborhood of $p \in \Lambda$ we have $\text{div} E = 2\pi(\delta_p - m\delta_0)$, curl $E = 0$, thus we have near $p$ the decomposition $E(x) = -\nabla \log |x - p| + f(x)$ where $f$ is smooth, and it easily follows that the limit (1.6) exists. It also follows that $E$ belongs to $L^q_{\text{loc}}$ for any $q < 2$.

An extra fact proven below is valid only in the one-dimensional case (and not the two-dimensional one):

**Lemma 1.4.** Let $E \in \mathcal{A}_m$ be such that $W(E) < +\infty$. Then any other $E'$ satisfying (1.3) – (1.4) with the same $\nu$ and $W(E') < +\infty$, is such that $E' = E$. In other words, $W$ only depends on the points.

By simple considerations similar to [SS2, Section 1.2] this makes $W$ a measurable function of the bounded Radon measure $\nu$.

This implies in particular
Corollary 1.5. Under the same assumptions, if \( S(x,y) = (x, -y) \) then \( E \circ S = S \circ E \).

Indeed, it is easy to check that \( E' = S \circ E \circ S \) satisfies (1.3) with the same \( \nu \) as \( E \), and obviously \( W(E') < +\infty \), hence \( E' = E \).

The following lemma is proven in [BSe], and shows that there is an explicit formula for \( W \) in terms of the points when the configuration is assumed to have some periodicity. Here we can reduce to \( m = 1 \) by scaling, as seen above.

Lemma 1.6. In the case \( m = 1 \) and when the set of points \( \Lambda \) is periodic with respect to some lattice \( \mathbb{N} \mathbb{Z} \), then it can be viewed as a set of \( N \) points \( a_1, \ldots, a_N \) over the torus \( \mathbb{T}_N := \mathbb{R}/(\mathbb{N} \mathbb{Z}) \). In this case, by Lemma 1.4 there exists a unique \( E \) satisfying (1.3) and for which \( W(E) < +\infty \). It is periodic and equal to \( E \{ a_i \} = \nabla H \), where \( H \) is the solution on \( \mathbb{T} \) to
\[
-\Delta H = 2\pi \left( \sum_i \delta_{a_i} - \delta_R \right),
\]
and we have the explicit formula:
\[
W(E \{ a_i \}) = -\pi \log \left( \frac{2}{N} \right) \sum_{i \neq j} \log \left| 2 \sin \left( \frac{\pi (a_i - a_j)}{N} \right) \right| - \pi \log \frac{2\pi}{N}.
\]

As in the two-dimensional case, we can prove that \( \min_{\Lambda} \) is achieved, but contrarily to the two-dimensional case, the value of the minimum can be explicitly computed: we will prove here the following

Theorem 1. \( \min_{\Lambda} W = -\pi m \log(2\pi m) \) and this minimum is achieved by the perfect lattice i.e. \( \Lambda = \frac{1}{m} \mathbb{Z} \).

We recall that in dimension 2, it was conjectured in [SS1] but not proven, that the minimum value is achieved at the triangular lattice with angles 60° (which is shown to achieve the minimum among all lattices), also called the Abrikosov lattice in the context of superconductivity.

The proof of Theorem 1 relies on showing that a minimizer can be approximated by configurations which are periodic with period \( N \to \infty \) (this result itself relies on the screening construction we mentioned above), and then using a convexity argument to find the minimizer among periodic configurations with a fixed period via (1.10).

1.2 The equilibrium measure and the splitting formula

The Hamiltonian (1.2) is written in the mean-field scaling. The limiting “mean-field” limiting energy and its minimizer are in fact well-known, and they tell us the optimal macroscopic distribution of points — what we investigate here is the microscopic distribution of points, as a correction to this optimal macroscopic behavior. More precisely, the mean-field limit energy is
\[
\mathcal{F}(\mu) = \int_{\mathbb{R} \times \mathbb{R}} -\log |x - y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x),
\]
and its unique minimizer among probability measures is called the equilibrium measure in the language of potential theory (cf. [SaTo]). We will denote it \( \mu_0 \). It is not hard to prove that the “spectral measure” (so-called in the context of random matrices) \( \nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) converge to \( \mu_0 \), the minimizer of \( \mathcal{F} \), itself also called Voiculescu’s noncommutative entropy in the context
of random matrices, cf. e.g. [AGZ] and references therein. The sense of convergence usually proven is

\[(1.12) \quad \mathbb{P}_n^\beta \left( \forall f \in C_b(\mathbb{C}, \mathbb{R}), \int f \, d\nu_n \to \int f \, d\mu_0 \right) = 1 \]

For example, for the case of the GUE i.e. when \( V(x) = |x|^2 \) and \( \beta = 1 \), the corresponding distribution \( \mu_0 \) is simply Wigner’s “semi-circle law” \( \rho(x) = \frac{1}{\pi} \sqrt{4 - x^2} 1_{|x|<2} \), cf. [Wi, Me]. A stronger result was proven in [BG] for all \( \beta \) (cf. [AGZ] for the case of general \( V \)): it estimates the large deviations from this convergence and shows that \( F \) is the appropriate rate function. The result can be written:

**Theorem 2** (Ben Arous - Guionnet [BG]). Let \( \beta > 0 \), and denote by \( \mathbb{P}_n^\beta \) the image of the law (1.1) by the map \((x_1, \ldots, x_n) \mapsto \nu_n\), where \( \nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \). Then for any subset \( A \) of the set of probability measures on \( \mathbb{R} \) (endowed with the topology of weak convergence), we have

\[ -\inf_{\mu \in A} \bar{F}(\mu) \leq \liminf_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A) \leq \limsup_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A) \leq -\inf_{\mu \in \bar{A}} \bar{F}(\mu), \]

where \( \bar{F} = F - \min F \).

Our result Theorem 5 below will be a sort of next-order extension of this result. Note that the Central Limit Theorem for (macroscopic) fluctuations from the limiting law was proved by Johansson [Jo].

Let us now state a few facts that we will need about the equilibrium measure \( \mu_0 \), for which we refer to [SaTo]: \( \mu_0 \) is characterized by the fact that there exists a constant \( c \) (depending on \( V \)) such that

\[(1.13) \quad U^{\mu_0} + \frac{V}{2} = c \quad \text{quasi-everywhere in the support of } \mu_0, \quad \text{and } U^{\mu_0} + \frac{V}{2} \geq c \quad \text{quasi-everywhere} \]

where for any \( \mu \), \( U^\mu \) is the potential generated by \( \mu \), defined by

\[(1.14) \quad U^\mu(x) = -\int_\mathbb{R} \log |x - y| \, d\mu(y). \]

(Note that looking in Fourier, this corresponds to \( U^\mu = -\Delta^{-1/2} \mu \)). Here and in all the paper, we denote by \( \Delta^{-1} \) the operator of convolution by \( \frac{1}{2\pi} \log |\cdot| \) in \( \mathbb{R}^2 \). It is such that \( \Delta \circ \Delta^{-1} = Id \), where \( \Delta \) is the usual Laplacian. We may view \( \mu_0 \) as a measure over \( \mathbb{R}^2 \) and extend \( U^\mu \) into a function of \( \mathbb{R}^2 \) by \( U^\mu(z) = -\int_\mathbb{R} \log |z - (y,0)| \, d\mu(y) \). Then \( U^\mu = -\Delta^{-1} \mu \) in \( \mathbb{R}^2 \).

We also define

\[ \zeta = U^{\mu_0} + \frac{V}{2} - c \]

where \( c \) is the constant in (1.13). From the above we know that \( \zeta \geq 0 \) in \( \mathbb{R} \) and \( \zeta = 0 \) in \( \Sigma := \text{Supp}(\mu_0) \). We will make the assumption that \( \mu_0 \) has a density \( m_0 \) with respect to the Lebesgue measure.

It’s now time to state our assumptions on \( V \): we assume in the sequel

\[(1.15) \quad \lim_{|x| \to +\infty} \frac{V(x)}{2} - \log |x| = +\infty. \]
(1.16) \( \Sigma \) is a finite union of closed intervals \( \Sigma_1, \ldots, \Sigma_M \).

(1.17) There exist \( \gamma, m > 0 \) such that \( \gamma \sqrt{\text{dist} \ (x, \mathbb{R} \setminus \Sigma)} \leq m_0(x) \leq m \) for all \( x \in \mathbb{R} \).

(1.18) \( m_0 \in C^{0,1/2}(\mathbb{R}) \).

(1.19) There exists \( \beta_1 > 0 \) such that \( \int_{\mathbb{R} \setminus [-1,1]} e^{-\beta_1 (V/2(x) - \log |x|)} \, dx < +\infty \).

The assumption (1.15) ensures (see [SaTo]) that (1.11) has a minimizer, and that its support \( \Sigma \) is compact. Assumptions (1.16)–(1.18) are needed for the construction in Section 4. They are certainly not optimal but are meant to include at least the model case of \( \mu_0 = \rho \), Wigner’s semi-circle law. Assumption (1.19) is a supplementary assumption on the growth of \( V \) at infinity, needed for the case with temperature. It only requires a very mild growth of \( V/2 - \log |x| \), i.e. slightly more than (1.15).

We are interested in examining the “next order” behavior, or that of fluctuations around the limiting law \( \mu_0 \). While \( F \) and \( \mu_0 \) are found through the large deviations at speed \( n^2 \), as seen in Theorem 2, we look into the speed \( n \) and show, not a large deviations principle, but that a suitably defined average \( \int W \, dP \) of the “renormalized energy” \( W \) defined above (1.8) acts as a sort of rate function, and that a “threshold phenomenon” holds: \( \int W \, dP \leq \min W + C_\beta \) except with exponentially small probability. The crucial fact is that \( C_\beta \to 0 \) as \( \beta \to \infty \), which shows the crystallization. The average of \( W \) corresponds to the average of \( W \) computed over all blown-up configurations of the points (with respect to all possible blow-up centers in \( \Sigma \)) at the scale \( n \). The measure \( P \) is like a Young measure on all the possible limits of blow-ups of \( E \).

The starting point of our analysis is to establish, through a simple but crucial exact “splitting formula”, a link between the law \( P^\beta_n \) and the energy \( w_n \), and \( W \):

**Lemma 1.7** (Splitting formula). For any \( x_1, \ldots, x_n \in \mathbb{R} \) the following holds

\[
(1.20) \quad w_n(x_1, \ldots, x_n) = n^2 F(\mu_0) - n \log n + \frac{1}{\pi} W(\nabla H'_n, 1_{\mathbb{R}^2}) + 2n \sum_{i=1}^{n} \zeta(x_i)
\]

where \( H'_n \) is the solution in \( \mathbb{R}^2 \) to

\[
(1.21) \quad H'_n(x') = -2\pi \Delta^{-1} \left( \sum_{i=1}^{n} \delta_{x'_i} - m_0(x'/n) \delta_{\mathbb{R}} \right)
\]

with \( x'_i = nx_i \) (viewed as points in \( \mathbb{R}^2 \)), and \( W(\cdot, \cdot) \) is as in (1.6).

We may then define

\[
(1.22) \quad F_n(\nu) = \begin{cases} 
\frac{1}{\pi} (\frac{1}{2} W(\nabla H'_n, 1_{\mathbb{R}^2}) + 2n \int_{\mathbb{R}} \zeta \, d\nu) & \text{if } \nu \text{ is of the form } \sum_{i=1}^{n} \delta_{x_i} \ 
+ \infty & \text{otherwise}
\end{cases}
\]

and also

\[
(1.23) \quad \bar{F}_n(\nu) = F_n(\nu) - 2 \int_{\mathbb{R}} \zeta \, d\nu \leq F_n(\nu)
\]
and we thus have the following rewriting of $w_n$:

(1.24) \[ w_n(x_1, \ldots, x_n) = n^2 F(\mu_0) - n \log n + n F_n(\nu). \]

This allows to separate orders in the limit $n \to \infty$ since one of the main outputs of our analysis is that $F_n(\nu)$ is of order 1. Moreover, $F_n(\nu)$ contains information about the microscopic structure of the configurations, its limit as $n \to \infty$ will be the average of $W$ over all blown-up points, $\int W \, dP$ mentioned above.

### 1.3 Main results

The first main result is the announced one on the next order asymptotic expansion of the partition function, which becomes sharp as $\beta \to \infty$.

**Theorem 3.** Let $V$ satisfy assumptions (1.15) -- (1.19). There exist functions $f_1, f_2$ depending only on $V$, such that for any $\beta_0 > 0$ and any $\beta \geq \beta_0$, and for $n$ larger than some $n_0$ depending on $\beta_0$, we have

(1.25) \[ n \beta f_1(\beta) \leq \log Z^\beta_n - \left( -\frac{\beta}{2} n^2 F(\mu_0) + \frac{\beta}{2} n \log n \right) \leq n \beta f_2(\beta), \]

with $f_1, f_2$ bounded in $[\beta_0, +\infty)$ and

(1.26) \[ \lim_{\beta \to \infty} f_1(\beta) = \lim_{\beta \to \infty} f_2(\beta) = \frac{\alpha}{2} \]

where

(1.27) \[ \alpha = \int_\Sigma \mu_0(x) \log(2\pi \mu_0(x)) \, dx. \]

**Remark 1.8.** In fact we prove that the statement holds with $f_2(\beta) = \frac{\alpha}{2} + \frac{C}{\beta}$ for any $C > \log |\Sigma|$.

The exact value for $Z^\beta_n$ is only known for arbitrary $\beta$’s for $V(x) = x^2/2$ via Selberg integrals cf. [Mel] AGZ. For some more general potentials there are formulas for $\beta = 2$ (e.g. [EML] [PAS] [GMS]). For general potential and $\beta$, this improves on the known results, which only gave the expansion $\log Z^\beta_n \sim \beta n^2 F(\mu_0)$.

Our next result is the one we mentioned about minimizers of $w_n$, or weighted Fekete sets. It identifies the $\Gamma$-limit (in the sense of $\Gamma$-convergence, cf. [Br]) of $\{F_n\}_n$, defined in (1.22) or (1.24). This allows a description of the minimizers at the microscopic level. Below we abuse notation by writing $\nu_n = \sum_{i=1}^n \delta_{x_i}$ when it should be $\nu_n = \sum_{i=1}^n \delta_{x_i,n}$. For such a $\nu_n$ we let $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ be the measure in blown-up coordinates $x' = nx$, and $E_{\nu'_n} = -\nabla H'_{\nu'_n}$, where $H'_{\nu'_n}$ is defined by (1.21). (To avoid confusion, we emphasize here that $\nu_n$ lives at the original scale while $E_{\nu'_n}$ lives at the blown-up scale.) We set $m_0'(x') = m_0(x'/n)$, where $m_0$ is the density of $\mu_0$. We also let

(1.28) \[ P_{\nu'_n} = \int_\Sigma \delta_{(x, E_{\nu'_n}(nx + \cdot))} \, dx, \]

i.e. the push-forward of the normalized Lebesgue measure on $\Sigma$ by $x \mapsto (x, E_{\nu'_n}(nx + \cdot))$. It is a probability measure on $\Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ (couples of (blow-up centers, blown-up electric...
field around this center)). In taking the $\Gamma$-limit of $F_n$, the limiting object is more complex, it is the limit $P$ of $P_n$, i.e. a Young measure akin to the Young measures on micropatterns introduced in [AM]. Note that in the rest of the paper, the probability $P$ has nothing to do with $\mathbb{R}^3_+$, and depends on the realizations of the configurations of points. We state the generalized invariance property satisfied by these limiting object as a definition.

**Definition 1.9.** We say a probability measure $P$ on $\Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ is $T_{\lambda(x)}$-invariant if $P$ is invariant by $$(x, E) \mapsto (x, E(\lambda(x) + \cdot)), \text{ for any } \lambda(x) \text{ of class } C^1 \text{ from } \Sigma \text{ to } \mathbb{R}.$$}

**Theorem 4** (Microscopic behavior of weighted Fekete sets). Let the potential $V$ satisfy assumptions (1.15)–(1.18). Fix $1 < p < 2$ and let $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.

**A. Lower bound.** Let $\nu_n = \sum_{i=1}^n \delta_{x_i}$ be a sequence such that $\tilde{F}_n(\nu_n) \leq C$. Then $P_{\nu_n}$ defined by (1.28) is a probability measure on $X$ and

1. Any subsequence of $\{P_{\nu_n}\}_n$ has a convergent subsequence converging to a probability measure on $X$ as $n \to \infty$. We denote by $P$ such a limit.
2. The first marginal of $P$ is the normalized Lebesgue measure on $\Sigma$. $P$ is $T_{\lambda(x)}$-invariant.
3. For $P$ almost every $(x, E)$ we have $E \in A_{\text{mo}(x)}$.
4. It holds that

\[ \liminf_{n \to \infty} \tilde{F}_n(\nu_n) \geq \frac{\|\Sigma\|}{\pi} \int W(E) \, dP(x, E) \geq \alpha, \]

where $\alpha$ is as in (1.27).

**B. Upper bound construction.** Conversely, assume $P$ is a $T_{\lambda(x)}$-invariant probability measure on $X$ whose first marginal is $\frac{1}{|E|} \, dx|\Sigma$ and such that for $P$-almost every $(x, E)$ we have $E \in A_{\text{mo}(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of empirical measures on $\Sigma$ and a sequence $\{E_n\}_n$ in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\text{div} \, E_n = 2\pi(\nu' - \nu_0), \delta_{\mathbb{R}}$ and such that defining $P_n$ as in (1.28), with $E_n$ replacing $E_{\nu_n}$, we have $P_n \to P$ as $n \to \infty$ and

\[ \limsup_{n \to \infty} F_n(\nu_n) \leq \frac{\|\Sigma\|}{\pi} \int W(E) \, dP(x, E). \]

**C. Consequences for minimizers.** If $(x_1, \ldots, x_n)$ minimizes $w_n$ for every $n$ and $\nu_n = \sum_{i=1}^n \delta_{x_i}$, then the limit $P$ of $P_{\nu_n}$ as defined in (1.28) satisfies the following.

1. For $P$-almost every $(x, E)$, the electric field $E$ minimizes $W$ over $A_{\text{mo}(x)}$.
2. We have

\[ \lim_{n \to \infty} F_n(\nu_n) = \lim_{n \to \infty} \tilde{F}_n(\nu_n) = \frac{\|\Sigma\|}{\pi} \int W(E) \, dP(x, E) = \alpha, \quad \lim_{n \to \infty} \sum_{i=1}^n \zeta(x_i) = 0. \]
Note that part B of the theorem is only a partial converse to part A because the constructed $E_n$ need not be curl-free, hence in general $E_n \neq E_{\nu_n}$.

It can be expected that $\zeta$ (which is positive exactly in the complement of $\Sigma$) controls the distance to $\Sigma$ (to some power). This can probably be proven under some strict growth assumption on $V$ and explicit formulas for the Poisson kernel of the half-Laplacian. We do not pursue this question here.

The next result is the announced next order result on deviations, to be compared to Theorem 2.

We let $A_n$ be a subset of $\mathbb{R}^n$. We identify points in $\mathbb{R}^n$ with measures $\nu_n$ of the form $\sum_{i=1}^n \delta_{x_i}$. We also embed $\mathbb{R}^n$ into the set of probabilities on $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ in the following way: For any $n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we let $\nu_n(x) = P_{\nu_n}$, where $\nu_n = \sum_{i=1}^n \delta_{x_i}$ and $P_{\nu_n}$ is as in (1.28), so that $\nu_n(x)$ is an element of $P(X)$, the set of probability measures on $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, which we consider endowed with the topology of weak convergence.

**Theorem 5.** Let $V$ satisfy (1.15)-(1.19). For any $\beta_0 > 0$ and any $\beta \geq \beta_0$ the following holds.

A. For any $n > 0$, let $A_n \subset \mathbb{R}^n$. Denote

$$A_{\infty} = \bigcap_{n>0} \bigcup_{m>n} i_m(A_m).$$

Then for any $\eta > 0$ there is $C_\eta > 0$ depending on $V$ and $\eta$ only such that $\alpha$ being as in (1.27),

$$\limsup_{n \to \infty} \frac{\log P^\beta_n(A_n)}{n} \leq -\frac{\beta}{2} \left( \frac{\|V\|}{\pi} \inf_{P \in A_{\infty}} \int W(E) dP(x, E) - \alpha - \eta - \frac{C_\eta}{\beta} \right).$$

Conversely, let $A \subset P(X)$ be a set of $T_{\nu_n}(x)$-invariant probability measures on $X$ and let $\hat{A}$ be the interior of $A$ for the topology of weak convergence. Then for any $\eta > 0$, there exists a sequence of subsets $A_n \subset \Sigma^n$ such that

$$-\frac{\beta}{2} \left( \frac{\|V\|}{\pi} \inf_{P \in \hat{A}} \int W(E) dP(x, E) - \alpha + \eta + \frac{C_\eta}{\beta} \right) \leq \liminf_{n \to \infty} \frac{\log P^\beta_n(A_n)}{n},$$

and such that for any sequence $\{
u_n = \sum_{i=1}^n \delta_{x_i}\}$ such that $(x_1, \ldots, x_n) \in A_n$ for every $n$ there exists a sequence of fields $E_n \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\text{div} E_n = 2\pi(\nu_n' - m_0/\delta_R)$ and such that — defining $P_n$ as in (1.28) with $E_n$ replacing $E_{\nu_n}$ — we have

$$\lim_{n \to \infty} P_n \in \hat{A}.$$

B. For each integer $n$, and given $\beta \geq \beta_0$ let $Q^\beta_n$ denote the push-forward of $P^\beta_n$ by $i_n$ defined above. Then $Q^\beta_n$ is tight in $P(P(X))$ and converges, up to a subsequence, to a probability measure $Q^\beta \in P(P(X))$.

Note that if $P_n = P_{\nu_n}$, then (1.34) would be equivalent to saying that $\cap_{n \geq m > n} i_n(A_m) \subset \hat{A}$. The difference between $P_{\nu_n}$ and $P_n$ is that the latter is generated by a field $E_n$ which is not necessarily curl free (we believe this gap cannot be bridged).

Part B of the result shows the existence of a limiting “electric field” process, hence equivalently, via projecting by $(x, E) \mapsto \frac{1}{2\pi} \text{div} E + m_0(x) \delta_R$, of a limiting point process.
As we said, Part A of this theorem means that events such that the average of $W$ is larger than $\alpha + \eta + C_\eta/\beta$ have exponentially decaying probability as $n \to \infty$, with $\eta > 0$ arbitrary. Note here the important fact that

\begin{align}
\min \frac{|\Sigma|}{\pi} \int W(E) dP(x,E) = \alpha
\end{align}

where the min is taken over $P$ probabilities satisfying $P$-a.e. $(x,E)$ satisfies $E \in A_{m_0(x)}$, a fact which simply follows from (1.9) and Theorem 1.

This means that configurations should have more and more order (evaluated via $W$) as $\beta$ gets large, and have to crystallize to minimizers of $W$ when $\beta \to \infty$ (the term crystallization can be legitimately used in view of Theorem 1). Note that we do not claim that this is a true large deviations result, since the corresponding lower bound is missing. In fact it is likely that the large deviations rate function is not truly $W$ but contains an extra entropy term. To demonstrate its existence is an open question.

Finally, let us mention that our method yields estimates on the probability of some rare events, typically the probability that the number of points in a microscopic interval deviates from the number given by $\mu_0$. We present them below, even though stronger results are obtained in [BEY1, BEY2]. The results below follow easily from the estimate (provided by Theorem 3) that

\begin{align}
\hat{F}_n \leq C \text{ except on a set of small probability. }
\end{align}

Theorem 6. Let $V$ satisfy assumptions (1.15)–(1.18). There exists a universal constant $R_0 > 0$ and $c, C > 0$ depending only on $V$ such that: For any $\beta > 0$, any $\beta \geq \beta_0$, any $n$ large enough depending on $\beta_0$, for any $x_1, \ldots, x_n \in \mathbb{R}$, any $R > R_0$, any interval $I \subset \mathbb{R}$ of length $R/n$, and any $\eta > 0$, letting $\nu_n = \sum_{i=1}^n \delta_{x_i}$, we have the following:

\begin{align}
\log P_{\beta n}(|\nu_n(I) - n\mu_0(I)| \geq \eta R) &\leq -c\beta \min(\eta^2, \eta^3) R^2 + C\beta(R + n) + Cn,
\end{align}

\begin{align}
\log P_{\beta n}((1 + R^2/n^2)^{1/2} - 1/\sqrt{\eta} \parallel \nu_n - n\mu_0 \parallel_{W^{-1,q}(I)} \geq \eta\sqrt{n}) &\leq -cn\beta \eta^2 + Cn(\beta + 1),
\end{align}

where $W^{-1,q}(I)$ is the dual of the Sobolev space $W^{1,q'}_0(I)$, with $1/q + 1/q' = 1$, in particular $W^{-1,1}$ is the dual of Lipschitz functions; and

\begin{align}
\log P_{\beta n} \left( \int \zeta d\nu_n \geq \eta \right) &\leq -\frac{1}{2} n\beta \eta + Cn(\beta + 1).
\end{align}

Note that in these results $R$ can be taken to depend on $n$.

(1.36) tells us that the density of eigenvalues is correctly approximated by the limiting law $\mu_0$ at all small scales bigger than $Cn^{-1/2}$ for some $C$. However this in fact should hold at all scales with $R \gg 1$, cf. [ESY, BEY1, BEY2]. (1.38) serves to control the probability that points are outside $\Sigma$ (since $\{\zeta > 0\} = \Sigma_c$).

The rest of the paper is organized as follows. In Section 2 we prove the preliminary results Lemma 1.4 and 1.7 as well as Theorem 1. In Section 3 we turn to the core of the proof, with the statement of the main screening result and construction. Sections 4 and 5 contain the proofs. Finally Section 6 gathers the proofs of all the remaining results which are adapted from [SS2].

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2 The first results

2.1 Proof of Lemma 1.4

Assume \( E \) and \( E' \) belong to \( A_m \) and satisfy [1.3] with the same \( \nu \). Then \( f = E - E' \) is divergence-free and curl-free, hence can be seen, identifying \( \mathbb{R}^2 \) and \( \mathbb{C} \), as an entire holomorphic function \( \sum_{n=0}^{\infty} a_n z^n \). If we assume that \( W(E) \) and \( W(E') \) are finite, then it follows from [SeTi], Corollary 1.2 that for \( p < 2 \), the growth of the \( L^1 \) norms of \( E \) and \( E' \) is no worse than \( R^{3/2} / \log R \) hence there exists \( C > 0 \) such that for any \( R > 2 \) we have \( \| f \|_{L^1(B_R)} \leq CR^{3/2} / \log R \). But by Cauchy’s formula we have, for any \( R > 0 \) and \( t \in [R, R+1] \)

\[
a_n = \frac{1}{2i\pi} \int_{\partial B(0,t)} \frac{f(z)}{z^{n+1}} \, dz = \frac{1}{2i\pi} \int_R^{R+1} \int_{\partial B(0,t)} \frac{f(z)}{z^{n+1}} \, dz.
\]

It follows with the above that \( |a_n| \leq CR^{3/2} / \log R R^{-n-1} \) which implies, letting \( R \to \infty \) that \( a_n = 0 \) for any \( n \geq 1 \), thus \( f \) is a constant. This constant must then be zero since both \( E \) and \( E' \) are square integrable on the infinite strips \( [a, b] \times [1, +\infty] \).

2.2 Proof of the splitting formula (Lemma 1.7)

Let \( \nu_n = \sum_{i=1}^{n} \delta_{x_i} \). First, letting \( \triangle \) denote the diagonal of \( \mathbb{R} \times \mathbb{R} \), we may rewrite \( w_n \) as

\[
w_n(x_1, \ldots, x_n) = \int_\triangle -\log |x-y| \, d\nu_n(x) \, d\nu_n(y) + n \int_{\mathbb{R}} V(x) \, d\nu_n(x).
\]

Splitting \( \nu_n \) as \( n\mu_0 + \nu_n - n\mu_0 \) and using the fact that \( \mu_0 \times \mu_0(\triangle) = 0 \), we obtain

\[
w(x_1, \ldots, x_n) = n^2 F(\mu_0) + 2n \int U^\mu_0(x) \, d(\nu_n - n\mu_0)(x) + n \int V(x) \, d(\nu_n - n\mu_0)(x) + \int_\triangle -\log |x-y| \, d(\nu_n - n\mu_0)(x) \, d(\nu_n - n\mu_0)(y).
\]

Since \( U^\mu_0 + \frac{V}{2} = c + \zeta \) and since \( \nu_n \) and \( n\mu_0 \) have same mass \( n \), we have

\[
2n \int U^\mu_0(x) \, d(\nu_n - n\mu_0)(x) + n \int V(x) \, d(\nu_n - n\mu_0)(x) = 2n \int \zeta \, d(\nu_n - \mu_0) = 2n \int \zeta \, d\nu_n,
\]

using the fact that \( \zeta = 0 \) on the support of \( \mu_0 \).

In addition, we have that

\[
(2.1) \quad \int_{(\mathbb{R} \times \mathbb{R}) \setminus \triangle} -\log |x-y| \, d(\nu_n - n\mu_0)(x) \, d(\nu_n - n\mu_0)(y) = \frac{1}{\pi} W(\nabla H_n, 1_{\mathbb{R}^2}),
\]

where we define \( H_n = -2\pi \Delta^{-1} (\sum_{i=1}^{n} \delta_{x_i} - n\mu_0) \). Indeed, the integral might as well be written as over \( \mathbb{R}^2 \times \mathbb{R}^2 \setminus \triangle \) with the diagonal in \( \mathbb{R}^2 \times \mathbb{R}^2 \); and then the identity is proven in [SS2], Section 2]. Combining all the above we find

\[
(2.2) \quad w(x_1, \ldots, x_n) = n^2 F(\mu_0) + 2n \int \zeta \, d\nu_n + \frac{1}{\pi} W(\nabla H_n, 1_{\mathbb{R}^2}).
\]
But, changing variables, we have
\[
\frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(x_i, \eta)} |\nabla H_n|^2 = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(x_i', \eta)} |\nabla H_n'|^2,
\]
and by adding \( \pi n \log \eta \) on both sides and letting \( \eta \to 0 \) we deduce that \( W(\nabla H_n, 1_{\mathbb{R}^2}) = W(-\nabla H_n', 1_{\mathbb{R}^2}) - \pi n \log n \). Together with (2.2) this proves the lemma.

### 2.3 Minimization of \( W \): proof of Theorem \ref{thm:boundary}

By scaling (cf. (1.9)), we reduce to \( m = 1 \). The result relies on the fact that there exists a minimizing sequence for \( \min_{A_1} W \) consisting of periodic vector-fields:

**Proposition 2.1.** For any large enough integer \( R \), there exists a sequence \( \{E_R\}_{R \in \mathbb{N}} \) in \( A_1 \) such that each \( E_R \) is 2\( R \)-periodic (with respect to the \( x \) variable) and
\[
\limsup_{R \to \infty} W(E_R) \leq \min_{A_1} W.
\]

The proof uses the main results of Section \ref{sec:section3} and is postponed to the end of Section \ref{sec:section6}.

The following proposition could be proven as in \cite{SS1}, however we omit the proof here.

**Proposition 2.2.** \( W : L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R} \cup \{+\infty\} \) is a Borel function. \( \inf_{A_1} W \) is achieved and is finite.

The result of Theorem \ref{thm:boundary} will follow from Proposition 2.1 combined with the following

**Proposition 2.3** (Minimization in the periodic case). Let \( a_1, \ldots, a_N \) be any points in \( [0, N] \) and \( E_{\{a_i\}} \) be the corresponding periodic vector field, as in Lemma \ref{lem:periodic}. Then
\[
W(E_{\{a_i\}}) \geq W(E_Z) = -\pi \log 2\pi
\]
where \( E_Z \) is the electric field associated to the perfect lattice \( \mathbb{Z} \).

**Proof.** \( W(E_Z) \) is immediately computed via (1.10), taking \( N = 1 \).

Let us now consider arbitrary points \( a_1, \ldots, a_N \) in \( [0, N] \), and assume \( a_1 < \cdots < a_N \). Let us also denote \( u_{1,i} = a_{i+1} - a_i \), with the convention \( a_{N+1} = a_1 + N \). We have \( \sum_{i=1}^N u_{1,i} = N \).

Similarly, let \( u_{p,i} = a_{i+p} - a_i \), with the convention \( a_{N+1} = a_1 + N \). We have \( \sum_{i=1}^N u_{p,i} = pN \).

By periodicity of sin, we may view the points \( a_i \) as living on the circle \( \mathbb{R}/(NZ) \). When adding the terms in \( a_i - a_j \) in the sum of (1.10), we can split it according to the difference \( p = j - i \) but modulo \( N \). This way, there remains (2.3)
\[
W(E_{\{a_i\}}) = -\frac{\pi}{N} \sum_{i \neq j} \log \left| 2 \sin \frac{\pi (a_i - a_j)}{N} \right| - \pi \log \frac{2\pi}{N} = -\frac{2\pi}{N} \sum_{p=1}^{[N/2]} \sum_{i=1}^N \log \left| 2 \sin \frac{\pi u_{p,i}}{T} \right| - \pi \log \frac{2\pi}{N},
\]
where \( [\cdot] \) denotes the integer part. But the function \( \log |2 \sin x| \) is strictly concave on \( [0, \pi] \). It follows that
\[
\frac{1}{N} \sum_{i=1}^N \log \left| 2 \sin \frac{\pi u_{p,i}}{N} \right| \leq \log \left| 2 \sin \left( \frac{\pi}{N^2} \sum_{i=1}^N u_{p,i} \right) \right| = \log \left| 2 \sin \frac{p\pi}{N} \right|
\]
with equality if and only if all the \( u_{p,i} \) are equal. Inserting into \( (2.3) \) we obtain

\[
(2.4) \quad W(E_{\{a_i\}}) \geq -2\pi \sum_{p=1}^{[N/2]} \log \left| 2 \sin \frac{p\pi}{N} \right| - \pi \log \frac{2\pi}{N}.
\]

On the other hand, if we take for the \( a_i \)'s the points of the lattice \( \mathbb{Z} \) viewing them as \( N \)-periodic, we have \( u_{p,i} = p \) for all \( p, i \), so if we compute \( W(E_Z) \) using \( (2.3) \), we find

\[
W(E_Z) = -2\pi \sum_{p=1}^{[N/2]} \log \left| 2 \sin \frac{p\pi}{N} \right| - \pi \log \frac{2\pi}{N} = -2\pi \sum_{p=1}^{[N/2]} \log \left| 2 \sin \frac{p\pi}{N} \right| - \pi \log \frac{2\pi}{N}.
\]

This is the right-hand side of \( (2.4) \), so \( (2.4) \) proves that \( W(E_{\{a_i\}}) \geq W(E_Z) \) with equality if and only if all the \( u_{p,i} \) are equal, which one can easily check implies that \( \{a_i\} = \mathbb{Z} \). \( \square \)

Combining with Corollary \( (2.1) \) this proves Theorem \( 1 \).

### 3 The main screening result and construction

This section contains the main “upper bound result” relying on explicit constructions, and requiring as a main ingredient the “screening result” mentioned in the introduction. We state the results, whose proofs will occupy the next section.

**Proposition 3.1.** Let \( P \) be a \( T_{\lambda(x)} \)-invariant probability measure (see Definition \( 1.9 \) on \( X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) with first marginal \( dx|_{\Sigma} / |\Sigma| \) and such that for \( P \) almost every \( (x, E) \) we have \( E \in \mathcal{A}_{m_0(x)} \). Then, for any \( \eta > 0 \), there exists \( \delta > 0 \) and for any \( n \) a subset \( A_n \subset \mathbb{R}^n \) such that \( |A_n| \geq n!(\delta/n)^n \) and for every sequence \( \{\nu_n = \sum_{i=1}^n \delta_{y_i}\} \) with \( (y_1, \ldots, y_n) \in A_n \) the following holds.

i) We have the upper bound

\[
(3.1) \quad \limsup_{n \to \infty} \frac{1}{n} \left( w_n(y_1, \ldots, y_n) - n^2 F(\mu_0) + n \log n \right) \leq \frac{|\Sigma|}{\pi} \int W(E) \, dP(x, E) + \eta.
\]

ii) There exists \( \{E_n\}_n \) in \( L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) such that \( \text{div} \, E_n = 2\pi (\nu'_n - m_0' \delta_\mathbb{R}) \) and such that the image \( P_n \) of \( dx|_{\Sigma} / |\Sigma| \) by the map \( x \mapsto (x, E_n(nx + \cdot)) \) is such that

\[
(3.2) \quad \limsup_{n \to \infty} \text{dist} (P_n, P) \leq \eta,
\]

where \( \text{dist} \) is a distance which metrizes the topology of weak convergence on \( \mathcal{P}(X) \).

Applying the above proposition with \( \eta = 1/k \) we get a subset \( A_{n,k} \) in which we choose any \( n \)-tuple \( (y_{i,k})_{1 \leq i \leq n} \). This yields in turn a family \( \{P_{n,k}\} \) of probability measures on \( X \). A standard diagonal extraction argument then yields

**Corollary 3.2** (Theorem \( 4 \) Part B). Under the same assumptions as Proposition \( 3.1 \) there exists a sequence \( \nu_n = \sum_{i=1}^n \delta_{y_i} \) and a sequence \( \{E_n\}_n \) in \( L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) such that \( \text{div} \, E_n = 2\pi (\nu'_n - m_0' \delta_\mathbb{R}) \) and

\[
(3.3) \quad \limsup_{n \to \infty} \frac{1}{n} \left( w(x_1, \ldots, x_n) - n^2 F(\mu_0) + n \log n \right) \leq \frac{|\Sigma|}{\pi} \int W(E) \, dP(x, E).
\]

Moreover, denoting \( P_n \) the image of \( dx|_{\Sigma} / |\Sigma| \) by the map \( x \mapsto (x, E_n(nx + \cdot)) \), we have \( P_n \to P \) as \( n \to +\infty \).
We may cancel out all leading order terms and rewrite the probability law (1.1) as

\begin{equation}
(3.4)
dP^\beta_n(x_1, \ldots, x_n) = \frac{1}{K_n^\beta} e^{-\frac{\beta}{2} F_n(\nu)} dx_1 \ldots dx_n
\end{equation}

where

\begin{equation}
(3.5)
K_n^\beta = Z_n^\beta e^{\frac{\beta}{2} (n^2 F(\mu_0) - n \log n)}.
\end{equation}

A consequence of Proposition 3.1 is, recalling (1.27) and (1.35):

**Corollary 3.3** (Lower bound part of Theorem 3). For any \( \eta > 0 \) there exists \( C_\eta > 0 \) such that for any \( \beta > 0 \) we have

\begin{equation}
(3.6)
\liminf_{n \to +\infty} \frac{\log K_n^\beta}{n} \geq -\frac{\beta}{2} (\alpha + \eta) - C_\eta.
\end{equation}

**Proof of the corollary.** It is exactly the same as in [SS2] but just letting

\begin{equation}
(3.7)
\sigma_m E(y) := m E(my).
\end{equation}

We now state our main screening result, on which the proof of Proposition 3.1 is based, and which is the main difference with the two-dimensional situation:

**Proposition 3.4.** Let \( I_R = [-R/2, R/2] \), let \( \chi_R \) satisfy (1.1).

Assume \( G \subset A_1 \) is such that there exists \( C > 0 \) such that for any \( E \in G \) and writing \( \nu = \frac{1}{2\pi} \text{div} E + \delta_R \) we have

\begin{equation}
(3.8)
\forall R > 1, \frac{\nu(I_R)}{R} < C,
\end{equation}

\begin{equation}
(3.9)
\lim_{R \to +\infty} \frac{W(E; \chi_R)}{R} = W(E) < C,
\end{equation}

\begin{equation}
(3.10)
\lim_{y_0 \to +\infty} \lim_{R \to +\infty} \int_{I_R} \int_{|y| > y_0} |E|^2 = 0,
\end{equation}

and such that moreover all the convergences are uniform w.r.t. \( E \in G \).

Then there exists for every \( 0 < \varepsilon < 1 \), there exists \( R_0 > 0 \) such that if \( R > R_0 \) with \( R \in \mathbb{N} \), then for every \( E \in G \) there exists a vector field \( E_R \in L^p_{\text{loc}}(I_R, \mathbb{R}^2) \) such that the following holds:

i) \( E_R \cdot \vec{\nu} = 0 \) on \( \partial I_R \), where \( \vec{\nu} \) denotes the outer unit normal.

ii) There is a discrete subset \( \Lambda \subset I_R \) such that

\[
\text{div } E_R = 2\pi \left( \sum_{\rho \in \Lambda} \delta_\rho - \delta_R \right) \text{ in } \overline{I}_R.
\]
iii) \( E_R(z) = E(z) \) for \( x \in [-R/2 + \varepsilon R, R/2 - \varepsilon R] \).

iv)

(3.11) \[
\frac{W(E_R, 1_{I_R})}{R} \leq W(E) + C\varepsilon.
\]

The assumption (3.10) is a supplementary assumption which is not satisfied for all vector-fields in \( A_m \), and which allows to perform the screening. We have not been able to show that screening is always possible without it. However we will see in Section 4 and in particular in Lemma 5.4 that this assumption is satisfied “generically” i.e. for a large set of vector-fields in the support of any invariant probability measure, and this will suffice for our purposes.

4 Proofs of Propositions 3.4 and 3.1

4.1 Preliminaries: a mass displacement result

In this subsection, we state the analogue in 1D of Proposition 4.9 in \( \text{SSI} \), a result we will need later. The point of this result is to say that even though the energy density associated to \( W: \frac{1}{2}|E|^2 + \pi \log \eta \sum_p \delta_p \), is not positive or even bounded below, it effectively behaves as if it were, because it can be replaced by an energy-density \( g \) which is pointwise bounded below, at the expense of a negligible error. The density \( g \) is obtained by displacing the negative part of the energy-density into the positive part. The proof is no different than in \( \text{SSI} \) once the one-dimensional setting has been embedded into the two-dimensional one as stated.

**Proposition 4.1.** Assume \( (\nu, E) \) are such that \( \nu = 2\pi \sum_{p \in \Lambda} \delta_p \) for some finite subset \( \Lambda \) of \( \mathbb{R} \), \( \text{div} \ E = 2\pi (\nu - a(x)\delta_R) \), for some \( a \in L^\infty(\mathbb{R}) \), and \( \text{curl} \ E = 0 \). Then, given \( 0 < \rho < \rho_0 \), where \( \rho_0 > 0 \) is universal, there exists a measure density \( g \) in \( \mathbb{R}^2 \) such that

i) There exists a family of disjoint closed balls \( B_\rho \) centered on the real line, covering \( \text{Supp}(\nu) \), such that the sum of the radii of the balls in \( B_\rho \) intersected with any segment of \( \mathbb{R} \) of length 1 is bounded by \( \rho \) and such that

(4.1) \[
g \geq -C(||a||_{L^\infty} + 1) + \frac{1}{4}|E|^2 1_{\mathbb{R}^2 \setminus B_\rho} \text{ in } \mathbb{R}^2,
\]

where \( C \) depends only on \( \rho \).

ii) \[
g = \frac{1}{2}|E|^2 \text{ in the complement of } \mathbb{R} \times [-1, 1].
\]

iii) For any function \( \chi \) compactly supported in \( \mathbb{R} \) we have, letting \( \bar{\chi}(x,y) = \chi(x) \),

(4.2) \[
\left| W(E, \chi) - \int \bar{\chi} \, dg \right| \leq CN(\log N + ||a||_{\infty})||\nabla \chi||_{\infty},
\]

where \( N = \# \{ p \in \Lambda \mid B(p, \lambda) \cap \text{Supp}(\nabla \bar{\chi}) \neq \emptyset \} \) and \( \lambda \) depends only on \( \rho \).
Proposition 4.9 of \cite{SS1} of which the above Proposition is a restatement, was stated for a fixed universal $\rho_0$, but we may use instead in its proof any $0 < \rho < \rho_0$, which makes the constant $C$ above depend on $\rho$. Another fact which is true from the proof of \cite[Proposition 4.9]{SS1} but not stated in the proposition itself is that in fact $g = \frac{1}{2}|E|^2$ outside $\cup_p B(p, r)$ for some constant $r > 0$ depending only on $\rho$, and if $\rho$ is taken small enough, then we may take $r = 1$, which yields item ii) of Proposition 4.1.

The next lemma shows that a control on $W$ implies a corresponding control on $\int g$ and $\int |E|^2$ away from the real axis, growing only like $R$.

**Lemma 4.2.** Assume that $G \subset A_1$ is such that (3.8) and (3.9) hold uniformly w.r.t. $E \in G$, for a certain constant $C_0$. Then for any $E \in G$, writing $\nu = \frac{1}{2m} \text{div } E + \delta_R$, for every $R$ large enough depending on $G$, we have

\begin{equation}
|\nu(I_R) - R| \leq C_1 R^{3/4} \log R,
\end{equation}

\begin{equation}
\int_{I_R \times \{|y| > 1\}} |E|^2 \leq CR(W(E) + 1),
\end{equation}

and denoting by $g$ the result of applying Proposition 4.1 to $E$ for some fixed value $\rho < 1/8$, we have

\begin{equation}
W(E, \chi_R) - C_1 R^{3/4} \log^2 R \leq \int_{I_R} dg \leq W(E, \chi_{R+1}) + C_1 R^{3/4} \log^2 R,
\end{equation}

where $\chi_R$ satisfies (1.7), $C_1$ depends only on $G$ and $C$ is a universal constant.

**Proof.** We denote by $C_1$ any constant depending only on $G$, and by $C$ any universal constant. From (3.8), (3.9) we have for any $E \in G$ that $\nu(I_R) \leq C_1 R$ and $W(E, \chi_R) \leq C_1 R$ if $R$ is large enough depending on $G$. Thus, applying (4.2) we have

\begin{equation}
\left| \int \chi_R \, dg \right| \leq C_1 R(\log R + 1)
\end{equation}

which, in view of the fact that $\chi_R = 1$ in $I_{R-1}$ and $g$ is positive outside $\mathbb{R} \times [-1, 1]$ and bounded below by a constant otherwise, yields that for every $R$ large enough,

\begin{equation}
\int_{I_{R-1}} \, dg \leq C_1 R(\log R + 1).
\end{equation}

This in turn implies — using (4.1) and the fact that $\frac{1}{2}|E|^2 = g$ outside $\mathbb{R} \times [-1, 1]$ — the first (unsufficient) control

\begin{equation}
\int \{ (x, y) | (x, 0) \notin \cup B_p \} |E|^2 \leq C_1 R(\log R + 1).
\end{equation}

Since the sum of the radii of the balls in $B_\rho$ intersected with any segment of $\mathbb{R}$ of length 1 bounded by $\rho < 1/8$, we deduce by a mean value argument with respect to the variable $x$ that there exists $t \in [0, 1]$ such that

\begin{equation}
\int_{\mathbb{R}} \left| E \left( \frac{-R}{2} - t, y \right) \right|^2 + \left| E \left( \frac{R}{2} + t, y \right) \right|^2 \, dy \leq C_1 R(\log R + 1).
\end{equation}
Using now a mean value argument with respect to \( y \), we deduce from (4.6) the existence of \( y_R \in [1, 1 + \sqrt{R}] \) such that

\[
(4.8) \quad \int_{-\frac{R}{2} - t}^{\frac{R}{2} + t} |E(x, y_R)|^2 + |E(x, -y_R)|^2 \, dx \leq C_1 \sqrt{R} (\log R + 1).
\]

Next, we integrate \( \text{div} \, E = 2\pi (\nu - \delta_x) \) on the square \([-\frac{R}{2} - t, \frac{R}{2} + t] \times [-y_R, y_R] \). We find using the symmetry property of Corollary 1.5 that

\[
|\nu(I_{R-\frac{1}{2}}) - R + 2t| \leq \int_{-y_R}^{y_R} \left| E \left( -\frac{R}{2} - t, y \right) \right| + \left| E \left( \frac{R}{2} + t, y \right) \right| \, dy + \int_{-\frac{R}{2} - t}^{\frac{R}{2} + t} |E(x, y_R)| \, dx.
\]

Using the Cauchy-Schwarz inequality and (4.7)-(4.8), this leads for \( R \) large enough to

\[
(4.9) \quad |\nu(I_{R-\frac{1}{2}}) - R| \leq 2 + C_1 R^{3/4} \sqrt{\log R + 1} + C_1 \sqrt{y_R} \sqrt{R (\log R + 1)} \leq C_1 R^{3/4} (\log R + 1),
\]

and then — since \( \nu(I_R) \geq \nu(I_{R-1/2}) \) — to

\[
\nu(I_R) - R \geq C_1 R^{3/4} \log R.
\]

To prove the same upper bound for \( R - \nu(I_R) \) we proceed in the same way, but using a mean value argument to find some \( t \in (-1, 0) \) instead of \((0, 1)\) such that (4.7) holds, and then also. We deduce as above that (4.9) holds and conclude by noting that since \( t \in (-1, 0) \) we have \( \nu(I_R) \leq \nu(I_{R-1/2}) \). This establishes (4.3).

We may bootstrap this information: Indeed (4.3) implies in particular that \( \nu(I_R) - \nu(I_{R-1}) \leq C_1 R^{3/4} \log R \) and thus we deduce from (4.2) that (4.5) holds:

\[
\left| W(E, \chi_R) - \int \bar{\chi}_R \, dg \right| \leq C_1 R^{3/4} \log^2 R.
\]

Then since \( W(E, \chi_R)/R \to W(E) \) as \( R \to \infty \) uniformly w.r.t. \( E \in G \) and since \( g \) is both bounded from below by a universal constant and equal to \( \frac{1}{2} |E|^2 \) outside \( \mathbb{R} \times [-1, 1] \), we deduce (4.4), for \( R \) large enough depending on \( G \).

\[\square\]

**Definition 4.3.** Assume \( \nu_n = \sum_{i=1}^n \delta_{x_i} \). Letting \( \nu'_n = \sum_{i=1}^n \delta_{x'_i} \) be the measure in blown-up coordinates and \( E_{\nu_n} = -\nabla H_{\nu_n}' \), where \( H_{\nu_n}' \) is defined by (1.21), we denote by \( g_{\nu_n} \) the result of applying Proposition 4.1 to \( (\nu'_n, E_{\nu_n}) \).

### 4.2 Some preliminary construction lemmas

The following lemmas serve to estimate the energy of explicit vector fields on boxes, which will be later combined to make test configurations. They are adaptations of [SS1] and rely on elliptic estimates.

**Lemma 4.4.** Let \( \mathcal{K} \) be the square \([-\frac{R}{2}, \frac{R}{2}]^2 \). Let \( \varphi \in L^2(\partial \mathcal{K}) \) and \( a \in L^\infty([-\frac{R}{2}, \frac{R}{2}]) \) be such that \( \int_{-\frac{R}{2}}^{\frac{R}{2}} a(x) \, dx = -\int_{\partial \mathcal{K}} \varphi \). Then, \( a_0 \) being the average of \( a \) over \([-\frac{R}{2}, \frac{R}{2}] \), the solution (well defined up to an additive constant) to

\[
(4.10) \quad \begin{cases} -\Delta u = a \delta_{\mathcal{K}} & \text{in } \mathcal{K} \\ \frac{\partial u}{\partial \nu} = \varphi & \text{on } \partial \mathcal{K}. \end{cases}
\]
satisfies for every \( q \in [1, 4] \)

\[
\int_\mathcal{K} |\nabla u|^q \leq C_q \left( a_0^q L^2 + L^2 \|a - a_0\|^q_{L^\infty([-\frac{L}{2}, \frac{L}{2}])} + L^2 \frac{q}{2} \|\varphi\|^q_{L^2(\partial\mathcal{K})} \right).
\]

Proof. We write the solution \( u \) of (4.10) as

\[
\begin{cases}
-\Delta u_1 = a_0 \delta_R & \text{in } \mathcal{K} \\
\frac{\partial u_1}{\partial \nu} = \bar{\varphi} & \text{on } \partial\mathcal{K}
\end{cases}
\]\n
(4.11)

where \( \bar{\varphi} \) is equal to 0 on the vertical sides of the square and to \( \frac{a_0}{2L} \) on both horizontal sides;

\[
\begin{cases}
-\Delta u_2 = (a - a_0) \delta_R & \text{in } \mathcal{K} \\
\frac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\mathcal{K}
\end{cases}
\]\n
(4.12)

and

\[
\begin{cases}
-\Delta u_3 = 0 & \text{in } \mathcal{K} \\
\frac{\partial u_3}{\partial \nu} = \varphi - \bar{\varphi} & \text{on } \partial\mathcal{K}.
\end{cases}
\]\n
The solution of (4.11) is (up to a constant) \( u_1(x, y) = \frac{a_0}{2} |y| \). Hence

\[
\int_\mathcal{K} |\nabla u_1|^q = \left( \frac{m_0}{2} \right)^q L^2.
\]

For \( u_2 \), we observe that \( \|a - a_0\|_{W^{-1,q}(\mathcal{K})} \) is controlled, for any \( q < \infty \), by \( \|a - a_0\|_{L^\infty([-\frac{L}{2}, \frac{L}{2}])} \).

Therefore, using elliptic regularity for (4.12), \( \|\nabla u_2\|_{L^q(\mathcal{K})} \) is controlled by \( \|a - a_0\|_{L^\infty([-\frac{L}{2}, \frac{L}{2}])} \)

and a scaling argument shows that for any \( q < \infty \)

\[
\int_\mathcal{K} |\nabla u_2|^q \leq C_q L^2 \|a - a_0\|^q_{L^\infty([-\frac{L}{2}, \frac{L}{2}])}.
\]

Finally, in the proof of Lemma 4.16 of [SST] it is shown that for any \( q \in [1, 4] \)

\[
\int_\mathcal{K} |\nabla u_3|^q \leq C_q L^{2-\frac{q}{2}} \|\varphi\|^q_{L^2(\partial\mathcal{K})}.
\]

Combining (4.13), (4.14) and (4.15), we obtain the result.

Lemma 4.5. Let \( m \) be a positive constant and let \( \mathcal{K} \) be a square of center \( 0 \), and sidelength \( 1/m \). Then the solution to

\[
\begin{cases}
-\Delta f = 2\pi(\delta_0 - m\delta_R) & \text{in } \mathcal{K} \\
\frac{\partial f}{\partial \nu} = 0 & \text{on } \partial\mathcal{K}
\end{cases}
\]\n
satisfies

\[
\lim_{\eta \to 0} \left| \int_{\mathcal{K}\setminus B(0, \eta)} |\nabla f|^2 + 2\pi \log \eta \right| = C - \pi \log m.
\]
where $C$ is universal, and for every $1 \leq q < 2$

\[(4.17) \quad \int_{K} |\nabla f|^q \leq C_q m^{q-2},\]

where $C_q$ depends only on $q$.

**Proof.** By scaling we can reduce to the case of $m = 1$. Then, it suffices to observe that $f(z) = -\log |z| + S(z)$ with $S \in W^{1,\infty}(K)$ and scale back.

We note that $W$ as defined in (1.6) still makes sense for vector fields satisfying \( \text{div} \ E = 2\pi(\sum \delta_{x_i} - m) \) as long as \( \text{curl} \ E = 0 \) in $\bigcup B(x_i, \eta_0)$ for some $\eta_0 > 0$. This is the notion we will use repeatedly below.

**Lemma 4.6.** Let $E_1$ and $E_2$ be two vector fields defined in a rectangle $\mathcal{R}$ of the plane which is symmetric with respect to the real axis, and satisfying

\[(4.18) \quad \text{div} \ E_1 = 2\pi(\sum_i \delta_{x_i} - a_1 \delta_R) \quad \text{in} \ \mathcal{R} \]
\[(4.19) \quad \text{div} \ E_2 = a_2 \delta_R \quad \text{in} \ \mathcal{R} \]

and \( \text{curl} \ E_1 \) and \( \text{curl} \ E_2 \) vanish near the $x_i$'s, for some distinct points $x_i \in \mathbb{R}$ and some bounded functions on the real line, $a_1$ and $a_2$. Then, for $q < 2$ and $q'$ its conjugate exponent, we have $E_1 \in L^q(\mathcal{R})$ and $E_2 \in L^{q'}(\mathcal{R})$ and

\[ W(E_1 + E_2, 1_\mathcal{R}) \leq W(E_1, 1_\mathcal{R}) + \|E_1\|_{L^q(\mathcal{R})} \|E_2\|_{L^{q'}(\mathcal{R})} + \frac{1}{2} \|E_2\|_{L^2(\mathcal{R})}^2, \]

where $W$ is still defined as in (1.6).

**Proof.** We have

\[ \int_{\mathcal{R} \setminus \bigcup_i B(x_i, \eta)} |E_1 + E_2|^2 = \int_{\mathcal{R} \setminus \bigcup_i B(x_i, \eta)} |E_1|^2 + |E_2|^2 + 2E_1 \cdot E_2. \]

By Hölder’s inequality we have

\[ \left| \int_{\mathcal{R} \setminus \bigcup_i B(x_i, \eta)} E_1 \cdot E_2 \right| \leq \|E_1\|_{L^q(\mathcal{R})} \|E_2\|_{L^{q'}(\mathcal{R})}. \]

The result easily follows. \(\square\)

### 4.3 Proof of the screening result Proposition 3.4

First we note that, in view of (3.10), if we assume $G \subset A_1$ satisfies the hypothesis of Proposition 3.4 and $0 < \varepsilon < 1$, then there exists $y_0 > 0$ and $R_0 > 0$ such that for all $E \in G$, we have

\[(4.20) \quad \forall R > R_0, \quad \int_{I_R \times \{|y| > y_0\}} |E|^2 < \varepsilon^{10} R. \]

This motivates the following:
Thus in view of (4.20), if
\[ |E|^2 \leq C \varepsilon^{10} \frac{|x| + |y|}{y^2}, \]
where \( C \) is universal.

Proof. Each of the coordinates of \( E \) is harmonic in the half plane \( \mathbb{R}^2_+ = \{ y > 0 \} \) since \( \text{div } E = \text{curl } E = 0 \) there. Therefore \( |E|^2 \) is sub-harmonic. Thus, if \( B(z, |y|/2) \subset \mathbb{R}^2_+ \) then by the maximum principle we have
\[ |E|^2(z) \leq \int_{B(z,|y|/2)} |E|^2. \]
If \( y > 2y_0 \), then \( B(z, |y|/2) \subset [x - \frac{|y|}{2}, x + \frac{|y|}{2}] \times [y_0, +\infty) \subset [-|x| - |y|, |x| + |y|] \times [y_0, +\infty). \) Thus in view of (4.20), if \( |x| + |y| > R_0/2 \) we have
\[ |E|^2(z) \leq \frac{8}{\pi} \varepsilon^{10} \frac{|x| + |y|}{y^2}, \]
and the result follows, by symmetry with respect to the \( x \)-axis. \( \square \)

Lemma 4.7. Let \( E \in A_1 \) satisfy (4.20), where \( 0 < \varepsilon < 1 \). Then, denoting \( z = (x, y) \), if \( |y| > \max(2y_0, R_0) \), we have
\[ |E|^2(z) \leq C \varepsilon^{10} \frac{|x| + |y|}{y^2}, \]
where \( C \) is universal.

Proof. Each of the coordinates of \( E \) is harmonic in the half plane \( \mathbb{R}^2_+ = \{ y > 0 \} \) since \( \text{div } E = \text{curl } E = 0 \) there. Therefore \( |E|^2 \) is sub-harmonic. Thus, if \( B(z, |y|/2) \subset \mathbb{R}^2_+ \) then by the maximum principle we have
\[ |E|^2(z) \leq \int_{B(z,|y|/2)} |E|^2. \]
If \( y > 2y_0 \), then \( B(z, |y|/2) \subset [x - \frac{|y|}{2}, x + \frac{|y|}{2}] \times [y_0, +\infty) \subset [-|x| - |y|, |x| + |y|] \times [y_0, +\infty). \) Thus in view of (4.20), if \( |x| + |y| > R_0/2 \) we have
\[ |E|^2(z) \leq \frac{8}{\pi} \varepsilon^{10} \frac{|x| + |y|}{y^2}, \]
and the result follows, by symmetry with respect to the \( x \)-axis. \( \square \)

Lemma 4.8. Let \( G \) satisfy the assumptions of Proposition 3.4. Then for any \( E \in G \), any \( 0 < \varepsilon < 1/2 \) and any \( R \) large enough depending on \( G, \varepsilon \), we may find \( t \in [R/2 - \varepsilon R, R/2 - \frac{1}{2} \varepsilon R] \) such that
\[ \int_{((-t)\cup(t))\times \mathbb{R}} |E|^2 \leq C \varepsilon^{-1} \]
where \( C \) depends only on \( G, \varepsilon \), and
\[ \lim_{R \to \infty} \frac{1}{2t} W(E, 1_{I_{2t}}) = W(E) \]
uniformly in \( G \).

Proof. Take \( E \in G \) and apply Proposition 4.1 to \( E \) for some fixed \( 0 < \rho < 1/8 \). We obtain a density \( g \) and balls \( B_\rho \). Now, using (4.5) in Lemma 4.2 together with the bound (3.9), we deduce that if \( R \) is large enough depending on \( G \) then for any \( E \in G \) and denoting \( g \) the result of applying Proposition 4.1 to \( E \) we have
\[ \int_{x=R/2-\varepsilon R}^{R/2-\varepsilon R} \int_{\mathbb{R}} (g(x, y) + g(-x, y)) \, dx \, dy \leq C R + \int_{I_R} d \rho \leq C R, \]
where \( C \) depends only on \( G \) and we have used the fact that \( g \geq -C \) everywhere and \( g \geq 0 \) on the set \( \{|y| > 1\} \). Then by using the fact that the radii of the balls in \( B_\rho \) which intersect any given interval of length \( 1 \) is bounded by \( 1/8 \) we deduce that if \( R \) is also large enough depending on \( \varepsilon \), the measure of the set \( A \) of \( x \in [R/2 - \varepsilon R, R/2 - \frac{1}{2} \varepsilon R] \) such that \( \{ x, -x \} \times \mathbb{R} \) does not intersect \( B_\rho \) is bounded below by \( \varepsilon R/4 \). This and (4.23) implies that the set \( T \) of \( t \in A \) such that \( \int_{I_t} (g(t, y) + g(-t, y)) \, dy < C/\varepsilon \) has measure at least \( \varepsilon R/8 \) if \( C \) is chosen
large enough depending on $G$, and (4.1) and the fact that $g = \frac{1}{2}|E|^2$ outside $\mathbb{R} \times [-1, 1]$ imply that (4.21) holds for $t \in T$. Thus

\[(4.24) \quad |\{t \in [R/2 - \varepsilon R, R/2 - \varepsilon R/2] \mid (4.21) \text{ holds}\}| \geq \frac{\varepsilon R}{8}.
\]

For (4.22) we argue as in [SS1], Lemma 4.14. We let $\chi : [0, +\infty) \to \mathbb{R}$ be a monotonic function with compact support and let $\bar{\chi}(x,y) = \chi(|x|)$. First we note that for any Radon measure $\mu$ in $\mathbb{R}^2$ we have

$$\int \bar{\chi} \, d\mu = -\int_{t=0}^{+\infty} \chi'(t)\mu(\bar{I}_t) \, dt = -2\int_{u=0}^{+\infty} \chi'(u/2)\mu(\bar{I}_u) \, du.$$\]

This implies straightforwardly using the definition of $W(E, \chi)$ that

$$2\int_{u=0}^{+\infty} (W(E, 1_{I_u}) - g(\bar{I}_u)) \chi'(u/2) \, dt = \int \bar{\chi} \, d\mu - W(E, \chi).$$

On the other hand, by (4.2) and applying Lemma 4.2 (4.3), if $\chi'$ is supported in $[x, y] \subset [R/2, R]$, then the right-hand side is bounded by $C(|x - y| + R^{3/4}\log^2 R)\|\chi'\|_\infty$ for any $R$ large enough depending on $G$. Given now any $\rho : \mathbb{R}^+ \to \mathbb{R}$ supported in $[x, y] \subset [R/2, R]$ we may consider the positive and negative parts $\rho_+$ and $\rho_-$, and and their primitives $\chi_+$ and $\chi_-$ with compact support, which are monotonic. Applying the above to $\chi_+$ and $\chi_-$ we find

$$\int_{u=0}^{+\infty} (W(E, 1_{I_u}) - g(\bar{I}_u)) \rho(u) \, du \leq C(|x - y| + R^{3/4}\log^2 R)\|\rho\|_\infty.$$

Since this is true for any $\rho$ supported in $[R - 2\varepsilon R, R - \varepsilon R]$, it follows by duality that

\[(4.25) \quad \int_x^y |W(E, 1_{I_u}) - g(\bar{I}_u)| \, du \leq C(|x - y| + R^{3/4}\log^2 R).
\]

Then we divide $[R - 2\varepsilon R, R - \varepsilon R]$ into, say, $[\sqrt{R}]$ intervals $I_1, \ldots, I_{[\sqrt{R}]}$ so that their length is equivalent to $\varepsilon \sqrt{R}$ for large $R$. Then, for large enough $R$, on each such interval (4.25) implies that

$$\int_{I_k} |W(E, 1_{I_u}) - g(\bar{I}_u)| \, du \leq CR^{3/4}\log^2 R.$$

Therefore the set of $u \in I_k$ such that $|W(E, 1_{I_u}) - g(\bar{I}_u)| \leq 8CR^{3/4}\log^2 R$ has measure at least $7|I_k|/8$ and since this is true on each interval, and changing variables, the set of $t \in [R/2 - \varepsilon R, R/2 - \varepsilon R/2]$ such that $|W(E, 1_{I_{\bar{t}}}) - g(\bar{I}_t)| \leq 8CR^{3/4}\log^2 R$ has measure at least $7\varepsilon R/16$. Together with (4.24), this implies the existence of $t \in [R/2 - \varepsilon R, R/2 - \varepsilon R/2]$ such that both (4.21) and (4.22) hold.

We now prove Proposition 3.4. Let $G$ satisfy its hypothesis and choose $0 < \varepsilon < 1$, and $E \in G$. Applying Lemma 4.8 we find that if $R$ is large enough depending on $G, \varepsilon$, then there exists $t \in [R/2 - \varepsilon R, R/2 - \varepsilon R/2]$ such that (4.21) and (4.22) hold. For any such integer $R \in \mathbb{N}$ we may also choose $y_R$ such that

\[(4.26) \quad \varepsilon^3 R < y_R < \varepsilon^{5/2} R.
\]
Finally we choose \( s > t \) such that \( s - t \in [y_R, y_R + 1] \) and \( \frac{R}{2} - s \in \mathbb{N} \), and start constructing the vector field \( E_R \).

- **Step 1: splitting the strip.** We split the strip \( I_R = [-\frac{R}{2}, \frac{R}{2}] \times \mathbb{R} \) into several rectangles and strips (see figure below): let

\[
\begin{align*}
D_0 &= [-t, t] \times [-y_R, y_R] \\
D_+ &= [t, s] \times [-y_R, y_R] \\
D_- &= [-s, -t] \times [-y_R, y_R] \\
D_0^+ &= [s, R/2] \times \mathbb{R} \\
D_0^- &= [-R/2, -s] \times \mathbb{R} \\
D_1 &= [-s, s] \times ([y_R, y_R + R] \cup (-R - y_R, -y_R)) \\
D_\infty &= [-s, s] \times ([R + y_R, +\infty) \cup (-\infty, -R - y_R)).
\end{align*}
\]

First we let \( E_R = E \) in \( D_0 \), \( E_R = 0 \) in \( D_\infty \) and below we are going to define \( E_R \) on each of the other sets.

Recall that from Corollary 1.5 we have that \( E(x, y) \) is the reflection of \( E(x, -y) \) with respect to the line \( \{y = 0\} \). We denote by \( \varphi_+ \) the trace \( E \cdot \vec{n} \) on the right-hand side of \( D_0 \) where \( \vec{n} \) is the outward-pointing normal to \( D_0 \), \( \varphi_- \) the same on the left-hand side, \( \varphi_h \) the trace on the upper side of \( D_0 \) (which by symmetry of the problem with respect to the real axis is equal to that on the lower side). From (4.21) and the Cauchy-Schwarz inequality, we have

\[
\int |\varphi_-|^2 + \int |\varphi_+|^2 \leq \frac{C}{\varepsilon} \quad \int |\varphi_-| + \int |\varphi_+| \leq C \sqrt{\frac{y_R}{\varepsilon}}.
\]

and from Lemma 4.7 and (4.26) we have

\[
\int_{[-t,t]} |\varphi_h|^2 \leq \frac{C\varepsilon^{10}R(R + y_R)}{y_R^2} < C\varepsilon^4 \quad \int_{[-t,t]} |\varphi_h| \leq C\varepsilon^2 \sqrt{R}.
\]
In addition, integrating the relation \( \text{div} \ E = 2\pi(\sum_{p \in \Lambda} \delta_p - \delta_R) \) over \( D_0 \) gives

\[
\frac{1}{2\pi} \left( \int \varphi_+ + \int \varphi_- + 2 \int \varphi_h \right) = \nu([-t, t]) - 2t.
\]

- **Step 2:** defining \( E_R \) in \( D_+ \). We first define \( \varphi_0 := E_R \cdot \vec{\nu} \) on the boundary \( \partial D_+ \), where \( \vec{\nu} \) is the outward normal to \( D_+ \). For a certain constant \( \varphi_h^+ \) to be chosen later we let

\[
\varphi_0 = \begin{cases} 
-\varphi_+ & \text{on } \partial D_+ \cap \partial D_0, \\
0 & \text{on } \partial D_+ \cap \{x = s\}, \\
\varphi_h^+ & \text{on } \partial D_+ \cap \{y = \pm y_R\}.
\end{cases}
\]

Then, inside \( D_+ \), we let \( E_R = E_1 + E_2 \), where

\[
\begin{cases} 
\text{div} \ E_1 = 2\pi \left( \sum_{i=1}^{n_+} \delta_{x_i} - m_+ \delta_R \right) & \text{in } D_+, \\
E_1 \cdot \nu = 0 & \text{on } \partial D_+.
\end{cases}
\]

\[
\begin{cases} 
\text{div} \ E_2 = 2\pi(m_+ - 1) \delta_R & \text{in } D_+, \\
E_2 \cdot \nu = \varphi_0 & \text{on } \partial D_+.
\end{cases}
\]

Here, \( n_+ \) is an integer and \( m_+ \) a real number which are defined by

\[
n_+ = \left( s - t \right) - \frac{1}{2\pi} \left( \int \varphi_+ + \int \varphi_h \right), \quad m_+ = \frac{n_+}{s-t},
\]

and for \( 1 \leq i \leq n_+ \) we have let

\[
x_i = t + \frac{s-t}{n_+} \left( i + \frac{1}{2} \right).
\]

Note that the above equations do not yield a uniquely defined \( E_1 \) and \( E_2 \). For (4.31) to make sense we need \( n_+ \geq 0 \) while for (4.32) to have a solution we need to have

\[
2\pi(n_+ - (s-t)) = \int \varphi_0 = 2(s-t)\varphi_h^+ - \int \varphi_+,
\]

which we take as the definition of \( \varphi_h^+ \). The fact that \( n_+ \geq 0 \) follows for \( R \) large enough depending on \( \varepsilon \) from the fact that \( s-t \geq \varepsilon^3 R \) and (4.27), (4.28).

- **Step 3:** Estimating the energy of \( E_R \) in \( D_+, D_- \). To compute the renormalized energy \( W(E_R, 1_{D_+}) \) we need to define \( E_1 \) and \( E_2 \) more precisely. For \( E_1 \) let us consider \( n_+ \) identical squares \( \{K_i\}_{i=1}^{n_+} \) with sidelength \( \frac{s-t}{n_+} = \frac{1}{m_+} \), sides parallel to the axes and such that \( K_i \) is centered at \( x_i \). We define \( E_1 \) restricted to \( K_i \) by applying Lemma 4.5 with \( m = m_+ \) and taking \( E_1 = -\nabla f \), while outside \( \cup \limits_{1}^{n_+} K_i \) we let \( E_1 = 0 \). Since from Lemma 4.5 we have \( E_{1|K_i} \cdot \vec{\nu} = 0 \) on \( \partial K_i \), it holds that \( \text{div} \ E_1 = \sum_i \text{div} \ E_{1|K_i} \) and therefore (4.31) is satisfied by
$E_1$. On the other hand, still from Lemma \[4.5\] we obtain by summing the bounds \[4.10\] and \[4.17\] on the $n_+$ rectangles

\[
\lim_{\eta \to 0} \left| \int_{D_+ \cup B(x, \eta)} |E_1|^2 + 2\pi \log \eta \right| \leq n_+ \left( C - \pi \log m_+ \right),
\]

and

\[
\forall 1 < q < 2, \quad \int_{D_+} |E_1|^q \leq C_q n_+.
\]

We define $E_2$ by applying Lemma \[4.4\] in $D_+$, hence with $L = s - t$, with the boundary data $\varphi_0$ and constant weight $m = 2\pi(m_+ - 1)$. From \[4.30\] and \[4.34\] the hypothesis $\int m(x) \, dx = -\int_{D_+} \varphi$ is satisfied and applying the lemma yields

\[
\forall 2 \leq q < 4, \quad \int_{D_+} |E_2|^q \leq C_q \left( |m_+ - 1|^q (s-t)^2 + (s-t)^{2-\frac{q}{2}} \| \varphi_0 \|_{L^2(\partial D_+)}^q \right).
\]

Using Lemma \[4.6\] we have, recalling that $E_R := E_1 + E_2$ in $D_+$ and using \[4.35\], \[4.36\], \[4.37\]:

\[
W(E_R, 1_{D_+}) \leq C n_+ + C_q n_+^{1/q} \left( |m_+ - 1|^q(s-t)^{2/q} + (s-t)^{2/q-1/2} \| \varphi_0 \|_{L^2(\partial D_+)}^q \right) + C \left( |m_+ - 1|^2(s-t)^2 + (s-t) \| \varphi_0 \|_{L^2(\partial D_+)}^2 \right),
\]

for any $1 < q < 2$ such that the conjugate exponent $q'$ is less than 4. Now, from \[4.33\], we have using \[4.27\], \[4.28\], \[4.26\] and the fact that $y_R \leq s - t \leq y_R + 1$ we deduce that

\[
|m_+ - 1| \leq C \left( 1 + \frac{1}{\sqrt{\varepsilon y_R}} \right),
\]

and thus for $R$ large enough depending on $\varepsilon$, since $s - t \simeq y_R$ for large $R$ and using \[4.26\] again as well as $\varepsilon < 1$,

\[
n_+ \leq C \varepsilon^{5/2} R, \quad |m_+ - 1| \leq \frac{C}{\varepsilon^2 \sqrt{R}}.
\]

Moreover, from \[4.33\], \[4.34\] and \[4.28\] we find

\[
|\varphi_h^+| \leq \frac{1}{2(s-t)} \left( 2\pi + \int \varphi_h \right) \leq C \frac{\varepsilon^2 \sqrt{R}}{y_R} \leq \frac{C}{\varepsilon \sqrt{R}}.
\]

Then, in view of \[4.30\], \[4.27\],

\[
\| \varphi_0 \|_{L^2(\partial D_+)} \leq \frac{C}{\varepsilon}.
\]

Now we fix for instance $q = 3/2$, so that $q' = 3$ and combining the above with \[4.38\], \[4.40\] we find that for $R$ large enough depending on $\varepsilon$, and denoting by $C_\varepsilon$ a positive constant depending on $\varepsilon$ but independent of $R$,

\[
W(E_R, 1_{D_+}) \leq C \varepsilon^{5/2} R + C_\varepsilon R^{2/3} \times R^{2/3-1/2} + C_\varepsilon^{3/2} R.
\]
Thus for \( R \) large enough depending on \( \varepsilon \) we find that

\[
(4.42) \quad W(E_R, 1_{D_+}) \leq C\varepsilon^{3/2}R.
\]

An almost symmetric computation yields the same bound for \( W(E_R, 1_{D_-}) \). It suffices to let

\[
(4.43) \quad n_- = 2(s-t) - \frac{1}{2\pi} \left( \int \varphi_+ + \int \varphi_- + 2 \int \varphi_h \right) - n_+,
\]

and carry on the proof with minuses instead of pluses. The fact that \( n_- \) is an integer follows from the identity \( (4.29) \), the fact that \( 2s \) is an integer and the fact that \( \nu([-t, t]) \in \mathbb{N} \). Moreover the definition of \( n_- \) implies that

\[
n_- = \left[ (s-t) - \frac{1}{2\pi} \left( \int \varphi_- + \int \varphi_h \right) \right], \text{ or } n_- = \left[ (s-t) - \frac{1}{2\pi} \left( \int \varphi_- + \int \varphi_h \right) \right] + 1,
\]

hence \( n_- \) is positive if \( R \) is large enough and \( (4.39) \) holds for \( n_- \) as well. The rest of the proof is unchanged.

- **Step 4:** defining \( E_R \) over \( D_1 \). We need only consider the intersection of \( D_1 \) with the upper half-plane (and then extend by reflection). We let \( \varphi_0 \) be equal to \( -\varphi_h, -\varphi_h, -\varphi_h \), respectively, on the intersection of \( \partial D_1 \) with \( \partial D_- \), \( \partial D_0 \), \( \partial D_+ \), respectively. On the remaining three sides of \( \partial D_1 \) we let \( \varphi_0 = 0 \). From \( (4.34) \) and its equivalent for \( n_- \) and the fact that \( n_\pm = (s-t)m_\pm \) we have

\[
- \int \varphi_0 = \pi(n_+ + n_- - 2(s-t)) + \frac{1}{2} \int \varphi_+ + \frac{1}{2} \int \varphi_- + \int \varphi_h,
\]

and then \( (4.33), (4.43) \) imply that the integral of \( \varphi_0 \) is zero.

Thus there exists a harmonic function \( u \) in \( D_1 \) with normal derivative \( \varphi_0 \) on \( \partial D_1 \), we let \( E_R = \nabla u \) on \( D_1 \). Using \( (4.41) \) — which holds for \( \varphi_h \) as well — and \( (4.28) \) we have \( C\varepsilon^4 + C\varepsilon^4 R/yR \) hence

\[
(4.44) \quad \|\varphi_0\|_{L^2(\partial D_1)}^2 \leq C \left( \varepsilon^4 + \varepsilon^4 R/yR \right) \leq C\varepsilon.
\]

Then standard elliptic estimates yield as in Lemma \( 4.4 \) that

\[
(4.45) \quad \int_{D_1} |E_R|^2 \leq CR\|\varphi_0\|_{L^2}^2 \leq CR\varepsilon,
\]

where we have concluded by \( (4.26) \).

- **Step 5:** defining \( E_R \) over \( D_+^\pm \). The construction will be entirely parallel in \( D_+^\pm \). We note that \( D_+^\pm \) is an infinite strip of width \( R/2 - s \) and we have chosen \( s \) so that this quantity is an integer. We can thus split this strip into exactly \( R/2 - s \) strips of width 1. On each of these strips we define \( E_R \) to be equal to 0 for \( |y| \geq \frac{1}{2} \) and for \( |y| \leq \frac{1}{2} \) (i.e. in a square of sidelength 1) we choose it to be \( \nabla f \) where \( f \) is given by Lemma \( 4.5 \) applied with \( m = 1 \). Since \( E_R \cdot \vec{v} = 0 \) on the boundary of each of these squares, no divergence is created at the interfaces, and the resulting \( E_R \) satisfies \( \text{div } E_R = 2\pi(\sum_{p \in \Lambda} \delta_p - \delta_\overline{p}) \). In addition in view of \( (4.17) \) the cost in energy is equal to a constant times the number of strips, that is

\[
(4.46) \quad W(E_R, 1_{D_+}) \leq C|R/2 - s| \leq C\varepsilon R.
\]
- Conclusion. We have now defined $E_R$ over the whole strip $I_R$. It satisfies items ii) and iii).

The main point is again that as long as $E \cdot \tilde{v}$ is continuous across an interface it creates no singular divergence there. Combining (4.40) with (4.22), (4.42) and (4.45), $E_R$ also satisfies (3.11). This concludes the proof of Proposition 3.4.

5 Proof of Proposition 3.1

The construction consists of the following. First we select a finite set of vector fields $J_1, \ldots, J_N$ ($N$ will depend on $\varepsilon$) which will represent the probability $P(x, E)$ with respect to its $E$-dependence, and whose renormalized energies are well-controlled. Since $P$ is $T_{x(\varepsilon)}$-invariant, we need it to be well-approximated by measures supported on the orbits of the $J_i$’s under translations. It is also during this approximation process that we manage to select the $J_i$’s as belonging to a part of the support of $P$ of almost full measure for which the extra assumption (3.10) holds and the screening can be made. Secondly, we work in blown-up coordinates and split the region $\Sigma'$ (of order $n$ size) into many intervals, and then select the proportion of the intervals that corresponds to the weight that the orbit of each $J_i$ carries in the approximation of $P$. In these rectangles we paste a (translated) copy of $J_i$ at the appropriate scale (approximating the density $m_0'$ by a piecewise constant one and controlling errors).

To conclude the proof of Proposition 3.1 we collect all of the estimates on the constructed vector field to show that its energy $w_n$ is bounded above in terms of $\int W \, dP$ and that the probability measures associated to the construction have remained close to $P$.

5.1 Abstract preliminaries

We repeat here the definitions of distances that we used in [SS2]. First we choose distances which metrize the topologies of $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $\mathcal{B}(X)$, the set of finite Borel measures on $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. For $E_1, E_2 \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ we let

$$d_p(E_1, E_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|E_1 - E_2\|_{L^p(B(0,k))}}{1 + \|E_1 - E_2\|_{L^p(B(0,k))}},$$

and on $X$ we use the product of the Euclidean distance on $\Sigma$ and $d_p$, which we denote $d_X$. On $\mathcal{B}(X)$ we define a distance by choosing a sequence of bounded continuous functions $\{\varphi_k\}_k$ which is dense in $C_b(X)$ and we let, for any $\mu_1, \mu_2 \in \mathcal{B}(X)$,

$$d_{\mathcal{B}}(\mu_1, \mu_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\langle \varphi_k, \mu_1 - \mu_2 \rangle|}{1 + |\langle \varphi_k, \mu_1 - \mu_2 \rangle|},$$

where we have used the notation $\langle \varphi, \mu \rangle = \int \varphi \, d\mu$.

We will use the following general facts, whose proofs are in [SS2, Sec. 7.1].

**Lemma 5.1.** For any $\varepsilon > 0$ there exists $\eta_0 > 0$ such that if $P, Q \in \mathcal{B}(X)$ and $\|P - Q\| < \eta_0$, then $d(P, Q) < \varepsilon$. Here $\|P - Q\|$ denotes the total variation of the signed measure $P - Q$, i.e. the supremum of $\langle \varphi, P - Q \rangle$ over measurable functions $\varphi$ such that $|\varphi| \leq 1$.

In particular, if $P = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ and $Q = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$, with $\sum_{i} |\alpha_i - \beta_i| < \eta_0$, then $d_{\mathcal{B}}(P, Q) < \varepsilon$. 

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Lemma 5.2. Let $K \subset X$ be compact. For any $\varepsilon > 0$ there exists $\eta_1 > 0$ such that if $x \in K, y \in X$ and $d_X(x, y) < \eta_1$ then $d_B(\delta_x, \delta_y) < \varepsilon$.

Lemma 5.3. Let $0 < \varepsilon < 1$. If $\mu$ is a probability measure on a set $A$ and $f, g : A \to X$ are measurable and such that $d_B(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for every $x \in A$, then

$$d_B(f \# \mu, g \# \mu) < C\varepsilon(|\log \varepsilon| + 1)$$

where $\#$ denotes the push-forward of a measure.

The next lemma shows how to, given a translation-invariant probability measure $\tilde{P}$ on $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, select a good subset $G_\varepsilon$ and vector fields $J_i$ of $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ to approximate it. It is essentially borrowed from [SS2] except it contains in addition the argument that ensures that we may choose $G_\varepsilon$ to satisfy the assumption (3.10) needed for the screening.

Lemma 5.4. Let $\tilde{P}$ be a translation invariant measure on $X$ such that, $\tilde{P}$-a.e., $E$ is in $A_1$ and satisfies $W(E) < +\infty$. Then, for any $\varepsilon > 0$ there exists a compact $G_\varepsilon \subset L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that

i) Letting $0 < \eta_0$ be as in Lemma 5.1 we have

$$\tilde{P}(\Sigma \times G_\varepsilon^c) < \min(\eta_0^2, \eta_0 \varepsilon).$$

ii) The convergence (1.8) is uniform with respect to $E \in G_\varepsilon$.

iii) Writing $\text{div } E = 2\pi(\nu_E - 1)$, both $W(E)$ and $\nu_E(I_R)/R$ are bounded uniformly with respect to $E \in G_\varepsilon$ and $R > 1$.

iv) Uniformly with respect to $E \in G_\varepsilon$ we have

$$\lim_{y_0 \to +\infty} \lim_{R \to +\infty} \int_{I_R} \int_{|y| > y_0} |E|^2 = 0$$

Moreover, (5.1) implies that for any $R_\varepsilon > 1$ there exists a compact subset $H_\varepsilon \subset G_\varepsilon$ such that

v) For every $E \in H_\varepsilon$, there exists $\Gamma(E) \subset I_{\bar{m}R_\varepsilon}$ such that

$$|\Gamma(E)| < CR_\varepsilon \eta_0 \text{ and } \lambda \notin \Gamma(E) \Rightarrow \theta_\lambda E \in G_\varepsilon.$$

vi) We have

$$d_B(\bar{P}, P') < C\varepsilon(|\log \varepsilon| + 1),$$

where

$$P' = \int_{\Sigma \times H_\varepsilon} \int_{m_0(x)I_{R_\varepsilon}} \delta_x \otimes \delta_{m_0(x)\theta_\lambda E} d\lambda d\bar{P}(x, E)$$

and

$$\bar{P} = \int_{\Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)} \delta_x \otimes \delta_{m_0(x)E} d\tilde{P}(x, E).$$
Finally, there exists a partition of $H_\varepsilon$ into $\bigcup_{i=1}^{N_\varepsilon} H^i_\varepsilon$ satisfying $\text{diam } (H^i_\varepsilon) < \eta_3$, where $\eta_3$ is such that

\begin{equation}
E \in H_\varepsilon, \ d_p(E, E') < \eta_3, \ m \in (0, \bar{m}], \ \lambda \in \bar{m}I_{R_\varepsilon} \setminus \Gamma(E) \implies d_B(\delta_{\sigma_0 E}, \delta_{\sigma_0 E'}) < \varepsilon; \end{equation}

and there exists for all $i, E_i \in H^i_\varepsilon$ such that

\begin{equation}
W(E_i) < \inf_{H^i_\varepsilon} W + \varepsilon. \end{equation}

\textbf{Proof.} The lemma is almost identical to Lemma 7.6 in [SS2], except for item iv). The proof in [SS2] is as follows: First one proves that there exists $G_\varepsilon$ satisfying items ii) and iii) with $\tilde{P}(\Sigma \times G_\varepsilon)$ arbitrarily small, in particular one can choose it so that (5.1) is satisfied. Then one deduces from (5.1) the existence, for any $R_\varepsilon > 1$, of a compact subset $H_\varepsilon \subset G_\varepsilon$ satisfying the remaining properties. The only difference here is that we must check that there exists $G_\varepsilon$ with $\tilde{P}(\Sigma \times G_\varepsilon)$ arbitrarily small satisfying not only items ii) and iii), but iv) as well. Then, the proof of the existence $H_\varepsilon \subset G_\varepsilon$ satisfying the remaining properties is exactly as in [SS2].

Of course, by intersecting sets, it is equivalent to prove that ii), iii), and iv) can be satisfied simultaneously or separately, on a set of measure arbitrarily close to full. The proof in [SS2] shows that this is possible for ii) or iii), it remains to check it for iv). For this we consider $G_n = \{E \mid W(E) < n\}$. Then $G_n$ is a translation-invariant set since $W$ is a translation-invariant function, and therefore by the multiparameter ergodic theorem (as in [Be]), and since $\tilde{P}$ is translation-invariant, we have

\begin{equation}
\int_{\Sigma \times G_n} \left( \int_{[-1,1] \times \{|y| > y_0\}} |E|^2 \right) d\tilde{P}(x, E) = \int_{\Sigma \times G_n} \left( \lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R} \times \{|y| > y_0\}} \chi_R |E|^2 \right) d\tilde{P}(x, E),
\end{equation}

where $\chi_R = 1_{I_R} * 1_{[-1,1]}$. Then, using Lemma 4.2 and using the fact that the $g$ there was defined in Proposition 4.1 hence is equal to $\frac{1}{2} |E|^2$ on $\mathbb{R} \times \{|y| > 1\}$ we deduce from (4.5) and the fact that $g$ is bounded below by a constant independent of $E$ that

\begin{equation}
\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R} \times \{|y| > y_0\}} \chi_R |E|^2 \leq C(1 + W(E)) \leq Cn
\end{equation}

holds for every $E \in G_n$ with $n \geq 1$.

It follows that for every fixed $n \geq 1$ the family of functions

\begin{equation}
\{ \varphi_{y_0} : (x, E) \mapsto \int_{[-1,1] \times \{|y| > y_0\}} |E|^2 \}_{y_0 > 1}
\end{equation}

decrees to 0 on $\Sigma \times G_n$ for $y_0 \to +\infty$, and is dominated by the bounded, hence $\tilde{P}$-integrable, function $\varphi_1$. Lebesgue's theorem then implies that their integrals on $\Sigma \times G_n$ converge to 0, hence in view of (5.7) that

\begin{equation}
\int_{\Sigma \times G_n} \left( \lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R} \times \{|y| > y_0\}} \chi_R |E|^2 \right) d\tilde{P}(x, E),
\end{equation}

tends to 0 as $y_0 \to +\infty$. Fatou's lemma then implies that (5.2) holds for $\tilde{P}$-almost every $(x, E) \in \Sigma \times G_n$.

Since $W(E) < +\infty$ holds for $\tilde{P}$-a.e. $(x, E)$, we know that $\tilde{P}(\Sigma \times G_n) \to 1$ as $n \to +\infty$ therefore the measure of $\Sigma \times G_n$ can be made arbitrarily close to 1, and then Egoroff's theorem implies that by restricting $G_n$ we can in addition require the convergence in (5.2) to be uniform. \hfill \Box
5.2 Construction

In what follows $\Sigma' = n\Sigma$, $m_0'(x) = m_0(x/n)$: we work in blown-up coordinates. In view of assumption (1.16), we may assume without loss of generality that $\Sigma$ is made of one closed interval $[a, b]$ (it is then immediate to generalize the construction to the case of a finite union of intervals). In that case $\Sigma' = [na, nb]$. Let $m > 0$ be a small parameter. For any integer $n$ we choose real numbers $a_n$ and $b_n$ (depending on $m$) as follows: Let $a_n$ be the smallest number and $b_n$ the largest such that

\[ a_n \geq na + \frac{nm}{\gamma^2} \quad b_n \leq nb - \frac{nm}{\gamma^2} \tag{5.8} \]

\[ \int_{na}^{a_n} m_0'(x) \, dx \in \mathbb{N} \tag{5.9} \]

\[ \int_{b_n}^{nb} m_0'(x) \, dx \in \mathbb{N} \tag{5.10} \]

\[ \int_{a_n}^{b_n} m_0'(x) \, dx \in q_\varepsilon \mathbb{N} \tag{5.11} \]

where $q_\varepsilon$ is an integer, to be chosen later, and $\gamma$ is the constant in (1.17). By (5.8) and assumption (1.16), we are sure to have $m_0' \geq m$ in $\Sigma'_m := [a_n, b_n]$. This fact also ensures that

\[ |a_n - na| \leq \frac{nm}{\gamma^2} + \frac{1}{m} \quad |b_n - nb| \leq \frac{nm}{\gamma^2} + q_\varepsilon m. \tag{5.12} \]

We also denote $\Sigma_{\overline{m}} := \frac{1}{n} \Sigma'_m$.

Let $P$ be a probability on $\Sigma \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is as in the statement of Proposition 3.1. Our goal is to construct a vector field $E_n$ whose $W$ energy is close to $\int W dP$ and such that the associated $P_n$ (defined as the push-forward of the normalized Lebesgue measure on $\Sigma$ by $x \mapsto (x, E(ax + \cdot))$) well approximates $P$, a probability measure as in the statement of Proposition 3.1.

In $[na, a_n]$ and $[b_n, nb]$, we approximate $m_0'(x) \, dx$ by a sum of Dirac masses at points appropriately spaced, and build an associated $E_n$, whose contribution to the energy will shown to be negligible as $\overline{m} \to 0$. We leave this part for the end.

For now we turn to $[a_n, b_n]$, where we will do a more sophisticated construction, approaching $P$ via Lemma 5.4 and using Proposition 3.4. The idea of the construction is to split the interval $[a_n, b_n]$ into intervals of width $\sim q_\varepsilon R_\varepsilon$, where $q_\varepsilon$ is an integer and $R_\varepsilon$ a number, both chosen large enough, and then paste in each of these intervals a large number of copies of the (rescaled) truncations of the $J_i$’s provided by Proposition 3.4 in a proportion following that of $P$.

**Step 1: Reduction to a density bounded below.** We have

\[ P = \int \delta_x \otimes \delta_{a_{m_0(x)}E} \, dQ(x, E), \quad \text{where} \quad Q = \int \delta_x \otimes \delta_{a_{1/m_0(x)}E} \, dP(x, E). \]

Moreover, since the first marginal of $P$ is the normalized Lebesgue measure on $\Sigma$ and since $|\Sigma_{\overline{m}}| \sim |\Sigma|$ as $\overline{m} \to 0$, we have

\[ \lim_{\overline{m} \to 0} d_B(P, \overline{P}) = 0, \tag{5.13} \]
where \( \tilde{P} \) is defined by

\[
(5.14) \quad \tilde{P} = \int \delta_{x} \otimes \delta_{\sigma_{m_{0}(\cdot)}E} \, d\tilde{P}(x, E), \quad \text{with} \quad \tilde{P} = \int_{\Sigma_{\text{m}} \times L^{P}_{\text{loc}}(\mathbb{R}^{2}, \mathbb{R}^{2})} \delta_{x} \otimes \delta_{\sigma_{1/m_{0}(\cdot)}E} \, dP(x, E).
\]

Clearly \( \tilde{P} \) is \( T_{\lambda(x)} \)-invariant since \( P \) is, and in particular it is translation-invariant. In addition, for \( \tilde{P}\text{-a.e.} \ (x, E) \), we have \( m_{0}(x) \in [m, \overline{m}] \), a situation similar to [SS2] where the density was assumed to be bounded below.

- **Step 2: Choice of the parameters.** Let \( 0 < \varepsilon < 1 \). We apply Lemma 5.4, which provides us with a compact set \( G_{\varepsilon} \subset L^{P}_{\text{loc}}(\mathbb{R}^{2}, \mathbb{R}^{2}) \). We may then apply Proposition 3.4 with the same \( \varepsilon \) to \( G_{\varepsilon} \). It provides us with \( R_{0} > 0 \) such that for any integer \( R > R_{0} \), and roughly speaking, any \( E \in G_{\varepsilon} \) may be truncated on \( \tilde{I}_{R} \) with an \( \varepsilon \)-cost in energy. Applying Lemma 5.2 on the compact set \( \{ \sigma_{m}E : m \in [m, \overline{m}], E \in G_{\varepsilon} \} \), there exists \( \eta_{1} > 0 \) such that

\[
(5.15) \quad m \in [m, \overline{m}], \ E \in G_{\varepsilon}, E' \in L^{P}_{\text{loc}}(\mathbb{R}^{2}, \mathbb{R}^{2}) \, \text{and} \, d_{P}(E, E') < \eta_{1} \implies d_{G}(\delta_{\sigma_{m}E}, \delta_{\sigma_{n}E'}) < \varepsilon.
\]

Then we define \( R_{\varepsilon} \) to be such that \( mR_{\varepsilon} > R_{0} \) and such that for any \( E, E' \in L^{P}_{\text{loc}}(\mathbb{R}^{2}, \mathbb{R}^{2}) \),

\[
(5.16) \quad E = E' \, \text{on} \, \tilde{I}_{mR_{\varepsilon}} \implies d_{P}(E, E') < \eta_{1}.
\]

Going back to Lemma 5.4, we deduce the existence of \( H_{\varepsilon} \subset G_{\varepsilon} \), of \( N_{\varepsilon} \in \mathbb{N} \) and of \( \{ E_{i} \}_{1 \leq i \leq N_{\varepsilon}} \) satisfying (5.3), (5.4), (5.5) and (5.6).

Finally, we choose \( q_{\varepsilon} \in \mathbb{N} \) sufficiently large so that

\[
(5.17) \quad \frac{N_{\varepsilon}}{q_{\varepsilon}} < \eta_{0}, \quad \frac{N_{\varepsilon}}{q_{\varepsilon}^{2}} \times \max_{m \in [m, \overline{m}]} W(\sigma_{m}E_{i}) < \varepsilon.
\]

- **Step 3: construction in \([a_{n}, b_{n}]\).** We start by splitting this interval into subintervals with integer “charge”. This is done by induction by letting \( t_{0} = a_{n} \) and, \( t_{k} \) being given, letting \( t_{k+1} \) be the smallest \( t \geq t_{k} + q_{\varepsilon}R_{\varepsilon} \) such that \( \int_{t_{k}-1}^{t_{k}} m_{0}'(x) \, dx \in q_{\varepsilon}\mathbb{N} \). By (5.11) there exists \( K \in \mathbb{N} \) such that \( t_{K} = b_{n} \), and

\[
(5.18) \quad K \leq \frac{b_{n} - a_{n}}{q_{\varepsilon}R_{\varepsilon}} < \frac{n(b - a)}{q_{\varepsilon}R_{\varepsilon}}.
\]

Since \( m_{0}' \geq m \) in \([a_{n}, b_{n}]\), it is clear that \( t_{k} - (q_{\varepsilon}R_{\varepsilon} + t_{k-1}) \leq q_{\varepsilon}m^{-1} \). To summarize and letting \( I_{k} = [t_{k-1}, t_{k}] \), we thus have

\[
(5.19) \quad |I_{k}| \in \left[q_{\varepsilon}R_{\varepsilon}, q_{\varepsilon}(R_{\varepsilon} + m^{-1})\right], \quad \int_{I_{k}} m_{0}'(x) \, dx \in q_{\varepsilon}\mathbb{N}.
\]

In each \( I_{k} \) we will paste \( n_{k,i} \) copies of a rescaled version of \( E_{i} \), where

\[
n_{i,k} = \left\lfloor \frac{q_{\varepsilon}(b_{n} - a_{n})}{|I_{k}|} \right\rfloor \quad \text{and} \quad p_{i,k} = \tilde{P} \left( \frac{1}{n} \overline{I}_{k} \times H_{\varepsilon}^{i} \right),
\]

\( [\cdot] \) denoting the integer part of a number. Because the first marginal of \( \tilde{P} \) is the normalized Lebesgue measure on \( \Sigma_{m} \) and since \([a_{n}, b_{n}] \subset \Sigma_{m} \subset [a, b]\), and \( \cup_{k} I_{k} = [a_{n}, b_{n}] \), we have that

\[
\frac{|I_{k}|}{n(b - a)} \leq \sum_{i=1}^{N_{\varepsilon}} p_{i,k} \leq \frac{|I_{k}|}{b_{n} - a_{n}},
\]

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and therefore $\sum_{i=1}^{N_{q}} n_{i,k} \leq q_{\varepsilon}$. Also, using in particular (5.17),

$$\sum_{i,k} \left| \frac{|I_{k}|}{q_{\varepsilon}(b_{n} - a_{n})} n_{i,k} - p_{i,k} \right| \leq \frac{N_{\varepsilon}}{q_{\varepsilon}} < \eta_{0}.$$ 

We divide $I_{k}$ into $q_{\varepsilon}$ subintervals with disjoint interiors, all having the same width $\in [R_{\varepsilon}, R_{\varepsilon} + m^{-1}]$. Then for each $1 \leq i \leq N_{\varepsilon}$ we let $I_{i,k}$ denote a family consisting of $n_{i,k}$ of these intervals. This doesn’t necessarily exhaust $I_{k}$ since $\sum_{i=1}^{N_{q}} n_{i,k} \leq q_{\varepsilon}$ so we let $n_{0,k} = q_{\varepsilon} - \sum_{i=1}^{N_{q}} n_{i,k}.

We define $m_{k}$ to be the average of $m_{0}'$ over $I_{k}$. From (5.19) we have $m_{k}|I_{k}| \in q_{\varepsilon}N$ hence for each $I \in I_{i,k}$ we have $R := |m_{k}| \in N$, and $R \in [m_{k}R_{\varepsilon}, m_{k}(R_{\varepsilon} + m^{-1})]$. We then apply Proposition 3.4 in $I_{R}$ to the vector field $E_{i,l}$ which yields a “truncated” vector field $E_{i,l}$ defined in $I_{R}$, where $\bar{R} = |m_{k}|I_{k}$. If $I \in I_{0,k}$ we apply the same procedure with an arbitrary current $E_{0} \in A_{1}$ fixed with respect to all the parameters of the construction.

We then set

$$E_{n}^{(1)}(x) = \sigma_{1/m_{k}} E_{i,l}(x_{l} + \cdot)$$

on each interval $I \in I_{i,k}$, where $x_{l}$ is the center of $I$. The next step is to rectify the weight in $E_{n}^{(1)}$. For this we let $R_{k}$ be the square $I_{k} \times (-|I_{k}|/2, |I_{k}|/2)$ and let $H_{k}$ be the solution to

$$\begin{cases}
-\Delta H_{k} = 2\pi(m_{0}' - m_{k}) & \text{in } R_{k} \\
\frac{\partial H_{k}}{\partial \nu} = 0 & \text{on } \partial R_{k}.
\end{cases}$$

From Lemma 4.4 applied with $\varphi$ and $m_{0}$ equal to zero, and using the fact that $m_{0}$ is assumed to belong to $C_{0}^{1/2}$, we have for any $q \in [1, 4]$,

$$\int_{R_{k}} |\nabla H_{k}|^{q} \leq C_{q}|I_{k}|^{2} \|m_{0}' - m_{k}\|^{q}_{L^{\infty}(I_{k})} \leq C_{q}|I_{k}|^{2} \|m_{0}\|^{q}_{C_{0,1/2}} n^{-\frac{q}{2}}.$$ 

We then define

$$E_{n}^{(2)} = \begin{cases}
\nabla H_{k} & \text{in } R_{k} \\
0 & \text{in } I_{k} \setminus R_{k}
\end{cases}$$

$$E_{n} = E_{n}^{(1)} + E_{n}^{(2)} \text{ in } I_{k}.$$ 

Using Lemma 4.6 and (5.20) we deduce using (5.19) that

$$W(E_{n}, 1_{I_{k}}) \leq W(E_{n}^{(1)}, 1_{I_{k}}) + o_{n}(1), \text{ as } n \to \infty,$$

where $o_{n}(1)$ tends to zero as $n \to \infty$ and depends on $\varepsilon, m > 0$ but not the interval $I_{k}$ we are considering. Summing (5.20) for $1 \leq k \leq K$ and in view of (5.18) we find that for any $q \in [1, 4]$

$$\int_{[a_{n}, b_{n}] \times \mathbb{R}} |E_{n}^{(1)} - E_{n}|^{q} \leq C_{q, \varepsilon, m} n^{1-\frac{q}{2}}.$$ 

On the other hand, in view of the construction and the result of Proposition 3.4 we have

$$W(E_{n}^{(1)}, 1_{I_{k}}) \leq |I_{k}| \left( \sum_{i=0}^{K} \frac{n_{i,K}}{q_{\varepsilon}} W(\sigma_{m_{K}} J_{i}) + C \varepsilon \right).$$
Then, following the exact same arguments as in [SS2, Sec. 7] which we do not reproduce here (the only difference is that the rescaling factors $\sqrt{n}$ there should be replaced by $n$), thanks to (5.16)–(5.15)–(5.17) we find that we can choose $C_1$ in (5.17) such that

$$d_B(\bar{P}, P') < C\varepsilon(|\log \varepsilon| + 1)$$

where

$$P' = \frac{1}{|\Sigma_m|} \sum_{k=1}^{K} \int_I \delta_{x_k} \otimes \delta_{\theta_1} E_n^{(1)} d\lambda$$

and stands for $P^{(6)}$ in [SS2, Sec. 7]. Also, and again as in [SS2], since (5.22) holds, and from Lemma 5.2, we may replace $E_n^{(1)}$ with $E_n$ at a negligible cost, more precisely for any large enough $n$ we have

$$d_B(\bar{P}, P'') < C\varepsilon(|\log \varepsilon| + 1)$$

where

$$P'' = \frac{1}{|\Sigma_m|} \sum_{k=1}^{K} \int_I \delta_{x_k} \otimes \delta_{\theta_1} E_n d\lambda.$$

- **Step 3: construction in $[b_n, nb]$.** The construction in $[na, an]$ is exactly the same hence will be omitted. We claim that there exists $E_n$ defined in $[b_n, nb] \times \mathbb{R}$ such that

$$\begin{cases} 
\text{div } E_n = 2\pi \left( \sum_i \delta_{x_i} - m_0' \delta_{\mathbb{R}} \right) & \text{in } [b_n, nb] \times \mathbb{R} \\
E_n \cdot \nu = 0 & \text{on } \partial([b_n, nb] \times \mathbb{R})
\end{cases}$$

and

$$W(E_n, 1_{[b_n, nb] \times \mathbb{R}}) \leq Cn \left( \frac{m}{\varepsilon} + o_n(1) \right),$$

where $C$ may depend on $\gamma, \bar{m}$ and $\varepsilon$. To prove this claim, let $s_0 = b_n$ and for every $l \geq 1$, let $s_l$ be the smallest $s \geq s_{l-1}$ such that $\int_{s_{l-1}}^{s_l} m_0'(x) dx = 1$. Since (5.10) holds, this terminates at some $s_L = nb$ with $L = \int_{b_n}^{nb} m_0' \leq \bar{m}|nb-b_n|$. We then set $x_l$ to be the middle of $[s_{l-1}, s_l]$. We let $u_l$ be the solution in the square $\mathcal{R}_l := [s_{l-1}, s_l] \times [-\frac{1}{2}(s_l - s_{l-1}), \frac{1}{2}(s_l - s_{l-1})]$ 

$$\begin{cases} 
-\Delta u_l = 2\pi (\delta_{x_i} - m_l' \delta_{\mathbb{R}}) & \text{in } \mathcal{R}_l \\
\frac{\partial u_l}{\partial \nu} = 0 & \text{on } \partial \mathcal{R}_l.
\end{cases}$$

This equation is solvable since, by construction of the $s_l$’s, the right-hand side has zero integral. Then for each $l$ we let $E_n = -\nabla u_l$ in $\mathcal{R}_l$, and let $E_n = 0$ in $[b_n, nb] \times \mathbb{R} \setminus \bigcup_l \mathcal{R}_l$. Clearly $E_n$ satisfies (5.26).

To estimate the energy of $u_l$ we let $u_l = v_l + w_l$ where, letting $m_l = \int_{[s_{l-1}, s_l]} m_0'$,

$$\begin{cases} 
-\Delta v_l = 2\pi (\delta_{x_i} - m_l' \delta_{\mathbb{R}}) & \text{in } \mathcal{R}_l \\
\frac{\partial v_l}{\partial \nu} = 0 & \text{on } \partial \mathcal{R}_l,
\end{cases}$$

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From Lemma 4.4 and Lemma 4.5 we find, choosing for instance $q = 4$ so that $q \in [1, 4]$ and $q' < 2$,

$$\int_{\mathcal{R}_l} |\nabla v_l|^q \leq C (s_l - s_{l-1})^2 \|m_l - m_0\|_{L^\infty([s_{l-1}, s_l])}^q,$$

and

$$W(v_l, 1_{\mathcal{R}_l}) = C \log m_l, \quad \int_{\mathcal{R}_l} |\nabla v_l|^{q'} \leq C m_l q'^{-2}.$$

From (1.17) and Lemma 4.6 since $E_n = - (\nabla v_l + \nabla w_l)$ in $\mathcal{R}_l$, we have

$$W(E_n, 1_{\mathcal{R}_l}) \leq C - \pi \log m_l + C \left\|m_l - m_0'\right\|_{L^\infty([s_{l-1}, s_l])} m_l^{1 - \frac{2}{q'} (s_l - s_{l-1})^2} + C \left\|m_l - m_0\right\|_{L^\infty([s_{l-1}, s_l])} (s_l - s_{l-1})^2.$$

Using (1.18),

$$\left\|m_l - m_0'\right\|_{L^\infty([s_{l-1}, s_l])} \leq C \left\|m_0\right\| C m_l \frac{(s_l - s_{l-1})^2}{\sqrt{n}}.$$

Replacing in (5.28) and letting $q = 4$ we deduce that

$$W(E_n, 1_{\mathcal{R}_l}) \leq C - \pi \log m_l + C \left( \frac{s_l - s_{l-1}}{\sqrt{n}} + \frac{(s_l - s_{l-1})^3}{n} \right).$$

Then, summing with respect to $l$ — using the fact that from (5.12) we have $\sum_1 |s_{l+1} - s_l| \leq C m_l (1 + o_n(1))$, the fact that the integral over $[s_{l-1}, s_l]$ of $m_0'$ is 1 and that from (1.17) we have $(s_l - s_{l-1}) \leq n^{\frac{1}{2}}$ — we find

$$W(E_n, 1_{[b_n, nb] \times \mathbb{R}}) \leq C \left( \int_{b_n} m_0'(x) - m_0'(x) \log m_0'(x) dx + n o_n(1) \right) \leq C n (m + o_n(1)),$$

since $m_0' - m_0' \log m_0'$ is bounded by a constant depending only on $m_0$ and using (5.12). This proves (5.27).

- **Step 4: conclusion.** Once the construction of $E_n$ is completed, the proof of Proposition 3.1 essentially identical to that of [SS2, Proposition 4.1], which is its 2-dimensional equivalent, except that the scaling factor $\sqrt{n}$ there must be replaced by $n$. We only sketch the proof below and refer to the specific part of [SS2] for the details.

The test vector-field $E_n$ has now been defined on all $[na, nb] \times \mathbb{R}$. It is extended by 0 outside, and is easily seen to satisfy the relation $\text{div } E_n = 2\pi (\nu_n' - m_0')$ for $\nu_n' = \sum_{i=1}^{N_n} \delta_{x_i'}$, a sum of Dirac masses on the real line. Combining (5.27) with (5.21), (5.23) and (5.6), we have

$$W(E_n, 1_{\mathbb{R}^2}) \leq \sum_k |I_k| \left( \sum_{i=0}^{N_k} \frac{n_i K}{q_c} W(\sigma_{m_i} E_i) + C \varepsilon + o_n(1) \right) + C n (m + o_n(1)).$$

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Letting $n \to \infty$ and then $m \to 0$, we see that the error term on the right-hand side can be made arbitrarily small, say smaller than $C\varepsilon$. On the other hand, the reasoning of [SS2], Step 2 in Paragraph 7.4, shows that

$$\sum_k |J_k| \left( \sum_{i=0}^{N_k} \frac{n_{iK}}{4\varepsilon} W(x_{mK} E_i) \right) \leq |\Sigma'| \int W(E) dP(x, E) + Cn (\varepsilon + o_n(1)),$$

so that taking $n$ larger if necessary we obtain

$$1 |\Sigma| W(E_n, 1_{\mathbb{R}^2}) \leq \int W(E) dP(x, E) + C\varepsilon. \tag{5.29}$$

Then arguing as in Paragraph 7.4, Step 3 of [SS2] we obtain that, letting $(x_1', \ldots, x_n)$ be the rescalings to the original scale of the points $x_i$, i.e. $x_i = x_i'/n$, we have for $n$ large enough

$$\limsup_{n \to \infty} \frac{1}{n} \left( w_n(x_1, \ldots, x_n) - n^2 F(\mu_0) + n \log n \right) \leq \frac{|\Sigma|}{\pi} \int W(E) dP(x, E) + C\varepsilon.$$

Also letting $P_n$ be the push-forward of $\frac{1}{|\Sigma|} dx |\Sigma|$ by the map $x \mapsto (x, E_n(nx + \cdot))$, it is easy to see that $d_B(P, P_n) < Cm$. In view of (5.13) and (5.25), and taking $m$ small enough, for any given $\varepsilon > 0$, we can achieve

$$d_B(P, P_n) < C\varepsilon.$$

This proves that items i) and ii) of Proposition 3.1 are satisfied by $(x_1', \ldots, x_n)$ and $E_n$. Then, the perturbation argument of Paragraph 7.4, Step 4 in [SS2] shows that there exists $\delta > 0$ and for each $n$ a subset $A_n \subset \mathbb{R}^n$ such that $|A_n| \geq n! (\delta/n)^n$ and such that for every $(y_1, \ldots, y_n) \in A_n$ there exists a corresponding $E_n$ satisfying (3.1) and (3.2). This concludes the proof of Proposition 3.1.

6 Proofs of the remaining theorems

6.1 Preliminary bounds on $Z_n^{\beta}$

The crucial fact is that there are good lower bounds for $F_n$. This follows from Proposition 4.1.

Even though we will not use the following result in the sequel, we state it to show how we can quickly derive a first upper bound on $Z_n^{\beta}$ from what precedes.

**Proposition 6.1.** We have

$$\log K_n^{\beta} \leq Cn \beta + n (\log |\Sigma| + o(1)) \tag{6.1}$$

and

$$\log Z_n^{\beta} \leq -\beta n^2 F(\mu_0) + \beta n \log n + Cn \beta + n (\log |\Sigma| + o(1)) \tag{6.2}$$

where $o(1) \to 0$ as $n \to \infty$ uniformly with respect to $\beta \geq \beta_0 > 0$, and $C$ depends only on $V$.

The proof uses two lemmas.
Lemma 6.2. For any $\nu_n = \sum_{i=1}^n \delta_{x_i}$, we have

\begin{equation}
\widehat{F}_n(\nu_n) = \frac{1}{n\pi} \int_{\mathbb{R}^2} dg_{\nu_n}
\end{equation}

where $\widehat{F}_n$ is as in (1.23) and $g_{\nu_n}$ is the result of applying Proposition 4.1 to $\nu_n$.

Proof. This follows from (4.2) applied to $\chi_R$, where $\chi_R$ is as in (1.7). If $R$ is large enough then $\#\{p \in \text{Supp}(\nu) \mid B(p, C) \cap \text{Supp}(\nabla \tilde{\chi}) \neq \emptyset\} = 0$ and therefore (4.2) reads

$$W(\nu_n, \chi_R) = \int \tilde{\chi} R \, dg_{\nu_n}.$$ 

Letting $R \to +\infty$ yields $W(\nu_n, 1_{\mathbb{R}^2}) = \int dg_{\nu_n}$ and the result, in view of (1.22).

The following lemma has the same proof as in [SS2, Lemma 3.5].

Lemma 6.3. Letting $\nu_n$ stand for $\sum_{i=1}^n \delta_{x_i}$ we have, for any constant $\alpha > 0$ and uniformly w.r.t. $\beta \geq \beta_0 > 0$,

\begin{equation}
\lim_{n \to \infty} \left( \int_{\mathbb{R}^n} e^{-\alpha \beta n} \int_0^{\nu_n} dx_1 \ldots dx_n \right)^{\frac{1}{n}} = |\Sigma|.
\end{equation}

Proposition 6.1 is then proved exactly as in [SS2, Proposition 3.3]. Note that a lower bound for $Z^\beta_n$ was obtained in Corollary 3.3. To complete the proof of Theorem 3 it then suffices to prove the corresponding upper bound.

6.2 Lower bounds for $\widehat{F}_n$ and consequences

We start with a result that shows how $\widehat{F}_n$ controls the fluctuation $\nu_n - n\mu_0$.

Lemma 6.4. Let $\nu_n = \sum_{i=1}^n \delta_{x_i}$. For any interval $I$ of width $R$ (possibly depending on $n$) and any $1 < q < 2$, we have

$$\|\nu_n - n\mu_0\|_{W^{-1,q}(I)} \leq C_q (1 + R^2)^{\frac{1}{q} - \frac{1}{2}} n^{\frac{1}{2}} \left( \widehat{F}_n(\nu_n) + 1 \right)^{\frac{1}{2}}.$$ 

Here $W^{-1,q}$ is the dual of the Sobolev space $W_0^{1,q'}$ with $1/q + 1/q' = 1$.

Proof. In [SS2, Lemma 5.1], we have the following statement

$$\|\nu_n - n\mu_0\|_{W^{-1,q}(B_R)} \leq C_q (1 + R^2)^{\frac{1}{q} - \frac{1}{2}} n^{\frac{1}{2}} \left( \widehat{F}_n(\nu_n) + 1 \right)^{\frac{1}{2}}.$$ 

The proof is based on [SeTi] which works in our one-dimensional context as well, thus the proof can be reproduced without change. It is immediate to deduce the result.

We now turn to bounding from below $\widehat{F}_n$.

The proof is the same as in [SS2, Sec. 6], itself following the method of [SS1] based on the ergodic theorem. We just state the main ingredients.

Let $\{\nu_n\}_n$ and $P_{\nu_n}$ be as in the statement of Theorem 1. We need to prove that any subsequence of $\{P_{\nu_n}\}_n$ has a convergent subsequence and that the limit $P$ is a $T_{\lambda(x)}$-invariant
probability measure such that $P$-almost every $(x, E)$ is such that $E \in \mathcal{A}_{m_{0}(x)}$ and (1.29) holds. Note that the fact that the first marginal of $P$ is $dx|_{\Sigma}/|\Sigma|$ follows from the fact that, by definition, this is true of $P_{\nu_{n}}$.

We thus take a subsequence of $\{P_{\nu_{n}}\}$ (which we don’t relabel). We may assume that it has a subsequence, still denoted $\nu_{n}$, which satisfies $\hat{F}_{n}(\nu_{n}) \leq C$, otherwise there is nothing to prove. This implies that $\nu_{n}$ is of the form $\sum_{i=1}^{n} \delta_{x_{i,n}}$. We let $E_{n}$ denote the electric field and $g_{n}$ the measures associated to $\nu_{n}$ as in Definition 4.3. As usual, $\nu_{n}' = \sum_{i=1}^{n} \delta_{nx_{i,n}}$.

A useful consequence of $\hat{F}_{n}(\nu_{n}) \leq C$ is that, using the result of Lemma 6.4, we have

$$\lim_{n} \frac{1}{n} \nu_{n} \to \mu_{0} \text{ on } \mathbb{R}. \tag{6.5}$$

We then set up the framework of Section 6.1 in [SS2]. We let $G = \Sigma$ and $X = \mathcal{M}_{+} \times L^{p}_{\text{loc}}(\mathbb{R}^{2}, \mathbb{R}^{2}) \times \mathcal{M}$, where $p \in (1, 2)$, where $\mathcal{M}_{+}$ denotes the set of positive Radon measures on $\mathbb{R}^{2}$ and $\mathcal{M}$ the set of those which are bounded below by the constant $-CV := -C(||m_{0}||_{\infty} + 1)$ of Proposition 4.1, both equipped with the topology of weak convergence.

For $\lambda \in \mathbb{R}$ and abusing notation we let $\theta_{\lambda}$ denote both the translation $x \mapsto x + \lambda$ and the action $\theta_{\lambda}(\nu, E, g) = \left(\theta_{\lambda}#\nu, E \circ \theta_{\lambda}, \theta_{\lambda}#g\right)$. Accordingly the action $T^{n}$ on $\Sigma \times X$ is defined for $\lambda \in \mathbb{R}$ by

$$T_{\lambda}^{n}(x, \nu, E, g) = \left( x + \frac{\lambda}{n}, \theta_{\lambda}#\nu, E \circ \theta_{\lambda}, \theta_{\lambda}#g \right).$$

Then we let $\chi$ be a smooth nonnegative cut-off function with integral 1 and support in $[-1, 1]$ and define

$$f_{n}(x, \nu, E, g) = \begin{cases} 
\frac{1}{\pi} \int_{\mathbb{R}^{2}} \chi(t) \, dg(t, s) & \text{if } (\nu, E, g) = (\nu_{n}', E_{n}, g_{n}), \\
+\infty & \text{otherwise}.
\end{cases} \tag{6.6}$$

Finally we let,

$$F_{n}(\nu, E, g) = \int_{\Sigma} f_{n}(x, \theta_{xn}(\nu, E, g)) \, dx. \tag{6.7}$$

We have the following relation between $F_{n}$ and $\hat{F}_{n}$, as $n \to +\infty$ (see [SS2, Sec. 6]):

$$F_{n}(\nu, E, g) \begin{cases} 
\leq \frac{1}{|\Sigma|} \hat{F}_{n}(\nu_{n}) + o(1) & \text{if } (\nu, E, g) = (\nu_{n}', E_{n}, g_{n}) \\
= +\infty & \text{otherwise}.
\end{cases} \tag{6.8}$$

The hypotheses in Section 6.1 of [SS2] are satisfied and applying the abstract result, Theorem 6 of [SS2], we conclude that letting $Q_{n}$ denote the push-forward of the normalized Lebesgue measure on $\Sigma$ by the map $x \mapsto (x, \theta_{nx}(\nu_{n}', E_{n}, g_{n}))$, and $Q = \lim_{n} Q_{n}$, we have

$$\liminf_{n} \frac{1}{|\Sigma|} \hat{F}_{n}(\nu_{n}) \geq \frac{1}{\pi} \int W(E) \, dQ(x, \nu, E, g) \tag{6.9}$$

and, $Q$-a.e. $(E, \nu) \in \mathcal{A}_{m_{0}(x)}$. 38
Now we let \( P_n \) (resp. \( P \)) be the marginal of \( Q_n \) (resp. \( Q \)) with respect to the variables \((x,E)\). Then the first marginal of \( P \) is the normalized Lebesgue measure on \( E \) and \( P \)-a.e. we have \( E \in \mathcal{A}_m(x) \); in particular

\[
W(E) \geq \min_{\mathcal{A}_m(x)} W = m_0(x) \left( \min_{\mathcal{A}_1} W - \pi \log m_0(x) \right).
\]

Integrating with respect to \( P \) and noting that since only \( x \) appears on the right-hand side we may replace \( P \) by its first marginal there, we find, in view of \((1.27)\) that the lower bound \((1.29)\) holds.

### 6.3 Proof of Theorems 4, 5 and 3 completed

As mentioned above, Part B of Theorem 4 is a direct consequence of Proposition 3.1, see Corollary 3.2.

Part C follows from the comparison of Parts A and B: for minimizers, the chains of inequalities \((1.29)\) and \((1.30)\) are in fact equalities and that \( \frac{\partial}{\partial x} \int W \, dP \) must be minimized hence equal to \( \alpha \), cf. \((1.35)\). Since \((1.29)\) follows from \((6.9)\), the inequality \((6.9)\) must also be an equality and moreover we must have \( \lim_{n \to \infty} \int \zeta \, d\nu_n = 0 \).

This completes the proof of Theorem 4. The proof of Theorems 5 and Theorem 3 is identical to SS2 except again the replacement of the scaling \( \sqrt{n} \) by \( n \) and \(|A_n| \geq n!(\pi \delta/n)^n\).

### 6.4 Proof of Proposition 2.1

The result of Proposition 2.1 is a consequence of Proposition 3.4.

First, applying Part C of Theorem 4 there exists a translation-invariant measure \( P \) on \( \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) such that \( P \)-a.e. \((x,E)\) is such that \( E \) minimizes \( W \) over \( \mathcal{A}_m(x) \). Then, taking the push-forward of \( P \) under \((x,E) \mapsto \sigma_{1/m_0(x)} E \), we obtain a probability \( Q \) on \( L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) such that \( Q \)-a.e. \( E \) minimizes \( W \) over \( \mathcal{A}_1 \).

Applying Lemma 5.4 to \( Q \), we find that \( Q \)-a.e. \( E \) is such that \( E \in \mathcal{A}_1 \), such that \((3.10)\) holds, and such that \( W(E) = \min_{\mathcal{A}_1} W \). Choose on such \( E_0 \) and apply Proposition 3.4 to \( G = \{E_0\} \). We find that for any given \( \varepsilon > 0 \) and any integer \( R \) large enough depending on \( \varepsilon \), there exists \( E_R \) defined on \( \bar{I}_R \) such that \( E_R \cdot \bar{v} = 0 \) on \( \partial \bar{I}_R \) and \( W(E_R, 1_{\bar{I}_R}) < R(W(E_0) + \varepsilon) \). This \( E_R \) can be extended periodically by letting \( E_R(x + kR, y) = E_R(x, y) \) for any \( k \in \mathbb{Z} \).

From the condition \( E_R \cdot \bar{v} = 0 \) on \( \partial \bar{I}_R \) we find that, letting \( \Lambda \subset I_R \) be the locations of the Dirac masses in \( \text{div} \ E_R \), we have \( \text{div} \ E_R = 2\pi \left( \sum_{p \in \Lambda_R} \delta_p - \delta_R \right) \), where \( \Lambda_R = \Lambda + R\mathbb{Z} \). Moreover

\[
W(E_R) = \frac{W(E_R, 1_{\bar{I}_R})}{|I_R|} \leq W(E_0) + C\varepsilon.
\]

There remains to make \( E_R \) curl-free. Following the proof of Corollary 4.4 of SS1 we let \( \tilde{E}_R = E_R + \nabla^\perp f_R \) in \( \bar{I}_R \) where \( f_R \) solves \( -\Delta f_R = \text{curl} E_R \) in \( \bar{I}_R \) and \( f_R = 0 \) on \( \partial \bar{I}_R \). Then, \( \text{div} \tilde{E}_R = \text{div} E_R \) and \( \text{curl} \tilde{E}_R = 0 \) in \( I_R \). We can thus find \( H_R \) such that \( \tilde{E}_R = \nabla H_R \) in \( \bar{I}_R \). It also satisfies \( \nabla H_R \cdot \bar{v} = \tilde{E}_R \cdot \bar{v} = E_R \cdot \bar{v} + \nabla^\perp f_R \cdot \bar{v} = 0 \) on \( \partial \bar{I}_R \). We may then extend \( H_R \) to a periodic function by even reflection, and take the final \( \tilde{E}_R \) to be \( \nabla H_R \). This procedure can only decrease the energy (arguing again as in SS1 SS2): we have \( W(\tilde{E}_R, 1_{\bar{I}_R}) \leq W(E_R, 1_{\bar{I}_R}) \).
We deduce that the variable data, we have circles" introduced in [SS3]: let \( \tau \) by assumption and by choice of \( \{ \}

\begin{align*}
\text{Proof.} \quad \text{Two cases can happen: either } c > 0 \text{ or } c < 0. \text{ We then follow the method of "integrating over circles" introduced in [SS3]: let } \tilde{C} \text{ denote } \{ x \in B_{\tau R}(x'_0) \setminus B_{\tau R}(x'_0), |x - x'_0| \notin \tilde{T} \}. \text{ For any } r \in T, \text{ since } \partial B_{\tau}(x'_0) \text{ does not intersect } \text{Supp}(\nu'_n), \text{ we have}

\begin{align*}
(6.10) \quad \int_{\partial B_{\tau}(x'_0)} E_{\nu_n} \cdot \nu = \int_{B_{\tau}(x'_0)} \text{div } E_{\nu_n} = 2\pi \nu'_n(B_{\tau}(x'_0)) - \int_{B_{\tau}(x'_0)} m_0 \left( \frac{x'}{n} \right) \delta_R &
\geq D(x_0, R) - 2(\tau - 1)R\|m_0\|_{L^\infty} \geq \frac{1}{2} D(x_0, R)
\end{align*}

by assumption and by choice of \( \tau \). Moreover, for any \( r \in T \), we have, by Cauchy-Schwarz,

\[ \int_{\partial B_{\tau}(x'_0)} |E_{\nu_n}|^2 \geq \frac{1}{2\pi r} \left( \int_{\partial B_{\tau}(x'_0)} E_{\nu_n} \cdot \nu \right)^2 \geq \frac{1}{8\pi r} D(x_0, R)^2. \]

\[ \text{The condition } R > 1 \text{ could be replaced by } R > R_0 \text{ for any } R_0 > 0 \text{ at the expense of a constant } c \text{ depending on } R_0. \]

6.5 Consequences for deviations: proof of Theorem 6

The proof relies on the following proposition, whose proof is much shorter than in [SS2], due to the simpler nature of the one-dimensional geometry.

**Proposition 6.5.** Let \( \nu_n = \sum_{i=1}^{n} \delta_{x_i} \), and \( g_{\nu_n} \) be as in Definition 4.3. For any \( R > 1 \), for any \( x_0 \in \mathbb{R} \), denoting

\[ D(x_0, R) = \nu_n \left( B_{\frac{R}{n}}(x_0) \right) - n\mu_0 \left( B_{\frac{R}{n}}(x_0) \right) \]

we have

\[ \int_{B_{2R}(x'_0)} dg_{\nu_n} \geq -CR + cD(x_0, R)^2 \min \left( 1, \frac{|D(x_0, R)|}{R} \right), \]

where \( c > 0 \) and \( C \) depend only on \( V \). \( \square \)

**Proof.** Two cases can happen: either \( D(x_0, R) \geq 0 \) or \( D(x_0, R) \leq 0 \).

We start with the first case. Let us choose \( \tau = \min \left( 2, 1 + \frac{D(x_0, R)}{2R\|m_0\|_{L^\infty}} \right) \) and denote \( T = \{ r \in [R, \tau R), B_r(x'_0) \cap B_{\rho} = \emptyset \} \), where \( B_{\rho} \) is as in Proposition 4.1. By construction of \( B_{\rho} \) and since \( \rho < \frac{1}{2} \), we have \( |T| \geq \frac{1}{2} (\tau - 1)R \). We then follow the method of "integrating over circles" introduced in [SS3]: let \( \tilde{C} \) denote \( \{ x \in B_{\tau R}(x'_0) \setminus B_{\tau R}(x'_0), |x - x'_0| \notin \tilde{T} \} \).

For any \( r \in T \), since \( \partial B_{\tau}(x'_0) \) does not intersect \( \text{Supp}(\nu'_n) \), we have

\[ \int_{\partial B_{\tau}(x'_0)} E_{\nu_n} \cdot \nu = \int_{B_{\tau}(x'_0)} \text{div } E_{\nu_n} = 2\pi \nu'_n(B_{\tau}(x'_0)) - \int_{B_{\tau}(x'_0)} m_0 \left( \frac{x}{n} \right) \delta_R \]

\[ \geq D(x_0, R) - 2(\tau - 1)R\|m_0\|_{L^\infty} \geq \frac{1}{2} D(x_0, R) \]

by assumption and by choice of \( \tau \). Moreover, for any \( r \in T \), we have, by Cauchy-Schwarz,
Integrating over $T$, using $|T| \geq \frac{1}{2}(\tau - 1)R$, we have
\[ \int_T \frac{dr}{r} \geq \int_{\tau R - \frac{1}{2}(\tau - 1)R}^{\tau R} \frac{dr}{r} = -\log \left(1 - \frac{\tau - 1}{2\tau}\right) \]
and thus
\[ \int_{B_r(x_0) \setminus B_r} |E_{\nu_n}|^2 \geq cD(x_0, R)^2 \min \left(1, \frac{D(x_0, R)}{R \|m_0\|_{L^\infty}}\right), \]
for some $c > 0$ depending only on $\|m_0\|_{L^\infty}$ hence on $V$. Inserting into (4.1), we are led to
\[ \int_{B_{2R}(x_0')} g_{\nu_n} \geq -C(|m_0|_{L^\infty} + 1)R + cD(x_0, R)^2 \min \left(1, \frac{D(x_0, R)}{R \|m_0\|_{L^\infty}}\right). \]
The case $D(x_0, R) \leq 0$ is essentially analogous.

We now proceed to the proof of Theorem 6, starting with (1.36). If $R > R_0$ and $|D(x'_0, R)| \geq \eta R$ then from Proposition 6.5 and using the fact — from Proposition 4.1 — that $g_{\nu_n}$ is positive outside $\bigcup_{i=1}^n B(x'_i, C)$ and that $g_{\nu_n} \geq -C$ everywhere, we deduce from (6.3) and (1.22), (1.23) that
\[ (6.11) \quad F_n(\nu_n) \geq \frac{1}{n} \left(-CR + c \min(\eta^2, \eta^3)R^2\right) + 2 \int \zeta d\nu_n. \]
Inserting into (3.4) we find
\[ \mathbb{P}_n^\beta(\int \xi d\nu_n \geq \eta) \leq \exp \left(C\beta R - c\beta \min(\eta^2, \eta^3)R^2\right) \int e^{-\beta \int \xi d\nu_n} d\xi, \]
Then, using the lower bound (3.6) and Lemma 6.3 we deduce that if $\beta \geq \beta_0$ and $n$ is large enough depending on $\beta_0$ then
\[ \log \mathbb{P}_n^\beta(\int |D(x'_0, R)| \geq \eta R) \leq -c\beta \min(\eta^2, \eta^3)R^2 + C\beta R + C n \beta + C n, \]
where $c, C > 0$ depend only on $V$. Thus (1.36) is established.

We next turn to (1.38). Arguing as above, from (6.3) we have $F_n(\nu_n) \geq -C + 2 \int \zeta d\nu_n$. Splitting $2 \int \zeta d\nu_n$ as $\int \xi d\nu_n + \int \zeta d\nu_n$, inserting into (3.4) and using (3.6) we are led to
\[ \mathbb{P}_n^\beta(\int \xi_n d\nu_n \geq \eta) \leq e^{-\frac{1}{2}n \beta \eta + C n (\beta + 1)} \int e^{-\beta \int \zeta d\nu_n} d\xi. \]
where $C$ depends only on $V$. Then, using Lemma 6.3 we deduce (1.38).

We finish with (1.37). Inserting the result of Lemma 6.4 into (3.4), we have, if $I$ is an interval of width $R/n$
\[ \mathbb{P}_n^\beta(\|\nu_n - n \mu_0\|_{W^{-1, q}(I)} \geq C_q \eta \sqrt{n}(1 + R^2/n^2)^{\frac{1}{2} - \frac{1}{2} q} \leq \frac{1}{K_n} e^{-\frac{1}{2}n \beta \eta} \int e^{-\beta \int \zeta d\nu_n} d\xi. \]
Arguing as before and rearranging terms yields (1.37).

This concludes the proof of Theorem 6.
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