QUADRATIC REVERSES OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some sharp quadratic reverses for the generalised triangle inequality in inner product spaces and applications are given.

1. Introduction

In 1966, J.B. Diaz and F.T. Metcalf [1] proved the following reverse of the triangle inequality in the general settings of inner product spaces:

**Theorem 1.** Let $a$ be a unit vector in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field $K$. Suppose that the vectors $x_i \in H \setminus \{0\}, i \in \{1, \ldots, n\}$ satisfy

$$(1.1) \quad 0 \leq r \leq \Re \langle x_i, a \rangle \|x_i\|, \quad i \in \{1, \ldots, n\}.$$ 

Then

$$(1.2) \quad r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|,$$

where equality holds if and only if

$$(1.3) \quad \sum_{i=1}^{n} x_i = r \left( \sum_{i=1}^{n} \|x_i\| \right) a.$$ 

For some similar results valid for semi-inner products in normed spaces, see [3] and [4].

In the same spirit, but providing a somewhat simpler sufficient condition with a clear geometrical meaning, we note the following result obtained by the author in [2]:

**Theorem 2.** Let $a$ be as above and $\rho \in (0, 1)$. If $x_i \in H, i \in \{1, \ldots, n\}$ are such that

$$(1.4) \quad \|x_i - a\| \leq \rho \quad \text{for each} \quad i \in \{1, \ldots, n\},$$

then we have the inequality

$$(1.5) \quad \sqrt{1 - \rho^2} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|.$$ 

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with equality if and only if

\[(1.6) \quad \sum_{i=1}^{n} x_i = \sqrt{1 - \rho^2} \left( \sum_{i=1}^{n} \|x_i\| \right) a.\]

In a complementary direction providing reverses of the triangle inequality in its additive form, i.e., upper bounds for the nonnegative difference

\[\sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\|,\]

we note the following recent result obtained in [2]:

**Theorem 3.** Let \(a\) be as above and \(x_i \in H, k_i \geq 0, i \in \{1, \ldots, n\}\) such that

\[(1.7) \quad \|x_i\| - \text{Re} \langle a, x_i \rangle \leq k_i \text{ for each } i \in \{1, \ldots, n\},\]

then we have the inequality

\[(1.8) \quad 0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} k_i.\]

The equality holds in (1.8) if and only if

\[(1.9) \quad \sum_{i=1}^{n} \|x_i\| \geq \sum_{i=1}^{n} k_i\]

and

\[(1.10) \quad \sum_{i=1}^{n} x_i = \left( \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i \right) a.\]

Another similar result but with a simpler condition, is the following one [2].

**Theorem 4.** Let \(a\) and \(x_i, i \in \{1, \ldots, n\}\) be as above. If \(r_i > 0, i \in \{1, \ldots, n\}\) are such that

\[(1.11) \quad \|x_i - a\| \leq r_i \text{ for each } i \in \{1, \ldots, n\},\]

then we have the inequality

\[(1.12) \quad 0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \sum_{i=1}^{n} r_i^2.\]

The equality holds in (1.12) if and only if

\[(1.13) \quad \sum_{i=1}^{n} \|x_i\| \geq \frac{1}{2} \sum_{i=1}^{n} r_i^2\]

and

\[(1.14) \quad \sum_{i=1}^{n} x_i = \left( \sum_{i=1}^{n} \|x_i\| - \frac{1}{2} \sum_{i=1}^{n} r_i^2 \right) a.\]
For other inequalities related to the triangle inequality, see Chapter XVII of the book [5].

The main aim of the present paper is to point out some quadratic reverses for the generalised triangle inequality, namely, sharp upper bounds for the nonnegative differences
\[
\left( \sum_{i=1}^{n} \| x_i \| \right)^2 - \left\| \sum_{i=1}^{n} x_i \right\|^2,
\]
under various assumptions for the vectors \( x_i \in H, i \in \{1,\ldots,n\} \) involved. Some related results are established. Applications for vector-valued integrals in Hilbert spaces are also given.

2. Quadratic Reverses of the Triangle Inequality

The following lemma holds:

Lemma 1. Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \( \mathbb{K} \), \( x_i \in H, i \in \{1,\ldots,n\} \) and \( k_{ij} > 0 \) for \( 1 \leq i < j \leq n \) such that
\[
0 \leq \| x_i \| \| x_j \| - \text{Re} \langle x_i, x_j \rangle \leq k_{ij}
\]
for \( 1 \leq i < j \leq n \). Then we have the following quadratic reverse of the triangle inequality
\[
\left( \sum_{i=1}^{n} \| x_i \| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + 2 \sum_{1 \leq i < j \leq n} k_{ij}.
\]

The case of equality holds in (2.2) if and only if it holds in (2.1) for each \( i, j \) with \( 1 \leq i < j \leq n \).

Proof. We observe that the following identity holds:
\[
\left( \sum_{i=1}^{n} \| x_i \| \right)^2 - \left\| \sum_{i=1}^{n} x_i \right\|^2
\]
\[
= \sum_{i,j=1}^{n} \| x_i \| \| x_j \| - \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j
\]
\[
= \sum_{i,j=1}^{n} \| x_i \| \| x_j \| - \sum_{i,j=1}^{n} \text{Re} \langle x_i, x_j \rangle
\]
\[
= \sum_{i,j=1}^{n} [\| x_i \| \| x_j \| - \text{Re} \langle x_i, x_j \rangle] - \sum_{1 \leq i < j \leq n} [\| x_i \| \| x_j \| - \text{Re} \langle x_i, x_j \rangle]
\]
\[
= 2 \sum_{1 \leq i < j \leq n} [\| x_i \| \| x_j \| - \text{Re} \langle x_i, x_j \rangle].
\]

Using the condition (2.1), we deduce that
\[
\sum_{1 \leq i < j \leq n} [\| x_i \| \| x_j \| - \text{Re} \langle x_i, x_j \rangle] \leq \sum_{1 \leq i < j \leq n} k_{ij},
\]
and by (2.3), we deduce the desired inequality (2.2).
The case of equality is obvious by the identity (2.3) and we omit the details.

**Remark 1.** From (2.2) one may deduce the coarser inequality that might be useful in some applications:

\[ 0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sqrt{2} \left( \sum_{1 \leq i < j \leq n} k_{ij} \right)^{1/2} \left( \leq \sqrt{2} \sum_{1 \leq i < j \leq n} \sqrt{k_{ij}} \right). \]

**Remark 2.** If the condition (2.1) is replaced with the following refinement of Schwarz’s inequality:

(2.4) \( (0 \leq \delta_{ij} \leq \|x_i\| \|x_j\| - \Re \langle x_i, x_j \rangle \) for \( 1 \leq i < j \leq n \),

then the following refinement of the quadratic generalised triangle inequality is valid:

(2.5) \( \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \geq \left\| \sum_{i=1}^{n} x_i \right\|^2 + 2 \sum_{1 \leq i < j \leq n} \delta_{ij} \left( \geq \left\| \sum_{i=1}^{n} x_i \right\|^2 \right). \)

The equality holds in the first part of (2.5) iff the case of equality holds in (2.4) for each \( 1 \leq i < j \leq n \).

The following result holds.

**Proposition 1.** Let \( (H; \langle \cdot, \cdot \rangle) \) be as above, \( x_i \in H, i \in \{1, \ldots, n\} \) and \( r > 0 \) such that

(2.6) \( \|x_i - x_j\| \leq r \)

for \( 1 \leq i < j \leq n \). Then

(2.7) \( \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n(n-1)}{2} r^2. \)

The case of equality holds in (2.7) if and only if

(2.8) \( \|x_i\| \|x_j\| - \Re \langle x_i, x_j \rangle = \frac{1}{2} r^2 \)

for each \( i, j \) with \( 1 \leq i < j \leq n \).

**Proof.** The inequality (2.6) is obviously equivalent to

\( \|x_i\|^2 + \|x_j\|^2 \leq 2 \Re \langle x_i, x_j \rangle + r^2 \)

for \( 1 \leq i < j \leq n \). Since

\( 2 \|x_i\| \|x_j\| \leq \|x_i\|^2 + \|x_j\|^2, \quad 1 \leq i < j \leq n \);

hence

(2.9) \( \|x_i\| \|x_j\| - \Re \langle x_i, x_j \rangle \leq \frac{1}{2} r^2 \)

for any \( i, j \) with \( 1 \leq i < j \leq n \).

Applying Lemma for \( k_{ij} := \frac{1}{2} r^2 \) and taking into account that

\[ \sum_{1 \leq i < j \leq n} k_{ij} = \frac{n(n-1)}{4} r^2, \]

we deduce the desired inequality (2.7). The case of equality is also obvious by the above lemma and we omit the details.
The constant 1

Applying Proposition 1 for \( r \) and \( a \) being suitable quantities.

Proof. Let \( \gamma \) be a constant

\( \leq \gamma \) for \( i < j \)

\( \gamma \) is best possible in the sense that it cannot be replaced in general by a smaller quantity.

The constant \( \frac{1}{2} \) is best possible in the sense that it cannot be replaced in general by a smaller quantity.

Proof. Let \( 1 \leq i < j \leq n \). Then, obviously,

\[
\|x_j - x_i\| = \left\| \sum_{k=i}^{j-1} \Delta x_k \right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\|.
\]

Applying Proposition \( \beta \) for \( r := \sum_{k=1}^{n-1} \|\Delta x_k\| \), we deduce the desired result \( (\text{2.10}) \).

To prove the sharpness of the constant \( \frac{1}{2} \), assume that the inequality \( (\text{2.10}) \) holds with a constant \( c > 0 \), i.e.,

\[
\left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + c n (n-1) \sum_{k=1}^{n-1} \|\Delta x_k\|.
\]

for \( n \geq 2, x_i \in H, i \in \{1, \ldots, n\} \).

If we choose in \( (\text{2.11}), n = 2, x_1 = -\frac{1}{2} e, x_2 = \frac{1}{2} e, e \in H, \|e\| = 1 \), then we get \( 1 \leq 2c \), giving \( c \geq \frac{1}{2} \).

The following result providing a reverse of the quadratic generalised triangle inequality in terms of the sup-norm of the forward differences also holds.

Proposition 2. Let \( (H; \langle \cdot, \cdot \rangle) \) be an inner product space and \( x_i \in H, i \in \{1, \ldots, n\} \).

Then we have the inequality

\[
\left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n^2 (n^2 - 1)}{12} \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2.
\]

The constant \( \frac{1}{12} \) is best possible in \( (\text{2.12}) \).

Proof. As above, we have that

\[
\|x_j - x_i\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq (j-i) \max_{1 \leq k \leq n-1} \|\Delta x_k\|,
\]

for \( 1 \leq i < j \leq n \).

Squaring the inequality, we get

\[
\|x_j\|^2 + \|x_i\|^2 \leq 2 \text{Re} \langle x_i, x_j \rangle + (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2
\]

for any \( i, j \) with \( 1 \leq i < j \leq n \), and since

\[
2 \|x_i\| \|x_j\| \leq \|x_j\|^2 + \|x_i\|^2,
\]

In the same spirit, and if some information about the forward difference \( \Delta x_k := x_{k+1} - x_k \) \( (1 \leq k \leq n-1) \) are available, then the following simple quadratic reverse of the generalised triangle inequality may be stated.

Corollary 1. Let \( (H; \langle \cdot, \cdot \rangle) \) be an inner product space and \( x_i \in H, i \in \{1, \ldots, n\} \).

Then we have the inequality

\[
(\text{2.10}) \quad \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n (n-1)}{2} \sum_{k=1}^{n-1} \|\Delta x_k\|.
\]

The constant \( \frac{1}{2} \) is best possible in the sense that it cannot be replaced in general by a smaller quantity.
hence
\[(2.13) \quad 0 \leq \|x_i\| \|x_j\| - \text{Re} \langle x_i, x_j \rangle \leq \frac{1}{2} (j - i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2\]
for any \(i, j\) with \(1 \leq i < j \leq n\).

Applying Lemma 1 for \(k_{ij} := \frac{1}{2} (j - i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2\), we can state that
\[
\left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j - i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2.
\]

However,
\[
\sum_{1 \leq i < j \leq n} (j - i)^2 = \frac{1}{2} \sum_{i,j=1}^{n} (j - i)^2 = n \sum_{k=1}^{n} k^2 - \left( \sum_{k=1}^{n} k \right)^2 = \frac{n^2 (n^2 - 1)}{12},
\]
giving the desired inequality.

To prove the sharpness of the constant, assume that \((2.12)\) holds with a constant \(D > 0\), i.e.,
\[(2.14) \quad \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + Dn^2 (n^2 - 1) \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2
\]
for \(n \geq 2, x_i \in H, i \in \{1, \ldots, n\}\).

If in \((2.14)\) we choose \(n = 2, x_1 = -\frac{1}{2}e, x_2 = \frac{1}{2}e, e \in H, \|e\| = 1\), then we get
\[1 \leq 12D \text{ giving } D \geq \frac{1}{12}.\]

The following result may be stated as well.

**Proposition 3.** Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space and \(x_i \in H, i \in \{1, \ldots, n\}\).

Then we have the inequality:
\[(2.15) \quad \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j - i)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}},\]
where \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\).

The constant \(E = 1\) in front of the double sum cannot generally be replaced by a smaller constant.

**Proof.** Using Hölder’s inequality, we have
\[
\|x_j - x_i\| \leq \sum_{k=1}^{j-1} \|\Delta x_k\| \leq (j - i)^{\frac{1}{p}} \left( \sum_{k=1}^{j-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \leq (j - i)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}},
\]
for \(1 \leq i < j \leq n\).
Squaring the previous inequality, we get
\[ \|x_j\|^2 + \|x_i\|^2 \leq 2 \Re \langle x_i, x_j \rangle + (j - i) \frac{2}{\ell} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}}, \]
for \(1 \leq i < j \leq n\).

Utilising the same argument from the proof of Proposition 2, we deduce the desired inequality (2.15).

Now assume that (2.15) holds with a constant \(E > 0\), i.e.,
\[ \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2 \leq \left( \sum_{i=1}^{n} x_i \right)^2 + E \sum_{1 \leq i < j \leq n} (j - i) \frac{2}{\ell} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}}. \]
for \(n \geq 2\) and \(x_i \in H, i \in \{1, \ldots, n\}\), \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\).

For \(n = 2, x_1 = -\frac{1}{2}e, x_2 = \frac{1}{2}e, \|e\| = 1\), we get \(1 \leq E\), showing the fact that the inequality (2.15) is sharp.

The particular case \(p = q = 2\) is of interest.

**Corollary 2.** Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space and \(x_i \in H, i \in \{1, \ldots, n\}\).

Then we have the inequality:
\[ \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2 \leq \left( \sum_{i=1}^{n} x_i \right)^2 + \sum_{1 \leq i < j \leq n} (j - i) \frac{2}{\ell} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}}. \]
The constant \(\frac{1}{6}\) is best possible in (2.16).

**Proof.** For \(p = q = 2\), Proposition 3 provides the inequality
\[ \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2 \leq \left( \sum_{i=1}^{n} x_i \right)^2 + \sum_{1 \leq i < j \leq n} (j - i) \sum_{k=1}^{n-1} \|\Delta x_k\|^2, \]
and since
\[ \sum_{1 \leq i < j \leq n} (j - i) = 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + n - 1) \]
\[ = \sum_{k=1}^{n-1} (1 + 2 + \cdots + k) = \sum_{k=1}^{n-1} \frac{k(k+1)}{2} \]
\[ = \frac{1}{2} \left( \frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2} \right) \]
\[ = \frac{n(n^2 - 1)}{6}, \]
hence the inequality (2.15) is proved. The best constant may be shown in the same way as above but we omit the details.

Finally, we may state and prove the following different result.

**Theorem 5.** Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space, \(y_i \in H, i \in \{1, \ldots, n\}\) and \(M \geq m > 0\) are such that either
\[ \Re \langle My_j - y_i, y_i - my_j \rangle \geq 0 \quad \text{for} \quad 1 \leq i < j \leq n, \]
(2.17)
or, equivalently,

\[
\left\| y_i - \frac{M + m}{2} y_j \right\| \leq \frac{1}{2} (M - m) \| y_j \| \quad \text{for } 1 \leq i < j \leq n.
\]

Then we have the inequality

\[
\left( \sum_{i=1}^{n} \| y_i \| \right)^2 \leq \left\| \sum_{i=1}^{n} y_i \right\|^2 + \frac{1}{2} \left( \frac{M - m}{M + m} \right)^2 \sum_{k=1}^{n-1} k \| y_{k+1} \|^2.
\]

The case of equality holds in (2.19) if and only if

\[
\| y_i \| \| y_j \| - \text{Re} \langle y_i, y_j \rangle = \frac{1}{4} \left( \frac{M - m}{M + m} \right)^2 \| y_j \|^2
\]

for each \( i, j \) with \( 1 \leq i < j \leq n \).

**Proof.** Firstly, observe that, in an inner product space \((H; \langle \cdot, \cdot \rangle)\) and for \(x, z, Z \in H\), the following statements are equivalent:

(i) \( \text{Re} \langle Z - x, x - z \rangle \geq 0 \)

(ii) \( \| x - \tfrac{Z + z}{2} \| \leq \tfrac{1}{2} \| Z - z \| \).

This shows that (2.17) and (2.18) are obviously equivalent.

Now, taking the square in (2.18), we get

\[
\| y_i \|^2 + \frac{(M - m)^2}{M + m} \| y_j \|^2 \leq 2 \text{Re} \langle y_i, \frac{M + m}{2} y_j \rangle + \frac{1}{n} (M - m)^2 \| y_j \|^2
\]

for \( 1 \leq i < j \leq n \), and since, obviously,

\[
2 \left( \frac{M + m}{2} \right) \| y_i \| \| y_j \| \leq \| y_i \|^2 + \frac{(M - m)^2}{M + m} \| y_j \|^2,
\]

hence

\[
2 \left( \frac{M + m}{2} \right) \| y_i \| \| y_j \| \leq 2 \text{Re} \langle y_i, \frac{M + m}{2} y_j \rangle + \frac{1}{n} (M - m)^2 \| y_j \|^2,
\]

giving the much simpler inequality

\[
\| y_i \| \| y_j \| - \text{Re} \langle y_i, y_j \rangle \leq \frac{1}{4} \left( \frac{M - m}{M + m} \right)^2 \| y_j \|^2,
\]

for \( 1 \leq i < j \leq n \).

Applying Lemma 1 for \( k_{ij} := \frac{1}{4} \left( \frac{M - m}{M + m} \right)^2 \| y_j \|^2 \), we deduce

\[
\left( \sum_{i=1}^{n} \| y_i \| \right)^2 \leq \left\| \sum_{i=1}^{n} y_i \right\|^2 + \frac{1}{2} \left( \frac{M - m}{M + m} \right)^2 \sum_{1 \leq i < j \leq n} \| y_j \|^2
\]

with equality if and only if (2.21) holds for each \( i, j \) with \( 1 \leq i < j \leq n \).
Since
\[\sum_{1 \leq i < j \leq n} \|y_j\|^2 = \sum_{1 < j \leq n} \|y_j\|^2 + \sum_{2 < j \leq n} \|y_j\|^2 + \cdots + \sum_{n-1 < j \leq n} \|y_j\|^2\]
\[= \sum_{j=2}^{n} \|y_j\|^2 + \sum_{j=3}^{n} \|y_j\|^2 + \cdots + \|y_j\|^2 + \|y_n\|^2\]
\[= \sum_{j=2}^{n} (j - 1) \|y_j\|^2 = \sum_{k=1}^{n-1} k \|y_{k+1}\|^2,\]
hence the inequality (2.19) is obtained.

3. Further Quadratic Refinements of the Triangle Inequality

The following lemma is of interest in itself as well.

Lemma 2. Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \(K\), \(x_i \in H, i \in \{1, \ldots, n\}\) and \(k \geq 1\) with the property that:

\[\|x_i\| \|x_j\| \leq k \Re \langle x_i, x_j \rangle,\]

for each \(i, j\) with \(1 \leq i < j \leq n\). Then

\[\left(\sum_{i=1}^{n} \|x_i\|^2\right)^2 + (k - 1) \sum_{i=1}^{n} \|x_i\|^2 \leq k \left(\sum_{i=1}^{n} x_i\right)^2.\]

The equality holds in (3.2) if and only if it holds in (3.1) for each \(i, j\) with \(1 \leq i < j \leq n\).

**Proof.** Firstly, let us observe that the following identity holds true:

\[\left(\sum_{i=1}^{n} \|x_i\|^2\right)^2 - k \left(\sum_{i=1}^{n} x_i\right)^2\]
\[= \sum_{i,j=1}^{n} \|x_i\| \|x_j\| - k \left(\sum_{i=1}^{n} x_i\right) \left(\sum_{j=1}^{n} x_j\right)\]
\[= \sum_{i,j=1}^{n} \|x_i\| \|x_j\| - k \Re \langle x_i, x_j \rangle\]
\[= 2 \sum_{1 \leq i < j \leq n} \|x_i\| \|x_j\| - k \Re \langle x_i, x_j \rangle + (1 - k) \sum_{i=1}^{n} \|x_i\|^2,\]
since, obviously, \(\Re \langle x_i, x_j \rangle = \Re \langle x_j, x_i \rangle\) for any \(i, j \in \{1, \ldots, n\}\).

Using the assumption (3.1), we obtain
\[\sum_{1 \leq i < j \leq n} \|x_i\| \|x_j\| - k \Re \langle x_i, x_j \rangle \leq 0\]
and thus, from (3.3), we deduce the desired inequality (3.2).

The case of equality is obvious by the identity (3.3) and we omit the details.
Remark 3. The inequality (3.2) provides the following reverse of the quadratic generalised triangle inequality:

\[ 0 \leq \left( \sum_{i=1}^{n} \|x_i\| \right)^2 - \sum_{i=1}^{n} \|x_i\|^2 \leq k \left( \left\| \sum_{i=1}^{n} x_i \right\| - \sum_{i=1}^{n} \|x_i\|^2 \right). \]

Remark 4. Since \( k = 1 \) and \( \sum_{i=1}^{n} \|x_i\|^2 \geq 0 \), hence by (3.2) one may deduce the following reverse of the triangle inequality

\[ \sum_{i=1}^{n} \|x_i\| \leq \sqrt{k} \left\| \sum_{i=1}^{n} x_i \right\|, \]

provided (3.1) holds true for \( 1 \leq i < j \leq n \).

The following corollary providing a better bound for \( \sum_{i=1}^{n} \|x_i\| \), holds.

Corollary 3. With the assumptions in Lemma 2, one has the inequality:

\[ \sum_{i=1}^{n} \|x_i\| \leq \sqrt{nk} \left( \sum_{i=1}^{n} \|x_i\| \right). \]

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality

\[ \frac{n \sum_{i=1}^{n} \|x_i\|^2}{n+k-1} \geq \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2 \]

we get

\[ (k-1) \sum_{i=1}^{n} \|x_i\|^2 + \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \geq \left( \frac{k-1}{n} + 1 \right) \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2. \]

Consequently, by (3.7) and (3.2) we deduce

\[ \sqrt{k} \left( \sum_{i=1}^{n} \|x_i\| \right)^2 \geq \frac{n+k-1}{n} \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^2 \]

giving the desired inequality (3.6).

The case of equality holds in (3.9) iff

\[ \left\| x_i - \frac{x_j}{\|x_j\|} \right\| = \rho \quad \text{for} \quad 1 \leq i < j \leq n. \]

Theorem 6. Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space and \(x_i \in H \setminus \{0\}, i \in \{1, \ldots, n\}, \rho \in (0,1), \) such that

\[ \left\| x_i - \frac{x_j}{\|x_j\|} \right\| \leq \rho \quad \text{for} \quad 1 \leq i < j \leq n. \]

Then we have the inequality

\[ \sqrt{1-\rho^2} \left( \sum_{i=1}^{n} \|x_i\| \right)^2 + \left( 1 - \sqrt{1-\rho^2} \right) \sum_{i=1}^{n} \|x_i\|^2 \leq \left( \sum_{i=1}^{n} x_i \right)^2. \]

The case of equality holds in (3.9) iff

\[ \|x_i\| \|x_j\| = \frac{1}{\sqrt{1-\rho^2}} \text{Re} \langle x_i, x_j \rangle \]

for any \( 1 \leq i < j \leq n. \)
Proof. The condition (3.1) is obviously equivalent to
\[ \|x_i\|^2 + 1 - \rho^2 \leq 2 \text{Re} \langle x_i, \frac{x_j}{\|x_j\|} \rangle \]
for each \(1 \leq i < j \leq n\).

Dividing by \(\sqrt{1 - \rho^2} > 0\), we deduce
\[ \frac{\|x_i\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2} \leq \frac{2}{\sqrt{1 - \rho^2}} \text{Re} \langle x_i, \frac{x_j}{\|x_j\|} \rangle, \]
for \(1 \leq i < j \leq n\).

On the other hand, by the elementary inequality
\[ \frac{p}{\alpha} + \frac{q\alpha}{\alpha} \geq 2\sqrt{pq}, \quad p, q \geq 0, \quad \alpha > 0 \]
we have
\[ 2\|x_i\| \leq \frac{\|x_i\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2}. \]

Making use of (3.11) and (3.13), we deduce that
\[ \|x_i\| \|x_j\| \leq \frac{1}{\sqrt{1 - \rho^2}} \text{Re} (x_i, x_j) \]
for \(1 \leq i < j \leq n\).

Now, applying Lemma 4 for \(k = \frac{1}{\sqrt{1 - \rho^2}}\), we deduce the desired result. \(\blacksquare\)

Remark 5. If we assume that \(\|x_i\| = 1, i \in \{1, \ldots, n\}\), satisfying the simpler condition
\[ \|x_j - x_i\| \leq \rho \quad \text{for } 1 \leq i < j \leq n, \]
then, from (3.9), we deduce the following lower bound for \(\|\sum_{i=1}^{n} x_i\|\), namely
\[ \left[ n + n (n - 1) \sqrt{1 - \rho^2} \right]^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|. \]
The equality holds in (3.15) iff \(\sqrt{1 - \rho^2} = \text{Re} (x_i, x_j)\) for \(1 \leq i < j \leq n\).

Remark 6. Under the hypothesis of Proposition 3 we have the coarser but simpler reverse of the triangle inequality
\[ \sqrt{1 - \rho^2} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \]

Also, applying Corollary 3 for \(k = \frac{1}{\sqrt{1 - \rho^2}}\), we can state that
\[ \sum_{i=1}^{n} \|x_i\| \leq \sqrt{\frac{n}{n\sqrt{1 - \rho^2} + 1 - \sqrt{1 - \rho^2}} \sum_{i=1}^{n} x_i}, \]
provided \(x_i \in H\) satisfy (3.8) for \(1 \leq i < j \leq n\).

In the same manner, we can state and prove the following reverse of the quadratic generalised triangle inequality.
Theorem 7. Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \(K\), \(x_i \in H, \ i \in \{1, \ldots, n\}\) and \(M \geq m > 0\) such that either
\[
\text{Re} \langle Mx_j - x_i, x_i - mx_j \rangle \geq 0 \quad \text{for} \ 1 \leq i < j \leq n,
\]
or, equivalently,
\[
\left\| x_i - \frac{M + m}{2} x_j \right\| \leq \frac{1}{2} (M - m) \| x_j \| \quad \text{for} \ 1 \leq i < j \leq n
\]
hold. Then
\[
2\sqrt{mM} \left( \sum_{i=1}^{n} \| x_i \| \right)^2 + \frac{\left( \sqrt{M} - \sqrt{m} \right)}{M + m} ^2 \sum_{i=1}^{n} \| x_j \|^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2.
\]
The case of equality holds in (3.20) if and only if
\[
\| x_i \| \| x_j \| = \frac{M + m}{2\sqrt{mM}} \text{Re} \langle x_i, x_j \rangle \quad \text{for} \ 1 \leq i < j \leq n.
\]

Proof. From (3.18), observe that
\[
\| x_i \|^2 + Mm \| x_j \|^2 \leq (M + m) \text{Re} \langle x_i, x_j \rangle,
\]
for \(1 \leq i < j \leq n\). Dividing (3.22) by \(\sqrt{mM} > 0\), we deduce
\[
\frac{\| x_i \|^2}{\sqrt{mM}} + \frac{\sqrt{mM} \| x_j \|^2}{\sqrt{mM}} \leq \frac{M + m}{\sqrt{mM}} \text{Re} \langle x_i, x_j \rangle,
\]
and since, obviously
\[
2 \| x_i \| \| x_j \| \leq \frac{\| x_i \|^2}{\sqrt{mM}} + \sqrt{mM} \| x_j \|^2,
\]
hence
\[
\| x_i \| \| x_j \| \leq \frac{M + m}{2\sqrt{mM}} \text{Re} \langle x_i, x_j \rangle, \quad \text{for} \ 1 \leq i < j \leq n.
\]
Applying Lemma 2 for \(k = \frac{M + m}{2\sqrt{mM}} \geq 1\), we deduce the desired result.

Remark 7. We also must note that a simpler but coarser inequality that can be obtained from (3.20) is
\[
\left( \frac{2\sqrt{mM}}{M + m} \right)^2 \sum_{i=1}^{n} \| x_i \|^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2,
\]
provided (3.21) holds true.

Finally, a different result related to the generalised triangle inequality is incorporated in the following theorem.

Theorem 8. Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over \(K\), \(\eta > 0\) and \(x_i \in H\), \(i \in \{1, \ldots, n\}\) with the property that
\[
\| x_j - x_i \| \leq \eta < \| x_j \| \quad \text{for each} \ i, j \in \{1, \ldots, n\}.
\]
Then we have the following reverse of the triangle inequality
\[
\sum_{i=1}^{n} \sqrt{\| x_i \|^2 - \eta^2} \leq \frac{\| \sum_{i=1}^{n} x_i \|^2}{\sum_{i=1}^{n} \| x_i \|^2}.
\]
The equality holds in \( (\mathbf{3.24}) \) iff
\[
\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} = \Re \langle x_i, x_j \rangle \quad \text{for each} \quad i, j \in \{1, \ldots, n\}.
\]

**Proof.** From \( (\mathbf{3.23}) \), we have
\[
\|x_i\|^2 - 2 \Re \langle x_i, x_j \rangle + \|x_j\|^2 \leq \eta^2,
\]
giving
\[
\|x_i\|^2 + \|x_j\|^2 - \eta^2 \leq 2 \Re \langle x_i, x_j \rangle, \quad i, j \in \{1, \ldots, n\}.
\]
On the other hand,
\[
2 \|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \|x_i\|^2 + \|x_j\|^2 - \eta^2, \quad i, j \in \{1, \ldots, n\}
\]
and thus
\[
\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \Re \langle x_i, x_j \rangle, \quad i, j \in \{1, \ldots, n\}.
\]
Summing over \( i, j \in \{1, \ldots, n\} \), we deduce the desired inequality \( (\mathbf{3.24}) \).

The case of equality is also obvious from the above, and we omit the details. \( \square \)

## 4. Applications for Vector-Valued Integral Inequalities

Let \( (H; \langle \cdot, \cdot \rangle) \) be a Hilbert space over the real or complex number field, \( [a, b] \) a compact interval in \( \mathbb{R} \) and \( \eta : [a, b] \to [0, \infty) \) a Lebesgue integrable function on \( [a, b] \) with the property that \( \int_a^b \eta(x) \, dx = 1 \). If, by \( L_\eta ([a, b]; H) \) we denote the Hilbert space of all Bochner measurable functions \( f : [a, b] \to H \) with the property that \( \int_a^b \eta(x) \| f(x) \|^2 \, dx < \infty \), then the norm \( \| \cdot \|_\eta \) of this space is generated by the inner product \( \langle \cdot, \cdot \rangle_\eta : H \times H \to \mathbb{K} \) defined by
\[
\langle f, g \rangle_\eta := \int_a^b \eta(x) \langle f(x), g(x) \rangle \, dx.
\]

The following proposition providing a reverse of the integral generalised triangle inequality may be stated.

**Proposition 4.** Let \( (H; \langle \cdot, \cdot \rangle) \) be a Hilbert space and \( \eta : [a, b] \to [0, \infty) \) as above. If \( g \in L_\eta ([a, b]; H) \) is so that \( \int_a^b \eta(x) \| g(x) \|^2 \, dx = 1 \) and \( f_i \in L_\eta ([a, b]; H), i \in \{1, \ldots, n\} \), and \( M \geq m > 0 \) are so that either
\[
\text{Re} \, (M f_j(t) - f_i(t), f_i(t) - m f_j(t)) \geq 0
\]
or, equivalently,
\[
\| f_i(t) - \frac{m + M}{2} f_j(t) \| \leq \frac{1}{2} (M - m) \| f_j(t) \|
\]
for a.e. \( t \in [a, b] \) and \( 1 \leq i < j \leq n \), then we have the inequality
\[
\left[ \sum_{i=1}^n \left( \int_a^b \eta(t) \| f_i(t) \|^2 \, dt \right)^{1/2} \right]^2 
\leq \int_a^b \eta(t) \left( \sum_{i=1}^n f_i(t) \right)^2 \, dt 
\leq \frac{1}{2} \cdot \frac{(M - m)^2}{m + M} \int_a^b \eta(t) \left( \sum_{k=1}^{n-1} k \| f_{k+1}(t) \|^2 \right) \, dt.
\]
The case of equality holds in (4.2) if and only if
\[
\left( \int_a^b \eta(t) \| f_i(t) \|^2 dt \right)^{1/2} \left( \int_a^b \eta(t) \| f_j(t) \|^2 dt \right)^{1/2} \\
- \int_a^b \eta(t) \text{Re} \langle f_i(t), f_j(t) \rangle dt \\
= \frac{1}{4} \frac{(M-m)^2}{m+M} \int_a^b \eta(t) \| f_j(t) \|^2 dt
\]
for each \( i, j \) with \( 1 \leq i < j \leq n \).

Proof. We observe that
\[
\text{Re} \langle Mf_j - f_i - mf_j \rangle_{\eta} \\
= \int_a^b \eta(t) \text{Re} \langle Mf_j(t) - f_i(t) - mf_j(t) \rangle dt \geq 0
\]
for any \( i, j \) with \( 1 \leq i < j \leq n \).

Applying Theorem 5 for the Hilbert space \( L_\eta([a,b];H) \) and for \( y_i = f_i, i \in \{1, \ldots, n\} \), we deduce the desired result.

Another integral inequality incorporated in the following proposition holds:

**Proposition 5.** With the assumptions of Proposition 4, we have
\[
(4.3) \quad \frac{2\sqrt{mM}}{m+M} \left[ \sum_{i=1}^n \left( \int_a^b \eta(t) \| f_i(t) \|^2 dt \right)^{1/2} \right]^2 \\
+ \frac{(\sqrt{M}-\sqrt{m})^2}{m+M} \sum_{i=1}^n \int_a^b \eta(t) \| f_i(t) \|^2 dt \\
\leq \int_a^b \eta(t) \left\| \sum_{i=1}^n f_i(t) \right\|^2 dt.
\]
The case of equality holds in (4.3) if and only if
\[
\left( \int_a^b \eta(t) \| f_i(t) \|^2 dt \right)^{1/2} \left( \int_a^b \eta(t) \| f_j(t) \|^2 dt \right)^{1/2} \\
= \frac{M+m}{2\sqrt{mM}} \int_a^b \eta(t) \text{Re} \langle f_i(t), f_j(t) \rangle dt
\]
for any \( i, j \) with \( 1 \leq i < j \leq n \).

The proof is obvious by Theorem 4 and we omit the details.

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