Long-time solvability for the 2D inviscid Boussinesq equations with critical regularity and dispersive effects

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Abstract
We are concerned with the long-time solvability for 2D inviscid Boussinesq equations. First we show the local solvability in Besov spaces uniformly with respect to a physical parameter $\kappa$ associated with the strength of gravity. After, employing a blow-up criterion and Strichartz-type estimates, the long-time solvability is obtained for large $\kappa$ and regardless of the size of initial data. Our results provide a larger class of initial data as well as cover borderline regularity for the system.

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1 Introduction
We consider the two-dimensional (2D) inviscid Boussinesq system with dispersive forcing

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u + \nabla p &= \kappa \theta e_2, \\
\partial_t \theta + (u \cdot \nabla)\theta &= 0, \\
\text{div } u &= 0, \\
u(x, 0) &= u_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{align*}
\]

(1.1)

where $(x, t) \in \mathbb{R}^2 \times (0, \infty)$, $u$ is the fluid velocity, $\theta$ denotes the temperature (or the density in geophysical flows), $p$ stands for the pressure, and the constant $\kappa$ is a parameter associated with the strength of gravity.

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The unit vector $e_2 = (0, 1)$ indicates the positive vertical direction. Applying the "curl" to the first equation in (1.1), and recalling that $\omega = \text{curl}(u) = \nabla^\perp \cdot u$, we arrive at the equivalent system

$$
\begin{aligned}
\partial_t \omega + (u \cdot \nabla)\omega &= \kappa \partial_1 \theta, \\
\partial_t \theta + (u \cdot \nabla)\theta &= 0, \\
u &= \nabla^\perp (-\Delta)^{-1/2}, \\
\nabla^\perp &= (-\partial_2, \partial_1), \\
\omega(x,0) &= \omega_0(x), \quad \theta(x,0) = \theta_0(x), \quad \text{in } \mathbb{R}^2.
\end{aligned}
$$

(1.2)

We wish to consider the initial temperature close to a nontrivial balance, namely $\theta_0(x) = \rho_0(x) - \kappa x_2$, and then look for the solution of (1.2) with the temperature in the form $\theta(t,x) = \rho(t,x) - \kappa x_2$; so that we will work with the following new system

$$
\begin{aligned}
\partial_t \omega + (u \cdot \nabla)\omega &= \kappa \partial_1 \rho, \\
\partial_t \rho + (u \cdot \nabla)\rho &= \kappa u_2, \\
u &= \nabla^\perp (-\Delta)^{-1/2}, \\
\omega(x,0) &= \omega_0(x), \quad \rho(x,0) = \rho_0(x), \quad \text{in } \mathbb{R}^2.
\end{aligned}
$$

(1.3)

The 2D Boussinesq equations arise as a model in lower dimensions for the 3D hydrodynamics equations by approximating the exact density of the fluid by a constant representative value [24]. In particular, these equations serve to model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see [23], [21]). Also, system (1.3) plays a relevant role in the analysis of the Rayleigh-Bénard convection [15].

From the mathematical point of view, the 2D Boussinesq equations are also important because, in the inviscid and viscous cases, they retain some structural ingredients of the 3D Euler and Navier-Stokes equations, respectively, such as the vortex stretching mechanism. In fact, in the context of 3D axisymmetric swirling flows and analyzing the dynamics away from the symmetry axis, the 2D inviscid Boussinesq system coincides with the 3D Euler equations. Given the importance of the global regularity problem for the 3D Euler and Navier-Stokes equations, which remains outstandingly open, this motivates the study of system (1.3) as a testbed for these equations.

Moreover, a number of applications have prompted the analysis for the viscous Boussinesq systems with nonlocal dissipative mechanisms for the velocity or the temperature (or both), which are represented by a fractional Laplacian operator. Global regularity has been shown in various scenarios of these viscous systems, but there is still quite a number of open problems; see for instance [8], [25], [30], and references therein.

Less is known for the regularity of the inviscid case (1.3), which in general appears to be more difficult. Recently, Elgindi and Jeong [18] showed the finite-time singularity formation of strong solutions of system (1.3) in certain domains by working with local solutions in spaces of critical regularity. They then extended their approach to show a similar result for the axisymmetric 3D Euler system. In this regard, it is worth observing that the structure of these equations is comparable to that of the inviscid SQG equations, so that local well-posedness results can be obtained with similar techniques ([11], [17], [14], [31]); even though long-time well-posedness for these problems in the whole space remains open.

The stability of solutions in fluid dynamics systems with the presence of viscous or thermal dissipation terms has been studied by several authors by means of different approaches, see, e.g., [16]. On the other
hand, the full inviscid case presents further difficulties and the stability property for (1.3) was addressed only recently in [17]. We observe that (1.3) models the setup where hot fluid is on top of cooler fluid, which is the opposite of that of the Rayleigh-Bénard convection (via instability) problem. In view of the proportionality between density and temperature in the Boussinesq approximation, the gravitational force is expected to stabilize such a density (or temperature) distribution, but it is conceivable that this might not be enough in the absence of dissipation.

However, by deriving a sharp dispersive estimate, Elgindi and Widmayer [17] are able to prove the non-linear stability of (1.3) about the stationary configuration given by $\rho_0$, on a time interval proportional to the power minus 4/3 of the perturbation size (see also [29] for related results). The dispersive estimate is also used to prove an increased existence time for the dispersive inviscid SQG equation; indeed, both proofs have a similar structure. We also quote [8] for a recent asymptotic stability result for a damped version of system (1.3) in the presence of boundaries.

In order to understand the stabilizing role of dispersion, it is useful to compare the results for these 2D models with those for their analogous 3D equations. For instance, the SQG equation with critical dissipation and dispersive forcing is the 2-dimensional analogue of the 3D Navier-Stokes-Coriolis (3DNSC) system [9]. The Coriolis term due to rotation in 3DNSC introduces a dispersive effect, such that in the limit of infinite dispersion (i.e. as the strength of rotation becomes large with the Rossby number going to zero) one obtains a (long-time well-posed) purely two-dimensional system: the velocity is independent of $x_3 \in \mathbb{R}$; this is known as columnar flow. In this address we refer the reader to [3, 13, 20] (see also their references), where the authors showed that high speeds of rotation tend to smooth out 3D Navier-Stokes and Euler flows. In analogy, it can be shown that this SQG system, where the dispersive forcing represents the meridional variation of the Coriolis force, the system is long-time solvable as the dispersive parameter tends to $\infty$, see [9, 19, 30] and [2] for results in Sobolev and Besov spaces, respectively.

In contrast, if one starts from the 3D Boussinesq system and rescales the temperature (to perform energy estimates) [31], the dispersive term is associated with a parameter $\sigma = g\lambda$ which combines the strength of gravity $g$ and that of stratification $\lambda$. It can then be shown [31] that, as $\sigma \to \infty$, the limiting system is a stratified system of 2D Euler equations, i.e. for any fixed $x_3 \in \mathbb{R}$ the velocity $u(x_1, x_2, x_3, t)$ solves a 2D Euler equation in the variables $(x_1, x_2, t)$.

The parameter $\kappa$ in system (1.3) can then be thought as analogous to $\sigma$ and we can think of the limit $\kappa \to \infty$ as the corresponding physical situation where the influence of stratification stabilizes the fluid; with $g$ fixed and $\lambda \to \infty$, or vice versa.

This idea was used by Wan and Chen [30] to show that system (1.3) has long-time solvability of strong solutions for large enough $\kappa$. The term solvability here refers to the pair of existence-uniqueness of solutions in an appropriate sense. The approach in [30] is based on generalizing the dispersive estimate of [17] and deriving the corresponding Strichartz estimates, to then show long-time solvability via a blow-up criterion in Sobolev spaces $(\dot{H}^s \cap \dot{H}^{-1}) \times H^{s+1}$, with $s > 2$. Note that $\dot{H}^s \cap \dot{H}^{-1} \hookrightarrow H^s$ for $s \geq 0$. We also refer the reader to [27] for a long-time existence result in $H^s$ ($s > 2$) for the 2D Boussinesq equations (1.1) with the stratification term $-\mathcal{N}^2 u_2$ ($\mathcal{N} >> 1$) in the right-hand side of the equation for $\theta$, by analyzing the time-evolution of the perturbations around a suitable stable state in hydrostatic balance.

Nevertheless, the methods of [30] do not reach the critical case $s = 2$; observe that a dimensional analysis gives that the critical value for long-time solvability of (1.3) in Sobolev spaces $H^s_p$ and Besov spaces $B^s_{p,q}$ is $s = \frac{p}{2} + 2$ for $p = 2$. Bearing in mind the importance of working with critical regularity
spaces for the issues mentioned above, our main goal is to improve these results to obtain local uniform solvability for \( s \geq 1 \) and long-time solvability for \( s \geq 2 \), notwithstanding we borrow from some previous known Strichartz estimates (see [20] and [30]). Our main theorem is as follows:

**Theorem 1.1.** Let \( s \) and \( q \) be real numbers such that \( s > 2 \) with \( 1 \leq q \leq \infty \) or \( s = 2 \) with \( q = 1 \).

(i) *(Local uniform solvability)* Let \( \omega_0 \in \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2) \) and \( \rho_0 \in B^s_{2,q}(\mathbb{R}^2) \). There exists \( T > 0 \) (depending of \( \|\omega_0\|_{\dot{B}^s_{2,q} \cap \dot{H}^{-1}} \) and \( \|\rho_0\|_{B^s_{2,q}} \)) such that (1.3) has a unique solution \((\omega, \rho)\) with \( \omega \in C([0, T]; \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, T]; \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \) and \( \rho \in C([0, T]; B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0, T]; B^s_{2,q}(\mathbb{R}^2)) \), for all \( \kappa \in \mathbb{R} \).

(ii) *(Long-time solvability)* Let \( T \in (0, \infty) \), \( \omega_0 \in \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2) \), and \( \rho_0 \in B^{s+1}_{2,q}(\mathbb{R}^2) \). There exists \( \kappa_0 = \kappa_0(T, \|\omega_0\|_{\dot{B}^s_{2,q} \cap \dot{H}^{-1}}, \|\rho_0\|_{B^{s+1}_{2,q}}) > 0 \) such that if \( |\kappa| \geq \kappa_0 \) then (1.3) has a unique solution \((\omega, \rho)\) such that \( \omega \in C([0, T]; \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, T]; \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \) and \( \rho \in C([0, T]; B^{s+1}_{2,q}(\mathbb{R}^2)) \cap C^1([0, T]; B^s_{2,q}(\mathbb{R}^2)) \).

In Theorem 1.1, we employ an intersection of spaces for the vorticity \( \omega \). In fact, \( \dot{H}^{-1} \) is important to control the influence of \( \kappa \) on the existence time \( T \), and then obtain a uniform time w.r.t \( \kappa \), via a cancellation effect involving the \( \dot{H}^{-1} \)-inner product, while the homogeneous Besov space \( \dot{B}^s_{2,q}(\mathbb{R}^2) \) provides the necessary control on the regularity, particularly for the critical case. It is worth noting that this difficulty does not appear in the context of Euler equations (with or without dispersive effects) when analyzing local solvability and critical regularity in Besov spaces (see [28, 22, 1]).

In order to prove Theorem 1.1 we first construct approximate solutions \((\omega_n, \rho_n)_{n \in \mathbb{N}}\) via a Picard iteration scheme, and show a priori estimates uniform with respect to the dispersive parameter \( \kappa \). This involves using commutator estimates for the Littlewood-Paley localizations so as to obtain a solution as the limit of \((\omega_n, \rho_n)\) in the Besov spaces with the critical regularity for (1.3) (see Section 4 and the proof of Theorem 1.1). Thus, due to the need for a uniform solvability, we prove some commutator estimates in the framework of homogeneous Besov spaces presented in a form which we could not find elsewhere in the literature (see Section 3). Subsequently, for large values of \( s \) and \( \kappa \geq 2 \), we obtain the long-time solvability by showing a blow-up criterion and handling globally the integral \( \int_0^T \| (\omega \pm \Lambda \rho)(\tau) \|_{B^s_{\infty,1}} \ d\tau \) using the Strichartz estimates.

The plan of the manuscript is as follows. In Section 2 we present some preliminaries about Besov spaces, Strichartz estimates, among others. Section 3 is devoted to the commutator estimates. In Section 4, we analyze the approximation scheme \((\omega_n, \rho_n)_{n \in \mathbb{N}}\) and obtain the local-in-time solvability of (1.3) uniformly with respect to the parameter \( \kappa \). The proof of Theorem 1.1 (ii) is carried out in Section 5.

## 2 Preliminaries

The purpose of this section is to provide some basic definitions and properties about Besov spaces as well as some estimates useful for our ends, such as product, embeddings, Strichartz estimates, among others. We refer the reader to the book [7] for more details on Besov spaces and their properties.

First, we denote the Schwartz space on \( \mathbb{R}^2 \) by \( \mathcal{S}(\mathbb{R}^2) \) and its dual by \( \mathcal{S}'(\mathbb{R}^2) \) (tempered distributions). For \( f \in \mathcal{S}'(\mathbb{R}^2) \), \( \hat{f} \) stands for the Fourier transform of \( f \). Select a radial function \( \psi_0 \in \mathcal{S}(\mathbb{R}^2) \) satisfying
0 \leq \hat{\psi}_0(\xi) \leq 1, \; \text{supp} \, \hat{\psi}_0 \subset \{ \xi \in \mathbb{R}^2 : \frac{5}{8} \leq |\xi| \leq \frac{7}{4} \} \text{ and }
\sum_{j \in \mathbb{Z}} \hat{\psi}_j(\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^2 \setminus \{0\},

where \( \psi_j(x) := 2^{2j}\psi_0(2^j x) \). For each \( k \in \mathbb{Z} \), we consider \( S_k, \hat{S}_k \in \mathcal{S} \) defined in Fourier variables as
\[
\hat{S}_k(\xi) = 1 - \sum_{j \geq k+1} \hat{\psi}_j(\xi) \quad \text{and} \quad \hat{S}_k(\xi) = \sum_{j \leq k} \hat{\psi}_j(\xi).
\]

We observe that
\[
\text{supp} \, \hat{\psi}_j \cap \text{supp} \, \hat{\psi}_{j'} = \emptyset \quad \text{if} \quad |j - j'| \geq 2.
\]

For \( f \in \mathcal{S}'(\mathbb{R}^2) \) and \( j \in \mathbb{Z} \), the Littlewood-Paley operator \( \Delta_j \) is the convolution \( \Delta_j f := \psi_j * f \) which works as a filter on the support of \( \psi_j \). We also consider the family of operators \( \{ \Delta_k \}_{k \in \{0\} \cup \mathbb{N}} \) defined as \( \Delta_0 = S_0 * f \) and \( \Delta_k = \Delta_k \) for every integer \( k \geq 1 \).

Let \( \mathcal{P} = \mathcal{P}(\mathbb{R}^2) \) denote the set of polynomials and consider \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \). The homogeneous Besov space \( \dot{B}^{s}_{p,q}(\mathbb{R}^2) \) is the set of all \( f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P} \) such that
\[
\|f\|_{\dot{B}^{s}_{p,q}} := \left\| \{ 2^{sj} \| \Delta_j f \|_{L^p} \} \right\|_{\ell^q(\mathbb{Z})} < \infty.
\]

The nonhomogeneous version of \( \dot{B}^{s}_{p,q}(\mathbb{R}^2) \), namely the Besov space \( \dot{B}^{s}_{p,q}(\mathbb{R}^2) \), is the space of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that the norm \( \|f\|_{\dot{B}^{s}_{p,q}} < \infty \), where
\[
\|f\|_{\dot{B}^{s}_{p,q}} = \begin{cases} \left( \sum_{k=0}^{\infty} 2^{ks} \| \Delta_k f \|_{L^p}^q \right)^{\frac{1}{q}}, & \text{if} \quad q < \infty, \\ \sup_{k \in \{0\} \cup \mathbb{N}} \{ 2^{ks} \| \Delta_k f \|_{L^p} \}, & \text{if} \quad q = \infty. \end{cases}
\]

The pairs \( (\dot{B}^{s}_{p,q}, \| \cdot \|_{\dot{B}^{s}_{p,q}}) \) and \( (B^{s}_{p,q}, \| \cdot \|_{B^{s}_{p,q}}) \) are Banach spaces. Also, for \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \), it follows that
\[
\|f\|_{\dot{B}^{s}_{p,q}} \leq C \left( \|f\|_{L^p} + \|f\|_{B^{s}_{p,q}} \right).
\]

For \( s > 0 \), we have the equivalence of norms
\[
\|f\|_{\dot{B}^{s}_{p,q}} \sim \|f\|_{L^p} + \|f\|_{\dot{B}^{s}_{p,q}}.
\]

In the case \( s = 0 \), we recall the inclusion \( B^0_{p,1} \hookrightarrow L^p \), for all \( p \in [1, \infty] \).

**Lemma 2.1** (Bernstein’s Lemma). Let \( 1 \leq p < \infty \) and \( f \in L^p \) be such that \( \text{supp} \, \hat{f} \subset \{ \xi \in \mathbb{R}^2 : 2^{j-2} \leq |\xi| \leq 2^j \} \). Then, we have the estimates
\[
C^{-1} 2^{jk} \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C 2^{jk} \|f\|_{L^p},
\]

where \( C = C(k) \) is a positive constant.

**Remark 2.2.** Using the above lemma, one can prove the equivalence
\[
\|f\|_{\dot{B}^{s+k}_{p,q}} \sim \|D^k f\|_{\dot{B}^{s}_{p,q}}.
\]
Moreover, considering $1 \leq p, q \leq \infty$ and $s \geq n/p$, with $q = 1$ if \( s = n/p \), we have that (see, e.g., \[7\])

\[
\|f\|_{L^\infty} \leq C\|f\|_{\dot{B}^s_{p,q}}.
\]

Then,

\[
\|\nabla f\|_{L^\infty} \leq C\|\nabla f\|_{\dot{B}^{s-1}_{p,q}} \leq C\|f\|_{\dot{B}^s_{p,q}},
\]

where $1 \leq p, q \leq \infty$ and $s \geq n/p + 1$ with $q = 1$ in the case $s = n/p + 1$.

Some Leibniz-type rules in Besov spaces are the subject of the lemma below (see [10]).

**Lemma 2.3.** Let $s > 0$, $1 \leq p_1, p_2 \leq \infty$, $1 \leq r_1, r_2 \leq \infty$ and $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, we have the estimates

\[
\|fg\|_{\dot{B}^s_{p,q}} \leq C(\|g\|_{L^{p_2}}\|f\|_{\dot{B}^{s}_{p_1,q}} + \|f\|_{L^{r_2}}\|g\|_{\dot{B}^{s}_{r_1,q}}),
\]

\[
\|fg\|_{\dot{B}^s_{p,q}} \leq C(\|g\|_{L^{p_2}}\|f\|_{\dot{B}^{s}_{p_1,q}} + \|f\|_{L^{r_2}}\|g\|_{\dot{B}^{s}_{r_1,q}}),
\]

where $C > 0$ is a universal constant.

We will employ the Strichartz estimates of [30], linked to the dispersive term $\kappa(\partial_1 \rho + \Lambda u_2)$ obtained from (1.3), which will allow us to obtain long-time solvability for (1.3). In particular, we will use the following results found in [2], [20] and [30].

**Lemma 2.4.** Let $\kappa \in \mathbb{R}$, $4 \leq \gamma \leq \infty$ and $2 \leq r \leq \infty$ be such that

\[
\frac{1}{\gamma} + \frac{1}{2r} \leq \frac{1}{4}, \tag{2.1}
\]

Then, there holds

\[
\|G_\pm(\kappa t)f\|_{L^{\gamma}(0,\infty;L^r)} \leq C\|\kappa\|^{-\frac{1}{4}}\|f\|_{L^2},
\]

for all $f \in L^2(\mathbb{R}^2)$, where

\[
G_\pm(t)f(x) := \int_{\mathbb{R}^2} e^{i\xi \cdot x \pm i\gamma t}\hat{\phi}(\xi)\hat{f}(\xi)\, d\xi
\]

and $\hat{\phi}$ is a compactly supported smooth function in $\mathbb{R}^2$.

**Lemma 2.5.** Let $s, t, \kappa \in \mathbb{R}$, $1 \leq q \leq \infty$, $4 \leq \gamma \leq q$, and $2 \leq r \leq \infty$. Assume also (2.1). Then,

\[
\|e^{\pm \kappa \rho \tau_1}f\|_{L^{\gamma}(0,\infty;\dot{B}^s_{r,q})} \leq C\|\kappa\|^{-\frac{1}{4}}\|f\|_{\dot{B}^{s+1-\frac{2}{r}}_{2,q}},
\]

for all $f \in \dot{B}^{s+1-\frac{2}{r}}_{2,q}(\mathbb{R}^2)$.

## 3 Commutator estimates

The present section is devoted to commutator estimates in $\dot{B}^s_{p,q}$ and $B^s_{p,q}$ that will be useful to obtain convergence of our approximate solutions. We state and prove some of them, as we have not been able to locate them in the literature with the needed hypotheses and conclusions for our purposes.

Recall the commutator operator

\[
[f \cdot \nabla, \Delta_j]g = f \cdot \nabla(\Delta_j g) - \Delta_j(f \cdot \nabla g).
\]

Using the Hölder inequality in a slightly different way than used in [10, 26, 32], it is possible to obtain the following estimates for the commutator:
Lemma 3.1. Let $1 < p < \infty$ and $1 \leq q, p_1, p_2, r_1, r_2 \leq \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

(i) Let $s > 0$, $f \in \dot{B}^s_{p_1,q}(\mathbb{R}^n)$ and $g \in \dot{B}^s_{r_1,q}(\mathbb{R}^n)$. Assume further that $\nabla f \in L^{r_2}(\mathbb{R}^n)$, $\nabla \cdot f = 0$ and $\nabla g \in L^{p_2}(\mathbb{R}^n)$. Then, we have the estimate

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [f \cdot \nabla, \Delta_j]g \|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^{r_2}} \| g \|_{\dot{B}^s_{r_1,q}} + \| \nabla g \|_{L^{p_2}} \| f \|_{\dot{B}^s_{p_1,q}} \right),$$

where $C > 0$ is a universal constant.

(ii) Let $s > -1$, $f \in \dot{B}^{s+1}_{p_1,q}(\mathbb{R}^n)$ and $g \in \dot{B}^s_{r_1,q}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Assume further that $\nabla f \in L^{r_2}(\mathbb{R}^n)$ and $\nabla \cdot f = 0$. Then, we have the estimate

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [f \cdot \nabla, \Delta_j]g \|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^{r_2}} \| g \|_{\dot{B}^s_{r_1,q}} + \| g \|_{L^{p_2}} \| f \|_{\dot{B}^{s+1}_{p_1,q}} \right),$$

where $C > 0$ is a universal constant.

Remark 3.2. With the help of Lemma 6.3 in [32] (see also [5]), the properties of the operator $\Lambda^{-1}$, considering the same hypotheses of Lemma 3.1 and using the same arguments in the proof of the same result, we obtain the following commutator estimate

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [f, \nabla, \Lambda^{-1}\Delta_j]g \|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^{r_2}} \| g \|_{\dot{B}^{s-1}_{r_1,q}} + \| g \|_{L^{p_2}} \| f \|_{\dot{B}^s_{p_1,q}} \right).$$

Recall that the Bony formula for the paraproduct of $f$ and $g$ is given by

$$fg = Tfg + Tg + R(f,g), \quad (3.1)$$

where

$$Tfg := \sum_{j \in \mathbb{Z}} S_{j-2}f \Delta_j g \quad \text{and} \quad R(f,g) := \sum_{|j-j'| \leq 1} \Delta_j f \Delta_j' g. \quad (3.2)$$

In the sequel we state and prove the following commutator-type estimates:

Lemma 3.3. Let $1 < p < \infty$ and $1 \leq q, p_1, p_2, r_1, r_2 \leq \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

(i) Let $s > 0$, $f \in \dot{B}^s_{p_1,q}(\mathbb{R}^n)$ with $\nabla f \in L^{r_2}(\mathbb{R}^n)$ and $\nabla \cdot f = 0$, and $g \in \dot{B}^s_{r_1,q}(\mathbb{R}^n)$ with $\nabla g \in L^{p_2}(\mathbb{R}^n)$. Then, there exists a universal constant $C > 0$ such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| (\dot{S}_{j-2}f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla)g \|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^{r_2}} \| g \|_{\dot{B}^s_{r_1,q}} + \| \nabla g \|_{L^{p_2}} \| f \|_{\dot{B}^s_{p_1,q}} \right).$$

(ii) Let $s > -1$, $f \in \dot{B}^{s+1}_{p_1,q}(\mathbb{R}^n)$ with $\nabla f \in L^{r_2}(\mathbb{R}^n)$ and $\nabla \cdot f = 0$, and $g \in \dot{B}^s_{r_1,q}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then, there exists a universal constant $C > 0$ such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| (\dot{S}_{j-2}f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla)g \|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^{r_2}} \| g \|_{\dot{B}^s_{r_1,q}} + \| g \|_{L^{p_2}} \| f \|_{\dot{B}^{s+1}_{p_1,q}} \right).$$
Proof. The proof of part (i), it follows from the calculations obtained by Chae in [10]. We show the part (ii). We follow closely the argument in [28] (see also [4, 12]). By Bony’s paraproduct formula (3.1), we can write
\[
(\hat{S}_{j-2} f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla) g = - \sum_{i=1}^{n} \Delta_j T_{i, g_k} f_i + \sum_{i=1}^{n} [T_{i, \partial_i \Delta_j g_k} f_i] - \sum_{i=1}^{n} \Delta_j T_{i-1} \partial_i \Delta_j g_k
\]
\[= - \sum_{i=1}^{n} \{ \Delta_j R(f_i, \partial_i g_k) - R(S_{j-2} f_i, \partial_i \Delta_j g_k) \} := I + II + III + IV.
\]
For $I$, in view of (3.2), it follows that
\[
I = - \sum_{j-j' \leq 3} \sum_{i=1}^{n} \Delta_j \hat{S}_{j-2} (\partial_i g_k) \Delta_j f_i.
\]
We observe that $\text{supp} \mathcal{F}(\hat{S}_{j-2} (\partial_i g_k) \Delta_j f_i) \subset \{ \xi : 2^{j'-3} \leq |\xi| \leq 2^{j'+1} \}$ and $\Delta_j \hat{S}_{j-2} (\partial_i g_k) \Delta_j f_i = 0$, if $|j-j'| \geq 4$. Then,
\[
I = - \sum_{j-j' \leq 3} \sum_{i=1}^{n} \Delta_j \hat{S}_{j-2} (\partial_i g_k) \Delta_j f_i.
\]
Using integration by parts, we arrive at
\[
I = - \sum_{j-j' \leq 3} \sum_{i=1}^{n} 2^{jn} \int_{\mathbb{R}^n} \psi_0(2^j (x - y)) (\hat{S}_{j-2} \partial_i g_k)(y) (\Delta_j f_i)(y) dy
\]
\[= - \sum_{j-j' \leq 3} \sum_{i=1}^{n} 2^j 2^{jn} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j (x - y)) (\hat{S}_{j-2} g_k)(y) (\Delta_j f_i)(y) dy
\]
\[= - \sum_{j-j' \leq 3} \sum_{i=1}^{n} 2^j \{ 2^{jn} \partial_i \psi_0(2^j \cdot) * ((\hat{S}_{j-2} g_k)(\Delta_j f_i)) \},
\]
which yields
\[
\|I\|_{L^p} \leq C \sum_{j-j' \leq 3} \sum_{i=1}^{n} 2^j \| (\hat{S}_{j-2} g_k)(\Delta_j f_i) \|_{L^p}
\]
\[\leq C \|g\|_{L^{p_2}} \sum_{j-j' \leq 3} 2^j \| \Delta_j f \|_{L^{p_1}}.
\]
(3.3)

For estimate $II$, by an argument similar to the one above, we first note that
\[
[\hat{S}_{j-2} f_i, \Delta_j] (\partial_i \Delta_j g_k) = 0, \text{ if } |j-j'| \geq 4.
\]
Then, using $\nabla \cdot \dot{S}_{j-2}f = 0$ and integration by parts, it holds that

$$II = \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} \{ (\dot{S}_{j-2}f_i) \Delta_j (\partial_i \Delta_j g_k) - \Delta_j (\dot{S}_{j-2}f_i) (\partial_i \Delta_j g_k) \}$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} 2^{jn} \int_{\mathbb{R}^n} \psi_0(2^j(x-y)) (\dot{S}_{j-2}f_i(x) - \dot{S}_{j-2}f_i(y)) (\partial_i \Delta_j g_k(y)) \, dy$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} 2^{jn+1} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j(x-y)) (\dot{S}_{j-2}f_i(x) - \dot{S}_{j-2}f_i(y)) (\Delta_j g_k(y)) \, dy$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} 2^{jn+1} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j(x-y)) \int_{0}^{1} ((x-y) \cdot \nabla) (\dot{S}_{j-2}f_i(x + \tau(y-x))) \, d\tau \, (\Delta_j g_k(y)) \, dy$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} \int_{\mathbb{R}^n} \partial_i \psi_0(z) \int_{0}^{1} (z \cdot \nabla) (\dot{S}_{j-2}f_i(x + \tau 2^{-j}z)) \, d\tau \, (\Delta_j g_k)(x - 2^{-j}z) \, dz.$$ 

Therefore,

$$|II| \leq \sum_{i=1}^{n} \sum_{|j-j'| \leq 3} \int_{\mathbb{R}^n} |\partial_i \psi_0(z)| \int_{0}^{1} |z| |\nabla (\dot{S}_{j-2}f_i(x + \tau 2^{-j}z))| \, d\tau \, |(\Delta_j g_k)(x - 2^{-j}z)| \, dz$$

$$\leq C \| \nabla f \|_{L^\infty} \sum_{|j-j'| \leq 3} \int_{\mathbb{R}^n} |z| |\nabla \psi_0(z)|||\Delta_j g_k)(x - 2^{-j}z)| \, dz,$$

which leads us to

$$\|II\|_{L^p} \leq C \| \nabla f \|_{L^\infty} \sum_{|j-j'| \leq 3} \int_{\mathbb{R}^n} |z| |\nabla \psi_0(z)|||\Delta_j g_k)(x - 2^{-j}z)||_{L^p} \, dz$$

$$\leq C \| \nabla f \|_{L^1} \sum_{|j-j'| \leq 3} \|\Delta_j g\|_{L^2}. \quad (3.4)$$

For $III$, we have that

$$III = \sum_{i=1}^{n} \sum_{|j-j'| \leq 1} \dot{S}_{j-2} (f_i - \dot{S}_{j-2}f_i) \partial_i \Delta_j \Delta_j g_k$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 1} \dot{S}_{j-2} \left( \sum_{m=j-1}^{j-1} \Lambda m f_i \right) \partial_i \Delta_j \Delta_j g_k$$

$$= \sum_{i=1}^{n} \sum_{|j-j'| \leq 1} \dot{S}_{j-2} (\Delta_{j-1} f_i + \Delta_j f_i) \partial_i \Delta_j \Delta_j g_k.$$ 

Applying the $L^p$-norm, we arrive at

$$\|III\|_{L^p} \leq \sum_{i=1}^{n} \sum_{|j-j'| \leq 1} \left( \|\Delta_{j-1} f_i\|_{L^\infty} + \|\Delta_j f_i\|_{L^\infty} \right) \|\partial_i \Delta_j \Delta_j g_k\|_{L^p}$$

$$\leq \sum_{i=1}^{n} \sum_{|j-j'| \leq 1} (2^{-j+1} \|\Delta_{j-1} \nabla f_i\|_{L^\infty} + 2^{-j} \|\Delta_j \nabla f_i\|_{L^\infty}) \|\Delta_j \Delta_j g_k\|_{L^p}$$

$$\leq C \| \nabla f \|_{L^1} \sum_{|j-j'| \leq 1} \|\Delta_j g\|_{L^2}. \quad (3.5)$$
For the parcel \( IV \), we can decompose

\[
IV = \sum_{i=1}^{n} \Delta_j \partial_i R(f_i - \hat{S}_{j-2} f_i, g_k) + \sum_{i=1}^{n} \{ \Delta_j R(\hat{S}_{j-2} f_i, \partial_i g_k) - R(\hat{S}_{j-2} f_i, \Delta_j \partial_i g_k) \} = IV_1 + IV_2.
\]

Since \( \sum_{i=1}^{n} \partial_i \Delta_j' (f_i - \hat{S}_{j-2} f_i) = 0 \), it follows that

\[
IV_1 = \sum_{i=1}^{n} \partial_i \Delta_j \left\{ \sum_{|j' - j''| \leq 1} \Delta_j' (f_i - \hat{S}_{j-2} f_i) \Delta_j'' g_k \right\}
\]

\[
= \sum_{i=1}^{n} \Delta_j \left\{ \sum_{|j' - j''| \leq 1} \Delta_j' (f_i - \hat{S}_{j-2} f_i) \Delta_j'' \partial_i g_k \right\}
\]

\[
= \sum_{i=1}^{n} \sum_{|j' - j''| \leq 1} \sum_{j' \geq j-3} 2^{jn} \int_{\mathbb{R}^n} \psi_0(2^j(x - y))(\Delta_j' f_i(y) - \hat{S}_{j-2} \Delta_j' f_i(y)) \Delta_j'' \partial_i g_k(y) dy
\]

\[
= \sum_{i=1}^{n} \sum_{|j' - j''| \leq 1} \sum_{j' \geq j-3} 2^{jn} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j(x - y))(\Delta_j' f_i(y) - \hat{S}_{j-2} \Delta_j' f_i(y)) \Delta_j'' g_k(y) dy
\]

\[
= \sum_{i=1}^{n} \sum_{|j' - j''| \leq 1} \sum_{j' \geq j-3} 2^j \{(2^{jn} \partial_i \psi_0(2^j \cdot)) * (\Delta_j' f_i - \hat{S}_{j-2} \Delta_j' f_i) \Delta_j'' g_k \}
\]

Then

\[
\|IV_1\|_{L^p} \leq C \sum_{i=1}^{n} \sum_{|j' - j''| \leq 1} \sum_{j' \geq j-3} 2^j \|\Delta_j' f_i\|_{L^p} + \|\hat{S}_{j-2} \Delta_j' f_i\|_{L^p} \|\Delta_j'' g_k\|_{L^\infty}
\]

\[
\leq C \|g\|_{L^{p2}} \sum_{j' \geq j-3} 2^j \|\Delta_j' f\|_{L^{p1}}.
\]

(3.6)

On the other hand, note that

\[
IV_2 = \sum_{i=1}^{n} \sum_{|j' - j''| \leq 1} \left[ \Delta_j((\Delta_j \hat{S}_{j-2} f_i) \Delta_j'' \partial_i g_k) - (\Delta_j \hat{S}_{j-2} f_i)(\Delta_j'' \Delta_j \partial_i g_k) \right]
\]

\[
= \sum_{i=1}^{n} \sum_{j - 1 \geq j' \geq j-3} \sum_{|j' - j''| \leq 1} \left[ \Delta_j, \Delta_j' \hat{S}_{j-2} f_i \Delta_j'' \partial_i g_k \right]
\]

Also, we can write

\[
[\Delta_j, \Delta_j' \hat{S}_{j-2} f_i] \Delta_j'' \partial_i g_k
\]

\[
= 2^{jn} \int_{\mathbb{R}^n} \psi_0(2^j(x - y))(\Delta_j' \hat{S}_{j-2} f_i(y) - \Delta_j \hat{S}_{j-2} f_i(x) \Delta_j'' \partial_i g_k(y) dy
\]

\[
= 2^{n+1} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j(x - y))(\Delta_j' \hat{S}_{j-2} f_i(y) - \Delta_j \hat{S}_{j-2} f_i(x)) \Delta_j'' g_k(y) dy
\]

\[
= 2^{n+1} \int_{\mathbb{R}^n} \partial_i \psi_0(2^j(x - y)) \int_0^1 ((y - x) \cdot \nabla)(\Delta_j' \hat{S}_{j-2} f_i(x + \tau(y - x)) d\tau \Delta_j'' g_k(y) dy
\]

\[
= \int_{\mathbb{R}^n} \partial_i \psi_0(z) \int_0^1 (z \cdot \nabla)(\Delta_j' \hat{S}_{j-2} f_i(x - \tau 2^{-j} z) d\tau \Delta_j'' g_k(x - 2^{-j} z) dz.
\]

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Hence,
\[
\| [\Delta_j, \Delta_j' \hat{S}_{j-2} f] \Delta_j'' \partial_z g_k \|_{L^p} \leq C \sum_{m=1}^{n} \| \Delta_j' \hat{S}_{j-2} \partial_m f_i \|_{L^\infty} \| \Delta_j'' g_k \|_{L^p}
\]
\[
\leq C \| \nabla f \|_{L^\infty} \| \Delta_j'' g_k \|_{L^p},
\]
and
\[
\| IV_2 \|_{L^p} \leq C \| \nabla f \|_{L^p} \sum_{|j-j'| \leq 5} \| \Delta_j' g_k \|_{L^p}.
\]

Summing up the estimates (3.3), (3.4), (3.5), (3.6) and (3.7), we obtain
\[
\| (\hat{S}_{j-2} f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla) g \|_{L^p}
\]
\[
\leq C \| \nabla f \|_{L^2} \sum_{|j-j'| \leq 5} \| \Delta_j' g_k \|_{L^1} + C \| g \|_{L^2} \left( \sum_{|j-j'| \leq 5} 2^j \| \Delta_j' f \|_{L^1} + \sum_{j' \geq j-3} 2^{j'} \| \Delta_j' f \|_{L^1} \right).
\]

Multiplying by $2^{j/2}$ and computing the $l^q(\mathbb{Z})$-norm, we can estimate
\[
\left( \sum_{j \in \mathbb{Z}} 2^{j/2} \| (\hat{S}_{j-2} f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla) g \|_{L^p}^q \right)^{1/q}
\]
\[
\leq C \| \nabla f \|_{L^2} \left( \sum_{j \in \mathbb{Z}} \sum_{|j-j'| \leq 5} 2^{j/2} \| \Delta_j' g_k \|_{L^1}^q \right)^{1/q} + C \| g \|_{L^2} \left( \sum_{j \in \mathbb{Z}} \sum_{|j-j'| \leq 5} 2^{j(s+1)/q} \| \Delta_j' f \|_{L^1}^q \right)^{1/q}
\]
\[
+ C \| g \|_{L^2} \left( \sum_{j \in \mathbb{Z}} \sum_{j' \geq j-3} 2^{j(s+1)/q} \| \Delta_j f \|_{L^1}^q \right)^{1/q}
\]
\[
:= K_1 + K_2 + K_3.
\]

Now, observing that
\[
\sum_{j \in \mathbb{Z}} \sum_{|j-j'| \leq 5} 2^{j/2} \| \Delta_j' h \|_{L_p}^q = 2^{kq} \sum_{j \in \mathbb{Z}} 2^{(j+k)/2q} \| \Delta_j + k h \|_{L_p}^q \leq C \sum_{j \in \mathbb{Z}} \| \Delta_j h \|_{L_p}^q,
\]
we have that
\[
K_1 \leq C \| \nabla f \|_{L^2} \| g \|_{B^{s+1}_{p,1,q}},
\]
and similarly $K_2 \leq C \| g \|_{L^2} \| f \|_{B^{s+1}_{p,1,q}}$.

For $K_3$, note that
\[
K_3 = C \| g \|_{L^2} \left( \sum_{j \in \mathbb{Z}} \sum_{j' \geq j-3} 2^{(j-j')(s+1)/2} \left( 2^{j'(s+1)/2} \| \Delta_j' f \|_{L^1} \right)^q \right)^{1/q}
\]
\[
= C \| g \|_{L^2} \left( \sum_{k \geq -3} 2^{-kq(s+1)} \left( \sum_{j \in \mathbb{Z}} \sum_{j' \geq -3} 2^{(j+k)/2q} \| \Delta_j + k f \|_{L^1} \right)^q \right)^{1/q}
\]
\[
= C \| g \|_{L^2} \left( \sum_{k \geq -3} 2^{-kq(s+1)} \left( \sum_{j \in \mathbb{Z}} 2^{jq(s+1)} \| \Delta_j f \|_{L^1} \right)^q \right)^{1/q}
\]
\[
\leq C \| g \|_{L^2} \| f \|_{B^{s+1}_{p,1,q}}.
\]
Therefore,
\[
\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \left\| (\mathcal{S}_{j-2} f \cdot \nabla) \Delta_j g - \Delta_j (f \cdot \nabla) g \right\|_{L^p}^q \right)^{1/q} \leq C \left( \| \nabla f \|_{L^2} \| g \|_{\dot{B}^s_{p,q}} + \| g \|_{L^p} \| f \|_{\dot{B}^{s+1}_{p,q}} \right).
\]
This completes the proof of (ii).

\[\hfill\]

4 An approximate linear iteration problem and local-in-time solvability

In order to prove the local existence to (1.3), we consider the approximate linear iteration problem

\[
\begin{aligned}
\partial_t \omega_{n+1} + (u_n \cdot \nabla) \omega_{n+1} &= \kappa \partial_1 \rho_{n+1} \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
\partial_t \rho_{n+1} + (u_n \cdot \nabla) \rho_{n+1} &= \kappa u_{2,n+1} \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
u_{n+1} &= \nabla^\perp (\Delta)^{-1} \omega_{n+1} \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
\omega_{n+1}|_{t=0} &= S_{n+2} \omega_0, \quad \rho_{n+1}|_{t=0} = S_{n+2} \rho_0 \quad \text{in} \quad \mathbb{R}^2.
\end{aligned}
\]

From (4.1), we provide uniform estimates for the sequence \( \{(\omega_n, \rho_n)\}_{n \in \mathbb{N}} \) and then obtain a solution for (1.3).

**Uniform estimates.** Applying \( \Delta_j \) in (4.1), taking the product with \( \Delta_j \omega_{n+1} \) in \( \dot{H}^{-1} \) and the product with \( \Delta_j \rho_{n+1} \) in \( L^2 \) in the first and second equations, respectively, and using the divergence-free condition \( \nabla \cdot \Delta_j u_n = 0 \), we obtain that

\[
\langle \Delta_j \partial_t \omega_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} = \langle (u_n \cdot \nabla) \Delta_j \omega_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} + \kappa \langle \Delta_j \partial_1 \rho_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}},
\]

\[
\langle \Delta_j \partial_t \rho_{n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} = \langle (u_n \cdot \nabla) \Delta_j \rho_{n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} + \kappa \langle \Delta_j u_{2,n+1}, \Delta_j \rho_{n+1} \rangle_{L^2}.
\]

Adding the two above inequalities and using the properties

\[
\langle (u_n \cdot \nabla) \Lambda^{-1} \Delta_j \omega_{n+1}, \Lambda^{-1} \Delta_j \omega_{n+1} \rangle_{L^2} = \langle (u_n \cdot \nabla) \Delta_j \rho_{n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} = 0,
\]

\[
\langle \Delta_j \partial_1 \rho_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle \Delta_j u_{2,n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} = 0,
\]

we arrive at

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta_j \omega_{n+1} \|_{\dot{H}^{-1}}^2 + \| \Delta_j \rho_{n+1} \|_{L^2}^2 \right) \leq \| (u_n \cdot \nabla) \Lambda^{-1} \Delta_j \omega_{n+1} \|_{L^2} \| \Delta_j \omega_{n+1} \|_{\dot{H}^{-1}} + \| (u_n \cdot \nabla) \Delta_j \rho_{n+1} \|_{L^2} \| \Delta_j \rho_{n+1} \|_{L^2}.
\]

Integrating over \((0, t)\), it follows that

\[
\frac{1}{2} \left( \| \Delta_j \omega_{n+1}(t) \|_{\dot{H}^{-1}}^2 + \| \Delta_j \rho_{n+1}(t) \|_{L^2}^2 \right) \leq \frac{1}{2} \left( \| \Delta_j \omega_{n+1}(0) \|_{\dot{H}^{-1}}^2 + \| \Delta_j \rho_{n+1}(0) \|_{L^2}^2 \right)
+ \int_0^t \left( \| (u_n(\tau) \cdot \nabla) \Lambda^{-1} \Delta_j \omega_{n+1}(\tau) \|_{L^2} + \| (u_n(\tau) \cdot \nabla, \Delta_j \rho_{n+1}(\tau) \|_{L^2} \right) \left( \| \Delta_j \omega_{n+1}(\tau) \|_{\dot{H}^{-1}}^2 + \| \Delta_j \rho_{n+1}(\tau) \|_{L^2}^2 \right)^{1/2} d\tau.
\]

Thus, by Grönwall’s inequality (see Proposition 1.2 in [6, page 24]), we have

\[
\| \Delta_j \omega_{n+1}(t) \|_{\dot{H}^{-1}} + \| \Delta_j \rho_{n+1}(t) \|_{L^2} \leq \| \Delta_j \omega_{n+1}(0) \|_{\dot{H}^{-1}} + \| \Delta_j \rho_{n+1}(0) \|_{L^2}
+ \int_0^t \left( \| (u_n(\tau) \cdot \nabla, \Lambda^{-1} \Delta_j \omega_{n+1}(\tau) \|_{L^2} + \| (u_n(\tau) \cdot \nabla, \Delta_j \rho_{n+1}(\tau) \|_{L^2} d\tau.
\]

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Combining \( \omega \parallel n + 1 \sum_{t} \left( \sum_{j} \omega \parallel n \cdot \nabla, \Lambda^{-1} \Delta \right) \omega_{n+1}(\tau) \), we get
\[
\left( \sum_{j \in \mathbb{Z}} 2^{sj} \left\| \left[ u_n \cdot \nabla, \Lambda^{-1} \Delta \right] \omega_{n+1} \right\|^q_{L^2} \right)^{\frac{1}{q}} \leq C \left( \left\| \nabla u_n \right\|_{L^\infty} \left\| \omega_{n+1} \right\|_{B^{s-1}_{2,q}} + \left\| \omega_{n+1} \right\|_{L^2} \left\| u_n \right\|_{B^{s}_{2,q}} \right)
\]
From Remark 3.2 and Lemma 3.1, we have
\[
\left( \sum_{j \in \mathbb{Z}} 2^{sj} \left\| \left[ u_n \cdot \nabla, \Delta_j \right] \rho_{n+1} \right\|^q_{L^2} \right)^{\frac{1}{q}} \leq C \left( \left\| \nabla u_n \right\|_{L^\infty} \left\| \rho_{n+1} \right\|_{B^{s-1}_{2,q}} + \left\| \nabla \rho_{n+1} \right\|_{L^\infty} \left\| u_n \right\|_{B^{s}_{2,q}} \right)
\]
Then,
\[
\left\| \omega_{n+1}(t) \right\|_{B^{s-1}_{2,q}} + \left\| \rho_{n+1}(t) \right\|_{B^{s-1}_{2,q}} \leq C \left\| \omega_{n+1}(0) \right\|_{B^{s-1}_{2,q}} + \left\| \rho_{n+1}(0) \right\|_{B^{s-1}_{2,q}}
\]
\[
+ C \int_0^t \left( \left\| \omega_n \right\|_{B^{s-1}_{2,q}} \left\| \omega_{n+1} \right\|_{B^{s-1}_{2,q}} + \left\| \rho_{n+1} \right\|_{B^{s-1}_{2,q}} \right) \, dt.
\]
On the other hand, taking the product with \( \omega_{n+1} \) in \( \dot{H}^{-1} \) and the product with \( \rho_{n+1} \) in \( L^2 \) in the first and second equations of (4.1), respectively, and using the divergence-free condition \( \nabla \cdot \Delta_j u_n = 0 \), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \omega_{n+1} \right\|_{H^{-1}}^2 + \left\| \rho_{n+1} \right\|_{L^2}^2 \right) \leq \left\| \nabla \cdot (u_n \otimes \omega_{n+1}) \right\|_{L^2} \left\| \omega_{n+1} \right\|_{H^{-1}}
\]
\[
\leq C \left\| u_n \right\|_{L^2} \left\| \omega_{n+1} \right\|_{L^\infty} \left\| \omega_{n+1} \right\|_{H^{-1}}
\]
\[
\leq C \left\| \omega_n \right\|_{B^{s-1}_{2,q}} \left\| \omega_{n+1} \right\|_{B^{s-1}_{2,q}} \left\| \omega_{n+1} \right\|_{H^{-1}}.
\]
Here, we have used the equality \( u_n = \nabla^\perp (-\Delta)^{-1} \omega_n \), Lemma 2.1, Remark 2.2, the embedding \( B^{s-1}_{2,q} \cap \dot{H}^{-1} \hookrightarrow B^s_{2,q} \) and the property (4.2). Now, we integrate over \((0,t)\) and we use a Grönwall-type inequality to get
\[
\left\| \omega_{n+1}(t) \right\|_{H^{-1}} + \left\| \rho_{n+1}(t) \right\|_{L^2} \leq C \left\| \omega_{n+1}(0) \right\|_{H^{-1}} + C \left\| \rho_{n+1}(0) \right\|_{L^2}
\]
\[
+ C \int_0^t \left( \left\| \omega_n(\tau) \right\|_{B^{s-1}_{2,q}} \left\| \omega_{n+1}(\tau) \right\|_{B^{s-1}_{2,q}} \right) \, d\tau.
\]
Combining (4.3) and (4.4), we obtain
\[
\left\| \omega_{n+1}(t) \right\|_{B^{s-1}_{2,q}} + \left\| \rho_{n+1}(t) \right\|_{B^{s-1}_{2,q}} \leq \left\| \omega_{n+1}(0) \right\|_{B^{s-1}_{2,q}} + \left\| \rho_{n+1}(0) \right\|_{B^{s-1}_{2,q}}
\]
\[
+ C \int_0^t \left( \left\| \omega_n(\tau) \right\|_{B^{s-1}_{2,q}} \left\| \omega_{n+1}(\tau) \right\|_{B^{s-1}_{2,q}} \right) \, d\tau.
\]
Doing $A_{n+1}(t) := \|\omega_{n+1}(t)\|_{B_{2,q}^{s-1} \cap H^{-1}} + \|\rho_{n+1}(t)\|_{B_{2,q}^s}$, we have

$$A_{n+1}(t) \leq A_{n+1}(0) + C \int_0^t A_n(\tau) A_{n+1}(\tau) \, d\tau.$$  

Since $\|\omega_{n+1}(0)\|_{B_{2,q}^{s-1} \cap H^{-1}} \leq C \|\omega_0\|_{B_{2,q}^{s-1} \cap H^{-1}}$ and $\|\rho_{n+1}(0)\|_{B_{2,q}^s} \leq C \|\rho_0\|_{B_{2,q}^s}$, Grönwall’s inequality yields

$$A_{n+1}(t) \leq C_0 A_0 \exp \left( C_1 \int_0^t A_n(\tau) \, d\tau \right),$$

where $A_0 = \|\omega_0\|_{B_{2,q}^{s-1} \cap H^{-1}} + \|\rho_0\|_{B_{2,q}^s}$ and the constants $C_0, C_1 > 0$ are independent of $n$. We wish to show that there exist $P > 0$ and $T > 0$ satisfying

$$A_{n+1}(t) \leq P A_0, \quad \text{for all } t \in (0, T) \quad \text{and } n \in \mathbb{N}. \tag{4.5}$$

In fact, for $n = 0$, note that

$$A_1(t) \leq C_0 A_0 \exp (C_1 t A_0) \leq P A_0, \quad \text{for all } t \in (0, T_1),$$

where $T_1 := \frac{1}{C_1 A_0} \log \left( \frac{P}{C_0} \right)$. Similarly, for $n = 1$ we have

$$A_2(t) \leq C_0 A_0 \exp \left( C_1 \int_0^t A_1(\tau) \, d\tau \right) \leq C_0 A_0 \exp (C_1 t P A_0) \leq P A_0,$$

for all $t \in (0, T_2)$, where $T_2 := \frac{1}{C_1 P A_0} \log \left( \frac{P}{C_0} \right)$. Denoting $T_3 := \min\{T_1, T_2\}$ and making the same calculations for $n = 2$, we arrive at

$$A_3(t) \leq P A_0, \quad \text{for all } t \in (0, T_3).$$

Thus, continuing in the same way, we obtain (4.5) by induction.

**Continuity of sequence.** Our intent now is to show that the sequences $\{\omega_n\}_{n \in \mathbb{N}}$ and $\{\rho_n\}_{n \in \mathbb{N}}$ belong to $C([0, T]; \dot{B}_{2,q}^{s-1}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2))$ and $C([0, T]; \dot{B}_{2,q}^s(\mathbb{R}^2))$, respectively. For that, considering the equality $(f \cdot \nabla) g = \nabla \cdot (f \otimes g)$ for all divergence free vector fields $f$, Remark 2.2, Lemma 2.3, the embedding $\dot{B}_{2,q}^{s-1} \subset H^{-1} \hookrightarrow \dot{B}_{2,q}^s$, the following estimates

$$\|(u_n \cdot \nabla) \omega_{n+1}\|_{\dot{B}_{2,q}^{s-2} \cap H^{-1}} \leq C \|u_n \otimes \omega_{n+1}\|_{\dot{B}_{2,q}^{s-1} \cap L^2} \leq C \|\omega_{n+1}\|_{\dot{B}_{2,q}^{s-1} \cap H^{-1}} \|\omega_{n+1}\|_{\dot{B}_{2,q}^{s-1} \cap H^{-1}},$$

$$\|(u_n \cdot \nabla) \rho_{n+1}\|_{\dot{B}_{2,q}^{s-1}} \leq C \|\omega_{n+1}\|_{\dot{B}_{2,q}^{s-1} \cap H^{-1}} \|\rho_{n+1}\|_{\dot{B}_{2,q}^{s}},$$

estimate (4.5), and the two first equations of (4.1), we have that $\partial_t \omega_{n+1} \in L^\infty(0, T; \dot{B}_{2,q}^{s-1}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2))$ and $\partial_t \rho_{n+1} \in L^\infty(0, T; \dot{B}_{2,q}^{s}(\mathbb{R}^2))$. Thus,

$$\omega_{n+1} \in W^{1,\infty}([0, T]; \dot{B}_{2,q}^{s-1}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)) \subset C([0, T]; \dot{B}_{2,q}^{s-1}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)),$$

$$\rho_{n+1} \in W^{1,\infty}([0, T]; \dot{B}_{2,q}^{s}(\mathbb{R}^2)) \subset C([0, T]; \dot{B}_{2,q}^{s}(\mathbb{R}^2)).$$

For $k \in \mathbb{N}$ and $n$ fixed, we denote $y_k^n := \dot{S}_k \omega_{n+1}$ and $z_k^n := S_k \rho_{n+1}$. We claim that $y_k^n \to \omega_{n+1}$ in $L^\infty(0, T; \dot{B}_{2,q}^{s-1}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2))$ and $z_k^n \to \rho_{n+1}$ in $L^\infty(0, T; \dot{B}_{2,q}^{s}(\mathbb{R}^2))$, respectively, as $k \to \infty$. Using the Littlewood-Paley operators, we can also write

$$\partial_j \Delta_j \omega_{n+1} + (\dot{S}_{j-2} u_n \cdot \nabla) \Delta_j \omega_{n+1} = \kappa \Delta_j \partial_1 \rho_{n+1} + (\dot{S}_{j-2} u_n \cdot \nabla) \Delta_j \omega_{n+1} - \Delta_j (u_n \cdot \nabla) \omega_{n+1},$$

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for each $j \in \mathbb{N}$. Since $\Delta_j \omega_{n+1}$ and $\Delta_j \rho_{n+1}$ are absolutely continuous functions from $[0, T]$ to $L^2(\mathbb{R}^2)$ and $\nabla \cdot S_{j-2} u_n = 0$, we obtain that

\[
\| \Delta_j \omega_{n+1}(t) \|_{H^{-1}} \leq \| \Delta_j \omega_{n+1}(0) \|_{H^{-1}} + |\kappa| \int_0^t \| \Delta_j \partial_1 \rho_{n+1}(\tau) \|_{H^{-1}} d\tau
\]

\[
+ \int_0^t \| (\dot{S}_{j-2} u_n(\tau) \cdot \nabla) \Delta_j \omega_{n+1}(\tau) - \Delta_j (u_n(\tau) \cdot \nabla) \omega_{n+1}(\tau) \|_{H^{-1}} d\tau,
\]

\[
\| \Delta_j \rho_{n+1}(t) \|_{L^2} \leq \| \Delta_j \rho_{n+1}(0) \|_{L^2} + |\kappa| \int_0^t \| \Delta_j u_{2,n+1}(\tau) \|_{L^2} d\tau + \int_0^t \| \Delta_j (u_n(\tau) \cdot \nabla) \rho_{n+1} \|_{L^2} d\tau.
\]

It follows that

\[
\| \omega_{n+1}(t) - y_k^n(t) \|_{B^{-1}_{2,q}} \leq C \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j \omega_{n+1}(t) \|_{L^2}^q \right)^{1/q} 
\]

\[
\leq C \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j \omega_{n+1}(0) \|_{L^2}^q \right)^{1/q} + C|\kappa| \int_0^t \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j \partial_1 \rho_{n+1}(\tau) \|_{L^2}^q \right)^{1/q} d\tau
\]

\[
+ C \int_0^t \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| (\dot{S}_{j-2} u_n(\tau) \cdot \nabla) \Delta_j \omega_{n+1}(\tau) - \Delta_j (u_n(\tau) \cdot \nabla) \omega_{n+1}(\tau) \|_{L^2}^q \right)^{1/q} d\tau,
\]

\[
\| \rho_{n+1}(t) - z_k^n(t) \|_{B^1_{2,q}} \leq C \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j \rho_{n+1}(t) \|_{L^2}^q \right)^{1/q}
\]

\[
\leq C \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j \rho_{n+1}(0) \|_{L^2}^q \right)^{1/q} + C|\kappa| \int_0^t \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j u_{2,n+1}(\tau) \|_{L^2}^q \right)^{1/q} d\tau
\]

\[
+ C \int_0^t \left( \sum_{j > k} 2^{j(\frac{1}{2} - \frac{1}{q})} \| \Delta_j (u_n(\tau) \cdot \nabla) \rho_{n+1}(\tau) \|_{L^2}^q \right)^{1/q} d\tau.
\]

As $\omega_{n+1}(t) \in B^{-1}_{2,q}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ and $\rho_{n+1}(t) \in B^1_{2,q}(\mathbb{R}^2)$, by Lemmas 2.3 and 3.3, the R.H.S. of the three above estimates go to zero as $k \to \infty$, and then the desired claim follows. Moreover, we get

\[
\| y_k^n(t') - y_k^n(t) \|_{B^{-1}_{2,q} \cap H^{-1}} = \| \dot{S}_k(\omega_{n+1}(t') - \omega_{n+1}(t)) \|_{B^{-1}_{2,q} \cap H^{-1}}
\]

\[
\leq \left( \sum_{j \leq k+1} 2^{(\frac{1}{2} - \frac{1}{q})j} \| \Delta_j (\omega_{n+1}(t') - \omega_{n+1}(t)) \|_{L^2}^q \right)^{1/q} + \sum_{j \leq k+1} \| \Delta_j (\omega_{n+1}(t') - \omega_{n+1}(t)) \|_{H^{-1}}
\]

\[
\leq C 2^{k+1} \| \omega_{n+1}(t') - \omega_{n+1}(t) \|_{B^{-2}_{2,q} \cap H^{-1}},
\]

\[
\| z_k^n(t') - z_k^n(t) \|_{B^1_{2,q}} = \| S_k(\rho_{n+1}(t') - \rho_{n+1}(t)) \|_{B^1_{2,q}} \leq \left( \sum_{j=-1}^{k+1} 2^{sj} \| \Delta_j (\rho_{n+1}(t') - \rho_{n+1}(t)) \|_{L^2}^q \right)^{1/q}
\]

\[
\leq C 2^{k+1} \| \rho_{n+1}(t') - \rho_{n+1}(t) \|_{B^{-1}_{2,q}}.
\]
Thus, \( \{y^n_k\}_{k \in \mathbb{N}} \subset C([0, T]; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \) and \( \{z^n_k\}_{k \in \mathbb{N}} \subset C([0, T]; B^s_{2,q}(\mathbb{R}^2)) \). Then

\[
\{\omega_n\}_{n \in \mathbb{N}} \subset C([0, T]; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \quad \text{and} \quad \{\rho_n\}_{n \in \mathbb{N}} \subset C([0, T]; B^s_{2,q}(\mathbb{R}^2)). \tag{4.7}
\]

**Convergence and local.** Now, we show that \( \{\omega_n\}_{n \in \mathbb{N}} \) and \( \{\rho_n\}_{n \in \mathbb{N}} \) converge in \( C([0, T]; \dot{B}^{s-2}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \) and \( C([0, T]; B^{s-1}_{2,q}(\mathbb{R}^2)) \), for some \( T > 0 \), respectively. For that, we consider the following system

\[
\begin{aligned}
\partial_t \omega_{n+1} + (\overline{u_n} \cdot \nabla) \omega_{n+1} + (u_{n-1} \cdot \nabla) \omega_{n+1} &= \kappa \partial_1 \rho_{n+1} \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
\partial_t \rho_{n+1} + (\overline{u_n} \cdot \nabla) \rho_{n+1} + (u_{n-1} \cdot \nabla) \rho_{n+1} &= \kappa |\omega_{n+1}|^2 \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
\omega_{n+1} &= \nabla (-\Delta)^{-\frac{1}{2}} \omega_{n+1} \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
\omega_{n+1} |_{t=0} &= \Delta_{n+1} \omega_0, \quad \rho_{n+1} |_{t=0} = \Delta_{n+1} \rho_0 \quad \text{in} \quad \mathbb{R}^2,
\end{aligned}
\tag{4.8}
\]

where \( \omega_{n+1} := \omega_{n+1} - \omega_n \) and \( \rho_{n+1} := \rho_{n+1} - \rho_n \).

We take the \( \dot{H}^{-1} \)-product with \( \omega_{n+1} \) and the \( L^2 \)-product with \( \rho_{n+1} \) in (4.8) to obtain

\[
\langle \partial_t \omega_{n+1}, \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle (\overline{u_n} \cdot \nabla) \omega_{n+1}, \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle (u_{n-1} \cdot \nabla) \omega_{n+1}, \omega_{n+1} \rangle_{\dot{H}^{-1}} = \langle \kappa \partial_1 \rho_{n+1}, \omega_{n+1} \rangle_{\dot{H}^{-1}},
\]

\[
\langle \partial_t \rho_{n+1}, \rho_{n+1} \rangle_{L^2} + \langle (\overline{u_n} \cdot \nabla) \rho_{n+1}, \rho_{n+1} \rangle_{L^2} = \langle \kappa |\omega_{n+1}|^2, \rho_{n+1} \rangle_{L^2}.
\]

Next, adding the above two equalities, using the property \( \langle \partial_1 \rho_{n+1}, \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle \omega_{n+1}, \rho_{n+1} \rangle_{L^2} = 0 \), the equality \( (f \cdot \nabla) g = \nabla (f \circ g) \) for all divergence free vector fields \( f \), Cauchy-Schwarz and Hölder inequalities, Lemma 2.1, Remark 2.2, the embedding \( \dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1} \hookrightarrow L^2 \) and the estimate (4.5) to arrive at

\[
\frac{1}{2} \frac{d}{dt} \left( \|\omega_{n+1}\|_{\dot{H}^{-1}}^2 + \|\rho_{n+1}\|_{L^2}^2 \right)
\leq \|\overline{u_n} \cdot \nabla \|_{L^2} \|\omega_{n+1}\|_{\dot{H}^{-1}} + \|\overline{u_n} \cdot \nabla \|_{L^2} \|\rho_{n+1}\|_{L^2} + \|\overline{u_n} \cdot \nabla \|_{L^2} \|\rho_{n+1}\|_{L^2}
\leq C \|\overline{u_n} \cdot \nabla \|_{L^2} \|\omega_{n+1}\|_{L^2} + C \|\overline{u_n} \cdot \nabla \|_{L^2} \|\rho_{n+1}\|_{L^2} + C \|\overline{u_n} \cdot \nabla \|_{L^2} \|\rho_{n+1}\|_{L^2}
\leq C P A_0 \left( \|\omega_{n+1}\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|\rho_{n+1}\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} \right)^{\frac{1}{2}}
\]

Integrating over \((0, t)\) and using a Grönewall-type inequality, we have that

\[
\|\omega_{n+1}(t)\|_{\dot{H}^{-1}} + \|\rho_{n+1}(t)\|_{L^2} \leq \|\omega_{n+1}(0)\|_{\dot{H}^{-1}} + \|\rho_{n+1}(0)\|_{L^2}
\leq + C P A_0 \int_0^t \|\overline{u_n}(\tau)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|\overline{\rho_{n+1}}(\tau)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} d\tau. \tag{4.9}
\]

On the other hand, we applies \( \Delta_j \) to the first and second equations in (4.8), and afterwards take the \( \dot{H}^{-1} \)-product with \( \Delta_j \omega_{n+1} \) and the \( L^2 \)-product with \( \Delta_j \rho_{n+1} \) in order to get

\[
\langle \partial_t \Delta_j \omega_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} = -\langle \Delta_j (\overline{u_n} \cdot \nabla) \omega_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle [u_{n-1} \cdot \nabla, \Lambda^{-\frac{1}{2}} \Delta_j] \overline{\omega_{n+1}}, \Lambda^{-\frac{1}{2}} \Delta_j \omega_{n+1} \rangle_{L^2}
\]

\[
+ \kappa \langle \Delta_j \partial_1 \rho_{n+1}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}},
\]

\[
\langle \partial_t \Delta_j \rho_{n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} = -\langle \Delta_j (\overline{u_n} \cdot \nabla) \rho_{n+1}, \Delta_j \rho_{n+1} \rangle_{L^2} + \langle [u_{n-1} \cdot \nabla, \Delta_j] \overline{\rho_{n+1}}, \Delta_j \rho_{n+1} \rangle_{L^2}
\]

\[
+ \kappa \langle \Delta_j \overline{\rho_{n+1}}, \Delta_j \rho_{n+1} \rangle_{L^2}.
\]

Adding the two previous equalities, employing the property

\[
\langle \Delta_j \partial_1 \overline{\rho_{n+1}}, \Delta_j \omega_{n+1} \rangle_{\dot{H}^{-1}} + \langle \Delta_j \overline{\rho_{n+1}}, \Delta_j \rho_{n+1} \rangle_{L^2} = 0,
\]

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and using Cauchy-Schwarz and Hölder inequalities, we obtain that

$$\frac{1}{2} \frac{d}{dt} \left( \| \Delta_j \omega_{n+1} \|^2_{H^{-1}} + \| \Delta_j \rho_{n+1} \|^2_{L^2} \right) \leq \left( \| \Delta_j (\bar{u}_n \cdot \nabla) \omega_{n+1} \|^2_{H^{-1}} + \| u_{n-1} \cdot \nabla, \Lambda^{-1} \Delta_j \omega_{n+1} \|^2_{L^2} \right)
$$

$$+ \| \Delta_j (\bar{u}_n \cdot \nabla) \rho_{n+1} \|^2_{L^2} + \| [u_{n-1} \cdot \nabla, \Delta_j] \rho_{n+1} \|^2_{L^2} \right) \left( \| \Delta_j \omega_{n+1} \|^2_{H^{-1}} + \| \Delta_j \rho_{n+1} \|^2_{L^2} \right)^{\frac{1}{2}}.$$

Integrating over (0, t) and using a Grönwall-type inequality, it follows that

$$\| \Delta_j \omega_{n+1}(t) \|^2_{H^{-1}} + \| \Delta_j \rho_{n+1}(t) \|^2_{L^2} \leq \| \Delta_j \omega_{n+1}(0) \|^2_{H^{-1}} + \| \Delta_j \rho_{n+1}(0) \|^2_{L^2}
$$

$$+ \int_0^t \| \Delta_j (\bar{u}_n(\tau) \cdot \nabla) \omega_{n+1}(\tau) \|^2_{H^{-1}} d\tau + \int_0^t \| [u_{n-1}(\tau) \cdot \nabla, \Lambda^{-1} \Delta_j \omega_{n+1}(\tau) \|^2_{L^2} d\tau
$$

$$+ \int_0^t \| \Delta_j (\bar{u}_n(\tau) \cdot \nabla) \rho_{n+1}(\tau) \|^2_{L^2} d\tau + \int_0^t \| u_{n-1}(\tau) \cdot \nabla, \Delta_j \rho_{n+1}(\tau) \|^2_{L^2} d\tau.$$

Taking into account Lemma 2.1, multiplying by $2^{(s-1)j}$ and taking the $l^q(\mathbb{Z})$-norm, we have that

$$\| \omega_{n+1}(t) \|^2_{B_{2,q}^{s-2}} + \| \rho_{n+1}(t) \|^2_{B_{2,q}^{s-1}} \leq C \| \omega_{n+1}(0) \|^2_{B_{2,q}^{s-2}} + C \| \rho_{n+1}(0) \|^2_{B_{2,q}^{s-1}}
$$

$$+ C \int_0^t \| (\bar{u}_n(\tau) \cdot \nabla) \omega_{n+1}(\tau) \|^2_{B_{2,q}^{s-2}} d\tau + C \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{(s-1)j} \| [u_{n-1}(\tau) \cdot \nabla, \Lambda^{-1} \Delta_j \omega_{n+1}(\tau) \|^2_{L^2} \right)^{\frac{1}{2}} d\tau
$$

$$+ C \int_0^t \| (\bar{u}_n(\tau) \cdot \nabla) \rho_{n+1}(\tau) \|^2_{B_{2,q}^{s-1}} d\tau + C \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{(s-1)j} \| [u_{n-1}(\tau) \cdot \nabla, \Delta_j \rho_{n+1}(\tau) \|^2_{L^2} \right)^{\frac{1}{2}} d\tau
$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

(4.10)

First, thanks to the embedding $B_{2,q}^{s-1} \cap H^{-1} \hookrightarrow L^2$ and the inequality

$$\| \omega_{n+1}(0) \|^2_{B_{2,q}^{s-2} \cap H^{-1}} + \| \rho_{n+1}(0) \|^2_{B_{2,q}^{s-1}} \leq C 2^{-n} (\| \omega_0 \|^2_{B_{2,q}^{s-2} \cap H^{-1}} + \| \rho_0 \|^2_{B_{2,q}^{s-1}}) = CA_0 2^{-n};$$

(4.11)

we can estimate $I_1$ and $I_2$. For $I_3$ and $I_5$, let us first note that by the equality $(f \cdot \nabla)g = \nabla \cdot (f \otimes g)$ for all divergence free vector fields $f$, Hölder inequality, Remark 2.2, the equality $u_n = \nabla^\bot (-\Delta)^{-1} \omega_n$, the embedding $B_{2,q}^{s-1} \cap H^{-1} \hookrightarrow B_{2,q}^{s-1}$ and estimate (4.5), we have that

$$\| \nabla \cdot (\bar{u}_n \otimes \omega_{n+1}) \|^2_{B_{2,q}^{s-2}} \leq C \| \bar{u}_n \|^2_{L^\infty} \| \omega_{n+1} \|^2_{B_{2,q}^{s-1}} \leq 2CPA_0 \| \bar{u}_n \|^2_{B_{2,q}^{s-2} \cap H^{-1}};$$

$$\| (\bar{u}_n \cdot \nabla) \rho_{n+1} \|^2_{B_{2,q}^{s-1}} \leq C \| \bar{u}_n \|^2_{L^\infty} \| \nabla \rho_{n+1} \|^2_{B_{2,q}^{s-1}} \leq 2CPA_0 \| \bar{u}_n \|^2_{B_{2,q}^{s-1} \cap H^{-1}}.$$

Then,

$$I_3 + I_5 \leq 4CPA_0 \int_0^t \| \bar{u}_n(\tau) \|^2_{B_{2,q}^{s-1} \cap H^{-1}} d\tau.$$

(4.12)
In view of Remark 3.2, Lemma 3.1 and using the same arguments to estimate $I_3$ and $I_5$, we see that

$$I_4 \leq C \int_0^t \| \nabla u_{n-1}(\tau) \|_{L^\infty} \| \varphi_{n-1}(\tau) \|_{B^2_{2,q}} + \| \varphi_{n-1}(\tau) \|_{L^2} \| u_{n-1}(\tau) \|_{\dot{B}^{3-1}_{\infty,q}} d\tau$$

$$\leq 2CPA_0 \int_0^t \| \varphi_{n-1}(\tau) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} d\tau,$$

$$I_6 \leq C \int_0^t \| \nabla u_{n-1}(\tau) \|_{L^\infty} \| \theta_{n-1}(\tau) \|_{B^2_{2,q}} + \| \theta_{n-1}(\tau) \|_{L^\infty} \| u_{n-1}(\tau) \|_{\dot{B}^3_{2,q}} d\tau$$

$$\leq 2CPA_0 \int_0^t \| \theta_{n-1}(\tau) \|_{B^3_{2,q}} d\tau.$$  

Thus,

$$I_4 + I_6 \leq 2CPA_0 \int_0^t \| \varphi_{n+1}(\tau) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} + \| \theta_{n+1}(\tau) \|_{B^3_{2,q}} d\tau. \quad (4.13)$$

Combining (4.9) and (4.10), and using (4.11), (4.12) and (4.13), it holds that

$$\| \varphi_{n+1}(t) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} + \| \theta_{n+1}(t) \|_{B^3_{2,q}} \leq CA_0 2^{-n} + 5CPA_0 \int_0^t \| \varphi_{n+1}(\tau) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} d\tau$$

$$\quad + 3CPA_0 \int_0^t \| \varphi_{n+1}(\tau) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} + \| \theta_{n+1}(\tau) \|_{B^3_{2,q}} d\tau. \quad (4.14)$$

By Grönwall inequality, it follows that

$$\| \varphi_{n+1}(t) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} + \| \theta_{n+1}(t) \|_{B^3_{2,q}} \leq \left( CA_0 2^{-n} + 5CPA_0 \int_0^t \| \varphi_{n+1}(\tau) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} d\tau \right) e^{3CPA_0t}.$$ 

Denoting $A_{n+1}(t) := \| \varphi_{n+1}(t) \|_{\dot{B}^{3-2}_2 \cap C^{1-1}_1} + \| \theta_{n+1}(t) \|_{B^3_{2,q}}$, $f(t) := CA_0 e^{3CPA_0t}$ and $g(t) := 5CPA_0 e^{3CPA_0t}$, we observe that

$$A_{n+1}(t) \leq 2^{-n} f(t) + g(t) \int_0^t A_n(\tau) d\tau.$$

Following the iterative process, we arrive at

$$A_{n+1}(t) \leq 2^{-n} f(t) + g(t) \int_0^t A_n(\tau) d\tau$$

$$\leq 2^{-n} f(t) + g(t) \int_0^t \left( 2^{-(n-1)} f(\tau) + g(\tau) \int_0^\tau A_{n-1}(\tau') d\tau' \right) d\tau$$

$$\leq 2^{-n} f(t) + g(t) \left( 2^{-(n-1)} f(t) + g(t) \int_0^t A_{n-1}(\tau) d\tau \right)$$

$$= 2^{-n} f(t) \left( 1 + 2g(t)t \right) + g(t)^2 t \int_0^t A_{n-1}(\tau) d\tau$$

$$\leq 2^{-n} f(t) \left( 1 + 2g(t)t + (2g(t)t)^2 \right) + g(t)^3 t^2 \int_0^t A_{n-2}(\tau) d\tau$$

$$\vdots$$

$$\leq 2^{-n} f(t) \sum_{i=0}^{n-1} (2g(t)t)^i + g(t)^n t^{n-1} \int_0^t A_1(\tau) d\tau$$

$$\leq 2^{-n} f(t) \sum_{i=0}^{n-1} (2g(t)t)^i + 2PA_0 g(t)^n t^n.$$
Let $T' \leq T$ be such that $g(t)t \leq \frac{1}{T}$ for all $t \in (0, T')$. Then, it is fulfilled that $\sum_{i=0}^{n-1} 2g(t)t^i \leq \sum_{i=0}^{n-1} \frac{1}{T} < \infty$ and $(g(t)t)^n \leq 4^n$ for all $n \in \mathbb{N}$ and $t \in (0, T')$. Moreover, there exists $C_3 > 0$ such that $f(t) \leq C_3$ for all $t \in (0, T')$. Therefore, for some constant $C_4 > 0$, we have

$$A_{n+1}(t) \leq C_3 2^{-n} + 2PA_0 4^{-n} \leq C_4 2^{-n}.$$ 

Let $n, m \in \mathbb{N}$ be such that $n > m$, then there exists $C_5 > 0$ satisfying

$$\|(\omega_n - \omega_m)(t)\|_{\dot{B}^{s-2}_2} + \|(\rho_n - \rho_m)(t)\|_{\dot{B}^{s-1}_2} \leq \sum_{i=m}^{n-1} A_{i+1}(t) \leq C_5 \sum_{i=m}^{n-1} 2^{-i}.$$ 

This implies that $\{\omega_n\}_{n \in \mathbb{N}}$ and $\{\rho_n\}_{n \in \mathbb{N}}$ are Cauchy in the spaces $L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $L^\infty(0, T'; \dot{B}^{s-1}_2)$, respectively. Therefore, there are $\omega \in L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $\rho \in L^\infty(0, T'; \dot{B}^{s-1}_2)$ such that if $n \to \infty$, then

$$\omega_n \to \omega \text{ in } L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1}) \text{ and } \rho_n \to \rho \text{ in } L^\infty(0, T'; \dot{B}^{s-1}_2).$$

Furthermore, since the sequences $\{\omega_n\}_{n \in \mathbb{N}}$ and $\{\rho_n\}_{n \in \mathbb{N}}$ belong to $C([0, T'); \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $C([0, T'); \dot{B}^{s-1}_2)$, respectively, we have that $\omega \in C([0, T'); \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $\rho \in C([0, T'); \dot{B}^{s-1}_2)$.

On the other hand, as $\{\omega_n\}_{n \in \mathbb{N}}$ and $\{\rho_n\}_{n \in \mathbb{N}}$ are bounded in $L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $L^\infty(0, T'; \dot{B}^{s}_2)$, respectively, we can extract subsequences $\{\omega_{n_j}\}_{j \in \mathbb{N}}$ and $\{\rho_{n_j}\}_{j \in \mathbb{N}}$ such that $\omega_{n_j} \rightharpoonup \omega$ and $\rho_{n_j} \rightharpoonup \rho$ in $L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1})$ and $L^\infty(0, T'; \dot{B}^{s}_2)$, respectively. Thus,

$$\omega \in C([0, T'); \dot{B}^{s-2}_2(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap L^\infty(0, T'; \dot{B}^{s-1}_2(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)),
\rho \in C([0, T'); \dot{B}^{s-1}_2(\mathbb{R}^2) \cap L^\infty(0, T'; \dot{B}^{s}_2(\mathbb{R}^2)),$$

(4.15)

with $\|\omega\|_{L^\infty(0, T'; \dot{B}^{s-2}_2 \cap \dot{H}^{-1})}, \|\rho\|_{L^\infty(0, T'; \dot{B}^{s}_2)} \leq PA_0$, where $P$ and $A_0$ are as in (4.5).

Now, with the above convergence in hand, we sketch the convergence of the nonlinearity of the second equation and the coupling term in (4.1). The others follow similarly and are left to the reader. By Hölder’s inequality, Remark 2.2, the identity $u_n - u = \nabla^\perp (-\Delta)^{-1}(\omega_n - \omega)$, the embedding $\dot{B}^{s-1}_2(\mathbb{R}^2) \hookrightarrow \dot{B}^{m+1}_2(\mathbb{R}^2)$ and $\dot{B}^{s}_2(\mathbb{R}^2) \hookrightarrow \dot{B}^{m}_\infty(\mathbb{R}^2)$ with $0 \leq m \leq s - 2$, Lemma 2.1 and (4.5), it follows that

$$\int_0^t \| (u_n(\tau) \cdot \nabla) \rho_{n+1}(\tau) - (u(\tau) \cdot \nabla) \rho(\tau) \|_{\dot{B}^{m+1}_2} d\tau \leq \int_0^t \| (u_n - u)(\tau) \cdot \nabla \|_{\dot{B}^{m+1}_2} + \| (u(\tau) \cdot \nabla) \|_{\dot{B}^{m+1}_2} d\tau \leq CPA_0 \int_0^t \| \omega_n - \omega \|_{\dot{B}^{s-2}_2} + \| \rho_{n+1} - \rho \|_{\dot{B}^{m+2}_2} d\tau \leq 2CPA_0 T \| \omega_n - \omega \|_{L^\infty(0, T; \dot{B}^{s-2}_2)} + \| \rho_{n+1} - \rho \|_{L^\infty(0, T; \dot{B}^{m+2}_2)} \to 0 \text{ as } n \to \infty,$$

which implies

$$\int_0^t (u_n(\tau) \cdot \nabla) \rho_{n+1}(\tau) d\tau \to \int_0^t (u(\tau) \cdot \nabla) \rho(\tau) d\tau \text{ in } \dot{B}^{m+1}_2(\mathbb{R}^2), \text{ as } n \to \infty.$$ 

Moreover, since $\rho_{n+1} \to \rho$ in $L^\infty(0, T; \dot{B}^{m+1}_2)$, we have

$$\left\| \int_0^t \kappa(\partial_1 \rho_{n+1}(\tau) - \partial_1 \rho(\tau)) d\tau \right\|_{L^\infty(0, T; \dot{B}^{m+1}_2 \cap \dot{H}^{-1})} \leq C|\kappa|T \| \rho_{n+1} - \rho \|_{L^\infty(0, T; \dot{B}^{m+1}_2)} \to 0, \text{ as } n \to \infty,$$
and then
\[
\int_0^t \kappa \partial_t \rho_{n+1}(\tau) \, d\tau \rightarrow \int_0^t \kappa \partial_t \rho(\tau) \, d\tau \text{ in } \dot{B}^{m}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2).
\]

So, we can pass the limit in the integral formulation of approximate system (4.1) and obtain that \((\omega, \rho)\) is an integral solution for (1.3) in \(L^\infty(0,T; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \times L^\infty(0,T; B^{m-1}_{2,q}(\mathbb{R}^2))\), namely
\[
\omega(t) - \omega_0 = \int_0^t (u(\tau) \cdot \nabla)\omega(\tau) + \kappa \partial_t \rho(\tau) \, d\tau,
\]
\[
\rho(t) - \rho_0 = \int_0^t (u(\tau) \cdot \nabla)\rho(\tau) + \kappa u_2(\tau) \, d\tau.
\]  

(4.16)

Considering \(y_k = \dot{S}_k \omega, z_k = S_k \rho\), using (4.15), and proceeding as in the proof of (4.6) and (4.7), it follows that \(\{y_k\}_{k \in \mathbb{N}} \subset C([0,T]; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2))\), \(\{z_k\}_{k \in \mathbb{N}} \subset C([0,T]; B^{s}_{2,q}(\mathbb{R}^2))\), \(y_k \rightarrow \omega\) in \(L^\infty(0,T; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2))\) and \(z_k \rightarrow \rho\) in \(L^\infty(0,T; B^{s}_{2,q}(\mathbb{R}^2))\) as \(k \to \infty\). Then, \(\omega \in C([0,T]; \dot{B}^{s}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2))\) and the integral system (4.16) is indeed verified in \((\dot{B}^{s-2}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \times B^{s-1}_{2,q}(\mathbb{R}^2)\), and consequently \(\omega \in C([0,T]; \dot{B}^{s-2}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2))\) and \(\rho \in C([0,T]; B^{s-1}_{2,q}(\mathbb{R}^2))\), as desired.

**Uniqueness.** In this part, we suppose that system (1.3) possesses two solutions \((\omega^1, \rho^1)\) and \((\omega^2, \rho^2)\) with the same initial data \((\omega_0, \rho_0)\) and we show that \(\omega^1 = \omega^2\) and \(\rho^1 = \rho^2\). For that, we set \(\bar{\omega} := \omega^2 - \omega^1\) and \(\bar{\rho} := \rho^2 - \rho^1\), respectively. Then, \((\bar{\omega}, \bar{\rho})\) satisfy the following system
\[
\begin{aligned}
\partial_t \bar{\omega} + (\bar{u} \cdot \nabla)\omega^2 + (u^1 \cdot \nabla)\bar{\omega} &= \kappa \partial_t \bar{\rho}, \\
\partial_t \bar{\rho} + (\bar{u} \cdot \nabla)\rho^2 + (u^1 \cdot \nabla)\bar{\rho} &= \kappa \bar{u}_2, \\
\bar{u} &= \nabla^\perp (-\Delta)^{-1} \bar{\omega}, \\
\bar{\omega} \big|_{t=0} &= 0, \quad \bar{\rho} \big|_{t=0} = 0.
\end{aligned}
\]  

(4.17)

Considering the spaces \(L^\infty(0,T; \dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1})\) and \(L^\infty(0,T; B^{s-1}_{2,q})\) in (4.17) and employing the argument used in (4.8) to estimate \(\bar{\omega}\) and \(\bar{\rho}\), we obtain an inequality similar to inequality (4.14) as follows
\[
\|
\bar{\omega}(t)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|
\bar{\rho}(t)\|_{B^{s-1}_{2,q}} \leq C \|\bar{\omega}(0)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|\bar{\rho}(0)\|_{B^{s-1}_{2,q}} \int_0^t \|\bar{\omega}(\tau)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|\bar{\rho}(\tau)\|_{B^{s-1}_{2,q}} \, d\tau.
\]

Therefore, by Grönwall’s inequality we obtain \(\|
\bar{\omega}(t)\|_{\dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1}} + \|
\bar{\rho}(t)\|_{B^{s-1}_{2,q}} \leq 0\) for all \(t \in [0,T]\). Thus, \(\|
\bar{\omega}\|_{L^\infty(0,T; \dot{B}^{s-2}_{2,q} \cap \dot{H}^{-1})} = \|
\bar{\rho}\|_{L^\infty(0,T; B^{s-1}_{2,q})} = 0\), implying that \(\omega^1 = \omega^2\) and \(\rho^1 = \rho^2\), which shows the uniqueness of solution to (1.3).

## 5 Long-time solvability

In this section we prove the global-in-time solvability of (1.3) for large values of \(\kappa\). We start with a proposition containing a blow-up criterion.

**Proposition 5.1.** Let \(s \) and \(q \) be such that \(s > 2\) with \(1 \leq q \leq \infty\) or \(s = 2\) with \(q = 1\). For \(\omega_0 \in \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)\) and \(\rho_0 \in B^{s}_{2,q}(\mathbb{R}^2)\), consider \((\omega, \rho)\) the corresponding solution of (1.3) satisfying
\[
\omega \in C([0,T]; \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0,T]; \dot{B}^{s-2}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)),
\]
\[
\rho \in C([0,T]; B^{s}_{2,q}(\mathbb{R}^2)) \cap C^1([0,T]; B^{s-1}_{2,q}(\mathbb{R}^2)),
\]

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where $T > 0$ is an existence time. If $\int_0^T \|\omega(t)\|_{L^\infty} + \|\nabla \rho(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} dt < \infty$, then there exists $T' > T$ such that $(\omega, \rho)$ can be extended to $[0, T')$ with

$$
\omega \in C([0, T'); \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, T'); \dot{B}^{s-2}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)),
$$

$$
\rho \in C([0, T'); B^{s-1}_{2,q}(\mathbb{R}^2)) \cap C^1([0, T'); B^{s-1}_{2,q}(\mathbb{R}^2)).
$$

**Proof.** By standard procedures used to estimate $\omega$ and $\rho$ in Besov norms, we obtain the following estimates:

$$
\|\omega(t)\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + \|\rho(t)\|_{\dot{B}^{s-1}_{2,q}} \leq C\|\omega(0)\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + C\|\omega(0)\|_{\dot{B}^{s}_{2,q}}
$$

$$
+ \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{sj} \| (u(\tau) \cdot \nabla, \Delta^{-1} \Delta_j \omega(\tau)) \|_{L^2} \right)^{\frac{1}{2}} d\tau + \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{sj} \| (u(\tau) \cdot \nabla, \Delta_j \rho(\tau)) \|_{L^2} \right)^{\frac{1}{2}} d\tau. \tag{5.1}
$$

If we denote by $I$ and $J$ the penultimate and last term of the previous inequality, respectively, by Lemma 3.1, we have that

$$
I \leq C\|\nabla u\|_{L^\infty} \|\omega\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + \|\omega\|_{L^\infty} \|u\|_{\dot{B}^{s-1}_{2,q}} \leq C\|\omega\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} (\|\nabla u\|_{L^\infty} + \|\omega\|_{L^\infty}),
$$

$$
J \leq C\|\nabla u\|_{L^\infty} \|\rho\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + \|\nabla \rho\|_{L^\infty} \|u\|_{\dot{B}^{s-1}_{2,q}} \leq C \left( \|\omega\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + \|\rho\|_{\dot{B}^{s}_{2,q}} \right) (\|\nabla u\|_{L^\infty} + \|\omega\|_{L^\infty}).
$$

Denoting $z_{s,q}(t) := \|\omega(t)\|_{\dot{B}^{s-1}_{2,q} \cap \dot{H}^{-1}} + \|\rho(t)\|_{\dot{B}^{s}_{2,q}}$ and employing the above inequalities in (5.1) and the Grönwall-type inequality, it holds

$$
z_{s,q}(t) \leq Cz_{s,q}(0) + C \int_0^t z_{s,q}(\tau) (\|\omega(\tau)\|_{L^\infty} + \|\nabla \rho(\tau)\|_{L^\infty} + \|\nabla u(\tau)\|_{L^\infty}) d\tau.
$$

Then, it follows that there exists a constant $C_6 > 0$ such that

$$
z_{s,q}(t) \leq z_{s,q}(0) \exp \left( C_6 \int_0^t \|\omega(\tau)\|_{L^\infty} + \|\nabla \rho(\tau)\|_{L^\infty} + \|\nabla u(\tau)\|_{L^\infty} d\tau \right), \tag{5.2}
$$

for all $t \in [0, T)$. Thus, by standard arguments, $(\omega, \rho)$ can be continued to $[0, T')$ for some $T' > T$, whenever

$$
\int_0^T \|\omega(t)\|_{L^\infty} + \|\nabla \rho(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} dt < \infty.
$$

**Long-time solvability.** For $\omega_0 \in \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$, $\rho_0 \in \dot{B}^{s+1}_{2,q}(\mathbb{R}^2)$, let $(\omega, \rho)$ be the solution of system (1.3) satisfying

$$
\omega \in C([0, T^*); \dot{B}^s_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, T^*); \dot{B}^{s-1}_{2,q}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)),
$$

$$
\rho \in C([0, T^*); B^{s+1}_{2,q}(\mathbb{R}^2)) \cap C^1([0, T^*); B^s_{2,q}(\mathbb{R}^2)),
$$

with maximal existence time $T^* > 0$. If $T^* = \infty$, we are done. Assume that $T^* < \infty$. Denoting $V^\pm := \omega \pm \Lambda \rho$, we can use Duhamel’s principle to get

$$
V^\pm(t) = e^{\pm \kappa R_1 t} V_0 - \int_0^t e^{\pm \kappa R_1 (\tau-t)} (f \pm \Lambda g)(\tau) \, d\tau,
$$

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where $f = (u \cdot \nabla) \omega$ and $g = (u \cdot \nabla) \rho$. For $0 \leq t \leq T^*$, we define

$$\mathcal{M}(t) := \int_0^t \|V^\pm(\tau)\|_{\dot{B}^{0}_{\infty, 1}} \, d\tau.$$ 

In what follows, we continue to use the notation $z_{s+1,q}(0) = \|\omega_0\|_{\dot{B}^{2}_{2,q} \cap \dot{H}^{-1}} + \|\rho_0\|_{\dot{B}^{s+1}_{2,q}}$. We first consider the case $s = 2$ with $q = 1$. We can estimate

$$\mathcal{M}(t) \leq C \int_0^t \|e^{\pm \kappa R_1 \tau} V_0\|_{\dot{B}^{0}_{\infty, 1}} \, d\tau + C \int_0^t \left\| \int_0^\tau e^{\pm \kappa R_1 (\tau' - \tau)} (f \pm \Lambda g)(\tau') \, d\tau' \right\|_{\dot{B}^{0}_{\infty, 1}} \, d\tau$$

$$:= K_1 + K_2.$$ 

For $K_1$, we use Hölder’s inequality and Lemma 2.5 with $r = \infty$ to get

$$K_1 \leq C t^{1 - \frac{1}{q}} \|e^{\pm \kappa R_1 \tau} V_0\|_{L^q(0, \infty; \dot{B}^{0}_{\infty, 1})} \leq C t^{1 - \frac{1}{q}} \|\nabla \omega_0\|_{\dot{B}^{1}_{2,1}} \leq C t^{1 - \frac{1}{q}} \|\nabla \omega_0\|_{\dot{B}^{2,1}}(0).$$

For $K_2$, we employ the Minkowski and Hölder inequalities, Lemma 2.5 and Remark 2.2 to obtain

$$K_2 \leq C \int_0^t \int_\tau^T \left\| e^{\pm \kappa R_1 (\tau' - \tau)} (f \pm \Lambda g)(\tau') \right\|_{\dot{B}^{0}_{\infty, 1}} \, d\tau' \, d\tau$$

$$= C \int_0^t \int_\tau^T \left\| e^{\pm \kappa R_1 (\tau' - \tau)} (f \pm \Lambda g)(\tau') \right\|_{\dot{B}^{0}_{\infty, 1}} \, d\tau \, d\tau'$$

$$\leq C t^{1 - \frac{1}{q}} \int_0^t \|f \pm \Lambda g)(\tau')\|_{\dot{B}^{1}_{2,1}} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{q}} \|\nabla \omega_0\|_{\dot{B}^{2,1}} + \|\rho\|_{\dot{B}^{2,1}}(\omega_0\|_{\dot{B}^{2,1}}) \, d\tau$$

Thus, for each $0 < t < T^*$, we use (5.2), the embedding $\dot{B}^{0}_{\infty, 1} \hookrightarrow L^\infty$ and the equality $u = \nabla^\perp (-\Delta)^{-1} \omega$ to get

$$\mathcal{M}(t) \leq C t^{1 - \frac{1}{q}} \|\nabla \omega_0\|_{\dot{B}^{2,1} \cap \dot{H}^{-1}} + \|\rho\|_{\dot{B}^{2,1}}(\omega_0\|_{\dot{B}^{2,1}}) \, d\tau$$

$$\leq C t^{1 - \frac{1}{q}} \|\nabla \omega_0\|_{\dot{B}^{2,1} \cap \dot{H}^{-1}} + \|\rho\|_{\dot{B}^{2,1}}(\omega_0\|_{\dot{B}^{2,1}}) \, d\tau$$

Now, we deal with the case $s > 2$ with $1 \leq q \leq \infty$. For each $1 \leq q \leq \infty$, we take $2 < r \leq \infty$ such that $q \leq r$. Note that since $s - 1 > 1$ we have the non-homogeneous embedding $\dot{B}^{s-1}_{\infty, \infty} \hookrightarrow W^{1,\infty}$, so that we can estimate $\|V\|_{\dot{B}^{0}_{\infty, 1}}$ by the $B^{s-2}_{\infty, \infty}$-norm of $V$. Thus, we have that

$$\mathcal{M}(t) \leq C \int_0^t \|e^{\kappa R_1 \tau} V_0\|_{\dot{B}^{s-2}_{\infty, \infty}} \, d\tau + C \int_0^t \left\| \int_0^\tau e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau') \, d\tau' \right\|_{\dot{B}^{s-2}_{\infty, \infty}} \, d\tau$$

$$:= K_3 + K_4.$$ 

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We now estimate $K_3$. Using the embedding $\dot{B}^{s-2}_{\infty,q} \hookrightarrow \dot{B}^{s-2}_{\infty,\infty}$, Hölder's inequality and Lemma 2.5, we can estimate

$$\int_0^t \|e^{\kappa R_1 t} V_0\|_{\dot{B}^{s-2}_{\infty,\infty}} \, d\tau \leq \int_0^t \|e^{\kappa R_1 t} V_0\|_{\dot{B}^{s-2}_{\infty,2}(\mathbb{R}^2)} \, d\tau$$

$$\leq t^{1 - \frac{1}{\gamma}} \|e^{\kappa R_1 t} V_0\|_{L^\gamma(0,\infty;\dot{B}^{s-2}_{\infty,\infty})}$$

$$\leq C t^{1 - \frac{1}{\gamma}} |\kappa|^{-\frac{1}{\gamma}} \|V_0\|_{\dot{B}^{s-1}_{2,2}}$$

$$\leq C t^{1 - \frac{1}{\gamma}} |\kappa|^{-\frac{1}{\gamma}} z_{s,q}(0).$$

Also, by Lemma 2.4, we have that

$$\int_0^t \|e^{\kappa R_1 t} V_0\|_{L^\infty} \, d\tau \leq t^{1 - \frac{1}{\gamma}} \|e^{\kappa R_1 t} V_0\|_{L^\gamma(0,\infty;L^\infty)}$$

$$\leq C t^{1 - \frac{1}{\gamma}} |\kappa|^{-\frac{1}{\gamma}} \|V_0\|_{L^2}.$$ 

Therefore,

$$K_3 \leq C t^{1 - \frac{1}{\gamma}} |\kappa|^{-\frac{1}{\gamma}} z_{s,q}(0).$$

We proceed to estimate $K_4$. Hölder’s inequality, the embedding $\dot{B}^{s-2}_{\infty,q} \hookrightarrow \dot{B}^{s-2}_{\infty,\infty}$ and Lemma 2.5 yield

$$\int_0^t \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau') \, d\tau' \|_{\dot{B}^{s-2}_{\infty,\infty}} \, d\tau \leq C \int_0^t \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{\dot{B}^{s-2}_{\infty,\infty}} \, d\tau' \, d\tau'$$

$$= C \int_0^t \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{\dot{B}^{s-2}_{\infty,\infty}} \, d\tau' \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\gamma(\tau',t;\dot{B}^{s-2}_{\infty,\infty})} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\gamma(\tau',t;\dot{B}^{s-2}_{\infty,\infty})} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\gamma(\tau',t;\dot{B}^{s-2}_{\infty,\infty})} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\gamma(\tau',t;\dot{B}^{s-2}_{\infty,\infty})} \, d\tau'$$

Here we have proceeded as for the term $K_2$ in the case $s = 2, q = 1$. Also, by Lemma 2.4, the continuity of $\mathcal{R}_i$ in $L^2$ and the embedding $\dot{B}^{s+1}_{2,q} \hookrightarrow \dot{B}^s_{2,q} \hookrightarrow L^2$, we arrive at

$$\int_0^t \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau') \, d\tau' \|_{L^\infty} \, d\tau' \leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^2} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'$$

It follows that

$$K_4 \leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau' + \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'.$$

Thus, for each $0 < t < T^*$, we have

$$\mathcal{M}(t) \leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau' + \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'.$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{\gamma}} \int_0^\tau \|e^{\kappa R_1 (\tau' - \tau)} (f + \Lambda g)(\tau')\|_{L^\infty} \, d\tau'.$$
Therefore, for both cases of $s$ and $q$ such that $s = 2$ with $q = 1$ or $s > 2$ with $1 \leq q \leq \infty$, we have that there exists $C_7 > 0$ such that

$$
\mathcal{M}(t) \leq C_7 t^{1-\frac{1}{s_{l+1,q}}|\kappa|^\frac{1}{\gamma}} z_{s_{l+1,q}}(0) \left( 1 + z_{s_{l+1,q}}(0) t e^{C_6 \mathcal{M}(t)} \right). 
$$

(5.3)

Next, for each $0 < T < \infty$ we define $\tilde{T} = \sup D_T$, where

$$
D_T = \{ t \in [0, T] \cap [0, T^*) \mid \mathcal{M}(t) \leq C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)} \}.
$$

We first show that $\tilde{T} = \min\{T, T^*\}$. We proceed by contradiction. So, assume on the contrary that $\tilde{T} < \min\{T, T^*\}$. We have that there exists $T_1$ such that $\tilde{T} < T < \min\{T, T^*\}$. It follows that

$$
\omega \in C([0, T_1]; B_{2,q}^s(\mathbb{R}^2) \cap \hat{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, T_1]; B_{2,q}^s(\mathbb{R}^2) \cap \hat{H}^{-1}(\mathbb{R}^2)),
$$

$$
\rho \in C([0, T_1]; B_{2,q}^{s+1}(\mathbb{R}^2) \cap C^1([0, T_1]; B_{2,q}^s(\mathbb{R}^2)).
$$

$\mathcal{M}(t)$ is uniformly continuous on $[0, T_1]$, and

$$
\mathcal{M}(\tilde{T}) \leq C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)}.
$$

(5.4)

We now take a $|\kappa|$ large enough so that

$$
|\kappa|^\frac{1}{\gamma} \geq 2 \left( 1 + (\|\omega\|_{B_{2,q}^s \cap \hat{H}^{-1}} + \|\rho\|_{B_{2,q}^{s+1}}) T \exp \left( C_6 C_7 T^{1-\frac{1}{s_{l+1,q}}(\|\omega\|_{B_{2,q}^s \cap \hat{H}^{-1}} + \|\rho\|_{B_{2,q}^{s+1}}) \right) \right).
$$

(5.5)

Using (5.3), (5.4) and (5.5), we obtain that

$$
\mathcal{M}(\tilde{T}) \leq C_7 (\tilde{T})^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)} \left( 1 + z_{s_{l+1,q}}(0) \tilde{T} \exp \left( C_6 \mathcal{M}(\tilde{T}) \right) \right)
$$

$$
\leq C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)} |\kappa|^{-\frac{1}{\gamma}} \left( 1 + z_{s_{l+1,q}}(0) \tilde{T} \exp \left( C_6 C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)} \right) \right)
$$

$$
\leq \frac{1}{2} C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)}.
$$

Thus, we can choose $T_2$ such that $\tilde{T} < T < T_1$ with $\mathcal{M}(T_2) \leq C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)}$. This contradicts the definition of $\tilde{T}$. It follows that $\tilde{T} = \min\{T, T^*\}$ when $\kappa$ verifies (5.5). If $T^* < T$, then $T^* = \tilde{T} = \sup D_T$ and

$$
\mathcal{M}(t) \leq C_7 T^{1-\frac{1}{s_{l+1,q}} z_{s_{l+1,q}}(0)} < \infty, \text{ for all } 0 \leq t < T^*.
$$

It follows that

$$
\int_0^{T^*} \|\omega(t)\|_{L^\infty} + \|\nabla \rho(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} \, dt \leq C \mathcal{M}(T^*) < \infty
$$

and, in view of the blow-up criterion, we obtain a contradiction with the maximality of $T^*$. This concludes the proof.

\[\blacksquare\]

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