Integration of trace anomaly in 6D

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Abstract. The trace anomaly in six-dimensional space is given by the local terms which have six derivatives of the metric. We find the effective action which is responsible for the anomaly. The result is presented in non-local covariant form and also in the local covariant form which employs two auxiliary scalar fields.

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1 Introduction

The interest to the higher-dimensional conformal theories is on rise since the advent of string/M-theory and the discovery of holography and AdS/CFT correspondence. It would be certainly useful to have an explicit form of the vacuum effective action for the conformal fields in dimensions $D$ higher than four. The simplest and practically working procedure to derive such an effective action is by integrating conformal anomaly. The two main examples of such integration are Polyakov action in $D = 2$ [1] and Riegert-Fradkin-Tseytlin action in $D = 4$ [2]. Both proved to be fruitful instruments for various applications (see, e.g., [3] for a review). The same integration in $D = 6$ attracts a great deal of attention, but until now there were only particular (albeit very interesting) results [4] (see further references therein) which do not enable one to obtain the anomaly-induced effective action in a closed form.

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In this Letter we report on a complete solution of the problem. The work is organized as follows. In Sect. 2 we briefly describe the scheme of integration which can be applied in any dimension $D$. As one can see there, the three necessary elements are conformal operator (analog of Paneitz operator in $D = 4$), modified topological invariant and its conformal transformation and, finally, the integration of surface terms. The part which requires the most significant efforts is the search of modified topological invariant with the simplest conformal property, and we have this problem solved for $D = 6$. The relevant building blocks of such an effective action in $D = 6$ are presented in Sect. 3. Finally, in Sect. 4 we draw our conclusions are describe some of the possible applications.

2 General scheme of integrating anomaly

Let us briefly summarize the general scheme of integrating anomaly, as it is described in the review paper [5] for $D = 4$. The changes which are requested in higher even dimensions are not relevant, regardless of the growth of technical difficulties.

The vacuum part of the trace anomaly in dimension $D \geq 4$ can be always written as [6, 7, 8]

$$T = \langle T^\mu_\mu \rangle = c_r W^r_D + a_E D + \Xi_D, \quad (1)$$

with the sum over $r$. Here $W^r_D$ are conformal invariant terms (typically constructed from Weyl tensor). In $D = 2$ there is no conformal term, and in $D = 4$ there is only one, the square of the Weyl tensor. In $D = 6$ there are three such terms, the explicit form can be found in [9]. Furthermore, $\Xi_D$ is a linear combination of the surface terms, $\Xi_D = \sum \gamma_k \chi_k$ in the corresponding dimension. The explicit form of the relevant $\chi_k$ terms in $D = 6$ will be given below in Eq. (12). Furthermore, $E_D$ is the integrand of the topological term,

$$E_D = \varepsilon^{\rho_1 \cdots \rho_D} \varepsilon^{\sigma_1 \cdots \sigma_D} R_{\rho_1 \sigma_1 \rho_2 \sigma_2} \cdots R_{\rho_{D-1} \sigma_{D-1} \rho_D \sigma_D}. \quad (2)$$

The classification (1) is a simple consequence of that the anomaly comes from the one-loop divergences and the last satisfy conformal Noether identity. It is easy to see that the terms which satisfy this identity should belong to the mentioned three categories.

The numerical coefficients $a$, $c$ and $\gamma_k$ depend on the number of massless conformal fields of different spins. These quantities have no real concern to us, because we will describe a general solution valid for any values of $a$, $c$ and $\gamma_k$.

Our purpose is to find the anomaly-induced effective action $\Gamma_{\text{ind}}$, such that

$$-\frac{2}{\sqrt{-g}} g_{\mu \nu} \frac{\delta \Gamma_{\text{ind}}}{\delta g_{\mu \nu}} = T. \quad (3)$$
As it was already mentioned, the integration of anomaly requires a modified topological invariant

\[ \tilde{E}_D = E_D + \sum_k \alpha_k \chi_k, \] (4)

where the values of \( \alpha_k \) are chosen to provide the special conformal property of the new topological term. Namely, we require that under the local conformal transformation

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\sigma(x)} \] (5)

there should be

\[ \sqrt{-\bar{g}} \tilde{E}_D = \sqrt{-\bar{g}} (\tilde{E}_D + \kappa \bar{\Delta}_D \sigma) , \] (6)

where \( \kappa \) is a constant and \( \Delta_D = \square^{D/2} + \ldots \) is the conformal operator acting on a conformally inert scalar. For example, in \( D = 4 \), the formulas have the well-known form, with \( \Delta_4 \) being the Paneitz operator [10, 2], \( \kappa = 4 \), and the surface term in (4) is \( \alpha_k \chi_k = -(2/3) \square R \). Some comment is in order. Of course, in \( D = 4 \) the \( \square R \) is the unique possible surface term, so this part is simple. However, the coefficient \(-2/3\) is a little bit mysterious, because it can be established only by a direct calculation. The details can be found in [11], where one can observe that the conformal transformation of each \( E_4 \) and \( \square R \) is quite complicated. Nevertheless, the particular combination with the mystic \(-2/3\) cancels all terms of second, third and fourth orders in \( \sigma \) and the remaining linear term involves the conformal operator. Indeed, we expect this symmetry in the general even \( D \) case, that means

\[ \sqrt{-\bar{g}} \Delta_D \varphi = \sqrt{-\bar{g}} \bar{\Delta}_D \bar{\varphi} \] (7)

with \( \varphi = \bar{\varphi} \) and all other quantities with bar are constructed with the fiducial metric \( \bar{g}_{\mu\nu} \).

In order to integrate the anomaly one needs the last element. Namely, there should be a set of local metric-dependent Lagrangians \( L_i \), providing that with some coefficients \( c_{ik} \) there is an identity

\[ -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \sum_i c_{ik} \int_x L_i = \chi_k , \] (8)

where \( \int_x \equiv \int d^D x \sqrt{-g} \), for each of the surface term components in (4). If the set \( L_i \) is found, the problem of solving (3) is reduced to integrating the first two terms in (1). And it is easy to see that this problem is easily solved by the use of identity (6). In order
to see this, let us follow \[2\] and introduce the conformal Green function \(G(x, x')\) of the operator \(\Delta_D\), where

\[
\sqrt{-g} \Delta_D^x G(x, x') = \delta^D(x, x'), \quad G = \bar{G}.
\] (9)

The complete solution for the anomaly-induced effective action can be written down in the form

\[
\Gamma_{\text{ind}} = S_c + \int\int \left\{ \frac{1}{4} c_r W^r_D + a \tilde{E}_D(x) \right\} G(x, y) \tilde{E}_D(y)
+ \sum_k (\gamma_k - \alpha_k) \sum_i c_{ik} \int_x L_i.
\] (10)

Here \(S_c = S_c[\bar{g}_{\mu\nu}]\) is an undefined conformal functional, which represents a boundary condition of the variational equation (3), and the modification of the coefficients \(\gamma_k\) of the anomaly (1) occurs because part of the surface terms were absorbed into \(\tilde{E}_D\).

Writing the non-local part of the expression (10) in the symmetric form, one can always present the effective action in the local covariant form which includes two auxiliary fields \(\psi\) and \(\varphi\), as it was suggested in \[12, 13\]

\[
\bar{\Gamma} = S_c + \sum_k (\gamma_k - \alpha_k) \sum_i c_{ik} \int_x L_i
+ \frac{1}{2} \int \left\{ \varphi \Delta_D \varphi - \psi \Delta_D \psi + \sqrt{-a} \varphi \tilde{E}_D + \frac{1}{\sqrt{-a}} (\psi - \varphi) c_r W^r(x) \right\}.
\] (11)

In these formulas we assume that \(a < 0\), as in the \(D = 4\) case. In case of \(a > 0\) the expression can be trivially modified by changing the sign \(\tilde{E}_D \to -\tilde{E}_D\). The last observation is that one can also write the action in terms of modified auxiliary fields \[13, 14\] or in the simplest non-covariant form in terms of \(\sigma\) and \(\bar{g}_{\mu\nu}\) \[2\]. Since the transition to these forms is not too different compared to the \(D = 4\) case, we will not consider these issues here.

All in all, it is clear that the integration of anomaly needs Eq. (6) at the first place and also Eq. (8) to deal with the local part of induced action. In the next section we present the result for (6) in \(D = 6\).

3 Conformal formulas in \(D = 6\)

The candidate terms to the total derivatives in (11) can be reduced to the form \[15\]

\[
\chi_1 = \Box^2 R, \quad \chi_{2;3;4} = \Box \left( R^2_{\mu\nu\alpha\beta}; R^2_{\mu\nu}; R^2 \right)
\]

\[
\chi_{5;6;7;8} = \nabla_{\mu} \nabla_{\nu} \left( R^\mu_{\lambda\alpha\beta} R^{\lambda\alpha\beta}; R_{\alpha\beta} R^{\mu\alpha\beta}; R_\alpha R^{\mu\alpha}; R R_{\mu\nu} \right).
\] (12)

4
After a very long and in fact complicated calculations, we arrived at the following coefficients which guarantee the equations (4) and (6) for \( D = 6 \),

\[
\begin{align*}
\alpha_1 &= \frac{3}{5}, \quad \alpha_2 = \frac{147}{20} + \xi, \quad \alpha_3 = -\frac{33}{5} - \frac{1}{2}\xi, \quad \alpha_4 = 0 \\
\alpha_5 &= -3 + 4\xi, \quad \alpha_6 = 6 - 4\xi, \quad \alpha_7 = 3 - 3\xi, \quad \alpha_8 = \xi.
\end{align*}
\]

(13)

Here \( \xi \) is a free parameter which remains undetermined by the condition (6). Assuming (13), all the non-linear in \( \sigma \) terms in (6) cancel, and the remaining linear term corresponds to \( \kappa = -6 \) and the conformal operator

\[
\Delta_6 = \Box^3 + 4R^{\mu\nu}\nabla_\mu \nabla_\nu \Box - R\Box^2 \\
+ 4\nabla_\alpha \left[ (\nabla^\alpha R^{\mu\nu}) \nabla_\mu \nabla_\nu \right] + V^{\mu\nu} \nabla_\mu \nabla_\nu + N^\lambda \nabla_\lambda,
\]

(14)

where

\[
V^{\mu\nu} = \frac{12 + 4\xi}{3} \left( R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} - R^{\mu}_{\alpha\gamma} R^{\nu\beta}_{\alpha\gamma} \right) + (9 + \xi) \left( \frac{R^{\mu\alpha} R^\nu_{\alpha} - \frac{1}{3} R R^{\mu\nu}}{R^\mu_{\alpha\beta}} \right) \\
+ g^{\mu\nu} \left[ \frac{81 + 20\xi}{15} R^{2}_{\mu\alpha\beta} - \frac{69 + 15\xi}{10} R^{\mu\nu}_\alpha + \frac{6 + \xi}{6} R^2 - \frac{3}{5} (\Box R) \right]
\]

and

\[
N^\lambda = \frac{44 + 10\xi}{5} R^{\mu\nu}_{\mu\alpha\beta} (\nabla^\lambda R^{\mu\nu}_{\mu\alpha\beta}) + \frac{12 + 4\xi}{3} R^{\mu\nu\alpha\lambda} (\nabla_\mu R^\nu_{\alpha\lambda}) - \left( \frac{49}{5} + \frac{5\xi}{3} \right) R^{\mu\nu}(\nabla^\lambda R^{\mu\nu}) \\
+ \frac{15 - \xi}{3} R^{\mu\nu}_{\mu\nu}(\nabla^\mu R^{\nu\lambda}) + \frac{9 + \xi}{6} R^{\mu\nu\lambda}(\nabla_\mu R^{\nu\lambda}) + \frac{3 + \xi}{6} R(\nabla^\lambda R) + \frac{2}{5} (\nabla^\lambda \Box R).
\]

One has to remember that here the covariant derivative does not act beyond the parenthesis.

Let us note that in the literature one can find a general theory for constructing conformal operators (see, e.g., [16, 17, 18, 4]), still the operator (14) is more general than the ones known before. The main relation (4) was not derived before, probably due to the complexity of calculations requested to get the coefficients (13). We could achieve it by combining hand-made work and the softwares Cadabra [19] and Mathematica [20]. The essential details will be published elsewhere [21], together with the solution for the local terms producing surface terms (8) in the anomaly.

Compared to the main calculation, it is much easier (but still consuming certain time and effort) to check that the operator \( \Delta_6 \) satisfies the conformal invariance (7) and is self-adjoint, \( \int_x \varphi \Delta_6 \chi = \int_x \chi \Delta_6 \varphi \). It is interesting that both conditions do not pose any restriction on the value of an arbitrary parameter \( \xi \). We shall discuss the physical consequence of this ambiguity in the last section.
4 Conclusions and discussions

The equations (13) and (14) form the full set of the building blocks for the non-local part of anomaly-induced action (11) in \( D = 6 \). Together with the previously known examples in \( D = 2, 4 \) this enables us to draw some general conclusions and discuss the similarities and differences between the new result and the previous one. One of the common points is that the anomaly-induced expression is an exact effective action for the homogeneous and isotropic metric, where the conformal functional \( S_c \) is irrelevant. Assuming that the space-time has six dimensions, and that there are massless conformal fields in the far IR, we arrive at the exact solution for anomaly-induced action in this particular class of metrics.

Qualitatively, the structure of (10) and (11) is the same in all even dimensions, but the complexity of the solution increases with dimension. On the transition from two to four dimensions the main complications were the integration constant \( S_c \) and the presence of the two different (conformal and topological) terms in (11) which produce non-local terms in the anomaly-induced action [6]. One of the consequences is that the integrated anomaly can be consistently written in local covariant form only by means of two auxiliary fields [12, 13, 5], while in \( D = 2, 4 \) one such field is sufficient. As we have seen in Sect. 2 the number of auxiliary fields remains the same in higher dimensions. At the same time the solution (13), (14) includes a qualitatively new arbitrary parameter \( \xi \). Nothing of this sort takes place in \( D = 2, 4 \). An interesting possibility is that the ambiguity can be fixed by imposing the consistency conditions [22, 23, 24], but it is not certain, of course. Another question is what could be the physical effect of an arbitrary parameter \( \xi \)?

Since the conformal anomaly is the same for any \( \xi \), one can simply ignore the ambiguity by fixing some particular value for this parameter. The difference between distinct values can be always absorbed into the conformal functional \( S_c \). The situation is technically similar to the one with the \( \psi \)-dependent part of (11), which can be also absorbed into conformal part. However, in the case of \( \psi \)-terms this would be a wrong idea. For instance, without the second auxiliary field one can not classify vacuum states in the vicinity of the spherically symmetric black holes [25]. There is no such a problem for the gravitational waves, but maybe only because all known calculations were done for the isotropic cosmological backgrounds [26, 27, 28, 29]. Concerning the role of \( \xi \), the question is whether it affects the relevant solutions, and this question will remain open until such solutions are explored for the action (11).

The last observation concerns the possible applications of the effective actions (10) and (11). One can imagine that the explicit form of effective vacuum action for the
conformal fields can be useful for verifying the calculations related to holography and AdS/CFT correspondence. Another application is related to the dimensional reduction to $D = 4$, expected to produce a four-dimensional action different from the one coming from integrating anomaly directly in $D = 4$. Due to the universality of the result, the calculation of such a reduced action and the study of its physically relevant solutions may be eventually useful in designing the experimental and/or observational tests for the existence of extra dimensions.

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