Circular solution of two unequal mass particles in post-Minkowski approximation

Matthew M. Glenz\textsuperscript{1} and Kōji Uryū\textsuperscript{1,2}

\textsuperscript{1} Department of Physics, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, WI 53201
\textsuperscript{2} Department of Physics, University of the Ryukyus, 1 Senbaru, Nishihara, Okinawa 903-0213 Japan

A Fokker action for post-Minkowski approximation with the first post-Newtonian correction is introduced in our previous paper, and a solution for the helically symmetric circular orbit is obtained. We present supplemental results for the circular solution of two unequal mass point-particles. Circular solutions for selected mass ratios are found numerically, and analytic formulas in the extreme mass ratio limit are derived. The leading terms of the analytic formulas agree with the first post-Newtonian formulas in this limit.

I. INTRODUCTION

In our previous paper \textsuperscript{1} (Paper I), a Fokker action that describes a time symmetric interaction of point-particles is introduced in the framework of a post-Minkowski approximation. From two types of Fokker action, a parametrization invariant action with a post-Newtonian correction and an affinely parametrized action, the equations of motion and expressions for conserved energy and angular momentum are derived following the variational calculation of Ref. \textsuperscript{2}. We find a solution describing a helically symmetric circular orbit in the post-Minkowski approximation (with post-Newtonian corrections) that is analogous to the circular solution of two charges obtained by Schild for the electromagnetic interaction \textsuperscript{3}. We report here results supplementing those of Paper I: numerically computed solution sequences for unequal mass particles, and analytic formulas in the extreme mass ratio limit. The latter results agree with the first post-Newtonian (1PN) formulas; hence a consistency of our model is confirmed in this limit.

II. FORMULAS FOR CIRCULAR SOLUTIONS

We present a set of formulas governing the helically symmetric circular orbits of two point particles, \{\bar{m}, \bar{v}\} and \{\bar{m}, \bar{v}\}, and derive analytic expressions in the extreme mass ratio limit \(q := m/\bar{m} \to 0\). The set of algebraic equations is solved numerically for a fixed binary separation to specify each circular orbit. The result for the unequal mass binary orbit is presented in Sec. IIII. Units of \(G = c = 1\) are used in this report.

A. Parametrization invariant model with post-Newtonian correction

1. Circular solution

We first list the result from Paper I for the parametrization invariant model with 1PN correction terms. The (integrated) equations of motion for particles \(m\) and \(\bar{m}\) are written in terms of the velocities, \(v\) and \(\bar{v}\), of particles \(m\) and \(\bar{m}\) which are related to the orbital radius by \(a := v/\Omega\) and \(\bar{a} := \bar{v}/\Omega\),

\[
- m\gamma^2 v\Omega = - m\bar{m}\gamma^2 \Omega^2 \left[ F(\varphi, v, \bar{v}) + (m + \bar{m})\Omega F_1(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) \right],
\]

(1)

\[
- \bar{m}\bar{\gamma}^2 \bar{v}\Omega = - \bar{m}\bar{m}\bar{\gamma}^2 \Omega^2 \left[ \bar{F}(\varphi, \bar{v}, \bar{\bar{v}}) + (m + \bar{m})\Omega \bar{F}_1(\varphi, \bar{v}, \bar{\bar{v}}, \gamma, \bar{\gamma}) \right].
\]

(2)

As shown below, \{\varphi, v, \bar{v}, \gamma, \bar{\gamma}\} are not independent. The functions \(F(\varphi, v, \bar{v}) = \bar{F}(\varphi, \bar{v}, \bar{\bar{v}})\) are the post-Minkowski terms, while \(F_1(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) = \bar{F}_1(\varphi, \bar{v}, \bar{\bar{v}}, \gamma, \bar{\gamma})\) is either of two alternative 1PN correction terms that agree at 1PN order: \(F_1 = F_{1PN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma})\) derived from a non-relativistic correction, or \(F_1 = F_{1SPN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma})\) derived from a special relativistically invariant correction.

\[
F(\varphi, v, \bar{v}) := - \frac{1}{(\varphi + v\bar{v}\sin \varphi)^2} \left[ (1 + \bar{v}\bar{\varphi})\bar{v}\right. \\
\times (\varphi \cos \varphi - v^2 \sin \varphi) + \frac{1}{2} v (1 - \bar{v}^2)(\varphi + v\bar{v}\sin \varphi) \\
\left. - \frac{1}{2} [\bar{v}\sin \varphi(\varphi + v\bar{v}\sin \varphi) + (1 + \bar{v}\bar{v}\cos \varphi)(\bar{v} + \bar{v}\bar{\varphi}) \right]
\]

(3)

\[
F_{1PN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) := - \frac{1}{\gamma^2(\varphi + v\bar{v}\sin \varphi)^2} \left[ 1 + \frac{1}{2} \gamma^2 v(\varphi + v\bar{v}\sin \varphi) \right],
\]

(4)

\[
F_{1SPN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) := - \frac{1}{\bar{\gamma}^2(\varphi + v\bar{v}\sin \varphi)^2} \left[ \frac{3}{4} \gamma^2 v + \frac{\bar{v}\sin \varphi}{\varphi + \bar{v}\sin \varphi} + \frac{(1 + \bar{v}\bar{v}\cos \varphi)(\varphi + \bar{v}\bar{\varphi})}{(\varphi + v\bar{v}\sin \varphi)^2} \right].
\]

(5)

The function \(\Phi(\varphi, v, \bar{v})\) is defined by

\[
\Phi(\varphi, v, \bar{v}) := \frac{(1 + \bar{v}\bar{v}\cos \varphi)^2 - \frac{1}{2}(1 - v^2)(1 - \bar{v}^2)}{\varphi + \bar{v}\bar{v}\sin \varphi}.
\]

(6)

For the parametrization invariant models, \(\gamma\) and \(\bar{\gamma}\) are derived from a flat-space normalization of the four-velocity,

\[
\gamma = (1 - v^2)^{-\frac{1}{2}}, \quad \bar{\gamma} = (1 - \bar{v}^2)^{-\frac{1}{2}}.
\]

(7)

The retarded angle \(\varphi\) is the positive root of \(\varphi^2 = v^2 + \bar{v}^2 + 2v\bar{v}\cos \varphi\).
2. Extreme mass ratio limit

The extreme mass ratio limit \( q := m/\tilde{m} \to 0 \) is identical to the limit \( \bar{v} \to 0 \) with \( \Omega \) fixed. In the limit \( \bar{v} \to 0 \), we may assume that \( v \) and \( \tilde{m} \) remain finite. Consequently, we have \( \gamma \to 1, \varphi \to v, \) and \( \tilde{m} \to M \), where \( M := m + \tilde{m} \) is the total mass. With \( v \) and \( \Omega \) regarded as independent variables, Eq. (11) is a quadratic equation for \( \Omega M \), whose \( q = 0 \) form is

\[
F_1 (\Omega M)^2 + F (\Omega M) - v = 0, \quad (8)
\]

with physical solution

\[
\Omega M = \frac{1}{2F_1} \left( -F + \sqrt{F^2 + 4F_1 v} \right). \quad (9)
\]

The functions \( F \) (the post-Minkowski term), \( F_1 = F_{PN} \) and \( F = F_{SPN} \) (the alternative forms of the 1PN correction) for \( q = 0 \) become

\[
F(\varphi, v, \bar{v}) = \frac{1 - 3v^2}{v^2(1 - v^2)}, \quad (10)
\]

\[
F_{PN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) = -\frac{1}{v^3} \left( 1 - \frac{1}{2} v^2 \right), \quad (11)
\]

\[
F_{SPN}(\varphi, v, \bar{v}, \gamma, \bar{\gamma}) = -\frac{(1 - v^2)^{1/4}}{v^3} \left( 1 - \frac{1}{4} v^2 \right), \quad (12)
\]

where \( \Phi \) has the form \( \Phi(\varphi, v, \bar{v}) = (1 + v^2)/(2v) \).

Note that the parametrization invariant post-Minkowski model is derived by setting \( F_1 = 0 \), and therefore \( \Omega M = v/F \). In the \( q \to 0 \) limit, this is written \( \Omega M = v^3(1 - v^2)/(1 - 3v^2) \).

3. Energy and angular momentum formulas

The conserved energy and angular momentum for the parametrization invariant model are written

\[
E = E_{PM} + e_1, \quad \text{and} \quad L = L_{PM} + \ell_1, \quad (13)
\]

where \( E_{PM} \) and \( L_{PM} \) are the post-Minkowski terms

\[
E_{PM} = \frac{m}{\gamma} + \frac{\tilde{m}}{\bar{\gamma}}, \quad (14)
\]

\[
L_{PM} = 2\tilde{m}m\gamma\bar{\gamma} \Phi(\varphi, v, \bar{v}), \quad (15)
\]

and \( e_1 \) and \( \ell_1 \) are the parametrization invariant 1PN corrections \( e_1 = e_{PN} \) and \( \ell_1 = \ell_{PN} \), or those of the special relativistically invariant model \( e_1 = e_{SPN} \) and \( \ell_1 = \ell_{SPN} \) given by

\[
e_1 = \frac{1}{2} \Omega \ell_1, \quad (for \ both \ e_1 = e_{PN} \ and \ e_{SPN}), \quad (16)
\]

\[
\ell_{PN} = -\frac{\tilde{m}m(m + \tilde{m})}{\gamma (v + \bar{v})}, \quad (17)
\]

\[
\ell_{SPN} = -\frac{\tilde{m}m(m + \tilde{m})}{(\gamma \bar{\gamma})^{3/2}} \left( \frac{1}{(v + \bar{v})^2} \right). \quad (18)
\]

In the \( q \to 0 \) limit, the conserved energy and angular momentum normalized by the mass remain finite. Subtracting the mass of the heavier particle from the post-Minkowski energy, \( \hat{E}_{PM} := E_{PM} - \tilde{m} \), and taking the limit \( \bar{v} \to 0 \) with \( \tilde{m} \to M \), we have

\[
\frac{\hat{E}_{PM}}{m} = m(1 - v^2)^{1/2}, \quad (19)
\]

\[
\frac{L_{PM}}{mM} = \frac{1 + v^2}{v(1 - v^2)^{1/2}}, \quad (20)
\]

\[
\frac{\ell_{PN}}{mM} = -\frac{(1 - v^2)^{1/2}}{v^2} \Omega M, \quad (21)
\]

\[
\frac{\ell_{SPN}}{mM} = -\frac{(1 - v^2)^{3/4}}{v^2} \Omega M. \quad (22)
\]

4. Solution sequence in \( q \to 0 \) limit

In Paper I, it is proved that the first law of thermodynamics that relates the changes in the conserved energy and the angular momentum, \( dE = \Omega dL \), is satisfied by binary solutions derived from the parametrization invariant Fokker action. This relation is used to cross check the analytic formula in the \( q \to 0 \) limit above as well as the numerical solutions shown in the next section by calculating \( dE/dv = \Omega dL/dv \), where \( E := \hat{E}_{PM} + e_1 \).

In the parametrization invariant post-Minkowski model, the normalized angular velocity of a particle \( m, \Omega M \), is defined in an interval \( 0 \leq v < 1/\sqrt{3} \), and it becomes infinite at \( v = 1/\sqrt{3} \). With the 1PN correction \( F_1 = F_{PN} \), the range of finite \( \Omega M \) is approximately \( 0 \leq v \lesssim 0.361598 \), and with the special relativistic invariant 1PN correction \( F_1 = F_{SPN} \), it is \( 0 \leq v \lesssim 0.36166 \). Newtonian point particles have no innermost stable circular orbit (ISCO), but adding a 1PN correction to the Newtonian orbit recovers the ISCO that is present in the exact theory. In the post-Minkowski framework, we find that the existence of an ISCO depends on our choice among actions that are equivalent to first post-Minkowski order. In particular, we will see that the parametrization-invariant action leads to sequences with no ISCO even when 1PN terms are included. This is plausibly due to the fact that the sequences associated with the parametrization-invariant action terminate before reaching the angular velocity of an ISCO. In fact, in the standard 1PN formalism, an ISCO occurs at an unrealistically high value of angular velocity, namely \( \Omega M = 0.544 \), where the 2PN and 3PN values are \( \Omega M = 0.124 \) and 0.0867 for \( q = 0 \), respectively [2].

Curiously, however, as we note in the next section, sequences associated with the affinely parametrized action do have an ISCO, in this case at an unrealistically small values of \( \Omega M \).

Finally, we show that these results of post-Minkowski plus 1PN corrections agree with the 1PN formula in the \( q \to 0 \) limit. In Eq. (19), an expansion of \( \Omega M \) in the
small $v$ limit becomes $\Omega M = v^3 + 3v^5 + O(v^7)$ for both PN and SPN models, and this is inverted to write $v$ in terms of small $\Omega M$ as $v = (\Omega M)^{1/3} - \Omega M + O((\Omega M)^{5/3})$. Substituting this into the energy and angular momentum formulas, the leading two terms agree with the post-Newtonian formulas (see e.g. [2]) up to the 1PN order for the extreme mass ratio $q \to 0$,

$$\frac{\hat{E}}{m} = -\frac{1}{2}(\Omega M)^{2/3} + \frac{3}{8}(\Omega M)^{4/3} + O((\Omega M)^2), \quad (23)$$

$$\frac{L}{mM} = \frac{1}{(\Omega M)^{1/3}} \left[ 1 + \frac{3}{2}(\Omega M)^{2/3} + O((\Omega M)^{4/3}) \right]. \quad (24)$$

**B. Affinely parametrized model**

### 1. Circular solution

For the affinely parametrized post-Minkowski model, analogous forms of Eqs. (1) and (2) are written

$$-m\gamma^2 v \Omega = -m\bar{m}\gamma^2 \Omega^2 F^A(\varphi, v, \bar{v}), \quad (25)$$

$$-\bar{m}\gamma^2 \bar{v} \Omega = -\bar{m}\gamma^2 \Omega^2 \bar{F}^A(\varphi, v, \bar{v}), \quad (26)$$

where the function $F^A(\varphi, v, \bar{v}) = \bar{F}^A(\varphi, v, \bar{v})$ is written

$$F^A(\varphi, v, \bar{v}) := -\frac{1}{4} \left( \frac{\varphi + v\bar{v} \sin \varphi}{(v \bar{v} \sin \varphi)^2} \right) \times \left\{ (1 + v\bar{v} \cos \varphi)\bar{v}(\varphi \cos \varphi - v^2 \sin \varphi) + \frac{1}{2}v(1 - \bar{v}^2)\varphi \sin \varphi - \frac{1}{2}\bar{v} \sin \varphi (v + v\bar{v} \sin \varphi) + (1 + v\bar{v} \cos \varphi)(v + v\bar{v} \cos \varphi) \right\} \Phi(\varphi, v, \bar{v}). \quad (27)$$

For the affinely parametrized world line, $\gamma$ and $\bar{\gamma}$ satisfy

$$-\gamma^2 (1 - v^2) + 4m\gamma^2 \gamma \Omega \Phi(\varphi, v, \bar{v}) = -1, \quad (28)$$

$$-\bar{\gamma}^2 (1 - \bar{v}^2) + 4\bar{m}\bar{\gamma}^2 \Omega \Phi(\varphi, v, \bar{v}) = -1. \quad (29)$$

In the limit of $q \to 0$ (or more directly $\bar{v} \to 0$), $F^A(\varphi, v, \bar{v}) = (1 - \bar{v}^2)/v^2$. From Eq. (25) and (26), we have $\gamma = (1 - v^2)^{1/2}/(1 - 4v^2 - v^4)^{1/2}$, while in Eq. (29), taking $\bar{v} \to 0$ and $m \to 0$ yields $\bar{\gamma} \to 1$. As a result we have in the extreme mass ratio $q \to 0$,

$$\Omega M = \frac{v^3}{1 - v^2}. \quad (30)$$

### 2. Energy and angular momentum formula

The conserved energy and angular momentum for the affinely parametrized model are written

$$E = \frac{m}{\gamma} + \frac{\bar{m}}{\bar{\gamma}} + 4m\bar{m}\gamma^2 \gamma \Omega \Phi(\varphi, v, \bar{v}), \quad (31)$$

$$L = 2m\bar{m}\gamma^2 \Phi(\varphi, v, \bar{v}), \quad (32)$$

**FIG. 1:** Angular velocity, in dimensionless form $\Omega M$, is plotted against the velocity of the lighter particle for 3 mass ratios and the $q \to 0$ limit for the parametrization invariant model with SPN correction. Curves of the analytic solution for $q \to 0$ and that of $q = 0.001$ overlap each other in the plot.

**FIG. 2:** Energy, in dimensionless form $\hat{E}/m$, where $\hat{E} = E - \bar{m}$, is plotted against $\Omega M$ for the same models as in Fig. 1, where the form of $\Phi(\varphi, v, \bar{v})$ is the same as that of the parametrization invariant model (6). Using Eq. (28) and (29), the energy can be rewritten

$$E = \frac{1}{2}m + \frac{1}{2}m\gamma(1 - v^2) + \frac{1}{2}m\bar{\gamma}(1 - \bar{v}^2). \quad (33)$$

In the $q \to 0$ limit, the energy without the rest mass of the heavier particle, $\hat{E} := E - \bar{m}$, and the angular momentum become

$$\frac{\hat{E}}{m} = \frac{(1 - 3v^2)}{[(1 - v^2)(1 - 4v^2 - v^4)^{1/2}]}, \quad (34)$$

$$\frac{L}{mM} = \frac{1 + v^2}{m} \left( \frac{1 - v^2}{1 - 4v^2 - v^4} \right)^{1/2}. \quad (35)$$
3. Solution sequence in $q \to 0$ limit

The first law $\delta E = \Omega \delta L$ is also satisfied for the affinely parametrized model, and hence one can cross check formulas in the $q \to 0$ limit using the relation $d\hat{E}/dv = \Omega dL/dv$. Although the normalized angular velocity of a particle $\Omega M$, is finite in an interval $v \in [0, 1)$, the redshift factor $\gamma$ as well as conserved quantities $E$ and $L$ become infinite at $v = \sqrt{\frac{5}{2}} - 2 \approx 0.485868$, which corresponds to $\Omega M = (\sqrt{5} - 2)^{3/2}/(3 - \sqrt{5}) \approx 0.150142$.

In this interval, $v \in [0, \sqrt{\frac{5}{2}} - 2)$, the energy and angular momentum have a simultaneous minima at $v = \sqrt{(1 + 2^{4/3} - 2^{8/3})/3} \approx 0.339136$, which corresponds to $\Omega M \approx 0.0440743$.

III. NUMERICAL SOLUTIONS FOR THE UNEQUAL MASS CIRCULAR ORBIT

A circular solution is calculated from algebraic equations given in Sec. II A 1 for the parametrization invariant model, and II B 1 for the affinely parametrized model. It turned out that a convenient way to find a solution is (1) fix the ratio of velocities $v/\bar{v}$ and determine the corresponding mass ratio from the equations of motion, then (2) change the velocity ratio to adjust the value of the mass ratio to a fixed value (using the bisection method, for example).

In Figs. 1 and 2, the plots of the parametrization invariant model with SPN correction terms are presented for three mass ratios, $q = 1.0, 0.1$, and $0.001$. Plots for the case with PN correction terms are not shown here, and they are qualitatively the same as SPN cases. The analytic solution in the $q \to 0$ limit is also plotted and it overlaps with the $q = 0.001$ line in the plots.

In Figs. 3 and 4 the plots of the affinely parametrized model are presented for the same mass ratios as above. The solutions of the affinely parametrized model are markedly different; for any mass ratio $q \in [0, 1)$, we found a simultaneous minima in the energy and angular momentum which corresponds to the ISCO.

IV. DISCUSSION

Agreement between the energy and angular momentum formulas of the 1PN circular solution, and those of the parametrization invariant post-Minkowski model with post-Newtonian correction, is exhibited only for the extreme mass ratio limit in this report. For an arbitrary mass ratio one needs to expand the retarded angle $\varphi$ to the next order in the velocities, $v$ and $\bar{v}$, as $\varphi \approx (v + \bar{v})(1 - v\bar{v}/2)$, and the rest of the calculation closely parallels that of the $q = 0$ case.

Acknowledgments

We thank John L. Friedman for discussions and careful reading of the manuscript. This work was supported by NSF grants Nos. PHY0071044 and PHY0503366, the Lynde and Harry Bradley Foundation, and the National Space Grant College and Fellowship Program and the Wisconsin Space Grant Consortium.

[1] J. L. Friedman and K. Uryu, Phys. Rev. D 73, 104039 (2006).
[2] J. W. Dettman and A. Schild, Phys. Rev. 95, 1057 (1954).
[3] A. Schild, Phys. Rev. 131, 2762 (1963).
[4] L. Blanchet, Phys. Rev. D 65, 124009 (2002).