A semi-classical versus quantum description of the ground state of three-level atoms interacting with a one-mode electromagnetic field

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Abstract

We consider three-level atoms (or systems) interacting with a one-mode electromagnetic field in the dipolar and rotating wave approximations. The order of the quantum phase transitions is determined explicitly for each of the configurations Ξ, Λ and V, with and without detuning. The semi-classical and exact quantum calculations for both the expectation values of the total number of excitations $M = \langle M \rangle$ and photon number $n = \langle n \rangle$ have excellent correspondence as functions of the control parameters. We prove that the ground state of the collective regime obeys sub-Poissonian statistics for the $M$ and $n$ distribution functions. Therefore, their corresponding fluctuations are not well described by the semi-classical approximation. We show that this can be corrected by projecting the variational state to a definite value of $M$.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The interaction of three-level atoms with a quantized electromagnetic field (QEMF), using dipolar and rotating wave approximations (RWAs), is described by the Tavis–Cummings model [1, 2], and has extensive use in quantum optics [3]. Recently, this model has been physically realized using a QED cavity with Bose–Einstein condensates [4, 5]. Particularly interesting has been the investigation of the phase transitions of the system in the thermodynamic limit [6, 7], and at zero temperature [8–10].

The system of three-level atoms interacting with a one-mode radiation field together with a dipole–dipole interaction between the atoms has been studied to determine the atomic squeezing [11, 12]. They consider Ξ and Λ configurations under initial conditions of the matter and field parts associated with $SU(2)$ and Heisenberg–Weyl coherent states, respectively. Spin
variances for the $V$ and $\Lambda$ configurations of an ensemble of atoms interacting with two light fields, a coherent pump state and a squeezed vacuum as a probe, have been calculated by means of the Langevin equations derived from the Bloch equations [13]. By using a Holstein–Primakoff mapping, two stable states, normal and superradiant (the latter in two colors), have been identified in the thermodynamic limit for the $\Lambda$ configuration [14].

More recently, we have analytically obtained the localization of the quantum phase transitions from the normal to the collective regimes for three-level atoms interacting with a one-mode field for the $\Xi$, $\Lambda$, and $V$ configurations in the RWA. These transitions appear in the ground state energy surface $E^c$ and the corresponding total number of excitations $M^c$, when plotted as functions of their corresponding dipole coupling constants (control parameters), are calculated using as a test function the direct product of the Heisenberg–Weyl (field contribution) and Gelfand–Tsetlin (matter contribution) coherent states. We found that the agreement of these quantities with the corresponding exact quantum calculations (namely $E^q$ and $M^q$) is remarkable [15].

In this paper we determine explicitly the order of the quantum phase transitions, and calculate the Mandel parameter of the $M$ distribution function and of the photon number distribution function of the ground state of the system. We find that first- and second-order transitions appear for atoms in the $\Xi$ configuration, and only second-order transitions appear for atoms in the $V$ configuration. Atoms in the $\Lambda$ configuration, depending on the detuning parameter, mimic the behavior of the $\Xi$ or the $V$ configuration. We find that in the collective regime, i.e., where the ground state possesses $M > 0$, the state obeys sub-Poissonian statistics while in the normal regime it satisfies Poissonian statistics.

While both the total number of excitations $M$ and the expectation value of the number of photons $\langle n \rangle$ are in agreement with their corresponding exact quantum calculation, we find that their fluctuations are not. This is because the semi-classical ground state has the contribution of an infinite number of photons in a Poissonian distribution. The above suggests a projection of the test function to a definite value of the total number of excitations. This we do by means of a discretization of $M$, according to its expectation value with respect to the test function. We prove that this projected state provides the appropriate correction, where now $M$, $\langle n \rangle$, and their corresponding fluctuations are in excellent agreement with the exact quantum calculation.

The paper is organized as follows: section 2 presents in general the problem for $N_a$ atoms of $N$-levels interacting with $L$-modes of a QEMF in the dipolar approximation. In section 3 we restrict ourselves to the problem of three-level atoms interacting with a one-mode QEMF in the RWA, and establish the corresponding constant of motion $M$ (total number of excitations) for each atomic configuration. In section 3.1 the test function as a direct product of the Heisenberg–Weyl (field contribution) and Gelfand–Tsetlin (matter contribution) coherent states is proposed for the semi-classical approximation. The corresponding semi-classical energy of the problem is calculated in section 3.2. In section 3.3 we provide an exact expression to evaluate the first-order derivatives of the ground state energy surface (as a function of the control parameters), so that the first-order transitions for each atomic configuration can be calculated in analytical form. For every value of the total number of excitations, the corresponding Mandel parameter of the semi-classical ground state, providing the kind of statistics that it satisfies, is evaluated in section 3.4. In section 3.5 we show the numerical results for both order transitions, the Mandel parameter and the photon expectation value, for all different atomic configurations. Section 4 presents the exact quantum calculations and compares them with the semi-classical ones. In section 5 the calculations obtained by using the projected variational state with the corresponding exact quantum results are compared. Finally, in section 6 we give some concluding remarks.
2. N-level atoms interacting with an L-mode QEMF

We consider, in the dipolar approximation, the Hamiltonian of \( N \) identical atoms of \( N \)-levels interacting with \( L \)-modes of a QEMF. Let \( A_{ij}^{(k)} \) denote the atomic operator of the \( k \)th atom. For each atom, these operators obey a unitary algebra \( u_k(N) \) in \( N \) dimensions, i.e.,

\[
\sum_{i=1}^{N} A_{ii}^{(k)} = 1, \tag{1}
\]

\[
[A_{ij}^{(k)}, A_{lm}^{(k)}] = \delta_{kk'} (\delta_{jl} A_{im}^{(k)} - \delta_{im} A_{lj}^{(k)}). \tag{2}
\]

Defining

\[
A_{ij} \equiv \sum_{k=1}^{N} A_{ij}^{(k)}, \tag{3}
\]

one can see that the following relationships are fulfilled

\[
n_a = \sum_{i=1}^{N} A_{ii}, \tag{4}
\]

\[
[A_{ij}, A_{lm}] = \delta_{jl} A_{im} - \delta_{im} A_{lj}. \tag{5}
\]

We have here defined the operator \( n_a \) representing the total number of atoms with eigenvalue \( n_a \), and equation (5) shows that the set of operators \( A_{ij} \) obey the commutation relations of a unitary algebra in \( N \) dimensions, \( U(N) = \bigoplus_{k=1}^{N} u_k(N) \).

Now, for \( L \)-modes of a quantized field and \( N_a \) atoms, the free Hamiltonian may be written as \( (\hbar = 1) \)

\[
H_0 = \sum_{\ell=1}^{L} \Omega_\ell a_\ell^\dagger a_\ell + \sum_{i=1}^{N} \omega_i A_{ii}, \tag{6}
\]

where \( \Omega_\ell \) and \( \omega_i \) correspond, respectively, to the frequencies of the \( \ell \)th field mode and the \( i \)th atomic level (we choose \( \omega_1 \leq \omega_2 \leq \cdots \leq \omega_N \)). Here \( a_\ell^\dagger, a_\ell \) are the usual creation and annihilation operators of the field obeying the boson algebra, i.e.,

\[
[a_i, a_j^\dagger] = \delta_{ij}, \tag{7}
\]

and \( A_{ij} \) are the atomic operators of equation (3).

The interaction Hamiltonian due to the dipole operator \( \vec{d} \) of atoms with the electromagnetic field \( \vec{E} \), reads as \( (16) \)

\[
H_{int} = -\vec{d} \cdot \vec{E}. \tag{8}
\]

\( \vec{d} \) may be written as

\[
\vec{d} = \sum_{i \neq j} d_{ij} A_{ij}, \tag{9}
\]

where \( d_{ij} \) represents the matrix elements of the vector dipole operator between the levels \( j \) and \( i \). Notice that \( \vec{d} \) has no diagonal contributions, because the dipolar interaction of a level with itself is zero. The corresponding quantized field may be written as

\[
\vec{E} = \sum_{\ell=1}^{L} [\vec{E}_\ell^\dagger (\vec{r}) a_\ell + \vec{E}_\ell (\vec{r}) a_\ell^\dagger], \tag{10}
\]
where $\vec{E}_E(\vec{r})$ obeys the Helmholtz equation for the $\ell$th field mode, providing the structure of the field in the cavity. Substituting equations (9) and (10) into equation (8), and reordering the different contributions, one may write the interaction Hamiltonian as

$$H_{\text{int}} = - \sum_{s=1}^{N-1} \sum_{\ell=1}^{L} (a_s^\dagger \vec{g}_{s\ell} \cdot \vec{\sigma}_{s-} + a_s \vec{\sigma}_{s+} \cdot \vec{g}_{s\ell}^T) - \sum_{s=1}^{N-1} \sum_{\ell=1}^{L} (a_s^\dagger \vec{g}_{s\ell} \cdot \vec{\sigma}_{s-} + a_s \vec{\sigma}_{s+} \cdot \vec{g}_{s\ell}^T),$$

(11)

where the vector operators were defined as

$$\vec{\sigma}_{s\pm} = (A_{1+s,1}, \ldots, A_{j+s,j}, \ldots, A_{(N-s)+s,N-s})$$

(12)

containing the set of operators $A_{ij}$ with transitions from the $j$th level of the atom to the $(j+s)$th level. Also, $\vec{\sigma}_{s\pm} = \vec{\sigma}^\dagger_{s\pm}$, and

$$\vec{g}_{s\ell} = \frac{1}{\sqrt{N_a}} (\mu_1^{(s)}, \ldots, \mu_{j+s}^{(s)}, \ldots, \mu_{N-s,(N-s)+s}^{(s)})$$

(13)

with $\mu_{ij}^{(s)}/\sqrt{N_a} = \vec{d}_{ij} \cdot \vec{E}_E^s$, the coupling parameter between levels $i$ and $j$, and where we have taken $\vec{d}_{ij} = \vec{d}_{ji}$. Here, we have eliminated the dependence on $\vec{r}$ of $\vec{E}_E^s$ by supposing that the $N_a$ atoms are stationary at the center of the cavity, and that the field is a smooth function in that region.

The second term in the rhs of equation (11) corresponds to the counter-rotating term, and when the RWA is considered this term is neglected. The interaction term in the RWA is therefore given by

$$H_{\text{int}} = - \sum_{s=1}^{N-1} \sum_{\ell=1}^{L} (a_s^\dagger \vec{g}_{s\ell} \cdot \vec{\sigma}_{s-} + a_s \vec{\sigma}_{s+} \cdot \vec{g}_{s\ell}^T).$$

(14)

Finally, the full Hamiltonian in the RWA reads as

$$H = \sum_{\ell=1}^{L} \Omega_{\ell} a_\ell^\dagger a_\ell + \sum_{j=1}^{N} \omega_j A_{jj} - \sum_{s=1}^{N-1} \sum_{\ell=1}^{L} (a_s^\dagger \vec{g}_{s\ell} \cdot \vec{\sigma}_{s-} + a_s \vec{\sigma}_{s+} \cdot \vec{g}_{s\ell}^T).$$

(15)

The Hamiltonian above shows the underlying structure of the unitary group in $N$ dimensions, $U(N)$, which makes the use of the Gelfand–Tsetlin states [18] natural. This allows for the description, in general, of systems with any kind of symmetry, including distinguishable particles.

### 3. Three-level atoms interacting with a one-mode QEMF

In what follows we consider $N_a$ three-level atoms interacting with a one-mode QEM field, i.e., we choose $N = 3$ and $L = 1$ in equation (15). Replacing the corresponding values of $\vec{\sigma}_{13}$ and $\vec{g}_{13}$ into equation (15) one finds the Hamiltonian of the system as

$$H = \Omega a^\dagger a + \omega_1 A_{11} + \omega_2 A_{22} + \omega_3 A_{33} - \frac{1}{\sqrt{N_a}} \mu_{12} (a A_{12} + a^\dagger A_{12})$$

$$- \frac{1}{\sqrt{N_a}} \mu_{13} (a A_{13} + a^\dagger A_{13}) - \frac{1}{\sqrt{N_a}} \mu_{23} (a A_{32} + a^\dagger A_{32}),$$

(16)

where the subscript on the field operators is no longer necessary, and without loss of generality we assume that the coupling constants obey $\mu_{ij} = \mu_{ji}^* = \mu_{ji}$. The only operator of the form

$$C = \lambda a^\dagger a + \lambda_1 A_{11} + \lambda_2 A_{22} + \lambda_3 A_{33}$$

that commutes with the Hamiltonian equation (16) is given by equation (4), i.e., the total number of atoms is conserved. However, if one allows one coupling term $\mu_{ij}$ to be zero, it is possible to find another operator that commutes with the
\[ \omega_1 \mu_{12} \mu_{23} \mu_{13} = 0 \]
\[ \Lambda \quad \omega_1 \omega_2 \omega_3 \mu_{12} \mu_{23} \mu_{13} = 0 \]
\[ V \quad \omega_1 \omega_2 \omega_3 \mu_{12} \mu_{23} \mu_{13} = 0 \]

Figure 1. Atomic configurations and dipolar coupling parameters.

Table 1. Values of \( \lambda_i \), \( i = 2, 3 \), which determine the constant of motion \( M \).

| Configuration | \( \lambda_2 \) | \( \lambda_3 \) |
|---------------|---------------|---------------|
| \( \Xi \)     | 1             | 2             |
| \( \Lambda \) | 0             | 1             |
| \( V \)       | 1             | 1             |

Hamiltonian equation (16). This operator, for each atomic configuration, is given by

\[ M_{\Xi} = a^\dagger a + A_{22} + 2A_{33} \quad (\mu_{13} = 0), \]  
(17)

\[ M_{\Lambda} = a^\dagger a + A_{33} \quad (\mu_{12} = 0), \]  
(18)

\[ M_V = a^\dagger a + A_{22} + A_{33} \quad (\mu_{23} = 0), \]  
(19)

which may be written in general as

\[ M = a^\dagger a + \lambda_2 A_{22} + \lambda_3 A_{33} \]  
(20)

with \( \lambda_i \) as in table 1.

The \( M \) operator corresponds to the total number of excitations for the different atomic configurations \( \Xi, \Lambda \) and \( V \) [17]. The condition \( \mu_{ij} = 0 \) implies that transitions between levels \( i \) and \( j \) are forbidden; a visual inspection of the different configurations (see figure 1) immediately suggests the expressions (17)–(19).

3.1. Semi-classical variational states

In the Hamiltonian that we have given above, for the description of three-level atoms interacting with an electromagnetic field, there naturally appear matter operators that generate the unitary algebra in three dimensions, \( U(3) \). This lends itself to description by the Gelfand–Tsetlin states [18] which carry the irreducible representations of \( U(3) \) and are in general denoted by

\[ \begin{pmatrix} h_1 & h_2 & h_3 \\ q_1 & q_2 & r \end{pmatrix} = \ket{h_1 h_2 h_3 q_1 q_2 r}, \]  
(21)

where the labels satisfy the inequalities \( q_1 \geq r \geq q_2 \) and \( h_i \geq q_i \geq h_{i+1} \), with \( i = 1, 2 \). The nine generators of \( U(3) \) can be classified into weight, raising, and lowering operators. The weight generators \( A_{ij} \) satisfy the eigenvalue equations

\[ A_{11} \ket{h_1 h_2 h_3 q_1 q_2 r} = r \ket{h_1 h_2 h_3 q_1 q_2 r}, \]
\[ A_{22} \ket{h_1 h_2 h_3 q_1 q_2 r} = (q_1 + q_2 - r) \ket{h_1 h_2 h_3 q_1 q_2 r}, \]
\[ A_{33} \ket{h_1 h_2 h_3 q_1 q_2 r} = (N_a - q_1 - q_2) \ket{h_1 h_2 h_3 q_1 q_2 r}, \]

with \( N_a = h_1 + h_2 + h_3 \).

For the values \( q_1 = h_1, q_2 = h_2 \), and \( r = h_1 \), one has the highest weight state (HWS), for which

\[ A_{ij} \ket{h_1 h_2 h_3 h_1 h_2 h_1} = 0, \quad \text{for } i < j, \]  
(22)
where \( A_{ij} \) are the raising weight generators. For this state, the eigenvalues of the weight generators determine the irreducible representation of \( U(3) \), i.e., \([h_1, h_2, h_3]\). Physically this means that we have \( h_i \) atoms in the level \( \omega_i \).

In this work we may consider as a variational test function the direct product of a Heisenberg–Weyl coherent state (field contribution) with a \( U(3) \) coherent state, because these generate a basis of the Hilbert space and let us obtain analytic expressions for the expectation values of matter and field observables, as was done for two-level systems in [9].

For the one-mode field we use the states \( |\alpha\rangle\), which satisfy \( a|\alpha\rangle = \alpha|\alpha\rangle \), while for the matter we follow the procedure established by Perelomov [19]. The unnormalized \( U(3) \) coherent states can be defined as

\[
|h_1, h_2, h_3; \vec{y}\rangle = O(\vec{y})|h_1, h_2, h_3, h_1, h_2, h_1\rangle,
\]

with \( \vec{y} = (\gamma_1, \gamma_2, \gamma_3) \), and where we have introduced the product of exponentials of lowering weight generators

\[
O(\vec{y}) = e^{\gamma_3 A_3} e^{\gamma_2 A_2} e^{\gamma_1 A_1}.
\]

Therefore, the variational test function is given by

\[
|h_1, h_2, h_3; \alpha \vec{y}\rangle \equiv |\alpha\rangle \otimes |h_1, h_2, h_3; \vec{y}\rangle.
\]

For the evaluation of the energy surface of the system, i.e., the expectation value of the Hamiltonian (16) with respect to the tensorial product \( |\alpha\rangle \otimes |h_1, h_2, h_3; \vec{y}\rangle \), we proceed as follows:

(i) Determine the coherent state representations of the generators \( A_{ij}, a \), and \( a^\dagger \).

(ii) Evaluate the kernel of the Heisenberg–Weyl and \( U(3) \) coherent states,

\[
|\alpha|\alpha\rangle \text{ and } |h_1 h_2 h_3, \vec{y}|h_1 h_2 h_3, \vec{y}\rangle.
\]

(iii) Apply the representation form of each operator of the Hamiltonian to the corresponding kernel evaluated at \( \alpha' = \alpha \) and \( \vec{y}' = \vec{y}\).

For the Heisenberg–Weyl case, it is well known that

\[
\alpha \rightarrow \frac{\partial}{\partial \alpha^*}, \quad a^\dagger \rightarrow a^* \quad \{\alpha|\alpha\rangle = \exp(a^* a')\}.
\]

Now, for the \( U(3) \) case, the first step is to determine the coherent state representation of the generators

\[
|h_1, h_2, h_3; \vec{y}|A_{ij}|\psi\rangle,
\]

where \( |\psi\rangle \) is an arbitrary state of the matter. Substituting the definition of the coherent state (23), one has

\[
\langle h_1, h_2, h_3, h_1, h_2, h_1 | G_{ij} O(\vec{y}) | \psi \rangle,
\]

where we define \( G_{ij} = O(\vec{y}) A_{ij} O^{-1}(\vec{y}) \). Using the expansion of \( e^A B e^{-A} \), it is straightforward that \( G_{ij} \) takes the form

\[
G_{ij} = A_{ij} + \delta_{3i} \gamma^*_3 A_{2j} + \left[ \delta_{i1} (\gamma^*_1 \gamma^*_3 - \gamma^*_2) - \gamma^*_i \delta_{2j} \right] A_{13}
\]

\[
+ \delta_{31} \left[ \delta_{i1} (\gamma^*_1 \gamma^*_3 - \gamma^*_2) - \delta_{2j} \gamma^*_i \right] A_{13}
\]

\[
+ \left( \delta_{31} \gamma^*_2 + \delta_{2i} \gamma^*_3 \right) A_{1j} - \delta_{1j} \gamma^*_i A_{12} - \delta_{3i} \delta_{1j} \gamma^*_1 \gamma^*_3 A_{22}
\]

\[
+ \left[ \delta_{i1} (\gamma^*_1 \gamma^*_3 - \gamma^*_2) - \delta_{2j} \gamma^*_i \right] A_{13}
\]

\[
- \delta_{1j} \left( \delta_{3i} \gamma^*_2 \gamma^*_3 + \delta_{2i} \gamma^*_3 \right) A_{12}.
\]
To apply \( G_{ij} \) to the bra associated to the HWS, we have to take into account that the weight generators are diagonal, the lowering generators yield zero, and the raising generators in (27) must be replaced by
\[
A_{23} \rightarrow \frac{\partial}{\partial \gamma_1}, \quad A_{13} \rightarrow \frac{\partial}{\partial \gamma_2}, \quad A_{12} \rightarrow \left( \frac{\partial}{\partial \gamma_3} + \gamma_1^* \frac{\partial}{\partial \gamma_2} \right).
\]
This yields the Gelfand–Tsetlin coherent representation of the \( U(3) \) generators as
\[
A_{ij}(h\vec{\gamma} | \psi) = \{ h\vec{\gamma} | A_{ij} | \psi \}.
\]

As an example, we give the \( U(3) \) coherent state representation of the \( U(2) \) subalgebra \( \{ A_{11}, A_{12}, A_{21}, A_{22} \} \): using (27), one writes
\[
\begin{align*}
G_{11} &= A_{11} - \gamma_1^* A_{12} + (\gamma_1^* \gamma_2^* - \gamma_2^*) A_{13} \\
G_{12} &= A_{12} - \gamma_1^* A_{13} \\
G_{21} &= A_{21} - \gamma_2^* A_{12} + (\gamma_1^* \gamma_2^* - \gamma_2^*) A_{13} + (\gamma_1^* \gamma_2^* - \gamma_2^*) A_{23} + \gamma_3^* (\alpha A_{11} - A_{22}) \\
G_{22} &= A_{22} + \gamma_2^* A_{12} - \gamma_1^* \gamma_2^* A_{13} - \gamma_1^* A_{23}.
\end{align*}
\]

Then we make the replacements indicated above, to get
\[
\begin{align*}
A_{11} &\rightarrow h_1 - \gamma_2^* \frac{\partial}{\partial \gamma_2} - \gamma_3^* \frac{\partial}{\partial \gamma_3} \\
A_{12} &\rightarrow \frac{\partial}{\partial \gamma_3} \\
A_{21} &\rightarrow \gamma_3^* \left( h_1 - h_3 + \gamma_1^* \frac{\partial}{\partial \gamma_1} - \gamma_2^* \frac{\partial}{\partial \gamma_2} - \gamma_3^* \frac{\partial}{\partial \gamma_3} \right) - \gamma_2^* \frac{\partial}{\partial \gamma_1} \\
A_{22} &\rightarrow h_2 - \gamma_1^* \frac{\partial}{\partial \gamma_1} + \gamma_3^* \frac{\partial}{\partial \gamma_3}.
\end{align*}
\]

It is straightforward to prove that the operators \( A_{ij} \) \((i, j = 1, 2)\) satisfy the commutation relations of a \( U(2) \) algebra.

The \( U(3) \) matter kernel is given by
\[
\langle h_1 h_2 h_3 | h_1 h_2 h_3 | h_1 h_2 h_3 \rangle = \{ h_1 h_2 h_3 | \psi \} O(\vec{\gamma}) | h_1 h_2 h_3 \rangle.
\]

To evaluate the expression it is convenient to rewrite the product of the operators as
\[
O(\vec{\gamma}) O(\vec{\gamma}') = O(\vec{\beta}) e^{i \lambda_{11} A_{11} + i \lambda_{12} A_{12} + i \lambda_{22} A_{22} + i \lambda_{23} A_{23}} O(\vec{\beta}'),
\]
because the matrix element, with respect to the Gelfand–Tsetlin HWS, of the operators \( O(\vec{\beta}) \) and \( O(\vec{\beta}') \) yield a result equivalent to the identity operator and the remaining exponential is diagonal.

To interchange the exponential operators, we use a faithful realization of the generators as \( A_{ij} = \{ i \} | \{ j \} \rangle \). One then finds the values of the \( \beta, \beta' \) and \( \lambda \) as functions of \( \gamma \) and \( \gamma' \), in such a way that expression (28) is satisfied. Following this procedure, one obtains the \( U(3) \) matter kernel
\[
\{ h_1 h_2 h_3, \vec{\gamma}' | h_1 h_2 h_3, \vec{\gamma} \} = \left( 1 + \gamma_2^* \gamma_2' + \gamma_3^* \gamma_3' \right)^{h_1 - h_2} \\
\times \left( 1 + \gamma_2^* \gamma_2' - \gamma_1^* \gamma_1' + \gamma_1^* (\gamma_1' - \gamma_2' \gamma_3') + \gamma_1^* (\gamma_2' \gamma_3' + \gamma_1^* \gamma_3') \right)^{h_3 - h_1}.
\]

The general case of distinguishable particles could be interesting in quantum information theory, for example, for the description of systems of q-trits. In our case of study we restrict ourselves to the totally symmetric configuration. For the symmetric basis the corresponding kernel of the matter contribution is obtained by taking \( h_2 = h_3 = 0 \). From here on we simplify the notation by omitting the values of \( h_2 \) and \( h_3 \) in the Gelfand–Tsetlin states. Therefore the kernel of the tensorial product of coherent states is
\[
K(h_1; \alpha, \alpha', \vec{\gamma}, \vec{\gamma}') = e^{\alpha' \alpha} \left( 1 + \gamma_2^* \gamma_2' + \gamma_3^* \gamma_3' \right)^{h_1}.
\]
3.2. The energy surface

Applying the corresponding coherent state representation of the Hamiltonian (16) on the kernel above, dividing by the scalar product of the coherent states, and replacing $\alpha' = \alpha$ and $\gamma' = \gamma$, the energy surface is

$$\mathcal{E}^c = \Omega r^2 + h \frac{\omega_1 + \omega_2 \phi_3^2 + \omega_3 \phi_2^2}{1 + \phi_1^2 + \phi_3^2} - 2\sqrt{\hbar_1 \mu_{12}} \frac{\rho \phi_3 \cos(\phi_3)}{1 + \phi_2^2 + \phi_3^2} - 2\sqrt{\hbar_1 \mu_{13}} \frac{\rho \phi_2 \cos(\phi_2)}{1 + \phi_2^2 + \phi_3^2} - 2\sqrt{\hbar_1 \mu_{23}} \frac{\rho \phi_2 \phi_3 \cos(\phi_2)}{1 + \phi_2^2 + \phi_3^2},$$

(31)

where we have rewritten the parameters in their polar form, i.e., $\phi = r e^{i\theta}$, $\gamma_i = r_i e^{i\psi_i}$ and identified $\phi_1 = \phi - \varphi_3$, $\phi_2 = \phi - \varphi_2$ and $\phi_3 = \phi - \varphi_2 - \varphi_3$.

Minimizing $\mathcal{E}^c$ with respect to the phases $\phi_i$ one finds that the critical values are given by $\phi_i = 0, \pi$. The minimum is obtained when $\mu_{ij} \cos(\phi_{ki}) > 0$ for cyclic indices $i, j$ and $k$. Since these values are independent of $\rho$, one may replace this condition in equation (31), and hence the energy surface is rewritten as

$$\mathcal{E}^c = \Omega r^2 + h \frac{\omega_1 + \omega_2 \phi_3^2 + \omega_3 \phi_2^2}{1 + \phi_1^2 + \phi_3^2} - 2\sqrt{\hbar_1 \mu_{12}} \frac{|\mu_{12}| \phi_3 + |\mu_{13}| \phi_2 + |\mu_{23}| \phi_2 \phi_3}{1 + \phi_2^2 + \phi_3^2}. \tag{32}$$

It is easy to see that the condition

$$\frac{\partial}{\partial \rho} \mathcal{E}^c = 0,$$

(33)

is satisfied when $\rho = \rho_c$ (critical value of the variable $\rho$) where $\rho_c$ is given by

$$\rho_c = \frac{\sqrt{\hbar_1} |\mu_{12}| \phi_3 + |\mu_{13}| \phi_2 + |\mu_{23}| \phi_2 \phi_3}{\Omega}.$$  \tag{34}

Here $\phi_2$ and $\phi_3$ stand for the critical values of $\phi_2$ and $\phi_3$, respectively.

It is worth stressing the fact that the energy surface given by equation (31) (or equivalently equation (32)) has no dependence on $\gamma_i = \phi_i e^{i\psi_i}$, because we are taking $h_2 = h_3 = 0$ in the definition of the Gelfand–Tsetlin coherent state.

For the semi-classical calculation of the ground state energy, it is worth referring to the intensive quantity $E^c = \mathcal{E}^c / \hbar_1$ which describes the energy per particle:

$$E^c = \Omega r^2 + h \frac{\alpha_1 + \omega_2 \phi_3^2 + \omega_3 \phi_2^2}{1 + \phi_1^2 + \phi_3^2} - 2r |\mu_{12}| \phi_3 + |\mu_{13}| \phi_2 + |\mu_{23}| \phi_2 \phi_3| \frac{1}{1 + \phi_2^2 + \phi_3^2}.$$  \tag{35}

where $r = \rho / \sqrt{\hbar_1}$. In a similar way we define the total number of excitations per particle $N' = N^c / N_c$.

An approximation to the ground state energy of the system is obtained by substituting the minima critical points into the energy surface. From (34) and (35) we obtain $E^c = E^c(\phi_2c, \phi_3c)$, whose minimum in general has no analytic solutions for arbitrary points in parameter space $(\mu_{ij})$ and a particular atomic configuration.

The critical points satisfy $\phi_2c, \phi_3c \geq 0$. To find these critical points numerically we proceed as follows, starting with the first quadrant in the $\phi_2c - \phi_3c$ plane:

- the area is divided into $N$ regions forming a lattice;
- the energy surface is evaluated at the central point of each of these regions;
- the region with minimum energy, together with its closest neighbors, is selected to build a new lattice;
- this method is iterated until the desired precision is reached.
If the area of the first set is $S$, the method establishes the critical point with a precision of $3^{m-1}\sqrt{N/m}$, where $m$ is the number of iterations.

Recently [15] we found the minimum energy surface $E^c$ as a function of the control parameters $\mu_{ij}$. It changes value from $E^c = 0$ to $E^c < 0$, when a transition from $M^c = 0$ (normal regime) to $M^c > 0$ (collective regime) in the total number of excitations of the corresponding semi-classical approximation to the ground state of the system takes place. This leads to the existence of a separatrix in parameter space, for which we were able to propose the following ansatz:

For the $\Sigma$ configuration,

$$\Omega_{\omega_{21}} = \mu_{12}^2 + [\mu_{23}] - \sqrt{\Omega_{\omega_{23}}} \Theta [\mu_{23}] - \sqrt{\Omega_{\omega_{31}}}$$

(36)

where the Bohr frequency $\omega_{ij} \equiv \omega_i - \omega_j$ is the energy shift between the atomic levels $i$ and $j$ and $\Theta[x]$ stands for the Heaviside theta function.

For the $\Lambda$ configuration,

$$\Omega_{\omega_{31}} = \mu_{13}^2 + [\mu_{23}] - \sqrt{\Omega_{\omega_{23}}} \Theta [\mu_{23}] - \sqrt{\Omega_{\omega_{31}}}$$

(37)

For the $V$ configuration,

$$\frac{\mu_{12}^2}{\Omega_{\omega_{21}}} + \frac{\mu_{13}^2}{\Omega_{\omega_{31}}} = 1$$

(38)

The separatrix of the different configurations correspond to the thermodynamic limit, that is, when the number of atoms $N_a \to \infty$.

### 3.3. Order of the transitions

A phase transition is of order $j$, according to the Ehrenfest classification [20], if $j$ is the lowest non-negative integer for which

$$\lim_{\epsilon \to 0} \frac{\partial^j E^c}{\partial s^j} \bigg|_{s=s_0} = \lim_{\epsilon \to 0} \frac{\partial^j E^c}{\partial s^j} \bigg|_{s=s_0-\epsilon},$$

where $s$ represents here any of the control parameters $\mu_{ij}$. In general we do not have analytical expressions for the critical points, so the order of the transitions must be obtained numerically. In the case of first-order transitions, however, we may use

$$dE^c = \left( \frac{\partial E^c}{\partial \rho} \right) \, d\rho + \left( \frac{\partial E^c}{\partial \varrho_2} \right) \, d\varrho_2 + \left( \frac{\partial E^c}{\partial \varrho_3} \right) \, d\varrho_3 + \sum_{i<j} \left( \frac{\partial E^c}{\partial \mu_{ij}} \right) \, d\mu_{ij}$$

which evaluated at the critical points reduces to

$$dE^c \bigg|_{\rho_1, \varrho_2, \varrho_3} = \sum_{i<j} \left( \frac{\partial E^c}{\partial \mu_{ij}} \right) \bigg|_{\rho_1, \varrho_2, \varrho_3} \, d\mu_{ij}$$

and this provides us with the following expressions.

For the $\Sigma$ configuration

$$\frac{\partial}{\partial \mu_{12}} E^c_{\Sigma} = -2 \frac{r_c \varrho_3}{1 + \varrho_2^2 + \varrho_3^2},$$

(39)

$$\frac{\partial}{\partial \mu_{23}} E^c_{\Sigma} = -2 \frac{r_c \varrho_2 \varrho_3}{1 + \varrho_2^2 + \varrho_3^2};$$

(40)

for the $\Lambda$ configuration

$$\frac{\partial}{\partial \mu_{13}} E^c_{\Lambda} = -2 \frac{r_c \varrho_2}{1 + \varrho_2^2 + \varrho_3^2},$$

(41)
\[ \frac{\partial}{\partial \mu} \frac{\partial E_c}{\partial \Lambda} = -2 \frac{r_c \varrho c}{1 + \varrho c^2 + \varrho c^3}; \]  
\[ \text{(42)} \]

and for the \( V \) configuration

\[ \frac{\partial}{\partial \mu} \frac{\partial E_v}{\partial \Lambda} = -2 \frac{r_c \varrho c}{1 + \varrho c^2 + \varrho c^3}; \]
\[ \text{(43)} \]

\[ \frac{\partial}{\partial \mu} \frac{\partial E_v}{\partial \Lambda} = -2 \frac{r_c \varrho c}{1 + \varrho c^2 + \varrho c^3}. \]
\[ \text{(44)} \]

For the second-order transitions one has to infer them through numerical differentiation of the equations, or through derivatives of second order when analytical expressions are available.

### 3.4. Statistics of the semi-classical ground state

The statistics of the semi-classical ground state is given by the well-known \( Q \)-Mandel parameter \[21\], defined for the field states as

\[ Q = \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle}. \]
\[ \text{(45)} \]

The photon distribution obeys \((\Delta n)^2 = \langle n \rangle\), and hence \( Q = 0 \) for any value of the control parameters, i.e., the contribution of the photons in the semi-classical ground state obeys Poissonian statistics.

On the other hand, one may study the statistics of the ground state as a function of the total number of excitations \( M \), i.e., consider both field and matter contributions. So one may define, in a similar way, the \( Q_M \)-Mandel parameter as

\[ Q_M = \frac{(\Delta M)^2 - \langle M \rangle}{\langle M \rangle}. \]
\[ \text{(46)} \]

To evaluate the expression \( (46) \) we use equation \( (20) \) together with

\[ M^2 = n^2 + \lambda_2^2 A_{22}^2 + \lambda_3^2 A_{33}^2 + 2n(\lambda_2 A_{22} + \lambda_3 A_{33}) + 2\lambda_2 \lambda_3 A_{22} A_{33}. \]
\[ \text{(47)} \]

For the totally symmetric coherent variational test function one may establish the following relations between expectation values for matter and field observables:

\[ \langle n^2 \rangle = \langle n \rangle^2 + \langle n \rangle, \]
\[ \text{(48)} \]

\[ \langle A_{22}^2 \rangle = \langle A_{22} \rangle + \left(1 - \frac{1}{N_n}\right) \langle A_{22} \rangle^2, \]
\[ \text{(49)} \]

\[ \langle A_{33}^2 \rangle = \langle A_{33} \rangle + \left(1 - \frac{1}{N_n}\right) \langle A_{33} \rangle^2, \]
\[ \text{(50)} \]

\[ \langle n A_{ii} \rangle = \langle n \rangle \langle A_{ii} \rangle, \]
\[ \text{(51)} \]

\[ \langle A_{22} A_{33} \rangle = \left(1 - \frac{1}{N_n}\right) \langle A_{22} \rangle \langle A_{33} \rangle. \]
\[ \text{(52)} \]

Hence, the fluctuation of the total number of excitations for the variational state, defined by \((\Delta M^c)^2 = \langle M^c \rangle - \langle M \rangle^2\), is given by

\[ (\Delta M^c)^2 = \langle M \rangle + \lambda_3 (\lambda_3 - 1) \langle A_{33} \rangle - \frac{1}{N_n}[\lambda_2 \langle A_{22} \rangle + \lambda_3 \langle A_{33} \rangle]^2, \]
\[ \text{(53)} \]

where we have used the fact that \( \lambda_2^2 = \lambda_2 \) and \( \lambda_3^3 = \lambda_3 \) or \( 2\lambda_3 \) to identify the appropriate value of \( \langle M \rangle \). Then the \( Q_M \)-Mandel parameter for this state reads

\[ Q_M^c = \frac{1}{2} \left[ \lambda_3 (\lambda_3 - 1) \langle A_{33} \rangle - \frac{1}{N_n}[\lambda_2 \langle A_{22} \rangle + \lambda_3 \langle A_{33} \rangle]^2 \right]. \]
\[ \text{(54)} \]
Figure 2. The first derivative of the ground state energy with respect to its control parameters, for atoms in Σ configuration under the double resonance condition $\Delta_{21} = \Delta_{32} = 0$ with respect to (a) $\mu_{12}$, and (b) $\mu_{23}$.

Note that the $Q_M$-Mandel parameter does not depend on the total number of atoms $N_a$, since both quantities $\langle M \rangle$ and $\langle A_{ii} \rangle$ are proportional to $N_a$.

Since $\lambda_3 = 1$ for the $\Lambda$ and $V$ configurations, one finds from equation (54) that in these cases $Q_M \leq 0$, and then the corresponding coherent state obeys only Poissonian ($Q_M = 0$) and sub-Poissonian ($Q_M < 0$) statistics. For the $\Xi$ configuration, however, $\lambda_3 = 2$ and hence the sign of $Q_M$ may be determined only via evaluation of the corresponding critical points.

3.5. Numerical results

As pointed out in [15], the minimization of the semi-classical energy $E^c$ provides analytic expressions for the phases and $r_c = \rho_c/\sqrt{N_a}$. There is not, in general, an analytic solution available for the minimum value of the energy surface with respect to the other two independent variables $\varphi_2$ and $\varphi_3$. This suggests the use of a numerical method to evaluate the critical points $\varphi_{2c}$, $\varphi_{3c}$, as functions of the control parameters $\mu_{ij}$.

To describe the levels of the atom we can use the detuning, defined by

$$\Delta_{ij} = \omega_{ij} - \Omega, \quad \omega_{ij} = \omega_i - \omega_j.$$  

(55)

Without loss of generality, we chose $\Omega = 1$ and $\omega_1 = 0$. So both the control parameters, atomic levels and the detuning are measured in units of the field frequency.

3.5.1. $\Xi$ configuration. The $\Xi$ configuration forbids the transition $\omega_1 \leftrightarrow \omega_3$, and this is introduced in the Hamiltonian by taking $\mu_{13} = 0$. Then $\Delta_{21}$ and $\Delta_{32}$ are related to the energy levels by

$$\omega_2 = \Delta_{21} + \omega_1 + \Omega,$$

$$\omega_3 = \Delta_{32} + \Delta_{21} + \omega_1 + 2\Omega.$$  

(56)  

(57)

Also, in the $\Xi$ configuration the condition $\omega_2 \approx \omega_3/2$ is fulfilled, and the detuning should satisfy $\Delta_{21} \approx \Delta_{32}$ with $|\Delta_{ij}| < 1$ to be consistent with the RWA approximation.

Figure 2 shows the first derivatives of the energy surface for the ground state in double resonance, i.e., when $\Delta_{21} = \Delta_{32} = 0$. The corresponding separatrix equation (36) is shown by a white line. One can observe that the derivative is continuous in the region $\mu_{23} \leq \sqrt{\Omega \omega_3}$, where the separatrix is given by $\mu_{12} = \sqrt{\Omega \omega_3}$; here a second-order transition occurs. For $\mu_{23} > \sqrt{\Omega \omega_3}$, the separatrix is given by $(|\mu_{23}| - \sqrt{\Omega \omega_3})^2 + \mu_{12}^2 = \Omega \omega_2$, the derivative is discontinuous and first-order transitions take place.
In figure 3(a) the $Q_M$-Mandel parameter equation (54) for this configuration is shown. One can observe that for

$$
\Omega \omega_{21} \geq \mu_{12}^2 + (|\mu_{23}| - \sqrt{\Omega \omega_{31}})^2 \Theta [|\mu_{23}| - \sqrt{\Omega \omega_{31}}]
$$

we have $Q_M = 0$, i.e., for this region in parameter space the semi-classical ground state has Poissonian statistics. On the other hand, when equation (58) is not satisfied one finds $Q_M < 0$, providing sub-Poissonian statistics.

Figures 3(b), (c) and (d), respectively, show the distribution of $M$ in the semi-classical ground state for $N_0 = 40$ atoms, using ($\mu_{12} = 1.01$, $\mu_{23} = 0.5$), ($\mu_{12} = 0.05$, $\mu_{23} = 2.45$) and ($\mu_{12} = 1.5$, $\mu_{23} = 2.5$) (solid bars) in comparison with the corresponding Poissonian distribution (dots). Notice that the first two points are very close to the separatrix, as shown in figure 3(a) (dotted data), but their corresponding $Q_M$-Mandel parameters are different. One may observe that in figure 3(b) both distributions are practically indistinguishable, since the average value of the total number of excitations is $M_{\Sigma} \approx 2.3 \times 10^{-2}$ (with the corresponding $Q_M$-Mandel parameter $Q_M \approx -4.7 \times 10^{-3}$), i.e., the contribution of the state with $M = 0$ dominates in the ground state, while in the other cases (figures 3(c), (d)) the average values
Figure 4. The average number of photons in units of the total number of atoms \( r_c^2 = \rho_c^2 / N_a \) is shown for atoms in \( \Xi_1 \) configuration in double resonance. Note that for greater values of \( \mu_{23} = \sqrt{2} \) along the separatrix there is coexistence between the two values of the number of photons.

are \( M_{\Xi_1}^c \approx 2.87 \) and 3 (with \( Q_{\Xi_1} \approx -0.41 \) and \(-0.17\), respectively), where the contribution of the state with \( M = 0 \) is negligible.

Figure 4 shows the average number of photons in units of the total number of atoms, \( r_c^2 = \rho_c^2 / N_a \equiv \langle n \rangle / N_a \). Since the field is a coherent state, the fluctuation of the number of photons satisfies \( (\Delta n)^2 = \langle n \rangle \).

### 3.5.2. \( \Lambda \) configuration

For atoms in the \( \Lambda \) configuration it is required that the transitions from \( \omega_1 \leftrightarrow \omega_2 \) be negligible, and so we take \( \mu_{12} = 0 \). The detuning for the corresponding values of the frequencies \( \omega_2 \) and \( \omega_3 \) are

\[
\omega_2 = \Delta_{31} - \Delta_{32} + \omega_1, \\
\omega_3 = \Delta_{31} + \omega_1 + \Omega.
\]

Because of the convention \( \omega_1 \leq \omega_2 \leq \omega_3 \) used in the labeling of the energy levels, the condition \( \omega_1 \approx \omega_2 \) requires \( \Delta_{31} - \Delta_{32} \approx 0 \) with \( \Delta_{31} \gg \Delta_{32} \).

First we consider the case of equal detuning, i.e., \( \Delta_{31} = \Delta_{32} \). In this case, the critical points may be calculated analytically as functions of the control parameters. These are given by \( \varrho_{2c} = \varrho_{3c} = 0 \) in the normal regime, with \( \mu_{13}^2 + \mu_{23}^2 \leq \Omega \omega_3 \), while in the collective regime we have

\[
\varrho_{2c} = \frac{1}{\mu_{13}} \sqrt{\frac{\left(\mu_{13}^2 + \mu_{23}^2\right)\left(\mu_{13}^2 + \mu_{23}^2 - \Omega \omega_3\right)}{\mu_{13}^2 + \mu_{23}^2 + \Omega \omega_3}}, \\
\varrho_{3c} = \frac{\mu_{23}}{\mu_{13}},
\]

where states with \( M > 0 \) contribute to the ground state.

Substituting the critical points in the expression for the energy, one finds that the minimum energy surface is given by \( E^{\text{c}}_{\lambda} = 0 \) for \( \mu_{13}^2 + \mu_{23}^2 < \Omega \omega_3 \) and

\[
E^{\text{c}}_{\lambda} = -\frac{1}{4\Omega} \left(\frac{\mu_{13}^2 + \mu_{23}^2 - \Omega \omega_3}{\mu_{13}^2 + \mu_{23}^2}\right)^2.
\]

in the collective region. Taking the first derivatives of the minimum energy surface and evaluating at the separatrix, one finds that only second-order transitions occur.
Figure 5. The first derivative of the ground state energy with respect to its control parameters, for atoms in the $\Lambda$ configuration with a non-resonant condition $\Delta_{31} = 0.3$ and $\Delta_{32} = -0.2$. (a) The derivative with respect to $\mu_{13}$, and (b) the derivative with respect to $\mu_{23}$.

In the collective regime the $Q_M$-Mandel parameter reads

$$Q_M = -\frac{\Omega^2(\mu_{13}^2 + \mu_{23}^2 - \omega_3)}{(\mu_{13}^2 + \mu_{23}^2)(\mu_{13}^2 + \mu_{23}^2 + \Omega \omega_3)}.$$  \hspace{1cm} (64)

One can show that, independently of the detuning values, $Q_M = 0$ in the normal regime ($\mathcal{M} = 0$), yielding Poissonian statistics, while in the collective regime ($\mathcal{M} > 0$) we have sub-Poissonian statistics, $Q_M < 0$. Also we notice that $Q_M \to 0$ when the control parameters go to infinity.

We now consider atoms in the $\Lambda$ configuration with $\Delta_{31} \neq \Delta_{32}$; we choose $\Delta_{31} = 0.3$ and $\Delta_{32} = -0.2$. In this case the problem does not have an analytic solution and one needs to consider numerical solutions as for the $\Xi$ configuration.

Figure 5 shows the first derivatives of the semi-classical energy surface for the ground state. These present discontinuities along the separatrix where $|\mu_{23}| > \sqrt{\Omega \omega_2}$ indicating first-order transitions. In the region where $|\mu_{23}| < \sqrt{\Omega \omega_2}$ with $\mu_{13} = \sqrt{\Omega \omega_3}$ the derivatives are continuous, and second-order transitions occur. The corresponding $Q_M$-Mandel parameter and the $M$-distribution of the coherent state for three values with $N_a = 40$ atoms is shown in figure 6. Here we compare the sub-Poissonian distribution of the state with its corresponding Poissonian distribution (dots). Finally the photon number distribution is shown in figure 7. One should compare the behavior of these quantities, figures 5, 6 and 7, with the corresponding ones for the $\Xi$ configuration, figures 2, 3 and 4, respectively. Notice that the behavior is very similar, i.e., for atoms in the $\Lambda$ configuration with unequal detuning, the physical quantities and properties (order of the transitions) resemble those of the atoms in the $\Xi$ configuration: they are both qualitatively equivalent.

3.5.3. V configuration. A system of atoms in the $V$ configuration requires $\mu_{23} = 0$, since transitions between the levels $\omega_3$ and $\omega_2$ are negligible. In this case, the detuning parameters are $\Delta_{21}$ and $\Delta_{31}$, given by

$$\omega_2 = \Delta_{21} + \omega_1 + \Omega,$$  \hspace{1cm} (65)

$$\omega_3 = \Delta_{31} + \omega_1 + \Omega.$$  \hspace{1cm} (66)

Notice that the condition $\omega_2 \approx \omega_3$ on $\omega_1 \leq \omega_2 \leq \omega_3$ reads, in terms of the detuning, as $\Delta_{21} \approx \Delta_{31}$ but satisfying $\Delta_{21} \leq \Delta_{31}$.

In a similar form to the atoms in the $\Lambda$ configuration, when the detuning parameters are equal, $\Delta_{21} = \Delta_{31}$, the problem has analytic solution. The critical points are $\varrho_{2c} = \varrho_{3c} = 0$.
Figure 6. (a) The $Q_M$-Mandel parameter as a function of the control parameters, for atoms in the $\Lambda$ configuration in a non-resonant condition $\Delta_{31} = 0.3$ and $\Delta_{32} = -0.2$. The separatrix is shown by a white line, and three points (dots) are shown where the corresponding $M$ distribution of the ground state for the $N_a = 40$ atom has been calculated (solid bars) and compared with its corresponding Poissonian distribution (dots). (b) $M$ distribution for $\mu_{13} = 1.15$, $\mu_{23} = 0.05$, for which $M^c_{\mu} \approx 1.98 \times 10^{-2}$ and $Q_M \approx -6.4 \times 10^{-3}$. (c) $M$ distribution for $\mu_{13} = 0.05$, $\mu_{23} = 1.85$, for which $M^c_{\mu} \approx 1.19$ and $Q_M \approx -0.12$. (d) $M$ distribution for $\mu_{13} = 1.5$, $\mu_{23} = 2.0$, for which $M^c_{\mu} \approx 1.92$ and $Q_M \approx -0.09$.

Figure 7. Average value of the photon distribution in units of the total number of atoms $r_2^e = \rho_2^e / N_a$ (see equation (34)), for atoms in the $\Lambda$ configuration in the non-resonant case $\Delta_{31} = 0.3$ and $\Delta_{32} = -0.2$. 


Figure 8. The first derivatives of the ground state energy with respect to its control parameters, for atoms in the V configuration with $\Delta_{21} = 0.2$ and $\Delta_{31} = 0.3$ detuning. (a) The derivative with respect to $\mu_{12}$ and (b) the derivative with respect to $\mu_{13}$.

for the normal regime implying an energy surface for the ground state equal to zero. For the collective regime, $\varrho_{2c}$ and $\varrho_{3c}$ take the values

$$
\varrho_{2c} = \mu_{13} \sqrt{\frac{\mu_{12}^2 + \mu_{13}^2 - \Omega \omega_3}{(\mu_{12}^2 + \mu_{13}^2)(\mu_{12}^2 + \mu_{13}^2 + \Omega \omega_3)}}.
$$

(67)

$$
\varrho_{3c} = \mu_{12} \sqrt{\frac{\mu_{12}^2 + \mu_{13}^2 - \Omega \omega_3}{(\mu_{12}^2 + \mu_{13}^2)(\mu_{12}^2 + \mu_{13}^2 + \Omega \omega_3)}}.
$$

(68)

Substituting these into the expression for the energy equation (35), one finds

$$
E_V^c = -\frac{1}{4\Omega} \left( \frac{\mu_{12}^2 + \mu_{13}^2 - \Omega \omega_3}{\mu_{12}^2 + \mu_{13}^2} \right)^2.
$$

(69)

This is similar to the case for atoms in the $\Lambda$ configuration, in fact, the expression is equal by just replacing $\mu_{23} \to \mu_{12}$ in equation (63). A similar situation occurs for the $Q_M$-Mandel parameter, which is given by

$$
Q_M = -\frac{\Omega^2 (\mu_{12}^2 + \mu_{13}^2 - \Omega \omega_3)}{(\mu_{12}^2 + \mu_{13}^2)(\mu_{12}^2 + \mu_{13}^2 + \Omega \omega_3)}.
$$

(70)

Hence, atoms in both configurations V and $\Lambda$ have similar properties under equal detuning considerations.

By considering the case of unequal detuning $\Delta_{21} \neq \Delta_{31}$, we choose to analyze the case $\Delta_{21} = 0.2$ and $\Delta_{31} = 0.3$. Figure 8 shows the first derivatives of the energy surface for the ground state as a function of the control parameters $\mu_{12}$, $\mu_{13}$. In both cases the first derivative is continuous, and so second-order transitions are present.

Similarly the $Q_M$-Mandel parameter is continuous (figure 9(a) in the vicinity of the separatrix (white line). The corresponding $\mathcal{M}$ distributions of the coherent state with $N_c = 40$ for three different points are shown in figures 9(b)–(d) (bars), and these are compared with their respective Poissonian distributions (dots). One can observe that the $\mathcal{M}$ distribution is very close to the corresponding Poissonian one, and this is due to the fact that $Q_M \sim 10^{-2}$ is close to zero for any considered value.
Figure 9. (a) The $Q_M$-Mandel parameter as a function of the control parameters, for atoms in the $V$ configuration in the non-resonant case $\Delta_{21} = 0.2$ and $\Delta_{31} = 0.3$. The separatrix is shown by a white line, and three points (dots) are displayed where the corresponding $A_M$ distribution of the ground state for the $N_a = 40$ atom has been calculated (solid bars) and compared with its corresponding Poissonian distribution (dots). (b) $A_M$ distribution for $\mu_{12} = 1.01$, $\mu_{13} = 0.5$, for which $M_c^V \approx 6.18 \times 10^{-2}$ and $Q_M \approx -1.15 \times 10^{-2}$. (c) $A_M$ distribution for $\mu_{12} = 0.5$, $\mu_{13} = 1.05$, for which $M_c^V \approx 4.56 \times 10^{-2}$ and $Q_M \approx -9.06 \times 10^{-3}$. (d) $A_M$ distribution for $\mu_{12} = 1.5$, $\mu_{13} = 1.5$, for which $M_c^V \approx -1.15 \times 10^{-2}$ and $Q_M \approx -9.33 \times 10^{-3}$.

Figure 10. Average value of the photon number distribution in units of the total number of atoms $r_c^2 = \rho_c^2 / N_a$ (see equation (34)), for atoms in the $V$ configuration considering the non-resonant case $\Delta_{21} = 0.2$ and $\Delta_{31} = 0.3$.

Finally, figure 10 shows the corresponding photon number distribution in units of the total number of atoms $r_c^2 = \rho_c^2 / N_a$. This quantity is a continuous smooth function around the separatrix, since this configuration presents only second-order transitions.

The same results are obtained for various values of the detuning parameters.
4. Comparison with the quantum solution

The exact numerical calculation of the ground state energy may be evaluated using the uncoupled basis given by the direct product between the field \( |n\rangle \) and matter states equation (21). Since we have chosen \( h_1 = N_a \) and \( h_2 = h_3 = 0 \) one may simplify the Gelfand–Tsetlin notation as

\[
|nq_1r\rangle = |n\rangle \otimes |q_1 0 r\rangle,
\]

(71)

\( q_2 \) is zero because it must satisfy \( h_2 \geq q_2 \geq h_3 \). The corresponding matrix elements of the operators \( A_{ij} \) (for this particular basis) are given in the appendix, which can be used to calculate the matrix elements of the Hamiltonian, equation (16), and to evaluate numerically its eigenvalues.

For each particular atomic configuration (Σ, Λ or V) there is an additional constant of motion \( M \), namely the total number of excitations (17)–(19). Taking a particular configuration, the Hamiltonian has a matrix representation as a block diagonal matrix, where the dimension of each matrix of the diagonal depends on \( Mq \) and \( N_a \). For large values of \( Mq \), however, the dimension depends only on \( N_a \) and is given by

\[
\frac{N_a(N_a+1)}{2} + N_a + 1;
\]

this occurs for \( Mq_{\Sigma} \geq 2N_a \) (Σ configuration), \( Mq_{\Lambda} \geq N_a \) (Λ configuration) and \( Mq_{V} \geq N_a \) (V configuration), relationships provided by the condition \( n \geq 0 \) in equation (17)–(19). For \( Mq_{\Sigma} < 2N_a \), \( Mq_{\Lambda} < N_a \) or \( Mq_{V} < N_a \) we could not find a simple relationship for the dimension of the matrix.

To find the quantum ground energy and its corresponding eigenstate, we proceed as follows. For each configuration of the atom, we take a value of \( Mq \), and for fixed parameters \( \Omega, \omega_1, \omega_2 \) and \( \omega_3 \) the eigenvalues and their corresponding eigenstates are evaluated numerically as functions of the control parameters \( \mu_{ij} \). This gives us the ground state energy for each corresponding total number of excitations.

It is worth mentioning that, for a fixed region of values of the interaction intensity, one may estimate the maximum value of \( Mq \) that is required to find the minimum energy; this value is provided by the semi-classical calculation.

In order to see how well the semi-classical results approximate the corresponding exact quantum ones, we consider atoms in the Σ configuration in a double resonance for \( N_a = 5 \) atoms. Notice that the quantum calculation of the ground state depends on the number of atoms \( N_a \) considered, and this is in contrast with the semi-classical one where this quantity plays the role of an extensive variable. Let us focus on the expectation values of the total number of excitations \( \langle M \rangle \), the number of photons \( \langle n \rangle \) and its fluctuations \( \langle \Delta n \rangle^2 = \langle n^2 \rangle - \langle n \rangle^2 \).

Figure 11 shows the expectation values of the total number of excitations (figure 11(a)), the number of photons (figure 11(b)) and the photon fluctuations (figure 11(c)) as a function of the intensity \( \mu_{12} \) for a fixed value \( \mu_{23} = 0.5 \). In all cases, the semi-classical calculation is represented by a continuous line while the corresponding exact quantum calculation is represented by dots. One may observe that in the case of the expectation values both calculations are in very good agreement (figures 11(a) and (b)). The fluctuation in the number of photons, however, fails to render the quantum results (figure 11(c)), except in the normal regime where \( \langle n \rangle = 0 \) in both cases. This difference is due to the fact that in the semi-classical ground state a coherent state for the photon contribution is considered, and hence, the fluctuations are equal to its expectation value, \( \langle \Delta n \rangle^2 = \langle n \rangle \), in other words, this possesses a Poissonian distribution. However, the photon distribution of the exact ground state does not have this property, because the total number of excitations is fixed for this state.
Figure 11. Properties of atoms in the $\Xi$ configuration in a double resonance condition, with fixed value $\mu_{12} = 0.5$ and $N_x = 5$, in the semi-classical (SC, continuous line) and exact quantum calculation (EQ, dots) are compared as functions of the control parameter $\mu_{12}$. (a) The expectation value of the total number of excitations $\langle M \rangle = \mathcal{M}_\Xi$, (b) the expectation value of the number of photons $\langle n \rangle$, and (c) the corresponding fluctuations of the number of photons $\langle (\Delta n)^2 \rangle$. The above comparison suggests that we should consider an additional correction to our semi-classical test state.

5. Projected variational state

The matter unnormalized $U(3)$ coherent state for the totally symmetric representation, i.e., $h_2 = h_3 = 0$, can be written as

$$|\nu, \chi\rangle = \sum_{n,m=0}^{\infty} \frac{\gamma_2^n \gamma_3^m}{n! m!} (A_{21})^n (A_{31})^m |\nu, 0, 0\rangle_F$$

(72)

because $A_{32}|h_1, 0, 0\rangle_F = 0$ and where $|h_1, 0, 0\rangle_F$ denotes the Gelfand–Tsetlin HWS. In this case, one can represent the $U(3)$ generators as follows: $A_{31} = b_1^\dagger b_1$ and $A_{21} = b_2^\dagger b_1$. Then the HWS can be written as

$$|h_1, 0, 0\rangle_F = \frac{1}{\sqrt{h_1!}} (b_1^\dagger)^{h_1} |0, 0, 0\rangle_F.$$  

(73)

where we are using the Fock vacuum state $|0, 0, 0\rangle_F$ defined by $b_k|0, 0, 0\rangle_F = 0$ with $k = 1, 2, 3$. The action of $A_{31}$, and $A_{21}$ on (73) is straightforward and results in

$$|\nu, \chi\rangle = \sum_{n=0}^{h_1} \sum_{m=0}^{h_1-n} \frac{h_1!}{(h_1-n-m)!n!m!} \gamma_2^n \gamma_3^m |\nu - n - m, n, m\rangle_F.$$  

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Therefore, the semi-classical variational state constructed by the tensor product of matter and field components is given by

$$|\alpha; h_1, \gamma\rangle = \frac{e^{-|\alpha|^2/2}}{(h_1; \gamma'|h_1; \gamma\rangle)^{1/2}} \sum_{n=0}^{\infty} \sum_{m=0}^{h_1 - n} \sum_{n=0}^{h_1 - n} \sqrt{h_1! \gamma_1^n \gamma_2^m} \times |v, h_1 - n - m, n, m\rangle_F,$$

(74)

where, by means of (29) with $h_2 = h_3 = 0$ and $\gamma' = \gamma$,

$$|h_1, \gamma'|h_1, \gamma\rangle = (1 + |\gamma_1|^2 + |\gamma_2|^2)^{h_1}$$

and one can thus write (74) in the form

$$|\alpha; h_1, \gamma\rangle = \frac{1}{(1 + |\gamma_1|^2 + |\gamma_2|^2)^{h_1/2}} \frac{e^{\alpha^* \hat{b}_1^\dagger + \alpha \hat{b}_2^\dagger + \gamma_2 \hat{b}_3^\dagger}}{\sqrt{\gamma_1^n \gamma_2^m}} |0, 0, 0, 0\rangle_F.$$

(75)

To obtain a variational state with a definite total number of excitations, we replace the eigenvalue of the number of photons by $\nu = M - \lambda_2 n - \lambda_3 m$. To select the atom configuration one uses the corresponding values of $\lambda_2$ and $\lambda_3$ in table 1. Then the unnormalized projected state is

$$|\alpha; h_1, \gamma\rangle_M = \sum_{n=0}^{h_1 - n} \sum_{m=0}^{h_1 - n} \sqrt{h_1! \alpha^M_{\lambda_2 n - \lambda_3 m}} \gamma_1^n \gamma_2^m \times |M - \lambda_2 n - \lambda_3 m, h_1 - n - m, n, m\rangle_F,$$

(76)

and contains only states with a fixed value of $M$, so that the semi-classical coherent state is written in simple form as

$$|\alpha; h_1, \gamma\rangle = \frac{e^{-|\alpha|^2/2}}{(1 + |\gamma_1|^2 + |\gamma_2|^2)^{h_1/2}} \sum_{M_0}^{M} |\alpha; h_1, \gamma\rangle_M.$$

(77)

The state $|\alpha; h_1, \gamma\rangle_M$ is the unnormalized projected state.

Since, the expectation value of the total number of excitations is very close to the exact one (see figure 11(a)), one may correct the semi-classical ground state by considering, for each value of $M$, the corresponding projected state $|h_1; \alpha \gamma\rangle_M$, but as the semi-classical calculation of $\langle M\rangle$ is a continuous function of the control parameters, it is necessary to discretize it. We do this by defining $M_{\text{dis}} = \lceil \langle M\rangle \rceil$, the ceiling of the expected $M$ value. So, for particular values of the control parameters we define the projected state as $|h_1; \alpha \gamma\rangle_{M_{\text{dis}}}$.

We will use these projected states to calculate the expectation values of observables. To this end, the overlap is given by (from equation (76))

$$\langle \alpha; h_1, \gamma'|\alpha; h_1, \gamma\rangle_{M_{\text{dis}}} = \sum_{n=0}^{h_1 - n} \sum_{m=0}^{h_1 - n} \rho_c^{2(M_{\text{dis}} - \lambda_2 n - \lambda_3 m)} \theta_3^{2n} \theta_2^{2m} (M_{\text{dis}} - \lambda_2 n - \lambda_3 m)! n! m! (h_1 - n - m)!,$$

(78)

where we have evaluated at the critical points of the semi-classical calculation. As an example, the unnormalized expectation value of the number of photons reads

$$\langle \alpha; h_1, \gamma'|n|\alpha; h_1, \gamma\rangle_{M_{\text{dis}}} = \sum_{n=0}^{h_1 - n} \sum_{m=0}^{h_1 - n} \rho_c^{2(M_{\text{dis}} - \lambda_2 n - \lambda_3 m)} \theta_3^{2n} \theta_2^{2m} (M_{\text{dis}} - \lambda_2 n - \lambda_3 m - 1)! n! m! (h_1 - n - m)!.$$

(79)

Figure 12 shows, in a form similar to figure 11 and for the same parameters and atomic configuration, the expectation values of $M$ (figure 12(a)), $n$ (figure 12(b)) and its fluctuations $(\Delta n)^2$ (figure 12(c)), comparing the exact calculation (EQ, darker dots) with the corresponding one using the projected state (SC, lighter dots). Notice that now the photon fluctuations provided by the projected state are comparable with the exact calculation, showing that the projected state corrects the wrong behavior of the fluctuations of the standard coherent state.

Figure 12 is shown for $N_a = 5$ atoms; for larger values of $N_a$ both calculations will be indistinguishable.
Figure 12. Properties of atoms in the $\Xi$ configuration in a double resonance condition, with the fixed value $\mu_{23} = 0.5$ and $N_\nu = 5$, in the projected semi-classical state (SC, lighter dots) and the exact quantum calculation (EQ, darker dots) are compared as functions of the control parameter $\mu_{12}$. (a) The expectation value of the total number of excitations $\langle M \rangle = M_{\Xi_1}$, (b) the expectation value of the number of photons $\langle n \rangle$, and (c) the fluctuations of the number of photons $\langle (\Delta n)^2 \rangle_{\Xi_1}$.

Figure 13. Exact and projected solutions compared for atoms in the $\Xi$ configuration in double resonance $\Delta_{21} = \Delta_{32} = 0$, considering $N_\nu = 40$ atoms. (a) The expectation value of the number of photons, showing no visual differences, and (b) the corresponding fluctuations. In both cases, the quantities were normalized to the number of atoms $N_\nu$.

5.1. The $\Xi$ configuration

For $N_\nu = 40$ atoms in the $\Xi$ configuration, in double resonance, i.e., $\Delta_{21} = \Delta_{32} = 0$, the expectation value of the number of photons and its fluctuations are compared for both the exact (mesh) and projected variational (continuous surface) states in figure 13. For $\langle n \rangle_{\Xi}/N_\nu$ (figure 13(a)) there are no visual differences. In fact, this figure is identical to figure 4 where
Figure 14. Exact and projected solutions compared for atoms in the Λ configuration in a non-resonant condition \( \Delta_{31} = 0.3 \), \( \Delta_{32} = -0.2 \), considering \( N_a = 40 \) atoms. (a) expectation value of the number of photons, showing no visual differences, and (b) the corresponding fluctuations. In both cases, the quantities were normalized to the number of atoms \( N_a \).

Table 2. The maximum and minimum difference between projected and exact quantum results equation (80) for the three configurations of the atom. The maximum difference is reached close to the separatrix due to the finite number of atoms; this value diminishes as we move away from the separatrix or as \( N_a \) is increased.

| Configuration | \( \min \delta \langle n \rangle / N_a \) | \( \max \delta \langle n \rangle / N_a \) |
|---------------|---------------------------------|---------------------------------|
| Ξ             | 0                               | \( \sim 3.3 \times 10^{-2} \)   |
| Λ             | 0                               | \( \sim 7.7 \times 10^{-1} \)   |
| V             | 0                               | \( \sim 2.4 \times 10^{-2} \)   |

the expectation value of the number of photons is shown for the semi-classical coherent state. Table 2 shows the minimum and maximum values of the difference between the projected and exact results given by

\[
\delta \langle n \rangle / N_a \equiv \left| \frac{\langle n \rangle_{\text{proj}} - \langle n \rangle_{\text{q}}}{N_a} \right|
\]

in absolute value and normalized by the number of atoms. In the normal regime the difference vanishes exactly, while in the collective regime it is of order \( \sim 10^{-2} \). Finally, figure 13(b) shows the corresponding fluctuations presenting very small differences in the collective regime.

5.2. The Λ configuration

For the Λ configuration we consider a non-resonant case \( \Delta_{31} = 0.3 \) and \( \Delta_{32} = -0.2 \), and \( N_a = 40 \) atoms. Under these conditions the behavior of the physical observables resembles that of the Ξ configuration by showing both first- and second-order phase transitions.

Figure 14(a) shows the comparison between the expectation values of the number of photons calculated with respect to the exact (mesh) and projected (continuous surface) states, where one may observe an excellent agreement between both surfaces. In the normal regime the difference \( \delta \langle n \rangle / N_a \) vanishes exactly, while in the collective regime the maximum value is of order \( \sim 10^{-1} \). As in the previous case, this diminishes as we move away from the separatrix or as \( N_a \) is increased. Figure 14(b) compares the fluctuations in the number of photons. In contrast to the Ξ configuration, here the fluctuations tend asymptotically to a constant value.
5.3. \( V \) configuration

Finally, we consider the expectation value of the number of photons for atoms in the \( V \) configuration, in a double resonance condition \( \Delta_{21} = \Delta_{31} = 0 \), with \( N_a = 40 \) atoms. As discussed in the semi-classical calculation of section 3.5.3, the qualitative behavior of the physical quantities for this configuration is independent of the detuning considered.

Figure 15(a) shows the comparison between the expectation value of the number of photons evaluated for the exact quantum (mesh) and projected (continuous surface) states. One may observe that there are no visual differences. Differences of order \( \sim 10^{-2} \) appear in the collective regime, as shown in table 2. The fluctuations are shown in figure 15(b), and once again these approach a constant in the collective regime, in a similar fashion to the \( \Lambda \) configuration.

6. Concluding remarks

The ground state of a system of \( N_a \) three-level atoms interacting via dipole interactions with a one-mode quantized electromagnetic field was described in the rotating wave approximation. The different atomic configurations \( \Xi \), \( \Lambda \), and \( V \) were considered.

The ground state was approximated by a test function (semi-classical state) constructed from the tensorial product of Heisenberg–Weyl and \( U(3) \) coherent states. There are two different behaviors, called normal, where the ground state is given by all the atoms in the lower energy level and without photons \( \langle M \rangle = 0 \), and collective, where the atoms are distributed amongst the three levels of the system, and with a corresponding number of excitations \( M \neq 0 \) and an average number of photons \( \langle n \rangle \neq 0 \).

The ground state of the system in the \( \Xi \) configuration exhibits first- and second-order transitions, independently of the detuning values (see figure 2). For atoms in the \( \Lambda \) configuration, one finds for equal detuning values that it can only present second-order transitions; this is shown analytically in equation (63). For different detuning parameters, this configuration yields first- and second-order transitions (see figure 5). For atoms in the \( V \) configuration, independent of the detuning, there are only second-order transitions, and this is shown analytically in equation (69) for equal detuning parameters and numerically in figure 8 for other cases.
For all atomic configurations, we have found that in the normal regime the expectation value of the total number of excitations with respect to the ground state is zero and it follows a Poissonian distribution. In the collective regime the total number of excitations for the ground state has a sub-Poissonian distribution as shown in figures 3, 6 and 9. The expectation values of the number of photons given in figures 4 and 7 display discontinuities where first-order transitions take place.

For \( N_p = 5 \) atoms in the \( \Xi \) configuration, the exact quantum calculation for the expectation values of the total number of excitations and of the number of photons were compared with the corresponding semi-classical ones. Both calculations agree, as shown in figures 11(a) and (b). Similar results can be obtained for the other configurations. However, the fluctuations in the number of photons are very different (figure 11(c)), which suggests that one should consider a new test function. We proposed projecting the semi-classical test function to a definite total number of excitations \( M \); this projected state was obtained by choosing the ceiling value of \( M \) together with the critical points for the semi-classical case. We showed that the photon fluctuations provided by the projected state are comparable with those of the exact calculation, so that the projected state corrects the wrong behavior of the standard coherent state (see figure 12).

Finally, for \( N_p = 40 \) atoms the expectation values and fluctuations of the number of photons were calculated. In all cases, we have found that the results for the projected state are indistinguishable from those of the exact one as can be seen in figures 13(a), 14(a) and 15(a). To have a quantitative estimation of the differences between these calculations, we used equation (80) observing the major differences along the separatrix (table 2). This is valid for all atomic configurations.

For the \( \Xi \) configuration, in the double resonance case and any number of atoms, we found a fixed point in the parameter space \((\mu_{12} = 1, \mu_{23} = \sqrt{2})\), in which there is coexistence between three different eigenstates associated with the same energy. They correspond to a total number of excitations of \( M = 0 \), \( M = 1 \), and \( M = 2 \), thus implying the presence of a triple point in the parameter space.

When more than one electromagnetic mode is present the physics can be much richer. Specific cases where each mode resonates with one and only one atomic energy transition, and where one considers only one atomic configuration, have been studied in the thermodynamic limit [14]. The general situation, however, is highly non-trivial and merits further study.

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Appendix. Matrix elements of the \( U(3) \) operators

The matrix elements of the generators of \( U(3) \), for a general irreducible representation \([h_1, h_2, h_3]\), can be found in [22]. For the totally symmetric representation, \([h_1, 0, 0]\), the Gelfand–Tsetlin states take the form

\[
|qr\rangle = \begin{pmatrix} q \\ r \end{pmatrix},
\]

(A.1)

where \( q \) and \( r \) take values from 0 to \( h_1 \).
In this representation, the matrix elements of the atomic operators $A_{ij}$ are given by

$$\langle qr|A_{11}|qr\rangle = r,$$  \hspace{1cm} (A.2)

$$\langle qr|A_{22}|qr\rangle = q - r,$$  \hspace{1cm} (A.3)

$$\langle qr|A_{33}|qr\rangle = h_1 - q,$$  \hspace{1cm} (A.4)

$$\langle qr + 1|A_{12}|qr\rangle = \sqrt{(q - r)(r + 1)},$$  \hspace{1cm} (A.5)

$$\langle q + 1r + 1|A_{13}|qr\rangle = \sqrt{(h_1 - q)(r + 1)},$$  \hspace{1cm} (A.6)

$$\langle q + 1r|A_{23}|qr\rangle = \sqrt{(h_1 - q)(q - r + 1)},$$  \hspace{1cm} (A.7)

and zero for other cases.

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