Renormalization by Projection:
On the Equivalence of the Bloch-Feshbach Formalism and Wilson’s Renormalization

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Abstract

We employ projection operator techniques in Hilbert space to derive a continuous sequence of effective Hamiltonians which describe the dynamics on successively larger length scales. We show for the case of $\phi^4$ theory that the masses and couplings in these effective Hamiltonians vary in accordance with 1-loop renormalization group equations. This is evidence for an intimate connection between Wilson’s renormalization and the venerable Bloch-Feshbach formalism.

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1 Introduction

The Bloch-Feshbach formalism, developed in the late 50’s to describe selected features of nuclear dynamics [1, 2], and Wilson’s renormalization group, developed in the early 70’s to better understand critical phenomena [3, 4, 5, 6], are based on the same physical idea: eliminate irrelevant modes in order to focus on the dynamics of few selected degrees of freedom. The Bloch-Feshbach formalism employs projection operators in Hilbert space in order to determine the effective Hamiltonian in some restricted model space, thereby discarding dynamical information that pertains to the irrelevant modes. In a similar spirit, renormalization –as originally conceived in the context of statistical physics– is a mathematical tool that allows one to iteratively eliminate short-wavelength modes and thus to arrive at effective (“renormalized”) theories which describe the dynamics on successively larger length scales. In both cases the irrelevant modes no longer appear explicitly in the effective theory, but their residual influence on the dynamics of the remaining modes is taken into account through adjustments of the effective interaction. The power and elegance of both methods derives from the fact that they thus allow one to study accurately selected features of the dynamics, such as its infrared limit, without ever having to solve the full underlying microscopic theory.

That renormalization and the old projection technique of the Bloch-Feshbach formalism are in fact closely related, and that in some cases the former can be regarded as a special case of the latter, has already been hinted at by Anderson in his “poor man’s scaling” approach to the Kondo problem [7], by Seke in his projection-method treatment of the nonrelativistic Lamb shift [8], and by more recent studies of a simple quantum mechanical model [9]; and it is clearly suggested by the modern view of renormalization as yielding a continuous sequence of effective theories [4, 5, 6]. In our letter we wish to supply further evidence for this connection by calculating the 1-loop renormalization of $\phi^4$ theory with the help of the old Bloch-Feshbach techniques.

We first introduce the basic mathematical framework. Let $H$ denote the original (full) Hamiltonian, $P$ a projection operator which projects the original Hilbert space onto some selected subspace, and $Q = 1 - P$ its complement. For $\phi^4$ theory, as for any many-particle theory, the elimination of short-wavelength modes corresponds to a projection in Fock space: lowering the momentum cutoff from some original value $\Lambda$ to

$$
\Lambda(\Delta s) := \exp(-\Delta s)\Lambda , \quad \Delta s \geq 0 ,
$$

(1)

is effected by a projection operator $P(\Delta s)$ which acts on $n$-particle states according to

$$
P(\Delta s)|k_1 \ldots k_n\rangle := \prod_{i=1}^{n} \theta(\Lambda - e^{\Delta s}|k_i\rangle)|k_1 \ldots k_n\rangle ,
$$

(2)
where \{k_1 \ldots k_n\} denote the particle momenta. Provided the eliminated modes \(Q\) have energies on some large characteristic scale \(\omega_\Lambda\), much larger than the energy scale at which we want to study the system’s physical properties, then in the reduced Hilbert space the dynamics is approximately\(^\text{1}\) governed by the effective Hamiltonian \(^\text{2}\)

\[
    H_{\text{eff}}(\Delta s) \approx P(\Delta s)H P(\Delta s) + \Sigma(\Delta s)
\]

with

\[
    \Sigma(\Delta s) := -P(\Delta s)H Q(\Delta s)\frac{1}{Q(\Delta s)H Q(\Delta s)} Q(\Delta s)H P(\Delta s)
\]

If the original Hamiltonian can be decomposed into a free and an interaction part, \(H = H^{(0)} + V\), where the free part \(H^{(0)}\) commutes with the projection, then to lowest nontrivial (i. e., second) order perturbation theory

\[
    \Sigma(\Delta s) \approx -P(\Delta s)V \frac{Q(\Delta s)}{H^{(0)}} V P(\Delta s)
\]

It is this approximation formula which we shall use in our subsequent calculations. As the parameter \(\Delta s\) increases, the masses and coupling constants in \(H_{\text{eff}}(\Delta s)\) vary. We claim that this flow is equivalent to a conventional renormalization. In particular, we claim that the above approximation formula yields the renormalization of \(\phi^4\) theory in agreement with diagrammatic 1-loop calculations. We will show that the projection \(PHP\) of the Hamiltonian is responsible for the renormalization of the mass, while the additional term \(\Sigma\) gives rise to the renormalization of the coupling constant.

The \(\phi^4\) Hamiltonian describes coupled anharmonic oscillators in spatial dimension \((d - 1)\). It reads

\[
    H = \frac{1}{2} \int d^{d-1}x : \left[ (\pi(x))^2 + |\nabla \phi(x)|^2 + m^2 \phi(x)^2 \right] + g \frac{\mu}{4!} \int d^{d-1}x \phi(x)^4 \]
\[
    =: H^{(0)}[\pi, \phi] + V[\phi],
\]

where \(\epsilon := (4 - d)\), \(\mu\) denotes a reference momentum scale of the interaction, \(m\) the mass, \(g\) the coupling constant, and :\:[\ldots]:: means normal ordering. The field \(\phi\) and its conjugate momentum \(\pi\) are time-independent (Schrödinger picture) operators which satisfy the commutation relations for bosons. They can be expressed in terms of annihilation and creation operators \(a, a^\dagger\); e. g.,

\[
    \phi(x) = \int_{|k| \leq \Lambda} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_k}} (a_k + a^\dagger_{-k}) \exp(ikx)
\]

\(^1\)In a relativistic theory this definition of \(P\) is not covariant.

\(^2\)up to corrections of order \(O(E/\omega_\Lambda)\), with \(E\) being the (low) physical scale.
with 
\[ [a_k, a_q^\dagger] = (2\pi)^{d-1} \delta^{d-1}(k - q) \]  
and 
\[ \omega_k := \sqrt{k^2 + m^2} \]  
In our subsequent calculations it will prove useful to decompose the field operators according to 
\[ \phi(x) = \phi_<(x) + \phi_>(x) \]  
where \( \phi_< \) contains all momentum modes up to the lower cutoff \( e^{-\Delta s} \Lambda \), whereas \( \phi_> \) contains the remaining modes in the “shell” \( [e^{-\Delta s} \Lambda, \Lambda] \); analogously for \( \pi = \pi_< + \pi_> \). Any power of \( \phi \) can then be written as a polynomial
\[ \phi^n = \sum_{m=0}^{n} \binom{n}{m} \phi_>^{n-m} \phi_<^m \]  
the binomial coefficients \( \binom{n}{m} \) counting the number of ways in which the \( \phi_< \) and \( \phi_> \) can be arranged. Since the slow field operators \( \phi_< \) act in the reduced Hilbert space only, they commute with the projection,
\[ [\phi_<, P(\Delta s)] = 0 \]  
The fast field operators \( \phi_> \), on the other hand, do not change the particle content within the reduced Hilbert space, and hence for an arbitrary polynomial \( f(\phi_>\, ) \) it is
\[ P(\Delta s)f(\phi_>)P(\Delta s) = \langle 0 | f(\phi_>) | 0 \rangle \cdot P(\Delta s) \]  
where \( | 0 \rangle \) denotes the vacuum. With the help of Wick’s theorem all such vacuum expectation values can be reduced to sums and products of
\[ \langle 0 | \phi_>(x) \phi_>(y) | 0 \rangle = \int_{\text{shell}} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\omega_k} \exp[ik \cdot (x - y)] \]  
\section{Renormalization Group Equations}

We first show that the projection \( PHP \) of the Hamiltonian gives rise to a renormalization of the mass. The unperturbed part of \( PHP \) simply contributes
\[ P(\Delta s)H^{(0)}[\pi, \phi]P(\Delta s) = H^{(0)}[\pi_<, \phi_<] \]  
(up to an additive constant) and thus does not entail any modification of the mass or coupling constant. The interaction, on the other hand, yields
\[ P(\Delta s)V[\phi]P(\Delta s) = \frac{g^2}{4!} \sum_{n=0}^{4} \binom{4}{n} \int d^{d-1}x \langle 0 | \phi_>(x)^{4-n} | 0 \rangle \phi_<^n(x) \]
Odd powers of $\phi_n$ have a vanishing vacuum expectation value, so only the terms with $n = 0, 2, 4$ survive. They contribute to the zero-point (vacuum) energy, mass term and $\phi^4$ interaction term, respectively, of the effective Hamiltonian. The latter contribution is just the original $\phi^4$ interaction term restricted to the slow modes, with no change in the associated coupling $g$. In contrast, the contribution to the mass term leads to a nontrivial modification

$$\frac{1}{2} \Delta m^2 = g \frac{\mu^e}{4!} \langle 0 | \phi_<(x)^2 | 0 \rangle .$$

Taking the flow parameter $\Delta s$ to be infinitesimal, the vacuum expectation value is given by

$$\langle 0 | \phi_<(x)^2 | 0 \rangle = \frac{S_{d-1}}{2(2\pi)^{d-1}} \Lambda^{d-2} \Lambda \Delta s ,$$

where $S_{d-1}$ denotes the surface of a unit shell in $(d-1)$-dimensional momentum space. Expanding

$$\frac{\Lambda}{\omega_\Lambda} = 1 - \frac{m^2}{2\Lambda^2} + O(m^4/\Lambda^4)$$

we obtain

$$\Delta m^2 \approx - \frac{S_{d-1} g}{8(2\pi)^{d-1}} \left( \frac{\mu^e}{\Lambda} \right) (m^2 - 2\Lambda^2) \Delta s .$$

For $d = 4$ and a spherical cut in 3-momentum space ($S_{d-1} = 4\pi$) this yields

$$\Delta m^2 = - \frac{g}{16\pi^2} (m^2 - 2\Lambda^2) \Delta s ,$$

in agreement with well-known 1-loop results.

Next we show that the additional term $\Sigma$ in the effective Hamiltonian yields a renormalization of the coupling constant. From the approximate expression (5) we read off that $\Sigma$ contains terms of order $\phi_0^0, \phi_2^2, \phi_4^4$ and $\phi_6^6$, contributing to vacuum energy, mass term, $\phi^4$ interaction term and a new $\phi^6$ interaction term, respectively, of the effective Hamiltonian. We denote the various contributions by $\Sigma_n$, $n \in \{0, 2, 4, 6\}$. It is

$$\Sigma_n(\Delta s) = - \left[ g \frac{\mu^e}{4!} \right]^2 \sum_{m=\max(0,n-3)}^{\min(3,n)} \binom{4}{m} \binom{4}{n-m} \int d^{d-1}x \int d^{d-1}y \phi_<(x)^m \phi_<(y)^{n-m} \times \left[ P\phi_>(x)^{4-m} \frac{Q}{H(0)} \phi_>(y)^{4+m-n} P \right] .$$

The numerical factor in front of the nonuniversal $\Lambda^2$-term may differ. This is due to different definitions of the cutoff in 3- or 4-momentum space, respectively. The negative overall sign stems from our definition of the flow parameter: for $\Delta s > 0$ we are lowering the cutoff, $d/ds = -\Lambda d/d\Lambda$.
The contribution $\Sigma_2$ to the mass term contains vacuum expectation values of the form $\langle 0 | \phi^2 Q \phi^2 | 0 \rangle$, $\langle 0 | \phi_1^2 Q \phi_2^2 | 0 \rangle$ and $\langle 0 | \phi_1^2 Q \phi_2^2 | 0 \rangle$, respectively, which are all at least of order $(\Delta s)^2$ and thus negligible for infinitesimal values of $\Delta s$. The contribution $\Sigma_6$ to the new $\phi^6$ interaction term, on the other hand, scales as

$$\Sigma_6 \sim \int d^{d-1} x \int d^{d-1} y \phi_<(x)^3 \phi_<(y)^3 \langle 0 | \phi_> \frac{Q}{H_0} \phi_> | 0 \rangle$$

$$\sim \frac{1}{\omega^2} \int d^{d-1} x \phi_<(x)^6$$

and is thus suppressed for large values of the cutoff. Within our approximations, therefore, the only contribution which could significantly alter the effective Hamiltonian is $\Sigma_4$; it could modify the $\phi^4$ interaction and hence the associated coupling constant $g$.

For $n = 4$ the sum over $m$ in Eq. (22) runs from 1 to 3; a non-vanishing contribution stemming, however, only from $m = 2$. In the remaining term we replace, according to Eq. (13),

$$P \phi_>(x)^2 \frac{Q}{H_0} \phi_>(y)^2 P = \langle 0 | \phi_>(x)^2 Q \phi_>(y)^2 | 0 \rangle \cdot \frac{P}{2\omega_0 + H_0} ;$$

where the vacuum expectation value can be further reduced to

$$\langle 0 | \phi_>(x)^2 Q \phi_>(y)^2 | 0 \rangle = 2 \langle 0 | \phi_>(x) \phi_>(y)| 0 \rangle^2$$

Provided the eliminated shell in momentum space is invariant under time reversal ($k \to -k$), the right-hand side is real and positive definite. It is a distribution in $(x - y)$ whose width scales as $1/\Lambda$. For large cutoff, therefore, it can be approximated by

$$2\langle 0 | \phi_>(x) \phi_>(y)| 0 \rangle^2 \approx 2 \langle 0 | \phi_>(x)^2 | 0 \rangle \cdot \frac{1}{2\omega_0} \delta^{d-1}(x - y)$$

$$= \frac{\delta^{d-1}(x - y)}{2(2\pi)^{d-1}\Lambda^\epsilon - 1} \delta^{d-1}(x - y) \cdot \Delta s + O(m^2/\Lambda^2)$$

As long as the mass $m$ and the energy $H_0$ of the external, slow modes are negligible compared to the cutoff $\Lambda$, we immediately find

$$\Sigma_4(\Delta s) = -g \frac{\mu^\epsilon}{4!} \left( \frac{4}{2} \right)^2 \frac{\delta^{d-1}}{4(2\pi)^{d-1}\Lambda^\epsilon} \Delta s \cdot V[\phi_>] .$$

The resultant modification of the coupling constant reads

$$\Delta g = -\frac{3S_d-1 g^2}{8(2\pi)^{d-1}} \left( \frac{\mu}{\Lambda} \right)^\epsilon \Delta s ;$$

which for $d = 4$ and a spherical cut in 3-momentum space reduces to

$$\Delta g = -\frac{3g^2}{16\pi^2} \Delta s ,$$

again in agreement with known 1-loop results [10].
3 Discussion

Renormalization is equivalent to determining a continuous sequence of effective Hamiltonians in smaller and smaller Hilbert spaces, obtained by successive elimination of short-wavelength modes. For the case of $\phi^4$ theory we have shown that, to 1-loop order, these effective Hamiltonians and hence the renormalization flow of the masses and couplings can be determined with the help of the venerable Bloch-Feshbach formalism. This finding might be interesting for several reasons:

1. Renormalization is often formulated in terms of functional integrals and diagrams, while projection techniques in Hilbert space are based on algebraic concepts such as linear subspaces and operators. Building a bridge between these different languages offers a new and interesting conceptual perspective and potentially broadens the range of available calculational tools.

2. Projection techniques permit the elimination not just of short-distance information, but also of other kinds of information deemed irrelevant, such as high angular momenta, spin degrees of freedom, or entire particle species. Adopting the projection approach may therefore open the way to new, more general renormalization schemes.

3. Finally, projection techniques provide a common framework for both renormalization and the transition to macroscopic transport theories [1]. They are thus a natural language to study issues such as the renormalization of macroscopic transport equations, or effective kinetic theory [2].

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