SHARP REGULARIZING ESTIMATES FOR THE GAIN TERM OF THE BOLTZMANN COLLISION OPERATOR

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Abstract. We prove the sharp regularizing estimates for the gain term of the Boltzmann collision operator including hard sphere, hard potential and Maxwell molecule models. Our new estimates characterize both regularization and convolution properties of the gain term and have the following features. The regularizing exponent is sharp both in the $L^2$ based inhomogeneous and homogeneous Sobolev spaces which is exact the exponent of the kinetic part of collision kernel. The functions in these estimates belong to a wider scope of (weighted) Lebesgue spaces than the previous regularizing estimates. Furthermore, for the estimates in homogeneous Sobolev spaces, we only need functions lying in Lebesgue spaces instead of weighted Lebesgue spaces, i.e., no loss of weight occurs in this case.

1. Introduction

The Boltzmann collision operator reads

$$Q(f, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2_+} (f' f'_* - f f_*) B(v - v_*, \omega) d\Omega(\omega) dv_*$$

where $d\Omega(\omega)$ is the solid element in the direction of $\omega$ and the abbreviations $f' = f(x, v', t)$, $f'_* = f(x, v'_*, t)$, $f_* = f(x, v_*, t)$, where

$$v' = v - [\omega \cdot (v - v_*)] \omega, \quad v'_* = v_* + [\omega \cdot (v - v_*)] \omega, \quad \omega \in \mathbb{S}^2_+$$

stand for the pre-collision velocities of particles which after collision have velocities $v$ and $v_*$. In the study of the Boltzmann equation, one of the most important tasks is to understand the properties of the collision operator.

When Grad’s cutoff assumption

$$\int_{\mathbb{S}^2_+} B(v - v_*, \omega) d\Omega(\omega) < \infty$$

is satisfied, we can split the collision operator into gain and loss terms, namely

$$Q^+(f, f) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2_+} B(v - v_*, \omega) f' f'_* d\Omega(\omega)$$

$$Q^-(f, f) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2_+} B(v - v_*, \omega) f f_* d\Omega(\omega).$$

The loss term is in fact

$$f(L f)$$
where $L$ is a convolution operator in velocity variable, while the gain term is more complicated. To study the gain term, it is convenient to consider the quadratic operator, i.e.

$$Q^+(f, g) = \int_{\mathbb{R}^3} \int_{S^2_+} B(v - v_*, \omega)f(v')g(v'_*)d\Omega(\omega)dv_*.$$  

In this paper, we consider the collision kernels of the form

$$B(v - v_*, \omega) = |v - v_*|^\gamma b(\cos \theta) = |v - v_*|\gamma \cos \theta,$$

where $0 \leq \gamma \leq 1$, $0 \leq \theta \leq \pi/2$. Thus (1.4) concludes the hard sphere, hard potentials and Maxwell molecule models.

In the study of the renormalized solutions of the Boltzmann equation, Lions [15] found that the gain operator $Q^+(f, g)$ acts like a regularizing operator on each of its components when the other is frozen, i.e.

$$\|Q^+(f, g)\|_{H^1} \leq C\|g\|_{L^1}\|f\|_{L^2}, \quad \|Q^+(f, g)\|_{H^1} \leq C\|g\|_{L^2}\|f\|_{L^1}$$

under the assumption that $B(|z|, \theta) \in C^\infty_c((\mathbb{R}^3 \setminus \{0\}) \times (0, \frac{\pi}{2}))$. The exponent 1 is due to that the compact support assumption. The definitions of Sobolev spaces and the other function spaces are give in the notation subsection in the end of this section. His proof of (1.5) based on a duality argument and the estimate of a Radon transform which relies the theory of Fourier integral operators (F.I.O.s).

It is worthwhile to mention that the regularity theory of the generalized Radon transform was studied in detail by Sogge and Stein [18, 19, 20] at the end of the eighties. Later Wennberg [23] gave a simplified proof of (1.5) by using the Carleman representation of $Q^+$ and classical Fourier transform. The estimates for full kernel without compactness assumption were given by Wennberg [23], Bouchut & Desvillettes [6]. Lu [16], Mouhot & Villani [17] in the forms different with (1.5), see [14] for more details. In [14], the author proved that the estimates of the form (1.5) for the full kernels (1.4) hold for lower regularity. More precisely, with assumption (1.4) with $\gamma > 0$ we have

$$\|Q^+(f, g)\|_{H^{\gamma-}} \leq C\|g\|_{L^1}\|f\|_{L^2}, \quad \|Q^+(f, g)\|_{H^{\gamma-}} \leq C\|g\|_{L^2}\|f\|_{L^1}$$

where $\gamma- = \gamma - \varepsilon > 0$ for arbitrary small $\varepsilon > 0$ and thus the constant $C$ depends on $\varepsilon$.

On the other hand, Gustafsson [11] proved that $Q^+(f, g)$ can be regarded as a convolution operator, and he used this fact to prove uniform $L^p$ estimates for solutions of the space homogeneous Boltzmann equation. The estimates by Duduchava, Kirsch and Rjasanow [9], Alonso and Carneiro [1], Alonso, Carneiro and Gamba [2] and Alonso and Gamba [3] are also of this type. Assume collision kernels

$$B(|z|, \theta) = |z|^\lambda b(\cos \theta)$$

with $b(\cos \theta)$ satisfies Grad’s cut-off assumption. Alonso, Carneiro and Gamba [2] obtained that if $\lambda, \alpha \geq 0$ and $1 \leq p, q, R \leq \infty$ with $1/p + 1/q = 1 + 1/R$, then

$$\|Q^+(f, g)\|_{L^R} \leq C\|g\|_{L^{p+\lambda}}\|f\|_{L^{q+\lambda}}.$$
where the explicit $C$ is given. For $-n < \lambda < 0$ ($n$ is the dimension of variable $z$) and $1/p + 1/q = 1 + \lambda/n + 1/R$, they also obtained
\begin{equation}
\|Q^+(f, g)\|_{L^R} \leq C \|g\|_{L^p} \|f\|_{L^q}.
\end{equation}

The main result of this paper is the new estimates in Theorem 1.1 below which not only improve (1.6), characterize both regularization and convolution properties of the gain term but also have the following features. The regularizing exponent is sharp both in the $L^2$ based inhomogeneous and homogeneous Sobolev spaces which is exact the exponent of the kinetic part of collision kernel. The functions in these estimates belong to a wider scope of (weighted) Lebesgue spaces than the previous regularizing estimates. Furthermore, for the estimates in homogeneous Sobolev spaces, we only need the functions lying in Lebesgue spaces instead of weighted Lebesgue spaces, i.e., no loss of weight occurs in this case. It might be reasonable to guess that the homogeneous Sobolev spaces are more suitable ones to study Boltzmann equation, at least for equation with cut-off assumption.

**Theorem 1.1.** Let $Q^+(f, g)$ be the operators defined by (1.3) with collision kernels (1.4). For $1 \leq p, q \leq 2, 1/p + 1/q = 3/2$ we have estimates in inhomogeneous Sobolev spaces as
\begin{equation}
\|Q^+(f, g)\|_{H^\gamma} \leq C \|g\|_{L^p} \|f\|_{L^q},
\end{equation}
and estimates in homogeneous Sobolev spaces as
\begin{equation}
\|Q^+(f, g)\|_{\dot{H}^\gamma} \leq C \|g\|_{L^p} \|f\|_{L^q}.
\end{equation}

The constants $C$ depend on $p, q$ and $\gamma$.

**Remark 1.2.** We note that the product of the angular function $b(\cos \theta) = \cos \theta$ and $\sin \theta$, Jacobin of solid element, has the feature that it decays to 0 when $\theta$ tends to 0 or $\pi/2$ which is needed for our proof of estimates. Indeed we can get the same estimates (1.9) and (1.10) when $b(\cos \theta) = \cos \theta$ is replaced by other angular function with the suitable decaying property near $\pi/2$, but we prefer to use $b(\cos \theta) = \cos \theta$ for the simplicity of representation.

**Remark 1.3.** In many papers, the authors consider the collision operator operator defined on $S^2$ instead of physical one $S^2_+$ by extending evenly the range of $\theta$ to $[0, \pi]$. This extension will not affect the results as we explained in the paragraph right after (3.24).

By the embedding theorem in homogeneous Sobolev space, see for example [4], we immediately obtain the following.

**Corollary 1.4.** Let $Q^+(f, g)$ be the same as Theorem 1.1. For $1 \leq p, q \leq 2, 1/p + 1/q = 3/2$ and $2 \leq R$ satisfies $1/2 = \gamma/3 + 1/R$, then we have
\begin{equation}
\|Q^+(f, g)\|_{L^R} \leq C \|g\|_{L^p} \|f\|_{L^q}.
\end{equation}

It is interesting to note that the relation of $p, q, R, \gamma$ in above corollary can be restated as
\begin{equation}
3/2 = 1/p + 1/q = 1 + \gamma/3 + 1/R.
\end{equation}
This relation is the same as that for the soft potential given by (1.8) except that the sum must be 3/2. The constraint is because our estimates in Theorem 1.1 is...
denote the gradient. A function \( p \) is called a symbol of order \( m \) \( \parallel f \parallel_{L^p} \leq C \parallel g \parallel_{L^q} \parallel f \parallel_{L^r} \),

and

\[
\parallel Q^r_\bullet (f, g) \parallel_{H^1} \leq C \parallel g \parallel_{L^q} \parallel f \parallel_{L^r}.
\]

The constants \( C \) depend on \( p, q, \) and \( \gamma \).

The proof of above corollary will be given in Section 4 since it is a consequence of the proof of the main Theorem.

**Notations**

We set Japanese bracket \( \langle v \rangle = (1 + |v|^2)^{1/2}, \) \( v \in \mathbb{R}^3, (a \cdot b) = \sum_{i=1}^3 a_i b_i \) the scalar product in \( \mathbb{R}^3 \) and \( \langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)dx \) the inner product in \( L^2(\mathbb{R}^3) \). The differential operator \( D^s, s \in \mathbb{R} \) is expressed through the Fourier transform:

\[
D^s f(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} |\xi|^s \hat{f}(\xi) d\xi
\]

where the Fourier transform \( \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx \). The weighted Lebesgue, fractional homogeneous and inhomogeneous Sobolev spaces are denoted by

\[
\parallel f \parallel_{L^p} = \left( \int_{\mathbb{R}^3} |f(v)|^p |v|^q dv \right)^{1/p},
\]

\[
\parallel f \parallel_{H^s} = \parallel |\xi|^s \hat{f}(\xi) \parallel_{L^2}, \quad \parallel f \parallel_{H^s} = \parallel |\xi|^s \hat{f}(\xi) \parallel_{L^2}
\]

We use the multi-indices notation \( \partial_{x_i}^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \). By \( \partial_x \) or by \( \nabla_x \), we will denote the gradient. A function \( p(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) satisfying

\[
\parallel \partial_{x_i}^\alpha \partial_{\xi}^\beta p(x, \xi) \parallel \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}
\]

for any multi-indices \( \alpha \) and \( \beta \) is called a symbol of order \( m \). The class of such function is denoted by \( S^m_{1,0} \). We will see that the symbols \( p(x, \xi) \) in this paper always enjoy the better decaying condition, i.e.,

\[
\parallel \partial_{x_i}^\alpha \partial_{\xi}^\beta p(x, \xi) \parallel \leq C_{\alpha,\beta} \langle x \rangle^{m_1-|\alpha|} \langle \xi \rangle^{m_2-|\beta|}
\]

for any multi-indices \( \alpha \) and \( \beta \) and which is called a \( SG \) symbol of order \( (m_1, m_2) \), used by Cordes \( \cite{core} \) and Coriasco \( \cite{C} \). We use \( SG^{m_1, m_2} \) to denote the set of such symbols. For each \( p(x, \xi) \in S^1_{1,0} \), the associate operators

\[
P(x, D)h(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi) \hat{h}(\xi) d\xi
\]
is called a pseudodifferential operator of order \( l \). The standard notation \( S^{−∞}_{1,0} = \cap_i S^{i}_{1,0}, l \in \mathbb{Z} \) is also used. If \( p(x,\xi) \in S^{−∞}_{1,0} \), it is called a symbol of the smooth operator. The operator

\[
Sf(x) = \int_{\mathbb{R}^3} e^{i\psi(x,\xi)} a(x,\xi) \hat{f}(\xi) d\xi
\]

with symbol \( a(x,\xi) \in S^{i}_{1,0} \) and the phase function \( \psi(x,\xi) \) satisfies non-degeneracy condition is called a Fourier integral operator of order \( l \). We say a phase function \( \psi(x,\xi) \) satisfies the non-degeneracy condition if there is a constant \( c > 0 \) such that

\[
| \det \left( \frac{\partial}{\partial x} \partial_{\xi} \psi(x,\xi) \right) | = | \det \left( \frac{\partial^2}{\partial x \partial_{\xi}} \psi(x,\xi) \right) | \geq c > 0
\]

for all \( (x,\xi) \in \text{supp} \ a(x,\xi) \).

2. Reduction and Almost Orthogonality

Inspired by Lions [15], we found that the proof of the Theorem [11] relies on the understanding of the following Radon transform

\[
\mathcal{T}h(x) = |x|^\gamma \int_{\omega \in S^2} b(\cos \theta) h(x - (x \cdot \omega) \omega) d\Omega(\omega), \quad 0 \leq \gamma \leq 1
\]

with \( \cos \theta = (x \cdot \omega)/|x|, x \neq 0, x = |x|(0,0,1) \) and \( \omega = (\cos \varphi \sin \gamma, \sin \varphi \sin \theta, \cos \theta), \quad 0 \leq \theta \leq \pi/2 \). We use the notation \( \| F(v, v_s) \|_{L^p(v)} \) to denote the \( L^p \) norm of \( F(v, v_s) \) with respect to variable \( v \) where variable \( v_s \) is regarded as the parameter. We will see soon that the estimate [1.10] is the consequence of the following Lemma which concerning the estimates of \( T \).

**Lemma 2.1.** Let \( \mathcal{T} \) be the operator defined by (2.1) and \( \tau_m \) be the translation operator \( \tau_m h(\cdot) = h(\cdot + m) \). We have

\[
(2.2) \quad \sup_{v_s} \| (\tau_{-v_s} \circ \mathcal{T} \circ \tau_{v_s}) h(v) \|_{L^2(v)} \leq C \| h \|_{H^{-\gamma}}
\]

and

\[
(2.3) \quad \sup_{v} \| (\tau_{-v} \circ \mathcal{T} \circ \tau_{v}) h(v) \|_{L^2(v_s)} \leq C \| h \|_{H^{-\gamma}}.
\]

Please note that the left hand side of (2.2) is an integration of variable \( v \) with parameter \( v_s \) while the roles of \( v \) and \( v_s \) are switched in (2.3). On the other hand an unified perspective is provided in the proof of these two estimates in section 3 with the aid of the structural understanding of \( \mathcal{T} \) in Lemma 3.1 below.

To find the key estimates for the proof of (1.9), we need a useful observation. Let \( \rho \in C^\infty(\mathbb{R}), 0 \leq \rho \leq 1 \) be supported in the open interval \((4,16)\) and satisfy

\[
(2.4) \quad 1 = \sum_{k \in \mathbb{Z}} \rho(2^{-k}r), \quad r > 0
\]

with property \( \text{supp} \ \rho(2^j r) \cap \text{supp} \ \rho(2^k r) = \emptyset \) if \( |j - k| \geq 2 \). Define \( s(0) = 0 \) and

\[
(2.5) \quad s(r) = \sum_{k \leq 0} \rho(2^{-k}r), \quad r > 0.
\]
Then $\mathfrak{s}$ is a smooth positive function on $[0, \infty)$ satisfying $\mathfrak{s}(r) = 1$ when $0 \leq r < 4$ and $\mathfrak{s}(r) = 0$ when $r > 16$. Let $\overline{\mathfrak{s}}(r) = 1 - \mathfrak{s}(r), r \geq 0$. Then we split the collision kernel $B$ into two parts by

$$B(z, \omega) = \mathfrak{s}(\|z\|)B(z, \omega) + \overline{\mathfrak{s}}(\|z\|)B(z, \omega) \overset{\text{def}}{=} B_{\mathfrak{s}}(z, \omega) + B_{\overline{\mathfrak{s}}}(z, \omega),$$

and split $Q^+$ into two parts by plugging above into (1.3) and write

$$(2.6) \quad Q^+(f, g) = Q^+_{\mathfrak{s}}(f, g) + Q^+_{\overline{\mathfrak{s}}}(f, g).$$

Clearly the estimates (1.9) follow from (2.6)

$$(2.7) \quad \|Q^+_{\mathfrak{s}}(f, g)\|_{H^{-\gamma}} \leq C\|g\|_{L^{p}}\|f\|_{L^{s}},$$

and

$$(2.8) \quad \|Q^+_{\overline{\mathfrak{s}}}(f, g)\|_{H^{-\gamma}} \leq C\|g\|_{L^{p}}\|f\|_{L^{s}}.$$ We define $|x|^\gamma = \mathfrak{s}(|x|)\|x\|^\gamma$ and follow (2.1) to define

$$(2.9) \quad T_{\mathfrak{s}}(x) = |x|^\gamma \int_{\omega \in S^2_{x}} b(\cos \theta) h(x - (x \cdot \omega)\omega) d\Omega(\omega),$$

and

$$(2.10) \quad H_{\mathfrak{s}}^w(v, v_{*}) = \int_{S^2_{x}}(v_{*})^\gamma \gamma(v_{*})^{-1} B_{\mathfrak{s}}(v - v_{*}, \omega) h(v') d\Omega(\omega)$$

whose structure is similar to that of $\tau_{-v_{*}} \circ T \circ \tau_{v_{*}}$. We will see soon that the estimates (2.7) and (2.8) follow the Lemma below.

**Lemma 2.2.** Let $T_{\mathfrak{s}}$ and $H_{\mathfrak{s}}^w$ be the defined by (2.9) and (2.10). We have

$$(2.11) \quad \sup_{v_{*}} \| (\tau_{-v_{*}} \circ T_{\mathfrak{s}} \circ \tau_{v_{*}}) h(v) \|_{L^2(v)} \leq C \|h\|_{H^{-\gamma}},$$

$$(2.12) \quad \sup_{v} \| (\tau_{-v} \circ T_{\mathfrak{s}} \circ \tau_{v}) h(v) \|_{L^2(v)} \leq C \|h\|_{H^{-\gamma}},$$

$$(2.13) \quad \sup_{v_{*}} \| H_{\mathfrak{s}}^w(v, v_{*}) \|_{L^2(v_{*})} \leq C \|h\|_{H^{-\gamma}},$$

and

$$(2.14) \quad \sup_{v} \| H_{\mathfrak{s}}^w(v, v_{*}) \|_{L^2(v_{*})} \leq C \|h\|_{H^{-\gamma}}.$$

Assume Lemma 2.1 and Lemma 2.2 hold temporarily, whose proofs are postponed to the Section 4, we can prove the Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the proof of the estimate (1.10). By the duality of homogeneous Sobolev spaces, see for example [4], we need to show that

$$\langle Q^+(f, g), h \rangle \leq C \|f\|_{L^{s}}\|g\|_{L^{p}}\|h\|_{H^{-\gamma}}$$

holds for any $h \in \dot{H}^{-\gamma}$. Using change of variables and Hölder inequality

$$\langle Q^+(f, g), h \rangle = \int \int \int f(v)g(v_{*})B(v - v_{*}, \omega) h(v')d\Omega(\omega)dv_{*}dv \leq \|f(v)\|_{L^{s}} \|Q^+_g(h)(v)\|_{L^{p}}$$

Then (1.10) follows.

...
where $q'$ is the conjugate exponent of $q$ and
\[
Q_g^{it}(h)(v) = \iint g(v_s)B(v - v_s, \omega)h(v')d\Omega(\omega)dv_s.
\]

Thus it is reduced to proving that
\[
\|Q_g^{it}(h)(v)\|_{L^{q'}} \leq C\|g\|_{L^{p'}}\|h\|_{H^{-\gamma}}.
\]
where $1/2 + 1/q' = 1/p$, $1 \leq p, q \leq 2$. Relabel these exponents of (2.15) as
\[
\|Q_g^{it}(h)(v)\|_{L^r} \leq C\|g\|_{L^p}\|h\|_{H^{-\gamma}}
\]
where $1/2 + 1/r = 1/q$, $1 \leq q \leq 2, r \geq 2$.

Defining the translation operator
\[
\tau_m h(\cdot) = h(\cdot + m),
\]
and following Lions [15], we rewrite
\[
Q_g^{it}(h)(v) = \int g(v_s)(\tau_{-v_s} \circ T \circ \tau_{v_s})h(v)dv_s.
\]

Define $H(v, v_s) = (\tau_{-v_s} \circ T \circ \tau_{v_s})h(v)$. By $1/r + (r - 2)/(2r) + (r - q)/(qr) = 1$ and Hölder inequality, we have
\[
|Q_g^{it}(h)(v)| \leq \int |H(v, v_s)||g(v_s)|dv_s.
\]
\[
= \int \left( |H(v, v_s)|^2|g(v_s)|^q \right)^{1/r} |H(v, v_s)|^{(r-2)/r}|g(v_s)|^{(r-q)/r} dv_s.
\]
\[
\leq \|(|H(v, v_s)|^2|g(v_s)|^q)^{1/r}\|_{L^{r}(v_s)} \times \|H(v, v_s)|^{(r-2)/r}|g(v_s)|^{(r-q)/r}\|_{L^{2r/q}(v_s)}.
\]

Here
\[
\|g(v_s)|^{(r-q)/r}\|_{L^{2r/q}(v_s)} = (\|g(v_s)\|_{L^{2r}(v_s)})^{r-q/r},
\]
\[
\|H(v, v_s)|^{(r-2)/r}\|_{L^{2r/q}(v_s)} = (\|\tau_{-v_s} \circ T \circ \tau_{v_s})h(v)\|_{L^{2r}(v_s)})^{r-2/r},
\]
and
\[
\|(|H(v, v_s)|^2|g(v_s)|^q)^{1/r}\|_{L^{r}(v_s)}
\]
\[
= \left( \int |g(v_s)|^q |(\tau_{-v_s} \circ T \circ \tau_{v_s})h(v)|^2 dv_s \right)^{1/2}.
\]

Applying (2.19) of Lemma 2.1 we have
\[
(2.21) \quad (2.19) \leq C\|h\|_{H^{-\gamma}}^{1/2}.
\]
where $C$ is independent of $v$. Combine (2.18), (2.21) and (2.20), we have

$$
\|Q^+_{w}(h)(v)\|_{L^r(v)} \\
\leq C(\|g(v_s)\|_{L^{q'(v)}})^{(r-q)}(\|h\|_{H^{-\gamma}(v_s)})^{(r-2)} \\
\int \int |g(v_s)|^q |(\tau_{-v_s} \circ T \circ \tau_{v_s})h(v)|^2 dv_s dv
$$

(2.22)

Then we conclude (2.16) by applying (2.2) of Lemma 2.1 to the last term of (2.22).

The proof of (1.10) is complete.

As we mentioned before the estimate (1.9) comes from (2.7) and (2.8). Clearly the above argument also indicates that the estimate (2.7) comes from (2.11) and (2.12).

Next we prove the estimate (2.8) is the consequence of (2.13) and (2.14). By the duality, it is equivalent show that for any $h \in H^{-\gamma}$ we have

$$
\|\langle Q^+_{w}(f,g), h \rangle\| \leq C \|f\|_{L^r} \|g\|_{L^p} \|h\|_{H^{-\gamma}},
$$

or equivalently

$$
\|\langle Q^+_{w}(f,g), h \rangle\| \leq C \|f^w\|_{L^r} \|g^w\|_{L^p} \|h\|_{H^{-\gamma}},
$$

where $f^w(v) = \langle v \rangle^\gamma \cdot f(v)$ is the product of $f$ and the weight function $\langle v \rangle^\gamma$ and similarly $g^w(v_s) = \langle v_s \rangle^\gamma \cdot g(v_s)$. Using change of variables and Hölder inequality,

$$
\|\langle Q^+_{w}(f,g), h \rangle\| = \int \int f^w(v)g^w(v_s) \frac{1}{\langle v \rangle^\gamma \langle v_s \rangle^\gamma} B^w(v - v_s, \omega) h(v') d\Omega(\omega) dv_s dv
$$

and

$$
Q^+_{w}(h)(v) = \int g^w(v_s) \frac{1}{\langle v \rangle^\gamma \langle v_s \rangle^\gamma} B^w(v - v_s, \omega) h(v') d\Omega(\omega) dv_s.
$$

It is reduced to proving that

(2.23)

$$
\|Q^+_{w}(h)(v)\|_{L^r} \leq C \|g^w\|_{L^p} \|h\|_{H^{-\gamma}}
$$

where $1/2 + 1/q' = 1/p$, $1 \leq p, q \leq 2$. Relabel these exponents of (2.15) as

(2.24)

$$
\|Q^+_{w}(h)(v)\|_{L^r} \leq C \|g^w\|_{L^p} \|h\|_{H^{-\gamma}}
$$

where $1/2 + 1/r = 1/q$, $1 \leq q \leq 2, r \geq 2$.

Recalling

$$
H^w_{w}(v, v_s) = \int_{S^2} \frac{1}{\langle v \rangle^\gamma \langle v_s \rangle^\gamma} B^w(v - v_s, \omega) h(v') d\Omega(\omega),
$$

thus we rewrite

$$
Q^+_{w}(h)(v) = \int g^w(v_s) H^w_{w}(v, v_s) dv_s.
$$
Applying the Hölder inequality as the previous argument, we see that the result follows from
\[
\sup_v \| H^w_w(v, v)^{\ast} \|_{L^2(v)} \leq C\| h \|_{H^{-\gamma}}
\]
\[
\sup_v \| H^w_w(v, v)^{\ast} \|_{L^2(v)} \leq C\| h \|_{H^{-\gamma}}
\]
which are the last two estimates of Lemma 2.2.

The proof of Lemma 2.1 and Lemma 2.2 depend on the understanding of the operator $T$ defined in (2.1), while it needs a lot of effort. For our purpose, we use inverse Fourier transform to rewrite
\[
T h(x) = |x|^\gamma \int_{\omega \in S_+^2} b(\cos \theta) h(x - (x \cdot \omega) \omega) d\Omega(\omega)
\]
\[
= (2\pi)^{-3} \int_{R^3} e^{ix \cdot \xi} A(x, \xi) \hat{h}(\xi) d\xi
\]
where
\[
A(x, \xi) = |x|^\gamma \int_{\omega \in S_+^2} e^{-i(x \cdot \omega)(\xi \cdot \omega)} b(\cos \theta) d\Omega(\omega).
\]
The operator $T$ originally defined on $S_+^2$ is turned to be an operator defined on $R^3$ and whose property is thus condensed in function $A(x, \xi)$. The function $A(x, \xi)$ is rather complicated and has differently properties on different portions of the phase space $(x, \xi)$. However the following estimate holds.

Lemma 2.3. The operator $T$ defined in (2.25) satisfies
\[
\| T h \|_{L^2} \leq C\| h \|_{H^{-\gamma}}.
\]

The proof of Lemma 2.3 can be reduced to a $L^2$ to $L^2$ estimate easily. Before that we introduce a preparation lemma which employs the almost orthogonality argument, see for example [22], to show the $L^2$ boundedness of a particular Fourier integral operator (F.I.O.) induced by $A(x, \xi)$.

We consider the cone defined in the phase space $(x, \xi) \in \{R^3 - \{0\}\} \times \{R^3 - \{0\}\}$ by
\[
\{(x, \xi) | \frac{\pi}{8} < \arccos\left(\frac{x \cdot \xi}{|x||\xi|}\right) < \frac{\pi}{2}\} \text{ def } \Gamma_x \times \Gamma_\xi.
\]
For each fixed $x$, we use $x \times \Gamma_\xi$ to denote the cone whose element $\xi \in R^3$ satisfies (2.27). The notation $\Gamma_\xi$ means the cone $x \times \Gamma_\xi$ where the vector $x$ is not specified. Also $\Gamma_x \times x, \Gamma_x$ are defined likewise. To estimate the F.I.O. whose amplitude function is defined on $\Gamma_x \times \Gamma_\xi$, we need a dyadic partition of unity on $R^3 - \{0\}$. For $x \in R^3$ and $k \in Z$ we define $\chi_k(x) = \rho(2^{-k}|x|)$ where $\rho$ is defined in (2.4), then we have dyadic partition of unity
\[
1 = \sum_{k \in Z} \chi_k(x), \ x \neq 0.
\]
We also need the definition
\begin{equation}
Q = \max\{ \int_{\Gamma_x} \chi_0(x) dx , \int_{\Gamma_x} \chi_0(x) \chi_1(x) dx \}.
\end{equation}
The lemma we need is the following.

\textbf{Lemma 2.4.} Let $F$ be defined by
\begin{equation}
Fu(x) = \int_{\mathbb{R}^3} e^{i\psi(x,\xi)} p(x,\xi) u(\xi) d\xi
\end{equation}
where $\psi(x,\xi)$ is homogeneous of degree 1 in $x$ and $\xi$, $p(x,\xi) \in C^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ and $\text{supp} \, p(x,\xi) \subset \Gamma_x \times \Gamma_\xi \cap \{ |x| > 8, |\xi| > 8 \}$ satisfies
\begin{equation}
|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta} |x|^{-|\alpha|} |\xi|^{-|\beta|},
\end{equation}
for $|\alpha|, |\beta| \leq 4$. Let
\begin{equation}
P = \sup_{|\alpha|,|\beta| \leq 4} \| \partial_x^\alpha \partial_\xi^\beta p(x,\xi) \|_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)}.
\end{equation}
When $(x,\xi) \in \text{supp} \, p(x,\xi)$, the phase function $\psi(x,\xi)$ satisfies
\begin{equation}
0 < C_1 < |\det [\partial_x \partial_\xi \psi(x,\xi)]| < C_2
\end{equation}
and
\begin{equation}
|\partial_x^\alpha \partial_\xi \psi(x,\xi)| \leq C_\alpha , \quad |\partial_x \partial_\xi^\beta \psi(x,\xi)| \leq C_\beta , \quad 1 \leq |\alpha|, |\beta| \leq 5
\end{equation}
Then $F$ is $L^2$ bounded and satisfies
\begin{equation}
\|F\|_{L^2 \rightarrow L^2} \leq C Q P
\end{equation}
where $C$ depends on the constants in (2.33) and (2.34) and $Q$ is defined in (2.29).

\textbf{Proof.} Note $\sum_{k=0}^{\infty} \chi_k(z) = 1$ when $|z| > 8$. Therefore we can decompose $F$ as
\begin{equation}
F = \sum_{(j,l) \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}} F_{(j,l)}
\end{equation}
where
\begin{equation}
F_{(j,l)} u(x) = \chi_j(x) \int_{\mathbb{R}^3} e^{i\psi(x,\xi)} \chi_l(\xi)p(x,\xi) u(\xi) d\xi
\end{equation}
The adjoint of $F_{j,l}$, denoted by $F_{j,l}^*$ is
\begin{equation}
F_{j,l}^* v(y) = \chi_j(\xi) \int e^{-i\psi(y,\xi)} \chi_l(\xi) p(y,\xi) v(y) dy.
\end{equation}
Then we have
\begin{equation}
F_{(j,l)} F_{(k,m)}^* u(x) = \int K_{(j,l),(k,m)} (x,y) u(y) dy,
\end{equation}
where
\begin{equation}
K_{(j,l),(k,m)} (x,y) = \chi_j(x) \chi_k(y) \int e^{i(\psi(x,\xi)-\psi(y,\xi))} \chi_l(\xi) \chi_m(\xi) p(x,\xi) p(y,\xi) dx d\xi.
\end{equation}
Since $\text{supp} \, \chi_l(\xi) \cap \text{supp} \, \chi_m(\xi) = \emptyset$ when $|l-m| \geq 2$, we only have to consider $l = m-1, m$ or $m+1$. Without loss of generality, we assume $l = m+1$ and $j \geq k$.
First we consider the sub-case $j \geq k + 3$. Let
\begin{equation}
\tilde{x} = 2^{-k} x, \tilde{y} = 2^{-k} y, \tilde{\xi} = 2^{-m} \xi = \tau^{-1}_2 \xi.
\end{equation}
Since $\psi$ is homogeneous of degree 1 in first and second variables, we have  

$$K_{(j,l)(k,m)}(x,y) = \tau_1^3 \chi_{j-k}(\tilde{x}) \chi_0(\tilde{y}) \int e^{i\tau_1 \tau_2 (\psi(\tilde{x},\tilde{\xi}) - \psi(\tilde{y},\tilde{\xi}))} \chi_0(\tilde{\xi}) \chi_1(\tilde{\xi}) p(\tau_1 \tilde{x}, \tau_2 \tilde{\xi}) \overline{p(\tau_1 \tilde{y}, \tau_2 \tilde{\xi})} d\tilde{\xi}$$

def = $\tau_2^3 K_{(j,l)(k,m)}(\tilde{x}, \tilde{y})$.

Hence  

$$(2.37) \quad F_{(j,l)} F^*_{{(k,m)}} u(\tau_1 \tilde{x}) = \int \tau_2^3 K_{(j,l)(k,m)}(\tilde{x}, \tilde{y}) u(\tau_1 \tilde{y}) \tau_1^3 d\tilde{y}.$$  

Define the operator  

$$(2.38) \quad L_{\tau_1 \tau_2} = \frac{1}{i \tau_1 \tau_2} \frac{\partial_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial_\xi \psi(\tilde{y}, \tilde{\xi})}{|\partial_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial_\xi \psi(\tilde{y}, \tilde{\xi})|^2} \partial_\xi$$

and observe that  

$L_{\tau_1 \tau_2}(e^{i\tau_1 \tau_2 (\partial_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial_\xi \psi(\tilde{y}, \tilde{\xi})))} = e^{i\tau_1 \tau_2 (\partial_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial_\xi \psi(\tilde{y}, \tilde{\xi}))}$.

Integration by parts yields  

$$(2.39) \quad \int e^{i\tau_1 \tau_2 (\psi(\tilde{x}, \tilde{\xi}) - \psi(\tilde{y}, \tilde{\xi}))} \chi_0(\tilde{\xi}) \chi_1(\tilde{\xi}) p(\tau_1 \tilde{x}, \tau_2 \tilde{\xi}) \overline{p(\tau_1 \tilde{y}, \tau_2 \tilde{\xi})} d\tilde{\xi} = \int e^{i\tau_1 \tau_2 (\psi(\tilde{x}, \tilde{\xi}) - \psi(\tilde{y}, \tilde{\xi}))} (L^*_{\tau_1 \tau_2})^4(\chi_0(\tilde{\xi}) \chi_1(\tilde{\xi}) p(\tau_1 \tilde{x}, \tau_2 \tilde{\xi}) \overline{p(\tau_1 \tilde{y}, \tau_2 \tilde{\xi})}) d\tilde{\xi},$$

where $L^*_{\tau_1 \tau_2}$ is the transpose of $L_{\tau_1 \tau_2}$. Since $p(x, \xi)$ satisfies (2.31), we have  

$$\sup_{|\alpha|, |\beta| \leq 4} \|\partial_\xi^\alpha \partial_\xi^\beta p(\tau_1 \tilde{x}, \tau_2 \tilde{\xi})\|_{L^\infty(\mathbb{R}^4)} \leq P.$$  

From the assumption (2.33) we obtain  

$$(2.40) \quad |\partial_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial_\xi \psi(\tilde{y}, \tilde{\xi})| \geq C|\tilde{x} - \tilde{y}|$$

($C$ depends on $C_1$, see [22] P.397 for obtaining (2.40) from (2.33) and  

$$|\partial^2_\xi \psi(\tilde{x}, \tilde{\xi}) - \partial^2_\xi \psi(\tilde{y}, \tilde{\xi})| \leq C_\beta |\tilde{x} - \tilde{y}|$$

for $1 \leq |\beta| \leq 5$ since phase function $\psi$ is homogeneous of degree 1 in first variable.  

Note $\tilde{x} \in \text{supp} \chi_{j-k}, \tilde{y} \in \text{supp} \chi_0$ and $j \geq k + 3$. Hence we have  

$$|K_{(j,l)(k,m)}(\tilde{x}, \tilde{y})| \leq C(\tau_1 \tau_2)^{-4} p^2 \frac{\chi_{j-k}(\tilde{x}) \chi_0(\tilde{y})}{1 + |\tilde{x} - \tilde{y}|^4} \times \int_{\mathbb{R}^4} \chi_0|_{y \times \Gamma_\xi} \chi_1|_{y \times \Gamma_\xi} \frac{d\tilde{\xi}}{d\xi}$$

where $C$ depends on the constants of (2.33) and (2.34) and $\chi_0|_{y \times \Gamma_\xi}, \chi_1|_{y \times \Gamma_\xi}$ are functions $\chi_0(\tilde{\xi}), \chi_1(\tilde{\xi})$ restricted to cones $\Gamma_\xi$ determined by $x, y$ respectively. We note that  

$$(2.42) \quad \int_{\mathbb{R}^4} \chi_0|_{y \times \Gamma_\xi} \chi_1|_{y \times \Gamma_\xi} d\tilde{\xi}$$

attains its maximum when $x = cy$ for $c > 0$. The maximum of (2.42) can be written as  

$$\int_{y \times \Gamma_\xi} \chi_0(\tilde{\xi}) \chi_1(\tilde{\xi}) d\tilde{\xi}$$
and note that its value depends on the the span of cone $\Gamma_\xi$. The corresponding part for the case $l = m - 1$ clearly has the same maximum as above. For $l = m$, the corresponding maximum will be
\[
\int_{x \times \Gamma_\xi} x_0(\xi) x_0(\hat{\xi}) d\xi.
\]
By definition of $\Gamma_x \times \Gamma_\xi$ and the fact $0 \leq x_0(\cdot) \leq 1$,
\[
\max \{ \int_{x \times \Gamma_\xi} x_0(\xi) x_0(\hat{\xi}) d\xi, \int_{x \times \Gamma_\xi} x_0(\xi) x_1(\hat{\xi}) d\xi \} \leq Q.
\]
We also note that (2.42) non-vanishes only when the angle spanned by $x, y$ is in a suitable range determined by the definition (2.27). From above and $\hat{x} \in \text{supp} \chi_{j-k}$, $\hat{y} \in \text{supp} \chi_0$, we have
\[
\sup_{\hat{x}} \int_{\xi} |\tau_3^2 K_{(j,i)(j,m)}(\hat{x}, \hat{y}) \tau_1^3| d\hat{y} \leq C 2^{-(j-3k)} 2^{-m} Q \int_{\Gamma_\hat{y}} x_0(\hat{y}) d\hat{y} \leq C 2^{-j} 2^{-m} Q^2 P^2
\]
\[
= C 2^{-(j-k)} 2^{-k} 2^{-m} Q^2 P^2,
\]
(2.43)
\[
\sup_{\hat{y}} \int_{\xi} |\tau_3^2 K_{(j,j)(k,m)}(\hat{x}, \hat{y}) \tau_1^3| d\hat{x} \leq C 2^{-(j-3k)} 2^{-m} Q \int_{\Gamma_\hat{x}} x_{j-k}(\hat{x}) d\hat{x} \leq C 2^{-j} 2^{-m} Q P^2 \int_{\Gamma_\hat{x}} \chi_{j-k}(\hat{x}) d\hat{x} \leq C 2^{-(j-k)} 2^{-k} 2^{-m} Q^2 P^2.
\]
Let $H(\cdot) : \mathbb{Z} \to \{0, 1\}$ be defined by $H(\cdot) = 1$ if $|\cdot| \leq 1$ and $H(\cdot) = 0$ if $|\cdot| > 1$. For the general $(j,l), (k,m)$ with $|j-k| \geq 3$, the $2^{-(j-k)} 2^{-k} 2^{-m}$ in the right hand side of (2.43) should be replaced by
\[
2^{-|j-k|} H(l - m) 2^{-\min(j,k)} 2^{-\min(l,m)}.
\]
We note that $2^{-\min(j,k)} 2^{-\min(l,m)} \leq 1$. By invoking Schur test lemma (lemma 5.2), there exists a constant $C$ such that
\[
\| F_{(j,l)} F_{(k,m)}^* \|_{L^2 \to L^2} \leq C 2^{-|j-k|} H(l - m) Q^2 P^2.
\]
Applying the same argument to the case $|l - m| \geq 3$, we have
\[
\| F_{(j,l)} F_{(k,m)}^* \|_{L^2 \to L^2} \leq C 2^{-|l-m|} H(j-k) Q^2 P^2.
\]
Next we prove that (2.44) and (2.45) also hold for $|j-k| < 3$ and $|l-m| < 3$ respectively. By symmetry it suffices to prove one of them. Thus we assume $k \leq j < k + 3$ and $l = m + 1$ and it remains to prove
\[
\| F_{(j,l)} F_{(k,m)}^* \|_{L^2 \to L^2} \leq C Q^2 P^2.
\]
When $|\hat{x} - \hat{y}| \geq (2 \tau_1 \tau_2)^{-1}$, similar to (2.44), we can derive
\[
|K_{(j,j)(k,m)}(\hat{x}, \hat{y})| \leq C (\tau_1 \tau_2)^{-1} 2^{-(j-k)} \chi_{j-k}(\hat{y}) \frac{x_{j-k}(\hat{x}) x_{j-k}(\hat{y})}{|\hat{x} - \hat{y}|^4}
\]
\[
\times \int_{\mathbb{R}^3} \chi_{|x \times \Gamma_\xi(\hat{\xi})} |x \times \Gamma_\xi(\hat{\xi})| d\xi.
\]
Applying (2.48) to the left hand side of (2.43) and estimating the integrals by considering the region $|\tilde{x} - \tilde{y}| \approx (2^{\tau_1} \tau_2)^{-1}$ for each $n \in \mathbb{N}$, we conclude that they have the upper bound $C Q^2 P^2$. When $|\tilde{x} - \tilde{y}| < (2^{\tau_1} \tau_2)^{-1}$, direct estimate gives (2.48) \[ |K((j,l),(k,m))(\tilde{x}, \tilde{y})| \leq C P^2 \chi_j(\tilde{x}) \chi_k(\tilde{y}) \int_{\mathbb{R}^3} \chi_0(y) \chi_1(z) \Gamma_{\xi}(\tilde{\xi}) \chi_0(z) d\tilde{\xi}. \]

Applying (2.48) to the left hand side of (2.43) and considering the support of $\tilde{x}$ and $\tilde{y}$, we conclude (2.46).

Now we have \[ \|F(j,l)F^*(k,m)\|_{L^2 \to L^2}, \|F^*(j,l)F(k,m)\|_{L^2 \to L^2} \leq C Q^2 P^2 \{\Theta(j - k, l - m)\}^2, \]
where \[ \Theta(j_1, j_2) = \sqrt{\frac{H(j_2)}{2^{j_1}}} + \frac{H(j_1)}{2^{j_2}}. \]

Since \[ \sum_{(j_1, j_2) \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}} \Theta(j_1, j_2) < \infty, \]
we can conclude the result by Colter-Stein lemma (Lemma 5.3). \[ \square \]

3. Proof of Lemma 2.3

First we reduce the estimate (2.26) to a $L^2$ to $L^2$ estimate which is more natural and simply. Recalling (2.25), we define (3.1) \[ a(x, \xi) \overset{\text{def}}{=} A(x, \xi)|\xi|^{\gamma} = (|x| |\xi|)^{\gamma} \int_{\omega \in S^2} e^{-i(x \cdot \omega)(\xi \cdot \omega)} b(\cos \theta) d\Omega(\omega) \]
and (3.2) \[ Th(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} a(x, \xi) \hat{h}(\xi) d\xi. \]

Recall \[ D^{-\gamma} f(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} |\xi|^{-\gamma} \hat{f}(\xi) d\xi. \]

From $Th(x) = (2\pi)^{-3} T(D^{-\gamma} h)(x)$, we see that the Lemma 2.3 is equivalent to Lemma 3.1. The operator $T$ defined in (3.2) satisfies (3.3) \[ \|Th\|_{L^2} \leq C \|h\|_{L^2}. \]

Proof. For the analysis of $a(x, \xi)$ we need a dyadic partition of unity on the phase space $\{(x, \xi) | x, \xi \in \mathbb{R}^3 \setminus \{0\}\}$. Using (2.28), we have the dyadic partition of unity by \[ 1 = \sum_{j \in \mathbb{Z}} \chi_j(x) \sum_{l \in \mathbb{Z}} \chi_l(\xi), x \neq 0, \xi \neq 0 \]
\[ = \chi_A(x, \xi) + \chi_B(x, \xi) + \chi_C(x, \xi) \]
where
\[
\chi_A(x, \xi) = \sum_{j \in \mathbb{N}} \chi_j(x) \sum_{l \in \mathbb{N}} \chi_l(\xi),
\]
\[
\chi_B(x, \xi) = \chi_{B,1}(x, \xi) + \chi_{B,2}(x, \xi)
\]
\[
\chi_C(x, \xi) = \chi_{C,1}(x, \xi) + \chi_{C,2}(x, \xi)
\]
(3.4)
\[
\begin{align*}
\text{Hence supp} \chi_A(x, \xi) \subset \{ |x| > 8, |\xi| > 8 \}, \text{ supp } \chi_{B,1}(x, \xi) \subset \{ |x| |\xi| > 64, |x| > 8, |\xi| < 16 \}, \text{ supp } \chi_{B,2}(x, \xi) \subset \{ |x| |\xi| > 64, |x| < 16 \}, \text{ supp } \chi_{C,1}(x, \xi) \subset \{ |x| |\xi| < 512, |x| > 8 \} \text{ and supp } \chi_{C,2}(x, \xi) \subset \{ |x| |\xi| < 512, |x| < 16 \}. \end{align*}
\]
Using (3.3), we decompose \(a(x, \xi)\) as
\[
\begin{align*}
a &= a_A + a_{B,1} + a_{B,2} + a_{C,1} + a_{C,2} \\
\text{def} &= \chi_A \cdot a + \chi_{B,1} \cdot a + \chi_{B,2} \cdot a + \chi_{C,1} \cdot a + \chi_{C,2} \cdot a.
\end{align*}
\]
Plugging this decomposition into (3.2), we write \(T = (T_A + T_{B,1} + T_{B,2} + T_{C,1} + T_{C,2})\) accordingly. It is reduced to proving that all these operators are \(L^2\) bounded.

**Part I. Estimate of \(T_A\)**

For the analysis of \(a_A(x, \xi)\), we need a dyadic decomposition in the interval \((0, \pi)\) which is constructed below. Let \(\zeta(\theta) \in C^\infty\) be supported in the interval \((\pi / 8, \pi / 2)\) and satisfy \(\sum_{n \in \mathbb{N}} \zeta(2^{-n} \theta) = 1\) for all \(\theta > 0\). Let \(\zeta_n(\theta) = \zeta(2^n \theta)\), if \(n \in \mathbb{N}\) and \(\tilde{\zeta}(\theta) = 1 - \sum_{n \in \mathbb{N}} \zeta_n(\theta)\). Define \(\zeta_0(\theta)\) equals \(\tilde{\zeta}(\theta)\), if \(0 < \theta \leq \pi / 2\) and \(\zeta_0(\pi - \theta)\), if \(\pi / 2 < \theta < \pi\). This extension of \(\zeta_0\) from \(0 < \theta \leq \pi / 2\) to \(\pi / 2 < \theta < \pi\) by reflection keeps \(\zeta_0\) a smooth function since \(\tilde{\zeta}\) equals 1 near \(\pi / 2\). We also define \(\zeta_{-n}(\theta) = \zeta_n(\pi - \theta)\) for \(n \in \mathbb{N}\). Then we have the dyadic decomposition
\[
1 = \zeta_0(\theta) + \sum_{n \in \mathbb{N}} (\zeta_n(\theta) + \zeta_{-n}(\theta))
\]
in the interval \(\theta \in (0, \pi)\). Abuse the notations, we define
\[
\zeta_0(x, \xi) = \zeta_0(\arccos(\frac{x \cdot \xi}{|x||\xi|})), \quad \zeta_{\pm n}(x, \xi) = \zeta_{\pm n}(\arccos(\frac{x \cdot \xi}{|x||\xi|})), \quad n \in \mathbb{N}.
\]
Thus the supports of \(\zeta_0(x, \xi), \zeta_n(x, \xi), \zeta_{-n}(x, \xi)\) lie respectively in the cones
\[
\Gamma_0 = \left\{(x, \xi) : \frac{\pi}{8} < \arccos\left(\frac{x \cdot \xi}{|x||\xi|}\right) < \frac{\pi}{8}\right\},
\]
\[
\Gamma_n = \left\{(x, \xi) : \frac{\pi}{2^n + 3} < \arccos\left(\frac{x \cdot \xi}{|x||\xi|}\right) < \frac{\pi}{2^n + 1}\right\}
\]
\[
\Gamma_{-n} = \left\{(x, \xi) : \pi(1 - \frac{1}{2^n + 1}) < \arccos\left(\frac{x \cdot \xi}{|x||\xi|}\right) < \pi(1 - \frac{1}{2^n + 3})\right\}.
\]
Define
\[
a_z(x, \xi) = \zeta_z(x, \xi)a_A(x, \xi), \quad z \in \mathbb{Z}
\]
and write \(a_A(x, \xi) = \sum_{z \in \mathbb{Z}} a_z(x, \xi)\). Then we have \(T_A = \sum_{z \in \mathbb{Z}} T_z\) where
\[
T_z h(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_z(x, \xi) \hat{h}(\xi) d\xi, \quad z \in \mathbb{Z}.
\]
We furthermore decompose each $a_z(x, \xi) = \zeta_z(x, \xi)\chi_A(x, \xi)a(x, \xi)$, $z \neq 0$ into three parts. More precisely, for each $z \neq 0$, we decompose $\chi_A(x, \xi)$ into three parts by selecting suitable disjoint subgroups of $(j, l) \in \mathbb{N} \times \mathbb{N}$ in (3.4) so that each has support lies respectively in

\begin{equation}
\Gamma_{z, I} = \{(x, \xi) \in \Gamma_z, |x| > 8 \cdot 2^{|z|}, |\xi| > 8 \cdot 2^{|z|}\}
\Gamma_{z, II} = \{(x, \xi) \in \Gamma_z, |x||\xi| > 8^2 \cdot 2^{|z|}, 8 < |\xi| < 4 \cdot 8 \cdot 2^{|z|}\}
\Gamma_{z, III} = \{(x, \xi) \in \Gamma_z, 64 < |x||\xi| < 16 \cdot 8^2 \cdot 2^{|z|}\}.
\end{equation}

Those partition functions are denoted by $\chi_{z, I}, \chi_{z, II}$ and $\chi_{z, III}$ respectively. Then we have the decomposition of $a_z(x, \xi), z \neq 0$ as

\begin{equation}
a_z(x, \xi) = \chi_{z, I}(x, \xi)a_z(x, \xi) + \chi_{z, II}(x, \xi)a_z(x, \xi) + \chi_{z, III}(x, \xi)a_z(x, \xi)
\end{equation}

def \quad = a_{z, I}(x, \xi) + a_{z, II}(x, \xi) + a_{z, III}(x, \xi)

and decomposition of $T_A$ as

\begin{equation}
T_A = T_0 + \sum_{z \neq 0} (T_{z, I} + T_{z, II} + T_{z, III}).
\end{equation}

We will see that $T_0$ is the sum of two Fourier integral operators and a smooth operator. Each operator $T_{z, I}$ is similar to $T_0$, each $T_{z, II}$ behaves like $T_{z, I}$ after change of variables and $T_{z, III}$ degenerates. We define $\Gamma_{0, A} = \{(x, \xi) \in \Gamma_0, |x| > 8, |\xi| > 8\}$ and

\begin{equation}
\begin{aligned}
\text{region I} &= \left( \bigcup_{z \neq 0} \Gamma_{z, I} \right) \cup \Gamma_{0, A}, \\
\text{region II} &= \bigcup_{z \neq 0} \Gamma_{z, II}, \quad \text{region III} = \bigcup_{z \neq 0} \Gamma_{z, III}.
\end{aligned}
\end{equation}

Let the angle spanned by $x$ and $\xi$ be $\theta_0$, we have $|x||\xi|\cos^2(\theta_0/2)\sin^2(\theta_0/2) > C_1 > 1$ on region I and II and $|x||\xi|\cos^2(\theta_0/2)\sin^2(\theta_0/2) < C_2$ on region III for fixed constants $C_1, C_2$.

(i). Estimate for $T_0 + \sum T_{z, I}$.

First we give the precise description of $a_0(x, \xi)$ and functions in (3.11). When $(x, \xi)$ lies in region I, we will calculate $a(x, \xi)$ given in (3.1) by using the stationary phase formula (Theorem 5.1 from [13] is recorded here). The calculation here is similar to that in [15]. We claim that on region I we have

\begin{equation}
a(x, \xi) = c_1 e^{-i|x||\xi|\sigma_+(x, \xi)} p_+(x, \xi) + c_2 e^{-i|x||\xi|\sigma_-(x, \xi)} p_-(x, \xi) + s(x, \xi)
\end{equation}

where

$$
\sigma_\pm(x, \xi) = \frac{1}{2}(\frac{x \cdot \xi}{|x||\xi|} \pm 1),
$$

$s \in SG^{-\infty, -\infty}$ is a SG symbol of the smooth operator and $p_{\pm} \in SG^{\gamma, \gamma-1} \subset SG^{0,0}$ are SG symbols of order $(0, 0)$ satisfying (2.31). Recall the SG symbols are defined in the notation subsection and $p(x, \xi) \in SG^{0,0}$ means

$$
|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha \beta} (x)^{0-|\alpha|}(\xi)^{0-|\beta|}.
$$
We remark that in this paper the term "symbol of order m" always means "SG symbols of order (m,m)". For our estimate, it is no harm to drop the symbol $s(x,\xi)$ in (3.13). Together with the construction of $a_0, a_{\alpha,\ell}, \alpha \neq 0$ in (3.8) and (3.11) we may write,

$$a_0(x,\xi) = c_1 e^{-i|x||\xi||\sigma_0(x,\xi) p_{00}(x,\xi) + c_2 e^{-i|x||\xi||\sigma_0(x,\xi) p_{0-}(x,\xi)}$$

$$a_{\alpha,\ell}(x,\xi) = c_1 e^{-i|x||\xi||\sigma_0(x,\xi) p_{00}(x,\xi) + c_2 e^{-i|x||\xi||\sigma_0(x,\xi) p_{0-}(x,\xi)}$$

where

$$p_{0\pm}(x,\xi) = \zeta \cdot \chi \cdot p_{\pm}(x,\xi), \quad p_{z,\pm}(x,\xi) = \zeta \cdot \chi_{z,\ell} \cdot p_{\pm}(x,\xi).$$

By the definitions of $\zeta, \chi\alpha$ and $\chi_{z,\ell}$, we see that $p_{0\pm}(x,\xi), p_{z,\pm}(x,\xi) \in SC^{0,0}$ satisfy (2.31) where $C_{\alpha,\beta}$ are independent of $z$.

Now we prove that (3.13) holds on region I. By parametrization

$$\begin{align*}
  x &= |x|(0,0,1) \\
  \xi &= |\xi|(\cos \varphi_0 \sin \theta_0, \sin \varphi_0 \sin \theta_0, \cos \theta_0) \\
  \omega &= (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta), \quad \theta \in [0, \pi/2], \varphi \in [0, 2\pi].
\end{align*}$$

the phase function in definition of $a(x,\xi)$ can be written as

$$(x \cdot \omega)(\xi \cdot \omega) = |x||\xi|((\cos \varphi_0 \sin \theta_0 \sin \varphi_0 \sin \theta_0 \cos \theta_0 \sin \theta) + \cos \theta_0 \cos \theta_0).$$

We regard $|x||\xi|$ as the parameter and define

$$\sigma(x,\xi;\omega) = \cos \varphi_0 \cos \theta_0 \sin \varphi_0 \cos \theta_0$$

be the phase function of integral (3.11) which is obvious smooth over $S^2$. In order to use the stationary phase formula, we need to find the critical point of the phase function $\sigma(x,\xi;\omega)$ and its Hessian over the critical point. It is known that we can calculate these quantities on the local coordinate or equivalently on manifold by using covariant derivatives (see, for example, [5]). We will use the later approach here. The parametrization (3.15) defines a mapping from $(\theta,\varphi)$ to sphere. Then $\{e_\theta = \frac{\partial}{\partial \theta}, e_\varphi = \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}\}$ is a basis on tangent bundle of sphere. Recall that the gradient and Hessian are given by the formulas,

$$\nabla_{S^2} := \left(\frac{\partial}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}\right)$$

and

$$H_{S^2} := \begin{bmatrix}
\frac{\partial^2}{\partial \theta^2} & \frac{\partial^2}{\sin \theta \partial \varphi} - \frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} & \frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} \\
\frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} & \frac{\partial^2}{\sin^2 \theta \partial \varphi^2} & \frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} \\
\frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} & \frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi} & \frac{\cos \theta}{\sin \theta \partial \varphi} \frac{\partial}{\partial \varphi}
\end{bmatrix}.$$

Applying (3.17) to $|x||\xi|\sigma(x,\xi;\omega)$, we see that the critical points of this function satisfy the formula

$$(x \cdot \omega)\xi + (\xi \cdot \omega)x = 2(x \cdot \omega)(\xi \cdot \omega)\omega, \quad \omega \in S^2.$$\)

Note $b(\cos \theta)\sin \theta$ vanishes when $x \cdot \omega = 0$, thus we only have to look for critical points $\omega$ such that $x \cdot \omega \neq 0$. By relation (3.19), we have $\xi \cdot \omega \neq 0$. Hence $\omega$ belongs to the plane generated by $x$ and $\xi$. A simple calculation yields four critical points $\omega_+, \omega_-, -\omega_+, -\omega_-$ where

$$\omega_+ = (|\xi||x|\xi + |x||\xi|)^{-1}$$

$$\omega_- = (|\xi||x| - |x||\xi|)^{-1}$$

and

$$\begin{align*}
\omega_+ = (|\xi||x| + |x||\xi|)^{-1} \\
\omega_- = (|\xi||x| - |x||\xi|)^{-1}
\end{align*}$$
if $x$ and $\xi$ are not collinear. We also express them as

$$
\omega_+ = (\cos \varphi_0 \sin(\theta_0/2), \sin \varphi_0 \sin(\theta_0/2), \cos(\theta_0/2)),
$$

(3.21)

$$
\omega_- = (\cos(\pi + \varphi_0) \sin((\pi - \theta_0)/2), 
\sin(\pi + \varphi_0) \sin((\pi - \theta_0)/2), 
\cos((\pi - \theta_0)/2)).
$$

If $x$ and $\xi$ are collinear, the four critical points are $\pm \frac{\xi}{|\xi|} = \pm \frac{x}{|x|}$ and $\pm \omega^\perp$ where $\omega^\perp$ is a unit vector orthogonal to $x$ (and $\xi$). We observe that if $\frac{|x\cdot\xi|}{|x||\xi|} = 0$ or 1, then for all these four critical points, $\frac{|x\cdot\omega|}{|x|}$ is 0 or 1. The case $x \cdot \omega = 0$ has been excluded, while we may disregard the case $|x\cdot\omega|/|x| = 1$ since $b(\cos \theta) \sin \theta = 0$ at $\theta = 0$ or $\pi$. Thus we only have to consider the critical points of the form (3.20). Furthermore, the support of $b(\cos \theta)$ is in $[0, \pi/2]$, thus we only have to consider the contribution from two critical points, $\omega_+$ and $\omega_-$ when applying the stationary phase formula.

The Hessians of the phase function $\sigma(x, \xi; \omega)$ at $\omega_+$ and $\omega_-$ are

$$
\sigma''(x, \xi; \omega_\pm) = \begin{bmatrix} -2 & 0 \\ 0 & -2\sigma_{\pm}(x, \xi) \end{bmatrix}
$$

where

$$
\sigma_{\pm}(x, \xi) = \frac{1}{2} \left( \frac{x \cdot \xi}{|x||\xi|} \pm 1 \right) = \sigma(x, \xi; \omega_{\pm}).
$$

Since

$$
|\det \sigma''(x, \xi; \omega_{\pm})| = 2^2 |\sigma_{\pm}|, \quad \text{sgn } \sigma''(\omega_{\pm}) = \mp 2,
$$

the critical points do not degenerate.

To calculate $a(x, \xi)$ we introduce a partition of unity on $S^2$ such that

$$
\kappa_i \in C^\infty(S^2), \quad 0 \leq \kappa_i \leq 1, \quad \sum_{i=1}^3 \kappa_i = 1
$$

and $\kappa_i \equiv 1$, $i = 1, 2$ in a neighborhood of $\omega_{\pm}$. By integration by parts, we see that

$$
(|x||\xi|)^\gamma \int_{\omega \in S^2} b(\cos \theta) e^{-i(x\cdot\omega)(\xi \cdot \omega)} \kappa_3(\omega) d\Omega(\omega) \in SG^{-\infty, -\infty}
$$

is the symbol of smooth operator, since the support of $b(\cos \theta)\kappa_3(\omega)$ is compact, the phase function has no critical point on this support. This gives the last term of (3.13). Note that if one extend $b(\cos \theta)$ evenly from $[0, \pi/2]$ to $[0, \pi]$, it again suffices to consider two critical points $\omega_+, \omega_-$ since $(x\cdot\omega)(\xi \cdot \omega)$ and $b(\cos \theta)$ are even in $\omega$, the contributions to $a(x, \xi)$ from the critical points $-\omega_+, -\omega_-$ are identical respectively to those from $\omega_+, \omega_-$. We see from (3.15), (3.20) that if the angle spanned by $x$ and $\xi$ is $\theta_0, 0 < \theta_0 < \pi$, then the angles spanned by $x$ and $\omega_{\pm}$ are $\theta_0/2$ and $(\pi - \theta_0)/2$ respectively. Applying
stationary phase formula to (3.24) with \( \kappa_3 \) replaced by \( \kappa_i \), \( 1 \leq i \leq 2 \), we have
\[
a(x, \xi) - s(x, \xi) = \pi(|x||\xi|)^\gamma \det(|x||\xi|\sigma''(x, \xi, \omega_+))^{-\frac{1}{2}} e^{\frac{i\pi}{4} \text{sgn} \sigma''(\omega_+)} e^{-i|x||\xi| \sigma''(x, \xi, \omega_+)} \times
\]
\[
e^{-i|x||\xi| \sigma''(x, \xi, \omega_+)} \left\{ b(\cos(\frac{\theta_0}{2})) \sin(\frac{\theta_0}{2}) + O(|x|^{-1}|\xi|^{-1}) \right\}
\]
(3.25)
\[
+ \pi(|x||\xi|)^\gamma \det(|x||\xi|\sigma''(x, \xi, \omega_-))^{-\frac{1}{2}} e^{\frac{i\pi}{4} \text{sgn} \sigma''(\omega_-)} e^{-i|x||\xi| \sigma''(x, \xi, \omega_-)} \times
\]
\[
e^{-i|x||\xi| \sigma''(x, \xi, \omega_-)} \left\{ b(\cos(\frac{\pi}{2} - \frac{\theta_0}{2})) \sin(\frac{\pi}{2} - \frac{\theta_0}{2}) + O(|x|^{-1}|\xi|^{-1}) \right\}
\]
(3.26)
\[
e^{-i|x||\xi| \sigma''(x, \xi, \omega_-)} p_+(x, \xi) + e^{-i|x||\xi| \sigma'(-x, \xi, \omega_-)} p_-(x, \xi).
\]

Also we note that the first term of \( p_\pm(x, \xi) \) are respectively
\[
-\pi (|x||\xi|)^\gamma \det(|x||\xi|\sigma''(x, \xi, \omega_+))^{-\frac{1}{2}} b(\cos(\theta_0/2)) \sin(\theta_0/2)
\]
(3.26)
\[
= \pi (|x||\xi|)^\gamma \cos(\theta_0/2) \sin(\theta_0/2)
\]
\[
= -2^{-3/2} i \pi \sin(\theta_0/2) (|x||\xi|)^{\gamma-1},
\]
\[
\pi (|x||\xi|)^\gamma \det(|x||\xi|\sigma''(x, \xi, \omega_-))^{-\frac{1}{2}} b(\cos(\pi/2 - \theta_0/2)) \sin(\pi/2 - \theta_0/2)
\]
(3.27)
\[
= \pi (|x||\xi|)^\gamma \cos(\pi/2 - \theta_0/2) \sin(\pi/2 - \theta_0/2)
\]
\[
= 2^{-3/2} i \pi \cos(\theta_0/2) (|x||\xi|)^{\gamma-1}.
\]

The terms (3.26) and (3.27) are symbols of order 0 (more precisely \( \gamma - 1 \)) when \((x, \xi) \in \text{supp} \chi_A \) as the \( x \) or \( \xi \) derivative of \( \cos(\theta_0/2) \) or \( \sin(\theta_0/2) \) descends one order in \( x \) or \( \xi \) respectively. For example,
\[
\frac{\partial}{\partial \xi_j} \cos(\frac{\theta_0}{2}) = -\frac{1}{2} \sin(\frac{\theta_0}{2}) \frac{\partial}{\partial \xi_j} \theta_0
\]

where
\[
\frac{\partial}{\partial \xi_j} \theta_0 = \frac{\partial}{\partial \xi_j} \arccos\left(\frac{x \cdot \xi}{|x||\xi|}\right)
\]
\[
= -\frac{1}{\sqrt{1 - (x \cdot \xi/|x||\xi|)^2}} \frac{\partial}{\partial \xi_j} \left(\frac{x \cdot \xi}{|x||\xi|}\right)
\]
\[
= -\frac{1}{\sin \theta_0} \frac{x_j - \cos \theta_0 |x||\xi|}{|x||\xi|}.
\]

Using (3.13), we see that
\[
(3.28) \quad \frac{\partial}{\partial \xi_j} \theta_0 = \begin{cases}
-\cos \theta_0 \cos \phi_0 |\xi|^{-1}, & j = 1 \\
-\cos \theta_0 \sin \phi_0 |\xi|^{-1}, & j = 2 \\
-\sin \theta_0 |\xi|^{-1}, & j = 3
\end{cases}
\]
The big O of (3.25) can be calculated explicitly using stationary phase asymptotics (5.1). When \( k \geq 2 \), the \( k \)-th term of asymptotics have the form
\[
(3.29) \quad \cos(\theta_0/2) \sin(\theta_0/2) \frac{q_{\pm k}(\cos(\theta_0/2), \sin(\theta_0/2))}{(|x||\xi| \cos^2(\theta_0/2) \sin^2(\theta_0/2))^{k-1}}(|x||\xi|)^{\gamma-1}
\]
where \( q_{\pm k}(t, s) \) are polynomials of \((t, s)\). The remainder term of the stationary phase formula also has the form as above. On the region I, we have
\[
|x||\xi| \cos^2(\theta_0/2) \sin^2(\theta_0/2) > C_1 > 1.
\]
Using this fact, definition of region I together with \( \sin(\gamma-1) \) on region I. By the asymptotic summability of symbols, we can conclude that \( p_{\pm}(x, \xi) \) of (3.14) are symbols of order 0 on region I. Furthermore \( p_{\pm}(x, \xi) \) satisfy (2.33) on region I. We remark that in order to estimate the \( L^2 \) bounds of the operators defined on region I or II induced by (3.13), one can use only the first four terms of the asymptotic expansion of the stationary phase formula instead of full series. Since the operator induced by the remainder term has kernel of the form (3.29) with \( k = 5 \) which are integrable with respect to \( x \) for fixed \( \gamma \) and vice versa. Thus the boundedness of the operator given by the remainder term follows from Schur test.

Using (3.14), we begin to estimate \( T_0 \) and \( T_{1,1} \), that is to consider the operators of the form (3.3) with \( a_z \) replaced by the right hand side of (3.3). Define
\[
\psi_{\pm}(x, \xi) = x \cdot \xi - |x||\xi|\sigma_{\pm}(x, \xi) = \frac{1}{2}(x \cdot \xi \mp |x||\xi|).
\]
By Plancherel theorem, we need to show that the operators
\[
(3.30) \quad F_{z\pm} h(x) = \int e^{i\psi_{\pm}(x, \xi)} p_{z\pm}(x, \xi) h(\xi) d\xi, \quad z \in \mathbb{Z}
\]
are \( L^2 \) bounded and their norms form a convergent series. We note that
\[
(3.31) \quad |\det \partial_x \partial_{\xi} \left[ \psi_{\pm}(x, \xi) \right]| = \left| \det \frac{1}{2} \left[ I \mp \frac{x}{|x|} \otimes \frac{\xi}{|\xi|} \right] \right| = \left( \frac{1}{2} \right)^3 |1 \mp \cos \theta_0|.
\]
When \((x, \xi) \in \text{supp } p_{0\pm}(x, \xi) \subset \Gamma_{0,A}\), there exists constants \( C_1, C_2 \) such that
\[
(3.32) \quad 0 < C_1 < \left( \frac{1}{2} \right)^3 |1 \mp \cos \theta_0| < C_2.
\]
The phase functions \( \psi_{\pm}(x, \xi) \) of \( F_{0\pm} \) satisfy (2.33) and (2.34). Also \( p_{0\pm} \) satisfy (2.31). Thus we conclude that \( F_{0\pm} \) are \( L^2 \) bounded by Lemma (2.4).

For the estimates of \( F_{z\pm} \), \( z \neq 0 \), we note that \( 1 - \cos \theta_0 \) tends to 0 on the support of \( p_{z\pm}(x, \xi) \subset \Gamma_{z,1} \) as \( z \) tends to \( \infty \), has uniform lower and upper bounds for \( z < 0 \). On the other hand, \( 1 + \cos \theta_0 \) tends to 0 on the support of \( p_{z\pm}(x, \xi) \) as \( z \) tends to \( -\infty \), has uniform lower and upper bounds for \( z > 0 \). This symmetry and the form of the symbols \( p_{z\pm}(x, \xi) \) indicate that we only have to consider \( F_{n\pm} \) with \( n \in \mathbb{N} \). We note that the phase function \( \psi_{\pm}(x, \xi) \) of \( F_{n\pm} \) satisfies (2.33) of Lemma (2.3) and has uniform lower and upper bounds on \( \Gamma_{n\pm} \). The amplitude functions \( p_{n\pm} \) of \( F_{n\pm} \) enjoy (2.31) with the same \( C_{\alpha, \beta} \). Thus we may sum up \( F_{n\pm} \) to obtain a new operator defined on region I, then Lemma (2.4) implies that this operator is \( L^2 \) bounded.
For the estimate of the $L^2$ norms of $F_{n+}$, we need to introduce some notations. In [2.35] of the Lemma 2.4, the upper bound of the $L^2$ norm of $F$ is denoted $\mathcal{CQP}$. When we applying Lemma 2.4 to estimate the upper bounds of $L^2$ norms of $F_{n+}$, we should use $\mathcal{C}_n Q_n P_{n+}$ to indicate its dependence on $n+$. The proof of the result that $F_{n+}, n \in \mathbb{N}$ are $L^2$ bounded and their norms form a convergent series is inferred by the following three results. The first one is that $F_{1+}$ is $L^2$ bounded, then the ratio $\mathcal{C}_2+Q_2+P_{2+}/C_1+Q_1+P_{1+}$ for $F_{2+}$ and $F_{1+}$ is less than 1 and finally the calculation of above ratio works for any pair $F_{(n+1)+}$ and $F_{n+}$ with $n \in \mathbb{N}$.

The first result that $F_{1+}$ is $L^2$ bounded follows from Lemma 2.4 directly since $\psi_+$ satisfies conditions (2.33), (2.34) on supp $p_{1+} \subset \Gamma_1$ and $p_{1+}$ satisfies (2.31). Next we calculate the ratio $\mathcal{C}_2+Q_2+P_{2+}/C_1+Q_1+P_{1+}$. The constants $C_1$ and $C_2$ are determined by the constant $C$ in (2.41) which is from the constants $C$ of (2.41), (2.47) and (2.48). Note that we do not need to consider the constant $C$ in (2.45) by symmetry. The constants $C$ of (2.44) come from (2.40), i.e., the $C_1$ of (3.32). The $C_1$ on $\Gamma_{2,l}$ is about 1/4 of that on $\Gamma_{1,l}$. On the other hand, the second term of (3.21) and the parametrization (3.19) indicate that this 1/4 decrease only occurs along the third component of $\xi$ (or $x$) when we use (2.38) and integration by parts. The calculation (3.28) for $j = 3$ suggests that actually we have 1/2 decrease since $\sin \theta_0$ on $\Gamma_{2,l}$ is about 1/2 of that on $\Gamma_{1,l}$. By these observations, it is easy to check that the constants $C$ of (2.41) and (2.47) on $\Gamma_{2,l}$ is 16 times of that on $\Gamma_{1,l}$ after applying integration by parts 4 times. From the definition of $\Gamma_{1,l}, \Gamma_{2,l}$ and the form of symbols (3.26), (3.29), we see that

$$P_{2+} \leq \frac{1}{2} P_{1+}$$

without counting the decrease of $\sin \theta_0$ from $\Gamma_{1,l}$ to $\Gamma_{2,l}$ mentioned before. Also by definition $Q_{2+}$ is about 1/4 of $Q_{1+}$. Combining these facts together, we see that the ratio $\mathcal{C}_2+Q_2+P_{2+}/C_1+Q_1+P_{1+}$ is about 1/2 in this case. The other case is that $C$ in (2.41) comes from $C$ of (2.45), where the later is independent of $n+$. Thus the the ratio $\mathcal{C}_2+Q_2+P_{2+}/C_1+Q_1+P_{1+}$ is about 1/8 in this case. By the telescopic definition of $\Gamma_{n+}$, the above argument works for ratio $\mathcal{C}_{(n+1)+} Q_{(n+1)+} P_{(n+1)+}/C_{n+} Q_{n+} P_{n+}$ as well for all $n$ and we conclude that $T_0 + \sum_{z \neq 0} T_{z,l}$ is $L^2$ bounded.

(ii) Estimate for $T_{z,II}, z \neq 0$.

We should prove that the upper bound of $L^2$ norm of $T_{z,II}$ is no more than 2 times that of $T_{z,l}$, then the result follows from the estimates of $T_{z,l}$ before. The factor 2 comes from the fact that the definition of $\Gamma_{z,II}$ contains 2 pieces. We should prove that when $T_{z,II}$ is restricted to one of these piece, its upper bound of $L^2$ norm is bounded by that of $T_{z,l}$. Recall the definition

$$T_{z,II} h(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_{z,II}(x; \xi) \hat{h}(\xi) d\xi$$

where $a_{z,II}(x; \xi) = \chi_{z,II}(x; \xi) a(x, \xi)$ and $\chi_{z,II}(x; \xi)$ has support on the region

$$\Gamma_{z,II} = \{(x, \xi) \in \Gamma_z, |x||\xi| > 8^2 \cdot 2^{2|z|}, 8 < |\xi| < 4 \cdot 8 \cdot 2^{|z|}\}$$

$$\cup \{(x, \xi) \in \Gamma_z, |x||\xi| > 8^2 \cdot 2^{2|z|}, 8 < |x| < 4 \cdot 8 \cdot 2^{|z|}\}.$$
We can write $\chi_{z,II} = \chi_{z,II}^1 + \chi_{z,II}^2$ according to each piece of $\Gamma_{z,II}$ as

$$\chi_{z,II}^1(x, \xi) \overset{\text{def}}{=} \zeta_z(x, \xi) \sum_{(j,l) \in I^1_{z,II}} \chi_j(x) \chi_l(\xi),$$

$$\chi_{z,II}^2(x, \xi) \overset{\text{def}}{=} \zeta_z(x, \xi) \sum_{(j,l) \in I^2_{z,II}} \chi_j(x) \chi_l(\xi),$$

where

$$I^1_{z,II} = \{(j, l) : j + l \geq 2|z| + 2, 1 \leq l \leq |z|\}$$

and

$$I^2_{z,II} = \{(j, l) : j + l \geq 2|z| + 2, 1 \leq j \leq |z|\}.$$ 

Following this, we have the decomposition

$$a_{z,II}(x, \xi) = a_{z,II}^1(x, \xi) + a_{z,II}^2(x, \xi) \overset{\text{def}}{=} \chi_{z,II}^1(x, \xi) a(x, \xi) + \chi_{z,II}^2(x, \xi) a(x, \xi)$$

and $T_{z,II} = T_{z,II}^1 + T_{z,II}^2$ accordingly. First we estimate $L^2$ norm of $T_{z,II}^1$. By Plancherel theorem, the $L^2$ norm of $T_{z,II}^1$ is equivalent to that of the operator

$$T_{z,II} u(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_{z,II}^1(x, \xi) u(\xi) d\xi.$$ 

We should estimate it using change of variables and Lemma 2.4. Write

$$T_{z,II} u(x) = \sum_{(j,l) \in I^1_{z,II}} T_{z,j,l} u(x) = \sum_{(j,l) \in I^1_{z,II}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_{z,j,l}(x, \xi) u(\xi) d\xi$$

with

$$a_{z,j,l}(x, \xi) = \zeta_z(x, \xi) \chi_j(x) \chi_l(\xi) a(x, \xi).$$

From Lemma 2.4, we know that $L^2$ norm of $T_{z,II}$ is determined by $\|T_{z,j,l} T_{z(k,m)}\|_{L^2 \to L^2}$ and $\|T_{z(j,l)} T_{z(k,m)}\|_{L^2 \to L^2}$ where $(j, l), (k, m) \in I^1_{z,II}$. By symmetry, it suffices to illustrate how to estimate $\|T_{z(j,l)} T_{z(k,m)}\|_{L^2 \to L^2}$ by change of variables.

We begin with a useful observation. Since $1 \leq l \leq |z|$, there exists $N \in \mathbb{N}, N \leq |z|$ so that $l + N = |z| + 1$. Let $x = 2^N \hat{x}, \xi = 2^{-N} \hat{\xi}$ and note $x \cdot \omega / |x| = \hat{x} \cdot \omega / |\hat{x}|$ for $\omega \in S^2_+$. Thus we have

$$a(x, \xi) = (|x||\xi|)^{\gamma} \int_{\omega \in S^2_+} e^{-i(x \cdot \omega)(\xi \cdot \omega)} b(\cos \theta) d\Omega(\omega)$$

$$= (|\hat{x}||\hat{\xi}|)^{\gamma} \int_{\omega \in S^2_+} e^{-i(\hat{x} \cdot \omega)(\hat{\xi} \cdot \omega)} b(\cos \theta) d\Omega(\omega)$$

$$= a(\hat{x}, \hat{\xi}).$$

From $(x \cdot \xi) / |x||\xi| = (\hat{x} \cdot \hat{\xi}) / |\hat{x}||\hat{\xi}|$, we have

$$a_{z,j,l}(x, \xi) = \zeta_z(x, \xi) \chi_j(x) \chi_l(\xi) a(x, \xi)$$

$$= \zeta_z(\hat{x}, \hat{\xi}) \chi_{j-N}(\hat{x}) \chi_{l+N}(\hat{\xi}) a(\hat{x}, \hat{\xi})$$

$$= a_{z(j-N,l+N)}(\hat{x}, \hat{\xi}).$$

Since

$$j + l \geq 2|z| + 2,$$
we have \( j - N \geq |z| + 1 \), \( l + N \geq |z| + 1 \) which means that new variables \((\tilde{x}, \tilde{\xi})\) lie in the region I. Thus we can calculate \( a_{z(j-N,l+N)}(\tilde{x}, \tilde{\xi})\) as (i) and conclude that they are the form of the second line of (3.14).

Note that
\[
T_z(j,l)T^*_z(k,m) u(x) = \int K_{z(j,l),(k,m)}(x,y)u(y)dy,
\]
where
\[
K_{z(j,l),(k,m)}(x,y) = \int e^{i(x-\xi-y)\zeta} a_{z(j,l)}(x,\xi) a_{z(k,m)}(y,\xi) d\xi.
\]
The kernel non-vanishes only \( l = m - 1, l = m \) or \( l = m + 1 \). Then the observation in the previous paragraph indicates that we can apply change of variables according to \( a_{z(k,m)} \), so that
\[
K_{z(j,l),(k,m)}(x,y) = \int e^{i(x-\xi-y)\zeta} a_{z(j-N,l+N)}(\tilde{x}, \tilde{\xi}) a_{z(k-N,m+N)}(\tilde{y}, \tilde{\xi}) 2^{-3N} d\tilde{x},
\]
where \( j - N, l + N, k - N, m + N \) are all greater than \( |z| + 1 \). This means the kernel \( K_{z(j-N,l+N),(k-N,m+N)}(\tilde{x}, \tilde{\xi}) \) can be calculated explicitly as \( a_{z(j-N,l+N)} \), \( a_{z(k-N,m+N)} \) have explicitly formulas by the calculation in (i). Note
\[
T_z(j,l)T^*_z(k,m) u(2^N \tilde{x}) = \int K_{z(j-N,l+N),(k-N,m+N)}(\tilde{x}, \tilde{y}) u(2^N \tilde{y}) d\tilde{y}.
\]
This means
\[
\|T_z(j,l)T^*_z(k,m)\|_{L^2(L^2)} \leq \|T_z(j-N,l+N)T^*_z(k-N,m+N)\|_{L^2(L^2)},
\]
and the later can be estimated as we did in Lemma 2.4 whose quantity is controlled by \( z, |(j-N)-(k-N)| = |j-k| \) and \( |(l+N)-(m+N)| = |l-m| \), independent of \( N \).

Thus \( \|T_z(j,l)\|_{L^2(L^2)} \leq \|T_z(j,l)\|_{L^2(L^2)} \leq \|T_z,j\|_{L^2(L^2)} \leq \|T_z,l\|_{L^2(L^2)} \). The proof of \( \|T^2_z,j\|_{L^2(L^2)} \leq \|T^2_z,l\|_{L^2(L^2)} \) is essential the same, we skip it.

(iii). Estimate for \( T_{z,III}, z \neq 0 \).

By Plancherel theorem, it is equivalent to write
\[
T_{z,III}h(x) = \int e^{i\zeta \cdot \xi} a_{z,III}(x,\xi) h(\xi) d\xi
= \int K_{z,III}(x,\xi) h(\xi) d\xi,
\]
where \( a_{z,III}(x,\xi) = \chi_{z,III}(x,\xi) a_z(x,\xi) = \chi_{z,III}(x,\xi) \zeta(x,\xi) a(x,\xi) \).

Recall that when \( (x,\xi) \in \Gamma_{z,III} \) we have
\[
|\xi| \cos^2(\theta_0/2) \sin^2(\theta_0/2) < C_2,
\]
and this means the stationary phase formula is not a good tool to calculate \( a_{z,III}(x,\xi) \). On the other hand the calculation of \( a_{z,l}(x,\xi) \) still gives some hint, thus we continue using the nations there. Recall
\[
a(x,\xi) = (|x|\xi)^{\gamma} \int_{\omega \in S^2_+} e^{-i(\xi \cdot \omega) b(\cos \theta) d\Omega(\omega)}
\]
and $\omega_{\pm}$ are the critical points of phase function $(x \cdot \omega) (\xi \cdot \omega)$. Use (3.15), (3.21) and consider a smooth partition on $S^2$ as

$$\pi_i \in C^\infty (S^2), \quad 0 \leq \pi_i \leq 1, \quad \sum_{i=1}^3 \pi_i = 1$$

where $\pi_1 \equiv 1$ when

$$(\theta, \varphi) \in (\varphi_0 - C_2(|x||\xi|)^{-1/2}, \varphi_0 + C_2(|x||\xi|)^{-1/2}) \times [0, C_2(|x||\xi|)^{-1/2})$$

and $\pi_1 \equiv 0$ when

$$(\theta, \varphi) \notin (\varphi_0 - 2C_2(|x||\xi|)^{-1/2}, \varphi_0 + 2C_2(|x||\xi|)^{-1/2}) \times [0, 2C_2(|x||\xi|)^{-1/2}).$$

And $\pi_2 \equiv 1$ when

$$(\theta, \varphi) \in (\varphi_0 - C_2(|x||\xi|)^{-1/2}, \varphi_0 + C_2(|x||\xi|)^{-1/2}) \times (\pi/2 - C_2(|x||\xi|)^{-1/2}, \pi/2]$$

and $\pi_2 \equiv 0$ when

$$(\theta, \varphi) \notin (\varphi_0 - 2C_2(|x||\xi|)^{-1/2}, \varphi_0 + 2C_2(|x||\xi|)^{-1/2}) \times (\pi/2 - 2C_2(|x||\xi|)^{-1/2}, \pi/2].$$

From the relation (3.34) and constructions of $\pi_1, \pi_2$, we know that $\omega_{\pm}$ must lie in the supports of $\pi_1, \pi_2$ respectively and the distances of $\omega_{\pm}$ to the boundary of support are of order $|x||\xi|^{-1/2}$. Hence there exists $C > 0$ such that for $\omega \in \text{supp} \pi_3$ we have

$$|\nabla S^2[(x \cdot \omega)(\xi \cdot \omega)]| \geq C(|x||\xi|)^{1/2}$$

where $\nabla S^2$ is the gradient on the sphere given by (3.17). Therefore the contribution from the support of $\pi_3$ to $a(x, \xi)$ is bounded by $|x||\xi|^{-n}$ for any $n \in \mathbb{N}$. On the supports of $\pi_1, \pi_2$ we have $\cos \theta \sin \theta \leq C(|x||\xi|)^{-1/2}$. Hence their contributions to $a(x, \xi)$ is bounded by $C(|x||\xi|)^{-1/2+\gamma(-1)}$. In summary, we have

$$|a(x, \xi)| \leq C(|x||\xi|)^{-1/2},$$

and thus

$$|K_{z, I, I I}(x, \xi)| \leq C \chi_{z, I, I I}(x, \xi) \zeta_z(x, \xi)|x||\xi|^{-1/2}.$$

Let $z = \pm 1$ temporarily. Let $p(\xi) = |\xi|^{-3/2}$ and $q(x) = |x|^{-3/2}$. For any fixed $x_0$ ($|x_0| > 8$), using spherically coordinate, we have

$$\int |K_{z, I, I I}(x_0, \xi)| p(\xi) d\xi \leq C_2 |x_0|^{-3/2} = C_2 q(x_0).$$

Similarly, For any fixed $\xi_0$ ($|\xi_0| > 8$) we have

$$\int |K_{z, I, I I}(x, \xi_0)| q(\xi) dx \leq C_2 |\xi_0|^{-3/2} = C_2 p(\xi_0).$$

where $C_1, C_2$ are the same as those in (3.37) by symmetry. Thus $T_{z, I, I I}$ are $L^2$ bounded by Schur test. To see that $T_{z, I, I I}, z \neq \pm 1$ are also $L^2$ bounded and their norms form a convergent series we need to track $C_1, C_2$ of (3.37) as $|z|$ varies. When
|z| goes from \( n, n \in \mathbb{N} \) to \( n + 1 \), the \( \zeta \) part of \((3.36)\) indicates that \( C_1 \) of the former is about \( 1/4 \) of that for later. On the other hand, the \( \chi_{z,II} \) part of \((3.36)\) or definition of \( \Gamma_{z,III} \) indicates the upper limit in the integration of the second line of \((3.37)\) increases 4 times from \( n \) to \( n + 1 \). Therefore \( C_2 \) of the former is the same as that for the later. To get a convergent series of \( C_2 \), we may adjust the definitions of \( \Gamma_{z,I}, \Gamma_{z,II}, \Gamma_{z,III} \) in \((3.10)\) by replacing \( 2^{|z|}, 2^2|z| \) with \( 2^{(1-\delta)|z|}, 2^{2(1-\delta)|z|} \) for some small positive \( \delta < 1/14 \). This adjustment will not affect the result of estimates in (i) and (ii). This is because as we remarked in the end of the paragraph after \((3.29)\), we may drop the lower order terms of the stationary phase asymptotics and the fact that we use integration by parts only 4 times in Lemma \( 2.4 \). Hence the ratio \( P_{(n+1)^+}/P_{n+} < 2^{-(1-12\delta)} \) after this adjustment. Also we see that \( C_2 \) for \( n + 1 \) is \( 2^{-2\delta} \) times of that for \( n \) after the adjustment and the result follows.

**Part II. Estimate of \( T_B \)**

The proof of the result that \( T_B \) is \( L^2 \) bounded is essential the reminiscence of Part I. Recall that

\[
T_B h(x) = \sum_{j=1}^{2} T_{B,j} h(x) = \sum_{j=1}^{2} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \chi_{B,j}(x, \xi) a(x, \xi) \hat{h}(\xi) d\xi
\]

where \( \chi_{B,1}(x, \xi) \subset \{|x||\xi| > 64, |x| > 8, |\xi| < 16\} \), \( \chi_{B,2}(x, \xi) \subset \{|x||\xi| > 64, |x| < 16\} \).

Similarly to \((3.10)\), we split operators \( T_{B,1}, T_{B,2} \) into sum of operators by decomposing the supports of \( \chi_{B,1}, \chi_{B,2} \) into cones according to \((3.7)\). Each cone is further split into three regions by letting \( z \in \mathbb{Z} \) and defining

\[
\Gamma_{z,I} = \{(x, \xi) \in \Gamma_z, |x| > 8 \cdot 2^{|z|}, |\xi| > 8 \cdot 2^{|z|}\},
\]

\[
\Gamma_{z,II} = \{(x, \xi) \in \Gamma_z, |x||\xi| > 8^2 \cdot 2^2|z|, |\xi| < 4 \cdot 8 \cdot 2^{|z|}\}
\cup \{(x, \xi) \in \Gamma_z, |x||\xi| > 8^2 \cdot 2^2|z|, |x| < 4 \cdot 8 \cdot 2^{|z|}\},
\]

\[
\Gamma_{z,III} = \{(x, \xi) \in \Gamma_z, 64 < |x||\xi| < 16 \cdot 8^2 \cdot 2^2|z|\}.
\]

Then we define

\[
\text{region I}_B = (\bigcup_z \Gamma_{z,I}) \bigcap (\text{supp } \chi_{B,1} \cup \text{supp } \chi_{B,2})
\]

\[
\text{region II}_B = (\bigcup_z \Gamma_{z,II}) \bigcap (\text{supp } \chi_{B,1} \cup \text{supp } \chi_{B,2})
\]

\[
\text{region III}_B = (\bigcup_z \Gamma_{z,III}) \bigcap (\text{supp } \chi_{B,1} \cup \text{supp } \chi_{B,2}).
\]

From the supports of \( \chi_{B,1}, \chi_{B,2} \) we know that on the region \( I_B \), only \( \Gamma_{\pm 1,I} \) has non-empty intersection with the set \( (\text{supp } \chi_{B,1} \cup \text{supp } \chi_{B,2}) \). Thus only two operators defined on region \( I_B \) have non-zero values. Also these two operators are \( L^2 \) bounded by the argument in (i) of Part I. For the operators defined on region \( II_B \), we note that the definition of \( \Gamma_{z,II} \) in \((3.38)\) is slightly different with that in \((3.10)\). The conditions \(|\xi| > 8\) in the first set and \(|\xi| > 8\) in the second set of the \( \Gamma_{z,II} \) in \((3.10)\) were removed now. However these two conditions in \((3.10)\) were used to emphasize the sets are part of support of \( \chi_A \) but not used in the proof of (ii) of Part I. Indeed, it is the condition \(|x||\xi| > 8^2 \cdot 2^2|z|\), i.e., \((3.33)\) ensures the argument in the proof
of (ii) works. Thus the operators defined on region \( II_B \) are \( L^2 \) bounded and their norms form a convergent series as before. Also the operators defined on region \( III_B \) can be treated exactly as (iii) of Part I and we finish the proof of Part II.

**Part III. Estimate of \( T_C \)**

By Plancherel Theorem, the \( L^2 \) boundedness of \( T_C \) is equal to that of

\[
\mathcal{T}h(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_C(x, \xi) h(\xi) d\xi.
\]

Recall that

\[
a_C(x, \xi) = a_{C,1}(x, \xi) + a_{C,2}(x, \xi)
\]

\[
= \chi_{C,1}(x, \xi)(|x| |\xi|)^\gamma \int_{\omega \in S^2} e^{-i(x \cdot \omega)(\xi \cdot \omega)} b(\cos \theta) d\Omega(\omega)
\]

\[
+ \chi_{C,2}(x, \xi)(|x| |\xi|)^\gamma \int_{\omega \in S^2} e^{-i(x \cdot \omega)(\xi \cdot \omega)} b(\cos \theta) d\Omega(\omega)
\]

where \( \text{supp} \chi_{C,j} \subset \{|x| |\xi| < 512, |x| > 8\} \) and \( \text{supp} \chi_{C,2} \subset \{|x| |\xi| < 512, |x| < 16\} \).

Write the operator \( \mathcal{T} \) as

\[
\mathcal{T}h(x) = \mathcal{T}_1 h(x) + \mathcal{T}_2 h(x)
\]

\[
= \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_{C,1}(x, \xi) h(\xi) d\xi + \int_{\mathbb{R}^3} e^{ix \cdot \xi} a_{C,2}(x, \xi) h(\xi) d\xi
\]

\[
= \int_{\mathbb{R}^3} K_1(x, \xi) h(\xi) d\xi + \int_{\mathbb{R}^3} K_2(x, \xi) h(\xi) d\xi
\]

It is clear from the support of \( \chi_{C,j} \) that \( K_j, i = 1, 2 \) satisfies

\[
|K_j(x, \xi)| \leq C \cdot \chi_{C,j}(x, \xi).
\]

We should employ the Schur test and consider two different sub-cases to prove that \( \mathcal{T} \) is \( L^2 \) bounded.

(a) boundedness of \( \mathcal{T}_1 \).

Let \( p(x) = (1 + |x|)^{-1} \) and \( q(\xi) = |\xi|^{-2} \). For any fixed \( |x_0| > 8 \), using spherically coordinate and (3.40), we have

\[
\int_{\mathbb{R}^3} |K_1(x_0, \xi)| q(\xi) d\xi \leq C_1 \int_0^{512|x_0|^{-1}} r^{-2} \cdot r^2 dr \\
\leq C_2 |x_0|^{-1} \leq C_3 p(x_0).
\]

And for any fixed \( |\xi_0| \leq 64 \), we have

\[
\int_{\{x \in \mathbb{R}^3, |x| > 8\}} |K_1(x, \xi_0)| p(x) dx \leq C_1 \int_8^{512|\xi_0|^{-1}} r^{-1} \cdot r^2 dr \\
\leq C_2 |\xi_0|^{-2} = C_2 q(\xi_0).
\]

By Schur test, we conclude the \( L^2 \) boundedness of this case.

(b) boundedness of \( \mathcal{T}_2 \).
Let $p(x) = |x|^{-1}$ and $q(\xi) = (1 + |\xi|)^{-2}$. For any fixed $|x_0| < 16$, using spherical coordinate and (3.40), we have
\[
\int_{\mathbb{R}^3} |K_2(x_0, \xi)| q(\xi) d\xi \leq C_1 \int_0^{512|x_0|^{-1}} r^{-2} \cdot r^2 dr \leq C_2 |x_0|^{-1} = C_2 p(x_0).
\]
For any fixed $|\xi_0| > 32$, we have
\[
\int_{\mathbb{R}^3} |K_2(x, \xi_0)| p(x) dx \leq C_1 \int_0^{512|\xi_0|^{-1}} r^{-1} \cdot r^2 dr \leq C_2 |\xi_0|^{-2} \leq C_3 q(\xi_0).
\]
And for any fixed $|\xi_0| \leq 32$, we have
\[
\int_{\{|x| \leq 16\}} |K_2(x, \xi_0)| p(x) dx \leq C_1 \int_0^{|x|} r^{-1} \cdot r^2 dr \leq C_2 \leq C_3 q(\xi_0).
\]
By Schur test, we conclude that $T$ is bounded.

\[\square\]

4. Proof of Lemma 2.1 and Lemma 2.2 and Corollary 1.5

With the help of the proof of Lemma 3.1, we can now prove Lemma 2.1 and Lemma 2.2 easily.

**Proof of Lemma 2.1**

Following (2.25), (3.1) and (3.2), we write

\[(\tau_{-v_*} \circ T \circ \tau_{v_*}) h(v) = |v - v_*|^\gamma \int_{\omega \in S^2_+} b(\cos \theta) h(v - ((v - v_*) \cdot \omega) \omega) d\Omega(\omega)\]

\[(4.1) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{iv \xi} \hat{A}(v - v_*, \xi) |\xi|\gamma (|\xi|^{-\gamma} \hat{h}(\xi)) d\xi\]

where

\[(4.2) \cos \theta = (v - v_*, \omega) / |v - v_*|, \ 0 \leq \theta \leq \pi / 2,\]

\[(4.3) \hat{A}(v - v_*, \xi) = |v - v_*|^\gamma \int_{S^2_+} b(\cos \theta) e^{-i((v - v_*) \cdot \omega) \xi = \omega)} d\Omega(\omega)\]

Let

\[(4.4) a(v - v_*, \xi) = \hat{A}(v - v_*, \xi) |\xi|\gamma.\]

Then we are reduced to proving the $L^2$ boundedness of the operator

\[(4.5) \int_{\mathbb{R}^3} e^{iv \xi} a(v - v_*, \xi) \hat{h}(\xi) d\xi\]

when it is regarded as an operator of $v$ variable with $v_*$ as a parameter and vice versa. Also we need to check the bounds are independent of the parameters.
The calculation of $a(x,\xi)$ in the proof of Lemma 3.1 can be applied to that of $a(v-v_*,\xi)$ with $v-v_*$ playing the role of $x$. We may summarize that calculation of $a$ in Lemma 3.1 as the following two cases. The first case is that $a$ has explicitly representation as (3.13) (may need change of variables) which induces the F.I.O.s and their $L^2$ bounds are obtained by Lemma 2.4. The second case is that $a(v-v_*,\xi)$ has explicitly upper bound as (3.35) or $a(v-v_*,\xi)$ is bounded as Part III of the proof of Lemma 3.1. Then the operator induced by that can be estimated by Schur test directly. The second case of $a(v-v_*,\xi)$ for our current estimates clearly give us the same result as the Lemma 3.1 no matter $v$ or $v_*$ is the variable of the operator. When the first case occurs, the operator (4.5) is the sum of the operators of the following two forms,

$$
(4.6) \int_{\mathbb{R}^3} e^{i\nu\cdot\xi} e^{-\frac{i}{2}(|v-v_*|\xi|v-v_*|\xi|\xi|\xi)} p_{\pm}(v-v_*,\xi) \hat{h}(\xi)d\xi
$$

For any fixed $v_*$, we write (4.6) as

$$
(4.7) \int_{\mathbb{R}^3} e^{\frac{i}{2}(v-v_*)\cdot\xi\pm(v-v_*)\cdot\xi} p_{\pm}(v-v_*,\xi) e^{i\nu\cdot\xi} \hat{h}(\xi)d\xi
$$

$$
= \int_{\mathbb{R}^3} e^{\frac{i}{2}(v-v_*)\cdot\xi\pm(v-v_*)\cdot\xi} p_{\pm}(v-v_*,\xi) \tau_{v_\pm}h(\xi)d\xi,
$$

and any fixed $v$, we write (4.6) as

$$
(4.8) \int_{\mathbb{R}^3} e^{\frac{i}{2}(v-v_*)\cdot\xi\pm(v-v_*)\cdot\xi} p_{\pm}(v-v_*,\xi) \tau_{v_\pm}h(\xi)d\xi.
$$

From the facts that the translation is $L^2$ invariant, (4.7) and (4.8) have the same forms as (3.30), we conclude that the no matter $v$ or $v_*$ is the variable of the operator, the proof of Lemma 3.1 still works here for first case.

**Proof of Lemma 2.2.** First we prove the estimates (2.11) and (2.12). Comparing (2.1) and (2.9) and recalling the proof of Lemma 2.3 and Lemma 3.1 we see that $T_s$ also satisfies (2.20). By the proof of Lemma 2.1 above we know that $T_s$ also satisfies (2.2) and (2.3). Thus we only need to prove that when $T_s$ is restricted to the low Fourier frequency ($|\xi| < C$ in (2.25)), denoted it by $T_sL$, then the followings hold

$$
\sup_{v_*} \| (\tau_{v_\pm} \circ T_sL \circ \tau_{v_*})h(v) \|_{L^2(v)} \leq C\| h \|_{L^2},
$$

$$
\sup_{v_*} \| (\tau_{v_\pm} \circ T_sL \circ \tau_{v_*})h(v) \|_{L^2(v_\pm)} \leq C\| h \|_{L^2}.
$$

Since $(\tau_{v_\pm} \circ T_sL \circ \tau_{v_*})h(v)$ has representation (4.1) with $v-v_*$ and $\xi$ being restricted to compact sets respectively, these operators are clearly $L^2$ bounded.

We turn to the proof of (2.13) and (2.14). We denote

$$
|x|^2 = (1-\delta)(|x|) \cdot |x|^\gamma
$$

where $\delta$ is given by (2.3). Then we have

$$
H^\nu_{w_s}(v, v_*) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\nu\cdot\xi} A^w_{\nu}(v-v_*, \xi) \hat{h}(\xi) d\xi
$$
where
\[ A^w_F(v - v_*, \xi) = \frac{|v - v_*|}{\langle v \rangle^\gamma} \int_{S^2_x} b(\cos \theta) e^{-i(v - v_*) \omega(\xi - \omega)} d\Omega(\omega) \]
(4.9)
\[ \text{def} = (1 - s)(|v - v_*|) A^w(v - v_*, \xi). \]

From the definitions of \( \chi, (2.28) \) and (3.4), we have the decomposition
\[ A^w_F(v - v_*, \xi) = (\chi_A(v - v_*, \xi) + \chi_B, (v - v_*, \xi) + \chi_C, (v - v_*, \xi) ) A^w(v - v_*, \xi) \]
\[ \text{def} = A^w_F(v - v_*, \xi) + A^w_B(v - v_*, \xi) + A^w_C(v - v_*, \xi) \]
and the corresponding decomposition
\[ H^w_F(v, v_*) = H^w_A(v, v_*) + H^w_B(v, v_*) + H^w_C(v, v_*) \]

By the supports of \( \chi_A, \chi_B, \) and \( \chi_C, \) we see that the estimates (2.13) and (2.14) follow from
\[ \sup \| H^w_A(v, v_*) \|_{L^2(v)} \leq C \| H \|_{\gamma} \]
(4.10)
\[ \sup \| H^w_B(v, v_*) \|_{L^2(v)} \leq C \| H \|_{L^2} \]
\[ \sup \| H^w_C(v, v_*) \|_{L^2(v)} \leq C \| H \|_{L^2} \]
and the other three estimates which switches the roles of \( v \) and \( v_* \) above estimates. By the argument of Lemma 2.1 we know that it suffices to consider above three inequalities out of six. From the support of \( \chi_A \), we know that the proof of the first inequality of (4.10) may follow from the first case in the proof of Lemma 2.1.

Compare the definition of \( A^w_F \) and \( A \) give by (4.3) and note that the additional factor \( \langle v \rangle^{-\gamma} \langle v_* \rangle^{-\gamma} \) does not affect the proof of first case there, i.e., the estimates for \( p \pm (v - v_*, \xi) \) of (4.6) during the almost argument remain the same after adding this factor, thus we conclude that result. For the second inequality of (4.10), we note that its proof may follow from Part II of the proof of Lemma 3.1. The main idea there is that after using change of variables the operator can be written as F.I.O.s on dyadic units. In the proof of Lemma 2.1 such argument works as well. Because \( a(v - v_*, \xi) \) give by (4.4) contains two components which are both invariant under change of variables. More precisely, \( |v - v_*| \langle \xi \rangle^\gamma = |2^{-N}(v - v_*)| \langle \xi \rangle^\gamma = |\tilde{v} - \tilde{v}_*| \langle \xi \rangle^\gamma \) for any \( N \in \mathbb{N} \) and
\[ \int_{S^2_x} b(\cos \theta) e^{-i((v - v_*) \omega(\xi - \omega)) d\Omega(\omega) = \int_{S^2_x} b(\cos \theta) e^{-i((\tilde{v} - \tilde{v}_*) \omega(\xi - \omega)) d\Omega(\omega).} \]
When we applying change of variables argument to \( A^w_B(v - v_*, \xi) \), the integration part remains invariant as (4.11). While the factor
\[ \frac{|v - v_*|^\gamma}{\langle v \rangle^\gamma \langle v_* \rangle^\gamma} = \frac{2^{N} |\tilde{v} - \tilde{v}_*|^\gamma}{2^{N} |\tilde{v} - \tilde{v}_*|^\gamma} \]
combines the estimate of the later of (4.11) again induces the F.I.O.s of order 0. Thus the arguments from Part II of the proof of Lemma 3.1 and Lemma 2.1 give us the second inequality of (4.10). To prove the third inequality of (4.10), we note that \( |v - v_*| / \langle v \rangle^{-\gamma} \langle v_* \rangle^{-\gamma} \) is bounded. Hence
\[ |A^w_C(v - v_*, \xi)| \leq C \chi_C, (v - v_*, \xi). \]
The above is parallel to (3.40), thus the argument of Part III of Lemma 3.1 gives us the desired inequality.

**Proof of Corollary 1.5.** For the homogeneous estimates, we note that in the proof of Lemma 3.1 the operators \( T_A \) is of order \( \gamma - 1 \) instead of 0. This means that if the reduction (3.1) of \( T \) is \( T_A \) instead, we can replace \( \gamma \) with 1. Since the estimates \( T_{B1} \) follows from \( T_A \), it enjoy the same property. And the operator \( T_{C1} \) is restricted to the low Fourier frequency, it is in our favor to rise the exponent. With these observations, we can derive the desired Lemma parallel to Lemma 2.1 to conclude the result.

The proof for inhomogeneous estimates is similarly, the key is that the first inequality of (4.10) can have \( -H^{-1} \) in the left hand side due to the operator is defined by \( \chi_A \).

\[ \square \]

5. Some tools

One of the most powerful tools in estimating the oscillatory integral

\[ I(\Lambda, f) = \int_{\mathbb{R}^n} e^{i\Lambda f(y)} u(y) dy, \]

for large \( \Lambda \) is the following lemma of stationary phase asymptotics. There are several versions used widely, here we only record one of these which is from Theorem 7.7.5 of Hörmander [13]. We use the notation \( D_j = -i\partial_j \).

**Theorem 5.1** (Stationary phase asymptotics). Let \( K \subset \mathbb{R}^n \) be a compact set, \( X \) an open neighborhood of \( K \) and \( k \) a positive number. If \( u \in C^2_k(\mathbb{R}^n) \), \( f \in C^{3k+1}(X) \) and \( \text{Im } f \geq 0 \) in \( X \), \( \text{Im } f(y_0) = 0 \), \( f'(y_0) = 0 \), \( \det f''(y_0) \neq 0 \), \( f' \neq 0 \) in \( K \setminus \{y_0\} \) then

\[
|I - e^{i\Lambda f(y_0)} (\det(\Lambda f''(y_0)/2\pi i))^{-1/2} \sum_{j<k} \Lambda^{-j} L_j u|\
\leq C\Lambda^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u| , \ \Lambda > 0
\]

(5.1)

Here \( C \) is bounded when \( f \) stays in a bounded set in \( C^{3k+1}(X) \) and \( |y - y_0|/|f'(y)| \) has a uniform bound. With

\[ g_{y_0}(y) = f(y) - f(y_0) - \langle f''(y_0)(y - y_0), y - y_0 \rangle/2 \]

which vanish of third order at \( x_0 \) we have

\[
L_j u = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} (f''(y_0)^{-1} D, D)^\nu (g_{y_0}^\nu u)(y_0)/\mu! \nu!
\]

which is a differential operator of order \( 2j \) acting on \( u \) at \( y_0 \). The coefficients are rational homogeneous functions of degree \(-j \) in \( f''(y_0), \ldots, f^{2j+2}(y_0) \) with denominator \( (\det f''(y_0))^3 \). In every term the total number of derivatives of \( u \) and \( f'' \) is at most \( 2j \).
In proof of lemma 2.4, we need the following Schur test lemma. See also Sogge’s book [21] Theorem 0.3.1 for the related Young’s inequality.

**Lemma 5.2** (Schur test lemma). Let $X, Y$ be two measurable spaces. Let $T$ be an integral operator with the non-negative Schwartz kernel, i.e.
$$ T f(x) = \int_Y K(x, y) f(y) \, dy. $$
If there exist functions $p(x) > 0$ and $q(x) > 0$ and numbers $\omega, \beta > 0$ such that
$$ \int_Y K(x, y) q(y) \, dy \leq \omega p(x) $$
for almost all $x$ and
$$ \int_X K(x, y) p(x) \, dx \leq \beta q(y) $$
for almost all $y$. Then $T$ is a continuous operator $L^2 \rightarrow L^2$ with the operator norm
$$ \|T\|_{L^2 \rightarrow L^2} \leq \sqrt{\omega \beta}. $$

We also need the following Coltar-Stein lemma. (See Stein’s book [22])

**Lemma 5.3** (Coltar-Stein lemma). Assume a family of $L^2$ bounded operators $\{T_j\}_{j \in \mathbb{Z}^n}$ and a sequence of positive constants $\{\gamma(j)\}_{j \in \mathbb{Z}^n}$ satisfy
$$ \|T_i^* T_j\|_{L^2 \rightarrow L^2} \leq \{\gamma(i-j)\}^2, \quad \|T_j^* T_j\|_{L^2 \rightarrow L^2} \leq \{\gamma(i-j)\}^2 $$
and
$$ M = \sum_{j \in \mathbb{Z}^n} \gamma(j) < \infty. $$
Then the operator $T = \sum_{j \in \mathbb{Z}^n} T_j$ satisfies
$$ \|T\|_{L^2 \rightarrow L^2} \leq M. $$

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