Note on 2d binary operadic harmonic oscillator

Eugen Paal and Jüri Virkepu

Department of Mathematics, Tallinn University of Technology
Ehitajate tee 5, 19086 Tallinn, Estonia
E-mails: eugen.paal@ttu.ee and jvirkepu@staff.ttu.ee

Abstract
It is explained how the time evolution of the operadic variables may be introduced. As an example, a 2-dimensional binary operadic Lax representation of the harmonic oscillator is found.

2000 MSC: 18D50, 70G60
Keywords: Operad, harmonic oscillator, operadic Lax pair

1 Introduction
It is well known that quantum mechanical observables are linear operators, i.e. the linear maps $V \to V$ of a vector space $V$ and their time evolution is given by the Heisenberg equation. As a variation of this one can pose the following question [7]: how to describe the time evolution of the linear algebraic operations (multiplications) $V^\otimes n \to V$. The algebraic operations (multiplications) can be seen as an example of the operadic variables [2, 3, 4, 5].

When an operadic system depends on time one can speak about operadic dynamics [7]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of operadic variables may be given by operadic Lax equation. In [8] it was shown how the dynamics may be introduced in 2d Lie algebra. In the present paper, an operadic Lax representation for harmonic oscillator is constructed in general 2d binary algebras.

2 Operad
Let $K$ be a unital associative commutative ring, and let $C^n$ ($n \in \mathbb{N}$) be unital $K$-modules. For $f \in C^n$, we refer to $n$ as the degree of $f$ and often write (when it does not cause confusion) $\hat{f}$ instead of $\deg f$. For example, $(-1)^f \hat{=} (-1)^n$, $C^f \hat{=} C^n$ and $o_f \hat{=} o_n$. Also, it is convenient to use the reduced degree $|f| \hat{=} n - 1$. Throughout this paper, we assume that $\otimes \hat{=} \otimes_K$.

Definition 2.1 (operad (e.g. [2, 3])). A linear (non-symmetric) operad with coefficients in $K$ is a sequence $C \hat{=} \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions are held to be true.

1. For $0 \leq i \leq m - 1$ there exist partial compositions $o_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1})$, $|o_i| = 0$
(2) For all \( h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g \), the composition (associativity) relations hold,

\[
(h \circ_i f) \circ_j g = \begin{cases} 
(-1)^{|f||g|}(h \circ_{i+|g|} g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\
 h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i + |f|, \\
(-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|.
\end{cases}
\]

(3) Unit \( I \in C^1 \) exists such that

\[
I \circ_0 f = f = f \circ_i I, \quad 0 \leq i \leq |f|
\]

In the second item, the first and third parts of the defining relations turn out to be equivalent.

**Example 2.2** (endomorphism operad [2]). Let \( V \) be a unital \( K \)-module and \( \mathcal{E}_V^n := \mathcal{E}_{\text{End}_V}^n := \text{Hom}(V^\otimes n, V) \). Define the partial compositions for \( f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g \) as

\[
f \circ_i g := (-1)^{|i||g|} f \circ (\text{id}_V^\otimes i \otimes g \otimes \text{id}_V^\otimes (|f|-i)), \quad 0 \leq i \leq |f|
\]

Then \( \mathcal{E}_V := \{ \mathcal{E}_V^n \}_{n \in \mathbb{N}} \) is an operad (with the unit \( \text{id}_V \in \mathcal{E}_V^1 \)) called the endomorphism operad of \( V \).

Therefore, algebraic operations can be seen as elements of an endomorphism operad.

Just as elements of a vector space are called vectors, it is natural to call elements of an abstract operad operations. The endomorphism operads can be seen as the most suitable objects for modelling operadic systems.

# 3 Gerstenhaber brackets and operadic Lax pair

**Definition 3.1** (total composition [2,3]). The total composition \( \bullet : C^f \otimes C^g \to C^{f+g} \) is defined by

\[
f \bullet g = \sum_{i=0}^{|f|} f \circ_i g \in C^{f+g}, \quad |\bullet| = 0
\]

The pair \( \text{Com} C = \{ C, \bullet \} \) is called the composition algebra of \( C \).

**Definition 3.2** (Gerstenhaber brackets [2,3]). The Gerstenhaber brackets \( [\cdot, \cdot] \) are defined in \( \text{Com} C \) as a graded commutator by

\[
[f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f = -(-1)^{|f||g|} [g, f], \quad |[\cdot, \cdot]| = 0
\]

The commutator algebra of \( \text{Com} C \) is denoted as \( \text{Com} \cdot C = \{ C, [\cdot, \cdot] \} \). One can prove that \( \text{Com} \cdot C \) is a graded Lie algebra. The Jacobi identity reads

\[
(-1)^{|f||h|}[[f, g], h] + (-1)^{|g||f|}[[g, h], f] + (-1)^{|h||g|}[[h, f], g] = 0
\]

Assume that \( K = \mathbb{R} \) and operations are differentiable. The dynamics in operadic systems (operadic dynamics) may be introduced by the
**Definition 3.3** (operadic Lax pair [7]). Allow a classical dynamical system to be described by the evolution equations

\[
\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n
\]

An *operadic Lax pair* is a pair \((L, M)\) of homogeneous operations \(L, M \in C\), such that the above system of evolution equations is equivalent to the *operadic Lax equation*

\[
\frac{dL}{dt} = [M, L] = M \bullet L - (-1)^{|M||L|} L \bullet M
\]

Evidently, the degree constraint \(|M| = 0\) gives rise to ordinary Lax pair \([6, 1]\).

### 4 Operadic harmonic oscillator

Consider the Lax pair for the harmonic oscillator:

\[
L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

Since the Hamiltonian is

\[
H(q, p) = \frac{1}{2} (p^2 + \omega^2 q^2)
\]

it is easy to check that the Lax equation

\[
\dot{L} = [M, L] = ML - LM
\]

is equivalent to the Hamiltonian system

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q
\]

If \(\mu\) is a homogeneous operadic variable one can use the above Hamilton’s equations to obtain

\[
\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p}
\]

Therefore, the linear partial differential equation for the operadic variable \(\mu(q, p)\) reads

\[
p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = M \bullet \mu - \mu \bullet M
\]

By integrating one gains sequences of operations called the *operadic (Lax representations of) harmonic oscillator*.

### 5 Example

Let \(A = \{V, \mu\}\) be a binary algebra with operation \(xy = \mu(x \otimes y)\). We require that \(\mu = \mu(q, p)\) so that \((\mu, M)\) is an operadic Lax pair, i.e the operadic Lax equation

\[
\dot{\mu} = [M, \mu] = M \bullet \mu - \mu \bullet M, \quad |\mu| = 1, \quad |M| = 0
\]

is equivalent to the Hamiltonian system of the harmonic oscillator.
Let $x, y \in V$. By assuming that $|M| = 0$ and $|\mu| = 1$, one has

$$M \bullet \mu = \sum_{i=0}^{0} (-1)^{|\mu|} M \circ_i \mu = M \circ_0 \mu = M \circ \mu$$

$$\mu \bullet M = \sum_{i=0}^{1} (-1)^{|M|} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M = \mu \circ (M \otimes \text{id}_V) + \mu \circ (\text{id}_V \otimes M)$$

Therefore, one has

$$\frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My)$$

Let $\dim V = n$. In a basis $\{e_1, \ldots, e_n\}$ of $V$, the structure constants $\mu_{jk}^i$ of $A$ are defined by

$$\mu(e_j \otimes e_k) \equiv \mu_{jk}^i e_i, \quad j, k = 1, \ldots, n$$

In particular,

$$\frac{d}{dt}(e_j e_k) = M(e_j e_k) - (Me_j)e_k - e_j(Me_k)$$

By denoting $M e_i = M_i^s e_s$, it follows that

$$\mu_{jk}^i = \mu_{jk}^s M^s_i - M^j_i \mu_{sk}^i - M^k_i \mu_{js}^i, \quad i, j, k = 1, \ldots, n$$

In particular, one has

**Lemma 5.1.** Let $\dim V = 2$ and $M \equiv (M_i^j) \equiv \frac{\omega}{2} \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. Then the 2-dimensional binary operadic Lax equations read

$$\begin{cases}
\mu_{11}^1 = -\frac{\omega}{2} (\mu_{11}^1 + \mu_{21}^1 + \mu_{12}^1), \\
\mu_{12}^1 = -\frac{\omega}{2} (\mu_{12}^1 + \mu_{22}^1 - \mu_{12}^1), \\
\mu_{21}^1 = -\frac{\omega}{2} (\mu_{21}^1 + \mu_{21}^1 + \mu_{21}^1), \\
\mu_{22}^1 = -\frac{\omega}{2} (\mu_{22}^1 - \mu_{12}^1 - \mu_{21}^1)
\end{cases}$$

For the harmonic oscillator, define its auxiliary functions $A_{\pm}$ and $D_{\pm}$ by

$$\begin{cases}
A_+^2 + A_-^2 = 2\sqrt{2}H \\
A_+^2 - A_-^2 = 2p \\
A_+ A_- = \omega q
\end{cases}$$

Then one has the following

**Theorem 5.2.** Let $C_\beta \in \mathbb{R}$ ($\beta = 1, \ldots, 8$) be arbitrary real-valued parameters, $M \equiv \frac{\omega}{2} \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ and

$$\begin{cases}
\mu_{11}^1(q, p) = C_5 A_- + C_6 A_+ + C_7 D_- + C_8 D_+ \\
\mu_{12}^1(q, p) = C_1 A_+ + C_2 A_- - C_7 D_+ + C_8 D_- \\
\mu_{21}^1(q, p) = -C_1 A_- - C_2 A_- - C_3 A_+ - C_4 A_+ - C_5 A_- - C_7 D_+ + C_8 D_- \\
\mu_{22}^1(q, p) = -C_3 A_- + C_4 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{11}^2(q, p) = C_3 A_+ + C_4 A_- - C_7 D_+ + C_8 D_- \\
\mu_{12}^2(q, p) = C_1 A_- - C_2 A_+ + C_3 A_- - C_4 A_+ + C_5 A_- - C_7 D_- - C_8 D_+ \\
\mu_{21}^2(q, p) = -C_1 A_+ - C_2 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{22}^2(q, p) = -C_5 A_+ + C_6 A_- + C_7 D_+ - C_8 D_- 
\end{cases}$$

Then $(\mu, M)$ is a 2-dimensional binary operadic Lax pair of the harmonic oscillator.
Idea of proof. Denote
\[
\begin{align*}
G_\pm^{\omega/2} & \doteq \dot{A}_\pm \pm \frac{\omega}{2} A_\pm \\
G_\pm^{3\omega/2} & \doteq \dot{D}_\pm \pm \frac{3\omega}{2} D_\pm
\end{align*}
\]
Define the matrix
\[
\Gamma = (\Gamma^\beta_\alpha) \doteq 
\begin{bmatrix}
0 & G_+^{\omega/2} & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 \\
0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 & 0 & -G_+^{\omega/2} & G_+^{\omega/2} & 0 \\
0 & 0 & -G_+^{\omega/2} & -G_-^{\omega/2} & G_+^{\omega/2} & G_-^{\omega/2} & 0 & 0 \\
G_+^{\omega/2} & 0 & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & 0 & -G_-^{\omega/2} \\
G_-^{\omega/2} & 0 & G_-^{\omega/2} & 0 & 0 & G_+^{\omega/2} & 0 & -G_+^{\omega/2} \\
G_+^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_+^{3\omega/2} & G_+^{3\omega/2} & G_-^{3\omega/2} \\
G_-^{3\omega/2} & G_-^{3\omega/2} & G_+^{3\omega/2} & G_+^{3\omega/2} & G_-^{3\omega/2} & G_+^{3\omega/2} & G_-^{3\omega/2} & -G_+^{3\omega/2}
\end{bmatrix}
\]
Then it follows from Lemma 5.1 that the 2-dimensional binary operadic Lax equations read
\[
C_\beta \Gamma^\beta_\alpha = 0, \quad \alpha = 1, \ldots, 8
\]
Since the parameters $C_\beta$ are arbitrary, the latter constraints imply $\Gamma = 0$. Thus one has to consider the following differential equations
\[
G_\pm^{\omega/2} = 0 = G_\pm^{3\omega/2}
\]
By direct calculations one can show that
\[
G_\pm^{\omega/2} = 0 \iff \begin{cases}
\dot{p} = -\omega^2 q \\
\dot{q} = p
\end{cases} \iff G_\pm^{3\omega/2} = 0 \quad \Box
\]

Acknowledgement

The research was in part supported by the Estonian Science Foundation, Grant 6912. More expanded version of the present paper will be published in [9].

References

[1] O. Babelon, D. Bernard, and M. Talon. Introduction to Classical Integrable Systems. Cambridge Univ. Press, 2003.
[2] M. Gerstenhaber. The cohomology structure of an associative ring. Ann. of Math. 78 (1963), 267-288.
[3] M. Gerstenhaber, A. Giaquinto, and S. D. Schack. Algebras, bialgebras, quantum groups, and algebraic deformations. Contemp. Math. 134 (1992), 51-92.
[4] L. Kluge and E. Paal. On derivation deviations in an abstract pre-operad. Comm. Algebra, 29 (2001), 1609-1626.
[5] L. Kluge, E. Paal, and J. Stasheff. Invitation to composition. Comm. Algebra, 28 (2000), 1405-1422.
[6] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. Comm. Pure Applied Math. 21 (1968), 467-490.
[7] E. Paal. Invitation to operadic dynamics. J. Gen. Lie Theory Appl. 1 (2007), 57-63.
[8] E. Paal and J. Virkepu. Note on operadic harmonic oscillator. Rep. Math. Phys. (to be published).
[9] E. Paal and J. Virkepu. 2d binary operadic Lax representation for harmonic oscillator, (in preparation).