Evolution of entanglement entropy in orbifold CFTs

Pawel Caputa\textsuperscript{1,2}, Yuya Kusuki\textsuperscript{2}, Tadashi Takayanagi\textsuperscript{2,3} and Kento Watanabe\textsuperscript{2}

\textsuperscript{1} Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden
\textsuperscript{2} Center for Gravitational Physics, Yukawa Institute for Theoretical Physics (YITP), Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{3} Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8582, Japan

E-mail: takayana@yukawa.kyoto-u.ac.jp

Received 4 February 2017, revised 7 April 2017
Accepted for publication 20 April 2017
Published 16 May 2017

Abstract

In this work we study the time evolution of the Rényi entanglement entropy for locally excited states created by twist operators in the cyclic orbifold $(T^2)^{n}/\mathbb{Z}_n$ and the symmetric orbifold $(T^2)^{n}/S_n$. We find that when the square of its compactification radius is rational, the second Rényi entropy approaches a universal constant equal to the logarithm of the quantum dimension of the twist operator. On the other hand, in the non-rational case, we find a new scaling law for the Rényi entropies given by the double logarithm of time \( \log \log t \) for the cyclic orbifold CFT.

Keywords: entanglement entropy, conformal field theory, quantum quenches

(Some figures may appear in colour only in the online journal)

1. Introduction

The most fundamental quantity which characterizes the degrees of freedom in conformal field theories (CFTs) is the thermal entropy, which can be computed universally in two dimensions thanks to the celebrated Cardy formula \cite{Cardy1986}. Entanglement entropy (EE) provides a more general probe of CFTs \cite{Calabrese2004, Calabrese2005} and can even capture various dynamical processes in them. In particular, as pioneered by Calabrese and Cardy, quantum quenches provide very important classes of excited states in CFTs, and beautiful results on the time evolution of entanglement entropy have been derived \cite{Calabrese2005b, Calabrese2006, Calabrese2007, Calabrese2008} in two dimensions. A quantum quench is triggered by a sudden shift of the Hamiltonian at a specific time. This shift can happen either globally or locally,
and we talk about global quench or local quench, respectively. To analyze quantum quenches in higher dimensional CFTs, holographic entanglement entropy [8–13] also provides a useful tool [14–16].

There is another interesting class of excited states in CFTs which is simple enough to be computed analytically. These are locally excited states obtained by a local operator $O(x)$ acting on the vacuum in a given CFT at the time $t = 0$, introduced in [17] for the purpose of the computation of entanglement entropy. The state at time $t$ is explicitly written as

$$|\psi\rangle = \mathcal{N} \cdot e^{-iHt} \cdot e^{-i\epsilon H} \cdot O(x_*)|0\rangle,$$

where $x_*$ represents the position of insertion of the operator, $\epsilon$ is a UV regularization of the local operator and $\mathcal{N}$ is a normalization factor so that $\langle \psi | \psi \rangle = 1$.

By applying the replica method, we can calculate the time evolution of entanglement entropy and its generalization called Rényi entanglement entropy (Rényi EE). To define these quantities we trace out a subsystem $B$, and define a reduced density matrix $\rho_B$ for the subsystem $A$, which is the complement of $B$. In this work, we will simply set $A$ to be a half space.

The $m$th Rényi entropy $S_A^{(m)}$ is defined by

$$S_A^{(m)} = -\frac{1}{m-1} \log \text{Tr} [\rho_B^m].$$

The limit $m \to 1$ defines the (von-Neumann) entanglement entropy $S_A$. For our excited state (1.1), the computation of $S_A^{(m)}$ is equivalent to that of the $2m$-point function on the $(m$ times) replicated space [17, 18]. Our main focus will be on the difference $\Delta S_A^{(m)}$ between the entropy for a given excited state and the vacuum state so that the area law UV divergences are cancelled.

In previous works, several interesting features of $\Delta S_A^{(m)}$ have been worked out for excited states in CFTs of the form (1.1), and our primary interest in this paper is to proceed more in this direction for two-dimensional CFTs. First, it was found that the Rényi EE growth $\Delta S_A^{(m)}$ approaches a finite value at a late time in free CFTs in any dimensions [17–21] and in (two-dimensional) rational CFTs (RCFTs) [22–26]. On the other hand, the holographic result [27, 28] for two-dimensional CFTs shows the logarithmic growth $\Delta S_A^{(m)} \sim \frac{c}{6} \log t$ under the time evolution. This holographic behavior was reproduced from a CFT computation by utilizing the spectrum gap and the known behavior of conformal blocks in the large central charge limit [29] (see also [30, 31] for further generalizations to finite temperature).

For two-dimensional CFTs, these previous results cover two extremal cases: rational CFTs and holographic CFTs. Note that the latter are expected both to be strongly coupled and to have large central charges, so that they are dual to classical gravity on AdS$_3$. One may naively think that the behavior of such time evolutions depends on whether the CFT is integrable or chaotic. This speculation raises a question: do we only have two possible behaviors of $\Delta S_A^{(m)}$ for any two-dimensional CFTs, i.e. (i) approaching a finite constant (as in RCFTs) and (ii) growing logarithmically (as in holographic CFTs)? This question motivates us to study integrable CFTs which are not rational. For this purpose we would like to study $\Delta S_A^{(m)}$ for a class of solvable CFTs defined by the sigma model whose target space is the cyclic orbifold:

$$(T^2)^n/\mathbb{Z}_n,$$

where $T^2 = S^1 \times S^1$ is the $c = 2$ CFT defined by two compact bosons $X_1$ and $X_2$, both of which are compactified on the same radius $R$. The $\mathbb{Z}_n$ action is defined by shifting $n$ copies of the two dimensional torus $T^2$ successively. We will choose the primary operator $O$ in (1.1), which creates the local excitation, to be the twist operator $\sigma_n$. Thanks to analytical results
by Calabrese, Cardy and Tonni [32], we have an analytical expression for $\Delta S_{A}^{(m)}$ in this CFT model.

When $R^2$ is rational, the $c = 2$ CFT becomes a rational CFT. Moreover, as we will see later, the cyclic orbifold CFT (1.3) also becomes rational for any $n$. However, if $R^2$ is irrational, these CFTs are irrational. Therefore, this offers an example of integrable but irrational CFTs. As we will show in this paper, the 2nd Rényi EE $\Delta S_{A}^{(2)}$ in this irrational CFT actually shows new behavior under the time evolution, which is different from (i) and (ii).

A similar statement is true for the symmetric orbifold CFT defined by a sigma model whose target space is given by

$$T^2/n/S_n,$$

where $S_n$ is the symmetric group. When the square of the compactification radius $R^2$ is rational (or irrational), this symmetric orbifold CFT is also rational (irrational). Note that it is only when $n = 2$ that (1.3) and (1.4) are equivalent. This class of CFTs is also motivated by the fact that typical examples of AdS$_3$/CFT$_2$ are given by symmetric orbifold CFTs of the form $M^n/S_n$ for various choices of 2d CFT $M$, though we need to deform them exactly by marginal operators to reach a CFT which has a classical gravity dual. Refer to [33] for an interesting generalization of entanglement entropy (called entwinement) of the ground state in a symmetric orbifold CFT and its connection to AdS$_3$/CFT$_2$. In this paper, we will indirectly compute $\Delta S_{A}^{(m)}$ for the excited state created by the twist operator in the rational case using the connection to the quantum dimensions [22], which in general leads to different results than those for the cyclic orbifold.

This paper is organized as follows: In section two, we review the calculation of the (2nd) Rényi entanglement entropy in two-dimensional CFTs and useful results in our symmetric orbifold CFTs. In section three, we present our results for Rényi EE when the cyclic orbifold CFT becomes rational and irrational respectively. In section four, we discuss the computation of Rényi EE in the symmetric orbifold CFT. In section five, we re-interpret the results in terms of mutual information for light-like separated intervals. In section six, we summarize our conclusions. In appendix A, we present a summary of the computation of quantum dimensions. In appendix B, we explain the details of the computation of a determinant.

2. $\Delta S_{A}^{(2)}$ and four-point functions

We begin by reviewing the computations of the second Rényi entropy $\Delta S_{A}^{(2)}$ [22] and the four-point functions [32] in cyclic orbifold CFTs in a way that is convenient for our later analysis.

2.1. 2nd Rényi EE from a four-point function

We would like to compute the growth of the 2nd Rényi entanglement entropy $\Delta S_{A}^{(2)}$ for an excited state of the form (1.1). We describe the 2d Euclidean space $\mathbb{R}^2$ by a complex coordinate $(w, \bar{w}) = (x + i t_E, x - i t_E)$, where $t_E$ is the Euclidean time and $x$ is the space coordinate.

We choose the location of the operator insertion to be $x_* = -l < 0$ at $t = 0$ and to be smeared by $\epsilon$. We take the subsystem $A$ to be half of the space $x > 0$. To calculate $\Delta S_{A}^{(2)}$, we replicate
the $w$-plane with two sheets, glue them along $A$ and uniformize them using the conformal map $w = z^2$, as in figure 1. As showed in [22], we can calculate $\Delta S_A^{(2)}$ for the excited state (1.1) from the four-point function of the operator $O$. The relevant four-point function in our CFT has the following structure:

$$\langle O(z_1, \bar{z}_1)\bar{O}(z_2, \bar{z}_2)O(z_3, \bar{z}_3)\bar{O}(z_4, \bar{z}_4)\rangle = |z_{13}z_{24}|^{-4\Delta}|z|^{-4\Delta}1 - z|^{-4\Delta}F_O(z, \bar{z}), \quad (2.1)$$

where $z$ is the cross ratio $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

The growth of the 2nd Rényi entanglement entropy for $t > l$ is given by

$$\Delta S_A^{(2)} = -\lim_{(z, \bar{z}) \to (1,0)} \log F_O(z, \bar{z}), \quad (2.2)$$

where the limit is understood more precisely as follows: for an infinitesimally small $\epsilon$, we have (assuming $t \gg l$)

$$z \simeq 1 - \frac{\epsilon^2}{4t^2}, \quad \bar{z} \simeq \frac{\epsilon^2}{4t^2}. \quad (2.3)$$

In 2d rational CFTs, we can take the $\epsilon \to 0$ limit directly and we end up with

$$\Delta S_A^{(2)} = \log d_O, \quad (2.4)$$

where $d_O$ is the quantum dimension of the primary operator $O$ [22]. This is defined by the ratio of elements of the modular S-matrix

$$d_O = \frac{S_{00}}{S_{00}}, \quad (2.5)$$

where 0 denotes the identity sector. Moreover, a more general analysis shows that for any $m$, the $m$th Rényi entropy growth $\Delta S_A^{(m)}$ also takes the same value

$$\Delta S_A^{(m)} = \log d_O, \quad (2.6)$$

as proved in [22].

In this paper, we focus on the cyclic orbifold CFTs given by (1.3) with the central charge $c = 2n$ and we choose the primary operator $O$ to be the twist operator

$$O(z, \bar{z}) = \sigma_n(z, \bar{z}), \quad (2.7)$$
where $\sigma_n$ is the twisted operator for the cyclic transformation $X_{1,2}^{(i)} \rightarrow X_{1,2}^{(i+1)}$ for $i = 1, 2, \cdots, n$.

The conformal dimension of $\sigma_n$ is given by $\Delta_n = \frac{1}{12}(n - 1/n)$.

Its four-point function takes the form:

$$
\langle \sigma_n(z_1, \xi_1) \sigma_n(z_2, \xi_2) \sigma_n(z_3, \xi_3) \sigma_n(z_4, \xi_4) \rangle = \langle z_1 z_2 z_3 z_4 \rangle \left| z_1 \right|^{-4\Delta} \left| z_2 \right|^{-4\Delta} \left| 1 - z_1 \right|^{-4\Delta} \left| 1 - z_2 \right|^{-4\Delta} F_n(z, \bar{z}),
$$

(2.8)

where $z$ is the cross ratio $z = \frac{z_1 z_2}{z_3 z_4}$. This way, in the states excited by twist operators, the growth of the 2nd Rényi EE for $i > i$ is found from

$$
\Delta S_{2,i} = - \lim_{(z; \bar{z}) \rightarrow (1, 0)} \log F_n(z, \bar{z}).
$$

(2.9)

To fix our conventions, we denote the radius of $T^2$ as $R$. As in [32], we introduce the parameter $\eta = R^2$, where we choose the action of the free scalar which describes $T^2$ as

$$
S = \frac{1}{4\pi} \int \! dx \! d\bar{x} \left( \partial_x \phi \right)^2.
$$

(2.10)

Our convention is then such that $\eta = 1$ corresponds to the self-dual radius and $\eta = 1/2$ is equivalent to a Dirac fermion.

### 2.2. Four-point functions in $(T^2)^n/\mathbb{Z}_n$

Here we summarize the known expression for the four-point function in the cyclic orbifold CFT $(T^2)^n/\mathbb{Z}_n$ following [32, 35] (for $n = 2$, see [36]). As in (2.8), the correlator is characterized by the function $F_n(z, \bar{z})$, which is expressed as follows [32, 35]:

$$
F_n(z, \bar{z}) \equiv \frac{2^{n-1} \eta^{n-1}}{\prod_{k=1}^{n-1} \Gamma_{k/n}(z, \bar{z})} \cdot \Theta(0|\eta\Gamma)^2,
$$

(2.11)

where the $2(n - 1) \times 2(n - 1)$ symmetric matrix $\Gamma$ is defined by

$$
\Gamma = \begin{pmatrix}
i\bar{\Omega} & -\Lambda/2 \\
-\Lambda^T/2 & i\Omega
\end{pmatrix},
$$

(2.12)

and the $p$-dimensional theta function is defined as

$$
\Theta(0|\Gamma) = \sum_{m \in \mathbb{Z}^p} e^{i\eta m^T \cdot \Gamma \cdot m}.
$$

(2.13)

We also introduced

$$
\Gamma_{k/n}(z, \bar{z}) = \delta_{k/n}(z) \bar{\delta}_{k/n}(1 - \bar{z}) + \bar{\delta}_{k/n}(z) \delta_{k/n}(1 - \bar{z}).
$$

(2.14)

where

$$
\delta_{k/n}(z) = 2 F_1(k/n, 1 - k/n, 1, z).
$$

(2.15)

Note the identities $\delta_{0/0}(0) = 1$ and $\delta_0(z) = 1$.

The $(n - 1) \times (n - 1)$ matrices $\Delta$, $\Omega$ and $\bar{\Omega}$ are defined by

---

5 In a recent work [34], similar four-point functions of twist operators were considered for the analysis of the Aharonov–Bohm effect on entanglement entropy in 2d CFT.

6 We set $g = 1$ in the notation of [32]. In other words, our convention corresponds to $\alpha' = 1$ in string theory.
\[ \Lambda_{r,s} = \frac{4}{n} \sum_{k=0}^{n-1} b_k \sin \left( \frac{\pi k}{n} \right) \cdot \sin \left( \frac{2\pi k}{n} (r - s + 1/2) \right), \]

\[ \Omega_{r,s} = \frac{2}{n} \sum_{k=0}^{n-1} a_k \sin \left( \frac{\pi k}{n} \right) \cdot \cos \left( \frac{2\pi k}{n} (r - s) \right), \]

\[ \tilde{\Omega}_{r,s} = \frac{2}{n} \sum_{k=0}^{n-1} \tilde{a}_k \sin \left( \frac{\pi k}{n} \right) \cdot \cos \left( \frac{2\pi k}{n} (r - s) \right), \] (2.16)

where \( r, s = 1, \ldots, n - 1 \) and

\[ a_k = \frac{2f_{k/n}(1 - z)f_{k/n}(1 - \bar{z})}{I_{k/n}(z, \bar{z})}, \]

\[ \tilde{a}_k = \frac{2f_{k/n}(z)f_{k/n}(\bar{z})}{I_{k/n}(z, \bar{z})}, \]

\[ b_k = \frac{f_{k/n}(z)f_{k/n}(1 - \bar{z}) - f_{k/n}(1 - z)f_{k/n}(\bar{z})}{I_{k/n}(z, \bar{z})}. \] (2.17)

The overall normalization of (2.11) is chosen such that

\[ F_n(0, 0) = 1. \] (2.18)

Also, the four-point function \( F_n(z, \bar{z}) \) manifestly satisfies the channel duality relation

\[ F_n(z, \bar{z}) = F_n(1 - z, 1 - \bar{z}), \] (2.19)

which is the invariance under the transformation \( z \leftrightarrow 1 - z, \bar{z} \leftrightarrow 1 - \bar{z} \). Under this transformation, the theta function \( \Theta(0|\Gamma) \) is itself invariant.

### 2.3. Another expression

It is possible to write the four-point function \( F_n(z, \bar{z}) \) in a manifestly invariant form under \( \eta \leftrightarrow 1/\eta \). By performing the Poisson resummation, we can derive the transformation of the theta function \( \Theta(0|\eta \Gamma) \) under \( \eta \leftrightarrow 1/\eta \) \( [35, 37] \)

\[ \Theta(0|\eta \Gamma) = \eta^{-(n-1)} \cdot \Theta(0|\Gamma/\eta). \] (2.20)

Applying this formula to (2.11), we can obtain the following interesting relation

\[ \frac{F_n(z, \bar{z})}{F_n^{(\eta=1)}(z, \bar{z})} = \frac{\Theta(0|\eta \Gamma) \cdot \Theta(0|\Gamma/\eta)}{(\Theta(0|\Gamma))^{2}}, \] (2.21)

where \( F_n^{(\eta=1)}(z, \bar{z}) \) is the function \( F_n(z, \bar{z}) \) at the self-dual point \( \eta = 1 \)

\[ F_n^{(\eta=1)}(z, \bar{z}) = \frac{2^{n-1}}{\prod_{k=1}^{n-1} I_{k/n}(z, \bar{z})} \cdot \Theta(0|\Gamma)^{2}. \] (2.22)

Also note the special property in which when \( z = \bar{z} = x \) (\( x \) is a real number), we simply find

\[ F_n^{(\eta=1)}(x, x) = 1. \] (2.23)
The expression (2.21) manifestly shows that $F_n(z, \bar{z})$ is invariant under the T-duality $\eta \leftrightarrow 1/\eta$ and the $\eta$ dependence of $F_n(z, \bar{z})$ comes from the ratio (2.21).

### 2.4. Generalization to two different radii

When the two radii of $T^2$ are different, we can generalize our previous result as follows:

$$F_n(z, \bar{z}) = \frac{2^{n-1}}{\prod_{k=1}^{n-1} R_k / \pi} \cdot \prod_{q=1,2} \left[ \eta_q^{\frac{\pi}{2^q}} \cdot \Theta(0|\eta_q \Gamma) \right] ,$$  

(2.24)

where $\eta_q = R_q^2$ ($q = 1, 2$). The overall normalization is fixed such that $F_n(0, 0) = 1$ as in the previous case ($\eta = \eta_1 = \eta_2$).

The four-point function manifestly satisfies the invariance under $z \leftrightarrow 1 - z, \bar{z} \leftrightarrow 1 - \bar{z}$ and also under $\eta_1 \leftrightarrow \eta_2$. By using the Poisson resummation formula, we can confirm the invariance under $\eta_q \leftrightarrow 1/\eta_q$. Our discussions in the following sections can be applied straightforwardly to the two different radii case.

### 3. Growth of Rényi entanglement entropy

In this section we will turn to the computation of $\Delta S^{(2)}_A$ in $(T^2)^n/Z_n$. We start by the analytical computation of $F_n(1 - c^2/4t^2, c^2/4t^2)$ in the limit $\epsilon \to 0$. We then find that the behavior of $F_n$ (i.e. $\Delta S^{(2)}_A$) for irrational $\eta$ is distinctly different from that for rational $\eta$.

For a rational $\eta$ ($= \frac{p}{q}$) the CFT is rational, and as expected from the result in [22], $\Delta S^{(2)}_A$ approaches a finite constant that we prove to be

$$\Delta S^{(2)}_A = (n - 1) \cdot \log(2pq).$$  

(3.1)

We derive this result by both analytical and numerical computations and ensure consistency with the formula (2.4) by using the quantum dimension of the twist operator $\sigma_n$ evaluated in the appendix A.

For an irrational $\eta$, we encounter a new late time behavior of $\Delta S^{(2)}_A$ in the form of the double logarithm

$$\Delta S^{(2)}_A \simeq (n - 1) \cdot \log \left( \log\left(t/\epsilon\right) \right) .$$  

(3.2)

This evolution belongs to neither the RCFT class nor the holographic CFT class encountered before and is the main new result of our work. This late time scaling suggests the existence of a third class—an irrational CFT class—from the perspective of the evolution of entanglement measures in excited states.

#### 3.1. Analytical computation of $F_n(1, 0)$

Now let us closely study the $\epsilon \to 0$ limit (2.3). It is useful to define an infinitesimal quantity $\delta$

$$\delta \equiv \frac{\pi}{\log(4t^2/c^2)} \to 0.$$  

(3.3)

In this limit (2.3) with $\delta \to 0$, (2.14) and (2.17) are approximated by
\[ I_{k/n} \simeq \sin^2 \left( \frac{\pi k}{n} \right) + O(\delta^0), \]

\[ a_k \simeq \frac{2\delta}{\sin \left( \frac{\pi k}{n} \right)} + O(\delta^2), \]

\[ \tilde{a}_k \simeq \frac{2\delta}{\sin \left( \frac{\pi k}{n} \right)} + O(\delta^2), \]

\[ b_k \simeq 1 + O(\delta^2), \]

and (2.16) are also approximated by

\[ \Omega_{rs} \simeq \frac{4}{n} \delta \cdot (-1 + n \cdot \delta_{rs}) \quad (\equiv \delta \cdot (\Omega_0)_{rs}), \]

\[ \tilde{\Omega}_{rs} \simeq \frac{4}{n} \delta \cdot (-1 + n \cdot \delta_{rs}) \quad (\equiv \delta \cdot (\Omega_0)_{rs}), \]

\[ \Lambda_{rs} \simeq 2(\delta_{rs} - \delta_{r,-1}) \quad (\equiv (\Lambda_0)_{rs}). \]

Therefore we can get the simple approximation form of the function \( F_n \) (2.11) as

\[ F_n \simeq 2^{n-1} \eta^{n-1} \cdot (g_n)^2, \]

with

\[ g_n \equiv \frac{2^{n-1} \delta^{n-1}}{n} \sum_{l,m \in \mathbb{Z}^{n-1}} e^{-\pi \delta \eta m^T \Omega_0 m + 2\pi \eta \eta (1/2) \Lambda_0} \eta m - \pi \delta \eta \Omega_0^{-1}. \]

By using the Poisson resummation formula

\[ \sum_{m \in \mathbb{Z}^{n-1}} e^{-\pi \delta \eta m^T \Omega_0 m + 2\pi \eta \eta (1/2) \Lambda_0} \eta m = \frac{1}{\sqrt{\det A}} \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} e^{-\pi \delta \eta (\tilde{m} + b)^T (A^{-1})(\tilde{m} + b)}, \]

we find

\[ g_n = \frac{2^{n-1} \delta^{n-1}}{n} \cdot \frac{1}{\sqrt{\det(\eta \Omega_0)}} \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} e^{-\pi \delta \eta (\tilde{m}^T 2 \Lambda_0) \Omega_0^{-1} \cdot (\tilde{m} - 2 \Lambda_0 \cdot 1) - \pi \delta \eta \Omega_0^{-1} \cdot (\tilde{m} + b)^T (\tilde{m} + b)}. \]

Since \( \Omega_0^{-1} \) is Hermitian it may be diagonalized by a unitary matrix \( U \): \( \Omega_0^{-1} = U \Omega U^\dagger \). If \( x = Uy \), then

\[ x^T \cdot \Omega_0^{-1} \cdot x = \sum_i D_i |y_i|^2, \]

hence if \( D_i > 0 \), then \( x^T \cdot \Omega_0^{-1} \cdot x > 0 \) for all \( x \neq 0 \) and we can easily show \( D_i > 0 \) (see appendix B). This discussion tells us that the quadratic form \( (\tilde{m}^T 2 \Lambda_0) \cdot \Omega_0^{-1} \cdot (\tilde{m} - 2 \Lambda_0 \cdot 1) \) in the exponent is minimized by the following condition:

\[ \text{Here we omit the } O(\delta) \text{ subleading corrections to the quadratic form on the exponent; they do not play an important role in the following discussion.} \]
\[ \tilde{m} - \frac{\eta}{2} \Lambda_0 \cdot 1 = 0. \] (3.12)

This gives us the dominant contribution in the \( \delta \to 0 \) limit. For rational \( \eta(= \frac{p}{q}) \), the condition is satisfied by
\[ \tilde{m}_r = pk_r \] (3.13)
and
\[ \left( \frac{1}{2} \Lambda_0 \cdot l \right)_r = l_r - l_{r+1} = qk_r \quad (r = 2, \ldots, n-2) \]
\[ -l_1 = qk_1 \]
\[ l_{n-1} = qk_{n-1}. \] (3.14)
where \( k_r \) is an arbitrary integer. Then, we can estimate \( g_n \) for rational \( \eta \) by applying this condition
\[ g_n = \frac{2^{n-1} \delta^{n-1}}{n} \cdot \frac{1}{\sqrt{\det(\eta \delta \Omega_0)}} \sum_{k \in \mathbb{Z}} e^{-\pi pq\delta k^2} \Omega_0 \cdot k \]
\[ = \frac{2^{n-1} \delta^{n-1}}{n} \cdot \frac{1}{\sqrt{\det(\eta \delta \Omega_0 \cdot pq \delta \Omega_0)}} \]
\[ = \frac{2^{n-1}}{n \eta^{n-1} \det \Omega_0} \]
\[ = \frac{1}{(2p)^{n-1}} \] (3.15)
where we used the following relation:
\[ \det \Omega_0 = \frac{2^{2(n-1)}}{n}. \] (3.16)

This relation is derived in appendix B. Finally, we can obtain the four-point function \( F_n \) for rational \( \eta(= \frac{p}{q}) \)
\[ F_n \sim 2^{n-1} \eta^{n-1} \cdot \frac{1}{(2p)^{2(n-1)}} = \frac{1}{(2pq)^{n-1}}. \] (3.17)

On the other hand, for irrational \( \eta \), only \( \tilde{m}_r = l_r = 0 \) satisfies the condition (3.12). Therefore, we have
\[ g_n = \frac{2^{n-1} \delta^{n-1}}{n \eta^{n-1}} \cdot \frac{1}{\sqrt{\det(\eta \delta \Omega_0)}} \]
\[ = \frac{\delta^{n-1}}{n \eta^{n-1}}. \] (3.18)
and we approximate \( F_n \) for an irrational \( \eta \) as
\[ F_n \sim 2^{n-1} \eta^{n-1} \cdot \frac{\delta^{n-1}}{n \eta^{n-1}} = \frac{2^{n-1}}{n} \delta^{n-1}. \] (3.19)

It is very important to stress that this approximation is only applicable when
If, for example, we consider the case (assuming $q \gg 1$)

$$1 \ll \frac{pq}{\delta} \ll 1,$$

we cannot justify the approximation that only $(\tilde{m}_r, l_r - l_{r+1}) = (p, q) k_r$ contributes.

3.2. Growth of the Rényi entanglement entropy for rational $\eta$

In the rational case, the four-point function $F_n(1, 0)$ is written by (3.17), hence the 2nd Rényi entropy increases by a constant that is equal to

$$\Delta S_A^{(2)} = (n - 1) \cdot \log(2pq),$$

which can be as large as the central charge $c = 2n$ in the large $n$ limit.

A few comments to support this result are in order at this point:

- First of all, we can check the formula (3.22) numerically for various values of $(z, \tilde{z})$ and $(p, q)$. For instance, figure 2 shows the behavior of $F_2$ for $\eta = \frac{10}{11}$ as a function of $\delta$. Indeed $F_2$ approaches $\frac{1}{220}$ in the $\delta \to 0$ limit. Also, there is a plateau $F_2 \simeq 1/2$ for $\delta \sim 0.3$ as $\eta$ is close to 1. A similar plateau can be observed more clearly for $\eta = \frac{10}{10p+1}$ for a large $p$.

- When $\eta$ is rational, we also expect the CFT $(T^2)^g/Z_n$ to be an RCFT. Indeed, as we explain in detail in appendix A, we can show that the quantum dimension of the twist operator $\sigma_n$ in $(T^2)^g/Z_n$ takes the value

$$d_{\sigma_n} = \frac{S_{0n\sigma}}{S_{00}} = \frac{1}{(s_{00})^{n-1}} = (2pq)^{n-1},$$

where $s_{00} = \frac{1}{2pq}$ is the vacuum S-matrix element in the $c = 2$ CFT at the radius of $T^2$ that is equal to $R = \sqrt{p/q}$. By using the formula (2.4), we can get the following expression of $\Delta S_A^{(2)}$:

$$\Delta S_A^{(2)} = \log d_{\sigma_n} = (n - 1) \cdot \log(2pq),$$

which perfectly matches (3.22).
For \( n \to 1 \), \( \Delta S_A^{(2)} \) reduces to zero. This is consistent with the fact that the twist operators become the identity in the \( n \to 1 \) limit (no orbifold).

Note that as shown in [22], for RCFTs, the growth of the \( m \)th Rényi EE \( \Delta S_A^{(m)} \) does not depend on \( m \), and therefore we also expect the same for the rational \( \eta \) in our setup

\[
\Delta S_A^{(m)} = (n - 1) \cdot \log(2pq).
\]

Moreover, we can extend the rational result to the case where we insert many twist operators. As shown in [25, 26], in RCFTs, entanglement is conserved after the scattering between the local operators. This means that the contribution from the local operators is summed up independently. Therefore, when the excited state is created by the insertion of \( K \) twist operators, we find the late time increase of the Rényi entropies

\[
\Delta S_A^{(m)} = K \cdot (n - 1) \cdot \log(2pq).
\]

### 3.3. The evolution of Rényi entanglement entropy for irrational \( \eta \)

When \( \eta \) is an irrational number, we simply find

\[
F_n(1, 0) = 0. \tag{3.27}
\]

This means that the 2nd Rényi EE diverges with time. To see the exact time dependence we need to find a first order correction in the \( \delta \to 0 \) limit.

By using the result (3.19), we may naively estimate

\[
F_n \left( 1 - \frac{\epsilon^2}{4\pi^2}, \frac{\epsilon^2}{4\pi^2} \right) \simeq \frac{2^n - 1}{n} \cdot \delta^{n-1} = \frac{1}{n} \cdot \left( \frac{2\pi}{\log(4\pi^2/\epsilon^2)} \right)^{n-1}. \tag{3.28}
\]

This leads to the following result of the time evolution of \( \Delta S_A^{(2)} \) when \( \eta \) is irrational

\[
\Delta S_A^{(2)} \simeq (n - 1) \cdot \log \left( \frac{\log(4\pi^2/\epsilon^2)}{2\pi} \right) + \log n \sim (n - 1) \cdot \log \left( \log(\delta/\epsilon) \right). \tag{3.29}
\]

This is proportional to the central charge \( c = 2n \) in the large \( n \) limit and independent of \( \eta \).

However, strictly speaking, there is a subtle problem with this argument, because we can make \( |\tilde{m} - \frac{\eta}{2}\Lambda_0 \cdot \tilde{l}| \) arbitrarily small by making \( \tilde{m}_r \) and \( \tilde{l}_r \) large enough, even if \( \eta \) is irrational. Since we do not have any analytical control of this problem, we performed numerical computations for various small values of \( \delta \). The upshot is that the estimation (3.28) is qualitatively correct. More explicitly, when \( \delta \) is small, we find that the ratio \( F_n/\delta^{n-1} \) is bounded both from below and above

\[
\frac{2^n - 1}{n} \leq F_n/\delta^{n-1} \leq A(\eta), \tag{3.30}
\]

where \( A(\eta) \) is a certain \( O(1) \) constant which depends on \( \eta \). Note that the lower bound is obvious because the following summation in (3.10)

\[
\sum_{1 \leq \tilde{m} \in \mathbb{Z}^{n-1}} e^{-\frac{\pi^2}{\epsilon^2} \left( (\tilde{m}_r - \frac{\eta}{2} \Lambda_0 \cdot \tilde{l}_r) \Omega_0^{-1} (\tilde{m}_r - \frac{\eta}{2} \Lambda_0 -1) \right) - \pi \eta \delta^{1/2} \Omega_0^{-1}} = 1 + \cdots \tag{3.31}
\]

is larger than 1. Indeed, in figures 3 and 4 we computed \( F_2 \) for \( \eta = \sqrt{2} \) numerically. It is clear that \( F_2/\delta \) approaches zero almost linearly, like \( F_2/\delta \sim 1.46 \).
Also note that figure 4 shows clear oscillations. This oscillation becomes more frequent as we approach $\delta = 0$. The reason why we have such oscillations is because we can approximate any irrational number by infinitely many different rational numbers, as in the continued fraction representation\(^8\), where $\delta$ measures the accuracy of the approximation.

In principle, we can extend our analysis to $n \geq 3$ in a straightforward way. However, our numerical analysis gets more involved as $n$ grows. We show our result for $n = 3$ in figures 5 and 6 for $\eta = \sqrt{2}$. The plots show that the ratio $F_3/\delta^2$ is bounded both from below and from above.

\(^8\) For example, the continued fraction expansion of $\sqrt{2} (= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}})$ generates rational numbers $\{\frac{p}{q}\} = 1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \cdots$ at each step. This approximation causes the oscillation of $F_3$ to be locally minimized at $\delta \sim \frac{1}{0.5, 0.0833\ldots, 0.0142\ldots, 0.00245\ldots, 0.000420\ldots, \cdots}$, as seen in figures 4 and 6. However, for irrational numbers with a complicated continued fraction expansion ($\pi, e, \ldots$), it seems to be hard to see such a structure.
Summarizing, our thorough analysis shows that for irrational \( \eta \), the growth of the Rényi entanglement entropy has the form of a double logarithm

\[
\Delta S^2_A \simeq (n - 1) \cdot \log \left( \log \left( \frac{t}{\epsilon} \right) \right),
\]

up to a constant term and ignoring the oscillating effect.

Note that similarly to the rational case, it is also natural to expect \( \Delta S^{(m)}_A \) not to depend on \( m \),

\[
\Delta S^{(m)}_A \simeq (n - 1) \cdot \log \left( \log \left( \frac{t}{\epsilon} \right) \right).
\]

To show this intuitively, in the \( \delta \to 0 \) limit, we pinch off as many nontrivial cycles of the replica manifold as the genus \( g = (m - 1)(n - 1) \). Then, \( F_n \) obtains a divergent factor \( \delta \) for each pinched cycle and the total divergent factor is \( \delta^g = \delta^{(m-1)(n-1)} \). This observation leads us to (3.33). We can confirm this explicitly for the \( m = 2 \) case, as discussed in this section, however the cautious reader should bear in mind potential subtleties that could remain for irrational
CFTs. It would be interesting to prove the independence of $m$ rigorously for irrational $\eta$, and we leave this as an open future problem.

4. Growth of the Rényi EE in symmetric orbifolds $(T^2)^n/S_n$

So far we have analyzed the Rényi EE for the cyclic orbifold $(T^2)^n/\mathbb{Z}_n$. Now we would like to turn to the symmetric orbifold $(T^2)^n/S_n$ with the same radii $R$. Except $n = 2$, they are different CFTs and unfortunately, no tractable formula of the four-point functions in the latter CFT is available at present. However, it is still correct for the symmetric orbifold CFT to be rational if the value of $R^2$ is rational, which we set as $R^2 = p/q$ again. Therefore, in this rational CFT case, we can apply (2.6) to calculate $\Delta S_A^{(m)}$. A detailed computation can be found in the appendix A. The upshot is that the quantum dimension takes the value

$$d_{\sigma_n} = (n-1)! \cdot (2pq)^{n-1},$$

(4.1)

because we have an additional prefactor of $|C^A| = (n-1)!$ in (A.13) for the symmetric orbifold. This leads to the growth of the Rényi entropy

$$\Delta S_A^{(m)} = (n-1) \log(2pq) + \log(n-1)!.$$

(4.2)

This result clearly shows that the four-point function of the twist operators in the cyclic orbifold CFT is different from that in the symmetric orbifold one. This difference is natural because the intermediate states in the computation of the four-point function are projected by two different orbifold groups. The additional term in (4.2) can be intuitively understood since the sizes of the two orbifold groups are different by a factor of $|S_n|/|\mathbb{Z}_n| = (n-1)!$. In future work, it will be interesting to confirm our prediction (4.2) by working out the four-point function in $(T^2)^n/S_n$ explicitly and also compute the time evolution in the irrational case in.

5. Re-interpretation in terms of mutual information

Since our analysis in the previous sections involves the four-point function of the twist operators, we can also interpret the result in terms of the Rényi entanglement entropy $S_{A,B}^{(n)}$ for two intervals $A$ and $B$ as in [3, 32, 38, 39], or the Rényi mutual information$^9$

$$I^{(n)}(A:B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}.$$  

(5.1)

Let us choose $A$ and $B$ to be $[x_1, x_2]$ and $[x_3, x_4]$ in the two-dimensional Lorentzian spacetime $\mathbb{R}^{1,1}$. Note that the Lorentzian time $t$ and space $x$ is related to the complex coordinate as $z = x + it_E = x - t$. In terms of the complex coordinate, the intervals are specified by the twist operators at

$$z_1 = \bar{z}_1 = x_1, \quad z_3 = \bar{z}_3 = x_3, \quad z_4 = \bar{z}_4 = x_4,$$

$$z_2 = x_2 - (-t), \quad \bar{z}_2 = x_2 + (-t),$$

(5.2)

where $x_1 < x_2 < x_3 < x_4$ and $t > 0$. Here we consider the simple case in which only $x_2$ goes away from the $t = 0$ slice. In this setup, the $(z, \bar{z}) \to (1, 0)$ limit corresponds to the ‘light-cone’ limit. In this limit, the interval $A$ is infinitely boosted and the Cauchy surface containing the

$^9$The time evolution of the mutual information after a global quench has been studied by another analytic continuation of the same four-point function in [40].
intervals becomes singular (figure 7). To regularize this null limit, let us introduce the regulators $\epsilon_{1,2}$ such that
\[
x_{32}^- = -\epsilon_2 (< 0) \to 0, \quad x_{21}^+ = \epsilon_1 (> 0) \to 0
\] (5.3)
where we defined the light-cone coordinate $x_i^\pm = t_i \pm x_i$ ($t_2 = -t, t_1 = t_3 = t_4 = 0$). The cross ratios are
\[
z = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{x_{31}x_{43}}{x_{31}x_{42}} = \frac{(x_{31} + t)x_{43}}{x_{31}(x_{42} - t)} = 1 - \frac{x_{32}x_{41}}{x_{31}(x_{41} + x_{32})} = 1 - \frac{(l + 2t + \epsilon_1 + \epsilon_2)\epsilon_2}{(2t + \epsilon_1 + \epsilon_2)(l + \epsilon_2)},
\] (5.4)
where $x_4 - x_3 = l$. They are expanded in $\epsilon_{1,2}$ as follows
\[
z \simeq 1 - \left( \frac{l + 2t}{2l} \right) \cdot \epsilon_2 \cdots, \quad \bar{z} \simeq \left( \frac{l}{2l(l + 2t)} \right) \cdot \epsilon_1 + \cdots.
\] (5.5)
In particular, we focus on the expansions for $l \to \infty$ and $\epsilon_1 = \epsilon_2 = \epsilon$ which have quite simple forms as with (2.3)
\[
z \simeq 1 - \frac{\epsilon}{2l} \cdots, \quad \bar{z} \simeq \frac{\epsilon}{2l} + \cdots.
\] (5.6)
Thus, if we boost $A$ to almost null, then the $n$th Rényi mutual information (3.17) can be computed by performing a similar analysis for the $(z, \bar{z}) \to (1, 0)$ limit of $F_n(z, \bar{z})$ in the previous sections.
\[
I^{(n)}(A : B) = \frac{1}{n - 1} \log \left[ |1 - z|^{-4\Delta_n} \cdot F_n(z, \bar{z}) \right],
\] (5.7)
where $\Delta_n = \frac{\epsilon}{2l}(n - 1/n)$. For any $\eta$, the mutual information (5.7) has the logarithmic divergent term coming from the first factor in the logarithm. The $\eta$-dependence comes from $F_n(z, \bar{z})$. When $\eta = p/q$ is rational, applying (3.17), we find the constant term

10 In [41], the light-cone limit of $S_{A,B}^{(n)}$ is discussed for globally excited states that detect the failure of the quasi-particle picture for the propagation of entanglement. Our setup is similar, but different in that the excitation is (quasi-)locally caused by boosting the interval $A$. Furthermore, in [42], other configurations of non-coplanar regions are discussed.
\begin{equation}
I^{(n)}(A, B) = \frac{c}{12} \left( 1 + \frac{1}{n} \right) \log \left( \frac{y}{\epsilon_2} \right) - \log(2pq),
\end{equation}

where \( y = 2t \). We expect this to be generalized into the following form in any rational CFT:

\begin{equation}
I^{(n)}(A, B) = \frac{c}{12} \left( 1 + \frac{1}{n} \right) \log \left( \frac{y}{\epsilon_2} \right) - \log d_{\text{tot}},
\end{equation}

where \( d_{\text{tot}} = 1/800 \) is the total quantum dimension of the (seed) CFT. When \( \eta \) is irrational, applying (3.19), we find the double logarithmic divergent term

\begin{equation}
I^{(n)}(A : B) = \frac{c}{12} \left( 1 + \frac{1}{n} \right) \log \left( \frac{y}{\epsilon_2} \right) - \log \left( \log \left( \frac{y}{\epsilon_2} \right) \right) - \log \left( \frac{\log n}{n - 1} + \log(2\pi) \right).
\end{equation}

6. Conclusions

In this work we studied the evolution of the Rényi entropy in cyclic and symmetric orbifold CFTs. As we showed, by considering twist operators as the primary excitations, we were able to extract various new and universal results from the four-point correlators developed by Calabrese, Cardy and Tonni [32] for the cyclic orbifold CFT. Here we summarize our main findings and list some open problems.

Depending on the compactification radius, our setup can be divided into rational and irrational models. In the rational case, the increase of the Rényi entropies approaches a universal constant at late times that is proportional to the logarithm of the total quantum dimension. This quantity is also defined as the inverse of the identity-identity component of the modular S-matrix of the seed theory. In our setup, for the rational radius \( \eta = p/q \), we showed both analytically and numerically that for twist operators in the \( n \)th cyclic orbifold CFT \( (T^2)^n/\mathbb{Z}_n \), the Rényi approaches the constant \( \Delta S^{(m)} = (n - 1) \log(2pq) \). In the case of the symmetric orbifold CFT \( (T^2)^n/\mathbb{S}_n \), we get \( \Delta S^{(m)} = (n - 1) \log(2pq) + \log(n - 1)! \). Except for \( n = 2 \), these do not agree with each other, because the intermediate states in the computation of four-point functions are different, due to different orbifold projections.

Moreover, we analyzed the time evolution of the Rényi entropy in the cyclic orbifold CFT when the square of the radius was irrational. In this irrational case, we found a universal growth at late times given by the double logarithm of time. This is slower than the holographic one [27, 28], but much faster and unconstrained than in RCFTs. This result strongly suggests that a breakdown of the quasi-particle picture is strongly related to the rationality of the underlying CFT. Motivated by our results we are tempted to conjecture that the logarithmic growth found in holographic CFTs is the fastest growth for the local excitations among all 2d CFTs. Such a systematical understanding of entanglement entropy growth certainly deserves future studies.

Moreover, interestingly, our results for the second Rényi entropy can be interpreted in terms of the mutual information between light-like separated intervals. We derived a universal answer for the mutual information in this limit and it would be interesting to explore the physical meaning of this quantity in more detail.

Another useful quantity which characterizes the evolution of excitations is the out-of-time order correlator (OTOC), which provides a new tool for classifying CFT from the perspective of the evolution of the quantum entanglement and information [43–45]. In our upcoming publication [46], we will present the results for OTOCs for our symmetric orbifold CFTs and compare the results with those for RCFTs [47, 48] as well as chaotic CFTs.
Acknowledgments

We are grateful to Chris Herzog, Alvaro Veliz-Osorio, Tokiro Numasawa and Noburo Shiba for useful discussions, and especially to Erik Tonni for detailed explanations of the relevant computations. PC is supported by the grant ‘Exact Results in Gauge and String Theories’ from the Knut and Alice Wallenberg foundation. TT and PC are supported by the Simons Foundation through the ‘It from Qubit’ collaboration. KW is supported by the JSPS fellowship. TT is supported by JSPS Grant-in-Aid for Scientific Research (A) no.16H02182 and the World Premier International Research Center Initiative (WPI Initiative) from the Japan Ministry of Education, Culture, Sports, Science and Technology (MEXT). TT is also very grateful to the workshop ‘Entanglement and Dynamical Systems’ held at Simons Center for Geometry and Physics, where this work was presented.

Appendix A. Quantum dimension from S-matrices in orbifold CFTs

A.1. Quantum dimension of twist operator

If $\mathcal{C}$ denotes an RCFT, $\mathcal{C}^\otimes n$ denotes an $n$th tensor product CFT of $\mathcal{C}$ and $\Omega$ denotes a permutation group in $S_n$, the characters of the primary fields in $\mathcal{C}^\otimes n/\Omega$ and their modular property (i.e. the S-matrix) are given in [49]. By using these expressions, we can get the quantum dimension of the twist operator in $\mathcal{C}^\otimes n/\Omega$. To explain this, we use the following notation.

(Notation)

$\mathcal{D}(\Omega) \cdots$ The Drinfeld double of the group $\Omega$, defined in [50].

$p \cdots$ Some representative of an orbit of $\Omega$ acting on the $n$-tuples $(p_1, p_2, \ldots, p_n)$ of the primaries $p_i$ of $\mathcal{C}$.

$\phi \cdots$ The irreducible character of the double $\mathcal{D}(\Omega_p)$ of the stabilizer $\Omega_p = \{ x \in \Omega | xp = p \}$ of the $n$-tuple $p$.

$p, \phi \cdots$ The primary fields of $\mathcal{C}^\otimes n/\Omega$.

$\chi_p(\tau) \cdots$ The genus one character of the primary field $p$ of $\mathcal{C}$.

$\omega_p \cdots$ The modular T-matrix of $\mathcal{C}$, in that $\omega_p = e^{2\pi i (\Delta_p - \frac{c}{24})}$.

$\mathcal{O}(x, y) \cdots$ The set of orbits of the subgroup generated by $x$ and $y$, where a pair of $x, y \in \Omega$ is of commuting permutations.

To each ordered triple $(x, y, \xi)$ with $\xi \in \mathcal{O}(x, y)$, we associate the following data:

1. $n_\xi$ (respectively $n_\xi^*$) is the length of any $x$ orbit (respectively $y$ orbit) contained in $\xi$.
2. $\mu_\xi$ (respectively $\mu_\xi^*$) is the number of the $x$ orbits (respectively $y$ orbits),
3. $\kappa_\xi$ (respectively $\kappa_\xi^*$) denotes the smallest non-negative integer for which $y^{\mu_\xi} = x^{\kappa_\xi}$ (respectively $y^{\mu_\xi^*} = x^{\kappa_\xi^*}$) holds on the points of $\xi$.

The characters of the primary fields $(p, \phi)$ in $\mathcal{C}^\otimes n/\Omega$ are written as follows:

$$\chi_{p,\phi}(\tau) = \frac{1}{|\Omega_p|} \sum_{x, y \in \Omega} \chi_p(x, y | \tau) \bar{\phi}(x, y),$$

(A.1)

where

$$\chi_p(x, y | \tau) = \begin{cases} \prod_{\xi \in \mathcal{O}(x, y)} \omega_{p_\xi}^{\frac{-\kappa_\xi}{\mu_\xi}} x_{p_\xi}(\tau_\xi) & \text{if } x, y \in \Omega_p \text{ commute}, \\ 0 & \text{otherwise}, \end{cases}$$

(A.2)
and $p_\xi$ is the component of $p$ associated with the orbit $\xi$. According to \cite{49}, the conformal dimensions of the primary fields $\langle p, \phi \rangle$ can be obtained from the following relation:

$$\frac{1}{d_\phi} \sum_{x \in \Omega} \phi(x,x) \prod_{\xi \in O(x,1)} \omega_{p_\xi}^{\frac{1}{\xi}} = \exp \left( 2\pi i \Delta_{(p,\phi)} - \frac{nc}{24} \right), \quad (A.4)$$

where $\Delta_{(p,\phi)}$ is the conformal dimension of the primary $\langle p, \phi \rangle$ of $C^{\otimes n}/\Omega$ and $d_\phi = \sum_{x \in \Omega} \phi(x,1)$. The irreducible character of the double $\phi$ can be labeled by the representative element $g^4$ of a conjugacy class $C^4$ of $\Omega$ and an irreducible character $\alpha$ of the centralizer $Z_A$ of $g^4$, so we denote by $\phi_\alpha^{g^4}$ the irreducible character of the quantum double.

Let us consider the special diagonal primary field $\langle p = (\Delta_p, \Delta_p, \cdots, \Delta_p), \phi_\alpha^{g^4} \rangle$ where $g^4$ is the cyclic permutation $(1, 2, \cdots, n)$ and $\alpha$ is the trivial irreducible representation of $Z_A$. In this case,

$$\phi_\alpha^{g^4}(x,y) = \begin{cases} 0, & \text{if } x \notin C^4 \text{ or } xy \neq yx \\ 1, & \text{otherwise} \end{cases} \quad (A.5)$$

and

$$\frac{1}{d_\phi} \sum_{x \in \Omega} \phi(x,x) \prod_{\xi \in O(x,1)} \omega_{p_\xi}^{\frac{1}{\xi}} \prod_{\xi \in O(x^n,1)} \omega_{p_\xi}^{\frac{1}{\xi}} = \prod_{\xi \in O(x^n,1)} \omega_{p_\xi}^{\frac{1}{\xi}}$$

$$= \omega_{p_\xi}^{\frac{1}{\xi}}$$

$$= \exp \left( 2\pi i \frac{\Delta_\phi}{n} - \frac{c}{24n} \right)$$

$$= \exp \left( 2\pi i \frac{\Delta_{(p,\phi)} - \frac{nc}{24}}{n} \right). \quad (A.6)$$

Therefore the conformal dimension of $\langle p = (\Delta_p, \Delta_p, \cdots, \Delta_p), \phi_\alpha^{g^4} \rangle$ is

$$\Delta_{(p,\phi)} = \frac{\Delta_\phi}{n} + \frac{c}{24} \left( n - \frac{1}{n} \right). \quad (A.7)$$
In particular,
\[ \Delta_{(0, \phi)} = \frac{c}{24} \left(n - \frac{1}{n}\right), \]  
(A.8)
where 0 is the set of the vacuum, \((0, 0, \cdots, 0)\). This is exactly the same as the weight of the twist operator.

Next, let us calculate the special element of the S-matrix, \(S_{0, \langle 0, \phi \rangle}\). In general, the elements of the S-matrix of \(C^\otimes n / \Omega\) are very complicated, but the special elements, \(S_{0, \langle p, \phi \rangle}\), have the following simple form:
\[ S_{0, \langle p, \phi \rangle} = \frac{1}{|\Omega|^p} \sum_{x \in \Omega_p} \phi(x, 1) \prod_{\xi \in \mathcal{O}(x, 1)} S_{\text{seed}}^{\langle 0, \phi \rangle_\xi}, \]  
(A.9)
where \(S_{\text{seed}}\) is the S-matrix of \(C\). Inserting (A.5) into (A.9), we can get this element, in that,
\[ S_{0, \langle 0, \phi \rangle_{gA_\alpha}} = \frac{|C|}{|\Omega|^p} \prod_{\xi \in \mathcal{O}(gA_\alpha, 1)} S_{\text{seed}}^{\langle 0, \phi \rangle_\xi}. \]  
(A.10)

On the other hand, the trivial character of the quantum double is
\[ \phi_c(x, y) = \delta_{x, y}. \]  
(A.11)
By the same calculation as above, we can get
\[ S_{0, 0} = \frac{1}{|\Omega|} (S_{\text{seed}}^{\langle 0, \phi \rangle_0})^n, \]  
(A.12)
hence the quantum dimension \(d_{\sigma_n}\) of the twist operator is
\[ d_{\sigma_n} = \frac{S_{0, \langle 0, \phi \rangle_{gA_\alpha}}}{S_{0, 0}} = \frac{|C|}{(S_{\text{seed}}^{\langle 0, \phi \rangle_0})^{n-1}}. \]  
(A.13)

This result holds in general RCFTs, in that the quantum dimension of the twist operator in \((\text{general RCFT})^\otimes n / \Omega\) is related to the \((n - 1)\)th power of the total dimension of its seed RCFT. But as is obvious from the above derivation, \(\Omega\) has to contain the group element \((1, 2, \cdots, n)\). \(\mathbb{Z}_n\) has the cyclic permutation \((1, 2, \cdots, n)\), so we can use the above formula (in this case, \(|C| = 1\)).

Let us focus on our case, where \(C\) is \(T^2\) with \(\eta = p/q\). As is well known, the characters of the primary fields in the free compactified boson theory with \(\eta = p/q\) are written as follows:
\[ \chi_{\ell}^{\langle p/q \rangle}(q) = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{\frac{\ell^2}{2} + 2\pi \ell pq} \]  
\[ = \frac{\Theta_{lpq}(\tau, 0)}{\eta(\tau)}, \quad -pq + 1 \leq l \leq pq, \]  
(A.14)
where \(q = e^{2\pi \tau}\) and \(\Theta_{lpq}(\tau, z)\) are the theta functions and \(\eta(\tau)\) is the eta function. Using the modular transformation law of the theta functions and the eta function, we can get the following modular properties:
\[ \chi_i^{(pq)} \left( \frac{-1}{\tau} \right) = \sum_{m=-(pq+1)}^{pq} \frac{e^{-\pi i m}}{\sqrt{2pq}} \chi_m(\tau). \]  

(A.15)

Therefore, the modular S-matrix of this theory is

\[ S_{lm}^i = e^{-\pi i lm} \sqrt{2pq}. \]  

(A.16)

We can get the modular matrix element \( S_{00}^{T^2} \) of \( T^2 \) just by squaring \( S_{00}^i \), so we get

\[ S_{00}^{seed} = S_{00}^{T^2} = \frac{1}{2pq}. \]  

(A.17)

Using (A.13), the quantum dimension of the twist operator in \((T^2)^n/Z_n\) is

\[ d_{\sigma_n} = (2pq)^{n-1}, \]  

(A.18)

which is exactly the (3.23) that we wanted to prove. Note that from the above derivation, we are able to understand that \( 2pq \) reflects the number of irreducible characters. Refer to the tables A1 and A2 for explicit examples.

**A.2. Application and another calculation for \( S_n \)**

We showed the quantum dimension for \((T^2)^n/Z_n\) by (A.18) in order to recover (3.24), but it is also interesting to consider the \( S_n \) case, as we said in the introduction. From (A.13), we can directly obtain the quantum dimension of the twist operator for \( S_n \):

\[ d_{\sigma_n} = \frac{(n-1)!}{(S_{00}^{seed})^{n-1}}, \]  

(A.19)

where we use the fact that \( |C_A| = (n-1)! \) for \( g_A = (1, 2, 3, \ldots, n) \) in \( S_n \).

In fact, we can check our result for the \( S_n \) case from another viewpoint. If we have the explicit expressions for the characters of the twist and vacuum operator, we can evaluate the quantum dimension of the twist field in the following way:

\[ \lim_{\tau \to i\infty} \frac{\chi_i \left( \frac{-1}{\tau} \right)}{\chi_0 \left( \frac{-1}{\tau} \right)} = \lim_{\tau \to i\infty} \frac{\sum_j S_{ij} \chi_j(\tau)}{\sum_j S_{0j} \chi_j(\tau)} = \frac{S_{i0}}{S_{00}} = d_i. \]  

(A.20)

The character of the \( S_3 \) orbifolds has been presented in [51], and in particular, the characters of the vacuum and twist operator are as follows:

If the character of the orbifold denotes \( \chi_i \), the characters corresponding to the vacuum (0) and twist (\( \sigma \)) operator are respectively

\[ \chi_0(\tau) = \frac{1}{6} \left[ (\chi_0(\tau))^3 + 3\chi_0(\tau) \chi_0(2\tau) + 2\chi_0(3\tau) \right] \]  

(A.21)

and

\[ \chi_1(\tau) = \frac{1}{6} \chi_0(\tau). \]
\[ X_\sigma(\tau) = \frac{1}{3} \left[ \chi_0 \left( \frac{\tau}{3} \right) + \chi_0 \left( \frac{\tau + 1}{3} \right) + \chi_0 \left( \frac{\tau + 2}{3} \right) \right]. \quad (A.22) \]

To apply these to (A.20), we need to evaluate the limit of the following quantities.
\[ \chi_i \left( -\frac{p}{\tau} \right) = \sum_j S_{ij} \chi_j \left( \frac{\tau}{p} \right) \]
\[ \tau \to i \infty \to S_{i0} \left( q^{-\frac{\tau}{\pi}} \right)^{\frac{1}{6}}, \quad (A.23) \]
\[ \chi_i \left( -\frac{1}{\tau} + \frac{1}{3} \right) = \sum_j S_{ij} \chi_j \left( \frac{3\tau}{1-\tau} \right) \]
\[ = \sum_j S_{ij} \chi_j \left( -3 + \frac{3}{1-\tau} \right) \]
\[ = \sum_j (ST^{-1})_{ij} \chi_j \left( \frac{3}{1-\tau} \right) \]
\[ = \sum_j (ST^{-1})_{ij} \chi_j \left( \frac{\tau - 1}{3} \right) \]
\[ \tau \to i \infty \to (ST^{-3})_{i0} \left( q^{-\frac{\tau}{\pi}} \right)^{\frac{1}{3}} \times (e^{-\frac{3\pi i}{2}})^{-\frac{\tau}{\pi}} \quad (A.24) \]

and
\[ \chi_i \left( -\frac{1}{\tau} + \frac{2}{3} \right) = \sum_j T_{ij} \chi_j \left( -\frac{1}{\tau} - \frac{1}{3} \right) \]
\[ = \sum_j (TST^{-1})_{ij} \chi_j \left( \frac{\tau + 1}{3} \right) \]
\[ \tau \to i \infty \to (TST^{-3})_{i0} \left( q^{-\frac{\tau}{\pi}} \right)^{\frac{1}{3}} \times (e^{\frac{3\pi i}{2}})^{-\frac{\tau}{\pi}}. \quad (A.25) \]

By using these, the leading term of \( X_0(\tau) (\tau \to i \infty) \) is
\[ X_0 \left( -\frac{1}{\tau} \right) \to i \infty \to \frac{1}{6} S_{00} \left( q^{-\frac{\tau}{\pi}} \right)^{\frac{1}{3}} \quad (A.26) \]
and that of \( X_\sigma \left( -\frac{1}{\tau} \right) \) is
\[ X_\sigma \left( -\frac{1}{\tau} \right) \to i \infty \to \frac{1}{3} S_{00} \left( q^{-\frac{\tau}{\pi}} \right)^{\frac{1}{3}}. \quad (A.27) \]

Inserting these into (A.20),
\[ d_{\sigma} = 2 \frac{1}{S_{00}}. \quad (A.28) \]

In the same way, we can also calculate the quantum dimension for the \( S_4 \) case.
According to the above table, the leading term of $\mathcal{X}_0(-\frac{1}{\tau})$ ($\tau \to i\infty$) is
\[
\mathcal{X}_0\left(-\frac{1}{\tau}\right) \tau \to i\infty \frac{1}{4\Pi} S^{00}(q^{-1})^4
\] (A.29)
and that of $\mathcal{X}_\sigma(-\frac{1}{\tau})$ is
\[
\mathcal{X}_\sigma\left(-\frac{1}{\tau}\right) \tau \to i\infty \frac{1}{4} S^{00}(q^{-1})^4.
\] (A.30)
Inserting these into (A.20),
\[
d_\sigma = \frac{3!}{S^{00}}.
\] (A.31)

The analysis of the higher $n$ proceeds along the same lines and the expression for a general $n$ is supposed as follows:
\[
d_\sigma = \frac{(n-1)!}{(S^{00})^{n-1}}.
\] (A.32)
This result is consistent with our general expression.

**Appendix B. The determinant of $\Omega_0$**

The characteristic polynomial of $\Omega_0$ is
\[
|\Omega_0 - \lambda| = \left(\frac{4}{\Pi} - \lambda\right)(4 - \lambda)^{n-2} = 0,
\] (B.1)
where we insert (3.5) into the following formula for the $(n-1) \times (n-1)$ matrices:
\[
\begin{vmatrix}
  a & b & b & \cdots & b \\
  b & a & b & \cdots & b \\
  b & b & a & \cdots & b \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & b & b & \cdots & a
\end{vmatrix} = (a + (n-2)b)(a - b)^{n-2}.
\] (B.2)
Therefore the eigenvalues of $\Omega_0^{-1}$ are \{ $\frac{4}{\Pi}, \frac{1}{\Pi}, \frac{1}{\Pi}, \ldots$ \} and these are clearly positive. We can also evaluate the determinant of $\Omega_0$ as follows:
\[
\det \Omega_0 = \frac{4^{n-1}}{n},
\] (B.3)
which is exactly the (3.16) that we wanted to prove.

**References**

[1] Cardy J L 1986 Operator content of two-dimensional conformally invariant theories *Nucl. Phys.* B 270 186–204
[2] Holzhey C, Larsen F and Wilczek F 1994 Geometric and renormalized entropy in conformal field theory *Nucl. Phys.* B 424 443–67
[3] Calabrese P and Cardy J L 2004 Entanglement entropy and quantum field theory *J. Stat. Mech.* P06002
[4] Calabrese P and Cardy J L 2005 Evolution of entanglement entropy in one-dimensional systems J. Stat. Mech. P04010
[5] Calabrese P and Cardy J L 2006 Time-dependence of correlation functions following a quantum quench Phys. Rev. Lett. 96 136801
[6] Calabrese P and Cardy J 2009 Entanglement entropy and conformal field theory J. Phys. A: Math. Theor 42 504005
[7] Calabrese P and Cardy J 2016 Quantum quenches in 1 + 1 dimensional conformal field theories J. Stat. Mech. 064003
[8] Ryu S and Takayanagi T 2006 Holographic derivation of entanglement entropy from AdS/CFT Phys. Rev. Lett. 96 181602
[9] Ryu S and Takayanagi T 2006 Aspects of holographic entanglement entropy J. High Energy Phys. JHEP08(2006)045
[10] Hubeny V E, Rangamani M and Takayanagi T 2007 A covariant holographic entanglement entropy proposal J. High Energy Phys. JHEP07(2007)062
[11] Nishioka T, Ryu S and Takayanagi T 2009 Holographic entanglement entropy: an overview J. Phys. A: Math. Theor. 42 504008
[12] Takayanagi T 2012 Entanglement entropy from a holographic viewpoint Class. Quantum Grav. 29 153001
[13] Rangamani M and Takayanagi T 2016 Holographic entanglement entropy (arXiv:1609.01287)
[14] Abajo-Arrastia J, Aparicio J and Lopez E 2010 Holographic evolution of entanglement entropy J. High Energy Phys. JHEP11(2010)149
[15] Balasubramanian V, Bernamonti A, de Boer J, Copland N, Craps B, Keski-Vakkuri E, Muller B, Schafer A, Shigemori M and Staessens W 2011 Thermalization of strongly coupled field theories Phys. Rev. Lett. 106 191601
[16] Hartman T and Maldacena J 2013 Time evolution of entanglement entropy from black hole interiors J. High Energy Phys. JHEP05(2013)014
[17] Nozaki M, Numasawa T and Takayanagi T 2014 Quantum entanglement of local operators in conformal field theories Phys. Rev. Lett. 112 111602
[18] Nozaki M 2014 Notes on quantum entanglement of local operators J. High Energy Phys. JHEP10(2014)147
[19] Nozaki M, Numasawa T and Matsura S 2016 Quantum entanglement of fermionic local operators J. High Energy Phys. JHEP02(2016)150
[20] Caputa P, Nozaki M and Numasawa T 2016 Charged entanglement entropy of local operators Phys. Rev. D 93 105032
[21] Nozaki M and Watamura N 2016 Quantum entanglement of locally excited states in Maxwell theory J. High Energy Phys. JHEP12(2016)069
[22] He S, Numasawa T, Takayanagi T and Watanabe K 2014 Quantum dimension as entanglement entropy in two dimensional conformal field theories Phys. Rev. D 90 041701
[23] Caputa P and Veliz-Osorio A 2015 Entanglement constant for conformal families Phys. Rev. D 92 065010
[24] Chen B, Guo W-Z, He S and Wu J-Q 2015 Entanglement entropy for descendent local operators in 2D CFTs J. High Energy Phys. JHEP10(2015)173
[25] Caputa P and Rams M M 2016 Quantum dimensions from local operator excitations in the Ising model (arXiv:1609.02428)
[26] Numasawa T 2016 Scattering effect on entanglement propagation in RCFTs (arXiv:1610.06181)
[27] Nozaki M, Numasawa T and Takayanagi T 2013 Holographic local quenches and entanglement density J. High Energy Phys. JHEP05(2013)080
[28] Caputa P, Nozaki M and Takayanagi T 2014 Entanglement of local operators in large-N conformal field theories Prog. Theor. Exp. Phys. 2014 093B06
[29] Asplund C T, Bernamonti A, Galli F and Hartman T 2015 Holographic entanglement entropy from 2d CFT: heavy states and local quenches J. High Energy Phys. JHEP02(2015)171
[30] Caputa P, Simon J, Stikonas A, Takayanagi T and Watanabe K 2015 Scrambling time from local perturbations of the eternal BTZ black hole J. High Energy Phys. JHEP08(2015)011
[31] Caputa P, Simon J, Stikonas A and Takayanagi T 2015 Quantum entanglement of localized excited states at finite temperature J. High Energy Phys. JHEP01(2015)102
[32] Calabrese P, Cardy J and Tonni E 2009 Entanglement entropy of two disjoint intervals in conformal field theory J. Stat. Mech. P11001
[33] Balasubramanian V, Bernamonti A, Craps B, De Jongheere T and Galli F 2016 Entwinement in discretely gauged theories J. High Energy Phys. JHEP12(2016)094
[34] Shiba N 2017 The Aharonov–Bohm effect on entanglement entropy in conformal field theory (arXiv:1701.00688)
[35] Calabrese P, Cardy J and Tonni E 2013 Entanglement negativity in extended systems: a field theoretical approach J. Stat. Mech. P02008
[36] Furukawa S, Pasquier V and Shiraishi J 2009 Mutual information and compactification radius in a \( c = 1 \) critical phase in one dimension Phys. Rev. Lett. 102 170602
[37] Coser A, Tagliazzo L and Tonni E 2014 On Rényi entropies of disjoint intervals in conformal field theory J. Stat. Mech. P01008
[38] Headrick M 2010 Entanglement Renyi entropies in holographic theories Phys. Rev. D 82 126010
[39] Hartman T 2013 Entanglement entropy at large central charge (arXiv:1303.6955)
[40] Coser A, Tonni E and Calabrese P 2014 Entanglement negativity after a global quantum quench J. Stat. Mech. P12017
[41] Asplund C T, Bernamonti A, Galli F and Hartman T 2015 Entanglement scrambling in 2d conformal field theory J. High Energy Phys. JHEP09(2015)110
[42] Blanco D D and Casini H 2011 Entanglement entropy for non-coplanar regions in quantum field theory Class. Quantum Grav. 28 215015
[43] Roberts D A and Stanford D 2015 Two-dimensional conformal field theory and the butterfly effect Phys. Rev. Lett. 115 131603
[44] Kitaev A 2014 Hidden correlation in the Hawking radiation and thermal noise Talk given at the Fundamental Physics Prize Symp. (10 November 2014)
[45] Maldacena J, Shenker S H and Stanford D 2016 A bound on chaos J. High Energy Phys. JHEP08(2016)106
[46] Caputa P, Kusuki Y, Takayanagi T and Watanabe K 2017 Out-of-time-ordered correlators in (T2)/Zn (arXiv:1703.09939)
[47] Caputa P, Numasawa T and Veliz-Osorio A 2016 Out-of-time-ordered correlators and purity in rational conformal field theories Prog. Theor. Exp. Phys. 2016 113B06
[48] Gu Y and Qi X-L 2016 Fractional statistics and the butterfly effect J. High Energy Phys. JHEP08(2016)129
[49] Bantay P 1998 Characters and modular properties of permutation orbifolds Phys. Lett. B 419 175–8
[50] Drinfeld V G 1988 Quantum groups J. Sov. Math. 41 898–915
Drinfeld V G 1986 Quantum groups Zap. Nauchn. Semin. 155 18
[51] Jevicki A and Yoon J 2016 S\( _N \) orbifolds and string interactions J. Phys. A: Math. Theor. 49 205401