Construction of a Kaluza-Klein type Theory from One Dimension

David J. Jackson

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Abstract

We describe how a physical theory incorporating the properties of fields deriving from extra-dimensional structures over a four-dimensional spacetime manifold can in principle be obtained through the analysis of a simple initial structure consisting of the one dimension of time alone, as represented by the real line. The simplicity of this starting point leads to symmetries of multi-dimensional forms of time, from which a geometrical structure can be derived which is similar to the framework employed in non-Abelian Kaluza-Klein theories. This leads to a relationship between the external and internal curvature on the spacetime manifold unified through the underlying constraint of the one dimension of time for the theory presented here. We also describe how the symmetry breaking structure is compatible with the Coleman-Mandula theorem for the subsequent quantisation of the theory.

Contents

1 Introduction and Motivation 2

2 Elementary Structure of the Theory 3
  2.1 Time and Spatial Dimensions 3
  2.2 Geometry of the Spacetime Manifold 5
  2.3 Extra Dimensions and Symmetry Breaking 9

3 Review of Kaluza-Klein Theories 13
  3.1 Riemannian Geometry and General Relativity 13
  3.2 Principle Fibre Bundle Structure 17
  3.3 General Relativity with Extra Dimensions 21
  3.4 Theories with Torsion on the Bundle 25

4 Geometric Unification through One Dimension 29
  4.1 Construction of a Linear Connection on $P \equiv M_4 \times G$ 29
  4.2 Perturbation to the Einstein-Hilbert Action on $M_4$ 37

5 Summary and Discussion 42
  5.1 Conceptual Picture 42
  5.2 Higher Symmetries 44
  5.3 Quantum Theory 47

1email: david.jackson.th@gmail.com
1 Introduction and Motivation

For many theories which aim to describe elementary empirical phenomena, such as summarised in the Standard Model of particle physics, a typical approach is to begin by postulating additional entities or structures on top of, or as an extension of, a 4-dimensional spacetime background. For example a wide ranging class of models invokes the introduction of extra spatial dimensions, as have developed from the original proposal of Kaluza and Klein [1, 2] and some of which we shall review in this paper.

While one goal of any unification scheme is to incorporate a broad range of empirical phenomena collectively within a single framework, it is also generally desirable that the initial framework itself should be as simple as possible. That is, the theory should ideally be largely devoid of apparently arbitrary assumptions or postulated features. This is the point of view adopted for the present work in which we argue that a unified framework, sharing many of the properties of Kaluza-Klein theories, can be founded simply upon the structure of the one dimension of time alone, as represented by the real line. While the full development of this theory has been presented in [3] with the uncovering of properties of the Standard Model described in [4], which summarises ([3] chapters 6–9), here we elaborate upon the foundations of the theory and the elementary geometric structures involved as arising from the simple starting point of one dimension. Being largely self-contained this paper both summarises and expands upon the contents of ([3] chapters 2–5), with further discussion of the underlying conception of the theory in section 2 and a more direct and explicit construction of the relation between the external and internal curvature presented here in section 4. The structure of the paper is further outlined below.

In the following section we describe how analysis of simple arithmetic decompositions implicit in intervals of the real line itself leads directly to elementary structures which exhibit a geometrical and spatial interpretation in several dimensions. Generalising from these observations the resulting multi-dimensional forms for the flow of time exhibit symmetry structures that allow both the identification of a 4-dimensional spacetime manifold together with apparent ‘extra dimensions’. This geometrical framework closely resembles that of non-Abelian Kaluza-Klein theories as constructed on the space of a principle fibre bundle. These latter structures are hence reviewed in section 3, in which much of our notation and general conventions will also be established. Combining the motivations of section 2 with geometric arguments adapted from the Kaluza-Klein theories of section 3, a means of constructing a relationship between the external spacetime geometry and the internal curvature for the present theory is proposed in section 4. We further compare and contrast the new approach with Kaluza-Klein theory in section 5, where we also relate this work to references [3] and [4] and allude to the further development of the theory.

While not describing details of the quantisation of the theory in this paper, in subsection 5.3, with reference to section 2, we also address the compatibility of the elementary symmetry breaking structures in the theory with the Coleman-Mandula theorem, which concerns the possible symmetries for a relativistic theory of interacting particles. It is necessary to address this question since both the external Lorentz symmetry and the internal gauge symmetry originate from a common unifying simple group in the mathematical construction of the theory, with physical structures derived through the breaking of the full symmetry.
2 Elementary Structure of the Theory

2.1 Time and Spatial Dimensions

We begin by considering a finite interval of time represented by the real number \( s \in \mathbb{R} \). Amongst the myriad of ways of expressing \( s \) in terms of other real numbers in accordance with the basic rules of arithmetic one possibility is the quadratic composition:

\[
S^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 = \eta_{ab} x^a x^b \tag{1}
\]

Here \( S^2 \) is the square of \( s \in \mathbb{R} \) while \( x^1 \), \( x^2 \) and \( x^3 \) are three further real numbers and \( \eta_{ab} \), with \( a, b = 1 \ldots 3 \), are the components of the \( 3 \times 3 \) diagonal unit matrix, with the conventional summation over repeated indices implied. In the above expression the left-hand side \( S^2 \) is invariant under the action of the orthogonal group \( SO(3) \) on the ordered set of numerical components \( (x^1,x^2,x^3) \), which can be interpreted as a 3-vector sweeping out a spherical shell in a 3-dimensional geometrical space.

This geometrical structure provides a simple example demonstrating how possible arithmetic compositions of a real interval \( s \in \mathbb{R} \) can exhibit a form and symmetry with a multi-dimensional spatial interpretation. A natural generalisation of equation 1 can also be considered with \( S^2 = \eta_{ab} x^a x^b \) for \( a, b = 1 \ldots n \) for arbitrary \( n \) and unit \( n \times n \) matrix \( \eta \). In this case the corresponding symmetry transformations \( SO(n) \) on the components \( (x^1, \ldots, x^n) \) describe an implicit \( n \)-dimensional spatial structure. The case with a metric \( \eta \) of arbitrary signature and an \( SO(p,q) \) symmetry, with \( p + q = n \), marks a further generalisation. We can also consider the case for the limit of an infinitesimal interval denoted by \( s \rightarrow \delta s \in \mathbb{R} \) for which equation 1 becomes simply:

\[
(\delta s)^2 = (\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2 = \eta_{ab} \delta x^a \delta x^b \tag{2}
\]

In this case the uniformity of the quadratic order of the expression is now required, to balance the order of the infinitesimal quantities in each term, while the property of the invariance of \( (\delta s)^2 \) under \( SO(3) \) transformations as described for equation 1, now applied to the components \( \delta x^a \) in equation 2, is retained.

However, a further symmetry is also apparent for the new expression, namely an invariance under the translation \( x^a \rightarrow x^a + r^a \) for any constant vector \( r_3 = (r^1,r^2,r^3) \in \mathbb{R}^3 \); with \( \delta x^a \rightarrow \delta(x^a + r^a) = \delta x^a \) under this simple transformation. That is, while for the original case with only one dimension the quantity \( \delta s \) can be conceived of as an infinitesimal interval anywhere on the real line by the symmetry of \( \mathbb{R} \), as depicted in figure 1(a), similarly each \( \delta x^a \) expresses an infinitesimal interval anywhere on the real line with each \( x^a \in \mathbb{R} \). Collectively the translation symmetry of the right-hand side of equation 2 over \( \mathbb{R}^3 \) describes the parameter space depicted in figure 1(b), with the interval \( \delta s \) and the relation \( \delta s^2 = \eta_{ab} \delta x^a \delta x^b \) composed at any point \((x^1,x^2,x^3) \in \mathbb{R}^3 \equiv M_3 \). Through this \( \mathbb{R}^3 \) translation symmetry both the metric structure with \( \eta = \text{diag}(+1,+1,+1) \) and the \( SO(3) \) rotation symmetry of equation 2 are exhibited locally throughout the extended manifold \( M_3 \).

Generalising from equation 1 a finite duration of time, represented by an interval of the real line \( s \in \mathbb{R} \), can be equated with an arithmetic composition of \( n \) further real numbers \( x^a \ (a = 1 \ldots n) \) in a large variety of ways, including inhomogeneous
polynomial expressions. However on taking the infinitesimal limit such expressions are constrained to homogeneous $p^{th}$ order polynomials of the form:

$$(\delta s)^p = \alpha_{abc...} \delta x^a \delta x^b \delta x^c \ldots$$

where $p$ is a power, $a, b, c \ldots$ are $p$ indices with values $1 \ldots n$, and with each coefficient $\alpha_{abc...} \in \{-1, 0, 1\}$, such that each non-zero term is of the same order in infinitesimal elements, generalising from equation 2. Further, it is possible to avoid dealing directly with such infinitesimal quantities and express equation 3 itself in terms of generally finite quantities by dividing both sides by $(\delta s)^p$ and defining the $n$-dimensional vector $v_n$ with components $v^a = \delta x^a / \delta s$ for the limit $\delta s \to 0$. This leads directly to the general homogeneous polynomial form (as described for [3] equation 2.9 and [4] equation 11):

$$L(v_n) := \alpha_{abc...} v^a v^b v^c \ldots = 1$$

For the trivial 1-dimensional case with $s = x^1$ and $v^1 = \frac{dx^1}{ds} \equiv \frac{\delta x^1}{\delta s} |_{\delta s \to 0} = 1$ the symmetry $x^1 \to x^1 + r^1$ can be readily visualised as a flow $v^1$ present everywhere on the real line parametrised by $x^1 \in \mathbb{R}$, by close analogy with figure 1(a). In the general case for $n$ dimensions the vector $v_n$ is invariant under translations of the form:

$$v_n = \left\{ \frac{d(x^1 + r^1)}{ds} , \frac{d(x^2 + r^2)}{ds} , \ldots , \frac{d(x^n + r^n)}{ds} \right\}$$

Since this equation is equally valid for all possible constant $r_n = (r^1, \ldots, r^n) \in \mathbb{R}^n$ the condition $L(v_n) = 1$ of equation 4 implicitly holds over the entire $\mathbb{R}^n \equiv M_n$ manifold. This structure is represented in figure 1(c) for the 3-dimensional case with

$$L(v_3) = (v^1)^2 + (v^2)^2 + (v^3)^2 = \eta_{ab} v^a v^b = 1$$

Figure 1: (a) The infinitesimal interval $\delta s$, represented by the short arrows, resides at any location on the real line $\mathbb{R}$. (b) Similarly, since the real variables $\{x^a\} \in \mathbb{R}^3$ of equation 2 are arbitrary this equation applies equally for the particular value $x_0 \in \mathbb{R}^3$ as for $x' = x_0 + r_3$ and over the range $-\infty < r^a < \infty$ for each of $a = 1, 2, 3$. (c) An equivalent observation is made for equation 6, with the finite 3-vector $v_3$ in $L(v_3) = 1$ represented here by the longer arrows. It is through this translation symmetry that a 'base manifold' $M_3 \equiv \mathbb{R}^3$ may be identified.
which is equivalent to equation 2 and with which it shares the same translation symmetry as described for figure 1(b). These figures exemplify the elementary structure of the present theory, here for the particular case with a base space $M_3$ arising from the symmetries of the quadratic form $L(v_3) = 1$ as a particular multi-dimensional expression for the original one-dimensional flow of time.

Further, consistent with the generalisation described immediately before equation 2 (and with the general form of equation 3) we can write a quadratic expression with Minkowski metric $\eta = \text{diag}(+1, -1, -1, -1)$ in four dimensions (with $a, b = 0 \ldots 3$):

\[(\delta s)^2 = (\delta x^0)^2 - (\delta x^1)^2 - (\delta x^2)^2 - (\delta x^3)^2 = \eta_{ab}\delta x^a\delta x^b \] (7)

\[\text{that is: } L(v_4) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = \eta_{ab}v^av^b = 1 \] (8)

For the corresponding 4-dimensional Lorentzian extension of figure 1(c) the Minkowski metric $\eta$ is imported throughout the space $M_4 \equiv \mathbb{R}^4$ through the translation symmetry $x^a \rightarrow x^a + r^a$ for $r_4 = (r^0, r^1, r^2, r^3) \in \mathbb{R}^4$ of $L(v_4) = 1$. The $\text{SO}^+(1, 3)$ symmetry of equations 7 and 8 is also everywhere imported locally onto the manifold $M_4$.

However, rather than taking quadratic space or spacetime forms as fundamental we take a different perspective and treat the time interval $\delta s$ on the left-hand side of equation 7 as the underlying basic entity of the theory. From this point of view equation 7 is interpreted as one of many possible arithmetic decompositions of a one-dimensional temporal interval within the more general form of equation 3. For any homogeneous form $L(v_n) = 1$, including the cubic and higher polynomial expressions implied in this generalisation to equation 4, the translation symmetry in $r_n \in \mathbb{R}^n$ of equation 5 can also be identified and represented by an $n$-dimensional parameter space, similarly as depicted in figure 1(c), however in general without a quadratic ‘metric’ structure as for the case of equations 6 and 8.

On the other hand even for cubic or higher degree polynomial forms $L(v_n) = 1$ a subspace structure for a subset of the components of $v_n$ may exhibit a metrical form, and this observation will be closely associated with the breaking of the symmetry of the full form $L(v_n) = 1$. Indeed if an underlying metric with components $\eta_{ab}$ can be identified within the coefficients $\alpha_{abc...}$ of equation 4 the manifold constructed from the translation symmetry associated with corresponding subcomponents of $v_n$ can be directly identified as an extended geometrical manifold with local metric $\eta$, or even in principle as a physical ‘spacetime’ manifold – similarly as for the full translation symmetry of the form $L(v_4) = 1$ with the metric of equation 8. We next further analyse the geometric structure of $M_4$ for this simplest case before considering a higher-dimensional extension in subsection 2.3.

### 2.2 Geometry of the Spacetime Manifold

The ‘base manifold’ $M_4$ originates from the continuous 4-dimensional parameter space of the 4-dimensional translational freedom of the form $L(v_4) = 1$ in equation 8, which is trivially invariant under $x^a \rightarrow x^a + r^a$ for the four components $v^a = dx^a/\delta s$ with $a = 0 \ldots 3$, as described more generally for equation 5. The set of four variables $\{r^a\} \in \mathbb{R}^4$ can be identified with an initial set of four coordinates $\{x^\mu\} \in \mathbb{R}^4$, with $x^\mu = \delta^a_\mu r^a$ (Greek indices $\{\mu, \nu, \ldots\}$ denote general coordinates while Latin indices
\{a, b, \ldots\} denote any frame in general or an orthonormal frame in particular on \(M_4\) depending on the context.

The Lorentzian structure of the vector space of \(v_4 \in \mathbb{R}^{1,3}\) is transferred onto the tangent space of the manifold \(M_4\), via the translation symmetry, and hence this space acquires the properties of a 4-dimensional pseudo-Riemannian manifold. That is, with the vector field \(v_4(x)\) naturally residing in the tangent space \(TM_4\) to the manifold, with components \(v^a = dx^a / ds\), a metric on \(M_4\) derives locally from the pseudo-Euclidean form \(L(v_4) = \eta_{ab}v^a v^b = 1\) of equation 8, as can be described by the metric components \(g_{\mu\nu}\) in a general coordinate system on \(M_4\) via a tetrad field \(e^a_\mu(x)\):

\[
g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{9}\]

Hence the manifold \(M_4\) inherits its pseudo-Riemannian structure from the Lorentz symmetry of \(L(v_4) = 1\). Incorporating a spatial \(SO(3)\) symmetry, similarly as described for figure 1(c), the structure of the form \(L(v_4) = 1\) and its symmetries contains the skeletal form of a mathematical framework for the description of an apparently external and extended geometrical structure with the Minkowski metric \(\eta_{ab}\) imported from equation 8.

For such a manifold in which there exist global coordinates such that \(g_{\mu\nu}(x) = \text{diag}(+1, -1, -1, -1)\) for all \(x \in M_4\), that is the constant Minkowski metric, we have the 4-dimensional spacetime of special relativity. In this case the local metric \(\eta_{ab}\) has been drawn out globally through the existence of large scale coordinates with respect to which the tetrad field can be expressed simply as \(e^a_\mu(x) = \delta^a_\mu\) in equation 9. In the following we formalise this notion of a flat manifold by considering the geometry of the Lorentz symmetry group \(SO^+(1,3)\) of the form \(L(v_4) = 1\) in relation to that of the manifold \(M_4\), which has been identified through the translation symmetry of the same 4-dimensional form of time.

We first note that in contrast to the ‘translational’ symmetries of a form \(L(v_n) = 1\), acting directly on the components of \(x_n\) as described in equation 5, groups such as \(SO^+(1,3)\) acting directly on the components of \(v_n\) will be referred to generically as ‘isochronal’ symmetries, which encompass both rotations and boosts as well as other transformations for higher symmetry groups leaving a higher-dimensional form of time \(L(v_n) = 1\) invariant. In general the highest-dimensional form under consideration will be denoted by \(L(\hat{\mathbf{v}}) = 1\) with the full isochronal symmetry \(\hat{G}\). The full set of isochronal \(\sigma_h\), with \(h \in \hat{G}\), and translational \(\{r^a\}\), with \(r_n \in \mathbb{R}^n\), symmetries act on the components of \(L(\hat{\mathbf{v}}) = 1\) as:

\[
L \left( \sigma_h \left\{ \frac{d(x^a + r^a)}{ds} \right\} \right) = 1 \tag{10}\]

In this subsection we are considering the relation between the full isochronal \(\hat{G} = SO^+(1,3)\) and full translational \(\{r^a\} \in \mathbb{R}^4\) symmetries of the form \(L(v_4) = 1\). The metric structure on the base manifold \((M_4, g)\) arising from the translation symmetry of equation 8, as described above for equation 9, is represented by the rectangular box employed for the base space in figures 2(a) and (b). It is the local Minkowski metric on \(M_4\) that allows this structure to be interpreted as an extended spacetime manifold rather than simply as a real parameter space \(\mathbb{R}^4\) alone.

The manifold of a Lie group, such as \(\hat{G} = SO^+(1,3)\), itself also exhibits a characteristic geometrical structure. The Maurer-Cartan 1-form \(\theta\) is a canonical object
on any Lie group manifold that, as a Lie algebra-valued 1-form, satisfies the Maurer-Cartan structure equation (see for example [5, 6, 7]):

\[ d\theta + \frac{1}{2}[\theta, \theta] = 0 \]  \hspace{1cm} (11)

where ‘d’ denotes the exterior derivative and the square brackets denote the exterior product for Lie algebra-valued 1-forms.

The two manifolds \( M_4 \) and \( \hat{G} = SO^+(1,3) \), respectively representing the translational and isochronal symmetries of the form \( L(v_4) = 1 \), are linked through the mapping \( h(x) : M_4 \rightarrow \hat{G} \) as depicted in figure 2(a). An initial orthonormal frame field \( \{e_a(x)\} \), with respect to the Minkowski metric \( \eta \) on \( M_4 \), can be transformed to any other orthonormal frame field \( \{e'_a(x)\} \) by the matrix action \( e'_a(x) = e_a(x)h^b_a(x) \) via the group elements \( h(x) \in SO^+(1,3) \), which can be considered as a ‘gauge’ freedom, at every \( x \in M \). Hence the map \( h(x) : M_4 \rightarrow \hat{G} \) expresses the local choice of an orthonormal frame field \( \{e_a(x)\} \) which, since it can be chosen arbitrarily, in general will not represent parallelism on the base manifold.

Since the operations of the exterior algebra of \( p \)-forms are preserved under the pull-back of forms through smooth maps between manifolds the Lie algebra-valued 1-form:

\[ A(x) = h^*\theta(h) \]  \hspace{1cm} (12)

on \( M_4 \) captures the structural properties of the Maurer-Cartan 1-form \( \theta \) on \( \hat{G} \) relative to the map \( h(x) : M_4 \rightarrow \hat{G} \). While on \( \hat{G} \) we have the linear map \( \langle \theta, V(h) \rangle \in L(\hat{G}) \) from \( V(h) \in T_h\hat{G} \) into the Lie algebra of \( \hat{G} \), on \( M_4 \) we have the linear map \( \langle A(x), u(x) \rangle \in L(\hat{G}) \) from \( u(x) \in T_xM_4 \) into the same Lie algebra. The Lie algebra-valued 1-form \( A(x) \) may be written as \( A(x) = A^\alpha_\mu(x) X_\alpha dx^\mu \) where \( \{dx^\mu\} \) is a coordinate basis of 1-forms on \( M_4 \) and \( \{X_\alpha\} \) is a basis for \( L(\hat{G}) \).

Unlike the canonical 1-form on \( \hat{G} \) (which can be written \( \theta = X_\alpha \theta^\alpha \) where \( \{\theta^\alpha\} \) is the basis of 1-forms on \( \hat{G} \) dual to the basis \( \{X_\alpha\} \)) the 1-form \( A \) of equation 12 on
$M_4$ has variable real coefficients $A_{\alpha \mu}^\alpha(x)$ which, however, are not arbitrary but depend upon the choice of gauge function $h(x)$ as well as upon the choice of coordinates $\{x^\mu\}$ on $M_4$. Explicitly, for the matrix group $\hat{G} = \text{SO}^+(1, 3)$, the 1-form $A = h^*\theta$ on $M_4$ can be written in terms of the matrices $h(x) \in \hat{G}$ as:

$$A(x) = h^{-1}dh = h^{-1}\frac{\partial h}{\partial x^\mu}dx^\mu \quad (13)$$

This canonical mathematical object can be interpreted as a connection 1-form on the base manifold $M_4$, to be described more generally in subsection 3.2, formalising the notion of parallelism in a manner which will naturally generalise for the case of finite curvature. Here it is possible to choose a gauge with $A(x) = 0$ everywhere on $M_4$, simply by taking $h(x)$ to be constant in equation 13, and hence we have a flat connection. Indeed, this connection can always be written in terms of ‘pure gauge’, as it is in equation 13, which is one way of defining a flat connection.

By the homomorphism of exterior algebra relations across the pull-back map the Lorentz Lie algebra-valued 1-form $A = h^*\theta$ is also subject to a structure equation corresponding to equation 11, that is:

$$dA + \frac{1}{2}[A, A] = 0 \quad (14)$$

In general the curvature 2-form $F$ on the base manifold can be expressed as:

$$F = dA + \frac{1}{2}[A, A] \quad (15)$$

which transforms under a gauge change $h(x)$ as $F \to F' = h^{-1}Fh = \text{Ad}(h^{-1})F$, that is under the adjoint representation. Equations 14 and 15 then immediately show that the curvature is equal to zero, with $F = 0$ in any gauge, and further expresses the global parallelism implied by the canonical flat connection of equation 12. Since here $F$ is the external Riemann curvature expressed in an orthonormal frame field, the full Riemann tensor vanishes in any general coordinate frame on the manifold $M_4$.

The group $\hat{G} = \text{SO}^+(1, 3)$ was introduced as the isochronal symmetry action on the form $L(v_4) = 1$ and hence the Lie algebra values of $A(x)$ and $F(x)$ are composed from a basis of 4×4 matrices $\{E_a\}$ in a representation of $L(\hat{G})$ acting naturally upon the vectors $u \in TM_4$, that is on the tangent space of the base manifold, and in particular on the vector $v_4 \in TM_4$ originating in the form $L(v_4) = 1$ of equation 8, as depicted in figure 2(b). The parameter space $x \in M_4$ itself arose from the translational symmetry of $L(v_4) = 1$ as described in equation 5. The mathematical objects involved are hence intimately associated with each other, deriving from the symmetries summarised in equation 10.

If the 4-dimensional form of time is embedded in a higher-dimensional form $L(\tilde{v}) = 1$ the full symmetry $\hat{G}$ of the larger form will be broken on employing only the 4-dimensional component of the full translational symmetry to generate the manifold $M_4$, exhibiting a structure that can be interpreted as an external spacetime arena with a local Minkowski metric. In either case the flow of time is not considered to be projected onto a pre-existing 4-dimensional spacetime manifold, rather the extended structure $M_4$ itself is implicit within the symmetries identified for a multi-dimensional form of temporal flow, originally deriving from the interval $\delta s \in \mathbb{R}$ itself as described for
equation 4. That is, the underlying geometrical structure derives from the isochronal and translational symmetries of the possible forms \( L(\hat{v}) = 1 \) in the infinitesimal limit \( s \to \delta s \) for an interval of time.

For the case of \( M_4 \) deriving from the symmetries of the form \( L(\hat{v}_4) = 1 \) while the connection form \( A(x) \) is gauge-dependent the external curvature is zero in any frame, as described for equations 13–15. However for the case in which the Lorentzian manifold is identified through a 4-dimensional subset of the translational symmetry of a higher-dimensional form \( L(\hat{v}) = 1 \) it will be possible, given the extra degrees of freedom, to obtain a finite external curvature. Further, the components of the higher dimensions and symmetries of \( L(\hat{v}) = 1 \), over and above those required to identify the Lorentz vector \( \mathbf{v}_4 \) and the 4-dimensional spacetime manifold, will give rise to fields on the extended space \( M_4 \) which can be interpreted as a ‘matter’ content. In this paper we focus upon the gauge field component arising from the internal symmetry \( G \) of the additional dimensions arising from the breaking of the full symmetry \( \hat{G} \) of \( L(\hat{v}) = 1 \) over the external spacetime \( M_4 \). For this extended case the geometry of the internal gauge curvature can also be investigated, and in fact exhibits a structure correlated with the finite external curvature as we explore in this paper.

2.3 Extra Dimensions and Symmetry Breaking

On considering a higher-dimensional temporal form \( L(\hat{v}) = 1 \) the mathematical basis for obtaining an extended base manifold is found in the application of the symmetry described in equation 5 and figure 1(c) to a 4-dimensional subset of the translational degrees of freedom exhibiting a Minkowski metrical form, as alluded to at the end of the previous subsection. This translation symmetry of such a multi-dimensional form is the underlying means through which the one-dimensional flow of time can in principle be exhibited simultaneously as a flow of physical fields in an extended spacetime, as we consider in this subsection.

We approach the generalisation for \( L(\hat{v}) = 1 \) via the group \( \text{SL}(2, \mathbb{C}) \) as the double cover of \( \text{SO}^+(1,3) \), which may be introduced by first mapping a Lorentz vector \( \mathbf{v}_4 \in \mathbb{R}^{1,3} \) into the space of \( 2 \times 2 \) complex Hermitian matrices \( h_2 \mathbb{C} \) as:

\[
\mathbf{v}_4 = (v^0, v^1, v^2, v^3) \rightarrow h_2 = \mathbf{v}_4 \cdot \sigma = \begin{pmatrix} v^0 + v^3 & v^1 - v^2 i \\ v^1 + v^2 i & v^0 - v^3 \end{pmatrix} \in h_2 \mathbb{C} \quad (16)
\]

where \( \sigma \) denotes the \( 2 \times 2 \) identity matrix \( \sigma^0 \) together with the three Pauli matrices, that is \( \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), as described for ([4] equation 17). Under the mapping of equation 16 we have \( L(\mathbf{v}_4) \to \det(h_2) \) for the quadratic form of equation 8.

While the fundamental representation of \( \text{SL}(2, \mathbb{C}) \) acts on the spinor space \( \mathbb{C}^2 \) the group action for elements \( S \in \text{SL}(2, \mathbb{C}) \) on the space \( h_2 \mathbb{C} \) describes the vector representation given by:

\[
h_2 \rightarrow h'_2 = S h_2 S^\dagger \quad (17)
\]

This maps \( h_2 \rightarrow h'_2 \) onto a new \( 2 \times 2 \) complex Hermitian matrix while preserving the value of the determinant; hence mapping the corresponding components \( v^a \to v'^a \) according to a Lorentz transformation of the real 4-vector \( \mathbf{v}_4 \in \mathbb{R}^{1,3} \). This \( \text{SL}(2, \mathbb{C}) \)
action expresses the symmetry of $L(v_4) = 1$ of equation 8 in a manner that naturally extends to an SL(3, $\mathbb{C}$) symmetry of the cubic polynomial form:

$$L(v_9) = \text{det}(v_9) = 1 \quad \text{with} \quad v_9 \in h_3\mathbb{C}$$  \hspace{1cm} (18)

The extension to this cubic expression, together with the set of symmetry transformations under which it is invariant, places the emphasis on generalising expressions for $\delta s$ on the left-hand side of equation 7, *without* being restricted to quadratic compositions as for the right-hand side of that equation. That is, equation 18 is considered necessarily as a higher-dimensional form of *time* rather than *spacetime*.

While the symmetry group SL(2, $\mathbb{C}$), as a subgroup of SL(3, $\mathbb{C}$), acts on the subspace $h_2\mathbb{C}$ embedded naturally within $h_3\mathbb{C}$ the action of SL(2, $\mathbb{C}$) on the full space $h_3\mathbb{C}$ can also be considered. The $2 \times 2$ matrices $S \in \text{SL}(2, \mathbb{C})$ can be embedded in $3 \times 3$ matrices acting on $v_9 \in h_3\mathbb{C}$ as:

$$v_9 \rightarrow \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_2 \\ \psi \end{pmatrix} = \begin{pmatrix} S^\dagger & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi^\dagger \\ n \end{pmatrix}$$  \hspace{1cm} (19)

This combines the vector representation of SL(2, $\mathbb{C}$) on $h_2 \in h_2\mathbb{C}$ and the spinor representation on $\psi \in \mathbb{C}^2$, together with the scalar denoted $n \in \mathbb{R}$ (in line with the notation used for the same expression in [4] equation 19), in a single symmetry transformation that preserves $L(v_9) = 1$ of equation 18. Hence the choice of a preferred SL(2, $\mathbb{C}$) implies a symmetry breaking pattern aligned with the isomorphism of vector spaces:

$$h_3\mathbb{C} \cong h_2\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{R}$$  \hspace{1cm} (20)

$$v_9 \rightarrow (h_2, \psi, n)$$  \hspace{1cm} (21)

$$9_{SL(3,\mathbb{C})} \rightarrow (4 + 4 + 1)_{SL(2,\mathbb{C})}$$  \hspace{1cm} (22)

where the first $4$ denotes the real dimension of the vector representation, the second $4$ that of the spinor and $1$ is the trivial scalar, in each case as representations of the Lorentz group. Applying the translation symmetry of equation 5 in four dimensions only, corresponding to the subspace of external vectors $v_4 \equiv h_2 \in h_2\mathbb{C} \subset h_3\mathbb{C}$, provides a natural mechanism for breaking the symmetry of the full group SL(3, $\mathbb{C}$) through the necessary identification of the extended 4-dimensional background manifold $M_4$, with a spacetime metric structure, upon which the Lorentz group acts locally similarly as for the base space in figure 2(b). This symmetry breaking is depicted in figure 3 (which is closely analogous to figure 1 of reference [4], here for the full isochronal symmetry group $\hat{G} = \text{SL}(3, \mathbb{C})$ in place of $\text{SO}^+(1,9)$).

The subgroup SL(2, $\mathbb{C}$), as the double cover of $\text{SO}^+(1,3)$, is distinguished in that it acts on tangent space vectors $v_4 \in TM_4$ on the base manifold, as depicted in figure 3(b), and is hence designated the *external* symmetry. While collectively deriving from the components of temporal flow $v_9$, the vector $v_4 \in TM_4$ is physically distinct from the spinor $\psi$ and scalar $n$ both in terms of their properties under external symmetry transformations and in terms of their relation to the external spacetime manifold.
Figure 3: (a) The gauge choice at each $x \in M_4$ depicted in figure 2(a) is extended to $\hat{G} = \text{SL}(3, \mathbb{C})$ as the full isochronal symmetry of the form $L(v_9) = 1$, with (b) only the subgroup $\text{SL}(2, \mathbb{C}) \subset \text{SL}(3, \mathbb{C})$ now acting on $TM_4$ as the double cover of the Lorentz group $SO^+(1, 3)$. A gauge field associated with the internal $U(1)$ symmetry, together with the components $\psi$ and $n$ of $v_9 \in h_3 \mathbb{C}$ in equation 19, can be interpreted as ‘matter fields’ over the external manifold $M_4$, augmenting the ‘vacuum’ case of figure 2(b).

$M_4$ itself. That is, the symmetry breaking is not only felt in the identification of the components of the external vector $v_4 \in TM_4$ projected onto the physical spacetime $M_4$ but in the partitioning of all the components of $v_9$ into irreducible representations of the preferred $\text{SL}(2, \mathbb{C}) \subset \text{SL}(3, \mathbb{C})$, including those with the physical properties of spinors and scalars.

These latter objects, such as $\psi$ and $n$ identified from the ‘internal’ components of $v_9$ in equations 19 and 21, will be associated with and underlie physical ‘particle’ states as deriving from the level of these elementary geometric structures of the theory through the symmetry breaking. In turn the internal symmetry $G$ is required to respect this partitioning into physical states and act collectively upon equivalent $\text{SL}(2, \mathbb{C})$ representation objects as individual entities, for example upon the set of spinors, rather than act more generally upon the individual real components of $v_9$ as for the original full $\text{SL}(3, \mathbb{C})$ symmetry. For the general case this implies that the representations of the internal symmetry $G$ are aligned with those of the external symmetry $\text{SL}(2, \mathbb{C})$, with $G$ consisting of remnant actions of the full isochronal symmetry $\hat{G}$ of $L(\hat{v}) = 1$ which survive the partitioning of $\hat{v}$ under the preferred $\text{SL}(2, \mathbb{C}) \subset \hat{G}$. Particle states hence transform in representation multiplets under a symmetry breaking structure of the direct product form:

$$\text{SL}(2, \mathbb{C}) \times G \subset \hat{G} \quad (23)$$

In the case of the $\hat{G} = \text{SL}(3, \mathbb{C})$ model the internal symmetry $G$ acts on the vector, spinor and scalar objects of equations 20–22 as individual entities, without any mixing between the different types of Lorentz representations. Since for this model there is only one of each of a vector, spinor and scalar we have only one-dimensional representations for $G$, which in turn are only non-trivial for an Abelian internal symmetry as identified.
for a U(1) subgroup in the breaking of the full SL(3, C) symmetry to:

$$\text{SL}(2, \mathbb{C}) \times U(1) \subset \text{SL}(3, \mathbb{C})$$

(24)

Here the external SL(2, C) symmetry ‘locks on’ to the tangent space $TM_4$ leaving the residual internal U(1) symmetry as depicted in figure 3(b). This structure resembles the standard geometric picture for which a bundle of frames is ‘soldered’ to the spacetime $M_4$, while a gauge symmetry bundle is arbitrarily attached, however here with the internal symmetry U(1) acting on the internal $\psi$ components. The internal symmetry U(1) and temporal components $\psi, n$, over and above those required to identify the base manifold $M_4$ as depicted in figure 3(b), give rise to gauge and further matter fields on the extended spacetime. Hence neither the extended spacetime $M_4$ nor the matter it contains are introduced independently of the flow of time itself, rather they derive collectively from the structure and symmetries of equation 18.

The internal one-parameter group U(1), with elements represented by the $3 \times 3$ matrices $U = \text{diag}(e^{i\alpha/2}, e^{i\alpha/2}, e^{-i\alpha})$ parametrised by $\alpha \in \mathbb{R}$, acts non-trivially on the spinor components $\psi$ of equations 19, through $\psi_9 \rightarrow U \psi_9 U^\dagger$, while leaving the components of the vector $\nu_4 \in TM_4$ and the scalar $n$ invariant. In having a trivial action on the external vectors $\nu_4 \in TM_4$ the group U(1) belongs to the stability subgroup $\text{Stab}(TM_4) \subset \hat{G}$, which could, more generally, itself be considered a requirement for a physical internal symmetry. For the case of the model considered only seven of the original sixteen generators of SL(3, C) survive the symmetry breaking to equation 24. The rank-4 Lie group SL(3, C) in fact contains a rank-4 subgroup decomposition, augmenting the rank-3 subgroup of equation 24 as:

$$\text{SL}(2, \mathbb{C}) \times U(1) \times \text{D}(1) \subset \text{SL}(3, \mathbb{C})$$

(25)

with the dilation symmetry D(1) associated with a non-compact generator of SL(3, C). The D(1) group action $\psi_9 \rightarrow D \psi_9 D^\dagger$, with $D = \text{diag}(e^{\lambda/2}, e^{\lambda/2}, e^{-\lambda})$ parametrised by $\lambda \in \mathbb{R}$, exhibits a complementary dilation effect on each of $\nu_4 \in TM_4$, $\psi$ and $n$ in equation 19 for this transformation. This dilation symmetry can be considered non-physical since it both relates different types of Lorentz representations and also does not respect the stability of the external spacetime with tangent space vectors $\nu_4 \in TM_4 \equiv \mathfrak{h}_2 \mathbb{C}$ as a distinguished set of components of $\nu_9 \in \mathfrak{h}_3 \mathbb{C}$, that is D(1) $\not\subset \text{Stab}(TM_4)$.

Identifying the background manifold $M_4$ within the symmetry structures of the mathematical form $L(\nu_4) = 1$ as depicted in figure 2 led to the 4-dimensional Minkowski spacetime of special relativity, that is with zero Riemannian curvature as implied by equations 14 and 15 for the $\hat{G} = \text{SO}^+(1, 3)$ case, as described in the previous subsection. Here breaking the full symmetry $\hat{G} = \text{SL}(3, \mathbb{C})$ of $L(\nu_9) = 1$, in extracting the spacetime base manifold $M_4$ through a subset of four translational degrees of freedom of the higher-dimensional form, leaves a space of symmetries associated with the product space $M_4 \times U(1)$ underlying figure 3(b) and will result in a more flexible and dynamic 4-dimensional spacetime structure as employed for general relativity.

Under the full form $L(\nu_9) = 1$ we also have a looser constraint on the four components $v^a(x)$ projected onto $TM_4$ with:

$$L(\nu_4) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = \eta_{ab} v^a v^b = \eta(\nu_4, \nu_4) = h^2.$$  

(26)

with $h \in \mathbb{R}$. With $M_4$ itself still originating locally out of a 4-dimensional translational symmetry of $L(\nu_9) = 1$ the Minkowski metric $\eta_{ab}$ implicit in the form $L(\nu_4) = h^2$
in equation 26 is sewn into the local tangent space structure everywhere on the base manifold. Since the manifold \( M_4 \) exists through the symmetries of \( L(\mathbf{v}_9) = 1 \) itself there are choices of local coordinates \( \{x^a\} \) such that the vector field \( \mathbf{v}_4(x) = v^a e_a \) has the components \( v^a = dx^a / ds \) with \( L(\mathbf{v}_4) = h^2 \) of equation 26 expressed in this basis. Hence there exists a frame field \( \{e_a(x)\} \) of local orthonormal basis vectors such that \( g(e_a, e_b) = \eta_{ab} \) with respect to the metric \( g(x) \). That is, we have a local Lorentzian structure on \( M_4 \) as for the original case based on the full form \( L(\mathbf{v}_4) = 1 \) of equation 8 as described in the previous subsection. The manifold \( M_4 \) is drawn as a rectangular box in figure 3 to represent the metric geometry of the base manifold \( (M_4, g) \), as it was in figure 2. As will be reviewed in the opening of the following subsection, such a metric structure \( g_{\mu\nu}(x) \) on \( M_4 \) is associated in a one-to-one manner with the existence of an \( \text{SO}^+(1, 3) \) orthonormal frame bundle \( OM_4 \) within the canonical \( \text{GL}^+(4, \mathbb{R}) \) general frame bundle \( FM_4 \) over the base manifold \( M_4 \).

The question then remains to identify a possible relation between the finite external Riemannian curvature of \( M_4 \) and the curvature of the internal gauge fields associated with the internal symmetry group \( G \) identified in equation 23. With the aim of identifying such a relation in both a generally covariant and gauge invariant manner the geometry arising from the structure depicted in figure 3(b) will be considered in more detail for the general case. With the external symmetry acting on \( FM_4 \) the broken symmetry structure of equation 23 deriving from a full higher-dimensional form \( L(\hat{\mathbf{v}}) = 1 \) can be accommodated on the product manifold \( M_4 \times G \). As a unifying framework for combining the external and internal symmetry this geometry is very similar to the principle fibre bundle structure employed in Kaluza-Klein theories, which might be adapted for the present theory as will be described in section 4. In the following section we first review both the textbook geometrical setting and several relevant Kaluza-Klein models in the literature (summarising [3] chapters 3 and 4).

3 Review of Kaluza-Klein Theories

3.1 Riemannian Geometry and General Relativity

In the previous section we have identified a Lorentzian manifold \( M_4 \) with the local Minkowski metric \( \eta \) deriving from the quadratic form \( L(\mathbf{v}_4) \) of equation 8 or 26. In the latter case the identification of this extended metrical manifold breaks the full symmetry of the higher-dimensional form, such as the \( \text{SL}(3, \mathbb{C}) \) symmetry of \( L(\mathbf{v}_9) = 1 \) as described for figure 3.

More generally, any differentiable manifold \( M \) is canonically associated with a linear frame bundle \( FM \), with structure group \( \text{GL}(n, \mathbb{R}) \) where \( n \) is the dimension of the base manifold \( M \), which is an example of a principle fibre bundle as described for the general case in the following subsection. If \( M \) is an \( n \)-dimensional Riemannian manifold \( (M, g) \), that is given a metric field with components \( g_{\mu\nu}(x) \) on the manifold, a subset of distinguished frames may be identified which are orthonormal with respect to the metric. This subset of frames over \( M \) reduces the total space of \( FM \) to a submanifold \( OM \subset FM \) which is itself a principle fibre bundle with structure group \( \text{SO}^+(p, q) \) (or more generally \( \text{O}(p, q) \)) with \( p + q = n \). There is a one-to-one correspondence between metric fields \( g_{\mu\nu}(x) \) on \( M \) and reductions of the structure group from \( \text{GL}^+(n, \mathbb{R}) \) to
SO\(^+(p,q)\) on \(FM\), with each choice of field \(g_{\mu\nu}(x)\) isolating one out of the many possible isomorphic copies of principle \(SO^+(p,q)\)-bundles.

Given a Lorentzian metric in 4-dimensional spacetime, an \(SO^+(1,3)\)-bundle \(OM_4\) can also be extended to the frame bundle \(FM_4\) with an \(SO^+(1,3)\)-valued Lorentz connection \(A(x)\) (an example of which was introduced in equation 12) uniquely inducing a linear connection \(\Gamma(x)\) for the extended bundle space. A \(GL^+(4,\mathbb{R})\)-valued linear connection \(\Gamma\) identified in this way is compatible with the metric, that is \(\nabla g = 0\). Here the kernel symbol \(\nabla\) denotes the covariant derivative on any differentiable manifold \((M,\Gamma)\) with a linear connection \(\Gamma\). While a manifold with both a metric and a linear connection can be denoted \((M,g,\Gamma)\) in the general case \(g(x)\) and \(\Gamma(x)\) need not be related and the two objects may be introduced independently.

With respect to a frame field \(\{e_a\}\) (here the indices \(\{a,b,c\ldots\}\) denote the use of a general frame field on the manifold) the components of a linear connection \(\Gamma^a_{\ bc}\) satisfy the relation \(\nabla e_b = \Gamma^a_{\ bc} e^c \otimes e_a\), that is:

\[
\nabla_c e_b = \Gamma^a_{\ bc} e_a
\]

with \(\nabla_c \equiv \nabla_{e_c}\). In particular this equation establishes the convention for the order of the three indices for \(\Gamma\). We also note here that the various possible sign conventions for general relativity can be distilled down to the \(+\) sign used for the right-hand side of just three expressions in the Riemannian geometry; in this paper we employ the following components:

1) The spacetime metric tensor:

\[
\eta_{ab} = \text{diag}(+1, -1, -1, -1)
\]

With ‘+1’ for the time component this is a natural convention for the present theory based on forms of temporal flow.

2) The Riemann curvature tensor:

\[
R^e_{\ bcd} = e_c \Gamma^a_{\ bd} - e_d \Gamma^a_{\ bc} + \Gamma^a_{\ ec} \Gamma^e_{\ bd} - \Gamma^e_{\ ed} \Gamma^a_{\ bc} - c^e_{\ cd} \Gamma^a_{\ be}
\]

Here \(c^e_{\ cd}\) are the structure coefficients in the general frame field employed (which vanish if a coordinate basis is adopted).

3) The Ricci tensor:

\[
R_{bc} = R^d_{\ bcd} \quad (=- R^d_{\ bdc})
\]

This is equivalent to choosing the sign convention for the Einstein field equation as \(C^e_{\ bc} = -\kappa T^e_{\ bc}\) with positive normalisation constant \(\kappa\).

The above three signs adopted here are ‘(+−−)’ relative to the original discussion of these conventions in [8]. Here a bold type \(\mathbf{R}\) will denote the Riemann tensor, with the components of equation 29, while \(R = g_{bc}R^e_{\ bc}\) is the scalar curvature. Further, for a general linear connection on the manifold \(M\) the components of the torsion tensor \(\mathbf{T}\) can be written as:

\[
T^a_{\ bc} = -\Gamma^a_{\ bc} + \Gamma^a_{\ eb} - c^a_{\ bc}
\]

where again the final term is zero for a general coordinate frame.
On a manifold with a metric the unique linear connection $\Gamma$ that is both torsion-free ($T = 0$) and metric compatible ($\nabla g = 0$) is called the Levi-Civita connection. This linear connection is employed in general relativity and can be written uniquely as a function of the metric tensor components $g_{\mu\nu}(x)$, expressed in a general coordinate frame (hence with $\{\mu, \nu, \rho \ldots\}$ indices) as:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

(32)

For a spacetime manifold $M_4$ with a metric $g_{\mu\nu}(x)$ equation 9 implies that in general a tetrad field $e^a_\mu(x)$ can also be interpreted as the gravitational field. If $A(x)$ is chosen to be the unique torsion-free Lorentz connection expressed in terms of a given tetrad field, then the associated linear connection $\Gamma(x)$ is the unique Levi-Civita connection. While gravitation in Einstein’s original theory of 1915 is described through the freedom of the metric field $g_{\mu\nu}(x)$, together with its relation to the Levi-Civita connection $\Gamma(x)$ of equation 32 on the spacetime manifold $M_4$, an equivalent formulation of general relativity can be given in terms of a tetrad field $e^a_\mu(x)$ together with a Lorentz connection $A(x)$. This latter approach was introduced in 1956 by Utiyama [9] in which general relativity is considered as a type of gauge theory invariant under local Lorentz transformations. (This is also the interpretation of the Lorentz connection $A(x)$ in equations 12–15 for the flat geometry of subsection 2.2).

As well as tensor representations the Lorentz group also has spinor representations via the group SL(2, $\mathbb{C}$) as the double cover of SO$^+(1,3)$. Hence spinor fields can be introduced on a spacetime manifold with an arbitrary metric $g_{\mu\nu}(x)$ via the tetrad field $e^a_\mu(x)$. (For the $L(v_9) = 1$ model described for the present theory in subsection 2.3 such a spinor field derives from the components $\psi \in \mathbb{C}^2$ of the space $\mathfrak{h}_3\mathbb{C}$ as introduced in equation 19 and pictured in figure 3(b)). This structure also permits gravitation to be expressed in terms of an sl(2, $\mathbb{C}$)-valued connection, accommodating a description of both vector and spinor objects in spacetime. However the dynamics of such an SL(2, $\mathbb{C}$) ‘gauge theory’ of gravitation (see for example [10]) are different to those of a standard Yang-Mills gauge theory. Such an approach is also in contrast with Kaluza-Klein theories for which an internal gauge theory itself derives from a structure of general relativity with extra spatial dimensions.

Finally in this subsection we review the standard use of the Lagrangian formalism to derive physical equations of motion as applied to general relativity and gauge theories. In the 4-dimensional spacetime of general relativity the scalar curvature $R = g_{\mu\nu}R^{\mu\nu}$ is adopted as the principle geometric contribution to the total scalar Lagrangian function, with the field equations determined from the Einstein-Hilbert action integral (see for example [11] page 75):

$$I = \int (\alpha(R - 2\Lambda) + \mathcal{L})\sqrt{|g|}d^4x$$

(33)

Here $\Lambda$ is the cosmological constant, $\mathcal{L}$ is the Lagrangian function for matter fields and $\alpha$ is a normalisation constant. The magnitude of the metric determinant $|g|$ is employed in the 4-dimensional invariant volume element $\sqrt{|g|}d^4x$. The vacuum equations for general relativity, that is with $\mathcal{L} = 0$ and $\Lambda = 0$, are obtained by requiring stationarity $\delta I = 0$ for the action in equation 33 under any variation $\delta g_{\mu\nu}$ of the metric components,
leading to the Einstein vacuum equation with the Einstein tensor:

$$G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 0$$  \hfill (34)

For the non-vacuum case the energy momentum tensor \( T^{\mu\nu} \) for a general matter Lagrangian \( \mathcal{L} \neq 0 \) can be defined under variation of the metric \( \delta g_{\mu\nu} \) through:

$$\delta I = \delta \int \mathcal{L} \sqrt{|g|} \, d^4x = \int \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} \sqrt{|g|} \, d^4x$$  \hfill (35)

Hence for the full action integral of equation 33 stationarity under the metric variation gives Einstein’s field equation for the general case, with \( \kappa = \frac{\Lambda}{2G} \) adopted as the normalisation constant (the \( \Lambda = 0 \) case was quoted in item ‘3)’ above):

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = - \kappa T^{\mu\nu}$$  \hfill (36)

The Maxwell Lagrangian for the electromagnetic field is constructed as:

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$  \hfill (37)

in terms of the electromagnetic field strength tensor components \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). Under variation \( \delta A_{\mu}(x) \) of the electromagnetic gauge field \( A_{\mu}(x) \) the Euler-Lagrange equation for \( \mathcal{L}_{\text{em}} \) yields Maxwell’s equation for the source-free case, that is \( \partial_{\mu} F^{\mu\nu} = 0 \) (which can be written \( \nabla_{\mu} F^{\mu\nu} = 0 \) in a curved spacetime). The form of the Lagrangian for a non-Abelian gauge theory is guided by the Abelian case of electromagnetism, motivating the Lorentz and gauge invariant Yang-Mills Lagrangian:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^{\alpha}_{\mu\nu} F_{\alpha}^{\mu\nu}$$  \hfill (38)

as a direct generalisation of equation 37. For the non-Abelian case there is a further contraction over the index \( \alpha = 1 \ldots n_G \), with \( n_G = \text{dim}(G) \) for the generators of the group \( G \), between the adjoint and coadjoint representations, which are related by the Killing metric \( K_{\alpha\beta} \) (which in a suitable basis is simply \( -\delta_{\alpha\beta} \) for the compact simple Lie groups relevant for the internal gauge symmetries in particle physics). In this case the Euler-Lagrange equation for \( \mathcal{L}_{\text{YM}} \) under variation of the gauge field components \( A^{\alpha}_{\mu}(x) \) yields the non-linear second order differential equation:

$$D_{\mu} F^{\alpha\mu\nu} = \partial_{\mu} F^{\alpha\mu\nu} + c^{\alpha}_{\beta\gamma} A^{\beta}_{\mu} F^{\gamma\mu\nu} = 0$$  \hfill (39)

Here \( D_{\mu} \) is the gauge covariant derivative and again \( \partial_{\mu} \) is replaced by \( \nabla_{\mu} \) for a curved spacetime.

In general relativity the energy-momentum tensor \( T^{\mu\nu} \) can be derived directly from the matter Lagrangian \( \mathcal{L} \) as described for equation 35. Substituting the Yang-Mills Lagrangian \( \mathcal{L}_{\text{YM}} \) of equation 38 into equation 35 results in:

$$T^{\mu\nu} = F^{\alpha\mu}_{\nu} F_{\alpha}^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha}_{\rho\sigma} F_{\alpha}^{\rho\sigma}$$  \hfill (40)

This yields an energy-momentum tensor \( T^{\mu\nu} \) which is symmetric, gauge invariant and compiles necessarily with the Einstein equation 36 since it derives from the Einstein-Hilbert action of equation 33 via equation 35. In equation 40 \( \alpha \) is a Lie algebra index (which is absent for the Abelian case of electromagnetism) while all other indices relate to spacetime coordinates on \( M_4 \).
3.2 Principle Fibre Bundle Structure

In subsection 2.2 we introduced two independent differentiable manifolds, the base space \( M_4 \) and Lie group \( \hat{G} = \text{SO}^+(1,3) \) with points labelled by \( x \in M_4 \) and \( h \in \hat{G} \), associated with the 4-dimensional form of temporal flow \( L(v_4) = 1 \) of equation 8 through the respective ‘translational’ \( \{ r^a \} \) and ‘isochronal’ \( \sigma_h \) symmetries as described for equation 10. The map between the manifolds \( h : M_4 \to \hat{G} \), mapping \( x \to h(x) \) as depicted in figure 2(a), represents a local choice of gauge, or orthonormal frame, in which to express the tangent vector \( v_4(x) \) on \( M_4 \). This association between \( M_4 \) and \( \hat{G} \) may be examined more concisely through the structure of a single differentiable manifold, namely a principle fibre bundle \( P \), which combines the geometric properties of a base space \( M \) and a Lie group \( G \) together with their mutual relation.

In the general case the structure group \( G \) of a principle fibre bundle \( P = (M, G) \) does not need to be related to a symmetry on the tangent space to the base manifold \( M \), as it is for the above case of figure 2. Indeed for the augmented case of figure 3(a) the full symmetry group \( \hat{G} = \text{SL}(3, \mathbb{C}) \) acts only partially on the tangent space of \( M_4 \) while the internal symmetry group \( G = \text{U}(1) \) does not act on the external tangent space at all, as pictured in figure 3(b). This latter figure contains information about both the external and internal symmetry, the geometries of which we are ultimately aiming to relate. Hence it is the generalisation in which \( M \) and \( G \) are initially introduced independently, and then related through the principle bundle space \( P = (M, G) \), that we shall review here for the benefit of the subsequent application to the case of a higher-dimensional form \( L(\hat{v}) = 1 \) such as presented in subsection 2.3. (For more details on principle bundle structures generally see for example [5, 6, 7]).

For the present theory it will be assumed that the structure of principle bundles with a trivial global topology will be sufficient. In this case the bundle \( P = (M, G) \) is diffeomorphic to the product space \( M \times G \), which can be expressed as \( P \equiv U \times G \) where a single ‘subset’ \( U \subset M \) may be identified with the entire base manifold \( M \). This triviality is implied in deriving the bundle structure through the symmetries of \( L(\hat{v}) = 1 \) as described for figures 1(c), 2 and 3. In this case a change of trivialisation, or gauge transformation, may apply over the entire volume of the base space \( M \).

The Lie algebra \( L(G) \) of a Lie group \( G \) can be represented by a basis of left-invariant vector fields \( \{ X_\alpha \} \), for \( \alpha = 1 \ldots n_G \), on the group manifold, that is with \( L_{h*}X_\alpha(h') = X_\alpha(hh') \) where \( L_{h*} \) is the differential of the left action \( L_h \) of \( G \) on itself and with \( h,h' \in G \). While the right action of \( G \) on the group manifold itself induces left-invariant vector fields, including any basis vector \( X_\alpha \), the right action \( R_{\exp(tA)} : p \to p\exp(tA) \) of \( G \) on points of a principle bundle manifold \( p \in P \equiv M \times G \) induces ‘vertical’ vector fields in the tangent space \( TP \) with:

\[
V_p^A(f(p)) = \frac{d}{dt} f(p \exp(tA)) \big|_{t=0}
\]  

(41)

where \( f(p) \) is any smooth real-valued function on the bundle space, \( A \in L(G) \) and \( V_p^A \) is a tangent vector to the fibre of \( P \) at the point \( p \in P \). The map \( A \to V_p^A \) described in equation 41 represents an isomorphism of the Lie algebra \( L(G) \) into the space of vector fields residing in the vertical tangent space \( VP \subset TP \).

While this structure relates to \( VP \), the space of vectors tangent to the individual fibres of \( P \), different fibres may be related by an additional structure called a connection.
on the principle bundle which, conceptually, is a smooth assignment of a ‘horizontal’ subspace $H_pP$ of the full tangent space $T_pP$ at each point $p \in P$ such that:

$$T_pP = V_pP \oplus H_pP \tag{42}$$

$$R_{h*}H_pP = H_{ph}P \tag{43}$$

Compatibility of the horizontal subspaces on $P$ with the right action $R_h$ of $G$ on the bundle space is described by the latter requirement, where $R_{h*}$ is the differential of this map for any $h \in G$. The tangent space decomposition of equation 42 is sketched in figure 4.

![Figure 4: Vertical and horizontal basis vectors in a local trivialisation $U \times G$ of a principle bundle $P$, together with their associated basis vectors on the group space $G$ and the base manifold $M$ respectively. (Since the bundle is trivial we take $U = M$.)](image)

At every point $p \in P$ a basis for the tangent space of the principle bundle can be expressed in terms of these complementary subspaces. Such a basis $\{\hat{e}_i\} = \{\hat{e}_\alpha, \hat{e}_a\}$ consists of the subset $\{\hat{e}_\alpha\} \in VP$, with $\alpha = 1 \ldots n_G$, (that is vectors of the form $V_pG$ in equation 41, tangent to the fibres $G_x$ over each point $x \in M$) and the subset $\{\hat{e}_a\} \in HP$, with $a = 1 \ldots n$, (where $\hat{e}_a$ is the ‘horizontal lift’ of the basis vector $e_a \in T_xM$ to the point $p \in P$ such that $\pi_*\hat{e}_a = e_a$, where $\pi$ is the projection between manifolds $\pi : P \to M$).

An ‘acute’ mark above a kernel symbol, such as for $\hat{e}$, denotes an object defined on a principle bundle space in the horizontal lift basis. In all cases the indices $\{i, j, k \ldots\}$ denote elements defined in the tangent space $TP$ on the principle bundle $P$ itself; $\{\alpha, \beta, \gamma \ldots\}$ in the vertical subspace on $P$ or on the manifold $G$; and $\{a, b, c \ldots\}$ in a complementary subspace on $P$ or on the base space $M$. There is a one-to-one correspondence between basis vector fields $\hat{e}_\alpha$ on $P$ and basis vector fields $X_\alpha$ on $G$, and similarly between basis vector fields $\hat{e}_a$ on $P$ and basis vector fields $e_a$ on $M$. The relations between these vector fields are implied in figure 4.

The defining structure for the horizontal subspace $HP$ of equation 42 and 43 can be specified via a smooth Lie algebra-valued 1-form $\omega \in L(G) \otimes TP$, mapping vectors $X \in TP$ into elements of $L(G)$. This connection 1-form $\omega$ on $P$ has the properties (the
right translation and hence equation 49 expresses the right-invariance of the fields
\[ R_h^* \omega = \text{Ad}(h^{-1}) \omega \quad \text{i.e.} \quad R_h^* \omega_p(X) = h^{-1} \omega_p(X) h \] where \( H_p \equiv \{ X \in T_p P \mid \omega(X) = 0 \} \)

is the horizontal subspace. Here \( R_h^* \) is the pull-back map associated with the right action \( R_h \) of the group on \( P \).

The specification of a connection \( \omega \) and corresponding horizontal subspace allows ‘parallel transport’ between the fibres to be defined. As depicted within figure 5 in subsection 4.1 given a point \( p_1 \in P \) with \( \pi(p_1) = x_1 \) and a curve \( C \) on the base space from \( x_1 \) to \( x_2 \) a connection on a principle bundle \( P \) specifies a unique horizontal lift of the curve \( C \) to the curve \( C' \) on \( P \), by advancing locally within the horizontal subspace \( H P \subset TP \). That is, the tangent vector to the curve \( C' \) at any \( p \in P \) always lies within the horizontal subspace \( H_p \) and projects under \( \pi \) to a tangent vector to \( C \) at \( x = \pi(p) \). The path \( C' \) then represents the parallel transport of \( p_1 \) mapped between fibres to the unique point \( p_2C' \in P \), with \( \pi(p_2C') = x_2 \). In subsection 4.1 this geometric structure of the gauge connection \( \omega \) on the bundle space will be modelled by a parallel transport according to a linear connection \( \check{\Gamma}(p) \) defined on \( P \) itself.

Since \( P \) itself is a differentiable manifold real-valued structure coefficients \( \check{c}_{jk}(p) \) can be defined for any frame field \( \{ \check{e}_i \} \) on \( P \) through the relation (with a ‘check’ on an object as for \( \check{e} \) denoting a field on the bundle space generally):

\[ [\check{e}_j, \check{e}_k] = \check{c}^i_{jk}(p) \check{e}_i \] (47)

The horizontal lift basis \( \{ \check{e}_i \} = \{ \check{e}_a, \check{e}_a \} \) for the tangent space \( TP \), introduced above, is adapted to a given connection \( \omega \) such that \( \check{e}_a \in V_p P \) and \( \check{e}_a \in H_p P \) at any \( p \in P \), as was depicted in figure 4, with \( \omega(\check{e}_a) = \check{X}_a \) and \( \omega(\check{e}_a) = 0 \), by the definition of the horizontal lift basis. In this basis the full set of structure coefficients on \( P \) are given by:

\[ [\check{e}_\alpha, \check{e}_\beta] = c^\gamma_{\alpha\beta} \check{e}_\gamma \] (48)
\[ [\check{e}_\alpha, \check{e}_b] = 0 \] (49)
\[ [\check{e}_a, \check{e}_b] = \check{c}^\alpha_{ab} \check{e}_\alpha = -F_{ab}(p) \check{e}_\alpha \] (50)

In equation 48 the \( c^\gamma_{\alpha\beta} \) are the structure constants of the Lie algebra of \( G \), expressing the Lie algebra isomorphism described after equation 41. Since right actions induce the basis vectors \( \check{e}_a \) of the subspace \( VP \), via equation 41, each \( \check{e}_a \) generates a right translation and hence equation 49 expresses the right-invariant of the fields \( \check{e}_b \in HP \), consistent with equation 43. For the third equation the structure coefficients \( \check{c}^\alpha_{ab} \) are zero since here a coordinate basis is taken for \( \{ e_a \} \) on the base manifold \( M \) in order to simplify the expressions. In fact with \( \text{d}e^a = 0 \) for the dual coframe basis \( \{ e^a \} \) on \( M \) and each \( \check{e}^a = \pi^* e^a \) on \( P \) we have \( \text{d}\check{e}^a = -\frac{1}{2} \check{c}^a_{jk} \check{e}_j \wedge \check{e}_k = 0 \) and hence each \( \check{e}^a_{jk} = 0 \). Equation 50 demonstrates the intimate relationship between the horizontal lift basis and the physical manifestation of the gauge connection in terms of the internal curvature components \( F^\alpha_{ab}(p) \) on the principle bundle.

Here the components of the Lie algebra-valued curvature 2-form \( \Omega(p) \) on a principle bundle \( P \) are denoted \( F^\alpha_{ab}(p) \) in order to match the notation of the references.
for the following two subsections. The curvature 2-form itself on \( P \) is defined by:

\[
\Omega = d\omega \circ \text{hor}
\]

where \( d\omega \) is the exterior derivative of the connection 1-from \( \omega \) and ‘hor’ maps vectors \( X \) in the tangent space \( TP \) to their horizontal components in the decomposition of equation 42, with the vertical component mapped to zero (that is, hor : \( X \rightarrow X_H \), such that \( X_H \subseteq HP \) with \( \omega(X_H) = 0 \) by equation 46).

For a particular trivialisation \( \psi : P \rightarrow M \times G \) on the principle bundle, with corresponding section \( \sigma(x) : M \rightarrow P \) mapping \( x \in M \) to \( \psi^{-1}(x, e) \) with \( e \in G \) the group identity element, a direct product basis \( \{ \hat{\epsilon}_i \} \) for the tangent space consists of the subset \( \{ \hat{e}_\alpha \} \in VP \), tangent to the fibres \( G_x \) over each point \( x \in M \), and the subset \( \{ \hat{\epsilon}_a \} \) with \( \hat{\epsilon}_a = \sigma_* e_a \) for each basis vector \( e_a \in T_x M \) (hence with \( \pi_* \hat{\epsilon}_a = e_a \)). Each vector \( \hat{\epsilon}_a \) defined on the section \( \sigma(x) \) is Lie transported via the right action of \( G \) on \( P \) such that the basis covers the entire principal bundle. The ‘double dot’ mark above the kernel symbol, such as for \( \hat{\epsilon} \), denotes an object defined on a principle bundle space in the direct product basis. Relative to the horizontal lift basis \( \{ \hat{\epsilon}_i \} \) we have:

\[
\hat{\epsilon}_\alpha = \hat{\epsilon}_\alpha \quad \text{and} \quad \hat{\epsilon}_a = \hat{\epsilon}_a + \omega^a_{\alpha} \hat{\epsilon}_\alpha
\]

where \( \omega^a_{\alpha}(p) \) are the components of the connection on \( P \) for this section. With \( \Omega = \Omega^x X_\alpha \) the curvature components at any \( p \in P \):

\[
F^a_{\alpha b} = \Omega^\alpha(\hat{\epsilon}_a, \hat{\epsilon}_b) = \Omega^\alpha(\hat{\epsilon}_a, \hat{\epsilon}_b)
\]

are identical in the horizontal lift basis and a direct product basis since the curvature vanishes on vertical components, equation 51, and these bases differ only by a vertical vector, equation 52. However, while the horizontal lift basis represents a physical geometrical structure a direct product basis represents a passive choice of gauge.

Given a section \( \sigma(x) \) on \( P \) the representative of the curvature \( \Omega \) on the base space is defined by the pull-back map as the 2-form \( F(x) = \sigma^* \Omega(p) \) while the representative of the connection \( \omega \) is the gauge field \( A(x) = \sigma^* \omega(p) \) on \( M \). Equation 51 above can be written as the structure equation \( \Omega = d\omega + \frac{1}{2}[\omega, \omega] \) on \( P \) which pulls-back to equation 15 on the base manifold \( M \). The curvature on the base manifold \( F(x) = F^\alpha(x) X_\alpha \) also takes values in the Lie algebra. With a coordinate basis employed on \( M \) the components of the curvature \( F \) are written \( F^\alpha_{\mu\nu}(x) \) on the base manifold, as is the case for equations 37–40. In general for a particular section \( \sigma : M \rightarrow P \), and corresponding trivialisation \( P \equiv M \times G \), the components of the curvature and gauge fields on \( M \) can be expressed respectively as:

\[
F^\mu_{\alpha\nu}(x) = \sigma^* \Omega^\alpha(e_\mu, e_\nu) = \Omega^\alpha(\sigma_* e_\mu, \sigma_* e_\nu) = \delta^\alpha_{\mu} \delta^\beta_{\nu} F^\beta_{ab}(x, e)
\]

\[
A^\mu_{\alpha}(x) = \sigma^* \omega^\alpha(e_\mu) = \omega^\alpha(\sigma_* e_\mu) = \delta^\alpha_{\mu} \omega^\alpha_{\alpha}(x, e)
\]

where \( \{ e_\mu \} \) is a coordinate basis on \( M \). The fibre dependence of the curvature components \( F^a_{\alpha b}(p) \) on \( P \) may be deduced by application of the Jacobi identity in the horizontal lift basis and use of equations 49 and 50:

\[
\begin{align*}
[\hat{\epsilon}_\alpha, [\hat{\epsilon}_a, \hat{\epsilon}_b]] + [\hat{\epsilon}_a, [\hat{\epsilon}_b, \hat{\epsilon}_\alpha]] + [\hat{\epsilon}_b, [\hat{\epsilon}_\alpha, \hat{\epsilon}_a]] &= 0 \\
\Rightarrow [\hat{\epsilon}_\alpha, F^\beta_{ab} \hat{\epsilon}_b] + 0 + 0 &= 0 \\
\Rightarrow (\hat{\epsilon}_\alpha F^\beta_{ab}) \hat{\epsilon}_b + F^\gamma_{ab} \epsilon_\alpha \epsilon_\beta &= 0 \\
\Rightarrow \hat{\epsilon}_\alpha F^\beta_{ab} &= -\epsilon_\alpha \epsilon_\gamma F^\gamma_{ab}
\end{align*}
\]
The final expression describes the directional derivative of the coefficients $F_{\beta}^{ab}$ with respect to the vector field $\dot{e}_a$, which generates right translations by the gauge group $G$, and hence expresses gauge transformations of the curvature $\Omega$. This is the expected transformation property for the components of the Lie algebra-valued curvature 2-form under infinitesimal gauge transformations, since the curvature transforms under the adjoint representation for finite gauge transformations, as expressed immediately after equation 15 for example.

### 3.3 General Relativity with Extra Dimensions

The unifying framework for gravitation and gauge theories reviewed here is constructed in the mathematical setting of a principal fibre bundle. Keeping within the spirit of Einstein’s original 4-dimensional spacetime theory of gravitation and the incorporation of electromagnetism in the extension to a 5-dimensional arena by Kaluza and Klein [1, 2], the generalisation for geometric unification with non-Abelian gauge theory is founded upon a metric tensor $\tilde{g}$, now defined upon the manifold of the principle bundle $P = (M_4, G)$ itself ([12], see also [13], [14] sections I–V and [15]).

We note that conventions vary in the literature – in particular with respect to the assignment of index labels such as \{a, b, ...\}, \{\alpha, \beta, ...\}, and \{i, j, ...\} which in this paper are associated with objects on the manifolds $M_4$, $G$ and $P$ respectively, in the manner described alongside and in figure 4. The conventional order of the indices for the linear connection coefficients $\Gamma^a_{bc}$ also varies, with the convention of equation 27 adopted here, while the sign of the Ricci tensor $R_{bc} = R^d_{bcd}$ of equation 30 also differs in some of the references. Hence in turn a number of derived expressions here will have signs differing to those in the literature.

In addition to the metric $g_{ab}(x)$ on the base manifold $M_4$ the Ad($G$)-invariant Killing form $K$ defines a natural bi-invariant metric on the group manifold $G$. That is, both the left $L_h$ and right $R_h$ group actions, for any $h \in G$, are isometries on $G$, with for example $(R^*_h K_{h'h'})(X,Y) = K_{h'}(X,Y)$ for all $X,Y \in T_h'G$ for the Killing metric $K$ at any point $h' \in G$. As a matrix of components $K$ is invertible provided $G$ is a semi-simple Lie group and negative definite if $G$ is compact. In the latter case a basis for the Lie algebra can be chosen such that the Killing form has components $K_{\alpha \beta} = -\delta_{\alpha \beta}$, as noted after equation 38. Here we choose metric components $g_{\alpha \beta} = +K_{\alpha \beta}$ in order to match the signature convention of equation 28, with spacelike components having a negative norm. In terms of the group structure constants $c^\sigma_{\alpha \beta \gamma}$ in a left-invariant basis \{${X}_\alpha$\} on the group manifold the components of the Killing metric are:

$$g_{\alpha \beta} = K_{\alpha \beta} = c^\rho_{\alpha \sigma} c^\sigma_{\beta \rho}$$ (57)

A gauge connection 1-form $\omega$ on a principle bundle $P$ specifies a right-invariant horizontal subspace $HP$ of the tangent space $TP$, as described for equations 44–46. A unique metric $\tilde{g}$ may be defined on such a principle bundle space, aligned with the gauge connection structure with:

$$\tilde{g}(X, Y) = g(\pi_\ast X, \pi_\ast Y) + K(\omega(X), \omega(Y))$$ (58)

where $X, Y \in TP$, while here $g$ and $K$ are the metrics on the base space $M_4$ and group space $G$ respectively. This construction yields an intuitively natural metric on...
the bundle space in the sense that the vertical VP and horizontal HP subspaces of the
tangent space of P, as depicted in figure 4, are then orthogonal with respect to \( \hat{g} \), with
\( \hat{g}(X, Y) = 0 \) if \( X \in VP \) and \( Y \in HP \) for example.

Alternatively, and perhaps more in the spirit of the original Kaluza-Klein theory, a metric \( \hat{g} \) rather than a connection \( \omega \) can be considered as the fundamental entity on \( P \). That is, the bundle is initially endowed with a pseudo-Riemannian metric \( \hat{g}(p) \) with certain restrictions – namely compatibility with a Lorentzian metric \( g_{ab}(x) \) for vectors projected onto \( M_4 \) and with the Killing metric \( g_{\alpha\beta} \) for vectors tangent to the fibres \( G_x \) and the requirement of invariance under the right action of \( G \) on \( P \):

\[
(R_h^* \hat{g})(X, Y) = \hat{g}_p(X, Y) = \hat{g}_{ph}(R_* X, R_* Y)
\]

for any \( p \in P, h \in G \) and \( X, Y \in TP \). This latter property then implies the existence of a subspace \( HP \), orthogonal to \( VP \), which is right-invariant and hence is equivalent to the existence of a connection 1-form \( \omega \) on the bundle \( P \), which is related to \( \hat{g} \) as described in equation 58.

From either perspective from the relation of \( \hat{g} \) to \( \omega \) in equation 58 in the horizontal lift basis \( \{ \hat{e}_i \} = \{ \hat{e}_\alpha, \hat{e}_a \} \), with \( \hat{e}_\alpha \in VP \) and \( \hat{e}_a \in HP \), for the tangent space on \( P \) the metric \( \hat{g} \), and its inverse, take respectively the simple forms:

\[
\hat{g}_{ij} = \begin{pmatrix} g_{ab} & 0 \\
0 & g_{\alpha\beta} \end{pmatrix} \quad \text{and} \quad \hat{g}^{ij} = \begin{pmatrix} g^{ab} & 0 \\
0 & g^{\alpha\beta} \end{pmatrix}
\]

(60)

That is with the components of the metric on the base space \( M_4 \) identified as \( g_{ab} = \hat{g}(\hat{e}_a, \hat{e}_b) \) and those of the Killing metric on the group space identified as \( g_{\alpha\beta} = \hat{g}(\hat{e}_\alpha, \hat{e}_\beta) \).

The off-diagonal block components in equation 60 are all zero, with for example \( \hat{g}_{a\beta} = \hat{g}(\hat{e}_a, \hat{e}_\beta) = 0 \) describing the orthogonality of any \( X = X^a \hat{e}_a \in H_p P \) to any \( Y = Y^\beta \hat{e}_\beta \in V_p P \) at any \( p \in P \) with respect to this right-invariant metric \( \hat{g} \).

Under a change of frame from a horizontal lift basis to a direct product basis
\( \{ \hat{e}_i \} \rightarrow \{ \check{e}_i \} \), aligned with a particular choice of trivialisation \( \psi : P \rightarrow M_4 \times G \) as described for equation 52, equation 60 is transformed to:

\[
\check{g}_{ij} = \begin{pmatrix} g_{ab} + g_{\alpha\beta} \omega^a_{\alpha} \omega^b_{\beta} & \omega^a_{\alpha} g_{\alpha\beta} \\
g_{\alpha\beta} \omega^a_{\beta} & g_{\alpha\beta} \end{pmatrix} \quad \text{and} \quad \check{g}^{ij} = \begin{pmatrix} g^{ab} & -g^{ab} \omega^\beta_{\beta} \\
-\omega^a_{\alpha} g^{ab} & g^{\alpha\beta} + g^{ab} \omega^\alpha_{\alpha} \omega^\beta_{\beta} \end{pmatrix}
\]

(61)

In this latter basis the trivialisation dependent components \( \omega_{\alpha}^a(p) \) of the connection 1-form on \( P \) for the non-Abelian internal symmetry are found alongside the Killing metric components \( g_{\alpha\beta} \) and the external spacetime metric components \( g_{ab} \), framed within the elements of the full metric \( \check{g}_{ij} \) on the bundle space. The form of equation 61 is preserved under any coordinate transformation on \( P \) corresponding to a different choice of gauge section \( \sigma : M_4 \rightarrow P \) (i.e. choice of trivialisation diffeomorphism \( \psi : P \rightarrow M_4 \times G \)) as well as under general coordinate transformations on \( M_4 \), but is not generally covariant under arbitrary changes of coordinates on \( P \).

As described in the opening of subsection 3.1 a principle bundle of linear frames can be constructed over any differentiable manifold, including the case for which the ‘base manifold’ is actually the space of a given principle fibre bundle \( P \) itself. While
the metrics $g$ and $K$ on the manifolds $M_4$ and $G$ can be naturally extended to the metric $\tilde{g}$ of equation 58 on the principle bundle $P = (M_4, G)$ with a gauge connection $\omega$, a linear connection $\Gamma(x)$ on the manifold $M_4$ can also be generalised to the domain of the larger manifold $P$. As described for equation 27 such a linear connection $\tilde{\Gamma}(p)$ will define covariant differentiation with $\tilde{\nabla}e_j = \tilde{\Gamma}^{i}{}_{j} \otimes \tilde{e}_{i}$ in a general tangent space basis $\{\tilde{e}_i\}$ for $TP$ with dual basis $\{\tilde{e}^{i}\}$ for $T^*P$, where

$$\tilde{\Gamma}^{i}{}_{j} = \tilde{\Gamma}^{i}{}_{j} e^{k}$$

are a set of linear connection 1-forms on $P$ as the base space of the frame bundle $FP$. The identification of the smooth symmetric gauge covariant rank-2 tensor field $\tilde{g}$ on $P$, equation 58, endows the principle bundle itself with the structure of a pseudo-Riemannian manifold $(P, \tilde{g})$. In turn a connection $\tilde{\Gamma}$ compatible with the metric $\tilde{g}$, and hence with the geometric structure of the underlying manifold $P$, may be extended from the notion of a metric connection on $M_4$, defining a structure denoted $(P, \tilde{g}, \tilde{\Gamma})$.

Further guided by Einstein’s general theory of relativity in 4-dimensional space-time, the unique linear connection which is torsion-free, $\tilde{T} = 0$, and compatible with the metric, $\tilde{\nabla}g = 0$, that is the Levi-Civita connection, may be defined on the bundle space $P$. The corresponding connection coefficients can be written, defining $\Gamma_{ijk} = g_{d} \Gamma^{l}{}_{jk}$ and $c_{ijk} = g_{d} c^{l}{}_{jk}$, as:

$$\tilde{\Gamma}_{ijk} = \frac{1}{2}(\tilde{e}_{j}(\tilde{g}_{ik}) + \tilde{e}_{k}(\tilde{g}_{ij}) - \tilde{e}_{i}(\tilde{g}_{jk})) - \frac{1}{2}(\tilde{c}_{ijk} + \tilde{c}_{kji} + \tilde{c}_{jki})$$

which expresses equation 32 in a general frame. These coefficients take a relatively simple form in the horizontal lift basis on $P$, as employed for the metric in equation 60 and the structure coefficients of equations 48–50, with a coordinate basis adopted on the base space $M_4$. In this basis the connection coefficients $\Gamma^{a}{}_{bc}(x)$ on the base space $M_4$ contribute to the set in equation 63 for $x = \pi(p)$, with:

$$\tilde{\Gamma}^{a}{}_{bc}(p) = \Gamma^{a}{}_{bc}(x) = \frac{1}{2} g^{ad}(e_{b}(g_{cd}) + e_{c}(g_{bd}) - e_{d}(g_{bc}))$$

which is simply equation 32, since the structure coefficients $\dot{e}_{bc}(p)$ on $P$ vanish in this basis. The connection coefficients $\tilde{\Gamma}_{ijk}(p)$ are also related to the internal curvature through equation 63 since in the horizontal lift basis, by equation 50, we have $\tilde{\dot{c}}_{ab}(p) = -F_{ab}(p)$. In fact via equation 63 we find in the horizontal lift basis on the bundle $P$ terms such as (see [12] equation 22):

$$\tilde{\Gamma}^{a}{}_{ab} = +\frac{1}{2} F^{a}{}_{ab} \quad \text{and} \quad \tilde{\Gamma}^{a}{}_{b\gamma} = \Gamma^{a}{}_{b\gamma} = +\frac{1}{2} g^{ac} g_{\gamma\beta} F^{\beta}{}_{bc}$$

The complete set of coefficients for the Levi-Civita connection on $P$ is listed under ‘Cho [12]’ as the first case in table 1 in the following subsection.

In turn the components of the Riemann curvature tensor $R_{ijkl}$ can be calculated for this Levi-Civita connection on $P$ via equation 29. Hence this Riemann curvature on the total bundle space $P$ is intimately related to both the external curvature on $M_4$ via equation 64 and the internal curvature, associated with gauge group $G$, which is drawn into the Riemannian geometry through equation 65.
Having the metric $\hat{g}_{ij}$ on $P$ the Ricci tensor $\hat{R}_{jk} = \hat{g}^{il} \hat{R}_{ijkl}$ (equation 30) and scalar curvature $\hat{R} = \hat{g}^{ij} \hat{R}_{ij}$ may also be computed, where here the latter is found in the horizontal lift basis to be (with differing sign convention to [12]):

$$\hat{R}(p) = R_M + R_G + \frac{1}{4} F^2$$  \hspace{1cm} (66)

Here $R_M$ is the usual scalar curvature on the base manifold (which varies with the point $x = \pi(p) \in M_4$ under $p \in P$) and $R_G$ is the constant scalar curvature on the group manifold $G$. The term $F^2 = F_{ab}^\alpha(p) F_{\alpha}^{ab}(p)$, constructed from the connection on $P$ can be expressed as $F^2 = F_{\mu\nu}^\alpha(x) F_{\alpha}^\mu\nu(x)$ in terms of the non-Abelian gauge fields on $M_4$, with the gauge covariant curvature components $F_{\mu\nu}^\alpha(x)$ on the base space $M_4$ deriving from $F(x) = \sigma^\alpha \Omega(p)$ as described for equation 54 towards the end of subsection 3.2. (In this sense the $F_{ab}^\alpha$ entries in table 1 can be interpreted as curvature components directly on $M_4$). Hence each term in equation 66 is gauge invariant.

As a scalar $\hat{R}(p)$ in equation 66 is a quantity which is also independent of the basis $\{\hat{e}_i\}$ in which it is determined. The equations of motion for the theory are then derived by adopting $\hat{R}(p)$ as the scalar Lagrangian function together with $\sqrt{|\hat{g}|} d^4x d^nG$ as the invariant volume element, where $|\hat{g}|$ is the magnitude of the determinant of the metric $\hat{g}_{ij}$ on $P$, in the Einstein-Hilbert action integral on the bundle space:

$$I_m = \int \hat{R} \sqrt{|\hat{g}|} d^4x d^nG$$  \hspace{1cm} (67)

with $m = 4 + n_G$. The integration over the group manifold $G$, with volume $V_G$, is trivial and the above expression reduces to the 4-dimensional action integral:

$$I_4 = V_G \int \hat{R} \sqrt{|g|} d^4x$$  \hspace{1cm} (68)

where $|g|$ is here the determinant of the metric $g_{ab}$ on $M_4$. The variational principle is then applied under the constraint $\delta I_m = 0$, and hence $\delta I_4 = 0$, with respect to restricted variations of the metric $\delta \hat{g}$ on the bundle space, consistent with equation 59, as will be discussed further before equation 74 in the following subsection. Within this restriction this again follows the prescription for the original theory of general relativity on a 4-dimensional spacetime manifold $M_4$ with scalar curvature $\hat{R} \equiv R_M$ for which the field equations can be determined from the Einstein-Hilbert action integral of equation 33.

By comparison of equations 66 and 68 with 33 the constant $R_G$ in this version of Kaluza-Klein theory appears as a cosmological constant term which, however, is problematically too large by a factor of $\sim 10^{120}$ if a natural normalisation is used with the length scale of the group space $G$ taken to be of order the Planck length [12]. On the other hand the $F^2$ term in equation 66 effectively contributes the content for the matter Lagrangian $\mathcal{L}$ in equation 33 in the form of equation 37 or 38. Hence, as a particularly elegant feature of Kaluza-Klein theory, the external geometry of the 4-dimensional spacetime manifold along with a matter contribution from the internal gauge fields is identified within a single geometrical object in the form of the scalar curvature $\hat{R}$ on the principle bundle space.
3.4 Theories with Torsion on the Bundle

The problematic cosmological term $R_G$ in equation 66 can be addressed by exploiting the flexibility within the Kaluza-Klein approach on a principle fibre bundle that opens up if the metric $\hat{g}_{ij}$ is not treated as the fundamental field of the theory, as it was from the perspective of the paragraph leading to equation 59. While the same natural metric $\hat{g}_{ij}(p)$ of equation 60 can be constructed on a bundle with a gauge connection $\omega$, a linear connection $\hat{\Gamma}^i_{jk}(p)$ on $P$ may be defined with some independence from $\hat{g}_{ij}(p)$, unlike the Levi-Civita connection of equation 63. In this case it is possible to derive a curvature scalar $\hat{R}$ on $P$ such that the cosmological term vanishes, that is with $R_G = 0$ in equation 66 (see for example [16, 17, 18, 19]).

One way to achieve this is to require the linear connection $\hat{\Gamma}^i_{jk}$ on $P$ to incorporate a description of absolute parallelism on the bundle fibres $G_x$ of figure 4. On the group manifold $G$ itself the list of canonical geometric objects includes a basis of left-invariant vector fields $\{X_\alpha\}$ and the Maurer-Cartan 1-form $\theta = X_\alpha \theta^\alpha$, which satisfies equation 11, as well as the structure constants $c^\alpha_{\beta\gamma}$ and the Killing form metric $g_\alpha^\beta$ of equation 57. Employing the derivative action of the left-invariant basis vectors $\{X_\alpha\}$ the right-invariance of the Killing metric implies $X_\alpha g^\beta_\gamma = 0$. In turn the covariant derivative, defined in terms of linear connection coefficients $\Gamma^\alpha_{\beta\gamma}$ on $G$, of the Killing metric vanishes:

$$\nabla_\alpha g^\beta_\gamma = X_\alpha g^\beta_\gamma - \Gamma^\delta_{\beta\alpha}g^\delta_\gamma - \Gamma^\delta_{\gamma\alpha}g^\delta_\beta = 0$$  \hspace{1cm} (69)

provided $\Gamma^\alpha_{\beta\gamma} = -\rho c^\alpha_{\beta\gamma}$ for any $\rho \in \mathbb{R}$  \hspace{1cm} (70)

by the antisymmetry in the indices of $c_{\alpha\beta\gamma}$. Hence for any value of $\rho$ this linear connection is metric compatible, with $\nabla g = 0$ on $G$. However the torsion is zero only for $\rho = \frac{1}{2}$ which hence represents the unique Levi-Civita connection on $G$ defined in terms of the Killing metric on the group manifold. For this case with the linear connection defined with components $\Gamma^\alpha_{\beta\gamma} = -\frac{1}{2}c^\alpha_{\beta\gamma}$, while the torsion vanishes by equation 31, the Riemann curvature is finite as can be seen from equation 29. In general the curvature and torsion on any manifold are independent geometric concepts where either one may be non-zero while the other is zero.

The Riemann curvature is zero on $G$ only for the case of $\rho = 0$ or $\rho = 1$ in equation 70, for which the torsion is finite, in contrast with the above Levi-Civita connection. The choice of linear connection coefficients $\Gamma^\alpha_{\beta\gamma} = 0$ is equivalent to inducing parallel transport on the group manifold via the left action $L_h$ of $G$ on itself, for any $h \in G$, that is with a complete parallelism on $G$ defined in terms of the self-parallel frame composed of left-invariant basis vector fields $\{X_\alpha\}$ on $G$. The other case with vanishing curvature for $\Gamma^\alpha_{\beta\gamma} = -c^\alpha_{\beta\gamma}$ in this left-invariant basis corresponds to a parallelism described by a right-invariant frame field under the action $R_h$. In either case the resulting Riemann curvature vanishes with $R^\alpha_{\beta\gamma\delta} = 0$, as can be shown using equation 29 together with the Jacobi identity expressed in terms of the structure constants. These latter two cases, while having finite torsion $T^\alpha_{\beta\gamma} = \mp c^\alpha_{\beta\gamma}$ by equation 31, in describing an absolute parallelism on $G$ can be considered as geometrically natural metric connections on the group manifold.

For a linear connection with components $\Gamma^\alpha_{\beta\gamma} = 0$ or $\Gamma^\alpha_{\beta\gamma} = -c^\alpha_{\beta\gamma}$ employed on the bundle fibres $G_x$ a subset of the torsion components on $P$ are also necessarily
non-zero, with $\mathring{T}_{\beta\gamma}(p) \neq 0$. Hence with the torsion allowed to be non-zero on the bundle space $P$ this version of Kaluza-Klein theory resembles the Einstein-Cartan theory on 4-dimensional spacetime for which $\Gamma$ and $g$ are treated as independent geometric objects. Here we briefly review four such approaches in the literature.

In Kopczyński [16] a linear connection $\mathring{\Gamma}$ with finite torsion is constructed in terms of the structure on a principle bundle with a gauge connection $\omega$ without reference to any metric. The ‘gravitational field’ on $P$ is described by the combination of both the metric $\mathring{g}$ of equation 58 and the components of $\mathring{\Gamma}$ as listed in the corresponding column under ‘Kop [16]’ in table 1. With these components the scalar curvature on $P$ is found to be $\mathring{R} = R_M + (\alpha - \alpha^2)K^2$, with $K^2 = K_{\alpha\beta}K_{\alpha\beta}$ (in [16] the metric on $G$ is $g_{\alpha\beta} = \lambda K_{\alpha\beta}$, here we take $\lambda = 1$ consistent with equation 57). For Einstein-Cartan theory the connection is compatible with the metric, which is achieved by setting $\beta = 0$ in row ‘5)’ of the ‘Kop [16]’ column of table 1. While this reference shows that the connection coefficients can be greatly simplified compared with the Levi-Civita case, as listed under ‘Cho [12]’ in the first column of table 1, in order to achieve the correct dynamics a more complicated Lagrangian function is postulated with the scalar $\mathring{R} + \frac{\lambda}{2}T_{jk}^iT_{ij}^k$, including a quadratic torsion term, employed in place of $\mathring{R}$ alone in equation 67. The cosmological constant $\Lambda$ obtained in this approach is arbitrary, and may be set to be zero or very small by a suitable choice of the parameters $\alpha$ and $\mu$.

In Orzalesi and Pauri [17] the main motivation is to describe a linear connection $\mathring{\Gamma}$ on the principle bundle which is gauge covariant. In particular requiring the Ricci curvature on the fibre space to be gauge invariant implies the adoption of zero curvature on the group manifold, that is the case $\rho = 0$ or $\rho = 1$ as described above after equation 70. This construction requires a relatively minimal modification of the Levi-Civita connection, as can be seen by comparing the entries of column ‘O+P [17]’ with column ‘Cho [12]’ in table 1. Here the simple scalar Lagrangian $\mathring{R}$ on the bundle space is again adopted, with the resulting vanishing of the $\Lambda \equiv R_G$ term (as seen in the bottom line of table 2) interpreted as a consequence of the underlying gauge $G$-symmetry of the Riemannian geometry on $P$. Without a finite $R_G$ term the vacuum solution corresponds to a zero Einstein tensor $G_{\mu\nu}(x) = 0$ together with zero internal curvature components $F_{\alpha\mu\nu}(x) = 0$ on the base space $M_4$.

In Kalinowski [18] the linear connection 1-forms $\mathring{\Gamma}_{i,j}^k = \tilde{\Gamma}_{i,j}^k \circ \text{hor}$ of equation 62 on $P$ are defined as the horizontal part of the Levi-Civita connection 1-forms, the latter here denoted $\mathring{\Gamma}_{i,j}$ with the components of equation 63 in general, as constructed in the horizontal lift basis. That is $\mathring{\Gamma}_{i,j}^k = \tilde{\Gamma}_{i,j}^k \circ \text{hor}$ in the notation of equation 51 (although here for the linear connection $\mathring{\Gamma}(p)$ the manifold $P$ is considered as the base space for the Riemannian geometry), with the components of this linear connection $\tilde{\Gamma}_{i,j}^k$ listed in column ‘Kal [18]’ of table 1. The factors of $\lambda$ arise as here the metric on $G$ is taken to be $g_{\alpha\beta} = \lambda^2 K_{\alpha\beta}$. This linear connection $\mathring{\Gamma}_{i,j}^k$ is metrical, invariant under the $G$-action, again with non-zero torsion and, while motivated in the context of gauge derivatives of spinor fields, again leads to a vanishing cosmological constant as seen in the bottom row of table 2.

In Katanaev [19] an initially completely general $\Gamma_{i,j}^k(p)$ on the principle bundle manifold is considered. Four conditions are postulated for $\Gamma$ in a geometrically meaningful way related to the structure group $G$ over $P$ and, as for the previous reference, with emphasis on horizontal propagation. In particular for column ‘Kat [19]’ of table 1
on taking \( c = 1 \) for entry ‘5)’ \( \hat{\Gamma}^{\alpha}_{ab} = cF^{\alpha}_{ab} \) the change in a tangent vector to \( P \) under parallel transport using these linear connection coefficients equals the change in the vector due to the basis transformation under parallel transport of the fibres using the gauge connection. Entry ‘4)’ in this column is included for compatibility with the natural metric of equation 58. The coefficients listed represent the case presented in [19] with finite torsion and the absence of a cosmological constant term, although a different choice of \( \hat{\Gamma} \) consistent with the postulates is possible. A small modification within this framework would be to set entries ‘1)’ and ‘2)’ equal to zero under column ‘Kat [19]’ in table 1, corresponding to taking \( \rho = 0 \) rather than \( \rho = 1 \) in equation 70. This reference is of significance for the present paper in that it highlights the possibility of a natural geometric origin for \( \hat{\Gamma} \) on \( P \) without any appeal to the Levi-Civita connection.

The complete set of linear connection coefficients for reference [12], augmenting equations 64 and 65, are collected in the first column of table 1. These are listed alongside the linear connection coefficients \( \Gamma^{i}_{jk} \) in the horizontal lift basis on the bundle space \( P \) for the above four cases with non-zero torsion. Where necessary signs have been aligned to the conventions used here, with the linear connection index order in \( \Gamma^{i}_{jk} \) as described for equation 27 together with the conventions listed for items 1) – 3) in subsection 3.1.

Only the first case in table 1 describes a torsion-free linear connection, yet each of the five cases is a Kaluza-Klein theory providing a unifying structure for general relativity together with gauge field theory. In part the purpose of collecting together this range of linear connection coefficients on \( P \) together with their motivating arguments is to demonstrate that a significant degree of flexibility is possible within Kaluza-Klein theory while still maintaining this unified framework.

For any linear connection on the bundle space \( P \), such as defined by any of the five sets of connection coefficients \( \hat{\Gamma}^{i}_{jk} \) listed in table 1, the Riemann curvature

| \( \hat{\Gamma}^{i}_{jk} \) | Cho [12] | Kop [16] | O+P [17] | Kal [18] | Kat [19] |
|----------------|--------|----------|---------|---------|---------|
| 1) \( \hat{\Gamma}^{\alpha}_{\beta\gamma} \) | \(-\frac{1}{2}c^{\alpha}_{\beta\gamma}\) | \(-\alpha c^{\alpha}_{\beta\gamma}\) | \(-c^{\alpha}_{\beta\gamma} \) or 0 | 0 | \(-c^{\alpha}_{\beta\gamma}\) |
| 2) \( \hat{\Gamma}^{\alpha}_{\gamma a} \) | 0 | 0 | 0 | 0 | \(-\omega^{\beta}_{a\alpha\gamma}\) |
| 3) \( \hat{\Gamma}^{\alpha}_{b\gamma} \) | \(\frac{1}{2}g^{ac}g_{\gamma\beta}F^{\beta}_{bc}\) | 0 | \(\frac{1}{2}g^{ac}g_{\gamma\beta}F^{\beta}_{bc}\) | 0 | 0 |
| 4) \( \hat{\Gamma}^{\alpha}_{ab} \) | \(\frac{1}{2}F^{\alpha}_{ab}\) | \(\beta F^{\alpha}_{ab}\) | \(\frac{1}{2}F^{\alpha}_{ab}\) | \(\frac{1}{2}F^{\alpha}_{ab}\) | \(cF^{\alpha}_{ab}\) |
| 5) \( \hat{\Gamma}^{\alpha}_{bc} \) | \(\Gamma^{\alpha}_{bc}\) | \(\Gamma^{\alpha}_{bc}\) | \(\Gamma^{\alpha}_{bc}\) | \(\Gamma^{\alpha}_{bc}\) | \(\Gamma^{\alpha}_{bc}\) |

Table 1: Linear connection components \( \hat{\Gamma}^{i}_{jk} \) on a principle bundle extracted from [12] equation 22, [16] page 367, [17] equation 19, [18] equation 29, and the case in [19] with non-zero torsion on \( G \). The \( \{\alpha, \beta, \gamma\ldots\} \) and \( \{a, b, c\ldots\} \) index convention is explained near the opening of subsection 3.3 with reference to figure 4. All components are expressed in the horizontal lift basis and \( \hat{\Gamma}^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{\alpha}_{\gamma\beta} = 0 \) in all five cases. Each of \( \lambda > 0, \alpha, \beta \) and \( c \), where used as coefficients, are real constant parameters.
tensor can be determined according to equation 29 applied in the horizontal lift basis on $P$. The corresponding Ricci curvature components $\hat{R}_{\alpha\beta}$ and $\hat{R}_{ab}$ are listed here in the first and fourth rows of table 2 for these five theories. In all cases the entries in this table calculated here agree with the corresponding equations of the respective references within the sign conventions adopted.

\[
\hat{R}_{\alpha\beta} = R_{(G)\alpha\beta} + \frac{1}{4} F_{\alpha b d} F_{\beta}^{\ b d}
\]

\[
R_{(G)\alpha\beta} = \frac{1}{4} K_{\alpha\beta} (\alpha - \alpha^2) K_{\alpha\beta}
\]

\[
K^{\alpha\beta} \hat{R}_{\alpha\beta} = R_{G} - \frac{1}{4} F^2
\]

\[
\hat{R}_{ab} = R_{(M)ab} + \frac{1}{2} F_{\alpha d}^{\ b} F_{\beta}^{\ d} F_{\beta b}^{\ d}
\]

\[
g^{ab} \hat{R}_{ab} = R_{M} + \frac{1}{2} F^2
\]

\[
\hat{R} = \hat{g}^{ij} \hat{R}_{ij} = R_{M} + R_{G} + \frac{1}{4} F^2
\]

Table 2: Composition of the scalar curvature $\hat{R}$ on the bundle space for the five cases of table 1. Contributions to the components of the Ricci curvature on the bundle include $R_{(G)\alpha\beta}$ and $R_{(M)ab}$ from the group manifold and base space respectively, with $R_{G} = K^{\alpha\beta} R_{(G)\alpha\beta}$ and $R_{M} = g^{ab} R_{(M)ab}$ being the respective scalar curvatures.

The scalar curvature constructed in the horizontal lift basis on the principle bundle space can be written as:

\[
\hat{R} = \hat{g}^{ij} \hat{R}_{ij} = g^{ab} \hat{R}_{ab} + K^{\alpha\beta} \hat{R}_{\alpha\beta}
\]

owing to the simple form of the metric $\hat{g}$ in this basis as expressed in equation 60. Hence the Ricci curvature components $\hat{R}_{\alpha\beta}$ and $\hat{R}_{ab}$ are not required in order to determine the scalar curvature on the bundle.

If the four factors of $\frac{1}{2}$ in the ‘Cho [12]’ column in table 1, for the case of the Levi-Civita connection coefficients $\hat{\Gamma}^{i}_{jk}$ on the bundle, listed in rows ‘1)’, ‘3)’, ‘4)’ and ‘5)’ are replaced by the real factors $f_1, f_3, f_4$ and $f_5$ respectively then the scalar curvature in the horizontal lift basis is found to be:

\[
\hat{R} = R_{M} + R_{G} + \chi F^2
\]

with \[
\chi = f_3 - f_3 f_4 + f_4 f_5 - f_3 f_5
\]

These equations reproduce the scalar curvature for the Levi-Civita case, with each $f_i = \frac{1}{2}$, as quoted originally in equation 66, and also apply to each subsequent case of table 1 as quoted in the final row of table 2.

Equations 72 and 73 show that $f_3$ is the only coefficient which is sufficient in itself to introduce a non-trivial $F^2$ term, alongside $R_{M}$, into the scalar curvature $\hat{R}$, and hence including $F_{\alpha b c} = \Gamma_{\alpha b c}^{\gamma}$ and $F_{\alpha b}^{\gamma} = f_3 g^{\alpha c} g^{\beta} F_{\beta}^{\ b c}$ as the only non-zero $\hat{\Gamma}^{i}_{jk}$ coefficients might be considered as a further, ‘minimal’, Kaluza-Klein model. While not developed here as a serious physical proposal this minimal model further demonstrates
the flexibility within the Kaluza-Klein framework, obtaining the appropriate link between the external geometry and internal curvature with a seemingly much simpler linear connection on the bundle compared with the Levi-Civita case. More generally, equations 72 and 73 display the mutual consequences of the non-zero $\Gamma_{ijk}$ terms for the models listed in table 1.

The derivation of Einstein’s equations in 4-dimensional spacetime from the Einstein-Hilbert action of equation 33 was described in subsection 3.1. In the vacuum case with $\mathcal{L} = 0$ and $\Lambda = 0$ variation of the metric $\delta g_{\mu\nu}$ on $M_4$ leads to the equation of motion $G^{\mu\nu} = 0$ of equation 34. For the Kaluza-Klein extension to a scalar curvature $\hat{R}$ on a principle bundle space the same steps lead to the requirement of the stationarity of the action integral over the full bundle space in equation 67, that is $\delta I_m = 0$, under variation of the extended metric $\delta \hat{g}_{ij}$ on $P$. However for the Kaluza-Klein theories described here the variations in the metric $\hat{g}_{ij}$ on the bundle space are not arbitrary since the right-invariance of equation 59 and the general form of the metric in equation 61 should be preserved on $P$. This limits the metric variations to the components $\delta g_{ab}$ and $\delta \omega^a$ in equation 61 and leads to two equations of motion on the base manifold $M_4$. Applying the variation $\delta g_{ab}$ under $\delta I_m = 0$ for the action in equation 67, with the curvature $\hat{R}$ of equation 72, leads to the generally non-zero solution expressed in a general coordinate basis on $M_4$ (see for example [18] page 394):

$$G^{\mu\nu} + R_G g^{\mu\nu} = 2\chi(-F^{\alpha\mu\rho\nu} F_{\alpha\rho\sigma} + \frac{1}{4}g^{\mu\nu} F^{\alpha\rho\sigma} F_{\alpha\rho\sigma}) = -\kappa T^{\mu\nu} \quad (74)$$

A necessarily finite cosmological term with $R_G \neq 0$ arises only for the case of the Levi-Civita connection in the ‘Cho [12]’ columns of tables 1 and 2, while the factor of $\chi$ is determined by equation 73. On the other hand the variation $\delta \omega^a$ on the bundle leads, on the spacetime manifold $M_4$, to:

$$D_\mu F^{\alpha\mu\nu} = 0 \quad (75)$$

Equation 74 with $R_G = 0$ is the Einstein field equation with an energy-momentum tensor $T^{\mu\nu}$, reproducing equation 40, composed purely from the gauge fields $A^{\alpha}(x)$, with the latter being subject to equation 75 which is the Yang-Mills field equation (or Maxwell’s equation $\nabla_\mu F^{\mu\nu} = 0$ in the case of the Abelian internal symmetry group $G = U(1)$). Hence the source-free Yang-Mills field equation 39 has been derived without the explicit introduction of the Yang-Mills Lagrangian of equation 38. Rather such a ‘Lagrangian term’ $F^2 = F^{\alpha}_{ab}(p)F_{\alpha}^{ab}(p) \equiv F^{\alpha\mu\nu}(x)F_{\alpha\mu\nu}(x)$ has been incorporated within an Einstein-Hilbert action deriving from the geometry of the bundle space. In this way non-Abelian Kaluza-Klein theory provides a unified framework for the combined Einstein-Yang-Mills field equations.

4 Geometric Unification through One Dimension

4.1 Construction of a Linear Connection on $P \equiv M_4 \times G$

In this section, guided by the framework of Kaluza-Klein theories described in the previous section, the aim is to determine a relation between the external and internal
geometry arising out of the symmetries of the full multi-dimensional form of temporal flow \( L(\hat{v}) = 1 \), as originally derived from the one dimension of time in subsection 2.1 and building upon the structures described in section 2 generally.

Initially in subsection 2.2 we considered the form \( L(v_4) = \eta_{ab} v^a v^b = 1 \) of equation 8, with Minkowski metric \( \eta = \text{diag}(+1, -1, -1, -1) \) and Lorentz \( SO^+(1, 3) \) symmetry. These structures are projected locally onto a 4-dimensional base space \( M_4 \), which itself derives from the translation symmetry of \( L(v_4) = 1 \), as depicted in figure 2. An extension from equation 8 to equation 18 for a full 9-dimensional form \( L(\hat{v}) = L(v_9) = \det(v_9) = 1 \) with \( v_9 \in h_3C \) and full symmetry group \( \hat{G} = SL(3, C) \), via \( SL(2, C) \) as the double cover of the Lorentz group, was then described in subsection 2.3. In this case the extended base manifold \( M_4 \) arises out of four of the nine translational degrees of freedom of \( L(v_9) = 1 \) leading directly to the symmetry breaking structure described for figure 3.

The breaking of the full symmetry through the extraction of a preferred subgroup \( SL(2, C) \subset SL(3, C) \) acting on the tangent space \( TM_4 \) of the external spacetime \( M_4 \), as described for equations 19–22, leaves a residual internal \( U(1) \) symmetry, as described for equation 24. Given that both \( SL(2, C) \) and \( U(1) \) are contained within the initial unbroken full symmetry \( SL(3, C) \) of figure 3(a) a correlation between the external curvature \( R \) and internal curvature \( F \) is implied in the symmetry breaking to the structure of figure 3(b), in particular with the case of both \( R = 0 \) and \( F = 0 \) simultaneously possible.

A principle bundle structure \( P \equiv M_4 \times U(1) \) underlying figure 3(b), representing the space of broken symmetries of the full form of time \( L(v_9) = 1 \), emerges in the identification of the base manifold \( M_4 \). The base space \( M_4 \) is also naturally associated with the frame bundle \( FM_4 \), which is itself a particular type of principle fibre bundle as described in the opening of subsection 3.1. In the symmetry breaking the degrees of freedom of the \( SL(2, C) \subset SL(3, C) \) subgroup part of the original full gauge connection are converted into the freedom of a linear connection on \( M_4 \). That is, an \( sl(2, C) \)-valued connection on \( P \) can be extended to a \( gl(4, R) \)-valued connection on the frame bundle, together with the associated tetrad \( e^\mu_a(x) \) and metric \( g_{\mu\nu}(x) \) fields on \( M_4 \), as related in equation 9 and familiar from the theory of general relativity, as also described in subsection 3.1.

For the generalisation with \( \hat{G} \) as the full isochronal symmetry of the full multi-dimensional form of time \( L(\hat{v}) = 1 \) the same symmetry breaking mechanism results in the bundle structure \( P \equiv M_4 \times G \), where \( G \) is the internal symmetry identified as described for equation 23. This principle bundle again implicitly combines the geometry of the frame bundle on \( M_4 \), described in terms of a metric \( g(x) \) and Riemann curvature \( R(x) \) with an underlying local \( SO^+(1, 3) \) symmetry, and the geometry of \( P \) itself, expressed in terms of a gauge field \( A(x) \) and gauge curvature \( F(x) \) on \( M_4 \) with an internal local \( G \) symmetry. Hence, in accommodating both the structure of the external geometry on \( M_4 \) and that of the internal gauge group \( G \) in the bundle space, in principle all the necessary geometric structures for relating the external and internal curvature can be identified on the bundle space \( P \equiv M_4 \times G \).

For the original case with a full form of simply \( L(v_4) = 1 \) a preferred globally defined orthonormal basis can be identified on the extended Riemannian manifold \( M_4 \), as described in subsection 2.2, supporting a linear connection with all coefficients \( \Gamma^a_{bc}(x) = 0 \). Such a global frame is adapted to the natural absolute parallelism and
canonical zero curvature described for figure 2. A Lie group manifold \( G \) also supports a natural absolute parallelism with all linear connection coefficients \( \Gamma^\alpha{}_{\beta\gamma}(h) = 0 \) for all \( h \in G \) in a left-invariant basis, also with zero Riemann curvature, as described following equation 70 in subsection 3.4.

The question can then be asked concerning the possible generalisation to a natural parallelism and corresponding linear connection \( \hat{\Gamma}^i_{jk}(p) \) on the larger space of symmetries \( P \equiv M_4 \times G \), and the manner in which this may perturb the external curvature on the base manifold \( M_4 \). In particular we seek a linear connection on \( P \), taking precedence over any possible metric structure, for which parallel transport in the horizontal and vertical directions on \( P \) closely reflects the geometry of the base manifold \( M_4 \). While the principle bundle space \( P \equiv M_4 \times G \) is not itself considered as a physical space, it opens up an ‘internal’ freedom in this larger space of symmetries for the broken form \( L(\hat{\nu}) = 1 \) that might in principle perturb the parallel transport on the external physical spacetime \( M_4 \) component.

Given a gauge connection \( \omega \) on \( P \equiv M_4 \times G \) associated with the internal symmetry \( G \), which in general is not flat, a distinguished horizontal subspace in the tangent space of \( P \) is identified. In turn any tetrad basis \( \{\epsilon_a\} \) on \( M_4 \) can be mapped from any point \( x \in M_4 \) into the natural horizontal components of the tangent space at any point \( p \in P \) with \( \pi(p) = x \), that is to the corresponding horizontal lift basis vectors \( \{\epsilon_a\} \), as related in figure 4. The horizontal lift basis is of direct physical significance for the internal geometry as described for equation 50, with \( F^{a\,b}_{\,c}(p) \) being the internal curvature components. In connecting the fibres of \( P \) over \( M_4 \) the geometry described by the horizontal subspace in principle provides a means of perturbing the geometry of the base manifold itself. In order to study this structure we introduce a linear connection \( \hat{\Gamma}(p) \) in the horizontal lift basis on \( P \) which defines a parallel transport in the tangent space \( TP \) describing the internal relation between the fibres over \( M_4 \) associated with the connection \( \omega \).

That is, the geometry on \( P \) is interpreted both in terms of a gauge connection \( \omega \) with curvature \( \Omega(p) \) on the principle bundle space and at the same time in terms of a linear connection \( \hat{\Gamma}(p) \) with \( P \) considered as the base space of \( FP \). Extending the structure of \( \Gamma(x) \) on \( M_4 \) to \( \hat{\Gamma}(p) \) on \( P \equiv M_4 \times G \) mirrors the extra-dimensional extension of general relativity in Kaluza-Klein theories, as described in section 3, relating the gauge symmetry structure to a structure of Riemannian geometry on \( P \) itself.

For completeness a full set of linear connection coefficients \( \hat{\Gamma}^i_{jk}(p) \) on the bundle space \( P \equiv M_4 \times G \) is constructed in the horizontal lift basis \( \{\hat{\epsilon}_i\} = \{\hat{\epsilon}_a, \hat{\epsilon}_d\} \), which was depicted in figure 4 after which we also described the index convention used here. With emphasis on parallel transport in the horizontal directions we take parallel propagation in the vertical directions to be trivial in this basis, with:

\[
\hat{\Gamma}^a_{\,b\gamma} = \hat{\Gamma}^a_{\,b\gamma} = \hat{\Gamma}^a_{\,\beta\gamma} = \hat{\Gamma}^a_{\,\beta\gamma} = 0
\] (76)

That is, with vanishing coefficients for propagation along the fibres of the bundle as indicated by the Greek \( \gamma \) for the third index. Here the final set \( \hat{\Gamma}^a_{\beta\gamma}(p) = 0 \) is essentially imported from the parallelism \( \Gamma^\alpha{}_{\beta\gamma}(h) = 0 \) on the manifold \( G \) and applied to vertical transport generally on \( P \). This vertical structure is similar to that for the linear connection on the bundle as described for the ‘Kal [18]’ column of table 1 in subsection 3.4, which was constructed with 1-forms \( \hat{\Gamma}^i_{\,j} \equiv \hat{\Gamma}^i_{\,j} \circ \text{hor} \) in the horizontal lift basis and hence with each \( \hat{\Gamma}^i_{\,j\gamma} = 0 \) as for equation 76.
With the focus upon parallel propagation in the horizontal directions on \( P \), conceived from a geometrical point of view as perturbing the geometry of the base manifold \( M_4 \), which is of primary physical interest as the external spacetime, the relevant structures are sketched in figure 5. In general, for finite external Riemann curvature \( R \neq 0 \) on \( M_4 \) the parallel transport of a given tangent vector \( u_1 \in T_{x_1}M_4 \) from \( x_1 \in M_4 \) along two different paths \( C \) and \( D \) to a point \( x_2 \in M_4 \) will result in two different vectors \( u_{2C}, u_{2D} \in T_{x_2}M_4 \). Similarly for non-trivial internal curvature \( \Omega \neq 0 \) on \( P \) the horizontal lifts of the curves \( C \) to \( C' \) and \( D \) to \( D' \) will generally lead from any given point \( p_1 \in P \), with \( \pi(p_1) = x_1 \), to two different points \( p_{2C'}, p_{2D'} \) on the fibre over \( x_2 \in M_4 \), as also depicted in figure 5. The aim is then to understand how the external and internal curvature might be mutually dependent for the space of symmetries and geometric structure \( P \equiv M_4 \times G \) arising from the symmetry of the full form \( L(\hat{v}) = 1 \) broken over \( M_4 \).

Figure 5: Parallel transport of \( u_1 \in T_{x_1}M_4 \) over two paths \( C \) and \( D \) between the same two points \( x_1, x_2 \) on the base space \( M_4 \) and the respective horizontal lifts \( C' \) and \( D' \) between the corresponding fibres in the principle bundle space \( P \). While the geometric structure is similar to that reviewed in subsection 3.2 here the bundle space \( P \equiv M_4 \times G \) derives from the broken symmetries of the full temporal form \( L(\hat{v}) = 1 \), as described for the example of figure 3(b).

We note here that the diagram in figure 5 might be interpreted quite literally for a toy model with \( P \equiv M_2 \times U(1) \), with for example a local \( SO^+(1,1) \) symmetry on \( M_2 \) and fibres with a single parameter \( \alpha \in \mathbb{R} \) for \( e^{i\alpha} \in U(1) = G \) as the internal gauge symmetry group. However the figure also holds metaphorically both for the bundle \( P \equiv M_4 \times U(1) \) arising from the \( L(v_0) = 1 \) model of figure 3(b) and for the more realistic case \( P \equiv M_4 \times G \) arising from a yet higher-dimensional form \( L(\hat{v}) = 1 \), where in general \( G \) is a non-Abelian internal symmetry group, and it is this more general case that we describe here.
Here the physical significance of the linear connection $\dot{\Gamma}(p)$ on $P$ derives from its relation to a linear connection $\Gamma(x)$ on the base manifold $M_4$, and the associated Riemannian geometry, when generalised for the larger space of symmetries of the full form $L(\hat{v}) = 1$ broken over the base manifold. With emphasis on horizontal propagation, generalised from the manifold $M_4$ into the space $P \equiv M_4 \times G$, a relation between the local structure of parallel transport on the bundle described by a linear connection $\dot{\Gamma}(p)$ and the geometry of an internal gauge connection relating the fibres in the horizontal directions will be established. The horizontal propagation on $P$ is described by the four sets of linear connection coefficients with a Latin $c$ replacing the $\gamma$ index for each set in equation 76. We consider each of these four cases in turn.

The parallel transport of $\dot{e}_b$ along $\dot{e}_a$ on the bundle $P$ when projected onto $M_4$ with $e_b = \pi_* (\dot{e}_b)$ and $e_a = \pi_* (\dot{e}_a)$ is taken to be equivalent to the parallel transport of $e_b$ along $e_a$ directly on the base space $M_4$. This is essentially the same motivation for these coefficients as for all of the references summarised in table 1 and, with $\pi(p) = x$, implies that:

$$\dot{\Gamma}^a_{bc}(p) = \Gamma^a_{bc}(x) \quad (77)$$

However the parallel transport of $\dot{e}_b$ along $\dot{e}_a$ also has freedom in the vertical directions of the bundle space, as parametrised by the connection coefficients $\dot{\Gamma}^\alpha_{ba}(p)$. That is, the covariant derivative for the vector field $\dot{e}_b$ with respect to $\dot{e}_a$ can, from equation 27, be written generally as:

$$\nabla_{\dot{e}_a} \dot{e}_b = \dot{\Gamma}^i_{ba} \dot{e}_i = \dot{\Gamma}^c_{ba} \dot{e}_c + \dot{\Gamma}^\alpha_{ba} \dot{e}_\alpha \quad (78)$$

In order to determine a natural expression for the coefficients $\dot{\Gamma}^\alpha_{ba}(p)$, corresponding to a natural parallelism on the bundle space, we first briefly review another standard structure of differentiable geometry. In general on any differentiable manifold $M$ there is a one-to-one correspondence between a vector field $V$ and its flow $\Phi_t$, associated with the integral curves of $V$. The flow $\Phi_t$ is a one-parameter group of transformations on $M$, and a diffeomorphism for each value of the parameter $t \in \mathbb{R}$. As a diffeomorphism $\Phi_t$ induces a pull-back map of tensor fields of any type in the space $T^r_s M$ on $M$, that is:

$$\Phi_t^* : T^r_s M \rightarrow T^r_s M \quad (79)$$

which is known as Lie transport or Lie dragging. While the geometric origin differs Lie transport in general shares a number of features in common with parallel transport via a linear connection. Both forms of transport preserve the $(r,s)$ degree of a tensor field and commute with tensor products and contractions. In addition both Lie transport and parallel transport give rise to derivatives that are derivations of the tensor algebra. The Lie derivative of a tensor field $X \in T^r_s M$ with respect to a vector field $V$ is defined by:

$$\mathcal{L}_V X := \left. \frac{d}{dt} \right|_{t=0} \Phi_t^* X = \lim_{t \to 0} \frac{\Phi_t^* X - X}{t} \quad (80)$$

The covariant derivative is defined by

$$\nabla_V X := \left. \frac{d}{dt} \right|_{t=0} T^\lambda_t X = \lim_{t \to 0} \frac{T^\lambda_t X - X}{t} \quad (81)$$

where $T^\lambda_t$ is the operation of parallel transport backwards by parameter distance $t$ along the integral curve $\lambda$ of the vector field $V$. 
Parallel transport is generally an independent structure on a manifold $M$ which can be introduced by endowing the manifold with a linear connection $(M, \Gamma)$. For the case of the bundle manifold $P \equiv M_4 \times G$ with a gauge connection $\omega$ the distinguished horizontal subspace has a geometric origin and physical meaning and hence Lie dragging along vector fields $\dot{e}_a \in HP$ describes a natural rule for transport in the bundle space. This structure can in turn be modelled in terms of a parallel transport of vector fields by introducing appropriate linear connection coefficients on the bundle space. This linear connection uniquely specifies a covariant derivative on the bundle via equation 27, as for example in equation 78 above in the horizontal lift basis on $P$.

On the other hand the Lie derivative of equation 80 of a vector field $X \in TM = T^1_0M$ with respect to the vector field $\dot{V}$ is given by the simple Lie bracket expression $\mathcal{L}_{\dot{V}}X = [V, X]$. Applying this to horizontal lift basis vector fields on the bundle $P \equiv M_4 \times G$ we have from equation 50 (hence adopting a coordinate basis on $M_4$):

$$\mathcal{L}_{\dot{e}_a} \dot{e}_b = [\dot{e}_a, \dot{e}_b] = -F^\alpha_{ab} \dot{e}_\alpha$$

(82)

We require that the vertical component in the parallel transport of a horizontal basis vector $\dot{e}_b$ along the field $\dot{e}_a$, as described by the coefficients $\Gamma^\alpha_{ba}$ in equation 78, should be directly associated with the natural Lie transport according to the local coefficients $-F^\alpha_{ab}$ of equation 82. Hence, taking into account the asymmetry in the $ab$ indices of the components $F^\alpha_{ab}$ of the gauge curvature 2-form $\Omega$, we have simply:

$$\Gamma^\alpha_{ab}(p) = F^\alpha_{ab}(p)$$

(83)

These linear connection coefficients derive from a similar argument as the corresponding components listed in ‘row 5)’ under ‘Kat [19]’ in table 1 for the case of the parameter $c = 1$ as described in subsection 3.4. Here however these coefficients are considered as inducing a perturbation of the original flat geometry of the external spacetime $M_4$, associated with the symmetries of the original form $L(v_4) = 1$, arising through the extra degrees of freedom of the internal gauge structure for the larger space $P \equiv M_4 \times G$ associated with the symmetries of the broken full form $L(\hat{v}) = 1$.

Unlike the Levi-Civita connection coefficients described in subsection 3.3, following Cho [12] and as listed in the first column of table 1, here the connection components are motivated directly in terms of a natural parallelism on the manifold $P \equiv M_4 \times G$ rather than via a metric structure on the bundle. However, since there is an internal gauge connection $\omega$ it is also the case here that the structure of the natural metric $\dot{g}(p)$ with the components $\dot{g}_{ij}$ of equation 60 can be identified in the horizontal lift basis on principle bundle $P$. In particular this metric encapsulates the decomposition of the tangent space $TP$ into vertical and horizontal subspaces, equation 42, in terms of the orthogonality of any two vectors – one in each of $VP$ and $HP$. Containing the components of the Killing form $g_{\alpha\beta} = K_{\alpha\beta}$ of equation 57 for the gauge symmetry group $G$ the metric $\dot{g}$ of equation 60 on $P$ is also right-invariant under the action of the gauge group, with the scalar product of any two right-invariant fields via $\dot{g}$ being independent of the fibre coordinates, as implied in equation 59.

Since $\dot{g}(p)$ is a non-degenerate symmetric rank-2 tensor field on $P$ the principle bundle space itself can be considered as a Riemannian manifold. Hence here we are considering the geometry of $(P, \dot{g}, \dot{\Gamma})$ with a pair of structures, both a metric and a linear connection, defined on $P$. After having determined a link between $\dot{\Gamma}$ and the
gauge curvature $\Omega$ on the bundle via equation 83 we next establish the relationship between $\tilde{\Gamma}$ and $\dot{g}$ in order to specify the remaining connection coefficients.

In particular we require the linear connection $\tilde{\Gamma}$ to be compatible with the metric $\dot{g}$ of equation 60 to the extent at least that parallel transport in the horizontal subspace of $P$, with local tangent space basis $\{\dot{e}_a\}$, should preserve the partition of $TP$ into vertical and horizontal subspaces as described by their orthogonality according to the metric components $\dot{g}_{a\beta} = \dot{g}_{ab} = 0$. This is reasonable since although $\dot{g}(p)$ is not considered a physical metric on $P$ (unlike $g_{ab}(x)$ which determines spacetime intervals in the appropriate units on $M_4$) the partition of $TP$ according to equation 42 does have a distinct physical meaning. Hence we require $\tilde{\nabla}_a \dot{g}_{ab} = 0$, that is:

$$\tilde{\nabla}_a \dot{g}_{ab} = \dot{e}_a \dot{g}_{ab} - \tilde{\Gamma}^c_{aa} \dot{g}_{cb} - \tilde{\Gamma}^\gamma_{ba} \dot{g}_{a\gamma} = 0 \quad (84)$$

via equation 83. The determination of these coefficients $\tilde{\Gamma}^d_{aa}(p)$ again follows by a similar argument that led to the entry of row ‘4’) for Katanaev [19] in table 1 as described in subsection 3.4. The generalisation of equation 84 for compatibility of the full metric $\dot{g}(p)$ with the covariant derivative in directions within the horizontal subspace can be written:

$$\tilde{\nabla}_a \dot{g}_{ij} = \dot{e}_a \dot{g}_{ij} - \tilde{\Gamma}^k_{ia} \dot{g}_{kj} - \tilde{\Gamma}^k_{ja} \dot{g}_{ik} = 0 \quad (86)$$

Since $\dot{g}$ is right-invariant on $P$, and given that $\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc}$ from equation 77 and $\dot{g}_{bc} = g_{bc}$ from equation 60, the $\tilde{\nabla}_a g_{bc} = 0$ components of equation 86 on the bundle space reduce to simply $\nabla_a g_{bc} = 0$ on the base space $M_4$. This just the statement of metric compatibility for the Levi-Civita connection $\Gamma^a_{bc}(x)$ employed on the base space, which is adopted as consistent with general relativity in 4-dimensional spacetime.

On the other hand the remaining components of equation 86 describe the preservation of Killing metric $\dot{g}_{a\beta} = g_{a\beta} = K_{a\beta}$ under parallel transport in the horizontal subspace on $P$. This in turn constrains the remaining linear connection components on the bundle space:

$$\tilde{\nabla}_a \dot{g}_{a\beta} = \dot{e}_a \dot{g}_{a\beta} - \tilde{\Gamma}^\gamma_{aa} \dot{g}_{a\beta} - \tilde{\Gamma}^\gamma_{a\beta} \dot{g}_{a\gamma} = 0$$

$$\Rightarrow \tilde{\Gamma}^\gamma_{aa} \dot{g}_{a\beta} + \tilde{\Gamma}^\gamma_{a\beta} \dot{g}_{a\gamma} = 0$$

$$\Rightarrow \tilde{\Gamma}^\gamma_{aa} = 0 \quad (87)$$

The adoption of $\tilde{\Gamma}_{aa} = 0$ in the final line as the simplest solution for the second line above for these metric compatible connection coefficients in the horizontal lift basis on $P$ is analogous to setting $\rho = 0$ in equation 70 for the simplest metric compatible connection in a left-invariant basis on a Lie group manifold $G$.

The full set of linear connection coefficients $\tilde{\Gamma}^i_{jk}(p)$ on the bundle space deduced above is in fact compatible with the metric $\dot{g}$ of equation 60 generally, that is the metric is covariantly constant with $\tilde{\nabla} \dot{g} = 0$ and the covariant derivative of $\dot{g}(p)$ is zero in any
direction on the bundle space, including along the vertical fibre components. This full set, that we have arrived at in the horizontal lift basis in equations 76, 77, 83, 85 and 87, is summarised here:

\[ 4) \hat{\Gamma}^a_{\gamma b} = g^{ac} g_{\gamma \beta} F_{\beta c}, \quad 5) \hat{\Gamma}^{\alpha}_{ab} = F^{\alpha}_{ab}, \quad 6) \hat{\Gamma}^a_{bc} = \Gamma^a_{bc} \] (88)

with all other \( \hat{\Gamma}^i_{jk} = 0 \), and may be compared with the models of table 1. Although the motivation differs significantly, this set is equivalent to that listed under ‘Kal [18]’ in the penultimate column of table 1 for \( \lambda = 2 \), with the motivation for employing this latter value here related to the geometrical argument in [19]. This latter argument also has the benefit of fixing the geometry of \( \hat{\Gamma} \), which has finite torsion as for the models of subsection 3.4, without any reference to the Levi-Civita connection on \( P \).

Since we have introduced the metric \( \hat{g} \) on the bundle space it is possible to define the unique metric compatible and torsion-free Levi-Civita connection on \( P \), as described in subsection 3.3, with the components of equation 63 as listed for the horizontal lift basis in the first column of table 1 under ‘Cho [12]’. However in the present theory at no stage is \( P \) considered to be an extended \textit{physical} space or spacetime structure, hence neither the metric \( \hat{g} \) nor a linear connection \( \hat{\Gamma}^i_{jk} \) on \( P \) have a physical meaning of the kind that such objects have on the external base space \( M_4 \). Hence the Levi-Civita connection is not here considered to be a natural structure on the bundle space as it is for the base manifold, and an alternative argument for the form of \( \hat{\Gamma} \) on \( P \) has been presented.

Observations regarding the structure of the theory presented here, together with the broader discussion of Kaluza-Klein theories in section 3, have led to the conjectured linear connection components on \( P \) as summarised in equation 88. This proposal for the properties of \( \hat{\Gamma} \), constructed in the distinguished horizontal lift basis on \( P \), appropriate for the present theory has been influenced by consideration of all cases collected in table 1, as we summarise here:

a) The bundle \( P \equiv M_4 \times G \) serves as an arena to relate the external and internal symmetry structures in a geometric framework closely analogous to that of [12], deriving from a generalisation of figure 3(b) for the present theory.

b) The linear connection \( \hat{\Gamma} \) on \( P \) may be compatible with the natural metric \( \hat{g} \) of equation 60, while the torsion can be finite as initially emphasised in [16], and here \( (P, \hat{g}, \hat{\Gamma}) \) does not represent a physical spacetime.

c) In deriving physical equations on the base space \( M_4 \) compatibility with gauge covariance should be observed, as emphasised in [17], as well as general covariance and the simultaneous possibility of \( R = 0 \) and \( F = 0 \) on \( M_4 \).

d) Parallel propagation via \( \hat{\Gamma} \) in the vertical subspace of equation 42 is taken to be trivial, with \( \hat{\Gamma}^i_{j} \equiv \hat{\Gamma}^i_{j} \circ \text{hor} \) similarly as for [18], that is any vector \( X(p) \in V_p P \) (i.e. with \( \pi_* X = 0 \)) is mapped to zero by the 1-forms \( \hat{\Gamma}^i_{j} \) on \( P \).

e) Parallel propagation via \( \hat{\Gamma} \) in the horizontal directions on \( P \) is determined by the geometry of the gauge curvature over the base space \( M_4 \), following [19] in the fifth column of table 1 for the case \( c = 1 \).
On the bundle space \((P, \hat{g}, \hat{\Gamma})\) both the metric \(\hat{g}\) of equation 60 and the components of the metric compatible linear connection \(\hat{\Gamma}\) of equation 88 transform in a gauge covariant manner on \(P\). This follows since both \(\hat{g}\) and \(\hat{\Gamma}\) are closely associated with the horizontal subspace \(HP\) of the tangent space on \(P\) which itself is right-invariant as described for equation 43, by the definition of a gauge connection on \(P\). That is, \(\hat{\Gamma}\) is defined in part in terms of a parallel transport on the bundle space that follows the contours of the horizontal lift subspace as described for figure 5, with the non-zero coefficients of equation 88 representing a gauge covariant perturbation from an initial structure, consistent with all \(\Gamma^a_{bc}(x) = 0\) throughout \(M_4\) under \(L(v_4) = 1\), in the extension onto \(P \equiv M_4 \times G\) for the symmetries of the broken full form \(L(\hat{v}) = 1\). The analysis of this perturbation led to equations 83 and 85, which via equation 56 imply for the respective coefficients of equation 88:

\[
4) \dot{e}_\alpha \hat{\Gamma}^a_{\beta b} = c^\gamma_{\alpha \beta} \hat{\Gamma}^a_{\gamma b}, \quad 5) \dot{e}_\alpha \hat{\Gamma}^\beta_{ab} = -c^\beta_{\alpha \gamma} \hat{\Gamma}^\gamma_{ab}, \quad 6) \dot{e}_\alpha \hat{\Gamma}^a_{bc} = 0 \quad (89)
\]

These transformation properties are a direct consequence of this construction of a linear connection on \(P\), inheriting this structure from the gauge curvature \(\Omega\) on the bundle. An alternative interpretation would be to first extend \(\Gamma\) on \(M_4\) to a linear connection \(\hat{\Gamma}\) on \(P \equiv M_4 \times G\) with trivial vertical propagation and parallel transport in the horizontal directions defined in gauge covariant manner, since \(G\) does not represent an extension into a physical space. The gauge covariant parallelism of \(\hat{\Gamma}\) itself then determines a local relation between the horizontal subspaces on \(P\) which is equivalent to the structure of a gauge curvature \(\Omega\). (This approach is analogous to taking the metric \(\hat{g}\) as the primary entity on \(P\) in conventional Kaluza-Klein theories, as described leading to equation 59). From either perspective, this gauge covariant parallelism via the set \(\hat{\Gamma}^i_{jk}\) of equation 88 on \(P\) provides an appropriate description of the geometry on the bundle as a means of deriving a gauge invariant relation between the external and internal curvature on the base space \(M_4\).

### 4.2 Perturbation to the Einstein-Hilbert Action on \(M_4\)

The main purpose of constructing a linear connection \(\hat{\Gamma}(p)\) on \(P\), as described in the previous subsection, is to provide a means through which a correlation between the external and internal curvature may be explicitly described. On the spacetime manifold \(M_4\) any relationship between the external geometry, expressed in terms of the Einstein tensor with components \(G_{\mu \nu}(x)\), and the internal geometry, expressed in terms of the gauge curvature with components \(F^\alpha_{\mu \nu}(x)\), must transform covariantly both under general coordinate transformations and under gauge transformations. One technique for obtaining such a relation is to first identify a scalar ‘Lagrangian’ function which has these invariance properties.

Towards this end a Riemann curvature tensor \(\bar{R}\) can be defined on the bundle space \(P\) itself in terms of the linear connection components \(\hat{\Gamma}^i_{jk}(p)\) of equation 88, with the components \(\bar{R}^i_{jk(l)}\) determined by equation 29. In turn the Ricci curvature \(\bar{R}_{jk} = \bar{R}^i_{jk(l)}\) and scalar curvature \(\bar{R} = \bar{R}^{ij} \bar{R}_{ij}\) can also be constructed. In these expressions the Killing form \(\hat{g}_{\alpha \beta} = g_{\alpha \beta} = K_{\alpha \beta}\) subcomponents of \(\hat{g}_{ij}\) relate the Lie algebra adjoint and coadjoint representations as usual, with for example \(F^\alpha_{ab} = g_{\alpha \beta} F^\beta_{ab}\), and may be employed as a mathematical structure on \(P\) in the derivation of scalar quantities.
through the contraction of indices associated with a basis for the Lie algebra of $G$, as for example in the second term of equation 71. That is, unlike the case of Kaluza-Klein theories for which $g_{\alpha\beta} \propto K_{\alpha\beta}$ is interpreted as a physical metric, here the components $g_{\alpha\beta} = K_{\alpha\beta}$ are employed in terms of the Killing form in the usual sense, both on the group manifold $G$ and the bundle space $P$. Further, the $\dot{g}_{\alpha\beta} = \dot{g}_{ab} = 0$ subcomponents of $\dot{g}_{ij}$ on $P$ simply express the physical distinction between the vertical and horizontal subspaces in the decomposition of equation 42. Only the remaining components $\dot{g}_{ab}(p)$ implicitly represent a physical metric in that they are inherited directly from the metric $g_{ab}(x)$ on the base space manifold $M_4$ as described for equation 60. This physical metric on the spacetime manifold $M_4$ itself derives from the local Minkowski metric $\eta_{ab}$ of equation 26 for the present theory.

The resulting scalar $\ddot{R}(p)$ can be determined directly from the more general case of equation 72, here with $\dot{R}_G = 0$ and with $\chi = 1$ from equation 73 via the non-zero coefficients $f_4 = f_5 = 1$ implied equation 88. As noted in the previous subsection, following equation 88, this set of $\dot{\Gamma}^i_{jk}(p)$ is equivalent to setting $\lambda = 2$ in the ‘Kal [18]’ column of table 1 and we have simply $\dot{R}(p) = R_M + F^2$, as can be read off directly from the bottom line of the ‘Kal [18]’ column of table 2. It can also be seen from that column that the finite scalar curvature $\ddot{R}$ on $P$ arises entirely from the $g^{ab}\ddot{R}_{ab}$ contribution in equation 71, consistent with an augmentation to the geometry of the base space $M_4$.

This scalar $\ddot{R}(p)$ on $P$ is independent of the location on the fibre over any $x \in M_4$ and as a scalar field for any given $p \in P$ it takes the same value in any tangent space basis for $FP$. Hence in a direct product basis associated with a section $\sigma : M_4 \rightarrow P$ this field is simply $\ddot{R}(p) = \ddot{R}(p)$ and we can unambiguously define:

$$\ddot{R}(x) = \sigma^* \ddot{R}(p) = R_M + F^2$$

(90)

as a gauge invariant scalar field on the base space $M_4$. The value of this scalar field $\ddot{R}(x)$ is equivalent to $\ddot{R}(p)$ for any $p \in P$ such that $\pi(p) = x \in M_4$. Hence $\ddot{R}(x)$ is a real scalar function on $M_4$ which contains information about both the external and internal geometry, is invariant under both coordinate and gauge transformations on the base space, and therefore makes a suitable ‘Lagrangian’ candidate on $M_4$. This expression derives from the geometry on the bundle $P \equiv M_4 \times G$ in a physically meaningful way that can be expressed in terms of entities defined on the base space $M_4$.

The means of constructing the scalar field of equation 90 can be considered as a perturbation to general relativity on the base space deriving from the need to take into account the symmetries and internal space of the full form $L(\hat{v}) = 1$ when broken over $M_4$ and the geometric structures entailed. This perturbation carries with it consequences for the Riemannian geometry on $M_4$ that follow from the embedding of the 4-dimensional spacetime manifold in the structures of the higher-dimensional form of temporal flow $L(\hat{v}) = 1$. We hence conjecture that a relation between the external and internal curvature can be determined via the scalar function $\ddot{R}(x)$ of equation 90, interpreted as a geometric perturbation to the vacuum case for the Einstein-Hilbert action on the base space $M_4$.

This approach is justified in part since the Einstein-Hilbert vacuum solution, $G^{\mu\nu} = 0$ of equation 34, correlates with the zero Riemann curvature, implied in equations 14 and 15, on $M_4$ for the original vacuum case of figure 2(b) deriving from the
symmetries of \( L(v_4) = 1 \) alone, as associated with the canonical flat Lorentz connection 
\( A(x) = h^*\theta(h) \) of equation 12. On generalising from that case, with a linear connection 
\( \Gamma(x) = 0 \) in suitable coordinates on \( M_4 \), to a linear connection \( \Gamma(p) \) defined on the 
space \( P \equiv M_4 \times G \) deriving from the broken symmetries of the higher-dimensional 
form \( L(\tilde{v}) = 1 \), as exemplified in figure 3(b), the geometry of the base manifold \( M_4 \) 
is taken to be determined by a perturbation to the Einstein-Hilbert action with the scalar \( R_M(x) \) augmented to \( \tilde{R}(x) \) in equation 90.

For Kaluza-Klein theory, reviewed in section 3, originating as a physical higher-
dimensional spacetime extension of general relativity, to be interpreted as a unified 
theory of gravitation and gauge fields in a 4-dimensional spacetime the symmetry of 
general coordinate transformations in the extended spacetime has to be \textit{broken} 
down to 4-dimensional general covariance together with the local gauge symmetry. This is 
equivalent to placing restrictions on the metric of the extended space, in conformity 
with equation 59, which then possesses a set of isometries described by Killing vector 
fields which have a one-to-one relationship with the left-invariant vector fields on the 
manifold of an apparent gauge group \( G \). In this way a principle fibre bundle structure 
emerges on the extended space, exhibiting symmetries such that the freedom in 
variation of the metric \( \tilde{g}_{ij} \), as expressed in a direct product basis in equation 61, is 
effectively reduced to the components \( g_{ab} \) and \( \omega_a^\alpha \). The construction of an Einstein-
Hilbert action integral on this bundle space \( P = (M_4, G) \) then leads to corresponding 
equations of motion as described for equations 74 and 75. A dynamical mechanism for 
this process in which an extended 4-dimensional base manifold \( M_4 \) of general relativity 
survives while the \textit{extra} dimensions apparently lose any sense of external spatial sig-
nificance, sometimes called ‘dimensional reduction’ or ‘spontaneous compactification’, 
then remains to be specified. That is, the origin of the above restrictions on the metric 
for the full space remains to be accounted for.

The origin of the bundle structure in Kaluza-Klein theories hence contrasts 
sharply with that for the present theory. While for Kaluza-Klein theory the degrees of 
freedom of the manifold of the internal symmetry group \( G \) can be interpreted as extra 
dimensions of space over the spacetime manifold \( M_4 \), here the full geometric structure 
\( P \equiv M_4 \times G \) arises out of the broken symmetries of the full form of temporal flow 
\( L(\tilde{v}) = 1 \) as described in subsection 2.3, however with only the degrees of freedom of \( M_4 \) 
ever interpreted as an extended physical space. For example for the 9-dimensional form 
\( L(v_9) = 1 \), as employed in figure 3, the base manifold \( M_4 \) arises out of a parametrisation 
of a 4-dimensional subset of the degrees of freedom of the ‘translational’ symmetry of 
the components \( v_9 \) under \( L(v_9) = 1 \), with an internal gauge field over the resulting base 
space deriving from the internal ‘isochronal’ \( G = U(1) \) symmetry of the same temporal 
form. With the only \textit{physical} space being the manifold \( M_4 \), providing the arena for 
general relativity in a 4-dimensional spacetime, no ‘compactification’ from a higher-
dimensional extended spacetime is required. The spacetime geometry on \( M_4 \) derives 
from the local Minkowski metric \( \eta_{ab} \) implicit in the 4-dimensional temporal form of 
equation 26 in the projection out of a higher-dimensional form such as \( L(v_9) = 1 \), 
breaking the symmetry of this full temporal form.

However, while the interpretation differs there is significant overlap between 
the geometric structure employed for the present theory and that of the Kaluza-Klein 
theories reviewed in section 3. Indeed, the argument here leading to the set of linear 
connection coefficients \( \Gamma_{ij}^k(p) \) of equation 88 is heavily influenced by the range of
models studied as summarised in points ‘a) – e)’ towards the end of the previous subsection. This argument focuses on the horizontal transport in $P$ skirting over the base manifold $M_4$ through the internal degrees of freedom, and in appealing in particular to references [18] and [19] meets halfway with Kaluza-Klein theory.

In standard Kaluza-Klein theory the action $I_m$ for the scalar curvature $\tilde{R}(p)$ defined on the bundle space $P$ in equation 67 reduces to the 4-dimensional action integral $I_4$ of equation 68 owing to the trivial integration over the fibre degrees of freedom. The point of view adopted here is that all fields in the expression $\tilde{R}(x) = R_M + F^2$ of equation 90 are defined directly on the base space $M_4$ itself, with the components of the gauge curvature on the base space related to those on the bundle space as described for equation 54. Being invariant under both gauge and coordinate transformations on $M_4$ this scalar $\tilde{R}(x)$ can be employed as a Lagrangian function in the simple action integral:

$$\tilde{I} = \int \tilde{R}\sqrt{|g|}\,d^4x = \int (R_M + F^2)\sqrt{|g|}\,d^4x \quad (91)$$

defined directly on the base space $M_4$. This is the action on the base manifold $M_4$ that arises through the external and internal symmetry degrees of freedom described by the bundle space $P \equiv M_4 \times G$. As denoted by the ‘tilde’ on $\tilde{I}$ this function is considered as a perturbation to the Einstein-Hilbert action for the vacuum case, namely equation 33 with $\Lambda = 0$ and $\mathcal{L} = 0$ as described in subsection 3.1. Equation 91 incorporates the perturbation to the effective scalar Lagrangian $R_M(x) \rightarrow \tilde{R}(x)$ on the base space $M_4$ reflecting the augmentation from the vacuum case of $L(v_4) = 1$, as pictured in figure 2, with zero external curvature as concluded in subsection 2.2, to the symmetries of the broken full form of $L(\hat{v}) = 1$, as exemplified in figure 3 and described in subsection 2.3. The full Einstein-Hilbert action of equation 33 can be written:

$$\mathcal{I} = \int (\alpha R_M + \mathcal{L})\sqrt{|g|}\,d^4x \quad (92)$$

where the cosmological constant $\Lambda$ has been dropped in correspondence with the lack of a finite $R_G$ term in equations 90 and 91. Further comparison between the above two equations shows that equation 91 describes a perturbation to general relativity equivalent to the introduction of a Lagrangian term $\mathcal{L} = \alpha F^2$ in the original Einstein-Hilbert action of equation 92. While the mathematical conclusion is identical to Kaluza-Klein theory, here the interpretation involves a more minimal impact on the arena of general relativity in 4-dimensional spacetime, namely without a physical augmentation into a higher-dimensional extended spacetime.

The equation of motion obtained by requiring $\delta \tilde{I} = 0$ for equation 91, under variations $\delta g_{\mu\nu}(x)$ of the metric on $M_4$, follows the derivation of equation 74 and can be written here as:

$$G^{\mu\nu} = 2\chi(-F^{\alpha\mu}_\rho F^{\rho\nu}_\alpha - \frac{1}{4}g^{\mu\nu}F^{\alpha}_{\rho\sigma}F^{\rho\sigma}_\alpha) =: -\kappa T^{\mu\nu} \quad (93)$$

with $\chi = 1$. At the purely theoretical level the factor of $\chi = 1$ in this relation between the external and internal geometry in the breaking of the full form $L(\hat{v}) = 1$ arises directly from equations 72 and 73 and the relation of the linear connection $\Gamma^{i}_{jk}(p)$ on the bundle to the gauge curvature components $F^{\alpha}_{ab}(p)$ as summarised in equation 88.
These geometric arguments establish a direct relationship between the external and the internal curvature, with energy-momentum here defined through $-\kappa T^{\mu\nu} := G^{\mu\nu}$ on the right-hand side of this expression.

Through equation 93 for the internal symmetry $G = U(1)$ case direct contact is made between gravitation in the form of the geometric curvature of spacetime and the familiar laboratory phenomena of the electromagnetic field. In practice powerful electromagnetic effects are generally observed for which the associated gravitational field is immeasurably small and the appropriate units and a normalisation factor connecting the left-hand side and central expression of equation 93 are a matter for empirical convention, as for the factor of $\kappa = \frac{8\pi G_{\text{N}}}{T}$ on the right-hand side of this equation. On the other hand the bare mathematical relations are needed to understand the theoretical basis of the unification.

By further considering the stationarity $\delta \tilde{I} = 0$ for the action in equation 91, now with respect to variation in the gauge field components $A^{\alpha\mu}(x)$, which are related to the connection components on the bundle space as explained for equation 55, leads, as described earlier for equation 75, to the Yang-Mills vacuum equation:

$$D^\mu F^{\alpha\mu\nu} = 0$$

(94)

that is equation 39, with $D_\mu$ the gauge covariant derivative in curved spacetime. For the case of an Abelian internal $U(1)$ symmetry this relation expresses Maxwell’s equation for a source-free electromagnetic field. Such a structure arises for example for the internal $U(1)$ symmetry of equation 24 from the breaking of the full $SL(3, \mathbb{C})$ symmetry for the $L(v) = 1$ model depicted in figure 3.

Given the relation of equation 93 itself, a number of further consequences for the equations of motion may be deduced without the need for the Lagrangian formalism. A fundamental difference between the Einstein equation $G^{\mu\nu} = -\kappa T^{\mu\nu}$ (with or without the cosmological term of equation 36) and other equations of motion is that, assuming that all fields are associated with energy-momentum $T^{\mu\nu}$, all fields are subject to the constraint of the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ and in principle ‘no physical entity escapes this surveillance’ ([8] page 475). For example the Einstein equation in the form of equation 93, for case of the $U(1)$ internal symmetry, mutually constrains the evolution of both the gravitational and electromagnetic field. In this way it can be shown ([8] page 472) that the source-free form of the Maxwell equation $\nabla_\mu F^{\mu\nu} = 0$ (that is, equation 94 for the $U(1)$ case) does not need to be derived independently, rather it may instead be deduced from $\nabla_\mu G^{\mu\nu} = 0$ applied to the Einstein equation for the electromagnetic field. This observation, together with further consequences for the equations of motion, is reviewed in more detail in ([3] section 5.2).

A further aim for the present theory would be to derive equation 93, or an equivalent expression, by a direct geometrical means without reference to a scalar Lagrangian function and the Einstein-Hilbert action. Here equation 93 has been considered to arise as a perturbation to the Einstein vacuum equations, derived for equation 34 in terms of the stationarity of the Einstein-Hilbert action under variations of the metric $\delta g_{\mu\nu}(x)$. With equation 93 itself conjectured to arise inevitably out of the geometric constraints implied in the breaking of the full $L(\hat{v}) = 1$ symmetry over $M_4$ any explicit reference to the Lagrangian formalism might in principle be avoided entirely.
In conclusion, the construction presented in this section leading to the relation between the external and internal curvature of equation 93 is based on the broken symmetry structure $P \equiv M_4 \times G$, which itself derives from the identification of the base manifold $M_4$ from a parametrisation of the translation symmetry of the full multi-dimensional form of time $L(\hat{v}) = 1$ as described in section 2. The mathematical framework for this geometric unification closely resembles that of the Kaluza-Klein theories reviewed in section 3, and which have provided an essential guide for the present theory. Attempting to justify all the steps along the way, via the linear connection on the bundle space of equation 88, scalar function on the base space of equation 90 and action integral of equation 91, the aim has been to arrive at the relation of equation 93 with minimal assumptions. This equation shows how a relation between the external and internal curvature can be achieved for the present theory through a multi-dimensional temporal form as an expression of the unifying structure inherent in basing the theory on the underlying one-dimensional flow of time.

5 Summary and Discussion

5.1 Conceptual Picture

The fundamental role of time in physical theories as well as the universal nature of time in infusing both experiments and our interactions with the world more generally is widely recognised. Physical theories themselves are typically constructed in terms of postulated particles, fields or other entities evolving in time according to equations of motion or consider time as an essential component of a 4-dimensional spacetime manifold, which may itself be embedded within a larger structure possessing ‘extra dimensions’.

The underlying idea described in this paper, developed in sections 2 and 4, contrasts with the Kaluza-Klein models, reviewed in section 3, since here ‘extra dimensions’ are not required to satisfy an explicitly geometric, or spacetime, symmetry. For the theory presented here we study a general higher-dimensional ‘symmetry of time’ and describe how a large scale extended 4-dimensional spacetime geometry supporting physical structures can derive from the underlying one-dimensional temporal flow itself. The means of obtaining multiple spatial, extended and also extra dimensions from a one-dimensional element of time has been described here in subsections 2.1, 2.2 and 2.3 respectively, elaborating upon ([3] chapter 2) and ([4] sections 2 and 3). These structures exploit the symmetries of the general multi-dimensional flow of time, expressed as $L(v_n) = 1$ as derived for equation 4.

We note that the starting point for the theory is the mathematical interval $\delta s \in \mathbb{R}$ as a purely one-dimensional entity, and hence any typical ‘picture’ of this element of time as seemingly embedded within a given two or three-dimensional space, as for figure 1(a) necessarily drawn on a two-dimensional page, is inevitably somewhat misleading. An accurate representation of the one-dimensional interval of time is provided however, mathematically and directly, by the real element $\delta s \in \mathbb{R}$, as for the left-hand side of equations 2 and 3. The mathematical structure of multi-dimensional space itself is then derived from the interval of time through the implicit arithmetic
structure and symmetries of this real interval, as for example for equations 2 and 6 as pictured in figures 1(b) and (c) respectively. That is, time is not considered here to flow through an independent geometric manifold or expand into a pre-existing space or spacetime, rather these multi-dimensional structures are implicit within the arithmetic forms and symmetries of one-dimensional time itself.

One of the attractions of using a symmetry of extra spatial dimensions in Kaluza-Klein theories, as well as its intuitive appeal as an extension to a 4-dimensional spacetime geometry, is that it limits the set of possible higher symmetries and mathematical structures to consider. For the present theory the construction of symmetries of multi-dimensional forms of time also greatly limits the choice of symmetry groups and their representations, based on the form of equation 4. However, while considering natural extensions to the full higher-dimensional form of time $L(\hat{v}) = 1$ we retain the significance and necessity of a $(1 + 3)$-dimensional metrical manifold as a background arena for observations. Indeed, while possessing different geometrical properties compared with other higher symmetries of $L(\hat{v}) = 1$, the symmetry of the space part of spacetime, that is the SO(3) subgroup of the Lorentz symmetry $SO^+(1, 3)$ acting approximately globally over extended regions of the base manifold, describes the symmetry of an extended 3-dimensional approximately Euclidean arena through which the physical world is actually observed.

Developing the theory as described for the three stages of figure 1 and given the symmetries of an appropriate full multi-dimensional form $L(\hat{v}) = 1$ the possibility of identifying an extended 4-dimensional space with a local Minkowski metric $\eta_{ab}(x)$ as well as a local $so^+(1, 3)$-valued connection 1-form gives a geometric meaning to $M_4$ as being not just a numerical parameter space for translational degrees of freedom, from equation 5, but rather implicitly possessing a Riemannian structure as an arena for the observation of physical objects in time and space, as described for figures 2 and 3. The identification of such an extended base space is possible given a ‘spacetime’ symmetry as a subgroup of the full symmetry of $L(\hat{v}) = 1$ which acts on the local tangent space of $M_4$. The interplay between the necessary geometrical form of spacetime and the general mathematical form of temporal flow breaks the symmetry of the latter structure, in turn shaping the physical form of the observed world. Both the extended manifold $M_4$, essential as the backdrop for observations in the world, and the local external symmetry group are identified through symmetries of the full form $L(\hat{v}) = 1$, leaving residual temporal components together with an internal symmetry $G$. In this way both the external 4-dimensional spacetime manifold and matter fields on the manifold arise together in the breaking of a higher-dimensional form of temporal flow without the need to introduce either the spacetime background or a postulated material substratum independently of time itself.

At the elementary level the equality in equation 3 signifies two different ways of expressing the same interval of time $\delta s$ by applying simple arithmetic rules, implicit in the structure of the real line $\mathbb{R}$ itself. In turn the flow of time can be considered to be intrinsically accompanied by structures which represent the geometric form of spacetime through the appropriate quadratic form, in the simplest case with $p = 2$ in equation 3 and $\alpha_{ab}$ identified with the Minkowski metric. Such a structure can be algebraically contained within a more general higher-dimensional form for equation 3, with for example $p > 2$ describing a cubic or higher polynomial form. In subsection 2.3 we presented these ideas through the example of an $SL(3, \mathbb{C})$ symmetry of the cubic
form $L(v_9) = 1$ projected over $M_4$ as depicted in figure 3 with an internal $G = U(1)$ symmetry identified. The symmetry breaking structure exemplified in figure 3(b) motivates the employment of the geometric framework of the principle bundle space $P \equiv M_4 \times G$ for the general case, as analysed in detail in section 4 (and building upon [3] section 5.1).

As described in section 4 the principle bundle $P \equiv M_4 \times G$ is not considered here to represent a physical space or spacetime. While in Kaluza-Klein theory the parameter space of the internal symmetry group $G$ augments the physical spacetime arena, here the physical spacetime $M_4$ itself derives from a 4-dimensional component of the translational symmetry of $L(\hat{v}) = 1$ that exhibits the appropriate metrical structure. For this theory the base space $M_4$, as the only physically extended manifold, arises spontaneously as a background arena for observations out of the symmetries inherent in the form $L(\hat{v}) = 1$ with no mechanism of ‘compactification’ from a larger manifold required. The necessary identification of the geometry of $M_4$ provides a natural mechanism for breaking the full symmetry of the form $L(\hat{v}) = 1$.

In subsection 4.1 we described the relation between the internal curvature $\Omega(p)$ on $P$ as a principle bundle and a natural linear connection $\Gamma(p)$ on $P$ as a base space for a Riemannian geometry. From this latter perspective the notion of full ‘general covariance’ is not useful owing to the physical significance of the product space structure $P \equiv M_4 \times G$ and with a distinguished reference frame provided by the horizontal lift basis associated with the gauge connection $\omega$ as a further physically meaningful structure on the bundle space. With horizontal basis vector fields $\dot{e}_a(p)$ on $P$ having a one-to-one correspondence with basis vector fields $e_a(x)$ on $M_4$, as described for figure 4, the horizontal lift basis reflects and augments the structure of the spacetime manifold $M_4$ which is the primary physical arena and upon which the full machinery of general relativity, including the properties of general covariance, can be fully deployed. Further, in linking the coefficients of the linear connection $\Gamma(p)$ with the gauge curvature structure on $P$, guided by figure 5 and leading to equation 88, the corresponding gauge covariant transformation properties of equation 89 lead to gauge invariant expressions on the base manifold $M_4$ culminating in equation 93, relating the external and internal curvature, as described in subsection 4.2.

5.2 Higher Symmetries

In the absence of a structure of extra spatial dimensions in general the full form of temporal flow $L(\hat{v}) = 1$ is not required to be associated with a metric geometry. It may be that higher symmetries, such as $SO^+(1,n-1)$ with $n > 4$ acting on a quadratic form $L(v_n) = 1$ as an $n$-dimensional extension of equation 8, could be interpreted in a geometrical spatial manner, but this feature is relatively incidental in comparison with the fundamental requirement that it must describe a symmetry of a general homogeneous polynomial form of time as derived for equation 4. In fact through investigating possible symmetries of time a significant example is identified for the symmetry group $SL(2,O)$ acting on the 10-dimensional space $h_2 O$, constructed in terms of the octonion algebra as described in ([3] section 6.3), as the two-to-one covering group of the 10-dimensional Lorentzian symmetry $SO^+(1,9)$.

The question concerning the natural mathematical augmentation to higher-
dimensional forms $L(\hat{v}) = 1$ and their connection with empirical observations of the physical world is considered in ([3] chapters 6–9) and summarised in [4]. In ([4] section 4) a 27-dimensional space of elements $v_{27} \equiv \mathcal{X} \in h_3\mathbb{O}$ with cubic temporal form $L(v_{27}) = \det(\mathcal{X}) = 1$ and the exceptional Lie group $\hat{G} = E_6 \equiv \text{SL}(3, \mathbb{O})$ as the full isochronal symmetry is described. Given the extra dimensions of the full vector object $v_{27} \in h_3\mathbb{O}$ the identification of a Riemannian geometry parametrised over an extended 4-dimensional spacetime manifold $M_4$ breaks the full $E_6$ symmetry ([4] section 5).

The form $L(v_{27}) = 1$ itself can be motivated as an augmentation from the 4-dimensional form $L(v_4) = 1$ of equation 8, building on $v_4 \equiv h_2 \in h_2\mathbb{C}$ in equation 16, via the action of $\text{SL}(3, \mathbb{C})$ on the form $L(v_9) = 1$ with $v_9 \in h_3\mathbb{C}$, hence with a $2 \times 2 \rightarrow 3 \times 3$ matrix structure as described here in subsection 2.3, or via the action of $\text{SL}(2, \mathbb{O})$ on the form $L(v_{10}) = 1$ with $v_{10} \in h_2\mathbb{O}$, that is with $\mathbb{C} \rightarrow \mathbb{O}$ where the octonions are the largest normed division algebra as presented in ([3] chapter 6); that is via either of two possible natural extensions:

$$
\begin{align*}
\text{SL}(2, \mathbb{C}) & \text{ on } h_2\mathbb{C} \rightarrow \text{SL}(3, \mathbb{C}) \text{ on } h_3\mathbb{C} \\
\downarrow & \downarrow \\
\text{SL}(2, \mathbb{O}) & \text{ on } h_2\mathbb{O} \rightarrow E_6 \text{ on } h_3\mathbb{O}
\end{align*}
$$

(95)

In all cases the elements $v_n$ of the representation space (such as $h_3\mathbb{O}$) for the $n$-dimensional flow of time, subject to the form $L(v_n) = 1$ of equation 4, are the starting point for the present theory. Certain forms have rich symmetry properties which then motivates the study of a group structure with actions defined on this space, as is the case for the $\hat{G} = E_6$ isochronal symmetry of $L(v_{27}) = 1$. The construction of an extended spacetime manifold $M_4$ from a 4-dimensional subset of the translation symmetries of $L(v_{27}) = 1$, similarly as described for figure 3, is a prerequisite for the identification of physical entities on the base manifold. The subgroup $\text{SL}(2, \mathbb{C}) \subset E_6$ acting on the projected $v_4 \in TM_4$ components of the physical manifold $M_4$ partitions all the components of $v_{27}$ into vector, spinor or scalar objects as irreducible elements of the structure of the physics on $M_4$. Interest then turns from the mathematical symmetry of $E_6$ on $L(v_{27}) = 1$ to the local symmetry of this physical structure, with the components of $v_{27}$ partitioned into representation multiplets under the broken symmetry in the form of a direct product of the external symmetry $\text{SL}(2, \mathbb{C})$ and an internal symmetry $G$ as described for equation 23.

Through the augmentation of the full symmetry $\hat{G}$ acting on $L(\hat{v}) = 1$ from $\text{SL}(3, \mathbb{C})$ on the vectors $v_9 \in h_3\mathbb{C}$ to $E_6$ acting on $v_{27} \in h_3\mathbb{O}$ the decomposition of equations 19–22 is augmented to a vector, four spinor and seven scalar pieces in total ([3] table 8.2 and [4] table 2) under the external $\text{SL}(2, \mathbb{C}) \subset E_6$ symmetry component of $L(v_{27}) = 1$. The internal symmetry group is found to be $G = \text{SU}(3)_c \times U(1)_Q \subset E_6$ ([4] equation 42), as an extension from the internal $U(1)$ identified in equation 24 in this paper. While acting trivially on the vector $v_4 \equiv h_2 \in h_2\mathbb{C} \subset h_3\mathbb{O}$ this internal $\text{SU}(3)_c \times U(1)_Q$ symmetry is found to act upon the set of four Weyl spinors in the manner of a colour singlet charged lepton together with a colour triplet of fractionally charged $d$-quarks. That is, the particle states associated with these components of $v_{27}$ transform in representation multiplets under the broken symmetry structure in a manner resembling known elements of the Standard Model of particle physics, which itself is based upon empirical observations.
For the augmented case the $G = U(1)_Q \subset E_6$ symmetry can still be considered as a component of the internal symmetry in itself, with the results of this paper applying for equation 24 here as well as for the embedding within $E_6$ as the full symmetry group, and also for non-Abelian internal symmetry groups such as $G = SU(3)_c$. While the geometry of the simpler case with $SL(3, \mathbb{C})$ as the full symmetry was introduced in subsection 2.3, in section 4 we determined the relationship between the external Riemannian curvature and internal gauge curvature in equation 93 for the full theory bearing in mind the need to incorporate also non-Abelian and product internal symmetries such as $G = SU(3)_c \times U(1)_Q$.

Further generalisation first to an $E_7$ symmetry of a 56-dimensional quartic form $L(v_{56}) = 1$ and on to a predicted $E_8$ symmetry of a hypothetical form $L(v_{248}) = 1$ is described in ([3] chapter 9) and summarised in ([4] sections 6 and 7). Internal components of $L(v_{56}) = 1$, including elements transforming as Dirac spinors under the external $SL(2, \mathbb{C}) \subset E_7$ symmetry, exhibit transformation properties under the internal $SU(3)_c \times U(1)_Q \subset E_7$ gauge group, incorporating fractional charges and also a left-right asymmetry, consistent with the distinctive properties of the Standard Model for a family of lepton and quark states, as summarised in ([4] equation 66). These internal components of $v_{56}$, in addition to the internal gauge fields themselves, are interpreted as ‘matter fields’ on the external spacetime $M_4$ and describe a non-trivial foothold in the structures of the Standard Model of particle physics as identified in the pattern of $E_7$ symmetry breaking for this theory.

In Kaluza-Klein theory, as reviewed here in section 3, while a unified framework is provided for gravity and gauge fields, equations 74 and 75, there are no spinor fields – hence the matter fields for the leptons and quarks are absent. These fields may be added by hand as sections of fibre bundles over $M_4$, associated to the principle bundle $P$, transforming as spinors under the external $SO^+(1,3)$ symmetry, via the $SL(2, \mathbb{C})$ covering group as described after equation 32, and in appropriately constructed representation multiplets of the internal gauge symmetry group. Coupling between the gauge fields and spinors may then be introduced through interaction terms, also added by hand for example via ‘minimal coupling’ involving covariant derivatives, in the Lagrangian constructed for the theory. Another approach to introducing fermion states is through a supersymmetric extension of Kaluza-Klein theory, as discussed in ([3] section 5.4, see for example [14] section VI).

For the present theory there is no need to postulate additional fields in spacetime or attempt a supersymmetric extension; rather it is the additional symmetries and components of the form $L(\psi) = 1$, over and above those required to describe the external geometry of the base space $M_4$, that give rise to matter fields in 4-dimensional spacetime. That is, out of the breaking of the full symmetry of the form $L(\psi) = 1$ in the geometric identification of the external symmetry and spacetime manifold $M_4$, the residual internal gauge fields and surplus temporal components collectively compose the apparent ‘matter’ content of the world with the characteristic physical properties observed. Through these structures the underlying one-dimensional flow of time is itself represented as a flow of physical entities in an extended spacetime. In the case of the $SL(3, \mathbb{C})$ model in figure 3(b) the vector $v_4 \in TM_4$ of the temporal flow is projected onto the tangent space of the base manifold $M_4$ while the residual temporal component $\psi$ transforms as a spinor under the external symmetry. In this model the internal $U(1)$ symmetry acts on the spinor field, leading to interactions between the
corresponding gauge field and spinor states through dynamic expressions deriving from the constraints of the theory. In the extension to the E_6 symmetry as described for equation 95, and onto the E_7 (and potential E_8) augmentation, the resulting elementary states and interactions closely resemble those of the Standard Model of particle physics. The quantum mechanical properties of the observed empirical particle states then remain to be accounted for.

5.3 Quantum Theory

While the construction presented in this paper has been developed from the properties of the local geometry on the bundle space P (in particular involving parallel transport as described for equations 78–83 for example) the global validity of equation 93 in principle should take into account the quantisation of the theory ([3] chapters 10 and 11). In this case the local relation of equation 93 can be interpreted as describing cubic and quartic self-couplings in the non-Abelian gauge field components \( A^\alpha_\mu(x) \), as a potential input for the set of local interaction terms in the structure of an effective quantum field theory. The global relation between the external and internal geometry may need to be reassessed in this context, potentially with a ‘renormalised’ value for \( \chi \) required in equation 93, relating to the macroscopic empirical value for \( \chi \) discussed after that equation.

As also discussed towards the end of subsection 4.2 the local relation of equation 93, together with the implied gauge field self-interactions \( A \leftrightarrow A \) for a non-Abelian internal symmetry, might itself in principle be derived through a purely geometric argument without reference to the Einstein-Hilbert action. In addition interactions between the gauge and spinor fields of the form \( A \leftrightarrow \psi \) arise through the constraints of the present theory, for example in the terms of the expression \( D_\mu L(\hat{v}) = 0 \) in the symmetry breaking over \( M_4 \), without the need to introduce terms of the Standard Model Lagrangian by hand. The origin of all of these interaction terms for the present theory is described in ([3] section 11.1; see in particular equations 11.29 and 11.36–11.38, and with further discussion alongside table 15.1 in section 15.2).

The structure of textbook quantum field theory itself is a critical guide for the quantisation of the present theory with for example the ‘optical theorem’ of QFT ([3] section 10.5) providing the link with the calculation of process probabilities for the new theory ([3] section 11.2). As well as the structure of cross-section calculations in high energy physics experiments the nature of the physical concepts of quantum and particle phenomena in general need to be addressed ([3] section 11.3). In the bigger picture the overall structure of a unified framework for both quantum theory and gravitation can be developed ([3] section 11.4). With regard to the quantum field theory limit, as one element of the question of identifying empirically observable physical particle states in further work it will be necessary to incorporate the ‘spin-statistics theorem’, or a related argument, to account for the fermionic properties of states associated with the spinor fields identified in the elementary structure of the theory, as for \( \psi(x) \) in figure 3(b) for the SL(3, \( \mathbb{C} \)) model, in contrast to the bosonic degrees of freedom of the gauge fields.

The ‘Coleman-Mandula theorem’ [20] for any relativistic theory of interacting particles demonstrates that the only possible Lie group symmetry of a non-trivial
The $S$-matrix is locally isomorphic to the direct product of the Poincaré group and an internal symmetry group, with any non-trivial combination of the external and internal symmetries prohibited. Since for the present theory the external and internal symmetries are identified as subgroups of the initial unifying simple group $\hat{G}$, as the full isochronal symmetry of the full temporal form $L(\hat{v}) = 1$, the compatibility of this approach with the Coleman-Mandula theorem will be addressed explicitly here. This compatibility rests on the nature of the symmetry breaking structure for the present theory as described in subsection 2.3 and also discussed in the two previous subsections.

The essential point is that the full symmetry $\hat{G}$ of $L(\hat{v}) = 1$ is not a symmetry of the resulting QFT, rather a quantum field theory itself can only be constructed on the base space $M_4$ after the full symmetry has been broken down to a direct product of the form $(\text{Lorentz} \times G) \subset \hat{G}$ as described for equation 23. That is, the full symmetry $\hat{G}$ cannot act on particle states in the quantum field theory limit of the theory since these latter structures are necessarily constructed upon the extended spacetime manifold $M_4$, the identification of which through the isochronal and translational symmetries of $L(\hat{v}) = 1$ in turn necessarily breaks the full symmetry. From the starting point of the one-dimension of time only, the group $\hat{G}$ describes the full mathematical isochronal symmetry applying to the full multi-dimensional form of temporal flow $L(\hat{v}) = 1$, as derived for the general $n$-dimensional case in equation 4. This symmetry $\hat{G}$ is broken absolutely to the form of equation 23 in the identification of the physical 4-dimensional spacetime manifold $M_4$ and the associated partitioning of $\hat{v}$ into irreducible representations of the external Lorentz symmetry.

This is unlike the case of symmetry breaking in a ‘Grand Unified Theory’, such as an SU(5) model as originally described in [21], for the unification of the internal gauge forces (or in the electroweak sector of the Standard Model), for which a Lorentz scalar Higgs field is introduced together with suitable Lagrangian potential terms to break the symmetry for the field configuration in the ground state. For such cases the spontaneous symmetry breaking occurs below a certain energy scale, with the gauge bosons of the broken symmetry generators gaining a mass and suppressing the corresponding interactions at low energy; leading for example to a prediction of proton decay that rules out the simplest SU(5) model. However for the symmetry breaking as depicted in figure 3 the full symmetry $\hat{G}$ is completely lost, and not just below a certain energy scale, but at all energy scales for all and any physics that can be defined on the 4-dimensional spacetime manifold, and in particular only the broken symmetry remnant of equation 23 can apply for relativistic interacting particles. Again while further work is needed, the quantisation of the present theory might not only be needed to fully establish the relation between internal components of the temporal flow $\hat{v}$ and empirically observed physical particle states but, following the discussion of equations 23–25, might itself also guide the identification of the full internal symmetry $G$ acting upon these states, in a manner consistent with the properties of an effective quantum field theory.

While appearing in the direct product decomposition of equation 25 the D(1) symmetry, associated with a non-compact generator of SL(3, $\mathbb{C}$), was provisionally considered non-physical in subsection 2.3 since it acts non-trivially on the external spacetime and hence does not belong to the subgroup Stab($TM_4$) $\subset$ SL(3, $\mathbb{C}$). However this symmetry relates different types of Lorentz representation subspaces of $\mathfrak{v}_9 \in \mathfrak{h}_3 \mathbb{C}$ via a dilation action rather than by mixing the components, and the possibility of such
transformations having an important physical role in the very early universe has been speculated in ([3] section 13.2), with the projected magnitude $h(t)$ from equation 26 considered to be dependent upon the cosmic time $t$ at the earliest epoch. (Potential cosmological applications for the present theory are described more generally in [3] chapters 12 and 13).

The possibility of identifying a compact internal symmetry group $G$ in a direct product decomposition $(\text{Lorentz} \times G) \subset \hat{G}$ but with $G \not\subset \text{Stab}(TM_4) \subset \hat{G}$ is also considered in ([3] section 8.3 and chapter 9). There it is suggested that an internal $G = \text{SU}(2)_L \times \text{U}(1)_Y$ symmetry (proposed to be fully identified within the extension to $\hat{G} = \text{E}_7$ or $\hat{G} = \text{E}_8$) may impinge upon the external spacetime components of $v_4 \in TM_4$, accounting for the properties of electroweak symmetry breaking in the Standard Model. As a preliminary requirement in ([3] section 8.2) internal symmetries were first considered to be of the form $G \subset \text{Stab}(TM_4) \subset \hat{G}$. However a more direct definition is given here in section 2.3 in terms actions identified within the original full symmetry $\hat{G}$ that respect the partitioning of the full temporal flow $\hat{v}$ into vector, spinor or scalar objects by the action of the external Lorentz symmetry, implying the direct product structure of equation 23.

Cases such as the internal $G = \text{U}(1)$ symmetry of equation 24, identified through its independent action on the three parts of equations 20–22, which also have the form $G \subset \text{Stab}(TM_4) \subset \hat{G}$ may be associated with massless gauge bosons such as the photon. This structure is augmented to the massless gluons and photons for the $\text{SU}(3)_c \times \text{U}(1)_Q \subset \text{Stab}(TM_4) \subset \text{E}_6$ symmetry under the extension of equation 95 and on to the $\text{E}_7$ case reviewed in the previous subsection. On the other hand, in the full theory, internal generators that impinge upon the components of $v_4 \in TM_4$ will be associated with massive gauge bosons, such as the $W^\pm$ and $Z^0$ states of an $\text{SU}(2)_L \times \text{U}(1)_Y$ electroweak theory ([3] subsection 8.3.3). Again the possible structure of all such interactions may be both consistent with and guided by the Coleman-Mandula theorem and properties of the quantum field theory limit for the theory generally.

Hence for the present theory there are two distinct elements to the breaking of the full symmetry $\hat{G}$ with two distinct types of consequences for the corresponding phenomenology. The first element is the absolute breaking of the full symmetry $\hat{G}$ of the form $L(\hat{v}) = 1$ into the external and internal components of equation 23. The second element is the apparent breaking of an internal symmetry through an impingement on the external spacetime component $v_4 \in TM_4$ of $\hat{v}$, which is associated with the phenomena of electroweak symmetry breaking in the Standard Model. In the discussion of ([3] equation 9.52) it is suggested that the vector $v_4 \equiv h_2 \in h_2 \mathbb{C}$ could itself be composed of spinors under an external $\text{SL}(2, \mathbb{C}) \subset \text{E}_8$ symmetry, with an internal $\text{SU}(2)_L \times \text{U}(1)_Y \subset \text{E}_8$ acting directly upon these as well as other spinor components of $v_{248}$. Hence the explicit structure of the full action of $\text{E}_8$ on the predicted form $L(v_{248}) = 1$ may be needed to fully explore electroweak theory and the associated Higgs sector within the context of the present theory.

All Grand Unified Theories, such as the $\text{SU}(5)$ model alluded to above, involve two widely differing energy scales and at least two multiplets of Higgs scalars with vacuum expectation values of around $10^{15}$ GeV at the GUT scale and $10^2$ GeV at the electroweak scale. Unavoidable interactions between the Higgses mixes the mass scales and destroys this hierarchy, with an aesthetically unappealing fine tuning of the potential parameters required at each order of perturbation theory to maintain the
light scalar Higgs. For the present theory while the resemblance between the ‘mock electroweak theory’ described in ([3] section 8.3) and standard electroweak theory is provisional, the empirically observed light scalar Higgs is associated with local fluctuations in the scalar magnitude $|v_4| = h$ of equation 26. However, the breaking of the full $\hat{G}$ symmetry in the identification of the base space $M_4$, proposed to ultimately provide the source of the full spectrum of Standard Model particle states, does not resemble the Higgs mechanism. Hence with two distinct mechanisms of symmetry breaking, and without any Higgs interactions associated directly with the breaking of $\hat{G}$, there is no interference with the electroweak Higgs and in principle the ‘hierarchy problem’ is avoided. This possibility provides a further area of study for the future development of the theory.

In this paper we have explored and developed the elementary geometric structure and physical interpretation associated with symmetries of a higher-dimensional form of time $L(\hat{v}) = 1$ when broken in the identification of the external spacetime manifold $M_4$. We note that beginning with a single dimension is perhaps the simplest conceivable starting point for a physical theory, and marks a significant contrast with Kaluza-Klein theories which generally begin with a complementary motivation in positing extra spatial dimensions over and above the 4-dimensional spacetime manifold. Unlike the unobserved extra dimensions of a Kaluza-Klein theory here founding the theory upon the one-dimensional flow of time offers both a conservative and an unambiguous starting point, with an interval of time modelled by an interval of the real line $\mathbb{R}$ which itself necessarily has zero intrinsic curvature. While setting out on a firm footing based on time as the fundamental entity the development of this theory exhibits a significant overlap with the geometrical framework of non-Abelian Kaluza-Klein theories (including [12, 13, 14, 15] and [16, 17, 18, 19]). The observations of these theories, adapted and reinterpreted for the present theory, help to motivate the construction of a linear connection $\hat{\Gamma}(p)$ as described for equation 88 defined on a principle fibre bundle $P \equiv M_4 \times G$, which itself is identified through the broken symmetries of the full form of temporal flow $L(\hat{v}) = 1$. This structure in turn leads to the relationship between the external and internal curvature described in equation 93, derived here for a theory developed from the single initial dimension of time alone.

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