Deformation independent open brane metrics and generalized theta parameters

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Abstract: We investigate the consequences of generalizing certain well established properties of the open string metric to the conjectured open membrane and open $D^p$-brane metrics. By imposing deformation independence on these metrics their functional dependence on the background fields can be determined including the notorious conformal factor. In analogy with the non-commutativity parameter $\Theta^{\mu\nu}$ in the string case, we also obtain ‘generalized’ theta parameters which are rank $(q + 1)$ antisymmetric tensors (polyvectors) for open $D^q$-branes and rank 3 for the open membrane case. The expressions we obtain for the open membrane quantities are expected to be valid for general background field configurations, while the open D-brane quantities are only valid for one parameter deformations. By reducing the open membrane data to five dimensions, we show that they, modulo a subtlety with implications for the relation between OM-theory and NCYMY, correctly generate the open string and open D2-data.
1 Introduction

Open string metrics play an important role in the study of open strings in non-trivial backgrounds. In particular, non-commutative Yang-Mills (NCYM) \([1]-[7]\) and non-commutative open string theories (NCOS) \([8]-[15]\) can arise on the world volume of D-branes, probing backgrounds with non-zero NS-NS two-form potentials. For constant background fields on the world volume of the D-brane a straightforward derivation of the two-point function between two string coordinates reveals that the closed string metric and NS-NS two-form potential get replaced by an effective open string metric and a \(\Theta\)-parameter governing the non-commutativity of world volume coordinates. We also get an effective open string coupling. These open string quantities determine in these backgrounds all the physical properties of the D-brane such as the open string spectrum.

When studying the physics of D-branes, one is often interested in describing the brane decoupled from the bulk closed string theory. This essentially means taking a scaling limit of the closed string quantities. One may then use the expressions of the open string quantities given in terms of the closed string data to determine the fate of the open strings on the D-brane \([16]\).

Similar decoupling limits have been studied not only in the case of open strings ending on D-branes, but also for the case of open membranes ending on five-branes in eleven dimensions \([16]-[20]\) and various situations with open D-branes ending on other D-branes \([21, 22]\) or NS5-branes \([16, 19]-[24]\).

In these other cases, involving the open membrane or other open \(p \geq 2\) branes, we do not know how to proceed with quantization. However, instead of trying to extract important quantities, for, e.g., open membranes, from \(n\)-point functions one may try to determine some general principles or key properties, which will make it possible to uniquely construct the open membrane analog of the open string metric and non-commutativity (or theta) parameter as done in \([17]\). The tensor structure of the metric (i.e., the metric modulo its conformal factor) could be determined from the equations of motion on the five-brane, while the overall conformal factor could be fixed, to leading order, by demanding that in the decoupled theory the dimensionless open membrane metric be fixed \([17]\). This was checked (in the limit) using IIB/M-theory duality. Further evidence for the scaling behavior of the conformal factor was put forth in \([20]\), where it was shown that it implies that the near 'horizon' region of the self-dual string soliton on the five-brane decouples from the 'bulk' five-brane theory in a Maldacena-like scaling limit of the five-brane world volume theory, and that this region has an interesting fixed OM-metric geometry.

The complete conformal factor was recently derived by requiring that the open membrane metric should reduce to the open string metric when reducing to ten dimensions \([25]\). The derivation in that work is based on an assumption regarding the
relation between the open string coupling and the component of the open string metric in the compactified direction, similar to how the string coupling arises from the eleven dimensional metric. One of the primary goals of this paper is to use an alternative method to determine the overall conformal factors of the various open brane metrics, based on non-trivial properties that one extrapolates from knowledge of the open string metric. It is then possible to check consistency, as we will show below in this paper, by examining the dimensional reduction of such quantities and see if they match to known results.

The central idea that will be used here is the property of deformation independence (described in detail below) [26, 27, 12, 13] to obtain explicit expressions for the open brane metrics and generalized theta ($\Theta$) parameters for the cases mentioned above. These theta parameters generalize the non-commutativity parameter $\Theta^{\mu\nu}$ appearing in the open string case to higher dimensional open branes. However, since here the connection to non-commutativity is unclear we will only refer to them as theta parameters. Theta parameters of rank larger than two have also been discussed in e.g., [28]. One might, however, speculate that these open brane quantities also for $q \geq 2$ contain some information about the two-point and $(q+1)$-point functions of the deformed quantum theory.

Starting with the observation that deformation independence is a property of the open string metric, we postulate that this principle extends to other brane metrics. Although we do not explore this avenue here, it may in fact be possible to derive these open brane metrics using various kinds of dualities. As will be clear in the later sections some of our arguments supporting the implementation of deformation independence for open branes are closely related to such duality arguments. For instance, T-duality relates as usual open D$q$-brane quantities to open D$(q \pm 1)$-brane quantities.

The paper is organized as follows. After explaining the concept of deformation independence in section two, we derive in section three the conformal factor for our proposed open D-brane metrics. In this section we also construct generalized theta parameters for open D$q$-branes, based on the idea that they must be scale independent. In section four we then repeat this exercise for the open membrane. By demanding deformation independence, we find an open membrane metric that agrees with the one obtained in [25] by reducing the open membrane metric to open string theory and by demanding that it correctly produces OM theory in the decoupling limit defined in [17]. Section four also contains a derivation, showing that the open membrane theta parameter is linearly self-dual with respect to the open membrane metric. In section five we check that the eleven dimensional quantities, the open membrane metric and theta parameter, correctly reduce to the expressions for the open string and open D2-brane in ten dimensions. Some concluding remarks are given in section six.
Open string data and deformation independence

The data governing the effective open string perturbation theory on a Dp-brane in a closed string background with string frame metric \( g_{MN} = g_{\mu\nu} \oplus g_{ij} \), dilaton \( e^{\phi} \) and two-form potential \( B_{\mu\nu} \) are given by the open string two-point function and the effective open string coupling. In this paper we use ten-dimensional spacetime indices \( M = 0, \ldots, 9 \), \( (p+1) \)-dimensional world volume indices \( \mu = 0, \ldots, p \), and \( (9-p) \)-dimensional transverse space indices \( i = p+1, \ldots, 9 \). The two-point function is

\[
< T[X^\mu(\tau)X^\nu(0)] > = -\alpha' G^{\mu\nu} \log |\tau| + i\pi \Theta^{\mu\nu} \epsilon(\tau),
\]

\[
< T[X^i(\tau)X^j(0)] > = -\alpha' g^{ij} \log |\tau|, \tag{1}
\]

where

\[
G_{\mu\nu} = g_{\mu\nu} + B_{\rho\mu} g^{\rho\sigma} B_{\sigma\nu}, \tag{2}
\]

\[
\Theta^{\mu\nu} = -\alpha' g^{\mu\rho} B_{\rho\sigma} G^{\sigma\nu}. \tag{3}
\]

The symmetric tensor \( G_{\mu\nu} \) is the open string metric governing the mass-shell condition for the open string states propagating on the D-brane, and the antisymmetric tensor \( \Theta^{\mu\nu} \) is the parameter of non-commutativity between the D-brane coordinates. The open string coupling constant is given by

\[
G^2_{\text{OS}} = e^{\phi} \left( \frac{\det G}{\det g} \right)^{1/4}. \tag{4}
\]

It is important to note that, as will be explained below, the open string metric and coupling are invariant under deformations of the tensor fields in the supergravity background that correspond to non-commutative deformations of a D-brane worldvolume (open string) theory. Such a supergravity solution is generated by a stack of \( N \) D-branes and corresponds to a U(N) gauge theory. Higgsing it to \( U(N-1) \times U(1) \) by giving a VEV to one of its scalars, corresponds to separating off one of the branes from the stack. Given that we are in a large \( N \) limit, the supergravity solution is unaffected and the separated brane that contains the U(1) degrees of freedom acts like a probe on the dual spacetime. The statement of deformation independence is then that the coupling and effective metric of these U(1) degrees of freedom as a function of the Higgs value are independent of non-commutative deformations of the field theory.

The non-commutativity parameter implies a deformation of the underlying algebra of functions, in terms of which the worldvolume theory is defined. From this point of view it is clear that it should not depend on the energy scale of the probe. This can be

\[\text{Our definition of the non-commutativity parameter } \Theta \text{ differs from the one in ref. by a factor } 2\pi.\]
verified explicitly using the supergravity dual description of the probe brane, \textit{i.e.,} the open string non-commutativity parameter given above does not depend on the location of the probe brane in the dual spacetime \cite{13, 7}.

The deformation independence implies that for both electric and magnetic deformations we have \cite{13}

\begin{alignat}{2}
G_{\mu\nu} \big|_{\text{deformed}} &= G_{\mu\nu} \big|_{\text{undeformed}} = g_{\mu\nu} \big|_{\text{undeformed}} = H^{-\frac{2}{p}} \eta_{\mu\nu}, \\
G_{\text{OS}}^2 \big|_{\text{deformed}} &= G_{\text{OS}}^2 \big|_{\text{undeformed}} = e^\phi \big|_{\text{undeformed}} = gH^{\frac{1-p}{p}},
\end{alignat}

where $H$ is a harmonic function. The NCOS (electric NS-NS two-form deformation) and NCYM (magnetic NS-NS two-form deformation) supergravity duals are obtained by taking the following near horizon limits ($\alpha' \to 0$):

\begin{alignat}{2}
\text{NCOS} & : & \quad x^M \bigg|_{\sqrt{\alpha'}} = g \text{ fixed}, & \\
\text{NCYM} & : & \quad x^\mu \bigg|_{\sqrt{\alpha'}} = g\beta^{\mu} \bigg|_{\text{undeformed}} = gH^{\frac{1-p}{p}} \text{ fixed}. &
\end{alignat}

The NCOS limit amounts to keeping fixed the ten scalars of the two-dimensional worldsheet theory, while the NCYM limit keeps fixed the transverse scalars of the probe brane gauge theory. Thus the former ‘sigma-model’ limit leads to the supergravity dual of a non-commutative open string theory, while the latter ‘field-theory’ limit yields the dual of a non-commutative Yang-Mills theory, even though (5) is valid in both cases (see \textit{e.g.,} \cite{13}).

Supergravity solutions corresponding to Dp-brane bound states can be obtained in many different ways, leading to equivalent solutions related by coordinate changes. In particular, by acting with transformations in the T-duality group $O(p+1, p+1)$ \cite{7, 13}, which act on the supergravity fields while keeping the spacetime coordinates fixed, the deformation independence of the open string metric and coupling constant becomes manifest, as written in (6). Other deformation methods that involve deformation dependent coordinate transformations do not lead to manifest deformation independence. Hence there exists a special choice of coordinates which implies (6). In these preferred coordinates the near horizon limits are of sigma-model and field theory type in the case of electric and magnetic NS-NS two-form deformations, respectively. The converse is not true, however, since there are deformation dependent reparametrizations that do not affect the nature of the near horizon limit while they upset (5), \textit{i.e.,} the preferred coordinates cannot be identified uniquely by examining the near horizon limits alone. In order to use the requirement of deformation independence to determine the open string metric, one has to identify the above-mentioned $O(p + 1, p + 1)$ action.
In [22] analogous preferred coordinates appropriate to open D(q)-branes on D(q + 2)-branes were identified by acting first with S-duality and then with T-duality transformations on the (D3,F1) bound state. In [22] it was shown that these coordinates indeed yield near horizon limits which are of sigma model type (i.e., all ten coordinates $x^M$ scale homogeneously in the near horizon limit) in the case of electric deformations of the RR $(q + 1)$-form potentials, and of field theory type in the case of magnetic such deformations.

Assuming that the open D-brane metrics are deformation independent (in these coordinates), we can use the form of the supergravity solutions to determine the covariant form of these quantities as functions of the closed D-brane data. Similarly, the generalized non-commutativity (or theta) parameters can be determined by requiring independence of energy scale. For simplicity, we shall apply these ideas in the next section in the case of minimal rank deformations.

The above approach can be ‘lifted’ to M-theory. The appropriate coordinates are given in section four below. As a result we find the deformation independent open membrane metric and scale independent theta-parameter as covariant expressions of the background three-form tensor field. These expressions are general since the three-form deformation in M-theory is general, even though the reductions to string theory may be restricted.

3 Open brane metrics and generalized theta parameters

3.1 The open Dq-brane metrics

In this subsection we will obtain the open Dq-brane metric that is valid for a background deformed by only one RR field, a $(q + 1)$-form. As discussed above, we commence with the assumption that deformation independence is also valid for open Dq-branes, similar to the open string case. We will obtain the open Dq-brane metric in the case of an electrically deformed D(q + 2)-brane background, corresponding to the $\overline{OD}_q$-theories, which are obtained in the decoupling limit of open Dq-branes ending on D(q + 2)-branes [22], and then show that it also holds in the case of a magnetically RR deformed background, corresponding to the generalized non-commutative gauge theories $D_q$-GT of [22], as well as for NS5-brane backgrounds, corresponding to the OD$p$-theories.

The relevant part of the supergravity solution, corresponding to a D(q + 2)-brane with an electric RR $(q + 1)$-form turned on, is given by [22]:

$$ds^2_{q+3} = (H h_-)^{-\frac{\theta}{2}} dx_{q+1}^2 + (H^{-1} h_-)^{\frac{\theta}{2}} dx_2^2,$$

Note that here $\theta$ is dimensionless, while in [22] $\theta$ was dimensionless.
\[ e^{2\phi} = g^2 h_0 H^{\frac{1-q}{2}} , \]
\[ C_{01 \cdots q} = - \frac{\theta}{gH h_-} , \]
\[ B_{q+1,q+2} = - \theta H^{-1} \]
\[ h_- = 1 - \theta^2 H^{-1} , \]
\[ H = 1 + \frac{gN(\alpha')^{\frac{5-q}{2}}}{r^{5-q}} , \]

where \[ dx_{q+1}^2 = -dx_0^2 + \cdots + dx_q^2 \] and \[ dx_3^2 = dx_{q+1}^2 + dx_{q+2}^2. \] To obtain this solution we have started with an electrically NS-NS deformed D3-brane (in the particular coordinates discussed in the previous section), i.e., a (D3,F1) bound state. Then we have used S-duality followed by T-duality to obtain the (D(q+2),Dq) bound state. It is now natural to assume that since we choose to start from a (D3,F1) bound state, given in coordinates such that we get a deformation independent open string metric, then after S-duality the new (D3,D1) bound state should give a deformation independent open D1-string metric (1). Finally, applying T-duality should provide us with solutions, giving deformation independent open Dq-brane metrics.

The open Dq-brane metric will now be derived by considering open Dq-branes ending on D(q+2)-branes. The expressions will also be correct for open Dq-branes \((q \neq 2)\) ending on NS5-branes, as will be shown below. We will make the following Ansatz for the open Dq-brane metric, which is similar in structure to the open string metric (3):

\[ G^{\tilde{O}Dq}_{\mu\nu} = A(X) g^{Dq}_{\mu\nu} + \frac{1}{q!} B(X) (C^2_{q+1})_{\mu\nu} , \]

where \( A(X) \) and \( B(X) \) are functions of

\[ X = \frac{1}{(q+1)!} C^2_{q+1} , \]

\[ g^{Dq}_{\mu\nu} = e^{-\frac{2\phi}{q+1}} g_{\mu\nu} \] is the closed Dq-brane metric, and

\[ (C^2)_{\mu\nu} = g^{\rho_1 \sigma_1} \cdots g^{\rho_q \sigma_q} C_{\rho_1 \cdots \rho_q \mu} C_{\sigma_1 \cdots \sigma_q \nu} , \quad C^2_{q+1} = g^{\mu\nu} (C^2_{q+1})_{\mu\nu} . \]

This Ansatz contains only one potential \( C_{\mu_1 \cdots \mu_{q+1}} \), corresponding to the fact that we only consider one-parameter deformations. \( A(X) \) and \( B(X) \) can now be obtained by inserting the background solution (8) and demanding deformation independence. This implies that the open Dq-brane metric \( G^{\tilde{O}Dq}_{\mu\nu} \) is equal to the closed Dq-brane metric in the original undeformed configuration,

\[ g^{Dq}_{\mu\nu}|_{\text{undeformed}} = (g^2 H)^{-\frac{1}{q+1}} \eta_{\mu\nu} . \]

This leads to two equations for \( A \) and \( B \), one in the 2 directions perpendicular to the electric flux, and one in the parallel directions (note the sign due to the lorentzian signature):
\[ A(X)(g^2 H)^{-1/2 + \frac{q-1}{2}} h_{-}^{-\frac{2}{q+1}} = (g^2 H)^{-1/2} , \]  
\[ A(X)(g^2 H)^{-1/2 + \frac{q-1}{2}} - B(X)(g^2 H)^{-1/2 + \frac{q-1}{2}} \theta^2 H^{-1} = (g^2 H)^{-1/2} . \]  
To obtain these equations we have used that 

\[ g_{\alpha \beta}^D = (g^2 H)^{-1/2 + \frac{q-1}{2}} \eta_{\alpha \beta} , \]  
\[ g_{ab}^D = (g^2 H)^{-1/2 + \frac{q-1}{2}} \delta_{ab} , \]  
\[ \frac{1}{q!} (C^2_{q+1})_{\alpha \beta} = -\theta^2 H^{-\frac{q+2}{q+1}} (gh_{-})^{-\frac{2}{q+1}} \eta_{\alpha \beta} , \]  
\[ \frac{1}{q!} (C^2_{q+1})_{ab} = 0 , \]  
\[ X = -\theta^2 H^{-1} , \]

where \( \alpha, \beta = 0, \ldots, q \) and \( a, b = q + 1, q + 2 \).

Solving for \( A(X) \) and \( B(X) \) from the above algebraic equations we find 

\[ A(X) = B(X) = (1 + X)^{\frac{1+q}{2+q}} , \]  
which gives the following open Dq-brane metric 

\[ G^\text{ODq}_{\mu \nu} = \left[ 1 + \frac{1}{(q+1)!} C^2_{q+1} \right]^{\frac{1+q}{2+q}} \left( g^D_{\mu \nu} + \frac{1}{q!} (C^2_{q+1})_{\mu \nu} \right) . \]  

Note that for \( q \neq 1 \) these open brane metrics are only expected to be valid for a one parameter deformation, i.e., only one RR field is turned on. For \( q = 1 \) we find agreement with the expected result for the open D-string \[14\], which we expect to be valid for any deformation, while for \( q = 2, 3 \) we have the following open D2-brane and D3-brane metrics:

\[ G^\text{OD2}_{\mu \nu} = \left[ 1 + \frac{1}{6} C^2_{3} \right]^{-\frac{1}{2}} \left( g^D_{\mu \nu} + \frac{1}{2} (C^2_{3})_{\mu \nu} \right) , \]  
\[ G^\text{OD3}_{\mu \nu} = \left[ 1 + \frac{1}{24} C^2_{4} \right]^{-\frac{1}{2}} \left( g^D_{\mu \nu} + \frac{1}{6} (C^2_{4})_{\mu \nu} \right) . \]  

As a first check of the expressions for the metrics we evaluate (17) in a magnetically (instead of electrically) RR deformed background \[22\]. The relevant part of this solution is 

\[ ds^2_{q+3} = h_+^{\frac{q}{2}} H^{\frac{q}{2}} (-dx_0^2 + dx_1^2) + h_+^{\frac{q}{2}} H^{\frac{q}{2}} (dx_2^2 + \cdots + dx_{q+2}^2) , \]  
\[ e^{2\phi} = g^2 H^{\frac{1+q}{2}} h_+^{\frac{3-q}{2}} , \]  
\[ C_{2q+2} = \theta g^{-1} H^{-1} h_+^{-1} , \]
where \( h_+ = 1 + \theta^2 H^{-1} \). This gives (12), which is the expected (deformation independent) result.

We can also check (17) in the case of a NS5-brane, deformed by an electric RR \((q + 1)\)-form, \( q \neq 2 \), where the relevant part of the solution is [22]

\[
d s_{1+5}^2 = h_+^{-2/3} \left(-d x_0^2 + \cdots + d x_p^2\right) + h_+^{-2/5} \left(dx_{p+1}^2 + \cdots + d x_5^2\right),
\]

\[
e^{2\phi} = g^2 H h_+^{-2/3}, \quad C_{01\ldots p} = -\theta g^{-1} H^{-1} h_+^{-1},
\]

or a magnetic RR \((q + 1)\)-form, \( q \neq 2 \), where the relevant part of the solution is [22]

\[
d s_{1+5}^2 = h_+^{-\frac{2}{5}} \left(-d x_0^2 + \cdots + d x_5^2\right),
\]

\[
e^{2\phi} = g^2 H h_+^{-\frac{2}{5}}, \quad C_{5-q\ldots 5} = -\theta g^{-1} H^{-1} h_+^{-1}.
\]

Also in these cases we get (12), which is the expected deformation independent result.

In the case \( q = 2 \) both \( C_{012} \) and \( C_{345} \) are non-zero due to the duality relation. This means that (17) is no longer valid (see also the final paragraph of section 5) and has to be replaced with the open membrane metric (50) below, where we of course must use the closed D2-brane metric instead of the closed M2-brane metric. This can easily be realized from the fact that the closed D2 and M2-brane metrics are ‘identical’ and that \( C_{\mu\nu\rho} = \tilde{g}^{-1} A_{\mu\nu\rho} \). Note also that \( C_{\mu\nu\rho} \) obeys the same self-duality relation as \( A_{\mu\nu\rho} \).

This self-duality has to be satisfied in order to obtain the correct number of degrees of freedom. It is also because of this self-duality constraint that we obtain a result which is different from the naive open D2-brane metric given by (15).

Next, we will take an electric near horizon limit of the solution (8), which gives the ‘relevant’ part of the supergravity dual of OD\(q\). We do this in order to obtain the OD\(q\) length scale \( \ell_{\text{OD}q} \), which we will then compare with the expression given in [22]. The electric near horizon limit is defined by keeping (22)

\[
\tilde{x}^\mu = \frac{\tilde{\ell}}{\sqrt{\alpha'}} x^\mu, \quad \tilde{r} = \frac{\tilde{\ell}}{\sqrt{\alpha'}} r, \quad \theta = 1, g, \tilde{\ell},
\]

fixed in the \( \alpha' \to 0 \) limit. This limit gives the following open D\(q\)-brane metric

\[
\frac{C_{\mu\nu}}{\alpha'} = \frac{1}{\ell_{\text{OD}q}^{q+1}} H^{-\frac{q+1}{q+1}},
\]

where \( \ell_{\text{OD}q} = g^{\frac{1}{q+1}} \tilde{\ell} \). This calculation gives exactly the same expression for the OD\(q\) length scale as in [22], which we expected. Note also that the metric is finite in the

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6Note that the harmonic functions are not the same, since the NS5-brane has four transverse directions while the M5-brane has five.
UV limit ($\hat{r} \to \infty$, $H \to 1$). Finally, the tension of the open D$q$-branes is fixed in the decoupling limit and can be calculated from

$$T = \left( \frac{\sqrt{-\det g^{Dq}}}{\alpha'^2 q} \right) - \epsilon^{01...q} \frac{C_{01...q}}{\alpha'^{q+1}}. \quad (24)$$

Inserting the solution (8), we get the following tension in the UV limit

$$T = \frac{1}{2\ell_{ODq}^{q+1}}. \quad (25)$$

### 3.2 Generalized theta parameters

Next we will obtain an expression for a generalized theta parameter for a theory of open D$q$-branes. Again these expressions will only be valid for a one parameter deformation (except for $q = 1$). In the string case, $\Theta^{\mu\nu}$ is given by the antisymmetric part of the two-point function, which should be expressed in terms of the background fields. When we have a $C_{q+1}$-form deformation with $q \neq 1$, we cannot construct an antisymmetric two-index tensor bilinear in the background fields, but one can instead try to write down quantities with more than two antisymmetrized indices that might in fact arise as an antisymmetric part of a $(q + 1)$-point function. Following the open string case where we make the following Ansatz for the generalized theta parameter

$$\Theta_{ODq}^{\mu_1...\mu_{q+1}} = -\alpha' g^{\mu\nu} B_{\rho\sigma} G^\sigma_{\nu} \Theta_{ODq}^{01...q}, \quad (26)$$

we have

$$\Theta_{ODq}^{\mu_1...\mu_{q+1}} = -\alpha' g^{\mu\nu} B_{\rho\sigma} G^\sigma_{\nu} \Theta_{ODq}^{01...q}, \quad (27)$$

where $\hat{A}(X)$ is a function of $X = \frac{1}{(q+1)!} C_{q+1}$. Now using (8) and the assumption that

$$\Theta_{ODq}^{0...q} = -\frac{\ell_{ODq}^{q+1}}{q+1} \quad (we choose the minus sign in order to have a minus sign in (30) below, similar to (26)), when we have taken the electric near horizon limit (22) gives

$$\hat{A}(X) = (1 + X)^{-\frac{1-q}{1+q}}. \quad (28)$$

The assumption above is motivated by the fact that the theta parameter in the NCOS case is constant and equal to the open string length scale $\Theta_{ODq}^{01} = \alpha'_{eff}$, i.e., $\Theta_{OS}^{01} = \alpha'_{eff}$. This yields the following generalized theta parameter

\[\text{Note that we ignore factors of } 2\pi.\]

\[\text{Note that this definition differ with a factor of } 2\pi \text{ from the usual definition } [3].\]

\[\text{Note also that this assumption implies that } \Theta_{ODq}^{0...q} = -\theta g(\alpha')^{\frac{q+1}{1+q}}, \text{ before one has taken the electric near horizon limit.}\]
\[ \Theta_{\sigma_{Dq}}^{\mu_1 \cdots \mu_{q+1}} = -\left(\alpha'\right)^{\frac{d+1}{2}} (1 + \frac{1}{(q + 1)!} (C_{q+1})^2)^{-\frac{1}{q+2}} g_{\sigma_{Dq}}^{\mu_1 \nu_1} C_{\nu_1 \cdots \nu_{q+1}} G_{\sigma_{Dq}}^{\mu_2 \mu_2} \cdots G_{\sigma_{Dq}}^{\mu_{q+1} \mu_{q+1}}. \quad (29) \]

In the case of a one parameter solution, this theta parameter can also be written as

\[ \Theta_{\sigma_{Dq}}^{\mu_1 \cdots \mu_{q+1}} = -\left(\alpha'\right)^{\frac{d+1}{2}} (1 + \frac{1}{(q + 1)!} (C_{q+1})^2)^{-\frac{1}{q+2}} g_{\sigma_{Dq}}^{\mu_1 \nu_1} \cdots g_{\sigma_{Dq}}^{\mu_q \nu_q} C_{\nu_1 \cdots \nu_{q+1}} G_{\sigma_{Dq}}^{\mu_{q+1} \mu_{q+1}}. \quad (30) \]

In fact, we have \( q + 2 \) possible expressions with the number of \( g_{\alpha \beta} \)'s ranging from zero to \( q + 1 \). The reason is that in the case of a one parameter solution, it can be written in terms of an \( \text{SO}(1, q+2)/\text{SO}(1, q) \times \text{SO}(2) \) parametrization for an electric \( C_{q+1} \) (similarly we have an \( \text{SO}(1, q+2)/\text{SO}(1,1) \times \text{SO}(q+1) \) parametrization for a magnetic \( C_{q+1} \) and in this case the formulas below hold when \( \alpha, \beta \) and \( a, b \) are exchanged)

\[ (C_{q+1})_{\alpha \beta} = \frac{1}{q + 1} C_{q+1}^2 g_{\alpha \beta}, \quad (C_{q+1})_{ab} = 0. \quad (31) \]

Inserting this in the open Dq-brane metric yields

\[ G_{\sigma_{Dq}}^{\alpha \beta} = (1 + \frac{1}{(q + 1)!} (C_{q+1})^2)^{\frac{2}{q+2}} g_{\sigma_{Dq}}^{\alpha \beta}, \quad G_{\sigma_{Dq}}^{ab} = (1 + \frac{1}{(q + 1)!} (C_{q+1})^2)^{\frac{2}{q+2}} g_{\sigma_{Dq}}^{ab}, \quad (32) \]

and we can therefore replace \( g_{\sigma_{Dq}}^{\alpha \beta} \) with \( C_{\sigma_{Dq}}^{\alpha \beta} \), but then we lose a factor of \( (1 + X)^{q+1} \).

It is not obvious that the expressions (29) and (30) are totally antisymmetric, but we can show that this indeed is the case. In the M2-brane case we have a three-form which obeys a non-linear self-duality relation on the M5-brane. Using dimensional reduction and T-duality, we get a relation between the \((q + 1)\)-form and the two-form on the D(\(q + 2\))-brane

\[ G_{\sigma_{Dq}}^{\mu_1 \mu_2 \cdots \mu_{q+1}} \sim \theta_{\sigma_{Dq}}^{\mu_1 \cdots \mu_{q+1}} B_{\mu_2 \cdots \mu_{q+1}}. \quad (33) \]

We thus see that (30) is proportional to the lefthand side of the relation above, and this theta is therefore totally antisymmetric. In general, the expressions for the theta parameters above will contain higher odd powers of \( C_{q+1}^{a_1 \cdots a_q+1} \). Since (30) only contains the powers one and three, we see that \( (C_{q+1}^3)^{a_1 \cdots a_q+1} \) is totally antisymmetric. In order to show the antisymmetry for (29), we only need to show that there is the highest power even in this case. To see this we use the following relation (which can be obtained by dimensional reduction of the M5-brane) on the D4-brane for the three-form

\[ (C_{3}^4)_{\mu \nu} = \frac{1}{3} C_{3}^{2} (C_{3}^{2})_{\mu \nu}. \quad (34) \]

T-dualizing this expression yields

\[ (C_{q+1}^{4})_{\mu \nu} = \frac{1}{q + 1} C_{q+1}^{2} (C_{q+1}^{2})_{\mu \nu}. \quad (35) \]
and therefore we do not get any powers of $C_{q+1}^{\mu_1 \cdots \mu_{q+1}}$ higher than three. Actually, due to the relation (31), the theta parameters will be linear in $C_{q+1}^{\alpha_1 \cdots \alpha_{q+1}}$

$$
\Theta_{\text{OD}_{q+1}}^{\alpha_1 \cdots \alpha_{q+1}} = - (\alpha')^{\frac{q+1}{2}} (1 + \frac{1}{(q+1)!} C_{q+1}^2)^{-1} C_{q+1}^{\alpha_1 \cdots \alpha_{q+1}}. \tag{36}
$$

As a test we now insert the magnetically RR deformed D$(q + 2)$-brane solution (19) in (29), followed by taking a magnetic near horizon limit [22]. This gives

$$
\Theta_{\text{OD}_{q+1}}^{2 \cdots (q+2)} = - \theta \ell^{q-1}, \tag{37}
$$

where $\theta$ has dimension $(\text{length})^2$ and $\ell$ has dimension length. This result for the ‘magnetic’ theta, i.e., the theta parameter for a magnetically deformed brane solution, is exactly the same as for the generalized non-commutativity parameter $\theta_{Dq-GT}$ for the non-commutative generalized gauge theories (D$q$-GT) defined in [22] (except for a trivial sign difference). This was expected since the supergravity dual of D$q$-GT is given by the magnetic near horizon limit of the (D$(q + 2), F1$) bound state, obtained by a magnetic RR $(q + 1)$-form deformation of a D$(q + 2)$-brane.

Next, we will check (29) for supergravity solutions which corresponds to NS5-branes with electric or magnetic RR $(p + 1)$-form deformations $p \neq 2$. Inserting (20) and (21) in (29) and taking electric and magnetic near horizon limits (see (20) in [22]) gives

$$
\Theta_{\text{OD}_{p}}^{0 \cdots p} = - \ell_{\text{OD}_{p}}^{p+1}, \tag{38}
$$

in the electric cases and

$$
\Theta_{\text{OD}_{p}}^{(5-p) \cdots 5} = \theta \ell^{q-1}, \tag{39}
$$

in the magnetic cases. As expected, (33) agree with $\theta_{Dq-GT}$ in [22]. Furthermore, just as in the string case, the theta parameter (38) of the light $Dp$-branes is given by the length scale.

In the special case $p = 2$, we have to use the open membrane generalized theta parameter (33) below (change $\ell_p^3$ to $(\alpha')^{3/2}$), since both $C_{012}$ and $C_{345}$ are non-zero due to the duality relation. This gives

$$
\Theta_{\text{OD}_{2}}^{012} = - \Theta_{\text{OD}_{2}}^{345} = - \ell_{\text{OD}_{2}}^3, \tag{40}
$$

and the theta parameter is again given by the length scale.

4 The open membrane metric and generalized theta parameter

In this section we will derive the open membrane metric and generalize the two-form non-commutativity parameter appearing in the string case to a three-form, which will
be referred to as the open membrane theta parameter\textsuperscript{10}. We will again assume deformation independence, now of the open membrane metric. The deformed solution we will use is the (M5,M2) solution, from which the supergravity dual of OM-theory can be constructed by taking the appropriate near horizon limit (see (54) below). Note that here, in contrast to the string case \cite{13, 7}, there is no known method that gives rise to a guaranteed deformation independent open membrane metric. Instead we will use duality relations between solutions to argue for the correct solution to start from.

The particular form of the (M5,M2) bound state solution we will make use of here is, in the five-brane directions, given by

\[
ds_{1+5}^2 = H^{-\frac{1}{2}}h^{-\frac{2}{3}}(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + H^{-\frac{4}{3}}h^{-\frac{2}{3}}((dx^3)^2 + (dx^4)^2 + (dx^5)^2),
\]

\[
A_{012} = \frac{\theta}{Hh}, \quad A_{345} = -\frac{\theta}{H}, \quad h = 1 - \theta^2 H^{-1}, \quad H = 1 + \frac{N\ell_p^3}{r^3}.
\] (41)

This solution can be obtained by lifting the (D4,F1) solution in the particular form given in (1) of \cite{22} to eleven dimensions. That solution is written in the proper coordinates to yield deformation independent open string metric and coupling constant, and we therefore expect that lifting that solution to 11 dimensions should provide a solution that can be used to obtain a deformation independent open membrane metric. For $\theta = 0$ the configuration is equal to the undeformed five-brane, and for $\theta = 1$ it defines the OM dual with critical scaling in the asymptotic region where $H \to 1$.

The open membrane metric $G_{\mu \nu}^{OM}$ is by definition a covariant function of a slowly varying background. Requiring the open membrane metric to be deformation independent implies that it should be equal to the undeformed metric

\[
g_{\mu \nu}|_{\text{undeformed}} = H^{-\frac{1}{2}}\eta_{\mu \nu}. \tag{42}
\]

We make the following Ansatz for the open membrane metric

\[
G_{\mu \nu}^{OM} = D(K^2)\left(g_{\mu \nu} + \frac{1}{4}(A^2)_{\mu \nu}\right), \quad K = \sqrt{1 + \frac{1}{24}A^2}, \tag{43}
\]

where

\[
(A^2)_{\mu \nu} = g^{\mu_1 \nu_1}g^{\mu_2 \nu_2}A_{\mu_1 \mu_2 \nu_1 \nu_2}, \quad A^2 = g^{\mu \nu}(A^2)_{\mu \nu}. \tag{44}
\]

The form of the tensorial factor can be argued for by noticing that it is this particular combination of $g_{\mu \nu}$ and $A_{\mu \nu \rho}$ that appears in the field equations on the M5-brane \cite{30}.

\textsuperscript{10}The three form theta parameters discussed here and elsewhere \cite{17, 28} can not, as in the string case, appear in the two point function but might arise in three point functions and in operator product expansions of two $x^{\mu}$’s. Therefore, strictly speaking, the theta parameter should probably be defined with upper indices and be referred to as a trivector in analogy with the bivector in the string case; see, e.g., \cite{29}.

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The conformal factor, on the other hand, is harder to derive and it has previously been obtained to leading order in \[17\]. However, more recently, in \[25\] its complete form was derived by reducing the open membrane metric to five dimensions and comparing it to the known answer for the open string metric. One of the main purposes of this paper is to derive the complete form of the open membrane metric without resorting to reductions to five dimensions. The derivation presented below therefore gives further and independent evidence for the correctness of the form of the open membrane metric obtained here and in \[25\].

Inserting \(41\) into \(43\) and equating the OM metric to the expression in \(42\) gives the following equation:

\[
D(K^2)H^{-1/3}h^{-2/3}\left(\frac{h+1}{2}\right) = H^{-1/3}, \tag{45}
\]

where

\[
K^2 = h^{-1}\left(\frac{h+1}{2}\right)^2, \tag{46}
\]

and hence the function \(D(K^2)\) is given by

\[
D(K^2) = h^{2/3}\left(\frac{h+1}{2}\right)^{-1}. \tag{47}
\]

Instead of now trying to find an expression for \(D(K^2)\), it is easier to first obtain an expression for \(\hat{D}(K^2)\) defined as

\[
\hat{D}(K^2) = [D(K^2)K^3] = h^{1/2}. \tag{48}
\]

Using \(44\), one finds the following expression for \(\hat{D}(K^2)\):

\[
\hat{D}(K^2) = K(1 - \sqrt{1 - K^{-2}}), \tag{49}
\]

where we should note that the minus sign in front of the second term must be chosen, since \(h < 1\) for all \(r\). Thus, extracting \(D(K^2)\) from \(48\) and \(49\) gives the following open membrane metric

\[
G_{\mu\nu}^{\text{OM}} = \left(\frac{1 - \sqrt{1 - K^{-2}}}{K^2}\right)^{1/3}\left(g_{\mu\nu} + \frac{1}{4}(A^2)_{\mu\nu}\right), \tag{50}
\]

which may also be written as

\[
G_{\mu\nu}^{\text{OM}} = [K^4(1 + \sqrt{1 - K^{-2}})]^{-1/3}\left(g_{\mu\nu} + \frac{1}{4}(A^2)_{\mu\nu}\right), \tag{51}
\]

or

\[
G_{\mu\nu}^{\text{OM}} = \left(\frac{(2K^2 - 1) - 2K^2\sqrt{1 - K^{-2}}}{K}\right)^{1/6}\left(g_{\mu\nu} + \frac{1}{4}(A^2)_{\mu\nu}\right). \tag{52}
\]
These expressions are valid for a generic three form $A_{\mu\nu\rho}$ in contrast to the open Dq metrics discussed in the previous section which are valid only for field configurations corresponding to one-parameter deformations. If we compare (52) for the open membrane metric with the open membrane metric (3.28) in [25], which was obtained using a completely different method, we see that the two expressions are identical. We therefore have two independent methods which give the same result, strongly indicating the correctness of the result, but the ultimate test is of course to obtain (50) from a microscopic formulation of OM theory, similar to how the open string metric has been obtained.

In the remainder of this section we will discuss the possibility to define a three form theta parameter by generalizing the open string two form non-commutativity parameter to the open membrane case [11]. To be physically acceptable, we demand the six dimensional three-form theta parameter to be constant, i.e., independent of $r$, when evaluated in terms of the solution (41), analogously to what happens in the case of the open string theta parameter, where it corresponds to theta being RG flow independent [26, 27, 13]. Using this assumption and (41) we find the following three-index theta parameter:

$$\Theta_{\text{OM}}^{\mu\nu\rho} = -\ell_3^3 p_1 [K(1 - \sqrt{1 - K - 2})]^{2/3} g_{\mu\nu\rho} A_{\mu\nu\rho} G_{\text{OM}}^{\mu\nu} G_{\text{OM}}^{\rho\nu}. \tag{53}$$

whose form is unique in the sense that no other combinations of $g_{\mu\nu}$ and $G_{\text{OM}}^{\mu\nu}$ would generate an acceptable result under the conditions stated above. Another way to see that there is only one acceptable form in this case is to note, that we can not rewrite this theta by exchanging the open membrane brane metric with the ordinary metric as we did for the theta parameter (29) in the case of open D-branes. The reason for this is the nonlinear self-duality condition for $A_{\mu\nu\rho}$, yielding components in both of the blocks in the SO(1,5)/SO(1,2) x SO(3) parametrization (see below). Furthermore, the theta parameter above is the only one reducing correctly to the open string and open D2-brane theta parameters, as will be shown in section 5.

A consequence of the nonlinear self-duality condition satisfied by $A_{\mu\nu\rho}$ is, that any theta constructed from it must itself satisfy some self-duality condition. In fact, the physics on the five brane should be describable in terms of only the open membrane quantities. As such one may replace $g_{\mu\nu}$ and $A_{\mu\nu\rho}$ with $G_{\text{OM}}^{\mu\nu}$ and $\Theta_{\text{OM}}^{\mu\nu\rho}$. The self-duality must then be expressible using only the appropriate metric, $G_{\text{OM}}^{\mu\nu}$. Therefore, one might suspect that the three form theta satisfies a linear self-duality condition with respect to the open membrane metric. Interestingly enough, enforcing such a linear condition leads exactly to the theta parameter with two explicit open membrane metrics as in the definition given above. Other definitions therefore can not satisfy this kind of simple self-duality condition. One way to make this plausible, is to consider the electric near

\footnote{For an independent discussion of theta parameters for the open membrane case, see [32].}
horizon limit of $\mathcal{H}$, i.e., the $\ell_p \to 0$ limit keeping the following quantities fixed

$$\tilde{x}^\mu = \frac{\ell_{\text{OM}}}{\ell_p} x^\mu, \quad \tilde{r} = \frac{\ell_{\text{OM}}}{\ell_p}, \quad \theta = 1, \ell_{\text{OM}}.$$  \hfill (54)

Taking this limit, we obtain the following open membrane metric and theta parameter

$$\frac{G_{\mu\nu}^{\text{OM}}}{\ell_p^2} = \frac{1}{\ell_{\text{OM}}^2} H^{-\frac{1}{3}},$$
$$G_0^{012} = \ell_{\text{OM}}^3,$$
$$G_0^{345} = \ell_{\text{OM}}^3,$$

and hence this theta parameter is linearly self-dual with respect to the open membrane metric in this limit. Note that also in the case of open membranes, the generalized theta parameter is given by the intrinsic length scale of the theory.

We end this section by proving that the theta defined in (53) does indeed satisfy a linear self-duality condition with respect to the open membrane metric given in (50). To do this we use a frame $(u_\alpha^\mu, v_a^\mu)$ where $\alpha = 0, 1, 2$ and $a = 3, 4, 5$ parametrized by the coset $\text{SO}(1,5)/\text{SO}(1,2) \times \text{SO}(3)$ as defined in [17]. With this parametrization we can write a generic three form $A_{\mu\nu\rho}$ in terms of one function as

$$A_{\mu\nu\rho} = a_1 u_3^{\mu\nu\rho} + a_2 v_3^{\mu\nu\rho}, \quad \hfill (56)$$

where, due to the non-linear self-duality [17], $a_1 = a_2 / \sqrt{1 + a_2^2}$. Then a general open membrane metric can be expressed as

$$G_{\alpha\beta}^{\text{OM}} = \left(1 + \frac{1}{6} A_1^2\right)^{2/3} g_{\alpha\beta}, \quad A_1^2 = g^{\alpha\beta} (A_1^2)_{\alpha\beta},$$

$$A_2 = g^{\alpha\beta} (A_2^2)_{\alpha\beta}, \quad \alpha, \beta = 0, 1, 2,$$

$$G_{ab}^{\text{OM}} = \left(1 + \frac{1}{6} A_2^2\right)^{1/3} g_{ab}, \quad A_2^2 = g^{ab} (A_2^2)_{ab}, \quad \hfill (57)$$

where we have used that

$$K^2 = \frac{(1 + \frac{1}{12} A_1^2)^2}{1 + \frac{1}{6} A_i^2}, \quad i = 1, 2,$$

$$1 + \frac{1}{6} A_2^2 = \left(1 + \frac{1}{6} A_1^2\right)^{-1}. \quad \hfill (58)$$

The last two relations follow directly from the relation between $a_1$ and $a_2$ given above, remembering that $u_3^{\mu\nu\rho}$ squares to $-6$ while $v_3^{\mu\nu\rho}$ squares to $+6$ (for more details see
The theta parameter can then be written as
\[ \Theta^{\alpha\beta\gamma} = (1 + \frac{1}{6} A_2^2) A^{\alpha\beta\gamma}, \quad \Theta^{abc} = (1 + \frac{1}{6} A_1^2) A^{abc}, \]  
and be shown to satisfy
\[ *(G \Theta)_{\mu\nu\rho} = \Theta_{\mu\nu\rho}, \]  
where the Hodge dual is defined as\(^\text{12}\)
\[ *(G \Theta)_{\mu\nu\rho} = \frac{1}{6} \frac{1}{\sqrt{-G}} \varepsilon^{\mu\nu\rho\kappa\lambda} \Theta_{\kappa\lambda}. \]  
Here \( G \) is the determinant of the open membrane metric and the indices on \( \Theta \) are lowered with \( G_{\mu\nu} \). We therefore see that \( \Theta^{\mu\nu\rho} \) is linearly self-dual with respect to the open membrane metric \( G_{\mu\nu} \). We also see from (59) that \( \Theta^{\mu\nu\rho} \) is completely antisymmetric.

5 Dimensional reduction of open membrane data

In this section we show that we can obtain the data for the open D2-brane corresponding to the OD2-theory as well as the open string data corresponding to NCOS by dimensional reduction of the open membrane data, i.e., the open membrane metric and theta parameter. When making these relations precise, we will find that only under certain conditions can the five-dimensional data be lifted to six dimensions. In particular, the open string metric with \( B^2 \geq 0 \), which we are accustomed to use in a magnetic limit to obtain NCYM, does not seem to have any direct connection to the six dimensional open membrane metric discussed in the previous section. The electric case with \( B^2 \leq 0 \) related to NCOS does, on the other hand, lift as expected.

We start from the open membrane data expressed in terms of a background metric and a three-form, which obeys a non-linear self-duality constraint \([30]\). When reducing from six to five dimensions, we obtain from the three-form \( A_{\hat{\mu}\hat{\nu}\hat{\rho}} \) (hatted indices are six dimensional in this section) both a two-form and a three-form which, however, are related due to the self-duality constraint. Here we will work with the supergravity fields, which according to the standard conventions reduce as follows (the far left hand sides of these equations are components of six dimensional quantities: \( \hat{\mu} = (\mu, y) \)):

\[ \frac{g_{\mu\nu}}{\ell_5^2} = e^{-\frac{2\phi}{\alpha'}} \frac{g^{\hat{\mu}\hat{\nu}}}{\alpha'}, \quad \frac{g_{yy}}{\ell_5^2} = \frac{e^{4\phi}}{R^2}, \]  
\[ \frac{A_{\hat{\mu}\hat{\nu}\hat{\rho}}}{\ell_5^3} = C_{\mu\nu\rho}, \quad \frac{A_{\mu\nu y}}{\ell_5^3} = B_{\mu\nu}, \]  
\[ \frac{A_{\mu\nu}}{\ell_6^3} = \frac{C_{\mu\nu}}{\alpha'^2}, \quad \frac{A_{\mu\nu y}}{\ell_6^3} = \frac{B_{\mu\nu}}{\alpha' R}. \]  
\(^\text{12}\)The Levi-Civita tensor is here defined with \( \varepsilon^{01...5} = 1. \)
where $R$ is the radius of the compactified direction labeled by $y$. The procedure is exactly the same as if we had chosen to reduce world volume fields; for instance the relations between the two-form and the three-form found in (25) hold, with a few minor modifications such as the inclusion of dilaton factors, for the supergravity fields as well.

The dimensional reduction is performed for the open membrane

$$G_{\hat{\mu} \hat{\nu}}^{\text{OM}} = z^{-1} K^{-1} \left( g_{\hat{\mu} \hat{\nu}} + \frac{1}{4} A_{\hat{\mu} \hat{\nu}}^2 \right), \quad (64)$$

where

$$z^{-1} = (K(1 - \sqrt{1 - K^{-2}}))^{\frac{1}{3}}, \quad (65)$$

and also for the theta parameter

$$\Theta_{\hat{\mu} \hat{\nu} \hat{\rho}}^{\text{OM}} = -\ell^3_p z^{-2} g^{\hat{\mu} \hat{\nu} \hat{\rho}} A_{\hat{\mu} \hat{\nu} \hat{\rho}} G_{\hat{\mu} \hat{\nu}}^{\text{OM}} G_{\hat{\mu} \hat{\nu}}^{\text{OM}}, \quad (66)$$

both discussed in detail in the previous section. Using the above reduction formulae, and the standard parameter relations $\ell^2_p = g^{\frac{2}{3}} \alpha'$ and $R = g^{\sqrt{\alpha'}}$, we get

$$A_{\mu \nu}^2 = g^{\frac{2}{3}} \left( C_{\mu \nu}^2 + 2 e^{-2\phi} B_{\mu \nu}^2 \right), \quad (67)$$

$$A^2 = C^2 + 3 B^2, \quad (68)$$

where contractions of $A, B$ and $C$ are done with $g_{\mu \nu}, g_{\mu \nu}^{s}$ and $g_{\mu \nu}^{D2}$, respectively. The choices of these particular metrics in the contractions of $B_{\mu \nu}$ and $C_{\mu \nu \rho}$ are dictated by the fact, that they will be used only in the context where they are relevant, i.e., the former in the open string context and the latter in the open D2-brane context. From the non-linear self-duality relation in six dimensions we get as in (25), provided we restrict ourselves to rank two fields in five dimensions, the following relations between the two-form and the three-form:

$$e^{-\frac{2\phi}{3}} B_{\mu \nu}^2 = \frac{1}{6} (1 + \frac{1}{6} C^2)^{-1} \left( 3 C_{\mu \nu}^2 - C^2 g_{\mu \nu}^{D2} \right), \quad (69)$$

$$1 + \frac{2}{3} B^2 = \left( 1 + \frac{1}{6} C^2 \right)^{-1}. \quad (70)$$

We will start by reducing on an electric circle defined by having a non-zero $B_{01}$ and hence $B^2 < 0$, which for a generic supergravity solution corresponds to a reduction from $(M5,M2)$ to $(D4,F1)$ under the rank two restriction (rank four is a two parameter deformation, corresponding to a more complicated bound state, see e.g. (25)). In the UV limit of these solutions, when interpreted as supergravity duals, we get light membranes in the $(M5,M2)$ case, i.e., OM-theory, and light strings in the $(D4,F1)$ case, corresponding to NCOS. This means that in this case, the open membrane metric should reduce to the open string metric, and we therefore express the reduced metric
in terms of the two-form $B_{\mu\nu}$ with $B^2 < 0$. In writing out the conformal factor in terms of the two-form, it turns out that the sign of $B^2$ (and of $C^2$ when using the three-form, since $B^2$ and $C^2$ always have opposite signs) becomes important. To be specific, the form of $z$ differs, when expressed in terms of the five-dimensional fields, for different signs of $B^2$, whereas the form of $K$ is the same for both signs. This difference is due to the second order equation that follows from (67) and (69), relating $B^2$ (or $C^2$) and $A^2$. For the electric reduction we get

\[ z = (1 + \frac{1}{2} B^2)^{-\frac{1}{2}} = (1 + \frac{1}{6} C^2)^{\frac{1}{2}}, \]  
\[ K = (1 + \frac{1}{2} B^2)(1 + \frac{1}{2} B^2)^{-\frac{1}{2}} = (1 + \frac{1}{12} C^2)(1 + \frac{1}{6} C^2)^{-\frac{1}{2}}. \]

Using (72), the open membrane metric reduces to

\[ \frac{G_{\mu\nu}^{OM}}{\ell_p^2} = \frac{g_2^{\phi_2}}{\ell_p^2} e^{-\frac{2\phi}{R^2}} (1 + \frac{1}{2} B^2)^{-\frac{1}{2}} \left( g^{\mu\nu} + B^2_{\mu\nu} \right). \]  

In analogy with (62) for the metric along the compactified direction, we can define the effective open string coupling from the open membrane metric

\[ \frac{G_{\mu\nu}^{OM}}{\ell_p^2} = \frac{(G_{OS})^{\phi_2}}{R^2}, \]

which yields

\[ G_{OS}^2 = e^{\phi} (1 + \frac{1}{2} B^2)^{\frac{1}{2}}, \]

which is the expected coupling in the open string case (analogous to the one found in [25]). We then define the open string metric as

\[ \frac{G_{\mu\nu}^{OM}}{\ell_p^2} = (G_{OS})^{-\frac{1}{2}} \frac{G_{\mu\nu}^{OS}}{\alpha'}, \]

also in analogy to the reduction formulae for the closed string quantities (62). Inserting (76) and (75) into (76) we get the expected open string metric

\[ G_{\mu\nu}^{OS} = g^{\mu\nu} + B^2_{\mu\nu}. \]

If one wants to relate open membrane and open string quantities without involving any closed quantities, one is naturally led to the above definitions. Thus by requiring deformation independence in six dimensions we can uniquely determine the open membrane metric and its connection via dimensional reduction to the open string metric. This argument can be turned around as done in [25], where the fundamental assumption
instead concerns the relations used in the dimensional reduction of the open membrane metric (74) and (76).

The expression for the open string metric was derived by performing an electric reduction, i.e., having $B^2 < 0$, but it can be analytically continued to $B^2 > 0$, which is needed in order to get NCYM in the decoupling limit. We have thus obtained the open string metric by electrically reducing the open membrane metric and then analytically continuing the obtained expression to all values of $B^2$. However, although it is generally considered a standard procedure, there seems to be an obstacle to lifting the five-dimensional open string metric to six dimensions for all values of $B^2$. The problem stems from the relation between the five-dimensional quantity $B^2$ and the six-dimensional one $K = \sqrt{1 + \frac{1}{24} A^2}$ discussed in the previous section. Using the relations between the fields in five and six dimensions, one finds the relation $B^2 = 4(K^2 - 1 \pm K^2 \sqrt{1 - K^{-2}})$ where $K > 1$. Thus $B^2 > 0$ corresponds uniquely to the plus sign and $B^2 < 0$ to the minus sign. But in the previous section (see comment below (49)) we showed that the minus sign is the only possible one, in agreement with the result of [25]. Our interpretation of this is that in the magnetic case with $B^2 > 0$ one should not use the open string metric to analyze the physics but rather the open D2-brane metric given in (17). Thus, as we will see below, it is in this case the open D2-metric that lifts naturally to six dimensions.

Having done the reduction on an electric circle, we now turn to the reduction on a magnetic circle, which for a generic solution means that (M5,M2) reduces to (D4,D2) under the restriction to rank two in five dimensions. We now have an electric three-form, i.e., $C^2 < 0$, and therefore also a magnetic two-form, i.e., $B^2 > 0$. The difference compared to the electric case above is that now $z = (1 + \frac{1}{2} B^2)^{-\frac{1}{2}} = (1 + \frac{1}{6} C^2)^{-\frac{1}{6}}$. (78)

The procedure is exactly as above, but since light membranes now reduce to light D2-branes, we expect to obtain the open D2-brane metric (18), and we should therefore express the reduced metric in terms of the three-form. Indeed, for the magnetic reduction we find the result

$$\frac{G^{OM}_{\mu\nu}}{\ell_p^2} = \frac{1}{\alpha'} (1 + \frac{1}{2} C^2)^{-\frac{1}{2}} (g^{D2}_{\mu\nu} + \frac{1}{6} C^2_{\mu\nu}) = \frac{G^{OD2}_{\mu\nu}}{\alpha'}.$$ (79)

As in the previous case we have to analytically continue the open D2-brane metric to the case $C^2 > 0$ relevant for D2-GT. One could speculate that $G^{OM}_{yy}$ in the above case is related to the open D2-brane coupling. However, as we have seen above, (73) can be obtained without knowing the exact relation between $G^{OM}_{yy}$ and the open D2-brane coupling.

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We now turn to the reduction of the eleven dimensional theta parameter. Using the reduction rules and the definitions above, we get after an electric reduction

\[ \Theta_{\text{OS}}^{\mu \nu} = \Theta_{\text{OM}}^{\mu \nu} \equiv -\alpha' g_s^{\mu \mu_1} B_{\mu_1 \nu_1} G_{\text{OS}}^{\nu_1} , \]  

and the theta parameter of OM-theory therefore reduces to the correct open string non-commutativity parameter. Instead, reducing on a magnetic circle we obtain the open D2-brane three-index theta

\[ \Theta_{\text{OM}}^{\mu \nu \rho} = \Theta_{\text{OD2}}^{\mu \nu \rho} = -\left( \alpha' \right)^3 (1 + \frac{1}{2} C^2)^{\frac{1}{2}} g_{D2}^{\mu \mu_1} C_{\mu_1 \nu_1 \rho_1} G_{\text{OD2}}^{\nu_1} G_{\text{OD2}}^{\rho_1} . \]  

Importantly the open string and open D2-brane theta parameters can be obtained in both the electric and magnetic reductions of (53), whereas an open membrane theta parameter defined in another way will reduce to two 2-index and two 3-index parameters. That we only get one 2-index and one 3-index theta, can be understood from the duality relation (60). Since all \( z \) dependence in the eleven-dimensional Hodge star cancels, it reduces in the same way in both the electric and magnetic case. Since it relates physical quantities in eleven dimensions, this should also be the case in ten dimensions, implying that we can only get one expression for the two-index theta and one for the three-index theta in ten dimensions. We have thus seen that the open membrane data correctly reduce to the open string and D2-brane data. In this way we can also relate the various decoupling limits including the NCYM and D2-GT cases. Note also that trying to obtain the open D2-brane metric from an electric reduction or an open string metric from the magnetic reduction will fail.

Finally, we note that if we reduce the open membrane data in a direction transverse to the M5-brane (\( i.e., (M5,M2) \) goes to (NS5,D2)), we get exactly the same expressions for the open 2-brane (with a self-dual three-form) metric and theta parameter, but with the identifications between the metrics and the three-forms (62).

6 Summary and discussion

In this paper we derive expressions for open brane metrics and theta parameters for open \( D_q \)-branes ending on NS5-branes or \( D(q+2) \)-branes, and for the open membrane ending on an M5-brane. For the open string these quantities can be read off from the two-point function, which is easily computed by quantizing the open string in the background of a constant metric and NS-NS field \( B_{\mu \nu} \). The open string quantities so obtained constitute the data needed in taking certain closed string decoupling limits and thus contain the information required to define new interesting theories like NCYM and NCOS.
In order to generalize these open string quantities to the open branes mentioned above, we have to rely on some indirect methods since in these cases there is no known method to compute two- or higher-point functions from basic principles. To proceed, one could imagine that the open brane quantities could be derived through S- and T-duality transformations, where the former may involve a dimensional lift, e.g., from five to six dimensions as discussed in this paper. Now, applying these transformations leads to ambiguities as to how the open scalar quantities (like the conformal factor and open brane coupling) after the transformations should be defined. This ambiguity can be eliminated only by invoking a further assumption tied to the relations between open brane quantities in different dimensions, or by generalizing some key properties exhibited by the open string data and regarding them as general principles.

In the recent work [25] the open membrane metric was determined, including the conformal factor, by invoking an explicit assumption concerning how the open string metric in five dimensions is related to the open membrane metric in six dimensions, namely eqs. (74) and (76). In fact, these relations follow also by assuming that they should only involve open string quantities.

In this paper we have instead chosen to elevate the facts that the open string metric is deformation independent and that $\Theta^{\mu\nu}$ is scale independent to general principles. This has the advantage that the argument can be applied to each case separately and may therefore have a more direct physical explanation. It also simplifies the computations considerably making the results more transparent. We have shown that the two approaches, relying on deformation independence and reduction respectively, yield the same open membrane metric.

The ultimate reason for the interest in these generalized open brane quantities is of course related to the possibility, that they contain information about the quantum theory in cases where we do not know how to extract such information by other means. In particular, the appearance of generalized theta parameters may indicate new algebraic structures, arising in operator products of fields generated by the open branes on the host branes. Related speculations have already appeared in the literature, e.g., in [28], and to find any kind of explicit realization of such structures is clearly an important problem.

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