FOURTH FUNDAMENTAL FORM AND $i$-TH CURVATURE FORMULAS
OF ROTATIONAL HYPERSURFACES IN 4-SPACE

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Abstract. We introduce fourth fundamental form $IV$, and $i$-th curvature formulas $C_i$ of rotational hypersurfaces in the four dimensional Euclidean geometry $E^4$. Defining fourth fundamental form and $i$-th curvatures for hypersurfaces, we calculate them on rotational hypersurface. In addition we study rotational hypersurface satisfying $\Delta IV x = Ax$ for some $4 \times 4$ matrix $A$.

1. Introduction

Refering Chen [10, 11, 12, 13], the researches of submanifolds of finite type whose immersion into $E^m$ (or $E^m_\nu$) by using a finite number of eigenfunctions of their Laplacian has been studied by geometers for almost half centuries.

Takahashi [38] proved that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $E^m$ or minimal in some hypersphere of $E^m$. Submanifolds of finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of 2-type spherical closed submanifolds were obtained by [7, 8, 11]. Garay gave [25] an extension of Takahashi’s theorem in $E^m$. Cheng and Yau worked hypersurfaces with constant scalar curvature; Chen and Piccinni [14] studied submanifolds with finite type Gauss map in $E^m$. Dursun [19] gave hypersurfaces with pointwise 1-type Gauss map in $E^{n+1}$.

Considering $E^3$; Takahashi [38] stated that minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay and Lucas [22] proved that the surfaces satisfying $\Delta H = AH$, $A \in \text{Mat}(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [17] characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind; Garay [24] worked a certain class of finite type surfaces of revolution; Dillen, Pas and Verstraelen [18] proved that the only surfaces satisfying $\Delta r = Ar + B$, $A \in \text{Mat}(3,3)$, $B \in \text{Mat}(3,1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [37] considered surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [36] studied helicoidal surfaces $M^2$ which are of finite type with respect to the fundamental forms $I, II$ and $III$, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in \text{Mat}(3,3)$; Kim, Kim and Kim [30] focused Cheng-Yau operator and Gauss map of surfaces of revolution.

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General rotational surfaces in 4-space were originated by Moore [34, 35]. Focusing on $\mathbb{E}^4$; Hasanis and Vlachos [29] considered hypersurfaces with harmonic mean curvature vector field; Cheng and Wan [15] considered complete hypersurfaces with CMC; Kim and Turgay [31] studied surfaces with $L_1$-pointwise 1-type Gauss map; Arslan et. al. [4] worked Vranceanu surface with pointwise 1-type Gauss map; Aksoyak and Yaylı [2] introduced flat rotational surfaces with pointwise 1-type Gauss map; Güler, Hacısalihoğlu and Kim [26] studied Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface; Güler and Turgay [28] introduced Cheng-Yau operator and Gauss map of rotational hypersurfaces.

In Minkowski 4-space $\mathbb{E}^4_1$; Ganchev and Milousheva [23] considered analogue of surfaces of [34, 35]; Arvanitoyeorgos, Kaimakamis and Magid [6] showed that if the mean curvature vector field of $M^3$ satisfies the equation $\Delta H = \alpha H$ (\(\alpha\) a constant), then $M^3$ has CMC; Arslan and Milousheva worked meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay gave some classifications of Lorentzian surfaces with finite type Gauss map; Dursun and Turgay studied space-like surfaces in with pointwise 1-type Gauss map. Aksoyak and Yaylı [3] focused general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Bektaş, Canfes and Dursun [9] classified surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$.

In literature, there is no any work about fourth fundamental form $f_{ij}$ (i.e. $IV$) and $i$-th curvature formulas $C_i$, where $i = 0, \ldots, 3$, of rotational hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$.

We consider fourth fundamental form $IV$, and $i$-th curvature formulas $C_i$ of rotational hypersurface in the four dimensional Euclidean geometry $\mathbb{E}^4$. In Section 2, we give some basic notions of the four dimensional Euclidean geometry. Defining fourth fundamental form and $i$-th curvature for hypersurfaces, we calculate $C_i$ and fourth fundamental form of rotational hypersurface in Section 3. Finally, in the last section, we study rotational hypersurface satisfying $\Delta IVx = Ax$ for some $4 \times 4$ matrix $A$ in $\mathbb{E}^4$.

2. Preliminaries

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let $\mathbb{E}^m$ denote the Euclidean $m$-space with the canonical Euclidean metric tensor given by $\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an $m$-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections of $\mathbb{E}^m$ and $M$ by $\tilde{\nabla}$ and $\nabla$, respectively. We shall use letters $X, Y, Z, W$ (resp., $\xi, \eta$) to denote vectors fields tangent (resp., normal) to $M$. The Gauss and Weingarten formulas are given, respectively, by

\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\tilde{\nabla}_X \xi &= -A_\xi(X) + D_X \xi,
\end{align*}

where $h$, $D$ and $A$ are the second fundamental form, the normal connection and the shape operator of $M$, respectively.
For each $\xi \in T^\perp_p M$, the shape operator $A_\xi$ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$ 

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y, Z, W) = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, (3)$$

$$\langle \nabla_X h(Y, Z) = \langle \nabla_Y h(X, Z) \rangle, (4)$$

where $R$, $R^D$ are the curvature tensors associated with connections $\nabla$ and $D$, respectively, and $\sqrt{\nabla} \nabla h$ is defined by

$$\langle \nabla_X h(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space. Now, let $M$ be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $S$ its shape operator (i.e. Weingarten map) and $x$ its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of consisting of principal directions of $M$ corresponding from the principal curvature $k_i$ for $i = 1, 2, \ldots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{j=1}^{n} \theta_j \wedge \omega_{ij}, \quad i = 1, 2, \ldots, n,$$ 

(5)

where $\omega_{ij}$ denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of $M$ and $\mathbb{E}^{n+1}$ by $\nabla$ and $\tilde{\nabla}$, respectively. Then, from the Codazzi equation (3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \quad (6)$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \quad (7)$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where $\sigma_j$ is the $j$-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$ 

We use following notation

$$r^j_i = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r^0_i = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function $s_k$ as the $k$-th mean curvature of $M$. We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of $M$, respectively. In particular, $M$ is said to be $j$-minimal if $s_j \equiv 0$ on $M$.

In $\mathbb{E}^{n+1}$, to find the $i$-th curvature formulas $C_i$ (Curvature formulas sometimes are represented as mean curvature $H_i$, and sometimes as Gaussian curvature $K_i$ by different writers, such as [1].
and [32]. We will call it just $i$-th curvature $\mathcal{C}_i$ in this paper.), where $i = 0, \ldots, n$, firstly, we use the characteristic polynomial of $S$:

$$P_S(\lambda) = 0 = \det(S - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \quad (8)$$

where $i = 0, \ldots, n$, $I_n$ denotes the identity matrix of order $n$. Then, we get curvature formulas $(n) \mathcal{C}_i = s_i$. That is, $(n) \mathcal{C}_0 = s_0 = 1$ (by definition), $(n) \mathcal{C}_1 = s_1, \ldots , (n) \mathcal{C}_n = s_n = K$.

$k$-th fundamental form of $M$ is defined by $I(S^{k-1}(X), Y) = \langle S^{k-1}(X), Y \rangle$. So, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{C}_i I(S^{n-i}(X), Y) = 0. \quad (9)$$

In particular, one can get classical result $III\mathcal{C}_0 - 2\mathcal{C}_1 II + \mathcal{C}_2 I = 0$ of surface theory for $n = 2$. See [32] for details.

For a Euclidean submanifold $x: M \rightarrow \mathbb{E}^m$, the immersion $(M, x)$ is called finite type, if $x$ can be expressed as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $(M, x)$, i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where $x_0$ is a constant map, $x_1, \ldots , x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, \ldots , k$.

If $\lambda_i$ are different, $M$ is called $k$-type. See [11] for details.

2.2. Rotational hypersurfaces. We will obtain a rotational hypersurface (rot-hypface for short) in Euclidean 4-space. Before we proceed, we would like to note that the definition of rot-hypfaces in Riemannian space forms were defined in [21]. A rot-hypface $M \subset \mathbb{E}^{n+1}$ generated by a curve $C$ around an axis $C$ that does not meet $C$ is obtained by taking the orbit of $C$ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\tau$ pointwise fixed (See [21] Remark 2.3).

Throughout the paper, we shall identify a vector $(a, b, c, d)$ with its transpose. Consider the case $n = 3$, and let $C$ be the curve parametrized by

$$\gamma(u) = (f(u), 0, 0, \varphi(u)). \quad (10)$$

If $\tau$ is the $x_4$-axis, then an orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\tau$ pointwise fixed has the form

$$Z(v, w) = \begin{pmatrix}
\cos v \cos w & -\sin v & -\cos v \sin w & 0 \\
\sin v \cos w & \cos v & -\sin v \sin w & 0 \\
\sin w & 0 & \cos w & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad v, w \in \mathbb{R}.$$

Therefore, the parametrization of the rot-hypface generated by a curve $C$ around an axis $\tau$ is given by

$$x(u, v, w) = Z(v, w) \gamma(u). \quad (11)$$

Let $x = x(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. Triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = (x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2, \\
-x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3y_4z_1 - y_1x_4z_3 + x_4y_3z_1, \\
x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1, \\
-x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1).$$
For a hypface $x$ in 4-space, we have

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \quad III = \begin{pmatrix} X & Y & O \\ Y & Z & S \\ O & S & U \end{pmatrix}, \quad (12)$$

and

$$\det I = (EG - F^2)C - EB^2 + 2FAB - GA^2, \quad \det II = (LN - M^2)V - LT^2 + 2MPT - NP^2, \quad \det III = (XZ - Y^2)U - ZO^2 + 2OSY - XS^2,$$

where $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$, $G = \langle x_u, x_w \rangle$, $A = \langle x_u, x_w \rangle$, $B = \langle x_v, x_w \rangle$, $C = \langle x_u, x_w \rangle$, $L = \langle x_vu, G \rangle$, $M = \langle x_uw, G \rangle$, $N = \langle x_vw, G \rangle$, $P = \langle x_uw, G \rangle$, $T = \langle x_uw, G \rangle$, $V = \langle x_vw, G \rangle$, $X = \langle G_u, G_u \rangle$, $Y = \langle G_u, G_w \rangle$, $Z = \langle G_v, G_v \rangle$, $O = \langle G_u, G_w \rangle$, $S = \langle G_v, G_w \rangle$, $U = \langle G_w, G_w \rangle$.

Here,

$$G = \frac{x_u \times x_w \times x_u}{\|x_u \times x_v \times x_w\|} \quad (13)$$

is unit normal (i.e. the Gauss map) of hypface $x$. On the other hand, $I^{-1}II$ gives shape operator matrix $S$ of hypface $x$ in 4-space. See [27, 28, 29] for details.

3. $i$-th Curvatures and the Fourth Fundamental Form

To compute the $i$-th mean curvature formula $\mathcal{C}_i$, where $i = 0, \ldots, 3$, we use characteristic polynomial $P_S(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_S(\lambda) = \det(S - \lambda I_3) = 0.$$  

Then, get $\mathcal{C}_0 = 1$ (by definition), $\langle \mathcal{C}_1 \rangle H = -\frac{d}{a}$, $\langle \mathcal{C}_2 \rangle = \frac{c}{a}$, $\langle \mathcal{C}_3 \rangle K = -\frac{b}{a}$.

Therefore, we reveal $i$-th curvature formulas depends on the coefficients of the $I$ and $II$ fundamental forms in 4-space (It also can write depends on the coefficients of the $II$ and $III$, or $III$ and $IV$):

**Theorem 1.** Any hypface $x$ in $\mathbb{E}^4$ has following curvature formulas, $\mathcal{C}_0 = 1$ (by definition),

$$\mathcal{C}_1 = \frac{(EN + GL - 2FM)C + (EG - F^2)V - LB^2 - NA^2 - 2(APG + BTE - ABM - ATF + BPF)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]} \quad (14)$$

$$\mathcal{C}_2 = \frac{(EN + GL - 2FM)V + (LN - M^2)C - ET^2 - GP^2 - 2(ANP - BMP - AMT + BLT - FPT)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]} \quad (15)$$

$$\mathcal{C}_3 = \frac{(LN - M^2)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \quad (16)$$

Proof. Solving $\det(S - \lambda I_3) = 0$ with some algebraic computations, we obtain coefficients $a, b, c, d$ of polynomial $P_S(\lambda)$.

**Corollary 1.** For any hypface $x$ in $\mathbb{E}^4$, the fourth fundamental form is related by

$$IV\mathcal{C}_0 - 3\mathcal{C}_1 III + 3\mathcal{C}_2 II - \mathcal{C}_3 I = 0. \quad (17)$$

Proof. Taking $n = 3$ in [9], it is clear.
Definition 1. For any hypface $x$ with its shape operator $S$ in 4-space, following relations hold

(a) the second fundamental form $(h_{ij}) = II$ is given by $II = I \cdot S$,
(b) the third fundamental form $(e_{ij}) = III$ is given by $III = II \cdot S$,
(c) the fourth fundamental form $(f_{ij}) = IV$ is given by $IV = III \cdot S$.

Corollary 2. For any hypface $x$ in $\mathbb{E}^4$, shape operator matrix has following relation

$$I \left( S^3 - 3c_1 S^2 + 3c_2 S - c_3 \right) = 0.$$  

Proof. Considering previous definition and previous corollary, we write $IV = III \cdot S = II \cdot S^2 = I \cdot S^3$. Then, it is clear.

Corollary 3. In $\mathbb{E}^4$, the fundamental forms of any hypface $x$ is related by

$$c_3 = \frac{\det III}{\det II} = \frac{\det IV}{\det III}.$$  

Proof. Since $c_3 = K = \frac{\det III}{\det I}$, and considering previous definition, it can be seen, easily.

Corollary 4. For any hypface $x$ in $\mathbb{E}^4$, the fourth fundamental form is given by

$$IV = III \cdot I^{-1} \cdot II.$$  

Proof. From $S = I^{-1} \cdot II = III^{-1} \cdot IV$, we get the result. So, coefficients of $IV$ are as follows

$$f_{11} = \frac{1}{\det I} \left\{ F^2 OP + B^2 LX + A^2 MY + BMOE - GOPE - CMYE + BPYE - AFMO + AGLO \right\},$$

$$f_{12} = \frac{1}{\det I} \left\{ B^2 MX + A^2 NY + F^2 OT + BNOE - CNYE - GOTE + BTYE - AFNO + AGMO \right\},$$

$$f_{13} = \frac{1}{\det I} \left\{ B^2 PX + F^2 OV + A^2 TY + BOTE - GOVE - CTYE + BVYE + AGOP - BFOP \right\},$$

$$f_{21} = \frac{1}{\det I} \left\{ B^2 LX + A^2 MZ + F^2 PS + BMSE - CMZE - GPSE + BPZE - AFMS + AGLS \right\},$$

$$f_{22} = \frac{1}{\det I} \left\{ B^2 MY + A^2 NZ + F^2 ST + BNSE - CNZE - GSTE + BTZE - AFNS + AGMS \right\},$$

$$f_{23} = \frac{1}{\det I} \left\{ B^2 MO + A^2 NS + F^2 TU - CNSE + BNUE + BSTE - GTUE - ABNO - ABMS \right\},$$

$$f_{31} = \frac{1}{\det I} \left\{ B^2 LO + A^2 MS + F^2 PU - CMSE + BMUE + BPSE - GPOE - ABMO - ABLS \right\},$$

$$f_{32} = \frac{1}{\det I} \left\{ B^2 PY + A^2 TZ + F^2 SV + BSTE - GSVE - CTZE + BVZE + AGPS - BFPS \right\},$$

$$f_{33} = \frac{1}{\det I} \left\{ B^2 OP + A^2 ST + F^2 UV - CSTE + BSVE + BTUE - GUVE - ABOT - ABPS \right\}.$$  

Here, $f_{ij} = f_{ji}$, where $i \neq j$, and $\det I = (EG - F^2)C - EB^2 + 2FAB - GA^2$. 

Corollary 5. For any hypface \( x \) in \( \mathbb{E}^4 \), determinant \( f = \det(f_{ij}) \) of the fourth fundamental form is given by
\[
\det IV = \frac{(XS^2 - 2YSO + ZO^2 - U(XZ - Y^2)) (LT^2 - 2MTP + NP^2 - V(LN - M^2))}{(EG - F^2)C - EB^2 + 2FAB - GA^2}.
\]
Proof. After some computations by using the right side of \( IV = III \cdot I^{-1} \cdot II \), it is clear.

3.1. Mean Curvatures and Fundamental Forms of Rotational Hypersurface. We consider the \( i \)-th curvatures of the rotational hypersurface (18), that is
\[
x(u, v, w) = (f(u) \cos v \cos w, f(u) \sin v \cos w, f(u) \sin w, \varphi(u)),
\]
where \( u \in \mathbb{R} - \{0\} \) and \( 0 \leq v, w \leq 2\pi \). Then, we obtain \( i \)-th curvatures of (11).

Using the first differentials of rot-hypface (18), we get the first quantities
\[
I = \text{diag} \left( W, f^2 \cos^2 w, f^2 \right),
\]
where \( W = f'^2 + \varphi'^2, f = f(u), f' = \frac{df}{du}, \varphi = \varphi(u), \varphi' = \frac{d\varphi}{du} \). The Gauss map of the rot-hypface is
\[
G = \frac{1}{\sqrt{W}} \left( \varphi' \cos v \cos w, \varphi' \sin v \cos w, \varphi' \sin w, -f' \right).
\]

With the second differentials and \( G \) of hypface (18), we have the second quantities
\[
II = \text{diag} \left( -\frac{f' \varphi'' - f'' \varphi'}{W^{1/2}}, -\frac{f' \varphi'^2}{W^{1/2}}, -\frac{f' \varphi'}{W^{1/2}} \right),
\]
Using the first differentials of (20), we find the third fundamental form matrix
\[
III = \text{diag} \left( \frac{(f' \varphi'' - f'' \varphi')^2}{W^2}, \frac{\varphi'^2 \cos^2 w}{W}, \frac{\varphi'^2}{W} \right).
\]
We calculate \( I^{-1} \cdot II \), then obtain shape operator matrix
\[
S = \text{diag} \left( -\frac{f' \varphi'' - f'' \varphi'}{W^{3/2}}, -\frac{\varphi'}{fW^{1/2}}, -\frac{\varphi'}{fW^{1/2}} \right).
\]

Finally, with all findings, we calculate \( i \)-th curvature \( \mathcal{C}_i \) of rot-hypface (18), and give in the following theorem

**Theorem 2.** Rot-hypface (18) has following curvatures
\[
\begin{align*}
\mathcal{C}_0 &= 1 \text{ (by definition)}, \\
\mathcal{C}_1 &= -\frac{2\varphi' W + f (f' \varphi'' - f'' \varphi')}{3fW^{3/2}}, \\
\mathcal{C}_2 &= \frac{\varphi' \left( 2f^2 (f' \varphi'' - f'' \varphi') - W^{3/2} \right)}{3f^3 W^2}, \\
\mathcal{C}_3 &= -\frac{\varphi^2 (f' \varphi'' - f'' \varphi')}{f^2 W^{5/2}}.
\end{align*}
\]
Corollary 6. Rot-hypface \([18]\) is 1-minimal iff

\[
\varphi = \mp i \frac{\text{EllipticF} \left[ i \sinh^{-1} \left( i (c_1)^{1/4} f \right), -1 \right]}{(c_1)^{1/4}} + c_2,
\]

where \(i = \sqrt{-1}, \) EllipticF[\(\phi, m\)] = \(\int_0^\phi (1 - m \sin^2 \theta)^{-1/2} d\theta\) is elliptic integral, \(\phi \in [-\pi/2, \pi/2],\)

\(0 \neq c_1, c_2\) are constants.

Proof. Solving differential eq. \(2\varphi W + f (f' \varphi'' - f'' \varphi') = 0,\) we find solutions.

Corollary 7. Rot-hypface \([18]\) is 2-minimal iff

\[
\varphi = c_1, \; \varphi = \mp i \left( \frac{c_1 \chi}{2\rho} + \log \left( \frac{-c_1 + 2\rho f + \rho^{1/2} \chi}{2\rho^{3/2}} \right) \right) + c_2,
\]

where \(\chi = \sqrt{1 - 4c_1 f + 4 \left( (c_1)^2 - 1 \right) f^2}, \rho = (c_1)^2 - 1, \) \(0 \neq c_1, c_2\) are constants.

Proof. Solving differential eq. \(2f^2 \varphi' (f' \varphi'' - f'' \varphi') - \varphi' W^{3/2} = 0,\) we have solutions.

Corollary 8. Rot-hypface \([18]\) is 3-minimal iff

\[
\varphi = c_1, \; \varphi = c_1 f + c_2.
\]

Proof. Solving differential eq. \(\varphi^2 (f' \varphi'' - f'' \varphi') = 0,\) we get the solutions.

Next, one can see some examples in \(\mathbb{E}^4.\)

**Example 1.** Catenoidal hypersurface Taking \(f (u) = a \cosh u\) and \(\varphi (u) = au,\) where \(-\infty < u < \infty,\)

\(0 < v, w \leq 2\pi,\) we get

\[
x(u, v, w) = (a \cosh u \cos v \cos w, a \cosh u \sin v \cos w, a \cosh u \sin w, au).
\]

\(x\) verifies \(\mathcal{C}_1 = \frac{1}{3a \cosh^2 u}, \mathcal{C}_2 = \frac{1}{3a \cosh^3 u}, \mathcal{C}_3 = \frac{1}{3a \cosh^4 u}.\)

**Example 2.** Hypersphere. Considering \(f (u) = r \cos u\) and \(\varphi (u) = r \sin u,\) where \(r > 0,\)

\(0 < u < \pi, \; 0 < v, w \leq 2\pi,\) we have

\[
x(u, v, w) = (r \cos u \cos v \cos w, r \cos u \sin v \cos w, r \cos u \sin w, r \sin u).
\]

\(x\) supplies \(\mathcal{C}_1 = -\frac{1}{r}, \mathcal{C}_2 = \frac{1}{r}, \mathcal{C}_3 = -\frac{1}{r}.\)

**Example 3.** Right Spherical Hypercylinder. Taking \(f (u) = r > 0\) and \(\varphi (u) = u,\) where \(0 < u < \pi, \; 0 < v, w \leq 2\pi,\) we obtain

\[
x(u, v, w) = (r \cos v \cos w, r \sin v \cos w, r \sin w, u).
\]

\(x\) has \(\mathcal{C}_1 = -\frac{2}{r}, \mathcal{C}_2 = \frac{1}{r}, \mathcal{C}_3 = 0.\) So, it is 3-minimal.

Let us see some results of the fourth fundamental form of \([18]\), and

**Corollary 9.** The fourth fundamental form matrix \((f_{ij})\) of rot-hypface \([18]\) is as follows

\[
IV = \text{diag} \left( -\frac{(f' \varphi'' - f'' \varphi')^3}{W^{7/2}}, -\frac{\varphi^3 \cos^2 w}{fW^{3/2}}, -\frac{\varphi^3}{fW^{3/2}} \right).
\]

Proof. Using Corollary 4 with rot-hypface \([18]\), we find the fourth fundamental form matrix.
Corollary 10. When the curve (10) of (18) is parametrized by the arc length, i.e. \( W = 1 \), then (18) has following relations

\[
\begin{align*}
\mathcal{C}_1 (9f^4 \mathcal{C}_2 + 6f^3 f'') & = 2 - ff'' - 2f'^2, \\
3f^3 \mathcal{C}_1 \mathcal{C}_3 & = -2f'' + 2f'^2 f'' + f f''', \\
-3f f'' \mathcal{C}_2 - \mathcal{C}_3 & = 2f'^2 
\end{align*}
\]

Proof. The curvatures (24), (25) and (26) of the rot-hypface (18) reduces to

\[
\begin{align*}
\mathcal{C}_0 & = 1, \quad \mathcal{C}_1 = -\frac{2 + 2f'^2 + ff''}{3f\varphi'}, \\
\mathcal{C}_2 & = -\frac{-2f'^2 f'' - \varphi'}{3f^3}, \quad \mathcal{C}_3 = \frac{\varphi' f''}{f^2}. 
\end{align*}
\]

Corollary 11. When \( f = u \neq 0 \) in the previous corollary, then (18) has following results, respectively,

1-minimal or 2-minimal, if (31) holds, \( (35) \)
1-minimal or 3-minimal, if (32) holds, \( (36) \)
3-minimal, if (33) holds. \( (37) \)

Proof. Taking \( f = u \), it is clear.

4. Fourth Fundamental Form of a Hypersurface

The fourth Laplace-Beltrami operator of a smooth function \( \phi = \phi(x^1, x^2, x^3) \mid \text{D} \subset \mathbb{R}^3 \) of class \( C^3 \) with respect to the fourth fundamental form of hyp-face \( x \) is the operator \( \Delta^IV \), defined by

\[
\Delta^IV \phi = \frac{1}{\sqrt{f}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} \left( \sqrt{f} f^{ij} \frac{\partial \phi}{\partial x^j} \right),
\]

where \( (f^{ij}) = (f_{ij})^{-1} \) and

\[
\begin{align*}
f & = \det (f_{ij}) \\
& = f_{11}f_{22}f_{33} - f_{11}f_{23}f_{32} - f_{12}f_{21}f_{33} + f_{12}f_{31}f_{23} + f_{21}f_{13}f_{32} - f_{13}f_{22}f_{31}.
\end{align*}
\]

4.1. Rotational Hypersurfaces Satisfying \( \Delta^IV x = Ax \). We now consider rot-hypface (18), (30) with (38), then have following theorem:

Theorem 3. The fourth Laplace-Beltrami operator of rot-hypface (18) is related by \( \Delta^IV x = Ax \), where \( A = \text{diag}(\Omega, \Omega, \Omega, \Phi) \),

\[
\begin{align*}
\frac{f^3 W^{13/4}}{\varphi^3 \psi^{3/2}} \left\{ f' \frac{\partial}{\partial u} \left( \frac{\varphi^3 W^{1/4}}{f \psi^{3/2}} \right) + \frac{f'' \varphi^3 W^{1/4}}{f \psi^{3/2}} - 2f^2 \varphi^3 W^{1/4} \right\} & = \Omega f, \\
\frac{f^3 W^{13/4}}{\varphi^3 \psi^{3/2}} \left( \varphi' \frac{\partial}{\partial u} \left( \frac{\varphi^3 W^{1/4}}{f \psi^{3/2}} \right) + \varphi'' \right) & = \Phi \varphi,
\end{align*}
\]

and \( W = f^2 + \varphi^2, \psi = f' \varphi'' - f'' \varphi' \).
Proof. By using (18), (30) with (38), we compute

\[
\Delta^IV \mathbf{x} = \frac{1}{\sqrt{|\mathbf{f}|}} \left\{ \frac{\partial}{\partial u} \left( \frac{f_{22} f_{33}}{\sqrt{|\mathbf{f}|}} \mathbf{x}_u \right) - \frac{\partial}{\partial v} \left( -\frac{f_{11} f_{33}}{\sqrt{|\mathbf{f}|}} \mathbf{x}_v \right) + \frac{\partial}{\partial w} \left( f_{11} f_{22} \sqrt{|\mathbf{f}|} \mathbf{x}_w \right) \right\},
\]

where \( f = \det IV \). If we assume that rot-hypface \( \mathbf{x} \) is constructed with component functions which are eigenfunctions of its Laplacian, we will have that \( \Delta^IV (f \cos v \cos w) = \Omega_1 f \cos v \cos w \), \( \Delta^IV (f \sin v \cos w) = \Omega_2 f \sin v \cos w \), \( \Delta^IV (f \sin w) = \Omega_3 f \sin w \), \( \Delta^IV (\varphi) = \Phi \varphi \). Hence, \( f(u) \cos v \cos w \), \( f(u) \sin v \cos w \) and \( f(u) \sin w \) are eigenfunctions of \( \Delta^IV \) for \( \Omega_1, \Omega_2, \Omega_3 \), respectively, iff \( f(u) \) supplies (39). So, \( \Omega_1 = \Omega_2 = \Omega_3 \) (= \( \Omega \) for short). Additionally, \( \varphi(u) \) is an eigenfunction with eigenvalue \( \Phi \) of \( \Delta^IV \) iff (40) holds.

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