DECAY ESTIMATES FOR THE WAVE AND DIRAC EQUATIONS WITH A MAGNETIC POTENTIAL

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Abstract. We study the electromagnetic wave equation and the perturbed massless Dirac equation on $\mathbb{R} \times \mathbb{R}^3$:

$$u_{tt} - (\nabla + iA(x))^2 u + B(x)u = 0, \quad iu_t - Du + V(x)u = 0$$

where the potentials $A(x), B(x), V(x)$ are assumed to be small but may be rough. For both equations, we prove the expected time decay rate of the solution

$$|u(t, x)| \leq \frac{1}{t} \|f\|_X$$

where the norm $\|f\|_X$ can be expressed as the weighted $L^2$ norm of a few derivatives of the data $f$.

1. Introduction

Dispersive properties of evolution equations play a crucial role in the study of nonlinear problems, and for this reason they have attracted a great deal of attention in recent years. In particular, for the Schrödinger and the wave equation a well established theory exists, see [14] and [22]. On the other hand, in the variable coefficient case the theory is very far from complete. The simplest situation is a perturbation with a term of order zero; this is already very interesting from the physical point of view (electrostatic potential). Several results are available for the equations

$$i\partial_t u - \Delta u + V(x)u = 0, \quad \Box u + V(x)u = 0.$$ We cite among the others [8], [15], [16], [19], [32] and the recent survey [33] for Schrödinger; and [5], [6], [10], [12], [13] for the wave equation. We must also mention the wave operator approach of Yajima (see [2], [39], [40], [41]) which permits to deal with the above equations in a unified way, although under nonoptimal assumptions on the potential in dimensions 1 and 3.

The next step in generality is a first order perturbation; from the physical point of view this corresponds to a magnetic potential. In this case only a handful of results are available: Strichartz estimates for the 3D wave equation [11], provided the coefficients are small and in the Schwartz class; and smoothing estimates for the 3D Schrödinger and wave operators [37]. The most general case of variable coefficients has been studied in [17], [31] and [35], where local Strichartz estimates have been proved, in various degrees of complexity; see also [7].

In the present paper, our main focus will be on the three dimensional wave equation with an electromagnetic potential

$$u_{tt} - (\nabla + iA(x))^2 u + B(x)u = 0, \quad u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C},$$

and the closely related massless Dirac system with a potential:

$$iu_t - Du + V(x)u = 0, \quad u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4.$$
Here \( A : \mathbb{R}^3 \to \mathbb{R}^3, \ B : \mathbb{R}^3 \to \mathbb{R}, \ V(x) = V^*(x) \) is a \( 4 \times 4 \) complex matrix on \( \mathbb{R}^3 \), and the symbol \( \mathcal{D} \) denotes the constant coefficient, elliptic, \( L^2 \) selfadjoint operator

\[
\mathcal{D} = \frac{1}{i} \sum_{j=1}^{3} \alpha_j \partial_k,
\]

where the Dirac matrices \( \alpha_1, \alpha_2, \alpha_3 \) have the following structure:

\[
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

We neglect the physical constants (i.e., we set \( c = \hbar = 1 \)), and we consider the zero mass case exclusively; the case of a positive mass, whose second order counterpart is the Klein-Gordon equation, has an additional term \( \alpha_4 u \) with

\[
\alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

The relation between massless Dirac and wave equation is readily explained: indeed, the Dirac matrices satisfy the commutation rules

\[
\alpha_i \alpha_k + \alpha_k \alpha_i = 2 \delta_{ik} I_4
\]

which imply immediately

\[
\mathcal{D}^2 = -\Delta I_4,
\]

where \( I_4 \) is the \( 4 \times 4 \) identity matrix. Thus we have the fundamental relation

\[
(i\partial_t - \mathcal{D})(i\partial_t + \mathcal{D}) = (\Delta - \partial_t^2) I_4,
\]

which can be interpreted as follows: squaring the Dirac system produces a diagonal system of wave equations (or, conversely: taking the square root of a wave equation produces a Dirac system. According to the folklore, this was the route that lead Dirac to his equation). When a potential is present in the Dirac system, the above reduction produces an electromagnetic wave equation in a natural way. A discussion of this can be found e.g. in [23] (Volume 4, Chapter 4); see also section 6 below.

Our goal here is to establish the decay rate of the spatial \( L^\infty \) norm of the solution, with minimal assumptions on the potentials. The expected decay rate is \( t^{-1} \), both for the wave equation and the Dirac system. Indeed, known results for hyperbolic systems (for constant coefficients see e.g. [24], [25], and for \( C^\infty_0 \) perturbations thereof see [20]) suggest a \( t^{-\frac{n}{2}} \) decay rate in \( n \) space dimensions.

Before stating our first result we introduce some basic notations. Under the assumptions of Theorem 1.1 below, the perturbed laplacian

\[
\mathcal{H} := -(\nabla + iA(x))^2 + B(x),
\]

where \( A(x) = (A_1(x), A_2(x), A_3(x)) : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( B(x) : \mathbb{R}^3 \to \mathbb{R} \), is a selfadjoint unbounded operator on \( \mathbb{R}^3 \); the explicit standard construction is recalled in Section 2. Spectral calculus allows us to define the operators \( \psi(\mathcal{H}) \) for any well behaved function \( \psi(s) \).

In particular, consider a (non-homogeneous) Paley-Littlewood partition of unity on \( \mathbb{R}^3 \), defined as follows: fix a radial nonnegative function \( \psi(r) \in C^\infty_0 \) with \( \psi(r) = 1 \) for \( r < 1 \), \( \psi(r) = 0 \) for \( r > 2 \), define \( \phi_j(r) = \psi(2^{-j+2}r) - \psi(2^{-j+1}r) \) for all \( j \geq 1 \), and \( \phi_0 = \psi \). Then \( 1 = \sum_{j \geq 0} \phi_j \) is the required partition of unity on \( \mathbb{R}^3 \). The
operators \( \phi_j(\sqrt{H}) \) will be used in the following to define suitable norms associated to the operator \( H \). We shall also use the notations

\[
\langle x \rangle = (1 + |x|^2)^{1/2}, \quad (D)^s f = (1 - \Delta)^{s/2} f \equiv \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f})
\]

Our first result concerns the Cauchy problem for the wave equation perturbed with a small rough electromagnetic potential

\[
\begin{align}
(1.6) & \quad u_{tt}(t,x) - (\nabla + iA(x))^2 u + B(x)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\
(1.7) & \quad u(0,x) = 0, \quad u_t(0,x) = g(x).
\end{align}
\]

We can prove:

**Theorem 1.1.** Assume the potentials \( A(x) \in \mathbb{R}^3, B(x) \in \mathbb{R} \) satisfy

\[
|A_j| \leq \frac{C_0}{|x|(|\log |x|| + 1)^\beta}, \quad \sum_{j=1}^3 |\partial_j A_j| + |B| \leq \frac{C_0}{|x|^2(|\log |x|| + 1)^\beta},
\]

for some constant \( C_0 > 0 \) sufficiently small and some \( \beta > 1 \). Then any solution of the Cauchy problem (1.6), (1.7) satisfies the decay estimate

\[
|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{2j} |\langle x \rangle w^{1/2}_\beta \varphi_j(\sqrt{H})g|_{L^2},
\]

where \( w_\beta(x) := |x|(|\log |x|| + 1)^\beta \). If in addition we assume that, for some \( \epsilon > 0 \),

\[
\langle D \rangle^{1+\epsilon} A_j \in L^\infty, \quad \langle D \rangle^\epsilon B \in L^\infty
\]

then \( u \) satisfies for any \( \delta > 0 \) the estimate

\[
|u(t,x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} g\|_{H^{2+\delta}}.
\]

**Remark 1.1.** The norm appearing in (1.9) can be regarded as a distorted analogue of a standard Besov norm, generated by the operator \( H \). Similar norms already appeared in [11] for magnetic potentials with coefficients in the Schwartz class; in that case, however, it was possible to prove the equivalence with standard Besov norms (see also [12], [13] for the analogous norms generated by \( -\Delta + V(x) \), which are also equivalent to the nondistorted norms). Under the slightly stronger assumptions (1.10) on the coefficients, it is possible to prove an estimate like (1.11) expressed in terms of standard weighted Sobolev norms.

Moreover, we remark that in our estimates we lose 2 derivatives; it is natural to conjecture that this is not optimal, and it should be possible to lose only one derivative as in the case of the free wave equation.

**Remark 1.2.** As an essential step in the proof of Theorem 1.1, we need to establish the *limiting absorption principle* (LAP) for the operator \( H \). This is obtained in Section 3 through several steps: starting from the “weak” LAP of [4] for the free resolvent, we first prove a strong version of the LAP for the free operator in the weighted spaces

\[
L^2(w_\beta(x)dx), \quad w_\beta(x) := |x|(|\log |x|| + 1)^\beta
\]

and then we get the LAP for the perturbed operator. For the precise statements see Proposition 3.4. See also [37] for related results.

**Remark 1.3.** When the initial data are of the form

\[
u(0,x) = f, \quad u_t(0,x) = 0,
\]

Theorem 1.1 implies, by standard arguments, the estimate

\[
|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{2j} |\langle x \rangle w^{1/2}_\beta \varphi_j(\sqrt{H})f|_{L^2}
\]
with an additional loss of one derivatives as expected. If in addition we assume that for some $\epsilon > 0$

$$\langle D \rangle^{2+\epsilon} A_j \in L^\infty, \quad \langle D \rangle^{1+\epsilon} B \in L^\infty$$

then also the simpler estimate

$$|u(t,x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} f\|_{H^{3+\epsilon}}.$$  

holds for all $\delta > 0$.

Our second result concerns the perturbed Dirac system

$$iu_t - D u + V(x) u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \quad u(0,x) = f(x).$$

By exploiting the above mentioned relation between the magnetic wave equation and the Dirac system, we can prove the following Theorem as a direct consequence of Theorem 1.1:

**Theorem 1.2.** Assume the $4 \times 4$ complex valued matrix $V(x) = V^*(x)$ satisfies

$$|V(x)| \leq \frac{C_0}{|x|} \langle x \rangle^2(|\log |x|| + 1)^\beta, \quad |DV(x)| \leq \frac{C_0}{|x|^2} (|\log |x|| + 1)^\beta,$$

for some $C_0 > 0$ small enough and some $\beta > 1$. Then the solution of the Cauchy problem (1.15), (1.16) satisfies the decay estimate

$$|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{3j} \|\langle x \rangle^{1/2} \varphi_j(D + V)f\|_{L^2},$$

where $w_\beta(x) = |x|(|\log |x|| + 1)^\beta$. If in addition we assume that, for some $\epsilon > 0$,

$$\langle D \rangle^{2+\epsilon} V \in L^\infty,$$

then $u$ satisfies for any $\delta > 0$ the estimate

$$|u(t,x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} f\|_{H^{3+\epsilon}}.$$}

Since Theorem 1.2 is proved essentially by “squaring” the perturbed Dirac operator, a condition on the derivative $DV$ is essential in order to apply Theorem 1.1 to the resulting wave equation. On the other hand, we can study the Cauchy problem (1.15), (1.16) by a direct application of the spectral calculus for the self-adjoint operator $D + V(x)$; this alternative approach allows us to consider much rougher potentials $V(x)$ (see (1.21)). The price to pay is an additional loss of one derivative, so that the total loss is 4 derivatives in our last result:

**Theorem 1.3.** Assume the $4 \times 4$ complex valued matrix $V(x) = V^*(x)$ satisfies

$$|V(x)| \leq \frac{C_0}{|x|^{1/2}} \langle x \rangle^{3/2} (|\log |x|| + 1)^{\beta/2},$$

for some $C_0 > 0$ small enough and some $\beta > 1$. Then the solution of the Cauchy problem (1.15), (1.16) satisfies for any $\epsilon > 0$ the decay estimate

$$|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{3j} \|\langle x \rangle^{3/2+\epsilon} \varphi_j(D + V)f\|_{L^2}.$$}

**Remark 1.4.** As a byproduct of our method of proof, we obtain the limiting absorption principle for the perturbed Dirac operator under assumption (1.21) (see Section 3.2). The LAP had been proved earlier for the free Dirac equation by Yamada [42], and for the Dirac equation with potential (and with mass) in [28] under quite stronger assumptions.
2. The self-adjointness of the perturbed operators

In this section we check the self-adjointness of the perturbed operators $\Delta_W$ and $D_V$ under quite general assumptions on the potentials $A, B, V$, which in particular are implied by the assumptions of Theorems 1.1, 1.2 and 1.3. Most of the material here is standard; however we decided to include a sketch of the proof for the sake of completeness. Moreover, the use of Lorentz spaces techniques (see the Appendix for a short review) makes the proofs quite straightforward.

It will be useful sometimes to express the magnetic laplacian both in the covariant form

$$H = -(\nabla + iA(x))^2 + B(x)$$

and in the expanded form

$$H = -\Delta + W(x, D), \quad W(x, D) = \sum_{j=1}^{3} a_j(x) \partial_j + b(x)$$

where

$$a_j(x) = -2iA_j(x), \quad b(x) = -i \sum_{j=1}^{3} \partial_j A_j(x) + |A(x)|^2 + B(x), \quad A_j, B \in \mathbb{R}.$$  

Then we have the following:

**Proposition 2.1.** Consider the operator on $C_0^\infty(\mathbb{R}^n)$

$$H = -(\nabla + iA(x))^2 + B(x),$$

where $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $B(x) : \mathbb{R}^n \to \mathbb{R}$ are measurable functions. Assume that the Lorentz (weak Lebesgue) norms of the coefficients

$$\|A\|_{L^{n,\infty}} \leq C_0, \quad \|B\|_{L^{n/2,\infty}} \leq C_0$$

are bounded by some constant $C_0 > 0$ small enough. Then $H$ has a (unique) self-adjoint extension to $H^2(\mathbb{R}^n)$.

**Proof.** Our proof is based on the standard results on quadratic forms, see e.g. the standard reference [29]. First of all we notice that by (2.5) we have immediately

$$|A(x)|^2 \in L^{n/2,\infty},$$

with a small norm. Now, the quadratic form $q(\phi, \psi)$ given by

$$q(\varphi, \psi) = ((\nabla + iA)\varphi, (\nabla + iA)\psi)_{L^2} + (B\varphi, \psi)_{L^2}$$

is well defined on the form domain $H^1$ under assumptions (2.5). Indeed, consider the identity

$$q(\psi, \psi) = \|\nabla \psi\|_{L^2}^2 + (|A|^2 + B)\|\psi\|_{L^{n/2,\infty}}^2 + 2\Im(A\nabla \psi, \psi)_{L^2};$$

using the embedding $H^1 \subset L^{2n/(n-2),2}$, the Hölder inequality in Lorentz spaces (see the Appendix at the end of the paper for a quick synopsis of the relevant results), and recalling assumption (2.5), we have easily

$$|q(\psi, \psi)| \leq \|\nabla \psi\|_{L^2}^2 + C\|A\|_{L^{n,\infty}}^2 \|\psi\|_{L^{n/2,\infty}}^2 + C\|A\|_{L^{n,\infty}} \|\nabla \psi \cdot \overline{\psi}\|_{L^{n/(n-2),1}} 
\leq \|\nabla \psi\|_{L^2}^2 + CC_0\|\psi\|_{L^{n/2,\infty}}^2 + CC_0\|\nabla \psi\|_{L^{n/2,\infty}} \|\psi\|_{L^{n/(n-2),1}}^2 \leq C\|\nabla \psi\|_{L^2}^2. $$

It is clear that the form is symmetric, since $A$ and $B$ are real valued. Now, recalling Theorem VIII.15 in [29], in order to prove that $q$ is the form associated to a (uniquely defined) self-adjoint operator, it will be sufficient to show that it is closed, i.e., its domain $H^1(\mathbb{R}^n)$ is complete under the norm

$$\|\psi\|^2 = q(\psi, \psi) + C\|\psi\|_{L^2}^2.$$
for some $C > 0$, and that it is *semibounded*, i.e.,
\begin{equation}
q(\psi, \psi) \geq -C\|\psi\|_{L^2}^2
\end{equation}
for some $C > 0$. Both properties follow from the identity (2.6); indeed, by estimating as above we obtain easily
\[ q(\psi, \psi) \geq \|\nabla \psi\|_{L^2}^2 - CC_0\|\nabla \psi\|_{L^2}^2. \]
In particular this implies that the norm (2.7) is *equivalent* to the $H^1(\mathbb{R}^n)$ norm, provided $C_0$ is small enough, so that the form is closed; and this implies also that (2.8) is satisfied with $C = 0$. \hfill \Box

For the perturbed Dirac operator we have a similar result:

**Proposition 2.2.** Let $V(x) = V^*(x)$ be a $4\times 4$ complex valued matrix on $\mathbb{R}^3$. Assume that
\begin{equation}
\|V\|_{L^{3,\infty}} \leq C_0,
\end{equation}
for some $C_0 > 0$ sufficiently small. Then the perturbed Dirac operator $\mathcal{D}_V = \mathcal{D} + V$ is self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

**Proof.** The proof is analogous to the proof of Theorem 2.1. We define the quadratic form $q : H^{1/2} \times H^{1/2} \rightarrow \mathbb{C}$ associated to the operator $\mathcal{D}_V$ as
\[ q(\varphi, \psi) := (\mathcal{D}\varphi, \psi) + (V\varphi, \psi). \]
First we prove that the domain of $q$ is $H^{1/2}$. With the same arguments of the previous theorem we estimate
\[ |q(\varphi, \varphi)| \leq \|\varphi\|_{H^{1/2}}^2 + C\|V\|_{L^{3,\infty}}\|\varphi\|_{L^{2/(n-1),1}}^2 \leq \|\varphi\|_{H^{1/2}}^2 + C\|V\|_{L^{3,\infty}}\|\varphi\|_{L^{2/(n-1),2}}^2 \leq (1 + C\|V\|_{L^{3,\infty}})\|\varphi\|_{H^{1/2}} \]
(where we used the embedding $H^{1/2} \subset L^{2/(n-1),2}$). From this point on, the proof proceeds exactly as in Proposition 2.1. \hfill \Box

3. The Limiting Absorption Principle

The essential tool in our proof will be the spectral theorem in the following version: given a selfadjoint (unbounded) operator $A$ on $L^2$ and a continuous bounded function $f(\lambda)$ on $\mathbb{R}$, the operator $f(A)$ can be defined as
\begin{equation}
f(A)\phi = -\frac{1}{\pi} \cdot L^2 - \lim_{\epsilon \downarrow 0} \int f(\lambda) \Im R(\lambda + i\epsilon) \phi d\lambda
\end{equation}
for any $\phi \in L^2$. Here $R(z) = (A - z)^{-1}$ denotes the resolvent operator of $A$ (see e.g. [38]). Under suitable assumptions on $H$, the limit operators $R(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon)$ are well defined as bounded operators in weighted $L^2$ spaces; this is usually called the *limiting absorption principle* (see below for details). Thus we have also the simpler representation
\begin{equation}
f(A)\phi = -\frac{1}{\pi} \cdot \int f(\lambda) \Im R(\lambda + i0) \phi d\lambda.
\end{equation}
Recalling the definition (6.3), consider now the operators
\[ H = -\Delta + W(x, \mathcal{D}) \equiv -\Delta + \sum_{j=1}^3 a_j(x) \partial_j + b(x) \]
and
\[ \mathcal{D}_V = \mathcal{D} + V(x). \]
In Section 2 we proved that, under assumptions (2.5) on $a_j, b$ and $V(x)$, both $H$ and $D_V$ are selfadjoint operators on $L^2$. In particular, the spectral formula (3.1) holds for both. We shall use the following notations: the free resolvents will be written as

$$R_0(z) = (-z - \Delta)^{-1} \quad \text{and} \quad R_{D}(z) = (-zI_4 + D)^{-1}$$

while we shall use the notation $R(z)$ for both perturbed resolvents:

$$R(z) = (-z - \Delta + W)^{-1} \quad \text{and} \quad R(z) = (-z + D + V)^{-1}.$$ 

From the context the meaning of $R(z)$ will always be clear. Note that $R_0(z)$ is defined for all $z \not\in \mathbb{R}^+$ while $R_{D}(z)$ is defined for $z \not \in \mathbb{R}$, and the same properties hold for the perturbed resolvents.

Our first task will be to show that the stronger representation (3.2), i.e., the limiting absorption principle, holds also for the perturbed operators. For $A = -\Delta$ this is a classical result (see e.g. Agmon [1]); here we shall use a very precise version of the principle, due to Barcelo, Ruiz and Vega [4]. On the other hand, for the Dirac operator only a few results are available, which concern the case with a nonzero mass term (see [28], [42]).

The classical results on $R_0$ (see [1]) state that the limits

$$\lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon) = R_0(\lambda \pm i0)$$

exist in the norm of bounded operators from $L^2(\langle x \rangle^s dx)$ to $H^2(\langle x \rangle^{-s} dx)$ for any $s > 1$; the convergence is uniform for $\lambda$ belonging to any compact subset of $]0, +\infty[$, and the following estimate holds

$$\|\langle x \rangle^{-s} R_0(\lambda \pm i0) \langle x \rangle^{-s} f\|_{L^2} \leq \frac{C(s)}{\sqrt{\lambda}} \|f\|_{L^2} \quad \forall \lambda > 0, \ s > \frac{1}{2}. $$

In $n = 3$ dimensions, the operators $R_0(\lambda \pm i0)$ have the explicit representation

$$R_0(\lambda \pm i0)g(x) = \frac{1}{4\pi} \int \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} g(y)dy, \quad \lambda \geq 0.$$

Recall also that for $\lambda < 0$ we have the similar formula

$$R_0(\lambda)g(x) = \frac{1}{4\pi} \int \frac{e^{-\sqrt{|\lambda|}|x-y|}}{|x-y|} g(y)dy, \quad \lambda \leq 0.$$

These results were extended in [4] to more general weights. Introduce the norm

$$\|a(x)\| = \sup_{\mu > 0} \int_{\mu}^{+\infty} \frac{h(r)r}{(r^2 - \mu^2)^{1/2}} dr \quad \text{where} \quad h(r) \equiv \sup_{|x|=r} |a(x)|.$$ 

For any measurable function on $\mathbb{R}^n$ such that $\sup f \subseteq \text{supp} a$, we can consider the (semi-)norm

$$\|f\|_{L^2(\text{supp } a(x)dx)} \equiv \|a(x)^{1/2}f\|_{L^2} < \infty$$

and we can define a Hilbert space $L^2(\text{supp } a(x)dx)$ as the closure in this norm of the subspace of $C_0^\infty$ functions with support contained in $\text{supp } a$. Then we can summarize Theorems 1 and 2 in [4] as follows:

**Theorem 3.1** ([4]). Let $a(x)$ be a nonnegative function on $\mathbb{R}^n$ with $\|a\| < \infty$, and denote by $R_0(\lambda \pm i0)$ the limit operators (3.3). Then the operators $R_0(z)$ for $z \not\in \mathbb{R}^+$ and $R_0(\lambda \pm i0)$ can be extended to bounded operators from $L^2(a(x)^{-1} dx)$ to $L^2(a(x)dx)$, and the following estimates hold:

$$\|R_0(\lambda \pm i0)f\|_{L^2(\text{supp } a(x)dx)} \leq \frac{C}{\sqrt{\lambda}} \|a\| \cdot \|f\|_{L^2(\text{supp } a(x)^{-1} dx)}, \quad \lambda \neq 0$$
(here of course $R_0(\lambda \pm i0) \equiv R_0(\lambda)$ for $\lambda < 0$)

\begin{equation}
\|\nabla R_0(\lambda \pm i0)f\|_{L^2(a(x) \, dx)} \leq C\|a\| \cdot \|f\|_{L^2(a(x)^{-1} \, dx)}.
\end{equation}

Moreover, the limiting absorption principle holds in the weak form: for all $f, g \in L^2(a(x)^{-1} \, dx)$

\begin{equation}
\lim_{\epsilon \to 0} (R_0(\lambda \pm i\epsilon)f, g) = (R_0(\lambda \pm i0)f, g).
\end{equation}

Remark 3.1. It is not difficult to extend the estimates (3.8) and (3.9) to the whole complex plane. Indeed, fix two functions $f, g \in C_0^\infty$ with support contained in $\operatorname{supp} a$ and consider on the half plane

\[ S = \{ z : \Im z > 0 \} \]

the holomorphic function

\begin{equation}
F(z) = z^{1/2} (R_0(z)f, g).
\end{equation}

It is clear that $F(z)$ is continuous on $\overline{S}$ up to the boundary, moreover it satisfies the estimate

\begin{equation}
|F(x)| \leq C\|a\| \cdot \|f\|_{L^2(a(x)^{-1} \, dx)} \|g\|_{L^2(a(x)^{-1} \, dx)}
\end{equation}

on the boundary $\Im z = 0$, and finally it has a polynomial growth for $|z| \to +\infty$, as it easily follows from the explicit expression of $R_0(z)$ as a convolution operator (see [4]). By the Phragmén-Lindelöf Theorem (see e.g., [36]) on the half plane we immediately obtain that estimate (3.12) holds on all of $\overline{S}$. A similar argument can be applied in the lower half plane $\Im z < 0$. In conclusion we obtain

\begin{equation}
\|R_0(z)f\|_{L^2(a(x) \, dx)} \leq \frac{C}{\sqrt{|z|}} \|a\| \cdot \|f\|_{L^2(a(x)^{-1} \, dx)}
\end{equation}

for all $f \in L^2(a(x)^{-1} \, dx)$ (see also part (ii) in Theorem 1, [4]). Notice that this estimate holds on the whole complex plane, in the sense that we apply it to $R_0(\lambda \pm i0)$ when $z \in \mathbb{R}^+$. If we apply the same argument to the function

\[ G(z) = (\nabla R_0(z)f, g) \]

we obtain in an analogous way the estimate

\begin{equation}
\|\nabla R_0(z)f\|_{L^2(a(x) \, dx)} \leq C \|a\| \cdot \|f\|_{L^2(a(x)^{-1} \, dx)}, \quad z \in \mathbb{C}.
\end{equation}

We now specialize the theorem to a particular choice of weights. Precisely, consider the family of functions

\begin{equation}
w_\beta(x) = |x|(|\log |x|| + 1)^\beta, \quad \beta > 1.
\end{equation}

As it is proved in [4] (see Proposition 1), the norms

\[ \|w_\beta^{-1}\| < +\infty \]

are finite for all $\beta > 1$, hence we can apply 3.1 with the choice

\[ a(x) = (w_\beta(x))^{-1} = \frac{1}{|x|(|\log |x|| + 1)^\beta}. \]

In this case it is possible to improve the above result and to obtain a stronger version of the limiting absorption principle. To this end, we need the following Lemma, which is inspired by [1]:

Lemma 3.2. Let $H$ be a Hilbert space, $H'$ its dual, and $H_0$ a second Hilbert space compactly embedded in $H'$. Let $T_j, T_j^* (j = 1, 2, \ldots)$ be bounded operators in $\mathcal{L}(H, H')$ such that
(i) $T_j, T$ are symmetric for the pairing $\langle \cdot, \cdot \rangle_{H' \times H}$, i.e.,
\[ \langle Tf, g \rangle_{H' \times H} = \langle Tg, f \rangle_{H' \times H} \quad \forall f, g \in H; \]

(ii) $T_j, T \in \mathcal{L}(H, H_0)$ and, for some constant $C$ independent of $j$,
\[ \|T_j\|_{\mathcal{L}(H, H_0)} \leq C. \]
Assume that
\[ T_j f \rightharpoonup Tf \quad \text{weakly in } H' \text{ for all } f \in H. \]
Then $T_j \to T$ in the operator norm of $\mathcal{L}(H, H')$.

Proof. Fix an $f \in H$; the sequence $T_j f$ converges weakly to $T f$ in $H'$, and
is bounded in $H_0$ by (ii), hence it admits a subsequence which converges in the norm
of $H'$, and the limit must be the same i.e. $T f$. By applying the same argument to
any subsequence of $T_j f$, we conclude that the entire sequence $T_j f$ converges to $T f$
in the norm of $H$.

Now, let $f_j$ be any sequence which converges to $f$ weakly in $H$. Then we have
for all $g \in H$
\[ \langle T_j f_j, g \rangle = \langle T g, f_j \rangle \to \langle T f, g \rangle \]
since $T_j g \to T g$ strongly in $H'$ and $f_j \to f$ weakly in $H$. In other words, for any
$f_j \to f$ weakly in $H$ we have that $T_j f_j \to T f$ weakly in $H'$. But, as in the first step,
we can remark that the sequence $T_j f_j$ is bounded in $H_0$ and by compact embedding
we obtain that the convergence is strong: $T_j f_j \to T f$ in the norm of $H'$.

By the same argument we obtain that, for any $f_j \to f$ weakly in $H$, the sequence
$T f_j$ converges to $T f$ in the norm of $H'$.

Finally, assume by contradiction that $T_j$ does not converge to $T$ in the operator
norm of $\mathcal{L}(H, H')$. This means that we can find a sequence $f_j \in H$ with norm

\[ \|f_j\|_H = 1 \]
such that
\[ \|T_j f_j - T f_j\|_{H'} > \epsilon > 0 \]
for some $\epsilon$ independent of $j$. By extracting a subsequence we can assume that $f_j \rightharpoonup f$
weakly in $H$, and by the above steps we immediately obtain a contradiction. \qed

Then we can prove:

**Proposition 3.3.** Let $w_\beta(x)$, $x \in \mathbb{R}^n$ one of the radial weights (3.15) for some
fixed $\beta > 1$. Then, for all $\lambda \neq 0$, the limits
\[ \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon) = R_0(\lambda \pm i0) \]
exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ to $H^2(w_\beta(x)^{-1}dx)$ and
satisfy the estimates
\[ \|R_0(\lambda \pm i0) f\|_{L^2(w_\beta^{-1}dx)} \leq \frac{C(b)}{\sqrt{|\lambda|}} \|f\|_{L^2(w_\beta dx)}, \quad \forall \lambda \neq 0, \]
\[ \|\nabla R_0(\lambda \pm i0) f\|_{L^2(w_\beta^{-1}dx)} \leq C(b) \|f\|_{L^2(w_\beta dx)}. \]

Proof. We apply Lemma 3.2 with the choices: $H = L^2(w_\beta(x)dx)$, and hence $H' = L^2(w_\beta(x)^{-1}dx)$ with the standard $L^2$ pairing; $H_0 = H^1(w_{\beta_0}(x)^{-1}dx)$ for some
arbitrary $\beta_0$ with $\beta > \beta_0 > 1$; the norm of $H_0$ of course is
\[ \|f\|_{H_0}^2 = \|w_{\beta_0}^{-1/2}f\|_{L^2}^2 + \|w_{\beta_0}^{-1/2}\nabla f\|_{L^2}^2. \]
Finally, as operators $T_j$ we shall take (any subsequence of) the resolvent operators
$R_0(\lambda \pm i\epsilon)$ as $\epsilon \downarrow 0$, while $T = R_0(\lambda \pm i0)$, for some fixed $\lambda \in \mathbb{R}$.

We now check the assumptions of the lemma. The compact embedding of $H_0$
into $H'$ is clear. Also the symmetry of the operators in the sense of (i) is evident.
The uniform bounds on $T_j, T$ as bounded operators from $H$ to $H'$ are simply the estimates (3.13), (3.14) applied with the choice $a(x) = w_\beta(x)^{-1}$. But it is clear that the estimate (3.13) implies also the following estimate

\[
\|R_0(z)f\|_{L^2(w_\beta^{-1}dx)} \leq C(\beta_0) \frac{\|f\|_{L^2(w_\beta dx)}}{|z|}, \quad \forall z \neq 0,
\]

which is only apparently stronger, in view of the trivial embedding $L^2(w_\beta dx) \subseteq L^2(w_\beta^{-1}dx)$.

In a similar way we have

\[
\|\nabla R_0(z)f\|_{L^2(w_\beta^{-1}dx)} \leq C(\beta_0) \|f\|_{L^2(w_\beta dx)}.
\]

These inequalities show that assumption (ii) of the Lemma is satisfied. Finally, assumption (3.16) is nothing but the weak limiting absorption principle of Barcelo, Ruiz, Vega (see (3.10)).

In conclusion, Lemma 3.2 implies that the limit (3.17) exists in the norm of bounded operators from $L^2(w_\beta dx)$ to $L^2(w_\beta^{-1}dx)$. Moreover, by the identity

$$\Delta R_0(z) = -I - zR_0(z)$$

we obtain that the limit exists also in the norm of bounded operators from $L^2(w_\beta dx)$ to $H^2(w_\beta^{-1}dx)$. The estimates (3.18) and (3.19) follow from the corresponding estimates for general $z$.

3.1. The limiting absorption principle for the magnetic Laplacian. In what follows, we shall focus on the case $n = 3$ exclusively. We follow the standard approach, based on the resolvent identity

$$R(z) = (-z - \Delta + W(x, D))^{-1} = R_0(z)(I + W R_0(z))^{-1}.$$ 

Thus the main step of the proof will consist in inverting the operator $I + WR_0$ in suitable weighted spaces. We shall assume that the coefficients $a_j(x)$ and $b(x)$ in $W(x, D)$, defined as in (2.3), satisfy the assumptions

\[
|a_j(x)| \leq \frac{C_0}{|x|^s(|\log |x|| + 1)\beta}, \quad |b(x)| \leq \frac{C_0}{|x|^2(|\log |x|| + 1)\beta}
\]

for some $s \in [0, 1]$, $\beta > 1$ and some constant $C_0$ small enough.

Our result is the following:

**Proposition 3.4.** Assume the coefficients of $W(x, D) = \sum a_j(x)\partial_j + b(x)$ satisfy (2.3) (3.22) for some $C_0$ small enough, some $s \in [0, 1]$ and some $\beta > 1$.

Then the operator $I + WR_0$ is invertible on the weighted space $L^2(w_\beta(x)|x|^{2s}dx)$, and the inverse operators $(I + WR_0(z))^{-1}$ are uniformly bounded for all $z \in \mathbb{C}$. Moreover, the strong limiting absorption principle holds for $R(z)$, in the following sense:

(i) the boundary values

\[
\lim_{\epsilon \to 0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0)
\]

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ to $H^2(w_\beta^{-1}(x)dx)$;

(ii) the following estimate

\[
\|R(z)f\|_{L^2(w_\beta(x)dx)} \leq \frac{C(\beta)}{\sqrt{|z|}} \cdot \|f\|_{L^2(w_\beta(x)^{-1}dx)}
\]

holds for all $z \in \mathbb{C}$, $z \neq 0$. 

Remark 3.2. In the case $s = 0$ we recover exactly the strong limiting absorption principle proved in Proposition 3.3 above for the free operator $R_0$. The additional weight $(x)^{s}$ was considered in view of the estimates that will be needed in the following section.

Proof. Consider the operator
\[ W(x, D)R_0(z)f = \sum a_j(x) \partial_j R_0(z)f + b(x)R_0(z)f; \]
we estimate the two terms separately.

First of all we have
\[ \|w_1^{1/2} \langle x \rangle^s a_j(x) \partial_j R_0f\|_{L^2} \leq \|w_1^{1/2} \langle x \rangle^s a_j f\|_{L^\infty} \|w_1^{-1/2} \partial_j R_0f\|_{L^2} \leq C_0 \|w_1^{1/2} f\|_{L^2} \]
by estimate (3.21), and this implies trivially
\[ (3.25) \quad \|w_1^{1/2} \langle x \rangle^s a_j(x) \partial_j R_0f\|_{L^2} \leq C_0 \|w_1^{1/2} \langle x \rangle^s f\|_{L^2}. \]
In order to estimate the electric term, we recall that, from the explicit expression of the free resolvent, we can write
\[ |R_0(z)f| \leq \frac{1}{4\pi} \left| \frac{1}{|x|} \ast |f| \right|. \]

Then we have
\[ (3.26) \quad \|w_1^{1/2} b(x)R_0(z)f\|_{L^2} \leq \|w_1^{1/2} b(x)\|_{L^2} \|R_0(z)f\|_{L^\infty} \leq \|w_1^{1/2} b(x)\|_{L^2} \cdot C \left\| \frac{1}{|x|} \ast |f| \right\|_{L^\infty}. \]
Recalling Young and H"older inequalities in Lorentz spaces (see Theorems A.2, A.3), we have
\[ \left\| \frac{1}{|x|} \ast |f| \right\|_{L^\infty} \leq C \|f\|_{L^{\infty/2,1}} = C \|w_1^{-1/2} w_1^{1/2} f\|_{L^{\infty/2,1}} \leq C \|w_1^{-1/2} \|_{L^{\infty/2,1}} \|w_1^{1/2} f\|_{L^2}. \]
Since $w_1^{-1/2} \in L^{6,2}$ for any $\beta > 1$ (Proposition A.4), (3.26) gives
\[ \|w_1^{1/2} b(x)R_0(z)f\|_{L^2} \leq C \|w_1^{1/2} b(x)\|_{L^2} \cdot \|w_1^{1/2} f\|_{L^2}. \]
Now, by assumption (3.22) on $b(x)$ we have easily
\[ \|w_1^{1/2} b(x)\|_{L^2} \leq CC_0 \]
and we conclude that
\[ (3.27) \quad \|w_1^{1/2} b(x)R_0(z)f\|_{L^2} \leq CC_0 \cdot \|w_1^{1/2} f\|_{L^2}. \]

In a similar way we have
\[ (3.28) \quad \|w_1^{1/2} \langle x \rangle bR_0(z)f\|_{L^2} \leq \|w_1^{1/2} \langle x \rangle b\|_{L^6} \|R_0(z)f\|_{L^3} \leq \|w_1^{1/2} \langle x \rangle b\|_{L^6} \cdot C \left\| \frac{1}{|x|} \ast |f| \right\|_{L^3} \]
and
\[ \left\| \frac{1}{|x|} \ast |f| \right\|_{L^3} \leq C \|f\|_{L^1} = C \|w_1^{-1/2} \langle x \rangle^{-1} w_1^{1/2} \langle x \rangle f\|_{L^1} \]
\[ \leq C \|w_1^{-1/2} \langle x \rangle^{-1} \|_{L^2} \|w_1^{1/2} \langle x \rangle f\|_{L^2}. \]
As above, we notice that $w_1^{-1/2} \langle x \rangle^{-1} \in L^2$ for any $\beta > 1$, hence we have from (3.28)
\[ \|w_1^{1/2} \langle x \rangle bR_0(z)f\|_{L^2} \leq C \|w_1^{1/2} \langle x \rangle b\|_{L^6} \cdot \|w_1^{1/2} \langle x \rangle f\|_{L^2}. \]
Assumption (3.22) guarantees that
\[ \|w_1^{1/2} \langle x \rangle b(x)\|_{L^6} \leq CC_0 \]
and, in conclusion,
\[(3.29) \quad \| \langle x \rangle w_{\beta}^{1/2} b(x) R_0(z) f \|_{L^2} \leq C C_0 \cdot \| \langle x \rangle w_{\beta}^{1/2} f \|_{L^2} \]
If we interpolate between (3.27) and (3.29), we obtain the estimate
\[(3.30) \quad \| \langle x \rangle^s w_{\beta}^{1/2} b(x) R_0(z) f \|_{L^2} \leq C C_0 \cdot \| \langle x \rangle^s w_{\beta}^{1/2} f \|_{L^2} \]
Summing up, from estimates (3.25) and (3.30) we get for all \( z \in \mathbb{C} \)
\[(3.31) \quad \| \langle x \rangle^s w_{\beta}^{1/2} W R_0(z) f \|_{L^2} \leq C C_0 \cdot \| \langle x \rangle^s w_{\beta}^{1/2} f \|_{L^2}. \]
Then it is clear that we can invert the operator \( I + WR_0 \) by a Neumann series on the space \( L^2(\langle x \rangle^{2s} w_{\beta} dx) \). Hence, the standard representation
\[(3.32) \quad R(z) = R_0(z)(I + WR_0(z))^{-1} \]

is valid. To conclude the proof of the Proposition, it is now sufficient to remark that, from property (3.17) of Proposition 3.3 and the uniform bounds on the norm of \( (I + WR_0(z))^{-1} \) we have just obtained (for \( s = 0 \)), the limits in 3.23 exist in a weak sense. Proceeding as in the proof of Proposition 3.3, using Lemma 3.2, we deduce (i). Finally, (ii) is a consequence of (3.32) and the corresponding estimate (3.20) for \( R_0 \).

\[\square\]

Remark 3.3. Note that the assumptions of the preceding proposition can be expressed in terms of the original coefficients \( A, B \) as follows:
\[(3.33) \quad |A(x)| \leq \frac{C_0}{|x|(|x|^{\beta}(|\log |x|| + 1)|}^{\beta}, \quad |\nabla A(x)| + |B(x)| \leq \frac{C_0}{|x|^2(|\log |x|| + 1)^\beta} \]
for some \( \beta > 1 \) and a constant \( C_0 > 0 \) small enough.

3.2. The limiting absorption principle for the Dirac operator and its perturbation. In this section we will study the limiting absorption principle for the massless Dirac operator \( D \); this property was studied by Yamada in [42] for the operator with mass. Moreover, as in the case of the magnetic Laplacian, we will extend this result to the perturbed operator \( D_V = D + V(x) \), under a suitable assumption on the potential \( V \).

It is well known that the spectrum of the free operator \( D \) is the whole real line. Due to the relation \( D^2 = -\Delta I_4 \), we immediately obtain the representation
\[(3.34) \quad R_D(z) = R_0(z^2)(D + zI_4), \]
for all \( z \in \mathbb{C} \) with \( \Re z = 0 \). Using this formula and the Proposition 3.3, we easily prove the following:

Proposition 3.5. Let \( w_\beta(x) \), \( x \in \mathbb{R}^3 \) be defined as in (3.15), for some fixed \( \beta > 1 \). Then, for all \( \lambda \in \mathbb{R} \), the limits
\[(3.35) \quad \lim_{\epsilon \downarrow 0} R_D(\lambda \pm i \epsilon) = R_D(\lambda \pm i 0) := R_0(\lambda^2 \pm i 0)(D + \lambda I_4) \]
exist in the norm of bounded operators from \( L^2(w_\beta(x)dx) \) to \( H^1(w_\beta(x)^{-1}dx) \) and satisfy the estimate
\[(3.36) \quad \| R_D(z) f \|_{L^2(w_\beta(x)^{-1}dx)} \leq \| f \|_{L^2(w_\beta(x)dx)}, \]
for all \( z \in \mathbb{C} \). Moreover, we have the explicit representation
\[(3.37) \quad R_D(\lambda \pm i 0) f = \frac{i|\lambda|}{4\pi} \int_{\mathbb{R}^3} e^{i|\lambda|x-y|} \left( I_4 - \sum_{j=1}^{3} \alpha_j \frac{x_j - y_j}{|x-y|} \right) f(y) \, dy + \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{i|\lambda|x-y|} \sum_{j=1}^{3} \alpha_j \frac{x_j - y_j}{|x-y|} f(y) \, dy.\]
Proof. The strong convergence of $R_D(\lambda \pm ic)$ to $R_D(\lambda \pm i0)$ in the space of bounded operators from $L^2(w_\beta(x)dx)$ to $H^1(w_\beta(x)^{-1}dx)$ is obtained by interpolation using the property (3.17) and the representation (3.34); estimate (3.36) immediately follows from (3.34) and the estimates (3.18), (3.19), (3.20), (3.21). In conclusion, recalling the explicit representation (3.5) for $R_0(\lambda \pm i0)$, after an integration by parts we get the formula (3.37) and this concludes the proof. □

At this point, we will proceed in a similar way to the case of the perturbed Laplacian and we will prove that it is possible to extend the above result to small electric perturbations of the free Dirac operator. As for the magnetic coefficients $W(x, D)$, we need to assume that the potential $V$ satisfies

$$
(3.38) \quad |V(x)| \leq \frac{C_0}{|x|^{s/2}(|\log |x|| + 1)^{\beta/2}}.
$$

for some $s \in [0, 1]$, $\beta > 1$ and some constant $C_0$ small enough. We prove the following result:

**Proposition 3.6.** Assume the potential $V$ satisfies (3.38) for some $C_0$ sufficiently small, some $s \in [0, 1)$ and some $\beta > 1$.

Then the operator $I + VR_D$ is invertible on the weighted space $L^2(w_\beta(x)^{2s}dx)$, and the inverse operators $(I + VR_D(z))^{-1}$ are uniformly bounded for all $z \in \mathbb{C}$. Moreover, the strong limiting absorption principle holds for $R(z)$, in the following sense:

(i) the limits

$$
(3.39) \quad \lim_{c \downarrow 0} R(\lambda \pm ic) = R(\lambda \pm i0)
$$

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ to $H^1(w_\beta^{-1}(x)dx)$;

(ii) the following estimate

$$
(3.40) \quad \|R(z)f\|_{L^2(w_\beta^{-1}dx)} \leq C(\beta) \cdot \|f\|_{L^2(w_\beta(x)dx)}
$$

holds for all $z \in \mathbb{C}$, $z \neq 0$.

**Proof.** The argument is the same of the proof of Proposition 3.4 for the magnetic part of $W$. First we observe that, by hypothesis (3.38), we have

$$
\|w_\beta^{1/2}(x)^{s}V(x)\|_{L^2} \leq \|w_\beta(x)^{s}V(x)\|_{L^\infty} \|w_\beta^{-1/2}R_D f\|_{L^2} \leq C_0 \cdot \|w_\beta^{-1/2}f\|_{L^2}.
$$

Hence we obtain the estimate

$$
\|w_\beta^{1/2}(x)^{s}V(x)R_D(z)f\|_{L^2} \leq \|w_\beta^{1/2}(x)^{s}f\|_{L^2},
$$

uniformly in $z \in \mathbb{C}$; thus we can invert the operator $I + VR_D$ by a Neumann series on the space $L^2(w_\beta dx)$. Again, we can exploit the representation

$$
R(z) = R_D(z)(I + VR_D(z))^{-1}.
$$

By property (3.35) of Proposition 3.5 and the uniform bounds of $(I + VR_D)^{-1}$, it follows that the limits in (3.39) exist in a weak sense. Then we can procede as in the previous cases, using Lemma 3.2 and obtain (i). In conclusion, the estimate (ii) is an immediate consequence of (3.41) and the inequality (3.36). This concludes the proof. □

In the following we shall also need a weaker version of the last result: we shall require that $V$ satisfies

$$
(3.42) \quad |V(x)| \leq \frac{C_0}{|x|^{s/2}(|\log |x|| + 1)^{\beta/2}},
$$

for some $s > \frac{1}{2}$, $\beta > 1$ and some constant $C_0$ small enough. Then we have
Corollary 3.7. Assume the potential $V$ satisfies (3.22) for some $C_0$ sufficiently small, $s > \frac{1}{2}$ and $\beta > 1$.

Then the operators $I + VR_{\mathcal{D}}$ are invertible on the space $L^2((x)^{2s}dx)$, and the inverse operators $(I + VR_{\mathcal{D}}(z))^{-1}$ are uniformly bounded for all $z \in \mathbb{C}$. Moreover, the strong limiting absorption principle holds for $R(z)$, in the following sense:

(i) the limits

$\lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0)$

exist in the norm of bounded operators from $L^2((x)^{2s}dx)$ to $H^1((x)^{-2s}dx)$;

(ii) the following estimate

$\|R(z)f\|_{L^2((x)^{2s}dx)} \leq C \cdot \|f\|_{L^2((x)^{2s}dx)}$

holds for all $z \in \mathbb{C}$, $z \neq 0$.

Proof. The proof is analogous to the proof of Proposition 3.6. Indeed, from estimate (3.36) and assumption (3.42) we have immediately

$\|\langle x \rangle^s VR_{\mathcal{D}}\|_{L^2} \leq \|\langle x \rangle^s w_\beta^{1/2} V\|_{L^\infty} \|w_\beta^{-1/2} R_{\mathcal{D}} f\|_{L^2} \leq C_0 \|w_\beta^{1/2} f\|_{L^2}$

and by the trivial inequality

$w_\beta^{1/2} \leq C_s(x)^s$,

valid for all $s > 1/2$, we conclude that

$\|\langle x \rangle^s VR_{\mathcal{D}}\|_{L^2} \leq C_0 \|\langle x \rangle^s f\|_{L^2}$.

Thus we can again invert $(I + VR_{\mathcal{D}})$ with a Neumann series, and proceeding exactly as before we obtain the proof of the Corollary. \qed

4. Resolvent Estimates

In this section we prepare the crucial resolvent estimates that will be used in the proof of the main results. In order to use the spectral formula, we need estimates on the perturbed resolvent operators and their derivatives with respect to $\lambda$ as bounded operators from suitable weighted $L^p$ spaces to $L^\infty$. We shall use the Hölder and Young inequalities in Lorentz spaces extensively; for the convenience of the reader, we give a sketch of the main useful results in the Appendix A.

We consider first the resolvent of the magnetic laplacian. We recall that, by Proposition 3.4, the operators $R(\lambda \pm i0) = R_0(\lambda \pm i0)(I + W(x, D)R_0(\lambda \pm i0))^{-1}$ are well defined as bounded operators from $L^2(w_\beta(x)dx)$ to $H^2(w_\beta(x)^{-1}dx)$; moreover, we have the explicit representation (3.5). Our first result is the following:

Lemma 4.1. Let $R(\lambda \pm i0) = R_0(\lambda \pm i0)(I + W(x, D)R_0(\lambda \pm i0))^{-1}$ be the resolvent of $-\Delta + W$ and assume the coefficients of $W(x, D) = \sum a_j(x) \partial_j + b(x)$ satisfy (3.22). Then, for all $\lambda \geq 0$, the following estimates hold:

$\|R(\lambda \pm i0)f\|_{L^\infty} \leq C \|w_\beta^{1/2} f\|_{L^2}$,

$\|\partial_\lambda R(\lambda \pm i0)f\|_{L^\infty} \leq C \left(1 + \frac{1}{\sqrt{\lambda}}\right) \|\langle x \rangle w_\beta^{1/2} f\|_{L^2}$.

Proof. The estimate (4.1) is the easiest one. In fact, by formula (3.32) and the explicit representation (3.5) for $R_0$, we obtain

$\|R(\lambda \pm i0)f\|_{L^\infty} \leq C \cdot \frac{1}{|x|} \|f\|_{L^\infty}$;
using Young inequality in Lorentz spaces, we get
\[ \|R(\lambda \pm i0)f\|_{L^\infty} \leq \|\langle x \rangle^{-\lambda}w_\beta(x)^{-1/2}(I + WR_0)^{-1}f\|_{L^2}, \]
\[ \leq \|w_\beta^{-1/2}w_\beta(x)^{1/2}(I + WR_0)^{-1}f\|_{L^3/2}, \]
\[ \leq \|w_\beta^{-1/2}\|_{L^{6,2}}\|w_\beta(x)^{1/2}(I + WR_0)^{-1}f\|_{L^2}. \]
The uniform bound for the operators \((I + WR_0)^{-1}\) proved in Proposition 3.4 and
the observation that \(w_\beta^{-1/2} \in L^{6,2}\), for all \(\beta > 1\) (see Proposition A.4) are sufficient
now to conclude the proof of estimate (4.1).

In order to proceed with the proof of (4.2) we observe that from (3.5) we imme-
diately obtain the following explicit representations, for all \(\lambda > 0\):
\[ \partial_\lambda R_0(\lambda \pm i0)f = R_0^2(\lambda \pm i0)f = \pm \frac{i}{8\pi\sqrt{\lambda}} \int_0^\infty e^{\pm i\sqrt{\lambda}|x-y|}f(y)dy, \]
\[ \partial_j R_0^2(\lambda \pm i0)f = \pm \frac{1}{8\pi} \int_0^\infty e^{\pm i\sqrt{\lambda}|x-y|}\sum \frac{x_j - y_j}{|x-y|}f(y)dy. \]
At this point, differentiating in (3.32) we get
\[ \partial_\lambda R(\lambda \pm i0) = A + B \]
where
\[ A = R_0^2(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1} \]
and
\[ B = R_0(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1}WR_0^2(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1}. \]
We treat separately the two terms. By (4.3), we estimate
\[ \|Af\|_{L^\infty} \leq \frac{C}{\sqrt{\lambda}}\|\langle x \rangle^{-1}w_\beta(x)^{-1/2}\|_{L^2}\|\langle x \rangle w_\beta(x)^{1/2}(I + WR_0)^{-1}f\|_{L^2}. \]
We observe (Proposition A.4) that \(\langle x \rangle^{-1}w_\beta(x)^{-1/2} \in L^2\) for all \(\beta > 1\) and, by
the uniform bound for the norms of \((I + WR_0)^{-1}\) in the space of bounded operators
onto \(L^2(\langle x \rangle w_\beta(x)dx)\) for (see Proposition 3.4), we conclude that, for some \(C > 0\)
\[ \|Af\|_{L^\infty} \leq \frac{C}{\sqrt{\lambda}}\|\langle x \rangle w_\beta(x)^{1/2}f\|_{L^2}. \]
For the estimate of the term \(B\), we start with some computation on the operator
\(WR_0^2\). Using the representation (4.4), we obtain
\[ \|w_\beta^{1/2}a_j\partial_j R_0^2f\|_{L^2} \leq \|w_\beta^{1/2}a_j\|_{L^\infty}\cdot \|\partial_j R_0^2f\|_{L^\infty} \leq C \cdot \|w_\beta^{1/2}a_j\|_{L^2}\|f\|_{L^1}. \]
By the above observation that
\[ \|f\|_{L^1} \leq \|\langle x \rangle w_\beta^{1/2}(x)f\|_{L^2}, \]
it turns out that, if \(w_\beta^{1/2}a_j \in L^2\), then
\[ \|w_\beta(x)^{1/2}a_j(x)\partial_j R_0^2f\|_{L^2} \leq C \cdot \|\langle x \rangle w_\beta^{1/2}(x)f\|_{L^2}. \]
In a similar way, using (4.3), we have
\[ \|w_\beta^{1/2}b R_0^2f\|_{L^2} \leq \|w_\beta^{1/2}b(x)\|_{L^\infty}\cdot \|R_0^2f\|_{L^\infty} \leq \frac{C}{\sqrt{\lambda}}\|w_\beta^{1/2}b\|_{L^2}\|f\|_{L^1}. \]
If we assume that \(w_\beta^{1/2}b \in L^2\), we conclude that
\[ \|w_\beta(x)^{1/2}b(x)R_0^2f\|_{L^2} \leq \frac{C}{\sqrt{\lambda}}\cdot \|\langle x \rangle w_\beta^{1/2}(x)f\|_{L^2}. \]
Inequalities (4.7) and (4.8) can be unified now, to show that, under the assumptions

\[ w^{1/2}_\beta a_j \in L^2, \quad w^{1/2}_\beta b \in L^2, \]

the estimate

\[ \| w_\beta(x)^{1/2}W(\lambda \pm i0)f \|_{L^2} \leq C \left( 1 + \frac{1}{\sqrt{\lambda}} \right) \| \langle x \rangle w_\beta(x)^{1/2}f \|_{L^2} \]

holds, for some \( C > 0 \). Observe that assumptions (4.9) are weaker than (3.22), so that they are obviously satisfied by the hypothesis of the Lemma.

Now we are ready for the estimate of the term \( B \). First, we use the representation (3.5) for \( R_0 \) to obtain

\[
\| Bf \|_{L^\infty} \leq \left\| \frac{1}{|x|} * |(I + WR_0)^{-1}WR_0^2(I + WR_0)^{-1}f| \right\|_{L^\infty}
\leq \| (I + WR_0)^{-1}WR_0^2(I + WR_0)^{-1}f \|_{L^{3/2,1}} =: \| Tf \|_{L^{3/2,1}}.
\]

As before, we use the properties of the weights \( w_\beta(x) \) to observe that

\[ \| g \|_{L^{3/2,1}} \leq \| w_\beta(x)^{1/2}g \|_{L^2}. \]

Then, the last series of inequalities gives

\[ \| Bf \|_{L^\infty} \leq \| w_\beta(x)^{1/2}Tf \|_{L^2}. \]

Now we use the uniform bounds for the inverse operators \( (I + WR_0)^{-1} \) (see Proposition 3.4) to proceed with

\[ \| Bf \|_{L^\infty} \leq \| w_\beta(x)^{1/2}WR_0^2(I + WR_0)^{-1}f \|_{L^2}; \]

finally, by inequality (4.10) and the above mentioned estimates on the norms of \( (I + WR_0)^{-1} \) in the space of bounded operators onto \( L^2(\langle x \rangle w_\beta(x)^{1/2}dx) \), we obtain the estimate

\[ \| Bf \|_{L^\infty} \leq C \left( 1 + \frac{1}{\sqrt{\lambda}} \right) \| \langle x \rangle w_\beta(x)^{1/2}f \|_{L^2}. \]

In conclusion, estimates (4.6), (4.11) and the representation (4.5) conclude the proof of (4.2) and the Lemma.

**Remark 4.1.** The limiting absorption principle allows us to rewrite the spectral formula in the following way: for any (smooth, compactly supported) function \( \phi(\lambda) \) on \( \mathbb{R} \), and any test function \( f \),

\[ \phi(-\Delta + W)f = \int_0^{+\infty} \phi(\lambda) \Im R(\lambda + i0) f d\lambda. \]

where the integral is restricted to the positive real axis since of course \( \Im R(\lambda) = 0 \) for negative \( \lambda \).

The resolvent estimates just proved imply that we can integrate by parts in the above formula, i.e., if

\[ \phi(\lambda) = \psi'(\lambda) \]

then

\[ \phi(-\Delta + W)f = \int_0^{+\infty} \psi'(\lambda) \Im R(\lambda + i0) f d\lambda = -\int_0^{+\infty} \psi(\lambda) \partial_\lambda \Im R(\lambda + i0) f d\lambda. \]
The problems arising from the singularity at $\lambda = 0$ are easily overcome. To prove this, consider a cutoff function $\chi(\lambda)$ supported in $[-L, L]$, and write

$$\phi(-\Delta + W)f = \lim_{L \to +\infty} \int_0^{+\infty} \phi(\lambda)(1 - \chi(\lambda L))\Im R(\lambda + i0)f d\lambda$$

whence

$$\phi(-\Delta + W)f = -\lim_{L \to +\infty} L \int_0^{1/L} \psi(\lambda)\chi'(\lambda L)\Im R(\lambda + i0)f d\lambda$$

$$-\lim_{L \to +\infty} \int_0^{+\infty} (1 - \chi(\lambda L))\psi(\lambda)\partial_\lambda \Im R(\lambda + i0)f d\lambda.$$ 

$$= u_L + v_L.$$ 

The last term $v_L$ converges to (4.13) uniformly, thanks to estimate (4.2) (and Lebesgue’s dominated convergence theorem), hence it is clear that $u_L = \phi(-\Delta + W)f - v_L$ also converges uniformly, and it will be sufficient to show that its limit is 0, e.g., in distribution sense. To estimate the integral

$$u_L = -L \int_0^{1/L} \psi(\lambda)\chi'(\lambda L)\Im R(\lambda + i0)f d\lambda$$

we can use the identity

$$\Im R(\lambda + i0) = (I + R_0(\lambda - i0)W)^{-1}\Im R_0(\lambda + i0)(I + WR_0(\lambda + i0))^{-1}.$$ 

Consider then the $L^2$ product

$$\langle \Im R(\lambda + i0)f, g \rangle = \langle \Im R_0(\lambda + i0)(I + WR_0(\lambda + i0))^{-1}f, (I + WR_0(\lambda + i0))^{-1}g \rangle.$$ 

From the explicit formula

$$\Im R_0(\lambda + i0)h = C \int \frac{\sin(\sqrt{x} y)}{|x - y|} h(y) dy$$

we have

$$|\Im R_0(\lambda + i0)h| \leq C \sqrt{x} \int |h(y)| dy$$

which implies

$$\|\Im R_0(\lambda + i0)h\|_{L^\infty} \leq C \sqrt{x} \|h\|_{L^1} \leq C \sqrt{x} \|\langle x \rangle w_{\beta}^{1/2} h\|_{L^2}$$

for any $\beta > 1$. Recalling now the uniform bound for $(I + WR_0(\lambda + i0))^{-1}$ in Proposition 3.4 in the weighted $L^2$ norms with weight $\langle x \rangle w_{\beta}^{1/2}$, we obtain easily

$$|\langle \Im R(\lambda + i0)f, g \rangle| \leq C \sqrt{x} \|\langle x \rangle w_{\beta}^{1/2} f\|_{L^2} \|\langle x \rangle w_{\beta}^{1/2} g\|_{L^2}.$$ 

From this estimate it is easy to prove that

$$(u_L, g) = -L \int_0^{1/L} \psi(\lambda)\chi'(\lambda L)(\Im R(\lambda + i0)f, g) d\lambda \to 0$$

as $L \to +\infty$, which concludes the argument.

We will prove now an analogue of Lemma 4.1 for the Dirac operator. In what follows, $R(z) = (-z I_4 + D + V)^{-1}$ denotes the resolvent of the perturbed Dirac operator. Our approach here will be slightly different: we shall use the formula

$$R(z) = R_D(z) + R_D(z)V(x)R_D(z)(I + V(x)R_D(z))^{-1},$$

valid for all $z \in \mathbb{C}$ (to be interpreted of course, for $z = \lambda \in \mathbb{R}$, as the extended resolvents $R(\lambda) := R(\lambda \pm i0)$ on the weighted $L^2$ spaces, as given by Proposition 3.6 and Corollary 3.7). When inserted in the spectral formula, the first term $R_D$
at the right hand side reproduces the solution to the free Dirac equation, and the main part of our proof will be the estimate of second term
\[ (4.16) \quad Q := R_D V R_D (I + V R_D)^{-1}. \]
To this end, we shall need an explicit representation for \( R_D (\lambda \pm i0) \), which is easily obtained from the formula
\[ (4.17) \quad R_D (\lambda \pm i0) = R_0 (\lambda^2 \pm i0) (D + \lambda I_4). \]
Recalling (3.5), after an integration by parts we obtain
\[ (4.25) \quad R_D (\lambda \pm i0) f = \frac{i \lambda}{4 \pi} \int_{\mathbb{R}^3} e^{\pm i \lambda |x - y|} \left( I_4 + \sum_{j=1}^{3} \alpha_j \frac{x_j - y_j}{|x - y|} \right) f(y) dy \]
\[ (4.21) \quad \parallel f \parallel_{L^1} \leq C_0. \]
\[ (4.22) \quad \parallel R_D (\lambda) V R_D (\lambda) f \parallel_{L^\infty} \leq C_\varepsilon \parallel \langle x \rangle ^{3/2 + \varepsilon} f \parallel_{L^2}, \]
\[ (4.23) \quad \parallel R_D^2 (\lambda) V R_D (\lambda) f \parallel_{L^\infty} + \parallel R_D (\lambda) V R_D^2 (\lambda) f \parallel_{L^\infty} \leq C_\varepsilon \parallel \langle x \rangle ^{3/2 + \varepsilon} f \parallel_{L^2} \]
for some \( C = C_\varepsilon \) independent of \( \lambda \).

**Proof.** In the following we shall use the shorthand notation, for \( s \in \mathbb{R} \),
\[ (4.24) \quad \parallel f \parallel_{L^s} := \parallel \langle x \rangle^s f \parallel_{L^2}. \]
From the explicit representations (4.18) and (4.19) we have the simple pointwise estimates
\[ (4.25) \quad |R_D (\lambda) f| \leq C (|\lambda| \cdot |x|^{-1} + |x|^{-2}) * f, \quad |R_D^2 (\lambda) f| \leq C (|\lambda| + |x|^{-1}) * f. \]
Since $|x|^{-1} \in L^{3,\infty}$, by the Young inequality in Lorentz spaces (see the Appendix) we get
\[
\|VR^2_D(\lambda)f\|_{L^2} \leq \|V\|_{L^2} \cdot |\lambda| \cdot \|1 \ast f\|_{L^\infty} + \|V\|_{L^2} \|x|^{-1} \ast f\|_{L^\infty}
\leq \|V\|_{L^2} (|\lambda| \cdot \|f\|_{L^1} + \|f\|_{L^{3/2,1}}).
\]
By the obvious inequalities valid for all $\epsilon > 0$
\[\tag{4.26}\|f\|_{L^{1}} \leq C(\epsilon)\|f\|_{L^{3/2,1}}, \quad \|f\|_{L^{3/2,1}} \leq C(\epsilon)\|f\|_{L^{3/2,1}},\]
we arrive at the first estimate
\[\tag{4.27}\|VR^2_D(\lambda)f\|_{L^2} \leq C(\epsilon)\|V\|_{L^2} \langle \lambda \rangle \cdot \|f\|_{L^{3/2,1}}.
\]
Since $\|V\|_{L^2} < \infty$ by assumption (4.20) as soon as $\gamma = 1/2 + \epsilon < s - 1$, we see that (4.21) follows provided $\epsilon$ is suitably small.

In a similar way, in order to prove (4.22) we use again (4.25) and we write (recall that $|x|^{-2} \in L^{3/2,\infty}$)
\[
\|R_D(\lambda)VR_D(\lambda)f\|_{L^\infty} \leq C (|\lambda| \cdot \||x|^{-1} \ast VR_Df\|_{L^\infty} + \|x|^{-2} \ast VR_Df\|_{L^\infty})
\leq C (|\lambda| \cdot \|VR_D(\lambda)f\|_{L^{3/2,1}} + \|VR_D(\lambda)f\|_{L^{3/1,1}}).
\]
For the first term we can write, recalling again (4.25),
\[\tag{4.28}\|VR_D(\lambda)f\|_{L^{3/2,1}} \leq \|\|V\|_{L^{3/2,1}}|\lambda| \cdot \||x|^{-1} \ast f\|_{L^\infty} + \|V\|_{L^2} \|x|^{-2} \ast f\|_{L^{6,2}}
\leq \|\|V\|_{L^{3/2,1}}|\lambda| \cdot \|f\|_{L^{3/2,1}} + \|V\|_{L^2} \|f\|_{L^2}
\leq (\|\|V\|_{L^{3/2,1}}|\lambda| + \|V\|_{L^2}) \|f\|_{L^{3/2,1}}
\]
(see (4.26)), while for the second term we have
\[\tag{4.29}\|VR_D(\lambda)f\|_{L^{3,1}} \leq \|\|V\|_{L^{3,1}}|\lambda| \cdot \|\|x|^{-1} \ast f\|_{L^\infty} + \|V\|_{L^{6,2}} \|x|^{-2} \ast f\|_{L^{6,2}}
\leq \|\|V\|_{L^{3,1}}|\lambda| \cdot \|f\|_{L^{3/2,1}} + \|V\|_{L^{6,2}} \|f\|_{L^2}
\leq (\|\|V\|_{L^{3,1}}|\lambda| + \|V\|_{L^{6,2}}) \|f\|_{L^{3/2,1}}
\]
where we have used (4.26) and the trivial inequality $\|f\|_{L^2} \leq \|f\|_{L^2}, \forall \gamma > 0.$

Summing up, we get
\[\tag{4.30}\|R_D(\lambda)VR_D(\lambda)f\|_{L^\infty} \leq C \cdot C(V)(\lambda)^2 \|f\|_{L^{3/2,1}}
\]
where the quantity
\[\tag{4.31}C(V) := \|\|V\|_{L^{3/2,1}} + \|V\|_{L^{3,1}} + \|V\|_{L^{6,2}} + \|V\|_{L^2} < \infty
\]
is finite by assumption (4.20) (see also the Appendix A).

The proof of (4.23) is similar: by (4.25) we get
\[
\|R^2_D(\lambda)VR_D(\lambda)f\|_{L^\infty} \leq C (|\lambda| \cdot \|1 \ast VR_Df\|_{L^\infty} + \|x|^{-1} \ast VR_Df\|_{L^\infty})
\leq C (|\lambda| \cdot \|VR_D(\lambda)f\|_{L^1} + \|VR_D(\lambda)f\|_{L^{3/2,1}}).
\]
We have already estimated the second term in (4.28), and for the first one we have
\[\tag{4.32}\|VR_D(\lambda)f\|_{L^1} \leq \|\|V\|_{L^{3/2}}|\lambda| \cdot \||x|^{-1} \ast f\|_{L^3} + \|V\|_{L^3} \|x|^{-2} \ast f\|_{L^{3/2}}
\leq (\|\|V\|_{L^{3/2}}|\lambda| + \|V\|_{L^3}) \|f\|_{L^1}
\leq (\|\|V\|_{L^{3/2}}|\lambda| + \|V\|_{L^3}) \|f\|_{L^{3/2,1}}
\]
and hence
\[\tag{4.33}\|R^2_D(\lambda)VR_D(\lambda)f\|_{L^\infty} \leq C \cdot C'(V)(\lambda)^2 \|f\|_{L^{3/2,1}}
\]
where the quantity
\[\tag{4.34}C'(V) := \|\|V\|_{L^{3/2}} + \|V\|_{L^{3/2,1}} + \|V\|_{L^3} + \|V\|_{L^2} < \infty
\]
is finite again by assumption (4.20).

Finally, the last estimate can be obtained as follows:
\[ \| R_D(\lambda) V R_D^2(\lambda) f \|_{L^\infty} \leq C (|\lambda| \cdot \|x|^{-1} \ast VR_D^2 f \|_{L^\infty} + \|x|^{-2} \ast VR_D^2 f \|_{L^\infty}) \]
\[ \leq C (|\lambda| \cdot \|VR_D(\lambda) f \|_{L^{3/2,1}} + \|VR_D(\lambda) f \|_{L^{3,1}}) . \]

Proceeding as above, we estimate
\[ \|VR_D^2(\lambda) f \|_{L^{3/2,1}} \leq \|V\|_{L^{3/2,1}} |\lambda| \cdot \|1 \ast f \|_{L^\infty} + \|V\|_{L^{3/2,1}} \|x|^{-1} \ast f \|_{L^\infty} \]
\[ \leq \|V\|_{L^{3/2,1}} 4(\lambda) (\|f\|_{L^1} + \|f\|_{L^{3/2,1}}) \]
\[ \leq \|V\|_{L^{3/2,1}} 4(\lambda) \|f\|_{L^{3/2,1}} \]
and
\[ \|VR_D^2(\lambda) f \|_{L^{3,1}} \leq \|V\|_{L^{3,1}} |\lambda| \cdot \|1 \ast f \|_{L^\infty} + \|V\|_{L^{3,1}} \|x|^{-1} \ast f \|_{L^\infty} \]
\[ \leq \|V\|_{L^{3,1}} 4(\lambda) (\|f\|_{L^1} + \|f\|_{L^{3/2,1}}) \]
\[ \leq \|V\|_{L^{3,1}} 4(\lambda) \|f\|_{L^{3,1}} \]
whence
\[ \|R_D(\lambda) VR_D^2(\lambda) f \|_{L^\infty} \leq C \cdot C''(V) \|\lambda\|^2 \|f\|_{L^{3,1}} \]
where the quantity
\[ C''(V) := \|V\|_{L^{3/2,1}} + \|V\|_{L^{3,1}} < \infty \]
is finite by assumption (4.20).

Remark 4.2. The same remark concerning the simpler version of the spectral formula (4.12) and the integration by parts formula (4.13) applies also to the Dirac resolvent, with obvious modifications in the proof.

5. Proof of Theorem 1.1

Let \((\varphi_j)_{j=0,1,...}\) be a standard Paley-Littlewood partition of the unity, with the properties
\[ \varphi_j(\lambda) = \varphi_0(2^{-j}\lambda), \quad \varphi_0 + \sum_{j \geq 1} \varphi_j = 1, \]
for a suitable \(\varphi_0 \in C^\infty_0\). We consider the Cauchy problem
\[ \begin{cases} u_t(t, x) - \Delta u(t, x) + W(x, D)u = 0 \\ u(0, x) = 0, \quad u_t(0, x) = \varphi_j(\sqrt{-\Delta + W})g(x), \end{cases} \]
The solution can be represented using the spectral formula as follows:
\[ u(t, x) = \frac{1}{2\pi i} \int_0^{+\infty} \varphi_j(\sqrt{\lambda}) \sin(t\sqrt{\lambda}) R(\lambda) g d\lambda, \]
and after an integration by parts (see Remark 4.1) this gives
\[ u(t, x) = \frac{C}{t} \int_0^{+\infty} \cos(t\sqrt{\lambda}) \left[ \partial_\lambda \varphi_j(\sqrt{\lambda}) R(\lambda) g + \varphi_j(\sqrt{\lambda}) \partial\lambda R(\lambda) g \right] d\lambda. \]

Thus, recalling estimates (4.1) and (4.2), we have
\[ |u(t, x)| \leq \frac{C}{t} \|x\|_{L^2} 1^{1/2} g \|_{L^2} \int_0^{+\infty} \left( |\partial_\lambda \varphi_j(\sqrt{\lambda})| + \left( 1 + \frac{1}{\sqrt{\lambda}} \right) |\varphi_j(\sqrt{\lambda})| \right) d\lambda \]
and a change of variables \(\lambda = 2^{2j} \mu\) in the integral gives immediately
\[ |u(t, x)| \leq \frac{C}{t} 2^{2j} \|x\|_{L^2} 1^{1/2} g \|_{L^2} \]
with some constant \(C\) independent of \(j\) and \(g\).
If we now define as usual
\[ \tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad \varphi_{-1} = 0, \]
so that \( \varphi_j = \varphi_j \tilde{\varphi}_j \), we see that the Cauchy problem (5.2) can be written equivalently
\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) + W(x, D)u &= 0 \\
u(0, x) &= 0, \quad u_t(0, x) = \varphi_j(\sqrt{-\Delta W})\tilde{\varphi}_j(\sqrt{-\Delta W})g(x),
\end{align*}
\]
hence our estimate (5.5) implies also the estimate
\[
|u(t, x)| \leq C t^2 \sum_{j \geq 0} 2^j \| (\langle x \rangle w^{1/2})_{\tilde{\varphi}_j(\sqrt{-\Delta W})g} \|_{L^2}.
\]
Finally, consider the original Cauchy problem (1.6), and decompose \( g \) as a sum
\[
g = \sum_{j \geq 0} \varphi_j(\sqrt{-\Delta W})g(x).
\]
By estimate (5.7) we obtain easily estimate (1.9).
\[
|u(t, x)| \leq C t \sum_{j \geq 0} 2^j \| (\langle x \rangle w^{1/2})_{\varphi_j(\sqrt{-\Delta W})g} \|_{L^2}.
\]
The computations in the case of initial data of the form
\[
u(0, x) = f, \quad u_t(0, x) = 0
\]
are completely analogous, and we thus obtain estimate (1.12).

Remark 5.1. In view of the application to the Dirac system, the following remark will be useful. If the initial datum \( g \) has the form
\[
g = (-\Delta W)^s h
\]
for some \( s > 0 \), a direct application of estimate (5.8) would give only
\[
|u(t, x)| \leq C t \sum_{j \geq 0} 2^j \| (\langle x \rangle w^{1/2})_{\varphi_j(\sqrt{-\Delta W})(-\Delta W)^s h} \|_{L^2}.
\]
Actually, if we go back to the spectral formula (5.4), we see that the solution can be written
\[
u(t, x) = C t \int_0^{+\infty} \lambda^{s/2} \cos(t \sqrt{\lambda}) \left[ \partial_\lambda \varphi_j(\sqrt{\lambda}) R(\lambda) h + \varphi_j(\sqrt{\lambda}) \partial_\lambda R(\lambda) h \right] d\lambda.
\]
with an additional factor \( \lambda^{s/2} \). Thus, proceeding as above, we arrive at the simpler estimate
\[
|u(t, x)| \leq C t \sum_{j \geq 0} 2^{j(s+2)} \| (\langle x \rangle w^{1/2})_{\varphi_j(\sqrt{-\Delta W})h} \|_{L^2}.
\]
We now prove estimate (1.11) under the stronger assumption (1.10) on the potential \( W(x, D) \). Consider first the case of initial data of the form
\[
u(0, x) = 0, \quad u_t(0, x) = g.
\]
We can write \( g \) as follows:
\[
g = (1 - \Delta + W)^{-1-\epsilon} (1 - \Delta + W)^{1+\epsilon} g
\]
for some fixed \( \epsilon > 0 \). Then the solution \( u \) can be represented as
\[
u(t, x) = \frac{1}{2\pi i} \int_0^{+\infty} \psi(\sqrt{\lambda}) \sin(t \sqrt{\lambda}) \lambda^{s/2} \left[ \partial_\lambda \varphi_j(\sqrt{\lambda}) R(\lambda) h + \varphi_j(\sqrt{\lambda}) \partial_\lambda R(\lambda) h \right] d\lambda
\]
where
\[
h = (1 - \Delta + W)^{1+\epsilon} g, \quad \psi(\sqrt{\lambda}) = (1 + \lambda)^{1+\epsilon}.
Proceeding as above, after an integration by parts we arrive at
\[ |u(t, x)| \leq \frac{C}{t} \| \langle x \rangle w_\beta^{1/2} h \|_{L^1} \int_0^{+\infty} \left((1 + \lambda)^{-1-\epsilon} + (1 + \lambda)^{-2-\epsilon}\right) d\lambda \]
and hence
\[ (5.13) \quad |u(t, x)| \leq \frac{C}{t} \| \langle x \rangle w_\beta^{1/2} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2} \leq \frac{C}{t} \| \langle x \rangle^{3/2+\epsilon} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2}. \]
To conclude the proof of the Theorem, it remains to show that
\[ (5.14) \quad \| \langle x \rangle^{3/2+\epsilon} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2} \leq \| \langle x \rangle^{3/2+\epsilon} g \|_{H^{2+\epsilon}}. \]
We start from the inequality
\[ \| \langle x \rangle^s (1 - \Delta + W) f \|_{L^2} \leq \| \langle x \rangle^s f \|_{H^{2\epsilon}} \]
which is obviously valid for any \( s \geq 0 \). By a standard complex interpolation argument, interpolating with the trivial inequality
\[ \| \langle x \rangle^s f \|_{L^2} \leq \| \langle x \rangle^s f \|_{H^{2\epsilon}} \]
we obtain that
\[ \| \langle x \rangle^s f \|_{L^2} \leq \| \langle x \rangle^s f \|_{H^{2\epsilon}} \]
for all \( 0 \leq \epsilon \leq 1 \) and all \( s \geq 0 \). This implies
\[ (5.15) \quad \| \langle x \rangle^s (1 - \Delta + W)^{1+\epsilon} f \|_{L^2} \leq \| \langle x \rangle^s (1 - \Delta + W) f \|_{H^{2\epsilon}} \leq \| \langle x \rangle^s f \|_{H^{2+2\epsilon}} + \| \langle x \rangle^s W f \|_{H^{2+2\epsilon}}. \]
The last term is of the form
\[ (5.16) \quad \| \langle x \rangle^s W(x, D) f \|_{H^{2\epsilon}} \leq \| \langle x \rangle^s a(x) D f \|_{H^{2\epsilon}} + \| \langle x \rangle^s b(x) f \|_{H^{2\epsilon}}; \]
in order to estimate it, we recall the Kato-Ponce inequality (see [21])
\[ (5.17) \quad \| \langle D \rangle^q (uv) \|_{L^p} \leq C \| \langle D \rangle^q v \|_{L^p} \| u \|_{L^2} + C \| v \|_{L^p} \| \langle D \rangle^q u \|_{L^2} \]
which is valid for all \( q \geq 0 \), \( p^{-1} = p_1^{-1} + p_2^{-1} = p_3^{-1} + p_4^{-1} \). With the choices \( v(x) = a(x), w(x) = \langle x \rangle^s D f(x), q = 2\epsilon, p_1 = p_3 = \infty \) and \( p_2 = p_4 = 2 \), we obtain
\[ \| \langle D \rangle^{2\epsilon} \langle x \rangle^s a(x) D f \|_{L^2} \leq C \| \langle D \rangle^{2\epsilon} a \|_{L^\infty} \| \langle x \rangle^s f \|_{L^2} + C \| a \|_{L^\infty} \| \langle D \rangle^{2\epsilon} \langle x \rangle^s D f \|_{L^2}. \]
Now it is clear that
\[ \| \langle D \rangle^{2\epsilon} \langle x \rangle^s D f \|_{L^2} \leq C \| \langle x \rangle^s f \|_{H^{1+2\epsilon}}. \]
( use again complex interpolation between the cases \( \epsilon = 0 \) and \( \epsilon = 1 \) ) and in conclusion we obtain
\[ \| \langle D \rangle^{2\epsilon} \langle x \rangle^s a(x) D f \|_{L^2} \leq C \| \langle D \rangle^{2\epsilon} a \|_{L^\infty} \| \langle x \rangle^s f \|_{H^{1+2\epsilon}}. \]
Here we have used the simple fact that
\[ \| a \|_{L^\infty} \leq C \| \langle D \rangle^{2\epsilon} a \|_{L^\infty}. \]
The corresponding estimate for the electric term is analogous (actually simpler):
\[ \| \langle D \rangle^{2\epsilon} \langle x \rangle^s b(x) f \|_{L^2} \leq C \| \langle D \rangle^{2\epsilon} b \|_{L^\infty} \| \langle x \rangle^s f \|_{H^{2\epsilon}}. \]
Recalling now (5.15) and (5.16) we conclude the proof of estimate (1.11).

On the other hand, when the data are of the form
\[ u(0, x) = f, \quad u_t(0, x) = 0 \]
the computations are completely analogous and we obtain estimate (1.14) under the stronger assumptions (1.13) on the coefficients.
6. Proof of Theorem 1.2

Remark 6.1. We notice that Theorem 1.1 (and Remark 1.3) can be trivially extended to a system of wave equations of the form
\begin{equation}
\frac{\partial^2 u}{\partial t^2} - \left( \nabla + iA(x) \right)^2 u + B(x) u = 0
\end{equation}
where $u(t, x)$ is a $\mathbb{C}^N$ valued function and $A_1(x)$, $A_2(x)$, $A_3(x)$, $B(x)$ are $\mathbb{C}^{N \times N}$ matrices whose coefficients satisfy the assumptions of the Theorem. The resulting dispersive estimates have exactly the same form as in the scalar case.

Consider now the Cauchy problem
\begin{equation}
\begin{cases}
 i u_t - D u - V(x) u = 0 \\
 u(0, x) = f(x).
\end{cases}
\end{equation}
If we apply to the perturbed Dirac system the operator $i\partial_t + D + V$ we obtain that $u$ is also a solution of a $4 \times 4$ system of perturbed wave equations of the form (6.1) with
\begin{equation}
A_j(x) = -\frac{1}{2}(\alpha_j V(x) + V(x) \alpha_j),
\end{equation}
\begin{equation}
B(x) = D V(x) + V(x)^2 + A_1^2 + A_2^2 + A_3^2 + i \sum \partial_j A_j
\end{equation}
and initial data
\begin{equation}
u(0, x) = f, \quad u_t(0, x) = i^{-1}(D + V)f.
\end{equation}
Note that the perturbed operator
\begin{equation} -\Delta_W = -(\nabla + iA(x))^2 + B(x) \end{equation}
is exactly the square of the operator $D + V$:
\begin{equation} -\Delta_W = (D + V)^2 \end{equation}
and hence the initial data for (6.1) can be written
\begin{equation} u(0, x) = f, \quad u_t(0, x) = i^{-1}(-\Delta_W)^{1/2}f. \end{equation}
We are in position to apply to the solution $u$ the estimates already proved in Theorem 1.1; keeping Remark 5.1 into account, we arrive easily at the estimate
\begin{equation} \|u(t, x)\| \leq C t \sum_{j \geq 0} 2^{3j} \|\langle x \rangle \frac{1}{2} \varphi_j(D + V) f\|_{L^2}, \end{equation}
provided the coefficients $a_j(x)$ and $b(x)$ satisfy the assumptions (1.8). Recalling the explicit form (6.3) of the coefficients in terms of $V(x)$, we see that $V$ must satisfy the conditions
\begin{equation} |V(x)| \leq \frac{C_0}{|x| \langle (|x| \log |x| + 1)\beta \rangle}, \end{equation}
from the magnetic term, and
\begin{equation} |V(x)| + |DV(x)| \leq \frac{C_0}{|x|^2 (|x| \log |x| + 1)\beta}, \end{equation}
from the electric term, for some $\beta > 1$ and some small constant $C_0$. Summing up, we obtain that (6.9) holds under assumption (1.17).

The estimate in terms of the Sobolev norm can be obtained in exactly the same way as for the perturbed wave equation. Indeed, proceeding as in (5.13) we arrive at the estimate
\begin{equation} \|u(t, x)\| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\epsilon} (-\Delta_W)^{3/2+\epsilon} f\|_{L^2}. \end{equation}
The same arguments used at the end of Section 5 give here

\[ |u(t, x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\varepsilon} f\|_{H^{3+2\varepsilon}} \]

provided

\[ \langle D \rangle^{1+2\varepsilon} A_j \in L^\infty, \quad \langle D \rangle^{1+2\varepsilon} B \in L^\infty, \]

which is implied by

\[ \langle D \rangle^{2+2\varepsilon} V \in L^\infty. \]

7. Proof of Theorem 1.3

By exploiting the connection between the massless Dirac and the wave equation, it is easy to obtain an optimal dispersive estimate in the unperturbed case. Indeed, let \( u(t, x) \) be a smooth solution of the free massless Dirac equation

\[ iu_t(t, x) = D u(t, x) \]

with initial data

\[ u(0, x) = f(x). \]

Recall now the identity

\[ (i\partial_t + D)(i\partial_t - D) = (\Delta - \partial^2_t)I_4; \]

if we apply the operator \( i\partial_t + D \) to the system (7.1) we immediately obtain that \( u \) solves the Cauchy problem for the wave equation

\[ u_{tt} - \Delta u = 0 \]

with initial data

\[ u(0, x) = f, \quad u_t(0, x) = i^{-1}D f. \]

Then, as a consequence of the well known decay estimates for solutions to the free wave equation (see e.g. [34]), we obtain

\[ |u(t, x)| \leq \frac{C}{t} \left( \|f\|_{\dot{B}_{1,1}^2} + \|D f\|_{\dot{B}_{1,1}^1} \right) \]

and hence

\[ |u(t, x)| \leq \frac{C}{t} \|f\|_{\dot{B}_{1,1}^2}. \]

Here \( \dot{B}_{1,1}^s \) is the homogeneous Besov space, with norm

\[ \|v\|_{\dot{B}_{1,1}^s} = \sum_{j \in \mathbb{Z}} 2^{js} \|\phi_j(\sqrt{-\Delta})v\|_{L^1} \]

where \( \phi_j \) now is a homogeneous Paley-Littlewood sequence, i.e., fixed a test function \( \psi(r) \in C_0^\infty \) such that \( \psi(r) = 1 \) for \( r < 1 \), \( \psi(r) = 0 \) for \( r > 2 \), we have \( \phi_j(r) = \psi(2^{-j+2}r) - \psi(2^{-j+1}r) \) for all \( j \in \mathbb{Z} \).

The proof of Theorem 1.3 follows the same lines as the proof of Theorem 1.1. Consider the Cauchy problem with frequency truncated data

\[ \begin{cases} iu_t(t, x) = D_V u(t, x) \\ u(0, x) = \varphi_j(D_V)f. \end{cases} \]

where \( (\varphi_j(\lambda))_{j=0,1,...} \) is the standard Paley-Littlewood partition of the unity defined in (5.1). By means of spectral formula, we can represent the solution of (7.4) as

\[ u(t, x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \varphi_j(\lambda)e^{i\lambda t} \mathcal{I}[R_V(\lambda)] f \ d\lambda. \]

Using the identity

\[ R_V(\lambda) = R_D - R_D V R_D (I + V R_D)^{-1}, \]
which is valid thanks to Corollary 3.7, we can split the integrals in (7.5) into two terms, the first one containing the contribution of the free resolvent $R_D$ and the second one containing the contribution of the operator $R_D V R_D (I + VR_D)^{-1}$. The first term

$$A := \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \Im \{ R_D(\lambda) \} f_d \lambda$$

was estimated above (see (7.3)); it remains to estimate the term

(7.7) $$B = - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \Im \{ Q(\lambda) \} f_d \lambda,$$

where

$$Q(\lambda) := R_D(\lambda) V R_D(\lambda) (I + VR_D(\lambda))^{-1}.$$

After an integration by parts, we obtain

(7.8) $$B = - \frac{1}{2\pi i} \left[ \int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \frac{\partial}{\partial \lambda} \Im(Q(\lambda)) f_d \lambda + \int_{-\infty}^{+\infty} \varphi_j'(\lambda) e^{i\lambda t} \Im [Q(\lambda)] f_d \lambda \right];$$

an explicit computation shows that

$$\frac{\partial Q}{\partial \lambda} = R_D^2 V R_D(I_4 + VR_D)^{-1} + R_D VR_D^2 (I_4 + VR_D)^{-1}$$

$$+ R_D VR_D (I_4 + VR_D)^{-1} VR_D^2 (I_4 + VR_D)^{-1}.$$ Now we can apply Lemma 4.2: under assumption (1.21), estimates (4.21), (4.22) and (4.23) are satisfied, and the Lemma gives

(7.9) $$\| Q(\lambda) f \|_{L^s} \leq C \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2},$$

(7.10) $$\left\| \frac{\partial}{\partial \lambda} Q(\lambda) f \right\|_{L^s} \leq C \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2},$$

for some $C > 0$. Using (7.9) and (7.10) in (7.8) we arrive at the estimate

$$|B| \leq \frac{C}{t} \| f \|_{L^{3/2+\epsilon}} \left[ \int_{-\infty}^{+\infty} (\langle \lambda \rangle^3 |\varphi_j(\lambda)| + \langle \lambda \rangle^2 |\varphi_j'(\lambda)|) d\lambda \right].$$

Recalling that $\varphi_j(\lambda) = \phi_0(2^{-j} \lambda)$, after a change of variables $2^{-j} \lambda = \mu$ we easily obtain

(7.11) $$|B| \leq \frac{C}{t} 2^{4j} \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2}.$$

From this point on, we can proceed as in the proof of Theorem 1.1 and complete the proof of Theorem 1.3.

**Appendix A. Lorentz Spaces**

For the convenience of the reader, we recall here the definitions and the main properties of the Lorentz spaces $L^{p,q}$, in view of the applications needed in the proof of our results.

For any measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and any $s \geq 0$ we define the upper-level $E^f_s$ as the set

$$E^f_s := \{ x : |f(x)| > s \}.$$ The non-increasing rearrangement of $f$ is then the function

$$f^*(t) := \inf \{ s > 0 : |E^f_s| \leq t \}, \quad t \in (0, +\infty).$$

It is also useful to consider the average of $f^*$ defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(r) \, dr.$$
The standard definition of the Lorentz spaces is the following:

**Definition A.1.** For any $1 \leq p < \infty$ and $1 \leq q \leq \infty$ we define the quasinorm $\|f\|_{L^{p,q}}$ as follows:

\[
\|f\|_{L^{p,q}} = \begin{cases} 
\left[ \int_0^\infty (t^{1/p} f^*(t))^{q \frac{dt}{t}} \right]^{1/q}, & 1 \leq q < \infty \\
\sup_{t>0} t^{1/p} f^*(t), & q = \infty.
\end{cases}
\]

When $p \neq 1$, if we replace $f^*$ with $f^{**}$ in the above definitions we obtain an equivalent quasinorm which is actually a norm (see [3], [9]). The Lorentz space $L^{p,q}$ is defined by

\[
L^{p,q} = \{ f : \|f\|_{L^{p,q}} < \infty \}.
\]

Moreover we define

\[
L^{1,1} := L^1, \quad L^{\infty,\infty} = L^\infty.
\]

The spaces $L^{q,q}$ for $1 \leq q < \infty$ are usually left undefined (although $L^{\infty,1}$ is defined in [9] as the closure of $L^\infty$ compactly supported functions in the $L^\infty$ norm).

With the above definitions, one obtains the elementary properties

\[
L^{p,p} = L^p, \quad 1 \leq p \leq \infty;
\]

\[
L^{p,q_1} \subseteq L^{p,q_2}, \quad 1 < p < \infty, \quad 1 \leq q_1 \leq q_2 \leq \infty
\]

(with continuous embedding). When the second index is $\infty$ we obtain the weak Lebesgue spaces (Marcinkiewicz spaces):

\[
L^{p,\infty} = L^p_w, \quad 1 \leq p \leq \infty.
\]

Moreover, the Lorentz spaces can be obtained by an equivalent construction using real interpolation:

\[
L^{p,q} = (L^{p_0}, L^{p_1})_{\theta,q}, \quad p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}
\]

provided

\[
p_0 < p_1, \quad p_0 < q \leq \infty, \quad 0 < \theta < 1.
\]

An alternative characterization of the Lorentz norm can be given using the so-called atomic decomposition:

**Lemma A.1.** Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function and let $1 \leq p < \infty$, $1 \leq q \leq \infty$; then $f \in L^{p,q}$ if and only if there exist a sequence of sets $(E_j)_{j \in \mathbb{Z}}$ and a sequence of numbers $a = (a_j)_{j \in \mathbb{Z}}$ such that $|E_j| = O(2^j)$, $a \in l^q$ and the following estimate

\[
|f(x)| \leq C \sum_{j \in \mathbb{Z}} a_j 2^{-j/p} \chi_{E_j}(x)
\]

holds, for some $C > 0$.

It is possible to see that the best constant $C$ in (A.3) is equivalent to the Lorentz norm of the function $f$.

The most useful properties of Lorentz spaces are the H"older and Young inequalities, which extend the classical ones for Lebesgue spaces. These were originally proved by O’Neill in [27]. We collect them in the following theorems:

**Theorem A.2** (H"older inequality). Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. The following estimates hold:

- if $p_1, p_2, p \in [1, \infty]$, $q_1, q_2, q \in [1, \infty]$, then

\[
\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \quad 1 > p_1^{-1} + p_2^{-1} = p^{-1}, \quad q_1^{-1} + q_2^{-1} \geq q^{-1};
\]

- if $p_1, p_2 \in [1, \infty]$, $q_1, q_2 \in [1, \infty]$, then

\[
\|fg\|_{L^1} \leq C\|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \quad p_1^{-1} + p_2^{-1} = 1, \quad q_1^{-1} + q_2^{-1} \geq 1.
\]
We remark that the above statement does not cover the trivial inequality
\[ (A.6) \quad \|fg\|_{L^{p,q}} \leq \|f\|_{L^\infty} \|g\|_{L^{p,q}} \]
which is easily proved to be true for all cases when \( L^{p,q} \) is defined.

**Theorem A.3** (Young inequality). Let \( f \in L^{p_1,q_1} \), \( g \in L^{p_2,q_2} \). Then the following estimates hold:

\[ (A.7) \quad \|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \]

- if \( p_1, p_2, p \in ]1, \infty[ \), \( q_1, q_2, q \in ]1, \infty[ \), then
  \[ p_1^{-1} + p_2^{-1} = 1 + p^{-1}, \quad q_1^{-1} + q_2^{-1} = q^{-1}; \]

- if \( p_1, p_2 \in ]1, \infty[ \), \( q_1, q_2 \in ]1, \infty[ \), then
  \[ p_1^{-1} + p_2^{-1} = 1, \quad q_1^{-1} + q_2^{-1} \geq 1. \]

As before, we remark that the above statement does not cover the inequality
\[ (A.9) \quad \|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \]
which is easily seen to be true in all cases when \( L^{p,q} \) is defined (e.g., by real interpolation).

We conclude this section by studying the weight functions \( w_\beta(x) = |x|(|\log |x|| + 1)^\beta \), with \( \beta > 1 \) which plays a crucial role in our results; in the following proposition we determine precisely to which Lorentz the powers \( w_\beta^{-s} \) belong.

**Proposition A.4.** For any \( s > 0 \), \( q \in ]1, \infty[ \) we have \( w_\beta^{-s} \in L^{n/s,q} \), provided \( \beta > 1/sq \).

**Proof.** We will use the equivalent Lorentz norm (A.3). For any \( j \in \mathbb{Z} \) consider the ball \( B_j := B_{2^j/n} = \{ x : |x| \leq 2^j/n \} \) and the rings \( E_j := B_{j+1} \setminus B_j \); it is clear that \( |E_j| = C_n 2^j \), where \( C_n \) depends only on the dimension \( n \). Then, for all \( x \in \mathbb{R}^n \) we have the estimate
\[ |w_\beta^{-s}(x)| = \left| \sum_{j \in \mathbb{Z}} \frac{1}{|x|^s(|\log |x|| + 1)^\beta} \chi_{E_j}(x) \right| \leq \sum_{j \in \mathbb{Z}} (|j| \log 2 + 1)^{-\beta s} 2^{-js/n} \chi_{E_j}(x). \]

The proof is concluded by the remark that the sequence \( a_j = (|j| \log 2 + 1)^{-\beta s} \) is in \( l^q \) if and only if \( \beta > 1/sq \).

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