Emergent Haldane phase in an alternating bond $Z_3$ parafermion chain

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The Haldane phase represents one of the most important symmetry protected states in modern physics. This state can be realized using spin-1 and spin-$\frac{1}{2}$ Heisenberg models and bosonic particles. Here we explore the emergent Haldane phase in an alternating bond $Z_3$ parafermion chain, which is different from the previous proposals from fundamental statistics and symmetries. We show that this emergent phase can also be characterized by a modified long-range string order, as well as four-fold degeneracy in the ground state energies and entanglement spectra. This phase is protected by both the charge conjugate and parity symmetry, and the edge modes are shown to satisfy parafermionic statistics, in which braiding of the two edge modes yields a $\frac{2\pi}{3}$ phase. This model also supports rich phases, including topological ferromagnetic parafermion (FP) phase, trivial paramagnetic parafermion phase, classical dimer phase and gapless phase. The boundaries of the FP phase are shown to be gapless and critical with central charge $c = 4/5$. Even in the topological FP phase, it is also characterized by the long-range string order, thus we observe a drop of string order across the phase boundary between the FP phase and Haldane phase. These phenomena are quite general and this work opens a new way for finding exotic topological phases in $Z_k$ parafermion models.

Topological phases and associated phase transitions beyond Landau paradigm of phase transition have been a major topic in modern physics [11-13]. These transitions are characterized by integer numbers in topological insulators [4, 5], or topological orders in symmetry protected topological phases [2, 6]. These phases can be used to realize exotic excitations with Abelian or non-Abelian statistics, which are building blocks for topological quantum computation. Along this line, the self-Hermitian Majorana zero modes [7] have been realized in experiments by several groups [8-13]. Moreover, the symmetry protected Haldane phase has been proposed to be realized in spin-1 and spin-$\frac{1}{2}$ Heisenberg models [14, 17], and strong interacting bosonic models [24-27].

Here we are interested in the realization of the Haldane phase using $Z_3$ parafermions [28-35], which are totally different from the above-mentioned models from their fundamental statistics and symmetries. We consider an alternating bond $Z_3$ parafermion model, which can be realized in superconductor and $\nu = \frac{2}{3}$ FQH hybrid structure [33, 36]. This Haldane phase is characterized by nonlocal string order and four-fold degeneracy in both ground state energies and entanglement spectra. We map out the whole phase diagram and find other exotic phases: ferromagnetic parafermion (FP) phase, paramagnetic parafermion (PP) phase, dimer phase and critical gapless phase. The boundaries for the FP phase are critical with central charge $c = \frac{4}{5}$. The edge modes in the Haldane phase exhibit fractional Abelian braiding, which are protected by charge conjugate symmetry and parity symmetry. This kind of emergent phenomena are rather general in parafermion models, thus we open a new avenue in searching of exotic symmetry protected topological phases in parafermion models.

Model. We consider the following alternating bond $Z_3$ parafermion chain ($\omega^3 = 1$),

$$H = -\omega^2 \sum_{j=1}^{L} J_j(\delta) \alpha_{2j}^\dagger \alpha_{2j+1} + h \alpha_{2j-1}^\dagger \alpha_{2j} + \text{h.c.} \tag{1}$$

where $J_j(\delta) = 1 + (-1)^j \delta$ and $\alpha_j$ are parafermions satis-
fying \(\alpha_i^3 = 1\), \(\alpha_i^1 = \alpha_i^2\) and \(\alpha_i \alpha_j = \alpha_j \alpha_i \omega^{\text{sgn}(i-j)}\). We propose to realize this model using the setup in Fig. 1 in superconductor and FQH hybrid structure \[39\], in which the alternating bonds are controlled by the distance between the superconducting islands. This model is related to the following Z\(_2\) clock model via Jordan-Wigner transformation: \(\alpha_{2j-1} = \prod_{k \leq j-1} \tau_k \sigma_j\) and \(\alpha_{2j} = \omega \sigma_j \prod_{k \leq j} \tau_k\), which yields \[33\] \[36\],

\[
H = -\sum_{i=1}^{L} [1 + (-1)^j \delta] \sigma_i^1 \sigma_{i+1} - h \sum_i \tau_i + \text{h.c.} \tag{2}
\]

We see that the second term plays the similar role as the magnetic field in Z\(_2\) spin models, thus \(h\), for convenience, is termed as Zeeman field. In above, the operators satisfy, \(\sigma_i^3 = \tau_i^3 = 1\), \(\sigma_i \tau_i = \omega \sigma_i \tau_i\), \(\sigma_i^1 = \sigma_i^2\) and \(\tau_i^1 = \tau_i^2\); all operators commute between different sites \[28\].

The Z\(_2\) symmetry in Heisenberg spin models is absent here, which is one critical difference between the Z\(_2\) spin models and Z\(_k\) parafermion models. However, this symmetry, can be recovered by a projection from Z\(_3\) to Z\(_2\) model, when \(h\) is negative enough, in which case the highest state in each site is unoccupied. In this condition, the above model may be projected to the following Z\(_2\) spin model to the leading term,

\[
H = -\sum_i [1 + (-1)^j \delta] s_i^1 s_{i+1}, \tag{3}
\]

where \(s_i = s_i^+ - is_i^z\), with \(s_i^z\), are Pauli matrices. This model has been investigated in literatures \[16\] \[40\] \[41\], which supports Haldane phase when \(\delta > 0\) and trivial dimer phase when \(\delta < 0\). We will show that this topological phase can be realized even with modest Zeeman field \(h\). This limiting case enables us to understand how the Haldane phase and related symmetries can emerge from the Z\(_3\), or even Z\(_k\) parafermions.

**Phase diagram.** We employ the density matrix renormalization group (DMRG) method implemented by ITENSOR project \[42\] and exact diagonalization (ED) method to understand the phase diagram in Fig. 2.

Firstly, we look at \(\delta = 0\), which supports several different critical points. The point at \(h = J\) is self-dual \[28\] \[43\], via \(\mu_j = \prod_{k \leq j} \tau_k\) and \(\nu_j = \sigma_j^1 \sigma_{j+1}^1\), which separate the three-fold degenerate FP phase from the PP phase. This boundary was studied in literatures \[28\] \[37\]. When \(h < 0\), we find another critical point at \(h_c = -4.6\). When \(h < h_c\), the model is equivalent to free fermion model or XX spin model, which is critical with central charge \(c = 1\) \[39\] \[44\] \[47\]. This point was unveiled in Ref. \[48\]. On the other hand when \(h = 0\), the corresponding clock model is purely classical, which supports critical points at \(\delta = \pm 1\). At these two points, the chain is divided into short segments with length \(L' = 4\), giving rise to an infinite-fold degenerate model. Similar infinite-fold degeneracy can be found when \(|\delta| > 1\) with eigenvectors as \(|m_1 m_2 m_3 m_4 \cdots\rangle\) with \(m_i \neq m_{i+1}\). This picture is similar to that in the extended parafermion model \[48\].

The line with \(\delta = \pm 1\) is essential for us to understand the phases in our model. We first focus on \(\delta = +1\). In this case the coupling between the parafermions is decoupled into the following segments,

\[
\alpha_1 - \alpha_2 \quad \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 \quad \alpha_7 - \alpha_8 = \alpha_9 - \alpha_{10} \cdots
\]

where the single and double bonds represent coupling strengths by \(h\) and \(J\), respectively. The two edge modes are fully decoupled from all the other sites. When \(h > 0\), the ground states for \(h \omega^2 \alpha_1^1 \alpha_2^1 + \text{h.c.} = -h(\tau^1 + \tau^1)\) is unique, corresponding to the PP phase. In contrast, when \(h < 0\), the ground states are two-fold degenerate. For this reason, it corresponds to the Haldane phase with four-fold degeneracy, taking into accounts the contribution from both ends. This picture is modified for the case with \(\delta = -1\), which is decoupled into segments,

\[
\alpha_1 - \alpha_2 \quad \alpha_3 - \alpha_4 \quad \alpha_5 - \alpha_6 \quad \alpha_7 - \alpha_8 \quad \alpha_9 - \alpha_{10} \cdots
\]

We only need to compute the eigenvalues of \(H_4 = -h \omega^2 \alpha_1^1 \alpha_2^1 - h \omega^2 \alpha_3^1 \alpha_4^1 - 2J \omega^2 \alpha_1^1 \alpha_3^1 + \text{h.c.}\). We always find the ground state of \(H_4\) to be unique, regardless of the sign of \(h\). When \(h \gg |J|\), it is \(|0 \otimes L\rangle\). For \(h \ll -|J|\), it is \(|\prod_{i}(2i+1)_{1}(2i+1)_{1} + (2i+1)(2i+1)_{2} + c(2i)(2i+1)_{0}\rangle\), where \(|0\rangle, |1\rangle, |2\rangle\) are eigenvectors of \(\tau\) with \(\tau^1|i\rangle = \omega^1|i\rangle\) and \(c = \frac{3}{4} \sqrt{36h^2 - 12h + 9 + 6h - 1} \approx 1/(3|h|) \ll 1\). This regime is termed as classical dimer phase, following Refs. \[16\] \[40\] \[41\], which is topological trivial.

**Emergent Haldane phase and topological transition.** Here we mainly focus on the properties of the emergent Haldane phase. From the simple picture at \(\delta = 1\), \(h < 0\), we see that the ground states are four-fold degenerate. We consider a more general case in Fig. 3a for a finite system with open boundary condition based on ED method.
The splitting between the fourth level and ground state, \( \delta E_{41} = E_4 - E_1 \), decreases exponentially to zero with the increasing of length \( L \), indicating of the exact four-fold degeneracy in infinite volume. This phase is protected by a finite energy gap between the fifth and ground state levels, i.e., \( \lim_{L \to \infty} \delta E_{51}(L) \) is finite. With close boundary condition \([39, 40]\), the edge modes are paired, leading to a unique ground state. In this case we perform the same analysis in Fig. 3, and show that the gap \( \delta E_{21} = E_2 - E_1 \) vanishes only at the phase boundaries. We show in the inset that at the phase boundary, \( \delta E_{a1} \propto 1/L \) for all \( a > 1 \) with finite \( L \), due to criticality.

We determine the phase boundaries for the FP phase using the ferromagnetic order \( \Delta = |\langle \sigma \rangle| \), which is realized in numerical simulation without \( \mathbb{Z}_3 \) symmetry restriction; see details in Ref. [48]. This method enables us to precisely determine the phase boundaries due to the sharp transitions from finite in FP phase to zero in all other phases, including that in Haldane phase and dimer phase. Notice that the case for \( \delta \) and \(-\delta\) give the same value for \( \Delta \), as expected, since these two cases can be made the same upon one lattice translation.

We now discuss the particular properties of the edge modes, which show some distinct features as compared with those from \( \mathbb{Z}_2 \) spin models. Let us again consider \( \delta = +1 \) and \( h < 0 \). In this case the left two parafermions, \( \alpha_1, \alpha_2 \), and right two parafermions, \( \alpha_{2L-1}, \alpha_{2L} \), are decoupled from the bulk \( H(\alpha_3, \alpha_4, \cdots, \alpha_{2L-3}, \alpha_{2L-2}) \). Taken the left two parafermions as an example, we may represent \( \alpha_1 = \sigma \) and \( \alpha_2 = \omega \sigma \tau \), then the Hamiltonian can be written as \( H = -h(\tau + \tau^\dagger) = -h \text{diag}(2, -1, -1) \).

Let us define left(right) projector \( P_L(P_r) = |1\rangle\langle 1| + |2\rangle\langle 2| = \text{diag}(0, 1, 1) \), which project any wave function to the lowest two eigenvectors of \( H \), then we can approximate the two edge modes as \( \alpha_1 \rightarrow \gamma_1 = P \alpha_1 P \). We find that \( \gamma_1 = \gamma_2 \), which are zero modes of the original Hamiltonian, i.e., \( \langle \gamma_1 | H | \gamma_1 \rangle = 0 \). The similar results can be found for the right zero modes for \( \gamma_{2L} = P \alpha_{2L} P \). These two modes satisfy fermionic relation at the same site, \( \gamma_1^\dagger \gamma_2^\dagger = \gamma_2 \gamma_1 = 0 \), and parafermionic commute relation between the two ends, \( \gamma_1 \gamma_{2L} = \omega \gamma_{2L} \gamma_1 \), which mark the important difference between our model and the previous fermionic edge modes [51]. If we denote this state as \( |1\rangle \), then we can construct three other zero modes as \( |2\rangle = \omega \gamma_{2L}^\dagger |1\rangle \), \( |21\rangle = \gamma_1^\dagger |11\rangle \), \( |22\rangle = \omega^2 \gamma_{2L}^\dagger |21\rangle \).

The zero modes are exact only in the limiting case of \( \delta = 1 \), \( h < 0 \). However, we expect these edge modes to be localized over a wide region, which extend to a distance of the order of one correlation length from the open ends[50]. In this case, the exact edge operators at the two ends have a finite overlap with \( \gamma_1 \) and \( \gamma_{2L} \). To this end we diagonalize the Hamiltonian with open boundary condition at \( h = -8 \), and calculate the matrix element of \( \gamma_{2L} \) between \( |22\rangle \) and \( |21\rangle \). This overlap depends only on the correlation length when \( L \) is large enough. For the result in Fig. 3, the correlation length \( \xi \sim 1 \) when \( \delta \to 1 \), thus the overlap is saturated even in a short chain.

**Long-range string order.** We define the string order as
\[
O_s = \lim_{|i-j| \to \infty} \langle \sigma_2 U_{2i+1} U_{2i+2} \cdots U_{2j-2} \sigma_{2j-1}^\dagger \rangle.
\]

which is generalized from the string order in previous Haldane phases [52, 53]. The center operator \( U_i \) in Eq. 4 is the generator of the charge conjugate symmetry, satisfying \( U_i \tau_i U_i^\dagger = \tau_i^\dagger \) and \( U_i \sigma_i U_i^\dagger = \sigma_i^\dagger \). This string order is not unique, nevertheless,
the one in Eq. 4 is enough to distinguish the topological states from the trivial phases. One should notice that due to the alternating bond strengths between the neighboring sites, the EE also exhibits oscillating behaviors, thus Eq. 5 should be fitted for the odd and even sites, respectively. With this technique, we find $c = 1$ in gapless phase, after breaking of infinite-fold degeneracy by the Zeeman field.

We provide more insight into the physics in these two topological phases from the entanglement spectra [50, 60–65], which is defined as $\rho_i \equiv -\ln(\rho_i)$. Here, $\rho_i$ is the eigenvalue of the reduced density matrix $\tilde{\rho}_A = \text{Tr}_B(\psi)(\psi|$, where $|\psi\rangle$ is the ground-state wave function, and $A, B$ are two partitions of the parafermion chain. As was unveiled in Ref. [50], the spectra $\xi_i$ is three-fold degenerate in FP phase. However, in Haldane phase, it is characterized by four-fold degeneracy [64]. Our results are presented in Fig. 4c, in which we used a closed parafermion chain [64, 65] with chain length $L = 80$ and the chain is cutted from the center for two partitions. We find that for the lowest eight eigenvalues, they are strictly four-fold degenerate in the Haldane phase. Nevertheless, for the higher spectra, their values are vanishing small, but no longer four-fold degenerate. The whole spectra are exactly four-fold degenerate in the limit when Zeeman field is negative enough. In FP phase, all the spectra are three-fold degenerate, in agreement with the previous observation [60]. In the PP phase, which is trivial, all spectra are not degenerate anymore.

Topological protection. We find that the Haldane phase is protected by both charge conjugate symmetry $C = \prod_i U_i K$, where $U_i$ was defined in Eq. 4 and $C^2 = +1$, and $Z_3$ parity $P = \prod_i \tau_i$, with $P^3 = 1$. The four-fold degeneracy will break down only for terms violating these two symmetries. Let us define two left(right) edge states, where $C|1\rangle = \omega|1\rangle$ and $C|2\rangle = \omega^2|2\rangle$. By projecting to these two edge modes, we find the projective representation: $\mathcal{P}_L = P_L \mathcal{P}_L P_L = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ and $\mathcal{C}_L = P_L \mathcal{C}_L P_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K$, which satisfy $[H, \mathcal{P}_L \mathcal{C}_L] = 0$ in the ground state subspace. We find that for any eigenvector $|\psi\rangle$ in the subspace, $\mathcal{P}_L \mathcal{C}_L |\psi\rangle$ is impossible to be identical to $|\psi\rangle$, since $(P_L \mathcal{C}_L)^2 = P_L \neq 1$ for this rea-
son, in each open end, the ground states should be two-fold degenerate, giving rise to the four edge modes $|i\rangle$, with parity $\omega^{i+2}$. This idea also applies well to other parafermion models.

**Conclusion.** To conclude, we investigate the emergent Haldane phase in the $Z_3$ alternating bond parafermion chain, which exhibits four-fold degeneracy in both ground states and entanglement spectra, and is characterized by nonzero long-range string order, generalized from the $Z_2$ spin models. Strikingly, we find that the FP phase is also characterized by nonzero string order and three-fold degeneracy in both ground states and entanglement spectra, which separate from the Haldane phase by a critical phase with central charge $c = \frac{3}{2}$. With our construction, the similar phases can also be found in other $Z_k$ parafermion chains, in regarding of the rather generality of emergent phenomena in these models. We expect the results in this work open a new way for finding of exotic topological phases in $Z_k$ parafermion models.

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[1] Xiao-Gang Wen, Quantum field theory of many-body systems (Oxford University Press on Demand, 2004).
[2] Xiao-Gang Wen, “Colloquium: Zoo of quantum-topological phases of matter,” Reviews of Modern Physics 89, 041004 (2017).
[3] Edward Witten, “Fermion path integrals and topological phases,” Reviews of Modern Physics 88, 035001 (2016).
[4] M Zahid Hasan and Charles L Kane, “Colloquium: topological insulators,” Rev. Mod. Phys. 82, 3045 (2010).
[5] Xiao-Liang Qi and Shou-Cheng Zhang, “Topological insulators and superconductors,” Rev. Mod. Phys. 83, 1057 (2011).
[6] Xiao-Gang Wen, “Topological orders in rigid states,” International Journal of Modern Physics B 4, 239–271 (1990).
[7] A Yu Kitaev, “Unpaired majorana fermions in quantum wires,” Physics-Uspekhi 44, 131 (2001).
[8] Jason Alicea, Yuval Oreg, Gil Refael, Felix Von Oppen, and Matthew PA Fisher, “Non-abelian statistics and topological quantum information processing in 1d wire networks,” Nat. Phys. 7, 412–417 (2011).
[9] Vincent Mourik, Kun Zuo, Sergey M Frolov, SR Plissard, EPAM Bakkers, and Leo P Kouwenhoven, “Signatures of majorana fermions in hybrid superconductor-semiconductor nanowire devices,” Science 336, 1003–1007 (2012).
[10] Anindya Das, Yuval Ronen, Yonatan Most, Yuval Oreg, Moty Heiblum, and Hadas Shtrikman, “Zero-bias peaks and splitting in an alinas nanowire topological superconductor as a signature of majorana fermions,” Nat. Phys. 8, 887–895 (2012).
[11] MT Deng, CL Yu, GY Huang, Marcus Larsson, Philippe Caroff, and HQ Xu, “Anomalous zero-bias conductance peak in a Nb-InSb Nanowire-Nb hybrid device,” Nano letters 12, 6414–6419 (2012).
[12] Qing Lin He, Lei Pan, Alexander L Stern, Edward C Burks, Xiaoyue Che, Gen Yin, Jing Wang, Biao Lian, Quan Zhou, Eun Sang Choi, et al., “Chiral majorana fermion modes in a quantum anomalous hall insulator-superconductor structure,” Science 357, 294–299 (2017).
[13] Hao Zhang, Chun-Xiao Liu, Sasa Gazibegovic, Di Xu, John A Logan, Guanzhong Wang, Nick van Loo, Jouri DS Bommer, Michiel WA de Moor, Diana Car, et al., “Quantized majorana conductance,” 10.1038/nature26142 (2017).
[14] P Duncan M Haldane, “Continuum dynamics of the 1-d heisenberg antiferromagnet: identification with the o(3) nonlinear sigma model,” Phys. Lett. A 93, 464–468 (1983).
[15] P Duncan M Haldane, “Nonlinear field theory of large-spin heisenberg antiferromagnets: semiclassically quantized solitons of the one-dimensional easy-axis néel state,” Phys. Rev. Lett. 50, 1153 (1983).
[16] Kazuo Hida, “Crossover between the haldane-gap phase and the dimer phase in the spin-\(\frac{1}{2}\) alternating heisenberg chain,” Phys. Rev. B 45, 2207 (1992).
[17] Kazuo Hida, “Ground-state phase diagram of the spin-\(\frac{1}{2}\) ferromagnetic-antiferromagnetic alternating heisenberg chain with anisotropy,” Phys. Rev. B 46, 8268 (1992).
[18] Emanuele G Dalla Torre, Erez Berg, and Ehud Altman, “Hidden order in 1d bose insulators,” Phys. Rev. Lett. 97, 260401 (2006).
[19] Erez Berg, Emanuele G Dalla Torre, Thierry Giamarchi, and Ehud Altman, “Rise and fall of hidden string order of lattice bosons,” Phys. Rev. B 77, 245119 (2008).
[20] Davide Rossini and Rosario Fazio, “Phase diagram of the extended bose-hubbard model,” New J. Phys. 14, 065012 (2012).
[21] GG Batrouni, RT Scalettar, VG Rousseau, and B Grémaud, “Competing supersolid and haldane insulator phases in the extended one-dimensional bosonic hubbard model,” Phys. Rev. Lett. 110, 265303 (2013).
[22] Satoshi Ejima, Florian Lange, and Holger Frohse, “Spectral and entanglement properties of the bosonic haldane insulator,” Phys. Rev. Lett. 113, 020401 (2014).
[23] Florian Lange, Satoshi Ejima, and Holger Frohse, “Anyonic haldane insulator in one dimension,” Phys. Rev. Lett. 118, 120401 (2017).
[24] Valentin Bois, Sylvain Capponi, Philippe Lecheminant, Marion Moliner, and Keisuke Totsuka, “Phase diagrams of one-dimensional half-filled two-orbital su (n) cold fermion systems,” Physical Review B 91, 075121 (2015).
[25] Sylvain Capponi, Philippe Lecheminant, and Keisuke Totsuka, “Phases of one-dimensional su (n) cold atomic fermi gasesfrom molecular luttinger liquids to topological phases,” Annals of Physics 367, 50–95 (2016).
[26] Luca Barbiero, Serena Fazzini, and Arianna Montorsi, “Non-local order parameters as a probe for phase tran-
sitions in the extended fermi-hubbard model,” The European Physical Journal Special Topics 226, 2697–2704 (2017).

[27] Masaya Nakagawa and Norio Kawakami, “Symmetry-protected topological phase transition in one-dimensional kondo lattice and its realization with ultracold atoms,” Physical Review B 96, 155133 (2017).

[28] Paul Fendley, “Parafermionic edge zero modes in Z_2-invariant spin chains,” J. Stat. Mech. Theory Exp. 2012, P11020 (2012).

[29] Adam S Jermyn, Roger SK Mong, Jason Alicea, and Paul Fendley, “Stability of zero modes in parafermion chains,” Phys. Rev. B 90, 165106 (2014).

[30] Ye Zhuang, Hitesh J Changlani, Norm M Tubman, and Taylor L Hughes, “Phase diagram of the Z_4 parafermionic chain with chiral interactions,” Phys. Rev. B 92, 035154 (2015).

[31] EM Stoudenmire, David J Clarke, Roger SK Mong, and Jason Alicea, “Assembling fibonacci anyons from a Z_3 parafermion lattice model,” Phys. Rev. B 91, 235112 (2015).

[32] Maissam Barkeshli and Xiao-Liang Qi, “Synthetic topological qubits in conventional bilayer quantum hall systems,” Phys. Rev. X 4, 041035 (2014).

[33] Roger SK Mong, David J Clarke, Jason Alicea, Netanel H Linhner, Paul Fendley, Chetan Nayak, Yuval Oreg, Ady Stern, Erez Berg, Kirill Shtengel, et al., “Universal topological quantum computation from a superconductor-abelian quantum hall heterostructure,” Phys. Rev. X 4, 011036 (2014).

[34] Abolhassan Vaezi and Maissam Barkeshli, “Fibonacci anyons from abelian bilayer quantum hall states,” Phys. Rev. lett. 113, 236804 (2014).

[35] Jason Alicea and Paul Fendley, “Topological phases with parafermions: theory and blueprints,” Annual Review of Condensed Matter Physics 7, 119–139 (2016).

[36] Abolhassan Vaezi, “Superconducting analogue of the parafermion fractional quantum hall states,” Phys. Rev. X 4, 031009 (2014).

[37] David J Clarke, Jason Alicea, and Kirill Shtengel, “Exotic non-abelian anyons from conventional fractional quantum hall states,” Nat. commun. 4, 1348 (2013).

[38] Ashley Milsted, Emilio Cobanera, Michele Burrello, and Gerardo Ortiz, “Commensurate and incommensurate states of topological quantum matter,” Phys. Rev. B 90, 195101 (2014).

[39] Yahya Alavirad, David Clarke, Amit Nag, and Jay D Sau, “Z_3 parafermionic zero modes without andreev backscattering from the 2/3 fractional quantum hall state,” Phys. Rev. Lett. 119, 217701 (2017).

[40] Yu-Chin Tzeng, Li Dai, Ming-Chiang Chung, Luigi Amico, and Leong-Chuan Kwek, “Entanglement convertibility by sweeping through the quantum phases of the alternating bonds xxz chain,” Sci. Rep. 6, 26453 (2016).

[41] Hai Tao Wang, Bo Li, and Sam Young Cho, “Topological quantum phase transition in bond-alternating spin-1/4 heisenberg chains,” Phys. Rev. B 87, 054402 (2013).

[42] ITensor library, version 1.2.0, http://itensor.org/.

[43] Wei Li, Shuo Yang, Hong-Hao Tu, and Meng Cheng, “Criticality in translation-invariant parafermion chains,” Phys. Rev. B 91, 115133 (2015).

[44] Ingo Peschel, “On the entanglement entropy for an XY spin chain,” J. Stat. Mech. Theory Exp. 2004, P12005 (2004).

[45] Paul Ginsparg, “Applied conformal field theory,” arXiv preprint hep-th/9108028 (1988).

[46] Pasquale Calabrese and John Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. Theory Exp. 2004, P06002 (2004).

[47] Frank Pollmann, Subroto Mukerjee, Ari M Turner, and Joel E Moore, “Theory of finite-entanglement scaling at one-dimensional quantum critical points,” Phys. Rev. Lett. 102, 255701 (2009).

[48] Shun-Yao Zhang, Hong-Ze Xu, Yue-Xin Huang, Guangcan Guo, Zhengwei Zhou, and Ming Gong, “Topological phase, supercritical point and emergent fermions in the extended parafermion chain,” arXiv preprint arXiv:1801.03269 (2018).

[49] A Alexandradinata, N Regnault, Chen Fang, Matthew J Gilbert, and B Andrei Bernevig, “Parafermionic phases with symmetry breaking and topological order,” Phys. Rev. B 94, 125103 (2016).

[50] Johannes Motruk, Erez Berg, Ari M Turner, and Frank Pollmann, “Topological phases in gapped edges of fractionalized systems,” Phys. Rev. B 88, 085115 (2013).

[51] W-P Su, JR Schrieffer, and Ao J Heeger, “Solitons in polyacetylene,” Phys. Rev. Lett. 42, 1698 (1979).

[52] Marcel den Nijs and Koos Rommelse, “Preroughening transitions in crystal surfaces and valence-bond phases in quantum spin chains,” Phys. Rev. B 40, 4709 (1989).

[53] Tom Kennedy and Hal Tasaki, “Hidden Z_2×Z_2 symmetry breaking in haldane-gap antiferromagnets,” Phys. Rev. B 45, 304 (1992).

[54] Tom Kennedy and Hal Tasaki, “Hidden symmetry breaking and the haldane phase in s = 1 quantum spin chains,” Communications in mathematical physics 147, 431–484 (1992).

[55] David Pérez-García, Michael M Wolf, M Sanz, Frank Verstraete, and J Ignacio Cirac, “String order and symmetries in quantum spin lattices,” Phys. Rev. Lett. 100, 167202 (2008).

[56] Frank Pollmann and Ari M Turner, “Detection of symmetry-protected topological phases in one dimension,” Phys. Rev. B 86, 125441 (2012).

[57] Ion P McCulloch, “Infinite size density matrix renormalization group, revisited,” arXiv preprint arXiv:0804.2509 (2008).

[58] Guifre Vidal, José Ignacio Latorre, Enrique Rico, and Alexei Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90, 227902 (2003).

[59] Pasquale Calabrese and John Cardy, “Entanglement entropy and conformal field theory,” Journal of Physics A: Mathematical and Theoretical 42, 504005 (2009).

[60] Guifre Vidal, José Ignacio Latorre, Enrique Rico, and Alexei Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90, 227902 (2003).

[61] Hui Li and F Duncan M Haldane, “Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-abelian fractional quantum hall effect states,” Phys. Rev. Lett. 101, 010504 (2008).

[62] Frank Pollmann, Ari M Turner, Erez Berg, and Masaki Oshikawa, “Entanglement spectrum of a topological phase in one dimension,” Phys. Rev. B 81, 064439 (2010).

[63] Lukasz Fidkowski and Alexei Kitaev, “Topological phases of fermions in one dimension,” Phys. Rev. B 83, 075103 (2011).

[64] Ari M Turner, Frank Pollmann, and Erez Berg, “Topo-
logical phases of one-dimensional fermions: An entanglement point of view,” Phys. Rev. B 83, 075102 (2011).

[65] Timothy H Hsieh and Liang Fu, “Bulk entanglement spectrum reveals quantum criticality within a topological state,” Phys. Rev. Lett. 113, 106801 (2014).

[66] Zi-Qi Wang, Guo-Yi Zhu, and Guang-Ming Zhang, “Decoding quantum criticalities from fermionic/parafermionic topological states,” arXiv preprint arXiv:1802.04542 (2018).