Non-integrability of Self-dual Yang-Mills-Higgs System

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Abstract

We examine integrability of self-dual Yang-Mills system in the Higgs phase, with taking simpler cases of vortices and domain walls. We show that the vortex equations and the domain-wall equations do not have Painlevé property. This fact suggests that these equations are not integrable.

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1 Introduction

Field theories in low dimensions as well as gauge theories (in Higgs phase in some cases) in $d = 3 + 1$ and other dimensions allow various kinds of solitons and soliton-like objects [1]–[3], see [4]. Study of such objects, particularly those of BPS class [2], has become a subject of growing interest in view of applications to particle physics and mathematical physics. One interesting recent development is the relationship of such objects of different dimensionality in gauge theories. This view was extended to a few other cases in susy gauge theories. Another development is the discovery of 1/4 BPS composites of such soliton-like objects [5]–[11] as well as 1/8 BPS states [12, 13], see [14, 15] as a review.

Solitons are solutions of non-linear partial differential equations (PDE). From mathematical point of view, an important question is whether these equations are integrable. One may take a different view and ask whether the solutions are unique in the sense of moduli space specification. The integrability question of nonlinear PDE has been addressed to in a few different approaches. One, not rigorous, but practical means of analysis is the one which uses the Painlevé property of non-linear ordinary differential equation (ODE) and it’s extension to PDE [16]–[21]. This method was applied to many well known soliton equations. From field theoretical points of view, more interesting applications would be those to the instanton equation and the monopole equation in Yang-Mills theory. The application was made to self-dual Yang-Mills [20, 21] paralleled the ADHM construction of instantons. The purpose of this paper is to study the integrability question (uniqueness question in other word) of the soliton equations in low dimensions which are derived from susy Yang-Mills theory in the Higgs phase, BPS vortex equation ($d = 2+1$) and domain-wall equation ($d = 1+1$).

The Painlevé property is seldom used in particle physics, and we find it desirable to review it briefly, which we will do in section 2. We then introduce the model of gauge theories in the Higgs phase in section 3. In section 4 and 5 we examine the Painlevé property of domain walls and vortices, respectively. Section 6 is devoted to conclusion and discussion.

2 Review of the Painlevé Test

Given a nonlinear equation, the first question one should address to is its (non)integrability. Integrability has been the notion for Hamiltonian systems of finite degrees of freedom. For the systems of $N$-degrees of freedom, if $N$ conserved quantities which are mutually involutive are found, the initial value problem can be solved by finite times of quadrature.

If the notion of integrability is extended to systems of infinite degrees of freedom, it does not have precise definition, although some sufficient conditions, like inverse scattering transformation class, are known. There are a few variations of it, which depend on the system in question. However, it has not been proven that they are equivalent to each other. The Painlevé property is one of such definitions, which we will use in this paper. We call the equation which has Painlevé property P-type.
A non-linear ordinary differential equation is said to have the Painlevé property if it has the solutions whose movable singularities, depending on the initial condition, are only poles. Regarding the ordinary differential equation (ODE for short), if it has the Painlevé property, its solution can be expanded in the Laurent series near the movable singularity. The analysis of the Painlevé test for ODE uses this property.

Regarding the partial differential equation (PDE), there is a conjecture, known as Ablowitz-Ramani-Segur conjecture (ARS conjecture), connecting its integrability to Painlevé property for ODE [16]–[21]. It says that every nonlinear ODE obtained by exact reduction of a nonlinear PDE which is integrable by inverse scattering method is of P-type. There are cases in which the ODE obtained in this way would not be of P-type at first look, even if the PDE is integrable. So we have to find certain transformation of variables which makes the ODE to pass the Painlevé test. This is not a convenient way to check whether the PDE is integrable.

Weiss, Tabor and Carnevale proposed an analytical way to study integrability of PDE [18], called Weiss-Tabor-Carnevale analysis. They have introduced the notion of integrable PDE in the following sense. If the PDE is integrable, it has a singularity manifold. It is written by

\[ \phi(z_1, z_2, \cdots, z_n) = 0 \quad (\phi: \text{arbitrary function}, \, \phi_{z_i} \neq 0). \]

We can expand integrable PDE near the singular manifold, in an analogous way that ODE can be expanded in a Laurent series near the movable singular point. We use this method to examine whether our models are integrable.

3 The Model and Soliton Equations

3.1 The Model and the Self-dual Yang-Mills-Higgs Equations

We consider \( \mathcal{N} = 1 \) supersymmetric Yang-Mills theory coupled with Higgs fields (Yang-Mills-Higgs system) in 5+1 dimensions. We limit our discussion to static solutions throughout the paper. We obtain Lagrangians in lower dimensions and their solutions by dimensional reduction of this system. Imposing the BPS condition on them, we obtain the instanton-like equation in 4+1 dimensions [7,9], the monopole-like equation in 3+1 [5,8], the wall web equation in 3+1 [11], the vortex equation in 2+1 [3,22–26], and the domain-wall equation in 1+1 [27,28,26].

In this paper we consider the \( SU(N_C) \times U(1) \) gauge theory coupled to \( N_F \) hypermultiplets as matter (Higgs) fields which are in the fundamental representation and constitute \( SU(2)_R \) doublet: the theory consists of a gauge field \( W_M (M = 0, 1, 2, 3, 4, 5) \) and Higgs fields \( H^i (i = 1, 2) \) in the form of \( N_C \) by \( N_F \) matrices. Supersymmetry forbids masses for hypermultiplets in \( d = 5 + 1 \) and there exist the \( SU(N_F) \) flavor symmetry. We will be concerned with the bosonic part, and we set the fermion fields to zero. The Lagrangian for the bosonic sector in \( d = 5 + 1 \) is given by

\[
L_{5+1} = \text{Tr} \left[ -\frac{1}{2g^2} F^{MN} F_{MN} + \mathcal{D}^M H^i (\mathcal{D}_M H^i)^\dagger \right] - V, \tag{3.1}
\]
where the gauge field strength and covariant derivative are given by

\[ F_{MN} = -i[D_M, D_N] = \partial_M W_N - \partial_N W_M + i[W_M, W_N], \quad (3.2) \]

\[ D_M H^i = \partial_M H^i + iW_M H^i, \quad (3.3) \]

and the potential \( V(H^i) \) is given by

\[ V = \frac{g^2}{4} \text{Tr} \left[ \left( c^a 1_{N_C} - (\sigma^a)^i H^i (H^j)^\dagger \right)^2 \right], \quad (a = 1, 2, 3). \quad (3.4) \]

Without loss of generality we may choose the Fayet-Iliopoulos parameter as \( c^a = (0, 0, c) \) with \( c > 0 \) by using \( SU(2)_R \). Nonzero value of \( c \) breaks \( SU(2)_R \) explicitly.

We consider static configuration. After dimensional reduction along \( x^5 \) and we ignore \( W^5 \). The second Higgs field \( H^2 \) vanishes for soliton configurations considered in this paper. We set \( H \equiv H^1 \). The potential reduces to

\[ V = \frac{g^2}{4} \text{Tr} \left[ (c 1_{N_C} - H H^\dagger)^2 \right]. \quad (3.5) \]

The energy density for static configuration \((\partial_{x^0} = 0)\) is written as follows:

\[
\mathcal{E} = \text{Tr} \left[ \frac{1}{2g^2} F_{mn} F_{mn} + D_m H (D_m H)^\dagger \right] + g^2 \frac{1}{4} (c 1_{N_C} - H H^\dagger)^2 \\
= \text{Tr} \left[ \frac{1}{g^2} \left\{ (F_{13} - F_{24})^2 + (F_{14} + F_{23})^2 + \left( F_{12} + F_{34} + g^2 \frac{1}{2} (c 1_{N_C} - H H^\dagger) \right)^2 \right\} \\
+ (D_1 H + iD_2 H)^\dagger (D_1 H + iD_2 H) + (D_3 H + iD_4 H)^\dagger (D_3 H + iD_4 H) \\
+ \frac{1}{2g^2} F_{mn} F_{mn} - c(F_{12} + F_{34}) + \partial_m J_m \right] \quad (3.6)
\]

where \( m, n = 1, 2, 3, 4 \). The energy density is minimized if the following set of BPS equations is satisfied,

\[ F_{13} - F_{24} = 0, \quad F_{14} + F_{23} = 0, \quad (3.7) \]

\[ F_{12} + F_{34} = -g^2 \frac{1}{2} (c 1_{N_C} - H H^\dagger), \quad (3.8) \]

\[ D_1 H + iD_2 H = 0, \quad D_3 H + iD_4 H = 0. \quad (3.9) \]

If we set the Fayet-Iliopoulos parameter \( c \) to zero and set \( H = 0 \), we obtain the well known self-dual Yang-Mills (SDYM) equations for instantons. We call Eqs. (3.7)–(3.9) the SDYM-Higgs equations. Several mathematicians have also studied these equations [29]. We will be concerned with the soliton equations in lower dimensions, \( d < 4 + 1 \), which are obtained by dimensional reduction from Eqs. (3.7)–(3.9). Now, we are going to show vortices and domain walls in \( d = 2 + 1, 1 + 1 \), respectively.

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\[ \text{The SDYM-Higgs equations (3.7)–(3.9) in } d = 4 + 0 \text{ can be derived from the Donaldson-Uhlenbeck-Yau equations in } d = 6 + 0 \text{ [30] by the } SU(2) \text{ equivariant dimensional reduction on } S^2, \text{ at least in the case of } U(1) \text{ gauge group [31]. We thank A. D. Popov to point this out.} \]
3.2 Vortex Equation

We first perform the trivial dimensional reduction of (3.1) to three dimensions. We choose the coordinates which are reduced to be \(x^3, x^4, x^5\). After dimensional reduction, gauge fields \(W_3, W_4, W_5\) are adjoint scalars (in \(d = 3\)) \(\Sigma_\alpha, (\alpha = 1, 2, 3)\). In the following we set \(\Sigma_\alpha = H^2 = 0\) because they do not contribute to vortex solutions (and set \(H = H^1\)). The Lagrangian (3.1) is now written as

\[
L_{2+1} = \text{Tr} \left[ -\frac{1}{2g^2} F_{mn} F^{mn} + \mathcal{D}_m H (\mathcal{D}^m H)^\dagger \right] - \frac{g^2}{4} \text{Tr} [(c \mathbf{1}_{N_C} - H H^\dagger)^2],
\]

with \(m = 0, 1, 2\).

We next consider the BPS states of our theory. We ignore the \(x^0\) dependence. The energy density of static configurations is given by

\[
E = \text{Tr} \left[ \frac{1}{g^2} (F_{12})^2 + \{ \mathcal{D}_1 H (\mathcal{D}_1 H)^\dagger + \mathcal{D}_2 H (\mathcal{D}_2 H)^\dagger \} + \frac{g^2}{4} (c \mathbf{1}_{N_C} - H H^\dagger)^2 \right]
\]

\[
= \text{Tr} \left[ \frac{1}{g^2} \left( F_{12} + \frac{g^2}{2} (c \mathbf{1}_{N_C} - H H^\dagger) \right)^2 + (\mathcal{D}_1 H + i \mathcal{D}_2 H) (\mathcal{D}_1 H + i \mathcal{D}_2 H)^\dagger \right] + \text{Tr} \left[ -c F_{12} + 2i \partial_1 H \mathcal{D}_2 H^\dagger \right].
\]

This energy is minimized if the following BPS equations for non-Abelian vortices are satisfied:

\[
0 = \mathcal{D}_1 H + i \mathcal{D}_2 H,
\]

\[
0 = F_{12} + \frac{g^2}{2} (c \mathbf{1}_{N_C} - H H^\dagger).
\]

These equations are called the vortex equations. These can also be obtained by ignoring \(x^3\) and \(x^4\) dependence and setting \(W_3 = W_4 = 0\) in Eqs. (3.7)–(3.9). In the case of \(U(1)\) gauge theory with a single Higgs field and hence \(U(1)\) flavor symmetry alone, this BPS state is known as Abrikosov-Nielsen-Olesen (ANO) vortex.

Following the same line as the Yang’s equation for self-dual Yang-Mills fields, we rewrite Eqs. (3.12) and (3.13) in the second order PDE. First we define complex notations

\[
z \equiv x^1 + ix^2, \quad \bar{W}_z \equiv \frac{W_1 + iW_2}{2}.
\]

Then note the first vortex equation (3.12) can be integrated as

\[
\bar{W}_z = -i S^{-1} \partial_z S, \quad H = S^{-1} H_0(z),
\]

where we have introduced \(S = S(z, z^*) \in GL(N_C, \mathbb{C})\) and \(H_0(z)\) is an \(N_C \times N_F\) matrix whose components are holomorphic with respect to \(z\). Constants in \(H_0\) are integration constants as moduli. Defining a gauge invariant

\[
\Omega(z, z^*) \equiv SS^\dagger,
\]

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the second vortex equation (3.12) can be rewritten as
\[ \partial_z (\Omega^{-1} \partial_z \Omega) = \frac{g^2}{4} (c1_{NC} - \Omega^{-1} H_0 H_0^\dagger). \]

We call this equation the master equation for vortices. We will examine if this equation has the Painlevé property in section 5.

3.3 Domain-Wall Equation

To obtain the domain-wall equation we have to give a mass to the Higgs fields $H$. This can be made by using the Scherk-Schwarz dimensional reduction \[33\] with respect the coordinate $x^2$, after the simple reduction to three dimensions of the previous sector.

We are considering Scherk-Schwarz dimensional reduction to a certain theory in 2 + 1 dimension; the coordinates are $x^0, x^1, x^2$. We would like to reduce one of those; we set it $x^2$. The dependence of the Higgs field on it will be compactified to $S^1$ by using twisted boundary condition

\[ H(x^\mu, x^2 + 2\pi R) = H(x^\mu, x^2) e^{i2\pi R M} \]

with $\mu = 0, 1$ and

\[ M = \text{diag.}(m_1, m_2, \cdots, m_{N_F}), \quad 0 \leq m_A < 1/R \quad (A = 1, 2, \cdots, N_F). \]

We take the lowest massive mode ignoring the infinite tower of higher Kaluza-Klein modes. We write

\[ H(x^\mu, x^2) = \frac{1}{\sqrt{2\pi R}} \hat{H}(x^\mu) e^{iM x^2}. \]

The other fields do not depend on the coordinate $x^2$:

\[ W_\mu(x^\mu, x^2) = W_\mu(x^\mu), \quad \Sigma(x^\mu, x^2) = \Sigma(x^\mu), \quad W_2(x^\mu, x^2) = -\hat{\Sigma}(x^\mu). \]

After the Lagrangian (3.1) is reduced by trivial dimensional reduction and Scherk-Schwarz dimensional reduction, we can write the Lagrangian in 1 + 1 dimensions as

\[ L_{1+1} = \text{Tr} \left[ -\frac{1}{2g^2} \{ F_{\mu\nu} F^{\mu\nu} - 2 D_{\mu} \hat{\Sigma} D^{\mu} \hat{\Sigma} \} + D_\mu H(D^\mu H) \dagger \right] - V, \]

where the potential is given by

\[ V = \frac{g^2}{4} \text{Tr} \left[ -(c1_{NC} - HH^\dagger)^2 \right] + \text{Tr} \left[ (\hat{\Sigma}H - HM)(\hat{\Sigma}H - HM) \dagger \right]. \]

Here, the covariant derivative of $\hat{\Sigma}$ is written by

\[ D_m \hat{\Sigma} = \partial_m \hat{\Sigma} + i [W_m, \hat{\Sigma}]. \]
The vacua of the theory are determined as the minimum of the potential $V$:

$$HH^\dagger = c1_{N_C}, \quad \hat{\Sigma}H - HM = 0.$$  

These can be solved as

$$H = \sqrt{c} \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & 0 & \vdots & 1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \vdots & \cdots & 0 & 1 \end{pmatrix}, \quad (3.22)$$

$$\hat{\Sigma} = \text{diag}(m_{A_1}, m_{A_2}, \cdots, m_{A_{NC}}). \quad (3.23)$$

The first form is constructed by $N_C$ unit vectors and $N_F - N_C$ zero vectors. Let us label the position of the unit vector as $[A_1N_1, A_2N_2, \cdots, A_{NC}N_{NC}]$. And the disposition of the unit vectors can be chosen arbitrarily, except to be satisfied $A_r < A_{r+1}$ ($r = 1, 2, \cdots, N_C$). Therefore the number of the vacua is $N_F C_{NC}$.

We next consider the BPS states of our theory. Their energy density is given by

$$\mathcal{E} = \text{Tr} \left[ \frac{1}{g^2} (D_y \hat{\Sigma})^2 + D_y H D_y H^\dagger + \frac{g^2}{4} (c1_{N_C} - HH^\dagger)^2 + (\hat{\Sigma}H - HM)(\hat{\Sigma}H - HM)^\dagger \right]$$

$$= \frac{1}{g^2} \text{Tr} \left[ D_y \hat{\Sigma} - \frac{g^2}{2} (c1_{N_C} - HH^\dagger) \right]^2$$

$$+ \text{Tr} \left[ (D_y H + \hat{\Sigma}H - HM)(D_y H + \hat{\Sigma}H - HM)^\dagger \right]$$

$$+ c \partial_y \text{Tr} \hat{\Sigma} - \partial_y \left\{ \text{Tr} \left[ (\hat{\Sigma}H - HM) H^\dagger \right] \right\}, \quad (3.24)$$

with $y \equiv x^1$. The energy density is minimized by imposing the BPS equations of domain walls (domain-wall equation)

$$D_y H = -\hat{\Sigma}H + HM, \quad (3.25)$$

$$D_y \hat{\Sigma} = \frac{g^2}{2} (c1_{N_C} - HH^\dagger). \quad (3.26)$$

These equations can be directly derived from vortex equations (3.12) and (3.13) by the Scherk-Schwarz dimensional reduction $H(x^0, x^1, x^2) \rightarrow H(x^0, x^1)e^{iMx^2}$.

We rewrite the set of domain-wall equations to a second order ODE in the form like the Yang’s equation [28, 15]. The first of the domain-wall equation (3.25) can be integrated as

$$H = S^{-1}H_0 e^{My}, \quad (3.27)$$

$$\hat{\Sigma} + iW_y = S^{-1}\partial_y S, \quad (3.28)$$

with an $N_C \times N_F$ matrix $H_0$ of integration constants, and $S = S(y) \in GL(N_C, \mathbb{C})$. Defining a gauge invariant

$$\Omega(y) \equiv SS^\dagger, \quad (3.29)$$
the second of the domain-wall equation (3.26) can be rewritten to the equation which we want:

$$\partial_y(\Omega^{-1} \partial_y \Omega) = g^2(c1_{NC} - \Omega^{-1} H_0 e^{2M y} (H_0)^†).$$  \tag{3.30}

We call this equation the master equation for domain walls. We will examine if this equation has the Painlevé property in the next section.

## 4 Painlevé Test for Domain Walls

In this section, we will check whether the master equation (3.30) of domain wall has Painlevé property. Since this master equation is the ODE, we can do it easier than PDE like the case of vortices.

From now on we consider the simple case of $N_C = 1, N_F = 2$. This is the simplest case admitting a single domain wall [35]. We set mass $M$ to be

$$M = \begin{pmatrix} 0 \\ -\Delta m \end{pmatrix}$$  \tag{4.1}

and put the moduli matrix

$$H_0 = \sqrt{c}(1, e^{\Delta m y_0})$$  \tag{4.2}

where $y_0$ is a constant. Defining $\psi$ by

$$\Omega \equiv e^{2\psi(y)}$$  \tag{4.3}

the master equation (3.30) can be rewritten as [27, 36]

$$\frac{d^2 \psi}{dy^2} = \frac{g^2 c}{2} \{1 - e^{-2\psi}(e^{-2\Delta m(y-y_0)} + 1)\}. \tag{4.4}$$

This equation has the factor $e^{-2\psi}$ which has to be expanded in a power series of $\psi$. It makes the Painlevé test complicated. Therefore we transform this equation into two coupled equations:

$$w \frac{dA}{dw} = AB, \tag{4.5}$$

$$w \frac{dB}{dw} = \frac{1}{2k^2} \{A(w + 1) - 1\}, \tag{4.6}$$

where we have defined

$$w \equiv \exp(-2\Delta m(y-y_0)), \tag{4.7}$$

$$A(w) \equiv \exp(-2\psi(y)) \quad B(w) \equiv \frac{\psi'(y)}{\Delta m}, \tag{4.8}$$

$$k \equiv \sqrt{\frac{2(\Delta m)^2}{g^2 c}}. \tag{4.9}$$
The ODE (4.5) and (4.5) have now only one dimensionless parameter $k$. The variable $y$ takes a value in $\mathbb{R}$ but we now consider $w$ to take a value in $\mathbb{C}$ by analytic continuation.

We assume that the solutions $A(w)$ and $B(w)$ have a movable singularity $w = w_*$. We then expand $A(w)$ and $B(w)$ in the Laurent series near $w_*:

\[
A(w) = (w - w_*)^{-\alpha} \sum_{l=0}^{\infty} A_l (w - w_*)^l, \quad B(w) = (w - w_*)^{-\beta} \sum_{l=0}^{\infty} B_l (w - w_*)^l.
\]

Here $\alpha$ and $\beta$ are positive constants and we assume that

\[
A_0 \neq 0, \quad B_0 \neq 0 \quad \text{and} \quad A_l = B_l = 0 \quad (l < 0).
\]

We substitute the expansion (4.10) into Eqs. (4.5) and (4.6). The leading order analysis yields

\[
\alpha = 2, \quad \beta = 1,
\]

\[
A_0 = 4k^2 \frac{w_*^2}{w_* + 1}, \quad B_0 = -2w_*.
\]

We also obtain the recursive relations

\[
(l - 3)A_{l-1} + w_*(l - 2)A_l = \sum_{j=0}^{l} A_{l-j}B_j,
\]

\[
(l - 2)B_{l-1} + w_*(l - 1)B_l = \frac{1}{2k^2}(w_* + 1)A_l - \frac{1}{2k^2} \delta_{n,2}.
\]

We now examine from these recursive relations whether certain degrees $l$ can become resonances, or not. The relations (4.13) can be written as the following form:

\[
(l + 1)(l - 2)w_*B_l = F_l(A_{l-1}, \ldots, A_0, B_{l-1}, \ldots, B_0),
\]

with some function $F_l$. We find that the degrees which are possible to be resonances are $l = -1, 2$ because $B_l$ (or $w_*$) can become arbitrary in these cases. The one of possibilities of resonance, $l = -1$, comes from the arbitrariness of $w_*$. If there would exist another resonance in Eqs. (4.5) and (4.6), it could be $l = 2$. In this case, $F_2$ must vanish from (4.14). However $F_2$ is obtained from (4.13) as

\[
F_2 = \frac{w_*}{(w_* + 1)^2} - \frac{1}{2k^2},
\]

and this is impossible to vanish due to the arbitrariness of $w_*$. Therefore $l = 2$ cannot become a resonance.

From these analyses, we have seen that there are no resonances. Therefore the master equation (3.30) for domain walls has no Painlevé property in the case of $N_F = 2, N_C = 1$. Since $U(N_C)$ gauge theory with $N_F (> N_C)$ flavors contains the $N_F = 2, N_C = 1$ case discussed above we conclude that the master equation (3.30) for domain walls has no Painlevé property in general cases. This implies that domain-wall equations are not integrable.
5 Painlevé Test for Vortices

In this section, we will check integrability of the master equation (3.17) for vortices. Since the master equation (3.30) for domain wall is not integrable and it is obtained from the master equation (3.17) of vortices by dimensional reduction, it is likely to be non-integrable from the ARS conjecture [16]. We will see that this is the case by directly performing the Painlevé test of the master equation (3.17) for vortices.

We consider the case of $N_c = 1, N_F = 1$, the simplest case of the ANO vortices [3]. We again define a new variable as

$$\Omega = e^{\psi(z, \bar{z})}. \quad (5.1)$$

We consider the single vortex case:

$$H_0 = \sqrt{c}(z - z_1). \quad (5.2)$$

If there were $n$ vortices, $H_0$ should become $H_0 = \sqrt{c} \prod_{k=1}^{n} (z - z_k)$ [24], but we can repeat the following analysis in the same way. Then, the master equation (3.17) is rewritten as

$$\partial_z \partial_{\bar{z}} \psi = \frac{g^2}{4} \{ c - e^{-\psi}(z - z_1)(\bar{z} - \bar{z}_1) \}. \quad (5.3)$$

It is impossible to check the integrability by the Painlevé test, when there exist exponential terms which depend on $z$ in PDE. In order to do it, we need to change the form of equation (5.3) again. The variables are changed as

$$A(z, \bar{z}) \equiv \exp[-\psi(z, \bar{z})],$$
$$B(z, \bar{z}) \equiv \partial_{\bar{z}} \psi(z, \bar{z}). \quad (5.4)$$

Then we obtain the two coupled equations, given by

$$\partial_{\bar{z}} A = -AB, \quad (5.5)$$
$$\partial_z B = \frac{g^2}{4} \{ c - A(z - z_1)(\bar{z} - \bar{z}_1) \}. \quad (5.6)$$

We assume that the solutions $A(z, \bar{z})$ and $B(z, \bar{z})$ can be expanded near a singularity manifold, given by $\phi(z, \bar{z}) = 0$, as

$$A(z, \bar{z}) = \sum_{l=0}^{\infty} A_l(z, \bar{z}) \phi^{l-m}(z, \bar{z}), \quad B(z, \bar{z}) = \sum_{l=0}^{\infty} B_l(z, \bar{z}) \phi^{l-n}(z, \bar{z}), \quad (5.7)$$

where, $m$ and $n$ are positive constants, and we assume that

$$A_0 \neq 0, \quad B_0 \neq 0 \quad \text{and} \quad A_l = B_l = 0 \quad (l < 0). \quad (5.8)$$

There is the element $(z - z_1)(\bar{z} - \bar{z}_1)$ in Eq. (5.6). This makes a difference of coefficient functions between a case (the case 1) that the function $\phi$, defining the singularity manifold, does not have
zeros at \( z = z_1 (\phi(z = z_1) \neq 0) \) and another case (the case 2) that it has zeros at least of first order at \( z = z_1 (\phi(z = z_1) = 0) \). In order to identify each other, we will write those singularity manifolds as \( \phi_1 = 0 \) and \( \phi_2 = 0 \), respectively. Also we check these Painlevé property separately.

**The case 1.** First we will consider the case in which the singularity manifold is given by \( \phi_1 = 0 \). If we substitute the expansion (5.7) into Eqs. (5.5) and (5.6), the leading order analysis yields

\[
m = 2 \quad , \quad n = 1,
A_0 = \frac{8}{g^2} \frac{(\partial_z \phi_1)(\partial_z \phi_1)}{(z - z_1)(\bar{z} - \bar{z}_1)} \quad , \quad B_0 = 2(\partial_z \phi_1).
\]  

(5.9)

We also obtain the recursive functions

\[
\partial_z A_{l-1} + (l - 2)A_l(\partial_z \phi_1) = - \sum_{j=0}^l A_{l-j}B_j \quad (l \geq 1),
\]

\[
\partial_z B_{l-1} + (l - 1)B_l(\partial_z \phi_1) = - \frac{g^2}{4}(z - z_1)(\bar{z} - \bar{z}_1)A_l + \frac{g^2}{4}c_1^2 \quad (l \geq 1).
\]  

(5.10)

We can find certain degrees \( l \) which are possible to become resonances from above recursive functions (5.10). Those functions can be written as

\[
\frac{4}{g^2}(l - 2)(l + 1) \frac{(\partial_z \phi_1)(\partial_z \phi_1)}{(z - z_1)(\bar{z} - \bar{z}_1)} B_l = F_l(A_{l-1}, \ldots, A_0, B_{l-1}, \ldots, B_0)
\]  

(5.11)

with some function \( F_l \). Now we find from the left hand side that the degrees which are possible to be resonances are \( l = -1, 2 \), because \( B_l \) (or \( \phi_1 \)) can be arbitrary function in these cases. The one of possibilities of resonance, \( l = -1 \), comes from the arbitrariness of \( \phi_1 \). If there exists another resonance in Eqs. (5.5) and (5.6), it should be \( l = 2 \). Therefore we now check whether it is the resonance, by using the recursive relations (5.10). If we think of the case \( l = 1 \) for Eq. (5.10), we obtain precise expression of \( A_1 \) and \( B_1 \):

\[
A_1 = \frac{A_0 B_1 + \partial_z A_0}{(\partial_z \phi_1) - B_0},
\]

\[
B_1 = - \left\{ \frac{g^2}{4}(z - z_1)(\bar{z} - \bar{z}_1)A_0}{\partial_z \phi_1} - (\partial_z \phi_1) \right\}^{-1} \left\{ \partial_z B_0 + \frac{g^2}{4}(z - z_1)(\bar{z} - \bar{z}_1)\frac{\partial_z A_0}{(\partial_z \phi_1) - B_0} \right\}
\]

\[
= - \left\{ 2(\partial_z \phi_1)(\partial_z \phi_1) \right\} + \left( \frac{\partial_z \phi_1}{\partial_z \phi_1} \right) \frac{(\partial_z \phi_1)(\partial_z \phi_1)}{\bar{z} - \bar{z}_1}.
\]  

(5.12)

For \( l = 2 \), the left hand side of (5.11) vanishes while we can show from these equations and the recursive functions (5.10) that the right hand side of (5.11) does not vanish, \( F_2 \neq 0 \), like the case of domain walls. It means that \( l = 2 \) is not a resonance. Hence, Eqs. (5.5) and (5.6) have no resonances in the case 1.

**The case 2.** Next we will consider the case that the singularity manifold is given by \( \phi_2 = 0 \). We write this singularity manifold as

\[
M \equiv \frac{\phi_2}{(z - z_1)(\bar{z} - \bar{z}_1)}
\]  

(5.13)
with a function $M(z, \bar{z})$, in order to clarify that the coefficient functions do not have any singularities. Here, $M$ could depend on $(z - z_1)(\bar{z} - \bar{z}_1)$. If we substitute (5.7) into Eqs. (5.5) and (5.6), the leading order analysis yields

$$m = 3, \quad n = 1,$$

$$A_0 = \frac{12}{g^2} M(\partial_z \phi_2)(\partial_{\bar{z}} \phi_2), \quad B_0 = 3(\partial_z \phi_2).$$  \tag{5.14}

We also obtain the recursive functions

$$\partial_{\bar{z}} A_{l-1} + (l - 3)A_l(\partial_z \phi_2) = - \sum_{j=0}^{l} A_{l-j}B_j \quad (l \geq 1),$$

$$\partial_z B_{l-1} + (l - 1)B_l(\partial_z \phi_2) = \frac{g^2}{4} c_{l,2} - \frac{g^2}{4} \frac{1}{M} A_l \quad (l \geq 1).$$  \tag{5.15}

We are concerned with whether there are some possibilities of resonance, or not. We check it by the same method as above. Then from (5.15) we get

$$\frac{4}{g^2} M(l^2 - l - 3)(\partial_z \phi_2)(\partial_{\bar{z}} \phi_2) B_l = F_l(A_{l-1}, \cdots, A_0, B_{l-1}, \cdots, B_0).$$  \tag{5.16}

Therefore $B_l$ can be an arbitrary function only when $l = \frac{1 + \sqrt{13}}{2}$. However the resonance must be an integer or at most rational number, so it is impossible that this number $l$ becomes a resonance. Therefore Eqs. (5.5) and (5.6) have no resonances.

From the above consideration, the cases 1 and 2, we conclude that the master equation (3.17) for $U(1)$, $N_F = 1$ vortex does not have the Painlevé property. Since equations for the theory with gauge group $U(N_C)$ and general number of flavor $N_F(> N_C)$ contain this case at least, we conclude that the master equation (3.17) for vortices does not have the Painlevé property in general. This fact implies that vortex equations are not integrable.

6 Conclusions and Discussion

The first (Higgs part) equations of vortex equation (3.12) and domain-wall equation (3.25) can be integrated (to give integration constants $H_0$) while the second (gauge part) of them, (3.13) and (3.26), are rewritten into the master equations (3.17) and (3.30) for vortices and domain walls, respectively. We have shown that the master equations for domain walls (3.30) and vortices (3.17) do not pass the Painlevé test and therefore that they do not have the Painlevé property. Of course there remains a possibility that better choice of variables might make the form of equations suitable to pass the Painlevé test. However we believe that it is not the case from many trials. Our results imply that the domain-wall equation and the vortex equation are not integrable, because most equations which do not pass Painlevé test are not integrable. This is in contrast to the cases of instantons and monopoles, whose equations are integrable and have Painlevé property.
Our results are related with uniqueness problem of moduli. Solution of the master equation of vortices in the case of $N_C = N_F = 1$ (equivalent to the Taubes equation) was shown to exist uniquely [34]. This implies that the master equation (3.17) does not contain moduli and therefore is not integrable. Thus all moduli are contained in $H_0(z)$. The uniqueness and existence of solution of the master equation of domain walls for $N_C = 1$ has also been proved recently [37].

Ward conjectured that all integrable equations are obtained from self-dual Yang-Mills (SDYM) equation [38] by some dimensional reductions. If this conjecture is true, the vortex equation and the domain-wall equation should not be integrable, because they are obtained as dimensional reduction of the SDYM coupled with the Higgs fields (3.7)–(3.9). In this sense our results support the Ward’s conjecture. Non-commutative version [39] of our analysis is also interesting to explore. In particular, the large non-commutativity limit may reduce equations integrable especially for vortices.

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*The uniqueness of solution of the master equation of arbitrary gauge group with arbitrary matter contents was also shown in the case that the base space is compact Kähler instead of $\mathbb{R}^2$ [29].
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