Abstract
We present the characteristic polynomial for the transition matrix of a vertex-face walk on a graph, and obtain its spectra. Furthermore, we express the characteristic polynomial for the transition matrix of a vertex-face walk on the 2-dimensional torus by using its adjacency matrix, and obtain its spectra. As an application, we define a new walk-type zeta function with respect to the transition matrix of a vertex-face walk on the two-dimensional torus, and present its explicit formula.

Keywords Quantum walk · Vertex-face walk · Transition matrix

1 Introduction
Recently, there were exciting developments between quantum walk [1, 8, 9, 12, 21] on a graph and the Ihara zeta function [2, 5–7, 14, 17–20] of a graph: Grover/Zeta correspondence [10]; Walk/Zeta correspondence [11].

In Grover/Zeta correspondence [10], a zeta function and a generalized zeta function of a graph $G$ with respect to its Grover matrix as analogue of the Ihara zeta function and the generalized Ihara zeta function [3] of $G$ were defined. By using the Konno-Sato theorem [13], the limits on the generalized zeta functions and the generalized Ihara zeta functions of a family of finite regular graphs were written as an integral expression, and contained the result on the generalized Ihara zeta function in Chinta...
et al. [3]. Furthermore, the limit on the generalized Ihara zeta functions of a family of finite tori was written as an integral expression, and contained the result on the Ihara zeta function of the two-dimensional integer lattice $\mathbb{Z}^2$ in Clair [4].

In Walk/Zeta correspondence [11], a walk-type zeta function was defined without use of the determinant expressions of zeta function of a graph $G$, and various properties of walk-type zeta functions of random walk (RW), correlated random walk (CRW) and quantum walk (QW) on $G$ were studied. Also, their limit formulas by using integral expressions were presented.

Recently, Zhan [22] introduced a vertex-face walk on an orientable embedding of a graph, and presented the spectra for its transition matrix.

In this paper, we treat a walk-type zeta function of a vertex-face walk on a graph defined by Zhan [22].

The rest of the paper is organized as follows. Section 2 gives a short review for Grover/Zeta correspondence. In Sect. 3, we state Walk/Zeta correspondence on a finite torus. In Sect. 4, we present an explicit formula for the characteristic polynomial of the transition matrix of a vertex-face walk on a graph, and obtain its spectra. In Sect. 5, we express the characteristic polynomial for the transition matrix of a vertex-face walk on the two-dimensional torus by using its adjacency matrix, and obtain its spectra. As an application, we define a new walk-type zeta function with respect to the transition matrix of a vertex-face walk on the two-dimensional torus, and present its explicit formula.

## 2 Grover/Zeta correspondence

All graphs in this paper are assumed to be simple. Let $G = (V(G), E(G))$ be a connected graph (without multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. Furthermore, let $n = |V(G)|$ and $m = |E(G)|$ be the number of vertices and edges of $G$, respectively. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Let $D_G$ be the symmetric digraph corresponding to $G$. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. For $v \in V(G)$, the degree $\deg_G v = \deg v = d_v$ of $v$ is the number of vertices adjacent to $v$ in $G$.

A walk $P$ of length $n$ in $G$ is a sequence $P = (e_1, \cdots, e_n)$ of $n$ arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1}) (1 \leq i \leq n-1)$ (see [2]). If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we write $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is called an $(o(P), t(P))$-walk. We say that a walk $P = (e_1, \cdots, e_n)$ has a backtracking if $e_{i+1} = e_i$ for some $i$ $(1 \leq i \leq n - 1)$. A $(v, w)$-walk is called a $v$-closed walk or closed walk if $v = w$. A closed walk $C = (e_1, \cdots, e_n)$ has a tail if $e_n^{-1} = e_1$. A closed walk $C$ is reduced if $C$ has neither a backtracking nor a tail. A walk $P$ is called a path if any two vertices of $P$ are distinct. A closed walk $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$ is called a cycle if any two vertices of $P$ except $v_0 = v_n$ are distinct.
The Ihara zeta function of a graph $G$ is a function of a complex variable $u$ with $|u|$ sufficiently small, defined by

$$Z(G, u) = \exp \left( \sum_{k=1}^{\infty} \frac{N_k}{k} u^k \right),$$

where $N_k$ is the number of reduced closed walks of length $k$ in $G$.

Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. The adjacency matrix $A = A(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. If $\deg v = k$ (constant) for each $v \in V(G)$, then $G$ is called $k$-regular.

**Theorem 2.1** (Ihara [7]; Bass [2]) Let $G$ be a connected graph. Then, the reciprocal of the Ihara zeta function of $G$ is given by

$$Z(G, u)^{-1} = \left(1 - u^2\right)^{r-1} \det(I - uA(G) + u^2(D - I)),$$

where $r$ is the Betti number of $G$, and $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0$, $i \neq j$. $(V(G) = \{v_1, \ldots, v_n\})$.

Let $G = (V(G), E(G))$ be a connected graph with $n$ vertices and $x_0 \in V(G)$ a fixed vertex. Then, the generalized Ihara zeta function $\zeta_G(u)$ of $G$ is defined by

$$\zeta_G(u) = \zeta(G, u) = \exp \left( \sum_{m=1}^{\infty} \frac{N^0_m}{m} u^m \right),$$

where $N^0_m$ is the number of reduced $x_0$-closed walks of length $m$ in $G$. Furthermore, the Laplacian of $G$ is given by

$$\Delta_v = \Delta(G) = D - A(G).$$

A formula for the generalized Ihara zeta function of a vertex-transitive graph is given as follows:

**Theorem 2.2** (Chinta, Jorgenson and Karlsson [3]) Let $G$ be a vertex-transitive $(q+1)$-regular graph with spectral measure $\mu_\Delta$ for the Laplacian $\Delta$. Then,

$$\zeta_G(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left( \int \log(1 - (q + 1 - \lambda)u + qu^2)d\mu_\Delta(\lambda) \right).$$

A graph $G$ is called vertex-transitive if for each $u, v \in V(G)$, there exists an automorphism $\phi$ of the automorphism group $Aut G$ of $G$ such that $\phi(u) = v$. Note that if $G$ is a vertex-transitive graph with $n$ vertices, then

$$\zeta_G(u) = Z(G, u)^{1/n}.$$
Let $G$ be a connected graph with $n$ vertices and $m$ edges. Set $V(G) = \{v_1, \ldots, v_n\}$ and $d_j = d_{v_j} = \deg v_j$, $j = 1, \ldots, n$. Then, the Grover matrix $U = U(G) = (U_{ef})_{e, f \in D(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} 2/d_{t(f)}(= 2/d_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete-time quantum walk with the matrix $U$ as a time evolution matrix is called the Grover walk on $G$.

Let $G$ be a connected graph with $\nu$ vertices and $m$ edges. Then, the $\nu \times \nu$ matrix $P = P(G) = (P_{uv})_{u, v \in V(G)}$ is given as follows:

$$P_{uv} = \begin{cases} 1/\deg_G u & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the matrix $P(G)$ is the transition probability matrix of the simple random walk on $G$.

We introduce the positive support $F^+ = (F^+_{ij})$ of a real matrix $F = (F_{ij})$ as follows:

$$F^+_{ij} = \begin{cases} 1 & \text{if } F_{ij} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Ren et al. [16] showed that the edge matrix of a graph is the positive support $(U^T)^+$ of the transpose of its Grover matrix $U$, i.e.,

$$Z(G, u)^{-1} = \det(I_{2m} - uU^+).$$

The Ihara zeta function of a graph is just a zeta function on the positive support of the Grover matrix of a graph.

Note that by Theorem 2.1,

$$\det(I_{2m} - uU^+) = (1 - u^2)^{m-\nu} \det((1 + qu^2)I_\nu - ((q + 1)I_\nu - \Delta_\nu)u)$$

for a $(q + 1)$-regular graph $G$.

Now, we propose a new zeta function of a graph. Let $G$ be a connected graph with $m$ edges. Then, we define a zeta function $\overline{Z}(G, u)$ of $G$ satisfying

$$\overline{Z}(u)^{-1} = Z(G, u)^{-1} = \det(I_{2m} - uU).$$

In Konno and Sato [13], they presented the following results in our setting.

**Theorem 2.3** (Konno and Sato [13]) Let $G$ be a connected graph with $\nu$ and $m$ edges. Then,

$$\det(I_{2m} - uU) = (1 - u^2)^{m-\nu} \det((1 + u^2)I_\nu - 2uP(G)).$$
We give a weight function $w : D(G) \times D(G) \rightarrow \mathbb{C}$ as follows:

$$w(e, f) = \begin{cases} 
\frac{2}{\deg t(e)} & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\
\frac{2}{\deg t(e)} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise.}
\end{cases}$$

For a closed walk $C = (e_1, e_2, \ldots, e_r)$, let

$$w(C) = w(e_1, e_2) \cdots w(e_{r-1}, e_r)w(e_r, e_1).$$

We define a generalized zeta function with respect to the Grover matrix of a graph. Let $G = (V(G), E(G))$ be a connected graph and $x_0 \in V(G)$ a fixed vertex. Then, the generalized zeta function $\xi_G(u)$ of $G$ is defined by

$$\xi_G(u) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r^0}{r} u^r \right),$$

where $N_r^0 = \sum \{w(C) \mid C: \text{an } x_0 \text{-closed walk of length } r \text{ in } G\}$.

Note that if $G$ is a vertex-transitive graph with $n$ vertices, then

$$\xi_G(u) = \mathbb{Z}(G, u)^{1/n}. \quad (1)$$

Then, we obtain the following results for a series of finite vertex-transitive $(q + 1)$-regular graphs.

**Theorem 2.4** (Grover/Zeta correspondence [10]) Let $\{G_n\}_{n=1}^{\infty}$ be a series of finite vertex-transitive $(q + 1)$-regular graphs such that

$$\lim_{n \to \infty} |V(G_n)| = \infty.$$

Then,

1. $\lim_{n \to \infty} \xi_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log((1 + u^2) - 2u\lambda) d\mu_P(\lambda) \right],$
2. $\lim_{n \to \infty} \xi_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log((1 - 2u + u^2) + \frac{2u}{q+1}\lambda) d\mu_\Delta(\lambda) \right],$
3. $\lim_{n \to \infty} \xi_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log((1 + qu^2) - (q + 1)u\lambda) d\mu_P(\lambda) \right],$
4. $\lim_{n \to \infty} \xi_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log((1 + qu^2) - ((q + 1) - \lambda)u) d\mu_\Delta(\lambda) \right],$

where $d\mu_P(\lambda)$ and $d\mu_\Delta(\lambda)$ are the spectral measures for the transition operator $P$ and the Laplacian $\Delta$.

We should note that the fourth formula in Theorem 2.4 is nothing but Theorem 1.3 in Chinta et al. [3].
Next, we obtain the following results for the generalized zeta function and the generalized Ihara zeta function of the $d$-dimensional integer lattice $\mathbb{Z}^d$ ($d \geq 2$).

Let $T^d_N$ ($d \geq 2$) be the $d$-dimensional torus (graph) with $N^d$ vertices. Its vertices are located in coordinates $i_1, i_2, \ldots, i_d$ of a $d$-dimensional Euclidian space $\mathbb{R}^d$, where $i_j \in \{0, 1, \ldots, N - 1\}$ for any $j$ from 1 to $d$. A vertex $v$ is adjacent to a vertex $w$ if and only if they have $d - 1$ coordinates that are the same, and for the remaining coordinate $k$, we have $|i^v_k - i^w_k| = 1$, where $i^v_k$ and $i^w_k$ are the $k$-th coordinate of $v$ and $w$, respectively. Remark that $V(T^d_N) = (\mathbb{Z} \mod N)^d$. Then, we have

$$|E(T^d_N)| = dN^d,$$

and $T^d_N$ is a vertex-transitive $2d$-regular graph.

**Theorem 2.5** (Grover/Zeta correspondence ($T^d_N$ case ) [10]) Let $T^d_N$ ($d \geq 2$) be the $d$-dimensional torus with $N^d$ vertices. Then,

$$\lim_{n \to \infty} \zeta(T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left\{ (1 + u^2) - \frac{2u}{d} \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi} \right],$$

$$\lim_{n \to \infty} \zeta(T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left\{ (1 + 2d - 1)u^2) - 2u \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi} \right],$$

where $\int_0^{2\pi} \cdots \int_0^{2\pi}$ is the $d$-th multiple integral and $\frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}$ is the uniform measure on $[0, 2\pi)^d$.

Specially, in the case of $d = 2$, we obtain the following result.

**Corollary 2.6** Let $T^2_N$ be the two-dimensional torus with $N^2$ vertices. Then,

$$\lim_{n \to \infty} \zeta(T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ (1 + u^2) - u \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right],$$

$$\lim_{n \to \infty} \zeta(T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ (1 + 3u^2) - 2u \sum_{j=1}^{2} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right].$$

The second formula corresponds to Eq. (10) in Clair [4].

### 3 Walk/Zeta correspondence on torus

We state the Walk/Zeta correspondence on a finite torus. At first, we give the definition of the $2d$-state discrete-time walk on $T^d_N$. The discrete-time walk is defined by using a *shift operator* and a *coin matrix* which will be mentioned below.
Let $f : V(T^d_N) \longrightarrow \mathbb{C}^{2d}$. For $j = 1, 2, \ldots, d$ and $x = (x_1, \ldots, x_d) \in V(T^d_N)$, the shift operator $\tau_j$ is defined by

$$(\tau_j f)(x) = f(x - e_j),$$

where $\{e_1, e_2, \ldots, e_d\}$ denotes the standard basis of $\mathbb{R}^d$.

Let $A = [a_{ij}], i,j=1,2,\ldots,2d$ be a $2d \times 2d$ matrix with $a_{ij} \in \mathbb{C}$ for $i, j = 1, 2, \ldots, 2d$. We call $A$ the coin matrix. If $a_{ij} \in [0, 1]$ and $\sum_{i=1}^{2d} a_{ij} = 1$ for any $j = 1, 2, \ldots, 2d$, then the walk is a CRW. In particular, when $a_{11} = a_{12} = \cdots = a_{2d} = 1$ for any $i = 1, 2, \ldots, 2d$, this CRW becomes a RW. If $A$ is unitary, then the walk is a QW. So our class of walks contains RWs, CRWs, and QWs as special models.

To describe the evolution of the walk, we decompose the $2d \times 2d$ coin matrix $A$ as

$$A = \sum_{j=1}^{2d} P_j A,$$

where $P_j$ denotes the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\eta_j$ in $\mathbb{C}^{2d}$. Here $\{\eta_1, \eta_2, \ldots, \eta_{2d}\}$ denotes a standard basis on $\mathbb{C}^{2d}$.

The discrete-time walk associated with the coin matrix $A$ on $T^d_N$ is determined by the $2dN^d \times 2dN^d$ matrix

$$M_A = \sum_{j=1}^{d} \left( P_{2j-1} A \tau_j^{-1} + P_{2j} A \tau_j \right).$$

(2)

Let $\mathbb{Z}_\geq = \mathbb{Z} \cup \{0\}$. Then, the state at time $n \in \mathbb{Z}_\geq$ and location $x \in V(T^d_N)$ can be expressed by a $2d$-dimensional vector:

$$\Psi_n(x) = \begin{bmatrix} \Psi^1_n(x) \\ \Psi^2_n(x) \\ \vdots \\ \Psi^{2d}_n(x) \end{bmatrix} \in \mathbb{C}^{2d}.$$

For $\Psi_n : V(T^d_N) \longrightarrow \mathbb{C}^{2d}$ ($n \in \mathbb{Z}_\geq$), from Eq. (2), the evolution of the walk is defined by

$$\Psi_{n+1}(x) \equiv (M_A \Psi_n)(x) = \sum_{j=1}^{d} \left( P_{2j-1} A \Psi_n(x + e_j) + P_{2j} A \Psi_n(x - e_j) \right).$$

(3)

Now, we define the walk-type zeta function by

$$\bar{\zeta} \left( A, T^d_N, u \right) = \det \left( I_{2dN^d} - u M_A \right)^{-1/N^d}.$$

(4)
In general, for a $d_c \times d_c$ coin matrix $A$, we put
\[
\zeta \left( A, T_N^d, u \right) = \det \left( I_{d_c N^d} - u M_A \right)^{-1/N^d}.
\]

Komatsu et al. [11] obtained the following result.

**Theorem 3.1** (Komatsu et al. [11])

\[
\zeta \left( A, T_N^d, u \right)^{-1} = \exp \left[ \frac{1}{N^d} \sum_{\tilde{k} \in \mathbb{Z}_N^d} \log \left\{ \det \left( F(\tilde{k}, u) \right) \right\} \right],
\]

\[
\lim_{N \to \infty} \zeta \left( A, T_N^d, u \right)^{-1} = \exp \left[ \int_{(0,2\pi)^d} \log \left\{ \det \left( F(\Theta^{(d)}, u) \right) \right\} d\Theta^{(d)} \right],
\]

where $\tilde{K}_N = \{0, 2\pi/N, \ldots, 2\pi(N - 1)/N\}$, $\Theta^{(d)} = (\theta_1, \theta_2, \ldots, \theta_d) \in [0, 2\pi)^d$ and $d\Theta^{(d)}_{\text{unif}}$ denotes the uniform measure on $[0, 2\pi)^d$, that is,

\[
d\Theta^{(d)}_{\text{unif}} = \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}.
\]

Furthermore,

\[
F \left( \mathbf{w}, u \right) = I_{2d} - u \hat{M}_A(\mathbf{w}) \quad \text{and} \quad \hat{M}_A(\mathbf{w}) = \sum_{j=1}^{d} \left( e^{iw_j} P_{2j-1} A + e^{-iw_j} P_{2j} A \right),
\]

with $\mathbf{w} = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d$.

Now, we give an example. Let $d = 1$ and $G = T_N^1 = (\mathbb{Z} \mod N)$, where $N \geq 3$. Note that $T_N^1$ is the cycle graph with $N$ vertices. Then, we have

\[
V(T_N^1) = \{0, 1, \ldots, N - 1\}.
\]

Furthermore, let $A$ be the $2 \times 2$ coin matrix as follows:

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C},
\]

and let

\[
P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then, we have

\[
P_1 A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad P_2 A = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}.
\]

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and so

\[ A = P_1 A + P_2 A. \]

Next, let \( e = e_1 = (1) \) be the standard basis of \( \mathbb{R}^1 \). Then, for \( f : V(T_{N}^1) \longrightarrow \mathbb{C}^2 \) and the location \( x \in V(T_{N}^1) \), the shift operator \( \tau = \tau_1 \) is defined by

\[ (\tau f)(x) = f(x - e). \]

Thus, we have

\[
\tau = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\tau^{-1} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

Therefore, it follows that

\[
M_A = P_1 A \tau^{-1} + P_2 A \tau = \tau^{-1} \otimes P + \tau \otimes Q
\]

\[
= \begin{bmatrix}
O & P & O & \cdots & \cdots & O & Q \\
Q & O & P & \cdots & \cdots & O & O \\
O & Q & O & P & \cdots & \cdots & O & O \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
P & O & O & \cdots & \cdots & Q & O
\end{bmatrix},
\]

where

\[ P = P_1 A, \quad Q = P_2 A. \]

Furthermore, the Kronecker product \( A \otimes B \) of matrices \( A \) and \( B \) is considered as the matrix \( A \) having the element \( a_{ij} \) replaced by the matrix \( a_{ij} B \).

Let the state at time \( n \in \mathbb{Z}_> \) and location \( x \in V(T_{N}^1) \) be expressed by

\[ \Psi_n(x) = \begin{bmatrix} \Psi_n^1(x) \\ \Psi_n^2(x) \end{bmatrix} \in \mathbb{C}^2. \]

Furthermore, let

\[ \Psi_n = \begin{bmatrix} \Psi_n(0) \\ \Psi_n(1) \\ \vdots \\ \Psi_n(N - 1) \end{bmatrix} \in \mathbb{C}^{2N}. \]
Then, the time evolution of the walk is given by
\[ \Psi_{n+1} = M_A \Psi_n. \]

If \( A \) is unitary, then the walk is called a *coined quantum walk* on \( T_N^1 \).

Let \( \text{Spec}(F) \) be the set of eigenvalues of a square matrix \( F \). Then, we have
\[ \text{Spec}(\tau^{-1}) = \{ e^{ik} \mid k \in \tilde{K}_N \}. \]

Therefore, it follows that
\[
\tilde{\zeta} (A, T_N^1, u)^{-1} = \det (I_{2N} - u M_A)^{1/N} \\
= \prod_{k \in \tilde{K}_N} \det \left( I_2 - u(e^{ik} P + e^{-ik} Q) \right)^{1/N} \\
= \exp \left[ \frac{1}{N} \sum_{k \in \tilde{K}_N} \log \left\{ \det \left( I_2 - u(e^{ik} P + e^{-ik} Q) \right) \right\} \right].
\]

4 A vertex-face walk on a graph

We introduce a discrete-time quantum walk on an orientable embedding of a graph. An embedding of a graph is *circular* if every face is bounded by a cycle.

Let \( G \) be a graph, and \( \mathcal{M} \) a circular embedding of \( G \) on some orientable surface. Then, we consider a consistent orientation of the faces, that is, for each edge \( e \) shared by two faces \( f \) and \( h \), the direction \( e \) receives in \( f \) is opposite to the direction it receives in \( h \). For such an orientation of faces, every arc belongs to exactly one face.

Let \( G \) have \( n \) vertices, \( m \) edges and \( k \) faces, and \( F(G) \) the set of faces of \( \mathcal{M} \). Then, let \( M = (M_{ef})_{e \in D(G); f \in F(G)} \) be the associated arc-face incidence matrix of \( G \) defined as follows:

\[
M_{ef} = \begin{cases} 
1 & \text{if } e \in f, \\
0 & \text{otherwise}.
\end{cases}
\]

Furthermore, let \( N = (N_{ev})_{e \in D(G); v \in V(G)} \) be the associated arc-origin incident matrix defined as follows:

\[
N_{ev} = \begin{cases} 
1 & \text{if } o(e) = v, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \hat{M} \) and \( \hat{N} \) be normalized versions of \( M \) and \( N \), respectively. Note that
\[
\hat{M}^T \hat{M} = I_k, \quad \hat{N}^T \hat{N} = I_n.
\]
Then, the following unitary matrix \( U \) is the transition matrix of a \textit{vertex-face walk} for \( \mathcal{M} \):

\[
U = (2 \hat{M} \hat{M}^T - I_{2m})(2 \hat{N} \hat{N}^T - I_{2m}).
\]

There are directly not related between a vertex-face walk and a coined quantum walk on a graph. A vertex-face walk on a graph is closely related to a staggered quantum walk on a it (see [15]). Thus, a vertex-face walk on a graph can be not caught from Grover/Zeta and Walk/Zeta Correspondences.

Now, we obtain the following formula for the transition matrix of a vertex-face walk on a graph.

\textbf{Theorem 4.1} Let \( G \) be a circular embedding graph on some orientable surface, and have \( n \) vertices, \( m \) edges and \( k \) faces. Then, for the transition matrix \( U \) of a vertex-face walk on \( G \),

\[
\det(I_{2m} - uU) = (1 - u)^{2m-n-k}(1 + u)^{n-k} \det((1 + u)^2 I_k - 4u \hat{M}^T \hat{N} \hat{N}^T \hat{M}).
\]

\textbf{Proof} By the definition of the transition matrix \( U \), we have

\[
\det(I_{2m} - uU) = \det(I_{2m} - u(2 \hat{M} \hat{M}^T - I_{2m})(2 \hat{N} \hat{N}^T - I_{2m}))
\]

\[
= \det((1 - u)I_{2m} + 2u \hat{N} \hat{N}^T - 2u \hat{M} \hat{M}^T (2 \hat{N} \hat{N}^T - I_{2m}))
\]

\[
= (1 - u)^{2m} \det \left( I_{2m} + \frac{2u}{1 - u} \hat{N} \hat{N}^T - \frac{2u}{1 - u} \hat{M} \hat{M}^T \left( 2 \hat{N} \hat{N}^T - I_{2m} \right) \right)
\]

\[
= (1 - u)^{2m} \det \left( I_{2m} - \frac{2u}{1 - u} \hat{M} \hat{M}^T \left( 2 \hat{N} \hat{N}^T - I_{2m} \right) \left( I_{2m} + \frac{2u}{1 - u} \hat{N} \hat{N}^T \right)^{-1} \right)
\]

\[
\times \det \left( I_{2m} + \frac{2u}{1 - u} \hat{N} \hat{N}^T \right).
\]

If \( A \) and \( B \) are an \( m \times n \) and \( n \times m \) matrices, respectively, then we have

\[
\det(I_m - AB) = \det(I_n - BA).
\]

Thus, we have

\[
\det(I_{2m} - uU)
\]

\[
= (1 - u)^{2m} \det \left( I_k - \frac{2u}{1 - u} \hat{M}^T \left( 2 \hat{N} \hat{N}^T - I_{2m} \right) \left( I_{2m} + \frac{2u}{1 - u} \hat{N} \hat{N}^T \right)^{-1} \hat{M} \right)
\]

\[
\times \det \left( I_{2m} + \frac{2u}{1 - u} \hat{N} \hat{N}^T \right).
\]
But,
\[
\det(I_{2m} + \frac{2u}{1-u} \tilde{N}\tilde{N}^T) = \det \left( I_n + \frac{2u}{1-u} \tilde{N}\tilde{N}^T \right) = \det \left( I_n + \frac{2u}{1-u} I_n \right) = \left( 1 + \frac{2u}{1-u} \right)^n = \frac{(1+u)^n}{(1-u)^n}.
\]
Furthermore, we have
\[
\left( I_{2m} + \frac{2u}{1-u} \tilde{N}\tilde{N}^T \right)^{-1} = I_{2m} - \frac{2u}{1-u} \tilde{N}\tilde{N}^T + \left( \frac{2u}{1-u} \right)^2 \tilde{N}\tilde{N}^T \tilde{N}\tilde{N}^T - \left( \frac{2u}{1-u} \right)^3 \tilde{N}\tilde{N}^T \tilde{N}\tilde{N}^T \tilde{N}\tilde{N}^T + \cdots
\]
\[
= I_{2m} - \frac{2u}{1-u} \tilde{N}\tilde{N}^T + \left( \frac{2u}{1-u} \right)^2 \tilde{N}\tilde{N}^T - \left( \frac{2u}{1-u} \right)^3 \tilde{N}\tilde{N}^T + \cdots
\]
\[
= I_{2m} - \frac{2u}{1-u} \left( 1 - \frac{2u}{1-u} + \left( \frac{2u}{1-u} \right)^2 - \cdots \right) \tilde{N}\tilde{N}^T
\]
\[
= I_{2m} - \frac{2u}{1-u} \left( 1 + \frac{2u}{1-u} \right) \tilde{N}\tilde{N}^T = I_{2m} - \frac{2u}{1+u} \tilde{N}\tilde{N}^T.
\]
Therefore, it follows that
\[
\det(I_{2m} - uU) = (1-u)^{2m} \det \left( I_k - \frac{2u}{1-u} \tilde{M}^T \left( 2\tilde{N}\tilde{N}^T - I_{2m} \right) \left( I_{2m} - \frac{2u}{1+u} \tilde{N}\tilde{N}^T \right) \tilde{M} \right) \frac{(1+u)^n}{(1-u)^n}
\]
\[
= (1-u)^{2m-n} (1+u)^n \det \left( I_k - \frac{2u}{1-u} \tilde{M}^T \left( -I_{2m} + \frac{2}{1+u} \tilde{N}\tilde{N}^T \right) \tilde{M} \right)
\]
\[
= (1-u)^{2m-n} (1+u)^n \det \left( I_k + \frac{2u}{1-u} \tilde{M}^T \tilde{M} - \frac{4u}{1-u^2} \tilde{M}^T \tilde{N}\tilde{N}^T \tilde{M} \right)
\]
\[
= (1-u)^{2m-n-k} (1+u)^{n-k} \det \left( (1-u^2)I_k + 2u(1+u)I_k - 4u \tilde{M}^T \tilde{N}\tilde{N}^T \tilde{M} \right)
\]
\[
= (1-u)^{2m-n-k} (1+u)^{n-k} \det \left( (1+u)^2I_k - 4u \tilde{M}^T \tilde{N}\tilde{N}^T \tilde{M} \right).
\]

\[\square\]

Substituting \( u = 1/\lambda \), we obtain the following result.

**Corollary 4.2** Let \( G \) be a circular embedding graph on some orientable surface, and have \( n \) vertices, \( m \) edges and \( k \) faces. Then, for the transition matrix \( U \) of a vertex-face walk on \( G \),

\[
\det(\lambda I_{2m} - U) = (\lambda - 1)^{2m-n-k} (\lambda + 1)^{n-k} \det((\lambda + 1)^2I_k - 4\lambda \tilde{M}^T \tilde{N}\tilde{N}^T \tilde{M}).
\]
\textbf{Proof} Let \( u = 1/\lambda \). Then, by Theorem 4.1, we have

\[
det(I_{2m} - 1/\lambda U) = (1 - 1/\lambda)^{2m-n-k}(1 + 1/\lambda)^{n-k} \det((1 + 1/\lambda)^2I_k - 4/\lambda \hat{M}^T \hat{N}^T \hat{M}),
\]

and so,

\[
det(\lambda I_{2m} - U) = (\lambda - 1)^{2m-n-k}(\lambda + 1)^{n-k} \det((\lambda + 1)^2I_k - 4\lambda \hat{M}^T \hat{N}^T \hat{M}).
\]

By Corollary 4.2, the following result holds.

\textbf{Corollary 4.3} Let \( G \) be a circular embedding graph on some orientable surface, and have \( n \) vertices, \( m \) edges and \( k \) faces. Then, the spectra of the transition matrix \( U \) are given as follows:

\begin{enumerate}
  \item \( 2k \) eigenvalues:
    \[
    \lambda = (2\mu - 1) \pm 2\sqrt{\mu(\mu - 1)}, \quad \mu \in \text{Spec}(\hat{M}^T \hat{N}^T \hat{M});
    \]
  \item \( 2m - n - k \) eigenvalues: \( 1 \);
  \item \( n - k \) eigenvalues: \( -1 \).
\end{enumerate}

\textbf{Proof} Let \( \text{Spec}(\hat{M}^T \hat{N}^T \hat{M}) = \{\lambda_1, \ldots, \lambda_k\} \). Since \( \hat{M}^T \hat{N}^T \hat{M} \) is symmetric, we have

\[
\lambda_1, \ldots, \lambda_k \in \mathbb{R}.
\]

Furthermore, by Corollary 4.2, we have

\[
det(\lambda I_{2m} - U) \\
= (\lambda - 1)^{2m-n-k}(\lambda + 1)^{n-k} \prod_{\mu \in \text{Spec}(\hat{M}^T \hat{N}^T \hat{M})} ((\lambda + 1)^2 - 4\mu \lambda) \\
= (\lambda - 1)^{2m-n-k}(\lambda + 1)^{n-k} \prod_{\mu \in \text{Spec}(\hat{M}^T \hat{N}^T \hat{M})} (\lambda^2 - 2(2\mu - 1)\lambda + 1).
\]

Solving \( \lambda^2 - 2(2\mu - 1)\lambda + 1 = 0 \), we obtain

\[
\lambda = (2\mu - 1) \pm 2\sqrt{\mu(\mu - 1)}.
\]

The result follows. \( \square \)
5 The vertex-face walk on the two-dimensional finite torus

At first, we state a result for the structure of the matrix $\hat{M}^T \hat{N} \hat{N}^T \hat{M}$ by Zhan [22].

Let $G$ be a circular embedding graph on some orientable surface, and have $n$ vertices, $m$ edges and $k$ faces. For a face $f \in F(G)$, let $|f|$ be the number of vertices (or arcs) contained in the boundary of $f$.

**Proposition 5.1** (Zhan [22]) Let $G$ be a circular embedding graph on some orientable surface, and have $n$ vertices, $m$ edges and $k$ faces. Furthermore, let $K = \hat{M}^T \hat{N} \hat{N}^T \hat{M}$. Then, for $f, h \in F(G)$, the $(f, h)$-entry of $K$ is

$$K_{fh} = \frac{1}{\sqrt{|f||h|}} \sum_{u \in f \cap h} \frac{1}{\deg u},$$

where $f \cap h$ denotes the set of vertices used by both $f$ and $h$.

Now, for a natural number $N \geq 2$, let $G = T_{2N}^2$ be the two-dimensional finite torus (graph). The two-dimensional finite torus $T_{2N}^2$ is circular embedded on the two-dimensional torus $T^2$, and isomorphic to itself on $T^2$. Then, we have

$$n = |V(T_{2N}^2)| = N^2, \quad m = |E(T_{2N}^2)| = 2N^2, \quad k = |F(T_{2N}^2)| = N^2.$$ 

Furthermore, $T_{2N}^2$ is a 4-regular graph, and the boundary of each face of $T_{2N}^2$ has four vertices.

Then, the following result holds for the transition matrix $U$ of the vertex-face walk on $T_{2N}^2$.

**Theorem 5.2** Let $G = T_{2N}^2$ be the two-dimensional finite torus. Then,

$$\det(I_{4N^2} - uU) = (1 - u)^{2N^2} \det((1 + u)^2 I_{N^2} - \frac{u}{4}(A^2 + 2A)).$$

**Proof** Let $n = |V(T_{2N}^2)| = N^2, \quad m = |E(T_{2N}^2)| = 2N^2, \quad k = |F(T_{2N}^2)| = N^2, \quad V(T_{2N}^2) = \{v_1, \ldots, v_n\}, \quad f_1, \ldots, f_k (k = N^2)$ be the faces of $T_{2N}^2$. For $i = 1, \ldots, k$, let $f_i = ((f_i)_u)_{u \in V(G)}$ be the $n$-dimensional vector such that

$$(f_i)_u = \begin{cases} 1 & \text{if } u \in f_i, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for each $f \in F(G)$ and $u \in V(G)$, we have

$$|f| = 4 \quad \text{and} \quad \deg u = 4.$$ 

By Proposition 5.1, we have

$$K = \hat{M}^T \hat{N} \hat{N}^T \hat{M} = \frac{1}{16} (f_i \cdot f_j)_{1 \leq i, j \leq k}.$$

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where $f_i \cdot f_j$ is the inner product of two vectors $f_i$ and $f_j$.

But, we have

$$f_i \cdot f_j = |f_i \cap f_j| \quad (1 \leq i, j \leq k).$$

If $i = j$, then

$$f_i \cdot f_j = 4.$$ 

If $|f_i \cap f_j| = 2$, then $f_i$ and $f_j$ are adjacent in the dual graph $G^*$. Thus, we have

$$\mathbf{A}(G^*)_{ij} = 1 \quad i f \quad |f_i \cap f_j| = 2.$$ 

Next, if $|f_i \cap f_j| = 1$, then $f_i$ and $f_j$ are not adjacent in the dual graph $(T^2_N)^*$, and there exist exactly two paths of length 2 from $f_i$ to $f_j$ in $(T^2_N)^*$. Let $A_2(G^*)$ be the an $n \times n$ matrix such that

$$(A_2(G^*))_{ij} = \begin{cases} 2 & \text{if there exists a path of length 2 from } f_i \text{ to } f_j \text{ in } G^*, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$(A_2(G^*))_{ij} = 2 \quad i f \quad |f_i \cap f_j| = 1.$$ 

Therefore, it follows that

$$K = \frac{1}{16}(4I_n + 2A(G^*) + A_2(G^*)).$$

But, since

$$A(G^*)^2 = A_2(G^*) + 4I_n,$$

we have

$$K = \frac{1}{16}(4I_n + 2A(G^*) + A_2(G^*))$$

$$= \frac{1}{16}(4I_n + 2A(G^*) + A^2(G^*) - 4I_n) = \frac{1}{16}A(G^*)^2 + \frac{1}{8}A(G^*).$$

Since $T^2_N$ and $(T^2_N)^*$ are isomorphic, we have

$$A(G^*) = A(G) = A.$$
up to reordering the rows and columns. Thus,

\[
\det(I_{4N^2} - uU) = (1 - u)^{2m-n-k}(1 + u)^{n-k} \det((1 + u)^2 I_k - 4u \left( \frac{1}{16} A^2 + \frac{1}{8} A \right))
\]

\[
= (1 - u)^{2N^2} \det((1 + u)^2 I_{N^2} - \frac{u}{4}(A^2 + 2A)).
\]

Therefore,

**Corollary 5.3** Let \( G = T^2_N \) be the two-dimensional finite torus. Then,

\[
\det(\lambda I_{4N^2} - U) = (\lambda - 1)^{2N^2} \prod_{k_1=0}^{N-1} \prod_{k_2=0}^{N-1} \left[ \lambda^2 - \lambda \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right)^2 + \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} - 2 \right] + 1 \right].
\]

**Proof** It is known that

\[
\text{Spec}(T^2_N) = \{ \cos \left( \frac{2\pi k_1}{N} \right) + 2 \cos \left( \frac{2\pi k_2}{N} \right) | k_1, k_2 = 0, 1, \ldots, N - 1 \}.
\]

By Theorem 5.2, we have

\[
\det(\lambda I_{4N^2} - U) = (\lambda - 1)^{2N^2} \det\left( (\lambda + 1)^2 I_{N^2} - \frac{\lambda}{4}(A^2 + 2A) \right)
\]

\[
= (\lambda - 1)^{2N^2} \prod_{k_1=0}^{N-1} \prod_{k_2=0}^{N-1} \left( (\lambda + 1)^2 - \frac{\lambda}{4} \left( 4 \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right)^2 + 2 \right) \right.
\]

\[
+ 2 \cdot 2 \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right) \right)
\]

\[
= (\lambda - 1)^{2N^2} \prod_{k_1=0}^{N-1} \prod_{k_2=0}^{N-1} \left( \lambda^2 - \lambda \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right)^2 + \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} - 2 \right) + 1 \right].
\]
Similarly to the Walk/Zeta correspondence of [11], we introduce a zeta function for the transition matrix $U$ of the vertex-face walk on $G = T_N^2$ as follows:

$$\bar{\zeta}(U, T_N^2, u) = \det(I_{4N^2} - uU)^{-1/N^2}.$$ 

Let $A = A(T_N^2)$. By Theorem 5.2 and Corollary 5.3, we have

$$\bar{\zeta}(U, T_N^2, u)^{-1} = \det(I_{4N^2} - uU)^{1/N^2}$$

$$= (1 - u)^{2N^2} \det((1 + u)^2 I_{N^2} - \frac{u}{4}(A^2 + 2A)))^{1/N^2}$$

$$= (1 - u)^2 \exp\left[\frac{1}{N^2} \log \det \left((1 + u)^2 I_{N^2} - \frac{u}{4}(A^2 + 2A)\right)\right]$$

$$= (1 - u)^2 \exp\left[\frac{1}{N^2} \prod_{k_1=0}^{N-1} \prod_{k_2=0}^{N-1} \log \left\{ u^2 - u \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right)^2 + \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} - 2 \right\} + 1 \right].$$

When $N \to \infty$, we have

$$\lim_{N \to \infty} \bar{\zeta}(U, T_N^2, u)^{-1}$$

$$= (1 - u)^2 \exp\left[\int_0^{2\pi} \int_0^{2\pi} \log\left\{ u^2 - u \left( \cos \theta_1 + \cos \theta_2 \right)^2 + \cos \theta_1 + \cos \theta_2 - 2 \right\} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right].$$

**Theorem 5.4** Let $G = T_N^2$ be the two-dimensional finite torus. Then,

$$\bar{\zeta}(U, T_N^2, u)^{-1} = (1 - u)^2 \exp\left[\frac{1}{N^2} \prod_{k_1=0}^{N-1} \prod_{k_2=0}^{N-1} \log \left\{ u^2 - u \left( \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} \right)^2 + \cos \frac{2\pi k_1}{N} + \cos \frac{2\pi k_2}{N} - 2 \right\} + 1 \right].$$
and

$$
\lim_{N \to \infty} \z(\mathbf{U}, T_N^2, u)^{-1} = (1 - u)^2 \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ u^2 - u((\cos \theta_1 + \cos \theta_2)^2 + \cos \theta_1 + \cos \theta_2 - 2) + 1 \right\} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right].
$$

6 Future works

In this paper, we presented a spectrum of the transition matrix of a vertex-face walk on a graph, and obtained an explicit formula for the characteristic polynomial of the transition matrix of a vertex-face walk on the two-dimensional torus as a corollary. As an application, we defined a new walk-type zeta function with respect to the transition matrix of a vertex-face walk on the two-dimensional torus, and presented its explicit formula.

As a vertex-face walk on a graph is closely related to a staggered quantum walk on it, we would like to study the details of the relationship between them, and formulate a new walk-type zeta function with respect to the transition matrix of a staggered quantum walk on a graph.

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