MAGNETAR GIANT FLARES IN MULTIPOLAR MAGNETIC FIELDS. I. FULLY AND PARTIALLY OPEN ERUPTIONS OF FLUX ROPES

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ABSTRACT

We propose a catastrophic eruption model for the enormous energy release of magnetars during giant flares, in which a toroidal and helically twisted flux rope is embedded within a force-free magnetosphere. The flux rope stays in stable equilibrium states initially and evolves quasi-statically. Upon the loss of equilibrium, the flux rope cannot sustain the stable equilibrium states and erupts catastrophically. During the process, the magnetic energy stored in the magnetosphere is rapidly released as the result of destabilization of global magnetic topology. The magnetospheric energy that could be accumulated is of vital importance for the outbursts of magnetars. We carefully establish the fully open fields and partially open fields for various boundary conditions at the magnetar surface and study the relevant energy thresholds. By investigating the magnetic energy accumulated at the critical catastrophic point, we find that it is possible to drive fully open eruptions for dipole-dominated background fields. Nevertheless, it is hard to generate fully open magnetic eruptions for multipolar background fields. Given the observational importance of the multipolar magnetic fields in the vicinity of the magnetar surface, it would be worthwhile to explore the possibility of the alternative eruption approach in multipolar background fields. Fortunately, we find that flux ropes may give rise to partially open eruptions in the multipolar fields, which involve only partial opening of background fields. The energy release fractions are greater for cases with central-arcaded multipoles than those with central-caved multipoles that emerged in background fields. Eruptions would fail only when the centrally caved multipoles become extremely strong.

Key words: instabilities – pulsars: general – stars: magnetars – stars: magnetic field – stars: neutron

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1. INTRODUCTION

Two small subsets of neutron stars—anomalous X-ray pulsars and soft gamma-ray repeaters (SGRs)—are interpreted as magnetars, which are neutron stars endowed with ultra-strong (10¹⁴–10¹⁵ G) magnetic fields (Mazets et al. 1979; Mereghetti & Stella 1995; Kouveliotou et al. 1998; Gavriil et al. 2002). Magnetic energy dissipation is commonly believed to account for their high energy persistent emissions and spasmodic outbursts (Duncan & Thompson 1992; Woods & Thompson 2006; Mereghetti 2008). One of the most intriguing phenomena related to magnetars is the episodic giant flare, which involves tremendous magnetic energy release (Hurley et al. 2005). The physical origin of giant flares still remains the bewildering enigma in high energy astrophysics. Magnetospheric eruption models (e.g., Lyutikov 2006; Gill & Heyl 2010; Yu 2012; Yu & Huang 2013) can naturally explain the short timescale, ~0.25 ms, of the rise time in giant flares (Palmer et al. 2005), although the precise mechanism of the eruption is still under debate (Mereghetti 2013). It is worthwhile to note that, in the magnetospheric eruption model, the magnetosphere stays in a stable equilibrium state at the pre-eruptive stage, in which the magnetosphere evolves quasi-statically. As a result, the magnetic energy released in an eruption is gradually accumulated on a timescale much longer than the dynamical timescale of giant flares. Since energy accumulation takes place very gradually, the answer to the question of how such long timescale events could initiate the sudden energy release, i.e., giant flares, on a very short dynamical timescale remains elusive for the magnetospheric eruption model.

To resolve this puzzle, a catastrophic flux rope eruption model has been put forward to explain the magnetar giant flares (Yu 2012; Yu & Huang 2013). In this particular scenario, the giant flare is considered to be driven by the destabilization of large-scale magnetospheric magnetic fields rather than the abrupt fracture of the neutron star crust. The most distinguishing feature in our model is that a helically twisted flux rope is embedded within the magnetosphere. Magnetic flux ropes could be naturally generated due to the magnetic helicity injections from the magnetar interior (Thompson et al. 2002; Lyutikov 2006; Götz et al. 2007; Gill & Heyl 2010). Such flux ropes are also an indispensable ingredient to explain the radio afterglow of SGR 1806 (Gaensler et al. 2005; Lyutikov 2006). It is interesting to note that the interior of the flux rope is helically twisted. The magnetic twist is locally confined within the flux rope, which is in variance with the global twist proposed by recent authors (Lyutikov 2013; Parfrey et al. 2013). These locally twisted magnetic features, when compared to the global twist, seem to be more relevant to the recent observations (e.g., Woods et al. 2007; Perna & Gotthelf 2008).

The flux rope eruption model tries to resolve a primary issue concerning the trigger mechanism of the magnetar eruption. The most appealing characteristic is that, during the flux rope’s evolution, it could make the catastrophic state transition from stable equilibrium states to unstable equilibrium states spontaneously, in accordance with the variations at the neutron star surface. The emergence of flux from the interior (Kluźniak & Ruderman 1998; Götz et al. 2007) and/or the shuffling of the crust (Ruderman 1991) cause the flux rope to evolve to
a critical loss of equilibrium point, beyond which no stable equi-
librium state can be sustained and the flux rope erupts abruptly.
It is widely accepted that the energy is progressively accumu-
lated in an initially closed force-free field before the flux rope
reaches the critical loss of equilibrium point. The flux rope’s
subsequent catastrophic eruption, beyond this particular point,
leads to the opening of an initially closed magnetic field config-
uration as well as a huge energy release. Since the stored
energy prior to an eruption is determined by the intrinsic prop-
erties of the magnetosphere rather than the tensile strength of
the crust (Yu 2012), a fundamental question is raised naturally
as to whether the magnetic energy in the force-free magneto-
sphere could build up enough energy to support eruptive giant
flares.

The fierce eruptive event is thought to open the pre-eruptive,
originally closed magnetic field lines. Observationally, in the
post-eruption epoch of giant flares, magnetic field configura-
tions are indeed inferred to stretch outward to form an open field
configuration. In other words, in addition to powering the giant
flares, the magnetic free energy has to be able to open the initially
closed magnetic fields. This constitutes a serious bottleneck
for magnetar giant flares because it is realized that to open
the initially closed field lines requires considerable extra work
to be done (Aly 1984, 1991; Sturrock 1991; Yu 2012). To
get over the energy threshold constrained by the post-eruptive
open field configurations, the pre-eruptive magnetosphere must
accumulate more energy in excess of the threshold set by the
open field configurations. It should be pointed out that the field
lines can also be opened at a greater distance from the star by
strong neutron star winds (e.g., Bucciantini et al. 2006). How
the wind affects the magnetar eruption energetics is still an open
issue. For simplicity, we only consider the case in which the field
lines are opened by the process of magnetic flux injection from
the neutron star interior.

The observational feature of a strong four-peaked pattern
in the pulse profile of the 1998 August 27 event from SGR
1900+14 indicates that the geometry of the magnetic field
was quite complicated in regions close to the star (Feroci
et al. 2001). Recent calculations also show that multipolar
magnetic fields may also have important effects on the emission
of the magnetars (Pavan et al. 2009). As a result, it is reasonable
to infer that, in the very vicinity of the magnetar surface,
the field configuration involves higher multipoles. The electric
currents formed during the birth of magnetars slowly push out
from within the magnetar and generate active regions on the
magnetar surface. These active regions manifest themselves
as multipolar regions on the magnetar surface. The flux rope
eruption in multipolar background fields, unlike the behavior in
dipole background fields, may just involve the opening of part
of the closed magnetic flux systems. Thus, it is interesting to
explore the physical behavior of the flux rope in response to
these more complex boundary conditions. However, for more
complex boundary conditions, no solid investigations about
the energy threshold specified by the fully/partially open field
configurations have been performed. A related question, which
is of crucial significance for the physical feasibility of the flux
rope eruption model, i.e., whether the flux rope could build
up enough energy to drive the giant flare with these complex
boundary conditions, also remains to be answered.

In this paper, we will establish a force-free magnetosphere
model with a helically twisted flux rope and examine the physi-
cal response of the flux rope to the variations of the background
multipolar magnetic fields. We perform rigorous calculations
on the magnetic energy accumulation in magnetospheres in various
configurations. Specifically, boundary conditions containing the
dipolar term and the high order multipolar term are considered
in the background closed magnetic field. Two kinds of open
configurations, the partially open and fully open fields, are con-
sidered to provide energy thresholds for eruptions. This paper
is structured as follows. The model of pre-eruptive magneto-
spheres with flux ropes and the multipolar boundary conditions
are introduced in Section 2. The physical behavior of the pre-
eruptive flux rope is described in Section 3, including the equi-
librium constraints and catastrophic loss of equilibrium of the
flux rope. The energetics of the flux rope in the multipolar back-
ground fields are investigated in Section 4. Conclusions and
discussions are provided in Section 5.

2. PRE-ERUPTIVE FORCE-FREE MAGNETOSPHERES
WITH EMBEDDED FLUX ROPES

2.1. Force-free Magnetospheres with Helically
Twisted Flux Ropes

In our model, the most distinctive characteristic is that there
exists a toroidal and helically twisted flux rope in the pre-
eruptive magnetosphere. It is possible that the precursor activity
of a giant flare may inject a certain amount of magnetic helicity
into the magnetosphere and generate such toroidal and helically
twisted flux ropes (Thompson et al. 2002; Görtz et al. 2007;
Gill & Heyl 2010). The toroidal flux rope has a major radius,
r, which can also be understood as the height of the flux rope
measured from the magnetar center, and a minor radius, r0,
which is small compared to r. The magnetic twist of the flux
rope is confined within the flux rope, which is unlike the globally
twisted magnetic field configurations proposed by some authors
(Thompson et al. 2002; Beloborodov 2009; Parfrey et al. 2013;
Lytikov 2013). These authors considered a non-potential force-
free field where the electric currents flow through the entire
magnetosphere. In comparison, our model only contains an
electric current in a spatially restricted region, viz., inside the
helically twisted flux rope. The magnetic field generated by
the current inside the flux rope can be represented by a wire
carrying a current I at the center of the flux rope (Forbes &
Priest 1995).

The presence of the flux rope separates the magnetosphere
into two regions; one is the region inside the flux rope. Further
details on the solution inside the flux rope are discussed in
Yu (2012). The other is the region outside the flux rope, in which
the steady state axisymmetric magnetic field B takes the follow-
ing form in spherical coordinates (r, θ, φ)

\[ B = -\frac{1}{r^2} \frac{\partial \Psi}{\partial \mu} \hat{e}_r - \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r} \hat{e}_\theta, \]

(1)

where \( \Psi(r, \mu) \) is the magnetic stream function and \( \mu = \cos \theta \).
Here \( \hat{e}_r \) and \( \hat{e}_\theta \) are the unit vectors along the radial and latitudinal
directions, respectively. The force-free condition can be expres-
sed in terms of the Grad–Shafranov (GS) equation, which explicit-
lly reads

\[ \frac{\partial^2 \Psi}{\partial r^2} + \frac{(1 - \mu^2)}{r^2} \frac{\partial^2 \Psi}{\partial \mu^2} = -r \sin \theta \frac{4\pi}{c} J_\phi, \]

(2)
Figure 1. Left: equilibrium curve of the flux rope in a dipole-dominated background with $a_1 = -1/4$ and $r_{00} = 0.01$, which shows the dependence of the height of the flux rope on the magnetic flux at the magnetar surface. The magnetospheric field configuration. The lower stable branch and the upper unstable branch are shown by the solid line and dotted line, respectively. The red dot marks the critical point. Right: magnetosphere with an embedded flux rope in the pre-eruptive critical state. The thick solid semi-circle represents the magnetar surface. The dashed line represents a circle with a radius equal to the critical height of the flux rope $h_c$. (A color version of this figure is available in the online journal.)

Figure 2. Same as Figure 1 but with a multipole-dominated background with $a_1 = 2/3$. (A color version of this figure is available in the online journal.)

where $c$ is the speed of light and the current density $J_\phi$ on the right-hand side in this inhomogeneous GS equation is induced by the toroidal flux rope, which is of the following form (Priest & Forbes 2000; Yu 2012):

$$J_\phi = \frac{I}{h} \delta(\mu) \delta(r - h), \quad (3)$$

where $I$ designates the electric current in the flux rope. According to the variable separation method, the general solution to the GS equation can be conveniently written as

$$\Psi(r, \mu) = \sum_{i=0}^{\infty} \left[ c_{2i+1} R_{2i+1}(r) + d_{2i+1} r^{-2i-1} \right] \times \left[ P_{2i}(\mu) - P_{2i+2}(\mu) \right], \quad (4)$$

where $P_{2i}(\mu)$ is the Legendre polynomial and $R_{2i+1}(r)$ is a continuous function of $r$ (see Yu 2011, 2012). The coefficients $c_{2i+1}$ are determined by the current inside the flux rope. Once the magnetar surface boundary conditions are fixed, the coefficients $d_{2i+1}$ can be readily specified in terms of $c_{2i+1}$ and the boundary conditions. More technical details to obtain solutions of the GS equation can be found in Yu (2012). Once we obtain the spatial distribution of the magnetic stream function, the magnetic field configuration in the magnetosphere can be determined. Illustrative figures of the magnetic field configurations are shown in panel (b) of Figures 1 and 2. The height of the flux rope (shown by the dashed line) in these two figures is $1.27 r_s$ and $2.20 r_s$, respectively ($r_s$ is the magnetar radius, shown by the thick solid line in these figures). In this paper we find that boundary conditions have important influences on the flux rope eruptions and in the following section we will further discuss the boundary conditions we adopt.

2.2. Multipolar Boundary Conditions and Post-eruptive Energy Thresholds

The GS equation is solved in the range $(r_s, \infty)$, where $r_s$ is the magnetar radius. The boundary conditions both at $r = r_s$ and $r \rightarrow \infty$ must be explicitly specified before we solve the GS equation (2). The physical requirement that $|\nabla \Psi| \rightarrow 0$ for $r \rightarrow \infty$ can be trivially satisfied (Yu 2012). At the magnetar surface $r = r_s$, we adopt the multipolar boundary condition as (e.g., Antiochos et al. 1999; Zhang & Low 2001)

$$\Psi(r_s, \mu) = \Psi_0 \sigma \Theta(\mu), \quad (5)$$

where $\Psi_0$ is a constant with magnetic flux dimension and the dimensionless variable $\sigma$ indicates the magnitude of magnetic flux at the magnetar surface. The large-scale field configuration of the neutron star is essentially a dipole field. However, in the vicinity of the neutron star surface, which is exactly the location where the catastrophic loss of equilibrium takes place, the magnetic field may be more complicated than a simple dipole (Feroci et al. 2001). To model multipolar regions on the neutron star surface, we include high order multipolar components in
addition to the dipole field and the angular dependence of the function \( \Theta(\mu) \) can be written explicitly as

\[
\Theta(\mu) = (1 - \mu^2) + a_1 (5\mu^2 - 1)(1 - \mu^2). \tag{6}
\]

The first term \((1 - \mu^2)\) denotes the dipolar component of the magnetic fields and the additional term represents the contributions from high order multipolar components. The parameter \( a_1 \) determines the strength of the multipoles. The value of \( a_1 \) can be either positive or negative; the schematic illustration of the field configurations for different signs of \( a_1 \) are shown in middle- and bottom-left panels of Figure 6 in Appendix A. In this paper, we confine \( a_1 \) in the range of \([-1, 1]\). Note that larger values of \(|a_1|\) may indicate stronger magnetic activity of the magnetar (Kluźniak & Ruderman 1998; Götz et al. 2007; Pavan et al. 2009).

In the post-eruption epoch of giant flares, magnetic field lines are stretched outward to form open field structures (Wood et al. 2001). The possible post-eruptive open field configurations are specified by the profile of the flux distribution at the magnetars surface. When the absolute value of the parameter \( a_1 \) is small, the background magnetic field is basically dipolar. The demarcation between dipole-dominated fields and multipole-dominated fields is determined by how many extremum points exist in the profile of the boundary flux distribution. More explicitly, if the parameter \( a_1 \) is in the range of \([-1/4, 1/6]\), only one extremum point exists at \( \mu = 0 \) in the boundary flux profile and the background magnetospheric field is essentially dipole-dominated. Otherwise, the boundary flux function we take has three extremum points at \( \mu_0 = 0, \mu_1 = \sqrt{(6a_1 - 1)/10a_1} \), and \( \mu_2 = -\mu_1 \), respectively (see Figure 6 in Appendix A). Due to the existence of the multiple extremum points, the multipolar configurations naturally arise in the background field. If \( a_1 \in (1/6, 1] \), the background field has a central-caved profile on the boundary. If \( a_1 \in (-1, -1/4) \), the background field has a central-arcaded profile on the boundary. Note that for a dipole-dominated background, there is only a single closed flux system in the magnetosphere (see top-left panel in Figure 6). The full opening of the magnetic field specifically means the opening of this particular closed flux system. However, for a multipole-dominated background, there are multiple closed flux systems in the magnetosphere (see the middle and bottom left panels in Figure 6). The full opening of the magnetic fields surely involves all the closed flux systems. However, it is possible that only part of the closed flux systems opens up and the post-eruptive field is then called a partially open field. Details to obtain both the fully and partially open field configurations according to the boundary conditions we adopt are further discussed in Appendix A.

Throughout this work, the magnetic energy is scaled by the energy of the potential field satisfying \( \nabla \times B = 0 \), which has minimum magnetic energy and is denoted by \( W_{\text{pot}} \). The magnetic energy of fully and partially open fields is denoted by \( W_{\text{open}} \) and \( W_{\text{open}} \), respectively. Further descriptions on how to calculate the magnetic energy of these two open states are given in Appendix B. The fully and partially open fields constitute two energy thresholds, \( W_{\text{open}}/W_{\text{pot}} \) and \( W_{\text{open}}/W_{\text{pot}} \), for the flux rope eruptions. It is conceivable that the flux rope must accumulate enough magnetic energy to either fully or partially open the initially closed fields. More specifically, the magnetic energy stored in the critical pre-eruptive state, \( W_{\text{pre}}(h_c)/W_{\text{pot}} \), should be larger than the fully or partially open threshold. Note that the magnetic energy of the critical pre-eruptive state, \( W_{\text{pre}}(h_c) \), is closely related to the catastrophic behavior of the flux rope, which will be discussed in the following section.

3. CATASTROPHIC RESPONSE OF FLUX ROPES TO VARIATIONS ON THE MAGNETAR SURFACE

It should be pointed out that, at the pre-eruptive stage, the flux rope stays in a stable equilibrium state and evolves quasi-statically with variations at the neutron star surface. During the pre-eruptive stage, the flux rope is unable to erupt and the magnetic energy is gradually accumulated in the magnetosphere. Upon the reaching the critical loss of equilibrium point, the quasi-static evolution is replaced by subsequent dynamical evolution (Yu 2012; Yu & Huang 2013). The accumulated energy at the catastrophic loss of equilibrium is of particular significance. This is because, beyond this point, no further gradual energy buildup is allowed and the flux rope’s dynamic behavior should be supported by the accumulated energy at this point. We denote the pre-eruptive energy at this loss of equilibrium point as \( W_{\text{pre}} \). To calculate the pre-eruptive state energy, \( W_{\text{pre}} \), it is necessary to know when the flux rope begins to lose its equilibrium.

3.1. Equilibrium Constraints of Flux Ropes

We adopt the Lundquist (1950) force-free solution to represent the current density and field inside the toroidal flux rope. This solution, though originally derived for a straight cylindrical twisted flux rope, is still valid as long as the minor radius \( r_0 \) is much smaller than the major radius, \( h \). The axial magnetic flux conservation of the flux rope suggests that the minor radius \( r_0 \) is inversely proportional to the current carried by the flux rope \( I \) (Yu 2012)

\[
r_0 = r_00 I_0/I = r_00/J, \tag{7}
\]

where the dimensionless current \( J \) is defined by \( J \equiv I/I_0 \). The scaling current \( I_0 = \Psi_0 c/\gamma \) is determined by the magnetic flux constant \( \Psi_0 \) in Equation (5), the magnetar radius \( r_s \), and the speed of light \( c \). For numerical convenience, we scale the length by the magnetar radius \( r_s \), magnetic flux by \( \Psi_0 \), and current by \( I_0 \) in our following calculations. The parameter \( r_{00} \) is the value of \( r_0 \) when \( J = 1 \). Typically for a flux rope with minor radius of 0.1 km, the value of the parameter \( r_{00} \sim 0.01 \). (We adopt the typical neutron star radius \( r_s \sim 10 \text{ km.} \))

In what follows, we consider slow responses of the flux rope to changes at the magnetar surface, and thus the flux rope is assumed to stay in a quasi-static equilibrium state over a sufficiently long timescale. Two aspects of the equilibrium constraint are considered, i.e., the force balance condition and the ideal frozen-flux condition. The force balance condition is satisfied when the total force exerted on the flux rope vanishes. The current inside the flux rope provides an outward force. The magnitude of this force is equal to the current, \( I \), times the magnetic field, \( B_r \) (Shafranov 1966):

\[
B_r = I/c h \left( \frac{8 h}{r_0} - 1 \right), \tag{8}
\]

where \( h \) and \( r_0 \) are the major and minor radii of the flux rope, respectively. This current-induced force must be balanced by the external magnetic field \( B_r \). To calculate the external magnetic field \( B_r \), the contribution from the current inside the flux rope must be excluded (Yu 2012). Finally we can arrive at the
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mechanical force balance condition

\[
f(\sigma, J, h) \equiv B_r - B_\ell = 0, \quad (9)
\]

where the function \( f(\sigma, J, h) \) can be written explicitly as

\[
f(\sigma, J, h) = \frac{J}{h} \left( \ln \frac{8 J h}{r_00} - 1 \right) - \sum_{i=0}^{\infty} (2i + 1) \frac{P_{2i}(0) - P_{2i+2}(0)}{4i + 3} \frac{d_{2i+1}}{h^{2i+3}}.
\]

We use \( \sigma \) to represent the dimensionless magnitude of the magnetic flux at the magnetar surface, \( J \) the dimensionless current, and \( h \) the dimensionless height of the flux rope. Here the coefficients \( d_{2i+1} \) are already determined in Equation (4). Note that \( P_{2i}(0) \) and \( P_{2i+2}(0) \) are values of the Legendre polynomials at \( \mu = 0 \).

The ideal frozen-flux condition must also be satisfied for the magnetic stream function, \( \Psi \). It demands that the stream function on the edge of the flux rope remain constant during the system’s evolution, which provides a connection between the variations at the magnetar surface and the current flowing inside the flux rope. Explicitly this condition can be written in the following form

\[
g(\sigma, J, h) \equiv \Psi(h - r_0, 0) = \text{const}, \quad (10)
\]

where the function \( g(\sigma, J, h) \) is defined by

\[
g(\sigma, J, h) = \sum_{i=0}^{\infty} \left[ \frac{P_{2i}(0) - P_{2i+2}(0)}{4i + 3} \right] \frac{d_{2i+1}}{h^{2i+3}}.
\]

Note that the symbols in this equation have the same meaning as those in Equation (10) and the coefficients \( c_{2i+1} \) and \( d_{2i+1} \) are specified in Equation (4). With these two equations, the current \( J \) and the height of the flux rope \( h \) can be determined numerically according to the Newton–Raphson method for any given value of \( \sigma \). In the following we will investigate how the equilibrium height of the flux rope behaves with variations at the magnetar surface.

### 3.2. Loss of Equilibrium of Flux Ropes in Response to Surface Variations

The loss of equilibrium of the flux rope is triggered by slow changes at the magnetar surface. There are two possible long timescale processes that could occur at the magnetar surface. One is that new magnetic flux, driven by plastic deformation of the neutron star crust, may be injected continuously into the magnetosphere (Kluźniak & Ruderman 1998; Thompson et al. 2002; Levin & Lyutikov 2012). Another interesting possibility is the crustal horizontal movement (Ruderman 1991; Jones 2003). The second possibility has been investigated in Yu (2012). Here we only consider the effects of flux injection on the behavior of the flux rope for simplicity. As new magnetic fluxes are injected, the background magnetic field would vary gradually. The background magnetic field would decrease (increase) if the opposite (same) polarity flux is injected. Note that there exist two possible field configurations, inverse and normal (Yu 2012). In the normal configuration, the critical equilibrium height is rather low, usually a few percent above the neutron star surface. Given the regular arrangements occurring at the magnetar surface, the small height of the normal configuration suggests that it may not survive those arrangements at the magnetar surface (Yu 2012). Hence, we will focus on the inverse configurations in this paper.

To be specific we fix the value of \( r_00 = 0.01 \) in the section. The effects of varying \( r_00 \) will be further discussed in Section 4. By solving Equations (9) and (10), we can get the flux rope’s equilibrium curves, which show the variations of the flux rope’s equilibrium height in response to the gradual background magnetic flux changes. The relevant results are shown in panel (a) of Figures 1 and 2. We show a dipole-dominated background field with \( a_1 = -1/4 \) in Figure 1 and a multipole-dominated field with \( a_1 = 2/3 \) in Figure 2. It can be found that each equilibrium curve contains two branches, the lower stable branch (solid line) and the upper unstable branch (dotted line). On the lower equilibrium branch, the total force on the flux rope \( F \propto h(B_r - B_\ell) \) shows a negative derivative with respect to \( h \), i.e., \( dF/dh < 0 \) (Forbes 2010). Physically speaking, the flux rope is stable if it lies on this branch, since a slight upward displacement would create an inward restoring force. The upper equilibrium branch, on the contrary, is unstable. Since the total force shows a positive derivative, i.e., \( dF/dh > 0 \), a slight upward displacement on the flux rope lying on this branch would generate an outward driving force. The two branches are joined together at the critical loss of equilibrium point (shown as red dot in these figures). In the left panel of Figures 1 and 2, we find that, with the decrease of the parameter \( \sigma \), the flux rope gradually approaches this critical point. At this point, the flux rope can no longer sustain the stable equilibrium and will erupt catastrophically. The quasi-static evolution of the flux rope is then replaced by the subsequent dynamical evolution.

In the right panel of the two figures, we show the critical pre-eruptive magnetic field configuration, which corresponds to the state represented by the red dot in the left panel. The magnetar surface is shown by the thick solid semi-circle and the critical height of the flux rope is shown by a dashed circle with a radius \( r = h_c \) in each case. The critical heights for Figures 1 and 2 are about 1.27r_c and 2.20r_c, respectively. The possible post-eruptive configurations for the critical pre-eruptive states in Figures 1 and 2 are shown in Appendix A. Note that whether or not the transition from the closed pre-eruptive state to the fully (or partially) open post-eruptive state is feasible is determined by the energy relations between the two states. When the magnetic energy accumulated at the critical pre-eruptive state is higher than the relevant post-eruptive state, such state transitions are physically favored. To check the feasibility of fully or partially open eruptions, we need to know the magnetic energy accumulated at the critical loss of equilibrium point. The energetics of the flux rope, or the feasibility of the state transition, will be further discussed in Section 4.

### 4. ENERGETICS OF FULLY AND PARTIALLY OPEN FLUX ROPE ERUPTIONS

We already know from the previous section that the flux rope presents a catastrophic behavior. This is consistent with the observational characteristics of giant flares. The flux rope initially stays on the stable branch and loses equilibrium after evolving to the critical loss of equilibrium point. The flux rope eruption would be physically favored if it contained magnetic energy over the fully or partially open field energy threshold, \( W_{\text{open}}^\theta \) or \( W_{\text{open}}^\rho \). In the following we will study the energetics of the flux rope eruption model and answer the question whether or not the state transition is possible.
4.1. Magnetic Energy Accumulation Prior to Catastrophe

To check whether or not the flux rope is able to drive giant, it is crucial to know the total magnetic energy that it could accumulate before the catastrophe. The total magnetic energy accumulated in the pre-eruptive state is the sum of the free energy and the potential magnetic energy, i.e., $W_{\text{pre}} = W_{\text{free}} + W_{\text{pot}}$. The free magnetic energy of the system prior to catastrophe is equal to the work required to move the flux rope from infinity to the location where the flux rope lies. Thus the free magnetic energy is given by

$$ W_{\text{free}} = - \int_\infty^h F dh' = - \int_\infty^h 2\pi \frac{h'}{c} (B_x - B_z) dh', \quad (11) $$

where $F$ is the total force exerted on the flux rope. The potential magnetic energy $W_{\text{pot}}$ of the magnetar is

$$ W_{\text{pot}} = \int \frac{B_{\text{pot}}^2}{8\pi} \, dV $$

$$ = \int_\partial V \frac{B_{\text{pot}}^2}{8\pi} (\mathbf{r} \cdot d\mathbf{S}) - \frac{1}{4\pi} \int_\partial V (\mathbf{B}_{\text{pot}} \cdot \mathbf{r})(\mathbf{B}_{\text{pot}} \cdot d\mathbf{S}), \quad (12) $$

where the volume integral is performed over the entire magnetosphere outside the magnetar, $\mathbf{r}$ is the position vector, and $d\mathbf{S}$ is the surface area element directed outward. Note that in Equation (12) the volume integral has been already transformed to the surface integral according to the magnetic virial theorem (Chandrasekhar 1961). In the above equation, the potential magnetic field $\mathbf{B}_{\text{pot}}$ can be obtained from the potential stream function $\Psi_{\text{pot}}$ via Equation (1). According to the boundary condition, i.e., Equation (5), the potential stream function $\Psi_{\text{pot}}$ can be explicitly written as

$$ \Psi_{\text{pot}}(r, \mu) = \Psi_0 \sigma \left[ \frac{r_x}{r} + a_1 (5\mu^2 - 1) \frac{r_y}{r} \right] (1 - \mu^2). \quad (13) $$

The total magnetic energy accumulated in the pre-eruptive state is the sum of the free energy and the potential magnetic energy, i.e., $W_{\text{pre}} = W_{\text{free}} + W_{\text{pot}}$. By examining the pre-eruptive energy accumulation process, we are able to figure out whether or not certain types of pre-eruptive states will support the giant flare. In the following we will investigate the energy accumulation process of the flux rope in greater detail.

4.2. Fully Open Eruptions in Dipole-dominated Backgrounds

The energy accumulation processes of the flux rope in the dipole-dominated and multipole-dominated backgrounds are shown in the left and right panels of Figure 3, respectively. These two panels correspond to the energy accumulation processes before the catastrophic loss of equilibrium shown in Figures 1 and 2, respectively. In this section we focus on the energy accumulation process in a dipole-dominated background field shown in the left panel. Since there is only one closed flux system in the dipole-dominated background for the case where $a_1 = -1/4$, the opening of the closed background always induces fully open eruptions. Fully open magnetic fields can be obtained by flipping the boundary flux distributions in the southern hemisphere, i.e., $\mu \in [-1, 0]$. A brief description of the procedures to obtain the field configuration and the energy threshold constrained by the fully open field is discussed in Appendices A and B, respectively. The fully open energy threshold for the case of $a_1 = -1/4$ is shown by the dashed line in Figure 3. We also present in this figure the normalized accumulated magnetic energy, $W_{\text{pre}}/W_{\text{pot}}$, versus the background flux, $\sigma$, before the catastrophe of flux ropes (in solid line). Note that the energy reaches maximum at the critical pre-eruptive point, marked by the red dot. Our calculations show that the accumulated energy of the critical pre-eruptive state, $W_{\text{pre}}(h_c)/W_{\text{pot}}$, is 2.17. The energy threshold constrained by the fully open state, $W_{\text{open}}/W_{\text{pot}}$, is 2.011. The energy release fraction of the pre-eruptive magnetosphere, $[W_{\text{pre}}(h_c) - W_{\text{open}}]/W_{\text{pre}}(h_c)$, is about 7%. Observationally,
the total magnetic energy in the magnetosphere is about $\sim 10^{40} (B/10^{14}\,G)^2 (r_s/10\,km)^3$ erg, the giant flare is typically $10^{44}$ erg, so only 1% of magnetic energy released in the magnetosphere could account for the giant flares (Woods & Thompson 2006; Mereghetti 2008). As a result, it is possible for the flux rope to fully open the background field and to provide abundant magnetic energy to drive a magnetar giant flares.

4.3. Partially Open Eruptions in Multipolar Backgrounds

In the previous section, we have established that for a dipole-dominated background it is possible to induce fully open eruptions. However, observation shows that multipolar magnetic fields may be involved for magnetar giant flares (Feroci et al. 2001; Pavan et al. 2009). It is natural to want to know whether fully open eruption is possible in the multipolar-dominated background. We now investigate the energy accumulation process of the flux rope in a multipole-dominated background with $a_1 = 2/3$, the field configuration of which is also shown in Figure 2. The detailed result is shown in the right panel of Figure 3. The accumulated energy, $W_{\text{pre}}(h_c)/W_{\text{pot}}$ in the critical pre-eruptive state is about 1.04. However, the energy threshold constrained by the fully open state, $W_{\text{open}}^{f}/W_{\text{pot}}$, is 2.090. It is clear that this flux rope eruption is impossible to fully open the multipole-dominated field.

However, it is interesting to note that the eruption may just involve partial opening of the closed flux systems for the multipole-dominated background. It is conceivable that the partial opening of the background field requires less work to be done than the full opening of background field. This will reduce the energy threshold constrained by the full opening eruption. Along this line we could find an alternative approach to the flux rope eruption. For this type of eruption, it may just involve the partial opening of the magnetic field, which has a lower threshold of 1.024 (shown by the dashed-dotted line in the right panel). It reduces the energy threshold constrained by the fully open field. The critical pre-eruptive state contains energy about 1.5% above the partially open threshold. The energy accumulation in the magnetosphere becomes sufficient to drive a partially open eruption.

4.4. Further Numerical Results

Two representative examples were given in previous sections to show the flux rope eruption from the perspective of the energy budget. Here we perform a comprehensive study on the flux rope energetics to check the possibility of the flux rope’s eruption in a different kind of background field. We investigate the accumulated magnetic energy, $W_{\text{pre}}(h_c)/W_{\text{pot}}$, for different values of $a_1 \in [-1, 1]$ and flux-rope radius $r_0 \in [10^{-4}, 0.1]$. The results are shown in Figure 4.

Let us first focus on the cases with positive values of $a_1$, i.e., the cases with a centrally caved background field. In the left panel of this figure, the black solid line represents the accumulated energy $W_{\text{pre}}(h_c)/W_{\text{pot}}$ in a centrally caved multipolar field with $a_1 = 0.2$. Since the energy threshold of both the fully and partially open fields are relevant in the multipole-dominated background, we show both energy thresholds in the same panel for comparison. The threshold value of the corresponding partially open field, $W_{\text{open}}^{p}/W_{\text{pot}}$, and the fully open field, $W_{\text{open}}^{f}/W_{\text{pot}}$, is shown respectively by the black dashed-dotted line and black dashed-three-dotted line. Detailed calculation of the two energy thresholds is given in Appendices A and B. We found that, when $r_00 \lesssim 0.0007$ or $r_00 \gtrsim 0.01$, the accumulated pre-eruptive energy could surpass the partially open threshold. When flux rope’s minor radius is larger, i.e., with $r_00 \gtrsim 0.02$, the accumulated pre-eruptive energy could even surpass the fully open threshold. We also show the comparison of the accumulated pre-eruptive energy (shown by the blue solid line) in a centrally caved background field for the stronger multipolar component $a_1 = 1/3$ with the relevant energy threshold. The corresponding partially open threshold is shown by the blue dashed-dotted line. We found that, for all possible values of $r_00$, the accumulated pre-eruptive energy is higher than the partially open threshold. However, none of them could surpass the fully open threshold $\sim 1.49$, which is beyond the scale of this figure and not shown. We choose $a_1 = 1$ to represent a centrally caved background field with a much stronger multipolar component. The dependence of the accumulated pre-eruptive energy on the flux rope minor radius for $a_1 = 1$ is shown by the red solid line. It is obvious that for all possible values of $r_00$, the flux rope cannot accumulate enough energy to surpass the corresponding partially open threshold, shown as the red dashed-dotted line. Since the fully open threshold is always higher than the partially open threshold, the flux rope could not support fully open eruption either in this case. It is clear that for a centrally caved multipole-dominated field, if the multipole components are too strong, the flux rope is not able to erupt.

Now let us turn to the centrally arcaded background fields, i.e., the cases with the multipole parameter $a_1$ with a negative sign. The black solid line in the right panel of Figure 4 represents the accumulated pre-eruptive energy in a dipole-dominated field with $a_1 = -0.2$. For the dipole-dominated background, the relevant energy threshold is only constrained by a fully open
field and this threshold is shown by the black dashed-three-dotted line in this figure. It is obvious that, for all possible values of \( r_{00} \), the flux rope could build up more magnetic energy before the catastrophe and give rise to fully open eruptions in the dipole-dominated background. The blue solid line in this figure represents the accumulated pre-eruptive magnetic energy in a centrally arcaded multipolar field with \( a_1 = -1/3 \). In this case, the relevant thresholds are restricted by either the fully or the partially open field. A detailed calculation shows that the pre-eruptive energy could surpass both the corresponding partially open threshold, shown by the blue dashed-dotted line, and the fully open threshold, shown by the blue dashed-three-dotted line. The red solid line represents the variation of the accumulated energy with the flux rope's minor radius in a background field with \( a_1 = -1 \), i.e., a centrally arcaded field with stronger multipolar component. The pre-eruptive energy could surpass the corresponding partially open threshold, shown by the dashed-dotted red line. However, the flux rope could not accumulate enough energy to get over the corresponding fully open threshold (~2.89, not shown in this figure).

Our previous results show that the possibility of the flux rope’s eruption depends on the parameter \( a_1 \) in a rather complex way. These complex behaviors can be more readily understood from Figure 5. We show in this figure the dependence of accumulated pre-eruptive magnetic energy on the parameter \( a_1 \) for different fixed values of \( r_{00} \). The relevant energy thresholds of fully and partially open fields are shown by the dashed-dotted and dashed-dotted lines, respectively. The comparison of the accumulated pre-eruptive energy and the energy thresholds can be made in a more straightforward way. The solid black, gray, and light gray lines represent the accumulated pre-eruptive magnetic energy for typical flux rope minor radii \( r_{00} = 0.01, 0.05, \) and 0.1, respectively. Note that partially open thresholds do not exist in dipole-dominated fields with \( a_1 \in [-1/4, 1/6] \), which is shown as two vertical lines in this figure.

We found that, in the dipole-dominated background fields, i.e., \( a_1 \in [-1/4, 1/6] \), it is always possible to drive a fully open eruption. However, in most cases of multipole background fields, the fully open energy threshold is always greater than the energy stored in the critical pre-eruptive state and it is impossible for the flux rope to induce fully open eruptions. Only in some special cases, when the multipole components are not so strong, i.e., \( -0.55 \lesssim a_1 \lesssim -0.25 \) or 1/6 \( \lesssim a_1 \lesssim 0.25 \), is it possible to induce fully open eruptions.

More importantly, we note that in these fields, the eruption may just involve a partial opening of the closed magnetic flux, which provides an alternative approach for the flux rope eruptions. It is clearly discernible in this figure that the energy release fraction, \( \frac{W_{\text{pre}}(h_c) - W_{\text{open}}^p}{W_{\text{pre}}(h_c)} \), is about 10% ~ 25% during the partially open eruptions in the centrally arcaded backgrounds, which is able to release and drive the giant flares. The energy release fraction for flux ropes embedded in the centrally caved backgrounds is in a smaller range, within ~5%. Note that there exists a special class of background fields with a very strong centrally caved multipole component, \( a_1 \gtrsim 0.75 \). The pre-eruptive energy possessed in flux ropes is always lower than the partially open thresholds. These kinds of background fields cannot be opened by eruptions of flux ropes.

In addition, note that the simplest case of \( a_1 = 0 \) has been investigated by Lin et al. (1998). However, their approximate treatment of the flux-frozen constraints has led them to the inappropriate result that the flux rope cannot support the fully open eruption. In our paper, we rigorously take into account the flux-frozen constraint and arrive at different results from them.6

5. CONCLUSIONS AND DISCUSSIONS

We propose a force-free magnetospheric model with an embedded helically twisted flux rope. With the gradual variations at the magnetar surface, the flux rope evolves quasi-statically in stable equilibrium states. When the point of equilibrium loss is reached, the global magnetosphere is then destabilized and the flux rope erupts catastrophically. During the process, the originally closed flux systems would be opened, accompanied by the rapid release of the magnetic energy stored in the magnetosphere. This energy release is of vital importance for the outbursts of magnetars. However, whether the flux systems’ opening could be achieved depends on whether the amount of energy accumulated prior to the flux rope eruptions could surpass the energy thresholds constrained by the post-eruptive open magnetic topologies.

In this paper, we adopt boundary conditions, which include both the contribution from a dipolar component and a high order multipolar component, to illustrate the complicated geometry of the magnetic field close to the magnetar surface. We establish a fully open field for the dipole-dominated magnetic fields, which involves opening the single closed flux system in the background. For a multipole-dominated closed background, we establish a partially open field, which involves opening part of the closed flux systems, as well as the fully open field, which involves opening all closed flux systems. The opening of closed magnetic fields requires a certain amount of work to be done to overcome the attractive magnetic tension force.

6 To double check our results, we also adopted the approximation in Lin et al. (1998) and reproduced their results.
Since partially open fields require fewer closed flux systems to be opened, the energy thresholds constrained by the partially open fields is lower than those by the fully open fields. Both the field configuration and the magnetic energy threshold of the two kinds of open fields are examined. Then we carefully investigated the magnetic energy accumulation process before the catastrophe, especially the magnetic energy stored at the critical catastrophic point.

We find that it is possible to fully open dipole-dominated background fields for catastrophic eruptions of flux ropes. However, it is generally difficult to fully open multipole-dominated background fields. In most cases with multipole-dominated backgrounds, the magnetic energy stored at the critical pre-eruptive point is significantly lower than the fully open thresholds, which suggests that the flux rope cannot support fully open eruptions. Fortunately, we find that the accumulated magnetic energy at the critical point is higher than the partially open thresholds. This provides an alternative opportunity for the flux rope to erupt in the multipolar magnetosphere. The multipole-dominated fields can be either centrally caved or centrally arcaded, depending on the flux profiles on the magnetar surface. Generally speaking, the magnetic energy stored in the critical pre-eruptive magnetosphere surpasses the partially open energy threshold by about 10% ~ 25%, if the flux rope is initially embedded in a centrally arcaded background field. For a flux rope initially embedded in a centrally caved background field, the magnetic energy stored in a critical pre-eruptive magnetosphere could surpass the partially open threshold, if the multipolar component is mildly strong, i.e., $1/6 < a_1 \lesssim 0.75$. The energy release fraction is within $\sim 5\%$. If the multipolar component becomes even stronger, $a_1 \gtrsim 0.75$, the accumulated magnetic energy cannot go beyond the partially open threshold and the partially open eruption of flux rope is not possible.

The magnetic energy of the critical pre-eruptive state in excess of the fully or partially open threshold is assumed to be released over a fast dynamical timescale. Observationally, the total magnetic energy in the magnetosphere is about $\sim 10^{46} (B/10^{14} \text{G})^2 (r_s/10 \text{km})^3 \text{erg}$, the giant flare is typically $10^{44} \text{erg}$, so only 1% of the magnetic energy released in the magnetosphere could account for the giant flares. Theoretically, all cases with a surplus energy fraction larger than 1% can possibly drive magnetar giant flares. Specifically for boundary conditions adopted in this paper, most cases with $-1 \leq a_1 \lesssim 0.75$ can possibly drive magnetar giant flares.

In addition to the physical processes considered in this paper, the magnetic field can also be opened by strong neutron star wind (Bucciantini et al. 2006). If the field lines are opened by neutron star wind at a larger distance from the neutron star, the amount of open magnetic flux would be reduced, and the magnetic energy to support the eruption would also decrease. To study how the neutron star wind affects the flux rope eruption energetics, we need to establish a model with a current sheet in the magnetosphere. Currently, we are now trying to construct a magnetosphere model to account for this important process. We leave the investigation into situations with current sheet formation to a companion paper (L. Huang & C. Yu, in preparation).

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**APPENDIX A**

**CONFIGURATIONS OF PARTIALLY OPEN AND FULLY OPEN MAGNETIC FIELDS**

In this Appendix, we describe the procedures to obtain the fully open magnetic field from the dipole-dominated field, and to obtain both fully and partially open fields from the multipole-dominated field. We show three illustrative examples in Figure 6.

In the upper row we show a dipolar field with $a_1 = 0$, as an example of a dipole-dominated field with $a_1 \in [-1/4, 1/6]$. The configuration of the potential background field is shown in the top-left panel. The thick solid semi-circle in these figures represents the magnetar surface. To be clear, we show the boundary flux distribution by the solid line in the sub-panel. Only one extremum appears in the boundary flux distribution. The fully open field is obtained by simply flipping the surface flux distribution of the potential field in the southern hemisphere, i.e., $-1 \leq \mu < 0$ or $\pi/2 < \theta \leq \pi$. The configuration of the fully open field is shown in the top-middle panel. The corresponding boundary flux distribution is shown by dashed-three-dotted line in the sub-panel. The boundary flux distribution of the original closed field, marked by the solid gray line, is also shown for comparison.

There appear three extremum points in the boundary flux distribution for parameters $a_1 \in [-1, -1/4]$ or $a_1 \in (1/6, 1)$, so that multipolar configurations arise in the background magnetic field. Under these circumstances, two kinds of open field configurations, i.e., partially open and fully open fields, can be obtained. The middle row is for a centrally caved field with $a_1 = 1$ and the lower row is for a centrally arcaded field with $a_1 = -1$. Configurations of the potential background fields, partially open fields, and fully open fields are shown in the left, middle, and right panels, respectively. In the following we describe the mathematical manipulations to obtain these two kinds of open fields in the case of $a_1 = 1$ as an example. The case of $a_1 = -1$ can be obtained in a similar way.

The partially open field is obtained by simply flipping the surface flux distribution of the potential field in the southern hemisphere. The corresponding boundary flux distribution is shown by the dashed-dotted line in the sub-panel of the middle-middle panel. It can be readily identified that only field lines near the central extremum around $\mu = 0$ are opened, while the other two closed flux systems around the non-zero extremum points remain closed. In this sense, we call the resulting magnetic field configurations as partially open fields. The fully open field configurations can be obtained from the original potential field in two steps. The details are clearly illustrated in the sub-panel of the middle-right panel. In the first step, the boundary flux between the two non-zero extremum points are reversed. After this step, the modified boundary flux contains only one extremum point at $\mu = 0$ (see the dashed line in the sub-panel). In the second step, the southern hemisphere boundary flux is...
flipped and the resulting boundary flux distribution is shown by the dashed-three-dots line. The closed configurations of the patterns caused by the high order multipolar terms are also opened, shown by the thick black lines near the two poles of the magnetar. There are three current sheets in total in the fully open field, together with the equatorial one.

**APPENDIX B**

**ENERGY OF PARTIALLY OPEN AND FULLY OPEN MAGNETIC FIELDS**

The boundary condition of the post-eruptive partially open field is obtained by flipping the flux function according to
the original boundary condition of the closed potential field (Yu 2011). Explicitly, the modified boundary flux distribution of the partially open fields can be written as (see the sub-panels of the partially open fields shown in Figure 6)

\[ \Theta^p_{\text{open}} = \begin{cases} \Theta(\mu), & 0 \leq \mu < 1 \\ 2\Theta(0) - \Theta(\mu), & -1 \leq \mu \leq 0 \end{cases} \]  

(B1)

where \( \Theta(\mu) \) is already defined in Equation (6). The general solutions to the GS equation are of the form

\[ \Psi^p_{\text{open}}(r, \mu) = P_1^p(\mu) + \sum_{k=1}^{\infty} a_{2k} P_{2k+1}^p(\mu) \]  

(B2)

To determine the partially open field, we have to specify the coefficients \( a_{2k} \) in the above equation. For convenience, we define the following flux function as

\[ \Phi(\mu) = \Psi^p_{\text{open}}(\mu) - \Psi(\mu) \]  

(B3)

According to the orthogonality of the associated Legendre polynomials \( P_{2k}^p(\mu) \) and \( \Psi(\mu) = \Psi_0^p \Theta(\mu) \), we can determine the coefficients \( a_{2k} \) as

\[ a_{2k} = \frac{4k + 1}{r_s^{2k+1}} \int_0^1 \Phi(r_s, \mu) P_{2k}^1(\mu) d\mu \]

\[ = \Psi_0^p \frac{4k + 1}{r_s^{2k+1}} \int_0^1 \left[ \Theta(\mu) - \Theta(0)(1 - \mu) \right] P_{2k}^1(\mu) d\mu. \]  

(B4)

Once these coefficients are fixed, we can obtain the partially open field configurations.

The boundary condition of the post-eruptive fully open field is obtained in a similar way to the partially open field. The difference is that we also need to flip the original boundary flux profile in the range \([0, 1]\). Hereafter we use \( \Theta^f(\mu) \) to denote the flipped boundary flux profile in the range \([0, 1]\). The modified boundary flux profile in the entire range \([-1, 1]\) can be expressed in terms of \( \Theta^f(\mu) \) as (see the sub-panels of the fully open fields shown in Figure 6)

\[ \Theta^f_{\text{open}} = \begin{cases} \Theta^f(\mu), & 0 \leq \mu < 1 \\ 2\Theta(0) - \Theta^f(\mu), & -1 \leq \mu \leq 0 \end{cases} \]  

(B5)

where the new surface flux distribution \( \Theta^f(\mu) \) in the range \([0, 1]\) is flipped as follows:

\[ \Theta^f(\mu) = \begin{cases} \Theta(\mu), & \mu_1 \leq |\mu| < 1 \\ \frac{(4a_1 + 1)^2}{(10a_1)} - \Theta(\mu), & 0 \leq |\mu| < \mu_1 \end{cases} \]  

where \( \mu_1 = \sqrt{(6a_1 - 1)/(10a_1)} \) is the non-zero extremum point in the range \([0, 1]\). Similarly, we can determine the coefficients \( a_{2k} \) in Equation (B2) in terms of the surface flux distribution of the fully open field, \( \Theta^f_{\text{open}} \). The fully open field subsequently can be determined in complete detail.

The magnetic energy possessed in the post-eruptive state, \( W^p_{\text{open}} \), for the partially open field state or \( W^f_{\text{open}} \) for the fully open field state in this paper, reads

\[ W^p_{\text{open}} = \int B^p_{\text{open}} f dV = \int_{\partial V} \frac{(B^p_{\text{open}} f)^2}{8\pi} (r \cdot dS) \]

\[ - \frac{1}{4\pi} \int_{\partial V} (B^p_{\text{open}} f) (r) (B^p_{\text{open}} f) (dS), \]  

(B7)

according to magnetic virial theorem. The energy thresholds in fully open fields and partially open fields are calculated using \( W^p_{\text{open}}/W_{\text{pot}} \) and \( W^p_{\text{open}}/W_{\text{pot}} \), respectively.

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ERRATUM: “GIANT MAGNETAR FLARES IN MULTIPOLAR MAGNETIC FIELDS. I. FULLY AND PARTIALLY OPEN ERUPTIONS OF FLUX ROPES” (2014, ApJL, 784, 168)

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