Sufficient conditions for optimality for stochastic evolution equations

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Abstract

In this paper we derive for a controlled stochastic evolution system on Hilbert space \( H \) a sufficient condition for optimality. Our result is derived by using its so-called adjoint backward stochastic evolution equation.

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1 Introduction

Consider a stochastic controlled problem governed by the following stochastic evolution equation (SEE):

\[
\begin{align*}
    dX(t) &= (AX(t) + b(X(t), \nu(t)))dt + \sigma(X(t), \nu(t))dW(t), \quad t \in [0, T], \\
    X(0) &= x_0.
\end{align*}
\]

This system is driven mainly by \( A \) is an unbounded linear operator on a separable Hilbert space \( H \), a cylindrical Wiener process \( W \) on \( H \). The control variable here is denoted by \( \nu(\cdot) \). Then the control problem is to minimize the cost functional, which is given by equation (3.2) in Section 3 over a set of admissible controls.

We shall concentrate in providing a sufficient condition for optimality of this optimal control problem, which gives this minimization. For this purpose we shall apply the theory of backward stochastic evolution equations (or shortly BSEE) as in equation (3.6) in Section 3, which together with backward stochastic differential equations have become nowadays of great interests in many different fields. For

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example one can see [9], [14], [12], [13] and [15] for the applications of backward stochastic differential equations in such optimal control problems.

Our work will not need studying Hamilton-Jacobi-Bellman equation either by using semi-group technique or the technique of viscosity solutions. We refer the reader to [5] and some of the related references therein for the semi-group technique.

Let us remark that necessary conditions for optimality of the control \( \nu(\cdot) \) and its corresponding solution \( X^{\nu(\cdot)} \) but for the case when \( \sigma \) does not depend on \( \nu \) can be found in [9]. This is also the case considered in our earlier work in [4]. So the present paper generalize the work in [4].

2 Notation

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and denote by \( \mathcal{N} \) the collection of \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Let \( \{W(t), 0 \leq t \leq T\} \) be a cylindrical Wiener process on \( H \) with its completed natural filtration \( \mathcal{F}_t = \sigma\{\ell \circ W(s), 0 \leq s \leq t, \ell \in H^*\} \cup \mathcal{N}, t \geq 0 \); cf. [1].

For a separable Hilbert space \( E \) let \( L^2(F(0,T;E)) \) denote the space of all \{\( \mathcal{F}_t, 0 \leq t \leq T \)\}-progressively measurable processes \( f \) with values in \( E \) such that

\[
\mathbb{E} \left[ \int_0^T |f(t)|^2_E \, dt \right] < \infty.
\]

Thus \( L^2_F(0,T;E) \) is a Hilbert space with the norm

\[
||f|| = \left( \mathbb{E} \left[ \int_0^T |f(t)|^2_E \, dt \right] \right)^{1/2}.
\]

It is known as in [6] that for \( f \in L^2_F(0,T;L_2(H)) \), where \( L_2(H) \) is the space of all Hilbert-Schmidt operators on \( H \), the stochastic integral \( \int f(t)dW(t) \) can be defined as a continuous stochastic process in \( H \). The inner product on \( L_2(H) \) will be denoted by by \( \langle \cdot, \cdot \rangle_2 \).

3 Results

Let \( \mathcal{O} \) be a separable Hilbert space equipped with an inner product \( \langle \cdot, \cdot \rangle_\mathcal{O} \), and let \( U \) be a convex subset of \( \mathcal{O} \). We say that \( \nu(\cdot) : [0,T] \times \Omega \to \mathcal{O} \) is admissible if \( \nu(\cdot) \in L^2_F(0,T;\mathcal{O}) \) and \( \nu(t) \in U \ a.e., \ a.s. \) The set of admissible controls will be
denoted by $U_{ad}$. Let $b : H \times \mathcal{O} \to H$ and $\sigma : H \times \mathcal{O} \to L_2(H)$ be two continuous mappings. Consider the following controlled system:

$$
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dX(t)}{dt} = (AX(t) + b(X(t), \nu(t)))dt + \sigma(X(t), \nu(t))dW(t), \\
X(0) = x_0,
\end{array}
\right.
\end{aligned}
$$

(3.1)

where $\nu(\cdot) \in U_{ad}$ represents a control variable. A solution of (3.1) will be denoted by $X_{\nu}^{(\cdot)}$ to indicate the presence of the control.

Let $\ell : H \times \mathcal{O} \to \mathbb{R}$ and $\phi : H \to \mathbb{R}$ be two measurable mappings such that the following cost functional is defined:

$$
J(\nu(\cdot)) := \mathbb{E} \left[ \int_0^T \ell(X_{\nu}^{(t)}(t), \nu(t))dt + \phi(X_{\nu}^{(T)}(T)) \right], \quad \nu(\cdot) \in U_{ad}.
$$

(3.2)

For example we can take $\ell$ and $\phi$ to satisfy the assumptions given in Theorem 3.3.

The optimal control problem of the system (3.1) is to find the value function $J^*$ and an optimal control $\nu^*(\cdot) \in U_{ad}$ such that

$$
J^* := \inf \{J(\nu(\cdot)) : \nu(\cdot) \in U_{ad}\} = J(\nu^*(\cdot)).
$$

(3.3)

If this happens, the corresponding solution $X_{\nu^*}^{(\cdot)}$ is called an optimal solution of the stochastic control problem (3.1)–(3.3) and $(X_{\nu^*}^{(\cdot)}, \nu^*(\cdot))$ is called an optimal pair.

Let us now state the following result.

**Theorem 3.1** Assume that $A$ is an unbounded linear operator on $H$ that generates a $C_0$-semigroup $\{S(t), t \geq 0\}$ on $H$, and $b, \sigma$ are continuously Fréchet differentiable with respect to $x$ and their derivatives $b_x, \sigma_x$ are uniformly bounded. Then for every $\nu(\cdot) \in U_{ad}$ there exists a unique mild solution $X_{\nu}^{(\cdot)}$ on $[0, T]$ to (3.1). That is $X_{\nu}^{(\cdot)}$ is a progressively measurable stochastic process such that $X(0) = x_0$ and for all $t \in [0, T]$,

$$
X_{\nu}^{(t)} = S(t)x_0 + \int_0^t S(t-s)b(X_{\nu}^{(s)}(s), \nu(s))ds \\
+ \int_0^t S(t-s)\sigma(X_{\nu}^{(s)}(s), \nu(s))dW(s).
$$

(3.4)

The proof of this theorem can be derived in a similar way to those in [7, Chapter 7] or [10].

From here on we shall assume that $A$ is the generator of a $C_0$-semigroup $\{S(t), t \geq 0\}$ on $H$. Its adjoint operator $A^* : \mathcal{D}(A^*) \subset H \to H$ is then the infinitesimal generator of the adjoint semigroup $\{S^*(t), t \geq 0\}$ of $\{S(t), t \geq 0\}$. 

As it is known that backward stochastic differential equations play an important role in deriving the maximum (or minimum) principle for SDEs, it is natural to search for such a role for SEEIs like (3.1). For this purpose, let us first consider the Hamiltonian:

\[ H : H \times O \times H \times L_2(H) \rightarrow \mathbb{R}, \]

\[ H(x, \nu, y, z) := \ell(x, \nu) + \langle b(x, \nu), y \rangle_H + \langle \sigma(x, \nu), z \rangle_2, \] (3.5)

Consider the following adjoint BSEE on \( H \):

\[
\begin{aligned}
- dY^{(\nu)}(t) &= \left( A^* Y^{(\nu)}(t) + \nabla_x H(X^{(\nu)}(t), \nu(t), Y^{(\nu)}(t), Z^{(\nu)}(t)) \right) dt \\
Y^{(\nu)}(T) &= \nabla \phi(X^{(\nu)}(T)),
\end{aligned}
\] (3.6)

where \( \nabla \phi \) denotes the gradient of \( \phi \), which is defined, by using the directional derivative \( D \phi(x)(h) \) of \( \phi \) at a point \( x \in H \) in the direction of \( h \in H \), as

\[ \langle \nabla \phi(x), h \rangle_H = D \phi(x)(h) \] (\( = \phi_x(h) \)).

A mild solution (or briefly a solution) of (3.6) is a pair \((Y, Z) \in L_2^2(0, T; H) \times L_2^2(0, T; L_2(H))\) such that we have \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \)

\[ Y^{(\nu)}(t) = S^*(T - t) \nabla \phi(X^{(\nu)}(T)) + \int_t^T S^*(s - t) \nabla_x H(X^{(\nu)}(s), \nu(s), Y^{(\nu)}(s), Z^{(\nu)}(s)) ds - \int_t^T S^*(s - t) Z^{(\nu)}(s) dW(s). \] (3.7)

The existence of such solutions can be obtained from the following theorem.

**Theorem 3.2 ([3] or [8])** Assume that \( b, \sigma, \ell, \phi \) are continuously Fréchet differentiable with respect to \( x \), the derivatives \( b_x, \sigma_x, \sigma \nu, \ell_x \) are uniformly bounded, and \( |\phi_x|_{L(H, H)} \leq C (1 + |x|_H) \) for some constant \( C > 0 \).

Then there exists a unique mild solution \((Y^{(\nu)}(\cdot), Z^{(\nu)}(\cdot))\) of the BSEE (3.6).

Let us now consider the following hypothesis:

\[ \sigma(X^{(\nu)}(t), \nu(t)) = S(t) \varphi(X^{(\nu)}(t), \nu(t)), \quad \forall \nu(\cdot) \in \mathcal{U}_{ad}, \forall t \in [0, T], \] (3.8)

for some mapping \( \varphi : H \times O \rightarrow H \) satisfying the same properties as \( \sigma \).

Now we state our main result.
**Theorem 3.3** For a given admissible control \( \nu^*(\cdot) \) let \( X^{\nu^*}(\cdot) \) and \( (Y^{\nu^*}(\cdot), Z^{\nu^*}(\cdot)) \) be the corresponding equations (3.1) and (3.6) respectively. Assume that (3.8) holds. Suppose that

(i) \( \phi \) is convex,
(ii) \( b, \sigma, \ell \) are continuously Fréchet differentiable with respect to \( x, \nu, \phi \) is continuously Fréchet differentiable with respect to \( x \), the derivatives \( b_x, b_\nu, \sigma_x, \sigma_\nu, \ell_x, \ell_\nu \) are uniformly bounded, and

\[
|\phi_x|_{L(K,K)} \leq C (1 + |x|_K)
\]

for some constant \( C > 0 \),
(iii) \( \mathcal{H}(\cdot, \cdot, Y^{\nu^*}(t), Z^{\nu^*}(t)) \) is convex for all \( t \in [0, T] \), \( \mathbb{P} \) - a.s., and
(iv) \( \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) = \inf_{\nu \in \mathcal{O}} \mathcal{H}(X^{\nu^*}(t), \nu, Y^{\nu^*}(t), Z^{\nu^*}(t)) \)

for a.e. \( t \in [0, T] \), \( \mathbb{P} \) - a.s.

Then \( (X^{\nu^*}(\cdot), \nu^*(\cdot)) \) is an optimal pair for the problem (3.1)–(3.3).

**4 Proofs**

In this section we shall establish the proof of Theorem 3.3. We need the following two lemmas.

**Lemma 4.1** Let \( \psi_1(t) := b(X^{\nu^*}(t), \nu^*(t)) - b(X^{\nu}(t), \nu(t)) \), where \( t \in [0, T] \). If assumption (ii) in Theorem 3.3 holds, then

\[
\mathbb{E} \left[ \left\langle Y^{\nu^*}(T), X^{\nu^*}(T) - X^{\nu}(T) \right\rangle \right] = \mathbb{E} \left[ \int_0^T \left\langle Y^{\nu^*}(t), \psi_1(t) \right\rangle \, dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)), X^{\nu^*}(t) - X^{\nu}(t) \right\rangle \, dt \right]
\]

\[
+ \mathbb{E} \left[ \left\langle Y^{\nu^*}(T), \int_0^T S(T - t) \left[ \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)) \right] \, dW(t) \right\rangle \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)), \int_s^t S(t - s) \left[ \sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu}(s), \nu(s)) \right] \, dW(s) \right\rangle \, dt \right].
\]

**Proof.** Thanks to (ii) and (3.1) we find that \( \psi_1 \in L^2_\mathbb{F}(0, T; H) \). Multiply equation (3.7) by \( \psi_1(t) \), integrate with respect to \( t \in [0, T] \), take the expectation and use
stochastic Fubini’s theorem to get

$$
\mathbb{E} \left[ \int_0^T \langle Y^{\nu^*}(t), \psi_1(t) \rangle \, dt \right]
= \mathbb{E} \left[ \int_0^T \langle S^*(T-t) Y^{\nu^*}(T), \psi_1(t) \rangle \, dt \right]
\geq \mathbb{E} \left[ \int_0^T \int_t^T S^*(s-t) \nabla_x \mathcal{H}(X^{\nu^*}(s), \nu^*(s), Y^{\nu^*}(s), Z^{\nu^*}(s)) \, ds, \psi_1(t) \rangle \, dt \right]
= \mathbb{E} \left[ \langle Y^{\nu^*}(T), \int_0^T S(T-t) \psi_1(t) \, dt \rangle \right]
+ \mathbb{E} \left[ \int_0^T \langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)), \int_0^t S(t-s) \psi_1(s) \, ds \rangle \, dt \right]
= \mathbb{E} \left[ \langle Y^{\nu^*}(T), X^{\nu^*}(T) - X^{\nu^*}(T) \rangle \right]
+ \mathbb{E} \left[ \int_0^T \langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)), X^{\nu^*}(t) - X^{\nu^*}(t) \rangle \, dt \right]
- \mathbb{E} \left[ \langle Y^{\nu^*}(T), \int_0^T S(T-t) \left[ \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu^*}(t), \nu(t)) \right] \, dW(t) \rangle \right]
- \mathbb{E} \left[ \int_0^T \langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)),\right.
\left. \int_0^t S(t-s) \left[ \sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu^*}(s), \nu(s)) \right] \, dW(s) \rangle \, dt \right].
$$

Here the following identity is used:

$$
X^{\nu^*}(t) - X^\nu(t) = \int_0^t S(t-s) \left[ b(X^{\nu^*}(s), \nu^*(s)) - b(X^{\nu^*}(s), \nu(s)) \right] \, ds \quad + \int_0^t S(t-s) \left[ \sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu^*}(s), \nu(s)) \right] \, dW(s),
$$

for $0 \leq t \leq T$. Thus the lemma follows. \( \square \)
Lemma 4.2 Under \((3.8)\) and assumption (ii) in Theorem 3.3 we have

\[
\mathbb{E} \left[ \left< Y^{\nu^*}(T), \int_0^T S(T-t) \left[ \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)) \right] dt \right> \right] \\
+ \mathbb{E} \left[ \left< \nabla \mathcal{H}(X^{\nu^*}(s), \nu^*(s), Y^{\nu^*}(s), Z^{\nu^*}(s)), \int_0^t S(t-s) \left[ \sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu}(s), \nu(s)) \right] dt \right> \right] \\
= \mathbb{E} \left[ \int_0^T \left< \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)), Z^{\nu^*}(t) \right>_2 dt \right].
\]

\((4.2)\)

Proof. From \((3.8)\) it follows that

\[
\sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)) = S(t) \psi(t) \text{ a.s., } 0 \leq t \leq T,
\]

for some \(\psi \in L^2_T(0, T; L^2(H))\). So by letting \(t = 0\) in \((3.7)\), multiplying it by \(\int_0^T \psi(t) \, dW(s)\) and then taking the expectation to the resulting equation we get \((4.2)\). \(\blacksquare\)

We are now ready to establish the proof of Theorem 3.3

Proof of Theorem 3.3. Let \(\nu(\cdot)\) be an arbitrary admissible control. From the definitions in \((3.3)\) and \((3.2)\) we obtain

\[
J(\nu^*(\cdot)) - J(\nu(\cdot)) = \mathbb{E} \left[ \int_0^T \ell(X^{\nu^*}(t), \nu^*(t)) \, dt + \phi(X^{\nu^*}(T)) \right] \\
- \mathbb{E} \left[ \int_0^T \ell(X^{\nu}(t), \nu(t)) \, dt + \phi(X^{\nu}(T)) \right] \\
= \mathbb{E} \left[ \int_0^T \left( \ell(X^{\nu^*}(t), \nu^*(t)) - \ell(X^{\nu}(t), \nu(t)) \right) \, dt \right] \\
+ \mathbb{E} \left[ \phi(X^{\nu^*}(T)) - \phi(X^{\nu}(T)) \right].
\]

\((4.3)\)

But

\[
\ell(X^{\nu^*}(t), \nu^*(t)) - \ell(X^{\nu}(t), \nu(t)) = \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) \\
- \mathcal{H}(X^{\nu}(t), \nu(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) \\
- \left< b(X^{\nu^*}(t), \nu^*(t)) - b(X^{\nu^*}(t), \nu(t)), Y^{\nu^*}(t) \right> \\
- \left< \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)), Z^{\nu^*}(t) \right>_2 \text{ a.s.}
\]
Therefore (4.3) becomes

\[ J(\nu^*(\cdot)) - J(\nu(\cdot)) = \mathbb{E} \left[ \int_0^T \left( \mathcal{H}(X^\nu(t), \nu^*(t), Y^\nu(t), Z^\nu(t)) - \mathcal{H}(X^\nu(t), \nu(t), Y^\nu(t), Z^\nu(t)) - \langle b(X^\nu(t), \nu^*(t)) - b(X^\nu(t), \nu(t)), Y^\nu(t) \rangle 
- \langle \sigma(X^\nu(t), \nu^*(t)) - \sigma(X^\nu(t), \nu(t)), Z^\nu(t) \rangle \right) dt \right] 
+ \mathbb{E} \left[ \phi(X^\nu^*(T)) - \phi(X^\nu(T)) \right]. \]  

By the convexity assumption on \( \phi \) in (i) we get

\[ \phi(X^\nu^*(T)) - \phi(X^\nu(T)) \leq \left\langle \nabla \phi(X^\nu^*(T)) , X^\nu^*(T) - X^\nu(T) \right\rangle \text{ a.s.} \]  

But \( \nabla \phi(X^\nu^*(T)) = Y^\nu^*(T) \). Thus

\[ \mathbb{E} \left[ \phi(X^\nu^*(T)) - \phi(X^\nu(T)) \right] \leq \mathbb{E} \left[ \left\langle Y^\nu^*(T) , X^\nu^*(T) - X^\nu(T) \right\rangle \right]. \]  

From (4.1) and (4.2) we find that

\[ \mathbb{E} \left[ \langle Y^\nu^*(T) , X^\nu^*(T) - X^\nu(T) \rangle \right] = \mathbb{E} \left[ \int_0^T \langle Y^\nu^*(t) , \psi_1(t) \rangle dt \right] 
- \mathbb{E} \left[ \int_0^T \langle \nabla_2 \mathcal{H}(X^\nu^*(t), \nu^*(t), Y^\nu(t), Z^\nu(t)), X^\nu^*(t) - X^\nu(t) \rangle dt \right] 
+ \mathbb{E} \left[ \int_0^T \langle \sigma(X^\nu^*(t), \nu^*(t)) - \sigma(X^\nu(t), \nu(t)), Z^\nu(t) \rangle dt \right]. \]  

Denote for \( t \in [0, T] \),

\[ \delta \mathcal{H}(t) = \mathcal{H}(X^\nu(t), \nu(t), Y^\nu(t), Z^\nu(t)) - \mathcal{H}(X^\nu(t), \nu(t), Y^\nu(t), Z^\nu(t)) \]

and

\[ \psi_2(t) = \delta \mathcal{H}(t) + \left\langle - \psi_1(t) , Y^\nu(t) \right\rangle 
- \langle \sigma(X^\nu(t), \nu(t)) - \sigma(X^\nu(t), \nu(t)), Z^\nu(t) \rangle \]  

By applying (4.4), (4.6) and (4.7) we obtain
\[ J(\nu^*(\cdot)) - J(\nu(\cdot)) \leq \mathbb{E} \left[ \int_0^T \langle Y^{(\cdot)}(t) , \psi_1(t) \rangle dt \right] \]
\[ - \mathbb{E} \left[ \int_0^T \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) , X^{\nu^*}(t) - X^{\nu}(t) \right\rangle dt \right] \]
\[ + \mathbb{E} \left[ \langle Y^{\nu^*}(T) , \int_0^T S(T-t) [\sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t))] dW(t) \rangle \right] \]
\[ + \mathbb{E} \left[ \langle \nabla_x \mathcal{H}(X^{\nu^*}(s), \nu^*(s), Y^{\nu^*}(s), Z^{\nu^*}(s)) , \int_0^t S(t-s) [\sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu}(s), \nu(s))] dW(s) \rangle dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T \psi_2(t) dt \right]. \quad (4.8) \]

Hence
\[ J(\nu^*(\cdot)) - J(\nu(\cdot)) \leq \mathbb{E} \left[ \int_0^T \delta \mathcal{H}(t) dt \right] \]
\[ - \mathbb{E} \left[ \int_0^T \left\langle \sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t)) , Z^{\nu^*}(t) \right\rangle dt \right] \]
\[ - \mathbb{E} \left[ \int_0^T \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) , X^{\nu^*}(t) - X^{\nu}(t) \right\rangle dt \right] \]
\[ + \mathbb{E} \left[ \langle Y^{\nu^*}(T) , \int_0^T S(T-t) [\sigma(X^{\nu^*}(t), \nu^*(t)) - \sigma(X^{\nu}(t), \nu(t))] dW(t) \rangle \right] \]
\[ + \mathbb{E} \left[ \langle \nabla_x \mathcal{H}(X^{\nu^*}(s), \nu^*(s), Y^{\nu^*}(s), Z^{\nu^*}(s)) , \int_0^t S(t-s) [\sigma(X^{\nu^*}(s), \nu^*(s)) - \sigma(X^{\nu}(s), \nu(s))] dW(s) \rangle dt \right]. \]

So from Lemma 4.2 this inequality becomes
\[ J(\nu^*(\cdot)) - J(\nu(\cdot)) \leq \]
\[ - \mathbb{E} \left[ \int_0^T \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) , X^{\nu^*}(t) - X^{\nu}(t) \right\rangle dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T \delta \mathcal{H}(t) dt \right]. \quad (4.9) \]

From the convexity assumption of \( \mathcal{H} \) in condition (iii) we see that the following inequality holds for a.e. \( t \in [0, T] \), \( \mathbb{P} \) - a.s.
\[ \delta \mathcal{H}(t) \leq \left\langle \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) , X^{\nu^*}(t) - X^{\nu}(t) \right\rangle \]
\[ + \left\langle \nabla_y \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) , \nu^*(t) - \nu(t) \right\rangle. \]
But the minimum condition (iv) implies
\[ \langle \nabla_{\nu} \mathcal{H}(X^{\nu^{*}}(t), \nu^{*}(t), Y^{\nu^{*}(\cdot)}(t), Z^{\nu^{*}(\cdot)}(t)), \nu^{*}(t) - \nu(t) \rangle_{\mathcal{O}} \leq 0 \]
for a.e. \( t \in [0, T], \mathbb{P}\text{-a.s.}; \) see e.g. [11]. Consequently, for a.e. \( t \in [0, T], \mathbb{P}\text{-a.s.}, \)
\[ \delta \mathcal{H}(t) - \langle \nabla_x \mathcal{H}(X^{\nu^{*}}(t), \nu^{*}(t), Y^{\nu^{*}(\cdot)}(t), Z^{\nu^{*}(\cdot)}(t)), X^{\nu^{*}(\cdot)}(t) - X^{\nu(\cdot)}(t) \rangle \leq 0. \]

Now by applying this result in (4.9) we deduce finally that \( J(\nu^{*}(\cdot)) \leq J(\nu(\cdot)). \) This completes the proof. \( \blacksquare \)

5 Conclusion

In Theorem 3.3 we derived sufficient conditions for optimality for the optimal control problem, which is governed by the SEE (3.1) when the mapping \( \sigma \) satisfies condition (3.8). It is needed as realized in the proof of Lemma 4.2. The general case, i.e. without this latter condition, will be the subject of our next paper. It is in fact a non-trivial generalization and to achieve it one needs to include the so-called second order BSEEs.

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