G-Actions on Riemann Surfaces and the Associated Group of Singular Orbit Data

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Mathematics Subject Classification (1991): Primary 57M60, 57R85, 20C15; Secondary 58G10

Abstract

Let $G$ be a finite group. To every smooth $G$-action on a compact, connected and oriented Riemann surface we can associate its data of singular orbits. The set of such data becomes an Abelian group $\mathbb{B}_G$ under the $G$-equivariant connected sum. The map which sends $G$ to $\mathbb{B}_G$ is functorial and carries many features of the representation theory of finite groups. In this paper we will give a complete computation of the group $\mathbb{B}_G$ for any finite group $G$.

There is a surjection from the $G$-equivariant cobordism group of surface diffeomorphisms $\Omega_G$ to $\mathbb{B}_G$. We will prove that the kernel of this surjection is isomorphic to $H_2(G;\mathbb{Z})$. Thus $\Omega_G$ is an Abelian group extension of $\mathbb{B}_G$ by $H_2(G;\mathbb{Z})$.

Finally we will prove that the group $\mathbb{B}_G$ contains only elements of order two if and only if every complex character of $G$ has values in $\mathbb{R}$. This property shows a strong relationship between the functor $\mathbb{B}$ and the representation theory of finite groups.

1 Introduction

Let $G$ be a finite group and $S_g$ a connected, oriented and compact Riemann surface of genus $g$. One can assign to every diffeomorphic $G$-action on a surface $S_g$ its data of singular orbits and with this define the singular orbit data. In [8] the author defines a group structure on the set of all singular orbit data and the resulting group is denoted by $\mathbb{B}_G$ (see section 2 for definitions). In the same paper it is proven that the correspondence

$$G \mapsto \mathbb{B}_G$$

is a covariant functor. For finite subgroups $H$ of $G$ there are also restriction and induction maps between the groups $\mathbb{B}_H$ and $\mathbb{B}_G$, together with a double coset formula, thus the functor $\mathbb{B}$ could be named geometric representation theory.

Another property of the group $\mathbb{B}_G$ which is proven in [8, section 6] states that there exists a surjection $\chi$ from $\Omega_G$, the $G$-equivariant cobordism group in dimension two, onto $\mathbb{B}_G$ (see also section 5).

The main purpose of this paper is to prove theorem 19 which gives a complete computation of the group $\mathbb{B}_G$ for any finite group $G$.

In section 6 we will prove that the kernel of the above map $\chi$ is isomorphic to $H_2(G;\mathbb{Z})$ and thus $\Omega_G$ is an Abelian group extension of $\mathbb{B}_G$ by $H_2(G;\mathbb{Z})$.

Moreover in section 6 we will show that the group $\mathbb{B}_G$ contains only elements of order two if and only if every complex character of $G$ has values in $\mathbb{R}$. This property
shows a strong relationship between the functor \( \mathbb{B} \) and the representation theory of finite groups.

Another reason for the interest in the group \( \mathbb{B}_G \) comes from the fact that it is possible to deduce information about the cohomology of the mapping class group.

The mapping class group \( \Gamma_g \) is defined as follows. We write \( \text{Diffeo}^+ (S_g) \) for the group of orientation preserving diffeomorphisms of \( S_g \) with the \( C^\infty \)-topology, and \( \text{Diffeo}_0^+ (S_g) \) for the connected component of the identity. Then we have \( \Gamma_g = \text{Diffeo}^+ (S_g) / \text{Diffeo}_0^+ (S_g) \). For a detailed discussion of the mapping class group see [10].

The singular orbit data is also an invariant of a diffeomorphic \( G \)-action up to isotopy, i.e., we can assign to every finite subgroup \( G \) of a mapping class group \( \Gamma_g \) its singular orbit data.

The mapping class group acts on the first homology group of \( S_g \). This action preserves the intersection pairing and thus gives rise to a symplectic representation \( \eta : \Gamma_g \to \text{Sp}_{2g}(\mathbb{R}) \). The unitary group \( U(g) \) is a maximal compact subgroup of \( \text{Sp}_{2g}(\mathbb{R}) \). Thus for any embedding \( \phi : G \to \Gamma_g \) the map \( \eta \circ \phi \) factors up to conjugation through \( U(g) \subset \text{Sp}_{2g}(\mathbb{R}) \) and this unitary representation is denoted by \( \varphi(\phi) \). Hence \( \varphi \) assigns to every embedding \( \phi \) a complex representation.

The important fact is now that the map \( \varphi \) induces a group homomorphism \( \theta \) from \( \mathbb{B}_G \) to a quotient of the complex representation ring of \( G \).

\[
\theta : \mathbb{B}_G \to R \mathbb{C}G / E_G
\]

With this homomorphism \( \theta \) we can now deduce results about the cohomology of the mapping class group.

In the case where \( G \) is the cyclic group of prime order \( p \), it is shown in [6] that the images of the symplectic classes \( d_i \in H^2i(BSp(\mathbb{R}); \mathbb{Z}) \) in \( H^2i(\Gamma; \mathbb{Z}) \) have infinite order. Here we consider the stable situation, i.e., \( \Gamma \) is the stable mapping class group.

Another result in [6] states that one can embed polynomial algebras in the cohomology of the stable mapping class group. For \( p \) a regular prime, we have \( H^*(BSp(\mathbb{R}); \mathbb{F}_p) \cong \mathbb{F}_p[d_1, d_2, \ldots] \), \( \text{deg } d_i = 2i \) and the map \( \eta^* : H^*(BSp(\mathbb{R}); \mathbb{F}_p) \to H^*(\Gamma; \mathbb{F}_p) \) is injective on the polynomial algebra \( \mathbb{F}_p[(d_i)_{i \in J}] \), \( J = \{ i \in \mathbb{N} | i \equiv 1 \pmod{2} \text{ or } i \equiv 0 \pmod{p-1} \} \).

## 2 The Group of Singular Orbit Data

Let \( \phi \) be an embedding of any finite group \( G \) into some mapping class group \( \Gamma_g \).

By Kerckhoff [4] any such embedding can be lifted to a homomorphism \( G \to \text{Diffeo}^+(S_g) \), i.e., to an action of \( G \) on the surface \( S_g \) such that the elements of \( G \) act by orientation preserving diffeomorphisms. Let \( Gx \) denote the orbit of \( x \in S_g \) under the action of \( G \). The orbit is called singular if \( |Gx| < |G| \), else regular. If the orbit is singular, then there are elements of \( G \) which stabilize the point \( x \in S_g \) and \( G_x \) denotes the stabilizer of \( G \) at \( x \). It is proven by Accola [4, Lemma 4.10] that the stabilizers are cyclic subgroups of \( G \). Let \( y \) be another element of the orbit \( Gx \). Thus there is an element \( a \in G \) such that \( ax = y \) and the stabilizer of \( y \) is conjugate
to \( G_x \), i.e., \( aG_xa^{-1} = G_y \). As the elements of \( G \) operate by orientation preserving diffeomorphisms there are only finitely many singular orbits. Let \( x_i \in S_g, \ i = 1, \ldots, q \) be representatives of these orbits and \( \nu_i \) the orders of the stabilizer groups \( G_x \). Every group \( G_x \) has a generator \( \gamma_i \in G \) such that \( \gamma_i \) acts by rotation through \( 2\pi/\nu_i \) on the tangent space at \( x_i \). Similarly \( a\gamma_i a^{-1}, a \in G \), generates \( aG_xa^{-1} \) and acts also by rotation through \( 2\pi/\nu_i \) on the tangent space at \( ax_i \). Thus in order to collect information about the singular orbits, it is enough to consider the conjugacy classes of the elements \( \gamma_i \). Let \( \gamma_i \) denote the conjugacy class of \( \gamma_i \) in \( G \). The singular orbit data of the embedding \( \phi \) is then the unordered collection

\[ \{\gamma_1, \ldots, \gamma_q\}_G. \]

By \( W_G \) we will denote the set of all singular orbit data of \( G \). This data depends a priori on the chosen lifting of \( \phi \), but by [3] Lemma 1] we know that this data is well defined for an embedding of \( G \). In the sequel we will omit the subscript \( G \) if it is clear with respect to which group the conjugacy classes are taken.

We have seen that every diffeomorphic \( G \)-action gives rise to a singular orbit data. The question is now, which \( q \)-tuple of conjugacy classes \( \gamma_1, \ldots, \gamma_q \) come form a \( G \)-action? The next proposition answers this question.

**Proposition 1** An unordered \( q \)-tuple of conjugacy classes \( \gamma_1, \ldots, \gamma_q \) is the singular orbit data of a \( G \)-action if and only if \( \gamma_1 \cdots \gamma_q \in [G, G] \).

Here \( [G, G] \) denotes the commutator subgroup of \( G \).

**Proof.** [3] proposition 4]

In the sequel we will denote by \([\gamma_1, \ldots, \gamma_q]_G \) an unordered \( q \)-tuple of conjugacy classes of \( G \) such that \( \gamma_1 \cdots \gamma_q \in [G, G] \) and by \( \Lambda_G \) the set of all such \( q \)-tuples. We have now by proposition[3] a one to one correspondence \([\gamma_1, \ldots, \gamma_q]_G \mapsto \{\gamma_1, \ldots, \gamma_q\}_G \) between the elements of \( \Lambda_G \) and \( W_G \). In view of this correspondence we will also call elements of \( \Lambda_G \) singular orbit data.

Next we will define an addition, the \( G \)-equivariant connected sum, on the set of singular orbit data \( W_G \). Let \( G \) act on a surface \( S_g \) with singular orbit data \( \{\gamma_1, \ldots, \gamma_q\} \) and on a surface \( S_h \) with singular orbit data \( \{\beta_1, \ldots, \beta_n\} \). Find discs \( D_1 \) in \( S_g \) and \( D_2 \) in \( S_h \) such that \( \{aD_j\}_{a \in G} \) are mutually disjoint for \( j = 1, 2 \). Then excise all discs \( \{aD_j\}_{a \in G} \) from \( S_g \) and \( S_h \) and take a connected sum by matching \( \partial(aD_1) \) to \( \partial(aD_2) \) for all \( a \in G \). The resulting surface \( S_{g+h+|G|-1} \) has \( |G| \) tubes joining \( S_g \) and \( S_h \). The actions of \( G \) on \( S_g \) and \( S_h \) can be extended to an action on \( S_{g+h+|G|-1} \) by permuting the tubes. The new action has a singular orbit data \( \{\gamma_1, \ldots, \gamma_q, \beta_1, \ldots, \beta_n\} \). This construction on surfaces defines an addition on the set \( W_G \).

\[ \{\gamma_1, \ldots, \gamma_q\} \oplus \{\beta_1, \ldots, \beta_n\} := \{\gamma_1, \ldots, \gamma_q, \beta_1, \ldots, \beta_n\} \]

After this geometric description we can give an algebraic description of the addition on \( \Lambda_G \). Let \([\gamma_1, \ldots, \gamma_q]_G \) and \([\beta_1, \ldots, \beta_r]_G \) be two elements of \( \Lambda_G \), the addition is now defined as follows:

\[ [\gamma_1, \ldots, \gamma_q]_G \oplus [\beta_1, \ldots, \beta_r]_G := [\gamma_1, \ldots, \gamma_q, \beta_1, \ldots, \beta_r]_G. \]
With this addition we have only a commutative monoid structure on \( \Lambda_G \), respectively \( W_G \), where the free actions represent the zero element. To obtain inverse elements we have to introduce the following relations.

Suppose we have an action of \( G \) on a surface \( S_g \) with singular orbit data \( \{ \hat{\gamma}_1, \hat{\gamma}_1^{-1}, \hat{\gamma}_2, \ldots, \hat{\gamma}_q \} \). Let \( \nu = |\langle \gamma_1 \rangle| \), then the conjugacy class \( \hat{\gamma}_1 \) gives rise to a singular orbit with representative \( x \) such that \( a\gamma_1a^{-1} \) acts by rotation through \( 2\pi/\nu \) on the tangent space at \( ax, a \in G \). On the other hand \( \hat{\gamma}_1^{-1} \) gives rise to another singular orbit with representative \( z \) such that \( a\gamma_1a^{-1} \) acts by rotation through \( -2\pi/\nu \) on the tangent space at \( az, a \in G \). Let \( T \) be a set of representatives for the \( \langle \gamma_1 \rangle \) left cosets of \( G \). Find discs \( D_1 \) and \( D_2 \) around \( x \) and \( z \) respectively such that \( D_j \) is fixed by \( \langle \gamma_1 \rangle \), \( j = 1, 2 \), and \( \cup_{j=1,2} \cup_{t \in T} \{ tD_j \} \) are mutually disjoint. Then excise all discs \( \{ tD_j \} \) \( t \in T \) for \( j = 1, 2 \), from \( S_g \) and connect the boundaries \( \partial(tD_1) \) with \( \partial(tD_2) \) by means of tubes \( S^1 \times [0, 1] \) for every \( t \in T \). The resulting surface \( S_{g+w} \) has \( w = |G|/\nu \) new handles. The action of \( G \) on \( S_g \) can be extended to \( S_{g+w} \) by permuting and rotating the new handles. This extended action yields the singular orbit data \( \{ \hat{\gamma}_2, \ldots, \hat{\gamma}_q \} \). Pairs of singular orbits which have opposite rotation on the tangent spaces will be called cancelling pairs. The above process of eliminating cancelling pairs will be called reduction and if there are no such cancelling pairs left the singular orbit data is said to be in reduced form.

Now we can define the relation.

\[
\{ \hat{\gamma}_1, \ldots, \hat{\gamma}_q \} \sim \{ \hat{\beta}_1, \ldots, \hat{\beta}_n \} : \iff \text{The two singular orbit data have the same reduced form}
\]

This relation defines an equivalence relation on the set \( W_G \) of singular orbit data.

\[
\mathbb{W}_G := W_G/\sim
\]

The set \( \mathbb{W}_G \) is not only a commutative monoid as \( W_G \) but contains also inverse elements and thus is a commutative group. The inverse element of \( \{ \hat{\gamma}_1, \ldots, \hat{\gamma}_q \} \) is \( \{ \hat{\gamma}_1^{-1}, \ldots, \hat{\gamma}_q^{-1} \} \) and the zero elements are cancelling pairs \( \{ \hat{\gamma}, \hat{\gamma}^{-1} \} \), \( \gamma \in G \), and \( \{ \varnothing \} \) the free action.

We can also give a purely algebraic description of this group in terms of \( \Lambda_G \)

\[
\mathbb{B}_G = \Lambda_G/\langle [\hat{\gamma}, \hat{\gamma}^{-1}]_G \mid \gamma \in G \rangle
\]

with the inverse elements \( \varnothing [\hat{\gamma}_1, \ldots, \hat{\gamma}_q] = [\hat{\gamma}_1^{-1}, \ldots, \hat{\gamma}_q^{-1}] \). It is obvious from the definitions that the groups \( \mathbb{B}_G \) and \( \mathbb{W}_G \) are canonically isomorphic (see also [8, proposition 4]). In the sequel we will only use the notation \([\hat{\gamma}_1, \ldots, \hat{\gamma}_q]\) for singular orbit data and \( \mathbb{B}_G \) for the group of singular orbit data. Even though elements of \( \mathbb{B}_G \) consist of classes of singular orbit data, we will by abuse of language also use the notation \([\hat{\gamma}_1, \ldots, \hat{\gamma}_q]\) for elements of \( \mathbb{B}_G \). But one has to keep in mind that with this notation there is always a choice of representative involved.

**Remark 1.** All free \( G \)-actions represent the zero element in \( \mathbb{B}_G \).

**Remark 2.** The addition doesn’t have any control on the genus. The genus can become arbitrarily large.

Because of the purely algebraic description of \( \mathbb{B}_G \) in equation (1), it is now possible
to give a complete computation of this group. This is done in theorem \[10\] for finite Abelian groups and for arbitrary finite groups in theorem \[19\].

The map \( \varphi \), which is defined in section \[3\], depends only on the singular orbit data and the genus of the surface. Two \( G \)-actions which have the same singular orbit data but not the same genus are mapped under \( \varphi \) to representations which differ only by a rational representation. Moreover the singular orbit data which form the relations, \([\hat{\gamma}, \hat{\gamma}^{-1}] \in G\), \( \gamma \in G \), are mapped under \( \varphi \) to rational representations and we obtain a well defined map

\[
\eta : \mathbb{B}_G \rightarrow R_C(G) \tag{2}
\]

\[
\alpha \mapsto \varphi(\phi_\alpha) - \overline{\varphi(\phi_\alpha)}.
\]

Here \( \phi_\alpha \) denotes a \( G \)-action which represents the element \( \alpha \in \mathbb{B}_G \) and \( \overline{\varphi(\phi_\alpha)} \) denotes the complex conjugate representation of \( \varphi(\phi_\alpha) \). In addition one can prove that the map \( \eta \) is actually a group homomorphism.

The homomorphism \( \eta \circ \chi \) is the \( G \)-signature defined by Atiyah and Singer in \[2\]. This \( G \)-signature was used in the case \( G \cong \mathbb{Z}/p\mathbb{Z} \), \( p \) a prime, by Ewing in \[5\] and in the case \( G \cong \mathbb{Z}/n\mathbb{Z} \), \( n \) an integer, by Edmonds/Ewing in \[4\] to prove their results.

There is another way to define a \( G \)-signature with the help of the map \( \varphi \). By factoring out the image under \( \varphi \) of all the relations of \( \mathbb{B}_G \), \( \varphi \) induces a group homomorphism \( \theta \).

\[
\theta : \mathbb{B}_G \rightarrow R_C(G)/E_G
\]

Here \( R_C(G)/E_G \) denotes the complex representation ring modulo the subgroup \( E_G \). The map \( \theta \) and the subgroup \( E_G \) are studied in some details in \[8, section 4\]. In this paper we want to concentrate on the group \( \mathbb{B}_G \). However, in section \[8\] we give a short overview of the map \( \theta \) and the subgroup \( E_G \).

There are other interesting facts about the groups \( \mathbb{B}_G \), which are proven in \[8\]. E.g. for a homomorphism of finite groups \( f : H \rightarrow G \) there is a homomorphism

\[
\mathbb{B}_f : \mathbb{B}_H \rightarrow \mathbb{B}_G \]

\[
[\hat{\gamma}_1, \ldots, \hat{\gamma}_q] \mapsto [\hat{f}(\gamma_1), \ldots, \hat{f}(\gamma_q)]
\]

such that \( \mathbb{B}_{f \circ g} = \mathbb{B}_f \circ \mathbb{B}_g \) and \( \mathbb{B}_{id} = id \) and thus the correspondence \( \mathbb{B} : G \rightarrow \mathbb{B}_G \) is functorial. This functor \( \mathbb{B} \) carries many features of the representation theory of finite groups. Let \( H \) be a subgroup of \( G \) and \( i \) its inclusion. The inclusion induces the homomorphism \( \mathbb{B}_i : \mathbb{B}_H \rightarrow \mathbb{B}_G \) which is called induction map. The terminology is motivated by the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{B}_H & \xrightarrow{\mathbb{B}} & \mathbb{B}_G \\
\downarrow & & \downarrow \\
R_C H/E_H & \xrightarrow{\text{Ind}_H^G} & R_C G/E_G
\end{array}
\]

Here \( \text{Ind}_H^G \) denotes the induction map restricted to the quotient. Let \( K \) be another subgroup of \( G \), then there is another map \( \mathbb{B}_{res_K}^G : \mathbb{B}_G \rightarrow \mathbb{B}_K \) called the restriction.
map. It is defined by restricting the singular orbits of $G$ to the subgroup $K$. This map satisfies also a commutative diagram

$$
\begin{array}{c}
B_K & \xrightarrow{\text{res}_K^G} & B_G \\
\downarrow & & \downarrow \\
RCK/E_K & \xleftarrow{\text{res}_K^G} & RG/E_G
\end{array}
$$

where $\text{res}_K^G$ is the restriction map restricted to the quotient. Furthermore the induction and restriction maps satisfy a double coset formula (see [8, proposition 5]). Thus the functor $B$ describes a geometric representation theory.

Before we can turn to the next section we will give five examples to illustrate the nature of the groups $B_G$. We will also prove two lemmata which will be useful throughout the paper.

In the following examples the cyclic group of order $m$ will be denoted by $C_m$.

**Example 1.** Let $G = C_2$; $B_G \cong 0$.

**Example 2.** Let $m$ be odd, $G = C_m = \langle x \rangle$ then $B_G \cong \mathbb{Z}^{m-1}$ and a basis for $B_G$ is given by $[x, x^i, x^{m-i-1}]$, $i = 1, \ldots, \frac{m-1}{2}$.

**Example 3.** Let $m$ be even, $G = C_m = \langle x \rangle$ then $B_G \cong \mathbb{Z}^{\frac{m}{2}-1}$ and a basis for $B_G$ is given by $[x, x^i, x^{m-i-1}]$, $i = 1, \ldots, \frac{m}{2}-1$.

**Example 4.** Let $p$ be an odd prime, $G = C_p \times C_p = \langle x \rangle \times \langle y \rangle$ then $B_G \cong \mathbb{Z}^{\frac{p^2-1}{2}}$ and a basis for $B_G$ is given by $[x^j, y^j, x^{p-j}y^{p-j}]$, $j = 1, \ldots, (p-1)/2$, $i = 1, \ldots, p-1$; $[x, x^k, x^{p-k}]$, $k = 1, \ldots, (p-1)/2$; $[y, y^l, y^{p-l}]$, $l = 1, \ldots, (p-1)/2$.

For $p$ the even prime, we have $B_G \cong C_2$ and the generator is $[x, y, xy]$.

**Example 5.** Let $G = S_3 = C_3 \rtimes C_2$, $C_3 = \langle a \rangle$ and $C_2 = \langle b \rangle$. Then we have $[G, G] = C_3$, $\hat{a} = \{a, a^2\}$ and $\hat{b} = \{ab, a^2b, b\}$. The singular orbit data are then $[\hat{a}]$, $2 \cdot [\hat{a}] = [\hat{a}, \hat{a}] = [\hat{a}, a^2] = 0$, $[\hat{b}, \hat{b}] = 0$. Thus the group of singular orbit data is generated by $[\hat{a}]$ and $B_{S_3} = \langle [\hat{a}] \rangle \cong C_2$.

As $B_{C_3} \cong 0$ the only maps which are of any interest are the maps $\text{res}_{C_3}^{S_3} : B_{S_3} \to B_{C_3}$ and $\text{res}_i : B_{C_3} \to B_{S_3}$, where $i$ is the inclusion $i : C_3 \hookrightarrow S_3$. The maps are given by $\text{res}_{C_3}^{S_3}([\hat{a}]_{S_3}) = [a, a^2]_{C_3} = 0$ and $\text{res}_i([a, a, a]_{C_3}) = [\hat{a}, \hat{a}, \hat{a}]_{S_3} = [\hat{a}]_{S_3}$. Thus $\text{res}_{C_3}^{S_3}$ is just the zero map and $\text{res}_i$ is the surjection of $\mathbb{Z}$ onto $C_2$.

**Lemma 2** Any element of $B_G$ can be written as a sum of triples $[\hat{x}, \hat{y}, \hat{z}] \in B_G$.

**Proof.** Let $[\hat{x}_1, \ldots, \hat{x}_n]$ be any element of $B_G$. Then we can reduce the length by splitting off a triple.

$$
[\hat{x}_1, \ldots, \hat{x}_n] = \bigoplus_{i=1}^{n-3} \left[ \hat{x}_1 \cdots \hat{x}_i \hat{x}_{i+1} \right] \oplus \left[ \hat{x}_1 \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_n \right] = \cdots
$$

\[= \bigoplus_{i=1}^{n-3} \left[ \hat{x}_1 \cdots \hat{x}_i \hat{x}_{i+1} \right] \oplus \left[ \hat{x}_1 \cdots \hat{x}_{n-2}, \hat{x}_{n-1}, \hat{x}_n \right].\]
Remark 3. The choice of the conjugacy class \( x_1 \cdots x_i, \ i = 2, \ldots , n - 2 \), in lemma 2 is not unique. Every conjugacy class which is mapped to the same element in \( G/[G,G] \) would also be a possible choice.

Lemma 3 For any group \( G \) let \( M = \{ [\hat{x}_{i,1}, \ldots , \hat{x}_{i,m_i}] | i = 1, \ldots , m \} \), \( Q = \{ [\hat{z}_{i,1}, \ldots , \hat{z}_{i,q_i}] | i = 1, \ldots , q \} \) and \( N = \{ [\hat{y}_{i,1}, \ldots , \hat{y}_{i,n_i}] | i = 1, \ldots , n \} \) be subsets of \( \mathbb{B}_G \). Let \( K \) be the subgroup generated by \( M \cup Q \) and \( H \) the subgroup generated by \( N \). If the restrictions
\[
\hat{x}_{i,m_i} \neq \hat{x}_{i,m_i}^{-1} \text{ for all } i = 1, \ldots , m \tag{3}
\]
\[
\hat{x}_{i,m_i} \neq \hat{x}_{k,l}^{-1} \text{ for all } (i, m_i) \neq (k, l) ; i, k = 1, \ldots , m ; l = 1, \ldots , m_k \tag{4}
\]
\[
\hat{z}_{i,j} = \hat{z}_{i,j}^{-1} \text{ for all } j = 1, \ldots , q_i ; i = 1, \ldots , q \tag{5}
\]
\[
\hat{z}_{i,q_i} \neq \hat{z}_{k,l} \text{ for all } (i, q_i) \neq (k, l) ; i, k = 1, \ldots , q ; l = 1, \ldots , q_k \tag{6}
\]
\[
\hat{x}_{i,m_i} \neq \hat{y}_{k,l}^{-1} \text{ for all } i = 1, \ldots , m ; k = 1, \ldots , n ; l = 1, \ldots , n_k \tag{7}
\]
\[
\hat{z}_{i,q_i} \neq \hat{y}_{k,l} \text{ for all } i = 1, \ldots , q ; k = 1, \ldots , n ; l = 1, \ldots , n_k \tag{8}
\]
apply, then the following hold.

(a) There are no relations between the elements of \( M \cup Q \) except the obvious ones, \( 2 \cdot \alpha = 0 \), \( \forall \alpha \in Q \), thus \( K \cong Z^m \oplus (Z/2Z)^q \).

(b) The intersection of \( K \) with \( H \) is the trivial group and thus \( K \oplus H < \mathbb{B}_G \).

Proof. Note that by equation \( \Box \) and \( \Box \) we have:
\[
\hat{x}_{i,m_i} \neq \hat{z}_{k,j} \text{ for all } i = 1, \ldots , m ; j = 1, \ldots , q_k ; k = 1, \ldots , q \tag{9}
\]
and every element of \( Q \) has order two.

First we prove statement \( \Box \). In a linear combination
\[
\bigoplus_{i=1}^{m} a_i [\hat{x}_{i,1}, \ldots , \hat{x}_{i,m_i}] \oplus \bigoplus_{i=1}^{q} \epsilon_i [\hat{z}_{i,1}, \ldots , \hat{z}_{i,q_i}] , \ a_i \in \mathbb{Z} \ , \ \epsilon_i = 0, 1 \tag{10}
\]
the conjugacy classes \( \hat{x}_{i,m_i}, \ i = 1, \ldots , m \), cannot cancel because of equations \( \Box \) and \( \Box \) and thus they appear exactly \( a_i \) times. Consequently to have the linear combination \( \Box \) equal zero, the coefficient \( a_i, i = 1, \ldots , m \), have to be trivial. Furthermore by equation \( \Box \) a similar argument about the conjugacy classes \( \hat{z}_{i,q_i}, \ i = 1, \ldots , q \), shows that the \( \epsilon_i, i = 1, \ldots , q \), have to be trivial.

Next we prove statement \( \Box \). Any nontrivial element \( \eta \) of the intersection \( K \cap H \) is a linear combination like in \( \Box \). Thus at least one of the conjugacy classes \( \hat{x}_{i,m_i}, \ i = 1, \ldots , m \), or \( \hat{z}_{i,q_i}, \ i = 1, \ldots , q \) has to appear. On the other hand \( \eta \) is also a linear combination of the \( [\hat{y}_{i,1}, \ldots , \hat{y}_{i,n_i}] \), \( i = 1, \ldots , n \), and thus by equations \( \Box \) and \( \Box \) the \( \hat{x}_{i,m_i}, \ i = 1, \ldots , m \) and \( \hat{z}_{i,q_i}, \ i = 1, \ldots , q \) cannot appear which yields a contradiction. \( \blacksquare \)
3 Finite Abelian Groups

In this section $G$ will always denote a finite Abelian group and the maximal rank of an elementary Abelian 2-subgroup of $G$ will be denoted by $n_G$. First let $G$ be a cyclic group of prime power order $p^r$ with generator $x$, then we can define the following subsets of $G$.

**Definition 4**

$$p = 2 \ , \ T_G^+ := \{x^i \mid i = 1, \ldots , p^r / 2 - 1\}$$
$$S_G := \{p^r / 2\}$$

$$p \neq 2 \ , \ T_G^+ := \{x^i \mid i = 1, \ldots , (p^r - 1)/2\}$$
$$S_G := \emptyset$$

With these sets we can define $T_G^- := \{x^{-1} \mid x \in T_G^+\}$ and thus $T_G := \{x \in G \mid x \neq x^{-1}\}$ is the disjoint union of $T_G^+$ with $T_G^-$. Furthermore we have also $S_G = \{x \in G \mid x = x^{-1}, x \neq 1\}$. For a product $G = G_1 \times G_2$ we introduce the following definitions.

**Definition 5**

$$T_G^+ := T_{G_1}^+ \cup T_{G_2}^+ \cup (T_{G_1}^+ \times S_{G_2}) \cup (T_{G_1}^+ \times T_{G_2}) \cup (S_{G_1} \times T_{G_2}^+)$$
$$= T_{G_1}^+ \cup T_{G_2}^+ \cup (T_{G_1}^+ \times G_2 \setminus \{1\}) \cup (S_{G_1} \times T_{G_2}^+)$$
$$S_G := S_{G_1} \cup S_{G_2} \cup (S_{G_1} \times S_{G_2})$$

In this definition elements $x$ of $G_1$ or $G_2$ are thought of as elements $(1,x)$, respectively $(1,x)$, of $G$.

With the set $T_G^+$ we can again define the set $T_G^- := \{x^{-1} \mid x \in T_G^+\}$ and $T_G := \{x \in G \mid x \neq x^{-1}\}$ is also the disjoint union of $T_G^+$ with $T_G^-$. It is again true that $S_G = \{x \in G \mid x = x^{-1}, x \neq 1\}$.

Every finite Abelian group $G$ is isomorphic to a product of cyclic groups of prime power order. By definitions 4 and 5 we can define recursively for every such factorization of $G$ together with a fixed choice of generator for each factor, different sets $T_G^+$, $T_G^-$ and $S_G$. These different sets however are isomorphic by the isomorphisms between the different factorizations. In the sequel we will fix for every finite Abelian group one factorization into cyclic subgroups of prime power order together with a generator for each factor. Thus from now on $T_G^+$ and $S_G$ are fixed subsets of $G$.

**Definition 6** For $G$ a cyclic group of order $p^r$, $p$ a prime, $x$ the fixed generating element, we define

$$W_G := \{[x, x^i, x^{p^r - 1 - i}] \in B_G \mid \forall i \in \mathbb{N} \text{ such that } x^i \in T_G^+\}$$
$$V_G := \emptyset$$

**Proposition 7** The set $W_G$ is a basis for $B_G$ for any cyclic group $G$ of prime power order.
Proof. To prove that $W_G$ generates $\mathbb{B}_G$, it is enough by lemma 8 to show that triples $[x^k, x^l, x^q]$ lie in the span of $W_G$.

Let $[x^k, x^l, x^q]$ be any triple in $\mathbb{B}_G$ with $k + l + q = sp^r$ and $1 \leq k, l, q < p^r$, then we have the following equations. (In the equations below the sums $q + k, q + t$ and $q + t + 1$ will always be modulo $p^r$.)

$$
[x^k, x^l, x^q] = [x^k, x^l, x^{q+1}, x^{p^r-1}, x, x^{p^r-(q+1)}, x^q] = \ldots = [x^k, x^l, x^{q+k}, x^{p^r-1}, \ldots, x^{p^r-1}] \oplus \bigoplus_{t=0}^{k-1} [x, x^{p^r-(q+t+1)}, x^{q+t}]
$$

$$
= \bigoplus_{j=1}^{k-1} [x, x^j, x^{p^r-j-1}] \bigoplus_{t=0}^{k-1} [x, x^{p^r-(q+t+1)}, x^{q+t}]
$$

We will prove the linear independence only in the case $p \neq 2$. Let

$$
\bigoplus_{i=1}^{(p^r-1)/2} a_i [x, x^i, x^{p^r-1-i}] \ , \ a_i \in \mathbb{Z}
$$

be any linear combination of elements in $W_G$. The elements $x^j, j = 2, \ldots, (p^r-3)/2$ and their inverse $x^{p^r-j}$ appear exactly $a_j$, respectively $a_{j-1}$ times, in this linear combination, namely in $a_j [x, x^j, x^{p^r-1-j}]$ and in $a_{j-1} [x, x^{j-1}, x^{p^r-j}]$. The element $x^{(p^r-1)/2}$ appears exactly $2a_{(p^r-1)/2}$ times namely in $a_{(p^r-1)/2} [x, x^{(p^r-1)/2}, x^{(p^r-1)/2}]$ and its inverse $a_{(p^r-3)/2}$ times in $a_{(p^r-3)/2} [x, x^{(p^r-3)/2}, x^{(p^r+1)/2}]$.

A necessary condition for the linear combination to be zero is such that every element $x^j, j = 1, \ldots, (p^r-1)/2$ cancels with its inverse. This can only happen when

$$
a_j = a_{j-1} \ , \ j = 2, \ldots, (p^r-3)/2 \ and \ a_{(p^r-3)/2} = 2a_{(p^r-1)/2}.
$$

This however implies that the element $x$ appears $p^r a_{(p^r-1)/2}$ times but its inverse never. Thus the linear combination can only be zero when $a_i = 0$ for all $i = 1, \ldots, (p^r-1)/2$.

\[ \Box \]

**Definition 8** Let $G$ be a product of two groups $G = G_1 \times G_2$. An element $x \in G_i, i = 1, 2$, will also be considered as an element of $G$. Then we define

$$
W_G := \left\{ [x, y, (x, y)^{-1}] \in \mathbb{B}_G \mid x \in T_G^1, y \in G_2 \setminus \{1\} \ or \ x \in S_{G_1}, y \in T_G^1 \right\}
$$

$$
\cup W_{G_1} \cup W_{G_2}
$$

$$
V_G := \left\{ [x, y, (x, y)] \in \mathbb{B}_G \mid x \in S_{G_1}, y \in S_{G_2} \right\} \cup V_{G_1} \cup V_{G_2}
$$
Remark 4. The set \( V_G \) contains only elements of order two, i.e., for all \( [x, y, (x, y)] \in V_G \) we have \( 2 \cdot [x, y, (x, y)] = 0 \). On the other hand \( W_G \) contains only elements of infinite order.

In the same way as we constructed the sets \( T_G^+ \) and \( S_G \) recursively for any finite Abelian group \( G \), we can construct by definitions 5 and 8 the subsets \( W_G \) and \( V_G \) of \( B_G \) recursively for any finite Abelian group \( G \).

**Theorem 9** The set \( W_G \cup V_G \) is a basis for \( B_G \) for any finite Abelian group \( G \).

**Proof.** We will take advantage of the recursive definition of the sets \( W_G \) and \( V_G \) and prove the theorem by induction on the factorization of \( G \) into a product of groups. For cyclic groups of prime power order the theorem follows from proposition 7.

Let now \( G \) be a product of \( G_1 \) and \( G_2 \) and assume that the theorem holds for the two groups \( G_1 \) and \( G_2 \).

First we prove that the set \( W_G \cup V_G \) generates \( B_G \). By lemma 2 it is enough to prove that triples lie in the span of \( W_G \cup V_G \). An arbitrary triple has the form \( [(x_1, y_1), (x_2, y_2), (x_1^{-1}x_2^{-1}, y_1^{-1}y_2^{-1})] \in B_G \), \( x_i \in G_1 \), \( y_i \in G_2 \), \( i = 1, 2 \), and we can write

\[
[(x_1, y_1), (x_2, y_2), (x_1^{-1}x_2^{-1}, y_1^{-1}y_2^{-1})] = [x_1^{-1}, y_1^{-1}, (x_1, y_1)] \\
\oplus [x_2^{-1}, y_2^{-1}, (x_2, y_2)] \oplus [x_1x_2, y_1y_2, (x_1^{-1}x_2^{-1}, y_1^{-1}y_2^{-1})] \\
\oplus [x_1, x_2, x_1^{-1}x_2^{-1}] \oplus [y_1, y_2, y_1^{-1}y_2^{-1}].
\]

The first three summand (resp. their inverse) of the right hand side of the equation are elements of \( W_G \cup V_G \). The forth summand is an element of \( B_G \), and thus by assumption a linear combination of elements in \( W_G \cup V_G \), which is contained in \( W_G \cup V_G \), the same holds for the fifth summand when we replace \( G_1 \) by \( G_2 \). Note that whenever \( x_i = 1 \) or \( y_i = 1 \) then we ignore the elements which are zero.

It remains to prove that there are no relations between the elements of \( W_G \cup V_G \) except the obvious ones, i.e., \( 2\alpha = 0 \), \( \forall \alpha \in V_G \). To apply lemma 8 we have to specify what are the sets \( M \), \( Q \), and \( N \). In our situation we have that \( M = W_G \setminus (W_G \cup W_G \cup V_G \cup V_G) \), \( Q = V_G \setminus (V_G \cup V_G \cup V_G) \) and \( N = W_G \cup W_G \cup V_G \cup V_G \). By definition 8 the equations 7 to 8 of lemma 3 are satisfied and therefore there are no relations, except the obvious ones, between the elements of \( W_G \cup V_G \setminus (W_G \cup W_G \cup V_G \cup V_G) \) and also no relations with elements of \( W_G \cup W_G \cup V_G \cup V_G \). On the other hand by assumption \( W_G \cup V_G \) is a basis of \( B_G \), \( i = 1, 2 \), and there are obviously no relations between \( W_G \cup V_G \) and \( W_G \cup V_G \) or \( W_G \cup V_G \).

**Theorem 10**

\[
B_G \cong \mathbb{Z}^{\mid T_G^+ \mid} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mid S_G \mid - n_G}
\]

**Proof.** By theorem 3 and remark 4 we have \( B_G = \mathbb{Z}^{\mid W_G \mid} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mid V_G \mid} \) thus it suffices to prove \( \mid W_G \mid = \mid T_G^+ \mid \) and \( \mid V_G \mid = \mid S_G \mid - n_G \). We will again proceed by induction on the product structure.
For a cyclic group of prime power order \( p^r \) we have by definitions \( 2 \) and \( 3 \)

\[
p = 2, \quad |W_G| = |T^+_G|, \\
|S_G| = 1 = n_G \text{ and thus } |V_G| = 0 = |S_G| - n_G;
\]

\[
p \neq 2, \quad |W_G| = |T^+_G|, \\
|S_G| = 0 = n_G \text{ and thus } |V_G| = 0 = |S_G| - n_G.
\]

Let now \( G \) be a product of \( G_1 \) and \( G_2 \) and assume that the theorem holds for the two groups \( G_1 \) and \( G_2 \), i.e.,

\[
|W_{G_1}| = |T^+_1|, \quad |V_{G_1}| = |S_{G_1}| - n_{G_1}; \ \\
|W_{G_2}| = |T^+_2|, \quad |V_{G_2}| = |S_{G_2}| - n_{G_2}.
\]

By definition \( 8 \), the equations \( 11 \) and \( 12 \) and the fact that \( n_G = n_{G_1} + n_{G_2} \) we have

\[
|W_G| = |T^+_G| \cdot (|G_2| - 1) + |S_{G_1}| \cdot |T^+_G| + |W_{G_1}| + |W_{G_2}| \\
= |T^+_G| \cdot (|G_2| - 1) + |S_{G_1}| \cdot |T^+_G| + |T^+_G| + |T^+_G| \\
= |T^+_G|, \\
|V_G| = |S_{G_1}| \cdot |S_{G_2}| + |V_{G_1}| + |V_{G_2}| \\
= |S_{G_1}| \cdot |S_{G_2}| + |S_{G_1}| - n_{G_1} + |S_{G_2}| - n_{G_2} \\
= |S_G| - n_G.
\]

Later on in the Non-Abelian case it will be more convenient to replace \( V_G \) by another basis. Let \( D_2 \) be the 2-subgroup of \( \mathbb{B}_G \), i.e., \( D_2 \cong (\mathbb{Z}/2\mathbb{Z})^{S_G-n_G} \). Furthermore let \( H_2 \) be the maximal elementary Abelian 2-subgroup of \( G \), i.e., \( H_2 \cong (\mathbb{Z}/2\mathbb{Z})^{n_G} \) and the elements \( \{z_1, \ldots, z_{n_G}\} \) denote a generating set of \( H_2 \). By theorem \( 10 \) and the fact that \( S_{H_2} = H_2 \setminus \{1\} \) and \( T^+_H = \emptyset \) we have

\[
(\mathbb{Z}/2\mathbb{Z})^{S_{H_2}-n_G} = (\mathbb{Z}/2\mathbb{Z})^{[H_2]-1-n_{H_2}} \cong \mathbb{B}_{H_2} < \mathbb{B}_G
\]

**Proposition 11**

\[
\mathbb{B}_{H_2} = D_2 < \mathbb{B}_G
\]

**Proof**. It suffices to prove that \( H_2 \setminus \{1\} = S_G \).

"\( \subseteq \)" Let \( x \in H_2 \setminus \{1\} \Rightarrow x = x^{-1}, x \neq 1 \Rightarrow x \in S_G \\
"\( \supseteq \)" Let \( x \in S_G \setminus H_2 \). The group generated by \( H_2 \) and \( x \) has to be an elementary Abelian 2-group. As \( H_2 \) is maximal it follows \( x \in H_2 \) which contradicts the assumption.

We have seen that \( V_G \) is a basis for \( D_2 \) and thus by proposition \( 11 \) also a basis for \( \mathbb{B}_{H_2} \). In the following we introduce another basis for \( \mathbb{B}_{H_2} \).
Definition 12

\[ L_G := \left\{ \left[ z_{i_1}, \ldots , z_{i_s}, z_{i_1} \cdots z_{i_s} \right] \in \mathbb{B}_{H_2} \mid (i_1, \ldots , i_s) \text{ an unordered } \right. \]
\[ \left. s \right. \text{-tuple, such that } i_j = 1, \ldots , n_G ; \ i_j \neq i_k , \ \forall \ j \neq k \]
\[ j, k = 1, \ldots , s ; \ 2 \leq s \leq n_G \right\} \]

The elements \( z_1, \ldots , z_{n_G} \) denote a generating set of \( H_2 \).

Proposition 13 \( L_G \) is a basis of \( \mathbb{B}_{H_2} \).

Proof. First we prove that \( L_G \) spans \( \mathbb{B}_{H_2} \). By lemma 2 every element of \( \mathbb{B}_{H_2} \) can be written as a sum of triples. The elements \( z_i, i = 1, \ldots , n_G \), generate \( H_2 \), thus an arbitrary triple has the form
\[
\left[ z_{i_1} \cdots z_{i_1}, z_{i_1} \cdots z_{i_1}, z_{i_1} \cdots z_{i_1} \right]
\]
with \( z_{i_k} \neq z_{i_h} \) and \( z_{j_r} \neq z_{j_q} \) for all \( k \neq h \) and \( r \neq q \). We can now write this element as a linear combination with elements from \( L_G \).

\[
\left[ z_{i_1} \cdots z_{i_1}, z_{i_1} \cdots z_{i_1}, z_{i_1} \cdots z_{i_1} \right] = \left[ z_{i_1}, \ldots , z_{i_1}, z_{i_1} \cdots z_{i_1} \right] \oplus \left[ z_{i_1}, \ldots , z_{i_1}, z_{i_1} \cdots z_{i_1} \right] \oplus \left[ z_{i_1}, \ldots , z_{i_1}, z_{i_1} \cdots z_{i_1} \right]
\]

If \( z_{i_k} = z_{j_r} \) for some \( k \) and \( r \) then reduce the last summand. Next we prove that there are no relations. Each element \( x \in H_2 \) has an unique presentation up to ordering \( x = z_{i_1} \cdots z_{i_s}, i_j = 1, \ldots , n_G, i_j \neq i_k, \ \forall \ j \neq k, \ j, k = 1, \ldots , s, \ 1 \leq s \leq n_G \). Thus there is a one to one correspondence between the elements of \( H_2 \) which are the product of at least two generators and the elements of \( L_G \). Applying lemma 3(a) with \( Q = L_G \) and \( M = \emptyset = N \) we deduce that there are no relations between the elements of \( L_G \) except the usual ones, i.e., \( 2\alpha = 0 \) for all \( \alpha \in L_G \).

The conclusion is that both sets \( L_G \) and \( V_G \) are a basis for the same subgroup \( \mathbb{B}_{H_2} \) of \( \mathbb{B}_G \).

4 Finite Groups

Let \( G \) be a finite group and \([G,G]\) denote the commutator subgroup of \( G \). We obtain a short exact sequence
\[
1 \to [G,G] \to G \xrightarrow{\phi} G' \to 1
\]
where \( G' = G/[G,G] \). We will write \( \hat{x} \) for the image of \( x \) under \( \phi \). The conjugacy class of \( x \) in \( G \) will be denoted by \( \hat{x} \) and the set of conjugacy classes of \( G \) by \( \hat{G} \). By the notation \( \hat{x}^{-1} \) we will mean \( x^{-1} \). This makes sense since \( x^{-1} = y^{-1} \) if and only if
$x$ and $y$ are conjugate. The homomorphism $\phi$ induces not only the homomorphism $\mathbb{B}_\phi$ but also the well defined set map

$$\hat{\phi} : \hat{G} \to \hat{G}' = G'.$$

$$\hat{x} \mapsto \hat{x}$$

In the sequel the symbol $\hat{x}$ will denote an element of $G'$ and also the subset $x : [G, G] \subset G$. Thus we can write for an element $\hat{x}$ of $G'$

$$\hat{x} = \bigcup_{i=0}^n \hat{x}_i, \quad \hat{x}_i \in \hat{G}, \ i = 0, \ldots, n.$$ 

For every element $\hat{x} \in G'$ we fix a numbering of its conjugacy classes $\hat{x}_i, \ i = 0, \ldots, n$, such that

- $\hat{x} = \bigcup_{i=0}^n \hat{x}_i$,
- if there are $k + 1$ conjugacy classes with $\hat{x}_j = \hat{x}_j^{-1}$ then $\hat{x}_i = \hat{x}_i^{-1}$ if and only if $i \in \{0, \ldots, k\}$,
- if $\hat{x} \neq \tilde{y}$ and $\hat{x}^{-1} = \tilde{y}$ then $\hat{x}_i^{-1} = \tilde{y}_i$ for all $i = 0, \ldots, n$.

**Remark 5.** If $\hat{x} \neq \hat{x}^{-1}$ then $\hat{x}_i^{-1} \neq \hat{x}_j, \forall j, i = 0, \ldots, n$.

**Remark 6.** If $\hat{x} = \hat{x}^{-1}$ and $\hat{x}_0^{-1} \neq \hat{x}_0$ then for every $i = 0, \ldots, n$, there exists some $j = 0, \ldots, n$, $j \neq i$ with $\hat{x}_i^{-1} = \hat{x}_j$.

**Remark 7.** If $\hat{x} = \hat{x}^{-1}$ and $\hat{x}_0^{-1} = \hat{x}_0$ then $\hat{x}_i^{-1} = \hat{x}_i, \forall i = 0, \ldots, k$, for some $0 \leq k \leq n$ and for every $k < i \leq n$, there exists some $k < j \leq n$, $j \neq i$ with $\hat{x}_i^{-1} = \hat{x}_j$.

With this numbering fixed we can define a well defined set map.

$$\hat{\psi} : G' \to \hat{G}$$

$$\hat{x} \mapsto \hat{x}_0$$

This map satisfies $\hat{\phi} \circ \hat{\psi} = id_{G'}$ and induces a map $\mathbb{B}_{\hat{\psi}}$

$$\mathbb{B}_{G'} \xrightarrow{\hat{\psi}} \mathbb{B}_G$$

$$[\hat{x}, \hat{y}, \ldots] \mapsto [\hat{\psi}(\hat{x}), \hat{\psi}(\hat{y}), \ldots].$$

The map $\mathbb{B}_{\hat{\psi}}$ is not well defined, indeed if $\hat{x} = \hat{x}^{-1}$ but $\hat{x}_0 \neq \hat{x}_0^{-1}$ then

$$0 = \mathbb{B}_{\hat{\psi}}(0) = \mathbb{B}_{\hat{\psi}}([\hat{x}, \hat{x}]) = [\hat{x}_0, \hat{x}_0] \neq 0.$$ 

We collect the images of the elements where the map $\mathbb{B}_{\hat{\psi}}$ fails to be well defined:

$$P_G = \{ [\hat{x}_0, \hat{x}_0] \mid \hat{x} = \hat{x}^{-1}, \ \hat{x}_0 \neq \hat{x}_0^{-1} \}.$$ 

**Proposition 14** The map $\mathbb{B}_{\hat{\psi}}$ is well defined and linear up to elements of $P_G$. 
Proof. The only relations in $\mathbb{B}_G'$ which are not satisfied in $\mathbb{B}_G$, under the map $\mathbb{B}_\psi$, are given by $\{ [\hat{x}, \check{x}] \mid \check{x} = \hat{x}^{-1}, \hat{x}_0 \neq \check{x}_0^{-1} \}$. Thus up to elements of $P_G$ the map $\mathbb{B}_\psi$ is well defined.

Let $[1\hat{x}, \ldots, i\check{x}]$ and $[1\hat{y}, \ldots, k\check{y}]$ be two elements in $\mathbb{B}_G'$. (We introduce here the indexing on the left, because we will need also the indexing on the right later on.) Their sum is given by $[1\hat{x}, \ldots, i\check{x}, 1\hat{y}, \ldots, k\check{y}]$ up to cancelling pairs. Under the map $\mathbb{B}_\psi$ we obtain

$$[1\hat{x}_0, \ldots, i\check{x}_0] \oplus [1\hat{y}_0, \ldots, k\check{y}_0] \equiv [1\hat{x}_0, \ldots, i\check{x}_0, 1\hat{y}_0, \ldots, k\check{y}_0]$$

up to the image of cancelling pairs in $\mathbb{B}_G'$. The only pairs which cancel in $\mathbb{B}_G'$ but their images under $\mathbb{B}_\psi$ do not cancel in $\mathbb{B}_G$ are the elements $\{ [\hat{x}, \check{x}] \mid \check{x} = \hat{x}^{-1}, \hat{x}_0 \neq \check{x}_0^{-1} \}$. Thus up to elements of $P_G$ the map $\mathbb{B}_\psi$ is linear.

Note that whenever $[\hat{x}_0, \check{y}_0, \ldots]$ is an element of $\mathbb{B}_G$ and $\hat{\phi}(\hat{x}_0) = \check{\phi}(\check{x}_i)$ and $\hat{\phi}(\check{y}_0) = \check{\phi}(\check{y}_j)$ then $[\hat{x}_i, \check{y}_j, \ldots]$ is also an element of $\mathbb{B}_G$ for any $i$ and $j$. Moreover they have the same image under $\mathbb{B}_\psi$, $\mathbb{B}_\phi([\hat{x}_0, \check{y}_0, \ldots]) = \mathbb{B}_\phi([\hat{x}_i, \check{y}_j, \ldots]) = [\hat{x}, \check{y}, \ldots]$.

We define again the sets $S_G$, $T_G$ and $T_G^\pm$ but with more conditions.

$$S_G = \{ \hat{x} \in \hat{G} \mid \hat{x}^{-1} = \check{x}, x \neq 1 \}$$

$$T_G = \{ \hat{x} \in \hat{G} \mid \hat{x}^{-1} \neq \check{x} \}$$

The set $T_G$ is again the disjoint union of $T_G^+$ with $T_G^-$ satisfying the following conditions.

1. If $\hat{x} \in T_G^+$ then $\hat{x}^{-1} \in T_G^-$.

2. If $\hat{x} \in T_G^+$ with $\hat{x} \neq \check{x}^{-1}$ then $\check{y} \in T_G^+$ for every $y \in x \cdot [G, G]$.

3. If $\hat{x} = \check{x}^{-1}$ with $\hat{x}_0 \neq \check{x}_0^{-1}$ then $\hat{x}_0 \in T_G^+$.

Note that if $\hat{x}_i \in S_G$ for some $i$ then $\hat{x} = \check{x}^{-1}$ and by the way we fixed the numbering of the conjugacy classes it follows $\hat{x}_0 \in S_G$.

With $H_2'$ we denote the maximal elementary Abelian 2-subgroup of $G'$ and with $n_{G'}$ its rank, i.e., $H_2' \cong (\mathbb{Z}/2\mathbb{Z})^{n_{G'}}$. Let

$$M_{G'} = \{ \hat{x} \in G' \mid \hat{x}_0^{-1} = \check{x}_0 \}$$

be a subset of $H_2'$ and $K'$ the subgroup generated by $M_{G'}$. For the rank of $K'$ we write $n_{K'}$ and then we have

$$(\mathbb{Z}/2\mathbb{Z})^{n_{K'}} \cong K' < H_2' \cong (\mathbb{Z}/2\mathbb{Z})^{n_{G'}}.$$
Now we have all the ingredients to construct a basis of \( \mathbb{B}_G \) and with this basis to prove theorem 9.

\[
N_1 = \left\{ [\hat{x}] \mid \hat{x} = 1, \hat{x} \in T_G^+ \cup S_G \right\}
\]

\[
N_2 = \left\{ [\hat{x}_0, \hat{x}_i^{-1}] \mid \hat{x} \neq \hat{x}^{-1}, \hat{x}_0 \in T_G^+ \cup S_G \right\}
\]

\[
N_3 = \left\{ [\hat{x}_0, \hat{x}_i] \mid \hat{x} = \hat{x}^{-1}, \hat{x} \neq 1, \hat{x}_0 \in T_G^+ \cup S_G \right\}
\]

\[
N_4 = \left\{ [\hat{x}_0, \hat{x}_i] \mid \hat{x} = \hat{x}^{-1}, \hat{x} \neq 1, \hat{x}_0 \in S_G \right\}
\]

\[
N_5 = \left\{ [\hat{x}_0, \hat{y}_0, \hat{z}_0] = \mathbb{B}_\hat{\psi}([\hat{x}, \hat{y}, \hat{z}]) \in \mathbb{B}_G \mid [\hat{x}, \hat{y}, \hat{z}] \in W_G \right\}
\]

For the last sets \( N_6 \) and \( N_7 \) we need the basis \( L_{G'} \) of \( \mathbb{B}_{H'} \) with a special choice of the generating elements of \( H'_2 \). Choose a generating set \( \{i_1 \hat{z}, \ldots, n_{K'}, \hat{z}\} \) of \( K' \) such that \( i_1 \hat{z} \in M_{G'}, i = 1, \ldots, n_{K'} \). Now find another set \( \{(n_{K'}+1) \hat{z}, \ldots, n_{H'_2} \hat{z}\} \) such that the union of both \( \{i_1 \hat{z}, \ldots, n_{H'_2} \hat{z}\} \) generates the group \( H'_2 \). With this generating set and definition 12 we can construct \( L_{G'} \):

\[
L_{G'} = \left\{ [i_1 \hat{z}, \ldots, i_s \hat{z}] \in \mathbb{B}_{H'_2} \mid (i_1, \ldots, i_s) \text{ an unordered } s\text{-tuple, such that } i_j = 1, \ldots, n_{G'}; i_j \neq i_k, \forall j \neq k \right\}
\]

\[
N_6 = \left\{ [i_1 \hat{z}_0, \ldots, i_s \hat{z}_0] = \mathbb{B}_\hat{\psi}([i_1 \hat{z}, \ldots, i_s \hat{z}]) \in \mathbb{B}_G \mid [i_1 \hat{z}, \ldots, i_s \hat{z}] \in L_{G'} \right\}
\]

\[
N_7 = \left\{ [i \hat{z}_0, i \hat{z}] = \left[ \hat{\psi}(i \hat{z}), \hat{\psi}(i \hat{z}) \right] \in \mathbb{B}_G \mid i = n_{K'} + 1, \ldots, n_{G'} \right\}
\]

Note that the product of conjugacy classes is not defined. The symbol \( i_1 \hat{z}_0 \cdots i_s \hat{z}_0 \) stands for the image of \( i_1 \hat{z} \cdots i_s \hat{z} \) under the map \( \hat{\psi} \). We haven’t added the elements \( [i \hat{z}_0, i \hat{z}_0], i = 1, \ldots, n_{K'} \), as they are all trivial. Indeed the elements \( i \hat{z}, i = 1, \ldots, n_{K'} \), belong to \( M_{G'} \) and thus \( i \hat{z}_0 = i \hat{z}_0^{-1} \).

**Proposition 15** The elements \([\hat{x}_0, \hat{x}_0] \in P_G\) lie all in the subgroup generated by \( N_6 \cup N_7 \).

**Proof.** Let \([\hat{x}_0, \hat{x}_0] \) be an element of \( P_G \), then \( \hat{\phi}(\hat{x}_0) = \hat{x} \) is an element of \( H'_2 \) and thus \( \hat{x} = i_1 \hat{z} \cdots i_t \hat{z} \) for some \( i_j = 1, \ldots, n_{H'_2}, j = 1, \ldots, t \), furthermore \( \hat{\psi}(\hat{x}) = \hat{x}_0 \).

Now we have two cases either \( t = 1 \) or \( t > 1 \).

For \( t = 1 \) we have \( i_1 \hat{z}_0 = \hat{\psi}(i \hat{z}) = \hat{\psi}(\hat{\phi}(\hat{x}_0)) = \hat{x}_0 \), thus \( n_{K'} < i_1 \leq n_{H'_2} \) and by the definition of \( N_7 \) it follows \([\hat{x}_0, \hat{x}_0] \in N_7\).
For \( t > 1 \) we obtain the following equation:
\[
[x_0, x_0] = 2 \cdot \left[ i_1 \hat{z}_0, \ldots, i_t \hat{z}_0, x_0 \right] \bigoplus \bigoplus_{j=1}^{t} \left[ i_j \hat{z}_0, i_j \hat{z}_0 \right]
\]

\[
= 2 \cdot \left[ i_1 \hat{z}_0, \ldots, i_t \hat{z}_0, i_1 \hat{z}_0 \cdots i_t \hat{z}_0 \right] \bigoplus \bigoplus_{j=1}^{t} \left[ i_j \hat{z}_0, i_j \hat{z}_0 \right]
\]

First note that the elements \( [i_j \hat{z}_0, i_j \hat{z}_0] \) are zero whenever \( 1 \leq i_j \leq n_{K'} \) and otherwise they belong to \( N_7 \). On the other hand the element \( [i_1 \hat{z}_0, \ldots, i_t \hat{z}_0, i_1 \hat{z}_0 \cdots i_t \hat{z}_0] \) belongs to \( N_6 \) and the proposition is proven. \[\blacksquare\]

**Proposition 16** The set \( \bigcup_{i=1}^{7} N_i \) generates \( B_G \).

**Proof.** By lemma 3 it suffices to prove that any triple \( [\hat{x}, \hat{y}, \hat{z}] \) is a linear combination of elements in \( \bigcup_{i=1}^{7} N_i \).

We fix the following notation: \( \hat{\phi}(\hat{x}) = \hat{x} \), \( \hat{\phi}(\hat{y}) = \hat{y} \), \( \hat{\phi}(\hat{z}) = \hat{z} \) and \( \hat{\psi}(\hat{x}) = \hat{x}_0 \), \( \hat{\psi}(\hat{y}) = \hat{y}_0 \), \( \hat{\psi}(\hat{z}) = \hat{z}_0 \).

We have to distinguish four cases:

(i) Let \( x, y, z \in [G, G] \) then we can write \( [\hat{x}, \hat{y}, \hat{z}] = [\hat{x}] \oplus [\hat{y}] \oplus [\hat{z}] \) and thus it is generated by \( N_1 \).

(ii) Let \( x, y \in [G, G] \) then we have \( x \cdot y = \hat{z} \in [G, G] \) and thus \( z \in [G, G] \) and we reduced this case to the first one.

(iii) Let \( x \in [G, G] \) and \( y, z \notin [G, G] \) then we can write \( [\hat{x}, \hat{y}, \hat{z}] = [\hat{x}] \oplus [\hat{y}] \oplus [\hat{z}] \) where \( [\hat{x}] \) lies in the span of \( N_1 \). Thus it is enough to show that \( \hat{y}, \hat{z} \) is in the span of \( \bigcup_{i=1}^{7} N_i \). Note that in this case \( \hat{z} = \hat{y}^{-1} \). Now assume that \( \hat{y}, \hat{z} \) isn’t a cancelling pair, then we have again different cases:

(a) Let \( \hat{z} \neq \hat{z}^{-1} \) which implies \( \hat{z}_0 \in T_G \) and \( \hat{z}_0 = \hat{y}_0^{-1} \), then we have either \( \hat{z}, \hat{z}_0 \in T_G^+ \) or \( \hat{y}, \hat{y}_0 \in T_G^- \).

- If \( \hat{z}, \hat{z}_0 \in T_G^+ \) then of course \( \hat{y}, \hat{y}_0 \in T_G^- \) and we obtain \( \hat{y}, \hat{z} = \hat{y}_0, \hat{z}_0 \) if \( \hat{z}_0 \in T_G^+ \) or \( \hat{y}, \hat{z} = \hat{y}_0, \hat{z}_0 \) if \( \hat{z}_0 \in T_G^- \). The last two summands are elements of \( N_2 \).

(b) Let \( \hat{z} = \hat{z}^{-1} \), which implies \( \hat{z}_0 = \hat{y}_0 \), and assume that \( \hat{z}_0 \in T_G^+ \). These assumptions lead to the following cases.

- \( \hat{y}, \hat{z} \in T_G^+ \Rightarrow [\hat{y}, \hat{z}] = [\hat{y}_0, \hat{y}] \oplus [\hat{z}_0, \hat{z}] \oplus [\hat{y}_0, \hat{z}_0] \)
- \( \hat{y} \in T_G^+, \hat{z} \in T_G^- \Rightarrow [\hat{y}, \hat{z}] = [\hat{y}_0, \hat{y}] \oplus [\hat{z}_0, \hat{z}^{-1}] \)
- \( \hat{y}, \hat{z} \in T_G^- \Rightarrow [\hat{y}, \hat{z}] = \hat{y}_0, \hat{y}^{-1} \oplus [\hat{z}_0, \hat{z}^{-1}] \oplus [\hat{y}_0, \hat{z}_0] \)
We see that in the above three cases the element \([\hat{y}, \hat{z}]\) is generated by \(N_3\) and by proposition \([13]\) also by the set \(N_6 \cup N_7\).

(c) Let \(\hat{z} = \hat{z}^{-1}\), which implies \(\hat{z}_0 = \hat{y}_0\), and assume that \(\hat{z}_0 \in S_G\). These assumptions lead to the following cases.

- \(\hat{y}, \hat{z} \in T_G^+ \cup S_G \Rightarrow [\hat{y}, \hat{z}] = [\hat{y}_0, \hat{y}] \oplus [\hat{z}_0, \hat{z}]\)
- \(\hat{y} \in T_G^+ \cup S_G, \hat{z} \in T_G^- \Rightarrow [\hat{y}, \hat{z}] = [\hat{y}_0, \hat{y}] \ominus [\hat{z}_0, \hat{z}^{-1}]\)
- \(\hat{y}, \hat{z} \in T_G^- \Rightarrow [\hat{y}, \hat{z}] = \ominus [\hat{y}_0, \hat{y}^{-1}] \oplus [\hat{z}_0, \hat{z}^{-1}]\)

For these three cases we can deduce that the element \([\hat{y}, \hat{z}]\) is generated by \(N_4\). Note that whenever \(\hat{z}_0 = \hat{z}\) or \(\hat{y}_0 = \hat{y}\) then we ignore the elements which are zero.

(iv) For the last case we have \(x, y, z \notin [G, G]\) which implies:

\[ [\hat{x}, \hat{y}, \hat{z}] = [\hat{x}_0, \hat{y}_0, \hat{z}_0] \oplus [\hat{x}_0^{-1}, \hat{x}] [\hat{y}_0^{-1}, \hat{y}] [\hat{z}_0^{-1}, \hat{z}] \cdot \]

The last three summands are pairs and by the third case lie in the span of \(\cup_{i=1}^{7} N_i\) and so we have to consider the element \([\hat{x}_0, \hat{y}_0, \hat{z}_0]\).

\[ [\hat{x}_0, \hat{y}_0, \hat{z}_0] = B_{\psi} \circ B_{\theta}([\hat{x}_0, \hat{y}_0, \hat{z}_0]) = B_{\psi}([\hat{x}, \hat{y}, \hat{z}]) \]

The element \([\hat{x}, \hat{y}, \hat{z}]\) belongs to \(B_{G'}\) and thus by theorem \([8]\) and proposition \([13]\) is a linear combination of elements \(\{\theta_i\}_{i \in I}\) in \(W_{G'} \cup L_{G'}\). By proposition \([14]\) the map \(B_{\psi}\) is linear up to elements of \(P_G\) and so we get:

\[ [\hat{x}_0, \hat{y}_0, \hat{z}_0] = B_{\psi}([\hat{x}, \hat{y}, \hat{z}]) = \bigoplus_{i \in I} B_{\psi}(\theta_i) \bigoplus_{t} \lambda_t \]

where \(\lambda_t\) are elements of \(P_G\). By proposition \([15]\) the elements of \(P_G\) are generated by \(N_6 \cup N_7\) and the elements \(B_{\psi}(\theta_i)\) lie in \(N_5\) or \(N_6\).

We have now proven the proposition by showing that any triple \([\hat{x}, \hat{y}, \hat{z}]\) is generated by elements in \(\cup_{i=1}^{7} N_i\). 

\[\]

**Proposition 17** There are no relations between the elements of \(\cup_{i=1}^{7} N_i\) except for some elements in \(N_1, N_4\) and \(N_6\) which have order two.

**Proof**. First we apply lemma \([3]\) to the sets \(M \cup Q = N_1 \cup N_2 \cup N_3 \cup N_4\) and \(N = N_5 \cup N_6 \cup N_7\) to deduce that there are no relations between the elements of \(N_1 \cup N_2 \cup N_3 \cup N_4\) and also no relations with elements of \(N_5 \cup N_6 \cup N_7\), besides of course the torsion.

Next we consider the set \(N_5\). Suppose we could express an element \(\alpha\) of the group generated by \(N_6 \cup N_7\) as a linear combination

\[ \alpha = \bigoplus_{i=1}^{k} a_i [i \hat{x}_0, i \hat{y}_0, i \hat{z}_0], \quad a_i \in \mathbb{Z} \]


where \([i\hat{x}_0, i\hat{y}_0, i\hat{z}_0], i = 1, \ldots, k\), are \(k\) different elements of \(N_5\). The map \(B_\phi\) sends \(\alpha\) to an element of the group generated by \(L_\mathcal{G}'\) and the linear combination to

\[
B_\phi(\alpha) = \bigoplus_{i=1}^{k} a_i [i\hat{x}, i\hat{y}, i\hat{z}]
\]

where \([i\hat{x}, i\hat{y}, i\hat{z}], i = 1, \ldots, k\), are \(k\) different elements of \(W_\mathcal{G}'\). By theorem \(\[\]\) and proposition \(\[\]\) we deduce that \(B_\phi(\alpha) = 0\) and as the elements of \(W_\mathcal{G}'\) form a basis the coefficient \(a_i, i = 1, \ldots, k\), have to be zero. Thus there are no relations among the elements of \(N_5\) and between \(N_5\) and \(N_6 \cup N_7\).

For the set \(N_6\) we apply again lemma \(\[\]\) with \(M \cup Q = N_6\) and \(N = N_7\) which shows that there are no relations among the elements of \(N_6\) and between the elements of \(N_6\) and \(N_7\), besides the two torsion.

Finally the set \(N_7\) remains. There again lemma \(\[\]\) can be applied where \(M = N_7\) and \(Q = N = \emptyset\) to show that the elements are linearly independent.

\[
\square
\]

**Theorem 18** The set \(\bigcup_{i=1}^{k} N_i\) is a basis for \(B_\mathcal{G}\).

**Proof.** By proposition \(\[\]\) the set \(\bigcup_{i=1}^{k} N_i\) generates the group \(B_\mathcal{G}\) and by proposition \(\[\]\) there are no relations among the elements of \(\bigcup_{i=1}^{k} N_i\), except for the two torsion and thus they form a basis.

**Theorem 19** The group of singular orbit data \(B_\mathcal{G}\) is isomorphic to

\[
\mathbb{Z}^{|T_\mathcal{G}^+|} \oplus (\mathbb{Z}/2\mathbb{Z})^{|S_\mathcal{G}| - n_{K'}}.
\]

**Proof.** We will prove this theorem by introducing a one to one correspondence between the elements of \(T_\mathcal{G}^+ \cup S_\mathcal{G}\), without the images under the map \(\psi\) of the \(n_{K'}\) generators of \(K'\), and the elements of \(\bigcup_{i=1}^{k} N_i\).

(i) To every element \(x \in [G,G]\) with \(\hat{x} \in T_\mathcal{G}^+ \cup S_\mathcal{G}\) corresponds the element \([\hat{x}] \in N_1\) and vice versa. The elements of \(T_\mathcal{G}^+\) give rise to copies of \(\mathbb{Z}\) and the elements of \(S_\mathcal{G}\) give rise to copies of \(\mathbb{Z}/2\mathbb{Z}\).

(ii) A conjugacy class \(\hat{x}_i \in T_\mathcal{G}^+, i > 0\), with \(\hat{x} \neq \hat{x}^{-1}\) corresponds to the element \([\hat{x}_0, \hat{x}_i^{-1}] \in N_2\) and vice versa. The elements of \(N_2\) give all rise to copies of \(\mathbb{Z}\).

(iii) Let \(\hat{x}_i \in T_\mathcal{G}^+, i > 0\), with \(\hat{x} = \hat{x}^{-1}\) and \(x \notin [G,G]\), then there are two possibilities.

(a) \(\hat{x}_0 \in T_\mathcal{G}^+\): Thus the conjugacy class \(\hat{x}_i\) corresponds to the element \([\hat{x}_0, \hat{x}_i] \in N_3\) and vice versa. The elements of \(N_3\) give rise to copies of \(\mathbb{Z}\).

(b) \(\hat{x}_0 \in S_\mathcal{G}\): Thus the conjugacy class \(\hat{x}_i\) corresponds to the element \([\hat{x}_0, \hat{x}_i] \in N_4\). These elements of \(N_4\) give also rise to copies of \(\mathbb{Z}\). For the other elements of \(N_4\) see the next item.
(iv) Let \( \hat{x}_i \in S_G \), \( i > 0 \), then it follows \( \hat{x}_0 \in S_G \) and the corresponding element \([\hat{x}_0, \hat{x}_i] \in N_4\) gives rise to a copy of \( \mathbb{Z}/2\mathbb{Z} \). With this we found a correspondence with every element of \( N_4 \).

(v) Now we want to look at the conjugacy classes \( \hat{x}_0 \in T_G^+ \) with \( \hat{x} \neq \hat{x}^{-1} \). Between the sets \( T_G^+ \) and \( T_G^+ \), we have the following relation:

\[
\# \{ \hat{x}_0 \in T_G^+ | \hat{x} \neq \hat{x}^{-1} \} = \# |T_G|.
\]

On the other hand by the construction of \( N_5 \) we have:

\[
\# |W_G'| = \# |N_5|.
\]

By theorem \( \ref{thm} \), we then obtain the equality:

\[
\# \{ \hat{x}_0 \in T_G^+ | \hat{x} \neq \hat{x}^{-1} \} = \# |T_G^+| = \# |W_G'| = \# |N_5|.
\]

The elements of \( N_5 \) give rise to copies of \( \mathbb{Z} \) because \( \mathbb{B}_\phi \) is a homomorphism and the images have infinite order.

Thus we have seen that all the elements \( \hat{x}_0 \in T_G^+ \) with \( \hat{x} \neq \hat{x}^{-1} \) give rise to copies of \( \mathbb{Z} \) in \( \mathbb{B}_G \).

(vi) Recall the notation we introduced to define the sets \( N_6 \) and \( N_7 \). In this notation an element \( \hat{x} \in H_2' \) can be written as a product

\[
\hat{x} = i_1 \hat{z} \cdots i_t \hat{z}
\]

with \( i_j = 1, \ldots, n_{G'} \), \( 1 \leq j \leq t \), \( t \geq 1 \), and we have \( \hat{x}_0 = \mathbb{B}_\psi(i_1 \hat{z} \cdots i_t \hat{z}) = i_1 \hat{z}_0 \cdots i_t \hat{z}_0 \). Note that the images under the map \( \psi \) of the \( n_{K'} \) generators of \( K' \) are denoted by \( i_0 \hat{z}_0, i = 1, \ldots, n_{K'} \).

Let now the conjugacy classes \( \hat{x}_0 \in S_{G_1} \) but \( \hat{x}_0 \neq i_0 \hat{z}_0 \) with \( i = 1, \ldots, n_{K'} \) (i.e., \( \hat{x}_0 \) is not the image under the map \( \psi \) of one of the \( n_{K'} \) generators of \( K' \)). Since \( \hat{x} \) is an element of \( M_{G'} \) but not a generator of \( K' \), we have the following presentation for \( \hat{x} \):

\[
\hat{x} = i_1 \hat{z} \cdots i_t \hat{z}
\]

with \( i_j = 1, \ldots, n_{K'}, 1 \leq j \leq t \) and \( t > 1 \). The element which corresponds to \( \hat{x}_0 \) is now \([i_1 \hat{z}_0, \ldots, i_t \hat{z}_0, \hat{x}_0] \in N_6 \) and this element has order two. The remaining elements of \( N_6 \) are covered by the next item.

(vii) The last conjugacy classes which remain from the set \( T_G^+ \cup S_G \), without the images under the map \( \psi \) of the \( n_{K'} \) generators of \( K' \), are the classes \( \hat{x}_0 \in T_G^+ \) with \( \hat{x} = \hat{x}^{-1} \) and \( \hat{x} \neq \hat{x} \) (i.e., \( \hat{x} \in H_2' - M_{G'} \)). With the notation of the previous item we have again two cases.

(a) \( t > 1 \); \( \hat{x}_0 \) corresponds to the element \([i_1 \hat{z}_0, \ldots, i_t \hat{z}_0, \hat{x}_0] \in N_6 \) which gives rise to a copy of \( \mathbb{Z} \). These are now all elements of \( N_6 \).

(b) \( t = 1 \); Then we have \( \hat{x}_0 = i_1 \hat{z}_0 \) and thus \( i_1 = n_{K'} + 1, \ldots, n_{G'} \) and the conjugacy class corresponds to the element \([\hat{x}_0, \hat{x}_0] \in N_7 \) and vice versa. Note that the elements of \( N_7 \) have all infinite order.
With this we have shown that every element of \( T^+_G \cup S_G \), without the images under the map \( \hat{\psi} \) of the \( n_{K'} \) generators of \( K' \), corresponds to exactly one element in \( \bigcup_{i=1}^7 N_i \) with the appropriate order.

\[ \square \]

**Corollary 20** The group \( B_G \) is trivial if and only if \( G \cong C_2 \) or \( G \) is trivial.

**Proof.** The case where \( G \) is trivial is trivial. If \( G \cong C_2 \) then by example 1, \( B_G \) is trivial.

On the other hand let \( B_G \cong 0 \), then \( T_G = \emptyset \) and \( |S_G| = n_{K'} \). From this we deduce that \( G \) consists only of elements which are conjugate to their inverse, i.e., \( |G| - 1 = |S_G| \) and thus \( n_{K'} = n_{G'} \). Moreover we have \( |G'| = 2^{n_{G'}} \). From this we conclude:

\[ 2^{n_{G'}} - 1 = |G'| - 1 \leq |G| - 1 = |S_G| = n_{G'} \]

This equation yields two cases either \( n_{G'} = 0 \) or \( n_{G'} = 1 \). If \( n_{G'} \) is zero then \( |S_G| = 0 \) and the group \( G \) is trivial. If \( n_{G'} \) is one then \( G/\lbrack G, G \rbrack \cong C_2 \) and \( |G| = 2 \). Thus \( [G, G] \) has to be trivial and \( G \cong C_2 \). \[ \square \]

5 Relation with Cobordism

Before we can talk about the relation with \( G \)-equivariant cobordism we give its definition.

**Definition 21** Let \( M_1 \) (resp. \( M_2 \)) be a compact, oriented, connected Riemann surface with smooth \( G \)-action \( \kappa_1 : G \to \text{Diffeo}_+(M_1) \) (resp. \( \kappa_2 : G \to \text{Diffeo}_+(M_2) \)) We say that \( \kappa_1 \) is \( G \)-equivariant cobordant to \( \kappa_2 \), written \( \kappa_1 \sim \kappa_2 \), if there exists a smooth, compact, oriented, connected 3-manifold \( V \) and a smooth \( G \)-action \( \Phi \) on \( V \) such that

(i) The boundary of \( V \) is the disjoint union of \( M_1 \) and \( -M_2 \), \( \partial(V) = M_1 \cup -M_2 \). The notation \( -M_2 \) denotes \( M_2 \) with opposite orientation. The orientations on \( M_1 \) and \( -M_2 \) coincide with the one induced by \( V \).

(ii) \( \Phi \) restricted to \( \partial(V) \) agrees with \( \kappa_1 \cup \kappa_2 \).

We also say that \( \kappa_1 \) is zero \( G \)-equivariant cobordant, written \( \kappa_1 \sim 0 \), if \( \partial(V) = M_1 \). \( \Omega_G \) will denote the set of \( G \)-equivariant cobordism classes and a class will be denoted by \( (\kappa, M) \).

The set \( \Omega_G \) forms an Abelian group where the addition is given as for \( B_G \) in section 2 by the \( G \)-equivariant connected sum. In \( \mathbb{B}_G \) the author shows that two \( G \)-actions which are cobordant have the same singular orbit data, thus there is a well defined homomorphism \( \chi : \Omega_G \to \mathbb{B}_G \) which sends every class \( (\kappa, M) \) to its singular orbit data. Moreover the map \( \chi \) is surjective. Indeed, take any \( G \)-action which represents a given singular orbit data \( \alpha \in \mathbb{B}_G \), the corresponding cobordism class will then be mapped to the same singular orbit data \( \alpha \) by \( \chi \). In the same paper it is also shown.
that the kernel of $\chi$ consists only of the cobordism classes of free $G$-actions. The next proposition proves that the subgroup of free actions is isomorphic to $H_2(G; \mathbb{Z})$. The proof is an easy consequence of a spectral sequence described in Conner and Floyd’s book $[3]$.

**Proposition 22** The kernel of the map $\chi$ is isomorphic to $H_2(G; \mathbb{Z})$.

**Proof.** In $[3, \text{Theorem 20.4}]$ Conner and Floyd prove that the subgroup of cobordism classes of free $G$-actions is isomorphic to $MSO_2(BG)$, the cobordism homology of the classifying space of $G$.

On the other hand there is a spectral sequence $\{E^r_{p,q}\}$ with $E^2_{p,q} = H_p(G; MSO_q)$ and whose $E^\infty$-term is associated to a filtration of $MSO_*(BG)$. It turns out that for $MSO_2(BG)$ the $E^2$-term already stabilizes and as $MSO_0 \cong \mathbb{Z}$ and $MSO_1 \cong MSO_2 \cong 0$ we have

$$MSO_2(BG) \cong \sum_{p+q=2} H_p(G; MSO_q) \cong H_2(G; \mathbb{Z}).$$

We can now conclude that the $G$-equivariant cobordism group $\Omega_G$ of surface diffeomorphisms is an Abelian group extension of $B_G$ by $H_2(G; \mathbb{Z})$.

**Corollary 23** Every $G$-action is cobordant to a free action if and only if $G \cong C_2$ or $G$ is the trivial group.

**Proof.** By corollary $20$ $B_G$ is the trivial group if and only if $G \cong C_2$ or $G$ is trivial. Thus the corollary follows as the kernel of $\chi$ consists of the cobordism classes of the free $G$-actions.

**Corollary 24** If there is no torsion in the group $B_G$, then $\Omega_G$ is isomorphic to the direct sum of $B_G$ with $H_2(G; \mathbb{Z})$.

**Proof.** The group $B_G$ consists only of copies of $\mathbb{Z}$ and $\Omega_G$ surjects onto this group. Thus as $\Omega_G$ is Abelian and finitely generated the short exact sequence $0 \to H_2(G; \mathbb{Z}) \to \Omega_G \to B_G \to 0$ splits.

### 6 Relation to Representation Theory of finite Groups

In section $2$ we introduced the $G$-signature of Atiyah and Singer $\eta : B_G \to \mathbb{C}(G)$ (equation $[2]$) and the $G$-signature $\theta : B_G \to \mathbb{C}(G)/E_G$, but we referred to this section for a short discussion of the subgroup $E_G$ and the properties of $\theta$.

The subgroup $E_G$ is defined as follows:

$$E_G := \langle \text{Ind}_H^G \rho_0 | H \leq G \rangle.$$
The representation $\rho_0$ denotes always the one dimensional trivial representation. The map $\theta$ is then just the map $\varphi$ followed by the surjection on to the quotient and it turns out that the resulting map is a homomorphism (see [8, section 4]).

It is proven in [8, theorem 21] that the map $\theta$ is injective on the copies of $\mathbb{Z}$ in $\mathbb{B}_G$ and moreover that $\theta(\beta) \neq \overline{\theta(\beta)}$ for every element of infinite order $\beta \in \mathbb{B}_G$ and $\theta(\alpha) = \overline{\theta(\alpha)}$ for every element of order two $\alpha \in \mathbb{B}_G$ [8, proposition 24]. By the way $\eta$ is defined in equation (2) it follows that $\eta$ is also injective on the elements of infinite order but zero on the elements of order two.

What can we say about $\theta$? Is it also zero on the elements of order two? The subgroup $E_G < R_C(G)$ is contained in $R_Q(G)$ which shows that $\theta(\alpha)$ might be zero for elements $\alpha$ of order two as $\theta(\alpha) = \overline{\theta(\alpha)}$ in this case.

For $G = S_3$ (see example 5), the symmetric group on three letters, we have $\mathbb{B}_{S_3} \cong C_2$ and on the other hand $E_G = R_Q(G) = R_C(G)$. From the second fact we deduce that the map $\theta$ is the zero map. This phenomenon where for a finite group $G$ its geometric representation theory $\mathbb{B}_G$ and its representation ring $R_C(G)$ both have special properties fits into a broader picture.

**Proposition 25** The group $\mathbb{B}_G$ contains only elements of order two if and only if every complex character of $G$ has values in $\mathbb{R}$.

**Proof.** The group $\mathbb{B}_G$ consists only of elements of order two if and only if every element of $G$ is conjugate to its inverse.

If every element $a$ of $G$ is conjugate to its inverse, then by the formula $\overline{\chi(a)} = \chi(a^{-1})$, $\chi$ a character, every character has values in $\mathbb{R}$. On the other hand if every character $\chi$ has values in $\mathbb{R}$, then the characters have the same value for an element $a$ of $G$ and their inverse $a^{-1}$. But the characters form a basis for the vector space of class functions and as such have to separate conjugacy classes. Thus every element has to be conjugate to its inverse. (See also Serre’s book [11].) ■

The proposition doesn’t say anything about $E_G$, so we don’t know in general whether the map $\theta$ is zero or not in this case.

The advantage of our approach is that in the case where $\theta$ is not injective, we can try to find a smaller subgroup $E'_G < E_G$ such that the new map

$$\theta' : \mathbb{B}_G \to R_C(G)/E'_G$$

is still a homomorphism and in addition becomes injective. We want to illustrate this idea with the example $G = S_3$.

Let $\chi_0$ be the trivial representation, $\chi_1$ the one dimensional non-trivial representation and $\chi_2$ the two dimensional irreducible representation of $S_3$. Recall the notation from example 5 and let

$$E'_S = \langle \text{Ind}_{S_3}^{S_3} \chi_0 \cdot V, \text{Ind}_{S_3}^{S_3} \chi_1 \cdot V, \text{Ind}_{S_3}^{S_3} \chi_2 \cdot V \rangle = \langle \chi_0 + \chi_1, \chi_0 + \chi_1 + 2\chi_2, \chi_0 \rangle.$$

Then we have that the class of $\chi_2$ generates $R_C S_3/E'_S \cong C_2$. We know that $\mathbb{B}_{S_3} \cong \langle [\hat{a}] \rangle \cong C_2$; thus it remains to show that $\theta' : \mathbb{B}_{S_3} \to R_C S_3/E'_S$ is non zero on $[\hat{a}]$.

Let $\phi_{[\hat{a}]} S_3$ be an embedding of $S_3$ into $\Gamma_3$ with singular orbit data $[\hat{a}] S_3$. Then by the method of [8, proposition 17] we find that $\varphi(\phi_{[\hat{a}]} S_3) = \chi_0 + \chi_2$ and thus $\theta'$ maps $[\hat{a}]$ to the class of $\chi_2$ and $\theta'$ is an isomorphism.
Note that $E'_{S_3}$ consists of the minimal relations in order to make $\theta'$ a group homomorphism. It consists of the representations coming from the free actions and the cancelling pair $[\hat{a}, \hat{a}]_{S_3}$.

In this situation the maps $B_1 : B_{C_3} \to B_{S_3}, \widetilde{Ind}^{S_3}_{C_3} : R\bar{C}_3 \to R\bar{C} S_3/E'_{S_3}$ and $\theta'$ still commute. This follows from the fact that

$$\widetilde{Ind}^{S_3}_{C_3}(\varphi(\phi[a,a,a]_{C_3})) \equiv Ind^{S_3}_{C_3} \rho_1 = \chi_2 \equiv \varphi(\phi_{[a]}_{S_3}) \equiv \varphi(\phi_{[a,a,a]_{C_3}})$$

where the equivalence is taken modulo $E'_{S_3}$ and $\rho_1$ denotes a one dimensional faithful representation of $C_3$. With $\phi_{[a,a,a]_{C_3}}$ and $\phi_{[a,a,a]_{C_3}}$ we denote embeddings of $C_3$ and $S_3$ respectively, into some mapping class groups with singular orbit data $[a,a,a]_{C_3}$ and $B_1([a,a,a]_{C_3})$ respectively.

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