Computing Symmetrized Weight Enumerators for Lifted Quadratic Residue Codes

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Abstract

The paper describes a method to determine symmetrized weight enumerators of \( \mathbb{Z}_p^m \)-linear codes based on the notion of a disjoint weight enumerator. Symmetrized weight enumerators are given for the lifted quadratic residue codes of length 24 modulo \( 2^m \) and modulo \( 3^m \), for any positive \( m \).

1 Introduction

Ring-linear codes have gained importance since the beginning of the last decade when it was discovered that certain non-linear binary codes are actually linear over \( \mathbb{Z}_4 \), the ring of integers modulo 4 (cf. [14, 11]). Since then a variety of papers has appeared dealing with different foundational, constructive and analytical aspects of ring-linear coding [16, 17, 10]. It appears that primary integer residue rings and more generally Galois rings form a most important class of ring alphabets for contemporary ring-linear coding theory [11, 6, 15]. For ring-linear codes, the Hamming weight generalizes to the symmetrized weight which uses a partition of the alphabet into equivalence classes such that two letters are equivalent if and only if they are associated, meaning they are unit multiples of each other. In particular, for a \( \mathbb{Z}_p^m \)-linear code, the symmetrized weight uses a partition of the alphabet into \( m + 1 \) equivalence classes and the symmetrized weight enumerators are polynomials in \( m + 1 \) variables.

The investigations in the current paper were motivated by the discovery of some high-quality codes over \( \mathbb{Z}_8 \), \( \mathbb{Z}_9 \) and \( \text{GR}(4,2) \). The paper [8] introduces a binary \( (96,2^{37},24) \) code derived from a \( \mathbb{Z}_8 \)-linear lift of the extended binary Golay code. The properties of the binary code follow from the symmetrized weight enumerator of the \( \mathbb{Z}_8 \)-linear code. Similarly, in [9] we describe a ternary \( (72,3^{25},24) \)-code derived from a \( \mathbb{Z}_9 \)-linear lift of the extended ternary QR-code of length 24. Again, the properties of the ternary code are derived from the symmetrized weight enumerator of the \( \mathbb{Z}_9 \)-linear code. Both the binary and the ternary code have more codewords than previously known codes of the same length and distance.

The investigation of these and other examples naturally revealed the feasibility limitation of a brute-force determination of symmetrized weight enumerators. At the same time the quality of the discovered codes suggested the development of a more rigid and theoretical
tool for the computation of structural parameters for ring-linear codes. This article therefore presents a method to efficiently compute weight enumerators of linear codes over primary integer residue rings. For the lifted QR-codes of length 24 over \( \mathbb{Z}_8 \) and \( \mathbb{Z}_9 \), respectively, the method reproduces weight enumerators that were obtained by brute-force computation in [8] and [9], respectively.

The main ingredient for our method is the notion of a disjoint weight enumerator \( A_{s,t} \). Given a family \( (C_s)_{s \in \mathbb{N}} \) of \( \mathbb{Z}_p \)-linear codes with the property that \( C_{s+1} \) is a lift of \( C_s \) for all \( s \in \mathbb{N} \), the weight enumerator \( A_{s,t} \) contains combined information of \( C_s \) and \( C_t^\perp \). Furthermore we will use what we call partial weight enumerators \( D_{i,j} \) in order to ease the computation of the \( A_{s,t} \). It turns out that only a finite number of \( D_{i,j} \) need to be computed in order to determine the entire collection \( (A_{s,t})_{s,t \in \mathbb{N}} \). This allows us to extend the results for codes over \( \mathbb{Z}_8 \) and \( \mathbb{Z}_9 \), and to describe the symmetrized weight enumerators for lifted QR-codes of length 24 over \( \mathbb{Z}_{2^m} \) and \( \mathbb{Z}_{3^m} \), for any \( m \). For further properties of disjoint weight enumerators and partial weight enumerators, we refer to [7].

2 Disjoint weight enumerators

Let \( C_m \) be a code with coefficients in \( \mathbb{Z}/p^m\mathbb{Z} \). The composition of a codeword \( c \in C_m \) is the \( m+1 \) tuple \( \text{wt}(c) := (n_0, n_1, \ldots, n_{m-1}, n_{\infty}) \), where \( n_i \) is the number of coefficients in \( c \) that have \( p \)-adic valuation \( i \).

**Definition 1** The symmetrized weight enumerator of the code \( C_m \) is the polynomial in \( m+1 \) variables

\[
A(x_0, x_1, \ldots, x_{m-1}; -; z) = \sum_{c \in C_m} x_0^{n_0} x_1^{n_1} \cdots x_{m-1}^{n_{m-1}} z^{n_{\infty}}
\]

The variables in the weight enumerator are divided over three groups. The second group of variables is reserved to give information about the dual code. Thus,

\[
A(-; y_0, y_1, \ldots, y_{m-1}; z) = \sum_{c \in C_m^\perp} y_0^{n_0} y_1^{n_1} \cdots y_{m-1}^{n_{m-1}} z^{n_{\infty}}
\]

will denote the dual weight enumerator.

**Example 2** The Octacode (cf. [11]) is defined as an extended cyclic code of length 8 over \( \mathbb{Z}_4 \). It is of rank 4 over \( \mathbb{Z}_4 \) and has weight enumerator

\[
A(u, v; -; z) = z^8 + 112z^3u^4v + 112zu^4v^3 + 16u^8 + 14z^4v^4 + v^8.
\]

The code is self-dual and the dual weight enumerator \( A(-; u, v; z) \) equals \( A(u, v; -; z) \).

**Definition 3** For \( m, \ell \geq 0 \), we define disjoint weight enumerators inductively by

\[
A_{m+1,\ell}(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_{\ell-1}; z) = \frac{1}{|C^\perp_1|} A_{m,\ell+1}(px_0, px_1, \ldots, px_m-1; y_0, y_1, \ldots, y_{\ell-1}, z-x_m; z + (p-1)x_m). \tag{1}
\]
The inverse transform

\[ A_{m,ℓ+1}(x_0, x_1, \ldots, x_{m-1}; y_0, y_1, \ldots, y_ℓ; z) = \frac{1}{|C|} A_{m+1,ℓ}(x_0, x_1, \ldots, x_{m-1}, z - y_ℓ; py_0, py_1, \ldots, py_{ℓ-1}; z + (p - 1)y_ℓ) \]  

(2)
is easily verified. In particular the transform is symmetric in the two groups of variables.

**Remark 4** The above definition is compatible with the MacWilliams’ transform for symmetrized weight enumerators over the alphabet \( \mathbb{Z}_{p^m} \) that transforms \( A_{m,0} \) into \( A_{0,m} \). Thus the above transformations give a decomposition of the MacWilliams’ transform into smaller steps, much like a Welsh-Hadamard transform. This is possible because the \( p \)-adic weight function corresponds to a subgroup filtration of the alphabet \( \mathbb{Z}_{p^m} \), such that at each step in the filtration the subgroups are of same index \( p \).

**Example 5** For a code over \( \mathbb{Z}_4 \) the transforms in the definiton become

\[
\begin{align*}
A_{2,0}(u, v; -; z) &= A_{1,1}(2u; z - v; z + v)/|C_1^+| \\
A_{1,1}(u; v; z) &= A_{0,2}(-; v, z - u; z + u)/|C_1^+|
\end{align*}
\]

with inverse transforms

\[
\begin{align*}
A_{1,1}(u; v; z) &= A_{2,0}(u, z - v; -; z + v)/|C_1| \\
A_{0,2}(-; u, v; z) &= A_{1,1}(z - v; 2u; z + v)/|C_1|
\end{align*}
\]

In particular for the Octacode,

\[ A_{1,1} = z^8 + 14u^4z^4 + u^8 + 14v^4z^4 + v^8 - 14u^4v^4 \]

### 3 Partial weight enumerators

The weight enumerator \( A_{m+1,0}(x_0, x_1, \ldots, x_m; -; z) \) of a \( \mathbb{Z}_{p^m} \)-linear code \( C \) contains a contribution \( A_{m,0}(x_1, \ldots, x_m; -; z) \) from the subcode \( pC \). We introduce the partial weight enumerator \( D_{m+1,0} \) that measures the difference. So that

\[
A_{m+1,0}(x_0, x_1, \ldots, x_m; -; z) = A_{m,0}(x_1, \ldots, x_m; -; z) + D_{m+1,0}(x_0, x_1, \ldots, x_m; -; z).
\]  

(3)

More generally we define an array of partial weight enumerators.

**Definition 6** For \( m, ℓ \geq 0 \) we define the partial weight enumerators \( D_{m,ℓ} \) as the sum of those monomials in \( A_{m,ℓ} \) that are divisible by \( x_0y_0 \).

Thus, the partial weight enumerators become the building blocks for the actual weight enumerators,

\[
A_{m,ℓ}(x_0, \ldots, x_{m-1}; y_0, \ldots, y_{ℓ-1}; z) = \sum_{i \leq m} \sum_{j \leq ℓ} D_{i,j}(x_{m-i}, \ldots, x_{m-1}; y_{ℓ-j}, \ldots, y_{ℓ-1}; z).
\]  

(4)
Example 7 For the Octacode, the decomposition $A_{1,1} = D_{0,0} + D_{0,1} + D_{1,0} + D_{1,1}$ of $A_{1,1}(u; v; z)$ becomes

$$
D_{0,0}(-; -; z) = z^8, \quad D_{0,1}(-; v; z) = 14v^4z^4 + v^8, \\
D_{1,0}(u; -; z) = 14u^4z^4 + u^8, \quad D_{1,1}(u; v; z) = -14u^4v^4.
$$

The terminology disjoint weight enumerator is motivated by the following theorem. It gives an explicit expression for $D_{i,j}$ as a sum over pairs of codewords with disjoint support.

Theorem 8 (see also [7]) Let $\{C_i : i \geq 0\}$ be a family of $\mathbb{Z}_p$-linear codes with the property that $C_{i+1}$ is a lift of $C_i$ for all $i \geq 0$, and let $\{C_j^\perp : j \geq 0\}$ be the family of dual codes. For each $j \geq 0$, there is a natural character family $\{\chi_v : v \in C_j^\perp\}$ such that the partial weight enumerator $D_{i,j}$, for $i \geq 0$, is given by

$$
D_{i,j}(x_0, \ldots, x_{i-1}; y_0, \ldots, y_{j-1}; z) = \sum_{u \in C_i, \, v \in C_j^\perp \atop u \neq 0 \neq v \atop u \cap v = \emptyset} \chi_v(u) x^{\text{wt}(u)} y^{\text{wt}(v)} z^{n - |u| - |v|}
$$

Here $x^{\text{wt}(u)} := x_0^{n_0} \cdots x_{i-1}^{n_{i-1}}$ where $\text{wt}(u) = (n_0, \ldots, n_{i-1})$. An analogous definition is understood for $y^{\text{wt}(v)}$. And finally $z$ homogenizes the polynomial $D_{i,j}$ by counting the zeros outside $u \cup v$.

The theorem shows that as $i, j$ increase we should expect fewer terms in the sum $D_{i,j}$ as the supports of codewords $u \in C_i$ and $v \in C_j^\perp$ are non-decreasing with $i$ and $j$. In the next sections, it will turn out that this is the main reason that we will be able to compute and describe symmetrized weight enumerators efficiently.

Example 9 The Octacode has empty sum $D_{2,1}$ since any word $u$ in $C_2$ with $\bar{u} \neq 0$ has Hamming weight at least five and any word $v$ in $C_1^\perp$ with $\bar{v} \neq 0$ has Hamming weight at least four. And the intersection of supports $v \cap u$ consists of at least one position. Thus we find

$$
A_{2,1}(u, v; w; z) = D_{0,0}(-; -; z) + D_{1,0}(v; -; z) + D_{2,0}(u, v; -; z) + D_{0,1}(-; w; z) + D_{1,1}(v; w; z) + D_{2,1}(u, v; w; z)
$$

$$
= (z^8) + (14v^4z^4 + v^8) + (112z^3u^4v + 112zu^4v^3 + 16u^8)
$$

$$
+ (14w^4z^4 + w^8) + (-14v^4w^4) + (0).
$$

Consequently $A_{3,0}(u, v, w; -; z)$ can be computed via

$$
A_{3,0}(u, v, w; -; z) = A_{2,1}(2u, 2v; z - w; z + w)/2^4
$$

$$
= z^8 + 14w^4z^4 + w^8 + 112z^3v^4w + 112zv^4w^3 + 16v^8
$$

$$
+ 224z^3u^4v + 672z^2u^4vw + 896zu^4v^3 + 672zu^4vw^2
$$

$$
+ 256u^8 + 896u^4v^3w + 224u^4vw^3 + 16u^8
$$

In particular, the information to compute the $\mathbb{Z}_8$ weight enumerator is already contained in the $\mathbb{Z}_4$ weight enumerator. We say the Octacode has $p$-adic depth two.
4 Computing symmetrized weight enumerators

First we observe that the transform relations (11) and (2) for the weight enumerators $A_{m,\ell}$ hold for the partial weight enumerators $D_{m,\ell}$ as well, provided both $m > 0$ and $\ell > 0$.

Lemma 10 For $m, \ell > 0$,

$$D_{m+1,\ell}(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_{\ell-1}; z) = \frac{1}{|C_1^*|} D_{m,\ell+1}(px_0, px_1, \ldots, px_{m-1}; y_0, y_1, \ldots, y_{\ell-1}, z - x_m; z + (p-1)x_m).$$  (5)

$$D_{m,\ell+1}(x_0, x_1, \ldots, x_{m-1}; y_0, y_1, \ldots, y_{\ell}; z) = \frac{1}{|C|} D_{m+1,\ell}(x_0, x_1, \ldots, x_m-1; z - y_\ell; py_0, py_1, \ldots, py_{\ell-1}; z + (p-1)y_\ell).$$  (6)

Proof: It suffices to consider (5). With (4) we obtain, for $\ell > 0$,

$$A_{m+1,\ell}(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_{\ell-1}; z) = A_{m,\ell}(x_1, \ldots, x_m; y_0, y_1, \ldots, y_{\ell-1}; z) + A_{m+1,\ell-1}(x_0, x_1, \ldots, x_m; y_0, \ldots, y_{\ell-1}; z) - A_{m,\ell-1}(x_1, \ldots, x_m; y_0, \ldots, y_{\ell-1}; z) + D_{m+1,\ell}(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_{\ell-1}; z).$$

All terms but $D_{m+1,\ell}$ can be rewritten using the transform (11), for $m > 0$. But then also $D_{m+1,\ell}$ obeys the transform. □

For the cases $\ell = 0$ or $m = 0$, respectively, an extra term needs to be included in the right hand side of equation (5) or (6), respectively.

Lemma 11 For $\ell = 0$ and $m > 0$,

$$D_{m+1,0}(x_0, \ldots, x_m; -; z) = \frac{1}{|C_1^*|} (D_{m,0}(px_0, \ldots, px_{m-1}; -; z + (p-1)x_m) + D_{m,1}(px_0, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m))$$  (7)

For $m = 0$ and $\ell > 0$,

$$D_{0,\ell+1}(-; y_0, \ldots, y_{\ell}; -; z) = \frac{1}{|C|} (D_{0,\ell}(-; py_0, \ldots, py_{\ell-1}; -; z + (p-1)y_\ell) + D_{1,\ell}(-; py_0, \ldots, py_{\ell-1}; z - y_\ell; z + (p-1)y_\ell))$$  (8)

Proof: It suffices to consider (7). First we write

$$A_{m+1,0}(x_0, \ldots, x_m; -; z) = \frac{1}{|C_1^*|} A_{m,1}(px_0, px_1, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m).$$

A decomposition similar to that used in the previous lemma applies to $A_{m,1}$, for $m > 0$,

$$A_{m,1}(px_0, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m) = A_{m-1,1}(px_1, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m) + A_{m,0}(px_0, \ldots, px_{m-1}; -; z + (p-1)x_m) - A_{m-1,0}(px_1, \ldots, px_{m-1}; -; z + (p-1)x_m) + D_{m,1}(px_0, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m)$$
For the contribution $A_{m-1,1}$ we have

\[ A_{m,0}(x_1, \ldots, x_m; -; z) = \frac{1}{|C_1|} A_{m-1,1}(px_1, \ldots, px_{m-1}; z - x_m; z + (p-1)x_m). \]

Collecting terms, using (3) twice, gives (7). The relation (8) follows by symmetry. □

In general, we obtain the following recursive procedure to compute $A_{m+\ell+2,0}$ from $A_{m+\ell+1,0}$ for a known partial weight enumerator $D_{m+1,\ell+1}$.

**Theorem 12 (Computing symmetrized weight enumerators)** Let the symmetrized weight enumerator $A_{m+\ell+1,0}$ be known together with a partial weight enumerator $D_{m+1,\ell+1}$, for some $m, \ell \geq 0$. Then $A_{m+\ell+2,0}$ is obtained as follows

1. Use the transform (2) repeatedly to obtain $D_{m+\ell+1,1}$ from $D_{m+1,\ell+1}$.
2. Use the transform (7) to obtain $D_{m+\ell+2,0}$ from $D_{m+\ell+1,1}$ and $D_{m+\ell+1,0}$.
3. Use (3) to obtain $A_{m+\ell+2,0}$ from $A_{m+\ell+1,0}$ and $D_{m+\ell+2,0}$.

**Remark 13** To compute the symmetrized weight enumerators $A_{m,0}$ for all $m \geq 0$, it suffices with the theorem to give one partial weight enumerator $D_{i,j}$ for each value of $i + j \geq 0$. The procedure succeeds in computing $A_{m,0}$ only in as far as we can determine $D_{i,j}$, for $i + j \leq m$. As indicated after Theorem 8, this actually becomes easier as $i$ and $j$ increase as the $D_{i,j}$ tend to have fewer terms for larger values of $i$ and $j$. This behaviour of course is directly opposite to a brute-force computation, for which the complexity of computing $A_{m,0}$ grows exponentially with $m$.

**Remark 14** The fact that the partial weight enumerators $D_{i,j}$ have few terms, especially for values of $i$ and $j$ that are comparable, makes them the better choice for storing and representing the weight enumerator $A$. With the transforms used in the theorem they expand rapidly.

**Remark 15** The best way to compute the $D_{i,j}$ is recursively. Pairs of words $(u', v')$ that occur in the sum $D_{i',j'}$, for large $i', j'$, reduce to pairs $(u, v)$ that occur in the sum $D_{i,j}$, for $i \leq i', j \leq j'$. Namely if $u', v'$ have disjoint support, then certainly their reductions $u, v$ have disjoint support. Thus a critical stage is to compute $D_{1,1}$, which runs over all pairs $(u, v) \in C \times C^\perp$ with disjoint support. A large automorphism group will help to reduce the computations. For a given complete list of disjoint words for the code and its dual over $\mathbb{Z}/p\mathbb{Z}$, it is relatively straightforward to compute the $D_{i,j}$ as $i$ and $j$ increase. They can be computed either directly, by lifting the words in the list, or by introducing unknowns for the coefficients in $D_{i,j}$ and then computing just enough terms in the weight enumerator $A_{i,j,0}$ to determine the unknowns. The last option worked well for the quadratic residue codes of length 24, for which the results are given in the following sections.

## 5 Lifts of the $[24, 12, 8]$ binary Golay code

In this section we will apply theorem 12 to the binary Golay code and verify results for the $\mathbb{Z}_4$ lifted code $3$ and the $\mathbb{Z}_8$ lifted code $8$. In fact we describe the symmetrized
Let Theorem 16 computation required more than a 24 hours, and is unfeasible for the larger alphabets. Weight enumerator for any lift of the binary Golay code by giving enough partial weight enumerators. Computations as described in Remark 15 were carried out in Magma [4] and required less than a few minutes on a Pentium III 750MHZ laptop. In comparison, computing the weight enumerator of just the \( \mathbb{Z}/8\mathbb{Z} \) lifted Golay code via an exhaustive computation required more than a 24 hours, and is unfeasible for the larger alphabets.

\textbf{Theorem 16} Let \( \{C_m : m \geq 0\} \) be the family of lifted extended cyclic quadratic residue codes of length 24 modulo \( 2^m \). The partial weight enumerators are

\[
\begin{align*}
D_{0,0} &= z^{24} \\
D_{1,0} &= 759 z^{16} u^8 + 2576 z^{12} u^{12} + 759 z^8 u^{16} + u^{24} \\
D_{1,1} &= 759 u^{16} v^8 - 2576 u^{12} v^{12} + 759 u^{8} v^{16} - 1518 z^8 u^8 v^8 \\
D_{2,1} &= 759 \cdot 16 \cdot u^8 w^8 \cdot (-v^2 z^6 + 2v^4 z^4 - v^6 z^2) \\
D_{2,2} &= 759 \cdot 64 \cdot u^8 w^8 \cdot (v^2 x^2 z^4 + v^4 x^2 z^2 + v^2 x^4 z^2 + v^4 x^4) \\
D_{3,2} &= 759 \cdot 128 \cdot u^8 x^8 \cdot (v^4 y^2 z^2 + 2v^4 w y^2 z - v^4 w^2 y^2 + v^4 y^4 - v^2 w^2 y^4) \\
D_{3,3} &= 759 \cdot 128 \cdot u^8 x^8 \cdot (2v^4 y^4 + v^4 y^2 t^2 - v^2 w^2 y^4) \\
D_{4,3} &= 759 \cdot 128 \cdot u^8 y^8 \cdot (v^4 t^2 s^2 - v^2 w^2 t^4 - 2v^2 w x t^2 s z + 2v^2 w x^2 t^2 s) \\
D_{4,4} &= 0, \text{ etc.}
\end{align*}
\]

In each of the \( D_{i,j} \)'s, the variables \( z, x_0, \ldots, x_{i-1}, y_0, \ldots, y_{j-1} \), are replaced with the variables \( z, u, v, x, y, t, s \), in that order.

We look at two special cases. The symmetrized weight enumerator \( A_{2,0}(u, v; -; z) \) is particularly important. It was computed in [3]. There it is shown that the theta series of the Leech lattice follows with the substitution

\[
z = \sum_{x \in 4\mathbb{Z}} q^{x^2}, \quad u = \sum_{x \in 4\mathbb{Z} + 1} q^{x^2}, \quad v = \sum_{x \in 4\mathbb{Z} + 2} q^{x^2}
\]

Similar to Example 7

\[A_{2,0}(u, v; -; z)\] has the following expression in terms of partial weight enumerators.

\textbf{Corollary 17} For the quaternary extended Golay code,

\[
A_{2,0}(u, v; -; z) = A_{1,1}(2u; z - v; z + v)/2^{12}
\]

\[
A_{1,1}(u; v; z) = D_{0,0}(-; -; z) + D_{1,0}(u; -; z) + D_{0,1}(-; v; z) + D_{1,1}(u; v; z)
\]

Thus, for a known weight enumerator \( A_{1,0} \) of the binary golay code, the only new information is given by the coefficients 759, -2576, 759, -1518 in \( D_{1,1} \). They are uniquely determined with the arguments from [3]. The first three coefficients are unique such that the Euclidean weights of the lifted code are divisible by eight. And the final coefficient is unique such that the lifted code contains no word with eight units and sixteen zeros. In [3], \( A_{2,0} \) is first expressed in terms of invariants before the coefficients can be determined. That part of the computation is avoided in our computation.
Another important weight enumerator is $A_{3,0}(u, v, w; -; z)$ which yields the Hamming weight enumerator for the non-linear binary code $(96, 2^{36}, 24)$ used in [3] after the substitution $(u, v, w, z) \mapsto (t^2, t^4, t, 1)$. We give the expression for $A_{3,0}(u, v; -; z)$ in terms of the partial weight enumerators. The expanded expression covers one page [8].

**Corollary 18** For the $\mathbb{Z}_8$-linear extended Golay code,

$$A_{3,0}(u, v, w; -; z) = A_{2,1}(2u, 2v; z - w; z + w)/2^{12}$$

$$A_{2,1}(u, v; w; z) = A_{2,0}(u, v; -; z) + A_{1,1}(v; w; z) - A_{1,0}(v; -; z) + D_{2,1}(u, v; w; z)$$

### 6 Lifts of the $[24, 12, 9]$ ternary QR code

We describe the symmetrized weight enumerator for any lift of the extended ternary quadratic residue code of length $24$.

**Theorem 19** Let $\{C_m : m \geq 0\}$ be the family of lifted extended cyclic quadratic residue codes of length $24$ modulo $3^m$. The partial weight enumerators are

$$D_{0,0} = z^{24}$$

$$D_{1,0} = 4048 \cdot z^{15} \cdot u^9 + 61824 \cdot z^{12} \cdot u^{12} + 242880 \cdot z^9 \cdot u^{15} + 198352 \cdot z^6 \cdot u^{18} + 24288 \cdot z^3 \cdot u^{21} + 48 \cdot u^{24}$$

$$D_{1,1} = -16192 \cdot z^6 \cdot u^6 \cdot v^9 + 12144 \cdot z^3 \cdot u^9 \cdot v^{12} + 12144 \cdot z^3 \cdot u^6 \cdot v^9 - 1104 \cdot u^{12} \cdot v^{12}$$

$$D_{2,1} = 4048 \cdot 18 \cdot u^9 \cdot w^9 \cdot (-u^2 z^4 + v^3 z^3 + v^5 z - v^6 - u^3 z^3 + 3 u^3 v^2 z + u^3 v^3 - v^3 w^3)$$

$$D_{2,2} = 4048 \cdot 117 \cdot u^9 \cdot w^9 \cdot (-v^3 x^3)$$

$$D_{3,2} = 4048 \cdot 27 \cdot u^9 x^9 \cdot (-6 v^2 y^2 w z + 6 v^2 y^2 w^3 - v^3 y^3)$$

$$D_{3,3} = 0, \text{ etc.}$$

In each of the $D_{i,j}$’s, the variables $z, x_0, \ldots, x_{i-1}, y_0, \ldots, y_{j-1}$, are replaced with the variables $z, u, v, w, x, y, t, s$, in that order.

The weight enumerator $A_{2,0}(u, v; -; z)$ yields the Hamming weight enumerator for the non-linear ternary code $(72, 3^{24}, 24)$ used in [3] after the substitution $(u, v, z) \mapsto (t^2, t^3, 1)$. The expanded expression for $A_{2,0}(u, v; -; z)$ covers half a page [9].

**Corollary 20** For the $\mathbb{Z}_9$-linear extended QR-code of length $24$,

$$A_{2,0}(u, v; -; z) = A_{1,1}(3u; z - w; z + 2w)/3^{12}$$

$$A_{1,1}(u; v; z) = D_{0,0}(-; -; z) + D_{1,0}(u; -; z) + D_{0,1}(-; v; z) + D_{1,1}(u; v; z)$$

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