Quaternionic Metrics From Harmonic Superspace:
Lagrangian Approach and Quotient Construction

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Abstract

Starting from the most general harmonic superspace action of self-interacting $Q^+$ hypermultiplets in the background of $N = 2$ conformal supergravity, we derive the general action for the bosonic sigma model with a generic $4n$ dimensional quaternionic-Kähler (QK) manifold as the target space. The action is determined by the analytic harmonic QK potential and supplies an efficient systematic procedure of the explicit construction of QK metrics by the given QK potential. We find out this action to have two flat limits. One gives the hyper-Kähler (HK) sigma model with a $4n$ dimensional target manifold, while another yields a conformally-invariant sigma model with $4(n + 1)$ dimensional HK target. We work out the harmonic superspace version of the QK quotient construction and use it to give a new derivation of QK extensions of four-dimensional Taub-NUT and Eguchi-Hanson metrics. We analyze in detail the geometrical and symmetry structure of the second metric. The QK sigma model approach allows us to reveal the enhancement of its $SU(2) \otimes U(1)$ isometry to $SU(3)$ or $SU(1, 2)$ at the special relations between its free parameters: the $Sp(1)$ curvature (“Einstein constant”) and the “mass”.

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1 Introduction

The explicit construction of metrics on the hyper-Kähler (HK) and quaternionic-Kähler (QK) manifolds is of interest both from the purely mathematical point of view and keeping in mind possible physical applications of these manifolds in the modern strings and branes stuff. Important physical multiplets of superbranes with $N = (1,0)$, $d = 6$ ($N = 2$, $d = 4$; $N = 4$, $d = 3$; ...) worldvolume supersymmetry obtained via appropriate compactifications of M-theory are hypermultiplets (see, e.g. [1], [2], [3] and references therein). Their bosonic fields parametrize HK manifolds in the case of rigid supersymmetry [4] and QK manifolds when couplings to the worldvolume supergravity are turned on [4]. As was shown in [3], the toric HK manifolds [8, 9] (4-dimensional HK manifolds with $n$ commuting triholomorphic $U(1)$ isometries) naturally arise in the M-theory context as the $D = 11$ supergravity solutions corresponding to the multiply-intersecting branes. It would be interesting to construct QK analogues of these HK manifolds, as well as to reveal their possible branes implications.

A universal method of explicit construction of HK and QK metrics is the twistor-harmonic one [8, 9, 10]. It emerged as a natural development of the basic ideas of the harmonic superspace (HSS) approach to $N = 2$ supersymmetric theories [11, 12]. In this method, the defining constraints of the HK or QK geometry are interpreted as integrability conditions for the existence of some analytic subspace in an extension of the given 4-dimensional HK or QK manifold by harmonic coordinates on an internal two-sphere $SU(2)/U(1)$ (in the HK and QK cases they parametrize, respectively, the space of non-equivalent covariantly constant complex structures and the $Sp(1)$ component of the holonomy group). The basic object of HK and QK geometries which fully specifies the relevant metric (at least, locally) is the analytic potential, a function of harmonic variables and other $2n$ coordinates of the analytic subspace. It is required to have the charge $+4$ with respect to the harmonic $U(1)$ (that is the denominator of the harmonic sphere $SU(2)/U(1)$). Otherwise, it is an arbitrary function of the involved variables. As the real advantage of the geometric HSS approach, it provides the practical recipes of how to construct most general HK or QK metrics, including those having no isometries. Other supersymmetry-inspired approaches to computing such metrics [13], [14] - [16] are restricted to the manifolds with isometries.

In refs. [8, 9] and [11] it was shown how to restore the HK or QK metrics by the known analytic potential. The road from HK or QK potentials to the metric, as compared, e.g., to that from the Kähler potential to the relevant metric, is somewhat obscured by the necessity to solve differential equations on the sphere $SU(2)/U(1)$ at the intermediate steps. In view of lacking of the systematic theory of such equations, this is the most difficult part of the twistor-harmonic formalism.

In all known examples where the twistor-harmonic approach was used to compute HK metrics [8, 9, 17, 18], it was rather easy to solve such equations because the corresponding HK potentials possessed some triholomorphic isometries. As for the QK metrics, the relevant equations remain rather complicated even in the presence of similar isometries (i.e. those which become triholomorphic in the HK limit, when the $Sp(1)$ curvature vanishes). In ref. [10], only the maximally symmetric case of homogeneous $H^P$ manifold was considered as an example. It corresponds to the vanishing QK potential and is an analog of the flat HK manifold (and goes into it in the HK limit). The first example of
QK manifold with a non-trivial potential, a quaternionic extension of the four-dimensional Taub-NUT (TN) manifold, was studied within the approach of [10] in [19]. Like in the HK Taub-NUT case [3, 4], the relevant harmonic equations can be easily solved due to an $U(1)$ isometry of the corresponding analytic QK potential. As the result, the explicit form of the metric can be readily found. However, while trying to compute in this way QK analogs of some other HK metrics allowing a nice description within the harmonic approach, e.g. the “dipolar breaking” metric [17, 18], a serious technical problem is encountered. It is related to the necessity to solve one more harmonic differential equation for the quantity specific just for the QK case. This is the “bridge” connecting harmonic variables in the original and analytic bases: whereas in the HK case these variables are the same, it is not so in the QK case.

In view of these subtleties, it is tempting to try some another strategy, and this is what we do in this paper, taking as the examples QK extensions of the Taub-NUT (TN) and Eguchi-Hanson (EH) metrics. These QK metrics, in a rather implicit form, were already constructed in [14] proceeding from the component $N = 2$ supergravity (SG)-matter action of ref. [20]. Their derivation within the HSS approach not only illustrates this general approach, but also clarifies some basic features of these metrics, e.g., makes manifest their symmetry properties.

First of all, we use the lagrangian version [21] of the geometric formalism of ref. [10]. Because of the one-to-one correspondence between the QK manifolds and general matter couplings in $N = 2$ SG background [3], an action of any such coupling yields a sigma model action with QK target manifold in the bosonic sector. Conversely, any QK sigma model action can be lifted to a locally $N = 2$ supersymmetric one. This correspondence becomes manifest when the $N = 2$ sigma model action is formulated via unconstrained harmonic analytic hypermultiplet superfields [21]. From the standpoint of the geometric approach of ref. [10], these superfields parametrize the analytic subspace of harmonic extension of the target QK manifold. The superfield interaction Lagrangian (which in general is an arbitrary charge +4 function of the hypermultiplet superfields and explicit harmonics) is just the analytic QK potential. The compensating hypermultiplet superfield which is needed to descend to Einstein $N = 2$ SG from conformal $N = 2$ SG is the aforementioned bridge relating harmonic variables in the initial and analytic bases. The bosonic part of the equations of motion which eliminate an infinite set of the auxiliary fields coming from the harmonic expansions of such superfields coincides with the already mentioned differential equations on $S^2$. Thus this approach can be regarded as a lagrangian version of the purely geometric approach of ref. [10]. As one of its merits, it directly yields the standard distance on the target bosonic QK space after elimination of auxiliary fields, viz. solving the above-mentioned differential equations on $S^2$. In the geometric approach, the basic objects are inverse vielbeins and metric.

One of the incentives of this paper is to deduce a convenient HSS form of the off-shell action of most general QK sigma model. It was not explicitly given in [21].

Another simplifying device we use here is the HSS version of the QK quotient construction (discussed at the component level in [14, 13]). It is a straightforward generalization of the analogous HSS construction for the HK case [22] which was used, in particular, to obtain the EH metric from a $N = 2$ supersymmetric HSS lagrangian[1]. Basically it amounts

*The $N = 1$ superfield version of the HK quotient approach was earlier employed in [13].
to gauging some isometries of a “free” hypermultiplet action by non-propagating $N = 2$ gauge superfields along the lines of ref. [23]. In a manifestly supersymmetric gauge, after elimination of the gauge superfields by their algebraic equations of motion, one ends up with a non-trivial analytic QK potential. On the other hand, in the Wess-Zumino gauge, the harmonic differential equations on $S^2$ become linear and can be immediately solved, thus avoiding the most difficult part of the geometric twistor-harmonic approach (this concerns as well the equations for the compensating hypermultiplet in some important cases). In this gauge, the metric is restored by solving a set of algebraic equations including some constraints on the original physical bosonic fields.

This approach is expected to work not only for the QK generalizations of 4-dimensional EH and TN manifolds, but also in higher-dimensional cases. So it can hopefully be used for explicit construction of the QK analogs of metrics on general toric HK manifolds. We are planning to perform this analysis in future publications. Our aim here is to explain the basic features of this approach on the simple 4-dimensional examples: QK extensions of the TN and EH metrics.

In Sect. 2 we give a short account of the HSS formulation of the off-shell hypermultiplets actions in the $N = 2$ SG background with focusing on the bosonic sector which is of primary interest for our purpose of extracting bosonic QK metrics. We present a convenient generic form of the off-shell action (with all fermions discarded) resulting in the most general target bosonic QK metric. As the simplest example we firstly derive the action of $\mathbb{H}P^n$ sigma model. Then in Sect. 3 we describe the HSS quotient construction on the examples of the quaternionic Taub-NUT and EH metrics. We demonstrate that various isometries of these metrics have a very transparent form in the HSS lagrangian language. We present the final bosonic actions from which the relevant distances can be read off. For the TN case the already known result [19] is recovered. In the EH case we still need to solve the appropriate algebraic constraint. Fortunately, it is the same as in HK case [24], [22]. We solve it in Sect. 4 and then study the local and global structure of the resulting quaternionic EH metric. It is Einstein with self-dual Weyl tensor and as such has already been studied in the physical and mathematical literature [23]–[31]. The conformal class encompasses two interesting Kähler metrics: the first one is the scalar-flat Le Brun metric [29], while the second one, which seems to be new, has a non-constant scalar curvature. Both families contain many complete metrics which are examined. In Conclusions we summarize the results and outline some problems for future study.

2 Quaternionic sigma models from harmonic superspace of local $N = 2$ SUSY

2.1 The HSS action of $N = 2$ SG with matter hypermultiplets

We start by recalling the salient features of the hypermultiplet $N = 2$ matter coupled to $N = 2$ SG in the HSS approach, basically following refs. [21], [11]. Further details of the HSS approach can be found in [11], [12].

The $N = 2, D = 4$ HSS in the analytic basis is represented by the following set of coordinates

$$\{Z^M\} = \{x^m, \theta^{\hat{\mu}+}, u^{+i}, u^{-j}, \theta^{\hat{\mu}-}\} \equiv \{\zeta^N, \theta^{\hat{\mu}-}\}, \quad \hat{\mu} = (\mu, \hat{\mu}), i, j = 1, 2.$$ (2.1)
The coordinate set \( \{ \zeta^N \} = \{ x^m, \theta^\mu, u^\pm i \} \) forms the analytic harmonic superspace, harmonic variables \( u^\pm i \) parametrize a sphere \( S^2 \), \( u^+iu^- = 1 \), \( u^+iu^- - u^+ku^- = \epsilon^{ki} \). HSS and its analytic sub-space are real with respect to a generalized conjugation \( \sim \) which does not change the harmonic \( U(1) \) charges of the HSS coordinates (it is the product of ordinary complex conjugation and a Weyl reflection of \( S^2 \)).

The fundamental group of conformal \( N = 2 \) SG is realized as the analyticity-preserving subclass of diffeomorphisms of \( Z^M \)

\[
\delta x^m = \lambda^m(\zeta) , \quad \delta \theta^\mu = \lambda^\mu(\zeta) ,
\]
\[
\delta u^+i = \lambda^++(\zeta)u^-i , \quad \delta u^-i = 0 , \quad (2.2)
\]
\[
\delta \theta^\mu = \lambda^-\mu(\zeta, \theta^-) . \quad (2.3)
\]

The matter superfields we shall deal with are the analytic hypermultiplet superfields \( Q^+r(\zeta) = F^{ri}(x)u^+_i + ... \), \( r = 1, ..., 2n \), \( F^{ri}(x) \) being the physical bosonic fields. These superfields satisfy the pseudo-reality condition

\[
(Q^+_r) = \Omega^r_s Q^+_s , \quad \Rightarrow \quad (F^{ri}(x)) = \Omega_{rs} \epsilon_{ik} F^{sk}(x) , \quad (2.4)
\]

where \( \Omega_{rs} = -\Omega_{sr} \), \( \Omega^{rs}\Omega_{st} = \delta^r_t \), is the totally antisymmetric constant \( Sp(n) \) metric. This condition leaves in \( F^{ri}(x) \) just \( 4n \) real fields which are identified with the coordinates of target QK manifold. The superfields \( Q^+r \) are assumed to transform as weight zero scalars under \( (2.2) \):

\[
Q^{+rr}(\zeta') = Q^+(\zeta) . \quad (2.5)
\]

The full invariant superfield action of the coupled system of \( Q \)-hypermultiplets and \( N = 2 \) Einstein SG consists of the two pieces

\[
S = S_{SG} + S_{q,Q} . \quad (2.6)
\]

We shall firstly explain the second piece. It can be written as an integral over the analytic HSS \( \{ \zeta^N \} \):

\[
S_{q,Q} = \frac{1}{2} \int d\zeta(\{-4\}) \left\{-a^+_a D^+q^+ + \frac{\kappa^2}{\gamma^2}(u^-q^+)^2 \left[ Q^+_r D^{++} Q^+ + L^4(Q^+, v^+, v^-) \right] \right\} . \quad (2.7)
\]

Here \( d\zeta(\{-4\}) \) is the measure of integration over \( \{ \zeta^N \} \) chosen so that \( d\zeta(\{-4\})(\theta^+)^4 = d^4x [du] \), \( \kappa \) is the Einstein constant, \( \gamma \) \( \left[ \gamma \right] = -1 \) is a sigma model constant which is set equal to 1 in what follows, \( D^{++} \) is the analyticity-preserving harmonic derivative covariantized with respect to the group \( (2.2), (2.3) \)

\[
D^{++} = \partial^{++} + H^{++m}(\zeta)\partial_m + H^{++\mu}(\zeta)\partial_{\mu} + H^{++\mu}(\zeta, \theta^-)\partial_{\mu} + H^+(\zeta)\partial^- \quad (2.8)
\]
\[
\partial^{++} = u_{+i} \frac{\partial}{\partial u_{+i}} , \quad \partial_m = \frac{\partial}{\partial x_m} , \quad \partial_{\mu\pm} = \frac{\partial}{\partial \theta_{\mu\pm}} . \quad (2.9)
\]

The transformation properties of the analytic vielbein \( H^{++m}(\zeta), H^{++\mu}(\zeta), H^+(\zeta) \) under \( (2.2) \) are uniquely fixed by the transformation law of \( D^{++} \)

\[
\delta D^{++} = -\lambda^{++}(\zeta) D^0 , \quad (2.10)
\]
\[
D^0 = \partial^0 + \theta^\mu \partial_{\mu} + \theta^- \partial_{\mu} - \partial^0 = u^+i \frac{\partial}{\partial u^+i} - u^-i \frac{\partial}{\partial u^-i} \quad (2.11)
\]

\[\dag\] We use the following conventions: \( \epsilon_{12} = 1 \), \( \epsilon_{ik} \epsilon^{kl} = \delta_i^l.\]
(the non-analytic component $H^{++\bar{\mu}-}$ is pure gauge, it can be gauged into $\theta^{\bar{\mu}+}$ by properly fixing the gauge freedom (2.3)). The irreducible $N = 2$ multiplet carried out by the components of this vielbeins (modulo the analytic gauge group freedom) is the gauge multiplet of conformal $N = 2$ SG, the $N = 2$ Weyl multiplet. The object $q^{+a}(\zeta)$ ($a = 1, 2; \widehat{g}_a^+ = e^{\theta a} \hat{q}_a^+$) is the compensating hypermultiplet superfield, one of the two compensating multiplets needed to descend from $N = 2$ conformal SG to $N = 2$ Einstein SG (to the most flexible version of the latter which allows the general $Q^+$ self-interactions [21]). It transforms so as to cancel the transformation of the analytic HSS integration measure in (2.7)

$$\delta q^{+a}(\zeta) \simeq q^{+a'}(\zeta') - q^{+a}(\zeta) = -\frac{1}{2} \Lambda(\zeta) q^{+a}(\zeta), \quad \Lambda(\zeta) = \partial_{\mu} \lambda^m + \partial^{-\lambda^{+\mu} - \partial_{\mu} \lambda^{+}}. \quad (2.12)$$

Note the wrong sign of the $q^+$ action compared to that of $Q^+$: it is the standard feature of the compensating superfields actions [32]. Also note that in the process of descending to Einstein $N = 2$ SG $q^{+i}$ compensates local conformal $SU(2)$ transformations.

The modified harmonics $v^{\pm i}$ present in the $Q^+$ self-interaction term in (2.7) are related to $u^{\pm i}$ as follows

$$v^{+i} = u^{+i} - N^{+\mp}(\zeta) u^{-i} = (u^{-q^+})^{-1} q^{+i}, \quad N^{+\mp} = \frac{(u^+ q^+)}{(u^- q^+)}; \quad (2.13)$$

$$v^{-i} = u^{-i}, \quad (2.14)$$

where, from now on,

$$(u^{\pm} b) \equiv u^\pm_i b^i.$$

They are invariant under $N = 2$ conformal SG group (2.2), (2.12) and so can appear inside $L^{+4}$ in any power consistent with the harmonic charge +4 of $L^{+4}$.

The flat limit is achieved by putting

$$q^{+i} = \frac{1}{|\kappa|} u^{+i} \Rightarrow (u^{-q^+}) = |\kappa|^{-1}, \quad N^{+\mp} = 0; \quad (2.15)$$

and by equating to zero all the analytic vielbein components except for

$$H^{++m}(\zeta) \Rightarrow -2i \theta^{\mu+} \sigma^m_{\mu \bar{\mu}} \bar{\theta}^{\bar{\mu}+}. \quad (2.16)$$

It should be pointed out that the action (2.7) with an arbitrary $L^{+4}$ yields the most general self-interaction of hypermultiplets in the conformal $N = 2$ SG background [21]. In the generic case it yields the bosonic sigma model action with the target QK manifold possessing no any isometries. On the other hand, the component approach of refs. [24, 14, 15] is limited to the hypermultiplet self-couplings which give rise to sigma models with certain isometries. This limitation is related to the use of the conformal compensators with finite sets of auxiliary fields. Only while using as a compensator the superfield $q^+$ with an infinite set of auxiliary fields, it becomes possible to construct the density $\sim (u^{-q^+})^2$ which compensates the local supersymmetry transformation of the analytic superspace integration measure in (2.7). Just this unique property of $N = 2$ SG with $q^+$ as the compensator allows one to arrange an arbitrary hypermultiplet self-interaction in (2.7).
After this short review of the action $S_{q,Q}$ let us turn to the pure SG action $S_{SG}$. It is the action of the compensating vector multiplet superfield $H^{++5}(\zeta)$ in the background of $N = 2$ Weyl multiplet (once again, with the “wrong” overall sign):

$$S_{SG} = -\frac{1}{4\kappa^2} \int dZ \ E \ H^{++5} H^{--5}. \quad (2.17)$$

Here $dZ$, $(dZ(\theta^+)^4(\theta^-)^4 = d^4x [du] \ )$, is the measure of integration over the full HSS. The non-analytic superfield $H^{--5}(\zeta, \theta^-)$ is defined by the non-linear harmonic differential equation

$$D^{++} H^{--5} - D^{--} H^{++5} + D^{--} H^{+4} H^{--5} = 0, \quad (2.18)$$

where $D^{--}$ is the appropriately covariantized second harmonic derivative

$$D^{--} = \partial^{--} + H^{--m} \partial_m + H^{--\mu} \partial_{\mu^+} + H^{--\mu} \partial_{\mu^-}. \quad (2.19)$$

The non-analytic vielbein components in (2.19) are determined by the following equations

$$D^{++} H^{--m} - D^{--} H^{++m} + D^{--} H^{+4} H^{--m} = 0, \quad (2.20)$$

$$D^{++} H^{--\mu^\pm} - D^{--} H^{++\mu^\pm} + D^{--} H^{+4} H^{--\mu^\pm} = \pm \theta^{\mu^\pm}. \quad (2.21)$$

The non-analytic density $E$ which transforms so as to cancel the transformations of both the HSS integration measure and the density $(H^{++5} H^{--5})$ is given by the expression

$$E = (-\det g^{mn})^{-1/2} \det \epsilon^\mu_{\dot{\alpha}} \quad (2.22)$$

with

$$\epsilon^\mu_{\dot{\alpha}} = \partial_{\dot{\alpha}^-} H^{--\mu^+}, \quad (2.23)$$

$$g^{mn} = \frac{1}{32} \epsilon^{\alpha^\beta\gamma\delta}_\epsilon \epsilon^{m}_{\dot{\alpha}\dot{\beta}} \epsilon^{n}_{\dot{\gamma}\dot{\delta}} = \frac{1}{32} \left( \epsilon^m_{\beta} \epsilon^n_{\gamma} \delta^\gamma_{\dot{\gamma}} - 2 \epsilon^m_{\alpha} \epsilon^n_{\alpha} \right) + (m \leftrightarrow n), \quad (2.24)$$

$$\epsilon^m_{\dot{\alpha}\dot{\beta}} = \partial_{\dot{\alpha}^-} \partial_{\dot{\beta}^-} H^{--m} - \partial_{\dot{\alpha}^-} \epsilon^\mu_{\dot{\beta}} (e^{-1})^\mu_{\dot{\mu}} \partial_{\dot{\mu}^-} H^{--m}. \quad (2.25)$$

The role of the vector multiplet $H^{++5}$ is to compensate local $\gamma_5$ transformations and dilatations.

### 2.2 Generic QK sigma model action

As was already mentioned, the action $S_{q,Q}$ (2.7) with arbitrary $L^{+4}(Q^+, v^+, v^-)$ is the most general off-shell action of $n$ self-interacting hypermultiplets coupled to $N = 2$ SG [21]. Hence, according to Bagger and Witten [3], in the sector of physical bosons $F^{ri}(x)$ it should yield the sigma model with a generic $4n$ dimensional QK manifold as the target space, $F^{ri}(x)$ being local coordinates of the latter.

Actually, this theorem is manifested by the above HSS formulation. This is much like the way how the one-to-one correspondence between $N = 1$ supersymmetric sigma models and Kähler manifolds [33] is visualized after writing the off-shell sigma model action in terms of complex chiral superfields. Similarly, an analogous correspondence between globally $N = 2$ supersymmetric sigma models and HK manifolds [4] is visualized
when writing the corresponding off-shell sigma model action via the harmonic analytic $Q^+$ superfields [11, 34].

The isomorphism between locally $N = 2$ supersymmetric sigma models and QK manifolds in the HSS formulation can be seen as follows. In ref. [10], starting from the standard definition of QK geometry as a constrained Riemannian geometry, it was found that the QK geometry constraints admit a general solution in terms of unconstrained potential. This potential is defined on an analytic subspace of harmonic extension of the initial QK manifold. All the basic geometric objects of this formulation of QK geometry were proved to be in the one-to-one correspondence with the quantities appearing in the action (2.7). The hypermultiplets $Q^{+} r$ are local coordinates of the QK analytic subspace, the harmonics $v^{\pm i}$ are the relevant harmonic variables, $u^{\pm i}$ are harmonic variables of the initial harmonic extension of QK manifold (with $q^{+ i}$ being basically the bridge relating these two harmonic sets), the interaction Lagrangian $L^{++ 4}$ is just the analytic QK potential which locally encodes the full information about the related QK metric. This completes the proof of the one-to-one correspondence between the action (2.7) and the unconstrained description of generic QK manifolds given in [10]. Hence (2.7) can be used as a tool of explicit construction of general QK metrics by the given $L^{++ 4}$.

The problem of extracting QK metric associated with some $L^{++ 4}$ can be solved in an algorithmic way: like in the HK case [8, 9, 22] it is reduced to the following two basic steps: (i) Discarding all fermions; (ii) Passing to the action of physical bosons $F^{ri}(x)$ by eliminating an infinite set of auxiliary bosonic fields present in $Q^{+ r}$ through their non-dynamical equations of motion. Important new features as compared to the HK case are, first, the presence of couplings to $N = 2$ SG in (2.7) and, secondly, the necessity to take into account the contribution from the purely SG action (2.17).

To see in detail how the QK sigma model action arises, let us choose the WZ gauge for the analytic vielbeins (2.8) as it was made in [21]. Discarding fermions, in this gauge

$$H^{++m}(\zeta) = -2i \theta^{+} a^{\alpha} \bar{\theta}^{+} e_{a}^{m}(x) + 6(\theta^{+})^{2} (\bar{\theta}^{+})^{2} V_{m}^{(ij)}(x) u_{i}^{-} u_{j}^{-},$$  (2.26)

$$H^{++\bar{\mu}+}(\zeta) = (\theta^{+})^{2} \bar{\sigma}_{\bar{\mu}}^{a} A^{\mu a}(x) + (\bar{\theta}^{+})^{2} \theta_{\nu}^{+} t^{(\nu \mu)}(x),$$  (2.27)

$$H^{++4}(\zeta) = (\theta^{+})^{2} (\bar{\theta}^{+})^{2} D(x)$$  (2.28)

(numerical coefficients are chosen for further convenience). In Appendix A we show that the fields $A^{\mu \bar{\mu}}$ (a complex gauge field for the residual local $\gamma_{5}$ symmetry and scale transformations) and $t^{(\nu \mu)}$ do not contribute to the final sigma model action of the physical bosons $F^{ri}(x)$. So from the beginning they can be put equal to zero. Also, we shall not be interested in couplings to gravity, so we put

$$e_{a}^{m}(x) = \delta_{a}^{m}$$

in (2.26). Thus the only components of the Weyl multiplet which couple to $q^{+ i}$ and $Q^{+ r}$ in (2.7) and can influence the structure of the final sigma model $F^{ri}(x)$ action are $D(x)$ and $V_{m}^{(ik)}(x)$. The former field is purely auxiliary, the second one is the gauge field for the local conformal $SU(2)$ transformations which form a residual symmetry of the WZ gauge (2.26)-(2.27) (this field is also non-propagating).

As the result of this discussion, the part $\tilde{S}_{Q}$ of the action (2.7) (with $\gamma = 1$) which is relevant to the calculation of the bosonic QK sigma model action is obtained from (2.7)
by discarding all fermionic components in \( q^+ (\zeta) , Q^{++} (\zeta) \) and making the substitution

\[
\mathcal{D}^{++} \Rightarrow \tilde{\mathcal{D}}^{++} = D^{++} + (\theta^+)^2 (\bar{\theta}^+)^2 \{ 6 V^m(ij)(x) u_i^- u_j^- \partial_m + D(x) \partial^- \},
\]

(2.29)

where \( D^{++} \) is the standard flat harmonic derivative

\[
D^{++} = \partial^{++} - 2i \theta^+ \sigma^a \partial_m .
\]

(2.30)

Thus

\[
\tilde{S}_{q,Q} = \frac{1}{2} \int d\zeta (-^4) \left\{ -q^a_a \tilde{D}^{++} q^+ a + \kappa^2 (u^- q^+)^2 \left[ Q^r_r \tilde{D}^{++} Q^{++} + L^{++} (Q^+, v^+, v^-) \right] \right\} .
\]

(2.31)

The terms proportional to \( D(x) \) and \( V^m(ij)(x) \) in (2.31) can be easily found. They have the following universal form which does not depend on details of \( Q^+ \) self-couplings and their coupling to the compensator \( q^+ a \)

\[
S^{D,V}_{q,Q} = \frac{1}{2} \int d^4 x \left\{ D(x) \int du \left[ f^+ a (x, u) \partial^- f^+_a (x, u) \right. \right.
\]

\[
- \kappa^2 (u^- f^+)^2 F^{++}(x, u) \partial^- F^+_r (x, u) \]

\[
+ 6 V^m(ij)(x) \int du (u_i^- u_j^-) \left[ f^+ a (x, u) \partial_m f^+_a (x, u) \right. \]

\[
- \kappa^2 (u^- f^+)^2 F^{++}(x, u) \partial_m F^+_r (x, u) \left. \right\} .
\]

(2.32)

Here

\[
f^+ a (x, u) \equiv q^+ a |_{\theta = 0} , \quad F^{++}(x, u) \equiv Q^{++} |_{\theta = 0} .
\]

(2.33)

Let us now turn to the supergravity action (2.17). It is convenient to choose the WZ gauge for \( H^{++5} (\zeta) \) and to fully fix the local \( \gamma_5 \) and dilatation symmetries for which \( H^{++5} \) serves as a compensator, by putting

\[
H^{++5} (\zeta) = i \left[ (\theta^+)^2 - (\bar{\theta}^+)^2 \right] \Rightarrow H^{-5} (\zeta) = i \left[ (\theta^-)^2 - (\bar{\theta}^-)^2 \right] + \ldots .
\]

(2.34)

The bosonic vector gauge field and an \( SU(2) \) triplet of auxiliary fields present in the bosonic part of \( H^{++5} \) in the WZ gauge are of no interest for us because they do not couple to the matter sector (such couplings could appear only for matter with non-trivial central charge; this is not the case for \( q^{++} \) and \( Q^{++} \)). The corresponding vielbeins \( H^{-5} \) and \( H^{-M} \) (entering the harmonic derivative \( \tilde{\mathcal{D}}^{--} \), eq. (2.19)) are computed using eqs. (2.20), (2.21), (2.18) with the particular expressions (2.29), (2.34) for the analytic harmonic vielbeins. Then after a straightforward though tedious computation one finds the only relevant bosonic term in \( S_{SG} \) to be

\[
S_{SG} \Rightarrow S^{D,V}_{SG} = -\frac{1}{2 \kappa^2} \int d^4 x \left[ D(x) + V^m(ij)(x) V_m(ij)(x) \right] .
\]

(2.35)

Now we are in a position to write the generic HSS action of QK sigma model. It is the sum of (2.31) and (2.35)

\[
S_{QK} = \tilde{S}_{q,Q} + S^{D,V}_{SG}
\]

\[
= \frac{1}{2} \int d\zeta (-^4) \left\{ -q^+ a D^{++} q^+ a + \kappa^2 (u^- q^+)^2 \left[ Q^r_r D^{++} Q^{++} + L^{++} (Q^+, v^+, v^-) \right] \right\}
\]

\[
+ S^{D,V}_{q,Q} + S^{D,V}_{SG} .
\]

(2.36)
Varying with respect to the non-propagating fields $D(x)$ and $V^m(x)$ in the last two terms in (2.36) yields, respectively, the relation between the compensating and matter hypermultiplets

$$\int du \left[ f^+ a \partial^- f^+_a - \kappa^2 (u^- f^+)^2 F^{+r} \partial^- F^+_r \right] = \frac{1}{\kappa^2}$$

and the expression for $V^m(x)$

$$V^m(x) = 3\kappa^2 \int du (u^- u^-) V^m(x, u) \equiv \kappa^2 \gamma^m(x)$$

$$V^m = f^+ a \partial_m f^+_a - \kappa^2 (u^- f^+)^2 F^{+r} \partial_m F^+_r$$

Substituting this back into (2.36) gives the convenient representation for the QK sigma model action containing only the hypermultiplets fields

$$S_{QK} = \frac{1}{2} \int d^{(-4)} \left\{ -q^+_a D^{++} q^+ + \kappa^2 (u^- q^+)^2 \left[ Q^+ D^{++} Q^+ + L^{++}(Q^+, v^+, v^-) \right] \right\}$$

$$+ \frac{\kappa^2}{2} \int d^4 x \nu^m(x) \nu_m(x) \equiv S_{QK}^{(1)} + S_{QK}^{(2)}$$

where one should keep only bosonic components in the superfields $q^+_a(\zeta), Q^+_r(\zeta)$. Also, the constraint (2.37) should be added.

Finally, modulo differences related to the presence of new terms $\sim \kappa^2$ in the action and the additional constraint (2.37), the recipe of deriving the QK metric from (2.40) is basically the same as in the HK case [8, 17, 18]. It consists of the following steps.

A. One substitutes into (2.40) the $\theta$ expansion of $q^+_a, Q^+_r$ with the fermions discarded

$$q^+_a(\zeta) = f^+_a(x, u) + (\theta^+)^2 m^-_a(x, u) + (\theta^+)^2 \tilde{m}^-_a(x, u)$$

$$Q^+_r(\zeta) = F^+_r(x, u) + (\theta^+)^2 M^-_r(x, u) + (\theta^+)^2 \tilde{M}^-_r(x, u)$$

and varies with respect to the unconstrained auxiliary fields $g^{(-3)}_a, G^{(-3)}_r, A^{(-3)}_{ma}, B^{(-3)}_{mr}, \overline{m}_a, M^-_r$. Then one inserts the result back into the action and integrates over the harmonics, thus expressing the action entirely in terms of the bosonic fields $f^+ a(x), F^r(x)$ and their $x$-derivatives. It is easy to show that the contribution from the fields $m^-_a, \overline{m}_a, M^-_r, \tilde{M}^-_r$ in the general case identically vanishes, so from the very beginning we can put them equal to zero in (2.41)

$$m^-_a = \overline{m}_a = M^-_r = \tilde{M}^-_r = 0$$

B. One fully fixes the gauge with respect to the residual local $SU(2)_c$ symmetry of the action (2.40) (see below) so as to gauge away the triplet part of $f^+ a(x)$:

$$f^+_a(x) \Rightarrow f^+_a(x) = \delta^+_a \sqrt{2} \omega(x)$$

and expresses $\omega(x)$ in terms of the target QK manifold coordinates $F^r(x)$ using the algebraic constraint (2.37).
C. One substitutes the expression for $\omega(x)$ into the previously obtained action and reads off the QK metric from the latter.

Note that the flat HK limit is achieved by putting altogether
\[ \sqrt{2} |\kappa| \omega = 1 \quad (2.44) \]
(recall (2.13)) and then setting $\kappa = 0$. The constraint (2.37) becomes identity $1 = 1$ in this limit, while (2.40) goes into the HSS action of the general off-shell HK sigma model with the $4n$ dimensional target space $[3,4,9]$: \[ S_{QK} \Rightarrow S_{HK} = \frac{1}{2} \int d\zeta(-4) \{ Q_{i}^{+} D^{++} Q^{++} + L^{++}(Q^{+}, u^{+}, u^{-}) \} . \quad (2.45) \]
We see that, at least at the level of local consideration, any HK manifold has its QK counterpart and vice-versa.

Before going further, we comment on the peculiarities of realization of residual local $SU(2)_{c}$ symmetry in the actions (2.36), (2.40). This is the only local symmetry from the SG sector which survives truncation (2.29). For our further purposes it will be enough to know its realization on the lowest components $f^{+a}(x,u)$, $F^{+r}(x,u)$ in the expansion (2.41). Taking into account the general transformation laws (2.5), (2.12), and the fact that in the WZ gauge the $SU(2)_{c}$ gauge parameter $\lambda^{ij}(x)$ appears in the residual coordinate transformations as $[21]$
\[ \delta u^{+i} = \lambda^{++} u^{-i} + O(\theta^{2}) , \quad \delta x^{m} = O(\theta^{2}) , \quad \delta \theta^{+\mu} = \lambda^{+\mu} - \theta^{+\mu} + O(\theta^{3}) , \quad (2.46) \]
\[ \lambda^{++} = \lambda^{ij}(x) u^{i+} u^{j+} , \quad \lambda^{++} = \lambda^{ij}(x) u^{i+} u^{j-} , \]
we find
\[ \delta^{*} f^{+a} = \lambda^{+\mu} f^{+a} - \lambda^{++} \partial^{--} f^{+a} , \quad \delta^{*} F^{+r} = -\lambda^{+\mu} \partial^{--} F^{+r} . \quad (2.47) \]
The invariance of the QK sigma model off-shell action (2.36) under $SU(2)_{c}$ is obvious because this action was obtained starting from the action (2.6) invariant under the full $N = 2$ SG group (2.2), (2.3), (2.5), (2.12). In particular, the invariance of the action $S^{D,V}_{SG}$ follows from the transformation rules
\[ \delta V^{ij}_{m} = -\partial_{m} \lambda^{ij}(x) + 2 \lambda^{[i}_{k}(x) V^{j]k}_{m}(x) , \quad \delta D(x) = 2 \partial_{m} \lambda^{ik}(x) V^{j}_{ik}(x) , \quad (2.48) \]
which can be easily derived from the condition of stability of the truncated WZ gauge (2.29) under the transformation (2.10) with $SU(2)_{c}$ as the only residual local symmetry.

It is somewhat more tricky to check the invariance of the pure hypermultiplets action (2.40) and the constraint (2.37). The latter turns out to be invariant only with taking account of the purely harmonic (i.e. non-dynamical) equations for $f^{+r}(x,u)$, $F^{+r}(x,u)$. These equations are obtained by varying with respect to the auxiliary fields $g_{a}^{-3}$, $G_{r}^{-3}$ defined in (2.41) and in the general case read
\[ \partial^{++} f^{+a} - \frac{\kappa^{2}}{2} f^{+a} \partial_{v}^{-} L^{+4} + \kappa^{2} u^{-a}(u^{-} f^{+}) \left( 1 - \frac{1}{2} F^{+r} \frac{\partial}{\partial F^{+r}} \right) L^{+4} = 0 , \quad (2.49) \]
\[ \partial^{++} F^{+r} + F^{+r} \left[ \left( u^{+} f^{+} \right) + \frac{\kappa^{2}}{2} \partial_{v}^{--} L^{+4} \right] + \frac{1}{2} \partial L^{+4}_{F^{+r}} = 0 , \quad (2.50) \]
where
\[ \partial_{v}^{-} = u^{-i} \partial_{v^{+}i} \]
acts only on the explicit harmonics \( v^{\pm i} \) in \( L^{+4}(F^{+}, v^{+}, u^{-}) \).

The result of varying the l.h.s. of the constraint (2.37) with respect to (2.47), up to a full harmonic derivative, is
\[ -\int du \lambda \, \partial^{++} \left[ f^{+a} \partial^{-} f_{a}^{+} - \kappa^{2} (u^{-} f^{+})^{2} F^{+r} \partial^{-} F_{r}^{+} \right] . \quad (2.51) \]

It is easy to check that eqs. (2.49), (2.50) imply the “conservation law”
\[ \partial^{++} \left[ f^{+a} \partial^{-} f_{a}^{+} - \kappa^{2} (u^{-} f^{+})^{2} F^{+r} \partial^{-} F_{r}^{+} \right] = 0 , \quad (2.52) \]
whence it follows that (2.51) is vanishing, i.e. the constraint (2.37) is invariant on the shell of non-dynamical \( S^{2} \) equations (2.49), (2.50). By the way, (2.52) means that the integrand in (2.37) does not depend on harmonics on-shell and, hence, the harmonic integral can be taken off in (2.37),
\[ f^{+a} \partial^{-} f_{a}^{+} - \kappa^{2} (u^{-} f^{+})^{2} F^{+r} \partial^{-} F_{r}^{+} = \frac{1}{\kappa^{2}} , \quad (2.53) \]
when computing the final form of the action in terms of the physical bosonic fields \( F^{ri}(x) \).

Quite analogously, one can check one more “conservation law”, this time for \( V_{m}^{++}(x, u) \) defined in (2.39):
\[ \partial^{++} V_{m}^{++} = 0 , \ \Rightarrow \ V_{m}^{++}(x, u) = v_{m}^{(kl)}(x) u_{k}^{+} u_{l}^{+} . \quad (2.54) \]

Using this property and eq. (2.53), it is easy to show that under the \( SU(2)_{c} \) transformations (2.47) the composite gauge field (2.38) is transformed just as in (2.48). This is a good check of self-consistency of our approach. It is also straightforward to check the \( SU(2)_{c} \) covariance of the system (2.49), (2.50).

Finally, we shall make a few further comments on the generality of the above action. When considering the coupled matter-SG actions within the field theory framework, the standard assumption is that the kinetic terms of the physical fields have the correct sign to ensure the positivity of energy. Otherwise the fields are ghosts. Just to have the standard signs of the Einstein action and the actions of other physical components of \( N = 2 \) SG and matter multiplets, the \( q \) compensating superfield action in (2.7) and that of \( H^{++5} \) (2.17) were chosen to have the “wrong” sign as compared to the sign of the \( Q^{+} \) action in (2.7). However, while using the \( N = 2 \) SG-matter system merely as a tool to recover the QK metrics, the above requirement is not longer compulsory: we wish to be able to reproduce the whole set of the QK metrics, with different signatures of the tangent space metric. In other words, we can choose other relative signs of the \( H^{++5} \), \( q^{+} \) and \( Q^{+} \) terms in the whole \( N = 2 \) QK sigma model action: local \( N = 2 \) supersymmetry is always preserved that guarantees the corresponding bosonic target metric to be QK. The only self-consistency condition we should care of is related to the constraint (2.37) (or (2.53)). Assuming that \( \kappa^{2} = |\kappa|^{2} > 0 \), eqs. (2.37) or (2.53) in the gauge (2.43) imply
\[ \omega^{2} + \ldots = \epsilon \frac{1}{2\kappa^{2}} , \]
where $\epsilon = \pm 1$ is the relative sign of the $H^{++5}$ and $q^+$ terms and “dots” stand for higher orders in $\omega(x)$ and $F^{ri}(x)$. Obviously, for this constraint to be solvable for $\omega(x) = \varpi(x)$, one should demand

$$\epsilon = +1.$$  

(2.55)

With this restriction taken into account, eq. (2.53) implies that

$$\omega(x) = \frac{1}{\sqrt{2|\kappa|}}[1 + \kappa^2 \tilde{\omega}(\kappa^2, F(x))] \quad \Rightarrow \quad \partial^m \omega \partial_m \omega = \frac{1}{2} \kappa^2 \partial^m \tilde{\omega} \partial_m \tilde{\omega},$$

(2.56)

where $\tilde{\omega}(\kappa^2, F(x))$ is a function of the physical bosonic fields which is non-singular at $\kappa^2 = 0$. Then a simple analysis shows that the above-mentioned uncertainty in the relative signs amounts to the freedom of changing $\kappa^2 \rightarrow -\kappa^2$ in (2.40) (up to the overall sign, and taking into account that $\kappa^2(u^- q^+)^2 = 1 + O(\kappa^2)$), in (2.49), (2.50), in the l.h.s. of (2.37) and (2.53), and in (2.39) . In the “maximally flat” case, with $L^{+4} = 0$, this substitution takes the non-compact homogeneous hyperbolic target QK manifold $\mathbb{H}H^n \sim Sp(1,n)/Sp(1) \otimes Sp(n)$ into its compact counterpart $\mathbb{H}P^n \sim Sp(1+n)/Sp(1) \otimes Sp(n)$ (see Subsect. 2.4).

An additional freedom is associated with the possibility to insert an arbitrary indefinite constant metric in the sum over different $Q^{+}_r$ (or $F^{+}_i$) in all formulas where such a sum is present. Let us equivalently replace the $Sp(n)$ index $r$ by the pair $(Ai)$ where $i = 1, 2$ is a $Sp(1)$ index and $A = 1, \ldots, n$ is the vector $SO(n)$ index. Then

$$Q^{+}_r D^{++} Q^{+}_r \sim Q^{+}_i A D^{++} Q^{+Ai}$$

and the generalization just mentioned amounts to the replacement

$$Q^{+}_i A D^{++} Q^{+Ai} \quad \Rightarrow \quad \eta^{AB} Q^{+}_i A D^{++} Q^{+Bi}$$

where $\eta^{AB}$ is the diagonal constant metric with an arbitrary signature. This corresponds to considering as the “flat” backgrounds the homogeneous QK manifolds $Sp(1+m, s)/Sp(1) \otimes Sp(m, s)$ or $Sp(m, 1+s)/Sp(1) \otimes Sp(m, s)$, $(m + s = n)$.

In what follows, for definiteness, we shall basically deal with the QK sigma models corresponding to the original $N = 2$ SG-matter action, i.e. those defined by the action (2.40) and constraint (2.37), (2.53) with the fixed signs and $\kappa^2 > 0$. This corresponds to $\mathbb{H}H^n$ as the “maximally flat” background.

As the last remark, let us notice the existence of another flat limit of the above general QK sigma model, besides (2.43). It yields the conformally-invariant HK sigma model with $4(n + 1)$ dimensional bosonic target space. One starts from the action (2.7) with the manifestly included sigma model constant $\gamma$, chooses $\gamma^2 = \kappa^2$ and assumes no flat limit (2.13) for $q^+_0$. Then putting all $N = 2$ SG fields equal to zero leaves us with the following $N = 2$ HK sigma model action

$$S'_{HK} = \frac{1}{2} \int d\zeta^{(-4)} \left\{ -q^{i+} D^{++} q^{+i} + \tilde{Q}^{+}_r D^{++} \tilde{Q}^{+r} + (u^- q^+)^2 L^{++} (u^- q^+)^{-1} \tilde{Q}^{+}, (u^- q^+)^{-1} q^{+}, u^- \right\},$$

(2.57)

where

$$\tilde{Q}^{+r} = (u^- q^+) Q^{+r}.$$
and $q^+$ and $Q^+$ are no longer related by any constraint. One can choose the correct sign for the $q^+$ kinetic term, since $q^+$ ceases to be the pure gauge compensating superfield after taking this limit. The action (2.54) (with any sign of the $q^+$ term) is invariant under the rigid $N = 2$ superconformal group the realization of which in the analytic HSS was given in [33]. It is the most general superconformally invariant off-shell HSS action of $(n + 1)$ self-interacting hypermultiplets. The on-shell bosonic geometry of such hypermultiplet self-couplings was recently discussed in [34], [37].

2.3 Isometries and Killing potential

The rigid isometries of HK or QK metrics are nicely represented in the language of the HSS actions as symmetries of the latter [23]. For further reference, we briefly describe their realization in the general QK sigma model HSS action following ref. [23].

Any isometry of the $Q^+$ off-shell action (2.7) including those which become triholomorphic in the HK limit (in the supersymmetric setting, the latter property amounts to the commutativity of the isometry transformations with the rigid $N = 2$ supersymmetry) is realized by the following transformations of $Q_i^{++}, q^a$

$$\delta Q^{++} = \epsilon A \lambda^{++A}(Q, v),$$

$$\delta q^a = \kappa^2 \left[ (u^+ q)^- u^- a \left( 1 - \frac{1}{2} Q^{++r} \frac{\partial}{\partial Q^{++r}} \right) \Lambda^{++A} - \frac{1}{2} q^a \partial^- \Lambda^{++A} \right] \epsilon A,$$

$$\delta v_i^+ = \kappa^2 \epsilon A v_i^- \left( 1 - \frac{1}{2} Q^{++r} \frac{\partial}{\partial Q^{++r}} \right) \Lambda^{++A},$$

where

$$\lambda^{++A} = \frac{1}{2} \left( \frac{\partial \Lambda^{++A}}{\partial Q^{++r}} + \kappa^2 Q^{++r} \partial^- \Lambda^{++A} \right),$$

and $\Lambda^{++A}(Q, v)$ satisfies the following equation

$$\partial^+ \Lambda^{++A} + \frac{1}{2} \frac{\partial \Lambda^{++A}}{\partial Q^{++r}} \left( \frac{\partial L^+}{\partial Q^{++r}} + \kappa^2 Q^{++r} \partial^- L^+ \right) + \kappa^2 \partial^- \Lambda^{++A} \left( 1 - \frac{1}{2} Q^{++r} \frac{\partial}{\partial Q^{++r}} \right) L^+ - \kappa^2 \Lambda^{++A} \partial^- L^+ = 0.$$ (2.62)

The objects $\lambda^{++A}(Q, v)$ and $\lambda^{++A}(Q, v)$ are called, respectively, the Killing vector and Killing potential [23]. All possible QK isometries are characterized by the HSS Killing potential satisfying eq. (2.62). It is straightforward to check the invariance of the $Q$ action (2.7) under these transformations. The $N = 2$ SG fields are of course inert under them. The constraint (2.37) and the composite $SU(2)_c$ gauge filed (2.38) can also be easily checked to be invariant under the corresponding transformations of $f^a$ and $F^{++}$ (actually without using the $S^2$ equations (2.49), (2.50) and the Killing potential equation (2.62); the latter is to be taken into account only when checking the invariance of the action and covariance of eqs. (2.49), (2.50)).
2.4 Example: $\mathbb{H}H^n$ and $\mathbb{H}P^n$ sigma models

To illustrate the above construction, let us consider the simplest case of the “flat” $4n$ dimensional QK manifold, that is $\mathbb{H}H^n \sim Sp(1, n)/Sp(1) \otimes Sp(n)$. It corresponds to the choice

$$L^{+4} = 0$$

in (2.40) [23, 10]. Redefining $Q^{+r}$ as

$$Q^{+r} = \frac{1}{\kappa (u^{-q+})} \hat{Q}^{+r},$$

we cast the superfield lagrangian of $S^{(1)}_{QK}$ in (2.40) into the simple form

$$-q_{a}^{+} D^{+a} q_{a}^{+} + \hat{Q}^{+}_{r} D^{+a} \hat{Q}^{+r}.$$  

(2.64)

After performing the integration over $\theta^+, \bar{\theta}^+$ the action $S^{(1)}_{QK}$ is reduced to

$$S^{(1)}_{QK} = \int d^{4}x du \left( \partial^{++} f^{+a} g_{(-3)a} - A_{ma} \partial^{m} f^{+a} + \frac{1}{4} A_{ma} \partial^{++} A^{ma} - \partial^{++} \hat{F}_{+}^{r} G^{(-3)r} + \hat{B}_{mr} \partial^{m} \hat{F}^{+r} - \frac{1}{4} \hat{B}^{mr} \partial^{++} \hat{B}^{-mr} \right).$$

(2.65)

Varying with respect to the auxiliary fields in this case yields

$$\delta g_{a}^{(-3)} : \quad \partial^{++} f^{+a}(x, u) = 0 \Rightarrow f^{+a}(x, u) = f^{+a}(x) u^{+}_{i} = \sqrt{2} e^{ai} \omega(x) u^{+}_{i},$$

(2.66)

$$\delta A_{ma} : \quad \partial^{++} A_{ma} - 2 \partial^{m} f^{+a} = 0 \Rightarrow A_{ma} = 2 \sqrt{2} u^{-a} \partial^{m} \omega(x),$$

(2.67)

$$\delta \hat{G}^{(-3)}_{r} : \quad \partial^{++} \hat{F}_{+}^{r}(x, u) = 0 \Rightarrow \hat{F}_{+}^{r}(x, u) = \hat{F}^{+r}(x) u^{+}_{i},$$

(2.68)

$$\delta \hat{B}_{mr} : \quad \partial^{++} \hat{B}^{m}_{r} - 2 \partial^{m} \hat{F}_{+}^{r} = 0 \Rightarrow \hat{B}^{m}_{r} = 2 \partial^{m} \hat{F}^{+r}(x) u^{+}_{i}.$$  

(2.69)

After substituting this back into the action and integrating over harmonics, one gets

$$S^{(1)}_{QK} = \int d^{4}x \left\{ \frac{1}{2} \partial_{m} \hat{F}_{+}^{m} \hat{F}^{+r} - 2 \partial^{m} \omega(x) \partial^{m} \omega(x) \right\}.$$  

(2.70)

The relevant expressions for $\omega(x)$ and $\nu^{(ij)}_m$ following from eqs. (2.37), (2.38), (2.39) are also easy to find

$$\omega(x) = \frac{1}{\sqrt{2\kappa}} \sqrt{1 + \frac{\kappa^{2}}{2} \hat{F}^{2}}, \quad \nu^{(ij)}_m = \hat{F}^{(i} \partial_{m} \hat{F}^{rj)}.$$  

(2.71)

Finally, the action (2.40) for this simple case is given by

$$S_{HP} = \frac{1}{2} \int d^{4}x \left\{ \partial_{m} \hat{F} \partial^{m} \hat{F} + \kappa^{2} (\hat{F}_{+}(i \partial_{m} \hat{F}^{r)}) (\hat{F}^{(i} \partial_{m} \hat{F}^{s)}j) \right\} - \frac{\kappa^{2}}{2} \frac{1}{1 + \frac{\kappa^{2}}{2} \hat{F}^{2}} (\hat{F}_{+} \hat{F} \hat{F}) (\hat{F} \partial^{m} \hat{F}) \right\},$$  

(2.72)
or, in terms of the original variables $F^{ri} = (\sqrt{2}K\omega)^{-1} \hat{F}^{ri}$,

$$S_{HP} = \frac{1}{2} \int d^4x \left\{ \frac{1}{1 - \frac{\kappa^2}{2} F^2} (\partial_m F^{\partial m} F) + \frac{\kappa^2}{[1 - \frac{\kappa^2}{2} F^2]^2} (F^{ri} F^{ri}_{\partial m} (\partial_m F^{rj} \partial^m F^{sj})) \right\} \quad (2.73)$$

The corresponding metric is recognized as that of the homogeneous non-compact space $\mathbb{H}^n \sim Sp(1, n)/Sp(1) \otimes Sp(n)$, or of its compact counterpart $\mathbb{H}^{n'} \sim Sp(1+n)/Sp(1) \otimes Sp(n)$ for $\kappa^2 \rightarrow -\kappa^2$.

The $Sp(1, n)$ isometry of the metric is originally realized as the following transformations of the superfields $q^+^a$ and $Q^+^a$:

$$Sp(1) : \quad \delta Q^+^r = \kappa^2 \beta^{(ij)} (v_i^+ v_j^-) Q^+^r, \quad \delta q^+_i = \kappa^2 \beta^{(ik)} q^+^k, \quad (2.74)$$

$$Sp(n) : \quad \delta Q^+^r = \lambda^{(rs)} Q^+_s, \quad \delta q^+_a = 0, \quad (2.75)$$

$$Sp(1, n)/Sp(1) \otimes Sp(n) : \quad \delta Q^+^r = \lambda^r v^+_r + \kappa^2 \lambda^s (Q^+_s v^-_r) Q^+^r, \quad \delta q^+_i = \kappa^2 \lambda^s Q^+_s (v^- q^+_i). \quad (2.76)$$

The corresponding Killing potentials are

$$\Lambda^{++(ik)} = v^+_i v^+_k, \quad \Lambda^{++(rs)} = Q^+^r Q^+^s, \quad \Lambda^{++ri} = 2 v^+_i Q^+^r. \quad (2.77)$$

They are evident solution of the Killing equation (2.62) which is simply

$$\partial^+\Lambda^{++}=0$$

since $L^+^4 = 0$ in this case. The $Sp(1, n)$ transformation properties of $f^+^a, F^+^r$ and, further, of the physical fields $F^{ri}(x)$ can be easily derived from (2.74) - (2.76) using (2.66), (2.68) and taking into account the compensating $SU(2)_c$ gauge (2.43).

### 3 QK quotient construction in the HSS approach

#### 3.1 HSS quotient for the quaternionic Taub-NUT

Having the general $N = 2$ SG inspired representation (2.40), (2.37) for the QK sigma model action, we are guaranteed, by the reasoning of [10], that the physical bosonic fields target metric is QK for any choice of the analytic QK potential $L^+^4(F, v)$, as soon as the infinite set of bosonic auxiliary fields has been eliminated by the non-dynamical harmonic equations (2.49), (2.50) and the $SU(2)_c$ gauge (2.43) has been imposed. Thus the action (2.40) augmented with the algebraic constraint (2.37) (or its on-shell form (2.53)) provides us with a general tool of computing QK metrics by the given $L^+^4$, in full analogy to the similar HSS approach to computing HK metrics [8, 9, 17, 18]. However, as was already mentioned earlier, for the generic $L^+^4$ eqs. (2.49), (2.50) are difficult to solve. The situation in the QK case is more complicated than in the HK one due to the presence of the extra harmonic equation (2.49) for the compensating hypermultiplet; in general, (2.49) and (2.50) form a highly nonlinear coupled system of differential equations on the sphere $S^2 = \{u^+_i\}$. 

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Like in the HK case, in a number of interesting examples there is a way round this difficulty. This is the HSS version of the quotient construction \([13, 14, 15]\). It is based on the observation that many non-trivial HSS actions with the isometries of the type discussed in Subsect. 2.3 (both in the HK and QK cases) can be thought of as gauge-fixed forms of more general gauge invariant actions including non-propagating HSS vector gauge multiplets. The simplest non-trivial example of this sort is provided by the QK extension of the four-dimensional Taub-NUT metric, and we shall illustrate the method just on this example.

The HSS action for this case was written for the first time in \([23]\). It is defined by the same quartic analytic potential \(L_{+4}\) as in the HK case \([8]\)

\[
L_{+4}^{TN} = \frac{1}{4} (c^{ab} Q^+_a Q^+ b)^2 \equiv \frac{1}{4} (J^{++})^2 , \quad a, b = 1, 2 .
\]  

(3.1)

Here \(c^{(ab)}\) is a constant vector which breaks the Pauli-Gursey \(SU(2)_{PG} \sim Sp(1)_{PG}\) isometry of the free \(Q^+ a\) action (with the Killing potential \(Q^+ a Q^+ b\) down to \(U(1)_{PG}\) (with the Killing potential \(c^{ab} Q^+_a Q^+_b\)). The coefficient \(1/4\) was chosen for further convenience. The full symmetry of the QK Taub-NUT HSS action is \(Sp(1) \times U(1)_{PG}\), with \(Sp(1)\) given by (2.74). Assuming \(c^{(ab)}\) to obey the standard pseudo-reality condition

\[
c^{ab} = \epsilon_{ad} \epsilon_{be} c^{(de)} , \tag{3.2}
\]

which together with (2.4) implies \(J^{++}\) to be real, \(\bar{J}^{++} = J^{++}\), and choosing the \(SU(2)_{PG}\) frame so that

\[
c^{11} = c^{22} = 0 , \quad \epsilon_{12} = i \lambda , \quad \bar{\lambda} = \lambda , \tag{3.3}
\]

one can rewrite (3.1) as

\[
L_{+4}^{TN} = -\lambda^2 (Q^+ \bar{Q}^+)^2 , \quad Q^+ \equiv Q^+_1 , \quad \bar{Q}^+ = -Q^+_2 .
\]  

(3.4)

This form of the QTN potential was used in \([23, 19]\).

The QTN metric in the harmonic approach was explicitly found in \([19]\) by making use of the pure geometric formalism of \([10]\). It is easy to show that, up to a slight difference in the notation, the general \(S^2\) equations (2.49), (2.50) in this case coincide with the equations for the harmonic and \(x^a, \tilde{x}^a\)-bridges employed in ref. \([19]\). Therefore the lagrangian approach we prefer to deal with here yields the same answer for the metric as the approach used in \([19]\). We are not going to compare both approaches in detail, we merely wish to show that the lagrangian approach suggests an alternative method to derive the QTN metric.

Let us start from the system of two free hypermultiplets \(Q^+ a, G^+ b\), namely from the superfield Lagrangian which corresponds to a 8-dimensional \(\mathbb{H}H^2\) manifold (cf. (2.64))

\[
\mathcal{L}^{+4} = -q^+ \mathcal{D}^{++} q^+ a + \kappa^2 (u^{-} q^+)^2 [Q^+_a \mathcal{D}^{++} Q^+ a + G^+_a \mathcal{D}^{++} G^+ a] . \tag{3.5}
\]

Now, let us specialize to the following one-parameter isometry of this action

\[
\delta Q^+ a = \epsilon [c^{(ab)} Q^+_b - \kappa^2 (u^{-} G^+) Q^+ a] ,
\]

\[
\delta G^+ a = \epsilon [v^+ a - \kappa^2 (u^{-} G^+) G^+ a] ,
\]

\[
\delta q^+ i = \epsilon \kappa^2 [(u^{-} G^+) q^{+} i - (v^+ G^+) (u^{-} q^+) u^{-} i] , \tag{3.6}
\]
The Killing potential for this particular case of the \( \mathbb{H}H^2 \) isometries (see (2.58) - (2.62)) can be easily found

\[
\Lambda^{++} = 2(v^+G^+) - c^{(ab)} Q^+_a Q^+_b = 2(v^+G^+) - J^{++} .
\] (3.7)

Let us now gauge this isometry by promoting \( \epsilon \) to an analytic gauge parameter, \( \epsilon \Rightarrow \epsilon(\zeta) \). As was shown in [23], the gauge invariant lagrangian is the following modification of (3.5)

\[
\mathcal{L}^{++} = -q^+_a D^{++} q^{++} + \kappa^2 (u - q^+)^2 \left[ Q^+_a D^{++} Q^{++} + G^+_a D^{++} G^{++} + V^{++} \Lambda^{++} \right] ,
\] (3.8)

where \( V^{++} = V^{++}(\zeta) \) is the standard analytic \( N = 2 \) gauge potential with the transformation law

\[
\delta V^{++} = D^{++} \epsilon(\zeta) .
\] (3.9)

It is straightforward to check invariance of the modified \( Q,G \) action under (3.6), (3.9) with local \( \epsilon(\zeta) \)

The \( N = 2 \) vector gauge multiplet is assumed to be non-propagating, i.e. having no kinetic term. In other words, it is the Lagrange multiplier for the constraint

\[
\Lambda^{++} = 2(v^+G^+) - J^{++} = 0 .
\] (3.10)

Further, the transformation law (3.6) shows that one can fully fix the \( \epsilon(\zeta) \) gauge freedom by choosing the gauge

\[
(u^- G^+) = 0 \Rightarrow G^{++} = -(v^+G^+) u^- .
\] (3.11)

In this gauge

\[
G^+_a D^{++} G^{++} = (v^+G^+)^2 .
\] (3.12)

Expressing \( (v^+G^+) \) from the constraint (3.10),

\[
(v^+G^+) = \frac{1}{2} J^{++} ,
\]

and substituting this back into (3.8) with taking account of (3.12), we recover the QTN potential (3.1). Note that we could choose the sign minus in front of the \( G \) part of the action (3.8) and properly modify the isometry group (it is related to (3.8) via the substitution \( \kappa^2 \rightarrow -\kappa^2 \)). This would give rise to QTN potential with the sign minus as compared to (3.1), (3.4). These two choices are related by the change \( \lambda^2 \rightarrow -\lambda^2 \) and yield the metrics with the positive and negative Taub-NUT “mass” which is basically just \( \lambda^2 \)

So we started from the system of two “free” \( Q \) hypermultiplets, gauged some \( U(1) \) isometry by the non-propagating gauge potential \( V^{++} \) and succeeded in eliminating one of these hypermultiplets by choosing the appropriate gauge and solving the constraint implied by \( V^{++} \) as the Lagrange multiplier. As the result we have gained the QTN action for the remaining hypermultiplet. In other words, the QTN action proves to be a special gauge of the more general action (3.8).

We are at freedom to choose other gauges in (3.8), the final bosonic sigma model action should evidently be the same irrespective of the gauge-fixing procedure. A convenient gauge is the Wess-Zumino (WZ) gauge for \( V^{++} \). Discarding fermions and the terms
\( \sim (\theta^+)^2, (\bar{\theta}^+)^2 \) (their contribution finally disappears from the action), \( V^{++} \) in this gauge reads

\[
V_{WZ}^{++}(\zeta) = i(\theta^+ \sigma^a \bar{\theta}^+) A_a(x) + (\theta^+)^2(\bar{\theta}^+)^2 S^{(ik)}(x) u_i^- u_k^- .
\] (3.13)

The residual gauge freedom is given by the transformations

\[
\delta A_m(x) = -2 \partial_m \epsilon(x) , \quad \epsilon(\zeta) = \epsilon(x) + ... .
\] (3.14)

Let us pass in (3.8) to the superfields \( \hat{\psi}^{(-3)}(x, u) \) and \( \hat{G}_r^{(-3)}(x, u) \) (see (2.41)) drop out from the interaction term in (3.15) in view of the structure of \( V_{WZ}^{++} \) (3.13). As the result, the \( S^2 \) equations obtained by varying with respect to these fields are linear in this gauge

\[
\partial^{++} f_a^+ = 0 , \Rightarrow f_a^+ = \sqrt{2} \delta^a_{\theta \omega} \omega(x) u_i^+ ,
\]

\[
\partial^{++} \hat{F}^{++} = \partial^{++} \hat{g}^{++} = 0 , \Rightarrow \hat{F}^{++} = \hat{F}^{ia}(x) u_i^+ , \quad \hat{g}^{++} = \hat{g}^{ia}(x) u_i^+ ,
\] (3.16)

where \( \hat{g}^{++} = \hat{G}^{++}|_{\theta = 0} \). After substituting this solution back into (2.36) (or (2.40) and (2.37)) the problem of finding the action of physical bosons is reduced to eliminating the remainder of auxiliary fields, fixing the residual gauge freedom (3.14) by the gauge condition

\[
\hat{g}^{ia}(x) = \hat{g}^{ia}(x) , \quad (\text{or} \quad \epsilon_{ia} \hat{g}^{ia} = 0)
\] (3.17)

and employing the algebraic constraint

\[
\hat{g}^{(ik)}(x) = \frac{1}{2} \sqrt{2} |\kappa| \omega(x) c^{(ab)} \hat{F}^{i}_a(x) \hat{F}^{k}_b(x) \quad \text{or} \quad g^{(ik)}(x) = \frac{1}{2} c^{(ab)} F^{i}_a(x) F^{k}_b(x) ,
\] (3.18)

which is obtained by varying with respect to the Lagrange multiplier \( S^{(ik)}(x) \) in the component form of the action corresponding to (3.15). The fields \( \omega(x) \) and \( \nu^{(ik)}(x) \) are expressed by the general relations (2.37), (2.38):

\[
\omega = \frac{1}{\sqrt{2} |\kappa|} \sqrt{1 + \frac{\kappa^2}{2} \hat{g}^2} , \quad \nu^{(ij)} = \hat{g}^{(i)} \partial_m \hat{g}^{(j)} a .
\] (3.19)

The \( S_{QK}^{(1)} \) part of the general action (2.40) in the present case,

\[
S_{QKN}^{(1)} = \frac{1}{2} \int d\zeta^{-4} \mathcal{L}_{\text{gauge}}^{++} ,
\] (3.20)

after integration over \( \theta \)s and eliminating auxiliary fields in \( q^+, Q^+, G^+ \) becomes

\[
S_{QTN}^{(1)} = \int d^4x \left( L_{\text{kin}} + L_{\text{vect}} \right) ,
\]

\[
L_{\text{kin}} = \frac{1}{2} \partial^i \hat{F}^{ia} \partial_m \hat{F}_{ia} + \frac{1}{2} \partial^m \hat{g}^{ik} \partial_m \hat{g}_{ik} - 2 \partial^m \omega \partial_m \omega ,
\] (3.21)

\[
L_{\text{vect}} = \frac{1}{2} \left\{ \nu^m c^{(ab)} \hat{F}^{ia}_a \partial_m \hat{F}^{ib}_b + V^2 \left[ \kappa^2 \omega^2 + \frac{1}{8} \epsilon^2 \hat{F}^2 - \frac{\kappa^2}{4} \hat{g}^2 \right] \right\} ,
\] (3.22)
where
\[ c^2 \equiv c^{(ab)}c_{(ab)} = 2\lambda^2. \]

Recall that the full QTN sigma model action is given by (see (2.40))
\[ S_{QTN} = S_{QTN}^{(1)} + S_{QTN}^{(2)} = \int d^4x \left( L^{kin} + L^{vec} + \frac{\kappa^2}{2} \nu^2 \right). \tag{3.23} \]

In order to get the final physical bosons action, one should pass to
\[ g^{(ik)} = \frac{1}{\sqrt{2|\kappa|\omega}} \hat{g}^{(ik)}, \quad F^{ai} = \frac{1}{\sqrt{2|\kappa|\omega}} \hat{F}^{ai}, \]
define
\[ F^{1i} \equiv \phi^i, \quad \bar{\phi}_i = F^{1i}, \quad s = \bar{s} \equiv \phi^i \bar{\phi}_i \Rightarrow \]
substitute all this (together with (3.19)) into (3.23), and finally eliminate the non-propagating field \( V_m(x) \) by its equation of motion. After these purely algebraic manipulations the final form of the action is as follows
\[ S_{QTN} = \frac{1}{2} \int d^4x \left( g_{1i1k} \partial^m \phi^i \partial_m \phi^k + g_{2i2k} \partial^m \bar{\phi}_i \partial_m \bar{\phi}_k \right. \]
\[ + g_{1i2k} \partial^m \phi^i \partial_m \bar{\phi}_k + g_{2i1k} \partial^m \bar{\phi}_i \partial_m \phi^k \right), \tag{3.25} \]
with
\[ g_{1i1k} = \frac{A}{BC^2} \phi^i \phi^k, \quad g_{2i2k} = \frac{A}{BC^2} \bar{\phi}_i \bar{\phi}_k, \]
\[ g_{1i2k} = g_{2k1i} = \frac{1}{BC^2} \left( \epsilon_{ik} B^2 + A \bar{\phi}_i \phi_k \right), \]
\[ A = \lambda^2 \left( 1 - \frac{\kappa^2}{2} s \right) \left( 1 + \frac{\lambda^2}{2} s \right), \quad B = 1 + \lambda^2 s - \frac{\kappa^2}{4} \lambda^2 s^2, \]
\[ C = 1 - \kappa^2 \left( s + \frac{\lambda^2}{4} s^2 \right). \tag{3.26} \]

This expression for the QTN metric coincides with the one obtained in [19] (up to the sign of \( \kappa^2 \) and a normalization of \( \lambda^2 \)) and with the metric following from the HSS sigma model action with \( L^{4+} \) defined by eqs. (3.1), (3.4). The computation using the HSS quotient approach turns out much simpler as it allows to avoid the problem of solving nonlinear differential equations on the harmonic sphere \( S^2 \).

Finally, we note that the “master” gauge-invariant action corresponding to the Lagrangian (3.8) possesses global \( SU(2) \otimes U(1) \) symmetry with the Killing potentials
\[ \Lambda_{ik}^{++} = v^+_i v^+_k - \kappa^2 G^+_i G^+_k, \quad \Lambda^{++} = J^{++} \tag{3.27} \]
(these combinations of the free action symmetries are singled out by the condition of invariance of the interaction term). This symmetry just amounts to the familiar \( SU(2) \times U(1) \) isometry of the QTN metric.

\[ \text{†Presumably, it is also related, via a change of parametrization, to the QK extension of TN metric derived in [14] within the component version of the QK quotient construction.} \]
3.2 Quaternionic EH metric from the HSS quotient

After explaining the basic points of the HSS quotient construction on the TN example let us discuss another interesting example of QK metric which was not still explicitly worked out in the HSS approach. It is QK generalization of another famous four-dimensional HK metric, the Eguchi-Hanson (EH) one [38, 39]. Actually, this HK metric was explicitly constructed in the HSS approach starting just from the appropriate quotient construction [22]. A QK generalization of the “master” EH Lagrangian of ref. [22] reads [23]

\[ \mathcal{L}_{EH}^{+4} = -q_i^+ D^{++} q^{++} + \kappa^2 (u^{-q^+})^2 \left[ Q_a^+ D^{++} Q^{++} + V^{++} \left( \epsilon^{AB} Q^{++} A Q^+_a + \xi^{++} \right) \right] , \]  

(3.28)

where

\[ \xi^{++} = \xi^{(ik)} v^+_i v^+_k , \quad \left( \xi^{(ik)} \right) = \epsilon_{il} \epsilon_{kj} \xi^{(lj)} . \]  

(3.29)

It is invariant under the following SO(2) gauge transformations with the local parameter \( \varepsilon(\zeta) \):

\[ \delta Q^+_a = \varepsilon \left[ \epsilon^{AB} Q^+_a - \kappa^2 \xi^{++} Q^{++}_a \right] , \]

\[ \delta q^{++} = \varepsilon \kappa^2 \left[ \xi^{++} q^{++} - \xi^{++} (u^{-q^+}) u^{-i} \right] , \]

\[ \delta V^{++} = D^{++} \varepsilon . \]  

(3.30)

It is straightforward to see that the supercurrent to which \( V^{++} \) couples in (3.28) is just the Killing potential for the rigid subgroup of these transformations.

The passing to the physical bosons component action follows the steps A - C of Subsect. 2.2, with

\[ \mathcal{L}_{EH}^{++(1)} = -q_i^+ D^{++} q^{++} + \hat{Q}_a^+ D^{++} \hat{Q}^{++} + V^{++} \left( \epsilon^{AB} \hat{Q}^{++} A \hat{Q}^+_a + \kappa^2 \xi^{(ik)} q_i^+ q_k^+ \right) . \]  

(3.32)

We have fixed the WZ gauge (3.13) for \( V^{++} \) and redefined the hypermultiplet superfields according to (2.63).

Now one should substitute the purely bosonic \( \theta \) expansions (2.41), (3.13) for the involved superfields (with (2.42) taken into account). After integrating over \( \theta \)s, the free part of this action precisely coincides with (2.65), so after varying with respect to \( \delta^{(-3)a} \), \( \hat{G}^{(-3)a} \) (these auxiliary fields do not appear in the interaction part) one gets for \( f^{+a} , \hat{f}^{+a} \) the linear harmonic equations like in the QTN case

\[ \partial^{++} f^{+a} = 0 \Rightarrow f^{+a} = \sqrt{2} \epsilon^{ai} \omega(x) u^+_i \quad \partial^{++} \hat{f}^{+a} = 0 \Rightarrow \hat{f}^{+a} = \hat{f}^{ai}(x) u^+_i . \]  

(3.33)

The full action \( S_{EH}^{(1)} \) then reads

\[ S_{EH}^{(1)} = \int d^3x du \left\{ -A_{ma} \partial^m f^{+a} + \frac{1}{4} A_{ma} \partial^{++} A^{-ma} + \hat{B}_{ma} \partial^m \hat{f}^{+a} + \frac{1}{4} \hat{B}_{ma} \partial^{++} \hat{B}^{-ma} A + S^{(ik)} u^+_i u^-_k \left[ \hat{f}^{+a b} \hat{f}^{+A} \epsilon^{BA} + 2 \kappa^2 \omega^2 \xi^{++} \right] \right\} \]  

(3.34)
where now \( \xi^{++} = \xi^{ik} u^+_i u^+_k, \xi^{+-} = \xi^{ik} u^+_i u^-_k \). Varying it with respect to the fields \( A^-_{ma}(x, u), \dot{B}_m^A(x, u) \) allows one to expresses them in terms of the remaining fields

\[
A_m^{-a} = 2 \sqrt{2} u^{-a} \partial_m \omega + \sqrt{2} \kappa^2 \omega \xi^{(ai)} u^-_i A_m \\
\dot{B}_m^A = 2 \partial_m \dot{F}^{aA}_i u^-_i - A_m \dot{F}^{aB}_i u^-_i \epsilon^{BA}, \tag{3.35}
\]

while varying with respect to \( S^{ik}(x) \) produces the algebraic constraint

\[
\dot{F}^{Aa(i} \dot{F}^{Bj)} + 2 \kappa^2 \omega^2 \xi^{(ij)} = 0. \tag{3.36}
\]

Substituting these expressions back into (3.34), we finally get

\[
S_{EH}^{(1)} = \int d^4x \left( \frac{1}{2} \nabla_m \dot{F}^{B}_ia \nabla_m \dot{F}^{aiB} - 2 \partial_m \dot{F}^{aiB}_i - \partial_m \dot{F}^{aB}_i \epsilon^{AB} - \kappa^2 \xi^{2} A^m A_m \right), \tag{3.37}
\]

where

\[
\xi^{2} = \xi^{ik} \xi^{ik}
\]

and

\[
\nabla_m \dot{F}^{B}_ia = \partial_m \dot{F}^{B}_ia - \frac{1}{2} A^m \dot{F}^{aB}_i \epsilon^{AB}. \tag{3.38}
\]

Further, the constraint (2.37) allows one to eliminate \( \omega(x) \)

\[
\omega(x) = \frac{1}{\sqrt{2} |\kappa|} \sqrt{1 + \frac{\kappa^2}{2} (\dot{F}^{B} \dot{F}^{B})}. \tag{3.39}
\]

and from (2.38) we find \( V^{(ik)}_m \)

\[
V^{(ik)}_m = \kappa^2 \dot{F}^{B}_a(i \partial_m F^{Bak}) = \kappa^2 V^{(ik)}_m. \tag{3.40}
\]

Putting all this together, passing to \( F^{Bai} = (\sqrt{2} |\kappa| \omega)^{-1} \dot{F}^{Bai} \) and expressing the auxiliary gauge field \( A_m \) from its algebraic equation of motion

\[
\delta A_m : \quad A_m = 2 \frac{F^{aiB} \partial_m F^{B}_{ai} \epsilon^{AB}}{(F^{B} F^{B}) - \kappa^2 \xi^{2}}, \tag{3.41}
\]

we obtain the following final form of the physical bosons action

\[
S_{QEH} = S_{EH}^{(1)} + \frac{\kappa^2}{2} \int d^4x V^{(ik)}_m(x) V^{(ik)}_m(x) \]
\[
= \frac{1}{2} \int d^4x \left\{ \partial_m F^{B}_{ai} \partial^n F^{B}_{ai} + \kappa^2 \left( \frac{F^{B}_a \partial^B_b (F^{B} F^{B}) (\partial_m F^{aB}_i \partial^n F^{b}^{B})}{1 - \frac{\kappa^2}{2} (F^{B} F^{B})} \right) \right\} . \tag{3.42}
\]

The constraint (3.36) rewritten in terms of \( F^{Bai} \) should be also added

\[
F^{a(i} F^{j)}_a \epsilon^{AB} + \xi^{ij} = 0. \tag{3.43}
\]
Note that it is the same as in the HK case \cite{24,22}. In what follows, we shall frequently use the frame in which (cf. (3.3))
\[ \xi^{11} = \xi^{22} = 0 \ , \ \xi^{12} = ia \ , \ \xi^2 = 2a^2 . \] (3.44)

Eqs. (3.42) and (3.43) fully specify the QEH target metric. We shall explicitly present it in the next Section by solving (3.43) in terms of four independent target space coordinates. Note that (3.43) itself eliminates 3 out of the original 8 bosonic component. One more degree of freedom is traded for the residual \( U(1) \) gauge symmetry of the action (3.42) which survives in the WZ gauge (3.13). It is realized by the transformations
\[ \delta F_{ai}^A = \varepsilon^{AB} F_{ai}^B + \varepsilon \kappa^2 \xi^{(i)} \xi^{(k)} F_{ak}^A , \]
\[ \delta A_m = -2 \partial_m \varepsilon \ , \ \delta V_{(ik)}^{(m)} = \kappa^2 \xi^{(ik)} \partial_m \varepsilon + \kappa^2 \varepsilon \left( \xi^{(l)} V_{lm}^k + \xi^{(k)} V_{lm}^i \right) . \] (3.45)

It is easy to check that the composite gauge fields (3.40), (3.41) possess just needed transformation properties (in checking this, one should take into account the constraint (3.36) or (3.43) and the expression (3.39) for \( \omega(x) \)). The constraint (3.43) is invariant in its own right.

Note that the \( SU(2) \otimes U(1) \) isometry of QEH metric is originally realized as the rigid symmetries of the “master” gauge-invariant Lagrangian (3.28) corresponding to the following Killing potentials
\[ \Lambda_{+}^{++} = Q_{+}^{+A} Q_{+}^{+B} , \quad \Lambda^{++} = \gamma \xi^{(ik)} q_i^k q_j^k , \quad \gamma = \gamma \neq 0 . \] (3.46)

It is also worth noting that at the special relation between \( \xi^2 = 2a^2 \) and Einstein constant \( \kappa \) there emerges an enhancement of this isometry up to \( SU(2,1) \) (or \( SU(3) \), depending on the sign of \( \kappa^2 a \)). To see this, it is convenient to deal with \( \hat{Q}_{+}^{+A} \). The supercurrent to which \( V^{++} \) couples in (3.28) exhibits an extra rigid symmetry
\[ \delta \hat{Q}_{+}^{+A} = \varepsilon^{+A} q_i^k \xi_{ik} , \quad \delta q_i^+ = \frac{1}{\kappa^2} \varepsilon_i^{+A} Q^{+ab} \varepsilon^{AB} . \] (3.47)

However, the kinetic part of (3.28) is invariant only providing that the additional “self-duality” constraint is imposed on the transformation parameters:
\[ \varepsilon^{AB} \varepsilon_{ij} = \kappa^2 \xi_{ij} \varepsilon^{+jA} . \] (3.48)

It leaves just 4 independent parameters in \( \varepsilon_{ij}^{+AB} \). For its self-consistency one should require
\[ \frac{1}{2} \kappa^4 \xi^2 = \kappa^4 a^2 = 1 \quad \Rightarrow \kappa^2 a = \pm 1 . \] (3.49)

It can be shown that under (3.48), (3.49) the transformations (3.47) close on \( SU(2) \times U(1) \) generated by the Killing potentials (3.46) and so, together with the latter, they form the \( SU(2,1) \) (or \( SU(3) \), depending on the sign of \( \kappa^2 a \)) isometry group. In other words, in this case four physical bosons parametrize the homogeneous quaternionic space \( SU(2,1)/SU(2) \otimes U(1) \) (or \( SU(3)/SU(2) \otimes U(1) \sim \mathbb{C}P^2 \)). This enhancement of isometry is specific just for the QK case: it does not allow HK limit in view of the presence of inverse powers of \( \kappa^2 \) in (3.47) \footnote{Actually, a similar enhancement of the isometry group to \( SU(2,1) \) or \( SU(3) \) emerges also in the QTN case, once again at the special relation between \( \kappa^2 \) and the TN “mass” parameter \cite{23,19}.}. 

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Finally, we recall that in the HK case \cite{22} the HSS action corresponding to EH metric can be concisely written in terms of one hypermultiplet superfield which comprises just 4 independent physical bosons parametrizing the target EH manifold. The same can be done in the QEH case. The relevant action looks rather complicated due to the presence of non-trivial dependence on the compensator superfield $q_i^+$. Nevertheless, such a representation is useful for understanding one important feature of the resulting QEH metric. Namely, this metric (see next Section) involves as an independent parameter not $a$ defined in (3.44), but its square $a^2$. The substitution $a^2 \to -a^2$ also yields an admissible QEH type metric. However, the latter cannot be reproduced using the $SO(2)$ quotient; to recover it within the quotient construction, one should start with $SO(1,1)$ as the gauge group. We explain this in Appendix B, specializing for simplicity to the HK case. The same reasoning equally applies to the QK case modulo $\kappa^2$ corrections.

4 Geometrical structure of the QEH metric

4.1 Coordinates choice and local structure

The next step in our analysis will be to solve the constraint (3.43) and to get from (3.42) the metric for some definite choice of coordinates.

To this end we choose for the $SU(2)_{\text{susy}}$ symmetry breaking triplet the form (3.44)

$$\xi^{11} = \xi^{22} = 0, \quad \xi^{12} = i a.$$ \hspace{1cm} (4.1)

The reality constraint (3.29) on the triplet $\xi^{ij}$ implies that the parameter $a$ has to be real. The constrained coordinates $F^{aiA}_{A=1}$ and $F^{aiA}_{A=2}$ are parametrized in terms of the Pauli-Gursey spinors, adapted to the expected $SU(2)_{\text{PG}}$ invariance, according to

$$F^{aiA}_{A=1} = \alpha(t) g^a, \quad F^{aiA}_{A=2} = i \beta(t) g^a, \quad t = g^a \varepsilon_a \geq 0,$$ \hspace{1cm} (4.2)

with real functions $\alpha(t)$ and $\beta(t)$. The remaining components follow from the reality conditions (2.4). As a consequence

$$F^{aiB} \epsilon_{ab} F^{Bj} + \xi^{(ij)} = 0 \quad \Rightarrow \quad t \alpha \beta = a/2.$$ \hspace{1cm} (4.3)

The scalar functions $A$ and $D$ defined by \cite{fg},

$$A \equiv 1 - \frac{\kappa^2}{2} (F^B F_B), \quad D \equiv (F^B F_B) - \kappa^2 \xi^2,$$ \hspace{1cm} (4.4)

become now

$$A = 1 - \kappa^2 t(\alpha^2 + \beta^2), \quad D = 2 t(\alpha^2 + \beta^2) - 2 \kappa^2 a^2.$$

To express the metric it is convenient to use the basis of one-forms such that

$$g^a d\varepsilon_a = \frac{1}{2} (dt - it \sigma_3), \quad g^a d\sigma_a = \frac{t}{2} (\sigma_2 + i \sigma_1),$$

$$d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k.$$ \hspace{1cm} (4.5)

\*We hope that the use of the same characters to denote the structure functions appearing in the QTN (eqs. (3.26)) and QEH metrics will not lead to any confusion.
For $\kappa^2 = 0$, the identification of the resulting metric with Eguchi-Hanson’s [40] requires the relation $t(\alpha^2 + \beta^2) = \sqrt{t^2 + a^2}$, which in turn implies

$$\alpha^2 = \frac{1}{2} \left( \sqrt{1 + \frac{a^2}{t^2}} - 1 \right), \quad \beta^2 = \frac{1}{2} \left( \sqrt{1 + \frac{a^2}{t^2}} + 1 \right).$$ \hspace{1cm} (4.6)

The same choice for $\kappa^2 \neq 0$ and the change of variable $s = \sqrt{t^2 + a^2}$ leads to the following local form of the metric

$$g = \frac{1}{4C^2} \left[ \frac{sB}{s^2 - a^2} ds^2 + sB (\sigma_1^2 + \sigma_2^2) + \frac{s^2 - a^2}{sB} \sigma_3^2 \right],$$ \hspace{1cm} (4.7)

where

$$C = 1 - \kappa^2 s, \quad sB = \frac{1}{2} D = s - \kappa^2 a^2.$$ \hspace{1cm} (4.8)

We see that the metric includes the real parameter $a^2$, and not $a$, in accord with the reasoning adduced at the end of the previous Section and in Appendix B. The choice of $a^2 < 0$ is equally admissible, it also yields a QK metric. The same is of course true for $\kappa^2$ which needs just to be real.

Using this explicit form of the metric we can check its geometrical structure. It is convenient to use the vierbein basis for which we have

$$g = \sum_{A=0}^{A=3} e_A^2.$$

To this end we take

$$e_0 = \frac{1}{2C} \sqrt{\frac{sB}{s^2 - a^2}} ds, \quad e_3 = \frac{1}{2C} \sqrt{\frac{s^2 - a^2}{sB}} \sigma_3, \quad e_1 = \frac{\sqrt{sB}}{2C} \sigma_1, \quad e_2 = \frac{\sqrt{sB}}{2C} \sigma_2.$$ \hspace{1cm} (4.9)

The spin connection $\omega_{AB}$ and the curvature $R_{AB}$ are defined [38] by

$$de_A + \omega_{AB} \wedge e_B = 0, \quad R_{AB} = d\omega_{AB} + \omega_{AC} \wedge \omega_{CB}, \quad A, B, C = 0, 1, 2, 3.$$ \hspace{1cm} (4.10)

The self-dual components are

$$R^\pm_i = R_{0i} \pm \frac{1}{2} \epsilon_{ijk} R_{jk}, \quad \lambda^\pm_i = e_0 \wedge e_i - \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k, \quad i, j, k = 1, 2, 3,$$

and similarly for the Weyl tensor. One then defines the matrices $A, B$ and $C$ by

$$\begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix},$$ \hspace{1cm} (4.10)

from which we deduce the scalar curvature and Weyl tensor by

$$\frac{R}{4} = \Lambda = \text{tr} A = \text{tr} C, \quad W^+ = A - \frac{\text{tr} A}{3} \mathbb{I}, \quad W^- = C - \frac{\text{tr} C}{3} \mathbb{I}.$$ \hspace{1cm} (4.11)

Easy computations give

$$B = 0, \quad \Lambda = -12\kappa^2, \quad W^+ = 0, \quad W^- = -4a^2 \frac{C^2}{(sB)^3} \text{diag} (1, 1, -2).$$ \hspace{1cm} (4.11)

The vanishing of $B$ implies [38] that the metric (4.8) is Einstein, and its Weyl tensor is anti-self-dual, as expected.
4.2 The hyperbolic monopole structure

Obviously, when the Einstein constant $\kappa^2 \to 0$, the metric (4.17) reduces to Eguchi-Hanson’s [39] which is a particular case of the multicentre metrics [38]. They can be written

$$\frac{1}{V} (d\tau + A)^2 + V h ,$$

(4.12)

where $A$ is some connection 1-form, $h$ the flat 3-space metric while $V$ and $A$ are related by the monopole equation

$$\ast_h dV = dA.$$

As shown by Pedersen in [30], as far as $a^2 > 0$, the monopole structure, properly generalized, survives at the level of the quaternionic extension (4.17). Indeed it can be written, using angular coordinates, as follows

$$\frac{1}{4C^2} \left\{ s^2 - a^2 (d\psi + \sin \theta d\phi)^2 + sB \left[ \frac{ds^2}{s^2 - a^2} + d\theta^2 + \sin \theta^2 d\phi^2 \right] \right\}.$$

The substitution $s/a = \coth \rho$ gives

$$\frac{a}{4(\sinh \rho - \kappa^2 a \cosh \rho)^2} \left\{ \frac{1}{V} (d\psi + A)^2 + V \gamma(\mathbb{H}_3) \right\},
\text{ (4.13)}$$

with

$$V = \coth \rho - \kappa^2 a , \quad A = \cos \theta d\phi , \quad \gamma(\mathbb{H}_3) = d\rho^2 + \sinh^2 \rho \left( d\theta^2 + \sin \theta^2 d\phi^2 \right),$$

(4.14)

where $\gamma(\mathbb{H}_3)$ is the metric of hyperbolic 3-space. The monopole equation reads now

$$\ast_\gamma dV = dA .$$

(4.15)

The quaternionic extension of Eguchi-Hanson retains some structure from its hyperkähler origin through the monopole equation, the main difference being that flat 3-space is replaced by hyperbolic 3-space.

The situation is somewhat different for $a^2 < 0$ since then we have to use the change of variables $s/a = 1/\sinh \rho$ to recover the hyperbolic 3-space. The metric can be written as

$$\frac{a \cosh^2 \rho}{4(\sinh \rho - \kappa^2 a)^2} \left\{ \frac{1}{V} (d\psi + A)^2 + V \tilde{\gamma}(\mathbb{H}_3) \right\},$$

(4.16)

with

$$V = \frac{1}{\sinh \rho} + \kappa^2 a , \quad A = \cos \theta d\phi , \quad \tilde{\gamma} = \frac{\gamma(\mathbb{H}_3)}{\cosh^2 \rho} .$$

(4.17)

The monopole equation is now

$$\ast_{\tilde{\gamma}} dV = dA .$$

(4.18)
4.3 Global structure: the complete metrics

Before considering generic values of the real parameters $\kappa^2$ and $a^2$, let us examine separately some special cases.

First, for $a^2 \to 0$ the metric (4.7) simplifies to

$$g = \frac{1}{(1 - \kappa^2 s)^2} \left[ \frac{ds^2}{4s} + \frac{s}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right],$$

in which we recognize the conformally flat structure of the sphere $S^4$ for $\kappa^2 > 0$, of the flat space for $\kappa^2 = 0$, and of the hyperbolic space $H^4$ for $\kappa^2 < 0$.

Second, when $sB$ and $A$ are proportional to each other, the Riemann tensor in the vierbein basis has constant components, showing that we deal with a symmetric space. This can happen only for $a^2 > 0$.

Defining $\kappa^2 a = \epsilon = \pm 1$ gives $sB = -\epsilon a C$. In this case the 2-form

$$\Omega^+ = e_0 \wedge e_3 + e_1 \wedge e_2 \equiv \frac{1}{2}(I^+)_\mu \nu dx^\mu \wedge dx^\nu \quad (4.19)$$

is closed. One can check that $(I^+)_\mu \nu$ defines a true complex structure and therefore the metric (4.7) is also kählerian. It must be either $\mathbb{CP}^2 \sim SU(3)/U(2)$ or its complete (albeit non-compact) dual $\tilde{\mathbb{CP}}^2 \sim SU(2,1)/U(2)$. To check this at the level of the metric we write it in the form

$$\frac{a^2}{4(s - \epsilon a)^2} \left\{ \frac{ds^2}{s + \epsilon a} + (s - \epsilon a)(\sigma_1^2 + \sigma_2^2) + (s + \epsilon a)\sigma_3^2 \right\}. \quad (4.20)$$

Simplifying the $a^2$ factor and the change of variable

$$\frac{1}{s - \epsilon a} = \frac{t^2}{1 + \lambda t^2/6}, \quad \lambda/6 = -2\epsilon a,$$  

bring the metric (4.11) to the form

$$\frac{dt^2}{(1 + \lambda t^2/6)^2} + \frac{t^2}{1 + \lambda t^2/6}(\sigma_1^2 + \sigma_2^2) + \frac{t^2}{(1 + \lambda t^2/6)^2}\sigma_3^2, \quad (4.21)$$

which is the standard form [38, p. 384] of $\mathbb{CP}^2$ for $\epsilon = -1$ and its non-compact dual for $\epsilon = +1$. These choices just amount to the isometry enhancement relation (3.49).

From now on we will exclude the previous special cases from the analysis and examine the global properties of the QEH metric

$$\frac{1}{4(1 - \kappa^2 s)^2} \left\{ \frac{s - \kappa^2 a^2}{s^2 - a^2} ds^2 + (s - \kappa^2 a^2)(\sigma_1^2 + \sigma_2^2) + \frac{s^2 - a^2}{s - \kappa^2 a^2}\sigma_3^2 \right\}. \quad (4.22)$$

As we will see there are several complete (but non-compact) metrics which we now enumerate:

- First case: $\kappa^2 < s < \infty$. Let us first observe that the metric positivity requires, for $a^2 > 0$, the additional constraint $\kappa^2 a < 1$. For large $s$ the metric is proportional to

$$g \approx d\tau^2 + \frac{\tau^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad \tau = 1/\sqrt{s},$$
a typical nut behavior in the terminology of [41]. This means that \( \tau = 0 \) is an apparent singularity which can be removed by going back to cartesian coordinates and the metric is smooth there.

The situation is fairly different at \( s = 1/\kappa^2 \) where the metric behaves like

\[
\frac{(1 - \kappa^4 a^2)}{4\kappa^2(1 - \kappa^2 s)^2} \left\{ \frac{\kappa^4}{1 - \kappa^4 a^2} d\tau^2 + \sigma_1^2 + \sigma_2^2 + \frac{\sigma_3^2}{1 - \kappa^4 a^2} \right\}.
\]

(4.23)

However this singularity is “infinitely far” since we have

\[
\int^{1/\kappa^2} \frac{ds}{1 - \kappa^2 s} = \infty,
\]

so it does not jeopardize completeness. We recognize as conformal structure, for fixed \( s \), the Berger (or squashed) sphere \( S^3 \) with the metric

\[
\sigma_1^2 + \sigma_2^2 + I \sigma_3^2, \quad I = \frac{1}{1 - \kappa^4 a^2},
\]

(4.24)

where the constant \( I \) is bigger or smaller than 1 according to the sign of \( a^2 \). If we set

\[
\rho^2 = \frac{1}{\kappa^2 s} \in [0, 1), \quad m^2 = -\kappa^4 a^2,
\]

the metric (4.22) becomes

\[
\frac{1}{\kappa^2 (1 - \rho^2)^2} \left\{ \frac{1 + m^2 \rho^2}{1 + m^2 \rho^4} d\rho^2 + \rho^2 (1 + m^2 \rho^2) \frac{\sigma_1^2 + \sigma_2^2}{4} + \rho^2 \frac{1 + m^2 \rho^4}{4} \sigma_3^2 \right\},
\]

(4.25)

which is, for \( \kappa^2 > 0 \), and for any real \( a^2 \), the complete metric given by Pedersen in [30].

- Second case : \( a \leq s < 1/\kappa^2 \).

This complete metric does exist only for \( a^2 > 0 \). Positivity requires the additional constraint \( \kappa^2 a < 1 \).

Near to the singularity \( s = a \) the metric behaves as

\[
g \approx \frac{1}{2(1 - \kappa^2 a)} \left\{ d\tau^2 + \frac{\tau^2}{(1 - \kappa^2 a)^2} \sigma_3^2 + \frac{a}{2} (\sigma_1^2 + \sigma_2^2) \right\}, \quad \tau = \sqrt{s - a}.
\]

(4.26)

Using angular coordinates, one has

\[
\sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad \sigma_3 = d\psi + \cos \theta d\phi, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad \psi \in [0, 4\pi].
\]

So we can impose the constraint

\[
\frac{1}{1 - \kappa^2 a} = k, \quad k = 2, 3, \ldots \quad \Rightarrow \quad \kappa^2 a = \frac{k - 1}{k} < 1,
\]

(4.27)

and take for \( \psi \) the interval of variation \([0, 4\pi/k]\). Then the metric can be smoothly continued through \( s = a \) which is a bolt [41] of twist \( k \).
Locally, the boundary at \( s = 1/\kappa^2 \) has the same structure (Berger metric) as in the first case considered previously. However, globally, the identification of \( \psi \) under the action of \( \mathbb{Z}_k \) gives for the conformal structure the Lens space \( S^3/\mathbb{Z}_k \).

Let us point out that our conclusions are in complete agreement with the full classification of self-dual Einstein metrics given by Hitchin in [25]. One will find in [27] an interesting discussion of various Einstein extensions of Eguchi-Hanson’s metrics in different coordinates. Beyond the quaternionic and the non-quaturnionic ones there is a Kähler-Einstein extension, first given in [42], whose completeness was discussed in [29].

### 4.4 Kähler metrics in the conformal class

It was first proved in [31] that the conformal class of Pedersen metric contains LeBrun metric, which can be written

\[
\frac{dr^2}{W} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4}W\sigma_3^2, \quad W = 1 + \frac{A}{r^2} + \frac{B}{r^4}.
\]

(4.28)

It is Kähler and scalar-flat, and for \( A = 0 \) it reduces to Eguchi-Hanson. LeBrun considered the special choice of parameters

\[
A = (n - 1), \quad B = -n, \quad n = 0, 1, 2, \ldots \quad r \geq 1,
\]

(4.29)

for which he proved completeness.

It follows that the QEH metric considered here should be (locally) conformal to (4.28). Imposing the Kähler constraint gives for the conformal factor \( \rho = C^2 \), and for the resulting metric

\[
\hat{g} = \frac{(s - \kappa^2 a^2)}{4(s^2 - a^2)}ds^2 + \frac{(s - \kappa^2 a^2)}{4}(\sigma_1^2 + \sigma_2^2) + \frac{(s^2 - a^2)}{4(s - \kappa^2 a^2)}\sigma_3^2.
\]

(4.30)

Its identification with the LeBrun metric is performed through the relations

\[
r^2 = s - \kappa^2 a^2, \quad W = \frac{s^2 - a^2}{(s - \kappa^2 a^2)^2}, \quad A = 2\kappa^2 a^2, \quad B = a^2(\kappa^4 a^2 - 1).
\]

(4.31)

An important observation is that, in our notations, its Weyl tensor which is self-dual, \( W^- = 0 \), and its complex structure given by

\[
\Omega = ds \wedge \sigma_3 - sB \sigma_1 \wedge \sigma_2,
\]

have the opposite orientation. However, in the same conformal class, we can find another Kähler metric, whose complex structure and Weyl tensor have the same orientation. A local computation gives for it

\[
\frac{dr^2}{V} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4}V\sigma_3^2, \quad V = \frac{1}{r^4} + \frac{A}{r^2} + B,
\]

(4.32)

that is indeed a close cousin of (4.28)! This shows that the LeBrun metric with \( B = 1 \) has two commuting complex structures of opposite self-duality and is scalar-flat. Its new partner is still Kähler but it has a non-constant scalar curvature \( R = -8(B - 1)/r^2 \).

Complete metrics are given, for instance, by

\[
A = -n - 2, \quad B = n + 1, \quad r \geq 1, \quad n = 1, 2, \ldots
\]

(4.33)

where \( r = 1 \) is a bolt of twist \( n \) and the metric is locally asymptotically flat at infinity. The scalar curvature is negative.
5 Conclusions

In this paper, starting from the most general HSS action of the coupled $N = 2$ SG-hypermultiplets system, we derived the most general bosonic QK sigma model action which is a useful tool for the explicit local construction of the metric on an arbitrary $4n$ dimensional QK manifold. We found that this action admits two flat limits. One of them yields a general HK sigma model action with $4n$ dimensional target, while the second gives rise to the HK sigma model with $4(n + 1)$ dimensional target space corresponding to the general superconformally-invariant self-coupling of $(n + 1)$ rigid hypermultiplets. We worked out the HSS version of the QK quotient approach and applied it to give a new derivation of the QK extensions of the four-dimensional Taub-NUT and Eguchi-Hanson metrics. We studied in detail local and global properties of the QEH metric and compared it with various examples of self-dual Einstein metrics known in literature. We showed that the HSS formulation allows one to readily reveal the enhancement of the $SU(2) \otimes U(1)$ isometry of the QEH metric to $SU(3)$ or $SU(1, 2)$ at the special ratio of its “mass” parameter and the Einstein constant $\kappa^2$ (or $Sp(1)$ curvature).

As possible directions of further study let us mention the explicit construction of metrics on QK analogs of the higher-dimensional toric HK manifolds, as well as four-dimensional QK metrics properly generalizing the multicentre ansatz for HK instantons [13]. The HK metrics to be generalized have a nice description in the HSS approach, both within the purely geometric setting of ref. [1] and in the lagrangian framework of refs. [8, 17, 18]. It is interesting to see how the relevant HSS actions are extended to the QK case, and how unique such extensions are. Being armed with the general HSS action for QK sigma models and the appropriate convenient quotient construction, we expect these problems to be amenable for solving. An interesting separate problem is to find out possible relationships of the QK sigma models in the HSS formulation with the brane-like solutions of higher-dimensional supergravities [44], as well as with the theory of intersecting branes (e.g., along the lines of ref. [3]).

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Appendix A

Here we show that the fields $A_{\mu \bar{\alpha}}$ and $t^{(\mu \nu)}$ defined in (2.27) do not contribute to the structure of the QK sigma model action of the fields $F^{ri}(x)$.

It is clear from the $\theta$ expansion (2.41) that $t^{(\mu \nu)}$ does not appear in the bosonic part of the action (2.7). The same is true for the real part of $A^{\mu \bar{\alpha}}$: it is the gauge field for the local $\gamma_5$ transformations and so cannot couple to $f_{ai}(x), F^{ri}(x)$ which are $\gamma_5$-neutral.
As for the imaginary part of $A^\mu$, the gauge field for the local scale transformations, its decoupling is not immediately obvious from (2.27), (2.28). The direct computation yields the following general expression for the relevant piece in the bosonic lagrangian (the precise normalization of $S^m \sim \text{Im}A^m$ is not crucial for our purposes)

$$L_S = S^m(x) J_m(x) ,$$

$$J_m(x) = \int du \left( - f_a^+ (x, u) A_m^a(x, u) + \kappa^2 (u^+ f^+)^2 F_r^+ (x, u) B_m^r (x, u) \right) . \quad (A.1)$$

However, $L_S \sim S_m S^m$ on the shell of eqs. (2.49), (2.50), (2.37), so $S^m$ decouples as well, in accord with the fact that the fields $F^{ri}(x)$ have zero conformal weight. This can be shown in the case of arbitrary $L^+Q, v^+, u^-$. For simplicity, just to give the idea of the proof, we specialize to the “maximally flat” $Sp(1, n)/Sp(1) \otimes Sp(n)$ case with $L^+ = 0$.

It is convenient to deal with the superfields $\hat{Q}^+ = |\kappa| (u^- q^+) Q^+$. The current $J_m(x)$ equals to

$$J_m(x) = \int du \left( - f_a^+ A_m^a + \hat{F}_r^+ \hat{B}_m^r \right) . \quad (A.2)$$

Using identities

$$f^+ = \partial^0 f^a = [\partial^{+\alpha}, \partial^{--}] f^a , \quad \hat{F}_r^{+\alpha} = \partial^0 \hat{F}_r^{+\alpha} = [\partial^{+\alpha}, \partial^{--}] \hat{F}_r^{+\alpha} ,$$

and integrating by parts, one can rewrite (A.2) as

$$J_m(x) = \int du \left( \partial^{--} f^+_a \partial^{++} A_m^a - \partial^{++} f^+_a \partial^{--} A_m^a \right.
\left. - \partial^{--} \hat{F}_r^+ \partial^{++} \hat{B}_m^r + \partial^{++} \hat{F}_r^+ \partial^{--} \hat{B}_m^r \right) . \quad (A.3)$$

Further, eqs. (2.49), (2.50) for $f^+_a, \hat{F}_r^{+\alpha}$ and the non-dynamical harmonic equations of motion for $A_m^a, \hat{B}_m^r$ read in this case

$$\partial^{++} f^+_a = \partial^{++} \hat{F}_r^{+\alpha} = 0 ,$$
$$\partial^{++} A_m^a = 2 (\partial_m - S_m) f^+_a , \quad \partial^{++} \hat{B}_m^r = 2 (\partial_m - S_m) \hat{F}_r^{+\alpha} . \quad (A.4)$$

Using them, one reduces (A.3) to

$$J_m(x) = (\partial_m - S_m) \int du \left( \partial^{--} f^+_a \partial^{--} f^+_a - \partial^{--} \hat{F}_r^{+\alpha} \partial^{--} \hat{F}_r^{+\alpha} \right) . \quad (A.5)$$

The expression on which $(\partial_m - S_m)$ acts is just the l.h.s. of the constraint (2.37), so

$$J_m(x) = - \frac{1}{\kappa^2} S_m(x) \quad \Rightarrow \quad L_S = - \frac{1}{\kappa^2} S_m(x) S^m(x) . \quad (A.6)$$

It is straightforward, though somewhat tiresome, to show that in the general case $J_m(x)$ and $L_S$ are given by the same expressions (A.5), (A.6). One should use the general $f^+, F^+$ equations (2.49), (2.50) and the corresponding harmonic equations for $A_m^a, \hat{B}_m^r$. 

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Appendix B

Here, on the simple example of the standard hyper-Kähler EH metric in the HSS approach [22], we explain the origin of the freedom in changing the sign of the EH “mass” parameter $a^2$. The same reasoning applies to the QEH case.

In the HK limit the QEH Lagrangian (3.28) becomes [22]

$$L^{++}_{EH} = Q^+_a D^{++} Q^{+aB} + V^{++} \left( \epsilon^{AB} Q^{+aA} Q^+_a + \xi^{(ik)} u^+_i u^+_k \right). \quad (B.1)$$

Passing to the complex combinations

$$Q^{+}_a = Q^{+1}_a + i Q^{+2}_a , \quad Q^+_a = Q^{+1}_a - i Q^{+2}_a ,$$

making the field redefinition

$$Q^{+}_a = u^{+_a} \frac{1}{\sqrt{2}} \omega + u^{-_a} L^{++} , \quad \bar{Q}^{+}_a = u^{+_a} \frac{1}{\sqrt{2}} \bar{\omega} + u^{-_a} \bar{L}^{++} ,$$

and, finally, eliminating the non-propagating superfields $L^{++}, \bar{L}^{++}$ from (B.1) by their algebraic equations of motion, we equivalently rewrite (B.1) as

$$L^{++}_{EH} = -\frac{1}{2} \left( D^{++} - i V^{++} \right) \omega \left( D^{++} + i V^{++} \right) \bar{\omega} + V^{++} \xi^{++} . \quad (B.2)$$

The initial gauge group now acts as $U(1)$ phase transformations of $\omega, \bar{\omega}$. Choosing the gauge

$$\omega = \bar{\omega} ,$$

and eliminating $V^{++}$ by its equation of motion

$$V^{++} = \frac{\xi^{++}}{\omega^2} ,$$

we obtain, for the HSS lagrangian corresponding to EH metric, the representation in terms of one hypermultiplet $\omega$ [22]

$$L^{++}_{EH} = -\frac{1}{2} \left( (D^{++}\omega)^2 - \frac{(\xi^{++})^2}{\omega^4} \right) . \quad (B.3)$$

From this form of the action it immediately follows that the relevant HK metric in the fixed frame (3.44) can depend only on $a^2$. Further, the action obtained from (B.3) by changing the sign of the “potential” term, or, equivalently, by the substitution $a^2 \to -a^2$, also yields some HK metric in the bosonic sector. These two metrics are related via the replacement $a^2 \to -a^2$. However, the second metric cannot be recovered from the above $SO(2)$ quotient construction. Indeed, this could be achieved only at cost of the change $a \to ia$, which would ruin the reality of the actions (B.1), (B.2).

Nevertheless, (B.3) with the “wrong” sign of the second term can be inferred from a proper quotient construction. One should start from the gauge-invariant action of two real $\omega$ hypermultiplets with $SO(1,1)$ as the gauge group:

$$L^{++}_{EH}' = -\frac{1}{2} \left( (D^{++} - V^{++}) \omega_1 (D^{++} + V^{++}) \omega_2 + V^{++} \xi^{++} \right) . \quad (B.4)$$
Here $\xi^{++}$ has the same reality properties as in the previous case, $\omega_{1,2}$ undergo scale transformations with the real analytic parameter $\varepsilon(\zeta)$:

$$
\delta \omega_1 = \varepsilon \omega_1, \quad \delta \omega_2 = -\varepsilon \omega_2 .
$$

(B.5)

The gauge superfield $V^{++}$ transforms in the same way as in the $SO(2)$ case, $\delta V^{++} = D^{++}\varepsilon$. Choosing the gauge

$$
\omega_1 = \omega_2 \equiv \omega
$$

and eliminating $V^{++}$,

$$
V^{++} = -\frac{\xi^{++}}{\omega^2} ,
$$

one obtains the desirable lagrangian

$$
L^{++}_{EH} = -\frac{1}{2} \left[ (D^{++}\omega)^2 + \frac{(\xi^{++})^2}{\omega^2} \right] .
$$

(B.6)

related to (B.3) just through the change $a^2 \to -a^2$.

A similar mechanism works in the QK case, demonstrating the existence of the two types of QEH metrics related by the same change of the parameter $a^2$. The QK analogs of the actions (B.3), (B.6) look rather complicated as they involve extra terms containing the compensator superfield $q^+$. However, the $SU(2)$ breaking parameters still appear as $(\xi^{++})^2 \sim a^2$.

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