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Status of the Clausius inequality in classical thermodynamics

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Abstract. We present an analysis of the foundations of the well known Clausius inequality. It is shown that, strictly speaking, the inequality is not a logical consequence of the Kelvin-Planck formulation of the second law of thermodynamics. Some thought experiments demonstrating the violation of the Clausius inequality are considered. Also, a reformulation of the Landauer’s principle in terms of the Clausius inequality is proposed. This version of the inequality may be considered a consequence of the fluctuation theorem.

Keywords: Landauer’s principle · Clausius inequality · Szilard engine

I. INTRODUCTION

This is a paper in classical thermodynamics. So, it is uncommon: classical (or macroscopic) thermodynamics, unlike statistical physics, is rarely a subject of a research paper today. Taking into account the treatment of the Clausius inequality, it must be more than uncommon. To avoid misunderstanding, the author has to say that to refute the inequality (or to prove it) is not a goal of this paper. The goal is to disclose a gap in the foundations of classical thermodynamics and to exploit it.

The paper consists of two parts. In the first one (Sec. II), we show that a hypothetical violation of the Clausius inequality does not contradict the basic principles of thermodynamics, like the second law. In other words, to test the Clausius inequality and to try to invent a perpetuum mobile of the second kind is not the same. The second part (Secs. III, IV) is devoted to another problems related to the Landauer’s principle. Despite the appearance,
these two parts are closely related. One of the reasons is that a hypothetical violation of the Clausius inequality is considered in both, although in different context. The summary is given below.

A. The Clausius inequality in macroscopic thermodynamics

We show in II A that the Clausius inequality is based on an implicit assumption about a process, which does not follow from any of the known principles. We call this assumption *environment independence*. Environment dependence is not exactly a new concept in thermodynamics. The general idea is simple: while a system undergoes a cycle, the environment is changed, the change should be taken into account to get the thermodynamics right. Bennett in [1] used essentially the same argument to resolve the Maxwell’s demon paradox, although in this case it is convenient to put it in terms of information.

If we do not assume environment independence, then a violation of the Clausius inequality does not contradict the laws of thermodynamics. It can be explained by *adiabatic entropy transfer* between thermodynamic systems (II C,D). A proper definition of Clausius entropy in this case is discussed in II D. Thermodynamics of an imaginary system capable of violating the Clausius inequality is considered in II B and II E. Whether it can be violated in a real process (by a macroscopic system) is a difficult question which is not addressed here.

B. The Clausius inequality in information thermodynamics

We call information thermodynamics the area of research related to the Landauer’s principle, Maxwell’s demon, and similar topics. It is almost obvious that the Clausius inequality *can* be violated in this context. Still, this fact does not attract much attention. The reason is, the only methods used in information thermodynamics today are the methods of statistical physics. In statistical physics, the inequality does not play a significant role, and it is possible to ignore its violation.

On the other hand, the Clausius inequality plays a fundamental role in classical thermodynamics, as a basis of Clausius entropy. The methods of classical thermodynamics can certainly be applied to Maxwell’s demon, to erasure of information etc., although not in a straightforward way. However, the decade old paper of Ishioka and Fuchikami [10] is the only attempt known to the author. So, he decided to make the second attempt.

The proposed theory is *nonstandard thermodynamics*, which is an extension of familiar (standard) classical thermodynamics. The formalism is explained in III B and the principles are discussed in Sec. IV. In terms of nonstandard thermodynamics, the Landauer’s principle has a natural interpretation as a generalization of the Clausius inequality (III C). This general inequality is proved in III D by means of Hamiltonian dynamics, as a consequence of the Crooks fluctuation theorem. (This is the only part of the paper where methods of statistical physics are employed. A more “classical” proof
is sketched in IV C.) The theory is applied to erasure of information (III E),
to the magnetization reversal (III F), and to the Szilard engine (IV B,E).

There are some new results: a formula for the area of the hysteresis loop
(18), another formula for the dispersion of work (23), as well as a specific
version of the Clausius inequality (17). (A similar but different inequality
was found by Sagawa and Ueda [21, Eq.(3)].) But the main goal was not
to obtain new results, it was to bridge the gap between modern statistical
mechanics and old-fashioned classical thermodynamics. So, we mostly use the
thermodynamic methods instead of traditional methods of statistical physics.
Technically, it means quite a different logic, but from a more pragmatic view-
point the main difference from the conventional approach is in the notion of
a process (discussed in III B).

II. THE STATUS OF THE CLAUSIUS INEQUALITY

A. The Clausius inequality and the second law

The Clausius inequality is a well known statement of classical thermodynam-
ics. Consider a system undergoing a cyclic process in contact with a heat bath
or with a sequence of baths, one at a time. The Clausius inequality gives an
upper bound for integrated reduced heat

$$\oint \frac{dQ}{T} \leq 0.$$  

Here $dQ$ is heat taken by the system from a bath at absolute temperature
$T$. (Note that in general the system itself is not supposed to have a well
deﬁned temperature.) The inequality was named after Rudolf Clausius, who
introduced it in the famous 1865 paper [4].

One can prove the inequality either by means of classical thermodynamics
or statistical physics. But in this section we consider thermodynamic methods
only. We follow the argument of Clausius, which may be found in many
textbooks [8][9]. (There is an alternative argument made by Caratheodory
[3]. It is mentioned in II D, but in general the Caratheodory’s method is too
narrow in scope for our purpose. We do not consider quasistatic processes
only.)

It is enough to consider a system undergoing a cyclic process in contact
with a single heat bath. In this case, the Clausius inequality is simply $Q \leq 0$,
where $Q$ is heat taken from the bath in a cycle. By the first law, $W = -Q$,
where $W$ is work done on the system. If $Q > 0$, then (positive) heat is
taken from a single bath and converted to work in a cycle. It looks like a
contradiction to the second law, but actually it is not yet.

The Kelvin-Planck formulation of the second law states that it is not possible
to take heat from a single heat bath and convert it to work in a cyclic
process. But a “cyclic process” in this statement is not a cyclic process we
have in mind. It is a process whose only net result is to take heat from a bath
and convert it to work [8][9]. That is, not just one system, but each system
(except for the bath) is supposed to undergo a cycle. Usually, competent
authors of textbooks point this out.
This ambiguity in the meaning of the term “cycle” is a potent source of confusion. To avoid it, we use the term *global process* when referring to all the thermodynamic systems, excluding the heat baths. A *process* or *local process* is always related to a single system. (Which may be a bath.) In this terminology, the Clausius inequality is about local cycles while the second law is about global ones. It makes a difference.

To complete the argument, we have to assume that it is possible to make the system undergo the process in such a way that all the environment, with the exception of a heat bath, remains unchanged. Call a process *environment independent* if this assumption is true and *environment dependent* otherwise. When a system $S$ undergoes an environment dependent cycle, some other system $S'$ undergoes a “parallel” process, which may be cyclic or not (Fig. 1 (a,b)). To make a global cycle we have to bring $S'$ to the original state. This may require some work to be done on $S'$ and some heat $-Q'$ to be taken from it (Fig. 1 (c)). What follows from the second law in this case is not the inequality $Q \leq 0$, it is the inequality $Q + Q' \leq 0$.

Certainly, environment dependence implies a sort of interaction between the system and the environment. (Which is shown on Fig. 1 (b) schematically by a dotted line.) From thermodynamic viewpoint, it is a peculiar interaction, for it does not involve heat or work. However, it is not excluded a *priori* by any of the known principles. If every process is supposed to be environment independent, we call it *standard thermodynamics*. Basically, it is thermodynamics familiar from textbooks. If this assumption is dropped, then what is left is *weak thermodynamics*, which is the subject of the rest of this section.

To make the argument down to earth, we have to consider at least one particular environment dependent process. A system considered below is the *xenium engine*, which is an imaginary device capable of violating the Clausius inequality. It is an interesting question to what extent it is realistic and whether a similar device can exist in nature, but this question is not relevant to the matter. The engine obeys the second law but violates the Clausius
inequality. Whether it is realistic or not, this is enough to conclude that the latter is not a consequence of the former.

B. The xenium engine

The xenium engine is a kind of heat engine, with an imaginary gas xenium as the working body. For the sake of convenience it is denoted by a “chemical” symbol Xe. Xenium is an ideal gas. A molecule of xenium can be in one of two states, denoted by \( \text{Xe}_a \) and \( \text{Xe}_b \). (With the same energy levels.) A single molecule can never change its state. However, two sufficiently close molecules may exchange their states

\[
\text{Xe}_a + \text{Xe}_b' \leftrightarrow \text{Xe}_b + \text{Xe}_a'.
\]

(Here Xe is a molecule and Xe' is another one.) The total number \( N_a(N_b) \) of the \( \text{Xe}_a(\text{Xe}_b) \) molecules remains constant, so xenium is a mixture of two gases to some extent.

The xenium engine is a cylinder with a piston which moves without friction (Fig. 2). The wall opposite to the piston is adiabatic. It is also thin (in a sense explained below). The cylinder is divided in two by a semipermeable partition. The \( \text{Xe}_a \) molecules can penetrate it while the \( \text{Xe}_b \) ones can not. The space between the thin wall and the partition, called a camera, is filled with a mixture of \( \text{Xe}_a \) and \( \text{Xe}_b \). Another part of the cylinder is filled with pure \( \text{Xe}_a \).

Consider first a single engine in contact with a heat bath at temperature \( T \). The piston moves in a quasistatic (hence reversible) process. The pressure \( P \) on the piston is then equal to the partial pressure of \( \text{Xe}_a \) in the camera. By the Gay-Lussac law,

\[
P = N_a k_B T/V,
\]

where \( k_B \) is the Boltzmann constant and \( V \) is the volume between the piston and the thin wall. It is convenient to introduce the dimensionless volume \( v = V/V_0 \geq 1 \), where \( V_0 \) is the (constant) volume of the camera. Work in the process is

\[
dW = -PdV = -N_a k_B T d\ln v.
\]

The internal energy of an ideal gas does not depend on volume, hence \( dQ = -dW \) and
Consider now two xenium engines connected as on Fig. 3. Each engine is in contact with a particular heat bath. The adiabatic wall separating the engines is so thin that xenium molecules in one camera may interact with molecules in another camera. The interaction looks somewhat similar to diffusion (Fig. 4). However, there is no real flow of molecules through the wall. Molecules simply change their states, so the numbers $N_a$ and $N_b$ aren’t actually conserved. (This will be transparent if we assume that xenium on the two sides of the wall are two different isotopes.)

To distinguish variables related to different engines we use subscripts ‘1’ and ‘2’. While the total number of Xe$_a$ molecules $N_{a,1} + N_{a,2}$ remains a constant, the summands became functions of the variables $v_1$ and $v_2$. It is not difficult to find this functions explicitly. Denote by $z$ the quotient of the concentrations in the camera:

$$z = \frac{[Xe_a]}{[Xe_b]} = \frac{N_a}{v N_b}.$$  

Then $N_a = N/(1 + v^{-1} z^{-1})$, where $N = N_a + N_b$ is a constant for each engine.

By symmetry, the equilibrium constant of the reaction (2) is one, hence it comes to equilibrium when $z_1 = z_2$. In a quasistatic process this equality must hold all the time. In the case $N_1 = N_2 = N_{a,1} + N_{a,2} = N$ we have

$$z_1 = z_2 = \frac{1}{\sqrt{v_1 v_2}} N_{a,1} = \frac{N \sqrt{v_1}}{\sqrt{v_1} + \sqrt{v_2}} \frac{dQ_1}{T_1} = \frac{2 k_B N}{\sqrt{v_1} + \sqrt{v_2}} d\sqrt{v_1}.$$
Clearly, \(dQ_1/T_1\) is not an exact differential. On the other hand, the sum

\[
\frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 2k_BNd\ln(\sqrt{v_1} + \sqrt{v_2})
\]

is an exact differential. (Which is also true for any choice of parameters.)

The conclusions are as follows. The xenium engine undergoes a reversible process, in which the reduced heat is not an exact differential. Thus, the Clausius inequality cannot be true for all the cycles. On the other hand, the sum \(\frac{dQ_1}{T_1} + \frac{dQ_2}{T_2}\) is an exact differential, which means that no contradiction to the second law is possible. So, in this situation the Clausius inequality is not really a consequence of the second law.

C. The weak Clausius inequality

In weak thermodynamics, the classical Clausius’s argument has to be modified. First of all, we have to consider a global cycle, not a local one. There can be many systems, interacting with each other in any possible way. Each of these systems is supposed to undergo a cyclic process, except for heat baths. Let us introduce one more heat bath at temperature \(T_0\). We can then restore every bath, save this one, to its initial state by giving it heat at the expense of heat taken from the exceptional bath. To make this process reversible we use Carnot engines (= environment independent reversible cyclic devices). Now, as the “intermediate” baths do not take essential part in the process, we can ignore them (Fig. 5).

By a simple calculation,

\[
\sum_i \oint \frac{dQ_i}{T_i} = \frac{Q_0}{T_0},
\]

where the sum is taken over all the systems and \(Q_0\) is heat taken from the exceptional bath. By the second law, the right hand side cannot be positive. Thus, we have the weak Clausius inequality

**Fig. 5** Three systems connected to a heat bath.
\[ \sum_i \oint \frac{dQ_i}{T_i} \leq 0. \] (4)

Call a global cyclic process *reversible* if (4) is an exact equality or the difference between the left hand side and zero is negligible. A local process, cyclic or not, is reversible if it may be a part of a reversible global cycle. (This is a broad definition of reversibility, which does not imply equilibrium. For example, a diamond at standard temperature and pressure undergoes a reversible process, despite being far from thermodynamic equilibrium.)

Following Clausius, we can define the total entropy by the equality

\[ dS = \sum_i \frac{dQ_i}{T_i}, \] (5)

in a reversible global process. (This is a consistent definition because of (4). This total entropy is the entropy of “genuine” systems; entropy of heat baths is not included. For this reason, it is not a constant even in a reversible process.) The definition of the entropy of an individual system is considered below.

D. The Clausius entropy

In classical thermodynamics, the very definition of entropy is based on the Clausius inequality. If the inequality is not valid, this definition has to be modified. Fortunately, the modification is rather obvious. All we have to do is to make explicit an implicit environment independence assumption. Thus, the Clausius entropy \( S \) of a system is defined by \( dS = dQ/T \) in a reversible environment independent process.

In general, we have \( dS \neq dQ/T \), even if a process is reversible. Call the difference \( d\bar{S} \) entropy, transferred adiabatically to the system. So,

\[ dS = \frac{dQ}{T} + d\bar{S}. \] (6)

The total entropy defined by (5) must be equal to the sum \( \sum_i S_i \) of “local” entropies. From (4), we have

\[ \sum_i d\bar{S}_i = 0. \]

The latter equality is a justification of the term “transfer”. Entropy does not come from nowhere, it comes from the environment. If \( T = \text{const} \), then

\[ \Delta F = W - T\bar{S}, \]

where \( F = U - TS \) is the Helmholtz free energy. Under close examination, the term \(-T\bar{S}\) on the right hand side is *chemical work*. (Note that it is done on a closed system, which is somewhat uncommon.)
For a general (not reversible) process the equality (6) turns into an inequality
\[ dS \geq \frac{dQ}{T} + d\mathcal{S}. \] (7)

Of course, in this case \( d\mathcal{S} \) cannot be defined as a difference between \( dS \) and \( dQ/T \). The definition is \( d\mathcal{S} = -d\mathcal{S}_{env} \), where \( d\mathcal{S}_{env} \) is entropy transferred to the environment (which may be supposed to undergo a reversible process). Taking the integral of the both sides of (7) over a cycle, we have the following generalization of the Clausius inequality
\[ \oint \frac{dQ}{T} \leq -\mathcal{S}. \] (8)

The standard Clausius inequality (1) can be violated if \( \mathcal{S} < 0 \), i.e. if entropy is transferred to the environment adiabatically.

So far, we followed Clausius. It is also instructive to take another view, more close to Caratheodory’s. The state of a system can be described by the entropy \( S \) and the parameters \( x_1, \ldots, x_n \). Let \( U(S, x_1, \ldots, x_n) \) be the internal energy as a function of these parameters. In an adiabatic process, \( dU = dW \). Caratheodory has assumed that work (in a reversible process) is
\[ dW = \sum_i p_i dx_i, \]
for some functions \( p_i \). It follows that
\[ TdS = \sum_i \left( p_i - \frac{\partial U}{\partial x_i} \right) dx_i. \] (9)

The point is, in an environment dependent adiabatic process, \( dS \neq 0 \), i.e. entropy is not conserved. In fact, nothing is conserved: any two states may be connected by an adiabatic process, as soon as environment dependent effects are present. This invalidates the central postulate of Caratheodory’s theory. To make sense of entropy, we have to consider environment independent processes separately; in this respect, there is no much difference with Clausius’s approach. In an environment independent adiabatic process \( dS = 0 \), hence the sum on the right hand side of (9) is zero. However, it does not follow that \( p_i = \frac{\partial U}{\partial x_i} \), because in this case the differentials \( dx_i \) are not linearly independent. For this reason, in a general process this sum (which is chemical work, with the opposite sign) is not zero.

In this argument, the difference between a reversible process (as it is defined in II C) and an equilibrium process is very important. Almost by definition, an equilibrium state may only depend on those parameters which can be changed by an external agent at will, such as volume, magnetic induction etc. It means that any equilibrium process is environment independent (and there can be no linear dependence between the differentials of the parameters). So, no adiabatic entropy transfer is possible in an equilibrium process.
E. The entropy of the xenium engine

The Clausius entropy of the xenium engine is more tricky to find than the Boltzmann-Gibbs entropy. (We have less information because of the black box method inherent in classical thermodynamics.) Of course, it is the same entropy in the end. We do the calculation below, in order to show the tools at work, and to point out some peculiarities. For the sake of simplicity, we consider isothermal processes only.

The system has two parameters. Convenient parameters are the dimensionless volume $v$, and the number of Xe$_a$ molecules $N_a$. We have a problem with the second one: in any environment independent process $N_a$ is a constant. So, we cannot measure the entropy difference between states with different $N_a$ values.

However, it is not a fault of the method, it is a fault of the model. Let us suppose that there exists a catalyst which makes xenium molecules undergo spontaneous transitions Xe$_a$ $\leftrightarrow$ Xe$_b$. We can add the catalyst to or remove it from the camera as necessary. With catalyst in the camera, we have $[\text{Xe}_a] = [\text{Xe}_b]$, which means $N_a = N/(1 + v^{-1})$. Otherwise, $N_a$ is a constant. (To reach states with $N_a > N/2$, we need a still more elaborated model, which is not considered here.)

In any quasistatic process, the equality $dS = N_a k_B d \ln v$, in an environment independent process. Environment independence in this case implies either $N_a = \text{const}$ or $N_a = N/(1 + v^{-1})$. The right hand side is not an exact differential, but this does not matter. (Because any environment independent cycle is trivial: it does not enclose positive area on the $v$ vs. $N_a$ plane.) Taking integral over the allowed processes, we have

$$S = N_a k_B \ln(v/N_a) - N_b k_B \ln N_b + C,$$

where $N_b = N - N_a$ and $C$ is a constant. This entropy coincides with the Boltzmann-Gibbs entropy

$$S = -N_a k_B \ln[\text{Xe}_a] - N_b k_B \ln[\text{Xe}_b],$$

as expected.

For an environment dependent process, $dS \neq dQ/T = N_a k_B d \ln v$. The difference is

$$d\mathcal{S} = dS - dQ/T = -k_B \ln zdN_a,$$

where $z = N_a/v N_b$.

Again, it was expected. The chemical work in the process is

$$-Td\mathcal{S} = \mu_a dN_a + \mu_b dN_b = (\mu_a - \mu_b) dN_a,$$

where $\mu_a$ and $\mu_b$ are the chemical potentials of Xe$_a$ and Xe$_b$ respectively. For an ideal gas

$$\mu_a - \mu_b = k_B T \ln[\text{Xe}_a] - k_B T \ln[\text{Xe}_b] = k_B T \ln z.$$
The author would like to make a remark. Once we introduce a catalyst, it becomes obvious that the xenium engine works out of equilibrium. Xenium, as a thermodynamic system, can only be in equilibrium if $[Xe_a] = [Xe_b]$. Otherwise, we can add a catalyst and observe an irreversible reaction. As it was noted at the end of II D, we could not expect a violation of the Clausius inequality in an equilibrium process.

III. THE LANDAUER’S PRINCIPLE

A. The original Landauer’s principle

The Landauer’s principle was proposed by Rolf Landauer in [16]. Apparently, this work did not attract much attention before 1982, when Bennett used the principle to explain the paradoxical behavior of the Szilard engine [1]. By now, there exists quite an extensive literature on this subject. Some of more recent or more relevant works are [2][6][20][17][21].

According to the principle, erasure of information is accompanied by heat generation [16]. The quantitative formulation may be presented as an inequality

$$Q_{dis} \geq k_B T I,$$

where $Q_{dis}$ is heat dissipated in the process and $I$ is the erased information (in nats). Of course, to make it a precise statement one has to specify the definitions of $Q_{dis}$ and $I$, but we avoid this technicalities. It is enough to say that $I$ is usually defined as Shannon entropy. (It may also be von Neumann entropy, but we only consider classical information here.)

The author is convinced that it is much more natural to interpret the Landauer’s principle as a statement in classical thermodynamics than a statement in statistical physics. But in view of thermodynamics, the original formulation is really weird. Classical thermodynamics does not deal with ensembles, so Shannon entropy does not make much sense. To give a reasonable thermodynamic formulation of the principle we need another measure of information. The proposed measure is thermodynamic information, introduced below.

B. Nonstandard thermodynamics

Apparently, classical thermodynamics is considered by many a kind of adaptation of statistical physics for engineers. The author does not share this opinion. Classical thermodynamics is a valuable part of theoretical physics, which proved to be useful on many occasions. It may be useful in information thermodynamics, too. To follow this route we have to adopt the black box method and to employ Clausius entropy instead of Boltzmann-Gibbs entropy.

However, there is an obstacle. One of the cornerstones of classical thermodynamics is determinism. Stochastic phenomena of any kind are treated as fluctuations, which are beyond the scope of thermodynamics itself. But many of the processes usually considered in information thermodynamics (such as
the Szilard engine cycle) are not deterministic. So, we have to extend classical thermodynamics.

The proposed extension is nonstandard thermodynamics. While the traditional, or standard classical thermodynamics deals with deterministic processes only, in nonstandard thermodynamics we may consider a process which can be observed with some probability. The fundamental characteristics of a process are work $W$ done by an external agent, heat $Q$ taken from a heat bath(s), and the probability of the process $P$. What is important, this probability can be measured, at least in principle, by trying the process many times. So, the probability of a process is a property of a system as a black box: we do not have to know anything about the internal physics to measure it. In this respect, it isn’t different from heat or work.

When we speak about a probability, we imply a number of alternatives. It should be noted that for a particular process the alternatives may be different, depending on the circumstances. (This is usually referred to as feedback control.) But the probability itself is supposed to be the same, so we may consider it a characteristic of a process.

The author would like to point out that this approach to a process is radically different from what one may find in the literature. The tradition is to consider all the possibilities at once [1][7][20][19][18], so “work (heat)” actually means “the mean value of work (heat)”, averaged over all the possible alternative processes. This is natural in the context of statistical physics, where the main tool is an ensemble. However, it would not be so natural in the context of classical thermodynamics. For this reason, in nonstandard thermodynamics we do not consider ensembles. We consider a single system undergoing a single process a single time. Moreover, the second time the outcome of the process is supposed to be exactly the same (with some exceptions discussed in IV A). In this respect, nonstandard thermodynamics is not very much different from the standard one.

For example, the Szilard engine, considered in IV B, can undergo one of two different cyclic processes, depending on where the working particle gets trapped. In the both cases, the probability is $P = 1/2$ and work is $W = -k_B T \ln 2$. But for a “skewed” version of the engine the works (and probabilities) are different, so it is more convenient to consider these two processes separately. Note that as far as we are interested in a single process, it makes no difference if there is any feedback control or not. (Occasionally, we consider a family of processes $A_i$, which may be observed in a certain situation with probabilities $P_i$, such that $\sum_i P_i = 1$. We call it a mixed process, denoted by $\sum_i P_i A_i$.)

We assume that for each process $A$ there exists an opposite process $A^\dagger$, which consists of the same states taken in the reverse order. The backward probability $P^\dagger$ of a process is the probability of the opposite process, $P^\dagger(A) = P(A^\dagger)$. A process is called balanced if $P \geq P^\dagger$, deterministic if $P = 1$ and bideterministic if $P = P^\dagger = 1$.

We assume also that heat taken by a system in a “trivial” cycle like $A^\dagger A$ cannot be positive. One can see that it is equivalent to the inequality $dQ + dQ^\dagger \leq 0$, where (infinitesimal) heat $dQ$ is taken in a part of the process and $dQ^\dagger$ is taken in the corresponding part of the opposite process. Call a
process $A$ reversible if no heat is taken in the cycle $A^\dagger A$. In this case, the opposite process a reverse process, $A^{-1} = A^\dagger$.

The author would like to point out that this definition is different from the definition of a reversible process in standard thermodynamics. (Formally, it is Carnot reversibility in terms of [13 Appendix C], taken in nonstandard context.) For example, a process with the probabilities $P = 1, P^\dagger < 1$ is irreversible in standard thermodynamics whether it is reversible in the above sense or not. (Parrondo [19] used the term quasiirreversible.) A process which is reversible in standard thermodynamics is a bideterministic reversible process.

If $AB$ is a composition of two processes ($A$ followed by $B$), then $P(AB) = P(A)P(B)$. It follows that the function of a process $I$, defined by

$$I = \ln \frac{P^\dagger}{P},$$

is additive, i.e. $I_{AB} = I_A + I_B$. We will often write it in the integral form, $I = \int dI$. Call it thermodynamic information created in the process (measured in nats). Thermodynamic information is not information in the sense of Shannon. We will see, however, that there is a good reason to consider it a thermodynamic analog of Shannon information.

At this point, the author would like to make a few remarks. To avoid possible confusion, in Secs. III, IV we do not take into account any environment dependence effects discussed in Sec. II. It simply means that in standard thermodynamics the Clausius inequality (1) is valid. However, it is not valid in nonstandard thermodynamics (for reasons explained below). Moreover, it is often convenient to interpret a violation as a consequence of a (fictional) adiabatic entropy transfer.

The author did his best to find an appropriate language for the classical counterpart of information thermodynamics, but the two main contributions were made by the others. Clausius entropy was introduced in information thermodynamics by Ishioka and Fuchikami [10]. The relation between Clausius entropy and thermodynamic information was, in essence, found by Kawai et.al. [14]. So, this work is not actually as original as it looks.

C. The Clausius inequality and the Landauer’s principle

In nonstandard thermodynamics, the Clausius inequality is different from the familiar one (1). It is

$$\oint \frac{dQ}{T} \leq k_B I,$$

where $I$ is thermodynamic information created in the cycle. The inequality (10) is both more general and more precise then (1). First of all, (11) is not true if $P < 1$ (some examples are considered below). It simply means that standard thermodynamics cannot be applied to a process which is not deterministic, which must not be a surprise. On the other hand, the both
inequalities can be applied to a deterministic cyclic process. But in this case (10) is stronger than (1), because $I \leq 0$.

If a cycle $A$ is reversible (in the sense of III B), then we have

$$k_B I_A = -k_B I_{A^{-1}} \leq -\oint_{A^{-1}} \frac{dQ}{T} = \oint_A \frac{dQ}{T} \leq k_B I_A.$$ 

So, in this case the inequalities are in fact equalities. Repeating the standard argument by Clausius, we have the following definition of the Clausius entropy $S$

$$dS = \frac{dQ}{T} - k_B dI.$$ 

(11)

In a general (not reversible) process, the equality turns into an inequality

$$dS \geq \frac{dQ}{T} - k_B dI.$$ 

(12)

From (12) and the first law of thermodynamics $dU = dW + dQ$, we have

$$dF \leq dW - SdT + k_B T dI,$$

where $F = U - T S$ is the Helmholtz free energy. In the case of an isothermal process it is equivalent to

$$W \geq \Delta F - k_B T I.$$ 

(13)

(This inequality is known [14, Eq. (8)].)

Consider a process which is isothermal and deterministic, but not bideterministic. The dissipated heat, as it is defined in standard thermodynamics is $Q_{\text{dis}} = W - \Delta F$. Thus,

$$Q_{\text{dis}} \geq k_B T |I|,$$

where $I = \ln P^{\dagger} < 0$. We will see in III E that in a standard erasure of information process $|I|$ is, basically, erased information in the sense of Landauer. This gives us a reason to consider (10) a reformulation of the Landauer’s principle.

The original Clausius inequality (1) can be derived from classical thermodynamics or from statistical physics [11]. The same is true for the general inequality (10). In this case, the author prefers statistical physics. (In IV C, the inequality is derived from the second law in a specific formulation. However, this formulation is by no means obvious or intuitive. So, this version of the second law needs a convincing proof for itself.)
D. The Clausius inequality and the Crooks fluctuation theorem

This is the only part of the paper devoted to statistical physics. We prove a version of the Jarzynski equality (or the integral fluctuation theorem), which is similar to the equality found recently by Sagawa and Ueda \[21, \text{Eq.}(3)\]. We use Hamiltonian dynamics instead of stochastic dynamics, but it is not the main difference. The result of Sagawa and Ueda is useless for our purpose, because their definition of information is completely different. (Apparently, this information cannot be defined properly in the black box conditions.) The principal tool is the Crooks fluctuation theorem, but, as was pointed out in \[21\], basically any version of the detailed fluctuation theorem can be employed. (The relevant references can be found in \[21\][12].)

A system in contact with a heat bath can be described by the Hamiltonian

\[ H(x, \lambda) + H_I(x, y) + H_B(y, \mu). \]

Here \(x\) and \(y\) are points in the phase space of the system and the bath respectively, and \(\lambda\) is a work parameter. In this sum, \(H\) is the energy of the system itself, \(H_I\) is the interaction energy and \(H_B\) is the energy of the bath.

We do not want to restrict ourselves to isothermal processes. The standard approach in this case is to introduce many baths, as in \[11\][21]. But this is not very convenient for technical reasons. Instead, we consider a single “bath” with variable temperature. It is a system with very short relaxation time and large specific heat. An external agent can change the temperature \(T\) of this system adiabatically, by varying the parameter \(\mu\).

Consider a protocol \((\lambda(t), T(t))\), where time \(t\) is in the interval \(0 \leq t \leq \tau\). In general, there are many processes with the same protocol. A system which is not ergodic can be in different (macroscopic) states at \(t = 0\). Moreover, the evolution of the system may be indeterministic; in this case there can be many processes with the same initial state.

A particular process \(\mathcal{A}\) can be described by the initial distribution \(\rho_F(x)\) at \(t = 0\), and some set of trajectories. A trajectory \(x \in \mathcal{A}\) if a system which goes along this trajectory is considered to be undergoing the process \(\mathcal{A}\). The opposite process \(\mathcal{A}^\dagger\) is described by the backward protocol \((\lambda(\tau - t), T(\tau - t))\), another initial distribution \(\rho_B(x)\), and the conjugate set of trajectories \(\mathcal{A}^\dagger = \{X^\dagger : X \in \mathcal{A}\}\). (Here \(X^\dagger = (x(\tau - t)^* : 0 \leq t \leq \tau\), where \(x^*\) is obtained from \(x\) by reversing the momenta.)

Let \(P_F[X]\) be the distribution of trajectories (under the forward protocol), given the initial phase space distribution \(\rho_F(x)\), and \(P_B[X]\) be the distribution of trajectories under the backward protocol, given the initial distribution \(\rho_B(x)\). Then

\[ P = P(\mathcal{A}) = \int_\mathcal{A} P_F[X], \quad P^\dagger = P(\mathcal{A}^\dagger) = \int_{\mathcal{A}^\dagger} P_B[X] = \int_\mathcal{A} P_B[X^\dagger]. \]

For a particular trajectory \(X\), denote by \(x_0 = x(0)\) the beginning and by \(x_1 = x(\tau)\) the end. Let \(P_F[X \mid x_0]\) and \(P_B[X \mid x_0]\) be the conditional distributions of trajectories, such that

\[ P_F[X] = P_F[X \mid x_0] \rho_F(x_0), \quad P_B[X] = P_B[X \mid x_0] \rho_B(x_0). \]
The fraction

\[ \mathcal{F}[\mathcal{X}] = \frac{\mathcal{P}_F[\mathcal{X} \mid x_0]}{\mathcal{P}_B[\mathcal{X}^\dagger \mid x_1]} \]

plays an important role in statistical mechanics. It is multiplicative in the following sense: if a trajectory \( \mathcal{X} = \mathcal{X}_1 \mathcal{X}_2 \) consists of two parts \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), then \( \mathcal{F}[\mathcal{X}] = \mathcal{F}[\mathcal{X}_1] \mathcal{F}[\mathcal{X}_2] \).

For an isothermal process, we have

\[ \mathcal{F}[\mathcal{X}] = e^{-Q/k_B T}, \quad Q = \int_X \frac{\partial H}{\partial x} dx, \quad (14) \]

by the Crooks fluctuation theorem [5, Eq. 9]. (Here \( Q \) is, technically, stochastic heat, not thermodynamic heat. The difference is explained in IV A.) A generalization to nonisothermal processes is completely obvious due to multiplicativity

\[ \mathcal{F}[\mathcal{X}] = \exp \left( -\frac{1}{k_B} \int_X \frac{dQ}{T} \right). \quad (15) \]

(Jarzynski used this equality in [11, Eq.(6)], but he did not write it down explicitly. Sagawa and Ueda used a similar equality [21, Eq.(11)].)

Denote by

\[ s = -k_B \ln(\rho(x)/\rho_0(x)) \]

a stochastic entropy, where \( x \) is a point in the phase space, \( \rho \) is a distribution and \( \rho_0 \) is the Liouville measure. (The measure is unique up to a factor, hence \( s \) is well defined up to a constant.) This “entropy” is a function of a microstate and a macroscopic state together. For example, if the distribution \( \rho \) is canonical then

\[ s = (H(x) - F)/T, \]

where \( F \) is the Helmholtz free energy.

The stochastic entropy production

\[ \sigma = \Delta s - \int_X \frac{dQ}{T} + k_B I \]

depends on the trajectory \( \mathcal{X} \) as well as the initial and final states. (Note that it is entropy production in nonstandard thermodynamics. In standard thermodynamics, the term \( k_B I \) is not included.) Assuming \( \rho_B(x^\dagger) = \rho_B(x) \), we have \( e^{-\Delta s/k_B} = \rho_B(x^\dagger)/\rho_F(x_0) \), hence by (15)

\[ e^{-\sigma/k_B + I} = \frac{\rho_B(x^\dagger)}{\rho_F(x_0)} \mathcal{F}[\mathcal{X}] = \frac{\mathcal{P}_B[\mathcal{X}^\dagger]}{\mathcal{P}_F[\mathcal{X}^\dagger]} \mathcal{F}[\mathcal{X}] . \]

Thus, the average of the exponent in the process is

\[ \left\langle e^{-\sigma/k_B + I} \right\rangle_A = \frac{1}{T} \int_A e^{-\sigma/k_B + I} \mathcal{P}_F[\mathcal{X}] = \frac{1}{T} \int_A \mathcal{P}_B[\mathcal{X}^\dagger] = \frac{P^\dagger}{P} . \]

We have the following version of the Jarzynski equality (compare to [11, Eq.(11)] and [21, Eq.(3)])

\[ \left\langle e^{-\sigma/k_B} \right\rangle = 1. \quad (16) \]
If the process is isothermal and the distributions $\rho_F, \rho_B$ are canonical (but not necessary ergodic), then $\sigma = (W - \Delta F)/T + k_B I$ and the equality become

$$\left\langle e^{-W/k_B T} \right\rangle = \frac{P}{P} e^{-\Delta F/k_B T}.$$  

In this form it was found by Kawai et.al. [14, Eq.(7)].

The Clausius inequality (10) is a simple consequence of (16). By the Jensen inequality, we have $\langle -\sigma \rangle \leq 0$. If the process is a cycle then $\langle \Delta s \rangle = 0$, hence

$$\left\langle \oint \frac{dQ}{T} \right\rangle \leq k_B I. \quad (17)$$

In classical thermodynamics, the fluctuation are supposed to be ignored, so we can replace the average by the actual value to obtain (10). (A bit more detailed argument is in IV A.)

E. Erasure of information

The applications of the inequality (10) in this section are two related but different processes: erasure of information and magnetization reversal. Erasure of information is a standard test of the Landauer’s principle, so we can make a comparison between the inequality and the principle to see what is common and what is different.

The system under consideration is a 1-bit memory device, which is normally in one of two states, called “zero” and “one”. A convenient model is a classical particle in a double-well potential [10][20][6], but actually the physics of the device is not relevant to the analysis. All we need is the assumption that it obeys the inequality (10).

Erasure of information is two different processes. One of the processes, denoted by $A$, is a cycle which begins and ends in the “zero” state. The other process $B$ has the initial state “one” and the final state “zero” (Fig. 6 (a)). The both processes are supposed to be isothermal and deterministic (that is, information is erased for sure).

However, the opposite processes $A^\dagger$ and $B^\dagger$ are not deterministic. The argument may be as follows. If an external agent “knows” which of the processes, $A$ or $B$, is going on, then this information remains somewhere after a process is complete. In this case, information would not be “erased”. It means that the processes must “look” the same for an external agent. The opposite processes look the same as well, and have a common initial state. So, an external agent does not have any means to make the system undergo, say, the process $A^\dagger$ for sure: the system has free choice between $A^\dagger$ and $B^\dagger$ (Fig. 6 (b)). Consequently, $P(A^\dagger) + P(B^\dagger) = 1$.

From (13), we have

$$W_A \geq -k_B I_A; W_B \geq \Delta F - k_B I_B.$$
where $\Delta F = F_0 - F_1$ is the difference in Helmholtz free energy between the states. In the both cases, the bound for the dissipated heat $Q_{\text{dis}} = W - \Delta F$ is

$$Q_{\text{dis}} \geq -k_B T I = -k_B T \ln P^\dagger.$$  

On the other hand, the Landauer’s principle gives us the inequality

$$\langle Q_{\text{dis}} \rangle = q_0 Q_{\text{dis}, A} + q_1 Q_{\text{dis}, B} \geq k_B T I_S$$

for a “mixed” process $q_0 A + q_1 B$. Here $q_0$ and $q_1$ are (arbitrary) probabilities, such that $q_0 + q_1 = 1$, and $I_S = -q_0 \ln q_0 - q_1 \ln q_1$ is the erased Shannon information. This inequality is supposed to be true for any probabilities $q_0$ and $q_1$, which is equivalent to the pair of inequalities

$$Q_{\text{dis}, A} \geq -k_B T \ln p_0, \quad Q_{\text{dis}, B} \geq -k_B T \ln p_1$$

for some positive numbers $p_0$ and $p_1$ satisfying $p_0 + p_1 = 1$. So, the Landauer’s principle gives us exactly the same information, except for it does not tell us where the mysterious numbers $p_0$ and $p_1$ come from. In fact, $p_0 = P(A^\dagger)$, $p_1 = P(B^\dagger)$.

If the process $B$ is reversible, then $W_B = \Delta F - k_B T \ln p_1$. But this process is not reversible in standard thermodynamics. We may as well consider a reversible in standard thermodynamics process $C$ with the initial state “one” and the final state “zero”, but it is a different process which is not erasure. (It means that after $C$ is complete, it is still possible to find out that the initial state was “one”.) In terms of nonstandard thermodynamics, $C$ is a bideterministic reversible process; the work done on the system is $W_C = \Delta F$.

The cycle $CB^{-1}$ deserves some attention (Fig. 6 (c)). It is a process in which heat is “antidissipated”, $W_C^{-1} = k_B \ln p_1 < 0$. Several authors have pointed out that this cycle is basically equivalent to the Szilard engine cycle [10][19]. In a sense, this is the opposite to erasure of information: we have $I > 0$ instead of $I < 0$. The original Landauer’s principle does not handle this case, but it is possible to consider it within a more general framework proposed by Maroney [17].

F. The magnetization reversal process

A small magnet is one of the standard implementations of a memory device. In view of the Landauer’s principle it was considered by Landauer himself
Such a system can be described by the Hamiltonian

$$H(x) - HM(x),$$

where $H$ is external magnetic field, $x$ is a point in the phase space and $M(x)$ is the (stochastic) magnetic moment. ($H$ is in units of magnetic induction, i.e. $\mu_0 = 1$.)

We do not need any specific information about this Hamiltonian, besides some properties. There is a symmetry $x \leftrightarrow x^*$ of the phase space (time reversal), such that $H(x^*) = H(x)$ and $M(x^*) = -M(x)$. At a fixed temperature $T$, the moment $M = \langle M(x) \rangle$ is a function of $H$. We assume that there is spontaneous magnetization. It means that in the region $|H| < H_c$ there are two (symmetric) phases with different magnetic moments, so we have a hysteresis loop on the $M$ vs. $H$ diagram (Fig. 7). (We assume a single domain; otherwise the loop might look quite different.)

Consider a quasistatic magnetization reversal process $A$, which begins at $H = H_0$ and ends at $H = -H_0$ (such that $H_0 > H_c$). In standard thermodynamics, it is an irreversible process. At $H = -H_c$ the system undergoes an irreversible first order phase transition, and its entropy increases by $\Delta_+ S$. Due to the symmetry, the entropy at $H = \pm H_0$ is the same, hence $\Delta_+ S + Q/T = 0$, where $Q$ is heat taken in the process.

The symmetric process $B$ begins at $H = -H_0$ and ends at $H = H_0$. Work done on the system in the cycle $AB$ is equal to the area $A$ of the hysteresis loop, because $\bar{d}W = -MdH$. Heat taken in $A$ is a half of heat taken in the cycle, i.e. $Q = -A/2$, hence

$$\Delta_+ S = A/2T.$$

Nonstandard thermodynamics has a somewhat deferent view on the process. It is postulated that there exists the opposite process $A^\dagger$. Its probability $p = P(A^\dagger)$ can be very small, but it cannot be zero. The opposite process is peculiar because at $H = -H_c$ the system undergoes a backward phase transition, and its entropy decreases. (Which may be interpreted as a sort of fluctuation.) Note that when $H$ increases, the system undergoes either $A^\dagger$ or $B$, hence $P(B) = 1 - p$.

But it means that, by symmetry, $A$ is not a deterministic process either: $P(A) = 1 - p$. At $H = H_c$ the system can undergo the same backward phase transition and follow the process $B^\dagger$ instead of $A$ (Fig. 7 (b)).

One can see that in the magnetization reversal process $A$ there are two points where $I \neq 0$. At $H = H_c$, we have the forward probability $1 - p$ and the backward probability one, hence $I = -\ln(1 - p)$. At $H = -H_c$ the forward probability is one, and the backward probability is $p$, hence $I = \ln p$. By (11), the entropy changes at these points are respectively

$$\Delta_- S = k_B \ln(1 - p), \quad \Delta_+ S = -k_B \ln p.$$ 

On the other hand, $\Delta_+ S + \Delta_- S + Q/T = 0$, and $Q = -A/2$. Thus,

$$A = 2k_B T \ln(p^{-1} - 1).$$

(18)
IV. PRINCIPLES OF NONSTANDARD THERMODYNAMICS

A. Heat and work

Interpretation of heat in nonstandard thermodynamics is a delicate matter. Consider the definition of Clausius entropy

\[ \Delta S = \int \frac{dQ}{T} - k_B I. \] (19)

To find the entropy difference between two states, we have to measure the right hand side in some reversible process. The question is, what exactly is heat in (19)? The standard interpretation of heat is simply energy taken from a heat bath. But in fact, energy goes constantly back and forth between a system and a bath due to heat fluctuations. Usually, the contribution of heat fluctuations to the reduced heat term in (19) is compatible to the information term, if not much bigger. So, it is not at all clear how to find entropy with high enough precision, so it would make sense to take the information term into account. In statistical physics it is not a problem, because the meaning of “heat” (or “work” etc.) is mostly “the average of heat”. Of course, to take
average is an easy way to get rid of heat fluctuation, but average is alien to classical thermodynamics.

There is another way: to measure work instead of heat. It makes a difference, because there are no work fluctuations in a reversible process. (Apparently, this fact is known to experts, but the author can’t produce a single reference where it is pointed out explicitly.) The thermodynamic meaning of this proposal is to use Helmholtz free energy $F$ instead of entropy as a basic thermodynamic function. Other thermodynamic functions are derived from $F$, for example

$$S = -\partial F/\partial T, \quad U = F + TS,$$

etc. Then, heat should be defined by $Q = \Delta U - W$. Call this kind of heat thermodynamic, to distinguish it from stochastic heat $Q^*$, which is just energy taken from a heat bath or baths. Stochastic heat is $Q^* = \Delta E - W$, where $E$ is the energy of a system. The difference between $U$ and $E$ is that the former is a function of a macroscopic state only while the latter is varying due to heat fluctuations.

In this scheme, (19) is not a definition of entropy, it is basically a definition of Helmholtz free energy, which can be rewritten as

$$dF - \frac{\partial F}{\partial T}dT = dW + k_B TdI. \tag{20}$$

In this equation, no average is necessary. Unfortunately, $F$ is not defined by (20) completely: the left hand side does not change if it is replaced by $F + f(T)$, where $f$ is an arbitrary function. (One cannot find specific heat by measuring work.) The only way to resolve this uncertainty is to measure the internal energy directly. To do this properly we have to detach the system from a heat bath and to measure its energy $E$ in an adiabatic process; then $U = \langle E \rangle$. At this point, one cannot avoid the average, but it is enough to measure specific heat in a single state, when the only varying parameter is temperature.

Now we can give a more accurate thermodynamic interpretation of the Jarzynski equality (16), then in III D. Consider a quasistatic process, reversible or not. In such a process, the distribution remains canonical (possibly not ergodic), hence $s = (E - F)/T$. By definition,

$$d\sigma = ds - \frac{dQ^*}{T} + k_B dI,$$

where $\sigma$ is stochastic entropy production and $Q^*$ is stochastic heat. (Which was denoted in III D by $Q$, but this must not led to a confusion.) The equivalent equation is

$$dF - \frac{\partial F}{\partial T}dT - k_B TdI = dW - Td\sigma + \left( s - \frac{\partial F}{\partial T} \right) dT.
\text{Note that in this equation the terms on the left hand side are thermodynamic while the terms on the right hand side are stochastic. In a quasistatic process, temperature is supposed to change gradually, hence the term } (s - \partial F/\partial T)dT \text{ can be replaced by its average value, which is zero.}$$
\[ dF - \frac{\partial F}{\partial T} dT - k_B T dI = dW - T d\sigma. \]  \hspace{1cm} (21)

Denote by \( \Sigma = \langle \sigma \rangle \) the macroscopic entropy production. From (21), we have

\[ \Delta S = \left\langle \int \frac{dQ}{T} \right\rangle - k_B I + \Sigma, \]  \hspace{1cm} (22)

where \( dQ \) is thermodynamic heat. For a cyclic process, the consequence of (22) is an analog of (17), because \( \Sigma \geq 0 \). The important difference is, in (22) the average is over work fluctuations while in (17) the average is over heat fluctuations. If a process is reversible, then \( \Sigma = 0 \) and (22) turns into (19); in this case we do not need any average at all.

Another important consequence of (21) is that the variables on the right hand side must have the same dispersion \( D(\bar{d}W) = T^2 D(\bar{d}\sigma) \). By assumption, the process is quasistatic, hence the contributions to \( \sigma \) from different parts of this process must be statistically independent. So, \( \sigma \) must have a normal distribution. From the Jarzynski equality (16), we have \( D\sigma = 2 k_B \langle \sigma \rangle \). Moreover, this argument is valid not only for the whole process, but also for any part of it, hence \( D(\bar{d}\sigma) = 2k_B\bar{d}\Sigma \). Taking into account that dispersion is additive, we have

\[ DW = 2k_B \int T^2 d\Sigma. \]  \hspace{1cm} (23)

For example, if the process is reversible, then \( DW = 0 \). (Presumably, a truly quasistatic process is reversible for a sensible model. But the equalities (22) and (23) must be valid for an irreversible process which is close to quasistatic.)

B. The Szilard engine

The Szilard engine is an imaginary device invented by L. Szilard in 1929 [22]. The thermodynamics of the engine was discussed in the literature: many times in terms of statistical physics [1,21,7,19,18] and one time in terms of classical thermodynamics (essentially) [10].

The engine consists of a box with a single particle. The box is provided with a thin piston which can be inserted to or removed from it as necessary (Fig 8). The engine undergoes an isothermal cyclic process in contact with a heat bath at temperature \( T \). At the beginning, the piston is out of the box. As the first step of the process it is inserted into the box at the middle, dividing it into two parts of equal volume. The particle gets trapped in one of the halves. After that, the piston moves into the empty half until it reaches the wall. The piston is then removed and the cycle is complete.

Under natural assumptions \( P = k_B T/V \) by the Gay-Lussac law, and the work done in the cycle on the system is negative

\[ W = - \int PdV = -k_B T \ln 2, \]
Fig. 8 The Szilard engine

in apparent contradiction to the second law. The correct solution of the paradox, found by Bennett [1], is well known, so there is no need to discuss it here in detail. Still, there are several important points apparently missed in the literature.

To begin with, there are two different “paradoxes” to deal with: a violation of the standard Clausius inequality (which is real) and a violation of the second law (only apparent). Admittedly, a violation of (1) is not really a problem. The cycle is not deterministic; we simply do not have a reason to expect the inequality to be true in such an exotic situation. Also, the explanation in nonstandard thermodynamics is straightforward. There are two different cyclic processes, each with probability \( P = \frac{1}{2} \) and the backward probability \( P^\dagger = 1 \), hence thermodynamic information created in any of the cycles is \( I = \ln 2 \). By (10), we have

\[
Q = k_B T \ln 2,
\]

in perfect agreement with the Gay-Lussac law. (Note that the processes are reversible.) One may also consider a “skewed” version of the engine, where the partition is not inserted in the middle [18]. If the particle occupies volume \( V_0 \) and the volume of the empty half is \( V_1 \), then \( P = V_0/(V_0 + V_1) \) and

\[
Q = k_B T \ln(1 + V_1/V_0),
\]

again in agreement with the Gay-Lussac law.

The apparent contradiction to the second law is a more delicate matter. In short, the Bennett’s argument is as follows. To work properly, the engine needs a controller and a controller needs a memory. After a cycle, 1 bit of information has to be erased from this memory, and all the heat returns to the bath by the Landauer’s principle. This is very similar to the environment dependence argument considered in II A: a memory plays the role of a “parallel” system \( \hat{S}' \). (We have \( Q = k_B T \ln 2 \), \( Q' = -k_B T \ln 2 \), and \( Q + Q' = 0 \).)
However, it is not exactly the same argument. In fact, the both Szilard engine cycles are environment independent. They only become “look” environment dependent when considered together, as a single mixed process. (The engine does not need any controller to work properly in one of the cases!) The Bennett’s explanation is completely sound, but it cannot be accepted as a proper explanation in nonstandard thermodynamics, because it is inherently probabilistic. To make sense of it, one has to consider ensembles instead of a single engine undergoing a single process. The right explanation in nonstandard thermodynamics is a reformulation of the second law, considered below.

Another subtle point is the behavior of the total entropy. In the Szilard engine cycle, thermodynamic information \( I = \ln 2 \) is created at the moment when the partition is inserted. By (11), the entropy of the engine decreases instantly by \( k_B \ln 2 \), which is what one might expect when the volume is halved. The problem is, the entropy of any other system in the environment remains the same, which means that the total entropy decreases as well. This paradox was discussed in [19] and in [10]. Parrondo proposed a redefinition of the entropy to solve the problem. This proposal (in different interpretation) is discussed in IV D.

C. The second law revised

It has already been mentioned that determinism is among the basic assumptions in standard classical thermodynamics. However, it does not mean that indeterministic phenomena cannot be considered at all: to some extent, they can. Apparently, the de facto interpretation of the second law in this situation is what one might call a statistical formulation of the law. It has never been formulated explicitly, for the best of author’s knowledge. We propose the following formulation: the average of heat taken from a single heat bath and converted to work in a mixed global cyclic processes is not positive. It means, if heat \( Q_i \) is taken in a process \( A_i \), which is observer with probability \( P_i \), then

\[
\sum_i P_i Q_i \leq 0
\]

(here \( P_i = P(A_i) \) and \( \sum_i P_i = 1 \).)

This formulation of the second law is not very satisfactory. To make any conclusion about a particular process which is not deterministic \( (P < 1) \), we have to take into account every possible alternative process. Moreover, for the same process the list of alternatives can be different in different situations.

A more reasonable formulation of the law is proposed below.

Consider a global cyclic process \( A \) with forward probability \( P \) and backward probability \( P^\dagger \), in which heat \( Q \) is taken from a single heat bath at temperature \( T \) and converted to work. It is formally possible to treat all the systems, except for this bath, as a single big system. The inequality (10) is valid for this big system, which means

\[
\frac{Q}{T} \leq k_B \ln \frac{P^\dagger}{P}. \tag{24}
\]
In the case $P \geq P^\dagger$ the right hand side is not positive. Thus, it is not possible to take heat from a single heat bath and convert it to work in a balanced global cyclic process. Call this statement a general Kelvin-Planck formulation of the second law. It is, in fact, very close to the original (special) formulation: it is enough to replace balanced by deterministic.

This formulation of the second law was obtained as a consequence of the fluctuation theorem. However, once it is taken for granted, we need no more appellations to statistical mechanics. Consider again the global process $\mathcal{A}$. If $P < P^\dagger$, then we cannot apply the (new formulation of) the second law directly. But we can introduce another process $\mathcal{C}$, which is the reverse to the “skewed” Szilard engine cycle described in IV B, such that $V_0/(V_0 + V_1) = P/P^\dagger$. We have $P(\mathcal{AC}) = P((\mathcal{AC})^\dagger)$, hence the cycle $\mathcal{AC}$ is balanced and $Q + Q_C \leq 0$. Heat $Q_C$, taken in the process $\mathcal{C}$, can easily be found from the Gay-Lussac law

$$Q_C = k_B T \ln(V_0/(V_0 + V_1)) = -k_B T \ln \frac{P^\dagger}{P},$$

and the inequality (24) follows. To prove the same inequality in the case $P > P^\dagger$, one can use the process $C^{-1}$.

Assuming, as usual, environment independence, we can deduce (10) from (24), following Clausius. (Note that the logic is reversed: the general case is a consequence of a special case.) So, we can use the general formulation of the second law as a fundament of nonstandard thermodynamics, basically repeating the standard scheme. In this argument the Szilard engine plays essentially the same role of a perfect machine as the Carnot engine in traditional thermodynamics. The statistical formulation of the second law should not be considered a fundamental principle, it is a consequence of the general Kelvin-Planck formulation of the law. From (24), we have the inequality

$$\left\langle e^{Q/k_B T}\right\rangle \leq \sum_i P_i^\dagger = 1,$$

which resembles the Jarzynski equality. By the Jensen inequality, $\langle Q \rangle \leq 0$.

Returning to the question considered in IV B, the Szilard engine cycle does not contradict any of the formulations of the second law. It does not contradict the standard Kelvin-Planck formulation, simply because it cannot be applied to a process which is not deterministic. It does not contradict the statistical formulation by the Bennett’s argument. And it obviously does not contradict the general Kelvin-Planck formulation.

D. The total entropy

It is often convenient not to call a global process cyclic unless $P = P^\dagger$. Apparently, this trick does not make much sense in general, but when it is possible, it allows to introduce a new thermodynamic function $H$, which can be interpreted as stored thermodynamic information.
The probability of a global process is equal to the product of the probabilities of the local ones. It follows that the condition $P = P^\dagger$ is a restatement of

$$\oint \sum_i \bar{d}I_i = 0,$$

where the sum is taken over all the systems. We can pretend that the sum is an exact differential and introduce a function $H$ by

$$dH = \sum_i \bar{d}I_i.$$

Following Parrondo [19, eq. (16)], the total entropy can then be redefined by

$$S_{tot} = \sum_i S_i + k_B H,$$  \hspace{1cm} (25)

where the sum is taken over all the systems, including heat baths.

We have the inequality [12] for a “genuine” system and the equality $dS = dQ/T$ for a heat bath. Thus, $dS_{tot} \geq \sum_i dQ_i/T_i$. But the sum on the right hand side is equal to zero. Heat taken by a system is heat taken from a bath, so all the terms in this sum are canceled.

So, $S_{tot}$ defined by (25) is a nondecreasing function, unlike the “naive” total entropy which does not include the term $k_B H$. This term can formally be interpreted as the entropy of a fictitious system, called a buffer. This is convenient, because we can use the language of weak thermodynamics; that is, we can pretend that the process is deterministic. Thermodynamic information must then be interpreted as entropy, transferred adiabatically from a system to the buffer

$$d\bar{S} = -k_B \bar{d}I,$$

and the general Clausius inequality [10] must be considered a special case of (8).

### E. Thermodynamics of the Szilard engine

Taking into account the entropy of the buffer allows us to take a fresh look at nonstandard thermodynamics. We consider below the Szilard engine as an example. The author would like to point out some essential differences from what one may find in the literature. We consider what happens if the particle gets into a particular part of the engine. What happens otherwise is ignored, as it is a different process. We follow [10], so the main subject is the Clausius entropy of the engine, but we give a much more detailed account of it.

The engine, which is connected to a heat bath at temperature $T_1$, is operated by a controller (called a Maxwell’s demon). There is no need to treat a controller as a thermodynamic system, it may well be considered a kind of external agent instead. But a controller needs a memory, which
certainly is a thermodynamic system. In the cycle, it undergoes an erasure of information process, accompanied by heat generation. The heat must be absorbed by a heat bath, connected to the memory. This is a different bath, at temperature $T_2$ (Fig. 9).

The memory is a usual 1-bit memory device considered in III E, which is initially in “zero” state. After the partition is inserted, the controller measures the position of the particle and brings the memory to “zero” or “one”, depending on where the particle happens to be. We consider the case when the memory remains in “zero” state all the time (but another case is not very different).

We do not assume that the engine or the memory are “straight”; it is possible that $V_0 \neq V_1$ and $p_0 \neq p_1$. However, we have to assume that $p_0 / p_1 = V_0 / V_1$. Otherwise, we would have $P \neq P^1$ for the global process, so it would not be a proper cycle. The natural entropy unit in this case is $u = -k_B \ln p_0$ (which is $k_B \ln 2$ if $p_0 = p_1$).

The cycle consists of four steps.

A. Insertion The piston is inserted into the box, such that the particle is trapped in the volume $V_0$. The entropy of the engine decreases by $u$. By now, $H = -\ln p_0$ (thermodynamic information is stored in the engine), hence the entropy of the buffer increases by $u$. The formal explanation is adiabatic entropy transfer to the buffer (Fig. 9).

B. Measurement Actually, nothing is changed. However, it is now more natural to assume that information is stored not in the engine alone, but in the pair engine + memory. (Because the states of two systems are correlated. Strictly speaking, this is not a meaningful fact when we consider a single process.)

C. Expansion The piston moves to the wall. The single-particle gas performs work at the expense of heat taken from the bath. The entropy of the engine increases by $u$ due to this heat. (Now information is stored in the memory only.)

D. Erasure This is the process $A$ considered in III E. In terms of “buffer thermodynamics”, entropy is transferred from the buffer to the memory. The entropy of the memory does not actually increase because of heat exchange with the bath.

The net result of the whole cycle is adiabatic entropy transfer between two heat baths, mediated by the buffer. Work done on the engine is $-uT_1$, work done on the memory is $uT_2$. Basically, what we have is a heat engine, but heat undergoes an unusual transformation

\[
\text{heat} \Rightarrow \text{work} + \text{thermodynamic information} \Rightarrow \text{heat}.
\]

V. CONCLUSIONS

In this paper the foundations of the Clausius inequality are discussed. It is shown that, strictly speaking, the inequality is not a consequence of first principles of thermodynamics. So, the validity of the inequality in general should be considered an open problem, not an established fact.
Fig. 9 The Szilard engine cycle. The engine is connected to a heat bath at temperature $T_1$. The memory on the right is connected to a bath at temperature $T_2$. The dotted rectangle represents the buffer. Entropy of each system at each step is shown in a square box. Entropy transfer is shown by arrows.

To put it into other words, if we do not make as many implicit assumptions as usual, then a hypothetical violation of the Clausius inequality does not contradict any of the known laws of physics. It can be explained by adiabatic entropy transfer between two systems (not accompanied by energy transfer or matter exchange). On this basis, one can build a consistent version of classical thermodynamics, called *weak thermodynamics*.

Whether weak thermodynamics has something to do with the physical reality or not, it may at least be useful for some purposes. The Clausius inequality is a falsifiable statement. One can test it in a laboratory; in this respect, weak thermodynamics may play a similar role to the post-Newtonian
formalism which is a tool in tests of general relativity. Also, it may be of help in analysis of exotic processes, which could not be analyzed properly within the standard thermodynamic framework.

In this paper, an extension of classical thermodynamics is proposed. This theory, called nonstandard thermodynamics, is supposed to describe thermodynamic processes which may be observed with some probability. This includes ergodicity breaking, some kind of fluctuations, and other phenomena beyond the scope of standard thermodynamics. The principal tool is a generalized Clausius inequality [10], which may be considered one of the possible formulations of the Landauer’s principle. To a large extent, nonstandard thermodynamics is an attempt to translate some well known results, like fluctuation theorems, from the language of statistical physics to the language of classical thermodynamics. It turns out that weak thermodynamics and nonstandard thermodynamics have much in common. In fact, it is sometimes convenient to take them for the same theory.

The generalized Clausius inequality [10] is a direct consequence of the generalized Jarzynski equality [14 Eq.(7)]. This equality has a solid theoretical basis and was tested in feedback control experiments [23]. For this reason, there is little doubt that the standard Clausius inequality (1) can be violated in a feedback control experiment, although it may be difficult to prove this directly. If this inequality can be violated on macroscale is a more difficult question, which deserves some attention.

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