Meromorphic Continuation of Weighted Zeta Functions on Open Hyperbolic Systems

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Abstract: In this article we prove meromorphic continuation of weighted zeta functions \(Z_f\) in the framework of open hyperbolic systems by using the meromorphically continued restricted resolvent of Dyatlov and Guillarmou (Ann Henri Poincaré 17(11):3089–3146, 2016). We obtain a residue formula proving equality between residues of \(Z_f\) and invariant Ruelle distributions. We combine this equality with results of Guillarmou et al. (Ann Henri Lebesgue 4:81–119, 2021) in order to relate the residues to Patterson–Sullivan distributions. Finally we provide proof-of-principle results concerning the numerical calculation of invariant Ruelle distributions for 3-disc scattering systems.

1. Introduction

1.1. Motivation. A closed hyperbolic surface \((M, g)\), i.e. a closed two-dimensional Riemannian manifold of constant negative curvature, can be considered as a paradigmatic mathematical model that allows the study of classically chaotic dynamics on the one hand and the investigation of spectral properties of the associated quantized system on the other hand: From the perspective of classical mechanics, the free motion of a particle on the curved configuration space \(M\) is described by the geodesic flow on the unit cosphere bundle \(SM := \{\xi \in T^*M; |\xi|^g = 1\}\), which is the unit energy shell in the classical phase space \(T^*M\). On the quantum mechanical side, the Hamiltonian operator is given by the positive Laplace–Beltrami operator \(\Delta\) acting as an unbounded self adjoint operator on \(L^2(M)\). It has discrete spectrum \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\), where we repeat the eigenvalues according to multiplicity, and we denote by \(\psi_i \in L^2(M)\) the associated eigenfunctions. On hyperbolic surfaces, it is convenient to parametrize the spectrum by \(\lambda_i = 1/4 + r_i^2\) (where \(r_i\) is real valued for eigenvalues \(\geq 1/4\) and imaginary for those \(< 1/4\)). If we fix a semiclassical Weyl quantization \(\text{Op}_h, h > 0\), sending test functions in \(C_\infty^\infty(T^*M)\) to bounded operators on \(L^2(M)\) (see e.g. [Zwo12]) then we can associate to any Laplace eigenfunction \(\psi_i\) its Wigner distribution \(W_{\psi_i} : C_\infty^\infty(T^*M) \ni \)
$g \mapsto \langle \text{Op}_{1/r_i}(g)\psi_i, \psi_i \rangle_{L^2(M)}$. This generalized function should be interpreted as a representation of the quantum state $\psi_i$ on the classical phase space $T^*M$.

In this setting, Anantharaman and Zelditch [AZ07] proved the remarkable result that one can define zeta functions that are given purely in terms of the classical geodesics but nevertheless allow one to express the quantum phase space distributions by means of residue formulae. More precisely, for any $f \in C^\infty(SM)$ they consider the following weighted zeta function

$$Z_{g}^{AZ}(s) := \sum_{\gamma} \frac{e^{-sT_{\gamma}}}{1 - e^{-T_{\gamma}}} \int_{\gamma}^\# f,$$  \hspace{1cm} (1.1)

where the sum runs over all closed oriented geodesics $\gamma \subset SM$, $T_{\gamma}$ is the period length of the closed geodesic and $\gamma^\#$ is the corresponding primitive geodesic. By an elementary estimate on the number of periodic geodesics with period length $\leq T$ one directly obtains that $Z_{g}^{AZ}(s)$ defines a holomorphic function for $\text{Re}(s) > 1$. Anantharaman and Zelditch now show the following:

**Theorem 1.1** (Anantharaman-Zelditch 2007). If $g \in C^\infty_c(T^*M)$ such that $g|_{SM}$ is real analytic then $Z_{g|_{SM}}^{AZ}(s)$ admits a meromorphic continuation to $\mathbb{C}$.

Furthermore for any $r > 0$, if $1/2 + ir$ is a pole of $Z_{g|_{SM}}^{AZ}(s)$ then $1/4 + r^2$ is a Laplace eigenvalue and one has the following spectral interpretation of the residue:

$$\text{Res}_{s=\frac{1}{4}+ir} \left[Z_{g|_{SM}}^{AZ}(s)\right] = \sum_{\lambda_i = \frac{1}{4}+r^2} W_{\psi_i}(g) + O(1/r),$$

For their proof, Anantharaman and Zelditch develop the notion of Patterson–Sullivan distribution and their techniques for meromorphic continuation and the residue formula heavily rely on the Lie-theoretic description of hyperbolic surfaces.

The aim of this article is to show that microlocal analysis allows the meromorphic continuation of weighted zeta functions very similar to (1.1) but in the considerably more general setting of open hyperbolic dynamical systems as studied in [DG16]. We also obtain a spectral interpretation of poles and residues of these weighted zeta functions in terms of Ruelle resonances and resonant states. Combining these results with previously obtained results on the quantum classical correspondence we recover the above mentioned theorem of Anantharaman and Zelditch in the special case of geodesic flows on closed hyperbolic surfaces.

### 1.2. Statement of results.

To formulate our main results let $\mathcal{M}$ be a smooth manifold with smooth (possibly empty) boundary, $X$ a smooth, nowhere vanishing vector field on $\mathcal{M}$, and $\varphi_t$ its flow. In addition, denote by $\mathcal{E}$ a smooth, complex vector bundle over $\mathcal{M}$ and by $X$ a first-order differential operator acting on sections of $\mathcal{E}$ which is related to $X$ via the Leibniz rule

$$X(fu) = (Xf)u + f(Xu), \quad f \in C^\infty(\mathcal{M}), \quad u \in C^\infty(\mathcal{M}, \mathcal{E}).$$

The dynamics of interest from the point of view of our application happen on the trapped set $K$ of $\varphi_t$, i.e. on the set

$$K := \{x \in \mathcal{M} \mid \varphi_t(x) \text{ defined } \forall t \in \mathbb{R} \text{ and } \exists \text{cpt. } A \subseteq \mathcal{M} \text{ with } \varphi_t(x)$$
Finally, we make the following dynamical assumptions:

(1) $K$ is compact,
(2) $K \subseteq \mathcal{M}$, with $\mathcal{M}$ the manifold interior of $\mathcal{M}$,
(3) $\varphi_t$ is hyperbolic on $K$.

For the formal definition of hyperbolicity see Sect. 2. Note that there are rich classes of dynamical systems that satisfy the above assumptions, e.g. Anosov flows on closed manifolds (in particular geodesic flows on closed manifolds with negative sectional curvature), Axiom A flows near a basic set (e.g. geodesic flows on asymptotically hyperbolic manifolds in the sense of Mazzeo-Melrose and more concretely geodesic flows on convex co-compact hyperbolic manifolds, see [DG16, Section 5.3]), or billiard flows obtained by non-grazing convex obstacles scattering (see [KSW21]).

In the setting just described one can define a discrete subset of the complex plane $\mathbb{C}$ called Pollicott-Ruelle resonances of $X$, directly based on the work of Dyatlov and Guillarmou [DG16]. They arise as the poles of the meromorphic continuation of the resolvent $(X + \lambda)^{-1}$ and any such resonance $\lambda_0$ is associated with a finite rank residue operator $\Pi_{\lambda_0}$. It is possible to obtain a precise wavefront set estimate for $\Pi_{\lambda_0}$ which in particular guarantees the existence of the flat trace $\text{tr}^b$ and therefore the well-definedness of the invariant Ruelle distributions $C^\infty_c(\mathcal{M}) \ni f \mapsto \text{tr}^b(\Pi_{\lambda_0} f)$. Here the flat trace is defined by restriction of the distributional kernel to the diagonal and subsequent integration.

With these ingredients we can formally define our object of main interest, namely the weighted zeta function with weight $f \in C^\infty(\mathcal{M})$ at $\lambda \in \mathbb{C}$:

$$Z^X_f(\lambda) := \sum_{\gamma} \left( \frac{\exp(-\lambda T_{\gamma}) \text{tr}(\alpha_{\gamma})}{|\det(\text{id} - \mathcal{P}_{\gamma})|} \int_{\gamma^\#} f \right),$$

where the sum is over all closed trajectories $\gamma$ of $\varphi_t$, $T_{\gamma}$ is its period, $\gamma^\#$ denotes the corresponding primitive closed trajectory, $\mathcal{P}_{\gamma}$ its linearized Poincaré map and $\alpha_{\gamma}$ the parallel transport map in $\mathcal{E}$ associated with $X$. For formal definitions of these objects we again refer the reader to Sect. 2. Our main result now reads as follows:

**Theorem 1.2** (Meromorphic Continuation of Weighted Zetas I). $Z^X_f$ converges absolutely in $\{\text{Re}(\lambda) \gg 1\}$ and continues meromorphically to $\mathbb{C}$. Any pole $\lambda_0$ of $Z_f$ is a Pollicott-Ruelle resonance of $X$ and if the resolvent has a pole of order $J(\lambda_0)$ at $\lambda_0$ then for $k \leq J(\lambda_0)$ we have

$$\text{Res}_{\lambda=\lambda_0} \left[ Z^X_f(\lambda)(\lambda - \lambda_0)^k \right] = \text{tr}^b \left( (X - \lambda_0)^k \Pi_{\lambda_0} f \right).$$

**Remark 1.3.** Note that in the particularly simple situation of a trivial, one-dimensional bundle $\mathcal{E} = \mathcal{M} \times \mathbb{C}$ and $X = X$ our weighted zeta simplifies and is given by the following expression:

$$Z_f(\lambda) := Z^X_f(\lambda) = \sum_{\gamma} \left( \frac{\exp(-\lambda T_{\gamma})}{|\det(\text{id} - \mathcal{P}_{\gamma})|} \int_{\gamma^\#} f \right).$$
If $G$ is a semisimple Lie group with finite center and real rank 1, $K \subset G$ a maximally compact subgroup and $\Gamma \subset G$ a discrete, cocompact and torsion free subgroup, then the rank one locally symmetric space $M = \Gamma \backslash G / K$ is a closed manifold of strictly negative curvature. Thus the results of Theorem 1.2 apply to this setting. Furthermore recent results on quantum classical correspondences [GHW21] building on ideas developed in [DFG15] show that the invariant Ruelle distribution $\mathcal{C}^\infty(SM) \ni f \mapsto \text{tr}^\rho(\Pi_{\lambda_0} f)$ is related to quantum phase space distributions in the semiclassical limit. Using these results we recover the result of Anantharaman and Zelditch for hyperbolic surfaces and generalize it to arbitrary rank one locally symmetric spaces (see Theorem 4.1 for a slightly more detailed version).

**Theorem 1.4** (Quantum Phase Space Distributions as Residues). Let $M = \Gamma \backslash G / K$ be a compact Riemannian locally symmetric space of rank one, $\Delta_M$ its Laplacian and $\varphi_t$ the geodesic flow on $SM$. Let $\rho > 0$ denote the half-sum of the restricted roots of $G$. Then the following holds:

Given $r > 0$ such that $-\rho + ir$ is a Ruelle resonance of $\varphi_t$ then $\rho^2 + r^2$ is an eigenvalue of $\Delta_M$ and for any $g \in \mathcal{C}^\infty_c(T^*M)$

$$\text{Res}_{\lambda = -\rho + ir} \left[ Z_g|_{SM}(\lambda) \right] = \sum_{l=1}^{m} W_{\varphi_l}(g) + O(1/r),$$

where the sum on the right-hand side extends over an orthonormal $L^2$-basis of the $\Delta_M$-eigenspace with eigenvalue $\rho^2 + r^2$.

### 1.3. Applications.

Let us mention two applications of our main Theorem 1.2: For certain dynamical systems that allow for a simple combinatorial coding of the closed orbits it has been observed that Ruelle resonances can be calculated numerically in a very efficient manner by finding zeros of a dynamical determinant [Bor14, BW16, BPSW21]. These dynamical determinants are closely related to our weighted zeta functions. In fact, the logarithmic derivative of the dynamical determinant yields the weighted zeta function for the constant weight $f \equiv 1$. For the same reasons that the dynamical determinant can be efficiently calculated from the periodic orbits, we can also calculate the poles and residues of our weighted zeta functions with adapted numerical algorithms. This allows not only to perform numerical experiments on how resonances are distributed in the complex plane, but also to study the behavior of the invariant Ruelle distributions for the first time. In the appendix we show some first numerical results for the obstacle scattering on three hard discs and already observe an interesting common localization pattern of the invariant Ruelle distributions of resonances that are very close to the spectral gap (see Fig. 4). Besides the 3-disc system there exists another practically feasible dynamical system where the numerical algorithms work very well, namely geodesic flows on convex co-compact hyperbolic surfaces. We conjecture that an analogous statement to Theorem 1.4 also holds for convex co-compact hyperbolic manifolds. This would then provide an efficient algorithm for the numerical calculation of localization properties of Laplace resonances on Schottky surfaces.

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1 Important special cases of such rank one locally symmetric spaces are $d$-dimensional manifolds of constant negative curvature (corresponding to $G = \text{PSO}(1, d)$, $K = \text{PSO}(d)$) and even more particularly hyperbolic surfaces ($G = \text{PSO}(1, 2) \cong \text{PSL}(2, \mathbb{R})$, $K = \text{PSO}(2)$). The latter coincides with the setting of Theorem 1.1.

2 Actually we even slightly improve it, by allowing the much more permissive class of smooth instead of real analytic test functions.
A second application of our main theorem is presented in [BSW22]: Three decades ago Eckhardt et al. [EFMW92] used semiclassical trace formulae to derive (in a mathematically non-rigorous way) weighted zeta functions and to predict that their residues are given by quantum phase space distributions in a semiclassical limit. In [BSW22] we explain how Theorem 1.4 can be interpreted as a rigorous version of these theoretical predictions on rank-one locally symmetric spaces. Furthermore we detail how for more general physical systems (such as convex obstacle scattering) Theorem 1.2 allows us to reinterpret the residue formulae of [EFMW92] in terms of rigorously defined twisted invariant Ruelle distributions which we investigate numerically. For symmetric 3-disc systems we find a surprisingly good agreement between the twisted invariant Ruelle distributions and quantum phase space distributions. This had been predicted by theoretical physics but does—to our best knowledge—still lack any mathematically rigorous justification until today.

1.4. Paper organization. We begin with a detailed introduction to our geometric setting in Sect. 2. This entails first the notion of open hyperbolic systems as used in [DG16] (Sect. 2.1) and second a setup quite similar to open hyperbolic systems but without the requirement of strict convexity (Sect. 2.2). In particular we define the restricted resolvent, Pollicott–Ruelle resonances and invariant Ruelle distributions. For our numerics it turns out to be useful to be able to restrict the invariant Ruelle distributions to certain hypersurfaces transversal to the flow (Sect. 2.3).

The following Sect. 3 introduces weighted zeta functions formally and proves their meromorphic continuation using a weighted trace formula (Sect. 3.1) and wavefront estimates for the restricted resolvent (Sects. 3.2–3.3). The proofs mostly follow the approaches of [DZ16] and [DG16] and in many parts we only outline how the arguments in these references must be adapted to fit our present situation. For full details we refer to the first author’s PhD thesis [Sch23].

In Sects. 2–3 we worked with the setting of general hyperbolic flows with compact trapped set. In Sect. 4 we restrict to the special case of geodesic flows on compact locally symmetric spaces of rank one. For these systems our residue formula (1.4) can be interpreted as a classical description of quantum mechanical phase space distributions known as Patterson–Sullivan distributions. This allows us to extend results of Anantharaman and Zelditch [AZ07] as well as Emonds [Emo14] to general rank one, compact, locally symmetric spaces and arbitrary smooth weight functions \( f \) on their unit tangent bundle.

One of our main motivations for proving Theorem 1.2 is the fact that dynamical zeta functions can often be calculated numerically and in a very efficient manner. Our theorem then immediately allows us to calculate the invariant Ruelle distributions numerically. So far these distributions have not been studied in depth and we hope that they encode certain aspects of the spectrum similar to the Laplace eigenfunctions in quantum chaos. As a first proof-of-principle we show in Appendix A some exemplary numerical calculations of invariant Ruelle distributions for the 3-disc scattering systems. Even in the few plots provided here one can already observe some interesting behavior which could serve as a starting point for more in depth numerical investigations.

2. Geometric Setup

First, we introduce the notion of open hyperbolic system as it was used in [DG16] in Sect. 2.1. This setting will allow us to employ the meromorphically continued restricted
resolvent which makes the upcoming proof of our main theorem in Sect. 3.2 rather straightforward later on. Afterwards in Sect. 2.2 we extend these results slightly by removing the requirement of strict convexity following ideas developed in [GMT21]. This will allow us to give a more practically feasible formulation of our main result in Sect. 3.3. The underlying idea was already used by the authors in their paper [KSW21] together with Benjamin Küster in the slightly more concrete setting of convex obstacle scattering. We finish this section with the proof of a dimensional reduction technique for invariant Ruelle distributions which is essential for the numerical calculations in “Appendix A”.

2.1. Open hyperbolic systems. Let $\overline{U}$ be a compact, $n$-dimensional smooth manifold with interior $U$ and smooth boundary $\partial U$. We may have $\partial U = \emptyset$, i.e. the case of a closed manifold. The dynamical object of main interest is a non-vanishing $C^\infty$-vector field $X$ on $\overline{U}$. We denote by $\varphi_t = e^{tX}$ the corresponding flow on $\overline{U}$, which for $\partial U \neq \emptyset$ will generally not be complete.

Before stating the dynamical requirements for $X$ we have to introduce some auxiliary objects. Concretely, let $\rho \in C^\infty(\overline{U}, [0, \infty])$ be a boundary defining function, that is $\rho^{-1}(0) = \partial U$ and $\partial \rho(x) \neq 0$ for any $x \in \partial U$. Boundary defining functions always exist [Lee12, p. 118]. Furthermore, let $M \supseteq \overline{U}$ be an embedding into a compact manifold without boundary, such that $X$ extends onto $M$ in a manner making $\overline{U}$ convex:

$$x, \varphi_T(x) \in U \text{ for some } T > 0 \implies \varphi_t(x) \in U \forall t \in [0, T].$$

Such an ambient manifold $M$ and continuation of $X$ always exist by [DG16, Lemma 1.1], given that (A1) below holds. Then [DG16, Lemma 1.2] asserts that the trapped set $K$ is contained in the interior $U$, i.e.

$$\Gamma_\pm := \bigcap_{t \geq 0} \varphi_t(U), \quad K := \Gamma_+ \cap \Gamma_- \subseteq U.$$

The set $K$, called the trapped set of $X$, is independent of $M$ and the extension of $X$. Finally, let $E$ be a complex $C^\infty$-vector bundle over $\overline{U}$ and $X : C^\infty(\overline{U}, E) \to C^\infty(\overline{U}, E)$ a first order differential operator.

With these preliminaries given we call the triple $(\mathcal{U}, \varphi_t, X)$ an open hyperbolic system if it satisfies the following dynamical requirements:

(A1) The boundary $\partial U$ is strictly convex, i.e.:

$$x \in \partial U, \ (X\rho)(x) = 0 \implies X(X\rho)(x) < 0.$$

This condition is independent of the choice of $\rho$.

(A2) The flow $\varphi_t$ is hyperbolic on $K$, i.e. for any $x \in K$ the tangent space splits as

$$T_x M = \mathbb{R} \cdot X(x) \oplus E_s(x) \oplus E_u(x),$$

where $E_s$ and $E_u$ are continuous in $x \in K$, invariant under $\varphi_t$ and there exist constants $C_0, C_1 > 0$ with

$$\|d\varphi_t(x)v\|_{T_{\varphi(t)x)M}} \leq C_0 \exp(-C_1 t)\|v\|_{T_x M}, \quad t \geq 0, \quad v \in E_s(x),$$

$$\|d\varphi_t(x)v\|_{T_{\varphi(t)x)M}} \geq C_0^{-1} \exp(C_1 t)\|v\|_{T_x M}, \quad t \geq 0, \quad v \in E_u(x).$$

We keep the notation $X$ and $\varphi_t$ for the continuations of our vector field and its flow; note that the continued flow is now complete by compactness of $M$ so the upcoming definitions make sense.
\| \cdot \| \text{ denotes any continuous norm on the tangent bundle.}\footnote{The definition does not depend on the specific norm as } K \text{ is compact. The same holds for upcoming arbitrary choices of densities, etc.}

(A3) \( X \) satisfies

\[
X(fu) = (Xf)u + f(Xu), \quad f \in C^\infty(\overline{U}), \quad u \in C^\infty(\overline{U}, \mathcal{E}).
\] (2.2)

We also denote by \( X \) an arbitrary extension onto \( M \) that still satisfies (2.2).

Within the setting just described we will need some additional dynamical objects derived from the flow \( \varphi_t \) and the operator \( X \). We start by fixing a smooth density on \( M \) and a smooth scalar product on \( \mathcal{E} \). That lets us consider the transfer operator \( \exp(-tX) : L^2(M, \mathcal{E}) \to L^2(M, \mathcal{E}) \) associated with \( X \), i.e. the solution semigroup of

\[
\frac{\partial}{\partial t} v(t, x) = (-Xv)(t, x), \quad v(0, \cdot) = u.
\]

This allows us to define the linear parallel transport map at \( x \in U \) and \( t > 0 \) such that \( \varphi_t(x) \in U \):

\[
\alpha_{x,t} : \mathcal{E}_x \longrightarrow \mathcal{E}_{\varphi_t(x)}
\]

\[
u \longmapsto (e^{-tX}u)(\varphi_t(x)),
\]

where \( u \) is some smooth section of \( \mathcal{E} \) with \( u(x) = u \). The definition does not depend on the choice of \( u \) because \( u(x) = 0 \) implies \( (e^{-tX}u)(\varphi_t(x)) = 0 \) by [DG16, Eq. (0.8)] and an expansion in terms of a local frame for \( \mathcal{E} \). In particular, if \( \gamma(t) = \varphi_t(x_0) \) is a closed orbit with period \( T_\gamma \) then \( \alpha_{x_0,T_\gamma} = \alpha_{x_0,t}^{-1} \circ \alpha_{\gamma(t),T_\gamma} \circ \alpha_{x_0,t} \) and the trace

\[
\text{tr}(\alpha_{\gamma}) := \text{tr}(\alpha_{\gamma(t),T_\gamma})
\]

is well-defined independent of \( t \).

Next we define the so-called linearized Poincaré map which at \( x \in U \) and \( t > 0 \) satisfying \( \varphi_t(x) \in U \) is defined as the linear map

\[
P_{x,t} := d\varphi_{-t}(x) \big|_{E_x(x) \oplus E_u(x)}.
\]

Again we will use this map if \( x \) is located on a closed trajectory \( \gamma \). In this case \( P_{x,T_\gamma} \) is an endomorphism and \( \det(\text{id} - P_{\gamma}) := \det(\text{id} - P_{x,T_\gamma}) \) is independent of \( x \) because similarly as for the parallel transport one finds that \( P_{x,T_\gamma} \) is conjugate to any \( P_{y,T_\gamma} \) provided \( y \) also belongs to \( \gamma \).

We finish this section by recasting the definition of invariant Ruelle distributions from [GHW21] into our setting: Let \( (\overline{U}, \varphi_t, X) \) be an open hyperbolic system, then we denote by \( R(\lambda) := 1_{\overline{U}}(X + \lambda)^{-1}1_{\overline{U}} : C^\infty_c(\overline{U}, \mathcal{E}) \to \mathcal{D}'(\overline{U}, \mathcal{E}) \) its restricted resolvent. In [DG16] it was proven that \( R(\lambda) \) continues meromorphically onto \( \mathbb{C} \) and its poles are called Ruelle resonances. Furthermore, the residue of \( R(\lambda) \) at a resonance \( \lambda_0 \) is a finite rank operator \( \Pi_{\lambda_0} : C^\infty_c(\overline{U}, \mathcal{E}) \to \mathcal{D}'(\overline{U}, \mathcal{E}) \) that satisfies [DG16, Thm. 2]

\[
X\Pi_{\lambda_0} = \Pi_{\lambda_0}X, \quad \text{supp}(K_{\Pi_{\lambda_0}}) \subseteq \Gamma_+ \times \Gamma_- \quad \text{WF}'(\Pi_{\lambda_0}) \subseteq E_+^* \times E_-^*,
\] (2.3)
support property in (2.3) shows that the restriction $K_{\Pi_{\lambda_0}}|_{\Delta}$ to the diagonal $\Delta \subseteq U \times U$ is supported in $\Gamma_+ \cap \Gamma_- = K$ and therefore yields an element of $\mathcal{E}'(U)$. Now combining this with the wavefront estimate [DG16, Eq. (0.14)] of $K_{\Pi_{\lambda_0}}$ guarantees that the following map is well-defined:

$$T_{\lambda_0} : \left\{ \begin{array}{l}
C^\infty(U) \longrightarrow \mathbb{C} \\
f \longmapsto \text{tr}^b (f \Pi_{\lambda_0} f)
\end{array} \right..$$

(2.4)

The generalised density $T_{\lambda_0}$ is called the invariant Ruelle distribution associated with $\lambda_0$. Note that we can re-write $T_{\lambda_0}(f) = \text{tr}^b (\Pi_{\lambda_0} f f^b) = \text{tr}^b (f \Pi_{\lambda_0})$ as the trace is cyclic.

The adjective invariant is justified because $T_{\lambda_0}$ is indeed invariant under the flow: The Leibniz rule (2.2) implies $\Pi_{\lambda_0}(X f) = \Pi_{\lambda_0} X f - f \Pi_{\lambda_0} f X$ which together with the vanishing of the commutator (2.3) and the cyclic property yields

$$T_{\lambda_0}(X f) = \text{tr}^b (\Pi_{\lambda_0} X f - f \Pi_{\lambda_0} f X)$$

$$= \text{tr}^b (X \Pi_{\lambda_0} f - \Pi_{\lambda_0} f X) = 0.$$

2.2. Removing strict convexity for resolvents. In practical applications it often turns out to be difficult to verify the strict convexity condition. To circumvent this difficulty we recast the meromorph continuation achieved in [DG16] into a simpler setting by constructing a perturbation of the generator $X$ of $\varphi_t$ and of the operator $X$. Our proof mostly follows [GMT21] which uses techniques developed in [CE71] and [Rob80]. For a very similar application but without the extension to the vector valued case see [DG18].

Let $\mathcal{M}$ be a smooth manifold with manifold interior $\tilde{\mathcal{M}}$ and smooth, possibly empty boundary, $X$ a smooth, nowhere vanishing vector field on $\mathcal{M}$, and $\varphi_t$ the flow associated with $X$. Furthermore, let a smooth, complex vector bundle $\mathcal{E}$ over $\mathcal{M}$ together with a first-order differential operator $X : C^\infty(\mathcal{M}, \mathcal{E}) \rightarrow C^\infty(\mathcal{M}, \mathcal{E})$ be given. We assume that $X$ satisfies the Leibniz rule (2.2), that the trapped set $K$ of $\varphi_t$ defined in (1.2) is compact and satisfies $K \subseteq \tilde{\mathcal{M}}$, and finally that $\varphi_t$ is hyperbolic on $K$ as defined in (2.1).

First we observe that we may assume $\mathcal{M}$ to be compact with smooth, non-empty boundary: If it is not then there exists a compact submanifold with smooth boundary of $\mathcal{M}$ which still contains $K$ in its manifold interior. This follows by standard tools in smooth manifold theory, namely via a smooth exhaustion function ([Lee12, Prop. 2.28]) combined with Sard’s theorem. If necessary, we may replace $\mathcal{M}$ with this submanifold.

Now [GMT21, Prop. 2.2 and Lemma 2.3] immediately yields the existence of a compact $\bar{U}_0 \subseteq \mathcal{M}$ with manifold interior $U_0$ and smooth boundary $\partial U_0$ such that $K \subseteq U_0$, and the existence of a smooth vector field $X_0$ on $\mathcal{M}$ such that $\partial U_0$ is strictly convex w.r.t. $X_0$ in the above sense and $X - X_0$ is supported in an arbitrarily small neighbourhood of $\partial U_0$. In addition, the trapped set of $X_0$ coincides with $K$. Finally, we may assume both the flow $\varphi_t$ as well as the flow $\varphi^0_t$ of $X_0$ to be complete by embedding $\mathcal{M}$ into an ambient closed manifold and extending $X, X_0$ arbitrarily. In this setting we define the escape times from some compact set $A \subseteq \mathcal{M}$ to be

$$\tau^\pm_A(x) := \pm \sup \left\{ t \geq 0 \mid \varphi_{\pm s}(x) \in A \forall s \in [0, t] \right\},$$

$$\tau^0_{A, \pm}(x) := \pm \sup \left\{ t \geq 0 \mid \varphi^0_{\pm s}(x) \in A \forall s \in [0, t] \right\}, \quad x \in A.$$

The respective forward and backward trapped sets are then given by

$$\Gamma_+(A) := \left\{ x \in A \mid \tau_A^-(x) = \mp \infty \right\}, \quad \Gamma_0^+(A) := \left\{ x \in A \mid \tau^0_{A, -}(x) = \mp \infty \right\},$$
such that in particular we have \( K = \Gamma_+ (\overline{U}_0) \cap \Gamma_- (\overline{U}_0) \).

Next we require an appropriate perturbation of \( X \) which satisfies the Leibniz rule (2.2) with respect to \( X_0 \) instead of \( X \). But this is straightforward: Given any section \( u \in C^\infty (\overline{U}, \mathcal{E}) \) and \( x \in \overline{U} \) we consider a local frame \( e_i \) in a neighbourhood of \( x \). Then \( u \) expands as \( u = \sum_1 u^i e_i \) in this neighbourhood and we define

\[
X_0 u := Xu + \sum_i \left( (X_0 - X) u^i \right) e_i.
\]

The second term on the right-hand side obviously yields a well-defined first-order differential operator and \( X - X_0 \) is supported near \( \partial U_0 \). Given \( f \in C^\infty (\overline{U}) \) we can verify the Leibniz rule via the following calculation in a neighbourhood of \( x \):

\[
X_0 (f \cdot u) = X(f \cdot u) + \sum_i \left( (X_0 - X) f u^i \right) e_i
\]

\[
= (Xf) \cdot u + f \cdot (Xu) + (X_0 f - Xf) \cdot u + f \cdot \sum_i \left( (X_0 - X) u^i \right) e_i
\]

\[
= f \cdot (X_0 u) + (X_0 f) u.
\]

We are now in a position to state and prove the main result of this section. For some open set \( \mathcal{O} \subseteq \mathcal{M} \), a section \( u \in C^\infty_c (\mathcal{O}, \mathcal{E}) \), and a spectral parameter \( \lambda \in \mathbb{C} \) consider the resolvent given by the following formal integral:

\[
R_{\mathcal{O}}(\lambda) u(x) := \int_0^{-\tau^- (x)} e^{-(X+\lambda) t} u(x) dt,
\]

where \( \exp (\tau^- X t) \) denotes the transfer operator associated with \( X \) as defined in the previous section. At this point, the expression in (2.5) remains formal because on the one hand \( \tau^- \) may be of low regularity and on the other hand the integral may not converge for \( \tau^- (x) = -\infty \). By choosing an appropriate \( \mathcal{O} \) we can get around these issues and obtain the following theorem:

**Theorem 2.1 (Meromorphic Resolvent Without Strict Convexity).** Let \( \mathcal{M} \) be a smooth manifold with smooth boundary, \( X \) a smooth, nowhere vanishing vector field on \( \mathcal{M} \), and \( K \) the trapped set of the flow \( \varphi_t \) of \( X \). Also, let \( \mathcal{E} \) be a smooth, complex vector bundle over \( \mathcal{M} \) and \( X \) a first-order differential operator on \( \mathcal{E} \).

Assume that \( X \) satisfies the Leibniz rule (2.2), \( K \) is compact, \( K \subseteq \mathcal{M} \), and \( \varphi_t \) is hyperbolic on \( K \). Then there exists an arbitrarily small compact \( \overline{U} \subseteq \mathcal{M} \) which contains \( K \) in its interior \( \mathcal{U} \) such that

1. \( R_{\mathcal{U}}(\lambda) \) is well-defined for \( \text{Re}(\lambda) \gg 0 \) as an operator on \( L^2 (\mathcal{M}, \mathcal{E}) \) and satisfies

\[
(X + \lambda) R_{\mathcal{U}}(\lambda) = \text{id}_{C^\infty_c (\mathcal{U}, \mathcal{E})},
\]

2. \( R_{\mathcal{U}}(\lambda) \) continues meromorphically to \( \mathbb{C} \) as a family of operators \( C^\infty_c (\mathcal{U}, \mathcal{E}) \rightarrow \mathcal{D}' (\mathcal{U}, \mathcal{E}) \) with poles of finite rank,

3. the residue \( \Pi_{\lambda_0} \) of \( R_{\mathcal{U}}(\lambda) \) satisfies (2.3).

**Proof.** For this proof we assume the setting and notation introduced in this section. Observe that both \( X = X_0 \) and \( X = X_0 \) on any compact \( \overline{U} \) with \( \overline{U} \subseteq U_0 \). If we additionally had for all \( t \geq 0 \) the property

\[
x, \varphi^0_t (x) \in \overline{U} \implies \varphi^0_s (x) \in \overline{U} \forall s \in [0, t],
\]

then we have

\[
R_{\mathcal{U}}(\lambda) u(x) := \int_0^{-\tau^- (x)} e^{-(X+\lambda) t} u(x) dt,
\]

where \( \tau^- \) denotes the transfer operator associated with \( X \) as defined in the previous section. At this point, the expression in (2.5) remains formal because on the one hand \( \tau^- \) may be of low regularity and on the other hand the integral may not converge for \( \tau^- (x) = -\infty \). By choosing an appropriate \( \mathcal{O} \) we can get around these issues and obtain the following theorem:

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\[
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\]

2. \( R_{\mathcal{U}}(\lambda) \) continues meromorphically to \( \mathbb{C} \) as a family of operators \( C^\infty_c (\mathcal{U}, \mathcal{E}) \rightarrow \mathcal{D}' (\mathcal{U}, \mathcal{E}) \) with poles of finite rank,

3. the residue \( \Pi_{\lambda_0} \) of \( R_{\mathcal{U}}(\lambda) \) satisfies (2.3).
then it would immediately follow for any \( x \in \mathcal{U}, u \in C_\infty^c(\mathcal{U}, \mathcal{E}) \) and \( \text{Re}(\lambda) \gg 0 \) that
\[
R_\mathcal{U}(\lambda)u(x) = \int_0^{-\tau_{\mathcal{U}}(x)} e^{-\mathbf{(X+\lambda)t}}u(x)dt = \int_0^\infty e^{-(X_0+\lambda)t}u(x)dt = 1_{\mathcal{U}}(X_0 + \lambda)^{-1} 1_{\mathcal{U}}u(x),
\]
and the claims (1)–(3) would be shown by a straightforward application of the material from \([DG16]\) recalled in Sect. 2.1.

It therefore remains to construct an arbitrarily small \( \overline{\mathcal{U}} \) satisfying (2.6). First note that we may assume (2.6) to hold on \( \overline{\mathcal{U}}_0 \) by choosing the extension of \( X_0 \) appropriately, see \([DG16, \text{Eq. (0.2)}]\). Given any open \( \mathcal{O} \) with \( K \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}_0 \) we can then set
\[
\overline{\mathcal{U}} := \phi_0^0(\overline{\mathcal{U}}_0) \cap \mathcal{U}_0 \cap \phi_T^0(\overline{\mathcal{U}}_0),
\]
where choosing \( T > 0 \) large enough guarantees \( \overline{\mathcal{U}} \subseteq \mathcal{O} \) by \([DG16, \text{Lemma 1.4}]\). It is now easily verified that \( \overline{\mathcal{U}} \) satisfies (2.6).

In Sect. 3 we will show that the poles of \( R_\mathcal{U}(\lambda) \) are independent of the choice of a set \( \overline{\mathcal{U}} \) (and also independent of \( \overline{\mathcal{U}}_0 \)). They are called the Pollicott-Ruelle resonances of \( \phi_t \) and will figure prominently below as the poles of our weighted zeta function. Furthermore we can define invariant Ruelle distributions \( T_{\lambda_0} \) in the setting of Theorem 2.1 by the same formula (2.4) presented in the previous Sect. 2.1.

**Remark 2.2.** The residue \( \Pi_{\lambda_0} \) at a resonance \( \lambda_0 \) is also independent of the set \( \overline{\mathcal{U}} \) in the sense that
\[
\Pi_{\lambda_0}|_{C_\infty^c(\mathcal{U}', \mathcal{E})} = \Pi'_{\lambda_0},
\]
where \( \overline{\mathcal{U}}' \subseteq \mathcal{U} \) denotes a second set on which the resolvent can be continued meromorphically and \( \Pi'_{\lambda_0} \) denotes the residue of this continuation \( R_{\mathcal{U}'}(\lambda) \). This follows immediately by uniqueness of meromorphic continuation and the fact that the restriction of \( R_{\mathcal{U}'}(\lambda) \) to \( C_\infty^c(\mathcal{U}', \mathcal{E}) \) coincides with \( R_{\mathcal{U}'}(\lambda) \) for \( \text{Re}(\lambda) \gg 0 \). In particular, note that this implies the independence of \( T_{\lambda_0} \) from \( \mathcal{U} \). An analogous independence statement holds for \( \overline{\mathcal{U}}_0 \).

### 2.3. Restricting Ruelle distributions.

One of the main applications of our weighted zeta functions \( Z_f \) is the concrete numerical calculation of invariant Ruelle distributions \( T_{\lambda_0} \) for certain 3-dimensional dynamical systems. For ease of calculation as well as plotting we would like to reduce this to a reasonable distribution on a 2-dimensional space. In this section we present a general theorem which allows just this.

For the general setup let \( T_{\lambda_0} \in \mathcal{D}'(\mathcal{M}) \) be an invariant Ruelle distribution in the setting of Theorem 2.1. We will call a smooth submanifold \( \Sigma \subseteq \mathcal{M} \) a Poincaré section for \( \phi_t \) if \( \Sigma \) has codimension one and is transversal to \( \phi_t \) on the trapped set \( K \), i.e. for each \( x \in K \) we have
\[
T_x\mathcal{M} = \mathbb{R} \cdot X(x) \oplus T_x\Sigma,
\]
where as above \( X \) denotes the generator of \( \phi_t \).
Lemma 2.3 (Restriction of Ruelle Distributions). Let $T_{\lambda_0}$ be an invariant Ruelle distribution for a dynamical system $(\mathcal{M}, \varphi_t, \mathbf{X})$ as in Theorem 2.1 and $\Sigma \subseteq \mathcal{M}$ a Poincaré section for $\varphi_t$. Then the pullback of $T_{\lambda_0}$ along $\iota : \Sigma \hookrightarrow \mathcal{U}$ is well-defined and will be called the restriction to $\Sigma$:

$$T_{\lambda_0}|_{\Sigma} := (\iota_{\Sigma})^* T_{\lambda_0} \in \mathcal{D}'(\Sigma).$$

(2.7)

Proof. For the existence of the pullback we use the classical Hörmander condition [Hör13, Theorem 8.2.4] similar to the proof of Theorem 1.2, i.e. what we need to show is

$$\WF(T_{\lambda_0}) \cap N^*\Gamma_{\iota\Sigma} = \{0\},$$

with $N^*\Gamma_{\iota\Sigma}$ the conormal bundle to the graph of $\iota_{\Sigma}$. Now the invariance property $XT_{\lambda_0} = 0 \in C^\infty(\mathcal{M})$ proven above immediately implies the following estimate on $\WF(T_{\lambda_0})$:

$$\WF(T_{\lambda_0}) \subseteq E^*_s \oplus E^*_u \subseteq T^*\mathcal{M}.$$

Now the conormal bundle is explicitly given by

$$N^*\Gamma_{\iota\Sigma} = \left\{ (x, \xi) \mid x \in \Sigma, \xi|_{T_x\Sigma} = 0 \right\} \subseteq T^*\mathcal{M},$$

and an element $(x, \xi) \in \WF(T_{\lambda_0}) \cap N^*\Gamma_{\iota\Sigma}$ therefore satisfies $x \in K \cap \Sigma, \xi(T_x\Sigma) = 0$ and $\xi(X_x) = 0$ by the definition of $E^*_s$ and $E^*_u$. Now the transversality condition $T_x\mathcal{M} = \mathbb{R} \cdot X(x) \oplus T_x\Sigma$ yields $\xi = 0$. □

3. Proof of the Main Theorem

In this section we prove our main theorem, namely the meromorphic continuation of the following weighted zeta function defined in terms of closed trajectories $\gamma$ of an open hyperbolic system $(\mathcal{U}, \varphi_t, \mathbf{X})$ and a weight $f \in C^\infty(\mathcal{U})$:

$$Z^\mathbf{X}_f(\lambda) := \sum_{\gamma} \left( \frac{\exp(-\lambda T_{\gamma}) \text{tr}(\alpha_{\gamma})}{|\det(\text{id} - P_{\gamma})|} \int_{\gamma^#} f \right), \quad \lambda \in \mathbb{C}.$$

In Sect. 3.2 we state and prove this meromorphic continuation result using a trace formula presented in Sect. 3.1. In practice the defining properties of open hyperbolic systems are rather cumbersome to handle. We therefore remove these requirements in Sect. 3.3 to obtain the result cited in the introduction.

Full details can be found either in an earlier arXiv version of this paper or in the first author’s PhD thesis [Sch23]. At several points where rather technical passages would only require straightforward modifications of existing results we leave out the details and point to the corresponding literature instead.
3.1. A weighted trace formula. Our main tool for connecting the restricted resolvent with our weighted zeta function is a weighted version of the Atiyah–Bott–Guillemin trace formula. In this section we present the statement together with a short sketch of the proof. A formulation for open systems but without weight function can be found in [DG16, Eq. (4.6)] and an analogous result for the compact case is presented in [DZ16, Eq. (2.4)].

Lemma 3.1 (Weighted Atiyah–Bott–Guillemin Trace Formula). For any cut-offs \( \chi \in C^\infty_c(\mathbb{R}\setminus \{0\}) \) and \( \tilde{\chi} \in C_c^\infty(\mathcal{U}) \), with \( \tilde{\chi} \equiv 1 \) near the trapped set \( K \), the following holds:

\[
\text{tr}^b \left( \int_{\mathbb{R}} \chi(t) \tilde{\chi} e^{-tX} f \tilde{\chi} dt \right) = \sum_{\gamma} \frac{\chi(T_\gamma) \text{tr}(\alpha_\gamma)}{|\text{det}(id - P_\gamma)|} \int_{\gamma}^\# f,
\]

where the sum is over all closed orbits \( \gamma \) of \( \varphi_t \).

Proof. The proof is directly adapted from [DG16, Sec. 4.1] and [DZ16, App. B] but the main points can already be found in [Gui77, §2 of Lecture 2].

For a given \( \chi \in C^\infty_c(\mathbb{R}\setminus \{0\}) \) one defines \( A_{f,\chi} := \int_{\mathbb{R}} \chi(t) \tilde{\chi} e^{-tX} f \tilde{\chi} dt \) as an operator

\[
A_{f,\chi} : C^\infty(\mathcal{M}, \mathcal{E}) \rightarrow C^\infty(\mathcal{M}, \mathcal{E}) \subseteq \mathcal{D}'(\mathcal{M}, \mathcal{E}),
\]

and via the Schwartz kernel theorem we can consider the integrand \( \tilde{\chi} e^{-tX} f \tilde{\chi} \) as an operator

\[
C^\infty_c(\mathbb{R}\setminus \{0\}) \rightarrow \mathcal{D}'(\mathcal{M} \times \mathcal{M}, \mathcal{E} \boxtimes \mathcal{E}^*).
\]

Applying the Schwartz kernel theorem once more we therefore obtain as its kernel a distribution \( K_f(x, y, t) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M} \times \mathbb{R}\setminus \{0\}, \mathcal{E} \boxtimes \mathcal{E}^*) \):

\[
A_{f,\chi}(u)(x) = \int_{\mathcal{M} \times \mathbb{R}} K_f(x, y, t) u(y) \chi(t) dy dt,
\]

where \( dy \) is the same (fixed but arbitrary) density on \( \mathcal{M} \) used to define the kernel of \( A_{f,\chi} \) and \( u \in C^\infty(\mathcal{M}, \mathcal{E}) \) is any smooth section of the bundle \( \mathcal{E} \). At this point the classical [Hör13, Thm. 8.2.12] immediately shows that

\[
\text{WF}(K_{f,\chi}) \subseteq \{(x, y, \xi, \eta) \mid \exists t \in \text{supp}(\chi) \text{ with } (x, y, t, \xi, \eta, 0) \in \text{WF}(K_f)\},
\]

where \( K_{f,\chi} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}, \mathcal{E} \boxtimes \mathcal{E}^*) \) denotes the kernel of \( A_{f,\chi} \). As \( K_{f,\chi} \) is compactly supported by virtue of \( \tilde{\chi} \), we are left with the task of estimating the wavefront set of the kernel \( K_f \).

As in the references cited above we can calculate a local coordinate expression for \( K_f \) to verify that this kernel is a delta distribution on the graph \( \Gamma_\varphi := \{(x, \varphi_{-t}(x), t) \mid x \in \mathcal{M}, t \in \mathbb{R}\} \) of \( \varphi_{-t} \), multiplied by smooth functions. The latter include suitable cutoffs, the coordinate expressions of the transfer operator \( \exp(-tX) \) and the weight \( f \).

The final wavefront set is therefore contained in the conormal bundle \( N^* \Gamma_\varphi \) given by ([Hör13, Example 8.2.5]):

\[
N^* \Gamma_\varphi := \{(x, y, t, \xi, \eta, \tau) \in T^*(\mathcal{M} \times \mathcal{M} \times \mathbb{R}) \mid y = \varphi_{-t}(x), (\xi, \eta, \tau) \mid_{T(\Gamma_\varphi)} = 0\}.
\]

Given bundles \( \mathcal{E}_i \rightarrow \mathcal{M}_i \) then \( \mathcal{E}_1 \boxtimes \mathcal{E}_2 \) denotes the tensor product of the pullbacks of the \( \mathcal{E}_i \) onto \( \mathcal{M}_1 \times \mathcal{M}_2 \). Note that [DG16] uses the notation \( \text{End}(\mathcal{E}) \) for the bundle \( \mathcal{E} \boxtimes \mathcal{E}^* \) over \( \mathcal{M} \times \mathcal{M} \).
and substituting into (3.2) we may estimate the wavefront set of the right-hand side of
the trace formula as follows:

\[
WF(K_{f,\chi}) \subseteq \{(x, y, \xi, \eta) \in T^* (\mathcal{M} \times \mathcal{M}) \mid \exists t \in \text{supp}(\chi) \ x, y \in \text{supp}(\tilde{\chi}) : \\
y = \varphi_{-t}(x), \ \eta \neq 0, \ \xi = -(d\varphi_{-t}(x))^T \eta, \ (\eta, X_\gamma) = 0\}.
\] (3.3)

This set does not intersect \( \{(x, x, \xi, -\xi) \mid \xi \in T^*_x \mathcal{M}\} \) by the hyperbolicity of \( \varphi_t \) on closed trajectories. Thus, the flat trace on the left-hand side is well defined.

The pullback of \( K_f \) is supported on the closed trajectories of \( \varphi_t \). As in [DZ16, Lemma B.1], it therefore makes sense to prove a local version of the final trace formula:

**Lemma 3.2** (Local Weighted Trace Formula). Suppose \( x_0 \in U \) and \( T \neq 0 \) is such that \( \varphi_T(x_0) = x_0 \). Then there are \( \varepsilon > 0 \) and an open neighbourhood \( x_0 \in U \subseteq \mathcal{U} \) with \( \varphi_\varepsilon(x_0) \in U \) for any \( |s| < \varepsilon \) and such that for any \( \rho(x, t) = \sigma(x) \chi(t) \in C^\infty_c(U \times]T - \varepsilon, T + \varepsilon[) \) the following holds:

\[
\text{tr}^\rho \left( \int_{\mathbb{R}} \rho(x, t) K_f(x, y, t) dt \right) = \int_{\mathbb{R} \times \mathcal{M}} \rho(x, t) K_f(x, x, t) dx dt
\]
\[
= \frac{\text{tr}(\alpha_\gamma)}{|\text{det}(\text{id} - \mathcal{P}_\gamma)|} \int_{-\varepsilon}^{\varepsilon} \rho(\varphi_\varepsilon(x_0), T) f(\varphi_\varepsilon(x_0)) ds
\] (3.4)

**Proof.** Using adapted coordinates similar to the proof of [DZ16, Lemma B.1] one may reduce the flat trace to an integral of delta distributions of the general form \( \delta(x - g(x)) \). Formula (3.4) then follows from a straightforward calculation.

Note that additional care must be taken to account for the weight function \( f \), but this basically amounts to the fact that the cutoffs present in [DZ16, Lemma B.1] are no longer constant on closed trajectories and therefore do not cancel.

Now a partition of unity argument combining the compactness of closed trajectories with the semi-group property of the flow finishes the proof of Lemma 3.1.

3.2. Continuation on open hyperbolic systems. We prove meromorphic continuation together with an explicit formula for the Laurent coefficients of \( Z_f \):

**Theorem 3.3** (Meromorphic Continuation of Weighted Zetas I). The weighted zeta function \( Z_f^X \) defined in (1.3) for open hyperbolic systems converges absolutely in \( \{\text{Re}(\lambda) \gg 1\} \) and continues meromorphically to \( \{\lambda \in \mathbb{C}\} \). Any pole \( \lambda_0 \) of \( Z_f \) is a Pollicott-Ruelle resonance of \( X \) and if the pole \( \lambda_0 \) has order \( J(\lambda_0) \) then for \( k \leq J(\lambda_0) \) we have

\[
\text{Res}_{\lambda = \lambda_0} \left[ Z_f^X(\lambda) (\lambda - \lambda_0)^k \right] = \text{tr}^\phi \left( (X - \lambda_0)^k \Pi_{\lambda_0} f \right).
\]

**Proof.** First, we prove that the formal expression (1.3) defines a holomorphic function in \( \{\text{Re}(\lambda) \gg 0\} \) by showing uniform convergence on compact sets. To this end, we treat every term separately and then combine the results for a final estimate:

1. \( N(T) := |\{\gamma \mid T_\gamma \leq T\}| \leq C_0 e^{C_1 T} \) for constants \( C_0, C_1 > 0 \) by [DG16, Lemma 1.17].
2. \( |\text{det}(\text{id} - \mathcal{P}_\gamma)| \geq C_2 > 0 \) by uniform contraction and expansion.
3. \( |\text{tr}(\alpha_\gamma)| \leq C_4 e^{C_3 T_\gamma} \) by the operator norm estimate on the transfer operator \( e^{\gamma X} \).
Combining (1.), (2.) and (3.) we get
\[ \sum_{\gamma} |e^{-\lambda T_{\gamma}} \text{tr}(\alpha_{\gamma})|_{\text{det}(\text{id} - \mathcal{P}_{\gamma})} \int_{\gamma} f | \leq \sum_{n \in \mathbb{N}} n C_{0} C_{4} e^{(C_{1} + C_{3})n} |e^{-(n-1)\lambda}| |f|_{K} \]
\[ \leq C \sum_{n \in \mathbb{N}} n \left( e^{C - \text{Re}(\lambda)} \right)^{n} \]

\( Z_f(\lambda) \) thus converges uniformly if \( \lambda \) varies in a compact subset of \( \text{Re}(\lambda) > C \). In conclusion, the function \( Z_f(\lambda) \) is holomorphic on some right halfplane.

The proof proceeds by expressing the weighted zeta function as the flat trace of an expression involving the \textit{restricted resolvent} and using the trace formula presented in Sect. 3.1 as the main tool. The procedure closely follows [DZ16, §4]: We begin by choosing \( 0 < t_0 < T_{\gamma} \) \( \forall \gamma, \chi T \in C_{c}^{\infty}(i(t_{0}/2, T+1]) \) and \( \chi T \equiv 1 \) on \( [t_0, T] \). Furthermore, we assume \( t_0 \) small enough such that \( \varphi_{-t_0}(\text{supp}(\widetilde{\chi})) \subseteq \mathcal{U} \). Then we can define the family of operators

\[ B_T := \int_{0}^{\infty} \chi T(t)e^{-\lambda t} \left( \widetilde{\chi} e^{-tX_{\mathcal{F}}} f \right) dt, \]

and Lemma 3.1 shows that, for \( \text{Re}(\lambda) \gg 1 \),

\[ \lim_{T \to \infty} \text{tr}^{\varnothing}(B_T) = \lim_{T \to \infty} \sum_{\gamma} \chi T(T_{\gamma}) e^{-\lambda T_{\gamma}} \text{tr}(\alpha_{\gamma}) \int_{\gamma} f = \sum_{\gamma} \frac{e^{-\lambda T_{\gamma}} \text{tr}(\alpha_{\gamma})}{|\text{det}(\text{id} - \mathcal{P}_{\gamma})|} \int_{\gamma} f. \]

By [DZ16, Lemma 2.8] we may re-express \( \text{tr}^{\varnothing}(B_T) \) using a family of smoothing operators \( E_{\varepsilon} \) and the limit \( \varepsilon \to 0 \). The scalar-valued version of [DZ16] is directly applicable by choosing our \( E_{\varepsilon} \) to be diagonal in the fiber variable.

As in [DZ16, §4] we split the resulting integrals in a vanishing part and a part strictly separated from \( t = 0 \). These calculations require no significant modifications as the weight \( f \) can simply be carried along together with the cutoffs necessary to account for the open dynamics.

A simple calculation combined with an adaptation of [DZ16, Lemma 4.1] shows that the limits in \( T \to \infty \) and \( \varepsilon \to 0 \) may be exchanged, and doing so yields

\[ Z^{X}_{f}(\lambda) = \lim_{T \to \infty} \text{tr}^{\varnothing}(B_T) \lim_{\varepsilon \to 0} \int_{0}^{\infty} e^{-\lambda(t+t_0)} \text{tr} \left( E_{\varepsilon} \widetilde{\chi} e^{-(t+t_0)X_{\mathcal{F}}} f E_{\varepsilon} \right) dt \]
\[ = e^{-\lambda t_0} \lim_{\varepsilon \to 0} \text{tr} \left( E_{\varepsilon} \widetilde{\chi} e^{-t_0X}(X + \lambda) - 1 \widetilde{\chi} f E_{\varepsilon} \right), \tag{3.5} \]

which at first only holds for \( \text{Re}(\lambda) \gg 1 \). Next we apply the meromorphic continuation of the \textit{restricted resolvent} \( R(\lambda) := \text{1}_{\mathcal{U}}(X + \lambda)^{-1} \text{1}_{\mathcal{U}} : \Gamma_{c}^{\infty}(\mathcal{U}, \mathcal{E}) \to \mathcal{D}'(\mathcal{U}, \mathcal{E}) \) achieved in [DG16]. To do so we first observe that \( (X + \lambda)^{-1} \widetilde{\chi} = (X + \lambda)^{-1} \text{1}_{\mathcal{U}} \widetilde{\chi} \) by \( \text{supp}(\widetilde{\chi}) \subseteq \mathcal{U} \). But we even demanded \( t_0 \) to be small enough that \( \varphi_{-t_0}(\text{supp}(\widetilde{\chi})) \subseteq \mathcal{U} \) holds, which by the support property of \( e^{-t_0X} \) lets us rewrite \( \widetilde{\chi} e^{-t_0X}(X + \lambda) - 1 \text{1}_{U} \widetilde{\chi} = \widetilde{\chi} e^{-t_0X} \text{1}_{U}(X + \lambda) - 1 \text{1}_{U} \widetilde{\chi} \).

Now if \( \lambda \in \mathbb{C} \) is not a resonance then the general wavefront estimates [Hör13, Example 8.2.5] and [Hör13, Thm. 8.2.14] together with the estimate of WF\((R(\lambda))\) in [DG16, Equation (3.43)] yield
where \( p(x, \xi) := \xi(X_x) \), \( e^{t_0 H_p}(x, \xi) = (\varphi_t(x), (d\varphi_t(x))^{-T}\xi) \) and the wavefront set of \( e^{-t_0 X} \) is contained in the graph of \( e^{t_0 H_p} \). The wavefront set (3.6) does not intersect the diagonal by \( t_0 < T_p \), the directness of the hyperbolic splitting and bijectivity of \( \text{id} - d\varphi_t(x) \) on \( E_s(x) \oplus E_u(x) \).

Now if \( \lambda_0 \) is a resonance, then the same argument using [DG16, Lemma 3.5] (and linearity of \( \text{tr}^p \)) shows that the right-hand side of (3.5) admits a Laurent expansion around \( \lambda_0 \) whose finitely many coefficients of negative order are the flat traces of the Laurent coefficients of \( R(\lambda) \). We therefore have a meromorphic continuation for \( Z_f^X(\lambda) \) onto \( \{ \lambda \in \mathbb{C} \} \), its poles are contained in the set of Pollicott-Ruelle resonances of \( X \) and for \( \lambda \) not a resonance we have

\[
Z_f^X(\lambda) = e^{-\lambda t_0} \text{tr}^p \left( \bar{\chi} e^{-t_0 X} R(\lambda) \bar{\chi} f \right), \tag{3.7}
\]

To complete our proof we show an explicit formula for the Laurent coefficients at a resonance \( \lambda_0 \). The starting point is (3.7) combined with (0.13) in [DG16, Thm. 2]. If we substitute the expansion in the second equation into the first we get

\[
Z_f^X(\lambda) = Z_{f,H}^X(\lambda) + \sum_{j=1}^{J(\lambda_0)} \text{tr}^p \left( \bar{\chi} e^{-t_0 X} (-\lambda - \lambda_0)^{-1} \Pi_{\lambda_0} \bar{\chi} f \right),
\]

where \( Z_{f,H}^X(\lambda) \) is holomorphic near \( \lambda_0 \).

For \( 0 \leq k < J(\lambda_0) \), one can use the Taylor expansion of the exponential around \( \lambda_0 \), i.e. \( \exp(-\lambda t_0) = \sum_{n=0}^{\infty} (-t_0)^n \exp(-\lambda_0 t_0)(\lambda - \lambda_0)^n/n! \), to obtain the weighted zeta function’s Laurent coefficient of order \( k \) at \( \lambda_0 \):

\[
\text{Res}_{\lambda=\lambda_0} \left[ Z_f^X(\lambda - \lambda_0)^k \right] = \text{Res}_{\lambda=\lambda_0} \left[ \sum_{j=k+1}^{J(\lambda_0)} \sum_{n=0}^{\infty} \text{tr}^p \left( \bar{\chi} e^{-t_0 X} (-\lambda - \lambda_0)^{-1} \Pi_{\lambda_0} \bar{\chi} f \right) \right]
= \sum_{n=0}^{J(\lambda_0) - k - 1} \frac{(-1)^n t_0^n}{n!} \text{tr}^p \left( \bar{\chi} e^{-t_0 X} (-\lambda - \lambda_0)^{k+n} \Pi_{\lambda_0} \bar{\chi} f \right) \tag{3.8}
\]

The operator \( X + \lambda \) is nilpotent on the image \( \text{im}(\Pi_{\lambda_0}) \) by Equations (0.12) and (0.15) in [DG16]. This simplifies the transfer operator \( e^{-t_0(X + \lambda_0)} \) drastically, and substituting this simplification into Equation (3.8) together with the abbreviation \( N := J(\lambda_0) - k - 1 \) yields:

\[
\text{Res}_{\lambda=\lambda_0} \left[ Z_f^X(\lambda - \lambda_0)^k \right] = \sum_{n=0}^{N} \sum_{m=0}^{N-n} (-1)^n \frac{t_0^{n+m}}{n!m!} \text{tr}^p \left( \bar{\chi} (-\lambda - \lambda_0)^{n+m+k} \Pi_{\lambda_0} \bar{\chi} f \right)
= \sum_{s=0}^{N} \sum_{n=0}^{s} (-1)^n \frac{t_0^n}{n!(s-n)!} \text{tr}^p \left( \bar{\chi} (-\lambda - \lambda_0)^{s+n+k} \Pi_{\lambda_0} \bar{\chi} f \right), \tag{3.9}
\]
where the second line is obtained by using the variable \( s := n + m \) as a reparametrization of the double sum. A close examination of Equation (3.9) reveals that the binomial theorem can be applied to show the vanishing of all coefficients with \( s > 0 \). We therefore have the following formula for the \( k \)-th Laurent coefficient:

\[
\text{Res}_{\lambda = \lambda_0} \left[ Z^X_f(\lambda)(\lambda - \lambda_0)^k \right] = \text{tr}^b \left( \tilde{\chi}(-X - \lambda_0)^k \Pi_{\lambda_0} \tilde{\chi} f \right),
\]

which finishes our proof because the restriction of the kernel of \( \Pi_{\lambda_0} \) to the diagonal is supported in \( \Gamma_+ \cap \Gamma_- = K \) and \( \tilde{\chi} \equiv 1 \) on \( K \), i.e. we can drop the cutoff functions.

\( \square \)

Remark 3.4. Note that by [DG16, Eq. (4.8)] we have \( \text{Res}_{\lambda = \lambda_0} \left[ Z^X_f(\lambda) \right] = \text{rank}(\Pi_{\lambda_0}) \) if \( f \equiv 1 \), the constant function on \( \mathcal{U} \). All Laurent coefficients of higher (negative) order vanish for this particular choice of test function.

Remark 3.5. For a slightly different formulation of the final formula for the Laurent coefficients, one could replace the flat trace by an ordinary trace in an appropriate anisotropic Sobolev space ( [DZ16, Lemma 4.2] and the proof of [DG16, Thm. 4]).

3.3. Removing strict convexity for zetas. The geometric setup of open hyperbolic systems and the requirement of strict convexity in particular are quite cumbersome to state and difficult to verify in practice. Given an arbitrary flow \( \varphi_t \) with compact trapped set \( K \) which is hyperbolic on \( K \) we therefore remove this requirement via the perturbations of \( X \) and \( \tilde{X} \) constructed in Sect. 2.2. This will complete the proof of the claims made in the introduction.

Theorem 3.6 (Meromorphic Continuation of Weighted Zetas II). Let the setting of Theorem 2.1 and a weight \( f \in C^\infty(\mathcal{M}) \) be given. Then the weighted zeta function \( Z^X_f \) defined in (1.3) continues meromorphically onto \( \mathbb{C} \) and its Laurent coefficients are given by (1.4).

Proof. Recall the discussion in Sect. 2.2 where we constructed a vector field \( X_0 \) and an operator \( X_0 \) which satisfy the requirements of open hyperbolic systems and coincide with \( X \) and \( \tilde{X} \) on a neighborhood of the trapped set. It is now clear that the weighted zeta function associated with \( X_0 \) and \( \tilde{X}_0 \) coincides with the weighted zeta function for \( X \) and \( \tilde{X} \) as both functions depend only on the dynamics near the trapped set. This yields the claim by an immediate application of Theorem 1.2.

\( \square \)

Remark 3.7. Theorem 3.6 implies the independence of the set of resonances from the open sets \( \mathcal{U} \) and \( \mathcal{U}_0 \) promised in Sect. 2.2: Choosing as our weight the constant function \( f \equiv 1 \) we obtain \( \text{Res}_{\lambda = \lambda_0}(Z_1(f)) = \text{tr}^b(\Pi_{\lambda_0}) = \text{rank}(\Pi_{\lambda_0}) \), where the second equality was proven in [DG16, proof of Thm. 4]. Now this implies that the resonances coincide exactly with the poles of the weighted zeta function with constant weight, but the definition of the latter only involves the dynamics on the trapped set.

4. Residue Formula for Patterson–Sullivan Distributions

In this section we relate the residues of \( Z^X_f \), defined in purely classical terms such as closed trajectories, with certain quantum mechanical phase space distributions called
Patterson–Sullivan distributions. For the general setup suppose $M$ is a Riemannian manifold, $\varphi_t$ its geodesic flow on the unit tangent bundle $SM$, $X$ the geodesic vector field, and $\Delta_M$ its Laplacian.

If $M$ is a compact hyperbolic surface, i.e. a compact surface of constant negative curvature equal to $-1$, the spectrum of $\Delta_M$ is purely discrete and consists only of eigenvalues: $\sigma(\Delta_M) = \{\lambda_i\}$, $0 = \lambda_0 < \lambda_1 \leq \ldots$ and $\Delta_M \psi_i = \lambda_i \psi_i$ for an orthonormal basis of real-valued eigenfunctions $\{\psi_i\} \subseteq C^\infty(M)$. It is convenient to parameterize the eigenvalues as $\lambda_i = 1/4 + r_i^2$. In this setting [AZ07] associated to any $\psi_i$ a Patterson–Sullivan distribution $PS_{\psi_i} \in D(SM)$ that are invariant under the geodesic flow $(\varphi_t)_* (PS_{\psi_i}) = PS_{\psi_i}$ and which, in the high frequency limit $r_i \to \infty$, fulfill [AZ07, Eq. (1.4) and Thm. 1.1]:

$$\forall a \in C^\infty(T^*M) : \hat{PS}_{\psi_i}(a|SM) = \langle Op_{1/r_i}(a) \psi_i, \psi_i \rangle_{L^2(M)} + O(1/r_i). \quad (4.1)$$

Here $Op_a$ denotes some fixed semiclassical Weyl quantization procedure (see e.g. [Zwo12]) on the classical phasespace $T^*M$ and $\hat{PS}_{\psi_i}$ denotes the Patterson–Sullivan distribution normalized to $\hat{PS}_{\psi_i}(1_{SM}) = 1$. The expression $W_{\psi_i}(a) := \langle Op_{1/r_i}(a) \psi_i, \psi_i \rangle$ is known under the name Wigner distribution and the second equation in (4.1) should be interpreted as the equivalence of Patterson–Sullivan and Wigner distributions in the high-frequency limit. [HHS12, Def. 4.8, Prop. 4.10] generalizes the construction of Patterson–Sullivan distributions to arbitrary compact locally symmetric spaces. These distributions still retain (generalizations of) the properties (4.1) [HHS12, Remark 4.11, Thm. 7.4]. Our results concerning the residues of weighted zeta functions provide a new view on these phasespace distributions. Concretely, we can use our main Theorem 1.2 and the results obtained by [GHW21] to prove the following residue formula for Patterson–Sullivan distributions in the closed case:

**Theorem 4.1 (Patterson–Sullivan Distributions as Residues).** Let $M = \Gamma \backslash G/K$ be a compact Riemannian locally symmetric space of rank one, $\Delta_M$ its Laplacian and $\varphi_t$ the geodesic flow on $SM$. Let $\rho > 0$ denote the half-sum of the restricted roots of $G$. Then the following holds:

Given $r > 0$ such that $-\rho + ir$ is a Ruelle resonance of $\varphi_t$ then $\rho^2 + r^2$ is an eigenvalue of $\Delta_M$ and for any $g \in C^\infty(T^*M)$

$$\text{Res}_{\lambda=-\rho+ir} \left[ Z_{g|SM}(\lambda) \right] = \sum_{l=1}^{m} \hat{PS}_{\psi_l}(g|SM) = \sum_{l=1}^{m} W_{\psi_l}(g) + O(1/r),$$

where the sum is over an orthonormal $L^2$-basis of the $\Delta_M$-eigenspace with eigenvalue $\rho^2 + r^2$.

**Proof.** By Theorem 1.2 we have $\text{Res}_{\lambda=-\rho+ir} \left[ Z_f(\lambda) \right] = T_{-\rho+ir}(f)$ and by [GHW21, Corollary 6.1] we have $T_{-\rho+ir}(f) = \sum_{l=1}^{m} \langle \hat{PS}_{\psi_l}, f \rangle$. Note that the constants $c(ir)$ appearing in [GHW21, Corollary 6.1] and defined in [GHW21, Eq. (6.1)] are the normalization factors for $PS_{\psi_l}$; the additional factor of $m^{-1}$ appearing in [GHW21, Corollary 6.1] are due to their slightly different definition of $T_{\lambda_0}$ in [GHW21, Eq. (2.1)]. Combining these two results proves our claim. $\square$

We would like to compare our result with previously known results obtained with different techniques. In their paper [AZ07], Anantharaman and Zelditch proved a similar close connection between slightly different weighted zeta functions and Patterson–Sullivan distributions: For $f \in C^\infty(SM)$ they define a weighted zeta function via [AZ07,
\[ Z^{AZ}_f(\lambda) := \sum_{\gamma} \left( \frac{\exp(-\lambda T_\gamma)}{1 - \exp(-T_\gamma)} \int_{\gamma} f \right), \quad (4.2) \]

where one sums over all closed geodesics \( \gamma \). In [AZ07, Thm. 1.3] they state that, provided \( f \) is real analytic, \( Z^{AZ}_f \) continues meromorphically onto \( \mathbb{C} \), its poles in \( \{ 0 < \text{Re}(\lambda) < 1 \} \) are of the form \( \lambda = 1/2 + ir \) where \( 1/4 + r^2 \) is an eigenvalue of \( \Delta_M \) and the following residue formula holds:

\[ \text{Res}_{\lambda=1/2+ir} \left[ Z^{AZ}_f(\lambda) \right] = \sum_{\psi_i: \lambda_i=1/4+r^2} (\hat{PS}\psi_i, f). \]

They give two different proofs with the first relying on the thermodynamic formalism and the second on representation theory and a version of Selberg’s trace formula. See also the later work [AZ12] for a generalization of the intertwining between Wigner and Patterson–Sullivan distributions to the non-diagonal case.

Note that for hyperbolic surfaces \( \text{det}(\text{id} - P_\gamma) = (1 - \exp(-T_\gamma))(1 - \exp(T_\gamma)) \), i.e. a simple calculation yields the following relation between \( Z^{AZ}_f \) and our weighted zeta:

\[ Z_f(\lambda) = \sum_{n=1}^{\infty} Z^{AZ}_f(\lambda + n). \]

We can therefore conclude that given an eigenvalue \( 1/4 + r^2 \) of \( \Delta_M \) the value \(-1/2 + ir\) is a Ruelle resonance of the geodesic flow on \( SM \) and

\[ \text{Res}_{\lambda=-1/2+ir} \left[ Z_f(\lambda) \right] = \sum_{\psi_i: \lambda_i=1/4+r^2} (\hat{PS}\psi_i, f). \]

In his thesis [Emo14], Emonds extended the residue formula of [AZ07] to the case of hyperbolic manifolds of arbitrary dimension. But this result again imposes a significant restriction on the space of test functions, namely that \( f \) be \( K \)-finite, and the proof heavily relies on techniques from representation theory. It appears that the methods of microlocal analysis are better suited for the meromorphic continuation of (weighted) zeta functions.

Let us finally note that our main result also holds for open systems, so we immediately obtain residue formulae for weighted zeta functions of geodesic flows on convex cocompact hyperbolic manifolds in terms of invariant Ruelle distributions. While there exists so far no theory of Patterson–Sullivan distributions for these systems, a quantum classical correspondence has been established on the level of resonances and resonant states [GHW18]. We thus conjecture that also in this setting the invariant Ruelle distributions are, in the high frequency limit, asymptotically equivalent to phase space distributions of quantum resonant states. Given the fact that the residues of \( Z_f \) can be numerically calculated quite efficiently, this would provide a method to study phase space distributions of quantum resonant states on Schottky surfaces numerically.

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Declarations

Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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Appendix A: Numerical Calculation of Invariant Ruelle Distributions (with Sonja Barkhofen)

In this appendix we provide a short outlook on a major application of the weighted zeta function developed in the main text, namely the numerical calculation of invariant Ruelle distributions $\mathcal{T}_{\lambda_0}$. While these distributions are interesting in their own right, they also provide a new perspective on quantum mechanical matrix coefficients, c.f. (4.1). This circumstance makes efficient tools for their calculation even more desirable. For a physics-oriented discussion of the connection between weighted zeta functions and quantum mechanical phase space distributions we refer to the companion article [BSW22].

Now in principle one wants to exploit Theorem 3.6 and more concretely the relation

$$\langle \mathcal{T}_{\lambda_0}, f \rangle = \text{Res}_{\lambda = \lambda_0} Z_f(\lambda).$$

To use this numerically we require an efficient method for calculating (the residues of) the weighted zeta function $Z_f$. While this endeavor is hopeless in the abstract setting of Theorem 2.1 there are concrete dynamical systems where this calculation is possible due to the availability of a symbolic encoding of the dynamics. Two such system which are well-known in the literature are convex obstacle scattering and geodesic flows on convex cocompact hyperbolic surfaces. For the sake of brevity we will focus on a particular instance of the former class of systems, namely so-called symmetric 3-disc systems.

This appendix is organized as follows: We begin by giving a short introduction to 3-disc systems in Section A.1. With this setup at hand we then provide some first numerical results in Section A.2.

A.1. Introducing 3-disc systems. The 3-disc system is a paradigmatic example of a convex obstacle scattering dynamics [Ika88, GR89]. It is given by three discs $D(x_i, r_i) \subseteq \mathbb{R}^2$,
Fig. 1. A symmetric 3-disc system with its defining parameters \( r \) and \( d \). The fundamental domain is given by the green region

\[ i \in \{1, 2, 3\} \], with radii \( r_i > 0 \), centers \( x_i \in \mathbb{R}^2 \) and disjoint closures. The dynamics takes place on the unit sphere bundle \( \mathcal{S} \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{D}(x_i, r_i) \right) \): In the interior \( \mathcal{S} \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{D}(x_i, r_i) \right) \) its trajectories coincide with the Euclidean geodesic flow, and upon boundary intersection the trajectories experience specular reflections. We will only consider fully symmetric 3-disc systems here, and for our purposes these are uniquely described by the quotient \( d/r \) of the common radius \( r = r_1 = r_2 = r_3 \) and the side length \( d > 0 \) of the equilateral triangle on which the centers of the discs are positioned. For a graphical illustration of this setup see Fig. 1.

Now the dynamics just described is obviously not smooth because of the instantaneous boundary reflections. If we assume that \( d/r \) is sufficiently large then this lack of smoothness is properly separated from the trapped set, though. In this case it can be dealt with via smooth models. To keep our presentation short we refer the reader to \([KSW21]\) where the construction of smooth models as well as the meromorphic continuation of weighted zeta functions was carried out in detail and in a more general setting.

What we will actually investigate numerically is not quite the 3-disc dynamics just described, but rather a symmetry reduced variant. A fundamental domain of this symmetry reduction is shown in green in Fig. 1. The reduced dynamics still admits a symbolic coding and is well understood on the classical as well as on the quantum side. It is used here as it provides the model of choice for experimental realizations \([PWB+12,BWP+13]\). For more details and references see \([WBK+14]\).

Our numerical algorithm itself resembles the algorithm developed by Cvitanovic and Eckhardt \([CE89,CE91]\) in physics and Jenkinson and Pollicott \([JP02]\) or Borthwick \([Bor14]\) in mathematics: One can derive a cycle expansion for the weighted zeta function \( \mathcal{Z}_f \) associated with a given 3-disc system. To calculate concrete summands in this expansion we make use of the symbolic encoding of closed trajectories available for sufficiently large \( d/r \). A detailed description of the algorithm will be presented elsewhere \([BSW22]\). We just want to mention the following two central simplifications:

1. To be able to plot the distributions \( \mathcal{T}_{\lambda_0} \) we calculate their convolution with Gaussians with variance \( \sigma > 0 \). In the limit \( \sigma \to 0 \) this convolution converges to \( \mathcal{T}_{\lambda_0} \) in \( \mathcal{D}' \)
and it is reasonable to expect the numerical results for small but positive $\sigma$ to reveal interesting properties of $\mathcal{T}_{\lambda_0}$ itself.

(2) While the convolutions discussed in (1) are smooth they still live on the 3-dimensional state space of the 3-disc system. To obtain 2-dimensional plots we restrict $\mathcal{T}_{\lambda_0}$ to a Poincaré section $\Sigma_1$ via Lemma 2.3.

For the numerics presented below we used a specific Poincaré section $\Sigma \subseteq S^2$ defined by so-called Birkhoff coordinates as follows: First, fix one of the discs and an origin on the boundary of this disc. Then a point $(q, p) \in [−\pi, \pi] \times [−1, 1]$ corresponds to the point $(x, v) \in S^2$ such that the boundary arc connecting the origin and $x$ has length $q \cdot r$ and such that the projection $\langle v, t(x) \rangle$ of $v$ onto the tangent $t(x)$ to the disc at $x$ equals $p$. For an illustration of these coordinates see Fig. 2. Now the transversality condition of Lemma 2.3 is obviously satisfied, making the restriction $\mathcal{T}_{\lambda_0}|_{\Sigma_1}$ a well-defined distribution. It is this object which will be plotted numerically in the following section.

A.2. Proof-of-principle results. In this section we present first numerical calculations of invariant Ruelle distributions. As already mentioned we restrict the distributions to the Poincaré section $\Sigma$ and then approximate by convolution with Gaussians of width $\sigma$. As the trapped set $K$ itself is fractal and therefore hard to visualize, we chose the following alternative: Denote by $K_1 \supseteq K$ those points of phase space which experience at least one disc reflection either in forward or backward time. We included the intersection $\Sigma_1 := K_1 \cap \Sigma$ in the figures below to give an idea of where the invariant Ruelle distributions are supposed to be supported in theory.

We begin the first series of illustrations by plotting four example resonances and associated distributions along the first clearly distinguishable resonance chain. In particular, we begin with the point closest to the spectral gap and continue towards the intersection with the second distinct chain. Our choices are marked in red in the first row of Fig. 3.

In the second and third rows of Fig. 3 we plotted the distribution associated with the marked resonance and the two choices $\sigma = 0.1$ and $\sigma = 0.001$, respectively. Going from left to right in either the first or the second row shows that the invariant Ruelle distributions clearly encode some kind of information regarding the location of their associated resonances. Especially the distribution at the point of intersection of the first two chains (fourth column in Fig. 3) differs significantly from the first three, which only exhibit a gradual reduction of intensity in the left and right component of $\Sigma_1$. 

\begin{figure}
\centering
\includegraphics[width=0.25\textwidth]{fig2}
\caption{The Birkhoff coordinates for the Poincaré section $\Sigma \subseteq S^2$ of a symmetric 3-disc system used in the numerics below.}
\end{figure}
Fig. 3. Comparison of invariant Ruelle distributions along a resonance chain for \( d/r = 6 \). The first row highlights in red the resonance at which the distributions in the second and third rows was evaluated. The second row was computed with \( \sigma = 0.1 \) and the third row with \( \sigma = 0.001 \). One clearly recognizes dynamical changes in \( T_{\lambda_0} \) as \( \lambda_0 \) varies along the chain. In addition, the reduction of \( \sigma \) seems to further localize the distribution on the trapped set, which theoretically contains the support of any \( T_{\lambda_0} \).

Going from top to bottom in Fig. 3 we see how the reduction of \( \sigma \) by two orders of magnitude significantly increases the localization of the distributions on the trapped set. This behavior is expected by the theory developed above and could allow a detailed numerical investigation of invariant Ruelle distributions on successively finer scales of the fractal trapped set.

Our second series of invariant Ruelle distributions is meant to give a first impression of a curious phenomenon which has not been understood theoretically yet: Calculating numerically resonances with imaginary part up to about 750 it would appear that the maximal real part which occurs becomes progressively smaller the larger \( \text{Im}(\lambda) \) becomes. If we proceed to even larger imaginary parts this progression reverses and at about \( \text{Im}(\lambda) = 1500 \) we observe several resonances with real parts close to the theoretical maximum of \( \lambda_1 \), where \( \lambda_1 \) denotes the first resonance on the real line. These observations are shown in the first row of Fig. 4. The same effect has been observed in even more pronounced fashion for resonances on Schottky surfaces [BW16].

As the question of asymptotic spectral gaps for such open systems is an important unsolved problem, it is interesting to understand such a recurrence of resonances to a neighborhood of the critical line. We therefore calculated the invariant Ruelle distributions for those resonances close to the critical line in the second and third rows of Fig. 4: The distribution plots from top left to bottom right belong to the resonances marked in red and ordered from small to large imaginary part. We immediately notice that all eight distributions while associated with different resonances appear exceedingly similar, even though we already calculated them with the rather small value of \( \sigma = 0.001 \). We have to admit that we cannot explain this observation so far but find it quite remarkable.
Fig. 4. Comparison of invariant Ruelle distributions for several different resonances near the line $\text{Re}(\lambda) = \lambda_1$, where $\lambda_1$ denotes the resonances with maximal real part. The resonances marked in red (from bottom to top) correspond to the plotted distributions (from top left to bottom right). Throughout we have $\sigma = 0.001$. Note how the distributions all appear very similar, even on this second level of the fractal trapped set.

Additional and more detailed illustrations can be found on the supplementary website https://go.upb.de/ruelle. In particular, it contains several additional distributions along the first chain discussed above, illustrations of further resonances near the spectral gap, and plots along a second resonance chain.

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