On Excesses and Duality in Woven Frames
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Abstract. Weaving frames in separable Hilbert spaces have been recently introduced by Bemrose et al. to deal with some problems in distributed signal processing and wireless sensor networks. In this paper, we study the notion of excess for woven frames and prove that any two frames in a separable Hilbert space that are woven have the same excess. We also show that every frame with a large class of duals is woven provided that its redundant elements have small enough norm. Also, we try to transfer the woven property from frames to their duals and vice versa. Finally, we look at which perturbations of dual frames preserve the woven property, moreover it is shown that under some conditions the canonical Parseval frame of two woven frames are also woven.

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1. Introduction

Frames which are introduced by Duffin and Schaefer provide robust, stable and usually non-unique representations of vectors [16]. The theory of frames has been growing rapidly [1 2 12 19] after the fundamental paper of Daubechies, Grossmann and Meyer [15]. they have seen great achievements in pure mathematics, science, and engineering such as image processing, signal processing, sampling and approximation theory [4 9 10 13 17 20].

Recently, due to application and theoretical goals, some generalizations of frames have been presented, such as g-frames [22], K-frames [18] and fusion frames [11 21]. In most of them and many application problems, reconstruction and duality play a key role. Hence, characterization, construction and in general survey of duality is a valuable discussion in frame theory [5 4 17].
In signal processing each signal is interpreted as a vector. In this interpretation, a vector expressed as a linear combination of the frame elements. Using a frame, it is possible to create a simpler, more sparse representation of a signal as compared with a family of elementary signals. The concept of woven frames is motivated by some problems in signal processing. For example; given two frames \( \Phi \) and \( \Psi \), At each sensor we measure a signal \( f \) either with \( \varphi_i \) or with \( \psi_i \), so that the collected information is the set of numbers \( \{(f, \varphi_i)\}_{i \in \sigma} \cup \{(f, \psi_i)\}_{i \in \sigma^c} \) for some subset \( \sigma \subset I \). Now we may recover \( f \) from these measurements, no matter which kind of measurement has been made at each sensors. In other words, is the set \( \{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c} \) a frame for all subset \( \sigma \subset I \)? This question led us to the definition of woven frames.

2. Preliminaries and Notations

A sequence \( \Phi = \{\varphi_i\}_{i \in I} \) in a separable Hilbert space \( H \) is called a frame for \( H \) if there exist constants \( 0 < A_\Phi \leq B_\Phi < \infty \) such that

\[
A_\Phi \|f\|^2 \leq \sum_{i \in I} |\langle f, \varphi_i \rangle|^2 \leq B_\Phi \|f\|^2, \quad (f \in H).
\] (2.1)

The constants \( A_\Phi \) and \( B_\Phi \) are called lower and upper frame bounds, respectively. The supremum of all lower frame bounds is called the optimal lower frame bound and the infimum of all upper frame bounds is called the optimal upper frame bound. A sequence \( \{\varphi_i\}_{i \in I} \) is called Bessel if the right inequality in (2.1) holds.

A sequence \( \Phi = \{\varphi_i\}_{i \in I} \) in Hilbert space \( H \) is called a Riesz sequence if there are constants \( 0 < A_\Phi \leq B_\Phi < \infty \) so that for all finite scalars \( c \) we have

\[
A_\Phi \sum_{i \in I} |c_i|^2 \leq \|\sum_{i \in I} c_i \varphi_i\|^2 \leq B_\Phi \sum_{i \in I} |c_i|^2.
\]

The constants \( A_\Phi \) and \( B_\Phi \) are called lower and upper Riesz bounds, respectively. In addition, if \( \Phi \) is complete in \( H \), then it is a Riesz basis for \( H \).

Given a Bessel sequence \( \Phi = \{\varphi_i\}_{i \in I} \), the synthesis operator \( T_\Phi : \ell^2(\mathbb{N}) \to H \) is defined by \( T_\Phi \{c_i\} = \sum_{i \in I} c_i \varphi_i \). Its adjoint, \( T_\Phi^* : H \to \ell^2(\mathbb{N}) \), which is called the analysis operator, is given by \( T_\Phi^* f = \{\langle f, \varphi_i \rangle\}_{i \in I} \). Moreover, \( S : H \to H \) the frame operator of \( \Phi \), is given by \( S_\Phi f = T_\Phi T_\Phi^* f \). If \( \Phi \) is a frame, then \( S_\Phi \) is invertible and \( A_\Phi \leq S_\Phi \leq B_\Phi \), see [14] for more details. The sequence \( \tilde{\Phi} = \{S_\Phi^{-1} \varphi_i\}_{i \in I} \), which is also a frame, is called the canonical dual frame. A frame \( \{\psi_i\}_{i \in I} \) is called a dual of \( \{\varphi_i\}_{i \in I} \) if

\[
f = \sum_{i \in I} \langle f, \psi_i \rangle \varphi_i, \quad (f \in H).
\]
Theorem 2.1. [5] If $\Phi = \{\varphi_i\}_{i \in I}$ be a frame. Then every dual frame of $\Phi$ is of the form of $\Phi^d = \{S_{\Phi}^{-1}\varphi + u_i\}_{i \in I}$ where $\{u_i\}_{i \in I}$ is a Bessel sequence such that

$$\sum_{i \in I} \langle f, \varphi_i \rangle u_i = 0, \quad (f \in \mathcal{H}). \quad (2.2)$$

2.1. Woven Frames

Throughout the paper, $\mathcal{H}$ is a separable Hilbert space, $I$ a countable index set and $I_\mathcal{H}$ the identity operator on Hilbert space $\mathcal{H}$. We denote $\Phi = \{\varphi_i\}_{i \in I}$ for a frame with bounds $A_\Phi$ and $B_\Phi$. Also we use of $[n]$ to denote the set $\{1, 2, \ldots, n\}$. We now review some definitions and primary results of woven frames, which are used in the present paper. For more information see [8].

A family of frames $\{\varphi_{ij}\}_{i \in I}$ for $j \in \{1, \ldots, m\}$ for a Hilbert space $\mathcal{H}$ is said to be woven if there are universal constants $A$ and $B$ so that for every partition $\{\sigma_j\}_{j=1}^m$ of $I$, the family $\{\varphi_{ij}\}_{i \in \sigma_j, j=1}^m$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $A$ and $B$, respectively [8]. Each family $\{\varphi_{ij}\}_{i \in \sigma_j, j=1}^m$ is called a weaving.

Two frames $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for Hilbert space $\mathcal{H}$ are weakly woven if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame for $\mathcal{H}$.

Two frames $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ are called Riesz woven if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz basis for $\mathcal{H}$. Combining Theorem 5.3 of [8] and Theorem 3.5 of [3], every Riesz basis with its canonical dual is Riesz woven. However, the canonical Parseval dual of two Riesz woven bases are not necessary woven. For example, $\{e_1, e_2\}$ and $\{e_1 + e_2, 2e_1 + e_2\}$ are Riesz woven bases with the canonical Parseval duals $\{e_1, e_2\}$ and $\{e_2, e_1\}$ which are clearly not woven [8]. In this article, we study woven frames from the duality approach. One of our aim is to find dual frame pairs which are woven.

Theorem 2.2. If $\{\varphi_{ij}\}_{j=1}^M_{i \in I}$ is a family of Bessel sequence for $\mathcal{H}$ with a Bessel bound $B_j$ for all $1 \leq j \leq M$ then every weaving is a Bessel sequence with the Bessel bound $\sum_{j=1}^M B_j$.

Proposition 2.3. Given two frames $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for $\mathcal{H}$, the following are equivalent:

1. The two frames are woven.
2. The two frames are weakly woven.

Proposition 2.4. Let $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be frames for Hilbert space $\mathcal{H}$ such that there is a $0 < \lambda < 1$ so that

$$\lambda(\sqrt{B_\Phi} + \sqrt{B_\Psi}) \leq \frac{A_\Phi}{2}.$$
and for all sequence of scalars \( \{a_i\}_{i \in I} \) we have
\[
\left\| \sum_{i \in I} a_i(\varphi_i - \psi_i) \right\| \leq \lambda \| \{a_i\}_{i \in I} \|.
\]
Then for every \( \sigma \subset I \), the family \( \{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c} \) is a frame for \( \mathcal{H} \) with frame bounds \( \frac{A \Phi}{2}, B \Phi + B \Psi \). That is, \( \Phi \) and \( \Psi \) are woven.

**Proposition 2.5.** Let \( \Phi = \{\varphi_i\} \) be a frame and \( U \) be a bounded operator such that \( \|I_H - U\|^2 < \frac{A \Phi}{B \Phi} \). Then \( \Phi \) and \( U \Phi \) are woven with the universal lower bound \( (\sqrt{A \Phi} - \sqrt{B \Phi})\|I_H - U\|^2 \).

We end this section with an example which shows that the woven property is not transitive, in general.

**Example 2.6.** Let \( \mathcal{H} = \mathbb{R}^2 \). Consider frames \( \Phi = \{e_1, e_1, e_2\}, \Psi = \{e_1, e_2, e_2\} \) and \( \eta = \{e_1, e_2, e_1\} \) on \( \mathcal{H} \) where \( \{e_1, e_2\} \) is the standard orthonormal basis of \( \mathcal{H} \). Then \( \Phi \) with \( \Psi \) and \( \Psi \) with \( \eta \) are woven with universal bounds \( A_1 = A_2 = 1 \) and \( B_1 = B_2 = 2 \). However, \( \Phi \) and \( \eta \) are not woven.

In order to achieve this, we take a restricted condition on bounds. More precisely, let \( \{\varphi_i\}_{i \in I} \), \( \{\psi_i\}_{i \in I} \) and \( \{\eta_i\}_{i \in I} \) be frames for Hilbert space \( \mathcal{H} \). If \( \{\varphi_i\}_{i \in I} \) and \( \{\psi_i\}_{i \in I} \) are woven frames with a universal lower bound \( A_1 \), and \( \{\psi_i\}_{i \in I} \) is woven with \( \{\eta_i\}_{i \in I} \) by a universal lower bound \( A_2 \) such that \( A_1 + A_2 - B_\psi > 0 \). Then for each \( \sigma \subset I \) and \( f \in \mathcal{H} \) we obtain
\[
(A_1 + A_2 - B_\psi) \| f \|^2 \leq \sum_{i \in \sigma} |\langle f, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 \\
+ \sum_{i \in \sigma^c} |\langle f, \eta_i \rangle|^2 - \sum_{i \in I} |\langle f, \psi_i \rangle|^2 \\
= \sum_{i \in \sigma} |\langle f, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \eta_i \rangle|^2 \\
\leq (B_1 + B_2) \| f \|^2.
\]
Hence, \( \{\varphi_i\}_{i \in I} \) and \( \{\eta_i\}_{i \in I} \) are woven.

### 3. Weaving and Excesses

The excess of a frame \( \varphi \), denoted by \( E(\varphi) \), is the greatest integer \( n \) so that \( n \) elements can be deleted from the frame and still leave a frame, or \( +\infty \) if there is no upper bound to the number of elements that can be removed. Two frames in a separable Hilbert space \( \mathcal{H} \) which are dual to each other have the same excess, see [6] for more details. Every frame \( \varphi \) with \( E(\varphi) = n \) can be written by \( \varphi = \)
\{ \varphi_i \}_{i \in \mathbb{I} \setminus \{ i_1, \ldots, i_n \}} \cup \{ \varphi_{i_1}, \ldots, \varphi_{i_n} \}, \text{ where } \varphi = \{ \varphi_i \}_{i \in \mathbb{I} \setminus \{ i_1, \ldots, i_n \}} \text{ is a Riesz basis for } \mathcal{H} \text{ and } \{ \varphi_{i_1}, \ldots, \varphi_{i_n} \} \text{ are redundant elements of } \varphi.

In the next theorem, we show that woven frames have the same excess.

**Theorem 3.1.** Let \( \Phi = \{ \varphi_i \}_{i \in \mathbb{I}} \) and \( \Psi = \{ \psi_i \}_{i \in \mathbb{I}} \) be woven. Then \( E(\Phi) = E(\Psi) \).

**Proof.** At first we suppose that \( E(\Phi) = 1 \). Without lose of generality we can assume that \( \{ \varphi_i \}_{i \in \mathbb{I}} \) is an orthonormal basis of \( \mathcal{H} \). If \( E(\Psi) > 1 \) then there exist \( r, s \in \mathbb{I} \) such that \( \psi_r, \psi_s \in \text{span} \{ \psi_i \}_{i \neq r, s} \). Given \( \epsilon > 0 \) and choose \( n \in \mathbb{N} \) so that
\[
d(\psi_r, K) < \epsilon, \quad d(\psi_s, K) < \epsilon.
\]
where
\[
K = \text{span} \{ \psi_i \}_{i = 1, i \neq r, s}^n.
\]
Clearly, \( \dim K \leq n - 2 \) and \( \dim (\text{span} \{ \varphi_i \}_{i = 2}^\infty) = n - 1 \). So, there exists \( f \in \text{span} \{ \varphi_i \}_{i = 2}^\infty \cap K^\perp \) with \( \| f \| = 1 \). Take \( \sigma = \{ 1, 2, \ldots, n \} \), then
\[
\sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \varphi_i \rangle|^2 = |\langle f, \psi_r \rangle|^2 + |\langle f, \psi_s \rangle|^2.
\]
(3.1)

For \( \psi_r \in \mathcal{H} \) we have \( \psi_r = \psi_{r_1} + \psi_{r_2} \) such that \( \psi_{r_1} \in K \) and \( \psi_{r_2} \in K^\perp \). Hence,
\[
|\langle f, \psi_r \rangle| = |\langle f, \psi_{r_1} \rangle + \langle f, \psi_{r_2} \rangle| = |\langle f, \psi_{r_2} \rangle| \leq \| \psi_{r_2} \| = d(\psi_r, K) < \epsilon.
\]
A similar calculation shows that \( |\langle f, \psi_s \rangle| < \epsilon \). Therefore, by (3.1) we obtain
\[
\sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \varphi_i \rangle|^2 < 2\epsilon.
\]
So, \( \{ \varphi_i \}_{i \in \sigma} \cup \{ \psi_i \}_{i \in \sigma^c} \) is not a frame and it is a contradiction. Thus, \( E(\Psi) = 0 \) or \( E(\Phi) = 1 \). If \( E(\Psi) = 0 \), then \( \Psi \) and \( \Phi \) are Riesz bases, by Theorem 5.4 of [8], which is contradiction. Therefore, \( E(\Phi) = 1 \). By induction, the result holds for all values of \( E(\Phi) = n \in \mathbb{N} \). Finally, assume that \( E(\Phi) = \infty \). We can certainly see that \( E(\Psi) = \infty \), since otherwise \( E(\Phi) < \infty \) which is contradiction. \( \square \)

In [3], it is shown that a frame with its canonical dual is woven provided that the norm of redundant elements are small enough. In the following, we prove this fact for alternate duals.

**Theorem 3.2.** Let \( \Phi = \{ \varphi_i \}_{i \in \mathbb{I}} \) be a frame for \( \mathcal{H} \) such that the norm of redundant elements of \( \Phi \) are small enough. Then there exist infinitely many dual frames which are woven with \( \Phi \).
Proof. Suppose first that \( E(\Phi) = n \). Without lose of generality we can write \( \Phi = \{ \varphi_i \}_{i \in [n]} \cup \{ \varphi_i \}_{i \in [n]} \), where \( \phi = \{ \varphi_i \}_{i \in [n]} \) is a Riesz basis for \( \mathcal{H} \). Using Theorem 3.5 of [3] follows that \( \phi \) and \( S_\phi^{-1} \phi \) are woven. In particular, \( \phi \) and \( S_\phi \phi \) are woven.

Denote its universal lower bound by \( A \) and suppose that \( \Phi^d = \{ S_\phi^{-1} \varphi_i + u_i \}_{i \in I} \) is a dual of \( \Phi \) such that \( \{ u_i \}_{i \in I} \) is a Bessel sequence satisfies in (2.2). Choose \( \epsilon > 0 \) so that

\[
\sum_{i \in [n]} \| \varphi_i \|^2 + 2\sqrt{\epsilon B_U B_\Phi} < \sqrt{\frac{A}{B_\Phi}}. \tag{3.2}
\]

Let \( \sigma \subset I \) and take \( \alpha_{i,k} = \langle \varphi_i, \varphi_k \rangle, 1 \leq k \leq n \). Then

\[
\sum_{i \in \sigma} \left| \sum_{j=1}^n \alpha_{i,j} \langle f, \varphi_j \rangle \right|^2 \leq \| f \|^2 B_\Phi \left( \sum_{j=1}^n \| \varphi_j \|^2 \right)^2, \quad (f \in \mathcal{H}). \tag{3.3}
\]

Indeed, for each \( f \in \mathcal{H} \) we have

\[
\sum_{i \in \sigma} \left| \sum_{j=1}^n \alpha_{i,j} \langle f, \varphi_j \rangle \right|^2 \\
= \sum_{i \in \sigma} \left| (\alpha_{i,1}, \ldots, \alpha_{i,n}) \cdot (\langle f, \varphi_1 \rangle, \ldots, \langle f, \varphi_n \rangle) \right|^2 \\
\leq \sum_{i=1}^\infty (|\alpha_{i,1}|^2 + \ldots + |\alpha_{i,n}|^2) \left( |\langle f, \varphi_1 \rangle|^2 + \ldots + |\langle f, \varphi_n \rangle|^2 \right) \\
\leq \left( \sum_{i=1}^\infty |\langle \varphi_i, \varphi_1 \rangle|^2 + \ldots + |\langle \varphi_i, \varphi_n \rangle|^2 \right) \| f \|^2 \left( \| \varphi_1 \|^2 + \ldots + \| \varphi_n \|^2 \right) \\
= (B_\Phi \| \varphi_1 \|^2 + \ldots + B_\Phi \| \varphi_n \|^2) \| f \|^2 \left( \| \varphi_1 \|^2 + \ldots + \| \varphi_n \|^2 \right) \\
= B_\Phi \left( \| \varphi_1 \|^2 + \ldots + \| \varphi_n \|^2 \right)^2 \| f \|^2.
\]

Putting \( \Phi^d_\epsilon = \{ S_\phi^{-1} \varphi_i + \epsilon u_i \}_{i \in I} \). We show that \( \Phi \) and \( \Phi^d_\epsilon \) are woven. For this, it is enough to show the existence of a universal lower bound for \( \{ S_\phi \varphi_i \}_{i \in I} \) and \( \{ \varphi_i + S_\phi \epsilon u_i \}_{i \in I} \) since the woven property is preserved under bounded invertible operators [8]. An easy computation shows that

\[
S_\phi \varphi_i = S_\phi \varphi_i + \sum_{k \in [n]} \alpha_{i,k} \varphi_k.
\]

Let \( \sigma \subset I \setminus [n] \), then we have
\[
\left( \sum_{i \in \sigma} |\langle f, S\phi \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \varphi_i + S\phi \epsilon u_i \rangle|^2 \right)^{1/2}
\]
\[
= \left( \sum_{i \in \sigma} |\langle f, S\phi \varphi_i \rangle|^2 + \sum_{i \in \sigma^c \setminus [n]} |\langle f, \varphi_i + S\phi \epsilon u_i \rangle|^2 + \sum_{i \in [n]} |\langle f, \varphi_i + S\phi \epsilon u_i \rangle|^2 \right)^{1/2}
\]
\[
= \left( \sum_{i \in \sigma} |\langle f, S\phi \varphi_i \rangle|^2 + \sum_{k \in [n]} \alpha_{i,k} |\langle f, \varphi_k \rangle|^2 \right) + \left( \sum_{i \in [n]} |\langle f, S\phi \epsilon u_i \rangle|^2 \right)^{1/2}
\]
\[
\geq \left( \sum_{i \in \sigma} |\langle f, S\phi \varphi_i \rangle|^2 + \sum_{i \in \sigma^c \setminus [n]} |\langle f, \varphi_i \rangle|^2 + \sum_{i \in [n]} |\langle f, \varphi_i \rangle|^2 \right)^{1/2}
\]
\[
- \left( \sum_{i \in \sigma} \sum_{k \in [n]} \alpha_{i,k} |\langle f, \varphi_k \rangle|^2 \right)^{1/2} - \left( \sum_{i \in [n]} |\langle f, S\phi \epsilon u_i \rangle|^2 \right)^{1/2}
\]

Applying (3.3) follows that
\[
\sum_{i \in \sigma} |\langle f, S\phi \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \varphi_i + S\phi \epsilon u_i \rangle|^2
\]
\[
\geq \left( \sqrt{A} - \sqrt{B \Phi} \sum_{i \in [n]} \| \varphi_i \|^2 - 2\sqrt{\epsilon U} \| S\phi \| \right)^2 \| f \|^2
\]
\[
= \left( \sqrt{A} - \sqrt{B \Phi} \sum_{i \in [n]} \| \varphi_i \|^2 - 2\sqrt{\epsilon U} B \psi \right)^2 \| f \|^2.
\]

The desired result follows provided that (3.2) holds. As a similar calculation shows that when \( \sigma^c \subset I \setminus [n] \) or \( \sigma \subset I \), then the result is true. If \( E(\Phi) = \infty \), we can write \( \Phi = \{ \varphi_i \}_{i \in I \setminus \sigma} \cup \{ \varphi_i \}_{i \in \sigma} \) where \( |\sigma| = \infty \) and \( \phi = \{ \varphi_i \}_{i \in I \setminus \sigma} \) is a Riesz basis for \( \mathcal{H} \). Similar to the above, we can easily check that if \( \sum_{i \in \sigma} \| \varphi_i \|^2 < \frac{A}{B \psi} \), where \( A \) is a universal lower bound of woven frames \( \Phi \) with \( S\phi \Phi \), then \( \Phi \) and a class of its duals are woven.

We point out that the converse of the previous theorem is not true. For instance, the frame \( \Phi = \{ ne_1, ne_1, e_2, e_3, \ldots \} \), where \( \{ e_i \}_{i \in I} \) is an orthonormal basis of \( \mathcal{H} \) and \( n \in \mathbb{N} \), is woven with all duals. Indeed, a straightforward calculation
immediately shows that every dual of $\Phi$ can be written as
\[
\{(1/2n + \alpha) e_1, (1/2n - \alpha) e_1, e_2, e_3, \ldots\},
\]
for any $\alpha \neq 0$. However, the norm of redundant elements are not small enough.

4. Weaving and Duality

The concept of weaving not only depends on the structure of two frames, but also on the order of their elements. For example, two orthonormal bases are not necessarily woven. Thus, dual frames and the images of a frame under bounded operators are the best candidates for weaving by the original frame. Obviously, the canonical dual of woven tight frames are woven. This leads to the following more general setting. In other word, the multiple of two woven frames are also woven.

**Proposition 4.1.** Let $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ be a pair of woven frames and $\{\alpha_i\}_{i \in I}$ and $\{\beta_i\}_{i \in I}$ be a sequence of scalers such that $0 < C < |\alpha_i| < D < \infty$ and $0 < C < |\beta_i| < D < \infty$ for every $i \in I$ and for some constants $C$ and $D$. Then $\{\alpha_i \varphi_i\}_{i \in I}$ and $\{\beta_i \psi_i\}_{i \in I}$ are woven.

In the next theorem, we show that if two frames $\Phi$ and $\Psi$ are woven then there are some duals $\Phi^d$ of $\Phi$ such that $\Psi$ and $S_{\Phi} \Phi^d$ are woven.

**Theorem 4.2.** Let $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be a pair of woven frames for $\mathcal{H}$ with universal bounds $A$ and $B$. Then there are infinitely many dual frames of $\Phi$ which are woven with $S_{\Phi}^{-1} \Psi$.

**Proof.** Let $U = \{u_i\}_{i \in I}$ be a Bessel sequence satisfies (2.2) and take $\alpha > 0$ such that
\[
A_\alpha := \alpha^2 \|S_{\Phi}\| B_U + 2\alpha \sqrt{B_U B_{\Phi} \|S_{\Phi}\|} < A.
\]

Then $\Phi^d = \{S_{\Phi}^{-1} \varphi_i + \epsilon u_i\}_{i \in I}$ is a dual frame of $\Phi$ for all $0 < \epsilon < \alpha$. It is enough to show that $\Psi$ and $S_{\Phi} \Phi^d$ are woven since the woven property is preserved under bounded invertible operators $\square$. To see this, we only need to prove the existence of a universal lower bound. Suppose $\sigma \subset I$, then for each $f \in \mathcal{H}$ we have
\[
\sum_{i \in \sigma \subset c} |\langle f, S \Phi \epsilon_i \rangle|^2 + \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 = \sum_{i \in \sigma \subset c} |\langle f, \varphi_i \rangle + \langle f, S \Phi \epsilon_i \rangle|^2 + \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 \\
= \sum_{i \in \sigma \subset c} |\langle f, \varphi_i \rangle + \langle f, S \Phi \epsilon_i \rangle|^2 + \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 \\
\geq \sum_{i \in \sigma \subset c} |\langle f, \varphi_i \rangle - \langle f, S \Phi \epsilon_i \rangle|^2 + \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 \\
\geq \sum_{i \in \sigma \subset c} |\langle f, \varphi_i \rangle|^2 - \sum_{i \in \sigma \subset c} |\langle f, S \Phi \epsilon_i \rangle|^2 \\
- 2 \sum_{i \in \sigma \subset c} |\langle f, \varphi_i \rangle| |\langle f, S \Phi \epsilon_i \rangle| + \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 \\
\geq (A - \epsilon^2 \| S \Phi \| B_U - 2 \epsilon \sqrt{B_U} \sqrt{\| S \Phi \| B_\Phi} ) \| f \|^2 \\
\geq (A - A_\alpha) \| f \|^2.
\]

By attention to (4.1) this completes the proof. □

Thus, it is very natural to ask whether every frame is woven with its canonical dual. The answer is affirmative when \( \mathcal{H} \) is a finite dimensional Hilbert space [3], however this fact is particularly true by the assumption of Proposition 2.5. Here, we obtain a sharper condition. More precisely, assume that \( \Phi = \{\varphi_i\}_{i \in I} \) is a frame such that

\[
\| I - S^{-1} \Phi \| \leq \frac{A_\Phi}{2(B_\Phi + \sqrt{B_\Phi})}.
\]

Hence,

\[
\| T_\Phi - T_\tilde{\Phi} \| = \| T_\Phi - S^{-1}_\Phi T_\Phi \| \\
\leq \| T_\Phi \| \| I - S^{-1}_\Phi \| \\
\leq \sqrt{B_\Phi} \| I - S^{-1}_\Phi \| \\
\leq \frac{A_\Phi \sqrt{B_\Phi}}{2(B_\Phi + \sqrt{B_\Phi})}.
\]

Combining Theorem 2.4 and Theorem 2.5 implies that \( \Phi \) and \( \tilde{\Phi} \) are woven.

**Theorem 4.3.** Let \( \Phi^d = \{S^{-1}_\Phi \varphi_i + u_i\}_{i \in I} \) be a dual frame of \( \Phi \) where \( U = \{u_i\}_{i \in I} \) is a Bessel sequence satisfies (2.2) such that

\[
\| I - S^{-1}_\Phi \| < \frac{A_\Phi}{2B_\Phi \left(1 + \sqrt{\| S^{-1}_\Phi \|}\right) + 2 \sqrt{B_U B_\Phi}}. \tag{4.2}
\]

Then \( \Phi \) with a family of its duals are woven.
Proof. By using (4.2), there exists $0 < \alpha < 1$ such that
\[
\|I - S^{-1}_\Phi\| + \alpha \sqrt{\frac{B_U}{B}\|B\|} \leq \frac{A\Phi}{2B\Phi \left(1 + \sqrt{\|S^{-1}_\Phi\|}\right) + 2\sqrt{B UB\Phi}} \leq \frac{A\Phi}{2B\Phi \left(1 + \sqrt{\|S^{-1}_\Phi\|}\right) + 2\alpha \sqrt{B UB\Phi}}.
\]

Putting $\Phi^d = \{S^{-1}_\Phi \varphi_i + \alpha u_i\}_{i \in I}$. Then $\Phi^d$ is a dual frame of $\Phi$ and an easy calculation follows that
\[
B_{\Phi^d} = \left(\sqrt{\|S^{-1}_\Phi\| B\Phi + \alpha \sqrt{B U}}\right)^2.
\]

Using this fact and the above inequality we obtain
\[
\|T_\Phi - T_{\Phi^d}\| = \|T_\Phi - S^{-1}_\Phi T_\Phi - \alpha T_U\| \\
\leq \|I - S^{-1}_\Phi\|\|T_\Phi\| + \alpha\|T_U\| \\
\leq \|I - S^{-1}_\Phi\| \sqrt{B\Phi} + \alpha \sqrt{B}\Phi \leq \frac{A\Phi}{2B\Phi \left(1 + \sqrt{\|S^{-1}_\Phi\|}\right) + 2\alpha \sqrt{B UB\Phi}} \\
= \frac{A\Phi}{2 \left(\sqrt{B\Phi} + \sqrt{B\Phi} \sqrt{\|S^{-1}_\Phi\| + \alpha \sqrt{B U}}\right)} \\
= \frac{A\Phi}{2 \left(\sqrt{B\Phi} + \sqrt{B\Phi^d}\right)}.
\]

Applying Theorem 2.4 follows that $\Phi$ and $\Phi^d$ are woven.

In the next theorem, we show that the wovenness property can be transferred to special dual frames. First, we prove it for Parseval frames.

Lemma 4.4. Let $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be Parseval woven frames. Then there are infinitely many dual frames $\Phi^d$ of $\Phi$ and $\Psi^d$ of $\Psi$ such that $\Phi^d$ and $\Psi^d$ are woven.

Proof. Using Theorem 2.4 follows that dual frames of $\Phi$ and $\Psi$ are of the form
\[
\{\varphi_i + u_i\}_{i \in I} \text{ and } \{\psi_i + v_i\}_{i \in I}, \text{ respectively, where } \{u_i\}_{i \in I} \text{ and } \{v_i\}_{i \in I} \text{ are Bessel sequences and for every } f \in \mathcal{H}
\]
\[
\sum_{i \in I} \langle f, \varphi_i \rangle u_i = 0, \quad \sum_{i \in I} \langle f, \psi_i \rangle v_i = 0.
\]
Choose $\alpha_0 > 0$ such that
\[
\alpha_0 B_U + \alpha_0 B_V + 2\sqrt{\alpha_0 B_U B_\Phi} + 2\sqrt{\alpha_0 B_V B_\Psi} < A,
\] (4.3)
where $A$ is a universal lower bound of weaving $\Phi$ and $\Psi$. Hence, for every $\sigma \subset I$ we have
\[
\sum_{\sigma} |\langle f, \varphi_i + \alpha_0 u_i \rangle|^2 + \sum_{\sigma^c} |\langle f, \psi_i + \alpha_0 v_i \rangle|^2
= \sum_{\sigma} |\langle f, \varphi_i \rangle + \langle f, \alpha_0 u_i \rangle|^2 + \sum_{\sigma^c} |\langle f, \psi_i \rangle + \langle f, \alpha_0 v_i \rangle|^2
\geq \sum_{\sigma} \left| \langle f, \varphi_i \rangle - \langle f, \alpha_0 u_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle f, \psi_i \rangle - \langle f, \alpha_0 v_i \rangle \right|^2
\geq \sum_{\sigma} |\langle f, \varphi_i \rangle|^2 - \sum_{\sigma} |\langle f, \alpha_0 u_i \rangle|^2 - 2 \sum_{\sigma} |\langle f, \varphi_i \rangle||\langle f, \alpha_0 u_i \rangle|
+ \sum_{\sigma^c} |\langle f, \psi_i \rangle|^2 - \sum_{\sigma^c} |\langle f, \alpha_0 v_i \rangle|^2 - 2 \sum_{\sigma^c} |\langle f, \psi_i \rangle||\langle f, \alpha_0 v_i \rangle|
\geq \left( A - \alpha_0 B_u - \alpha_0 B_v - 2\sqrt{\alpha_0 B_u B_\varphi} - 2\sqrt{\alpha_0 B_v B_\psi} \right) \|f\|^2
\]
This shows that $\Phi^d_\alpha = \{ \varphi_i + \alpha u_i \}_{i \in I}$ and $\Psi^d_\alpha = \{ \psi_i + \alpha v_i \}_{i \in I}$ which are dual frames of $\Phi$ and $\Psi$, respectively, are woven for all $0 < \alpha < \alpha_0$.

5. Stability of dual woven frames

In this section, we state some general stability results for woven frames, compare to the facts in [3, 7].

**Theorem 5.1.** Let $\Phi = \{ \varphi_i \}_{i \in I}$ and $\Psi = \{ \psi_i \}_{i \in I}$ be a pair of woven frames such that for every finite scalar sequence $\{ c_i \}$
\[
\left\| \sum_{i \in I} c_i \varphi_i - c_i \psi_i \right\| < \frac{\sqrt{A}}{\sqrt{B_\Psi} \| S_\Phi^{-1} \| \| S_\Psi^{-1} \| \left( \sqrt{B_\varphi} + \sqrt{B_\psi} \right)} \sum_{i \in I} \left| c_i \right|^2,
\] (5.1)
where $A$ is a universal lower bound of weaving $S_\Phi^{-1} \Phi$ and $S_\Psi^{-1} \Psi$. Then there are infinitely many dual frames $\Phi^d$ of $\Phi$ and $\Psi^d$ of $\Psi$ such that $\Phi^d$ and $\Psi^d$ are woven.

**Proof.** By the assumption $S_\Phi^{-1} \Phi$ and $S_\Psi^{-1} \Psi$ are also woven. Denote its universal bounds by $A$ and $B$. Choose arbitrary dual frames $\{ S_\Phi^{-1} \varphi_i + u_i \}_{i \in I}$ and $\{ S_\Psi^{-1} \psi_i + v_i \}_{i \in I}$ where $U = \{ u_i \}_{i \in I}$ and $V = \{ v_i \}_{i \in I}$ are Bessel sequences satisfying (2.2), see Theorem 2.1. Then
\[
\| S_\Phi - S_\Psi \| = \| T_\Phi T_\Phi^* - T_\Psi T_\Psi^* \| = \| T_\Phi T_\Phi^* - T_\Phi T_\Psi^* + T_\Phi T_\Psi^* - T_\Psi T_\Psi^* \|
\leq \| T_\Phi - T_\Psi \| (\| T_\Phi \| + \| T_\Psi \|).
\]
By (5.1), we obtain 
\[ \|S_\Phi - S_\Psi\| < \frac{1}{\|S_\Phi^{-1}\| \|S_\Psi^{-1}\|} \sqrt{\frac{A}{B_\Psi}}. \]
So,
\[ \|S_\Phi^{-1} - S_\Psi^{-1}\| = \|S_\Psi^{-1}(S_\Phi - S_\Psi)S_\Phi^{-1}\| < \sqrt{\frac{A}{B_\Psi}}. \] (5.2)
Choose \(0 < \alpha < 1\) such that
\[ \|S_\Phi^{-1} - S_\Psi^{-1}\| + \alpha \left( \frac{\sqrt{B_U} + \sqrt{B_V}}{B_\Psi}\right) < \sqrt{\frac{A}{B_\Psi}}. \] (5.3)
Then \(\Phi_\alpha^d = \{S_\Phi^{-1}\varphi_i + \alpha u_i\}_{i \in I}\) and \(\Psi_\alpha^d = \{S_\Psi^{-1}\psi_i + \alpha v_i\}_{i \in I}\) are duals of \(\Phi\) and \(\Psi\), respectively. Hence, for every \(\sigma \subset I\) we have
\[
\left( \sum_\sigma |\langle f, \Phi_\alpha^d \rangle|^2 + \sum_{\sigma^c} |\langle f, \Psi_\alpha^d \rangle|^2 \right)^{\frac{1}{2}}
\]
\[= \left( \sum_\sigma \left| \langle S_\Phi^{-1}f, \varphi_i \rangle + \langle f, \alpha u_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle S_\Psi^{-1}f, \psi_i \rangle + \langle f, \alpha v_i \rangle \right|^2 \right)^{\frac{1}{2}}
\]
\[= \left( \sum_\sigma \left| \langle S_\Phi^{-1}f, \varphi_i \rangle + \langle f, \alpha u_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle S_\Phi^{-1}(S_\Psi^{-1}f, \psi_i) + \langle f, \alpha v_i \rangle \right|^2 \right)^{\frac{1}{2}}
\]
\[\geq \left( \sum_\sigma |\langle S_\Phi^{-1}f, \varphi_i \rangle|^2 + \sum_{\sigma^c} |\langle S_\Psi^{-1}f, \psi_i \rangle|^2 \right)^{\frac{1}{2}} - \left( \sum_\sigma |\langle f, \alpha u_i \rangle|^2 \right)^{\frac{1}{2}}
\]
\[\geq \left( \sqrt{A} - \alpha \sqrt{B_U} - \alpha \sqrt{B_V} - \sqrt{B_\Psi} \parallel S_\Psi^{-1} - S_\Phi^{-1} \parallel \right) \parallel f \parallel
\]
\[\geq \sqrt{B_\Psi} \left( \sqrt{\frac{A}{B_\Psi}} - \alpha \left( \frac{\sqrt{B_U} + \sqrt{B_V}}{\sqrt{B_\Psi}} \right) - \parallel S_\Psi^{-1} - S_\Phi^{-1} \parallel \right) \parallel f \parallel.
\]
Applying (5.3), we obtain a universal lower bound for \(\Phi_\alpha^d\) and \(\Psi_\alpha^d\). This completes the proof. \(\square\)

The next result, which is proved similarly, shows that two frames are woven from the wovenness of their duals.

**Corollary 5.2.** Let \(\Phi = \{\varphi_i\}_{i \in I}\) and \(\Psi = \{\psi_i\}_{i \in I}\) be frames with dual frames \(\Phi^d = \{S_\Phi^{-1}\varphi_i + u_i\}_{i \in I}\) and \(\Psi^d = \{S_\Psi^{-1}\psi_i + v_i\}_{i \in I}\), respectively. If \(\Phi^d\) and \(\Psi^d\) are woven with universal bounds \(A\) and \(B\) such that
\[ \sqrt{B_U} + \sqrt{B_\Psi} \parallel S_{\Phi^d} - S_{\Psi^d} \parallel < \sqrt{A}. \] (5.4)
Then $\Phi$ and $\Psi$ are woven.

Now, let us restrict our attention to the canonical duals, we obtain the same conditions.

**Corollary 5.3.** Let $\Phi$ and $\Psi$ be two frames. The following assertions are hold:

1. If $\Phi$ and $\Psi$ are woven frames with a universal lower bound $A$ such that
   \[ \| S^{-1}_\Phi - S^{-1}_\Psi \| \leq \sqrt{\frac{A}{B_\Psi}}. \]
   Then $\tilde{\Phi}$ and $\tilde{\Psi}$ are woven frames.

2. If $\tilde{\Phi}$ and $\tilde{\Psi}$ are woven frames with a universal lower bound $A$ such that
   \[ \| S^{-1}_\Phi - S^{-1}_\Psi \| \leq \sqrt{\frac{A}{B_\Psi}}. \]
   Then $\Phi$ and $\Psi$ are woven frames.

For every frame $\Phi = \{ \varphi_i \}_{i \in I}$, the Parseval frame $\{ S^{-\frac{1}{2}}_\Phi \varphi_i \}_{i \in I}$ is called the canonical Parseval frame of $\Phi$. In [3], it is represented an example that two frames are woven but their canonical Parseval frames are not woven. In the next theorem, we show that this fact is true under some condition.

**Theorem 5.4.** Let $\Phi$ and $\Psi$ be a pair of woven frames with a universal lower bound $A$ such that

\[
\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \| < \frac{1}{\| S^{-\frac{1}{2}}_\Phi \| \| S^{-\frac{1}{2}}_\Psi \|} \left( \sqrt{\| S^{-\frac{1}{2}}_\Psi \|^2 + AB_\Phi} - \| S^{-\frac{1}{2}}_\Psi \| B_\Phi \right). \quad (5.5)
\]

Then their canonical Parseval frames $\tilde{\Phi}$ and $\tilde{\Psi}$ are also woven.

**Proof.** It is not difficult to see that

\[
B_\Phi \left( \| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \|^2 + 2 \| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \| \| S^{-\frac{1}{2}}_\Psi \| - A \right) < 0, \quad (5.6)
\]

whenever we have

\[
\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \| < \frac{-\| S^{-\frac{1}{2}}_\Psi \| + \sqrt{\| S^{-\frac{1}{2}}_\Psi \|^2 + AB_\Phi}}{B_\Phi}.
\]
Also, by the assumption we have
\[ \left\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right\| = \left\| S^{-\frac{1}{2}}_\Phi \left( S^{-\frac{1}{2}}_\Psi - S^\frac{1}{2}_\Phi \right) S^{-\frac{1}{2}}_\Psi \right\| \]
\[ \leq \left\| S^{-\frac{1}{2}}_\Phi \right\| \left\| S^{-\frac{1}{2}}_\Psi - S^\frac{1}{2}_\Phi \right\| \left\| S^{-\frac{1}{2}}_\Psi \right\| \]

(Due to (5.5))
\[ < \left\| S^{-\frac{1}{2}}_\Phi \right\| \left\| S^{-\frac{1}{2}}_\Psi \right\| \frac{1}{\left\| S^{-\frac{1}{2}}_\Phi \right\| \left\| S^{-\frac{1}{2}}_\Psi \right\|} \left( \sqrt{\left\| S^{-\frac{1}{2}}_\Psi \right\|^2 + A B_\Phi} - \left\| S^{-\frac{1}{2}}_\Psi \right\| \right) \]
\[ = -\left\| S^{-\frac{1}{2}}_\Psi \right\| + \frac{\sqrt{\left\| S^{-\frac{1}{2}}_\Psi \right\|^2 + A B_\Phi}}{B_\Phi} . \]

So,
\[ \left\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right\| < \frac{-\left\| S^{-\frac{1}{2}}_\Psi \right\| + \sqrt{\left\| S^{-\frac{1}{2}}_\Psi \right\|^2 + A B_\Phi}}{B_\Phi} . \]

In particular, (5.6) is valid. Since \( \Phi \) and \( \Psi \) are woven so for every \( \sigma \subset I \) we have
\[ \sum_{\sigma} |\langle f, S^{-\frac{1}{2}}_\Phi \varphi_i \rangle|^2 + \sum_{\sigma^c} |\langle f, S^{-\frac{1}{2}}_\Psi \psi_i \rangle|^2 \]
\[ = \sum_{\sigma} \left| \langle S^{-\frac{1}{2}}_\Phi f, \varphi_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle S^{-\frac{1}{2}}_\Psi f, \psi_i \rangle \right|^2 \]
\[ = \sum_{\sigma} \left| \left( \left( S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right) f, \varphi_i \right) + \langle S^{-\frac{1}{2}}_\Psi f, \varphi_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle S^{-\frac{1}{2}}_\Psi f, \psi_i \rangle \right|^2 \]
\[ \geq \sum_{\sigma} \left| \langle S^{-\frac{1}{2}}_\Psi f, \varphi_i \rangle \right|^2 + \sum_{\sigma^c} \left| \langle S^{-\frac{1}{2}}_\Psi f, \psi_i \rangle \right|^2 - \sum_{\sigma} \left| \left( \left( S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right) f, \varphi_i \right) \right|^2 \]
\[ \geq \left( A - B_\Phi \left\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right\|^2 - 2 \left\| S^{-\frac{1}{2}}_\Phi - S^{-\frac{1}{2}}_\Psi \right\| \right) \left\| f \right\|^2 . \]

By attention to (5.6), Parseval frames \( \{ S^{-\frac{1}{2}}_\Phi \varphi_i \}_{i \in I} \) and \( \{ S^{-\frac{1}{2}}_\Psi \psi_i \}_{i \in I} \) are woven. □

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