Distributed Secure Storage: Rate-Privacy Trade-Off and XOR-Based Coding Scheme

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Abstract—We consider the problem of storing data in a distributed manner over \( T \) servers. We require the data to be recoverable from the \( T \) servers, and to remain private from any \( T - 1 \) colluding servers, where privacy is quantified in terms of mutual information between the data and all the information available at the \( T - 1 \) colluding servers. For this model, we determine (i) the fundamental trade-off between storage size and the level of desired privacy, (ii) the optimal amount of local randomness necessary at the encoder, and (iii) an explicit low-complexity coding scheme that solely relies on XOR operations and that asymptotically (with the data size) matches the fundamental limits found.

I. INTRODUCTION

Secure distributed storage schemes, e.g., [1]–[5], often rely on the idea of secret sharing as introduced in [6], [7]. Hence, there is a fundamental lower bound on the required storage space necessary to securely store information in a distributed manner. Specifically, in any secret sharing scheme, the total amount of information that needs to be stored must at least be equal to the entropy of the secret times the number of participants, see e.g., [8], and it is thus impossible to reduce the storage space without any changes to the model assumptions.

To this end, we propose to determine the optimal cost reduction, in terms of storage space, that can be obtained in exchange of tolerating a controlled amount of reduced privacy. This idea is closely related to non-perfect secret sharing [9], [10] with a non-linear access function. Unfortunately, for large secrets, as required for data storage, no low-complexity coding scheme is known to implement non-perfect secret sharing. However, we note that in the absence of a privacy constraint, a low-complexity secret sharing scheme based on XOR operations has been proposed in [11].

We aim to fill this void in this paper and focus on a setting where any two servers and the file \( F \) must not exceed \( L \triangleq \frac{1}{4}H(F) \). Assuming that \( F \) is a sequence of uniformly distributed bits, we split \( F \) in four parts \((F_1, F_2, F_3, F_4)\) of equal length (for simplicity we assume here that \(|F|\) is a multiple of four), and we store in the three servers the shares

\[
M_1 \triangleq (F_1 \oplus K_1) (F_2 \oplus K_2) (F_3 \oplus K_3) (F_4 \oplus K_4),
\]

\[
M_2 \triangleq (F_1 \oplus K_3) (F_2 \oplus K_4) (F_3 \oplus K_1) (F_4 \oplus K_2),
\]

\[
M_3 \triangleq (F_1 \oplus K_2) (F_2 \oplus K_3) (F_3 \oplus K_4) (F_4 \oplus K_1),
\]

where \((K_1, K_2, K_3, K_4, K_5)\) are four sequences of uniformly distributed bits with size \(|F|/4\), \(\oplus\) denotes the XOR operation, and \(\parallel\) denotes concatenation. We remark that all the four parts \((F_1, F_2, F_3, F_4)\) are either stored in clear or encrypted through a one-time pad. By inspection, one easily sees in this example that \(F\) can be recovered from \((M_1, M_2, M_3)\) and any two shares leak at most \(\frac{1}{2}H(F)\) bits about \(F\). As it will be shown in the following, the size of the shares is optimal as well as the amount of local randomness, i.e., the length of \((K_1, K_2, K_3, K_4, K_5)\).

Our main contribution is the design of a low-complexity coding scheme for this problem with arbitrary parameters \(L \leq T\) that solely relies on XOR operations and that is asymptotically (with file size) optimal in terms of data storage and the required amount of local randomness at the encoder.

The remainder of the paper is organized as follows. We formalize the problem in Section II and state our main results in Section III. Our coding scheme is presented in Section IV. We present the proofs of our results in the appendix. Finally, we provide concluding remarks in Section V.

II. PROBLEM STATEMENT

Notation: For \(a, b \in \mathbb{R}\), define \([a, b] \triangleq [\lfloor a \rfloor, \lceil b \rceil] \cap \mathbb{N}\). Let \(\oplus\) denote the XOR operator. For \(x \in \mathbb{R}\), define \([x]^+ \triangleq \max(0, x)\).

Consider \(T \geq 2\) servers and define \(\mathbb{T} \triangleq [1, T]\). Consider a file \(F\) which is a sequence of \(|F|\) bits uniformly distributed over \(\{0, 1\}^{|F|}\).

Definition 1. A \((\lambda, \rho)\) coding scheme consists of

- A stochastic encoder \(e : \{0, 1\}^{|F|} \times \{0, 1\}^\rho \to \{0, 1\}^{\lambda T}, (F, R) \mapsto (M_t)_{t \in \mathbb{T}}\), which takes as input the

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file $F$ to store and a sequence $R$ of bits uniformly distributed over \{0, 1\} and independent of $F$, and outputs $T$ sequences (referred to as shares in the sharing) $(M_t)_{t \in T}$ of length $\lambda$, where $M_t$ is stored in Server $t \in T$.

- A decoder $d : \{0, 1\}^{|F|} \rightarrow \{0, 1\}^{|T|}, (M_t)_{t \in T} \rightarrow \hat{F}$, which takes as input all the $T$ sequences stored at the servers, and outputs an estimate $\hat{F}$ of the file $F$.

**Definition 2.** Fix $L \geq 0$. A $(\lambda, \rho)$ coding scheme is said to be $L$-private if

\begin{align*}
H(F | M_T) &= 0, \quad \text{(Decodability)} \\
I(F; M_S) &\leq L, \forall S \subsetneq T, \quad \text{(Privacy)}
\end{align*}

where we have used the notation $M_S \triangleq (M_t)_{t \in S}, \forall S \subsetneq T$.

The objective is to design $L$-private $(\lambda, \rho)$ coding schemes with minimal storage size requirement, i.e., minimal $\lambda$, and minimal amount of local randomness requirement at the encoder, i.e., minimal $\rho$.

**III. MAIN RESULTS**

Let $L \in [0, H(F)]$. Define $\alpha \triangleq L / H(F) \in [0, 1]$.

**Theorem 1 (Minimal storage size requirement).** For any $L$-private $(\lambda, \rho)$ coding scheme, we have

$$\lambda \geq (1 - \alpha)H(F).$$

**Theorem 2 (Minimal local randomness requirement).** For any $L$-private $(\lambda, \rho)$ coding scheme, we have

$$\rho \geq H(F)[T(1 - \alpha) - 1]^+. $$

**Theorem 3 (Achievability).** Assume that $\alpha = 0$ or $\alpha = \frac{1}{k}$, for some $k, l \in \mathbb{N}^+$ with $k$ and $l$ coprime. By density of $\mathbb{Q}$ in $\mathbb{R}$, such an $\alpha$ can be chosen arbitrarily close to any point in $[0, 1]$. The coding scheme in Section IV is an $L$-private $(\lambda, \rho)$ coding scheme with

\begin{align*}
\lambda &\leq (1 - \alpha)H(F) + \epsilon_\lambda, \\
\rho &\leq H(F)[T(1 - \alpha) - 1]^+ + \epsilon_\rho,
\end{align*}

where $\epsilon_\lambda \triangleq (k-1)\alpha, \epsilon_\rho \triangleq (k-1)[T(1 - \alpha) - 1]^+ + \epsilon_\rho$ are constant and thus negligible compared to $H(F)$, i.e., $\frac{\epsilon_\lambda}{H(F)} \xrightarrow{|F| \to \infty} 0$ and $\frac{\epsilon_\rho}{H(F)} \xrightarrow{|F| \to \infty} 0$.

**IV. CODING SCHEME**

Consider $\alpha \in [0, 1]$ such that $\alpha = 0$ or $\alpha = \frac{1}{k}$, for some $k, l \in \mathbb{N}^+$ with $k$ and $l$ coprime.

**A. Preliminaries**

- In the following, one can assume

$$\alpha < 1/T,$$

i.e., $lT < k$. Indeed, if $\alpha \geq 1/T$, i.e., $L \geq H(F)/T$, then the privacy constraint is trivially satisfied if one splits the file in $T$ parts of size $H(F)/T$ and store one part in each server.

- There exists $q \in \mathbb{N}, r \in [0, T-2]$ such that

$$l = q(T-1) + r,$$

and one can assume

$$qT + r < k.$$

Otherwise, if $qT + r = k$ (one has a similar argument if $qT + r > k$), then one can split the file $F$ in $k$ parts of equal size and store $q+1$ parts of $F$ in the first $r$ servers and $q$ parts of $F$ in the remaining servers. The privacy constraint is satisfied by (6) because any $N-1$ servers have at most $q(T-1) + r = l$ parts of $F$, and (2) holds.

- Note that $T(k-l)-k = (T-2)k + k - lT > (T-2)k \geq 0$, where we have used (5), and there exists $u \in \mathbb{N}, v \in [1, T-1]$ such that

$$T(k-l) - k = uT + v.$$

We emphasize that $l, k$, and $T$ are the only parameters of the coding scheme; $q, r, u$, and $v$ are obtained as sole functions of $l, k$, and $T$.

**B. Coding Scheme**

**Step 1.** Divide the file in $k$ parts $(F_i)_{i \in [1,k]}$ (if necessary, add $\beta$ zeros to $F$, where $\beta$ is the smallest integer in $[1,k-1]$ such that $(H(F) + \beta)/k \in \mathbb{N}$. For convenience, we write $F_{i,j} \triangleq (F_{i'})_{i' \in [i,j]}$ for $i, j \in [1,k]$.

**Step 2.** Generate $N_{\text{keys}} \triangleq T(k-l) - k (> 0)$ by (8) keys $(K_{i,j})_{i \in [1,N_{\text{keys}}]}$, each uniformly distributed over $\{0, 1\}^{(H(F) + \beta)/k}$ and independent of all other random variables. For convenience, we write $K_{i,j} \triangleq (K_{i,j'})_{j' \in [i,j]}$ for $i, j \in [1, N_{\text{keys}}]$.

**Step 3.** We now describe how to design the shares $M_T$. Each share $M_t$, stored in Server $t \in T$, is a vector of $(k-l)$ sequences (labeled from 1 to $k-l$) of size $(H(F) + \beta)/k$.

For convenience, for $t \in T$, and $i, j \in [1, k-1]$ such that $i \leq j$, we write $M_t[i : j]$ to designate the sequences of $M_t$ labeled from $i$ to $j$, and $M_t[i]$ to designate the sequences of $M_t$ labeled by $i$. For $t \in T$, the elements of the vector $M_t$ are of one of the following types.

- An unencrypted part of $F$, i.e., an element of $\{F_i : i \in [1,k]\}$.
- A key, i.e., an element of $\{K_{i,j} : i \in [1,N_{\text{keys}}]\}$.
- An encrypted part of $F$, obtained by XORing a part of $F$ with one or several keys.

To precisely describe how the shares $M_T$ are chosen, we distinguish two cases.

**Case 1.** Assume that $r + v < T$. The unencrypted parts of $F$ and keys are assigned according to Algorithms 1, 2, respectively. The encrypted parts of $F$ are defined and assigned according to Algorithm 4. Note that we have the following.

- For $t \in [1,r]$, $M_t$ consists of $k-l$ sequences: $q+1$ unencrypted parts of $F$, $u$ keys, and

$$x \triangleq k - l - q - 1 - u$$

encrypted parts of $F$. 
For $t \in [r+1, T-v]$, $M_t$ consists of $k-l$ sequences: $q$ unencrypted parts of $F$, $u$ keys, and $x+1$ encrypted parts of $F$.

For $t \in [T-v+1, T]$, $M_t$ consists of $k-l$ sequences: $q$ unencrypted parts of $F$, $u+1$ keys, and $x$ encrypted parts of $F$.

**Case 2.** Assume that $r+v \geq T$. The unencrypted parts of $F$ and keys are assigned according to Algorithms 1, 3, respectively. The encrypted parts of $F$ are defined and assigned according to Algorithm 5. Note that we have the following.

For $t \in [1, T-v]$, $M_t$ consists of $k-l$ sequences: $q+1$ unencrypted parts of $F$, $u$ keys, and $x$ encrypted parts of $F$.

For $t \in [T-v+1, r]$, $M_t$ consists of $k-l$ sequences: $q+1$ unencrypted parts of $F$, $u+1$ keys, and $x-1$ encrypted parts of $F$.

For $t \in [r+1, T]$, $M_t$ consists of $k-l$ sequences: $q$ unencrypted parts of $F$, $u+1$ keys, and $x$ encrypted parts of $F$.

**Remark 1.** In Case 1, we have $x \geq 0$, otherwise $k-r-Tq = N_{\text{Encrypted}} = (r+v)x + (T-r-v)(x+1) < 0$, which contradicts (7). In Case 2, we also have $x \geq 0$, otherwise $k-r-Tq = N_{\text{Encrypted}} = (T-v+T-r)x + (r-T+v)(x-1) < 0$, which again contradicts (7).

**Remark 2.** In both cases, the first $N_{\text{Plain}} \triangleq qT + r$ parts $F_i \in [1, qT+r]$ are stored unencrypted in the servers, and the $N_{\text{Encrypted}} \triangleq k-r-Tq (> 0)$ by (7) remaining parts $F_i \in [qT+r+1, k]$ are first encrypted using the keys $(K_i)_{i \in [1, T(k-t)-k]}$ before being stored in the servers.

### Algorithm 1 Assignment of unencrypted parts

**Inputs:** File $F$

1: for $t \in T$ do
2: $M_t[1: q] \triangleq F_t q+1:q$
3: if $t \leq r$ then
4: $M_t[q+1] \triangleq F_{T+q+t}$
5: end if
6: end for

### Algorithm 2 Assignment of keys when $r+v < T$

**Inputs:** Keys $(K_i)_{i \in [1, N_{\text{keys}}]}$

1: for $t \in T$ do
2: if $t \leq r$ then
3: $M_t[q+2: q+1+u] \triangleq K_{u(t-1)+1:ut}$
4: else if $r < t \leq T-v$ then
5: $M_t[q+1: q+u] \triangleq K_{u(t-1)+1:ut}$
6: else if $t > T-v$ then
7: $M_t[q+1: q+u] \triangleq K_{u(t-1)+1:ut}$
8: $M_t[q+u+1] \triangleq K_{T_u(t-(T-v))}$
9: end if
10: if $t \leq r$ then
11: $I_t \triangleq \{j \in [1, N_{\text{keys}}]: \exists i \in [1, I-k], M_t[i] = K_j\}$
12: end for

### Algorithm 3 Assignment of keys when $r+v \geq T$

**Inputs:** Keys $(K_i)_{i \in [1, N_{\text{keys}}]}$

1: for $t \in T$ do
2: if $t \leq T-u$ then
3: $M_t[q+2: q+1+u] \triangleq K_{u(t-1)+1:ut}$
4: else if $T-u < t \leq r$ then
5: $M_t[q+2: q+u+2] \triangleq K_{u(T-u)+(u+1)(t-(T-u)-1)+1:u(T-u)+(u+1)(t-(T+v))}$
6: else if $t > r$ then
7: $M_t[q+1: q+u+1] \triangleq K_{u(T-u)+(u+1)(t-(T-u)-1)+1:u(T-u)+(u+1)(t-(T+v))}$
8: end if
9: end if
10: if $t \leq r$ then
11: $I_t \triangleq \{j \in [1, N_{\text{keys}}]: \exists i \in [1, I-k], M_t[i] = K_j\}$
12: end for

After running Algorithm 5, we obtain the encrypted parts of the file $F$ as

$M_1[6] \triangleq F_5 \oplus K_4 \oplus K_8,$
$M_1[7] \triangleq F_6 \oplus K_5 \oplus K_{10},$
$M_2[6] \triangleq F_7 \oplus K_1 \oplus K_9,$
$M_2[7] \triangleq F_8 \oplus K_3 \oplus K_{11},$
$M_3[6] \triangleq F_9 \oplus K_2 \oplus K_5,$
$M_3[7] \triangleq F_{10} \oplus K_7.$

**Example 2** ($T = 5, L = 7H(F)/11$). Define $\alpha \triangleq \frac{7}{11}$, i.e., $(l, k) \triangleq (3, 10)$. By (6), we have $(q, r) = (1, 1)$ and by (8), we have $(u, v) = (3, 2)$. We have $r+v = T$, so we are in Case 2. Moreover, $N_{\text{Keys}} = 11, N_{\text{Plain}} = 4, N_{\text{Encrypted}} = 6, x = 2$. After running Algorithms 1 and 3 we have

$M_1 \triangleq (F_1 \oplus F_6 \oplus K_1 \oplus M_1[4]),$
$M_2 \triangleq (F_2 \oplus F_7 \oplus K_2 \oplus K_3),$
$M_3 \triangleq (F_3 \oplus F_8 \oplus K_4 \oplus K_5),$
$M_4 \triangleq (F_9 \oplus K_5 \oplus K_7 \oplus M_4[4]),$
$M_5 \triangleq (F_5 \oplus K_8 \oplus K_9 \oplus M_5[4]).$
Algorithm 4 Creation and assignment of encrypted parts when $r + v < T$

Inputs: $F$ and keys $(K_i)_{i \in [1:N]}$

1: for $i \in [1,x+1]$ do
2:    for $t \in T$ do
3:       if $t < r$ then
4:          $j \triangleq N_{\text{Plain}} + (t-1)x + i$
5:          $z \triangleq \begin{cases} 
           q + u + 1 + i & \text{if } i \neq x + 1 \\
           \emptyset & \text{if } i = x + 1 
          \end{cases}
6:    \end{if}
7:    \else if $r < t \leq T - v$ then
8:       $j \triangleq N_{\text{Plain}} + rx + (t - r - 1)(x + 1) + i$
9:       $z \triangleq \begin{cases} 
           q + u + 1 + i & \text{if } i \neq x + 1 \\
           \emptyset & \text{if } i = x + 1 
          \end{cases}
10: \end{if}
11: \else if $t > T - v$ then
12:    \end{end if}
13: end if
14: \end for
15: end for

where after running Algorithm 5 the encrypted parts of $F$ are fixed as
\begin{align*}
M_1[4] & \triangleq F_1 \oplus K_2 \oplus K_4 \oplus K_6 \oplus K_8, \\
M_4[4] & \triangleq F_10 \oplus K_1 \oplus K_3 \oplus K_5 \oplus K_9, \\
M_5[4] & \triangleq F_{11} \oplus K_7.
\end{align*}

Example 3 ($T = 3, L = 5H(F)/17$). Define $\alpha \triangleq \frac{5}{17}, i.e., (l, k) \triangleq (5, 17)$. By (6), we have $(q, r) = (2, 1)$ and by (8), we have $(u, v) = (6, 1)$. We have $r + v < T$, so we are in Case 1. Moreover, $N_{\text{Keys}} = 19, N_{\text{Plain}} = 7, N_{\text{Encrypted}} = 10, x = 3$. After running Algorithms 1 and 2, we have
\begin{align*}
M_1 & \triangleq (F_1 \| F_2 \| F_7 \| K_1 \| K_2 \| K_3 \| K_4 \| K_5 \| K_6 \\
    & \| M_1[10] \| M_1[11] \| M_1[12]), \\
M_2 & \triangleq (F_3 \| F_4 \| F_7 \| K_8 \| K_9 \| K_{10} \| K_{11} \| K_{12} \\
    & \| M_2[9] \| M_2[10] \| M_2[11] \| M_2[12]), \\
M_3 & \triangleq (F_5 \| F_6 \| K_{13} \| K_{14} \| K_{15} \| K_{16} \| K_{17} \| K_{18} \| K_{19} \\
    & \| M_3[10] \| M_3[11] \| M_3[12])
\end{align*}

After running Algorithm 5, the encrypted parts of $F$ are obtained as follows:
\begin{align*}
M_1[10] & \triangleq F_3 \oplus K_7 \oplus K_{13}, \\
M_1[11] & \triangleq F_9 \oplus K_9 \oplus K_{15}, \\
M_1[12] & \triangleq F_{10} \oplus K_{11} \oplus K_{17}, \\
M_2[9] & \triangleq F_{11} \oplus K_1 \oplus K_{14}, \\
M_2[10] & \triangleq F_{12} \oplus K_4 \oplus K_{16}, \\
M_2[11] & \triangleq F_{13} \oplus K_5 \oplus K_{18}, \\
M_2[12] & \triangleq F_{14} \oplus K_9, \\
M_3[10] & \triangleq F_{15} \oplus K_2 \oplus K_8, \\
M_3[12] & \triangleq F_{16} \oplus K_9 \oplus K_{10}, \\
M_3[12] & \triangleq F_{17} \oplus K_6 \oplus K_{12}.
\end{align*}

V. CONCLUDING REMARKS

We have studied the problem of storing a file in $T$ servers such that the privacy leakage generated by $T - 1$ colluding servers with respect to the contents of the file is bounded. The main contribution of this paper is a coding scheme for this problem that (i) achieves the asymptotically (with the file size) optimal storage space at the servers, (ii) uses the optimal amount of local randomness at the encoder, (iii) solely relies on XOR operations and is thus suited to handle large amount of data with low-complexity. Generalization of our XOR-based coding scheme to a threshold access structure, i.e., when decodability in (1) and the privacy constraint in (2) are replaced by
\begin{align*}
H(F|M_A) = 0, \forall A \subset T \text{ s.t. } |A| \geq t, \\
I(F; M_t) \leq L, \forall U \subset T \text{ s.t. } |A| \leq t - 1
\end{align*}
for some $t \in [1, T - 1]$, is under investigation.
APPENDIX A  
PROOF OF THEOREM 1

Let $S \subseteq T$. For any $L$-private $(\lambda, \rho)$ coding scheme, we have
\[
\rho \geq H(M_t) \\
\geq I(F; M_t | M_{T \setminus \{t\}}) \\
= H(F | M_{T \setminus \{t\}}) - H(F | M_T) \\
\underset{(a)}{\geq} H(F | M_{T \setminus \{t\}}) \\
= H(F) - I(F; M_{T \setminus \{t\}}) \\
\underset{(b)}{\geq} H(F) - L \\
\underset{(c)}{=} (1 - \alpha)H(F),
\]
where (a) holds by (1), (b) holds by (2), (c) holds by definition of $\alpha$.

APPENDIX B  
PROOF OF THEOREM 2

For any $L$-private $(\lambda, \rho)$ coding scheme, we have
\[
\rho + H(F) \overset{(a)}{=} H(R) + H(F) \\
\overset{(b)}{=} H(RF) \\
\underset{(c)}{\geq} H(M_T) \\
\overset{(d)}{=} \sum_{t \in T} H(M_t | M_{T \setminus \{t\}}) \\
\underset{(e)}{\geq} \sum_{t \in T} I(M_t; F | M_{T \setminus \{t\}}) \\
\overset{(f)}{\geq} \sum_{t \in T} (H(F) - L) \\
\overset{(g)}{=} TH(F)(1 - \alpha),
\]
where (a) holds by uniformity of $R$, (b) holds by independence between $F$ and $R$, (c) holds because $M_T$ is a deterministic function of $(R, F)$, (d) holds by the chain rule, (e) holds because conditioning reduces entropy, (f) holds by the proof of Theorem 1, (g) holds by definition of $\alpha$.

APPENDIX C  
PROOF OF THEOREM 3

We will use the following definition in our analysis of the coding scheme of Section IV.

**Definition 3.** Consider $M_t[z] = F_t \oplus \bigoplus_{t' \in T \setminus \{t\}} \kappa_{t'}$, $z \in \left\{ [q + u, k - l] \mid t \in [r + 1, T - v] \right\}$, an encrypted part of $F$ stored in Server $t \in T$ as in Line 14 of Algorithm 4, in Line 15 of Algorithm 5. The encrypted part $M_t[z]$ is said to be protected by a key of Server $t' \in T$, if there exists $j \in \mathcal{I}_{t'}$ such that $\kappa_{t'} = j$.

By construction, it is straightforward to verify that the storage size in (3) and the required amount of local randomness at the encoder in (4) are satisfied. Next, decodability (1) holds because all the parts $(F_i)_{i \in [1, k]}$ of $F$ are stored in $M_T$: $N_{\text{Plain}}$ parts are unencrypted and the $N_{\text{Encrypted}}$ encrypted parts can be decrypted from modulo-2 addition with keys that are all stored in $M_T$. Finally, it is sufficient to prove that the privacy constraint in (2) holds for all the subsets of $T$ with size $T - 1$, since $I(F; M_{S_T}) \leq I(F; M_S)$ if $S' \subseteq S \subseteq T$. We will thus prove that (2) holds for the sets $S_t = T \setminus \{t\}$, $t \in T$. We first define $K_{S_t}$, $F_{S_t}$, and $E_{S_t}$ as all the keys, encrypted parts of $F$, and encrypted parts of $F$, respectively, stored in the servers in $S_t$. We next consider two cases.

A. Case 1. $r + v < T$

Remark that
\[
(r - 1)x + (T - v - r)(x + 1) + vx = (T - 1)x + T - v - r \overset{(a)}{=} (T - 1)(k - l - q - 1 - u) + T - v - r \overset{(b)}{=} 1 + u,
\]
where (a) holds by (9), (b) holds by (6) and (8). We next consider three cases depending on the value of $t \in T$.

- **Assume** $t \leq r$. We have
  \[
  I(M_{S_t}; F) = I(K_{S_t}, F_{S_t}, E_{S_t}; F) \overset{(a)}{=} I(K_{S_t}, F_{S_t}, F_j; F_{s(i)})_{i \in \mathcal{I}_t} \\
  = I(K_{S_t}, F_{S_t}, F_j; F) + I((F_{s(i)} \oplus K_i)_{i \in \mathcal{I}_t}; F | K_{S_t}, F_{S_t}, F_j) \\
  \overset{(b)}{=} I(F_{S_t}; F) + I((F_{s(i)} \oplus K_i)_{i \in \mathcal{I}_t}; F | K_{S_t}, F_{S_t}) \\
  \overset{(c)}{\leq} I(F_{S_t}; F) + I((F_{s(i)} \oplus K_i)_{i \in \mathcal{I}_t}; (F_{s(i)})_{i \in \mathcal{I}_t}) \\
  \overset{(d)}{=} I(F_{S_t}; F) + \sum_{i \in \mathcal{I}_t} I(F_{s(i)} \oplus K_i; F_{s(i)}) \\
  \overset{(e)}{=} L,
  \]
  where (a) holds for some $j \in [N_{\text{Plain}} + 1, k]$ and $(s(i))_{i \in \mathcal{I}_t} \in [N_{\text{Plain}} + 1, k]^{|\mathcal{I}_t|}$ as follows. Note indeed that Server $t$ stores $|\mathcal{I}_t| = u$ keys and all the other servers store $(r - 1)x + (T - v - r)(x + 1) + vx = u + 1$ (by (11)) encrypted parts of $F$, hence, considering the encrypted parts $E_{S_t}$ of all the servers in $S_t$, all but one are protected by a key of Server $t$ by Line 14 of Algorithm 4.

(b) holds because $K_{S_t}$ is independent of $F$, (c) holds by the chain rule and because $(K_{S_t}, F_j)_{j \in [1, k]} \setminus \{(s(i))_{i \in \mathcal{I}_t}\}$ is independent from $(F_{s(i)}; F_j)_{i \in \mathcal{I}_t}$, (d) holds because $|\mathcal{I}_t| = (r - 1)(q + 1) + (T - r)q = l - 1$ by (6), (e) holds by the one-time pad (12).

- **Assume** $r + 1 \leq t \leq T - v$. We have
  \[
  I(M_{S_t}; F) = I(K_{S_t}, F_{S_t}, E_{S_t}; F) \overset{(a)}{=} I(K_{S_t}, F_{S_t}, (F_{s(i)} \oplus K_i)_{i \in \mathcal{I}_t}; F) \\
  \overset{(b)}{=} L,
  \]
where (a) holds for some \((s(i))_{i \in I_t} \in [N_{\text{Plain}} + 1, k]^{|I_t|}\) as follows. Note indeed that Server \(t\) stores \(|I_t| = u\) keys, and all the other servers store \(rx + (T - v - r - 1)(x + 1) + vx = u\) (by (11)) encrypted parts of \(F\). Hence, all the encrypted parts \(E_{S_i}\) of all the servers in \(S_t\) are protected by a key of Server \(t\) by Line 14 of Algorithm 4. (b) holds similar to (13) because \(|S_t| = r(q + 1) + (T - r - 1)q = l\) by (6).

- Assume \(t \geq T - 1\). The proof of (13) is identical to Subcase ii by remarking that (i) Server \(t\) stores \(|I_t| = u\) keys, and all the other servers store \(rx + (T - v - r - 1)(x + 1) + (v - 1)x = u\) (by (11)) encrypted parts of \(F\), and (ii) \(|S_t| = r(q + 1) + (T - r - 1)q = l\) by (6).

**B. Case 2.** \(r + v \geq T\)

Remark that

\[
\begin{align*}
(T - v - 1)x + (r - T + v)(x - 1) + (T - r)x \\
= (r - 1)x + (T - v - r)(x + 1) + vx \\
= 1 + u,
\end{align*}
\]

where the last equality holds by (11). We next consider three cases depending on the value of \(t \in T\).

- Assume \(t \leq T - v\). We have

\[
I(M_{S_t}; F) = I(K_{S_t}, F_{S_t}, E_{S_t}; F) \\
= I(K_{S_t}, F_{S_t}, F_j (F_{s(i)} \oplus K_i)_{i \in I_t}; F) \\
\leq L,
\]

(b) holds similar to (13) because \(|S_t| = (r - 1)(q + 1) + (T - r)q = l - 1\) by (6).

- Assume \(r < t \leq T - v\). We have

\[
I(M_{S_t}; F) = I(K_{S_t}, F_{S_t}, E_{S_t}; F) \\
= I(K_{S_t}, F_{S_t}, F_j (F_{s(i)} \oplus K_i)_{i \in I_t}; F) \\
\leq L,
\]

where (a) holds for some \((s(i))_{i \in I_t} \in [N_{\text{Plain}} + 1, k]^{|I_t|}\) as follows. Note indeed that Server \(t\) stores \(|I_t| = u + 1\) keys, and all the other servers store \((T - v) + (r - T + v)(x - 1) + (T - r)x = u + 2\) (by (14)) encrypted parts of \(F\), hence, considering the encrypted parts \(E_{S_t}\) of all the servers in \(S_t\), all but one are protected by a key of Server \(t\) by Line 14 of Algorithm 5. (b) holds similar to (13) because \(|S_t| = (r - 1)(q + 1) + (T - r)q = l - 1\) by (6).

- Assume \(t \geq T - v + 1\). We have

\[
I(M_{S_t}; F) = I(K_{S_t}, F_{S_t}, E_{S_t}; F) \\
\leq I(K_{S_t}, F_{S_t}, (F_{s(i)} \oplus K_i)_{i \in I_t}; F) \\
\leq L,
\]

where (a) holds for some \((s(i))_{i \in I_t} \in [N_{\text{Plain}} + 1, k]^{|I_t|}\) as follows. Note indeed that Server \(t\) stores \(|I_t| = u + 1\) keys, and all the other servers store \((T - v)x + (r - T + v)(x - 1) + (T - r)\) (by (11)) encrypted parts of \(F\). Hence, all the encrypted parts \(E_{S_t}\) of all the servers in \(S_t\) are protected by a key of Server \(t\) by Line 14 of Algorithm 4. (b) holds similar to (13) because \(|S_t| = r(q + 1) + (T - r - 1)q = l\) by (8).

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