Connection between topology and statistics

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Abstract

We introduce topological magnetic field in two-dimensional flat space, which admits a solution of scalar monopole that describes the nontrivial topology. In the Chern-Simons gauge field theory of anyons, we interpret the anyons as the quasi-particles composed of fermions and scalar monopoles in such a form that each fermion is surrounded by infinite number of scalar monopoles. It is the monopole charge that determines the statistics of anyons. We re-analyze the conventional arguments of the connection between topology and statistics in three-dimensional space, and find that those arguments are based on the global topology, which is relatively trivial compared with the monopole structure. Through a simple model, we formulate the three-dimensional anyon field using infinite number of Dirac’s magnetic monopoles that change the ordinary spacetime topology. The quasi-particle picture of the three-dimensional anyons is quite similar to that of the usual two-dimensional anyons. However, the exotic statistics there is not restricted to the usual fractional statistics, but a functional statistics.

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I. Introduction

More than fifty years ago, based on local relativistic quantum field theory, Pauli\(^1\) proved that spin-integral particles obey bose statistics, and spin-half integral particles obey fermi statistics. Since then, the connection between spin and statistics has become an interesting topic and acquired a lot of studies. Among the earlier attempts to study physics using topology, Ref. [2] is probably the first to adopt homotopy group to describe the intrinsic invariant for example *kink* in some nonlinear systems embedding soliton. Through an analysis of the significance of topology for quantization of nonlinear classical fields, one of the authors and Rubinstein\(^3\) clarified that in the Skyrme’s model\(^4\), the half-integral spin solitons obey fermi statistics, and showed that statistics is related with topology. Leinaas and Myrheim,\(^5\) and later Wilczek\(^6\) demonstrated that in two spatial dimensions, the possibilities for quantum statistics are not limited to bosons and fermions, but rather allow continuous interpolation between these two extremes, called exotic statistics, which is defined by the phase of the amplitude associated with slow motion of distance particles around one another. The particles following the exotic statistics are generally called *anyons*. The discovery of anyons in two spatial dimensions opens a new ground for the study in condensed matter system. The manifestations of anyons as quasi-particles in fractional quantum Hall effect\(^7\) and anyon superconductivity\(^8\) have been intensively investigated for years.

Since the two-dimensional Abelian rotation group allows fractal angular momentum, in quantum mechanics exotic statistics is interpreted through the trajectories of the wavefunction of the objects with fractal angular momentum\(^5,6\). Nevertheless, there are at least three spatial dimensions in the physical world, it is clear that anyons are not real particles. In quantum field theory, the anyons are interpreted as the quasi-particles through a system of *spinless* fermionic field which minimally couples to the Chern-Simons gauge field\(^9\). It is the Chern-Simons term, a topological invariant of winding number one, that transforms the fermions into anyons. Recently, the Chern-Simons gauge theory was found to embed the solution of solitons, which still follow an exotic statistics\(^11\). There is a further question:
How to interpret the exotic statistics from the viewpoint of topology\textsuperscript{12}?

In the Chern-Simons gauge theory of anyons, the above question becomes more concrete: What kind of topology does the Chern-Simons theory describe in the system of anyons? In this paper we attempt to answer this question by studying a system of nonrelativistic fermionic field, which minimally couples to the Chern-Simons gauge field. Through looking into the two-dimensional topology, we find that the system embeds a solution of topological monopole, called \textit{scalar monopole} (its potential is a scalar), which describes the two-dimensional nontrivial topology analogous to the magnetic monopole of three-dimensional space. As a result, the anyons are found to be such combinations of fermions and scalar monopoles that each fermion is surrounded by infinite number of scalar monopoles in the whole space. Within this quasi-particle picture, the exotic statistics is easily interpreted: When two anyons exchange their positions, each of them must pass through the scalar monopole of the others, and result in a nontrivial phase factor in term of the \textit{charge} of the scalar monopole that determines the statistics of the anyons.

Another topic discussed in the present paper is about the statistics in three-dimensional space. It has been argued\textsuperscript{5} from the viewpoints of both topology and quantum field theory that there are only bose and fermi statistics in three spatial dimensions. Nevertheless, there are still some interesting discoveries worth notice: Both the Dirac’s magnetic monopole and t’Hooft-Polyakov’s non-Abelian monopole\textsuperscript{13} are described by the bosonic fields, however, they obey fermi statistics\textsuperscript{14}; The bosonization scheme was realized in four-dimensional quantum electrodynamics\textsuperscript{15} using the Wilson loop; The Jordan-Wigner transformation was generalized to three spatial dimensions by $SU(2)$ doublet Heisenberg spin operators\textsuperscript{16}, etc. We notice that though these discoveries enrich the three-dimensional statistics, they are still consistent with the usual topological argument of statistics. But, there is one exception as far as we know: Libby, Zou and Laughlin\textsuperscript{17} formulated fractional statistics in three-dimensional chiral spin liquid state through $SU(2)$ non-Abelian gauge interaction. Unfortunately, the fractional statistics there, defined by Berry’s phase, breaks the three-dimensional rotational invariance.
In this paper, we re-analyze the conventional arguments of the three-dimensional statistics, and find that the usual conclusion results from the relatively trivial topology, in other words, there is no nontrivial topological structure like the scalar monopole of two dimensions involved in the system as the background topology. Then, are there other possibilities of statistics besides bosons and fermions in three-dimensional space with nontrivial topology? To answer this question, we consider a system of non-relativistic fermion field, which couples to an Abelian gauge field. The dynamics of the gauge field is determined by the Pontrjagin term multiplying a scalar field. By solving the Lagrangian equations, we find that the gauge field can be formally expressed as a pure gauge in terms of the vector potential of Dirac’s magnetic monopole. Then through a peculiar gauge transformation, we construct an anyon field by fermions and magnetic monopoles, where each fermion is surrounded by infinite number of magnetic monopoles in the whole space. This quasi-particle picture of anyons is similar to that of the usual two-dimensional anyons. However, the exotic statistics is not restricted to be the fractional statistics, but becomes functional statistics. Same as the case in Ref. [17], the anyons in our model break the rotational invariance too, which implies that the three-dimensional anyons can only be realized as the quasi-particles in condensed matter system, where the rotation symmetry is not necessarily preserved in many cases. We further point out that the action of the anyons results from a low-energy approximation for a quantum system with axial chiral current anomaly. The scalar field is then consistently interpreted as the chiral mode of the system.

This paper is organized as follows: In Section II, we first introduce topological magnetic field and scalar monopole in two-dimensional space, then show how the scalar monopole enters the usual Chern-Simons gauge field and determines the statistics of anyons. In addition, we point out that the Chern-Simons field can be realized by Berry’s potential. Section III is devoted to the connection between statistics and three-dimensional topology. We present a simple model to formulate the exotic statistics in three-dimensional space. Section IV is the conclusion.
II. Topology and statistics in (2+1)-dimensional spacetime

As we know in quantum mechanics, the exotic statistics in two-dimensional space is interpreted by the objects with fractal angular momenta; On the other hand, in the Chern-Simons gauge field theory, it is interpreted by the gauge flux produced by the permutation of particles. In this section, we provide an alternative interpretation for the exotic statistics from the viewpoint of topology. Usually, one studies the topology through the physical models. For an example, topological defect in three-dimensional space is modeled by Dirac's magnetic monopole. In the following Subsection 2.1, we introduce a new object—scalar monopole in two-dimensional flat space to represent the point defect there. Using the scalar monopole, we re-approach the usual Chern-Simons field theory of anyons in Subsection 2.2, and show that the statistics of anyons is determined by the charge of the scalar monopole.

2.1. Scalar monopole in two dimensional space

In two-dimensional flat space $\mathbf{x} = (x_1, x_2)$, there is no magnetic field in the usual sense, since when charged particles move in a plane, the induced magnetic field is always perpendicular to the plane. Nevertheless, to describe the two-dimensional topology conveniently, we introduce topological magnetic field from the purely mathematical consideration as follows.

For simplicity, we consider only the static case, therefore neglect the time dimension. Different from the vector potential of magnetic field in three-dimensional space, the potential of the topological magnetic field here is assumed to be scalar, denoted as $a(\mathbf{x})$. The topological magnetic field $\mathbf{b} = (b_1, b_2)$ is then defined as

$$
\begin{align*}
    b_1 &= \frac{\partial}{\partial x_2} a, \\
    b_2 &= -\frac{\partial}{\partial x_1} a.
\end{align*}
$$

For a regular potential $a(\mathbf{x})$ without singularity, the above $\mathbf{b}(\mathbf{x})$ satisfies the relation:
\[ \nabla \cdot \mathbf{b} = 0, \]  
(2)

which is consistent with the usual definition of magnetic field. If we regard \( a(\mathbf{x}) \) as a gauge field, we may study the gauge invariance of \( a(\mathbf{x}) \): Let \( a'(\mathbf{x}) = a(\mathbf{x}) + f \), if \( \mathbf{b}(\mathbf{x}) \) is invariant under this transformation, following Eq. (1) we have

\[
\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0,
\]

thus, \( f \) can only be a constant (this holds for the static case only).

It is evident that the topology in the above Eq. (2) is trivial. To describe the nontrivial topology, we consider such a case that \( \mathbf{b}(\mathbf{x}) \) is generated by a point-like source:

\[ \nabla \cdot \mathbf{b} = \delta(\mathbf{x}). \]

(4)

Analogous to the definition of Dirac’s magnetic monopole in three-dimensional space, we see that Eq. (4) also defines a topological monopole in a plane with the monopole charge equal to one. One infers that the potential \( a(\mathbf{x}) \) in this case must be singular. To solve the monopole, we turn to the method of Green function: Let \( \mathbf{b} = \nabla G \), equation (4) becomes

\[ \nabla^2 G(\mathbf{x}) = \delta(\mathbf{x}), \]

(5)

which is solved as

\[ G(\mathbf{x}) = \frac{1}{2\pi} \ln x, \]

(6)

where \( x = |\mathbf{x}| \). Then

\[ \mathbf{b}(\mathbf{x}) = \frac{1}{2\pi x^2} \mathbf{x}. \]

(7)

Using the relation between \( a(\mathbf{x}) \) and \( G(\mathbf{x}) \):

\[ \partial_i a(\mathbf{x}) = -\epsilon_{ij} \partial_j G(\mathbf{x}), \]

(8)

we obtain the scalar potential of the monopole Eq. (4) as
\[ a(x) = \frac{1}{2\pi} \tan^{-1}\left(\frac{x_2}{x_1}\right), \quad (9) \]

where the "gauge" condition \( f = 0 \) is chosen. The above expression shows that \( a(x) \) is singular on the \( x_2 \)-axis. For convenience, we write \( a(x) \) into the polar coordinates \((r, \theta)\) as

\[ a(x) = \frac{1}{2\pi} \theta(x). \quad (10) \]

In the meantime we should define a branch cut for instance \( \theta = 0 \) to eliminate the ambiguity in the expression (10). Since the topological monopole in two-dimensional space is described by the scalar potential, we call it \textit{scalar monopole}.

In order to describe the above nontrivial topology without concerning the singularity in potential, we adopt an often-used quantity—\textit{Winding number}, which is defined through a loop \( l \) as:

\[ W = \oint_l (\epsilon_{ij} b_i) \, dl_j. \quad (11) \]

For the loop without encircling a monopole, we use Eq. (1) to write the winding number as: \( W = \oint_l \partial_i a(x) \, dl_i \), which gives: \( W = 0 \); For the loop encircling a scalar monopole as shown in Fig. 1 (a), we obtain \( W = 1 \), and further \( W = n \) for the loop in Fig. 1 (b). These results is easily understood through a homotopic analysis: A plane having a defect is homotopic to a ring \( S^1 \). It is the homotopy group,

\[ \Pi_1(S^1) = Z, \quad (12) \]

that accounts for the winding number.

As a topological invariant, \( W \) is evidently independent of the definition of branch cut. We now briefly show how to evaluate \( W \) by taking into account the branch cut. Before doing this, we first introduce the following function, which is related to the branch cut \( \theta = 0 \),

\[ \Delta(x) = \theta(-x) - \theta(x) = \begin{cases} \pi \text{ sign}(x_2), & x_2 \neq 0; \\ \pi \text{ sign}(x_1), & x_2 = 0. \end{cases} \quad (13) \]
We divide the loop into two parts: $l_1$ and $l_2$ as shown in Fig. 1 (a). Since $l_1$ does not pass through the branch cut, we may use Eq. (13) to obtain:

$$W_1 = \int_{l_1} \partial_i a(x) dl_i = \frac{1}{2\pi} \left[ \theta(-x') - \theta(x') \right] = \frac{1}{2}. \tag{14}$$

While $l_2$ pass through the cut branch anti-clockwise, we must add $2\pi$ to the integral,

$$W_2 = \int_{l_2} \partial_i a(x) dl_i = \frac{1}{2\pi} \left[ 2\pi + \theta(x') - \theta(-x') \right] = \frac{1}{2}. \tag{15}$$

Sum up $W_1$ and $W_2$, we still get to $W = 1$.

Since the scalar monopole is a mathematical description of the nontrivial two-dimensional topology, we do not need to quantize it here. We will show below that the scalar monopole plays a crucial role in the statistics of identical particles in (2+1)-dimensional spacetime.

### 2.2. Quantum field theory of anyons in (2+1)-dimensional spacetime

Consider a system of nonrelativistic spinless fermion field $\psi(x,t)$ of mass $m$ and charge $e$, which minimally couples to an Abelian Chern-Simons gauge field $A_i(x,t)$. The action is written as ($\hbar = c = 1$)

$$S = \int d^3x \left\{ i\psi^\dagger D_0 \psi + \frac{1}{2m} \psi^\dagger D^2 \psi + \frac{\lambda}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \right\}, \tag{16}$$

where $D_\alpha = \partial_\alpha + ieA_\alpha$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\lambda$ is a constant, and the metric is chosen to be $\eta = (+ - -)$. The action is easily quantized. For example, for the fermion field, it is

$$\left\{ \psi(x,t), \psi^\dagger(x',t) \right\} = \delta(x - x'). \tag{17}$$

Varying $S$ with respect to $A_\alpha$ gives:

$$\epsilon^{\alpha\beta\gamma} \partial_\beta A_\gamma(x) = \frac{e}{\lambda} j^\alpha(x), \tag{18}$$

where the current $j^\alpha$ are given by: $j^0 = \rho = \psi^\dagger \psi$, $j^i = \frac{i}{2m} [\psi^\dagger D^i \psi - (D^i \psi^\dagger) \psi]$, $i = 1, 2$, and they satisfy the continuity equation: $\partial_\alpha j^\alpha = 0$. Equation (18) implies that the Chern-Simons field is determined by the particle current. This can be seen clearly by the following calculations.
We first look at the $\alpha = 0$ component of equation (18):

$$\partial_1 A_2(x) - \partial_2 A_1(x) = \frac{e}{\lambda} \rho(x),$$  \hspace{1cm} (19)

which is in fact a constraint equation. To solve $A^i, i = 1, 2$, we impose the Coulomb’s gauge condition: $\partial_i A^i = 0$, which enables us to let $A^i = \epsilon^{ij} \partial_j \Omega$. Equation (19) then becomes:

$$\nabla^2 \Omega(x) = \frac{e}{\lambda} \rho(x).$$  \hspace{1cm} (20)

Employing the Green function as in Eq. (5), we can formally solve Eq. (20) and obtain further:

$$A^i(x) = \epsilon^{ij} \frac{\partial}{\partial x^j} \left[ \frac{e}{\lambda} \int d^2 y \ G(x - y) \rho(y) \right].$$  \hspace{1cm} (21)

Using Eq. (8), the relation between Green function and the scalar monopole, we express the above $A^i(x)$ as the following compact form,

$$A^i(x) = \frac{\partial}{\partial x^i} \Theta(x),$$  \hspace{1cm} (22)

where

$$\Theta(x) = \frac{e}{2\pi \lambda} \int d^2 y \ \theta(x - y) \rho(y).$$  \hspace{1cm} (23)

One observes that the scalar monopole enters the solution of $A^i(x)$ naturally. This makes it clear that the Chern-Simons gauge theory embeds the nontrivial topological structure. Notice that it must be sure that $\Theta(x)$ is a well-defined function when moving the derivative out of $\Theta(x)$. There is no problem here, since the density of fermion field $\rho(y)$ is a point-like function in the nonrelativistic case.

To solve $A_0(x)$, we look at the $\alpha = i, \ i = 1, 2$ components of equation (18):

$$\partial_i A_0(x) - \partial_0 A_i(x) = -\frac{e}{\lambda} \epsilon_{ij} j^j(x).$$  \hspace{1cm} (24)

After a further derivative $\partial/\partial x_i$ acting on Eq (24) and using Coulomb’s gauge condition, we turn Eq. (24) into a Laplacian equation. Then adopting the Green function, we solve $A^0(x)$ as
\[ A_0(x) = -\frac{e}{\lambda} \epsilon_{ij} \int d^2y \ G(x - y) \frac{\partial}{\partial y^i} j^j(y) \]  

(25)

Using the integration in parts and the continuity equation of the current, we get to [See Appendix A]:

\[ A_0(x) = \partial_0 \Theta(x). \]  

(26)

Combining Eqs. (22) and (26), we arrive at a remarkable conclusion that the Chern-Simons gauge field can be formally solved as a pure gauge:

\[ A_\alpha(x) = \partial_\alpha \Theta(x), \]  

(27)

where \( \Theta(x) \) is the \( x \)-site scalar potential of all the topological monopoles in two-dimensional space.

The above solution of the gauge field allows us to remove the Chern-Simons gauge field from the action \( S \) by the following singular gauge transformations,

\[
\begin{aligned}
A'_\alpha(x) &= A_\alpha(x) - \partial_\alpha \Theta(x) = 0, \\
\psi'(x) &= e^{ie\Theta(x)} \psi(x), \\
\psi'^\dagger(x) &= \psi^\dagger(x) e^{-ie\Theta(x)}.
\end{aligned}
\]

(27)

Under these transformations, the action \( S \) turns out to be

\[ S' = \int d^3x \left[ i\psi'^\dagger \partial_0 \psi' + \frac{1}{2m} \psi'^\dagger \nabla^2 \psi' \right]. \]  

(28)

Before discussing the physical meaning of the field \( \psi'(x) \) from the above action, we first look at the function \( \Theta(x) \). From the expression (23), we know that \( \Theta(x) \) represents the total potential of the scalar monopoles with charge density \( \frac{e}{2\pi\lambda} \rho(y) \) in the whole space as shown in Fig. 2 (a). Moreover, the equation

\[ [\Theta(x, t), \ \psi'(x', t)] = -\theta(x - x') \psi'(x', t), \]  

(29)

indicates that even if \( \Theta(x, t) \) and \( \psi'(x', t) \) are space-likely separated, they are still related with each other. Therefore, \( \Theta(x) \) is essentially a nonlocal quantity. We further infer that the new field \( \psi'(x) \) is a nonlocal field that combines the topological monopoles in whole space with the fermi field at \( x \). In this sense, \( \psi'(x) \) describes a set of quasi-particles only, which
is far from free, though these quasi-particles have the free-form action (28). In the presence of topological monopoles, the statistics of these quasi-particles differs from the that of the fermions. A simple calculation by using Eqs. (17) and (27) gives

\[ \psi'(x) \psi'(x') = e^{-i\kappa \Delta(x-x')} \psi'(x') \psi'(x), \quad t = t', \quad (30) \]

where \( \kappa = e^2/\lambda \). The multi-valued function \( \Delta(x-x') \) is defined in Eq. (13), its values (\( \pm \pi \)) are specified by the relative positions of \( x \) and \( x' \). The above commutation rule is understood by the way drawn in Fig. 2.b that when two quasi-particles exchange their positions, one of them must pass through the monopole of each others, this will result in a nontrivial phase factor in terms of the monopole charge (\( e/\lambda \)) that determines statistics of the quasi-particles.

In general, the nonlocal field \( \psi'(x) \) obey an anyonic statistics interpolated between bosons and fermions. It should be emphasized that even if \( \kappa \) is an add number, namely, \( \psi'(x) \) satisfies a bosonic commutation relation, \( \psi'(x) \) can not be taken as a boson field, it still describes the quasi-particles, and does not commute with the fermion field \( \psi(x) \). Likewise, \( \psi'(x) \) is not the real fermion field for the even \( \kappa \).

We conclude that the anyon field embeds not only the property of nonlocality in physical interest, but also the nontrivial two-dimensional topology. It is the topological monopole that gives rise to the exotic statistics of the anyon field.

### 2.3. Chern-Simons field as a realization of Berry’s potential

Consider a quantum mechanical system in which the Hamiltonian \( H \) evolves adiabatically with parameters \( R_i \equiv R_i(t) \), and has discrete eigenvalues \( E_n(R) \):

\[ H(R)|\Psi_n(R)\rangle = E_n(R)|\Psi_n(RR)\rangle, \quad (31) \]

where \( |\Psi_n(R)\rangle \) are eigenstates. If \( R \) executes a closed loop in parameter space: \( R(0) = R(T) \), Berry\(^{18} \) proved that after a round trip along the loop, the state of the system will
acquire a geometric phase: $\gamma = \oint A_i(R) dR^i$, which is attributed to the holonomy in the parameter space, where

$$A_i(R) \equiv \langle \Psi_n(R) | \frac{\partial}{\partial R^i} | \Psi_n(R) \rangle,$$

is called Berry’s potential and has wide application in physics.

It is well known that the vector potential of Dirac’s magnetic monopole in three-dimensional space can be realized by Berry’s potential of the system of spin-1/2 particle moving in the magnetic field which traces out a circuit slowly without changing the magnitude. In two-dimensional space, the object analogous to Dirac’s magnetic monopole is evidently the Chern-Simons gauge field. Choosing the classical limit to the density of fermions in Eq. (21): $\rho(x) = \delta(x)$, we get to the usual expression of the Chern-Simons field for single particle:

$$A_i(x) = \frac{e}{2\pi \lambda} \epsilon_{ij} x^j x^2.$$  

(33)

One may ask: Can the Chern-Simons field be realized as Berry’s potential?

Since there is no the usual magnetic field in two spatial dimensions, we have to turn to the topological magnetic field introduced in Subsection 2.1. Then the above question becomes the purely mathematical scheme.

Let us consider a constant topological magnetic field $b = (b_1, b_2)$, the corresponding scalar potential is simply: $a(x) = b_1 x_2 - b_2 x_1$. We notice that this magnetic field can also be realized as the long distance approximation of the topological monopole Eq. (7). Assuming a spin-1/2 particle moving in this field, the simplified Hamiltonian is

$$H = \eta (\sigma_1 b_1 + \sigma_2 b_2),$$  

(34)

where $\sigma_{1,2}$ are two Pauli matrices, $\eta$ accounts for the magnetic moment and Lander factor. The eigenkets of the Hamiltonian for spin up and down are respectively,

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\theta} \end{pmatrix},$$

(35)
\[ \theta = \tan^{-1}(b_2/b_1). \]

For the case that the magnetic field \( b = (b_1, b_2) \), taken as the parameter space, traces out a circuit slowly without changing the magnitude, we use Eq. (32) to obtain the Berry’s potential for the ket for instance \( |\phi_1\rangle \) as

\[ A'_i(b) = i\langle\phi_1| \frac{\partial}{\partial b_i} |\phi_1\rangle = \frac{1}{2} \epsilon_{ij} b_j. \]  

(36)

The above result is identified with \( A_i \) in Eq. (33) up to a constant factor.

### III. Topology and statistics in (3+1)-dimensional spacetime

In last section we have shown that topology determines the statistics of identical particles in two-dimensional flat space. The naive model Eq. (34) indicates that the Chern-Simons field is analogous to the vector field of Dirac’s monopole from the viewpoints of both topology and Berry’s potential. This observation inspires us to investigate further the role topology plays in three-dimensional statistics.

In this section, we first recapitulate the conventional conclusion of that there are only bose and fermi statistics in (3+1)-dimensional spacetime, and point out that this conclusion essentially results from the global and relatively trivial topology. We then present a simple model of anyons in three-dimensional space, in which infinite number of Dirac’s magnetic monopoles appear, they change the topology, and give rise to the exotic statistics\(^{20}\).

#### 3.1. Statistics in trivial topology

It was believed that identical particles in three-dimensional space follow either bose or fermi statistics. This belief generally comes from the following two arguments:

(i) The non-Abelian nature of three-dimensional Rotation group implies that its representation is labeled by a discrete index, integer or half-integer angular momentum. It was proved\(^1\) by the local relativistic quantum field theory that the particles having integer angu-
lar momenta obey the bose statistics, and the particles having half-integer angular momenta
obey the fermi statistics. However, the proof becomes invalid in the case of nonrelativistic
spinless fields, especially for the quasi-particles described by nonlocal fields.

(ii) Another argument is based on topology. In three-dimensional Euclidean space $\mathbb{R}^3$, a
closed curve encircling a singularity twice can be continuously contracted to a point without
passing through the singularity. Therefore, for a system of two identical particles at $x_1,
x_2$, the wave function $\Psi(x_1, x_2)$ should satisfy

$$\hat{P}_x^2 \Psi(x_1, x_2) = \Psi(x_1, x_2),$$

where $\hat{P}_x$ is the permutation operator defined by the parallel displacement, and the point

$$x = x_1 - x_2 = 0$$

is taken as the singularity. Following Eq. (37), one gets to that $\hat{P}_x$ has constant eigenvalues
$p = \pm 1$ only, corresponding to bose and fermi cases, respectively.

In fact, the above argument (ii) can be alternatively achieved through a homotopic
analysis of $\mathbb{R}^3$. For this purpose, we still adopt the above example of two identical particles
at $x_1$ and $x_2$. Excluding the singularity as in Eq. (38) in the state of the system, $\mathbb{R}^3$ turns
out to be:

$$\mathbb{R}^3 - \{0\} \approx (0, \infty) \times \varphi^2,$$

where $\varphi^2$ is the two-dimensional projective plane for the direction $\pm x/|x|$ of $x$, it is also
topologically equivalent to a doubly connected surface of a three-dimensional sphere with
diametrically opposite points identified. Then the homotopy group of the system becomes:

$$\Pi_1(\varphi^2) = \Pi_1 [SO(3)] = Z_2.$$ (40)

This result was frequently used to interpret that there are only bose and fermi statistics in
the ordinary three-dimensional space $\mathbb{R}^3$.

It is evident that the above argument (ii) (also as those in Refs. [3] and [19]) is based
on the global topology, which does not concern the concrete structure of the system. The
singularity shown in Eq. (38) is independent of the system itself, it comes from the permutation of two particles only. However, for the two-dimensional anyon system in last section, the nontrivial topology (the scalar monopole) is not caused by permutation of two anyons, but by the Chern-Simons term, which determines the permutation rule as the result. Further, there are infinite number of scalar monopoles associate with the anyon field in its construction as Eq. (27).

The above discussion makes it clear that the usual argument of three-dimensional statistics is based on the relatively trivial topology, compared with the scalar monopole of two dimensions. In other words, the usual conclusion holds only in the case that there is no other nontrivial topological singularity involved in the system.

3.2. A simple model for exotic statistics in (3+1)-dimensional spacetime

In this section we attempt to investigate how the topological singularity for example magnetic monopole affects the statistics of a physical system. For the purpose, we construct a model by a system of non-relativistic spinless fermionic matter field \( \psi(x) \) of mass \( m \) and charge \( e \), which couples to an Abelian gauge field \( A_\mu(x) \). The gauge field \( A_\mu(x) \) interacts with a scalar field \( \chi(x) \) through the Pontrjagin term. Without loss of generality, the action is:

\[
S = \int d^4x \left[ i \psi^\dagger D_0 \psi + \frac{1}{2m} \psi^\dagger D^2 \psi + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \chi \right].
\]  

(41)

Since the dynamical term of \( \chi(x) \) is not included in action, \( \chi(x) \) is taken as an external source, its physical implication will be discussed later.

Before studying the above action, we first present two remarks on it:

(i) The term \( L_0(x) = \frac{e^2}{16\pi} \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \chi \) in the action is not a purely topological invariant since \( \chi(x) \) is a spacetime-dependent function. \( L_0(x) \) has been extensively used in quantum field theory to deal with the problems such as the CP violation in strong interaction and
\(\theta\)-vacuum\(^{22}\), etc. However, the physical meaning of the gauge field \(A_\mu(x)\) in our model is essentially different from that in Refs. [21, 22]: \(A_\mu(x)\) here is not the true electromagnetic potential, it is entirely determined by the particle current and \(\chi(x)\) (see the following Eq. (42)), therefore has no independent dynamics. In this sense, \(A_\mu(x)\) is similar to the Chern-Simons field of the anyons, whose purpose in life is to implement exotic statistics to particles\(^8\).

(ii) The above action is different from the one used by Goldhaber et al. in Ref. [23], where they showed that the Pontrjagin term (\(\chi\) is a constant in \(L_0(x)\)) does not contribute to the statistics of particles, since it is a total derivative\(^{23}\), and vanishes after integrating over the gauge field. Moreover, the gauge field in Ref. [23] has Maxwell-like dynamics, which does not appear in our model.

To deal with the above action, we follow the procedure of treating the action (16) in Section II. The Lagrangian equation of \(A_\mu\) is obtained to be:

\[
\epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho} \partial_\nu \chi = 4\pi^2 e^{-1} j^\mu, \tag{42}
\]

where the current \(j^\mu\) are given by:

\[
j^0 = \psi^\dagger \psi = \rho, \quad j^i = \frac{i}{2m} [\psi^\dagger D^i \psi - (D^i \psi^\dagger) \psi],
\]

and they satisfy the continuity equation: \(\partial_\mu j^\mu = 0\). For convenience, we further separate Eq. (42) into two equations with respect to \(\mu = 0\) and \(\mu = i = 1, 2, 3\):

\[
\partial_i C^i(x) = 4\pi^2 e^{-1} \rho(x), \tag{43}
\]

\[
-\partial_0 C^i(x) - \epsilon^{ijk} \partial_j D_k(x) = 4\pi^2 e^{-1} j^i(x), \tag{44}
\]

where

\[
C^i(x) = \epsilon^{ijk} \partial_j A_k(x) \chi(x), \quad D_i(x) = [\partial_i A_0(x) - \partial_0 A_i(x)] \chi(x). \tag{45}
\]

Equation (43) is in fact a constraint among the charge density, gauge flux and the scalar field. In the lattice formulation, equation (43) represents a duality transformation between the current in the lattice and field strength in the dual lattice. Our calculation is carried in the continuous space, it is easily copied to the lattice.
To solve the vector component of the gauge field, we choose the subsidiary condition: \( \partial_i A^i = 0 \), and let \( A_i = \epsilon_{ijk} \partial^j \Phi^k \), while \( \partial_i \Phi^i = 0 \). We then obtain from Eq. (45) that:

\[
\nabla^2 \Phi^i(x) = -\frac{C^i(x)}{\chi(x)},
\]

(46)
a Laplacian equation, which can be solved by using the Green function \( G(x - x') \): \( \nabla^2 G(x - x') = \delta(x - x') \), and gives further:

\[
A_i(x) = -\int d^3x' \epsilon_{ijk} \frac{\partial}{\partial x_j} G(x - x') \left[ \frac{C^k(x')}{\chi(x')} \right].
\]

(47)

To simplify Eq. (47), we introduce the vector potential of Dirac’s magnetic monopole \( M \) by the equation

\[
\frac{\partial}{\partial x_j} G(x - x') = \epsilon^{jkl} \frac{\partial}{\partial x_k} M_l(x - x').
\]

(48)

Notice that there are many \( M \) satisfying Eq. (48), and they are connected with each other by gauge transformations. With the help of Eq. (48), we turn Eq. (47) into

\[
A_i(x) = \int d^3x' \left[ \frac{\partial}{\partial x^i} M_j(x - x') \frac{C^j(x')}{\chi(x')} - \frac{\partial}{\partial x^l} M_i(x - x') \frac{C^l(x')}{\chi(x')} \right].
\]

(49)

We observe that \( \frac{\partial}{\partial x^l} M_i(x - x') = -\frac{\partial}{\partial x^i} M_l(x - x') \). After a integration in parts and using Eq. (45), we obtain that second term in the right of Eq. (49) vanishes. Then,

\[
A_i(x) = \partial_i \Omega(x),
\]

(50)

where

\[
\Omega(x) \equiv \int d^3x' \ M_j(x - x') \frac{C^j(x')}{\chi(x')}.
\]

(51)

Using Eq. (45), one can prove that \( \Omega(x) \) is invariant under the transformation

\[
M'_i(x) = M_i(x) - \partial_i \beta(x),
\]

(52)

which implies that the above results are principally independent of a particular \( M \).

We now solve \( A_0(x) \) by Eq. (45). A left-derivative acting on \( D(x)/\chi(x) \) gives:
\[ \nabla^2 A_0(x) = \frac{\partial}{\partial x_i} \left[ \frac{D_i(x)}{\chi(x)} \right]. \]  

(53)

Employing the Green function, we obtain \(A_0(x)\) as

\[ A_0(x) = \int d^3x' G(x - x') \frac{\partial}{\partial x_i'} \left[ \frac{D_i(x')}{\chi(x')} \right] \]  

(54)

Using Eqs. (44) and (45), and integration in parts, we get to [See Appendix B]:

\[ A_0(x) = \partial_0 \Omega(x). \]  

(55)

Notice that some techniques adopted in the above calculations such as integral in part, removing the derivative \(\partial_i\) out of Eq. (49) to obtain Eq. (50), are feasible by checking the behaviors of quantities \(M, C/\chi\), etc., around the origin \(x = 0\) and in long distance as well. Since \(M\) is singular, one has to assume a branch cut in space, which will be discussed later.

Combining Eqs. (50) and (55) into a covariant expression: \(A_\mu(x) = \partial_\mu \Omega(x)\), we arrive an conclusion that \(A_\mu(x)\) is formally solved as a pure gauge, albeit of a nonstandard form, because of the singularity (or multi-valued) in the vector potential of magnetic monopole.

It is interesting to notice that the way of the Dirac’s magnetic monopole’s entering the gauge field \(A_\mu(x)\) is exactly the same way as the scalar monopole’s entering the Chern-Simons gauge field in two-dimensional space. This similarity leads to the following analysis about the three-dimensional statistics.

Let us return to the action \(S\) as in Eq. (41). Since \(A_\mu\) is formally solved as a pure gauge, it can be removed from the action through a peculiar gauge transformation,

\[ A'_\mu(x) = A_\mu(x) - \partial_\mu \Omega(x) = 0, \]  

(56)

The corresponding transformations to the fermion field are

\[ \varphi(x) = e^{ie\Omega(x)} \psi(x), \quad \varphi^\dagger(x) = \psi^\dagger(x) e^{-ie\Omega(x)}. \]  

(57)

By Eqs. (56) and (57), \(S\) is transformed into the following free-form:

\[ S = \int d^4x [i\varphi^\dagger \partial_0 \varphi + \frac{1}{2m} \varphi^\dagger \partial_i^2 \varphi]. \]  

(58)
However, $\varphi(x)$ is far from a free field. In order to understand the physical meaning of $\varphi(x)$, we first make some analysis to the function $\Omega(x)$:

From Eq. (51), we know that $\Omega(x)$ represents the vector potential of Dirac's magnetic monopoles in the whole space along the direction $C(x')$, where the charge density of the monopoles is $C(x')/\chi(x')$. Thus, similar to the function $\Theta(x)$ in Eq. (23), $\Omega(x)$ is a nonlocal quantity. Further, equation (50) indicates that the term $\partial_i \Omega(x)$ is not a trivial quantity: The vector field $A(x)$ describes a point-like “magnetic” flux localized at $C(x)/\chi(x)$, where $C(x)$ is associated with particle density, and the “magnetic” flux is measured through the Aharonov-Bohm effect:

$$
\int_{\partial \omega} d\vec{l} \cdot A = \int_{\omega} d\vec{\omega} \cdot \frac{C}{\chi},
$$

(59)

where $\partial \omega$ is the boundary of area $\omega$. These facts suggest that $\Omega(x)$ seems to be the three-dimensional duplication of $\Theta(x)$. However, they are quite different: In two-dimensional space, when the charged particles move in a plane, the induced “magnetic” flux is always perpendicular to the plane, thus it can be treated as a scalar and directly associated with the particle density as shown in Eq. (19); In three-dimensional space, we have to adopt a “bridge” to connect the vector–“magnetic” fluxes to the scalar–particle density. The scalar field $\chi(x)$ in $S$ is really a simple (also cheap) “bridge” that works in Abelian gauge theory.

From the above discussions, we infer that $\varphi(x)$, constructed by $\Omega(x)$, is essentially a nonlocal field, which describes a system of quasi-particles composed of fermions and infinite number of magnetic monopoles in space. Therefore, when two quasi-particles exchange their positions, one of them must pass through the monopole of others. This permutation will produce an unusual phase factor in terms of magnetic monopoles, which leads to that the field $\varphi(x)$ follows a different commutation rule from that of the fermion field $\psi(x)$. Choosing a particular source function $\chi(x)$, one can compute the commutation rule of $\varphi(x)$ using the following equation which is proved in Appendix C,

$$
\frac{\partial \chi(x)}{\partial x_i} \left[ \frac{C_i(x)}{\chi(x)}, \psi(x') \right]_{t=t'} = -\frac{4\pi^2}{e} \psi(x') \delta(x - x').
$$

(60)
We now present an explicit example to illustrate that \( \varphi(x) \) generally obeys an exotic commutation rule. To avoid the singularity in the vector potential of magnetic monopole, we choose the vector potential as in Ref. [24],

\[
\begin{align*}
M_1 &= (4\pi r)^{-1} \tan\left(\frac{\theta}{2}\right) \hat{\phi}, & 0 \leq \theta < \frac{\pi}{2} + \delta_1, \\
M_2 &= -(4\pi r)^{-1} \cot\left(\frac{\theta}{2}\right) \hat{\phi}, & \frac{\pi}{2} - \delta_2 < \theta \leq \pi.
\end{align*}
\]

In the overlap region \( \frac{\pi}{2} - \delta_2 < \theta < \frac{\pi}{2} + \delta_1 \), \( M_1 \) and \( M_2 \) are connected by the transformation: \( M_2 = M_1 + \nabla \beta \), where \( \beta = -\phi/2\pi \). Notice that this \( U(1) \) transformation does not require to quantize the monopole charge, because the field \( \varphi(x) \) is invariant under this local transformation. Along with above choice of \( M(x) \), we need the \( \hat{\phi} \) component of \( C(x) \). This can be achieved by choosing

\[
\chi = g(\phi) \neq 0,
\]

where \( \partial_\phi g(\phi) \neq 0 \). Equation (60) is then reduced to be

\[
\left[ \frac{C_\phi(x)}{\chi(x)}, \psi(x') \right]_{t=t'} = -\frac{4\pi^2}{e} h(x) \psi(x') \delta(x - x'),
\]

where \( h = r \sin \theta/\partial_\phi g(\phi) \).

Unfortunately, the above particular choice of \( \chi(x) \) breaks the \textit{rotation} symmetry of the system, which can be seen from Eq. (63). Moreover, there raises a problem in the meantime that the local \( U(1) \) symmetry between \( M_1 \) and \( M_2 \) is broken when we compute the commutation role of the fields \( \varphi(r) \) in the overlap region \( \frac{\pi}{2} - \delta_2 < \theta < \frac{\pi}{2} - \delta_2 \). This \( U(1) \) symmetry breaking leads to a headache ambiguity to the commutation rule in the overlap region. To resolve this ambiguity, we set up a \textit{branch cut} for instance \( x \)-axis in three-dimensional space, then the whole space is divided into two regions \( R_1 \) and \( R_2 \) as shown in Fig. 3:

\[
R_1 = \begin{cases} 
\theta \in [0, \pi/2), \phi \in [0, 2\pi], \\
\theta = \frac{\pi}{2}, \phi \in [0, \pi),
\end{cases} \quad R_2 = \begin{cases} 
\theta \in (\pi/2, \pi], \phi \in [0, 2\pi], \\
\theta = \frac{\pi}{2}, \phi \in [\pi, 2\pi).
\end{cases}
\]

20
We require that \( M_2 \) be restricted in \( R_1 \) and \( M_2 \) in \( R_2 \), instead of the regions defined in Eq. (61).

With the above choices of \( C \) and \( M \), we can compute the permutation rule of \( \varphi(r) \) and obtain finally: (The symbol \( x \) is replaced by \( r \) in follows)

\[
\varphi(r_1, t) \varphi(r_2, t) = -\varphi(r_2, t) \varphi(r_1, t) \ e^{i\Gamma(r_1, r_2)}, \quad r_1 \neq r_2,
\]

(65)

where

\[
\Gamma(r_1, r_2) = \begin{cases} 
\pi r_{12}^{-1} \tan(\frac{\theta_{12}}{2}) \ [h(r_1) + h(r_2)], & r_{12} \in R_a, \\
-\pi r_{12}^{-1} \cot(\frac{\theta_{12}}{2}) \ [h(r_1) + h(r_2)], & r_{12} \in R_b,
\end{cases}
\]

(66)

where \( r_{12} = |r_1 - r_2|, \theta_{12} = \arg(r_{12}, n_z) \). It follows from the relation \( \theta_{12} = \pi - \theta_{21} \) that

\[
\Gamma(r_1, r_2) = -\Gamma(r_2, r_1),
\]

(67)

which is consistent with the permutation rule (65). Equation (65) explicitly shows that the new field \( \varphi(x) \) obeys an exotic statistics, which is not confined in the domain of fractional statistics, but becomes a space-dependent functional statistics determined by \( \Gamma(r_1, r_2) \). \( \varphi(x) \) is still called \textit{anyon} field.

We see that the three-dimensional rotation symmetry is inevitably sacrificed in the above example, which has similarly appeared in Ref. [17]. In this consideration, our model is only appropriate to the \textit{condensed matter} system, where the \textit{SO}(3) symmetry is not necessarily preserved in many cases. Certainly, one can choose other \( \chi(x) \) to obtain the permutation rules different from Eq. (65). This flexibility of \( \chi(x) \) exceeds our usual understanding of the exotic statistics, and adds a new variable to the system as well. For the case that

\[
\chi = \epsilon(z) = \begin{cases} 
1, & z \geq 0, \\
0, & z < 0,
\end{cases}
\]

(68)

the system [mainly Eq. (42)] is reduced into (2+1)-dimensional case discussed in Section \textbf{II}, where the Green function and topological monopole are different from those in three-dimensional space. This reduction has been pointed out by Wilczek in Ref. [25].
The wave functions of anyons here can not be imitated by the Laughlin’s expression. However, one can easily write down the state for a many-anyon system in the Fock space.

The equivalence of two actions [Eqs. (41) and (58)] leads to a conclusion: Anyons in four dimensional spacetime can be realized through a system of fermion field, Abelian gauge field and scalar field; It is infinite number of magnetic monopoles embodied in the anyon field that bring in system the nontrivial topology, and give rise to the exotic statistics of anyons.

### 3.3. Physical meaning of the scalar field $\chi$

In above Subsection 3.2 we have seen that the scalar field $\chi(x)$ plays an important role in the model of anyons. There it is interpreted as the external source function. In this subsection, we hope to understand deeply the implication of $\chi(x)$ in our model by approaching a chiral system.

For simplicity in calculation, we consider a system of relativistic massless fermion field $\psi(x)$, which minimally couples to a gauge field $A_\mu(x)$. The chirality is introduced by the interaction of axial chiral current with the chiral mode $\chi(x)$ of the system. The effective action is

$$S_{\text{eff}} = \int d^4x \left\{ \bar{\psi}' [i \gamma^\mu (\partial_\mu + ieA_\mu)] \psi' + \partial_\mu \chi' j_5^\mu \right\},$$

where $\gamma_\mu, \mu = 0, 1, 2, 3$ are gamma matrices, $j_5^\mu = \bar{\psi}' \gamma^\mu \gamma_5 \psi'$ are the axial chiral currents, and the dynamical term of the chiral mode is neglected. It is easily proved that $S_{\text{eff}}$ is invariant under the local $U(1)$ chiral transformations:

$$\begin{cases}
\psi'' = e^{i\alpha_5} \psi', \\
\bar{\psi}'' = \bar{\psi}' e^{i\alpha_5}, \\
\chi'' = \chi' - \alpha
\end{cases},$$

where $\alpha \equiv \alpha(x)$ is an infinitesimal real parameter. In the case of low energy, we can safely integrate over the fermion fields in functional integral. $S_{\text{eff}}$ then becomes:
\[ S'_{\text{eff}} = \int d^4 x \ln \{ \det [i\gamma^\mu (\partial_\mu + ieA_\mu) + \partial_\mu \chi' \gamma^\mu \gamma_5] \}. \]  

(71)

By the perturbation theory, we expand \( S_{\text{eff}} \) in powers of \( A_\mu \). To the second order, there are two non-vanishing terms:

\[ S^{(2)}_1 = \frac{e^2}{4} \int d^4 (xy) \langle j^\mu (x) j^\nu (y) \rangle A_\mu (x) A_\nu (y), \]  

(72)

\[ S^{(2)}_2 = -\frac{e^2}{4} \int d^4 (xyz) \langle \partial_\lambda j_5^\lambda (x) j_5^\mu (y) j_5^\nu (z) \rangle \chi' (x) A_\mu (y) A_\nu (z), \]  

(73)

where \( j^\mu = \bar{\psi}_s \gamma^\mu \psi_s \). The first term \( S^{(2)}_1 \) is a Maxwell-like term, which can be canceled by adding the same term but with opposite sign to the action \( S_{\text{eff}} \). Our interest is in the second term \( S^{(2)}_2 \). At one-loop level, there appear triangle graphs that break down the usual Ward-Takahashi identity of the chiral current \( j_5^\mu \), and leads to the triangle anomaly\(^{26}\). A detailed calculation gives

\[ S^{(2)}_2 = \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\sigma\rho} \int d^4 x \ F_{\mu\nu} F_{\sigma\rho} \chi'. \]  

(74)

This result can be alternatively obtained using the technique introduced by Fujikawa\(^{27}\).

Comparing the above term \( S^{(2)}_2 \) with the term \( L_0 (x) \) in the action \( S \) in Eq.(41), we identify immediately the scalar field \( \chi (x) \) in \( S \) with the chiral mode \( \chi' (x) \) in \( S_{\text{eff}} \). These results imply that the anyons in (3+1)-dimensional spacetime are originated from the chiral structure of the quantum system as Eq.(69). It is interesting to recall that the anyons in (2+1)-dimensional spacetime are associated with a chiral spin system\(^{28,29}\), where the chiral mode is simply the mass of particles. This similarity of anyons in different dimensions suggests that there presumably exists a deeper connection among chirality, topology and statistics, which is still unclear.

### IV. Conclusion

In summary, we have discussed the connection between topology and statistics in two and three-dimensional space, respectively. With the introduction of scalar monopole in
two-dimensional space, we re-interpret the usual anyons, described by the Chern-Simons
gauge field theory, as the quasi-particles constructed by fermions and the scalar monopoles
in such a way that each fermion is surrounded by infinite number of scalar monopoles in
whole space. By this quasi-particle picture, we observe that when two anyons exchange their
positions, each of them must pass through the monopole of others. This exchange will lead
to a nontrivial phase factor in term of monopole charge. It is the monopole charge that
determines the statistics of the anyons.

By re-analyzing the conventional arguments about the connection between topology and
statistics in three spatial dimensions, we find that the usual conclusion of the bose and fermi
statistics as the only possibilities there are based on the global topology, which is relatively
trivial compared with the scalar monopole of two spatial dimensions. We construct a simple
model of three-dimensional anyons by a system of non-relativistic fermion field, Abelian
gauge field, and a scalar field which is taken as the external source. The Dirac’s magnetic
monopole enters the solution of the gauge field and changes the three-dimensional topo-
logy. As a result, the anyon field is formulated by the fermion field and Dirac’s magnetic
monopoles in such a form that each fermion are surrounded by infinite number of magnetic
monopoles in the whole space. This quasi-particle picture of anyons is similar to that of
two-dimensional anyons. However, the exotic statistics is not restricted to be the fractional
statistics, but becomes a generalization from fractions to space-dependent functions. We
further show that the scalar field is interpreted as the chiral mode of the system with chiral
anomaly.

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Appendix: A

After a integral in part in Eq. (25), we have

\[ A_0(x) = \frac{e}{\lambda} \epsilon_{ij} \int d^2 y \frac{\partial}{\partial y^i} G(x - y) j^j(y). \]  

(1)

Using \( \frac{\partial}{\partial y^i} G(x - y) = -\frac{\partial}{\partial x^i} G(x - y) \), and Eq. (8), we get

\[ A_0(x) = -\frac{e}{2\pi\lambda} \int d^2 y \frac{\partial}{\partial x^j} \theta(x - y) j^j(y). \]  

(2)

Since \( \frac{\partial}{\partial y^i} \theta(x - y) = -\frac{\partial}{\partial x^i} \theta(x - y) \), a further integral in part and using continuity equation of the current, we get

\[ A_0(x) = \frac{e}{2\pi\lambda} \int d^2 \theta(x - y) \partial_0 j^0(y), \]  

(3)

which is exactly Eq. (26).

Appendix: B

After a integral in part in Eq. (54), and using Eq. (48), we get

\[ A_0(x) = -\int d^3 x' \epsilon^{ijk} M_k(x - x') \left[ \frac{D_i(x')}{\chi(x')} \right]. \]  

(1)

After a further integral in part in above equation and using Eq. (45), we get

\[ A_0(x) = -\int d^3 x' \epsilon^{ijk} M_k(x - x') \frac{\partial}{\partial x'^j} \partial_0 A_i(x'). \]  

(2)

Using again Eq. (45) the expression of \( C^i(x) \), we directly obtain Eq. (55).

Appendix: C

It follows from Eq. (45) that

\[ 0 = \frac{\partial}{\partial x^i} \left[ C^i(x) / \chi(x) \right] = \left[ \frac{\partial}{\partial x^i} C^i(x) \right] / \chi(x) - \frac{\partial}{\partial x^i} \chi(x) C^i(x) / \chi(x)^2. \]  

(1)
Using Eq. (43), we get

\[ \frac{\partial \chi(x)}{\partial x_i} \left[ \frac{C_i(x)}{\chi(x)}, \psi(x') \right]_{t=t'} = \frac{4\pi^2}{e} \left[ \rho(x), \psi(x') \right]_{t=t'}, \quad (2) \]

which is easily reduced to be Eq. (60) by adopting the anticommutation relation of fermion field:

\[ \{ \psi^\dagger(x), \psi(x') \}_{t=t'} = \delta(x - x'). \quad (3) \]
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FIGURES

Fig. 1. (a) shows that a closed loop $l$ encircles a scalar monopole at $O$, $x$-axis is the branch cut. (b) shows a closed loop that winds the scalar monopole at $O$ $n$ circles.

Fig. 2. (a) The quasi-particle picture of the anyons: a fermion (bigger dot) at $x$ is surrounded by infinite number scalar monopoles (smaller dot) in the whole space. (b) When two anyons at $x$ and $x'$ exchange their positions, each of them must pass through a monopole of others.

Fig. 3. (a) indicates the regions: $R_1$ for $z > 0$, and $R_2$ for $z < 0$; For $z = 0$, i.e., $x - y$ section, $R_1$ and $R_2$ are shown in (b).