EQUIVARIANT OPERATIONAL CHOW RINGS OF T-LINEAR SCHEMES

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Abstract. We study \( T \)-linear schemes, a class of objects that includes spherical and Schubert varieties. We provide a Künneth formula for the equivariant Chow groups of these schemes. Using such formula, we show that equivariant Kronecker duality holds for the equivariant operational Chow rings (or equivariant Chow cohomology) of \( T \)-linear schemes. As an application, we obtain:

(i) a localization theorem for the equivariant Chow cohomology of projective \( T \)-linear schemes that does not depend on resolution of singularities, and (ii) a presentation of the equivariant Chow cohomology of possibly singular complete spherical varieties admitting a smooth equivariant envelope (e.g. group embeddings).

1. Introduction and motivation

Let \( k \) be an algebraically closed field. Let \( G \) be a connected reductive linear algebraic group (over \( k \)). Let \( B \) be a Borel subgroup of \( G \) and \( T \subset B \) be a maximal torus of \( G \). An algebraic variety \( X \), equipped with an action of \( G \), is spherical if it contains a dense orbit of \( B \). (Usually spherical varieties are assumed to be normal but this condition is not needed here.) Spherical varieties have been extensively studied in the works of Akhiezer, Brion, Knop, Luna, Pauer, Vinberg, Vust and others. For an up-to-date discussion of spherical varieties, as well as a comprehensive bibliography, see [Ti] and the references therein. If \( X \) is spherical, then it has a finite number of \( B \)-orbits, and thus, also a finite number of \( G \)-orbits (see e.g. [Vin], [Kn2]). In particular, \( T \) acts on \( X \) with a finite number of fixed points. These properties make spherical varieties particularly well suited for applying the methods of Goresky-Kottwitz-MacPherson [GKM], nowadays called GKM theory, in the topological setup, and Brion’s extension of GKM theory [Br3] to the algebraic setting of equivariant Chow groups, as defined by Totaro, Edidin and Graham [EG1]. Through this method, substantial information about the topology and geometry of a spherical variety can be obtained by restricting one’s attention to the induced action of \( T \).

Examples of spherical varieties include \( G \times G \)-equivariant embeddings of \( G \) (e.g., toric varieties are spherical) and the regular symmetric varieties of De Concini-Procesi [DP1]. The equivariant cohomology and equivariant Chow groups of smooth complete spherical varieties have been studied by Bifet, De Concini and Procesi [DP2], [BCP], De Concini-Littelmann [LP], Brion [Br3] and Brion-Joshua [BJ2]. In these cases, there is a comparison result relating equivariant cohomology with equivariant Chow groups: for a smooth complete spherical variety, the equivariant

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cycle map yields an isomorphism from the equivariant Chow group to the equivariant (integral) cohomology (e.g. Proposition 2.11). As for the study of the equivariant Chow groups of possibly singular spherical varieties, some progress has been made by Brion [Bl3], Payne [P] and the author [G2,G5].

The problem of developing intersection theory on singular varieties comes from the fact that the Chow groups $A_*(\cdot)$ do not admit, in general, a natural ring structure or intersection product. But when singularities are mild, for instance when $X$ is a quotient of a smooth variety $Y$ by a finite group $F$, then $A_*(X) \otimes \mathbb{Q} \simeq (A_*(Y) \otimes \mathbb{Q})^F$, and so $A_*(X) \otimes \mathbb{Q}$ inherits the ring structure of $A_*(Y) \otimes \mathbb{Q}$. To simplify notation, if $A$ is a $\mathbb{Z}$-module, we shall write hereafter $A_\mathbb{Q}$ for the rational vector space $A \otimes \mathbb{Q}$ (the tensor product is understood to be taken over $\mathbb{Z}$).

In order to study more general singular schemes, Fulton and MacPherson [FM] introduced the notion of operational Chow groups or Chow cohomology. Similarly, Edidin and Graham defined the equivariant operational Chow groups [EG1], which we briefly recall. (For our conventions on varieties and schemes, see Section 2.1.) Let $X$ be a $T$-scheme. The $i$-th $T$-equivariant operational Chow group of $X$, denoted $\text{op}A^i_T(X)$, is defined as follows: an element $c \in \text{op}A^i_T(X)$ is a collection of homomorphisms $c_f^{(m)} : A^m_T(Y) \to A^{m-i}_T(Y)$, written $z \mapsto f^*c \cap z$, for every $T$-equivariant map $f : Y \to X$ and all integers $m$ (the underlying category is the category of $T$-schemes). Here $A^*(Y)$ denotes the equivariant Chow groups of $Y$ (Section 2.1). As in the case of ordinary operational Chow groups ([FM], Chapter 17), these homomorphisms must satisfy three conditions of compatibility: with proper pushforward (resp. flat pull-back, resp. intersection with a Cartier divisor) for $T$-equivariant maps $Y' \to Y \to X$, with $Y' \to Y$ proper (resp. flat, resp. determined by intersection with a Cartier divisor); see [FM], Chapter 17 for precise statements. The homomorphism $c_f^{(m)}$ determined by an element $c \in \text{op}A^i_T(X)$ is usually denoted simply by $c$, with an indication of where it acts. For any $X$, the ring structure on $\text{op}A^*_T(X) := \oplus_i \text{op}A^i_T(X)$ is given by composition of such homomorphisms. The ring $\text{op}A^*_T(X)$ is graded, and $\text{op}A^i_T(X)$ can be non-zero for any $i \geq 0$. The most salient functorial properties of equivariant operational Chow groups are summarized below:

(i) Cup products $\text{op}A^a_T(X) \otimes \text{op}A^b_T(X) \to \text{op}A^{a+b}_T(X)$, $a \otimes b \mapsto a \cup b$, making $\text{op}A^*_T(X)$ into a graded associative commutative ring.

(ii) Contravariant graded ring maps $f^* : \text{op}A^*_T(X) \to \text{op}A^*_T(Y)$ for arbitrary equivariant morphisms $f : Y \to X$.

(iii) Cap products $\text{op}A^i_T(X) \otimes A^j_T(X) \to A^j_{m-i}(X)$, $c \otimes z \mapsto c \cap z$, making $A^*_T(X)$ into an $\text{op}A^*_T(X)$-module and satisfying the projection formula.

(iv) If $X$ is a nonsingular $n$-dimensional variety, then the Poincaré duality map from $\text{op}A^*_T(X)$ to $A^{n-i}_T(X)$, taking $c$ to $c \cap [X]$, is an isomorphism, and the ring structure on $\text{op}A^*_T(X)$ is that determined by intersection products of cycles on the mixed spaces $X_T$ [EG1, Proposition 4].

(v) Equivariant vector bundles on $X$ have equivariant Chern classes in $\text{op}A^*_T(X)$.

(vi) Localization theorems of Borel-Atiyah-Segal type and GKM theory (with rational coefficients) for possibly singular complete $T$-varieties in characteristic zero. See [G3] or the Appendix for details.

In [FMSS], Fulton, MacPherson, Sottile and Sturmfels succeed in describing the non-equivariant operational Chow groups of complete spherical varieties. Indeed,
they show that the Kronecker duality homomorphism

\[ K: op A^*(X) \longrightarrow \text{Hom}(A_*(X), \mathbb{Z}), \quad \alpha \mapsto (\beta \mapsto \deg(\beta \cap \alpha)) \]

is an isomorphism for complete spherical varieties. Here \( \deg(\cdot) \) is the degree homomorphism \( \mathbb{A}^0(X) \to \mathbb{Z} \). Moreover, they show that \( A_*^T(X) \) is finitely generated by the classes of \( T \)-orbit closures, and with the aid of the map \( K \), they provide a combinatorial description of \( op A^*(X) \) and the structure constants of the cap and cup products \([FMSS]\). In addition, if \( X \) is nonsingular and complete, they show that the cycle map \( cl_X: A_*(X) \to H_*(X) \) is an isomorphism. Although we stated their results in the case of spherical varieties, these hold more generally for complete schemes with a finite number of orbits of a solvable group. In particular, the conclusions of \([FMSS]\) hold for Schubert varieties. The results of \([FMSS]\) are quite marvelous in that they give a presentation of a rather abstract ring, namely \( op A^*(X) \), in a very combinatorial manner.

The aim of this paper is to extend the results of \([FMSS]\) to the setting of equivariant operational Chow rings for a natural class of schemes with torus actions, namely, \( T \)-linear schemes. Briefly, a \( T \)-linear scheme is a \( T \)-scheme that can be obtained by an inductive procedure starting with a finite dimensional \( T \)-representation, in such a way that the complement of a \( T \)-linear scheme equivariantly embedded in affine space is also a \( T \)-linear scheme, and any \( T \)-scheme which can be stratified as a finite disjoint union of \( T \)-linear schemes is a \( T \)-linear scheme. See Section 2.3 for a precise definition. \( T \)-linear schemes have been studied by Jannsen \([Jann]\), Totaro \([T]\), and Joshua-Krishna \([JK1]\). Remarkably, by a result of Brion (Theorem 2.5), the class of \( T \)-linear schemes includes all spherical varieties in characteristic zero. Additionally, by a result of Renner (see the comments after Theorem 2.5), affine and projective group embeddings are \( T \)-linear in arbitrary characteristic. We expect that all spherical varieties are \( T \)-linear in positive characteristic, though we do not have a proof of this at the moment. Nevertheless, this is not a major issue for us, because the homotopy invariance of equivariant higher Chow groups allows to transfer results about \( T \)-linear schemes to spherical varieties quite easily (Theorem 3.5). The results of this article yield a presentation of the rational equivariant Chow cohomology of complete possibly singular spherical varieties admitting an equivariant smooth envelope (Theorems 3.12 and 4.8), thus increasing the applicability of Brion’s techniques \([Br3, Section 7]\) from the smooth to the singular setup.

This article is organized as follows. Section 2 reviews the results from \([Br3, FMSS] \) needed in our study. Section 3 is the conceptual core of this article. We start by defining equivariant Kronecker duality schemes. These objects are projective \( T \)-schemes \( X \) which satisfy two conditions: (i) \( A^*_T(X) \) is finitely generated over \( S = A^*_T(pt) \), and (ii) the equivariant Kronecker duality map

\[ K_T: op A^*_T(X) \longrightarrow \text{Hom}_S(A^*_T(X), A^*_T(pt)), \quad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha)) \]

is an isomorphism of \( S \)-modules. Here \( p_{X*}: A^*_T(X) \to A^*_T(pt) \) is the map induced by pushforward to a point. Also, \( S \) is isomorphic to \( A^*_T(pt) \) with the opposite grading. As an important example, we show that this class of objects includes all complete special \( T \)-schemes, i.e. projective \( T \)-linear schemes and complete spherical varieties in arbitrary characteristic (Definition 3.4 and Theorem 3.6). This is done by showing that special \( T \)-linear schemes satisfy the equivariant Künneth formula (Theorem 3.5). Furthermore, we describe the cap and cup product structures in
Corollaries 3.7 and 3.8. These formulas are equivariant analogues of the ones in [FMSS, Theorem 4]. Later in that section we state a localization theorem for equivariant Kronecker duality schemes, a theorem of independent interest that holds with integral coefficients. In general, such theorems hold in characteristic zero, and with \( \mathbb{Q} \)-coefficients (cf. Appendix). We conclude Section 3 by showing that projective group embeddings in arbitrary characteristic satisfy equivariant localization (Theorem 3.12). This extends well-known results on torus embeddings [P] to more general compactifications of connected reductive groups.

Finally, in Section 4, we apply the machinery just developed to spherical varieties in characteristic zero. If \( X \) is a complete, possibly singular, \( G \)-spherical variety, then our findings describe the image of the injective map \( i_T^* : \text{op} \mathcal{A}_T^*(X)_\mathbb{Q} \to \text{op} \mathcal{A}_T^*(X^T)_\mathbb{Q} \) by congruences involving pairs, triples or quadruples of \( T \)-fixed points (Theorem 4.8). This extends [Br3, Theorem 7.3] to the singular setting.

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## 2. Definitions and basic properties

### 2.1. Conventions and notation.
Throughout this paper, we fix an algebraically closed field \( k \) (of arbitrary characteristic, unless stated otherwise). All schemes and algebraic groups are assumed to be defined over \( k \). By a scheme we mean a separated scheme of finite type over \( k \). A variety is a reduced scheme. Observe that varieties need not be irreducible. A subvariety is a closed subscheme which is a variety. Unless explicit mention is made to the contrary, we will assume all schemes are equidimensional. A point on a scheme will always be a closed point.

We denote by \( T \) an algebraic torus. We write \( \Delta \) for the character group of \( T \), and \( S \) for the symmetric algebra over \( \mathbb{Z} \) of the abelian group \( \Delta \). We denote by \( \mathbb{Q} \) the quotient field of \( S \). A scheme \( X \) provided with an algebraic action of \( T \) is called a \( T \)-scheme. For a \( T \)-scheme \( X \), we denote by \( X^T \) the fixed point subscheme and by \( i_T : X^T \to X \) the natural inclusion. If \( H \) is a closed subgroup of \( T \), we similarly denote by \( i_H : X^H \to X \) the inclusion of the fixed point subscheme. When comparing \( X^T \) and \( X^H \) we write \( i_{T,H} : X^T \to X^H \) for the natural (\( T \)-equivariant) inclusion.

A \( T \)-scheme \( X \) is called locally linearizable (and the \( T \)-action is called locally linear) if \( X \) is covered by invariant affine open subsets upon which the action is linear. For instance, \( T \)-stable subschemes of normal \( T \)-schemes are locally linearizable. \( T \)-schemes are called \( T \)-quasiprojective if it has an ample \( T \)-linearized invertible sheaf. This assumption is satisfied, e.g. for \( T \)-stable subschemes of normal quasiprojective \( T \)-schemes. Similar notions are given for actions of more general (connected) linear algebraic groups, see [Fu] for many details. Recall that an envelope \( p : \bar{X} \to X \) is a proper map such that for any subvariety \( W \subset X \) there is a subvariety \( \bar{W} \) mapping birationally to \( W \) via \( p \) [Fu, Definition 18.3]. In the case
of \(T\)-actions, we say that \( p : \tilde{X} \to X \) is an equivariant envelope if \( p \) is \( T \)-equivariant, and if we can take \( \tilde{W} \) to be \( T \)-invariant for \( T \)-invariant \( W \). If there is an open set \( U \subseteq X \) over which \( p \) is an isomorphism, then we say that \( p : \tilde{X} \to X \) is a birational envelope. By [Su Theorem 2], if \( X \) is a \( T \)-scheme, then there exists a \( T \)-equivariant birational envelope \( p : \tilde{X} \to X \), where \( \tilde{X} \) is a \( T \)-quasiprojective scheme. Moreover, if \( \text{char}(k) = 0 \), then we may choose \( \tilde{X} \) to be nonsingular. This holds more generally for actions of linear algebraic groups, see [Su Theorem 2] and [EG2 Proposition 7.5]. If \( p : \tilde{X} \to X \) is a \( T \)-equivariant envelope and \( H \subseteq T \) is a closed subgroup, then the induced map \( \tilde{X}^H \to X^H \) is also a \( T \)-equivariant envelope [EG2 Lemma 7.2].

Let \( X \) be a \( T \)-scheme of dimension \( n \). Following Edidin and Graham [EG1], we denote by \( A^T_*(X) \) the equivariant Chow group of \( X \). For the sake of completeness, we briefly recall the definition of this (graded) group. Let \( V \) be a finite dimensional \( T \)-module, and let \( U \subseteq V \) be an invariant open subset such that a principal bundle quotient \( U \to U/T \) exists. Then \( T \) acts freely on \( U \times X \) and the quotient scheme \( X_T := (U \times X)/T \) exists. We define the \( i \)-th equivariant Chow group \( A^T_i(X) \) by \( A^T_i(X) := A_{i+\dim T}^-(U, X) \), if \( V \setminus U \) has codimension more than \( n - i \). Such pairs \((V, U)\) always exist, and the definition is independent of the choice of \((V, U)\), see [EG1]. Finally, \( A^T_*(X) := \oplus_i A^T_i(X) \). Unlike ordinary Chow groups, \( A^T_*(X) \) can be non-zero for any \( i \leq n \), including negative \( i \). If \( X \) is a \( T \)-scheme, and \( Y \subseteq X \) is a \( T \)-stable closed subscheme, then \( Y \) defines a class \([Y]\) in \( A^T_*(X) \). If \( X \) is smooth, then so is \( X_T \), and \( A^T_*(X) \) admits an intersection pairing; in this case, denote by \( A^T_*(X) \) the corresponding ring graded by codimension. The equivariant Chow ring \( A^T_*(X) \) identifies to \( S \), and \( A^T_*(X) \) is a \( S \)-module, where \( \Delta \) acts on \( A^T_*(X) \) by homogeneous maps of degree \(-1\). This module structure is induced by pullback through the flat map \( p_{X,T} : X_T \to U/T \). Restriction to a fiber of \( p_{X,T} \) gives \( i^* : A^T_*(X) \to A_*(X) \), and this map is surjective (Theorem 2.6). If \( X \) is complete, we denote by \( p_{X,T*} \) (or simply by \( p_{X*}(\alpha) \)) the proper pushforward to a point of a class \( \alpha \in A^T_*(X) \). We may also write \( \int_X(\alpha) \) or \( \deg(\alpha) \) for this pushforward. Notice that \( A^G_*(pt) \) is isomorphic to \( A^G_*(-pt) \) with the opposite grading. We usually write \( A^T_*(X) \) for the negatively graded \( S \)-module \( A^T_*(pt) \).

Let \( X \) be a quasiprojective scheme. We denote by \( A_*(X, j) \) the higher Chow groups of Bloch [Bl] (indexed by dimension). These groups have been extended to the equivariant setting by Edidin and Graham [EG1]. Let \( X \) be a \( T \)-quasiprojective scheme. Then \( X_T \), as given above, is quasiprojective (e.g. [EG1 Proposition 23]). Define equivariant higher Chow groups by setting \( A^T_*(X, m) := A_{i+\dim U - \dim T}^-(U, X_T, m) \), where \( X_T \) is formed from a pair \((V, U)\) such that \( \text{codim}(V \setminus U) > n - i \). The homotopy property of higher Chow groups shows that \( A^T_*(X, m) \) is well defined [EG1]. One reason for constructing equivariant higher Chow groups is to extend the localization short exact sequence of equivariant Chow groups: if \( X \) is a \( T \)-quasiprojective scheme, and \( Y \subseteq X \) is an invariant closed subscheme, then there is a long exact sequence of equivariant higher Chow groups

\[
\cdots \to A^T_*(Y, j) \to A^T_*(X, j) \to A^T_*(X - Y, j) \to \cdots \to A^T_*(Y) \to A^T_*(X) \to A^T_*(X - Y) \to 0.
\]
Finally, let $X$ be a $T$-scheme, and let $(V,U)$ be a pair as above. If $X$ has a $T$-equivariant smooth envelope (in particular, if $\text{char}(k) = 0$), then there is an isomorphism $\text{op}A^T_\ast(X) \simeq \text{op}A^\ast(X \times U/T)$ provided $V \setminus U$ has codimension more than $j$ [EG1 Theorem 2]. So in this case restriction to a fiber of $p_{X,T} : X_T \rightarrow U/T$ induces a canonical map $i^* : \text{op}A^T_\ast(X) \rightarrow \text{op}A^\ast(X)$. But this map, unlike its counterpart for equivariant Chow groups, is not surjective in general, and its kernel is not necessarily generated in degree one, not even for toric varieties [PK]. This becomes an issue when translating results from equivariant to non-equivariant Chow cohomology. In Corollary 3.3 we give some conditions under which $i^*$ is surjective and yields an isomorphism $\text{op}A^T_\ast(X)/\Delta\text{op}A^T_\ast(X) \simeq \text{op}A^\ast(X)_Q$. Such conditions are fulfilled e.g. by complete $\mathbb{Q}$-filtrable spherical varieties.

2.2. The Białynicki-Birula decomposition. Let $X$ be a $T$-scheme. For simplicity, assume that $X$ is locally linearizable. Recall that a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$ is called generic if $X^{G_m} = X^T$, where $\mathbb{G}_m$ acts on $X$ via $\lambda$. Generic one-parameter subgroups always exist. Now fix a generic one-parameter subgroup $\lambda$ of $T$, and let $X^T = \bigsqcup_{i=1}^m F_i$ be the decomposition of $X^T$ into connected components. For each $F_i$, we define the subset

$$X_+(F_i, \lambda) = \{ x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \text{ exists and is in } F_i \}. $$

We denote by $\pi_i : X_+(F_i, \lambda) \rightarrow F_i$, the map $x \mapsto \lim_{t \to 0} \lambda(t) \cdot x$. Then $X_+(F_i, \lambda)$ is a locally closed $T$-invariant subscheme of $X$, and $\pi_i$ is a $T$-equivariant morphism. The (disjoint) union of the locally closed $T$-invariant subspaces $X_+(F_i, \lambda)$, where $\lambda$ is a fixed generic one-parameter subgroup, might not cover all of $X$, but when it does (for instance, when $X$ is complete), the decomposition $\{X_+(F_i, \lambda)\}_{i=1}^m$ is called the Białynicki-Birula decomposition, or BB-decomposition, of $X$ associated to $\lambda$. Each $X_+(F_i, \lambda)$ is referred to as a stratum of the decomposition. If, moreover, all fixed points of the given $T$-action on $X$ are isolated (i.e. $X^T$ is finite), the corresponding $X_+(F_i, \lambda)$ are simply called cells of the decomposition.

Usually the BB-decomposition of a complete $T$-scheme is not a Whitney stratification; that is, it may happen that the closure of a stratum is not the union of strata, even when the scheme is assumed to be smooth. For a justification of this claim, see [B2 Example 1].

**Definition 2.1.** Let $X$ be a $T$-scheme endowed with a BB-decomposition $\{X_+(F_i, \lambda)\}$, for some generic one-parameter subgroup $\lambda$ of $T$. The decomposition is said to be filtrable if there exists a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \ldots \subset \Sigma_m$ of $T$-invariant closed subspaces of $X$ such that:

a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,
b) $\Sigma_j \setminus \Sigma_{j-1}$ is a stratum of the decomposition $\{X_+(F_i, \lambda)\}$, for each $j = 1, \ldots, m$.

In this context, it is common to say that $X$ is filtrable. If, moreover, $X^T$ is finite and the cells $X_+(F_i, \lambda)$ are isomorphic to affine spaces $\mathbb{A}^n$, then $X$ is called $T$-cellular. The following result is due to Białynicki-Birula ([B1], [B2]).

**Theorem 2.2.** Let $X$ be a complete $T$-scheme, and let $\lambda$ be a generic one-parameter subgroup. If $X$ admits an ample $T$-linearized invertible sheaf, then the associated BB-decomposition $\{X_+(F_i, \lambda)\}$ is filtrable. Furthermore, if $X$ is smooth, then $X^T$ is also smooth, and for any component $F_i$ of $X^T$, the map $\pi_i : X_+(F_i, \lambda) \rightarrow F_i$ makes $X_+(F_i, \lambda)$ into a $T$-equivariant vector bundle over $F_i$.
Consequently, smooth projective $T$-schemes with finitely many fixed points are $T$-cellular. On the other hand, in the category of $T$-quasiprojective schemes, BB-decompositions of projective schemes are always filtrable.

2.3. $T$-linear schemes. We introduce here the main objects of our study and outline some of their relevant features.

Definition 2.3. Let $T$ be an algebraic torus and let $X$ be a $T$-scheme.

1. We say that $X$ is $T$-equivariantly 0-linear if it is either empty or isomorphic to $\text{Spec}(\text{Sym}(V^*))$, where $V$ is a finite-dimensional rational representation of $T$.

2. For a positive integer $n$, we say that $X$ is $T$-equivariantly $n$-linear if there exists a family of $T$-schemes $\{U, Y, Z\}$, such that $Z \subseteq Y$ is a $T$-invariant closed immersion with $U$ its complement, $Z$ and one of the schemes $U$ or $Y$ are $T$-equivariantly $(n-1)$-linear and $X$ is the other member of the family $\{U, Y, Z\}$.

3. We say that $X$ is $T$-equivariantly linear (or simply, $T$-linear) if it is $T$-equivariantly $n$-linear for some $n \geq 0$.

4. $T$-linear varieties are varieties that are $T$-linear schemes.

It is immediate from the above definition that if $T \to T'$ is a morphism of algebraic tori, then every $T'$-equivariantly linear scheme is also $T$-equivariantly linear. Notice that $T$-linear schemes are also linear schemes in the sense of Jann and [J]. From the inductive definition of $T$-linear schemes it follows that the fixed point subscheme $X^H$ of any subtorus $H \subset T$ is $T$-equivariantly $n$-linear, provided $X$ is $T$-equivariantly $n$-linear. The result below is recorded in [JK1].

Proposition 2.4. Let $T$ be an algebraic torus and let $T'$ be a quotient of $T$. Let $T$ act on $T'$ via the quotient map. Then the following hold:

1. $T'$ is $T$-linear.
2. A $T$-cellular scheme is $T$-linear.
3. Every $T$-scheme with finitely many $T$-orbits is $T$-linear. In particular, a toric variety with dense torus $T$ is $T$-linear. □

The previous assertions, together with the Bruhat decomposition, imply that flag varieties, partial flag varieties and Schubert varieties are all $T$-linear, as they come with a paving by affine spaces (in fact, they are $T$-cellular).

Notably, by a result of Brion (personal communication), $T$-linear schemes in characteristic zero include a very rich class of geometric objects: spherical varieties. Next we present Brion’s proof of this important fact.

Theorem 2.5. Assume $\text{char}(k) = 0$. Let $B$ be a connected solvable linear algebraic group with maximal torus $T$. Let $X$ be a $B$-scheme. If $B$ acts on $X$ with finitely many orbits, then $X$ is $T$-linear.

Proof. Brion’s proof is as follows. Since $X$ is a disjoint union of $B$-orbits and these are $T$-stable, it suffices to show that every $B$-orbit is $T$-linear. Write this orbit as $B/H$ where $H$ is a closed subgroup of $B$. Let $U$ be the unipotent radical of $B$. Then, we have a natural map $f : B/H \to B/UH$ and the right-hand side is a torus (for it is a homogeneous space under the torus $T = B/U$). Moreover, $f$ is a $B$-equivariant fibration with fiber $UH/H = U/(U \cap H)$, which is an affine space (as it is homogeneous under $U$).
We will show that the fibration \( f \) is \( T \)-equivariantly trivial by factoring it into \( T \)-equivariantly trivial fibrations with fiber the affine line. For this, we argue by induction on the dimension of \( UH/H \). If this dimension is zero, then there is nothing to prove. If it is positive, then \( U \) acts non-trivially on \( B/H \); replacing \( U \) and \( B \) with suitable quotients, we may assume that \( U \) acts faithfully. Because \( U \) is a unipotent group normalized by \( T \), we may find a one-dimensional unipotent subgroup \( V \) of the center of \( U \), which is normalized by \( T \) (see Lemma 6.3.2). Then \( V \) acts freely on \( B/H \) by left multiplication, and the quotient map is the natural map \( B/H \to B/VH \) which is a principal \( V \)-bundle. Since the variety \( B/VH \) is affine (e.g. by the induction assumption) and \( V \) is the additive group \( \mathbb{G}_a \), this bundle is trivial. Hence \( B/H \) is an affine variety and we have an isomorphism of coordinate rings \( k[B/H] = k[B/VH][t] \) where \( t \) is an indeterminate. We claim that \( t \) may be chosen an eigenvector of \( T \). To prove the claim, note that the additive group \( V \) acts on \( B/H \) and on \( k[B/H] \) by automorphisms. So the Lie algebra of \( V \) yields a derivation \( D \) of the algebra \( k[B/H] \), and the subalgebra of zeros of \( D \) is just \( k[B/VH] \) (given that we are in characteristic 0, this subalgebra consists of the \( V \)-invariants in \( k[B/H] \)). As \( V \) is normalized by \( T \), we see that \( D \) is an eigenvector of \( T \), with weight, say, \( \alpha \). Because the principal \( V \)-bundle \( B/H \to B/VH \) is trivial, the isomorphism \( k[B/H] = k[B/VH][t] \) can be chosen so that \( V \) acts on the right-hand side via its action by translations on \( t \), that is, \( v \cdot t = v + t \) (indeed, this is equivalent to having a \( V \)-equivariant isomorphism \( B/H \cong B/VH \times V \)). By differentiating, we get \( D(t) = 1 \). Now let \( t = \sum \lambda \lambda \lambda \) be the decomposition of \( t \) into \( T \)-eigenvectors. Then \( 1 = D(t) = \sum \lambda \lambda \lambda \lambda \lambda \) is the decomposition of 1 into \( T \)-eigenvectors. Looking at weights, we get \( D(t-\lambda) = 1 \) and \( D(t_\lambda) = 0 \) for all \( \lambda \neq -\alpha \). This means that \( t_\lambda \in k[B/VH] \) for all \( \lambda \neq -\alpha \). Thus, \( k[B/H] = k[B/VH][u] \), where \( u = t_{-\alpha} \) satisfies \( D(u) = 1 \); hence \( v \cdot u = v + u \) for all \( v \in V \). So \( B/H = B/VH \times V \), equivariantly for the actions of \( T \) and \( V \); here \( T \) acts linearly on \( V \) with weight \( \alpha \). Now that the claim has been verified, we conclude by induction. \( \square \)

We expect that Theorem 2.5 also holds for spherical varieties in positive characteristic, though we do not have a proof of this at the moment. Nevertheless, for the subclass of group embeddings, we proceed to check this condition directly, using some results of Renner. Let \( G \) be a connected reductive linear algebraic group with Borel subgroup \( B \) and maximal torus \( T \subset B \). Recall that a normal irreducible variety \( X \) is called an embedding of \( G \), or a group embedding, if \( X \) is a \( G \times G \)-variety containing an open orbit isomorphic to \( G \). Due to the Bruhat decomposition, group embeddings are spherical \( G \times G \)-varieties. Briefly, group embeddings are classified as follows (see [1] for details).

(i) Affine case: Let \( M \) be a reductive monoid with unit group \( G \) (implicit in the definition is that \( M \) is affine, irreducible and normal as a variety). Then \( G \times G \)-acts naturally on \( M \) via \((g, h) \cdot x = gxh^{-1}\). The orbit of the identity is \( G \times G/\Delta(G) \simeq G \). Thus \( M \) is an affine embedding of \( G \). By a result of Rittatore, reductive monoids are exactly the affine embeddings of reductive groups.

(ii) Projective case: Let \( M \) be a reductive monoid with zero and unit group \( G \). Then there exists a central one-parameter subgroup \( \epsilon : \mathbb{G}_m^* \to T \), with image \( Z \), such that \( \lim_{t \to 0} \epsilon(t) = 0 \). Moreover, the quotient space

\[
\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z
\]
is a normal projective variety on which \( G \times G \) acts via \((g, h) \cdot [x] = [g x h^{-1}]\). Hence, \( \mathbb{P}_x(M) \) is a normal projective embedding of the quotient group \( G/Z \). Notably, projective embeddings of connected reductive groups are exactly the projectivizations of reductive monoids.

Now let \( M \) be a reductive monoid with unit group \( G \). Then the \( B \times B \)-orbits of \( M \) are indexed by the Renner monoid \( R \), a finite semigroup whose group of units is the Weyl group \( W \), and contains \( E(T) = \{ e \in T | e^2 = e \} \) as idempotent set [R1] (here \( T \subset M \) is the associated affine torus embedding). Given \( r \in R \), let \( BrB \subset M \) be the corresponding \( B \times B \)-orbit. By [R1] Lemma 13.1, \( BrB \cong r T \times \mathbb{A}^\ell \), for some \( \ell \geq 0 \). It follows from the proof of [R1] Lemma 13.1 that this splitting is \( T \times T \)-equivariant. In other words, the orbit \( BrB \) is \( T \times T \)-linear. Thus \( M \) is \( T \times T \)-linear as well. Similarly, one checks that \( \mathbb{P}_x(M) \) is \( T \times T \)-linear. In conclusion, affine and projective group embeddings are \( T \times T \)-linear in arbitrary characteristic.

2.4. Description of equivariant Chow groups. Next we state Brion’s presentation of the equivariant Chow groups of schemes with a torus action in terms of invariant cycles [Br3, Theorem 2.1]. It also shows how to recover usual Chow groups from equivariant ones.

**Theorem 2.6.** Let \( X \) be a \( T \)-scheme. Then the \( S \)-module \( A^T_*(X) \) is defined by generators \( [Y] \) where \( Y \) is an invariant irreducible subvariety of \( X \) and relations \([\text{div}_Y(f)] - \chi[Y]\) where \( f \) is a rational function on \( Y \) which is an eigenvector of \( T \) of weight \( \chi \). Moreover, the map \( A^T_*(X) \to A_*(X) \) vanishes on \( \Delta A^T_*(X) \), and it induces an isomorphism \( A^T_*(X)/\Delta A^T_*(X) \to A_*(X) \). \( \square \)

Now let \( \Gamma \) be a connected solvable linear algebraic group with maximal torus \( T \). If \( X \) is a \( \Gamma \)-scheme, then the generators of \( A^T_*(X) \) in Theorem 2.6 can be taken to be \( \Gamma \)-invariant ([Br3, Proposition 6.1]). In particular, if \( X \) has finitely many \( \Gamma \)-orbits, then the \( S \)-module \( A^T_*(X) \) is finitely generated by the classes of the \( \Gamma \)-orbit closures. This holds e.g. when \( X \) is a spherical variety.

**Lemma 2.7.** Let \( X \) be a \( T \)-linear scheme. Then \( A^T_*(X) \) is a finitely generated \( S \)-module. Moreover, \( A_*(X) \) is a finitely generated abelian group.

**Proof.** This is a consequence of the inductive definition of \( T \)-linear schemes and the fact that for \( 0 \)-linear schemes, i.e., the \( T \)-equivariant linear representations \( \mathbb{A}^n \) of \( T \), one has \( A^*_T(\mathbb{A}^n) \cong S \) (by homotopy invariance). Now we argue by induction. Assume the result for \( T \)-equivariantly \((n - 1)\)-linear schemes. Let \( X \) be a \( n \)-linear scheme. By definition, two localization sequences can occur. In the first case,

\[
A^T_*(Z) \to A^T_*(X) \to A^T_*(U) \to 0,
\]

where \( Z \) and \( U \) are \((n - 1)\)-linear. By the inductive hypothesis, the terms on both ends are finitely generated, hence so is \( A^T_*(X) \). In the second case,

\[
A^T_*(Z) \to A^T_*(Y) \to A^T_*(X) \to 0,
\]

where \( Z \) and \( Y \) are \((n - 1)\)-linear. Clearly, in this case, it follows that \( A^T_*(X) \) is also finitely generated. The second assertion of the lemma is proved similarly. \( \square \)

**Corollary 2.8.** Let \( X \) be a \( T \)-linear scheme. If \( X \) is complete, then rational equivalence and algebraic equivalence coincide on \( X \).
Proof. Observe that the kernel of the natural morphism $A_*(X) \to B_*(X)$ is divisible (Fu, Example 19.1.2), and hence trivial, for $A_*(X)$ is finitely generated by Lemma 2.7.

The next lemma will become relevant later, when integrality of the equivariant operational Chow rings is discussed (cf. Lemma 3.3). It is essentially due to Brion, del Baño and Karpenko.

**Lemma 2.9.** Let $X$ be nonsingular projective $T$-variety. Then the following are equivalent.

(i) $A_*(X^T)$ is $\mathbb{Z}$-free.

(ii) $A^T_*(X)$ is $S$-free.

(iii) $A_*(X)$ is $\mathbb{Z}$-free.

If moreover $X$ is $T$-linear, then any (and hence all) of these conditions hold.

**Proof.** The implication (i)$\Rightarrow$(ii) follows from [Br3, Corollary 3.2.1], as any smooth projective variety is filtrable (Theorem 2.2). That (ii) implies (iii) is a consequence of Theorem 2.6. To show that (iii) implies (i) we use a result of del Baño [dB, Theorem 2.4] and Karpenko [Kar, Section 6]. Namely, let $\lambda$ be a generic one-parameter subgroup of $T$, and let $X^T = \sqcup F_i$ be the decomposition of $X^T$ into connected components. Then, for every non-negative integer $j \leq \dim(X)$, there is a natural isomorphism

$$
\oplus_i A^{j-d_i}(F_i) \cong A^j(X),
$$

where $d_i$ is the codimension of $W_i(\lambda)$ in $X$ (all spaces involved are smooth, so there is an intersection product on the Chow groups graded by codimension). These isomorphisms yield the assertion (iii)$\Rightarrow$(i).

Finally, if $X$ is a nonsingular projective $T$-linear variety, then, in particular, it is a projective linear variety, and so it satisfies the Künneth formula. Now Theorem 2.10(ii) implies that condition (iii) of the lemma holds for $X$. This concludes the argument.

Over the rationals, the corresponding conditions of Lemma 2.9 are always satisfied, i.e. if $X$ is a nonsingular projective $T$-variety, then the $S_\mathbb{Q}$-module $A^T_*(X)_\mathbb{Q}$ is free [Br3, Corollary 3.2.1].

For any schemes $X$ and $Y$, one has a Künneth map

$$A_*(X) \otimes A_*(Y) \to A_*(X \times Y),$$

taking $[V] \otimes [W]$ to $[V \times W]$, where $V$ and $W$ are subvarieties of $X$ and $Y$. This is an isomorphism only for very special schemes, but when it is, strong consequences can be derived from it, as we shall illustrate below. Let us start with the following remarkable result due to Ellingsrud and Stromme [ES, Theorem 2.1].

**Theorem 2.10.** Let $X$ be a nonsingular complete variety. Assume that the rational equivalence class $\delta$ of the diagonal $\Delta(X) \subseteq X \times X$ is in the image of the Künneth map $A_*(X) \otimes A_*(X) \to A_*(X \times X)$. Let

$$\delta = \sum u_i \otimes v_i$$

be a corresponding decomposition of $\delta$, where $u_i, v_i \in A_*(X)$. Then

(i) The $v_i$ generate $A_*(X)$, i.e. any $z \in A_*(X)$ has the form $\sum (u_i \cdot z)v_i$. 

(ii) Numerical and rational equivalence coincide on $X$. In particular, $A_*(X)$ is a free $\mathbb{Z}$-module.

(iii) If $k = \mathbb{C}$, then the cycle map $cl_X : A_*(X) \to H_*(X, \mathbb{Z})$ is an isomorphism. In particular, the homology and cohomology groups of $X$ vanish in odd degrees.

Consider now a nonsingular complex algebraic variety $X$ with an action of a complex algebraic torus $T$. Together with a cycle map $cl_X : A_*(X) \to H_*(X, \mathbb{Z})$ (which doubles degrees, see [Fu, Corollary 19.2]), there is also an equivariant cycle map $cl_T^X : A_*(T \times X) \to H_*^T(X, \mathbb{Z})$ where $H_*^T(X, \mathbb{Z})$ denotes equivariant cohomology with integral coefficients, see [EG1, Section 2.8]. If $X$ is also complete and the class of the diagonal is in the image of the Künneth map, then $cl_X$ is an isomorphism (Theorem 2.10 (iii)). We prove here an analogous result for $cl_T^X$.

**Proposition 2.11.** Let $X$ be a nonsingular complete complex algebraic variety with a $T$-action. If the class of the diagonal in $X \times X$ is in the image of the Künneth map $A_*(X) \otimes A_*(X) \to A_*(X \times X)$, then the equivariant cycle map

$$cl_T^X : A_*(T \times X) \to H_*^T(X, \mathbb{Z})$$

is an isomorphism. In particular, $cl_T^X$ is an isomorphism in the following cases:

(i) $X$ is a nonsingular complete complex variety on which a connected solvable linear algebraic group $B$ acts with finitely many orbits, and $T$ is a maximal torus of $B$.

(ii) $X$ is a nonsingular projective complex $T$-linear variety.

**Proof.** In view of Theorem 2.10 (iii), the given hypothesis on $X$ imply that $cl_X$ is an isomorphism, and that $X$ has no integral cohomology in odd degrees. Then the spectral sequence associated to the fibration $X \times_T ET \to BT$ collapses, where $ET \to BT$ is the universal $T$-bundle. So the $S$-module $H_*^{T}(X, \mathbb{Z})$ is free, and the map

$$H_*^{T}(X, \mathbb{Z})/\Delta H_*^{T}(X, \mathbb{Z}) \to H_*^{*}(X, \mathbb{Z})$$

is an isomorphism. These results, together with the graded Nakayama Lemma, yield surjectivity of the equivariant cycle map $cl_T^X : A_*(T \times X) \to H_*^{T}(X, \mathbb{Z})$. To show injectivity, we proceed as follows. First, choose a basis $z_1, \ldots, z_n$ of $H^*(X, \mathbb{Z})$. Now identify that basis with a basis of $A_*(X)$ (via $cl_X$) and lift it to a generating system of the $S$-module $A_*(T \times X)$. Then this generating system is a basis, since its image under the equivariant cycle map is a basis of $H_*^{T}(X, \mathbb{Z})$.

Finally, the second assertion of the proposition follows easily from the first one, for in cases (i) and (ii) the class of the diagonal is known to be in the image of the Künneth map (Section 2.3).

Later, in Theorem 3.5, we show that special $T$-schemes (Definition 3.4) satisfy the equivariant Künneth formula. Our techniques are a blend of those from [T] and [FMSS]. We also obtain an equivariant version of the Kronecker duality isomorphism (Theorem 3.6).

### 2.5. Equivariant Localization

Let $T$ be an algebraic torus. The following is the localization theorem for equivariant Chow groups [Br3, Corollary 2.3.2].
**Theorem 2.12.** Let $X$ be a $T$-scheme and let $i_T : X^T \to X$ be the inclusion of the fixed point subscheme. If $X$ is locally linearizable, then the $S$-linear map $i_{T*} : A_k^T(X^T) \to A_k^T(X)$ becomes an isomorphism after inverting all non-zero elements of $\Delta$.

For later use, we record here a slightly more general version of the previous localization theorem.

**Proposition 2.13.** Let $X$ be a $T$-scheme, let $H \subset T$ be a closed subgroup, and let $i_H : X^H \to X$ be the inclusion of the fixed point subscheme. Then the induced morphism of equivariant Chow groups

$$i_{H*} : A_k^T(X^H) \to A_k^T(X)$$

becomes an isomorphism after inverting finitely many characters of $T$ that restrict non-trivially to $H$.

Before proving this proposition, we need a technical lemma. We would like to thank M. Brion for suggesting us a simplified proof of this fact.

**Lemma 2.14.** Let $X$ be an affine $T$-scheme. Let $H$ be a closed subgroup of $T$. Then the ideal of the fixed point subscheme $X^H$ is generated by all regular functions on $X$ which are eigenvectors of $T$ with a weight that restricts non-trivially to $H$.

**Proof.** Recall that $X^H$ is the largest closed subscheme of $X$ on which $H$ acts trivially. In other words, the ideal $I$ of $X^H$ is the smallest $H$-stable ideal of $k[X]$ such that $H$ acts trivially on the quotient $k[X]/I$. So $I$ is $T$-stable and hence the direct sum of its $T$-eigenspaces. Moreover, if $f \in k[X]$ is a $T$-eigenvector of weight $\chi$ which restricts non-trivially to $H$, then $f \in I$. Indeed, let $\overline{f}$ be the image of $f$ in $k[X]/I$. Notice that $\overline{f}$ is a $T$-eigenvector of the same weight $\chi$ as $f$. Since $H$ acts trivially on $k[X]/I$, we obtain the identity $\overline{f} = h \cdot \overline{f} = \chi(h) \overline{f}$, valid for all $h \in H$. Nevertheless, there exists $h_0 \in H$ such that $\chi(h_0) \neq 1$, by our assumption on $\chi$. Substituting this information into the above identity yields $\overline{f} = 0$, equivalently, $f \in I$. Thus, $I$ contains the ideal $J$ generated by all such functions $f$. But $k[X]/J$ is a trivial $H$-module by construction, and hence $I = J$ by minimality. □

**Proof of Proposition 2.13.** In virtue of Lemma 2.14, our proof is an adaptation, almost word for word, of Brion’s proof of [Br3, Corollary 2.3.2]. So we provide only a sketch of the crucial points. First, assume that $X$ is locally linearizable. From Theorem 2.12 we know that $A_k^T(X)$ is generated by the classes of $T$-invariant subvarieties $Y \subseteq X$. Moreover, by assumption, $X$ is a finite union of $T$-stable affine open subsets $X_i$. Now Lemma 2.14 implies that the ideal of each fixed point scheme $X_i^H$ is generated by all regular functions on $X_i$ which are eigenvectors of $T$ with a weight that restricts non-trivially to $H$. We can choose a finite set of such generators $(f_{ij})$, with respective weights $\chi_{ij}$. Now let $Y \subseteq X$ be a $T$-invariant subvariety of positive dimension. If $Y$ is not fixed pointwise by $H$, then one of the $f_{ij}$ defines a non-zero rational function on $Y$. Then, in the Chow group, we have $\chi_{ij}[Y] = [\text{div}_Y f_{ij}]$. So after inverting $\chi_{ij}$, we get $[Y] = \chi_{ij}^{-1}[\text{div}_Y f_{ij}]$. Arguing by induction on the dimension of $Y$, we obtain that $i_*$ becomes surjective after inverting the $\chi_{ij}$’s. A similar argument, using these $\chi_{ij}$’s in Brion’s proof of [Br3, Corollary 2.3.2], shows that $i^*$ is injective after localization.

Finally, if $X$ is not locally linearizable, let $\pi : \tilde{X} \to X$ be the disjoint union of the normalizations of the irreducible components of $X$. Then $\pi$ is an equivariant
birational envelope. Let $U \subset X$ be the open subset where $\pi$ is an isomorphism. Set $Z = X \setminus U$ and $E = \pi^{-1}(Z)$. Then, by [FMSS Lemma 2] and [EG2 Lemma 7.2], there is a commutative diagram

$$
\begin{array}{c}
A^*_T(E) \longrightarrow A^*_T(Z) \oplus A^*_T(\tilde{X}) \longrightarrow A^*_T(X) \longrightarrow 0 \\
\downarrow_{i_{\tilde{X}}^*} \quad \downarrow_{i^*_H} \quad \downarrow_{i_H^*} \\
A^*_T(E^H) \longrightarrow A^*_T(Z^H) \oplus A^*_T(\tilde{X}^H) \longrightarrow A^*_T(X^H) \longrightarrow 0.
\end{array}
$$

Observe that $E$ and $Z$ have strictly smaller dimension than $X$. Moreover, $E$ and $\tilde{X}$ are locally linearizable. Applying Noetherian induction and the previous part of the proof, we get that the first two left vertical maps become isomorphisms after localization; hence so does the third one. □

When $X$ is projective and smooth, the localization theorem yields a GKM description of the image of the natural map $i^*_T : A^*_T(X) \rightarrow A^*_T(X^H)$ [Br3 Theorem 3.4]. Here $A^*_T(X)$ denotes the rational equivariant Chow group graded by codimension (equivalently, the rational equivariant operational Chow group of $X$ [EG1 Proposition 4]). In characteristic zero this description extends to the rational equivariant Chow cohomology of possibly singular varieties [EG3 Section 7]. See the Appendix for a review of the main results in this regard.

3. Equivariant Kronecker duality and Küneth formulas

3.1. Equivariant Kronecker duality schemes.

**Definition 3.1.** Let $X$ be a complete $T$-scheme. We say that $X$ satisfies $T$-equivariant Kronecker duality if the following conditions hold:

(i) $\text{op}A^*_T(X)$ is a finitely generated $S$-module.

(ii) The equivariant Kronecker duality map

$$
\kappa_T : \text{op}A^*_T(X) \longrightarrow \text{Hom}_S(A^*_T(X), A^*_T(pt)) \\
\alpha \mapsto (\beta \mapsto p_{X^*}(\beta \cap \alpha))
$$

is an isomorphism of $S$-modules.

Likewise, we say that $X$ satisfies rational $T$-equivariant Kronecker duality if the $S_Q$-modules $A^*_T(X)_Q$ and $\text{op}A^*_T(X)_Q$ satisfy the conditions (i) and (ii) with rational coefficients.

**Remark 3.2.** Notice that the equivariant Kronecker duality map is functorial for morphisms between complete $T$-schemes. Indeed, let $f : \tilde{X} \rightarrow X$ be an equivariant (proper) morphism of complete $T$-schemes. It is important to notice that

$$
\int_{\tilde{X}} f^*(\xi) \cap z = \int_X f_*(f^*(\xi) \cap z) = \int_X (\xi \cap f_*(z)),
$$

where $\xi \in A^*_T(X)$, due to the projection formula [FM]. This formula implies the commutativity of the diagram

$$
\begin{array}{c}
\text{Hom}_S(A^*_T(X), A^*_T) \xrightarrow{(f_*)^!} \text{Hom}_S(A^*_T(\tilde{X}), A^*_T) \\
\kappa_T \downarrow \quad \kappa_T \downarrow \\
\text{op}A^*_T(X) \xrightarrow{f^*} \text{op}A^*_T(\tilde{X})
\end{array}
$$
where \((f_\ast)^t\) is the transpose of \(f_\ast : A^T_\ast(\hat{X}) \to A^T_\ast(X)\).

It follows from the definition that if \(X\) satisfies \(T\)-equivariant Kronecker duality, then the \(S\)-module \(\mathrm{op}A^T_\ast(X)\) is finitely generated and torsion free. In particular, if \(T\) is one dimensional, i.e. \(T = \mathbb{G}_m\), then \(A^T_{\mathbb{G}_m}(X)\) is a finitely generated free module over \(A^T_{\mathbb{G}_m} = \mathbb{Z}[t]\). Moreover, if \(X\) is projective and smooth, then \(A_* (X_{\mathbb{G}_m})\) is a finitely generated free abelian group (Lemma 2.9).

As it stems from the previous paragraph, not all smooth varieties with a torus action satisfy Equivariant Kronecker duality. For a more concrete example, consider the trivial action of \(T\) on a projective smooth curve. In this case, one checks that \(K^T\) is an isomorphism. These facts, together with the commutativity of the diagram below

\[
\begin{array}{ccc}
A^T_\ast(X) & \xrightarrow{\kappa_T} & \mathrm{Hom}_S(A^T_\ast(X), A^T_\ast) \\
\downarrow & & \downarrow \\
A^\ast(X) & \xrightarrow{\kappa} & \mathrm{Hom}(A_\ast(X), \mathbb{Z}),
\end{array}
\]

yield the content of the lemma. \(\square\)

**Lemma 3.3.** Let \(X\) be a smooth projective \(T\)-variety. Then \(X\) satisfies rational \(T\)-equivariant Kronecker duality if and only if it satisfies the rational non-equivariant Kronecker duality, i.e. \(\kappa : \mathrm{op}A^i(X) \to \mathrm{Hom}(A_i(X), \mathbb{Q})\) is an isomorphism for all \(i\). If moreover \(A_* (X^{\mathbb{G}_m})\) is \(\mathbb{Z}\)-free, then the equivalence holds over the integers.

**Proof.** Both assertions are proved similarly, so we focus on the second one. Since \(X\) is smooth and projective, the assumption on \(A_* (X^T)\) implies that \(A^T_\ast(X)\) is a free \(S\)-module (Lemma 2.9). Now, by [EG1] Proposition 4, \(\mathrm{op}A^T_\ast(X)\) is isomorphic to the equivariant Chow group \(A^T_\ast(X)\) graded by codimension, and so it is also a free \(S\)-module. By the graded Nakayama lemma, \(\kappa_T\) is an isomorphism if and only if

\[
\overline{\kappa_T} : A^T_\ast(X)/\Delta A^T_\ast(X) \to \mathrm{Hom}_S(A^T_\ast(X), A^T_\ast)/\Delta \mathrm{Hom}_S(A^T_\ast(X), A^T_\ast)
\]

is an isomorphism. But freeness of \(A^T_\ast(X)\) yields an isomorphism

\[
\mathrm{Hom}_S(A^T_\ast(X), A^T_\ast)/\Delta \mathrm{Hom}_S(A^T_\ast(X), A^T_\ast) \simeq \mathrm{Hom}(A^T_\ast(X)/\Delta A^T_\ast(X), \mathbb{Z})
\]

and the later identifies to \(\mathrm{Hom}(A_* (X), \mathbb{Z})\) by Theorem 2.6. On the other hand, by Theorem 2.6 again, the map

\[
A^T_\ast(X)/\Delta A^T_\ast(X) \to A^\ast(X)
\]

is an isomorphism. These facts, together with the commutativity of the diagram above

\[
\begin{array}{ccc}
A^T_\ast(X) & \xrightarrow{\kappa_T} & \mathrm{Hom}_S(A^T_\ast(X), A^T_\ast) \\
\downarrow & & \downarrow \\
A^\ast(X) & \xrightarrow{\kappa} & \mathrm{Hom}(A_\ast(X), \mathbb{Z}),
\end{array}
\]

yield the content of the lemma. \(\square\)

**Definition 3.4.** Let \(X\) be a \(T\)-scheme. We say that \(X\) is **special** if one of the following hold:

(i) \(X\) is \(T\)-linear and \(T\)-quasiprojective.

(ii) the \(T\)-action comes induced from the action of a connected reductive group \(G\) with Borel subgroup \(B\) and maximal torus \(T\), and \(X\) is \(G\)-scheme with finitely many \(B\)-orbits (cf. Theorem 2.5). In particular, each irreducible component of \(X\) is \(G\)-spherical.
In view of Theorem \[2.6\] and Lemma \[2.7\] if \(X\) is a special \(T\)-scheme, then the \(S\)-module \(A^T_{\ast}(X)\) is finitely generated.

Our goal is to show that complete special \(T\)-schemes satisfy equivariant Kronecker duality. For this, the main ingredient is the following result, due to Joshua and Krishna [JK1] for equivariant \(K\)-theory, and to Totaro [T] and Jannsen [Jann] for the non-equivariant Chow groups. It states that special \(T\)-schemes satisfy the equivariant Künneth formula. Recall that a \(T\)-scheme \(X\) is said to satisfy the equivariant Künneth formula if the Künneth map (or exterior product, see [EG1])

\[
A^T_{\ast}(X) \otimes_A A^T_{\ast}(Y) \to A^T_{\ast}(X \times Y)
\]

is an isomorphism for any \(T\)-scheme \(Y\). Although a proof of this fact can be obtained using equivariant motivic cohomology (and will appear in [JK2]), we give here a somewhat simpler proof.

**Theorem 3.5.** Let \(X\) be a \(T\)-scheme. If \(X\) is special, then \(X\) satisfies the equivariant Künneth formula.

**Proof.** As a first reduction, we claim that \(X\) satisfies the equivariant Künneth formula if (and only if) the exterior product

\[
A^T_{\ast}(X) \otimes_A A^T_{\ast}(\hat{Y}) \to A^T_{\ast}(X \times \hat{Y})
\]

is an isomorphism for any \(T\)-quasiprojective scheme \(\hat{Y}\). Indeed, let \(Y\) be a \(T\)-scheme. Then we can find a \(T\)-equivariant envelope \(\pi : \hat{Y} \to Y\), where \(\hat{Y}\) is \(T\)-quasiprojective (Section 2.1). So we have the exact sequence [FMSS, Lemma 2]

\[
A^T_{\ast}(Y') \to A^T_{\ast}(\hat{Y}) \to A^T_{\ast}(Y) \to 0,
\]

where \(Y' = \hat{Y} \times_Y \hat{Y}\). Notice that \(Y'\) is \(T\)-quasiprojective too. By naturality of the exterior product, and the fact that \(\pi \times \text{id}_X : \hat{Y} \times X \to Y \times X\) is also an envelope, we have the commutative diagram

\[
\begin{array}{ccc}
A^T_{\ast}(Y') & \to & A^T_{\ast}(\hat{Y}) \\
\otimes_A A^T_{\ast}(X) & \to & A^T_{\ast}(X) \\
\downarrow & & \downarrow \\
A^T_{\ast}(Y' \times X) & \to & A^T_{\ast}(Y \times X)
\end{array}
\]

where the first two vertical maps on the left are isomorphisms by hypothesis. Hence the right vertical map is also an isomorphism, proving the claim.

Keeping the claim in mind, we proceed to prove the theorem.

**Case I: \(X\) is \(T\)-quasiprojective and \(T\)-linear.** We follow closely Totaro’s proof of the corresponding statement in the non-equivariant case ([T, Proposition 1]). Accordingly, we will show by induction on \(n\) that for a \(T\)-equivariantly \(n\)-linear scheme \(X\) the Künneth map \(A^T_{\ast}(X) \otimes_A A^T_{\ast}(Y) \to A^T_{\ast}(X \times Y)\) is an isomorphism, and the map

\[
\oplus A^T_{\ast}(X, Y, 1) \to A^T_{\ast}(X \times Y, 1)
\]

is surjective, where \(Y\) is any \(T\)-quasiprojective scheme. Here, the groups \(A^T_{\ast}(-, \ast)\) appearing on the right hand side are the equivariant higher Chow groups, whereas the groups \(A^T_{ij}(X, Y, \ast)\) appearing on the left are the homology groups of the complex \(Z^j(X_T, \ast) \otimes Z^j(Y_T, \ast)\) (see, e.g., [T]). As pointed out by Totaro [T, p.
6], the multiplicativity of higher Chow groups \[E3\], yields a map \(A^T_{i+j}(X, Y, *) \to A^T_{i+j}(X \times Y, *)\).

Let us start the induction. For the base case \(n = 0\), we have \(X \simeq \mathbb{A}^N\), where \(\mathbb{A}^N\) is some finite-dimensional rational representation of \(T\). In this case, the claim is an immediate consequence of the homotopy invariance of the equivariant higher Chow groups \[E1,GT\]. Next, assuming the statement holds for \(n\), we need to show it also holds for \(n + 1\). Because \(X\) is \(T\)-equivariantly \((n+1)\)-linear, two subcases can occur:

(a) There is a \(T\)-scheme \(W\) which contains \(X\) as a \(T\)-invariant open subscheme such that \(W\) and \(Z = W \setminus X\) are \(T\)-equivariantly \(n\)-linear. In that case, arguing as in \[H\], we have, for any \(T\)-quasiprojective scheme \(Y\), an exact sequence of equivariant higher Chow groups:

\[
\begin{align*}
A^T_{*}((Z, Y, 1) & \to A^T_{*}(W, Y, 1) \to A^T_{*}(X, Y, 1) \to A^T_{*}(Z) \oplus A^T_{*}(Y) \to A^T_{*}(W) \oplus A^T_{*}(Y) \to A^T_{*}(X) \oplus A^T_{*}(Y) \to 0 \\
& \to A^T_{*}(Z \times Y, 1) \to A^T_{*}(W \times Y, 1) \to A^T_{*}(X \times Y, 1) \to A^T_{*}(Z \times Y) \to A^T_{*}(W \times Y) \to A^T_{*}(X \times Y) \to 0
\end{align*}
\]

By the inductive hypothesis, the first two vertical arrows are surjective and the fourth and fifth vertical maps are isomorphisms. Now diagram chasing shows that the third map is surjective and the sixth one is an isomorphism.

(b) There exists a \(T\)-invariant closed subscheme \(Z\) of \(X\) with complement \(U\) so that \(Z\) and \(U\) are \(T\)-equivariantly \(n\)-linear. Thus, for any \(T\)-quasiprojective scheme \(Y\), we are in the presence of the following exact sequence:

\[
\begin{align*}
A^T_{*}((Z, Y, 1) & \to A^T_{*}(X, Y, 1) \to A^T_{*}(U, Y, 1) \to A^T_{*}(Z) \oplus A^T_{*}(Y) \to A^T_{*}(X) \oplus A^T_{*}(Y) \to A^T_{*}(U) \oplus A^T_{*}(Y) \to 0 \\
& \to A^T_{*}(Z \times Y, 1) \to A^T_{*}(X \times Y, 1) \to A^T_{*}(U \times Y, 1) \to A^T_{*}(Z \times Y) \to A^T_{*}(X \times Y) \to A^T_{*}(U \times Y) \to 0
\end{align*}
\]

This time, the first and third vertical maps are surjective and the fourth and sixth vertical maps are isomorphisms. Once again, diagram chasing shows that the second vertical map is surjective and the fifth one is an isomorphism. This proves the inductive step and concludes the argument.

Case II: \(X\) satisfies condition (ii) of Definition \[3.4\]. To begin with, assume that \(X\) is \(T\)-quasiprojective. Let \(O \subset X\) be a \(B\)-orbit. Write this orbit as \(B/H\), where \(H \subset B\) is a closed subgroup. Let \(U\) be the unipotent radical of \(B\). Then, as in the proof of Theorem \[2.5\], we have a \(B\)-equivariant fibration \(f : B/H \to B/UH\), with base space the torus \(B/UH\) and fiber the affine space \(UH/H\). By homotopy invariance of the equivariant higher Chow groups, we get the natural isomorphism

\[
A^T_{*}(B/H \times Y, *) \simeq A^T_{*}(B/UH \times Y, *)
\]

for any \(T\)-quasiprojective scheme \(Y\). This identification and Case I applied to \(B/UH\) imply that (i) the Künneth map \(A^T_{*}(B/H) \otimes_S A^T_{*}(Y) \to A^T_{*}(B/H \times Y)\) is an isomorphism, and (ii) the map

\[
\oplus A^T_{*}(B/H, Y, 1) \to A^T_{*}(B/H \times Y, 1)
\]

is surjective. Next recall that given a nonnegative integer \(j\), the union of all \(B\)-orbits whose codimensions are greater than \(j\) is a closed subscheme of \(X\). It is denoted by
Note that \( X^j \setminus X^{j+1} \) is a disjoint union of all \( B \)-orbits of codimension \( j \). Let \( n \) be the dimension of \( X \), then we have the sequence of closed subschemes of \( X \):

\[
\emptyset = X^{n+1} \subset X^n \subset X^{n-1} \subset \cdots \subset X^0 = X.
\]

Arguing by induction on \( n \), as in (b) above, we easily get that \( X \) satisfies the equivariant K"unneth formula.

Finally, for general \( X \), choose a \( G \)-equivariant birational envelope \( p : \tilde{X} \to X \), where \( \tilde{X} \) is \( G \)-quasiprojective [Su, Theorem 2]. We may arrange so that \( \tilde{X} \) is also a \( G \)-scheme with finitely many \( B \)-orbits. Suppose that \( Z \) is a \( G \)-invariant closed subscheme of \( X \) such that \( \pi \) is an isomorphism outside \( Z \). Let \( E = \pi^{-1}(Z) \). Observe that we can adjust the construction so that \( Z \) and \( E \) have strictly smaller dimension than \( X \). Therefore, by [FMSS, Lemma 2], we have the exact sequence

\[
A^T_X(E) \to A^T_X(Z) \oplus A^T_X(Y) \to A^T_X(Y) \to 0.
\]

Now let \( Y \) be a \( T \)-scheme. Naturality of the exterior product yields the commutative diagram

\[
\begin{array}{ccc}
A^T_X(E) \otimes_S A^T_Y & \to & (A^T_X(Z) \oplus A^T_X(\tilde{X})) \otimes_S A^T_Y \\
\downarrow & & \downarrow \\
A^T_X(E \times Y) & \to & A^T_Z(\tilde{X} \times Y)
\end{array}
\]

By Noetherian induction (and the fact that \( \tilde{X} \) is \( T \)-quasiprojective), the first two left vertical maps are isomorphisms, hence so is the third one.

With the aid of the previous result, we now establish Equivariant Kronecker duality for complete special \( T \)-schemes. For the analogous result in the non-equivariant case, see [FMSS, Theorem 3].

**Theorem 3.6.** Let \( X \) be a complete \( T \)-scheme. If \( X \) is special, then the equivariant Kronecker map

\[
K_T : \text{op}A_T^\tau(X) \to \text{Hom}_S(A_T^\tau(X), A_T^\tau(pt))
\]

is an isomorphism.

**Proof.** Our arguments are quite close to those of Fulton, MacPherson, Sottile, and Sturmfels ([FMSS, Theorem 3]) in the non-equivariant case. We construct a formal inverse to \( K_T \), i.e., given a \( S \)-module homomorphism \( \varphi : A_T^\tau(X) \to A_T^\tau \), we need to construct an element \( c_\varphi \in \text{op}A_T^\tau(X) \). Because \( A_T^\tau(X) \) is finitely generated over \( S \), we can further assume, without loss of generality, that \( \varphi \) is homogeneous. Indeed, any \( \varphi \in \text{Hom}_S(A_T^\tau(X), A_T^\tau) \) decomposes as \( \varphi = \sum \varphi_\tau \), where \( \varphi_\tau \) is a homogeneous homomorphism of degree \( \tau \) (see e.g., [Bour] Part II, Section 11.6]). Bearing this in mind, given a homomorphism \( \varphi : A_T^\tau(X) \to A_T^{\tau'} \) of degree \(-\lambda \), we build \( c_\varphi \) in \( \text{op}A_T^\lambda(X) \) as follows: for every \( T \)-map \( f : Y \to X \), the corresponding homomorphism \( f^*c_\varphi := c_\varphi(f)^\lambda : A_T^\lambda(Y) \to A_T^{-\lambda}(Y) \) is defined to be the composite

\[
A_T^\tau(Y) \xrightarrow{(\gamma_f)_*} A_T^\tau(X \times Y) \xrightarrow{=} A_T^\tau(X) \otimes_S A_T^\tau(Y) \xrightarrow{\varphi \otimes \text{id}} A_T^\tau \otimes_S A_T^\tau(Y) \cong A_T^\tau(Y),
\]

where \((\gamma_f)_*\) denotes the proper pushforward along the graph of \( f \). Notice that the second displayed map is an isomorphism in view of the equivariant K"unneth formula (Theorem 3.5).
Now we must verify that the above composite map actually sends an element of \( A_{m}^{T}(Y) \) to an element of \( A_{m-\lambda}^{T}(Y) \). If \( z \in A_{m}^{T}(Y) \), then \((\gamma f)_{*}(z) = \sum_{i} u_{i} \otimes v_{i}\), where \( u_{i} \in A_{p(i)}^{T}(X) \) and \( v_{i} \in A_{m-p(i)}^{T}(Y) \). Consequently,

\[
f^{*}c_{\varphi} \cap z = (\varphi \otimes \text{id}) \circ (\gamma f)_{*}(z) = \sum_{i} \varphi(u_{i}) \otimes v_{i} = \sum_{i} 1 \otimes \varphi(u_{i})v_{i},
\]

because \( \varphi(u_{i}) \) lies in \( A_{p(i)-\lambda}^{T} \), the \((p(i)-\lambda)\)-th homogeneous component of \( A_{i}^{T} \). Therefore, \( f^{*}c_{\varphi} \cap z \) sits inside \( \mathbb{Z} \otimes (A_{p(i)-\lambda}^{T}A_{m-p(i)}^{T}(Y)) \subseteq \mathbb{Z} \otimes A_{m-\lambda}^{T}(Y) \) and the latter is isomorphic to \( A_{m-\lambda}^{T}(Y) \). Observe that the above sums are finite, for the degrees in \( A_{i}^{T}(X) \) (resp. \( A_{i}^{T}(Y) \)) are at most the dimension of \( X \) (resp. dimension of \( Y \)).

One easily checks (as in [FMSS Theorem 3]) that these maps \( c_{\varphi}(f)^{\lambda} \) (for different \( Y \)'s) satisfy the compatibility conditions of [FMSS Chapter 17], to give an element of \( \text{op}A_{\lambda}^{T}(X) \).

Next we verify that the composite \( \text{op}A_{\lambda}^{T}(X) \to \text{Hom}_{\mathbb{S}}(A_{i}^{T}(X),A_{i}^{T}) \to \text{op}A_{\lambda}^{T}(X) \) is the identity. Suffices to check this on homogeneous elements. Given \( c \in \text{op}A_{\lambda}^{T}(X) \), \( f : Y \to X \) and \( z \in A_{m}^{T}(Y) \), let \( \gamma f : Y \to X \times Y \) be the graph of \( f \), and let \( \pi_{1} \) and \( \pi_{2} \) be the two projections from \( X \times Y \) to \( X \) and \( Y \). Since \( \pi_{1} \circ \gamma f = f \) and \( \pi_{2} \circ \gamma f = \text{id}_{Y} \), the fact that operational classes commute with proper pushforward implies that

\[
f^{*}c \cap z = (\pi_{2})_{*}(\gamma f)_{*}(\pi_{1}^{*}c \cap z) = (\pi_{2})_{*}(\pi_{1}^{*}c \cap (\gamma f)_{*}(z)).
\]

Write \((\gamma f)_{*}(z) = \sum u_{i} \otimes v_{i} \) with \( u_{i} \in A_{p(i)}^{T}(X) \) and \( v_{i} \in A_{m-p(i)}^{T}(Y) \). Since \( c \) commutes with flat pullback, then \( \pi_{1}^{*}c \cap (u_{i} \otimes v_{i}) = (c \cap u_{i}) \otimes v_{i} \). The projection \( (\pi_{2})_{*} \) maps such a class to zero unless \( p(i) \leq \lambda \), and in this case,

\[
(\pi_{2})_{*}((c \cap u_{i}) \otimes v_{i}) = \text{deg}(c \cap u_{i})v_{i}.
\]

Therefore \( f^{*}c \cap z \) is the sum of the terms \( \text{deg}(c \cap u_{i})v_{i} \) for which \( p(i) \leq \lambda \). This shows that \( c \) can be recovered from the functional \( \text{deg}(c \cap \cdot) \) on \( A_{\lambda}^{T}(X) \) by applying the above sequence of maps, which is the required assertion. A similar computation shows that the other composite

\[
\text{Hom}_{\mathbb{S}}(A_{i}^{T}(X),A_{i}^{T}) \to \text{op}A_{\lambda}^{T}(X) \to \text{Hom}_{\mathbb{S}}(A_{i}^{T}(X),A_{i}^{T})
\]

is indeed the identity and finishes the proof.

For \( c \in \text{op}A_{\lambda}^{T}(X), z \in A_{\lambda}^{T}(X) \), we write \( c(z) \) for \( \text{deg}(c \cap z) \). The next result was obtained along the proof of the previous proposition.

**Corollary 3.7.** Let \( f : Y \to X, c \in \text{op}A_{\lambda}^{T}(X), z \in A_{\lambda}^{T}(Y) \). Suppose \((\gamma f)_{*}(z) = \sum u_{i} \otimes v_{i} \) with \( u_{i} \in A_{p(i)}^{T}(X) \) and \( v_{i} \in A_{m-p(i)}^{T}(Y) \). Then

\[
f^{*}c \cap z = \sum_{p(i) \leq \lambda} c(u_{i})v_{i}.
\]

**Corollary 3.8.** Let \( c \in \text{op}A_{\lambda}^{T}(X), c' \in \text{op}A_{\mu}^{T}(X), \) and \( z \in A_{\mu}^{T}(X) \), where \( m \leq \lambda + \mu \). Write \( \delta_{*}(z) = \sum u_{i} \otimes v_{i} \) with \( u_{i} \in A_{p(i)}^{T}(X) \) and \( v_{i} \in A_{m-p(i)}^{T}(X) \). Then

\[
(c \cup c')(z) = \sum_{m-\mu \leq p(i) \leq \lambda} c(u_{i})c'(v_{i}).
\]
Proof. As in [FMSS] proof of Theorem 4, we apply the previous corollary with the identify on $X$ and $\gamma_f = \delta$ the diagonal embedding. Observing that the cup product in $\text{op}A^*_T(X)$ is defined by composition of the corresponding operators and that $\delta_*(z) = \sum v_i \otimes u_i$ (by permuting the two factors) yields

$$(c \cup c') \cap z = c \cap (c' \cap z) = c \cap \left( \sum_{m - \mu \leq \rho(i)} c'(v_i)u_i \right) = \sum_{m - \mu \leq \rho(i)} c'(v_i)(c \cap u_i).$$

The sought-after formula is obtained by taking the degrees of both sides of this equation. \hfill $\square$

As stated in Section 2.1, if $X$ has an equivariant resolution of singularities, we have a map of restriction to the fiber $i^* : \text{op}A^*_T(X) \to \text{op}A^*(X)$. When $X$ is a complete special $T$-scheme and $A^*_T(X)$ is $S$-free, we describe below the image and kernel of $i^*$. This yields the compatibility of our product formulas with those of [FMSS] 3.9. Assume $\text{char}([k]) = 0$. Let $X$ be a complete $T$-scheme. If $X$ is special and $A^*_T(X)$ is $S$-free, then the map $\text{op}A^*_T(X)/\Delta \text{op}A^*_T(X) \to \text{op}A^*(X)$ is an isomorphism, where $\Delta$ is the character group of $T$.

Proof. Theorem 3.6 together with freeness of $A^*_T(X)$ yield

$$\text{op}A^*_T(X)/\Delta \text{op}A^*_T(X) \cong \text{Hom}_S(A^*_T(X), A^*_T)/\Delta \text{Hom}_S(A^*_T(X), A^*_T)$$

$$\cong \text{Hom}_Z(A^*_T(X)/\Delta A^*_T(X), Z).$$

Furthermore, by Theorem 2.6, the term on the right hand side corresponds to $\text{Hom}(A_*(X), Z)$, which, in turn, is isomorphic to $\text{op}A^*(X)$, due to the non-equivariant version of Kronecker duality ([FMSS Theorem 3] and [T Proposition 1]). Considering this information alongside the commutative diagram

$$\begin{array}{ccc}
\text{op}A^*_T(X) & \xrightarrow{\kappa_T} & \text{Hom}_S(A^*_T(X), A^*_T) \\
\downarrow & & \downarrow \\
\text{op}A^*(X) & \xrightarrow{\kappa} & \text{Hom}(A_*(X), Z),
\end{array}$$

produces the content of the corollary. \hfill $\square$

The conditions of Corollary 3.9 are satisfied by possibly singular $T$-cellular varieties (e.g. Schubert varieties). With rational coefficients, the corresponding version is satisfied by $\mathbb{Q}$-filtrable spherical varieties [G4]. This class includes all rationally smooth projective equivariant embeddings of reductive groups (op. cit.).

We conclude this section by exploring some of the appealing features of $T$-equivariant Kronecker duality schemes, a class of spaces that, in view of our previous results, includes all complete special $T$-schemes. From the viewpoint of algebraic torus actions, the main attribute of equivariant Kronecker duality schemes is that they supply a, somewhat more intrinsic, background for establishing localization theorems on integral equivariant Chow cohomology.

**Theorem 3.10.** Let $X$ be a complete $T$-scheme satisfying $T$-equivariant Kronecker duality. Let $H \subset T$ be a subtorus of $T$ and let $i_H : X^H \to X$ be the inclusion of the
fixed point subscheme. If \(X^H\) also satisfies \(T\)-equivariant Kronecker duality, then the morphism
\[
i_H^* : \text{op}\ A^*_T(X) \to \text{op}\ A^*_T(X^H)
\]
becomes an isomorphism after inverting finitely many characters of \(T\) that restrict non-trivially to \(H\). In particular, \(i_H^*\) is injective over \(\mathbb{Z}\).

**Proof.** By Proposition 2.14 the localized map \((i_H)_*: A^*_T(X^H)_F \to A^*_T(X)_F\) is an isomorphism, where \(F\) is a finite family of characters of \(T\) that restrict non-trivially to \(H\).

Now consider the commutative diagram
\[
\begin{array}{c}
\text{op}\ A^*_T(X) \\
\downarrow \\
\text{Hom}_S(A^*_T(X), A^*_T) \\
\downarrow \\
\text{Hom}_S(A^*_T(X^H), A^*_T),
\end{array}
\]

where \((i_H)_*^t\) represents the transpose of \(i_H^* : A^*_T(X^H) \to A^*_T(X)\) (commutativity follows from Remark 3.2 because \(i_H^*\) is proper). By our assumptions on \(X\) and \(X^H\), both vertical maps are isomorphisms. Moreover, after localization at \(F\), the above commutative diagram becomes
\[
\begin{array}{c}
\text{op}\ A^*_T(X)_F \\
\downarrow \\
\text{Hom}_S(A^*_T(X), A^*_T)_F \\
\downarrow \\
\text{Hom}_S(A^*_T(X^H), A^*_T)_F.
\end{array}
\]

Since \(A^*_T(X)\) is a finitely generated \(S\)-module (as \(X\) satisfies equivariant Kronecker duality), localization commutes with formation of \(\text{Hom}\) (see [53, Prop. 2.10, p. 69]), and so
\[
\text{op}\ A^*_T(X)_F \simeq (\text{Hom}_S(A^*_T(X), A^*_T))_F \simeq \text{Hom}_{S_F}(A^*_T(X)_F, A^*_T)_F.
\]

Similarly, for \(X^H\) we obtain
\[
\text{op}\ A^*_T(X^H)_F \simeq (\text{Hom}_S(A^*_T(X^H), A^*_T))_F \simeq \text{Hom}_{S_F}(A^*_T(X^H)_F, A^*_T)_F.
\]

In other words, the bottom map from the preceding diagram fits in the commutative square
\[
\begin{array}{c}
(\text{Hom}_S(A^*_T(X), A^*_T))_F \\
\downarrow \\
\text{Hom}_{S_F}(A^*_T(X)_F, A^*_T)_F \\
\downarrow \\
(\text{Hom}_S(A^*_T(X^H), A^*_T))_F
\end{array}
\]

where the vertical maps are natural isomorphisms. But we already know that \((i_H)_*^t\) is an isomorphism, hence so are \((i_H)_*^t\) and \((i_H)_*^t\).

Finally, to prove the last assertion of the theorem, recall that the \(S\)-module \(\text{op}\ A^*_T(X)\) is finitely generated and torsion free (Definition 3.1). Hence the natural map \(\text{op}\ A^*_T(X) \to \text{op}\ A^*_T(X) \otimes_S Q\) is injective, where \(Q\) is the quotient field of \(S\). In particular, the (also natural) map \(\text{op}\ A^*_T(X) \to \text{op}\ A^*_T(X)_F\) is injective. This, together with the first part of the theorem, yields injectivity of \(i_H^*\). We are done. \(\square\)
Corollary 3.11. Let $X$ be a complete $T$-scheme. Let $H$ be a codimension-one subtorus of $T$, and let $i_H : X^H \to X$ be the natural inclusion. If $X$ is special, then the pullback $i^*_H : \text{op} A^*_T(X) \to \text{op} A^*_T(X^H)$ is injective over $\mathbb{Z}$.

Proof. If $X$ is special, then so is $X^H$. Now use Theorems 3.6 and 3.10.

Let $X$ be a complete special $T$-linear scheme. It follows from Corollary 3.11 that the image of the injective map $i^*_T : \text{op} A^*_T(X) \to \text{op} A^*_T(X^T)$ is contained in the intersection of the images of the (also injective) maps $i^*_{T,H} : \text{op} A^*_T(X^H) \to \text{op} A^*_T(X^T)$, where $H$ runs over all subtori of codimension one in $T$. When the image of $i^*_T$ is exactly the intersection of the images of the maps $i^*_{T,H}$ we say, following [G3], that $X$ has the Chang-Skjelbred property (or CS property). If the defining condition holds over $\mathbb{Q}$ rather than $\mathbb{Z}$, we say that $X$ has the rational CS property. By Theorem A.6, any complete $T$-scheme in characteristic zero has the rational CS property.

It would be interesting to determine which complete special $T$-schemes satisfy the CS property over $\mathbb{Z}$ and in arbitrary characteristic. For instance, toric varieties are known to have this property [P]. We anticipate that this also holds for projective embeddings of semisimple groups of adjoint type (this shall be pursued elsewhere).

Theorem 3.12. Let $X$ be a complete special $T$-scheme. If either one of the following conditions holds

(a) $\text{char}(k) = 0$.

(b) $\text{char}(k) \geq 0$ and there is a $T$-equivariant envelope $\pi : \tilde{X} \to X$, where $\tilde{X}$ is special, projective and smooth.

Then $X$ has the rational CS property. In particular, projective embeddings of connected reductive linear algebraic groups have the rational CS property in arbitrary characteristic.

Proof. If (a) holds, then the result is a consequence of Theorem A.6 (cf. [G3, Section 7]). Now assume that (b) holds. Let $u \in \text{op} A^*_T(X^T)_\mathbb{Q}$ be such that $u \in \bigcap_{H \subset T} \text{Im}(i^*_{T,H}) : \text{op} A^*_T(X^H)_\mathbb{Q} \to \text{op} A^*_T(X^T)_\mathbb{Q}$, where the intersection runs over all codimension-one subtori $H$ of $T$. Our task is to show that $u \in \text{Im}(i^*_T)_\mathbb{Q}$. First, observe that there is a commutative diagram

![Diagram](image)

obtained by combining and comparing the sequences that [EG2, Lemma 7.2] and Theorem A.1 assign to the envelopes $\pi : \tilde{X} \to X$, $\pi_H : \tilde{X}^H \to X^H$ and $p_T : \tilde{X}^T \to X^T$. 


From the diagram it follows that $\pi_T^*(u)$ is in the image of $i_T^{*H}$. Hence, $\pi_T^*(u)$ is in the intersection of the images of all $i_T^{*H}$, where $H$ runs over all codimension-one subtori of $T$. Since $X$ is known to have the rational CS property [Br3, Theorem 3.3], $\pi_T^*(u)$ is in the image of $i_T^*$. So let $y \in A_T^*(X)$ be such that $i_T^*(y) = \pi_T^*(u)$. To conclude the proof, we need to check that $y$ is in the image of $\pi^*$. In view of equivariant Kronecker duality (Theorem 3.6), this is equivalent to checking that the dual of $y$, namely, $\hat{\varphi} := K_T(y)$ is in the image of $\pi^*_*$. The transpose of the surjective morphism $\pi_* : A_T^*(X)_Q \to A_T^*(X)_Q$. Also, we should observe that the functor $K_T(\cdot)$ transforms the previous commutative diagram into another one involving the corresponding dual modules $\text{Hom}(A_T^*(-)_Q, S_Q)$. Now set $\varphi_u := K_T(u)$. By construction, for every codimension-one subtorus $H$, there exists $\varphi_u^H : A_T^*(XH)_Q \to S_Q$ such that $\varphi_u = \varphi_u^H \circ i_{T,H}$. In fact, we can place this information into a commutative diagram:

\[
\begin{array}{ccc}
A_T^*(XT)_Q & \overset{i_{T,H}^*}{\longrightarrow} & A_T^*(XH)_Q \\
\text{In this form, our task reduces to showing that there exists } \varphi \text{ making the dotted arrow into a solid arrow. Bearing this in mind, we claim that } \hat{\varphi} \text{ is zero on the kernel of } \pi_* \text{. Indeed, let } v \in A_T^*(X)_Q \text{ be such that } \pi_*(v) = 0. \text{ By the localization theorem there exists a product of non-trivial characters } \chi_1 \cdots \chi_m \text{ such that } \chi_1 \cdots \chi_m \cdot v \text{ is in the image of } i_T^* : A_T^*(XT)_Q \to A_T^*(X)_Q. \text{ As both of these } S\text{-modules are free, then, unless } v \text{ is zero, we have } \chi_1 \cdots \chi_m \cdot v \neq 0. \text{ Let } w \text{ be such that } i_T^*(w) = \chi_1 \cdots \chi_m \cdot v. \text{ By commutativity of the diagram, } i_T^*(\pi_*^T(w)) = 0. \text{ But } i_T^* \text{ is injective by Theorem 3.10} \text{ so } \pi_*^T(w) = 0. \text{ Thus}
\end{array}
\]

\[
\hat{\varphi}(\chi_1 \cdots \chi_m \cdot v) = \chi_1 \cdots \chi_m \cdot \hat{\varphi}(v) = \varphi_u(\pi_T^*(w)) = 0.
\]

As $S_Q$ has no torsion, we get $\hat{\varphi}(v) = 0$, which proves the claim. Using this, one easily defines $\varphi$ with the sought-after properties.

Finally, for the last assertion of the theorem, recall that group embeddings are a special class of spherical varieties known to have resolutions of singularities in arbitrary characteristic [BK, Chapter 6].

**Remark 3.13.** Theorem 3.12 and its proof can be readily translated into the language of equivariant operational $K$-theory with $\mathbb{Z}$-coefficients. See [G3 Section 6] for a presentation of the (integral) equivariant operational $K$-theory of projective group embeddings.

4. **APPLICATION: RATIONAL EQUIVARIANT CHOW COHOMOLOGY OF SPHERICAL VARIETIES**

Throughout this section we work in characteristic zero. The aim is to describe the rational equivariant Chow cohomology of spherical varieties by comparing it with that of an equivariant resolution, and using the rational CS property (Theorem 3.12).
alongside equivariant Kronecker duality (Theorem 4.6). The main result (Theorem 4.8), inspired by [Br3, Theorem 7.3], is an extension of Brion’s description to the setting of equivariant operational Chow groups.

In what follows, we denote by $G$ a connected reductive linear algebraic group with Borel subgroup $B$ and maximal torus $T \subset B$. We denote by $W$ the Weyl group of $(G, T)$. Observe that $W$ is generated by reflections $\{s_\alpha\}_{\alpha \in \Phi}$, where $\Phi$ stands for the set of roots of $(G, T)$. Recall that $S^W_\mathbb{Q} = (A_T^\mathbb{Q})^W = A_T^\mathbb{Q} \otimes \mathbb{Q}$. For the purposes of this section, we shall assume that $G$-spherical varieties are locally linearizable.

4.1. Preliminaries. Recall that any spherical $G$-variety contains only finitely many $G$-orbits; as a consequence, it contains only finitely many fixed points of $T$. Moreover, since char($k$) = 0, any spherical $G$-variety $X$ admits an equivariant resolution of singularities, i.e., there exists a smooth $G$-variety $\tilde{X}$ together with a proper birational $G$-equivariant morphism $\pi : \tilde{X} \to X$. Then the $G$-variety $\tilde{X}$ is also spherical; if moreover $X$ is complete, we may arrange so that $\tilde{X}$ is projective. Notice that, in general, a resolution of singularities need not be an equivariant envelope. The next result gives an important class of spherical varieties for which equivariant resolutions are equivariant envelopes. We thank M. Brion for leading us to the following proof.

**Proposition 4.1.** Let $X$ be a normal simply-connected spherical $G$-variety (i.e. the $B$-isotropy group of its dense orbit is connected). Let $f : \tilde{X} \to X$ be a proper birational morphism. Then $f$ is an equivariant envelope.

**Proof.** Let $p : \tilde{X} \to X$ be a toroidal resolution of $X$. It suffices to show that every $B$-orbit in $X$ is the isomorphic image via $p$ of a $B$-orbit in $\tilde{X}$. So let $O = (B) \cdot x = B/B_x$ be an orbit in $X$. It follows from [BJ1] that $O$ has a connected isotropy group. The preimage $p^{-1}(O) \subset \tilde{X}$ is of the form $B \times B_x F$, where $F$ denotes the fiber $p^{-1}(x)$. Since $F$ is connected and complete (by Zariski’s main theorem), it contains a fixed point $y$ of the connected solvable group $B_x$. Then the orbit $B \cdot y$ in $\tilde{X}$ is mapped isomorphically to $B \cdot x$. \qed

**Remark 4.2.** Examples of simply-connected spherical varieties include (normal) $G \times G$-equivariant embeddings of $G$.

Now we record a few notions and results from [Br3, Section 7] needed in our task. A subtorus $H \subset T$ is called regular if its centralizer $C_G(H)$ is equal to $T$; otherwise $H$ is called singular. A subtorus of codimension one is singular if and only if it is the kernel of some positive root $\alpha$. In this case, $\alpha$ is unique and the group $C_G(H)$ is the product of $H$ with a subgroup $\Gamma$ isomorphic to $SL_2$ or $PSL_2$. Furthermore, if $X$ is a $G$-variety, then $X^H$ inherits an action of $C_G(H)/H$, a quotient of $\Gamma$.

**Proposition 4.3** ([Br3 Proposition 7.1]). Let $X$ be a spherical $G$-variety. Let $H \subset T$ be a subtorus of codimension one. Then each irreducible component of $X^H$ is a spherical $C_G(H)$-variety. Moreover,

1. If $H$ is regular, then $X^H$ is at most one-dimensional.
2. If $H$ is singular, then $X^H$ is at most two-dimensional. If moreover $X$ is complete and smooth, then any two-dimensional connected component of $X^H$ is (up to a finite, purely inseparable equivariant morphism) either a...
rational ruled surface
\[ \mathbb{F}_n = \mathbb{P}(\mathcal{O}_p \oplus \mathcal{O}_p(n)) \]
where \( C_G(H) \) acts through the natural action of \( SL_2 \), or the projective plane
where \( C_G(H) \) acts through the projectivization of a non-trivial \( SL_2 \)-module
of dimension three.

For later use, we record Brion’s presentation of the rational equivariant Chow rings of nonsingular ruled surfaces. We follow closely the notation and conventions of [Br3, Section 7]. Let \( D \) be the torus of diagonal matrices in \( SL_2 \), and let \( \alpha \) be the character of \( D \) given by
\[
\alpha \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right) = t^2.
\]
We identify the character ring of \( D \) with \( \mathbb{Q}[\alpha] \). Now consider a rational ruled surface \( \mathbb{F}_n \) with ruling \( \pi : \mathbb{F}_n \to \mathbb{P}^1 \). Notice that \( \mathbb{F}_n \) has exactly four fixed points \( x, y, z, t \) of \( D \), where \( x, y \) (resp. \( z, t \)) are mapped to 0 (resp. \( \infty \)) by \( \pi \). Moreover, we may assume that \( x \) and \( z \) lie in one \( G \)-invariant section of \( \pi \), and that \( y \) and \( t \) lie in the other \( G \)-invariant section. With this ordering of the fixed points, we identify \( opA^*_T(\mathbb{F}_n^D)\mathbb{Q} \) with \( \mathbb{Q}[\alpha]^4 \). In contrast, denote by \( \mathbb{P}(V) \) the projectivization of a nontrivial \( SL_2 \)-module \( V \) of dimension three. The weights of \( D \) in \( V \) are either \(-2\alpha, 0, 2\alpha \) (in the case when \( V = sl_2 \)) or \(-\alpha, 0, \alpha \) (in the case when \( V = k^2 \oplus k \)). We denote by \( x, y, z \) the corresponding fixed points of \( D \) in \( \mathbb{F}(V) \), and we identify \( A^*_T(\mathbb{P}(V)^D)\mathbb{Q} \) with \( \mathbb{Q}[\alpha]^3 \).

**Proposition 4.4.** Notation being as above, the image of
\[
i_D : A^*_D(\mathbb{F}_n)\mathbb{Q} \to \mathbb{Q}[\alpha]^4
\]
consists of all \((f_x, f_y, f_z, f_t)\) such that \( f_x \equiv f_y \equiv f_z \equiv f_t \mod \alpha \) and \( f_x - f_y + f_z - f_t \equiv 0 \mod \alpha^2 \). On the other hand, the image of
\[
i_D : A^*_D(\mathbb{P}(V))\mathbb{Q} \to \mathbb{Q}[\alpha]^3
\]
consists of all \((f_x, f_y, f_z)\) such that \( f_x \equiv f_y \equiv f_z \mod \alpha \) and \( f_x - 2f_y + f_z \equiv 0 \mod \alpha^2 \). \( \square \)

4.2. \( T \)-equivariant Chow cohomology. Let \( X \) be a complete possibly singular spherical \( G \)-variety. Let \( H \subset T \) be a singular subtorus of codimension one. In order to obtain an explicit description of \( opA^*_T(X)\mathbb{Q} \) out of Theorem 3.12 we need to determine which \( C_G(H) \)-spherical surfaces could appear as irreducible components of \( X^H \). We will do this by means of Proposition 4.3. (This is the only case of interest to us, for if \( H \) is regular, then \( X^H \) is \( T \)-skeletal, and GKM theory applies, see Appendix.) So let \( Y \) be a two-dimensional irreducible component of \( X^H \). By [Su, Theorem 2] we may find a proper birational equivariant morphism \( X' \to X \) with \( X' \) smooth and \( G \)-spherical. Thus \( Y \) is the image of some irreducible component \( Y' \) of \( X'^H \) (by Borel’s fixed point theorem). Given that \( X \) is complete, so is \( X' \) and, under our considerations, \( Y' \) is a two-dimensional complete \( C_G(H) \)-spherical variety. Hence \( Y' \) is either the projective plane or a rational ruled surface (up to a finite, purely inseparable equivariant morphism). We inspect these two cases in more detail.

(a) If \( Y' = \mathbb{P}^2 \), then the normalization \( \tilde{Y} \) of \( Y \) is also \( \mathbb{P}^2 \) (up to a finite, purely inseparable equivariant morphism, which is, in particular, bijective).
(b) If $Y'$ is a rational ruled surface, then the normalization $\tilde{Y}$ of $Y$ is either

(i) $Y'$ or (ii) the surface obtained by contracting the unique section $C$ of negative self-intersection in $Y'$ (this is a very special weighted projective plane).

Notice that, except for case (b)-(ii), the normalization $\tilde{Y}$ of $Y$ is a smooth projective surface with finitely many $T$-fixed points. In such cases, it readily follows that $\text{op}A^*_T(\tilde{Y})_\mathbb{Q}$ is free of rank $|\tilde{Y}^T|$ (cf. [Br3, Corollary 3.2.1]). We show that this property also holds in case (b)-(ii).

**Lemma 4.5.** Let $P_n = \mathbb{F}_n/C$ be the weighted projective plane obtained by contracting the unique section $C$ of negative self-intersection in $\mathbb{F}_n$. Then $A^*_T(P)_\mathbb{Q}$ is a free $S_{\mathbb{Q}}$-module of rank three. Hence, $\text{op}A^*_T(P)_\mathbb{Q}$ is also $S_{\mathbb{Q}}$-free of rank three.

**Proof.** Clearly, $|P_n^T| = 3$. The associated BB-decomposition of $P_n$ consists of three cells: a point, a copy of $\mathbb{A}^1$ and an open cell, say $U$, isomorphic to $\mathbb{A}^2/\mu_n$, where $\mu_n \subset D$ is the cyclic group with eigenvalues $(\xi, \xi^{-1})$, where $\xi$ is a $n$-th root of unity. Note that $A_*(U)_\mathbb{Q} \cong A_*(\mathbb{A}^2/\mu_n)$, and the latter identifies to $A_*(\mathbb{A}^2)_\mathbb{Q}$, because the action of $\mu_n$ on $\mathbb{A}^2$ is induced by the action of $D$ (a connected group). So $A_*(U)_\mathbb{Q} \cong \mathbb{Q}$. This yields the isomorphism $A^*_T(U) \cong S_{\mathbb{Q}}$ (see e.g. [G4]). From this, and the fact that the BB-decomposition is filtrable, it easily follows that the $S_{\mathbb{Q}}$-module $A^*_T(P)_\mathbb{Q}$ is free of rank 3. Finally, the second assertion of the lemma follows from Theorem [5.6].

**Corollary 4.6.** Notation being as above, assume that $C$ joins the fixed points $y$ and $t$ of $\mathbb{F}_n$, so that the fixed points of $P_n$ are identified with $x, y, z$. Then the image of $i_D^* : \text{op}A^*_T(P_n)_\mathbb{Q} \rightarrow \mathbb{Q}[\alpha]^3$ consists of all $(f_x, f_y, f_z)$ such that $f_x \equiv f_y \equiv f_z \mod (\alpha)$ and $f_x - 2f_y + f_z \equiv 0 \mod (\alpha^2)$.

**Proof.** Observe that $q : \mathbb{F}_n \rightarrow P_n$ is an envelope. By Theorem [A.1] and Proposition [A.2] the problem reduces to find the image of $q^*$. By Theorem [A.1] again, an element $(f_x, f_y, f_z, f_t) \in \text{op}A^*_T(\mathbb{F}_n)_\mathbb{Q}$ is in the image of $q^*$ if and only if it satisfies the usual relations $f_x \equiv f_y \equiv f_z \equiv f_t \mod \alpha$ and $f_x - f_y + f_z - f_t \equiv 0 \mod (\alpha^2)$, plus the extra relation $f_y = f_t$ (which accounts for the fact that $C$ is collapsed to a fixed point in $P_n$). Hence, the relation $f_x - f_y + f_z - f_t \equiv 0 \mod (\alpha^2)$ reduces to $f_x - 2f_y + f_z \equiv 0 \mod (\alpha^2)$, finishing the argument.

Back to the general setup, let $X$ be a $G$-spherical variety and let $H$ be a singular subtorus of codimension one. Let $Y$ be an irreducible component of $X^H$, and let $\pi : \tilde{Y} \rightarrow Y$ be the normalization map. By the previous analysis, we know the relations that define the image of $i_T^* : \text{op}A^*_T(\tilde{Y})_\mathbb{Q} \rightarrow \text{op}A^*_T(Y^T)_\mathbb{Q}$. We claim that $\pi^* : \text{op}A^*_T(Y)_\mathbb{Q} \rightarrow \text{op}A^*_T(\tilde{Y})_\mathbb{Q}$ is in fact an isomorphism. First, consider the commutative diagram

$$
\begin{array}{ccc}
\text{op}A^*_T(Y)_\mathbb{Q} & \xrightarrow{\pi^*} & \text{op}A^*_T(\tilde{Y})_\mathbb{Q} \\
\downarrow \kappa_T & & \downarrow \kappa_T \\
\text{Hom}_{S_{\mathbb{Q}}}(A^*_T(Y), S_{\mathbb{Q}}) & \xrightarrow{(\pi^*)'} & \text{Hom}_{S_{\mathbb{Q}}}(A^*_T(\tilde{Y}), S_{\mathbb{Q}}).
\end{array}
$$

where the vertical maps are isomorphisms because of equivariant Kronecker duality (Theorem [3.3]), and $(\pi^*)'$ represents the transpose of the surjective map $\pi_* : A^*_T(\tilde{Y})_\mathbb{Q} \rightarrow A^*_T(Y)_\mathbb{Q}$ (commutativity follows from the projection formula). Thus, to prove our claim, it suffices to show that $\pi_*$ is injective. In fact, since $\pi_*$ is
a surjective map of free $S_Q$-modules, the problem reduces to comparing the ranks of $A_T^r(\hat{Y}_Q)$ and $A_T^r(Y)_Q$. If these ranks agree, we are done, for a surjective map of free $S_Q$-modules of the same rank is an isomorphism. Bearing this in mind, we invoke the localization theorem (Theorem A.9): the ranks of $A_T^r(\hat{Y}_Q)$ and $A_T^r(Y)_Q$ are $|\hat{Y}^T|$ and $|Y^T|$ respectively. But $|\hat{Y}^T| = |Y^T|$ by Lemma 4.7. This yields the claim.

**Lemma 4.7.** Let $Y$ be a complete $T$-variety with finitely many fixed points. Let $p : \hat{Y} \to Y$ be the normalization. If $Y$ is locally linearizable and $\hat{Y}$ is projective, then the normalization $p$ induces a bijection $p_T : \hat{Y}^T \to Y^T$ of the fixed point sets.

**Proof.** Clearly, $p$ induces a surjection $p_T : \hat{Y}^T \to Y^T$. Arguing by contradiction, suppose that $p_T$ is not injective. Then there are at least two different fixed points $x, y \in \hat{Y}$ such that $p(x) = p(y)$. Now choose a $T$-invariant curve $\ell \subset \hat{Y}$ passing through $x$ and $y$. It follows that the image $\pi(\ell)$ is an invariant curve on $Y$ with exactly one fixed point. But this is impossible, for the action on $Y$ is locally linearizable (cf. [Ti, Example 4.2]).

With all the ingredients at our disposal, we are now ready to state the main result of this section. This builds on and extends Brion’s result ([Br3, Theorem 7.3]) to the rational equivariant Chow cohomology of possibly singular complete spherical varieties. Our findings complement Brion’s deepest results ([Br3]).

**Theorem 4.8.** Let $X$ be a complete $G$-spherical variety. The image of the injective map

$$i_T^* : \text{op}A_T^*(X)_Q \to \text{op}A_T^*(X^T)_Q$$

consists of all families $(f_x)_{x \in X^T}$ satisfying the relations:

(i) $f_x \cong f_y \mod \chi$, whenever $x, y$ are connected by a $T$-invariant curve with weight $\chi$.

(ii) $f_x - 2f_y + f_z \equiv 0 \mod \alpha^2$ whenever $\alpha$ is a positive root, $x, y, z$ lie in an irreducible component of $X^{\ker \alpha}$ whose normalization is isomorphic to $\mathbb{P}^2$ or the weighted projective plane $\mathbb{P}_n$, and $x, y, z$ are ordered as in Section 4.1.

(iii) $f_x - f_y + f_z - f_t \equiv 0 \mod \alpha^2$ whenever $\alpha$ is a positive root, $x, y, z, t$ lie in an irreducible component of $X^{\ker \alpha}$ whose normalization is isomorphic to $\mathbb{F}_n$, and $x, y, z, t$ are ordered as in Section 4.1.

**Proof.** In light of Theorem 3.12 and Theorem A.9 it suffices to consider the case when $H$ is a singular codimension one subtorus, i.e. $H = \ker \alpha$, for some positive root $\alpha$. Let $X^H = \bigcup_j X_j$ be the decomposition into irreducible components. Notice that each $X_j$ is either a fixed point, a $T$-invariant curve or a possibly singular rational surface (by Proposition 4.3 and our previous analysis). Now, with by Remark A.4 we have the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \text{op}A_T^*(X^{\ker \pi})_Q & \to & \bigoplus_i \text{op}A_T^*(X_j)_Q & \to & \bigoplus_{i,j} \text{op}A_T^*(X_{i,j})_Q \\
\downarrow{\iota_{T,H}} & & \downarrow{\text{id}} & & \downarrow{\text{id}} & & \\
0 & \to & \text{op}A_T^*(X^T)_Q & \to & \bigoplus_i \text{op}A_T^*(X_j^T)_Q & \to & \bigoplus_{i,j} \text{op}A_T^*(X_{i,j}^T)_Q,
\end{array}
$$

where each $X_{i,j}$ is at worst a union of fixed points and $T$-invariant curves. The image of the middle vertical map is completely characterized by our previous analysis,
4.3. G-equivariant Chow cohomology. Let X be a G-spherical variety. The purpose of this subsection is to describe $A_{G}^*(X)_Q$ and its relation to $A_T^*(X)_Q$. To adapt the definition of G-equivariant operational Chow groups, in this subsection we work in the category of G-quasiprojective schemes.

As it occurs in equivariant cohomology, one can often simplify the computation of equivariant Chow groups for schemes with a reductive group action by focusing one’s attention on the induced action of a maximal torus.

**Proposition 4.9** ([EG1, Proposition 6]). Let G be a connected reductive group with maximal torus T and Weyl group W. Let X be a G-scheme. Then

$$A_{G}^*(X)_Q \simeq A_T^*(X)_Q^W.$$  

We shall show that if X is a spherical G-variety, then $\text{op}A_{G}^*(X)_Q \simeq \text{op}A_T^*(X)_Q^W$. This does not follow immediately from Theorem 4.9 because the techniques of [Vis] only apply to the Chow groups of the fibration

$$G/B \hookrightarrow X_B \rightarrow X_G,$$

where the equivariant map $X_B \rightarrow X_G$ is smooth. In our setting of operational Chow rings we proceed as follows.

**Theorem 4.10.** Let X be a projective G-variety with a finite number of B-orbits. Then the G-equivariant Kronecker map

$$K_G : \text{op}A_{G}^*(X)_Q \longrightarrow \text{Hom}_{S_Q^W}(A_{G}^*(X)_Q, S_Q^W) \quad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

is a $S_Q^W$-linear isomorphism.

**Proof.** First, we consider the case when X is projective and smooth. Then, because T acts on X with finitely many fixed points, X is T-cellular. It follows that $A_T^*(X)_Q$ is a free $S_Q$-module ([Br3, Corollary 3.2.1]). Moreover, $A_T^*(X)_Q$ carries an intersection product making it into a graded ring $A_T^*(X)_Q$ (graded by codimension) isomorphic the corresponding equivariant operational Chow group ([EG1, Proposition 4]). Proposition 1.3 gives, in turn, the isomorphism $A_{G}^*(X)_Q \simeq (A_T^*(X)_Q)^W$ and so $A_{G}^*(X)_Q$ is a free $S_Q^W$-module. Also, $A_{G}^*(X)_Q/S_Q^W + A_{G}^*(X)_Q \simeq A^*_G(X)_Q$ [Br3, Corollary 6.7.1], where $S_Q^W +$ denotes the ideal of $S_Q^W$ generated by homogeneous elements of positive degree. Now freeness of $A_{G}^*(X)_Q$ yields the identifications

$$\text{Hom}_{S_Q^W}(A_{G}^*(X)_Q, S_Q^W)/S_Q^W + \cdot \text{Hom}_{S_Q^W}(A_{G}^*(X)_Q, S_Q^W) \simeq \text{Hom}_{Q}(A_{*}(X)_Q, Q).$$
Consider the commutative diagram
\[
\begin{array}{ccc}
A^*_G(X) & \xrightarrow{\mathcal{K}_G} & \text{Hom}_{S^W}(A^*_G(X), S^W) \\
\downarrow & & \downarrow \\
A^*(X) & \xrightarrow{\mathcal{K}} & \text{Hom}_Q(A^*(X), Q).
\end{array}
\]

In order to prove that $\mathcal{K}_G$ is an isomorphism, it suffices, by the graded Nakayama lemma, to show that the non-equivariant Kronecker duality map $\mathcal{K}$ is an isomorphism, but this has already been established in [FMSS Theorem 3].

In the general case, choose a $G$-equivariant birational envelope $p : \tilde{X} \to X$, where $\tilde{X}$ is smooth, projective and $G$-spherical. Suppose that $Z$ is a $G$-invariant closed subscheme of $X$ such that $p$ is an isomorphism outside $Z$ (note that we may arrange so that $\dim Z < \dim X$). Let $E = p^{-1}(Z)$. By Theorem $4.10$ we have the exact sequence
\[
0 \to \text{op}A^*_G(X) \to \text{op}A^*_G(Z) \oplus \text{op}A^*_G(\tilde{X}) \to \text{op}A^*_G(E).
\]

Hence, arguing by induction on the dimension of $X$ yields the result, because $\tilde{X}$ satisfies $G$-equivariant Kronecker duality (by the previous case) and the map $\mathcal{K}_G$ is functorial for proper morphisms (Remark $3.2$).

**Corollary 4.11.** Under the hypothesis of Theorem $4.10$
\[
\text{op}A^*_G(X)_Q \cong \text{op}A^*_G(X)_Q \otimes_{S^W} S_Q.
\]
Consequently, $\text{op}A^*_G(X)_Q \cong \text{op}A^*_G(X)_Q$.

**Proof.** Simply notice that
\[
\text{Hom}_Q(A^*_G(X)_Q \otimes_{S^W} S_Q, S^W_Q) \cong S_Q \otimes_{S^W} \text{Hom}_{S^W}(A^*_G(X)_Q, S^W_Q),
\]
because $S_Q$ is free over $S^W_Q$ and $A^*_G(X)_Q$ is finitely generated. Now observe that the term on the left hand side identifies in turn to $\text{op}A^*_G(X)_Q$, due to Theorem $3.3$ and the fact that $A^*_G(X)_Q \cong A^*_G(X)_Q \otimes_{S^W} S_Q$. Finally, notice that the right hand side corresponds to $S_Q \otimes_{S^W} \text{op}A^*_G(X)_Q$ by Theorem $4.10$.

Arguing as in [Br3 Corollary 6.7.1], one obtains the next result.

**Corollary 4.12.** Let $X$ be a spherical $G$-variety. If $\text{op}A^*_G(X)_Q$ is $S_Q$-free, then $\text{op}A^*_G(X)_Q$ is $S^W_Q$-free and restriction to the fiber induces an isomorphism
\[
\text{op}A^*_G(X)_Q/S^W_Q \cong \text{op}A^*_G(X)_Q \cong \text{op}A^*_G(X)_Q.
\]

Corollary 4.11 is satisfied by $Q$-filtrable spherical $G$-varieties [G4].

5. Further remarks

1. *Description of the image of restriction to the fiber $i^*: \text{op}A^*_G(X) \to \text{op}A^*(X)$ by using equivariant multiplicities.* So far, this has been carried out for singular toric varieties [PK]. It would be interesting to obtain similar formulas for more general, possibly singular, projective group embeddings.

2. *Understand the action of $PP^*_T(X)$ on $\text{op}A^*_T(X)$ for $T$-skeletal spherical varieties,* in light of Brion’s description of the intersection pairing between curves and
divisors on spherical varieties [Br2]. This should also provide a geometric interpretation of the coefficients arising from the cap and cup product formulas (Corollaries 3.7 and 3.8). This will be pursued in a subsequent paper.

Appendix A. Localization theorem and GKM theory for rational equivariant Chow cohomology

Here we translate the results of [G3] into the language of equivariant Chow cohomology. In that paper we studied equivariant operational $K$-theory, but as stated in [G3] Section 7 the results readily extend to equivariant Chow cohomology with rational coefficients. The purpose of this appendix is to supply a detailed proof of this claim, merely for the sake of completeness.

Let $p : \tilde{X} \to X$ be a $T$-equivariant birational envelope which is an isomorphism over an open set $U \subset X$. Let $\{Z_i\}$ be the irreducible components of $Z = X - U$, and let $E_i = p^{-1}(Z_i)$, with $p_i : E_i \to Z_i$ denoting the restriction of $p$. The next theorem is Kimura’s fundamental result adapted to our setup.

**Theorem A.1** ([Ki] Theorem 3.1). Let $p : \tilde{X} \to X$ be a $T$-equivariant envelope. Then the induced map $p^* : \text{op}A^*_T(X) \to \text{op}A^*_T(\tilde{X})$ is injective. Furthermore, if $p$ is birational (and notation is as above), then the image of $p^*$ is described inductively as follows: a class $\tilde{c} \in \text{op}A^*_T(\tilde{X})$ equals $p^*(c)$, for some $c \in \text{op}A^*_T(X)$ if and only if, for all $i$, we have $\tilde{c}|_{E_i} = p_i^*(c_i)$ for some $c_i \in \text{op}A^*_T(Z_i)$. □

Since $E_i$ and $Z_i$ have smaller dimension than $X$, we can use this result to compute $\text{op}A^*_T(X)$ using a resolution of singularities (if e.g. $\text{char}(k) = 0$) and induction on dimension. In fact, if $\tilde{X}$ can be chosen to be smooth, then $\text{op}A^*_T(\tilde{X}) \simeq A^*_T(\tilde{X})$ and thus $\text{op}A^*_T(X) \subset A^*_T(\tilde{X})$ sits inside a more geometric object. Theorem A.1 is one of the reasons why Kimura’s results [Ki] make operational Chow groups more computable.

Hereafter we assume $\text{char}(k) = 0$.

In Proposition A.2 we state another crucial consequence of Kimura’s work. Put in perspective, it asserts that the rational equivariant operational Chow ring $\text{op}A^*_T(X)_\mathbb{Q}$ of any complete $T$-scheme $X$ is a subring of $\text{op}A^*_T(X^T)_\mathbb{Q}$. Moreover, there is a natural isomorphism (with $\mathbb{Z}$ coefficients)

$$\text{op}A^*_T(X^T) \simeq \text{op}A^*(X^T) \otimes_\mathbb{Z} S.$$

Indeed, for a fixed degree $j$, [EG1] Theorem 2, yields the identifications $\text{op}A^j_T(X^T) \simeq \text{op}A^j(\langle X^T \times U \rangle/T) \simeq \text{op}A^j(X^T \times (U/T))$, where $U$ is an open $T$-invariant subset of a $T$-module $V$, so that the quotient $U \rightarrow U/T$ exists and is a principal $T$-bundle, and the codimension of $V \setminus U$ is large enough. Additionally, we can find $U$ such that $U/T$ is a product of projective spaces (see e.g. [EG1]). It follows that $\text{op}A^*(X^T \times U/T) \simeq \text{op}A^*(X) \otimes \text{op}A^*(U/T)$, by the projective bundle formula (see [Fu] Chapter 17, Example 17.5.1 (b))). In many cases of interest, $X^T$ is finite (e.g. for spherical varieties) and so one has $\text{op}A^*_T(X)_\mathbb{Q} \subseteq \bigoplus_{\ell \in \{X^T\}} \text{op}A^*_T(pt)_\mathbb{Q} = S^\ell_\mathbb{Q}$, where $\ell = |X^T|$. This motivated our introduction of localization techniques, and ultimately GKM theory, into the study of rational equivariant operational Chow rings and integral equivariant operational $K$-theory [G3].
Proposition A.2. Let $X$ be a complete $T$-scheme and let $i_T : X^T \to X$ be the inclusion of the fixed point subscheme. Then the pull-back map

$$i_T^* : \text{op}A_T^*(X)_Q \to \text{op}A_T^*(X^T)_Q$$

is injective.

Proof. The argument is essentially that of [G3 Proposition 3.7]. We include it here for convenience. Choose a $T$-equivariant envelope $p : \tilde{X} \to X$, with $X$ projective and smooth. It follows that $p^* : \text{op}A_T^*(X) \to \text{op}A_T^*(\tilde{X})$ is injective (Theorem A.1). Since $\tilde{X}$ is smooth and projective, $i_T^* : \text{op}A_T^*(\tilde{X})_Q \to \text{op}A_T^*(\tilde{X}^T)_Q$ is injective (by [EG1 Proposition 4] together with [Br3 Corollary 3.2.1]). Now the chain of inclusions $\tilde{X}^T \subset p^{-1}(X^T) \subset \tilde{X}$ indicate that $i_T^*$ factors through $i^* : \text{op}A_T^*(\tilde{X}) \to \text{op}A_T^*(p^{-1}(X^T))$, where $i : p^{-1}(X^T) \hookrightarrow \tilde{X}$ is the natural inclusion. Thus, $i^*$ is injective over $\mathbb{Q}$ as well. Finally, adding this information to the commutative diagram below

$$\begin{array}{ccc}
\text{op}A_T^*(X)_Q & \xrightarrow{p^*} & \text{op}A_T^*(\tilde{X})_Q \\
\downarrow{i_T^*} & & \downarrow{i^*} \\
\text{op}A_T^*(X^T)_Q & \xrightarrow{p^*} & \text{op}A_T^*(p^{-1}(X^T))_Q.
\end{array}$$

renders $i_T^* : \text{op}A_T^*(X)_Q \to \text{op}A_T^*(X^T)_Q$ injective. \hfill \Box

Corollary A.3 ([G3 Corollary 3.8]). Let $X$ be a complete $T$-scheme. Let $Y$ be a $T$-invariant closed subscheme containing $X^T$. Let $i : Y \to X$ be the natural inclusion. Then the pullback $i^* : \text{op}A_T^*(X)_Q \to \text{op}A_T^*(Y)_Q$ is injective. In particular, if $H$ is a closed subgroup of $T$, then $i_H^* : \text{op}A_T^*(X)_Q \to \text{op}A_T^*(X^H)_Q$ is injective. \hfill \Box

Remark A.4. Of particular interest is the case $Y = \cup_{i=1}^{n} Y_i$, where $Y_i$ are the irreducible components of $Y$. Let $Y_{ij} = Y_i \cap Y_j$. By Theorem A.1 the following sequence is exact

$$0 \to \text{op}A_T^*(Y)_Q \to \bigoplus_i \text{op}A_T^*(Y_i)_Q \to \bigoplus_{i,j} \text{op}A_T^*(Y_{ij})_Q.$$

When $Y^T$ is finite, the sequence above yields the commutative diagram [G3 Corollary 3.6]:

$$\begin{array}{ccccccc}
0 & \to & \text{op}A_T^*(Y)_Q & \to & \bigoplus_i \text{op}A_T^*(Y_i)_Q & \to & \bigoplus_{i,j} \text{op}A_T^*(Y_{ij})_Q \\
\downarrow{i_T^*} & & \downarrow{i_T^*} & & \downarrow{i_T^*} & & \\
0 & \to & \text{op}A_T^*(Y^T)_Q & \to & \bigoplus_i \text{op}A_T^*(Y_{i}^T)_Q & \to & \bigoplus_{i,j} \text{op}A_T^*(Y_{ij}^T)_Q
\end{array}$$

Since all vertical maps are injective (Proposition A.2), it is important to observe that we can describe the image of the first vertical map in terms of the image of the second vertical map and the kernel of $q$. In other words, the map

$$p : \text{im}(i_T^*,Y) \to \{w \in \bigoplus_i \text{op}A_T^*(Y_i)_Q | w \in \text{im}(\sum_i i_T^*) \text{ and } q(w) = 0\}$$

sending $u \to p(u)$ is an isomorphism. Now, since $Y^T$ is finite, the kernel of the map $q$ consists of all families $(f_i)_i$ such that $f_i(x_k) = f_j(x_k)$ (equality of $k$-components), whenever $x_k$ is in the intersection of $Y_i$ and $Y_j$. 


Back to the general case, let \( X \) be a complete \( T \)-scheme. One wishes to describe the image of the injective map
\[
i_T^*: \text{op} A_T^*(X) \otimes \mathbb{Q} \to \text{op} A_T^*(X^T) \otimes \mathbb{Q}.
\]
For this, let \( H \subset T \) be a subtorus of codimension one. Observe that \( i_T^* \) factors as
\[
i_T^*: X^T \to X^H \quad \text{followed by} \quad i_H^*: X^H \to X^T.
\]
Thus, the image of \( i_T^* \) is contained in the image of \( i_{T,H}^* \). In symbols, \( \text{Im}(i_T^*) \subseteq \bigcap_{H \subset T} \text{Im}(i_{T,H}^*) \), where the intersection runs over all codimension-one subtori \( H \) of \( T \). This observation leads to a complete description of the image of \( i_T^* \) over the rationals. Before stating it, we recall a definition from [G3, Section 4].

**Definition A.5.** Let \( X \) be a complete \( T \)-scheme. We say that \( X \) has the Chang-Skjelbred property (or CS property, for short) if the map
\[
i_T^*: \text{op} A_T^*(X) \to \text{op} A_T^*(X^T)
\]
is injective, and its image is exactly the intersection of the images of
\[
i_{T,H}^*: \text{op} A_T^*(X^H) \to \text{op} A_T^*(X^T),
\]
where \( H \) runs over all subtori of codimension one in \( T \). When the defining conditions hold over \( \mathbb{Q} \) rather than \( \mathbb{Z} \), we say that \( X \) has the rational CS property.

Some obstructions for the CS property to hold are e.g. (i) \( A_*(X^T) \) could have \( \mathbb{Z} \)-torsion, (ii) \( \dim T \geq 2 \) and there exist a \( T \)-orbit on \( X \) whose stabilizer is not connected, for instance, if \( X \) is nonsingular, \( T \)-skeletal (Definition A.7) and the weights of the \( T \)-invariant curves are not primitive.

By [Br3, Theorem 3.3], every nonsingular projective \( T \)-scheme has the rational CS property. We extend this result to include all possibly singular complete \( T \)-schemes. See [G3, Theorem 4.4] for the corresponding statement in equivariant operational \( K \)-theory with integral coefficients.

**Theorem A.6.** Let \( X \) be a complete \( T \)-scheme. Then \( X \) has the rational CS property.

**Proof.** Simply argue as in [G3, Theorem 4.4], using [EG2, Lemma 7.2], Theorem A.1 and Proposition A.2. \( \square \)

Before stating our version of GKM theory, let us recall a few definitions from [GKM], [G1] and [G3].

**Definition A.7.** Let \( X \) be a complete \( T \)-variety. Let \( \mu: T \times X \to X \) be the action map. We say that \( \mu \) is a \( T \)-skeletal action if
\begin{enumerate}
\item \( X^T \) is finite, and
\item The number of one-dimensional orbits of \( T \) on \( X \) is finite.
\end{enumerate}

In this context, \( X \) is called a \( T \)-skeletal variety. The associated graph of fixed points and invariant curves is called the GKM graph of \( X \). We shall denote this graph by \( \Gamma(X) \).

Notice that, in principle, Definition A.7 allows for \( T \)-invariant irreducible curves with exactly one fixed point (i.e. the GKM graph \( \Gamma(X) \) may have simple loops). In [G3, Proposition 5.3] we show that the functor \( \text{op} A_T^*(-) \otimes \mathbb{Q} \) “contracts” such loops to a point. The proof there is given in the context of operational \( K \)-theory, but the proof easily extends to our current setup.
Proposition A.8 ([G3 Proposition 5.3]). Let $X$ be a complete $T$-variety and let $C$ be a $T$-invariant irreducible curve of $X$ which is not fixed pointwise by $T$. Then the image of the injective map $i^*_T: \text{op}A^*_T(C)_Q \to \text{op}A^*_T(C^T)_Q$ is described as follows:

(i) If $C$ has only one fixed point, say $x$, then $i^*_T: \text{op}A^*_T(C)_Q \to \text{op}A^*_T(x)_Q$ is an isomorphism; that is, $\text{op}A^*_T(C)_Q \simeq S_Q$.

(ii) If $C$ has two fixed points, then

$$\text{op}A^*_T(C)_Q \simeq \{(f_0, f_\infty) \in S_Q \oplus S_Q | f_0 \equiv f_\infty \mod \chi\},$$

where $T$ acts on $C$ via the character $\chi$. 

Let $X$ be a complete $T$-skeletal variety. It is possible to define a ring $PP^*_T(X)_Q$ of (rational) piecewise polynomial functions. Indeed, let $\text{op}A^*_T(X^T)_Q = \bigoplus_{x \in X^T} S_Q$. We then define $PP^*_T(X)_Q$ as the subalgebra of $\text{op}A^*_T(X^T)_Q$ defined by

$$PP^*_T(X)_Q = \{(f_1, \ldots, f_m) \in \bigoplus_{x \in X^T} S_Q | f_i \equiv f_j \mod (\chi_{i,j})\}$$

where $x_i$ and $x_j$ are the two (perhaps equal) fixed points in the closure of the one-dimensional $T$-orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of $T$ associated with $C_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points $\chi_{i,j}$ to its opposite). Invariant curves with only one fixed point do not impose any relation, and this is compatible with Proposition A.8. Now we are ready to state our version of GKM theory.

Theorem A.9 ([G3 Theorem 5.4]). Let $X$ be a complete $T$-skeletal variety. Then the pullback

$$i^*_T: \text{op}A^*_T(X)_Q \longrightarrow \text{op}A^*_T(X^T)_Q = \bigoplus_{x_i \in X^T} S_Q$$

induces an algebra isomorphism between $\text{op}A^*_T(X)_Q$ and $PP^*_T(X)_Q$. 

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EQUIVARIANT OPERATIONAL CHOW RINGS

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