COMBINATORIAL INTERPRETATIONS OF THE $q$-FAULHABER AND $q$-SALIÉ COEFFICIENTS

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Dedicated to Xavier Viennot on the occasion of his sixtieth birthday

Abstract. Recently, Guo and Zeng discovered two families of polynomials featuring in a $q$-analogue of Faulhaber’s formula for the sums of powers and a $q$-analogue of Gessel-Viennot’s formula involving Salié’s coefficients for the alternating sums of powers. In this paper, we show that these are polynomials with symmetric, nonnegative integral coefficients by refining Gessel-Viennot’s combinatorial interpretations.

1. Introduction

In the early seventeenth century, Johann Faulhaber [1] (see also [5]) considered the sums of powers $S_{m,n} = \sum_{k=1}^{n} k^m$ and provided formulas for the coefficients $f_{m,k}$ ($0 \leq m \leq 8$) in

$$S_{2m+1,n} = \frac{1}{2} \sum_{k=1}^{m} f_{m,k} (n(n+1))^{k+1},$$

(1)

In 1989, Ira Gessel and Xavier Viennot [3] studied the alternating sum $T_{2m,n} = \sum_{k=1}^{n} (-1)^{n-k} k^{2m}$ and showed that there exist integers $s_{m,k}$ such that

$$T_{2m,n} = \frac{1}{2} \sum_{k=1}^{m} s_{m,k} (n(n+1))^{k}.$$  

(2)

In particular, they proved that the Faulhaber coefficients $f_{m,k}$ and the Salié coefficients $s_{m,k}$ count certain families of non-intersecting lattice paths.

Recently, two of the authors [4], continuing work of Michael Schlosser [7], Sven Ole Warnaar [8] and Kristina Garrett and Kristen Hummel [2], have found $q$-analogues of (1) and (2). More precisely, setting $[k] = \frac{1-q^k}{1-q}$, $[k]! = \prod_{i=1}^{k} [i]$, and

$$S_{m,n}(q) = \sum_{k=1}^{n} \frac{[2k]}{[2]} [k]^{m-1} q^{\frac{m+1}{2}(n-k)},$$

(3)

$$T_{m,n}(q) = \sum_{k=1}^{n} (-1)^{n-k} [k]^{m} q^{\frac{m}{2}(n-k)},$$

(4)

for $m, n \in \mathbb{N}$, they proved the following results:
Theorem 1.1. There exist polynomials $P_{m,k}, Q_{m,k}, G_{m,k}$ and $H_{m,k}$ in $\mathbb{Z}[q]$ such that

\[ S_{2m+1,n}(q) = \sum_{k=0}^{m} (-q^n)^{m-k} \frac{[k]!}{[m+1]!} P_{m,m-k}(q) \frac{([n][n+1])^{k+1}}{[2]}, \]

(5)

\[ S_{2m,n}(q) = (1 - q^{n+\frac{1}{2}}) \sum_{k=0}^{m} (-q^n)^{m-k} (1 - q^{\frac{3}{2}})^{m-k} Q_{m,m-k}(q^{\frac{3}{2}}) \frac{([n][n+1])^k}{[2]}, \]

(6)

\[ T_{2m,n}(q) = \sum_{k=1}^{m} (-q^n)^{m-k} \frac{G_{m,m-k}(q)}{\prod_{i=0}^{m-k} (1 + q^{m-i})} ([n][n+1])^k, \]

(7)

and

\[ T_{2m-1,n}(q) = (-1)^{m+n} H_{m,m-1}(q^{\frac{3}{2}}) \frac{q^{m-\frac{3}{2}}n}{(1 + q^{\frac{3}{2}})m \prod_{i=0}^{m-1} (1 + q^{m-i-\frac{3}{2}})} \]

\[ + \frac{1 - q^{n+\frac{3}{2}}}{1 - q^{n+\frac{3}{2}}} \sum_{k=1}^{m} (-q^n)^{m-k} \frac{H_{m,m-k}(q^{\frac{3}{2}})([n][n+1])^{k-1}}{(1 + q^{\frac{3}{2}})^{m-k+1} \prod_{i=0}^{m-k} (1 + q^{m-i-\frac{3}{2}})}, \]

(8)

Comparing with (3) and (4), we have

\[ f_{m,k} = (-1)^{m-k} \frac{k!}{(m+1)!} P_{m,m-k}(1) \]

and

\[ s_{m,k} = (-1)^{m-k} 2^{k-m} G_{m,m-k}(1), \]

but the numbers corresponding to $Q_{m,k}(1)$ and $H_{m,k}(1)$ do not seem to be studied in the literature. The first values of $P_{m,k}, Q_{m,k}, G_{m,k}$ and $H_{m,k}$ are given in Tables 1–4, respectively.

**Table 1.** Values of $P_{m,k}(q)$ for $0 \leq k < m \leq 5$.

| $k \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----------------|---|---|---|---|---|---|
| 0               | 0 | 1 | 1 | 1 | 1 | 1 |
| 1               | 2(q+1) | 3q^2 + 4q + 3 | 2(q+1) | 2q^2 + q + 2 |
| 2               | (q+1)(5q^2 + 8q + 5) | (q+1)(9q^4 + 19q^2 + 20q^2 + 3q + 9) | (q+1)(9q^4 + 19q^2 + 20q^2 + 3q + 9) |
| 3               | (q+1)(5q^2 + 8q + 5) | (q+1)(9q^4 + 19q^2 + 20q^2 + 3q + 9) |
| 4               | 2(q+1)(q^2 + q + 1)(7q^4 + 11q^2 + 7) |

**Table 2.** Values of $Q_{m,k}(q)$ for $0 \leq k < m \leq 4$.

| $k \setminus m$ | 0 | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|---|
| 0               | 0 | 1 | 1 | 1 | 1 |
| 1               | 2q^2 + q + 2 | 3q^4 + 2q^4 + 4q^2 + 2q + 3 |
| 2               | 2q^2 + q + 2 | (q^2 + q + 1)(5q^2 + q^2 + 9q^2 + q + 9) |
| 3               | (q^2 + q + 1)(5q^2 + q^2 + 9q^2 + q + 9) |

| $k \setminus m$ | 0 | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|---|
| 0               | 0 | 1 | 1 | 1 | 1 |
| 1               | 2q^2 + q + 2 | 3q^4 + 2q^4 + 4q^2 + 2q + 3 |
| 2               | 2q^2 + q + 2 | (q^2 + q + 1)(5q^2 + q^2 + 9q^2 + q + 9) |
| 3               | (q^2 + q + 1)(5q^2 + q^2 + 9q^2 + q + 9) |
Lemma 2.1. By convention, $a$ and $b$ are $\geq 0$. The tables above suggest that the coefficients of the polynomials $P_{m,k}$, $Q_{m,k}$, $G_{m,k}$ and $H_{m,k}$ are nonnegative and symmetric. The aim of this paper is to prove this fact by showing that the coefficients count certain families of non-intersecting lattice paths.

2. Inverses of Matrices

Recall that the $n$-th complete homogeneous functions in $r$ variables $x_1, x_2, \ldots, x_r$ has the following generating function:

$$
\sum_{n \geq 0} h_n(x_1, \ldots, x_r) t^n = \frac{1}{(1 - x_1 t)(1 - x_2 t) \ldots (1 - x_r t)}.
$$

For $r, s \geq 0$, let $h_n(\{1\}^r, \{q\}^s)$ denote the $n$-th complete homogeneous functions in $r+s$ variables, of which $r$ are specialized to 1 and the others to $q$, i.e.,

$$
\sum_{n \geq 0} h_n(\{1\}^r, \{q\}^s) z^n = \frac{1}{(1 - z)^r (1 - qz)^s}.
$$

By convention, $h_n(\{1\}^r, \{q\}^s) = 0$ if $r < 0$ or $s < 0$. For convenience, we also write $h_n(\{1, q\}^r)$ instead of $h_n(\{1\}^r, \{q\}^r)$.

We first prove the following result.

Lemma 2.1. Let $a$ and $b$ be non-negative integers, then

$$
\sum_{m \geq 0} \sum_{k \geq 0} h_{m-2k}(\{1\}^{k+a}, \{q\}^{k+b}) \left( \frac{q^k}{[l]^z} \right)^k z^m = \frac{[l]^2}{[l]z} \left\{ \begin{array}{ll} 
\frac{[l+1]}{[l]-[l+1]z} - \frac{q[l-1]}{[l]-q[l-1]z} & \text{for } a = 1, b = 1, \\
\frac{1}{[l]-[l+1]z} - \frac{q[l-1]}{[l]-q[l-1]z} & \text{for } a = 1, b = 0, \\
\frac{q^k}{[l]-[l+1]z} + \frac{1}{[l]-q[l-1]z} & \text{for } a = 0, b = 1.
\end{array} \right.
$$

Recall that a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $n$ has symmetric coefficients if $a_i = a_{n-i}$ for $0 \leq i \leq n$. The tables above suggest that the coefficients of the polynomials $P_{m,k}$, $Q_{m,k}$, $G_{m,k}$ and $H_{m,k}$ are nonnegative and symmetric. The aim of this paper is to prove this fact by showing that the coefficients count certain families of non-intersecting lattice paths.

Table 3. Values of $G_{m,k}(q)$ for $0 \leq k < m \leq 5$.

| $k \setminus m$ | 1   | 2   | 3       | 4       | 5       |
|-----------------|-----|-----|---------|---------|---------|
| 0               | 1   | 1   | 4       | 1       | 1       |
| 1               | 2   | 3(q+1) | 4(q^2 + q + 1) | 5(q+1)(q^2 + 1) |
| 2               | 6(q+1) | 2(q+1)(5q^2 + 7q + 5) | 5(q+1)(3q^4 + 4q^3 + 8q^2 + 4q + 3) |
| 3               | 4(q+1)(5q^2 + 7q + 5) | 5(q+1)(7q^4 + 14q^3 + 20q^2 + 14q + 7) |
| 4               |     |     | 10(q+1)(7q^4 + 14q^3 + 20q^2 + 14q + 7) |

Table 4. Values of $H_{m,k}(q)$ for $0 \leq k < m \leq 4$.

| $k \setminus m$ | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 |
| 1               | 2 | 3q^2 + 2q + 3 | 4q^4 + 3q^3 + 4q^2 + 3q + 4 |
| 2               | 2(3q^3 + 2q + 3) | 10q^6 + 15q^5 + 30q^4 + 26q^3 + 30q^2 + 15q + 10 |
| 3               | 2(10q^6 + 15q^5 + 30q^4 + 26q^3 + 30q^2 + 15q + 10) |
Proof. Using the definition (21) of the complete homogeneous functions we have
\[
\sum_{m \geq 0} \sum_{k \geq 0} h_{m-2k}(\{1\}^{k+a}, \{q\}^{k+b}) x^k z^m
= \sum_{k \geq 0} \frac{x^k z^{2k}}{(1-z)^{k+a}(1-qz)^{k+b}}
= \frac{1}{(1-z)^{a-1}(1-qz)^{b-1}(1-z)(1-qz) - xz^2}.
\]
Setting \(x = \frac{q^j}{[j]^2}\), a little calculation shows that the denominator of the second fraction factorizes:
\[
\frac{1}{(1-z)(1-qz) - xz^2} = \frac{1}{(\lfloor l \rfloor - qz[l-1])(\lfloor l \rfloor - z[l+1])}.
\]
The result then follows from the standard partial fraction decomposition. \(\square\)

Let \(X_n = \frac{[n][n+1]}{q^{\binom{n}{2}}}.\) The following lemma might be interesting per se. When \(q = 1\) it reduces to simple applications of the binomial theorem.

**Lemma 2.2.** For \(k, m \geq 1\), set
\[
c_{k,m}(q) := h_{2m-k}(\{1\}^{2}k^{m+1}) + q h_{2m-k-1}(\{1\}^{2}k^{m+1},)
\]
\[
g_{k,m}(q) := h_{2m-k}(\{1\}^{k-m+1}, \{q\}^{k-m}) + h_{2m-k}(\{1\}^{k-m}, \{q\}^{k-m+1}),
\]
\[
d_{k,m}(q) := g_{k,m}(q^2) + qg_{k-1,m-1}(q^2).
\]
For \(m, l \geq 1\), we have
\[
X_{l}^{m+1} - X_{l-1}^{m+1} = \sum_{k} h_{m-2k}(\{1\}^{k+1}, [2l])[l]^{2(m-k)} q^{-l(m-k+1)}, \tag{10}
\]
\[
\frac{1-q_{l+\frac{1}{2}}}{(1-q^2)q_{l+\frac{1}{2}}} X_{l}^{m} - \frac{1-q_{l-\frac{1}{2}}}{(1-q^2)q_{l-\frac{1}{2}}} X_{l-1}^{m} = \sum_{k} c_{m,m-k}(q^{\frac{1}{2}})[2l][l]^{2(m-k-\frac{1}{2})} q^{-l(m-k+\frac{1}{2})}, \tag{11}
\]
\[
X_{l}^{m} + X_{l-1}^{m} = \sum_{k} g_{m,m-k}(q)[l]^{2(m-k)} q^{-l(m-k)}, \tag{12}
\]
\[
\frac{1-q_{l+\frac{1}{2}}}{(1-q^2)q_{l+\frac{1}{2}}} X_{l}^{m-1} + \frac{1-q_{l-\frac{1}{2}}}{(1-q^2)q_{l-\frac{1}{2}}} X_{l-1}^{m-1} = \sum_{k} d_{m,m-k}(q^{\frac{1}{2}})[l]^{2(m-k-\frac{1}{2})} q^{-l(m-k+\frac{1}{2})}. \tag{13}
\]

**Proof.** The proof rests on the previous lemma.

- Applying Lemma 2.1 with \(a = 1\) and \(b = 1\) yields that the coefficient of \(z^m\) in \(\sum_k h_{m-2k}(\{1\}^{k+1}, q^k[l]^{-2k}\) is

\[
\frac{[l]}{[2l]} \left( \binom{l+1}{[l+1]} \right)^m - q[l-1] \left( \frac{q[l-1]}{[l]} \right)^m.
\]

Multiplying this expression with \([2l][l]^{2m} q^{\binom{m}{2}}\), we obtain (10).

- Since \(c_{m,m-k}(q^{\frac{1}{2}}) = h_{m-2k}(\{1\}^{k+1}, q^{\frac{1}{2}}) + q^{\frac{1}{2}} h_{m-2k}(\{1\}^{k+1})\), Equation 11 follows directly from the previous calculation.
• As \( g_{m,m-k}(q) = h_{m-2k}([1]^{k+1}, \{q\}^k) + h_{m-2k}([1]^k, \{q\}^{k+1}) \), applying Lemma \(2.4\) with \( a = 1, b = 0 \) and \( a = 0, b = 1,\)

\[
\sum_k \left( h_{m-2k}([1]^{k+1}, \{q\}^k) + h_{m-2k}([1]^k, \{q\}^{k+1}) \right) q^{lk} [l]^{-2k}
\]

\[
= \frac{[l]!^2}{[2l]!} \left( \frac{1 + q^l}{[l] - [l + 1]z} + \frac{1 + q^l}{[l] - q[l - 1]z} \right)
\]

Multiplying the coefficient of \( z^m \) of this expression with \([l]!^2 q^{-lm}\) we obtain \(12\).

• Since \( d_{m,m-k}(q) = g_{m,m-k}(q) + \frac{1}{q} g_{m-1,m-k-1}(q) \), Equation \(13\) follows directly from the previous calculation.

\[\square\]

The following is the main result of this section. Note that together with Theorems \(3.2\) and \(3.6\) it also provides an alternative proof of Theorem \(1.1\).

**Theorem 2.3.** The inverses of the lower triangular matrices

\((h_{2m-k}([1,q]^{k-m+1}))_{0 \leq k,m \leq n}, (c_{k,m}(q))_{1 \leq k,m \leq n}, (g_{k,m}(q))_{1 \leq k,m \leq n}, (d_{k,m}(q))_{1 \leq k,m \leq n}\)

are respectively the lower triangular matrices

\[
(-1)^{k-m} \frac{[m]!}{[k + 1]!} P_{k,m}(q),
\]

\[
(-1)^{k-m} \frac{(1 - q)^{k-m+1} Q_{k,k-m}(q)}{\prod_{i=0}^{k-m} (1 - q^{2k-2i+1})},
\]

\[
(-1)^{k-m} \frac{G_{k,k-m}(q)}{\prod_{i=0}^{k-m} (1 + q^{k-i})},
\]

\[
(-1)^{k-m} \frac{H_{k,k-m}(q)}{(1 + q)^{k-m+1} \prod_{i=0}^{k-m} (1 + q^{2k-2i-1})}.
\]

**Proof.**

• Summing Equation \(10\) over \( l \) from 1 to \( n \) and applying Equation \(8\), we obtain

\[
X_n^{m+1} = [2] \sum_{k=0}^{\lfloor m/2 \rfloor} h_{m-2k}([1,q]^{k+1}) S_{2m-2k+1,n}(q) q^{-n(m-k+1)}.
\]

Plugging \(5\) in Equation \(15\), the right-hand side becomes

\[
\sum_{k=0}^{\lfloor m/2 \rfloor} \sum_{l=0}^{m-k} h_{m-2k}([1,q]^{k+1}) (-1)^{m-k-l} \frac{[l]!}{[m-k-l]!} P_{m-k,m-k-l}(q) X_n^{l+1}.
\]

Comparing the coefficients of \( X_n^{l+1} \) we see that \((h_{2m-k}([1,q]^{k-m+1}))_{0 \leq k,m \leq n}\) and \(14\) are indeed inverses.
• Summing Equation (11) over \( l \) from 1 to \( n \) and applying Equation (3), we obtain

\[
\frac{1 - q^{n+\frac{1}{2}}}{(1 - q^2)q^\frac{3}{2}} X_n^m = \sum_{k=0}^{[m/2]} \sum_{l=1}^{m-k} c_{m,m-k}(q^\frac{1}{2}) S_{2m-2k,n}(q) q^{-n(m-k+\frac{1}{2})}. \tag{19}
\]

Substituting (3) into (19) and dividing both sides by \( \frac{1 - q^{n+\frac{1}{2}}}{(1 - q^2)q^\frac{3}{2}} \), we get

\[
X_n^m = \sum_{k=0}^{[m/2]} \sum_{l=1}^{m-k} c_{m,m-k}(q^\frac{1}{2})(-1)^{m-k-l} \frac{(1 - q^2)^{m-k-l} Q_{m-k,m-k-l}(q^\frac{1}{2})}{\prod_{i=0}^{m-k-l}(1 - q^{m-k+i+\frac{1}{2}})} X_n^l. \tag{20}
\]

Comparing the coefficients of \( X_n^l \), we see that \( (c_{k,m}(q))_{1 \leq k,m \leq n} \) and (15) are indeed inverses.

• Equation (12) may be written as

\[
(-1)^{n-l} X_l^m - (-1)^{n-l+1} X_{l-1}^m = (-1)^{n-l} \sum_{k=0}^{[m/2]} g_{m,m-k}(q) \frac{(1 - q^l)^2 m-k}{(1 - q)^{2m-k-2k}} q^{-l(m-k)}. \tag{20}
\]

Summing Equation (20) over \( l \) from 1 to \( n \) and applying Equation (4), we obtain

\[
X_n^m = \sum_{k=0}^{[m/2]} g_{m,m-k}(q) T_{2m-2k,n}(q) q^{-n(m-k)}. \tag{21}
\]

Substituting (7) into (21), the right-hand side becomes

\[
\sum_{k=0}^{[m/2]} \sum_{l=1}^{m-k} g_{m,m-k}(q)(-1)^{m-k-l} \frac{G_{m-k,m-k-l}(q)}{\prod_{i=0}^{m-k-l}(1 + q^{m-k+i})} X_n^l. \tag{22}
\]

Comparing the coefficients of \( X_n^l \), we see that \( (g_{k,m}(q))_{1 \leq k,m \leq n} \) and (16) are inverse to each other.

• Equation (13) may be written as

\[
(-1)^{n-l} \frac{1 - q^{l+\frac{1}{2}}}{(1 - q^2)q^\frac{3}{2}} X_l^{m-1} - (-1)^{n-l+1} \frac{1 - q^{l-\frac{1}{2}}}{(1 - q^2)q^\frac{3}{2}} X_{l-1}^{m-1} = (-1)^{n-l} \sum_k d_{m,m-k}(q^\frac{1}{2})[q^{2(m-k-\frac{1}{2})}] q^{-l(m-k-\frac{1}{2})}. \tag{23}
\]

Summing Equation (23) over \( l \) from 1 to \( n \) and applying Equation (4), we obtain

\[
\frac{1 - q^{n+\frac{1}{2}}}{(1 - q^2)q^\frac{3}{2}} X_n^{m-1} = \sum_k d_{m,m-k}(q^\frac{1}{2}) T_{2m-2k-1,1}(q) q^{-n(m-k-\frac{1}{2})}, \quad m \geq 2. \tag{24}
\]
Substituting (26) into (24) yields
\[
\frac{1 - q^{n+\frac{1}{q}}}{(1 - q^2)q^{\frac{n}{q}}}
\left( X_{n}^{m-1} - \sum_{k=1}^{m-k} \sum_{l=1}^{m-k-l} (-1)^{m-k-l} d_{m,m-k}(q^{\frac{l}{2}}) H_{m-k,m-k-l}(q^{\frac{l}{2}}) X_{n}^{l-1}\right)
\]
\[
= (-1)^{n} \sum_{k} \frac{(-1)^{m-k} d_{m,m-k}(q^{\frac{k}{2}}) H_{m-k,m-k-1}(q^{\frac{k}{2}})}{(1 + q^{\frac{k}{2}}) H_{m-k,m-k-1}(1 + q^{m-k-1})}.
\]  (25)

We now show that the right-hand side of (25) must vanish. Suppose $0 < q < 1$. Denote the left-hand side of (25) by $L_n$. If there exists an $n \in \mathbb{N}$ such that $L_n = 0$ we are done. Suppose $L_n \neq 0$ for all $n \geq 1$, then $L_n$ is a rational function in $t = q^{\frac{n}{2}}$ and can be written as
\[
L_n = t^s f(t) \quad \text{with} \quad t = q^{\frac{n}{2}},
\]
where $s$ is an integer and $f(t)$ a rational function with $f(0) \neq 0$. Since $f(q^{\frac{n}{2}}) \neq 0$, the right-hand side of (25) implies that
\[
f(q^{\frac{n}{2}}) f(q^{\frac{n-s}{2}}) < 0 \quad \forall n \geq 1.
\]
Taking the limit as $n \to \infty$ we get $(f(0))^2 \leq 0$, which is impossible. Hence $L_n = 0$ and (25) reduces to
\[
X_{n}^{m-1} = \sum_{k} d_{m,m-k}(q^{\frac{k}{2}}) \sum_{l=1}^{m-k-l} \frac{(-1)^{m-k-l} H_{m-k,m-k-l}(q^{\frac{l}{2}}) X_{n}^{l-1}}{(1 + q^{\frac{l}{2}}) H_{m-k,m-k-l}(1 + q^{m-k-1})}.
\]  (26)

Comparing the coefficients of $X_{n}^{l-1}$ on both sides of (25), we see that $(d_{k,m}(q))_{1 \leq k,m \leq n}$ and (17) are indeed inverses.

The following easily verified result has been given by Gessel and Viennot [8].

**Lemma 2.4.** Let $(A_{ij})_{0 \leq i,j \leq m}$ be an invertible lower triangular matrix, and let $(B_{ij}) = (A_{ij})^{-1}$. Then for $0 \leq k \leq n \leq m$, we have
\[
B_{n,k} = \frac{(-1)^{n-k}}{A_{k,k} A_{k+1,k+1} \cdots A_{n,n}} |A_{k+i+1,k+j} |_{0 \leq i,j \leq n-k-1}.
\]

Using the above lemma we derive immediately from Theorem 2.3 the following determinant formulas:
\[
P_{m,k}(q) = \det_{0 \leq i,j \leq k-1} (h_{m-k-i+2j-1}(\{1, q\}^{i-j+2})),
\]  (27)
\[
Q_{m,k}(q) = \det_{0 \leq i,j \leq k-1} (c_{m-k+i+1,m-k+j}(q)),
\]  (28)
\[
G_{m,k}(q) = \det_{0 \leq i,j \leq k-1} (g_{m-k+i+1,m-k+j}(q)),
\]  (29)
\[
H_{m,k}(q) = \det_{0 \leq i,j \leq k-1} (d_{m-k+i+1,m-k+j}(q)).
\]  (30)
3. Combinatorial interpretations

A lattice path or path $s_0 \to s_n$ is a sequence of points $(s_0, s_1, ..., s_n)$ in the plane $\mathbb{Z}^2$ such that $s_i - s_{i-1} = (1, 0)$, $(0, 1)$ for all $i = 1, ..., n$. Let us assign a weight to each step $(s_i, s_{i+1})$ of $s_0 \to s_n$. We define the weight $N(s_0 \to s_n)$ of the path $s_0 \to s_n$ to be the product of the weights of its steps. Let $s_0 = (a, b)$ and $s_n = (c, d)$, if we weight each vertical step with $x$-coordinate $i$ by $x_i$ and all horizontal steps by 1 then

$$N(s_0 \to s_n) = h_{d-b}(x_a, x_{a+1}, ..., x_c).$$

Now consider two sequences of lattice points $u := (u_1, u_2, ..., u_n)$ and $v := (v_1, v_2, ..., v_n)$ such that for $i < j$ and $k < l$ any lattice path between $u_i$ and $v_j$ has a common point with any lattice path between $u_j$ and $v_k$. Set

$$N(u, v) := \sum N(u_1 \to v_1) \cdots N(u_n \to v_n),$$

where the sum is over all families of non-intersecting paths $(u_1 \to v_1, ..., u_n \to v_n)$.

The following remarkable result can be found in Gessel and Viennot [3]. For historical remarks see also Krattenthaler [6].

Theorem 3.1. [Lindström-Gessel-Viennot] We have

$$N(u, v) = \det_{1 \leq i,j \leq n} (N(u_j \to v_i)).$$

We are now ready to exhibit the combinatorial interpretation of the $q$-Faulhaber numbers.

Theorem 3.2. Let $u = (u_0, ..., u_{k-1})$ and $v = (v_0, ..., v_{k-1})$, where $u_i = (2i, -2i)$ and $v_i = (2i + 3, m - k - i - 1)$ for $0 \leq i \leq k - 1$.

(i) The polynomial $P_{m,k}(q)$ is the sum of the weights of $k$-non-intersecting paths from $u$ to $v$, where a vertical step with an even $x$-coordinate has weight $q$, and all the other steps have weight 1.

(ii) The polynomial $Q_{m,k}(q)$ is the sum of the weights of $k$-non-intersecting paths from $u$ to $v$, where the weight of the individual steps is the same as before with the exception that $q$ is replaced with $q^2$ and the vertical step starting from any $u_j$ has weight $q^2 + q$ instead of $q^2$.

Proof. For (i), by means of (31) we have

$$N(u_j \to v_i) = h_{m-k-i+2j-1}(\{1, q\}^{i-j+2}).$$

The result then follows from (31) and Theorem 3.1.

For (ii), assume that $u_j' = (2j+1, -2j)$ and $u_j'' = (2j, 1 - 2j)$. The first step of a lattice path from $u_j$ to $v_i$ is either $u_j \to u_j'$ or $u_j \to u_j''$. As $N(u_j \to u_j') = 1$, $N(u_j \to u_j'') = q^2 + q$ and $h_n(x_1, ..., x_{r-1}) + x_r h_{n-1}(x_1, ..., x_r) = h_n(x_1, ..., x_r)$, we have

$$N(u_j \to v_i) = N(u_j \to u_j') N(u_j' \to v_i) + N(u_j \to u_j'') N(u_j'' \to v_i)$$

$$= N(u_j' \to v_i) + (q^2 + q) N(u_j'' \to v_i)$$

$$= h_{m-k-i+2j-1}(\{1\}^{i-j+2}, \{q^2\}^{i-j+1})$$

$$+ (q^2 + q) h_{m-k-i+2j-2}(\{1, q^2\}^{i-j+2})$$

$$= h_{m-k-i+2j-1}(\{1, q^2\}^{i-j+2}) + q h_{m-k-i+2j-2}(\{1, q^2\}^{i-j+2})$$.
The result then follows from \(28\) and Theorem 3.1.

\[\square\]

**Corollary 3.3.** The polynomials \(P_{m,k}(q)\) and \(Q_{m,k}(q)\) have symmetric coefficients.

**Proof.** A combinatorial way to see the symmetry of the coefficients of \(P_{m,k}(q)\) is as follows: Modifying the weights such that vertical steps with an odd \(x\)-coordinate have weight \(q\) and all the others weight 1 does not change the entries of the determinant.

However, consider any given family of paths with weight \(q^w\), when vertical steps with even \(x\)-coordinate have weight \(q^w\). After the modification of the weights it will have weight \(q^{\text{max} - w}\), where max is the total number of vertical steps in such a family of paths, which implies the claim.

For the polynomials \(Q_{m,k}\), we use the following alternative weight: vertical steps with odd \(x\)-coordinate have weight \(q^2\), vertical steps with starting point \(u_i\) have weight \(1 + q\) and all others have weight 1. \[\square\]

When \(k = m - 1\), there is only one lattice path from \(u_0 = (0, 0)\) to \(v_0 = (3, 0)\), which has weight 1. This establishes the following result:

**Corollary 3.4.** For \(m \geq 2\), we have \(P_{m,m-1}(q) = P_{m,m-2}(q)\) and \(Q_{m,m-1}(q) = Q_{m,m-2}(q)\).

For the combinatorial interpretation of the \(q\)-Salié numbers, we need an auxiliary lemma:

**Lemma 3.5.** Let \((A_{ij})_{1 \leq i,j \leq n}\) and \((B_{ij})_{1 \leq i,j \leq n}\) be two matrices. Then

\[
\det_{1 \leq i,j \leq n} (A_{ij} + B_{ij}) = \sum_{I \subseteq \{1, \ldots, n\}} \det_{1 \leq i,j \leq n} (D_{ij}^{(I)}),
\]

where

\[
D_{ij}^{(I)} = \begin{cases} 
A_{ij}, & \text{if } j \in I, \\
B_{ij}, & \text{otherwise}. 
\end{cases}
\]

**Theorem 3.6.** Let \(u = (u_0, \ldots, u_{k-1})\) and \(v = (v_0, \ldots, v_{k-1})\), where \(u_i = (2i, -2i)\) and \(v_i = (2i + 2, m - k - 1 - i)\) for \(0 \leq i \leq k - 1\).

(i) The polynomial \(G_{m,k}(q)\) is the sum of the weights of \(k\)-non-intersecting lattice paths \(L\) from \(u\) to \(v\) with the weight of \(L\) being

\[
\sum_{I \subseteq \{0, 1, \ldots, k-1\}} w_I(L),
\]

where \(w_I\) is defined as follows: for each \(i \in I\), vertical steps with \(x\)-coordinate \(2i - 1\) have weight \(q\), and for any integer \(i \notin I\), vertical steps with \(x\)-coordinate \(2i\) have weight \(q\). All other steps have weight 1.

(ii) The polynomial \(H_{m,k}(q)\) is the sum of the weights of \(k\)-non-intersecting lattice paths \(L\) from \(u\) to \(v\), with the weight of \(L\) being

\[
\sum_{I \subseteq \{0, 1, \ldots, k-1\}} \overline{w}_I(L),
\]

where \(\overline{w}_I\) is the same as \(w_I\) – replacing \(q\) with \(q^2\) – with the exception of vertical steps starting from one of the points \(u_i\), which have an additional weight of \(q\). More precisely,
Suppose that $j$ is the weighted sum of lattice paths from $u$ to $v$. Meanwhile, for $j$, the sum of weights of lattice paths from $u$ to $v$ is the sum of weights of lattice paths from $u$ to $v$, where the vertical steps have the weight given in the claim. To end note, that $h_{m-k+i+2j-1}(1)^{i-j+2}, q^{i-j+1})$ counts lattice paths from $u_j$ to $v_i$, when steps on $i-j+1$ given vertical lines have weight $q$, those steps on the remaining $i-j+2$ vertical lines have weight 1.

By the construction in the claim, steps on exactly one of the vertical lines with $x$-coordinates $2r - 1$ and $2r$ have weight $q$. Since $j \in I$, steps on the vertical line with $x$-coordinate $2j$, i.e., with the $x$-coordinate of $u_j$, have weight 1.

Similarly, if $j \notin I$ we can verify that there are exactly $i-j+2$ vertical lines between $u_j$ and $v_i$ with steps thereon having weight $q$.

(ii) In the same way, we can show that for $j \in I$ and $0 \leq i \leq k-1$.

$$h_{m-k+i+2j-1}(1)^{i-j+2}, q^{i-j+1}) + q\cdot h_{m-k+i+2j-2}(1)^{i-j+2}, q^{i-j+1})$$

is the sum of weights of lattice paths from $u_j$ to $v_i$, where the vertical steps have the weight given in the claim. Meanwhile, for $j \notin I$ and $0 \leq i \leq k-1$,

$$h_{m-k+i+2j-1}(1)^{i-j+1}, q^{i-j+2}) + q\cdot h_{m-k+i+2j-2}(1)^{i-j+1}, q^{i-j+2})$$

is the sum of weights of lattice paths from $u_j$ to $v_i$. □

As an illustration of the underlying configurations in Theorem 3.6, we give an example in Figure 1 for $m = 7$ and $k = 4$.

**Corollary 3.7.** The polynomials $G_{m,k}(q)$ and $H_{m,k}(q)$ have symmetric coefficients.

**Proof.** A combinatorial way to see the symmetry of the coefficients of $G_{m,k}(q)$ is as follows: Modifying $w_i$ such that for each $i \in I$, vertical steps with $x$-coordinate $2i$ have weight $q$, and for any integer $i \notin I$, vertical steps with $x$-coordinate $2i - 1$ have weight 1 does not change the entries of the determinant.

However, consider any given family of paths with weight $q^w$ with weight by Theorem 3.6(i). After the modification of the weights it will have weight $q^{\max-u}$, where max is the total number of vertical steps in such a family of paths, which implies the claim.

We omit the proof of the symmetry of the coefficients of $H_{m,k}(q)$. □

**Corollary 3.8.** Let $u = (u_0, \ldots, u_{k-1})$ and $v = (v_0, \ldots, v_{k-1})$, where $u_i = (2i, -2i)$ and $v_i = (2i + 2, m - k - 1 - i)$ for $0 \leq i \leq k-1$.

(i) The polynomial $G_{m,k}(q)$ is the sum of the weights of $k$-non-intersecting lattice paths $L$ from $u$ to $v$ with the weight of $L$ being

$$q^{\sigma_k(L)} \prod_{i=0}^{k-1} \left(q^{\sigma_{2i-1}(L)} + q^{\sigma_{2i}(L)}\right),$$

where $\sigma_j$ denotes the number of vertical steps with $x$-coordinate $j$.
Figure 1. Example for $w_I$ in Theorem 3.6.

\[ I = \{1, 2\}, \quad w_I(L) = q^8 \text{ and } \bar{w}_I(L) = q^{14}(q + q^2)(q + 1)^2 \]

(ii) The polynomial $H_{m,k}(q)$ is the sum of the weights of $k$-non-intersecting lattice paths $L$ from $u$ to $v$ with the weight of $L$ being

\[
(1 + q)^{f(L)} q^{2\sigma_{2k}(L)} \prod_{i=0}^{k-1} \left( q^{2\sigma_{2i-1}(L)} + q^{2\sigma_{2i}(L) - f_i(L)} \right),
\]

where $\sigma_j$ is as in (i) and $f$ (resp. $f_i$) denotes the number of vertical steps starting from $u$ (resp. $u_i$).

Proof. (i) By the definition of $w_I$, for $0 \leq i \leq k - 1$, if $i \in I$, then vertical steps on the line with $x$-coordinates $2i - 1$ have weight $q$ and vertical steps on the line with $x$-coordinates $2i$ have weight 1; and if $i \notin I$, the case is just contrary. Note that steps on the vertical line with $x$-coordinates $2k$ always have weight $q$ and steps on the vertical line with $x$-coordinates $2k - 1$ always have weight 1. This implies that

\[
\sum_{I \subseteq \{0, 1, \ldots, k-1\}} w_I(L) = q^{\sigma_{2k}(L)} \prod_{i=0}^{k-1} \left( q^{\sigma_{2i-1}(L)} + q^{\sigma_{2i}(L)} \right).
\]
(ii) Notice that for \(0 \leq i \leq k - 1\), we have \(f_i(L) = 1\) if \(L\) contains a vertical step starting from the point \(u_i\), and \(f_i(L) = 0\) otherwise. Similarly, we have

\[
\sum_{I \subseteq \{0,1,\ldots,k-1\}} \overline{w}_I(L) = q^{2\sigma_{2k}(L)} \prod_{i=0}^{k-1} \left( q^{2\sigma_{2i-1}(L)} (1 + q) f_i(L) + q^{2\sigma_{2i}(L)-2f_i(L)} (q^2 + q) f_i(L) \right),
\]

\[
= (1 + q) f(L) q^{2\sigma_{2k}(L)} \prod_{i=0}^{k-1} \left( q^{2\sigma_{2i-1}(L)} + q^{2\sigma_{2i}(L)-f_i(L)} \right).
\]

This completes the proof.

The computation of \(G_{4,2}(q)\) is illustrated in Figure 2 while the value of \(H_{4,2}(q)\) as given in Table 4 is computed in Table 5.

**Table 5.** Values of \(\sum_{I \subseteq \{0,1,\ldots,k-1\}} \overline{w}_I(L)\) corresponding to Figure 2

| \(1 + q)q(1 + q^2)\) | \(2q^2(1 + q^2)\) | \(1 + q^2\) | \(2q(1 + q^2)\) |
|-------------------------|------------------|-----------|----------------|
| \(2(1 + q)(1 + q^2)\)  | \(q^2(1 + q)^2\) | \(2q^2(1 + q^2)\) | \(2q^2(1 + q)(1 + q^2)\) |
| \(2(1 + q)^2\)         | \(2(1 + q^2)^2\) | \(2(1 + q^2)^2\) | \(2q^2(1 + q)^2\) |
| \(2q^2(1 + q^2)\)      | \(2q^2(1 + q^2)\) | \(2q^2(1 + q)^2\) | \(2q^2(1 + q^2)\) |
| \(2q^2(1 + q^2)\)      | \(2q^2(1 + q^2)\) | \(2q^2(1 + q)^2\) | \(2q^2(1 + q^2)\) |

**Remark.** Since

\[
\det (A_{ij} + B_{ij}) = \sum_{I \subseteq \{1,\ldots,n\}} \det (C_{ij}^{(I)}),
\]

where

\[
C_{ij}^{(I)} = \begin{cases} A_{ij}, & \text{if } i \in I, \\ B_{ij}, & \text{otherwise}, \end{cases}
\]

we may also define \(w_I\) in Theorem 3.6(i) as follows: for each \(i \in I\), vertical steps with \(x\)-coordinate \(2i + 3\) have weight \(q\), and for any integer \(i \notin I\), vertical steps with \(x\)-coordinate \(2i + 2\) have weight \(q\). All other steps have weight 1. In this case, for each \(i \in I\) and \(0 \leq j \leq k - 1\), we can show that \(h_{m-k-i+2j-1}(\{1\}^{i-j+2}, \{q\}^{i-j+1})\) is the weighted sum of lattice paths from \(u_j\) to \(v_i\). Moreover,

\[
\sum_{I \subseteq \{0,1,\ldots,k-1\}} w_I(L) = q^{\sigma_0(L)} \prod_{i=1}^{k} \left( q^{\sigma_{2i}(L)} + q^{\sigma_{2i+1}(L)} \right).
\]

Similarly, we may define \(\overline{w}_I\) in Theorem 3.6(ii) as follows: for each \(i \in I\), a vertical step toward the point \(v_i\) has weight \(q + 1\), vertical steps with \(x\)-coordinate \(2i + 3\) have weight \(q^2\). For any integer \(i \notin I\), a vertical step toward the point \(v_i\) has weight \(q^2 + q\), and vertical steps with \(x\)-coordinate \(2i + 2\) not toward \(v_i\) have weight \(q^2\). All other steps have weight 1. In this case, we have

\[
\sum_{I \subseteq \{0,1,\ldots,k-1\}} \overline{w}_I(L) = (1 + q)^{\overline{f}(L)} q^{2\sigma_0(L)} \prod_{i=1}^{k} \left( q^{2\sigma_{2i}(L)-\overline{f}_i(L)} + q^{2\sigma_{2i+1}(L)} \right),
\]

where \(\overline{f}\) (resp. \(\overline{f}_i\)) denotes the number of vertical steps ending in \(v\) (resp. \(v_i\)).
Figure 2. An illustration for \( G_{4,2}(q) = 10q^3 + 24q^2 + 24q + 10 \).

When \( k = m - 1 \), there is only one lattice path from \( u_0 = (0,0) \) to \( v_0 = (2,0) \), which has weight 1. This establishes the following result:

**Corollary 3.9.** \( G_{m,m-1}(q) = 2G_{m,m-2}(q) \) and \( H_{m,m-1}(q) = 2H_{m,m-2}(q) \).
4. Open problems

We would like to point out three directions of possible further research: It appears that the polynomials $P_{m,k}$ and $G_{m,k}$ are log-concave, however, we did not pursue this question further. Note that the polynomials $Q_{m,k}$ and $H_{m,k}$ are not even unimodal.

Victor Guo and Jiang Zeng gave in [4] even finer $q$-analogues of the polynomials considered here, replacing (3) and (4) by

$$S_{m,n,r}(q) = \sum_{k=1}^{n} \left[ \frac{2rk}{2r} \right] m_{k-1} q^{m+2r-1} \left( n-k \right),$$

$$T_{m,n,r}(q) = \sum_{k=1}^{n} (-1)^{n-k} \left[ \frac{(2r-1)k}{2r-1} \right] m_{k-1} q^{m+2r-1} \left( n-k \right),$$

where $r \geq 1$.

Although the coefficients of the corresponding polynomials $P_{m,k,r}$, $Q_{m,k,r}$, $G_{m,k,r}$ and $H_{m,k,r}$ are not positive anymore, one might hope for a refinement of Theorem 2.3.

Finally, we should point out that Ira Gessel and Xavier Viennot [3] also presented nice generating functions for their coefficients $f_{m,k}$ and $s_{m,k}$, namely

$$\sum_{m,k} s_{m,k} x^{2n} \left( \frac{k}{2n} \right) \frac{t^{k}}{(2n)!} = \frac{\cosh \sqrt{1 + 4t} - \cosh \frac{\sqrt{1 + 4t}}{2}}{t \sinh \frac{\sqrt{1 + 4t}}{2}}.$$

It would be interesting to find the corresponding refinements.

5. Epilogue

One may wonder how these results were discovered. The truth is, that at first “only” formula [3] was known. Using this formula, Table 1 was computed. Then, in analogy to [3], the matrix

$$\left( (-1)^{k-m} \left[ \frac{m}{k+1} \right] P_{k,k-m}(q) \right)_{0 \leq k,m \leq n}$$

was inverted and, since we were looking for a lattice path interpretation, the entry in row $i$ and column $j$ of the inverse matrix had to be the weighted number of lattice paths from $u_j$ to $v_i$. This given, it was easy to find the correct weights. Finally, we read the proof given in [3] backwards, its first line corresponding to our Lemma 2.2.

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