Random time averaged diffusivities for Lévy walks

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Abstract. We investigate a Lévy-Walk alternating between velocities \( \pm v_{0} \) with opposite sign. The sojourn time probability distribution at large times is a power law lacking its mean or second moment. The first case corresponds to a ballistic regime where the ensemble averaged mean squared displacement (MSD) goes linearly with time, \( \langle x^2 \rangle \propto t^2 \), the latter to enhanced diffusion with \( \langle x^2 \rangle \propto t^\nu \), \( 1 < \nu < 2 \). The correlation function and the time averaged MSD are calculated. In the ballistic case, the deviations of the time averaged MSD from a purely ballistic behavior are shown to be distributed according to a Mittag-Leffler density function. In the enhanced diffusion regime, the fluctuations of the time averages MSD vanish at large times, yet very slowly. In both cases we quantify the discrepancy between the time averaged and ensemble averaged MSDs.

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1 Introduction

Dispersion of Brownian particles is described on the macroscopic level by Fick’s second law which states that the local change in particle density is proportional to the negative gradient of the local particle flux, where the diffusion constant \( D \) is the proportionality factor. Accordingly, the density of the particles is a spreading Gaussian with the mean squared displacement (MSD) going linearly with time, \( \langle x^2 \rangle = 2Dt \). Einstein [1] established a relationship between the macroscopic quantity \( D = \langle (\partial x)^2 \rangle / 2\langle \tau \rangle \) and the underlying stochastic process with jump lengths \( \partial x \) of variance \( \langle (\partial x)^2 \rangle \) and the average time passing between two jumps \( \langle \tau \rangle \). For a single Brownian particle, the time averaged MSD (TAMSD) \( \overline{\delta^2} \) has to be considered,

\[
\overline{\delta^2} = \frac{1}{t-\Delta} \int_{0}^{t-\Delta} [x(t'+\Delta) - x(t')]^2 \, dt'.
\]

(1)

Here the averaging was performed over all displacements along the single trajectory occurring during a fixed lag time \( \Delta \) within the measurement time \( t \). Due to the stationary increments, for a Brownian motion the time averaged MSD will attain the limit \( \overline{\delta^2} = 2D\Delta \) for large times \( t \). Brownian motion is therefore ergodic, ensemble and time averages are equal, \( \langle x^2 \rangle = \overline{\delta^2} \). It is important to note that in practice an estimation of ensemble averaged diffusion constants from single particle trajectories even for normal diffusion is a subtle issue due to the lack of statistics. However one can exploit the ergodic property of such processes in order to find suitable estimators [2], [3].

Unlike in normal diffusion, in strongly disordered media the mean squared displacement often does not grow linearly with time, but according to a power law \( \langle x^2 \rangle \propto t^\nu \). The case \( 0 < \nu < 1 \) corresponds to subdiffusion, \( 1 < \nu < 2 \) indicates enhanced diffusion or superdiffusion. In disordered and complex systems time averages can differ considerably from the corresponding ensemble averages. The increasing employment of single particle tracking techniques makes it indispensable to understand these differences in order to interpret the experiments and gain insight to the mechanisms underlying anomalous transport.

Examples for enhanced diffusion are experiments on active transport of microspheres [4], of polymeric particles [5] or of pigment organelles [6] in living cells. Enhanced diffusion in living cells is promoted by molecular motors that move along microtubules or the cytoskeleton [7], [8]. In vivo experiments usually examine the trajectories \( x(t) \) of single particles and hence assess the TAMSD Eq. (1) instead of ensemble averages. Measurements often find this quantity to be a random variable, so that ensemble average and (single trajectory) time average differ. For this behavior several reasons come into question, either variations in the probed environments or cells, ergodicity breaking or too short measurement times. In Refs. [4], [6] the observed enhanced diffusion was described within the framework of generalized Langevin equations (GLE), which is Gaussian and ergodic [9]. Using this theory, the experimentally observed exponents characterizing the anomalous transport were reproduced. However, this Gaussian approach cannot explain the multiscaling of moments found for the enhanced motion of polymeric particles in living cells [5], a feature that seems to be more consistent with a Lévy walk scheme.
Lévy flights describe enhanced diffusion in terms of random walk processes where the distribution of the particle displacements lack the second or even the first moment (i.e. give rise to a Lévy statistics). Lévy flights have been used in the past to describe phenomena as diverse as the dispersal of bank notes [10], tracer diffusion in systems of breakable elongated micelles [11] or animal foraging patterns [12]. However, Lévy Flight models can lead to unphysical behavior with regard to velocities since Lévy flights are characterized by extremely large jump lengths, corresponding to the heavy tails in the jump length distributions. Lévy walks address finite velocities by either penalizing long instantaneous jumps with long resting times (jump models), or by letting the particles move at a certain velocity for a certain time or displacement, and choosing a new direction (or velocity) and sojourn time according to given probabilities (velocity models) [13, 14].

The theory of Lévy walks finds a wide range of applications. Experiments with passive tracer particles in a laminar flow have shown that the flight times and hence displacements within the resultant chaotic trajectories of the tracer particles can exhibit power-law distributions [15]. Likewise, the motion of tracers in turbulent flows can be described as Lévy walks [16]. Another example is the stochastic description of on- off-times in blinking quantum dots, where the intensity corresponds to the velocity of a particle alternating between states of zero and constant nonzero velocity. Nonergodic behavior was found in the correlation functions for sojourn time distributions lacking their mean [17],[18]. Another example is the dynamics of cold atoms in optical traps [19] and the related Brownian motion in shallow logarithmic potentials [20], or perturbation spreading in many-particle systems [21]. Moreover, also deterministic systems such as certain classes of iterated nonlinear maps may show enhanced diffusion. The chaotic behavior of resistively shunted Josephson junctions manifests itself in an anomalous (deterministic) phase diffusion, which can therefore be modelled by means of such maps [22]. In turn, the enhanced diffusion behavior emerging from such iterated maps can be modelled stochastically using the Lévy walk approach (in particular velocity models) [13].

In this article we study Lévy walks in one dimension where the persistence times in the positive- or negative- velocity state are drawn according to a probability density function $\psi(\tau)$ with either first or second moment lacking. The first case is referred to as the ballistic case, the latter as the subballistic or enhanced case. Recently, the dynamics emerging from nonlinear map similar to the Lévy walk was investigated [23]. However, this work did not address the fluctuations of the TAMSD. Fluctuations were considered in a very recent numerical study for the special case of Lévy walks exhibiting enhanced diffusion [24] and numerically and analytically for enhanced and ballistic case in a brief publication of the authors [25].

The article is divided into four parts. The first one is dedicated to the ballistic case of a Lévy walk, the second one to a Lévy flight with a step size distribution lacking the second moment, and the third one to the enhanced Lévy walk case. For both the ballistic and the enhanced case we first review briefly occupation times, propagators and ensemble averaged mean squared displacements. Then we turn to the ensemble averaged quantities such as correlation functions and ensemble averages of the TAMSDs. Finally, we investigate the distributions and properties of the fluctuations of the TAMSDs. In the second part, we investigate for comparison the Lévy flight corresponding to the enhanced case. We also provide the fluctuations of time averages of Lévy flights, using simple arguments, thus adding to the work in [23] who addressed this problem rigorously and more generally.

2 Ballistic regime

We consider a one-dimensional motion of a particle with a two-state-velocity $\pm v_0$ where the sojourn times in the states are drawn from a probability density function (PDF) $\psi(\tau)$. Hence the particle has a velocity $+ v_0$ for period $\tau_1$ drawn from $\psi(\tau)$, after that switches to velocity $- v_0$ and remains in this state for another period $\tau_2$ also drawn from $\psi(\tau)$. This process is then renewed. In particular, this PDF is chosen such that it lacks its first moment with a power-law decay at large times, $\psi(\tau) \sim A/\Gamma(-\alpha)\tau^{-1-\alpha}$ with $0 < \alpha < 1$. Particularly in the simulations we will use

$$\psi(\tau) = \begin{cases} \alpha\tau^{-1-\alpha} & \tau \geq 1 \\ 0 & \text{else} \end{cases} .$$

The Laplace transform of Eq. (2) in the small-$u$-limit is

$$\tilde{\psi}(u) \simeq 1 - Au^\alpha$$

with $A = \Gamma(1-\alpha)$ and $u$ being the Laplace conjugate of $\tau$. This relation can easily be derived via Laplace transformation of $\int_0^\infty \psi(\tau)d\tau \simeq 1 - \tau^{-\alpha}$, which yields $\frac{1}{\alpha}(1 - \Gamma(1-\alpha)u^\alpha)$ by using convolution and Tauberian theorems.

In the limit of long times $t$ the particle moves ballistically so that the mean-squared displacement is $\langle x^2 \rangle = (1 - \alpha)t^2$ [23, 27]. Recently, a similar system had been generated in the context of deterministic superdiffusion and the ensemble average of the time averaged mean squared displacement $\langle \delta^2 \rangle$ was derived [23]. Processes whose temporal dynamics is governed by heavy-tailed PDFs lacking their mean exhibit ageing [28] which manifests itself in a deviation of the ensemble averaged mean-squared displacement from the TAMSDs which are random themselves. Therefore, the fluctuations of TAMSDs are an important signature of this process.

2.1 Occupation fraction and particle position

The distribution for the fraction of time $z_k = t_k/t$ spent in one state (positive or negative velocity) after a large
time $t$ is given by Lamperti’s law \[29\],
\[
p_{occ}(z_\pm) = \frac{\sin \pi \alpha}{\pi} \frac{z_\pm^{\alpha-1} (1 - z_\pm)^{\alpha-1}}{z_\pm^{2\alpha} + (1 - z_\pm)^{2\alpha} + 2 \cos \pi \alpha z_\pm^2 (1 - z_\pm)^\alpha}
\] (4)
a generalization of the arcsine law which is reproduced for the case $\alpha = 1/2$. The particle position $x(t)$ is given by the integral over the velocities $\int_0^t v(t') dt'$, and the temporal mean of the velocity is $x/t = v_0(t_+ - t_-)/t$. The distribution of this quantity in the limit of large times had been calculated in the context of the mean magnetization of a two-state system \[33\]. A related problem is the calculation of the integral over the intensity of emitted light in blinking quantum dots. In that case one state corresponds to the “on”-state, i.e. the emitting state, and the other state to the “off”-state where no light is emitted \[17\].

The probability to find the particle at position $x$ at time $t$ for large times can be obtained in terms of the scaling variable $z = x/(v_0 t)$ by using Eq. (4) and change of variables. We have $x/v_0 = 2z_+ - 1$, hence $\frac{dx}{dz} = 2$ and
\[
p(z) = \left| \frac{dz}{dx} \right| p_{occ}(z_+) = \frac{2 \sin \pi \alpha (1 - z^2)^{\alpha-1}}{\pi (1 + z^{2\alpha} + (1 - z)^{2\alpha} + 2 \cos \pi \alpha (1 - z^2)^\alpha)}
\] (5)
Fig. 1 shows this distribution of the scaled particle position for two different values of $\alpha$.

![Distribution of scaled particle position](image)

Fig. 1: Distribution of the position of particles at $t = 10^7$ for $\alpha = 0.5$ (left) and $\alpha = 0.7$ (right panel) in terms of the scaling variable $z$ ($v_0 = 1$). Sample size $10^4$, $\psi(t)$ is given by Eq. (5).

### 2.2 Ensemble average of $\langle \delta x^2 \rangle$

First we will analyze the ensemble averaged TAMSD:
\[
\langle \delta x^2 \rangle = \frac{1}{t - \Delta} \left( \int_0^{t - \Delta} [x(t' + \Delta) - x(t')]^2 dt' \right). 
\] (6)

Changing the order of integration and ensemble averaging in Eq. (6), we get
\[
\langle \delta x^2 \rangle = \int_0^{t - \Delta} \frac{\langle x^2(t' + \Delta) \rangle + \langle x^2(t') \rangle - 2\langle x(t')(t' + \Delta) \rangle}{t - \Delta} dt'. 
\] (7)

In order to find $\langle \delta x^2 \rangle$, we first derive the Lévy walk correlation function $\langle x(t_1)x(t_2) \rangle$, which turns out to exhibit ageing. The position correlation function $\langle x(t_1)x(t_2) \rangle$ is related to the velocity correlation function $\langle v(s_1)v(s_2) \rangle$ via
\[
\langle x(t_1)x(t_2) \rangle = \int_0^{t_1} v(s_1) ds_1 \int_0^{t_2} v(s_2) ds_2
\]
\[
= \int_0^{t_1} ds_1 \int_0^{t_2} \langle v(s_1)v(s_2) \rangle ds_2
\]
\[
+ \int_0^{t_1} ds_2 \int_0^{t_2} \langle v(s_1)v(s_2) \rangle ds_1
\] (8)
where we took into account that $t_2 > t_1$. Using the approach of Ref. \[29\] we obtain for the velocity correlation function $\langle v(t_1)v(t_2) \rangle$
\[
\langle v(t_1)v(t_2) \rangle = v_0^2 \sum_{n=0}^{\infty} (-1)^n p_n(t_1, t_2)
\] (9)
where $p_n(t_1, t_2)$ is the probability of the velocity to switch its sign $n$ times within the time interval $[t_1, t_2]$, i.e. for even $n$ between $t_1$ and $t_2$ we have $v(t_1)v(t_2) = v_0^2$, and for odd $n$, $v(t_1)v(t_2) = -v_0^2$.

In the scaling limit where $t_1$ and $t_2$ are large and Eq. (9) applies, the particle gets stuck in the $+$ or $-$ state for times of the order of the measurement time due to the lacking first moment of the sojourn time distribution $\psi(t)$. Therefore, only the first term $n = 0$ is relevant in Eq. (9). The corresponding probability $p_0(t_1, t_2)$ of the velocity not switching its sign from a given $t_1$ on up to $t_2$ is called the persistence probability, to which the velocity correlation function is proportional, $\langle v(t_1)v(t_2) \rangle = v_0^2 p_0(t_1, t_2)$. Let us denote the first waiting time from an arbitrary time $t_1$ up to the next switching event by $\tau_f$, see Fig. 2. This first waiting time is called the forward recurrence time, and its PDF $\psi_{f,t_1}(\tau_f)$ differs from $\psi(t)$ since $t_1$ does not necessarily coincide with a renewal event.

![Sketch of forward recurrence time](image)

Fig. 2: Sketch of the forward recurrence time $\tau_f$. Ticks symbolize renewal events.

With $t_2 = t_1 + \Delta$, we express the persistence probability as \[30\]
\[
p_0(t_1, t_1 + \Delta) = \int_\Delta^\infty \psi_{f,t_1}(\tau_f) d\tau_f.
\] (10)
In terms of the scaling variable $\theta = \Delta/t_1$, the pdf of the forward recurrence time to take a value of $\Delta$ at time $t_1$ reads in the scaling limit where also $\Delta$ is large:

$$\lim_{t_1 \to \infty} \psi_f(\theta) = \frac{\sin \pi \alpha}{\pi} \frac{1}{\theta^{\alpha}(1 + \theta)} \tag{11}$$

The limit theorem for forward recurrence times $\psi_{f,t_1}$ Eq. (11) is due to Dynkin [31]. Hence,

$$\langle v(t_1) v(t_1 + \Delta) \rangle = v^2_0 \int_0^\infty \frac{\sin \pi \alpha}{\pi} \frac{1}{\theta^{\alpha}(1 + \theta)} d\theta = v^2_0 \int_0^\infty \frac{\sin \pi \alpha}{\pi} \frac{1}{\xi^{\alpha}(1 - \xi)^{1-\alpha}} d\xi$$

$$= v^2_0 \frac{\sin \pi \alpha}{\pi} B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right) \tag{12}$$

where $B(y; a, b) = \int_0^y du u^{a-1}(1 - u)^{b-1}$ denotes the incomplete Beta-function [32]. Note that this expression yields only real values for $t_2 \geq t_1$. In the case of $t_1 > t_2$, the $t_1$ and $t_2$ in Eq. (12) have to be interchanged. Inserting Eq. (12) into Eq. (8) and using integration by parts we find

$$\langle x(t_1) x(t_2) \rangle = v^2_0 \sin \pi \alpha \pi \left[ t_1 t_2 B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right) - \frac{1}{2} t_1^2 B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right) - \frac{1}{2} t_2^2 B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right) - \frac{\alpha}{2} \frac{v^2_0}{t_1} \right] \tag{13}$$

In particular, for $t_1 = t_2$ we find the MSD

$$\langle x^2(t_1) \rangle = (1 - \alpha) v^2_0 \frac{t_1^2}{2}, \tag{14}$$

in agreement with [26], [27], [30]. The theoretical autocorrelation functions increase in this case with increasing time difference. For normal diffusion we have $\langle x(t_1) x(t_2) \rangle = 2D\min(t_1, t_2)$, so that $\langle x(t_1) x(t_2) \rangle/\langle x^2(t_1) \rangle = 1$ for $t_2 \geq t_1$. In contrast, the behavior of the Lévy walk is governed by long periods of ballistic motion. Thus, it exhibits strong correlations compared to normal diffusion which are due to the long sticking times in the positive or negative velocity states. In the limiting case $\alpha \to 0$ the particle remains in state $+v_0$ or $-v_0$ throughout the measurement so that $x(t) = \pm v_0 t$ with probability $1/2$ for either sign. Therefore we purely expect ballistic, deterministic behavior $\langle x(t_1) x(t_2) \rangle = v^2_0 t_1 t_2$ for the position-position correlation function, and hence $\langle x(t_1) x(t_2) \rangle/\langle x^2(t_1) \rangle = t_2/t_1$. To see this, note that for $\alpha \to 0$, $B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right)$ diverges and the first term in Eq. (13) is the only term that remains:

$$\frac{v^2_0 \sin \pi \alpha}{\pi} B \left( t_1, t_2; 1 - \alpha, 1 - \alpha \right) = \frac{v^2_0 \sin \pi \alpha}{\pi} \left( \frac{t_1}{t_2} \right)^\alpha \sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)} \frac{1}{(\alpha + i)!} \left( \frac{t_1}{t_2} \right)^i,$$

recalling that $t_1/t_2 < 1$ and taking the limit $\alpha \to 0$ we are left with

$$\lim_{\alpha \to 0} \langle x(t_1) x(t_2) \rangle = v^2_0 t_1 t_2$$

where we used de l’Hôpital’s rule. Simulations of the system for moderate $\alpha$ show a good agreement with theory Eq. (13), see Fig. 3.

Fig. 3: Simulational results for $\langle x(t_1) x(t_2) \rangle$ for $\alpha = 0.5$ (upper) and $\alpha = 0.7$ (lower graph), and the respective theoretical predictions (solid lines), Eq. (13). $t_2$ was fixed at 100, $v_0 = 1$. Sample size $10^5$.

The asymptotic behavior of the position-autocorrelation function Eq. (13) is

$$\langle x(t_1) x(t_1 + \Delta) \rangle = \left\{ \begin{array}{ll}
\frac{v^2_0 \sin \pi \alpha}{\pi \alpha(1-\alpha)} t_1^{1-\alpha} & , t_1 \ll \Delta \\
v^2_0 (1-\alpha) (t_1^2 + \Delta t_1) & , t_1 \gg \Delta .
\end{array} \right. \tag{15}$$

Note that we made again the transformation from $(t_1, t_2)$ to $t_1, \Delta = t_2 - t_1$ and $t_1 \leq t_2$. Inserting the above results for the correlation function Eq. (15) and mean squared displacement Eq. (13) into Eq. (7), integrating by parts and using again the integral definition of the incomplete Beta function, we obtain the ensemble averaged TAMSD. In the limit $\Delta/t \ll 1$ we get

$$\langle \delta^2 \rangle \approx v^2_0 \left[ \Delta^2 - \frac{\sin \pi \alpha}{\pi \alpha} \frac{2 \Delta^2}{6 - 11 \alpha + 6 \alpha^2 - \alpha^3} \right]. \tag{16}$$

Note that in fact the short time behavior of the correlation function is not negligible in the integral Eq. (7) since it affects the long-time behavior of the ensemble-averaged TAMSD $\langle \delta^2 \rangle$. It is important to point out that even the first term of $\langle \delta^2 \rangle (\Delta)$, Eq. (14), differs from $\langle x^2(t) \rangle = v^2_0 (1 - \alpha) t^2$ by a factor. This term was also found recently in a different context of deterministic maps by [25].

2.3 Fluctuations of the time averages

More important are the fluctuations of the TAMSDs, since these allow conclusions to be drawn with respect to the ergodic properties of the system. Our simulations revealed...
that these fluctuations are quite small compared to the value of the TAMSDs, and become even smaller with time relative to the ballistic contribution $(v_0 \Delta)^2$. The fluctuations are more pronounced if one looks at the shifted TAMSD $v_0^2 \Delta^2 - \overline{\Delta^2}$, which is the natural random variable of this process as will turn out soon. In Fig. 4 we plot $v_0^2 \Delta^2 - \overline{\Delta^2}$ versus $\Delta$ for ten different trajectories. $v_0^2 \Delta^2 - \overline{\Delta^2}$ remains visibly random.

**Small $\Delta$ limit:** In order to specify the distribution of $v_0^2 \Delta^2 - \overline{\Delta^2}$, we construct the following special case. Consider again the sojourn time PDF Eq. (16) with $0 < \alpha < 1$. Moreover, let us for now only adhere to small $\Delta < 1$, so that at maximum one change of direction takes place during the interval $\Delta$. In this case, the TAMSD Eq. (16) can be explicitly calculated. Clearly, if there were no changes of direction, the TAMSD would be $\overline{\Delta^2} = v_0^2 \Delta^2$. For a single switching event at time $t_s$ within a time interval $(t, t + \Delta)$, the integrand in Eq. (16) becomes

$$
[x(t + \Delta) - x(t)]^2 =
\begin{cases}
   v_0^2 \Delta^2 & \text{for } t \leq (t_s - \Delta) \text{ and } t_s \leq t \\
   (2v_0 t_s - v_0 \Delta - 2v_0 t_s)^2 & \text{for } (t_s - \Delta) \leq t \leq t_s.
\end{cases}
$$

Hence, using Eq. (11), one change of direction within the observation time $t$ reduces the TAMSD to a value of $\overline{\Delta^2} = v_0^2 \Delta^2 - \frac{2v_0^2 \Delta^2}{3}$ for $\Delta \gg \Delta (t - \Delta \approx t)$, as shows integration of Eq. (17). Two changes of direction result in $\overline{\Delta^2} = v_0^2 \Delta^2 - 2 \frac{2v_0^2 \Delta^2}{3} + \frac{4v_0^4 \Delta^4}{9}$ and so forth. Altogether we find for an amount $n_t$ of switching events within the observation time $t$

$$
\overline{\Delta^2} \approx \frac{v_0^2}{t} \left[ \Delta^2 t - \frac{2}{3} \Delta^3 n_t \right],
$$

where the amount of direction changes $n_t$ within $(0, t)$ is a random variable. The probability $p_n(t)$ of the number of events $n$ within $(0, t)$ is determined by the convolution of $n$ sojourn time PDFs with the probability of no event after the $n$th one \[31\]. and is well investigated. Taking $\tilde{\psi}(u) \approx 1 - A u^n$ in Laplace domain and using the convolution theorem results in $\tilde{p}_n(u) = A u^n - 1 \cdot [n \ln(1 - A u^n)]$. Laplace inversion yields

$$
p_n(t) = \frac{1}{\alpha A^{1/(\alpha + 1)} + 1} \left[ \frac{t}{A^{1/(\alpha + 1)}} \right].
$$

$l_{a,1}(t)$ denotes the one-sided Lévy density whose Laplace transform is given by $\exp[-u^\alpha]$ \[31\]. Hence, with $(n_t) \sim t^\alpha(\mathcal{A}\Gamma(1 + \alpha))$ the ensemble average of Eq. (18) becomes

$$
\langle \overline{\Delta^2} \rangle = v_0^2 \Delta^2 - \frac{2v_0^2}{3\mathcal{A}} \langle n_t \rangle.
$$

Therefore, in terms of a new variable

$$
\xi = \frac{v_0^2 \Delta^2 - \overline{\Delta^2}}{v_0^2 \Delta^2 - \langle \overline{\Delta^2} \rangle} = \frac{n_t}{\langle n_t \rangle},
$$

and using Eq. (19) the rescaled distribution of the TAMSD becomes

$$
p(\xi) = \frac{\Gamma^{1/\alpha}(1 + \alpha)}{\alpha \xi^{1/\alpha + 1/\alpha}} \left[ \frac{\Gamma^{1/\alpha}(1 + \alpha)}{\xi^{1/\alpha}} \right].
$$

This PDF is the density of the Mittag-Leffler distribution, a distribution already encountered in the context of TAMSD fluctuations in the subdiffusive continuous time random walk \[33\], \[34\]. Fig. 5 shows the PDF (23) and the respective results for simulations of the Lévy-Walk for two different values of $\alpha$, but for large $\Delta$. It is interesting to note that $\langle v_0^2 \Delta^2 - \overline{\Delta^2} \rangle$ differs for small and large $\Delta$ regimes, however the distribution of the rescaled variable \[20\] does not depend on $\Delta$.

**Crossover to the large $\Delta$ regime:** The ensemble average of the fluctuations of $(v_0^2 \Delta^2 - \overline{\Delta^2})$ is given by Eq. (16) for large $1 \ll \Delta \ll t$. In the present case where $\psi(\tau) = 0$ at short times $\tau < 1$, the behavior of the ensemble averaged TAMSD at $\Delta \ll 1$ is given by Eq. (20). This behavior at small $\Delta$ constitutes the lower bound for more general $\psi(\tau)$ with arbitrary shape at small $\tau$. However, the fluctuations of the time averages Eq. (20) are governed only by the tail...
times. For this purpose, we calculate the ergodicity breaking (EB) parameter \[ \text{EB} = \lim_{t \to \infty} \frac{\langle \xi^2 \rangle - \langle \xi \rangle^2}{\langle \xi \rangle^2} = \frac{2\tau^2(1+\alpha)}{T(1+2\alpha)} - 1, \quad (25) \]
where we used Eq. (23). Numerics for \( \alpha = 0.5 \) show that the EB-parameter for \( \xi \) tends indeed to the predicted finite value \( \text{EB} = 0.571 \) (Fig. 7, i.e. the variable \( \xi \) remains distributed according to Eq. (25)).

![Fig. 6: \( \langle (v_0 \Delta)^2 - \langle \bar{\delta}^2 \rangle \rangle \) versus \( \Delta \). The small–Δ region is sensitive to the shape of \( \psi(\tau) \). Lines indicate theory Eq. (20) (dashed) matching numerical data (dots) at small \( \Delta \), and Eq. (16) (solid) for large \( \Delta \). Note that the crossover takes place in the region of the small-time cutoff of the sojourn time PDF Eq. (2), \( \Delta_c \approx 1 \).](image)

![Fig. 7: EB-parameter of \( \xi \) for \( \alpha = 0.5 \). Red circles indicate \( \Delta = 0.5 \), blue diamonds \( \Delta = 100 \), green triangles \( \Delta = 5000 \). The grey solid line indicates the theoretical value \( \text{EB} = 0.571 \), Eq. (25). Sample size 5000.](image)

### 3 Lévy flight TAMSD

Before we turn to the behavior of the TAMSD of the Lévy walk in the enhanced diffusion regime, let us illustrate the situation in the related Lévy flight. We present a rather illustrative than rigorous argument, to avoid complicated math. A more general and rigorous treatment can be found in Ref. [25]. The Lévy flight is a random walk process where at each renewal the displacement \( x \) of the walker is drawn according to a jump PDF \( \lambda(x) \), but in contrast to the normal random walk the jump PDF lacks the second moment. With a unit time span passing between consecutive renewal events, the number of jumps acts as the (discrete) time variable \( t \). Hence consider the coordinate of a Lévy flight after \( t \) steps as a sum of independent identically distributed (i.i.d.) random variables or displacements
\( x_i \)

\[ X_t = \sum_{i=1}^{t} x_i. \]

Let the \( \{x_i\} \) be distributed according to a two-sided symmetric distribution \( \lambda(x) = \lambda(-x) \) falling off as a power law for large \( |x| \),

\[ \lambda(x) \propto \frac{A_0}{2} |x|^{-1-\alpha}, \quad 1 < \alpha < 2, \quad (26) \]

In particular, for our simulations we used Eq. (26) with \( A_0 = \alpha \) for \( |x| \geq 1 \), and \( \lambda(x) = 0 \) for \( |x| < 1 \). Then, the distribution of the sum \( X_t \) will yield a two-sided Lévy law for large \( t \), \( L_{\alpha,0}(t/X_t/(tA_0/\alpha(1-\alpha))) \), according to the generalized central limit theorem \([28,31]\). The function \( L_{0,\alpha}(x) \) is defined as the inverse Fourier transform of

\[ \tilde{L}_{\alpha,0}(k) = \exp\left[-|k|^\alpha\right] \quad (27) \]

with \( k \) being the Fourier variable \([31]\). This Green function of the Lévy flight has a similar behavior as the central part of the Lévy walk Green function when \( 1 < \alpha < 2 \), as will become obvious in the next section.

The time-averaged mean squared displacement (TAMSD) is defined as \([35]\)

\[ \delta^2 = \frac{1}{t-\Delta} \sum_{i=1}^{t-\Delta} [X_{i+\Delta} - X_i]^2 \]

\[ = \frac{1}{t-\Delta} \sum_{i=1}^{t-\Delta} \left( \sum_{k=1}^{i+\Delta} x_k \right)^2 \quad (28) \]

where the integer \( \Delta \) is the lag time. We have

\[ \delta^2 = \frac{1}{t-\Delta} \sum_{i=1}^{t-\Delta} \sum_{k=i}^{i+\Delta} x_k^2 + \sum_{k=i}^{i+\Delta} \sum_{j} x_k x_j \quad (29) \]

The mixed terms \( \sum_{k=i}^{i+\Delta} \sum_{j} x_k x_j \) on average cancel out for large enough \( \Delta \), hence we omit them. Moreover we assume \( 1 \ll \Delta \ll t \) so that

\[ \delta^2 \simeq \frac{1}{t-\Delta} \sum_{i=1}^{t-\Delta} \sum_{k=1}^{i+\Delta} x_k^2 \]

\[ \simeq \frac{1}{t} \sum_{i=1}^{t} \sum_{k=1}^{i+\Delta} x_k^2 \]

\[ \approx \frac{\Delta}{t} \sum_{k=1}^{t} x_k^2 \quad (30) \]

We find for the distribution of the \( y = x^2 \)

\[ p(x^2) = p(x) \frac{dx}{dy} \propto \frac{A_0}{2} y^{1-\frac{\alpha}{2}}. \quad (31) \]

Note that the transition from \( x \) to the positive valued \( y \) results in a factor 2 in the normalization. The large \((y = x^2)\) asymptotics can be obtained in Laplace domain, using the Tauberian theorem:

\[ \tilde{p}(u_y) \simeq 1 - \frac{A_0}{\alpha} t^{1/\alpha - \frac{\alpha}{2}} u_y^\frac{\alpha}{2}. \quad (32) \]

Hence, the sum over these \( x_i^2 = y_k \) in Eq. (30) is a sum over positive i.i.d. random variables distributed according to a PDF with a power law tail of exponent \(-1 - \alpha/2\). Hence, due to the generalized central limit theorem we find in the large \( t \) limit \([28,31]\) the PDF of \( \zeta = t\delta^2/\Delta \):

\[ p(\zeta) = \frac{1}{(K^{LF}_\alpha t)^{\frac{\alpha}{2}}} L^{\frac{\alpha}{2}}_\Delta \left\{ \frac{\zeta}{(K^{LF}_\alpha t)^{\frac{\alpha}{2}}} \right\}. \quad (33) \]

where \( K^{LF}_\alpha = \frac{2\alpha}{\Gamma(1-\frac{\alpha}{2})} \) and \( L_{\alpha/2,1}(\cdot) \) is the one-sided Lévy PDF given by \( \exp[-u^\alpha/2]\) in Laplace domain \([31]\). A similar result was obtained in \([34]\) though a slightly different scaling was reported. Fig. 8 shows the PDF Eq. (33) obtained with Mathematica, and the corresponding simulational results which perfectly match the theory.

**Fig. 8:** Lévy flight simulation results for the distribution of \( \frac{1}{t} \) for \( \alpha = 1.5 \) and \( K^{LF}_\alpha = \Gamma(1 - \frac{\alpha}{2}) \), \( \Delta = 1, t = 10^5 \) (left) and \( \Delta = 100, t = 10^6 \) (right), and theoretical predictions according to Eq. (33) (solid curves).

Hence, this example illustrates that the TAMSD of Lévy Flights with jump length distributions without second moment is a one-sided Lévy density lacking the first moment. As will be shown later, the TAMSD distribution for the corresponding Lévy walk is completely different despite the similarity in the central part of the propagator, see Fig. 9

### 4 Enhanced diffusion regime

In this section we consider the regime of sojourn times distributed according to a PDF with existing mean \( \langle \tau \rangle \), but diverging second moment, i.e. Eq. (2) with \( 1 < \alpha < 2 \). The expansion in Laplace domain is hence

\[ \tilde{\psi}(u) \simeq 1 - \langle \tau \rangle u + Au^\alpha. \quad (34) \]

In our case Eq. (2) the average sojourn time is \( \langle \tau \rangle = \alpha/(\alpha - 1) \) and \( A = |\Gamma(1-\alpha)| \).
4.1 Particle position distribution

Again, the particle position is given by the integral over the velocities \( v_0(t_+ - t_-) \). The MSD is well known \[36\]:

\[
\langle x^2(t) \rangle \simeq \frac{2v_0^2 A (\alpha - 1)}{\langle \tau \rangle F(4 - \alpha)} \tau^{3-\alpha},
\]

as well as the propagator which is given by a two-sided Lévy-distribution for the central part where \(|x| < v_0 t\),

\[
p(x,t) \simeq \frac{1}{(K_\alpha t)^{1/\alpha}} \hat{L}_{\alpha,0} \left( \frac{x}{(K_\alpha t)^{1/\alpha}} \right),
\]

where \( \hat{L}_{\alpha,0}(k) = \exp[-k^\alpha] \) and

\[
K_\alpha = \frac{v_0^2 A(\alpha - 1)}{\langle \tau \rangle F(4 - \alpha)}.
\]

Due to the finite velocity the Lévy walk propagator exhibits a cutoff so that \( p(x,t) = 0 \) at \(|x| > v_0 t\), similarly to other Lévy walk models \[36, 38\]. Moreover, we expect those particles that have never changed their direction of motion to form a delta-peak at the edge of the cutoff \[37\]. In the following we will calculate the time averaged mean squared displacement (TAMSD) of the Lévy Walk described above.

4.2 Ensemble average of \( \langle \Delta^2 \rangle \)

It is important to note that also in the subballistic regime \( 1 < \alpha < 2 \), the velocity correlation is governed by the persistence probability \( p_0(t) \) to stay in one state for a time \( t \). For sojourn time PDFs \( \psi(\tau) \) with existing mean the corresponding forward recurrence time PDF \( \psi_{f,t_1}(\tau_f) \) reaches stationarity at large \( t_1 \) and obeys the limiting distribution \[30\]

\[
\lim_{t_1 \to \infty} \psi_{f,t_1}(\tau_f) = \frac{1}{\langle \tau \rangle} \int^\infty_0 \psi(\tau) d\tau \simeq \frac{A}{\langle \tau \rangle F(1 - \alpha)^{4-\alpha}}.
\]

This first waiting time in turn has no mean and therefore \( p_0 \) dominates \( \langle v(t_1)v(t_1 + \Delta) \rangle \). Again, with Eqs. \(39, 40\), \( \langle v(t_1)v(t_1 + \Delta) \rangle = v_0^2 p_0(t_1, t_1 + \Delta) \) holds for the velocity correlation function for \( t_1 \) and \( \Delta \) both large. The \( p_0 \) for the present process is well known, so that following e.g. the procedure presented in \[30\] we have

\[
\langle v(t_1)v(t_1 + \Delta) \rangle = \frac{v_0^2 A}{\langle \tau \rangle F(1 - \alpha)^{4-\alpha}} \times \left( \frac{t_1}{t_1 + \Delta} \right)^{\alpha - 1} \left( 1 - \frac{t_1}{t_1 + \Delta} \right)^{1 - \alpha} - 1.
\]

In the equilibrated regime (or stationary state) \( t_1 \gg \Delta \), \( \langle v(t_1)v(t_1 + \Delta) \rangle \) becomes independent of time \( t_1 \) so that

\[
\langle v(t_1)v(t_1 + \Delta) \rangle_{eq} \simeq \frac{v_0^2 A}{\langle \tau \rangle F(1 - \alpha)^{4-\alpha}} \Delta^{1 - \alpha}.
\]

By integrating Eq. \[38\] as in Eq. \[8\] we obtain the position autocorrelation

\[
\langle x(t_1)x(t_1 + \Delta) \rangle = \frac{A v_0^2}{\langle \tau \rangle F(4 - \alpha)} \Delta^\alpha \times \left[ - (y)^{3-\alpha} + (1 + y)^{3-\alpha} + (\alpha - 3)(1 + y)^{2-\alpha} + \alpha \right]
\]

where \( y = \Delta/t_1 \). Our simulations have shown that this estimation reproduces the large \( t_1 \) behavior of the position correlation \( \langle x(t_1)x(t_2) \rangle \) quite well. For \( \Delta \to 0 \) the behavior of the MSD Eq. \[35\] is reproduced.

Note that in the subballistic case the MSD for a process starting with the beginning of the measurement differs from the MSD for a process that started a long time before the beginning of the measurement time \( t_0 \), as was found earlier in the context of a stochastic collision model \[38\]. This behavior is due to the predominant role of the persistence probability \( p_0 \) in the correlation functions discussed above. In what follows the MSD for the process that started long before \( t_0 \) will be called the equilibrium MSD \( \langle x^2 \rangle_{eq} \), alluding to the fact that this process has no memory of its starting time. The situation is sketched in Fig. \[10\]. It is important to introduce the equilibrium MSD at this point for the following reason: Since the time averaging procedure comprises averaging over all continuously shifted time lags, and not only over those starting at a switching event, there is an inherent averaging over disorder. The equilibrium MSD accounts for this averaging over disorder and is therefore the natural ensemble averaged quantity to later compare the time averaged MSD to. Note also that such a definition of an equilibrium MSD is not possible in the ballistic case since there a stationary state does not exist – the respective MSD would never become independent of the time difference between start of
the process in the past and the actual start of the measurement.

Using Eq. (39) and \( \langle x^2 \rangle_{eq} = 2 \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2) \rangle_{eq} \) we obtain the known equilibrium MSD \[38\], and hence differs from the MSD \[35\] by lacking a factor:

\[
\lim_{t \to \infty} \frac{\langle \delta^2 \rangle}{\langle x^2 \rangle} = \frac{1}{\alpha - 1}.
\] (44)

Numerical simulations also indicate that these two averages appear to differ by a factor (Fig. 11). Further numerical evidence for this behavior was found in \[23\], \[24\]. Especially for small \( \alpha \) the convergence of \( \langle \delta^2 \rangle \) is extremely slow. Moreover, in our simulations the large time behavior of the \( \langle x^2 \rangle \) is not represented very accurately at the corresponding relatively small times (up to \( 10^4 \)).

\[4.3 \text{ Fluctuations of the TAMSD}\]

The TAMSDs of trajectories measured up to a certain observation time appear to be distributed. Fig. 12 shows the TAMSD evolution of some sample trajectories. The fluctuations of the TAMSDs decrease with increasing total measurement time. However, since in experiments the observation time is always finite, these fluctuations may play a role in practice. In our simulations for example we find large fluctuations among the \( \langle \delta^2 \rangle \) for \( \alpha = 1.25 \) and \( \Delta = 100 \), \( t = 10^5 \) (see Fig. 12).

Although the propagators of the Lévy walk and flight look very similar in the central part \( |x| < \nu_0 t \), unlike the flight case simulations suggest that the TAMSD distribution for the Lévy walk cannot be expressed by a Lévy distribution of stability index \( \alpha/2 \). For the Ergodicity Breaking (EB) parameter of the subballistic Lévy Walk regime,

\[
EB = \lim_{t \to \infty} \frac{\langle \delta^2 \rangle - \langle \delta^2 \rangle}{\langle \delta^2 \rangle^2}.
\] (45)

we find a steady decay with \( t \), which is yet very slow with an exponent of roughly \( 1 - \alpha \), to the value zero indicating that the width of the distribution tends to zero (Figs. 13 for \( \alpha = 1.5 , 1.25 \)). Hence, the TAMSDs do not remain
ergodic systems such as relaxation of confined fractional Brownian motion \(^{39}\). Interestingly, this distribution is controlled by rare events, since the tails of the mentioned distributions are where one finds differences between the models. Thus taking into consideration finite velocity (like in the Lévy walk model) is crucial for our understanding of the ergodic properties of these processes. In the future it might be worth while checking the time averages of lower order moments, since they might exhibit behavior very different the second moment considered here. We note that for finite times the fluctuations of TAMSDs are large. Consequently, in the laboratory where experiments are made for finite time the process may seem non ergodic, but this is only a finite time effect. Moreover, in this sub-ballistic case \(\langle \delta^2 \rangle\) is equal to the equilibrium MSD \(\langle x^2 \rangle_{eq}\), but not to \(\langle x^2 \rangle\). If one wishes to compare time and ensemble averages, the conclusion on equality of these two averages will depend on how the ensemble is prepared. To attain ergodicity we need to start the ensemble in a stationary state, which is not so surprising. The point is that for normal processes, e.g. the case where all moments of the waiting time PDF \(\psi(t)\) exists, it does not matter how we start the process, in the long time limit the time and ensemble average procedures are all identical. In that sense the Lévy walk, in the enhanced regime, is unique. A similar effect might be found also for sub-diffusive CTRW, for the case of finite average waiting time, but with an infinite variance, but that is left for future work. Further time averaged drifts, when a bias is present, also exhibit interesting ergodic features and Einstein relations, as we discussed recently in \(^{25}\).

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### 5 Summary

In this article we have shown that the shifted time averaged MSD of the ballistic Lévy walk is described by the Mittag-Leffler distribution, similar to the distribution of the TAMSD in the sub-diffusive continuous time random walk (CTRW). This distribution describes the fluctuations of the time averages and is universal. The TAMSD averaged over an ensemble of trajectories \(\langle \delta^2 \rangle\) is not equal to the ensemble average \(\langle x^2 \rangle\) as already pointed out by Akimoto \(^{23}\). For Lévy flights the TAMSDs are random with a PDF given by the one sided Lévy PDF, which is in agreement with rigorous results (though note that our coefficients are different than those reported in \(^{35}\), possibly due to a typo). In the enhanced diffusion regime, the PDF of the particle position of the Lévy walk is similar to the Lévy flight case, at least in its center. However, the TAMSD of the two models is vastly different, and for Lévy walks no fluctuations are found for \(\vec{a}^2\). This indicates that the TAMSD is controlled by rare events, since the tails of the mentioned distributions are where one finds differences between the models. Thus taking into consideration finite velocity (like in the Lévy walk model) is crucial for our understanding of the ergodic properties of these processes. In the future it might be worth while checking the time averages of lower order moments, since they might exhibit behavior very different the second moment considered here. We note that for finite times the fluctuations of TAMSDs are large. Consequently, in the laboratory where experiments are made for finite time the process may seem non ergodic, but this is only a finite time effect. Moreover, in this sub-ballistic case \(\langle \delta^2 \rangle\) is equal to the equilibrium MSD \(\langle x^2 \rangle_{eq}\), but not to \(\langle x^2 \rangle\). If one wishes to compare time and ensemble averages, the conclusion on equality of these two averages will depend on how the ensemble is prepared. To attain ergodicity we need to start the ensemble in a stationary state, which is not so surprising. The point is that for normal processes, e.g. the case where all moments of the waiting time PDF \(\psi(t)\) exists, it does not matter how we start the process, in the long time limit the time and ensemble average procedures are all identical. In that sense the Lévy walk, in the enhanced regime, is unique. A similar effect might be found also for sub-diffusive CTRW, for the case of finite average waiting time, but with an infinite variance, but that is left for future work. Further time averaged drifts, when a bias is present, also exhibit interesting ergodic features and Einstein relations, as we discussed recently in \(^{25}\).

This work was supported by the Israel Science Foundation.

### A Correlation functions

For the derivation of the velocity correlation function we follow \(^{30}\). Hence, for the time variables \(t_1, \Delta\) with \(t_2 - t_1 = \Delta\) and going to Laplace domain with respect to \(\Delta\) we have

\[
\tilde{\rho}_n(t_1, u_\Delta) = \left\{ \begin{array}{ll} \psi_{t_1}(u_\Delta)\psi^{n-1}(u_\Delta) - u_\Delta^{-2}\psi(t_1) & n \geq 1 \\ 1 - \psi_{t_1}(u_\Delta) & n = 0 \end{array} \right.
\]

(46)

where \(\psi_{t_1}\) is the forward recurrence time PDF, i.e. the PDF of the time it takes to encounter the next event after a given
time $t_1$ which in double-Laplace domain reads

$$\tilde{\psi}_{f,u_1}(u_\Delta) = \frac{1}{1 - \psi(u_1)} \hat{\psi}(u_\Delta) - \hat{\psi}(u_1). \quad (47)$$

Here $u_\Delta$ and $u_1$ are the Laplace variables conjugate to $\Delta$ and $t_1$, respectively. Inserting (40) and (47) back into (9) we get

$$L_\Delta \{ (v(t_1)v(t_1 + \Delta))|u_\Delta \} = v_0 \frac{1 - \tilde{\psi}_{f,t_1}(u_\Delta)}{u_\Delta}$$

and repeated integration by parts we find

$$\langle x(t_1)x(t_2) \rangle = v_0^2 \frac{\sin \pi \alpha}{\pi} \times \left[ \right.$$

$$\left. \int_0^{t_1} ds_1 \int_{s_1}^{t_2} B \left( \frac{t_1}{t_2}; \alpha, 1 - \alpha \right) ds_2 + \int_0^{t_1} ds_2 \int_0^{s_2} B \left( \frac{t_1}{t_2}; \alpha, 1 - \alpha \right) ds_1 \right]$$

$$= v_0^2 \frac{\sin \pi \alpha}{\pi} \times \left[ \right.$$

$$\left. \int_0^{t_1} ds_1 \left[ t_2 B \left( \frac{t_1}{t_2}; \alpha, 1 - \alpha \right) - s_1 B \left( \frac{s_1}{t_2}; 1 + \alpha, 1 - \alpha \right) + s_1 B \left( 1, -1 + \alpha, 1 - \alpha \right) \right] \right]$$

$$= v_0^2 \frac{\sin \pi \alpha}{\pi} \times \left[ \right.$$

$$\left. \int_0^{t_1} ds_1 \left[ t_2 B \left( \frac{t_1}{t_2}; \alpha, 1 - \alpha \right) - t_2^2 B \left( \frac{t_1}{t_2}; 1 + \alpha, 1 - \alpha \right) - \frac{1}{2} t_2^2 B \left( \frac{t_1}{t_2}; 1 + \alpha, 1 - \alpha \right) + \frac{1}{2} t_2^2 B \left( 1, -1 + \alpha, 1 - \alpha \right) \right] \right]$$

$$+ \frac{v_0^2}{(1 - \alpha)^2} t_1^2 \right]$$

which with $B(1, -1 + \alpha, 1 - \alpha) = 0$ for $0 < \alpha < 1$ finally yields Eq. (49).

### A.2 Subballistic phase

Here the Laplace transform of the sojourn time of the particle in a velocity state for small $v(\eta) = 1 - (\tau) u + A u^\alpha$ with $(\tau) = \alpha/\alpha - 1$ and $A = \Gamma(1 - \alpha)$. For $t_1, \Delta$ large, i.e. $u_1, u_\Delta \to 0$ in Laplace domain, Eq. (49) becomes again

$$L_{t_1,\Delta} \{ (v(t_1)v(t_1 + \Delta))|u_1, u_\Delta \} =$$

$$v_0^2 \left[ \frac{1}{u_\Delta u_1} - \frac{1}{u_1 - u_\Delta} \tilde{\psi}(u_\Delta) - \hat{\psi}(u_1) \right]$$

and after Laplace inversion

$$\langle v(t_1)v(t_1 + \Delta) \rangle = v_0^2 \left[ 1 - \int_0^{\Delta} \psi_{f,t_1}(\Delta') d\Delta' \right]$$

$$= v_0^2 \int_0^{\Delta} \psi_{f,t_1}(\Delta') d\Delta', \quad (50)$$

which we will use in the following since we know the scaling form of $\psi_{f,t_1}(\Delta)$ for large $t_1$ and $\Delta$ due to Dynkin’s theorem Eq. (11). leading to (12).

Inserting Eq. (12) into Eq. (8) and using the definition of the incomplete Beta function $B(y; a, b) = \int_0^y du a^{a-1} (1 - u)^{b-1}$

$$B(x, y; \alpha, \beta) = \int_0^1 du \left[ \frac{\beta}{\beta + u} \right]^\alpha \left[ \frac{\alpha}{\alpha + u} \right]^\beta \left[ \frac{1}{\alpha + \beta + u} \right]^\gamma$$

and repeated integration by parts we find

$$\langle x(t_1)x(t_2) \rangle =$$

$$v_0^2 \frac{\sin \pi \alpha}{\pi} \times \left[ \right.$$
B Ensemble averaged TAMSD, ballistic phase

Inserting Eq. (13) and Eq. (14) into Eq. (7) and again using integration by parts and the integral definition of the incomplete Beta function yields

\[
\langle \Delta t \rangle = \frac{v_0^3}{3} \left[ \frac{1}{t - \Delta} \left( \frac{1 - \alpha}{3} t^3 - \Delta^3 \right) \right] - \frac{\sin \alpha}{\pi (t - \Delta)^2} \left[ \left( \frac{1}{3} t^3 - \frac{1}{2} \Delta t^2 \right) B \left( \frac{t - \Delta}{t}, \alpha, 1 - \alpha \right) - \frac{1}{3} \Delta^3 B \left( \frac{t - \Delta}{t}, \alpha, -2 - \alpha \right) + \frac{1}{2} \Delta^3 B \left( \frac{t - \Delta}{t}, \alpha, -1 - \alpha \right) \right] - \frac{1}{6} \Delta^3 B \left( \frac{t - \Delta}{t}, 1 + \alpha, 1 - \alpha \right) + \frac{1}{6} \Delta^3 B \left( \frac{t - \Delta}{t}, 1 + \alpha, -2 - \alpha \right) + \left( \frac{1}{6} t^3 + \frac{1}{2} \Delta t^2 - \frac{1}{2} \Delta^2 t \right) B \left( \frac{t - \Delta}{t}, -1 + \alpha, 1 - \alpha \right) - \frac{1}{2} \Delta^3 B \left( \frac{t - \Delta}{t}, -1 + \alpha, -1 - \alpha \right) + \frac{1}{2} \Delta^3 B \left( \frac{t - \Delta}{t}, -1 + \alpha, -2 - \alpha \right) \right],
\]

(55)

For the small $\Delta$ expansion Eq. (15) we used the identity

\[
B \left( \frac{t_i}{t_2}, a, b \right) = B \left( 1, b, a \right) - B \left( \frac{\Delta}{t_2}, b, a \right)
\]

(56)

where $\Delta = t_2 - t_1$ and the expansion of the incomplete Beta function for small arguments $y$

\[
B \left( y, a, b \right) = y^a \sum_{n=0}^{\infty} \frac{(1 - b)^n}{n! (a + n)}
\]

(57)

where $(c)_n = \Gamma(c + n)/\Gamma(c)$ is the Pochhammer symbol.

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