INFINITE-DIMENSIONAL FROBENIUS MANIFOLDS
UNDERLYING THE UNIVERSAL WHITHAM HIERARCHY

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Abstract. We construct a class of infinite-dimensional Frobenius manifolds on
the spaces of pairs of meromorphic functions with a pole at infinity and a movable
pole. Such Frobenius manifolds are shown to be underlying the universal Whitham
hierarchy, which is an extension of the dispersionless Kadomtsev-Petviashvili hi-
erarchy.

To the memory of Professor Boris Dubrovin

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1. Introduction

1.1. Background. The concept of Frobenius manifold was introduced by Dubrovin [10] to exhibit the geometry behind the WDVV (Witten-Dijkgraaf-E. Verlinde-H. Verlinde) system of nonlinear differential equations [7, 26] (see also [20]). This concept plays a significant role in several branches of mathematical physics, including the theory of singularities, Gromov-Witten invariants and integrable systems e.t.c., see, for example, [10, 12, 13, 16, 18, 19, 30] and references therein.

A Frobenius manifold of charge $d$ is an $n$-dimensional manifold $M$ equipped on each tangent space $T_v M$ with a structure of Frobenius algebra $(A_v = T_v M, \circ, e, < , >)$ depending smoothly on $v \in M$, such that the following three axioms are satisfied:

(FM1) The nondegenerate bilinear form $< , >$ is a flat metric on $M$, and the unity vector field $e$ is covariantly constant, i.e., $\nabla e = 0$ (here $\nabla$ denotes the Levi-Civita connection for the flat metric);

(FM2) Let $c$ be a 3-tensor defined by $c(X,Y,Z) := < X \cdot Y, Z >$ with $X, Y, Z \in T_v M$, then the 4-tensor $(\nabla W c)(X,Y,Z)$ is symmetric in $X, Y, Z, W \in T_v M$;

(FM3) There exists a vector field $E$, called the Euler vector field, which satisfies the conditions $\nabla \nabla E = 0$ and

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = X \circ Y,$$

$$E(< X, Y >) - < [E, X], Y > - < X, [E, Y] > = (2 - d) < X, Y >$$

for any vector fields $X, Y$ on $M$.

According to the axiom (FM1), on the Frobenius manifold $M$ one can choose a system of flat local coordinates $v = (v^1, \ldots, v^n)$ such that the unity vector field reads $e = \frac{\partial}{\partial v^1}$. By using such flat coordinates, a constant non-degenerate $n \times n$ matrix is given by

$$\eta_{\alpha\beta} = < \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} >,$$

and its inverse is denoted as $(\eta^{\alpha\beta})$. The matrices $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$ will be used to lower and to lift indices, respectively, and summations over repeated Greek indices are assumed. Let

$$c_{\alpha\beta\gamma} = c\left( \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\gamma} \right),$$

then the product $\circ$ of the Frobenius algebra $T_v M$ is given by

$$\frac{\partial}{\partial v^\alpha} \circ \frac{\partial}{\partial v^\beta} = c_{\alpha\beta\gamma} \frac{\partial}{\partial v^\gamma}, \quad c_{\alpha\beta}^\gamma := \eta^{\sigma\tau} c_{\alpha\beta\sigma\tau}.$$

The structure constants of the product satisfy

$$c_{\beta\gamma}^\alpha = \delta_{\beta\gamma}^\alpha, \quad c_{\alpha\beta}^\sigma c_{\sigma\gamma}^\tau = c_{\alpha\gamma}^\epsilon c_{\epsilon\beta}^\sigma.$$

(1.1)
According to the axioms (FM2) and (FM3), there locally exists a so-called potential function $F(v)$ such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma},$$

(1.2)

$$\text{Lie}_E F = (3 - d) F + \text{quadratic terms in } v.$$  

(1.3)

In other words, the function $F$ solves the WDVV equation (1.1)–(1.3), and its third-order derivatives $c_{\alpha\beta\gamma}$ are called the 3-point correlation functions in topological field theory [9]. Conversely, given a solution $F$ of the WDVV equation (1.1)–(1.3) (including a flat metric, a unity vector field and a Euler vector field), one can recover the structure of a Frobenius manifold.

Similar to the tangent space $T_v M$, the cotangent space $T^*_v M$ of $M$ also carries a Frobenius algebra structure with a product $\ast$ and an invariant bilinear form $<, >^\ast$ given respectively by

$$dv^\alpha \ast dv^\beta = \eta^{\alpha\epsilon} c^\beta_\epsilon dv^\gamma, \quad <dt^\alpha, dt^\beta>^\ast = \eta^{\alpha\beta}.$$  

Moreover, on $T^*_v M$ there is another symmetric bilinear form, named as the intersection form, defined by

$$(dv^\alpha, dv^\beta)^\ast = g^{\alpha\beta}, \quad g^{\alpha\beta} := i_E (dv^\alpha \circ dv^\beta).$$

It is known that the bilinear forms $g^{\alpha\beta}$ and $\eta^{\alpha\beta}$ on $T^* M$ compose a covariant flat pencil of metrics. Namely, any linear combination of these two metrics is still a flat metric, and their covariant Levi-Civita symbols obey the same linear combination relation. Based on this fact and the pioneering work of Dubrovin and Novikov on Hamiltonian structures of hydrodynamics type [11], Dubrovin established a link between Frobenius manifolds and certain $(1 + 1)$-dimensional dispersionless integrable hierarchies. Such a seminal theory has been achieved in essentially the finite-dimensional case, namely, on an $n$-dimensional Frobenius manifold it is associated with a system of evolutionary equations of hydrodynamic type of $n$ unknown functions.

The first trial to extend the theory of Frobenius manifolds to the case of infinite dimension was done by Carlet, Dubrovin and Mertens [3]. More exactly, they discovered an infinite-dimensional Frobenius manifold structure on a space of pairs of certain meromorphic functions with single poles at the origin and at infinity. Such a Frobenius manifold, in contrast to the finite-dimensional case, is associated with an integrable hierarchy of $2 + 1$ evolutionary equations [1, 25]. Following this approach, more infinite-dimensional Frobenius manifolds were constructed in [27, 29] by considering pairs of meromorphic functions with higher-order poles at the origin and at infinity, and such (formal) manifolds underly the two-component BKP hierarchy.
and the Toda lattice hierarchy [25] respectively. In particular, when the meromorphic functions are with single poles at the origin and at infinity, the Frobenius manifold in [28] is similar, but not exactly the same, with that given in [3].

As it is known, a fundamental role in the theory of integrable systems is played by the Kadomtsev-Petviashvili (KP) hierarchy [6] of 2 + 1 evolutionary equations. By using a method different from that in [3], Raimondo [21] proposed an infinite-dimensional Frobenius manifold for the dispersionless KP equation via the theory of Schwartz functions. What is more, a scheme was proposed by Szablowski [24] to construct Frobenius manifolds based on the Rota-Baxter identity and a counterpart of the modified Yang-Baxter equation for the classical \( r \)-matrix, and this scheme was applied to a number of models including the dispersionless KP hierarchy. The relation between the constructions in [21, 24] and that in [3, 27, 29] is not clear yet. In fact, it is still open how to derive Frobenius manifolds underlying the dispersionless KP hierarchy via the approach originated in [3].

Towards a possible solution to the above problem, the aim of the present paper is to study along the line of [3, 27, 29] infinite-dimensional Frobenius manifolds related to a certain extension of the dispersionless KP hierarchy. Such an extension (see (3.2)–(3.3) below for the definition) will be referred as the (special) universal Whitham hierarchy. In fact, Krichever investigated in [14, 15] the universal Whitham hierarchy and their reductions in moduli spaces of (formal) meromorphic functions on Riemann surfaces of all genera. In stead of the general setting in [14, 15], this paper only concerns the case (3.2)–(3.3), or precisely, its bi-Hamiltonian structures of hydrodynamic type derived in [28].

1.2. Main results. Let \( U \) be a neighborhood of \( z = 0 \) on the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \), and we assume that there is one circle \( \Gamma \) around \( U \) and another circle \( \Gamma' \) outside \( \Gamma \), which satisfy

\[
|z_1| > |z_2| \quad \text{and} \quad |z_1 - \varphi| > |z_2 - \varphi| \quad \text{for any} \quad z_1 \in \Gamma', \ z_2 \in \Gamma, \ \varphi \in U. \quad (1.4)
\]

For any \( \varphi \in U \), we consider two sets of holomorphic functions on closed disks as follows:

\[
\mathcal{H}^-_{\varphi} = \left\{ f(z) = \sum_{i \geq 0} f_i (z - \varphi)^{-i} \mid f \text{ holomorphic outside } \Gamma \right\},
\]

\[
\mathcal{H}^+_{\varphi} = \left\{ f(z) = \sum_{i \geq 0} f_i (z - \varphi)^{+i} \mid f \text{ holomorphic inside } \Gamma' \right\}.
\]

Here by the words “outside/inside” we mean to include the boundaries \( \Gamma \) and \( \Gamma' \) respectively, namely, the holomorphic functions can be extended analytically beyond such boundaries.
Given two arbitrary positive integers \(m\) and \(n\), let
\[
\tilde{M}_{m,n} = \bigcup_{\varphi \in U} \left( (z^m + (z - \varphi)^{m-2}H_\varphi^-) \times (z - \varphi)^{-n}H_\varphi^+ \right),
\]
whose elements are written as pairs of Laurent series of the form
\[
\tilde{a} = \left( z^m + \sum_{i \leq m-2} a_i (z - \varphi)^i, \sum_{i \geq -n} \hat{a}_i (z - \varphi)^i \right).
\] (1.5)
For any point \(\tilde{a} = (a(z), \hat{a}(z)) \in \tilde{M}_{m,n}\), we introduce
\[
\zeta(z) = a(z) - \hat{a}(z), \quad \ell(z) = a(z)_+ + \hat{a}(z)_-,
\] (1.6)
where the subscripts ‘\(\pm\)’ mean as before to take the nonnegative and the negative powers in \((z - \varphi)\) (one notes that \(z^m = \sum_{i=0}^{m} \binom{m}{i} \varphi^{m-i} (z - \varphi)^i\)). Clearly, the function \(\zeta(z)\) is holomorphic in a neighborhood of the closed bend bounded by \(\Gamma\) and \(\Gamma'\), while \(\ell(z)\) can be extended analytically to \(\mathbb{C} \setminus \{\varphi\}\). Conversely, given such two functions \(\zeta(z)\) and \(\ell(z)\), one can recover \((a(z), \hat{a}(z))\) of the form (1.5) by
\[
a(z) = \zeta(z)_+ + \ell(z), \quad \hat{a}(z) = -\zeta(z)_+ + \ell(z).
\] (1.7)
Hence each point in \(\tilde{M}_{m,n}\) can be represented equivalently by the pair of series \((\zeta(z), \ell(z))\).

Let \(M_{m,n}\) be a subset of \(\tilde{M}_{m,n}\) consisting of points \(\tilde{a} = (a(z), \hat{a}(z))\) such that the following conditions are fulfilled:

(C1) the coefficient \(\hat{a}_{-n} \neq 0\);

(C2) at any point of the circle \(\Gamma\), it holds that
\[
\zeta'(z) \neq 0, \quad \ell'(z) \neq 0, \quad a'(z) \hat{a}(z) - a(z) \hat{a}'(z) \neq 0;
\] (1.8)
(C3) the function \(\zeta(z)\) has winding number 1 around \(z = 0\), such that it maps the circle \(\Gamma\) biholomorphically to a simple smooth curve \(\Sigma\) around the origin.

Such a subset \(M_{m,n}\) can be viewed as an infinite-dimensional manifold, whose coordinates can be chosen as \(\{a_i\}_{i \leq m-2} \cup \{\hat{a}_j\}_{j \geq -n} \cup \{\varphi\}\).

On \(M_{m,n}\), let us introduce the set of variables
\[
t = \{t_i\}_{i \in \mathbb{Z}}, \quad h = \{h_j\}_{j=1}^{m-1}, \quad \hat{h} = \{\hat{h}_k\}_{k=0}^{n},
\] (1.9)
where
\[
t_i = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{i\zeta(z)^i} \, dz, \quad i \in \mathbb{Z} \setminus \{0\}; \quad t_0 = \frac{1}{2\pi i} \oint_{\Gamma} \log \frac{z}{\zeta(z)} \, dz;
\] (1.10)
\[
h_j = -\frac{1}{j} \operatorname{Res}_{z=\varphi} \ell(z)^j \, dz, \quad j = 1, \ldots, m-1;
\] (1.11)
\[
\hat{h}_0 = \varphi; \quad \hat{h}_k = \frac{1}{k} \operatorname{Res}_{z=\varphi} \ell(z)^k \, dz, \quad k = 1, \ldots, n.
\] (1.12)
Main Theorem 1. For any two positive integers $m$ and $n$, the set $\mathcal{M}_{m,n}$ is an infinite-dimensional Frobenius manifold with a system of flat coordinates (1.9) such that

(i) the unity vector field is

\[ \vec{e} = \begin{cases} \frac{\partial}{\partial t_0} + \frac{\partial}{\partial h_0}, & m = 1, \\
1 \frac{\partial}{m \partial h_{m-1}}, & m \geq 2; \end{cases} \]

(ii) the potential $\mathcal{F}$, up to quadratic terms in the flat coordinates, is

\[ \mathcal{F} = \frac{1}{(2\pi i)^2} \oint \oint \left( \frac{1}{2} \zeta(z_1)\zeta(z_2) + \zeta(z_1)\ell(z_2) - \ell(z_1)\zeta(z_2) \right) \log \left( \frac{z_1 - z_2}{z_1} \right) dz_1 dz_2 \\
+ \left( \frac{1}{2} \ell_{-1} - n\hat{h}_n \right) V_1(t) - t_{-1} V_2(h) + G(h, \hat{h}). \] (1.13)

where the functions $V_1(t), V_2(h)$ and $G(h, \hat{h})$ are given in Lemma 2.15 below;

(iii) the Euler vector field is

\[ \vec{E} = \sum_{i \in \mathbb{Z}} \left( \frac{1}{m} - i \right) t_i \frac{\partial}{\partial t_i} + \sum_{j=1}^{m-1} \frac{j+1}{m} h_j \frac{\partial}{\partial h_j} + \sum_{k=0}^{n} \left( \frac{1}{m} + \frac{k}{n} \right) \hat{h}_k \frac{\partial}{\partial \hat{h}_k}, \]

and the charge is $d = 1 - \frac{2}{m}$.

Main Theorem 2. The bi-Hamiltonian structure induced from the flat pencil of metrics on the Frobenius manifold $\mathcal{M}_{m,n}$ coincides with that given in Proposition 3.1 below for the universal Whitham hierarchy (3.2) – (3.3).

1.3. Organization. This paper is arranged as follows. In the next section, we are to prove Main Theorem 1, namely, to construct an infinite-dimensional Frobenius manifold structure on $\mathcal{M}_{m,n}$. In Section 3, we will study the relationship between the infinite-dimensional Frobenius manifold $\mathcal{M}_{m,n}$ and the universal Whitham hierarchy (Main Theorem 2). The last section is devoted to some concluding remarks.

2. The infinite-dimensional Frobenius manifold structure on $\mathcal{M}_{m,n}$

In this section, our aim is to construct an infinite-dimensional Frobenius manifold structure on $\mathcal{M}_{m,n}$, including the flat metric, the potential function, the unity and the Euler vector fields.

2.1. The tangent space and the cotangent space. Let us describe the tangent and the cotangent spaces on $\mathcal{M}_{m,n}$ with Laurent series. Firstly, at each point $\vec{a} = \ldots$
with respect to the pairing of $\vec{a}$ accordingly, the cotangent space at $\vec{a}$ is eliminated by a factor of $\alpha$.

Vector $\vec{ξ}$ and introduce two generating functions

\[ \text{Lemma 2.1.} \quad \text{For any cotangent vector } \vec{ω}, \text{ tangent vector } \vec{ξ} = (ξ(z), \hat{ξ}(z)) \in T_{\vec{a}}M_{m,n}, \text{ it holds that} \]

\[
\langle \vec{ω}, \vec{ξ} \rangle := \frac{1}{2πi} \oint_{\Gamma} (ω(z)ξ(z) + \hat{ω}(z)\hat{ξ}(z)) dz
\]

for $(ω(z), \hat{ω}(z)) \in T_{\vec{a}}^*M_{m,n}$ and $(ξ(z), \hat{ξ}(z)) \in T_{\vec{a}}M_{m,n}$.

Let us fix a basis of $T^*_a\mathcal{M}_{m,n}$ as

\[
\{e^*_i = ((z - \varphi)^i, 0), e^*_j = (0, (z - \varphi)^j) \mid i \geq -m + 1, j \leq n\}.
\]

and introduce two generating functions

\[
da(p) := \sum_{i \geq -m+1} (p - \varphi)^{-i-1}e^*_i = \left(\frac{(p - \varphi)^{m-1}}{(p - z)(z - \varphi)^{m-1}}, 0\right), \quad |p - \varphi| > |z - \varphi|,
\]

\[
d\hat{a}(q) := \sum_{j \leq n} (q - \varphi)^{-j-1}e^*_j = \left(0, \frac{(z - \varphi)^{n+1}}{(z - q)(q - \varphi)^{n+1}}\right), \quad |q - \varphi| < |z - \varphi|.
\]

The following lemma can be checked by using the Cauchy integral formula.

\textbf{Lemma 2.1.} For any cotangent vector $\vec{ω} = (ω(z), \hat{ω}(z)) \in T_{\vec{a}}^*\mathcal{M}_{m,n}$ and tangent vector $\vec{ξ} = (ξ(z), \hat{ξ}(z)) \in T_{\vec{a}}\mathcal{M}_{m,n}$, it holds that

\[
\langle \vec{ω}(z), 0 \rangle = \frac{1}{2\pi i} \oint_{|p - \varphi| > |z - \varphi|} ω(p)da(p) \, dp,
\]

\[
(0, \hat{ω}(z)) = \frac{1}{2\pi i} \oint_{|q - \varphi| < |z - \varphi|} \hat{ω}(q)d\hat{a}(q) \, dq
\]

and

\[
\langle da(p), \vec{ξ} \rangle = ξ(p), \quad \langle d\hat{a}(q), \vec{ξ} \rangle = \hat{ξ}(q).
\]

On the cotangent space $T^*_a\mathcal{M}_{m,n}$, let us introduce a symmetric bilinear form

\[
\langle da(p), dβ(q) \rangle := \frac{α'(p)}{p - q} + \frac{β'(q)}{q - p}, \quad α, β \in \{a, \hat{a}\},
\]

where the prime means to take the derivative with respect to the argument. For the right hand side of (2.7), we remark that whenever $α = β$, the denominator $(p - q)$ is eliminated by a factor of $α'(p) - α'(q)$; otherwise, suppose $α = a$ and $β = \hat{a}$,
then $\frac{1}{p-q}$ is expanded according to $|p-\varphi| > |q-\varphi|$ as in $\eqref{2.4}$–$\eqref{2.5}$. Such kind of convention will be used below.

Clearly, the pairing $\eqref{2.3}$ is nondegenerate, hence there is a linear map

$$\eta : T_{\hat{a}}^* \mathcal{M}_{m,n} \to T_{\hat{a}} \mathcal{M}_{m,n}$$

\text{(2.8)}

defined by

$$\langle \bar{\omega}_1, \eta \cdot \bar{\omega}_2 \rangle = \langle \bar{\omega}_1, \bar{\omega}_2 \rangle^*, \quad \forall \bar{\omega}_1, \bar{\omega}_2 \in T_{\hat{a}}^* \mathcal{M}_{m,n}.$$ \text{(2.9)}

We proceed to describe the map $\eta$ more explicitly.

**Lemma 2.2.** The linear map $\eta$ defined above can be represented as, for any cotangent vector $\bar{\omega} = (\omega(z), \hat{\omega}(z)) \in T_{\hat{a}}^* \mathcal{M}_{m,n},$

$$\eta \cdot \bar{\omega} = (a'(z)[\omega(z) + \hat{\omega}(z)]_+ - [\omega(z)a'(z) + \hat{\omega}(z)a'(z)]_+, -a'(z)[\omega(z) + \hat{\omega}(z)]_+ + [\omega(z)a'(z) + \hat{\omega}(z)a'(z)]_+).$$ \text{(2.10)}

**Proof.** Firstly, let us state the following two equalities that will be used below. For any function $f(z) = \sum_{i \in \mathbb{Z}} f_i(z - \varphi)^i$ holomorphic on a neighbourhood of $\Gamma$, it holds that

$$\frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \frac{1}{p-q} f(q) dq = f(p)_-, \quad \text{(2.11)}$$

$$\frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \frac{1}{p-q} f(p) dp = f(q)_+. \quad \text{(2.12)}$$

For $\bar{\omega} = (\omega(z), \hat{\omega}(z)) \in T_{\hat{a}}^* \mathcal{M}_{m,n}$, we denote $\eta \cdot \bar{\omega} = (\xi(z), \hat{\xi}(z)) \in T_{\hat{a}} \mathcal{M}_{m,n}$.

- If $\bar{\omega} = (\omega(z), 0)$, by using Lemma 2.1 we have

$$\xi(p) = \langle da(p), (\omega(z), 0) \rangle^*$$

$$= \frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \langle da(p), da(q) \rangle^* \omega(q) dq$$

$$= \frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \left( \frac{a'(p)}{p-q} + \frac{a'(q)}{q-p} \right) \omega(q) dq$$

$$= a'(p)\omega(p)_- - (a'(p)\omega(p))_-,$$

$$\hat{\xi}(q) = \langle d\hat{a}(q), (\omega(z), 0) \rangle^*$$

$$= \frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \langle d\hat{a}(q), da(p) \rangle^* \omega(p) dp$$

$$= \frac{1}{2\pi i} \oint_{|p-\varphi| > |q-\varphi|} \left( \frac{\hat{a}'(q)}{q-p} + \frac{\hat{a}'(p)}{p-q} \right) \omega(p) dp$$

$$= -\hat{a}'(q)\omega(q)_+ + (a'(q)\omega(q))_+.$$

Hence, we obtain

$$\eta \cdot (\omega(z), 0) = (a'(z)\omega(z)_- - (a'(z)\omega(z))_-, -a'(z)\omega(z)_+ + (a'(z)\omega(z))_+).$$
\begin{itemize}
\item If \( \bar{\zeta} = (0, \hat{\omega}(z)) \), similarly we have
   \[
   \eta \cdot (0, \hat{\omega}(z)) = (a'(z)\hat{\omega}(z)_- - (\hat{a}'(z)\hat{\omega}(z))_-, -\hat{a}'(z)\hat{\omega}(z)_+ + (\hat{a}'(z)\hat{\omega}(z))_+).
   \]
   Thus the lemma is proved due to the linearity of \( \eta \).
\end{itemize}

\textbf{Lemma 2.3.} The map \( \eta \) in (2.8) is a bijection.

\textit{Proof.} It follows from Lemma 2.2 that \( \eta \) is surjective, so it suffices to show \( \eta \) to be injective. For any \( (\omega(z), \hat{\omega}(z)) \in T^*_a \mathcal{M}_{m,n} \), denote \( (\xi(z), \hat{\xi}(z)) = \eta \cdot (\omega(z), \hat{\omega}(z)) \).

With the use of (2.10), one gets
\[
\xi(z) = a'(z)[\omega(z) + \hat{\omega}(z)]_+ + [\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_+.
\]
Furthermore, by using (1.6) one has
\[
\xi(z) - \hat{\xi}(z) = (a'(z) - \hat{a}'(z))(\hat{\omega}(z)_- - \omega(z)_+) = \zeta'(z)(\hat{\omega}(z)_- - \omega(z)_+).
\]
Since \( \zeta'(z) \neq 0 \) for \( z \in \Gamma \) (recall the condition (C3) in Section 1.2), then
\[
\omega(z)_+ = -\left( \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)} \right)_+ , \quad \hat{\omega}(z)_- = \left( \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)} \right)_-.
\]

On the other hand, since
\[
\frac{1}{a'(z)} = \frac{1}{m} z^{-m+1} + \mathcal{O}((z - \varphi)^{-m}) \in (z - \varphi)^{-m+1}\mathcal{H}^-,-
\]
\[
\frac{1}{\hat{a}'(z)} = -\frac{1}{n\hat{a}_{-n}}(z - \varphi)^{n+1} + \mathcal{O}((z - \varphi)^{n+2}) \in (z - \varphi)^{n+1}\mathcal{H}^+,-
\]
then from (2.13) and (2.14) it follows that
\[
([\omega(z) + \hat{\omega}(z)]_-)_{\geq -m+1} = \left( \frac{\xi(z)}{a'(z)} \right)_{\geq -m+1} , \quad ([\omega(z) + \hat{\omega}(z)]_+)_n = -\left( \frac{\xi(z)}{\hat{a}'(z)} \right)^n.
\]
Here for a Laurent series \( f(z) = \sum_{i \in \mathbb{Z}} f_i(z - \varphi)^i \), the following notations are used:
\[
f(z)_{\leq k} = \sum_{i \leq k} f_i(z - \varphi)^i , \quad f(z)_{\geq k} = \sum_{i \geq k} f_i(z - \varphi)^i.
\]
Recall \( T^*_a \mathcal{M}_{m,n} \) in (2.2), in combination of (2.15) and (2.16) one obtains
\[
\omega(z)_- = ([\omega(z) + \hat{\omega}(z)]_- - \hat{\omega}(z)_-)_{\geq -m+1} = \left( \frac{\xi(z)}{a'(z)} - \left( \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)} \right)_- \right)_{\geq -m+1},
\]
\[
\hat{\omega}(z)_+ = ([\omega(z) + \hat{\omega}(z)]_+ - \omega(z)_+)_{\leq n} = \left( -\frac{\xi(z)}{\hat{a}'(z)} + \left( \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)} \right)_+ \right)_{\leq n}.
\]
These equalities together with (2.15) mean that the cotangent vector \((\omega(z), \hat{\omega}(z))\) can be uniquely solved. Thus the proof of Lemma 2.3 is completed. \(\square\)

From the above proof, one also knows that

**Corollary 2.4.** For any vector \((\xi(z), \hat{\xi}(z)) \in T_{\tilde{a}}\mathcal{M}_{m,n}\), it holds that

\[
\eta^{-1}\left(\xi(z), \hat{\xi}(z)\right) = \begin{pmatrix}
\frac{\xi(z)}{a'(z)} - \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)} \\
\frac{\hat{\xi}(z)}{a'(z)} + \frac{\xi(z) - \hat{\xi}(z)}{\zeta'(z)}
\end{pmatrix}_{\geq m+1} \left(\begin{pmatrix}
\xi(z) - \hat{\xi}(z) \\
\xi(z) - \hat{\xi}(z)
\end{pmatrix}_{\leq n}\right). \tag{2.20}
\]

2.2. The flat metric. With the help of the bijection \(\eta\) in (2.8), let us introduce a bilinear form on the tangent space \(T_{\tilde{a}}\mathcal{M}_{m,n}\) as

\[
\langle \partial_1, \partial_2 \rangle \eta := \langle \eta^{-1}(\partial_1), \eta^{-1}(\partial_2) \rangle, \quad \partial_1, \partial_2 \in T_{\tilde{a}}\mathcal{M}_{m,n}. \tag{2.21}
\]

**Lemma 2.5.** For any vectors \(\partial_1, \partial_2 \in T_{\tilde{a}}\mathcal{M}_{m,n}\), the bilinear form (2.21) is given by

\[
\langle \partial_1, \partial_2 \rangle \eta = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{\partial_1 \zeta(z) \cdot \partial_2 \zeta(z)}{\zeta'(z)} dz - \left(\text{Res}_{z=\infty} + \text{Res}_{z=\varphi}\right) \frac{\partial_1 \ell(z) \cdot \partial_2 \ell(z)}{\ell'(z)} dz. \tag{2.22}
\]

**Proof.** Suppose that \(\partial_{\nu}\) are identified with \((\xi_{\nu}(z), \hat{\xi}_{\nu}(z))\) for \(\nu = 1, 2\), it is easy to see

\[
\partial_{\nu} \zeta(z) = \xi_{\nu}(z) - \hat{\xi}_{\nu}(z), \quad \partial_{\nu} \ell(z) = \xi_{\nu}(z)_+ + \hat{\xi}_{\nu}(z)_-, \quad \nu = 1, 2.
\]

Denote

\[
\eta^{-1}(\partial_1) = (\omega(z), \hat{\omega}(z)),
\]

then, according to the definition (2.21), one has

\[
\langle \partial_1, \partial_2 \rangle \eta = \frac{1}{2\pi i} \oint_{\Gamma} (\omega(z) \xi_2(z) + \hat{\omega}(z) \hat{\xi}_2(z)) dz = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\]

where

\[
\mathcal{I}_1 = \frac{1}{2\pi i} \oint_{\Gamma} (\omega(z)_+ - \hat{\omega}(z)_-) \left(\xi_2(z) - \hat{\xi}_2(z)\right) dz,
\]

\[
\mathcal{I}_2 = \frac{1}{2\pi i} \oint_{\Gamma} (\omega(z)_+ + \hat{\omega}(z)_-) \xi_2(z) dz,
\]

\[
\mathcal{I}_3 = \frac{1}{2\pi i} \oint_{\Gamma} (\omega(z)_+ + \hat{\omega}(z)_+) \hat{\xi}_2(z) dz.
\]

Let us proceed to calculate these integrals. Firstly, by using (2.15) one has

\[
\mathcal{I}_1 = \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\xi_1(z) - \hat{\xi}_1(z))(\xi_2(z) - \hat{\xi}_2(z))}{\zeta'(z)} dz = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{\partial_1 \zeta(z) \cdot \partial_2 \zeta(z)}{\zeta'(z)} dz.
\]

Secondly, observe that when \(|z| \to \infty\) and \(\nu = 1, 2\), one has

\[
\partial_{\nu} a(z), \partial_{\nu} \ell(z) \in (z - \varphi)^{m-2} \mathcal{H}_{\varphi}^-; \quad \partial_{\nu} a(z) - \partial_{\nu} \ell(z) \in (z - \varphi)^{-1} \mathcal{H}_{\varphi}^-; \quad \frac{1}{a'(z)} - \frac{1}{\ell'(z)} \in (z - \varphi)^{-2m+1} \mathcal{H}_{\varphi}^-.
\]
With the use of (2.16) and \( \ell(z) = a(z) + \hat{a}(z)_- \), one obtains

\[
\mathcal{I}_2 = \frac{1}{2\pi i} \oint_{\Gamma} \left( \left[ \omega(z) + \hat{\omega}(z)_- \right] \right)_{\geq-m+1} \xi_2(z) \, dz
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 a(z)}{a'(z)} \right)_{\geq-m+1} \partial_2 a(z) \, dz
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 \ell(z)}{\ell'(z)} \right)_{\geq-m+1} \partial_2 \ell(z) \, dz
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 \ell(z)}{\ell'(z)} \right)_{\geq-m+1} \partial_2 \ell(z) \, dz
\]

\[
= - \mathrm{Res}_{z=\infty} \frac{\partial_1 \ell(z)_+ \partial_2 \ell(z)}{\ell'(z)} \, dz.
\]

Similarly, when \( |z - \varphi| \to 0 \) and \( \nu = 1, 2 \), one has

\[
\partial_{\nu} \hat{a}(z), \partial_{\nu} \hat{\ell}(z) \in (z - \varphi)^{-n+1} \mathcal{H}^+_{\varphi}; \quad \partial_{\nu} a(z) - \partial_{\nu} \ell(z) \in \mathcal{H}^+_{\varphi};
\]

\[
\frac{1}{\partial'(z)}, \frac{1}{\ell'(z)} \in (z - \varphi)^{n+1} \mathcal{H}^+_{\varphi}, \quad \frac{1}{\partial'(z)} - \frac{1}{\ell'(z)} \in (z - \varphi)^{2n+2} \mathcal{H}^+_{\varphi}
\]

and

\[
\mathcal{I}_3 = \frac{1}{2\pi i} \oint_{\Gamma} \left( \left[ \omega(z) + \hat{\omega}(z)_+ \right] \right)_{\leq n} \hat{\xi}_2(z) \, dz
\]

\[
= - \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 \hat{a}(z)}{\hat{\partial}'(z)} \right)_{\leq n} \partial_2 \hat{a}(z) \, dz
\]

\[
= - \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 \ell(z)}{\hat{\partial}'(z)} \right)_{\leq n} \partial_2 \ell(z) \, dz
\]

\[
= - \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{\partial_1 \ell(z)}{\ell'(z)} \right)_{\leq n} \partial_2 \ell(z) \, dz
\]

\[
= - \mathrm{Res}_{z=\varphi} \frac{\partial_1 \ell(z)_+ \partial_2 \ell(z)}{\ell'(z)} \, dz.
\]

Therefore, the lemma is proved. \( \square \)

The bilinear form (2.21) can be viewed as a metric on the manifold \( \mathcal{M}_{m,n} \) (not necessary to be positively definite). We proceed to show that this metric is a flat one. According to the condition (C3) in Section 1.2, we can consider the inverse function of \( \zeta(z) \), i.e.,

\[
z = z(\zeta) : \Sigma \to \Gamma.
\]

This function can be extended holomorphically to a neighborhood of \( \Sigma \) that surrounds \( \zeta = 0 \). Let us take the Laurent expansion

\[
z(\zeta) = \sum_{i \in \mathbb{Z}} t_i \zeta^i, \quad \zeta \in \Sigma, \quad (2.23)
\]
where
\[ t_i = \frac{1}{2\pi i} \oint_{\Sigma} z(\zeta)\zeta^{-i-1}d\zeta, \quad i \in \mathbb{Z}. \] (2.24)

On the other hand, we introduce the following two functions in a punctured neighborhood of \( \infty \) and of \( \varphi \) respectively,
\[ \chi(z) := \ell(z) \frac{1}{z} = z + \chi_1 z^{-1} + \chi_2 z^{-2} + \cdots \quad \text{near } \infty; \] (2.25)
\[ \hat{\chi}(z) := \ell(z) \frac{1}{z} = \hat{\alpha}_{-n}(z - \varphi)^{-1} + \hat{\chi}_0 + \hat{\chi}_1 (z - \varphi) + \cdots \quad \text{near } \varphi. \] (2.26)

The inverse functions of them can be represented as
\[ z(\chi) = \chi - h_1 \chi^{-1} - h_2 \chi^{-2} - \cdots - h_{m-1} \chi^{-m+1} + \mathcal{O}(\chi^{-m}), \quad |\chi| \to \infty; \] (2.27)
\[ z(\hat{\chi}) = \hat{h}_0 + \hat{h}_1 \hat{\chi}^{-1} + \hat{h}_2 \hat{\chi}^{-2} + \cdots + \hat{h}_n \hat{\chi}^{-n} + \mathcal{O}(\hat{\chi}^{-n-1}), \quad |\hat{\chi}| \to \infty. \] (2.28)

In particular, one has \( \hat{h}_0 = \varphi \). It is easy to see that the correspondence
\[ (\zeta(z), \ell(z)) \longleftrightarrow \{ t_i \}_{i \in \mathbb{Z}} \cup \{ h_j \}_{j=1}^{m-1} \cup \{ \hat{h}_k \}_{k=0}^{n} \] (2.29)

is one-to-one. In other words, we obtain a system of coordinates of \( \mathcal{M}_{m,n} \). Clearly, such coordinates can be represented equivalently as (1.10)–(1.12), and in what follows we will use the notations \( t, h \) and \( \hat{h} \) given in (1.2). Moreover, unless otherwise stated, the following convention of indices will be assumed below
\[ i, i_1, i_2, i_3 \in \mathbb{Z}, \quad j, j_1, j_2, j_3 \in \{1, 2, \ldots, m - 1\}, \quad k, k_1, k_2, k_3 \in \{0, 1, \ldots, n\}. \]

**Proposition 2.6.** The metric defined by (2.21) is flat, and \( t \cup h \cup \hat{h} \) of \( \mathcal{M}_{m,n} \) serves as a system of flat coordinates. More precisely, these flat coordinates satisfy
\[ \left\langle \frac{\partial}{\partial h_{i_1}}, \frac{\partial}{\partial h_{i_2}} \right\rangle_{\eta} = -\delta_{i_1+i_2}, \] (2.30)
\[ \left\langle \frac{\partial}{\partial h_{j_1}}, \frac{\partial}{\partial h_{j_2}} \right\rangle_{\eta} = m\delta_{m,j_1+j_2}, \] (2.31)
\[ \left\langle \frac{\partial}{\partial h_{k_1}}, \frac{\partial}{\partial h_{k_2}} \right\rangle_{\eta} = n\delta_{n,k_1+k_2}, \] (2.32)

and
\[ \left\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial h_j} \right\rangle_{\eta} = \left\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial \hat{h}_j} \right\rangle_{\eta} = \left\langle \frac{\partial}{\partial h_j}, \frac{\partial}{\partial \hat{h}_k} \right\rangle_{\eta} = 0. \] (2.33)
Proof. For preparation, let us show the following equalities:

\[
\frac{\partial \zeta(z)}{\partial t_i} = -\zeta(z)^i\zeta'(z), \quad \frac{\partial \ell(z)}{\partial t_i} = 0, \quad (2.34)
\]

\[
\frac{\partial \zeta(z)}{\partial h_j} = 0, \quad \frac{\partial \ell(z)}{\partial h_j} = (\ell'(z)\chi(z)^{-j})_+, \quad (2.35)
\]

\[
\frac{\partial \zeta(z)}{\partial \hat{h}_k} = 0, \quad \frac{\partial \ell(z)}{\partial \hat{h}_k} = -(\ell'(z)\hat{\chi}(z)^{-k})_- \quad (2.36)
\]

Firstly, the inverse function \((2.23)\) of \(\zeta(z)\) satisfies \(z = z(\zeta)\mid_{\zeta \mapsto \zeta(z)}\). Taking its derivative with respective to \(t_i\), one has

\[
0 = \left. \frac{\partial z(\zeta)}{\partial t_i} \right|_{\zeta \mapsto \zeta(z)} + z'(\zeta)\left. \frac{\partial \zeta(z)}{\partial t_i} \right|_{\zeta \mapsto \zeta(z)},
\]

which leads to the first equality in \((2.34)\), namely,

\[
\frac{\partial \zeta(z)}{\partial t_i} = -\frac{1}{z'(\zeta)} \left. \zeta(z)^i \right|_{\zeta \mapsto \zeta(z)} = -\zeta'(z)\zeta(z)^i .
\]

Similarly, from \((2.25)\) one has

\[
\frac{\partial \chi(z)}{\partial h_j} = \chi'(z) \cdot (\chi^{-j} + O(\chi^{-m}))|_{\chi \mapsto \chi(z)} = \chi'(z) (\chi(z)^{-j} + O(\chi^{-m})) , \quad |z| \to \infty,
\]

hence

\[
\frac{\partial \ell(z)}{\partial h_j} = \frac{\partial \ell(z)_+}{\partial h_j} = \left( m\chi(z)^{m-1} \frac{\partial \chi(z)}{\partial h_j} \right)_+ = (m\chi(z)^{m-1-j} \chi'(z))_+ = (\ell'(z)\chi(z)^{-j})_+ ,
\]

which is the second equality in \((2.35)\). In the same way when \(|z - \varphi| \to 0\), for \((2.26)\) one gets

\[
\frac{\partial \hat{\chi}(z)}{\partial \hat{h}_k} = -\hat{\chi}'(z) \cdot (\hat{\chi}^{-k} + O(\hat{\chi}^{-n}))|_{\hat{\chi} \mapsto \hat{\chi}(z)} = -\hat{\chi}'(z) (\hat{\chi}(z)^{-k} + O((z - \varphi)^{n+1})),
\]

hence

\[
\frac{\partial \ell(z)}{\partial \hat{h}_k} = \frac{\partial \ell(z)_-}{\partial \hat{h}_k} = \left( n\hat{\chi}(z)^{n-1} \frac{\partial \hat{\chi}(z)}{\partial \hat{h}_k} \right)_- = -(n\hat{\chi}(z)^{n-1-k} \hat{\chi}'(z))_- = -(\ell'(z)\hat{\chi}(z)^{-k})_- ,
\]

which is just the second equality in \((2.36)\). The other cases in \((2.34) - (2.36)\) are trivial.
Now we are ready to verify the proposition based on Lemma 2.5. We have that
\[
\left\langle \frac{\partial}{\partial t_{i}}, \frac{\partial}{\partial t_{j}} \right\rangle_{\eta} = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{\zeta^{i+i+j}\zeta'}{\zeta'} dz = -\delta_{-i, i+j},
\]
\[
\left\langle \frac{\partial}{\partial h_{j}}, \frac{\partial}{\partial h_{j}} \right\rangle_{\eta} = -\text{Res}_{z=\infty} \frac{(\ell'\chi^{-j})_+ (\ell'\chi^{-j})_+}{\ell'} dz
\]
\[
= -m \text{Res}_{z=\infty} \chi^{-j} \chi^{-j} dz = m\delta_{m,j1+j2},
\]
\[
\left\langle \frac{\partial}{\partial h_{k}}, \frac{\partial}{\partial h_{k}} \right\rangle_{\eta} = -\text{Res}_{z=\infty} \frac{(\ell'\chi^{-k})_+ (\ell'\chi^{-k})_+}{\ell'} dz
\]
\[
= -n \text{Res}_{z=\infty} \chi^{-k} \chi^{-k} dz = n\delta_{n,k1+k2},
\]
and that all other pairings of such vectors vanish. Thus the proposition is proved. □

The above proof together with (1.6) also implies

**Corollary 2.7.** In the tangent space $T_{a}^{\ast} \mathcal{M}_{m,n}$, there is a basis consisting of vectors represented as
\[
\frac{\partial}{\partial t_{i}} = (-\zeta^{i}(z)^{-i}(z))_-, \quad i \in \mathbb{Z}; \quad (2.37)
\]
\[
\frac{\partial}{\partial h_{j}} = ((\ell' \chi(z)^{-j})_+, (\ell' \chi(z)^{-j})_+), \quad 1 \leq j \leq m-1; \quad (2.38)
\]
\[
\frac{\partial}{\partial h_{k}} = (-\ell' (\chi(z)^{-k})_-, -\ell' (\chi(z)^{-k})_-), \quad 0 \leq k \leq n. \quad (2.39)
\]

Furthermore, this result together with Corollary 2.4 leads to

**Corollary 2.8.** In the cotangent space $T_{a}^{\ast} \mathcal{M}_{m,n}$, there is a basis consisting of vectors represented as
\[
\eta^{-1} \frac{\partial}{\partial t_{i}} = ((\zeta(z)^{i})_{\geq -m+1}, -\zeta(z)^{i})_{\leq n}), \quad i \in \mathbb{Z}; \quad (2.40)
\]
\[
\eta^{-1} \frac{\partial}{\partial h_{j}} = ((\chi(z)^{-j})_{\geq -m+1}, 0), \quad 1 \leq j \leq m-1; \quad (2.41)
\]
\[
\eta^{-1} \frac{\partial}{\partial h_{k}} = (0, (\chi(z)^{-k})_{\leq n}), \quad 0 \leq k \leq n. \quad (2.42)
\]
Proof. We substitute (2.37)–(2.39) into the formula (2.20) and observe the following equalities:

\[
\left( \frac{f(z)-}{a'(z)} \right)_{ \geq -m+1} = 0, \quad \left( \frac{f(z)+}{a'(z)} \right)_{ \leq n} = 0 \quad \text{with arbitrary } f(z) = \sum_{i \in \mathbb{Z}} f_i (z - \varphi)^i;
\]

\[
(\ell'(z)\chi(z)^{-j})_+ = (a'(z)\chi(z)^{-j})_+, \quad (\ell'(z)\chi(z)^{-k})_- = (a'(z)\chi(z)^{-k})_-,
\]

then the corollary is achieved after a straightforward calculation. \(\square\)

### 2.3. The Frobenius algebra structure.

For any \(\vec{a} = (a(z), \hat{a}(z)) \in \mathcal{M}_{m,n}\), let us introduce a product on the cotangent space \(T^*_a \mathcal{M}_{m,n}\) as

\[
d\alpha(p) \star d\beta(q) = \frac{\beta'(q)}{q-p}d\alpha(p) + \frac{\alpha'(p)}{p-q}d\beta(q), \quad \alpha, \beta \in \{a, \hat{a}\}. \tag{2.43}
\]

**Lemma 2.9.** The product on \(T^*_a \mathcal{M}_{m,n}\) defined by (2.43) is commutative, associative and invariant with respect to the bilinear form given in (2.7).

**Proof.** The product in (2.43) is clearly commutative. What is more, for \(\alpha_1, \alpha_2, \alpha_3 \in \{a, \hat{a}\}\), one has

\[
d\alpha_1(p_1) \star d\alpha_2(p_2) \star d\alpha_3(p_3)
\]

\[
= \left( \frac{\alpha'_1(p_1)}{p_1 - p_2}d\alpha_2(p_2) + \frac{\alpha'_2(p_2)}{p_2 - p_1}d\alpha_1(p_1) \right) \star d\alpha_3(p_3)
\]

\[
= \frac{\alpha'_1(p_1)}{p_1 - p_2} \frac{\alpha'_2(p_2)}{p_2 - p_3}d\alpha_3(p_3) + \frac{\alpha'_1(p_1)}{p_1 - p_2} \frac{\alpha'_3(p_3)}{p_3 - p_2}d\alpha_2(p_2)
\]

\[
+ \frac{\alpha'_2(p_2)}{p_2 - p_1} \frac{\alpha'_1(p_1)}{p_1 - p_3}d\alpha_3(p_3) + \frac{\alpha'_2(p_2)}{p_2 - p_1} \frac{\alpha'_3(p_3)}{p_3 - p_1}d\alpha_1(p_1)
\]

\[
= \left\{ \alpha'_1(p_1)\alpha'_2(p_2)d\alpha_3(p_3)(p_1 - p_2) + c.p.(1,2,3) \right\}
\]

where c.p. means “cyclic permutation”. So this product is associative. Finally, by using (2.7) it is straightforward to verify

\[
\langle d\alpha_1(p_1) \star d\alpha_2(p_2), d\alpha_3(p_3) \rangle^*
\]

\[
= \left( \frac{\alpha'_1(p_1)}{p_1 - p_2}d\alpha_2(p_2) + \frac{\alpha'_2(p_2)}{p_2 - p_1}d\alpha_1(p_1), d\alpha_3(p_3) \right)^*
\]

\[
= \frac{\alpha'_1(p_1)}{p_1 - p_2} \left( \frac{\alpha'_2(p_2)}{p_2 - p_3} + \frac{\alpha'_3(p_3)}{p_3 - p_2} \right) + \frac{\alpha'_2(p_2)}{p_2 - p_1} \left( \frac{\alpha'_1(p_1)}{p_1 - p_3} + \frac{\alpha'_3(p_3)}{p_3 - p_1} \right)
\]

\[
= \frac{\alpha'_1(p_1)\alpha'_2(p_2)}{(p_1 - p_3)(p_2 - p_3)} + c.p.(1,2,3),
\]

which implies the invariance of the product. The lemma is proved. \(\square\)
Lemma 2.10. The product defined by \((\omega(z), 0) \ast (\omega(z), 0)\) can be represented as follows: for any \(\omega_1 = (\omega_1(z), \omega_1(z))\), \(\omega_2 = (\omega_2(z), \omega_2(z)) \in T_{\alpha}^* M_{m,n}\),
\[
\omega_1 \ast \omega_2 = ([\omega_2(\omega_1a')]_+ - \omega_1(\omega_2a')_+ + \omega_1(\omega_2a')_+ + \omega_1(\omega_2a')_+)_1 \leq m+1,
\]
\[
[\omega_2(\omega_1a')]_+ - \omega_1(\omega_2a')_+ + \omega_1(\omega_2a')_+ + \omega_1(\omega_2a')_+)_2 \leq n).
\]
\[(2.44)\]

Proof. In order to simplify the calculation, let us check the case \((\omega_1, 0) \ast (\omega_2, 0)\). By using Lemma \[\ref{lem:2.10}\] and the equalities \((\ref{eq:2.11})-(\ref{eq:2.12})\), we have
\[
(\omega_1(z), 0) \ast (\omega_2(z), 0)
= \frac{1}{(2\pi i)^2} \int_{|q-z|>|p-z|}^\omega \omega_1(p) da(p) \ast da(q) \omega_2(q) dp dq
= \frac{1}{(2\pi i)^2} \int_{|q-z|>|p-z|}^\omega \omega_1(p) \left( \frac{a'(p)}{p-q} da(q) + \frac{a'(q)}{q-p} da(p) \right) \omega_2(q) dp dq, 0
= \frac{1}{2\pi i} \left( \int_{|q-z|>|p-z|}^\omega \omega_1(q) a'(q) da(q) dq
- \int_{|p-z|>|q-z|}^\omega \omega_1(p) \omega_2(p) a'(p) da(p) dp, 0 \right)
= \left( (\omega_1(z) a'(z))_1 + \omega_2(z) - \omega_1(z) (\omega_2(z) a'(z))_2 \right)_{1 \leq m+1, 0}.
\]

Observe that the result is indeed symmetric with respect to the indices 1 and 2. The cases \((0, \omega_1) \ast (\omega_2, 0), (\omega_1, 0) \ast (0, \omega_2)\) and \((0, \omega_1) \ast (0, \omega_2)\) are similar. The lemma is proved. \(\square\)

As an immediate application of Lemma \[\ref{lem:2.10}\], we have the following result.

Corollary 2.11. For the product defined by \((\ref{eq:2.13})\), there is a unity cotangent vector as
\[
\hat{e}^* = \left( \frac{1}{m} (z - \varphi)^{-m+1}, 0 \right) \in T_{\alpha}^* M_{m,n}.
\]
\[(2.45)\]

Proof. The corollary is verified by using the property \(\left( \frac{1}{m} (z - \varphi)^{-m+1} a'(z) \right) = 1. \) \(\square\)

Summarizing the above discussions, one knows that \((T_{\alpha}^* M_{m,n}, \ast, \hat{e}^*, \langle \cdot, \cdot \rangle^*)\) is a Frobenius algebra. With the help of the bijection \(\eta : T_{\alpha}^* M_{m,n} \rightarrow T_{\alpha}^* M_{m,n}\) defined in \[\ref{eq:2.30}\], it is induced a Frobenius algebra structure on \(T_{\alpha}^* M_{m,n}\). More precisely, we arrive at the following result.

Proposition 2.12. The quadruple \((T_{\alpha}^* M_{m,n}, \circ, \hat{e}, \langle \cdot, \cdot \rangle_\eta)\) is a Frobenius algebra, in which the product \(\circ\) is given by
\[
\vec{\xi}_1 \circ \vec{\xi}_2 := \eta \cdot \left( \eta^{-1}(\vec{\xi}_1) \ast \eta^{-1}(\vec{\xi}_2) \right), \quad \vec{\xi}_1, \vec{\xi}_2 \in T_{\alpha}^* M_{m,n}
\]
\[(2.46)\]
and the unit vector field $\vec{e}$ is
\begin{equation}
\begin{cases}
\frac{\partial}{\partial t_0} + \frac{\partial}{\partial h_0}, & m = 1, \\
\frac{1}{m} \frac{\partial}{\partial h_{m-1}}, & m \geq 2.
\end{cases}
\end{equation}

(2.47)

Proof. It suffices to verify the formula (2.47), say $\vec{e} = \eta \cdot \vec{e}^*$. To this end, let us check $\eta^{-1}(\vec{e}) = \vec{e}^*$. When $m \geq 2$, by using (2.41) one has
\begin{equation}
\eta^{-1} \left( \frac{1}{m} \frac{\partial}{\partial h_{m-1}} \right) = \left( \frac{1}{m} (\chi(z)^{-m+1})_{z_{m+1},0} \right) \in T_{\bar{a}} \mathcal{M}_{m,n}.
\end{equation}

(2.48)

This fact together with (2.25), i.e.,
\begin{equation}
\chi(z) = z + \chi_1 z^{-1} + \chi_2 z^{-2} + \cdots = (z - \varphi) + \varphi \chi_1 (z - \varphi)^{-1} + \cdots \text{ near } \infty,
\end{equation}

(2.49)

leads to
\begin{equation}
\eta^{-1} \left( \frac{1}{m} \frac{\partial}{\partial h_{m-1}} \right) = \left( \frac{1}{m} (z - \varphi)^{-m+1}, 0 \right) = \vec{e}^*.
\end{equation}

(2.50)

When $m = 1$, from (2.40) and (2.42) it follows that
\begin{equation}
\eta^{-1} \left( \frac{\partial}{\partial t_0} + \frac{\partial}{\partial h_0} \right) = (1, 0) = \vec{e}^*.
\end{equation}

(2.51)

The proposition is proved. □

Note that one can also represent the unity vector field in the form $\vec{e} = \frac{\partial}{\partial v}$ with some flat coordinate $v$ via a linear transformation of $t \cup h \cup \hat{h}$.

2.4. The symmetric 3-tensor. In this subsection, we want to compute a 3-tensor $c$ defined by
\begin{equation}
c(\partial_u, \partial_v, \partial_w) := \langle \partial_u \circ \partial_v, \partial_w \rangle_\eta, \quad u, v, w \in t \cup h \cup \hat{h},
\end{equation}

(2.52)

where $\partial_u = \frac{\partial}{\partial u} \in T_{\bar{a}} \mathcal{M}_{m,n}$. Clearly, the right hand side of (2.52) is symmetric with respect to $\partial_u, \partial_v$, and $\partial_w$. 
Proposition 2.13. The symmetric 3-tensor in (2.32) can be represented as
\[
c(\partial_{h_{k1}}, \partial_{h_{k2}}, \partial_{h_{k3}}) = \frac{1}{2\pi i} \int_{\Gamma} (\zeta^{i_1+i_2} \zeta' + \zeta' + \zeta'') + \{\zeta^{i_1+i_2} \zeta' (\zeta'^{i_1} \zeta'^{i_2}) - a \text{ c.p.}(i_1, i_2, i_3)\} \, dz,
\]
where \(\chi\) and \(\hat{\chi}\) are given in (2.25) and (2.26) respectively.

Before proving this proposition, let us show a lemma as follows.

Lemma 2.14. The following equalities hold:
\[
\eta^{-1} \frac{\partial}{\partial t_{11}} \star \eta^{-1} \frac{\partial}{\partial t_{12}} = \left(\left[\zeta^{i_1+i_2} \zeta' - \zeta^{i_1} \zeta'^{i_2} \zeta'^{i_1} \zeta'^{i_2}\right]_{\frac{1}{2}} \right)_{\geq m+1},
\]
\[
\eta^{-1} \frac{\partial}{\partial h_{11}} \star \eta^{-1} \frac{\partial}{\partial h_{12}} = \left(\left[\zeta^{i_1+i_2} \zeta' - \zeta^{i_1} \zeta'^{i_2} \zeta'^{i_1} \zeta'^{i_2}\right]_{\frac{1}{2}} \right)_{\geq m+1},
\]
(2.53)
\[
\eta^{-1} \frac{\partial}{\partial h_{k1}} \star \eta^{-1} \frac{\partial}{\partial h_{k2}} = \left(\left[\hat{\chi}^{k_1} \hat{\chi}^{k_2} \hat{\chi}' \hat{\chi}' \hat{\chi}' \hat{\chi}'\right]_{\frac{1}{2}} \right)_{\leq n},
\]
\[
\eta^{-1} \frac{\partial}{\partial h_{11}} \star \eta^{-1} \frac{\partial}{\partial h_{12}} = \left(\left[\hat{\chi}^{k_1} \hat{\chi}^{k_2} \hat{\chi}' \hat{\chi}' \hat{\chi}' \hat{\chi}'\right]_{\frac{1}{2}} \right)_{\leq n}.
\]

Proof. This lemma is verified by taking Corollary 2.8 and Lemma 2.10 together, with the help of the following equalities for any series \(f(z)\) and \(g(z)\) in \((z - \varphi)\),
\[
(f_{\geq m+1} a')_+ = |f a' + f g|_{\geq m+1} = (f g)_+ \geq m+1,
\]
\[
(f_{\leq n} a')_+ = (f a')_+, \quad (f_{\leq n} g)_+ \leq n = (f g)_+ \leq n.
\]

For instance, one has
\[
\eta^{-1} \frac{\partial}{\partial t_{11}} \star \eta^{-1} \frac{\partial}{\partial t_{12}} = ((\zeta^{i_1})_{\geq m+1}, - (\zeta^{i_1})_{\leq n}) \star ((\zeta^{i_2})_{\geq m+1}, - (\zeta^{i_2})_{\leq n}) = (\omega, \hat{\omega}),
\]
where
\[\omega = [((\zeta^{i_2})_{\geq -m+1}(\zeta^{i_1}a')_+ - \zeta^{i_1}((\zeta^{i_2})_{\geq -m+1}a')_+ \\
+ ((\zeta^{i_2})_{\geq -m+1}(\zeta^{i_1}a')_- + (\zeta^{i_2})_{\geq -m+1}(\zeta^{i_2}a')_-)]_{\geq -m+1} \\
= [-((\zeta^{i_2})_{\geq -m+1}(\zeta^{i_1}a')_- + (\zeta^{i_2})_{\geq -m+1}(\zeta^{i_1}a')_+ - \zeta^{i_1}((\zeta^{i_2})_{\geq -m+1}a')_+ \\
+ (\zeta^{i_2}((\zeta^{i_1}a')_- + \zeta^{i_1}(\zeta^{i_2}a')_-)]_{\geq -m+1} \\
= [\zeta^{i_1}(\zeta^{i_2}a')_- - \zeta^{i_1}(\zeta^{i_2}(a' - a'))_+ - \zeta^{i_2}(\zeta^{i_1}(a' - a'))_-]_{\geq -m+1} \\
= [\zeta^{i_1}+12a' - \zeta^{i_1}(\zeta^{i_2}c')_- - \zeta^{i_2}(\zeta^{i_1}c')_-]_{\geq -m+1}, \\
\check{\omega} = [\zeta^{i_2}((\zeta^{i_1})_{\leq n}(\zeta^{i_2}a')_- - \zeta^{i_1}((\zeta^{i_2})_{\leq n}a')_+ - \zeta^{i_1}((\zeta^{i_2})_{\leq n}a')_+]_{\leq n} \\
= [-\zeta^{i_2}((\zeta^{i_1})_{\leq n}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}a')_- + (\zeta^{i_2})(\zeta^{i_1}a')_- - \zeta^{i_2}(\zeta^{i_1}a')_+ - \zeta^{i_1}(\zeta^{i_2}a')_+]_{\leq n} \\
= [-\zeta^{i_1}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}(a' - a'))_- + \zeta^{i_2}(\zeta^{i_1}(a' - a'))_+]_{\leq n} \\
= [-\zeta^{i_1}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}(a' - a'))_- - \zeta^{i_2}(\zeta^{i_1}c')_+]_{\leq n}.
\]

So the first equality in (2.58) is valid. The other cases are similar, thus the lemma is proved. \(\square\)

**Proof of Proposition 2.15** According to (2.3), (2.21) and (2.46), we have
\[\langle \partial_u \circ \partial_v, \partial_w \rangle_\eta = \langle \eta^{-1} \partial_u, \eta^{-1} \partial_v, \partial_w \rangle, \quad u, v, w \in t \cup h \cup \hat{h}. \quad (2.54)\]

Let us substitute into it with the data in Lemma 2.14 and Corollary 2.7. For instance,
\[c(\partial_{i_1}, \partial_{i_2}, \partial_{i_3}) = \left\langle \frac{\partial}{\partial t_{i_1}} \circ \frac{\partial}{\partial t_{i_2}} : \frac{\partial}{\partial t_{i_3}} \right\rangle_\eta \]
\[= \frac{1}{2\pi i} \int_\Gamma \left\{ [\zeta^{i_1}(\zeta^{i_2}a')_- - \zeta^{i_1}(\zeta^{i_2}c')_- - \zeta^{i_2}(\zeta^{i_1}c')_-](-\zeta^{i_3}c')_- \\
+ [-\zeta^{i_1}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}c')_- - \zeta^{i_2}(\zeta^{i_1}c')_-](-\zeta^{i_3}c')_- \right\} dz \]
\[= \frac{1}{2\pi i} \int_\Gamma \left\{ [-\zeta^{i_1}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}c')_- + \zeta^{i_2}(\zeta^{i_1}c')_-](-\zeta^{i_3}c')_- \\
+ [-\zeta^{i_1}(\zeta^{i_2}a')_- + \zeta^{i_1}(\zeta^{i_2}(a' - a'))_- + \zeta^{i_2}(\zeta^{i_1}(a' - a'))_-](-\zeta^{i_3}c')_- \right\} dz \]
\[= \frac{1}{2\pi i} \int_\Gamma \left\{ [-\zeta^{i_1}(\zeta^{i_2}(a' - a'))_- + \zeta^{i_1}(\zeta^{i_2}(a' - a'))_- - \zeta^{i_2}(\zeta^{i_1}(a' - a'))_-](-\zeta^{i_3}c')_- \right\} dz \]
\[= \frac{1}{2\pi i} \int_\Gamma \left\{ [-\zeta^{i_1}(\zeta^{i_2}(a' - a'))_- + \zeta^{i_1}(\zeta^{i_2}(a' - a'))_- - \zeta^{i_2}(\zeta^{i_1}(a' - a'))_- + c.p.(i_1, i_2, i_3)] \right\} dz. \]

The other cases are similar. Thus the proposition is proved. \(\square\)
2.5. The potential function. We proceed to show that the values \(< \partial_u \circ \partial_v, \partial_w >_\eta\) in Proposition 2.13 can be represented as the third-order derivatives of a certain potential function. To this end, let us state the following lemma first.

Lemma 2.15. There locally exist three functions \(V_1(t), V_2(h)\) and \(G(h, \hat{h})\) such that

\[
\frac{\partial^2 V_1}{\partial t_{i_1} \partial t_{i_2}} = \frac{1}{2\pi i} \oint \frac{\zeta'(z)\zeta(z)^{i_1+i_2}}{z} dz, \tag{2.55}
\]

\[
\frac{\partial^2 V_2}{\partial h_{j_1} \partial h_{j_2}} = -\operatorname{Res}_{z=\infty} \ell'(z)\chi(z)^{-j_1-j_2} dz, \tag{2.56}
\]

\[
\frac{\partial^3 G}{\partial u \partial v \partial w} = -\left(\operatorname{Res}_{z=\infty} + \operatorname{Res}_{z=\varphi}\right) \frac{\partial_u \ell(z) \cdot \partial_v \ell(z) \cdot \partial_w \ell(z)}{\ell'(z)} dz. \tag{2.57}
\]

with \(u, v, w \in h \cup \hat{h}\).

Proof. Firstly, it follows from (2.34) that

\[
\frac{1}{2\pi i} \oint \frac{\zeta'(z)\zeta(z)^{i_1+i_2}}{z} dz = \frac{1}{2\pi i} \oint \frac{-(\zeta'(z)\zeta(z)^{i_1})' \cdot \zeta(z)^{i_1+i_2} - \zeta'(z) \cdot (i_1 + i_2)\zeta'(z)\zeta(z)^{i_1+i_2+i_3-1}(z)}{z} dz
\]

which is symmetric with respect to the indices \(i_1, i_2\) and \(i_3\). So the existence of \(V_1\) is verified.

Secondly, in consideration of the form of \(\ell(z)\) and \(\chi(z)\), we observe that the right hand side of (2.56) depends only on \(\{h_j\}_{j=1}^{m-1}\). In the same way as above, by using (2.35) we have

\[
-\frac{\partial}{\partial h_{j_1}} \operatorname{Res}_{z=\infty} \frac{\ell'(z)\chi(z)^{-j_1-j_2}}{z} dz
\]

\[
= -\operatorname{Res}_{z=\infty} \left( (\ell'(z)\chi(z)^{-j_1})' \chi(z)^{-j_1-j_2} - \ell'(z) \cdot \frac{j_1 + j_2}{m} \chi(z)^{-j_1-j_2-m} (\ell'(z)\chi(z)^{-j_1})' \right) \frac{dz}{z}
\]

\[
= -\operatorname{Res}_{z=\infty} \left( (\ell'(z)\chi(z)^{-j_1})' \chi(z)^{-j_1-j_2} + (\chi(z)^{-j_1-j_2(z)})' \ell'(z)\chi(z)^{-j_3} \right) \frac{dz}{z}
\]

which is symmetric with respect to the indices \(j_1, j_2\) and \(j_3\). Hence the function \(V_2\) exists.

The existence of \(G\) has been proved by Krichever in [15] (see Theorem 5.6 therein for a more general conclusion). For convenience of the readers, let us verify it in brief now. To this end, denote the right hand side of (2.57) as \(G_{u,v,w}\), then it is
sufficient to show that \( \partial_s G_{u,v,w} \) is symmetric with respect to \( u, v, w, s \in \mathbf{h} \cup \hat{\mathbf{h}} \). For instance, we have

\[
G_{h_{j_1}, h_{j_2}, h_{j_3}} = - \text{Res}_{z = \infty} \left( \frac{\partial h_{j_1}}{\ell} \cdot \frac{\partial h_{j_2}}{\ell} \cdot \frac{\partial h_{j_3}}{\ell} \right) dz
\]

Furthermore, from (2.36) it follows that

\[
\text{Res}_{z = \infty} \left( \frac{\ell' \chi^{-j_1}}{\ell} + (\ell' \chi^{-j_2})_+ + (\ell' \chi^{-j_3})_+ \right) dz
\]

where \( R_j(z) = \ell'(z) \chi(z)^{-j} \). With the help of (2.35), it is easy to see that

\[
R_{j_1} R_{j_2 + j_3} = R_{j_2} R_{j_3 + j_1}, \quad \frac{\partial R_{j_1}}{\partial h_{j_2}} = (\chi^{-j_1} (R_{j_2})_+)' \quad \left( \frac{\partial R_{j_1}}{\partial h_{j_2}} \right)_+ = (R_{j_1 + j_2})_+'. \tag{2.59}
\]

In order to avoid lengthy notations, let us write \( f \prec g \) if \( f - g \) is symmetric with respect to the indices \( j_3 \) and \( j_4 \), then it is straightforward to calculate

\[
\frac{G_{h_{j_1}, h_{j_2}, h_{j_3}}}{\partial h_{j_4}} \prec \text{Res}_{z = \infty} \left( (\chi^{-j_1} (R_{j_4})_+)' (R_{j_3})_+ - (R_{j_1 + j_3})_+ (\chi^{-j_2} (R_{j_4})_+)' \right.

- (R_{j_2 + j_3})_+ (\chi^{-j_1} (R_{j_4})_+)' \right) dz

\[
\prec \text{Res}_{z = \infty} \left( \chi^{-j_1} (R_{j_4})_+ + (\chi^{-j_2} (R_{j_4})_+)' \right) \left( (\chi^{-j_1} (R_{j_3})_+)' (R_{j_1})_+ - (R_{j_1 + j_3})_+ (\chi^{-j_2} (R_{j_4})_+)' \right.

- (R_{j_2 + j_3})_+ (\chi^{-j_1} (R_{j_4})_+)' \right) dz

\[
\prec \text{Res}_{z = \infty} \left( \chi^{-j_1} (R_{j_4})_+ + (\chi^{-j_2} (R_{j_4})_+)' \right) \left( \chi^{-j_1} (R_{j_3})_+ + (\chi^{-j_2} (R_{j_4})_+)' \right)_+ \chi^{-j_1} (R_{j_4})_+ dz

\[
\prec 0,
\]

which leads to

\[
\frac{G_{h_{j_1}, h_{j_2}, h_{j_3}}}{\partial h_{j_4}} = \frac{G_{h_{j_1}, h_{j_2}, h_{j_4}}}{\partial h_{j_3}}.
\]

Furthermore, from (2.36) it follows that

\[
\partial_{h_k} \ell(z) = \partial_{h_k} \ell(z)_-, \quad \left( \frac{\partial h_{k}}{\ell} \right)_- = 0 \text{ as } |z - \varphi| \to 0.
\]
The first equality together with \( \partial_{h_j} \ell(z) = \partial_{h_i} \ell(z)_+ \) implies \( \partial_{h_j} \partial_{h_k} \ell = 0 \), hence we have

\[
G_{h_{j_1}, h_{j_2}, h_k} = - (\text{Res}_{z = \infty} + \text{Res}_{z = \varphi}) \frac{\partial_{h_{j_1}} \ell_+ \cdot \partial_{h_{j_2}} \ell_+ \cdot \partial_{h_k} \ell_-}{\ell'} dz
\]

\[
= - \text{Res}_{z = \infty} \frac{[\ell' \chi^{j_1} - (\ell' \chi)^{-j_1}][\ell' \chi^{j_2} - (\ell' \chi)^{-j_2}][\ell' \chi^{j_3} - (\ell' \chi)^{-j_3}]}{\ell'} dz
\]

\[
= - \text{Res}_{z = \infty} (\ell' \chi^{j_1-j_2}) \partial_{h_k} \ell_- dz,
\]

(2.60)

From the first and the second equalities in (2.58) it follows that

\[
\frac{\partial G_{h_{j_1}, h_{j_2}, h_k}}{\partial h_k} = \text{Res}_{z = \infty} \frac{\partial_{h_{j_1}} \ell_+ \cdot \partial_{h_{j_2}} \ell_+ \cdot \partial_{h_{j_3}} \ell_+ \cdot \partial_{h_k} \ell_-}{\ell'} dz
\]

\[
= - \text{Res}_{z = \infty} \frac{[\ell' \chi^{j_1-j_2-j_3} \partial_{h_k} \ell_-]}{\ell'} dz
\]

\[
= - \text{Res}_{z = \infty} (\ell' \chi^{j_1-j_2-j_3}) \partial_{h_k} \ell_- dz
\]

\[
= - \text{Res}_{z = \infty} \frac{\partial (\ell' \chi^{j_1-j_2-j_3})}{\partial h_{j_3}} \partial_{h_k} \ell_- dz
\]

\[
= \frac{\partial G_{h_{j_1}, h_{j_2}, h_k}}{\partial h_{j_3}},
\]

where the last equality is due to (2.60) and (2.59). The other cases are similar. Therefore the lemma is proved.

**Remark 2.16.** Each function of \( V_1 \) and \( V_2 \) is determined up to a linear function of its arguments, while the function \( G \) is determined up to a quadratic function of its arguments. One refers to, for example [2] for some concrete examples of \( G \).

Under the assumptions in (1.4), for any \( z_1 \in \Gamma', z_2 \in \Gamma, \varphi \in U \), one has

\[
\log \left( \frac{z_1 - z_2}{z_1} \right) = - \sum_{i \geq 1} \frac{1}{i} \left( \frac{z_2}{z_1} \right)^i,
\]

\[
\frac{1}{z_1 - z_2} = \frac{1}{(z_1 - \varphi) - (z_2 - \varphi)} = \sum_{i \geq 0} \frac{(z_2 - \varphi)^i}{(z_1 - \varphi)^{i+1}}.
\]

Let \( f(z) = \sum_{i = -\infty}^{+\infty} f_i (z - \varphi)^i \) be an arbitrary holomorphic function in a neighborhood of the closed bend with boundaries \( \Gamma \) and \( \Gamma' \), then it is easy to verify the following
equalities:

\[
\frac{1}{2\pi i} \oint_{\Gamma} f'(z_2) \log \left( \frac{z_1 - z_2}{z_1} \right) \, dz_2 = \frac{1}{2\pi i} \oint_{\Gamma} f(z_2) \, dz_2 = f(z_1), \quad (2.61)
\]

\[
\frac{1}{2\pi i} \oint_{\Gamma} f'(z_1) \log \left( \frac{z_1 - z_2}{z_1} \right) \, dz_1 = \frac{1}{2\pi i} \oint_{\Gamma} f(z_1) \left( \frac{-1}{z_1 - z_2} + \frac{1}{z_1} \right) \, dz_1
\]

\[= - f(z_2) + \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z} \, dz. \quad (2.62)
\]

Now we are ready to introduce a function of \( t \cup h \cup \hat{h} \) as

\[
F = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \left( \frac{1}{2} \zeta(z_1) \zeta(z_2) + \zeta(z_1) \ell(z_2) - \ell(z_1) \zeta(z_2) \right) \log \left( \frac{z_1 - z_2}{z_1} \right) \, dz_1 \, dz_2
\]

\[- \frac{V_1(t)}{2\pi i} \oint_{\Gamma} \left( \frac{1}{2} \zeta(z) + \ell(z) \right) \, dz + \frac{V_2(h)}{2\pi i} \oint_{\Gamma} \zeta(z) \, dz + G(h, \hat{h}). \quad (2.63)
\]

In fact, from the equalities (2.24) and (2.26) it follows that

\[
t_{-1} = \frac{1}{2\pi i} \oint_{\Sigma} z(\zeta) \, d\zeta = - \frac{1}{2\pi i} \oint_{\Gamma} \zeta(z) \, dz, \quad (2.64)
\]

\[
nh_n = - n \operatorname{Res}_{\hat{\chi} = \infty} z(\hat{\chi}) \hat{\chi}^{n-1} d\hat{\chi} = \operatorname{Res}_{z = \phi} \hat{\chi}^n(z) \, dz = \frac{1}{2\pi i} \oint_{\Gamma} \ell(z) \, dz, \quad (2.65)
\]

hence the function \( F \) can be represented equivalently as (1.13). Clearly, by virtue of Remark 2.16 the function \( F \) is determined up to a quadratic function of the flat coordinates.

**Proposition 2.17.** The function \( F \) given above satisfies, for any \( u, v, w \in t \cup h \cup \hat{h} \),

\[
\frac{\partial^3 F}{\partial u \partial v \partial w} = \langle \partial_u \circ \partial_v, \partial_w \rangle_\eta. \quad (2.66)
\]

**Proof.** Let us verify the equalities (2.66) in a straightforward way, with the help of (2.34)–(2.36), (2.61) and (2.62). For convenience, denote \( Z_i(z) = \zeta(z) \zeta'(z) \), then from (2.34) it is easy to see that

\[
\frac{\partial \zeta(z)}{\partial t_i} = -Z_i(z), \quad \frac{\partial^2 \zeta(z)}{\partial t_{i_1} \partial t_{i_2}} = Z_{i_1 + i_2}'(z), \quad \frac{\partial^3 \zeta(z)}{\partial t_{i_1} \partial t_{i_2} \partial t_{i_3}} = -Z_{i_1 + i_2 + i_3}''(z). \quad (2.67)
\]
By using these equalities and (2.55), we have

\[
\frac{\partial^3 F}{\partial t_1 \partial t_2 \partial t_3} = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \left( (-Z_{i_1+i_2+i_3}''(z_1) \left( \frac{1}{2} \zeta(z_2) + \ell(z_2) \right) - \left( \frac{1}{2} \zeta(z_1) - \ell(z_1) \right) Z_{i_1+i_2+i_3}''(z_2) \right) 

- \frac{1}{2} \left\{ Z_{i_1+i_2}(z_1)Z_{i_3}(z_2) + Z_{i_3}(z_1)Z_{i_1+i_2}(z_2) + \text{c.p.}(i_1, i_2, i_3) \right\} \log \left( \frac{z_1 - z_2}{z_1} \right) dz_1 dz_2

- \frac{1}{2} \left\{ -Z_{i_1+i_2}(z_1)Z_{i_3}(z_2) + Z_{i_3}(z_1)Z_{i_1+i_2}(z_2) + \text{c.p.}(i_1, i_2, i_3) \right\} \right) \] 

\[
= \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{1}{2} \zeta(z) + \ell(z) \right) dz \cdot \frac{1}{2\pi i} \oint_{\Gamma} -Z_{i_1+i_2+i_3}'(z) \frac{dz}{z}

- \left\{ \frac{1}{2\pi i} \oint_{\Gamma} \left( -\frac{1}{2} Z_{i_3}(z) \right) dz \cdot \frac{1}{2\pi i} \oint_{\Gamma} Z_{i_1+i_2}(z) dz + \text{c.p.}(i_1, i_2, i_3) \right\}

= \frac{1}{2\pi i} \oint_{\Gamma} \left( Z_{i_1+i_2+i_3}'(z) \right) + \left( \frac{1}{2} \zeta(z) + \ell(z) \right) - \left( \frac{1}{2} \zeta(z) - \ell(z) \right) Z_{i_1+i_2+i_3}'(z) \] 

\[
- \frac{1}{2} \left\{ -Z_{i_1+i_2}(z_1)Z_{i_3}(z_2) + Z_{i_3}(z_1)Z_{i_1+i_2}(z_2) + \text{c.p.}(i_1, i_2, i_3) \right\} \left( \frac{1}{2} \zeta(z) + \ell(z) \right) Z_{i_1+i_2+i_3}'(z) - \frac{1}{2} \left\{ -Z_{i_1+i_2}(z_1)Z_{i_3}(z_2) + Z_{i_3}(z_1)Z_{i_1+i_2}(z_2) + \text{c.p.}(i_1, i_2, i_3) \right\} \right) \] 

where the second equality is due to (2.61) and (2.62), and the fourth equality is due to \( Z_{i_3}(z) Z_{i_1+i_2}(z) = Z_{i_1+i_2+i_3}(z) \zeta'(z) \).

Similarly, by using (2.35) and (2.56) we have

\[
\frac{\partial^3 F}{\partial t_1 \partial t_2 \partial h_j} = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \left( Z_{i_1+i_2}'(z_1) (\ell'(z_2) \chi(z_2)^{-j})_+ \right)

\]
Moreover, by using (2.59) and (2.58) we have

\[ -(\ell'(z_1)\chi(z_1)^{-j})_+ Z_{i_1+i_2} (z_2) \log \left( \frac{z_1 - z_2}{z_1} \right) d\bar{z}_1 dz_2 \]

\[ = - \frac{1}{2\pi i} \oint_{\Gamma} \left( Z_{i_1+i_2} (z_1) - \frac{1}{2\pi i} \oint_{\Gamma'} Z_{i_1+i_2} (z_1) \right) (\ell'(z_2)\chi(z_2)^{-j})_+ \]

\[ - (\ell'(z)\chi(z)^{-j}) Z_{i_1+i_2} (z^-) dz \]

\[ = - \frac{1}{2\pi i} \oint_{\Gamma} (\ell'(z)\chi(z)^{-j}) + \zeta(z)^{i_1+i_2} \zeta'(z) dz \]

\[ = \left\langle \frac{\partial}{\partial h_{i_1}}, \frac{\partial}{\partial h_{i_2}}, \frac{\partial}{\partial h_{j}} \right\rangle_\eta. \]

The other cases are similar. Therefore the proposition is proved. \(\square\)

2.6. The Euler vector field. On \(\mathcal{M}_{m,n}\), let us introduce a vector field as

\[ \hat{E} = \sum_{i \in \mathbb{Z}} \left( \frac{1}{m} - i \right) t_i \frac{\partial}{\partial h_i} + \sum_{j=1}^{m-1} j + \frac{1}{m} \hat{h}_j \frac{\partial}{\partial h_j} + \sum_{k=0}^{n} \left( \frac{1}{m} + \frac{k}{n} \right) \hat{h}_k \frac{\partial}{\partial h_k}. \quad (2.68) \]

Proposition 2.18. The function \(\mathcal{F}\) defined by (2.69) satisfies

\[ \text{Lie}_{\hat{E}} \mathcal{F} = \left( 2 + \frac{2}{m} \right) \mathcal{F} + \text{quadratic terms in flat coordinates}. \quad (2.69) \]

Proof. If we assign \(\text{deg } z = \frac{1}{m}\), and assume each of the functions \(\zeta(z)\) and \(\ell(z)\) to be homogeneous of degree 1, then from (2.34)–(2.36) one sees that the flat coordinates
have degrees as follows

\[ \deg t_i = \frac{1}{m} - i, \quad i \in \mathbb{Z}; \quad (2.70) \]

\[ \deg h_j = \frac{j + 1}{m}, \quad 1 \leq j \leq m - 1; \quad (2.71) \]

\[ \deg \hat{h}_k = \frac{k + 1}{n}, \quad 0 \leq k \leq n. \quad (2.72) \]

Accordingly, one observes that \( V_1 \) and \( V_2 \) defined by (2.55) and (2.56) are homogeneous of degree \( 1 + \frac{1}{m} \) (up to a linear terms in the flat coordinates) while \( G \) defined by (2.57) is homogeneous of degree \( 2 + \frac{2}{m} \) (up to a quadratic terms in the flat coordinates). Hence, up to quadratic terms the function \( F \) is homogeneous of degree

\[ \deg F = 2 + \frac{2}{m}. \]

Therefore the proposition is proved. \( \square \)

2.7. **Proof of Main Theorem 1.** It follows from Propositions 2.6, 2.12, 2.17 and 2.18 that equipped to \( \mathcal{M}_{m,n} \) there is an infinite-dimensional Frobenius manifold structure. More precisely, for the Frobenius manifold the flat metric \( \langle \cdot, \rangle_\eta \) is given in (2.21), the product is defined by (2.46), the unity vector field \( \vec{e} \) is given in (2.47), the potential function \( F \) is given in (2.63) and the Euler vector field \( \vec{E} \) is given in (2.68).

**Remark 2.19.** Recall the definition of \( \mathcal{M}_{m,n} \) in Section 1.2. Suppose that the condition (C3) is replace by a general setting:

(C3)’ The winding number of \( \zeta(z) \) around 0 is an arbitrary positive integer \( r \), such that \( \zeta(z)^{1/r} \) maps the circle \( \Gamma \) biholomorphically to a simple smooth curve \( \Sigma \) around 0.

Then we can still carry out the above approach and obtain a Frobenius manifold structure on \( \mathcal{M}_{m,n} \) under certain assumptions, with \( \zeta(z) \) being replaced by \( \zeta(z)^{1/r} \) when introducing the flat coordinates \( t_i \). In particular, if we take

\[ r = 2, \quad \varphi = 0, \quad m = 2m', \quad n = 2n' \]

with arbitrary positive integers \( m' \) and \( n' \), and assume \( (a(z), \hat{a}(z)) \) to be a pair of even functions of \( z \), then the Frobenius manifold \( \mathcal{M}_{2m',2n'} \) coincides with the one constructed in [27] underlying the two-component BKP hierarchy.

3. **Proof of Main Theorem 2**

In this section, we want to clarify the relationship between the infinite-dimensional Frobenius manifold \( \mathcal{M}_{m,n} \) and the universal Whitham hierarchy.
3.1. The universal Whitham hierarchy and its bihamiltonian structures.

As mentioned before, a general version of the universal Whitham hierarchy was investigated by Krichever [14, 15], and now we only focus on the case of meromorphic functions with a fixed pole at infinity and a movable pole. More precisely, let us consider two meromorphic functions of $z$ on the Riemann sphere of the form

$$
\lambda(z) = z + \sum_{i \geq 1} v_i(x)(z - \varphi(x))^{-i}, \quad \hat{\lambda}(z) = \sum_{i \geq -1} \hat{v}_i(x)(z - \varphi(x))^i,
$$

with $x \in S^1$ being a loop parameter. In this paper by the (special) universal Whitham hierarchy we mean the following system of evolutionary equations:

$$
\frac{\partial \lambda(z)}{\partial s_k} = \left[ (\lambda(z)^k)_+, \lambda(z) \right], \quad \frac{\partial \hat{\lambda}(z)}{\partial s_k} = \left[ (\hat{\lambda}(z)^k)_+, \hat{\lambda}(z) \right],
$$

(3.2)

$$
\frac{\partial \lambda(z)}{\partial \hat{s}_k} = \left[ -(\hat{\lambda}(z)^k)_-, \lambda(z) \right], \quad \frac{\partial \hat{\lambda}(z)}{\partial \hat{s}_k} = \left[ - (\hat{\lambda}(z)^k)_-, \hat{\lambda}(z) \right],
$$

(3.3)

where the Lie bracket $[ , ]$ reads

$$
[f, g] := \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial x}
$$

(3.4)

and $k = 1, 2, 3, \ldots$. Here it is omitted some additional flows constructed via logarithm functions in the general version of [15], for they do not concern to what we study below.

Observe that, when $|z| \to \infty$ the function $\lambda(z)$ in (3.1) can also be expanded as $\lambda(z) = z + \sum_{i \geq 1} \hat{v}_i z^{-i}$, thus the flows $\partial \lambda(z)/\partial s_k$ in (3.2) indeed compose the dispersionless KP hierarchy. That is to say, the hierarchy (3.2)–(3.3) is an extension of the dispersionless KP hierarchy. In fact, the universal Whitham hierarchy admits other reductions and the corresponding Frobenius manifolds/WDVV equations have been constructed in the context of Landau-Ginsburg topological models. For example, when $\hat{\lambda}(z) \to 0$ and $(\lambda(z)^m)_- = 0$ with some positive integer $m$, the hierarchy (3.2)–(3.3) is reduced to the dispersionless Gelfand-Dickey hierarchy, and its bihamiltonian structure was involved by Dubrovin [8] in the $A$-series Landau-Ginsburg topological models. When $\lambda(z)^m = \hat{\lambda}(z)^n$ with positive integers $m$ and $n$, it was achieved by Krichever a residue formula for the solution of WDVV equations (see Theorem 5.6 in [15] for details, and see also [1, 2]), which can be regarded as the potential function of a Frobenius manifold. We hope that our construction below for the universal Whitham hierarchy (3.2)–(3.3) would help us to understand such works.

Let us recall the bihamiltonian structures derived in [28] for the universal Whitham hierarchy (3.2)–(3.3). Given two arbitrary positive integers $m$ and $n$, we consider the loop space, denoted as $\mathcal{LM}_{m,n}$, of smooth maps from the unit circle $S^1$ to the
manifold $M_{m,n}$ defined in (1.2). A point in the loop space $LM_{m,n}$ is written as
\[
\vec{a} = (a, \hat{a}) = \left( z^m + \sum_{i \leq m-2} a_i(x)(z - \varphi(x))^i, \sum_{i \geq -n} \hat{a}_i(x)(z - \varphi(x))^i \right).
\]

At this point, the tangent space $T_{\vec{a}}LM_{m,n}$ and the cotangent space $T^*_{\vec{a}}LM_{m,n}$ of the loop space take a similar form as (2.1) and (2.2) respectively.

On the loop space $LM_{m,n}$, we introduce a ring of formal differential polynomials as
\[
A := C^\infty(a) \left[ \partial_x a, \partial^2_x a, \ldots \right],
\]
where
\[
a(x) = (\varphi(x), a_{m-2}(x), a_{m-3}(x), \ldots, \hat{a}_{-n}(x), \hat{a}_{-n+1}(x), \ldots).
\]
Let us consider the quotient space $F := A/\partial_x A$, whose elements are called local functionals written in the form
\[
F = \int f(a, \partial_x a, \partial^2_x a, \ldots) \, dx \in F, \quad f \in A.
\]
Given a local functional $F \in F$, its variational gradient at $\vec{a}$ means a cotangent vector $dF \in T^*_{\vec{a}}LM_{m,n}$ such that
\[
\delta F = \int \langle dF, \delta \vec{a} \rangle \, dx,
\]
with the nondegenerate pairing $\langle \cdot, \cdot \rangle$ given in (2.3).

Proposition 3.1 (28). For any positive integers $m$ and $n$, there is a bihamiltonian structure on $LM_{m,n}$ given by the follows two compatible Poisson brackets
\[
\{ F, H \}_\nu = \int \langle dF, \mathcal{P}_\nu \cdot dH \rangle \, dx, \quad \nu = 1, 2.
\]
Here the Poisson tensors $\mathcal{P}_\nu : T^*_{\vec{a}}LM_{m,n} \to T_{\vec{a}}LM_{m,n}$ read
\[
\mathcal{P}_1 \cdot \vec{w} = \left( [\omega, a]_+ + [\hat{\omega}, \hat{a}]_+ - [\omega_+ + \hat{\omega}_+, a], -[\omega, a]_+ - [\hat{\omega}, \hat{a}]_+ + [\omega_+ + \hat{\omega}_+, \hat{a}]_+ \right), \quad (3.6)
\]
\[
\mathcal{P}_2 \cdot \vec{w} = \left( ( [\omega, a]_+ + [\hat{\omega}, \hat{a}]_+) a - ( [\omega a + \hat{\omega} a], a) - \sigma a', \right.

\[
- ( [\omega, a]_+ + [\hat{\omega}, \hat{a}]_+) \hat{a} + ([\omega a + \hat{\omega} a], \hat{a}) - \sigma \hat{a}', \right)
\]
(3.7)

with the Lie bracket $[\cdot, \cdot]$ given in (3.4) and
\[
\sigma = \frac{1}{m} \text{Res}_{z=\varphi}([\omega, a] + [\hat{\omega}, \hat{a}]) \, dz.
\]

Moreover, let
\[
a = \lambda(z)^m, \quad \hat{a} = \hat{\lambda}(z)^n,
\]
then the universal Whitham hierarchy (3.2)–(3.3) can be represented in a bihamiltonian recursive form as
\[
\frac{\partial F}{\partial s_k} = \{ F, H_{k+m} \}_1 = \{ F, H_k \}_2, \quad \frac{\partial F}{\partial \hat{s}_k} = \{ F, \hat{H}_{k+n} \}_1 = \{ F, \hat{H}_k \}_2,
\]
(3.9)
where the Hamiltonian functionals are given by
\[
H_k = -\frac{m}{k} \int \text{Res}_{z=\infty} \lambda(z)^k \, dx, \quad \hat{H}_k = \frac{n}{k} \int \text{Res}_{z=\varphi} \hat{\lambda}(z)^k \, dx,
\]
(3.10)
with \( k = 1, 2, 3, \ldots \).

Remark 3.2. A dispersionful version of the hierarchy (3.2)–(3.3) was proposed in [22, 28] by using scalar pseudo-differential operators, which is an extension of the full KP hierarchy. In analogue of the KP hierarchy, such an extension of it can be represented equivalently as a bilinear equation of Baker-Akhiezer functions [17], and it is equipped with a series of bihamiltonian structures [28] whose dispersionless limits are given in the above proposition.

3.2. The bihamiltonian structure induced from \( \mathcal{M}_{m,n} \). In order to study the relation between the infinite-dimensional Frobenius manifold \( \mathcal{M}_{m,n} \) and the universal Whitham hierarchy (3.2)–(3.3), we continue to study the intersection form on \( \mathcal{M}_{m,n} \).

Lemma 3.3. The Euler vector field \( \vec{E} \) defined in (2.68) can be represented in Laurent series as
\[
\vec{E} = \left( a(z) - \frac{za'(z)}{m}, \hat{a}(z) - \frac{z\hat{a}'(z)}{m} \right).
\]
(3.11)
Proof. It suffices to show
\[
\vec{E}(\zeta(z)) = \zeta(z) - \frac{z\zeta'(z)}{m}, \quad \vec{E}(\ell(z)) = \ell(z) - \frac{z\ell'(z)}{m}.
\]
(3.12)
Firstly, substituting \( \zeta = \zeta(z) \) into (2.23), one has
\[
z = \sum_{i \in \mathbb{Z}} t_i \zeta(z)^i,
\]
whose derivative with respect to \( z \) is
\[
1 = \sum_{i \in \mathbb{Z}} it_i \zeta(z)^{i-1} \zeta'(z).
\]
Thus, by using (2.34) one obtains
\[
\zeta(z) - \frac{z\zeta'(z)}{m} = \sum_{i \in \mathbb{Z}} it_i \zeta(z)^i \zeta'(z) - \sum_{i \in \mathbb{Z}} t_i \zeta(z)^i \frac{\zeta'(z)}{m} = \sum_{i \in \mathbb{Z}} \left( i - \frac{1}{m} \right) t_i \zeta(z)^i \zeta'(z) = \vec{E}(\zeta(z)).
\]
With the same method, by using (2.35) and (2.36) one gets

\[
\left( \ell(z) - \frac{z\ell'(z)}{m} \right)_+ = \sum_{j=1}^{m-1} \frac{\ell_j(z)}{m} \hat{h}_i(\ell(z)\chi(z)^{-i})_+ = E(\ell(z)_+), \\
\left( \ell(z) - \frac{z\ell'(z)}{m} \right)_- = -\sum_{i \geq 0} \left( \frac{1}{m} + \frac{i}{n} \right) \hat{h}_i(\ell(z)\chi(z)^{-i})_- = E(\ell(z)_-).
\]

Therefore the lemma is proved. \(\square\)

Recall the generating functions (2.4) and (2.5) of covectors on \(T^*_a\mathcal{M}_{m,n}\). With the help of the Euler vector field, we define the intersection form on the contangent space by

\[(d\alpha(p), d\beta(q))^* := i_E (d\alpha(p) \star d\beta(q)), \quad \alpha, \beta \in \{a, \hat{a}\}. \tag{3.13}\]

**Proposition 3.4.** The intersection form (3.13) can be represented as

\[ (d\alpha(p), d\beta(q))^* = \frac{\alpha'(q)\beta(q)}{p - q} + \frac{\beta'(q)\alpha(p)}{q - p} + \frac{\alpha'(p)\beta'(q)}{m}, \quad \alpha, \beta \in \{a, \hat{a}\}. \tag{3.14} \]

**Proof.** For any \(\alpha, \beta \in \{a, \hat{a}\}\), by using (2.6), (2.43) and (3.11) one has

\[
\langle d\alpha(p) \star d\beta(q), E(z) \rangle = \frac{\beta'(q)}{q - p} \langle d\alpha(p), E(z) \rangle + \frac{\alpha'(p)}{p - q} \langle d\beta(q), E(z) \rangle
\]

\[
= \frac{\beta'(q)}{q - p} \left( \alpha(p) - \frac{pa'(p)}{m} \right) + \frac{\alpha'(p)}{p - q} \left( \beta(q) - \frac{q\beta'(q)}{m} \right)
\]

\[
= \frac{\alpha'(p)\beta(q)}{p - q} + \frac{\beta'(q)\alpha(p)}{q - p} + \frac{\alpha'(p)\beta'(q)}{m}.
\]

Thus the proposition is proved. \(\square\)

Similar as before, based on the nondegenerate pairing (2.3), there is a linear map

\[ g : T^*_a\mathcal{M}_{m,n} \to T_{\hat{a}}\mathcal{M}_{m,n} \tag{3.15} \]

defined by

\[ \langle \tilde{\omega}_1, g \cdot \tilde{\omega}_2 \rangle = (\tilde{\omega}_1, \tilde{\omega}_2)^*, \quad \tilde{\omega}_1, \tilde{\omega}_2 \in T_{\hat{a}}\mathcal{M}_{m,n}. \tag{3.16} \]

**Lemma 3.5.** The map \(g\) defined above is a bijection. More precisely, for any \((\omega(z), \hat{\omega}(z)) \in T_{\hat{a}}\mathcal{M}_{m,n}\) it holds that \(g \cdot (\omega(z), \hat{\omega}(z)) = (\xi(z), \hat{\xi}(z))\) with

\[
\xi(z) = a'(z)(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z))_+ - a(z)(a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z))_+ + \rho a'(z), \tag{3.17}
\]

\[
\hat{\xi}(z) = -\hat{a}'(z)(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z))_+ + \hat{a}(z)(a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z))_+ + \rho \hat{a}'(z), \tag{3.18}
\]

where

\[ \rho = \frac{1}{m} \text{Res} \left( a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z) \right) dz. \]
Proof. Firstly, one can verify (3.17) and (3.18) in the same way as in Lemma 2.2, with \( (\cdot, \cdot)^* \) replaced by \( (\cdot, \cdot)^\ast \). Now let us show that the map \( g \) is bijective. In fact, according to (3.17) and (3.18) one has
\[
\hat{a}' \xi - a' \hat{\xi} = a' \hat{a}(a\omega + \hat{a}\tilde{\omega}) - a' a(a' \omega + a' \hat{\omega})_+ - a' \hat{a}(a' \omega + \hat{a}' \omega)_+ \\
= (a' a - a' \hat{a})(a' \omega)_+ - (a' \hat{\omega})_+,
\]
which leads to (recall the third inequality in (1.8))
\[
(a' \omega)_+ = \left( \frac{\hat{a}' \xi - a' \hat{\xi}}{a' \hat{a} - a' a} \right)_+,
\]
\[
(a' \hat{\omega})_- = - \left( \frac{\hat{a}' \xi - a' \hat{\xi}}{a' \hat{a} - a' a} \right)_-.
\] (3.19)

To simplify notations, let us denote
\[
K(z) = \frac{a'(z)\xi(z) - a'(z)\hat{\xi}(z)}{a'(z)a(z) - a'(z)\hat{a}(z)}.
\]

By using \( \frac{1}{a'} \in (z - \varphi)^{-m+1}H^{-} \) and \( \frac{1}{a'} \in (z - \varphi)^{n+1}H^{+} \), one obtains
\[
\omega = (\omega)_{\geq -m+1} = \left( \frac{1}{a'} (a' \omega)_+ \right)_{\geq -m+1} = \left( \frac{1}{a'} K_+ \right)_{\geq -m+1} = \left( \frac{K}{a'} \right)_{\geq -m+1},
\]
\[
\hat{\omega} = (\hat{\omega})_{\leq n} = \left( \frac{1}{a'} (a' \hat{\omega})_- \right)_{\leq n} = \left( \frac{1}{a'} K_- \right)_{\leq n} = \left( \frac{K}{a'} \right)_{\leq n}.
\] (3.20) (3.21)

Conversely, for any \( (\xi, \hat{\xi}) \in T_{a}\mathcal{M}_{m,n} \), one has \( (\omega, \hat{\omega}) \) by the formulae (3.20) and (3.21). Now let us compute \( (\psi, \hat{\psi}) := g \cdot (\omega, \hat{\omega}) \). To this end, one has
\[
\rho = \frac{1}{m} \text{Res} (a' \omega + a' \hat{\omega}) dz \\
= \frac{1}{m} \text{Res}_{z = \varphi} \left( a' \left( \frac{K}{a'} \right)_{\geq -m+1} - \hat{a}' \left( \frac{K}{a'} \right)_{\leq n} \right) dz \\
= \frac{1}{m} \text{Res}_{z = \varphi} \left( a' \frac{K}{a'} - a' \frac{K}{a'} \right) dz - \text{Res}_{z = \varphi} \frac{K}{a'} (z - \varphi)^{m-1} dz \\
= - \text{Res}_{z = \varphi} \frac{K}{a'} (z - \varphi)^{m-1} dz,
\]
\[
(a\omega)_+ = \left( a \left( \frac{K}{a'} \right)_{\geq -m+1} \right)_+ = \left( \frac{K}{a'} \right)_+ + - \text{Res}_{z = \varphi} \frac{K}{a'} (z - \varphi)^{m-1} dz \\
= \left( \frac{\xi}{a'} + \frac{\hat{a}' \xi - a' \hat{\xi}}{a' a - a' \hat{a}} \right)_+ + \rho = \left( \frac{\hat{a}' \xi - a' \hat{\xi}}{a' a - a' \hat{a}} \right)_+ + \rho,
\]
\[
(\hat{a}\hat{\omega})_- = - \left( \hat{a} \left( \frac{K}{a'} \right)_{\leq n} \right)_- = - \left( \frac{K}{a'} \right)_-.
\]
\[ = - \left( \frac{\hat{a} \xi - a \hat{\xi}}{\hat{a}' a - a' \hat{a}} + \frac{\hat{\xi}}{\hat{a}'} \right) + = - \left( \frac{\hat{a} \xi - a \hat{\xi}}{\hat{a}' a - a' \hat{a}} \right). \]

Hence, by using (3.17), (3.18) and (3.19), we obtain

\[
\psi = a ((a' \omega)_+ - (\hat{a}' \hat{\omega})_-) - a' ((a \omega)_+ - (\hat{a} \hat{\omega})_- - \rho) = a \frac{\hat{a}' \xi - a' \hat{\xi}}{\hat{a}' a - a' \hat{a}} - a' \frac{\hat{a} \xi - a \hat{\xi}}{\hat{a} a' - a' \hat{a}} = \xi;
\]

\[
\hat{\psi} = \hat{a} ((\hat{a}' \omega)_+ - (\hat{a}' \hat{\omega})_-) - \hat{a}' ((\hat{a} \omega)_+ - (\hat{a} \hat{\omega})_- - \rho) = \hat{a} \frac{\hat{a}' \xi - \hat{a}' \hat{\xi}}{\hat{a}' a - a' \hat{a}} - \hat{a}' \frac{\hat{a} \xi - \hat{a} \hat{\xi}}{\hat{a} a' - a' \hat{a}} = \hat{\xi}.
\]

Thus the lemma is proved. \hfill \square

With the help of the bijection \( g \), let us introduce another metric on \( M_{m,n} \) as

\[ (\partial_1, \partial_2)_g := (g^{-1}(\partial_1), g^{-1}(\partial_2))^\ast, \quad \partial_1, \partial_2 \in T_{\hat{a}}M_{m,n}. \] (3.22)

**Theorem 3.6.** The compatible Poisson brackets (3.5) are the ones induced by the metrics \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_g \) given in (2.21) and in (3.22) respectively.

**Proof.** Denote \( \tilde{\omega}_x = (\omega_x, \hat{\omega}_x) = (\frac{\partial \omega}{\partial x}, \frac{\partial \hat{\omega}}{\partial x}) \) for \( \tilde{\omega} \in T_{\hat{a}}^1 \mathcal{L}M_{m,n} \). Observe that the formulae (3.6) and (3.7) can be represented as

\[
P_\nu \cdot \tilde{\omega} = P_\nu(\tilde{\omega}_x) + Q_\nu(\tilde{\omega}), \quad \nu = 1, 2,
\]

where \( P_\nu \) and \( Q_\nu \) are functions depending linearly on \( \omega_x \) and on \( \hat{\omega} \) respectively. According to the theory of hamiltonian structures of hydodynamic type [11], we only need to compare the functions \( P_\nu (\nu = 1, 2) \) with \( \eta \cdot \omega_x \) and \( g \cdot \tilde{\omega}_x \) (recall (2.8) and (3.15)). More explicitly, by using Lemmas 2.2 and 3.5 it is straightforward to check

\[
P_1(\tilde{\omega}_x) = (- (a' \omega_x)_+ - (\hat{a}' \hat{\omega}_x)_+ + a'(\omega_x)_+ + \hat{a}'(\hat{\omega}_x)_+) = \eta \cdot \tilde{\omega}_x,
\]

\[
P_2(\tilde{\omega}_x) = \left( - a(a' \omega_x + \hat{a}' \hat{\omega}_x)_+ + a'(a \omega_x + \hat{a} \hat{\omega}_x)_+ - \frac{a}{m} \operatorname{Res}_{z = \varphi}(-a' \omega_x - \hat{a}' \hat{\omega}_x) dz, \right. \]
\[
\left. \hat{a}(a' \omega_x + \hat{a}' \hat{\omega}_x)_+ + \hat{a}'(a \omega_x + \hat{a} \hat{\omega}_x)_+ - \frac{\hat{a}'}{m} \operatorname{Res}_{z = \varphi}(-a' \omega_x - \hat{a}' \hat{\omega}_x) dz \right)
\]

\[
= g \cdot \tilde{\omega}_x.
\]

Therefore the theorem is proved. \hfill \square

So we complete the proof of Main Theorem 2.
4. Concluding remarks

In this paper we have constructed a class of infinite-dimensional Frobenius manifolds underlying the universal Whitham hierarchy (3.2)–(3.3), whose dispersionful analogue is an extension of the KP hierarchy. More exactly, on such a Frobenius manifold $\mathcal{M}_{m,n}$ with positive integers $m$ and $n$, its flat coordinates for the metric, the product with unity vector field, the potential function $\mathcal{F}$, the Euler vector field and the intersection form are obtained. Moreover, the flat pencil of metrics on the Frobenius manifold is proved to induce the bihamiltonian structure (3.5) for the universal Whitham hierarchy (3.2)–(3.3). In other words, the method initiated in [3] and developed in [27, 29] is applied successfully to a wider range of concrete examples.

Similar to the finite-dimensional case, if we take the following Hamiltonian functionals given by the derivatives of the potential function as

$$H_{u,0} = \int \frac{\partial \mathcal{F}}{\partial u} \bigg|_{v \rightarrow v(x)} \ dx, \quad u, v \in t \cup h \cup \hat{h},$$

then the Hamiltonian equations given by the first Poisson bracket in (3.5) are up to constant factors with those flows of the nondecendant level in the principal hierarchy associated to $\mathcal{M}_{m,n}$. We expect that the bihamiltonian recursion relation in (3.9) might help us to obtain the complete principal hierarchy. Furthermore, a tau function of the principal hierarchy is expected to be introduced, which may help us to understand the tau function with that solving the string equation for the Whitham hierarchy studied in [2, 15]. We will consider it in other occasions.

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References

[1] Aoyama, S., & Kodama, Y. (1994). Topological conformal field theory with a rational $W$ potential and the dispersionless KP hierarchy. Modern Physics Letters A, 9(27), 2481-2492.

[2] Aoyama, S., & Kodama, Y. (1996). Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy. Communications in mathematical physics, 182(1), 185-219.

[3] Carlet, G., Dubrovin, B., & Mertens, L. P. (2011). Infinite-dimensional Frobenius manifolds for 2+ 1 integrable systems. Mathematische Annalen, 349(1), 75-115.

[4] Carlet, G., & Mertens, L. P. (2015). Principal hierarchies of infinite-dimensional Frobenius manifolds: the extended 2D Toda lattice. Advances in Mathematics, 278 137–181.
[5] Date, E., Jimbo, M., Kashiwara, M., & Miwa, T. (1982). Transformation groups for solitons equations: IV. A new hierarchy of soliton equations of KP-type. *Physica D: Nonlinear Phenomena, 4*(3), 343-365.
[6] Jimbo, M., Kashiwara, M., & Miwa, T. (1982). *Transformation groups for soliton equations* (No. RIMS-394).
[7] Dijkgraaf, R., Verlinde, H., & Verlinde, E. (1991). Topological strings in d<1. *Nuclear Physics B, 352*(1), 59-86.
[8] Dubrovin, B. (1992). Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models. *Communications in mathematical physics, 145*(1), 195-207.
[9] Dubrovin, B. (1992). Integrable systems in topological field theory. *Nuclear physics B, 379*(3), 627-689.
[10] Dubrovin, B. (1996). Geometry of 2D topological field theories. In *Integrable systems and quantum groups* (pp. 120-348). Springer, Berlin, Heidelberg.
[11] Dubrovin, B., & Novikov, S. P. (1984). Poisson brackets of hydrodynamic type. In *Doklady Akademii Nauk* (Vol. 279, No. 2, pp. 294-297). Russian Academy of Sciences.
[12] Dubrovin, B., & Zhang, Y. (2001). Normal forms of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants. *Preprint* arXiv: [math.DG/0108160]
[13] Dubrovin, B., & Strachan, I. A. B., & Zhang, Y., & Zuo, D. (2019). Extended affine Weyl groups of BCD-type: their Frobenius manifolds and Landau–Ginzburg superpotentials, *Advances in Mathematics, 351* 897–946.
[14] Krichever, I. M. (1988). Method of averaging for two-dimensional” integrable” equations. *Functional analysis and its applications, 22*(3), 200–213.
[15] Krichever, I. M. (1996). The $\tau$-function of the universal Whitham hierarchy, matrix models and topological field theories. In *30 Years Of The Landau Institute selected Papers* (pp. 477–515).
[16] Liu, S.-Q., & Ruan, Y.-B., & Zhang, Y. (2015). BCfg Drinfeld-Sokolov hierarchies and FJRW-theory, *Invent. Math.*, 201, no. 2, 711–772.
[17] Lu, J., & Wu, C.-Z. (2019). Bilinear equation and additional symmetries for an extension of the Kadomtsev-Petvisahvili hierarchy. arXiv:1911.12727.
[18] Manin,Y.,(1999). Frobenius manifolds, quantum cohomology and moduli spaces, AMS Colloquium Publ.47, Providence, RI.
[19] T. Milanov and H.-H. Tseng,(2008) The spaces of Laurent polynomials, Gromov-Witten theory of $P^1$-orbifolds, and integrable hierarchies, *J. Reine Angew. Math.*, (622)189–235.
[20] Saito, K. (1983). Period mapping associated to a primitive form. *Publications of the Research Institute for Mathematical Sciences, 19*(3), 1231-1264.
[21] Raimondo, A. (2012). Frobenius manifold for the dispersionless Kadomtsev-Petviashvili equation. *Communications in Mathematical Physics, 311*(3), 557-594.
[22] Szablikowski, B. M., & Blaszak, M. (2008). Dispersionful analog of the Whitham hierarchy. *Journal of mathematical physics, 49*(8), 082701.
[23] Strachan, I. A. B. (2001). Frobenius submanifolds. *Journal of Geometry and Physics, 38*(3-4), 285-307.
[24] Szablikowski, B. M. (2015). Classical r-matrix like approach to Frobenius manifolds, WDVV equations and flat metrics. *Journal of Physics A: Mathematical and Theoretical, 48*(31), 315203.
[25] Ueno, K., & Takasaki, K. (1984). Toda lattice hierarchy. Group representations and systems of differential equations (Tokyo, 1982), *Adv. Stud. Pure Math.*, 4.

[26] Witten, E. (1990). On the structure of the topological phase of two-dimensional gravity. *Nuclear Physics B*, 340(2-3), 281-332.

[27] Wu, C.-Z., & Xu, D. (2012). A class of infinite-dimensional Frobenius manifolds and their submanifolds. *International Mathematics Research Notices, 2012*(19), 4520-4562.

[28] Wu, C.-Z., & Zhou, X. (2016). An extension of the Kadomtsev-Petviashvili hierarchy and its hamiltonian structures. *Journal of Geometry and Physics, 106*, 327-341.

[29] Wu, C.-Z., & Zuo, D. (2014). Infinite-dimensional Frobenius manifolds underlying the Toda lattice hierarchy. *Advances in Mathematics, 255*, 487-524.

[30] Zuo, D. (2020). Frobenius manifolds and a new class of Extended affine Weyl groups of A-type, To appear in *Letters in Mathematical Physics*, [arXiv:1905.09470](https://arxiv.org/abs/1905.09470).

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