On Finite Type Invariants of Links and Rational Homology Spheres Derived from the Jones Polynomial and Witten-Reshetikhin-Turaev Invariant.

L. Rozansky

School of Mathematics, Institute for Advanced Study
Princeton, NJ 08540, U.S.A.

E-mail address: rozansky@math.ias.edu

Abstract

We define an infinite set of invariants of rational homology spheres by presenting a link surgery formula which expresses them in terms of the derivatives of the colored Jones polynomial of the link. We study the properties of this formula and prove its invariance under the Kirby moves.
1 Introduction

It has been established a while ago by D. Bar-Natan [3], J. Birman, X. S. Lin [7] and P. Melvin, H. Morton [19] that the derivatives of the Jones polynomial with respect to the variable $1/K \left( q = e^{2\pi i K} \right)$ at $K \to \infty$ are Vassiliev (i.e. finite type) invariants of knots and links. D. Bar-Natan [4] and M. Kontsevich [13] showed that these derivatives are related to the Feynman diagram calculations of the Jones polynomial in the framework of the quantum Chern-Simons theory proposed by E. Witten [33]. Guided by the principle that what is good for knots is good for 3d manifolds, one might look for the finite type invariants of manifolds among the Feynman diagram contributions to the large $K$ limit of their Witten-Reshetikhin-Turaev (WRT) invariant.

A definition of finite type invariants of integer homology spheres was given by T. Ohtsuki [24] (S. Garoufalidis gave an alternative definition in [10]). He demonstrated that Casson’s invariant was of finite type of order 3. The definition of finite type invariants was later extended [11] to rational homology spheres (RHS) (see also [30]). A particularly promising place to look for these finite type invariants is the $1/K$ expansion of the trivial connection contribution to the WRT invariant. We conjectured a surgery formula for this contribution [27], [29], [30]. This formula generates explicit surgery formulas for the individual coefficients of the $1/K$ expansion. We showed at the physical level of rigor that the second coefficient is proportional to the Casson-Walker invariant.

In this paper we take a mathematically rigorous approach by using the surgery formula of [27], [29] as a definition of the generating function of an infinite sequence of invariants $S_n(M)$ of a RHS. We explore some of their properties and demonstrate their invariance under the meridian Kirby move. The rigorous proofs of the propositions stated in this paper will be presented elsewhere.

In accordance with the physical considerations of [27] we expect the invariants $S_n(M)$ to coincide with $(n+1)$-loop corrections studied by S. Axelrod and I. Singer [1], [2]. However the results of this paper are independent of this identification or of any physical arguments used in our previous papers. The proofs are based upon Kontsevish’s integral representation [13]
of the $1/K$ expansion of the Jones polynomial and on the fusion properties of the local cables established by N. Reshetikhin and V. Turaev [26].

The propositions presented in this paper validate the claims of our previous papers [30] and [31]. We sum up this claims in the last section of this paper.

2 The $K^{-1}$ Expansion of the Colored Jones Polynomial

Let $L$ be a framed $N$-component link in $S^3$. We assign the $\alpha_j$-dimensional representations of $SU(2)$ to its components. The colored Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(L; K)$ of $L$ is normalized in such a way that

$$J_\alpha(\text{unknot}; K) = \frac{\sin \left( \frac{\pi}{K} \alpha \right)}{\sin \left( \frac{\pi}{K} \right)}, \quad J_{\alpha_1,\alpha_2}(L_1 \# L_2; K) = J_{\alpha_1}(L_1; K)J_{\alpha_2}(L_2; K)$$

($L_1 \# L_2$ denotes here a disconnected sum of $L_1$ and $L_2$) and changing the framing (i.e. self-linking number) of a component $L_j$ of $L$ by one unit leads to the multiplication of the whole polynomial by a factor $\exp \left( \frac{i\pi}{2K}(\alpha^2_j - 1) \right)$. P. Melvin and H. Morton proved [19] that the polynomial $J_{\alpha_1,\ldots,\alpha_N}(L; K)$ can be expanded in powers of colors and $K^{-1}$:

$$J_{\alpha_1,\ldots,\alpha_N}(L; K) = \left( \prod_{j=1}^N \alpha_j \right) \sum_{n \geq 0} D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2)K^{-n}, \quad (2.1)$$

here $D_{m,n}$ are homogeneous polynomials of order $m$. whose coefficients are finite type invariants of $L$ of order $n$.

The expansion (2.1) can be presented in an alternative form:

**Proposition 2.1** For any link $L \in S^3$ there exists a set of $SU(2)$-invariant homogeneous polynomials $L_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_N)$, $P_{m,l}(\vec{\alpha}_1, \ldots, \vec{\alpha}_N)$, $m, l \geq 0$ of order $m$ on the Lie algebra $su(2)$ (we denote the elements of $su(2)$ as vectors $\vec{\alpha}$) such that the colored Jones polynomial can be presented as a multiple integral over the co-adjoint orbits (i.e. spheres in $\mathbb{R}^3$ of radii $\alpha_j$).
in the case of SU(2) corresponding to the representations assigned to the link components:

\[
J_{\alpha_1, \ldots, \alpha_N}(L; K) = \int_{|\bar{\alpha}_j| = \alpha_j} \left( \prod_{j=1}^{N} \frac{d^2 \bar{\alpha}_j}{4\pi \alpha_j} \right) \exp \left( \frac{i\pi}{2} \sum_{m \geq 2} L_m(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)K^{1-m} \right) \tag{2.2}
+ \sum_{m, l \geq 0, m \neq l \neq 0} P_{m,l}(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)K^{-l-m}\right).
\]

This equation should be understood in the following way: for any \(n_0 > 0\)

\[
\int_{|\bar{\alpha}_j| = \alpha_j} \left( \prod_{j=1}^{N} \frac{d^2 \bar{\alpha}_j}{4\pi \alpha_j} \right) \exp \left( \frac{i\pi}{2} \sum_{m = 2}^{n_0+1} L_m(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)K^{1-m} + \sum_{m, l \geq 0, 0 < l + m \leq n_0} P_{m,l}(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)K^{-l-m} \right)
= \left( \prod_{j=1}^{N} \alpha_j \right) \sum_{0 \leq v \leq n_0} \sum_{0 \leq m \leq n} D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2)K^{-n} + O(K^{-n_0-1}). \tag{2.3}
\]

The idea to present the Jones polynomial as an integral over the coadjoint orbits corresponding to the colors was first proposed by N. Reshetikhin [25]. We proved the formula (2.2) at the physical level of rigor in [28]. In fact, that proof becomes rigorous as soon as one uses Kontsevich’s integral [13] instead of a generic Chern-Simons perturbation theory. We sketch this procedure in Appendix. The detailed proof will be presented in [32].

The next proposition was also proved in [28] (see the comments in Appendix and [32]).

**Proposition 2.2** The polynomials \(L_m(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)\) satisfy the following properties:

1. \[
L_2 = \sum_{i,j=1}^{N} l_{ij} \bar{\alpha}_i \cdot \bar{\alpha}_j, \tag{2.4}
\]
   here \(l_{ij}\) are the linking numbers of \(L\).

2. If for the three numbers \(1 \leq j_1 \leq j_2 \leq j_3 \leq N\) all linking numbers \(l_{jm_1jm_2}, 1 \leq m_1 < m_2 \leq 3\) are zero then the sum of monomials of \(L_3(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)\) which depend only on vectors \(\bar{\alpha}_{jm}, 1 \leq m \leq 3\) is

\[
-4\pi l_{j_1j_2j_3}^{(\mu)} \bar{\alpha}_{j_1} \cdot (\bar{\alpha}_{j_2} \times \bar{\alpha}_{j_3}). \tag{2.5}
\]
   here \(l_{ijk}^{(\mu)}\) are triple Milnor linking numbers.
3. If for four numbers \(1 \leq j_1 \leq j_2 \leq j_3 \leq j_4 \leq N\) all the off-diagonal linking and triple Milnor linking numbers are zero then the sum of monomials of \(L_4(\vec{\alpha}_1, \ldots, \vec{\alpha}_N)\) containing only the vectors \(\vec{\alpha}_{jm}, 1 \leq m \leq 4\) is

\[
\frac{\pi^2}{3} \sum_{1 \leq m_1, m_2, m_3, m_4 \leq N} \left( l^{(\mu)}_{jm_1j_2m_3j_4} - l^{(\mu)}_{jm_3j_1j_2m_4} \right) (\vec{\alpha}_{jm_1} \times \vec{\alpha}_{jm_2}) \cdot (\vec{\alpha}_{jm_3} \times \vec{\alpha}_{jm_4}),
\]

(2.6)

here \(l^{(\mu)}_{ijkl}\) are quartic Milnor linking numbers.

4. A polynomial \(L_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_N)\) is a linear combination of tree monomials coming from tree graphs with trivalent vertices and \(m\) legs as described in [28]. A tree monomial is formed from a tree graph by placing the Lie algebra structure constants (\(\epsilon_{\mu\nu\rho}\) for \(su(2)\)) at trivalent vertices, the Killing scalar product (\(\delta_{\mu\nu}\) for \(su(2)\)) at edges and Lie algebra elements \(\vec{\alpha}_j, 1 \leq j \leq N\) at legs.

A finite type nature of Milnor’s linking numbers was observed by D. Bar-Natan [5] and X-S. Lin [14].

It is instructive to see what happens to eq. (2.2) when \(L\) has only one component, i.e. when \(L\) is a knot \(K\). First, since all the polynomials \(L_m, P_{m,l}\) are \(SU(2)\)-invariant, the integral over the only variable \(\vec{\alpha}\) becomes trivial: \(\int_{|\vec{\alpha}|=\alpha} \frac{d^2\vec{\alpha}}{4\pi\alpha} = \alpha\). Second, the property 4 of the Proposition 2.2 implies that all the polynomials \(L_m(\vec{\alpha}), m \geq 3\) are equal to zero due to the antisymmetric structure constants \(\epsilon_{\mu\nu\rho}\) in the vertices of the tree graphs. As a result, we end up with the expansion

\[
e^{-\frac{2\pi}{l_{11}}\alpha^2} J_\alpha(\mathcal{K}; K) = \alpha \left( 1 + \sum_{m,l \geq 0} \frac{D_{m,l} \alpha^{2m} K^{-2m-l}}{m+l \neq 0} \right)
\]

(2.7)

with some coefficients \(D_{m,l}\). This expansion is equivalent to the Melvin-Morton bound [19] proved by D. Bar-Natan and S. Garoufalidis [6] (for a simple “path integral” proof see [27]).

The formula (2.2) can be put into a more suggestive form if we introduce new variables

\[
\vec{\alpha}_j = \frac{\vec{\alpha}_j}{K}.
\]

(2.8)
In these variables

\[ J_{\alpha_1,\ldots,\alpha_N}(L; K) = \int_{|\vec{a}_j| = \frac{\alpha}{\pi}} \left( \prod_{j=1}^{N} \frac{K}{4\pi} d^2\vec{a}_j \right) \exp \left( \frac{i\pi K}{2} \sum_{m \geq 2} L_m(\vec{a}_1, \ldots, \vec{a}_N) \right) \]

(2.9)

\[ + \sum_{m,l \geq 0 \atop m \neq 0} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_N)K^{-l} \].

The integral decomposes into a product of the rapidly oscillating exponential

\[ \exp \left( \frac{i\pi K}{2} \sum_{m \geq 2} L_m \right) \]

whose exponent is proportional to \( K \) and a slowly varying preexponential factor

\[ \exp \left( \sum_{m,l \geq 0 \atop m \neq 0} P_{m,l}K^{-l} \right) \]

whose expansion contains only negative powers of \( K \). As a result, the integral (2.9) can be calculated in the stationary phase approximation when \( K \to \infty \). In [28], [29] we discuss at physical level of rigor a relation between the stationary phase configurations of the vectors \( \vec{a}_j \) and the homomorphisms of the link group into \( SU(2) \). We also establish (at the same rigor level) a Melvin-Morton [19] type relation between the contribution of configurations when all vectors \( \vec{a}_j \) are (anti-)parallel and the Alexander polynomial of \( L \).

### 3 Perturbative Invariants

We will use eq. (2.9) in order to define “perturbative” invariants of RHS. We recall that if a 3d manifold \( M \) is constructed by a surgery on a framed link \( L \in S^3 \) (we denote this as \( M = \chi_L(S^3) \)) then its WRT invariant \( Z(M; K) \) is given by the surgery formula

\[ Z(M; K) = Z(S^3; K)e^{i\phi_n} \sum_{1 \leq \alpha_1,\ldots,\alpha_N \leq K-1} J_{\alpha_1,\ldots,\alpha_N}(L; K) \prod_{j=1}^{N} \sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \alpha_j \right). \]

(3.1)

Here \( Z(S^3; K) \) is the WRT invariant of \( S^3 \):

\[ Z(S^3; K) = \sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right) \]

(3.2)
and $\phi_{fr}$ is a “framing correction” which depends on the surgery data:

$$\phi_{fr} = -\frac{3}{4}\pi \frac{K - 2}{K} \text{sign} (l_{ij}), \quad (3.3)$$

sign $(l_{ij})$ is the signature of the linking matrix $l_{ij}$.

We are going to use the formula (2.9) for the Jones polynomial in eq. (3.1) and substitute a sum $\sum_{1 \leq \alpha_1, \ldots, \alpha_N \leq K - 1}$ in that equation by an integral $\int_0^\infty d\alpha_1 \cdots d\alpha_N$. The integrals over $d\alpha_j$ and $d^3\vec{a}_j$ combine into 3d integrals $\int d^3\vec{a}_j$ which we will calculate in a stationary phase approximation.

**Definition 3.1** Let $M$ be a RHS constructed by a surgery on a framed link $L \in S^3$. We define an infinite sequence of its invariants $\Delta_n(M)$ (or, equivalently, $S_n(M)$) by expressing their generating function

$$Z^{(tr)}(M; K) = \sqrt{2\pi} K^{\frac{3}{2}} \text{ord} |H_1(M, \mathbb{Z})|^\frac{3}{2} \sum_{n=0}^\infty \Delta_n(M) K^{-n} \quad (3.4)$$

by the surgery formula

$$Z^{(tr)}(M; K) = Z(S^3; K) e^{i\phi_{fr} \int [\vec{a}_j = 0] \left( \prod_{j=1}^N \left( \frac{K}{2} \right)^{\frac{3}{2}} d^3\vec{a}_j \frac{\sin(\pi |\vec{a}_j|)}{\pi |\vec{a}_j|} \right) \left( \sum_{m \geq 2} L_m(\vec{a}_1, \ldots, \vec{a}_N) + \sum_{m,l \geq 0, m+l \neq 0} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-l} \right)}.$$ (3.5)

Here the symbol $[\vec{a}_j = 0]$ means that we are taking only the contribution of the stationary phase point $\vec{a}_j = 0$ to the integral in the stationary phase approximation.

Equation (3.5) should be understood in the following way. Consider the formal power series expansion

$$\left( \prod_{j=1}^N \frac{\sin(\pi |\vec{a}_j|)}{\pi |\vec{a}_j|} \right) \exp \left( \frac{i\pi K}{2} \sum_{m=3}^\infty L_m(\vec{a}_1, \ldots, \vec{a}_N) + \sum_{m,l \geq 0, m+l \neq 0} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-l} \right) \quad (3.6)$$

$$= \sum_{m \geq 0} \tilde{P}_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-l},$$
here $\tilde{P}_{m,l}(\vec{a}_1, \ldots, \vec{a}_N)$ are homogeneous $SU(2)$-invariant polynomials of degree $m$. Then for any $n_0 \geq 0$

$$\sum_{n=0}^{n_0} \Delta_n(M) K^{-n} = \exp \left[ \sum_{n=1}^{n_0} S_n(M) \left( \frac{i\pi}{K} \right)^n \right] + O(K^{-n_0-1})$$

$$= \frac{K}{\pi} \sin \left( \frac{\pi}{K} \right) e^{i\phi_0} |\det(l_{ij})|^\frac{1}{2} \left( \frac{K}{2} \right)^\frac{3N}{2}$$

$$\times \int d^3\vec{a}_1 \cdots d^3\vec{a}_N \exp \left( \frac{i\pi K}{2} \sum_{1 \leq i,j \leq N} l_{ij} \vec{a}_i \cdot \vec{a}_j \right) \sum_{0 \leq m \leq n_0} \sum_{\frac{n_0}{2} \leq l \leq n_0 - \frac{m}{2}} \tilde{P}_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-l} + O(K^{-n_0-1})$$

(note that $\text{ord} |H_1(M, \mathbb{Z})| = |\det(l_{ij})|$). Since only a finite number of polynomials $\tilde{P}_{m,l}$ participate in the preexponential sum of the r.h.s. of eq. (3.7) for a given $n_0$, the integral there is well defined.

**4 A Step-by-Step Procedure**

For a given link $\mathcal{L}$ eq. (2.2) does not determine the polynomials $L_m$ and $P_{m,l}$. In other words, different sets of polynomials $L_m, P_{m,l}$ may lead to the same Jones polynomial through eq. (2.2). However the invariants $\Delta_n(M)$ and $S_n(M)$ depend actually only on the derivatives of the coefficients in the $1/K$ expansion (2.1) of the Jones polynomial $J_{\alpha_1, \ldots, \alpha_N}(\mathcal{L}; K)$.

**Proposition 4.1** For a given $n_0 > 0$ the invariants $\Delta_{n_0}(M)$ and $S_{n_0}(M)$ can be expressed in terms of the linking numbers $l_{ij}$ of $\mathcal{L}$ and the polynomials $D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2)$ of the expansion (2.1) for which

$$n \leq 2^N n_0,$$

(4.1)

here $N$ is the number of components of $\mathcal{L}$.

To prove this proposition we will use an alternative method of calculating $Z^{(\text{tr})}(M; K)$ by integrating over the colors of $\mathcal{L}$ step-by-step\footnote{I am thankful to D. Thurston for suggesting to try this approach.}. It follows from the property 4 of the Proposition\footnote{1} that the maximum power of a given vector $\vec{a}_j$ in any monomial of the polynomial
$L_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_N)$ is $m-2$ (at least two legs in a tree with at least one vertex should be assigned to different link components in order to get a non-zero monomial). As a result, an expansion of the colored Jones polynomial of a link satisfies the Melvin-Morton bound in individual colors:

**Proposition 4.2** The $1/K$ expansion (2.1) can be rewritten as

$$e^{-\frac{i\pi}{2K}l_{11}a_1^2}J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L}; K) = \left(\prod_{j=1}^{N} \alpha_j\right) \sum_{m,n \geq 0 \atop m \leq n} d_{m,n}^{(1)}(\alpha_2^2, \ldots, \alpha_N^2)\alpha_1^{2m}K^{-n}, \quad (4.2)$$

here $d_{m,n}^{(1)}$ are polynomials of a degree not greater than $n$.

We can use this fact in order to substitute an integral instead of a sum in the Reshetikhin-Turaev surgery formula for the surgery on $L_1$. Then by switching a sum over $\alpha_1$ to an integral we arrive at the definition which is similar to the Definition 3.1:

**Definition 4.1** If $l_{11} \neq 0$, then we define an infinite sequence of invariants of a link $\mathcal{L} \setminus L_1$ in a RHS $M_2 = \chi_{\mathcal{L}_1}(S^3)$ by expressing their generating function

$$J_{\alpha_2,\ldots,\alpha_N}^{(tr)}(M_2, \mathcal{L} \setminus L_1; K) = \left(\prod_{n=2}^{N} \alpha_j\right) \sum_{m,n \geq 0 \atop m \leq n} D_{m,n}^{(2)}(\alpha_1^2, \ldots, \alpha_N^2)K^{-n} \quad (4.3)$$

(here $D_{m,n}^{(2)}$ are homogeneous polynomials of degree $m$) by a stationary phase contribution of the point $a_1 = 0$ to the integral

$$J_{\alpha_2,\ldots,\alpha_N}^{(tr)}(M_2, \mathcal{L} \setminus L_1; K) = e^{i\phi_{tr}^{(1)}} \sqrt{2K} \int_{[a_1=0]}^{\infty} da_1 e^{\frac{iK}{2}l_{11}a_1^2} \sin(\pi a_1) \times \left(e^{-\frac{iK}{2}l_{11}a_1^2}J_{K\alpha_1,\alpha_2,\ldots,\alpha_N}(\mathcal{L}; K)\right), \quad (4.4)$$

here

$$\phi_{tr}^{(1)} = -\frac{3}{4} \pi \frac{K - 2}{K} \text{sign}(l_{11}). \quad (4.5)$$

Equation (4.4) should be understood in the following way: for any $n_0 > 0$

$$e^{i\phi_{tr}^{(1)}} \sqrt{2K} \int_{0}^{\infty} da_1 e^{\frac{iK}{2}l_{11}a_1^2} \sin(\pi a_1) \sum_{0 \leq m \leq n_0 \atop 2m \leq n \leq m+n_0} d_{m,n}^{(1)}(\alpha_2^2, \ldots, \alpha_N^2)\alpha_1^{2m}K^{2m-n} \quad (4.6)$$

$$= \left(\prod_{n=2}^{N} \alpha_j\right) \sum_{0 \leq n \leq n_0 \atop 0 \leq m \leq n} D_{m,n}^{(2)}(\alpha_2^2, \ldots, \alpha_N^2)K^{-n}.$$
Conjecture 4.1 \( \frac{Z(S^3,K)}{Z^{(0)}(M,K)} J_{\alpha_2,\ldots,\alpha_N}^{(tr)}(M_2, L \setminus L_1; K) \) is the trivial connection contribution to the colored Jones polynomial of the link \( L \setminus L_1 \) in the RHS \( M_2 = \chi_{L_1}(S^3) \).

We want to relate \( J_{\alpha_2,\ldots,\alpha_N}^{(tr)}(M_2, L \setminus L_1; K) \) to the integral in eq. (3.5).

Proposition 4.3 The same generating function (4.3) can be expressed as an integral over \( d^3 \vec{a}_1 \):

\[
J_{\alpha_2,\ldots,\alpha_N}^{(tr)}(M_2, L \setminus L_1; K) = \int_{|\vec{a}_j|=\alpha_j} \left( \prod_{j=2}^{N} \frac{K d^2 \vec{a}_j}{4\pi |\vec{a}_j|} \right) e^{i\phi^{(1)}_{\alpha_j}} \left( \frac{K}{2} \right)^\frac{3}{2} \]

\times \int \left[ d^3 \vec{a}_1 \frac{\sin(\pi|\vec{a}_1|)}{\pi|\vec{a}_1|} \exp \left( \frac{i\pi K}{2} \sum_{m \geq 2} L_m(\vec{a}_1, \ldots, \vec{a}_N) + \sum_{m,l \geq 0 \atop m+l \neq 0} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-l} \right) \right] \]  

There are two distinct ways of understanding eq. (4.7) because there are two ways of calculating the stationary phase integrals in its r.h.s.. The first way is to substitute the condition \( |\vec{a}_j| = \frac{\alpha_j}{K}, 2 \leq j \leq N \) inside the integral over \( \vec{a}_1 \). Consider the expansion

\[
e^{-\frac{i\pi}{2} l^0 a^2} \exp \left( \frac{i\pi K}{2} \sum_{m \geq 2} L_m(\vec{a}_1, \ldots, \vec{a}_N) K^{1-m} + \sum_{m,l \geq 0 \atop m+l \neq 0} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_N) K^{-m-l} \right) \]

\[
= \sum_{m,l \geq 0} \tilde{P}_{m,l}^{(1)}(\vec{a}_1, \ldots, \vec{a}_N) K^{-m-l},
\]

here \( \tilde{P}_{m,l}^{(1)} \) are invariant polynomials of total order at most \( 2(m+l) \) and of the homogeneous order \( m \) in \( \vec{a}_1 \). The fact that \( l \geq 0 \) in the r.h.s. of eq. (4.8) follows from the property 4 of the Proposition 2.2 which limits the maximum power of \( \vec{a}_1 \) in the polynomials \( L_m(\vec{a}_1, \ldots, \vec{a}_N) \).

In view of expansion (4.8), eq. (4.7) reads in the first approximation for any \( n_0 > 0 \):

\[
\left( \prod_{j=2}^{N} \alpha_j \right) \sum_{0 \leq l \leq n_0} D^{(2)}_{m,n}(\alpha_2, \ldots, \alpha_N) K^{-n} = e^{i\phi^{(1)}_{\alpha}} \left( \frac{K}{2} \right)^\frac{3}{2} \int d^3 \vec{a}_1 \frac{\sin(\pi|\vec{a}_1|)}{\pi|\vec{a}_1|} e^{\frac{i\pi K}{4} l^0 a^2}
\]

\times \int_{|\vec{a}_j|=\alpha_j} \left( \prod_{j=2}^{N} \frac{d^2 \vec{a}_j}{4\pi \alpha_j} \right) \sum_{0 \leq m \leq 2n_0} \tilde{P}_{m,l}^{(1)}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_N) K^{-l} + O(K^{-n_0-1}).
\]
This equation is easy to prove. We split the integral over \( \bar{a}_1 \): 
\[
\int d^3 \bar{a}_1 = \int_0^\infty \! \! da_1 \int_{|\bar{a}_1|=a_1} \! d^2 \bar{a}_1
\]
and then integrate over \( d^2 \bar{a}_1 \) and \( d^2 \bar{a}_j, \) \( 2 \leq j \leq N. \) According to the Proposition 2.1 and eq. (4.2) we recover all polynomials \( d^{(1)}_{m,n}(\alpha_2^2, \ldots, \alpha_N^2)\alpha_1^{2m}K^{-n}, 0 \leq m \leq n_0, 2m \leq n \leq m+n_0 \)
from the polynomials \( \hat{P}^{(1)}_{m,n}(\bar{a}_2, \ldots, \bar{a}_N)K^{-l} \) if we set \( a_1 = \frac{m}{K}. \) Therefore the subsequent integral over \( da_1 \) gives the same results in eqs. (4.6) and (4.9).

The second way of calculating the integral over \( d^3 \bar{a}_1 \) in the r.h.s. of eq. (4.7) is to pretend that \( |\bar{a}_j|, \) \( 2 \leq j \leq N \) are of order 1 and impose the condition \( |\bar{a}_j| = \frac{\alpha_i}{K} \) only after the integral over \( d^3 \bar{a}_1 \) is already calculated. The following lemma results from a Feynman diagram calculation\(^2\) of the 3d integral over \( \bar{a}_1 \) in eq. (4.7).

**Lemma 4.1** There exists a set of \( SU(2) \)-invariant homogeneous polynomials \( L^{(2)}_m(\bar{a}_2, \ldots, \bar{a}_N), \)
\( P^{(2)}_{m,l}(\bar{a}_2, \ldots, \bar{a}_N), \) \( m, l \geq 0 \) of order \( m \) such that

\[
e^{i\phi^{(1)}(\bar{a}_1)} \left( \frac{K}{2} \right)^3 \int_{|\bar{a}_1|=1} d^3 \bar{a}_1 \frac{\sin(\pi |\bar{a}_1|)}{\pi |\bar{a}_1|} \exp \left[ \frac{i\pi K}{2} \sum_{m \geq 2} L_m(\bar{a}_1, \ldots, \bar{a}_N) + \sum_{m, l \geq 0 \atop m+l \neq 0} P_{m,l}(\bar{a}_1, \ldots, \bar{a}_N)K^{-l} \right] = \exp \left[ \frac{i\pi K}{2} \sum_{m \geq 0 \atop m+l \neq 0} L^{(2)}_m(\bar{a}_2, \ldots, \bar{a}_N) + \sum_{m \geq 0} P^{(2)}_{m,l}(\bar{a}_2, \ldots, \bar{a}_N)K^{-l} \right].
\]

(4.10)

The polynomials \( L^{(2)}_m \) share the property 4 of the polynomials \( L_m \) in the Proposition 4.2 and

\[
L^{(2)}_2(\bar{a}_2, \ldots, \bar{a}_N) = \sum_{2 \leq i,j \leq N} l'_{ij} \bar{a}_i \cdot \bar{a}_j, \quad l'_{ij} = l_{ij} - \frac{l_{1i}l_{1j}}{l_{11}}.
\]

(4.11)

Combining eqs. (4.7) and (4.10) we find that a corollary of this lemma is the following

**Proposition 4.4** The invariant \( J_{\alpha_2, \ldots, \alpha_N}^{(tr)}(M_2, \mathcal{L}\setminus\mathcal{L}_1; K) \) can be presented as an integral over the coadjoint orbits in the way similar to the Proposition 2.1

\[
J_{\alpha_2, \ldots, \alpha_N}^{(tr)}(M_2, \mathcal{L}\setminus\mathcal{L}_1; K) = \int_{|\bar{a}_j|=\alpha_j} \left( \prod_{j=2}^N \frac{d^2 \bar{a}_j}{4\pi \alpha_j} \right) \exp \left[ \frac{i\pi K}{2} \sum_{m, l \geq 0 \atop m+l \neq 0} L^{(2)}_m(\bar{a}_2, \ldots, \bar{a}_N)K^{-l-m} \right. \\
\left. + \sum_{m, l \geq 0} P^{(2)}_{m,l}(\bar{a}_2, \ldots, \bar{a}_N)K^{-l-m} \right].
\]

(4.12)

\(^2\)This calculation has nothing to do with path integrals and is absolutely rigorous.
The interpretation of this equation is similar to eq. (2.3).

As we see, the function \( J_{\alpha_2,\ldots,\alpha_N}(M_2, \mathcal{L} \setminus \mathcal{L}_1; K) \) shares all the main properties of the original Jones polynomial \( J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L}; K) \). Therefore the expansion of \( e^{-\frac{i\pi}{2}l_{11}^2 \alpha_2^2} J_{\alpha_2,\ldots,\alpha_N}(M_2, \mathcal{L} \setminus \mathcal{L}_1; K) \) in powers of \( 1/K \) satisfies the same Melvin-Morton bound on individual powers of \( \alpha_2 \) as the expansion of \( e^{-\frac{i\pi}{2}l_{11}^2 \alpha_2^2} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L}; K) \) in the Proposition 4.2. This allows us to define the next invariant \( J_{\alpha_3,\ldots,\alpha_N}(M_3, \mathcal{L} \setminus \mathcal{L}_1 \setminus \mathcal{L}_2; K) \) for \( M_3 = \chi_{\mathcal{L}_2}(M_2) \) in the way similar to eq. (4.4):

\[
J_{\alpha_3,\ldots,\alpha_N}(M_3, \mathcal{L} \setminus \mathcal{L}_1 \setminus \mathcal{L}_2; K) = e^{i\phi^{(2)}_{\mathcal{L}_1}} \sqrt{2K} \int_{[a_1=0]} \cdots \int_{[a_N=0]} da_1 e^{\frac{i\pi}{2}l_{11}^2 \alpha_2^2} \sin(\pi\alpha_2^2) \times \left( e^{-\frac{i\pi}{2}l_{22}^2 \alpha_2^2} J_{\alpha_2,\ldots,\alpha_N}(M_2, \mathcal{L} \setminus \mathcal{L}_1; K) \right),
\]

\[
\phi^{(2)}_{\mathcal{L}_1} = -\frac{3}{4} \pi \frac{K - 2}{K} \text{sign}(l_{22}^2). \tag{4.14}
\]

Repeating this procedure of step-by-step 1d stationary phase integrations over \( da_j \) \( N \) times we end up with \( J^{(tr)}(M_{N+1}; K) \) such that

\[
Z^{(tr)}(M; K) = Z(S^3; K) J^{(tr)}(M_{N+1}; K). \tag{4.15}
\]

It follows from eq. (4.6) that if we want to determine all the polynomials \( D_{m,n}^{(2)} \) of eq. (4.3) for \( n \leq n_0 \) then it is enough to know all the polynomials \( d_{m,n}^{(1)} \) of eq. (4.2) for \( n \leq 2n_0 \). This means that at each step of consecutive stationary phase integrations we reduce the precision in terms of powers of \( K^{-1} \) by a factor of 2. Therefore in order to calculate all the invariants \( S_n(\chi_{\mathcal{L}}(S^3)) \), \( n \leq n_0 \) by the step-by-step procedure it is enough to know all the polynomials \( D_{m,n} \) of eq. (2.1) for \( n \leq 2^N n_0 \), \( N \) being the number of components in \( \mathcal{L} \). This estimate may be an overkill, but it almost proves the essential claim of the Proposition 4.1 that one has to know only a finite number of coefficients in the expansion of the Jones polynomial in order to find \( Z^{(tr)}(M; K) \) with any given precision.

To complete the proof of the Proposition 4.1 we have to decide what to do if one of the self-linking numbers that appear in the exponents of stationary phase integration formulas \( (l_{11} \text{ in eq. (4.4) or } l_{22} \text{ in eq. (4.13))} \) is zero so that the stationary phase approximation
calculation fails. We suggest to use a “regularized” linking matrix depending on parameters $\epsilon_j$

$$l_{ij}^{(\epsilon)} = l_{ij} + \epsilon_j \delta_{ij} \quad (4.16)$$

instead of the actual linking matrix $l_{ij}$ of $L$ in all the stationary phase calculations. Since we changed only the diagonal part of $l_{ij}$, the “regularized” Jones polynomial is simply related to the original one:

$$J_{\alpha_1, ..., \alpha_N}^{(\epsilon)}(L; K) = e^{\frac{2\pi i}{K} \sum_{j=1}^{N} \epsilon_j \alpha_j^2} J_{\alpha_1, ..., \alpha_N}(L; K). \quad (4.17)$$

There are no zero self-linking numbers in the step-by-step procedure applied to $J_{\alpha_1, ..., \alpha_N}^{(\epsilon)}(L; K)$ for general values of $\epsilon_j$. This means that the analytic expression for the “regularized” invariant $Z_{\epsilon}^{(tr)}(M; K)$ obtained through the step-by-step procedure in the assumption that no self-linking numbers are zero, coincides with the expression coming from eq. (3.5). At the same time, since the matrix $l_{ij}$ is non-degenerate, eq. (3.5) indicates that $Z_{\epsilon}^{(tr)}(M; K)$ is a continuous function at $\epsilon = 0$. Therefore we can perform a step-by-step calculation while keeping $\epsilon_j$ as variables and take a limit $\epsilon_j \to 0$ only at the very end when all the integrals have been calculated. The limit exists and is equal to the r.h.s. of eq. (3.5). The possible singularities of intermediate invariants $J^{(tr)}$ at $\epsilon_j = 0$ are ultimately canceled.

5 Invariance under the Kirby Meridian Move

The step-by-step procedure allows for a seemingly simple proof of the invariance of $S_n(M)$ under the Kirby meridian move. We recall that we have to prove that performing a surgery on a 1-framed meridian of a local cable is equivalent to adding an extra negative twist to that cable. We will follow the idea of the Reshetikhin-Turaev proof [26] of the invariance of the WRT invariant as given by the surgery formula (3.1). They use the “fusion” properties of a local cable.

Let $J_{\alpha_1, ..., \alpha_N}$ be the colored Jones polynomial of a framed link with an $N$-strand local cable, $\alpha_j$ being the colors of the cable strands. We denote the Jones polynomial of the same
link with an extra negative twist added to the local cable as $J^{\text{twist}}_{\alpha_1,\ldots,\alpha_N}$. $J^{\text{meridian}}_{\alpha_1,\ldots,\alpha_N;\beta}$ denotes the Jones polynomial of the link if a 0-framed meridian of the local cable is added as an extra component carrying the color $\beta$. In accordance with the surgery formula (3.1) Reshetikhin and Turaev had to show that

$$J^{\text{twist}}_{\alpha_1,\ldots,\alpha_N} = e^{-i\frac{\pi}{2}K^{-2}K} \sum_{1 \leq \beta \leq K-1} e^{i\pi K (\beta^2 - 1)} J^{\text{meridian}}_{\alpha_1,\ldots,\alpha_N;\beta} \sqrt{2 \over K} \sin \left( {\pi \over K} \beta \right).$$

If we apply the step-by-step formula (4.4) to the surgery on the meridian, then we see that the equation

$$J^{\text{twist}}_{\alpha_1,\ldots,\alpha_N} = e^{-i\frac{\pi}{2}K^{-2}K} \sqrt{2K} \int_{[b=0]}^{\infty} db e^{i\pi K b^2 - {i\pi \over 2} K} J^{\text{meridian}}_{\alpha_1,\ldots,\alpha_N;\beta} \sin(\pi b)$$

would demonstrate the invariance of $Z^{\text{tr}}(M; K)$.

Reshetikhin and Turaev used the fact that the polynomial $J_{\alpha_1,\ldots,\alpha_N}$ of a link with a local cable can be decomposed into a sum over the color $\alpha$ of a strand into which the local cable is fused:

$$J_{\alpha_1,\ldots,\alpha_N} = \sum_{1 \leq \alpha \leq \alpha_1 + \ldots + \alpha_N} J_{\alpha_1,\ldots,\alpha_N|\alpha}.$$  

This decomposition satisfies two properties:

$$J^{\text{twist}}_{\alpha_1,\ldots,\alpha_N} = \sum_{1 \leq \alpha \leq \alpha_1 + \ldots + \alpha_N} J_{\alpha_1,\ldots,\alpha_N|\alpha} e^{-i\pi K (\alpha^2 - 1)},$$

$$J^{\text{meridian}}_{\alpha_1,\ldots,\alpha_N;\beta} = \sum_{1 \leq \alpha \leq \alpha_1 + \ldots + \alpha_N} J_{\alpha_1,\ldots,\alpha_N|\alpha} \frac{\sin \left( {\pi \over K} \alpha \beta \right)}{\sin \left( {\pi \over K} \alpha \right)}.$$  

A substitution of these expressions into eqs. (5.1) and (5.2) reveals that it is enough to prove the relations

$$e^{-i\pi K (\alpha^2 - 1)} = -i e^{i\pi K} \sqrt{2 \over K} \sin \left( {\pi \over K} \alpha \right) \sum_{1 \leq \beta \leq K-1} e^{i\pi K \beta^2} \sin \left( {\pi \over K} \alpha \beta \right) \sin \left( {\pi \over K} \beta \right),$$

$$e^{-i\pi K (\alpha^2 - 1)} = -i e^{i\pi K} \sqrt{2 \over K} \sin \left( {\pi \over K} \alpha \right) \int_{[b=0]}^{\infty} db e^{i\pi K b^2} \sin(\pi \alpha b) \sin(\pi b).$$
for eqs. (5.1) and (5.2) respectively. Note that the point $b = 0$ is indeed the stationary phase point of the integral in eq. (5.7) because $\alpha \ll K$ in view of the bound $1 \leq \alpha \leq \alpha_1 + \cdots \alpha_N$ coming from the summation range in eq. (5.3).

Equation (5.7) can be checked by a straightforward calculation while eq. (5.6) might require an application of the Poisson resummation. It follows from comparing eqs. (5.6) and (5.7) that the identity

$$\sum_{-K \leq \beta \leq K-1} e^{i\pi K \beta^2} = \int_{-\infty}^{+\infty} d\beta e^{i\pi K \beta^2}$$

is the reason why the substitution of a stationary phase integral instead of a sum in the Reshetikhin-Turaev surgery formula (3.1) does not destroy the invariance of its l.h.s. under the meridian Kirby move.

### 6 Other Results

In this final section we want to list the results of [30] and [31] which are validated by the Propositions 2.1, 2.2 and by the invariance of $Z^{(tr)}(M; K)$ under the Kirby move.

A link $\mathcal{L}$ is called algebraically split (ASL) if all of its non-diagonal linking numbers are zero: $l_{ij} = 0$ for $i \neq j$. The next proposition is an easy corollary of the Proposition 2.1:

**Proposition 6.1** Let $\mathcal{L}$ be an $N$-component ASL in $S^3$. Then its colored Jones polynomial has the following $1/K$ expansion (cf. eq. (4.2)):

$$\exp \left( -\frac{i\pi}{2K} \sum_{j=1}^{N} l_{jj} \alpha_j^2 \right) J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L}; K) = \left( \prod_{j=1}^{N} \alpha_j \right) \sum_{n \geq 0} D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2) K^{-n}. \quad (6.1)$$

The coefficients of the polynomials $D_{3n,4n}(\alpha_1^2, \ldots, \alpha_N^2)$ are expressed in terms of triple Milnor linking numbers $l_{ijk}^{(\mu)}$ with the help of closed diagrams with trivalent vertices.

Although the bound $m \leq \frac{3}{4}n$ in eq. (6.1) is weaker than the Melvin-Morton bound $m \leq \frac{n}{2}$ of eq. (4.2), still it is better than the trivial bound $m \leq n$. This allows us to perform the integrals over $da_j$ of the step-by-step procedure in one scoop:
Proposition 6.2 If $L$ is an ASL in $S^3$, then

$$Z^{(tr)}(\chi_L(S^3); K) = e^{i\phi_R(2K)\frac{N}{2}} \int_0^\infty da_1 \cdots da_N e^{i\pi K \sum_{j=1}^{N} t_{jj} a_j^2} \times \left( e^{-i\pi K \sum_{j=1}^{N} t_{jj} a_j^2} J_{K_{a_1}, \ldots, K_{a_N}}(L; K) \right)^N \prod_{j=1}^{N} \sin(\pi a_j).$$

(6.2)

T. Ohtsuki defined in [22] the finite type invariants of integer homology spheres. Here is the trivial extension of his definition to rational homology spheres. Let $\#L$ denote a number of components of a link $L \in S^3$. If $\lambda(M)$ is an invariant of r.h.s. then we denote by $\tilde{\lambda}(L)$ an associated invariant of ASL $L$ with non-zero self-linking numbers defined by the formula

$$\tilde{\lambda}(L) = \sum_{L' \in L} (-1)^{\#L'} \lambda(\chi_{L'}(S^3)).$$

(6.3)

Definition 6.1 An invariant $\lambda$ of RHS is of finite type of at most order $n$ if $\tilde{\lambda}(L) = 0$ for any ASL $L$ with $\#L = n + 1$. An invariant $\lambda$ is of order $n$ if it is of at most order $n$ and not of at most order $n - 1$.

Proposition 6.3 The invariants $S_n(M)$ defined by eqs. (3.4), (3.5) are of finite type $3n$.

The proof presented in [30] is based on the fact that eq. (6.2) expresses $S_{n_0}(M)$ in terms of the polynomials $D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2)$ with $n - m \leq n_0$. In view of the bound $m \leq \frac{3}{4}n$ in the sum of the r.h.s. of eq. (6.1) this inequality implies that only the polynomials $D_{m,n}$ with $m \leq 3n_0$ participate in the expression for $S_{n_0}(M)$. Each monomial of the polynomial $D_{m,n}(\alpha_1^2, \ldots, \alpha_N^2)$ can depend on at most $m$ different colors $\alpha_j$. It turns out that if a monomial does not depend on a particular color $\alpha_j$, then its contribution to $Z^{(tr)}(\chi_{L'}; K)$ does not depend on whether the sublink $L' \in L$ contains the link component $L_j$. As a result, the contribution of $D_{m,n}$ is canceled in the alternating sum (6.3) if $\#L > m$ because for each monomial of $D_{m,n}$ there will be a component of $L$ whose color is not represented in that monomial. This proves that $S_n(M)$ is of at most order $3n$. The diagrammatic rules for the calculation of $\tilde{S}_n(L)$
for \( \# \mathcal{L} = 3n \) developed in [30] demonstrate that there is a \( 3n \)-component link \( \mathcal{L} \) for which \( \tilde{S}_n(\mathcal{L}) \neq 0 \). This proves that \( S_n(M) \) is of exactly order \( 3n \).

In their most recent preprint [11], S. Garoufalidis and T. Ohtsuki gave a modified definition of the finite type invariants of RHS. In addition to the requirement that \( \tilde{\lambda}(\mathcal{L}) = 0 \) for any ASL \( \mathcal{L} \) with \( \# \mathcal{L} = n + 1 \) they also demanded that \( \tilde{\lambda}(\mathcal{L}) \) for \( \# \mathcal{L} = n \) should have a specific dependence on the self-linking numbers \( l_{jj} \), namely, it should be inversely proportional to the product \( \prod_{j=1}^{n} l_{jj} \). It is easy to see that \( S_n(M) \) satisfy this extra requirement. Indeed, the most color diverse monomials of \( D_{3n,4n} \) which contribute to \( \tilde{S}_n(\mathcal{L}) \) for \( \# \mathcal{L} = 3n \) contain all the colors in the minimal square power. As a result, the integral in eq. (6.2) produces exactly the factors of \( l_{jj}^{-1} \).

An alternative to look at the structure of \( Z(M; K) \) as \( K \to \infty \) is to study this invariant for prime values of \( K \). This program was developed by R. Kirby and P. Melvin [16], S. Garoufalidis [9], L. Jeffrey [12], H. Murakami [20], [21], T. Ohtsuki [22], [23], R. Lawrence [17], X-S. Lin and Z. Wang [15], G. Masbaum and J. Roberts [18], and others.

**Proposition 6.4 (Murakami)** If \( K \) is prime, then the modified WRT invariant of a RHS \( M \)

\[
Z'(M; K) = \begin{cases} 
\frac{Z(M; K)}{Z(M; 3)}, & \text{if } K = -1 \text{ (mod 4)} \\
\frac{Z(M; K)}{Z(M; 3)}, & \text{if } K = 1 \text{ (mod 4)},
\end{cases}
\]

(6.4)
defined by Kirby and Melvin [16] \((Z'(M; K) = Z(M; K) \text{ if } M \text{ is an integer homology sphere})\) belongs to the cyclotomic ring \( \mathbb{Z}[e^{2\pi i / K}] \).

For \( p, q \in \mathbb{Z}, q \not\equiv 0 \) (mod \( K \)) define \((p/q)^\vee = pq^*, \text{ here } qq^* \equiv 1 \) (mod \( K \)).

**Proposition 6.5 (Ohtsuki)** For a RHS \( M \), let \( K \) be a prime number such that

\[ \text{ord } |H_1(M, \mathbb{Z})| \not\equiv 0 \text{ (mod } K) \].

Then there exists an infinite sequence of rational invariants of \( M \): \( \lambda_0(M), \lambda_1(M), \ldots \) such that if we present \( Z'(M; K) \) as a polynomial in \( (e^{2\pi i / K} - 1) \):

\[
Z'(M; K) = \sum_{n=0}^{K-2} a_n(e^{2\pi i / K} - 1)^n,
\]

(6.5)
then
\[ a_n \equiv (\lambda_n)^\nu \pmod{K} \quad \text{for} \quad 0 \leq n \leq \frac{K-3}{2}. \quad (6.6) \]

In [31] we used the bound \( m \leq \frac{3}{4}n \) in the expansion (6.1) of the Jones polynomial of an ASL in order to provide a conceptually simple proof of the Propositions 6.4 and 6.5. We also established a relation between the “perturbative” invariants \( S_n(M) \) and Ohtsuki’s invariants \( \lambda_n(M) \):

**Proposition 6.6**  \( The \) generating function \( Z^{(tr)}(M;K) \) is proportional to Ohtsuki’s polynomial
\[ \sum_{n=0}^{\infty} \lambda_n(M) (e^{\frac{2\pi i}{K}} - 1)^n = \frac{Z^{(tr)}(M;K)}{Z(S^3;K)}. \quad (6.7) \]
Equation (6.7) should be understood in the following way: for any \( n_0 > 0 \)
\[ \sum_{n=0}^{n_0} \lambda_n(M) (e^{\frac{2\pi i}{K}} - 1)^n = \frac{\left( \frac{i\pi}{K} \right)}{\sin \left( \frac{i\pi}{K} \right)} \exp \left[ \sum_{n=1}^{n_0} S_n(M) \left( \frac{i\pi}{K} \right)^n \right] + O(K^{-n_0-1}). \quad (6.8) \]
The proof of [31] is based on an apparent similarity between a gaussian integral and a gaussian sum for prime \( K \):
\[ \int_{-\infty}^{+\infty} e^{\frac{2\pi i}{K} \left( \frac{p}{q} \alpha^2 + 2n\alpha \right)} d\alpha = e^{\frac{i\pi}{4} \text{sign}(\frac{p}{q})} \left( \frac{K}{2} \left| \frac{q}{p} \right| \right)^{\frac{1}{2}} e^{-\frac{2\pi i}{K} p \cdot n^2}, \quad (6.9) \]
\[ \sum_{\alpha=0}^{K-1} e^{\frac{2\pi i}{K} \left( pq^* \alpha^2 + 2n\alpha \right)} = e^{i\pi (1-\kappa)} K^{\frac{1}{2}} \left( \frac{pq^*}{K} \right) e^{-\frac{2\pi i}{K} p^* q n^2}, \quad (6.10) \]
here \( p, q, n \in \mathbb{Z} \),
\[ \kappa = \begin{cases} 
1 & \text{if } K \equiv -1 \pmod{4} \\
-1 & \text{if } K \equiv 1 \pmod{4}, 
\end{cases} \quad (6.11) \]
and \( \left( \frac{pq^*}{K} \right) \) in eq. (6.10) is Legendre’s symbol.

**Acknowledgements**

I am very thankful to the organizers and participants of the School on Quantum Invariants of 3-manifolds for a wonderful opportunity to discuss these subjects. I especially appreciate
the numerous comments made by J. Andersen, S. Axelrod, D. Bar-Natan, S. Garoufalidis, G. Masbaum, D. Thurston and V. Turaev.

This work was supported by the National Science Foundation under Grants No. PHY-9209978 and DMS 9304580. Part of this work was done during my visit to Argonne National Laboratory. I want to thank Professor C. Zachos for his hospitality and advice.

Appendix

We are going to comment on the proof of Propositions 2.1 and 2.2 in [28]. We will sketch how this proof can be made rigorous by the use of Kontsevich’s integral. A detailed explanation will be provided in [32].

Let \( \mathcal{L} \) be an \( N \)-dimensional link \( \mathcal{L} \) in \( S^3 \). Let its components \( \mathcal{L}_j \) be fixed by smooth maps \( x_j(t), 0 \leq t \leq 1 \):

\[
x_j : [0,1] \to S^3, \quad x_j(0) = x_j(1). \tag{A.1}
\]

We also introduce the following notation: \( J(x) \) is a 2-form on \( S^3 \) taking values in the Lie (co-)algebra \( su(2) \) (since \( su(2) \) has a standard Killing form, we will not distinguish between the algebra and its conjugate). In order to simplify combinatorics in our calculations we will use variational derivatives in the simplest way possible: if \( G(x) \) is a 1-form taking values in \( su(2) \), then

\[
\frac{\delta}{\delta J(x)} \int_{S^3} d^3y G(y) J(y) = G(x) \tag{A.2}
\]

(we dropped the wedge: \( G(y)J(y) = G(y) \wedge J(y) \), and assumed an implicit \( su(2) \) contraction between \( G(y) \) and \( J(y) \)). According to eq. (A.2), \( \frac{\delta}{\delta J(x)} \) behaves as a 1-form on \( S^3 \), so we can pull it back with the map (A.1) and integrate it along the link components.

The proof of the Proposition 2.1 in [28] was based essentially on the following assumption about the expansion (2.1) (unfortunately, we did not emphasize this fact in [28]):
Assumption A.1 There exist ‘multi-local’ 1-forms (i.e. 1-forms in all their arguments) \( G_{m,n}(y_1, \ldots, y_m), \ m \geq 2, \ n \geq 0 \) on \( S^3 \) taking values in \((su(2))^{\otimes m}\) such that the expansion (2.1) comes from the formula

\[
J_{\alpha_1, \ldots, \alpha_N} (\mathcal{L}; K) = \prod_{j=1}^{N} \left[ \sum_{n_j \geq 0} \int_{0 \leq t_1 < \cdots < t_n_j < 1} dt_1 \cdots dt_n_j \ Tr_{\alpha_j} \left( \frac{\delta}{\delta J(x_j(t_1))} \cdots \frac{\delta}{\delta J(x_j(t_n_j))} \right) \right] (A.3)
\]

\[
\times \exp \left[ \sum_{m \geq 2, n \geq 0} K^{1-m-n} \int_{S^3} dy_1 \cdots dy_m G_{m,n}(y_1, \ldots, y_m) J(y_1) \cdots J(y_m) \right] \bigg|_{J=0}.
\]

This equation should be understood in the following way: for any \( n_0 > 0 \)

\[
\left( \prod_{j=1}^{N} \alpha_j \right) \sum_{0 \leq n \leq n_0} D_{m,n}(\alpha_1, \ldots, \alpha_N) K^{-n} = \prod_{j=1}^{N} \left[ \sum_{n_j \geq 0} \int_{0 \leq t_1 < \cdots < t_n_j < 1} dt_1 \cdots dt_n_j \ Tr_{\alpha_j} \left( \frac{\delta}{\delta J(x_j(t_1))} \cdots \frac{\delta}{\delta J(x_j(t_n_j))} \right) \right]
\]

\[
\times \exp \left[ \sum_{m \geq 2, n \geq 0} K^{1-m-n} \int_{S^3} dy_1 \cdots dy_m G_{m,n}(y_1, \ldots, y_m) J(y_1) \cdots J(y_m) \right] \bigg|_{J=0} + \mathcal{O}(K^{-n_0-1})
\]

It is clear from eq. (A.4) that we treat the sum \( \sum_{m \geq 2, n \geq 0} \) in the exponential of the r.h.s. of eq. (A.3) as a formal power series in \( K^{-1} \). We do not need to know whether it is convergent, because only a finite number of terms in this sum contribute to any particular term in the r.h.s. of eq. (2.1).

Equation (A.3) is the standard assumption of the quantum field theory applied to the Chern-Simons action in the framework of Witten’s description [33] of the colored Jones polynomial. The set of the forms \( G_{m,n}(y_1, \ldots, y_m) \) is not unique, it depends on the choice of gauge fixing required for the calculation of the Chern-Simons path integral.

M. Kontsevich proved the Assumption A.1 (modulo some minor corrections, see e.g. [4] and references therein) without any reference to path integrals. He obtained a particularly
simple set of forms $G_{m,n}$:

$$G_{m,n}(y_1, \ldots, y_m) = 0 \quad \text{if either} \quad m \geq 3 \quad \text{or} \quad n \geq 1,$$

so that only $G_{2,0} \neq 0$. The approach of D. Bar-Natan [3] and R. Bott, C. Taubes [8] produces different forms $G_{m,n}$.

The particular form of the forms $G_{m,n}$ played no role in the proof of the Proposition 2.1 in [28], because that proof involved only manipulations with path ordered integrals

$$\int_{0 \leq t_1 < \ldots < t_n < 1} dt_1 \cdots dt_n$$

and Lie algebra traces $\text{Tr}_{\alpha_j}$. Indeed, the proof of [28] was based on the following general fact which can be derived from the Campbell-Hausdorff formula (cf. Proposition 2.2 of [28], there we used the notation $A_{\alpha}(x)$ for the 1-form $\frac{\delta}{\delta J(x)}$):

**Proposition A.1** There exist $SU(2)$-equivariant multilinear forms $C^{(n)}(v_1, \ldots, v_n)$, $v_1, \ldots, v_n \in su(2)$ with values in $su(2)$:

$$C^{(n)} : (su(2))^\otimes n \to su(2)$$

(in particular, $C^{(1)}(v) = v$, $C^{(2)}(v_1, v_2) = \frac{1}{2}[v_1, v_2]$), such that

$$\sum_{n_j \geq 0} \int_{0 \leq t_1 < \ldots < t_n < 1} dt_1 \cdots dt_n \frac{\delta}{\delta J(x_j(t_1))} \cdots \frac{\delta}{\delta J(x_j(t_n_j))} = e^{v_j},$$

$$v_j = \sum_{n \geq 1} \int_{0 \leq t_1 < \ldots < t_n < 1} dt_1 \cdots dt_n C^{(n)} \left( \frac{\delta}{\delta J(x_j(t_1))}, \ldots, \frac{\delta}{\delta J(x_j(t_n))} \right), \quad v_j \in su(2).$$

Another ingredient in the proof of [28] was Kirillov’s trace formula

$$\text{Tr}_{\alpha_j} e^{v_j} = \frac{|v_j|}{\sin |v_j|} \int_{|\alpha_j| = \alpha_j} \frac{d^2 \alpha_j}{4\pi \alpha_j} e^{iv_j \cdot \alpha_j}.$$
formula (2.2) through the application of rigorous combinatorial rules (which are associated with Feynman rules in quantum field theory) to the calculation of the action of derivatives \( \frac{\delta}{\delta j} \) contained in \( e^{\vec{v}_j \cdot \vec{\alpha}_j} \), on the exponential in the r.h.s. of eq. (A.3). We will provide the details in [32].

The proof of the Proposition 2.2 in [28] was based on two assumptions about the properties of the forms \( G_{m,0}, m \geq 2 \).

**Assumption A.2** For a bi-local \((1,1)\)-form

\[
\Omega(y_1, y_2) = \frac{2}{i\pi} G_{2,0}(y_1, y_2),
\]

(A.10)

there exists a bi-local \((0,2)\)-form \( \tilde{\Omega}(y_1, y_2) \) such that

\[
d_{y_2}\Omega(y_1, y_2) = \delta^{(3)}(y_2 - y_1) + d_{y_1}\tilde{\Omega}(y_1, y_2).
\]

(A.11)

**Assumption A.3** The forms \( G_{m,0}, m \geq 3 \) come from tree Feynman diagrams with propagator (A.10) and triple vertices of the Chern-Simons action (see, e.g. [3] and references therein for details).

In [28] we derived the set of forms \( G_{m,n} \) with the help of the quantum field theory perturbation theory described e.g. in [3]. Therefore Assumption A.3 was satisfied automatically. Also, it was explained in [3] that in the notations \( S^3 = \mathbb{R}^3 \cup \{\infty\}, y_{1,2} \in \mathbb{R}^3 \) the form (A.10) was equal to

\[
\Omega(y_1, y_2) = \frac{1}{4\pi} \epsilon_{\mu
u\rho} \frac{y_2^\rho - y_1^\rho}{|y_2 - y_1|^3} dy_1^\mu dy_2^\nu.
\]

(A.12)

and the Assumption A.2 was satisfied with the choice of the form

\[
\tilde{\Omega}(y_1, y_2) = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{y_2^\rho - y_1^\rho}{|y_2 - y_1|^3} dy_2^\mu \wedge dy_2^\nu.
\]

(A.13)

The two Assumptions A.2 and A.3 are also satisfied for the set of forms \( G_{m,0} \) coming from Kontsevich integral derivation of eq. (A.3). We use the notations \( S^3 = \mathcal{C}^1 \times \mathbb{R}^1 \cup \{\infty\}, (z, t) \in \mathcal{C}^1 \times \mathbb{R}^1 \), so Kontsevich’s form \( \Omega \) is

\[
\Omega(z_1, \bar{z}_1; z_2, \bar{z}_2; t_1, t_2) = \frac{1}{2\pi i} \frac{\delta(t_2 - t_1)}{z_2 - z_1} (dt_1 dz_2 - dt_2 dz_1).
\]

(A.14)
It satisfies Assumption A.2 if we choose
\[ \tilde{\Omega}(z_1, \bar{z}_1, t_1; z_2, \bar{z}_2, t_2) = -\frac{1}{2\pi i} \frac{\delta(t_2 - t_1)}{z_2 - \bar{z}_1} dt_2 \wedge dz_2. \] (A.15)

The Feynman diagrams with cubic vertices built upon the propagator (A.14) are all zero, because the form (A.14) does not contain \( dz \) so that its triple wedge products are zero. Therefore Assumption A.3 is satisfied in the trivial way due to eq. (A.5). Hence the proof of Proposition 2.2 remains valid verbatim if we use the set of forms \( G_{m,n} \) coming from Kontsevich’s integral instead of the ones coming from the \( \text{[3]-style} \) perturbation theory.

References

[1] S. Axelrod, I. Singer, Chern-Simons Perturbation Theory, Proceedings of XXth Conference on Differential Geometric Methods in Physics (New York, 1991) (S. Catto and A. Rocha, eds) World Scientific, 1992, 3-45.

[2] S. Axelrod, I. Singer, Chern-Simons Perturbation Theory II, Jour. Diff. Geom. 39 (1994) 173-213.

[3] D. Bar-Natan, Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory, Ph.D. thesis, Princeton University, 1991.

[4] D. Bar-Natan, On the Vassiliev Knot Invariants, Topology 34 (1995) 423-472.

[5] D. Bar-Natan, Vassiliev Homotopy String Link Invariants, Harvard University preprint, 1993.

[6] D. Bar-Natan, S. Garoufalidis, On the Melvin-Morton-Rozansky Conjecture, preprint 1994.

[7] J. S. Birman, X-S. Lin, Knot polynomials and Vassiliev’s invariants, Invent. Math. 111 (1993) 225-270.

[8] R. Bott, C. Taubes, On the Self-Linking of Knots, J. Math. Phys. 35 (1994) 5247-5287.
[9] S. Garoufalidis, *Relations among 3-Manifold Invariants*, Ph.D. Thesis, University of Chicago, August 1992.

[10] S. Garoufalidis, *On Finite Type 3-Manifold Invariants I*, M.I.T. preprint, January 1995, to appear in J. Knot Theory and its Ramifications.

[11] S. Garoufalidis, T. Ohtsuki, *On Finite Type Invariants V: Rational Homology 3-Spheres*, preprint.

[12] L. Jeffrey, *Chern-Simons-Witten Invariants of Lens Spaces and Torus Bundles, and the Semiclassical Approximation*, Commun. Math. Phys. 147 (1992) 563-604.

[13] M. Kontsevich, *Vassiliev’s Knot Invariants*, Adv. in Sov. Math. 16(2) (1993) 137-150.

[14] X-S. Lin, *Milnor Link Invariants Are All of Finite Type*, Columbia University preprint, 1992.

[15] X-S. Lin, Z. Wang, *On Ohtsuki’s Invariants of Integral Homology 3-Spheres, I*, preprint, q-alg/9509004.

[16] R. Kirby, P. Melvin, *The 3-Manifold Invariants of Witten and Reshetikhin-Turaev for sl(2,C)*, Invent. Math. 105 (1991) 473-545.

[17] R. Lawrence, *Asymptotic Expansions of Witten-Reshetikhin-Turaev Invariants for Some Simple 3-Manifolds*, preprint IHES/M/95/39, to appear in J. Mod. Phys.

[18] G. Masbaum, J. Roberts, *A Simple Proof of Integrality of Quantum Invariants at Prime Roots of Unity*, preprint, September 28, 1995.

[19] P. Melvin, H. Morton, *The Coloured Jones Function*, Commun. Math. Phys. 169 (1995) 501-520.

[20] H. Murakami, *Quantum SU(2)-Invariants Dominate Casson’s SU(2)-Invariant*, Math. Proc. Camb. Phil. Soc. 115 (1993) 253-281.
[21] H. Murakami, *Quantum SO(3)-Invariants Dominate the SU(2)-Invariant of Casson and Walker*, Math. Proc. Camb. Phil. Soc. **117** (1995) 237-249.

[22] T. Ohtsuki, *A Polynomial Invariant of Integral Homology 3-Spheres*, Math. Proc. Camb. Phil. Soc. **117** (1995) 83-112.

[23] T. Ohtsuki, *A Polynomial Invariant of Rational Homology 3-Spheres*, preprint UTMS 94-49, August 12, 1994.

[24] T. Ohtsuki, *Finite Type Invariants of Integral Homology 3-Spheres*, preprint UTMS 94-42, June 1, 1994.

[25] N. Reshetikhin, private communications.

[26] N. Reshetikhin, V. Turaev, *Invariants of 3-Manifolds via Link Polynomials and Quantum Groups*, Invent. Math. **103** (1991) 547-597.

[27] L. Rozansky, *A Contribution of the Trivial Connection to the Jones Polynomial and Witten’s Invariant of 3d Manifolds I.*, preprint UMTG-172-93, UTTHG-30-93, [hep-th/9401061](http://arxiv.org/abs/hep-th/9401061), to be published in Commun. Math. Phys.

[28] L. Rozansky, *Reshetikhin’s Formula for the Jones Polynomial of a Link: Feynman Diagrams and Milnor’s Linking Numbers*, Journ. Math. Phys. **35** (1994) 5219-5246.

[29] L. Rozansky, *A Contribution of the Trivial Connection to the Jones Polynomial and Witten’s Invariant of 3d Manifolds II.*, preprint UMTG-187-94, [hep-th/9403021](http://arxiv.org/abs/hep-th/9403021), to be published in Commun. Math. Phys.

[30] L. Rozansky, *The Trivial Connection Contribution to Witten’s Invariant and Finite Type Invariants of Rational Homology Spheres*, preprint UMTG-182, [q-alg/9503011](http://arxiv.org/abs/q-alg/9503011), submitted to Commun. Math. Phys.

[31] L. Rozansky, *Witten’s Invariants of Rational Homology Sheres at Prime Values of K and Trivial Connection Contribution*, preprint UMTG-183, [q-alg/9504013](http://arxiv.org/abs/q-alg/9504013), submitted to Commun. Math. Phys.
[32] L. Rozansky, *Kontsevich’s Integral and the proof of Reshetikhin’s Formula*, in preparation.

[33] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. 121 (1989) 351.