A CONTINUUM VERSION OF THE KUNZ-SOUILLARD APPROACH TO LOCALIZATION IN ONE DIMENSION

DAVID DAMANIK AND GÜNTER STOLZ

ABSTRACT. We consider continuum one-dimensional Schrödinger operators with potentials that are given by a sum of a suitable background potential and an Anderson-type potential whose single-site distribution has a continuous and compactly supported density. We prove exponential decay of the expectation of the finite volume correlators, uniform in any compact energy region, and deduce from this dynamical and spectral localization. The proofs implement a continuum analog of the method Kunz and Souillard developed in 1980 to study discrete one-dimensional Schrödinger operators with potentials of the form background plus random.

1. Introduction

In their work [9], Kunz and Souillard introduced the first method which allowed one to give a rigorous proof of localization for the one-dimensional discrete Anderson model; see also [2] for a presentation of the method. Since then, other, more versatile, methods have been developed which allowed one to establish Anderson localization also for multi-dimensional and continuum Anderson models. However, the Kunz-Souillard method has several important features which still deserve interest. First, it directly establishes a strong form of dynamical localization which was proven with other methods only much later and with more effort.

Second, and most importantly, among the available methods which establish one-dimensional localization, it is the only one which shows localization at all energies, arbitrary disorder, and without requiring ergodicity. It completely avoids the use of Lyapunov exponents and provides an extremely direct path to dynamical localization.

The virtues of the Kunz-Souillard methods were demonstrated in the work [4], which proves localization for one-dimensional discrete Anderson models with arbitrary bounded background potential, and in [12], which applies the method to decaying random potentials.

It is our goal here to extend the Kunz-Souillard method to one-dimensional continuum Anderson-type models and provide a localization proof which allows for a rather general class of bounded, not necessarily periodic, background potentials.

We mention one earlier work of Royer [11] which carried over many of the features of the Kunz-Souillard method to a continuum model. There the Brownian motion generated random potential from the work [7] by Goldsheid, Molchanov and Pastur is considered. The argument in [11] does not carry over to Anderson-type models and does not include deterministic background potentials.

---

D. D. was supported in part by NSF grants DMS–0653720 and DMS–0800100.
G. S. was supported in part by NSF grant DMS–0653374.
Specifically, we consider the random Schrödinger operator
\begin{equation}
H_\omega = -\frac{d^2}{dx^2} + W_0(x) + V_\omega(x)
\end{equation}
in $L^2(\mathbb{R})$, where the random potential is given by
\begin{equation}
V_\omega(x) = \sum_{n \in \mathbb{Z}} \omega_n f(x - n).
\end{equation}
For the single-site potential $f$, the random coupling constants $\omega_n$, and the background potential $W_0$, we fix the following assumptions:

(i) The coupling constants $\omega = (\omega_n)_{n \in \mathbb{Z}}$ are i.i.d. real random variables, whose distribution has a continuous and compactly supported density $r$.

(ii) For the single-site potential $f$, we assume that
\begin{equation}
c \chi_I(x) \leq f(x) \leq C \chi([-1, 0])
\end{equation}
for constants $0 < c \leq C < \infty$ and a non-trivial subinterval $I$ of $[-1, 0]$. We also assume that there exists an interval $[a, b] \subset [-1, 0]$ such that
\begin{equation}
f(x) > 0 \text{ for a.e. } x \in [a, b], \quad f(x) = 0 \text{ for } x \in [-1, 0] \setminus [a, b].
\end{equation}

(iii) The background potential $W_0$ is bounded and such that
\begin{equation}
\{ W_0(\cdot - n) |_{[-1, 0]} : n \in \mathbb{Z} \}
\end{equation}
is relatively compact in $L^\infty(-1, 0)$.

This includes 1-periodic potentials $W_0$, but, much more generally, allows for many situations where $H_\omega$ is not ergodic with respect to 1-shifts, examples being

(i) almost periodic potentials $W_0$, that is, $\{ W_0(\cdot - \tau) : \tau \in \mathbb{R} \}$ is relatively compact in $L^\infty(\mathbb{R})$,

(ii) any $W_0$ that is uniformly continuous on $\mathbb{R}$, in which case (5) follows from equicontinuity and the Arzela-Ascoli theorem.

Let $\chi_x = \chi_{[x-1,x]}$ and, for $x, y \in \mathbb{Z}$ and $E_{\max} > 0$, define
\begin{equation}
\rho(x, y; E_{\max}) := E \left( \sup_{t \in \mathbb{R}} \| \chi_x e^{-itH_\omega} \chi_y \| : g : \mathbb{R} \to \mathbb{C} \text{ Borel measurable,} \quad \| g \| = 1, \text{ supp } g \subset [-E_{\max}, E_{\max}] \right).
\end{equation}

Our main result is

**Theorem 1.** For every $E_{\max} > 0$, there exist $C < \infty$ and $\eta > 0$ such that for all $x, y \in \mathbb{Z}$,
\begin{equation}
\rho(x, y; E_{\max}) \leq C e^{-\eta|x-y|}.
\end{equation}

Let us discuss some immediate consequences of this result. First, Theorem 1 implies a strong form of dynamical localization. Indeed, choosing $g_t(E) = e^{-it E} \chi_{[-E_{\max}, E_{\max}]}(E)$, we find that
\begin{align*}
E \left( \sup_{t \in \mathbb{R}} \| \chi_x e^{-itH_\omega} \chi_{[-E_{\max}, E_{\max}]}(H_\omega) \chi_y \| \right) \leq C e^{-\eta|x-y|}.
\end{align*}
Dynamical localization implies spectral localization. Explicitly, we can deduce the following result.

**Corollary 1.1.** For almost every $\omega$, $H_\omega$ has pure point spectrum with exponentially decaying eigenfunctions.
A CONTINUUM VERSION OF THE KUNZ-SOUILLARD APPROACH

We refer the reader to [1] for an explicit derivation of this consequence and other interesting forms and signatures of localization from a statement like Theorem 1. Note that spectral localization throughout the spectrum is obtained by taking a countable intersection over an unbounded sequence of values of $E_{\text{max}}$.

Spectral localization and a weaker form of dynamical localization with periodic background potential can be established by different methods for more general single-site distributions. The most general result, which merely assumes non-trivial bounded support, can be found in [3].

It has also been shown recently in [6] how the fractional moment method can be used to show the exponentially decaying dynamical localization bound (7) for the model (1), (2). This work requires periodicity of the background potential and the existence of a bounded and compactly supported density for the single-site distribution.

Given these results for the case of periodic background and the limitation of the methods used in their proofs to this particular choice of background, we see that the method we present here should be regarded as the primary tool to perturb a given Schrödinger operator by the addition of a (preferably small) random potential to generate pure point spectrum. In fact, a question of this kind posed by T. Colding and P. Deift triggered our work. Thus, as a sample application, we state the following corollary, which answers the specific question we were faced with.

**Corollary 1.2.** Given $W_0 : \mathbb{R} \to \mathbb{R}$ with $C^1$-norm $\|W_0\|_\infty + \|W_0'\|_\infty < \infty$, there exists a $V : \mathbb{R} \to \mathbb{R}$ with arbitrarily small $C^1$-norm such that $-d^2/dx^2 + W_0 + V$ has pure point spectrum.

This is an immediate consequence of Theorem 1 and Corollary 1.1, since $W_0$ is uniformly continuous and we may choose $V$ of the form (2) with single-site potential $f$ of small $C^1$-norm and $\omega_n$ supported in $[0, 1]$. 

Apart from the ability to handle quite general background potentials, the flexibility of the Kunz-Souillard method manifests itself in the discrete case in another way: it allows one to prove powerful results for random decaying potentials; see Simon [12] and the follow-up paper [5]. We have not been able to establish a continuum analogue of Simon’s work using our method. The obstacle to doing this is in making Proposition 4.1 in Section 4 quantitative. We regard it as an interesting open problem to establish such a quantitative estimate.

The remainder of the paper is organized as follows. Section 2 describes the overall strategy of the proof of Theorem 1, adapting the strategy of [9] and culminating in Section 2.5. In the remaining sections we prove various norm bounds on integral operators which are used in this argument, with the crucial contraction property being established in Section 4. A technical result on the large coupling limit of Prüfer amplitudes, used in Section 4, is proven in Appendix A while Appendix B summarizes a number of standard ODE facts.

Acknowledgements: We would like to thank B. Simon for useful discussions at an early stage of this work, as well as T. Colding and P. Deift for posing a question which motivated us to finish it.

\footnote{Incidentally, it was realized sixteen years after Simon’s 1982 work that there is a different approach to random decaying potentials which is applicable in more general situations and which does extend easily to the continuum [5]. Another approach to continuum random decaying potentials was given in [10].}
2. REDUCTION TO INTEGRAL OPERATOR BOUNDS

2.1. Finite Volume Correlators. Consider finite-volume restrictions of $H_\omega$, that is, for $L \in \mathbb{Z}_+$, we denote by $H^L_\omega$ the restriction of $H_\omega$ to the interval $[-L, L]$ with Dirichlet boundary conditions. For fixed $L$ we will use the abbreviation $\omega = (\omega_{-L+1}, \ldots, \omega_L)$, as these are the only coupling constants which $H^L_\omega$ depends on. Moreover, for $x, y \in \mathbb{Z}$ and $E_{\text{max}} > 0$, we introduce the finite volume correlators

$$\rho_L(x, y; E_{\text{max}}) := \mathbb{E}\left( \sup\{||x g(H^L_\omega) \chi_y|| : |g| \leq 1, \text{supp}\ g \subset [-E_{\text{max}}, E_{\text{max}}]\} \right).$$

**Lemma 2.1.** We have

$$\rho(x, y; E_{\text{max}}) \leq \liminf_{L \to \infty} \rho_L(x, y; E_{\text{max}}).$$

**Proof.** See [1] Eq. (2.28)] and its discussion there. \hfill \Box

We will often suppress the dependence of these quantities on $E_{\text{max}}$ in what follows. Moreover, we can without loss of generality restrict our attention to the case $x = 1$ and $y = n \in \mathbb{Z}_+$. Thus, we aim to estimate $\rho(1, n)$ by means of finding estimates for $\rho_L(1, n)$ that are uniform in $L$. Explicitly, our goal is to show the following:

**Proposition 2.2.** There exist $C < \infty$ and $\eta > 0$ such that, for all $n, L \in \mathbb{Z}_+$ with $n \leq L$, we have

$$\rho_L(1, n) \leq C e^{-\eta n}. \tag{8}$$

In order to estimate $\rho_L(1, n)$, we consider the eigenfunction expansion of $H^L_\omega$. Thus, for $L \in \mathbb{Z}_+$ and $\omega = (\omega_{-L+1}, \ldots, \omega_L)$, we denote the (simple) eigenvalues of $H^L_\omega$ by $\{E_k : k \geq 1\}$ and the associated normalized eigenvectors by $\{v_k : k \geq 1\}$. Here we leave the dependence of these quantities on $\omega$ and $L$ implicit.

The proof of Proposition 2.2 starts with the following observation:

$$\rho_L(1, n) \leq \mathbb{E}\left( \sum_{|E_k| \leq E_{\text{max}}} \|\chi_n v_k\| \cdot \|\chi_1 v_k\| \right) \tag{9}$$

$$= \sum_{k=1}^{\infty} \mathbb{E}\left( \chi_{\{\omega:|E_k(\omega)| \leq E_{\text{max}}\}} \|\chi_n v_k\| \cdot \|\chi_1 v_k\| \right)$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}^{2L}} \chi_{\{\omega:|E_k(\omega)| \leq E_{\text{max}}\}} \|\chi_n v_k\| \cdot \|\chi_1 v_k\| \prod_{j=-L+1}^{L} r(\omega_j) d\omega_j$$

In the next subsection we will rewrite the integral in the last line of (9) by introducing a change of variables based on the Prüfer phase of the eigenfunctions $v_k$ at the integer sites in $[-L, L]$. The Jacobian of this change of variables will be computed in Subsection 2.3. This will then lead to a formula for $\rho_L(1, n)$ involving integral operators, which will be made explicit in Subsection 2.3.

2.2. Change of Variables. In this subsection we introduce the change of variables according to which we will rewrite the integrals appearing in the last line of (9).

To introduce Prüfer variables, let $u_{-L}(\cdot, \omega, E)$ be the solution of $-u'' + (W_0 + V_\omega)u = Eu$ satisfying $u(-L) = 0$ and $u'(-L) = 1$. The corresponding Prüfer phase $\varphi_{-L}(\cdot, \omega, E)$ and amplitude $R_{-L}(\cdot, \omega, E)$ are defined by

$$u_{-L} = R_{-L} \sin \varphi_{-L}, \quad u'_{-L} = R_{-L} \cos \varphi_{-L}$$
normalized so that \( \varphi_L(-L) = 0 \) and \( \varphi_L(\cdot, \omega, E) \) continuous to get uniqueness of the phase. The Prüfer phase satisfies the first order equation (cf. Lemma \[B.4\])

\[
\varphi_L' = 1 - (1 + W_0 + V_0 - E) \sin^2 \varphi_L. 
\]

Fix an \( M > 0 \) such that

\[
supp r \subseteq [-M, M].
\]

Thus

\[
\|\varphi'\|_\infty \leq 2 + \|W_0\|_\infty + M\|f\|_\infty + E_{max} < \infty.
\]

Choose \( N \in \mathbb{Z}_+ \) such that

\[
2 + \|W_0\|_\infty + M\|f\|_\infty + E_{max} < N \pi.
\]

Hence the change of the Prüfer phase over any interval of length one is uniformly bounded in absolute value by \( N \pi \).

With the circle

\[ T_N := \mathbb{R}/(2\pi N \mathbb{Z}) \]

and

\[ \Omega := \{ (\omega, k) \in [-M, M]^{2L} \times \mathbb{Z}_+: E_k(\omega) \in [-E_{max}, E_{max}] \}\]

we can now define our change of variables

\[ C: \Omega \rightarrow T_N^{2L-1} \times [-E_{max}, E_{max}] \times \{0, \ldots, 2N - 1\} \]

\[ (\omega, k) \mapsto (\theta_{-L+1}, \ldots, \theta_{L-1}, E, j) \]

as follows:

- For \( i = -L + 1, \ldots, L - 1, \theta_i \in T_N \) is chosen so that
  \[ \varphi_L(i, \omega, E_k(\omega)) \equiv \theta_i \mod 2\pi N. \]

- \( E \in [-E_{max}, E_{max}] \) is given by
  \[ E = E_k(\omega). \]

- Finally, \( j \in \{0, \ldots, 2N - 1\} \) is defined so that
  \[ k \equiv j \mod 2N. \]

Lemma 2.3. The change of variables \( C \) is one-to-one.

Proof. The key point is that the Prüfer phase of the \( k \)-th eigenfunction of \( H_\omega \) runs from 0 (at \(-L\)) to \( k\pi \) (at \( L\)), i.e. \( \varphi_L(L, \omega, E_k(\omega)) = k\pi \). Since we are taking phases modulo \( 2N\pi \), we need to ensure that no ambiguities are generated. The desired uniqueness follows from our choice of \( N \), which can be seen as follows.

Suppose that \( (\theta_{-L+1}, \ldots, \theta_{L-1}, E, j) \) belongs to the range of \( C \) and that \( C(\omega, k) = (\theta_{-L+1}, \ldots, \theta_{L-1}, E, j), (\omega, k) \in \Omega \). Let \( \theta_{-L} := 0, \theta_L := j\pi \). Thus, by definition of \( C \), for \( i = -L + 1, \ldots, L \),

\[ \varphi_L(i, \omega, E) \equiv \theta_i \mod 2\pi N. \]

This determines \( \omega_i, i = -L + 1, \ldots, L \), uniquely, as is seen iteratively in \( i \): For fixed \( i \), \( \varphi_L(i - 1, \omega, E) \) is determined by \( \omega_{-L+1}, \ldots, \omega_{i-1} \) and, by \( (12) \) and \( (13) \),

\[ |\varphi_L(i, \omega, E) - \varphi_L(i - 1, \omega, E)| < N\pi. \]

Actually, we need \( N\pi \) to bound the growth of the Prüfer phase over a unit interval, but it follows from the differential equation that it can never decrease by more than \( \pi \).
We also know from Appendix [11 (cf. Lemma [3.6]) that \( \varphi_{-L}(i, \omega, E) \) is strictly decreasing in \( \omega \). Therefore, given \( \omega_{-L+1}, \ldots, \omega_{i-1} \), there can be at most one \( \omega_i \in [-M, M] \) satisfying (13).

Finally, with the unique values of \( \omega_{-L+1}, \ldots, \omega_L \) reconstructed, \( k \in \mathbb{Z}_+ \) is uniquely determined by \( \varphi_{-L}(L, \omega, E) = k\pi \).

Now we carry out the change of variables in (9) and consider the resulting integral.

Similarly to the definition of \( u_{-L} \), let \( u_L(\cdot, \omega, E) \) be the solution of \(-u'' + (W_0 + V_\omega) u = Eu\) determined by \( u_L(L) = 0, u_L'(L) = 1 \), with corresponding Prüfer variables \( \varphi_L \) and \( R_L \), where \( \varphi_L(L) = 0 \).

For \( g \in L^\infty([-1, 0]) \) and \( E, \lambda, \theta, \eta \in \mathbb{R} \), let \( u_0(\cdot, \theta, \lambda, g, E) \) be the unique solution of

\[
-u'' + gu + \lambda fu = Eu
\]

with \( u_0'(0) = \cos \theta, u_0(0) = \sin \theta \) and let \( u_{-1}(\cdot, \eta, \lambda, g, E) \) be the unique solution of (15) with \( u_{-1}'(-1) = \cos \eta, u_{-1}(-1) = \sin \eta \).

Let \( \varphi_0(\cdot, \theta, \lambda, g, E) \) and \( R_0(\cdot, \theta, \lambda, g, E) \) be the Prüfer phase and amplitude, respectively, for \( u_0(\cdot, \theta, \lambda, g, E) \). Similarly, let \( \varphi_{-1}(\cdot, \eta, \lambda, g, E) \) and \( R_{-1}(\cdot, \eta, \lambda, g, E) \) be the Prüfer variables for \( u_{-1}(\cdot, \eta, \lambda, g, E) \).

For the next definition, in order to make use of (12) and (13), we assume \( \|g\|_\infty \leq \|W_0\|_\infty \) and \( |E| \leq E_{\max} \).

Note that \( \varphi_0(\cdot, \theta + \pi, \lambda, g, E) = \varphi_0(\cdot, \theta, \lambda, g, E) + \pi \) and \( \varphi_{-1}(\cdot, \eta + \pi, \lambda, g, E) = \varphi_{-1}(\cdot, \eta, \lambda, g, E) + \pi \). Thus, in particular, \( \varphi_0(-1, \cdot, \lambda, g, E) \) and \( \varphi_{-1}(0, \cdot, \lambda, g, E) \) induce well-defined mappings from \( \mathbb{T}_N \) to \( \mathbb{T}_N \), which is how we use them below.

If \( \alpha, \beta \in \mathbb{T}_N \) are such that there exists a coupling constant \( \lambda \in [-M, M] \) with \( \varphi_0(-1, \alpha, \lambda, g, E) = \beta \) (or, equivalently, \( \varphi_{-1}(0, \beta, \lambda, g, E) = \alpha \)), we define \( \lambda(\beta, \alpha, g, E) = \lambda \). Note that by (12) and (13), using the same argument as in the proof of Lemma 2.3, this \( \lambda \) is uniquely determined if it exists.

Finally, write \( f_i := f(-i) \) and \( g_i := W_0(-i) \).

Lemma 2.4. We have

\[
\rho_L(1, n) \leq \int_{E_{\max}}^{E_{\max}} \rho_L(1, n, E) \, dE,
\]

where, for \( E \in [-E_{\max}, E_{\max}] \), we write

\[
\rho_L(1, n, E) := \sum_{j=0}^{2N-1} \int_{\mathbb{S}_N} r(\lambda(\theta_{L-1}, j\pi, g_L, E)) \cdot r(\lambda(0, \theta_{-L+1}, g_{-L+1}, E)) \cdot \\
\left( \prod_{i=-L+2}^{L-1} r(\lambda(\theta_{i-1}, \theta_i, g_i, E)) \right) \left( \int_{\theta_{-L+1}}^{\theta_{L+2}} u_L^2 \right)^{1/2} \left( \int_{0}^{1} u_L^2 \right)^{1/2} \\
\frac{R_{-L}^2(-L+1) \cdots R_{-L}^2(0) \cdot R_{L}^2(1) \cdots R_{L}^2(L-1)}{\prod_{i=-L+1}^{L+2} \int f_{L+1} u_{L-1}^2 \cdots \int f_0 u_L^2 \cdot \int f_{L} u_{L-1}^2 \cdots \int f_{-L} u_{L}^2} \\
d\theta_{-L+1} \cdots d\theta_{L-1}
\]

and interpret \( r(\lambda(\cdot, \cdot)) \) as zero if \( \lambda(\cdot, \cdot) \) does not exist. Here the argument \( \omega \) in the functions \( u_{\pm L} = u_{\pm L}(\cdot, \omega, E) \) and \( R_{\pm L} = R_{\pm L}(\cdot, \omega, E) \) is the one uniquely determined via Lemma 2.3 by \( \theta_{-L+1}, \ldots, \theta_L, E \) and \( j \).
Proof. For \( k \in \mathbb{Z}_+ \), let
\[
\Omega_k := \{ \omega \in [-M, M]^{2L} : |E_k(\omega)| \leq E_{\text{max}} \}
\]
and write
\[
A_k := \int_{\mathbb{R}^{2L}} \chi_{\Omega_k} \|\chi_n v_k\| \|\chi_1 v_k\| \prod_{i=-L+1}^L r(\omega_i) \, d\omega_i.
\]
On \( \Omega_k \) we change variables by the map
\[
C_k : \Omega_k \to \mathbb{T}_N^{2L-1} \times [-E_{\text{max}}, E_{\text{max}}] \quad \omega \mapsto (\theta_{-L+1}, \ldots, \theta_{L-1}, E_k(\omega)).
\]
Let \( J_k = \partial C_k / \partial \omega \) be its Jacobian. Pick \( j \in \{0, \ldots, 2N-1\} \) so that \( j \equiv k \mod 2N \). Noting that, in terms of the new variables on \( C_k(\Omega_k) \),
\[
v_k = \frac{u_L(\cdot, \omega(\theta_{-L+1}, \ldots, \theta_L, E), j)}{\|u_L(\cdot, \omega(\theta_{-L+1}, \ldots, \theta_L, E), j)\|},
\]
we get
\[
A_k = \int_{-E_{\text{max}}}^{E_{\text{max}}} \int_{\mathbb{T}_N^{2L-1}} \chi_{C_k(\Omega_k)} \det J_k^{-1} r(\lambda(0, \theta_{-L+1}, g_{-L+1}, E))
\]
\[
\left( \prod_{i=-L+2}^{L-1} r(\lambda(\theta_{i-1}, \theta_i, g_i, E)) \right) r(\lambda(\theta_{L-1}, j\pi, g_L, E))
\]
\[
\left( \int_{n-1}^n u_L^2 \right)^{1/2} \left( \int_0^1 u_L^2 \right)^{1/2}
\]
\[
\int_{-L}^L u_L^2 \, d\theta_{-L+1} \ldots d\theta_{L-1} \, dE.
\]
The Jacobian determinant is calculated in Lemma 2.6 of the next subsection. Inserting the result yields
\[
A_k = \int_{-E_{\text{max}}}^{E_{\text{max}}} \int_{\mathbb{T}_N^{2L-1}} \chi_{C_k(\Omega_k)} r(\lambda(\theta_{L-1}, j\pi, g_L, E)) r(\lambda(0, \theta_{-L+1}, g_{-L+1}, E))
\]
\[
\left( \prod_{i=-L+2}^{L-1} r(\lambda(\theta_{i-1}, \theta_i, g_i, E)) \right) \left( \int_{n-1}^n u_L^2 \right)^{1/2} \left( \int_0^1 u_L^2 \right)^{1/2}
\]
\[
\frac{R_{-L}^2(\theta_{-L+1}) \ldots R_L^2(\theta_{L-1})}{\int_{-L+1}^{-L} u_L^2 \ldots \int_0 u_L^2 \ldots \int_{L} u_L^2} \, d\theta_{-L+1} \ldots d\theta_{L-1} \, dE.
\]
Let \( k_1 \neq k_2 \) and \( j_1, j_2 \in \{0, \ldots, 2N-1\} \) with \( j_\ell \equiv k_\ell \mod 2N \). Then Lemma 2.3 says that
\[
(C_{k_1}(\Omega_{k_1}) \times \{j_1\}) \cap (C_{k_2}(\Omega_{k_2}) \times \{j_2\}) = \emptyset.
\]
Thus it follows that \( \sum_k A_k \leq \int_{-E_{\text{max}}}^{E_{\text{max}}} \rho_L(1, n, E) \), with \( \rho_L(1, n, E) \) defined in (13). But by (9) we have \( \rho_L(1, n) \leq \sum_k A_k \), which completes the proof.

2.3. Calculation of the Jacobian. It will turn out that the Jacobians arising above, up to constant row and column multipliers, have the simple structure considered in the next lemma.
Lemma 2.5. We have
\[
\det \begin{pmatrix} a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ b_2 & a_2 & a_2 & \cdots & a_2 & a_2 \\ b_3 & a_3 & a_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_n & b_n & b_n & \cdots & b_n & a_n \end{pmatrix} = a_1(a_2 - b_2)(a_3 - b_3) \cdots (a_n - b_n).
\]

Proof. Observe that
\[
\det \begin{pmatrix} a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ b_2 & a_2 & a_2 & \cdots & a_2 & a_2 \\ b_3 & b_3 & a_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_n & b_n & b_n & \cdots & b_n & a_n \end{pmatrix} = \det \begin{pmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ b_2 - a_2 & 0 & 0 & \cdots & 0 \\ b_3 & b_3 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & b_n \end{pmatrix} = (a_2 - b_2) \det \begin{pmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ b_3 & b_3 & a_3 & \cdots & a_3 \\ b_4 & b_4 & a_4 & \cdots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & b_n \end{pmatrix}
\]
and then obtain the result by iteration (or induction).

Lemma 2.6. With the conventions for the arguments of \(u_{\pm L}\) and \(R_{\pm L}\) made in Lemma 2.4 we have
\[
\det \mathcal{J}_k = \int_{-L}^L u_{-L}^2 \cdots \int_{-L}^L u_L^2 \cdot \int_{-L}^L u_{-L}^2 \cdots \int_{-L}^L u_L^2 \left( \int_{-L}^L u_L^2 \right)^{-1}.
\]

Proof. With a slight adjustment of the notation introduced above, we have for \(i = -L + 1, \ldots, 0\),
\begin{align*}
\theta_i \equiv \varphi_{-L}(i, 0, (\omega_{-L+1}, \ldots, \omega_i), E_k(\omega)) \mod 2\pi N,
\end{align*}
and for \(i = 1, \ldots, L - 1\),
\begin{align*}
\theta_i \equiv \varphi_{L}(i, j\pi, (\omega_{i+1}, \ldots, \omega_L), E_k(\omega)) \mod 2\pi N.
\end{align*}
The notational adjustment made here consists in stressing that \(\varphi_{-L}(i)\) depends explicitly on \(\omega_n\) only for \(n = -L + 1, \ldots, i\) (we only need to know the potential on \([-L, i]\) to calculate it), while it depends on all \(\omega_n\) implicitly through \(E_k(\omega)\). Similar reasoning applies to \(\varphi_{L}(i)\) in \([\text{18}]\).

Thus, using \([\text{18}]\) for \(1 \leq i \leq L - 1\) and \(n \leq i\), we have by Corollary \([\text{B.7}]\) and the Feynman-Hellmann formula
\[
\frac{\partial \theta_i}{\partial \omega_n} = \frac{\partial \varphi_{L}(i)}{\partial E} \cdot \frac{\partial E_k}{\partial \omega_n}
= -\frac{1}{R_k^L(i)} \int_i^L u_L^2 \cdot \int_{-L}^L u_L^2
= -\frac{1}{R_k^L(i)} \oint_{-L}^L u_L^2 \cdot \int_{-L}^L u_L^2;
\]
while for $1 \leq i \leq L - 1$ and $n > i$, we have
\[
\frac{\partial \theta_i}{\partial \omega_n} = \frac{\partial \varphi_L(i)}{\partial \omega_n} + \frac{\partial \varphi_L(i)}{\partial E} \cdot \frac{\partial E_k}{\partial \omega_n}
\]
\[
= \frac{1}{R_L^2(i)} \int_{f_n u_L^2}^{f_n u_L^2} - \frac{1}{R_L^2(i)} \int_{f_n u_L^2}^{f_n u_L^2} u^2_L \cdot \int f_n v_k^2
\]
\[
= \frac{1}{R_L^2(i)} \int_{f_n u_L^2}^{f_n u_L^2} \int f_n u_L^2.
\]

Analogous formulae, based on (17), hold in the case $-L + 1 \leq i \leq 0$. In this case we get for $n > i$ that
\[
\frac{\partial \theta_i}{\partial \omega_n} = \frac{1}{R_L^2(i)} \int_{f_n u_L^2}^{f_n u_L^2} \int f_n u_L^2
\]
and, for $n \leq i$,
\[
\frac{\partial \theta_i}{\partial \omega_n} = -\frac{1}{R_L^2(i)} \int_{f_n u_L^2}^{f_n u_L^2} \int f_n u_L^2.
\]

Also, writing $E$ as the first of the new variables, the first row of the Jacobian has entries $\frac{\partial E}{\partial \omega_n} = \int f_n v_n^2 = \int f_n u_L^2 / \int f_n u_L^2$.

Using these formulae and factoring out common factors in rows and columns, we find that
\[
\det J_k = \frac{\int f_{-L+1} u_L^2 \cdots \int f_L u_L^2}{R_L^2(-L + 1) \cdots R_L^2(0) \cdot R_L^2(1) \cdots R_L^2(L - 1)} \left( \int f_{-L} u_L^2 \right)^{-2L} \cdot \det A
\]

where
\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-I_{L+1,L} & I_{L+1,L} & I_{L,L+1} & \cdots & \cdots & I_{L,L+1} & I_{L,L+1} \\
-I_{L+1,L} & -I_{L+1,L} & I_{L,L+1} & \cdots & \cdots & -I_{L,L+1} & I_{L,L+1} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-I_{L,L} & \cdots & I_{L,L} & I_{L,L} & \cdots & \cdots & I_{L,L} \\
-I_{L,L} & \cdots & -I_{L,L} & -I_{L,L} & \cdots & \cdots & I_{L,L} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
-I_{L-1,L} & -I_{L-1,L} & \cdots & -I_{L-1,L} & -I_{L-1,L} & -I_{L-1,L} & -I_{L-1,L} \\
\end{pmatrix}
\]

and
\[
I_{m,t}^\pm := \int_m^{1} u_{m,L}^2.
\]
Applying Lemma 2.5, we obtain
\[
\det A = (I_{L,-L+1} - L, -L + 1 + I_{L-1,1} + I_{L-1,0}) \cdot \cdots \cdot (I_{L,0} - L, 0 + I_{L,1}) \cdot (I_{L,1} + I_{L-1,0} + I_{L-1,1}) \cdot \cdots \cdot (I_{L,1} + I_{L-1,1} + I_{L-1,0}).
\]
Plugging this into the formula for \( \det J_k \) obtained above, the result follows. \( \square \)

2.4. The Integral Operator Formula. The expression in Lemma 2.4 may be written in a more succinct form once we have introduced a number of quantities.

If, for given \( g \in L^\infty(-1,0) \) with \( \|g\|_\infty \leq \|W_0\|_\infty \), \( E \in [-E_{\max}, E_{\max}] \) and \( \beta, \alpha \in \mathbb{T}_N \), \( \lambda(\beta, \alpha, g, E) \) as defined in Section 2.2 exists, we write

\[
\begin{align*}
  u_+((\cdot, \beta, \alpha, g, E)) &:= u_0((\cdot, \alpha, \lambda(\beta, \alpha, g, E), g, E)), \\
  u_-((\cdot, \beta, \alpha, g, E)) &:= u_{-1}((\cdot, \beta, \lambda(\beta, \alpha, g, E), g, E)) \\
\end{align*}
\]
and

\[
\begin{align*}
  R_+((\cdot, \beta, \alpha, g, E)) &:= R_0((\cdot, \alpha, \lambda(\beta, \alpha, g, E), g, E)), \\
  R_-((\cdot, \beta, \alpha, g, E)) &:= R_{-1}((\cdot, \beta, \lambda(\beta, \alpha, g, E), g, E)).
\end{align*}
\]

For later use, note that

\[
u_+^2((\cdot, \beta, \alpha, g, E)) = R_+^2(-1, \beta, \alpha, g, E) \cdot u_+^2((\cdot, \beta, \alpha, g, E)) = R_-^2(0, \beta, \alpha, g, E) \cdot u_-^2((\cdot, \beta, \alpha, g, E).
\]

We introduce the following integral operators defined on functions \( F \) on \( \mathbb{T}_N \):

\[
\begin{align*}
  (T_0(g, E)F)(\beta) &= \int \frac{R_+^2(-1, \beta, \alpha, g, E)}{\int f u_+^2((\cdot, \beta, \alpha, g, E)} r(\lambda(\beta, \alpha, g, E)) F(\alpha) \, d\alpha, \\
  (T_0(g, E)F)(\alpha) &= \int \frac{R_+^2(0, \beta, \alpha, g, E)}{\int f u_+^2((\cdot, \beta, \alpha, g, E)} r(\lambda(\beta, \alpha, g, E)) F(\beta) \, d\beta, \\
  (T_1(g, E)F)(\beta) &= \int \frac{R_+^2(-1, \beta, \alpha, g, E)}{\int f u_+^2((\cdot, \beta, \alpha, g, E)} r(\lambda(\beta, \alpha, g, E)) F(\alpha) \, d\alpha,
\end{align*}
\]
and, for \( j = 0, \ldots, 2N-1 \), the functions

\[
\begin{align*}
  (\Psi_j(g, E))(\theta) &= \frac{R_+^2(-1, \theta, j\pi, g, E)}{\int f u_+^2((\cdot, \theta, j\pi, g, E)} r(\lambda(\theta, j\pi, g, E)), \\
  (\Phi(g, E))(\theta) &= \frac{R_+^2(0, 0, \theta, g, E)}{\int f u_+^2((\cdot, 0, \theta, g, E)} r(\lambda(0, \theta, g, E)).
\end{align*}
\]

Now we are finally in a position to state the integral operator formula, which bounds \( \rho_L(1, n, E) \) from above.
Lemma 2.7. There exists a constant $C = C(E_{\text{max}})$ such that for every $E \in [-E_{\text{max}}, E_{\text{max}}]$, we have
\begin{equation}
(22) \quad \rho_L(1, n, E) \leq C \sum_{j=0}^{2N-1} \left\langle \hat{T}_0(g_0, E) \cdot \hat{T}_0(g_{-L+2}, E) \Phi(g_{-L+1}, E), T_1(g_1, E) \cdot \cdots \cdot T_1(g_n, E) T_0(g_{n+1}, E) \cdot \cdots \cdot T_0(g_{L-1}, E) \Psi_j(g_L, E) \right\rangle.
\end{equation}
Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{T}_N)$.

Proof. It follows from the a priori bounds in Lemma 3.1 that there exists a constant $C = C(E_{\text{max}})$ such that
\[
\int_{n=1}^{N} u_L^2 \leq CR_1^2(n), \quad \int_{0}^{1} u_L^2 \leq CR_2^2(0)
\]
uniformly in $E \in [-E_{\text{max}}, E_{\text{max}}]$. After using these bounds in the integrand on the right hand side of (10) we rearrange the terms in the integrand as
\[
R_L(n) R_L(0) \frac{R^2_L(-L+1) \cdot \cdots \cdot R^2_L(0) R^2_L(1) \cdot \cdots \cdot R^2_L(L-1)}{\int f_{-L+1} u_L^2 \cdot \cdots \cdot \int f_0 u_L^2 \cdot \int f_1 u_L^2 \cdot \cdots \cdot \int f_n u_L^2}
\]
\[
= \left( \prod_{i=-L+1}^{0} R_L^2(i) \int f_i u_L^2 \cdot \left( \prod_{i=1}^{n} R_L(i-1) R_L(i) \int f_i u_L^2 \right) \prod_{i=n+1}^{L} R_L^2(i) \int f_i u_L^2 \right).
\]
Taking into account the scaling properties of Prüfer amplitudes, we get the following relations between $u_{\pm L}$, $R_{\pm L}$ and $u_{\pm}$, $R_{\pm}$:
\[
\frac{R^2_L(i)}{\int f_i u_L^2} = \frac{R^2_L(0, \theta_{i-1}, \theta_i, g_i, E)}{\int f u^2 (\theta_{i-1}, \theta_i, g_i, E)},
\]
\[
\frac{R^2_L(i-1)}{\int f_i u_L^2} = \frac{R^2_L(-1, \theta_{i-1}, \theta_i, g_i, E)}{\int f u^2 (\theta_{i-1}, \theta_i, g_i, E)},
\]
and
\[
\frac{R_L(i-1) R_L(i)}{\int f_i u_L^2} = \frac{R^2_L(i-1)}{\int f u^2 (\theta_{i-1}, \theta_i, g_i, E)} \cdot \frac{R_L(i)}{R_L(i-1)} = \frac{R^2_L(-1, \theta_{i-1}, \theta_i, g_i, E)}{\int f u^2 (\theta_{i-1}, \theta_i, g_i, E)} \cdot \frac{1}{R_+(-1, \theta_{i-1}, \theta_i, g_i, E)} = \frac{R_+(-1, \theta_{i-1}, \theta_i, g_i, E)}{\int f u^2 (\theta_{i-1}, \theta_i, g_i, E)}.
\]
Plugging all this into (10) and using the definitions of the integral operators $T_0$, $\tilde{T}_0$, $T_1$ as well as the functions $\Psi_j$ and $\Phi$, we obtain (22). \hfill \Box

2.5. Proof of Proposition 2.2 and Theorem 1. We are now in a position to describe how our main result Theorem 1 follows from norm bounds for the operators $T_1$, $T_0$ and $\tilde{T}_0$, which we will establish in the remaining sections of this paper.

As was explained in Section 2.1 it suffices to prove Proposition 2.2 By Lemma 2.4 it suffices to establish a bound
\begin{equation}
(23) \quad \rho_L(1, n, E) \leq Ce^{-\eta n}
\end{equation}
Thus we can change variables and find that the right-hand side of (24) is equal to
\[ C \quad \text{or fixed} \quad \alpha \]

Denote the norm of a linear operator \( T \) from \( L^p(T_N) \) to \( L^q(T_N) \) by \( \| T \|_{p,q} \).

By Lemmas 3.1 and 3.2 we have \( \| T_0(g, E) \|_{1,1} = 1 \) as well as \( \| T_0(g, E) \|_{1,2} \leq C \) and \( \| \Psi_j(g, E) \|_1 \leq C \) uniformly in \( E \in [-E_{max}, E_{max}] \). \( g \|_\infty \leq \| W \|_\infty \) and \( j = 0, \ldots, 2N - 1 \). Thus
\[ \| T_0(g_{n+1}, E) \cdots T_0(g_{L-1}, E) \Psi_j(g_L, E) \|_2 \leq C \]
uniformly in \( E \in [-E_{max}, E_{max}] \), \( L \in \mathbb{N} \) and \( j = 0, \ldots, 2N - 1 \). Similarly, also uniformly,
\[ \| \hat{T}_0(g_0, E) \cdots \hat{T}(g_{-L+2}, E) \Phi(g_{-L+1}, E) \|_2 \leq C. \]
This yields, by (22) and Cauchy-Schwarz, that there is \( C = C(E_{max}, W_0, N) \) such that
\[ \rho_L(1, n, E) \leq C \int_1^n \| T_1(g_i, E) \|_{2,2} \]
In Section 4 we will show that \( \| T_1(g, E) \|_{2,2} \leq 1 \) for every \( g \in L^\infty(-1, 0) \) and \( E \in \mathbb{R} \). Finally, we establish in Section 5 that \( \| T_1(g, E) \|_{2,2} = \| T_1(g-E, 0) \|_{2,2} \) is continuous in \( (g, E) \in L^\infty(-1, 0) \times \mathbb{R} \). By assumption 5 it is guaranteed that \( \{ g_i : i \in \mathbb{Z} \} \) is relatively compact in \( L^\infty(-1, 0) \) and thus \( \{ g_i : i \in \mathbb{Z} \} \times [-E_{max}, E_{max}] \) is relatively compact in \( L^\infty(-1, 0) \times \mathbb{R} \). Thus
\[ \| T_1(g_i, E) \|_{2,2} \leq \gamma < 1 \]
uniformly in \( i \) and \( E \in [-E_{max}, E_{max}] \). This proves (23) with \( \eta = \ln(1/\gamma) \).

3. Elementary Results for the Integral Operators

In this section we consider the integral operators introduced in Section 2.4 and establish several elementary results for them.

We begin with \( T_0 \) and \( \hat{T}_0 \).

**Lemma 3.1.** We have
\[ \| T_0(g, E) \|_{1,1} = \| \hat{T}_0(g, E) \|_{1,1} = 1. \]

**Proof.** As \( g, E \) are fixed here, we will suppress them in this proof. Suppose \( F \in L^1(T_N) \). Then,
\begin{align*}
\| T_0 F \|_1 & \leq \int_{T_N} \int_{T_N} \frac{R_{\alpha}^2(-1, \beta, \alpha)}{f u_+^2(-, \beta, \alpha)} \rho(\lambda(\beta, \alpha)) |F(\alpha)| \, d\alpha \, d\beta \\
& = \int_{T_N} \int_{T_N} \frac{R_{\alpha}^2(-1, \beta, \alpha)}{f u_+^2(-, \beta, \alpha)} \rho(\lambda(\beta, \alpha)) \, d\beta \, |F(\alpha)| \, d\alpha.
\end{align*}
For fixed \( \alpha, \lambda(\beta, \alpha) \) is strictly increasing in \( \beta \) and the inverse function satisfies, see Lemma 3.6 b,
\[ \frac{\partial \beta}{\partial \lambda} = -R_{\alpha}^2(-1, \beta, \alpha) \int_0^{-1} f u_+^2(-, \beta, \alpha) = R_{\alpha}^2(-1, \beta, \alpha) \int_{-1}^0 f u_+^2(-, \beta, \alpha). \]

Thus we can change variables and find that the right-hand side of (24) is equal to
\[ \int_{T_N} \int_{\mathbb{R}} \rho(\lambda) \, d\lambda \, |F(\alpha)| \, d\alpha = \| r \|_1 \| F \|_1 = \| F \|_1, \]

where we also used the fact that $r$ is the density of a probability distribution. This shows that $\|T_0\|_{1,1} \leq 1$. Since the first step in (24) becomes an identity when $F \geq 0$, we get $\|T_0\|_{1,1} = 1$.

Using
\begin{equation}
\frac{d\alpha}{d\lambda} = -R^{-2}(0, \beta, \alpha) \int_{-1}^{0} f u^2_\beta(\cdot, \beta, \alpha). \tag{26}
\end{equation}

instead of (25), the proof of $\|\tilde{T}_0\|_{1,1} = 1$ is completely analogous. \hfill \Box

**Lemma 3.2.** We have
\[ \|T_0(g, E)\|_{1,2} \leq C < \infty \]
and
\[ \|\tilde{T}_0(g, E)\|_{1,2} \leq C < \infty \]
uniformly in $E \in [-E_{\text{max}}, E_{\text{max}}]$ and $\|g\|_\infty \leq \|W_0\|_\infty$.

Denoting by $\| \cdot \|_1$ the $L^1$-norm on $T_N$, we also have
\[ \|\Psi_j(g, E)\|_1 \leq C < \infty \]
and
\[ \|\Phi(g, E)\|_1 \leq C < \infty \]
uniformly in $E \in [-E_{\text{max}}, E_{\text{max}}]$, $\|g\|_\infty \leq \|W_0\|_\infty$ and $j = 0, \ldots, 2N - 1$.

**Proof.** Lemmas B.1 and B.3 provide bounds $C_1 < \infty$ and $C_2 > 0$ such that
\begin{equation}
R^2_\lambda(-1, \beta, \alpha) = R^2_0(-1, \alpha, \lambda(\beta, \alpha)) \leq C_1 \tag{27}
\end{equation}
and
\begin{equation}
\int f u^2_\beta(\cdot, \beta, \alpha) = \int f u^2_0(\cdot, \alpha, \lambda(\beta, \alpha)) \geq C_2 \tag{28}
\end{equation}
uniformly in $E \in [-E_{\text{max}}, E_{\text{max}}]$, $\|g\|_\infty \leq \|W_0\|_\infty$ and $\alpha, \beta$ such that $\lambda(\beta, \alpha) \in \text{supp } r$. In (28) we have also exploited the assumption (3) on the single site potential $f$.

Thus we get for $F \in L^1(T_N)$ that
\[ \|T_0 F\|_2^2 = \int_{T_N} \left| \int_{T_N} R^2_\lambda(-1, \beta, \alpha) f(\lambda(\beta, \alpha)) F(\alpha) d\alpha \right|^2 d\beta \leq 2\pi N(C_1/C_2)^2 \|r\|_\infty^2 \|F\|_2^2, \]
resulting in the required norm bound for $T_0$. The bound for $\tilde{T}_0$ is found similarly.

From (27) and (28) we also get the bounds for $\Psi_j$ and $\Phi$, first in the $L^\infty$-norm and then in the $L^1$-norm since $T_N$ has finite volume. \hfill \Box

Let us now turn to $T_1$. For $\alpha, \beta \in T_N$, we write
\begin{equation}
T_1(\beta, \alpha) = \begin{cases} \frac{R_{\alpha}(-1, \beta, \alpha) r(\lambda(\beta, \alpha))}{\int f(x) u^2(\beta, \alpha) dx} & \text{if } \lambda(\beta, \alpha) \text{ exists}, \\ 0 & \text{otherwise} \end{cases} \tag{29}
\end{equation}
for its integral kernel.

**Proposition 3.3.** We have
\begin{equation}
T_1(\beta + \pi, \alpha + \pi) = T_1(\beta, \alpha) \text{ for all } \beta, \alpha \in T_N, \tag{30}
\end{equation}
\begin{equation}
T_1(\cdot, \cdot) \text{ is continuous on } T_N^2. \tag{31}
\end{equation}
Proof. Note that \( \varphi(x, \alpha + \pi, \lambda) = \varphi(x, \alpha, \lambda) + \pi \) for every \( x \). This follows from \( u_0(x, \alpha + \pi, \lambda) = -u_0(x, \alpha, \lambda) \) for every \( x \), along with the initial condition \( \varphi(0, \alpha, \lambda) = \alpha \).

To derive (30) from this, consider a pair \((\beta, \alpha)\) such that \( \lambda(\beta, \alpha) \) exists. Then \( \lambda(\beta + \pi, \alpha + \pi) \) exists, too, and is equal to \( \lambda(\beta, \alpha) \). Moreover, it then follows readily from the definition that \( T_1(\beta + \pi, \alpha + \pi) = T_1(\beta, \alpha) \).

To show (31), we will need the following:

\[
D := \{ (\beta, \alpha) \in \mathbb{T}_N^2 : \lambda(\beta, \alpha) \text{ exists} \} \text{ is open and } \lambda(\cdot, \cdot) \text{ is continuous on } D.
\]

To see this, fix some \((\beta, \alpha) \in D\) and \( \varepsilon > 0 \). Keep \( \beta \) initially fixed and increase \( \alpha \) a bit. Clearly, there will still be a corresponding \( \lambda \) that sends \( \beta \) to \( \alpha + \delta_1, \delta_1 > 0 \). Choose \( \delta_1 \) small enough so that \( \lambda(\beta, \alpha + \delta_1) \leq \lambda(\beta, \alpha) + \frac{\varepsilon}{2} \). Similarly, keeping \( \alpha + \delta_1 \) fixed and decreasing \( \beta \) a bit, we find \( \delta_2 > 0 \) so that \( \lambda(\beta - \delta_2, \alpha + \delta_1) \leq \lambda(\beta, \alpha) + \varepsilon \).

Similarly, we can choose \( \delta_3, \delta_4 > 0 \) with \( \lambda(\beta + \delta_4, \alpha - \delta_3) \geq \lambda(\beta, \alpha) - \varepsilon \). It then follows, again by the monotonicity properties, that the set \( \{(\beta + \delta, \alpha + \delta) : -\delta_2 \leq \delta \leq \delta_4, -\delta_3 \leq \delta \leq \delta_1 \} \) is contained in \( D \) and \( \lambda \) restricted to this set takes values in the interval \([\lambda(\beta, \alpha) - \varepsilon, \lambda(\beta, \alpha) + \varepsilon]\). The assertion (32) follows.

With the closed subset \( A := \lambda^{-1}(\text{supp } r) \) of \( D \), we can rewrite \( T_1(\beta, \alpha) \) as

\[
T_1(\beta, \alpha) = \begin{cases} 
\frac{R_1(-1, \beta, \alpha)r(\lambda(\beta, \alpha))}{\int f(x)\bar{u}_+^2(x, \beta, \alpha)\,dx} & \text{if } (\beta, \alpha) \in D, \\
0 & \text{if } (\beta, \alpha) \in \mathbb{T}_N^2 \setminus A.
\end{cases}
\]

Notice that this is well-defined. Since \( \{ D, \mathbb{T}_N^2 \setminus A \} \) is an open cover of \( \mathbb{T}_N^2 \), it suffices to check continuity for each of these two open sets. Continuity on \( \mathbb{T}_N^2 \setminus A \) is obvious. Continuity on \( D \) follows by (32), the continuity of \( \lambda \) in \((\beta, \alpha)\), and (via (19) and (20)) the joint continuity of \( R_0(-1, \alpha, \lambda) \) and \( \int f u_+^2(\cdot, \alpha, \lambda) \) in \((\alpha, \lambda)\). The latter is a consequence of the a priori bound provided in Lemma 3.2. This concludes the proof of (31). \( \square \)

**Lemma 3.4.** We have

\[
\|T_1(g, E)\|_{2,2} \leq 1.
\]

**Proof.** Let

\[
K_1(\beta, \alpha) = \frac{R_2(-1, \beta, \alpha)r(\lambda(\beta, \alpha))}{\int f u_+^2(\cdot, \beta, \alpha)}
\]

and

\[
K_2(\beta, \alpha) = \frac{R_2^2(-1, \beta, \alpha)r(\lambda(\beta, \alpha))}{\int f u_+^2(\cdot, \beta, \alpha)}
\]

if \( \lambda(\beta, \alpha) \) exists and \( K_1(\beta, \alpha) = K_2(\beta, \alpha) = 0 \) otherwise.
Thus, using (21) and the change of variables (26),

\[ \int_{T_N} K_1(\beta, \alpha) \, d\alpha = \int_{T_N} \frac{r(\lambda(\beta, \alpha))}{\int f u_1^2(\cdot, \beta, \alpha)} \, d\alpha \]

\[ = \int_{T_N} \frac{R^2_1(0, \beta, \alpha) \cdot r(\lambda(\beta, \alpha))}{\int f u_1^2(\cdot, \beta, \alpha)} \, d\alpha \]

\[ = \int_{T_N} \frac{R^2_{-1}(0, \beta, \lambda) \cdot r(\lambda)}{\int f u_1^2(\cdot, \beta, \lambda)} \left| \frac{d\alpha}{d\lambda} \right| \, d\lambda \]

\[ = \int_{T_N} r(\lambda) \, d\lambda \]

\[ = 1. \]

Similarly, using (25),

\[ \int_{T_N} K_2(\beta, \alpha) \, d\beta = \int_{T_N} \frac{R^2_2(-1, \beta, \alpha) \cdot r(\lambda(\beta, \alpha))}{\int f u_1^2(\cdot, \beta, \alpha)} \, d\beta \]

\[ = \int_{T_N} \frac{R^2_0(-1, \alpha, \lambda) \cdot r(\lambda)}{\int f u_0^2(\cdot, \alpha, \lambda)} \left| \frac{d\beta}{d\lambda} \right| \, d\lambda \]

\[ = \int_{T_N} r(\lambda) \, d\lambda \]

\[ = 1. \]

We have \( T_1(\beta, \alpha) = \sqrt{K_1(\beta, \alpha)} \sqrt{K_2(\beta, \alpha)} \), so that the Schur Test, e.g. [14], immediately gives \( \|T_1\|_{2,2} \leq 1. \)

\[ \square \]

4. The Operator \( T_1 \) Has \( \| \cdot \|_{2,2} \)-Norm Less Than One

The purpose of this section is to establish to following strengthening of Lemma 3.4, which is the key technical result of our work.

**Proposition 4.1.** We have \( \|T_1(g, E)\|_{2,2} < 1. \)

We will suppress the \((g, E)\)-dependence in our notation for the remainder of this section.

We have already seen that \( T_1 \) is a bounded operator on \( L^2(T_N) \). Moreover, (30) suggests that we decompose \( T_1 \) as a direct sum of integral operators on \( L^2(0, \pi) \). Let us implement this:

**Lemma 4.2.** (a) Suppose \( h \) is continuous on \((\pi n, \pi(n+1))\) for \( n = 0, 1, \ldots, 2N-1, \) \( j \in \{0, 1, \ldots, 2N-1\} \), and \( x \in (0, \pi) \), and let

\[ (Uh)_j(x) = \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} e^{-\frac{i j \pi n}{2N}} h(x + \pi n). \]

Then \( U \) extends to a unitary operator

\[ U : L^2(T_N) \to \bigoplus_{j=0}^{2N-1} L^2(0, \pi). \]

(b) We have

\[ UT_1 U^{-1} = \bigoplus_{j=0}^{2N-1} L_j, \]
where $L_j$ is the integral operator in $L^2(0, \pi)$ with kernel

$$L_j(\beta, \alpha) = \sum_{n=0}^{2N-1} T_1(\beta, \alpha + n\pi) e^{\frac{i\pi j n}{N}}.$$ 

(c) We have $\|T_1\| = \|L_0\|$, with both norms being the operator norm in the respective $L^2$ space.

Proof. (a) Suppose $h$ is continuous on $(\pi n, \pi(n+1))$ for $n = 0, 1, \ldots, 2N - 1$. Then,

$$\|Uh\|^2 = \sum_{j=0}^{2N-1} \int_0^\pi \left| \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} e^{-\frac{i\pi j n}{N}} h(x + \pi n) \right|^2 dx$$

$$= \int_0^\pi \sum_{j=0}^{2N-1} \left| \sum_{n=0}^{2N-1} \frac{e^{-\frac{i\pi j n}{N}}}{\sqrt{2N}} h(x + \pi n) \right|^2 dx$$

$$= \int_0^{2N\pi} |h(x + \pi j)|^2 dx$$

$$= \int_0^{2N\pi} |h(x)|^2 dx$$

$$= \|h\|^2.$$ 

Here, all steps save the third follow by simple rewriting and the third step follows from the Parseval identity for $\mathbb{C}^{2N}$.

For a continuous $g = (g_j) \in \bigoplus_{j=0}^{2N-1} L^2(0, \pi)$, we define

$$h(x + \pi n) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} e^{\frac{i\pi j n}{N}} g_j(x),$$

where $x \in (0, \pi)$ and $n \in \{0, 1, \ldots, 2N - 1\}$ and note that $g = Uh$. Since $h$ is continuous on $(\pi n, \pi(n+1))$ for $n = 0, 1, \ldots, 2N - 1$, we may conclude that $U$ is a densely defined isometry with dense range, and hence $U$ extends to a unitary operator from $L^2(T_N)$ onto $\bigoplus_{j=0}^{2N-1} L^2(0, \pi)$.

(b) We have

$$\left( \bigoplus_{k=0}^{2N-1} L_k Uh \right)_j(\beta) = \int_0^\pi L_j(\beta, \alpha)(Uh)_j(\alpha) \, d\alpha$$

$$= \int_0^{2N\pi} \sum_{n=0}^{2N-1} T_1(\beta, \alpha + n\pi) e^{\frac{i\pi j n}{N}} \frac{1}{\sqrt{2N}} \sum_{m=0}^{2N-1} e^{-\frac{i\pi j m}{N}} h(\alpha + \pi m) \, d\alpha$$

$$= \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} \sum_{m=0}^{2N-1} e^{-\frac{i\pi(j-n)}{N}} \int_0^{2\pi} T_1(\beta, \alpha + n\pi) h(\alpha + \pi m) \, d\alpha$$
A CONTINUUM VERSION OF THE KUNZ-SOUILLARD APPROACH

and

\[(UT_1h)_j(\beta) = (U \int_0^{2N\pi} T_1(\cdot, \alpha)h(\alpha) d\alpha)_j(\beta)\]
\[= \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} e^{-i\pi/n} \int_0^{2N\pi} T_1(\beta + \pi n, \alpha)h(\alpha) d\alpha\]
\[= \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} \sum_{m=0}^{2N-1} \int_0^{\pi} T_1(\beta + \pi n, \alpha + \pi m)h(\alpha + \pi m) d\alpha\]
\[= \frac{1}{\sqrt{2N}} \sum_{\tilde{n}=0}^{2N-1} \sum_{m=0}^{2N-1} e^{-i\pi/2} \int_0^{\pi} T_1(\beta, \alpha + \pi n)h(\alpha + \pi m) d\alpha,\]

from which the asserted identity follows.

(c) From the decomposition established above, we get \(\|T_1\| = \max_{0 \leq j \leq 2N-1} \|L_j\|\). As \(T_1(\beta, \alpha) \geq 0\), we have \(|L_j(\beta, \alpha)| \leq L_0(\beta, \alpha)\) and therefore \(\|L_j\| \leq \|L_0\|\) for every \(j\). This yields the claim. □

Proof of Proposition 4.1. By Lemma 4.2.(c), it suffices to show \(\|T_1\| = \max_{0 \leq j \leq 2N-1} \|L_j\|\). By Lemma 3.4 and Lemma 4.2.(c), \(\|L_0\| \leq 1\). Suppose that \(\|L_0\| = 1\). By compactness, there exists \(f \neq 0\) such that \(\|L_0f\| = \|f\|\) (choose \(f\) as an eigenvector to the eigenvalue 1 = \(\|L_0\| = \|L_0\|\) of \(L_0\)) and use \(\|L_0f\| = \|L_0f\|\). The operator \(L_0\) has a positive kernel and we may therefore assume that \(f \geq 0\).

Consider the \(\pi\)-periodic extension \(\tilde{f}\) of \(f\) to \(\mathbb{T}_N\). Then,

\[(L_0f)(\beta) = \int_0^{\pi} \sum_{n=0}^{2N-1} T_1(\beta, \alpha + n\pi)f(\alpha) d\alpha\]
\[= \sum_{n=0}^{2N-1} \int_0^{\pi} T_1(\beta, \alpha + n\pi)\tilde{f}(\alpha + n\pi) d\alpha\]
\[= \int_{\mathbb{T}_N} T_1(\beta, \alpha)\tilde{f}(\alpha) d\alpha.\]
By (33), (34) along with \( K_1(\beta + \pi, \alpha + \pi) = K_1(\beta, \alpha) \) and \( K_2(\beta + \pi, \alpha + \pi) = K_2(\beta, \alpha) \), we find

\[
\| L_0 f \|^2 = \int_0^\pi |(L_0 f)(\beta)|^2 \, d\beta = \int_0^\pi \left| \int_{\mathbb{T}_N} T_1(\beta, \alpha) \tilde{f}(\alpha) \, d\alpha \right|^2 \, d\beta = \int_0^\pi \left| \int_{\mathbb{T}_N} \sqrt{K_1(\beta, \alpha)} \sqrt{K_2(\beta, \alpha)} \tilde{f}(\alpha) \, d\alpha \right|^2 \, d\beta \leq \int_0^\pi \left( \int_{\mathbb{T}_N} K_1(\beta, \alpha) \, d\alpha \int_{\mathbb{T}_N} K_2(\beta, \alpha) |\tilde{f}(\alpha)|^2 \, d\alpha \right) \, d\beta = \int_0^\pi \int_{\mathbb{T}_N} K_2(\beta, \alpha) |\tilde{f}(\alpha)|^2 \, d\alpha \, d\beta = \int_0^\pi 2^{N-1} \sum_{n=0}^\pi K_2(\beta, \alpha - \pi n) |\tilde{f}(\alpha - \pi n)|^2 \, d\alpha \, d\beta = \int_0^\pi 2^{N-1} \sum_{n=0}^\pi K_2(\beta + \pi n, \alpha) |\tilde{f}(\alpha)|^2 \, d\alpha \, d\beta = \int_0^\pi \left( \sum_{n=0}^{2^{N-1}} \int_0^\pi K_2(\beta + \pi n, \alpha) \, d\beta \right) |\tilde{f}(\alpha)|^2 \, d\alpha = \int_0^\pi \left( \int_{\mathbb{T}_N} K_2(\beta, \alpha) \, d\beta \right) |\tilde{f}(\alpha)|^2 \, d\alpha = \int_0^\pi |\tilde{f}(\alpha)|^2 \, d\alpha = \int_0^\pi |f(\alpha)|^2 \, d\alpha = \| f \|^2 = \| L_0 f \|^2.
\]

Thus, we have equality in the application of the Cauchy-Schwarz inequality, which implies that for almost every \( \beta \in (0, \pi) \), the functions \( \sqrt{K_1(\beta, \cdot)} \) and \( \sqrt{K_2(\beta, \cdot)}\tilde{f}(\cdot) \) are linearly dependent in \( L^2(\mathbb{T}_N) \). Since they are both non-negative and non-zero, we see that for \( \beta \in (0, \pi) \setminus N \), \( \text{Leb}(N) = 0 \), there is \( C_\beta > 0 \) such that

\[
C_\beta K_1(\beta, \cdot) = K_2(\beta, \cdot)\tilde{f}(\cdot)^2.
\]

Fix \( \beta \in [0, \pi) \setminus N \) and let

\[
M_\beta := \{ \alpha : \lambda(\beta, \alpha) \in \text{supp } r \}.
\]

Then, for almost every \( \alpha \in M_\beta \), we have \( C_\beta = R_+^2(-1, \beta, \alpha)\tilde{f}(\alpha)^2 \), or

\[
(35) \quad \tilde{f}(\alpha)^2 = C_\beta R_+^2(0, \beta, \lambda(\beta, \alpha)).
\]

Let \([A, B]\) be a non-trivial interval that is contained in the support of \( r \) (recall that \( r \) is continuous). If \( c_\beta \) and \( d_\beta \) are the unique phases determined by \( \lambda(\beta, c_\beta) = B \) and \( \lambda(\beta, d_\beta) = A \) (i.e., \( c_\beta = \varphi_1(0, \beta, B) \) and \( d_\beta = \varphi_1(0, \beta, A) \)), then \( c_\beta < d_\beta \).
and \([c_\beta, d_\beta] \subset M_\beta\). Moreover, \(c_\beta\) and \(d_\beta\) are strictly increasing and continuous in \(\beta\) (cf. Lemma \[13\]) and we have \([c_{\beta + \pi}, d_{\beta + \pi}] = [c_\beta + \pi, d_\beta + \pi]\).

It follows that

\[
I := \bigcup_{\beta \in [0, \pi) \setminus \mathbb{N}} (c_\beta, d_\beta)
\]

is an open interval of length greater than \(\pi\). For fixed \(\beta \in (0, \pi) \setminus \mathbb{N}\), \(\tilde{f}^2\) is real-analytic on \((c_\beta, d_\beta)\) by \([38]\). This uses that (i) \(\alpha(\lambda)\) is analytic in \(\lambda\) with \(\alpha'(\lambda) < 0\) (and thus its inverse function \(\lambda(\beta, \alpha)\) is analytic in \(\alpha\)) and (ii) \(R^2_{-1}(0, \beta, \lambda)\) is analytic in \(\lambda\). Property (ii) follows from part (a) of Lemma \[13\]. This also implies the analyticity of the right hand side of \([21]\) in \(\lambda\) which in turn gives (i).

We conclude that \(\tilde{f}^2\) is analytic on \(I\) and, due to \(\pi\)-periodicity, on all of \(\mathbb{R}\).

Now, we again fix a \(\beta \in [0, \pi) \setminus \mathbb{N}\) and conclude by analytic continuation that \([35]\) holds for all \(\alpha\) for which \(\lambda(\beta, \alpha)\) exists. Therefore, \(R^2_{-1}(0, \beta, \lambda(\beta, \alpha))\) is bounded in \(\alpha\) (as this holds for the \(\pi\)-periodic analytic function on the LHS of \([35]\)). But \(\lambda(\beta, \alpha)\) takes on arbitrary real values as \(\alpha\) varies and hence

\[
\sup_{\lambda \in \mathbb{R}} R_{-1}(0, \beta, \lambda) < \infty,
\]

which is impossible by Proposition \[A.1\]. This contradiction completes the proof of \(||T_1|| = ||L_0|| < 1\). \(\square\)

5. The Dependence of \(T_1\) on the Background

In this section we study the map \((g, E) \mapsto T_1(g, E)\). Note that the energy \(E\) can be absorbed in \(g\), that is, \(T_1(g, E) = T_1(g - E, 0)\). For this reason, we will consider without loss of generality the case \(E = 0\). Consequently, in this section, \(E\) is dropped from the notation and assumed to be zero. For example, we write \(T_1(g)\) for \(T_1(g, 0)\) and \(\lambda(\beta, \alpha, g)\) for \(\lambda(\beta, \alpha, g, 0)\).

Write

\[
D(g) = \{(\beta, \alpha) \in \mathbb{R}^2 : \lambda(\beta, \alpha, g) \text{ exists}\}
\]

and

\[
A(g) = \lambda(\cdot, \cdot, g)^{-1}([-M, M]) \subset D(g),
\]

where, as in Section \[2\], \(\text{supp } r \subset [-M, M]\).

**Lemma 5.1.** Suppose \(g_n \to g\) in \(L^\infty(-1, 0)\). Then, we have

\[
D(g) \subseteq \liminf_{n \to \infty} D(g_n)
\]

and

\[
\mathbb{R}^2 \setminus A(g) \subseteq \liminf_{n \to \infty} \mathbb{R}^2 \setminus A(g_n).
\]

**Proof.** Let \((\beta, \alpha) \in D(g)\) so that \(\tilde{\lambda} := \lambda(\beta, \alpha, g)\) exists. Fix some \(\varepsilon > 0\). Then, by monotonicity,

\[
\varphi_0(-1, \alpha, \tilde{\lambda} - \varepsilon, g) < \beta < \varphi_0(-1, \alpha, \tilde{\lambda} + \varepsilon, g).
\]

It follows from \(L^1_{\text{loc}}\)-continuity of solutions in \(g\), specifically the bound provided in Lemma \[B.2\] that for \(n\) sufficiently large,

\[
\varphi_0(-1, \alpha, \tilde{\lambda} - \varepsilon, g_n) < \beta < \varphi_0(-1, \alpha, \tilde{\lambda} + \varepsilon, g_n).
\]

Thus, for such values of \(n\), there is \(\tilde{\lambda}_n \in (\tilde{\lambda} - \varepsilon, \tilde{\lambda} + \varepsilon)\) with \(\varphi_0(-1, \alpha, \tilde{\lambda}_n, g_n) = \beta\).

In particular, \(\lambda(\beta, \alpha, g_n)\) exists (and is given by \(\tilde{\lambda}_n\)). This proves \([37]\). For later use, we note that the proof also shows \(\lambda(\beta, \alpha, g_n) \to \lambda(\beta, \alpha, g)\).
Now consider \((\beta, \alpha) \in \mathbb{R}^2 \setminus A(g)\). That is, either \(\lambda(\beta, \alpha, g)\) does not exist or it does exist but lies outside the interval \([-M, M]\). Suppose there is a sequence \(n_k \to \infty\) such that \(\lambda(\beta, \alpha, g_{n_k})\) exists and belongs to \([a, b]\) for every \(k\). This means that \(\varphi_{-1}(0, \beta, \lambda(\beta, \alpha), g_{n_k}) = \alpha\). By monotonicity, this gives

\[
\varphi_{-1}(0, \beta, b, g_{n_k}) \leq \alpha \quad \text{and} \quad \varphi_{-1}(0, \beta, a, g_{n_k}) \geq \alpha.
\]

Taking \(k \to \infty\), we find

\[
\varphi_{-1}(0, \beta, b, g) \leq \alpha \quad \text{and} \quad \varphi_{-1}(0, \beta, a, g) \geq \alpha.
\]

This means, however, that there exists \(\lambda \in [-M, M]\) such that \(\varphi_{-1}(0, \beta, \lambda, g) = \alpha\), which is a contradiction. This proves (39).

**Lemma 5.2.** Suppose \(g_n \to g\) in \(L^\infty(-1, 0)\). Then,

\[
\lim_{n \to \infty} T_1(\beta, \alpha, g_n) = T_1(\beta, \alpha, g).
\]

for every \((\beta, \alpha) \in \mathbb{R}^2\).

**Proof.** We first consider the case \((\beta, \alpha) \in \mathbb{R}^2 \setminus A(g)\). As seen above, this implies \((\beta, \alpha) \in \mathbb{R}^2 \setminus A(g_n)\) for \(n \geq N_1\). Consequently, \(T_1(\beta, \alpha, g_n) = T_1(\beta, \alpha, g) = 0\) for \(n \geq N_1\), which trivially implies convergence.

If \((\beta, \alpha) \in D(g)\), we know that \((\beta, \alpha) \in D(g_n)\) for \(n \geq N_2\). Then, using continuous dependence of solutions on the potential again, it is readily seen that

\[
T_1(\beta, \alpha, g_n) = \frac{R_0(-1, \alpha, \lambda(\beta, \alpha, g_n), g_n) r(\lambda(\beta, \alpha, g_n))}{\int f(x) u_0^2(x, \alpha, \lambda(\beta, \alpha, g_n), g_n) \, dx} \to T_1(\beta, \alpha, g).
\]

Here we also used that \(\lambda(\beta, \alpha, g_n) \to \lambda(\beta, \alpha, g)\), which was proven above. \(\square\)

**Proposition 5.3.** The real-valued map \(g \mapsto \|T_1(g)\|_{2,2}\) is continuous on the ball of radius \(\|W_0\|_\infty\) in \(L^\infty(-1, 0)\).

**Proof.** We have to show that for \(g, g_n \in L^\infty(-1, 0)\), \(\|g\|_\infty, \|g_n\|_\infty \leq \|W_0\|_\infty, n \geq 1\), with \(\|g_n - g\|_\infty \to 0\), we have

\[
\lim_{n \to \infty} \|T_1(g_n)\|_{2,2} = \|T_1(g)\|_{2,2}.
\]

By Lemma 4.2(c), it suffices to show that

\[
(39) \quad \lim_{n \to \infty} \|L_0(g_n)\|_{2,2} = \|L_0(g)\|_{2,2}.
\]

Recall that

\[
L_0(\beta, \alpha, \cdot) = \sum_{n=0}^{2N-1} T_1(\beta, \alpha + \pi n, \cdot);
\]

compare Lemma 4.2(b). Using (39) and the a priori bounds Lemma 3.1 and Lemma 3.3 this implies that \(L_0(\beta, \alpha, \cdot)\) is uniformly bounded, uniformly for \(\{g\} \cup \{g_n\}_{n \geq 1}\). By Lemma 5.2, the functions \(L_0(\cdot, \cdot, g_n)\) converge pointwise to \(L_0(\cdot, \cdot, g)\). Consequently,

\[
\|L_0(g_n) - L_0(g)\|_{2,2}^2 \leq \|L_0(g_n) - L_0(g)\|_{HS}^2
\]

\[
= \int_0^\pi \int_0^\pi |L_0(\beta, \alpha, g_n) - L_0(\beta, \alpha, g)|^2 \, d\alpha \, d\beta
\]

\[
\to 0
\]

by dominated convergence. This proves (39) and hence the theorem. \(\square\)
APPENDIX A. LARGE COUPLING LIMIT OF THE PRÜFER AMPLITUDE

Here we establish a technical fact which was used in the proof of Proposition 4.1.

**Proposition A.1.** It holds that

\[ \lim_{\lambda \to \infty} R_{-1}(0, \beta, \lambda) = \infty. \]

**Proof.** For \([a, b]\) from (31) let \(\theta \in [0, \pi)\) be such that \(\theta = \varphi_{-1}(a, \beta, \lambda, g) \mod \pi\) and denote by \(\varphi(x, \lambda) := \varphi_a(x, \theta, \lambda, g)\) and \(R(x, \lambda) := R_a(x, \theta, \lambda, g)\) the Prüfer phase and amplitude for the solution of \(-u'' + gu + \lambda fu = 0\) with \(u'(a) = \cos \theta, u(a) = \sin \theta\).

It follows from (4) that

\[ \lim_{\lambda \to \infty} (\ln \lambda) = \infty. \]

This follows as \(\text{supp } f \cap [(-1, a) \cup (b, 0)] = \emptyset\) and therefore, by Lemma B.1, \(R_{-1}(a, \alpha, \lambda, g) \approx 1\) and \(R_{-1}(0, \alpha, \lambda, g) \approx R_{-1}(b, \alpha, \lambda, g) \approx R(b, \lambda)\).

Note that \(\varphi\) and \(R\) satisfy the Prüfer differential equations

\[ \varphi' = 1 - (1 + g + \lambda f) \sin^2 \varphi \]

and

\[ (\ln R)' = \frac{1}{2}(1 + g + \lambda f) \sin 2\varphi. \]

We will first show that there exists \(\lambda_0 \in \mathbb{R}\) and \(\eta \in (0, \pi)\) such that

\[ \varphi(x, \lambda) < \eta \quad \text{for all } \lambda \geq \lambda_0 \text{ and all } x \in [a, b]. \]

By (42) and \(\theta \geq 0\) we know that \(\varphi(b, \lambda) > 0\) for all \(\lambda\). Sturm comparison Lemma B.8 or, more directly, Lemma B.5 shows that in proving (44) it suffices to assume that \(\theta \in [\pi/2, \pi)\). Choose \(\eta \in (\theta, \pi)\) with \(\sin^2 \eta = \frac{1}{2} \sin^2 \theta\) and let

\[ M_\lambda := \left\{ x \in [a, b] : 1 + g(x) + \lambda f(x) \geq \frac{1}{\sin^2 \eta} \right\}. \]

It follows from (4) that \(|M_\lambda| \to b - a\) as \(\lambda \to \infty\). To show (44) for the given choice of \(\eta\), we assume, by way of contradiction, that there are arbitrarily large \(\lambda > 0\) for which the set \(\{x \in [a, b] : \varphi(x, \lambda) \geq \eta\}\) is non-empty and thus has a minimum \(b_\lambda\) with \(\varphi(b_\lambda, \lambda) = \eta\). Also, let \(a_\lambda := \max\{x \in [a, b] : \varphi(x, \lambda) = \theta\}\). Thus \(\varphi(x, \lambda) \in [\theta, \eta]\) for all \(x \in [a_\lambda, b_\lambda]\).

By (42) we have \(\varphi'(x) \leq 1 + \|g\|_\infty\) for all \(x \in [a_\lambda, b_\lambda]\) and \(\varphi'(x) \leq 1 - \sin^2 \phi(x) / \sin^2 \eta \leq 0\) for \(x \in M_\lambda \cap [a_\lambda, b_\lambda]\). Thus

\[ \eta - \theta = \varphi(b_\lambda, \lambda) - \varphi(a_\lambda, \lambda) = \int_{a_\lambda}^{b_\lambda} \varphi'(x, \lambda) \, dx \]

\[ \leq \int_{[a_\lambda, b_\lambda] \setminus M_\lambda} (1 + \|g\|_\infty) \, dx \leq (1 + \|g\|_\infty)(b - a - |M_\lambda|). \]

Choosing a sufficiently large \(\lambda > 0\), we can make the right-hand side arbitrarily small and hence we obtain the contradiction \(\eta - \theta \leq 0\), proving (44).

Next, consider the set

\[ N_\lambda := \left\{ x \in [a, b] : \varphi(x, \lambda) \geq \frac{\pi}{4} \right\}. \]
As \( f \geq 0 \), we see by Lemma B.6(b) that \( N_\lambda \) is decreasing for increasing \( \lambda \). We will show that

\[
\lim_{\lambda \to \infty} |N_\lambda| = 0
\]

and

\[
\sup_{\lambda > 0} \lambda \int_{N_\lambda} f(x) \, dx < \infty.
\]

By (42) we have for all \( \lambda \) that

\[
\pi > |\varphi(b, \lambda) - \varphi(a, \lambda)| = \left| b - a - \int_a^b (1 + g(x) + \lambda f(x)) \sin^2 \varphi(x, \lambda) \, dx \right|.
\]

For \( \lambda \geq \lambda_0 \) from (44) we can bound

\[
\int_a^b (1 + g(x) + \lambda f(x)) \sin^2 \varphi(x, \lambda) \, dx \geq |b - a| (1 - \|g\|_\infty) + \lambda \min\{\sin^2 \eta, 1/2\} \int_{N_\lambda} f(x) \, dx.
\]

By (47) it follows that \( \lambda \int_{N_\lambda} f(x) \, dx \) is bounded in \( \lambda \), proving (46). This implies (45) as \( |N_\lambda| \to |N| \), where \( N = \bigcap_\lambda N_\lambda \) and \( f > 0 \) almost everywhere on \( N \).

We will now use (43) to prove (41). We estimate

\[
\int_a^b (1 + g(x) + \lambda f(x)) \sin 2\varphi(x, \lambda) \, dx \geq - (1 + \|g\|_\infty)(b - a)
\]

\[
+ \lambda \int_a^b f(x) \sin 2\varphi(x, \lambda) \, dx.
\]

By (46),

\[
\lambda \int_a^b f(x) \sin 2\varphi(x, \lambda) \, dx \geq - \lambda \int_{N_\lambda} f(x) \, dx \geq - C
\]

uniformly in \( \lambda \). Moreover, Sturm comparison (Lemma B.8) with the solution \( u(x) = \exp(\sqrt{K}x) \) of \( -u'' + Ku = 0 \), where \( K = \lambda \|f\|_\infty + \|g\|_\infty \), gives for \( \theta \neq 0 \) and \( \lambda \) sufficiently large (such that \( \theta > 1/\sqrt{K} \)) that \( \varphi(x, \lambda) \geq 1/\sqrt{K} \). This shows, using the definition of \( N_\lambda \) and (45),

\[
\lambda \int_{[a,b]\setminus N_\lambda} f(x) \sin 2\varphi(x, \lambda) \, dx \geq \sqrt{\lambda} \int_{[a,b]\setminus N_\lambda} f(x) \, dx \geq \sqrt{\lambda}.
\]

Finally, (48), (49) and (50) yield

\[
\ln R(b, \lambda) = \int_a^b (1 + g(x) + \lambda f(x)) \sin 2\varphi(x, \lambda) \, dx \to \infty \quad \text{as} \quad \lambda \to \infty.
\]

Thus we have shown (41) for \( \theta \neq 0 \). The case \( \theta = 0 \) is slightly different. Comparing with the solution of \( -u'' + Ku = 0 \), \( u(a) = 0 \), in this case gives, for suitable \( C_1 > 0 \) and \( C_2 > 0 \),

\[
\varphi(x, \lambda) \geq \frac{C_1}{\lambda} \quad \text{if} \quad |x - a| \geq \frac{C_2}{\sqrt{\lambda}}.
\]

This suffices to get the bound (50) and thus lets the above argument go through. 

\[\square\]
Appendix B. Prüfer Variables and A Priori Bounds

This appendix contains a brief summary of standard facts on Prüfer variables and a priori bounds on solutions which have been used throughout the paper. Proofs can for example be found in the appendix of [6] (for Lemma B.1, Lemma B.3, Lemma B.4, Lemma B.6(b) and Corollary B.7) and the appendix of [13] (Lemma B.5). Lemma B.6(a) and Lemma B.8 are special cases of Theorems 2.1 and 13.1 in [15]. Lemma B.2 is proven as Lemma A.2 in [3] for the special case that $u_1$ and $u_2$ satisfy the same initial condition at $y$. The proof given there extends easily to give the result we need here.

Lemma B.1. For every $q \in L^1_{\text{loc}}(\mathbb{R})$, every solution $u$ of $-u'' + qu = 0$, and all $x,y \in \mathbb{R}$ one has

$$
(|u(y)|^2 + |u'(y)|^2) \exp \left( -\int_{\min(x,y)}^{\max(x,y)} |1 + q(t)| \, dt \right) 
\leq |u(x)|^2 + |u'(x)|^2 \leq (|u(y)|^2 + |u'(y)|^2) \exp \left( \int_{\min(x,y)}^{\max(x,y)} |1 + q(t)| \, dt \right).
$$

Lemma B.2. For $i = 1, 2$, let $q_i \in L^1_{\text{loc}}(\mathbb{R})$ and let $u_i$ be solutions of $-u_i'' + q_i u_i = 0$. Then for any $x \in \mathbb{R}$,

$$(|u_1(x) - u_2(x)|^2 + |u_1'(x) - u_2'(x)|^2)^{1/2} \leq \left( |u_1(y) - u_2(y)|^2 + |u_1'(y) - u_2'(y)|^2 \right)^{1/2} \exp \left\{ \int_{\min(x,y)}^{\max(x,y)} (|q_1(t)| + |q_2(t)| + 2) \, dt \right\} 
$$

$$
+ \left( |u_1(y)|^2 + |u_1'(y)|^2 \right) \exp \left\{ \int_{\min(x,y)}^{\max(x,y)} (|q_1(t)| + |q_2(t)| + 2) \, dt \right\} 
\times \int_{\min(x,y)}^{\max(x,y)} |q_1(t) - q_2(t)| \, dt.
$$

Lemma B.3. For any positive real numbers $\ell$ and $M$ there exists $C > 0$ such that

$$\int_c^{c+\ell} |u(t)|^2 \, dt \geq C \left( |u(c)|^2 + |u'(c)|^2 \right)$$

for every $c \in \mathbb{R}$, every $L^1_{\text{loc}}$ function $q$ with $\int_c^{c+\ell} |q(t)| \, dt \leq M$, and any solution $u$ of $-u'' + qu = 0$ on $[c, c + \ell]$.

Our remaining results relate to Prüfer variables. In general, for any real potential $q \in L^1_{\text{loc}}(\mathbb{R})$ and real parameters $c$ and $\theta$ let $u_c$ be the solution of

$$-u'' + qu = 0$$

with $u_c(c) = \sin \theta$, $u'_c(c) = \cos \theta$. By regarding this solution and its derivative in polar coordinates, we define the Prüfer amplitude $R_c(x)$ and the Prüfer phase $\phi_c(x)$ by writing

$$u_c(x) = R_c(x) \sin \phi_c(x) \quad \text{and} \quad u'_c(x) = R_c(x) \cos \phi_c(x).$$

For uniqueness of the Prüfer phase we declare $\phi_c(\cdot) = \theta$ and require continuity of $\phi_c$ in $x$. In what follows the initial phase $\theta$ will be fixed and we thus leave the dependence of $u_c, R_c$ and $\phi_c$ on $\theta$ implicit in our notation.
In the new variables $R$ and $\phi$ the second order equation $-u'' + qu = 0$ becomes a system of two first order equations, where the equation for $\phi$ is not coupled with $R$:

**Lemma B.4.** For fixed $c$ and $\theta$, one has that

$$ (\ln R^2_c(x))' = \left(1 + q(x)\right) \sin (2 \phi_c(x)), $$

and

$$ \phi'_c(x) = 1 - \left(1 + q(x)\right) \sin^2(\phi_c(x)). $$

When considering $\phi_c(x)$ at fixed $x$ as a function of the initial phase $\theta$ one can show

**Lemma B.5.** For fixed $c$ and $x$, one has

$$ \frac{\partial}{\partial \theta} \phi_c(x, \theta) = \frac{1}{R^2_c(x, \theta)}. $$

Next we provide some results about the dependence of solutions on a coupling constant at a potential.

**Lemma B.6.** Let $W$ and $V$ be real valued functions in $L^1_{loc}(\mathbb{R})$. For real parameters $c$, $\theta$ and $\lambda$, let $u_c(\cdot, \lambda)$ be the solution of

$$ -u'' + Wu + \lambda Vu = 0, $$

normalized so that $u_c(c, \lambda) = \sin \theta$ and $u'_c(c, \lambda) = \cos \theta$. Denote the Prüfer variables of $u_c(\cdot, \lambda)$ and $u'_c(\cdot, \lambda)$ by $\phi_c(x, \lambda)$ and $R_c(x, \lambda)$.

(a) For fixed $x$, $u_c(x, \lambda)$ and $u'_c(x, \lambda)$ (and thus also $\phi_c(x, \lambda)$ and $R_c(x, \lambda)$) are entire functions of $\lambda$.

(b) One has that

$$ \frac{\partial}{\partial \lambda} \phi_c(x, \lambda) = -R^{-2}_c(x, \lambda) \int^x V(t) u^2_c(t, \lambda) \, dt. $$

As a special case one finds the energy derivative of the Prüfer phase.

**Corollary B.7.** Let $u$ be the solution of $-u'' + Wu = Eu$ normalized so that $u(c) = \sin \theta$ and $u'(c) = \cos \theta$, and let $\phi_c(x, E)$ and $R_c(x, E)$ be the corresponding Prüfer variables. Then

$$ \frac{\partial}{\partial E} \phi_c(x, E) = R^{-2}_c(x, E) \int^x u^2(t) \, dt. $$

Finally, we state a version of Sturm’s comparison theorem.

**Lemma B.8.** For $i = 1, 2$, let $u_i$ be the solution of $-u^{(i)'} + q_i u_i = 0$ with $u_i(c) = \sin \theta_i$ and $u'_i(c) = \cos \theta_i$. Define the Prüfer phases $\phi_i(x)$ to $(u_i, u'_i)$ as in [52].

Suppose that $q_1(t) \geq q_2(t)$ for all $t \in [c, x]$ and $\theta_2 \geq \theta_1$, then $\phi_2(x) \geq \phi_1(x)$.

**References**

[1] M. Aizenman, A. Elgart, S. Naboko, J. Schenker, G. Stolz, Moment analysis of localization in random Schrödinger operators, *Invent. Math.* 163 (2006), 343–413

[2] H. Cycon, R. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987

[3] D. Damanik, R. Sims, G. Stolz, Localization for one-dimensional, continuum, Bernoulli-Anderson models, *Duke Math. J.* 114 (2002), 59–100
A CONTINUUM VERSION OF THE KUNZ-SOUILARD APPROACH

[4] F. Delyon, H. Kunz, B. Souillard, One-dimensional wave equations in disordered media, *J. Phys. A* **16** (1983), 25–42

[5] F. Delyon, B. Simon, B. Souillard, From power pure point to continuous spectrum in disordered systems, *Ann. Inst. Henri Poincaré Phys. Théor.* **42** (1985), 283–309

[6] E. Hamza, R. Sims, G. Stolz, A note on fractional moments for the one-dimensional continuum Anderson model, preprint [arXiv:0907.4771](http://arxiv.org/abs/0907.4771), to appear in *J. Math. Anal. Appl.*

[7] I. Goldsheid, S. Molchanov, L. Pastur, A pure point spectrum of the stochastic and one dimensional Schrödinger equation, *Funct. Anal. Appl.* **11** (1977), 1–10

[8] A. Kiselev, Y. Last, B. Simon, Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, *Commun. Math. Phys.* **194** (1998), 1–45

[9] H. Kunz, B. Souillard, Sur le spectre des opérateurs aux différences finies aléatoires, *Commun. Math. Phys.* **78** (1980), 201–240

[10] S. Kotani and N. Ushirobayashi, One-dimensional Schrödinger Operators with Random Decaying Potentials, *Commun. Math. Phys.* **115** (1988), 247–266

[11] G. Royer, Études des opérateurs de Schrödinger à potentiel aléatoire en dimension 1, *Bull. Soc. Math. France* **110** (1982), 27–48

[12] B. Simon, Some Jacobi matrices with decaying potential and dense point spectrum, *Comm. Math. Phys.* **87** (1982), 253–258

[13] G. Stolz, Localization for random Schrödinger operators with Poisson potential, *Ann. Inst. Henri Poincaré* **63** (1995), 297–314

[14] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics **68**, Springer-Verlag, New York, 1980

[15] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics **1258**, Springer, Berlin, Heidelberg 1987

Department of Mathematics, Rice University, Houston, TX 77005, USA

*E-mail address: damanik@rice.edu*

Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294, USA

*E-mail address: stolz@math.uab.edu*