Morse-like squeezed coherent states and some of their properties

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Abstract

Using the f-deformed oscillator formalism, we introduce a class of squeezed coherent states for a Morse potential system (Morse-like squeezed coherent states) defined as approximate eigenstates of a linear combination of f-deformed ladder operators. For the states thus constructed, we analyze their statistical properties, their uncertainty relations, their Wigner function and their temporal evolution on phase space.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In recent times, the concept of coherent states has received a great deal of interest in many fields of physics. The coherent states of the quantum harmonic oscillator were first proposed by Schrödinger [1] in 1926, as minimum uncertainty product states, being thus the quantum states which most closely resemble classical ones from the dynamical viewpoint, i.e., they remain well localized around their corresponding classical trajectory and fulfil the minimum uncertainty product for the position and momentum canonical variables, when the two standard variances are equal. After Schrödinger’s proposal, almost four decades elapsed before the notion of coherent states was widely recognized as an important and convenient tool for the description of the quantum properties of the electromagnetic field. This was done by Glauber [2] in 1963, who first coined the term coherent states. More specifically, Glauber constructed what are now called field coherent states as eigenstates of the annihilation operator of the harmonic oscillator so as to study the correlation and coherence properties of the electromagnetic field, a theme of great relevance in quantum optics [1, 3].

Coherent states have been applied not only in quantum optics but in many fields of physics such as, for instance, the study of the forced harmonic oscillator [4], quantum oscillators [5], quantum nondemolition measurements [6], path integrals [7, 8], anharmonic crystals [9],
the theory of superfluidity [10, 11], nonequilibrium statistical mechanics [12], the quantized Hall effect [13], thermodynamics [14, 15], atomic physics [16, 17], nuclear physics [18, 19], elementary particle physics [20, 21] and quantum field theory [22, 23]. Excellent reviews (though not new) of the coherent states and their applications can be found in [24–26].

Because of their increasing theoretical and experimental applicability in different aspects of modern physics, numerous generalizations of coherent states for systems other than the harmonic oscillator have been undertaken, either from the dynamical point of view, from symmetry considerations, or from proposals based on deformed or nonlinear algebras [27–33]. A nonlinear algebraic theory that has attracted much attention concerns a quantum deformation of the harmonic oscillator, the f-deformed oscillator formalism proposed by Man’ko and co-workers [34, 35]. The corresponding f-coherent states display nonclassical features such as squeezing and anti-bunching effects, and they have also found interesting applications in the description of quantum dots [36] and the quantization of the center-of-mass motion of a laser-driven trapped ion [37–39].

Another class of quantum states that has attracted much attention, and which will be the central theme of this work, are the so-called squeezed coherent states. Besides being minimum uncertainty product states, they are characterized by the property that one of the variances of two observables is smaller than the vacuum (or coherent-state) noise level allowed by Heisenberg’s uncertainty principle, at the expense of enhanced noise in the other observable [40, 41]. Reduced fluctuations below the standard quantum limit in a given observable have found a manifold of applications in noise-free processes of measurement including spectroscopic techniques [42], optical interferometry [43] and gravitational wave detection [44], among others.

As stated by Glauber, the field coherent states may be obtained as eigenstates of the annihilation operator of the harmonic oscillator algebra \( \hat{a}(\alpha) = \alpha |\alpha\rangle \) with \( \alpha \) a complex number, by application of a displacement operator upon the vacuum state \( D(\alpha)|0\rangle = |\alpha\rangle \) with \( D(\alpha) = \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] \) and as those states for which \( (\Delta \hat{\phi})^2 (\Delta \hat{\chi})^2 = (1/2)^2 \) with the operators \( \hat{\phi} = -i/\sqrt{2}(\hat{a} - \hat{a}^\dagger) \) and \( \hat{\chi} = 1/\sqrt{2}(\hat{a} + \hat{a}^\dagger) \) and \( \Delta \hat{\phi} = \Delta \hat{\chi} = 1/2 \). They are given in the number space representation as

\[
|\alpha\rangle = e^{-1/2|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
\]  

These states are overcomplete, nonorthogonal and possess temporal stability.

The squeezed coherent states of the harmonic oscillator, also called two-photon coherent states, result from the application of a unitary operator to the coherent state \( |\alpha\rangle \) [41], i.e.,

\[
|\xi, \alpha\rangle \equiv \hat{S}(\xi)|\alpha\rangle,
\]  

where \( \hat{S}(\xi) \), known as the squeeze operator, is defined as

\[
\hat{S}(\xi) = \exp \left[ \frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^2) \right],
\]  

with \( \xi = re^{i\phi} \). Indeed, the squeeze operator acting upon the coherent state reduces the dispersion of any one of the \( \hat{x} \) and \( \hat{p} \) conjugate variables of an electromagnetic field mode (or of its mechanical analogue, a harmonic oscillator) in such a way that

\[
\langle \xi, \alpha |(\Delta \hat{x})^2|\xi, \alpha\rangle = e^{-2r^2} \langle \alpha |(\Delta \hat{x})^2|\alpha\rangle,
\]  

\[
\langle \xi, \alpha |(\Delta \hat{p})^2|\xi, \alpha\rangle = e^{2r^2} \langle \alpha |(\Delta \hat{p})^2|\alpha\rangle,
\]  

with \( r \) being the squeeze parameter, while the uncertainty product takes on its smallest value

\[
\langle \xi, \alpha |(\Delta \hat{x})^2|\xi, \alpha\rangle \langle \xi, \alpha |(\Delta \hat{p})^2|\xi, \alpha\rangle = \langle \alpha |(\Delta \hat{x})^2|\alpha\rangle \langle \alpha |(\Delta \hat{p})^2|\alpha\rangle = \frac{1}{2}.
\]
A more general way of defining minimum uncertainty squeezed states of any pair of physical quantities is to build them as solutions of the eigenvalue equation

\[(\hat{X}_1 + \imath \lambda \hat{X}_2) |\psi\rangle = z |\psi\rangle,\]

(7)

where \(\hat{X}_1\) and \(\hat{X}_2\) are noncommuting Hermitian operators in a given algebra, \(\lambda\) is a generalized squeeze parameter and \(z\) is a complex number. It is known that the \(|\psi\rangle\) states satisfying the above equation, also known as intelligent states [45], reduce the corresponding uncertainty relation to the equality

\[\langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle = \frac{1}{2} \langle [\hat{X}_1, \hat{X}_2] \rangle^2.\]

(8)

It can be found in [46] that the squeezed coherent states defined by (2) indeed satisfy the eigenvalue equation (7), on the condition that the operators \(\hat{X}_1\) and \(\hat{X}_2\) constitute Hermitian generators of the harmonic oscillator algebra. Thus, by suitably redefining \(z\) in terms of the parameters \(\lambda\) and \(\xi\) (see [46] for more details), such states may be viewed as eigenstates of a linear combination of the ladder operators \(\hat{a}\) and \(\hat{a}^\dagger\), i.e.,

\[(1 + \lambda) \hat{a} + (1 - \lambda) \hat{a}^\dagger |\psi\rangle = \sqrt{2z} |\psi\rangle,\]

(9)

when the Hermitian operators \(\hat{X}_1 = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger)\) and \(\hat{X}_2 = \frac{1}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger)\) are inserted into (7). Further, it was shown in [47] that knowing the underlying dynamical algebra of the system under study, one can extend the idea of employing linear combinations of ladder operators to cope with the problem of constructing squeezed states for general systems. Generalizations of squeezed coherent states for different algebras [48–51] and different potentials [52–56] have already been explored along this line. For instance, intelligent states associated with the SU(2) or an SU(1,1) algebra, each case leading to one group or the other of such states.

Generalized minimum uncertainty squeezed states related to nonlinear algebras, as well as some considerations about how to perform their experimental realization as quantum states of the center-of-mass motion of a laser-driven trapped ion were introduced in [58]. More specifically, two eigenvalue problems for two general noncommuting operators,

\[(1 + \lambda) g(\hat{n}) \hat{a} + (1 - \lambda) \hat{a}^\dagger g(\hat{n}) |\psi\rangle = z |\psi\rangle,\]

(10)

\[(1 + \lambda) f(\hat{a}) + (1 - \lambda) f(\hat{a}^\dagger) |\psi\rangle = z |\psi\rangle,\]

(11)

where both \(g\) and \(f\) are arbitrary functions, were solved by formulating its corresponding differential equation within the Fock–Bargmann representation. The first type of generalization (equation (10)) yields what are known as nonlinear squeezed states in the context of the quadratures of a deformed algebra, whereas the second one (equation (11)) is a nonlinear generalization of the quadrature squeezed states.

A very recent and successful attempt to construct squeezed coherent states of a quantum system with a finite discrete spectrum modeled by the Morse potential was undertaken by Angelova et al [59]. By considering two different types of ladder operators, the authors were able to construct two different types of squeezed coherent states called oscillator-like and energy-like states. These states turned out to be almost eigenstates of a linear combination of the ladder operators for the said potential in the sense that they are approximate solutions to the eigenvalue equation

\[(\hat{A}^- + \gamma \hat{A}^+) |\psi\rangle = z |\psi\rangle,\]

(12)

with \(\gamma\) being a squeezing parameter. Here, the ladder operators \(\hat{A}^-\) and \(\hat{A}^+\) satisfy either an SU(2) or an SU(1,1) algebra, each case leading to one group or the other of such states.

The Morse oscillator has been demonstrated to be a particularly useful anharmonic potential for the description of systems that deviate from the ideal harmonic oscillator conduct
under certain conditions. In molecular physics, for instance, this model is widely used to model the vibrations of a diatomic molecule. In nonlinear optics, it has been found that the discrete spectrum of certain nonlinear models such as the optical Kerr Hamiltonian (whose energy spectrum is unequally spaced) can be described as a Morse-like oscillator; in this sense, it is worth mentioning an interesting physical application introduced in [60] in which a Morse-type Hamiltonian with a finite spectrum is intended for modeling the properties of what is called a finite Kerr medium.

Motivated by the aforementioned studies, we shall consider a new class of squeezed coherent states associated with the Morse algebra (Morse-like squeezed coherent states) on the basis of the f-deformed oscillator formalism. We choose a deformation function \( f(\hat{n}) \) depending on the number operator such that the energy spectrum of our deformed Hamiltonian yields an energy spectrum similar to that of a Morse Hamiltonian. We shall adopt the ladder-operator technique outlined above for the construction of our squeezed states as eigenstates of a linear combination of lowering and raising f-deformed operators \( \hat{A} = \hat{a} f(\hat{n}), \hat{A}^\dagger = f(\hat{n})\hat{a}^\dagger \).

This paper is organized as follows. In section 2, the Hamiltonian model that describes a Morse-like f-oscillator, the corresponding f-deformed ladder operators and the squeezed coherent states cited above are introduced. We devote section 3 to a broad discussion of the numerical results focused on the statistical properties, the phase space behavior and the uncertainty relations of such states; in all calculations, we constrain our approach to the low-lying region of the discrete energy spectrum. Finally, in section 4, some conclusions are given.

2. The Morse-like f-deformed oscillator and its squeezed coherent states

According to Man’ko et al [34, 35], an f-oscillator is a non-harmonic quantum system characterized by a Hamiltonian of the form

\[
\hat{H}_D = \frac{\hbar \Omega}{2} (\hat{A}^\dagger \hat{A} + \hat{A}\hat{A}^\dagger),
\]  

with deformed creation and annihilation operators \( \hat{A}, \hat{A}^\dagger \) defined by

\[
\hat{A} = \hat{a} f(\hat{n}) = f(\hat{n} + 1)\hat{a}, \quad \hat{A}^\dagger = f(\hat{n})\hat{a}^\dagger = \hat{a}^\dagger f(\hat{n} + 1),
\]  

where \( \hat{n} = \hat{a}^\dagger \hat{a} \) is the usual number operator. The number operator function \( f(\hat{n}) \) is a deformation function to be specified later.

The deformed Hamiltonian (13) written in terms of the deformation function and the number operator is given by

\[
\hat{H}_D = \frac{\hbar \Omega}{2} ((\hat{n} + 1) f^2(\hat{n} + 1) + \hat{n} f^2(\hat{n})).
\]  

It is not, in general, a linear function of the number operator, which explains why such systems are usually referred to as nonlinear. Note that in the limit \( f(\hat{n}) \rightarrow 1 \) the harmonic oscillator algebra is completely recovered.

Additionally, it is important to note that the set of operators \( \{\hat{A}, \hat{A}^\dagger, \hat{n}\} \) obeys the commutation relations

\[
[\hat{A}, \hat{n}] = \hat{A}, \quad [\hat{A}^\dagger, \hat{n}] = -\hat{A}^\dagger, \quad [\hat{A}, \hat{A}^\dagger] = (\hat{n} + 1) f^2(\hat{n} + 1) - \hat{n} f^2(\hat{n}).
\]  

It is worth commenting that this framework has allowed us to describe the modified and the trigonometric Pöschl–Teller potentials, and to construct their associated coherent states as well [61].

If we choose the deformation function [62]:

\[
f^2(\hat{n}) = 1 - \chi_0 \hat{n},
\]  

4
where $\chi_d$ is an anharmonicity parameter, and substitute it in the deformed Hamiltonian, we obtain

$$\hat{H}_D = \hbar \Omega \left[ \hat{n} + \frac{1}{2} - \chi_d \left( \hat{n} + \frac{1}{2} \right)^2 - \frac{\chi_d}{4} \right],$$

(18)

whose spectrum is in essence identical to that of the Morse potential [63]

$$E_n = \hbar \omega_c \left( n + \frac{1}{2} \right) - \frac{\hbar \omega_c}{2N + 1} \left( n + \frac{1}{2} \right)^2,$$

(19)

provided that $\omega_c = \Omega$ and $\chi_d = 1/(2N + 1)$, with $N$ being the number of bound states corresponding to the integers $0 \leq n \leq N - 1$.

In addition, for this particular choice of deformation function, the commutator between $\hat{A}$ and $\hat{A}^\dagger$ is explicitly given by

$$[\hat{A}, \hat{A}^\dagger] = 1 - \chi_d (2\hat{n} + 1) = \frac{2N - 2\hat{n}}{2N + 1}.$$  

(20)

The action of the deformed operators on the number states $|n\rangle$ is

$$\hat{A}|n\rangle = \sqrt{n} f(n)|n-1\rangle = \sqrt{n(1-\chi_d)}n|n-1\rangle,$$

$$\hat{A}^\dagger|n\rangle = \sqrt{n+1} f(n+1)|n+1\rangle = \sqrt{(n+1)(1-\chi_d(n+1))}|n+1\rangle.$$  

(21)

The deformed operators change the number of quanta in $\pm 1$ and their corresponding matrix elements depend on the deformation function $f(n)$. Furthermore, from the commutation relations, it is clear that the set $\{\hat{A}, \hat{A}^\dagger, \hat{n}, 1\}$ is closed under the operation of commutation, and, as such, they may be considered as elements of the underlying dynamical algebra for the system being described.

Therefore, by virtue of the equivalence between (18) and (19), we shall employ the former as a suitable algebraic Hamiltonian to describe the discrete part of the spectrum of a Morse system and evaluate the temporal evolution of its associated nonlinear coherent states.

Having established the ladder operators associated with the Morse-like system, we define the squeezed coherent states as those states that fulfil the equation

$$(\hat{A} + \gamma \hat{A}^\dagger)|\alpha, \gamma\rangle = \alpha |\alpha, \gamma\rangle.$$  

(22)

Here, the parameter $\gamma$ is a non-negative real number that takes on the role of a squeezing parameter, while $\alpha$ is called the size of the coherent state (it suffices for our purposes to assume $\alpha$ to be also real).

In order to solve equation (22), let the state $|\alpha, \gamma\rangle$ be expressed as a weighted superposition of number eigenstates:

$$|\alpha, \gamma\rangle = \sum_{m=0}^{\infty} C_m |m\rangle.$$  

(23)

The substitution of (23) in (22) gives the following recursion relation for $C_m$:

$$C_{m+1} \sqrt{(m+1)(2N-m)} + \gamma C_{m-1} \sqrt{m(2N+1-m)} = \sqrt{2N+1} \alpha C_m.$$  

(24)

In the case under consideration, the recursion relation given above must be truncated at $m = N - 1$ for a Morse potential possessing $N$ bound states. Now, it is convenient to define

$$C_m = \frac{C_m}{\sqrt{m!(2N-m)!}}.$$  

On inserting this in (24) we obtain

$$C_{m+1} (m+1) + \gamma C_{m-1} (2N-m+1) = \sqrt{2N+1} \alpha C_m.$$  

(25)
Equation (25) turns out to be a special case of an exactly solvable three-term recursion relation of the type (see equation (10.66) of [46])

\[ m\alpha_1 C_m + (m + 1)\gamma_1 C_{m+1} - \beta_1 (M - m + 1) C_{m-1} = \lambda C_m, \]  

(26)

where \( \alpha_1, \beta_1 \) and \( \gamma_1 \) are fixed constants, \( \lambda \) is an eigenvalue, \( m = 0, 1, \ldots, M \) (\( M \) being a finite number describing the upper limit on the allowed values of \( m \), i.e. \( C_{M+j} = 0 \) for \( j > 1 \)) and \( C_{-1} = 0 \). Then, in accordance with [46], the solution of (26) assumes the general form

\[ C_m = (-1)^{M+m} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{M - n}{m - k} \right)^{n-k} x_{n-k}^{M-n-m+k}, \]  

(27)

where

\[ x_{\pm} = \frac{1}{2\beta_1} \left[ -\alpha_1 \pm \sqrt{\alpha_1^2 - 4\gamma_1 \beta_1} \right], \]  

(28)

\[ n = \frac{\lambda + M \beta_1 x_+}{\beta_1 (x_+ - x_-)}. \]  

(29)

So, taking \( \alpha_1 = 0, \gamma_1 = 1, \beta_1 = -\gamma, M = 2N \) and \( \lambda = \sqrt{2N + 1} \alpha \) in equation (26), it is effortless to obtain the corresponding analytical solution of (25) when referring to equation (27), which, under the normalization condition \( \langle \alpha, \gamma | \alpha, \gamma \rangle = 1 \), takes the form

\[ C_{mn} = N_n \sqrt{m! (2N - m)!} C_m \]

\[ = N_n (-1)^m \sqrt{m! (2N - m)!} \sum_{k=0}^{[n]} \binom{n}{k} \left( \frac{2N - n}{m - k} \right) (-1)^k, \]  

(30)

with \( N_n \) being a normalization constant given by

\[ N_n = \left\{ \frac{\gamma^m m! (2N - m)!}{\sum_{m=0}^{\infty} \frac{\gamma^m m! (2N - m)!}{\sum_{k=0}^{[n]} \binom{n}{k} \left( \frac{2N - n}{m - k} \right) (-1)^k}} \right\}^{-1/2}, \]  

(31)

where, in turn, \( n = N + \frac{a}{2\sum_{m=0}^{\infty} \gamma^m m! (2N - m)!} \). Finally, on substituting (30) in (23), our first group of Morse-like squeezed coherent states is explicitly written as

\[ |\alpha, \gamma \rangle = N_n \sum_{m=0}^{N-1} (-1)^m \sqrt{m! (2N - m)!} \sum_{k=0}^{[n]} \binom{n}{k} \left( \frac{2N - n}{m - k} \right) (-1)^k |m\rangle, \]  

(32)

and in the summations above, \([n]\) means the greatest integer less than or equal to \( n \). In equation (32), we have taken into account the fact that the index \( m \) runs only on the bound part of the spectrum. For this approximation to be valid, it is important to be careful in using suitable values for \( \alpha \) and/or \( \gamma \) so as to constrain the evolution of our coherent states to the low-lying region of the spectrum. The states thus obtained will be called deformed ladder-operator quasi-coherent states (LOQCSS).

3. Numerical results

We now proceed with the calculation of some physical quantities of interest related to the Morse-like squeezed coherent states obtained above. To begin with, we examine their statistical properties via the normalized variance of the number operator. Next, we analyze their temporal behavior on phase space and dispersion relations. From now on, we shall consider a Morse-like oscillator possessing \( N = 10 \ (\chi_a \approx 0.05) \) bound states.
Figure 1. Plot of the normalized variance of the number operator, \( \langle (\Delta n)^2 \rangle / \langle n \rangle \), as a function of \( \gamma \) for the Morse-like squeezed vacuum state \( |\alpha = 0, \gamma \rangle \) (frame (a)) compared with the result obtained after numerically solving equation (24) (frame (b)).

3.1. Statistical properties

Deviations of the occupation number distribution from Poissonian statistics can be evaluated by means of the normalized variance of the number operator [64]:

\[
\frac{\langle (\Delta \hat{n})^2 \rangle}{\langle \hat{n} \rangle} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle},
\]

where the average values are taken between the deformed LOQCS. Values of this quantity such that \( \langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle < 1 \) or \( \langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle > 1 \) correspond to sub-Poissonian or super-Poissonian statistics, respectively; the former is known to be a nonclassical feature [65]. Poissonian distribution means that \( \langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle = 1 \).

Figure 1 shows the normalized variance, equation (33), as a function of the squeezing parameter \( \gamma \) for the Morse-like squeezed vacuum state which corresponds to the particular case of the LOQCS, defined by equation (32), with \( \alpha = 0 \). In order to confirm the feasibility of this analytic expression, we have added in figure 1(b) the result from the numerical solution of the recursion relation given by equation (24). Both calculations lead to the same statistical behavior, as expected. We can see that such a behavior is super-Poissonian in the interval considered. Its ascending profile is closely related to spreading effects in the number-occupation probability distribution associated with such states. The probability that the state \( |\alpha = 0, \gamma \rangle \) has the excitation number \( n \), namely \( p_n = |\langle n |\alpha = 0, \gamma \rangle|^2 \), is evaluated using (32). In figure 2, we have plotted \( p_n \) as a function of \( n \) for \( \gamma = 0.1, 0.3, 0.5 \), and 0.8. We see that as the squeeze parameter \( \gamma \) increases, the occupation number distribution \( p_n \) becomes wider, which is consistent with the fact that the dispersion in the number operator \( \hat{n} \) is larger than its mean. In conjunction with this tendency, \( p_n \) reveals an oscillatory behavior, being zero for odd \( n \). On the other hand, figure 3 shows the more general case for which the normalized variance of the LOQCSs is plotted as a function of \( \alpha \) for \( \alpha = 0 \) and 0.3. Clearly, in such a case, the statistical behavior seems very different from the one displayed in figure 1, ranging from super-Poissonian to sub-Poissonian conduct; the plots appearing in figures 3(a) and (c) owe their stepped shape to the particular way in which the LOQCSs are algebraically constructed (see equation (32)). Again, the corresponding numerical results calculated from equation (24) (see figures 3(b) and (d)) yield a confirmation of the validity of equation (32). In figure 4, we have also plotted \( p_n \) for the coherent state \( |\alpha, \gamma = 0.3 \rangle \) (equation (32)) as a function of \( n \) for \( \alpha = 0.2, 0.4, 0.8 \), and 1.6. It is found that the occupation number distribution is a decreasing function of \( n \) for values
of $\alpha$ in the interval $0.2 < \alpha < 0.8$. For larger values of $\alpha$, the distribution takes its maximum at approximately $\langle n \rangle = \alpha^2$. Note that we have focused our attention on those energy regions where the influence of the continuum can be discarded.

3.2. Phase space and uncertainty relations

In order to compute the temporal evolution on the phase space of our squeezed coherent states, as well as their dispersion and uncertainty relations, we will make use of a convenient algebraic representation for the position and momentum operators put forward by Carvajal et al [66] in their work on the SU(2) model of vibrational excitations for Morse oscillators. In this work, the authors express the Morse coordinate and momentum operators as an expansion in terms of SU(2) operators. It was shown in [67] that when the number of bound states supported by the potential is large enough, the algebra describing the SU(2) system and that describing the deformed Morse-like potential is the same, which allowed the use of their expressions to compute the average values of coordinate and momentum operators. It was also found that the use of up to second-order terms in $\hat{A}$ and $\hat{A}^\dagger$ gives a very reliable description of the coordinate and momentum.

According to [66], the position and momentum variables are given as a series expansion involving all powers of the deformed creation and annihilation operators. Keeping up to second-order terms, we obtain

$$x_D \approx \sqrt{\frac{\hbar}{2\mu\Omega}} (f_{00} + f_{10}\hat{A}^\dagger + \hat{A} f_{01} + f_{20}\hat{A}^{12} + \hat{A}^2 f_{02}).$$

Figure 2. Plots of the number-state probability distribution for the Morse-like squeezed vacuum state $|\alpha = 0, \gamma \rangle$. Frames (a)–(d) correspond to $\gamma = 0.1, 0.3, 0.5$ and 0.8, respectively.
Figure 3. Plots of the normalized variance of the number operator, $\langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle$, as a function of $\alpha$ for Morse-like squeezed states with $\gamma = 0.1$ (frames (a), (b)) and $\gamma = 0.3$ (frames (c), (d)). The results appearing in frames (a) and (c) were obtained by using the analytical expression of $|\alpha, \gamma \rangle$ defined in (32), while those results displayed in frames (b) and (d) come from solving equation (24) numerically.

$p_D \approx i \sqrt{\frac{\hbar \mu \Omega}{2}} (g_{01} \hat{A} + \hat{A} g_{01} + g_{20} \hat{A}^2 + \hat{A}^2 g_{02})$. \quad (35)

Here, the expansion coefficients $f_{ij}$ and $g_{ij}$ are functions of the number operator $\hat{n}$ and are given by

\begin{align*}
    f_{00}(\hat{n}) &= \sqrt{k} \left[ f_0 + \ln \frac{(k - 2)(k - \hat{n} - 1)}{(k - 1 - 2\hat{n})(k - 2\hat{n})} \right] (1 - \delta_{\hat{n},0}), \quad (36) \\
    f_{10}(\hat{n}) &= f_{01}(\hat{n}) = \sqrt{\frac{k - 1}{k}} \left( 1 + \frac{\hat{n}}{k - \hat{n}} \right), \quad (37) \\
    f_{20}(\hat{n}) &= f_{02}(\hat{n}) = \frac{k - 1}{2k \sqrt{k}} \left( \frac{-1}{(1 - (\hat{n} - 1)/k)(1 - \hat{n}/k)} \right), \quad (38) \\
    g_{10}(\hat{n}) &= -g_{01}(\hat{n}) = \sqrt{\frac{k - 1}{k}} \left( \frac{k - 2\hat{n}}{k - \hat{n}} \right), \quad (39) \\
    g_{20}(\hat{n}) &= -g_{02}(\hat{n}) = -\frac{k - 1}{k \sqrt{k}} \left( \frac{k - (2\hat{n} - 1)}{k(1 - (\hat{n} - 1)/k)(1 - \hat{n}/k)} \right). \quad (40)
\end{align*}
Figure 4. Plots of the number-state probability distribution for Morse-like squeezed coherent states with \( \gamma = 0.3 \). Frames (a)-(d) correspond to \( \alpha = 0.2, 0.4, 0.8 \) and 1.6, respectively.

where, in turn,

\[
f_0 = \ln k - \left( \sum_{p=1}^{k-2} \frac{1}{p} - \Gamma \right),
\]

(41)

with

\[
\Gamma = \lim_{m \to \infty} \left( \sum_{p=1}^{m} \frac{1}{p} - \ln m \right) = 0.577216
\]

(42)

being the Euler constant, and \( k = 2N + 1 = 1/\chi_a \).

In what follows, we will define the deformed position and momentum operators \( \hat{x}_D \) and \( \hat{p}_D \) taking \( \hbar = \mu = \Omega = 1 \) in (34) and (35). When the deformation function is equal to 1, these expressions take the harmonic value.

Application of the time evolution operator to the initial state \( |\alpha, \gamma; t = 0\rangle \) yields, at time \( t \),

\[
|\alpha, \gamma; t\rangle = \hat{U}(t)|\alpha, \gamma; 0\rangle = e^{-i\Omega t(\hat{p} + 1/2 - (\hat{p} + 1/2)^2)} e^{-\chi_a t/4}|\alpha, \gamma; 0\rangle.
\]

(43)

Using these states, we can evaluate the temporal average of \( x_D \) and \( p_D \), their dispersions, and, using equation (8), the normalized uncertainty product

\[
\Delta_{x_D,p_D} = \frac{4\langle (\Delta x_D)^2 \rangle \langle (\Delta p_D)^2 \rangle}{(|\langle [x_D, p_D] \rangle|^2)^{1/2}}.
\]

(44)

If \( \langle (\Delta O)^2 \rangle < 0.5 \), where \( O \) is either \( x_D \) or \( p_D \), the deformed coherent state is squeezed, and if \( \Delta_{x_D,p_D} = 1 \), it is a minimum uncertainty state.
Figure 5. Phase space trajectories of the Morse-like squeezed vacuum state $|\alpha \approx 0, \gamma \rangle$. Frames (a)–(d) correspond to $\gamma = 0.1, 0.3, 0.5$ and 0.8, respectively.

Phase space trajectories $x_D$ versus $p_D$ for the LOQCS with $\alpha = 0$ (the Morse-like vacuum state $|\alpha = 0, \gamma \rangle$) are shown in figure 5 for the squeezing parameter $\gamma = 0.1, 0.3, 0.5$ and 0.8. It is found that if the squeezing parameter is small, say $\gamma = 0.1$ (figure 5(a)), the evolution of the state appears to be well localized on the phase space, the amplitude of the oscillations being practically constant, which is consistent with the harmonic result where for a fixed energy one has a single elliptical trajectory. However, as the value of $\gamma$ increases (figure 5(b) to (c)), the state exhibits less stable behavior, and there are several trajectories with different amplitudes.
Figure 6. Temporal evolution of the uncertainties in the coordinate (frames (a), (c)) and momentum (frames (b), (d)) for the Morse-like squeezed vacuum state $|\alpha = 0, \gamma \rangle$. Frames (a) and (b) correspond to $\gamma = 0.1$ (continuous line) and $\gamma = 0.3$ (dashed line). Frames (c) and (d) correspond to $\gamma = 0.5$ (continuous line) and $\gamma = 0.8$ (dashed line).

Figure 7. Temporal evolution of the normalized uncertainty product $\Delta_{1p} = 4\langle (\Delta x_D)^2 \rangle \langle (\Delta p_D)^2 \rangle / |\langle [x_D, p_D] \rangle|^2$ for the Morse-like squeezed vacuum state $|\alpha = 0, \gamma \rangle$. Frame (a) corresponds to $\gamma = 0.1$ (continuous line) and $\gamma = 0.3$ (dashed line). Frame (b) corresponds to $\gamma = 0.5$ (continuous line) and $\gamma = 0.8$ (dashed line).

of oscillation for $x_D$ and $p_D$, which is a signature of the nonlinear terms in the Hamiltonian governing the temporal evolution [68]. For $\gamma = 0.8$ (see figure 5(d)), we note the appearance of trajectories where $x_D$ and $p_D$ both take rather small values at the same time. This last fact is closely related to spreading effects of the associated coherent state wave packet, which is manifested in figures 6 and 7 where we display the corresponding temporal behavior of the variances, and the normalized uncertainty product (44), respectively.
Following the sequence of graphs in figure 6, we can see that the larger the value of the squeeze parameter γ, the larger the oscillatory amplitude of both fluctuations \((\langle \Delta x^2 \rangle)^2\) and \((\langle \Delta p^2 \rangle)^2\). The increase in γ gives rise to a temporal behavior consisting in a sequence of beats due to the different effective frequencies involved in the evolution (note the presence of the number operator in the exponent of (43)). We observe that for γ = 0.1, there is squeezing in both x₀ and p₀ and the amplitude of the oscillations is rather small (of the order of 0.2); for γ = 0.3, the amplitude of the oscillations in the dispersions is larger (of the order of 0.6) and there is squeezing in both x₀ and p₀. Further increasing γ to 0.5 and 0.8, the state ceases to be squeezed with respect to x₀ (figure 6(c)), but it is still squeezed with respect to p₀ for γ = 0.5 (see figure 6(d)), and the squeezing is lost for γ = 0.8. From the lower graph of figure 7(a), we can observe that the squeezed vacuum state is a minimum uncertainty state at certain time instants (i.e., Δxp ≈ 1) for γ = 0.1; however, this is not so for larger values of γ.

In figure 8, we have plotted the behavior of the dispersions, \((\langle \Delta x^2 \rangle)^2\) and \((\langle \Delta p^2 \rangle)^2\), and the normalized uncertainty product Δxp at t = 0 as functions of γ for the vacuum state |α, γ⟩. We see that for γ < 0.4, the state can show squeezing in x₀ (figure 8(a)). With respect to p₀, the squeezing is restricted to a much smaller region of γ. And, at least for the interval 0 < γ < 0.2, the uncertainty product Δxp is almost 1 (see figure 8(c)) and the state may be regarded as a minimum uncertainty coherent state.

Another measure for the nonclassicality of a system is provided by its Wigner function. We follow the work of [69, 70] where the Wigner function is written as an infinite series in terms of the displaced number-state expectation values |α, k⟩:

\[
W(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \hat{\rho} | \alpha, k \rangle,
\]

(45)

and where the states |α, k⟩ = D(α)|k⟩ are the displaced number states [71, 72]. The Wigner function corresponding to the LOQCSs is obtained from

\[
W(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \hat{\beta} | \beta, \gamma \rangle | \beta, \gamma | \alpha, k \rangle.
\]

(46)

In figure 9, we show the Wigner function for a density matrix \(\hat{\rho} = |\beta = 0.001, \gamma = 0.1\rangle \langle \beta = 0.001, \gamma = 0.1|\) corresponding to the parameters used in figure 5(a). For this set of parameters, the Wigner function is always positive, corresponding to a classical distribution in agreement with the phase space trajectory which is similar to that expected for a harmonic oscillator.

On the other hand, in figure 10, we show the Wigner function for a density matrix \(\hat{\rho} = |\beta = 0.001, \gamma = 0.5\rangle \langle \beta = 0.001, \gamma = 0.5|\) corresponding to the parameters used in figure 5(c). Here, the presence of negativities in the Wigner function is evident, and this is a
signature of the nonclassicality of the state. Note that in this case, the phase space trajectories display a very different conduct as compared with the linear case.

As a function of $\alpha$ and for a fixed value of $\gamma$, the phase space trajectories show an ovoid shape and nonclassical behavior (several trajectories for a given set of parameters) due to the nonlinear terms in the Hamiltonian. The system shows squeezing but as the value of $\alpha$ is increased, the time intervals at which squeezing is present is smaller. For most of the time, the states $|\alpha,\gamma\rangle$ are not minimum uncertainty states.
4. Conclusions

Based on the f-deformed oscillator formalism, we have introduced squeezed coherent states associated with a Morse system (Morse-like squeezed coherent states) and have examined them from a statistical and dynamical viewpoint. Such states were constructed by making use of the ladder-operator technique, i.e., as approximate eigenstates of a linear combination of f-deformed ladder operators, called deformed ladder-operator quasi-coherent states (LOQCSs). For these squeezed states, we have illustrated their statistical properties and have evaluated the temporal evolution of some quantities of physical interest such as phase space trajectories, dispersion relations, and squeezing. In short, we emphasize the main features of the introduced states.

• Concerning their statistical properties, we have found that, depending on the values of the parameters $\alpha$ and $\gamma$, the LOQCSs $|\alpha, \gamma\rangle$ can exhibit both sub-Poissonian and super-Poissonian statistics. In particular, for $\alpha \approx 0$, the Morse-like vacuum state $|0, \gamma\rangle$ always exhibits super-Poissonian statistics as a function of $\gamma$, which leads to a faster spreading of a state’s occupation number distribution function.

• Only for sufficiently small values of $\gamma$ does the evolution of the LOQCSs show a somewhat well-localized behavior on phase space. As we have seen, these states are almost minimum uncertainty coherent states under these conditions. In contrast, for larger values of $\gamma$, the dispersion of both the coordinate and momentum increases and, accordingly, the deformation and collapse of the states’ phase space trajectories become much more significant. This effect is due to the influence of the nonlinearity of the Hamiltonian. The LOQCSs also display squeezing in the deformed coordinate and momentum variables at certain stages of their evolution and for certain restricted values of $\gamma$.

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