An investigation for the appearance of long range nuclear potential on the ultra low energy nuclear synthesis

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Motivation & Starting Point

Phys. Rev. C86, 044001 (2012),
Few—Body Syst.59, 51 (2018).

The author proposed a Long Range (GPT) Potential which appears, not only at the 3-body Break-up Threshold (3BT), but also at the Quasi 2-body Threshold (Q2T).

The GPT Potential represents
1) a Yukawa-type potential for shorter range,
2) but a $1=\frac{1}{r^n}$ -type potential for longer range.
Energy dependent quasi 2-body AGS-Born (E2Q)

\[ Z_{\alpha\beta}(q,q';E) = \frac{g_\alpha(p)g_\beta(p')(1-\delta_{\alpha\beta})}{(E+\varepsilon_B)-q^2/2\mu} \to \infty \]

at (Q2T): \( p = p' = 0, \quad E_{cm} = (E + \varepsilon_B) = 0, \quad q = 0 \)

For \( E_{cm} < 0 \): with \( \sigma^2 = 2\mu|E_{cm}|, \quad 0 < C_{\alpha\beta} \)

\[ Z_{\alpha\beta} \to \frac{g_\alpha(0)g_\beta(0)(1-\delta_{\alpha\beta})}{-|E_{cm}| - q^2/2\mu} = -\frac{C_{\alpha\beta}}{q^2 + \sigma^2} \]

Therefore, the Fourier transformation becomes

\[ F \left[ Z_{\alpha\beta}(q,q';E) \right] = V_0 \frac{e^{-\sigma r}}{r} : \quad V_0 < 0 \]
To avoid the divergence at $Q2T$, adopt a statistical average with a weight $P$:

$$P = \frac{\sigma^{2\gamma+1} e^{-a\sigma}}{\rho}$$

$$\rho = \int_{0}^{\infty} \sigma^{2\gamma+1} e^{-a\sigma} d\sigma = \frac{\Gamma(2\gamma + 2)}{a^{2\gamma+2}}$$

$$L \{U^{(0)}(\Delta, \sigma; r)\} \equiv V_0 \frac{1}{\rho} \int_{0}^{\infty} \sigma^{2\gamma+1} e^{-a\sigma} \frac{e^{-\sigma r/2}}{r} d\sigma$$

$$= V_0 \frac{a^{2\gamma+2}}{r(r/2 + a)^{2\gamma+2}}$$
GPT-potential: below the Q2T, or 3BT ($E_{cm} < 0$) with parameters $a$ and $\gamma$, where $V_0(\leq 0)$ is a potential depth.

| $\gamma$ | $r << a$ | GPT-potential | $a << r$ |
|----------|----------|---------------|----------|
| $-1$ | $V_0/r$ | $V_0/r$ | $V_0/r$ |
| $-1/2$ | $V_0e^{-r/2a}/r$ | $V_0(2a)/\left[r(r + 2a)\right]$ | $V_0(2a)/r^2$ |
| $0$ | $V_0e^{-2r/2a}/r$ | $V_0(2a)^2/\left[r(r + 2a)^2\right]$ | $V_0(2a)^2/r^3$ |
| $1/2$ | $V_0e^{-3r/2a}/r$ | $V_0(2a)^3/\left[r(r + 2a)^3\right]$ | $V_0(2a)^3/r^4$ |
| $1$ | $V_0e^{-4r/2a}/r$ | $V_0(2a)^4/\left[r(r + 2a)^4\right]$ | $V_0(2a)^4/r^5$ |
| $3/2$ | $V_0e^{-5r/2a}/r$ | $V_0(2a)^5/\left[r(r + 2a)^5\right]$ | $V_0(2a)^5/r^6$ |
| $2$ | $V_0e^{-6r/2a}/r$ | $V_0(2a)^6/\left[r(r + 2a)^6\right]$ | $V_0(2a)^6/r^7$ |
The GPT-potential includes automatically the Efimov-like potential in any systems.

In order to confirm the GPT potential, we investigate ultra low energy reactions $^{135}\text{Cs}(2d,\gamma)^{139}\text{La};~^{137}\text{Cs}(2d,\gamma)^{141}\text{La}$

by the three-ion quasi-molecule CsD$_2$
in the Pd-cluster with a form: CsD$_2$Pd$_{12}$. 
Define quasi molecules $\text{CsD}_2$, $\text{CsD}_4$ and $\text{CsD}_6$. 

$\text{Cs-d}_6-\text{Pd}_{12}$  
Cub-octahedron
Our calculation:

\[ ^{137}_{55}\text{Cs} (7/2^+:30.07\text{y}) + 2\text{d} \rightarrow ^{141}_{57}\text{La} (7/2^+:3.2\text{h}) + \beta^- \]
\[ \rightarrow ^{141}_{58}\text{Ce} (7/2^-:32.5\text{d}) + \beta^- \rightarrow ^{141}_{59}\text{Pr} (5/2^+:\text{stable}) \]

\[ ^{135}_{55}\text{Cs} (7/2^+:2.3 \times 10^6\text{y}) + 2\text{d} \rightarrow ^{139}_{57}\text{La} (7/2^+:\text{stable}) \]

\[ ^{135}_{55}\text{Cs} (7/2^+:2.3 \times 10^6\text{y}) + 4\text{d} \rightarrow ^{143}_{59}\text{Pr} (5/2^+:13.57\text{d}) + \beta^- \]
\[ \rightarrow ^{143}_{60}\text{Nd} (7/2^-:\text{stable}) \]

Calculate \( D-\text{Cs-D (d-Cs-d)} \) three-body bound states and wave functions.
Cs-D$_6$-Pd$_{12}$ Cub-octahedron

Number of surface:
8-regular triangles + 6-regular squares = 14 surfaces

24-sides
12-apexes

Surface area $S = (6 + 2\sqrt{3})a^2$

Volume $V = \frac{5\sqrt{2}}{3}a^3$

Radius of the circumscribed sphere $a$

Pd - Pd distance $R_{\text{Pd}} = a = 5.2 \text{ au} \approx 2.75172144 \times 10^5 \text{ fm}$

D - Cs distance $R_{\text{D}} = 3.1 \text{ au} \approx 1.64044932 \times 10^5 \text{ fm}$

$1\text{ au} = 0.591772\text{ Å}$

D exists inside of the surface
Total potential (red)

3-body force pot.

Total potential

Ion-ion Coulomb pot.

GPT-Long range pot.

WS-potential

Pd-border

fm
Potentials in the three-body system

1) Nuclear potential:

\[ V_{W_{ij}}^{N_iN_j} (r_{ij}) = \frac{V_{W0}^{N_iN_j}}{1 + \exp \left( \frac{r_{ij} - R_{W_{ij}}^{N_iN_j}}{a_{W_{ij}}^{N_iN_j}} \right)} : \text{WS – potential} \]

\[ V_{W0}^{Csd} = -79.30 \text{ MeV}, \quad V_{W0}^{dd} = -27.57 \text{ MeV}, \quad R_{W}^{Csd} = 10.21 \text{ fm}, \]

\[ R_{W}^{dd} = 1.49 \text{ fm}, \quad a_{W}^{Csd} = 0.4 \text{ fm}, \quad a_{W}^{dd} = 0.3 \text{ fm}, \]

2) Coulomb potential:

\[ V_{c}^{N_iN_j} (r_{ij}) = \begin{cases} \frac{Z_i Z_j e^2}{8 \pi R} \left[ 3 - \left( \frac{r_{ij}}{R_{c}^{N_iN_j}} \right)^2 \right] & \text{for } r \leq R \\ \frac{Z_i Z_j e^2}{8 \pi r_{ij}} & \text{for } R \leq r \end{cases} \]

\[ R_{c}^{CsH} = 10.21 \text{ fm}, \quad R_{c}^{H_1H_2} = 10.21 \text{ fm}. \]
3) **Pd-N$_i$ potential:** one-body potential

$$V_{c}^{PdN_i} (r_i) = V_{c0}^{Pd} \left( \frac{r_i}{a_{c}^{Pd}} \right)^{10} \exp \left\{ -\left( \frac{r_i - a_{c}^{Pd}}{b_{c}^{Pd}} \right)^2 \right\}$$

$$V_{c0}^{Pd} = 1.0 \times 10^{-4} \text{ MeV}, \quad a_{c0}^{Pd} = 5.0 \times 10^5 \text{ fm},$$

$$b_{c0}^{Pd} = 3.1623 \times 10^5 \text{ fm}.$$ 

Pd position $1.57 \times 10^6 \text{ fm}$, $2.73 \text{ MeV}$ height.

4) **Three-cluster potential:**

$$V_t (r_1, r_2, r_3) = V_{t0} \exp \left[ -\left( \frac{r_{12}}{a_t} \right)^2 - \left( \frac{r_{23}}{a_t} \right)^2 - \left( \frac{r_{31}}{a_t} \right)^2 \right]$$

$$V_{t0} = 1800 \text{ MeV}, \quad a_t = 3.0 \text{ fm}.$$
5) Long range potential:

\[ V_e(r_1, r_2, r_3) = \frac{V_{e0}}{l} \left( \frac{r_{12}}{a_e} \right)^l + \left( \frac{r_{23}}{a_e} \right)^m + \left( \frac{r_{31}}{a_e} \right)^n + 1 \]

\[ \text{take } l = m = n = 2 \ (Efimov \ case). \]

\[ V_e(r_1, r_2, r_3) = \frac{V_{e0}}{2} \left( \frac{r_{12}}{a_e} \right)^2 + \left( \frac{r_{23}}{a_e} \right)^2 + \left( \frac{r_{31}}{a_e} \right)^2 + 1 \]

\[ V_{e0} = -80000 \ \text{MeV}, \quad a_e = 500 \text{fm}. \]

\[ V_e(r_1, r_2, r_3) = \frac{V_{e0}a_e^2}{r_{12}^2 + r_{23}^2 + r_{31}^2 + a_e^2} \]

\[ V_{e0} = -80000 \ \text{MeV}, \quad a_e = 5000 \text{fm}. \]
For the 3rd particle transfer:

A) \( r_{23} = r_2 - r_3 = 0 \) or \( r_{31} = r_3 - r_1 = 0 \)

\[
V_e(r_1, r_2, r_3) = \frac{V_{e0}a_e^2}{r_{12}^2 + r_{23}^2 + r_{31}^2 + a_e^2} \Rightarrow \frac{V_{e0}a_e^2}{r_{12}^2 + a_e^2}
\]

three-body long range (two-body long range pot. = 0)

B) \( r_{12} = r_1 - r_2 = 0 \)

\[
V_e(r_1, r_2, r_3) = \frac{V_{e0}a_e^2}{r_{12}^2 + r_{23}^2 + r_{31}^2 + a_e^2} \Rightarrow \frac{V_{e0}a_e^2}{2r_{23}^2 + a_e^2} = \frac{V_{e0}a_e^2}{2r_{31}^2 + a_e^2}
\]

two-body long range (three-body long range pot. = 0)
The wave function overlap value (WFO: $W_{n;m}$) between
1) the La highest nuclear excited state with the quantum number $n = 5$,

2) and the lowest CsD$_2$ quasi molecular states is of critical importance for the existence of the electro-magnetic (EM) transition in the Cs(2d,$\gamma$)La reaction.
$n = 1 \sim 12$
$n = 13 \sim 24$
\[ n = 25 \sim 36 \]
$n = 37 \sim 48$
$n = 49 \sim 60$
Without long range
Short range Nuclear potential

\( R \): transition rate

\( T = 10^9 \) K

\( E \approx 3.4 \times 10^{-4} \) MeV
Short range

$T = 10^9 \text{ K}$

With long range

$T = 500 \text{ K}$

$R$

$n$

With long range

$T = 500 \text{ K}$ (red)
Transition probability from $|\psi_i\rangle$ to $|\psi_f\rangle$ by the spontaneous emission in the vacuum

**E1- transition**

$$W_{i\rightarrow f}^{E1} (E_i, E_f) = \frac{(E_i - E_f)^3}{3\pi\varepsilon_0 h_b^4 c^3} \sum_{k=1}^{3} \left| \langle \Psi_f | Z_k e \vec{r}_k | \Psi_i \rangle \right|^2 = \frac{4}{3} \frac{(E_i - E_f)^3}{h_b^3 c^2} \left( \frac{1}{4\pi\varepsilon_0 h_b c} \right) \sum_{k=1}^{3} \left| \langle \Psi_f | Z_k e \vec{r}_k | \Psi_i \rangle \right|^2$$

$$= \left\{ \frac{4}{3} \frac{(E_i - E_f)^3}{h_b^3 c^2} \alpha \sum_{k=1}^{3} \left| \langle \Psi_f | Z_k e \vec{r}_k | \Psi_i \rangle \right|^2 \right\}$$

**E2-transition**

$$W_{i\rightarrow f}^{E2} (E_i, E_f) = \frac{1}{20} \frac{4}{3\pi\varepsilon_0 h_b^6 c^3} \sum_{k=1}^{3} \left| \langle \Psi_f | \frac{1}{Z_k e \vec{r}_k | \Psi_i \rangle \right|^2 \left( \frac{1}{2} (3z_k^2 - x_k^2 - y_k^2) Z_k e | \Psi_i \rangle \right|^2$$

$$\rightarrow W_{i\rightarrow f}^{E2'} = \frac{1}{20} \frac{4}{3\pi\varepsilon_0 h_b^6 c^3} \sum_{k=1}^{3} \left| \langle \Psi_f | \frac{1}{2} r_k^2 Z_k e | \Psi_i \rangle \right|^2$$

**M1-transition**

$$W_{i\rightarrow f}^{M1} (E_i, E_f) = \frac{4}{3\pi\varepsilon_0 h_b^4 c^3} \sum_{k=1}^{3} \left| \langle \Psi_f | \frac{Z_k e g_k}{2m_k} (\vec{L}_k + \vec{S}_k) | \Psi_i \rangle \right|^2$$

**transition time**

$$\tau_{i\rightarrow f} = \frac{1}{W_{i\rightarrow f}}$$

$g_k$ : gyromagnetic ratio of nucleus
• Let us obtain the transition probability from |ψ_i⟩ to |ψ_f⟩ by photon emission from CsD_{2}Pd_{12} in the thermal equilibrium of temperature T. The average La number at the energy E_i and with the temperature T is given by the Maxwell-Boltzmann distribution;

\[ f_{MB}(E_i,T) = \exp \left[ -\frac{E_i}{k_B T} \right] / Z \]

\[ Z = \sum_{j=1}^{\infty} \exp \left[ -\frac{E_j}{k_B T} \right] \]

\[ k_B = 1.380649 \times 10^{-23} \text{ J/K} \approx 8.6171 \times 10^{-8} \text{ MeV/K} \]

In the radiation field of the thermal equilibrium with the temperature T and the energy E_i, the average photon number is give by the Bose-Einstein statistics,

\[ f_{BE}(E_i - E_f) = \frac{1}{\exp \left[ \frac{E_i - E_f}{k_B T} \right] - 1} \quad \text{for Black body radiation} \]

Therefore, the transition probability for the unit time, and the unit number of La is given by

\[ \frac{dN_{i\rightarrow f}^{E_1}}{dt} = f_{MB}(E_i,T) \left[ \frac{4(E_i - E_f)^{3}}{3\pi \varepsilon_0 \hbar^4 c^3} \sum_{k=1}^{3} \left| \langle \psi_f | Z_k e^{\vec{r}_k} | \psi_i \rangle \right|^2 + f_{BE}(E_i - E_f,T) \frac{4(E_i - E_f)^{3}}{3\pi \varepsilon_0 \hbar^4 c^3} \sum_{k=1}^{3} \left| \langle \psi_f | Z_k e^{\vec{r}_k} | \psi_i \rangle \right|^2 \right] \]

\[ = f_{MB}(E_i,T) \left[ W_{ij}^{E_1}(E_i, E_f) + f_{BE}(E_i - E_f,T)W_{ij}^{E_1}(E_i, E_f) \right] = \frac{W_{ij}^{E_1}(E_i, E_f) / Z}{\exp \left[ \frac{E_i}{k_B T} \right] - \exp \left[ \frac{E_f}{k_B T} \right]} \]

\[ E2、M1 \text{ transitions as well} \]
Recent Experimental Results:

By Iwamura et al. (MHI), (2002)

\[ ^{133}_{55} \text{Cs}(7/2^+:\text{stable}) + 4d \rightarrow ^{141}_{59} \text{Pr}(5/2^+:\text{stable}) \]

Praseodymium

\[ ^{88}_{38} \text{Sr}(0^+:82.58\%) + 4d \rightarrow ^{96}_{42} \text{Mo}(0^+:16.68\%) \]

Strontium Molybdenum

\[ ^{88}_{38} \text{Sr}(0^+:82.58\%) + 2d \rightarrow ^{92}_{40} \text{Zr}(0^+:17.15\%) \]

Zirconium

Hioki et al, Toyota-Nagoya univ. group confirmed (2013)
Cs planting

- Thin Pd Film: 400 Å
- CaO and Pd Layers: 1000 Å
- Bulk Pd: 0.1 mm

25mm × 25mm
Fig. 1. $D_2$ gas permeation through the Pd complex.
$^{133}\text{Cs} + 4\text{d} \rightarrow ^{141}\text{Pr}$
$n = 1.4 \times 10^{15} \text{ /cm}^2$

$T = 343K \approx 70^\circ C$

Transition Probability for:

$^{135}\text{Cs}+2d \rightarrow ^{139}\text{La}$

1) No long range & electron-ion pot.

$$\sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx 7.1 \times 10^7 \text{ /cm}^2$$

2) With long range, No electron-ion pot.

$$W_{i\rightarrow f}^{E2'} (L) = \sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx 1.5 \times 10^{16} \text{ /cm}^2$$

3) No long range, with electron-ion pot.

$$W_{i\rightarrow f}^{E2'} (S) = \sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx 1.1 \times 10^8 \text{ /cm}^2$$

Our result is 36 times larger than Experiment.
\[ ^{88}_{38}\text{Sr} + 4d \rightarrow ^{96}_{42}\text{Mo} \]
Conclusion

1) Wave function overlapping:

\[ W_{56}^L \bigg/ W_{56}^S \approx 10^7 \]

2) Transition probability for an approximated E2 gives

\[
W_{i\rightarrow f}^{L} \equiv W_{i\rightarrow f}^{E2'} (L) = \sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx 1.5 \times 10^{16} / \text{cm}^2
\]

\[
W_{i\rightarrow f}^{S} \equiv W_{i\rightarrow f}^{E2'} (S) = \sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{i\rightarrow f}^{E2'}}{dt} \times 120 \times 3600 \approx 1.1 \times 10^{8} / \text{cm}^2
\]
3) **Long Range Potential** is essential to obtain the ultra low energy nuclear synthesis.

4) The **GPT** $1/r^n$-type potentials are promising for the D-particle transfer potential in D-Cs-D of $1/r^2$ (or $1/r^3$) – type, while D$_2$-transfer in D$_2$-Cs-D$_2$ could be $1/r^3$ (or $1/r^4$) – type potential etc.
5) Therefore, pure D-absorption into Pd complex never occur the $D+D \rightarrow ^4\text{He}$ fusion, because $D\text{-Pd}_n$ is not a three-body system but a many-body system, then no GPT potential could be made.

6) Our theoretical calculation is the first success for the description of ultra low energy nuclear synthesis after the Experimental breakthrough was done.

7) As a conclusion, our Few-Body community could contribute to ultra low energy nuclear synthesis by the GPT long range potential.
Thank you very much for your attention!
XPS: X-ray photoelectron spectroscopy

Fig. 3. (a) Experimental apparatus, (b) Schematic of test setup in the vicinity of Pd complex test piece, (c) Path of D₂ gas flowing through Pd complex test piece and chamber wall.
Fig. 4. Experimental results obtained by D₂ gas permeation through Pd complex (Pd/CaO/Pd) deposited with Cs: (a) Time variation in number of Cs and Pr atoms (number of atoms per cm²), (b) XPS spectrum of Cs for experiment run #1, (c) XPS spectrum of Pr for experiment run #1, (d) XPS spectrum of Pd for experiment run #1, (e) Wide-range XPS spectrum for experiment run #1.
Fig. 9. Anomalous isotopic composition of detected Mo: (a) Isotopic composition of detected Mo for run #1, (b) Isotopic composition of detected Mo for run #2, (c) Isotopic composition of detected Mo for run #3, (d) SIMS analysis for Pd complex test piece with added Sr without D$_2$ gas permeation, (e) Natural abundance of Mo analyzed by SIMS.
Fig. 10. Relationship of mass numbers between given Sr and detected Mo: (a) Isotopic composition of detected Mo, (b) Isotopic composition of given Sr.
Fig. 1. $D_2$ gas permeation through the Pd complex.

$D_2 \rightarrow D + D$

Dissociative absorption

Vacuum

$D + D \rightarrow D_2$

Recombination
XPS: X-ray photoelectron spectroscopy

Fig. 3. (a) Experimental apparatus, (b) Schematic of test setup in the vicinity of Pd complex test piece, (c) Path of D₂ gas flowing through Pd complex test piece and chamber wall.
Fig. 4. Experimental results obtained by D$_2$ gas permeation through Pd complex (Pd/CaO/Pd) deposited with Cs: (a) Time variation in number of Cs and Pr atoms (number of atoms per cm$^2$), (b) XPS spectrum of Cs for experiment run #1, (c) XPS spectrum of Pr for experiment run #1, (d) XPS spectrum of Pd for experiment run #1, (e) Wide-range XPS spectrum for experiment run #1.
Fig. 5. Experimental results obtained by D₂ gas permeation through thin film and bulk Pd with added Cs: (a) Time variation in number of Cs and Pr atoms, (b) XPS spectrum of Cs, (c) XPS spectrum of Pr.

Fig. 6. Experimental results obtained by H₂ gas permeation through Pd complex (Pd/CaO/Pd) with added Cs: (a) Time variation in number of Cs and Pr atoms, (b) XPS spectrum of Cs, (c) XPS spectrum of Pr.
Fig. 8. Experimental results obtained by D$_2$ gas permeation through thin film and bulk Pd deposited with Sr: (a) Time variation in number of Sr and Mo atoms, (b) XPS spectrum of Sr, (c) XPS spectrum of Mo.
Fig. 9. Anomalous isotopic composition of detected Mo: (a) Isotopic composition of detected Mo for run #1, (b) Isotopic composition of detected Mo for run #2, (c) Isotopic composition of detected Mo for run #3, (d) SIMS analysis for Pd complex test piece with added Sr without D$_2$ gas permeation, (e) Natural abundance of Mo analyzed by SIMS.
Fig. 10. Relationship of mass numbers between given Sr and detected Mo: (a) Isotopic composition of detected Mo, (b) Isotopic composition of given Sr.
\[ ^{135}\text{Cs} + 2d \rightarrow ^{139}\text{La} \]

\[ n = 1.4 \times 10^{15} \text{ / cm}^2 \]
\[ T = 343\text{K} \approx 70\text{°C} \]

- No long range pot.  No electron-ion pot.

\[
\sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx 7.1 \times 10^7 / \text{cm}^2
\]

- With long range  No electron-ion pot.

\[
\sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx 1.5 \times 10^{16} / \text{cm}^2
\]

- No long range pot.  With electron-ion pot.

\[
\sum_{i=6}^{n_{\text{max}}} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx \sum_{i=6}^{60} \sum_{f=1}^{5} \frac{n}{100} \frac{dN_{E2'}}{dt} \times 120 \times 3600 \approx 1.1 \times 10^8 / \text{cm}^2 \sim \frac{7 \times 10^{14}}{73 \times 10^8} \approx 10^5 \text{ y}
\]
GPT-potential is given by parameters $a$ and $\gamma$ and a potential depth $V_0(<0)$. 

$$V^{GPT}(r) = V_0 \frac{a^{2\gamma+2}}{r(r / 2 + a)^{2\gamma+2}}$$

| $\gamma$ | $r \ll a$ | GPT-potential | $a \ll r$ |
|-------|--------|---------------|---------|
| $-1$  | $V_0/r$ | $V_0/r$       | $V_0/r$ |
| $-1/2$| $V_0 e^{-r/2a}/r$ | $V_0(2a)/\left[ r(r + 2a) \right]$ | $V_0(2a)/r^2$ |
| $0$   | $V_0 e^{-2r/2a}/r$ | $V_0(2a)^2/\left[ r(r + 2a)^2 \right]$ | $V_0(2a)^2/r^3$ |
| $1/2$ | $V_0 e^{-3r/2a}/r$ | $V_0(2a)^3/\left[ r(r + 2a)^3 \right]$ | $V_0(2a)^3/r^4$ |
| $1$   | $V_0 e^{-4r/2a}/r$ | $V_0(2a)^4/\left[ r(r + 2a)^4 \right]$ | $V_0(2a)^4/r^5$ |
| $3/2$ | $V_0 e^{-5r/2a}/r$ | $V_0(2a)^5/\left[ r(r + 2a)^5 \right]$ | $V_0(2a)^5/r^6$ |
| $2$   | $V_0 e^{-6r/2a}/r$ | $V_0(2a)^6/\left[ r(r + 2a)^6 \right]$ | $V_0(2a)^6/r^7$ |

...
第1章

Efimov effect (エフィモフ効果)と長距離ハドロンポテンシャルの予言
Review of Efimov-effect

Efimov V. Energy levels arising from resonant two-body forces in a three-body system, Phys. Lett. B33, 563 (1970)

Efimov V. Energy levels of three resonantly interacting particles, Nucl. Phys. A210 157 (1973)
1) the scattering length of the sub-system should be $a \rightarrow \infty$ (the first criterion)

2) three-body binding energies condense on the three-body break-up threshold (3BT) where the energy level structure is given by

$$\frac{E_n}{E_{n+1}} = \text{constant} > 1 \quad (n: \text{quantum number})$$

(the second criterion)

3) energy level can be obtained by

$r^{-2}$ potential (the third criterion)

Nicholson A.F., Bound states and scattering in an $r^{-2}$ potential
Australian J. Phys 15, 174-179 (1962)
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Efimov V. Energy levels arising from resonant two-body forces in a three-body system, Phys. Lett. **B33**, 563 (1970)

Efimov V. Energy levels of three resonantly interacting particles, Nucl. Phys. **A210** 157 (1973)

Kraemer T. et al., Evidence for Efimov quantum states in an ultracold gas of caesium atoms. *Nature* vol. **440**, pages 315–318 (2006)
In the hadron systems:

1) The first criterion: $a_{NN} \neq \infty$, $a_{N\pi} \neq \infty$

2) The second criterion is that there are some instances that energy levels come near the threshold region. However, it is very hard to confirm whether they are Efimov levels or not.

3) The third criterion is that the nuclear potential is usually a short-range: one pion exchange Yukawa potential etc.

エネルギー0近傍では実験的検証が困難！

$1/ r^2$のようなポテンシャルは考えられない！
I. General particle transfer (GPT)-potential

We reevaluate the Efimov physics by the thresholds.

1) Pay attention to the three-body break-up thresholds (3BT): appear in reactions:

$$d + p \rightarrow n + p + p,$$

$$N + N' \rightarrow N + N + \pi, \text{ etc.}$$

2) From 3-body bound state to the quasi two-body system:

quasi two-body threshold (Q2T):

$$^3He \rightarrow d + p,$$

$$D \rightarrow N + N' \equiv N + (N\pi), \text{ etc.}$$
$(a)$ Q2T $\rightarrow$ 3BT

(b) $G_0(q, q'; E)$

(c) $Z_{\alpha\beta}(q, q'; E) \quad (E - q'^2/2\mu)$
1) At the 3BT, the Born term $Z$ of the Faddeev or the Alt-Grassberger-Sandhas (AGS) equation, and the propagator have singularities;

At 3BT ($E = 0, q = p = 0$) : by using $E = q^2 / 2\mu + z$

$$Z_{\alpha\beta}(q, q'; E) = \frac{g_\alpha(p)g_\beta(p')(1 - \delta_{\alpha\beta})}{E - q^2 / 2\mu - p^2 / 2\nu} \to \infty$$

$$\tau(z) = \tau(E - q^2 / 2\mu) = \frac{f(z)}{\varepsilon_B + z} \Rightarrow \frac{f(z)}{z} = \frac{f(z)}{E - q^2 / 2\mu} \to \infty.$$  

or $\tau(z) \propto \frac{1}{-1/\alpha - ik} \to \lim_{a \to \pm\infty} i \frac{\sqrt{2\nu}}{\sqrt{E - q^2 / 2\mu}} \to i\infty.$
2) At the Q2T:

a) Propagator: at Q2T, with \( E = q^2 / 2 \mu + z \)

\[
\tau_B(z) = \frac{f(z)}{\varepsilon_B + z} = \frac{f(z)}{(\varepsilon_B + E) - q^2 / 2 \mu} \rightarrow \infty
\]

for \( E_{cm} = E + \varepsilon_B = 0, \quad q = 0 \)

Apart from AGS, an Energy dependent Two-body Quasi (E2Q) potential with two-body bound state (or \( a \neq \infty \)) becomes by using on-shell condition for Q2T.
Our analytic prediction fits to the numerical solution.

| $n$ | $E_n$      | $E_n/E_{n+1}$ | $\langle r_n^2 \rangle^{1/2}$ | $\langle r_{n+1}^2 \rangle^{1/2}$ | $\langle r^2 \rangle^{1/2}$ |
|-----|------------|----------------|-------------------------------|----------------------------------|----------------------------|
| 1   | -2.222     |                | 2.516                         |                                  |                            |
| 2   | -1.271 x 10^{-2} | 174.8     | 3.652 x 10^{1}               | 14.52                            |                            |
| 3   | -7.433 x 10^{-5} | 171.0     | 4.812 x 10^{2}               | 13.18                            |                            |
| 4   | -4.347 x 10^{-7} | 171.0     | 6.296 x 10^{3}               | 13.08                            |                            |
| 5   | -2.543 x 10^{-9} | 171.0     | 8.233 x 10^{4}               | 13.08                            |                            |
| 6   | -1.487 x 10^{-11}| 171.0    | 1.077 x 10^{6}               | 13.08                            |                            |
| 7   | -8.697 x 10^{-14}| 171.0    | 1.408 x 10^{7}               | 13.08                            |                            |
| 8   | -5.087 x 10^{-16}| 171.0    | 1.841 x 10^{8}               | 13.08                            |                            |
| 9   | -2.975 x 10^{-18}| 171.0    | 2.407 x 10^{9}               | 13.08                            |                            |
| 10  | -1.740 x 10^{-20}| 171.0    | 3.147 x 10^{10}              | 13.08                            |                            |
