An Average Curvature Accelerated Composite Gradient Method for Nonconvex Smooth Composite Optimization Problems

Jiaming Liang∗ Renato D.C. Monteiro∗

September 9, 2019

Abstract

This paper presents an accelerated composite gradient (ACG) variant, referred to as the AC-ACG method, for solving nonconvex smooth composite minimization problems. As opposed to well-known ACG variants that are either based on a known Lipschitz gradient constant or a sequence of maximum observed curvatures, the current one is based on a sequence of average observed curvatures. More specifically, AC-ACG uses a positive multiple of the average of all observed curvatures until the previous iteration as a way to estimate the “function curvature” at the current point and then two resolvent evaluations to compute the next iterate. In contrast to other variable Lipschitz estimation variants, e.g., the ones based on the maximum curvature, AC-ACG always accepts the aforementioned iterate regardless how poor the Lipschitz estimation turns out to be. Finally, computational results are presented to illustrate the efficiency of AC-ACG on both randomly generated and real-world problem instances.

1 Introduction

In this paper, we study an ACG-type algorithm for solving a nonconvex smooth composite optimization (N-SCO) problem

\[ \phi^* := \min \{ \phi(z) := f(z) + h(z) : z \in \mathbb{R}^n \} \]  

where \( f \) is a real-valued differentiable (possibly nonconvex) function with an \( M \)-Lipschitz continuous gradient on \( \text{dom } h \) and \( h : \mathbb{R}^n \to (-\infty, \infty] \) is a proper lower semicontinuous convex function with a bounded domain.

A large class of algorithms for solving (1) computes the next iterate \( y_{k+1} \) by solving a linearized prox subproblem of the form

\[ y_{k+1} = \arg \min \{ l_f(x, \tilde{x}_k) + h(x) + \frac{M_k}{2} \| x - \tilde{x}_k \|^2 : x \in \mathbb{R}^n \} \]  

where \( \tilde{x}_k \) is chosen as either the current iterate \( y_k \) (as in unaccelerated algorithms) or a convex combination of \( y_k \) and another auxiliary iterate \( x_k \) (as in accelerated algorithms), and \( M_k \) is an upper estimation of the “local function curvature” of \( f \) at \( \tilde{x}_k \). More specifically, letting

\[ C(y; \tilde{x}) := \frac{2 \left( f(y) - \ell_f(y, \tilde{x}) \right)}{\| y - \tilde{x} \|^2} \]  

∗School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, 30332-0205. (email: jiaming.liang@gatech.edu and renato.monteiro@isye.gatech.edu). This work was partially supported by ONR Grant N00014-18-1-2077.
More specifically, if \( M_{\text{ACG}} \) always computes a new step regardless of whether \( M_{\text{opposed to ACG variants based on the scheme outlined above as well as other ACG variants, AC-}} \) as a positive multiple of the average of all observed curvatures up to the previous iteration. As \( h \) on dom \( y \) set to be a convex combination of evaluation of \( h \) on \( f \) of the method becomes. Hence, it is desirable to choose \( M_k = \bar{M} \) where \( \bar{M} \), referred to as the local curvature of \( f \) at \( \tilde{x}_k \), is the smallest value of \( M_k \) satisfying (4). However, since finding \( \bar{M} \) is generally time-consuming, alternative strategies that upper estimate \( \bar{M} \) are used. A common one is a backtracking procedure that tentatively sets \( M_k \) to be the maximum of all the observed curvatures \( C_1, \ldots, C_{k-1} \) (see (4)) and if the corresponding \( y_{k+1} \) satisfies (4) then the successful \( y_{k+1} \) is taken as the next step; otherwise, \( y_{k+1} \) is rejected, \( M_k \) is updated as \( M_k \leftarrow \eta M_k \) for some parameter \( \eta > 1 \) and the above test is repeated until a successful \( y_{k+1} \) is found. Such an approach has been used extensively in the literature dealing with composite gradient methods both in the context of convex and nonconvex SCO problems (see for example [1, 2, 3, 16]) and can be efficient particularly for those SCO instances where a sharp upper bound \( M \) on the smallest Lipschitz constant \( M \) of \( \nabla f \) on dom \( h \) is not available.

This paper investigates an ACG variant for solving the N-SCO problem where \( M_k \) is computed as a positive multiple of the average of all observed curvatures up to the previous iteration. As opposed to ACG variants based on the scheme outlined above as well as other ACG variants, AC-ACG always computes a new step regardless of whether \( M_k \) overestimates or underestimates \( C_k \). More specifically, if \( M_k \) overestimates \( C_k \) then a composite step as in (2) is taken; otherwise, \( y_{k+1} \) is set to be a convex combination of \( y_k \) and an auxiliary iterate \( \tilde{x}_{k+1} \), which is obtained by a resolvent evaluation of \( h \). It is worth noting that both of these steps are used in previous ACG variants but only one of them is used at a time. The main result of the paper establishes a convergence rate for AC-ACG. More specifically, it states that \( k \) iterations of the AC-ACG method generates a pair \((y, v)\) satisfying \( v \in \nabla f(y) + \partial h(y) \) and \( \|v\|^2 = O(M_k/k) \) where \( M_k \) is as in the beginning of this paragraph. Since \( M_k \) is usually much smaller than \( \bar{M} \) or even \( M_k \), this convergence rate bound explains the efficiency of AC-ACG to solve both randomly generated and real-world problem instances of (1) used in our numerical experiments. Finally, it is shown that AC-ACG also has similar iteration-complexity as previous ACG variants (e.g., [6, 11, 16, 15]).

**Related works.** The first complexity analysis of an ACG algorithm for solving (1) under the assumption that \( f \) is a nonconvex differentiable function whose gradient is Lipschitz continuous and that \( h \) is a simple lower semicontinuous convex function is established in the novel work [6]. Inspired by [6], many papers have proposed other ACG variants for solving (1) under the aforementioned assumptions (see e.g. [5, 7, 16]) or even under the relaxed assumption that \( h \) is nonconvex (see e.g. [13, 14, 21]). It is worth mentioning that: i) in contrast to [6, 16], other works deal with hybrid-type accelerated methods that resort to unaccelerated composite gradient steps whenever a certain descent property is not satisfied; and ii) in contrast to the methods of [7, 16] that choose \( M_k \) adaptively in a manner similar to that described in the second paragraph in Section 1, the methods in [5, 6, 13, 14, 21] works with a constant sequence \( \{M_k\} \), namely, \( M_k = M \) for some \( M \geq \bar{M} \).

Other approaches towards solving (1) use an inexact proximal point scheme where each prox subproblem is constructed to be (possibly strongly) convex and hence efficiently solvable by a convex ACG variant. Papers [4, 11, 18] propose a descent unaccelerated inexact proximal-type method, which works with a larger prox stepsize and hence has a better outer iteration-complexity than the approaches in the previous paragraph. [15] presents an accelerated inexact proximal point method that performs an accelerated step with a large prox stepsize in every outer iteration and requires a prox subproblem to be approximately solved by an ACG variant in the same way as in the papers [4, 11].
Definitions and notations. The set of real numbers is denoted by \( \mathbb{R} \). The set of non-negative real numbers and the set of positive real numbers are denoted by \( \mathbb{R}_+ \) and \( \mathbb{R}_{++} \), respectively. Let \( \mathbb{R}^n \) denote the standard \( n \)-dimensional Euclidean space with inner product and norm denoted by \( \langle \cdot , \cdot \rangle \) and \( \| \cdot \| \), respectively. The Frobenius inner product and Frobenius norm in \( \mathbb{R}^{m \times n} \) are denoted by \( \langle \cdot , \cdot \rangle_F \) and \( \| \cdot \|_F \), respectively. The cardinality of a finite set \( \mathcal{A} \) is denoted by \( |\mathcal{A}|\).

Let \( \psi : \mathbb{R}^n \to (-\infty, +\infty] \) be given. The effective domain of \( \psi \) is denoted by \( \text{dom} \psi := \{ x \in \mathbb{R}^n : \psi(x) < \infty \} \) and \( \psi \) is proper if \( \text{dom} \psi \neq \emptyset \). Moreover, a proper function \( \psi : \mathbb{R}^n \to (-\infty, +\infty] \) is said to be \( \mu \)-strongly convex for some \( \mu \geq 0 \) if

\[
\psi(\alpha z + (1-\alpha)u) \leq \alpha \psi(z) + (1-\alpha)\psi(u) - \frac{\alpha(1-\alpha)\mu}{2}\|z-u\|^2
\]

for every \( z , u \in \text{dom} \psi \) and \( \alpha \in [0,1] \). If \( \psi \) is differentiable at \( \bar{z} \in \mathbb{R}^n \), then its affine approximation \( \ell_\psi(\cdot; \bar{z}) \) at \( \bar{z} \) is defined as

\[
\ell_\psi(z; \bar{z}) := \psi(\bar{z}) + \langle \nabla \psi(\bar{z}), z - \bar{z} \rangle \quad \forall z \in \mathbb{R}^n.
\]

The subdifferential of \( \psi \) at \( z \in \mathbb{R}^n \) is denoted by \( \partial \psi(z) \). The set of all proper lower semi-continuous convex functions \( \psi : \mathbb{R}^n \to (-\infty, +\infty] \) is denoted by \( \text{Conv} (\mathbb{R}^n) \).

Organization of the paper. Section 2 describes the N-SCO problem and the assumptions made on it. It also presents the AC-ACG method for solving the N-SCO problem and describes the main result of the paper, which establishes a convergence rate bound for AC-ACG in terms of the average of observed curvatures. Section 3 provides the proof of the main result stated in Section 2. Section 4 presents computational results illustrating the efficiency of the AC-ACG method.

2 The AC-ACG method for solving the N-SCO problem

This section presents the main algorithm studied in this paper, namely, an ACG method based on a sequence of average curvatures, and derives a convergence rate for it expressed in terms of this sequence. More specifically, it describes the N-SCO problem and the assumptions made on it, presents the AC-ACG method and states the main result of the paper, i.e. the convergence rate of the AC-ACG method.

The problem of interest in this paper is the N-SCO problem (1), where the following conditions are assumed to hold:

(A1) \( h \in \text{Conv}(\mathbb{R}^n) \);

(A2) \( f \) is a nonconvex differentiable function on \( \text{dom} h \) and there exist scalars \( m \geq 0 \), \( M \geq 0 \) such that for every \( u , u' \in \text{dom} h \),

\[
-\frac{m}{2}\|u - u'\|^2 \leq f(u) - \ell_f(u; u'), \quad \|\nabla f(u) - \nabla f(u')\| \leq M\|u - u'\|; \tag{5}
\]

(A3) the diameter \( D := \sup\{\|u - u'\| : u , u' \in \text{dom} h\} \) is bounded.

Throughout the paper, we let \( \tilde{m} \) (resp., \( \tilde{M} \)) denote the smallest scalar \( m \geq 0 \) (resp., \( M \geq 0 \)) satisfying the first (resp., second) inequality in (5).

We now make some remarks about the above assumptions. First, the set of optimal solutions \( X^* \) is nonempty and compact in view of (A1)-(A3). Second, the second inequality in (5) implies

\[
-\frac{M}{2}\|u - u'\|^2 \leq f(u) - \ell_f(u; u') \leq \frac{M}{2}\|u - u'\|^2 \quad \forall u , u' \in \text{dom} h. \tag{6}
\]
Third, the last remark together with the fact that $f$ is nonconvex on $\text{dom } h$ due to assumption (A2) implies that $0 < \bar{m} \leq \bar{M}$.

A necessary condition for $\hat{y}$ to be a local minimum of (1) is that $0 \in \nabla f(\hat{y}) + \partial h(\hat{y})$, i.e., $\hat{y}$ be a stationary point of (1). More generally, given a tolerance $\hat{\rho} > 0$, a pair $(\hat{y}, \hat{v})$ is called a $\hat{\rho}$-approximate stationary point of (1) if

$$\hat{v} \in \nabla f(\hat{y}) + \partial h(\hat{y}), \quad \|\hat{v}\| \leq \hat{\rho}. \tag{7}$$

The AC-ACG method stated below stops whenever a $\hat{\rho}$-approximate stationary point of (1) is computed.

We now state the AC-ACG method for approximately solving (1).

**Average Curvature - Accelerated Composite Gradient (AC-ACG)**

0. Let a parameter $\gamma \in (0, 1)$, a scalar $M_0 = \gamma \bar{M}$, a tolerance $\hat{\rho} > 0$ and an initial point $y_0 \in \text{dom } h$ be given and set $A_0 = 0$, $x_0 = y_0$, $k = 0$ and

$$\alpha = \frac{0.9}{8} \left( 1 + \frac{1}{0.9\gamma} \right)^{-1}; \tag{8}$$

1. compute

$$a_k = \frac{1 + \sqrt{1 + 4M_k^2A_k^2}}{2M_k}, \quad A_{k+1} = A_k + a_k, \quad \tilde{x}_k = \frac{A_k y_k + a_k x_k}{A_{k+1}}; \tag{9}$$

2. compute

$$x_{k+1} = \arg\min_u \left\{ a_k (\ell_f(u; \tilde{x}_k) + h(u)) + \frac{1}{2} \|u - x_k\|^2 \right\}, \tag{10}$$

$$y_{k+1}^g = \arg\min_u \left\{ \ell_f(u; \tilde{x}_k) + h(u) + \frac{M_k}{2} \|u - \tilde{x}_k\|^2 \right\}, \tag{11}$$

$$v_{k+1} = M_k (\tilde{x}_k - y_{k+1}^g) + \nabla f(y_{k+1}^g) - \nabla f(\tilde{x}_k). \tag{12}$$

if $\|v_{k+1}\| \leq \hat{\rho}$ then output $(\hat{y}, \hat{v}) = (y_{k+1}^g, v_{k+1})$ and **stop**; otherwise, compute

$$C_k = \max \left\{ \frac{2 [f(y_{k+1}^g) - \ell_f(y_{k+1}^g; \tilde{x}_k)]}{\|y_{k+1}^g - \tilde{x}_k\|^2}, \frac{\|\nabla f(y_{k+1}^g) - \nabla f(\tilde{x}_k)\|}{\|y_{k+1}^g - \tilde{x}_k\|} \right\}; \tag{13}$$

$$C_{k}^{avg} = \frac{1}{k+1} \sum_{j=0}^{k} C_j, \tag{14}$$

$$M_{k+1} = \max \left\{ \frac{1}{\alpha} C_{k}^{avg}, \gamma \bar{M} \right\}; \tag{15}$$

3. set

$$y_{k+1} = \begin{cases} y_{k+1}^h := \frac{A_k y_k + a_k x_{k+1}}{A_{k+1}}, & \text{if } C_k > 0.9M_k; \\ y_{k+1}^g, & \text{otherwise} \end{cases} \tag{16}$$

and $k \leftarrow k + 1$, and go to step 1.
We add a few observations about the AC-ACG method. First, the first two identities in (9) imply that
\[ A_{k+1} = M_k u_k^2. \]  
Second, the AC-ACG method evaluates one gradient of \( f \) and exactly two resolvents of \( h \), (i.e., an evaluation of \((I + \lambda \partial h)^{-1}\) for some \( \lambda > 0 \)) per iteration, namely, one in (10) and the other one in (11). Third, Theorem 2.1 below guarantees that AC-ACG always terminates and outputs a \( \hat{c} \)-approximate solution \((\hat{y}, \hat{v})\) (see step 2). Fourth, \( C_k \) is the most recent observed curvature, \( C_{k}^{\text{avg}} \) is the average of all observed curvatures obtained so far and \( M_{k+1} \) is a modified average curvature that will be used in the next iteration to compute \( y_{k+2}^g \). Fifth, the observed curvature \( C_k \) used here is different from the one mentioned in the Introduction (see (4)) since it is the most suitable for our theoretical analysis. However, the computational results reported in Section 4 are based on a variant of AC-ACG that uses \( C_k \) (see (4)) instead of \( C_k \) (see the paragraph following (10)). Sixth, every iteration starts with a triple \((A_k, x_k, y_k)\) and obtains the next one \((A_{k+1}, x_{k+1}, y_{k+1})\) as in (9), (10) and (11). The iterate \( y_{k+1} \) is chosen to be either \( y_{k+1}^g \) obtained in (11) or the convex combination \( y_{k+1}^g \) defined in (13) depending on whether the current curvature \( C_k \) is smaller than or equal to a multiple (e.g., 0.9) of the modified average curvature \( M_k \) or not, respectively. Seventh, in those iterations (called good) in which \( C_k \leq 0.9 M_k \), we have \( y_{k+1} = y_{k+1}^g \), which, together with (11) and the definition of \( C_k \), then implies that (2) hold. Thus, in view of the discussion after (2) in the Introduction, the size of \( M_k \) and the frequency of good iterations should dictate the rate of convergence of the AC-ACG method. In fact, it is proved in the results of Section 3 that the number of good iterations is relatively large and that the overall effect of the bad ones are nicely under control.

We now discuss the likelihood of \( \gamma_{k+1} := M_{k+1}/\bar{M} \) being small. For that, we make the blanket, but quite reasonable assumption, that \( C_{k}^{\text{avg}} \) is small. More specifically, it is assumed that \( C_{k}^{\text{avg}} \) is such that \( M_{k+1} \) in (13) is equal to \( \gamma \bar{M} \), or equivalently,
\[ C_{k}^{\text{avg}} \leq \alpha \gamma \bar{M} = \frac{0.9}{8} \left(1 + \frac{1}{0.9 \gamma} \right)^{-1} \gamma \bar{M}, \]  
which in turn implies that \( C_{k}^{\text{avg}} \leq O(\gamma^2 \bar{M}) \). Under this assumption, we have \( \gamma_{k+1} = \gamma \) and hence the size of \( \gamma_{k+1} \) depends on how small \( \gamma \) is chosen. However, it should be noted that choosing \( \gamma \) too small makes the likelihood of (18) occurring small too. However, it is argued in the paragraph following Lemma 3.5 that assumption (18) can actually be replaced by
\[ C_{k}^{\text{avg}} \leq \frac{0.9}{8} (1 + \bar{\gamma}_k)^{-1} \gamma \bar{M}, \]  
where \( \bar{\gamma}_k \) is a quantity that is expected to behave as \( \Theta(1) \). In view of (19), such behavior actually implies that \( C_{k}^{\text{avg}} \leq O(\gamma \bar{M}) \), which is a less restrictive condition on \( C_{k}^{\text{avg}} \) than the inequality \( C_{k}^{\text{avg}} \leq O(\gamma^2 \bar{M}) \) implied by (18).

We will now comment on the practical aspect of the AC-ACG method. First, AC-ACG requires as input the constant \( \bar{M} \), which is generally difficult to compute. It can be easily shown that the version of AC-ACG with a scalar \( \bar{M} \geq \bar{M} \) replacing \( \bar{M} \) satisfies the same results obtained for the AC-ACG method. Second, we have implemented a variant of AC-ACG where neither \( \gamma \in (0, 1) \) nor \( \bar{M} \) nor any scalar \( M \geq \bar{M} \) is needed as input, namely, it is similar to AC-ACG except that, for some \( \alpha \in (0, 1) \), \( C_k \) and \( M_{k+1} \) are set to
\[ C_k = C_k, \quad M_{k+1} = \frac{1}{\alpha} C_{k}^{\text{avg}}, \]
where \( c_k \) is defined in (4). Note that since this variant does not use \( \gamma \), the restrictive condition (8) is no longer needed. The numerical results reported in Section 4 show that the latter approach performs substantially better than previous ACG approaches as well as the AC-ACG method studied in this paper. However, the theoretical analysis of this variant is an open problem and AC-ACG is the closest method to it for which we could establish a relatively fast convergence rate.

We now state the main result of the paper that describes how fast one of the iterates \( y_1, \ldots, y_k \) approaches the stationary condition \( 0 \in \nabla f(y) + \partial h(y) \). A remarkable feature of its convergence rate bound is that it is expressed in terms of \( M_k \) rather than a scalar \( M \geq \bar{M} \).

**Theorem 2.1.** The following statements hold:

(a) for every \( k \geq 1 \), we have \( v_k \in \nabla f(y_k) + \partial h(y_k) \);

(b) for every \( k \geq 12 \), we have

\[
\min_{1 \leq i \leq k} \|v_i\|^2 \leq O \left( \frac{M_k^2 D^2}{\gamma^2} + \frac{\theta_k \bar{m} M_k D^2}{k} \right)
\]

where

\[
\theta_k := \max \left\{ \frac{M_k}{M_i} : 0 \leq i \leq k \right\} \geq 1.
\]

We now make two remarks about Theorem 2.1. First, the result also leads to a worst-case iteration complexity bound as follows. The assumption that \( \gamma < 1 \) and relations (8) and (15) imply that \( \gamma \leq M_{k+1}/\bar{M} \leq 1/\alpha = O(1/\gamma) \), and hence that \( \theta_k \leq 1/(\alpha \gamma) = O(1/\gamma^2) \). Using these estimates, we then easily see that AC-ACG computes a pair \((y_k, v_k)\) satisfying (7) in

\[
O \left( \frac{M D}{\gamma^{3/2} \hat{\rho}^2} + \frac{\bar{m} M D^2}{\gamma^3 \hat{\rho}^2} + 1 \right)
\]

iterations. Hence, for small values of \( \gamma \), the worst-case iteration complexity of AC-ACG is high but, if \( \gamma \) is viewed as a constant, i.e., \( 1/\gamma = O(1) \), then the above complexity is as good as any other ACG method found in the literature for solving the nonconvex SCO problem as long as the second term in (21) is the dominant one. In particular, in terms of \( \hat{\rho} \) only, its worst-case iteration complexity for solving a nonconvex SCO is \( O(1/\hat{\rho}^2) \), which is identical to that of any other known ACG method (see e.g. [6, 11, 16, 15]).

Second, the dependence of the worst-case iteration complexity (21) on \( \gamma \) is not so good because it is obtained based on the conservative estimate \( M_{k+1}/\bar{M} \leq 1/\alpha \) but, in practice, \( M_{k+1}/\bar{M} \) is substantially smaller than \( 1/\alpha \). We will now examine the convergence rate bound under the assumption that \( M_{k+1}/\bar{M} = \gamma \) for all \( k \geq 0 \), or equivalently, (18) holds. In this case, \( \theta_k = 1 \) and the convergence rate bound easily leads to the inner iteration complexity bound

\[
O \left( \frac{\gamma^{1/2} M D}{\hat{\rho}} + \frac{\gamma \bar{m} M D^2}{\hat{\rho}^2} + 1 \right),
\]

for AC-ACG, which improves as \( \gamma \) decreases. This contrasts with the bound (21), which becomes worse as \( \gamma \) decreases.
3 Proof of Theorem 2.1

This section presents the proof of Theorem 2.1. We start with the following technical result, which assumes that all sequences start with $k = 0$.

**Lemma 3.1.** The following statements hold:
(a) the sequences $\{x_k\}$, $\{y_k\}$, $\{y^g_{k+1}\}$ and $\{\tilde{x}_k\}$ are all contained in $\text{dom } h$;
(b) for every $u \in \text{dom } h$ and $k \geq 0$, we have
\[
A_k \|y_k - \tilde{x}_k\|^2 + a_k \|u - \tilde{x}_k\|^2 \leq \left(\frac{1}{M_k} + a_k\right) D^2;
\]
(c) for every $k \geq 0$, $C_k \leq \bar{M}$ and $F_k \leq \bar{M}$, where
\[
F_k := C(y_{k+1}; \tilde{x}_k);
\]
(d) for every $k \geq 0$, we have
\[
v_{k+1} \in \nabla f(y^g_{k+1}) + \partial h(y^g_{k+1}), \quad \|v_{k+1}\| \leq (M_k + C_k) \|y^g_{k+1} - \tilde{x}_k\|.
\]

**Proof:**
(a) The sequences $\{x_k\}$ and $\{y^g_{k+1}\}$ are contained in $\text{dom } h$ in view of (10), (11) and step 0 of AC-ACG. Hence, using step 0 of AC-ACG again, (16) and the convexity of $\text{dom } h$, we easily see by induction that $\{y_k\}$ and $\{y^b_{k+1}\}$ are contained in $\text{dom } h$. Finally, $\{\tilde{x}_k\} \subset \text{dom } h$ follows from the third identity in (9) and the convexity of $\text{dom } h$.

(b) Using the definition of $\tilde{x}_k$ in (9), (17), Lemma 3.1(a), (A3) and the fact that $A_{k+1} \geq A_k$ in view of the second equality in (9), we obtain
\[
A_k \|y_k - \tilde{x}_k\|^2 + a_k \|u - \tilde{x}_k\|^2 \leq \left(\frac{1}{M_k} + a_k\right) D^2.
\]
(c) The conclusion follows from definitions of $C_k$ in (13) and $F_k$ in (22) and the fact that $\bar{M}$ satisfies both the second inequality in (5) and (13).

(d) The inclusion in (23) follows from the optimality condition of (11) and the definition of $v_{k+1}$ in (12). Moreover, the inequality in (23) follows from definitions of $C_k$ in (13) and $v_{k+1}$, and the triangle inequality.

The next result provides an important recursive formula involving a certain potential function $\eta_k(u)$ and the quantity $\|y_{k+1} - \tilde{x}_k\|$ that will later be related to the residual vector $\|v_{k+1}\|$ (see the proof of Lemma 3.3(a)).

**Lemma 3.2.** For every $k \geq 0$ and $u \in \text{dom } h$, we have
\[
\frac{M_k - F_k}{2} A_k \|y_{k+1} - \tilde{x}_k\|^2 \leq \eta_k(u) - \eta_{k+1}(u) + \frac{1}{2} \left(\frac{1}{M_k} + a_k\right) D^2
\]
where $M_k$ is as in (15) and
\[
\eta_k(u) := A_k (\phi(y_k) - \phi(u)) + \frac{1}{2} \|u - x_k\|^2.
\]
**Proof:** Let \( k \geq 0 \) and \( u \in \text{dom } h \) be given and define \( \gamma_k(u) := \ell_f(u; \bar{x}_k) + h(u) \). Using the fact \( x_{k+1} \) is an optimal solution of (10) and \( \gamma_k \) is a convex function, the second and third identities in (9), and relations (17) and (16), we conclude that

\[
A_k \gamma_k(y_k) + a_k \gamma_k(u) + \frac{1}{2} \| u - x_k \|^2 - \frac{1}{2} \| u - x_{k+1} \|^2 \geq A_k \gamma_k(y_k) + a_k \gamma_k(x_{k+1}) + \frac{1}{2} \| x_{k+1} - x_k \|^2 \\
\geq A_{k+1} \gamma_k(y_{k+1}) + \frac{1}{2} A_k^2 a_k \| y_{k+1} - \bar{x}_k \|^2 \\
= A_{k+1} \left[ \gamma_k(y_{k+1}) + \frac{M_k}{2} \| y_{k+1} - \bar{x}_k \|^2 \right].
\]

Moreover, (16), (11), (22) and the fact that \( \{ y_k^b \} \subset \text{dom } h \) imply that

\[
\gamma_k(y_{k+1}^b) + \frac{M_k}{2} \| y_{k+1}^b - \bar{x}_k \|^2 \geq \gamma_k(y_{k+1}) + \frac{M_k}{2} \| y_{k+1} - \bar{x}_k \|^2 = \phi(y_{k+1}) + \frac{M_k - F_k}{2} \| y_{k+1} - \bar{x}_k \|^2.
\]

Using the above two inequalities, the definition of \( \eta_k(u) \) in (24) and the first inequality in (5), we easily see that

\[
\frac{M_k - F_k}{2} A_{k+1} \| y_{k+1} - \bar{x}_k \|^2 - \eta_k(u) + \eta_{k+1}(u) \leq A_k (\gamma_k(y_k) - \phi(y_k)) + a_k (\gamma_k(u) - \phi(u)) \\
\leq \frac{\bar{m}}{2} (A_k \| y_k - \bar{x}_k \|^2 + a_k \| u - \bar{x}_k \|^2),
\]

which, together with Lemma 3.1(b), then immediately implies the lemma.

For the purpose of stating the next results, we define the set of good and bad iterations as

\[
\mathcal{G} := \{ k \geq 0 : C_k \leq 0.9 M_k \}, \quad \mathcal{B} := \{ k \geq 0 : C_k > 0.9 M_k \},
\]

respectively. The following result specializes the bound derived in Lemma 3.2 to the two exclusive cases in which \( k \in \mathcal{G} \) and \( k \in \mathcal{B} \). More specifically, it derives a controllable bound on the residual vector \( v_{k+1} \) and the potential function difference \( \eta_{k+1}(u) - \eta_k(u) \) in the good iterations and a controllable bound only on \( \eta_{k+1}(u) - \eta_k(u) \) in the bad iterations.

**Lemma 3.3.** The following statements hold for every \( u \in \text{dom } h \) and \( k \geq 0 \):

(a) if \( k \in \mathcal{G} \) then

\[
\frac{A_{k+1}}{72.2 M_k} \| v_{k+1} \|^2 \leq \eta_k(u) - \eta_{k+1}(u) + \frac{\bar{m}}{2} \left( \frac{1}{M_k} + a_k \right) D^2;
\]

(b) if \( k \in \mathcal{B} \) then

\[
0 \leq \eta_k(u) - \eta_{k+1}(u) + \frac{\bar{m}}{2} \left( \frac{1}{M_k} + a_k \right) D^2 + \frac{1 - \gamma}{2 \gamma} D^2.
\]

**Proof:** (a) Let \( k \in \mathcal{G} \) be given and note that (25) and (16) imply that \( 0.9 M_k \geq C_k \) and \( y_{k+1} = y_{k+1}^0 \) where \( y_{k+1}^0 \) is as in (11). Hence, using the inequality in (24), and the definitions of \( C_k \) in (13) and \( F_k \) in (22), we conclude that \( \| v_{k+1} \| \leq 1.9 M_k \| y_{k+1} - \bar{x}_k \| \) and \( F_k \leq C_k \leq 0.9 M_k \). The latter two conclusions and Lemma 3.2 then immediately imply that (26) holds.
Lemma 3.4. For every $u \in \text{dom } h$ and $k \geq 1$, we have

\[
\eta_k(u) - \eta_{k+1}(u) + \frac{\bar{m}}{2} \left( \frac{1}{M_k} + a_k \right) D^2 \geq \frac{(M_k - F_k)}{2} \left\| A_{k+1} y_{k+1}^b - \bar{x}_k \right\|^2_A
\]

\[
= \frac{(M_k - F_k)}{2} \left\| \frac{A_k y_k + a_k x_{k+1}}{A_{k+1}} - \frac{A_k y_k + a_k x_k}{A_{k+1}} \right\|^2_A
\]

\[
= \frac{(M_k - F_k) a_k^2}{2A_{k+1}} \left\| x_{k+1} - x_k \right\|^2
\]

\[
\geq \frac{1}{2} \left( 1 - \frac{1}{n} \right) \left\| x_{k+1} - x_k \right\|^2
\]

and hence that (27) holds in view of Lemma 3.1(a) and (A3).

As a consequence, the next lemma provides the result of summation of inequalities for $k \in G$ and $k \in B$ in Lemma 3.3.

Lemma 3.5. For every $u \in \text{dom } h$ and $k \geq 1$, we have

\[
\left( \frac{1}{36.1} \sum_{i \in G_k} \frac{A_{i+1}}{M_i} \right) \min_{1 \leq i \leq k} \| v_i \|^2 \leq \| u - x_0 \|^2 - 2\eta_k(u) + \bar{m} D^2 \sum_{i=0}^{k-1} \frac{1}{M_i} + \bar{m} D^2 A_k + \frac{1 - \gamma}{\gamma} D^2 |B_k|.
\]

(28)

where $G_k$ and $B_k$ are defined as

\[
G_k = \{ i \in G : i \leq k - 1 \}, \quad B_k := \{ i \in B : i \leq k - 1 \}.
\]

Proof: First, note that

\[
\sum_{i \in G_k} \frac{A_{i+1}}{M_i} \| v_{i+1} \|^2 \geq \left( \sum_{i \in G_k} \frac{A_{i+1}}{M_i} \right) \min_{i \in G_k} \| v_{i+1} \|^2 \geq \left( \sum_{i \in G_k} \frac{A_{i+1}}{M_i} \right) \min_{1 \leq i \leq k} \| v_i \|^2.
\]

The conclusion follows by adding (25) and (27) both with $k = i$ as $i$ varies in $G_k$ and $B_k$, respectively, and using the above inequality, the definition of $\eta_k(u)$ in (24), and the facts that $A_k = A_0 + \sum_{i=0}^{k-1} a_i$ and $A_0 = 0$, which are due to (9) and step 0 of the AC-ACG method, respectively.

Note that the left hand side of (28) is actually zero when $G_k = \emptyset$, and hence (28) is meaningless in this case. The result below, which plays a major role in our analysis, uses for the first time the fact that $M_k$ is chosen as in (15) and shows that $G_k$ is nonempty and well-populated. This fact in turn implies that the term inside the parenthesis in the left hand side of (28) is sufficiently large (see Lemma 3.3 below). The proof of Theorem 2.1 will then follow by combining these observations.

Lemma 3.6. For every $k \geq 1$, $|B_k| \leq k/4 + 1$ where $B_k$ is defined in (29). As a consequence, $|B_k| \leq k/3$ for every $k \geq 12$.

Proof: Let $k \geq 1$ be given. In view of (15) and the definition of $B_k$ in (29), it follows that for every $i \in B_k$,

\[
\alpha \frac{9}{0.9} C_i > \alpha M_i \geq C_i^{\text{avg}},
\]

and hence that

\[
\alpha \frac{9}{0.9} \sum_{i \in B_k} C_i > \sum_{i \in B_k} C_i^{\text{avg}}.
\]

(30)
Using Lemma 3.1(c) and the facts that $C_i > 0.9M_i$ for every $i \in B_k$ and that $M_i \geq \gamma\bar{M}$ (see (15) and step 0 of the AC-ACG method) for every $i \geq 0$, we have
\[ 0.9\gamma\bar{M} \leq C_i \leq \bar{M} \quad i \in B_k. \quad (31) \]

Let $l := |B_k|$ and let $i_1 < \cdots < i_l$ denote the indices in $B_k$. Clearly, in view of (14) and the fact that $i_j \leq k$ for every $j = 1, \ldots, l$, we have
\[ C_{i_1-1}^{\text{avg}} \geq 0, \quad C_{i_2-1}^{\text{avg}} \geq \frac{1}{k}C_{i_1}, \quad \ldots, \quad C_{i_l-1}^{\text{avg}} \geq \frac{1}{k}(C_{i_1} + \cdots + C_{i_{l-1}}). \]

Summing these inequalities, we obtain
\[ \sum_{i \in B_k} C_{i-1}^{\text{avg}} \geq \frac{1}{k} \sum_{j=1}^{l} (l-j)C_{i_j} \geq \frac{1}{k} \sum_{j=1}^{[l/2]} (l-j)C_{i_j} \geq \frac{1}{k} \left\lfloor \frac{l}{2} \right\rfloor \sum_{j=1}^{[l/2]} C_{i_j}. \]

Combining (30) and the last inequality, we then conclude that
\[ \alpha (S_1 + S_2) \geq \frac{1}{k} \left\lfloor \frac{l}{2} \right\rfloor S_1 \]
where
\[ S_1 := \sum_{j=1}^{[l/2]} C_{i_j}, \quad S_2 := \sum_{j=[l/2]+1}^{l} C_{i_j}. \quad (32) \]

Since (31) and the above definitions of $S_1$ and $S_2$ immediately imply that $S_2/S_1 \leq 1/(0.9\gamma)$, we then conclude from the above inequality that
\[ |B_k| = l \leq \left( \frac{2\alpha k}{0.9} \right) \left( 1 + \frac{S_2}{S_1} \right) + 1 \leq \left( \frac{2\alpha k}{0.9} \right) \left( 1 + \frac{1}{0.9\gamma} \right) + 1 \quad (33) \]
and hence that $|B_k| \leq k/4 + 1$ in view of the definition of $\alpha$ in (8). The last conclusion of the lemma follows straightforwardly from the first one.

We now make some remarks about choosing $\alpha$ more aggressively, i.e., larger than the value in (8). Recall from the discussion in the second paragraph following the AC-ACG method that choosing $\alpha$ larger makes the assumption (18) in that discussion more plausible to happen. First, in view of their definition in (32), the quantities $S_1$ and $S_2$ are actually quantities that depend on the iteration index $k$ and hence should have been denoted by $S_k^1$ and $S_k^2$. Second, it follows from the first inequality in (33) that
\[ |B_k| \leq \left( \frac{2\alpha k}{0.9} \right) (1 + \tilde{\gamma}_k) + 1 \]
where $\tilde{\gamma}_k := S_k^2/S_k^1$. Third, we have used in the proof of lemma that $\tilde{\gamma}_k$ is bounded above by $1/(0.9\gamma)$, which is a very conservative bound for this quantity. We strongly believe that in practice it should behave as $O(1)$ (if not for all $k$, then at least for a substantial number of iterations).

Before presenting Lemma 3.8, we first state two technical results about the sequences $\{M_k\}$ and $\{A_k\}$.

**Lemma 3.6.** For every $0 \leq i < k$, we have
\[ M_k \geq \frac{i}{k} M_i. \]
Proof: From the definition of $C_{k}^{avg}$ in (14), for every $i < k$, we have
\[ kC_{k-1}^{avg} - iC_{i-1}^{avg} = C_i + \ldots + C_{k-1} \]
and thus
\[ \frac{C_{k-1}^{avg}}{C_{i-1}^{avg}} = \frac{i}{k} + \frac{C_i + \ldots + C_{k-1}}{kC_{i-1}^{avg}} \geq \frac{i}{k}. \]
The conclusion follows from the above inequality, the definition of $M_k$ in (15) and the fact that $\max\{a, c\} \geq \max\{b, d\}$ for $a, b, c, d \in \mathbb{R}$ such that $a \geq b$ and $c \geq d$.

The following result describes bounds on $A_k$ in terms of the first $k$ elements of the sequence $\{M_i\}$ and also in terms of $M_k$ alone.

Lemma 3.7. Consider the sequences $\{A_k\}$ and $\{M_i\}$ defined in (9) and (15), respectively. For every $k \geq 12$, we have
\[ A_k \leq \left( \sum_{i=0}^{k-1} \frac{1}{\sqrt{M_i}} \right)^2 \leq k \sum_{i=0}^{k-1} \frac{1}{M_i} \leq k^2 \frac{\theta_k}{M_k} \tag{34} \]
and
\[ A_k \geq \frac{1}{4} \left( \sum_{i=0}^{k-1} \frac{1}{\sqrt{M_i}} \right)^2 \geq \frac{k^2}{12M_k} \tag{35} \]
where $\theta_k$ is as in (20).

Proof: We first establish the inequalities in (34). Using the first two identities in (9) and the fact $\sqrt{b_1} + b_2 \leq \sqrt{b_1 + b_2}$ for any $b_1, b_2 \in \mathbb{R}_+$, we conclude that for any $k \geq 0$,
\[ \sqrt{A_{i+1}} = \left( A_i + \frac{1 + \sqrt{1 + 4M_iA_i}}{2M_i} \right)^{1/2} \leq \left( A_i + \frac{1 + \sqrt{M_iA_i}}{M_i} \right)^{1/2} \leq \sqrt{A_i + \frac{1}{M_i}}. \]
Now, the first inequality in (34) follows by summing the above inequality from $i = 0$ to $k - 1$ and using the assumption that $A_0 = 0$. Moreover, the second and third inequalities in (34) follow straightforwardly from the Cauchy-Schwartz inequality and the definition of $\theta_k$ in (20), respectively.

We now establish the inequalities in (35). Using the first two identities in (9), we have
\[ \sqrt{A_{i+1}} = \left( A_i + \frac{1 + \sqrt{1 + 4M_iA_i}}{2M_i} \right)^{1/2} \geq \left( A_i + \frac{1 + 2\sqrt{M_iA_i}}{2M_i} \right)^{1/2} \geq \sqrt{A_i + \frac{1}{2\sqrt{M_i}}}. \]
The first inequality in (35) now follows by summing the above inequality from $i = 0$ to $k - 1$ and using the assumption that $A_0 = 0$. For every $k \geq 12$, we have
\[ \sum_{i=0}^{k-1} \sqrt{i} \geq \int_0^{k-1} \sqrt{x} \, dx = \frac{2}{3}(k - 1)^{3/2} \geq \frac{2}{3} \left( \frac{11}{12} k \right)^{3/2} \geq 0.58k^{3/2}, \]
which, together with Lemma 3.6, then implies that
\[ \sum_{i=0}^{k-1} \frac{1}{\sqrt{M_i}} \geq \frac{1}{\sqrt{kM_k}} \sum_{i=0}^{k-1} \sqrt{i} \geq \frac{0.58k}{\sqrt{M_k}}. \]
The second inequality in (35) now follows immediately from the one above.

The following result provides a lower bound on the term inside the parenthesis of the left hand side of (25).
Lemma 3.8. For every $k \geq 12$, we have
\[ \sum_{i \in G_k} \frac{A_{i+1}}{M_i} \geq \frac{k^3}{3402M_k^2}. \]

Proof: Let $k \geq 12$ be given and define
\[ \tilde{G}_k := \{ i \in G_k : i \geq \lfloor k/3 \rfloor \}, \quad \tilde{B}_k := \{ i \in B_k : i \geq \lfloor k/3 \rfloor \}. \] (36)

Using Lemma 3.6, the facts that $\tilde{G}_k \subset G_k$, $\{A_k\}$ is strictly increasing, and $i/k \geq 2/7$ for any $i \in \tilde{G}_k$ and $k \geq 12$, and inequality (35), we conclude that
\[
\sum_{i \in G_k} \frac{A_{i+1}}{M_i} \geq \sum_{i \in \tilde{G}_k} \frac{iA_{i+1}}{kM_k} \geq \frac{2|\tilde{G}_k| A_{[k/3]+1}}{7M_k} \geq \frac{|\tilde{G}_k|^2}{42M_kM_{[k/3]}} \geq \frac{|\tilde{G}_k|^2}{378M_kM_{[k/3]}} \geq \frac{|\tilde{G}_k|^2}{378M_kM_{[k/3]}}.
\] (37)

On the other hand, Lemma 3.6 with $i = \lfloor k/3 \rfloor$ implies that
\[ M_k \geq \frac{[k/3]}{k} M_{[k/3]} \geq \frac{1}{3} M_{[k/3]}. \]
Moreover, the definition of $G_k$ in (36), the fact that $\tilde{B}_k \subset B_k$ and Lemma 3.5 imply that
\[ |\tilde{G}_k| = k - \lfloor k/3 \rfloor - |\tilde{B}_k| \geq k - \lfloor k/3 \rfloor - |B_k| \geq k/3. \]

The conclusion of the lemma now follows by combining (37) with the last two observations. \hfill \blacksquare

4 Numerical results

This section presents computational results showing the performance of the AC-ACG method on a collection of nonconvex optimization problems that are either in the form of or can be easily
reformulated into (1). All the computational results were obtained using MATLAB R2017b on a
MacBook Pro with a quad-core Intel Core i7 processor and 16 GB of memory.

We compare AC-ACG with two other methods: (i) the AG method proposed in [6], which
uses the Lipschitz gradient constant (i.e. \( M \) in the second inequality in (5)) and performs two
resolvent evaluations in each iteration; and (ii) the ADAP-NC-FISTA method proposed in [16],
which works with any input pair of positive scalars \((m, M)\) (see (5)) and uses a backtracking
procedure on observed curvatures. We observe that the latter two methods are natural extensions
of ACG variants for solving convex programs to the context of N-SCO problems.

The following five subsections report computational results on the following classes of non-
convex optimization problems: (a) support vector machine (SVM, Subsection 4.1); (b) quadratic
programming (Subsection 4.2); (c) sparse PCA (Subsection 4.3); (d) matrix completion (Subsec-
tion 4.4); and (e) nonnegative matrix factorization (Subsection 4.5). Note that nonnegative matrix
factorization and sparse PCA are problems for which \( \text{dom} h \) is unbounded.

We will now provide some details of our implementation of the three methods considered in our
numerical benchmark. All three methods terminate with a pair \((z, v)\) satisfying
\[
v \in \nabla f(z) + \partial h(z), \quad \frac{\|v\|}{\|\nabla f(z_0)\| + 1} \leq \hat{\rho},
\]
where \( \hat{\rho} = 5 \times 10^{-4} \) in the matrix completion problem and \( \hat{\rho} = 10^{-7} \) in all the other problems.

AG was implemented as described in Algorithm 1 of [6] with sequences \(\{\alpha_k\}, \{\beta_k\}\) and \(\{\lambda_k\}\)
chosen as \(\alpha_k = 2/(k + 1), \beta_k = 0.99/M\) and \(\lambda_k = k\beta_k/2\) for \(k \geq 1\), where \(M\) will be specified
in each problem. We note that the choice \(\beta_k = 0.99/M\) used in our implementation differs from
the one suggested in [6], namely, \(\beta_k = 0.5/M\) (see (2.27) of [6]), and consistently improves the
practical performance of AG. ADAP-NC-FISTA uses a triple \((\lambda_0, \xi_0, \theta)\) as input and its choice,
which depends on the problem under consideration, is specified inside the five subsections below.
In regards to AC-ACG, we have implemented its variant described in the third paragraph following
the AC-ACG method. Our implementation of this variant sets \(M_0 = 0.01M\) and \(\alpha\) to values, which
depend on the problem under consideration and are specified in the subsections below.

4.1 Support Vector Machine

This subsection presents the performance of AC-ACG for solving a support vector machine problem
[7]. Given data points \(\{(x_i, y_i)\}_{i=1}^p\), where \(x_i \in \mathbb{R}^n\) is a feature vector and \(y_i \in \{-1, 1\}\) denotes
the corresponding label, we consider the SVM problem defined as
\[
\min_{z \in \mathbb{R}^n} \frac{1}{p} \sum_{i=1}^p \ell(x_i, y_i; z) + \frac{\lambda}{2} \|z\|^2 + I_{B_r}(z) \tag{38}
\]
for some \(\lambda, r > 0\), where \(\ell(x_i, y_i; \cdot) = 1 - \tanh(y_i(x_i, \cdot))\) is a nonconvex sigmoid loss function
and \(I_{B_r}(\cdot)\) is the indicator function of the ball \(B_r := \{z \in \mathbb{R}^n : \|z\| \leq r\}\), i.e. \(I_{B_r}(z) = 0\) inside of
\(B_r\) and \(I_{B_r}(z) = \infty\) outside of \(B_r\). The SVM problem (38) is an instance of N-SCO problems (11)
where
\[
f(z) = \frac{1}{p} \sum_{i=1}^p \ell(x_i, y_i; z) + \frac{\lambda}{2} \|z\|^2, \quad h(z) = I_{B_r}(z).
\]
Clearly, \(f\) is differentiable everywhere and its gradient is \(M\)-Lipschitz continuous where
\[
M = \frac{1}{p} \sum_{i=1}^p L_i + \lambda, \quad L_i = \frac{4\sqrt{3}}{9} \|x_i\|^2 \quad \forall i = 1, \ldots, p. \tag{39}
\]
We generate synthetic data sets as follows: for each data point \((x_i, y_i), x_i\) is drawn from the uniform distribution on \([0, 1]^n\) and is sparse with 5% nonzero components, and \(y_i = \text{sign}(\langle \tilde{z}, x_i \rangle)\) for some \(\tilde{z} \in B_r\). We consider four different problem sizes \((n, p)\), i.e. \((1000, 500)\), \((2000, 1000)\), \((3000, 1000)\) and \((4000, 500)\). The corresponding \(M\) is listed in Table 1. We set \(\lambda = 1/p\) and \(r = 50\).

We start all three methods from the same initial point \(z_0\) that is chosen randomly from the uniform distribution within the ball \(B_r\). The input triple of ADAP-NC-FISTA is set to \((\lambda_0, \xi_0, \theta) = (1, 1, 1.25)\) and the parameter of AC-ACG is set to \(\alpha = 0.5\).

Numerical results of all three methods are summarized in Table 1. Specifically, the first column gives the value of \(M\) computed according to [39], the second column provides the function values of \([35]\) at the last iteration, the third to fifth (resp., sixth to eighth) columns provide numbers of iterations (resp., running times) for the three methods, the ninth to tenth columns specify the largest observed curvature in ADAP-NC-FISTA and the average of all observed curvatures in AC-ACG, respectively, and the eleventh column gives the percentage of good iterations in view of \([25]\) among all iterations of AC-ACG. The bold numbers highlight the method that has the best performance in an instance of the problem. Note that all three methods obtain the same objective function value for each instance.

| \(M\) | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|------|----------------|-----------------|-----------------|-----------|------|
|      | AG  | ADAP | AC | AG  | ADAP | AC | ADAP | AC | AC |
| 13   | 1.62E-1 | 37384 | 4475 | 546 | 639 | 173 | 14 | 0.04 | 0.05 | 68% |
| 25   | 9.98E-2 | 112562 | 9231 | 1131 | 4419 | 967 | 81 | 0.05 | 0.06 | 65% |
| 38   | 7.96E-2 | 155503 | 40212 | 1032 | 12636 | 7346 | 101 | 0.08 | 0.07 | 63% |
| 50   | 6.31E-2 | 79752 | 4564 | 615 | 4406 | 538 | 54 | 0.12 | 0.07 | 71% |

Table 1: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

### 4.2 Quadratic Programming

This subsection discusses the performance of the AC-ACG method for solving a class of quadratic programming problems

\[
\min \left\{ f(Z) := -\frac{\alpha_1}{2} \|DB(Z)\|^2 + \frac{\alpha_2}{2} \|A(Z) - b\|^2 : Z \in P_n \right\},
\]

where \((\alpha_1, \alpha_2) \in \mathbb{R}^{2}_{++}\), linear operators \(A : S^n_+ \to \mathbb{R}^l\) and \(B : S^n_+ \to \mathbb{R}^p\) are defined by

\[
[A(Z)]_i = \langle A_i, Z \rangle_F \text{ for } A_i \in \mathbb{R}^{n \times n} \text{ and } 1 \leq i \leq l,
\]

\[
[B(Z)]_j = \langle B_j, Z \rangle_F \text{ for } B_j \in \mathbb{R}^{n \times n} \text{ and } 1 \leq j \leq p,
\]

\(b \in \mathbb{R}^l\) is a vector such that its entries are sampled from the uniform distribution \(U(0, 1)\), \(D \in \mathbb{R}^{n \times n}\) is a diagonal matrix whose nonzero entries are sampled from the discrete uniform distribution \(U\{1, 1000\}\), and \(P_n := \{Z \in S^n_+ : \text{tr}(Z) = 1\}\) denotes the spectraplex.

The quadratic programming problem (40) is an instance of N-SCO problems \([11]\) where \(h(Z) = I_{P_n}(Z)\) is an indicator function of the spectraplex \(P_n\). For chosen curvature pairs \((M, m) \in \mathbb{R}^{2}_{++}\), the scalars \(\alpha_1\) and \(\alpha_2\) are set such that \(M = \lambda_{\max}(\nabla^2 f)\) and \(-m = \lambda_{\min}(\nabla^2 f)\) in which \(\lambda_{\max}(\cdot)\) and \(\lambda_{\min}(\cdot)\) denote the largest and smallest eigenvalue functions, respectively.

We start all three methods with the centroid of \(P_n\) being the initial point \(Z_0\), namely \(Z_0 = I_n/n\) where \(I_n\) is an \(n \times n\) identity matrix. We set \(n = 1000\) in the problem. The input triple \((\lambda_0, \xi_0, \theta)\) is set to \((1, 1000, 1.25)\) in ADAP-NC-FISTA and the parameter \(\alpha\) is set to 1 in AC-ACG.
Numerical results of all three methods are summarized in Tables 2-4. The explanation of their columns excluding the first one is the same as those of Table 1 (see the paragraph preceding Table 1). Their first columns differ from that of Table 1 in that they list not only $M$ but also a scalar $m$ that is used to generate the corresponding instance as explained in the first paragraph of this subsection. Tables 2-4 differ in dimensions $l$ and $p$, and the sparsity of $A_i$ and $B_j$ for $1 \leq i \leq l$ and $1 \leq j \leq p$. The bold numbers highlight the method that has the best performance in an instance of the problem. Note that all three methods obtain the same objective function value for each instance.

Table 2: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

In Table 2, the sparsity of $A_i, B_j$ is set to 2.5% and the dimensions are set to $(l, p) = (50, 200)$.

| $(M,m)$ | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|---------|----------------|-----------------|-----------------|-----------|------|
| $(10^6, 10^6)$ | -2.06E5 | 46 14 8 | 1.56 0.44 **0.36** | **1.17E4** **3.04E4** | 88% |
| $(10^6, 10^5)$ | -3.65E3 | 3089 2209 **883** | 130.33 84.90 **36.83** | 6.26E5 **8.84E4** | 64% |
| $(10^6, 10^4)$ | -1.74E2 | 5400 2595 **1760** | 188.31 100.42 **73.08** | 6.72E5 **1.24E5** | 65% |
| $(10^6, 10^3)$ | 2.05E1 | 4621 2641 **1508** | 176.06 104.97 **68.03** | 6.76E5 **1.20E5** | 68% |
| $(10^6, 10^2)$ | 3.67E1 | 4476 2643 **1472** | 175.77 113.24 **65.44** | 6.77E5 **1.20E5** | 68% |
| $(10^6, 10)$ | 3.82E1 | 4461 2644 **1485** | 170.67 109.37 **66.13** | 6.77E5 **1.18E5** | 70% |

Table 3: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

In Table 3, the sparsity of $A_i, B_j$ is set to 0.5% and the dimensions are set to $(l, p) = (50, 400)$.

| $(M,m)$ | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|---------|----------------|-----------------|-----------------|-----------|------|
| $(10^6, 10^6)$ | -1.78E5 | 44 10 8 | 4.36 **0.88** 1.03 | **1.49E4** **3.22E4** | 88% |
| $(10^6, 10^5)$ | -4.41E3 | 1411 534 **131** | 134.39 54.32 **15.92** | 1.88E5 **1.98E4** | 76% |
| $(10^6, 10^4)$ | 2.12E3 | 1963 872 **237** | 195.25 84.33 **28.42** | 2.48E5 **2.50E4** | 72% |
| $(10^6, 10^3)$ | 2.54E3 | 1935 904 **245** | 192.94 90.21 **30.43** | 2.54E5 **2.55E4** | 69% |
| $(10^6, 10^2)$ | 2.58E3 | 1934 907 **242** | 189.45 91.06 **29.31** | 2.54E5 **2.55E4** | 71% |
| $(10^6, 10)$ | 2.59E3 | 1934 908 **246** | 194.31 88.03 **29.39** | 2.54E5 **2.51E4** | 68% |

Table 4: Numerical results for AG, ADAP-NC-FISTA and AC-ACG method

In Table 4, the sparsity of $A_i, B_j$ is set to 0.1% and the dimensions are set to $(l, p) = (50, 800)$.

| $(M,m)$ | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|---------|----------------|-----------------|-----------------|-----------|------|
| $(10^6, 10^6)$ | -7.55E4 | 69 10 8 | 21.99 **2.69** 3.50 | **6.25E2** **1.70E4** | 88% |
| $(10^6, 10^5)$ | 1.02E3 | 277 26 7 | 118.97 9.31 **3.09** | 1.84E4 **2.84E3** | 86% |
| $(10^6, 10^4)$ | 8.21E3 | 491 61 **11** | 173.33 20.71 **4.62** | 3.90E4 **3.89E3** | 91% |
| $(10^6, 10^3)$ | 8.86E3 | 531 69 **12** | 168.91 21.73 **6.78** | 4.13E4 **3.73E3** | 92% |
| $(10^6, 10^2)$ | 8.93E3 | 535 70 **12** | 171.76 21.75 **5.50** | 4.15E4 **3.75E3** | 92% |
| $(10^6, 10)$ | 8.93E3 | 536 71 **12** | 172.08 22.12 **5.23** | 4.15E4 **3.75E3** | 92% |
4.3 Sparse PCA

This subsection considers a penalized version of the sparse PCA problem studied in [9], namely,

\[
\min_{X,Y \in \mathbb{R}^{p \times p}} -\langle \hat{\Sigma}, X \rangle_F + \frac{\mu}{2} \|X\|_F^2 + Q_{\lambda,b}(Y) + \lambda\|Y\|_1 + \frac{\beta}{2} \|X - Y\|_F^2 + I_{F^r}(X),
\]

(41)

where the dataset consists of an empirical covariance matrix \(\hat{\Sigma} \in \mathbb{R}^{p \times p}\), two regularization parameters \(\mu > 0\) and \(\lambda > 0\), a penalty parameter \(\beta > 0\) and two scalars \(b > 0\) and \(r \in \mathbb{N}_+\). Moreover, \(\|\cdot\|_1\) and \(Q_{\lambda,b}(\cdot)\) are the matrix 1-norm and a decomposable nonconvex penalty function defined as

\[
\|Y\|_1 := \sum_{i,j=1}^p |Y_{ij}|, \quad Q_{\lambda,b}(X) := \sum_{i,j=1}^p q_{\lambda,b}(X_{ij})
\]

where

\[
q_{\lambda,b}(t) := \begin{cases} 
-\frac{t^2}{b^2}, & \text{if } |t| \leq b\lambda; \\
b\lambda^2 - \lambda|t|, & \text{otherwise}
\end{cases}
\]

and \(I_{F^r}(\cdot)\) is the indicator function of the Fantope

\[
F^r := \{X \in S^n : 0 \preceq X \preceq I \text{ and } \text{tr}(X) = r\}.
\]

Clearly, problem (41) is an instance of the N-SCO problem (1) where

\[
f(X,Y) = -\langle \hat{\Sigma}, X \rangle_F + \frac{\mu}{2} \|X\|_F^2 + Q_{\lambda,b}(Y) + \frac{\beta}{2} \|X - Y\|_F^2, \quad h(X,Y) = I_{F^r}(X) + \lambda\|Y\|_1.
\]

Moreover, it is easy to see that the pair

\[
(M,m) = \left(\max \left\{\mu + 2\beta, \frac{1}{b}\right\}, \frac{1}{b}\right)
\]

(42)

satisfies assumption (A2).

We discuss how synthetic datasets are generated. Let \(\Sigma \in \mathbb{R}^{p \times p}\) be an unknown covariance matrix and \(X^*\) be the projection matrix onto the \(r\)-dimensional principal subspace of \(\Sigma\). In the sparse PCA problem, we seek an \(s\)-sparse approximation \(X\) of \(X^*\) in the sense that \(\|\text{diag}(X)\|_0 \leq s\), where \(s \in \mathbb{N}_+\). We generate four datasets by designing four covariance matrices \(\Sigma\) as described in [9] and list all required parameters in Table 5. For each covariance matrices \(\Sigma\), we sample \(n = 80\) i.i.d. observations from the normal distribution \(\mathcal{N}(0, \Sigma)\) and then calculate the sample covariance matrix \(\hat{\Sigma}\).

| dataset | \(s\) | \(r\) | \(p\) | \(b\) | \(\beta\) | \(\mu\) | \(\lambda\) |
|---------|------|------|------|------|--------|--------|--------|
| I       | 5    | 1    | 128  | 3    | 0.33   | 33.33  | 0.1    |
| II      | 10   | 5    | 128  | 3    | 0.33   | 3.33   | 0.25   |
| III     | 5    | 1    | 128  | 0.2  | 100    | 0      | 0.1    |
| IV      | 10   | 5    | 800  | 3    | 0.33   | 3.33   | 0.25   |

Table 5: Synthetic datasets for the sparse PCA problem

All three methods are started from the same initial point \((X_0,Y_0)\) computed as follows. For dataset I, II and IV, the initial point \((X_0,Y_0)\) is set to \((\mathbf{1}^{p \times p}, \mathbf{1}^{p \times p})\). For dataset III, the initial point
\((X_0, Y_0)\) is set to \((\mathbf{0}^{p \times p}, \mathbf{0}^{p \times p})\). The input triple of ADAP-NC-FISTA is set to \((\lambda_0, \xi_0, \theta) = (1, 1, 1.25)\) and the parameter of AC-ACG is set to \(\alpha = 0.8\).

Numerical results of all three methods are summarized in Table 6, whose format is similar to that of Table 1 with the exception that the second to fourth columns provide the function values of (41) at the last iteration for all three methods. Note that the first column gives the value of \(M\) computed according to (42). The bold numbers highlight the method that has the best performance in an instance of the problem.

| \(M\) | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|-------|----------------|-----------------|------------------|-----------|------|
| AG    | -107           | 105             | 0.38             | 33.7      | 8.9  | 74%  |
| ADAP  | -107           | 78              | 0.33             | 0.22      | 3.7  | 26%  |
| AC    | -107           | 31              | 0.14             | 165.9     | 24.33| 58%  |
| AG    | -420           | 119             | 1.35             | 2.6       |      |      |
| ADAP  | -420           | 64              | 0.67             |           |      |      |
| AC    | -420           | 48              | 0.22             |           |      |      |
| AG    | -123           | 200             | 36.47            | 27%       |      |      |
| ADAP  | -123           | 1000*           | 14.33            |           |      |      |
| AC    | -123           | 40              | 3.7              |           |      |      |
| AG    | -425           | 200             | 2.5E3            | 45%       |      |      |
| ADAP  | -425           | 40              | 1.4E3            |           |      |      |
| AC    | -425           | 44              | 1.7              |           |      |      |

Table 6: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

4.4 Matrix Completion

This subsection focuses on a constrained version of the nonconvex low-rank matrix completion problem considered in [20].

Before stating the problem, we first give a few definitions. Let \(\Omega\) be a subset of \(\{1, \ldots, l\} \times \{1, \ldots, n\}\) and let \(\Pi_\Omega\) denote the linear operator that maps a matrix \(A\) to the matrix whose entries in \(\Omega\) have the same values of the corresponding ones in \(A\) and whose entries outside of \(\Omega\) are all zero. Also, for given parameters \(\beta > 0\) and \(\theta > 0\), let \(p : \mathbb{R} \to \mathbb{R}_+\) denote the log-sum penalty defined as

\[
p(t) = p_{\beta, \theta}(t) := \beta \log \left(1 + \frac{|t|}{\theta}\right).
\]

The constrained version of the nonconvex low-rank matrix completion problem considered in this subsection is

\[
\min_{Z \in \mathbb{R}^{l \times n}} \left\{ \frac{1}{2} \|\Pi_\Omega(Z - O)\|_F^2 + \mu \sum_{i=1}^{r} p(\sigma_i(Z)) : Z \in \mathcal{B}(R) \right\}
\]

(43)

where \(R\) is a positive scalar, \(\mathcal{B}(R) := \{Z \in \mathbb{R}^{l \times n} : \|Z\|_F \leq R\}\), \(O \in \mathbb{R}^{\Omega}\) is an incomplete observed matrix, \(\mu > 0\) is a parameter, \(r := \min\{l, n\}\) and \(\sigma_i(Z)\) is the \(i\)-th singular value of \(Z\). The above problem differs from the one considered in [20] in that it adds the constraint \(\|Z\|_F \leq R\) to the latter one.

The matrix completion problem in (43) is equivalent to

\[
\min_{Z \in \mathbb{R}^{l \times n}} f(Z) + h(Z),
\]

(44)

where

\[
f(Z) = \frac{1}{2} \|\Pi_\Omega(Z - O)\|_F^2 + \mu \sum_{i=1}^{r} [p(\sigma_i(Z)) - p_0 \sigma_i(Z)],
\]

\[
h(Z) = \mu p_0 \|Z\|_* + I_{\mathcal{B}(R)}(Z), \quad p_0 = p'(0) = \frac{\beta}{\theta}
\]
and \( \| \cdot \|_* \) denotes the nuclear norm defined as \( \| \cdot \|_* := \sum_{i=1}^{r} \sigma_i(\cdot) \). It is proved in [20] that the second term in the definition of \( f \), i.e., \( \mu \sum_{r=1}^{r} [p(\sigma_r(\cdot)) - p_0\sigma_r(\cdot)] \), is concave and \( 2\mu\tau \)-smooth where \( \tau = \beta/\theta^2 \), so \( f \) is nonconvex and smooth. Since \( h \) is convex and nonsmooth, the problem in (44) falls into the general class of N-SCO problems (1). It is easy to see that the pair
\[
(M, m) = (\max\{1, 2\mu\tau\}, 2\mu\tau)
\]
satisfies assumption (A2).

We use the MovieLens dataset\(^1\) to obtain the observed index set \( \Omega \) and the incomplete observed matrix \( \mathcal{O} \). The dataset includes a sparse matrix with 100,000 ratings of \( \{1,2,3,4,5\} \) from 943 users on 1682 movies, namely \( l = 943 \) and \( n = 1682 \). The radius \( R \) is chosen as the Frobenius norm of the matrix of size \( 943 \times 1682 \) containing the same entries as \( \mathcal{O} \) in \( \Omega \) and 5 in the entries outside of \( \Omega \).

We start all three methods from the same initial point \( Z_0 \) that is sampled from the standard Gaussian distribution. The input triple of ADAP-NC-FISTA is set to \( (\lambda_0, \xi_0, \theta) = (1, 1, 1.25) \) and the parameter of AC-ACG is set to \( \alpha = 0.5 \).

Numerical results for all three methods are summarized in Table 7 whose format is the same as that of Table 6 with the exception that the scalar \( M \) is computed according to (45). The bold numbers highlight the method that has the best performance in an instance of the problem.

### Table 7: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

| M        | Function Value | Iteration Count | Running Time (s) | Curvature | Good |
|----------|----------------|-----------------|-----------------|-----------|------|
| AG       | ADAP          | AC              | AG              | ADAP      | AC   |
| 4.4      | 2.26E3        | 2.62E3          | 2.29E3          | 3856      | 1675 | 765 | 4568 | 1339 | 923 | 0.98 | 0.31 | 96% |
| 8.9      | 3.89E3        | 4.20E3          | 3.88E3          | 9158      | 1795 | 968 | 10251 | 1471 | 1173 | 0.95 | 0.28 | 94% |
| 20       | 4.28E3        | 4.58E3          | 4.27E3          | 22902     | 1843 | 1079 | 29274 | 1441 | 1236 | 0.93 | 0.25 | 91% |
| 30       | 5.97E3        | 6.29E3          | 5.97E3          | 37032     | 1651 | 1085 | 41673 | 1278 | 1263 | 0.89 | 0.23 | 89% |

#### 4.5 Nonnegative Matrix Factorization

This subsection focuses on the following nonnegative matrix factorization (NMF) problem
\[
\min \left\{ f(X, Y) := \frac{1}{2}\|A - XY\|_F^2 : X \geq 0, Y \geq 0 \right\},
\]
where \( A \in \mathbb{R}^{n \times l} \), \( X \in \mathbb{R}^{n \times p} \) and \( Y \in \mathbb{R}^{p \times l} \), which have been thoroughly studied in the literature (see e.g. [8, 12]).

This subsection reports the efficiency of directly using the AG, ADAP-NC-FISTA and AC-ACG to solve (46) without making use of its two-block structure. We use the facial image dataset provided by AT&T Laboratories Cambridge\(^2\) to construct the matrix \( A \). More specifically, this dataset consists of 400 images, and each of those contains \( 92 \times 112 \) pixels with 256 gray levels per pixel. It results in an \( n \times l = 10,304 \times 400 \) matrix \( A \) whose columns are the vectorized images. The dimension \( p \) is set to 20.

We start all three methods from the same initial point \( (X_0, Y_0) = \left( \frac{1}{n \times p}/(np), \frac{1}{p \times l}/(pl) \right) \), where \( 1_{n \times p} \) and \( 1_{p \times l} \) are matrices of all ones of sizes \( n \times p \) and \( p \times l \), respectively. We estimate \( M \) in [5] as

---

1. http://grouplens.org/datasets/movielens/
2. https://www.cl.cam.ac.uk/research/dtg/attarchive/facedatabase.html
\[ M = 100 \times C ((X_0, Y_0), (0, 0)) \] where \( C(\cdot, \cdot) \) is defined in (3). The input triple of ADAP-NC-FISTA is set to \((\lambda_0, \xi_0, \theta) = (1, 100, 1.25)\) and the parameter of AC-ACG is set to \(\alpha = 0.7\).

Numerical results for all three methods are summarized in Table 8. The bold numbers highlight the method that has the best performance in the problem. Note that all three methods obtain the same objective function value.

| Method | Function Value | Iteration Count | Running time(s) |
|--------|---------------|-----------------|-----------------|
| AG     | 2.80E9        | 786             | 73.03           |
| ADAP   | 2.80E9        | 87              | 12.47           |
| AC     | 2.80E9        | 37              | 4.70            |

Table 8: Numerical results for AG, ADAP-NC-FISTA and AC-ACG

References

[1] A. Beck. *First-order methods in optimization*, volume 25. SIAM, 2017.

[2] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.

[3] A. Beck and M. Teboulle. Gradient-based algorithms with applications to signal recovery. *Convex optimization in signal processing and communications*, pages 42–88, 2009.

[4] Y. Carmon, J. C. Duchi, O. Hinder, and A. Sidford. Accelerated methods for nonconvex optimization. *SIAM Journal on Optimization*, 28(2):1751–1772, 2018.

[5] D. Drusvyatskiy and C. Paquette. Efficiency of minimizing compositions of convex functions and smooth maps. *Mathematical Programming*, pages 1–56, 2018.

[6] S. Ghadimi and G. Lan. Accelerated gradient methods for nonconvex nonlinear and stochastic programming. *Math. Programming*, 156:59–99, 2016.

[7] S. Ghadimi, G. Lan, and H. Zhang. Generalized uniformly optimal methods for nonlinear programming. *Journal of Scientific Computing*, 79(3):1854–1881, 2019.

[8] N. Gillis. The why and how of nonnegative matrix factorization. *Regularization, Optimization, Kernels, and Support Vector Machines*, 12(257):257–291, 2014.

[9] Q. Gu, Z. Wang, and H. Liu. Sparse pca with oracle property. In *Advances in neural information processing systems*, pages 1529–1537, 2014.

[10] O. Güler. New proximal point algorithms for convex minimization. *SIAM Journal on Optimization*, 2(4):649–664, 1992.

[11] W. Kong, J. G. Melo, and R. D. C. Monteiro. Complexity of a quadratic penalty accelerated inexact proximal point method for solving linearly constrained nonconvex composite problem. *Available on arXiv:1802.03504*, 2018.

[12] D. D. Lee and H. S. Seung. Learning the parts of objects by non-negative matrix factorization. *Nature*, 401(6755):788, 1999.
[13] H. Li and Z. Lin. Accelerated proximal gradient methods for nonconvex programming. *Adv. Neural Inf. Process. Syst.*, 28:379–387, 2015.

[14] Q. Li, Y. Zhou, Y. Liang, and P. K. Varshney. Convergence analysis of proximal gradient with momentum for nonconvex optimization. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 2111–2119, 2017.

[15] J. Liang and R. D. C. Monteiro. A doubly accelerated inexact proximal point method for nonconvex composite optimization problems. *Available on arXiv:1811.11378*, 2018.

[16] J. Liang, R. D. C. Monteiro, and C.-K. Sim. A FISTA-type accelerated gradient algorithm for solving smooth nonconvex composite optimization problems. *Available on arXiv:1905.07010*, 2019.

[17] R. D. C. Monteiro and B. F. Svaiter. An accelerated hybrid proximal extragradient method for convex optimization and its implications to second-order methods. *SIAM Journal on Optimization*, 23(2):1092–1125, 2013.

[18] Courtney Paquette, Hongzhou Lin, Dmitriy Drusvyatskiy, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for gradient-based non-convex optimization. *Available on arXiv:1703.10993*, 2017.

[19] S. Salzo and S. Villa. Inexact and accelerated proximal point algorithms. *Journal of Convex Analysis*, 19(4):1167–1192, 2012.

[20] Q. Yao and J. T. Kwok. Efficient learning with a family of nonconvex regularizers by redistributing nonconvexity. *Journal of Machine Learning Research*, 18:179–1, 2017.

[21] Q. Yao, J. T. Kwok, F. Gao, W. Chen, and T.-Y. Liu. Efficient inexact proximal gradient algorithm for nonconvex problems. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence*, pages 3308–3314. IJCAI, 2017.