ANOTHER ORDERING OF THE TEN CARDINAL CHARACTERISTICS IN CICHÓN’S DIAGRAM

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Dedicated to the memory of Bohuslav Balcar (1943–2017)

ABSTRACT. It is consistent that
\[ \aleph_1 < \text{add}(\mathcal{N}) < \text{add}(\mathcal{M}) = b < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = 2^\aleph_0. \]

Assuming four strongly compact cardinals, it is consistent that
\[ \aleph_1 < \text{add}(\mathcal{N}) < \text{add}(\mathcal{M}) = b < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = b < \text{cof}(\mathcal{N}) < 2^\aleph_0. \]

INTRODUCTION

We assume that the reader is familiar with basic properties of Amoeba, Hechler, random and Cohen forcing, and with the cardinal characteristics in Cichoń’s diagram, given in Figure 1: An arrow between \( \mathfrak{x} \) and \( \mathfrak{y} \) indicates that ZFC proves \( \mathfrak{x} \leq \mathfrak{y} \). Moreover, \( \max(b, \text{non}(\mathcal{M})) = \text{cof}(\mathcal{M}) \) and \( \min(b, \text{cov}(\mathcal{M})) = \text{add}(\mathcal{M}) \). These (in)equalities are the only one provable. More precisely, all assignments of the values \( \aleph_1 \) and \( \aleph_2 \) to the characteristics in Cichoń’s diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, ch. 7].)

In the following, we will only deal with the ten “independent” characteristics listed in Figure 2 (they determine \( \text{cof}(\mathcal{M}) \) and \( \text{add}(\mathcal{M}) \)).

**Figure 1.** Cichoń’s diagram
Regarding the left hand side, it was shown in [GMS16] that consistently

\begin{align*}
\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \text{add}(\mathcal{M}) = b < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = 2^{\aleph_0}.
\end{align*}

(This corresponds to \(\lambda_1\) to \(\lambda_5\) in Figure 3.) The proof is repeated in [GKS], in a slightly different form which is more convenient for our purpose. Let us call this construction the “old construction”.

In this paper, building on [She00], we give a construction to get a different order for these characteristics, where we swap \(\text{cov}(\mathcal{N})\) and \(b\):

\begin{align*}
\aleph_1 < \text{add}(\mathcal{N}) < b < \text{cov}(\mathcal{N}) < \text{add}(\mathcal{M}) = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = 2^{\aleph_0}.
\end{align*}

(This corresponds to \(\lambda_1\) to \(\lambda_4\) in Figure 4.)

This construction is more complicated than the old one. Let us briefly describe the reason: In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation \(R\). E.g., we use the “eventually different” relation \(R_4 \subseteq \omega^\omega \times \omega^\omega\) for \(\text{non}(\mathcal{M})\). We can then show that the characteristic remains “small” (i.e., is at most the intended value \(\lambda\) in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than \(\lambda\)) or are “\(R\)-good”. However, \(b\) (with the “eventually dominating” relation \(R_2 \subseteq \omega^\omega \times \omega^\omega\)) is an exception: We do not know any variant of an eventually different forcing (which we need to increase \(\text{non}(\mathcal{M})\)) which satisfies that all of its subalgebras are \(R_2\)-good. Accordingly, the main effort (in both constructions) is to show that \(b\) remains small.

In the old construction, each non-small forcing is a \((\sigma\text{-centered})\) subalgebra of the eventually different forcing \(\mathcal{E}\). To deal with such forcings, ultralimit limits of sequences of \(\mathcal{E}\)-conditions are introduced and used (and we require that all \(\mathcal{E}\)-subforcings are basically \(\mathcal{E}\) intersected with some model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of “large” forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [She00], we can use finite additive measures (FAMs) and interval-FAM-limits of random conditions. But now \(\mathcal{E}\) doesn’t seem to work with interval-FAM-limits any more, so we replace it with a creature forcing notion \(\tilde{\mathcal{E}}\).
We also have to show that \( \text{cov}(\mathcal{N}) \) remains small. In the old construction, we could use a rather simple (and well understood) relation \( R^{\text{old}} \) and use the fact that all \( \sigma \)-centered forcings are \( R^{\text{old}, \text{good}} \): As all large forcings are subalgebras of either eventually different forcing or of Hechler forcing, they are all \( \sigma \)-centered. In the new construction, the large forcings we have to deal with are subforcings of \( \mathcal{E} \). But \( \mathcal{E} \) is not \( \sigma \)-centered, just \( (\rho, \pi) \)-linked for a suitable pair \( (\rho, \pi) \) (a property between \( \sigma \)-centered and \( \sigma \)-linked, first defined in [OK14], see Def. 1.18). So we use a different (and more cumbersome) relation \( R_3 \), introduced in [OK14], where it is also shown that \( (\rho, \pi) \)-linked forcings are \( R_3 \)-good.

**Regarding the whole diagram:** In [GKS], starting with the iteration for \( (\text{left}_{\text{old}}) \), a new iteration is constructed to get simultaneously different values for all characteristics: Assuming four strongly compact cardinals, the following is consistent (cf. Figure 3):

\[
\mathbb{N}_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < b < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < b < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) < 2^{\aleph_0}.
\]

The essential ingredient is the concept of the Boolean ultrapower of a forcing notion.

In exactly the same way we can expand our new version (left$_{\text{new}}$) to the right hand side, where also the characteristics dual to \( b \) and \( \text{cov}(\mathcal{N}) \) are swapped. So we get: If four strongly compact cardinals are consistent, then so is the following (cf. Figure 4):

\[
\mathbb{N}_1 < \text{add}(\mathcal{N}) < b < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < b < \text{non}(\mathcal{N}) < b < \text{cof}(\mathcal{N}) < 2^{\aleph_0}.
\]

We closely follow the presentation of [GKS]. Several times, we refer to [GKS] and to [She00] for details in definitions or proofs. We thank Martin Goldstern and Diego Mejia for valuable discussions, and an anonymous referee for a very detailed and helpful report pointing out (and even fixing) several mistakes in the first version of the paper.

1. Finitely additive measure limits and the \( \mathcal{E} \)-forcing.

1.1. FAM-limits and random forcing. We briefly list some basic notation and facts around finite additive measures. (A bit more details can be found in Section 1 of [She00].)

**Definition 1.1.**

- A “partial FAM” (finitely additive measure) \( \Xi' \) is a finitely additive probability measure on a sub-Boolean algebra \( B \) of \( P(\omega) \), the power set of \( \omega \), such that \( \{n\} \in B \) and \( \Xi'((n)) = 0 \) for all \( n \in \omega \). We set \( \text{dom}(\Xi') = B \).
- \( \Xi \) is a FAM if it is a partial FAM with \( \text{dom}(\Xi) = P(\omega) \).
- For every FAM \( \Xi \) and bounded sequence of non-negative reals \( \bar{a} = (a_n)_{n \in \omega} \) we can define in the natural way the average (or: integral) \( \text{Av}_\Xi(\bar{a}) \), a non-negative real number.

[She00, 1.2] lists several results that informally say:

There is a FAM \( \Xi \) that assigns the values \( a_i \) to the sets \( A_i \) (for all \( i \) in some index set \( I \)) iff for each \( I' \subseteq I \) finite and \( \epsilon > 0 \) there is an arbitrary large\(^1\) finite \( u \subseteq \omega \) such that the counting measure on \( u \) for \( A_i \) approximates \( a_i \) with an error of at most \( \epsilon \), for all \( i \in I' \).

For the size of such an “\( \epsilon \)-good approximation” \( u \) to some FAM \( \Xi \) we can give an upper bound for \( |u| \) which only depends on \( |I'| \) and \( \epsilon \) (and not on \( \Xi \)):

**Lemma 1.2.** Given \( N, k^* \in \omega \) and \( \epsilon > 0 \), there is an \( M \in \omega \) such that: For all FAMs \( \Xi \) and \( (A_n)_{n < N} \) there is a nonempty \( u \subseteq \omega \) of size \( \leq M \) such that \( \min(u) > k^* \) and \( \Xi(A_n) - \epsilon < \frac{|A_n \cap u|}{|u|} \leq \Xi(A_n) + \epsilon \) for all \( n < N \).

\(^1\)Equivalently: “a finite \( u \) with arbitrary large minimum”, which is the formulation actually used in most of the results.
Proof. We can assume that \( \epsilon = \frac{1}{L} \) for an integer \( L \). \([A_n : n \in \mathbb{N}]\) generates the set algebra \( \mathcal{B} \subseteq \mathcal{P}(\omega) \). Let \( \mathcal{X} \) be the set of atoms of \( \mathcal{B} \). So \( \mathcal{X} \) is a partition of \( \omega \) of size \( \leq 2^N \).

Set \( \mathcal{X}' = \{ x \in \mathcal{X} : \Xi(x) > 0 \} \). Every \( x \in \mathcal{X}' \) is infinite, and \( \sum_{x \in \mathcal{X}'} \Xi(x) = 1 \).

Round \( \Xi(x) \) to some number \( \Xi'(x) = \ell_x \cdot \frac{1}{L \cdot 2^N} \) for some integer \( 0 \leq \ell_x \leq L \cdot 2^N \), such that \( |\Xi(x) - \Xi'(x)| < \frac{1}{L \cdot 2^N} \) and \( \sum_{x \in \mathcal{X}'} \Xi'(x) \) is still 1. So \( \sum_{x \in \mathcal{X}'} \ell_x = L \cdot 2^N \), and we construct \( u \) consisting of \( \ell_x \) many points that are bigger than \( k^\ast \) and in \( x \) (for each \( x \in \mathcal{X}' \)).

We will use the following variants of (\( \ast \)), regarding the possibility to extend a partial FAM \( \Xi' \) to a FAM \( \Xi \). The straightforward, if somewhat tedious, proofs are given in [She00, 1.3(G) and 1.7].

Fact 1.3. Let \( \Xi' \) be a partial FAM, and \( I \) some index set.

(a) Fix for each \( i \in I \) some \( A_i \subseteq \omega \).

If \( A \cap \bigcap_{i \in I} A_i \neq \emptyset \) for all \( I' \subseteq I \) finite and \( A \in \text{dom}(\Xi') \) with \( \Xi'(A) > 0 \),

then \( \Xi' \) can be extended to a FAM \( \Xi \) such that \( \Xi(A_i) = 1 \) for all \( i \in I \).

(b) Fix for each \( i \in I \) some real \( b^i \) and some bounded sequence of non-negative reals

\( \bar{a}^i = (a^i_k)_{k \in \omega} \).

If for each finite partition \( (B_m)_{m \in \omega} \) of \( \omega \) into elements of \( \text{dom}(\Xi') \), for each \( \epsilon > 0 \),

\( k^\ast \in \omega \), and \( I' \subseteq I \) finite there is a finite \( u \subseteq \omega \setminus k^\ast \) such that

- for all \( m < m^\ast \), \( \Xi'(B_m) - \epsilon \leq \frac{|B_m \cap u|}{|u|} \leq \Xi'(B_m) + \epsilon \), and

- for all \( i \in I' \), \( \frac{1}{|u|} \sum_{k \in u} a^i_k \geq b^i - \epsilon \),

then \( \Xi' \) can be extended to a FAM \( \Xi \) such that \( \Xi \xi (\bar{a}^i) \geq b^i \) for all \( i \in I \).

We first define what it means for a forcing \( Q \) to have FAM limits.

Remark 1.4. Intuitively, this means (in the simplest version): Fix a FAM \( \Xi \). We can define for each sequence \( q_k \) of conditions that are all “similar” (e.g., have the same stem and measure) a limit \( \lim_{\Xi} \xi q \).

We find in the \( Q \)-extension a FAM \( \Xi' \) extending \( \Xi \), such that \( \lim_{\Xi} \xi q \) forces that the set of \( k \) satisfying \( P(k) \equiv \xi q_k \in G \) has “large” \( \Xi' \)-measure. Up to here, we get the notion used in [GMS16] and [GKS] (but there we use ultrafilters instead of FAMs, and “large” means being in the ultrafilter). However, we need a modification: Instead of single conditions \( q_k \), we use a finite sequence \( (p_{\ell} : \ell \in I_k) \) (where \( I_k \) is a fixed, finite interval); and the condition \( P(k) \), which we want to satisfy on a large set, now is \( \frac{|\{ \ell \in I_k : p_{\ell} \in G \}|}{|I_k|} > b \) for some suitable \( b \). This is the notion used implicitly in [She00].

Notation. Let \( T^\ast \) be a compact subtree of \( \omega^{<\omega} \), for example \( T^\ast = 2^{<\omega} \). Let \( s, t \in T^\ast \). Let \( S \) be a subtree of \( T^\ast \).

- \( t \triangleright s \) means “\( t \) is immediate successor of \( s \)”.

- \( |s| \) is the length of \( s \) (i.e.: the height, or level, of \( s \)).

- \( [t] \) is the set of nodes in \( T^\ast \) comparable with \( t \).

- We set \( \text{lim}(S) = \{ x \in \omega^\omega : (\forall n \in \omega) x \upharpoonright n \in S \} \).

- \( \text{trunk}(S) \) is the smallest splitting node of \( S \). With “\( t \in S \) above the stem” we mean that \( t \in S \) and \( t \geq \text{trunk}(S) \); or equivalently: \( t \in S \) and \( |t| \geq |\text{trunk}(S)| \).

- \( \text{Leb} \) is the canonical measure on the Borel subsets of \( \text{lim}(T^\ast) \). We also write \( \text{Leb}(S) \) instead of \( \text{Leb}(\text{lim}(S)) \).

2I.e., we define \( \text{Leb}(s) \) by induction on the height of \( s \in T^\ast \) as follows: \( \text{Leb}(T^\ast) = 1 \), and if \( s \) has \( n \) many immediate successors in \( T^\ast \), then \( \text{Leb}(t) = \frac{\text{Leb}(s)}{n} \) for any such successor. This defines a measure on each basic clopen set, which in turn defines a (probability) measure on the Borel subsets of \( \text{lim}(T^\ast) \) (a closed subset of \( \omega^\omega \)).
We fix, for the rest of the paper, an interval partition $I = (I_k)_{k \in \omega}$ of $\omega$ such that $|I_k|$ converges to infinity. We will use forcing notions $Q$ satisfying the following setup:

**Assumption 1.5.**
- $Q' \subseteq Q$ is dense and the domain of functions trunk and loss, where $\text{trunk}(q) \in H(\mathbb{N}_0)$ and $\text{loss}(q)$ is a non-negative rational.
- For each $\varepsilon > 0$ the set $\{q \in Q' : \text{loss}(q) < \varepsilon\}$ is dense in $Q'$ and thus in $Q$.
- $\{p \in Q' : (\text{trunk}(p), \text{loss}(p)) = (\text{trunk}^*, \text{loss}^*)\}$ is $\lfloor \frac{1}{\text{loss}^*} \rfloor$-linked. I.e., each $\lfloor \frac{1}{\text{loss}^*} \rfloor$ many such conditions are compatible.\(^3\)

In this paper, $Q$ will be one of the following two forcing notions: random forcing, or $\vec{\Xi}$ (as defined in Definition 1.12). We will now specify the instance of random forcing that we will use:

**Definition 1.6.**
- A random condition is a tree $T \subseteq 2^{<\omega}$ such that $\text{Leb}(T \cap \{t\}) > 0$ for all $t \in T$.
- trunk$(T)$ is the stem of $T$ (i.e., the shortest splitting node).
- If $\text{Leb}(T) = \text{Leb}([\text{trunk}(T)])$, we set $\text{loss}(T) = 0$. Otherwise, let $m$ be the maximal natural number such that
  
  $$\text{Leb}(T) > \text{Leb}([\text{trunk}(T)])(1 - \frac{1}{m})$$

  and set\(^4\) $\text{loss}(T) = \frac{1}{m}$.

  Note that $\text{Leb}(T) \geq 2^{-\text{trunk}(T)}(1 - \text{loss}(T))$ (and the inequality is strict if $\text{loss}(T) > 0$).

  Note that this definition of random forcing satisfies Assumption 1.5 (with $Q' = Q$).

**Definition 1.7.** Fix $Q$ and functions $(\text{trunk}, \text{loss})$ as in Assumption 1.5, a FAM $\Xi$ and a function $\lim_\Xi : Q^\omega \to Q$. Let us call the objects mentioned so far a "limit setup". Let a $(\text{trunk}^*, \text{loss}^*)$-sequence be a sequence $(q_\ell)_{\ell \in \omega}$ of $Q$-conditions such that $\text{trunk}(q_\ell) = \text{trunk}^*$ and $\text{loss}(q_\ell) = \text{loss}^*$ for all $\ell \in \omega$.

We say "$\lim_\Xi$ is a strong FAM limit for intervals", if the following is satisfied: Given
- a pair $(\text{trunk}^*, \text{loss}^*)$, $j^* \in \omega$, and $(\text{trunk}^*, \text{loss}^*)$-sequences $\bar{q}^j$ for $j < j^*$,
- $\varepsilon > 0$, $k^* \in \omega$,
- $m^* \in \omega$ and a partition of $\omega$ into sets $B_m$ ($m \in m^*$), and
- a condition $q$ stronger than all $\lim_\Xi(q^j)$ for all $j < j^*$,

there is a finite $u \subseteq \omega \setminus k^*$ and a $q^j$ stronger than $q$ such that
- $\Xi(B_m) - \varepsilon < \frac{|\cap_{[u \cap B_m]}|}{|u|} < \Xi(B_m) + \varepsilon$ for $m < m^*$,
- $\frac{1}{|u|} \sum_{k \in u} |\{\ell \in I_k : q^j \leq q_\ell\}| \geq 1 - \text{loss}^* - \varepsilon$ for $j < j^*$

(We are only interested in $\lim_\Xi(q)$ for $\bar{q}$ as above, so we can set $\lim_\Xi(q)$ to be undefined or some arbitrary value for other $\bar{q} \in Q^\omega$.)

The motivation for this definition is the following:

**Lemma 1.8.** Assume that $\lim_\Xi$ is such a limit. Then there is a $Q$-name $\Xi^+$ such that for every $(\text{trunk}^*, \text{loss}^*)$-sequence $\bar{q}$ the limit $\lim_\Xi(q)$ forces $\Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\text{loss}^*}$, where

$$A_{\bar{q}} = \{k \in \omega : |\{\ell \in I_k : q_\ell \in G\}| \geq |I_k| \cdot (1 - \sqrt{\text{loss}^*})\}$$

\(^3\)In [She00, 2.9], trunk and loss are called $h_2$ and $h_1$; and instead of $I_k$ the interval is called $[n^*_k, n^*_k + 1]$. Moreover, in [She00] the sequence $(n^*_k)_{k \in \omega}$ is one of the parameters of a "blueprint", whereas we assume that the $I_k$ are fixed.

\(^4\)In [She00], this is implicit in 2.11(f).
we set $\Omega$ vals, if for every $FAM$ ∗

Definition 1.14.

\[\text{Lemma 1.11.} \] Random forcing has strong FAM-limits for intervals.

\[\text{Proof.} \] $\lim_{\Xi}$ is implicitly defined in [She00, 2.18], in the following way: Given a sequence $r_{\xi}$ with $(\text{trunk}(p_{\xi}), \text{loss}(p_{\xi})) = (\text{trunk}^{*}, \text{loss}^{*})$, we can set $r^{*} = [\text{trunk}^{*}]$ and $b = 1 - \text{loss}^{*}$; and we set $n_{k}^{*}$ such that $I_{k} = [n_{k}^{*}, n_{k+1}^{*}] - 1$. We now use these objects to apply [She00, 2.18] (note that (c)(*) is satisfied). This gives $r^{\text{FAM}}$, and we define $\lim_{\Xi}(r)$ to be $r^{\text{FAM}}$.

In [She00, 2.17], it is shown that this $r^{\text{FAM}}$ satisfies Definition 1.7, i.e., is a limit: If $r$ is stronger than all limits $r^{\text{FAM}}$, then $r$ satisfies [She00, 2.17(*)]. □

1.B. The forcing $\tilde{\Xi}$. We now define $\tilde{\Xi}$, a variant of the forcing notion $Q^{2}$ defined in [HS]:

Definition 1.12. By induction on the height $h \geq 0$, we define a compact homogeneous tree $T^{*} \subset \omega^{\omega^{\omega}}$, and set

\[\rho(h) = \text{max}\{(T^{*} \cap \omega^{h}), h + 2\} \quad \text{and} \quad \pi(h) = ((h + 1)^{2}\rho(h)^{h+1})^{\rho(h)^{h}},\]

we set $\Omega_{h}$ to be the set $\{t \triangleright s : t \in T^{*}\}$, i.e., the set of immediate successors of $s$, and define for each $s$ a norm $\mu_{s}$ on the subsets of $\Omega_{h}$. In more detail:

- The unique element of $T^{*}$ of height 0 is $\langle \rangle$, i.e., $T^{*} \cap \omega^{0} = \{\langle \rangle\}$.
- We set $a(h) = \pi(h)^{h+2}$, $M(h) = a(h)^{2}$, and $\mu_{h}(n) = \log_{a(h)} \left( \frac{M(h)}{M(h) - n} \right)$ for natural numbers $0 \leq n < M(h)$, and we set $\mu_{h}(M(h)) = \infty$.
- For any $s \in T^{*} \cap \omega^{h}$, we set $\Omega_{s} = \{s \triangleright \xi : \xi \in M(h)\}$ (which defines $T^{*} \cap \omega^{h+1}$).

For $A \subset \Omega_{h}$, we set $\mu_{h}(A) = \mu_{h}(|A|)$. So $|\Omega_{h}| = M(h)$, $\mu_{h}(\emptyset) = 0$ and $\mu_{h}(\Omega_{h}) = \infty$. Note that $|A| = |\Omega_{h}| \cdot (1 - a(h)^{-\mu_{h}(A)})$.

We can now define $\tilde{\Xi}$:

Definition 1.14. For a subtree $p \subseteq T^{*}$, the stem of $p$ is the smallest splitting node. For $s \in p$, we set $\mu_{s}(p) = \mu_{s}(\{t \in p : t \triangleright s\})$.

$\tilde{\Xi}$ consists of subtrees $p$ with some stem $s^{*}$ of height $h^{*}$ such that $\mu_{s}(p) \geq 1 + \frac{1}{m}$ for all $t \in p$ above the stem. (So the only condition with $h^{*} = 0$ is the full condition, where all norms are $\infty$.)

$\tilde{\Xi}$ is ordered by inclusion.

- trunk($p$) is the stem of $p$.
- loss($p$) is defined if there is an $m \geq 2$ satisfying the following, and in that case $\text{loss}(p) = \frac{1}{m}$ for the maximal such $m$:
  - $p$ has stem $s^{*}$ of height $h^{*} > 3m$,
  - $\mu_{s}(p) \geq 1 + \frac{1}{m}$ for all $s \in p$ of height $\geq h^{*}$.

We set $Q' = \text{dom(loss)}$.  

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By simply extending the stem, we can find for any \( p \in \hat{E} \) and \( \epsilon > 0 \) some \( q \leq p \) in \( Q' \) with \( \text{loss}(q) < \epsilon \); i.e., one of the assumptions in 1.5 is satisfied. (The other one is dealt with in Lemma 1.19(a).) In particular \( Q' \subseteq \hat{E} \) is dense.

We list a few trivial properties of the loss function:

**Facts 1.15.** Assume \( p \in Q' \) with \( s = \text{trunk}(p) \) of height \( h \).

(a) \( \text{loss}(p) < 1, \mu_s(p) \geq 1 + \text{loss}(p) \) for any \( s \) above the stem, and \( \text{loss}(p) > \frac{3}{h} \).

(b) If \( q \) is a subtree of \( p \) such that all norms above the stem \( s \) are \( \geq 1 + \text{loss}(p) - \frac{2}{h} \), then \( q \) is a valid \( \hat{E} \)-condition.

(c) \( \prod_{\ell = h}^{\infty} \left( 1 - \frac{1}{\ell^2} \right) = 1 - \frac{1}{h} > 1 - \frac{\text{loss}(p)}{3} \).

**Lemma 1.16.** Let \( s \in T^* \) be of height \( h \) and \( A \subseteq \Omega_s \).

(a) If \( \mu_s(A) \geq 1 \), then \( |A| \geq |\Omega_s| \cdot \left( 1 - \frac{1}{h^2} \right) \).

(b) If \( A \subseteq \Omega_s \), i.e., \( A \) is a proper subset, then \( \mu_s(\Omega_s \setminus \{ t \}) > \mu_s(A) - \frac{1}{h^2} \) for \( t \in A \).

(c) For \( i < \pi(h) \), assume that \( A_i \subseteq \Omega_s \) satisfies \( \mu_s(A_i) \geq x \). Then \( \mu_s(\bigcap_{i \in \pi(h)} A_i) > x - \frac{1}{h} \).

(d) For \( i \leq 1 \) (an arbitrary finite index set) pick proper subsets \( A_j \subseteq \Omega_s \) such that \( \mu_s(A_j) \geq x \), and assign weights \( a_i \) to \( A_j \) such that \( \sum_{i \in A_i} a_i = 1 \). Then

\[
(1.17) \quad \mu_s(B) > x - \frac{1}{h} \quad \text{for} \quad B := \left\{ t \in \Omega_s : \sum_{i \in A_i} a_i > 1 - \frac{1}{h^2} \right\}.
\]

**Proof.** (a) Trivial, as \( \alpha(h)^{-\mu_s(A)} \leq \frac{1}{\alpha(h)} < \frac{1}{h^2} \).

(b) \( \mu_s(A \setminus \{ t \}) = \log_{\alpha(h)}(|\Omega_s|) - \log_{\alpha(h)}(|\Omega_s| - |A| + 1) \geq \log_{\alpha(h)}(|\Omega_s|) - \log_{\alpha(h)}(2(|\Omega_s| - |A|)) \geq \mu_s(A) - \log_{\alpha(h)}(2) > \mu_s(A) - \frac{1}{h} \).

(c) \( \mu_s(\bigcap_{i \in \pi(h)} A_i) = \log_{\alpha(h)}(|\Omega_s|) - \log_{\alpha(h)}(|\Omega_s| - |\bigcap_{i \in \pi(h)} A_i|) = \geq \log_{\alpha(h)}(|\Omega_s|) - \log_{\alpha(h)}(|\bigcup_{i \in \pi(h)} (\Omega_s \setminus A_i)|) \geq \log_{\alpha(h)}(|\Omega_s|) - \log_{\alpha(h)}(\pi(h) \cdot \max_{i \in \pi(h)} |\Omega_s \setminus A_i|) \geq x - \log_{\alpha(h)}(\pi(h)) > x - \frac{1}{h} \).

(d) Set \( y = \sum_{i \in A_i} a_i \cdot |A_i| \). On the one hand, \( y \geq |\Omega_s| \cdot (1 - a(h)^{-s}) \). On the other hand, \( y = \sum_{i \in \Omega_s} \sum_{j \in A_i} a_j \leq |B| + (|\Omega_s \setminus B|) \cdot (1 - \frac{1}{h^2}) \).

So \( |B| \geq |\Omega_s| \cdot (1 - a(h)^{-s}) > |\Omega_s| \cdot (1 - a(h)^{-s}) \cdot \frac{2}{h^2} \), as \( a(h)^{-s} > \pi(h) > h^2 \).  

\( \hat{E} \) is not \( \sigma \)-centered, but it satisfied a property, first defined in [OK14], which is between \( \sigma \)-centered and \( \sigma \)-linked:

**Definition 1.18.** Fix \( f, g \) functions from \( \omega \) to \( \omega \) converging to infinity. \( Q \) is \((f, g)\)-linked if there are \( g(i)\)-linked \( Q'_i \subseteq Q \) for \( i < \omega, f < f(i) \) such that each \( q \in Q \) is in every \( \bigcup_{j < f(i)} Q'_j \) for sufficiently large \( i \).

Recall that we have defined \( \rho \) and \( \pi \) in (1.13).

**Lemma 1.19.** (a) If \( \pi(h) \) many conditions \( (p_i)_{i \in \pi(h)} \) have a common node \( s \) above their stems, \( |s| = h \), then there is a \( q \) stronger than each \( p_i \).

(b) \( \hat{E} \) is \((\rho, \pi)\)-linked (in particular it is ccc).

(c) The \( \hat{E} \)-generic real \( \eta \) is eventually different (from every real in \( \text{lim}(T^*) \), and therefore from every real in \( \omega^\omega \) as well).

(d) \( \text{Leb}(\rho) \geq \text{Leb} \left( |\text{trunk}(p)| \cdot (1 - \frac{1}{2} \text{loss}(p)) \right) \); more explicitly: for any \( h > |\text{trunk}(p)| \),

\[
\frac{|p \cap \omega^h|}{|T^* \cap \omega^h \cap |\text{trunk}(p)||} \geq 1 - \frac{1}{2} \text{loss}(p).
\]
(e) \( Q' \) (which is a dense subset of \( \mathcal{E} \)) is an incompatibility-preserving subforcing of random forcing, where we use the variant of random forcing on \( \text{lim}(T^*) \) instead of \( 2^\omega \). Let \( B' \) be the the sub-Boolean-algebra of Borel/Null generated by \( \{ \text{lim}(q) : q \in Q' \} \). Then \( Q' \) is dense in \( B' \).

(Here, Borel refers to the set of Borel subsets of \( \text{lim}(T^*) \). In the following proof, we will denote the equivalence class of a Borel set \( A \) by \( [A]_{\mathcal{B}'} \).)

**Proof.** (a) Set \( S = [s] \cap \bigcap_{s < \rho(h)} \mathcal{P}_l \). According to 1.16(c), for each \( t \in S \) of height \( h' \geq h \), the successor set has norm bigger than \( 1 + 1/h - 1/h' > 1 \), so in particular there is a branch \( x \in S \), and \( S \cap [x \uparrow 2h] \) is a valid condition stronger than all \( p_i \).

(b) For each \( h \in \omega \), enumerate \( T^* \cap \omega^h \) as \( \{ s^h_1, \ldots, s^h_{\rho(h)} \} \), and set \( Q_i^h = \{ p \in \mathcal{E} : s^h_i \in p \text{ and } |\text{trunk}(p)| \leq h \} \). So for all \( h \), \( Q_i^h \) is \( \pi(h) \)-linked, and \( p \in \bigcup_{s < \rho(h)} Q_i^h \) for all \( p \in Q \) with \( |\text{trunk}(p)| \leq h \).

(c) Use 1.16(b).

(d) Use 1.16(a) and the definition of loss.

(e) As in the previous item, we get that \( \text{Leb}(p \cap [t]) > 0 \) whenever \( p \in Q' \) and \( t \in p \). So \( Q' \) is a subset of random forcing. As both sets are ordered by inclusion, \( Q' \) is a subforcing. If \( q_1, q_2 \in Q' \) and \( q_1, q_2 \) are compatible as a random conditions, then \( q_1 \cap q_2 \) has arbitrary high nodes, in particular a node above both stems, which implies that \( q_1 \) is compatible with \( q_2 \) in \( \mathcal{E} \) and therefore in \( Q' \). It remains to show that \( Q' \) is dense in \( B' \). It is enough to show: If \( x \neq 0 \) in \( B' \) has the form \( x = \bigwedge_{i < \iota} \text{lim}(q_i) \) of \( \mathcal{B}' \) then there is some \( q \in Q' \) with \( \text{lim}(q) < x \). Note that \( 0 \neq x = [A]_{\mathcal{B}'} \) for \( A = \text{lim}(\bigcap_{i < \iota} q_i) \) \( \in \bigcup_{s < \rho(h)} \text{lim}(q_i) \), so pick some \( r \in A \) and pick \( h > i^* \) large enough such that \( s = r \uparrow h \) is not in any \( q_i \). Then any \( q \in Q' \) stronger than all \( q_i \cap [s] \) (for \( i < i^* \)) is as required.

**Lemma 1.20.** \( \mathcal{E} \) has strong FAM-limits for intervals.

**Proof.** Let \( (p_{\ell})_{\ell \in \omega} \) be a \((s^*, \text{loss}^*)\)-sequence, \( s^* \) of height \( h^* \). Set \( \zeta^{h^*} = 0 \) and

\[
\zeta^h := 1 - \prod_{m=h}^{h^*} (1 - \frac{1}{m}) \text{ for } h > h^*.
\]

This is a strictly increasing sequence below \( \frac{1}{3} \text{loss}^* \), cf. Fact 1.15(c). Also, all norms in all conditions of the sequence are at least \( 1 + \text{loss}^* \), cf. Fact 1.15(a).

We will first construct \( (q_k)_{k \in \omega} \) with stem \( s^* \) and all norms \( > 1 + \text{loss}^* - \frac{1}{h^*} \) such that \( q_k \) forces \( \lfloor \ell \in I_k : p_{\ell} \in G \rfloor \frac{1}{|I_k|} > 1 - \frac{1}{3} \text{loss}^* \). We will then use \( \bar{q} \) to define \( \text{lim}_\mathcal{E}(\bar{p}) \), and in the third step show that it is as required.

**Step 1:** So let us define \( q_k \). Fix \( k \in \omega \).

- Set \( X_i = \{ \ell \in I_k : \ell \in p_{\ell} \} \) and \( Y_h = \{ t \in [s^*] \cap \omega^h : |X_i| \geq |I_k| \cdot (1 - \zeta^h) \} \).

- We define \( q_k \) by induction on the level, such that \( q_k \cap \omega^h \subseteq Y_h \). The stem is \( s^* \). (Note that \( X_{s^*} = I_k \) and so \( s^* \in Y_h \).) For \( s \in q_k \cap \omega^h \) (and thus, by induction hypothesis, in \( Y_h \)), we set \( q_k \cap [s] \cap \omega^{h+1} = [s] \cap Y_{h+1} \), i.e., a successor \( t \) of \( s \) is in \( q_k \) iff it is \( Y_{h+1} \). Then \( \mu_s(q_k) > 1 + \text{loss}^* - \frac{1}{h} \).

**Proof:** Set \( I = X_i \). By induction, \( |X_i| \geq |I_k| \cdot (1 - \zeta^h) \). For \( \ell \in I \), set \( A_{\ell} = p_{\ell} \cap [s] \cap \omega^{h+1} \), i.e., the immediate successors of \( s \) in \( p_{\ell} \). Obviously \( \mu_s(A_{\ell}) \geq \frac{3}{4} \text{loss}^* \) because \( \text{lim}(q) \cap \omega^h \subseteq Y_h \).
1 + loss. We give each $A_\ell$ equal weight $a_\ell = \frac{1}{|I|}$. According to (1.17), the set $B = \{ t \mapsto s : |\{ \ell \in \mathcal{X}_s : t \in A_\ell \}| \geq |I| \cdot (1 - \frac{\epsilon}{h}) \}$ has norm $\geq 1 + loss - \frac{1}{h}$.

- $q_k$ forces that $p_\ell \in G$ for $|I| \cdot (1 - \frac{1}{3} loss^*)$ many $\ell \in I_k$.

Proof: Let $r < q_k$, have stem $s'$ of length $h'$, without loss of generality $h' > |I_k| + 1$. As $s' \in \mathcal{Y}_h$, there are $|I_k| \cdot (1 - \frac{1}{3} loss^*)$ many $\ell \in I_k$ such that $s' \in p_\ell$. So we can find a condition $r'$ stronger than $r$ and all these $p_\ell$ (as these are at most $|I_k| + 1 \leq h'$ many conditions all containing $s'$ above the stem).

**Step 2:** Now we use $(q_k)_{k \in \omega}$ to construct by induction on the height $q^* = \lim_{\omega}(\bar{p})$, a condition with stem $s^*$ and all norms $\geq 1 + loss - \frac{2}{h}$ such that for all $s \in q^*$ of height $h \geq h^*$,

\[(*) \quad \Xi(Z_s) \geq 1 - \zeta^h, \quad \text{for} \quad Z_s := \{ k \in \omega : s \in q_k \}. \quad \text{So} \quad \Xi(Z_s) > 1 - \frac{1}{3} loss^*.
\]

Note that $Z_{s^*} = \omega$, so (*) is satisfied for $s^*$. Fix an $s \geq s^*$ satisfying (*). Set $A(k)$ to be the $s$-successors in $q_k$ for each $k \in \mathcal{Z}_s$. Enumerate the (finitely many) $A(k)$ as $(A_i)_{i \in I}$. Clearly $\mu_i(A_i) > 1 + loss - \frac{1}{h}$. Assign to $A_i$ the weight $a_i = \frac{1}{h} \Xi((k \in \mathcal{Z}_s : A(k) = A_i))$.

Again using (1.17), $\mu_i(B) \geq 1 + loss - \frac{2}{h}$, where $B$ consists of those successors $t$ of $s$ such that

\[1 - \frac{1}{h^2} < \sum_{i \in A_i} a_i = \frac{1}{\Xi(Z_s)} \Xi((k \in \mathcal{Z}_s : t \in q_k)) \leq \frac{1}{\Xi(Z_s)} \Xi(Z_s).
\]

So every $t \in B$ satisfies $\Xi(Z_s) > 1 - \frac{1}{h} \Xi(Z_s)(1 - \frac{1}{h}) \geq \zeta^{h+1}$, i.e., satisfies (*). So we can use $B$ as the set of $s$-successors in $q^*$.

This defines $q^*$, which is a valid condition by Fact 1.15(b).

**Step 3:** We now show that this limit works: As in Definition 1.7, fix $m^*$, $(B_m)_{m < m^*}$, $\epsilon$, $k^*$, $i^*$ and sequences $(p_\ell)_{\ell \in \omega}$ for $i < i^*$, such that $(\text{trunk}(p_\ell), \text{loss}(p_\ell)) = (\text{trunk}^*, \text{loss}^*)$.

For each $i < i^*$, $q^* = (q^*_k)_{k \in \omega}$ is defined from $\bar{p}^* = (p^*_\ell)_{\ell \in \omega}$, and in turn defines the limit $\lim_{\omega}(\bar{p})$. Let $q$ be stronger than all $\lim_{\omega}(\bar{p})$.

Let $M$ be as in Lemma 1.2, for $N = m^* + i^*$. So for any $N$ many sets there is a $u$ of size at most $M$ (above $k^*$) which approximates the measure well. We use the following $N$ many sets:

- $B_m$ (for $m < m^*$).
- Fix an $s \in q$ of height $h > M \cdot i^*$; and use the $i^*$ many sets $\mathcal{Z}_s^i \subseteq \omega$ defined in (*).

Accordingly, there is a $u$ (starting above $k^*$) of size $\leq M$ with

- $\Xi(B_m) - \epsilon \leq \frac{|B_m \cap u|}{|u|} \leq \Xi(B_m) + \epsilon$ for each $m < m^*$, and
- $\frac{|Z_m^i \cap u|}{|u|} \geq 1 - \frac{1}{3} loss^* - \epsilon$ for each $i < i^*$.

So for each $i < i^*$ there are at least $|u| \cdot (1 - \frac{1}{3} loss^* - \epsilon)$ many $k \in u$ with $s \in q^*_k$. There is a condition $r$ stronger than $q$ and all those $q^*_k$ (as $\leq M \cdot i^*$ + 1 many conditions of height $h > M \cdot i^*$ with common node $s$ above their stems are compatible). So $r$ forces, for all $i < i^*$ and $k \in u \cap \mathcal{Z}_s^i$ that $q^*_k \in G$ and therefore that $|\{ \ell \in I_k : p_\ell \in G \}| \geq |I_k|(1 - \frac{1}{3} loss^*)$.

By increasing $r$ to some $q'$, we can assume that $r$ decides which $p_\ell$ are in $G$ and that $r$ is actually stronger than each $p_\ell$ decided to be in $G$. So all in all we get $q' \leq q$ such that

\[\frac{1}{|u|} \sum_{k \in u} \frac{|\{ \ell \in I_k : q' \leq p_\ell \}|}{|I_k|} \geq \frac{1}{|u|} \sum_{k \in u} \left( \frac{|k \in u : k \in \mathcal{Z}_s^i|}{|I_k|} \right) \geq 1 - \frac{1}{3} loss^* - \epsilon.
\]
as required.

\[\begin{aligned}
\text{2. THE LEFT HAND SIDE OF CICHON’S DIAGRAM}

We write \( \mathfrak{f}_1 \) for \( \text{add}(\mathcal{N}) \), \( \mathfrak{f}_2 \) for \( b \) (which will also be \( \text{add}(\mathcal{M}) \)), \( \mathfrak{f}_3 \) for \( \text{cov}(\mathcal{N}) \) and \( \mathfrak{f}_4 \) for \( \text{non}(\mathcal{M}) \).

2.A. **Good iterations and the LCU property.** We want to show that some forcing \( \mathbb{P}^5 \) results in \( \mathfrak{f}_i = \lambda_i \) (for \( i = 1 \ldots 4 \)). So we have to show two “directions”, \( \mathfrak{f}_i \leq \lambda_i \) and \( \mathfrak{f}_i \geq \lambda_i \).

For \( i = 1, 3, 4 \) (i.e., for all the characteristics on the left hand side apart from \( b = \text{add}(\mathcal{M}) \)), the direction \( \mathfrak{f}_i \leq \lambda_i \) will be given by the fact that \( \mathbb{P}^5 \) is \( (R_i, \lambda_i) \)-good for a suitable relation \( R_i \). (For \( i = 2 \), i.e., the unbounding number, we will have to work more.)

We will use the following relations:

**Definition 2.1.**

1. Let \( C \) be the set of strictly positive rational sequences \( (q_n)_{n \in \omega} \) such that \( \sum_{n \in \omega} q_n \leq 1. \)
2. \( R_2 \subseteq (\omega^\omega)^2 \) is defined by: \( \lim f \geq \lim g \).
3. \( R_4 \subseteq (\omega^\omega)^2 \) is defined by: \( \lim f n \leq \lim g n \).

So far, these relations fit the usual framework of goodness, as introduced in [JS90] and [Bre91] and summarized, e.g., in [BJ95, 6.4] or [GMS16, Sec. 3] or [Mej13, Sec. 2]. For \( \mathfrak{f}_3 \), i.e., \( \text{cov}(\mathcal{N}) \), we will use a relation \( R_3 \) that does not fit this framework (as the range of the relation is not a Polish space). Nevertheless, the property “\( (R_3, \lambda) \)-good” behaves just as in the usual framework (e.g., finite support limits of good forcings are good, etc.). The relation \( R_3 \) was implicitly used by Kamo and Osuga [OK14], who investigated \( (R_3, \lambda) \)-goodness. \(^7\) It was also used in [BM14]; a unifying notation for goodness (which works for the usual cases as well as relations such as \( R_3 \)) is given in [MC, §4].

**Definition 2.2.** We call a set \( E \subseteq \omega^\omega \) an \( R_3 \)-parameter, if for all \( e \in E \)

- \( \lim e(n) = \infty \), \( e(n) \leq n \), \( \lim(n - e(n)) = \infty \),
- there is some \( e' \in E \) such that \( (\forall n)(e(n) + 1 \leq e'(n)) \), and
- for all countable \( E' \subseteq E \) there is some \( e \in E' \) such that for all \( e' \in E' \) \( (\forall n)(e(n) \geq e'(n)). \)

Note that such an \( R_3 \)-parameter of size \( \aleph_1 \) exists. This is trivial if we assume CH (which we could in this paper), but also true without this assumption, see [MC, 4.20]. Recall that \( \rho \) and \( \pi \) were defined in (1.13).

**Definition 2.3.** We fix, for the rest of the paper, an \( R_3 \)-parameter \( E \) of size \( \aleph_1 \), and set

\[
\psi(h) = (h + 1)^2 \rho(h)^{h+1}, \quad S = \\{ \psi \in \prod_{h \in \omega} P(b(h)) : (\forall h \in \omega) |\psi(h)| \leq \rho(h)^h \},
\]

\[
S_e = \{ \phi \in \prod_{h \in \omega} P(b(h)) : (\forall h \in \omega) |\phi(h)| \leq \rho(h)^{\varphi(h)} \} \quad \text{and} \quad \hat{S} = \bigcup_{e \in E} S_e.
\]

We can now define the relation for \( \text{cov}(\mathcal{N}) \):

- \( R_3 \subseteq \mathcal{N} \times \hat{S} \) is defined by: \( \psi R_3 \phi \text{ iff } (\forall n \in \omega) \phi(n) \not\subseteq \psi(n). \)

---

\(^6\)It is easy to see that \( C \) is homeomorphic to \( \omega^\omega \), when we equip the rationals with the discrete topology and use the product topology.

\(^7\)They use the notation \( (\ast_{c,h}) \), cf. [OK14, Def. 6].
Note that \( S_e \subset \tilde{S} \subset S \) and that \( S_e \) and \( S \) are Polish spaces. Assume that \( M \) is a forcing extension of \( V \) by either a ccc forcing (or by a \( \sigma \)-closed forcing). Then \( E \) is an “\( R_i \)-parameter” in \( M \) as well, and we can evaluate in \( M \) for each \( e \in E \) the sets \( S_e^M \) and \( S^M \), as well as \( \tilde{S}^M = \bigcup_{e \in E} S_e^M \). Absoluteness gives \( S_e^V = S_e^M \cap V \) and \( \tilde{S}^V = \tilde{S}^M \cap V \).

**Definition 2.4.** Fix one of these relations \( R \subseteq X \times Y \).

- We say “\( f \) is bounded by \( g \)” if \( f \) \( R \) \( g \); and, for \( Y \subseteq \omega^\omega \), “\( f \) is bounded by \( Y \)” if \( (\exists y \in Y) f \) \( R \) \( y \). We say “unbounded” for “not bounded”. (I.e., \( f \) is unbounded by \( Y \) if \( (\forall y \in Y) \neg f \) \( R \) \( y \).)
- We call \( \mathcal{X} \) an \( R \)-unbounded family, if \( (\exists y) (\forall x \in \mathcal{X}) f \) \( R \) \( y \), and an \( R \)-dominating family if \( (\forall f) (\exists x \in \mathcal{X}) f \) \( R \) \( x \).
- Let \( b_i \) be the minimal size of an \( R_i \)-unbounded family,
- and let \( b_i \) be the minimal size of an \( R_i \)-dominating family.

We only need the following connection between \( R_i \) and the cardinal characteristics:

**Lemma 2.5.** 1. \( \text{add}(\mathcal{N}) = b_1 \) and \( \text{cof}(\mathcal{N}) = b_1 \).
2. \( b = b_3 \) and \( \mathcal{b} = b_2 \).
3. \( \text{cov}(\mathcal{N}) \leq b_3 \) and \( \text{non}(\mathcal{N}) \geq b_3 \).
4. \( \text{non}(\mathcal{M}) = b_4 \) and \( \text{cov}(\mathcal{M}) = b_4 \).

**Proof.** (2) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82; Bar87], cf. [BJ95, 2.4.1 and 2.4.7].

To see (3), we work in the space \( \Omega = \prod_{h \in \omega} \beta(h) \), with the \( h \) defined in Definition 2.3 and the usual (uniform) measure. It is well known that we get the same values for the characteristics \( \text{cov}(\mathcal{N}) \) and \( \text{non}(\mathcal{N}) \) whether we define them using \( \Omega \) or, as usual, \( 2^{\omega} \) (or \([0,1]\) for that matter, etc). Given \( \psi \in S \), note that

\[ N_\psi = \{ \eta \in \Omega : (3^{\omega} h) \eta(h) \in \psi(h) \} \]

is a Null set, as \( \{ \eta \in \Omega : (\forall h > k) \eta(h) \notin \psi(h) \} \) has measure \( \prod_{h > k} (1 - \frac{|\psi(h)|}{\beta(h)}) \geq \prod_{h > k} (1 - \frac{1}{(h+1)^{\omega}}) \), which converges to 1 for \( k \to \infty \).

Let \( A \subseteq S \) be an \( R_3 \)-unbounded family. So for every \( \phi \in \tilde{S} \) there is some \( \psi \in A \) such that \( (3^{\omega} h) \phi(h) \geq \phi(h) \). In particular, for each \( \eta \in \Omega \), there is a \( \psi \in A \) with \( \eta \notin N_\psi \); i.e., \( \text{cov}(\mathcal{N}) \leq |\mathcal{A}| \).

Analogously, let \( X \) be a non-null set (in \( \Omega \)). For each \( \psi \) there is an \( x \in X \setminus N_\psi \), so \( \phi_x(n) = \{ x(n) \} \) satisfies \( \psi \) \( R_4 \) \( \phi_x \).

**Remark 2.6.** As shown implicitly in [OK14], and explicitly in [MC, 4.22], we actually get \( \text{cov}(\mathcal{N}) \leq c_{b_3}^3 \leq b_3 \).

**Definition 2.7.** Let \( P \) be a ccc forcing, \( \lambda \) an uncountable regular cardinal, and \( R_i \subseteq X \times Y \) one of the relations above (so for \( i = 1, 2, 4, Y = X \), and for \( i = 3 Y = \tilde{S} \)). \( P \) is \( (R_i, \lambda) \)-good, if for each \( P \)-name \( r \) for an element of \( Y \) there is (in \( V \)) a nonempty set \( Y \subseteq Y \) of size \( \lambda \) such that every \( f \in X \) (in \( V \)) that is \( R_i \)-unbounded by \( Y \) is forced to be \( R_i \)-unbounded by \( r \) as well.

Note that \( \lambda \)-good trivially implies \( \mu \)-good if \( \mu \geq \lambda \) are regular.

**Lemma 2.8.** Let \( \lambda \) be uncountable regular.

a. Forcings of size \( \lambda \) are \( (R_i, \lambda) \)-good. In particular, Cohen forcing is \( (R_i, \mathcal{N}) \)-good.
b. A FS ccc iteration of \((R_i, \lambda)\)-good forcings (and in particular, a composition of two such forcings) is \((R_i, \lambda)\)-good.
1. A sub-Boolean-algebra of the random algebra is \((R_1, \mathbb{N}_1)\)-good. Any \(\sigma\)-centered forcing notion is \((R_1, \mathbb{N}_1)\)-good.
2. A \((\rho, \pi)\)-linked forcing is \((R_1, \mathbb{N}_1)\)-good (for the \(\rho, \pi\) of Definition 1.12).

Proof. \((a\&b):\) For \(i = 1, 2, 4\) this is proven in [JS90], cf. [BJ95, 6.4]. The same proof works for \(i = 3\), as shown in [OK14, Lem. 12,13]. The proof for the uniform framework can be found in [MC, 4.10,4.14].

(1) follows from [JS90] and [Kam89], cf. [BJ95, 6.5,17–18].
(3) is shown in [OK14, Lem. 10], cf. [MC, Lem. 4.24]; as our choice of \(\pi, \rho\) and \(b\) (see Definition 2.3) satisfies \(\pi(h) \supseteq b(h)^{\rho(h)} = ((h + 1)^2\rho(h^{b+1}))^{\rho(h)}\).

Each relation \(R_i\) is a subset of some \(X \times Y\), where \(X\) is either \(2^\omega\), \(\omega^\omega\) (or homeomorphic to it) or \(S\), and \(Y\) is the range of \(R_i\).

Lemma 2.9. For each \(i\) and each \(g \in Y\), the set \(\{ f \in X : f \restriction R_i \subseteq X \}\) is meager.

Proof. We have explicitly defined each \(f \restriction R_i\) as \(\forall n \exists R^n_i(f, g)\) for some \(R^n_i\). The lemma follows easily from the fact that for each \(n \in \omega\), the set \(\{ f \in X : R^n_i(f, g) \}\) is closed nowhere dense.

Lemma 2.10. Let \(\lambda \leq \kappa \leq \mu\) be uncountable regular cardinals. Force with \(\mu\) many Cohen reals \(c_{\alpha} : \alpha \in \mu\), followed by an \((R_i, \lambda)\)-good forcing. Note that each Cohen real \(c_{\alpha}\) can be interpreted as element of the Polish space \(X\) where \(R_i \subseteq X \times Y\). Then we get: For every real \(r\) in the final extension’s \(Y\), the set \(\{ \alpha \in \kappa : c_{\alpha}\) is \(R_i\)-unbounded by \(r\}\) is cobounded in \(\kappa\). I.e., \((\exists \alpha \in \kappa) (\forall \beta \in \kappa \setminus \alpha) \neg c_{\alpha} \restriction R_i\).

Proof. Work in the intermediate extension after \(\kappa\) many Cohen reals, let us call it \(V_{\alpha}\). The remaining forcing (i.e., \(\mu \setminus \kappa\) many Cohens composed with the good forcing) is good; so applying the definition we get (in \(V_{\alpha}\)) a set \(Y \subseteq Y\) of size \(\kappa\).

As the initial Cohen extension is ccc, and \(\kappa \geq \lambda\) is regular, we get some \(\alpha \in \kappa\) such that each element \(y\) of \(Y\) already exists in the extension by the first \(\alpha\) many Cohens, call it \(V_{\alpha}\).

Fix some \(\beta \in \kappa \setminus \alpha\) and \(y \in Y\). As \(\{ x \in X : x \restriction R_i y\}\) is a meager set already defined in \(V_{\alpha}\), we get \(\neg c_{\beta} \restriction R_i y\). Accordingly, \(c_{\beta}\) is unbounded by \(Y\); and, by the definition of good, unbounded by \(\alpha\) as well.

In the light of this result, let us revisit Lemma 2.5 with some new notation, the “linearly cofinally unbounded” property LCU:

Definition 2.11. For \(i = 1, 2, 3, 4, \gamma\) a limit ordinal, and \(P\) a ccc forcing notion, let \(\text{LCU}_i(P, \gamma)\) stand for:

There is a sequence \((x_{\alpha})_{\alpha \in \gamma}\) of \(P\)-names such that for every \(P\)-name \(y\) \(\exists \alpha \in \gamma\) \(\forall \beta \in \gamma \setminus \alpha\) \(P \not\vDash \neg x_{\beta} \restriction R_i y\).

Lemma 2.12. \(\text{LCU}_i(P, \delta)\) is equivalent to \(\text{LCU}_i(P, \text{cf}(\delta))\).

If \(\lambda\) is regular, then \(\text{LCU}_i(P, \lambda)\) implies \(\text{b}_{i} \leq \lambda\) and \(\text{c}_{i} \geq \lambda\).

In particular:

1. \(\text{LCU}_1(P, \lambda)\) implies \(P \vDash (\text{add}(\mathcal{N}) \leq \lambda \;\&\; \text{cof}(\mathcal{N}) \geq \lambda)\).
2. \(\text{LCU}_2(P, \lambda)\) implies \(P \vDash (\text{b} \leq \lambda \;\&\; \text{c} \geq \lambda)\).
3. \(\text{LCU}_3(P, \lambda)\) implies \(P \vDash (\text{cov}(\mathcal{N}) \leq \lambda \;\&\; \text{non}(\mathcal{N}) \geq \lambda)\).
4. \(\text{LCU}_4(P, \lambda)\) implies \(P \vDash (\text{non}(\mathcal{M}) \leq \lambda \;\&\; \text{cov}(\mathcal{M}) \geq \lambda)\).
Proof. Assume that \((a_\beta)_{\beta \in \delta}\) is increasing continuous and cofinal in \(\delta\). If \((x_\alpha)_{\alpha \in \delta}\) witnesses \(\text{LCU}(P, \delta)\), then \((x_{a_\beta})_{\beta \in \delta}\) witnesses \(\text{LCU}(P, \text{cf}(\delta))\). And if \((x_\beta)_{\beta \in \delta}\) witnesses \(\text{LCU}(P, \text{cf}(\delta))\), then \((y_\alpha)_{\alpha \in \delta}\) witnesses \(\text{LCU}(P, \text{cf}(\delta))\), where \(y_\alpha := x_\beta\) for \(\alpha, \beta \in [a_\beta, a_{\beta+1})\).

The set \(\{x_\alpha : \alpha \in \lambda\}\) is certainly forced to be \(R_i\)-unbounded; and given a set \(Y = \{y_j : j < \theta\}\) of \(\theta < \lambda\) many \(P\)-names, each has a bound \(a_j \in \lambda\) so that \((\forall \beta \in \lambda \setminus a_j) P \not\models \neg x_\beta R_i y_j\), so for any \(\beta \in \lambda\) above all \(a_j\) we get \(P \models \neg x_\beta R_i y_j\) for all \(j\); i.e., \(Y\) cannot be dominating. \(\Box\)

2.B. The initial forcing \(\mathbb{P}^5\) and the COB property. We will make the following throughout the paper:

Assumption 2.13. \(\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5\) are regular uncountable cardinals such that \(\mu < \lambda_i\) implies \(\mu^{\lambda_i} < \lambda_i\).

- We set \(\delta_5 = \lambda_3 + \lambda_5\), and partition \(\delta_5 \setminus \lambda_5\) into unbounded sets \(S_i\) for \(i = 1, \ldots, 4\). Fix for each \(\alpha \in \delta_5 \setminus \lambda_5\) a \(w_\alpha \subseteq \alpha\) such that \(\{w_\alpha : \alpha \in S_i\}\) is cofinal\(^8\) in \([\delta_5]^{<\lambda_i}\) (for each \(i = 1, \ldots, 4\)).

The reader can assume that \((\lambda_i)_{i=1, \ldots, 5}\) and \((S_i)_{i=1, \ldots, 4}\) have been fixed once and for all (let us call them “fixed parameters”), whereas we will investigate various possibilities for \(\bar{w} = (w_\alpha)_{\alpha \in \delta_5 \setminus \lambda_5}\) in the following. (We will call a \(\bar{w}\) which satisfies the assumption a “cofinal parameter”.)

We define by induction:

Definition 2.14. We define the FS iteration \((P_\alpha, Q_\alpha)_{\alpha \in \delta_5}\) and, for \(\alpha > \lambda_5\), \(P_\alpha'\) as follows:

If \(\alpha \in \lambda_5\), then \(Q_\alpha\) is Cohen forcing. In particular, the generic at \(\alpha\) is determined by the Cohen real \(\eta_\alpha\). For \(\alpha \in \delta_5 \setminus \lambda_5\):

\[
\begin{align*}
Q_\alpha^{\text{full}} := \begin{cases}
\text{Amoeba} & \text{for } \alpha \in S_1 \\
\text{Hechler} & \text{for } \alpha \in S_2 \\
\text{Random} & \text{for } \alpha \in S_3 \\
\mathcal{E} & \text{for } \alpha \in S_4
\end{cases}
\]

So \(Q_\alpha^{\text{full}}\) is a Borel definable subset of the reals, and the \(Q_\alpha^{\text{full}}\)-generic is determined, in a Borel way, by the canonical generic real \(\eta_\alpha\).

\(P_\alpha'\) is the set of conditions \(p \in P_\alpha\) satisfying the following, for each \(\beta \in \text{supp}(p)\):

- \(\beta \in w_\alpha\) and there is \((\text{in the ground model})\) a countable \(u \subseteq w_\alpha \cap \beta\) and a Borel function \(B : (\omega^{<\omega})^\alpha \to Q_\beta^{\text{full}}\) such that \(p \upharpoonright \beta\) forces that \(p(\beta) = B(\eta_\gamma)_{\gamma \in u}\). We assume that

\[
P_\alpha'\]
is a complete subforcing of \(P_\alpha\).

(3) In the \(P_\alpha\)-extension, let \(M_\alpha\) be the induced \(P_\alpha'\)-extension of \(V\). Then \(Q_\alpha\) is the \(M_\alpha\)-evaluation of \(Q_\alpha^{\text{full}}\). Or equivalently (by absoluteness): \(Q_\alpha = Q_\alpha^{\text{full}} \cap M_\alpha\). We call \(Q_\alpha\) a “partial \(Q_\alpha^{\text{full}}\) forcing” (e.g.: a “partial random forcing”).

Some notes:

- For item (3) to make sense, (2.15) is required.
- We do not require any “transitivity” of the \(w_\alpha\), i.e., \(\beta \in w_\alpha\) does generally not imply \(w_\beta \subseteq w_\alpha\).
- We do not require (and it will generally not be true) that \(P_\alpha\) forces that \(Q_\alpha\) is a complete subforcing of \(Q_\alpha^{\text{full}}\).

A simple absoluteness argument (between \(M_\alpha\) and \(V[G_\alpha]\)) shows:

\(^8\)i.e., if \(\alpha \in S_i\) then \(|w_\alpha| < \lambda_i\), and for all \(u \subseteq \delta_5\), \(|u| < \lambda_i\) there is some \(\alpha \in S_i\) with \(w_\alpha \supseteq u\).
Lemma 2.16. $P_a$ forces:

(a) $Q_a$ is an incompatibility preserving subforcing of $Q_a^{\text{full}}$ and in particular ccc. (And so, $P_a$ itself is ccc for all $\alpha$.)

(b) For $a \in S^i$, $|Q_a| < \lambda_i$.

(c) $Q_a$ forces that its generic filter $G(\alpha)$ is also generic over $M_a$. So from the point of view of $M_a$, $M_a[G(\alpha)]$ is a $Q_a^{\text{full}}$-extension.

(2) For $a \in S^2$: The partial Hechler forcing $Q_a$ is $\sigma$-centered.

(3) For $a \in S^3$: The partial random forcing $Q_a$ is equivalent to a subalgebra of the random algebra.

(4) For $a \in S^4$: A partial $\bar{E}$ forcing is $(\rho, \pi)$-linked and basically equivalent to a subalgebra of the random algebra (as in Lemma 1.19(e)).

Proof. (b): $|P_a^\alpha| \leq |w_a|^{\aleph_0} \times 2^{\aleph_0} < \lambda_i$ by Assumption 2.13. There is a set of nice $P_a^\alpha$-names of size $< \lambda_i$ such that every $P_a^\alpha$-name for a real has an equivalent name in this set. Accordingly, the size of the reals in $M_a$ is forced to be $< \lambda_i$.

(c) is trivial, as $Q_a$ is element of the transitive class $M_a$.

(4): By Lemma 1.19(b) we know that $M_a$ thinks that $\bar{E}$ is $(\rho, \pi)$-linked; i.e., that there is a family $^9 Q'_i$ as in Definition 1.18. Being $\ell'$-linked is obviously absolute between $M_a$ and $V[G_a]$ (for any $\ell' < \omega$); and $M_a \models \bigcup_{h \in \eta, \in \rho(h)} Q_i^h = Q_a^{\text{full}}$ translates to $V[G_a] \models \bigcup_{h \in \eta, \in \rho(h)} Q_i^h = Q_a$. So from the point of view of $V[G_a]$, there is a $Q'$ dense in $\bar{E} \cap M_a$ and a dense embedding of $Q'$ into some $B'$, which is a subalgebra of the random algebra in $M_a$ and therefore of the random algebra in $V[G_a]$.

It is easy to see that (2.15) is a “closure property” of $w_a$:

Lemma 2.17. Assume we have constructed (in the ground model) $(P_\beta, Q_\beta)_{\beta \ll a}$ and $w_a$ according to Definition 2.14; for some $a \in S^i$, $i = 1, \ldots, 4$. This determines the (limit or composition) $P_a$.

(a) For every $P_a$-name $\tau$ of a real, there is (in $V$) a countable $u \subseteq a$ and a Borel function $B : (\omega^\omega)^u \to \omega^\omega$ such that $P_a$ forces $\tau = B(\eta_\tau)_{\tau \in w_a}$.

(b) The set of $w_a$ satisfying (2.15) is an $\omega_1$-club in $[a]^{< \omega_1}$ (in the ground model).

(A set $A \subseteq [a]^{< \omega_1}$ is an $\omega_1$-club, if for each $a \in [a]^{< \omega_1}$ there is a $b \supseteq a$ in $A$, and if $(a\rho)_{\rho \in \omega_1}$ is an increasing sequence of sets in $A$, then the limit $b := \bigcup_{\rho \in \omega_1} a\rho$ is in $A$ as well.)

Proof. The first item follows easily from the fact that we are dealing with a FS ccc iteration where the generics of all iterands $Q_\beta$ are Borel-determined by some generic real $\eta_\beta$. (See, e.g., [KTT18, 1.2], for more details.)

Any $w \in [a]^{< \omega_1}$ defines some $P_w^\alpha$. We first define $w'$ for such a $w$:

Set $X = [P_w^\alpha]^{\aleph_0}$, as set of size at most $(2^{\aleph_0} \times |w|^{\aleph_0})^{\aleph_0} < \lambda_i$. For $x \in X$, pick some $p \in P_a$ stronger than all conditions in $x$ (if such a condition exists), and some $q \in P_a$ incompatible to each element of $x$ (again, if possible). There is a countable $w_x \subseteq a$ such that $p, q \in P_w^x$. Set $w' := w \cup \bigcup_{x \in X} w_x$.

---

9 Actually there is even a Borel definable family $Q'_i$, see the proof of Lemma 1.19(a), but this is not required here.
Start with any $w_0 \in [\alpha]^{<\lambda}$. Construct an increasing continuous chain in $[\alpha]^{<\lambda}$ with $w^{k+1} = (w^k)'$. Then $w^{\omega_1} \supseteq w_0$ is in the set of $w$ satisfying (2.15); which shows that this set is unbounded. It is equally easy to see that it is closed under increasing sequences of length $\omega_1$.

For later reference, we explicitly state the assumption we used (for every $\alpha \in \delta_3 \setminus \lambda_3$):

**Assumption 2.18.** $w_a$ is sufficiently closed so that (2.15) is satisfied.

Let us also restate Lemma 2.17(a):

**Lemma 2.19.** For each $\mathbb{P}^5$-name $f$ of a real, there is a countable set $u \subseteq \delta_3$ such that $w_a \supseteq u$ implies that ($\mathbb{P}^5$ forces that) $f \in M_a$.

**Lemma 2.20.** $\text{LCU}_i(\mathbb{P}^5, \kappa)$ holds for $i = 1, 3, 4$ and each regular cardinal $\kappa$ in $[\lambda_1, \lambda_3]$.

**Proof.** This follows from Lemma 2.16:

For $i = 1$: Partial random and partial $\mathbb{E}$ forcings are basically equivalent to a sub-Boolean algebra of the random algebra; and partial Hechler forcings are $\sigma$-centered. The partial amoeba forcings are small, i.e., have size $<\lambda_1$. So according to Lemma 2.8, all iterands $Q_\alpha$ (and therefore the limits as well) are $(R_1, \lambda_1)$-good.

For $i = 3$, note that partial $\mathbb{E}$ forcings are $(\rho, \pi)$-linked. All other iterands have size $<\lambda_3$, so the forcing is $(R_3, \lambda_3)$-good.

For $i = 4$ it is enough to note that all iterands are small, i.e., of size $<\lambda_4$.

We can now apply Lemma 2.10. \qed

So in particular, $\mathbb{P}^5$ forces $\text{add}(\mathcal{N}) \leq \lambda_1$, $\text{cov}(\mathcal{N}) \leq \lambda_3$, $\text{non}(\mathcal{M}) \leq \lambda_4$ and $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \lambda_3 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. We now show that they are also large, using the “cone of bounds” property $\text{COB}$:

**Definition 2.21.** For a ccc forcing notion $P$, regular uncountable cardinals $\lambda, \mu$ and $i = 1, 2, 4$, let $\text{COB}_i(P, \lambda, \mu)$ stand for:

There is a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s \in S}$ of $P$-names for reals such that for each $P$-name $f$ of a real

$(\exists s \in S) (\forall t \succ s) \ P \vDash f \in R_s g_t$.

For $i = 3$, let $\text{COB}_3(P, \lambda, \mu)$ stand for:

There is a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s \in S}$ of $P$-names for reals such that for each $P$-name $f$ of a null-set

$(\exists s \in S) (\forall t \succ s) \ P \vDash g_t \notin f$.

So $s$ is the tip of a cone that consists of elements bounding $f$, where in case $i = 3$ we implicitly use an additional relation $N R_3' r$ expressing that the null-set $N$ doesn’t contain the real $r$. Note that $\text{cov}(\mathcal{N})$ is the bounding number $\mathbb{b}_3'$ of $R_3'$, and $\text{non}(\mathcal{N})$ the dominating number $\mathbb{d}_3$. So $\text{add}(\mathcal{N}) = \mathbb{b}_3' \leq \mathbb{b}_3$ and $\text{non}(\mathcal{N}) = \mathbb{d}_3' \geq \mathbb{d}_3$ (as defined in Lemma 2.5).

$\text{COB}_1(P, \lambda, \mu)$ implies that $P$ forces that $\mathbb{b}_1 \geq \lambda$ and that $\mathbb{d}_1 \leq \mu$ (for $i = 1, 2, 4$, and the same for $i = 3$ and $\mathbb{b}_1', \mathbb{d}_1'$). Clearly $P$ forces that $\{g_s : s \in S\}$ is dominating. And if $A$ is set of names of size $\kappa < \lambda$, then for each $f \in A$ the definition gives a bound $s(f)$ and directedness some $t > s(f)$ for all $f$, i.e., $g_t$ bounds all elements of $A$. So we get:

**Lemma 2.22.**

1. $\text{COB}_1(P, \lambda, \mu)$ implies $P \vDash (\text{add}(\mathcal{N}) \geq \lambda \& \text{cof}(\mathcal{N}) \leq \mu)$.
2. $\text{COB}_2(P, \lambda, \mu)$ implies $P \vDash (\mathbb{b} \geq \lambda \& \mathbb{d} \leq \mu)$.
3. $\text{COB}_3(P, \lambda, \mu)$ implies $P \vDash (\text{cov}(\mathcal{N}) \geq \lambda \& \text{non}(\mathcal{N}) \leq \mu)$.
4. $\text{COB}_4(P, \lambda, \mu)$ implies $P \vDash (\text{non}(\mathcal{M}) \geq \lambda \& \text{cov}(\mathcal{M}) \leq \mu)$.
Lemma 2.23. \( \text{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_s) \) holds (for \( i = 1, 2, 3, 4 \)).

Proof. We use the following facts (provable in ZFC, or true in the \( P_a \)-extension, respectively):

1. Amoeba forcing adds a sequence \( \bar{b} \) which \( R_1 \)-dominates the old elements of \( C \).
   
   (The simple proof can be found in [GKS, Lem. 1.4], a slight variation in [BJ95].)
   
   Accordingly (by absoluteness), the generic real \( \eta_a \) for partial amoeba forcing
   
   \[ Q_a \ R_1 \)-dominates \( C \cap M_a \).
2. Hechler forcing adds a real which \( R_2 \)-dominates all old reals.
   
   Accordingly, the generic real \( \eta_a \) for partial Hechler forcing \( Q_a \ R_2 \)-dominates
   
   all reals in \( M_a \).
3. Random forcing adds a random real.
   
   Accordingly, the generic real \( \eta_a \) for partial random forcing \( Q_a \) is not in any
   
   nullset whose Borel-code is in \( M_a \).
4. The generic branch \( \eta \in \lim(T^*) \) added by \( \bar{E} \) is eventually different to each old real,
   
   i.e., \( R_4 \)-dominates the old reals.
   
   (This was shown in Lemma 1.19(c).)
   
   Accordingly, the generic branch \( \eta_a \) for partial \( \bar{E} \)-forcing \( Q_a \ R_4 \)-dominates the
   
   reals in \( M_a \).

Fix \( i \in \{1, 2, 3, 4\} \), and set \( S = S_i \) and \( s < t \) if \( w_s \not\subseteq w_t \), and let \( g_i \) be \( \eta_i \), i.e., the generic
   
   added at \( s \) (e.g., the partial random real in case of \( i = 3 \), etc).
   
   Fix a \( \mathbb{P}^5 \)-name \( f \) for a real. It depends (in a Borel way) on a countable index set \( w^* \subseteq \delta_S \).
   
   Fix some \( s \in S \) such that \( w_s \supseteq w^* \). Pick any \( i > s \). Then \( w_i \supseteq w_s \supseteq w^* \), so (\( \mathbb{P}^5 \) forces
   
   that) \( f \in M_i \), so, as just argued, \( \mathbb{P}^5 \models \neg \ R_i g_i \) (or: \( \mathbb{P}^5 \models \neg \ R_i' g_i \) for \( i = 3 \)).

So to summarize what we know so far about \( \mathbb{P}^5 \): Whenever we choose (in addition to the “fixed” \( \lambda, S \)) a cofinal parameter \( \bar{w} \) satisfying Assumptions 2.13 and 2.18, we get

Fact 2.24.

- \( \text{COB}_i \) holds for \( i = 1, 2, 3, 4 \). So the left hand side characteristics are
   
   large.
- \( \text{LCU}_i \) holds for \( i = 1, 3, 4 \). So the left hand side characteristics other than \( \mathbb{b} \) are
   
   small.

What is missing is “\( \mathbb{b} \) small”. We do not claim that this will be forced for every \( \bar{w} \) as above; but we will show in the rest of Section 2 that we can choose such a \( \bar{w} \).

2.C. FAMs in the \( P_a \)-extension compatible with \( M_a \), explicit conditions. We first investigate sequences \( \bar{q} = (q_\alpha)_{\alpha \in \omega} \) of \( Q_a \)-conditions that are in \( M_a \), i.e., the (evaluations of)
   
   \( P_a^t \)-names for \( o \)-sequences in \( Q^a \). \( \alpha \in S^3 \cup S^4 \), \( M_a \) thinks that \( Q_a \) (i.e., \( Q^a \) has FAM-limits. So if \( M_a \) thinks that \( \Xi_a \) is a FAM, then for any sequence \( \bar{q} \) in \( M_a \) there is a
   
   condition \( \lim_{\Xi_a}(\bar{q}) \) in \( M_a \) (and thus in \( Q_a \)). We can relativize Lemma 1.8 to sequences in \( M_a \):

Lemma 2.25. Assume that \( \alpha \in S^3 \cup S^4 \), that \( \Xi \) is a \( P_a \)-name for a FAM and that \( \Xi_0 \), the
   
   restriction of \( \Xi \) to \( M_a \), is forced to be in \( M_a \). Then there is a \( P_{a+1} \)-name \( \Xi^+ \) for a FAM
   
   such that for all (trunk*, loss*\()-sequences \( \bar{q} \) in \( M_a \),
   
   \[ \lim_{\Xi_a}(\bar{q}) \in G(\alpha) \implies \Xi^+(A_\bar{q}) \geq 1 - \sqrt{\text{loss}^*}. \]

\( A_\bar{q} \) was defined in (1.9) (here we use \( G(\alpha) \) instead of \( G \), of course).
Proof. This Lemma is implicitly used in [She00]. Note that $P'_a$ is a complete subforcing of $P_a$, and so there is a quotient $R$ such that $P_a = P'_a \ast R$. We consider the following (commuting) diagram:

$$
\begin{array}{ccc}
V & \xrightarrow{P_a} & V_a \\
\downarrow & & \downarrow \\
V_a & \xrightarrow{Q_a} & V_{a+1} \\
\uparrow & & \\
M_a & \xrightarrow{R} & Q_a
\end{array}
$$

Note that ($P'_a$ forces that) $R \ast Q_a = R \times Q_a$. So from the point of view of $M_a$:

- $Q_a = Q_a^{\text{full}}$ has FAM limits, and $\Xi_0$ is a FAM. So there is a $Q_a$-name for a FAM $\Xi_0^+$ satisfying Lemma 1.8.
- $R$ is a ccc forcing, and there is an $R$-name$^{10}$ $\Xi$ for a FAM extending $\Xi_0$.
- So there is $R \times Q_a$-name $\Xi^+$ for a FAM extending both $\Xi_0^+$ and $\Xi$ (cf. [She00, Claim 1.6]).

Back in $V$, this defines the $P_{a+1}$-name $\Xi^+$. Let $\tilde{q} = (q_\ell)_{\ell \in \omega}$ be a sequence in $M_a$. Then $M_a[G(\alpha)]$ thinks: If $\lim_{\Xi_0}(\tilde{q}) \in G(\alpha)$, then $\Xi_0^+(A_{\tilde{q}})$ is large enough. This is upwards absolute to $V[G_{a+1}]$ (as $A_{\tilde{q}}$ is absolute).

For later reference, we will reformulate the Lemma for a specific instance of “sequence in $M_a$”. Recall that a sequence in $M_a$ corresponds to a “$P'_a$-name of a sequence in $Q_a^{\text{full}}$”. This is not equivalent to a “$P'_a$-name for a sequence in $Q_a$”, which would correspond to an arbitrary sequence in $Q_a$ (of which there are $|\alpha + \aleph_0|^{\aleph_0}$ many, while there are only less than $\lambda_\alpha$ many sequences in $M_a$). However, we can define the following:

Definition 2.26.  
- An explicit $Q_a$-condition (in $V$) is a $P'_a$-name for a $Q_a^{\text{full}}$-condition.
- A condition $p \in \mathbb{B}^5$ is explicit, if for all $\alpha \in \text{supp}(p) \cap (S^4 \cup S^5)$, $p(\alpha)$ is an explicit $Q_a$-condition.

Here we mean that for $p(\alpha)$ there is a $P'_a$-name $q_\alpha$ such that $p \upharpoonright \alpha \forces p(\alpha) = q_\alpha$ (and the map $\alpha \mapsto q_\alpha$ exists in the ground model, i.e., we do not just have a $P'_a$-name for a $P'_a$-condition $q_\alpha$).

Lemma 2.27. The set of explicit conditions is dense.

Proof. We show by induction that the set $D_a$ of explicit conditions in $P_a$ is dense in $P_a$. As we are dealing with FS iterations, limits are clear. Assume that $(p,q) \in P_{a+1}$. Then $p$ forces that there is a $P'_a$-name $q'$ such that $q' = q$. Strengthen $p$ to some $p' \in D_a$ deciding $q'$. Then $(p', q') \leq (p,q)$ is explicit.

Note that any sequence in $V$ of explicit $Q_a$-conditions defines a sequence of conditions in $M_a$ (as $V \subseteq M_a$). So we get:

Lemma 2.28. Let $\alpha$, $\Xi$, and $\Xi^+$ be as in Lemma 2.25, and let $(p_\ell)_{\ell \in \omega}$ be (in $V$) a sequence of explicit conditions in $\mathbb{B}^5$ such that $\alpha \in \text{supp}(p_\ell)$ for all $\ell \in \omega$. Set $q_\ell := p_\ell(\alpha)$ and $\tilde{q} := (q_\ell)_{\ell \in \omega}$ and assume that $(\text{trunk}(q_\ell), \text{loss}(q_\ell))$ is forced to be equal to some constant (trunk$^+$, loss$^+$).

Then there is a $P'_a$-name for a $Q_a^{\text{full}}$-condition (and thus a $P'_a$-name for a $Q_a$-condition) $\lim_{\Xi_0}(\tilde{q})$ such that $\lim_{\Xi_0}(\tilde{q})$ forces that $\Xi^+(A_{\tilde{q}}) \leq 1 - \sqrt{\text{loss}^+}$.

$^{10}$We identify the $P_a$-name $\Xi$ in $V$ and the induced $R$-name in $M_a = V[G_a]$. 

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2.D. Dealing with b (without GCH). In this section, we follow [GKS, 1.3], additionally using techniques inspired by [She00].

We assume the following (in addition to Assumption 2.13):

**Assumption 2.29.** (This section only.) $\chi < \lambda_3$ is regular such that $\chi^{\aleph_0} = \chi$, $\chi^+ \geq \lambda_2$ and $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2$. So $\delta_5 = S^0 \cup S^3 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along $\delta_5$ such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_2$, i.e., $|Q_\alpha| \leq \chi$ (and $Q_\alpha$ is a partial random forcing for $\alpha \in S^1$ and a partial $\mathcal{E}$-forcing for $\alpha \in S^4$).

Let us fix, for each $\alpha \in S^0$, a $P_\alpha$-name

\[
i_\alpha : Q_\alpha \rightarrow \chi \text{ injective.}
\]

**Definition 2.31.**
- A “partial guardrail” is a function $h$ defined on a subset of $\delta_5$ such that, for $\alpha \in \text{dom}(h)$: $h(\alpha) \in \chi$ if $\alpha \in S^0$; and $h(\alpha)$ is a pair $(x, y)$ with $x \in H(\aleph_0)$ and $y$ a rational number otherwise. (Any (trunk, loss)-pair is of this form.)
- A “countable guardrail” is a partial guardrail with countable domain. A “full guardrail” is a partial guardrail with domain $\delta_5$.

We will use the following lemma, which is a consequence of the Engelking-Karlowicz theorem [EK65] on the density of box products (cf. [GMS16, 5.1]):

**Lemma 2.32.** (As $|\delta_5| \leq 2^\chi$.) There is a family $H^*$ of full guardrails of cardinality $\chi$ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an $H^*$.

Note that the notion of guardrail (and the density property required in Lemma 2.32) only depends on the “fixed” parameters $\chi, \delta_5, S^0, S^3$ and $S^4$; so we can fix an $H^*$ that will work for all these fixed parameters and all choices of the cofinal parameter $\vec{a}$.

Once we have decided on $\vec{a}$, and thus have defined $\mathbb{P}^5$, we can define the following:

**Definition 2.33.** $D^* \subseteq \mathbb{P}^5$ consists of $p$ such that there is a partial guardrail $h$ (and we say: “$p$ follows $h$”) with $\text{dom}(h) \supseteq \text{supp}(p)$ and, for all $\alpha \in \text{supp}(p)$,

- If $\alpha \in S^0$, then $p \nmid \alpha \vdash i_\alpha(p(\alpha)) = h_\alpha$.
- If $\alpha \in S^3 \cup S^4$, the empty condition of $P_\alpha$ forces $p(\alpha) \in Q_\alpha$ and $(\text{trunk}(p(\alpha)), \text{loss}(p(\alpha))) = h(\alpha)$.
- Furthermore, $\sum_{\alpha \in \text{supp}(p) \cap (S^3 \cup S^4)} \text{loss}(p(\alpha)) < \frac{1}{2}$.
- $p$ is explicit (as in Definition 2.26).

**Lemma 2.34.** $D^* \subseteq \mathbb{P}^5$ is dense.

**Proof.** By induction we show that for any sequence $(\epsilon_i)_{i \in \omega}$ of positive numbers the following set of $p$ is dense: If $\text{supp}(p) = \{a_0, \ldots, a_m\}$, where $a_0 > a_1 > \ldots$ (i.e., we enumerate downwards), $\text{loss}_{\hat{p}}^{\hat{\alpha}_n} < \epsilon_n$ whenever $\alpha_n \in S^3 \cup S^4$. For the successor step, we use that the set of $q \in Q_\alpha$ such that $\text{loss}(q) < \epsilon_0$ is forced to be dense. \hfill $\square$

**Remark 2.35.** So the set of conditions following some guardrail is dense. For each fixed guardrail $h$, the set of all conditions $p$ following $h$ is $n$-linked, provided that each loss in the domain of $h$ is $\leq \frac{1}{n}$ (cf. Assumption 1.5).

**Definition 2.36.** A “$\Delta$-system with heart $V$ following the guardrail $h$” is a family $\vec{p} = (p_\gamma)_{\gamma \in I}$ of conditions such that

- all $p_\gamma$ are in $D^*$ and follow $h$,
• (supp(\(p_i\)))\(_{i \in I}\) is a \(\Delta\) system with heart \(V\) in the usual sense (so \(V \subseteq \delta_\kappa\) is finite)
• the following is independent of \(i \in I\):
  - \(|\text{supp}(p_i)|\), which we call \(m^\beta\).
  - Let \((\bar{\alpha}_i)_{n \leq m^\beta}\) increasingly enumerate \(\text{supp}(p_i)\).
  - Whether \(\bar{\alpha}_i\) is less than, equal to or bigger than the \(k\)-th element of \(V\).
  - In particular it is independent of \(i\) whether \(\bar{\alpha}_i \in V\), in which case we call \(i\) a "heart position".
  - Whether \(\bar{\alpha}_i\) is in \(S^0\), in \(S^3\) or in \(S^4\).
  - If \(\bar{\alpha}_i \in S^1\), we call \(i\) an "\(S^1\)-position".
  - If \(n\) is not an \(S^0\)-position: \(^{11}\) The value of \(h(\bar{\alpha}_i) (= (\text{trunk}^\beta, \text{loss}^\beta))\) if \(n\) is an \(S^0\)-position, we set \(\text{loss}^\beta := 0\).

A "countable \(\Delta\)-system" \(\bar{\rho} = (\rho_\ell : \ell \in \omega)\) is a \(\Delta\) system that additionally satisfies:

• For each non-heart position \(^{12}\) \(\beta = m^\beta\), the sequence \((\bar{\alpha}_\ell)_{\ell \in \omega}\) is strictly increasing.

**Fact 2.37.**
• Each infinite \(\Delta\)-system \((p_i)_{i \in I}\) contains a countable \(\Delta\)-system. I.e., there is a sequence \(i_\ell \in I\) such that \((p_{i_\ell})_{\ell \in \omega}\) is a countable \(\Delta\)-system.
• If \(\bar{\rho}\) is a \(\Delta\)-system (or: a countable \(\Delta\)-system) following \(h\) with heart \(V\), and \(\beta \in V \cup (\max(V + 1))\), then \(\bar{\rho} \uparrow \beta := (p_i \uparrow \beta)_{i \in I}\) is again a \(\Delta\)-system (or: a countable \(\Delta\)-system, respectively) following \(h\), now with heart \(V \cap \beta\).

**Definition 2.38.** Let \(\bar{\rho}\) be a countable \(\Delta\)-system, and assume that \(\bar{\Xi} = (\Xi_{a\in\mathcal{V}\cap(S^1\cup S^3)})\) is a sequence such that each \(\Xi_a\) is a \(P^a\)-name for a FAM and \(P_a\) forces that \(\Xi_a\), restricted to \(M_a\), is in \(M_a\). Then we can define \(q = \lim_{\Xi}\bar{\rho}\) to be the following \(P^5\)-condition with support \(\mathcal{V}\):

  - If \(a \in \mathcal{V} \cap S^0\), then \(q(a)\) is the common value of all \(p_n(a)\). (Recall that this value is already determined by the guardrail \(h\).
  - If \(a \in \mathcal{V} \cap (S^1 \cup S^3)\), then \(q(a)\) is (forced by \(P^5\) to be) \(\lim_{\Xi}(p_\ell(a))_{\ell \in \omega}\), see Lemma 2.28.

We now give a specific way to construct such \(\bar{u}\), which allows to keep \(b\) small.

**Lemma/Construction 2.39.** We can construct by induction on \(a \in \delta_\kappa\) for each \(h \in H^*\) some \(\Xi_{a^h}\) and, if \(a > \kappa_\delta\), also \(w_{a^h}\), such that:

(a) Each \(\Xi_{a^h}\) is a \(P^a\)-name of a FAM extending \(\bigcup_{\beta < a} \Xi_{\beta}\).
(b) If \(a\) is a limit of countable cofinality: Assume \(\bar{\rho}\) is a countable \(\Delta\)-system in \(P_a\) following \(h\), and \(n < m^\beta\) such that \((\bar{\alpha}_\ell)_{\ell \in \omega}\) has supremum \(a\). Then \(A_{\beta,n}\) is forced to have \(\Xi_{a^h}\)-measure 1, where

\[A_{\beta,n} = \left\{ k \in \omega : \left| \{ \ell \in I_k : p_\ell(\bar{\alpha}_\ell) \in G(\bar{\alpha}_\ell) \} \right| \geq |I_k| \cdot \left( 1 - \sqrt{\text{loss}^\beta} \right) \right\}\]

\(^{11}\)If \(n\) is a \(S^0\)-position, \(h(\bar{\alpha}_\ell)\) will generally not be independent of \(i\); unless of course \(n\) is a heart position.

\(^{12}\)For a heart position \(n\), \((\bar{\alpha}_\ell)_{\ell \in \omega}\) is of course constant.
(c) For each countable $\Delta$-system $\bar{p}$ in $P_a$ following $h$, the $P_a$-condition $\lim_{e_{\beta}p_{\alpha}}(\bar{p})$ is well-defined and forces

$$\Xi^h(A_{\beta}) \geq 1 - \sum_{n<m^\beta} \sqrt{\text{loss}^{h,n}}, \text{ where}$$

$$A_{\beta} := \left\{ k \in \omega : \left| \{ \ell \in I_k : p_{\ell} \in G_\alpha \} \right| \geq |I_k| \cdot \left( 1 - \sum_{n<m^\beta} \sqrt{\text{loss}^{h,n}} \right) \right\}. $$

(d) For $\alpha > \kappa_S$: $w_\alpha$ is “sufficiently closed”. More specifically: It satisfies Assumptions 2.13 and 2.18, and if $\alpha \in S^3 \cap S^4$ then $P_a$ forces that $\Xi^h$ restricted to $M_a$ is in $M_a$.

Actually, the set of $w_\alpha$ satisfying this is an $\omega_1$-club set.

\textbf{Proof. (a&c) for cf$(\alpha) > \omega$:} We set $\Xi^h_a = \bigcup_{\beta < \alpha} \Xi^h_\beta$. As there are no new reals at uncountable cofinalities, this is a FAM. Each countable $\Delta$-system is bounded by some $\beta < \alpha$, and, by induction, (c) holds for $\beta$; so (c) holds for $\alpha$ as well.

\textbf{(a&b) for cf$(\alpha) = \omega$:} Fix $h$. We will show that $P_\alpha$ forces $A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} \neq \emptyset$, where $A$ is a $\Xi^h_{\beta}$-positive set for some $\beta < \alpha$, and each $(\bar{p}^j, n^j)$ is as in (b).

Then we can work in the $P_\alpha$-extension and apply Fact 1.3(a), using $\bigcup_{\beta < \alpha} \Xi^h_\beta$ as the partial FAM $\Xi'$. This gives an extension of $\Xi'$ to a FAM $\Xi^h_a$ that assigns measure one to all $A_{\bar{p}^\beta,n^\beta}$, showing that (a) and (b) are satisfied.

So assume towards a contradiction that some $p \in P_a$ forces

$$A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} = \emptyset. $$

We can assume that $p$ decides the $\beta$ such that $A \in V_\beta$, that $\beta$ is above the hearts of all $\Delta$-sequences $\bar{p}^j$ involved, and that $\text{supp}(p) \subseteq \beta$. We can extend $p$ to some $p^* \in P_\beta$ to decide $k \in A$ for some “large” $k$: By large, we mean:

- Let $F(l,n,p)$ (the cumulative binomial probability distribution) be the probability that $n$ independent experiments, each with success probability $p$, will have at most $l$ successful outcomes. As $\lim_{n \to \infty} F(n \cdot p'; n, p) = 0$ for all $p' < p$, and as $\lim_{k \to \infty} |I_k| = \infty$, we can find some $k$ such that

$$(2.40) \quad F(|I_k|p^*_j, |I_k|, p_j) < \frac{1}{2 \cdot j^*}$$

for all $j < j^*$, where we set $p^*_j := 1 - \sqrt{\text{loss}^{\bar{p}^j,n^j}}$ and $p_j := 1 - \frac{1+\sqrt{2}}{2} \cdot \text{loss}^{\bar{p}^j,n^j}$.

(Note that $p^*_j < p_j$, as $\text{loss}^{\bar{p}^j,n^j} < \frac{1}{2}$.)

- All elements of $Y = \{ a_{\bar{p}^j,n^j}^\ell : j < j^* \text{ and } \ell \in I_k \}$ are larger than $\beta$. (This is possible as each sequence $(a_{\bar{p}^j,n^j}^\ell)_{\ell \in \omega}$ has supremum $\alpha$.) We enumerate $Y$ by the increasing sequence $(\bar{p}_j)_{j \in M_a}$, and set $\beta_{-1} = \beta$.

We will find $q \leq p^*$ forcing that $k \in \bigcap_{j<j^*} A_{\bar{p}^j,n^j}$. To this end, we define a finite tree $T$ of height $M_a$, and assign to each $s \in T$ of height $i$ a condition $q_s \in P_{\beta_{i+1}}$ (decreasing along each branch) and a probability $p_r_s \in [0, 1]$, such that $\sum_{r \in s} p_r_s = 1$ for all non-terminal nodes $s \in T$. For $s$ the root of $T$, i.e., for the unique $s$ of height 0, we set $q_s = p^* \in P_{\beta_{-1}}$ and $p_r_s = 1$. 


So assume we have already constructed \( q_s \in P_{\beta_i+1} \) for some \( s \) of height \( i < M \). We will now take care of index \( \beta_i \) and construct the set of successors of \( s \), and for each successor \( t \), a \( q_t \leq q_s \) in \( P_{\beta_i+1} \).

- If \( \beta_i \in S^0 \), the guardrail guarantees that \( \beta_i \in \text{supp}(p'_\ell) \) implies \( p'_\ell \upharpoonright \beta_i \models i_{\beta_i}(p'_\ell(\beta_i)) = h(\beta_i) \). In that case we use a unique \( T \)-successor \( t \) of \( s \), and we set \( q_t = q_s^{-}(\beta_i, i_{\beta_i}^{-1} h(\beta_i)) \), and \( p_r = 1 \).

  In the following we assume \( \beta_i \notin S^0 \).

- Let \( J_i \) be the set of \( j < j^* \) such that there is an \( \ell \in I_k \) with \( \alpha^\ell_{\beta_i} = \beta_i \) (there is at most one such \( \ell \)). For \( j \in J_i \), set \( r^j_i = p'_\ell(\beta_i) \) for the accordingly \( \ell \). So each \( r^j_i \) is a \( P_{\beta_i} \)-name for an element of \( Q_{\beta_i} \).

  The guardrail gives us the constant value \( (\text{trunk}^*_s, \text{loss}^*_s) := h(\beta_i) \) (which is equal to \( (\text{trunk}^*_{pr^i}, \text{loss}^*_{pr^i}) \) for all \( j \in J_i \)).

- The case \( \beta_i \in S^3 \), i.e., the case of random forcing, is basically [She00, 2.14]:

  For \( x \subseteq [\text{trunk}^*_s] \), set \( \text{Leb}^{\text{rel}}(x) = \frac{\text{Leb}(x)}{\text{Leb}([\text{trunk}^*_s]_0)} \). Note that the \( r^j_i \) are closed subsets of \([\text{trunk}^*_s]\) and \( \text{Leb}^{\text{rel}}(r^j_i) \geq 1 - \text{loss}^*_s \).

  Let \( B^* \) be the power set of \([\text{trunk}^*_s] \); and let \( B \) be the sub-Boolean-algebra generated by \( r^j_i \) (\( j \in J_i \)), let \( \mathcal{X} \) be the set of atoms and \( \mathcal{X}' = \{ x \in \mathcal{X} : \text{Leb}^{\text{rel}}(x) > 0 \} \). So \(|\mathcal{X}'| \leq 2^{J_i} \leq 2^{j^*} \cdot \sum_{x \in \mathcal{X}'} \text{Leb}^{\text{rel}}(x) = 1 \), and \( \sum_{x \in \mathcal{X}' \cap \text{trunk}^*_s} \text{Leb}^{\text{rel}}(x) = \text{Leb}^{\text{rel}}(r^j_i) \).

  So far, \( \mathcal{X}' \) is a \( P_{\beta_i} \)-name. Now we increase \( q_s \) inside \( P_{\beta_i} \) to some \( q^+ \) deciding which of the (finitely many) Boolean combinations result in elements of \( \mathcal{X}' \), and also deciding rational numbers \( y_x \) \((x \in \mathcal{X}')\) with sum \( 1 \) such that \( |\text{Leb}^{\text{rel}}(x) - y_x| < \frac{\sqrt{2} - 1}{2} \cdot \text{loss}^*_s \cdot 2^{-j^*} \).

  We can now define the immediate successors of \( s \) in \( T \): For each \( x \in \mathcal{X}' \), add an immediate successor \( t_x \) and assign it to the probability \( p_r = y_x \) and the condition \( q_t = q^+\upharpoonright (\beta_i, r_x) \), where \( r_x \) is a (name for a) partial random condition below \( x \) (such a condition exists, as the Lebesgue positive intersection of finitely many partial random condition contains a partial random condition).

  Note that when we choose a successor \( t \) randomly (according to the assigned probabilities \( p_r \)), then for each \( j \in J \) the probability of \( q^+ \models q_t(\beta_i) \leq r^j_i \) is at least

\[
\sum_{x \in \mathcal{X}' \cap \text{trunk}^*_s} p_r x \geq \sum_{x \in \mathcal{X}' \cap \text{trunk}^*_s} \left( \text{Leb}^{\text{rel}}(x) - \frac{\sqrt{2} - 1}{2} \cdot \text{loss}^*_s \cdot 2^{-j^*} \right) \geq
\geq \left( \sum_{x \in \mathcal{X}' \cap \text{trunk}^*_s} \text{Leb}^{\text{rel}}(x) - \frac{\sqrt{2} - 1}{2} \cdot \text{loss}^*_s = \text{Leb}^{\text{rel}}(r^j_i) - \frac{\sqrt{2} - 1}{2} \cdot \text{loss}^*_s \geq
\geq 1 - \text{loss}^*_s - \frac{\sqrt{2} - 1}{2} \cdot \text{loss}^*_s = 1 - \frac{1 + \sqrt{2}}{2} \cdot \text{loss}^*_s .
\]

- The case \( \beta_i \in S^4 \), i.e., the case of \( \overline{E} \):

  Recall that \( \overline{E} \)-conditions are subtrees of some basic compact tree \( T^* \), and there is a \( h \) such that: if \( \max \{ |I_k|, j^* \} \) many conditions share a common node (above their stems) at height \( h \), then they are compatible.

  All conditions \( r^j_i \) have the same stem \( s^* = \text{trunk}^*_s \). For each \( j \in J_i \), set \( d(j) = r^j_i \cap \omega^h \). Note that \( (P_{\beta_i} \text{ forces that}) \) \( d(j) \) is a subset of \( T^* \cap [s^*] \cap \omega^h \) of relative size
\[ \geq 1 - \frac{1}{2} \cdot \text{loss}^* \] (according to Lemma 1.19(d)). First find \( q^+ \leq q_j \) in \( P_{\beta_i} \) deciding all \( d(j) \).

We can now define the immediate successors of \( s \) in \( T \): For each \( x \in T^* \cap \{s^*\} \cap \omega^h \) add an immediate successor \( t_x \), and assign to it the uniform probability (i.e., \( \Pr_{t_x} = \frac{1}{|T^* \cap \omega^h|} \)) and the condition \( q_{t_x} = q^+ - (\beta_i, r_x) \), where \( r_x \) is a partial \( \vec{E} \)-condition stronger than all \( r'_j \) that satisfy \( x \in d(j) \). (Such a condition exists, as we can intersect \( \leq j^* \) many conditions of height \( h \).

If we chose \( t \) randomly, then for each \( j \in J \) the probability of \( q^+ \models q_i \leq r'_j \) is at least \( 1 - \frac{1}{2} \cdot \text{loss}^* \geq 1 - \frac{1+\sqrt{2}}{2} \cdot \text{loss}^* \).

In the end, we get a tree \( T \) of height \( M \), and we can chose a random branch through \( T \), according to the assigned probabilities. We can identify the branch with its terminal node \( t^* \), so in this notation the branch \( t^* \) has probability \( \prod_{n \leq M} \Pr_{r^* \upharpoonright n} \).

Fix \( j < j^* \). There are \( |I_k| \) many levels \( i < M \) such that at \( \beta_i \) we deal with the \( (\bar{\beta}^i, n_{i'}) \)-case. Let \( M^j \) be the set of these levels. For each \( i \in M^j \), we perform an experiment, by asking whether the next step \( t \in T \) (from the current \( s \) at level \( i \)) will satisfy \( q_{1, \beta_i} \models \beta_i \leq r'_j \). While the exact probability for success will depend on which \( s \) at level \( i \) we start from, a lower bound is given by \( 1 - \frac{1+\sqrt{2}}{2} \cdot \text{loss}^* \). Recall that \( \text{loss}^* = \text{loss}^{\bar{\beta}, n_i} \), and that we set \( p_{j, i} = 1 - \frac{1+\sqrt{2}}{2} \cdot \text{loss}^* \) and \( p_{j, i}' = 1 - \sqrt{\text{loss}^{\bar{\beta}, n_i}} \) in (2.40). So the chance of our branch \( t^* \) having success fewer than \( |I_k| \cdot (1 - \sqrt{\text{loss}^{\bar{\beta}, n_i}}) \) many times, out of the the \(|I_k|\) many tries, (let us call such a \( t^* \) “bad for \( j^* \)) is at most \( F(|I_k|, p'; |I_k|, p) \leq \frac{1}{2} \).

Accordingly, the measure of branches that are not bad for any \( j < j^* \) is at least \( \frac{1}{2} \). Fix such a branch \( t^* \). Then for each \( j < j^* \),

\[
\left| \left\{ i \in M^j : q_{1, \beta_i} \models \beta_i \leq r'_j \right\} \right| \geq |I_k| \cdot \left( 1 - \sqrt{\text{loss}^{\bar{\beta}, n_i}} \right),
\]

and thus \( q_{1, \beta_i} \) forces that

\[
\left| \left\{ \ell' \in I_k : p_{\ell'}(a^{\bar{\beta}, n_i}) \in G(a^{\bar{\beta}, n_i}) \right\} \right| \geq |I_k| \cdot \left( 1 - \sqrt{\text{loss}^{\bar{\beta}, n_i}} \right).
\]

(c) for \( \text{cf}(\alpha) = \omega \):

Fix \( \bar{\beta} \) as in the assumption of (c). To simplify notation, let us assume that \( V \neq \emptyset \) and that \( \sup(V) < \sup(\text{sup}(p_{\ell'})) \) (for some, or equivalently: all \( \ell' \in \omega \)). Let \( 0 < n_0 < m^\beta \) be such that \( \sup(V) \) is at position \( n_0 - 1 \) in \( \text{sup}(p_{\ell'}) \), i.e., \( \sup(V) = a^{\bar{\beta}, n_{0} - 1} \) (independent of \( \ell' \)), and set \( \beta := \sup(V) + 1 \).

\( \bar{\beta} \upharpoonright \beta \) is again a countable \( \Delta \)-system following the same \( h \), and \( \lim_{(\equiv^h_{\ell'})_{\ell' \in \omega}} (\bar{\beta} \upharpoonright \beta) \) is by definition identical to \( \lim_{(\equiv^h_{\ell'})_{\ell' \in \omega}} (\bar{\beta} \upharpoonright \beta) \), which by induction is a valid condition and forces (c) for \( \bar{\beta} \upharpoonright \beta \). This gives us the set \( A_{\bar{\beta} \upharpoonright \beta} \) of measure at least \( 1 - \sum_{n < n_0} \sqrt{\text{loss}^{\bar{\beta}, n}} \).

For the positions \( n_0 \leq n < m^\beta \), all \( (a^{\bar{\beta}, n})_{\ell' \in \omega} \) are strictly increasing sequences above \( \beta \) with some limit \( a_{\bar{\beta}} \leq \alpha \). Then (b) (applied to \( a_{n} \)) gives us an according measure-1-set \( A_{\bar{\beta}, n} \).

So \( \lim_{(\equiv^h_{\ell'})_{\ell' \in \omega}} (\bar{\beta} \upharpoonright \beta) \) forces that \( A' = A_{\bar{\beta} \upharpoonright \beta} \cap \bigcap_{n_0 \leq n < m^\beta} A_{\bar{\beta}, n} \) has measure \( \Xi_{\alpha}(A') \geq 1 - \sum_{n < n_0} \sqrt{\text{loss}^{\bar{\beta}, n}} \geq 1 - \sum_{n < m^\beta} \sqrt{\text{loss}^{\bar{\beta}, n}} \).

Note that \( p_{\ell'} \in G \iff p_{\ell'} \upharpoonright \beta \in G_{\bar{\beta}} \) and \( p_{\ell'}(a^{\bar{\beta}, n}) \in G(a^{\bar{\beta}, n}) \) for all \( n_0 \leq n < m^\beta \).
Fix $k \in A'$. As $k \in A_{\beta_1\beta}$, the relative frequency for $\gamma' \in I_k$ to not satisfy $p\gamma' \upharpoonright \beta \in G_\beta$ is at most $\sum_{n<\eta_0} \sqrt{\text{loss}^\beta_n}$. For any $n_0 \leq n < m^\beta$, as $k \in A_{\hat{\beta},n}$, the relative frequency for not $p\gamma'(a_n^\beta) \in G(a_n^\beta)$ is at most $\sqrt{\text{loss}^\beta_n}$. So the relative frequency for $p\gamma' \in G$ to fail is at most $\sum_{n<\eta_0} \sqrt{\text{loss}^\beta_n} + \sum_{n_0 \leq n < m^\beta} \sqrt{\text{loss}^\beta_n}$, as required.

(a&c) for $\alpha = \gamma + 1$ successor:

For $\gamma \in S^0$ this is clear: Let $\Xi^h_\gamma$ be the name of some FAM extending $\Xi^h_\gamma$. Let $\tilde{\rho}$ be as in (c), without loss of generality $\gamma \in V$. Then $q^+ := \lim_{\gamma \leq \tilde{\rho} \gamma < \gamma} (\tilde{\rho} \gamma) = q^- (\gamma, r)$, where $q := \lim_{\Xi^h_\gamma \epsilon \prec \gamma} (\tilde{\rho} \gamma) \gamma < \gamma$ and $r$ is the condition determined by $h(\gamma)$, i.e., each $p_\xi \upharpoonright \gamma$ forces $p_\xi (\gamma) = r$. In particular, $q^+$ forces that $p_\xi \in G_\xi$ iff $p_\xi \upharpoonright \gamma \in G_\xi$. By induction, (c) holds for $\gamma$, and therefore we get (c) for $\alpha$.

Assume $\gamma \in S^3 \cup S^4$. By induction we know that (d) holds for $\gamma$, i.e., that $\Xi^h_\gamma$ restricted to $M_\gamma$ (call it $\Xi_\gamma$) is in $M_\gamma$. So the requirement in the definition 2.38 of the limit is satisfied, and thus the limit $q^+ := \lim_{\gamma \upharpoonright \gamma < \gamma} (\tilde{\rho} \gamma)$ is well defined for any countable $\Delta$-system $\tilde{\rho}$ as in (c): $q^+$ has the form $q^- (\gamma, r)$ with $q := \lim_{\gamma \upharpoonright \gamma < \gamma} p_\xi \gamma < \gamma$ and $r = \lim_{\Xi_\gamma \epsilon \prec \gamma} (p_\xi (\gamma) \gamma < \gamma)$. Now Lemma 2.28 gives us the $P_a$-name $\Xi^+$, which will be our new $\Xi^h_\gamma$.

This works as required: Again without loss of generality we can assume $\gamma \in V$. By induction, $q$ forces that $\Xi^h_\gamma (A_{\epsilon}(\alpha)) \geq 1 - \sum_{n<\eta_0-1} \sqrt{\text{loss}^\beta_n}$. According to Lemma 2.28, $r$ forces that $\Xi^+ (A_{\epsilon}(\alpha)) \geq 1 - \sqrt{\text{loss}^\beta_n}$. So $q^+ = q^- r$ forces that $\Xi^h_\gamma (A_{\epsilon}) \geq 1 - \sum_{n<\eta_0-1} \sqrt{\text{loss}^\beta_n}$.

(d):

So we have (in V) the $P_a$-name $\Xi^h_\gamma$. We already know that there is (in V) an $\omega_1$-club set $X_0$ in $[\alpha]^{<\beta}$ (for the appropriate $i \in \{3, 4\}$) such that $v \in X_0$ implies that $v$ satisfies Assumptions 2.13 and 2.18. So each such $v \in X_0$ defines a complete subforcing $P^v_\alpha$ of $P_\alpha$ and the $P_a$-name for the according $P^v_\alpha$-extension $M^v_\alpha$.

Fix some $v \in X_0$. We will define $v' \supseteq v$ as follows: For a $P^v_\alpha$-name (and thus a $P_\alpha$-name) $r \in 2^{<\alpha}$, let $s$ be the name of the $E_\gamma (r) \in \{0, 1\}$. As in Lemma 2.17(a), we can find a countable $w_\alpha$, determining $s$. (I.e., there is a Borel function that calculates the real $s$ from the generics at $w_\alpha$; moreover we know this Borel function in the ground model.) Let $u' \supseteq u$ be in $X_0$ and contain all these $w_\gamma$, for a (small representative set of) all $P^v_\alpha$-names for reals.

Iterating this construction $\omega_1$ many steps gives us a suitable $u_\gamma$: Note that the assignment of a name $r$ to the $\Xi_\gamma$-value $s$ can be done in $V$, and thus is known to $M_\gamma$. In addition, $M_\gamma$ sees that for each “actual real” (i.e., element of $M_\gamma$), the value $s$ is already determined (by $P^v_\alpha$). So the assignment $r \mapsto s$, which is $\Xi_\gamma$ restricted to $M_\gamma$, is in $M_\gamma$.

Note that in (c), when we deal with a countable $\Delta$-system $\tilde{\rho}$ following the guardrail $h \in H^*$, the condition $\lim_{\Xi_\gamma} \tilde{\rho}$ forces in particular that infinitely many $p_\xi$ are in $G$. So after carrying out the construction as above, we get a forcing notion $P^5$ satisfying the following (which is actually the only thing we need from the previous construction, in addition to the fact that we can choose each $\xi_\alpha$ in an $\alpha_1$-club):

**Lemma 2.41.** For every countable $\Delta$-system $\tilde{\rho}$ there is some $q$ forcing that infinitely many $p_\xi$ are in the generic filter.

**Proof.** According to Lemma 2.32, $\tilde{\rho}$ follows some $h \in H^*$; so $q = \lim_{\Xi_\gamma} (\tilde{\rho})$ will work. □

**Lemma 2.42.** $\text{LCU}_\alpha (P^5, \kappa)$ for $\kappa \in [\lambda^2, \lambda^+]$ regular, witnessed by the sequence $(c_\alpha)_{\alpha < \kappa}$ of the first $\kappa$ many Cohen reals.
Proof. Fix a $\mathbb{P}^5$-name $y \in \omega^{\omega}$. We have to show that $(\exists \alpha \in \kappa) (\forall \beta \in \kappa \setminus \alpha) [\mathbb{P}^5] \vdash \neg c_\beta \leq^* y$.

Assume towards a contradiction that $p^*$ forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{\alpha \in \kappa}$. Pick $p^i \leq p^*$ deciding $\alpha_i$ to be some $\beta_i$, and also deciding $n_i$ such that $(\forall m \geq n_i) c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta_i \in \text{supp}(p^i)$.

Note that $\beta_i$ is a Cohen position (as $\beta_0 < \kappa \leq \lambda_5$), and we can assume that $p'(\beta_i)$ is a Cohen condition in $V$ (and not just a $P_{\beta_i}$-name for such a condition). By strengthening and thinning out, we may assume:

- $(p'_i)_{i \in \kappa}$ forms a $\Delta$ system with heart $\forall$.
- All $n_i$ are equal to some $n^*$.
- $p'_i(\beta_i)$ is always the same Cohen condition $s \in \omega^{<\omega}$, without loss of generality of length $|s| = n^* + n^*$.
- For some position $n < m^*$, $\beta_i$ is the $n$-th element of $\text{supp}(p')$.

Note that this $n$ cannot be a heart condition: For any $\beta \in \kappa$, at most $|\beta|$ many $p'$ can force $\alpha_i = \beta$, as $p'$ forces that $\alpha_i \geq i$ for all $i$.

Pick a countable subset of this $\Delta$-system which forms a countable $\Delta$-system $\bar{p} := (p'_\iota)_{\iota \in \omega}$. So $p'_\iota = p'^\iota$ for some $\iota \in \kappa$, and we set $p_{\iota} = p'^\iota$. In particular all $p_{\iota}$ are distinct. Now extend each $p_{\iota}$ to $p'_\iota$ by extending the Cohen condition $p_{\iota}(\beta_i) = s$ to $s^\iota \leq \iota$ (i.e., forcing $c_{\beta_i}(n^*) = \iota$). Note that $p' := (p'_i)_{i \in \kappa}$ is still a countable $\Delta$-system, and by Lemma 2.41 some $q$ forces that infinitely many of the $p'_i$ are in the generic filter. But each such $p'_i$ forces that $c_{\beta_i}(n^*) = \iota \leq \gamma(n^*)$, a contradiction. \qed

2E. The left hand side. We have now finished the consistency proof for the left hand side:

**Theorem 2.43.** Assume GCH and let $\lambda_i$ be an increasing sequence of regular cardinals, none of which is a successor of a cardinal of countable cofinality for $i = 1, \ldots, 5$. Then there is a cofinalities-preserving forcing $P$ resulting in

$$\begin{align*}
\text{add}(\mathcal{N}) &= \lambda_1 < \text{add}(\mathcal{M}) = b = \lambda_2 < \text{cov}(\mathcal{N}) = \lambda_3 < \text{non}(\mathcal{M}) = \lambda_4 < \\
&< \text{cov}(\mathcal{M}) = 2^\aleph_0 = \lambda_5.
\end{align*}$$

**Proof.** Set $\chi = \lambda_2$, and let $R$ be the set of partial functions $f : \chi \times \lambda_5 \to 2$ with $|\text{dom}(f)| < \chi$ (ordered by inclusion). $R$ is $<\chi$-closed, $\chi^+$-cc, and adds $\lambda_3$ many new elements to $2^\chi$. So in the $R$-extension, Assumption 2.29 is satisfied, and we can construct $\mathbb{P}^5$ according to Assumption 2.13 and Construction 2.39. Fact 2.24 gives us all inequalities for the left hand side, apart from $b \leq \lambda_2$, which we get from 2.42.

In the $R$-extension, CH holds and $P$ is a FS ccc iteration of length $\delta_5$, $|\delta_5| = \lambda_5$, and each iterand is a set of reals; so $2^\aleph_0 \leq \lambda_5$ is forced. Also, any FS ccc iteration of length $\delta$ (of nontrivial iterands) forces $\text{cov}(\mathcal{M}) \geq \text{cf}(\delta)$: Without loss of generality $\text{cf}(\delta) = \lambda$ is uncountable. Any set $A$ of (Borel codes for) meager sets that has size $<\lambda$ already appears at some stage $\alpha < \delta$, and the iteration at state $\alpha + \omega$ adds a Cohen real over the $V_\alpha$, so $A$ will not cover all reals. \qed

**Remark 2.44.** So this consistency result is reasonably general, we can, e.g., use the values $\lambda_i = \aleph_{i+1}^+$. This is in contrast to the result for the whole diagram, where in particular the small $\lambda_i$ have to be separated by strongly compact cardinals.

13Note that $p'$ will not follow the same guardrail as $\bar{p}$. 
3. Ten different values in Cichoń’s diagram

We can now apply, with hardly any change, the technique of [GKS] to get the following:

**Theorem 3.1.** Assume GCH and that

\[ \aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9 \] are regular, \( \lambda_i \) is not a successor of a cardinal of countable cofinality for \( i = 1, \ldots, 5 \), \( \lambda_2 = \chi^+ \) with \( \chi \) regular, and \( \kappa_i \) strongly compact for \( i = 6, 7, 8, 9 \).

Then there is a ccc forcing notion \( P^5 \) resulting in:

\[
\begin{align*}
\text{add}(\mathcal{N}) &= \lambda_1 < b = \text{add}(\mathcal{M}) = \lambda_2 < \text{cov}(\mathcal{N}) = \lambda_3 < \text{non}(\mathcal{M}) = \lambda_4 < \\
&< \text{cov}(\mathcal{M}) = \lambda_5 < \text{non}(\mathcal{N}) = \lambda_6 < d = \text{cof}(\mathcal{M}) = \lambda_7 < \text{cof}(\mathcal{N}) = \lambda_8 < \beth_0 = \lambda_9.
\end{align*}
\]

To do this, we first have to show that we can achieve for the left hand side, i.e., Theorem 2.43, starting with GCH and using a FS ccc iteration \( \mathbb{P} / \mathcal{U} \):

(a) If \( \mathbb{P} \) is a ccc forcing notion then there is a ccc forcing notion \( \mathbb{Q} \) with

\[ \text{add}(\mathcal{N}) = \lambda_1 < b = \text{add}(\mathcal{M}) = \lambda_2 < \text{cov}(\mathcal{N}) = \lambda_3 < \text{non}(\mathcal{M}) = \lambda_4 < \\
< \text{cov}(\mathcal{M}) = \lambda_5 < \text{non}(\mathcal{N}) = \lambda_6 < d = \text{cof}(\mathcal{M}) = \lambda_7 < \text{cof}(\mathcal{N}) = \lambda_8 < \beth_0 = \lambda_9. \]

(b) If \( \mathbb{P} = R \ast \mathbb{P}^5 \), where \( R \) is not ccc. This is the only argument that requires \( \lambda_2 = \chi^+ \). We will just briefly sketch it here, as it can be found with all details in [GKS, 1.4]:

- We already know that in the \( R \)-extension, (where \( R \) is \( \chi \)-closed, \( \chi^+ \)-cc and forces \( 2^\chi = \lambda_5 \)) we can find by the inductive construction 2.39 suitable \( w_a \) such that \( R \ast \mathbb{P}^5 \) works.

- We now perform a similar inductive construction in the ground model: At stage \( \alpha \), we know that there is an \( R \)-name for a suitable \( w_a^1 \) of size \( < \lambda_i \) (where \( i \) is 3 in the random and 4 in the \( \mathbb{E} \)-case). This name can be covered by some set \( \tilde{w}_a^1 \) in \( V \), still of size \( < \lambda_i \), as \( R \) is \( \chi^+ \)-cc. Moreover, in the \( R \)-extension, the suitable parameters form an \( \omega_1 \)-club; so there is a suitable \( w_a^2 \supseteq \tilde{w}_a^1 \), etc. Iterating \( \omega_1 \) many times and taking the union at the end leads to \( w_a \) in \( V \) which is forced by \( R \) to be suitable.

- Not only \( w_a \) is in \( V \), but the construction for \( w_a \) is performed in \( V \), so we can construct the whole sequence \( \vec{w} = (w_a)_{a \in \delta} \) in \( V \).

- We now know that in the \( R \)-extension, the forcing \( P^5 \) defined from \( \vec{w} \) will satisfy \( \text{LCU}_2(P^5, \kappa) \) in the form of Lemma 2.42.

- By an absoluteness argument, we can show that actually in \( V \) the forcing \( P^5 \) defined form \( \vec{w} \) will satisfy Lemma 2.42 as well.

The rest of the proof is the same as in [GKS, Sec. 2], where we interchange \( b \) and \( \text{cov}(\mathcal{N}) \) as well as \( b \) and \( \text{non}(\mathcal{N}) \). We cite the following facts from [GKS, 2.2–2.5]:

(a) If \( \kappa \) is a strongly compact cardinal and \( \theta > \kappa \) regular, then there is an elementary embedding \( j_{\kappa, \theta} : V \rightarrow M \) (in the following just called \( j \)) such that

- the critical point of \( j \) is \( \kappa \), \( \text{cf}(j(\kappa)) = |j(\kappa)| = \theta \),
- \( \max(\theta, \lambda) = j(\lambda) < \max(\theta, \lambda)^+ \) for all \( \lambda \geq \kappa \) regular, and
- \( \text{cf}(j(\lambda)) = \lambda \) for \( \lambda \neq \kappa \) regular,

and such that the following is satisfied:

(b) If \( P \) is a FS ccc iteration along \( \delta \), then \( j(P) \) is a FS ccc iteration along \( j(\delta) \).

(c) \( \text{LCU}_1(P, \lambda) \) implies \( \text{LCU}_1(j(P), \text{cf}(j(\lambda))) \), and thus \( \text{LCU}_1(j(P), \lambda) \) if \( \lambda \neq \kappa \) regular.\(^{14}\)

(d) If \( \text{COB}_1(P, \lambda, \mu) \), then \( \text{COB}_1(j(P), \lambda, \mu') \), for \( \mu' = \begin{cases} |j(\mu)| & \text{if } \kappa > \lambda \\ \mu & \text{if } \kappa < \lambda. \end{cases} \)

\(^{14}\) In [GKS], we only used “classical” relations \( R \), that are defined on a Polish space in an absolute way. In this paper, we use the relation \( R \), which is not of this kind. However, the proof still works without any change: The parameter \( E \) used to define the relation \( R \), cf. Definition 2.2, is a set of reals. So \( j(E) = E \), and we can still use the usual absoluteness arguments between \( M \) and \( V \). (A parameter not element of \( H(\kappa_9) \) might be a problem.)
Using these facts, it is easy to finish the proof.\footnote{This is identical to the argument in [GKS], with the roles of $b$ and $\text{cov}(\mathcal{N})$, as well as their duals, switched.}

**Proof of Theorem 3.1.** Recall that we want to force the following values to the characteristics of Figure 2 (where we indicate the positions of the $\kappa_i$ as well):

\begin{center}
\begin{tikzpicture}
  \node (N1) at (0,0) {$\aleph_0$};
  \node (k0) at (1,0) {$\kappa_0$};
  \node (l1) at (2,1) {$\lambda_1$};
  \node (l2) at (2,0) {$\lambda_2$};
  \node (l3) at (2,-1) {$\lambda_3$};
  \node (l4) at (3,1) {$\lambda_4$};
  \node (l5) at (3,0) {$\lambda_5$};
  \node (l6) at (3,-1) {$\lambda_6$};
  \node (l7) at (4,0) {$\lambda_7$};
  \node (l8) at (4,1) {$\lambda_8$};
  \node (l9) at (4,2) {$\lambda_9$};

  \draw[->] (N1) -- (l1);
  \draw[->] (l1) -- (l2);
  \draw[->] (l2) -- (l3);
  \draw[->] (l3) -- (k0);
  \draw[->] (k0) -- (l4);
  \draw[->] (l4) -- (l5);
  \draw[->] (l5) -- (l6);
  \draw[->] (l6) -- (l7);
  \draw[->] (l7) -- (l8);
  \draw[->] (l8) -- (l9);
\end{tikzpicture}
\end{center}

**Step 5:** Our first step, called “Step 5” for notational reasons, just uses $\mathbb{P}^5$. This is an iteration of length $\delta_6$ with $\text{cf}(\delta_6) = |\delta_6| = \lambda_6$, satisfying:

(3.3) For all $i$: $\text{LCU}_i(\mathbb{P}^5, \mu)$ for all $\mu \in [\lambda_i, \lambda_3]$ regular, and $\text{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_2)$.

As a consequence, the characteristics are forced by $\mathbb{P}^5$ to have the following values\footnote{These values, and the ones forced by the “intermediate forcings” $\mathbb{P}^6$ to $\mathbb{P}^8$, are not required for the argument; they should just illustrate what is going on.} (we also mark the position of $\kappa_6$, which we are going to use in the following step):

\begin{center}
\begin{tikzpicture}
  \node (N1) at (0,0) {$\aleph_0$};
  \node (k6) at (1,0) {$\kappa_6$};
  \node (l1) at (2,1) {$\lambda_1$};
  \node (l2) at (2,0) {$\lambda_2$};
  \node (l3) at (2,-1) {$\lambda_3$};
  \node (l4) at (3,1) {$\lambda_4$};
  \node (l5) at (3,0) {$\lambda_5$};
  \node (l6) at (3,-1) {$\lambda_6$};
  \node (l7) at (4,0) {$\lambda_7$};
  \node (l8) at (4,1) {$\lambda_8$};
  \node (l9) at (4,2) {$\lambda_9$};

  \draw[->] (N1) -- (l1);
  \draw[->] (l1) -- (l2);
  \draw[->] (l2) -- (l3);
  \draw[->] (l3) -- (k6);
  \draw[->] (k6) -- (l4);
  \draw[->] (l4) -- (l5);
  \draw[->] (l5) -- (l6);
  \draw[->] (l6) -- (l7);
  \draw[->] (l7) -- (l8);
  \draw[->] (l8) -- (l9);
\end{tikzpicture}
\end{center}

**Step 6:** Consider the embedding $j_6 := j_{\kappa_6}(\mathcal{N})$. According to 3.2(b), $\mathbb{P}^6 := j_6(\mathbb{P}^5)$ is a FS ccc iteration of length $\delta_6 := j_6(\delta_5)$. As $|\delta_6| = \lambda_6$, the continuum is forced to have size $\lambda_6$.

For $i = 1$, we have $\text{LCU}_i(\mathbb{P}^6, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_3]$, so using 3.2(c) we get $\text{LCU}_i(\mathbb{P}^6, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_3]$ different to $\kappa_6$; as well as $\text{LCU}_1(\mathbb{P}^6, \lambda_6)$ (as $\text{cf}(j(\kappa_6)) = \lambda_6$). For $\mu = \lambda_1$ the former implies $\mathbb{P}^6 \not\models \text{add}(\mathcal{N}) \leq \lambda_1$, and the latter $\mathbb{P}^6 \models \text{cof}(\mathcal{N}) \geq \lambda_6 = 2^{\aleph_0}$.

More generally, we get from (3.3) and 3.2(c)

(3.4) For all $i$: $\text{LCU}_i(\mathbb{P}^6, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_3] \setminus \{\kappa_6\}$.

For $i < 4$: $\text{LCU}_i(\mathbb{P}^6, \lambda_6)$.

So in particular for $\mu = \lambda_i$, we see that the characteristics on the left do not increase; for $\mu = \lambda_3$ that the ones on the right are still at least $\lambda_5$; and for $i < 4$ an $\mu = \lambda_6$ that the according characteristics on the right will have size continuum. (But not for $i = 4$, as $\kappa_4 < \lambda_4$. And we will see that $\text{cov}(\mathcal{M})$ is at most $\lambda_5$.)

Dually, because $\lambda_3 < \kappa_6 < \lambda_4$, we get from (3.3) and 3.2(d)

(3.5) For $i < 4$: $\text{COB}_i(\mathbb{P}^6, \lambda_i, \lambda_6)$. For $i = 4$: $\text{COB}_4(\mathbb{P}^6, \lambda_4, \lambda_5)$.

(The former because $|j_6(\lambda_3)| = \max(\lambda_6, \lambda_4) = \lambda_6$.) So the characteristics on the left do not decrease, and $\mathbb{P}^6 \models \text{cov}(\mathcal{M}) \leq \lambda_5$.\footnote{…}
Accordingly, \( P^6 \) forces the following values:

\[
\lambda_3 \rightarrow \lambda_4 \rightarrow \cdots \rightarrow \lambda_6
\]

\[
N_1 \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_5 \rightarrow \lambda_6
\]

**Step 7:** We now apply a new embedding, \( j_7 = j_{\kappa_7, \lambda_7} \), to the forcing \( P^6 \) that we just constructed. (We always work in \( V \), not in any inner model \( M \) or any forcing extention.) As before, set \( P^7 := j_7(P^6) \), a FS ccc iteration of length \( \delta_7 = j_7(\delta_6) \), forcing the continuum to have size \( \lambda_7 \).

Now \( \kappa_7 \in (\lambda_2, \lambda_3) \), so arguing as before, we get from (3.4)

\[
\text{for all } i: \text{LCU}_i(P^7, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7\}.
\]

(3.6)

For \( i < 4 \): LCU\(_i\)(P\(^7\), \( \lambda_6 \)). For \( i < 3 \): LCU\(_i\)(P\(^7\), \( \lambda_7 \)).

and from (3.5)

For \( i < 3 \): COB\(_i\)(P\(^7\), \( \lambda_1, \lambda_7 \)).

(3.7)

For \( i = 3 \): COB\(_3\)(P\(^7\), \( \lambda_3, \lambda_6 \)). For \( i = 4 \): COB\(_4\)(P\(^7\), \( \lambda_4, \lambda_5 \)).

Accordingly, \( P^7 \) forces the following values:

\[
\lambda_3 \rightarrow \lambda_4 \rightarrow \cdots \rightarrow \lambda_7 \rightarrow \lambda_7
\]

\[
N_1 \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_5 \rightarrow \lambda_6
\]

**Step 8:** Now we set \( P^8 := j_{\kappa_8, \lambda_8} (P^7) \), a FS ccc iteration of length \( \delta_8 \). Now \( \kappa_8 \in (\lambda_1, \lambda_2) \), and as before, we get from (3.6)

\[
\text{for all } i: \text{LCU}_i(P^8, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_6] \setminus \{\kappa_6, \kappa_7, \kappa_8\}.
\]

(3.8)

For \( i < 4 \): LCU\(_i\)(P\(^8\), \( \lambda_6 \)). For \( i < 3 \): LCU\(_i\)(P\(^8\), \( \lambda_7 \)).

For \( i < 2 \) (i.e., \( i = 1 \)): LCU\(_1\)(P\(^8\), \( \lambda_8 \)).

and from (3.7)

For \( i = 1 \): COB\(_1\)(P\(^8\), \( \lambda_1, \lambda_8 \)). For \( i = 2 \): COB\(_2\)(P\(^8\), \( \lambda_2, \lambda_7 \)).

(3.9)

For \( i = 3 \): COB\(_3\)(P\(^8\), \( \lambda_3, \lambda_6 \)). For \( i = 4 \): COB\(_4\)(P\(^8\), \( \lambda_4, \lambda_5 \)).

Accordingly, \( P^8 \) forces the following values:

\[
\lambda_3 \rightarrow \lambda_4 \rightarrow \cdots \rightarrow \lambda_8 \rightarrow \lambda_8
\]

\[
N_1 \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_5 \rightarrow \lambda_6
\]

**Step 9:** Finally we set \( P^9 := j_{\kappa_9, \lambda_9} (P^8) \), a FS ccc iteration of length \( \delta_9 \) with \( |\delta_9| = \lambda_9 \), i.e., the continuum will have size \( \lambda_9 \). As \( \kappa_9 \neq \lambda_1 \), (3.8) and (3.9) also hold for \( P^9 \) instead.
of $\mathcal{P}^8$. Accordingly, we get the same values for the diagram as for $\mathcal{P}^8$, apart from the value for the continuum, $\lambda_9$. □

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