Strongly nonequilibrium Bose-condensed atomic systems

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Abstract A trapped Bose-Einstein condensate, being strongly perturbed, exhibits several spatial structures. First, there appear quantum vortices. Increasing the amount of the injected energy leads to the formation of vortex tangles representing quantum vortex turbulence. Continuing energy injection makes the system so strongly perturbed that vortices become destroyed and there develops another kind of spatial structures with essentially heterogeneous spatial density. These structures consist of high-density droplets, or grains, surrounded by the regions of low density. The droplets are randomly distributed in space, where they can move; however they live sufficiently long time to be treated as a type of metastable creatures. Such structures have been observed in nonequilibrium trapped Bose gases of $^{87}\text{Rb}$ subject to the action of an oscillatory perturbation modulating the trapping potential. Perturbing the system even stronger transforms the droplet structure into wave turbulence, where Bose condensate is destroyed. Numerical simulations are in good agreement with experimental observations.

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1 Introduction

Bose atoms in a trap can be cooled down to such low temperatures when almost all of them pile down to the ground state corresponding to Bose-Einstein condensate. If the trapped Bose-condensed cloud is slightly perturbed, it exhibits elementary collective excitations, similarly to other finite systems, such as quantum dots, atomic nuclei, or clusters\textsuperscript{[1,2,3,4,5]}. But what happens when the Bose-condensed cloud is strongly perturbed?

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We know that, when the external perturbation imposes a rotation moment, there can arise quantum vortices that may form a vortex lattice \[6,7\]. But if the external perturbation imposes no rotation moment, but just strongly shakes the condensate, as has been suggested in Refs. \[8,9\], then no vortex lattices arise. Instead, the increasing number of vortices forms a random tangle corresponding to quantum turbulence \[10,11,12,13\]. This vortex turbulence of trapped atoms has been observed in three-dimensional traps \[14,15,16\] as well as in quasi-two-dimensional traps \[17\], and is summarized in the review articles \[18,19,20\].

In the present paper, we study what could be other possible nonequilibrium states in a trapped strongly perturbed Bose-Einstein condensate. We investigate this problem both in experiments with trapped atoms of \(^{87}\)Rb and accomplishing computer simulations for the setup exactly corresponding to these experiments. Observed experimental data are in good agreement with numerical simulations.

2 Methods of strong perturbation

Generally, a Bose-Einstein condensate is characterized by an equation for the condensate wave function \(\eta = \eta (r,t)\). For a system of weakly interacting atoms at temperatures close to zero, the condensate-function equation takes the form of the nonlinear Schrödinger equation (NLS)

\[
i \partial_t \eta = \left( -\frac{\nabla^2}{2m} + U + \Phi |\eta|^2 \right) \eta.
\]  

(1)

Here and in what follows, the Planck constant is set to one, \(U(r,t)\) is the total external potential, and the atomic interaction strength is

\[
\Phi = 4\pi \frac{a_s}{m},
\]

(2)

with \(a_s\) being scattering length and \(m\), atomic mass.

When the system is in equilibrium, the condensate is in its ground state. To perturb the condensate, it is necessary to inject energy into the system. The amount of injected energy, per atom, during the excitation time \(t\), can be represented as

\[
E_{inj} = \frac{1}{N} \int_0^t \left\langle \frac{\partial H}{\partial t} \right\rangle dt,
\]

(3)

where \(N\) is the total number of atoms, and the energy Hamiltonian is

\[
\hat{H} = \int \eta(r,t) \left( -\frac{\nabla^2}{2m} + U \right) \eta(r,t) \, dr + \frac{1}{2} \int |\eta(r,t)|^4 \, dr.
\]

(4)

This tells us that energy can be injected through one of two ways, either adding to the trapping potential \(U(r)\) a time-dependent perturbative term, so that the total external potential be the sum

\[
U(r,t) = U(r) + V(r,t),
\]

(5)

or by varying the scattering length, so that the interaction strength be time-dependent,

\[
\Phi(t) = 4\pi \frac{a_s(t)}{m}.
\]

(6)

The perturbations are assumed to be organized in such a way that no rotation moment is imposed onto the system.
If the trap modulation is done by means of an alternating potential
\[ V(\mathbf{r}, t) \sim A \cos(\omega t) \quad (A \geq 0) , \]
as is advanced in Refs. [8,9], then the injected energy, for sufficiently long time \( t \gg 2\pi/\omega \), takes the form
\[ E_{\text{inj}} \sim \frac{2}{\pi} A \omega t , \quad (7) \]
which makes it possible to study the amplitude-time phase diagram through the relation
\[ A \sim \frac{\pi E_{\text{inj}}}{2\omega t} . \quad (8) \]

The other possibility of injecting energy is by modulating the scattering length, making it time-dependent with the help of Feshbach resonance techniques, as has been suggested in Refs. [21,22,23]. The time dependence of the scattering length is realized through the oscillating magnetic field, when
\[ a_s(t) = \bar{a}_s \left[ 1 - \frac{\Delta B}{B(t) - B_\infty} \right] , \quad (9) \]
where \( \bar{a}_s \) is a background scattering length, \( \Delta B \), resonance width, and \( B_\infty \) is a resonance field. The modulating magnetic field oscillates by the law
\[ B(t) = B_0 + B_1 \cos(\omega t) . \quad (10) \]
For a small oscillation amplitude, such that \( |B_1/B_0| \ll 1 \), the effective scattering length is
\[ a_s(t) \simeq a_0 + a_1 \cos(\omega t) , \quad (11) \]
with the notation
\[ a_0 \equiv \bar{a}_s \left( 1 - \frac{\Delta B}{B_0 - B_\infty} \right) , \quad a_1 \equiv \bar{a}_s \frac{B_1 \Delta B}{(B_0 - B_\infty)^2} . \]
Then the interaction strength takes the form
\[ \Phi(t) = \Phi_0 + \Phi_1 \cos(\omega t) , \quad (12) \]
in which
\[ \Phi_0 \equiv 4\pi \frac{a_0}{m} , \quad \Phi_1 \equiv 4\pi \frac{a_1}{m} . \]
The injected energy reads as
\[ E_{\text{inj}} \sim \frac{1}{\pi} \rho \Phi_1 \omega t , \quad (13) \]
where \( \rho \) is average atomic density. Denoting the effective amplitude
\[ A \equiv \frac{1}{2} \rho \Phi_1 = 2\pi \rho \frac{a_1}{m} , \]
we get the same amplitude-time relation (8) as in the case of the trap modulation. Thus, these two methods of perturbing condensate should lead to the same consequences, producing similar strongly excited nonequilibrium states.
The nature of the possible nonequilibrium states essentially depends on the atomic scattering length as well as on trap geometry. Important quantities quantifying these characteristics are the aspect ratio

\[ \alpha \equiv \frac{\omega_z}{\omega_\perp} = \left( \frac{l_\perp}{l_z} \right)^2, \quad (14) \]

in which \( l_\perp \equiv 1/\sqrt{m\omega_\perp} \), \( l_z \equiv 1/\sqrt{m\omega_z} \) are the oscillator lengths, and the effective coupling parameter

\[ g \equiv 4\pi N a_s l_\perp. \quad (15) \]

It is possible to use other methods for perturbing Bose-Einstein condensate, for instance, by creating laser-induced obstacles [24], which is equivalent to the perturbation by external potentials, that is, to trap modulation.

### 3 Nonequilibrium atomic states

The system is initially prepared in an equilibrium state, at temperature close to zero, with almost all atoms being in Bose-Einstein condensate. Energy is injected into the system by modulating the trapping potential as described in Refs. [14][15][16][20]. Injecting energy into the trap produces a sequence of nonequilibrium states that we have studied experimentally as well as by computer simulations for the order-function equation (1), adding there the attenuation by replacing in Eq. (1) \( i \) by \( i - \gamma \), which imitates the loss of energy and atoms [18][19]. The experimental results are in very good agreement with computer simulations. Physical effects, occurring during the process of perturbing the system can be illustrated by theoretical estimates that are in close agreement with both experimental results and numerical simulations.

**Weak perturbation.** Starting pumping into the trap the injected energy (3), first, creates elementary collective excitations corresponding to small fluctuations around the equilibrium state. This regime of weak perturbation lasts in the interval of energies

\[ 0 < E_{\text{inj}} < 2E_{\text{vor}}, \quad (16) \]

where the injected energy is smaller than the energy required for exciting at least a pair of vortices with opposite vorticities. Such a pair makes the total vorticity of the atomic cloud zero, as it should be, when the perturbing potential does not impose rotation. It is worth stressing that the created pair does not form a bound state, but the vortices are rather independent of each other. The energy of a vortex per atom can be defined [25][26] as

\[ E_{\text{vor}} = \frac{0.9\omega_z}{(9g)^{3/4}} \ln(0.8\alpha g). \quad (17) \]

Keeping in mind the setup of experiments [14][15][16][20] with \(^{87}\text{Rb}\), we have the trap characteristics

\[ \omega_\perp = 1.32 \times 10^3 \text{s}^{-1}, \quad \omega_z = 1.45 \times 10^2 \text{s}^{-1}, \]

\[ l_\perp = 0.74 \times 10^{-4} \text{cm}, \quad l_z = 2.25 \times 10^{-4} \text{cm} \quad (^{87}\text{Rb}), \]

which implies the aspect ratio \( \alpha = 0.11 \). The \(^{87}\text{Rb}\) atoms have the scattering length \( a_s = 0.577 \times 10^{-6} \text{ cm} \). With the number of atoms \( N \approx 2 \times 10^5 \), the effective coupling (15) is \( g = 1.96 \times 10^4 \). This gives the pair vortex energy as \( 2E_{\text{vor}} = 0.566 \times 10^{-12} \text{ eV} \). It is convenient
to measure the energies in units of the transverse trap energy $E_\perp \equiv \hbar \omega_0$ that in the present case is $E_\perp = 0.869 \times 10^{-12}$ eV. Then $2E_{\text{vor}} = 0.65E_\perp$.

In order to stress that the appearance of vortices as well as of other nonequilibrium states essentially depends on the involved sort of atoms and on the trap geometry, we compare the typical values for $^{87}\text{Rb}$ atoms with the setup employed in the group of Hulet\cite{27,28} dealing with $^7\text{Li}$ atoms and a more elongated trap, where
\begin{align*}
  \omega_\perp &= 1.48 \times 10^3 s^{-1}, \quad \omega_c = 0.304 \times 10^2 s^{-1}, \\
  l_\perp &= 2.5 \times 10^{-4} \text{cm}, \quad l_c = 1.7 \times 10^{-3} \text{cm} \quad (^7\text{Li}),
\end{align*}
which gives the aspect ratio $\alpha = 0.021$, an order smaller than in the case of $^{87}\text{Rb}$. The scattering length of $^7\text{Li}$ is tuned by Feshbach resonance to $a_c = 3.2 \times 10^{-8}$ cm, which is two orders shorter than that of $^{87}\text{Rb}$. The effective coupling $g = 0.48 \times 10^3$ is much smaller than for $^{87}\text{Rb}$. The energy, required for vortex generation, is $2E_{\text{vor}} = 1.45 \times 10^{-12}$ eV. The transverse energy here is $E_\perp = 0.97 \times 10^{-12}$ eV. Hence $2E_{\text{vor}} = 0.49E_\perp$. In this setup, it is more difficult to create vortices, since the required injected energy $E_{\text{inj}}$ should be about three times larger than in the case of $^{87}\text{Rb}$.

In the regime of weak perturbation (16), vortices and other topological modes are not created, unless the modulating potential is tuned to a transition frequency $\omega_n \equiv E_n - E_0$, corresponding to a transition between the ground-state condensate, with energy $E_0$, and a coherent topological mode \cite{30,31,32,33,34,35}, with energy $E_n$. The condition $\omega = \omega_n$, when topological modes are generated, even under weak perturbation, can be called\textit{topological resonance}. Note that this is not the standard parametric resonance \cite{30}, when $\omega = 2\omega_0$, where $\omega_0$ is a system natural frequency, or more generally, when $k\omega = 2\omega_n$, where $k = 1, 2, \ldots$. The natural frequency $\omega_0$ characterizes the motion of the system as a whole and, for trapped atoms, is defined by the effective oscillator trap frequency. However the transition frequencies $\omega_n$ describe the internal transitions between the collective energy levels and, for an interacting system, $\omega_n$ can be very different from the effective oscillator trap frequency \cite{8,9,10,11,12}. Topological modes can also be created under harmonic generation, when $k\omega = \omega_n$, where $k = 1, 2, \ldots$, under parametric conversion, when there are two external modulating fields, with the frequencies $\omega_1$ and $\omega_2$ such that $\omega_1 \pm \omega_2 = \omega_n$, and under combinations of these resonance conditions. But if there is no a kind of a resonant tuning, the topological modes do not arise.

When considering nonequilibrium states, it is necessary to distinguish the atomic cloud itself and the subsystem of its elementary excitations. The atomic cloud can be slightly perturbed, but, nevertheless, such a perturbation can result in the creation of many elementary excitations. The waves, corresponding to these excitations, are involved in wave interactions and are described by quantum kinetic equations \cite{33,34,35}. In the subsystem of these excitations, there can arise weak wave turbulence \cite{36,37}, which is inhomogeneous in the presence of a trap \cite{38}.

\textbf{Vortex formation.} As soon as the injected energy surpasses the vortex energy (17), vortices start being created. The regime of vortex\textit{formation} corresponds to the interval
\begin{equation}
  2E_{\text{vor}} < E_{\text{inj}} < E_{\text{tur}}, \tag{18}
\end{equation}
until the number of vortices is so high that the system turns into turbulent regime, which happens when the number of vortices reaches a critical value $N_c$, corresponding to the energy
\begin{equation}
  E_{\text{tur}} = N_c E_{\text{vor}}. \tag{19}
\end{equation}
The critical number can be defined by the condition that the mean distance between the outer parts of the vortices becomes equal to their diameter $2\xi$, given by twice the coherence length \( \xi \sim \frac{\hbar}{mc} \), where \( c \sim \sqrt{\frac{\hbar}{ma}} \) is the sound velocity. This means that the distance between the centers of the vortices is \( 4\xi \), which gives

\[
N_c = \left(\frac{r_\perp}{4\xi}\right)^2.
\]

(20)

For the experiments with $^{87}\text{Rb}$, the atomic cloud radius and length are \( r_\perp \approx 4 \times 10^{-4} \) cm and \( L \approx 6 \times 10^{-3} \) cm, respectively. The coherence length is \( \xi \approx 2 \times 10^{-5} \) cm. Thus, the critical vortex number is \( N_c \approx 25 \), which is in perfect agreement with experiments and simulations. This corresponds to the vortex turbulence energy \( E_{\text{tur}} = 0.71 \times 10^{-11} \) eV, or in units of the trap energy, \( E_{\text{tur}} = 8.2E_L \).

It is much more difficult, if possible at all, to create vortex turbulence in an elongated trap with trapped $^7\text{Li}$, as in the experiments of the Hulet group [27,28]. In the latter case, the cloud radius and length are \( r_\perp \approx 3 \times 10^{-4} \) cm and \( L \approx 2 \times 10^{-2} \) cm, while the coherence length is \( \xi \approx 2 \times 10^{-4} \) cm, so that a single vortex would fill almost the whole trap. Formally, the critical number (20) would be 0.14. However, there is no turbulence with a single vortex.

Typical vortex states are illustrated in Fig. 1 showing the results of numerical calculations for the atomic density in a transverse cross-section of the trap, for the setup characterized by the experimental data with $^{87}\text{Rb}$ atoms. The higher density corresponds to brighter colour. So that vortices are shown as black spots. By calculating the vorticity, we find that practically all vortices have vorticity plus or minus one. Typical vortex states observed in the time-of-flight experiment are presented in Fig. 2. Again, brighter colour corresponds to higher density.

Fig. 1 (Color online) Vortex state. Typical distributions of the density of $^{87}\text{Rb}$ atoms in a transverse cross-section of the trap, found by numerical simulations. Brighter colour corresponds to higher density. So that vortices are seen as black spots.

Increasing the amount of the injected energy produces a larger number of vortices. Since, as is clear from relation (7), the injected energy is proportional to the modulation time, the number of vortices, in the vortex state, increases with time. After their number reaches the critical number 25, the regime of vortex turbulence comes into play, when the number of vortices yet increases, but slower than before. This behavior is illustrated by numerical simulations presented in Fig. 3, showing the number of vortices as a function of time, under fixed amplitude. The results are in good agreement with experimental data. We see that at some moment of time the number of vortices abruptly diminishes. Then the system passes to another regime, termed grain turbulence or droplet turbulence, to be described below.
Fig. 2 (Color online) Vortex state. Experimental observation of vortices in a transverse cross-section of the cloud of $^{87}$Rb atoms in time-of-flight measurements. Brighter colour corresponds to higher density. So that vortices are seen as black spots.

Fig. 3 (Color online) Number of vortices, found by numerical simulations, as a function of time, that is, as a function of injected energy.

Vortex turbulence. The regime of vortex turbulence lasts till the number of vortices becomes sufficiently high, so that the mean distance between vortices is such that their interactions become rather strong, when the interaction energy of two vortices at the distance $\delta$ from each other becomes equal to the vortex energy. Equating these energies, given in Ref. 26, we get

$$\ln \left(1.4 \frac{R}{\xi}\right) = 2 \ln \frac{R}{\delta},$$

where $R$ is an effective radius of the cloud. This yields

$$\delta = 1.19 \sqrt{R\xi}.$$  \hspace{1cm} (21)

Taking the Thomas-Fermi radius

$$R = l_\perp \left(\frac{15}{4\pi} \alpha R\right)^{1/5}$$
results in the critical distance
\[ \delta = 0.86 \sqrt{l_\perp \xi (eg)^{1/10}}. \]  
\hspace{1cm} (22)

For \(^{87}\text{Rb}\), the latter is \(\delta \sim 0.71 \times 10^{-4} \text{ cm}\), so that \(\delta \sim 3.6\xi\).

One distinguishes random vortex turbulence, or Vinen turbulence, as opposed to the correlated vortex turbulence, or Kolmogorov turbulence [19,39,40,41,42]. In Vinen tangles, far-field effects tend to cancel out, the motion of a vortex line is mainly determined by its local curvature, and the vortex tangle is homogeneous. The intervortex distance in Vinen turbulence is much larger than the coherence length. Therefore the summary energy of vortices in the random Vinen tangle can be approximated by the product of the vortex number and a single vortex energy.

In the regime of vortex turbulence, we have observed, the distance between the vortices is yet such that the energy of vortex interactions is smaller than the energy of a vortex, although interactions can be noticeable. The vortices are not correlated by the used trap modulation that does not impose such correlations, nor imposes a rotational polarization. However, to distinguish precisely what type of turbulence has been realized requires the study of energy spectra, which is yet in progress. In the present publication, we concentrate on the study of density distributions that can be directly observed and which allows for the qualitative classification of the observed nonequilibrium regimes.

The regime of vortex turbulence is realized in the interval of the injected energy
\[ E_{\text{tur}} < E_{\text{inj}} < E_{\text{fog}}, \]  
\hspace{1cm} (23)
where the upper energy boundary
\[ E_{\text{fog}} = N_c^* E_{\text{vor}} \]  
\hspace{1cm} (24)
corresponds to the critical number of vortices, when interactions become rather strong, which means that the distance between the vortices is smaller than the critical distance (22). This critical number can be estimated as
\[ N_c^* = \left( \frac{r_\perp}{\delta} \right)^2. \]  
\hspace{1cm} (25)
For the considered case of \(^{87}\text{Rb}\), we have \(N_c^* = 1.2N_c \approx 30\). This defines the energy \(E_{\text{fog}} = 0.84 \times 10^{-11} \text{ eV}\), corresponding to \(E_{\text{fog}} = 9.7E_\perp\).

It is clear that the estimates for the energies \(E_{\text{tur}}\) and \(E_{\text{fog}}\) give their approximate values. It turns out, these values are close to each other. Although the absolute range between the energies \(E_{\text{tur}}\) and \(E_{\text{fog}}\), of course, is approximate, but its narrowness means that the region of vortex turbulence is quite narrow. This has really been observed in experiments as well as in numerical modelling.

The typical cross-section of the random vortex tangle, obtained in numerical simulations, is demonstrated in Fig. 4. And Fig. 5 shows the vortex turbulent state observed in the experiment.

**Grain turbulence.** After the number of vortices in the random tangle grows to the critical number \(N_c^*\), the vortices start strongly interacting with each other by colliding and destroying each other. Their number sharply decreases, as is shown in Fig. 3. The remnants of the destroyed vortices form the pieces of the granulated Bose condensate, which can be called droplets or grains. These droplets, being multiscale in the interval \((1-5) \times 10^{-5} \text{ cm}\), have the typical size \(l_{dr} \sim 3 \times 10^{-5} \text{ cm}\), which is close to the coherence length \(\xi \sim l_{dr}\), as it
should be for a coherent object. The coherence of a droplet is confirmed by calculating its phase that is constant through the droplet volume. The droplets are surrounded by a rarefied gas of much lower density; the ratio of the droplet density to the density of surrounding $\rho_{dr}/\rho_{sur}$ varies in a wide range: from a few tens to 100. At each snapshot, the droplets are randomly distributed in space, forming no spatial structures, reminding water droplets in fog. Therefore, this regime can be termed droplet turbulence or grain turbulence [43]. The droplets are compact objects, with the linear sizes in different directions being close to each other. The typical droplet lifetime is of order $10^{-2}$ s. The droplets do not form any regular structure, since the average atomic density is not high, being much lower than that necessary for creating a kind of crystalline order [44,45]. The random spatio-temporal distribution of the droplets makes it admissible to call this state as grain turbulence.

In numerical simulations, considering the opposite process of Bose condensate formation from an uncondensed gas, the regime of grain turbulence corresponds to that of strong turbulence [46,47]. We prefer to call it grain turbulence, since this term reflects the physical nature of the system consisting of dense droplets, or grains, in a rarefied surrounding.

The equivalent interpretation of the granular state could be by imagining it as a collection of dark solitons, typical of defocusing NLS, inside a Bose-condensed surrounding. The granular state is similar to the critical balanced state that is a state where the linear and nonlinear terms are balanced for a wide range of scales [48,49].

It is worth emphasizing that the transition from one dynamical regime to another is not an abrupt phase transition, but a gradual crossover, although it can be very sharp. Therefore the transition lines are, of course, conditional, showing where one dynamic behavior changes to the other. At the same time, some features of one dynamic regime can survive to the

Fig. 4 (Color online) Vortex turbulence. Cross-section of a random vortex tangle obtained by numerical simulations. Darker coulor corresponds to lower density, thus vortices are shown as black spots.

Fig. 5 (Color online) Vortex turbulence. Experimental observation of a vortex turbulent state in the cloud $^{87}$Rb atoms. Multiple dark spots correspond to vortices.
other region. For instance, in the regime of grain turbulence, some occasional vortices can occur, although their number is very small and does not change the overall picture. However, eventually a few vortices can arise and then annihilate in various locations at random.

Continuing pumping energy into the trapped system destroys Bose condensed droplets. So that the grain turbulence exists in the interval of energies

\[ E_{\text{fog}} < E_{\text{inj}} < E_c, \]  

(26)

until the injected energy per atom is so high that all Bose condensate becomes destroyed, when the energy reaches the value

\[ E_c = k_B T_c, \]  

(27)

where \( T_c \) is the critical temperature of Bose-Einstein condensation. For the treated case of \(^{87}\text{Rb}\), the critical temperature is \( T_c = 2.76 \times 10^{-7} \) K. Then we have \( E_c = 0.238 \times 10^{-10} \) eV. This gives \( E_c = 27.4E_\perp \).

Figure 6 presents the cross-section of the atomic cloud in the regime of grain turbulence, found numerically. Droplets are shown as bright spots surrounded by dark rarefied regions. Each droplet is formed by Bose-condensed atoms, which is confirmed by a constant phase inside a droplet. In Fig. 7, an experimentally observed granular state is shown.

**Wave turbulence.** Increasing the amount of injected energy should finally lead to a complete destruction of condensate \([50]\). This happens after the injected energy surpasses the critical energy \( E_c \),

\[ E_{\text{inj}} > E_c, \]  

(28)
Fig. 8 (Color online) Wave turbulence. Cross-section of the trapped cloud in the regime of wave turbulence, found numerically. The left figure shows the density distribution, where brighter colour corresponds to higher density. The right figure shows the spatial phase distribution, where the larger phase is of brighter colour.

Fig. 9 (Color online) Characteristic features of the regimes of grain turbulence and wave turbulence.

| Grain turbulence | Wave turbulence |
|------------------|-----------------|
| $\rho_{gr} / \rho_{sur} \sim 100$ | $\rho_{w} / \rho_{sur} \sim 3$ |
| constant phase inside | random phase inside |
| $E_{kin} / E_{int} \sim 3$ | $E_{kin} / E_{int} \sim 300$ |
| $l_{gr} \sim 10^{-5}$ cm | $l_{w} \sim 10^{-4}$ cm |
| $l_{gr} \sim \xi$ | $l_{w} \gg \xi$ |

when all Bose-Einstein condensate is getting destroyed. Then the system is represented by small-amplitude waves, because of which it is termed wave turbulence, or weak turbulence. The linear sizes of waves are in the interval $(0.5 - 1.5) \times 10^{-4}$ cm, so that the typical wave size is $l_{w} \sim 10^{-4}$ cm. The wave density is only slightly greater than that of their surrounding, $\rho_{w} / \rho_{sur} \sim 3$. The phase inside a wave, as well as between the waves, is completely random. The system kinetic energy is much higher than the interaction energy, so that the system can be represented as a collection of almost independent modes described by a kinetic equation with four-wave processes [37]. The dynamic transition from grain turbulence to wave turbulence is a crossover, such that in a wide region grains coexist with waves.

Also, in the weak turbulence regime, there exist the so-called ghost vortices that are nodal points of the wave field, which are present in abundance if the waves are linear [37].

The cross-section of the trapped cloud in the regime of wave turbulence is shown in Fig. 8, where the system density and phase are presented. This figure is obtained by numerical simulations. Because of the large amount of injected energy, required for creating this regime, it has not yet been reached in experiments.

To stress the principal difference between the regimes of grain turbulence and wave turbulence, their characteristic features are compared in Fig. 9.

In conclusion, we have studied both experimentally and by means of computer simulations strongly nonequilibrium states of trapped $^{87}$Rb atoms, prepared in Bose-condensed
state and subject to strong perturbations realized by modulating the trapping potential. The sequence of the nonequilibrium regimes, obtained by computer simulations, is summarized in Fig. 10. And the numerical amplitude-time diagram is presented in Fig. 11. The latter is in good agreement with the experimentally found amplitude-time diagram discussed earlier [15,16]. It is interesting to note that the sequence of the nonequilibrium states, generated in the process of strong perturbation, which includes weakly nonequilibrium state, vortex state, vortex turbulence, grain turbulence, and wave turbulence, repeats the analogous states, although in the reverse order, as those occurring during the process of equilibration [51,52] through a nonequilibrium Bose-condensation phase transition from a strongly nonequilibrium uncondensed state to an equilibrium condensed state [37,46,47,53]. Thus, we generate the states typical of the Kibble-Zurek mechanism [54,55], but in the opposite temporal or-

**Fig. 10** (Color online) Sequence of nonequilibrium states produced by modulating the trapping potential for $^{87}$Rb atoms.

**Fig. 11** (Color online) Numerical amplitude-time phase diagram for $^{87}$Rb.
More discussions on such an inverse Kibble-Zurek scenario will be given in a separate paper.

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