A note on the orientation covering number

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Abstract

Given a graph $G$, its orientation covering number $\sigma(G)$ is the smallest non-negative integer $k$ with the property that we can choose $k$ orientations of $G$ such that whenever $x, y, z$ are vertices of $G$ with $xy, xz \in E(G)$ then there is a chosen orientation in which both $xy$ and $xz$ are oriented away from $x$. Esperet, Gimbel and King showed that $\sigma(G) \leq \sigma(K_{\chi(G)})$, where $\chi(G)$ is the chromatic number of $G$, and asked whether we always have equality. In this note we prove that it is indeed always the case that $\sigma(G) = \sigma(K_{\chi(G)})$. We also determine the exact value of $\sigma(K_n)$ explicitly for 'most' values of $n$.

1 Introduction

Given a non-empty graph $G$ and $k$ orientations $\vec{G}_1, \ldots, \vec{G}_k$ of $G$, we say that $\vec{G}_1, \ldots, \vec{G}_k$ is an orientation covering of $G$ if whenever $x, y, z \in V(G)$ with $xy, xz \in E(G)$ then there is an orientation in which both $xy$ and $xz$ are oriented away from $x$ (i.e., there is some $i$ such that $(x, y), (x, z) \in E(\vec{G}_i)$). The orientation covering number $\sigma(G)$ of $G$ is the smallest positive integer $k$ such that there is a list of $k$ orientations forming an orientation covering of $G$. Orientation coverings were introduced by Esperet, Gimbel and King [2], who used them to study the minimal number of equivalence subgraphs needed to cover a given graph.

Esperet, Gimbel and King [2] showed that $\sigma(G) \leq \sigma(K_{\chi(G)})$ for any graph $G$, where $\chi$ denotes the chromatic number. They asked whether we always have $\sigma(G) = \sigma(K_{\chi(G)})$. In this note we answer this question in the positive.

Theorem 1. For any non-empty graph $G$, we have $\sigma(G) = \sigma(K_{\chi(G)})$.

The value of $\sigma(K_n)$ has been investigated by Esperet, Gimbel and King [2], who determined its order of magnitude and the exact values for small values of $n$. An observation of Gyárfás (see [2]) shows that we have $\chi(DS_n) \leq \sigma(K_n) \leq \chi(DS_n) + 2$, where $DS_n$ is the double-shift graph on $n$ vertices. Using the results of Füredi, Hajnal, Rödl and Trotter [3] on the chromatic number of $DS_n$, this gives $\sigma(K_n) = \log \log n + \frac{1}{2} \log \log \log n + O(1)$. (All logarithms in this paper are base 2.) In this note we will also determine the value of $\sigma(K_n)$ exactly in terms of a certain sequence of positive integers sometimes called the Hoşten–Morris numbers. As a corollary, we get the following improved estimate.

Theorem 2. We have $\sigma(K_n) = \lceil \log \log n + \frac{1}{2} \log \log \log n + \frac{1}{2} \log \pi + 1 + o(1) \rceil$ as $n \to \infty$.

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Given a positive integer \( k \), let \([k]\) denote \{1, \ldots, k\}, as usual. Given a family \( \mathcal{A} \subseteq \mathcal{P}([k]) \) of subsets of \([k]\), we say that \( \mathcal{A} \) is intersecting if whenever \( S, T \in \mathcal{A} \) then \( S \cap T \neq \emptyset \). We say that \( \mathcal{A} \) is maximal intersecting if \( \mathcal{A} \) is intersecting and whenever \( \mathcal{B} \supseteq \mathcal{A} \) and \( \mathcal{B} \) is intersecting then \( \mathcal{B} = \mathcal{A} \). (Equivalently, if \( \mathcal{A} \) is intersecting and \( |\mathcal{A}| = 2^{k-1} \).) The following characterisation of \( \sigma(G) \) is the key to our results.

**Theorem 3.** For any non-empty graph \( G \), \( \sigma(G) \) is the smallest positive integer \( k \) such that there are at least \( \chi(G) \) maximal intersecting families over \([k]\).

Clearly, Theorem 3 implies Theorem 1. Let \( \lambda(k) \) denote the number of maximal intersecting families over \([k]\). The numbers \( \lambda(k) \) are sometimes called Hosten–Morris numbers, after a paper of Hošten and Morris [4] in which they showed that the order dimension of \( K_n \) is the smallest positive integer \( k \) with \( \lambda(k) \geq n \). An equivalent formulation of their result is that the minimal number of linear orders on \([n]\) with the property that the induced orientations of \( K_n \) form an orientation covering is the smallest positive integer \( k \) with \( \lambda(k) \geq n \). Note that by Theorem 3 this number is the same as the orientation covering number of \( K_n \).

Although no exact or asymptotic formula is known for \( \lambda(k) \), it was shown by Brouwer, Mills, Mills and Verbeek [1] that

\[
\log \lambda(k) \sim \frac{2^k}{\sqrt{2\pi k}}
\]

Furthermore, the exact values of \( \lambda(k) \) are known [1] for \( k \) up to 9, with \( \lambda(9) \approx 4 \times 10^{20} \).

Theorem 2 follows from Theorem 3 and (1). Indeed, taking logarithms in (1) shows that \( \sigma(K_n) \) is the smallest positive integer \( k \) with \( \log \log n \leq k - \frac{1}{2} (\log \pi + 1) - \frac{1}{2} \log k + o(1) \), which gives \( \sigma(K_n) = \lceil \log \log n + \frac{1}{2} \log \log \log n + \frac{1}{2} (\log \pi + 1) + o(1) \rceil \).

## 2 Proof of Theorem 3

The proof is based on the following observation.

**Lemma 4.** For any non-empty graph \( G \), \( \sigma(G) \) is the smallest positive integer \( k \) with the property that there is a collection \( \{ A_v \}_{v \in V(G)} \) of subsets of \( \mathcal{P}([k]) \) (i.e., \( A_v \subseteq \mathcal{P}([k]) \) for all \( v \)) such that the following two conditions hold.

1. If \( uv \in E(G) \), then there exists \( S \in A_u \) and \( T \in A_v \) such that \( S \cap T = \emptyset \).
2. For all \( v \in V(G) \) and \( S, T \in A_v \), we have \( S \cap T \neq \emptyset \). (i.e., \( A_v \) is intersecting.)

**Proof.** First assume that \( \sigma(G) = k \) and \( \vec{G}_1, \ldots, \vec{G}_k \) form an orientation cover of \( G \). For each directed edge \((x, y)\) of \( G \), let \( S_{(x, y)} = \{ i \in [k] : (x, y) \in E(\vec{G}_i) \} \). Let \( A_u = \{ S_{(v, w)} : uv \in E(G) \} \). Clearly \( S_{(v, w)} \cap S_{(w, u)} = \emptyset \), so Condition 1 holds. Also, we have \( S_{(v, w)} \cap S_{(w, v')} \neq \emptyset \) whenever \( vw, vw' \in E(G) \), since by assumption there is an \( i \) such that \((v, w), (v, w') \in E(\vec{G}_i) \). So Condition 2 holds as well.

Conversely, suppose that we have such a collection \( \{ A_v \}_{v \in V(G)} \) with \( A_v \subseteq \mathcal{P}([k]) \) for all \( v \). For each \( uv \in E(G) \), pick \( S_{(u, v)} \in A_u \) and \( S_{(v, u)} \in A_v \) such that \( S_{(u, v)} \cap S_{(v, u)} = \emptyset \). Define the orientations \( \vec{G}_1, \ldots, \vec{G}_k \) of \( G \) by orienting the edge \( uv \) from \( u \) to \( v \) in \( \vec{G}_i \) if \( i \in S_{(u, v)} \), from \( v \) to \( u \) if \( i \in S_{(v, u)} \), and arbitrarily otherwise. This is clearly well-defined, and whenever \( uv, uw \in E(G) \), then \( S_{(u, v)} \cap S_{(u, w)} \neq \emptyset \) (by Condition 2). This gives \( \sigma(G) \leq k \), as claimed. \( \square \)
Proof of Theorem 3. We first show the lower bound for $\sigma(G)$. Let $G$ be any non-empty graph, and let $(A_v)_{v \in V(G)}$ be as in Lemma 4 for $k = \sigma(G)$. For each $v \in V(G)$, let $B_v$ be a maximal intersecting family with $B_v \supseteq A_v$. Note that the families $(B_v)_{v \in V(G)}$ still satisfy both conditions in Lemma 4. Furthermore, $v \mapsto B_v$ is a proper vertex-colouring (since each $B_v$ is intersecting but $B_v \cup B_w$ is not whenever $vw \in E(G)$). It follows that the number of maximal intersecting families over $[k]$ is at least $\chi(G)$.

Conversely, assume that $k$ is a positive integer such that there are at least $\chi(G)$ distinct maximal intersecting families $B_1, \ldots, B_k$ over $[k]$. Let $c : V(G) \mapsto [\chi(G)]$ be a proper vertex-colouring of $G$, and set $A_v = B_{c(v)}$ for each $v$. Certainly each $A_v$ is intersecting. Furthermore, by maximality, no $A_v \cup A_w$ can be intersecting when $c(v) \neq c(w)$, and hence $A_v \cup A_w$ is not intersecting when $vw \in E(G)$. It follows that $(A_v)_{v \in V(G)}$ satisfies both conditions in Lemma 4 and so $\sigma(G) \leq k$.

References

[1] A. E. Brouwer, C. F. Mills, W. H. Mills, and A. Verbeek. Counting families of mutually intersecting sets. Electron. J. Combin., 20(2):Paper 8, 2013.

[2] L. Esperet, J. Gimbel, and A. King. Covering line graphs with equivalence relations. Discrete Appl. Math., 158(17):1902–1907, 2010.

[3] Z. Füredi, P. Hajnal, V. Rödl, and W. T. Trotter. Interval orders and shift graphs. In Sets, graphs and numbers (Budapest, 1991), volume 60 of Colloq. Math. Soc. János Bolyai, pages 297–313. North-Holland, Amsterdam, 1992.

[4] S. Hoşten and W. D. Morris, Jr. The order dimension of the complete graph. Discrete Math., 201(1-3):133–139, 1999.