BLOW-UP FORMULAE FOR TWISTED COHOMOLOGIES WITH SUPPORTS

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Abstract. We study twisted cohomologies with paracompactifying families of supports. The K"unneth theorems, Leray-Hirsch theorems and self-intersection formulae are established. Based on these results, we eventually give explicit expressions of complex blow-up formulae for twisted Dolbeault cohomology on arbitrary complex manifolds and the ones of generalized blow-ups formulae for twisted de Rham cohomology on arbitrary oriented smooth manifolds. These expressions are induced by the morphisms of (simple or double) complexes of spaces of forms and currents rather than just the maps between cohomologies, which help us to obtain the corresponding results for twisted Bott-Chern, Aeppli cohomologies and hypercohomologies of truncated twisted holomorphic de Rham complexes.

1. Introduction

Unless stated otherwise, all manifolds are assumed to be connected, paracompact, all submanifolds (resp. complex submanifolds) are assumed to be closed (in the topological sense) embedded smooth (resp. complex) submanifolds without boundary and set $k = \mathbb{R}$ or $\mathbb{C}$.

The concept of blow-ups was invented by algebraic geometers in the study of birational transformations. O. Zariski [39] first defined it in modern language and used it to study singularities. In complex geometry, the corresponding notion was first introduced by H. Hopf [16], which is said to be the complex blow-up in the present paper. For smooth complex algebraic varieties, blow-ups in complex setting coincide with the ones in algebraic setting. Blow-up transformations play an important role in complex and algebraic geometries. Using them, K. Kodaira [18] proved the well-known embedding theorem and H. Hironaka [15] constructed the first example of non-algebraic Moishezon threefold. J.-P. Demailly and M. Paun [10] obtained a characterization of the Fujiki class $C$ via the complex blow-up operations. Besides algebraic and complex settings, blow-ups can also be defined in other geometric categories. D. McDuff [19] defined the symplectic blow-ups and used it to construct the examples of simply-connected non-K"ahlerian symplectic manifolds. Inspired by [19], S. Yang, X.-D. Yang and G. Zhao [36] defined the blow-up of a locally conformally symplectic manifold along a compact induced symplectic submanifold and proved it admits a locally conformally symplectic structure. Generalized complex geometry unified complex geometry and symplectic geometry in one framework, which plays a significant role in string theory. To find more examples of generalized complex manifolds, the ones have been trying to construct blow-ups in this setting. G. Cavalcanti and M. Gualtieri [5] showed that a blow-up exists for a generically symplectic 4-manifold along a non-degenerate point of complex type. This was used

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to produce new examples of generalized complex structures on \( m\mathbb{C}P^2 \# n\mathbb{C}P^2 \) for \( m \) odd. In [6, 34], the condition of generalized Kählerian blow-ups were studied. M. Bailey, G. Cavalcanti and J. van der Leer Durán [2] introduced the concept of holomorphic ideals to define a blow-up in the category of smooth manifolds. They proved that a generalized Poisson submanifold carries a canonical holomorphic ideal and gave a necessary and sufficient condition for the blow-up of a generalized complex manifold along a generalized Poisson submanifold to be generalized complex. They also proved a normal form theorem for a neighborhood of a generalized Poisson transversal and used it to define a generalized complex blow-up of a generalized complex manifold along a generalized Poisson transversal. Forgetting additional structures, the underlying manifolds of blow-ups in all above settings are roughly viewed as a union of two pieces: the complement of blow-up center and the complex projectivization of normal bundle of blow-up center. Based on this observation, we define the concept of generalized blow-ups (see Sect. 5.2.1) and investigate it.

How invariants vary under blow-up transformations is a natural and important question. Recently, there are some progress on the complex blow-up formulae for twisted cohomologies. On compact complex manifolds, complex blow-up formulae were established for twisted Dolbeault cohomologies [28, 29, 1, 31, 32] and twisted de Rham cohomologies [37, 36, 8, 40]. In different approaches, we [20, 21, 22, 23] established and explicitly expressed these formulae on arbitrary complex manifolds without the hypothesis of compactness.

Our first goal of the present paper is to establish complex blow-up formulae for twisted Dolbeault cohomology with supports in a paracompactifying family.

**Theorem 1.1.** Let \( \pi : \tilde{X} \to X \) be the complex blow-up of a complex manifold \( X \) along a complex submanifold \( Y \) of complex codimension \( r \) with the exceptional divisor \( E \). Suppose that \( E \) is a locally free sheaf of \( O_X \)-modules of finite rank on \( X \) and \( \Phi \) is a paracompactifying family of supports on \( X \). Denote by \( i_E : E \to \tilde{X} \) the inclusion and by \( h \in H^{1,1}(E) \) the Dolbeault class of a first Chern form of the universal line bundle \( O_E(\pi) \rightarrow \pi^* E \) over the projective bundle \( \pi|_E : E = \mathbb{P}(N_{Y/X}) \rightarrow Y \) associated to the normal bundle \( N_{Y/X} \) of \( Y \) in \( X \). Then

\[
\pi^* + \sum_{i=1}^{r-1} i_{E*} \circ (h^{i-1}) \circ (\pi|_E)^* \quad (1.1)
\]
gives an isomorphism

\[
H_{\Phi}^{\bullet, \bullet}(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H_{\Phi, Y}^{\bullet-i, \bullet-i}(Y, i_Y^* \mathcal{E}) \rightarrow H_{\pi^{-1}, \Phi}^{\bullet, \bullet}(\tilde{X}, \pi^* \mathcal{E})
\]

and

\[
(\pi_*, \ G_{h, \Phi}^{-1} \circ i_Y^*, \ ..., \ G_{h, \Phi}^{-r+1} \circ i_Y^*)
\]

(1.2)
gives an isomorphism

\[
H_{\pi^{-1}, \Phi}^{\bullet, \bullet}(\tilde{X}, \pi^* \mathcal{E}) \rightarrow H_{\Phi}^{\bullet, \bullet}(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H_{\Phi, Y}^{\bullet-i, \bullet-i}(Y, i_Y^* \mathcal{E}), \quad (1.3)
\]

where \( G_{h, \Phi}^{i} : H_{(\pi^{-1}, \Phi)|_{E}}^{\bullet-i, \bullet-i}(E, (\pi|_E)^* i_Y^* \mathcal{E}) \rightarrow H_{\Phi, Y}^{\bullet-i, \bullet-i}(Y, i_Y^* \mathcal{E}) \) is defined in Sect. 4.3.
On compact complex manifolds, S. Rao, S. Yang and X.-D. Yang [29] gave an expression of (1.3) for $\Phi = c\tau_X$ in the form (4.15) (see Sect. 4.4.1), which is exactly (1.2) via $\tau^{Dol, c\tau_X}$ (see Sect. 4.4.1). An advantage of the expressions given here is to help us understand (1.3) on the level of forms and currents rather than just on the level of cohomologies. The twisted Bott-Chern cohomology is a useful tool in the study of locally conformally Kählerian geometry [26, 27]. Utilizing (1.1) (1.2) and a Stelzig’s result [32], we give the explicit complex blow-up formulae for twisted Bott-Chern, Aeppli cohomologies in Sect. 4.5.1. Hypercohomology of a truncated holomorphic de Rham complex is an important invariant in Hodge theory. For instance, it is used to compute the Hodge filtration on de Rham cohomology of a compact Kähler manifold [35]; it connects with singular cohomology and Deligne cohomology [11], singular cohomology and integral Bott-Chern cohomology [30]. By use of (1.1) (1.2), we obtain the complex blow-up formula for the hypercohomology of a truncated twisted holomorphic de Rham complex on an arbitrary complex manifold, which extends an open question of Y. Chen, S. Yang [8] and answers it in Sect. 4.5.2.

Our second goal is to establish the formulae of generalized blow-ups for twisted de Rham cohomologies with supports in paracompactifying families on (not necessarily compact) oriented smooth manifolds, which apply to complex, symplectic, locally conformally symplectic and generalized complex blow-ups.

**Theorem 1.2.** Let $\pi: \tilde{X} \to X$ be a generalized blow-up of an oriented smooth manifold $X$ along an oriented smooth submanifold $Y$ of codimension $2r$ with the exceptional divisor $E$. Assume that $\mathcal{L}$ is a local system of $k$-modules with finite rank on $X$ and $\Phi$ is a paracompactifying family of supports on $X$. Denote by $i_{\Phi}: E \to \tilde{X}$ the inclusion and by $h = c_1(\mathcal{O}_E(-1)) \in H^2_{dR}(E, k)$ the first Chern form of the universal line bundle $\mathcal{O}_E(-1)$ on the complex projectivization $\pi|_E : E = \mathbb{P}(N_{Y/X}) \to Y$ associated to the normal bundle $N_{Y/X}$ of $Y$ in $X$. Then

$$\pi^* + \sum_{i=1}^{r-1} i_{E, \pi} \circ (h^{i-1}) \circ (\pi|_E)^* : \quad (1.4)$$

$$H^k_{\Phi}(X, \mathcal{L}) \oplus \bigoplus_{i=1}^{r-1} H^{k-2i}_{\Phi|_Y}(Y, \mathcal{L}|_Y) \to H^k_{\pi-1}\Phi(\tilde{X}, \pi^{-1}\mathcal{L}) \quad (1.5)$$

and

$$(\pi_* , G_{h, \Phi}^{-1} \circ i_{E, \pi}^{*}, \ldots , G_{h, \Phi}^{-r+1} \circ i_{E, \pi}^{*}) : \quad (1.6)$$

$$H^\bullet_{\pi-1}\Phi(\tilde{X}, \pi^{-1}\mathcal{L}) \to H^\bullet_{\Phi}(X, \mathcal{L}) \oplus \bigoplus_{i=1}^{r-1} H^{\bullet-2i}_{\Phi|_Y}(Y, \mathcal{L}|_Y), \quad (1.7)$$

are inverse isomorphisms, where $G_{h, \Phi}^{-i} : H^\bullet_{(\pi-1)\Phi|_E}(E, (\pi|_E)^{-1}(\mathcal{L}|_Y)) \to H^\bullet_{\Phi|_Y}(Y, \mathcal{L}|_Y)$ is defined as in Sect. 5.1.

It is worthy to notice that, self-intersection formulae (Propositions 4.7, 5.4) play a key role for giving the expressions (1.1) (1.4) (Proposition 4.8, Sect. 5.2.3) and studying the inverse relationships of (1.1) and (1.2), (1.4) and (1.6) (Proposition 4.11, Theorem 1.2).

All these formulae are established on the cohomologies with supports in quite general families, which apply to the usual cohomologies and the cohomologies with compact supports. They may be useful to study the topological properties of blow-up manifolds.
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2. Preliminaries

2.1. Families of supports. For the readers’ convenience, we collect some terminology and results on families of supports, refer to [4, I. §6, II. §9, IV. §5].

A family $\Phi$ of supports on a topological space $X$ means a family $\Phi$ of closed subsets of $X$ satisfying that:

1. any closed subset of a member of $\Phi$ is a member of $\Phi$,
2. $\Phi$ is closed under finite unions.

If in addition:

3. each element of $\Phi$ is paracompact,
4. each element of $\Phi$ has a closed neighborhood which is in $\Phi$,

then $\Phi$ is said to be a paracompactifying family of supports on $X$.

Let $\Phi$ be a family of supports on $X$. For a subset $A \subseteq X$, set $\Phi|_A := \{ K \subseteq A | K \in \Phi \}$ and $\Phi \cap A := \{ K \cap A | K \in \Phi \}$, which are two families of supports on $A$. If $B \subseteq A$, $(\Phi|_A)|_B = \Phi|_B$. For the families $\Phi$ and $\Psi$ of supports on $X$, $\Phi \cap \Psi$ denotes the family of all closed subsets of sets of the form $K \cap L$ for $K \in \Psi$ and $L \in \Phi$. For the families $\Phi$ and $\Psi$ of supports on $X$, $Y$ respectively, $\Phi \times \Psi$ means the family of all closed subsets of sets of the form $K \times L$ for $K \in \Psi$ and $L \in \Phi$. Let $f : Y \to X$ be a continuous map of topological spaces and let $\Phi$ be a family of supports on $X$. Denote by $f^{-1}(\Phi)$ the family of all closed subsets of sets of the form $f^{-1}(K)$ for $K \in \Phi$. For a continuous map $g : Z \to Y$ of topological spaces, $g^{-1}(f^{-1}(\Phi)) = (f \circ g)^{-1}(\Phi)$.

Denote by $clt_X$ and $c_X$ the families of all closed subsets and all compact subsets of $X$ respectively. If $X$ is paracompact, $clt_X$ is paracompactifying. If $X$ is locally compact, $c_X$ is paracompactifying. Let $f : X \to Y$ be a continuous map of topological spaces. Then $f^{-1}clt_Y = clt_X$. Moreover, if $f$ is proper, $f^{-1}c_Y = c_X$. Let $pr_1 : X \times Y \to X$ be the first projection and $\Phi$ a family of supports on $X$. Then $pr_1^{-1}(\Phi) = \Phi \times clt_Y$, which is denoted by $\Phi \times Y$.

Proposition 2.1. (1) Let $\Phi$ be a family of supports on a topological space $X$. For a closed subset $A \subseteq X$, $\Phi|_A = \Phi \cap A = i^{-1}\Phi$, where $i : A \to X$ is the inclusion.

(2) Let $f : X \to Y$ be a proper map of locally compact Hausdorff spaces and $\Phi$ a family of supports of $Y$. For any subset $A \subseteq Y$, $(f|_{f^{-1}(A)})^{-1}(\Phi|_A) = (f^{-1}\Phi)|_{f^{-1}(A)}$.

(3) Assume that $\Phi$ and $\Psi$ are families of supports on a topological space $X$. Then $\Phi \cap \Psi = l^{-1}(\Phi \times \Psi)$, where $l : X \to X \times X$ is the diagonal map.

(4) Let $\Phi$ be a paracompactifying family of supports on $X$. For a locally closed subset $A \subseteq X$, $\Phi|_A$ is paracompactifying on $A$.

(5) Let $f : X \to Y$ be a continuous map of locally compact Hausdorff spaces and $\Psi$ a paracompactifying family of supports on $Y$. Then $f^{-1}\Psi$ is paracompactifying on $X$. 


(6) Let $\Phi$ and $\Psi$ be paracompactifying families of supports on locally compact Hausdorff spaces $X$ and $Y$ respectively. Then $\Phi \times \Psi$ is paracompactifying on $X \times Y$.

Proof. By the definitions, we immediately get (1) (3) (6). For (4), see [4, I. 6.5]. By [4, IV. 5.4 (3), 5.5], (5) holds. For $K \in (f|_{f^{-1}(A)})^{-1}(\Phi|A)$, $K$ is a closed subset of $(f|_{f^{-1}(A)})^{-1}(L) = f^{-1}(L)$ for a set $L \subseteq A$ with $L \in \Phi$. Then $K \in f^{-1}\Phi$ and $K \subseteq f^{-1}(A)$, i.e., $K \in (f^{-1}\Phi)|_{f^{-1}(A)}$. So $(f|_{f^{-1}(A)})^{-1}(\Phi|A) \subseteq (f^{-1}\Phi)|_{f^{-1}(A)}$. For $K \in (f^{-1}\Phi)|_{f^{-1}(A)}$, $K \subseteq f^{-1}(A)$ and there is a set $L$ such that $L \in \Phi$ and $K$ is a closed subset of $f^{-1}(L)$. Set $Z = f(K)$. Then $Z \subseteq L \in \Phi$ and $Z \subseteq A$. Since $f$ is proper, $Z$ is closed in $Y$. So $Z \in \Phi|A$. Since $K$ is a closed subset of $(f|_{f^{-1}(A)})^{-1}(Z)$, $K \in (f|_{f^{-1}(A)})^{-1}(\Phi|A)$. We proved (2).

Suppose that $\mathcal{F}$ is a sheaf on $X$ and $\Phi$ is a family of supports on $X$. Denote by $\Gamma_\Phi(X, \mathcal{F})$ the group of sections of $\mathcal{F}$ on $X$ with supports in $\Phi$ and by $H^p_\Phi(X, \mathcal{F})$ the cohomology of $\mathcal{F}$ with supports in $\Phi$. The sheaf $\mathcal{F}$ is said to be $\Phi$-acyclic, if $H^p_\Phi(X, \mathcal{F}) = 0$ for $p > 0$. The sheaf $\mathcal{F}$ is called a $\Phi$-soft sheaf, if the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F})$ is surjective for any $Z \in \Phi$. Let $\Phi$ be a paracompactifying family of supports on a compact manifold $X$. The sheaf $C^\infty_X$ of germs of (real or complex valued) smooth functions on $X$ is $\Phi$-soft ([4, II. 9.4]), so are $\mathcal{A}_X^{p,q}$ and $\mathcal{D}_X^{p,q}$ for any $p, q$ by [4, II. 9.16]. Hence $\mathcal{A}_X^{p,q}$ and $\mathcal{D}_X^{p,q}$ are $\Phi$-acyclic by [4, II. 9.11].

2.2. Sheaf Theory. We recall some notations and results in sheaf theory, refer to [9, IV. Sects. 2, 8, 9]. Suppose that $U$ is any open set of $X$. For a sheaf $\mathcal{F}$ on $X$, $\Gamma(U, \mathcal{F}^{[0]})$ is the set of all maps $f : U \to \mathcal{F}$ such that $f(x) \in \mathcal{F}_x$ for all $x \in U$. For any $s \in \Gamma(U, \mathcal{F})$, let $\tilde{s}(x) = s_x$ for any $x \in U$, where $s_x \in \mathcal{F}_x$ is the stalk of $s$ over $x$. Then $s \to \tilde{s}$ define a natural injection $j : \mathcal{F} \to \mathcal{F}^{[0]}$ of sheaves. Set $\mathcal{F}^{[p]} = (\mathcal{F}^{[p-1]})^{[0]}$ for $p \geq 0$. For all $p$, $\mathcal{F}^{[p]}$ are flabby sheaves. The stalk $\mathcal{F}_{x}^{[p]}$ can be considered as the set of equivalence classes of maps $f : X^{p+1} \to \mathcal{F}$ such that $f(x_0, \ldots, x_p) \in \mathcal{F}_{x_p}$ for any $(x_0, \ldots, x_p) \in X^{p+1}$, with such two maps are equivalent if and only if they coincide on a set of the form

$$x_0 \in V, \ x_1 \in V(x_0), \ldots, x_p \in V(x_0, \ldots, x_{p-1}),$$

where $V$ is an open neighborhood of $x$ and $V(x_0, \ldots, x_j)$ an open neighborhood of $x_j$, depending on $x_0, \ldots, x_j$. Similarly, $\Gamma(U, \mathcal{F}^{[p]})$ can be considered as the set of equivalence classes of maps $f : X^{p+1} \to \mathcal{F}$ such that $f(x_0, \ldots, x_p) \in \mathcal{F}_{x_p}$ for any $(x_0, \ldots, x_p) \in X^{p+1}$, with such two maps are equivalent if and only if they coincide on a set of the form

$$x_0 \in U, \ x_1 \in V(x_0), \ldots, x_p \in V(x_0, \ldots, x_{p-1}).$$

For $f \in \Gamma(U, \mathcal{F}^{[p]})$, denote by $\text{supp} f$ the support of $f$. Then $U - \text{supp} f$ is just the set of the point $x \in U$ satisfying that there exists an open neighborhood $V \subseteq U$ of $x$ such that $f : X^p \to \mathcal{F}$ is zero on a set of the form

$$x_0 \in V, \ x_1 \in V(x_0), \ldots, x_p \in V(x_0, \ldots, x_{p-1}).$$

For $t \in \mathcal{F}_x$, let $S(t)$ be any element in $\mathcal{F}^{[0]}(X)$ whose restriction on some neighborhood $U$ of $x$ is in $\Gamma(U, \mathcal{F})$ and $S(t)(x) = t$. Define $d^p : \mathcal{F}^{[p]} \to \mathcal{F}^{[p+1]}$ as

$$(d^p f)(x_0, \ldots, x_{p+1}) = \sum_{0 \leq i \leq p} (-1)^i f(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) + (-1)^{p+1} S(f(x_0, \ldots, x_p))(x_{p+1}).$$
The definition is independent of the choice of $S$. Moreover, $d^j \circ j = 0$ and $d^{p+1} \circ d^p = 0$ for any $p \geq 0$. The complex $(\mathcal{F}^\bullet, d)$ is a flabby resolution of the sheaf $\mathcal{F}$, which is called the simplicial flabby resolution of $\mathcal{F}$ and is briefly denoted as $\mathcal{F}^\bullet$.

For any exact sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$, their resolutions $\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet$ is exact. So $\mathcal{F} \to \mathcal{F}^p$ is an exact functor for any $p$. Any flabby sheaf on $X$ is $\Phi$-acyclic for any family $\Phi$ of supports on $X$, so the functor $\mathcal{F} \mapsto \Gamma_\Phi(X, \mathcal{F}^p)$ is exact for any $p$. In particular,

$$H^q \left( \Gamma_\Phi(X, (\mathcal{F}^\bullet)[p]) \right) = \Gamma_\Phi \left( X, (\mathcal{H}^q(\mathcal{F}^\bullet))[p] \right)$$

for a complex $\mathcal{F}^\bullet$ of sheaves on $X$, where $\mathcal{H}^q(\mathcal{F}^\bullet)$ denotes the $q$-th cohomological sheaf of $\mathcal{F}^\bullet$.

Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules (resp. $k$-modules) on a complex manifold (resp. topological space) $X$ and let $\Phi$, $\Psi$ be two families of supports on $X$. For $u \in \Gamma(X, \mathcal{F}^p)$ and $v \in \Gamma(X, \mathcal{G}^q)$, the cup product $u \cup v \in \Gamma(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{p+q})$ (resp. $\Gamma(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{p+q})$) is defined as

$$u \cup v(x_0, \ldots, x_{p+q}) = S(u(x_0, \ldots, x_q)) \otimes v(x_p, \ldots, x_{p+q})$$

$$\in \mathcal{F}_{x_{p+q}} \otimes_{\mathcal{O}_X} \mathcal{G}_{x_{p+q}}$$

In such way, we get a $\mathbb{C}$-bilinear (resp. $k$-bilinear) map

$$\Gamma_\Phi(X, \mathcal{F}^p) \times \Gamma_\Psi(X, \mathcal{G}^q) \to \Gamma_{\Phi \cap \Psi}(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{p+q})$$

(resp. $\Gamma_\Phi(X, \mathcal{F}^p) \times \Gamma_\Psi(X, \mathcal{G}^q) \to \Gamma_{\Phi \cap \Psi}(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{p+q})$),

which maps $(u, v)$ to $u \cup v$. It induces a cup product

$$\cup : H^p_\Phi(X, \mathcal{F}) \times H^q_\Psi(X, \mathcal{G}) \to H^{p+q}_{\Phi \cap \Psi}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

(resp. $H^p_\Phi(X, \mathcal{F}) \times H^q_\Psi(X, \mathcal{G}) \to H^{p+q}_{\Phi \cap \Psi}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$).

The wedge product of holomorphic forms gives an embedding $\wedge : \Omega^r_X \otimes_{\mathcal{O}_X} \Omega^s_X \hookrightarrow \Omega^{r+s}_X$ for $r + s \leq \dim \mathbb{C} X$. Via this embedding, we furthermore define a second type of cup product

$$\cup : H^p_\Phi(X, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^r_X) \times H^q_\Psi(X, \mathcal{G} \otimes_{\mathcal{O}_X} \Omega^s_X) \to H^{p+q}_{\Phi \cap \Psi}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \otimes_{\mathcal{O}_X} \Omega^r_X \otimes_{\mathcal{O}_X} \Omega^s_X)$$

$$\hookrightarrow H^{p+q}_{\Phi \cap \Psi}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \otimes_{\mathcal{O}_X} \Omega^{r+s}_X).$$

Let $f : Y \to X$ be a holomorphic (resp. continuous) map of complex manifolds (resp. topological spaces) and $\mathcal{F}$ a sheaf of $\mathcal{O}_Y$-modules (resp. $k$-modules) on $Y$. For any $u \in \Gamma(X, \mathcal{F}^p)$, define $f^* u \in \Gamma(Y, (f^* \mathcal{F})^p)$ (resp. $\Gamma(Y, (f^{-1} \mathcal{F})^p)$) as

$$(f^* u)(y_0, \ldots, y_p) = u(f(y_0), \ldots, f(y_p)) \otimes 1 \in (f^* \mathcal{F})_{y_p} = \mathcal{F}_{f(y_p)} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y, y_p}$$

(resp. $(f^* u)(y_0, \ldots, y_p) = u(f(y_0), \ldots, f(y_p)) \in (f^{-1} \mathcal{F})_{y_p} = \mathcal{F}_{f(y_p)}$).

Clearly, $\text{supp}(f^* u) \subseteq f^{-1}(\text{supp}(u))$. Suppose that $\Phi$ is a family of supports. For a morphism $f^* : \Gamma_\Phi(X, \mathcal{F}^\bullet) \to \Gamma_{f^{-1}\Phi}(Y, (f^* \mathcal{F})^\bullet)$ (resp. $f^* : \Gamma_\Phi(X, \mathcal{F}^\bullet) \to \Gamma_{f^{-1}\Phi}(Y, (f^{-1} \mathcal{F})^\bullet)$) of complexes, which induces a pullback

$$f^* : H^p_\Phi(X, \mathcal{F}) \to H^p_{f^{-1}\Phi}(Y, f^* \mathcal{F}).$$

(resp. $f^* : H^p_\Phi(X, \mathcal{F}) \to H^p_{f^{-1}\Phi}(Y, f^{-1} \mathcal{F})$).
The pullback of holomorphic forms gives a morphism $f^* \Omega^0_Y \to \Omega^0_X$, hence we can define a second type of pullback

$$
f^*: H^p_{\Phi}(X, F \otimes_{\mathcal{O}_X} \Omega^0_X) \to H^p_{\Phi}(Y, f^* F \otimes_{\mathcal{O}_Y} f^* \Omega^0_X) \to H^p_{\Phi}(Y, f^* F \otimes_{\mathcal{O}_Y} \Omega^0_Y).
$$

(2.4)

**Lemma 2.2.** Suppose that $X$ is a topological space.

1. Let $j: U \to X$ be an inclusion of an open subset. Assume that $F$ is a sheaf on $X$ and $G$ is a sheaf on $U$. Then $(j^{-1} F)[p] = j^{-1} (F[p])$ and $(j_* G)[p] = j_*(G[p])$ for any $p$.

2. Let $i: Z \to X$ be an inclusion of a closed subset and $H$ a sheaf on $Z$. Then $(i_* H)[p] = i_*(H[p])$ for any $p$.

**Proof.** We only prove $(j_* G)[p] = j_*(G[p])$ and the other two conclusions can be obtained similarly. For any open set $W \subseteq X$,

$$
\Gamma(W, j_!(G^{(0)})) = \{ t \in \Gamma(W \cap U, G^{(0)}) | \text{ suppt is closed in } W \}.
$$

For any $s \in \Gamma(W, (j_! G)^{(0)})$, $s(x) = 0$ for any $x \in W - U$, so supp$s \subseteq W \cap U$. Set $s = s|_{W \cap U}$. Then supp$s = \text{ supps is closed in } W$. Hence $s \mapsto \bar{s}$ for all $s \in \Gamma(W, (j_! G)^{(0)})$ give a morphism $\Gamma(W, (j_! G)^{(0)}) \to \Gamma(W, j_!(G^{(0)}))$. For $t \in \Gamma(W, (j_! G)^{(0)})$, set $t = t(x)$ for $x \in W \cap U$ and $0$ for $x \in W - U$. Then $t \mapsto \bar{t}$ for all $t \in \Gamma(W, j_!(G^{(0)}))$ give a morphism $\Gamma(W, j_!(G^{(0)})) \to \Gamma(W, (j_! G)^{(0)})$. We easily see that the two morphisms are inverse to each other. So $(j_! G)^{(0)} = j_!(G^{(0)})$. By the induction, we complete the proof. \qed

Assume that $X$ is a topological space and $F$ is a sheaf on $X$. Let $Z$ be a closed subset of $X$ and set $U = X - Z$. Denote by $i: Z \to X$ and $j: U \to X$ the inclusions. Suppose that $\Phi$ is a family of supports on $X$. As we know, $0 \to j_! j^{-1} F \to F \to i_* i^{-1} F \to 0$ is exact, so is $0 \to \Gamma_{\Phi}(X, (j_! j^{-1} F)[p]) \to \Gamma_{\Phi}(X, F[p]) \to \Gamma_{\Phi}(X, (i_* i^{-1} F)[p]) \to 0$. By Lemma 2.2, $\Gamma_{\Phi}(X, (i_* i^{-1} F)[p]) = \Gamma_{\Phi|_Z}(Z, (i^{-1} F)[p])$ and $\Gamma_{\Phi}(X, (j_! j^{-1} F)[p]) = \Gamma_{\Phi|_U}(U, F[p])$. So we have the short exact

$$
0 \longrightarrow \Gamma_{\Phi|_U}(U, F[p]) \xrightarrow{j_*} \Gamma_{\Phi}(X, F[p]) \xrightarrow{i^*} \Gamma_{\Phi|_Z}(Z, (i^{-1} F)[p]) \longrightarrow 0
$$

for any $p$, where $j_*$ is the extension by zero and $i^*$ is the pullback. Furthermore, we have

**Lemma 2.3.** Let $f: Y \to X$ be a continuous map of topological spaces. Suppose that $F$ a sheaf on $X$ and $\Phi$ is a family of supports on $X$. Put $A$ a closed subset of $X$ and set $B = f^{-1}(A)$. Denote by $j: X - A \to X$, $j: Y - B \to Y$, $i: A \to X$, $i: B \to Y$ the inclusions. Then there exists a commutative diagram

$$
\begin{array}{cccccccc}
\cdots & H^k_{\Phi|_{X - A}}(X - A, F) & \xrightarrow{j_*} & H^k_{\Phi}(X, F) & \xrightarrow{i^*} & H^k_{\Phi|_A}(A, i^{-1}_A F) & \xrightarrow{\delta} & H^{k+1}_{\Phi|_{X - A}}(X - A, F) & \cdots \\
\cdots & H^k_{(f^{-1} \Phi)|_{Y - B}}(Y - B, f^{-1} F) & \xrightarrow{j_*} & H^k_{f^{-1} \Phi}(Y, f^{-1} F) & \xrightarrow{i^*} & H^k_{(f^{-1} \Phi)|_A}(B, i^{-1}_A f^{-1} F) & \xrightarrow{\delta} & H^{k+1}_{(f^{-1} \Phi)|_{Y - B}}(Y - B, f^{-1} F) & \cdots \\
\end{array}
$$

of long exact sequences, where $f_Z: f^{-1}(Z) \to Z$ denotes the restriction of $f$ for any $Z \subseteq X$. 
Proof. We easily check the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Gamma_{\phi|X-A}(X-A, \mathcal{F}[\bullet]) & \xrightarrow{f^*} & \Gamma_{\phi}(X, \mathcal{F}[\bullet]) & \xrightarrow{\iota^*} & \Gamma_{\phi|A}(A, (\mathcal{F}^{-1}[\bullet]) & \rightarrow & 0 \\
0 & \rightarrow & \Gamma_{(\mathcal{F}^{-1})^{-1}}(Y-B, (\mathcal{F}^{-1}[\bullet]) & \xrightarrow{f^*} & \Gamma_{(\mathcal{F}^{-1})^{-1}}(Y, (\mathcal{F}[\bullet]) & \xrightarrow{\iota^*} & \Gamma_{(\mathcal{F}^{-1})^{-1}}(B, (\mathcal{F}^{-1}[\bullet]) & \rightarrow & 0
\end{array}
\]

of short exact sequences of complexes, which implies the conclusion.

Lemma 2.4. Assume that \( \pi : X \rightarrow Y \) is a continuous map of topological spaces and \( \mathcal{F} \) is a sheaf of on \( X \). Let \( \mathcal{H} \) be the group consisting of the map \( h : Y^{p+1} \times X^{q+1} \rightarrow \mathcal{F} \) satisfying \( h(y_0, \ldots, y_p; x_0, \ldots, x_q) \in \mathcal{F}_x \) for any \( (y_0, \ldots, y_p; x_0, \ldots, x_q) \in Y^{p+1} \times X^{q+1} \) and let \( \mathcal{H}_0 \) be the subgroup of \( \mathcal{H} \) consisting of the map \( h \) satisfying that \( h = 0 \) on a set of the form

\[
y_0 \in Y, \ y_i \in V(y_0, \ldots, y_{i-1}), \ 1 \leq i \leq p, \quad x_0 \in \pi^{-1}(V(y_0, \ldots, y_p)), \quad x_j \in V(y_0, \ldots, y_p; x_0, \ldots, x_{j-1}), 1 \leq j \leq q, \tag{2.5}
\]

where \( V(y_0, \ldots, y_{i-1}) \) is an open neighborhood of \( y_i \) in \( Y \) depending on \( y_0, \ldots, y_{i-1} \) for \( 1 \leq i \leq p \) and \( V(y_0, \ldots, y_p; x_0, \ldots, x_{j-1}) \) is an open neighborhood of \( x_j \) in \( X \) depending on \( y_0, \ldots, y_p, x_0, \ldots, x_{j-1} \) for \( 1 \leq j \leq q \). Then \( \mathcal{H}_0 \) can be viewed as a subgroup of \( \Gamma(Y, (\pi_*(\mathcal{F}[\bullet]))[\bullet]) \).

Proof. Let \( \mathfrak{P} \) be the group consisting of the map \( f : Y^{p+1} \rightarrow \pi_*(\mathcal{F}[\bullet]) \) satisfying that \( f(y_0, \ldots, y_p) \in (\pi_*(\mathcal{F}[\bullet]))_{y_p} \) for any \( (y_0, \ldots, y_p) \in Y^{p+1} \). Let \( \mathfrak{P}_0 \) be the subgroup of \( \mathfrak{P} \) consisting of the map \( f \) which is zero on a set of the form \( y_0 \in Y, \ y_i \in V(y_0), \ldots, y_p \in V(y_0, \ldots, y_{p-1}) \). Then \( \Gamma(Y, (\pi_*(\mathcal{F}[\bullet]))[\bullet]) \cong \mathfrak{P}/\mathfrak{P}_0 \). For any \( h \in \mathcal{H} \), set

\[
G(h)(y_0, \ldots, y_p) = h(y_0, \ldots, y_p; \bullet) : X^{q+1} \rightarrow \mathcal{F}
\]

for any \( (y_0, \ldots, y_p) \in Y^{p+1} \). Then \( G(h)(y_0, \ldots, y_p) \) can be viewed as a section of \( \mathcal{F}[\bullet] \) on \( X \). Set

\[
P(h)(y_0, \ldots, y_p) = [G(h)(y_0, \ldots, y_p)]_{y_p} \in \lim_{V \ni y_p} \Gamma(V, \mathcal{F}[\bullet]) = \left( \pi_*(\mathcal{F}[\bullet]) \right)_{y_p},
\]

where \([\bullet]_{y_p}\) denotes the equivalent class under the direct limit. Then \( P(h) \in \mathfrak{P} \). Define \( \mathcal{H} \rightarrow \mathfrak{P}/\mathfrak{P}_0 \) as \( h \mapsto P(h) \) modulo \( \mathfrak{P}_0 \). We only need to prove that the kernel of this morphism is \( \mathcal{H}_0 \).

Assume that \( h \in \mathcal{H} \) satisfies that \( P(h) \in \mathfrak{P}_0 \), i.e., \([G(h)(y_0, \ldots, y_p)]_{y_p} = 0\) on the set of the form \( y_0 \in Y, \ y_i \in V(y_0), \ldots, y_p \in V(y_0, \ldots, y_{p-1}) \). This is equivalent to say that, there exists an open neighborhood \( V \) of \( y_p \) such that \( G(h)(y_0, \ldots, y_p) \mid_{\pi^{-1}(V)} = 0 \) on the set of the form \( y_0 \in Y, \ y_i \in V(y_0, \ldots, y_{i-1}), 1 \leq i \leq p \). We can write \( V = V(y_0, \ldots, y_p) \), since \( y_p \in V(y_0, \ldots, y_{p-1}) \) and \( V \) also depends on \( y_p \). Hence \( h(y_0, \ldots, y_p; x_0, \ldots, x_q) = 0 \) on a set of the form (2.5), i.e., \( h \in \mathcal{H}_0 \). Inversely, if \( h \in \mathcal{H}_0 \), \( P(h) \in \mathfrak{P}_0 \) from above arguments. We complete the proof.
2.3. Gluing principle. Recall the gluing principle, which will be used in the following part.

**Lemma 2.5** ([21, Lemma 2.1]). Let $X$ be a smooth manifold and denote by $\mathcal{P}(U)$ a statement on any open subset $U$ in $X$. Assume that $\mathcal{P}$ satisfies conditions:

(i) (local condition) There exists a basis $\mathcal{U}$ of the topology of $X$ such that $\mathcal{P}(U_1 \cap \ldots \cap U_l)$ holds for any finite many $U_1, \ldots, U_l \in \mathcal{U}$.

(ii) (disjoint condition) Let $\{U_n|n \in \mathbb{N}^+\}$ be any collection of disjoint open subsets of $X$. If $\mathcal{P}(U_n)$ hold for all $n \in \mathbb{N}^+$, $\mathcal{P}(\bigcup_{n=1}^{\infty} U_n)$ holds.

(iii) (Mayer-Vietoris condition) For open subsets $U, V$ of $X$, $\mathcal{P}(U \cup V)$ holds if $\mathcal{P}(U)$, $\mathcal{P}(V)$ and $\mathcal{P}(U \cap V)$ hold. Then $\mathcal{P}(X)$ holds.

3. Twisted forms and currents with supports

3.1. Locally free sheaves on complex manifolds. Let $X$ be a complex manifold and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_X$-modules of rank $m$ on $X$. An open subset $U$ of $X$ is said to be $\mathcal{E}$-free, if the restriction $\mathcal{E}|_U$ is a free sheaf of $\mathcal{O}_U$-modules. An open covering $\mathcal{U}$ of $X$ is said to be $\mathcal{E}$-free, if all $U \in \mathcal{U}$ are $\mathcal{E}$-free. An open covering of $X$ is called an $\mathcal{E}$-free basis, if it is both a basis of the topology and an $\mathcal{E}$-free covering of $X$. For an open set $U \subseteq X$, the elements of $\Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,q})$ and $\Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q})$ are called $\mathcal{E}$-valued $(p,q)$-forms and currents on $U$, respectively. In Sects. 3 and 4, the tensor $\otimes_{\mathcal{O}_X}$ of sheaves of $\mathcal{O}_X$-modules will be simply denoted by $\otimes$.

3.1.1. Local representations. Let $U$ be an $\mathcal{E}$-free open subset of $X$ and $e_1, \ldots, e_m$ a basis of $\Gamma(U, \mathcal{E})$ as an $\mathcal{O}_X(U)$-module. For $\omega \in \Gamma(X, \mathcal{E} \otimes \mathcal{A}_X^{p,q})$, the restriction $\omega|_U$ to $U$ can be written as $\sum_{i=1}^{m} e_i \otimes \alpha_i$, where $\alpha_1, \ldots, \alpha_m \in \mathcal{A}_X^{p,q}(U)$. Similarly, for $S \in \Gamma(X, \mathcal{E} \otimes \mathcal{D}_X^{p,q})$, $S|_U = \sum_{i=1}^{m} e_i \otimes T_i$ for some $T_1, \ldots, T_m \in \mathcal{D}_X^{p,q}(U)$. We easily get

**Lemma 3.1.** For any $1 \leq i \leq m$, $\text{supp} \alpha_i \subseteq \text{supp} \omega \cap U$ and $\text{supp} T_i \subseteq \text{supp} S \cap U$.

3.1.2. Extensions by zero and restrictions. Let $\Phi$ be a paracom pactifying family of supports on $X$. Assume that $j : V \rightarrow X$ is the inclusion of the open subset $V$ into $X$. Let $\mathcal{U}$ be an $\mathcal{E}$-free covering of $X$ and $e_1^U, \ldots, e_m^U$ a basis of $\Gamma(U, \mathcal{E})$ as an $\mathcal{O}_X(U)$-module for any $U \in \mathcal{U}$.

For $\omega \in \Gamma_{\Phi|_V}(V, \mathcal{E} \otimes \mathcal{A}_X^{p,q})$ and $U \in \mathcal{U}$, the restriction $\omega|_{V \cap U}$ to $U \cap V$ is $\sum_{i=1}^{m} e_i^U|_{V \cap U} \otimes \alpha_i$, where $\alpha_1, \ldots, \alpha_m \in \mathcal{A}_X^{p,q}(V \cap U)$. Clearly, $\alpha_i = 0$ on $(V \cap U) \cap (U - \text{supp} \alpha_i) = V \cap U - \text{supp} \alpha_i$. So $\alpha_i$ can be extended on $(V \cap U) \cup (U - \text{supp} \alpha_i) = U$ by zero, denoted by $\alpha_i$. Set $\tilde{\omega}_U = \sum_{i=1}^{m} e_i^U \otimes \tilde{\alpha}_i$ in $\Gamma(U, \mathcal{E} \otimes \mathcal{A}_X^{p,q})$. Then $\{\tilde{\omega}_U|U \in \mathcal{U}\}$ can be glued as a global section of $\mathcal{E} \otimes \mathcal{A}_X^{p,q}$ on $X$, denoted by $j_* \omega$. It is noteworthy that $j_* \omega$ doesn’t depend on the choice of the $\mathcal{E}$-free open covering $\mathcal{U}$. Since $\text{supp} j_* \omega = \text{supp} \omega \in \Phi$, we get a map $j_* : \Gamma_{\Phi|_V}(V, \mathcal{E} \otimes \mathcal{A}_X^{p,q}) \rightarrow \Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{A}_X^{p,q})$. 
Similarly, we can define

\[ j_* : \Gamma_{\Phi, Y}(V, E \otimes D_{X}^{p,q}) \to \Gamma_{\Phi}(X, E \otimes D_{X}^{p,q}). \]  

(3.1)

Let \( j^* : \Gamma(X, E \otimes D_X^{p,q}) \to \Gamma(V, E \otimes D_X^{p,q}) \) be the restriction of the sheaf \( E \otimes D_X^{p,q} \). For any \( S \in \Gamma(X, E \otimes D_X^{p,q}) \) and \( U \in \mathcal{U} \), if the restriction \( S|_U = \sum e_i^U \otimes T_i \) for some \( T_1, \ldots, T_m \in D^{p,q}(U) \), then \( (j^* S)|_{V \cap U} = \sum_{i=1}^m e_i^U|_{V \cap U} \otimes T_i|_{V \cap U} \).

3.1.3. **Pullbacks and pushforwards.** Let \( f : Y \to X \) be a holomorphic map of complex manifolds and \( r = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X \). Put \( f^* E = f^{-1} E \otimes_{\mathcal{O}_X} \mathcal{O}_Y \) the inverse image of \( E \) by \( f \). The adjunction morphism \( E \to f_* f^* E \) induces \( f_U^* : \Gamma(U, E) \to \Gamma(f^{-1}(U), f^* E) \) for any open set \( U \subseteq X \), where \( f_U : f^{-1}(U) \to U \) is the restriction of \( f \) to \( f^{-1}(U) \).

**Pullbacks.** The pullback \( A_X^{p,q} \to f_* A_Y^{p,q} \) induces a morphism of sheaves

\[ E \otimes A_X^{p,q} \to E \otimes f_* A_Y^{p,q} = f_*(E \otimes A_Y^{p,q}), \]

hence induces a pullback of \( E \)-valued \((p, q)\)-forms

\[ f^* : \Gamma(X, E \otimes A_X^{p,q}) \to \Gamma(Y, f^* E \otimes A_Y^{p,q}). \]  

(3.2)

Suppose that \( U \) is an \( E \)-free open set in \( X \) and \( e_1, \ldots, e_m \) is a basis of \( \Gamma(U, E) \) as an \( \mathcal{O}_X(U) \)-module. Obviously, \( f_U^* e_1, \ldots, f_U^* e_m \) is a basis of \( \Gamma(f^{-1}(U), f^* E) \) as an \( \mathcal{O}_Y(f^{-1}(U)) \)-module. For an \( E \)-valued \((p, q)\)-form \( \omega \), set \( \omega|_U = \sum_i e_i \otimes \alpha_i \) for some \( \alpha_1, \ldots, \alpha_m \in A^{p,q}(U) \). Then

\[ (f^* \omega)|_{f^{-1}(U)} = \sum_{i=1}^m f_U^* e_i \otimes f_U^* \alpha_i. \]  

(3.3)

We have

**Lemma 3.2.** \( \text{supp} f^* \omega \subseteq f^{-1}(\text{supp} \omega) \).

**Proof.** For any \( y \in Y - f^{-1}(\text{supp} \omega) \), \( f(y) \in X - \text{supp} \omega \). There exists an \( E \)-free open neighborhood \( V \) of \( f(y) \) such that \( V \subseteq X - \text{supp} \omega \), and then \( \omega|_V = 0 \). By the local representation (3.3) of \( f^* \omega, (f^* \omega)|_{f^{-1}(V)} = 0 \), i.e., \( \text{supp} f^* \omega \cap f^{-1}(V) = \emptyset \). So \( y \) is not in \( \text{supp} f^* \omega \). We proved the lemma.

By Lemma 3.2, the pullback (3.2) gives

\[ f^* : \Gamma_{\Phi}(X, E \otimes A_X^{p,q}) \to \Gamma_{\Phi}(Y, f^* E \otimes A_Y^{p,q}) \]  

(3.4)

for a paracompactifying family \( \Phi \) of supports on \( X \).

**Pushforwards.** Assume that \( S \) is an \( f^* E \)-valued \((p, q)\)-current on \( Y \) satisfying that \( f|_{\text{supp} S} : \text{supp} S \to X \) is proper. Let \( \mathcal{U} \) be an \( E \)-free covering of \( X \) and \( e_1^U, \ldots, e_m^U \) a basis of \( \Gamma(U, E) \) as an \( \mathcal{O}_X(U) \)-module for any \( U \in \mathcal{U} \). For \( U \in \mathcal{U} \), \( S \) can be written as \( \sum_{i=1}^m f_U^* e_i^U \otimes T_i \) on \( f^{-1}(U) \), where \( T_1, \ldots, T_m \in D^{p,q}(f^{-1}(U)) \). By Lemma 3.1, \( \text{supp} T_i \subseteq \text{supp} S \cap f^{-1}(U) \), and then \( f_U|_{\text{supp} T_i} : \text{supp} T_i \to U \) is proper. So \( f_{U*} T_i \) is well defined. Define an \( E \)-valued \((p - r, q - r)\)-current

\[ \tilde{S}_U = \sum_{i=1}^m e_i^U \otimes f_{U*} T_i \]  

(3.5)
on $U$. For $U, U' \in \mathcal{U}$ satisfying $U \cap U' \neq \emptyset$, $\tilde{S}_U = \tilde{S}_{U'}$ on $U \cap U'$. We obtain an $\mathcal{E}$-valued $(p - r, q - r)$-current on $X$, denoted by $f_*S$, such that $(f_*S)|_U = \tilde{S}_U$. The definition of $f_*S$ is independent of the choice of $\mathcal{E}$-free coverings.

**Lemma 3.3.** Suppose that $f|_{\text{supp}\mathcal{S}} : \text{supp}\mathcal{S} \to X$ is proper. Then $\text{supp}f_*\mathcal{S} \subseteq f(\text{supp}\mathcal{S})$.

**Proof.** For any $x \in X - f(\text{supp}\mathcal{S})$, there exists an $\mathcal{E}$-free open neighborhood $U$ of $x$ such that $U \subseteq X - f(\text{supp}\mathcal{S})$. Clearly, $f^{-1}(U) \cap \text{supp}\mathcal{S} = \emptyset$, and then $S|_{f^{-1}(U)} = 0$. By the definition (3.5) of $f_*S$, $(f_*S)|_U = 0$, i.e., $\text{supp}f_*\mathcal{S} \cap U = \emptyset$. Hence, $x$ is not in $\text{supp}f_*\mathcal{S}$. The lemma follows.

Let $\Phi$ be a paracompactifying family of supports on $X$. Then

$$\Phi(c) := \{K \in f^{-1}\Phi : f|_K : K \to X \text{ is proper}\}$$

is paracompactifying on $Y$ by [4, IV, 5.3 (b), 5.5]. By Lemma 3.3, we get a pushforward

$$f_* : \Gamma_{\Phi(c)}(Y, f^*\mathcal{E} \otimes \mathcal{D}^{p,q}_X) \to \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{D}^{p-r,q-r}_X).$$

(3.6)

If $f$ is proper, $\Phi(c) = f^{-1}\Phi$ and hence (3.6) is

$$f_* : \Gamma_{f^{-1}\Phi}(Y, f^*\mathcal{E} \otimes \mathcal{D}^{p,q}_V) \to \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{D}^{p-r,q-r}_X).$$

Let $j : V \to X$ be the inclusion of an open subset $V$ into $X$. Clearly, $\Phi|_V \subseteq \Phi(c)$ in such case. For $K \in \Phi(c), K \cap S$ is compact for any compact set $S \subseteq X$, so $K$ is closed in $X$. Since $K \in j^{-1}\Phi, K = L \cap V \subseteq L$ for some $L \in \Phi$, which implies $K \in \Phi|_V$. So $\Phi(c) \subseteq \Phi|_V$. Hence $\Phi(c) = \Phi|_V$. The pushforward $j_*$ is just the extension by zero of sections of the sheaf $\mathcal{E} \otimes \mathcal{D}^{p,q}_X$, i.e., (3.1).

We easily check that

$$\Gamma_{\Phi|_V}(V, \mathcal{E} \otimes \mathcal{A}^{p,q}_X) \xrightarrow{j_*} \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{A}^{p,q}_X) \xrightarrow{j^*} \Gamma_{j^{-1}\Phi}(V, \mathcal{E} \otimes \mathcal{A}^{p,q}_X),$$

$$\Gamma_{\Phi|_V}(V, \mathcal{E} \otimes \mathcal{D}^{p,q}_X) \xrightarrow{j_*} \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{D}^{p,q}_X) \xrightarrow{j^*} \Gamma_{j^{-1}\Phi}(V, \mathcal{E} \otimes \mathcal{D}^{p,q}_X)$$

(3.7)

are both inclusions and

**Proposition 3.4.** Let $f : Y \to X$ be a proper holomorphic map of complex manifolds and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$. For an open set $V \subseteq X$, denote by $f_V : f^{-1}(V) \to V$ the restriction of $f$ to $f^{-1}(V)$ and by $j : V \to X, j' : f^{-1}(V) \to Y$ the inclusions. Assume that $\Phi$ is a paracompactifying family of supports on $X$. Then $j_*f_* = f_*j_*$ on $\Gamma_{\Phi|_V}(V, \mathcal{E} \otimes \mathcal{A}^{\bullet, \bullet}_X)$ and $f_*j'^* = j'^*f_*$ on $\Gamma_{j^{-1}\Phi}(Y, f^*\mathcal{E} \otimes \mathcal{D}^{\bullet, \bullet}_Y)$.

We have the Mayer-Vietoris sequences as follows.

**Proposition 3.5.** Let $X$ be a complex manifold and $X = U \cup V$ for open sets $U, V$. Denote the corresponding inclusions by $j_1 : U \to X, j_2 : V \to X, j'_1 : U \cap V \to V, j'_2 : U \cap V \to U$ respectively. Assume that $\Phi$ is a paracompactifying family of supports on $X$. Then

$$\cdots \to \Gamma_{\Phi|_{U \cap V}}(U \cap V, \mathcal{E} \otimes \mathcal{A}^{p,q}_X) \xrightarrow{(j'_2, j'_1)} \Gamma_{\Phi|_{U \cup V}}(U \cup V, \mathcal{E} \otimes \mathcal{A}^{p,q}_X) \xrightarrow{j_2 - j_1} \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{A}^{p,q}_X) \to \cdots,$$

$$\cdots \to \Gamma_{\Phi|_{U \cap V}}(U \cap V, \mathcal{E} \otimes \mathcal{D}^{p,q}_X) \xrightarrow{(j'_2, j'_1)} \Gamma_{\Phi|_{U \cup V}}(U \cup V, \mathcal{E} \otimes \mathcal{D}^{p,q}_X) \xrightarrow{j_2 - j_1} \Gamma_{\phi}(X, \mathcal{E} \otimes \mathcal{D}^{p,q}_X) \to \cdots$$

are exact sequences for any $p, q$. 
Proof. The proofs of the two conclusions are similarly and we only give the proof of the first one. Clearly, \((j'_{2*}, j'_{1*})\) is injective and \((j_{1*} - j_{2*}) \circ (j'_{2*}, j'_{1*}) = 0\). Suppose that \(\alpha_1 \in \Gamma_{\Phi \mid U} (V, E \otimes \mathcal{A}^{p,q}_X)\) and \(\alpha_2 \in \Gamma_{\Phi \mid U} (U, E \otimes \mathcal{A}^{p,q}_X)\) satisfy \(j_{1*}\alpha_2 - j_{2*}\alpha_1 = 0\). Then supp\(\alpha_2 = \text{supp} \alpha_2 \subseteq U \cap Y\) is in \(\Phi\). Set \(\alpha = \alpha_1 |_{U \cap Y}\). Then supp\(\alpha \in \Phi |_{U \cap Y}\). Moreover, \(j'_{1*}\alpha = \alpha_i\) for \(i = 1, 2\). Hence ker\((j_{1*} - j_{2*})\) \(\subseteq \text{Im}(j'_{2*} - j'_{1*})\). Let \(\{\rho_U, \rho_V\}\) be a partition of unity subordinate to \(\{U, V\}\). For any \(\beta \in \Gamma_{\Phi} (X, E \otimes \mathcal{A}^{p,q}_X)\), supp\((\rho_U \cdot \beta)\) \(\subseteq \text{supp} \beta \subseteq \Phi\), so supp\((\rho_U \cdot \beta)\) \(\in \Phi |_{U \cap Y}\). Moreover, \(j_{1*}((\rho_U \cdot \beta)|_U) = \rho_U \cdot \beta\). Similarly, supp\((\rho_V \cdot \beta)\) \(\in \Phi |_{V}\) and \(j_{2*}((\rho_V \cdot \beta)|_V) = \rho_V \cdot \beta\). Then \(\beta = (j_{1*}, j_{2*})((\rho_U \cdot \beta)|_U, -(\rho_V \cdot \beta)|_V)\). Hence \(j_{1*} - j_{2*}\) is surjective. We finish the proof. \(\square\)

3.1.4. Twisted Dolbeault cohomology. We still denote by \(\tilde{\partial}\) the differentials \(1 \otimes \tilde{\partial} : \mathcal{E} \otimes \mathcal{A}_X^{p,*} \rightarrow \mathcal{E} \otimes \mathcal{A}_X^{p,*+1}\) and \(1 \otimes \tilde{\partial} : \mathcal{E} \otimes \mathcal{D}_X^{p,*} \rightarrow \mathcal{E} \otimes \mathcal{D}_X^{p,*+1}\). Let \(\Phi\) be a p-paracompactifying family of supports on \(X\). Then \(\mathcal{E} \otimes \mathcal{O}_X^p\) has two \(\Phi\)-soft resolutions

\[
0 \rightarrow \mathcal{E} \otimes \Omega_X^p \rightarrow \cdots \rightarrow \mathcal{E} \otimes \mathcal{A}_X^p \rightarrow \mathcal{E} \otimes \mathcal{A}_X^{p,n} \rightarrow 0,
\]

\[
0 \rightarrow \mathcal{E} \otimes \Omega_X^p \rightarrow \cdots \rightarrow \mathcal{E} \otimes \mathcal{D}_X^p \rightarrow \mathcal{E} \otimes \mathcal{D}_X^{p,n} \rightarrow 0,
\]

where \(\dim_{\mathbb{C}} X = n\). So

\[H^q_{\Phi}(X, \mathcal{E} \otimes \mathcal{O}_X^p) \cong H^q(\Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{A}_X^{p,*})) \cong H^q(\Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{D}_X^{p,*})).\]

The later two are uniformly called the twisted Dolbeault cohomology with supports in \(\Phi\) and denoted by \(H^p_{\Phi}(X, \mathcal{E})\). Assume that \(E\) is the holomorphic vector bundle associated to \(\mathcal{E}\). Then \(H^p_{\Phi}(X, \mathcal{E})\) coincides with the bundle-valued Dolbeault cohomology \(H^p_{\Phi}(X, \mathcal{E})\) (see [9, V. Proposition 11.5]). Clearly, all operators defined in Sects 3.1.1-3.1.3 commute with \(\tilde{\partial}\), hence induce the corresponding morphisms at the level of cohomology.

For a complex manifold \(X\), denote by \(\text{Mod}(\mathcal{O}_X)\) the category of sheaves of \(\mathcal{O}_X\)-modules on \(X\).

Proposition 3.6. Let \(f : Y \rightarrow X\) be a flat holomorphic map (e.g., holomorphic submersion, open embedding) of complex manifolds, i.e., \(\mathcal{O}_Y\) is a flat \(f^{-1}\mathcal{O}_X\)-module sheaf. Assume that \(\mathcal{E}\) is a locally free sheaf of \(\mathcal{O}_X\)-modules of finite rank on \(X\) and \(\Phi\) is a paracompactifying family of supports on \(X\). Then the pullback \(f^* : H^q_{\Phi}(X, \mathcal{E}) \rightarrow H^q_{\Phi}(Y, f^* \mathcal{E})\) defined via (3.4) is compatible with the second type of pullback defined via (2.4).

Proof. Let \(\mathcal{I}^*\) and \(\mathcal{J}^*\) be injective resolutions of \(\mathcal{E} \otimes \mathcal{O}_X^p\) and \(f^* \mathcal{E} \otimes \mathcal{O}_Y^p\) in \(\text{Mod}(\mathcal{O}_X)\) and \(\text{Mod}(\mathcal{O}_Y)\) respectively. By [17, I. Theorem 6.2], there exist quasi-isomorphisms \(\mathcal{E} \otimes \mathcal{A}_X^{p,*} \rightarrow \mathcal{I}^*\) and \(f^* \mathcal{E} \otimes \mathcal{A}_Y^{p,*} \rightarrow \mathcal{J}^*\), which are unique up to chain homotopy. Any injective sheaf is flabby ([4, II. Proposition 5.3]) and hence is \(\Phi\)-acyclic ([4, II. Proposition 5.5]). We have the isomorphisms \(H^q_{\Phi}(X, \mathcal{E}) \cong H^q(\Gamma_{\Phi}(Y, \mathcal{I}^*))\) and \(H^p_{\Phi}(Y, f^* \mathcal{E}) \cong H^q(\Gamma_{f^{-1}\Phi}(Y, \mathcal{J}^*))\) by [4, II. 4.2]. Since \(f\) is flat, \(f^* : \text{Mod}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_Y)\) is an exact functor, so \(f^*(\mathcal{E} \otimes \mathcal{A}_X^{p,*}) \rightarrow f^* \mathcal{I}^*\) is a quasi-isomorphism of complexes of sheaves of \(\mathcal{O}_Y\)-modules. By [17, I. Theorem 6.2], there is a unique morphism \(\varphi : f^* \mathcal{I}^* \rightarrow \mathcal{J}^*\) of complexes of sheaves of \(\mathcal{O}_Y\)-modules up to chain
homotopy such that the diagram
\[
\begin{array}{ccc}
\Phi & \xrightarrow{f^*} & \mathcal{A}_Y^p \\
\downarrow \quad & \quad \downarrow \varphi & \quad \downarrow \quad \\
\mathcal{T} & \quad \quad \quad \quad & \mathcal{J}.
\end{array}
\]
is commutative up to chain homotopy, where the upper map is induced by the pullback \(f^* \mathcal{A}_X^p \to \mathcal{A}_Y^p\). On cohomologies, we have the commutative diagram
\[
\begin{array}{ccc}
H^p_{\Phi}(X, \mathcal{E}) & \xrightarrow{l} & H^p_{f^{-1}\Phi}(Y, f^* \mathcal{E}) \\
\downarrow \quad & \quad \downarrow \quad & \quad \downarrow \quad \\
H^q(\Gamma_f(Y, \mathcal{T})) & \xrightarrow{\delta} & H^q(\Gamma_{f^{-1}\Phi}(Y, f^* \mathcal{J})).
\end{array}
\]
where the maps in the two rows of the left square are induced by adjunctions \(\mathcal{E} \otimes \mathcal{A}_X^p \to f^* \mathcal{E} \otimes \mathcal{A}_Y^p\) and \(\mathcal{T} \to f^* \mathcal{J}\) respectively. The composition of the two maps in the lower row of (3.8) is induced by \(\mathcal{J} \to f_* f^* \mathcal{J} \to f_* \mathcal{J}\), denoted by
\[
\delta : H^q(\Gamma_{\Phi}(Y, \mathcal{T})) \to H^q(\Gamma_{f^{-1}\Phi}(Y, \mathcal{J})).
\]
Notice that \(\delta\) is independent of the choice of \(\varphi\), since \(H^q(\Gamma_{f^{-1}\Phi}(\varphi))\) is. Clearly, the composition of the two maps in the upper row of (3.8) is just the pullback \(f^*\) defined by (3.4). Hence (3.4) is compatible with \(\delta\). By similar arguments, we can prove that the pullback \(f^*\) defined by (2.4) is compatible with \(\delta\) using \((\mathcal{E} \otimes \Omega_X^p)\) and \((f^* \mathcal{E} \otimes \Omega_Y^p)\) instead of \(\mathcal{E} \otimes \mathcal{A}_X^p\) and \(f^* \mathcal{E} \otimes \mathcal{A}_Y^p\) respectively. We complete the proof. 

Suppose that \(\Phi\) and \(\Psi\) are paracompactifying families of supports on \(X\) with \(\Phi \subseteq \Psi\). The inclusion \(\Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{A}_X^p) \hookrightarrow \Gamma_{\Psi}(X, \mathcal{E} \otimes \mathcal{A}_X^p)\) naturally induces a morphism \(l : H^q(\Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{A}_X^p)) \to H^q(\Gamma_{\Psi}(X, \mathcal{E} \otimes \mathcal{A}_X^p))\). Let \(f : Y \to X\) be a holomorphic map of complex manifolds. If \(f^{-1}\Phi = f^{-1}\Psi\), there is a commutative diagram
\[
\begin{array}{ccc}
H^p_{\Phi}(X, \mathcal{E}) & \xrightarrow{l} & H^p_{f^{-1}\Phi}(Y, f^* \mathcal{E}) \\
\downarrow \quad & \quad \downarrow \quad & \quad \downarrow \quad \\
H^p_{f^{-1}\Phi}(Y, f^* \mathcal{E}) & \xrightarrow{\phi} & H^p_{f^{-1}\Phi}(Y, f^* \mathcal{E}).
\end{array}
\]

3.1.5. **Cup products.** Assume that \(\mathcal{E}, \mathcal{F}\) are locally free sheaves of \(\mathcal{O}_X\)-modules of rank \(m, n\) on \(X\) respectively, \(\mathcal{U}\) is an \(\mathcal{E}\)- and \(\mathcal{F}\)-free open covering of \(X\) and \(\Phi, \Psi\) is a paracompactifying family of supports on \(X\). Let \(e_1^U, \ldots, e_m^U\) and \(f_1^U, \ldots, f_n^U\) be bases of \(\Gamma(U, \mathcal{E})\) and \(\Gamma(U, \mathcal{F})\) as \(\mathcal{O}_X(U)\)-modules, respectively.

For \(S \in \Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{D}_X^{p,q})\), \(\omega \in \Gamma_{\Psi}(X, \mathcal{F} \otimes \mathcal{A}_X^{p,q})\) and \(U \in \mathcal{U}\), \(S\) and \(\omega\) are represented by \(\sum_{i=1}^m e_i^U \otimes T_i\) and \(\sum_{j=1}^n f_j^U \otimes \alpha_j\) on \(U\) respectively, where \(T_i \in \mathcal{D}_X^{p,q}(U)\) and \(\alpha_j \in \mathcal{A}_X^{p,q}(U)\) for any \(i\). Then
\[
\sum_{1 \leq i \leq m} e_i^U \otimes \sum_{1 \leq j \leq n} f_j^U \otimes (T_i \wedge \alpha_j)
\]
gives an $\mathcal{E} \otimes \mathcal{F}$-valued $(r + s, p + q)$-current on $U$. They are glued as a global section of $\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{D}_X^{r+s,p+q}$ on $X$, which is independent of the choice of open coverings. Denote it by $S \wedge \omega$. We easily check that $\text{supp}(S \wedge \omega) \subseteq \text{supp} S \cap \text{supp} \omega$, hence $S \wedge \omega \in \Gamma_{\Phi \cap \Psi}(X, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{D}_X^{r+s,p+q})$. Similarly, the $\mathcal{F} \otimes \mathcal{E}$-valued current $\omega \wedge S$ and the $\mathcal{E} \otimes \mathcal{F}$-valued form $\psi \wedge \omega$ on $X$ can be defined well, where $\psi \in \Gamma_{\Phi}(X, \mathcal{E} \otimes \mathcal{A}_X^p)$. Clearly, $\partial(S \wedge \omega) = \partial S \wedge \omega + (-1)^{p+q}S \wedge \partial \omega$.

Denote by $[\alpha]_\Phi$ the Dolbeault class with support in $\Phi$ of the $\partial$-closed form or current $\alpha$. For paracompactifying families $\Phi$, $\Psi$ of supports on $X$, define a cup product

$$
\cup : H^p_{\Phi, \mathbb{C}}(X, \mathcal{E}) \times H^q_{\Psi, \mathbb{C}}(X, \mathcal{F}) \rightarrow H^{p+q}_{\Phi \cap \Psi, \mathbb{C}}(X, \mathcal{E} \otimes \mathcal{F})
$$

as $[\psi]_\Phi \cup [\omega]_\Psi = [\psi \wedge \omega]_{\Phi \cap \Psi}$ or $[S]_\Phi \cup [\omega]_\Psi = [S \wedge \omega]_{\Phi \cap \Psi}$.

**Remark 3.7.** There is a commutative diagram

$$
\begin{array}{ccc}
(E \otimes \Omega^p_X) \otimes (F \otimes \Omega^q_X) & \rightarrow & \wedge \mathcal{E} \otimes \mathcal{F} \otimes \Omega^{p+q}_X \\
(1\otimes i) \otimes (1\otimes i) & & (1\otimes i) \otimes (1\otimes i) \\
(E \otimes \mathcal{A}_X^p) \otimes (F \otimes \mathcal{A}_X^q) & \rightarrow & \wedge \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{A}_X^{p+q}.
\end{array}
$$

By [13, II. Théorème 6.6.1], the cup product (3.10) coincides with the second type of cup product (2.3).

Suppose that $f : Y \rightarrow X$ is a holomorphic map of complex manifolds. Then $f^*(\psi \wedge \omega) = f^*\psi \wedge f^*\omega$. In addition, let $T$ be an $f^*\mathcal{E}$-valued current on $Y$ satisfying that $f|_{\text{supp} T} : \text{supp} T \rightarrow X$ is proper. Then

$$
f_*(T \wedge f^*\omega) = f_*T \wedge \omega.
$$

Indeed, it is the classical projection formula locally. For $\varphi \in H^{\bullet,\bullet}_{\Phi(c)}(Y, f^*\mathcal{E})$ and $\eta \in H^{\bullet,\bullet}_{\Psi}(X, \mathcal{F})$, $f_*(\varphi \cup f^*\eta) \in H^{\bullet,\bullet}_{\Phi(c) \cap f^{-1}\Psi}(X, \mathcal{E} \otimes \mathcal{F})$ and $f_*\varphi \cup \eta \in H^{\bullet,\bullet}_{\Phi \cap \Psi}(X, \mathcal{E} \otimes \mathcal{F})$. By [4, IV. 5.4 (7)], $\Phi(c) \cap f^{-1}\Psi = (\Phi \cap \Psi)(c)$. By (3.11),

$$
f_*(\varphi \cup f^*\eta) = f_*\varphi \cup \eta.
$$

**Proposition 3.8.** Let $f : Y \rightarrow X$ be a proper surjective holomorphic map between complex manifolds and let $\Phi$ be a paracompactifying family of supports on $X$. Set $r = \dim_Y Y - \dim_X X$ and assume that there exists a closed current $T \in \mathcal{D}_r^{r,r}(Y)$ such that $f_*T \neq 0$. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$. Then $f^* : H^{\bullet,\bullet}_{\Phi}(X, \mathcal{E}) \rightarrow H^{\bullet,\bullet}_{f^{-1}\Phi}(Y, f^*\mathcal{E})$ is injective and $f_* : H^{\bullet,\bullet}_{f^{-1}\Phi}(Y, f^*\mathcal{E}) \rightarrow H^{\bullet,\bullet}_{\Phi\cap\Psi}(X, \mathcal{E})$ is surjective. In particular, if $X$ and $Y$ have the same dimensions, then $f^* : H^{\bullet,\bullet}_{\Phi}(X, \mathcal{E}) \rightarrow H^{\bullet,\bullet}_{f^{-1}\Phi}(Y, f^*\mathcal{E})$ is injective and $f_* : H^{\bullet,\bullet}_{f^{-1}\Phi}(Y, f^*\mathcal{E}) \rightarrow H^{\bullet,\bullet}_{\Phi}(X, \mathcal{E})$ is surjective.

**Proof.** Since $c = f_*T$ is a closed current of degree 0, hence a constant. By (3.12), $f_*([T] \cup f^*\eta) = c \cdot \eta$, where $[T] \in H^{r,r}(Y)$ and $\eta \in H^{p,q}_{\Phi}(X, \mathcal{E})$. The proposition follows.

### 3.2. Local systems on smooth manifolds

For a topology space $X$, a **local system of $k$-modules** on $X$ refers to a locally constant sheaf of $k$-modules on $X$, or equivalently, a locally free sheaf of $k_X$-modules on $X$, where $k_X$ is the constant sheaf with stalk $k$ on $X$. Assume that $\mathcal{L}$ is a local system of $k$-modules on $X$. An open subset $U$ of $X$ is said to be $\mathcal{L}$-**constant**,
if the restriction $L|_{U}$ is a constant sheaf. An open covering $\mathcal{U}$ of $X$ is said to be $\mathcal{L}$-constant, if all $U \in \mathcal{U}$ are $\mathcal{L}$-constant.

For a smooth map $f : Y \to X$ of smooth manifolds and local systems $\mathcal{L}, \mathcal{H}$ of $k$-modules of finite ranks on $X$, all notions can be similarly defined as those in Section 3.1 and the corresponding results are also true, where we only need to replace $O_X, \mathcal{E}, \mathcal{L}$-free, $A^*_X, D^*_X, f^*\mathcal{E}, H^*_L(X, \mathcal{E})$ and $\mathcal{E} \otimes O_X \mathcal{F}$ with $k_X, \mathcal{L}, \mathcal{L}$-constant, $A^*_X, D^*_X, f^{-1}\mathcal{L}, H^*_L(X, \mathcal{L})$ and $\mathcal{L} \otimes k_X \mathcal{H}$, respectively. It is noteworthy that, if the definitions of notions involve the currents, then the related manifolds must be oriented.

Now, we list partial results as follows, which will be frequently used in Sect. 5.

1. Let $\mathcal{L}$ and $\mathcal{H}$ be local systems of $k$-modules of finite rank on $X$. Suppose that $f : Y \to X$ is a smooth map of oriented smooth manifolds. In addition, let $T$ be an $f^{-1}\mathcal{L}$-valued current on $Y$ satisfying that $f|_{\text{supp}T} : \text{supp}T \to X$ is proper. Then

$$
f_* (T \wedge f^* \omega) = f_* T \wedge \omega. \quad (3.13)
$$

For $\varphi \in H^*_\Psi(Y, f^{-1}\mathcal{L})$ and $\eta \in H^*_\Phi(X, \mathcal{H})$,

$$
f_* (\varphi \cup f^* \eta) = f_* \varphi \cup \eta. \quad (3.14)
$$

2. Suppose that $\Phi$ and $\Psi$ are paracompactifying families of supports on $X$ with $\Phi \subseteq \Psi$. The inclusion $\Gamma_\Phi(X, \mathcal{L} \otimes A^*_X) \hookrightarrow \Gamma_\Psi(X, \mathcal{L} \otimes A^*_X)$ naturally induces a morphism $l : H^p_\Phi(X, \mathcal{L}) \to H^p_\Psi(X, \mathcal{L})$. Let $f : Y \to X$ be a smooth map of smooth manifolds. If $f^{-1}_* \Phi = f^{-1}_* \Psi$, there is a commutative diagram

$$
\begin{array}{ccc}
H^p_{f^{-1}_* \Phi}(Y, f^{-1}\mathcal{L}) & \xrightarrow{l} & H^p_\Phi(X, \mathcal{L}) \\
\downarrow f^* & & \downarrow f_* \mathcal{L} \\
H^p_{f^{-1}_* \Psi}(Y, f^{-1}\mathcal{L}) & \xleftarrow{f^*} & H^p_\Psi(X, \mathcal{L})
\end{array}
\quad (3.15)
$$

4. Twisted Dolbeault cohomology with supports

4.1. Künneth theorems. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are coherent analytic sheaves on complex manifolds $X$ and $Y$ respectively. The analytic external tensor product of $\mathcal{F}$ and $\mathcal{G}$ is defined as

$$
\mathcal{F} \boxtimes \mathcal{G} = pr^*_1 \mathcal{F} \otimes_{O_{X \times Y}} pr^*_2 \mathcal{G},
$$

where $pr_1$ and $pr_2$ are projections from $X \times Y$ onto $X, Y$, respectively. The cartesian product

$$
\times : H^p_\Psi(X, \mathcal{F}) \times H^q_\Phi(Y, \mathcal{G}) \to H^{p+q}_{\Phi \times \Psi}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \quad (4.1)
$$

is defined as $pr^*_1(\bullet) \cup pr^*_2(\bullet)$.

Denote by $K^\bullet$ the associated simple complex of a double complex $K^{\bullet, \bullet}$. For a double complex $K^{\bullet, \bullet}$, there are two spectral sequences $K E^{p, q}_r \Rightarrow K H^p$ and $K E^{q, p}_r \Rightarrow K H^q$, where $K E^{p, q}_1 = H^q(K^{p, \bullet}), K E^{p, q}_2 = H^p(E^{q, \bullet}), K E^{p, q}_1 = H^q(E^{p, \bullet}), K E^{p, q}_2 = H^p(E^{q, \bullet})$, $K H^k = K \bar{H}^k = H^k(K^\bullet)$.

**Proposition 4.1.** Let $\mathcal{F}, \mathcal{G}$ be coherent analytic sheaves on complex manifolds $X, Y$ respectively and let $\Phi$ be a family of supports on $X$. Suppose that $H^\bullet(Y, \mathcal{G})$ is finite dimensional.
Then \((\alpha^p \otimes \beta^q)_{p+q=k} \mapsto \sum_{p+q=k} \alpha^p \times \beta^q\) gives an isomorphism
\[
\bigoplus_{p+q=k} H^p_\Phi(X, \mathcal{F}) \otimes_C H^q(Y, \mathcal{G}) \cong H^k_{\Phi \times Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}).
\]

**Proof.** Let \(C^{\bullet \bullet} = \Gamma_\Phi(X, \mathcal{F}^{\bullet \bullet}) \otimes_C \Gamma(Y, \mathcal{G}^{\bullet \bullet})\) be the double complex associated to the complexes \(\Gamma_\Phi(X, \mathcal{F}^{\bullet \bullet})\) and \(\Gamma(Y, \mathcal{G}^{\bullet \bullet})\). By [9, IV, (15.8)], \(C^{1,0}_{E_1, q} = \Gamma_\Phi(X, \mathcal{F}^{[p]} \otimes_C H^q(Y, \mathcal{G}))\) and \(C^{0,1}_{E_2, q} = H^p_\Phi(X, \mathcal{F}) \otimes_C H^q(Y, \mathcal{G})\). Let \(\pi \colon X \times Y \to X\) be the first projection of \(X \times Y\) onto \(X\). Set \(D^{p,q} = \Gamma_\Phi(X, (\pi_*((\mathcal{F} \boxtimes \mathcal{G})^{[q]}))^{[p]})\). Then
\[
D^{p,q} = H^q\left(\Gamma_\Phi(X, (\pi_*((\mathcal{F} \boxtimes \mathcal{G})^{[q]}))^{[p]})\right)
\]
\[
= \Gamma_\Phi\left(X, \left(H^q\left((\pi_*((\mathcal{F} \boxtimes \mathcal{G})^{[q]}))\right)\right)^{[p]}\right) \quad \text{(by (2.1))}
\]
\[
= \Gamma_\Phi\left(X, (R^p\pi_*((\mathcal{F} \boxtimes \mathcal{G}))^{[p]})\right)
\]
and \(D^{p,q} = H^p_\Phi(X, R^q\pi_*((\mathcal{F} \boxtimes \mathcal{G})))\). Since \(H^\bullet(Y, \mathcal{G})\) is finite dimensional, \(R^q\pi_*((\mathcal{F} \boxtimes \mathcal{G})) = \mathcal{F} \boxtimes C H^q(Y, \mathcal{G})\) by [9, IX, 5.22 (c)]. So \(D^{p,q} = H^p_\Phi(X, \mathcal{F} \boxtimes C H^q(Y, \mathcal{G})) = H^p_\Phi(X, \mathcal{F} \otimes C H^q(Y, \mathcal{G})).\)

Define a morphism \(\varphi : C^{\bullet \bullet} \to D^{\bullet \bullet}\) of double complexes as \(f \otimes g \mapsto h\), where
\[
h(\xi_0, \ldots, \xi_p; (x_0, y_0), \ldots, (x_q, y_q)) = S(f(\xi_0, \ldots, \xi_p))(x_q) \otimes 1 \otimes g(y_0, \ldots, y_q) \otimes 1
\]
\[
e(\mathcal{F} \boxtimes \mathcal{G})_{(x_q, y_q)} = (\mathcal{F}_{x_q} \otimes \mathcal{O}_{x \times Y, (x_q, y_q)}) \otimes \mathcal{O}_{x \times Y, (x_q, y_q)}
\]
for any \((\xi_0, \ldots, \xi_p; (x_0, y_0), \ldots, (x_q, y_q)) \in X^{p+1} \times (X \times Y)^{q+1}\). By Lemma 2.4, the map is defined well. It induces the isomorphism \(C^{p,q} \to D^{p,q}\) for any \(p, q\), hence induces the isomorphism \(C^{p,q} \to D^{p,q}\) for \(r \geq 2\). Clearly, \(C^{p,q} \to D^{p,q}\) degenerates at \(2\)-page, so does \(D^{p,q}\). So \(\varphi\) induces the isomorphism \(\varphi : CH^k \to DH^k\). By [9, IV, (15.5)], \(CH^k = H^k_\Phi(X, \mathcal{F} \otimes C H^q(Y, \mathcal{G})).\) Moreover,
\[
\tilde{D}^{p,q} = \begin{cases} 
\Gamma_{\Phi \times Y} (X \times Y, (\mathcal{F} \boxtimes \mathcal{G})^{[q]}), & p = 0, \\
0, & p \geq 1,
\end{cases}
\]
\[
\tilde{D}^{p,q} = \begin{cases} 
H^p_{\Phi \times Y} (X \times Y, \mathcal{F} \boxtimes \mathcal{G}), & p = 0, \\
0, & p \geq 1,
\end{cases}
\]
where we use [4, IV, 5.2]. Hence \(\tilde{D}^{p,q}\) degenerates at \(2\)-page, so \(DH^k = H^k_{\Phi \times Y} (X \times Y, \mathcal{F} \boxtimes \mathcal{G}).\)

From the definition of \(\varphi\), \(H^\bullet(\varphi)\) is just the cartesian product. We complete the proof.

For bigraded vector spaces \(K^{\bullet \bullet}\) and \(L^{\bullet \bullet}\) over \(C\), the associated bigraded space \(K^{\bullet \bullet} \otimes_C L^{\bullet \bullet}\) over \(C\) is defined as
\[
(K^{\bullet \bullet} \otimes_C L^{\bullet \bullet})^{p,q} = \bigoplus_{k+l=p, r+s=q} K^{k,r} \otimes_C L^{l,s}
\]
for any \(p, q\).

Let \(X\) and \(Y\) be two complex manifolds. Suppose that \(H^{\bullet \bullet}(Y)\) is finite dimensional and \(\Phi\) is a paracompact family of supports on \(X\). Notice that \(\Omega^p_{X \times Y} = \bigoplus_{r+s=p} \Omega^r_X \boxtimes \Omega^s_Y\). By
Proposition 3.6 and Remark 3.7,
\[ pr_1^*(\bullet) \cup pr_2^*(\bullet) : H^\bullet_{\Phi}(X) \otimes_C H^\bullet(Y) \rightarrow H^\bullet_{\Phi \times Y}(X \times Y) \] (4.2)
coincides with the cartesian product (4.1), which is an isomorphism by Proposition 4.1.

4.2. Leray-Hirsch theorem. Using Borel’s spectral sequence, L. Cordero et al. [7] established a version of Leray-Hirsch theorem for Dolbeault cohomology and S. Rao et al. [29] extend this result to the twisted cases with the similar way. We obtained a version of these results with a different way [21, 23]. Now, we further generalize the Leray-Hirsch theorems on the twisted Dolbeault cohomologies with supports.

**Theorem 4.2.** Let \( \pi : E \rightarrow X \) be a holomorphic fiber bundle over a complex manifold \( X \) and let \( E \) be a locally free sheaf of \( \mathcal{O}_X \)-modules of finite rank on \( X \). Assume that there exist \( e_i \in H^\bullet_{\Phi}(E) \) with degree \( (u_i, v_i) \) for \( 1 \leq i \leq r \) such that their restrictions \( e_i|_{E_x}, \ldots, e_r|_{E_x} \) freely linearly generate \( H^\bullet_{\Phi}(E_x) \) for every \( x \). Then

\[
\sum_{i=1}^r \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^r H^\bullet_{\Phi^{u_i \cdot v_i}}(X, \mathcal{E}) \rightarrow H^\bullet_{\Phi^{u_i \cdot v_i}}(E, \pi^*\mathcal{E})
\]
is an isomorphism for a paracompactifying family \( \Phi \) of supports on \( X \).

**Proof.** Let \( t_i \) be a \( \bar{\partial} \)-closed form of degree \( (u_i, v_i) \) in \( A^\bullet_{\Phi}(E) \), such that \( e_i = |t_i| \) for \( 1 \leq i \leq r \).
For any open set \( U \subseteq X \), set \( E_U = \pi^{-1}(U) \) and \( B^\bullet_{\Phi}(U) = \bigoplus_{i=1}^r A^\bullet_{\Phi^{u_i \cdot v_i}}(U) \). For any \( p \), the \( q \)-th cohomology of the complex \( B^{p \cdot \bullet}(U) \) is \( D^{p, q}(U) = \bigoplus_{i=1}^r H^{p-1-q-\Phi}_{\Phi^{u_i \cdot v_i}}(U) \). The morphism

\[
f^p_U = \sum_{i=1}^r \pi^*(\bullet) \cup t_i : B^{p \cdot \bullet}(U) \rightarrow A^\bullet_{(\pi|_{E_U})^{-1}(\Phi|_{E_U})}(E_U) = A^\bullet_{(\pi^{-1}(\Phi)|_{E_U})}(E_U)
\]
of complexes is defined well by Proposition 2.1 (3), which induces a morphism

\[
F^{p, q}_U = \sum_{i=1}^r \pi^*(\bullet) \cup e_i : D^{p, q}(U) \rightarrow G^{p, q}(U) := H^{p, q}_{(\pi^{-1}(\Phi)|_{E_U})}(E_U).
\]

Denoted by \( \mathcal{P}(U) \) the statement that \( F^{p, q}_U \) are isomorphisms for all \( p, q \). The theorem is equivalent to say that \( \mathcal{P}(X) \) holds. We only need to check the three conditions in Lemma 2.5. Clearly, \( \mathcal{P} \) satisfies the disjoint condition.

We fix some notations. For the inclusion \( j : U \subseteq V \) of open sets in \( X \), denote by \( j : E_U \subseteq E_V \) the corresponding inclusion and denote by \( j_* : A^\bullet_{\Phi^{u_i \cdot v_i}}(U) \rightarrow A^\bullet_{\Phi^{u_i \cdot v_i}}(V) \), \( j_* : A^\bullet_{(\pi^{-1}(\Phi)|_{E_U})}(E_U) \rightarrow A^\bullet_{(\pi^{-1}(\Phi)|_{E_V})}(E_V) \), \( J = (j_1, \ldots, j_r) : B^\bullet_{\Phi}(U) \rightarrow B^\bullet_{\Phi}(V) \) the extensions by zero. Now, we go back to the proof. Fix an integer \( p \). For open sets \( U, V \subseteq X \), let \( j_1 : U \cap V \rightarrow U, j_2 : U \cap V \rightarrow V, j_3 : U \rightarrow U \cup V, j_4 : V \rightarrow U \cup V \) be inclusions. By Propositions 3.5 and 3.4, there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B^{p \cdot \bullet}(U \cap V) & \rightarrow & (j_1, j_2) & B^{p \cdot \bullet}(U) \oplus B^{p \cdot \bullet}(V) & \rightarrow & (j_3, j_4) & B^{p \cdot \bullet}(U \cup V) & \rightarrow & 0 \\
0 & \rightarrow & A^\bullet_{(\pi^{-1}(\Phi)|_{E_U \cap V})}(E_U \cap V) & \rightarrow & (j_{1, 3 \cdot j_2}) & A^\bullet_{(\pi^{-1}(\Phi)|_{E_U})}(E_U) \oplus A^\bullet_{(\pi^{-1}(\Phi)|_{E_V})}(E_V) & \rightarrow & (j_{3 \cdot j_4}) & A^\bullet_{(\pi^{-1}(\Phi)|_{E_U \cup V})}(E_U \cup V) & \rightarrow & 0 \\
\end{array}
\]
of exact sequences of complexes. Therefore, we have a commutative diagram

$$
\cdots \longrightarrow D^{p,q}(U \cap V) \longrightarrow D^{p,q}(U \oplus D^{p,q}(V) \longrightarrow D^{p,q}(U \cup V) \longrightarrow D^{p,q+1}(U \cap V) \longrightarrow \cdots \\
\cdots \longrightarrow G^{p,q}(U \cap V) \longrightarrow G^{p,q}(U \oplus G^{p,q}(V) \longrightarrow G^{p,q}(U \cup V) \longrightarrow G^{p,q+1}(U \cap V) \longrightarrow \cdots 
$$

of long exact sequences. If $F^{p,q}_U$, $F^{p,q}_V$ and $F^{p,q}_{U \cap V}$ are isomorphisms for all $p, q$, then so are $F^{p,q}_{U \cup V}$ for all $p, q$ by the five-lemma. Hence $\mathcal{P}$ satisfies the Mayer-Vietoris condition.

To check the local condition, we first verify the following claim:

$(\bigcirc)$ Assume that $U$ is an $\mathcal{E}$-free open set of $X$ satisfying that $E_U$ is holomorphically trivial, then $\mathcal{P}(U)$ holds.

Without generality, assume that $\mathcal{E}|_U = \mathcal{O}_U$. Suppose that $F$ is the general fiber of $E$ and $\varphi_U : U \times F \to E_U$ is a holomorphic trivialization. Let $p_1$ and $p_2$ be projections from $U \times F$ to $U$ and $F$ respectively, which satisfy $\pi \circ \varphi_U = p_1$. Given a point $o \in U$, set $j_o : F \to U \times F$ as $f \mapsto (o, f)$. Clearly, $p_2 \circ j_o = id_F$ and $i_o := \varphi_U \circ j_o$ is the embedding $F \to E_U$ of the fiber $E_o \cong F$ over $o$ into $E_U$. Set $e'_i = (\varphi_U^*)^{-1} p_2^* i_o^* e_i \in H^{u_i,v_i}(E_U)$, $1 \leq i \leq r$. Then $i_o^* e_i = i_o^* e_i$ for any $i$. Since $i_o^* e_1$, $\ldots$, $i_o^* e_r$ is linearly independent, mapping $e_i$ to $e'_i$ for $1 \leq i \leq r$ give an isomorphism $\text{span}_C \{e_1, \ldots, e_r\} \to \text{span}_C \{e'_1, \ldots, e'_r\}$. For any $p, q$, we have a commutative diagram

$$
\begin{align*}
(H^{p,q}_{\Phi_{U \cap V}}(U) \otimes_C \text{span}_C \{e_1, \ldots, e_r\})^{p,q} & \cong (H^{p,q}_{\Phi_{U \cap V}}(U) \otimes_C \text{span}_C \{e'_1, \ldots, e'_r\})^{p,q} \\
\cong (H^{p,q}_{\Phi_{U \cap V}}(U) \otimes_C \text{span}_C \{e_1, \ldots, e_r\})^{p,q} & \cong (H^{p,q}_{\Phi_{U \cap V}}(U) \otimes C \text{span}_C \{e'_1, \ldots, e'_r\})^{p,q} \\
\end{align*}
$$

By the assumption, $i_o^* : \text{span}_C \{e_1, \ldots, e_r\} \to H^{p,q}(F)$ is an isomorphism, so is $i_o^* : \text{span}_C \{e'_1, \ldots, e'_r\} \to H^{p,q}(F)$. By (4.2), $p_{2U}^*(\bullet) \cup p_{2U}^*(\bullet)$ is isomorphic, so is $\pi^*(\bullet) \cup \bullet$ in (4.3). Mapping $(\alpha_1, \ldots, \alpha_r)$ to $\sum_{i=1}^r \alpha_i \otimes e_i$ gives a morphism

$$
\bigoplus_{i=1}^r H^{p,-u_i,v_i}_{\Phi_{U \cap V}}(U) \to (H^{p,q}_{\Phi_{U \cap V}}(U) \otimes_C \text{span}_C \{e_1, \ldots, e_r\})^{p,q},
$$

which is clearly isomorphic. Then $F^{p,q}_{U \cup V}$ is the composition of (4.4) and the two vertical maps in the first column of (4.3), hence an isomorphism. We proved $(\bigcirc)$. Let $\mathcal{U}$ be an $\mathcal{E}$-free basis of the topology of $X$ such that $E_U$ is holomorphically trivial for any $U \in \mathcal{U}$. For $U_1, \ldots, U_l \in \mathcal{U}$, $E_{U_1 \cap \ldots \cap U_l}$ is holomorphically trivial, then $(\bigcirc)$ asserts that $\mathcal{P}(\bigcap_{i=1}^l U_i)$ is an isomorphism. Hence $\mathcal{P}$ satisfies the local condition.

We complete the proof.
4.3. Projective bundle formulae. We successively define \( P_{r-1}, P_{r-2}, P_{r-3}, \ldots, P_1, P_0 \) by recursion relations

\[
P_i(T_1, \ldots, T_{r-1}) = \begin{cases} 
(-1)^{r-1-i} \sum_{k=1}^{r-1-i} T_k P_{k+i}(T_1, \ldots, T_{r-1}), & 0 \leq i < r-1 \\
(-1)^{r-1}, & i = r-1,
\end{cases}
\]

for \( 0 \leq i \leq r-1 \). For example,

\[
P_{r-1} = (-1)^{r-1}, \quad P_{r-2} = -T_1, \quad P_{r-3} = (-1)^{r-1}T_1^2 - T_2, \ldots
\]

Clearly, \( P_i(T_1, \ldots, T_{r-1}) \in \mathbb{Z}[T_1, \ldots, T_{r-1}] \). By (4.5), it is easy to prove the following two lemmata by the induction. We will only give the details of the proof of Lemma 4.4.

**Lemma 4.3.** Suppose that

\[
P_i(T_1, \ldots, T_{r-1}) = \sum_{d_1, \ldots, d_{r-1} \geq 0} \sum_{k=1}^{r-1} a_{i,d_1,\ldots,d_{r-1}} T_1^{d_1} \cdots T_{r-1}^{d_{r-1}}.
\]

For any nonzero \( a_{i,d_1,\ldots,d_{r-1}} \), \( \sum_{k=1}^{r-1} kd_k = r - 1 - i \).

**Lemma 4.4.** Set \( T_0 = (-1)^{r-1} \). For \( k \in \{0, 1, \ldots, r-1\} \), put

\[
H_k(T_1, \ldots, T_{r-1}) = \sum_{i=-r-1}^{-r+1} T_{i-(r-1)} P_{i-k}(T_1, \ldots, T_{r-1}).
\]

Then

\[
H_k(T_1, \ldots, T_{r-1}) = \begin{cases} 
1, & k = 0 \\
0, & 1 \leq k \leq r - 1,
\end{cases}
\]

**Proof.** For \( k = 0 \), the lemma holds clearly. For \( l > 0 \), assume that the lemma holds for any \( k < l \). Then

\[
H_l = (-1)^{r-1}T_l + (-1)^r \sum_{i=-r-1}^{-r+1} T_{i-(r-1)} \left( \sum_{s=1}^{r-1} T_s P_{s+i-l} \right) \quad \text{(by (4.5))}
\]

\[
= (-1)^{r-1}T_l + (-1)^{r} \sum_{s=1}^{l} T_s \left( \sum_{i=-r-1}^{r-1} T_{i-(r-1)} P_{i-(l-s)} \right) \quad \text{(exchange sums)}
\]

\[
= (-1)^{r-1}T_l + (-1)^{r} \sum_{s=1}^{l} T_s H_{l-s} \quad \text{(by the definition of } H_k) 
\]

\[
= 0 \quad \text{(by the inductive assumption)}.
\]

We complete the proof. \( \square \)

Suppose that \( \pi : \mathbb{P}(E) \rightarrow X \) is the projective vector bundle associated to a holomorphic bundle \( E \) of rank \( r \) over a complex manifold \( X \). \( E \) is a locally free sheaf of \( \mathcal{O}_X \)-modules of finite rank and \( \Phi \) is a paracompactifying family of supports. Let \( t \in A^{1,1}(\mathbb{P}(E)) \) be a Chern
form of the universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$. Notice that $dt = 0$, i.e., $\partial t = \bar{\partial} t = 0$.

The morphism $\sum_{i=0}^{r-1} \pi^*(\cdot) \wedge t^i$ defines

$$
\bigoplus_{i=0}^{r-1} \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right) \to \Gamma_{\pi-1}(\mathbb{P}(E), \pi^*\mathcal{E} \otimes \mathcal{A}_{\pi}(\mathbb{P}(E))) .
$$

(4.6)

By Lemma 4.3,

$$
G_{t,\Phi}^{-i}(\cdot) = \sum_{j=0}^{r-1-i} P_{i+j}(\pi_*t^r, \ldots, \pi_*t^{2r-2}) \wedge \pi_*(t^j \wedge \cdot)
$$

defines $\Gamma_{\pi-1}(\mathbb{P}(E), \pi^*\mathcal{E} \otimes \mathcal{A}_{\pi}(\mathbb{P}(E))) \to \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right)$ for $0 \leq i \leq r-1$. The morphism

$(G_{t,\Phi}^{-i}(\cdot), G_{t,\Phi}^{-1}(\cdot), \ldots, G_{t,\Phi}^{-r+1}(\cdot))$

defines

$$
\Gamma_{\pi-1}(\mathbb{P}(E), \pi^*\mathcal{E} \otimes \mathcal{A}_{\pi}(\mathbb{P}(E))) \to \bigoplus_{i=0}^{r-1} \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right) .
$$

(4.7)

Denote (4.6) and (4.7) by $\mu^\Phi_{t}$ and $\tau^\Phi_{t}$ respectively. Now we prove

$$
\tau^\Phi_{t} \circ \mu^\Phi_{t} = id.
$$

(4.8)

Clearly, $\pi_*t^i = 0$, for $0 \leq i \leq r - 2$. Moreover, $\pi_*t^{r-1} = (-1)^{r-1}$. Actually, $\pi_*t^{r-1}$ is a $d$-closed smooth 0-form on $X$, hence a constant. For any $x \in X$,

$$
\pi_*t^{r-1} = \int_{\mathbb{P}(E_x)} t^{r-1} = \int_{\mathbb{P}^{r-1}} c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(-1))^{r-1} = (-1)^{r-1}.
$$

For $\alpha^{p-i,q-i} \in \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right)$, $0 \leq i \leq r - 1$, we have

$$
\pi_*(t^j \wedge \mu^\Phi_{t}(\alpha^{p,q}, \alpha^{p-1,q-1}, \ldots, \alpha^{p-r+1,q-r+1})) = \sum_{l=r-1-j}^{r-1} \pi_*(t^{j+l} \wedge \alpha^{p-l,q-l})
$$

(4.9)

by (3.11). Then

$$
G_{t,\Phi}^{-i} \circ \mu^\Phi_{t}(\alpha^{p,q}, \alpha^{p-1,q-1}, \ldots, \alpha^{p-r+1,q-r+1})
$$

$$
= \sum_{j=0}^{r-1-i} \sum_{l=r-1-j}^{r-1} P_{i+j}(\pi_*t^r, \ldots, \pi_*t^{2r-2}) \wedge \pi_*(t^{j+l} \wedge \alpha^{p-l,q-l}) \quad (\text{by (4.9)})
$$

$$
= \sum_{l=i}^{r-1} \left[ \sum_{j=r-1-l}^{r-1-i} P_{i+j}(\pi_*t^r, \ldots, \pi_*t^{2r-2}) \wedge \pi_*(h^{j+l} \wedge \alpha^{p-l,q-l}) \right] \quad (\text{exchange sums})
$$

$$
= \sum_{l=i}^{r-1} H_{l-i}(\pi_*t^r, \ldots, \pi_*t^{2r-2}) \wedge \alpha^{p-l,q-l} \quad (\text{by the definition of } H_k)
$$

$$
= \alpha^{p-i,q-i} , \quad (\text{by Lemma 4.4})
$$

i.e.,

$$
G_{t,\Phi}^{-i} \circ \mu^\Phi_{t} = \text{pr}_i
$$

(4.10)

for any $0 \leq i \leq r - 1$, where $\text{pr}_i : \bigoplus_{i=0}^{r-1} \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right) \to \Gamma_\Phi \left( X, \mathcal{E} \otimes \mathcal{A}_X^{*-i,*-i} \right)$ is the $i$-th projection. So $\tau^\Phi_{t} \circ \mu^\Phi_{t} = id.$
Denote by \( h \in H^{1,1}(\mathbb{P}(E)) \) the Dolbeault class of \( t \) and denote by \( G_{h,\Phi}^{-i}(\bullet) \) the morphism \( H^{*\bullet-1}_{\Phi}((\mathbb{P}(E), \pi^*\mathcal{E})) \to H^{*\bullet-1}_{\Phi}(X, \mathcal{E})) \) induced by \( G_{h,\Phi}^{-i}(\bullet) \) for \( 0 \leq i \leq r - 1 \). Denote by \( \mu_{\text{Dol},\Phi}^\varepsilon \) and \( \tau_{\text{Dol},\Phi}^\varepsilon \) the morphisms on twisted Dolbeault cohomologies induced by \( \mu_{\Phi}^\varepsilon \) and \( \tau_{\Phi}^\varepsilon \) respectively. We have

**Proposition 4.5.** \( \mu_{\text{Dol},\Phi}^\varepsilon \) and \( \tau_{\text{Dol},\Phi}^\varepsilon \) are inverse isomorphisms.

**Proof.** For every \( x \in X, 1, h, \ldots, h^{-1} \) restricted to the fibre \( \pi^{-1}(x) = \mathbb{P}(E_x) \) freely linearly generate \( H^{*\bullet}(\mathbb{P}(E_x)) \). By Theorem 4.2, \( \mu_{\text{Dol},\Phi}^\varepsilon \) is an isomorphism. By (4.8), we easily conclude it. \( \square \)

### 4.4. Blow-up formulae.

Let \( X \) be a complex manifold and \( i : Y \to X \) the inclusion of a complex submanifold \( Y \) into \( X \). Suppose that \( \Phi \) is a paracompactifying family of supports on \( X \). For any \( p, q \), set \( \mathcal{F}_{X,Y}^{p,q} = \ker(\mathcal{A}_{X}^{p,q} \to i_*\mathcal{A}_{Y}^{p,q}) \). There is an exact sequence of sheaves

\[
0 \to \mathcal{F}_{X,Y}^{p,q} \to \mathcal{A}_{X}^{p,q} \to i_*\mathcal{A}_{Y}^{p,q} \to 0 \tag{4.11}
\]

for any \( p \) ([29, Sect. 4.2] or [21, Sect. 4]). Define \( \mathcal{F}_{X,Y}^p = \ker(\bar{\partial} : \mathcal{F}_{X,Y}^{p,0} \to \mathcal{F}_{X,Y}^{p,1}) \). There is an exact sequence

\[
0 \to \mathcal{F}_{X,Y}^p \to \mathcal{F}_{X,Y}^{0,p} \to \mathcal{F}_{X,Y}^{1,p} \to \cdots \to \mathcal{F}_{X,Y}^{n,p} \to 0, \tag{4.12}
\]

see [29, Sect. 4.2] or [21, Sect. 4]. Since \( \mathcal{F}_{X,Y}^p \) is a sheaf of \( C^\infty \)-modules, it is \( \Phi \)-soft by [4, II. 9.16], so (4.12) is a resolution of \( \Phi \)-soft sheaves of \( \mathcal{F}_{X,Y}^p \).

Suppose that \( \mathcal{E} \) is a locally free sheaf of \( \mathcal{O}_X \)-modules of finite rank on \( X \). We get an exact sequence of sheaves

\[
0 \to \mathcal{E} \otimes \mathcal{F}_{X,Y}^{p,q} \to \mathcal{E} \otimes \mathcal{A}_{X}^{p,q} \to i_*\mathcal{E} \otimes \mathcal{A}_{Y}^{p,q} \to 0 \tag{4.13}
\]

by (4.11) and the projection formula of sheaves. Since \( \mathcal{E} \otimes \mathcal{F}_{X,Y}^{p,q} \) is \( \Phi \)-soft,

\[
0 \to \Gamma_\Phi(X, \mathcal{E} \otimes \mathcal{F}_{X,Y}^{p,q}) \to \Gamma_\Phi(X, \mathcal{E} \otimes \mathcal{A}_{X}^{p,q}) \to \Gamma_\Phi(Y, i^*_\mathcal{E} \otimes \mathcal{A}_{Y}^{p,q}) \to 0 \tag{4.14}
\]

is exact.

Go back to the cases of complex blow-ups. Suppose that \( \pi : \tilde{X} \to X \) is the complex blow-up of a complex manifold \( X \) along a complex submanifold \( Y \) of complex codimension \( r \) with the exceptional divisor \( E \). By [21, Lemma 4.2],

\[
R^q\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}_{\tilde{X},E}^p) = \mathcal{E} \otimes R^q\pi_*\mathcal{F}_{\tilde{X},E}^p = \begin{cases} \mathcal{E} \otimes \mathcal{F}_{X,Y}^p, & q = 0, \\ 0, & q \geq 1. \end{cases} \tag{4.15}
\]

By [4, IV. 6.1], \( \pi^* \) induces an isomorphism

\[
H^q_{\Phi}(X, \mathcal{E} \otimes \mathcal{F}_{X,Y}^p) \cong H^q_{\pi^{-1}\Phi}(\tilde{X}, \pi^*\mathcal{E} \otimes \mathcal{F}_{\tilde{X},E}^p). \tag{4.16}
\]

For a given \( p \), we have a commutative diagram of exact sequences of complexes

\[
\begin{CD}
0 @>>> \Gamma_\Phi(X, \mathcal{E} \otimes \mathcal{F}_{X,Y}^{p,*}) @>>> \Gamma_\Phi(X, \mathcal{E} \otimes \mathcal{A}_{X}^{p,*}) @>>> \Gamma_\Phi(Y, i^*_\mathcal{E} \otimes \mathcal{A}_{Y}^{p,*}) @>>> 0 \\
@V{\pi^*}VV @V{\pi^*}VV @V{(\pi|_Z)^*}VV @V{(\pi|_Z)^*}VV \\
0 @>>> \Gamma_{\pi^{-1}\Phi}(\tilde{X}, \pi^*\mathcal{E} \otimes \mathcal{F}_{\tilde{X},E}^{p,*}) @>>> \Gamma_{\pi^{-1}\Phi}(\tilde{X}, \pi^*\mathcal{E} \otimes \mathcal{A}_{\tilde{X}}^{p,*}) @>>> \Gamma_{(\pi^{-1}\Phi)|_E}(E, i^*_E\pi^*\mathcal{E} \otimes \mathcal{A}_E^{p,*}) @>>> 0.
\end{CD}
\]
It induces a commutative diagram of long exact sequences

\[
\begin{array}{ccccccc}
\cdots & H^p_q(X, \mathcal{E}) & \xrightarrow{\pi^*} & H^p_q(Y, \tilde{\mathcal{E}}) & \xrightarrow{i^*_Y} & H^p_q(\tilde{X}, \pi^*\mathcal{E}) & \cdots \\
\searrow & \downarrow & | & \downarrow & | & \downarrow & | \\
\cdots & H^p_q(X, \mathcal{E}) & \xrightarrow{\pi^*} & H^p_q(Y, \tilde{\mathcal{E}}) & \xrightarrow{i^*_Y} & H^p_q(\tilde{X}, \pi^*\mathcal{E}) & \cdots \\
\end{array}
\]

By Proposition 3.8, \(\pi^*\) is injective. By Proposition 4.5, \(\pi|_\mathcal{E}^*\) is injective. By the snakelemma, \(i^*_E\) induces an isomorphism \(\mathrm{coker}\pi^* \xrightarrow{\sim} \mathrm{coker}(\pi|_\mathcal{E}^*)\). We get a commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
0 & \xrightarrow{0} & H^p_q(X, \mathcal{E}) & \xrightarrow{\pi^*} & H^p_q(Y, \tilde{\mathcal{E}}) & \xrightarrow{i^*_Y} & \mathrm{coker}\pi^* & \xrightarrow{0} \\
\downarrow & | & \downarrow & | & \downarrow & | & \downarrow & | \\
0 & \xrightarrow{0} & H^p_q(Y, i^*_Y \mathcal{E}) & \xrightarrow{(\pi|_\mathcal{E}^*)^*} & H^p_q(E, i^*_E \pi^*\mathcal{E}) & \xrightarrow{\mathrm{coker}(\pi|_\mathcal{E}^*))} & 0 \\
\end{array}
\]

(4.14) for any \(p, q\).

Denote (1.1) and (1.2) by \(\psi^\Phi_{\mathrm{Dol}, \Phi}\) and \(\phi^\Phi_{\mathrm{Dol}, \Phi}\) respectively.

4.4.1. \(\phi^\Phi_{\mathrm{Dol}, \Phi}\) is an isomorphism. By (3.12), \(\pi_*\pi^* = \text{id}\), namely, the upper row of (4.14) is a splitting sequence. So \((\pi_*, i^*_E)\) gives an isomorphism

\[H^p_q(\tilde{X}, \pi^*\mathcal{E}) \xrightarrow{\sim} H^p_q(X, \mathcal{E}) \oplus \mathrm{coker}(\pi|_\mathcal{E}^*)\]

By Proposition 4.5, it is

\[H^p_q(\tilde{X}, \pi^*\mathcal{E}) \xrightarrow{\sim} H^p_q(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H^p_{q-i}(Y, i^*_Y \mathcal{E})\]

\[\alpha \mapsto (\pi_*\alpha, \alpha^{p-1, q-1}, \ldots, \alpha^{p-r+1, q-r+1}), \]

(4.15) where \(\alpha^{p-i, q-i} \in H^p_{q-i}(Y, i^*_Y \mathcal{E})\) for \(0 \leq i \leq r - 1\) satisfy

\[i^*_E \alpha = \sum_{i=0}^{r-1} h^i \cup (\pi|_\mathcal{E}^*)^* \alpha^{p-i, q-i}.\]

By Proposition 4.5, \(\alpha^{p-i, q-i} = G^i_{h, \Phi} \circ i^*_E(\alpha)\) for \(0 \leq i \leq r - 1\). So (4.15) is just \(\phi^\Phi_{\mathrm{Dol}, \Phi}\).

Remark 4.6. On compact complex manifolds, the expression (4.15) was first obtained in [29, Sect. 5.2] for \(\Phi = \text{clit}_X\). The compactness of \(X\) is necessary there, since the finiteness of dimensions of cohomologies was used.

4.4.2. \(\psi^\Phi_{\mathrm{Dol}, \Phi}\) is an isomorphism.

Lemma 4.7. Let \(\pi : \tilde{X} \to X\) be the complex blow-up of \(X\) along a complex submanifold \(Y\) with the exceptional divisor \(E\). Assume that \(Y\) has a holomorphically contractible neighborhood in \(X\). Then the composite map

\[H^{q+1}_p(\pi|_{\tilde{X}})_E \xrightarrow{i^*_E} H^{q+1}_p(X, \pi^*\mathcal{E}) \xrightarrow{i^*_E} H^{q+1}_p(\tilde{X}, \pi^*\mathcal{E}) \xrightarrow{i^*_E} H^{q+1}_p(\pi|_{\tilde{X}})_E \]

is \(h \cup \bullet\), where \(h\) is the Dolbeault class of a Chern form of the universal line bundle \(\mathcal{O}_E(-1)\) over \(E\) and \(i^*_E : E \to \tilde{X}\) is the inclusion.
Proof. Let \( U \) be a neighborhood of \( Y \) with a holomorphic map \( \tau : U \to Y \) such that \( \tau \circ t_Y = id_Y \), where \( t_Y : Y \to U \) is the inclusion. Denote by \( l_E : E \to \tilde{U} \) and by \( j : \tilde{U} \to \tilde{X} \) the inclusions. Set \( \tilde{U} = \pi^{-1}(U) \). By (4.14), \( l_E^* \) induces a surjective map

\[
H^{p,q}(\tilde{U}) \to H^{p,q}(E)/(\pi|E)^* H^{p,q}(Y).
\]

Since \((\pi|E)^* H^{p,q}(Y) = l_E^*(\pi|\tilde{U})^* \tau^* H^{p,q}(Y) \subseteq \text{Im} l_E^* \), \( l_E^* : H^{p,q}(\tilde{U}) \to H^{p,q}(E) \) is surjective. For any \( \alpha \in H^{p,q}(E) \), \( \alpha = l_E^* \beta \) for some \( \beta \in H^{p,q}(\tilde{U}) \). Denote by \( T_E \) the current on \( \tilde{U} \) defined by the integral along the divisor \( E \). Clearly, \( l_{E*}(1) = T_E \). By the Lelong-Poincaré equation (see [9, p. 271 (13.2), (13.5)]),

\[
T_E = \frac{i}{2\pi} \Theta(O_E(E)) + \bar{\partial}\partial T
\]

for some \( T \in \mathcal{D}^{0,0}(\tilde{U}) \), where \( \Theta(O_E(E)) \) denotes a Chern form of \( O_E(E) \). By (3.12), \( l_{E*}\alpha = l_{E*}l_E^* \beta = l_{E*}(1) \cup \beta \). So

\[
l_E^* [l_{E*}(\tau^* \eta)] = l_E^* l_{E*} \eta = i \left[ \frac{i}{2\pi} \Theta(O_E(E)) \right] \cup \alpha = h \cup \alpha,
\]

(4.16)

where we used the fact that \( O_E(E(-1)) = O_E(E)|E \).

For any \( \sigma \in H^r(\pi|\tilde{U}) \), there exists \( \alpha_i \in H^r(\pi|\tilde{U}) \) for \( 0 \leq i \leq r - 1 \) such that \( \sigma = \sum_{i=0}^{r-1} h^i \cup (\pi|E)^* \alpha_i \) by Proposition 4.5. Let \( u_i \in \mathcal{A}^r(\pi|\tilde{U}) \) be a representative of \( \alpha_i \). Set \( \Omega = j^{-1}(\pi|\tilde{U})^{-1}(\Phi|Y) \). By Proposition 2.1 (3) – (6), \( \Omega \) is a paracompactifying family of supports on \( \tilde{U} \). By Proposition 2.1 (1) (2),

\[
(\pi|E)^{-1}(\Phi|Y) = l_{E*}^{-1}(j^{-1}(\pi|E)\Phi|Y) = l_{E*}^{-1}(j^{-1}((\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))) = l_{E*}^{-1}(\Omega).
\]

(4.17)

Since \( \text{supp}(t^i \land (\pi|E)^* u_i) \in (\pi|E)^{-1}(\Phi|Y) \),

\( \text{supp}(l_{E*}(t^i \land (\pi|E)^* u_i)) \in j^{-1}(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y)) = \Omega \).

Then

\[
l_E^*[l_{E*}(t^i \land (\pi|E)^* u_i)]_{j^{-1}(\pi|\tilde{U})^{-1}(\Phi)} = l_E^*[l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|E)^{-1}(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y)) \cup l_E^* l_{E*}(t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (4.17) and (3.9))

\[
= l_E^*[l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (4.17) and (3.9))

\[
= l_E^*[l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (3.11))

\[
= l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (4.16))

Since the support of \( l_{E*}(t^i \land (\pi|E)^* u_i) \) is closed in \( \tilde{X} \), \( j_* l_{E*}(t^i \land (\pi|E)^* u_i) \) is defined well and is just \( i_{E*}(t^i \land (\pi|E)^* u_i) \). By (3.7),

\[
j^* i_{E*}(t^i \land (\pi|E)^* u_i) = l_{E*}(t^i \land (\pi|E)^* u_i).
\]

(4.18)

Then

\[
i_{E*} l_{E*}(h^i \cup (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\Phi)} = i_{E*} l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\Phi)}
\]

(by (5.9))

\[
= l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (3.11))

\[
= l_{E*}((t^i \land (\pi|E)^* u_i)]_{(\pi|\tilde{U})^{-1}(\tau^{-1}(\Phi|Y))}
\]

(by (4.16)).
So \( i_E^* i_{E*} \sigma = \sum_{i=0}^{r-1} h^{i+1} \cap (\pi|_E)^* \alpha_i = h \cup \sigma \).

With the similar proof of [21, Proposition 4.5], we have

**Lemma 4.8.** Suppose that \( Y \) has a holomorphically contractible neighborhood in \( X \). Then \( \psi_{\text{Dol}, \Phi}^O \) is an isomorphism.

**Proof.** Suppose that \( \pi^* \alpha + \sum_{i=1}^{r-1} i_{E*} (h^{i-1} \cup (\pi|_E)^* \beta_i) = 0 \), where \( \alpha \in H^*_Y(X) \) and \( \beta_i \in H^{\ast - i - i}_{\Phi|Y}(Y) \) for \( 1 \leq i \leq r - 1 \). Pull it back by \( i_E^* \), we get

\[
(\pi|_E)^* i_Y^* \alpha + \sum_{i=1}^{r-1} h^i \cup (\pi|_E)^* \beta_i = 0
\]

by Lemma 4.7. By Proposition 4.5, \( \beta_i = 0 \) for all \( i \). So \( \pi^* \alpha = 0 \). By Proposition 3.8, \( \pi^* \) is injective, which implies that \( \alpha = 0 \). Then \( \psi_{\text{Dol}, \Phi}^O \) is injective.

For any \( \gamma \in H^*_Y(X) \), by Proposition 4.5, there exist \( \beta_i \in H^{\ast - i - i}_{\Phi|Y}(Y) \) for \( 0 \leq i \leq r - 1 \) such that \( i_E^* \gamma = \sum_{i=0}^{r-1} h^i \cup (\pi|_E)^* \beta_i \). By Lemma 4.7,

\[
i_E^* \gamma = \sum_{i=1}^{r-1} i_{E*} (h^{i-1} \cup (\pi|_E)^* \beta_i) = (\pi|_E)^* \beta,
\]

which is zero in coker\((\pi|_E)^*\). By (4.14),

\[
\gamma = \sum_{i=1}^{r-1} i_{E*} (h^{i-1} \cup (\pi|_E)^* \beta_i) = \pi^* \alpha,
\]

for some \( \alpha \in H^*_Y(X) \). So \( \psi_{\text{Dol}, \Phi}^O \) is surjective.

We complete the proof. \( \square \)

Set

\[
\mathcal{F}^{\ast, \ast} = (\mathcal{E} \otimes \mathcal{A}_X^{\ast, \ast}) \oplus \bigoplus_{i=1}^{r-1} i_Y^* (i_Y^* \mathcal{E} \otimes \mathcal{A}_Y^{\ast, \ast})[-i, -i].
\]

and \( \mathcal{G}^{\ast, \ast} = \pi^* \mathcal{E} \otimes \mathcal{D}_X^{\ast, \ast} \). Let \( t \in \mathcal{A}_E^{1, 1}(E) \) be a Chern form of the universal line bundle \( \mathcal{O}_E(-1) \) over \( E \). Set \( \tilde{U} = \pi^{-1}(U) \) for any open subset \( U \subseteq X \). Notice that \( (\pi^{-1} \Phi)|_{\tilde{U}} = (\pi|_{\tilde{U}})^{-1}(\Phi|_U) \).

Define a morphism

\[
\Gamma_{\Phi|\tilde{U}}(U, \mathcal{F}^{\ast, \ast}) = \Gamma_{\Phi|U}(U, \mathcal{E} \otimes \mathcal{A}_X^{\ast, \ast}) \oplus \bigoplus_{i=1}^{r-1} \Gamma_{\Phi|Y \cap U}(Y \cap U, i_Y^* \mathcal{E} \otimes \mathcal{A}_Y^{\ast, \ast})
\]

\[
\Rightarrow \Gamma_{(\pi|_{\tilde{U}})^{-1}(\Phi|_{\tilde{U}})}(\tilde{U}, \pi^* \mathcal{E} \otimes \mathcal{D}_X^{\ast, \ast}) = \Gamma_{(\pi^{-1} \Phi)|_{\tilde{U}}}(\tilde{U}, \mathcal{G}^{\ast, \ast})
\]

of double complexes as

\[
\psi_{\Phi|\tilde{U}} = (\pi|_{\tilde{U}})^* \gamma + \sum_{i=1}^{r-1} i_{E \cap \tilde{U}} \circ (t^{i-1}|_{E \cap \tilde{U}}) \circ (\pi|_{E \cap \tilde{U}})^* \gamma,
\]

where \( i_{E \cap \tilde{U}} : E \cap \tilde{U} \to \tilde{U} \) is the inclusion. On cohomologies, it induces a morphism

\[
L^{\ast, \ast}(U) := H^{\ast, \ast}_{\Phi|U}(U, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H^{\ast - i, \ast - i}_{\Phi|Y \cap U}(Y \cap U, i_Y^* \mathcal{E}) \to K^{\ast, \ast}(\tilde{U}) := H^{\ast, \ast}_{(\pi^{-1} \Phi)|_{\tilde{U}}}(\tilde{U}, \pi^* \mathcal{E}),
\]
which is just \( \psi_{clt}^L \). Briefly set \( F_U^{\bullet, \bullet} = \psi_{clt}^L \). Denote by \( \mathcal{P}(U) \) the statement that \( F_U^{\bullet, \bullet} \) is an isomorphism. The theorem is equivalent to say that \( \mathcal{P}(X) \) holds.

First, we fix some notations. For the inclusion \( j : U \rightarrow V \) of open sets in \( X \), denote by \( \tilde{j} : \tilde{U} \rightarrow \tilde{V} \) and \( j' : U \cap Y \rightarrow V \cap Y \) the corresponding inclusions. Denote by \( J = \langle j_1, j_2, \ldots, j_r \rangle : \Gamma_{\Phi}|_U(U, F^{\bullet, \bullet}) \rightarrow \Gamma_{\Phi}|_V(V, F^{\bullet, \bullet}) \) the extension by zero. Now, we go back to the proof. Fix an integer \( p \). For any open sets \( U, V \subseteq X \), let \( j_1 : U \cap V \rightarrow U, j_2 : U \cap V \rightarrow V, j_3 : U \rightarrow U \cup V, j_4 : V \rightarrow U \cup V \) be inclusions. By Propositions 3.5 and 3.4, there is a commutative diagram of exact sequences of complexes

\[
0 \xrightarrow{} \Gamma_{\Phi}|_{U \cap V}(U \cap V, F^{p, \bullet}) \xrightarrow{(j_1, j_2)} \Gamma_{\Phi}|_{U}(U, F^{p, \bullet}) \oplus \Gamma_{\Phi}|_{V}(V, F^{p, \bullet}) \xrightarrow{j_3 - j_4} \Gamma_{\Phi}|_{U \cup V}(U \cup V, F^{p, \bullet}) \xrightarrow{} 0.
\]

Therefore, we have a commutative diagram

\[
\cdots \xrightarrow{L^{p,q}(U \cap V)} L^{p,q}(U) \oplus L^{p,q}(V) \xrightarrow{F^{p,q}_{U,V}} L^{p,q}(U \cup V) \xrightarrow{F^{p,q+1}_{U,V}} L^{p,q+1}(U \cup V) \xrightarrow{} \cdots
\]

\[
\cdots \xrightarrow{K^{p,q}(U \cap \tilde{V})} K^{p,q}(\tilde{U}) \oplus K^{p,q}(\tilde{V}) \xrightarrow{F^{p,q}_{\tilde{U},\tilde{V}}} K^{p,q}(\tilde{U} \cup \tilde{V}) \xrightarrow{F^{p,q+1}_{\tilde{U},\tilde{V}}} K^{p,q+1}(\tilde{U} \cup \tilde{V}) \xrightarrow{} \cdots
\]

of long exact sequences. If \( F^{p,q}_U, F^{p,q}_V \), and \( F^{p,q}_{U \cap V} \) are isomorphisms for all \( p, q \), then \( F^{p,q}_{U \cup V} \) are also isomorphisms for all \( p, q \) by the five-lemma. Thus \( \mathcal{P} \) satisfies the Mayer-Vietoris condition in Lemma 2.5. Let \( \mathcal{U} \) be an \( \mathcal{E} \)-free basis of the topology of \( X \) such that every \( U \in \mathcal{U} \) is Stein. For \( U_1, \ldots, U_l \in \mathcal{U}, \bigcap_{i=1}^l U_i \) is \( \mathcal{E} \)-free and \( Y \cap \bigcap_{i=1}^l U_i \) is Stein. By [12, Theorem 3.3.3], any Stein complex submanifold has a holomorphically contractible neighborhood. By Lemma 4.8, \( F^{p,q}_{\cup_{i=1}^l U_i} = \psi_{clt}^L \cup_{i=1}^l U_i \) is an isomorphism, so \( \mathcal{P} \) satisfies the local condition in Lemma 2.5. Obviously, \( \mathcal{P} \) satisfies the disjoint condition in Lemma 2.5. By Lemma 2.5, \( \mathcal{P}(X) \) holds.

We complete the proof.

**Remark 4.9.** In general, \( clt_X|_U \neq clt_U \) for an open set \( U \subseteq X \). We used a type of the Mayer-Vietoris sequences (Proposition 3.5) here and used another type of the ones in the proof of [21, Theorem 1.2] for \( \Phi = clt_X \), where the later one seems difficult to be used to prove the case with supports in a paracompactifying family \( \Phi \).

### 4.4.3. Relationship of (1.1) and (1.2).

A natural question is:

**Question 4.10.** Are \( \psi_{Dol, \Phi}^L \) and \( \phi_{Dol, \Phi}^L \) inverse to each other?

This question has an affirmative answer in the following case.

**Proposition 4.11.** Suppose that

\[
i_E^*i_E^*\sigma = h \cup \sigma
\]

holds for any \( \sigma \in H^{\bullet, \bullet}_{clt}(E, i_E^*\pi^*\mathcal{E}) \). Then \( \psi_{Dol, \Phi}^L \) and \( \phi_{Dol, \Phi}^L \) are inverse isomorphisms.
Proof. For $\alpha^{p,q} \in H^{p,q}_\Phi(X,E)$ and $\beta^{p-i,q-i} \in H^{p-i,q-i}_{\Phi}\bigl(Y,i_Y^*\mathcal{E}\bigr)$, $1 \leq i \leq r-1$,

$$i_Y^*\psi_{\text{Dol},\Phi}^E(\alpha^{p,q},\beta^{p-1,q-1},\ldots,\beta^{p-r+1,q-r+1})$$

$$=\mu_{\text{Dol},\Phi}(i_Y^*\alpha^{p,q},\beta^{p-1,q-1},\ldots,\beta^{p-r+1,q-r+1})$$

by (4.19). Hence

$$G_{h,\Phi}^{-i} \circ i_E^*\bigl(\psi_{\text{Dol},\Phi}^E(\alpha^{p,q},\beta^{p-1,q-1},\ldots,\beta^{p-r+1,q-r+1})\bigr) = \beta^{p-i,q-i}$$

by (4.10), which implies that $\phi_{\text{Dol},\Phi}^E \circ \psi_{\text{Dol},\Phi}^E = \text{id}$, i.e., $\phi_{\text{Dol},\Phi}^E$ is the inverse isomorphism of $\psi_{\text{Dol},\Phi}^E$.

\textbf{Proposition 4.12.} Let $\Phi$ be a paracompactifying family of supports on $X$. Assume that one of the following conditions is satisfied:

1. $X$ and $Y$ are compact complex manifolds satisfying the $\partial\bar{\partial}$-lemma and $\Phi = \text{clt}_X$,
2. $Y$ has a holomorphically contractible neighborhood in $X$,
3. $E$ has a holomorphically contractible neighborhood in $\tilde{X}$.

Then $i_E^*i_{E,*}\sigma = h \cup \sigma$ for any $\sigma \in H^{**}_{(\pi-1)E}(E)$.

Proof. If $X$ and $Y$ satisfy the $\partial\bar{\partial}$-lemma, so do $E = \mathbb{P}(N_X/Y)$ and $\tilde{X}$ (see [1, Corollary 3], [32, Corollary 26] or [24, Theorems 1.1, 1.2]). Let $t \in A^{1,1}(E)$ be a Chern form of $\mathcal{O}_E(-1)$. Then $h = |t|_{\text{Dol}}$ and $c_1(\mathcal{O}_E(-1)) = |t|_{dR}$. Since $\mathcal{O}_E(-1) = \mathcal{O}_X(E)|_E$, $|t|_{dR} = |E||_E$. By Proposition 5.4 (see Sect. 5.1), $i_E^*i_{E,*}\sigma = |t|_{dR} \cup \sigma$ for any $\sigma \in H^*(E,\mathbb{C})$. The $\partial\bar{\partial}$-manifolds satisfy Hodge decompositions, which concludes the first case. The second case is just Lemma 4.7. With some modification of the proof of Lemma 4.7, we prove the third case as follows:

Let $W$ be a neighborhood of $E$ with a holomorphic map $\tau : W \to E$ such that $\tau \circ l_E = id_E$, where $l_E : E \to W$ is the inclusion. Denote by $j : W \to \tilde{X}$ the inclusion. Let $u \in \Gamma_{(\pi-1)E}(E,A^*)$ be any $\partial$-closed form. Set $\Omega = j^{-1}\pi^{-1}E \cap \tau^{-1}(\pi^{-1}E)|_E$. By Proposition 2.1, $\Omega$ is a paracompactifying family of supports on $W$ and

$$\left(\pi^{-1}E\right)|_E = l^{-1}_E \left(j^{-1}\pi^{-1}E\right) = l^{-1}_E \left(\tau^{-1}(\pi^{-1}E)|_E\right) = l^{-1}_E \Omega.$$  \hspace{1cm} (4.20)

Since $\text{supp} u \in (\pi^{-1}E)|_E$, supp $(l_E u) \in j^{-1}\pi^{-1}E \cap \tau^{-1}(\pi^{-1}E)|_E = \Omega$. By (4.20) and (3.9),

$$l^*_E[l_E u]_{\tau^{-1}(\pi^{-1}E)} = l^*_E[l_E u]|_{\Omega} = l^*_E[l_E u]_{\tau^{-1}(\pi^{-1}E)|_E}.$$  \hspace{1cm} (4.21)

By (3.11), $l_{E,*}u = l_{E,*}(1) \cap \tau^*u$. So

$$l^*_E[l_E u]_{\tau^{-1}(\pi^{-1}E)|_E} = l^*_E \left([l_E u](1) \cup [\tau^*u]_{\tau^{-1}(\pi^{-1}E)|_E}\right) = h \cup [u]_{(\pi-1)E}|_E.$$  \hspace{1cm} (4.22)

Then

$$l^*_E[\tau^*i_E u]_{(\pi-1)E}|_E = l^*_E[j^*i_E u]_{\tau^{-1}(\pi^{-1}E)}$$

$$= l^*_E[l_E u]_{\tau^{-1}(\pi^{-1}E)} \quad \text{(by (3.7))}$$

$$= h \cup [u]_{(\pi-1)E}|_E.$$  \hspace{1cm} (by (4.21) and (4.22)) \hspace{1cm} \square

4.5. Applications.
4.5.1. Conjugate local systems. Let \((V, \cdot)\) be a complex vector space with the scalar multiplication \(\cdot\). The action \(c \cdot v = \overline{c} \cdot v\) for any \(c \in \mathbb{C}\) and \(v \in V\) defines the conjugate vector space \((V, \ast)\) of \((V, \cdot)\). We shortly write \((V, \cdot)\) and \((V, \ast)\) as \(V\) and \(\overline{V}\) respectively. Notice that \(\overline{V}\) has the same underlying space with \(V\) and the identity map \(V \to \overline{V}\) is a real isomorphism.

Let \(\mathcal{L}\) be a local system of \(\mathbb{C}\)-modules of finite rank on \(X\). Set \(\Gamma(U, \overline{\mathcal{L}}) = \overline{\Gamma(U, \mathcal{L})}\) for all open set \(U \subseteq X\), which define a local system of \(\mathbb{C}\)-modules of finite rank on \(X\). Suppose that a trivialization \(\mathcal{L}|_U \to U \times \mathbb{C}^r\) is defined as \(\sum_{i=1}^r c_i \cdot v_i(x) \mapsto (x, c_1, \ldots, c_r)\), where \(v_1, \ldots, v_r\) is a basis of \(\Gamma(U, \mathcal{L})\). Then \(\sum_{i=1}^r c_i \ast v_i(x) \mapsto (x, c_1, \ldots, c_r)\) give a trivialization \(\overline{\mathcal{L}}|_U \to U \times \mathbb{C}^r\). For any open set \(U \subseteq X\), \(s \mapsto s\) for all \(s \in \Gamma(U, \mathcal{L})\) define a real isomorphism \(\Gamma(U, \mathcal{L}) \to \Gamma(U, \overline{\mathcal{L}})\), which gives an isomorphism \(\mathcal{L} \to \overline{\mathcal{L}}\) of real local systems.

Suppose that \(X\) is a complex manifold. Then \(\mathcal{L} \otimes_{\mathbb{C}_X} A^{p,q}_X = (\mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X) \otimes_{\mathcal{O}_X} A^{p,q}_X\) and \(\mathcal{L} \otimes_{\mathbb{C}_X} D^{p,q}_X = (\mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X) \otimes_{\mathcal{O}_X} D^{p,q}_X\), where \(\mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X\) is a locally free sheaf of \(\mathcal{O}_X\)-modules. For a holomorphic map \(f : Y \to X\) of complex manifolds, \(f^{-1}\mathcal{L} \otimes_{\mathbb{C}_X} A^{p,q}_Y = f^*(\mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X) \otimes_{\mathcal{O}_Y} A^{p,q}_Y\) and \(f^{-1}\mathcal{L} \otimes_{\mathbb{C}_X} D^{p,q}_Y = f^*(\mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X) \otimes_{\mathcal{O}_Y} D^{p,q}_Y\). Mapping \(\sum_{i=1}^r v_i \otimes \alpha_i\) to \(\overline{\sum_{i=1}^r v_i \otimes \alpha_i}\) define \(\mathcal{L} \otimes_{\mathbb{C}_X} A^{p,q}_X \to \overline{\mathcal{L}} \otimes_{\mathbb{C}_X} A^{p,q}_X\) and \(\mathcal{L} \otimes_{\mathbb{C}_X} D^{p,q}_X \to \overline{\mathcal{L}} \otimes_{\mathbb{C}_X} D^{p,q}_X\), which are said to be the complex conjugation maps.

4.5.2. Projective bundle and blow-up formulae on the \(E_1\)-level. We recall some notions of the double complex and related structures, see \([32, \text{Sect.}\ 2]\) for more details. All the double complexes here are assumed to be of vector spaces over \(\mathbb{C}\) and bounded.

Let \((K^{\bullet, \bullet}, \partial_1, \partial_2)\) be a double complex with two endomorphisms \(\partial_1, \partial_2\) of bidegree \((1,0)\) and \((0,1)\), which satisfy that \(\partial_1 \circ \partial_2 = 0\) for \(i = 1, 2\) and \(\partial_1 \circ \partial_2 + \partial_2 \circ \partial_1 = 0\). For convenience, we briefly write it as \(K^{\bullet, \bullet}\) sometimes. Let \(\partial_1^{p,q} : K^{p,q} \to K^{p+1,q}\) and \(\partial_2^{p,q} : K^{p,q} \to K^{p,q+1}\) be the restrictions of \(\partial_1\) and \(\partial_2\) respectively. Recall the following constructions.

- **The row and column cohomologies**
  
  \[
  R_{\partial_1}(K^{\bullet, \bullet}) = H^p(K^{\bullet, q}, \partial_1) \quad \text{and} \quad R_{\partial_2}(K^{\bullet, \bullet}) = H^q(K^{p, \bullet}, \partial_2).
  \]

- **The Bott-Chern and Aeppli cohomologies**
  
  \[
  H_{BC}^{p,q}(K^{\bullet, \bullet}) = \frac{\ker \partial_1^{p-1,q} \cap \ker \partial_2^{0,q}}{\text{im} \partial_1^{p-1,q} \circ \partial_2^{0,q-1}} \quad \text{and} \quad H_{A}^{p,q}(K^{\bullet, \bullet}) = \frac{\ker \partial_1^{p,q+1} \circ \partial_2^{p,q}}{\text{im} \partial_1^{p,q} + \text{im} \partial_2^{p,q}}.
  \]

A morphism of double complexes is called an \(E_1\)-isomorphism, if it induces an isomorphism on both row and column cohomologies. J. Stelzig obtained the following result.

**Theorem 4.13** (\([32, \text{Corollary}\ 13]\)[31, Lemma 1.3]). Any \(E_1\)-isomorphism of double complexes induces isomorphisms on Bott-Chern and Aeppli cohomologies respectively.

Let \(X\) be a complex manifold and let \(\mathcal{L}\) be a local system of \(\mathbb{C}\)-modules of finite rank on \(X\). Set \(A^{p,q}_\phi(X, \mathcal{L}) = \Gamma_\phi(X, \mathcal{L} \otimes_{\mathbb{C}_X} A^{p,q}_X)\) and \(D^{p,q}_\phi(X, \mathcal{L}) = \Gamma_\phi(X, \mathcal{L} \otimes_{\mathbb{C}_X} D^{p,q}_X)\). Denote by \(A^{p,q}_\phi(X, \mathcal{L})\) and \(D^{p,q}_\phi(X, \mathcal{L})\) the double complexes \((A^{p,q}_\phi(X, \mathcal{L}), \partial, \overline{\partial})\) and \((D^{p,q}_\phi(X, \mathcal{L}), \partial, \overline{\partial})\).

On column cohomologies, the inclusion \(i : A^{p,q}_\phi(X, \mathcal{L}) \to D^{p,q}_\phi(X, \mathcal{L})\) induces an isomorphism.
$H^0_\partial(A^{\bullet}_{\phi}(X, \mathcal{L})) \to H^0_\partial(D^{\bullet}_{\phi}(X, \mathcal{L}))$, which are just $H^0_{\partial}(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X)$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^0_\partial(A^{\bullet}_{\phi}(X, \overline{Z})) & \longrightarrow & H^0_\partial(D^{\bullet}_{\phi}(X, \overline{Z})) \\
\downarrow & & \downarrow \\
H^0_\partial(A^{\bullet}_{\phi}(X, \mathcal{L})) & \longrightarrow & H^0_\partial(D^{\bullet}_{\phi}(X, \mathcal{L})),
\end{array}
$$

where the horizontal maps are induced by the inclusion $i$ and the vertical maps are defined by complex conjugation maps. The vertical maps are real isomorphisms and the upper map is an (complex) isomorphism, so the lower map is a real isomorphism. Since it is a complex linear map, the lower map is an (complex) isomorphism, i.e., an isomorphism on row cohomologies. Hence the inclusion $i : A^{\bullet}_{\phi}(X, \mathcal{L}) \to D^{\bullet}_{\phi}(X, \mathcal{L})$ is an $\mathcal{E}_1$-isomorphism.

By Theorem 4.13, the Bott-Chern cohomologies of $A^{\bullet}_{\phi}(X, \mathcal{L})$ and $D^{\bullet}_{\phi}(X, \mathcal{L})$ coincide with each other via the inclusion, which are both denoted by $H^0_{\partial, \mathcal{E}_1}(X, \mathcal{L})$. Similarly, we write their Aeppli cohomologies as $H^0_{\partial, \mathcal{E}_1}(X, \mathcal{L})$.

Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle associated to a holomorphic bundle $E$ of rank $r$ over a complex manifold $X$ and let $\mathcal{L}$ be a local system of $\mathbb{C}$-modules of finite rank on $X$. Denote by $t \in A^{1,1}(\mathbb{P}(E))$ a Chern form of the universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$.

By Proposition 4.5, $\mu_{\mathcal{L} \otimes \mathcal{O}_X}$ and $\mu_{\mathcal{L} \otimes \mathcal{O}_X}$ are isomorphisms. By similar arguments with those of $i$, $\mu_{\mathcal{L} \otimes \mathcal{O}_X} : \bigoplus_{i=1}^{r-1} A^{\bullet}_{\phi}(X, \mathcal{L}) \to A^{\bullet}_{\mathcal{E}_1}(\mathbb{P}(E), \pi^{-1}\mathcal{L})$ is an $\mathcal{E}_1$-isomorphism.

Similarly, $\tau_{\phi}$ is an $\mathcal{E}_1$-isomorphism. Denote by $\mu_{BC, \phi}$, $\tau_{BC, \phi}$ the morphisms on Bott-Chern cohomologies and by $\mu_{A, \phi}$, $\tau_{A, \phi}$ the morphisms on Aeppli cohomologies induced by $\mu_{\mathcal{L} \otimes \mathcal{O}_X}$ and $\tau_{\mathcal{L} \otimes \mathcal{O}_X}$.

By Theorem 4.13, $\mu_{BC, \phi}$, $\tau_{BC, \phi}$, $\mu_{A, \phi}$, $\tau_{A, \phi}$ are isomorphisms. As Proposition 4.5, we easily check that $\mu_{BC, \phi}$ and $\tau_{BC, \phi}$, $\mu_{A, \phi}$ and $\tau_{A, \phi}$ are inverse isomorphisms.

We summarize these results as follows.

**Proposition 4.14.** $\mu_{\mathcal{L} \otimes \mathcal{O}_X}$ and $\tau_{\mathcal{L} \otimes \mathcal{O}_X}$ are $\mathcal{E}_1$-isomorphisms. Moreover, $\mu_{BC, \phi}$ and $\tau_{BC, \phi}$, $\mu_{A, \phi}$ and $\tau_{A, \phi}$ are inverse isomorphisms.

Let $\pi : \tilde{X} \to X$ be the complex blow-up of a complex manifold $X$ along a complex submanifold $Y$ with the exceptional divisor $E$. Suppose that $E$ is a locally free sheaf of $\mathcal{O}_X$-module of finite rank on $X$. Denote by $i_E : E \to \tilde{X}$ the inclusion. Let $t \in A^{1,1}(E)$ be a Chern form of the universal line bundle $\mathcal{O}_E(-1)$ on $E = \mathbb{P}(N_{Y/X})$. The morphism $\pi^* + \sum_{i=1}^{r-1} i_{E*}(t) \wedge (\pi|E)^* (\bullet)$ defines

$$
\Gamma_\phi(X, \mathcal{E} \otimes A_X^{\bullet}) \oplus \bigoplus_{i=1}^{r-1} \Gamma_\phi(Y, i_Y^{*} \mathcal{E} \otimes A_Y^{-i} \otimes A_X^{\bullet+i}) \to \Gamma_\phi(\tilde{X}, \pi^* \mathcal{E} \otimes A_{\tilde{X}}^{\bullet}).
$$

(4.23)

The morphism $\left( \Gamma_\phi(\bullet), G_{t, \phi} \circ i_{E*}(\bullet), \ldots, G_{t, \phi} \circ i_{E*}(\bullet) \right)$ defines

$$
\Gamma_\phi(\tilde{X}, \pi^* \mathcal{E} \otimes A_{\tilde{X}}^{\bullet}) \to \Gamma_\phi(X, \mathcal{E} \otimes D_X^{\bullet}) \oplus \bigoplus_{i=1}^{r-1} \Gamma_\phi(Y, i_Y^{*} \mathcal{E} \otimes A_Y^{-i} \otimes A_X^{\bullet+i}).
$$

(4.24)

Denote (4.23) and (4.24) by $\psi_{BC, \phi}^t$ and $\phi_{BC, \phi}^t$, respectively. Denote by $\psi_{A, \phi}^t$, $\phi_{A, \phi}^t$ the morphisms on Bott-Chern cohomologies and denote by $\psi_{BC, \phi}^t$, $\phi_{BC, \phi}^t$ the morphisms on Aeppli cohomologies
induced by \( \psi_\phi^L \otimes_{\mathcal{O}_X} \mathcal{O}_X \) and \( \phi_\phi^L \otimes_{\mathcal{O}_X} \mathcal{O}_X \). As Proposition 4.14, we have the following result by Theorems 1.1 and 4.13.

**Proposition 4.15.** \( \psi_\phi^L \otimes_{\mathcal{O}_X} \mathcal{O}_X \) and \( \phi_\phi^L \otimes_{\mathcal{O}_X} \mathcal{O}_X \) are \( E_1 \)-isomorphisms. Moreover, \( \psi_{BC,\phi}^L \), \( \phi_{BC,\phi}^L \), \( \psi_{A,\phi}^L \), \( \phi_{A,\phi}^L \) are isomorphisms.

**Remark 4.16.** On compact complex manifolds, S. Yang, X.-D. Yang [38, Theorem 1.2] and J. Stelzig [31, Corollary 12, Theorem 23] [32, Proposition 4, Theorem 8] showed the existence of the isomorphism \( H^{p,q}_{BC}(X) \oplus \bigoplus_{i=1}^{r-1} H^{p-i,q-i}_{BC}(Y) \cong H^{p,q}_{BC}(\tilde{X}) \), which was not expressed explicitly.

**Question 4.17.** Are \( \psi_{BC,\phi}^L \) and \( \phi_{BC,\phi}^L \) (resp. \( \psi_{A,\phi}^L \) and \( \phi_{A,\phi}^L \)) inverse to each other?

Analogue to Proposition 4.11, if \( i_E^*i_{E*}(\bullet) = h_{BC} \cup \bullet \) (resp. \( i_E^*i_{E*}(\bullet) = h_{A} \cup \bullet \)) holds on \( H_{BC,(\pi^{-1}\Phi)|\tilde{Y}}^p(E, (\pi_E)^{-1}(\mathcal{L}|Y)) \) (resp. \( H_{A,(\pi^{-1}\Phi)|\tilde{Y}}^p(E, (\pi_E)^{-1}(\mathcal{L}|Y)) \)), \( \psi^L_{BC,\phi} \) and \( \phi^L_{BC,\phi} \) (resp. \( \psi^L_{A,\phi} \) and \( \phi^L_{A,\phi} \)) are inverse isomorphisms.

4.5.3. Hypercohomologies of truncated twisted holomorphic de Rham complexes. For a complex \( \mathcal{F}^* \) of sheaves on a topological space \( X \), set \( H^i_{\Phi}(X, \mathcal{F}^*) = R^i\Gamma_{\Phi}(\mathcal{F}^*) \), which is called the \( k \)-th hypercohomology with supports in \( \Phi \). Using the results in the earlier version [23] of the present paper, we generalized several classical results for the hypercohomologies of truncated twisted holomorphic de Rham complexes in [25]. Based on the present results, we can extend them to the ones with supports in paracompactifying families. We only write out the blow-up formula and give a detailed proof here.

Let \( X \) be an \( n \)-dimensional complex manifold and let \( \mathcal{L} \) be a local system of \( \mathbb{C} \)-modules of finite rank on \( X \). Given any integers \( s \) and \( t \), the truncated twisted holomorphic de Rham complex \( \Omega^{[s,t]}_X(\mathcal{L}) \) is defined as the zero complex if \( s > t \) and as the complex

\[
0 \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^s_X \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^{s+1}_X \longrightarrow \cdots \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^t_X \longrightarrow 0
\]

if \( s \leq t \), where \( \mathcal{L} \otimes \Omega^k_X \) is placed in degree \( k \) for \( s \leq k \leq t \) and zeros are placed in other degrees. In particular, \( \Omega^{[0,n]}_X(\mathcal{L}) = \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^n_X \) is the twisted holomorphic de Rham complex on \( X \) and \( \Omega^{[p,p]}_X(\mathcal{L}) = (\mathcal{L} \otimes_{\mathcal{O}_X} \Omega^p_X)[-p] \), where \( \Omega^p_X \) denote the complex with \( \Omega^p_X \) in degree 0 and zeros in other degrees.

Set

\[
S^{p,q}_X(\mathcal{L}, s, t) = \begin{cases} 
\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A}^{p,q}_X, & s \leq p \leq t \\
0, & \text{others.}
\end{cases}
\]

\[
d^p_{1,q} = \begin{cases} 
\partial, & s \leq p < t \\
0, & \text{others.}
\end{cases}
\quad d^p_{2,q} = \begin{cases} 
\bar{\partial}, & s \leq p \leq t \\
0, & \text{others.}
\end{cases}
\]

Then \( (S^{s,*}_X(\mathcal{L}, s, t), d_1, d_2) \) is a double complex of sheaves, which is shortly denoted by \( \mathcal{S}^{s,*}_X(\mathcal{L}, s, t) \). Let \( \mathcal{S}^{s,*}_X(\mathcal{L}, s, t) \) be the simple complex associated to \( \mathcal{S}^{s,*}_X(\mathcal{L}, s, t) \). For any \( p \in \mathbb{Z} \), \( \Omega^{[s,t]}_X(\mathcal{L})^p \rightarrow (S^{s,*}_X(\mathcal{L}, s, t), d_1^{p,*}, d_2^{p,*}) \) given by the inclusion is a resolution of \( \Omega^{[s,t]}_X(\mathcal{L})^p \). By [35, Lemma 8.5], the inclusion gives a quasi-isomorphism \( \Omega^{[s,t]}_X(\mathcal{L}) \rightarrow S^{s,*}_X(\mathcal{L}, s, t) \) of complexes of sheaves. Suppose that \( \Phi \) is a paracompactifying family of supports on \( X \). Set \( S^{p,q}_\phi(X, \mathcal{L}, s, t) = \)
The sheaf $C^\infty_X$ is $\Phi$-soft ([4, II. 9.4]), so are $S^p_X(L, s, t)$ by [4, II. 9.16]. Hence $S^p_X(L, s, t)$ are $\Phi$-acyclic by [4, II. 9.11]. By [35, Proposition 8.12], the hypercohomology

$$H^k_{\Phi}(X, \Omega^{|s,t|}_X(L)) \cong H^k(S^p_X(L, s, t))$$

(4.26)

for any $k \in \mathbb{Z}$. For example, $H^k_{\Phi}(X, \Omega^{[0,n]}(L)) \cong H^k_{\Phi}(X, L)$ and $H^k_{\Phi}(X, \Omega^{[p]}(L)) \cong H^k_{\Phi}(X, L \otimes_{\mathcal{O}_X} \mathcal{O}_X)$.

Similarly, we can define $T^*_{\Phi}(L, s, t)$, $T^*_{\Phi}(L, s, t)$, $T^*_{\Phi}(X, L, s, t)$ and $T^*_{\Phi}(X, L, s, t)$, where

$$T^q_{\Phi}(L, s, t) = \begin{cases} L \otimes_{\mathcal{O}_X} \mathcal{D}^{p,q}_{\mathcal{O}_X}, & s \leq p \leq t \\ 0, & \text{others.} \end{cases}$$

The inclusion gives a quasi-isomorphism $\Omega^{|s,t|}_X(L) \to T^*_{\Phi}(L, s, t)$ of complexes of sheaves. There is an isomorphism

$$H^k_{\Phi}(X, \Omega^{|s,t|}_X(L)) \cong H^k_{\Phi}(T^*(X, L, s, t))$$

(4.27)

for any $k \in \mathbb{Z}$.

The following result extends [8, Question 10] and gives it a positive answer.

**Theorem 4.18.** Let $\pi : \tilde{X} \to X$ be the complex blow-up of a complex manifold $X$ along a complex submanifold $Y$ and $L$ a local system of $\mathbb{C}$-modules of finite rank on $X$. Denote by $i_Y : Y \to X$ the inclusion and set $r = \text{codim}_C Y \geq 2$. Suppose that $\Phi$ is a paracontracting family of supports on $X$. Then there exists an isomorphism

$$\mathbb{H}^k_{\pi-\Phi}(\tilde{X}, \Omega^{|s,t|}_{\tilde{X}}(\pi^{-1}L)) \cong \mathbb{H}^k_{\Phi}(X, \Omega^{|s,t|}_X(L)) \oplus \bigoplus_{i=1}^{r-1} \mathbb{H}^k_{\Phi}(Y, \Omega^{|s-i-i|}_Y(\mathcal{L}|_Y)),
$$

for any $k$, $s$, $t$.

**Proof.** Fix two integers $s$ and $t$. Consider the complexes $K^{*}(s, t) = S^*_{\pi^{-1} \Phi}(\tilde{X}, \pi^{-1}L, s, t)$ and

$$L^{*}(s, t) = T^*_{\Phi}(X, L, s, t) \oplus \bigoplus_{i=1}^{r-1} T^*_{\Phi}(Y, L|_Y, s - i, t - i)[-i, -i].$$

We have the first pages

$$\kappa E^p_{1,q} = \mathbb{H}^q(S^p_{\pi^{-1} \Phi}(\tilde{X}, \pi^{-1}L, s, t))$$

$$= \begin{cases} \mathbb{H}^p_{\pi^{-1} \Phi}(\tilde{X}, \pi^{*}(L \otimes_{\mathcal{O}_X} \mathcal{O}_X)), & s \leq p \leq t \\ 0, & \text{others,} \end{cases}$$

and

$$\lambda E^p_{1,q} = \mathbb{H}^q(T^p_{\Phi}(X, L, s, t)) \oplus \bigoplus_{i=1}^{r-1} \mathbb{H}^{q-i}(T^p_{\Phi}(Y, L|_Y, s - i, t - i))$$

$$= \begin{cases} \mathbb{H}^p_{\Phi}(X, L \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus \bigoplus_{i=1}^{r-1} \mathbb{H}^{p-i,q-i}_{\Phi}(Y, i_Y^{*}(L \otimes_{\mathcal{O}_X} \mathcal{O}_X)), & s \leq p \leq t \\ 0, & \text{others} \end{cases}$$
of the spectral sequences associated to $K^\bullet$, $L^\bullet$ respectively. Let $t \in \mathcal{A}^1(E)$ be a first Chern form of the universal line bundle $\mathcal{O}_E(−1)$ on $E \cong \mathbb{P}(N_Y/X)$ and let $i_E : E \to X$ be the inclusion. By Theorem 1.1, the morphism

$$(\pi_*, G^{-1}_{t,\Phi} \circ i_E^*, \ldots, G^{-r+1}_{t,\Phi} \circ i_E^*): K^\bullet \to L^\bullet$$

induces an isomorphism $K E^*_{-1,\Phi} \to L E^*_{-1,\Phi}$ at $E_1$-pages, hence induces an isomorphism $H^k(K^\bullet) \to H^k(L^\bullet)$ for any $k$, where $K^\bullet$ and $L^\bullet$ are the associated complex to $K^\bullet$ and $L^\bullet$ respectively. By (4.26) and (4.27), $H^k(K^\bullet) \cong H^k_{\pi-1,\Phi}(X, \Omega^k_{\pi^{-1}}(\mathcal{L}^*))$ and

$$H^k(L^\bullet) \cong H^k_{\Phi}(X, \Omega^k_{\pi^{-1}}(\mathcal{L}^*)) \oplus \bigoplus_{i=1}^{r-1} H^{k-2i}_{\Phi}(Y, \Omega^k_{\pi^{-1}}(\mathcal{L}|_Y)),$$

from which the theorem follows.

\[\square\]

5. Twisted de Rham cohomology with supports

5.1. General results. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathbb{k}$-modules on topological spaces $X$ and $Y$ respectively. The external tensor product of $\mathcal{F}$ and $\mathcal{G}$ is defined as

$$\mathcal{F} \boxtimes \mathcal{G} = pr_1^{-1} \mathcal{F} \otimes_{\mathbb{k}^X \times Y} pr_2^{-1} \mathcal{G},$$

where $pr_1$ and $pr_2$ are projections from $X \times Y$ onto $X$, $Y$, respectively.

**Proposition 5.1.** Let $\mathcal{L}$, $\mathcal{H}$ be local systems of $\mathbb{k}$-modules of finite ranks on smooth manifolds $X$, $Y$ respectively and let $\Phi$ be a family of supports on $X$. Suppose that $H^\bullet(Y, \mathcal{H})$ has finite dimension. Then $(\alpha^p \otimes \beta^q)_{p+q=k} \mapsto \sum_{p+q=k} \alpha^p \otimes \beta^q$ gives an isomorphism

$$\bigoplus_{p+q=k} H^p_{\Phi}(X, \mathcal{L}) \otimes_{\mathbb{k}} H^q(Y, \mathcal{H}) \to H^k_{\Phi \times Y}(X \times Y, \mathcal{L} \boxtimes \mathcal{H}).$$

**Proof.** Let $\mathfrak{U}$ be a good covering of $Y$ such that $U$ is $\mathcal{H}$-constant for any $U \in \mathfrak{U}$. Assume that $V \subseteq X$ is an $\mathcal{L}$-constant open subset of $X$. The cartesian product gives an isomorphism $\Gamma(V, \mathcal{L}) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{H}) \cong \Gamma(V \times U, \mathcal{L} \boxtimes \mathcal{H})$ for any $\mathcal{H}$-constant open set $U \subseteq Y$. Then the Čech complex $\check{C}^\bullet(V \times \mathfrak{U}, \mathbb{C} \boxtimes \mathfrak{H}) = \Gamma(V, \mathcal{L}) \otimes_{\mathbb{C}} \check{C}^\bullet(\mathfrak{U}, \mathcal{H})$. Since $V \times \mathfrak{U}$ and $\mathfrak{U}$ are acyclic with respect to $\mathcal{L} \boxtimes \mathcal{H}$ and $\mathcal{H}$,

$$H^p(V \times Y, \mathcal{L} \boxtimes \mathcal{H}) = H^p(\check{C}^\bullet(V \times \mathfrak{U}, \mathcal{L} \boxtimes \mathcal{H})) = \Gamma(V, \mathcal{L}) \otimes_{\mathbb{C}} H^p(\check{C}^\bullet(\mathfrak{U}, \mathcal{H})) = \Gamma(V, \mathcal{L} \boxtimes \mathcal{H}).$$

So $\mathcal{L} \boxtimes \mathcal{H}$ is the sheaf associated to the presheaf $W \mapsto H^p(W \times Y, \mathcal{L} \boxtimes \mathcal{H})$ for any open set $W \subseteq X$. Namely,

$$R^p\pi_*(\mathcal{L} \boxtimes \mathcal{H}) = \mathcal{L} \boxtimes \mathcal{H}, \quad (5.1)$$

where $\pi : X \times Y \to X$ is the first projection of $X \times Y$ onto $X$.

Define $\Gamma_{\Phi}(X, \mathcal{L}^{[q]}) \otimes_{\mathbb{C}} \Gamma(Y, \mathcal{H}^{[q]}) \to \Gamma_{\Phi}(X, (\pi_*(\mathcal{L} \boxtimes \mathcal{H})^{[q]}))^{[p]}$ as $f \otimes g \mapsto h$, where

$$h((\xi_0, \ldots, \xi_p; (x_0, y_0), \ldots, (x_q, y_q)) = S(f(\xi_0, \ldots, \xi_p))(x_q) \otimes g(y_0, \ldots, y_q)$$

for any $(\xi_0, \ldots, \xi_p; (x_0, y_0), \ldots, (x_q, y_q)) \in X^{p+1} \times (X \times Y)^{q+1}$. Using (5.1) instead of [9, IX. 5.22 (c)], we easily prove this theorem as Theorem 4.1.

\[\square\]

With the similar proof of Theorem 4.2, we have
Theorem 5.2. Let \( \pi : E \to X \) be a smooth fiber bundle over a smooth manifold \( X \) and let \( \mathcal{L} \) be a local system of \( k \)-modules of finite rank on \( X \). Assume that there exist \( e_i \in H^*_{dR}(E, k) \) with degree \( u_i \) for \( 1 \leq i \leq r \) such that their restrictions \( e_i|_{E_x}, \ldots, e_r|_{E_x} \) freely linearly generate \( H^*_{dR}(E_x, k) \) for every \( x \in X \). Then

\[
\sum_{i=1}^{r} \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^{r} H^*_{\Phi} - u_i(X, \mathcal{L}) \to H^*_{\pi - 1\Phi}(E, \pi^{-1}\mathcal{L})
\]

is an isomorphism for any paracompactifying family \( \Phi \) of supports on \( X \).

Let \( \pi : \mathbb{P}(E) \to X \) be the complex projectivization of a complex vector bundle \( E \) of complex rank \( r \) over a smooth manifold \( X \) and let \( t \in \mathcal{A}^2(\mathbb{P}(E)) \) be a Chern form of the universal line bundle \( \mathcal{O}_{\mathbb{P}(E)}(-1) \) over \( \mathbb{P}(E) \). Then \( \mathbb{P}(E) \) is an orientable fiber bundle and \( t^{r-1} \) represents an orientation of \( \mathbb{P}(E) \) (see [14, VII., 7.4] for definitions). Define the pushforward \( \pi_* \) as the integral over the fiber, refer to [14, VII., 7.12]. Notice that \( X \) is not necessarily orientable here. As those in Sect. 4.3, \( \pi_* t^i = 0 \) for \( 0 \leq i \leq r-2 \) and \( \pi_* t^{r-1} = (-1)^{r-1} \). By [14, VII., Proposition X (3)], \( \pi_* : \mathcal{A}^*(\mathbb{P}(E)) \to \mathcal{A}^{*-2r}(X) \) is a morphism of complexes.

Suppose that \( \mathcal{L} \) is a local system of \( k \)-modules of finite rank on \( X \) and \( \Phi \) is a paracompactifying family of supports of \( X \). Analogous to Sect. 3.1.3, we can further define \( \pi_* : \Gamma_{\pi - 1\Phi}(\mathbb{P}(E), \pi^{-1}\mathcal{L} \otimes \mathcal{A}^*_{\mathbb{P}(E)}) \to \Gamma_{\Phi}(X, \mathcal{L} \otimes \mathcal{A}^{*-2r}_X) \). By [14, VII., Proposition IX],

\[
\pi_*(\alpha \wedge \pi^* \beta) = \pi_* \alpha \wedge \beta
\]

for \( \alpha \in \Gamma_{\pi - 1\Phi}(\mathbb{P}(E), \pi^{-1}\mathcal{L} \otimes \mathcal{A}^*_{\mathbb{P}(E)}) \) and \( \beta \in \Gamma_{\Phi}(X, \mathcal{L} \otimes \mathcal{A}^{*-2r}_X) \).

Put \( h = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in H^{2}_d(\mathbb{P}(E), k) \) the first Chern class of \( \mathcal{O}_{\mathbb{P}(E)}(-1) \). Denote by \( \mu_{dR, \Phi} \) the morphism

\[
\sum_{i=0}^{r-1} \pi^*(\bullet) \cup h^i : \bigoplus_{i=0}^{r-1} H^*_{\Phi} - 2i(X, \mathcal{L}) \to H^*_{\pi - 1\Phi}(\mathbb{P}(E), \pi^{-1}\mathcal{L})
\]

and by \( \tau_{dR, \Phi} \) the morphism

\[
(G^0_{h, \Phi}(\bullet), G^1_{h, \Phi}(\bullet), ..., G^{r+1}_{h, \Phi}(\bullet)) : H^*_{\pi - 1\Phi}(\mathbb{P}(E), \pi^{-1}\mathcal{L}) \to \bigoplus_{i=0}^{r-1} H^*_{\Phi} - 2i(X, \mathcal{L})
\]

where

\[
G^{-i}_{h, \Phi}(\bullet) = \sum_{j=0}^{r-i-1} P_{i+j}(\pi_* h^r, ..., \pi_* h^{2r-2}) \cup \pi_*(h^j \cup \bullet) : H^*_{\pi - 1\Phi}(\mathbb{P}(E), \pi^{-1}\mathcal{L}) \to H^*_{\Phi} - 2i(X, \mathcal{L})
\]

for \( 0 \leq i \leq r - 1 \). As Proposition 4.5, we have

Proposition 5.3. \( \mu_{dR, \Phi} \) and \( \tau_{dR, \Phi} \) are inverse isomorphisms.

Now, we prove the self-intersection formula.

Proposition 5.4. Let \( Y \) be an oriented submanifold in an oriented smooth manifold \( X \) with codimension \( r \) and \( \mathcal{L} \) a local system of \( k \)-modules of finite rank on \( X \). Denote by \( i : Y \to X \) the inclusion and by \( [Y] \in H^r_{dR}(X, k) \) the fundamental class of \( Y \) in \( X \). Suppose that \( \Phi \) is a paracompactifying family of supports on \( X \). Then the composite map

\[
H^*_{\Phi|_Y}(Y, \mathcal{L}|_Y) \xrightarrow{i^*} H^*_{\Phi} + r(X, \mathcal{L}) \xrightarrow{i^*} H^*_{\Phi|_Y}(Y, \mathcal{L}|_Y)
\]
is just $[Y]|_Y \cup \bullet$. Moreover, if the normal bundle $N_{Y/X}$ of $Y$ in $X$ has a complex vector bundle structure, then $[Y]|_Y = c_r(N_{Y/X})$ is the $r$-th Chern class of $N_{Y/X}$.

**Proof.** First, we have the following claim.

**Claim 1.** There exist an open neighborhood $U$ of $Y$ in $X$, a smooth map $\tau : U \to Y$ and an open covering $\mathcal{U} = \{U_\alpha\}$ of $U$ satisfying that $L|_{U_\alpha}$ is constant, $U_\alpha \subseteq \alpha := g^{-1}(U_\alpha)$ and $Y \cap \alpha \cap U_\alpha$, where $l : Y \to U$ is the inclusion and $g = l \circ \tau : U \to U$.

Let $N$ be a tubular neighborhood of $Y$ in $X$. Denote by $\tau$ the projection of the vector bundle $N$ onto $Y$ and by $l : Y \to N$ the inclusion. Then $\tau \circ l = id_Y$. For any $y \in Y$, choose an open neighborhood $V_y \subseteq N$ of $y$ such that $L|_{V_y}$ is constant. Set $U_y = \tau^{-1}(V_y \cap Y) \cap V_y$, which is an open neighborhood of $y$. Then

$$l \circ \tau(U_y) \subseteq l(V_y \cap Y) \subseteq V_y \cap \tau^{-1}(V_y \cap Y) = U_y. \quad (5.2)$$

Set $U = \bigcup_{y \in Y} U_y$. Clearly, $Y \subseteq U$. Still denote by $l : Y \to U$ the inclusion and by $\tau : U \to Y$ the projection. Set $g = l \circ \tau : U \to U$. By $(5.2)$, $g(U_y) \subseteq U_y$, i.e., $U_y \subseteq g^{-1}(U_y)$. Evidently, $g \circ l = l$. For any $x \in Y \cap g^{-1}(U_y)$, $x = l(x) = g \circ l(x) = g(x) \in U_y$, so $x \in Y \cap U_y$. Hence $Y \cap g^{-1}(U_y) = Y \cap U_y$. Then $U$, $\tau$ and $\{U_y|y \in Y\}$ satisfy the conditions in this claim.

Now, we choose $U$, $\tau$ and $\mathcal{U}$ as the ones in Claim 1. Denote by $j : U \to X$ the inclusion. Set $\Omega = j^{-1}\Phi \cap \tau^{-1}(\Phi|_Y)$. By Proposition 2.1 (1) (2), $l^{-1}(j^{-1}\Phi) = \Phi|_Y$, $l^{-1}(\tau^{-1}(\Phi|_Y)) = \Phi|_Y$ and $l^{-1}\Omega = l^{-1}(j^{-1}\Phi) \cap l^{-1}(\tau^{-1}(\Phi|_Y)) = \Phi|_Y$. Since $l$ is proper and $l^{-1}(L|_U) = l^{-1}(g^{-1}(L|_U)) = L|_Y$, $l$ induces four pushforwards $l_*$ satisfying the commutative diagrams

$$\begin{align*}
\Gamma_{\Phi|_Y}(Y, \mathcal{L}|_Y \otimes \mathcal{A}_Y^\bullet) & \xrightarrow{l_*} \Gamma_{\Phi|_Y}(U, \mathcal{L}|_U \otimes \mathcal{D}_U^{\bullet+r}) \\
\Gamma_{\Phi|_Y}(Y, \mathcal{L}|_Y \otimes \mathcal{A}_Y^\bullet) & \xrightarrow{l_*} \Gamma_{\Phi|_Y}(U, g^{-1}(\mathcal{L}|_U) \otimes \mathcal{D}_U^{\bullet+r})
\end{align*}$$

where the vertical maps are inclusions. For clearness, the pushforwards in the first, second diagrams are written as $l_*", "l_*$ respectively.

For any $U_\alpha \in \mathcal{U}$, denote by $g_\alpha : W_\alpha = g^{-1}(U_\alpha) \to U_\alpha$ the restriction of $g$. Since $L|_{U_\alpha}$ is a constant local system, $(g^{-1}(\mathcal{L}|_U)|_{U_\alpha}) = g_\alpha^{-1}(\mathcal{L}|_{U_\alpha})$ is constant with the same rank with $L|_{U_\alpha}$. Suppose that $e_1^\alpha, \ldots, e_m^\alpha$ is a basis of $\Gamma(U_\alpha, \mathcal{L})$. Then $g_\alpha^* e_1^\alpha, \ldots, g_\alpha^* e_m^\alpha$ is a basis of $\Gamma(W_\alpha, g^{-1}(\mathcal{L}|_U))$. Let $u \in \Gamma_{\Phi|_Y}(Y, \mathcal{L}|_Y \otimes \mathcal{A}_Y^\bullet)$ be any closed form. Denote by $l_V : Y \cap V \to V$ the restriction of $l$ for any open set $V \subseteq U$. On $Y \cap U_\alpha$, $u = \sum_{i=1}^m l_{U_\alpha}^* e_i^\alpha \otimes u_i^\alpha$ with $u_i^\alpha \in$
\( \Gamma(Y \cap U_\alpha, \mathcal{A}_U^r) \) for \( i = 1, \ldots, m \). Meanwhile, \( u = \sum_{i=1}^{m} l_{W_\alpha}(g^*_\alpha e_i^\alpha) \otimes u_i^\alpha \) on \( Y \cap W_\alpha = Y \cap U_\alpha \), since \( l_{U_\alpha} = g_\alpha \circ l_{W_\alpha} \). Then

\[
{l}_u' = \sum_{i=1}^{m} e_i^\alpha \otimes l_{U_\alpha} u_i^\alpha \quad \text{on} \quad U_\alpha, \\
{l}_u'' = \sum_{i=1}^{m} g^*_\alpha e_i^\alpha \otimes l_{W_\alpha} u_i^\alpha \quad \text{on} \quad W_\alpha.
\]

By Proposition 2.1, \( \Omega \) is paraanalyzing, so we may assume that \( l_u' = v'' + dT'' \) for a closed \( v'' \in \Gamma(\Omega(U, g^{-1}(\mathcal{L}|U) \otimes \mathcal{A}_U^{r+r}) \) and \( T'' \in \Gamma(\Omega(U, g^{-1}(\mathcal{L}|U) \otimes \mathcal{D}_U^{r+r-1}) \). On \( W_\alpha \), set \( v'' = \sum_{i=1}^{m} g^*_\alpha e_i^\alpha \otimes v_i'' \) and \( T'' = \sum_{i=1}^{m} g^*_\alpha e_i^\alpha \otimes T_i'' \), where \( v_i'' \in \Gamma(W_\alpha, \mathcal{A}_U^{r+r}) \) and \( T_i'' \in \Gamma(W_\alpha, \mathcal{D}_U^{r+r-1}) \). Set \( v_i'' = v_i''|_{U_\alpha} \) and \( T_i'' = T_i''|_{U_\alpha} \). Then

**Claim 2.** \( \{ \sum_{i=1}^{m} e_i^\alpha \otimes v_i'' \mid U_\alpha \in \Omega \} \) and \( \{ \sum_{i=1}^{m} e_i^\alpha \otimes T_i'' \mid U_\alpha \in \Omega \} \) respectively piece together to give \( v' \in \Gamma(\Omega(U, \mathcal{L}|U) \otimes \mathcal{A}_U^{r+r}) \) and \( T' \in \Gamma(\Omega(U, \mathcal{L}|U) \otimes \mathcal{D}_U^{r+r-1}) \) satisfying that \( l_u' = v' + dT' \).

Assume that \( e_i^\alpha = \sum_{j=1}^{m} c_{ij} e_j^\beta \) on \( U_\alpha \cap U_\beta \) for the matrix \( (c_{ij}^\alpha)_{1 \leq i,j \leq m} \in GL_m(\mathbb{R}) \). Clearly,

\[
g^*_\alpha e_i^\alpha = \sum_{j=1}^{m} c_{ij}^\alpha g_j^\alpha e_j^\beta \quad \text{on} \quad W_\alpha \cap W_\beta.
\]

Let \( (d_{ij}^\alpha)_{1 \leq i,j \leq m} \) be the inverse matrix of \( (c_{ij}^\alpha)_{1 \leq i,j \leq m} \). Since \( v'' \) is a global form on \( U \), \( v_i'' = \sum_{j=1}^{m} d_{ji}^\alpha v_j'' \) on \( W_\alpha \cap W_\beta \), which implies that \( v_i'' = \sum_{j=1}^{m} d_{ji}^\alpha v_j'' \) on \( U_\alpha \cap U_\beta \). Hence \( \{ \sum_{i=1}^{m} e_i^\alpha \otimes v_i'' \mid U_\alpha \in \Omega \} \) define \( v' \in \Gamma(U, \mathcal{L}|U) \otimes \mathcal{A}_U^{r+r} \) well. Moreover,

\[
\text{supp} v' \cap U_\alpha = \bigcup_{i=1}^{m} \text{supp} v_i'' \cap U_\alpha = \bigcup_{i=1}^{m} (\text{supp} u_i'' \cap U_\alpha) = \text{supp} v'' \cap U_\alpha,
\]

so \( \text{supp} v' = \text{supp} v'' \in \Omega \), i.e., \( v' \in \Gamma(\Omega(U, \mathcal{L}|U) \otimes \mathcal{A}_U^{r+r}) \). Similarly, \( T' \in \Gamma(\Omega(U, \mathcal{L}|U) \otimes \mathcal{D}_U^{r+r-1}) \) is defined well. Since \( U_\alpha \subseteq W_\alpha \), \( l_{U_\alpha} v_i'' = (l_{W_\alpha} v_i''|_{U_\alpha}) \) for any \( i \). Notice that \( l_{U_\alpha} v_i'' = v_i'' + dT_i'' \). So \( l_{U_\alpha} u_i'' = v_i'' + dT_i'' \) for all \( i \) and \( \alpha \), which implies that \( l_u' = v' + dT' \). The claim follows.

By (3.15),

\[
l^r[l_u j-1_\Phi] = l^r[l_u|\Omega], \tag{5.3}
\]

\[
l^r[l_u|\Omega] = l^r[l_u|\Omega]. \tag{5.4}
\]

Since \( l_{U_\alpha} v_i'' = l_{W_\alpha} v_i'' \) and \( l_{U_\alpha} = g_\alpha \circ l_{W_\alpha} \),

\[
l^r v' = \sum_{i=1}^{m} l_{U_\alpha}^r e_i^\alpha \otimes l_{U_\alpha} v_i'' = \sum_{i=1}^{m} l_{W_\alpha} g^*_\alpha e_i^\alpha \otimes l_{W_\alpha} v_i'' = l^r v''
\]
on \( Y \cap U_\alpha = Y \cap W_\alpha \) for all \( \alpha \), hence \( l^r v' = l^r v'' \). So

\[
l^r[l_u|\Omega] = [l^r v']|_Y = [l^r v'']_{|Y} = l^r[l_u|\Omega]. \tag{5.5}
\]

By (5.3)-(5.5),

\[
l^r[l_u j-1_\Phi] = l^r[l_u|\Omega]_{j-1|\Phi}. \tag{5.6}
\]
By (3.13),
\[ l_* u = l_*(l^* \tau^* u) = l_*(1) \wedge \tau^* u \in \Gamma_{\tau-1(\Phi|_Y)}(U, g^{-1}(L|_U) \otimes D_U^{\bullet+r}). \]  
(5.7)
Notice that \( l_*(1) \) is just the current on \( U \) defined by the integral along \( Y \). We have \( [l_*(1)] = [Y]|_U \). Then
\[
l^*[l_* u] _{\tau-1(\Phi|_Y)} = l^*[l_*(1) \wedge \tau^* u] _{\tau-1(\Phi|_Y)} \quad \text{(by (5.7))}
\]
\[
= l^*[l_*(1)] \cup [\tau^* u] _{\tau-1(\Phi|_Y)} \quad \text{(by (5.8))}
\]
Since the support of \( l_* u \) is closed in \( X \), \( j_* l_* u \) is defined well and is just \( i_* u \). By (3.13),
\[
j^* i_* u = l_* u \in \Gamma_{\tau-1(\Phi)}(U, L|_U \otimes D_U^{\bullet+r}). \]  
(5.9)

Then
\[
i^* j^* i_* u|_{\Phi|_Y} = l^*[j^* i_* u]_{\tau-1(\Phi|_Y)} \quad \text{(by (5.9))}
\]
\[
= l^*[l_*(1)] \cup [\tau^* u] _{\tau-1(\Phi|_Y)} \quad \text{(by (5.6) and (5.8))}
\]

Suppose that \( N_{Y/X} \) has a complex vector bundle structure and denote by \( e(N_{Y/X}) \) the Euler class of \( N_{Y/X} \). By [3, (20.10.6)], \( e(N_{Y/X}) = c_r(N_{Y/X}) \) and by [3, Proposition 12.4], \( e(N_{Y/X}) = [Y]|_Y \). So \( c_r(N_{Y/X}) = [Y]|_Y \). We complete the proof. \( \square \)

5.2. Generalized blow-ups. For convenience, still denote by \( X \) the zero section of the bundle \( F \) over \( X \).

5.2.1. Generalized blow-ups. Recall a McDuff’s construction [19, Definition 2.2] with a slight modification as follows: For a submanifold \( Y \) of a smooth manifold \( X \), assume that the normal bundle \( N = N_{Y/X} \) is equipped with a complex vector bundle structure. Let \( U \subseteq N \) be an open or a closed neighbourhood of the zero section \( Y \) of \( N \) and let \( l : U \to X \) be a smooth embedding of \( U \) onto an open or a closed neighbourhood \( W \) of \( Y \) with \( l|_Y = id_Y \). Such \( U \) and \( l \) always exist (but not unique) by the tubular neighborhood theorem. Set \( N_0 = N - Y \) and \( O_{\mathbb{P}(N)}(-1)_0 = O_{\mathbb{P}(N)}(-1) - \mathbb{P}(N) \). There is a commutative diagram
\[
\begin{array}{c}
\mathbb{P}(N)(-1)_0 \injarrow \mathbb{P}(N)(-1) \xrightarrow{\psi} \mathbb{P}(N), \\
N_0 \downarrow \cong \phi \downarrow \tau \xrightarrow{\varphi} N \xrightarrow{\pi} Y
\end{array}
\]
where \( \psi \) and \( \varphi \) are induced by the projections from \( O_{\mathbb{P}(N)}(-1) \subseteq \mathbb{P}(N) \times N \) onto \( \mathbb{P}(N) \) and \( N \) respectively. Set \( \tilde{U} = \varphi^{-1}(U) \subseteq O_{\mathbb{P}(N)}(-1) \). The composition of \( \varphi|_{\tilde{U}} \) and \( l|_{U-Y} \) gives a diffeomorphism \( \phi : \tilde{U} - \mathbb{P}(N) \to U - Y \to W - Y \). Define
\[
\tilde{X} = (X - Y) \cup_{\phi} \tilde{U},
\]
where \( W - Y \subseteq X - Y \) is identified with \( \tilde{U} - \mathbb{P}(N) \subseteq \tilde{U} \) via \( \phi \). Gluing the inclusion \( X - Y \to X \) and \( l \circ \varphi|_{\tilde{U}} : \tilde{U} \to U \to X \) via \( \phi \), we get a smooth map \( \pi : \tilde{X} \to X \). The smooth manifold \( \tilde{X} \) and the map \( \pi : \tilde{X} \to X \) depends on the choices of the complex vector bundle structure of \( N_{Y/X} \) and the embedding \( l : U \to X \). We say that \( \pi : \tilde{X} \to X \) is the generalized blow-up of \( X \) along \( Y \) associated to the complex vector bundle structure of \( N_{Y/X} \) and the embedding.
Let \( U : U \to X \). Moreover, \( E = \pi^{-1}(Y) \) is said to be the exceptional divisor. Clearly, symplectic blow-ups \([19, \text{Definition 2.2}]\) and locally conformal symplectic blow-ups \([36, \text{Definition 3.4}]\) are generalized blow-ups.

The following lemma may be well known for experts and we don’t find the references. For readers’ convenience, we will give a proof.

**Lemma 5.5.** Let \( F \) be a smooth vector bundle over a smooth manifold \( X \). Then the normal bundle \( N_{X/F} \) is isomorphic to \( F \) over \( X \).

**Proof.** Let \( U_\alpha \) be a coordinate chart of \( X \) with a trivialization \( F|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r \). Denote by \( g_{\alpha\beta} \) the translation functions of the vector bundle \( F \) over \( U_\alpha \cap U_\beta \). Suppose that \( x^1_\alpha, \ldots, x^n_\alpha \) and \( v^1_\alpha, \ldots, v^n_\alpha \) are coordinates of \( U_\alpha \) and \( \mathbb{R}^r \) respectively. Then \( (v^1_\mu, \ldots, v^n_\mu) = (v^1_\alpha, \ldots, v^n_\alpha) \cdot g_{\alpha\beta} \). The tangent bundles \( TF|_{U_\alpha} \) and \( TX|_{U_\alpha} \) are linearly generated by \( \frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^n_\alpha} \) and \( \frac{\partial}{\partial v^1_\alpha}, \ldots, \frac{\partial}{\partial v^n_\alpha} \) respectively. Hence \( N_{X/F}|_{U_\alpha} \) is linearly generated by \( \frac{\partial}{\partial v^1_\alpha}, \ldots, \frac{\partial}{\partial v^n_\alpha} \) modulo \( TX|_{U_\alpha} \), which gives a natural trivialization \( N_{X/F}|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r \). Under such trivializations, the transition functions of \( N_{X/F} \) is just \( g_{\alpha\beta} \) on \( U_\alpha \cap U_\beta \). So \( N_{X/F} \cong F \). \( \square \)

**Lemma 5.6.** Let \( F \) be a smooth fiber bundle over an oriented smooth manifold \( X \). Assume that there exists an open covering \( \mathcal{U} \) with trivializations \( h_U : F|_U \to U \times Z \) for all \( U \in \mathcal{U} \) satisfying that:

(i) \( Z \) is a complex manifold

(ii) For any \( U, V \in \mathcal{U} \) and any \( x \in U \cap V \), \( pr_2 \circ h_{UV}(x, \cdot) : Z \to Z \) is holomorphic, where \( h_{UV} = h_V \circ h_U^{-1} : (U \cap V) \times Z \to (V \cap U) \times Z \) and \( pr_2 : (V \cap U) \times Z \to Z \) is the second projection.

Then \( F \) is orientable. In particular, a complex vector bundle \( F \) on an oriented smooth manifold and its complex projectivization \( \mathbb{P}(F) \) are both orientable.

**Proof.** Let \( \{V_\mu | \mu \in I\} \) be an open covering of \( Z \) such that \( V_\mu \) are holomorphic coordinate charts for all \( \mu \in I \). Without loss of generality, assume that \( \mathcal{U} = \{U_\alpha | \alpha \in J\} \) consists of coordinate charts of \( X \). Since \( X \) is orientable, we can choose coordinates \( x^1_\alpha, \ldots, x^n_\alpha \) for \( U_\alpha \) such that \( \det \begin{pmatrix} \frac{\partial x^i_\alpha}{\partial x^j_\alpha} \\
_{1 \leq i \leq n} \\
_{1 \leq j \leq n} \end{pmatrix} > 0 \) on \( U_\alpha \cap U_\beta \) for any \( \alpha, \beta \in J \). Let \( z^1_\mu = u^1_\mu + \sqrt{-1}v^1_\mu, \ldots, z^r_\mu = u^r_\mu + \sqrt{-1}v^r_\mu \) be the holomorphic coordinates of \( V_\mu \). Set \( h_{\alpha\beta} = h_{U_\alpha \cap U_\beta} \). By the assumption (ii), \( \frac{\partial (x^i_\mu h_{\alpha\beta})}{\partial x^j_\alpha} = 0 \), i.e., \( \frac{\partial (v^i_\mu h_{\alpha\beta})}{\partial v^j_\alpha} = \frac{\partial (v^i_\mu)}{\partial v^j_\alpha} \) and \( \frac{\partial (v^i_\mu)}{\partial x^j_\alpha} = -\frac{\partial (v^i_\mu h_{\alpha\beta})}{\partial x^j_\alpha} \). Notice that \( x^i_\alpha \circ h_{\alpha\beta} \) is only dependent on \( x^1_\alpha, \ldots, x^n_\alpha \), hence \( \frac{\partial (x^i_\mu h_{\alpha\beta})}{\partial x^j_\alpha} = 0 \) and \( \frac{\partial (x^i_\mu)}{\partial x^j_\alpha} = 0 \). Consider the coordinate charts \( (h^{-1}_{U_\alpha}(U_\alpha \times V_\mu), x^1_\alpha, \ldots, x^n_\alpha, u^1_\mu, \ldots, u^r_\mu, v^1_\mu, \ldots, v^r_\mu) \) for \( \alpha \in J \),
\[ \mu \in I \text{ of } F. \] The Jacobi of \( h_{\alpha \beta} \) is

\[
\det \begin{pmatrix}
\frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial x^i_{\alpha}} & \frac{\partial (x^i_{\beta} \circ h_{\alpha \beta})}{\partial x^i_{\alpha}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial x^i_{\alpha}} \\
\frac{\partial (x^i_{\beta} \circ h_{\alpha \beta})}{\partial x^i_{\beta}} & \frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial x^i_{\beta}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial x^i_{\beta}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial x^i_{\alpha}} & \frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial x^i_{\beta}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial x^i_{\mu}}
\end{pmatrix}
\begin{pmatrix}
O_{n \times r} \\
O_{n \times r}
\end{pmatrix}
\]

\[
= \det \left( \frac{\partial x^i_{\beta}}{\partial x^i_{\alpha}} \right) \cdot 2^{2r} \cdot \det \begin{pmatrix}
\frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \frac{\partial (x^i_{\beta} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} \\
\frac{\partial (x^i_{\beta} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \frac{\partial (x^i_{\alpha} \circ h_{\alpha \beta})}{\partial z^k_{\mu}} & \cdots & \frac{\partial (x^i_{\mu} \circ h_{\alpha \beta})}{\partial z^k_{\mu}}
\end{pmatrix}
\begin{pmatrix}
O_{r \times r}
\end{pmatrix}
\]

where we use the fact that \( \det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A - iB)|^2 \) for any \( r \times r \) matrices \( A, B \).

Hence \( F \) is orientable. \( \square \)

Now, we consider the properties of generalized blow-ups.

**Proposition 5.7.** Let \( \pi : \tilde{X} \to X \) be a generalized blow-up of \( X \) along \( Y \) associated to a complex vector bundle structure of \( N_{Y/X} \) and an embedding \( l : U \to X \). Denote by \( E \) its exceptional divisor. They satisfy the properties:

1. For any open set \( V \subseteq X \), \( \pi : \pi^{-1}(V) = V \) is the generalized blow-up of \( V \) along \( Y \cap V \) associated to the complex vector bundle structure of \( N_{Y \cap V/V} = N_{Y/X} \) induced by \( N_{Y/X} \) and the embedding \( l' : l^{-1}(V) \to V \) induced by \( l \). In particular, \( \pi|_{\tilde{X} - E} : \tilde{X} - E \to X - Y \) is a diffeomorphism.

2. The map \( \pi \) is surjective and proper.

3. \( E \) is a submanifold of \( \tilde{X} \) with codimension 2 and \( \pi|_{\tilde{E}} : \tilde{E} \to Y \) is diffeomorphic to the complex projectivization \( \mathbb{P}(N_{Y/X}) \) over \( Y \).

4. The normal bundle \( N_{E/\tilde{X}} \) is isomorphic to \( \mathcal{O}_E(-1) \) as smooth vector bundles over \( E = \mathbb{P}(N_{Y/X}) \) and hence \( N_{E/\tilde{X}} \) has a natural complex vector bundle structure induced by \( \mathcal{O}_E(-1) \).

5. Assume that \( U' \subseteq U \) is an open or a closed neighbourhood of the zero section of \( N_{Y/X} \). Let \( \pi' : \tilde{X}' \to X \) be the blow-up of \( X \) along \( Y \) associated to the given complex vector bundle structure of \( N_{Y/X} \) and the embedding \( l|_{U'} : U' \to X \). Then \( \tilde{X}' \) is diffeomorphic to \( \tilde{X} \) over \( X \).

6. If \( X \) and \( Y \) are orientable, then \( \tilde{X} \) and \( E \) are orientable.

**Proof.** Evidently, (1) holds and \( \pi \) is surjective by the definition. Assume that \( N, \varphi, \psi, \phi, \tilde{U} \) and \( \tilde{W} \) are the ones in the definition of generalized blow-ups. Let \( T \subseteq W \) be a tubular neighborhood of \( Y \). Given a metric on the smooth vector bundle \( T \) over \( Y \) and denote by \( T_r, \overline{T}_r \) the open, closed disc bundles with radius \( r \) respectively. Then \( X - T_{1/2} \subseteq X - Y \) and \( \overline{T}_1 \subseteq W \). For any compact set \( K \subseteq X, K \cap (X - T_{1/2}) \) and \( (l \circ \varphi_{\tilde{U}})^{-1}(K \cap \overline{T}_1) \) are compact. Under the quotient map \( (X - Y) \sqcup \tilde{U} \to \tilde{X}, \pi^{-1}(K) \) is the image of \( (K \cap (X - T_{1/2})) \sqcup (l \circ \varphi_{\tilde{U}})^{-1}(K \cap \overline{T}_1) \)
and then is compact. Hence \( \pi \) is proper. We proved (2). By the definition, \( \psi \) restricts to a diffeomorphism \( E = \varphi^{-1}(Y) \to \mathbb{P}(N) \) over \( Y \), which implies (3). By Lemma 5.5,

\[
N_{E/\tilde{X}} = N_{E/U} = N_{\mathbb{P}(N)/\mathbb{P}(N)(-1)} \cong \mathcal{O}_{\mathbb{P}(N)}(-1),
\]

i.e., (4) holds. For (5), gluing the identity \( id : X - Y \to X - Y \) and the inclusion \( \tilde{U}' = \varphi^{-1}(U') \hookrightarrow \tilde{U} \) gives a diffeomorphism between \( \tilde{X}' \) and \( \tilde{X} \) over \( X \). Assume that \( X \) and \( Y \) are orientable. By Lemma 5.6, \( E = \mathbb{P}(N) \) and then \( \mathcal{O}_{\mathbb{P}(N)}(-1) \) are orientable, so is \( \tilde{U} \subseteq \mathcal{O}_{\mathbb{P}(N)}(-1) \). Clearly, \( X - Y \) is orientable. Choose suitable orientations for \( \mathcal{O}_{\mathbb{P}(N)}(-1) \) and \( X - Y \) such that \( \phi : \tilde{U} - \mathbb{P}(N) \to W - Y \) preserves orientations, which gives an orientation on \( \tilde{X} \). We obtain (6). \( \square \)

5.2.2. A class of generalized blow-ups. We recall the blow-up of a holomorphic ideal in a smooth manifold defined in [2]. Denote by \( C_X^\infty \) the sheaf of germs of complex valued smooth functions on a smooth manifold \( X \).

**Definition 5.8.** (1) ([2, Definition 3.1]) Let \( Y \) be a submanifold of a smooth manifold \( X \) with codimension \( 2r \) for \( r \geq 1 \). A **holomorphic ideal** for \( Y \) in \( X \) is an ideal sheaf \( I_Y \subseteq C_X^\infty \) satisfying that:

(i) \( I_Y|_{X - Y} = C_X^\infty|_{X - Y} \).

(ii) For any \( y \in Y \), there exists an open neighborhood \( U \) of \( y \) and \( z_1, \ldots, z_r \in I_Y(U) \), such that \( I_Y|_U \) is generated by \( z_1, \ldots, z_r \) and \( z = (z_1, \ldots, z_r) : U \to \mathbb{C}^r \) is a submersion with \( z^{-1}(a) = Y \cap U \).

(2) ([2, Definition 3.4]) A **divisor** on a smooth manifold \( X \) is an ideal sheaf \( I \subseteq C_X^\infty \) which can be locally generated by a single function and whose zero set is nowhere dense in \( X \).

Suppose that \( f : Y \to X \) is a smooth map of smooth manifolds and \( I \) is an ideal sheaf of \( C_X^\infty \). Denote by \( f^{-1}I \cdot C_Y^\infty \subseteq C_Y^\infty \) the image of the natural morphism \( f^*I = f^{-1}I \otimes f^{-1}C_X^\infty \to f^{-1}C_X^\infty \otimes f^{-1}C_X^\infty \cong C_Y^\infty \) induced by the inclusion \( I \hookrightarrow C_X^\infty \). For the inclusion \( j : U \hookrightarrow X \) of an open set \( U \), \( j^{-1}I \cdot C_U^\infty = I|_U \). Assume that \( g : Z \to Y \) is a smooth map of smooth manifolds. By \( g \circ f \) being an isomorphism of sheaves on \( X \), we easily get

\[
g^{-1}(f^{-1}I \cdot C_Y^\infty) \cdot C_Z^\infty \cong (f \circ g)^{-1}I \cdot C_Z^\infty. \tag{5.11}
\]

**Definition 5.9** ([2, Definition 3.6]). Let \( I_Y \) be a holomorphic ideal for a submanifold \( Y \) in a smooth manifold \( X \). The **blow-up** of \( I_Y \) in \( X \) is defined as a smooth map \( \pi : \tilde{X} \to X \) between smooth manifolds such that \( \pi^{-1}I_Y \cdot C_{\tilde{X}}^\infty \) is a divisor on \( \tilde{X} \) and the following universal property holds: For any smooth map \( f : Z \to X \) such that \( f^{-1}I_Y \cdot C_Z^\infty \) is a divisor, there is a unique smooth map \( g : Z \to \tilde{X} \) such that \( \pi \circ g = f \).

**Theorem 5.10** ([2, Theorem 3.7]). Given a holomorphic ideal \( I_Y \) for a submanifold \( Y \) in a smooth manifold \( X \), there exists a unique blow-up \( \pi : \tilde{X} \to X \) of \( I_Y \) in \( X \) up to unique isomorphism.

We have the following properties.

**Proposition 5.11.** Suppose that \( X \) is a smooth manifold and \( I_Y \subseteq C_X^\infty \) is a holomorphic ideal for a submanifold \( Y \) in \( X \).
(1) Let $\pi : \tilde{X} \to X$ be the blow-up of $\mathcal{I}_Y$ in $X$.

(i) For any open set $U \subseteq X$, $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is the blow-up of $\mathcal{I}_Y|_U$ in $U$.

(ii) Assume that $Z$ is a smooth manifold and $\mathcal{I}_X \to Z \times X \to X$ is the second projection. Then $\mathcal{I}_X \to Z \times X$ is a holomorphic ideal for $Z \times Y \times Z \times X$ and $id_Z \times \pi : Z \times \tilde{X} \to Z \times X$ is the blow-up of $\mathcal{I}_X \to Z \times X$.

(2) Suppose that $\pi : \tilde{X} \to X$ is a smooth map and $\mathcal{I}_X$ is an open covering of $X$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is the blow-up of $\mathcal{I}_Y|_U$ in $U$ for any $U \in \mathcal{I}_X$. Then $\pi : \tilde{X} \to X$ is the blow-up of $\mathcal{I}_Y$ in $X$.

Proof. (1) (i) Evidently, $(\pi|_{\pi^{-1}(U)})^{-1}(\mathcal{I}_Y|_U) \cdot C_{X,\pi^{-1}(U)} = (\mathcal{I}_Y \cdot C_{\tilde{X}})|_{\pi^{-1}(U)}$ is a divisor on $\pi^{-1}(U)$. Suppose that $j_U : U \to X$ and $\tilde{j}_U : \pi^{-1}(U) \to \tilde{X}$ are the inclusions. Let $f : Z \to U$ be any smooth map of smooth manifolds such that $f^{-1}(\mathcal{I}_Y|_U) \cdot C_{\tilde{X}}$ is a divisor on $Z$. By (5.11), $(j_U \circ f) (\mathcal{I}_Y \cdot C_{\tilde{X}}) = f^{-1}(\mathcal{I}_Y|_U) \cdot C_{\tilde{X}}$ is a divisor on $Z$. By the universal property of the blow-up $\pi$, there is a smooth map $\tilde{f}_1 : Z \to \tilde{X}$ such that $\tilde{j} \circ \tilde{f}_1 = j_U \circ f$. So $\tilde{f}_1(Z) \subseteq \pi^{-1}(U)$. Then $\tilde{f}_1$ induces a smooth map $\tilde{f} : Z \to \pi^{-1}(U)$, i.e., $\tilde{j}_U \circ \tilde{f} = \tilde{f}_1$. Since $j_U$ is injective, $\pi|_{\pi^{-1}(U)} \circ \tilde{f} = f$. Let $g : Z \to \pi^{-1}(U)$ be another smooth map such that $\pi|_{\pi^{-1}(U)} \circ g = \tilde{f}$. Then $\pi \circ (j_U \circ g) = j_U \circ f$. By the universal property of the blow-up $\pi$, $j_U \circ g = \tilde{f}_1$, so $g = \tilde{f}$.

(ii) By (5.11), $(pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi})|_{Z \times X, \pi^{-1}(Y)} = pr_2^{-1}(\mathcal{I}_X \cdot C_{Z,\pi})|_{Z \times X}$ for any $(z, y) \in Z \times Y$, there exists an open neighborhood $U$ of $y$ and $u_1, \ldots, u_r \in \mathcal{I}_Y(U)$ such that $pr_2^{-1}(\mathcal{I}_X \cdot C_{Z,\pi})|_{Z \times X, \pi^{-1}(Y)} = \langle u_1 \ldots, u_r \rangle : U \to \mathbb{C}^r$ is a submersion with $u^{-1}(o) = Y \cap U$. Set $v_i = pr_2^{-1}u_i \in C_{Z,\pi}(Z \times U)$. Then $(pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi})|_{Z \times X, \pi^{-1}(Y)} = \langle v_1, \ldots, v_r \rangle : Z \times U \to \mathbb{C}^r$ is a submersion with $u^{-1}(o) = Z \times (Y \cap U)$. So $pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi}$ is a holomorphic ideal for $Z \times Y \times Z \times X$. Let $pr_1^\prime$ and $pr_2^\prime$ be the projections from $Z \times X$ onto $Z$ and $\tilde{X}$ respectively. Since $\pi^{-1}(\mathcal{I}_Y \cdot C_{Z,\pi})$ is a divisor on $\tilde{X}$, so is $(id_Z \times \pi)^{-1}(pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi}) \cdot C_{Z,\tilde{X}} \cdot C_{Z,\tilde{X}} = pr_2^{-1}(\pi^{-1}(\mathcal{I}_Y \cdot C_{Z,\pi}) \cdot C_{Z,\tilde{X}})$ on $Z \times \tilde{X}$. Let $f : M \to Z \times X$ be any smooth map of smooth manifolds such that $f^{-1}(pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi}) \cdot C_M$ is a divisor on $M$. Then $(pr_2 \circ f) (\mathcal{I}_Y \cdot C_{Z,\pi}) = f^{-1}(pr_2^{-1} \mathcal{I}_Y \cdot C_{Z,\pi}) \cdot C_M$ is a divisor on $M$. By the universal property of the blow-up $\pi$, there is a unique smooth map $\tilde{f}_1 : M \to \tilde{X}$ such that $\pi \circ \tilde{f}_1 = pr_2 \circ f$. Define $\tilde{f} : M \to Z \times \tilde{X}$ as $m \mapsto (pr_1 \circ f(m), \tilde{f}_1(m))$. Clearly, $(id_Z \times \pi) \circ \tilde{f} = f$. Suppose that $g : M \to Z \times \tilde{X}$ is another smooth map satisfying that $(id_Z \times \pi) \circ g = f$. Then $\pi \circ (pr_2 \circ g) = pr_2 \circ (id_Z \times \pi) \circ g = pr_2 \circ f = \pi \circ \tilde{f}_1$. So $pr_2 \circ g = \tilde{f}_1$ by the universal property of the blow-up $\pi$. Moreover, $pr_1^\prime \circ g = pr_1 \circ (id_Z \times \pi) \circ g = pr_1 \circ f$. Hence $g = \tilde{f}$.

(2) By the assumption, $(\pi^{-1}(\mathcal{I}_Y \cdot C_{\tilde{X}})|_{\pi^{-1}(U)} = (\pi|_{\pi^{-1}(U)})^{-1}(\mathcal{I}_Y|_U) \cdot C_{\pi^{-1}(U)}$ is a divisor on $\pi^{-1}(U)$ for any $U \in \mathcal{I}_X$, so is $\pi^{-1}(\mathcal{I}_Y \cdot C_{\tilde{X}})$ on $\tilde{X}$. Let $f : Z \to X$ be any smooth map of smooth manifolds such that $f^{-1} \mathcal{I}_Y \cdot C_{\tilde{X}}$ is a divisor on $\tilde{X}$. By (5.11), $(f|_{f^{-1}(U)})^{-1}(\mathcal{I}_Y|_U) \cdot C_{f^{-1}(U)} = (f^{-1} \mathcal{I}_Y \cdot C_{f^{-1}(U)})|_{f^{-1}(U)}$ is a divisor on $f^{-1}(U)$. By the universal property of the blow-up $\pi|_{\pi^{-1}(U)}$, there is a unique smooth map $\tilde{f}_1 : f^{-1}(U) \to \pi^{-1}(U)$ such that $\pi|_{\pi^{-1}(U)} \circ \tilde{f}_1 = f|_{f^{-1}(U)}$. For any $U, \mathcal{I}_X \in \mathcal{I}_X$, $\pi|_{\pi^{-1}(U \cap V)} \circ \tilde{f}_1|_{f^{-1}(U \cap V)} = f|_{f^{-1}(U \cap V)}$ and $\pi|_{\pi^{-1}(U \cap V)} \circ \tilde{f}_1|_{f^{-1}(U \cap V)} \circ f|_{f^{-1}(U \cap V)} = f|_{f^{-1}(U \cap V)}$. By (1) (i), $\pi|_{\pi^{-1}(U \cap V)} : \pi^{-1}(U \cap V) \to U \cap V$ is the blow-up of $\mathcal{I}_Y|_{U \cap V}$ in $U \cap V$. Notice that $(f|_{f^{-1}(U \cap V)})^{-1}(\mathcal{I}_Y|_{U \cap V}) \cdot C_{f^{-1}(U \cap V)} = (f^{-1} \mathcal{I}_Y \cdot C_{\tilde{X}})|_{f^{-1}(U \cap V)}$ is a divisor on $f^{-1}(U \cap V)$.}
\[ \pi \circ g = f. \] By the universal property of the blow-up \( \pi|_{x^{-1}(U)}, g|_{f^{-1}(U)} = \tilde{f}_U \) for any \( U \in \mathfrak{U} \), which implies that \( g = \tilde{f}. \]

M. Bailey, G. Cavalcanti and J. van der Leer Durán [2, p. 2114] defined a canonical holomorphic ideal for the zero section in the normal bundle \( N_Y/X \). We generalize their definition on general complex vector bundles. Suppose that \( F \) is a complex vector bundle over a smooth manifold \( X \). A canonical holomorphic ideal \( \mathcal{I}_{0,F} \) for the zero section \( X \) in \( F \) is constructed as follows: Suppose that \( U \subseteq X \) is an open set with a trivialization \( \phi_U : F|_U \cong U \times \mathbb{C}^r \). Let \( z_1, \ldots, z_r \) be the canonical holomorphic coordinates of \( \mathbb{C}^r \). For every \( i, \) \( z_i \) can be viewed as a complex valued smooth function on \( F|_U \) via the projection \( U \times \mathbb{C}^r \to \mathbb{C}^r \). Set \( \mathcal{I}_U = \langle z_1, \ldots, z_r \rangle \subseteq C^\infty_F|_U \). On such two charts \( U \) and \( V \), the generators of \( \mathcal{I}_U \) and the ones of \( \mathcal{I}_V \) defined as above can represent each other via a translation matrix \( \phi_V \circ \phi_U^{-1} \) of \( F \) over \( U \cap V \). So \( \mathcal{I}_U|_{U \cap V} = \mathcal{I}_V|_{U \cap V} \). Hence the definition of \( \mathcal{I}_U \) is independent of the choice of the trivialization \( \phi_U \) and \( \{\mathcal{I}_U\} \) define an ideal \( \mathcal{I}_{0,F} \subseteq C^\infty_F \) such that \( \mathcal{I}_{0,F}|_{F|_U} = \mathcal{I}_U \). Clearly, \( \mathcal{I}_{0,F} \) is a holomorphic ideal for the zero section \( X \).

**Proposition 5.12.** Let \( F \) be a complex vector bundle over a smooth manifold \( X \).

1. For any open set \( U \subseteq X \), \( \mathcal{I}_{0,F}|_{F|_U} = \mathcal{I}_{0,F}|_{F|_U} \).
2. If \( F \) is a complex line bundle, \( \mathcal{I}_{0,F} \) is a divisor on \( F \).
3. Suppose that \( Y \) is a smooth manifold and \( \text{pr}_2 : Y \times X \to X \) is the second projection. Denote by \( \text{pr}_2 \) the second projection of \( \text{pr}_2^*F = Y \times X \to F \). Then
   \[
   \text{pr}_2^{-1}\mathcal{I}_{0,F} \cdot C^\infty_{\text{pr}_2^*F} = \mathcal{I}_{0,\text{pr}_2^*F}. \tag{5.12}
   \]
4. The blow-up of \( \mathcal{I}_{0,F} \) in \( F \) is the projection \( \varphi : \mathcal{O}_{\mathbb{P}(F)}(1) \to F \). Moreover, \( \varphi^{-1}\mathcal{I}_{0,F} \cdot C^\infty_{\mathcal{O}_{\mathbb{P}(F)}(1)} = \mathcal{I}_{0,\mathcal{O}_{\mathbb{P}(F)}(1)}. \)

**Proof.** Clearly, (1) and (2) hold by the definition of \( \mathcal{I}_{0,F} \). Let \( U \subseteq X \) be any open set with a trivialization \( \phi_U : F|_U \to U \times \mathbb{C}^r \). There is the trivialization \( \text{id}_Y \times \phi_U : (\text{pr}_2^*F)|_{Y \times U} \to Y \times U \times \mathbb{C}^r \). Denote by \( \text{pr}_2^\prime : U \times \mathbb{C}^r \to \mathbb{C}^r \) the second projection and by \( \text{pr}_3^\prime : Y \times U \times \mathbb{C}^r \to \mathbb{C}^r \) the third projection. Let \( z_1, \ldots, z_r \) be the canonical holomorphic coordinates of \( \mathbb{C}^r \). Then \( \mathcal{I}_{0,F}|_{F|_U} = \langle z_1, \ldots, z_r \rangle \) and \( (\mathcal{I}_{0,\text{pr}_2^\prime}|_{F|_U} \cap \mathcal{I}_{0,\text{pr}_3^\prime}|_{F|_Y \times U})(z_i) = \langle z_1, \ldots, z_r \rangle \), where \( z_i \) is viewed as the complex valued smooth function on \( F|_U \) via \( \text{pr}_2^\prime \) and on \( \text{pr}_2^\prime|_{Y \times U} \) and \( \text{pr}_3^\prime \) respectively for \( 1 \leq i \leq r \). Evidently, (5.12) holds on \( (\text{pr}_2^*F)|_{Y \times U} \), so does (5.12) on \( \text{pr}_2^*F \). We proved (3). Obviously, \( \langle z_1, \ldots, z_r \rangle \subseteq C^\infty_F \) is a holomorphic ideal for the original point \( o \) in \( \mathbb{C}^r \).

By [33, Proposition 2.3.1], the blow-up of \( \langle z_1, \ldots, z_r \rangle \) in \( \mathbb{C}^r \) is just the complex blow-up \( \varphi_o : \mathcal{O}_{\mathbb{C}^{pr-1}}(-1) \to \mathcal{O}_{\mathcal{C}^{pr-1}}(1) \) along \( o \), which is naturally induced by the projection \( \mathbb{C}^{pr-1} \to \mathbb{C}^r \to \mathbb{C}^r \). As we know, \( \mathcal{O}_{\mathbb{C}^{pr-1}}(-1) \) consists of \( \{(w_1, \ldots, w_r), (z_1, \ldots, z_r)\} \in \mathbb{C}^{pr-1} \times \mathbb{C}^r \) such that \( w_i z_j = w_j z_i \) for \( 1 \leq i, j \leq r \). Set \( U_i = \{[w_1, \ldots, w_r] \in \mathbb{C}^{pr-1}|w_i \neq 0\} \) for \( 1 \leq i \leq r \). Then \( ([w_1, \ldots, w_r], (z_1, \ldots, z_r)) \to (u_1, \ldots, \hat{u_i}, \ldots, u_r, t) = \left( \frac{w_1}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_r}{w_i}, z_i \right) \) gives a trivialization \( \mathcal{O}_{\mathbb{C}^{pr-1}}(-1)|_{U_i} \to U_i \times \mathbb{C}^r \), where \( (u_1, \ldots, \hat{u_i}, \ldots, u_r) \) is the coordinates of \( U_i \cong \mathbb{C}^{r-1} \). Then
   \[
   \left( \varphi^{-1}_o(z_1, \ldots, z_r) \cdot C^\infty_{\mathcal{O}_{\mathbb{C}^{pr-1}}(-1)} \right)|_{\mathcal{O}_{\mathbb{C}^{pr-1}}(-1)|_{U_i}} = \langle u_1 t, \ldots, \hat{u_i} t, \ldots, u_r, t \rangle = (t). \tag{5.13}
   \]

By the definition,
   \[
   \varphi^{-1}(z_1, \ldots, z_r) \cdot C^\infty_{\mathcal{O}_{\mathbb{C}^{pr-1}}(-1)} = \mathcal{I}_{0,\mathcal{O}_{\mathbb{C}^{pr-1}}(-1)}.
   \]
By (2),
\[ p^n_{2}^{-1}(z_1, \ldots, z_r) \cdot C^n_{\mathbb{C}^r} = \mathcal{I}_{0, U \times \mathbb{C}^r}. \] (5.14)

Hence \( id_U \times \varphi_o : \mathcal{O}_{U \times \mathbb{C}^{p-1}}(-1) = U \times \mathcal{O}_{\mathbb{C}^{p-1}}(-1) \rightarrow U \times \mathbb{C}^r \) is the blow-up of \( \mathcal{I}_{0, U \times \mathbb{C}^r} \) in \( U \times \mathbb{C}^r \) by Proposition 5.11 (1) (ii). Denote by \( p^n_{2} : U \times \mathcal{O}_{\mathbb{C}^{p-1}}(-1) \rightarrow \mathcal{O}_{\mathbb{C}^{p-1}}(-1) \) the second projection. Then
\[
(id_U \times \varphi_o)^{-1} \mathcal{I}_{0, U \times \mathbb{C}^r} \cdot C^n_{\mathbb{C}^r} = p^n_{2}^{-1}(z_1, \ldots, z_r) \cdot C^n_{\mathbb{C}^r} \] (by (5.14))
\[
= p^n_{2}^{-1} \left( \varphi_o^{-1}(z_1, \ldots, z_r) \cdot C^n_{\mathbb{C}^r} \right) \quad (by \ (5.11))
\[
= p^n_{2}^{-1} \mathcal{I}_{0, \mathcal{O}_{\mathbb{C}^{p-1}}(-1)} \cdot C^n_{\mathbb{C}^r} \quad (by \ (5.13))
\[
= \mathcal{I}_{0, \mathcal{O}_{\mathbb{C}^{p-1}}(-1)} \quad (by \ (5.12)).
\]

Via the diffeomorphism \( \phi_U, \varphi : \mathcal{O}_{\mathcal{O}(F|U)}(-1) : \mathcal{O}_{\mathcal{O}(F|U)}(-1) \rightarrow F|U \) is the blow-up of \( \mathcal{I}_{0, F|U} = \mathcal{I}_{0, F|U} \) in \( F|U \) with \( \left( \varphi \mathcal{O}(F|U)(-1) \right)^{-1} \mathcal{I}_{0, F|U} \cdot C^n_{\mathcal{O}(F|U)(-1)} = \mathcal{I}_{0, \mathcal{O}(F|U)(-1)} \). By Proposition 5.11 (2), we get (4).

If \( I_Y \) is a holomorphic ideal for \( Y \), then the complexification of the conormal bundle of \( Y \) in \( X \) has the decomposition \( N^*_Y/X \otimes_R \mathbb{C} = N^*_{r, 0} \oplus N^*_{r, 1} \), where \( N^*_{r, 0} = \{(df)_{y} \mid f \in I_Y \} \) and \( N^*_{r, 1} = \{(df)_{y} \mid f \in I_Y \} \) for any \( y \in Y \). Then \( N^*_{r, 1} \) gives a complex vector bundle structure on \( N^*_{Y/X} \) via the isomorphism \( N^*_{Y/X} \cong N^*_{r, 1} \) of smooth vector bundles. Hence the normal bundle \( N^*_{Y/X} \) has a complex vector bundle structure, which is said to be the complex vector bundle structure of \( N^*_{Y/X} \) induced by \( I_Y \).

For complex vector bundles, Lemma 5.5 is strengthen as follows.

**Lemma 5.13.** Let \( F \) be a complex vector bundle over a smooth manifold \( X \). Equip \( N_{X/F} \) with the complex vector bundle structure induced by \( I_{0,F} \). Then there exists an isomorphism \( N_{X/F} \rightarrow F \) of complex vector bundles over \( X \).

**Proof.** Let \( \phi_U : F|U \rightarrow U \times \mathbb{C}^r \) be a trivialization of \( F \) over an open set \( U \subseteq X \) given by the sections \( e_1, \ldots, e_r \) of \( F|U \), i.e., \( \phi_U \left( \sum_{i=1}^{r} z_i \cdot e_i(x) \right) = (x, z_1, \ldots, z_r) \). For the complex vector bundle structure induced by \( I_{0,F} \), \( N^*_{X/F}|U = \{(df) \mid f \in I_Y \} = \text{span}_{\mathbb{C}} \{dz_1, \ldots, dz_r \} \), where \( z_1, \ldots, z_r \) are viewed as complex valued smooth functions on \( F|U \). Define \( \Psi_U : N^*_{X/F}|U \rightarrow F|U \) as \( \sum_{i=1}^{r} a_i \cdot \frac{\partial}{\partial z_i} \) modulo \( TX|U \rightarrow \sum_{i=1}^{r} a_i \cdot e_i(x) \), which is an isomorphism of complex vector bundles over \( U \). Assume that \( f_1, \ldots, f_r \) give a trivialization \( \Psi_V \) of \( F \) over an open set \( V \subseteq X \) and \( e_i = \sum_{j=1}^{r} g_{ij} \cdot f_j \) with \( g_{ij} \in C^\infty(U \cap V) \) for \( 1 \leq i, j \leq r \) on \( U \cap V \). Then
\[ (x, w_1, \ldots, w_r) = \phi_V \circ \phi^{-1}_U (x, z_1, \ldots, z_r) = (x, \sum_{i=1}^{r} z_i g_{1i}(x), \ldots, \sum_{i=1}^{r} z_i g_{ri}(x)). \]

So \( \sum_{i=1}^{r} a_i \cdot \frac{\partial}{\partial z_i} = \sum_{i,j=1}^{r} a_i g_{ij} \cdot \frac{\partial}{\partial z_j} \) modulo \( TX|U \cap V \), by which we easily check that \( \{\Psi_U\} \) give a global isomorphism \( N_{X/F} \rightarrow F \) of complex vector bundles over \( X \).
Example 5.14. (1) Let $Y$ be a complex submanifold of a complex manifold $X$. Denote by $\mathcal{I}_Y \subseteq \mathcal{C}_X^\infty$ the ideal generated by the holomorphic functions which vanish on $Y$. Clearly, $\mathcal{I}_Y$ is a holomorphic ideal for $Y$. The blow-up of $\mathcal{I}_Y$ in $X$ is just the classical complex blow-up of $X$ along $Y$, see [33, Proposition 2.3.1].

(2) Let $Y$ be a generalized Poisson submanifold of a generalized complex manifold $X$. There exists an canonical holomorphic ideal $\mathcal{I}_Y$ for $Y$ ([2, Proposition 3.12]) and we can get the blow-up $\tilde{X}$ of $\mathcal{I}_Y$ in $X$. Under suitable conditions, $\tilde{X}$ has a generalized complex structure ([2, Theorem 3.16]).

(3) Suppose that $Y$ is a submanifold of $X$ such that normal bundle $N_{Y/X}$ has a complex vector bundle structure and we choose a tubular embedding $l : N_{Y/X} \to X$. Let $\mathcal{I}_{Y,W} \subseteq \mathcal{C}_X^\infty$ be the ideal on $W = l(N_{Y/X})$ satisfying that $\mathcal{I}_{0,N_{Y/X}} = l'^{-1}\mathcal{I}_{Y,W}$, where $l' : N_{Y/X} \to W$ is the diffeomorphism induced by $l$. Since $\mathcal{I}_{Y,W}|_{W-Y} = \mathcal{C}_W^\infty|_{Y}$, we obtain a sheaf $\mathcal{I}_Y \subseteq \mathcal{C}_X^\infty$ by gluing $\mathcal{I}_{Y,W}$ and $\mathcal{C}_X^\infty|_{Y}$. Clearly, $\mathcal{I}_Y$ is a holomorphic ideal for $Y$ satisfying $l^{-1}\mathcal{I}_Y = \mathcal{I}_{0,N_{Y/X}}$. Then the complex vector bundle structure of $N_{Y/X}$ induced by $\mathcal{I}_Y$ coincides with the one of $N_{Y/N_{Y/X}}$ induced by $\mathcal{I}_{0,N_{Y/X}}$ via the diffeomorphism $l'$, hence is just the given one by Lemma 5.13. Moreover, if $Y$ is compact, such holomorphic ideal is unique up to (non-canonical) diffeomorphism by [2, Corollary 3.3]. In such way, the blow-up of a compact generalized Poisson transversal $Y$ in a generalized complex manifold $X$ is defined well, which carries a generalized complex structure, see [2, Theorem 3.34].

Proposition 5.15. The blow-up of a holomorphic ideal for $Y$ in $X$ is a generalized blow-up of $X$ along $Y$.

Proof. Let $\mathcal{I}_Y$ be a holomorphic ideal for $Y$ and let $\pi : \tilde{X} \to X$ be the blow-up of $\mathcal{I}_Y$ in $X$. Then $N = N_{Y/X}$ has the complex vector bundle structure induced by $\mathcal{I}_Y$. By [2, Proposition 3.2], there is a tubular embedding $l : N \to X$ such that $l^{-1}\mathcal{I}_Y = \mathcal{I}_{0,N}$. Set $W = l(N)$. We use the notations in the diagram (5.10). Let $\pi' : \tilde{X'} = (X - Y) \cup_{\phi} \mathcal{O}_{P(N)}(-1) \to X$ be the blow-up of $X$ along $Y$ associated to the complex vector bundle structure of $N$ induced by $\mathcal{I}_Y$ and the embedding $l$. Denote by $[x]$ the class in $\tilde{X'}$ of $x \in (X - Y) \cup \mathcal{O}_{P(N)}(-1)$. Then $\pi'([x]) = x$ for $x \in X - Y$ and $\pi'([x]) = l(x)$ for $x \in \mathcal{O}_{P(N)}(-1)$. Let $l' : N \to W$ be the diffeomorphism induced by $l$. Since $l'^{-1}(\mathcal{I}_Y|_W) = \mathcal{I}_{0,N}$, $l' \circ \varphi : \mathcal{O}_{P(N)}(-1) \to W$ is the blow-up of $\mathcal{I}_Y|_W$ in $W$ with $(l' \circ \varphi)^{-1}(\mathcal{I}_Y|_W) : \mathcal{C}_W^\infty|_{\mathcal{O}_{P(N)}(-1)} = \mathcal{I}_{0,N} \cap \mathcal{O}_{P(N)}(-1)$ by Proposition 5.12 (4). By Proposition 5.11 (1) (i), $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$ is the blow-up of $\mathcal{I}_Y|_W$ in $W$. By Theorem 5.10, there exists a diffeomorphism $a : \pi^{-1}(W) \to \mathcal{O}_{P(N)}(-1)$ such that $(l' \circ \varphi) \circ a = a|_{\pi^{-1}(W)}$. Define $f : \tilde{X} \to \tilde{X'}$ as $f(\tilde{x}) = [\pi(\tilde{x})]$ for $\tilde{x} \in \pi^{-1}(X - Y)$ and $f(\tilde{x}) = [a(\tilde{x})]$ for $\tilde{x} \in \pi^{-1}(W)$. Obviously, $f$ is defined well and $\pi' \circ f = \pi$. The restrictions $f : f^{-1}(\pi'^{-1}(X - Y)) = \pi^{-1}(X - Y) \to \pi'^{-1}(X - Y)$ and $f : f^{-1}(\pi'^{-1}(W)) = \pi^{-1}(W) \to \pi'^{-1}(W)$ are induced by diffeomorphisms $\pi^{-1}(X - Y) \to X - Y$ and $a$ respectively, hence they are both diffeomorphisms. Then $f$ is a diffeomorphism over $X$. So $\pi : \tilde{X} \to X$ is a generalized blow-up.

\[5.23. \text{A proof of Theorem 1.2.}\] By Proposition 5.7 (6), $\tilde{X}$ and $E$ are oriented, so the push-forward $i_{E*}$ is defined well. Set $U = X - Y$ and $\tilde{U} = \tilde{X} - E$. By Proposition 5.7 (1), $\pi|_{\tilde{U}} : \tilde{U} \to U$ is a diffeomorphism. By Proposition 2.3, there is a commutative diagram of
long exact sequences
\[ \cdots H^k_{\Phi|i|}(U, L) \longrightarrow H^k_{\Phi}(X, L) \longrightarrow i^*_{Y} \longrightarrow H^k_{\Phi|i|}(Y, L|_{Y}) \longrightarrow H^{k+1}_{\Phi}(U, L) \cdots \]
\[ \cong (\pi_{|i|})^* \]
\[ \cdots H^k_{(\pi^{-1}-\Phi)|}(U, \pi^{-1}L) \longrightarrow H^k_{\pi^{-1}-\Phi}(\tilde{X}, \pi^{-1}L) \longrightarrow i^*_{E} \longrightarrow H^k_{(\pi^{-1}-\Phi)|}(E, (\pi^{-1}L)|_{E}) \longrightarrow H^{k+1}_{(\pi^{-1}-\Phi)|}(U, \pi^{-1}L) \cdots . \]

By (3.14), \( \pi_{*}\pi^* = id \), so \( \pi^* \) is injective. By the snake lemma, \( i^*_{E} \) induces an isomorphism \( \text{coker}(\pi^*) \cong \text{coker}(\pi|E|^*) \). We get a commutative diagram of short exact sequences
\[ \begin{array}{c}
0 \\
\longrightarrow H^k_{\Phi}(X, L) \overset{\pi^*}{\longrightarrow} H^k_{\pi^{-1}-\Phi}(\tilde{X}, \pi^{-1}L) \overset{k}{\longrightarrow} \text{coker}(\pi^*) \longrightarrow 0 \\
\end{array} \quad \text{(5.15)} \]
\[ \begin{array}{c}
0 \\
\longrightarrow H^k_{\Phi|i|}(Y, i^{-1}_Y L) \overset{(\pi|E|^*)}{\longrightarrow} H^k_{(\pi^{-1}-\Phi)|}(E, i^{-1}_E \pi^{-1}L) \overset{\cong}{\longrightarrow} \text{coker}(\pi|E|^*) \longrightarrow 0 .
\end{array} \]

By Proposition 5.7 (4), \( c_1(N_{E|X}) = h \), hence \( i^*_{E}i_{E*}(\bullet) = h \cup \bullet \) on \( H^*_{(\pi^{-1}-\Phi)|}(E, i^{-1}_E \pi^{-1}L) \) by Proposition 5.4. With the similar proofs of Lemma 4.8 and Proposition 4.11, we easily show that (1.4) and (1.6) are inverse isomorphisms. We prove Theorem 1.2.

Proposition 5.16. Let \( \pi : \tilde{X} \rightarrow X \) be a proper, surjective, preserving orientations, smooth map of oriented smooth manifolds with degree one and let \( L \) be a local system of \( k \)-modules of finite rank on \( X \). Assume that the set \( E \) of critical points of \( \pi \) and \( Y = \pi(E) \) are (not necessarily orientable) smooth manifolds. Then there exists an short exact sequence
\[ \begin{array}{c}
0 \\
\longrightarrow H^k_{\Phi}(X, L) \overset{(\pi^{-1}-\Phi)^*}{\longrightarrow} H^k_{\pi^{-1}-\Phi}(\tilde{X}, \pi^{-1}L) \oplus H^k_{\Phi|i|}(Y, L|_{E}) \overset{i^*_{\Phi-\pi^{-1}}}{\longrightarrow} H^k_{(\pi^{-1}-\Phi)|}(E, i^{-1}_E \pi^{-1}L) \longrightarrow 0
\end{array} \]
for any \( k \), where \( i_Y : Y \rightarrow X \) and \( i_E : E \rightarrow X \) are inclusions.

Proof. With the same proof, (5.15) still holds in such case, which easily imply the conclusion by diagram chasing. \( \square \)

As an application of Theorem 1.2, we have the excess intersection formula.

Corollary 5.17. Under the assumptions in Theorem 1.2, let \( Q \) be the quotient bundle \( (\pi|E|^*)^N_{Y/X}/O_{E}(-1) \) over \( E \). Then
\[ H^*_{\Phi|i|}(Y, L|_{Y}) \overset{i_Y^*}{\longrightarrow} H^*_{\Phi}(X, L) \overset{c_{r-1}(Q)\cup(\pi|E|^*)}{\longrightarrow} H^{2r}_{(\pi|E|^*)^{-1}(\Phi|Y|)}(E, (\pi|E|)^{-1}(L|_{Y})) \overset{i^*_{E}}{\longrightarrow} H^{2r}_{\pi^{-1}-\Phi}(Y, \pi^{-1}L). \]
is a commutative diagram.

Proof. The total Chern classes satisfy \( c(O_{E}(-1))\cup c(Q) = (\pi|E|^*)c(N_{Y/X}) \), which implies that
\[ h \cup c_{r-1}(Q) = (\pi|E|^*)c_{r}(N_{Y/X}), \]
\[ c_{r-1}(Q) = \sum_{i=0}^{r-1}(-1)^ih^i \cup (\pi|E|^*)c_{r-1-i}(N_{Y/X}). \]
Notice that \((\pi|_E)_*h^i = 0\) for \(0 \leq i \leq r - 2\) and \((\pi|_E)_*h^{r-1} = (-1)^{r-1}\), see Sect. 4.3. By (3.14),
\[
(\pi|_E)_*(c_{r-1}(Q)) = 1. \tag{5.19}
\]
Fix an integer \(k\). For \(\sigma \in H^*_Y(Y, L|_Y)\), set \(\alpha = i_{E*}(c_{r-1}(Q) \cup (\pi|_E)^*\sigma)\). By Proposition 5.4 and (5.17),
\[
i^*_E \alpha = h \cup c_{r-1}(Q) \cup (\pi|_E)^*\sigma = (\pi|_E)^*(c_r(N_{Y/X}) \cup \sigma). \tag{5.20}
\]
By Theorem 1.2, \(\alpha = \pi^*\beta + \sum_{i=1}^{r-1} i_{E*}(h^{i-1} \cup (\pi|_E)^*\gamma_i)\) for unique \(\beta \in H^{k+2r}_{\Phi}(X, L)\) and \(\gamma_i \in H^{k+2r-2i}_{\Phi Y} (Y, L|_Y)\). By Proposition 5.4,
\[
i^*_E \alpha = (\pi|_E)_*i^*_Y \beta + \sum_{i=1}^{r-1} h^i \cup (\pi|_E)^*\gamma_i. \tag{5.21}
\]
Comparing (5.20) and (5.21), \(\gamma_i = 0\) for \(1 \leq i \leq r - 1\) by Proposition 5.3. Then \(\alpha = \pi^*\beta\). So
\[
\beta = \pi_* \alpha \quad \text{(by (3.14))}
\]
\[
=i_{Y*}(\pi|_E)_*(c_{r-1}(Q) \cup (\pi|_E)^*\sigma) \quad \text{(by the definition of \(\alpha\))}
\]
\[
= i_{Y*} \alpha \quad \text{(by (3.14) and (5.19)).}
\]
Hence, \(\alpha = \pi^* i_{Y*} \sigma\). We complete the proof. \(\square\)

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