Analyzing Guarded Protocols: Better Cutoffs, More Systems, More Expressivity

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Abstract. We study cutoff results for parameterized verification and synthesis of guarded protocols, as introduced by Emerson and Kahlon (2000). Guarded protocols describe systems of processes whose transitions are enabled or disabled depending on the existence of other processes in certain local states. Cutoff results reduce reasoning about systems with an arbitrary number of processes to systems of a determined, fixed size. Our work is based on the observation that existing cutoff results for guarded protocols are often impractical, since they scale linearly in the number of local states of processes in the system. We provide new cutoffs that scale not with the number of local states, but with the number of guards in the system, which is in many cases much smaller. Furthermore, we consider natural extensions of the classes of systems and specifications under consideration, and present results for problems that have not been known to admit cutoffs before.

1 Introduction

Concurrent systems are notoriously hard to get correct, and are therefore a promising application area for formal methods like model checking or synthesis. However, while such general-purpose formal methods can give strong correctness guarantees, they have two drawbacks: i) the state explosion problem prevents us from using them for systems with a large number of components, and ii) correctness properties are often expected to hold for an arbitrary number of components, which cannot be guaranteed without an additional argument that extends a proof of correctness to systems of arbitrary size. Both problems can be solved by approaches for parameterized model checking and synthesis, which give correctness guarantees for systems with any number of components without considering every possible system instance explicitly.

While parameterized model checking (PMC) is undecidable even if we restrict systems to uniform finite-state components [22], there exist a number of methods that decide the problem for specific classes of systems [1, 8, 10–13, 17], some of which have been collected in surveys of the literature recently [5, 14]. Additionally, there are semi-decision procedures that are successful in many interesting cases [6, 7, 19, 21]. In this paper, we consider the cutoff approach to PMC, that can guarantee properties of systems of arbitrary size by considering only systems of up to a certain fixed size, thus providing a decision procedure for PMC if components are finite-state.
Guarded protocols, the systems under consideration, are composed of an arbitrary number of processes, each an instance of a finite-state process template. Processes communicate by guarded updates, where guards are statements about other processes that are interpreted either conjunctively (“every other process satisfies the guard”) or disjunctively (“there exists a process that satisfies the guard”). Conjunctive guards can be used to model atomic sections or locks, while disjunctive guards can model pairwise rendezvous or token-passing.

This class of systems has been studied by Emerson and Kahlon [10, 11], and cutoffs that depend on the size of process templates are known for specifications of the form \( \forall \vec{p}. \Phi(\vec{p}) \), where \( \Phi(\vec{p}) \) is an LTL\( \setminus \)X property over the local states of one or more processes \( \vec{p} \). Außerlechner et al. [3] have extended and improved these results, but a number of open issues remain. We will explain some of them in the following.

**Motivating Example** As an example, consider the reader-writer protocol on the right, modeling access to data shared between processes. A process can signal that it wants to read the data by entering state \( tr \) (“try-read”). From \( tr \), it can move to the reading state \( r \). However, this transition is guarded by a statement \( \neg w \), meaning that no other process should currently be in state \( w \), i.e., writing the data. Similarly, a process that wants to enter \( w \) has to go through \( tw \), and the transition into \( w \) is guarded by \( \neg w \land \neg r \), i.e., no state should be either reading or writing.

The cutoff results by Emerson and Kahlon [10] allow us to check parameterized safety conditions such as

\[
\forall i \neq j. \, \mathcal{G} \left( \neg (w_i \land w_j) \land \neg (w_i \land r_j) \right),
\]

where indices \( i \) and \( j \) refer to different processes in the system. In particular, they provide a cutoff that is linear in the size of the process template for detecting the absence of global deadlocks, and (assuming that deadlocks are not possible) an efficient cutoff of 2 for 1-indexed LTL\( \setminus \)X formulas, which can be generalized to a cutoff of \( k + 1 \) for \( k \)-indexed properties.

However, when considering a liveness property such as

\[
\forall i. \, \mathcal{G} \left( (tr_i \rightarrow F r_i) \land (tw_i \rightarrow F w_i) \right),
\]

then their cutoff results are not very useful, since they do not consider fairness assumptions on the scheduling of processes, and there obviously exists a run with unfair scheduling that violates the property.

Außerlechner et al. [3] have looked at this problem, and divided it into two aspects: i) cutoffs for the detection of local deadlocks under the assumption of strong fairness, and ii) cutoffs for LTL\( \setminus \)X properties under the assumption of unconditional fairness. Since strong fairness and absence of local deadlocks
imply unconditional fairness, this enables the verification of liveness properties under the assumption of strong fairness. For ii), the provided cutoff is the same as for the non-fair case. For i), they give a cutoff that is linear in the size of the process template, but only for a restricted class of process templates.

A number of limitations of the existing results is highlighted by the example above. First, the existing cutoff results for local deadlock detection do not support the given process template. More specifically, they only support 1-conjunctive systems, i.e., systems where each guard can only exclude a single state. In this paper, we consider generalizations of this restricted class of process templates, and provide cutoffs for a class that includes examples such as the given one. Furthermore, we show that the general problem is very hard.

Another drawback of the existing results is that they use only minimal knowledge about the process templates: the size of templates and the type of guards. As a result, many cutoffs are linear in the size of the process template. Intuitively, the communication between processes should be more important for the cutoff than their internal state space. This can be seen in the example above: out of the 5 states, only 2 can be observed by the other processes, and can thus influence their behavior. In this paper, we investigate how cutoff results change when we also consider communication-related measures of the process templates, such as the number of different guards, or the number of states that appear in guards.

Contributions We provide new cutoff results for guarded protocols:

1. We show that by closer analysis of process templates, in particular the number and the form of transition guards, we can get smaller cutoffs in many cases. This circumvents the tightness results of Außerlechner et al. [3], which state that no smaller cutoffs can exist for the class of all processes of a given size.

2. For conjunctive systems, we additionally extend the class of process templates that are supported by cutoff results. In particular, we provide cutoff results for local deadlock detection in classes of templates that are not 1-conjunctive. However, we do not solve the general problem, and instead show that a cutoff for arbitrary conjunctive systems would at least be quadratic in the size of the template.

3. For disjunctive systems, we additionally extend both the class of process templates and the class of specifications that are supported by cutoff results. In particular, we show that systems with finite conjunctions of disjunctive guards are also supported by many of the existing proof methods, or variations of them. Based on this observation, we obtain cutoff results for these systems. Furthermore, we give cutoffs that support checking the simultaneous reachability (and repeated reachability) of a target set by all processes in a disjunctive system.
2 Preliminaries

2.1 System Model

We consider systems $A||B^n$, usually written $(A,B)^{(1,n)}$, consisting of one copy of a process template $A$ and $n$ copies of a process template $B$, in an interleaving parallel composition.\footnote{Process template $A$ may be a trivial process that does nothing if we want to just consider a system $B^n$, as in the example in Section 1.} We distinguish objects that belong to different templates by indexing them with the template. E.g., for process template $U \in \{A,B\}$, $Q_U$ is the set of states of $U$. For this section, fix two disjoint finite sets $Q_A$, $Q_B$ as sets of states of process templates $A$ and $B$, and a positive integer $n$.

Processes. A process template is a transition system $U = (Q, \text{init}, \Sigma, \delta)$ with

- $Q$ is a finite set of states including the initial state $\text{init}$,
- $\Sigma$ is a finite input alphabet,
- $\delta : Q \times \Sigma \times (Q_A \cup Q_B) \times Q$ is a guarded transition relation.

A process template is closed if $\Sigma = \emptyset$, and otherwise open.

For $U \in \{A,B\}$, define the size $|U| = |Q_U|$. We write $G_U$ for the set of non-trivial guards that are used in $\delta_U$, i.e., guards different from $Q_A \cup Q_B$ and $\emptyset$. Then, let $G = G_A \cup G_B$.

A copy of template $U$ will be called a $U$-process. Different $B$-processes are distinguished by subscript, i.e., for $i \in [1..n]$, $B_i$ is the $i$th copy of $B$, and $q_{B_i}$ is a state of $B_i$. A state of the $A$-process is denoted by $q_A$.

For the rest of this subsection, fix templates $A$ and $B$. We assume that $\Sigma_A \cap \Sigma_B = \emptyset$. We will also write $p$ for a process in $\{A,B_1,\ldots,B_n\}$, unless $p$ is specified explicitly.

Disjunctive and Conjunctive Systems. In a system $(A,B)^{(1,n)}$, consider global state $s = (q_A,q_{B_1},\ldots,q_{B_n})$ and global input $e = (\sigma_A,\sigma_{B_1},\ldots,\sigma_{B_n})$. We also write $s(p)$ for $q_p$, and $e(p)$ for $\sigma_p$. A local transition $(q_p,\sigma_p,g,q'_p) \in \delta_U$ of $p$ is enabled for $s$ and $e$ if its guard $g$ is satisfied for $p$ in $s$, written $(s,p) \models g$.

Disjunctive and conjunctive systems are distinguished by the interpretation of guards:

In disjunctive systems: $(s,p) \models g$ iff $\exists p' \in \{A,B_1,\ldots,B_n\} \setminus \{p\} : q_p' \in g$.

In conjunctive systems: $(s,p) \models g$ iff $\forall p' \in \{A,B_1,\ldots,B_n\} \setminus \{p\} : q_p' \in g$.

Note that we check containment in the guard (disjunctively or conjunctively) only for local states of processes different from $p$. A process is enabled for $s$ and $e$ if at least one of its transitions is enabled for $s$ and $e$, otherwise it is disabled.

Like Emerson and Kahlon \cite{EmersonK91}, we assume that in conjunctive systems $\text{init}_A$ and $\text{init}_B$ are contained in all guards, i.e., they act as neutral states. For conjunctive systems, we call a guard $n$-conjunctive if it is of the form $(Q_A \cup Q_B) \setminus \{q_1,\ldots,q_n\}$ for some $q_1,\ldots,q_n \in Q_A \cup Q_B$. A state $q$ is 1-conjunctive if all
non-trivial guards of transitions from \( q \) are 1-conjunctive. A conjunctive system is 1-conjunctive if every state is 1-conjunctive.

Then, \((A, B)^{(1,n)}\) is defined as the transition system \((S, \text{init}_S, E, \Delta)\) with

- set of global states \( S = (Q_A) \times (Q_B)^n \),
- global initial state \( \text{init}_S = (\text{init}_A, \text{init}_B, \ldots, \text{init}_B) \),
- set of global inputs \( E = (\Sigma_A) \times (\Sigma_B)^n \),
- and global transition relation \( \Delta \subseteq S \times E \times S \) with \((s, e, s') \in \Delta\) iff
  
  i) \( s = (q_A, q_{B_1}, \ldots, q_{B_n}) \),
  
  ii) \( e = (\sigma_A, \sigma_{B_1}, \ldots, \sigma_{B_n}) \), and
  
  iii) \( s' \) is obtained from \( s \) by replacing one local state \( q_p \) with a new local state \( q'_p \), where \( p \) is a \( U \)-process with local transition \((q_p, \sigma_p, g, q'_p) \in \delta_U \)
  
  and \((s, p) \models g \).

We say that a system \((A, B)^{(1,n)}\) is of type \((A, B)\). A system is closed if all of its templates are closed. We often denote the set \( \{B_1, \ldots, B_n\} \) as \( B \).

**Runs.** A configuration of a system is a triple \((s, e, p)\), where \( s \in S \), \( e \in E \), and \( p \) is either a system process, or the special symbol \( \perp \). A path of a system is a configuration sequence \( x = (s_1, e_1, p_1), (s_2, e_2, p_2), \ldots \) such that for all \( m < |x| \) there is a transition \((s_m, e_m, s_{m+1}) \in \Delta \) based on a local transition of process \( p_m \). We say that process \( p_m \) moves at moment \( m \). Configuration \((s, e, \perp)\) appears iff all processes are disabled for \( s \) and \( e \). Also, for every \( p \) and \( m < |x| \): either \( e_{m+1}(p) = e_m(p) \) or process \( p \) moves at moment \( m \). That is, the environment keeps input to each process unchanged until the process can read it.\(^2\)

A system run is a maximal path starting in the initial state. Runs are either infinite, or they end in a configuration \((s, e, \perp)\). We say that a run is initializing if every process that moves infinitely often also visits its init infinitely often.

Given a system path \( x = (s_1, e_1, p_1), (s_2, e_2, p_2), \ldots \) and a process \( p \), the local path of \( p \) in \( x \) is the projection \( x(p) = (s_1(p), e_1(p)), (s_2(p), e_2(p)), \ldots \) of \( x \) onto local states and inputs of \( p \). \( x(p) \) is a local run if \( x \) is a run. Similarly define the projection on two processes \((p_1, p_2)\) denoted by \( x(p_1, p_2) \).

**Deadlocks and Fairness.** A run is globally deadlocked if it is finite. An infinite run is locally deadlocked for process \( p \) if there exists \( m \) such that \( p \) is disabled for all \( s_{m'}, e_{m'} \) with \( m' \geq m \). A run is deadlocked if it is locally or globally deadlocked. A system has a (local/global) deadlock if it has a (locally/globally) deadlocked run. Note that absence of local deadlocks for all \( p \) implies absence of global deadlocks, but not the other way around.

A run \( (s_1, e_1, p_1), (s_2, e_2, p_2), \ldots \) is unconditionally-fair if every process moves infinitely often. A run is strong-fair if it is infinite and for every process \( p \), if \( p \) is enabled infinitely often, then \( p \) moves infinitely often.

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\(^2\) By only considering inputs that are actually processed, we approximate an action-based semantics. Paths that do not fulfill this requirement are not very interesting, since the environment can violate any interesting specification that involves input signals by manipulating them when the corresponding process is not allowed to move.
2.2 Specifications

Fix templates \((A, B)\). We consider formulas in \text{LTL}\backslash X$, i.e., \text{LTL} without the next-time operator \(X\). Let \(h(A, B_{i_1}, \ldots, B_{i_k})\) be an \(\text{LTL}\backslash X\) formula over atomic propositions from \(Q_A \cup Q_B\) and indexed propositions from \((Q_B \cup \Sigma_B) \times \{i_1, \ldots, i_k\}\). For a system \((A, B)^{(1,n)}\) with \(n \geq k\) and \(i_j \in [1..n]\), satisfaction of \(A h(A, B_{i_1}, \ldots, B_{i_k})\) and \(E h(A, B_{i_1}, \ldots, B_{i_k})\) is defined in the usual way (see e.g. [4]).

**Parameterized Specifications.** A parameterized specification is a temporal logic formula with indexed atomic propositions and quantification over indices. A \(k\)-indexed formula is of the form \(\forall i_1, \ldots, i_k. A h(A, B_{i_1}, \ldots, B_{i_k})\) or \(\forall i_1, \ldots, i_k. E h(A, B_{i_1}, \ldots, B_{i_k})\). For given \(n \geq k\),

\[
(A, B)^{(1,n)} \models \forall i_1, \ldots, i_k. A h(A, B_{i_1}, \ldots, B_{i_k})
\]

iff

\[
(A, B)^{(1,n)} \models \bigwedge_{j_1 \neq \ldots \neq j_k \in [1..n]} A h(A, B_{j_1}, \ldots, B_{j_k}).
\]

By symmetry of guarded protocols, this is equivalent (cp. [10]) to \((A, B)^{(1,n)} \models A h(A, B_{i_1}, \ldots, B_{i_k})\). The latter formula is denoted by \(A h(A, B^{(k)})\), and we often use it instead of the original \(\forall i_1, \ldots, i_k. A h(A, B_{i_1}, \ldots, B_{i_k})\). For formulas with path quantifier \(E\), satisfaction is defined analogously, and equivalent to satisfaction of \(E h(A, B^{(k)})\).

**Specification of Fairness and Local Deadlocks.** It is often convenient to express fairness assumptions and local deadlocks as parameterized specifications. To this end, define auxiliary atomic propositions \(\text{move}_p\) and \(\text{en}_p\) for every process \(p\) of system \((A, B)^{(1,n)}\). At moment \(m\) of a given run \((s_1, e_1, p_1), (s_2, e_2, p_2), \ldots\), let \(\text{move}_p\) be true whenever \(p_m = p\), and let \(\text{en}_p\) be true if \(p\) is enabled for \(s_m, e_m\). Note that we only allow the use of these propositions to define fairness, but not in general specifications. Then, an infinite run is

- **local-deadlock-free** if it satisfies \(\forall p. \text{GF en}_p\), abbreviated as \(\Phi_{\text{dead}}\).
- **strong-fair** if it satisfies \(\forall p. \text{GF en}_p \rightarrow \text{GF move}_p\), abbreviated as \(\Phi_{\text{strong}}\), and
- **unconditionally-fair** if it satisfies \(\forall p. \text{GF move}_p\), abbreviated as \(\Phi_{\text{uncond}}\).

2.3 Model Checking Problems and Cutoffs

For a given system \((A, B)^{(1,n)}\) and specification \(h(A, B^{(k)})\) with \(n \geq k\),

- the model checking problem is to decide whether \((A, B)^{(1,n)} \models A h(A, B^{(k)})\),
- the (global/local) deadlock detection problem is to decide whether \((A, B)^{(1,n)}\) has (global/local) deadlocks,
- the parameterized model checking problem (PMCP) is to decide whether \(\forall m \geq n: (A, B)^{(1,m)} \models A h(A, B^{(k)})\), and
- the parameterized (local/global) deadlock detection problem is to decide whether for some \(m \geq n\), \((A, B)^{(1,m)}\) does have (global/local) local deadlocks.
These definitions can be flavored with different notions of fairness, and with the E path quantifier instead of A. Also, corresponding problems for the synthesis of process templates can be defined (compare Außerlechner et al. [3]). Parameterized synthesis based on cutoffs [18] is also supported by our cutoff results, but the details will not be necessary for understanding the results presented here.

**Cutoffs.** We define cutoffs with respect to a class of systems (either disjunctive or conjunctive), a class of process templates \( P \), and a class of properties, which can be \( k \)-indexed formulas for some \( k \in \mathbb{N} \) or the existence of (local/global) deadlocks.

A cutoff for a given class of properties and a class of systems with processes from \( P \) is a number \( c \in \mathbb{N} \) such that for all \( A, B \in P \) and all properties \( \varphi \) in the given class:

\[
(A, B)^{(1, n)} \models \varphi \iff (A, B)^{(1, c)} \models \varphi.
\]

Like the problem definitions above, cutoffs may additionally be flavored with different notions of fairness.

**Cutoffs and Decidability.** Note that the existence of a cutoff implies that the parameterized model checking and parameterized deadlock detection problems are decidable iff their non-parameterized versions are decidable.

3 Better Cutoffs for Disjunctive Systems

In this section, we state our new cutoff results for disjunctive systems, and compare them to the previously known results in Table 1. Full proofs can be found in Appendix A.

To state our first theorem, we need the following additional definitions.

Fix process templates \( A, B \) with \( G = G_A \cup G_B \). Let \( |B|_G = \{|q \in Q_B | \exists g \in G : q \in g\}|. \) For a state \( q \in Q_B \) in a disjunctive system, define \( \text{Enable}_q = \{q' \in Q_A \cup Q_B | \exists (q, \sigma, g, q'') \in \delta_B : q' \in g\} \), i.e., the set of states of \( A \) and \( B \) that enable a transition from \( q \). Furthermore, let \( N = \{q \in Q_B | q \in \text{Enable}_q\} \), and let \( N^* \) be the maximal subset (wrt. number of elements) of \( N \) such that \( \forall q_i, q_j \in N^* : q_i \notin \text{Enable}_{q_j} \land q_j \notin \text{Enable}_{q_i} \). Then we obtain:

**Theorem 1 (Disjunctive Cutoff Theorem).** For disjunctive systems and process templates \( A, B \) with \( G = G_A \cup G_B \):

- \( |B|_G + k + 1 \) and \( |G| + k + 1 \) are cutoffs for \( k \)-indexed properties in non-fair executions,
- \( |B| + |G| + k \) is a cutoff for \( k \)-indexed properties in unconditionally fair executions,
- \( m + |G| + 1 \) is a cutoff for local deadlock detection in non-fair executions, where \( m = \max_{q \in Q_B} \{|\text{Enable}_q|\} \) for \( Q_B^* = \{q \in Q_B | |\text{Enable}_q| < |B|\}, \)
- \( |B| + |G| \) is a cutoff for local deadlock detection in unconditionally fair executions,
- \( |B| + |N^*| \) is a cutoff for global deadlock detection.
Proof Ideas. We explain our proof ideas as modifications of the original proofs by Außerlechner et al. [2], for the results given in the second results column of Table 1.

In the original proofs corresponding to the first four items, to simulate a given run of an arbitrarily large system, up to $|B|$ processes of the cutoff system are moved into the states that appear in the original run, in the same order. This ensures that all transitions will also be enabled in the cutoff system. Based on our knowledge about guards, we guarantee the same effect by moving into one representative state per guard. In this way, we can replace (one occurrence of) $|B|$ by $|G|$ in the cutoff.

By a similar argument, in the first item we can also replace $|B|$ by $|B|_G$ (this does not work for the other items since additional processes may be needed to ensure fairness or preserve the deadlock).

For local deadlocks, there is an additional construction in the proofs where a process in the cutoff system has to move into some state and then leave it again, because otherwise the deadlock would not be possible. We compute $m$ as an upper bound for the number of states for which this is necessary, which replaces an occurrence of $|B| - 1$ in the cutoff.

Finally, for global deadlocks the original proof distinguishes between states in $\mathcal{N}$ and other states. To construct a simulating run in the cutoff system, for each state in $\mathcal{N}$ that appears in the deadlocked global state it uses one process that exactly mimics the behavior of one process that moved there in the original run. For the processes that do deadlock in local states that are not in $\mathcal{N}$, a construction similar to the local deadlocks is needed, moving processes into all states that are visited in the original run, and possibly moving them out of these states again if they are not part of the deadlock. Our improvement concerns only the first set of processes: we compute $\mathcal{N}^*$ in order to find out how many states from $\mathcal{N}$ can appear together in a global deadlock. Then, we can replace one occurrence of $|B| - 1$ with $|\mathcal{N}^*|$ in the cutoff.

Remark. To compute $\mathcal{N}^*$ exactly, we need to find the smallest set of states in $\mathcal{N}$ that do not satisfy the additional condition. This amounts to finding the minimum vertex cover (MVC) for the graph with vertices from $\mathcal{N}$ and edges from $q_i$ to $q_j$ if $q_i \in \text{Enable}_{q_j}$. This problem is itself $NP$-hard. This effort is justified since model checking complexity is in general exponential in the number of components. On the other hand, the MVC can be approximated in $PTIME$ such that at least half of the unnecessary nodes are removed.

4 Better Cutoffs for Conjunctive Systems

In this section, we state our new cutoff results for conjunctive systems, and compare them to the previously known results in Table 2. Full proofs can be found in Appendix B.

For conjunctive systems, the cutoffs for $\text{LTL}\setminus X$ properties cannot be improved. We give improved cutoffs for global deadlock detection in general, and
Table 1: Cutoff Results for Disjunctive Systems

|                | EK [10]          | AJK [3]          | our work                       |
|----------------|------------------|------------------|-------------------------------|
| k-indexed LTL | $|B| + k + 1$     | $|B| + k + 1$     | $|B| + k + 1$ and $|G| + k + 1$ |
| X non-fair    |                  |                  |                               |
| k-indexed LTL | -                | $2|B| + k - 1$    |                               |
| X fair        |                  |                  |                               |
| Local Deadlock| -                | $|B| + 2$         | $m + |G| + 1$, with $m < |B|$   |
| non-fair      |                  |                  |                               |
| Local Deadlock| -                | $2|B| - 1$        |                               |
| fair          |                  |                  |                               |
| Global Deadlock| -               | $2|B| - 1$        |                               |
|                |                  |                  | $|B| + |N^*|$, with $|N^*| < |B|$ |

for local deadlock detection for the restricted case of 1-conjunctive systems. After that, we explain why local deadlock detection in general is hard, and identify a number of cases where we can solve the problem even for systems that are not 1-conjunctive.

To state our theorems for conjunctive systems, we define the following for a given conjunctive system $(A, B)^{(1,n)}$:

We say that $D \subseteq (Q_A \cup Q_B)$ is a deadset of $q \in (Q_A \cup Q_B)$ if $\forall(q, \sigma, g, q') \in \delta : \exists q'' \in D : q'' \notin g$ and $\forall q'' \in D \exists(q, \sigma, g, q') \in \delta : q'' \notin g$, and $D$ contains at most one state from $Q_A$.

For a given $q$, $\text{dead}^\wedge_q$ is the set of all deadsets of $q$: $\text{dead}^\wedge_q = \{D \subseteq (Q_A \cup Q_B) \mid D \text{ is a deadset of } q\}$.

If $\text{dead}^\wedge_q = \emptyset$, then we say $q$ is free. If a state $q$ does not appear in $\text{dead}^\wedge_q$ for any $q' \in Q_A \cup Q_B$, then we say $q$ is non-blocking. If a state $q$ does not appear in $\text{dead}^\wedge_q$, then we say $q$ is not self-blocking.

**Theorem 2 (Conjunctive Cutoff Theorem).** For conjunctive systems and process templates $A, B$:

- let
  - $k_1 = |D_1|$, where $D_1 \subseteq Q_B$ is the set of free states in $B$,
  - $k_2 = |D_2 \setminus D_1|$, where $D_2 \subseteq Q_B$ is the set of non-blocking states in $B$, and
  - $k_3 = |D_3 \setminus (D_1 \cup D_2)|$, where $D_3 \subseteq Q_B$ is the set of not self-blocking states in $B$.

  Then $2|B| - 2k_1 - 2k_2 - k_3$ is a cutoff for global deadlock detection.

- if process template $U$ is 1-conjunctive, then
  - $|G_U| + 2$ is a cutoff for local deadlock detection in a $U$-process and non-fair executions,
  - $2|G_U| + 1$ is a cutoff for local deadlock detection in an initializing $U$-process and fair executions.

**Proof Ideas.** Again, we explain our proof ideas as modifications of the original proofs by Außerlechner et al. [2], in this case for the results given in the second results column of Table 2.
In order to simulate a global deadlock of a large system in the cutoff system, the original proof uses up to 2 processes that move into each of the states — except for the initial state, which is assumed to be included in every conjunctive guard, and therefore cannot contribute to a deadlock. A generalization of this idea is our notion of non-blocking states, which can further reduce the cutoff. In part, this also applies to states that are not self-blocking: for these, we need at most 1 copy, since the second copy can only be useful for blocking transitions from the same state. Finally, also states that are free can never contribute to a deadlock, since they are never deadlocked themselves.

Regarding local deadlocks in 1-conjunctive systems, the idea is similar to the basic idea described in the proof of Theorem 1: where the original proof needs up to one copy of every state (except init) to ensure that the deadlock is preserved, we need at most one copy for every guard in the template. Therefore, we can replace $|B| - 1$ by $|G_U|$ in the cutoff. In the fair case, by a similar argument we can even replace $2|B| - 2$ by $2|G_U| + 1$.

\[ \text{Table 2: Cutoff Results for Conjunctive Systems} \]

|                  | EK [10] | AJK [3] | our work |
|------------------|---------|---------|----------|
| $k$-indexed LTL\X non-fair | $k + 1$ | $k + 1$ | unchanged |
| $k$-indexed LTL\X fair        | -       | $k + 1$ | unchanged |
| Local Deadlock non-fair       | -       | $|B| + 1^*$ | $|G_U| + 2^*$ |
| Local Deadlock fair           | -       | $2|B| - 2^*$ | $2|G_U| + 1^*$ |
| Global Deadlock              | $2|B| + 1$ | $2|B| - 2$ | $2|B| - 2k_1 - 2k_2 - k_3$ |

$^*$: systems have to be 1-conjunctive; in fair case, they additionally have to be initializing; 
$k_1$: number of free states; 
$k_2$: number of non-blocking states (that are not free); 
$k_3$: number of not self-blocking states (that are not free or non-blocking)

**Local Deadlock Detection: Beyond 1-conjunctive Systems** While we improve on the local deadlock detection cutoff for conjunctive systems in some cases, the results above still have the same restriction as in Außerlechner et al. [3]: process template $B$ has to be 1-conjunctive. The reason for this restriction is that when going beyond 1-conjunctive systems, the local deadlock detection cutoff (even without considering fairness) can be shown to grow at least quadratic in the number of states or guards, and it becomes very hard to determine a cutoff.

To analyze these cases, define the following: A sequence of states $q_1 \ldots q_n$ is **connected** if $\forall q_i \in \{q_1, \ldots, q_n\} : \exists(q_i, \sigma, g, q_{i+1}) \in \delta$. A **cycle** is a connected sequence of states $q q_1 \ldots q_n q$ such that $\forall q_i, q_j \in \{q_1, \ldots, q_n\} : q_i \neq q_j$. We denote such a cycle by $C_q$. (By abuse of notation, $C_q$ is also used for the set of states on $C_q$.) We denote the set of guards of the transitions on $C_q$ as $G_{C_q}$. A
cycle $C_q$ is called *free* if $\forall p \in C_q \setminus q \ \forall g \in G_{C_q} : p \in g$. We denote such a cycle by $C_q^{\text{free}}$.

**Example 1.** If we consider the process template in Figure 1 without the parts in blue, then it exhibits a local deadlock in state $q_l$ for 9 processes, but not for 8 processes: one process has to move to $q_l$, and for each cycle that starts and ends in states $a, b, c, d$, we need 2 processes that move along the cycle to keep all guards of $q_l$ covered at all times. Intuitively, one copy per cycle has to be in the state of interest, or ready to enter it, and the other copy is traveling on the cycle, waiting until the guards are satisfied.

Fig. 1: Process Template with Quadratic Cutoff for Local Deadlocks

Now, consider the modified template (as depicted in blue in Figure 1) where we i) add two states $e, f$ in a similar way as $a, b, c, d$, ii) add a new state connected to $q_l$ with guard $\neg e \land \neg f$, and iii) change the guards in the sequence from $u_1$ to init to $\neg a \land \neg e$ and $\neg b \land \neg d \land \neg f$, respectively. Then we have 6 cycles that need 2 processes each, and we need 13 processes to reach a local deadlock in $q_l$.

Moreover, consider the modified template where we increase the length of the sequence from $u_1$ to init by adding additional states $u_3$ (which is connected to $u_2$ instead of init) and $u_4$ (which is connected to $u_3$ and init with transitions that have the same guards as those from $u_1$ to $u_2$ and from $u_2$ to $u_3$, respectively). Then, for every cycle we need 3 processes instead of 2, as otherwise they cannot
traverse the cycle fast enough to ensure that the local deadlock is preserved infinitely long. That is, the template with both modifications now needs 19 processes to reach a local deadlock. Observe that by increasing the height of the template, we increase the necessary number of states without increasing the number of different guards.

Moreover, when increasing both the width and height of the template, we observe that the number of processes that are necessary for a local deadlock increases quadratically with the size of the template.

This example leads us to the following result.

**Theorem 3.** For conjunctive systems, a cutoff for local deadlock detection must grow at least quadratically in the number of states. Furthermore, it cannot be bounded by the number of guards at all.

*Proof Idea.* For a system that does exhibit a local deadlock for some size $n$, but not for $n - 1$, the cutoff cannot be smaller than $n$. Thus, the example shows that a cutoff for local deadlock detection in general is independent of the number of guards, and must grow at least quadratic in the size of the template.

Cutoffs that can in the best case be bounded by $|B|^2$ will not be very useful in practice. Therefore, instead of solving the general problem we identify in the following a number of cases where the cutoff remains small (i.e., linear in the number of states or guards).

When comparing the proof of the second item of Theorem 2 to the example above, we note that the reason that the cutoff in Theorem 2 does not apply is the following: while in 1-conjunctive systems every state has a unique deadset, in the general case every state may have many deadsets, and the structure of the process template may require infinitely many alternations between different deadsets to preserve the local deadlock. Moreover, as shown in the example, the number of processes needed to alternate between deadsets may increase with the size of the template, even if the set of guards (and thus, the number of different deadsets) remains the same.

We say that a locally deadlocked run is *alternation-free* if it does not alternate infinitely often between different deadsets. In the following, we will first show that for certain systems with alternation-free local deadlocks, the cutoff for 1-conjunctive systems applies. After that, we consider a (still restricted) class of systems that does not have alternation-free local deadlocks, and give a local deadlock detection cutoff for this class.

**Systems with Alternation-Free Local Deadlocks.** To identify systems with alternation-free deadlocks, we need some additional definitions.

We say that a conjunctive process template $U$ is *effectively 1-conjunctive* if every $q \in Q_U$ is either 1-conjunctive or free.

A *lasso* $lo$ is a connected sequence of states $q_0 \ldots q_i \ldots q_n$ such that $q_0$ is an initial state, $q_i = q_n$, and $q_i \ldots q_n$ is a cycle. We denote by $G_{lo}$ the set of guards of the transitions on $lo$. We say that a conjunctive process template $U$ is *freely
traversable if for every non-free state \( q \in Q_U \), and every set of states \( \{q_1, \ldots, q_n\} \) that disables the \( n \)-conjunctive guards with \( n > 1 \) in transitions from \( q \), there exists a lasso \( lo \) that is free of \( \neg q \), free of all \( \neg q_i \), and free of all 1-conjunctive guards in transitions from \( q \).

Intuitively, in a freely traversable process template there is always an infinite local run that can start from \( \text{init} \) when a single other process is already in a local deadlock. The example process in Section 1 is not freely traversable, since there is a lasso that is free of \( \neg tw \) and \( \neg r \), but no lasso that is free of \( \neg tw \) and \( \neg w \).

We say that a conjunctive process template \( U \) is alternation-free if one of the following holds:

- for every non-free state \( q \in Q_U \), and every set of states \( D = \{q_1, \ldots, q_n\} \) that disables the \( n \)-conjunctive guards with \( n > 1 \) in transitions from \( q \), there is at most one \( q_i \) for which the following does not hold:
  
  \[
  \text{for all cycles } C_{q_i} = q_i \ldots q_i \in U : C_{q_i} \cap (C_{q_i} \cup \neg q) \neq \emptyset
  \]

- for every non-free state \( q \in Q_U \), and every \( n \)-conjunctive guard \( g = \neg q_1 \land \ldots \land \neg q_n \) with \( n > 1 \), \( G_q \cap \{\neg q_1, \ldots, \neg q_n\} \neq \emptyset \).

Intuitively, in an alternation-free process template there can never be an infinite alternation between different deadsets of a single locally deadlocked process (without releasing the deadlock). The process template from Section 1 is alternation-free, since: i) \( tw \) is the only non-free state with guards that are not 1-conjunctive, ii) \( \{w, r\} \) is the set of states that disables the only guard that is not 1-conjunctive, and iii) all cycles that contain \( w \) also contain a guard that is in \( G_{tw} \) (since all these cycles move through \( tw \)).

**Observation 1.** If a process template \( U \) is either effectively 1-conjunctive, freely traversable, or alternation-free, then for every locally deadlocked run there exists a locally deadlocked run that is alternation-free.

**Theorem 4 (Local Deadlock Detection in Conjunctive Systems).** For conjunctive systems and process templates \( A, B \), for \( U \in \{A, B\} \) the respective cutoff for local deadlock detection in 1-conjunctive systems applies in the following cases:

- for non-fair executions if \( U \) is effectively 1-conjunctive, freely traversable, or alternative-free
- for unconditionally fair executions if \( U \) is effectively 1-conjunctive or alternative-free.

**Proof Ideas.** The statement follows from the observation above, and from the proof of Theorem 2. Only the notion of freely traversable process templates is not compatible with the proof for local deadlocks under fairness.
Systems without Alternation-free Local Deadlocks. To demonstrate the complexity of the problem in general, let us analyze a non-trivial, but still strongly restricted case where alternation between deadsets may be necessary. Consider a system where all non-trivial guards are 1-conjunctive, except for a single 2-conjunctive guard $g_2 = \neg a \land \neg b$ that is used in a single transition from state $q_l$. To simplify the analysis, assume that the process template has unique cycles $C_a$ and $C_b$, i.e., no other cycles pass through $a$ or $b$. Assume that both cycles are free of 1-conjunctive guards that are necessary to deadlock $q_l$, and free of $\neg q_l$ (otherwise, the template would be alternation-free).

To state the cutoff result, define the following: A segment $S_{ga-b}$ is a connected sequence of states $q_i \ldots q_j$ where:

- $q_i$ has an incoming transition with guard $\neg a$
- $q_j$ has an outgoing transition with guard $\neg b$
- $\forall q_m \in S_{ga-b} \exists (q_m, \sigma, g, q_{m+1}) \in \delta :$ if $q_{m+1} \in S_{ga-b}$ then $b \in g$

For a cycle $C_q$, we denote by $|S_{ga-b}|_{C_q}$ the total number of segments $S_{ga-b}$ on $C_q$.

**Theorem 5.** For a system with process templates $A, B$ and the restrictions described above, let $n_a = \max(|S_{ga-b}|_{C_a}, |S_{gb-a}|_{C_b})$ and $n_b = \max(|S_{ga-b}|_{C_b}, |S_{gb-a}|_{C_a})$. Then:

$(A, B)^{(1,n)}$ has a local deadlock in $q_l \implies (A, B)^{(1,|G_A|+n_a+n_b+5)}$ has a local deadlock in $q_l$.

That is, already for this restricted class of systems, the available proof methods only give us a cutoff that increases with the number of segments $S_{ga-b}$ and $S_{gb-a}$ on the cycles. For systems with multiple $n$-conjunctive guards, both the complexity of the analysis and the size of the cutoff grow quickly (and Example 1 shows that this may indeed be necessary).

## 5 Verification of the Reader-Writer Example

We consider again the reader-writer example from Section 1, and show how our new results allow us to check correctness, find a bug, and check a fixed version.

With our results, we can for the first time check this liveness property in a meaningful way, i.e., under the assumption of fair scheduling. Since the process template is alternation-free, by Theorems 2 and 4 the local deadlock detection cutoff for the system is $2|G_B| + 1 = 5$. Moreover, compared to previous results we reduce the cutoff for global deadlock detection by recognizing that $k_1 = 3$ states can never be deadlocked, and $k_2 = 2$ additional states never appear in any guard. This reduces the cutoff to $2|B| - 2k_1 - 2k_2 = 10 - 6 - 4 = 0$, i.e., we detect that there can be no global deadlocks by analyzing only a single process template.

However, checking the system for local deadlocks shows that a local deadlock is possible: a process may forever be stuck in $tw$ if the other processes move in a loop $(\text{init}, tr, r)^\omega$ (and always at least one process is in $r$). To fix this, we can add an additional guard $\neg tw$ to the
transition from init to tr, as shown in the process template to the right. For the resulting system, our results give a local deadlock detection cutoff of $2|G_B| + 1 = 7$, and a global deadlock detection cutoff of $2|B| - 2k_1 - 2k_2 - k_3 = 10 - 6 - 2 - 1 = 1$ (where $k_3$ is the number of states that do appear in guards and could be deadlocked themselves, but do not have a transition that is blocked by another process in the same state).

6 More Disjunctive Systems and More Specifications

We show two further extensions of the class of problems for which cutoffs are available:

1. systems where transitions are guarded with a conjunction of disjunctive guards
2. two important classes of specifications that cannot be expressed in prenex indexed temporal logic.

6.1 Systems with Conjunctions of Disjunctive Guards

We consider systems where a transition can be guarded by a set of sets of states, interpreted as a conjunction of disjunctive guards. I.e., a guard $\{D_1, \ldots, D_n\}$ is satisfied in a given global state if for all $i = 1, \ldots, n$, there exists another process in a state $D_i$.

We observe that for this class of systems, most of the original proof ideas still work. For results that depend on the number of guards, we have to count the number of different conjuncts in guards.

**Theorem 6.** For systems with conjunctions of disjunctive guards, cutoff results for disjunctive systems that do not depend on the number of guards still hold (first and second column of results in Table 1).

Cutoff results that depend on the number of guards (last column of Table 1) hold if we consider the number of conjuncts in guards instead. For results that additionally refer to some measure of the sets of enabling states ($m$ and $|N^*|$, respectively), we obtain a valid cutoff for systems with conjunctions of disjunctive guards if we replace this measure by $|B| - 1$.

**Proof Ideas.** The cutoff results that are independent of the number of guards still hold since all of the original proof constructions still work. To simulate a run $x$ of a large system in a run $y$ the cutoff system, one task is to make sure that all necessary transitions are enabled in the cutoff system. To this end, the original construction of $y$ moves one process into each state that appears in $x$, as soon as possible. This ensures that if we only want to enter states that appear in the original run, disjunctive guards of all necessary transitions will be satisfied.
However, the same holds for transitions with conjunctions of disjunctive guards — if the set of states that appear in the other processes is the same at a given time, then the same conjunctions of disjunctive guards will be satisfied.

By a similar argument, we can always move out of a state if necessary for the construction, and deadlocks are preserved in the same way as for disjunctive systems.

For cutoffs that depend on the number of guards, transitions with conjunctions of disjunctive guards require us to use one representative for each conjunct in a guard, in the construction explained in the proof idea of Theorem 1.

Finally, the reductions of the cutoff based on the analysis of states that can or cannot appear together in a deadlock do not work in these extended systems, and we have to replace $m$ and $|X^*|$ by $|B| − 1$ in the cutoffs. The reason is that $\text{Enable}_q$ is now not a set of states anymore, but a set of sets of states. A more detailed analysis based on this observation may be possible, but is left open for now.

### 6.2 Simultaneous Reachability of Target States

An important class of properties for parameterized systems asks for the reachability of a global state where all processes of type $B$ are in a given local state $q$ (compare Delzanno et al. [9]). This can be written in indexed LTL\X as $F \forall i.q_i$, but is not expressible in the fragment where index quantifiers have to be in prenex form. We denote this class of specifications as $\text{Target}$. Similarly, repeated reachability of $q$ by all states simultaneously can be written $GF \forall i.q_i$, and is also not expressible in prenex form. We denote this class of specifications as $\text{Repeat-Target}$.

**Theorem 7 (Disjunctive Target and Repeat-Target).** For disjunctive systems: $|B|$ is a cutoff for checking $\text{Target}$ and $\text{Repeat-Target}$.

**Proof Ideas.** We can simulate a run $x$ in a large system where all processes are in $q$ at time $m$ in the cutoff system by first moving one process into each state that appears in $x$ before $m$, in the same order as in $x$. To make all processes reach $q$, we move them out of their respective states in the same order as they have moved out of them in $x$. For this construction, we need at most $|B|$ processes.

If in $x$ the processes reach are repeatedly in $q$ at the same time, then we can simulate this also in the cutoff system: if $m' > m$ is a point in time where this happens again, then we use the same construction as above, except that we consider all states that are visited between $m$ and $m'$, and we move to these states from $q$ instead from init. The correctness argument is the same, however.

Finally, if the run with $\text{Repeat-Target}$ should also be fair, then we do not simply select any $m'$ with the property above, but we choose it such that all processes move between $m$ and $m'$. If the original run $x$ is fair, then such an $m'$ must exist.
**7 Conclusion**

We have shown that better cutoffs for guarded protocols can be obtained by analyzing properties of the process templates, in particular the number and form of transition guards. We have further shown that cutoff results for disjunctive systems can be extended to a new class of systems with conjunctions of disjunctive guards, and to specifications \textsc{Target} and \textsc{Repeat-Target}, that have not been considered for guarded protocols before.

For conjunctive systems, previous works have treated local deadlock detection only for the restricted case of systems with 1-conjunctive guards. We have considered the general case, and have shown that it is very difficult — the cutoffs grow independently of the number of guards, and at least quadratically in the size of the process template. To circumvent this worst-case behavior, we have identified a number of conditions under which a small cutoff can be obtained even for systems that are not 1-conjunctive.

By providing cutoffs for systems and specifications that were previously not known to have cutoffs or to be decidable, we have in particular proved decidability of the respective problems.

Our work is inspired by applications in parameterized synthesis [18], where the goal is to automatically construct process templates such that a given specification is satisfied in systems with an arbitrary number of components. In this setting, deadlock detection and expressive specifications are particularly important, since all relevant properties of the system have to be specified, in contrast to verification, where a partial specification may be acceptable. The results of this paper can be seen as a continuation of our research on efficient parameterized synthesis, orthogonal to the approaches like modular application of cutoffs presented in earlier work [20].

Besides making verification and synthesis more efficient through smaller cutoffs, our results can also be used to guide synthesis algorithms towards “simple” implementations, that have additional benefits such as being easier to understand, verify, and maintain (by humans and machine alike). This approach has been used by others before: \textit{bounded synthesis} [15] prefers implementations with a small number of states, \textit{bounded cycle synthesis} [16] prefers implementations with a small number of cycles. Investigating the applications of our results in parameterized synthesis is one of our goals in future work.

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A Appendix: Proofs and Proof Methods for Disjunctive Systems

In this section, we present lemmas and proof methods that allow us to obtain our cutoff results for disjunctive systems. Note that usually we only state a bounding lemma, which states that any behavior in a large system can be replicated in the cutoff system. For the opposite direction, we can use existing monotonicity lemmas from previous work [3, 10] (see in particular the full version of Außerlechner et al. [2]). Also, in many cases we only consider a problem for a copy of template $B$, but not for $A$. The case of $A$ can be obtained by minor modifications of the proofs.

A.1 Definitions

Given a run $x = x_0, x_1, \ldots$ of a system $(A, B)^{(1,n)}$ and a state $q \in Q_B$, we define the following notation:

- **appears**$_q$ is the set of all moments where at least one copy of $B$ is in state $q$: $\text{appears}_q = \{m \in \mathbb{N} \mid \exists i \in [n] : x_m(B_i) = q\}$
- $f_q$ is the first moment where $q$ appears: $f_q = \min(\text{appears}_q)$
- $\text{first}_q \in [n]$ is the process index with $x_{f_q}(B_{\text{first}_q}) = q$
- if $\text{appears}_q$ is finite, then $l_q$ is the last moment where $q$ appears: $l_q = \max(\text{appears}_q)$
- $\text{last}_q \in [n]$ is the process index with $x_{l_q}(B_{\text{last}_q}) = q$
- given a guard $g \in G$, its **representative** is a tuple that contains the state from $g$ that first appears in $x$, and the local run in which this state appears first: a tuple $(x(B_{\text{first}_g}), q_r)$ is a representative for $g$ iff the following holds: $\forall q_i \in g : f_{q_i} \leq f_g$. Note that multiple guards might have the same representative.
- $\text{occurs}_m(q)$ is the number of processes that are in state $q$ at moment $m$: $\text{occurs}_m(q) = |\{B_i \in B \mid x_m(B_i) = q\}|$

A.2 LTL\(\setminus\)X Properties, Without Fairness

In this section, we show how to obtain a cutoff for LTL\(\setminus\)X properties in disjunctive systems without fairness. As mentioned before, we only need to show that a behaviour from a large system can be replicated in the cutoff system.

**Lemma 1 (Bounding Lemma, LTL\(\setminus\)X, disjunctive, non-fair).** For process templates $A, B$ with $G = G_A \cup G_B$ and $n \geq |G| + 1$:

$$(A, B)^{(1,n)} \models E h(A, B^{(1)}) \implies (A, B)^{(1,|G|+1)} \models E h(A, B^{(1)})$$

**Proof.** Let $x = x_0, x_1, \ldots$ be a run of $(A, B)^{(1,n)}$ that satisfies $h(A, B^{(1)})$. We construct a run $y = y_0, y_1, \ldots$ of $(A, B)^{(1,c)}$ that satisfies $h(A, B^{(1)})$ as follows:

1. $y(A) = x(A)$
2. \( y(B_1) = x(B_1) \)
3. for each \( g_i \in G = \{g_1, \ldots, g_k\} \), let \( x(B_{\text{first}_{g_i}}), q_r \) be the representative for \( g_i \), then \( y(B_{i+1}) = x(B_{\text{first}_{g_i}})|[1 : f_{q_r}](q_r)^w \). In other words, \( B_{j+1} \) imitates \( B_{\text{first}_{g_i}} \) until it reaches \( q_r \) then it stays in \( q_r \) forever. This is called flooding of a local state \( q_r \).

With this construction, it might happen that the run \( y \) violates the interleaving semantics requirement (i.e., that only one process moves at a time), because it is possible that two different guards have the same process representative (i.e., that only one process moves at a time), because two or more processes move at the same time.

The intuition behind the construction is that instead of flooding all states (that appear in the given run), we only flood at most one per guard — the one that appears first in \( x \).

To prove correctness, it is enough to prove that at any moment \( m \), if a transition \( t \) for a process is enabled in \( x \) then it is enabled in \( y \). Now suppose at time \( m \) a transition \( t \) is enabled in \( x \), then \( \exists q \in g_t \) (guard of transition \( t \)) and \( \exists p \) such that \( x_m(p) = q \), then \( q \) enables \( g_t \) but it is not necessarily a representative.

In case it is a representative then by construction \( g_t \) is enabled in \( y \). In case it is not, then either \( q \in Q_A \) or \( \exists q_r \in g_t \) such that \( f_{q_r} \leq f_q \), and by construction \( \exists B_{r} \) where \( y_m(B_r) = q_r \). In both cases, \( g_t \) is enabled in \( y \) at time \( m \).

### A.3 LTL\( \setminus X \) Properties, With Fairness

**Lemma 2 (Bounding Lemma, LTL\( \setminus X \), disjunctive, fair).** For process templates \( A, B \) with \( G = G_A \cup G_B \) and \( n \geq |B| + |G| + 1 \):

\[
(A, B)^{(1, n)} \models E(\Phi_{\text{uncond}} \land h(A, B^{(1)})) \implies (A, B)^{(1, |B| + |G| + 1)} \models E(\Phi_{\text{uncond}} \land h(A, B^{(1)}))
\]

**Proof.** Let \( x = x_0, x_1 \ldots \) be a run of \( (A, B)^{(1, n)} \) that satisfies \( h(A, B^{(1)}) \) and unconditional fairness. Given a subset \( F \subseteq B \), define

\[
\text{Visited}^{\text{inf}}_F = \{ q \in Q_B \mid \text{appears}_q \text{ is infinite} \}
\]

\[
\text{Visited}^{\text{fin}}_F = \{ q \in Q_B \mid \text{appears}_q \text{ is finite} \}
\]

A tuple \( (x(B_{\text{first}_{g_i}}), q_r) \) is an infinite representative for a guard \( g \in G \) if \( q_r \in \text{Visited}^{\text{inf}}_F \) and \( \forall q_t \in g, q_t \in \text{Visited}^{\text{inf}}_F : f_{q_t} \leq f_{q_r} \).

**Construction:** We construct a run \( y = y_0, y_1 \ldots \) of \( (A, B)^{(1, c)} \) that satisfies \( h(A, B^{(1)}) \) and unconditional fairness:

1. \( y(A) = x(A) \).
2. \( y(B_1) = x(B_1) \).
3. to every \( q \in \text{Visited}^{\text{fin}}_{B_{\text{first}_{g_i}}} \) devote one process \( B_{i_q} \) such that \( y(B_{i_q}) = x(B_{\text{first}_{g_i}})|[1 : f_{q}](q)^{1 - f_{q_i}}x(B_{\text{last}_{g_i}})[l_{q_i} + 1 :] \)

This is called flooding of state \( q \) with evacuation into \( \text{Visited}^{\text{inf}}_F \) (since \( B_{\text{last}_{g_i}} \) has to move into \( \text{Visited}^{\text{inf}}_F \) eventually).
4. for each \( g \in G_B \), let \((x(B_{\text{first}_{g}}, q_r), q_r)\) be the infinite representative for \( g \), and devote two processes \( B_{q_l} \) and \( B_{q_r} \) to \( g \), such that \( y(B_{q_l}) \) and \( y(B_{q_r}) \) imitate \( x(B_{\text{first}_{g}}) \) until the first occurrence of \( q_r \), then they take turns: always one process copies \( x(B_{\text{first}_{g}}) \) while the other stutters in \( q_r \), and they switch roles every time \( x(B_{\text{first}_{g}}) \) visits \( q_r \).

The local runs of the processes devoted to states in \( \text{Visited}_{B_2 \ldots B_n}^{\text{fin}} \) ensure that at any moment the subset of \( \text{Visited}_{B_2 \ldots B_n}^{\text{fin}} \) that appears in \( y \) is a superset of the subset of \( \text{Visited}_{B_2 \ldots B_n}^{\text{inf}} \) that appears in \( x \). Together with the local runs of the processes devoted to the guards’ infinite representatives, this ensures that any transition enabled in \( x \) is also enabled in \( y \): at any moment, if a state \( q_i \) appears in \( x \) and either \( q_i \in \text{Visited}_{B_2 \ldots B_n}^{\text{inf}} \) or \( q_i \) is an infinite representative of some guard, then it also appears in \( y \).

Note that for the finite part we cannot use the guard representative, because its “life span” may be shorter than we need, and we cannot flood it as we need to preserve fairness.

To see how many copies we need in the worst case, note that every process is visited finitely or infinitely often, and from the latter there may be up to \( k \) states for which we need two instances. Let’s denote \( |\text{Visited}_{B_2 \ldots B_n}^{\text{fin}}| \) by \( \text{fin} \). Then we need at most \( \text{fin} + 2k + 1 \) instances (including one instance for \( B_1 \)). However, if we write \( \text{inf} \) for \( |\text{Visited}_{B_2 \ldots B_n}^{\text{inf}}| \), then we have \( \text{fin} = |B| - \text{inf} \). Then, since we know that \( k \leq \text{inf} \), we have \( \text{fin} + 2k + 1 = |B| - \text{inf} + 2k + 1 \leq |B| + k + 1 \).

### A.4 Local Deadlocks, Without Fairness

We give a bounding lemma for local deadlocks without fairness, using a new construction.

**Lemma 3 (Bounding Lemma, local deadlocks, disjunctive, non-fair).** Let \( A, B \) be process templates with \( G = G_A \cup G_B \). Let \( Q_B^1 = \{ q \in Q_B \mid |\text{Enable}_q| < |B| \} \) and \( m = \max_{q \in Q_B^1} (|\text{Enable}_q|) \). Then, for \( c = m + |G| + 1 \leq n \):

\[(A, B)^{(1,n)} \text{ has a local deadlock} \implies (A, B)^{(1,c)} \text{ has a local deadlock.}\]

Note that if a local deadlock is possible in \( q \), then \( |\text{Enable}_q| < |B| \), i.e., \( Q_B^1 \) is the set of states in which a local deadlock could occur.

**Proof.** Given a locally deadlocked run \( x = x_0, x_1 \ldots \) of \( (A, B)^{(1,n)} \), we construct a locally deadlocked run \( y = y_0, y_1 \ldots \) of \( (A, B)^{(1,c)} \).

**Construction:**

Assume \( B_1 \) is locally deadlocked in state \( q_l \) (other cases are similar):

1. set \( y(A) = x(A) \) and \( y(B_1) = x(B_1) \)
2. for every \( q \in \text{Enable}_{q_l} \), if \( q \) appears in the run (i.e., \( \exists j, m : x_m(j) = q \)), devote one process \( B_{i_q} \) such that \( y(B_{i_q}) = x(B_{\text{first}_{i_q}})[1 : f_q](q)^{l_q} - f_q \cdot x(B_{\text{first}_{i_q}})[l_q + 1 :] \)
3. for every guard $g \in G$, let $(x(B_i), q_r)$ be the representative for $g$, then devote one process $B_j$ of $(A, B)^{(1,c)}$ such that: $y(B_j) = x(B_i)[1 : f_q,](q_r)^w$. Note that if $q_r \in \text{Enable}_{q_l}$ then we must choose the next representative of $g$. If we cannot find a representative that is not in $\text{Enable}_{q_l}$, then we simply disregard the guard.

Suppose the deadlock in the original run occurred at time $d$, then the construction ensures that, at any time $t \geq d$ we have $\neg \exists q_i \in \text{Enable}_{q_l}$ and $q_i \in y(t)$. Therefore the local deadlock is preserved in the constructed run $y$ at any time greater than $d$. Furthermore, all transitions in $y$ are enabled by a similar argument as in the proof of Lemma 1.

A.5 Local Deadlocks, With Fairness

**Lemma 4 (Bounding Lemma, local deadlocks, disjunctive, fair).** For process templates $A, B$ with $G = G_A \cup G_B$ and $n \geq |B| + |G| + 1$, and strong-fair runs:

$$(A, B)^{(1,n)} \text{ has a local deadlock } \implies (A, B)^{(1,|B|+|G|+1)} \text{ has a local deadlock}$$

**Proof.** We can use the same construction as for Lemma 2, where either process $A$ or process $B_1$ is now the process that is eventually locally deadlocked. The local deadlock is preserved since states that appear finitely often in the original run, also appear also finitely often in the constructed run. Fairness holds by construction.

A.6 Global Deadlocks

For Theorem 1, we defined $\mathcal{N} = \{q \in Q_B \mid q \in \text{Enable}_q\}$, and $\mathcal{N}^*$ as the maximal subset (wrt. number of elements) of $\mathcal{N}$ such that $\forall q_i, q_j \in \mathcal{N}^* : q_i \notin \text{Enable}_{q_j} \land q_j \notin \text{Enable}_{q_i}$. To prove the part of the theorem that regards global cutoffs, we need to prove the following lemma.

**Lemma 5 (Bounding Lemma, global deadlocks, disjunctive).** For disjunctive systems and $n \geq |B| + |\mathcal{N}^*|:

$$(A, B)^{(1,n)} \text{ has a global deadlock } \implies (A, B)^{(1,|B|+|\mathcal{N}^*|)} \text{ has a global deadlock}$$

**Proof.** For a state $q \in Q_A \cup Q_B$, let $\text{dead}_q^y = Q_A \cup Q_B \setminus \text{Enable}_q$. Given a run $x = x_0, x_1, \ldots$, a state $q \in Q_B$ is disabled at time $m$ if all of the following hold:

- $q \in \text{Set}_m(x)$,
- $\text{Set}_m(x) \setminus \{q\} \subseteq \text{dead}_q^y$, and
- if $q \in \text{Enable}_q$ then $\text{occurs}_m(q) = 1$.

A state $q \in Q_A$ is disabled at time $m$ if the first two conditions above hold. Then, a run $x$ is globally deadlocked at time $m$ iff all $q \in \text{Set}_m(x)$ are disabled at time $m$. Note that this holds iff the following two conditions hold:
∀q_i ≠ q_j ∈ Set(x_m) : q_i ∈ dead^q_j and q_j ∈ dead^q_i,
and
∀q_i ∈ (Set_m(x) ∩ N) : occurs_{x_m}(q_i) = 1.

These conditions determine the configurations of a system \((A,B)^{(1,n)}\) in which a global deadlock is possible. This observation is crucial to obtain smaller cutoffs for global deadlock detection.

The cutoff obtained previously was \(c = 2|B| - 1\). In the proof of this result [2], the processes are divided into two sets: \(C\) and \(B \setminus C\), where \(B\) is the set of all \(B\)-processes and \(C\) is the set of processes deadlocked in a state from \(N\). In the following, let \(\text{Visited}_{B \setminus C}^{inf}\) be the set of states in which the processes from \(B \setminus C\) are deadlocked, and let \(\text{Visited}_{B \setminus C}^{fin}\) be the states that are only visited on the path to the deadlock. Then, a run of \((A,B)^{(1,c)}\) is constructed as follows:

1. Copy (in addition to process \(A\)) all local runs of processes in \(C\).
2. Flood all deadlocked states of processes \(B \setminus C\), i.e., that are in \(\text{Visited}_{B \setminus C}^{inf}\).
3. All remaining states that appear in the processes \(B \setminus C\), i.e., that are in \(\text{Visited}_{B \setminus C}^{fin}\), are flooded with evacuation into \(\text{Visited}_{B \setminus C}^{inf}\).

Therefore, \(|C| + |\text{Visited}_{B \setminus C}^{fin}| + |\text{Visited}_{B \setminus C}^{inf}|\) is a cutoff. Since \(|\text{Visited}_{B \setminus C}^{fin}| + |\text{Visited}_{B \setminus C}^{inf}| \leq |B|\), also \(|C| + |B|\) is a cutoff. Thus, we can obtain cutoffs smaller than \(2|B| - 1\) in case \(|C|\) is smaller than \(|B|\). Indeed, we know that \(|C| \leq |N|\), which is in many cases much less than the size of \(B\). Thus, the cutoff can be reduced to \(|N| + |B|\).

If we consider in addition to the properties of single states also the properties of pairs of states, then the cutoff can be minimized further: if two states are not in the \(dead^v\) sets of each other, they can never be together part of a global deadlock. Thus, a sufficient size for any subset of \(N\) that can be in a global deadlock together can be found by computing the maximal subset \(N^* \subseteq N\) such that \(∀q_i, q_j \in N^* : q_i \notin \text{Enable}_{q_j} \land q_j \notin \text{Enable}_{q_i}\). □

Remark. Computing \(N^*\) exactly amounts to computing the minimal vertex cover \(mvc\) of the undirected graph \(G = (V,E)\), where:

- \(V = N\)
- \(E = \{(q_1, q_2) | q_1 \notin \text{dead}^q_{q_2}\}\)

The vertex cover problem is \(NP\)-Complete, but it can be safely underapproximated in the following way: first we sort the states by their number of edges in descending order, then starting from the top, we compute minimum number of states \(U\) such that the sum of their edges is greater or equal to \(|E|\). The correctness of this method stems from the fact that any set of states with size less than \(U\) can never be a vertex cover.
B Appendix: Proofs and Proof Methods for Conjunctive Systems

In this section, we present lemmas and proof methods that allow us to obtain our cutoff results for local and global deadlock detection in conjunctive systems. For LTL \( \neg X \) properties, we do not give new cutoff results, since the existing ones are already optimal (see Table 2).

B.1 Definitions

Given a system \( (A, B)^{(1,n)} \), we define the following:

- We say that \( D \subseteq (Q_A \cup Q_B) \) is a deadset of \( q \in (Q_A \cup Q_B) \) if \( \forall (q, \sigma, g, q') \in \delta : \exists q'' \in D : q'' \not\in g \) and \( \forall q'' \in D \exists (q, \sigma, g, q') \in \delta : q'' \not\in g \), and \( D \) contains at most one state from \( Q_A \).
- \( \text{dead}_q^\wedge \) is the set of all deadsets of \( q : \text{dead}_q^\wedge = \{ D \subseteq (Q_A \cup Q_B) \mid D \text{ is a deadset of } q \} \).

B.2 Global Deadlocks

Recall that \( \text{dead}_q^\wedge = \emptyset \), then we say \( q \) is free. If a state \( q \) does not appear in any \( \text{dead}_q^\wedge \), then we say \( q \) is non-blocking. If a state \( q \) does not appear in \( \text{dead}_q^\wedge \), then we say \( q \) is not self-blocking.

Lemma 6 (Bounding Lemma, global deadlocks, conjunctive). In a conjunctive system, where

- \( D_1 \subseteq Q_B \) is the set of free states in \( B \),
- \( D_2 \subseteq Q_B \) is the set of non-blocking states in \( B \), and
- \( D_3 \subseteq Q_B \) is the set of not self-blocking states in \( B \).

Let \( c = 2|B| - 2|D_1| - 2|D_2 \setminus D_1| - |D_3 \setminus (D_1 \cup D_2)| \). Then, for \( n \geq c \):

\( (A, B)^{(1,n)} \) has a global deadlock \( \implies (A, B)^{(1,c)} \) has a global deadlock

Proof. Given a run \( x = x_0, x_1... \), a state \( q \in \text{Set}_m(x) \) is disabled at time \( m \) iff:

- \( \exists D \in \text{dead}_q^\wedge : D \subseteq \text{Set}_m(x) \)
- if \( q \in D \) then \( \text{occurs}_{x_m}(q) \geq 2 \).

A run \( x \) is globally deadlocked at time \( m \) iff all \( q \in \text{Set}_m(x) \) are disabled.

For a deadlocked run \( x \) of \( (A, B)^{(1,n)} \), let \( \text{Visited}^{m}_{x_f} = \text{Set}_m(x) \cap Q_B \), i.e., the set of states of \( B \) that appear in the deadlock. Außerlechner et al. [2] have shown that then the global deadlock can be replicated in \( (A, B)^{(1,c)} \) by copying, for each \( q \in \text{Visited}^{m}_{x_f} \), at most two local runs that end in \( q \). Since \( \text{init} \) is assumed to appear in every guard, the resulting cutoff is \( 2|B| - 2 \).

By a similar argument as for \( \text{init} \), we can obtain an even smaller cutoff if any of the other states in process template \( B \) satisfy one of the properties defined before this lemma. In particular, \( \text{init} \) is an example of a non-blocking state. If
there are other non-blocking states in $B$, then the cutoff can be reduced by the same argument as for $\text{init}$: since such states do not block any transitions, local runs that end in these states can just be removed from the system, and the run will still be deadlocked. Moreover, we can also reduce the cutoff if there are states that are not self-blocking: the reason why we may need 2 copies of a state $q$ is that the second copy may be needed to block a transition of another process that also is in $q$. However, if $q$ is not self-blocking, then this second copy is not necessary. Finally, if $q$ is not self-blocking, then this second copy is not necessary. Finally, if $q$ is free, then $q$ cannot be part of a deadlocked configuration at all, since $q$ always has at least one transition that can be taken. Thus, copied local runs for free states will never be necessary.

Thus, we can reduce the cutoff to $2|B| - 2|D_1| - 2|D_2 \setminus D_1| - |D_3 \setminus (D_1 \cup D_2)|$. Note that if this results in a cutoff of 0 or 1, then we have statically detected that a global deadlock is not possible.

Example 2. Consider the process templates in Figure 2.

A. \hspace{1cm} B.

The deadsets of the local states are:

- $\text{dead}_{1_B} = \{1_B, 2_B, 3_B\}$
- $\text{dead}_{2_B} = \text{dead}_{1_B}$
- $\text{dead}_{3_B} = \text{dead}_{1_B}$
- $\text{dead}_{in_A} = \text{dead}_{1_B}$
- $\text{dead}_{in_B} = \emptyset$

The state $in_B$ can not be part of any global deadlock because its deadset is empty. On the other hand the deadset $D = \{1_B, 2_B, 3_B\}$ can be a part of a global deadlock and it is reachable. According to the definition of the global deadlock all the states of this set must be duplicated in the run.
B.3 Local Deadlocks

Local deadlock detection in conjunctive systems is not an easy task even for the unfair case. The main problem is to find the minimum number of processes needed that can provide an infinite behavior while preserving the deadlock. In some special cases, this number can be found by fetching special lassos from the process templates.

Definitions

Given a system \((A, B)^{(1,n)}\) and a run \(x = x_1, x_2, \ldots\), we define the following:

- A sequence of states \(q_1 \ldots q_n\) is connected if \(\forall q_i \in \{q_1, \ldots, q_n\} : \exists (q_i, \sigma, g, q_{i+1}) \in \delta\)
- A cycle is a connected sequence of states \(q_1 \ldots q_n \) such that \(\forall q_i, q_j \in \{q_1, \ldots, q_n\} : q_i \neq q_j\). We denote such a cycle by \(C_q\). (By abuse of notation, \(C_q\) is also used for the set of states on \(C_q\).) We denote the set of guards of the transitions on \(C_q\) as \(G_{C_q}\) if \(G_{C_q}\) is called free if \(\forall p \in C_q \setminus q \forall g \in G_{C_q} : p \in g\). We denote such a cycle by \(C_{q}^{free}\).
- A covered alternation between two states \(p\) and \(q\) occurs iff \(\exists m, m'\) where \(m + 1 < m', p \in x_m, q \notin x_m, \forall i \in [m + 1, m']\ \{p, q\} \subseteq x_i, p \notin x_{m'}\) and \(q \in x_{m'}\).
- A lasso \(lo\) is a connected sequence of states \(q_0 \ldots q_i \ldots q_n\) such that:
  - \(q_0\) is an initial state
  - \(q_i = q_n\), and \(q_i \ldots q_n\) is a cycle.
  - We denote by \(G_{lo}\) the set of guards of the transitions on \(lo\).

Local Deadlocks in 1-conjunctive Systems

Lemma 7 (Bounding Lemma, local deadlocks, 1-conjunctive, non-fair).

For a 1-conjunctive system \((A, B)^{(1,n)}\) and \(n \geq |G_B| + 2\):

\((A, B)^{(1,n)}\) has a local deadlock \(\implies (A, B)^{(1,|G_B|+2)}\) has a local deadlock

Proof. This result follows from Außerlechner et al. [2, Lemma 12]. The proof construction in a nutshell was that if in a run of \((A, B)^{(1,n)}\), process \(B_1\) is locally deadlocked in some state \(q_i\) at time \(d\), then we construct a run of \((A, B)^{(1,c)}\) by computing \(q_i\)’s deadset and for each state \(q \in Q_B\) in the deadset we copy one local run until it visits \(q\), and then we let it stay in \(q\) forever. In addition, we copy the local runs of \(B_1\) and some process that moves infinitely often. Since our system is 1-conjunctive, the size of any deadset is always less or equal to \(|G_B|\).

Lemma 8 (Bounding Lemma, local deadlocks, 1-conjunctive, fair).

For a 1-conjunctive system \((A, B)^{(1,n)}\) and \(n \geq 2|G_B| + 1\) and strong-fair runs:

\((A, B)^{(1,n)}\) has a local deadlock \(\implies (A, B)^{(1,2|G_B|+1)}\) has a local deadlock
Proof. Similar to what we have described above, we get this result by inspection of the proof of Außerlechner et al. [2, Lemma 16]. The original construction includes 2 local runs for every state in the deadset, and one additional state that is locally deadlocked. Since the size of the deadset is bounded by $|G_B|$, we get that $2|G_B| + 1$ processes are sufficient to replicate the local deadlock.

Local Deadlocks: Beyond 1-conjunctive Systems In this sections we will show how to obtain cutoffs for conjunctive systems that are not 1-conjunctive. First, we will consider a number of cases that can be reduced the 1-conjunctive case, and therefore have the same cutoff. Then, we will consider a case that cannot be reduced to the 1-conjunctive case, and show that it already requires a significantly larger cutoff. Example 1 shows that the cutoff for local deadlock detection in general conjunctive systems is at least quadratic in the number of states, and can grow independently of the number of guards. Since a general cutoff results are very hard to obtain, and would not be very useful because of their size, we restrict ourselves to these partial results.

Below, for simplicity we explain one case in detail: a system $(A, B)^{(1,n)}$ where a single guard, say $(g^2_{q_l} = \neg a \land \neg b)$, is 2-conjunctive, and all other guards are 1-conjunctive. We further assume that $g^2_{q_l}$ only appears in transitions from $q_l$ to some other state.

We then explain how this case can be generalized.

Systems with Alternation-free Local Deadlocks If any of the following holds, then for non-fair runs we can reduce the problem to the 1-conjunctive case:

1. If the deadlock is not possible on $q_l$, either because $q_l$ is not reachable, or because $q_l$ is free. Since we assumed that all other processes have only 1-conjunctive guards, the problem reduces to the 1-conjunctive case, and the same cutoff applies. This also holds for the fair case.
2. If there exists a lasso $l_{q_l}$ such that $\forall(q_l, \sigma, g, q'') \in \delta, \forall g_{l_{q_l}} \in G_{l_{q_l}}$ we have $g_{l_{q_l}} \neq \neg q_l$ and $g_{l_{q_l}} \cap g = g_{l_{q_l}}$, then the 1-conjunctive cutoff applies.

The idea of this restriction is that we need one process that can move infinitely often, after the deadlocked process enters $q_l$ and we have other processes in all the states that disable $q_l$. Since one representative per guard is enough for this, we need at most $|G_B|$ processes to disable $q_l$. The two additional processes are the deadlocked one and the one that moves through the lasso. This process waits in init until all other processes have reached their destination. Then, by construction, it can take transitions along the lasso until infinity. Since this construction is inherently not fair, it does not give a cutoff for the fair case.

3. The requirement above can be relaxed, in that not a single lasso must be free of both $\neg a$ and $\neg b$, but it is sufficient if two separate lassos exist, one that is free of $\neg a$, and one that is free of $\neg b$:

If there exist two lassos $l_{q_l}$ and $l_{q_2}$ such that $\forall(q_l, \sigma, g, q'') \in \delta, \forall g_{l_{q_l}} \in G_{l_{q_l}}$ we have $g_{l_{q_l}} \neq \neg q_l$ and $a \in (g_{l_{q_l}} \cap g)$ and $\forall(q_l, \sigma, g, q'') \in \delta, \forall g_{l_{q_2}} \in G_{l_{q_2}}$ we have $g_{l_{q_2}} \neq \neg q_l$ and $b \in (g_{l_{q_2}} \cap g)$, then the 1-conjunctive cutoff applies.
The cutoff does not increase compared to the previous case, since the construction will only use one of the lassos, depending on whether \( a \) or \( b \) are present in the local deadlock state of the other processes. Again, the construction is inherently not fair.

4. If for all cycles \( C_a \) that traverse \( a \) we have \( G_{C_a} \cap (G_q \cup \neg q) \neq \emptyset \), or for all cycles \( C_b \) that traverse \( b \) we have \( G_{C_b} \cap (G_q \cup \neg q) \neq \emptyset \), or if we have \( G_q \cap \{\neg a, \neg b\} \neq \emptyset \), then the cutoff for 1-conjunctive systems applies both in the non-fair and the fair case.

The idea is that under each of this assumptions, an infinite alternation between \( \{a, \neg b\} \in x_i \) and \( \{\neg a, b\} \in x_i \) is not possible. Then we simply copy one process for every 1-conjunctive guard of \( q \), and one process for either \( a \) or \( b \), as well as one more process that moves infinitely often in the original run.

For the fair case, we need up to 2 processes to ensure that every process that is enabled can also move eventually, similar to the 1-conjunctive fair case.

\[
\text{Example 3.}
\]

\[
\begin{array}{c}
  r \\
  \rightarrow \\
  \neg w \\
  \text{init} \\
  \rightarrow \\
  w \\
  \text{tw} \\
  \rightarrow \\
  \neg w \land \neg r \\
  \text{tr} \\
\end{array}
\]

- if the local deadlock is in a node that has no 2-conjunctive guard, then the problem is reduced to 1-conjunctive system.
- if the local deadlock is in \( tw \), and as all the cycles that contain \( w \) contain also \( tw \) then the covered alternation is not possible. But as there is a lasso \( lo = [\text{init}, \text{tr}, r, \text{init}] \) that is free of the guard \( \neg r \), then the 1-conjunctive cutoff for local deadlock detection can be used.

The special cases above can be generalized in the following way, which in many cases results in strong restrictions on the process template:

1. If the deadlock is not possible in any state that has guards that are not 1-conjunctive (either because they are not reachable, or because they are free), then the problem reduces to the 1-conjunctive case, and the same cutoff applies.

2. If for every state \( q_i \) with a set of transitions with not 1-conjunctive guards \( G_{q_i} = \{g_1, \ldots, g_n\} \), there exists a lasso \( lo_i \) such that \( \forall (q_i, \sigma, g, q') \in \delta, \forall g_{lo_i} \in G_{lo_i} \) we have \( g_{lo_i} \neq \neg q_i \) and \( g_{lo_i} \cap g_i = g_{lo_i} \) for all \( g_i \in G_{q_i} \), then the 1-conjunctive cutoff applies. The idea of this restriction is a straightforward generalization of what is described above.

3. As above, we can have several lassos instead of a single one: if for every state with a set of not 1-conjunctive guards \( G_{q_i} = \{g_1, \ldots, g_n\} \), and for every state
q ∈ g_i, there exists a lasso that is free of ¬q_i and ¬q, then the 1-conjunctive cutoff applies.

4. Similar to what we had for the lassos, for every state q_i with a set of transitions with not 1-conjunctive guards G_{q_i} = \{q_1, \ldots, q_n\}, for every q_i = ¬q_1 \land \ldots \land ¬q_k there must exist k − 1 cycles that are not traversable during the local deadlock. In this case, we know exactly which states can appear infinitely often during a local deadlock, and the cutoff for 1-conjunctive systems applies in both the non-fair and the fair case.

**Systems without Alternation-free Local Deadlocks** In this section we will assume that special cases do not hold, and we have to consider the case that, for a 2-conjunctive guard g = ¬a \land ¬b, alternating infinitely often between a and b is necessary to obtain a locally deadlocked run.

We need the following additional definitions:

- A segment \( S_{g_{a \rightarrow b}} \) is a connected sequence of states \( q_1 \ldots q_j \) where:
  - \( q_i \) has an incoming transition with guard ¬a
  - \( q_j \) has an outgoing transition with guard ¬b
  - \( \forall q_m \in S_{g_{a \rightarrow b}} \exists (q_m, \sigma, g, q_{m+1}) \in \delta : q_{m+1} \in S_{g_{a \rightarrow b}} \) then \( b \in g \)
- For a cycle \( C_q \), we denote by \( |S_{g_{a \rightarrow b}}|_{C_q} \) the total number of segments \( S_{g_{a \rightarrow b}} \) on \( C_q \)

- A segment transition on some cycle \( C_x \) is a path \((s_1, e_1, p)(s_2, e_2, p)\ldots(s_n, e_n, p)\) such that \( s_1(p) \in S_{g_{a \rightarrow b}} \) and \( s_n(p) \in S_{g_{b \rightarrow a}} \) and \( \forall i \ s_i(p) \in C_x \) and \( \exists p' \neq p : s_1(p') = a \) and \( b \not\in s_1 \).

For systems with a single 2-conjunctive guard that need to alternate between a and b to obtain a local deadlock, we state the following.

**Lemma 9.** Given a single 2-conjunctive system \((A, B)^{(1, n)}\) deadlocked locally in state \( q_i \), \( g_{q_i}^2 = ¬a \land ¬b \), with unique cycles \( C_a \) and \( C_b \) where these cycles are free and \( G_{(C_a \cup C_b)} \cap (G_{q_i} \cup ¬q_i) = \emptyset \). Let

\[
\begin{align*}
n_a &= \max(|S_{g_{a \rightarrow b}}|_{C_a}, |S_{g_{b \rightarrow a}}|_{C_a}) \\
n_b &= \max(|S_{g_{a \rightarrow b}}|_{C_b}, |S_{g_{b \rightarrow a}}|_{C_b})
\end{align*}
\]

Then:

\((A, B)^{(1, n)}\) has a local deadlock in \( q_i \) \( \implies \) \((A, B)^{(1, |G_B|+n_a+n_b+5)}\) has a local deadlock in \( q_i \).

To prove the lemma, we will use the following observation on transitions between segments on free cycles.

**Observation 2.** Given a single 2-conjunctive system \((A, B)^{(1, n)}\) deadlocked locally in state \( q_i \), \( g_{q_i}^2 = ¬a \land ¬b \), if there exist two cycles \( C_{a_{free}} \) and \( C_{b_{free}} \) where \( \forall g \in G_{C_{a_{free}}} : C_{b_{free}} \subseteq g \) and \( \forall g \in G_{C_{b_{free}}} : C_{a_{free}} \subseteq g \) then at any moment \( m \), if \( \text{Set}(x_m) \subseteq (C_{a_{free}} \cup C_{b_{free}}) \) then:
if \( a \in x_m \) and \( b \notin x_m \) then:

\[ \exists \text{ segment transition } S_{g_{a-b}} \text{ to } S_{g_{b-a}} \]

\[ \neg \exists \text{ segment transition } S_{g_{b-a}} \text{ to } S_{g_{a-b}} \]

if \( b \in x_m \) and \( a \notin x_m \) then :

\[ \exists \text{ segment transition } S_{g_{b-a}} \text{ to } S_{g_{a-b}} \]

\[ \neg \exists \text{ segment transition } S_{g_{a-b}} \text{ to } S_{g_{b-a}} \]

**Proof of Lemma 9.** First we need to prove that if the number of processes on \( C_a \) is less than \( n_a + 1 \), then the deadlock cannot be preserved. Suppose we have \( n_a \) processes on \( C_a \) at some time \( m \), we distinguish three cases:

1. All processes are in \( S_{g_{a-b}} \) and \( a \in S_{g_{a-b}} \): In this case \( b \in x_m \). According to Observation 2, all processes can make a segment transition, then at some time \( m' \), assuming all processes move whenever possible, all processes are in \( S_{g_{b-a}} \) and in particular a process must be in \( a \). Now after another covered alternation, all processes can make a segment transition except the one in \( a \), then the number of processes in \( S_{g_{b-a}} = |S_{g_{b-a}}| - 1 \), then at some point in time \( > m' \), by pigeonhole principle, neither \( a \) nor \( b \) will be covered and thus the deadlock can not be preserved.

2. All processes are in \( S_{g_{a-b}} \) and \( a \in S_{g_{a-b}} \): similar argument to the above.

3. Processes are scattered between \( S_{g_{b-a}} \) and \( a \in S_{g_{a-b}} \): If this was the case and as we only have \( n_a \) process, then we will have at least two empty consecutive segments \( S_{g_{b-a}} \) and \( S_{g_{a-b}} \) then at some time in the future a covered alternation is not possible.

We can deduce from the above that at least \( n_a + 1 \) processes can reach \( C_a \) and at least \( n_b + 1 \) processes can reach \( C_b \). Note that we might need one additional process for \( C_a \) cycle if \( \exists q_1, q_2 \) in \( a \)’s segment where these two states have outgoing transitions on the cycle with guard \( \neg b \) and one of them appears before \( a \) and the other after it (same applies for \( C_b \)). In the following we will denote by \( k_a \) either \( n_a + 1 \) or \( n_a + 2 \) and by \( k_b \) either \( n_b + 1 \) or \( n_b + 2 \), depending whether the special case applies or not.

**Construction.** Given a run \( x = x_1, x_2, \ldots \), let the process \( B_1 \) be the deadlocked process in state \( q_1 \), we construct the run \( y = y_1, y_2, \ldots \) as follows:

- \( y(B_1) = x(B_1) \)
- let \( D \in \text{dead}^a_\ast \) then \( \forall q \in D \setminus \{a, b\} : y(B_{i_q}) = x(B_{f_{irrst_q}})[1 : f_q(q)] \)
- \( \exists m_1, \ldots, m_{k_a} \) where \( x_{m_i}(B_{m_i}) = q : q \in C_a \) then \( y(B_j) = x(B_{m_i})[1 : m_i] \)
- \( \exists t_1, \ldots, t_{k_b} \) where \( x_{t_i}(B_{t_i}) = q : q \in C_b \) then \( y(B_a) = x(B_{t_i})[1 : t_i] \)

**Starting Positions**
let all processes move outside $a$ or $b$
for each segment $S_{b-a}$ in $C_a$ or $C_b$ let one process reaches it
let remaining processes in the closest position to $a$ or $b$

**Infinite Behavior Loop.** In the following loop, we require that no process leaves the cycle that was assigned for it in the start position.

1. let a single process moves into $b$
2. leave $a$
3. let all processes take all possible transitions except those that enters $a$ or $b$
4. let a single process moves into $a$
5. leave $b$
6. let all processes take all possible transitions except those that enters $a$ or $b$
7. go to 1

Starting positions are valid as we assumed that the cycles are free and their guards are independents of both cycles states. The infinite behavior loop chosen ensures continues covered alternation between $a$ and $b$, this is due to the fact that the loop has the following two invariants:

1. At anytime $m$ after starting the loop, there is always a process in $a$, or a process with enabled transitions to reach $a$ (while $b$ is occupied).
2. At anytime $m$ after starting the loop, there is always a process in $b$, or a process with enabled transitions to reach $b$ (while $a$ is occupied).

\[\square\]

C Appendix: Proofs and Proof Methods for Extensions

**Lemma 10 (Bounding Lemma for Disjunctive Target).** For disjunctive systems and process templates $A, B$ with $q \in Q_B$: $$(A, B)^{(1,n)} \models \text{Target}(q) \implies (A, B)^{(1,|B|)} \models \text{Target}(q)$$

*Proof.* Given a run $x$ of $(A, B)^{(1,n)}$ where eventually all $B$-processes are in $q$ at the same time $m$, let $D \subseteq Q_B$ be the set of all states of $B$ that appears in $x$ up to time $m$. To construct a run $y$ that satisfies $\text{Target}(q)$ in $(A, B)^{(1,|B|)}$, we flood all states in $D$, and evacuate them to $q$ at the time they occur for the last time before moment $m$. Since neither flooding of a state, nor evacuation from a state can depend on another process in the same state, $|B|$ processes are sufficient, at most one per state.

\[\square\]

**Lemma 11 (Bounding Lemma for Disjunctive Repeat-Target).** For disjunctive systems and process templates $A, B$ with $q \in Q_B$: $$(A, B)^{(1,n)} \models \text{Repeat-Target}(q) \implies (A, B)^{(1,|B|)} \models \text{Repeat-Target}(q)$$

This result holds with or without restriction to fair runs.
Proof. To construct a run $y$ of $(A, B)^{|1..|B|]}$, we essentially use the construction from above twice. The construction is the same up to moment $m$. Then, in the original run there must be a time $m'$ such that all processes are again in $q$. Let $D'$ be the set of all states that appear between $m$ and $m'$ in $x$, and use the same construction as above to extend the run $y$ until all processes visit $q$ again. This construction can then be repeated to obtain an infinite run that satisfies Repeat-Target$(q)$. To obtain a fair run, we may have to consider not a simple loop from $\forall i.q_i$ to $\forall i.q_i$, but we have to find a loop such that every process moves at least once. If the original run was fair, such a loop must exist. The cutoff remains the same. \hfill $\square$