Possibilities of Recursive GPU Mapping for Discrete Orthogonal Simplices

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Abstract—The problem of parallel thread mapping is studied for the case of discrete orthogonal m-simplices. The possibility of a $O(1)$ time recursive block-space map $\lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ is analyzed from the point of view of parallel space efficiency and potential performance improvement. The 2-simplex and 3-simplex are analyzed as special cases, where constant time maps are found, providing a potential improvement of up to $2 \times$ and $6 \times$ more efficient than a bounding-box approach, respectively. For the general case it is shown that finding an efficient recursive parallel space for an $m$-simplex depends on the choice of two parameters, for which some insights are provided which can lead to a volume that matches the $m$-simplex for $n > n_0$, making parallel space approximately $m!$ times more efficient than a bounding-box.

Keywords—GPU computing; thread mapping; discrete orthogonal simplices;

I. INTRODUCTION

The field of GPU computing has become a well established research area in the last ten years [20], [18], [17] thanks to the high performance of programmable graphics hardware and the release of a generic GPU programming model, being CUDA [19] and OpenCL [10] the most known implementations. In the GPU programming model there are three constructs that allow the execution of highly parallel algorithms; (1) thread, (2) block and (3) grid. Threads are the smallest elements and they are in charge of executing the instructions of the GPU kernel. A block is an intermediate structure that contains a set of threads organized as an Euclidean box. Blocks provide fast shared memory access as well as local synchronization for all of its threads. The grid is the largest construct of all three and it keeps all blocks together spatially organized for the execution of a GPU kernel. These three constructs play an important role when mapping the execution resources to the problem domain.

For every GPU computation there is a stage where threads are mapped from parallel to data space. A map, defined as $f : \mathbb{Z}^k \rightarrow \mathbb{Z}^m$, transforms each $k$-dimensional point $x = (x_1, x_2, \ldots, x_k)$ in parallel space into a unique $m$-dimensional point $f(x) = (y_1, y_2, \ldots, y_m)$ in data space. GPU parallel spaces are defined as orthotopes $\Pi^m_n$ in $m = 1, 2, 3$ dimensions. A known approach for mapping threads is to build a bounding-box (BB) type of orthotope, sufficiently large to cover the data space and map threads using the identity $f(x) = x$. Such map is highly convenient and efficient for the class of problems where data space is also defined by an orthotope; such as vectors, tables, matrices and box-shaped volumes. But there is a different class of problems where data space follows a discrete orthogonal $m$-simplex organization (see Figure 1).

Figure 1: Discrete orthogonal $m$-simplices up to $m = 3$ dimensions.

Problems such as the Euclidean distance matrix (EDM) [13], [12], [14], collision detection [1], adjacency matrices [9], cellular automata simulation on triangular/tetrahedral spatial domains [4], matrix inversion [21], LU/Cholesky decomposition [5] and the $n$-body problem [23], [2], [7], among others, follow the shape of a discrete orthogonal 2-simplex, $\Delta_2^m$, with a volume of $V(\Delta_2^m) = n(n+1)/2 \in \mathcal{O}(n^2)$. The default bounding-box (BB) approach turns out to be inefficient because the volume of its parallel space, $V(\Pi_2^m)$, produces $n(n-1)/2 \in \mathcal{O}(n^2)$ unnecessary threads (see Figure 2).

Figure 2: For $m = 2$, the bounding-box strategy generates a parallel space $P_2$ that approaches $2 \times$ the required volume.

Higher dimensional orthotopes can be still be represented by linearizing to a three-dimensional one.
Problems such as the triple-interaction n-body problem [11] and triple correlation analysis [6] are represented with a discrete orthogonal $3$-simplex. In the $3$-simplex class, data space has a size of $V(\Delta_n^3) = n(n+1)(n+2)/6 \in \mathcal{O}(n^3)$ elements, organized in a tetrahedral way. Once again, the default bounding-box (BB) approach is inefficient as it generates a parallel volume $V(\Pi_n^m)$ with $\mathcal{O}(n^3)$ unnecessary threads (see Figure 3).

Figure 3: Bounding-box approach mapped to a discrete orthogonal tetrahedron.

In general, an orthogonal $m$-simplex is by definition an $m$-dimensional polytope where its facets define a convex hull, with one vertex having all of its adjacent facets orthogonal one to each other. A discrete orthogonal $m$-simplex, denoted as $\Delta_n^m$, is the analog of the continuous one, but volumetric and composed of a finite number of discrete elements $\vec{x} = \{x_1, x_2, ..., x_m\}$ that can be characterized as

$$\Delta_n^m \equiv \{\vec{x} \in \mathbb{Z}_n^n | 0 \leq x_1 \leq n \land x_1 + x_2 + \ldots x_m \leq n\}.$$ (1)

which establishes an upper bound for the absolute Manhattan distance from any element $\vec{x}$ to the orthogonal corner of the $m$-simplex. The expression for the volume of an $m$-simplex is well defined by the Simplicial polytopic numbers

$$V(\Delta_n^m) = \binom{n + m - 1}{m} = \frac{n(n+1)...(n+m-1)}{m!}$$ (2)

which can be proved using an induction [3] on the fact that the volume of $\Delta_n^{m+1}$ is the sum of the volumes of $n$ stacked $m$-simplexes of lengths $\{1, 2, 3, ..., n\}$, i.e.,

$$V(\Delta_n^{m+1}) = \sum_{i=1}^{n} V(\Delta_i^m)$$ (3)

which when combined with the properties of sums of binomial coefficients, leads to expression [2]. When using a bounding-box approach, the fraction of extra volume of $V(\Pi_n^m)$ that lies outside of the $m$-simplex approaches to

$$\lim_{n \to \infty} \alpha(\Pi, \Delta)_n = \frac{V(\Pi_n^m)}{V(\Delta_n^m)} = 1 = m! - 1$$ (4)

making it an inefficient approach for large $n$ as $m$ increases.

A natural enumeration approach can be used by expanding expression [3] and indexing the elements in a linear way. Such approach allows to formulate a map of the form $g : \mathbb{Z}^1 \mapsto \mathbb{Z}^m$ with $V(\Pi_n^m) = V(\Delta_n^m)$. Although $g(\vec{x})$ may be computable by arithmetic and elementary functions, its complexity increases with $m$ as it requires the solution of an $m$-th order equation. Furthermore, the method is limited to $m \leq 4$ as no analytical solutions exist for polynomials of $m \geq 5$. It is of interest then to find a different kind of map, free of such problems.

The limitations of the enumeration principle can be overcome, in great part, by taking advantage of the dimensionality available in the parallel space. Although parallel spaces in GPU cannot have a geometry different from an orthotope, they can be $m$-dimensional which makes them topologically equivalent to an $m$-simplex. Finding an homeomorphism of the form $\lambda : \mathbb{Z}^m \mapsto \mathbb{Z}^m$ would produce zero dimensional distance between parallel and data spaces which would free it from the computation of $m$-th roots.

This work presents a study of the possibilities of recursive GPU mapping of thread-blocks onto $m$-simplices. A dedicated analysis is devoted to the special cases of 2-simplex and 3-simplex, where $O(1)$ time maps are found and described, offering a space improvement of $2 \times$ and $6 \times$, respectively, that results in a potential performance improvement given that no $m$-roots are required. For general $m$ it is shown that building an efficient set of recursive orthotopes requires finding optimal values for the reduction factor $r$ and the arity $b$. Values for both parameters are analyzed, giving the possibility to build highly tight recursive volumes for $n \geq n_0$, making an improvement of $m!$ in parallel space efficiency with respect to the bounding-box approach.

The rest of the manuscript presents related work (Section [1]), a formal definition and analysis of $\lambda(\vec{x})$ (Section [1]) for the different cases and finally the main results are discussed including future work (Section [IV]).

II. RELATED WORK

Ying et. al. have proposed a GPU implementation for parallel computation of DNA sequence distances [22] which is based on the Euclidean distance maps (EDM), a problem in the 2-simplex class. The authors mention that the problem domain is indeed symmetric and they do realize that only the upper or lower triangular part of the interaction matrix is sufficient. Li et. al. [12] have also worked on GPU-based EDMs on large data and have also identified the symmetry involved in the computation.

Jung et. al. [8] proposed packed data structures for representing triangular and symmetric matrices with applications to LU and Cholesky decomposition [5]. The strategy is based on building a rectangular box strategy (RB) for accessing and storing a triangular matrix (upper or lower). Data structures become practically half the size with respect to classical methods based on the full matrix. The strategy was originally intended to modify the data space (i.e., the matrix), however one can apply the same concept to the parallel space.
Ries et al. contributed with a parallel GPU method for the triangular matrix inversion [21]. The authors identified that the parallel space indeed can be improved by using a recursive partition (REC) of the grid, based on a divide and conquer strategy. The approach takes \( O(\log_2(n)) \) time by doing a balanced partition of the structure, from the orthogonal point to the diagonal.

Q. Avril et al. proposed a GPU mapping function for collision detection based on the properties of the upper-triangular map [1]. The map is a thread-space function \( u(x) \rightarrow (a, b) \), where \( x \) is the linear index of a thread \( t_x \) and the pair \((a, b)\) is a unique two-dimensional coordinate in the upper triangular matrix. Since the map works in thread space, the map is accurate only in the range \( n \in [0, 3000] \) of linear problem size.

Navarro, Hichtfeld and Bustos have proposed a block-space map function for 2-simplices and 3-simplices [16], [13], based on the solution of an \( m \) order equation that is formulated from the linear enumeration of the discrete elements. The authors report performance improvement for 2-simplices. For the 3-simplex case, the mapping technique is extended to discrete orthogonal tetrahedron, where the parallel space usage can be \( 6 \times \) more efficient. However the authors clarify that it is difficult to translate such space improvement into performance improvement, as the map requires the computation of several square and cubic roots that introduce a significant amount of overhead to the process. From the point of view of data-reorganization, a succinct blocked approach can be combined along with the block-space thread map, producing additional performance benefits with a sacrifice of \( o(n^3) \) extra memory.

The present work proposes a new type of map \( \lambda(\tilde{w}) \) that uses a recursive organization of blocks but does not require multiple passes to map threads onto the data space. Instead, it maps all blocks directly to the data space by using a flat expression of lower computational cost than the non-linear maps proposed in the past, which were based on the enumeration principle [1, 16, 15].

III. FORMULATION OF \( \lambda(\tilde{w}) \)

Note: for practical purposes, a discrete orthogonal \( m \)-simplex will be just referred as an \( m \)-simplex.

The formulation of \( \lambda(\tilde{w}) \) begins by considering the special cases \( m = 2, 3 \), where the mapping is graphically represented. Case \( m = 1 \) is not considered as both orthotopes and simplices match in geometry.

A. Mapping to 2-Simplices

For a 2-simplex the volume of \( \Delta^2_n \) is given by the triangular numbers

\[
V(\Delta^2_n) = \frac{n(n + 1)}{2}
\]

An orthotope \( \Pi^2_n \) can be subdivided by a set \( S^2_n \) of self-similar sub-orthotopes with a recursive structure, giving a volume of

\[
V(\Pi^2_n) = V(S^2_n) = \left(\frac{n}{2}\right)^2 + 2V(S^2_{n/2})
\]

with a boundary condition of \( V(S^2_2) = 1 \) and \( n = 2^k \) with \( k \in \mathbb{Z}_+ \). Its expanded form produces the sum

\[
V(S^2_n) = 2^0 \left(\frac{n}{2}\right)^2 + 2^1 \left(\frac{n}{2}\right)^2 + \cdots + 2^{\log_2 n} \left(\frac{n}{2^{\log_2 n}}\right)^2
\]

\[
= n^2 \sum_{i=1}^{\log_2 n} \left(\frac{1}{2^i}\right).
\]

where its reduction via the geometric series \( \sum_{i=0}^{k} a^i = \frac{a^{k+1} - 1}{a - 1} \), results in

\[
V(S^2_n) = \frac{n^2}{2} \left(1 + \sum_{i=0}^{\log_2 n} \left(\frac{1}{2^i}\right)\right)
\]

\[
= \frac{n^2}{2} \left(1 + \frac{(1/2)^{\log_2 n + 1} - 1}{1/2 - 1}\right)
\]

\[
= \frac{n(n - 1)}{2} = \Delta^2_{n-1}.
\]

The result from expression (11) is equivalent to

\[
V(S^2_n) + n = V(S^2_{n+1}) = V(\Delta^2_n).
\]

which means that set \( S^2_{n+1} \) can be organized both as an orthotope \( \Pi^2_n \) as well as a 2-simplex \( \Delta^2_n \). Therefore, a proper block-space homeomorphism \( \lambda : \mathbb{Z}^2_n \rightarrow \mathbb{Z}^2_n \) could map \( \Pi^2_n \) onto \( \Delta^2_n \) and provide an improvement in parallel space efficiency, as shown in Figure 4.

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Figure 4: Both \( \Pi^2_n \) and \( \Delta^2_n \) can be defined by a recursive set \( S^2_{n+1} \).
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Let \( \tilde{w} = (w_x, w_y) \) be a block of threads in parallel space \( \Pi^2_n \) (each block is illustrated as a gray lined square in Figure 4) located at \((x, y)\), with the origin at the top-left corner. The value \( w_y \) of a block can be used to obtain the recursion level, \( i.e., \log_2 y \), which is used to define the starting height value \( b = 2^{\log_2 y} \) for the type of orthotope whom \( w_x, y \) belongs to. The value \( q = \lfloor w_x / b \rfloor \) provides a way to know which of the sub-orthotopes of the same level \( w_x, y \) belongs to.
B. Mapping to 3-Simplices

For a 3-simplex of size \( n \) per dimension, denoted as \( \Delta_3^n \), its volume is given by the tetrahedral numbers

\[
V(\Delta_3^n) = \frac{n(n+1)(n+2)}{6}.
\]

It is important to identify that there are multiple ways of extending the two-dimensional approach to three dimensions. One way to formulate \( S_m^{3n=3} \) is to extend the binary approach used in 2-simplices, now to half-cubes with an arity of \( \beta = 3 \) for the recursion, as the illustration of Figure 5.

![Figure 5: Two different views of how the orthotope set (red) maps to the tetrahedron (white cells) using an arbitrary number of recursions.](image)

From the illustration, the red sub-volumes that form a structure similar to the Sierpinski gasket correspond to the parallel space that lies outside of the tetrahedron. It is relevant to know what is the volume of this fractal structure, relative to the tetrahedron volume.

The recursive orthotope set has the volume expression

\[
V(S_3^n) = \left(\frac{n}{2}\right)^3 + 3V(S_3^{n/2}) = \frac{n^3}{3} \sum_{i=1}^{\log_2(n)} \left(\frac{3}{2^3}\right)^i
\]

where its reduction via geometric series is

\[
V(S_3^n) = \frac{n^3}{5} - 3^{\log_2(n)}.
\]

In the infinite limit of \( n \), the extra volume approaches to

\[
\lim_{n \to \infty} \alpha(S, \Delta)_n^m = \lim_{n \to \infty} \frac{n^3 - 3^{\log_2(3n)}}{n(n+1)(n+2)} - 1 = \frac{1}{5}
\]

Considering that the extra volume constitutes no more than 20% of the volume of the tetrahedron, one can consider that this recursive strategy does not suffer from significant extra volume problems in \( m = 3 \). However, organizing the set \( S_3^n \) into a single major orthotope \( \Pi_3^n \) of dimensions \((n-1) \times n/2 \times (n+1)/3\) to match \( \Delta_3^n \) is not trivial as the largest sub-orthotope is already greater than \((n+1)/3\) and each recursion adds three sub-structures, leaving a gap when trying to close \( \Pi_3^n \). Forcing the sub-orthotopes to fit through deformation is neither an efficient approach, as it would introduce greater complexity to the map \( \lambda(\bar{w}) \). For
this reason mapping in $O(\log_2(n))$ recursive levels is re-
considered for 3-simplices, as it is a practical approach that
allows to keep the arithmetic computations simple. The map
$\lambda^{n=3}$ can be formulated as

$$
\lambda(\vec{\omega}, \vec{c}, n)^{m=3} = \varphi(\vec{\omega}, \vec{c})_{n/2} + \lambda(\vec{\omega}, \vec{c} + (\frac{n}{2}, 0, 0))_{n/2} + \\
\lambda(\vec{\omega}, \vec{c} + (0, n/2, 0))_{n/2} + \\
\lambda(\vec{\omega}, \vec{c} + (0, 0, n/2))_{n/2} +
$$

where $\vec{c}$ is the relative center and $\varphi(\vec{\omega}, \vec{c})_n = w + \vec{c}$ for a
cube of $n^3$ blocks. The map begins mapping the major cube
of $\frac{(n/2)^3}{(n/2)^3}$ blocks to the initial origin $\vec{c} = (0, 0, 0)$, then it
recursively calls three more maps with the corresponding
new relative origins which are located at the top and the
sides of the cube. This process is repeated until the smallest
sized block is reached. In the end, the total number of map
calls must be at least

$$
\sum_{i=1}^{\log_2(n)} 3^i \geq \frac{2^{\log_2(n)+1} - 1}{3 - 1} = \frac{n - 1}{2} = \Theta(n).
$$

(20)

Although the number of map calls is at least linear, in
practice it turns out to be an excessive number of parallel
calls for the GPU computing model, which at the present
time can handle up to 32 concurrent kernels. A more efficient
map free of $O(n)$ recursive calls can be formulated by doing
a small modification to the strategy.

**C. Alternative Map for 3-Simplices**

Although the previous map works in $O(\log_2(n))$ time,
its main disadvantage is the number of recursive calls to the
map, making it unlikely to work efficient when implemented
on a GPU.

It is possible to modify the strategy and improve the effi-
ciency of the parallel space, as well as the map, by realizing
that the recursive set $S^3_n$ can actually match the volume
of the tetrahedron $\Delta^3_n$ by taking out one of the recursion
branches initially established. By doing this, the red sub-
tetrahedrons lying in the empty spaces of the Sierpinski
gasket volume can correspond to a unique uncovered sub-
tetrahedron of data-space lying inside $\Delta^3_n$. The process
is done recursively, making it an effective optimization. The
modified strategy is illustrated in Figure [6].

With the new approach, the volume of the redefined set
$S^3_n$ becomes

$$
V(S^3_n) = \left(\frac{n}{2}\right)^3 + 2V(S^3_{n/2}) = \frac{n^4}{2} \sum_{i=1}^{\log_2(n)} \left(\frac{1}{4}\right)^i
$$

(21)

which can be reduced using the geometric series

$$
V(S^3_n) = \frac{n^3 - n}{6} = V(\Delta^3_{n-1}).
$$

(22)

As in the 2-simplex map, the diagonal plane is not consid-
ered, therefore the relation of data coverage is $V(S^3_{n+1}) =

$$
V(\Delta^3_n). \text{ With this new organization it is now possible to build a } O(1) \text{ time map free of the problems found in the original formulation and free of square roots. The mapping works with a main orthotope of dimensions } V(\Pi^3_n) = \left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right) \times 3(n-1) 4 \text{ for } x, y, z, \text{ respectively. Figure } 7 \text{ illustrates the new map.}
$$

![Figure 6: Two different views of how the orthotope set (red) maps to the
tetrahedron (white cells) with only two recursion branches.](image)

![Figure 7: Two different views of how the orthotope set (red) maps to the
tetrahedron (white cells) with only two recursion branches.](image)

Function $\lambda(\vec{\omega})$ assumes the origin of $\Pi^3_n$ at the bottom-
right corner from Figure [7] and the the origin of $\Delta^3_n$ at
the bottom right corner too, with the axes aligned to its
orthogonal sides. The mapping begins by moving the main
sub-orthotope of $(n/2)^3$ directly onto the center of the
tetrahedron with a simple map of the type

$$
h(\vec{\omega}) = \vec{\omega} + 0, \frac{n}{2}, 0).
$$

(23)

At the same time the rest of the sub-orthotopes map as

$$
\lambda(\vec{\omega}) = \begin{cases} 
(\omega_x + q \beta, \omega_y + 2q \beta, \omega_z - n/2), & \text{inside} \\
(b(1 + 2q) - \omega_x, 2b(1 + q) - \omega_y, 2b - \omega_z + \frac{n}{2}), & \text{outside}
\end{cases}
$$

Parameters $q, b$ have the same definitions as in the 2-
simplex map and the total cost is constant in time, i.e.,
$T(\lambda(\vec{\omega})) = O(1)$ even if done in sequence. The extra
volume introduced by this approach is

$$
\alpha(\Pi, \Delta)^3_n = \frac{V(\Pi^m=3)}{V(\Delta^m=3)} - 1 = \frac{3n^2(n-1)}{16(n-1)n(n+1)} - 1 = \frac{2}{16}
$$

(24)
making $\Pi_n^{m=3}$ only 12.5% larger than $\Delta_n^3$. Such amount of extra volume constitutes a small fraction of the bounding-box that surrounds the tetrahedron, which is practically 600% the volume of $\Delta_n^m$ for large $n$. For this reason, there is a potential performance improvement that can be exploited by GPUs when using the optimized version of $\lambda(\omega)$ on 3-simplices.

D. Considerations for m-Simplices.

The maps proposed for the 2-simplex and 3-simplex followed specific designs for their corresponding dimensions. Although the maps take constant time for both cases, it is important to note that for the 3-simplex it was necessary to introduce 12% of extra parallel volume in order to fit the set $S_n^m$ on both $\Pi_n$ and $\Delta_n$ and produce a single-pass map. When generalizing the approach to m-simplices, it is important to first verify if $V(S_n^m) \geq V(\Delta_n^m)$ satisfies as well as to find out how much extra space is introduced.

The task is to build a set of recursively organized orthotopes $S_n^m$, where the following satisfies $\forall \bar{x} \in \Delta_n^m, \bar{x} \in S_n^m$. For general $m$, the volume of a set of recursive orthotopes $S_n^m$ is defined as

$$V(S_n^m) = (rn)^m + \beta V(S_{rn}^m) = (rn)^m \sum_{i=0}^{\log_1/r(n)-1} (\beta r^m)^i$$

where $r$ is the scaling factor and $\beta$ the arity of the recursion. Applying the geometric series, the expression becomes

$$V(S_n^m) = (nr)^m \left(\frac{(\beta r^m)^{\log_1/n} - 1}{\beta r^m - 1}\right)$$

$$= \frac{n^m - \beta^{\log_1/r(n)}}{1/r^m - \beta}$$

where at least $V(S_n^m) \geq V(\Delta_n^m)$ must hold. For $m = 2$, it is possible verify that setting $r = 1/2$ and $\beta = 2$ leads to equations (11) and (22) for $m = 2,3$, respectively. For $m = 4$ the total volume is

$$V(S_n^m) = \frac{n^4 - n}{14} > \frac{(n-1)n(n+1)(n+2)}{24}, n \geq 2$$

and for large $n$ the extra volume introduced approaches $5/7$ of $\Delta_n^m$. For large $n$ in higher dimensions, the recursive strategy of using $r = 1/2$, $\beta = 2$ produces a fraction of extra volume of

$$\lim_{n \to \infty} \alpha(S, \Delta_n^m) = \lim_{n \to \infty} \frac{n^m - n}{2^{m+1} - 2} = 1 - \frac{m!}{2m - 2} - 1$$

which makes it inefficient for higher dimensions, i.e., for $m = 5$ and $m = 7$ it produces $3\times$ and $39\times$ the volume of $\Delta_n^m$.

A more efficient set $S_n^m$ can be found by searching the right values for $r$ and $\beta$ in order to satisfy $1/(r^m) - \beta = m!$ or at least approach it from below. The restriction however is that the term $\beta^{\log_1/r(n)}$ needs to be positive and should not grow too fast as it has an impact on the efficiency of the parallel space.

For example, a value of $r = 1/(m^{-1/m})$ produces the required $m!$, making $\beta$ a free parameter to be adjusted, with $\beta \in \mathbb{Z}_+$ and $\beta \geq 2$. Choosing $\beta = 2$ provides a set $S_n^m$ that covers $\Delta_n^m$ from a certain $n \geq n_0$, where $n_0$ is a value that increases with $m$. It is possible to bring $n_0$ closer to the origin by increasing $\beta$, however the extra volume increases as well. What is interesting is that from $n \geq n_0$, the parallel space is practically $m!$ times more efficient than a bounding box approach, presenting a great potential for transforming this space improvement into a performance one. Studying how parameters $r$ and $\beta$ can be set and relate to each other is indeed an interesting open question, since finding the best set becomes an optimization problem where the the difference $1/(r^m) - \beta - m!$ and the term $\beta^{\log_1/r(n)}$ are to be minimized.

IV. Discussion

The results from the analysis on recursive GPU mapping for discrete orthogonal $m$-simplices can serve as a guide for implementing efficient GPU computations for interaction and simulation problems which are often parallelized using a bounding-box approach due to its simplicity in implementation. The 2-simplex and 3-simplex were studied as special cases, re-defining them as a set of recursive orthotopes. From the analysis it was possible to formulate new $O(1)$ time maps with a potential improvement of $2\times$ and $6\times$ respectively.

The generalization to $m$-simplices presents a greater challenge, as it has been shown that obtaining an optimal set $S_n^m$ of orthotopes with minimal extra volume becomes an optimization problem where the the scaling and arity parameters, $r, \beta$ respectively, have to be chosen carefully in order to find a small value $n_0$ from which the mapping can take place and obtain a volume function that introduces a moderate amount of extra volume. Knowing what parameters are the optimal for building a recursive set of orthotopes for any $m$-simplex, as well as provide a rule for the shape of the orthotope container of $S_n^m$ in any dimension, are indeed interesting questions that require further study in order to be answered.

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References

[1] Quentin Avril, Valérie Gouranton, and Bruno Arnaldi. Fast collision culling in large-scale environments using gpu mapping function. In EGPGV, pages 71–80, 2012.
[2] Jeroen Bédorf, Evghenii Gaburov, and Simon Portegies Zwart. A sparse octree gravitational n-body code that runs entirely on the GPU processor. *J. Comput. Phys.*, 231(7):2825–2839, April 2012.

[3] J. Costello. On the number of points in regular discrete simplexes (corresp.). *IEEE Transactions on Information Theory*, 17(2):211–212, Mar 1971.

[4] M. Gardner. The fantastic combinations of John Conway’s new solitaire game “life”. *Scientific American*, 223:120–123, October 1970.

[5] Fred Gustavson. New generalized data structures for matrices lead to a variety of high performance algorithms. In Roman Wyrzykowski, Jack Dongarra, Marcin Paprzycki, and Jerzy Wasniewski, editors, *Parallel Processing and Applied Mathematics*, volume 2328 of *Lecture Notes in Computer Science*, pages 418–436. Springer Berlin / Heidelberg, 2006.

[6] Dale A. Huckaby and Lesser Blum. Effect of triplet correlations on the adsorption of a dense fluid onto a crystalline surface. *The Journal of Chemical Physics*, 97(8):5773–5776, 1992.

[7] Lubomir Ivanov. The n-body problem throughout the computer science curriculum. *J. Comput. Sci. Coll.*, 22(6):43–52, June 2007.

[8] Jin Hyuk Jung and Dianne P. OLeary. Exploiting structure of symmetric or triangular matrices on a gpu. Technical report, University of Maryland, 2008.

[9] J. Kepner and J. Gilbert. *Graph Algorithms in the Language of Linear Algebra*. Society for Industrial and Applied Mathematics, 2011.

[10] Khronos OpenCL Working Group. *The OpenCL Specification, version 1.0.29*, 8 December 2008.

[11] Penporn Koanantakool and Katherine Yelick. A computation- and communication-optimal parallel direct 3-body algorithm. In *Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis*, SC ’14, pages 363–374, Piscataway, NJ, USA, 2014. IEEE Press.

[12] Qi Li, Vojislav Kecman, and Raied Salman. A chunking method for euclidean distance matrix calculation on large dataset using multi-gpu. In *Proceedings of the 2010 Ninth International Conference on Machine Learning and Applications*, ICMLA ’10, pages 208–213, Washington, DC, USA, 2010. IEEE Computer Society.

[13] D. Man, K. Uda, H. Ueyama, Y. Ito, and K. Nakano. Implementations of parallel computation of euclidean distance map in multicore processors and gpus. In *Networking and Computing (ICNC), 2010 First International Conference on*, pages 120–127, 2010.

[14] DuHu Man, Kenji Uda, Yasuaki Ito, and Koji Nakano. A gpu implementation of computing euclidean distance map with efficient memory access. In *Proceedings of the 2011 Second International Conference on Networking and Computing*, ICNC ’11, pages 68–76, Washington, DC, USA, 2011. IEEE Computer Society.

[15] Cristóbal A. Navarro, Benjamín Bustos, and Nancy Hitschfeld. Potential benefits of a block-space GPU approach for discrete tetrahedral domains. In *CLEI-2016, XLII Conferencia Latinoamericana de Informática, Valparaíso, Chile, October 10-14, 2016*, 2016.

[16] Cristobal A. Navarro and Nancy Hitschfeld. GPU maps for the space of computation in triangular domain problems. In *2014 IEEE International Conference on High Performance Computing and Communications, 6th IEEE International Symposium on Cyberspace Safety and Security, 11th IEEE International Conference on Embedded Software and Systems, HPCC/CSS/ICCESS 2014, Paris, France, August 20-22, 2014*, pages 375–382, 2014.

[17] Cristobal A. Navarro, Nancy Hitschfeld-Kahler, and Luis Mateu. A survey on parallel computing and its applications in data-parallel problems using GPU architectures. *Commun. Comput. Phys.*, 15:285–329, 2014.

[18] John Nickolls and William J. Dally. The gpu computing era. *IEEE Micro*, 30(2):56–69, March 2010.

[19] Nvidia-Corporation. *Nvidia CUDA C Programming Guide*, 2011.

[20] J.D. Owens, M. Houston, D. Luebke, S. Green, J.E. Stone, and J.C. Phillips. Gpu computing. *Proceedings of the IEEE*, 96(5):879–899, May 2008.

[21] Florian Ries, Tommaso De Marco, Matteo Zivieri, and Roberto Guerrieri. Triangular matrix inversion on graphics processing unit. In *Proceedings of the Conference on High Performance Computing Networking, Storage and Analysis*, SC ’09, pages 9:1–9:10, New York, NY, USA, 2009. ACM.

[22] Zhi Ying, Xinhua Lin, Simon Chong-Wee See, and Minglu Li. Gpu-accelerated dna distance matrix computation. In *Proceedings of the 2011 Sixth Annual ChinaGrid Conference, CHINAGRID ’11*, pages 42–47, Washington, DC, USA, 2011. IEEE Computer Society.

[23] Rio Yokota and Lorena A. Barba. Fast n-body simulations on GPUs. *CoRR*, abs/1108.5815, 2011.