ON A POINCARÉ POLYNOMIAL FROM KHOVANOV HOMOLOGY AND VASSILIEV INVARIANTS

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Abstract. We introduce a Poincaré polynomial with two-variable \( t \) and \( x \) for knots, derived from Khovanov homology, where the specialization \((t, x) = (1, -1)\) is a Vassiliev invariant of order \( n \). Since for every \( n \), there exist non-trivial knots with the same value of the Vassiliev invariant of order \( n \) as that of the unknot, there has been no explicit formulation of a perturbative knot invariant which is a coefficient of \( y^n \) by the replacement \( q = e^y \) for the quantum parameter \( q \) of a quantum knot invariant, and which distinguishes the above knots together with the unknot. The first formulation is our polynomial.

1. Introduction

Vassiliev [6] introduces his ordered invariants by using singularity theory. For the space \( \mathcal{M} \) of all smooth maps from \( S^1 \) to \( \mathbb{R}^3 \), let \( \Sigma \) be the set of maps which are not embeddings. Then, a filtration of subgroups \( \{G_n\}_{n=1}^\infty \) of the reduced cohomology \( \tilde{H}^0(\mathcal{M} \setminus \Gamma) \) is introduced. An element in \( G_n \setminus G_{n-1} \) corresponds to an oriented knot \( K \) gives us a knot invariant, which is so-called a Vassiliev invariant of order \( n \). Birman and Lin [2] give a relation between the Jones polynomial and the Vassiliev invariant, i.e., for a one-variable polynomial \( U_x(K) \) obtained from the Jones polynomial by replacing the variable with \( e^x \), they show that for a power series \( U_x(K) = \sum_{n=0}^{\infty} u_n(K)x^n \), each \( u_n \) is a Vassiliev invariant of order \( n \) (Fact 1).

In this paper, we consider an analogue of this Birman-Lin argument using Khovanov homology as follows. For an oriented link \( L \), Khovanov [4] defines groups that are knot invariants and are so-called Khovanov homology \( H_{i,j}(L) \) such that \( J(L)(q) = \sum_{i,j} (-1)^i q^j \text{ rank } H_{i,j}(L) \), where \( J(L)(q) \) is a version of the Jones polynomial of \( L \). It implies the Khovanov polynomial \( \sum_{i,j} t^i q^j \text{ rank } H_{i,j}(L) = Kh(L)(t,q) \).

Using each coefficient \( v_n(K)(t, x) \) of \( y^n \) in \( Kh(L)(t,q)|_{q=xe^y} \), we have:

**Theorem 1.** Let \( l, m, \) and \( n \) be integers where \( 2 \leq l < m < n \). Let \( v_n(K)(t, x) \) be a function as in Definition 4. Then, \( v_n(K)(-1,1) \) is a Vassiliev invariant of order \( n \) and there exists a set \( \{K_\mu\}_{\mu \geq 2} \) consisting of oriented knots such that for a given tuple \((l, m, n)\), \( v_n(K_l)(-1,1) = v_n(K_m)(-1,1) = v_n(\text{unknot})(-1,1) \) but \( v_n(K_l)(t, x) \neq v_n(K_m)(t, x), \) \( v_n(K_l)(t, x) \neq v_n(\text{unknot})(t, x), \) and \( v_n(K_m)(t, x) \neq v_n(\text{unknot})(t, x). \)

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Remark 1. If $n = 3$, $l = 2$, there exists an oriented knot $K_2$ such that $v_3(K_2)(-1, 1) = v_3(\text{unknot})(-1, 1)$ where $v_3(K_2)(t, x) \neq v_3(\text{unknot})(t, x)$. The proof is placed on the end of Section 3.

Remark 2. This $v_n(K)(t, x)$ equals $\sum_{i,j} \frac{j^n}{n!} t^i \text{rank } H^{i,j}(K) x^j$ (Lemma 1), which implies a triply graded homology $H^{i,j}(K)$ by assigning $n$ to $H^{i,j}(K)$ that belongs to the coefficient of $y^n$, i.e., a formula

$$Kh(L)(t, xe^y) = \sum_{n=0}^{\infty} \sum_{i,j} \frac{j^n}{n!} t^i \text{rank } H^{i,j}(L) x^j y^n,$$

satisfying $Kh(L)(-1, e^y) = J_L(q)$ holds (cf. 11 of the proof of Lemma 1).

To the best our knowledge, there has been no explicit formulation of a perturbative knot invariant which is a coefficient of $y^n$ obtained from the replacement $q = e^y$ of the quantum parameter $q$ of a quantum knot invariant, and which distinguishes $K_m$ ($m \leq n - 1$) of Theorem 1 (Figure 2) together with the unknot where the Vassiliev invariant cannot. The first formulation is our two-variable Poincaré polynomial $v_n(K_n)(t, x)$ which is introduced in this paper, and which is the coefficient of $y^n$ and satisfies that the specialization $v_n(K_n)(-1, 1)$ is a Vassiliev invariant of order $n$. Further, it is interesting that though this polynomial invariant $v_n(K)(t, x)$ can detect the difference between $K_m$ ($m \leq n - 1$) and the unknot, essentially, there exists a fixed number $j_0$ such that the coefficient $x^{j_0}y^n$ detect them (here, $j_0$ is actually the lowest degree of $x$ in $v_n(K)(t, x)$). It implies that an information of the $j$-grade of $v_n(K)(t, x)$ is useful (for the detail, see Section 3). In the literature, this usefulness of the grade implicitly appeared in a work of Kanenobu-Miyazawa [3], they showed that $V^{(n)}_K(1)$ is a Vassiliev invariant by using the $n$th derivative of the Jones polynomial $V_K(q)$.

The plan of the paper is as follows. We will prove Theorem 1 (Section 3) after we obtain definitions and notations (Section 2). In Section 4 we give a table of our function $v_{n,j}(K)(t, x)$ and its sum $v_n(K)(t, x)$.

2. Preliminaries

2.1. The Jones polynomial and the Vassiliev invariant.

Definition 1 (normalized Jones polynomial). Let $L$ be an oriented link. The Jones polynomial $V_L(r)$ is well-known, which is a polynomial in $\mathbb{Z}[r^{1/2}, r^{-1/2}]$ that is determined by an isotopy class of $L$. The Jones polynomial $V_L(r)$ is defined by

$$V_{\text{unknot}}(r) = 1,$$

$$r^{-1}V_{L_+}(r) - rV_{L_-}(r) = (r^{1/2} - r^{-1/2})V_{L_0}(r),$$

where links $L_+$, $L_-$, and $L_0$ are defined by Figure 4 and where Figure 4 corresponds to local figures are included on a neighborhood and the exteriors of the three neighborhoods are the same.
Definition 2 (unnormalized Jones polynomial). Letting $q = -r^{1/2}$, we define an unnormalized Jones polynomial $J_L(q)$ by

$$J_L(q)|_{q=-r^{1/2}} = (-r^{1/2} - r^{-1/2})V_L(r).$$

By definition, $J_L(q)$ is a polynomial in $\mathbb{Z}[q, q^{-1}]$ that is determined by an isotopy class of $L$. Let $L_+, L_-$, and $L_0$ be as in Definition 1. Then, the polynomial $J_L(q)$ satisfies

$$J_{\text{unknot}}(q) = q + q^{-1},$$

$$q^{-2}J_{L_+}(q) - q^{2}J_{L_-}(q) = (q^{-1} - q)J_{L_0}(q).$$

Fact 1 (Birman-Lin, Theorem of [2]). Let $K$ be a knot and let $V_K(r)$ be its Jones polynomial as in Definition 1. Let $U_x(K)$ be obtained from $V_K(r)$ by replacing the variable $r$ by $e^x$. Express $U_x(K)$ as a power series in $x$:

$$U_x(K) = \sum_{i=0}^{\infty} u_i(K)x^i.$$

Then, $u_0(K) = 1$ and each $u_i(K)$, $i \geq 1$ is a Vassiliev invariant of order $i$.

2.2. A polynomial invariant from Khovanov polynomial.

Definition 3. Let $L$ be a link and $H_{i,j}(L)$ the Khovanov homology group of $L$. The Khovanov polynomial is defined by

$$Kh(L)(t, q) = \sum_{i,j} t^i q^j \text{rank } H_{i,j}(L).$$

Definition 4 (two-variable polynomials). Let $Kh(K)(t, q)|_{q=xe^y}$ be a polynomial obtained from the Khovanov polynomial $Kh(K)(t, q)$ by replacing the variable $q$ with $xe^y$. Then, let $v_n(K)(t, x)$ the coefficient of $y^n$ and let $v_{n,j}(K)(t, x)$ be (the coefficient of $x^j y^n$) · $x^j$.

By definition, $v_n(K)(t, x) = \sum_{j} v_{n,j}(K)(t, x)$. It is clear that every $v_{n,j}(K)(t, x)$ is a link invariant, which implies that $v_n(K)(t, x)$ is also a link invariant. Definition 3 and Definition 4 imply Lemma 1.

Lemma 1.

$$v_{n,j}(K)(t, x) = \frac{j^n}{n!} \sum_i t^i \text{rank } H_{i,j}(K) x^i.$$

As a corollary,

$$v_n(K)(t, x) = \sum_{i,j} \frac{j^n}{n!} t^i \text{rank } H_{i,j}(K) x^j.$$
Proof:

\[ Kh(L)(t, xe^y) = \sum_j (e^y)^j x^j \sum_i t^i \text{rank } \mathcal{H}^{i,j}(L) \]

\[ = \sum_j \sum_{n=0}^{\infty} \frac{(jy)^n}{n!} x^j \sum_i t^i \text{rank } \mathcal{H}^{i,j}(L) \]

\[ = \sum_j \sum_{n=0}^{\infty} \frac{j^n}{n!} \sum_i t^i \text{rank } \mathcal{H}^{i,j}(L) x^j y^n. \]

Then, the coefficient of \( x^j y^n \) is \( \sum_{n=0}^{\infty} \frac{j^n}{n!} \sum_i t^i \text{rank } \mathcal{H}^{i,j}(L) \). This fact together with Definition 3 of \( v_{n,j}(K)(t, x) \), we have

\[ v_{n,j}(K)(t, x) = \frac{j^n}{n!} \sum_i t^i \text{rank } \mathcal{H}^{i,j}(L) x^j. \]

As a corollary,

\[ v_n(K)(t, x) = \sum_j v_{n,j}(K)(t, x) = \sum_{i,j} \frac{j^n}{n!} t^i \text{rank } \mathcal{H}^{i,j}(K) x^j. \]

\[ \square \]

Lemma 2. The integer \( v_n(K)(-1, 1) \) is a Vassiliev invariant of order \( n \).

As a corollary, every Vassiliev invariant of order \( n \) has a presentation

\[ v_n(K)(-1, 1) = \sum_j v_{n,j}(K)(-1, 1). \]

Proof. Using the above proof of Lemma 3 setting \( x = 1 \) and \( t = -1 \), we have

\[ J(L)(q)|_{q=e^y} = Kh(L)(-1, e^y) = \sum_j \sum_{n=0}^{\infty} \frac{j^n}{n!} \sum_i (-1)^i \text{rank } \mathcal{H}^{i,j}(L) y^n. \]

The coefficient of \( y^n \) is \( \sum_j \sum_{n=0}^{\infty} \frac{j^n}{n!} \sum_i (-1)^i \text{rank } \mathcal{H}^{i,j}(L) \), which is \( v_n(K)(-1, 1) \). Then, by the same argument as [2, Proof of Theorem 4.1] of Birman-Lin, it is elementary to prove that the coefficient of \( y^n \) of \( J(L)(q)|_{q=e^y} \) is a Vassiliev invariant of order \( n \). This fact and Lemma 3 imply the formula of the claim. \( \square \)

3. A proof of Theorem 3

Since Lemma 2 holds, we should the latter part of the claim. For this proof, we use notations and definitions of Khovanov homology as in [7]. Although it is sufficient to use \( \mathbb{Z}/2\mathbb{Z} \)-homology, here we use \( \mathbb{Z} \)-homology to avoid adding notations of symbols. We recall that a chain group \( C^{i,j}(D) \) of an oriented link diagram \( D \). In particular, for each enhanced state of \( C^{i,j}(D) \), \( i(S) = \frac{w(D) - \sigma(s)}{2} \) and \( j(S) = w(D) + i(S) + \tau(S) \) (for definition of a state \( s \), an enhanced state \( S \), the writhe number \( w(D) \), a sum \( \sigma(s) \) of signs, and a sum \( \tau(S) \) of signs, see [7]).

Let \( m \) be a positive integer \( (m \geq 2) \) and \( K_n \) a knot with a fixed \( m \) that is defined by Figure 2. It is well-known that for every Vassiliev invariant \( v_n \) of order \( n \), \( v_n(\text{unknot}) = 0 \) and \( v_n(K_m) = 0 \) \( (m \leq n - 1) \) [5], which implies that \( v_n(K_1)(-1, 1) = v_n(K_0)(-1, 1) = v_n(\text{unknot})(-1, 1) \) (\( \therefore \) Lemma 2).

Let \( D_m \) be a knot diagram defined by Figure 2, \( s_n \) a state defined by Figure 3(a), and \( S_n \) a state defined by Figure 3(b). 


Figure 2. A diagram of a knot $K_m$.

Figure 3. (a) the state $s$ of $K_m$ (each short edge indicates the direction of smoothing of a crossing) and (b) the enhanced state $S$ of $K_m$ (each sign indicates a sign of a circle).

Note that by the definition of this $\mathbb{Z}$-homology, $S_m$ obtains the minimum number of degree $i$ is $-2m$ and the minimum number of degree $j$ is $-4m - 1$ as follows:

$$w(D_m) = 0,$$

$$i(S_m) = \frac{0 - 4m}{2} = -2m, \text{ and}$$

$$j(S_m) = 0 + (-2m) + (-1 - 2m) = -4m - 1.$$

Note also that by the definition of the differential $d : C^{i,j}(D) \to C^{i+1,j}(D)$, $d^{-2m}(S_m) = 0$ and $\text{Im} d^{-2m-1} = 0$. Then, for each $m \geq 2$,

$$(2) \quad \mathcal{H}^{-2m,-4m-1}(K_m) = \mathbb{Z}.$$

By Lemma 1,

$$v_{n,j}(K)(t, x) = \sum_i t^i \text{ rank } \mathcal{H}^{i,j}(K) x^i.$$

We focus on the minimum number of $i$ that is $-2m$, and the minimum number of $j$ that is $-4m - 1$. Setting $i = -2m$ and $j = -4m - 1$, the coefficient of $t^{-2m}x^{-4m-1}$
in $v_{n,-4m-1}(K)(t,x)$ is

\[
\frac{(-4m-1)^n}{n!} \text{rank } \mathcal{H}^{-2m,-4m-1}(K).
\]

Then, (2) implies

\[
\frac{(-4m-1)^n}{n!} \text{rank } \mathcal{H}^{-2m,-4m-1}(K_m) = \frac{(-4m-1)^n}{n!}.
\]

Thus, for every pair $l, m$ ($2 \leq l < m$), $v_{n,-4l-1}(K_l)(t,x) = \frac{(-4l-1)^n}{n!} \neq \frac{(-4m-1)^n}{n!} = v_{n,-4m-1}(K_m)(t,x)$. Here, recall that for the unknot, it is well-known that $Kh(\text{unknot})(t,q) = q^{-1} + q$, which implies that there is no non-trivial coefficient of $t^k$ ($k \neq 0$), i.e., any non-trivial part corresponds to the coefficient $q + q^{-1}$, which belongs to the coefficient of $t^0$. It implies $v_{n,-4l-1}(K_l)(t,x) \neq v_{n,-4l-1}(\text{unknot})(t,x)$ and $v_{n,-4m-1}(K_m)(t,x) \neq v_{n,-4m-1}(\text{unknot})(t,x)$.

Note that for every knot $K_j$ ($2 \leq l$), the minimum number of $j$ is $-4l - 1$, by always focusing on the lowest degree of $x$ in $v_n(K)(t,x) = \sum_j v_{n,j}(K)(t,x)$, the above argument works since the coefficient of the lowest degree of $x$ exactly equals $v_{n,-4l-1}(K)(t,x)$. Therefore, by focusing the case $j = -4l - 1$ or the case $j = -4m - 1$, for every pair $l, m$ ($2 \leq l < m$), $v_{n}(K_l)(t,x) \neq v_{n}(K_m)(t,x)$, $v_{n}(K_l)(t,x) \neq v_{n}(\text{unknot})(t,x)$ and $v_{n}(K_m)(t,x) \neq v_{n}(\text{unknot})(t,x)$ since for each case, two lowest degrees are different. It completes the proof of Theorem 1.

**Proof of Remark 1** Note that the coefficient of $t^{-2}2x^{-4}2^{-1}$ in $v_{3,-9}(K_2)(t,x)$ is $\frac{(-4-2)^{3}}{3!}$. Thus, $v_{3,-9}(K_2)(t,x) \neq v_{3,-9}(\text{unknot})(t,x)$. By focusing on the lowest degree of $x$ in $v_{3}(K)(t,x) = \sum_j v_{3,j}(K)(t,x)$, we have the statement of Remark 1.

4. Table

We give some examples of the Khovanov polynomial and the two-variables polynomials for a few prime knots. We use the data of the Khovanov polynomial in the Mathematica package KnotTheory [1] and attach a Mathematica file to arXiv page.

| Knot | $3_1$ |
|------|-------|
| $Kh$ | $q^3t^3 + q^4t^4 + q^4 + q$ |
| $v_0$ | $t^3x^3 + t^4x^4 + x^3 + x$ |
| $v_1$ | $9t^2x^5 + 5t^2x^6 + 3x^5 + x$ |
| $v_2$ | $81t^2x^7 + 25t^2x^8 + 9x^7 + \frac{x}{6}$ |
| $v_3$ | $216t^2x^8 + 129t^2x^9 + 9x^8 + \frac{x}{2}$ |
| $v_4$ | $2187t^2x^9 + 6225t^2x^{10} + 27x^9 + \frac{x}{2}$ |
| $v_5$ | $19683t^2x^{10} + 6225t^2x^{11} + 81x^{10} + \frac{x}{10}$ |
### Knot $4_1$

| $\text{Kh}$ | $q^4t^2 + \frac{1}{q}t^2 + qt + \frac{1}{q}t + q + \frac{q}{t}$ |
|------------|---------------------------------------------------------------|
| $v_0$      | $t^4x^3 + \frac{2}{2}tx + tx + \frac{1}{2}x + \frac{1}{2}$   |
| $v_1$      | $5t^2x^5 - \frac{9}{2}tx^2 + tx - \frac{1}{2}x - \frac{1}{2}$ |
| $v_2$      | $\frac{25t^2x^5}{2} + 25tx^2 + tx + \frac{1}{2}x + \frac{1}{2}$ |
| $v_3$      | $\frac{125t^2x^5}{4} - \frac{25}{2}tx^2 + \frac{1}{2}x + \frac{1}{2}$ |
| $v_4$      | $\frac{625t^2x^5}{8} + 625t^2tx^2 + tx + \frac{1}{2}x + \frac{1}{2}$ |
| $v_5$      | $\frac{625t^2x^5}{24} + 625t^2tx^2 + tx + \frac{1}{2}x + \frac{1}{2}$ |

### Knot $5_1$

| $\text{Kh}$ | $q^4t^4 + q^3t^3 + q^2t^2 + q^2t^2 + q^2t + q^3 + q$ |
|------------|---------------------------------------------------------------|
| $v_0$      | $t^4x^{15} + t^3x^{11} + t^2x^{11} + t^2x^3 + x^3 + x^2$   |
| $v_1$      | $15t^5x^{15} + 11t^4x^{11} + 11t^3x^{11} + 7t^2x^3 + 5tx^3 + 3x^3$ |
| $v_2$      | $25t^5x^5 - 121t^4x^{11} + 121t^3x^{11} + 49t^2x^3 + 25x^3 + 9x^2$ |
| $v_3$      | $1125t^5x^5 + 1331t^3x^{11} + 1331t^3x^{11} + 54t^2x^3 + 125x^3 + 9x^2$ |
| $v_4$      | $16875t^5x^{11} + 14641t^3x^{11} + 14641t^3x^{11} + 240t^2x^3 + 625x^3 + 27x^2$ |
| $v_5$      | $50625t^5x^{11} + 161051t^3x^{11} + 161051t^3x^{11} + 16807t^2x^3 + 625t^2x^3 + 81t^2x^3$ |

### Knot $5_2$

| $\text{Kh}$ | $q^4t^5 + q^5t^4 + q^2t^3 + q^2t^2 + q^2t + q^3 + q$ |
|------------|---------------------------------------------------------------|
| $v_0$      | $t^5x^3 + t^4x^3 + t^3x^3 + t^4x^3 + t^2x^3 + x^3 + x^2$   |
| $v_1$      | $13t^5x^3 + 9t^4x^3 + 9t^3x^3 + 7t^2x^3 + 5t^2x^3 + 3tx^3 + 3x^3 + x$ |
| $v_2$      | $169t^5x^3 + 81t^4x^3 + 81t^3x^3 + 49t^2x^3 + 25t^2x^3 + 9x^3 + 3x^2 + \frac{x^2}{2}$ |
| $v_3$      | $1297tx^{11} + 243tx^{11} + 243tx^{11} + 343tx^{11} + 125tx^{11} + 9tx^{11} + 9x^2 + 3x^2 + \frac{x^2}{2}$ |
| $v_4$      | $1297tx^{11} + 243tx^{11} + 243tx^{11} + 343tx^{11} + 125tx^{11} + 9tx^{11} + 9x^2 + 3x^2 + \frac{x^2}{2}$ |
| $v_5$      | $51293t^5x^{11} + 19683t^5x^{11} + 19683t^5x^{11} + 16807t^2x^3 + 625t^2x^3 + 31t^2x^3 + \frac{x^2}{2}$ |

### Knot $6_1$

| $\text{Kh}$ | $q^6t^4 + q^5t^4 + q^3t^4 + q^3t^4 + q^3t + q + \frac{q}{t}$ |
|------------|---------------------------------------------------------------|
| $v_0$      | $t^4x^3 + t^3x^3 + t^3x^3 + t^3x^3 + t^3x^3 + 0 + x^3 + \frac{x^3}{2}$ |
| $v_1$      | $9t^5x^3 + 5t^5x^3 + 5t^5x^3 - \frac{15}{2}tx^3 + 3tx^3 + tx - \frac{1}{2}x + \frac{1}{2}$ |
| $v_2$      | $81t^2x^3 + 25tx^3 + 25tx^3 + \frac{25}{2}tx^3 + 9tx^3 + \frac{tx}{2} + \frac{x}{2} + \frac{1}{2} + \frac{1}{2}$ |
| $v_3$      | $243t^2x^3 + 129tx^3 + 129tx^3 - 125 \frac{9tx}{2} + tl + \frac{1}{2} + \frac{1}{2}$ |
| $v_4$      | $2187tx^{11} + 625tx^{11} + 625tx^{11} + 625tx^{11} + 27tx^3 + \frac{tx}{2} + \frac{x}{2} + \frac{1}{2}$ |
| $v_5$      | $1683t^5x^{11} + 625tx^{11} + 625tx^{11} + 625tx^{11} + 27tx^3 + \frac{tx}{2} + \frac{x}{2} + \frac{1}{2}$ |
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