REALIZING FINITE GROUPS AS AUTOMIZERS

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Abstract. It is shown that any finite group $A$ is realizable as the automizer in a finite perfect group $G$ of an abelian subgroup whose conjugates generate $G$. The construction uses techniques from fusion systems on arbitrary finite groups, most notably certain realization results for fusion systems of the type studied originally by Park.

1. Introduction

Not every finite group is realizable as $\text{Aut}(U)$ for some finite group $U$. For example, no nontrivial cyclic group of odd order is the automorphism group of a group. We study here the realization of finite groups by automizers of subgroups of finite groups. That is, given a finite group $A$, we study when it is possible to find a finite group $G$ and a subgroup $U \leq G$ such that $A \cong \text{Aut}_G(U) = N_G(U)/C_G(U)$. As it stands, the answer to this question is “always possible” for trivial reasons: choose a faithful action of $A$ on an elementary abelian $p$-group (for some prime $p$), and take for $G$ the semidirect product of $U$ by $A$. In this case, $U$ is normal in $G$. Our main result shows that it is possible to realize $A$ as $\text{Aut}_G(U)$ where $U$ is very far from being normal.

Theorem 1.1. For each finite group $A$, there exist a finite perfect group $G$ and a homocyclic abelian subgroup $U$ of $G$ such that $\langle U^G \rangle = G$ and $\text{Aut}_G(U) \cong A$.

Here, we write $\langle U^G \rangle$ for the normal closure of $U$ in $G$, the subgroup of $G$ generated by the $G$-conjugates of $U$. A group $G$ is perfect if it coincides with its commutator subgroup. A homocyclic abelian group is a direct product of isomorphic cyclic groups.

We do not know whether more restrictions can be placed on $G$, up to and including whether $G$ can be taken to be simple. Likewise, we do not know if whether more restrictions can be placed on $U$, such as requiring $U$ to be an elementary abelian $p$-group for some prime $p$.

Ultimately, the group $G$ is constructed fairly explicitly as the commutator subgroup of a wreath product of the form $(U \rtimes A) : \Sigma_n$, but the embedding of $U$ in $G$ is not an obvious one. The method for constructing $G$ and the embedding of $U$ relies on certain basic constructions in fusion systems on arbitrary finite groups. A fusion system on a finite group $S$ (not necessarily a $p$-group) is a category with objects the subgroups of $S$, and with morphism sets consisting of injective homomorphisms between subgroups, subject to two weak axioms which we recall in Section 2. The standard example is the fusion system $F_S(G)$ of the group $G$ on the finite subgroup $S$ in which the morphisms are the $G$-conjugation homomorphisms between subgroups of $S$. The most important ingredient in the construction here is a result of Sejong Park realizing fusion on finite $p$-groups [Par10, Par16], but which we require in the more general setting of fusion systems on finite groups, where it is due to Warraich [War19].

Theorem 1.2 ([War19 Section 4], c.f. [Par16 Theorem 1.1]). For each fusion system $\mathcal{F}$ on a finite group $S$, there is a finite group $G$ containing $S$ as a subgroup and such that $\mathcal{F} = \mathcal{F}_S(G)$.
Ünlü and Yalçın also considered fusion systems on finite groups with an eye on Park’s result \cite{UYn12}, although they did not prove Theorem 1.2. For the convenience of the reader, we provide a sketch of proof of Theorem 1.2 which is modeled closely on Park’s proof for $S$ a $p$-group. For example, the $G$ of Theorem 1.2 is the group of automorphisms as a right $S$-set of a certain $S$-$S$ biset associated with $F$, similarly as in \cite{Par10} and \cite{Par16}.

In order to use Theorem 1.2 to prove Theorem 1.1, we need to be able to construct a suitable finite group $S$ and fusion system $F$ on $S$. One consequence of the way this fusion system is built is the following result.

**Theorem 1.3.** For each finite group $A$, there are a finite group $S$, a homocyclic abelian subgroup $U$ of $S$, and a fusion system $F$ on $S$ such that $\text{foc}(F) = S$, $Q(F) = 1$, and $\text{Aut}_F(U) \cong A$.

The definition of the focal subgroup $\text{foc}(F)$ of an arbitrary fusion system is given in Section 2 and is identical to the definition for fusion systems on $p$-groups. The definition of the subgroup $Q(F)$ of $S$, which is a sort of replacement for $O_p(F)$ in a fusion system over an arbitrary finite group when compared with a saturated fusion system over a $p$-group, is also given there.

When $A$ is a $p$-group for some prime $p$, $S$ is also a $p$-group in the construction we present. But we do not know whether it is possible to choose $F$ to be a fusion system on a $p$-group independently of $A$ in Theorem 1.3, much less whether $F$ can be taken to be a saturated fusion system on a $p$-group.

A MathOverflow question of Peter Mueller asks \cite{Mue20}: is every finite group of the form $N_{\Sigma_n}(U)/U$ for a subgroup $U$ of some finite symmetric group $\Sigma_n$? This work arose out of an attempt to say something about that question.

Here is a brief outline of the paper and some remarks on notation. In Section 2 we give some background on fusion systems and semicharacteristic bisets and give a definition of $Q(F)$. We also write down a proof of the existence of semicharacteristic bisets for fusion systems on arbitrary finite groups and provide a discussion of Theorem 1.2. In Section 3 we prove a slightly more detailed version of Theorem 1.3 and combine it with Theorem 1.2 to prove Theorem 1.1. We use left-handed notation for conjugation $x \mapsto gxg^{-1}$. Our iterated commutators are right-associated: $[X, Y, Z] = [X, [Y, Z]]$, etc. We sometimes write $G'$ for the commutator subgroup of a group $G$.

2. **Fusion systems on finite groups, semicharacteristic bisets, and the Park embedding**

2.1. **Fusion systems.**

**Definition 2.1.** Let $S$ be a finite group. A fusion system on $S$ is a category $\mathcal{F}$ with objects the set of subgroups of $S$, subject to the following two axioms: for all $P, Q \leq S$,

1. $\text{Hom}_\mathcal{F}(P, Q)$ consists of a set of injective homomorphisms from $P$ to $Q$, including all such morphisms induced by $S$-conjugation.

2. Each $\varphi \in \text{Hom}_\mathcal{F}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism from $P$ to $\varphi(P)$ and the inclusion from $\varphi(P)$ to $Q$.

Axiom (1) implies that any inclusion $\iota_P^Q$ of subgroups $P \leq Q$ is a morphism in $\mathcal{F}$ from $P$ to $Q$ (being conjugation by $1 \in S$). Therefore, a morphism can be restricted to any subgroup of the source. Axiom (2) then implies for example that the target of any morphism can be restricted to a subgroup containing the image.

If $G$ is a group and $S$ is a finite subgroup of $G$, there is a fusion system $\mathcal{F}_S(G)$ of $G$ on $S$ with morphism sets $\text{Hom}_\mathcal{F}(P, Q) = \{c_g : t \mapsto g t | g P \leq Q\}$ consisting of the $G$-conjugation homomorphisms mapping $P$ into $Q$. This is the standard example of a fusion system. Theorem 1.2 shows that indeed every fusion system on $S$ is of this form, and $G$ can be taken finite.
The notation $\text{Aut}_F(P)$ is short for $\text{Hom}_F(P, P)$ in a fusion system $F$ on $S$. When $F = F_S(G)$ for some group $G$ and $P \leq S$, then $\text{Aut}_F(P) = \text{Aut}_G(P)$ from the definitions.

We introduce now several properties of subgroups and morphisms in a fusion system that we will need, many of which are identical to their counterparts for fusion systems on $p$-groups [AKO11, Cra11].

**Definition 2.2** (Generation of fusion systems). Let $S$ be a finite group and let $X$ be a set of injective homomorphisms between subgroups of $S$. The **fusion system on $S$ generated by $X$**, denoted $\langle X \rangle_S$, is the intersection of the fusion systems on $S$ containing $X$.

If $F_1$ and $F_2$ are two fusion systems on the finite group $S$, then the category $F_1 \cap F_2$ with objects the subgroups of $S$ and with morphism sets $\text{Hom}_{F_1 \cap F_2}(P, Q) := \text{Hom}_{F_1}(P, Q) \cap \text{Hom}_{F_2}(P, Q)$ is again a fusion system on $S$. Thus, the definition makes sense. As in the case of fusion systems on finite $p$-groups, it is easy to see that an injective group homomorphism is in $\langle X \rangle_S$ if and only if it can be written as a composition of restrictions of homomorphisms in $\text{Inn}(S) \cup X$.

**Definition 2.3** (Direct products). Let $S_1$ and $S_2$ be finite groups, and let $F_i$ be a fusion system on $S_i$, $i = 1, 2$. The direct product $F_1 \times F_2$ is the fusion system over $S_1 \times S_2$ generated by the homomorphisms $(\varphi_1, \varphi_2): P_1 \times P_2 \to S_1 \times S_2$, where $\varphi_i \in \text{Hom}_{F_i}(P_i, S_i)$.

We also need the definition of the focal subgroup of a fusion system.

**Definition 2.4** (The focal subgroup). Let $F$ be a fusion system on the finite group $S$. The **focal subgroup of $F$** is the subgroup of $S$ generated by elements of the form $[\varphi, s] := \varphi(s)s^{-1}$, where $s \in S$ and $\varphi: (s) \to S$ is a morphism in $F$.

**Remark 2.5**. By the Focal Subgroup Theorem [Gor80, 7.3.4], if $G$ is a finite group and $S$ is a Sylow $p$-subgroup of $G$, then $\text{foc}(F_S(G)) = S \cap [G, G]$. When $S$ is an arbitrary subgroup of $G$, there is the obvious inclusion $\text{foc}(F_S(G)) \leq S \cap [G, G]$ since each generating element $\varphi(s)s^{-1} \in S$ is a commutator $gsg^{-1}s^{-1} = [g, s]$ for some $g \in G$, but in general the reverse inclusion need not hold.

### 2.2. Nonextendable morphisms and the subgroup $Q(F)$.

**Definition 2.6** (Nonextendable morphisms). Let $F$ be a fusion system on the finite group $S$, and let $P, Q \leq S$. A morphism $\varphi \in \text{Hom}_F(P, Q)$ is said to be **nonextendable** if it does not extend to a morphism defined on any subgroup of $S$ properly containing $P$. That is, whenever $P \leq R \leq S$ and $\hat{\varphi} \in \text{Hom}_F(R, S)$ is such that $i^R_P \circ \varphi = \hat{\varphi} \circ i^S_P$, then $R = P$.

**Definition 2.7**. For a fusion system $F$ on a finite group $S$, define $Q(F)$ to be the set of all subgroups $Q$ of $S$ for which there is a nonextendable morphism $\varphi: Q \to S$ in $F$, and let $Q(F)$ be the intersection of the family $Q(F)$.

The relevance of the subgroup $Q(F)$ will be seen later in Lemma 2.14. By the same proof as for fusion systems on $p$-groups, if $F$ is a fusion system on a finite group $S$, there is a unique largest subgroup $N$ of $S$ having the property that each morphism $\varphi: P \to Q$ in $F$ extends to a morphism $\hat{\varphi}: PN \to QN$ with $\hat{\varphi}|_N(P) = N$, which we might denote by $O_S(F)$. (If $S$ is a $p$-group, then this is the largest normal $p$-subgroup $O_p(F)$ of $F$.) It follows from the definitions that $O_S(F)$ is a subgroup of each member of $Q(F)$, and so $O_S(F) \leq Q(F)$. Thus, the requirement of $Q(F) = 1$ in Theorem 1.3 is stronger than a requirement of $O_S(F) = 1$.

**Remark 2.8**. The direct product $\text{Aut}_F(S) \times \text{Aut}_F(S)$ acts on the set of pairs $(Q, \varphi)$ consisting of a subgroup $Q \in Q(F)$ and a nonextendable morphism $\varphi: Q \to S$ via $(\alpha, \beta) \cdot (Q, \varphi) = (\alpha(Q), \beta \varphi \alpha^{-1})$. In particular, $Q(F)$ is $\text{Aut}_F(S)$-invariant.
2.3. Semicharacteristic bisets. For a finite group $S$, an $S$-$S$-biset $X$ is a set with left and right $S$-action such that $(ux)v = u(xv)$ for all $u, v \in S, x \in X$. An $S$-$S$-biset can also be viewed as an $(S \times S)$-set via $(u, v) \cdot x = u x v^{-1}$. For a subgroup $Q \leq S$ and a homomorphism $\varphi : Q \to S$, let

$$S \times_{(Q, \varphi)} S = (S \times S)/\sim$$

where $(xu, y) \sim (x, \varphi(u)y)$ for $x, y \in S, u \in Q$, and let $\langle x, y \rangle$ be the equivalence class of $(x, y)$. The group action is defined for $t \in S$ by $t \langle x, y \rangle = \langle tx, y \rangle$ and $(x, y) t = \langle x, yt \rangle$. Importantly, $S \times_{(Q, \varphi)} S$ is also isomorphic to $(S \times S)/\Delta(Q, \varphi)$ as $S \times S$-sets, where $\Delta(Q, \varphi) := \{ (u, \varphi(u)) : u \in Q \}$. We refer to $\Delta(Q, \varphi)$ as a twisted diagonal subgroup.

Definition 2.9 (cf. [Par16 Definition 1.2]). Let $F$ be a fusion system on a finite group $S$. A left semicharacteristic biset for $F$ is a finite $S$-$S$-biset $X$ satisfying the following properties.

- $X$ is $F$-generated, i.e., every transitive subbiset of $X$ is of the form $S \times_{(Q, \varphi)} S$ for some $Q \leq S$ and some $\varphi \in \text{Hom}_F(Q, S)$.
- $X$ is left $F$-stable, i.e., $Q X \cong \varphi X$ as $Q$-$S$-bisets for every $Q \leq S$ and every $\varphi \in \text{Hom}_F(Q, S)$, where $\varphi X$ is the $Q$-$S$-biset whose left action is induced by $\varphi$.

In [Par16], Park showed that each fusion system over a $p$-group has a semicharacteristic biset. Then Warraich [War19] extended this to fusion systems on arbitrary finite groups. We give a proof here for the convenience of the reader.

Theorem 2.10. Let $F$ be a fusion system on the finite group $S$. Then there exists a left semi-characteristic biset $X$ for $F$, and $X$ can be chosen to include at least one $S$-$S$ orbit of the form $S \times_{(S, \alpha)} S$ for each $[\alpha] \in \text{Out}_F(S)$.

Proof. The construction of $X$ is the same as that of [BLO03], similarly as in [Par16] and does not depend on $S$ being a $p$-group. We give the details for the convenience of the reader.

Let $F' = F \times F_S(S)$ be the product fusion system. Observe that the set of subgroups of the form $\Delta(P, \varphi)$ with $P \leq S$ and $\varphi \in \text{Hom}_F(P, S)$ is closed under $F'$-conjugacy and taking subgroups. For example, $(\psi, c_s) \in \text{Hom}_{F'}(\Delta(P, S), S \times S)$ sends $\Delta(P, S)$ to $\Delta(\psi(P), c_s \varphi \psi^{-1})$.

Let

$$X_0 = \sum_{[\alpha] \in \text{Out}_F(S)} S \times_{(S, \alpha)} S,$$

where the sum denotes disjoint union. This is a $F$-generated virtual $S$-$S$-biset with nonnegative rational coefficients having the property that $|X_0^{\Delta(S, \beta)}| = |N_{S \times S}(\Delta(S, \text{id}))/\Delta(S, \text{id})| = |Z(S)|$ for all $\beta \in \text{Aut}_F(S)$. Thus, the fixed point sizes in $X_0$ are constant on $F'$-conjugacy classes of twisted diagonal subgroups $\Delta(S, \beta)$ with $\beta \in \text{Aut}_F(S)$.

Let $H$ be a set of subgroups of $S \times S$ of the form $\Delta(P, \varphi)$ with $\varphi : P \to S$ in $F$ such that $H$ is closed under $F'$-conjugacy and taking subgroups. Assume given inductively an $F'$-generated virtual $S$-$S$-biset $X_0$ with nonnegative rational coefficients such that fixed point sizes on $X_0$ are constant on $F'$-conjugacy classes of twisted diagonal subgroups which are not in $H$. Let $P$ be an $F'$-conjugate class in $H$ whose members are maximal under inclusion among the subgroups in $H$, and let $\Delta(P, \varphi) \in P$ be a subgroup for which the fixed point set $X_0^{\Delta(P, \varphi)}$ has largest size (among the elements of $P$). Define

$$X_1 = X_0 + \sum_{\Delta} \frac{|X_0^{\Delta(P, \varphi)}| - |X_0^{\Delta(S)}|}{|N_{S \times S}(\Delta)|/|\Delta|} (S \times S)/\Delta$$

where the sum runs over a set of representatives $\Delta$ for the subgroups in $P$ up to $S \times S$-conjugacy. Thus, $X_1$ is an $F$-generated virtual $S$-$S$-biset with nonnegative rational coefficients. A subgroup $D \notin H - P$ has a fixed point on $(S \times S)/\Delta$ if and only if $D$ is $S \times S$-conjugate to some $\Delta$, and in this case the number of
such fixed points is \(|N_{S \times S}(\Delta)/\Delta|\). So by construction, \(|X_P^D| = |X_0^D|\) for each subgroup \(D \leq S \times S\) which is not in \(\mathcal{H}\), and \(|X_P^F| = |X_0^\Delta(P,\varphi)|\) for each \(D \in \mathcal{P}\). In particular, fixed point sizes on \(X_1\) are constant on \(\mathcal{F}'\)-conjugacy classes of twisted diagonal subgroups which are not in \(\mathcal{H} - \mathcal{P}\).

By induction there is thus an \(\mathcal{F}\)-generated virtual \(S\)-\(S\) biset \(X_Q\) with nonnegative rational coefficients such that fixed points sizes are constant on \(\mathcal{F}'\)-conjugacy classes of twisted diagonal subgroups. Let \(m\) be a positive integer such that \(X := mX_Q\) has integer coefficients. Then \(X\) is an \(\mathcal{F}\)-generated \(S\)-\(S\) biset with the same property.

It remains to show that \(X\) is left \(\mathcal{F}\)-stable. Let \(\varphi : Q \to S\) be a morphism in \(\mathcal{F}\), and let \(D \leq Q \times S\) be a subgroup. Then either \(D\) and \((\varphi, \text{id})(D)\) are not twisted diagonal subgroups in which case they have no fixed points on \(X\), or \(|(QX)^D| = |X^D| = |X^{(\varphi, \text{id})(D)}| = |(\varphi X)^D|\) by construction of \(X\). This shows the fixed points on \(QX\) and on \(\varphi X\) are equal for any subgroup of \(Q \times S\). So \(QX \cong \varphi X\) as \(Q \times S\)-sets. \(\blacksquare\)

The inductive nature of the proof in the proof of Theorem 2.4 sometimes makes it difficult to understand precisely which orbits of the form \(S \times (Q, \varphi)\) \(S\) actually occur in a semicharacteristic biset. The following lemma gives a sufficient condition on a pair \((Q, \varphi)\) which forces the inclusion of the corresponding orbit.

**Lemma 2.11.** Let \(X\) be a left semicharacteristic biset for \(\mathcal{F}\) containing an orbit of the form \(S \times (S, \text{id})\) \(S\). If \(\varphi \in \text{Hom}_\mathcal{F}(P, S)\) is nonextendable, then \(X\) contains an orbit isomorphic to \(S \times (P, \varphi)\) \(S\).

**Proof.** Since \(\Delta(P, \text{id}) \leq \Delta(S, \text{id})\) and \(\Delta(S, \text{id})\) fixes a point in \(S \times (S, \text{id})\) \(S\), \(\Delta(P, \text{id})\) fixes a point in \(X\). So \(\Delta(P, \varphi)\) fixes a point, say \(x \in X\), because \(X\) is left \(\mathcal{F}\)-stable. Let its orbit be isomorphic to \((S \times S)/\Delta(Q, \gamma)\), and identify \(x\) with the coset \((x_1, x_2)\), \(x_2\Delta(Q, \gamma)\). The stabilizer of \(x\) is then \(\Delta(Q^\gamma, c_{x_2}\gamma c^{-1}_{x_2})\), so \(\Delta(P, \varphi) \leq \Delta(Q^\gamma, c_{x_2}\gamma c^{-1}_{x_2})\). This means that \(\varphi\) extends to the morphism \(c_{x_2}\gamma c^{-1}_{x_2}\) in \(\mathcal{F}\) defined on \(Q^\gamma\). Since \(\varphi\) is nonextendable, \(P = Q^\gamma\) and \(\varphi = c_{x_2}\gamma c^{-1}_{x_2}\). Thus, \(X\) contains the \(S\)-\(S\) orbit \(S \times (Q, \gamma)\) \(S \cong S \times (P, \varphi)\) \(S\). \(\blacksquare\)

**Lemma 2.12.** Let \(X = \bigcup_{i=0}^{k} S \times (Q_i, \varphi_i)\) \(S\) be a left semicharacteristic biset for a fusion system \(\mathcal{F}\) on a finite group \(S\). Then

\[
\bigcap_{i=1}^{k} \bigcap_{s \in S} *Q_i \leq Q(\mathcal{F}).
\]

**Proof.** Let \(Q(X) = \{*Q_i : 1 \leq i \leq k, s \in S\}\) and \(Q(X) = \bigcap Q(X)\) for short. Thus, we must show \(Q(\mathcal{F}) \leq Q(\mathcal{F})\). By Lemma 2.11 for each nonextendable morphism \(\varphi : Q \to S\) in \(\mathcal{F}\), there is some point of \(X\) with stabilizer \(\Delta(Q, \varphi)\) \(S \times S\). So for each \(s \in S\), there is some point in \(X\) with stabilizer \(\Delta(*Q, c_s\varphi c^{-1}_s)\) and \(c_s\varphi c^{-1}_s\) is nonextendable by Remark 2.8. This shows \(Q(\mathcal{F}) \subseteq Q(X)\), so \(Q(X) \leq Q(\mathcal{F})\). \(\blacksquare\)

The reverse inclusion in Lemma 2.12 need not hold. If \(X\) is a left semicharacteristic biset for \(\mathcal{F}\), then the disjoint union of \(X\) with a number of free \(S \times S\)-orbits \((S \times S)/\Delta(1, i_{i=1}^\mathcal{F})\) is again left semicharacteristic. So there is always a semicharacteristic biset with some \(Q_i = 1\).

### 2.4 The Park embedding

Let \(\mathcal{F}\) be a fusion system on the finite group \(S\), and let \(X\) be a left semicharacteristic biset for \(\mathcal{F}\) which contains an orbit of the form \(S \times (S, \text{id})\) \(S\). Consider the group \(G = \text{Aut}(1X)\) of automorphisms of \(X\) as a right \(S\)-set. We explain briefly Park’s embedding of \(S\) into \(G\) with respect to which conjugation in \(G\) on the subgroups of \(S\) realizes the fusion system \(\mathcal{F}\).

Fix a decomposition

\[
X = \sum_{i=1}^{k} S \times (Q_i, \varphi_i) S
\]

such that \(Q_i \leq S\) and \(\varphi_i \in \text{Hom}_\mathcal{F}(Q_i, S)\) for all \(1 \leq i \leq k\) and such that \(Q_1 = S\) and \(\varphi_1 = \text{id}_S\). Following \([\text{Par10}]\), define \(\iota\) as:

\[
S \xrightarrow{\iota} \text{Aut}(1X) = G
\]
This is indeed an injection because each orbit \( S \times (Q_i, \varphi_i) \) is free as a left \( S \)-set. The same argument from [Par10] Theorem 3 copied verbatim shows that \( \iota \) induces an isomorphism of fusion systems \( \mathcal{F} \cong \mathcal{F}_\iota(S)(G) \). This gives Theorem 1.2.

**Theorem 2.13** ([War19] Chapter 4, c.f. [Par10]). Let \( \mathcal{F} \) be a fusion system on the finite group \( S \), and let \( X \) be any left semicharacteristic biset for \( \mathcal{F} \) which contains the orbit \( S \times_{(S, \text{id})} S \). Let \( G = \text{Aut}(1X) \), the group of automorphisms of \( X \) as a right \( S \)-set. Then \( G \cong S \wr \Sigma_n \) for some natural number \( n \), and there is an injection \( \iota : S \rightarrow G \) such that \( \mathcal{F} \cong \mathcal{F}_\iota(S)(G) \).

We next set up notation that will be needed later, looking more closely at the structure of \( G \) and the embedding \( \iota \). For each \( i \), fix a collection \( \{t_{ij}\}_{j \in J_i} \) of representatives of the left cosets of \( Q_i \), and set \( n_i = |S : Q_i| = |J_i| \). The action of \( u \in S \) on the coset representatives is given by \( ut_{ij}Q_i = t_{i\sigma_i(u)(j)}Q_i \), where \( \sigma_i(u) : J_i \rightarrow J_i \) is the induced permutation on \( J_i \). As a right \( S \)-set, the biset \( S \times_{(Q_i, \varphi_i)} S \) decomposes as

\[
S \times_{(Q_i, \varphi_i)} S = \sum_{j \in J_i} \langle t_{ij}, S \rangle,
\]

where \( \langle t_{ij}, S \rangle := \{t_{ij}, y \mid y \in S \} \) is the set of ordered pairs with free and transitive right \( S \)-action given by \( (t_{ij}, y) \cdot s = (t_{ij}, ys) \). Hence, also

\[
X = \sum_{i=1}^k \sum_{j \in J_i} \langle t_{ij}, S \rangle
\]
as a right \( S \)-set.

Since the right action of \( S \) on \( \langle t_{ij}, S \rangle \) is regular, each automorphism of \( \langle t_{ij}, S \rangle \) as a right \( S \)-set is left multiplication by an element of \( S \), i.e., of the form \( \langle t_{ij}, y \rangle \rightarrow \langle t_{ij}, sy \rangle \). Thus, \( \text{Aut}(1 \langle t_{ij}, S \rangle) \cong S \). It therefore follows from the above decompositions that

\[
G_i := \text{Aut}(1\langle S \times_{(Q_i, \varphi_i)} S \rangle) \cong S \wr \Sigma_{n_i},
\]

and

\[
G = \text{Aut}(1X) \cong S \wr \Sigma_n.
\]

where \( n = \sum_{i=1}^k n_i \).

We examine more closely the map \( \iota \). Now \( S \) acts from the left on each \( S \times_{(Q_i, \varphi_i)} S \), so \( \iota(S) \leq \prod G_i \leq G \). Let \( u \in S \). Since \( ut_{ij} \in t_{i\sigma_i(u)(j)}Q_i \), we have \( (t_{i\sigma_i(u)(j)})^{-1}ut_{ij} \in Q_i \), and

\[
u(t_{ij}, y) = (ut_{ij}, y) = (t_{i\sigma_i(u)(j)} \cdot (t_{i\sigma_i(u)(j)})^{-1}ut_{ij}, y) = (t_{i\sigma_i(u)(j)}, \varphi_i((t_{i\sigma_i(u)(j)})^{-1}ut_{ij})y).
\]

Thus, writing \( \pi_i \) for the projection \( II_G = G_i \), we have

\[
\pi_i(\iota(u)) = ((\varphi_i((t_{i\sigma_i(u)(j)})^{-1}ut_{ij}) \cdot j \in J_i ; \sigma_i(u) \in S \wr \Sigma_{n_i}.
\]

The following lemma gives some information on the intersection of \( \iota(S) \) with the base subgroup of \( G \).

**Lemma 2.14.** Let \( \mathcal{F} \) be a fusion system on the finite group \( S \) with left semicharacteristic biset \( X \) containing \( S \times_{(S, \text{id})} S \) and Park embedding \( \iota : S \rightarrow G = \text{Aut}(1X) \cong S \wr \Sigma_n \). Let \( B = S^n \) be the base subgroup of \( G \). Then

\[
B \cap \iota(S) \leq \iota(Q(\mathcal{F})).
\]

**Proof.** Write \( X = \bigcup_{i=1}^k S \times_{(Q_i, \varphi_i)} S \). For each \( u \in S \), the image \( \iota(u) \in B \) if and only if \( \sigma_i(u) = 1 \) for all \( 1 \leq i \leq k \) in the notation above. That is, \( \iota(u) \in B \) if and only if \( u \) fixes all cosets \( tQ_i \), that is, if and only if \( u \in \bigcap_i \bigcap_{t \in S} Q_i \). The result now follows from Lemma 2.12. □
We now state and prove a slightly more detailed version of Theorem 1.3.

**Theorem 3.1.** Let $A$ be a finite group. Then there are a finite group $S$, a fusion system on $S$, and a homocyclic abelian subgroup $U$ of $S$ such that $Q(F) = 1$, $|o_F| = S$, and $\text{Aut}_F(U) = A$. Moreover, $S$, $U$, and $F$ can be chosen so as to satisfy the following additional properties.

(i) $S$ is the semidirect product of $U$ by $A$ with respect to a faithful action of $A$ on $U$,

(ii) the exponent of $U$ is the exponent of $A$, and

(iii) if $A > 1$, then there is $Q \in Q(F)$ such that $|S : Q| > 2|A|$. 

**Proof.** Let $G = S = U = 1$ and $\mathcal{F} = F_S(G)$. So we may assume $A \neq 1$. Let $e$ be the exponent of $A$. Consider the homocyclic group $U = C_e[|A|] \times C_e[|A|]$ with free action of $A$ on each $C_e[|A|]$ factor, and let $S := UA$ be the semidirect product with respect to this action. Thus, $\text{Aut}_S(U) \cong A$ and (i) and (ii) are satisfied. Let $\mathcal{V}$ be the collection of all rank 2 homocyclic subgroups of $S$ of order $e^2$, and define

$$\mathcal{F} = \langle \text{Aut}(V) \mid V \in \mathcal{V} \rangle_S.$$

We first prove that $\text{Aut}_F(U) = \text{Aut}_S(U) \cong A$. By definition of a fusion system, $\text{Aut}_S(U) \subseteq \text{Aut}_F(U)$. Let $\psi \in \text{Aut}_F(U)$. By construction of $\mathcal{F}$ we may choose a natural number $n$ and automorphisms $\psi_1, \ldots, \psi_n$ of subgroups $T_i \leq S$ such that $\psi = \psi_n^{|U|} \circ \cdots \circ \psi_1^{|U|}$ for certain subgroups $U_i$ which are isomorphic to $U$, and such that either $T_i = S$ or $T_i \in \mathcal{V}$ for each $i = 1, \ldots, n$. Since $A$ is not the trivial group, $U \cong U_i$ has rank at least 4, and so is not isomorphic to a subgroup of any $V \in \mathcal{V}$. Thus, we must have $T_i = S$ for each $i$, and hence $\psi \in \text{Aut}_S(U)$.

Consider the collection of triples $(V, R, \alpha)$ such that $V < R \leq S$, $V \in \mathcal{V}$, and $\alpha \in \text{Hom}_F(R, S)$, and such that there is an element $c \in V$ with $\alpha(c)$ not $S$-conjugate to $c$. We claim this is the empty collection. Assume false, and among all such triples, choose one such that $\alpha$ has a decomposition with a minimal number $n$ of morphisms, and then choose such a minimal decomposition $\alpha = \alpha_n|_{R_{n-1}} \circ \cdots \circ \alpha_1|_{R_0}$ with $R_0 = R$ and $\alpha_i \in \text{Aut}(T_i)$ ($T_i \in \{S \cup \mathcal{V}\}$). By definition of $\mathcal{V}$ and assumption on the structure of $V, U$ is maximal under inclusion in $\mathcal{V}$. So $T_0 = S$ and $\alpha_1 \in \text{Inn}(S)$. Replace the triple $(V, R, \alpha)$ by $(\alpha_1(V), R_1, \alpha_1|_{R_1})$. As $\alpha_1 \in \text{Inn}(S)$, we have that $\alpha_1(c)$ is not $S$-conjugate to $\alpha_1^{-1}(\alpha_1(c)) = \alpha(c)$. Moreover, $\alpha_1|_{R_1}$ extends to $\alpha_n|_{R_{n-1}} \circ \cdots \circ \alpha_2|_{R_1}$ on $R_1 = \alpha_1(R) > \alpha_1(V)$. Thus, $(\alpha_1(V), R_1, \alpha_1|_{R_1})$ is another counterexample in which the morphism has a shorter decomposition. This contradicts the choice of the triple $(V, R, \alpha)$. In particular, this shows that if $V \in \mathcal{V}$ and a morphism $\alpha \in \text{Hom}_F(V, S)$ has the property that $c$ and $\alpha(c)$ are not $S$-conjugate, then $\alpha$ is nonextendable.

Let

$$\mathcal{V}_1 = \{V \in \mathcal{V} \mid V \text{ supports an nonextendable automorphism}\}.$$

We next claim that $\langle V \mid V \in \mathcal{V}_1 \rangle = S$ and $\bigcap_{V \in \mathcal{V}_1} V = 1$. Since the focal subgroup of $\mathcal{F}$ contains $[V, \text{Aut}(V)] = V$ for each $V \in \mathcal{V}$, this will also show $\text{foc}(\mathcal{F}) = S$ and thus complete the proof.

By the structure of $U$ as an $A$-module, $C_U(A) = Z_1Z_2$ with $\langle z_1 \rangle = Z_1 \cong C_e \cong Z_2 = \langle z_2 \rangle$ and $Z_1 \cap Z_2 = 1$. Since $|A| \geq 2$, $U$ has rank at least 4. So there is a choice of a pair of cyclic subgroups $W_1, W_2 \leq U$ of order $e$ such that $W_1 \cap C_U(A) = 1$ and $W_1Z_1 \cap W_2Z_2 = 1$. For any such choice, there is an automorphism $\alpha_i$ of $W_iZ_i$ which interchanges $W_i$ and $Z_i$ and thus does not extend to an $S$-automorphism of $W_iZ_i$ (because $Z_i \leq Z(S)$). So by the above, we see that $W_iZ_i \in \mathcal{V}_1$ for $i = 1, 2$. In particular this shows that $U \leq \langle \mathcal{V}_1 \rangle$ and $\bigcap_{V \in \mathcal{V}_1} V = 1$. The method of proof also shows (iii) is satisfied, since $Z_1Z_2 \in Q(F)$ and $|S : Z_1Z_2| \geq |U : Z_1Z_2| \geq e^2 |A| > 2|A|$.

Let $p$ be a prime dividing $|A|$, let $p^a$ be the $p$-part of the exponent of $A$, and let $C$ be any cyclic subgroup of $A$ with generator $c$ of order $p^b$. We claim that there is $V \in \mathcal{V}_1$ with $UC/U \leq UV/U$. Let $u \in U - [C, U]$ be any element of order $p^b$. Then $uc$ has order $p^a$. Fix an element $w \in C_U(C)$ of order $e/p^a$ and set
Let $J$ and an $S$ find a cyclic subgroup $Z$ of order $e$. Since the rank of $C_V(A)$ is 2, we can again find a cyclic subgroup $Z \leq C_V(A)$ of order $e$ with $W \cap Z = 1$, and then $V = WZ$ is homomorphic of order $e^2$. As before, there is an automorphism of $V$ interchanging $W$ and $Z$, which therefore does not extend to an $S$-automorphism of $V$. This shows that $V \in \mathcal{V}_1$. By construction $UC/U \leq UV/U$, and we saw above that $U \leq (V_1)$. Since $C$ was an arbitrary cyclic subgroup of $p$-power order, and the set of such subgroups generates $A$ as $p$ ranges over the primes dividing $A$, it follows that $\langle V_1 \rangle = S$. 

Before giving the proof of Theorem 1.1, we prove a specialized lemma about the commutator subgroup of a wreath product.

**Lemma 3.2.** Let $S$ be a group, let $K$ be a subgroup of $\Sigma_n$ with $n > 1$, and let $\Gamma = S \wr K$ with base subgroup $B$ and $G = \Gamma' = [\Gamma, \Gamma]$. Assume that $K'$ is perfect and transitive. Then $[B, B] \leq [K, B] = [K', B] = [K', K', B]$ and $G = [K', B]$ is perfect.

**Proof.** Although $n > 1$ was assumed initially, the further assumptions give implicitly that $n \geq 5$. Write $e_i(s)$ for the element $(1, \ldots, 1, s, 1, \ldots, 1) \in B = S_1 \times \cdots \times S_n$ (with $s$ in the $i$-th place), and $c_j^i(s) = e_j(s)e_i(s)^{-1}$.

Let $J$ be any transitive subgroup of $\Sigma_n$. Then $c_j^i(s) = [g, e_i(s)] \in [J, B]$ for each element $g \in J$ which sends $i$ to $j$, so $c_j^i(S) \in [J, B]$ for each $i$ and $j$. For $i$ an index taken modulo $n$ and for $s, t \in S$,

$$[c_j^1(s)c_i^1(t^{-1})] = e_i([s, t]).$$

This shows that $[S_i, S_j] \leq [J, B]$ for each $i$, and hence $[B, B] = [S, S]^n \leq [J, B]$. Under the same assumptions on $J$, we just saw $[J, B]$ contains all $c_j^i(s)$ with $s \in S$ and $1 \leq i, j \leq n$. These generate $\ker(\pi)$, where $\pi : B \to S' \times S'$ is the homomorphism sending an element of $B$ to the product of its components. Since each generating element of $[J, B]$ is clearly in this kernel, we have $[J, B] = \ker(\pi)$. In particular, $[B, B] \leq [K', B] = [K, B]$.

Next, for any subgroup $J$, we have $[J, B, J] = [J, J, B]$. So if $J' = J$, then $[B, B, J] = [B, J] = [B, J'] = [B, J, J] \leq [J, J, B] \leq [J, B, J]$, the first inclusion by the Three Subgroups Lemma [Gor80, Theorem 2.3(ii)].

So $[J, J, B] = [J, B]$. In particular, $[K', K', B] = [K', B]$ since $K'$ was assumed perfect.

Applying [Gor80] Theorem 2.1 for example to $\Gamma = KB$, we see that $G = \Gamma' = K'[K, B][B, B]$, and then $G = K'[K', B]$ as $[B, B] \leq [K, B] = [K', B]$. Keeping in mind that $[[K', B], [K', B]] \leq [B, B]$ since $B/[B, B]$ is abelian, a similar argument gives $G' = [K', K'][K', B, B] = K'[KB'] = G$, so $G$ is perfect. 

**Proof of Theorem 1.1.** Let $A$ be any finite group. If $A = 1$, then we take $G = U = 1$, so we may assume $A > 1$. Fix a fusion system $\mathcal{F}$ on a finite group $S = U \rtimes A$ with $U$ homocyclic and $A$ faithful on $U$, satisfying the conclusion of Theorem 3.1. Let $X = \sum_{i=1}^{k} S \times (Q_i, \varphi_i)$ $S$ any left semicharacteristic biset for $\mathcal{F}$ as in Theorem 2.11 and write $\iota : S \to \Gamma = \Aut(\iota \chi) \cong S \wr \Sigma_n$ for the Park embedding, so that $\mathcal{F} \cong \mathcal{F}_\iota(\iota \chi)$ via $\iota$. Set $G = \Gamma'$. To ease notation, we identify $S$ with its image in $\Gamma$, and so we identify $\mathcal{F}$ and $\mathcal{F}_\bar{\iota}(\overline{\iota})$. By choice of $\mathcal{F}$, we know $S \cap B \leq Q(\mathcal{F}) = 1$ by Lemma 2.11 where $B$ is the base subgroup of $\mathcal{F}$ as usual, while also $S = \fix(\mathcal{F}) \leq S \cap G$ by Remark 2.3. Thus, $U \leq S \leq G$. Since $F = \mathcal{F}_\bar{\iota}(\overline{\iota})$, we have $\Aut_G(U) = \Aut_F(U) \cong A$ again by choice of $\mathcal{F}$, and $\Aut_S(U) \cong A$ by construction. Since $S \leq G$, this shows $\Aut_G(U) \cong A$. We want to verify that $G$ satisfies the conclusion of the theorem.

Let $H$ be the alternating subgroup of $\Sigma_n$, the usual complement of $B$ in $\Gamma$, and let $N = \langle U^G \rangle$ be the normal closure of $U$ in $G$. We will see below that $n \geq 5$, so $H$ is simple. By Lemma 3.2 $G$ is perfect and $G = H[H, B]$. Thus, it remains to show that $N = G$.

Recall from the discussion of the Park embedding that $n = \sum_{i=1}^{k} |S : Q_i|$, so by Theorem 3.1(iii) and Lemma 2.11 there is $i$ such that $n \geq |S : Q_i| > 2|A|$. So indeed, $n \geq 5$ and $H$ is simple. Use Bertrand’s postulate to get a prime $p$ with $|A| < p < 2|A|$, and so a prime $p$ that divides $|H|$ but not $|A|$. By Theorem 3.1(ii), $p$ divides $|H|$ but not $|S|$, so $|H|^2$ does not divide $|G|$. On the other hand, $U \cap B \leq S \cap B \leq Q(F) = 1$, so as $H$ is simple, $N$ projects modulo $B$ onto $H$. Thus $|H|$ divides $|N|$. Since
N ∩ H is normal in H, we have H ≤ N or H ∩ N = 1. In the latter case, G contains the subgroup HN of order divisible by |H|², a contradiction, and hence H ≤ N. As H ≤ N, N contains the normal closure of H in G, which is H[H, B] = G, and this completes the proof of the theorem. ■

References

[AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, Fusion systems in algebra and topology, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), no. 4, 779–856 (electronic).

[Cra11] David A. Craven, The theory of fusion systems, Cambridge Studies in Advanced Mathematics, vol. 131, Cambridge University Press, Cambridge, 2011, An algebraic approach. MR 2808319

[Gor80] Daniel Gorenstein, Finite groups, second ed., Chelsea Publishing Co., New York, 1980.

[Mue20] Peter Mueller, Normalizers in symmetric groups, MathOverflow, 2020, URL:https://mathoverflow.net/q/102532 (version: 2020-06-15).

[Par10] Sejong Park, Realizing a fusion system by a single finite group, Arch. Math. (Basel) 94 (2010), no. 5, 405–410. MR 2643975

[Par16] ———, Realizing fusion systems inside finite groups, Proc. Amer. Math. Soc. 144 (2016), no. 8, 3291–3294. MR 3503697

[UYe12] Özgür Ünlü and Ergün Yalçın, Fusion systems and constructing free actions on products of spheres, Math. Z. 270 (2012), no. 3-4, 939–959. MR 2892931

[War19] Athar Ahmad Warraich, Realizing infinite families of fusion systems over finite groups, Ph.D. thesis, The University of Birmingham, 2019.

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