A New Proof of a Theorem in Analysis by Generating Integrals and Fractional Calculus

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Abstract

The idea of generating integrals analogous to generating functions is first introduced in this paper. A new proof of the well-known Finite Harmonic Series Theorem in Analysis and Analytical Number Theory is then obtained by the method of Generating Integrals and Fractional Calculus. A generalization of the Riemann zeta function up to non-integer order is derived.

1 The Finite Harmonic Series Theorem

Definition 1 The finite harmonic series is

\[ h(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \]

Theorem 1 (The Finite Harmonic Series Theorem)

It is well known [4, p. 16, (1.7.9)] that

\[ h(n) = \psi(1+n) + \gamma \]

\[ = \log n + \gamma + O(1/n) \quad (1) \]

where the digamma function \( \psi(z) = \frac{d}{dz} \log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)} \), the Euler constant \( \gamma = -\Gamma'(1) = 0.577215\cdots \), and the prime denotes differentiation.

The first part (1) of the Theorem can be proved by differentiating the recurrence relation of \( \Gamma(n+1) = n \Gamma(n) \) and summing over \( n \) [2, p. 256]. The second part (2) can be proved by using the Euler-Maclaurin summation formula [4, p. 629].

In this paper, we shall present an alternative elementary proof [3] of the first part (1) of the Theorem by using Fractional Calculus and the idea of generating integrals.
2 Introduction & Motivation of Fractional Calculus

“Thus it follows that $d^{1/2}x$ will be equal to ...from which one day useful consequences will be drawn.” — Leibniz in a letter [5] to L’Hospital.

“When $n$ is an integer, the ratio $d^n p$, $p$ a function of $x$, to $dx^n$ can always be expressed algebraically. Now it is asked: what kind of ratio can be made if $n$ is a fraction?” — Euler [6].

“The idea of an integral or derivative, of arbitrary non-integral order, was introduced into analysis by Liouville and Riemann. Such integrals and derivatives may be, and have been by different writers, defined in a variety of manners, and different systems of definitions may be the most useful in different fields of analysis.” — Hardy and Littlewood [7].

In this paper, we shall consider $D^\sigma$, $\sigma \in \mathbb{R}$, an operator of non-integer order, a notion first pondered upon by Leibniz [5, pp. 301-302], Euler [6, p. 55], Lagrange [8], Laplace [9, p. 85 and p. 186], Fourier [10] and Abel [11] during the late 17th century to early 19th century. We shall briefly review several known and equally valid definitions for $D^\sigma$, and then focus on one of the definitions, the Riemann-Liouville (R-L) Fractional Calculus. The foundation of R-L Fractional Calculus was laid by Riemann [12] and Liouville [13] in the late 19th century, and then subsequently developed by Cayley [14], Laurent [15], Heaviside [16], Hardy and Littlewood [7, 17, 18, 19], and many others. It was largely regarded as a mathematical curiosity until only recently when Mandelbrot, the discoverer of Fractals, found an application of the R-L Fractional Calculus in the Brownian motion in a fractal medium, and speculated a possible connection between the analysis of Fractional Calculus and the geometry of Fractals [20].

We shall also introduce the idea of generating integrals by analogy to generating functions. As we shall see, just as a certain generating function is useful for generating a certain desired sequence of numbers, a certain generating integral is similarly useful. While a generating function $f(z)$ generates a sequence of numbers $\{p_n\}$ in the coefficients of the terms of different orders in its power series expansion $\sum_n p_n z^n$, a generating integral generates a sequence of numbers $q_n$ in the coefficient of a term in the result of an $n$-fold integration.

However, a generating integral has one unique advantage over a generating function. The R-L Fractional Calculus can be used to analytically extend a generating integral of iteration order $n \in \mathbb{Z}^+$ to order $\rho \in \mathbb{R}$. The result is that the sequence of numbers $\{q_n\}$ is in turn analytically extended to a function $q(\rho)$, $\rho \in \mathbb{R}$. 
We shall then show how the Riemann-Liouville Fractional Calculus and the idea of generating integrals can be used to prove the well-known Finite Harmonic Series Theorem.

3 Differential-Integral Operator $D^n$

**Definition 2** Let the operator $D_n^x|a$, $n \in \mathbb{Z}$, acting on a function $f$ at the point $x$ be defined as

$$D_n^x|a f(x) = \begin{cases} \frac{d^n}{dx^n} f(x) & (n > 0) \\ f(x) & (n = 0) \\ \int_a^x f(\hat{x})(d\hat{x})^{-n} & (n < 0) \end{cases}$$

(3)

where the $n$-fold integration is defined inductively as

$$\int_a^x f(\hat{x})(d\hat{x})^{-n} = \int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1) \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n.$$ 

As an example, consider $f(x) = x^m$.

$$D_n^x|a x^m = \begin{cases} 0 & (m \geq 0, m < n) \\ \frac{m!}{(m-n)!} x^{m-n} = \Gamma(1+m) \Gamma(1+m-n) x^{m-n} & (m \geq 0, m \geq n) \\ (-1)^n \frac{(|m|-1+n)!}{(|m|-1)!} x^{m-n} \bigg|_a^x & (m \leq 0, m < n) \\ = \lim_{\epsilon \to 0} \frac{\Gamma(1+m+\epsilon) \Gamma(1+m+\epsilon-n)}{\Gamma(1+m+\epsilon-n)} x^{m-n} \bigg|_a^x \\ = \int_a^x \left( \int_a^{\hat{x}} \hat{x}^m (d\hat{x})^{|m|} \right) (d\hat{x})^{(m-n)} & (m \leq 0, m \geq n) \\ = (-1)^{m+1} \frac{1}{(|m|-1)!} \int_a^x \log(\hat{x}) (d\hat{x})^{(m-n)} & \end{cases}$$

(4)

where $\hat{x}^{m-n} \bigg|_a^x = (x^{m-n} - a^{m-n})$.

If we tabulate $D_n^x|a x^m$ for $n, m \in \mathbb{Z}$ and omit the constant terms containing $a$, we can observe a pattern emerges as in Table 1.
4 Riemann-Liouville (R-L) Fractional Calculus

The R-L Fractional Calculus [21] begins with
\[
\int_a^x f(\hat{x}) (d\hat{x})^n \equiv \underbrace{\int_a^x \int_a^{x_2} \cdots \int_a^{x_n} \int_a^{x_1}}_{n\text{-times}} f(x_1) \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n \tag{5}
\]
for \( n \in \mathbb{Z^+} \) as the fundamental defining expression, and it can be shown [21, p. 38] to be equal to the Cauchy formula for repeated integration,
\[
\frac{1}{\Gamma(n)} \int_a^x \frac{f(t)}{(x-t)^{1-n}} \, dt . \tag{6}
\]

Definition 3 (R-L Fractional Calculus)

The R-L fractional integral is analytically extended from (6) as
\[
D_{x|a}^\sigma f(x) = \frac{d^\sigma}{dx^\sigma} f(x) = \int_a^x f(x)(dx)^{-\sigma} \text{ by extending (3)}
= \frac{1}{\Gamma(-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma}} \, dt \quad (\sigma < 0, \sigma, a \in \mathbb{R}) \text{ by (8)} ,
\tag{7}
\]
and the R-L fractional derivative is in turn derived from the R-L fractional integral (8) by ordinary differentiation:
\[
D_{x|a}^\sigma f(x) = D_{x|a}^m \left( D_{x|a}^{-(m-\sigma)} f(x) \right) \quad (\sigma > 0, m \in \mathbb{Z^+})
\tag{8}
\]
where \( m \) is chosen such that \( m > 1 + \sigma, \sigma > 0 \).

Lemma 1 The equation (8) is independent of the choice of \( m \) for \( m > 1 + \sigma, \sigma, m \in \mathbb{Z^+}, \sigma \in \mathbb{R}, \sigma > 0 \).
Proof

For $m > 1 + \sigma$, $m \in \mathbb{Z}^+$, $\sigma > 0$, we have $-(m-\sigma) < -1 < 0$ and $(m-\sigma-1) > 0$. The first condition, $-(m-\sigma) < 0$, allows us to use the equation (7) to write

$$D_{x|[a]}^{-1}(m-\sigma) f(x) = \frac{1}{\Gamma(m-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma-m}} dt.$$  

From (8),

$$D_{x|[a]}^{-m} \left( D_{x|[a]}^{-1}(m-\sigma) f(x) \right) = \frac{d^m}{dx^m} \left( \frac{1}{\Gamma(m-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma-m}} dt \right) = \frac{1}{\Gamma(m-\sigma)} \int_a^x f(t) \left( \frac{d^m}{dx^m} (x-t)^{m-\sigma-1} \right) dt.$$  

The second condition, $(m-\sigma-1) > 0$, and the condition $m > 0$ allow us to use the second case of (8). Thus, (9) becomes

$$\frac{1}{\Gamma(m-\sigma)} \int_a^x f(t) \frac{\Gamma(m-\sigma)}{\Gamma(-\sigma)} (x-t)^{-(1+\sigma)} dt = \frac{1}{\Gamma(-\sigma)} \int_a^x f(t) \frac{(x-t)^{1+\sigma}}{dt} = D_{x|[a]}^{-\sigma} f(x). \quad \Box$$

When $a = 0$, (9) for $f(x) = x^r$ is well-defined only for the half plane $r > -1$. Consequently, in the R-L Fractional Calculus, $D_{x|[a]}^{-\sigma} x^r$ is well-defined only for the half plane $r > -1$.

$$D_{x|[a]}^{-\sigma} x^r = \begin{cases} \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} x^{r-\sigma} & (\sigma > 0, \ r > -1) \\ x^r & (\sigma = 0, \ \forall \ r) \\ \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} x^{r-\sigma} \bigg|_a^x & (\sigma < 0, \ r > -1) \end{cases}. \quad (10)$$

5 Fractional Calculus by Cauchy Integral

The Cauchy Integral for an analytic function $f(z)$ in the complex plane [22, p. 120] is

$$f^{(n)}(z_0) = \frac{\Gamma(1+n)}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{1+n}} dz. \quad (11)$$

Analytic extension of the Cauchy Integral from $n \in \mathbb{Z}^+$ to $s \in \mathbb{R}^+$ gives an analytic extension of $D^n f(z_0)$. However, the analytic extension is not trivial. The term $(z-z_0)^{1+\sigma}$ will become multi-valued and thus the result may depend on the choice of branch cut and integration path.
6 Fractional Calculus by Fourier Transform

In the theory of Fourier Transforms,

\[ \tilde{f}(x) = \int_{-\infty}^{+\infty} f(x) e^{ikx} \, dx, \]
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(x) e^{-ikx} \, dk, \]

and

\[ D_\sigma^\sigma f(x) = \int_{-\infty}^{+\infty} \tilde{f}(k) D_\sigma^\sigma (e^{-ikx}) \, dk \quad (\sigma \in \mathbb{R}) \]
\[ = \int_{-\infty}^{+\infty} (-ik)^\sigma \tilde{f}(k) e^{-ikx} \, dk. \]

This approach is often known as the pseudo-differential operator approach. It was shown by Závada [23] to be equivalent to the Riemann-Liouville Fractional Calculus and the Fractional Calculus by Cauchy Integral.

7 Functional Analytic Approach

In the functional analytic approach, an example of a functional integral of an operator \( A \) is

\[ (-A)^a = -\frac{\sin a\pi}{\pi} \int_0^{\infty} \lambda^{a-1} (\lambda \mathbb{1} - A)^{-1}A \, d\lambda \quad (0 < a < 1, \lambda \in \mathbb{R}). \quad (12) \]

(\( \lambda \mathbb{1} - A \)) is called the kernel of the functional integral. The evaluation of the integral with respect to real variable \( \lambda \) requires various conditions on the spectrum of the operator \( A \).

The analytic extension of \( D \) in the functional approach is then obtained from replacing \( A \) by \( D \) in (12).

For details of this well-developed functional analysis approach, see [24].

8 Differentiating and Integrating in non-integer \( s \)-dimensions

The differential of an integer \( n \)-dimensional function in \( n \)-dimensions can be expressed as

\[ \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} f(x_1, x_2, \ldots, x_n). \]

The corresponding integral can be expressed as

\[ \int f(x_1, x_2, \ldots, x_n) \, d^n x \equiv \underbrace{\int x_n \cdots \int x_2}_{n \text{-times}} \int f(x_1, x_2, \ldots, x_n) \, dx_1 dx_2 \cdots dx_n. \]
If $f$ is spherically symmetric, $f = f(r)$, then
\[
\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \ldots \frac{\partial}{\partial x_n} f = \frac{\partial^{n-1}}{\partial r^{n-1}} \frac{\partial}{\partial \Omega_{n-1}} f = \frac{\Gamma(n/2)}{2 \pi^{n/2}} \frac{\partial^{n-1}}{\partial r^{n-1}} f(r) \tag{13}
\]
can then be analytically extended to the differential of a non-integer $s$-dimensional function in $s$-dimensions,
\[
\frac{\partial^{s-1}}{\partial r^{s-1}} \frac{\partial}{\partial \Omega_{s-1}} f(r) = \frac{\Gamma(s/2)}{2 \pi^{s/2}} \frac{\partial^{s-1}}{\partial r^{s-1}} f(r) \tag{14}
\]
where $s \in \mathbb{R}$ or $s \in \mathbb{C}$, and $\Omega$ is the $n$-dimensional solid angle.

Similarly, the corresponding integral
\[
\int f \, d^n x \equiv \int_0^\infty r^{n-1} f(r) \, dr \int_0^{2\pi} d\theta_1 \int_0^\pi \sin \theta_2 d\theta_2 \cdots \int_0^{\pi} \sin^{n-2} \theta_{n-1} d\theta_{n-1}
\]
\[
= \int_0^\infty r^{n-1} f(r) \, dr \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty r^{n-1} f(r) \, dr \tag{15}
\]
can be analytically extended to
\[
\int f \, d^s x = \frac{2\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty r^{s-1} f(r) \, dr . \tag{16}
\]

This method was developed by 't Hooft and Veltman \[25\] in 1960’s. The method was central to an important technique called Dimensional Regularization in Quantum Field Theory where the method is used to isolate singularities in divergent integrals.

9 Generating Integral of the Finite Harmonic Series

**Theorem 2 (Generating Integral of $h(n)$)**

\[
h(n) = \log x - \frac{\Gamma(1+n)}{x^n} \int_0^x \log \hat{x} (d\hat{x})^n \quad (n \in \mathbb{Z}) \tag{17}
\]

where
\[
\int_a^x f(\hat{x}) (d\hat{x})^n \equiv \int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1) \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n .
\]

**Proof**

We observe that $-h(n)/n!$ appears in the coefficient of the $x^n$ term when we repeatedly integrate $\log x$:
\[
\int_0^x \log \hat{x} (d\hat{x}) = x(\log x - 1) ,
\]
\[
\int_0^x \log \hat{x} (d\hat{x})^2 = \frac{x^2}{2} (\log x - \frac{3}{2}) ,
\]
\[
\vdots
\]
\[
\int_0^x \log \hat{x} (d\hat{x})^n = \frac{x^n}{n!} (\log x - h(n)) . \tag{18}
\]
We can prove this observation by induction:

\[
\int_0^x \log \hat{x} (d\hat{x})^{n+1} = \int_0^x \left( \int_0^x \log \hat{x} (d\hat{x})^n \right) (d\hat{x})
\]

\[
= \int_0^x \frac{\hat{x}^n}{n!} (\log \hat{x} - h(n)) (d\hat{x})
\]

\[
= \frac{x^{n+1}}{(n+1)!} (\log x - h(n)) - \frac{x^n}{(n+1)!} (d\hat{x})
\]

\[
= \frac{x^{n+1}}{(n+1)!} \left( \log x - h(n) - \frac{1}{n+1} \right)
\]

\[
= \frac{x^{n+1}}{(n+1)!} (\log x - h(n+1)) .
\]

Rearrangement of (18) yields the Theorem. □

**Theorem 3 (Generating Integral of \(h(\rho)\))**

\[
h(\rho) = \log x - \frac{\Gamma(1+\rho)}{x^\rho} \int_0^x \log \hat{x} (d\hat{x})^\rho \quad (\rho \in \mathbb{R}) .
\]  

(19)

**Proof**

By analogy to generating functions, we take

\[
\int_0^x \log \hat{x} (d\hat{x})^n
\]

as the *generating integral* of the finite harmonic series \(h(n)\), and so the natural analytic extension of the generating integral takes the form of

\[
\int_0^x \log \hat{x} (d\hat{x})^\rho = \frac{x^\rho}{\Gamma(1+\rho)} (\log x - h(\rho)) .
\]  

(20)

Noting that \(\log x\) may be expressed as

\[
\log x = \int_1^x \hat{x}^{-1} d\hat{x} = \lim_{\epsilon \to 0} \int_1^x \hat{x}^{-1+\epsilon} d\hat{x} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (x^\epsilon - 1) ,
\]  

(21)

we can now evaluate the fractional integral in (20) by the **R-L Fractional Calculus** (Definition 3).

\[
\int_0^x \log \hat{x} (d\hat{x})^\rho = \int_0^x \lim_{\epsilon \to 0} \frac{1}{\epsilon} (x^\epsilon - 1) (d\hat{x})^\rho = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^x (\hat{x}^\epsilon - 1) (d\hat{x})^\rho
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_0^x \hat{x}^\epsilon (d\hat{x})^\rho - \int_0^x (d\hat{x})^\rho \right]
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ D_{\hat{x}}^\rho \hat{x}^\epsilon [0] - D_{\hat{x}}^\rho 1 [0] \right]
\]

\[
= \lim_{\epsilon \to 0} \frac{x^\rho}{\epsilon} \left[ \frac{\Gamma(1+\epsilon) x^\epsilon}{\Gamma(1+\epsilon+\rho)} - \frac{1}{\Gamma(1+\rho)} \right]
\]  

(22)
where the interchange of the integral and the limit is justified by Arzelà’s theorem on bounded convergence [26, pp. 405-406] as \((x^\varepsilon - 1)/\varepsilon\) is integrable in \(\hat{x} \in [0, x]\).

Combining (22) with (20) gives the analytic extension of the finite harmonic series:

\[
h(\rho) = \log x - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \frac{\Gamma(1+\varepsilon)}{\Gamma(1+\varepsilon+\rho)} x^\varepsilon - 1 \right]
\]

\[
= \log x - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (x^\varepsilon - 1) \left[ \frac{\Gamma(1+\varepsilon)\Gamma(1+\rho)}{\Gamma(1+\varepsilon+\rho)} \right]
\]

\[
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ 1 - \frac{\Gamma(1+\varepsilon)\Gamma(1+\rho)}{\Gamma(1+\varepsilon+\rho)} \right]
\]

\[
= \left[ \log x - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (x^\varepsilon - 1) \right] + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ 1 - \frac{\Gamma(1+\varepsilon)\Gamma(1+\rho)}{\Gamma(1+\varepsilon+\rho)} \right]
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ 1 - \frac{\Gamma(1+\varepsilon)\Gamma(1+\rho)}{\Gamma(1+\varepsilon+\rho)} \right]
\]

\[
= \frac{\Gamma'(1+\rho)}{\Gamma(1+\rho)} - \Gamma'(1)
\]

\[
= \psi(1+\rho) + \gamma
\]  \hspace{1cm} (23)

where the limit has been taken with L’Hospital rule.

\[
\int_0^x \log \hat{x} \, (d\hat{x})^\rho = \frac{x^\rho}{\Gamma(1+\rho)} (\log x - \psi(1+\rho) - \gamma) \quad (\rho \in \mathbb{R}).
\]  \hspace{1cm} (24)

Figure 1: The curve \(h(\rho) = \psi(1+\rho) + \gamma\) passes through the points \((n, h(n))\) where \(n \in \mathbb{Z}^+, \ h(n) = \sum_{k=1}^n \frac{1}{k}\).

Hence, by the application of the R-L Fractional Calculus to analytically extend the generating integral, we have found an alternative elementary proof of the first part (ii) of the Finite Harmonic Series Theorem.

**Theorem 4**

\[
\int_0^x \log \hat{x} \, (d\hat{x})^\rho = \frac{x^\rho}{\Gamma(1+\rho)} (\log x - \psi(1+\rho) - \gamma) \quad (\rho \in \mathbb{R}).
\]  \hspace{1cm} (24)
Proof
Replacing the \( h(\rho) \) in (24) by (23) gives the Theorem.

\[ \square \]

10 The Riemann Zeta Function up to Order \( n \)

The analytic extension (23) can be generalized.

Definition 4 The Riemann zeta function \( [27] \)

\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\text{Re}(s) > 1), \]  

(25)

the polygamma functions \( [2, \text{p. 260, (6.4.1)}] \)

\[ \psi^{(m)}(x) = \frac{d^m}{dx^m} \psi(x) = \frac{d^{m+1}}{dx^{m+1}} \log \Gamma(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}, \]  

(26)

and the Riemann zeta function up to order \( n \)

\[ \zeta(s | n) = \sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) - \sum_{k=n+1}^{\infty} \frac{1}{k^s} \quad (\text{Re}(s) > 1) \]  

(27)

may be combined to write

\[ \zeta(m | n) = \sum_{k=1}^{n} \frac{1}{k^m} = \frac{(-1)^m}{(m-1)!} \left( \psi^{(m-1)}(1+n) - \psi^{(m-1)}(1) \right) \]

\[ = \frac{(-1)^m}{(m-1)!} \frac{d^m}{dx^m} \log(\Gamma(1+x)) \bigg|_{x=0}^{x=n}. \]  

(28)

The analytic extension is then obtained by replacing the derivative in (28) with a fractional derivative:

\[ \zeta(s | z) = \frac{w(s)}{\Gamma(s)} D_x^s \log(\Gamma(1+x)) \bigg|_{x=0}^{x=z} (s, z \in \mathbb{C}) \]  

(29)

which can be evaluated when \( \log(\Gamma(1+x)) \) is expressed in the form of an asymptotic series \( [2, \text{p. 257, (6.1.41)}] \). However, \( w(s) \) depends on the choice of extension to the R-L Fractional Calculus (Definition 3) into the other half plane, \( r \leq -1, \sigma \in \mathbb{R} \).

11 Analytic Extension of R-L Fractional Calculus

Consider the case of \( D_x^n \) where \( a = 0 \). We shall introduce the \( (\sigma, r) \) diagram in which the numerical factor of \( D_x^a x^r \) is mapped to the point at coordinate \( (\sigma, r) \) of the diagram. The \( (\sigma, r) \) diagram of \( D_x^a x^r \) can be characterised into 4 regions as in Figure 3.
Definition 5 (Regions of \((\sigma, r)\))

The zero region \(Z_{zer} = \{ (\sigma, r) : r < \sigma, r \geq 0 \} \);
the upper region \(U_{up} = \{ (\sigma, r) : r \geq \sigma, r \geq 0 \} \);
the lower region \(L_{ow} = \{ (\sigma, r) : r < \sigma, r < 0 \} \);
the log region \(L_{og} = \{ (\sigma, r) : r \geq \sigma, r < 0 \} \).

A point lying on the right of the \(r\)-axis \((\sigma > 0)\) is a differentiation; a point on the left \((\sigma < 0)\) is an integration.

The \((\sigma, r)\) diagram at integer grid points everywhere except in the log region gives numerical factors identical to those in Table 1.

An extension of R-L Fractional Calculus to the other half plane \(r \leq -1\) is given by

\[
D_x^\sigma x^r = \begin{cases} 
\lim_{\epsilon \to 0} \frac{\Gamma(1+r+\epsilon)}{\Gamma(1+r+\epsilon-\sigma)} x^{r-\sigma} & \text{elsewhere} \\
\lim_{\epsilon \to 0} \frac{\Gamma(1+r+\epsilon)}{\Gamma(1+r+\epsilon-\sigma)} x^{r-\sigma} & \text{in } \Omega
\end{cases}
\]  

where \(\Omega = \{ (\sigma, r) : \sigma \in \mathbb{R}, r \in \mathbb{Z}^- \}\), the set of horizontal lines in lower and log regions.

\(\Gamma(1+r)/\Gamma(1+r-\sigma)\) is finite everywhere in the zero and upper regions.
\(\lim_{\epsilon \to 0} \Gamma(1+r+\epsilon)/\Gamma(1+r+\epsilon-\sigma)\) is well-defined everywhere in the lower and log regions except in \(\Omega \setminus (\mathbb{Z}^- \times \mathbb{Z}^-)\). Following Theorem 4, the R-L fractional integral of \(\log x\) can be evaluated exactly and expressed in only elementary functions,

\[
D_x^\sigma \log x = \begin{cases} 
\frac{x^{-\sigma}}{\Gamma(1-\sigma)} \left( \log x - \psi(1-\sigma) - \gamma \right) & (\sigma \in \mathbb{R} \setminus \mathbb{Z}) \\
\lim_{\epsilon \to 0} \frac{\Gamma(1+r+\epsilon)}{\Gamma(1+r+\epsilon-\sigma)} x^{r-\sigma} & (\sigma \in \mathbb{Z})
\end{cases}
\]
is thus well-defined.

To analytically extend from $D^\sigma_x x^r$ on the real plane ($\sigma, r$) to $D^s_x x^r$ on the product of complex plane and real line, $(s, r) \in \mathbb{C} \times \mathbb{R}$, we simply replace $\sigma \in \mathbb{R}$ in (30) by $s \in \mathbb{C}$.

12 Open Problems

1. Generalize (30) for $D^s_{z-c} z^w$, $s, w, z, c \in \mathbb{C}$. For the complex function $z^w$, one has to specify, in addition, the integration contour for $\text{Re}(s) < 0$.

2. Find the exact expression for $\log(\Gamma(1+x))$ and $w(s)$ in terms of elementary functions in analytic extension of the R-L Fractional Calculus given by (30).

13 Tables of Generating Integrals

“Nature laughs at the difficulties of integration.” — Laplace

An interesting consequence is that objects of the form $\int x^r (\log x)^a (dx)^\rho$, $r, a, \rho \in \mathbb{R}$, exist and can be generating integrals for certain functions. Perhaps it may be worthwhile to introduce, in the future editions of Tables of Integrals, a new section which gives the coefficient of the $x^k$ term, $w(\rho, r, a, k)$, corresponding to these generating integrals to facilitate the evaluation of integrals of similar forms.

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