An analogue of the Conjecture of Dixmier is true for the algebra of polynomial integro-differential operators

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Abstract

Let $A_1 := K\langle x, \frac{d}{dx} \rangle$ be the Weyl algebra and $I_1 := K\langle x, \frac{d}{dx}, \int \rangle$ be the algebra of polynomial integro-differential operators over a field $K$ of characteristic zero. The Conjecture/Problem of Dixmier (1968) [still open]: is an algebra endomorphism of the Weyl algebra $A_1$ an automorphism? The aim of the paper is to prove that each algebra endomorphism of the algebra $I_1$ is an automorphism. Notice that in contrast to the Weyl algebra $A_1$ the algebra $I_1$ is a non-simple, non-Noetherian algebra which is not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals.

Key Words: the Weyl algebra, the Conjecture/Problem of Dixmier, the algebra of polynomial integro-differential operators, the Jacobian Conjecture.

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1 Introduction

In this paper, $A_1 := K\langle x, \frac{d}{dx} \rangle$ is the Weyl algebra (i.e. the algebra of polynomial differential operators) and $I_1 := K\langle x, \frac{d}{dx}, \int \rangle$ is the algebra of polynomial integro-differential operators over a field $K$ of characteristic zero ($A_1, I_1 \subseteq \text{End}_K(K[x])$ where $K[x]$ is a polynomial algebra in one variable $x$), and $\int : K[x] \to K[x]$, $x^n \mapsto \frac{x^{n+1}}{n+1}$, $n \geq 0$, is the integration.

Six Problems of Dixmier, [18], for the Weyl algebra $A_1$: In 1968, Dixmier [18] posed six problems for the Weyl algebra $A_1$.

The First Problem/Conjecture of Dixmier, [18]: is an algebra endomorphism of the Weyl algebra $A_1$ an automorphism?

Dixmier writes in his paper [18], p. 242: “A. A. Kirillov informed me that the Moscow school also considered this problem”.

In 1975, the Third Problem of Dixmier was solved by Joseph and Stein [19] (using results of McConnel and Robson [21]); and using his (difficult) polarization theorem for the Weyl algebra $A_1$ Joseph [19] solved the Sixth Problem of Dixmier (a short proof to this problem is given in [4]; moreover, an analogue of the Sixth Problem of Dixmier is true for the ring of differential operators on an arbitrary smooth irreducible algebraic curve [4]). In 2005, the Fifth Problem of Dixmier was solved in [3]. Problems 1, 2, and 4 are still open. The Fourth Problem of Dixmier has positive solution for all homogeneous elements of the Weyl algebra $A_1$ (Theorem 2.3, [3]).

The aim of the paper is to prove an analogue of the First Problem/Conjecture of Dixmier for the algebra $I_1$ (Theorem 1.1). The proof is not straightforward and several key results of the papers [10], [11] and [12] are used. To make the proof more accessible for the reader we use a ‘zoom in’ way of presenting it: in the Introduction we explain the structure of the proof, it consists of nine steps; in Section 3 each steps is proved using some of the results of [10], [11] and [12].

Theorem 1.1 Each algebra endomorphism of $I_1$ is an automorphism.
Structure of the Proof. Let \( \sigma \) be an algebra endomorphism of \( I_1 \). Since \( I_1 = K[H, \int, \partial] \) where \( H := \partial x \) (notice that \( x = \int H \)), the endomorphism \( \sigma \) is uniquely determined by the elements

\[
H' := \sigma(H), \quad \int' := \sigma(\int), \quad \partial' := \sigma(\partial).
\]

Step 1. \( \sigma \) is a monomorphism.

Step 2. \( \sigma(F) \subseteq F \), where \( F \) is the only proper ideal of the algebra \( I_1 \). Therefore, there is a commutative diagram of algebra homomorphisms:

\[
\begin{array}{ccc}
I_1 & \xrightarrow{\sigma} & I_1 \\
\downarrow{\pi} & & \downarrow{\tau} \\
B_1 & \xrightarrow{\varpi} & B_1
\end{array}
\]

where \( B_1 := I_1/F \cong K[H][\partial, \partial^{-1}; \tau] \), \( \tau(H) = H + 1 \), is a simple algebra, and so \( \varpi \) is an algebra monomorphism.

Step 3. \( H' = \lambda H + \mu + h \) for some elements \( \lambda \in K^* := K\setminus\{0\}, \mu \in K \) and \( h \in F \).

Step 4. \( H' = \frac{1}{n}H + \mu + f, \int' = \nu \int^n + f \) and \( \partial' = \nu^{-1} \partial^n + g \) for some elements \( \nu \in K^*, n \geq 1 \) and \( h, f, g \in F \).

Step 5. \( ^{\sigma}K[x] \cong K[x]^n \), an isomorphism of \( I_1 \)-modules where \( n \) is as in Step 4 and \( ^{\sigma}K[x] := I_1/I_1 \partial, ^{\sigma}K[x] \) is the twisted \( I_1 \)-module \( K[x] \) by the algebra endomorphism \( \sigma \).

Step 6. \( n = 1 \), i.e. \( ^{\sigma}K[x] \cong K[x] \).

Step 7. Up to the algebraic torus action \( T^1 (\subseteq \text{Aut}_{K-\text{alg}}(I_1)) \), \( \nu = 1 \), i.e.

\[
H' = H + \mu + h, \quad \int' = \int + f, \quad \partial' = \partial + g.
\]

Step 8. \( \mu = 0 \).

Step 9. \( \sigma \) is an inner automorphism \( \omega_u \) of the algebra \( I_1 \) for some unit \( u \in (1 + F)^* \) of the algebra \( I_1 \).

Remark. The algebra \( B_1 \) (see Step 3) is the left and right localization of the Weyl algebra \( A_1 \) at the powers of the element \( \partial \), i.e. the algebra \( B_1 \) is obtained from \( A_1 \) by adding the two-sided inverse \( \partial^{-1} \) of the element \( \partial \) (the algebra \( B_1 \) is also a left (but not right) localization of the algebra \( I_1 \) at the powers of the element \( \partial \)), \( 10 \), but in contrast to the Weyl algebra \( A_1 \) the element \( \partial \) is not regular in \( I_1 \). An analogue of the Conjecture/Problem of Dixmier fails for the algebra \( B_1 \): for each natural number \( n \geq 2 \), the algebra monomorphism

\[
\sigma_n : B_1 \rightarrow B_1, \quad H \mapsto \frac{1}{n}H, \quad \partial \mapsto \partial^n,
\]

is obviously not an automorphism (use the \( \mathbb{Z} \)-grading of the algebra \( B_1 = \bigoplus_{i \in \mathbb{Z}} K[H]\partial^i \), \( \partial^i \alpha = \tau^i(\alpha)\partial^i \) for all \( \alpha \in K[H] \) and \( i \in \mathbb{Z} \)). In view of existence of this counterexample for the algebra \( B_1 \) it looks surprising that Theorem11 is true as the algebra \( I_1 \) is obtained from the Weyl algebra \( A_1 \) by adding a right, but not two-sided, inverse of the element \( \partial : \partial \int = 1 \) but \( \int \partial \neq 1 \). Theorem11 can be seen as a sign that the Conjecture/Problem of Dixmier is true.
Conjecture. Each algebra endomorphism of \( \mathbb{I}_n \) is an automorphism.

Ideas behind the proof of Theorem 1.1. This is a combination of old ideas/approach due to Dixmier [18] of using the eigenvalues of certain inner derivations (this was a key moment in finding the group \( \text{Aut}_{K-\text{alg}}(A_1) \) in [18] modulo many technicalities) and new ideas/approach of using (i) the Fredholm operators and their indices based on the fact that for the algebra \( \mathbb{I}_1 \) the (Strong) Compact-Fredholm Alternative holds [12] (which says that the action of each polynomial integro-differential operator of \( \mathbb{I}_1 \) on each simple \( \mathbb{I}_1 \)-module is either compact or Fredholm) and (ii) the structure of the centralizers of elements of \( \mathbb{I}_1 \) [12].

The Problem/Conjecture of Dixmier: recent progress. In 1982, it was proved that a positive answer to the Problem/Conjecture of Dixmier for the Weyl algebra \( A_n \) implies the Jacobian Conjecture for the polynomial algebra \( P_n \) in \( n \) variables, see Bass, Connel and Wright [1]. In 2005, it was proved independently by Tsuchimoto [22] and Belov-Kanel and Kontsevich [13], see also [13] for a short proof, that these two problems are equivalent. The Problem/Conjecture of Dixmier can be formulated as a question of whether certain modules \( \mathcal{M} \) over the Weyl algebras are simple [2] (recall that due to Inequality of Bernstein [15] each simple module over the Weyl algebra \( A_n \) has the Gelfand-Kirillov dimension which is one of the natural numbers \( n, n+1, \ldots, 2n-1 \); Bernstein and Lunts [16], [17] showed that ‘generically’ a simple \( A_n \)-module has the Gelfand-Kirillov dimension \( 2n-1 \)). It is not obvious from the outset that the modules \( \mathcal{M} \) are even finitely generated. In 2001, giving a positive answer to the Question of Rentschler on the Weyl algebra it was proved that the modules \( \mathcal{M} \) are finitely generated and have the smallest possible Gelfand-Kirillov dimension, i.e. \( n \) (i.e. they are holonomic) and as the result they have finite length, [2]. This means that the next step, as far as the Jacobian Conjecture and the Problem/Conjecture of Dixmier are concerned, is either to prove the conjectures or to give a counter-example.

One may wonder that for two different classes of algebras, the polynomial algebras and the Weyl algebras, seemingly unrelated and formulated in completely different ways conjectures, the Jacobian Conjecture and the Conjecture of Dixmier, turned out to be equivalent. It is obvious that there is a phenomenon not yet well understood. One may wonder that there are more algebras for which one can formulate ‘similar’ conjectures. Surprisingly, there is a definite answer to this question: in the class of all the associative algebras conjecture like the two mentioned conjectures makes sense only for the algebras \( P_m \otimes A_n \) as was proved in [8] (where \( P_m \) is a polynomial algebra in \( m \) variables; the two conjectures can be reformulated in terms of locally nilpotent derivations that satisfy certain conditions, and the algebras \( P_m \otimes A_n \) are the only associative algebras that have such derivations). This general conjecture for the algebras \( P_m \otimes A_n \) is true iff either the Jacobian Conjecture or the Problem/Conjecture of Dixmier is true, see [8].

Meaning of the Problem/Conjecture of Dixmier and the Jacobian Conjecture, the groups of automorphisms. The groups of automorphisms of the polynomial algebras \( P_n = P^n_{K^n} \), the Weyl algebra \( A_n = A^n_1 \) and the algebra \( \mathbb{I}_n := \mathbb{I}^{\{n\}}_1 = K[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1^\infty, \ldots, \int_n^\infty] \) of polynomial integro-differential operators are huge infinite-dimensional algebraic groups. The groups of automorphisms are known only for the polynomial algebras when \( n = 1 \) (trivial) and \( n = 2 \) (Jung (1942) [23] and Van der Kulk (1953) [24]); and for the Weyl algebra \( A_1 \) (Dixmier (1968) [18]) (in characteristic \( p > 0 \), the group \( \text{Aut}_{K-\text{alg}}(A_1) \) was found by Makar-Limanov (1984) [20], see also [9] for further developments and another proof). In 2009, the group \( G_n := \text{Aut}_{K-\text{alg}}(\mathbb{I}_n) \) of automorphisms of the algebra \( \mathbb{I}_n \) was found for all \( n \geq 1 \), [11];

\[
G_n = S_n \ltimes T^n \ltimes \text{Im}(\mathbb{I}_n) \supseteq S_n \ltimes T^n \ltimes \underbrace{\text{GL}(K) \times \cdots \times \text{GL}(K)}_{2^n-1 \text{ times}},
\]

\[
G_1 \simeq T^1 \ltimes \text{GL}(K),
\]

where \( S_n \) is the symmetric group, \( T^n \) is the \( n \)-dimensional algebraic torus, \( \text{Im}(\mathbb{I}_n) \) is the group of inner automorphisms of \( \mathbb{I}_n \) (which is huge). The ideas and approach in finding the groups \( G_n \)
are completely different from that of Jung, Van der Kulk and Dixmier: the Fredholm operators, $K_1$-theory, indices. On the other hand, when we look at the groups of automorphisms of the algebras $F_2, A_1$ and $I_1$ (the only cases where we know explicit generators) we see that they have the ‘same nature’: they are generated by affine automorphisms and ‘transvections.’

The Jacobian Conjecture and the Problem/Conjecture of Dixmier (if true) would give the ‘defining relations’ for the infinite dimensional algebraic groups of automorphisms as infinite dimensional varieties in the same way as the condition $\det = 1$ defines the special linear (finite dimensional) algebraic group $SL_n(K)$. Even true the conjectures would tell us nothing about generators of the groups of automorphisms (i.e. about the solutions of the defining relations, in the same way and the defining relation $\det = 1$ tells nothing about generators for the group $SL_n(K)$).

More obvious meaning of the Problem/Conjecture of Dixmier is that the Weyl algebras $A_n$, which are simple infinite dimensional algebras, behave like simple finite dimensional algebras (each algebra endomorphism of a simple finite dimensional algebra is, by a trivial reason, an automorphism). For a polynomial algebra $P_n$ there are plenty algebra endomorphisms that are not automorphisms. Recall that the Jacobian Conjecture claims that each algebra endomorphism $\sigma$ of the polynomial algebra $P_n$ with the Jacobian $\text{Jac}(\sigma) := \det(\frac{\partial \sigma(x)}{\partial x}) \in K^* := K\setminus \{0\}$ is necessarily an automorphism. The Jacobian condition is obviously holds for all automorphisms of $P_n$ and the Jacobian Conjecture implies that $\sigma$ is a monomorphism. So, the Jacobian Conjecture (if true) means that each algebra monomorphism of $P_n$ which is as close as possible to be an automorphism is, in fact, an automorphism.

The paper is organized as follows. In Section 2 necessary facts for the algebra $I_1$ are gathered which are used later in the paper. In Section 3 the proof of Theorem 1.1 is given.

2 The algebra $I_1$

In this section, we collect necessary (mostly elementary) facts on the algebra $I_1$ from [10], [11], and [12] that are used later in the paper.

The algebra $I_1$ is generated by the elements $\partial, H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations (Proposition 2.2, [10]):

$$\partial \int = 1, \ [H, \int] = \int, \ [H, \partial] = -\partial, \ H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial,$$

(1)

where $[a, b] := ab - ba$ is the commutator of elements $a$ and $b$. The elements of the algebra $I_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \ i, j \in \mathbb{N},$$

(2)

satisfy the relations $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where $\delta_{jk}$ is the Kronecker delta function and $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers. Notice that $e_{ij} = \int^i e_{00} \partial^j$. The matrices of the linear maps $e_{ij} \in \text{End}_K(K[x])$ with respect to the basis $\{x^s := \frac{x^s}{s!} \}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^s = \begin{cases} x^i & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \text{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js} x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij},$$

(3)

$K e_{ij} = K E_{ij}$, and $F := \bigoplus_{i,j \geq 0} K e_{ij} = \bigoplus_{i,j \geq 0} K E_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices. $F$ is the only proper ideal (i.e. $\neq 0, I_1$) of the algebra $I_1$ [10]. Using
induction on \( i \) and the fact that \( \int^j e_{kk} \partial^j = e_{k+j,k+j} \), we can easily prove that

\[
\int^i \partial^j = 1 - e_{00} - e_{11} - \cdots - e_{i-1,i-1} = 1 - E_{00} - E_{11} - \cdots - E_{i-1,i-1}, \quad i \geq 1. \tag{4}
\]

The monoid \( 1 + F = 1 + \bigoplus_{i,j \in \mathbb{N}} K E_{ij} = 1 + \bigoplus_{i,j \in \mathbb{N}} K e_{ij} \) admits the determinant map:

\[
\det : 1 + F \to K, \quad 1 + \sum_{i,j=0}^d \lambda_{ij} E_{ij} \mapsto \det(\sum_{i=0}^d E_{ii} + \sum_{i,j=0}^d \lambda_{ij} E_{ij}). \tag{5}
\]

By (3), this map can be defined as follows

\[
\det : 1 + F \to K, \quad 1 + \sum_{i,j=0}^d \lambda_{ij} e_{ij} \mapsto \det(\sum_{i=0}^d e_{ii} + \sum_{i,j=0}^d \lambda_{ij} e_{ij}). \tag{6}
\]

For all elements \( a, b \in 1 + F \), \( \det(ab) = \det(a)\det(b) \) and \( \det(1) = 1 \). Therefore, an element \( a \in 1 + F \) is a unit iff \( \det(u) \neq 0 \) (use the fact that \( F \) is an ideal of \( \mathbb{I}_1 \)).

**Z-grading on the algebra \( \mathbb{I}_1 \) and the canonical form of an integro-differential operator** [10], [12]. The algebra \( \mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i} \) is a \( \mathbb{Z} \)-graded algebra (\( \mathbb{I}_{1,i} \subseteq \mathbb{I}_{1,i+j} \) for all \( i, j \in \mathbb{Z} \)) where

\[
\mathbb{I}_{1,i} = \begin{cases} D_1 f^i = f^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^i D_1 = D_1 \partial^i & \text{if } i < 0, \end{cases}
\]

the algebra \( D_1 := K[H] \bigoplus \bigoplus_{i \in \mathbb{N}} K e_{ii} \) is a commutative non-Noetherian subalgebra of \( \mathbb{I}_1 \), \( H e_{ii} = e_{ii} H = (i + 1)e_{ii} \) for \( i \in \mathbb{N} \) (and so \( \bigoplus_{i \in \mathbb{N}} K e_{ii} \) is the direct sum of non-zero ideals \( K e_{ii} \) of the algebra \( D_1 \)); \( \int^i D_1 = D_1, \int^i d \mapsto d; D_1(\partial^i) \simeq D_1, \partial^i d \mapsto d, \) for all \( i \geq 0 \) since \( \partial^i \int^i = 1 \). Notice that the maps \( \cdot \int^i : D_1 \to D_1 \int^i, d \mapsto d \int^i, \partial^i : D_1 \to \partial^i D_1, d \mapsto \partial^i d, \) have the same kernel \( \bigoplus_{i=0}^{i-1} K e_{ij} \).

Each element \( a \) of the algebra \( \mathbb{I}_1 \) is the unique finite sum

\[
a = \sum_{i > 0} a_{-i} \partial^i + a_0 + \sum_{i > 0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \tag{7}
\]

where \( a_k \in K[H] \) and \( \lambda_{ij} \in K \). This is the **canonical form** of the polynomial integro-differential operator [10].

**Definition.** Let \( a \in \mathbb{I}_1 \) be as in (7) and let \( a_F := \sum \lambda_{ij} e_{ij} \). Suppose that \( a_F \not= 0 \) then

\[
\deg_F(a) := \min\{ n \in \mathbb{N} \mid a_F \in \bigoplus_{i,j=0}^n K e_{ij} \} \tag{8}
\]

is called the **F-degree** of the element \( a \); \( \deg_F(0) := -1 \).

Let \( v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0 \end{cases} \) Then \( \mathbb{I}_{1,i} = D_1 v_i = v_i D_1 \) and an element \( a \in \mathbb{I}_1 \) is the unique finite sum

\[
a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \tag{9}
\]
where \( b_i \in K[H] \) and \( \lambda_{ij} \in K \). So, the set \( \{H^j \partial^i, H^j, \int H^j, e_{st} | i \geq 1; j, s, t \geq 0\} \) is a \( K \)-basis for the algebra \( I_1 \). The multiplication in the algebra \( I_1 \) is given by the rule:

\[
\int H = (H-1) \int, \quad \partial H = \partial(H-1), \quad \int e_{ij} = e_{i+1,j}, \quad e_{ij} \int = e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j} \quad e_{ij} \partial = \partial e_{i,j+1}.
\]

\[He_{ii} = e_{ii}H = (i+1)e_{ii}, \quad i \in \mathbb{N},\]

where \( e_{-1,j} := 0 \) and \( e_{i,-1} := 0 \).

The factor algebra \( B_1 := I_1/F \) is the simple Laurent skew polynomial algebra \( K[H][\partial, \partial^{-1}; \tau] \) where the automorphism \( \tau \in \text{Aut}_{K\text{-alg}}(K[H]) \) is defined by the rule \( \tau(H) = H + 1, [10] \). Let

\[
\pi : I_1 \to B_1, \quad a \mapsto \overline{a} : a + F,
\]

be the canonical epimorphism.

The groups of units \( I^n_1 \) and automorphisms \( \text{Aut}_{K\text{-alg}}(I^n_1) \) of the algebra \( I^n_1 \). For a group \( G \), let \( Z(G) \) denote its centre. Let \( I^n_1 \) be the group of units of the algebra \( I_1 \). Since \( F \) is an ideal of the algebra \( I_1 \), the intersection \((1 + F)^* := I^n_1 \cap (1 + F)\) is a subgroup of the group \( I^n_1 \). Moreover,

\[
(1 + F)^* = \{u \in 1 + F | \det(u) \neq 0\} \simeq \text{GL}_\infty(K).
\]

The group \( \text{Aut}_{K\text{-alg}}(I^n_1) \) of automorphisms of the algebra \( I_1 \) contains the algebraic torus

\[
T^1 := \{t_\lambda | \lambda \in K^*, \quad t_\lambda(\int) = \lambda \int, \quad t_\lambda(\partial) = \lambda^{-1} \partial, \quad t_\lambda(H) = H \} \simeq K^*, \quad t_\lambda \leftrightarrow \lambda,
\]

and the group of inner automorphisms \( \text{Inn}(I_1) = \{\omega_u : a \to uau^{-1} | u \in I^n_1\} \) of the algebra \( I_1 \).

**Theorem 2.1**

1. (Theorem 4.5, [10]) \( I^n_1 = K^* \times (1 + F)^* \simeq K^* \times \text{GL}_\infty(K) \) and \( Z(I^n_1) = K^* \).

2. (Theorem 5.5.(1), [11]) \( \text{Aut}_{K\text{-alg}}(I^n_1) = T^1 \ltimes \text{Inn}(I_1) \).

3. (Theorem 3.1.(2), [11]) The map \((1 + F)^* \to \text{Inn}(I_1), \quad u \mapsto \omega_u\), is a group isomorphism.

### 3 Proof of Theorem 1.1

This entire section is the proof of Theorem 1.1. We follow the steps outlined in the Introduction.

Let \( \sigma \) be an algebra endomorphism of \( I_1 \). We have to show that \( \sigma \) is an automorphism. The endomorphism \( \sigma \) is uniquely determined by its action on the generators \( H, \int \) and \( \partial \) of the algebra \( I_1 \):

\[
H' := \sigma(H), \quad \int' := \sigma(\int), \quad \partial' := \sigma(\partial).
\]

**Step 1.** \( \sigma \) is a monomorphism.

Suppose that \( \sigma \) is not a monomorphism, we seek a contradiction. Then \( \ker(\sigma) = F \) since \( F \) is the only proper (i.e. \( \neq 0, I_1 \)) ideal of the algebra \( I_1 \), [10], and so there is the algebra homomorphism

\[
\overline{\sigma} : B_1 := I_1/F \to I_1, \quad a + F \mapsto \sigma(a).
\]

Since the algebra \( B_1 \) is a simple algebra, \( \overline{\sigma} \) is a monomorphism. The element \( \partial \) of the algebra \( B_1 \) is an invertible element and \( \dim_K(K[\partial]) = \infty \). Then \( \overline{\sigma} = \sigma(\partial) \) is an invertible element of the algebra \( I_1 \) and \( \dim_K(K[\overline{\sigma}(\partial)]) = \dim_K(K[\overline{\sigma}]) = \dim_K(K[\partial]) = \infty \) since \( \overline{\sigma} \) is a monomorphism. This contradicts the following lemma.

**Lemma 3.1** For all units \( u \in I^n_1 \), \( \dim_K(K[u]) < \infty \).
\textbf{Proposition 3.3} \(K\) and \(\text{dim} K\) and \(Cen\) the inclusion \(\subseteq\) is obvious. To show that the inverse inclusion holds it suffices to prove that, for all elements \(a \in K\). \(I\) and \(\lambda\) holds we have to show that \(\sigma(e_{ij}) \in F\) for all \(i, j \in \mathbb{N}\). If \(i = j\) then \(e_{ii}^2 = e_{ii}\). If \(\sigma(e_{ii}) \notin F\) then necessarily \(\sigma(e_{ii}) \in \lambda_i + F\) for some \(\lambda_i \in K^*\) such that \(\lambda_i^2 = \lambda_i\), i.e. \(\lambda_i = \pm 1\), we seek a contradiction. Since \(\sigma(K + F) \subseteq K + F\) and

\[
\infty = \dim_K(\ker_{K + F}(e_{ii})) = \dim_K(\ker_{\sigma(K + F)}(\cdot \sigma(e_{ii}))) \quad \text{(by Step 1)}
\leq \dim_K(\ker_{K + F}(\sigma(e_{ii}))) = \dim_K(\ker_{K + F}(\cdot (\pm 1 + f)))
\leq \infty,
\]

a contradiction. Then \(\sigma(e_{ii}) \in F\) for all \(i \in \mathbb{N}\).

For all \(i \neq j\), \(e_{ij}^2 = 0\), hence \(\sigma(e_{ij})^2 = 0\), and so \(\sigma(e_{ij}) \in F\) since \(I_1 / F\) is a domain. This proves that the inclusion \(\sigma(F) \subseteq F\) holds. Therefore, there is a commutative diagram of algebra homomorphisms:

\[
\begin{array}{ccc}
\mathbb{I}_1 & \xrightarrow{\sigma} & \mathbb{I}_1 \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{B}_1 & \xrightarrow{\overline{\sigma}} & \mathbb{B}_1
\end{array}
\]

where \(\overline{\sigma}(a + F) = \sigma(a) + F\) for all \(a \in \mathbb{I}_1\); \(\pi : \mathbb{I}_1 \to \mathbb{B}_1 = \mathbb{I}_1 / F\), \(a \mapsto a + F\); and so \(\overline{\sigma}\) is an algebra monomorphism since \(\mathbb{B}_1\) is a simple algebra.

\textbf{Step 3.} \(H' = \lambda H + \mu + h\) for some elements \(\lambda \in K^* := K \setminus \{0\}\), \(\mu \in K\) and \(h \in F\) where \(F\) is the only proper ideal of the algebra \(\mathbb{I}_1\).

For an element \(a \in \mathbb{I}_1\), let \(\text{Cen}_{\mathbb{I}_1}(a) = \{b \in \mathbb{I}_1, |ab = ba\}\) be its centralizer in the algebra \(\mathbb{I}_1\), and \(\text{Cen}_{F}(a) := F \cap \text{Cen}_{\mathbb{I}_1}(a)\).

\textbf{Proposition 3.3} \(\text{[2]}\) Let \(a \in \mathbb{I}_1\). Then \(\dim_K(\text{Cen}_F(a)) = \infty\) iff \(a \in K[H] + F\).

By Proposition \(\text{[3]}\), \(H' \in K[H] + F\), i.e. \(H' = \alpha + h\) for unique elements \(\alpha \in K[H]\) and \(h \in F\) since \(K[H] \cap F = 0\) (see \(\text{[3]}\)). Since, for each element \(\theta \in \{H, \int, \partial\}\),

\[
\infty = \dim_K(K[\theta]) = \dim_K(\sigma(K[\theta]) = \dim_K(K[\sigma(\theta)]) \quad \text{and} \quad \dim_K(K(\lambda + f)) < \infty,
\]

for all elements \(\lambda \in K\) and \(f \in F\), we must have

\[
\alpha \in K[H] \setminus K \quad \text{and} \quad \int', \partial' \notin K + F.
\quad \text{(11)}
\]

Using \(\text{[1]}\) and the direct sum decomposition

\[
\mathbb{I}_1 = \bigoplus_{i \geq 1} D_1 \partial^i \bigoplus_{i \geq 1} D_1 \bigoplus_{i \geq 1} \int^i D_1,
\]

\textbf{Proof.} The result follows from the equality \(\mathbb{I}_1^* = K^*(1 + F)^*\) (Theorem 4.5, \(\text{[1]}\)). \(\square\)

Therefore, \(\sigma\) is a monomorphism.

\textbf{Step 2.} \(\sigma(F) \subseteq F\).

\textbf{Lemma 3.2} \(K + F = \{a \in \mathbb{I}_1 \mid \dim_K(K\langle a \rangle) < \infty\}\).

\textbf{Proof.} The inclusion \(\subseteq\) is obvious. To show that the inverse inclusion holds it suffices to prove that, for all elements \(a \notin K + F\), \(\dim_K(K\langle a \rangle) = \infty\), but this is obvious since \(\overline{\pi} := a + F \in \mathbb{B}_1 \setminus K\) and \(\dim_K(K(\overline{\pi})) = \infty\). \(\square\)
we see that the set of eigenvalues of the inner derivation \(\text{ad}(H) : \mathbb{I}_1 \rightarrow \mathbb{I}_1\), \(\Rightarrow [H, a] := Ha - aH\), of the algebra \(\mathbb{I}_1\) is \(\text{Ev}(\text{ad}(H)) = \mathbb{Z}\), and, for each eigenvalue \(\lambda \in \mathbb{Z}\),

\[
\ker_{\mathbb{I}_1}(\text{ad}(H) - i) = \begin{cases} 
\int D_1 & \text{if } i \geq 1, \\
D_1 & \text{if } i = 0, \\
D_1 \partial^{|i|} & \text{if } i \geq -1.
\end{cases}
\]

Since \(\sigma\) is a monomorphism (by Step 1), \(\text{Ev}(\text{ad}(H')) \geq \mathbb{Z}\). By (11), \(\pi'(f) \neq 0\) and \(\pi(\partial f) \neq 0\) for all \(i \in \mathbb{N}\) where \(\pi\) is defined in (10). Since, by (1),

\[
[\pi(H'), \pi(\int f)] = i\pi(\int f) \quad \text{and} \quad [\pi(H'), \pi(\partial f)] = -i\pi(\partial f), \quad \text{for all } i \geq 1,
\]

we see that \(\text{Ev}(\pi(H'), B_1) \geq \mathbb{Z}\). By (11), \(\pi(H') = \alpha \in K[H]\backslash K\).

By Lemma 3.4

\[
\alpha = \lambda H + \mu \quad (12)
\]

for some \(\lambda \in K^*\) and \(\mu \in K\).

**Lemma 3.4** Let \(a \in K[H]\backslash K\). Then \(\text{Ev}(\text{ad}(a), B_1) \neq 0\) iff \(a = \lambda H + \mu\) where \(\lambda \in K^*\) and \(\mu \in K\).

**Proof.** (⇐) Obvious: \(|\lambda H + \mu, \partial| = -\lambda \partial \).

(⇒) It suffices to show that if \(\deg_H (a) > 1\) then \(\text{Ev}(\text{ad}(a), B_1) = 0\). The algebra \(B_1 = \bigoplus_{i \in \mathbb{Z}} K[H] \partial^i\) is a \(\mathbb{Z}\)-graded algebra where \(K[H] \partial^i\) is the \(i\)th graded component of the algebra \(B_1\). The element \(a \in K[H]\) is a homogeneous element of the algebra \(B_1\). Therefore, for each eigenvalue \(\nu \in \text{Ev}(\text{ad}(a), B_1)\),

\[
\ker_{B_1}(\text{ad}(a) - \nu) = \bigoplus_{i \in \mathbb{Z}} (\ker_{B_1}(\text{ad}(a) - \nu) \cap K[H] \partial^i).
\]

Suppose that \(\nu \neq 0\), then \([a, \beta \partial^i] = \nu \beta \partial^i\) for some elements \(0 \neq \beta \in K[H]\) and \(i \in \mathbb{Z}\), necessarily \(i \neq 0\) since \(\nu \neq 0\). The equality can be written as \((a - \tau(a)) \beta \partial^i = \nu \beta \partial^i\), and so \(a - \tau(a) = \nu\) since \(B_1\) is a domain. Since \(\deg_H (a - \tau(a)) = \deg_H(a) - 1 \geq 1\), this is impossible. Therefore, \(\text{Ev}(\text{ad}(a), B_1) = 0\). \(\square\)

**Step 4.** \(H' = \frac{1}{n} H + \mu + h, f' = \nu f^n + h + \partial' = \nu^{-1} \partial^n + g\) for some elements \(\nu \in K^*, n \geq 1\) and \(h, f, g \in F\).

By Step 3, \(\text{Ev}(\pi(H') = \lambda H + \mu, B_1) = \lambda \text{Ev}(H, B_1) = \lambda \mathbb{Z}\) and, for each element \(\nu \in \mathbb{Z}\),

\[
\ker_{B_1}(\text{ad}(\pi(H')) - i\lambda) = B_1 \partial^{-i}.
\]

Applying the algebra homomorphism \(\pi\sigma\) to the relations \([H, f] = f, [H, \partial] = -\partial\) and \(\partial f = 1\) yields the equalities

\[
[\pi(H'), \pi(f')] = \pi(f'), \quad [\pi(H'), \pi(\partial')] = -\pi(\partial'), \quad \pi(\partial') \pi(f') = 1.
\]

By (11), \(\pi(f') \neq 0\) and \(\pi(\partial') \neq 0\). Therefore, by Lemma 3.3 there are two options

(i) \(\pi(H') = \frac{1}{n} H + \mu, \pi(f') = \nu \partial^{-n}, \pi(\partial') = \nu^{-1} \partial^n;\)

(ii) \(\pi(H') = -\frac{1}{n} H + \mu, \pi(f') = \nu^{-1} \partial^n, \pi(\partial') = \nu \partial^{-n};\)

for some natural number \(n \geq 1\) and \(\nu \in K^*\) since

\[
B_1 = \bigoplus_{i \in \mathbb{Z}} K[H] \partial^i, \quad \text{Ev}(\text{ad}(H), B_1) = \mathbb{Z}, \quad \ker(\text{ad}(H) - i) = K \partial^{-i}, \quad i \in \mathbb{Z}.
\]
Therefore, there are elements $h, f, g \in F$ such that

\[
(i) \quad H' = \frac{1}{n} H + \mu + h, \quad \int' = \nu \int^n + f, \quad \partial' = \nu^{-1} \partial^n + g;
\]

\[
(ii) \quad H' = -\frac{1}{n} H + \mu + h, \quad \int' = \nu^{-1} \partial^n + g, \quad \partial' = \nu \int^n + f.
\]

We are going to show that the case (ii) is not possible. For we need some results.

Since $\partial' \int' = 1$, the map $\partial' : K[x] \to K[x]$, $p \mapsto \partial' * p$, is a surjection, and so

\[
\dim_K(\ker_{K[x]}(\partial')) = \text{ind}_{K[x]}(\partial')
\]

(13)

where $\text{ind}_{K[x]}(\varphi) := \dim_K(\ker_{K[x]}(\varphi)) - \dim_K(\coker_{K[x]}(\varphi))$ is the index of a linear map $\varphi \in \text{End}_{K}(K[x])$ provided the kernel and cokernel of the map $\varphi$ are finite dimensional.

**Theorem 3.5** \cite{12} Let $a \in \mathbb{I}_1$, $M$ be a nonzero $\mathbb{I}_1$-module of finite length and $a_M : M \to M$, $m \mapsto am$. Then $\dim_K(\ker(a_M)) < \infty$ iff $\dim_K(\coker(a_M)) < \infty$ iff $a \not\in F$.

**Lemma 3.6** \cite{12} Let $a \in \mathbb{I}_1 \setminus F$ and $f \in F$. Then $\text{ind}_M(a + f) = \text{ind}_M(a)$ for all left or right $\mathbb{I}_1$-modules $M$ of finite length where $\text{ind}_M(a) := \dim_K(\ker(a_M)) - \dim_K(\coker(a_M))$.

**Step 5.** $\sigma K[x] \simeq K[x]^n$, an isomorphism of $\mathbb{I}_1$-modules where $n$ is as in Step 4.

Here $1_1 K[x] := 1_1 \mathbb{I}_1 \partial$ is a faithful $1_1$-module (since $1_1 \subseteq \text{End}_{K}(K[x])$), and the action of an element $a \in 1_1$ on a polynomial $p \in K[x]$ is denoted by $a * p$. $\sigma K[x]$ is the twisted by the algebra endomorphism $\sigma \mathbb{I}_1$-module $K[x]$: as vector spaces $\sigma K[x] = K[x]$ but the action of the algebra $\mathbb{I}_1$ on $\sigma K[x]$ is given by the rule, $a \cdot p := \sigma(a) * p$ for all elements $a \in 1_1$ and $p \in K[x]$. The $1_1$-module $K[x]$ is a simple (since $1_1 \subseteq 1_1$ and the $1_1$-module $K[x]$ is simple), and $\partial' \not\in F$, by \cite{11}. By \cite{13} and Lemma 3.6

\[
\ker_{K[x]}(\partial') = \text{ind}_{K[x]}(\partial') = \text{ind}_{K[x]}((\nu^{-1} \partial^n + g) \cdot) = \text{ind}_{K[x]}(\partial^n) = \ker_{K[x]}(\partial^n) = n
\]

since $\ker_{K[x]}(\partial^n) = \bigoplus_{i=0}^{n-1} K x^i$. Recall that $1_1 K[x] \simeq 1_1 \mathbb{I}_1 \partial$ is a simple $1_1$-module such that

\[
K[x] = \bigcup_{i \geq 1} \ker_{K[x]}(\partial^i), \quad \ker_{K[x]}(\partial^i) = \bigoplus_{j=0}^{i-1} K x^j, \quad \ker_{K[x]}(\partial) = K.
\]

(14)

Similarly, for a natural number $n \geq 1$, the direct sum $K[x]^n$ of $n$ copies of the simple $\mathbb{I}_1$-module $K[x]$ is a semi-simple $1_1$-module of finite length $n$, $K[x]^n = \bigcup_{i \geq 1} \ker_{K[x]^n}(\partial^i)$ and $\dim_K(\ker_{K[x]^n}(\partial)) = n \dim_K(\ker_{K[x]}(\partial)) = n$. It follows that the $1_1$-module epimorphism

\[
\varphi : K[x]^n \to V := 1_1 \cdot \ker_{K[x]}(\partial) = \sigma(1_1) * \ker_{K[x]}(\partial)
\]

(where $1_1 V \subseteq 1_1 (\sigma K[x]))$ given by the rule

\[
\varphi : (1, 0, \ldots, 0) \mapsto v_1, \ldots, (0, \ldots, 0, 1) \mapsto v_n,
\]

is an isomorphism where $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is the standard free $K[x]$-basis for the $\mathbb{I}_1$-module $K[x]^n$ and $v_1, \ldots, v_n$ is a $K$-basis for the vector space $\ker_{K[x]}(\partial)$ (otherwise, $1_1 V \simeq K[x]^m$ for some $m < n$, and so $n = \dim_K(\ker_V(\partial')) = \dim_K(\ker_{K[x]^n}(\partial')) = m$, a contradiction.)

Fix $s \in \mathbb{N}$ such that

\[
s > \max\{n, d, \deg_F(h), \deg_F(f), \deg_F(g)\}
\]
where \( \ker_{K[x]}(\partial') \subseteq K[x] \leq d := \bigoplus_{i=0}^d Kx^i \) for some number \( d \in \mathbb{N} \). Then, for all integers \( i \geq s \), by Step 4,

\[
H' \ast x^i = \left( \frac{1}{n} (i + 1) + \mu \right)x^i,
\]

\[
\int' \ast x^i = \nu x^{i+n},
\]

\[
\partial' \ast x^i = \nu^{-1}x^{i-n},
\]

where \( x^{|j|} := 0 \) for all integers \( j < 0 \). For each integer \( i \in \mathbb{N} \), let \( K[x] \leq i := \bigoplus_{j=0}^i Kx^j \). Then \( K[x] = \bigcup_{i \in \mathbb{N}} K[x] \leq i \). Consider the ascending chain of vector spaces in \( K[x] \):

\[
V_0 := K[x] \leq s \subset V_1 := K[x] \leq s+n \subset \cdots \subset V_t := K[x] \leq s+nt \subset \cdots,
\]

\[
\dim_K(V_t) = 1 + s + nt.\]

Then, for all \( t \in \mathbb{N} \),

\[
H' \ast V_t \subseteq V_t,
\]

\[
\int' \ast V_t \subseteq V_{t+1},
\]

\[
\partial' \ast V_t \subseteq V_t.
\]

Since \( \ker_{K[x]}(\partial') \subseteq V_0 \) and \( K[x] \leq i = \sum_{i=0}^i K f^i \ast 1 \), we see that \( \varphi(K[x] \leq t) \subseteq \sum_{i=0}^t K f^i \ast V_0 \subseteq V \leq t \), and so

\[
\dim_K(V \leq t) - \dim_K(\varphi(K[x] \leq t)) = 1 + s + nt - n(t + 1) = 1 + s - n = \text{const}.
\]

This means that the factor \( \mathbb{I}_1 \)-module \( \sigma K[x] / \ker(\varphi) \) is finite dimensional. Therefore,

\[
\ker(\varphi) = \sigma K[x],
\]

since the only finite dimensional \( \mathbb{I}_1 \)-module is the zero one (the algebra \( \mathbb{I}_1 \) contains the simple infinite dimensional algebra \( A_1 \), and the only finite dimensional \( A_1 \)-module is the zero one), i.e. the \( \mathbb{I}_1 \)-modules \( K[x] \) and \( \sigma K[x] \) are isomorphic via \( \varphi \).

**Step 6.** \( n = 1 \), i.e. \( \sigma K[x] \simeq K[x] \).

By Step 5, \( \sigma K[x] \simeq K[x] \). Notice that \( K[x] = \bigoplus_{i \in \mathbb{N}} Kx^i \) and \( Kx^i = \ker_{K[x]}(H - i - 1) \), i.e. the linear map \( H : K[x] \rightarrow K[x], p \mapsto H \ast p \), is semi-simple. Therefore, the map \( H : K[x]^n \rightarrow K[x]^n, p \mapsto H \ast p \), is semi-simple and each of its eigenvalues has multiplicity (i.e. the dimension of the corresponding eigenspace) \( n \). Since \( \mathbb{I}_1(\sigma K[x]) \simeq K[x] \) (Step 5), the linear map \( H' : K[x] \rightarrow K[x], p \mapsto H' \ast p \), is semi-simple and each of its eigenvalues has multiplicity \( n \). Since

\[
H' \ast x^i = \left( \frac{1}{n} (i + 1) + \mu \right)x^i, \quad i \geq s,
\]

\[
H' \ast V_0 \subseteq V_0, \quad \dim_K(V_0) < \infty,
\]

we must have

\[
n = 1,
\]

(since the eigenvalues \( \{ \frac{1}{n} (i + 1) + \mu \mid i \geq s \} \) of the linear map \( H' \) acting in \( K[x] \) are all distinct) and so

\[
H' = H + \mu + h, \quad \int' = \nu \int + f, \quad \partial' = \nu^{-1} \partial + g.
\]
Up to the algebraic torus action $T^1 (\subseteq \text{Aut}_{\text{alg}}(\mathbb{I}_1))$, we may assume that $\nu = 1$, i.e.

**Step 7.** $H' = H + \mu + h$, $f' = f + f$ and $\partial' = \partial + g$.

**Step 8.** $\mu = 0$.

For the $\mathbb{I}_1$-module $K[x]$ and for all natural numbers $i \geq 1$,

$$K[x]_{\leq i-1} = \ker_{K[x]}(\partial^i) = \bigoplus_{j=1}^i \ker_{K[x]}(H - j) = \partial K[\partial] \ast \ker_{K[x]}(H - (i + 1)) \quad (15)$$

and

$$i = \dim_K(\partial K[\partial] \ast \ker_{K[x]}(H - (i + 1))) \quad \text{for all } i \in \text{Ev}(H', K[x]) = \{1, 2, \ldots\}. \quad (16)$$

Since the vector space $U := V_0 \bigoplus K x^{[s+1]} = K[x]_{\leq s+1}$ is $\partial'$-invariant, $\partial' \ast V_0 \subseteq V_0$, $\partial' \ast x^{[s+1]} = x^{[s]} \in V_0$,

$$H' \ast x^{[s+1]} = (s + 1 + 1 + \mu)x^{[s+1]} \text{ and } \mathbb{I}_1(\nu K[x]) \simeq K[x] \quad (17)$$

we must have, by (15),

$$V_0 = \partial' K[\partial'] \ast \ker_{K[x]}(H' - (s + 2 + \mu)) = \partial' K[\partial'] \ast x^{[s+1]}.$$ 

By (15) and since $\mathbb{I}_1(\nu K[x]) \simeq K[x]$, $(s + 2 + \mu) - 1 = \dim_K(\partial' K[\partial'] \ast \ker_{K[x]}(H' - (s + 2 + \mu))) = \dim_K(V_0) = \dim_K(K[x]_{\leq s} = s + 1$.

Therefore, $\mu = 0$.

**Step 9.** $\sigma$ is an inner automorphism $\omega_u$ of the algebra $\mathbb{I}_1$ for some unit $u \in (1 + F)^\ast$ of the algebra $\mathbb{I}_1$.

Notice that

$$K[x] = \bigoplus_{i \geq 1} \ker_{K[x]}(H' - i), \quad \text{Ev}(H', K[x]) = \{1, 2, \ldots\},$$

$$\dim_K(\ker_{K[x]}(H' - i)) = 1 \quad \text{for all } i \in \text{Ev}(H', K[x]),$$

$$\int \ast \ker_{K[x]}(H' - i) = \ker_{K[x]}(H' - (i + 1)) \quad \text{and}$$

$$\partial' \ast \ker_{K[x]}(H' - i) = \ker_{K[x]}(H' - (i - 1)) \quad \text{for all } i \in \text{Ev}(H', K[x]).$$

Since

$$K[x] = V_0 \bigoplus (x^{s+1}) = \bigoplus_{i=0}^s K x^{[i]} \bigoplus K x^{[s+1]} \bigoplus K x^{[s+2]} \bigoplus \ldots$$

where $(x^{s+1}) = K[x] x^{[s+1]}$, $x^{[i]} := \partial'(s+1-i) \ast x^{[x+1]}$ and $\mathbb{I}_1(\nu K[x]) \simeq K[x]$, we see that (by (17))

$$\ker_{K[x]}(H' - i - 1) = \begin{cases} K x^{[i]} & \text{if } i = 0, 1, \ldots, s, \\ K x^{[i]} & \text{if } i > s. \end{cases}$$

Let $x^{[i]} := x^{[i]}$ for all $i > s$. Then

$$\partial' \ast x^{[i]} = \begin{cases} x^{[i-1]} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Then necessarily,

$$\int \ast x^{[i]} = x^{[i+1]}, \quad i \geq 0,$$
using the facts that $\partial' \int' = 1$, $\int' \ast \ker_{K[x]}(H'-i) = \ker_{K[x]}(H'-i-1)$ and $\ker_{K[x]}(H'-i) = K x'^{i-1}$ for all $i \geq 1$. The $K$-linear map

$$u : K[x] \to K[x], \quad x[i] \mapsto x'[i],$$

is an $\mathbb{I}_1$-module isomorphism $u : K[x] \to \sigma K[x]$ since

$$u a * x[i] = a' * (u x'[i])$$

for all elements $a \in \{H, \int, \partial\}$ and $i \in \mathbb{N}$, i.e. $u a = \sigma(a) u$, and so $\sigma(a) = u a u^{-1} = \omega_u(a)$ for all elements $a \in \{H, \int, \partial\}$. Notice that $u \in (1 + F)^*$, i.e. $\sigma = \omega_u \in \text{Inn}(\mathbb{I}_1)$. □

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