Application of the Nonlinear Steepest Descent Method to the Coupled Sasa-Satsuma Equation

Xianguo Geng, Mingming Chen* and Kedong Wang

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, China.

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Abstract. We use spectral analysis to reduce Cauchy problem for the coupled Sasa-Satsuma equation to a $5 \times 5$ matrix Riemann-Hilbert problem. The upper and lower triangular factorisations of the jump matrix and a decomposition of the vector-valued spectral function are given. Applying various transformations related to the Riemann-Hilbert problems and suitable decompositions of the jump contours and the nonlinear steepest descent method, we establish the long-time asymptotics of the problem.

AMS subject classifications: 35Q53, 35B40, 35Q55

Key words: Coupled Sasa-Satsuma equation, nonlinear steepest descent method, long-time asymptotics.

1. Introduction

The Sasa-Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u \left(|u|^2\right)_x = 0 \tag{1.1}$$

also called the higher-order nonlinear Schrödinger equation, was originally aimed to describe the propagation of pulses in optical fiber [18, 19]. It attracted a considerable attention and has been extensively studied because of significant applications. The inverse scattering method [34] and the Hirota bilinear method [12] were used to obtain $N$-soliton solution of this equation. On the other hand, by linearising the corresponding spectral operator it was shown that the squared eigenfunctions of the Sasa-Satsuma equation can be represented as the sums of two terms, each of which is a product of Jost and adjoint Jost functions [43]. Akhmediev et al. [2] studied the rogue wave spectra of the Eq. (1.1) and its presence in the spectra of chaotic wave fields produced by the modulation instability. Ling [22] obtained high order solution formulas in the determinant form by using a generalised Darboux transformation and the formal series method. In [44], finite genus solutions

*Corresponding author. Email addresses: xggeng@zzu.edu.cn (X.G. Geng), chenmmzzu@163.com (M.M. Chen), wangkdmath@163.com (K.D. Wang)
of the Sasa-Satsuma hierarchy, associated with a $3 \times 3$ matrix spectral problem, are obtained by using asymptotic expansions of the Baker-Akhiezer function and its Riemann theta function representation [37]. The Riemann-Hilbert approach, Darboux transformation and Riccati equation are employed in investigating the integrability of multi-coupled nonlinear integrable equations and finding their exact solutions — cf. Refs. [9, 11, 15, 20, 21, 27, 38, 41].

Let

$$\mathcal{S}(\mathbb{R}) = \left\{ f(x) \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{N} \right\}$$

be the Schwartz class. In this work, we use the nonlinear steepest descent method in order to study the long-time asymptotic behavior of the Cauchy problem for the coupled Sasa-Satsuma equation

$$u_t + u_{xxx} + 6 \left( |u|^2 + |v|^2 \right) u_x + 3u \left( |u|^2 + |v|^2 \right)_x = 0,$$

$$v_t + v_{xxx} + 6 \left( |u|^2 + |v|^2 \right) v_x + 3v \left( |u|^2 + |v|^2 \right)_x = 0,$$

(1.2)

where $u(x, t)$ and $v(x, t)$ are complex-valued potentials, $u_0(x), v_0(x) \in \mathcal{S}(\mathbb{R})$ and are generic in the sense that the below defined determinant $\det(a(k))$ does not vanish in the lower complex half $k$-plane $\mathbb{C}_-$. The coupled Sasa-Satsuma equation can describe the simultaneous propagation in birefringent or two-mode fibers [32]. In [40], multi-soliton solutions of the coupled Sasa-Satsuma equation are derived by solving a Riemann-Hilbert problem. Besides, infiniteness of conserved quantities of the Eqs. (1.2) is discussed in [33], the Painlevé property in [36], and some other characteristics in [24, 28, 45]. The Deift-Zhou nonlinear steepest descent method introduced in [7] is aimed to study the long-time asymptotic behavior of solutions for the mKdV equation. The method was subsequently applied to a number of integrable nonlinear evolution equations associated with numerous matrix spectral problems [4–6, 8, 10, 13, 16, 17, 23, 25, 26, 29–31, 35, 42]. However, to the best of author’s knowledge, the nonlinear steepest descent method has not been used in the study of long-time asymptotics for integrable equation associated with $5 \times 5$ matrix Lax pairs and the aim of this work is to extend the Deift-Zhou method to the Eqs. (1.2) associated with such Lax pairs. The main result of this paper is the following theorem.

**Theorem 1.1.** Let $(u(x, t), v(x, t))$ be the solution for the Cauchy problem of the coupled Sasa-Satsuma equation (1.2) with $u_0(x)$ and $v_0(x) \in \mathcal{S}(\mathbb{R})$. If $x < 0$ and $|x/t| \leq C$, as $t \to \infty$, then the leading asymptotics of $(u(x, t), v(x, t))$ has the form

$$\left( u(x, t), v(x, t) \right)$$

$$= -\frac{\nu e^{\pi \nu^2}}{\sqrt{24 t k_0 \pi}} \left[ \delta_A^s e^{-\pi i/4} \Gamma(-i \nu) (\gamma_2(k_0), \gamma_4(k_0)) + \delta_A^{-2} e^{\pi i/4} \Gamma(i \nu) (\gamma_1^s(k_0), \gamma_3^s(k_0)) \right]$$

$$+ \Theta \left( c(k_0) t^{-1} \log t \right),$$

where $C$ is a constant, $\Gamma$ the Gamma function, $\gamma(k) = (\gamma_1(k), \gamma_2(k), \gamma_3(k), \gamma_4(k))$ the vector-