Lattice points in d-dimensional spherical segments

Martin Ortiz Ramirez

Abstract

We study lattice points in \(d\)-dimensional spheres, and count their number in thin spherical segments. We found an upper bound depending only on the radius of the sphere and opening angle of the segment. To obtain this bound we slice the segment by hyperplanes of rational direction, and then cover an arbitrary segment with one having rational direction. Diophantine approximation can be used to obtain the best rational direction possible.

Keywords: discrete geometry, lattice points on spheres, Diophantine approximation.

MSC(2010): 11P21, 11K60.

1 Introduction

The study of lattice points in \(d\)-spheres has been extensively studied across history, starting with the two dimensional case of Gauss’s circle problem, that asks for the number of lattice points \(N(r)\) inside a circle of radius \(r\), with \(r^2 = n\) an integer. Gauss already knew that by geometrical considerations \(N(r)\) is close to the area of the circle \(\pi r^2\), so the problem quickly became about the asymptotic behaviour of \(E(r) = |N(r) - \pi r^2|\). A natural question is to ask what is the least \(\alpha\) such that \(E(r) = O(r^\alpha)\). Gauss proved that \(\alpha \leq 1\) by considering the length of the circumference, Sierpinski improved it to \(\alpha \leq 2/3\), and Van der Corput lowered it to \(\alpha < 2/3\) [Gro85, p.20-22]. They also conjectured that \(\alpha\) could get arbitrarily close to \(1/2\). The best upper bound to date is \(\alpha \leq 131/208\) by Huxley [Hux03]. As for a lower bound, Hardy and Landau independently proved that \(\alpha > 1/2\).

Gauss’s circle problem could be considered as the average of all the \(r_2(m)\) for \(m \leq n\), where \(r_2(m)\) is the number of integral solutions to \(x^2 + y^2 = m\). Unlike Gauss’s problem, where obtaining the correct order of magnitude is easy, here it is more difficult since \(r_2(m)\) vanishes for arbitrarily large \(m\). Relating \(r_2(m)\) to the number of divisors of \(m\) [HW08, Ch 16.10] yields \(r_2(m) \ll m^\epsilon\) for any \(\epsilon > 0\)\(^1\). Consider the number of lattice points on a short circular arc. Cilleruelo and Cordoba [CC92] proved that on a circle of radius \(R\), an arc of length no greater than \(\sqrt{2}R^{1/2-1/[(4l/2)+2]}\) contains at most \(l\) lattice points. Cilleruelo and Granville [CG09] proved that in an arc of length

\(^1\)Here and in what follows \(f(m) \ll g(m)\) means that \(f(m)/g(m)\) is bounded as \(m\) tends to infinity.
less than \((40 + \frac{40}{3}\sqrt{10})^{1/3}R^{1/3}\) there are at most 3 lattice points, an improvement on a result by Jarnik [Jar26], that in an arc of length less than or equal to than \(R^{1/3}\) there are at most two lattice points.

Moving to 3 dimensions, \(S(R)\) denoting the number of lattice points inside a sphere of radius \(R\), we have the analogue of Gauss’s circle problem. By the same geometrical considerations \(1.3\] Szego [Sze26] proved that \(E(R) = o(R(\log R)^{1/2})\) is not true. Regarding upper bounds \(E(R) = O(R^\theta)\), Heath-Brown [HB99] proved that \(\theta \leq 21/16\).

As in the two dimensional case, the number \(r_3(R^2)\) of lattice points on the sphere of radius \(R\), \(R^2 = n\) still vanishes for arbitrarily large \(n\), of the form \(4^n(8k + 7)\) (Legendre’s theorem [Gro85, Ch. 4]), although Walfisz [Wal42] proved that \(r_3(R^2) \propto R\log R\) holds for infinitely many \(n\). We also have the upper bound \(r_3(R^2) \ll R^{1+\epsilon}\) for any \(\epsilon > 0\) and the lower bound \(r_3(R^2) \gg R^{1-\epsilon}\) for \(n \neq 0, 4, 7 \mod 8\) [BSR12, 1] i.e. when there are primitive solutions to \(x^2 + y^2 + z^2 = n\) (solutions where the greatest common factor of \(x, y\) and \(z\) is 1). The distribution of lattice points on the sphere has also been a topic of study. In the 1950s Linnik proved that as \(n\) tends to infinity amongst \(n\) square-free and \(n \equiv \pm 1 \mod 5\), the projections of the lattice points onto the unit sphere becomes equidistributed. Duke [Duk88] and Golubeva-Fomenko [GF90] independently proved the result removing the constraints on \(n\) after thirty years.

A three dimensional analogue of an arc is a cap, the surface obtained by cutting the sphere with a plane, or equivalently, the intersection of \(RS^2\) and sphere of radius \(\lambda\) centered at a point of the sphere. Bourgain and Rudnick [BR12] proved using a theorem by Jarnik [Jar26] that the maximal number \(F_3(R, \lambda)\) of lattice points in a cap of radius \(\lambda\) satisfies

\[
F_3(R, \lambda) \ll R^\epsilon \left(1 + \frac{\lambda^2}{R^{1/2}}\right) \text{ for all } \epsilon > 0.
\]

This is only useful for small caps having \(\lambda \ll R^{1/2}\), because slicing methods already give \(F_3(R, \lambda) \ll R^\epsilon (1 + \lambda)\) [BR12, Lemma 2.2]. For bigger caps Bourgain and Rudnick also proved [BR12, Proposition 1.3]

\[
F_3(R, \lambda) \ll R^\epsilon \left(1 + \lambda \left(\frac{\lambda}{R}\right)^\eta\right) \text{ for any } 0 < \eta < \frac{1}{15}.
\]

If we slice the sphere by two parallel planes we obtain a spherical segment. Maffucci [Ma17, Proposition 6.2] gave a bound for the number of lattice points in segments given their opening angle, a theorem that we generalise in this paper.

The four dimensional case still shows an erratic behaviour, for instance \(r_4(2^n) = 24\) for all \(n\). For \(d\) greater than 4 everything behaves much more nicely, and there are sharp bounds for \(r_d\) i.e. \(r_d(R^2) \approx R^{d-2}\), via the circle method [Gro85, Chapters 10-12]. By similar methods Bourgain-Rudnick [BR12, Appendix A] also proved bounds for the number of lattice points in \(d\)-dimensional spherical caps:

\[
F_d(R, \lambda) \ll R^\epsilon \left(\frac{\lambda^{d-1}}{R} + \lambda^{d-3}\right) \text{ for } d \geq 5.
\]

\(^2\)Here and elsewhere \(f(R) = o(g(R))\) means that \(f(R)/g(R)\) tends to 0 as \(R\) tends to infinity.
The proof of the 3 dimensional case is considerably different from higher dimensions, as it uses the technique of slicing by rational planes, and then Diophantine approximation, a technique we also use in this paper.

While interesting in its own right, lattice points problems have found applications in other fields. Recently there have been applied in arithmetic waves, with results by Oravecz-Rudnick-Wigman [ORW08], Krishnapur-Kurlberg-Wigman [KKW13], Rossi-Wigman [RW18] and Benatar-Maffucci [BM17] among others.

Before stating the main results, we first need to define some key concepts. Let \( S^{d-1} \) denote the unit sphere in \( \mathbb{R}^d \), and \( B(\alpha, r) \) the closed ball of radius \( r \) around \( \alpha \in \mathbb{R}^d \).

**Definition 1.1.** A spherical cap \( T \subseteq RS^{d-1} \) of direction \( \beta \in \mathbb{R}^d \), \( |\beta| = 1 \), and radius \( r \) is defined as
\[
T = RS^{d-1} \cap B(R\beta, r).
\]
We then define a spherical segment \( S \subseteq RS^{d-1} \) as \( S = T_1 \setminus T_2 \), where each \( T_i \) is a spherical cap of direction \( \beta \) and radius \( r_i \), \( r_1 > r_2 \).

We will soon see that the boundary of \( T \) contains a base which is a \((d-2)\)-sphere in a hyperplane orthogonal to \( \beta \), hence the following definition.

**Definition 1.2.** The opening angle of a spherical cap is \( \angle POQ \), where \( O \) is the origin and \( P,Q \) are antipodal points of the \((d-2)\)-sphere that is the base. The opening angle \( \theta \) of a segment \( S \) is \( \theta_1 - \theta_2 \), where \( \theta_i \) is the opening angle of \( T_i \).

We define \( \psi(R, \theta) \) to be the maximal number of lattice points in segments of opening angle \( \theta \). Our method will involve slicing the segment by hyperplanes. In this context, \( \kappa_d(R) \) is the maximal number of integer points on the intersection of \( RS^{d-1} \) and a hyperplane. We are now in position to state the main results of the paper. The first one is a generalisation of the following result by Maffucci [Maf17, Proposition 6.2] for the three dimensional case:
\[
\psi \ll \kappa_3(R)(1 + R^{1/3}) \quad \text{as} \quad \theta \to 0.
\]

**Theorem 1.3.** Let \( \psi(R, \theta) \) be the maximal number of lattice points on spherical segments in \( RS^{d-1} \) with opening angle \( \theta \). Then
\[
\psi \ll \kappa_d(R)(1 + R^{1/d})
\]
as \( \theta \to 0 \).

We can improve Theorem 1.3 if we are given more information about the direction of the segment.

**Definition 1.4.** For \( \beta \in \mathbb{R}^d \), \( |\beta| = 1 \), \( \beta \) is said to have \( s \) rational quotients if \( \frac{1}{k} \beta \) has exactly \( s + 1 \) rational coordinates, where \( k = \max\{ |\beta_i| : i = 1, 2, \ldots, d \} \).
We then fix some direction $\beta$ and ask for the number of lattice points in segments of radius $R$ and opening angle $\theta$. The case $s = 0$ is already covered in Theorem 1.3.

**Theorem 1.5.** Let $S \in RS^{d-1}$ be a spherical segment of opening angle $\theta$ and direction $\beta$ having $s$ rational quotients, $1 \leq s \leq d - 2$. Let $\psi(R, \theta, \beta)$ be the number of lattice points in $S$. Then

$$\psi \ll \kappa_d(R) \left(1 + R\theta^\frac{s-1}{d-2}\right)$$

as $\theta \to 0$. \(^3\)

Next, we prove some general geometrical facts about segments that will be useful later on. On Section 3 we will state all the relevant lemmas and prove Theorem 1.3. Section 4 is devoted to the proof of such lemmas and we prove Theorem 1.5 on Section 5.

## 2 Some geometrical considerations

Recall Definition 1.1. We now prove that the bases of $S$ are indeed $(d - 2)$-spheres. Let $v \cdot w$ denote the dot product of two vectors in $\mathbb{R}^d$. The intersection of $\{|x - R\beta| = r_i\}$ and $|x| = R$ lies in the plane $\beta \cdot x = R - \frac{r^2}{2R} = \lambda_i$ since

$$|x - R\beta|^2 = R^2 + |x|^2 - 2R\beta \cdot x = r_i^2$$

holds on the intersection. Then

$$|\lambda_i \beta - x|^2 = \lambda_i^2 + R^2 - 2\lambda_i \beta \cdot x = R^2 - \lambda_i^2 = r_i^2 - \frac{r_i^4}{4R^2}$$

so that $S$ has two bases $B_1$ and $B_2$ that are $(d - 2)$-spheres lying on the hyperplanes $\beta \cdot x = \lambda_i$, of radii

$$k_i^2 = r_i^2 - \frac{r_i^4}{4R^2}.$$

In Definition 1.2, $POQ$ is an isosceles triangle of side lengths $R$ and $2k_i$, so $\theta$ is well-defined. Because $P$ and $Q$ are antipodal points, $O, P, Q$ and $R\beta$ are coplanar, hence by basic geometry we have that

$$r_i = 2R \sin(\theta_i/4). \quad (2.1)$$

Another parameter that will be useful is the height of the segment, the distance between the two bases $B_1$ and $B_2$

$$h = \frac{|\beta \lambda_1 - \beta \lambda_2|}{|\beta|^2} = \frac{1}{|\beta|} |\lambda_1 - \lambda_2| = \frac{1}{2R} (r_1^2 - r_2^2). \quad (2.2)$$

\(^3\)Here $\ll_\beta$ means that the implied constant depends on $\beta$.\[4]
3 Proof of Theorem 1.3

We will first state the necessary lemmas, that will be proved in the next section.

**Lemma 3.1.** Let $S \subseteq RS^{d-1}$ be a spherical segment of direction $\frac{b}{|b|}$, with $b \in \mathbb{Z}^d$ and height $0 \leq h \leq 2R$. Then

$$\psi \leq \kappa_d(R)(1 + |b|h).$$

Lemma 3.1, although simple, provides the best upper bound for a segment of rational direction; we will use it to prove the next lemma. Lemma 3.1 will be proved in the next section.

**Lemma 3.2.** Let $S \subseteq RS^{d-1}$ be a spherical segment with direction $\beta$ and opening angle $\theta$. For any $a \in \mathbb{Z}^d$, the maximal number of lattice points lying on $S$ satisfies

$$\psi \ll \kappa_d(R)(1 + R|a|(|\theta + \phi)|) \quad \text{as} \quad \theta \to 0$$

where $\phi$ is the angle between $\beta$ and $a$. The implied constant is absolute.

We will prove Lemma 3.2 in the next section. We will try to find an $a \in \mathbb{Z}^d$ that optimises $|a|$ and $\phi$ simultaneously using Diophantine approximation. The main ingredient is the following.

**Theorem 3.3.** (Dirichlet’s theorem) [Sch93, Theorem 1A, p.27].

Let $\xi_i \in \mathbb{R}$ for $i = 1, 2 \ldots d$, and $H \in \mathbb{N}$. Then there exist $p_i \in \mathbb{N}$ and $1 \leq q \leq H^d$ such that

$$\left| \xi_i - \frac{p_i}{q} \right| \leq \frac{1}{qH} \quad \forall i = 1, 2 \ldots d.$$

Note that $\left| \frac{a}{|a|} - \beta \right| = 2 \sin (\phi/2) \sim \phi$ as $\phi \to 0$. The next lemma will provide a good simultaneous bound for $|a|$ and $\phi$.

**Lemma 3.4.** For all $\beta \in \mathbb{R}^d$, $|\beta| = 1$, and integers $H \geq 1$, there exists some $a \in \mathbb{Z}^d$ such that

$$|a| \ll H^{d-1} \quad \text{and} \quad \left| \frac{a}{|a|} - \beta \right| \ll \frac{1}{|a|H}$$

where $\ll$ is understood with respect to $H$, and the implied constants are absolute.

We will prove Lemma 3.4 in the next section. Finally, for Lemma 3.4 we will need an auxiliary lemma (proven in Section 4).

**Lemma 3.5.** Let $\alpha, \beta$ be two non-zero vectors of $\mathbb{R}^n$. Then

$$\left| \frac{\alpha}{|\alpha|} - \frac{\beta}{|\beta|} \right| \leq 2 \frac{|\alpha - \beta|}{|\alpha|}. $$
Proof of Theorem 1.3 assuming all the lemmas. We follow the proof of the 3-dimensional case by Maffucci [Maf17, Prop. 6.2]. Given a segment with direction $\beta$ we choose $a \in \mathbb{Z}^d$ satisfying Lemma 3.4 for some yet to be determined $H$. Then one has

$$\left| \beta - \frac{a}{|a|} \right| = 2\sin(\phi/2) \sim \phi \text{ as } \phi \to 0.$$ 

Assuming that $\theta \to 0$ implies $\phi \to 0$, thus by Lemma 3.2

$$\psi \ll \kappa_d(R) \left( 1 + R|a|\theta + R|a| \left| \beta - \frac{a}{|a|} \right| \right) \quad (3.1)$$

and by Lemma 3.4

$$|a|\theta + |a| \left| \beta - \frac{a}{|a|} \right| \ll \theta H^{d-1} + \frac{1}{H} \quad (3.2)$$

Setting $H = \left\lceil \theta^{-1/d} \right\rceil$ in (3.2), so that $H = \mathcal{O}(\theta^{-1/d})$, we obtain

$$|a|\theta + |a| \left| \beta - \frac{a}{|a|} \right| \ll \theta^{1/d}.$$ 

Therefore, since the implied constants in (3.1) and (3.2) are absolute, we arrive at

$$\psi \ll \kappa_d(R)(1 + R\theta^{1/d}).$$

To complete the proof we have that for our choice of $H$,

$$2\sin(\phi/2) = \left| \beta - \frac{a}{|a|} \right| \leq \frac{1}{|a|H} \leq \frac{1}{H} \sim \theta^{1/d} \to 0 \text{ as } \theta \to 0.$$ 

Therefore $\phi \to 0$ as $\theta \to 0$, because $0 \leq \phi \leq \pi$. \hfill \Box

4 Proofs of the Lemmas

Proof of Lemma 3.1. Since $b \in \mathbb{Z}^d$, all lattice points in the segment lie on a hyperplane of the form $b \cdot x = n \in \mathbb{Z}$. The distance between planes is

$$\left| \frac{nb}{|b|^2} - \frac{(n+1)b}{|b|^2} \right| = \frac{1}{|b|}$$

and there are at most

$$1 + \left| \frac{h}{1/|b|} \right| \leq 1 + |b|h$$

hyperplanes intersecting the segment, each of them with at most $\kappa_d(R)$ lattice points in them; thus the lemma follows. \hfill \Box
A slightly different proof of Lemma 3.1 when $d = 3$ can be found in [Maf17, Proposition 6.3].

**Proof of Lemma 3.2** Let $S = T_1 \setminus T_2$ with respective radii $r_1$ and $r_2$. We will construct $S' = T'_1 \setminus T'_2$ having direction proportional to $a \in \mathbb{Z}^d$, such that $S \subseteq S'$, then $\psi$ can be bounded above using Lemma 3.1 on $S'$. For this it will suffice to construct $S'$ so that $T'_2 \subseteq T_2$ and $T_1 \subseteq T'_1$. We claim that taking

$$r'_2 = r_2 - 2R \sin(\phi/2)$$

and

$$r'^2_1 = r_1^2 + 4R_1 \sin(\phi/2) + 4R^2 \sin(\phi/2)^2$$

satisfies the conditions. A point inside $T'_2$ is of the form $R_2 \frac{a}{|a|} + v$ with $|v| \leq r'_2$, hence

$$\left| R_2 \frac{a}{|a|} + v \right|^2 = R^2 \left| \frac{a}{|a|} \right|^2 + |v|^2 + 2R \cdot \left( \frac{a}{|a|} - \frac{a}{|a|} \right)$$

$$\leq 4R^2 \sin(\phi/2)^2 + |v|^2 + 2R |v| \left| \beta - \frac{a}{|a|} \right|$$

$$\leq 4R^2 \sin(\phi/2)^2 + r'^2_2 + 4R \sin(\phi/2) r'_2 = r'^2_2$$

where we have used that $|\beta - \frac{a}{|a|}| = 2 \sin(\phi/2)$. This shows that $T'_2 \subseteq T_2$. Now let $R_2 + v$ with $|v| \leq r_2$. We have

$$\left| R_2 \frac{a}{|a|} + v \right|^2 \leq 4R^2 \sin(\phi/2)^2 + |v|^2 + 2R |v| \left| \beta - \frac{a}{|a|} \right|$$

$$\leq 4R^2 \sin(\phi/2)^2 + r'^2_2 + 4r_1 \sin(\phi/2) = r'^2_1$$

so that $T_1 \subseteq T'_1$ as desired. Now, according to (2.2), the height of $S'$ is

$$h' = \frac{1}{2R}(r'^2_1 - r'^2_2) = \frac{1}{2R} \left( r'^2_1 - r'^2_2 + 4R \sin(\phi/2) (r_1 + r_2) \right) .$$

Recall that $r_1 = 2R \sin(\theta_i/4) \ (2.1)$, and $\theta = \theta_1 - \theta_2$. Therefore,

$$h' = 2R (\sin(\theta_1/4)^2 - \sin(\theta_2/4)^2) + 4R \sin(\phi/2) (\sin(\theta_1/4) + \sin(\theta_2/4))). \quad (4.1)$$

Now, $\theta_1/4 = \theta_2/4 + \theta/4$, so we can Taylor expand to obtain

$$\sin(\theta_1/4) = \sin(\theta_2/4) \cos(\theta/4) + \cos(\theta_2/4) \sin(\theta/4) \frac{\theta}{4} + O(\theta^2),$$

implying

$$\sin(\theta_1/4)^2 = \sin(\theta_2/4)^2 + \sin(\theta_2/4)^2 \frac{\theta}{4} + O(\theta^2). \quad (4.2)$$

It follows from (4.1) and (4.2) that
\[ h' = 2R \left( \sin(\theta_2/2) \frac{\theta}{4} + 2 \sin(\phi/2) (\sin(\theta_1/4) + \sin(\theta_2/4)) + O(\theta^2) \right). \]  

(4.3)

We have \( \sin(\phi/2) \leq \phi/2 \), and all the other sines in (4.3) are bounded above by 1, thus

\[ h' \ll R(\theta + \phi) \]

as \( R \) tends to infinity and \( \theta \) tends to 0, and the implied constant is absolute. By Lemma 3.1, the number of lattice points in \( S' \) is no greater than

\[ \kappa_d(R)(1 + |a|h') \ll \kappa_d(R)(1 + R|a|(|\theta + \phi|)) \]

which is an upper bound for \( \psi \) since \( S \subseteq S' \).

We now prove Lemma 3.5, which will be used in the proof of Lemma 3.4.

Proof of Lemma 3.5. This is an easy application of the triangle inequality. We have

\[ \left| \frac{\alpha}{|\alpha|} - \frac{\beta}{|\beta|} \right| = \frac{1}{|\alpha|} \left| \alpha - \frac{|\alpha|}{|\beta|} \beta \right| \]

and

\[ \left| \alpha - \frac{|\alpha|}{|\beta|} \beta \right| \leq |\alpha - \beta| + \left| \beta - \frac{|\alpha|}{|\beta|} \beta \right| = |\alpha - \beta| + ||\beta| - |\alpha|| \leq 2|\alpha - \beta|. \]

(4.4)

Next, we prove Lemma 3.4.

Proof of Lemma 3.4. Without loss of generality we assume that \( |\beta_1| = \max(|\beta_i| \ i = 1, 2 \ldots d) \). Define \( \xi_i = \frac{\beta_i}{\beta_1} \) for \( i = 2, 3 \ldots d \), so that we have \( |\xi_i| \leq 1 \) for all \( i \). By Dirichlet’s theorem (Theorem 3.3) there exist \( 1 \leq q \leq H^{d-1} \) and \( p_i \leq q \) for \( i = 2, 3, \ldots, d \) such that

\[ \left| \xi_i - \frac{p_i}{q} \right| \leq \frac{1}{qH} \text{ for } i = 2, 3, \ldots, d. \]  

(4.4)

Let \( a = (q, p_2, \ldots, p_d) \). We have \( \left| \frac{\beta_i}{q} \right| \leq 1 + \frac{1}{qH} \leq 2 \), so that \( |p_i| \leq 2q \). Hence

\[ |a|^2 = q^2 + p_2^2 + \ldots + p_d^2 \leq (4d - 3)q^2 \quad \Rightarrow \quad |a| \leq (4d - 3)^{1/2}q \ll H^{d-1}. \]  

(4.5)

Now let \( d = \frac{\beta_1}{q} a \), thus \( \frac{d}{|a|} = \frac{a}{|a|} \), so that by Lemma 3.5 and (4.4)

\[ \left| \beta - \frac{a}{|a|} \right| = \left| \beta - \frac{d}{|d|} \right| \leq 2|\beta - d| = 2|\beta_1| \left( \sum_{i=2}^{d} \left( \xi_i - \frac{p_i}{q} \right)^2 \right)^{1/2} \leq \frac{2\sqrt{d-1}}{qH}. \]
Since \(|a| \leq (4d - 3)^{1/2}q\), we have
\[
\left| \beta - \frac{a}{|a|} \right| \leq \frac{2(4d^2 - 7d + 3)^{1/2}}{|a|H} \ll \frac{1}{|a|H}.
\] (4.6)
Both implied constants in (4.5) and (4.6) are absolute.

\[\square\]

5 Rational quotients: proof of Theorem 1.5

We now prove a generalisation of Lemma 3.4 for the case when the direction of the segment has rational quotients (recall Definition 1.4). We are only interested in the case when the number of rational quotients \(s\) is less than or equal to \(d - 2\), since for \(s = d - 1\) the proof Lemma 3.1 can be easily modified to give the best upper bound for \(\psi\) using this method, and no Diophantine approximation is needed.

**Corollary 5.1.** Let \(\beta \in \mathbb{R}^d\) have \(s\) rational quotients, \(1 \leq s \leq d - 2\), and let \(H \geq 1\) be an integer. Then there exists \(a \in \mathbb{Z}^d\) such that
\[
|a| \ll_{\beta} H^{d-1-s} \quad \text{and} \quad \left| a - \frac{a}{|a|} - b \right| \ll_{\beta} \frac{1}{|a|H}
\]
where \(\ll_{\beta}\) is understood with respect to \(H\).

**Proof.** Without lost of generality assume that \(|\beta_1| = \max(|\beta_i| \mid i = 1, 2 \ldots d)\). Define \(\xi_i = \frac{\beta_i}{\beta_1}\) for \(i = 2, 3 \ldots d\), so that we have \(|\xi_i| \leq 1\) for all \(i\). If there are \(1 \leq s \leq d - 2\) rational quotients \(\xi_i\), say
\[
\xi_i = \frac{m_i}{n_i} \in \mathbb{Q} \quad \text{for} \quad 2 \leq i \leq s + 1
\]
then by Dirichlet’s theorem (Theorem 3.3) there exist \(1 \leq q' \leq H^{d-1-s}\) and \(p_i \leq q'\) such that
\[
\left| \xi_i - \frac{p_i}{q'} \right| \leq \frac{1}{q'H} \quad \text{for} \quad s + 2 \leq i \leq d.
\] (5.1)

Let \(m = \prod_{i=2}^{s+1} n_i\), \(q = q'm\), and \(a = (q, \frac{am_2}{n_2}, \ldots, \frac{am_{s+1}}{n_{s+1}}, mp_{s+2}, \ldots, mp_d) \in \mathbb{Z}^d\). Then
\[
|a| \leq \sqrt{dq} \leq \sqrt{dm}H^{d-1-s}.
\] (5.2)

Now let \(d = \frac{\beta_1}{q} a\), then \(\frac{d}{|a|} = \frac{a}{|a|}\), so that by Lemma 3.3 and (5.1)
\[
\left| \beta - \frac{a}{|a|} \right| = \left| \beta - \frac{d}{|d|} \right| \leq 2|\beta - d|
\]
\[
= 2|\beta_1| \left( \sum_{i=2}^{s+1} \left( \xi_i - \frac{m_i}{n_i} \right)^2 + \sum_{i=s+2}^{d} \left( \xi_i - \frac{p_i}{q'} \right)^2 \right)^{1/2} \leq 2|\beta_1| \frac{\sqrt{d-1-s} 1}{q'H}.
\]
Therefore, using (5.2), we obtain

\[ |\beta - \frac{a}{|a|}| \leq \frac{2m\sqrt{d(d-1-s)}}{|a|H} \ll_{\beta} \frac{1}{|a|H}. \]

This enables us to prove Theorem 1.5 in essentially the same way as Theorem 1.3.

**Proof of Theorem 1.5.** Given a segment with direction $\beta$ having $s$ rational quotients, we choose $a \in \mathbb{Z}^d$ satisfying Corollary 5.1 for some yet to be determined $H$. Then

\[ |\beta - \frac{a}{|a|}| = 2\sin(\phi/2) \sim \phi \quad \text{as } \phi \to 0. \]

Assuming that $\theta \to 0$ implies $\phi \to 0$. We have, by Lemma 3.2

\[ \psi \ll_{\beta} \kappa_d(R) \left( 1 + R|a|\theta + R|a| \left| \beta - \frac{a}{|a|} \right| \right), \]

and by Corollary 5.1

\[ |a|\theta + |a| \left| \beta - \frac{a}{|a|} \right| \ll_{\beta} \theta H^{d-1-s} + \frac{1}{H}. \] (5.3)

Setting $H = \left\lceil \frac{1}{\theta \pi^{-s}} \right\rceil$ in (5.3), so that $H = \mathcal{O} \left( \frac{1}{\theta \pi^{-s}} \right)$, we obtain

\[ |a|\theta + |a| \left| \beta - \frac{a}{|a|} \right| \ll_{\beta} \theta \frac{1}{\pi^{-s}}. \]

Therefore,

\[ \psi \ll_{\beta} \kappa_d(R) \left( 1 + R\theta \frac{1}{\pi^{-s}} \right). \]

Finally,

\[ 2\sin(\phi/2) = \left| \beta - \frac{a}{|a|} \right| \ll_{\beta} \frac{1}{|a|H} \leq \frac{1}{H} \sim \theta \pi^{-s} \to 0 \quad \text{as } \theta \to 0. \]

Thus $\phi \to 0$ as $\theta \to 0$ since $0 \leq \phi \leq \pi$. \qed

**Acknowledgements**

Thanks to Riccardo Maffucci for being my supervisor in this project, who has been indispensable in every sense. Thank you also to the LMS and Oxford University Math Institute for providing funding for this project. I would also like to thank Zeév Rudnick and R. Heath-Brown for generously answering my questions about their work.
References

[BM17] Jacques Benatar and Riccardo W Maffucci. Random Waves On $\mathbb{T}^3$: Nodal Area Variance and Lattice Point Correlations. *International Mathematics Research Notices*, 2019(10):3032–3075, 09 2017.

[BR12] Jean Bourgain and Zeév Rudnick. Restriction of toral eigenfunctions to hypersurfaces and nodal sets. *Geometric and Functional Analysis*, 22:878–937, 2012.

[BSR12] Jean Bourgain, Peter Sarnak, and Zeév Rudnick. Local statistics of lattice points on the sphere. *arXiv e-prints*, page arXiv:1204.0134, Mar 2012.

[CC92] Javier Cilleruello and Antonio Cordoba. Trigonometric polynomials and lattice points. *Proceedings of the American Mathematical Society*, 114(4), 1992.

[CG09] Javier Cilleruelo and Andrew Granville. Close lattice points on circles. *Canadian Journal of Mathematics-journal Canadien De Mathematiques - CAN J MATH*, 61, 12 2009.

[Duk88] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Inventiones mathematicae*, 92(1):73–90, Feb 1988.

[GF90] E. P. Golubeva and O. M. Fomenko. Asymptotic distribution of integral points on the three-dimensional sphere. *Journal of Soviet Mathematics*, 52(3):3036–3048, Nov 1990.

[Gro85] Emil Grosswald. *Representations of Integers as Sums of Squares*. Springer-Verlag, 1985.

[HB99] R. Heath-Brown. Lattice points in the sphere. *Number theory in progress*, (2), 1999.

[Hux03] M. N. Huxley. Exponential sums and lattice points iii. *Proceedings of the London Mathematical Society*, 87(3):591–609, 2003.

[HW08] G.H. Hardy and E.M. Wright. *An introduction to the theory of numbers*. Oxford University Press, 2008.

[Jar26] V. Jarnik. Uber die gitterpunkte auf konvexen kurven. *Math. Z*, 24, 1926.

[KKW13] Manjunath Krishnapur, Pär Kurlberg, and Igor Wigman. Nodal length fluctuations for arithmetic random waves. *Ann. Math. (2)*, 177(2):699–737, 2013.

[Maf17] Riccardo W. Maffucci. Nodal intersections for random waves against a segment on the 3-dimensional torus. *Journal of Functional Analysis*, 272(12):5218–5254, 2017.

[ORW08] Ferenc Oravecz, Zeév Rudnick, and Igor Wigman. The Leray measure of nodal sets for random eigenfunctions on the torus. *Annales de l’Institut Fourier*, 58(1):299–335, 2008.

[RW18] Maurizia Rossi and Igor Wigman. Asymptotic distribution of nodal intersections for arithmetic random waves. *Nonlinearity*, 31(10):4472, Oct 2018.

[Sch93] Wolfgang M. Schimdt. *Diophantine approximation*. Springer-Verlag, 2 edition, 1993.
[Sze26] G. Szego. Beitrage zur theorie de laguerreschen polynome. *Zahlentheoretische Anwendungen*, 1926.

[Wal42] A. Walfisz. On the class-number of binary quadratic forms. *Math. Tbilissi*, 11, 1942.