HOCHSCHILD COHOMOLOGY AND ORBIFOLD JACOBIAN ALGEBRAS ASSOCIATED TO INVERTIBLE POLYNOMIALS

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ABSTRACT. Let \( f \) be an invertible polynomial and \( G \) a group of diagonal symmetries of \( f \). This note shows that the orbifold Jacobian algebra \( \text{Jac}(f, G) \) of \( (f, G) \) defined by \([BTW16]\) is isomorphic as a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra to the Hochschild cohomology \( \text{HH}^*(\text{MF}_G(f)) \) of the dg-category \( \text{MF}_G(f) \) of \( G \)-equivariant matrix factorizations of \( f \) by calculating the product formula of \( \text{HH}^*(\text{MF}_G(f)) \) given by Shklyarov \([S17]\). We also discuss the relation of our previous results to the categorical equivalence.

1. Introduction

To a polynomial \( f = f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) which defines an isolated singularity only at the origin, one can associate a finite dimensional \( \mathbb{C} \)-algebra called the Jacobian algebra \( \text{Jac}(f) := \mathbb{C}[x_1, \ldots, x_n]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \) of \( f \), which is an important and interesting invariant of \( f \).

In our previous paper \([BTW16]\), when \( f \) is an invertible polynomial and \( G \) is a finite abelian group acting diagonally on variables which respects \( f \) we show the existence and the uniqueness of the \( G \)-twisted version of the Jacobian algebra of \( f \), a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( G \)-twisted commutative algebra denoted by \( \text{Jac}'(f, G) \). The expected Jacobian algebra \( \text{Jac}(f, G) \) of the pair \( (f, G) \), a natural generalization of the Jacobian algebra \( \text{Jac}(f) \) of \( f \) to the pair \( (f, G) \), is then given as the \( G \)-invariant subalgebra of \( \text{Jac}'(f, G) \) and it is called the orbifold Jacobian algebra of \( (f, G) \).

Another important invariant associated to the pair \( (f, G) \) is the dg-category \( \text{MF}_G(f) \) of \( G \)-equivariant matrix factorizations of \( f \). To this category, one can associate the Hochschild cohomology \( \text{HH}^*(\text{MF}_G(f)) \), which is equipped with a \( \mathbb{Z}/2\mathbb{Z} \)-graded commutative cup product. Actually, our axiomatization of \( \text{Jac}'(f, G) \) and \( \text{Jac}(f, G) \) is motivated by the algebraic structure of the pair \( (\text{HH}^*(\text{MF}_G(f)), \text{HH}_*(\text{MF}_G(f))) \). It is natural to show now that \( \text{Jac}(f, G) \) is isomorphic to \( \text{HH}^*(\text{MF}_G(f)) \).

Recently, Shklyarov \([S17]\) developed a method to compute \( \text{HH}^*(\text{MF}_G(f)) \). He introduces a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( G \)-twisted commutative algebra \( \mathcal{A}^*(f, G) \) such that its \( G \)-invariant part \( \mathcal{A}^*(f, G)^G \) is isomorphic as a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra to the Hochschild cohomology \( \text{HH}^*(\text{MF}_G(f)) \) \([S17]\), Theorem 3.1 and Theorem 3.4.

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It is natural to expect the isomorphism between $\Jac'(f, G)$ and $\A^*(f, G)$ since they have the same underlying $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-vector spaces with similar algebraic structures. We show this by calculating the Shklyarov’s product formula of $\A^*(f, G)$.

**Theorem** (Theorem 8). Let $f$ be an invertible polynomial and $G$ a subgroup of the group $G_f$ of maximal diagonal symmetries. We have a $\mathbb{Z}/2\mathbb{Z}$-graded algebra isomorphism

$$\Jac'(f, G) \cong \A^*(f, G),$$  

which is compatible with the $G$-actions on both sides. In particular, by taking the $G$-invariant part, we have a $\mathbb{Z}/2\mathbb{Z}$-graded algebra isomorphism

$$\Jac(f, G) \cong \HH^*(\MF_G(f)).$$

When $f$ is of chain type in two variables, this is done in [S17] Appendix A.1.

**Remark 1.** While preparing the manuscript, a closely related work by He–Li–Li [HLL] has appeared. They seem to give another method to calculate the cup product of the Hochschild cohomology $\HH^*(\mathbb{C}[x] \rtimes G, f)$ of $G$-equivariant curved algebra (cf. [S17, Section 2]) isomorphic to $\A^*(f, G)$. Their product formula coincides (up to sign) with the Shklyarov’s formula.

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2. Notations and terminologies

For a non-negative integer $n$ and $f = f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ a polynomial, the *Jacobian algebra* $\Jac(f)$ of $f$ is a $\mathbb{C}$-algebra defined as

$$\Jac(f) = \mathbb{C}[x_1, \ldots, x_n] / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

If $\Jac(f)$ is a finite-dimensional, then set $\mu_f := \dim_{\mathbb{C}} \Jac(f)$ and call it the *Milnor number* of $f$. In particular, if $n = 0$ then $\Jac(f) = \mathbb{C}$ and $\mu_f = 1$. The *Hessian* of $f$ is defined as

$$\hess(f) := \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,n}.$$  

In particular, if $n = 0$ then $\hess(f) = 1$. 
A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called a weighted homogeneous polynomial if there are positive integers $w_1, \ldots, w_n$ and $d$ such that $f(\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n) = \lambda^df(x_1, \ldots, x_n)$ for all $\lambda \in \mathbb{C}^*$. A weighted homogeneous polynomial $f$ is called non-degenerate if it has at most an isolated critical point at the origin in $\mathbb{C}^n$, equivalently, if the Jacobian algebra $\text{Jac}(f)$ of $f$ is finite-dimensional.

**Definition 2.** A non-degenerate weighted homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called invertible if the following conditions are satisfied.

- The number of variables coincides with the number of monomials in $f$:
  
  $$f(x_1, \ldots, x_n) = \sum_{i=1}^n c_i \prod_{j=1}^n x_{ij}^E$$

  for some coefficients $c_i \in \mathbb{C}^*$ and non-negative integers $E_{ij}$ for $i, j = 1, \ldots, n$.

- The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.

Let $f = \sum_{i=1}^n c_i \prod_{j=1}^n x_{ij}^E$ be an invertible polynomial. Without loss of generality one may assume that $c_i = 1$ for all $i$ by rescaling the variables. According to [KS], an invertible polynomial $f$ can be written as a Thom–Sebastiani sum $f = f_1 \oplus \cdots \oplus f_p$ of invertible ones (in groups of different variables) $f_\nu$, $\nu = 1, \ldots, p$ of the following types:

1. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_m^{a_m-1} x_m + x_m^{a_m}$ (chain type, $m \geq 1$);
2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_m^{a_m-1} x_m + x_m^{a_m} x_1$ (loop type, $m \geq 2$).

**Remark 3.** In [KS] the authors distinguished also polynomials of the so called Fermat type: $x_1^{a_1}$, which is regarded as a chain type polynomial with $m = 1$ in this paper.

**Definition 4.** The group of maximal diagonal symmetries of an invertible polynomial $f(x_1, \ldots, x_n)$ is defined as

$$G_f := \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \mid f(\lambda_1 x_1, \ldots, \lambda_n x_n) = f(x_1, \ldots, x_n)\}.$$

Each element $g \in G_f$ has a unique expression of the form $g = (e[\alpha_1], \ldots, e[\alpha_n])$ with $0 \leq \alpha_i < 1$, where $e[\alpha] := \exp(2\pi i \sqrt{-1} \alpha)$. The age of $g$ is defined as the rational number

$$\text{age}(g) := \sum_{i=1}^n \alpha_i.$$

For each $g \in G_f$, let $I_g := \{i_1, \ldots, i_{n_g}\}$ be a subset of $\{1, \ldots, n\}$ such that $\text{Fix}(g) = \{x \in \mathbb{C}^n \mid x_j = 0, j \notin I_g\}$. In particular, $I_{id} = \{1, \ldots, n\}$ and $n_g = \dim_\mathbb{C} \text{Fix}(g)$. Denote by $I_g^c$ the complement of $I_g$ in $I_{id}$ and set $d_g := n - n_g$, the codimension of $\text{Fix}(g)$.

**Proposition 5.** For $f = x_1^{a_1} x_2 + \cdots + x_m^{a_m-1} x_m + x_m^{a_m}$ of chain type, for each $g \in G_f \backslash \{\text{id}\}$ there exists $1 \leq k \leq n$, such that $I_g^c = \{1, \ldots, k\}$. For $f = x_1^{a_1} x_2 + \cdots + x_m^{a_m} x_1$ of loop type, for each element $g \in G_f \backslash \{\text{id}\}$ has $I_g^c = \{1, \ldots, n\}.$
3. Orbifold Jacobian algebra

We briefly recall our orbifold Jacobian algebras. From now on, let \( f = f(x_1,\ldots,x_n) \) be an invertible polynomial and \( G \) a subgroup of \( G_f \). In order to simplify the notation, we often write \( f(x_1,\ldots,x_n) \) as \( f(x) \), \( \mathbb{C}[x_1,\ldots,x_n] \) as \( \mathbb{C}[x] \) and so on.

A \( G \)-twisted Jacobian algebra \( \text{Jac}'(f,G) \) of a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra characterized by a set of axioms [BTW16, Section 3] motivated by properties satisfied by the pair of the Hochschild cohomology and homology. It exists and it is uniquely defined up to an isomorphism [BTW16, Theorem 21]. This allows us to describe it by the explicit formula.

As a \( G \times \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathbb{C} \)-vector space, we have

\[
\text{Jac}'(f,G) = \bigoplus_{g \in G} \text{Jac}(f^g) \tilde{v}_g, \quad f^g := f|_{\text{Fix}(g)},
\]

where \( \tilde{v}_g \) is a generator (a formal letter) attached to each \( g \in G \). The \( \mathbb{Z}/2\mathbb{Z} \)-grading of the element \( [\phi(x)] \tilde{v}_g \) is defined by the parity of \( d_g \). \( \text{Jac}'(f,G) \) is endowed with a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( G \)-twisted commutative\(^1\) product \( \circ \) with the unit \( \tilde{v}_{\text{id}} = [1] \tilde{v}_{\text{id}} \). Namely, for all \( g,h \in G \)

\[
[\psi(x)] \tilde{v}_g \circ [\phi(x)] \tilde{v}_h \in \text{Jac}(f^{gh}) \tilde{v}_{gh},
\]

\[
[\psi(x)] \tilde{v}_g \circ [1] \tilde{v}_{\text{id}} = [\psi(x)] \tilde{v}_g = [1] \tilde{v}_{\text{id}} \circ [\psi(x)] \tilde{v}_g,
\]

and

\[
[\psi(x)] \tilde{v}_g \circ [\phi(x)] \tilde{v}_h = (-1)^{d_g d_h} \cdot g^*([\phi(x)] \tilde{v}_h) \circ [\psi(x)] \tilde{v}_g
\]

where \( g^* \) is the \( G \)-action, given on the subspace \( \text{Jac}(f^h) \tilde{v}_h \) by

\[
g^*([\phi(x)] \tilde{v}_h) = \prod_{i \in I_h^n} g_i^{-1} \cdot [\phi(g \cdot x)] \tilde{v}_h, \quad g = (g_1,\ldots,g_n).
\]

Note that \( f^g \) is also an invertible polynomial and there is a surjective map \( \text{Jac}(f) \to \text{Jac}(f^g) \) ([ET13, Proposition 5] and [BTW16, Proposition 7]). The product is compatible with this map and, in particular, we have \([\psi(x)] \tilde{v}_{\text{id}} \circ [\phi(x)] \tilde{v}_g = [\psi(x) \phi(x)] \tilde{v}_g\).

In order to explain the explicit product formula for \( \text{Jac}'(f,G) \) simpler, we give the following Künneth property of \( \text{Jac}'(f,G) \).

**Proposition 6.** Let \( f_1, f_2 \) be invertible polynomials and \( G_1 \subseteq G_{f_1}, G_2 \subseteq G_{f_2} \) be subgroups. We have an algebra isomorphism compatible with \((G_1 \times G_2)\)-actions on both sides:

\[
\text{Jac}'(f_1 \oplus f_2, G_1 \times G_2) \cong \text{Jac}'(f_1, G_1) \otimes_{\mathbb{C}} \text{Jac}'(f_2, G_2).
\]

**Proof.** This is a direct consequence of a set of axioms [BTW16, Section 3] or the product formula [BTW16, Corollary 43]. The key fact is that for any \( g_1 \in G_1 \) and \( g_2 \in G_2 \) we

\(^1\)in [SI17] it is called braided commutative
have \( \tilde{v}_{g_1} \circ \tilde{v}_{g_2} := \text{sgn}(\sigma_{g_1,g_2})\tilde{v}_{g_1g_2} \) where \( \sigma_{g_1,g_2} \) is the permutation that turns the ordered sequence \( I_{c_{g_1}} \sqcup I_{c_{g_2}} \) to \( I_{c_{g_1g_2}} \). In [BTW16], \( \text{sgn}(\sigma_{g_1,g_2}) \) is denoted by \( \tilde{\epsilon}_{g_1,g_2} \).

By this Künneth property, we assume \( f \) is of chain type or of loop type. For each pair \((g,h)\) of elements in \( G \) and \( \phi(x), \psi(x) \in \mathbb{C}[x] \), the product formula is given as follows.

- If \( gh \neq id \), \( g \neq id \) and \( h \neq id \), then \([\phi(x)]\tilde{v}_g \circ [\psi(x)]\tilde{v}_h = 0\).
- If \( gh = id \), then \([\phi(x)]\tilde{v}_g \circ [\psi(x)]\tilde{v}_g^{-1} = (\frac{-1}{\mu_f}) \cdot \tilde{\epsilon}_{g,g^{-1}} \cdot \tilde{v}_{id}, \) where

\[
H_{g,g^{-1}} := \tilde{m}_{g,g^{-1}} \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j \in I_{c_g}}
\]

and \( \tilde{m}_{g,g^{-1}} \) is a constant uniquely determined by the following equation in \( \text{Jac}(f) \)

\[
\frac{1}{\mu_f} \text{[hess}(f^g)H_{g,g^{-1}} = \frac{1}{\mu_f} \text{[hess}(f)].
\]  

The product \( \circ \) is invariant under the \( G \)-action of (11) (cf. [BTW16 Proposition 58]) and hence \( G \)-invariant subspace \( \text{Jac}^*(f,G) \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded commutative algebra.

**Definition 7.** Let \( f \) and \( G \) be as above. The \( G \)-invariant \( \mathbb{Z}/2\mathbb{Z} \)-graded subalgebra \( \text{Jac}(f,G) := (\text{Jac}^*(f,G))^G \) is called the **orbifold Jacobian algebra** of \((f,G)\).

### 4. Shklyarov’s description of Hochschild cohomology

Shklyarov [S17] introduces a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( G \)-twisted commutative algebra \( A^*(f,G) \) whose underlying \( \mathbb{C} \)-vector space is given by

\[
A^*(f,G) = \bigoplus_{g \in G} \text{Jac}(f^g)\xi_g,
\]

where \( \tilde{\xi}_g \) is a generator (a formal letter) attached to each \( g \in G \). It is required that the group \( G \) acts naturally on \( \text{Jac}(f^g) \) for each \( g \in G \) and \( \xi_g \) transforms as

\[
G \ni h = (h_1, \ldots, h_n) : \tilde{\xi}_g \mapsto \prod_{i \in I_{c_g}} h_i^{-1} \cdot \tilde{\xi}_g,
\]

so that the product structure of \( A^*(f,G) \) is invariant under the \( G \)-action. In particular, its \( G \)-invariant part \( A^*(f,G)^G \) is isomorphic as a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra to the Hochschild cohomology \( \text{HH}^*(\text{MF}_G(f)) \) of \( \text{MF}_G(f) \) equipped with the cup product [S17 Theorem 3.1 and Theorem 3.4]. We shall recall the product structure of \( A^*(f,G) \) [S17 Section 3].

Define the \( n \)-th \( \mathbb{Z} \)-graded **Clifford algebra** \( \text{Cl}_n \) as the quotient algebra of

\[
\mathbb{C}\langle \theta_1, \ldots, \theta_n, \partial_{\theta_1}, \ldots, \partial_{\theta_n} \rangle
\]
modulo the ideal generated by

\[ \theta_i\theta_j = -\theta_j\theta_i, \quad \partial_{\theta_i}\partial_{\theta_j} = -\partial_{\theta_j}\partial_{\theta_i}, \quad \partial_{\theta_i}\theta_j = -\theta_j\partial_{\theta_i} + \delta_{ij}, \]

where \( \theta_i \) is of degree \(-1\) and \( \partial_{\theta_i} \) is of degree \( 1 \). For \( I \subseteq \{1, \ldots, n\} \) write

\[ \partial_{\theta_I} := \prod_{i \in I} \partial_{\theta_i}, \quad \theta_I := \prod_{i \in I} \theta_i, \]

(18)

where in both cases the multipliers are taken in increasing order of the indices. The subspaces \( \mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n] \) and \( \mathbb{C}[\theta_0] = \mathbb{C}[\partial_{\theta_1}, \ldots, \partial_{\theta_n}] \) of \( \text{Cl}_n \) have the left \( \mathbb{Z} \)-graded \( \text{Cl}_n \)-module structures via the isomorphisms

\[ \mathbb{C}[\theta] \cong \text{Cl}_n / \text{Cl}_n(\partial_{\theta_1}, \ldots, \partial_{\theta_n}), \quad \mathbb{C}[\theta_0] \cong \text{Cl}_n / \text{Cl}_n(\theta_1, \ldots, \theta_n). \]

Write \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) as \( \mathbb{C}[x, y] \) and \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n] \) as \( \mathbb{C}[x, y, z] \). For each \( 1 \leq i \leq n \), there is a map

\[ \nabla_i^{x \rightarrow (x, y)} : \mathbb{C}[x] \to \mathbb{C}[x, y], \quad \nabla_i(p) := \frac{l_i(p) - l_{i+1}(p)}{x_i - y_i}. \]

(19)

where \( l_i(p) := p(y_1, \ldots, y_{i-1}, x_i, \ldots, x_n) \), \( l_1(p) = p(x) \) and \( l_{n+1}(p) = p(y) \) \footnote{17}. They are called the difference derivatives, whose key property is the following:

\[ \sum_{i=1}^{n} (x_i - y_i)\nabla_i(p) = p(x) - p(y). \]

(20)

The difference derivatives can be applied consecutively. In particular, we shall use \( \nabla_j^{y \rightarrow (y, x)}\nabla_i^{x \rightarrow (x, y)}(p) \), which is an element of \( \mathbb{C}[x, y, z] \). For an \( \mathbb{C} \)-algebra homomorphism \( \psi : \mathbb{C}[x] \to \mathbb{C}[x] \), write \( \nabla_i^{x \rightarrow (x, \psi(x))}(p) := \nabla_i^{x \rightarrow (x, y)}(p)|_{y = \psi(x)} \in \mathbb{C}[x] \).

Now we are ready to describe the product structure of \( \mathcal{A}^*(f, G) \). For each pair \((g, h)\) of elements in \( G \), define the class \( \sigma_{g,h} \in \text{Jac}(f^{gh}) \) as follows.

- If \( d_{g,h} := \frac{1}{2}(d_g + d_h - d_{gh}) \) is not a non-negative integer, set \( \sigma_{g,h} = 0 \).
- If \( d_{g,h} \) is a non-negative integer, define \( \sigma_{g,h} \) to be the class of the coefficient of \( \partial_{\theta_{c_{gh}}} \)

in the expression

\[ \frac{1}{d_{g,h}} ! \nabla \left( \left[ H_f(x, g(x), x) \right]_g + [H_{f,g}(x)]_{gh} \otimes 1 + 1 \otimes [H_{f,h}(g(x))]_{gh} \right) \partial_{\theta_{c_{gh}}} \]

(21)

where

\[ H_f(x, g(x), x) \]

(1) is an element of \( \mathbb{C}[x] \otimes \mathbb{C}[\theta] \) defined as the restriction to the set \( \{ y = g(x), z = x \} \) of the following element of \( \mathbb{C}[x, y, z] \otimes \mathbb{C}[\theta] \)

\[ H_f(x, y, z) := \sum_{1 \leq j \leq n} \nabla_j^{y \rightarrow (y, x)}\nabla_i^{x \rightarrow (x, y)}(f) \theta_i \otimes \theta_j; \]

(22)
(2) $H_{f,g}(x)$ is the element of $\mathbb{C}[x] \otimes \mathbb{C}[\theta]$ given by

$$H_{f,g}(x) := \sum_{i,j \in I_g, j < i} \frac{1}{1-g_j} \nabla_j^{x \rightarrow (x, x^q)} \nabla_i^{x \rightarrow (x, g(x))} (f) \theta_j \theta_i;$$

(23)

where $x^q$ is defined as $(x^q)_i = x_i$ if $i \in I_g$ and $(x^q)_i = 0$ if $i \notin I_g$;

(3) $[-]_{gh} : \mathbb{C}[x] \otimes V \rightarrow \text{Jac}(f^{gh}) \otimes V$ for $V = \mathbb{C}[x] \otimes \mathbb{C}[\theta] \otimes^2$ or $V = \mathbb{C}[x] \otimes \mathbb{C}[\theta]$ is a $\mathbb{C}$-linear map defined as the extension of the quotient map $\mathbb{C}[x] \rightarrow \text{Jac}(f^{gh})$;

(4) the $d_{g,h}$-th power in Equation (21) is computed with respect to the natural product on $\mathbb{C}[x] \otimes \mathbb{C}[\theta] \otimes \mathbb{C}[\theta]$;

(5) $\Upsilon$ is the $\mathbb{C}[x]$-linear extension of the degree zero map $\mathbb{C}[\theta] \otimes^2 \mathbb{C}[\partial_\theta] \otimes^2 \rightarrow \mathbb{C}[\partial_\theta]$ defined by

$$p_1(\theta) \otimes p_2(\theta) \otimes q_1(\partial_\theta) \otimes q_2(\partial_\theta) \mapsto (-1)^{|q_1||p_2|} p_1(\sigma_1) \cdot p_2(\sigma_2)$$

(24)

where $p_i(q_i)$ denotes the action of $p_i(\theta)$ on $q_i(\partial_\theta)$ via the Cl$_n$-module structure on $\mathbb{C}[\partial_\theta]$ defined above and $\cdot$ is the natural product in $\mathbb{C}[\partial_\theta]$.

Then the product of $A^*(f, G)$ is given by

$$[\phi(x)]e_g \cup [\psi(x)]e_h = [\phi(x)\psi(x)]e_{g,h}, \quad \phi(x), \psi(x) \in \mathbb{C}[x].$$

(25)

5. Results

Now we can state our main theorem in this paper.

**Theorem 8.** Let $f$ be an invertible polynomial and $G$ a subgroup of $G_f$. We have a $\mathbb{Z}/2\mathbb{Z}$-graded algebra isomorphism $\text{Jac}'(f, G) \cong A^*(f, G)$ compatible with the $G$-actions on both sides.

One sees immediately that $\text{Jac}'(f, G)$ and $A^*(f, G)$ have isomorphic underlying $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. It is also clear that the $G$-actions are compatible on both sides (recall (11) and (17)). It remains to check that the products agree. One only needs to do this for $f$ of chain type or of loop type since $\text{Jac}'(f, G)$ and $A^*(f, G)$ have the same Künneth property (12) and [S17 Proposition 2.6]. The rest of this section is devoted to the proof of the theorem.

**Proof.** First, similarly to the case $\text{Jac}'(f, G)$, we have the following

**Proposition 9.** If $gh \neq id$, $g \neq id$ and $h \neq id$, then $\sigma_{g,h} = 0$.

**Proof.** Since the algebra $A^*(f, G)$ is also $\text{Aut}(f, G)$-invariant due to [S17 Theorem 3.1], we may apply [BTW16 Proposition 34], which yields the statement. We can also show the vanishing of $\sigma_{g,h}$ due to degree reason by direct calculation, which is elementary.
Hence, we only need to calculate $\sigma_{g,g^{-1}}$ for each $g \in G_f\setminus\{id\}$.

**Proposition 10.** For each $g = (e[\alpha_1], \ldots, e[\alpha_n]) \in G_f\setminus\{id\}$ with $0 \leq \alpha_i < 1$, we have

$$\sigma_{g,g^{-1}} = (-1)^{\frac{d_g(d_g-1)}{2}} \cdot e \left[ -\frac{1}{2} \text{age}(g) \right] \cdot \left( \prod_{i=1}^{d_g} e \left[ \frac{-1}{2 \sin(\alpha_i \pi)} \right] \right) [H_{g,g^{-1}}]. \quad (26)$$

**Proof.** For $f$ of chain type one applies Lemma 11 and Lemma 12 (see Section 5.1 below) and for $f$ of loop type one does Lemma 13 and Lemma 14 (see Section 5.2 below). \qed

Note that for $i = 1, \ldots, d_g$ we have $0 < \alpha_i < 1$ and that $\prod_{i=1}^{d_g} 2 \sin(\alpha_i \pi)$ is invariant under taking $g$ to its inverse $g^{-1}$ since this is equivalent to substituting $\alpha_i$ with $1 - \alpha_i$.

The algebra isomorphism $A^*(f, G) \rightarrow \text{Jac}'(f, G)$ reads:

$$\tilde{v}_g \rightarrow e \left[ \frac{-d_g}{8} \right] \left( \prod_{i=1}^{d_g} 2 \sin(\alpha_i \pi) \right)^{-1/2} \xi_g. \quad (27)$$

5.1. Chain type $f = x_1^{a_1} x_2 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$. First, we list the ingredients for the product formula (25):

$$H_f(x, y, z) = \sum_{i=1}^{n} \frac{x_i+1}{y_i - z_i} \left( \frac{x_i^{a_i} - y_i^{a_i}}{x_i - y_i} - \frac{x_i^{a_i} - z_i^{a_i}}{x_i - z_i} \right) \theta_i \otimes \theta_i + \sum_{i=1}^{n-1} \frac{y_i^{a_i} - z_i^{a_i}}{y_i - z_i} \theta_{i+1} \otimes \theta_i, \quad x_{n+1} := 1,$n

$$H_{f,g}(x) = \sum_{i=1}^{d_g-1} x_i^{a_i-1} g_i^{a_i} \theta_i \theta_{i+1}, \quad g = (g_1, \ldots, g_n) \in G_f\setminus\{id\}.$$ They give

$$H_f(x, g(x), x) = \sum_{i=1}^{n} \frac{x_i^{a_i-2} x_i+1}{g_i - 1} \left( \frac{1 - g_i^{a_i}}{1 - g_i} - a_i \right) \theta_i \otimes \theta_i + \sum_{i=1}^{n-1} \frac{g_i^{a_i-1}}{g_i - 1} x_i^{a_i-1} \theta_{i+1} \otimes \theta_i,$$

$$H_{f,g^{-1}}(g(x)) = \sum_{i=1}^{d_g-1} x_i^{a_i-1} \frac{1}{g_i - 1} \theta_i \theta_{i+1}, \quad g = (g_1, \ldots, g_n) \in G_f\setminus\{id\}.$$ Consider a square matrix $M_g$ of size $d_g$ whose $(i, j)$-th entry is given by

$$(M_g)_{ij} := -\frac{x_i^{a_i-2} x_i+1}{g_i - 1} \left( \frac{1 - g_i^{a_i}}{1 - g_i} - a_i \right) \delta_{i,j} + x_i^{a_i-1} g_i^{a_i} \delta_{i+1,j} + x_i^{a_i-1} - \frac{1}{g_i - 1} \delta_{i-1,j}. \quad (28)$$

**Lemma 11.** For each $g \in G_f\setminus\{id\}$, we have the following equality in Jac$(f)$:

$$\sigma_{g,g^{-1}} = (-1)^{\frac{d_g(d_g-1)}{2}} [\det(M_g)]. \quad (29)$$

**Proof.** Since $I_{id}^c = \emptyset$ and due to (24), we only consider the coefficient of $\theta_{I_g} \otimes \theta_{I_g}^*$ in

$$\frac{1}{d_g} \left[ [H_f(x, g(x), x)]_{id} + [H_{f,g}(x)]_{id} \otimes 1 + 1 \otimes [H_{f,h}(g(x))]_{id} \right]^{d_g}.$$
It is clear from this and also the specific form of $H_f(x, g(x), x)$, $H_{f,g}(x)$ and $H_{f,g^{-1}}(g(x))$ that the coefficients of $\theta_{i+1} \otimes \theta_i$ in $H_f(x, g(x), x)$ do not contribute to $\sigma_{g,g^{-1}}$. Due to the same reason one sees that $H_{f,g}$ and $H_{f,g^{-1}}$ contribute to $\sigma_{g,g^{-1}}$ in pairs. Taking care of the Clifford algebra coefficients, one obtains a recurrence formula for $\sigma_{g,g^{-1}}$ in terms of $\det(M_g)$ via its minors.

**Lemma 12.** For each $g \in G_f \setminus \{\text{id}\}$, we have

$$[\det(M_g)] = \left( \prod_{i=1}^{d_g} \frac{-a_i}{1 - g_i} \right) [x_1^{a_1-2} x_2^{a_2-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1}] = \left( \prod_{i=1}^{d_g} \frac{1}{g_i - 1} \right) [H_{g,g^{-1}}]. \quad (30)$$

**Proof.** The proof is done inductively as follows:

$$[\det(M_g)] = \left[ \frac{-x_1^{a_1-2} x_2^{a_2-2}}{g_1 - 1} \left( \frac{1 - g_1}{1 - g_1} - a_1 \right) \left( \prod_{i=2}^{d_g} \frac{-a_i}{1 - g_i} \right) x_2^{a_2-2} x_3^{a_3-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1} \right]$$

$$+ \left[ \frac{x_1^{a_1-2} x_2^{a_2-2}}{(1 - g_1)^2} \left( \prod_{i=3}^{d_g} \frac{-a_i}{1 - g_i} \right) x_3^{a_3-2} x_4^{a_4-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1} \right]$$

$$= \left( \prod_{i=1}^{d_g} \frac{-a_i}{1 - g_i} \right) [x_1^{a_1-2} x_2^{a_2-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1}]$$

$$+ \left( \frac{-1 - g_1}{1 - g_2} - g_1^{a_1} \right) \left( \prod_{i=3}^{d_g} \frac{-a_i}{1 - g_i} \right) [x_1^{a_1-2} x_2^{a_2-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1}]$$

where we used the relations $[x_1^{a_1}] = -a_2[x_2^{a_2-1} x_3]$ and $g_1^{a_1} g_2 = 1$. Since the element $[H_{g,g^{-1}}]$ is easily calculated and it is given by $(\prod_{i=1}^{d_g} a_i) [x_1^{a_1-2} x_2^{a_2-1} \cdots x_{d_g}^{a_{d_g}-1} x_{d_g+1}]$ (cf. [BTW16, Proof of Lemma 29]), the statement follows.

Thus we have finished the proof of Proposition 10 for $f$ of chain type.

5.2. **Loop type** $f = x_1^{a_1} x_2 + \cdots + x_n^{a_n} x_1$. First, we list the ingredients for the product formula (25):

$$H_f(x, y, z) = \sum_{i=1}^{n-1} \frac{x_i + 1}{y_i - z_i} \left( \frac{x_i - y_i}{x_i - z_i} - \frac{x_i - z_i}{x_i - z_i} \right) \theta_i \otimes \theta_i + \sum_{i=1}^{n-1} \frac{y_i - z_i}{y_i - z_i} \theta_i \otimes \theta_i$$

$$+ \frac{z_1}{y_n - z_n} \left( \frac{x_n - z_n}{x_n - y_n} - \frac{x_n - y_n}{x_n - y_n} \right) \theta_n \otimes \theta_n + \frac{x_n - y_n}{x_n - y_n} \theta_n \otimes \theta_1.$$

$$H_{f,g}(x) = \sum_{i=1}^{n-1} \frac{g_i}{1 - g_i} \theta_i \otimes \theta_{i+1} + \frac{1}{g_n - 1} \theta_1 \otimes \theta_n, \quad g = (g_1, \ldots, g_n) \in G_f \setminus \{\text{id}\}.$$
They give
\[
H_f(x, g(x), x) = \sum_{i=1}^{n-1} \frac{x_i^{a_i-2} x_{i+1}}{g_i-1} \left( \frac{1 - g_i^{a_i}}{1 - g_i} - a_i \right) \theta_i \otimes \theta_i + \sum_{i=1}^{n-1} \frac{g_i^{a_i} - 1}{g_i-1} x_i^{a_i-1} \theta_{i+1} \otimes \theta_i \\
+ \frac{x_n^{a_n-2} x_1}{g_n-1} \left( \frac{1 - g_n^{a_n}}{1 - g_n} - a_n \right) \theta_n \otimes \theta_n + \frac{g_n^{a_n} - 1}{g_n-1} x_n^{a_n-1} \theta_n \otimes \theta_n.
\]
\[
H_{f,g^{-1}}(g(x)) = \sum_{i=1}^{n-1} \frac{x_i^{a_i-1}}{g_i-1} \theta_i \theta_{i+1} + \frac{g_n^{a_n} - 1}{g_n-1} \theta_n \theta_n, \quad g = (g_1, \ldots, g_n) \in G_f \setminus \{1\}.
\]
Consider a square matrix \( M_g \) of size \( n \) whose \((i,j)\)-th entry is given by
\[
(M_g)_{ij} := -x_i^{a_i-2} x_{i+1} \left( \frac{1 - g_i^{a_i}}{1 - g_i} - a_i \right) \delta_{i,j} + x_i^{a_i-1} \frac{g_i^{a_i}}{1 - g_i} \delta_{i+1,j} + x_i^{a_i-1} \frac{1}{g_i-1} \delta_{i,j} \\
+ \frac{x_n^{a_n-1} g_n^{a_n}}{g_n-1} \delta_{i,n} \delta_{j,1} + x_n^{a_n-1} \frac{1}{g_n-1} \delta_{i,j} \delta_{j,n}.
\] (31)

**Lemma 13.** For each \( g \in G_f \setminus \{1\} \), we have the following equality in \( \text{Jac}(f) \):
\[
\sigma_{g,g^{-1}} = (-1)^{\frac{n(n-1)}{2}} [\det(M_g)].
\] (32)

**Proof.** The proof is similar to that of Lemma 11. The only differences are the term \( \theta_i \otimes \theta_i \) in \( H_f(x, g(x), x) \) and \( \theta_i \theta_n \) in \( H_{f,g^{-1}}(g(x)) \) (and also in \( H_{f,g^{-1}}(g(x)) \)) that give some additional contributions to \( \sigma_{g,g^{-1}} \) (in particular the third and fourth summands in proof of Lemma 14 below).

**Lemma 14.** For each \( g \in G_f \setminus \{1\} \), we have
\[
[\det(M_g)] = \left( \prod_{i=1}^{n} \frac{1}{g_i-1} \right) \left( \prod_{l=1}^{n} a_l \right) (-1)^n \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] = \left( \prod_{i=1}^{n} \frac{1}{g_i-1} \right) \left[ H_{g,g^{-1}} \right].
\] (33)

**Proof.** First, we have the following

**Lemma 15.** Let \( M'_g \) be a square matrix of size \( n \) whose \((i,j)\)-th entry is given by
\[
(M'_g)_{ij} := -x_i^{a_i-2} x_{i+1} \left( \frac{1 - g_i^{a_i}}{1 - g_i} - a_i \right) \delta_{i,j} + x_i^{a_i-1} \frac{g_i^{a_i}}{1 - g_i} \delta_{i+1,j} + x_i^{a_i-1} \frac{1}{g_i-1} \delta_{i,j} \\
+ \frac{x_n^{a_n-1} g_n^{a_n}}{g_n-1} \delta_{i,n} \delta_{j,1} + x_n^{a_n-1} \frac{1}{g_n-1} \delta_{i,j} \delta_{j,n}.
\]
We have
\[
[\det(M'_g)] = \left( \prod_{i=1}^{n} \frac{1}{g_i-1} \right) \left( \sum_{k=0}^{n} (-1)^k \left( \prod_{l=1}^{n-k} a_l \right) \left( \prod_{l=n-k+1}^{n} \frac{1 - g_l^{a_l}}{1 - g_l} \right) \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right].
\]

**Proof.** One can show easily the statement by induction where we use the relations \( [x_1^{a_1}] = -a_2[x_2^{a_2-1} x_3], \ldots, [x_{n-1}^{a_{n-1}^{-1}}] = -a_n[x_n^{a_n-1} x_1] \) and \( g_1^{a_1} g_2 = \cdots = g_{n-1}^{a_{n-1}} g_n = 1 \).

The statement follows from a direct calculation of the determinant:
\[
\left( \prod_{i=1}^{n} (g_i - 1) \right) [\det(M_g)]
\]
\[= \left( \sum_{k=0}^{n} (-1)^k \left( \prod_{l=1}^{n-k} a_l \right) \left( \prod_{l=n-k+1}^{n} \frac{1-g_l}{1-g_l} \right) \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] \]
\[+ a_1 \frac{1-g_n}{1-g_n} \left( \sum_{k=0}^{n-2} (-1)^k \left( \prod_{l=2}^{n-k-1} a_l \right) \left( \prod_{l=n-k}^{n-1} \frac{1-g_l}{1-g_l} \right) \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] \]
\[+ (-1)^n \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] - \left( \prod_{i=1}^{n} g_i^{a_i} \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] \]
\[= \left( \sum_{k=0}^{n} (-1)^k \left( \prod_{l=1}^{n-k} a_l \right) \left( \prod_{l=n-k+1}^{n} \frac{1-g_l}{1-g_l} \right) \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right] \]
\[= \left( a_1 \cdots a_n - (-1)^n \right) \left[ x_1^{a_1-1} \cdots x_n^{a_n-1} \right], \]

where we used the relations \( [x_n^{a_n}] = -a_1 [x_1^{a_1-1} x_2], 1-g_1^{a_1} = -g_2^{a_1} (1-g_2), \ldots, 1-g_{n-1}^{a_n} = -g_n (1-g_n), 1-g_n^{a_n} = g_1 (1-g_1) \) and \( g_1^{a_1} \cdots g_n^{a_n} = g_1^{a_1-1} \cdots g_n^{-1} \). Since the element \([H_{g,g-1}]\) is nothing but \([\text{hess}(f)]\) and it is given by \((a_1 \cdots a_n - (-1)^n) [x_1^{a_1-1} \cdots x_n^{a_n-1}]\).

Thus we have finished the proof of Proposition \([\text{III}]\) also for \( f \) of loop type.

6. CATEGORICAL EQUIVALENCE

In our previous papers \([\text{BTW16, BTW17}]\), we found an algebra isomorphism
\[
\text{Jac}(\overline{f}) = \text{Jac}(\overline{f}, \{ \text{id} \}) \cong \text{Jac}(f, G), \quad (34)
\]
for some invertible polynomials \( f, \overline{f} \) and subgroups \( G \subseteq G_f \cap \text{SL}(3, \mathbb{C}) \). More precisely, \( f \) is an invertible polynomial defining an ADE singularity or an exceptional unimodal singularity and \( G \) is any subgroup of \( G_f \cap \text{SL}(3, \mathbb{C}) \). The polynomial \( \overline{f} \) is defined as the restriction of the map \( \widehat{f} : \mathbb{C}^3 \to \mathbb{C} \) to a chart \( U \) isomorphic to \( \mathbb{C}^3 \) containing all the critical points of \( \widehat{f} \) where \( \mathbb{C}^3 / G \) is a crepant resolution of \( \mathbb{C}^3 / G \). See \([\text{BTW16, Theorem 63}]\) and \([\text{BTW17, Theorem 1}]\).

On the other hand, one may ask whether the (quasi-)equivalence of categories of matrix factorizations holds:
\[
\text{MF}(\overline{f}) = \text{MF}_{\{ \text{id} \}}(\overline{f}) \cong \text{MF}_G(f). \quad (35)
\]

\(^2\)Theorem 1 in \([\text{BTW17}]\) is stated in a different way. Namely, we used as \( \overline{f} \) the Berghlund–Hübsch transpose of \( f \). However, it is easy to check that two different \( \overline{f} \)'s give isomorphic Jacobian algebras.
This is true from the construction of $\tilde{f}$ due to the local property of the category of singularities of $\{\hat{f} = 0\} \subset \mathbb{C}^3/G$ which is equivalent to $\text{MF}_G(f)$ [Or, Proposition 1.14]. It is worth mentioning that in [CRCR16, RCN16] the equivalence is given as their “orbifold equivalence” $\tilde{f} \sim_{\text{orb}} f$.

The Hochschild cohomology and the cup product is invariant under categorical equivalences. Theorem 8 of this paper confirms the compatibility of the isomorphism (34) and the equivalence (35) as expected.

**References**

[AGV85] V. Arnold, A. Gusein-Zade, A. Varchenko, *Singularities of Differentiable Maps*, vol I
Monographs in Mathematics, 82. Birkhäuser Boston, Inc., Boston, MA, 1985

[BTW16] A. Basalaev, A. Takahashi, E. Werner, *Orbifold Jacobian algebras for invertible polynomials*, arXiv preprint: 1608.08962.

[BTW17] A. Basalaev, A. Takahashi, E. Werner, *Orbifold Jacobian algebras for exceptional unimodal singularities*, Arnold Math J. (2017). https://doi.org/10.1007/s40598-017-0076-8.

[CRCR16] N. Carqueville, A.R. Camacho, I. Runkel, *Orbifold equivalent potentials*, Journal of Pure and Applied Algebra, 220(2), (2016). 759781. http://doi.org/10.1016/j.jpaa.2015.07.015

[ET13] W. Ebeling, A. Takahashi, *Variance of the exponents of orbifold Landau–Ginzburg models*, Math. Res. Lett. 20 (1) (2013), 51–65.

[HLL] W. He, S. Li, Y. Li, *G-twisted braces and orbifold Landau-Ginzburg Models*, arXiv:1801.04560.

[KS] M. Kreuzer, H. Skarke: *On the classification of quasihomogeneous functions*, Commun. Math. Phys. 150, 137–147 (1992).

[Or] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Proc. Steklov Inst. Math. 2004, no. 3(246), 227–248.

[RCN16] A.R. Camacho, R. Newton, *Orbifold autoequivalent exceptional unimodal singularities*, arXiv preprint: 1607.07081

[S17] D. Shklyarov, *On Hochschild invariants of Landau-Ginzburg orbifolds*, arXiv preprint: 1708.06030v1.

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