Sally Modules and Associated Graded Rings

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1 Introduction

To frame and motivate the goals pursued in the present article we recall that, loosely speaking, the most common among the blowup algebras are the Rees algebra \( R[I] = \bigoplus_{n=0}^{\infty} I^n t^n \) and the associated graded ring \( \text{gr}_I(R) = \bigoplus_{n=0}^{\infty} I^n / I^{n+1} \) of an ideal \( I \) in a commutative Noetherian local ring \((R, m)\). The three main clusters around which most of the current research on blowup algebras has been developed are: \((a)\) the study of the depth properties of \( R[I] \), or of an appropriate object related to it such as its Proj; \((b)\) the comparison between the arithmetical properties of \( R[I] \) and \( \text{gr}_I(R) \); \((c)\) the correspondence between the Hilbert/Hilbert–Samuel functions and the properties of \( \text{gr}_I(R) \) for an \( m \)-primary ideal \( I \).

In this paper we address the relation mentioned in \((c)\). To make the terminology more precise, the Hilbert–Samuel function is the numerical function that measures the growth of the length of \( R/I^n, \lambda(R/I^n) \), for all \( n \geq 1 \). It is well known that for \( n \gg 0 \) this function is a polynomial in \( n \) of degree \( d \), namely

\[
e_0 \binom{n+d-1}{d} - e_1 \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d,
\]

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where $d$ denotes the dimension of the ring $R$ and $e_0, e_1, \ldots, e_d$ are the normalized coefficients of the Hilbert–Samuel polynomial of $I$.

Pioneering work on the interplay described in (c) was made by Judith Sally in a sequence of papers \cite{22, 23, 24, 25, 26, 27, 28}. A major recognition of her important contribution came with the introduction of the Sally module (see \cite{29, 30}). Its definition requires a notion that has proved to be quite useful in the theory of Rees algebras: We say that the ideal $J \subset I$ is a reduction of $I$ if $I^{n+1} = JI^n$ for some $n \geq 0$. The least such $n$ is called the reduction number of $I$ with respect to $J$ and denoted by $r_J(I)$. A minimal reduction is a reduction which is minimal with respect to inclusion. Minimal reductions always exist and if the residue field of the ring $R$ is infinite the number of generators of any minimal reduction of $I$ equals the analytic spread of the ideal $I$, namely $\dim(R[It] \otimes R/m)$. If $I$ is an $m$-primary ideal with a minimal reduction $J$, the Sally module of $I$ with respect to $J$, $S_J(I)$, is the graded $R[It]$-module, of dimension $d$ whenever $S_J(I) \neq 0$, defined by the short exact sequence

$$0 \rightarrow IR[It] \rightarrow IR[It] \rightarrow S_J(I) = \bigoplus_{n=2}^{\infty} I^n/J^{n-1}I \rightarrow 0. \quad (1)$$

This new object is the outgrowth of a successful attempt made by W.V. Vasconcelos to give a unified and at the same time simplified treatment of several results by J. Sally and others. This comes about as follows. For $n \gg 0$, the growth of the length of the graded pieces of the Sally module $S_J(I)$ is also measured by a polynomial in $n$ of degree $d - 1$

$$s_0\left(\frac{n + d - 2}{d - 1}\right) - s_1\left(\frac{n + d - 3}{d - 2}\right) + \cdots + (-1)^{d-1}s_{d-1}.$$ 

The $e_i$’s relate to the $s_i$’s in the following manner (see \cite{31, 32})

$$e_0 = \lambda(R/J) \quad e_1 = \lambda(I/J) + s_0 \quad e_i = s_{i-1} \quad \text{for } i = 2, \ldots, d$$

so that: (a) $e_0 - e_1 \leq \lambda(R/I)$ (due to Northcott, see \cite{16}); (b) $e_0 - e_1 = \lambda(R/I)$ if and only if $I^2 = JI$ (due to Huneke, see \cite{17}, and Ooishi, see \cite{18}); (c) $e_1 \geq 0$. In this spirit, one of the results we give in Section 2 is a simple proof of the positivity of $e_2$ (due to Narita, see \cite{15}) and another lower bound for $e_1$ (see Proposition \cite{25}). We also show the independence from the minimal reduction of the length of the graded components of the Sally module (see Proposition \cite{24}).

A recurring theme in the work of J. Sally is the discovery of conditions on the multiplicity $e$ of the local ring $(R, m)$ that assure that $\text{gr}_m(R)$ is Cohen–Macaulay. By \cite{1}, the multiplicity $e$ of $R$ satisfies the inequality $e \geq \mu(m) - d + 1$, where $\mu(m)$ denotes the minimal number of generators of $m$. More precisely, the closed formula is: $e = \mu(m) - d + 1 + \lambda(m^2/Jm)$, where $J$ is a minimal reduction of $m$. If
$\lambda(m^2/Jm) = 0$, i.e., $R$ has minimal multiplicity. J. Sally proved in [22] that $\text{gr}_m(R)$ is always Cohen–Macaulay. After this case was settled it was natural to investigate the case in which $\lambda(m^2/Jm) = 1$, i.e., $e = \mu(m) - d + 2$. In [26] she proved that if in addition $R$ is Gorenstein then $\text{gr}_m(R)$ is also Gorenstein. Later, in [28] she established the Cohen–Macaulayness of $\text{gr}_m(R)$ for an arbitrary Cohen–Macaulay ring $R$ having type $s$ different from $\mu(m) - d$ (we recall that the type of a Cohen–Macaulay local ring $(R, m)$ of dimension $d$ is given by $\dim_{R/m}(\text{Ext}_R^d(R/m, R))$). Still in [28] she exhibited examples of rings having type $\mu(m) - d$ and with $\text{gr}_m(R)$ not Cohen–Macaulay; however, in all the given examples depth($\text{gr}_m(R)$) always turned out to be $d - 1$. Therefore the conjecture that arose from this kind of scenario was whether, in the critical case, the depth of $\text{gr}_m(R)$ is always at least $d - 1$. A simpler proof of Sally’s results was given in the Ph.D. theses of two of the authors (see [33, 19]). There, they also verified the conjecture with the additional assumption that the reduction number of $m$ with respect to $J$ is at most 4. Finally, in 1996, M.E. Rossi and G. Valla (see [21]) and H. Wang (see [33]) positively solved, at the same time, Sally’s conjecture using two different methods. Based on the proof of Rossi–Valla, later S. Huckaba proved that if $\lambda(m^3/Jm^2) = 1$ then $\text{gr}_m(R)$ has depth at least $d - 1$. In fact, he showed that the same conclusion holds for any $m$-primary ideal $I$ such that $J \cap I^2 = JI$ and $\lambda(I^3/JI^2) \leq 1$.

The original trust of our work was to see to which extent one could generalize the above results. The main theorems of the paper appear in the third section and, roughly speaking, deal with the class of $m$-primary ideals $I$ in a Cohen–Macaulay (sometimes even Gorenstein) local ring $(R, m)$ such that: (a) $J \cap I^k = JI^{k-1}$ for $k = 1, \ldots, n$; and (b) $\lambda(I^{n+1}/JI^n) = 1$. To be more precise, we show:

**Theorem 3.6** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field. Let $I$ be an $m$-primary ideal of $R$ and let $J$ be a minimal reduction of $I$ with $\lambda(I^{n+1}/JI^n) = 1$ for some $n \geq 1$. If the following hold

(a) $J \cap I^k = JI^{k-1}$ for all $k = 1, \ldots, n$;

(b) the vector space dimension of $V = I + J: I^n/J: I^n$ is at least 2;

then the associated graded ring of $I$ is Cohen–Macaulay.

**Theorem 3.10** Let $(R, m)$ be a Gorenstein local ring of dimension $d > 0$ and infinite residue field. Let $J$ be a minimal reduction of $m$ with $\lambda(m^3/Jm^2) = 1$. Then the associated graded ring of $m$ is Cohen–Macaulay.

**Theorem 3.14** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field. Let $I$ be an $m$-primary ideal of $R$ with a minimal reduction $J$ satisfying the following conditions

(a) $J \cap I^k = JI^{k-1}$ for all $k = 1, \ldots, n$;
(b) $\lambda(I^{n+1}/JI^n) \leq 1$.

Then $\text{depth}(\text{gr}_I(R)) \geq d - 1$.

The results stated in the previous theorems require conditions on the length of $I^{n+1}/JI^n$, where $I$ is an $m$-primary ideal with minimal reduction $J$. It is therefore natural to investigate the independence, from the minimal reduction, of such lengths. We show in Proposition 2.3 that the depth of $\text{gr}_I(R)$ being at least $d - 1$ is a sufficient condition. The independence of these lengths was first observed by T. Marley in [11, Corollary 2.9] and then later recovered by S. Huckaba in [7, Corollary 2.6]. In each case, the means of the proof are different: Our proof is a consequence of the existence and the properties of a natural filtration of the Sally module $S_I(I)$ introduced in [33, 34]. The result about the independence of $\lambda(R/J \cap I^n)$ is instead still open.

Section 4 ends the paper by describing various classes of ideals where the hypotheses required in Theorem 5.10 are satisfied.

Throughout the paper, the notation and terminology are the ones of [3] and [13].

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2 On the Hilbert–Samuel coefficients, Sally modules, and independence of lengths

Given any minimal reduction $J$ of an $m$-primary ideal $I$ one can define, using (1), the Sally module of $I$ with respect of $J$, $S_I(I)$. The following proposition shows, in particular, that the length of each graded component is in fact an invariant of $I$.

**Proposition 2.1** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field $R/m$. Let $I$ be an $m$-primary ideal of $R$ and let $J$ be a minimal reduction of $I$. Then the following lengths are independent of $J$

(a) $\lambda(I/J)$ and $\lambda(I^n/J^{n-1}I)$ for $n \geq 2$;
(b) $\lambda(R/J:I)$;
(c) $\lambda(S_2(I/J))$, where $S_2(I/J)$ denotes the second component of the symmetric algebra of $I/J$. 

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Proof. (a) We can write \( \lambda(I/J) = \lambda(R/J) - \lambda(R/I) = e_0 - \lambda(R/I) \), where \( e_0 \) denotes the multiplicity of \( I \). For \( n \geq 2 \) one has that \( I^n/J^{n-1} \) is the component of degree \( n-1 \) of the Sally module of \( I \) with respect to \( J \). From \([31],[32],[33]\) it follows that

\[
\lambda(I^n/J^{n-1}) = e_0 \left( \frac{n+d-2}{d} \right) + \lambda(R/I) \left( \frac{n+d-2}{d} \right) - \lambda(R/I^n).
\]

(b) The claim follows from the fact that \( \lambda(R/J:J) = \lambda(R/J) - \lambda(J:I/J) = e_0 - \lambda(H_{m-d}(I)) \), where \( m \) denotes the minimal number of generators of \( I \) and \( H_{m-d}(I) \) is the last nonvanishing Koszul homology of \( I \).

(c) One has the following short exact sequence

\[
0 \rightarrow \delta(I) \rightarrow S_2(I/J) \rightarrow I^2/JI \rightarrow 0,
\]

where \( \delta(I) \) denotes the kernel of the natural surjection from \( S_2(I) \) to \( I^2 \) (see \([8]\) Proof of Remark 2.7]). Now again the assertion follows from (a).

RemarK 2.2 Proposition \([24]\) shows that \( \lambda(I/J) \) and \( \lambda(I^2/JI) \) never depends on the minimal reduction \( J \) of \( I \). It is natural then to address the issue of the independence from \( J \) of \( \lambda(I^n/JI^{n-1}) \) for any \( n \). In general, though, such an independence fails for \( n \geq 3 \) as the reduction number \( r \) of \( I \) may depend on the chosen minimal reduction \( J \), unless the depth of the associated graded ring of \( I \) is at least \( d-1 \) (see \([3],[12],[29]\]). On the other hand, the sum of the previous lengths gives an upper bound for the coefficient \( e_1 \) of the Hilbert–Samuel polynomial of \( I \). To be more precise one has the inequalities

\[
\sum_{n=1}^{r} \lambda(I^n,J)/J \leq e_1 \leq \sum_{n=1}^{r} \lambda(I^n/JI^{n-1}) \tag{2}
\]

(see \([9],[7],[33]\)) and \([7]\) shows that \( e_1 = \sum_{n=1}^{r} \lambda(I^n/JI^{n-1}) \) if and only if the depth of the associated graded ring of \( I \) is at least \( d-1 \). In Proposition \([2,3]\) we show that this lower bound on the depth of the associated graded ring of \( I \) is also sufficient for the independence of each single length and not just of the entire sum (see also \([11]\) Corollary 2.9) and \([7]\) Corollary 2.6).

Proposition 2.3 Let \((R,\mathfrak{m})\) be a Cohen–Macaulay local ring of dimension \( d > 0 \) and infinite residue field \( R/\mathfrak{m} \). Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( R \) and let \( J \) be a minimal reduction of \( I \). If \( \text{depth}(\text{gr}_J(R)) \geq d-1 \) then \( \lambda(I^n/JI^{n-1}) \) does not depend on \( J \), for all \( n \geq 1 \).

Proof. By \([33]\), \( \text{depth}(\text{gr}_J(R)) \geq d-1 \) is equivalent to \( s_0 = \sum_{n=2}^{r} \lambda(I^n/JI^{n-1}) \), where \( s_0 \) is the multiplicity of \( S_J(I) \). Hence the Hilbert–Poincaré series of \( \text{gr}_J(R) \) has the form

\[
\text{HP}(\text{gr}_J(R),t) = \frac{\lambda(R/I) + \sum_{n=1}^{r} \left[ \lambda(I^n/JI^{n-1}) - \lambda(I^{n+1}/JI^n) \right] t^n}{(1-t)^d} \tag{3}
\]
(see [34, Corollary 1.2]). In particular, the numerator of (3) is a polynomial, say $p(t)$, with coefficients independent from $J$.

We now proceed by induction on $n$. The cases $n = 0, 1$ are taken care of by Proposition 2.1. For $n \geq 2$ consider the identity

$$\lambda(I^{n+1}/JI^n) = \lambda(I^n/JI^{n-1}) - \left[\lambda(I^n/JI^{n-1}) - \lambda(I^{n+1}/JI^n)\right];$$

the assertion now follows by induction.

\[\square\]

**Remark 2.4** The first inequality in (4) also raises the issue of whether or not the condition $\text{depth} (\gr_I(R)) \geq d - 1$ guarantees that $\lambda((I^n,J)/J) = \lambda(I^n/J \cap I^n)$ does not depend on $J$ for $n \geq 0$. By Proposition 2.3, the question has a positive answer if the associated graded ring of $I$ is Cohen–Macaulay (see [30, Corollary 2.7]). The case in which $\text{depth} (\gr_I(R)) = d - 1$ is still open.

We end this section by proving two results on the normalized Hilbert–Samuel coefficients of $I$ by means of the Sally module. The first result gives a lower bound for $e_1$; it is in general less sharp than the one in (2) but it has the advantage of being independent of the minimal reduction of $I$. The second one is a simpler proof of the positivity of $e_2$.

**Proposition 2.5** Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then the following hold

1. $e_1 \geq 2e_0 - \lambda(I/I^2)$;
2. (Narita, see [15]) if $d \geq 2$ then $e_2 \geq 0$. Moreover, if $d = 2$ then $e_2 = 0$ if and only if $I^q$ has reduction number one for $q \gg 0$.

**Proof.** (a) By passing to the faithfully flat extension $R \longrightarrow R[X]/\mathfrak{m}[X]$ we may assume that the residue field of $R$ is infinite and that $J$ is a minimal reduction of $I$. After tensoring with $R/I$ the defining sequence (4) of $S_J(I)$ one has

$$I/I^2R[Jt] \longrightarrow \gr_I(R)_+ = \bigoplus_{i=1}^{\infty} I^i/I^{i+1} \longrightarrow S_J(I) \otimes R/I \rightarrow 0.$$

Notice that the dimension of $S_J(I) \otimes R/I$ is either $d$ or $0$, since the set of associated primes of $S_J(I)$, as $R[Jt]$-module, is either given by $\mathfrak{m}R[Jt]$ or it is empty (see [51, Proposition 2.2]). Thus we obtain the following multiplicity estimate

$$e_0 = e(\gr_I(R)_+) \leq e(I/I^2R[Jt]) + e(S_J(I) \otimes R/I).$$

Observe that $e(I/I^2R[Jt])$ is less than or equal to $\lambda(I/I^2)$, as $I/I^2R[Jt]$ is the homomorphical image of the polynomial module $I/I^2[T_1,\ldots,T_d]$. On the other hand,
$e(S_J(I) \otimes R/I)$ is less than or equal to $s_0 = e(S_J(I))$. Hence $e_0 \leq \lambda(I/I^2) + s_0$. The asserted inequality now follows from the fact that $s_0 = e_1 - e_0 + \lambda(R/I)$ (see [31, Corollary 3.3]).

(b) By [14, Section 22], we only need to show the statement in the case $d = 2$. For $n \gg 0$ the Hilbert–Samuel function of $I$ can be written as

$$\lambda(R/I^n) = e_0 \binom{n+1}{2} - e_1 \binom{n}{1} + e_2.$$  

(4)

Let $q$ be an integer large enough so that $\lambda(R/I^q)$ is given by (4) and consider the Hilbert function of $I^q$. For $n \gg 0$ one has that

$$\lambda(R/(I^q)^n) = \bar{e}_0 \binom{n+1}{2} - \bar{e}_1 \binom{n}{1} + \bar{e}_2.$$  

(5)

As $\lambda(R/(I^q)^n) = \lambda(R/I^{nq})$, an easy comparison between (4), with $nq$ in place of $n$, and (5) yields

$$e_0 \binom{nq+1}{2} - e_1 \binom{nq}{1} + e_2 = \bar{e}_0 \binom{n+1}{2} - \bar{e}_1 \binom{n}{1} + \bar{e}_2$$

or, equivalently,

$$\frac{1}{2} q^2 e_0 n^2 + \left( \frac{1}{2} q e_0 - q e_1 \right) n + e_2 = \frac{1}{2} \bar{e}_0 n^2 + \left( \frac{1}{2} \bar{e}_0 - \bar{e}_1 \right) n + \bar{e}_2.$$  

Hence one concludes that

$$\bar{e}_0 = q^2 e_0 \quad \bar{e}_1 = q e_1 + \frac{1}{2} q^2 e_0 - \frac{1}{2} q e_0 \quad \bar{e}_2 = e_2.$$  

Let $\tilde{s}_0$ denote the multiplicity of the Hilbert–Samuel polynomial of the Sally module of $I^q$. By [31, Corollary 3.3] it satisfies the identity $\tilde{s}_0 = \bar{e}_1 - \bar{e}_0 + \lambda(R/I^q)$. Hence the following calculation goes through

$$\tilde{s}_0 = \bar{e}_1 - \bar{e}_0 + \lambda(R/I^q)$$

$$= \left( q e_1 + \frac{1}{2} q^2 e_0 - \frac{1}{2} q e_0 \right) - q^2 e_0 + \left( e_0 \left( \frac{q+1}{2} \right) - e_1 \left( \frac{q}{1} \right) + e_2 \right)$$

$$= e_2.$$  

This implies that $e_2 = \tilde{s}_0 \geq 0$, as $\tilde{s}_0$ is the leading coefficient of a polynomial that measures lengths. Moreover, $e_2 = \tilde{s}_0 = 0$ if and only if the Sally module of $I^q$ is zero, i.e., $I^q$ has reduction number one.
3 On the depth properties of the associated graded ring of a class of $m$-primary ideals

In this section we study the depth properties of the associated graded ring of any $m$-primary ideal $I$ in a Cohen–Macaulay local ring $(R, m)$ for which there exists a positive integer $n$ such that $J \cap I^k = JI^{k-1}$ for all $k = 1, \ldots, n$ and $\lambda(I^{n+1}/JI^n) \leq 1$. In Theorem 3.10 we show that the associated graded ring of any such ideal $I$ has always depth at least $d-1$, where $d$ is the dimension of the ring $R$, while in Proposition 3.1 we single out those ideals whose associated graded ring is Cohen–Macaulay.

Proposition 3.1 Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field. Let $I$ be an $m$-primary ideal of $R$ and let $J$ be a minimal reduction of $I$ with $\lambda(I^{n+1}/JI^n) = 1$ for some $n \geq 0$. Then the following conditions are equivalent

(a) $\text{gr}_j(R)$ is Cohen–Macaulay;

(b) $J \cap I^k = JI^{k-1}$ for all $k = 1, \ldots, n$, $I^{n+1} \not\subset J$, and $I^{n+2} = JI^{n+1}$.

Proof. Suppose that $\text{gr}_j(R)$ is Cohen–Macaulay. Then by [30, Corollary 2.7] one has that $J \cap I^k = JI^{k-1}$ for all $k$. In particular, $J \cap I^{n+1} = JI^n$ and $I^{n+1} \not\subset J$, as $JI^n \subset I^{n+1}$. Moreover, from the fact that $\lambda(I^{n+1}/JI^n) = 1$ one concludes that $I^{n+2} \subset mI^{n+1} \subset JI^n \subset J$. Hence $I^{n+2} = J \cap I^{n+2} = JI^{n+1}$.

Conversely, from the short exact sequence

$$0 \to J \cap I^{n+1}/JI^n \to I^{n+1}/JI^n \to I^{n+1}/J \cap I^{n+1} \to 0$$

together with the fact that $\lambda(I^{n+1}/JI^n) = 1$ and $I^{n+1}/J \cap I^{n+1} \neq 0$ (as $I^{n+1} \not\subset J$) it follows that $J \cap I^{n+1} = JI^n$. However, $I^{n+2} = JI^{n+1}$ implies that $J \cap I^k = JI^{k-1}$ for all $k \geq n+2$. Hence by [30, Corollary 2.7] we conclude that the associated graded ring of $I$ is Cohen–Macaulay.

Theorem 3.3 and Theorem 3.6 below describe interesting cases in which condition (b) of Proposition 3.1 is verified. A lemma is needed first.

Lemma 3.2 Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field. Let $I$ be an $m$-primary ideal of $R$ and let $J$ be a minimal reduction of $I$. Suppose $\lambda(I^{n+1}/JI^n) = 1$ for some $n \geq 1$ and $I^{n+1} \not\subset J$. Then

(a) $I^{n+1} \subset (\alpha) + J$ for some $\alpha \in I^{n+1} \setminus J$;

(b) $V = \tilde{I} = I + J; I^n/J; I^n$ is a finite dimensional $R/m$-vector space;
(c) by letting
\[ V^{n+1} \ni (\tilde{t}_1, \ldots, \tilde{t}_{n+1}) \mapsto f(i_1, \ldots, i_{n+1}) + m, \]
where \( i_1 \cdots i_{n+1} - f(i_1, \ldots, i_{n+1}) \alpha \in J \), one defines a non-degenerate, symmetric, \((n+1)\)-linear form \( f \) on \( V^{n+1} \).

If, in addition, the vector space dimension of \( V \) is at least 2 then \( \alpha I \subseteq J^{n+1} \).

**Proof.** (a) By the proof of Proposition [3.3] we have that \( I^{n+1} \cap J = J^p \), hence \( \lambda(I^{n+1} + J/J) = \lambda(J^{n+1}/J^n) = 1 \). Therefore \( I^{n+1} + J/J = (\alpha) + J/J \) for some \( \alpha \in I^{n+1} \setminus J \).

(b) \( V = \tilde{V} = I + J: I^n / J : I^n \) is an \( R/m \)-vector space since \( I^{n+1}/J^n \) is.

(c) For any given \((\tilde{t}_1, \ldots, \tilde{t}_{n+1}) \in V^{n+1}\) it follows that \( i_1 \cdots i_{n+1} \in I^{n+1} \subseteq (\alpha) + J \).

Thus one has \( i_1 \cdots i_{n+1} - f(i_1, \ldots, i_{n+1}) \alpha \in J \), for some \( f(i_1, \ldots, i_{n+1}) \in R \). By letting
\[ (\tilde{t}_1, \ldots, \tilde{t}_{n+1}) \mapsto f(i_1, \ldots, i_{n+1}) + m \]
one obtains a well-defined, non-degenerate, symmetric \((n+1)\)-linear form on \( V^{n+1} \).

We only check that it is well-defined and non-degenerate, the other properties being trivial. For the well-definedness it is enough to show that if for all \( t = 1, \ldots, n+1 \) one has that \((\tilde{t}_1, \ldots, \tilde{t}_{n+1}) \) and \((\tilde{t}_1, \ldots, \tilde{t}_{n+1}) \) are two representatives of the same \((n+1)\)-tuple of \( V^{n+1} \), i.e., \( i_t - l_t \in J: I^n \), then \( f(i_1, \ldots, i_t, \ldots, i_{n+1}) - f(i_1, \ldots, l_t, \ldots, i_{n+1}) \in m \). By assumption we have that
\[ i_1 \cdots i_t - i_{t+1} \cdots i_{n+1} (i_t - l_t) - (f(i_1, \ldots, i_t, \ldots, i_{n+1}) - f(i_1, \ldots, l_t, \ldots, i_{n+1})) \alpha \]
is an element in \( J \). But \( i_1 \cdots i_{t-1} i_{t+1} \cdots i_{n+1} (i_t - l_t) \in J \), so that
\[ (f(i_1, \ldots, i_t, \ldots, i_{n+1}) - f(i_1, \ldots, l_t, \ldots, i_{n+1})) \alpha \in J \]
as well. Therefore, \( f(i_1, \ldots, i_t, \ldots, i_{n+1}) - f(i_1, \ldots, l_t, \ldots, i_{n+1}) \) cannot be an invertible element of \( R \), as otherwise this implies \( \alpha \in J \). For the non-degeneracy of the form, suppose that for any \( t = 1, \ldots, n+1 \) one has \( f(i_1, \ldots, i_t, \ldots, i_{n+1}) \in m \) for all \( i_j \in I \) with \( j \neq t \). By the definition of the form, this implies that \( i_1 \cdots i_t \cdots i_{n+1} \in J \) for all \( i_j \in I \) with \( j \neq t \). Hence \( i_t \in J: I^n \) or, equivalently, \( \tilde{t}_t = 0 \).

Finally, if the vector space dimension of \( V \) is at least 2 for any \( c \in I \) one can find an element \( \tilde{d}_2 \in V \) such that \( f(c, \tilde{d}_2, \ldots, \tilde{d}_n) \in m \). By the non-degeneracy of the form we can also find \( \tilde{d}_1, \ldots, \tilde{d}_{n+1} \) in \( V \) such that \( f(d_1, d_2, d_3, \ldots, d_{n+1}) = 1 \); it follows that \( d_1 d_2 d_3 \cdots d_{n+1} - \alpha \in I^{n+1} \cap J = J^p \). Hence \( cd_1 d_2 d_3 \cdots d_{n+1} - c \alpha \in J^{n+1} \). On the other hand, \( f(c, d_2, d_3, \ldots, d_{n+1}) \in m \) implies \( cd_2 d_3 \cdots d_{n+1} \in I^{n+1} \cap J = J^n \). Hence \( d_1 (cd_2 d_3 \cdots d_{n+1}) \in J^{n+1} \), thus yielding \( c \alpha \in J^{n+1} \) as desired. \( \square \)

**Theorem 3.3** Let \((R, m)\) be a Cohen–Macaulay local ring of dimension \( d \geq 0 \) and infinite residue field. Let \( I \) be an \( m \)-primary ideal of \( R \) and let \( J \) be a minimal reduction of \( I \) with \( \lambda(I^{n+1}/J^n) = 1 \) for some \( n \geq 1 \). If the following hold
(a) $J \cap I^k = JI^{k-1}$ for all $k = 1, \ldots, n$;

(b) the vector space dimension of $V = I + J : I^n / J^n$ is at least 2;

then the associated graded ring of $I$ is Cohen–Macaulay.

**Proof.** By Proposition [3.3] we only need to check that $I^{n+1} \not\subseteq J$ and $I^{n+2} = JI^{n+1}$. If $I^{n+1} \subseteq J$ then $I \subseteq J : I^n$; this forces the vector space $V$ to be zero thus contradicting the assumption on its dimension. Let now $i_1, i_2, \ldots, i_{n+1}, i_{n+2}$ be $n + 2$ arbitrary elements of $I$. If one of them, say $i_1$, belongs to $J : I^n$ then we have that $i_1(i_2 \cdots i_{n+1}) \in J \cap I^{n+1} = JI^n$ (by the proof of Proposition [3.1]) and $i_1i_2 \cdots i_{n+1}i_{n+2} \in JI^{n+1}$. Therefore, we may assume that none of the $i_k$’s is in $J : I^n$. The first $n + 1$ of them, for example, define a non-zero element $(\bar{i}_1, \ldots, \bar{i}_{n+1})$ of $V_{n+1}$. Making use of the terminology and the results in Lemma [3.2] we have that $i_1 \cdots i_{n+1} - f(i_1, \ldots, i_{n+1}) \alpha \in J \cap I^{n+1} = JI^n$. Therefore, $i_1 \cdots i_{n+1} i_{n+2} - f(i_1, \ldots, i_{n+1}) \alpha i_{n+2} \in JI^{n+1}$. However, the dimension of $V$ is at least 2 so that $\alpha I \subseteq JI^{n+1}$ by Lemma [3.2]; in particular $\alpha i_{n+2} \in JI^{n+1}$. Thus $i_1 \cdots i_{n+1} i_{n+2} \in JI^{n+1}$ as well. □

**Remark 3.4** In the case of Theorem [3.3] with $I = m$ and $n = 1$ (so that $e = \mu(m) - d + 2$), an easy length comparison in the short exact sequence

$$0 \to J : m / J \longrightarrow m / J \longrightarrow V = m / J : m \to 0$$

yields the following set of equalities

$$\dim (m / J : m) = \lambda(m / J : m) = \lambda(m / J) - \lambda(J : m / J) = e - 1 - \text{type}(R) = \mu(m) - d + 1 - \text{type}(R).$$

Hence $\dim (m / J : m) \geq 2$ is equivalent to $\mu(m) - d > \text{type}(R)$. Therefore Theorem [3.3] recovers the result in [28].

**Lemma 3.5** Let $(R, m)$ be a Gorenstein local ring of dimension $d > 0$ and infinite residue field. Let $J$ be a minimal reduction of $m$ such that $m^{n+1} + J / J$ is the socle of the Gorenstein ring $R / J$. Then

$$\lambda(m / J : m^n) = \lambda(m^n + J / m^{n+1} + J).$$

**Proof.** As $R$ is Gorenstein one has that $J : (m^n + J) / J$ is isomorphic to the canonical module of $R / (m^n + J)$. Hence it follows that $\lambda(J : m^n / J) = \lambda(J : (m^n + J) / J) = \lambda(R / m^n + J) = \lambda(m / m^n + J) + 1$. Thus, an easy diagram chase together with the fact that $m^{n+1} + J / J$ is the socle of $R / J$ yields

$$\lambda(m / J : m^n) = \lambda(m^n + J / J) - 1 = \lambda(m^n + J / m^{n+1} + J) + \lambda(m^{n+1} + J / J) - 1 = \lambda(m^n + J / m^{n+1} + J),$$

as claimed. □
**Theorem 3.6** Let \((R, \mathfrak{m})\) be a Gorenstein local ring of dimension \(d > 0\) and infinite residue field. Let \(J\) be a minimal reduction of \(\mathfrak{m}\) with \(\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1\). Then the associated graded ring of \(\mathfrak{m}\) is Cohen–Macaulay.

**Proof.** We first observe that \(J \cap \mathfrak{m}^2 = J\mathfrak{m}\), by the analytic independence of the generators of \(J\), and \(\mathfrak{m}^3 \not\subset J\). Indeed, if \(\mathfrak{m}^3 \subset J\) then \(R\) has multiplicity \(e = n - d + 2\), where \(n\) denotes the embedding dimension of \(R\). In this case \(\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 0\) by [26], thus contradicting our assumption.

By Lemma 3.5, with \(n = 2\), we have that \(\lambda(\mathfrak{m}/J: \mathfrak{m}^2) = \lambda(\mathfrak{m}^2 + J/\mathfrak{m}^3 + J)\). By Theorem 3.3 the statement is taken care of if the previous length is greater than or equal to 2. Hence we only have to consider the case in which \(\lambda(\mathfrak{m}^2 + J/\mathfrak{m}^3 + J) = 1\). But this condition implies that \(R\) has multiplicity \(e = n - d + 3\); in this case the Cohen–Macaulayness of \(\text{gr}_\mathfrak{m}(R)\) follows from [27, Theorem 1].

In Theorem 3.6, the hypothesis of \(R\) being Gorenstein cannot be dropped; moreover, there are examples of Gorenstein rings with \(\text{gr}_\mathfrak{m}(R)\) not Gorenstein.

**Example 3.7**

(a) Let \(k\) be a field. The ring \(k[[t^6, t^7, t^9, t^{17}]]\) is Cohen–Macaulay, but not Gorenstein, with \(\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1\), where \(J = (t^6)\). In this case, the associated graded ring of \(\mathfrak{m}\) is not Cohen–Macaulay.

(b) Let \(k\) be a field. The ring \(k[[t^5, t^6, t^9]]\) is Gorenstein with \(\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = 1\), where \(J = (t^5)\). By Theorem 3.4 the associated graded ring of \(\mathfrak{m}\) is Cohen–Macaulay. However, it is not Gorenstein.

The lemmata below are inspired by and at the same time generalize the following list of results: [7, Lemma 2.1 and Lemma 2.2], [27, Lemma 1.1, Proposition 1.2, and Corollary 2.3], [33, Lemma 2.1.2], and [35, Lemma 2.1, Lemma 2.3, and Corollary 2.7].

**Lemma 3.8** Let \((R, \mathfrak{m})\) be a Cohen–Macaulay local ring and let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\). Let \(J\) be a minimal reduction of \(I\) such that \(\lambda(I^{n+1}/JI^n) = 1\). Then either there exists \(z \in I\) such that \(I^{t+1} = JI^t + (z^{t+1})\) for all \(t \geq n\), or \(I^{n+2} = JI^{n+1}\). In particular, \(\lambda(I^{t+1}/JI^t) \leq 1\) for all \(t \geq n\).

**Proof.** Let us write \(I = (J, z_1, \ldots, z_l)\) for some \(l \geq 1\). If there exists a \(k\) such that \(z_k^{n+1} \not\in JI^n\) then we may set \(z = z_k\) and we are done. Otherwise, suppose that \(z_i^{n+1} \in JI^n\) for all \(i = 1, \ldots, l\). Since \(JI^n \not\subset I^{n+1}\) there exists \(f\) such that \(I^{n+1} = JI^n + (f)\) of the form \(f = z_1^{p_1} \cdots z_l^{p_l}\) with \(p_1 + \cdots + p_l = n + 1\). Choose \(i\) such that \(p_i > 0\) is maximal with respect to the property that \(f \not\in JI^n\). By assumption \(p_j < n + 1\) so that there exists \(j \neq i\) with \(p_j > 0\). Note that \((z_1, \ldots, z_l) f \in z_i I^{n+1} = z_i (JI^n + (f)) \subseteq I^{n+2} = JI^{n+1}\).
This implies that

\[ JI^{n+1} \subseteq I^{n+2} = \lambda (J, z_1, \ldots, z_l) (JI^n + (f)) \subseteq JI^{n+1} + (z_1, \ldots, z_l) f = JI^{n+1}, \]

as claimed.

To complete the proof, let us assume that there exists \( z \in I \) such that \( I^{n+1} = JI^n + (z^{n+1}) \). We will show that \( I^{t+1} = JI^t + (z^{t+1}) \) for any \( t \geq n \) by inducting on the difference \( t - n \geq 0 \). If \( t - n = 0 \) there is nothing to prove. Hence by inductive hypothesis we have

\[ JI^{t+1} + (z^{t+2}) \subseteq I^{t+1} = I(JI^t + (z^{t+1})). \]

Since \( I = (J, z_1, \ldots, z_l) \) and \( (z_1, \ldots, z_l) z^n \subseteq I^{n+1} = JI^n + (z^{n+1}) \) one can also write

\[ I(JI^t + (z^{t+1})) = J(JI^t + (z^{t+1})) + (z_1, \ldots, z_l) z^n \subseteq JI^{t+1} + z^{t+1-n}(z_1, \ldots, z_l) z^n \]

\[ \subseteq JI^{t+1} + z^{t+1-n}(JI^n + (z^{n+1})) = JI^{t+1} + (z^{t+2}). \]

Hence \( I^{t+1} = JI^{t+1} + (z^{(t+1)+1}) \) as requested. The assertion on \( \lambda(I^{t+1}/JI^t) \) for all \( t \geq n \) is now obvious.

For the definition and properties of superficial elements/sequences see [14, Section 22].

**Lemma 3.9** Let \((R, m)\) be a two dimensional Cohen–Macaulay local ring and let \( I \) be an \( m \)-primary ideal of \( R \). Let \( J \) be a minimal reduction of \( I \), assume that \( J = (x, y) \) where both \( x \) and \( y \) are superficial for \( I \), and set \( r = r_J(I), \ s = r_J/(x)(I/(x)) \). If \( \lambda(I^{n+1}/JI^n) = 1 \) and \( I \cap I^k = JI^{k-1} \) for all \( k = 1, \ldots, n \), then the following statements hold

(a) \( e_1 = \sum_{t=1}^s \lambda(I^t/JI^{t-1}) \);

(b) \( \text{depth}(\text{gr}_J(R)) \geq 1 \) if and only if \( s = r \).

**Proof.** (a) As \( x \) is a superficial element \( e_1 = e_1(I) = e_1(I/(x)) \). Moreover, by [9, Corollary 4.13] one has that

\[ e_1(I/(x)) = \sum_{t=1}^s \lambda((I/(x)) I^t/(J/(x))(I/(x)) I^{t-1}) = \sum_{t=1}^s \lambda(I^t + (x)/JI^{t-1} + (x)) \]

\[ = \sum_{t=1}^s \lambda(I^t/JI^{t-1} + ((x) \cap I^t)). \]
However, by assumption one has that \((x) \cap I^t \subseteq JI^{t-1}\) for \(t = 1, \ldots, n\). On the other hand, for \(t = n + 1, \ldots, s\) it follows from Lemma 3.8 that 
\[
0 < \lambda((I/(x))^t/(J/(x))(I/(x))^{t-1}) = \lambda(I'/JI^{t-1} + ((x) \cap I')) \leq \lambda(I'/JI^{t-1}) \leq 1,
\]
which implies that \((x) \cap I^t \subseteq JI^{t-1}\) for \(t = n + 1, \ldots, s\) as well. This yields the conclusion.

(b) The statement follows from [4, Theorem 3.1].

The next theorem contains the third main result of this paper. Its proof is a simplified version of the one of [5, Theorem 2.6], which in turn was inspired by and follows the steps of the one of [21, Theorem 2.5]. The result requires three ingredients: (a) a reduction to the two dimensional case; (b) the fact that \(e_1\) can be written in two different ways (one using the \(I\)-adic filtration of \(I\) and the other using the filtration given by the Ratliff–Rush closure of the powers of \(I\), see [20]); and (c) a key reduction bound due to Rossi–Valla.

**Theorem 3.10** Let \((R, \mathfrak{m})\) be a Cohen–Macaulay local ring of dimension \(d > 0\) and infinite residue field. Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\) with a minimal reduction \(J\) satisfying the following conditions

(a) \(J \cap I^k = JI^{k-1}\) for all \(k = 1, \ldots, n\);

(b) \(\lambda(I^{n+1}/JI^n) \leq 1\).

Then \(\text{depth}(\text{gr}_I(R)) \geq d - 1\).

**Proof.** By [5, Lemma 2.2] the conclusion of the theorem holds in \(R\) if and only if it holds in \(R/(x_1, \ldots, x_{d-2})\), where \(x_1, \ldots, x_{d-2} \in J\) is a superficial sequence for \(I\); this is the so called Sally machine. Moreover, conditions (a) and (b) are preserved modulo \(x = (x_1, \ldots, x_{d-2})\). Clearly, \(\lambda((I/x)^{n+1}/(J/x)(I/x)^n) \leq 1\) (see the proof of Lemma 3.9) so we only need to verify that \((J/x) \cap (I/x)^k = (J/x)(I/x)^{k-1}\) for all \(k = 1, \ldots, n\). But for that it will be enough to show that \((J/(x)) \cap ((I/(x))^k = JI^{k-1} + (x)/(x)\) holds for any \(x \in J\) and for all \(k = 1, \ldots, n\). Let \(J = j + ax = i_k + bx\) for some \(j \in J\), \(i_k \in I^k\), and \(a, b \in R\). Thus \(i_k \in J\) and then, by assumption, \(i_k \in J \cap I^k = JI^{k-1}\). Hence \(J \in JI^{k-1} + (x)/(x)\).

Therefore we may assume \(R\) to be two dimensional and \(J = (x, y)\) with \(x, y\) superficial elements for \(I\). Let \(s = r_{J/(x)}((I/(x)))\) and \(r = r_I(I)\) as in Lemma 3.9. If \(r \leq n\) the associated graded ring of \(I\) is Cohen–Macaulay by Valabrega–Valla (see [30, Corollary 2.7]). If \(r = n + 1\) then the associated graded ring has depth at least 1 (or \(d - 1\) after lifting back) by [5, Theorem 3.2]. Thus we may assume \(r \geq n + 2\). The proof will be completed once we show that \(s = r\) (see Lemma 3.9(b)).
By Lemma 3.8, there exists $z \in I$ such that $I_{t+1} = JI_t + (z^{t+1})$ for all $t \geq n$ and $\lambda(I_{t+1}/JI_t) \leq 1$ for all $t \geq n$ (equality holds if in addition $t < r$). The integers

$$p = \inf\{k : J\tilde{I}^k = \tilde{I}^{k+1}\} \quad q = \inf\{k : I_{k+1} \subseteq J\tilde{I}\}$$

satisfy the following inequalities

$$n \leq q \leq p \leq s.$$

Indeed, if $q < n$ we have that $I_{q+1} = J \cap I_{q+1} = JI_q$ as $I_{q+1} \subseteq J\tilde{I} \subseteq J$. But this contradicts the fact that $r \geq n + 2$. Hence $n \leq q$. Since $I_{p+1} \subseteq I_{p+1} = J\tilde{I}$ it also follows that $q \leq p$. In order to prove the last inequality notice that $J\tilde{I}^{k-1} \cap I_k = JI^{k-1}$ for $1 \leq k \leq n$. Hence, we obtain the following family of short exact sequences

$$0 \to J\tilde{I}^{k-1}/JI^{k-1} \xrightarrow{\phi_k} \tilde{I}^k/I^k \longrightarrow \tilde{I}^k/J\tilde{I}^{k-1} + I^k \to 0.$$  

Therefore for $k = 2, \ldots, n$ we have that the following expression

$$\lambda(\tilde{I}^k/J\tilde{I}^{k-1} + I^k) = \lambda(\tilde{I}^k/I^k) - \lambda(J\tilde{I}^{k-1}/JI^{k-1}) = \lambda(\tilde{I}^k/J\tilde{I}^{k-1}) - \lambda(I^k/JI^{k-1})$$

is positive. Moreover, $\lambda(\tilde{I}/I) = \lambda(\tilde{I}/J) - \lambda(I/J) \geq 0$, as $I \subseteq \tilde{I}$. Consider now the identity

$$e_1 = \sum_{k=1}^{s} \lambda(I^k/JI^{k-1}) = \sum_{k \geq 1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1})$$

that holds by [7, Corollary 2.10] and Lemma 3.9(a). We can rewrite the previous formula as

$$\sum_{k=1}^{n} \left(\lambda(\tilde{I}^k/J\tilde{I}^{k-1}) - \lambda(I^k/JI^{k-1})\right) = \sum_{k=n+1}^{s} \lambda(I^k/JI^{k-1}) - \sum_{k \geq n+1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1}). \quad (6)$$

From (6) and Lemma 3.8 one concludes that

$$0 \leq s - n - \sum_{k \geq n+1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1}).$$

Hence $s \geq p$. Let $\mu_k$ denote the minimal number of generators of $\tilde{I}^k/J\tilde{I}^{k-1} + I^k$, for each $k \geq 1$. We have that

$$\mu_k < \lambda(\tilde{I}^k/J\tilde{I}^{k-1}) \quad \text{for all } k = 1, \ldots, q$$

and also

$$\mu_k \leq \lambda(\tilde{I}^k/J\tilde{I}^{k-1}) - \lambda(I^k/JI^{k-1}) \quad \text{for all } k = 1, \ldots, n.$$
By letting $\mu = \mu_1 + \cdots + \mu_q$ it follows from \cite{[21], Proposition 2.4 and Proof of Theorem 2.5} (see also \cite{[8], Proposition 2.3 and Proof of Theorem 2.6}) that
\[
\mu^{\mu+q+1} = JJ^{\mu+q},
\]
this means that $r \leq \mu + q$. We now show that $\mu + q \leq s$ thus yielding $s = r$. From \cite{[8]} we have that
\[
\mu = (\mu_1 + \cdots + \mu_n) + (\mu_{n+1} + \cdots + \mu_q)
\leq \sum_{k=n+1}^{s} \lambda(I^k/JI^{k-1}) - \sum_{k \geq n+1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1}) + (\mu_{n+1} + \cdots + \mu_q)
\leq s - n + \sum_{k=n+1}^{q} \left( \mu_k - \lambda(I^k/JI^{k-1}) \right) - \sum_{k \geq q+1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1})
\leq s - n - (q - n) - \sum_{k \geq q+1} \lambda(\tilde{I}^k/J\tilde{I}^{k-1})
\leq s - q
\]
or, equivalently, $\mu + q \leq s$.

It is worth pointing out the following consequence of Theorem \cite{[3.10]} as it describes a situation that quite frequently occurs in nature. We note that this result was previously known only in the case of an ideal $I$ with reduction number two (see \cite{[32], Proposition 5.1.4(a)}).

**Corollary 3.11** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and infinite residue field. Let $I$ be an $m$-primary ideal of $R$ with a minimal reduction $J$ such that $\lambda(I^2/JI) = 1$. Then the associated graded ring of $I$ has depth at least $d - 1$.

## 4 Classes of Examples

We now describe two situations where the previous results apply.

### 4.1 Stretched Cohen–Macaulay rings

Let $(R, m)$ be a Cohen–Macaulay local ring with dimension $d$, infinite residue field, multiplicity $e$, and embedding dimension $\mu(m) = e + d - n$ for some $n \geq 1$. The ring $R$ is said to be stretched if there exists a minimal reduction $J$ of $m$ such that $m^n \not\subset J$. As $R/J$ is an Artin local ring of length $e$ and embedding dimension $e - n$ one has that $m^{n+1} \subset J$.

By combining Theorem \cite{[3.10]} and some results of J. Sally, one obtains the following
Proposition 4.1 Let \((R, m)\) be a Cohen–Macaulay local ring with dimension \(d\), infinite residue field, and embedding dimension \(e + d - n\). Let \(J\) be a minimal reduction of \(m\) such that \(m^n \not\subset J\).

(a) (Sally, see [25, Corollary 2.4]) The associated graded ring \(\text{gr}_m(R)\) of \(m\) is Cohen–Macaulay if and only if \(m^n + 1 = Jm^n\).

(b) If \(\lambda(m^n/Jm^n - 1) = 1\) then the depth of the associated graded ring \(\text{gr}_m(R)\) of \(m\) is at least \(d - 1\).

Proof. (b) By [25, Theorem 2.3] it follows that \(J \cap m^k = Jm^{k-1}\) for all \(k = 1, \ldots, n\), since \(m^{n+1} \subset Jm^n\). Hence Theorem 3.10 applies. \(\square\)

4.2 Ideals arising from graphs

Let \(k\) be a field and let \(R\) be the polynomial ring over \(k\) in the \(d = 2n + 1\) variables \(x_1, \ldots, x_{n+2}, y_1, \ldots, y_{n-1}\), where \(n \geq 1\). Let

\[
M = (x_1, \ldots, x_{n+2}, y_1, \ldots, y_{n-1}) \quad J = (x_1^2, \ldots, x_{n+2}^2, y_1^2, \ldots, y_{n-1}^2)
\]

and define a new family of ideals, say \(J_{(3,2(n-1))}\), as follows

\[
J_{(3,2(n-1))} = (J, M^3, f),
\]

where \(f\) is the form of degree two given by

\[
f = \sum_{i=1}^{n+1} x_ix_{i-1} + x_{n+2}x_1 + \sum_{i=1}^{n-1} x_ix_i.
\]

The ideal \(J_{(3,2(n-1))}\), or \(J\) for short, has a simple combinatorial description which arises from the following graph

![Graph Diagram](image-url)
More precisely, the element \( f \) is obtained by adding together all the products \( x_ix_{i+1} \), where the pair \( (x_i, x_{i+1}) \) consists of adjacent vertices on the cycle, and all the products \( x_iy_i \), where the pair \( (x_i, y_i) \) is a whisker.

**Example 4.2** If \( n = 1 \) then \( 2(n - 1) = 0 \) and the ideal \( J = J_{(3, 0)} \) corresponds to the case of a cycle with three vertices and no whisker. More precisely

\[
J = (x_1^2, x_2^2, x_3^2, f),
\]

where \( f = x_1x_2 + x_2x_3 + x_3x_1 \). It turns out that \( \lambda(R/J) = 6, \lambda(R/J) = 8, \lambda(J/J) = 2, \lambda(J^2/JJ) = 1, J^2 \subset J \) so that \( J \cap J^2 = J^2 \neq JJ \). Hence the depth of the associated graded ring of \( J \) is exactly \( d - 1 = 3 - 1 = 2 \).

**Example 4.3** If \( n = 2 \) then \( 2(n - 1) = 2 \) and the ideal \( J = J_{(3, 2)} \) corresponds to the case of a cycle with four vertices and one whisker. More precisely

\[
J = (x_1^2, x_2^2, x_3^2, x_4^2, y_1^2, x_1x_3x_4, x_2x_3x_4, x_1x_4y_1, x_3x_4y_1, x_2x_4y_1, f),
\]

where \( f = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 + x_1y_1 \). This example appears in [8, Example 2.13]. It turns out that \( \lambda(R/J) = 15, \lambda(R/J) = 32, \lambda(J/J) = 17, \lambda(J^2/JJ) = 2, J \cap J^2 = JJ, \lambda(J^3/JJ^2) = 1, J^3 \subset J \) so that \( J \cap J^3 = J^3 \neq JJ^2 \). Hence the depth of the associated graded ring of \( J \) is exactly \( d - 1 = 5 - 1 = 4 \).

**Example 4.4** If \( n = 3 \) then \( 2(n - 1) = 4 \) and the ideal \( J = J_{(3, 4)} \) corresponds to the case of a cycle with five vertices and two whiskers. More precisely

\[
J = (x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, y_1^2, y_2^2, \mathcal{M}^3, f),
\]

where \( f = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 + x_1y_1 + x_2y_2 \). It turns out that \( \lambda(R/J) = 28, \lambda(R/J) = 128, \lambda(J/J) = 100, \lambda(J^2/JJ) = 30, J \cap J^2 = JJ, \lambda(J^3/JJ^2) = 2, J \cap J^3 = JJ^2, \) and \( \lambda(J^4/JJ^3) = 1, J^4 \subset J \) so that \( J \cap J^4 = J^4 \neq JJ^3 \). Hence the depth of the associated graded ring of \( J \) is exactly \( d - 1 = 7 - 1 = 6 \).

**Question 4.5** It is natural then to ask whether or not in general the ideals \( J = J_{(3, 2(n - 1))} \) and \( J \) satisfy the following properties: (a) \( J \cap J^k = JJ^{k-1} \) for all \( k = 1, \ldots, n \); (b) \( \lambda(J^{n+1}/JJ^n) = 1 \); (c) \( J^{n+1} \subset J \).

**Note added in proof**

After this paper was completed, the authors learned that both M.E. Rossi and J. Elias independently wrote articles that partially overlap with ours.
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