Theoretical Analysis of Divide-and-Conquer ERM: Beyond Square Loss and RKHS

Yong Liu, Lizhong Ding and Weiping Wang

Abstract—Theoretical analyses on the divide-and-conquer based distributed learning with least square loss in the reproducing kernel Hilbert space (RKHS) have recently been explored within the framework of learning theory. However, studies on learning theory for general loss functions and hypothesis spaces remain limited. To fill this gap, we study the risk performance of distributed empirical risk minimization (ERM) for general loss functions and hypothesis spaces. Our main contributions are two-fold. First, we derive two tight risk bounds under certain basic assumptions on the hypothesis space, as well as the smoothness, Lipschitz continuity, and strong convexity of the loss function. Second, we further develop a more general risk bound for distributed ERM without the restriction of strong convexity. These results fill the gap in learning theory of distributed ERM for general loss functions and hypothesis spaces.

Index Terms—Kernel Methods, Empirical Risk Minimization, Distributed Learning, Divide-and-Conquer, Risk Analysis.

I. INTRODUCTION

The rapid expansion in data size and complexity has introduced a series of scientific challenges to the era of big data, such as storage bottlenecks and algorithmic scalability issues [1], [2], [3]. Distributed learning is the most popular approach for handling these challenges. Among many strategies of distributed learning, the divide-and-conquer approach has been shown most simple and effective, while also being able to preserve data security and privacy by minimizing mutual information communications [2], [4].

This paper aims to study the theoretical performance of the divide-and-conquer based distributed learning for Empirical Risk Minimization (ERM) within a learning theory framework. Given

\[ S = \{ (x_i, y_i) \}_{i=1}^N \in (\mathcal{Z} = \mathcal{X} \times \mathcal{Y})^N, \]

drawn identically and independently (i.i.d) from an unknown probability distribution \( \mathbb{P} \) on \( \mathcal{Z} = \mathcal{X} \times \mathcal{Y} \), the ERM can be defined as

\[ \hat{f} = \arg \min_{f \in \mathcal{H}} \hat{R}(f) := \frac{1}{N} \sum_{j=1}^{N} \ell(f, z_j), \quad (1) \]

where \( \ell(f, z) \) is a loss function and \( \mathcal{H} \) is a Hilbert space. In distributed learning, the data set \( S \) is partitioned into \( m \) disjoint subsets \( \{S_i\}_{i=1}^m \), and \( |S_i| = \frac{N}{m} =: n \). The \( i \)th local estimator \( \hat{f}_i \) is produced on each data subset \( S_i \):

\[ \hat{f}_i = \arg \min_{f \in \mathcal{H}} \hat{R}_i(f) := \frac{1}{|S_i|} \sum_{z_j \in S_i} \ell(f, z_j). \quad (2) \]

The final global estimator \( \hat{f} \) is then obtained by

\[ \hat{f} = \frac{1}{m} \sum_{i=1}^{m} \hat{f}_i. \]

The theoretical foundations of distributed learning for (regularized) ERM have received increasing attention within the machine learning community, and have recently been explored within the framework of learning theory [2], [3], [5], [6], [4], [7], [8], [9], [10]. However, most existing risk analyses are based on the closed form of the least square solution and the properties of the reproducing kernel Hilbert space (RKHS), which is only suitable when the distributed learning uses a least square loss in the RKHS. Studies on establishing the risk bounds of distributed learning for general loss functions and hypothesis spaces remain limited.

In this paper, we study the risk performance of distributed ERM based on the divide-and-conquer approach for general loss functions and hypothesis spaces. Specifically, we use the proof techniques from stochastic convex optimization for general loss functions and the covering number for general hypothesis space. Note that the proof techniques of stochastic convex optimization and covering numbers are usually two significantly different paths for theoretical analysis. The main technical difficulty of this paper is thus integrating these two different proof techniques for distributed learning.

The main contributions of the paper include:

- **Result I.** If the number of processors \( m \leq O(\sqrt{N}) \), we present a tight risk bound with order \( O(\frac{1}{m}) \), assuming there is a logarithmic covering number of hypothesis space \( (\log C(\mathcal{H}, \epsilon) \simeq h \log \frac{1}{\epsilon}, h > 0 \), see Assumption 1 for details), and a smooth, Lipschitz continuous and strongly convex loss function.

- **Result II.** Under another basic assumption that there exists a hypothesis space of polynomial covering number \( (\log C(\mathcal{H}, \epsilon) \simeq (\frac{1}{\epsilon})^h \), see Assumption 2 for details), and if the number of processors \( m \leq O(N^{\frac{1}{2h+1}}) \), another tight risk bound of order \( O(N^{-\frac{1}{2h+1}}) \) is established.

- **Result III.** Without Result I’s restriction of a strong convexity of loss function, a more general risk bound of

\[ \Omega(\frac{1}{m}) \]

By using this work, we use \( \Omega \) to hide constant factors as well as polylogarithmic factors in \( N \) or \( m \).

1If \( \ell \) is a regularizer loss function, that is \( \ell'(f, \cdot) = \ell(f, \cdot) + r(f) \), \( r(f) \) is a regularizer, then \( \Omega \) is related to a regularizer ERM.
order $O\left(\frac{1}{\epsilon^2}\right)$ is derived when the number of processors $m \leq O(N^r)$, $0 \leq r < \frac{1}{2}$, and the optimal risk is small.

Since $0 \leq r < \frac{1}{2}$, the rate is faster than $O\left(\frac{1}{\sqrt{N}}\right)$.

Overall, these results fill the gap in learning theory of distributed ERM for general loss functions and hypothesis spaces.

The rest of the paper is organized as follows. In Section 2, we discuss our main results. In Section 3, we compare against related work. Section 4 is the conclusion. All the proofs are given in the last part.

II. MAIN RESULTS

In this section, we provide and discuss our main results. To this end, we first introduce several notations.

**Definition 1 (ε-covering).** Let $(H, \| \cdot \|_H)$ be a Hilbert space, $N(H, \epsilon)$ is an ε-covering of the Hilbert space $H$ if for all $f \in H$, $\exists \tilde{f} \in N(H, \epsilon)$ such that $\| f - \tilde{f} \|_H \leq \epsilon$.

**Definition 2 (ε-covering number).** Let $C(H, \epsilon)$ be the ε-covering number of $H$, that is, the smallest number of cardinality for an ε-covering of $H$ which can written as $C(H, \epsilon) := \min \left\{ k : \exists k\text{-covering over } H \text{ of size } k \right\}$.

The performance of a function $f$ given by the learning machine is usually measured by the risk $R(f) := E_x[\ell(f, x)]$. We denote the optimal function and risk of $H$, respectively, as $f_* := \arg \min_{f \in H} R(f)$ and $R_* := R(f_*)$.

A. Assumptions

In this subsection, we introduce some basic assumptions of the hypothesis space and loss function.

**Assumption 1 (logarithmic covering number).** There exists some $h > 0$ such that

$$\forall \epsilon \in (0, 1), \log C(H, \epsilon) \simeq h \log(1/\epsilon).$$

Many popular function classes satisfy the above assumption when the hypothesis $H$ is bounded:

- Any function space with finite VC-dimension [12], including linear functions and univariate polynomials of degree $k$ (for which $h = k + 1$) as special cases;
- Any RKHS based on a kernel with rank $h$ [13].

**Assumption 2 (polynomial covering number).** There exists some $h > 0$ such that

$$\forall \epsilon \in (0, 1), \log C(H, \epsilon) \simeq (1/\epsilon)^{1/h}. \quad (4)$$

If $H$ is bounded, this type of covering number is satisfied by many Sobolev/Besov classes [14]. For instance, if the kernel eigenvalues decay at a rate of $k^{-2h}$, then the RKHS satisfies Assumption [12] [13]. For the RKHS of a Gaussian kernel, the kernel eigenvalues decay at a rate of $h \to \infty$.

**Remark 1.** To derive the risk bounds for divide-and-conquer ERM without specific assumptions on the type of hypothesis, we adopt the covering number to measure the complexity of the hypothesis. To use the covering number in learning theory, an assumption on the bounded hypothesis is usually needed (see [13], [14] for details). In fact, ERM usually includes a regularizer, that is

$$\min_{f \in H} \frac{1}{N} \sum_{j=1}^{N} \ell(f, z_j) + \lambda \| f \|_{H_0}^2,$$

which is equivalent to the following optimization for a constant $c$ related to $\lambda$,

$$\min_{f \in H} \frac{1}{N} \sum_{j=1}^{N} \ell(f, z_j), \text{ s.t. } \| f \|_{H_0}^2 \leq c.$$

Thus, the assumption for the bounded hypothesis is usually implied in (regularized) ERM.

**Assumption 3.** The loss function $\ell(f, z)$ is non-negative, $G$-smooth, $L$-Lipschitz continuous, and convex w.r.t $f$ for any $z \in Z$.

Assumption 3 is satisfied by several popular losses when $H$ and $Y$ are bounded, such as the square loss $\ell(f, z) = (f(x) - y)^2$, logistic loss $\ell(f, z) = \ln(1 + \exp(-y f(x)))$, square Hinge loss $\ell(f, z) = \max(1 - y f(x))^2$, square ε-loss $\ell(f, z) = \max(0, |y - f(x)| - \epsilon)^2$, and so on.

**Assumption 4.** The loss function $\ell(f, z)$ is an $\eta$-strongly convex function w.r.t $f$ for any $z \in Z$.

Note that $\ell(f, \cdot)$ usually includes a regularizer, e.g. $\ell(f, \cdot) = \hat{\ell}(f, \cdot) + \eta \| f \|_{H_0}^2$. In this case, $\hat{\ell}(f, \cdot)$ is a strongly convex function which only requires $\ell(f, \cdot)$ to be a convex function.

**Assumption 5 ($\tau$-diversity).** There exists some $\tau > 0$ such that

$$\frac{1}{4m^2} \sum_{i,j=1,i\neq j}^{m} \| \hat{f}_i - \hat{f}_j \|_{H_0}^2 \geq \tau,$$

where $\frac{1}{4m^2} \sum_{i,j=1,i\neq j}^{m} \| \hat{f}_i - \hat{f}_j \|_{H_0}^2$ is the diversity between all partition-based estimates, and $\hat{f}_i$ is the $i$th local estimator, $i = 1, \ldots, m$.

If not all the partition-based estimates $\hat{f}_i$, $i = 1, \ldots, m$, are almost the same, Assumption 5 is satisfied.

B. Risk Bounds

In the following, we first derive two tight risk bounds with a smooth, Lipschitz continuous and strongly convex function. Then, we further consider the more general case by removing the restriction of strong convexity.

**Theorem 1.** Under Assumptions 3, 4, 5 if the number of processors $m$ satisfies the bound:

$$m \leq \min \left\{ \frac{N \eta}{8Gh \log \frac{\sqrt{N}h\eta}{\delta}}, \frac{\sqrt{N}h\eta}{L \log \frac{\delta}{\eta}}, \frac{\eta \tau}{128G\log \frac{\delta}{\eta}}, \frac{N \eta}{GL \log \frac{\delta}{\eta}} \right\},$$
then, with probability at least $1 - \delta$, we have:

$$R(\bar{f}) - R(f_*) \leq O\left(\frac{h}{N}\right).$$

The above theorem implies that when $\ell$ is smooth, Lipschitz continuous and strongly convex, the distributed ERM achieves a risk bound in the order of $R(\bar{f}) - R(f_*) = O\left(\frac{h}{N}\right)$. This rate in Theorem 3 is minimax-optimal for some cases:

- **Finite VC-dimension.** If the VC-dimension of $\mathcal{H}$ is bounded by $h$, which is a special case of Assumption 1, 2, 3, 4, 15, 16, 17 showed that there exists a constant $c' \geq 0$ and a function $f \in \mathcal{H}$, such that

$$R(f) - R(f_*) \geq c' \frac{h}{N},$$

where $\mathcal{B}_h(1)$ is the 1-norm ball in $\mathcal{H}$.

From Theorem 1, we know that, to achieve the tight risk bound, the number of processors $m$ should satisfy the restriction

$$m \leq \Omega \left(\min \left\{ \frac{N h}{L \log(2/\delta)} \right\} \right).$$

Thus, $m$ can reach $\Omega(\sqrt{N})$, which is sufficient for using distributed learning in practical applications.

**Theorem 2.** Under Assumptions 2, 3, 4, 5 if the number of processors $m$ satisfies the bound:

$$m \leq \min \left\{ \frac{N h}{L \log(2/\delta)} \right\} \sqrt{n N \frac{2h}{\eta} \log(2/\delta)} + \frac{\eta N \frac{2h}{\eta} + N \eta^2 \tau}{GL \log(2/\delta)},$$

then, with probability at least $1 - \delta$, we have

$$R(\bar{f}) - R(f_*) \leq \tilde{O}\left(\frac{h}{N^{1/r}}\right).$$

From Theorem 2(b) of 18 with $s = d = 1$, we know that, under Assumption 1, there is a universal constant $c' > 0$ such that

$$\inf_{f} \sup_{f_* \in \mathcal{B}_N(1)} \mathbb{E} \left[ \frac{1}{N} \right] \leq c' N^{-\frac{2h}{\eta}}.$$

Thus, our risk bound of order $O\left(N^{-\frac{2h}{\eta}}\right)$ is minimax-optimal in this case.

From Theorem 2, we know that, to achieve the tight risk bound, the number of processors $m$ should satisfy the restriction

$$m \leq \tilde{O}\left(\min \left\{ N^{\frac{2h}{\eta}}, N^{\frac{h}{2h/\eta} + \frac{h}{2h/\eta}}, N^{\frac{h}{2h/\eta}} \right\} \right) = \tilde{O}\left(\frac{h}{N}\right).$$

Note that $\frac{h}{N^{r}} \leq \frac{1}{2}$, thus the number of processors $m \leq \tilde{O}(\sqrt{N})$, which is smaller than in Theorem 3. This is because the restriction of the polynomial covering number is looser than that of the logarithmic one. When $h \to \infty$ (satisfied by the Gaussian kernel), $m$ can reach $\tilde{O}(\sqrt{N})$.

### C. Risk Bounds without Strong Convexity

As follows, we provide a more general risk bound without the restriction of strong convexity.

**Theorem 3.** Under Assumptions 2, 3 and assuming that $\forall f \in \mathcal{H}$, $\|f\|_{\mathcal{H}} \leq B$, if the number of processors $m \leq O(N^r)$, $0 \leq r \leq \frac{1}{2}$, then, with a probability of at least $1 - \delta$, we have

$$R(\bar{f}) - R(f_*) \leq \tilde{O}\left(\frac{h \log(N/\delta)}{N^{1-r}} + \sqrt{\frac{R_* \log N}{N^{1-r}}}\right).$$

If the optimal risk $R_*$ is small, that is $R_* \leq O(N^{r-1})$, we have

$$R(\bar{f}) - R(f_*) \leq O\left(\frac{h \log(N/\delta)}{N^{1-r}}\right).$$

From the above theorem, one can see that:

1. A general risk bound without the restriction of strong convexity is established. The rate of this theorem is $\tilde{O}\left(\frac{h \log(N/\delta)}{N^{1-r}} + \sqrt{\frac{R_* \log N}{N^{1-r}}}\right)$, which is worse than that of Theorem 3. This is due to the relaxation of the loss function restriction.

2. The above theorem implies that, when the optimal risk $R_*$ is small, the risk bound is in the order of $\tilde{O}\left(\frac{h \log(N/\delta)}{N^{1-r}}\right)$. Note that $0 \leq r \leq \frac{1}{2}$, so in this case, the rate is faster than $O\left(\frac{h \log N}{N}\right)$.

3. In the central case, that is $m = 1$, the order of the risk can reach

$$R(\bar{f}) - R(f_*) = \tilde{O}\left(\frac{h}{N}\right),$$

which is nearly optimal. To the best of our knowledge, such a fast rate of ERM for the central case has never been given before for general loss function and hypothesis space.

**Remark 2.** In Theorem 3 the risk bound is satisfied for all $m \leq O(N^r)$, $r \in [0, 1/2]$. Parameter $r$ is used to balance the tightness of the bound and number of processors. The smaller the $r$, the tighter the bound and the fewer the processors.
III. COMPARISON WITH RELATED WORK

In this section, we compare our results with related work. Risk analyses for the original (regularized) ERM have been extensively explored within the framework of learning theory [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. Recently, divide-and-conquer based distributed learning with ridge regression [5], [2], [4], [7], gradient descent algorithms [30], [9], online learning [31], local average regression [10], spectral algorithms [32], [33], semi-supervised learning [19] and the minimum error entropy principle [34], have been proposed and their learning performances have been observed in many practical applications. For point estimation, [3] showed that the distributed moment estimation is consistent if an unbiased estimate is obtained for each of the subproblems. For the distributed regularized least square in RKHS, [4] showed that the distributed ERM leads to an estimator that is consistent with the unknown regression function. Under local strong convexity, smoothness and a reasonable set of other conditions, an improved bound was established in [6].

Optimal learning rates for divide-and-conquer kernel ridge regression in expectation were established in [4], under certain eigenfunction assumptions. Removing these assumptions, an improved bound was derived in [3] using a novel integral operator method. Using similar proof techniques as [3] or [4], optimal learning rates were established for distributed spectral algorithms [32], kernel-based distributed gradient descent algorithms [9], kernel-based distributed semi-supervised learning [19], distributed local average regression [10], and distributed KRR with communications [35].

Among these works, [4], [3], [19] are the three most relevant papers. Thus, as follows, we will compare our results with those in [4], [3], [19]. The seminal work of [4] considered the learning performance of divide-and-conquer kernel ridge regression. Using a matrix decomposition approach, [4] derived two optimal learning rates of order \(O\left(\frac{1}{n}\right)\) and \(O\left(N^{-\frac{s}{2s+1}}\right)\), respectively, for the \(h\)-finite-rank kernels and \(h\) polynomial eigen-decay kernels, under the assumption that, for some constants \(k \geq 2\) and \(A < \infty\), the normalized eigenfunctions \(\{\phi_i\}_k\) satisfy

\[
\forall j = 1, 2, \ldots, \mathbb{E}[\phi_j(X)^{2h}] \leq A^{2h}.
\]  

The condition in [5] is possibly too strong, and it was thus removed in [3], which used a novel integral operator approach under the regularity condition:

\[
f_\rho = L_K h_\rho, \text{ for some } 0 < s \leq 1 \text{ and } h_\rho \in L^2_h,
\]

where \(L_K\) is the integral operator induced by the kernel function \(K\):

\[
L_K(f)(x) := \int_X K(x, x') f(x') d\mathbb{P}(x'), \ x \in X,
\]

and \(f_\rho = \int y d\mathbb{P}(y|x)\) is the regression function. However, the analysis in [3] only works for \(s > 1/2\). In [19], they generalized the results of [3], and derived the optimal learning rate for all \(1/2 \leq s \leq 1\) under the restriction \(m \leq N^{\frac{s}{2s+1}}\) for bounded kernel functions. Thus, we find that, for the special case of \(s = 1/2\), the number of local processors \(m \to \Omega(1)\), does not increase with \(N\). Note that \(1/2 \leq s \leq 1\), so the largest number of local processors can only reach \(m = \Omega(N^{1/3})\), which may limit the applicability of distributed learning.

Our and the most related previous results are summarized in Table I. Compared with previous works, there are two main novelties of our results.

1) The proof techniques of this paper are based on the general properties of loss functions and hypothesis spaces, while for [4], [3], [19], the proofs depend on the special properties of the square loss and RKHS. Thus, our results generalized the results of [4], [3], [19].

2) To derive the optimal rates, [3], [19] show that the number of local processors should be less than \(\Omega(N^{-\frac{s}{2s+1}})\), \(1/2 \leq s \leq 1\). Thus, the highest number \(m\) will be restricted by a constant for \(s = 1/2\), and the best result is \(\Omega(N^{1/3})\) (for \(s = 1\)). However, in this paper, the number of processors that our result can reach is \(\Omega(\sqrt{N})\). Thus, our result can relax the restriction on the number of processors form [3], [19].

IV. CONCLUSION

In this paper, we studied the risk performance of the divide-and-conquer ERM and derived tight risk bounds for general loss functions and hypothesis spaces. To make our results suitable for general loss functions and hypothesis spaces, we used the proof techniques from stochastic convex optimization and the covering number, which are usually two significantly different paths for theoretical analysis. These results fill the
gap in learning theory of distributed ERM, and the proof techniques we used may provide a new path for theoretical analysis.

There are some work worth studying in the future.

1) In our analysis, we assume that the loss function is a (strong) convex function. How to extend our results to a non-convex function is an interesting direction.

2) In [19], they showed that the number of processors can be improved using the unlabeled samples. To use the unlabeled samples to improve our results may be a good question.

3) In this paper, we only considered the simple divide-and-conquer based distributed learning. To extend our results to other distributed learning scenarios, such as distributed KRR with communications [35], is worthy of attention.

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V. PROOF

In this section, we first introduce the key idea of proof, and then provide proofs for Theorems 1, 2 and 3.

A. The Key Idea

Note that if \( \ell \) is an \( \eta \)-strongly convex function, then, \( R(f) \) is also \( \eta \)-strongly convex. According to the properties of a strongly convex function, \( \forall f, f' \in \mathcal{H}, \) we have

\[
\langle \nabla R(f'), f - f' \rangle_{\mathcal{H}} + \frac{\eta}{2} \| f - f' \|^2_{\mathcal{H}} \leq R(f) - R(f'),
\]

or \( \forall f, f' \in \mathcal{H}, t \in [0, 1], \)

\[
R(tf + (1-t)f') \leq tR(f) + (1-t)R(f') - \frac{\eta t(1-t)}{2} \| f - f' \|^2_{\mathcal{H}}.
\]

By (8), one can see that

\[
\begin{align*}
\mathbb{E}(R(\hat{f})) &= R\left( \frac{1}{m} \sum_{i=1}^{m} \hat{f}_i \right) \\
&\leq \frac{1}{m} \sum_{i=1}^{m} R(\hat{f}_i) - \frac{\eta}{4m^2} \sum_{i,j=1, i \neq j} \| \hat{f}_i - \hat{f}_j \|^2_{\mathcal{H}} \\
&\leq \frac{1}{m} \sum_{i=1}^{m} R(\hat{f}_i) - \eta \tau \quad \text{(by Assumption 5)}.
\end{align*}
\]

Therefore, we have

\[
R(\hat{f}) - R(f_*) \leq \frac{1}{m} \sum_{i=1}^{m} \left[ R(\hat{f}_i) - R(f_*) \right] - \eta \tau.
\]

As follows, we will estimate \( R(\hat{f}_i) - R(f_*) \):

\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{2} \| \hat{f}_i - f_* \|^2_{\mathcal{H}}
\]

\[
\leq \langle \nabla R(\hat{f}_i), \hat{f}_i - f_* \rangle_{\mathcal{H}}
\]

\[
= \langle \nabla R(\hat{f}_i) - \nabla R(f_*), \hat{f}_i - f_* \rangle_{\mathcal{H}}
+ \langle \nabla R(f_*), \hat{f}_i - f_* \rangle_{\mathcal{H}} + \langle \nabla R_* f(\hat{f}_i), \hat{f}_i - f_* \rangle_{\mathcal{H}}.
\]

\[
(10)
\]

Note that \( \ell(\cdot, z) \) is convex, thus \( \hat{R}_t(\cdot) \) is convex. By the convexity of \( \hat{R}_t(\cdot) \) and the optimality condition of \( \hat{f}_i \), we have

\[
\langle \nabla \hat{R}_t(\hat{f}_i), f - \hat{f}_i \rangle_{\mathcal{H}} \geq 0, \forall f \in \mathcal{H}.
\]

Thus, we get

\[
\langle \nabla \hat{R}_t(\hat{f}_i), \hat{f}_i - f_* \rangle_{\mathcal{H}} \leq 0.
\]

Substituting the above equation into (10), we have

\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{2} \| \hat{f}_i - f_* \|^2_{\mathcal{H}}
\]

\[
\leq \langle \nabla R(\hat{f}_i) - \nabla R(f_*), \hat{f}_i - f_* \rangle_{\mathcal{H}}
+ \langle \nabla R(f_*), \hat{f}_i - f_* \rangle_{\mathcal{H}}
\]

\[
\leq \left\| \nabla R(\hat{f}_i) - \nabla R(f_*) \right\|_{\mathcal{H}} \cdot \left\| \hat{f}_i - f_* \right\|_{\mathcal{H}}
+ \left\| \nabla R(f_*) - \nabla R_* f(\hat{f}_i) \right\|_{\mathcal{H}} \cdot \left\| \hat{f}_i - f_* \right\|_{\mathcal{H}}.
\]

\[
(12)
\]

As follows, we utilize the covering number to establish an upper bound for the first term in the last line of (12). The second term in the last line of (12) is upper bounded by the concentration inequality.

B. Proof of Theorem 1

To prove Theorem 1, we first introduce a lemma of [27], and then provide two other lemmas.

Lemma 1 (Lemma 2 of [27]). Let \( \mathcal{H} \) be a Hilbert space and \( \xi \) be a random variable on \( (\mathcal{Z}, \rho) \) with values in \( \mathcal{H} \). Assume

\[
\| \xi \|_{\mathcal{H}} \leq \tilde{M} < \infty
\]

almost surely. Denote

\[
\sigma^2(\xi) = \mathbb{E}(\| \xi \|^2_{\mathcal{H}}).
\]

Let \( \{z_i\}_{i=1}^l \) be independent random drawers of \( \rho \). For any \( 0 < \delta < 1 \), with confidence \( 1 - \delta \),

\[
\left\| \frac{1}{l} \sum_{i=1}^{l} [\xi_i - \mathbb{E}(\xi_i)] \right\| \leq \frac{2\tilde{M} \log(2/\delta)}{l} + \sqrt{\frac{2\sigma^2(\xi) \log(2/\delta)}{l}}.
\]
Lemma 2. If the loss function $\ell$ is a $G$-smooth and convex function, then for any $f \in \mathcal{N}(\mathcal{H}, \epsilon)$, with a probability of at least $1 - \delta$, we have

$$
\left\| \nabla R(f) - \nabla R(f_*) - \left[ \nabla \hat{R}_s(f) - \nabla \hat{R}_s(f_*) \right] \right\|_{\mathcal{H}} \\
\leq \frac{Gm\|f - f_*\|_{\mathcal{H}D_{\mathcal{H}, \delta, \epsilon}}}{N} + \sqrt{\frac{Gm(R(f) - R(f_*))D_{\mathcal{H}, \delta, \epsilon}}{N}},
$$

where $D_{\mathcal{H}, \delta, \epsilon} = 2\log(2C(\mathcal{H}, \epsilon)/\delta)$.

Proof. Note that $\ell$ is $G$-smooth and convex, so by (2.1.7) of [37], $\forall z \in \mathcal{Z}$, we have

$$
\|\nabla \ell(f, z) - \nabla \ell(f_*, z)\|^2_{\mathcal{H}} \\
\leq G(\ell(f, z) - \ell(f_*, z) - \langle \nabla \ell(f, z), f - f_*\rangle_{\mathcal{H}}).
$$

Taking the expectation over both sides, we have

$$
\mathbb{E}_z \left[ \|\nabla \ell(f, z) - \nabla \ell(f_*, z)\|^2_{\mathcal{H}} \right] \\
\leq G \left( R(f) - R(f_*) - \langle \nabla R(f_*) - \nabla R(f_*) \rangle_{\mathcal{H}} \right) \quad (14)
$$

where the last inequality follows from the optimality condition of $f_*$, i.e.,

$$
\langle \nabla R(f_*), f - f_*\rangle_{\mathcal{H}} \geq 0, \forall f \in \mathcal{H}.
$$

Note that $\ell(f, z)$ is $G$-smooth, thus we have

$$
|\nabla \ell(f, z) - \nabla \ell(f_*, z)| \leq G\|f - f_*\|_{\mathcal{H}}, \forall f \in \mathcal{H}. \quad (15)
$$

Substituting (14) and (15) into Lemma 1 with $\xi_i = \nabla \ell(f, z_i) - \nabla \ell(f_*, z_i)$, we have

$$
\left\| \nabla R(f) - \nabla R(f_*) - \left[ \nabla \hat{R}_s(f) - \nabla \hat{R}_s(f_*) \right] \right\|_{\mathcal{H}} \\
= \frac{1}{N} \sum_{i=1}^{n} \left\| \mathbb{E}[\xi_i] - \xi_i \right\| \\
\leq \frac{2mG\|f - f_*\|_{\mathcal{H}D_{\mathcal{H}, \delta, \epsilon}}}{N} + \sqrt{\frac{2mG(R(f) - R(f_*))D_{\mathcal{H}, \delta, \epsilon}}{N}}.
$$

We obtain Lemma 2 by taking the union bound over all $f \in \mathcal{N}(\mathcal{H}, \epsilon)$.

Lemma 3. Under Assumption 3 with a probability of at least $1 - \delta$, we have

$$
\left\| \nabla R(f_*) - \nabla \hat{R}_s(f_*) \right\|_{\mathcal{H}} \leq \frac{2Lm \log(2/\delta)}{N} + \sqrt{\frac{8G\epsilon \log(2/\delta)}{N}}. \quad (16)
$$

Proof. Since $\ell(f, \cdot)$ is $G$-smooth and non-negative, from Lemma 4 of [38], we have

$$
\|\nabla \ell(f_*, z_i)\|^2_{\mathcal{H}} \leq 4G\ell(f_*, z_i)
$$

and thus we can get

$$
\mathbb{E}_z \left[ \|\nabla \ell(f_*, z)\|^2_{\mathcal{H}} \right] \\
\leq 4G\mathbb{E}_z[\ell(f_*, z)] = 4G\epsilon.
$$

Since $\ell(f, \cdot)$ is a $L$-Lipschitz continuous function, we have

$$
\|\ell(f_*, z) - \ell(f_*, z)\|_{\mathcal{H}} \leq L\|\delta f\|_{\mathcal{H}}, \forall \delta f \in \mathcal{H}.
$$

If $\|\delta f\|_{\mathcal{H}} \rightarrow 0$, from the definition of differential of $\ell(f_*, z)$, we can obtain that

$$
\|\nabla \ell(f_*, z)\|_{\mathcal{H}} \leq L.
$$

Lemma 4. If the loss function $\ell$ is $G$-smooth and convex function, then for any $f \in \mathcal{N}(\mathcal{H}, \epsilon)$, with a probability of at least $1 - \delta$, we have

$$
\left\| \nabla R(f) - \nabla R(f_*) - \left[ \nabla \hat{R}_s(f) - \nabla \hat{R}_s(f_*) \right] \right\|_{\mathcal{H}} \\
\leq \frac{2Lm \log(2/\delta)}{N} + \sqrt{\frac{8G\epsilon \log(2/\delta)}{N}}.
$$

Proof of Theorem 7. From the properties of $\epsilon$-covering, we know that there exists a function $\hat{f} \in \mathcal{N}(\mathcal{H}, \epsilon)$ such that

$$
\|\hat{f} - \hat{f}\|_{\mathcal{H}} \leq \epsilon.
$$

(17)
Thus, we have
\[
\left\| \nabla R(\hat{f}_i) - \nabla R(f_*) - \left[ \nabla \hat{R}_i(\hat{f}_i) - \nabla \hat{R}_i(f_*) \right] \right\|_H \\
\leq \left\| \nabla R(\hat{f}) - \nabla R(f_*) - \left[ \nabla \hat{R}_i(\hat{f}) - \nabla \hat{R}_i(f_*) \right] \right\|_H \\
+ \left\| \nabla R(f_*) - \nabla R(\hat{f}) \right\|_H \\
\leq \left\| \nabla R(\hat{f}) - \nabla R(f_*) - \left[ \nabla \hat{R}_i(\hat{f}) - \nabla \hat{R}_i(f_*) \right] \right\|_H \\
+ 2G\|\hat{f}_i - \hat{f}\|_H \quad \text{(by } G\text{-smooth)} \\
\leq \frac{GD_{\mathcal{H}, \delta, \epsilon}\|\hat{f}_i - f_*\|_\mathcal{H} m}{N} \\
+ \frac{GD_{\mathcal{H}, \delta, \epsilon}\|\hat{f}_i - f_*\|_\mathcal{H} m}{N} \\
+ \frac{G\|\hat{f}_i - \hat{f}\|_\mathcal{H} m}{N} + 2G\epsilon \quad \text{(by } L\text{-Lipschitz)}.
\]
Substituting (18) and (16) into (12), with a probability of at least 1 - 2\delta, we have
\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{2}\|\hat{f}_i - f_*\|_\mathcal{H}^2 \\
\leq \frac{GD_{\mathcal{H}, \delta, \epsilon}\|\hat{f}_i - f_*\|_\mathcal{H}^2 m}{N} \\
+ \frac{GD_{\mathcal{H}, \delta, \epsilon}\|\hat{f}_i - f_*\|_\mathcal{H} m}{N} \\
+ 2G\|\hat{f}_i - f_*\|_\mathcal{H} + 2L\|\hat{f}_i - f_*\|_\mathcal{H} \sqrt{\frac{GD_{\mathcal{H}, \delta, \epsilon}(R(\hat{f}_i) - R(f_*))}{N}} \\
+ \|\hat{f}_i - f_*\|_\mathcal{H} \sqrt{\frac{G\|\hat{f}_i - f_*\|_\mathcal{H} m}{N}} \\
+ \|\hat{f}_i - f_*\|_\mathcal{H} \sqrt{\frac{2L\log^2(\frac{2}{\delta})}{N}} \\
+ \|\hat{f}_i - f_*\|_\mathcal{H} \sqrt{\frac{8GR_\epsilon m\log(\frac{1}{\epsilon})}{N}}.
\]
Note that
\[
\sqrt{ab} \leq \frac{a}{2c} + \frac{bc}{2}, \forall a, b, c > 0.
\]
Therefore, we have
\[
\|\hat{f}_i - f_*\|_\mathcal{H} \sqrt{\frac{G\log D_{\mathcal{H}, \delta, \epsilon}(R(\hat{f}_i) - R(f_*))}{N}} \\
\leq \frac{2GD_{\mathcal{H}, \delta, \epsilon}(R(\hat{f}_i) - R(f_*))}{N\eta} + \frac{\eta}{8}\|\hat{f}_i - f_*\|_\mathcal{H}^2; \\
2L\log^2(\frac{2}{\delta})\|\hat{f}_i - f_*\|_\mathcal{H} m \leq \frac{16L^2m^2\log^2(\frac{2}{\delta})}{N^2\eta} + \frac{\eta}{16}\|\hat{f}_i - f_*\|_\mathcal{H}^2; \\
2G\|\hat{f}_i - f_*\|_\mathcal{H} \leq \frac{64G^2\log(\frac{2}{\delta})m}{N\eta} + \frac{\eta}{32}\|\hat{f}_i - f_*\|_\mathcal{H}^2; \\
2G\|\hat{f}_i - f_*\|_\mathcal{H} \leq \frac{32G\|\hat{f}_i - f_*\|_\mathcal{H} m}{N\eta} + \frac{\eta}{128}\|\hat{f}_i - f_*\|_\mathcal{H}^2.
\]
Substituting the above inequalities into (19), we have
\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{2}\|\hat{f}_i - f_*\|_\mathcal{H}^2 \\
\leq \frac{GD_{\mathcal{H}, \delta, \epsilon}\|\hat{f}_i - f_*\|_\mathcal{H}^2 m}{N} \\
+ \frac{2GD_{\mathcal{H}, \delta, \epsilon}(R(\hat{f}_i) - R(f_*))}{N\eta} \\
+ \frac{16L^2m^2\log^2(\frac{2}{\delta})}{N^2\eta} + \frac{64G^2\log(\frac{2}{\delta})m}{N\eta} \\
+ \frac{32G\|\hat{f}_i - f_*\|_\mathcal{H} m}{N\eta} + \frac{32G^2\|\hat{f}_i - f_*\|_\mathcal{H} m}{N^2\eta}. \quad \text{(20)}
\]
From Assumption [1] we know that $\log C(\mathcal{H}, \epsilon) \approx h \log(1/\epsilon)$. Thus, we can obtain that
\[
D_{\mathcal{H}, \delta, \epsilon} = 2h \log \left( \frac{2}{\delta \epsilon} \right). \quad \text{(21)}
\]
If we set \( \epsilon = \frac{3}{4} \), substituting (21) into (20), we have

\[
\frac{R(\hat{f}_i) - R(f_*) + \frac{\eta}{4} \| \hat{f}_i - f_* \|_H^2}{N^2} \leq \frac{2Gh log(2N/\delta) \| \hat{f}_i - f_* \|_H^2}{N} + \frac{4Gh log(2N/\delta)(R(\hat{f}_i) - R(f_*))}{N^2} + \frac{16L^2 m^2 log^2(2\delta)}{N^2} + \frac{64GR_m log(\frac{2}{\delta})}{N^2} + \frac{64G^2}{N^2} + \frac{64GLm h log(2N/\delta)}{N^2} + \frac{128Gm^2 h^2 log^2(2N/\delta)}{N^4}.
\]

Thus, when \( m \leq \frac{N\eta}{6Gh log(2N/\delta)} \), one can obtain that

\[
R(\hat{f}_i) - R(f_*) \leq \frac{\eta}{4} \| \hat{f}_i - f_* \|_H^2 + \frac{1}{2}(R(\hat{f}_i) - R(f_*)) + \frac{16L^2 m^2 log^2(\frac{2}{\delta})}{N^2} + \frac{64GR_m log(\frac{2}{\delta})}{N^2} + \frac{64G^2}{N^2} + \frac{64GLh log(2N/\delta)m}{N^2} + \frac{128Gh log^2(2N/\delta)m^2}{N^4}.
\]

Thus, we have

\[
R(\hat{f}_i) - R(f_*) \leq \frac{32L^2 m^2 log^2(\frac{2}{\delta})}{N^2} + \frac{128GR_m log(\frac{2}{\delta})}{N^2} + \frac{128G^2}{N^2} + \frac{128GLh log(2N/\delta)m}{N^2} + \frac{256Gh log^2(2N/\delta)m^2}{N^4}.
\]

Substituting (22) into (9), we have

\[
R(\hat{f}) - R(f_*) \leq \frac{32L^2 m^2 log^2(\frac{2}{\delta})}{N^2} + \frac{128GR_m log(\frac{2}{\delta})}{N^2} + \frac{128G^2}{N^2} + \frac{128GLh log(2N/\delta)m}{N^2} + \frac{256Gh log^2(2N/\delta)m^2}{N^4} - \eta \tau.
\]

Note that when

\[
m \leq \min \left\{ \frac{h\eta + N\eta^2 \tau}{128GR_m log(2N/\delta)}, \sqrt{\frac{N\eta}{4}}, \frac{N\eta}{2GLh log(2N/\delta)} \right\},
\]

one can obtain

\[
\frac{128GR_m log \frac{2}{\delta}}{N^2} - \eta \tau \leq \frac{h}{N}, \quad \frac{L^2 m^2 log^2 \frac{2}{\delta}}{N^2} \leq \frac{h}{N}, \quad \frac{GLh log(2N/\delta)m}{N^2} \leq \frac{h}{N}.
\]

Therefore, substituting the above equations into (23), we have

\[
R(\hat{f}) - R(f_*) \leq O \left( \frac{h}{N} + \frac{1}{N^2} + \frac{hm^2 log^2(N)}{N^4} \right) = O \left( \frac{h}{N} \right).
\]

C. Proof of Theorem 3

Proof. According to Assumption 2, we know that

\[
log C(H, \epsilon) \simeq (1/\epsilon)^{1/2}.
\]

Thus, when setting \( \epsilon = N^{-\frac{1}{2\alpha + 1}} \), one can see that

\[
D_{H, \epsilon, \delta} = 2 \left( \log C(H, \epsilon) + \log \frac{2}{\delta} \right) = 2 \left( N^{\frac{1}{2\alpha + 1}} + \log \frac{2}{\delta} \right).
\]

From (20) with \( \epsilon = N^{-\frac{1}{2\alpha + 1}} \), we have

\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{4} \| \hat{f}_i - f_* \|_H^2 \leq \frac{GD_{H, \epsilon, \delta} \| \hat{f}_i - f_* \|_H^2}{N} + \frac{2GD_{H, \epsilon, \delta}(R(\hat{f}_i) - R(f_*))}{N^2} + \frac{16L^2 m^2 log^2(\frac{2}{\delta})}{N^2} + \frac{64GR_m log(\frac{2}{\delta})}{N^2} + \frac{64G^2}{N^2} + \frac{64GL_{H, \epsilon, \delta}}{N^2} \quad \eta N^{\frac{1}{2\alpha + 1}} + \frac{32GLmD_{H, \epsilon, \delta}}{\eta N^{\frac{1}{2\alpha + 1}}} + \frac{32Gm^2 D_{H, \epsilon, \delta}}{\eta N^{\frac{1}{2\alpha + 1}}.}
\]

Thus, when

\[
m \leq \frac{N\eta}{4D_{H, \epsilon, \delta}} = \frac{N\eta}{4(N^{\frac{1}{2\alpha + 1}} + \log(2/\delta))},
\]

we have

\[
R(\hat{f}_i) - R(f_*) + \frac{\eta}{4} \| \hat{f}_i - f_* \|_H^2 \leq \frac{\eta}{4} \| \hat{f}_i - f_* \|_H^2 + \frac{1}{2} \left( R(\hat{f}_i) - R(f_*) \right) + \frac{16L^2 m^2 log^2(\frac{2}{\delta})}{N^2} + \frac{64GR_m log(\frac{2}{\delta})}{N^2} + \frac{64G^2}{N^2} + \frac{64GLmD_{H, \epsilon, \delta}}{N^2} \quad \eta N^{\frac{1}{2\alpha + 1}} + \frac{32GLmD_{H, \epsilon, \delta}}{N^2} \quad \eta N^{\frac{1}{2\alpha + 1}} + \frac{32Gm^2 D_{H, \epsilon, \delta}}{N^2} \quad \eta N^{\frac{1}{2\alpha + 1}}.
\]
Thus, one can obtain that

\[ R(\hat{f}) - R(f_*) \leq \frac{32L^2m^2 \log^2(\frac{2}{\delta})}{N^2 \eta} + 128GR_* m \log(\frac{2}{\delta}) \]
\[ + \frac{128G^2}{N \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}} + \frac{64GLmD_{H,\delta,\epsilon}}{N \frac{\log(\frac{2}{\delta})}{N^2 \eta}} + \frac{64Gm^2D_{H,\delta,\epsilon}^2}{N^2 \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}}. \]

Substituting the above inequality into (29), we have

\[ R(\hat{f}) - R(f_*) \leq \frac{32L^2m^2 \log^2(\frac{2}{\delta})}{N^2 \eta} + 128GR_* m \log(\frac{2}{\delta}) \]
\[ + \frac{128G^2}{N \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}} + \frac{64GLmD_{H,\delta,\epsilon}}{N \frac{\log(\frac{2}{\delta})}{N^2 \eta}} \]
\[ + \frac{64Gm^2D_{H,\delta,\epsilon}^2}{N^2 \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}} - N^2 \eta. \]

Note that,

\[ D_{H,\delta,\epsilon} = 2 \left( N \frac{\log^2(\frac{2}{\delta})}{N^2 \eta} + \log \frac{2}{\delta} \right) \leq 2N \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}. \]

Thus, from (26), we can obtain that

\[ R(\hat{f}) - R(f_*) \leq \frac{32L^2m^2 \log^2(\frac{2}{\delta})}{N^2 \eta} + 128GR_* m \log(\frac{2}{\delta}) \]
\[ + \frac{128G^2}{N \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}} + \frac{64GLmD_{H,\delta,\epsilon}}{N \frac{\log(\frac{2}{\delta})}{N^2 \eta}} \]
\[ + \frac{64Gm^2D_{H,\delta,\epsilon}^2}{N^2 \frac{\log^2(\frac{2}{\delta})}{N^2 \eta}} - \eta N. \]

Note that, when

\[ m \leq \min \left\{ \frac{\sqrt{\eta N \frac{\log^2(2/\delta)}{L \log(2/\delta)}}}{\eta N \frac{\log^2(2/\delta)}{L \log(2/\delta)}}, \frac{\eta N \frac{\log^2(2/\delta)}{L \log(2/\delta)} + \eta N \frac{\log^2(2/\delta)}{L \log(2/\delta)}}{128R_* m \log(2/\delta)} \right\}, \]

one can obtain that

\[ L^2m^2 \log^2(2/\delta) \leq N^{-\frac{m}{2N \eta}}, \]
\[ GLm \log(2/\delta) \leq N^{-\frac{m}{2N \eta}}, \]
\[ 128GR_* m \log(2/\delta) \leq N^{-\frac{m}{2N \eta}}. \]

Substituting (28) into (27), we have

\[ R(\hat{f}) - R(f_*) \leq O \left( N^{-\frac{2m}{2N \eta}} + \frac{m^2 \log^2(2/\delta)}{N^2 \eta} \right). \]

By (28), we know that

\[ O \left( \frac{m \log^2(2/\delta)}{N^2 \eta} \right) \leq O \left( N^{-\frac{m}{2N \eta}} \right). \]

Thus, we have

\[ R(\hat{f}) - R(f_*) \leq O \left( N^{-\frac{2m}{2N \eta}} + N^{-\frac{m}{2N \eta}} \right) = O \left( N^{-\frac{m}{2N \eta}} \right). \]

\[ \square \]

D. Proof of Theorem 5

Proof. We set \( \eta = 0 \) in (19), and obtain that

\[ R(\hat{f}) - R(f_*) \leq \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_2^2}{N} + \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H^2}{N} \]
\[ + 2GL \| \hat{f} - f_* \|_H \left( \sqrt{GD_{H,\delta,\epsilon}(R(\hat{f}) - R(f_*))} m \right) \]
\[ + \frac{LmD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{N} \]
\[ + \frac{2LmD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{N}. \]

(29)

Note that

\[ \sqrt{ab} \leq \frac{a}{2c} + \frac{bc}{2}, \forall a, b, c \geq 0. \]

Thus, we have

\[ \| \hat{f} - f_* \|_H \leq \frac{\sqrt{GD_{H,\delta,\epsilon}(R(\hat{f}) - R(f_*))} m}{N} \]
\[ + \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_2^2}{2N} + \frac{R(\hat{f}) - R(f_*)}{2N} \]
\[ \leq \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_2^2}{2N} + \frac{R(\hat{f}) - R(f_*)}{2N} \]
\[ \leq \frac{LmD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{2N} + \frac{Ge}{2}. \]

(30)

Substituting (30) into (29), we get

\[ \frac{1}{2} (R(\hat{f}) - R(f_*)) \leq \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_2^2}{N} + \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{N} \]
\[ + 2GL \| \hat{f} - f_* \|_H + \frac{GD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_2^2}{2N} \]
\[ + \frac{LmD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{2N} + \frac{Ge}{2} \]
\[ + \frac{2LmD_{H,\delta,\epsilon} \| \hat{f} - f_* \|_H}{N} \]
\[ + \frac{\| \hat{f} - f_* \|_H}{N}. \]

(31)
Thus, from (31), we have

\[ R(\tilde{f}_i) - R(f_*) \leq \frac{8G^2 B^2 m D_{\mathcal{H}, \delta, \epsilon}}{N} + \frac{4BG m D_{\mathcal{H}, \delta, \epsilon}}{N} + \frac{8BG \epsilon}{N} + \frac{4BG m D_{\mathcal{H}, \delta, \epsilon}}{N} + \frac{4B^2 G m D_{\mathcal{H}, \delta, \epsilon}}{N} + \frac{G \epsilon}{N} \quad (32) \]

Note that

\[ D_{\mathcal{H}, \delta, \epsilon} = \frac{2}{\delta} \left( \log C(\mathcal{H}, \epsilon) + \log 2/\delta \right) = \frac{2h \log(2/\delta)}{N} \]

From (32) with \( \epsilon = 1/N \), we have

\[ R(\tilde{f}_i) - R(f_*) \leq O \left( \frac{mh \log N}{N^2} + \frac{mR_* \log \frac{1}{\delta}}{N} \right). \]

Substituting the above equation into (9) with \( \eta = 0 \), we have

\[ R(\tilde{f}) - R(f_*) \leq R(\tilde{f}_i) - R(f_*) \leq O \left( \frac{mh \log N}{N^2} + \frac{mR_* \log \frac{1}{\delta}}{N} \right). \]

So, when \( m \leq O(N^r) \), we can get,

\[ R(\tilde{f}) - R(f_*) \leq O \left( h \log \frac{N}{N^{1-r}} + \frac{R_* \log \frac{1}{\delta}}{N^{1-r}} \right). \]

If the optimal risk \( R_* \leq O \left( N^{r-1} \right) \), then

\[ \frac{R_* \log \frac{1}{\delta}}{N^{1-r}} \leq O \left( \frac{1}{N^{1-r}} \right). \]

Thus, in this case, we have

\[ R(\tilde{f}) - R(f_*) \leq O \left( h \log \frac{N}{N^{1-r}} \right). \]

\[ \square \]

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