Anisotropic Inverse Problems for Quasilinear Elliptic Equations

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Abstract. In this paper we discuss inverse boundary value problems for quasilinear elliptic equations in connection with the nonlinear conductivity problem and inverse problems in nonlinear material sciences.

1. Introduction
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $A(x, z, p) \in C^{1,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ be an elliptic quasilinear coefficient with appropriate structure conditions that guarantee the unique solvability of a solution $u \in C^{2,\alpha}(\bar{\Omega})$ for the boundary value problem

$$L_A(u) = \nabla \cdot A(x, u, \nabla u) = 0, \quad u|_{\partial \Omega} = f \in C^{2,\alpha}(\Omega). \quad (1.1)$$

The Dirichlet to Neumann (DN) map for the boundary value problem (1.1) is defined as the map: $f \rightarrow \Lambda_A f$:

$$\Lambda_A f = \nu \cdot A(x, f, \nabla u)|_{\partial \Omega}, \quad (1.2)$$

where $\nu$ is the outer normal at $\partial \Omega$. The inverse problem is to recover information about $A$ from $\Lambda_A$.

Physically, this problem is a mathematical model for several inverse problems in nonlinear material sciences. The DN map encodes the physical measurements at the boundary, and the inverse problem is to recover interior physical information about the material.

However, it is impossible to recover the coefficient $A$ itself because of the following invariance property of the DN map: the DN map is invariant under a $C^{2,\alpha}(\bar{\Omega})$ diffeomorphism $\Phi : \Omega \rightarrow \Omega$ with $\Phi|_{\partial \Omega} = Identity$. Indeed, for a given diffeomorphism $\Phi$,

$$\Lambda_{\Phi \circ A} = \Lambda_A, \quad (1.3)$$

where $\Phi \circ A$ is the push forward of $A$ defined by

$$\Phi \circ A(x, z, p) = [(D\Phi)^{-1} (D\Phi)^T A(x, z, (D\Phi)p)] \circ \Phi^{-1}(x). \quad (1.4)$$

We have the following
Conjecture:

Let $A_1$ and $A_2$ be two elliptic quasilinear coefficients in $C^{1,\alpha}(\Omega)$. If

$$\Lambda_{A_1} = \Lambda_{A_2},$$

then there exists a unique diffeomorphism $\Phi : \Omega \to \Omega$ in the $C^{2,\alpha}$ class with $\Phi|_{\partial \Omega} = \text{identity}$ so that

$$A_2 = \Phi_* A_1.$$

We now give a brief review of the above conjecture in the linear case. In the linear case the boundary value problem (1.1) is given by

$$L_A(u) = \nabla \cdot A(x) \nabla u = 0, \quad u|_{\partial \Omega} = f \in C^{2,\alpha}(\Omega)$$

and the Dirichlet to Neumann (DN) map: $f \to \Lambda_A f$:

$$\Lambda_A f = \nu \cdot A(x) \nabla u|_{\partial \Omega}.$$

Then

$$\Lambda_{\Phi_* A} = \Lambda_A,$$

where $\Phi_* A$ is the push forward of $A$ defined by

$$\Phi_* A(x) = [[D\Phi]^{-1}(D\Phi)^T A(x)(D\Phi)] \circ \Phi^{-1}(x).$$

In the linear case, this conjecture was proved in dimension two by Sylvester [S] and Nachman [N] for coefficient $A$ in the $C^{2,\alpha}$ category and by Brown-Uhlmann [BU] and Sun-Uhlmann [SuU2] in the $W^{1,p}$ category. In the case of dimension three, if $\Omega$ is simply connected and the coefficients are real analytic, with one of analytically extendable to $\mathbb{R}^n$, then the conjecture was proved by Lee-Uhlmann [LU]. We refer the reader to [U] for the history of inverse boundary value problems originated in early 80’s from the contribution of A.P. Calderon [C].

In this paper, we shall discuss the above conjecture for the inverse boundary value problem for the quasilinear equation (1.1) with certain special structures. We shall discuss the nonlinear conductivity problem in Section 2 and inverse problems in nonlinear material sciences in Section 3.

2. The Nonlinear Conductivity Problem

For the nonlinear conductivity problem [KV2] we consider the boundary value problem

$$L_A(u) = \nabla \cdot A(x, u) \nabla u = 0, \quad u|_{\partial \Omega} = f \in C^{2,\alpha}(\Omega)$$

and the DN map

$$\Lambda_A f = \nu \cdot A(x, f) \nabla u|_{\partial \Omega}.$$

The transformation (1.4) is now given by

$$\Phi_* A(x, z) = [[D\Phi]^{-1}(D\Phi)^T A(x, z)(D\Phi)] \circ \Phi^{-1}(x).$$

It has been proven in [SuU1] that all the known results for the linear case also hold in this quasilinear case.
**Theorem 2.1.** For $n = 2$: if $A_1, A_2$ in $C^2,\alpha(\Omega \times R)$ with $\Lambda_{A_1} = \Lambda_{A_2}$, then there exists a unique $\Phi \in C^3,\alpha(\Omega)$ such that $A_2 = \Phi_* A_1$. For $n \geq 3$: if $\Omega$ simply connected with real analytic boundary and $A_1, A_2$ real analytic (extendable to $R^n \times R$), then $\Lambda_{A_1} = \Lambda_{A_2}$ implies the existence of a unique $\Phi$ in the analytic category such that $A_2 = \Phi_* A_1$.

**Remark:** The uniqueness of $\Phi$ follows directly from the linear result. The main issue is the existence of $\Phi$.

**An Outline of Proof**

- **Linearization:** We collect information about $A(x, z)$ by differentiating the nonlinear map $\Lambda_A$. Let $t \in R$ be a fixed constant. We have, for any $f \in C^2,\alpha(\partial \Omega)$:
  \[
  \frac{d}{ds} \Lambda_A(t + sf)|_{s=0} = \lim_{s \to 0} \frac{1}{s} \Lambda_A(t + sf) = \Lambda_{A'}(f),
  \]
  where $\Lambda_{A'}$ is the DN map for the linearized equation with the linear coefficient $A'(x) = A(x, t)$: $\nabla \cdot A'(x) \nabla u = 0$.
  So, from $\Lambda_A$ one can recover $\Lambda_{A'}$.

**Remark:** The linearization technique was introduced by Isakov in [I1] and further developed in [I2][IS][I3][Su1,3].

- Thus, $\Lambda_{A_1} = \Lambda_{A_2} \Rightarrow \Lambda_{A'_1} = \Lambda_{A'_2}, \forall t \in R.$ By the result in the linear case, there exists a unique $\Phi^t$ such that
  \[
  A'_2 = \Phi^t_* A'_1, \forall t \in R.
  \]
  One can show that $\Phi^t$ depends smoothly on $t$ [SuU1][AB]. Thus, it is reduced to prove that $\Phi^t$ is independent of $t$ by showing $\Phi^0 = 0$, where $\cdot$ represents the derivative in $t$. However, the information provided by the first derivative of the DN map is insufficient to achieve this goal.

- We look for more information about $A(x, z)$ by computing the 2nd derivative of $\Lambda_A(t + sf)$:
  \[
  \frac{d}{ds} \frac{1}{s} \Lambda_A(t + sf)|_{s=0} = \lim_{s \to 0} \frac{1}{s} \frac{1}{s} \Lambda_A(t + sf) - \Lambda_{A'}(f) = K_{A,t}(f),
  \]
  where the operator $K_{A,t}$ contains rich information about the derivative $A^t = \frac{d}{dt} A'(t = 0)$ and is implicitly defined by
  \[
  \int_{\partial \Omega} f_1 K_{A,t}(f_2) ds = \frac{1}{2} \int_{\Omega} \nabla v_1 \cdot A^0 \nabla (v_2^2) dx
  \]
  with $v_i$ the unique solution to the linear equation $\nabla \cdot A^0(x) \nabla v_i = 0, v_i|_{\partial \Omega} = f_i, i = 1, 2$.

- Given $A_1, A_2$ with $\Lambda_{A_1} = \Lambda_{A_2}$. Without the loss of generality, assume $A^0_1 = A^0_2 = A$ (one can achieve this by redefining $A_1$ by $\Phi^t_* A_1$ and replacing the family of diffeomorphisms $\Phi^t$ by $\Phi^t \circ (\Phi^0)^{-1}$). Then, for $B(x) = A^0_1(x) - A^0_2(x)$, we obtain by polarization a control over the function $B$:
  \[
  \int_{\Omega} \nabla v_1 \cdot B(x) \nabla (v_2 v_3) dx = 0
  \]
  with $v_i$ the unique solution to the linear equation $\nabla \cdot A^0(x) \nabla v_i = 0, v_i|_{\partial \Omega} = f_i, i = 1, 2, 3.$
We view $\nabla v_1 \cdot B(x)$ as a vector function on $\bar{\Omega}$. By using singular solution method [Al], one can show that this vector function, when restricted on $\partial \Omega$, is tangential to $\partial \Omega$. Thus, one can integrate by part to get
\[
\int_{\Omega} \nabla \cdot (B(x)\nabla v_1)v_2 v_3 dx = 0.
\]

By the completeness of solutions for the linear equations (See [No] for $n = 2$ and [KV1] for real analytic case with $n \geq 3$), one then obtain
\[
\nabla \cdot (B(x)\nabla v) = 0
\]
for solution $v$ solving $\nabla \cdot A^0(x)\nabla v = 0$ with $v|_{\partial \Omega} = f$.

Consider the solution flow $v^t_i$, $i = 1, 2$,
\[
\nabla \cdot A^t_i(x)\nabla v^t_i = 0, \quad v^t_1|_{\partial \Omega} = f \in C^{2,\alpha}(\partial \Omega).
\]
Then the above control over $B$ translates to a control over the solution flow:
\[
\nabla \cdot (B(x)\nabla v) + \nabla \cdot A(x)\nabla (v^0_i - v^0_2) = 0, \quad \text{and thus } v^0_i = v^0_2.
\]

Since the transformation $\Phi^t$ also transforms the solution:
\[
v^t_1 = v^t_2 \circ \Phi^t, \quad \forall t,
\]
we have by differentiation that $\dot{v}^t_1 - \dot{v}^t_2 = \dot{\Phi}^t \cdot \nabla v$, which gives that
\[
\dot{\Phi}^0 \cdot \nabla v = 0
\]
for every solution $v$. By the Runge Property of elliptic equations, we conclude that $\dot{\Phi}^0 = 0$.

3. The Inverse Problem in Nonlinear Material Sciences

The boundary value problem involved in nonlinear material sciences takes the form
\[
L_A(u) = \nabla \cdot A(x, \nabla u) = 0, \quad u|_{\partial \Omega} = f \in C^{2,\alpha}(\partial \Omega).
\]
where $A$ satisfies the following structure conditions

A1) $A_p$ is a symmetric matrix and $A_p \in C^{2,\alpha}(\bar{\Omega} \times R^n)$.

A2) $A(x, 0) = 0$, $\forall x \in \bar{\Omega}$.

A3) $\nu (1 + \|p\|)^{m-2} \|\xi\|^2 \leq \xi^T A_p(x, p) \xi \leq \mu (1 + \|p\|)^{m-2} \|\xi\|^2$, $\forall \xi \in R^n$ and $\forall (x, p) \in \bar{\Omega} \times R^n$, for some $m > 1$, where $\nu, \mu \in R^+$.

A4) $\|A\| (1 + \|p\|) + \|A_x\| \leq \mu (1 + \|p\|)^m$, $\forall \xi \in R^n$ and $\forall (x, p) \in \bar{\Omega} \times R^n$.

The DN map and the transformation are in the following forms:
\[
\Lambda_A f = \nu \cdot A(x, \nabla u)|_{\partial \Omega},
\]
\[
\Phi_* A(x, p) = [\|D\Phi|^{-1}(D\Phi)^T A(x, (D\Phi)p)] \circ \Phi^{-1}(x).
\]
The conjecture: given $A_1, A_2$ in $C^{2,\alpha}(\bar{\Omega} \times R^n)$ with $\Lambda_{A_1} = \Lambda_{A_2}$, then there exists a unique $\Phi \in C^{3,\alpha}(\bar{\Omega})$ such that $A_2 = \Phi_* A_1$. This conjecture is open in general. We shall discuss an
approach toward proving this conjecture [HSu].

**An Outline of Proof**

- **Linearization:** Let \( f \in C^{2,\alpha}(\partial \Omega) \) be a reference point. We have, for any \( g \in C^{2,\alpha}(\partial \Omega) \):
  \[
  \frac{d}{ds} \Lambda_A(f + sg)|_{s=0} = \Lambda_{A_p}(x, \nabla u_f) \cdot g,
  \]
  where \( \Lambda_{A_p}(x, \nabla u_f) \) is the DN map for the linearized equation \( \nabla \cdot A_p(x, \nabla u_f) \nabla v = 0 \). So, from \( \Lambda_A \) one can recover \( \Lambda_{A_p}(x, \nabla u_f) \).

- Thus, \( \Lambda_A = \Lambda_B \Rightarrow \Lambda_{A_1,p}(x, \nabla u_{1,f}) = \Lambda_{A_2,p}(x, \nabla u_{2,f}), \forall f \in C^{2,\alpha}(\partial \Omega) \). By the result in the linear case, there exists a unique \( \Phi_f \) such that
  \[
  A_{2,p}(x, \nabla u_{2,f}) = (\Phi_f)_* A_{1,p}(x, \nabla u_{1,f}), \forall t \in \mathbb{R}.
  \]
  Again, one can show that \( \Phi_f \) depends smoothly on \( f \). Thus, it reduced to prove that \( \Phi_f \) is independent of \( f \) by showing
  \[
  \dot{\Phi}_{f,h} = \frac{d}{ds} \Phi_{f+sh}|_{s=0} = 0, \quad \forall h \in C^{2,\alpha}(\partial \Omega).
  \]

**Remark:** In this case, \( (\Phi_f, f \in C^{2,\alpha}(\partial \Omega)) \) is an infinite dimensional family of diffeomorphisms rather than an one-dimensional one in the previous case, the derivative at constant \( f \) will no longer be sufficient. Also, an immediate second differentiation would not provide sufficient new information about the quasilinear coefficient. In a recent work of Kang and Nakamura [KaNa], it has been shown that some partial information about \( A \) can be obtained through the second differentiation.

- Given a quasilinear coefficient matrix \( A(x, p) \) and \( f \in C^{2,\alpha}(\partial \Omega) \), we define a Riemannian metric
  \[
  g_f = A_p^{-1}(x, \nabla u_f)
  \]
on \( \bar{\Omega} \), where \( A_p(x, \nabla u_f) \) is the linearized coefficient matrix. Then the diffeomorphism \( \Phi_f \) becomes a conformal diffeomorphism from \( (\bar{\Omega}, g_{1,f}) \) to \( (\bar{\Omega}, g_{2,f}) \). In fact,
  \[
  \Phi_f^* g_{2,f} = |D\Phi_f| g_{1,f}.
  \]

- For a fixed \( f \), we may assume without the loss of generality (by a transform if needed) that \( \Phi_f = \text{identity} \) and set \( g_{1,f} = g_{2,f} = g_f \) and \( u_{1,f} = u_{2,f} = u_f \). By differentiating the above equality in a fixed direction \( h \in C^{2,\alpha}(\partial \Omega) \), we have
  \[
  \dot{g}_{1,f,h} - \dot{g}_{2,f,h} = L_X g_f - (e^\sigma \nabla g_f \cdot (e^{-\sigma} X)) g_f,
  \]
  where \( \dot{g}_{f,h} = \frac{d}{ds} g_{f+sh}|_{s=0}, \sigma = \frac{1}{2} \log|g|, X = \dot{\Phi}_{f,h}, \) and \( L_X \) stands for the Lie derivative under the vector field \( X \). From this equality we wish to prove that \( X = 0 \). This equation is the starting point of the rest of our analysis.
• However, this equality has no real implication if only one direction \( h \) is considered. The observation is that one needs to consider a pair of directions \( h_1, h_2 \in C^{2,\alpha}(\partial\Omega) \) because of the following intertwining relation through the gradients of solutions:

\[
\dot{g}_{f,h_1}l_{f,h_2} = \dot{g}_{f,h_2}l_{f,h_1},
\]

where \( g = g_1 \) or \( g_2 \) and \( l_{f,h} = g_f^{-1}\nabla u_{f,h} \) is the gradient of solution under the metric \( g \).

• Thus, for a pair of directions \( h_1, h_2 \in C^{2,\alpha}(\partial\Omega) \),

\[
\dot{l}_{f,h_1}(L X_1 g_f - (e^\sigma \nabla g_f \cdot (e^{-\sigma} X_1)) g_f) = \dot{l}_{f,h_2}(L X_2 g_f - (e^\sigma \nabla g_f \cdot (e^{-\sigma} X_2)) g_f),
\]

where \( X_i = \dot{\Phi}_{f,h_i}, \ i = 1, 2 \).

• Consider the inhomogeneous conformal Killing field equation:

\[
l (L X - (e^\sigma \cdot (e^{-\sigma} X)) g) = F
\]

where the \( F \) is a 1-form. A geometrical analysis is needed here to find information about the vector field \( X \) and \( l \) from the given 1-form \( F \). It has been shown that both inner products \( \langle X, l \rangle_g \) and \( \langle X, l^\perp \rangle_g \) are uniquely determined by \( F \), where \( l^\perp \) stands for the unique vector perpendicular to \( l \) with \( \|l^\perp\| = \|l\| \) in the counterclockwise direction under the metric \( g \) [Su2].

**Remark:** In fact, by using the moving frame method, one can show that for the function \( p = \langle X, l \rangle_g \) and \( q = \langle X, l^\perp \rangle_g \), the following system hold:

\[
\Delta_g q + \nabla_g \cdot (pq) = -\nabla_g \cdot F^\perp,
\]

\[
\Delta_g p + \nabla_g \cdot (q^\perp) = \nabla_g \cdot F.
\]

Since \( p = q = 0 \) on \( \partial\Omega \), we have that both \( p \) and \( q \) are determined by \( F \).

• Hence, we conclude that the vector fields \( X_i \) and \( l_{f,h_i} \) must satisfy the following system of equations:

\[
\begin{align*}
\langle X_1, l_{f,h_2} \rangle_{g_f} &= \langle X_2, l_{f,h_1} \rangle_{g_f} \\
\langle X_1, l_{f,h_2}^\perp \rangle_{g_f} &= \langle X_2, l_{f,h_1}^\perp \rangle_{g_f}.
\end{align*}
\]

• Consider a two-parameter family of conformal diffeomorphisms \( \Phi_{f+\eta_1 h_1+\eta_2 h_2} \) with parameters \( \eta_1 \) and \( \eta_2 \) in \( R \). Fixed \( x \in \Omega \) and define

\[
\omega(\eta_1, \eta_2) = \Phi_{f+\eta_1 h_1+\eta_2 h_2}(x) : R^2 \to \bar{\Omega}
\]

as a function of \( (\eta_1, \eta_2) \). Then, for \( i = 1, 2 \),

\[
\omega_{\eta_i} = \dot{\Phi}_{f+\eta_1 h_1+\eta_2 h_2, h_i}(x).
\]
By replacing $f$ by $f + \eta_1 h_1 + \eta_2 h_2$ one can show that the function $\omega$ satisfies the following first order system:

$$\begin{cases}
\langle \omega, l_1 \rangle_g = \langle \omega, \eta_1 \rangle_g \\
\langle \omega, l_2 \rangle_g = \langle \omega, \eta_2 \rangle_g,
\end{cases}$$

where $l_j = l_{f + \eta_1 h_1 + \eta_2 h_2, j} \circ \Phi_{f + \eta_1 h_1 + \eta_2 h_2}$, $j = 1, 2$.

Here the additional term $\Phi_{f + \eta_1 h_1 + \eta_2 h_2}$ is needed once one removes the assumption $\Phi_f = \text{identity}$.

• This system can be viewed as a generalized Cauchy-Riemann system with respect to the vector fields $h_1$ and $h_2$. One can show that this system is elliptic if $|\langle l_1, l_2 \rangle_g| \geq c > 0$ holds uniformly in $(\eta_1, \eta_2) \in \mathbb{R}^2$ for some constant $c$. We are now reduced to show that this system admits no global bounded nonconstant solution on $\mathbb{R}^2$. This would lead to $X_i = 0$, $i = 1, 2$.

• One reasonable way to achieve this is to use the Liouville type theorems. However, it is still open in general in this approach how to construct the directions $h_1$ and $h_2$ with the above uniform independence condition. One can construct such directions if one of the quasilinear coefficient is sufficiently close to a quasilinear coefficient that is independent on $x$ or its $p$ derivative is independent on $p$. In both cases, one can use the Liouville type theorem to prove the absence of global bounded nonconstant solution. This leads to $X_1 = X_2 = 0$, which we shall assume in the remaining discussion.

• Consider the example in which $D_x A(x, p)$ is small in an appropriate topology [Su4]. In this case the boundary value problems

$$L_A(u_i) = \nabla \cdot A(x, \nabla u_i) = 0, \quad u_i|_{\partial \Omega} = x_i, \quad i = 1, 2$$

have unique solutions. The smallness of $D_x A(x, p)$ guarantees the map

$$\Phi = (u_1, u_2) : \bar{\Omega} \rightarrow \bar{\Omega},$$

which is the identity map when restricted on the boundary, is a diffeomorphism. Under this diffeomorphism $\Phi$, we can transform the equation so that the new equation carries the special solutions $x_1$ and $x_2$. Thus, in this case, one can simply assume that one of the quasilinear equation carries the special solutions $x_1$ and $x_2$ and choose

$$h_i = x_i|_{\partial \Omega}, \quad i = 1, 2.$$ 

Under such a choice one can show that the above system of $\omega$ is equivalent to

$$\bar{\omega}_z = \text{Re} (\mu \bar{\omega}_z), \quad \mu = \frac{g^{22} - g^{11} + 2ig^{12}}{2(g^{11} + g^{22})}.$$ 

Thus $\omega$ is quasiconformal and the Liouville theorem is available in this case.
The above analysis shows that \( X_i = \hat{\Phi}_{f,h_i} = 0 \) for \( i = 1, 2 \). To finish the argument, we need to show that

\[
\hat{\Phi}_{f,h} = 0, \quad \forall h \in C^{2,\alpha}(\partial \Omega).
\]

We shall see that this is true for any direction \( h \) as long as it is true for two special directions \( h_1 \) and \( h_2 \) with the above independence condition.

- Assume \( \Phi = id \) as we did earlier, and recall

\[
\hat{g}_{1,f,h} - \hat{g}_{2,f,h} = L_{X,f,h}(g) - \left[ e^\sigma \nabla_g \cdot (e^{-\sigma} X_{f,h}) \right] g.
\]

Since \( \hat{\Phi}_{f,h_1} = \hat{\Phi}_{f,h_2} = 0 \), i.e. \( X_{f,h_1} = X_{f,h_2} = 0 \), we have

\[
\hat{g}_{1,f,h} - \hat{g}_{2,f,h} = 0 \quad \text{for} \quad i = 1, 2.
\]

- To understand this equality better, we take a closer look at the structure of \( \hat{g}_{1,f,h} - \hat{g}_{2,f,h} \), \( i = 1, 2 \). Recall that \( g_{1,f} \) and \( g_{2,f} \) are given by

\[
g_{1,f} = A^{-1}_{1,p}(x, \nabla u_{1,f}) = \left( a^{ij}_{1}(x, \nabla u_{1,f}) \right)_{1 \leq i,j \leq 2},
\]

\[
g_{2,f} = A^{-1}_{2,p}(x, \nabla u_{2,f}) = \left( a^{ij}_{2}(x, \nabla u_{2,f}) \right)_{1 \leq i,j \leq 2}.
\]

So, if we write \( A^{-1}_{i,p} = \left( a^{ij}_{i} \right)_{1 \leq i,j \leq 2} \), then

\[
\hat{g}_{1,f,h} = \left( \sum_{k=1}^{2} \frac{\partial a^{ij}_{1}(x, \nabla u_{1,f})}{\partial p_k} \frac{\partial \hat{u}_{1,f,h}}{\partial x_k} \right)_{1 \leq i,j \leq 2},
\]

\[
\hat{g}_{2,f,h} = \left( \sum_{k=1}^{2} \frac{\partial a^{ij}_{2}(x, \nabla u_{2,f})}{\partial p_k} \frac{\partial \hat{u}_{2,f,h}}{\partial x_k} \right)_{1 \leq i,j \leq 2}.
\]

Since \( \Phi = identity \), we have \( \hat{u}_{1,f,h} = \hat{u}_{2,f,h} \). Thus,

\[
\hat{g}_{1,f,h} - \hat{g}_{2,f,h} = \left( \sum_{k=1}^{2} \left[ \frac{\partial a^{ij}_{1}(x, \nabla u_{1,f})}{\partial p_k} - \frac{\partial a^{ij}_{2}(x, \nabla u_{2,f})}{\partial p_k} \right] \frac{\partial \hat{u}_{2,f,h}}{\partial x_k} \right)_{1 \leq i,j \leq 2}.
\]

Remember that \( \hat{g}_{1,f,h_i} = \hat{g}_{2,f,h_i} \) for \( l = 1, 2 \), so

\[
\left( \sum_{k=1}^{2} \left[ \frac{\partial a^{ij}_{1}(x, \nabla u_{1,f})}{\partial p_k} - \frac{\partial a^{ij}_{2}(x, \nabla u_{2,f})}{\partial p_k} \right] \frac{\partial \hat{u}_{2,f,h_l}}{\partial x_k} \right)_{1 \leq i,j \leq 2} = 0
\]

for \( l = 1, 2 \). For each fixed \( i, j \), define

\[
A^{ij}_k = \left[ \frac{\partial a^{ij}_{1}(x, \nabla u_{1,f})}{\partial p_k} - \frac{\partial a^{ij}_{2}(x, \nabla u_{2,f})}{\partial p_k} \right], \quad k = 1, 2.
\]

Then we have a \( 2 \times 2 \) linear system of \( A^{ij}_k \) with coefficients \( \frac{\partial \hat{u}_{2,f,h_l}}{\partial x_k} \), \( 1 \leq k, l \leq 2 \):

\[
\begin{pmatrix}
\frac{\partial \hat{u}_{2,f,h_1}}{\partial x_1} & \frac{\partial \hat{u}_{2,f,h_2}}{\partial x_1} \\
\frac{\partial \hat{u}_{2,f,h_2}}{\partial x_2} & \frac{\partial \hat{u}_{2,f,h_2}}{\partial x_2}
\end{pmatrix}
\begin{pmatrix}
A^{ij}_1 \\
A^{ij}_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
The independence condition for $h_1$ and $h_2$ implies the unique solution to this homogeneous system is the zero solution, i.e. $A_{ij}^{1} = A_{ij}^{2} = 0$ for $i, j = 1, 2$, which implies

$$\frac{\partial a_{ij}^{1}}{\partial p_{k}} (x, \nabla u_{1,f}) - \frac{\partial a_{ij}^{2}}{\partial p_{k}} (x, \nabla u_{2,f}) = 0$$

for $1 \leq i, j \leq 2$. Therefore for any $h \in C^{2,\alpha}(\partial \Omega)$

$$\breve{g}_{1,f,h} = \breve{g}_{2,f,h},$$

which implies $X_{f,h}$ now satisfies the homogeneous conformal Killing field equation

$$L_{X_{f,h}} (g) - \left[ e^{\sigma} \nabla g \cdot (e^{-\sigma} X_{f,h}) \right] g = 0.$$ 

One can show from here that this equation has only trivial solution: $X_{f,h} = 0$.

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