A probabilistic angle on one-loop scalar integrals

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Abstract

Recasting the $N$-point one-loop scalar integral as a probabilistic problem allows the derivation of integral recurrence relations, as well as exact analytical expressions in the most common cases. $\epsilon$ expansions are derived by writing a formula that relates an $N$-point function in decimal dimensions to an $N$-point function in integer dimensions. As an example, we give relations for the massive five-point function in $n = 4 - 2\epsilon$ and $n = 6 - 2\epsilon$ dimensions. The reduction of tensor integrals of rank two with $N = 5$ is achieved showing the method’s potential. No hypergeometric functions are involved. Results are expressed as integrals of arcsine functions, whose analytical continuation is well known.

Keywords: Feynman diagrams, one-loop scalar integral, tensor integral, $\epsilon$ expansion, probability, Fourier transform

1. Introduction

There exists a large amount of literature on the problem of $N$-point one-loop scalar integrals. The challenge is to compute the $N$-point function for a large $N$ in $n = 4 - 2\epsilon$ dimensions, but also for $n > 4$, as planned experiments will probe processes like $e^+ e^- \rightarrow b\bar{f}_1 f_1 \bar{b}\bar{f}_2 f_2$. The so-called non-factorisable corrections to $e^+ e^- \rightarrow 6f$ require five-, six- and seven-point functions [1]. In [2], an application of the Mellin–Barnes lemma allowed a general series representation for the $N$-point function in terms of the Lauricella hypergeometric function $F_D^N$ or its generalised version. In [3], it is shown that the negative dimension approach, the Mellin–Barnes representation and the Feynman parameters yield the same expression for the $N$-point function. [4] used a recurrence relation between the $N$-point functions in different dimensions to give an explicit representation of the two-, three- and four-point functions in arbitrary $n$ dimensions. Specifically, the four-point function (or box diagram) has received a lot of attention. [5–9] perform the integration of the Feynman parameters to produce results in terms of a varying number of polylogarithms with real or complex values of the masses. References
show how to find expressions that are also numerically stable for computer implementation. The five point function has also been studied extensively. References [10, 14] obtained decomposition formulae for the $N$-point function ($N > 5$) in terms of four point functions. Halpern [10] gave an explicit decomposition for the five-point function in 4D in terms of the four-point functions in 4D, while the others [11–14] gave more general decomposition formulæ. References [15, 16] are more specific studies of the massless five-point function. The dimensional recurrence proposed in [17, 18] is used in [15] to obtain an analytic result for the massless pentagon diagram in general D dimensions. Kozlov and Lee [16] apply the differential equation technique to get a one-dimensional integral representation and also discuss the $\epsilon$ expansion. In [19] the problem is written as a problem of hyperbolic geometry involving the computation of the volume of a simplex in D dimensions. This is obtained when the $N$-point function is cast into the problem of finding the volume of the positive part of an ellipsoid (the volume for which the coordinates are positive). The same problem can actually be understood probabilistically and describe the probability that a Gaussian random vector has all its components positive. In the following, we rewrite the $N$-point function to make this probabilistic interpretation explicit. This allows us to derive an $N$ dimensional series expansion in terms of the multivariate Hermite polynomials, unlike [2] where there are $N + N(N − 1)/2$ summation variables. Unfortunately the sum of the Hermite series can not be easily written down in the general case because the explicit form of the multivariate Hermite polynomials is still a challenge. Introducing Gaussian random variables, the $N$-point function can be written as the expectation of a Lauricella hypergeometric function $F_D^N$ whose arguments depend on the Gaussian variables. Exploiting known relations between $F_D^N$ and $F_D^{N−1}$, it is possible to write a relation between the $N$-point functions, e.g. an integral relation linking the $N$-point function in $n$ dimensions to the $(N − 1)$-point function in $n − 2$ dimensions. In [19], the geometrical interpretation was pushed as far as $N = 4$. Here, we show that our methods can easily handle cases of $N > 4$. We give explicit computations for $N = 2, 3, 4, 5, 6, 7$ in integer dimensions. No hypergeometric functions are required to express the results which are given by integrals of arcsine functions. When the dimension $n$ is smaller than $N$, the $N$-point function can be given by a large number of terms, as is seen for the six-point function in four dimensions. An $N$-point function is also characterised by the powers $\nu_i$ of each propagator. When $\nu = \sum_i \nu_i$ is greater (smaller) than the space-time dimension $n$, the $N$-point function in $n$ dimensions can be obtained by integrating an $N$-point function in $N$ dimensions in the complex plane (the real positive line). This integral representation is a compact way of expressing the $N$-point function in $n \neq N$ dimensions. All order $\epsilon$ expansions can be obtained from this integral representation, as the quantity $n$ is decoupled from the other variables in the integrand. Starting from the more familiar Feynman representation, the $\epsilon$ expansion is obtained as an integral relating a general $N$-point function in $d − 2\epsilon$ dimensions to the same $N$-point function in $d − 2k$ dimensions with $d − 2k \in \mathbb{N}$ and modified mass parameters. Each term of the epsilon expansion is an integral of an $N$-point function in integer dimensions. Having shown how to perform the $\epsilon$ expansion as well as how to compute the diagrams with the power $\nu_i > 1$, using Davydychev’s formula [20] allows us to completely reduce tensor integrals. This is shown explicitly for tensor integrals of rank $r = 2$ with $N = 5$. The extension of the reduction program to $N > 5$ and $r > 2$ is left for the future.

2. Probability theory

In this section, we summarise some results from probability theory [21, 22]. This will set some notations as well as help those less familiar with the concepts used in probability theory.
We can consider a random variable as a function that associates a numerical value with the given outcome of an experiment. If the set of possible outcomes is finite (e.g. as in throwing a dice), the random variable is said to be discrete. When the set of possible outcomes is continuous, the random variable is said to be continuous.

2.1. Density and distribution functions

To study continuous random variables, the concept of probability density function is introduced. The probability density function \( f_X(x) \) gives the probability that the random variable \( X \) will fall within a certain interval. We have

\[
\text{Prob}(x < X < x + dx) = f_X(x) \, dx.
\]

(1)

Let us consider a continuous random variable \( X \) for which the set of possible outcomes is \( \mathbb{R} \). Because any outcome should be in \( \mathbb{R} \), the probability density function is normalised as follows:

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = 1.
\]

(2)

By integrating equation (1), we can find out the probability of a random variable \( X \) falling in a given interval \([u, v]\)

\[
\text{Prob}(u < X < v) = \int_{u}^{v} f_X(x) \, dx.
\]

(3)

The ‘cumulative distribution function’ of \( X \), \( F_X(x) \), gives the probability that the random variable will be smaller than a threshold \( x \). It is defined as follows:

\[
F_X(x) = \int_{-\infty}^{x} f_X(u) \, du,
\]

(4)

so the density function is obtained from the distribution function by differentiation

\[
f_X(x) = \frac{d}{dx} F_X(x).
\]

(5)

For continuous random variables \( X_1, X_2, \ldots, X_n \), we define the ‘joint probability density function’ \( f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) \). For a domain \( D \in \mathbb{R} \), the probability that each random variable \( X_i, i = 1, \ldots, n \) falls in the domain \( D \) is

\[
\text{Prob} \left[ (X_1, X_2, \ldots, X_n) \in D \right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \mathbf{1}_{D_i} f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n).
\]

(6)

To simplify the notation, we introduce the following vector notation

\[
X = (X_1, X_2, \ldots, X_n),
\]

\[
x = (x_1, x_2, \ldots, x_n),
\]

\[
f_X(x) = f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n).
\]

(7)

It should be clear from the context whether we are working with unit or multivariate densities. The most ubiquitous random variable is the so-called normal (or Gaussian) random variable, whose density function \( f^G_X(x) \) is given by
\[ f_X^G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \]  
(8)

where \( \mu \in \mathbb{R} \) and \( \sigma > 0 \) are two parameters whose meaning will be given later.

### 2.2. Expected value, moments, variance and covariance

The expected value of a random variable \( X \) can be intuitively understood as the average value of the outcomes after a large number of experiments has been performed. Given the density function \( f_X(x) \), the expected value \( \mathbb{E}[X] \) is computed as follows:

\[ \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx. \]  
(9)

Given a functional form \( g(X) \), which can represent the quantity one is interested in, the expected value \( \mathbb{E}[g(X)] \) of \( g(X) \) is

\[ \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx. \]  
(10)

For \( g(X) = X^n, n \in \mathbb{N} \), we obtain the raw moments \( \mu'_n \)

\[ \mu'_n(X) = \int_{-\infty}^{\infty} x^n f_X(x) \, dx. \]  
(11)

For \( g(X) = (X - \mathbb{E}[X])^n, n \in \mathbb{N} \), we obtain the central moments \( \mu_n \)

\[ \mu_n(X) = \int_{-\infty}^{\infty} (x - \mu)^n f_X(x) \, dx \]  
(12)

with \( \mu = \mathbb{E}[X] \). The second central moment \( \mu_2 \) is called the variance of the random variable \( X \)

\[ \mu_2 = \text{Var}(X), \]

\[ = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) \, dx. \]  
(13)

The variance \( \text{Var}(X) \) gives the averaged squared deviation from the expected value \( \mathbb{E}[X] \).

Given a joint probability density function \( f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) \) the raw moments are denoted as \( \mu_{k_1, \ldots, k_n} \)

\[ \mu_{k_1, \ldots, k_n} = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n x_1^{k_1} \cdots x_n^{k_n} f_{X_1, \ldots, X_n}(x_1, \ldots, x_n). \]  
(14)

Truncated moments \( \mu_{k_1, \ldots, k_n}^{\leq z_1, \ldots, z_n} \) are obtained when some or all of the integrations are truncated by introducing some thresholds \( z_i \) for the \( X_i \) variables

\[ \mu_{k_1, \ldots, k_n}^{\leq z_1, \ldots, z_n}(z_1, \ldots, z_n) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n x_1^{k_1} \mathbb{I}_{x_1 > z_1} \cdots x_n^{k_n} \mathbb{I}_{x_n > z_n} f_{X_1, \ldots, X_n}(x_1, \ldots, x_n). \]  
(15)

In general, the expectation operator \( \mathbb{E} \) and the functions of the random variables do not commute; that is for a non-linear function \( g(X) \) we have

\[ \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} dx \, g(x) f_X(x), \]

\[ \neq g(\mathbb{E}[X]). \]  
(16)
If the functional form $g(X)$ is given by an integral, we can still commute the integral and the expectation operator $E$, as it corresponds to interchanging the order of integration

$$
E \left[ \int_a^b du \, g(x, u) \right] = \int_{-\infty}^\infty dx \left[ \int_a^b du \, g(x, u) \right] f_X(x),
$$

$$
= \int_a^b du \left[ \int_{-\infty}^{\infty} dx \, g(x, u) f_X(x) \right],
$$

$$
= \int_a^b du \, E \{ g(x, u) \}. \tag{17}
$$

Two random variables $X$ and $Y$ are said to be independent if the events $X \leq x$ and $Y \leq y$ are independent; i.e. one event has no influence on the other, and vice versa. For independent random variables $X$ and $Y$, both the distribution $F_{X,Y}(x, y)$ and density functions $f_{X,Y}(x, y)$ satisfy

$$
F_{X,Y}(x, y) = F_X(x)F_Y(y),
$$

$$
f_{X,Y}(x, y) = f_X(x)f_Y(y).
$$

Also, the expected value of the product is the product of the expected values

$$
E[XY] = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \, x f_{X,Y}(x, y),
$$

$$
= \int_{-\infty}^\infty dx \, x f_X(x) \int_{-\infty}^\infty dy \, y f_Y(y),
$$

$$
= E[X]E[Y]. \tag{18}
$$

The covariance $COV(X, Y)$ between two jointly distributed random variables $X$ and $Y$ is defined as

$$
COV(X, Y) = E \left[ (X - E[X]) (Y - E[Y]) \right],
$$

$$
= E[XY] - E[X]E[Y],
$$

$$
= COV(Y, X). \tag{19}
$$

The covariance measures how much the changes in one random variable will affect the other. When $X$ and $Y$ are independent, the covariance is null. The covariance between $X$ and itself is the variance

$$
COV(X, X) = Var(X). \tag{20}
$$

From the covariance, we define the linear correlation coefficient $\rho_{XY}$ as

$$
\rho_{XY} = \frac{COV(X, Y)}{\sqrt{Var(X)Var(Y)}},
$$

$$
= \rho_{YX}. \tag{21}
$$

Let $X$ and $Y$ be random variables with non-zero variances. Then, the absolute value of the correlation coefficient $|\rho_{XY}|$ is less than or equal to one. If $|\rho_{XY}| = 1$, then for some constants, $a$ and $b$, the equality $Y = aX + b$ holds almost surely.

Let $X = (X_1, \ldots, X_n)$ be a random vector. The covariance matrix $\Sigma$ is a matrix whose $(i, j)$ component is the covariance between $X_i$ and $X_j$. The components of the matrix $(i \neq j)$ are
\[ \Sigma_{ii} = \text{COV}(X_i, X_i) = \text{Var}(X_i), \]
\[ \Sigma_{ij} = \text{COV}(X_i, X_j), \]
\[ = \sqrt{\text{Var}(X_i)} \sqrt{\text{Var}(X_j)} \rho_{X_i X_j}. \quad (22) \]

### 2.3. Characteristic functions and multivariate normal distribution

The characteristic function \( \phi_X(s) \), with \( s \in \mathbb{R} \), of a random variable \( X \) is given by the expected value of \( e^{-isX} \)
\[
\phi_X(s) = \mathbb{E} \left[ e^{-isX} \right],
\]
\[
= \int_{-\infty}^{\infty} dx \, e^{-isx} f_X(x). \quad (23)
\]

The characteristic function of a normal random variable with the parameters \( \mu \) and \( \sigma^2 \) is
\[
\phi_{\mu, \sigma^2}(s) = e^{-\frac{1}{2} \sigma^2 s^2 - is\mu}. \quad (24)
\]

Given a vector \( s = (s_1, \ldots, s_n) \), we define the multivariate characteristic function as the following expectation value:
\[
\phi_{X_1, \ldots, X_n}(s_1, \ldots, s_n) = \mathbb{E} \left[ e^{-i(s_1X_1 + \cdots + s_nX_n)} \right]. \quad (25)
\]

Introducing the vector \( X = (X_1, \ldots, X_n) \), we can rewrite it as
\[
\phi_{X_1, \ldots, X_n}(s) = \mathbb{E} \left[ e^{-i(s^T X)} \right], \quad (26)
\]
where \( T \) is the transpose operator and the dot is the matrix multiplication.

A random vector \( X = (X_1, \ldots, X_n) \) is said to have the multivariate normal distribution if there is an \( n \)-vector \( \mu \) and a symmetric, positive semi-definite \( n \times n \) matrix \( \Sigma \), such that the characteristic function of \( X \) is
\[
\phi_X(s) = e^{-\frac{1}{2} s^T \Sigma s - i\mu^T s}. \quad (27)
\]

The joint density of \( X_i \) is
\[
f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu)}, \quad (28)
\]
where \( |\Sigma| \) is the determinant of \( \Sigma \).

The orthant probability \( P_0(\Sigma) \) is the probability that a multivariate normal random vector with a zero mean has all its components positive (equivalently negative)
\[
P_0(\Sigma) = \mathbb{E} \left[ \prod_{i=1}^{n} 1_{X_i > 0} \right],
\]
\[
= \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \int_{0}^{\infty} dx_1 \cdots \int_{0}^{\infty} dx_\nu e^{\frac{1}{2} x^T \Sigma^{-1} x}. \quad (29)
\]

It will be seen in the next sections that the \( N \)-point function in \( n = N \) dimensions can be interpreted as an orthant probability. The matrix \( \Sigma \) has components that depend on the kinematical variables of the \( N \)-point diagram.
3. N-point one-loop scalar integral

We consider the N-point one-loop scalar integral in n dimensions and with \( \nu_i \geq 1 \) the power of each propagator

\[
J^N(n; \nu_1, \nu_2, \ldots, \nu_N) = \int \frac{d^n q}{i \pi^2} \frac{1}{\prod_{i=1}^{N} [(p_i + q)^2 - m_i^2]^{\nu_i}}
\]

(30)

which after the introduction of the Feynman parameters takes the following form [19]:

\[
J^N(n; \nu; \Sigma) = \frac{1}{(-1)^n} \frac{\prod \Gamma(\nu - n/2)}{\prod \Gamma(\nu_i)} \int_0^1 du_1 u_1^{\nu_1-1} \cdots \int_0^1 du_N u_N^{\nu_N-1} \frac{\delta \left( \sum_{i=1}^{N} u_i - 1 \right)}{\left( - \sum_{j<\ell} u_j u_\ell k_{j\ell}^2 + \sum u_i m_i^2 \right)^{\nu-\frac{n}{2}}},
\]

(31)

\[
= (-1)^n \frac{\prod \Gamma(\nu - n/2)}{\prod \Gamma(\nu_i)} \int_0^1 du_1 u_1^{\nu_1-1} \cdots \int_0^1 du_N u_N^{\nu_N-1} \frac{\delta \left( \sum_{i=1}^{N} u_i - 1 \right)}{(\nu^T \Sigma \nu)^{\nu-\frac{n}{2}}},
\]

(32)

where \( \sum_i \nu_i = \nu \) and \( \Sigma \) is a matrix with components

\[
\Sigma_{ii} = m_i^2,
\]

\[
\Sigma_{ij} = m_i m_j c_{j\ell},
\]

\[
c_{j\ell} = \frac{m_i^2 + m_j^2 - k_{j\ell}^2}{2 m_i m_j},
\]

\[
k_{j\ell}^2 = (p_j - p_\ell)^2.
\]

The \( c_{j\ell} \) can be understood as the cosines of some angles \( \tau_{j\ell} \) [19]

\[
c_{j\ell} = \cos \tau_{j\ell} = \begin{cases} 1, & k_{j\ell}^2 = (m_j - m_\ell)^2 \\ -1, & k_{j\ell}^2 = (m_j + m_\ell)^2 \end{cases}.
\]

(33)

The corresponding angles \( \tau_{j\ell} \) are [19]

\[
\tau_{j\ell} = \arccos(c_{j\ell}) = \arccos \left( \frac{m_i^2 + m_j^2 - k_{j\ell}^2}{2 m_i m_j} \right) = \begin{cases} 0, & k_{j\ell}^2 = (m_j - m_\ell)^2 \\ \pi, & k_{j\ell}^2 = (m_j + m_\ell)^2 \end{cases}.
\]

(34)

The angles \( \tau_{j\ell} \) can be analytically continued when the \( c_{j\ell} \) are not in the range \([-1, 1]\). When \( k_{j\ell}^2 < (m_j - m_\ell)^2 \), the \( c_{j\ell} \) are greater than one and the angles \( \tau_{j\ell} \) are given by [19]

\[
\tau_{j\ell} = -i \text{ Arch}(c_{j\ell}),
\]

(35)

\[
= -\frac{i}{2} \ln \left( \frac{m_i^2 + m_j^2 - k_{j\ell}^2 + \sqrt{\lambda(m_i^2, m_j^2, k_{j\ell}^2)}}{m_i^2 + m_j^2 - k_{j\ell}^2 - \sqrt{\lambda(m_i^2, m_j^2, k_{j\ell}^2)}} \right),
\]

(36)

where \( \lambda(x, y, z) \) is the Källen function

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.
\]

(37)
When $k_{J}^2 > (m_j + m_e^2)^2$, the $c_{J}^{\ell}$ are smaller than minus one and the angles $\tau_{J}^{\ell}$ are given by [19]

$$\tau_{J}^{\ell} = \pi + i \text{Arch}(-c_{J}^{\ell}),$$

(38)

$$= \pi + \frac{i}{2} \ln \left( \frac{k_{J}^2 - m_j^2 - m_e^2 + \sqrt{\lambda(m_j^2, m_e^2, k_{J}^2)}}{k_{J}^2 - m_j^2 - m_e^2 - \sqrt{\lambda(m_j^2, m_e^2, k_{J}^2)}} \right).$$

(39)

When the angles $\tau_{J}^{\ell}$ are real (as in equation (34)), the $c_{J}^{\ell}$ are in the range $[-1, 1]$ and the matrix $\Sigma$ in equation (32) can be interpreted as a covariance matrix. Let us consider a random vector $Z$ of size $N$, whose components are normal random variables $Z_j$ with zero mean and variance $\sigma_j^2 = m_j^2 > 0$. When the $c_{J}^{\ell}$ are in the range $[-1, 1]$, we can interpret the $c_{J}^{\ell}$ as the correlation between $Z_j$ and $Z_e$.

Using the relation

$$\Gamma(\alpha) / A^\alpha = \int_0^\infty d\tau \tau^{\alpha-1} e^{-\tau A}$$

(40)

equation (32) is rewritten as

$$\mathcal{J}^N(\nu; \Sigma) = \frac{(-1)^{-\nu}}{\prod_i \Gamma(\nu_i)} \int_0^1 \cdots \int_0^1 d\nu_i d\nu_N \delta \left( \sum_{i=1}^N u_i - 1 \right) \int_0^\infty \cdots \int_0^\infty d\tau \tau^{\nu - \nu - 1} e^{-\tau A} \Sigma_{\nu}. \Sigma_{\nu}.$$  

(41)

We change the variable $\tau$ to $v$, such that $\tau = \frac{v^2}{2}$, and obtain

$$\mathcal{J}^N(\nu; \Sigma) = (-1)^{-\nu} \int_0^1 \cdots \int_0^1 d\nu_i d\nu_N \delta \left( \sum_{i=1}^N u_i - 1 \right) \int_0^\infty \cdots \int_0^\infty dv \nu^{2\nu - n - 1} e^{\nu \Sigma_{\nu} \Sigma_{\nu}}.$$  

(42)

Let $Z = (Z_1, \ldots, Z_N)$ be an $N$-dimensional random vector distributed according to the multivariate normal distribution with zero mean vector and covariance matrix $C$; then its characteristic function $\phi(s_1, \ldots, s_n) = \phi(s)$ is

$$\phi(s) = \mathbb{E} \left[ e^{-i s^T Z} \right] = e^{-\frac{1}{2} s^T C s}. \quad \quad (43)$$

Choosing an $N$-dimensional random vector $Z$ distributed normally with zero mean and covariance matrix $\Sigma$, we can linearise the quadratic form $(\nu u)^T \Sigma (\nu u)$, which appears in the exponential, and perform the integration over $\nu$ using equation (40)

$$\mathcal{J}^N(\nu; \Sigma) = (-1)^{-\nu} \int_0^1 \cdots \int_0^1 d\nu_i d\nu_N \delta \left( \sum_{i=1}^N u_i - 1 \right) \frac{\Gamma(2\nu - n)}{\prod_i \Gamma(\nu_i)} \int_0^\infty \cdots \int_0^\infty dv \nu^{2\nu - n - 1} \mathbb{E} \left[ e^{-\nu \Sigma_{\nu} Z Z} \right].$$

(44)

Let us write

$$iu^T Z = i \sum_{j=1}^N u_j Z_j = 1 + \sum_{j=1}^N u_j(-1 + i Z_j).$$

(8)
and use the Mellin–Barnes representation [23, equation (3.4)]

\[ \frac{\Gamma(c)}{(A_1 + A_2 + \ldots + A_n)^\nu} = \int_{-\infty}^{\infty} \frac{ds_1}{2\pi i} \cdots \int_{-\infty}^{\infty} \frac{ds_{n-1}}{2\pi i} \Gamma(-s_1) \cdots \Gamma(-s_{n-1}) \Gamma(c + s_1 + \ldots + s_{n-1}), \]

\[ A_1^a A_2^b \cdots A_n^{a_i} = \frac{\Gamma(c) \Gamma(\nu + \sum j) \prod_{j=1}^{N} \Gamma(\nu_j + k_j) \prod (1 - iZ_j)_{k_j}}{\prod_{j=1}^{N} \Gamma(\nu_j + k_j)} \]

(45)

to get

\[ F^N(n; \nu; \Sigma) = \mathbb{E} \left[ (-1)^{-\nu} \int_0^1 du_1 u_1^{\nu-1} \int_0^1 du_N u_N^{\nu-1} \frac{\delta \left( \sum_{j=1}^N u_j - 1 \right)}{2^{\nu + \frac{1}{2} \sum_j k_j}} \prod_{j=1}^N \Gamma(\nu_j + k_j) \prod_{j=1}^N \frac{(1-iZ_j)^{k_j}}{k_j!} \right]. \]

(46)

which can be written as a Lauricella $F_D$ hypergeometric function

\[ F^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu} \Gamma(2\nu - n)}{2^{\nu - \frac{1}{2} \sum k_j} \Gamma(\nu)} \mathbb{E} \left[ F_D(2\nu - n; \nu_1, \ldots, \nu_N; \nu; 1-iZ_1, \ldots, 1-iZ_N) \right]. \]

(47)

The Lauricella hypergeometric function $F_D$ can be expressed as a series

\[ F_D(a; b_1, b_2, \ldots, b_N; c; x_1, x_2, \ldots, x_N) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_N=0}^{\infty} \frac{(a)_{\sum k}}{(c)_{\sum k}} \prod_{i=1}^{N} (b_i)_{k_i} \prod_{i=1}^{N} x_i^{k_i}, \]

\[ |x_i| < 1, i = 1, \ldots, n, \]

(48)

and it is this form that first occurred in the computation of the $N$-point function [23, equation (2.9)]. It also has an integral representation (that can be easily derived from the series representation)

\[ F_D(a; b_1, b_2, \ldots, b_N; c; x_1, x_2, \ldots, x_N) = \frac{\Gamma(c) \Gamma(a) \Gamma(c-a) \prod_{i=1}^{N} (1-x_i)^{b_i}}{\prod_{i=1}^{N} (1-x_i)^{b_i}}, \]

\[ \Re(c) > \Re(a) > 0. \]

(49)

In equation (46), if we take the expectation over the normally distributed random variables, we obtain a series in terms of the multivariate Hermite polynomials. In [24], it was shown that the multivariate Hermite polynomials $H_{a_1a_2\ldots a_p}(x, \Sigma)$ in $p$ dimensions have the probabilistic representation
\[ H_{e_{k_1}...k_p}(x, \Sigma) = \mathbb{E} \left[ \prod_{j=1}^{\mu} ((\Sigma^{-1} x)_j + iZ)_j^k \right], \] (50)

where \( Z \) is a \( p \)-dimensional random vector normally distributed with zero mean and covariance matrix \( \Sigma^{-1} \). We have

\[ f^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu-n-1} \prod \Gamma(\nu_{ij})} \sum_{k=0}^{\infty} \frac{\Gamma(2\nu - n + k)}{\Gamma(\nu + k)} \sum_{k_i=0}^{\infty} \frac{\Gamma(n + k_i)}{k_i!} H_{e_{k_1}...k_p}(\Sigma, 1, \Sigma), \] (51)

where \( 1 \) is a vector with all components equal to one. This is a multi-sum with \( N \) indices to be contrasted with the series obtained in [2], which involves \( N^2 \) summation indices. As the computation of the multivariate Hermite polynomial is still a challenge, this series seems at first to be of limited numerical interest, but it might be useful to obtain an asymptotic expansion by considering the first few terms \( k = 0, 1, 2 \) of the series. The Hermite polynomials are given by expectation of the products of normal random variables, which can be computed by application of the Wick Theorem, as is done in the path integral perturbation theory for connected \( N \)-point functions.

In the following, we exploit the form given by the expectation of the Lauricella hypergeometric function \( F_D \) with the parameters

\[ a = 2\nu - n, \]
\[ b_i = \nu_i, \]
\[ c = \nu. \]

We distinguish three cases:

(\( \nu - n < 0 \)). In this case \( c > a \), so we use the one-dimensional integral representation equation (49) to write

\[ f^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu-n-1} \prod \Gamma(n - \nu i)} \mathbb{E} \left[ \int_0^1 dt t^{2\nu - n - 1} (1 - t)^{n - \nu - 1} \frac{1}{\prod_{i=1}^{N} (1 - \theta_i)^{\nu_i}} \right], \] (52)

with \( \theta_i = 1 - iZ_i \).

(\( \nu = n \)). In this case, the two gamma functions in equation (46) disappear and using

\[ \frac{\Gamma(\nu)}{(1-x)^{\nu}} = \sum_{k=0}^{\infty} \frac{\Gamma(\nu + k)}{k!} x^k, \] (53)

we obtain

\[ f^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu-n-1} \prod \Gamma(\nu_i)} \mathbb{E} \left[ \prod_{i=1}^{N} \frac{1}{(iZ_i)^{\nu_i}} \right]. \] (54)

(\( \nu - n = k > 0 \)). In this case, we go back to the Feynman representation equation (44)

\[ f^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu-n-1} \prod \Gamma(\nu_i)} \mathbb{E} \left[ \int \prod_i du_i u_i^{\nu_i - 1} \delta \left( \sum_i u_i - 1 \right) \frac{1}{(i\mu^T \Sigma)^{2\nu - n}} \right]. \] (55)
We insert the number 1 into the numerator

\[ 1 = \left( \sum_{i=1}^{N} u_i \right)^k = \sum_{k=1}^{k_i} \frac{k!}{k_i!} \prod_i u_i^k, \] (56)

and obtain

\[ J_N(n; \nu; \Sigma) = \left( -1 \right)^{-\nu} \frac{\Gamma(2\nu - n)}{\Gamma(\nu)} \frac{1}{2^{\nu + \frac{1}{2}}} \sum_{k=1}^{k_i} \frac{k!}{k_i!} \prod_i \Gamma(\nu_i + k_i) \prod_i \Gamma(\nu + k_i + Z_i) \right], \] (57)

\[ = (-1)^k \sum_{k=1}^{k_i} \frac{k!}{k_i!} \prod_i \frac{\Gamma(\nu_i + k_i)}{\Gamma(\nu + k_i + Z_i)} J_N(\nu + k; \nu + k; \Sigma). \] (58)

where \( k \) is a vector whose \( i \)th component is \( k_i \) and \( (a)_x \) is the Pochhammer symbol

\[ (a)_x = \frac{\Gamma(a + x)}{\Gamma(a)}. \] (59)

If we apply this for \( k = 1 \), we find

\[ J_N(\nu - 1; \nu; \Sigma) = -\sum_{j=1}^{N} \nu_j J_N(\nu + 1; \nu + \delta_j; \Sigma), \] (60)

where \( \delta_j \) indicates a vector whose \( j \)th component is one. This particular case can also be obtained from a formula derived in [20] and later in [17].

4. Relations between the \( N \)-point functions

By exploiting the different representation of the \( F_D \) function, it is possible to derive the relations between the \( N \)-point functions. The parameter \( c \) of the \( F_D \) function is equal to the sum of the \( b_i \) parameters, so it can be written as the so-called Carlson \( R \) function [25]

\[ R(a; b; z) = R(a; b_1, b_2, \ldots, b_N; z_1, z_2, \ldots, z_N), \]

\[ = F_D(a; b_1, b_2, \ldots, b_N; b_1 + b_2 + \ldots + b_N; 1 - z_1, 1 - z_2, \ldots, 1 - z_N). \] (61)

So

\[ J_N(n; \nu; \Sigma) = \left( -1 \right)^{-\nu} \frac{\Gamma(2\nu - n)}{\Gamma(\nu)} \frac{1}{2^{\nu + \frac{1}{2}}} E[R(2\nu - n; \nu; iZ)], \] (62)

\[ = \left( -1 \right)^{-\nu} \frac{\Gamma(2\nu - n)}{\Gamma(\nu)} \frac{1}{2^{\nu + \frac{1}{2}}} \bar{J}_N(n; \nu; \Sigma), \] (63)

where we have defined the rescaled \( N \)-point function \( \bar{J}_N(n, \nu, \Sigma) \) for ease of use.

In [26–28], explicit representations for the scalar and tensor two- and three-point functions were given in terms of the \( R \) function. In our representation, we have the \( R \) function but we still need to compute the expectation over the Gaussian random variables, which is a
non-trivial task in the general case. Instead, we are going to exploit known relations satisfied by \( R \) functions of different parameters to write new relations between the \( N \)-point functions. Let us define

\[
B(b_1, b_2, \ldots, b_N) = \frac{\Gamma(b_1) \cdots \Gamma(b_N)}{\Gamma(b_1 + b_2 + \cdots + b_N)}.
\]  

(64)

An interesting relation for the \( R \) function is an integral representation that gives \( R \) as the integral of an \( R \) function with one less variable [25, equation (7.4)]

\[
B(c - b_N, b_N)R(a; b_1, \ldots, b_{N-1}; z_1 + tz_N, \ldots, z_{N-1} + tz_N) = \int_0^\infty dt \frac{t^{b_N-1}}{(1 + t)^a} R(a; b_1, \ldots, b_{N-1}; z_1 + tz_N, \ldots, z_{N-1} + tz_N),
\]  

(65)

for \( \Re(c) > \Re(b_N) > 0 \). Let \( \eta_i, i = 1, \ldots, N - 1 \) be normal random variables defined as

\[
\eta_i = Z_i + tz_N.
\]  

(66)

The \( \eta_i \) have a covariance matrix \( \Sigma_\eta(t) \) with the components

\[
\Sigma_{\eta j}(t) = \mathbb{E}[(Z_j + tZ_N)(Z_\ell + tZ_N)], \quad 1 \leq j, \ell \leq N - 1,
\]  

(67)

\[
= \Sigma_{\eta j} + t(\Sigma_{\eta N} + \Sigma_{\ell N}) + t^2\Sigma_{NN}.
\]  

(68)

So

\[
\bar{J}^N(n; \nu; \Sigma) = \frac{1}{B(n - \nu_N, \nu_N)} \int_0^\infty dt \frac{t^{n-1}}{(1 + t)^{a-\nu}} \bar{J}^{N-1}(n - 2\nu_N; \nu; \Sigma_\eta(t)).
\]  

(69)

For example, the four-point function in four dimensions will be given by the integral of the three-point function in two dimensions. The right-hand side contains the vector \( \nu \), but it is understood that only \( N - 1 \) components are used.

An iteration of equation (65) gives

\[
R(a; b; z) = \int_0^\infty dt_1 \cdots dt_N \frac{dt_{k+1} \cdots dt_N}{B(b_1 + \cdots + b_k, b_{k+1} + \cdots, b_N)} \left( \prod_{j=k+1}^N \int_{t_j=0}^{t_j=1} \right)^{-(a'-1)} \left( 1 + \sum_{j=k+1}^N t_j \right)^{-a'} R(a; b_1, \ldots, b_{k-1}; z_1 + \sum_{j=k+1}^N t_j z_j).
\]  

(70)

If we take \( k = 1 \), we obtain [25, equation (7.7)]

\[
R(a; b; z) = \int_0^\infty dt_2 \cdots dt_N \frac{dt_2 \cdots dt_N}{B(b_1, \ldots, b_N)} \left( \prod_{j=2}^N \int_{t_j=0}^{t_j=1} \right)^{-(a'-1)} \left( 1 + \sum_{j=2}^N t_j \right)^{-a'} \left( z_1 + \sum_{j=2}^N t_j z_j \right)^{-a},
\]  

so
\[ \bar{J}^N(n; \nu; \Sigma) = \frac{1}{B(\nu_1, \ldots, \nu_N)} \mathbb{E} \left[ \int_0^\infty dt_2 \cdots dt_N \left( \prod_{j=2}^N f_j^{\nu_j-1} \right) \left( \frac{1 + \sum_{j=2}^N t_j}{iZ_1 + \sum_{j=2}^N iZ_j} \right)^{\nu-n} \right], \]

\[ = \int_0^\infty \frac{dt_2 \cdots dt_N}{2\Gamma(2\nu-n)B(\nu_1, \ldots, \nu_N)} \left( \prod_{j=2}^N f_j^{\nu_j-1} \right) \frac{\Gamma(\nu-n/2)}{\left( \Sigma_{11} + 2 \sum_{j=2}^N t_j \Sigma_{ij} + \sum_{j=k=2}^N t_j \Sigma_{jk} \right)^{\nu-n/2}}, \]

where we have used equation (40) to introduce an integration variable \( \tau \), taken the expectation numerically.

Another representation for the \( R \) function is given in [25, equation (7.8)]

\[ R(a; b; z) = \int_0^1 \frac{du_1 b_1^{a_1-1}(1-u)b_2^{a_2-1}}{B(b_1, b_2)} R(a; b_1 + b_2, b_3, \ldots, b_N; u z_1 + (1-u) z_2, \ldots, z_N), \]

provided that \( b_1 \) and \( b_2 \) have positive parts. The iteration of this integral representation gives

\[ R(a; b; z) = \int_0^1 \frac{du_1 \cdots du_k}{B(b_1, b_2, \ldots, b_k)} \delta \left( \sum_{j=1}^k u_j - 1 \right) \left( \prod_{j=1}^k u_j^{b_j-1} R(a; b_1, b_{k+1}, \ldots, b_N; \sum_{j=1}^k u_j z_j, \ldots, z_N), \right. \]

(73)

and when \( k = N \)

\[ R(a; b; z) = \int_0^1 \frac{du_1 \cdots du_N}{B(b_1, b_2, \ldots, b_N)} \delta \left( \sum_{j=1}^N u_j - 1 \right) \left( \prod_{j=1}^N u_j^{b_j-1} \frac{1}{\left( \sum_{j=1}^N u_j z_j \right)^a} \right). \]

(74)

which is the standard form in term of the Feynman parameters. Using equation (72) we get

\[ \bar{J}^N(n; \nu; \Sigma) = \int_0^1 \frac{du^{\nu-n-1}(1-u)^\nu-1}{B(\nu_1, \nu_2)} \bar{J}^{N-1}(n; \nu_1, \nu_2, \ldots, \nu_{N-2}, \nu_{N-1} + v_N; \Sigma(u)). \]

(75)

The components of the matrix \( \Sigma \) are

\[ \Sigma_{ij} = \Sigma_{ji}, \quad i, j < N - 1, \]

\[ \Sigma_{N-1,j} = u \Sigma_{N-1,j} + (1-u) \Sigma_{Nj}, \quad j < N - 1, \]

\[ \Sigma_{N-1,N-1} = u^2 \Sigma_{N-1,N-1} + (1-u)^2 \Sigma_{NN} + 2u(1-u) \Sigma_{N-1,N}. \]

(76)

Because of the term \( \Sigma_{N-1,N} \), which can be negative, the new mass squared \( \Sigma_{N-1,N-1} \) can be negative.
5. Probabilistic interpretation

In equation (42) we make the change of variable \( x_i = \tau u_i \),
\[
J^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu+\frac{n}{2}-1} \prod \Gamma(\nu_i)} \int_0^\infty dx_1 x_1^{\nu_1-1} \cdots \int_0^\infty dx_N x_N^{\nu_N-1} \\
\times \int_0^\infty d\tau \tau^{n-1} \delta \left( \frac{1}{\tau} \sum_{i=1}^N x_i - 1 \right) e^{-\frac{1}{2} \tau^T \Sigma \tau}.
\]
(77)

We can rewrite the delta function as
\[
\delta \left( \frac{1}{\tau} \sum_{i=1}^N x_i - 1 \right) = \tau \delta \left( \sum_{i=1}^N x_i - \tau \right),
\]
(78)

which forces us to set \( \tau = \sum_{i=1}^N x_i \) in the integrand. We have
\[
J^N(n; \nu; \Sigma) = (-1)^{-\nu} N_x^{N} \int_0^\infty dx_1 \prod_{i=1}^N \Gamma(\nu_i) \left( \sum_{i=1}^N x_i \right)^{\nu-n} e^{-\frac{1}{2} x^T R^{-1} x} \frac{\tau}{N_x^{N}},
\]
(79)

with \( R = \Sigma^{-1} \) and \( N_x^{N} = (2\pi)^{\frac{N_x}{2}} |R|^{\frac{1}{2}} \). The exponential in the integrand is the probability density of the multivariate normal distribution, so the \( N \)-point function in \( n \) dimensions is given by the product of the truncated moments of the correlated normal random variables times the power of their sum.
\[
J^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu} N_x^{N}}{2^{\nu+\frac{n}{2}-1} \prod \Gamma(\nu_i)} \mathbb{E} \left[ \prod_{i=1}^N \left( x_i^{\nu_i-1} \mathbb{1}_{\{x_i > 0\}} \right) \left( \sum_{i=1}^N \epsilon_i \right)^{\nu-n} \right],
\]
(80)

where \( \epsilon_i, i = 1, \ldots, N \) are normal random variables with the covariance matrix \( R = \Sigma^{-1} \).

We now consider some special cases.

\( (\nu_i = 1 \text{ and } \nu - n < 0) \)
\[
J^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu} N_x^{N}}{2^{\nu+\frac{n}{2}-1} \prod \Gamma(\nu_i)} \mathbb{E} \left[ \prod_{i=1}^N \mathbb{1}_{\{x_i > 0\}} \left( \sum_{i=1}^N \epsilon_i \right)^{\nu-n} \right].
\]
(81)

\( (\nu_i = 1 \text{ and } \nu = N = n) \) In this case we have
\[
J^N(n; 1; \Sigma) = \frac{(-1)^{-N} N_x^{N}}{2^{\nu+\frac{n}{2}-1}} \mathbb{Prob} \left( \epsilon_1 > 0, \ldots, \epsilon_N > 0 \right).
\]
(82)

\( (\nu_i = 1 \text{ and } \nu - n = k > 0) \) We expand
\[
\left( \sum_{i=1}^N x_i \right)^k = \sum_{k_1+k_2+\cdots+k_l = k} k_1! \prod_{i} x_i^{k_i}.
\]
(83)

and get
\[
J^N(n; \nu; \Sigma) = \frac{(-1)^{-N} N_x^{N}}{2^{\nu+\frac{n}{2}-1} \prod \Gamma(\nu_i)} \sum_{k_1+k_2+\cdots+k_l = k} k_1! \prod_{i} \mathbb{E} \left[ \prod_{i=1}^N \mathbb{1}_{\{x_i > 0\}} x_i^{\nu_i+k_i+1} \right].
\]
(84)
6. Explicit evaluation of the $N$-point functions

To find explicit expressions we follow the method of [30], where the Fourier transform is used. Recall equation (79)

$$J^N(n; \nu; \Sigma) = \left( -1 \right)^{-\nu} \frac{(2\pi)^{\frac{N}{2}}}{2^{\nu - \frac{N}{2} - 1} \prod_i \Gamma(\nu_i)} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dz_j \prod_{j=1}^{N} U(z_j) \frac{\prod_j z_j^{\nu_j - 1} (\sum_j z_j)^{\nu - n}}{(2\pi)^{2} |\Sigma|^2} e^{-\frac{1}{2} z^T \Sigma^{-1} z},$$

where $U(z_j) = 1_{z_j > 0}$. The $N$-dimensional Fourier transform $\hat{f}(w)$ of a function $f(x)$ is defined as

$$\hat{f}(w) = \int_{-\infty}^{\infty} d^N x f(x) e^{-i w^T x},$$

and its inverse is

$$f(x) = \frac{1}{(2\pi)^{N}} \int_{-\infty}^{\infty} d^N w \hat{f}(w) e^{+i w^T x}.$$

Both Fourier transforms of the unit step (unidimensional) and Gaussian functions (multi-dimensional) are known [30–32]

$$\hat{U}(w) = \frac{1}{1 - \omega \epsilon},$$

$$= \frac{1}{i} \left( i \pi \delta(w) + \text{PV} \frac{1}{w} \right),$$

$$= \pi \delta(w) + \text{PV} \left( \frac{1}{iw} \right),$$

$$= \pi \delta(w) - i \lim_{\epsilon \to 0} \frac{w}{w^2 + \epsilon^2},$$

$$F \left( \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^\frac{1}{2}} e^{-\frac{1}{2} w^T \Sigma^{-1} w} \right)(w) = e^{-\frac{1}{2} w^T R w}.$$  

In the following, the limit ($\epsilon \to 0$) is assumed, even though it is not written explicitly. Moreover, the Fourier transform satisfies

$$\mathcal{F} [x_k f(x)] (w) = i \frac{\partial}{\partial \omega_k} \hat{f}(w).$$

The Parseval relation states

$$\int_{-\infty}^{\infty} d^N x f(x) g(x) = \frac{1}{(2\pi)^{N}} \int_{-\infty}^{\infty} d^N w \hat{f}(w) \hat{g}(w),$$

so we get

$$J^N(n; \nu; \Sigma) = \frac{(-1)^{-\nu}}{2^{\nu - \frac{N}{2} - 1} \prod_i \Gamma(\nu_i)} \frac{(2\pi)^{\frac{N}{2}}}{(2\pi)^{2N}} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dw_j \prod_{j=1}^{N} \left( \pi \delta(\omega_j) - i \omega_j \right)$$

$$\times \prod_{j=1}^{N} \left( i \frac{\partial}{\partial \omega_j} \right)^{\nu_j - 1} \left( i \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} \right)^{\nu - n} e^{-\frac{1}{2} w^T \Sigma^{-1} w}.$$  

(92)
We introduce the following notation for the integral of the Gaussian kernel weighted by the denominators
\[ I_{ij...}^{kl...} = \int_{-\infty}^{\infty} d\omega_i d\omega_j d\omega_k ... \frac{1}{\omega_i \omega_j ...} e^{\omega^T R^{ij...} \omega}, \] (93)

which means that we integrate the \((i, j, k, l)\) \(\omega\) variables with \(\omega_i\) and \(\omega_j\) appearing in the denominator with \(R^{ij}\) as the inverse of the covariance matrix \(\Sigma\).

6.1. Two-point function in 2D

The Fourier integral we need to compute is (with \(R = \Sigma^{-1}\))
\[ F_2 = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dw_1 dw_2 \left( \pi \delta(w_1) + \frac{1}{iw_1} \right) \left( \pi \delta(w_2) + \frac{1}{iw_2} \right) e^{-\frac{1}{2}(R_{11}w_1^2 + 2R_{12}w_1w_2 + R_{22}w_2^2)}. \] (94)

The components of the matrix \(R\) are
\[ R_{11} = \frac{m_1^2}{\Delta^{(2)}}, \]
\[ R_{22} = \frac{m_2^2}{\Delta^{(2)}}, \]
\[ R_{12} = -\frac{m_1 m_2 c_{12}}{\Delta^{(2)}}, \]
\[ \Delta^{(2)} = m_1^2 m_2^2 (1 - c_{12}^2) = m_1^2 m_2^2 \sin^2 \tau_{12}. \]

Two terms are null because they involve an odd power of \(w_i\) and the product of the two delta functions gives a constant
\[ F_2 = \frac{1}{4} - \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dw_1 dw_2 \frac{e^{-\frac{1}{2}(R_{11}w_1^2 + 2R_{12}w_1w_2 + R_{22}w_2^2)}}{w_1 w_2}. \] (95)

We have [30, reproduced here in the appendix]
\[ \int_{-\infty}^{\infty} dw_1 dw_2 \frac{e^{-\frac{1}{2}(R_{11}w_1^2 + 2R_{12}w_1w_2 + R_{22}w_2^2)}}{w_1 w_2} = -2\pi \arcsin \left( \frac{R_{12}}{\sqrt{R_{11} R_{22}}} \right). \] (96)

The two-point function is
\[ J^2(2; 1; \Sigma) = \frac{2\pi}{m_1 m_2 \sin \tau_{12}} \left( \frac{1}{4} + \frac{1}{2\pi} \arcsin \left( \frac{R_{12}}{\sqrt{R_{11} R_{22}}} \right) \right), \] (97)
\[ = \frac{\tau_{12}}{m_1 m_2 \sin \tau_{12}}, \] (98)

where we have used
\[ \arcsin(z) = \frac{\pi}{2} + \arccos(-z), \] (99)

to get the last line, which is exactly the same expression as equation (4.3) derived in [19] (up to an \(i \pi \frac{1}{2}\) factor that we included in our definition of \(J^N(n, \nu, \Sigma)\)).
6.2. Three-point function in 3D

The components of the matrix $R$ are

\[ R_{11} = \frac{m_1^2 m_2^2 (1 - c_{12}^2)}{\Delta(3)}, \]
\[ R_{22} = \frac{m_1^2 m_2^2 (1 - c_{13}^2)}{\Delta(3)}, \]
\[ R_{33} = \frac{m_1^2 m_2^2 (1 - c_{23}^2)}{\Delta(3)}, \]
\[ R_{12} = \frac{m_1^2 m_2^2 (c_{12} c_{23} - c_{12})}{\Delta(3)}, \]
\[ R_{13} = \frac{m_1^2 m_2^3 (c_{12} c_{23} - c_{13})}{\Delta(3)}, \]
\[ R_{23} = \frac{m_1^2 m_2^3 (c_{12} c_{13} - c_{23})}{\Delta(3)}, \]
\[ \Delta(3) = m_1^2 m_2^2 m_3^2 \left(1 - c_{12}^2 - c_{13}^2 - c_{23}^2 + 2c_{12} c_{13} c_{23}\right), \]
\[ \Delta(3) = m_1^2 m_2^2 D(3). \]

In equation (92), four terms are null and we get a constant as well as three terms that are exactly like the two-point integral

\[ J^3(3; 1; \Sigma) = \frac{-1}{2m_1 m_2 m_3} \sqrt{\Delta(3)} \left[ \frac{1}{8} + \frac{1}{4\pi} \arcsin \left( \frac{R_{12}}{\sqrt{R_{11} R_{22}}} \right) \right] + \frac{1}{4\pi} \arcsin \left( \frac{R_{13}}{\sqrt{R_{11} R_{33}}} \right) + \frac{1}{4\pi} \arcsin \left( \frac{R_{23}}{\sqrt{R_{22} R_{33}}} \right) \].

Using the relation equation (99) and the notation in [19]

\[ \cos \Psi_{12} = \frac{R_{12}}{\sqrt{R_{11} R_{22}}} = \frac{c_{12} - c_{13} c_{23}}{\sin \tau_{13} \sin \tau_{23}}, \]
\[ \cos \Psi_{13} = \frac{R_{13}}{\sqrt{R_{11} R_{33}}} = \frac{c_{13} - c_{12} c_{23}}{\sin \tau_{12} \sin \tau_{23}}, \]
\[ \cos \Psi_{23} = \frac{R_{23}}{\sqrt{R_{22} R_{33}}} = \frac{c_{23} - c_{12} c_{13}}{\sin \tau_{13} \sin \tau_{12}}, \]
\[ \Omega(3) = \Psi_{12} + \Psi_{13} + \Psi_{23} - \pi, \]

we get

\[ J^3(3; 1; \Sigma) = \frac{-\pi^2}{2m_1 m_2 m_3} \sqrt{\Delta(3)}, \]

exactly as equation (5.6) in [19].

6.3. The three-point function in 2D

In this case, in equation (79), the term \( (\sum_{i=1}^{3} x_i)^{\nu-n} \) contributes to the integral.

\[ J^3(2; 1; \Sigma) = \frac{(-1)(2\pi)^{\frac{3}{2}}}{2m_1 m_2 m_3 \sqrt{D(3)}} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} d^3 \omega \prod_{j=1}^{3} \left[ \pi \delta(i\omega_j) + \frac{1}{\omega_j} \right] \left( \sum_{j=1}^{3} \frac{\partial}{\partial \omega_j} \right) e^{i\omega \cdot \mathbf{k}}. \]
We expand and keep the non-zero terms
\[
J^3(2; 1; \Sigma) = \frac{(-1)(2\pi)^\frac{3}{2}}{2m_1m_2m_3\sqrt{D^{(3)}}} \left( (R_{11} + R_{12} + R_{13})(-\pi^2 I_1 + I_{23}^1) \\
+ (R_{12} + R_{22} + R_{23})(-\pi^2 I_2 + I_{13}^2) \\
+ (R_{13} + R_{23} + R_{33})(-\pi^2 I_3 + I_{12}^3) \right),
\]
with
\[
I_j = \sqrt{2\pi} \sqrt{R_{jj}},
\]
\[
I^i_j = \frac{(2\pi)^\frac{1}{2}}{\sqrt{R_{kk}}} \arcsin \left( \frac{\rho_{ij} - \rho_{ik}\rho_{jk}}{\sqrt{1 - \rho_{ij}^2} \sqrt{1 - \rho_{ik}^2}} \right),
\]
\[
\rho_{ij} = \frac{R_{ij}}{\sqrt{R_{ii}R_{jj}}},
\]
Explicitly
\[
J^3(2; 1; \Sigma) = \frac{(2\pi)^\frac{3}{2}}{2^\frac{1}{2} m_1m_2m_3\sqrt{D^{(3)}}} \left[ \frac{(R_{11} + R_{12} + R_{13})}{\sqrt{\pi R_{11}}} \left( \frac{1}{8} + \frac{1}{4\pi} \arcsin \left( \frac{\rho_{33} - \rho_{13}\rho_{23}}{\sqrt{1 - \rho_{13}^2} \sqrt{1 - \rho_{23}^2}} \right) \right) \\
+ \frac{(R_{12} + R_{22} + R_{23})}{\sqrt{\pi R_{22}}} \left( \frac{1}{8} + \frac{1}{4\pi} \arcsin \left( \frac{\rho_{13} - \rho_{12}\rho_{23}}{\sqrt{1 - \rho_{12}^2} \sqrt{1 - \rho_{23}^2}} \right) \right) \\
+ \frac{(R_{13} + R_{23} + R_{33})}{\sqrt{\pi R_{33}}} \left( \frac{1}{8} + \frac{1}{4\pi} \arcsin \left( \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{1 - \rho_{13}^2} \sqrt{1 - \rho_{23}^2}} \right) \right) \right].
\]
(107)

In [33, section 4.2], the three-point function in 2D is also written as a sum of three terms. Each term looks like a two-point function in 2D. The three-point function in 3D is a linear combination of the 2D two-point functions with the coefficients written using the \( \tau_\mu \) variables. Equation (107) looks very similar, but is written explicitly in terms of the \( c_\mu \) variables, which are directly linked to the kinematical invariants. Davydychev [33] also presents results for the three-point function in 4 and 5D.

6.4. Four-point function in 4D

We write
\[
|\Sigma|^\frac{1}{2} = m_1m_2m_3m_4\sqrt{D^{(4)}}.
\]
(108)

In equation (92), eight terms are null and we get a constant, as well as six terms that are exactly like the two-point integral, plus an integral that involves all the variables \( \omega_i \). We have
\[
J^4(4; 1; \Sigma) = \frac{2\pi^2}{m_1 m_2 m_3 m_4 \sqrt{D^{(4)}}} \left[ 1 + \frac{2}{\pi} \arcsin \left( \frac{R_{12}}{\sqrt{R_{11} R_{22}}} \right) \right]
+ \frac{2}{\pi} \arcsin \left( \frac{R_{13}}{\sqrt{R_{11} R_{33}}} \right)
+ \frac{2}{\pi} \arcsin \left( \frac{R_{14}}{\sqrt{R_{11} R_{44}}} \right)
+ \frac{2}{\pi} \arcsin \left( \frac{R_{23}}{\sqrt{R_{22} R_{33}}} \right)
+ \frac{2}{\pi} \arcsin \left( \frac{R_{24}}{\sqrt{R_{22} R_{44}}} \right)
+ \frac{2}{\pi} \arcsin \left( \frac{R_{34}}{\sqrt{R_{33} R_{44}}} \right) + \frac{1}{\pi} I_{1234}. \tag{109}
\]

In the appendix, it is shown that \( I_{1234} \) is the sum of three integrals

\[
\frac{1}{\pi^2} I_{1234} = \frac{4\rho_{12}}{\pi^2} \int_0^1 \frac{du}{\sqrt{1 - \rho_{12}^2 u^2}} \arcsin \left( \frac{\alpha_{34}(u)}{\sqrt{1 - \rho_{12}^2 u^2}} \right)
+ \frac{4\rho_{13}}{\pi^2} \int_0^1 \frac{du}{\sqrt{1 - \rho_{13}^2 u^2}} \arcsin \left( \frac{\beta_{34}(u)}{\sqrt{1 - \rho_{13}^2 u^2}} \right)
+ \frac{4\rho_{14}}{\pi^2} \int_0^1 \frac{du}{\sqrt{1 - \rho_{14}^2 u^2}} \arcsin \left( \frac{\gamma_{34}(u)}{\sqrt{1 - \rho_{14}^2 u^2}} \right), \tag{110}
\]

with

\[
\alpha_{33} = (1 - \rho_{23}^2) + u^2(2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2),
\alpha_{34} = (\rho_{14} - \rho_{23}\rho_{24}) + u^2(\rho_{12}\rho_{13}\rho_{24} + \rho_{12}\rho_{14}\rho_{23} - \rho_{12}^2\rho_{34}),
\alpha_{44} = (1 - \rho_{24}^2) + u^2(2\rho_{12}\rho_{14}\rho_{24} - \rho_{12}^2 - \rho_{24}^2),
\beta_{22} = (1 - \rho_{23}^2) + u^2(2\rho_{13}\rho_{12}\rho_{23} - \rho_{13}^2 - \rho_{12}^2),
\beta_{24} = (\rho_{24} - \rho_{23}\rho_{34}) + u^2(\rho_{13}\rho_{12}\rho_{34} + \rho_{13}\rho_{14}\rho_{23} - \rho_{12}\rho_{14} - \rho_{13}^2\rho_{24}),
\beta_{44} = (1 - \rho_{24}^2) + u^2(2\rho_{13}\rho_{14}\rho_{34} - \rho_{13}^2 - \rho_{14}^2),
\gamma_{22} = (1 - \rho_{23}^2) + u^2(2\rho_{14}\rho_{12}\rho_{24} - \rho_{14}^2 - \rho_{12}^2),
\gamma_{23} = (\rho_{23} - \rho_{24}\rho_{34}) + u^2(\rho_{14}\rho_{12}\rho_{34} + \rho_{14}\rho_{13}\rho_{24} - \rho_{12}\rho_{13} - \rho_{14}^2\rho_{23}),
\gamma_{33} = (1 - \rho_{24}^2) + u^2(2\rho_{14}\rho_{13}\rho_{34} - \rho_{14}^2 - \rho_{13}^2).
\]

where

\[
\rho_{ij} = \frac{R_{ij}}{\sqrt{R_{ii} R_{jj}}}, \tag{111}
\]

\( \Delta_{ij} \) is the \((i, j)\) co-factor of the matrix \( \Sigma \). This formula is not explicitly symmetric with respect to the indices \((1, 2, 3, 4)\), because we have chosen to transform the denominator \( \omega_i \).

We could have symmetrised the final expression by cyclic permutation of the indices. The symmetrised result represents 12 integrals of the arcsine and square root functions whose values in the complex plane are known. In [19], the geometrical interpretation worked easily for the two- and three-point functions. For the four-point function in four dimensions, the
computation of the volume of a four-dimensional tetrahedron is quite complicated. In general a symmetric result can be written by decomposing the tetrahedron into 12 so-called bi-rectangular tetrahedrons. The volume of a bi-rectangular tetrahedron can be expressed in terms of the Lobachevsky or Schafli functions, which can be related to dilogarithms. The volume of the four-dimensional tetrahedron is, according to our probabilistic interpretation, related to the quadrivariate normal orthant probability. This fact has already been exploited in \cite{34}, where the quadrivariate normal orthant probability is computed using the decomposition of the four-dimensional tetrahedron into the bi-rectangular tetrahedron, which are called orthoschemes in \cite{34}. Using Fourier transforms allows us to circumvent the difficult geometrical decomposition. For $N \geq 5$, the geometrical interpretations seem to become unrealisable, while the Fourier transform still performs well. Besides the geometrical approach to the four-point function, direct integration of the Feynman parameters in equation (31), as done in \cite{5, 7, 9}, has produced results in terms of a varying number of dilogarithms, depending on the assumptions of the internal masses and the momenta. Several transformations are applied to the Feynman representation equation (31) to write the final results as a sum of dilogarithms. The Fourier approach disentangles the singularities in Fourier space to produce contributions to the four-point function that can be recognised as two-point functions in 2D. From a first look at the remaining integrals, it looks difficult to cast them in terms of dilogarithms to make contact with the Feynman parameter integration. The presence of the square root and the arcsine function suggests the integrals might be expressible in terms of elliptic functions. References \cite{6, 8} present expressions that are also numerically stable. More efforts are required to implement the above formula in a computer program and study its numerical stability.

6.5. The five-point function in 5D

There are $2^5$ terms of which only 16 survive. With

$$\det \Sigma = \prod_{i=1}^{5} m_i^2 D^{(5)},$$

$$J^5(5; \mathbf{1}; \Sigma) = \frac{2\pi^2}{5} \frac{1}{\prod_{i=1}^{5} m_i \sqrt{D^{(5)}}} \left[ 1 + \frac{2}{\pi} \sum_{i<j} \arcsin \rho_{ij} + \frac{1}{\pi^3} \sum_{i=1}^{5} I_{12345\mathbf{1}} \right],$$

where $I_{12345\mathbf{1}}$ contains four denominators and is thus computed as $I_{ijkl}$, with a correlation matrix $R^{(5)}$ obtained from the matrix $R$ by removing the column and row $i$

$$\frac{1}{\pi^2} I_{12345\mathbf{1}} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \prod_{j \neq i} d \omega_j \prod_{j \neq i} \omega_j e^{-\frac{1}{2} \omega_i^2 R^{(5)} \omega}. \tag{113}$$

6.6. The six-point function in 6D

There are $2^6$ terms of which only 32 survive.

$$J^6(6; \mathbf{1}; \Sigma) = \frac{2\pi^3}{6} \frac{1}{\prod_{i=1}^{6} m_i \sqrt{D^{(6)}}} \frac{1}{64} \left[ 1 + \frac{2}{\pi} \sum_{i<j} \arcsin \rho_{ij} + \frac{1}{\pi^3} \sum_{i<j} I_{ij}^4 - \frac{1}{\pi^2} I_{ijklm} \right],$$

$$\tag{114}$$
with $I_{pq} = I_{ijklmn}(p,q)$. $I_{pq}$ is computed exactly as $I_{ijkl}$ as it contains four $\omega$ in the denominator. Regarding $I_{ijklmn}$, we proceed as for $I_{ijkl}$, except that in this case, we will need two integrations to decrease the number of $\omega$ in the denominator. $I_{ijklmn}$ is computed in appendix C

$$\frac{1}{\pi^6} I_{ijklmn}(R) = - \frac{2}{\pi^3} \int_0^1 du \sum_{r \neq l, q} \frac{\rho_{rj}}{\sqrt{1 - \rho_{rj}^2 u^2}} I_{ijklmn}(\tilde{\rho}(u)), \quad (115)$$

with the matrix $\tilde{\rho}$

$$\tilde{\rho}_{rs}(u) = \rho_{rs} - \rho_{ir} \rho_{js} u^2 + \frac{1}{1 - \rho_{ij}^2 u^2} \left( \rho_{jr} - \rho_{ij} \rho_{ir} u^2 \right) \left( \rho_{js} - \rho_{ij} \rho_{is} u^2 \right). \quad (116)$$

In [35–37], the hexagonal diagram in six dimensions was considered with zero, one and three masses. The results were given in terms of polylogarithms. The three-mass case result is expressed as a sum of 24 terms involving only one basic function, which is a simple linear combination of logarithms, dilogarithms and trilogarithms. In our case, the result is also compact and derived straightforwardly. The interest in the six-point function in six dimensions stems from its relation to the MHV amplitudes in $\mathcal{N} = 4$ SYM theory [38].

### 6.7 The five-point function in 4D

The term $\left( \sum_{i=1}^5 x_i \right)^{\nu-n} = \left( \sum_{i=1}^5 x_i \right)$ in equation (79) contributes to the integral. We replace each $x_i$ by $i \frac{\partial}{\partial \omega_i}$. We get

$$J_5^5(4; 1; \Sigma) = - \frac{2^\frac{5}{2} \pi^\frac{5}{2}}{\prod_{i=1}^5 m_i \sqrt{D(5)}} \frac{1}{(2\pi)^5} \int_{-\infty}^{\infty} d^5 \omega \prod_{j=1}^5 \left( \pi \delta(\omega_j) - \frac{1}{\omega_j} \right) \left( i \sum_{j=1}^5 \frac{\partial}{\partial \omega_j} \right) e^{-\frac{1}{2} \omega^T R \omega}. \quad (117)$$

Each derivative with respect to $\omega_j$ generates five terms, so in total we have $2^5 \times 5 \times 5 = 800$ terms, where only 200 survive

$$J_5^5(4; 1; \Sigma) = - \frac{2^\frac{5}{2} \pi^\frac{5}{2}}{\prod_{i=1}^5 m_i \sqrt{D(5)}} \frac{1}{32} \left[ \frac{1}{\pi} \sum_{i=1}^5 \left( \sum_{j \neq i} R_{ij} + \sum_{j > i} R_{ij} \right) I_i 
+ \frac{1}{\pi^2} \sum_{j > i} \sum_{k \neq i,j} t_{ij} \left( \sum_{l \leq i} R_{ik} + \sum_{l > i} R_{ik} \right) I_{(12345 \setminus \{i\})} \right]. \quad (118)$$

This last formula can be interpreted as a decomposition of the five-point function in 4D in terms of the four-point functions in 4D. Reduction formulae have been presented in [10–14]. The decomposition of the five-point function in 4D in terms of a sum of the four-point functions in 4D was first achieved in [10]. Melrose [11] has generalised the approach to the $N$-point ($N \geq 5$) functions and presented a detailed decomposition for the five-, six- and seven-point functions in 4D. It is remarkable that for $N = 5$, the Fourier transform produces rather simple expressions that can also be interpreted as decomposition formulae. References [13, 14] have also presented original decomposition formulae.
6.8. The six-point function in 4D

The term \( \left( \sum_{i=1}^{6} x_i \right)^{\nu-n} \) in equation (79) contributes to the integral. We write
\[
\left( \sum_{i=1}^{6} x_i \right)^{2} e^{-\frac{1}{2} x^T R^{-1} x} = \left( \sum_{i=1}^{6} x_i^2 + 2 \sum_{i<j} x_i x_j \right) e^{-\frac{1}{2} x^T \Sigma x},
\]
so that
\[
\mathcal{J}^{(4;1)}(\Sigma) = -\frac{2}{6} \prod_{i=1}^{6} m_i \sqrt{D(6)} \left( \sum_{i=1}^{6} \frac{\partial}{\partial \Sigma_{ii}} + \sum_{i<j} \frac{\partial}{\partial \Sigma_{ij}} \right) e^{-\frac{1}{2} x^T \Sigma x},
\]

Unlike [12], which produced a decomposition formula in the case \( N > n \), where the six-point function in 4D is an example, our formula is written in terms of the six-point function in 6D. It is rather compact, but taking the derivatives will likely generate a large number of terms. It is unclear if the double integrals appearing in the six-point function in 6D can be simplified so that the six-point function in 6D can be explicitly decomposed in terms of the four-point functions.

6.9. \( \nu_i \geq 1 \) and \( \nu > n \)

In the general case where \( \nu_i > 1 \), the \( x_i \) variables in the integrand of equation (79) will contribute the term \( \prod_{i=1}^{N} \left( \sum_{j=1}^{N} x_j \right)^{\nu-n} \) which is also dealt with by replacing each \( x_i \) by \( i \frac{\partial}{\partial \omega_i} \), so that
\[
\mathcal{J}^{(N;\nu)}(\Sigma) = -\frac{(-1)^{\nu-n}(2\pi)^{\frac{N}{2}}}{2^{\nu-n-1} \prod_{i=1}^{N} \Gamma(\nu_i) m_i \sqrt{D(N)}} \left( \int_{-\infty}^{\infty} d^N \omega \right) \left( \prod_{j=1}^{N} \left( \pi \delta(\omega_j) - i \frac{1}{\omega_j} \right) \right) \times \prod_{j=1}^{N} \left( i \frac{\partial}{\partial \omega_j} \right)^{\nu_j-1} \left( i \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} \right)^{\nu-n} e^{-\frac{1}{2} \omega^T R \omega}.
\]

Unlike [12], which produced a decomposition formula in the case \( N > n \), where the six-point function in 4D is an example, our formula is written in terms of the six-point function in 6D. It is rather compact, but taking the derivatives will likely generate a large number of terms. It is unclear if the double integrals appearing in the six-point function in 6D can be simplified so that the six-point function in 6D can be explicitly decomposed in terms of the four-point functions.

6.10. \( \nu - n > 0 \)

In this case, we can introduce the inverse Laplace transform result
\[
x^{q} \mathbb{1}_{x > 0} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(q+1)}{s^{q+1}} e^{sx},
\]
with \( c > 0 \) and \( \Re(q) > -1 \). Plugging this result, with \( q = \frac{\nu-n}{2} \), in equation (79), we obtain
\[ J^N(n < \nu; \nu; \Sigma) = \frac{(-1)^{n}}{2^{n-3} - 1} \prod_{\nu} \Gamma(\nu) \int_{-\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{s^{n-1} + 1} \int_{0}^{\infty} ds \prod_{i} x_{i}^{n-1} e^{-\frac{1}{2} \Sigma_{\nu,\nu}(\Sigma_{\nu})^{1}}, \]

The diagonal elements become
\[ \Sigma_{ii} = 2s = m_{i}^{2} - 2s = \tilde{m}_{i}^{2}, \]
and the non-diagonal elements become \((j \neq \ell)\)
\[ \Sigma_{ij} = 2s = \frac{m_{j}^{2} + m_{\ell}^{2} - k_{ij}^{2}}{2m_{j}m_{\ell}} = 2s, \]

\[ = \tilde{m}_{i} \tilde{m}_{\ell} \tilde{m}_{\ell} = \tilde{m}_{i}^{2} + \tilde{m}_{\ell}^{2} - k_{ij}^{2}. \]

We introduce the matrix \(\tilde{\Sigma}\), which is the same as the matrix \(\Sigma\), but with the mass parameters \(m_{j}^{2}\) replaced by \(\tilde{m}_{j}^{2}\). In this case, we see that the \(N\)-point function with \(\nu > n\) is obtained by performing the inverse Laplace transform of the \(N\)-point function for which \(n = \nu\) and the matrix \(\tilde{\Sigma}\)

\[ J^{N}(n < \nu; \nu; \Sigma) = \frac{(-1)^{n}}{2^{n-3} - 1} \prod_{\nu} \Gamma(\nu) \int_{-\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{s^{n-1} + 1} J^{N}(\nu; \nu; \Sigma(s)), \]

with \(\tilde{m}_{i}^{2} = m_{i}^{2} - s\) after rescaling \(s\) to \(s/2\). In the previous section, we have seen that some computations contain many terms. Using this inverse Laplace transform, we can compute the six-point function in four dimensions using the six-point function in six dimensions, but with the complex mass squared \(m_{j}^{2} - s\). This inverse Transform also frees us from being careful in the bookkeeping, which is necessary when the number of terms increases, as seen above. This result does not depend on probabilistic interpretation as it can also be easily derived from the Feynman parameter representation equation (31).

### 6.11. \(\nu_{l} \geq 1\) and \(\nu < n\)

In this case, we use equation (40), and doing as above we end up with the following relation

\[ J^{N}(n > \nu; \nu; \Sigma) = \frac{1}{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)} \int_{0}^{\infty} d\tau \frac{\tau^{s-1}}{s^{n-1}} J^{N}(\nu; \nu; \tilde{\Sigma}(\tau)). \]

The matrix \(\tilde{\Sigma}(\tau)\) is the same as the matrix \(\Sigma\) but with the mass parameters \(m_{j}^{2}\) replaced by \(m_{j}^{2} + \tau\).

### 6.12. \(\nu_{l} > 1, n = \nu\)

Also of interest are diagrams with some propagator power that is larger than one, i.e. \(\nu_{k} > 1\). Let us consider the case of an \(N\)-point function in \(n = \nu = N - 1 + \nu_{k}\) dimensions, with the \(k\)th propagator having the power \(\nu_{k} > 1\). In equation (79), the term \(\sum_{i} x_{i}\) disappears but remains as a term \(x_{k}^{\nu_{k}-1}\). We use equation (121)
\[
\chi_k^{\nu_k-1} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \Gamma\left(\frac{\nu_k-1}{2}\right) e^{s\chi_k}.
\]

We end up with
\[
J^N(N - 1 + \nu_k; \nu_j = 1 + (\nu_k - 1)\delta_{kj}; \Sigma) = (-1)^{\nu_k - 1} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \Gamma\left(\frac{\nu_k}{2}\right) e^{s\chi_k}.
\]

The matrix \(\Sigma^{k\nu} \) is the same as the matrix \(\Sigma \) with the mass parameters \(m_k^2 \) replaced by \(m_k^2 - s \), but only in the \(\Sigma_{kk} \) component. The other components remain unchanged; that is, \(\Sigma_{kk} \) is changed into
\[
\Sigma_{kk} = m_k^2 \to (\Sigma^k)_{kk} = m_k^2 - s.
\]

Another important case is when two propagators (with the index e.g. \(k \) and \(k' \)) have a power greater than one in \(n = \nu = N - 2 + 2\nu_k \) dimensions. We consider the case when the powers are equal, i.e. \(\nu_k = \nu_{k'} \). In equation (79), we have the term \((\chi_k \chi_{k'})^{\nu_k-1}\), which we write as
\[
(\chi_k \chi_{k'})^{\nu_k-1} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(\nu_k)}{s^{\nu_k}} e^{s\chi_k \chi_{k'}}.
\]

We end up with
\[
J^N(N - 2 + 2\nu_k; \nu_j = 1 + (\nu_k - 1)\delta_{kj} + (\nu_k - 1)\delta_{kj' }; \Sigma)
\]

\[
= \frac{(-1)^{-2(\nu_k-1)}}{2^{2\nu_k} \Gamma(\nu_k)} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(\nu_k)}{s^{\nu_k}} J^N(N; \nu_j = 1; \Sigma^{k\nu}).
\]

The matrix \(\Sigma^{k\nu} \) is the same as the matrix \(\Sigma \) but with the component \(\Sigma_{kl} \) changed as follows:
\[
(\Sigma^{k\nu})_{kl} = m_k m_l \frac{m_k^2 + m_l^2 - k_{kl}^2}{2m_k m_l} - 2s,
\]

\[
= m_k m_l \frac{m_k^2 + m_l^2 - (k_{kl}^2 + 4s)}{2m_k m_l},
\]

which is the kinematical invariant \(k_{kl}^2 \) shifted by the quantity \(4s \)
\[
k_{kl}^2 \to k_{kl}^2 + 4s.
\]

Equations (129) and (126) show that the \(N\)-point functions with higher powers (and \(n = \nu \)) by integrating it in the complex plane \(N\)-point function (with \(\nu = N \)) with a modified \(\Sigma \) matrix. Another way to proceed, if we want to avoid integration in the complex plane, is to consider that a propagator with a power greater than one is the same as the product of the standard propagators, i.e.
\[
\frac{1}{(p_j + q)^2 - m_j^2} = \prod_{j=1}^{\nu_j} \frac{1}{(k_j + q)^2 - M_j^2},
\]

with \(M_j^2 = m_j^2 \) and \(k_j = p_j \). This means that we can use equation (79) again and obtain
\[
J^N(N - 2 + \nu_k + \nu_k; \nu_j = 1 + (\nu_k - 1)\delta_{kj} + (\nu_k - 1)\delta_{kj' }; \Sigma) = J^{N-2+\nu_k+\nu_{k'}}(N - 2 + \nu_k + \nu_{k'}; \nu_j = 1; \Sigma^{k\nu'}).
\]
where the matrix $\Sigma^{kk'}$ is obtained from the matrix $\Sigma$ by adding $\nu_k + \nu_{k'} - 2$ columns. In these columns, the component to be added is $m_j m_n c_{jm}$ for $j = 1, \ldots, N - 2 + \nu_k + \nu_{k'}$ and $n = N - 1, \ldots, N - 2 + \nu_k + \nu_{k'}$.

7. $\epsilon$ expansion

To show that equation (124) does not depend on the probabilistic interpretation, we start from the Feynman parametrisation equation (31)

$$J_N(n; \nu; \Sigma) = \frac{1}{\Gamma(\nu - n/2)} \prod_i \Gamma(\nu_i) \int \frac{d^\nu u_i}{(2\pi)^{\nu/2}} \delta \left( \sum_{i=1}^N u_i - 1 \right) \frac{1}{\left( -\sum_{j \neq \ell} u_j u_{\ell} k_{j\ell}^2 + \sum u_i m_i^2 \right)^{\nu - \frac{d}{2}}}.$$  

(134)

and apply the relation

$$\frac{1}{A^{\lambda_1} B^{\lambda_2}} = \frac{\Gamma(\lambda_1 + \lambda_2)}{\Gamma(\lambda_1) \Gamma(\lambda_2)} \int_0^\infty dx \frac{x^{\lambda_2 - 1}}{(A + x B)^{\lambda_1 + \lambda_2}}$$

(135)

with

$$A = -\sum_{j \neq \ell} u_j u_{\ell} k_{j\ell}^2 + \sum u_i m_i^2,$$

$$\lambda_1 = \nu - \frac{n}{2} = \nu - \frac{d}{2} + \epsilon > 0,$$

$$B = 1,$$

$$\lambda_2 = k - \epsilon > 0.$$  

(136)

$$J_N(n; \nu; \Sigma) = \frac{1}{\Gamma(\nu - n/2)} \prod_i \Gamma(\nu_i) \int \frac{d^\nu u_i}{(2\pi)^{\nu/2}} \delta \left( \sum_{i=1}^N u_i - 1 \right)$$

$$\times \frac{1}{\left( -\sum_{j \neq \ell} u_j u_{\ell} k_{j\ell}^2 + \sum u_i (m_i^2 + x) \right)^{\nu - \frac{d}{2}}}$$

$$= \frac{1}{\Gamma(k - \epsilon)} \int_0^\infty dx x^{k - \epsilon - 1} J_N^N(d - 2\epsilon; \nu; \Sigma(x)),$$  

(137)

where $\Sigma(x)$ as before but with the squared mass $m_i^2 + x$ for each line in the diagram. We have obtained a way to express an $N$-point function in decimal dimensions as a sum of $N$-point functions in integer dimensions, with all possible masses from $m_i^2$ to $\infty$. For example, with $N = 5$ in $n = 6 - 2\epsilon$ dimensions we can write

$$J_5^5(6 - 2\epsilon; \nu_j = 1; \Sigma) = \frac{1}{\Gamma(1 - \epsilon)} \int_0^\infty dx x^{1 - \epsilon} J_5^5(4; \nu_j = 1; \Sigma(x)),$$

$$= \frac{1}{\Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty dx x^{\frac{1}{2} - \epsilon - 1} J_5^5(5; \nu_j = 1; \Sigma(x)).$$  

(138)
We prefer the expression which involves \( J^5(5; \nu_j = 1; \Sigma(x)) \) as it is an easier one to compute in this framework, as well as being the one reproduced exactly by equation (124). The \( \epsilon \) expansion is given by

\[
J^5(6 - 2\epsilon; \nu_j = 1; \Sigma) = \frac{2}{\Gamma(\frac{1}{2} - \epsilon)} \sum_{k=0}^{\infty} \frac{(-2\epsilon)^k}{k!} \int_0^\infty du \ln(u)^k J^5(5; \nu_j = 1; \Sigma(u)),
\]

(139)

with \( u = \sqrt{x} \).

8. Tensor integrals

In [20], a formula was derived for the reduction of tensor integrals. We take, for example, the case of an tensor \( N \). In [20], a formula was derived for the reduction of tensor integrals. We take, for example, the case of an tensor \( N \) with \( n = 4 - 2\epsilon \) we need to compute three diagrams, i.e.

\[
\begin{align*}
J^5(6 - 2\epsilon; \nu_j = 1; \Sigma), \\
J^5(8 - 2\epsilon; \nu_j = 1 + 2\delta_k; \Sigma), \\
J^5(8 - 2\epsilon; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma).
\end{align*}
\]

(141)

Using equation (137) we have

\[
\begin{align*}
J^5(6 - 2\epsilon; \nu_j = 1; \Sigma) &= \frac{1}{\Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty dx x^{\frac{1}{2} - \epsilon - 1} J^5(5; \nu_j = 1; \Sigma(x)), \\
J^5(8 - 2\epsilon; \nu_j = 1 + 2\delta_k; \Sigma) &= \frac{1}{\Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty dx x^{\frac{1}{2} - \epsilon - 1} J^5(7; \nu_j = 1 + 2\delta_k; \Sigma(x)), \\
J^5(8 - 2\epsilon; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma) &= \frac{1}{\Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty dx x^{\frac{1}{2} - \epsilon - 1} J^5(7; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma(x)),
\end{align*}
\]

(142, 143, 144)

where we have used \( k = \frac{1}{2} \). \( J^5(5; \nu_j = 1; \Sigma(x)) \) is computed using equation (112), \( J^5(7; \nu_j = 1 + 2\delta_k; \Sigma(x)) \), using equations (126) or (133) and \( J^5(7; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma(x)) \) using equations (129) or (133)

\[
J^5(7; \nu_j = 1 + 2\delta_k; \Sigma(x)) = J^5(7; \nu_j = 1; [\Sigma(x)]^{kk}),
\]

(145)
\[ J^5(7; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma(x)) = J^7(7; \nu_j = 1; [\Sigma(x)]^{kk'}), \]  
(146)

or

\[ J^5(7; \nu_j = 1 + 2\delta_k; \Sigma(x)) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{1}{s^2} J^5(5; \nu_j = 1; [\Sigma(x)]^{kk'}(s)), \]  
(147)

\[ J^5(7; \nu_j = 1 + \delta_k + \delta_{k'}; \Sigma(x)) = \frac{1}{2} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{1}{s^2} J^5(5; \nu_j = 1; [\Sigma(x)]^{kk'}(s)). \]  
(148)

By applying two transformations on the matrix \( \Sigma \), an \( N \)-point function with higher powers of the propagators, and in decimal dimensions, is reduced to the computation of an \( N' \)-point function in integer dimensions. Moreover, the \( \epsilon \) expansion is explicit and easy to derive. In the case of \( N = 5 \), we end up computing the complex integral of a five-point function or a seven-point function

\[ J^7(7; \nu_j = 1; \Sigma) = \pi \omega \prod_{i=1}^{m_i} \sqrt{D^7} \int_{c-i\infty}^{c+i\infty} \frac{d7\omega}{2\pi} \prod_{j=1}^{7} \left( \delta(\omega_j) - \frac{i}{\omega_j} \right) e^{-\frac{1}{2}x^T \Sigma^T \Sigma x}, \]

(149)

This expression only involves quantities we know how to compute.

9. Conclusion

Recasting the standard Feynman parameter expression for the \( N \)-point function into a probability problem allowed us to find new integral recurrence relations between the \( N \)-point functions. We have also derived a Hermite polynomial expansion as a general result. However, the multivariate polynomials are still a challenge for numerical evaluation. The series might still be of interest for asymptotic expansions and we hope to spend more time on the analysis of the series. We have also derived a multi-fold integral representation for the \( N \)-point function that admits a probabilistic interpretation. The \( N \)-point function is related to the truncated moments of the multivariate normal distribution. This distribution has been extensively studied and many numerical and mathematical methods have been developed for its computation. Other methods can be borrowed from the statistical literature to compute the \( N \)-point function. Using Fourier transforms, it was possible to compute several \( N \)-point functions in integer dimensions. The extension to the case of decimal dimensions was made possible by introducing an extra variable. The \( N \)-point function in decimal dimensions is given by integrating the \( N \)-point functions in integer dimensions, but with mass parameters that depend on the integration variable. The case of tensor integrals of rank \( r = 2 \) with \( N = 5 \) was treated explicitly. A reduction program was achieved in this case. We leave it to the future to extend the reduction program to tensors of rank \( r > 2 \), with \( N > 5 \).
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Appendix A. Computation of $I_{ij}$

We define

$$
\rho_{ij} = \frac{R_{ij}}{\sqrt{R_{ii} \sqrt{R_{jj}}}}. \quad (A.1)
$$

$I_{ij}(R)$ is given by

$$
I_{ij}(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 \omega d^2 \omega' \frac{1}{\omega_i \omega_j} e^{-\frac{1}{2} \sum_{m,n} \omega_m R_{mn} \omega_n}. \quad (A.2)
$$

We change variable $w_m \to \sqrt{\omega_i} w_m$

$$
I_{ij}(R) = -\pi \int_{-\infty}^{\infty} d^2 \omega \int_{0}^{\rho_{ij}} d\rho \ e^{-\frac{1}{2} (\omega_i^2 + \omega_j^2 + 2\rho_{ij} \omega_i \omega_j)}, \quad (A.3)
$$

For the function $f(\rho)$ we can write

$$
f(\rho) = f(0) + \int_{0}^{\rho} f'(u) du, \quad (A.4)
$$

where in our case the function $f(\rho)$ is $I_{ij}$ as a function of $\rho_{ij}$. For $\rho_{ij} = 0$, $I_{ij}$ is zero, so we are left with

$$
I_{ij}(R) = -\pi \int_{-\infty}^{\infty} d^2 \omega \int_{0}^{\rho_{ij}} d\rho \ e^{-\frac{1}{2} (\omega_i^2 + \omega_j^2 + 2\rho_{ij} \omega_i \omega_j)}. \quad (A.5)
$$

$$
= -2\pi \int_{0}^{\rho_{ij}} d\rho \ e^{-\frac{1}{2} (\omega_i^2 + \omega_j^2 + 2\rho_{ij} \omega_i \omega_j)}. \quad (A.6)
$$

$$
= -2\pi \arcsin \rho_{ij}. \quad (A.7)
$$

$$
= -2\pi \arcsin \frac{R_{ij}}{\sqrt{R_{ii} R_{jj}}}. \quad (A.8)
$$

Appendix B. Computation of $I_{ijkl}$

$I_{ijkl}(R)$ is given by

$$
\frac{1}{\pi^4} I_{ijkl}(R) = \frac{1}{\pi^4} \int_{-\infty}^{\infty} d^4 \omega d^4 \omega' \frac{1}{\omega_i \omega_j \omega_k \omega_l} e^{-\frac{1}{2} \sum_{m,n} \omega_m R_{mn} \omega_n}. \quad (B.1)
$$

We rescale the $\omega$ variable so that the matrix $R$ has unit numbers in the diagonal $\omega_i = \frac{\sqrt{\rho}}{\sqrt{R_{ii}}}$ and use
\[
\frac{1}{\omega_i} = \frac{\omega_j}{\omega_j^2} = \omega_j \int_0^\infty d\tau e^{-\tau \omega_j^2}, \tag{B.2}
\]

which combined with the already existing \(\omega_i^2\) in the exponential gives

\[
\frac{1}{\pi} I_{ijkl}(R) = \frac{1}{\pi^3} \int_1^\infty d\tau \int_{-\infty}^\infty d^4\omega \frac{\omega_i}{\omega_j \omega_k \omega_l} e^{-\tau \omega_i^2 - \sum_{\alpha \neq i} \omega_\alpha^2 - 2 \sum_{\alpha \beta} \omega_\alpha \rho_{\alpha \beta} \omega_\beta}, \tag{B.3}
\]

with (no assumed summation of repeated indices)

\[
\rho_{mn} = \frac{R_{mn}}{\sqrt{R_{mm} R_{nn}}},
\]

\[
= \frac{\Delta_{mn}}{\sqrt{\Delta_{mm} \Delta_{nn}}}, \tag{B.4}
\]

where \(\Delta_{mn}\) is the \((m, n)\) co-factor of the matrix \(\Sigma\).

We set the functions

\[
f(\omega_i) = \omega_i e^{-\tau \omega_i^2},
\]

\[
g(\omega_i) = e^{-2\omega_i (\rho_{ij} \omega_j + \rho_{ik} \omega_k + \rho_{il} \omega_l)},
\]

and perform an integration by part whose first contribution is zero and the remaining integral

\[
\frac{1}{\pi^3} I_{ijkl}(R) = -\frac{1}{\pi^3} \int_1^\infty d\tau \int_{-\infty}^\infty d^4\omega \left( \frac{\rho_{ij}}{\omega_i \omega_j} + \frac{\rho_{ik}}{\omega_i \omega_k} + \frac{\rho_{il}}{\omega_i \omega_l} \right) e^{-\tau \omega_i^2 - \sum_{\alpha \neq i} \omega_\alpha^2 - 2 \sum_{\alpha \beta} \omega_\alpha \rho_{\alpha \beta} \omega_\beta},
\]

\[
= -\frac{1}{\pi^3} \int_1^\infty d\tau \left( \rho_{ij} F_{ij}^k(\tau) + \rho_{ik} F_{ik}^j(\tau) + \rho_{il} F_{il}^j(\tau) \right),
\]

with

\[
F_{ij}^k(\tau) = \int_{-\infty}^\infty d^4\omega \frac{1}{\omega_i \omega_j} e^{-\tau \omega_i^2 - \sum_{\alpha \neq i} \omega_\alpha^2 - 2 \sum_{\alpha \beta} \omega_\alpha \rho_{\alpha \beta} \omega_\beta}. \tag{B.5}
\]

We integrate first the \(\omega\) variables that do not appear in the denominator and get

\[
F_{ij}^k(\tau) = \int_{-\infty}^\infty d\omega_i \int_{-\infty}^\infty d\omega_j \frac{e^{-\omega_i^2 - \omega_j^2 - 2\rho_{ij} \omega_i \omega_j}}{\omega_i \omega_j} \int_{-\infty}^\infty d\omega_j e^{-\omega_j^2 - 2\omega_j (\rho_{ij} \omega_j + \rho_{ij} \omega_i)}
\]

\[
\times \int_{-\infty}^\infty d\omega_i e^{-\tau \omega_i^2 - 2\omega_i (\rho_{ij} \omega_j + \rho_{ij} \omega_i) + \rho_{ij} \omega_i}. \tag{B.6}
\]

With

\[
\int_{-\infty}^\infty d\omega e^{-a \omega^2 + bw} = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{b^2}{4a}}, \tag{B.7}
\]

we obtain
$F_{ij}^{bl}(\tau) = \int_{-\infty}^{\infty} d\omega_i \int_{-\infty}^{\infty} d\omega_k \frac{e^{-\omega_i^2 - \omega_k^2 - 2\rho_i\omega_i\omega_k}}{\omega_i\omega_k} \\
\int_{-\infty}^{\infty} d\omega_j e^{-\omega_j^2 - 2\rho_j(\rho_i\omega_i + \rho_k\omega_k)\sqrt{\frac{\pi}{\tau}} e^{\frac{(\rho_j\omega_j + \rho_i\omega_i + \rho_k\omega_k)^2}{\tau}}}$

$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \int_{-\infty}^{\infty} d\omega_i \int_{-\infty}^{\infty} d\omega_k \frac{e^{-\omega_i^2 - \omega_k^2 - 2\rho_i\omega_i\omega_k}}{\omega_i\omega_k} e^{\frac{(\rho_i\omega_i + \rho_k\omega_k + \rho_j\omega_j)^2}{\tau}} (\omega_i(\rho_j\omega_j + \omega_k(\tau\rho_k - \rho_i\omega_i)))$

$= \frac{\pi}{\sqrt{\tau - \rho_j^2}} \int_{-\infty}^{\infty} d\omega_i \int_{-\infty}^{\infty} d\omega_k \frac{1}{\omega_i\omega_k} e^{-\omega_i^2 - \omega_k^2 - 2\rho_i\omega_i\omega_k} \left(\frac{\alpha_i(\tau)}{\sqrt{\alpha_i(\tau)\alpha_i(\tau)}}\right)$, (B.8)

with

\[
\alpha_k(\tau) = \frac{\tau}{2} - \frac{\rho_k^2}{2} + 2\rho_i\rho_k\rho_j - \rho_j^2 - \rho_k^2, = \alpha_0^0 + \alpha_1^1, \\
\alpha_{kl} = \frac{\tau}{2} + \frac{\rho_k^2 - \rho_j^2}{2} + \rho_i\rho_k\rho_j + \rho_j\rho_k\rho_l - \rho_k^2\rho_l, = \alpha_0^0 + \alpha_1^1, \\
\alpha_l = \frac{\tau}{2} + 2\rho_j\rho_k\rho_l - \rho_j^2 - \rho_k^2, = \alpha_0^0 + \alpha_1^1.
\]

Bringing everything together we get

\[
\frac{1}{\pi^2} I_{ijkl}(R) = \frac{4}{\pi^2} \int_0^1 du \left( \frac{\rho_j}{\sqrt{1 - \rho_j^2 u^2}} \arcsin \left( \frac{\alpha_0^0 + \alpha_1^1 u^2}{\sqrt{(\alpha_0^0 + \alpha_1^1 u^2)(\alpha_0^0 + \alpha_1^1 u^2)}} \right) + \frac{\rho_k}{\sqrt{1 - \rho_k^2 u^2}} \arcsin \left( \frac{\alpha_0^0 + \alpha_1^1 u^2}{\sqrt{(\alpha_0^0 + \alpha_1^1 u^2)(\alpha_0^0 + \alpha_1^1 u^2)}} \right) + \frac{\rho_l}{\sqrt{1 - \rho_l^2 u^2}} \arcsin \left( \frac{\alpha_0^0 + \alpha_1^1 u^2}{\sqrt{(\alpha_0^0 + \alpha_1^1 u^2)(\alpha_0^0 + \alpha_1^1 u^2)}} \right) \right), \tag{B.9}
\]

where we have made a change to the variable $\tau = 1/u^2$.

**Appendix C. Computation of $I_{ijklmn}$**

$I_{ijklmn}(R)$ is given by
\[
\frac{1}{\pi^6}I_{ijklmn}(R) = \frac{1}{\pi^6} \int_{-\infty}^{\infty} d^6\omega \frac{1}{\omega^2} e^{-\frac{i}{\pi^6} \sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}. \tag{C.1}
\]

We proceed as for \(I_{ijkl}(R)\) by rescaling the \(\omega\) variables, which make us use the matrix \(\rho\) of equation (B.4), use equation (B.2) and get (after integration by part as above)

\[
\frac{1}{\pi^6}I_{ijklmn}(R) = -\frac{1}{\pi^6} \int_{1}^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} d^6\omega \left( \frac{\rho_{ij}}{\omega^2} + \frac{\rho_{ik}}{\omega^2} + \frac{\rho_{il}}{\omega^2} \right) \frac{1}{\omega^2} e^{-\frac{\tau}{\pi^6} \sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} + \frac{\rho_{im}}{\omega^2} \frac{1}{\omega^2} e^{-\frac{\tau}{\pi^6} \sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} + \frac{\rho_{i\omega}}{\omega} \frac{1}{\omega^2} e^{-\frac{\tau}{\pi^6} \sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}},
\]

\[
= -\frac{1}{\pi^6} \int_{1}^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} d^6\omega \left( \sum_{\rho \neq i} \frac{\rho_{i\rho}}{\omega} \right) \frac{1}{\omega^2} e^{-\frac{\tau}{\pi^6} \sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}, \tag{C.2}
\]

where \(jklnm \setminus r\) means we remove the index that is equal to \(r\). Taking \(r = j\),

\[
H_{klmn}^{ij}(\tau) = \int_{-\infty}^{\infty} d^4\omega e^{-\frac{\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}{\omega^2}} \int_{-\infty}^{\infty} d\omega_j e^{-\omega_j^2 - 2\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} \int_{-\infty}^{\infty} d\omega_i e^{-\omega_i^2 - 2\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}, \tag{C.4}
\]

where \(\rho_{ij}\) is the matrix \(\rho\) with the column and row \((i, j)\) removed so that it is a \((4, 4)\) matrix for the variables \((k, l, m, n)\). Because the indices \((i, j)\) are always different from the indices \((r, s)\), we have \(\rho_{ij}^{(r,s)} = \rho_{rs}\). The indices \((q, r, s)\) are in the set \((k, l, m, n)\) and the index \(p\) goes over \((j, k, l, m, n)\). The variables \(\omega_i\) and \(\omega_j\) do not appear in the denominator and are integrated first

\[
H_{klmn}^{ij}(\tau) = \int_{-\infty}^{\infty} d^4\omega e^{-\frac{\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}{\omega^2}} \int_{-\infty}^{\infty} d\omega_j e^{-\omega_j^2 - 2\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} \int_{-\infty}^{\infty} d\omega_i e^{-\omega_i^2 - 2\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} \sqrt{\frac{\pi}{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}},
\]

\[
= \int_{-\infty}^{\infty} d^4\omega e^{-\frac{\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}{\omega^2}} \int_{-\infty}^{\infty} d\omega_j e^{-\omega_j^2 - 2\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}} \sqrt{\frac{\pi}{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}} \frac{\pi}{\sqrt{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}},
\]

\[
= \frac{\pi}{\sqrt{\tau}} \int_{-\infty}^{\infty} d^4\omega e^{-\frac{\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}{\omega^2}} \frac{\pi}{\sqrt{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}} \frac{\pi}{\sqrt{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}},
\]

\[
= \frac{\pi}{\sqrt{\tau}} \int_{-\infty}^{\infty} d^4\omega e^{-\frac{\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}}{\omega^2}} \frac{\pi}{\sqrt{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}} \frac{\pi}{\sqrt{\tau}} e^{\frac{\left(\sum_{\omega} \omega_{\rho,\omega_j} R_{\omega,\omega_j}\right)^2}{\tau}}, \tag{C.5}
\]

\[
\]
where the matrix $\tilde{\rho}$ has the components ($\tau = \frac{1}{n^2}$)

$$
\tilde{\rho}_{\tau}(u) = \rho_{\tau} - \rho_{\tau} \rho_{\tau} u^2 + \frac{1}{1 - \rho_{\tau} u^2} \left( \rho_{\tau} - \rho_{\tau} \rho_{\tau} u^2 \right) \left( \rho_{\tau} - \rho_{\tau} \rho_{\tau} u^2 \right).
$$

(C.6)

The final result is

$$
\frac{1}{\pi^6} I_{ijklmn}(R) = -\frac{2}{\pi^5} \int_0^1 du \sum_{r \neq j} \rho_{ij} \rho_{ij} u^2 \sqrt{1 - \rho_{ij} u^2} I_{ijklmn}\tau(\tilde{\rho}(u)).
$$

(C.7)

$I_{ijklmn}(R)$ is an integral with four denominators and a $4 \times 4$ matrix $\tilde{\rho}$. We compute $I_{ijklmn}(\tilde{\rho}(u))$ using equation (B.9). $I_{ijklmn}(R)$ is thus given by 15 double integrals, with the integrand containing square roots and $\text{arcsin}$ functions whose analytical continuation in the complex plane is known.

References

[1] Staron N A 2006 Calculations of One-Loop N-Point Scalar Integrals for $N > 5$ (Janvier) (http://supgow.us.edu.pl/~ztpec/images/stories/seminaria/staron.pdf)

[2] Davydychev A I 1992 General results for massive n-point Feynman diagrams with different masses J. Math. Phys. 33 358–69

[3] Suzuki A T, Santos E S and Schmidt A G M 2003 Large loop integrals

[4] Fleischer J, Jegerlehner F and Tarasov O V 2003 A new hypergeometric representation of one-loop scalar integrals in $d$ dimensions Nucl. Phys. B 672 303–28

[5] t’Hooft G and Veltman M 1979 Scalar one-loop integrals Nucl. Phys. B 153 365–401

[6] van Oldenborgh G J and Vermaseren J A M 1990 New algorithms for one-loop integrals Z. Phys. C 46 425–37

[7] Denner A, Nierste U and Scharf R 1991 A compact expression for the scalar one-loop four-point function Nucl. Phys. B 367 637–56

[8] van Oldenborgh G J 1992 The complex four point function for arbitrary masses Phys. Lett. B 282 185–9

[9] Denner A and Dittmaier S 2011 Scalar one-loop 4-point integrals Nucl. Phys. B 844 199–242

[10] Halpern F R 1963 Reduction formula for the five-point function Phys. Rev. Lett. 10 310–2

[11] Melrose D B 1965 Reduction of Feynman diagrams Il Nuovo Cimento A 40 181–213

[12] Petersson B 1965 Reduction of a one-loop Feynman diagram with $n$ vertices in $m$-dimensional Lorentz space J. Math. Phys. 6 1955–9

[13] van Neerven W L and Vermaseren J A M 1984 Large loop integrals Phys. Lett. B 137 241–4

[14] Bern Z, Dixon L and Kosower D A 1993 Dimensionally regulated one-loop integrals Phys. Lett. B 302 299–308

[15] Kneihi B A and Tarasov O V 2010 Analytic result for the one-loop scalar pentagon integral with massless propagators Nucl. Phys. B 833 298–319

[16] Kozlov M G and Lee R N 2016 One-loop pentagon integral in $d$ dimensions from differential equations in $s$-form J. High Energy Phys. JHEP16(2016)21

[17] Tarasov O V 1996 Connection between Feynman integrals having different values of the space-time dimension Phys. Rev. D 54 6479–90

[18] Tarasov O V 2000 Application and explicit solution of recurrence relations with respect to space-time dimension Nucl. Phys. B 59 237–45

[19] Davydychev A I and Delbourgo R 1998 A geometrical angle on Feynman integrals J. Math. Phys. 39 4299–334

[20] Davydychev A I 1991 A simple formula for reducing Feynman diagrams to scalar integrals Phys. Lett. B 263 107–11

[21] Korolov L and Sinai Y G 2012 Theory of Probability and Random Processes 2nd edn (New York: Springer)
[22] Venkatesh S S 2012 *The Theory of Probability: Explorations and Applications* (Cambridge: Cambridge University Press)
[23] Davydychev A I 1991 Some exact results for \( n \)-point massive Feynman integrals *J. Math. Phys.* 32 1052–60
[24] Withers C S 2000 A simple expression for the multivariate Hermite polynomials *Stat. Probab. Lett.* 47 165–9
[25] Carlson B C 1963 Lauricella’s hypergeometric function \{FD\} *J. Math. Anal. Appl.* 7 452–70
[26] Kreimer D 1992 One-loop integrals revisited *Phys. C* 54 667–71
[27] Kreimer D 1993 One-loop integrals revisited—the three-point functions *Int. J. Mod. Phys.* A 08 1797–814
[28] Brucher L, Franzkowski J and Kreimer D 1994 Loop integrals, \( r \)-functions and their analytic continuation *Mod. Phys. Lett.* A 09 2335–45
[29] van Hameren A 2011 One-loop: for the evaluation of one-loop scalar functions *Comput. Phys. Commun.* 182 2427–38
[30] Childs D R 1967 Reduction of the multivariate normal integral to characteristic form *Biometrika* 54 293–300
[31] Hansen E W 2014 *Fourier Transforms: Principles and Applications* 1st edn (New York: Wiley)
[32] Butz T 2015 *Fourier Transformation for Pedestrians* (Undergraduate Lecture Notes in Physics) 2nd edn (New York: Springer)
[33] Davydychev A I 2006 Geometrical methods in loop calculations and the three-point function *Nucl. Instrum. Methods Phys. Res.* 559 293–97
[34] Abrahamsson I G 1964 Orthant probabilities for the quadrivariate normal distribution *Ann. Math. Stat.* 35 1685–703
[35] Del Duca V D, Duhr C and Smirnov V A 2011 The massless hexagon integral in \( D = 6 \) dimensions *Phys. Lett.* B 703 363–5
[36] Del Duca V D, Duhr C and Smirnov V A 2011 The one-loop one-mass hexagon integral in \( D = 6 \) dimensions *J. High Energy Phys.* JHEP07(2011)1064
[37] Del Duca V, Dixon L J, Drummond J M, Duhr C, Henn J M and Smirnov V A 2011 The one-loop six-dimensional hexagon integral with three massive corners *Phys. Rev.* D 84 045017
[38] Dixon L J, Drummond J M and Henn J M 2011 The one-loop six-dimensional hexagon integral and its relation to MHV amplitudes in \( \mathcal{N} = 4 \) SYM *J. High Energy Phys.* JHEP06(2011)100