MINIMUM DOMINATING DISTANCE ENERGY OF A GRAPH

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Abstract. Recently we introduced the concept of minimum dominating energy[19]. Motivated by this paper, we introduced the concept of minimum dominating distance energy \( E_{Dd}(G) \) of a graph \( G \) and computed minimum dominating distance energies of a star graph, complete graph, crown graph and cocktail party graphs. Upper and lower bounds for \( E_{Dd}(G) \) are also established.

Key words and Phrases: Minimum dominating set, dominating distance matrix, dominating distance eigenvalues, dominating distance energy.

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [9] in the year 1978. Let \( G \) be a graph with \( n \) vertices \( \{v_1, v_2, ..., v_n\} \) and \( m \) edges. Let \( A = (a_{ij}) \) be the adjacency matrix of the graph. The eigenvalues \( \lambda_1, \lambda_2, \cdots, \lambda_n \) of \( A \), assumed in non increasing order, are the eigenvalues of the graph \( G \). As \( A \) is
real symmetric, the eigenvalues of $G$ are real with sum equal to zero. The energy $E(G)$ of $G$ is defined to be the sum of the absolute values of the eigenvalues of $G$. i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$
Minimum Dominating Distance Energy of A Graph

Figure 1

i) $A_{Dd_1}(G) = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$

Characteristic equation is $\rho^6 - 2\rho^5 - 43\rho^4 - 114\rho^3 - 94\rho^2 - 8\rho + 8 = 0$. Minimum dominating distance eigen values are $\rho_1 \approx -3.0257, \rho_2 \approx -2, \rho_3 \approx -1.3386, \rho_4 \approx -0.5067, \rho_5 \approx 0.2255, \rho_6 \approx 8.6456$. Minimum dominating distance energy, $E_{Dd_1}(G) \approx 15.7420$

ii) $A_{Dd_2}(G) = \begin{bmatrix} 0 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$

Characteristic equation is $\rho^6 - 2\rho^5 - 43\rho^4 - 100\rho^3 - 41\rho^2 + 36\rho - 4 = 0$. Minimum dominating distance eigen values are $\rho_1 \approx -3.3028, \rho_2 \approx -2, \rho_3 \approx -1.6445, \rho_4 \approx 0.1431, \rho_5 \approx 0.3028, \rho_6 \approx 8.5015$. Minimum dominating distance energy, $E_{Dd_2}(G) \approx 15.8946$. Therefore, minimum dominating distance energy depends on the dominating set.
3. Minimum Dominating Distance Energy of Some Standard Graphs

**Definition 3.1.** The cocktail party graph, is denoted by $K_{n \times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set $E = \{u_iu_j, v_iv_j : i \neq j\} \cup \{u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$.

**Theorem 3.2.** The minimum dominating distance energy of cocktail party graph $K_{n \times 2}$ is $4n$.

**Proof.** Let $K_{n \times 2}$ be the cocktail party graph with vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$. The minimum dominating set of $K_{n \times 2}$ is $D = \{u_1, v_1\}$. Then

$$A_{Dd}(K_{n \times 2}) = \begin{pmatrix}
1 & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 2 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 0 & \ldots & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \ldots & 0 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 2 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 2 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 2 & 0
\end{pmatrix}_{n \times n}$$

Characteristic equation is $\rho^n - 2(\rho + 1)(\rho + 2)^{(n-1)}[\rho^2 - (2n + 1)\rho + (2n - 2)] = 0$

Minimum dominating distance eigenvalues are $\rho = 0$ [$(n-2)$ times], $\rho = -1$ [one time], $\rho = -2$ [$(n-1)$ times], $\rho = \frac{(2n+1) \pm \sqrt{4n^2 - 4n + 9}}{2}$ [one time each]. So, minimum dominating distance energy is $E_{Dd}(K_{n \times 2}) = 4n$.

**Theorem 3.3.** For any integer $n \geq 3$, the minimum dominating distance energy of star graph $K_{1,n-1}$ is equal to $4n - 7$.

**Proof.** Consider the star graph $K_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, where $\text{deg}(v_0) = n - 1$. Minimum dominating set $D = \{v_0\}$. Then

$$A_{Dd}(K_{1,n-1}) = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 0 & 2 & \ldots & 2 \\
1 & 2 & 0 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 2 & \ldots & 0
\end{pmatrix}_{n \times n}$$

Characteristic equation is $(\rho + 2)^{n-2}(\rho^2 - (2n - 3)\rho + (n - 3)) = 0$
The minimum dominating distance eigenvalues are $\rho = -2 \ [(n-2) \ \text{times}], \rho = (2n - 3) \pm \sqrt{4n^2 - 16n + 21} \ [\text{one time each}].$ So, minimum dominating distance energy is $E_{Dd}(K_{1,n-1}) = 4n - 7.$ ■

**Definition 3.4.** The crown graph $S^0_n$ for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$. Hence $S^0_n$ coincides with the complete bipartite graph $K_{n,n}$ with horizontal edges removed.

**Theorem 3.5.** For any integer $n \geq 2$, the minimum dominating distance energy of the crown graph $S^0_n$ is equal to $7(n-1) + \sqrt{n^2 - 2n + 5}.$

**Proof.** For the crown graph $S^0_n$ with vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$, minimum dominating set is $D = \{u_1, v_1\}$. Then

$$A_{Dd}(S^0_n) = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 & 3 & 1 & 1 & \ldots & 1 \\ 2 & 0 & 2 & \ldots & 2 & 1 & 3 & 1 & \ldots & 1 \\ 2 & 2 & 0 & \ldots & 2 & 1 & 1 & 3 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \ldots & 0 & 1 & 1 & 1 & \ldots & 3 \\ 3 & 1 & 1 & \ldots & 1 & 1 & 2 & 2 & \ldots & 2 \\ 1 & 3 & 1 & \ldots & 1 & 2 & 0 & 2 & \ldots & 2 \\ 1 & 1 & 3 & \ldots & 1 & 2 & 2 & 0 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 3 & 2 & 2 & 2 & \ldots & 0 \end{pmatrix}_{(2n \times 2n)}$$

Characteristic equation is

$$\rho^{2(n-2)}(\rho + 4)^{n-2}[(\rho^2 + (7 - n)\rho + (11 - 3n)][\rho^2 - (3n + 1)\rho + (3n - 3)] = 0$$

Minimum dominating distance eigenvalues are $\rho = 0\ [(n-2)\ \text{times}], \rho = -4 \ [(n-2)\ \text{times}], \rho = (n - 7) \pm \sqrt{n^2 - 2n + 5} \ [\text{one time each}], \rho = (3n + 1) \pm \sqrt{9n^2 - 6n + 13} \ [\text{one time each}].$ So, minimum dominating distance energy is $E_{Dd}(S^0_n) = 7(n-1) + \sqrt{n^2 - 2n + 5}.$ □

**Theorem 3.6.** For any integer $n \geq 2$, the minimum dominating distance energy of complete graph $K_n$ is $(n - 2) + \sqrt{n^2 - 2n + 5}.$

**Proof.** For complete graphs the minimum dominating distance matrix is same as minimum dominating matrix [19], therefore the minimum dominating distance energy is equal to minimum dominating energy. □
4. Properties of Minimum Dominating Eigenvalues

**Theorem 4.1.** Let $G$ be a simple graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, edge set $E$ and $D = \{u_1, u_2, ..., u_k\}$ be a minimum dominating set. If $\rho_1, \rho_2, ..., \rho_n$ are the eigenvalues of minimum dominating distance matrix $A_{Dd}(G)$ then

(i) $\sum_{i=1}^{n} \rho_i = |D|

(ii) $\sum_{i=1}^{n} \rho_i^2 = 2m + 2M + |D|$ where $M = \sum_{i<j, d(v_i,v_j) \neq 1} d(v_i,v_j)^2$ and $m = |E|$.

**Proof.**

1) We know that the sum of the eigenvalues of $A_{Dd}(G)$ is the trace of $A_{Dd}(G)$. Therefore,

$$\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} d_{ii} = |D| = k.$$ 

(ii) Similarly, the sum of squares of the eigenvalues of $A_{Dd}(G)$ is trace of $[A_{Dd}(G)]^2$ Therefore,

$$\sum_{i=1}^{n} \rho_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^2

= \sum_{i=1}^{n} (d_{ii})^2 + \sum_{i \neq j} d_{ij}^2

= \sum_{i=1}^{n} (d_{ii})^2 + 2\sum_{i<j} (d_{ij})^2

= |D| + 2\sum_{i<j} d(v_i,v_j)^2

= k + 2m + 2M.$$ 

Corollary 4.2. Let $G$ be a $(n,m)$ simple graph with diameter 2 and $D = \{u_1, u_2, ..., u_k\}$ be a minimum dominating set. If $\rho_1, \rho_2, ..., \rho_n$ are the eigenvalues of minimum dominating distance matrix $A_{Dd}(G)$ then

$$\sum_{i=1}^{n} \rho_i^2 = k + 2(2n^2 - 2n - 3m).$$

**Proof.** We know that in $A_{Dd}(G)$ there are $2m$ elements with 1 and $n(n-1) - 2m$ elements with 2 and hence corollary follows from the above theorem. □
5. Bounds for Minimum Dominating Energy

Similar to McClelland’s [17] bounds for energy of a graph, bounds for \( E_{Dd}(G) \) are given in the following theorem.

**Theorem 5.1.** Let \( G \) be a simple \((n,m)\) graph. If \( D \) is the minimum dominating set and \( P = |detA_{Dd}(G)| \) then

\[
\sqrt{(2m + 2M + k) + n(n-1)P^2} \leq E_{Dd}(G) \leq \sqrt{n(2m + 2M + k)}
\]

where \( k \) is a domination number.

**Proof.**

Cauchy Schwarz inequality is

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

If \( a_i = 1, b_i = |\rho_i| \) then

\[
\left( \sum_{i=1}^{n} |\rho_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \rho_i^2 \right) \]

\[|E_{Dd}(G)|^2 \leq n(2m + 2M + k) \quad \text{[Theorem 4.1]} \]

\[\Rightarrow E_{Dd}(G) \leq \sqrt{n(2m + 2M + k)}\]

Since arithmetic mean is not smaller than geometric mean we have

\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\rho_i| |\rho_j| \geq \left[ \prod_{i \neq j} |\rho_i| |\rho_j| \right] \left[ \frac{1}{n(n-1)} \right]
\]

\[
= \left[ \prod_{i=1}^{n} |\rho_i| \right]^{2(n-1)} \left[ \frac{1}{n(n-1)} \right]
\]

\[
= \left( \prod_{i=1}^{n} |\rho_i| \right)^{2(n-1)}
\]

\[
= \prod_{i=1}^{n} \rho_i^{2(n-1)}
\]

\[
= |detA_{Dd}(G)|^{2(n-1)} = P^{2(n-1)}
\]

\[\therefore \sum_{i \neq j} |\rho_i| |\rho_j| \geq n(n-1)P^{2(n-1)} \quad (1)\]
Now consider, \[ |E_{Dd}(G)|^2 = \left( \sum_{i=1}^{n} | \rho_i |^2 \right)^2 = \sum_{i=1}^{n} | \rho_i |^2 + \sum_{i \neq j} | \rho_i \rho_j | \]
\[ : |E_{Dd}(G)|^2 \geq (k + 2m + 2M) + n(n - 1)P^2 \quad \text{[From (1)]} \]
\[ i.e., E_{Dd}(G) \geq \sqrt{(k + 2m + 2M) + n(n - 1)P^2} \]

\[ \square \]

**Theorem 5.2.** If \( \rho_1(G) \) is the largest minimum dominating distance eigenvalue of \( A_{Dd}(G) \), then

\[ \rho_1(G) \geq \frac{2W(G) + k}{n} \]

where \( k \) is the domination number and \( W(G) \) is the Wiener index of \( G \).

**Proof.** Let \( X \) be any nonzero vector. Then by [1], we have

\[ \rho_1(A_{Dd}) = \max_{X \neq 0} \left\{ \frac{X' A_{Dd} X}{X' X} \right\}. \]

Therefore,

\[ \rho_1(A_{Dd}) \geq \frac{\sum_{i \neq j} d(v_i, v_j) + k}{n} = \frac{2W(G) + k}{n} \]

where \( J \) is a unit matrix.

\[ \square \]

**Lemma 5.3.** Let \( G \) be a graph of diameter 2 and \( \rho_1(G) \) is the largest minimum dominating distance eigenvalue of \( A_{Dd}(G) \), then

\[ \rho_1(G) \geq \frac{2n^2 - 2m - 2n + k}{n} \]

where \( k \) is the domination number.

**Proof.** Let \( G \) be a connected graph of diameter 2 and \( d_i \) denotes the degree of vertex \( v_i \). Clearly \( i \)-th row of \( A_{Dd} \) consists of \( d_i \) one’s and \( n - d_i - 1 \) two’s. By using Raleigh’s principle, for \( J = [1, 1, 1, \ldots, 1] \) we have

\[ \rho_1(A_{Dd}) \geq \frac{\sum_{i=1}^{n} [d_i \times 1 + (n - d_i - 1)2] + k}{n} = \frac{2n^2 - 2m - 2n + k}{n}. \]

Similar to Koolen and Moulton’s [15] upper bound for energy of a graph, upper bound for \( E_{Dd}(G) \) is given in the following theorem.
Theorem 5.4. If $G$ is a $(m, n)$ graph with diameter 2 and $\frac{k + 2n^2 - 2n - 2m}{n} \geq 1$ then

$$E_{Dd}(G) \leq \frac{k + 2n^2 - 2n - 2m}{n} + \sqrt{(n - 1)\left[ k + 4n^2 - 4n - 6m - \frac{(k + 2n^2 - 2n - 2m}{n}\right]^2].$$

Proof. Cauchy-Schwartz inequality is

$$\left[\sum_{i=2}^{n} a_i b_i\right]^2 \leq \left(\sum_{i=2}^{n} a_i^2\right) \left(\sum_{i=2}^{n} b_i^2\right).$$

Put $a_i = 1, b_i = |\rho_i|$, then

$$\left(\sum_{i=2}^{n} |\rho_i|\right)^2 \leq \sum_{i=2}^{n} 1 \sum_{i=2}^{n} \rho_i^2$$

Then,

$$|E_{Dd}(G) - \rho_1|^2 \leq (n - 1)(k + 4n^2 - 4n - 6m - \rho_1^2).$$

We have

$$E_{Dd}(G) \leq \rho_1 + \sqrt{(n - 1)(k + 4n^2 - 4n - 6m - \rho_1^2}).$$

Let

$$f(x) = x + \sqrt{(n - 1)(k + 4n^2 - 4n - 6m - x^2}).$$

For decreasing function, $f'(x) \leq 0$ Then,

$$1 - \frac{x(n - 1)}{\sqrt{(n - 1)(k + 4n^2 - 4n - 6m - x^2)}} \leq 0.$$ 

We have

$$x \geq \sqrt{\frac{k + 4n^2 - 4n - 6m}{n}}.$$ 

Therefore, $f(x)$ is decreasing in

$$\left[ \sqrt{\frac{k + 4n^2 - 4n - 6m}{n}}, \sqrt{k + 4n^2 - 4n - 6m} \right].$$

Clearly,

$$\sqrt{\frac{k + 2n^2 - 2n - 2m}{n}} \in \left[ \sqrt{\frac{k + 4n^2 - 4n - 6m}{n}}, \sqrt{k + 4n^2 - 4n - 6m} \right].$$

Since $\frac{k + 2n^2 - 2n - 2m}{n} \geq 1$, we have

$$\sqrt{\frac{k + 2n^2 - 2n - 2m}{n}} \leq \frac{k + 2n^2 - 2n - 2m}{n} \leq \rho_1 \text{(by lemma 5.3)}.$$
Therefore, \( f(\rho_1) \leq f \left( \frac{k+2n^2-2n-2m}{n} \right) \). Then,
\[
E_{Dd}(G) \leq f(\rho_1) \\
\leq f \left( \frac{k+2n^2-2n-2m}{n} \right) \\
\leq \frac{k+2n^2-2n-2m}{n} \\
+ \sqrt{(n-1) \left[ k+4n^2-4n-6m - \left( \frac{k+2n^2-2n-2m}{n} \right)^2 \right]}.
\]

\[\square\]

Bapat and S. Pati [2] proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum dominating energy is given in the following theorem.

**Lemma 5.5.** Let \( G \) be a graph with a minimum dominating set \( D \). If the minimum dominating distance energy \( E_{Dd}(G) \) is a rational number, then \( E_{Dd}(G) \equiv \vert D \vert \mod 2 \).

**Proof.** Proof is similar to Theorem 5.4 of [19]. \[\square\]

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