Entanglement meter: estimation of entanglement with single copy in interferometer

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Abstract
Efficient certification and quantification of high dimensional entanglement of composite systems are challenging both theoretically as well as experimentally. Here, we demonstrate how to measure the linear entropy, negativity and the Schmidt number of bipartite systems from the visibility of Mach–Zehnder interferometer using single copies of the quantum state. Our result shows that for any two qubit pure bipartite state, the interference visibility is a direct measure of entanglement. We also propose how to measure the mutual predictability experimentally from the intensity patterns of the interferometric set-up without having to resort to local measurements of mutually unbiased bases. Furthermore, we show that the entanglement witness operator can be measured in an interference setup and the phase shift is sensitive to the separable or entangled nature of the state. Our proposal bring out the power of Interferometric set-up in entanglement detection of pure and several mixed states which paves the way towards design of entanglement meter.

1. Introduction

Since its discovery by Einstein et al [1], entanglement has been firmly established as one of the most important features of quantum theory. Quantum entanglement has been helpful in some important discoveries in quantum information like quantum teleportation [2], quantum dense coding [3], quantum cryptography [4], remote state preparation [5], where entanglement acts as a resource. Recently, interest has been devoted towards the possibility of generating high dimensional entangled states. Such states can, in principle, contain a large amount of entanglement, which is not only conceptually interesting but also offers novel perspectives for applications in quantum information, particularly in quantum communications [6–10]. There are several experimental implementations of high dimensional entanglement. In particular, using photonic system, high dimensional entanglement can be created using energy time [11–13], time bins [14–16], orbital angular momentum [17–19], and frequency modes [20–22]. Also, the entanglement of these states can be detected experimentally, via the use of entanglement witnesses or Bell inequalities [23, 24].

However, efficient experimental certification and quantification of high dimensional entanglement is still a challenging problem. There are mainly two reasons why this is demanding. Firstly, the characterization of a high dimensional entangled state via standard methods (e.g. quantum tomography) typically requires the estimation of a large number of independent parameters, which in turn requires a large number of different measurements to be performed. Furthermore, it has also been comprehensively argued that, even with full knowledge of the density matrix of the state obtained from tomography, the computational complexity of evaluating any entanglement monotone increases so rapidly with increasing dimension making it a NP-hard problem [25, 26]. Secondly, the types of measurements that can actually be performed in a real experiment are typically limited.

To circumvent the above-mentioned difficulties various methods have been proposed in recent years for states with high purity [27–30]. Other approaches include determination of Schmidt number [31, 32], comparison of measurement results corresponding to two or more mutually unbiased bases [33–37], using...
the violation of the entropic Einstein–Podolsky–Rosen-steering inequality [38] or other suitably defined statistical correlators [39]. A method for entanglement detection and quantification via multiple copies of the input state is presented in [40–42]. Entanglement detection method based on structural approximations was discussed in [43, 44]. Also, a probabilistic technique for entanglement detection from a single copy of the input state as an information processing task in an adversarial scenario was introduced in [45–48].

Against this backdrop, we explore whether one can use the Mach–Zehnder interferometer to design entanglement meter which can detect if a bipartite quantum state is entangled or not. We also explore how to estimate the entanglement content in terms of certain entanglement measures directly from the interference pattern. It is known that linear and non-linear functionals of the density matrix can be measured from the Mach–Zehnder interferometric set-up [49, 50], where the functionals of the density operator are taken to be various measures and witnesses of entanglement. In [51], it has been shown that the average visibility in interference is related to concurrence of the quantum state. However, the proposal described in [51] involves averaging over multiple execution instances. It has been recently shown that, compared to quantum state tomography, one can determine the quantum state more efficiently using a process called quantum state interferography [52]. Inspired by these, we attempt to see whether one can determine entanglement of two-qubit states as well as higher dimensional states from the Mach–Zehnder interferometer [51, 53] more efficiently and more economically without quantum state tomography and classical post processing.

In this article, we propose protocols that use copies of single instances of state $\rho$ (single copy) and a single unitary operation to determine whether the bipartite state is entangled or not and the degree of entanglement from the interference visibility. It was shown in [49, 50, 54, 55] that the linear entropy [56] can be measured via interference setup using copies of $\rho \otimes \rho$ (two copies) as input of the experiment. We show that a single copy of the input state (or even a subsystem of the input state) is sufficient to measure the linear entropy in interference setup. In addition, we also provide a scheme to determine the negativity [57] of entanglement using single copy of the joint state at the input port of the interferometer. In particular, our method can determine the negativity of bipartite pure state and certain mixed state from the interference visibility using a single unitary operation acting on a single copy of the state. Further, we have also shown a process to determine Schmidt number [58] for a quantum state $\rho_{AB}$ using our set-up.

We also propose methods to detect whether a state is entangled of separable using a single copy of the state. In particular, we propose how to measure the mutual predictability [59] directly from the intensity patterns of the interferometric set-up without having to resort to local measurements of mutually unbiased bases. We also show that the entanglement witness operator [60] can be measured in an interference setup and the phase shift is sensitive to the separable or entangled nature of the state. Our methods can have an interesting application in design of phase sensitive entanglement meter where one can have a portable device to detect entangled or separable states using the phase shift (similar to the concept of voltmeter which measures the voltage between two points). The notion of entanglement meter will have several applications in quantum technology as this can directly display how much entanglement is there between a pair of systems.

2. Preliminaries

We will consider that $\mathcal{H}$ represent a separable Hilbert space with $\dim(\mathcal{H})$ denoting the dimension of the Hilbert space. We let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the Hilbert space associated with quantum system $A$ and $B$, respectively. The Hilbert space of composite system $AB$ is denoted by $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We denote $|\Psi\rangle_{AB}$ as pure entangled state and $\rho_{AB}$ as a mixed entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Consider a pure bipartite state $|\Psi\rangle_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite dimensional Hilbert spaces. Then, using the Schmidt decomposition theorem, $|\Psi\rangle_{AB}$ can be written in the form $|\Psi\rangle_{AB} = \sum_{j=0}^{n-1} \sqrt{\lambda_j} |j\rangle_A \otimes |j\rangle_B$, where $\{|j\rangle_A, |j\rangle_B\}$ are orthonormal vectors corresponding to subsystems $A$ and $B$, respectively and $\{|\lambda\rangle\}$ are the Schmidt coefficients (non-negative real numbers) with $\sum_{\lambda} \lambda_j = 1$, and $d < \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$. For mixed state, the definition of entanglement is more involved. Consider a density operator $\rho_{AB}$ which is a positive, Hermitian operator with unit trace acting on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state $\rho_{AB}$ is called separable if there exists an ensemble $\{p_i, \rho_A \otimes \rho_B\}$ with $\rho_{AB} = \sum_i p_i \rho_A \otimes \rho_B$. If $\rho_{AB}$ cannot be expressed as a separable form then it is entangled.

For the sake of completeness, we first describe the interference of mixed states as formalized in [53]. A typical interferometric setup for a system is a sequence of the following operations: taking the arms of the beam splitter as the basis states $|0\rangle$ and $|1\rangle$ and the incoming quantum state as $\rho$, first we apply the Hadamard gate $H$ (which corresponds to the first beam splitter), followed by a phase shift $\phi$ in one of the arms (say the arm corresponding to $|0\rangle$), Hadamard gate $H$ (which corresponds to the final beam splitter) and finally a measurement in the computational basis. We insert a controlled-unitary operation between the Hadamard gates, with its control on the qubit and with the unitary acting on a quantum system described by some unknown density operator $\rho$. The above sequence of gates results in the evolution operator

\[
\rho' = U \rho U^\dagger
\]
are proved. In particular, the following results are contained in the state, or (2) whether the quantum state is entangled from the interferometric pattern. Given

Detecting when a state is entangled

As mentioned before, given a single copy of quantum state (or even its subsystem) as input of the interferometer we will construct unitary operators $U$ such that one infer (1) the amount of entanglement present in the bipartite system.

3. Results

As mentioned before, given a single copy of quantum state (or even its subsystem) as input of the interferometer we will construct unitary operators $U$ such that one infer (1) the amount of entanglement contained in the state, or (2) whether the quantum state is entangled from the interferometric pattern. Given a quantum state as input such unitary operators are prepared by an oracle. In particular, the following results are proved

| Estimating entanglement from interferometer | Quantity of interest | Relationship |
|--------------------------------------------|---------------------|-------------|
| Linear entropy ($\mathcal{E}(\cdot)$)       | $\mathcal{E}(\Psi) = 1 - \frac{d}{2}(1 - V)$ for pure states (8) and $\mathcal{E}_c(\rho_{AB}) \leq 1 - \frac{d(1 - V)}{V}$ with $\mathcal{E}_c(\rho_{AB})$ as convex-roof extended linear entropy (13), $V = \frac{1}{tr} (8N + (d - 2)^2)$ for pure states (20), $0 \leq \frac{1 + V}{2} \leq SN(\rho)\leq \frac{2}{d}$ (16) |
| Negativity ($\mathcal{N}(\cdot)$)            |                     |
| Schmidt number ($SN(\rho)$)                |                     |

| Detecting when a state is entangled from interferometer | Mutual predictability ($C_{O_1,\alpha_2}$) | Witness |
|---------------------------------------------------------|-----------------------------------------|---------|
|                                                         | $C_{O_1,\alpha_2} = \frac{1}{2} (1 \pm V)$ | For entangled Werner state phase is shifted by angle $\pi$ (48) |
We assume that given a quantum state $\rho$ the corresponding oracles produces the relevant unitary operations. For Werner state, there exists a unitary Witness operator. For the other cases, the structure of the unitary operations are presented below

| Quantity of interest | Nature of oracle |
|----------------------|------------------|
| Linear entropy       | Given a state $|\Psi\rangle$ or $\rho U = I - 2P$ where $P$ is projection operation for $|\Psi\rangle$ or purification of $\rho$ |
| Negativity           | $U = U_A \otimes U_B$ where $U_{AB} = e^{i \frac{\pi}{d} X_A}$ and $X_k = \sum_{j=0}^{d-1} |j\rangle \langle k|$ |
| Schmidt number       | $U^* = I - |\Psi^*\rangle \langle \Psi^*|$ where $\max_{|\Psi\rangle \in S_B} \langle \Psi^* | \rho | \Psi^* \rangle$ and $S_B$ is the set of maximally entangled states. |
| Mutual predictability | $U = (I \otimes I - 2 \sum_k |k\rangle_A \langle k| \otimes |k\rangle_B \langle k|$ |

In the following section we will give the mathematical details pertaining to each of the quantity of interest.

4. Methods

4.1. Measuring entanglement of pure state from single copy

To measure the purity of a state, usually one has to estimate the quantum state and from the classical description of the density operator one can extract the purity by classical evaluations. However, it should be noted that the estimation of purity does not require the knowledge of full density operator. Therefore, prior state estimation procedure followed by classical calculations, in general, are inefficient. To overcome this, it was proposed that direct measurement of purity can be carried out with the help of quantum networks which is essentially an interferometric scheme [49]. However, this proposal needs two copies of the input state $\rho$ to measure the purity. If one has a limited supply of quantum systems, it is desirable to have more efficient scheme which can reduce the number of copies at the input port of the interferometer.

Here, we achieve this precisely where having access to one copy of the density operator $\rho$, we can measure the purity $\text{tr}(\rho^2)$, and the linear entanglement entropy of any pure bipartite quantum state. Given a pure bipartite state $|\Psi\rangle_{AB}$, the entanglement content can be quantified using the linear entropy [56] which is given by

$$E(|\Psi\rangle) = (1 - \text{tr}(\rho_A^2)),$$

where $\rho_A$ is the reduced density matrix of the subsystem $A$. The linear entropy is a valid measure of entanglement similar to the von Neumann entropy of entanglement. This is also concave and unitarily invariant. If one can measure $\text{tr}(\rho_A^2)$ in an experiment, then it is possible to measure the linear entropy. For $d = 2$, the linear entropy is the same as the concurrence of the bipartite quantum state. Here, we propose a direct method of measuring the linear entropy with single copy of the input state $\rho_A$ along with a single unitary operation. This method works even if we do not know the pure bipartite entangled state.

To achieve this, we need a unitary operator where the additional power comes from the ability to verify a state instead of knowing the state completely. Similar argument has been used in the quantum state restoration method using single-copy tomography [61]. For an unknown pure bipartite state $|\Psi\rangle_{AB}$ this algorithm takes as input state a local subsystem $A$ and a random state for the subsystem $B$. On using the projection operator $P = |\Psi\rangle_{AB} \langle \Psi|$, one can restore the state probabilistically in time that scales as $O(\text{poly}(\dim(H_B)))$. However, if we do not want to restore the entangled state, rather measure the entanglement, then our protocol is more efficient and there is no chance of failure. For an unknown quantum state $|\Psi\rangle_{AB}$ a black-box (oracle) can exist which can implement the unitary

$$U = (I \otimes I - 2P),$$

where $P = |\Psi\rangle_{AB} \langle \Psi|$. It has been shown that [61] $P$ can be implemented for an unknown quantum pure state $|\Psi\rangle_{AB}$ using additional quantum registers. In particular, the method prescribed in [61] measures the operator $O = (P - P^\perp)$, where $P^\perp$ is the projection operator onto the orthogonal subspace of $|\Psi\rangle_{AB}$.

The unitary given by (5) is same as $O$ with a minus sign. Thus, our oracle can use similar procedure to implement $U$. Since, we are not interested in knowing the state, the no-cloning theorem [62] does not apply here. Using this oracle operator $U$, we will show how one can measure the linear entropy $(1 - \text{tr}(\rho_A^2))$ from the interference visibility. Here, one should not tend to think that there is some observable and by measuring the observable one can measure the entanglement. In quantum mechanics, the notion of measuring something has gone beyond the standard notion of measuring Hermitian operator. In fact, recent papers show that one
Using the interference visibility obtained in equation (8), we can measure any operator (need not be Hermitian) using quantum interference setup [63, 64]. However, we are not going to discuss those schemes here.

Suppose, we want to measure the entanglement content of a pure bipartite state $|\Psi\rangle_{AB}$. Instead of sending the full state, we send one of the local subsystem $\rho_A$ and another system which is prepared in a maximally mixed state $\frac{I}{d}$, where $d = \text{dim}(H_B)$. Let $U = (I - 2P)$ acts on one arm of the interferometer where we have fed the input state $\rho_{AB} = \rho_A \otimes \frac{I}{d}$. Now, if we start with a state $|0\rangle_0 \otimes \rho_{AB}$ and feed this through the interferometer, then the final state after a sequence of Hadamard gate, controlled oracle gate ($U$) and Hadamard gate is given by $|0\rangle_0 \otimes \rho_{AB} \xrightarrow{H \otimes I_{AB}} |+\rangle |+\rangle \otimes \rho_{AB} \xrightarrow{U} U(|+\rangle |+\rangle \otimes \rho_{AB}) U^\dagger \xrightarrow{H \otimes I_{AB}} \rho_f$, where

$$
\rho_f = \frac{1}{2} \left[ |+\rangle \langle +| \otimes U \left( \rho_A \otimes \frac{I}{d} \right) U^\dagger + |−\rangle \langle −| \otimes \left( \rho_A \otimes \frac{I}{d} \right) \right] + e^{-i\phi} \left[ |−\rangle \langle −| \otimes U \left( \rho_A \otimes \frac{I}{d} \right) U^\dagger \right] + e^{i\phi} \left[ |+\rangle \langle +| \otimes U \left( \rho_A \otimes \frac{I}{d} \right) U^\dagger \right],
$$

(6)

$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ and $|−\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$.

For this choice of the oracle unitary operator and the initial state, the interferometric visibility is modified as

$$
V = |\text{tr} \left( (I - 2P) \left( \rho_A \otimes \frac{I}{d} \right) \right)|
= 1 - \frac{2}{d} \text{tr}(\rho_f^2).
$$

(7)

The protocol to measure the linear entanglement entropy described above can be depicted in the figure 2. Using the interference visibility obtained in equation (7), we can measure the linear entanglement entropy as given by

$$
E(|\Psi\rangle) = 1 - \frac{d}{2} (1 - V).
$$

(8)

Our method works even for unknown pure entangled state. As explained, the oracle has the capability of implementing the unitary $U = (I - 2P)$ even for an unknown state $P = |\Psi\rangle_{AB} \langle \Psi|$. We only need the quantum state corresponding to subsystem $A$, at the input port of the interferometer along with an additional maximally mixed state. This shows that for any arbitrary two qubit pure entangled state, we have $E(|\Psi\rangle) = V$. Thus, for any pure bipartite two qubit state, the interference visibility is a direct measure of entanglement. It may be noted that, in general, to evaluate the spectrum of any density matrix we need to estimate $(d^2 - 1)$ parameters. However, our method can estimate the linear entropy directly without full quantum state tomography. Another important feature of this method is that we need single copy of the input state $\rho$ at the input port of the interferometer, and therefore it is also resource economical. Our proposal can be designed as a portable device to measure the entanglement of any pure bipartite state from the visibility, thus paving towards the notion of an entanglement meter.

Now, we extend our analysis for the case of mixed states. Consider a bipartite system $AB$ in mixed state. Then, the density operator $\rho_{AB}$ can be written as convex mixture of pure states $\psi_i$ with mixing probabilities $p_i$, i.e. $\rho_{AB} = \sum_i p_i |\psi_{iAB}\rangle \langle \psi_{iAB}|$, where each $|\psi_{iAB}\rangle$ is one dimensional projector, i.e. $\psi_i = |\psi_i\rangle_{AB} \langle \psi_i|_{AB}$, and $\sum_i p_i = 1$. Figure 2. Here we depict the Interference setup for measurement of linear entanglement entropy. Once we obtain the visibility, using equation (8), linear entropy can be calculated. 

\[ 
\begin{array}{c}
|0\rangle \\
H \\
\phi \\
H \\
M \\
\rho_A \\
I_B \\
I - 2P
\end{array} 
\]
Using the convex roof construction, the linear entropy can also be used as a measure of entanglement for mixed states \[65\].

**Definition 1.** For a bipartite mixed state \(\rho_{AB}\) with decomposition \(\{p_i, |\Psi_i\rangle\}\), the convex roof extended linear entropy is defined as \[65\]

\[
E_{c}(\rho_{AB}) = \min_{\{p_i, |\Psi_i\rangle\}} \left( p_i E_c(|\Psi_i\rangle) \right),
\]

(9)

here minimum is taken over all decomposition of \(\rho_{AB}\), \(E(|\Psi_i\rangle) = (1 - \text{tr}(\rho_{A|i}^2))\) is the linear entropy of pure state \(|\Psi_i\rangle\), and \(\rho_{A|i} = \text{tr}_B(|\Psi_i\rangle\langle\Psi_i|)\).

The reduced density operator \(\rho_A\) is given by \(\rho_A = \text{tr}_B(\sum_i p_i |\Psi_i\rangle\langle\Psi_i|_A\rho_{AB} |\Psi_i\rangle)\). From the definition of convex roof extended linear entropy, we have the following inequality

\[
E_{c}(\rho_{AB}) \leq \sum_i p_i (1 - \text{tr}(\rho_{A|i}^2)).
\]

(10)

Now, we show that the term on the right hand side of this inequality is upper bounded by \(1 - \text{tr}(\rho_{A|^c}^2)\). Using the Cauchy–Schwarz inequality and the fact that geometric mean is less than or equal to arithmetic mean we obtain

\[
\text{tr}(AB) \leq \sqrt{\text{tr}(A^2)\text{tr}(B^2)} \leq \frac{1}{2} (\text{tr}(A^2) + \text{tr}(B^2)),
\]

(11)

where \(A\) and \(B\) are Hermitian operators. It then follows that

\[
\text{tr}(\rho_{A|^c}^2) \leq \frac{1}{2} \sum_{ij} p_i p_j \left( \text{tr}(\rho_{A|i}^2) + \text{tr}(\rho_{A|^c}^2) \right).
\]

(12)

Summing over \(j|i\) in the first(second) term of the previous equation, we finally obtain \(\text{tr}(\rho_{A|^c}^2) \leq \sum_i p_i \text{tr}(\rho_{A|i}^2)\). This shows that the convex roof extended linear entropy is upper bounded as \(E_{c}(\rho_{AB}) \leq 1 - \text{tr}(\rho_{A|^c}^2)\). From the definition of convex roof extended linear entropy one can then obtain

\[
E_{c}(\rho_{AB}) \leq 1 - \frac{d(1 - V)}{2}.
\]

(13)

From the above inequality we then obtain an upper bound on convex roof extended linear entropy in terms of interferometric visibility \(V\) and \(d\). This shows that, for an arbitrary two-qubit mixed state \(\rho_{AB}\), we have \(E_{c}(\rho_{AB}) \leq V\), i.e. the visibility gives an upper bound to the convex roof extended linear entropy for \(\rho_{AB}\). This shows the true power of quantum interference in determining the entanglement content.

### 4.2. Determination of Schmidt number from interference visibility

**Definition 2.** A bipartite density matrix \(\rho_{AB}\) in Hilbert space \(\mathcal{H}_A \otimes \mathcal{H}_B\) has Schmidt number \(k\) \([58]\) if (a) for any decomposition of \(\rho_{AB} = \sum_i |\Psi_i\rangle\langle\Psi_i|\), at least one of the vectors \(|\Psi_i\rangle\) has at least Schmidt rank \(k\) and (b) there exists a decomposition of \(\rho\) with all vectors \(|\Psi_i\rangle\) of Schmidt rank at most \(k\). In other words, let Schmidt rank of a pure state \(|\Psi\rangle\) is denoted as \(\text{SR}(|\Psi\rangle)\). The Schmidt number of a mixed state is defined as

\[
\text{SN}(\rho) = \min_{\{p_i, |\Psi_i\rangle\}, \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|} \max_{\rho} \text{SR}(|\Psi_i\rangle).
\]

(14)

The Schmidt number of a density matrix does not increase under local quantum operations and classical communication (LOCC). In addition, the Schmidt number for mixed states, in general, does not increase under tensor product of states and have important bearing on the inter-convertibility of mixed states under LOCC.

Now, we will show how this important notion can be measured in interference set-up. Let us denote the set of all states with Schmidt number less than or equal to \(k\) as \(S_k\). Then, \(S_k := \{\rho : \text{SN}(\rho) \leq k\}\). It immediately follows that \(S_{k-1} \subset S_k\). \(S_k\) is the set of separable states. Considering the set of maximally entangled states as \(S_1\), for a state \(\rho \in S_1\) the following relation holds \([58]\]

\[
\max_{|\Psi\rangle \in S_0} \langle \Psi | \rho | \Psi \rangle \leq \frac{k}{d}.
\]

(15)

Now, consider a bipartite state \(\rho_{AB}\) with local Hilbert space dimension \(d\) and unknown Schmidt number \(k\) is used as input of the interferometric setup. The oracle produces an unitary \(U^* = \mathbb{I} - 2|\Psi^*\rangle\langle\Psi^*|\) where...
max $\langle \Psi | \rho | \Psi \rangle = \langle \Psi^* | \rho | \Psi^* \rangle$. In other words, the oracle correctly identify the maximally entangled state, for which, the maximum is reached for the lhs of equation (15).

For such a controlled oracle one can have $\text{tr}(U_{PAB}) = 1 - 2 \langle \Psi^* | \rho_{AB} | \Psi^* \rangle$. Since $\text{tr}(U_{PAB})$ is real and $0 \leq \langle \Psi^* | \rho_{AB} | \Psi^* \rangle \leq 1$ we need to consider two regions of the inner product separately.

- If $0 < \langle \Psi^* | \rho_{AB} | \Psi^* \rangle < \frac{1}{2}$ then
  \[
  V = |\text{tr}(U_{PAB})| = 1 - 2 \langle \Psi^* | \rho_{AB} | \Psi^* \rangle.
  \]

- On the other hand if $1 > \langle \Psi^* | \rho_{AB} | \Psi^* \rangle > \frac{1}{2}$ then we have
  \[
  V = |\text{tr}(U_{PAB})| = 2 \langle \Psi^* | \rho_{AB} | \Psi^* \rangle - 1.
  \]

Now, using equation (15) and the above two results, for $\rho_{AB} \in S_k$ we then obtain
\[
0 \leq 1 + \frac{V}{2} \leq k \frac{d_{V}}{d}.
\]

here $+$ comes when $\langle \Psi^* | \rho_{AB} | \Psi^* \rangle > \frac{1}{2}$ and $-$ comes when $\langle \Psi^* | \rho_{AB} | \Psi^* \rangle < \frac{1}{2}$. Equation (16) tells us that the lowest integer larger than $\frac{d_{V} \rho_{AB}}{d}$ is the Schmidt rank $k$. Thus, we have the following relation which determines the Schmidt rank $k$ of the state $\rho_{AB}$.
\[
\left[ \frac{d_{V}}{2} \right] = k
\]

here $[.]$ is the ceiling function.

4.3. Measuring negativity with single copy
Note that the above protocol depends on an oracle which produces a unitary $U = (I - 2P)$ given a density matrix of a pure state $P$. In what follows, we provide another method where, given a bipartite state $\rho_{AB}$ and a unitary operation $U = e^{i\theta X_A} \otimes e^{i\theta X_B}$ such that $\text{tr}(U \rho)$ contains the signature of entanglement in terms of negativity measure of entanglement. Since the linear entropy and the negativity only differ by a multiplicative factor for any pure state, one can implement either the first or the second protocol depending upon the resource one has at the disposal. However, in case of mixed states, we will show that there exists class of mixed state, for which, the second method provides the exact value of the negativity whereas the first method only provides a lower bound.

The negativity is an entanglement monotone which is derived from the positive partial transpose (PPT) criterion for the separability of a bipartite quantum states. For a pure bipartite state $|\Psi\rangle = \sum_{j=0}^{d-1} \sqrt{\lambda_j} |j\rangle_A |j\rangle_B$, its negativity is given by [57]
\[
\mathcal{N}(|\Psi\rangle) = \frac{1}{2} \sum_{i \neq j} \sqrt{\lambda_i \lambda_j},
\]

where $\lambda_i$'s are Schmidt coefficients. Now, consider an operator
\[
X = \sum_{i,j=0}^{d-1} |i\rangle \langle j| = \sum_{i < j} (|i\rangle \langle j| + |j\rangle \langle i|).
\]

This operator is Hermitian in any finite dimension. In addition, for $d = 2$, this operator is also unitary as $X^2 = I$. First, we describe how to measure the negativity of any pure bipartite two qubit state directly using the interference visibility. For example, consider the input state as an arbitrary pure two qubit state $|\Psi\rangle_{AB}$ in the Mach–Zehnder interferometer and apply the unitary operator $U = X_A \otimes X_B$ in one arm of the interferometer. The visibility is given by $V = |\text{tr}(U_{PAB})| = \langle \Psi | X_A \otimes X_B | \Psi \rangle = 2 \mathcal{N}(|\Psi\rangle)$. Thus, by looking at the interference visibility we can infer the negativity and hence the entanglement content of any pure bipartite two-qubit state. In general, given a $d$-dimensional bipartite system $|\Psi\rangle_{AB} = \sum_{j} \sqrt{\lambda_j} |j\rangle_A |j\rangle_B$, one can define $X_A$ and $X_B$ given by equation (19) i.e., $X_A = \sum_{i<j} C_{ij}^{A}$ and $X_B = \sum_{i<j} C_{ij}^{B}$ for subsystem $A$ and $B$, respectively. For these observables, we have $\langle \Psi | X_A \otimes X_B | \Psi \rangle = 2 \mathcal{N}(|\Psi\rangle)$ and $\langle \Psi | X_A \otimes I_B | \Psi \rangle = \langle \Psi | I_A \otimes X_B | \Psi \rangle = 0$. However, the observable $X$ is unitary only in the case of $d = 2$ thus we can directly
measure their expectation value from the interferometric set-up. We provide a generalization of the above result which is valid for arbitrary finite dimensions. In particular, considering local unitary operators, \( U_A = e^{i \pi X_A} \) and \( U_B = e^{i \pi X_B} \) we obtain

\[
V = |\text{tr}(\rho_{AB} U_{AB})| = \frac{1}{d^2} (8N + (d-2)^2),
\]

where \( U_{AB} = U_A \otimes U_B \). In the appendix, we have provided a detailed proof of equation (20). Here, we will discuss the main steps of the result. Our result hinges on the fact that, for any finite dimension \( d \geq 2 \), the \( n \)th power of the observable \( X \) can be written as

\[
X^n = f_n X + g_n I,
\]

where \( \{f_n\} \) follows Lucas sequence of first kind \([66]\) and given by

\[
f_n = \frac{(d-1)^n - (-1)^n}{d}.
\]

Similarly, the sequence \( \{g_n\} \) is related to \( \{f_n\} \) as follows

\[
g_n = (d-1)f_{n-1} = \frac{d-1}{d} [(d-1)^{n-1} - (-1)^{n-1}].
\]

Note that, for \( d = 2 \), \( f_n = g^{2n+1} = 0 \) and \( f^{2n+1} = g^{2n} = 1 \) yielding the familiar relations for Pauli matrices. After that, we can expand the exponential in \( U_{A(B)} = e^{i X_{A(B)}} \) and use equations (21)–(23) to obtain the following result

\[
e^{i\theta X} = \frac{e^{-i\theta}}{d} \left[ (e^{i\theta d} - 1) X + (e^{i\theta d} + (d-1)) I \right].
\]

Note that the quantity \( \text{tr}(U \rho) \) for pure state \( \rho_{AB} = |\Psi\rangle \langle \Psi| \) where \( |\Psi\rangle_{AB} = \sum_j \sqrt{\lambda_j} |j\rangle_A |j\rangle_B \) and \( U_{AB} = U_A \otimes U_B \) with \( U_M = e^{i X_M} \) contains the following three terms \( \langle \Psi | X_A \otimes X_B | \Psi \rangle \), \( \langle \Psi | X_A \otimes I_B | \Psi \rangle \) and \( \langle \Psi | I_A \otimes X_B | \Psi \rangle \). The first term is proportional to negativity, second and third terms are zero. We then have the following relation

\[
\text{tr}(\rho_{AB} U_{AB}) = \frac{e^{-2i\theta}}{d^2} \left[ 2 (e^{i\theta d} - 1)^2 N + (e^{i\theta d} + (d-1))^2 \right],
\]

where \( U_{AB} = U_A \otimes U_B \) and \( U_{A(B)} = e^{i X_{A(B)}} \). If we take \( \theta = \frac{\pi}{d} \) in the above equation then we have

\[
\text{tr}(\rho_{AB} U_{AB}) = \frac{e^{-2i\pi/d}}{d^2} \left[ 8N + (d-2)^2 \right].
\]

The interference visibility is then given by equation (20).

The above protocol is not restricted to pure state. To see that, consider the following mixed state which is a mixture of maximally entangled state and a classical–classical state \( \rho_{AB} = x |\Phi^+\rangle \langle \Phi^+ | + \frac{(1-x)}{d^2} \sum_{j=1}^{d-1} |j\rangle_A \langle j| \otimes |j\rangle_B \), where \( x \) is a real number and \( x \in [0, 1] \). Here, \( |\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d-1} |j\rangle_A \otimes |j\rangle_B \) is the maximally entangled state of two qudit. The entanglement of the above state is given by \([67]\]

\[
N(\rho_{AB}) = \frac{x(d-1)}{2}.
\]

Now, it can be shown that the joint observable \( X_A \otimes X_B \) where \( X_A = \sum_{i<j} G^A_{ij} \) and \( X_B = \sum_{i<j} G^B_{ij} \) we have \( \text{tr}(X_A \otimes X_B \rho_{AB}) = 2N(\rho_{AB}) \), and \( \text{tr}(X_A \otimes I_B \rho_{AB}) = \text{tr}(I_A \otimes X_B \rho_{AB}) = 0 \). Therefore, the entanglement of the mixed state can be determined from the interferometric setup.
4.4. Entanglement detection from interferometric set-up

Finding out whether a given state is entangled or separable is one of the fundamental and crucial problems in the theory of entanglement. For qubit–qubit and qubit–qutrit systems there exist a necessary and sufficient condition for separability based on PPT criteria \[ [68–71]. For multipartite systems or higher dimensional systems the most general method to detect entangled states is via entanglement witness \[ [60]. An alternative approach to detect entanglement is via measuring suitably defined statistical correlations (for example, mutual predictability \[ [59] or Pearson correlation \[ [72]) in complementary observables. The notion of ‘mutual predictability’ is a complementary correlation which can be used to detect entanglement in multipartite systems or higher dimensional systems as well as for continuous variable systems \[ [59]. Another approach to detect entanglement is to use a modification of the Bell scenario \[ [73] which demonstrate that all entangled state violates an inequality involving joint probabilities \[ [47]. In this section our objective is to demonstrate how some of the existing entanglement detection methods can be implemented in an interferometric set-up. In particular, we have chosen the entanglement detection process involving mutual predictability \[ [59]. Instead of performing local measurements in complementary bases we have constructed a suitable unitary which yields the value of mutual predictability directly. Furthermore, we will also discuss how some class of entanglement witness operators can be implemented in a interferometric setup.

4.4.1. Determination of mutual predictability from interferometric set-up

In \[ [33, 59], it was shown that there is intimate connection between the mutually unbiased bases (MUBs) and detection of entanglement for composite systems in high-dimensional Hilbert space. In particular by performing local measurements in MUBs one can detect whether a state is entangled or not by using a statistical quantity called the mutual predictability. In addition, the approach given in \[ [59] can provide necessary and sufficient criteria for separability if a complete set of MUBs is available for the local subsystems. In this section, we show that without doing explicit local measurements of complete set of MUBs, we can construct a unitary operation which yields the value of mutual predictability in terms of the interferometric visibility and hence can detect entanglement.

Consider a bipartite state \( \ket{\Psi}_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B \) and observables \( \mathcal{O}_A \) and \( \mathcal{O}_B \) for each subsystem. The eigenvectors of \( \mathcal{O}_A \) and \( \mathcal{O}_B \) can be taken as \( \{ \ket{I}_A \} \) and \( \{ \ket{I}_B \} \), where \( l, k \in \{0, 1, 2, 3, \ldots, d - 1 \} \) and \( d \leq \min\{ \dim(\mathcal{H}_A), \dim(\mathcal{H}_B) \} \). We can define mutual predictability as

\[
C_{\mathcal{O}_A, \mathcal{O}_B} = \sum_k \mathcal{A}_k \langle k|_A \langle \mathcal{B}_k|_B | \mathcal{B}_k \rangle_A | \mathcal{A}_k \rangle_B. \tag{27}
\]

Now, if we consider \( m \) mutually unbiased observables for each subsystem as \( \{ \mathcal{O}_A, \mathcal{O}_B, \ldots, \mathcal{O}_{A_m} \} \) and \( \{ \mathcal{O}_{B_1}, \mathcal{O}_{B_2}, \ldots, \mathcal{O}_{B_m} \} \), then for separable state we have \[ [59]

\[
\sum_{i=1}^m C_{\mathcal{O}_{A_i}, \mathcal{O}_{B_i}} \leq 1 + \frac{m - 1}{d}. \tag{28}
\]

If any state violates the upper bound given in equation (28) then it is entangled. If \( m = d + 1 \), then the above criteria a necessary and sufficient to detect separability of any bipartite state \[ [59]. In what follows, we will provide a scheme to obtain \( C_{\mathcal{O}_A, \mathcal{O}_B} \) from the interference experiment. Note that given any projector \( \Pi \), the oracle can be represented by a unitary operator

\[
U = I - 2\Pi. \tag{29}
\]

Now let us consider two observable \( \mathcal{O}_A \) and \( \mathcal{O}_B \) with the basis vectors are \( \{ \ket{I}_A \} \) and \( \{ \ket{I}_B \} \). Now, Let us define \( U_k = I \otimes I - 2|k \rangle_A \langle k | \otimes | k \rangle_B \langle k | \). We can then consider the following unitary operator

\[
U_{\mathcal{O}_A, \mathcal{O}_B} = \prod_k U_k = \prod_k (I \otimes I - 2|k \rangle_A \langle k | \otimes | k \rangle_B \langle k |) = \left( I \otimes I - 2 \sum_k |k \rangle_A \langle k | \otimes | k \rangle_B \langle k | \right). \tag{30}
\]

Consider the state \( \rho_{AB} \) at the input port of the interferometer. Let \( U_{\mathcal{O}_A, \mathcal{O}_B} \) act on one arm of the interferometer. Now we calculate \( \text{tr}(\rho_{AB} U_{\mathcal{O}_A, \mathcal{O}_B}) \) which is given by
\[
\text{tr}(\rho_{AB} U_{C_A, C_B}) = \text{tr}\left( \mathbb{1} \otimes 1 - 2 \sum_k |k\rangle_A \langle k| \otimes |k\rangle_B \langle k| \rho_{AB} \right)
\]
\[
= 1 - 2 \sum_k \text{tr}\left( |k\rangle_A \langle k| \otimes |k\rangle_B \langle k| \rho_{AB} \right)
\]
\[
= 1 - 2 \sum_k \langle k| \rho_{AB} |k\rangle_A |k\rangle_B \langle k|_B
\]
\[
= 1 - 2C_{C_A, C_B}.
\]

The above equation shows that the mutual predictability can be obtained without doing measurement on the system itself and reveals another power of interferometer setup. Since \(0 \leq C_{C_A, C_B} \leq 1\), we have
\[
|\text{tr}(\rho_{AB} U_{C_A, C_B})| = |1 - 2C_{C_A, C_B}|.
\]

If \(C_{C_A, C_B} > \frac{1}{2}\), then the visibility will be \((2C_{C_A, C_B} - 1)\), if it is less than half then visibility will be \((1 - 2C_{C_A, C_B})\). Furthermore, the phase shift, \(\alpha = \arg[\text{tr}(U_{C_A, C_B} \rho_{AB})]\) will be zero in this case. We can rewrite above equation as
\[
\begin{align*}
C_{C_A, C_B} &= \frac{1}{2} \left( 1 \pm |\text{tr}(\rho_{AB} U_{C_A, C_B})| \right).
\end{align*}
\]

Similarly, if we consider \(m\) mutually unbiased observables for each subsystem as \(\{O_{A_1}, O_{A_2}, \ldots, O_{A_n}\}\) and \(\{O_{B_1}, O_{B_2}, \ldots, O_{B_n}\}\), then for each pair of observables \(\{O_{A_i}, O_{B_i}\}\) we have a unitary \(U_{O_{A_i}, O_{B_i}}\) similar to equation (30) and we can obtain mutual predictability in terms of \(|\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})|\) as
\[
\begin{align*}
C_{O_{A_i}, O_{B_i}} &= \frac{1}{2} \left( 1 \pm |\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})| \right),
\end{align*}
\]
like equation (31). Taking summation over \(i\) from 1 to \(m\) on both the sides of the above equation, we then obtain
\[
\begin{align*}
\sum_i^m C_{O_{A_i}, O_{B_i}} &= \frac{1}{2} \left( m \pm m \sum_i |\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})| \right).
\end{align*}
\]

Since \(|\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})|\) is real and positive, it implies that \(\arg[\text{tr}(U_{O_{A_i}, O_{B_i}} \rho_{AB})]\) will vanish and phase will be constant for all unitary operators \(\{U_{O_{A_i}, O_{B_i}}\}\). Let us denote \(\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})| = V_i\). Now, using inequality (28), equation (3) and the above equation, we obtain the separability criteria in terms of \(|\text{tr}(\rho_{AB} U_{O_{A_i}, O_{B_i}})|\)
\[
\begin{align*}
2 \left( 1 + \frac{(m - 1)}{d} \right) - m \geq \sum_{i=1}^m V_i \geq m - 2 \left( 1 + \frac{(m - 1)}{d} \right).
\end{align*}
\]

Here \(V_i\) is the visibility obtained after implementation of \(i\)th unitary. If a state \(\rho_{AB}\) violates the above inequality then the state is entangled. Let us study the above inequality in some detail. Note that \(\sum_i V_i\) is always positive. However, it may happen that either the lower or upper bound can be negative. In that case, only one part of the inequality is valid for a separable state.

To give an example, consider the case where there are \(d + 1\) mutually unbiased bases, i.e. \(m = d + 1\). In that case, the mutual predictability is a necessary and sufficient condition for separability [59] and equation (34) can be modified as
\[
(d - 3) \leq \sum_{i=1}^{d+1} V_i \leq (3 - d).
\]

We can see that the upper bound is less than zero for \(d > 3\), thus it is unattainable by \(\sum_{i=1}^{d+1} V_i\) and the entanglement detection can be done through the lower bound on \(\sum_{i=1}^{d+1} V_i\) i.e.
\[
\sum_{i=1}^{d+1} V_i \geq (d - 3).
\]
If the quantum state violates the above inequality for \((d + 1)\) mutually unbiased observables then it is entangled. Similarly, in the case of \(m = 2\), the separable states satisfy

\[
V_1 + V_2 \leq \frac{1}{d}. \tag{36}
\]

As an example of how equation (34) can detect entanglement consider the \(d\) dimensional isotropic state given by \(\rho_I = x|\Phi^+\rangle \langle \Phi^+| + \frac{(1-x)}{d^2} I \otimes I\), where \(|\Phi^+\rangle = \frac{1}{\sqrt{2}} \sum |j\rangle_A |j\rangle_B\) and \(x\) is a real number between zero to one. The isotropic state is separable if \(x \leq 1/(d + 1)\). Note that isotropic state is \(U \otimes U^*\) invariant. Given an arbitrary choice of basis \(\{|j\rangle\}_A\) for subsystem \(A\), we choose \(|j^*\rangle_B\) as the basis for subsystem \(B\), where \(|j^*\rangle\) denotes the vector \(|j\rangle\) with the corresponding complex conjugate amplitudes. For such a choice of basis the invariance of isotropic state guarantees the following

\[
b_i (|j\rangle_A \langle j| \rho |j\rangle_B |j\rangle_B) = \frac{x}{d} + \frac{(1-x)}{d^2}. \tag{37}
\]

Considering \(O_{A_i}\) and \(O_{A_i}^*\) to be the observables whose eigenvectors are \(\{|j\rangle\}_A\) and \(\{|j^*\rangle\}_A\), respectively. Then the mutual predictability is given as

\[
C_{A_iB_i} = x + \frac{(1-x)}{d}. \tag{38}
\]

For a set of choice of mutually unbiased observables \(\{O_{A_i}\}\) and \(\{O_{B_i}^*\}\) we then have

\[
\sum_{i=1}^{m} C_{A_iB_i} = m \left( x + \frac{(1-x)}{d} \right). \tag{39}
\]

The R.H.S of equation (39) exceeds the upper bound for separability given by equation (38) if \(x > \frac{1}{m}\). Thus, if there exists \(m = (d + 1)\) mutual unbiased observables then the mutual predictability is necessary and sufficient to detect entanglement of isotropic states. For \(m = (d + 1)\) and \(x > \frac{1}{m} = \frac{1}{d+1}\), we have

\[
\sum_{i=1}^{d+1} C_i > 2.
\]

Using equation (33) for \(m = (d + 1)\), we then obtain

\[
\sum_{i=1}^{d+1} V_i < (d - 3),
\]

thus, violating equation (35) for entangled isotropic state. Therefore, by suitable choice of unitary as given in equation (30) one can detect whether the state is entangled or not.

### 4.4.2. Determination of entanglement witness from interferometric set-up

For multipartite systems, there are entanglement witnesses which can detect quantum entanglement. These witness operators have positive average values on all separable states and negative on some entangled states.

**Definition 3.** A state \(\rho\) is entangled if and only if there exists a Hermitian operator \(W\) such that \(\text{tr}(W\rho) < 0\) and for any separable state \(\sigma\) we have \(\text{tr}(W\sigma) \geq 0\). The operator \(W\) is known as entanglement witness.

Let us consider the following entanglement witness in \(H_A \otimes H_B\) with \(\dim(H_A) = \dim(H_B) = d\)

\[
F_{AB} = \sum_{i,j} |i\rangle \langle j| \otimes |j\rangle \langle i|. \tag{40}
\]

We can show that \(F_{AB}\) acts as entanglement witness for the Werner state defined as follows

\[
\rho_w = xQ_+ + (1-x)Q_d, \tag{41}
\]

where

\[
Q_+ = \frac{2}{d(d+1)} (\mathbb{I}_A \otimes \mathbb{I}_B + F_{AB}) \tag{42}
\]
Figure 3. Schematic of the entanglement meter. By looking at the interference pattern we can say how much entanglement is present in the two-qubit pure state.

\[ Q_e = \frac{2}{d(d-1)} \left( \mathbb{I}_A \otimes \mathbb{I}_B - F_{AB} \right). \] (43)

It is well known that the state (41) is separable if and only if \( x \geq \frac{1}{2} \). From equations (40) and (41) we can obtain

\[ \text{tr}(F_{AB} \rho_w) = 2(2x - 1). \] (44)

Thus, \( \text{tr}(F_{AB} \rho_w) < 0 \) if and only if \( x < \frac{1}{2} \) i.e. \( \rho_w \) is entangled.

We can obtain the expectation value of a witness operator by considering \( U = \exp[i \theta W] \), where \( \theta \) is a real parameter. Let us apply the above unitary in one arm of the interferometer for an infinitesimal angle \( \theta \). In this limit one can write

\[ \text{tr}(U \rho) \simeq (1 + i \theta \text{tr}(W \rho)), \] (45)

\[ \simeq \exp[i \pi \theta \text{tr}(W \rho)]. \] (46)

From equation (46) we can infer that the phase of the intensity given by equation (3) is changed by an amount proportional to \( \text{tr}(W \rho) \). If \( \text{tr}(W \rho) < 0 \), then the state is entangled and the phase in equation (3) changed from \( \phi \) to \( \phi - \theta \text{tr}(W \rho) \). On the other hand, if \( \text{tr}(W \rho) > 0 \), then the phase changed from \( \phi \) to \( \phi + \theta \text{tr}(W \rho) \). Thus, from observing the change in phase, we can determine whether the quantum state is entangled.

Note that the above formalism of detecting entanglement is limited to small values of \( \theta \). However, we can go beyond it for certain types of witness operator. To see this, consider the SWAP operator defined in equation (40). Note that \( \text{tr}(F_{AB}) = d, F_{AB}^\dagger = F_{AB} \) and \( F^2 = \mathbb{I}_A \otimes \mathbb{I}_B \), where \( \mathbb{I}_A \) and \( \mathbb{I}_B \) are identity operators associated with \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively. Thus, \( F_{AB} \) is Hermitian as well as unitary operator. Use of SWAP operator to detect entanglement in an interferometric set-up was proposed in [49]. It was shown that an unitary involving SWAP operation acting on the two copies of the state shows the presence of entanglement in the interferometric visibility for pure state. On the other hand, we consider the capability of witnessing entanglement from the SWAP operator and use it as an example where we can go beyond the small values of \( \theta \) in equations (45) and (46). To determine expectation value of \( F_{AB} \) from an unitary operation let us take our unitary operator \( U \) to be \( F_{AB} \). We then have

\[ \text{tr}(U \rho_w) = \text{tr}(F \rho_w) = 2(2x - 1). \] (47)

If \( \rho_w \) is entangled then the right hand side of equation (47) is negative otherwise it is positive. In other words

\[ \text{tr}(U \rho_w) = \begin{cases} 2(2x - 1)e^{i\pi} & \text{when } \rho_w \text{ is entangled} \\ 2(2x - 1) & \text{when } \rho_w \text{ is separable}. \end{cases} \] (48)

Thus, if \( \rho_w \) is entangled then the phase in equation (3) changes from \( \phi \) to \( \phi + \pi \). If \( \rho_w \) is separable then the phase in equation (3) does not change. Thus, our proposal provides a phase sensitive method to detect entangled and separable states for the Werner state. In future, it will be worth exploring how to design a phase-sensitive entanglement meter which measures entanglement based on the above result.
Figure 4. A photon in state $|\psi\rangle$ is inserted in an Mach Zehnder interferometer via a 50 – 50 beam splitter (BS1). Totally reflecting mirrors M1 and M2 are placed in the reflected and transmitted paths of BS1. If the photon passes through the transmitting paths then the unitary $U$ is acted on the quantum state $|\psi\rangle$. Placing a detector in one of the arms corresponding to the second 50–50 beam splitter (BS2) and measuring the intensity the corresponding intensity would yield $\text{Tr}(U|\psi\rangle\langle\psi|)$.

The results presented here can pave the way towards design of an entanglement meter where we can have a portable device inside which we have the Mach–Zehnder interferometer with suitable unitary operators in place. By sending single copy of the local system or joint system at the input port of the entanglement meter, we can directly measure the entanglement between the pairs of subsystems. This can have several industrial applications in quantum technology. For example, if a private company is supplying maximally entangled pairs, then to convince a buyer, company can give one of the pair to the user along with the entanglement meter which has the sequence of unitaries along with the oracle. Using the entanglement meter, the end user can verify that indeed the supplied pair is maximally entangled state. Schematic of entanglement meter is given in figure 3.

The scheme is experimentally feasible and readily applicable in current quantum optics laboratories. For example, our scheme can be implemented by optical means using the paths of the Mach Zehnder interferometer as control qubits [75–78]. Such schemes usually apply the unitary when the quantum system inside the interferometer passes through one of the arms. A schematic is given in the figure 4.

5. Conclusion

To summarize, we have proposed several methods to efficiently certify and quantify entanglement of composite systems for qubits as well as higher dimensional systems using Mach–Zehnder interferometer. We have shown that having access to single copy of one of the subsystem and a suitable oracle, we can measure the linear entanglement entropy for any bipartite pure states in any finite dimension. In particular, our result shows that for any two qubit pure bipartite state, the interference visibility is a direct measure of entanglement. For arbitrary bipartite mixed states, using convex roof construction, we provide an upper bound to the linear entropy of entanglement. We have proposed another interferometric scheme to measure the negativity of any pure bipartite state and some mixed states from the visibility. We have also proposed a method to determine the Schmidt number of a quantum state from interference visibility. In addition to measuring entanglement content, we have shown how to detect entangled states using the interference visibility and phase shift. Furthermore, we have proposed how to measure mutually predictability experimentally from the intensity patterns of the interferometric set-up without having to resort to local measurements of mutually unbiased basis. Towards the end, we have proposed how to measure the average of the witness operator in Mach–Zehnder interferometer and argued that the phase shift in the interference pattern can be a signature of entangled or separable nature of the input state. Thus, our proposal can have wide variety of applications in detection and quantification of entanglement in pure as well as mixed bipartite states. In addition, results presented in this paper can have interesting applications in design of entanglement meter where one can have a portable device to measure the entanglement content of bipartite states. The proposal can also be exploited to design phase-sensitive entanglement meter. In future, it will be worth generalizing these results for multipartite systems. We believe that our proposal can be experimentally tested with the existing technology.

Data availability statement

No new data were created or analysed in this study.
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Appendix. Proof of equations (21)–(24)

As explained in the main text, the proof of \( V = \frac{1}{d} (8N + (d-2)^2) \) hinges on proving equations (21)–(24). Before proving that, we will briefly discuss Lucas sequence [66].

**Definition 4.** Given integers \( P, Q \) with \( D = P^2 - 4Q \neq 0 \), the Lucas sequences of the first kind, \( U_n = U_n(P, Q) \) and of the second kind \( V_n = V_n(P, Q) \) are the recursive sequences defined by the same linear recurrence relation

\[
X_{n+1} = P X_n - Q X_{n-1}
\]

but with starting values \( U_0 = 0, U_1 = 1 \) and \( V_0 = 2, V_1 = 1 \).

The characteristic equation of the recurrence relation is \( X^2 = P X - Q \) with roots \( \alpha, \beta = (P \pm \sqrt{D})/2 \). In terms of \( \alpha, \beta \) we have

\[
U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \frac{\alpha^n + \beta^n}{\alpha - \beta}
\]

Now, we are in a position to prove the following result.

**Theorem 1.** For a \( d \) dimensional Hilbert space, given a computational basis \( \{|i\rangle\}_{i=0}^{d-1} \), the observable \( X = \sum_{i,j=0; i\neq j}^{d-1} |i\rangle \langle j| = \sum_{i<j}^{d-1}(|i\rangle \langle j| + |j\rangle \langle i|) \) satisfies the following relation

\[
X^n = f_n X + g_n I,
\]

where \( f_n \) is Lucas sequence of first kind and \( g_n = (d-1) f_{n-1} \) satisfying

\[
f_n = \frac{(d-1)^n - (-1)^n}{d}
\]

and

\[
g_n = \frac{d-1}{d} \left[ (d-1)^{n-1} - (-1)^{n-1} \right].
\]

**Proof.** We first proof that the observable \( X \) defined above has the following property for any positive integer \( n > 2 \), i.e.

\[
X^n = (d-1)X^{n-2} + (d-2)X^{n-1}.
\]

To prove this, we start with the relation

\[
X^2 = \sum_{i,j=0; i\neq j}^{d-1} |i\rangle \langle j| \delta_{jm}.
\]

Now, if we take \( i = l \) in the previous equation, then it can be easily checked that there are \( (d-1) \) values of \( j \) and \( m \), for which \( \delta_{jm} = 1 \). Similarly, if we take \( i \neq l \), then there are \( (d-2) \) values of \( j \) and \( m \) for which \( \delta_{jm} = 1 \). Therefore, the previous equation can be rewritten as

\[
X^2 = (d-1) \sum_{i=0; i\neq l}^{d-1} |i\rangle \langle l| + (d-2) \sum_{i=0; i\neq l}^{d-1} |i\rangle \langle l|
\]

\[
= (d-1)I + (d-2)X.
\]
Multiplying both sides of equation (A3) with $X^{n-2}$ we obtain equation (A2). Note that equation (A2) already demonstrates that the $n$th power of $X$, satisfies the functional relationship $X^n = f_n X + g_n \mathbb{I}$ for $n > 2$. Here, $f_n$ and $g_n$ are real-valued numbers which depend on the dimension of the Hilbert space $\dim(\mathcal{H})$ and the power of $X$, $n$, extending the functional relationship for $n = 2$ and comparing the equation (A3), we obtain $f_2 = (d - 2)$ and $g_2 = (d - 1)$. Extending it again for $n = 1$, we then obtain $f_1 = 1$ and $g_1 = 0$. Thus, we have shown that $X^n = f_n X + g_n \mathbb{I}$ for $n \geq 1$. In what follows we will show that $f_n$ has a Fibonacci-type recursive relationship known as Lucas sequence [66].

Consider the functional relationship for $(n-1)$th power, i.e. $X^{n-1} = f_{n-1} X + g_{n-1} \mathbb{I}$. Multiplying it with $X$ on both the sides and using equation (A3) we obtain the following relations for $g_n$ and $f_n$, respectively, i.e. $g_n = (d - 1)f_{n-1}$ and $f_n = (d - 2)f_{n-1} + (d - 1)f_{n-2}$. Since, $f_1 = 1$, if we consider $f_0 = 0$ then equation (A1) implies that $f_n$ is the Lucas sequence of first kind with $P = (d - 2)$ and $Q = -(d - 1)$. Thus, we have

$$f_n = \frac{(d - 1)^n - (-1)^n}{d},$$

(A4)

and

$$g_n = \frac{d - 1}{d} \left[ (d - 1)^{n-1} - (-1)^{n-1} \right].$$

(A5)

Now, consider the unitary operator $U = e^{i\theta X}$. On expanding the exponential and using equations (A4) and (A5) we obtain

$$e^{i\theta X} = \sum_{n=0}^{\infty} \frac{\theta^n X^n}{n!},$$

(A6)

$$= \mathbb{I} + \frac{1}{d} \sum_{i=1}^{n} \frac{\theta^n}{n!} \left( (d - 1)^n (X + \mathbb{I}) + (-1)^n ((d - 1)\mathbb{I} - X) \right)$$

$$= \frac{1}{d} \sum_{i=0}^{n} \frac{\theta^n}{n!} \left( (d - 1)^n (X + \mathbb{I}) + (-1)^n ((d - 1)\mathbb{I} - X) \right)$$

$$= \frac{1}{d} \left[ e^{i\theta(d-1)}(X + \mathbb{I}) + e^{-i\theta}((d - 1)\mathbb{I} - X) \right]$$

$$= \frac{e^{-i\theta}}{d} \left[ (e^{i\theta d} - 1) X + (e^{i\theta d} + (d - 1)) \mathbb{I} \right].$$

(A7)

Thereby, obtaining equation (24) in the main text.

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