EQUIVARIANT BURNSIDE GROUPS: STRUCTURE AND OPERATIONS

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Abstract. We introduce and study functorial and combinatorial constructions concerning equivariant Burnside groups.

1. Introduction

Let $G$ be a finite group, and $k$ a field of characteristic zero containing all roots of unity of order dividing $G$. In this paper, we continue the study of a new invariant in $G$-equivariant birational geometry over $k$, the equivariant Burnside group

$$\text{Burn}_n(G),$$

introduced in [7], and building on [6], [5], [8], and [3].

The class of an $n$-dimensional $G$-variety in this group is computed on an appropriate smooth $G$-birational model $X$, called standard form: after a sequence of $G$-equivariant blowups we may assume that [12]:

- there exists a Zariski open $U \subset X$ such that the $G$-action on $U$ is free,
- the complement $X \setminus U$ is a $G$-invariant simple normal crossing divisor,
- for every $g \in G$ and every irreducible component $D$ of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

The standard form is preserved under $G$-equivariant blowups with smooth centers which have normal crossings with respect to the components of $D$. Moreover, the stabilizer of every $x$ on such $X$ is an abelian subgroup of $G$ [12, Thm. 4.1]. On such a model, the class of $X \circlearrowleft G$ is defined by:

$$[X \circlearrowleft G] := \sum_{H \leq G} \sum_{F} \mathcal{S}_F \in \text{Burn}_n(G),$$

with summation over conjugacy classes of abelian subgroups $H \subseteq G$ and strata $F \subseteq X$ with generic stabilizer $H$; the symbol

$$\mathcal{S}_F := (H, N_G(H)/H \circlearrowleft k(F), \beta_F(X))$$

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records the action of the normalizer $N_G(H)$ of $H$ on $k(F)$, the product of the function fields of the components of $F$, as well as the generic normal bundle representation $\beta_F(X)$ of $H$. The class in (1.1) takes values in a quotient of the free abelian group generated by such symbols, by certain blow-up relations, spelled out in [7, Definition 4.2], and ensuring that this expression is a well-defined $G$-birational invariant [7, Theorem 5.1].

In [3], we presented first geometric applications of this invariant. Here, we continue to explore functorial and combinatorial properties of $\text{Burn}_n(G)$. We introduce and study:

- filtrations on $\text{Burn}_n(G)$,
- the restriction homomorphism
  $$\text{Burn}_n(G) \to \text{Burn}_n(G'),$$
  where $G' \subseteq G$ is any subgroup,
- products,
- a combinatorial analog $\mathcal{BC}_n(G)$ of $\text{Burn}_n(G)$, obtained by forgetting field-theoretic information, while keeping only discrete invariants encoded in a symbol (1.2).

One of the motivating problems in this field is to distinguish equivariant birational types of (projectivizations of) linear actions (see, e.g., [11], [4]). A sample question, raised in [2, Section 8], is: Are there isomorphic finite subgroups of $\text{PGL}_3$ which are not conjugate in the plane Cremona group?

Examples of equivariantly nonbirational representations considered in [13] required that $G$ contains an abelian $p$-subgroup of rank equal to the dimension of the representation. Our formalism yields new examples without the rank condition on the group, see Example 5.3.

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2. Generalities

We adopt notational conventions from [7]:

- $G$ is a finite group,
- $k$ is a field of characteristic 0, containing roots of unity of order $|G|$,
- $H \subseteq G$ is an abelian subgroup, with character group
  $$H^\vee := \text{Hom}(H, k^\times),$$
• Bir\(_d(k)\) is the set of birational equivalence classes of \(d\)-dimensional algebraic varieties over \(k\), i.e., the set of finitely generated fields of transcendence degree \(d\) over \(k\); we identify a field with its isomorphism class in Bir\(_d(k)\),

• \(\text{Alg}_N(K_0)\) is the set of isomorphism classes of Galois algebras \(K\) over \(K_0 \in \text{Bir}_d(k)\) for the group

\[
N := N_G(H)/H,
\]
satisfying

**Assumption 1:** the composite homomorphism

\[
H^1(N_G(H), K^\times) \to H^1(H, K^\times)^N \to H^\vee
\]

is surjective (see [7, Section 2] for more details).

• More generally, for a subgroup \(M \subset N\) we denote by \(\text{Alg}_M(K_0)\) the set of isomorphism classes of \(M\)-Galois algebras \(K/K_0\) (i.e., Galois algebras \(K\) over \(K_0\) for the group \(M\)), such that \(\text{Ind}_M^N(K)\) satisfies Assumption 1. Of particular interest is

\[
Z := Z_G(H)/H \subseteq N = N_G(H)/H.
\]

**Lemma 2.1.** Let \(K \in \text{Alg}_N(K_0)\). Then

\[
K \cong \text{Ind}_Z^K(K')
\]

for some \(K' \in \text{Alg}_Z(K_0)\).

**Proof.** With notation of Assumption 1, we have trivial \(H\)-action on \(K^\times\), so

\[
H^1(H, K^\times) = \text{Hom}(H, K^\times).
\]
Writing \(K \cong K^1 \times \cdots \times K^\ell\), where each \(K^i\) is a field, as rightmost map in (2.1) we take

\[
H^1(H, K^\times) \to H^1(H, (K^1)^\times) \cong H^\vee.
\]
Projection \(K^\times \to (K^1)^\times\) is equivariant for the subgroup \(Y \subseteq N\), defined by the condition of sending \(K^1\) to \(K^1\), where the action on \(H^1(H, (K^1)^\times)\) is given just by conjugation on \(H\). Assumption 1 implies that the conjugation action is trivial, i.e., \(Y \subseteq Z\). So the result holds with

\[
K' = \text{Ind}_Y^K(K^1).
\]

\[\Box\]

**Remark 2.2.** Assumption 1, for an \(N\)-Galois algebra \(K/K_0\), of the form \(\text{Ind}_Z^K(K')\), where \(K'/K_0\) is a \(Z\)-Galois algebra, may be expressed as the surjectivity of

\[
H^1(Z_G(H), K'^\times) \to H^\vee.
\]

(2.2)
Given this, the proof of [7, Prop. 2.2] supplies an equivalence of categories between

- $H$-Galois algebras over étale $K_0$-algebras and
- $Z_G(H)$-Galois algebras with equivariant homomorphism from $K'$;

in particular, there is then a $Z_G(H)$-Galois algebra $L/K_0$ with homomorphism $K' \to L$, compatible with the structure of Galois algebra for the group $Z$, respectively $Z_G(H)$. Assumption 1 is also implied by the existence of such a Galois algebra $L$ and homomorphism $K' \to L$, as we see by using the Hochschild-Serre spectral sequence and Hilbert’s Theorem 90 to identify $H^1(Z_G(H), K'^\times)$ with $H^1(H \times Z_G(H), L^\times)$. This allows us to view Assumption 1 as a lifting problem of Galois cohomology $H^1(\text{Gal}_{K_0}, Z_G(H)) \to H^1(\text{Gal}_{K_0}, Z)$ and remark that the machinery of nonabelian cohomology (cf. [10 §1.3.2]) supplies an obstruction to lifting in $H^2(\text{Gal}_{K_0}, H)$.

We now recall the definition of the equivariant Burnside group

$\text{Burn}_n(G) = \text{Burn}_{n,k}(G)$

following [7, Section 4]: it is a $\mathbb{Z}$-module, generated by symbols

$s := (H, N \triangleleft K, \beta)$,

where

- $H \subseteq G$ is an abelian subgroup,
- $K \in \text{Alg}_N(K_0)$, with $K_0 \in \text{Bir}_d(k)$, and $d \leq n$,
- $\beta = (b_1, \ldots, b_{n-d})$, a sequence of nonzero elements of the character group $H^\vee$, that generate $H^\vee$.

The sequence of characters $\beta$ determines a faithful $(n - d)$-dimensional representation of $H$ over $k$, with trivial space of invariants. As every $(n - d)$-dimensional representation of $H$ over $k$ splits as a sum of one-dimensional representations, any faithful $(n - d)$-dimensional representation of $H$ over $k$ determines a sequence of characters, generating $H^\vee$, up to order. The ambiguity of order gives us the first of several relations that we impose on symbols:

(\textbf{O}): $(H, N \triangleleft K, \beta) = (H, N \triangleleft K, \beta')$ if $\beta'$ is a reordering of $\beta$.

The further relations are conjugation and blowup relations:

(\textbf{C}): $(H, N \triangleleft K, \beta) = (H', N' \triangleleft K, \beta')$, when $H' = gHg^{-1}$ and $N' = N_G(H')/H'$, with $g \in G$, and $\beta$ and $\beta'$ are related by conjugation by $g$.

(\textbf{B1}): $(H, N \triangleleft K, \beta) = 0$ when $b_1 + b_2 = 0$. 

(B2): \((H, N \subset K, \beta) = \Theta_1 + \Theta_2\), where

\[
\Theta_1 = \begin{cases} 
0, & \text{if } \beta_1 = \beta_2, \\
(H, N \subset K, \beta_1) + (H, N \subset K, \beta_2), & \text{otherwise,}
\end{cases}
\]

with

\[
\beta_1 := (b_1, b_2 - b_1, b_3, \ldots, b_{n-d}), \quad \beta_2 := (b_2, b_1 - b_2, b_3, \ldots, b_{n-d}),
\]

\[(2.3)\]

and

\[
\Theta_2 = \begin{cases} 
0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\
(H, N \subset K, \bar{\beta}), & \text{otherwise,}
\end{cases}
\]

with

\[
\bar{H} := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \ldots, \bar{b}_{n-d}), \quad \bar{b}_i \in \bar{H}^\vee,
\]

and \(K\) carries the action described in Construction (A) in [7, Section 2], applied to the character \(b_1 - b_2\).

We permit ourselves to write a symbol in the form

\[
(H, M \subset K, \beta)
\]

\[(2.4)\]

with a subgroup \(M \subset N\) and \(K \in \text{Alg}_M(K_0)\), with

\[
(H, M \subset K, \beta) := (H, N \subset \text{Ind}_M^N(K), \beta).
\]

We further allow \(K_0\) to be a product of fields; then \[(2.4)\] will denote the corresponding sum of symbols, one for each factor.

By Lemma 2.1 any symbol in \(\text{Burn}_n(G)\) is of the form

\[
(H, Z \subset K, \beta),
\]

with \(K \in \text{Alg}_Z(K_0)\). In this notation, Construction (A) has a compact formulation. Applied to a single character \(b \in H^\vee\), this yields the subgroup

\[
\bar{H} := \ker(b) \subset H
\]

and the symbol

\[
(\bar{H}, Z_G(H) / \bar{H} \subset K(t), \bar{\beta}),
\]

where a \(Z_G(H)\)-action on \(K(t)\) arises by lifting \(b\) via \[(2.2)\] and is trivial on \(\bar{H}\), and \(\bar{\beta}\) is obtained from \(\beta\) by applying the map \(H^\vee \to \bar{H}^\vee\), as above.

Remark 2.3. Construction (A) may be applied to a collection of characters, yielding the same outcome as when applied iteratively, one character at a time.
A $G$-action on $X$ in standard form always satisfies

**Assumption 2:** The stabilizers for the $G$-action on $X$ are abelian, and for every $H$ and $F$ in (1.1) the composite homomorphism

$$\text{Pic}^G(X) \to H^1(N_G(H), k(F)^{\times}) \to H^\vee$$

is surjective, where the first map is given by restriction and the second is the map from Assumption 1, with $K = k(F)$.

Note that Assumption 2 implies Assumption 1, for every $H$ and every $N_G(H)/H \subset k(F)$ (see [7, Rmk. 3.2(i)]).

A variant, that will occur below, is the requirement of surjectivity, when we restrict to a given subgroup of $\text{Pic}^G(X)$. Given this, we will say that Assumption 2 holds for the given subgroup of $\text{Pic}^G(X)$.

3. Filtrations

In this section, we explore additional combinatorial constructions on equivariant Burnside groups $\text{Burn}_n(G)$, reflecting the geometry of the $G$-action on strata with given generic stabilizers.

**Definition 3.1.** A $G$-prefilter is a collection $H$ of pairs $(H,Y)$ consisting of an abelian subgroup $H \subseteq G$ and a subgroup

$$Y \subseteq Z = Z_G(H)/H,$$

such that $H$ is closed under conjugation, i.e., for $(H,Y) \in H$ we have

$$(gHg^{-1}, gYg^{-1}) \in H,$$

for all $g \in Z_G(H)$ satisfying $\bar{g} \in Y$ and $Y \subseteq Z_G(g)/H$.

**Definition 3.2.** Given a $G$-prefilter $H$, we let

$$\text{Burn}^H_n(G)$$

be the quotient of $\text{Burn}_n(G)$ by the subgroup generated by classes of the form

$$(H,Y \subset K, \beta),$$

where $K \in \text{Alg}_Y(K_0)$ is a field, and

$$(H,Y) \notin H.$$ 

**Proposition 3.3.** Let $H$ be a $G$-prefilter such that if $(H,Y) \in H$, with $H$ nontrivial, then $((H,g), Y/\langle \bar{g} \rangle) \in H$ for all $g \in Z_G(H)$ satisfying

$$\bar{g} \in Y$$

and

$$Y \subseteq Z_G(g)/H.$$ 

Then $\text{Burn}^H_n(G)$ is generated by triples

$$(H,Y \subset K, \beta),$$
where $K \in \text{Alg}_Y(K_0)$ is a field and
\[(H,Y) \in \mathbf{H},\]
subject to relations (O), (C), (B1), and (B2) applied to these triples.

**Proof.** For any
\[(H,Y \varsubsetneq K, \beta)\]
with $K \in \text{Alg}_Y(K_0)$ a field, the term $\Theta_2$ from (B2), when nontrivial, consists of a subgroup $\ker(b)$ of $H$ for some $b \in H'$, a field $K(t)$ with action of the pre-image of $Y$ in $Z_G(H)/\ker(b)$, and a sequence of characters. If $(H,Y) \notin \mathbf{H}$, then by hypothesis the pair consisting of $\ker(b)$ and the pre-image of $Y$ is not in $\mathbf{H}$. This observation establishes the proposition, since (B1) involves just one triple, and in $\Theta_1$ from (B2) the group and algebra do not change. \qed

**Example 3.4.**
- For $G$ abelian, we have
  \[\text{Burn}_n^G(G) = \text{Burn}_n^{(G,\text{triv})}(G),\]
  where the left side was introduced in [7, §8]: this is the quotient of $\text{Burn}_n(G)$ by all triples whose first entry is a proper subgroup of $G$.
- For $\mathbf{H}$ consisting of all $(H,Y)$ with $H$ nontrivial cyclic and $Y$ noncyclic, and $k$ algebraically closed, $\text{Burn}_n^H(G)$ appeared in [3, §7.4].

**Remark 3.5.** One can additionally suppress the field information, which will lead to combinatorial analogues of Burnside groups. We will explore this in Section 8.

4. **Nontrivial generic stabilizers**

In this section, we introduce a version of the equivariant Burnside group, relevant for considerations of actions with nontrivial generic stabilizer.

Let $G$ be a finite group. A variant of the equivariant Burnside group takes the additional data of a finite index set
\[I \subset \mathbb{N}.\]

The **equivariant indexed Burnside group**
\[\text{Burn}_{n,I}(G),\]
is defined as a quotient of the $\mathbb{Z}$-module generated by symbols
\[(H \subseteq H', N' \varsubsetneq K, \beta, \gamma),\]
where

- $H \subseteq H'$ are abelian subgroups of $G$,
- $N' := N_{N_G(H)}(H')/H'$,
- $K \in \text{Alg}_{N'}(K_0)$, with $K_0 \in \text{Bir}_d(k)$, and $d \leq n - |I|$,
- $\beta = (b_1, \ldots, b_{n-d-|I|})$, a sequence of nonzero characters of $H'$, trivial upon restriction to $H$, that generate $(H'/H)\wedge$,
- $\gamma = (c_i)_{i \in I}$ is a sequence of elements of $H'\vee$, such that the images of $c_i$ in $H^\vee$ generate $H^\vee$.

As in Section 2, we permit ourselves to write a symbol in the form

$$(H \subseteq H', M' \subset K, \beta, \gamma),$$

where $M' \subset N'$ is a subgroup. Every symbol may be expressed as

$$(H \subseteq H', Z' \subset K, \beta, \gamma), \quad Z' := Z_G(H')/H'.$$

(Notice that $Z_G(H') = Z_{N_G(H)}(H').$)

These symbols are subject to relations:

- **(O):** $(H \subseteq H', N' \subset K, \beta, \gamma) = (H \subseteq H', N' \subset K, \beta', \gamma)$ if $\beta'$ is a reordering of $\beta$.

- **(C):** $(H \subseteq H', N' \subset K, \beta, \gamma) = (gHg^{-1} \subseteq gH'g^{-1}, gN'g^{-1} \subset K, \beta', \gamma')$ for $g \in G$, with $\beta$ and $\beta'$, respectively $\gamma$ and $\gamma'$, related by conjugation by $g$.

- **(B1):** $(H \subseteq H', N' \subset K, \beta, \gamma) = 0$ when $b_1 + b_2 = 0$.

- **(B2):** $(H \subseteq H', N' \subset K, \beta, \gamma) = \Theta_1 + \Theta_2$, where $\Theta_1$ and $\Theta_2$ are as in Section 2, with $H$ prepended and $\gamma$, respectively $\bar{\gamma}$, appended to the corresponding symbols.

**Remark 4.1.** By analogy with Remark (2.2), we may express Assumption 1, for the Galois algebra $K$, as the surjectivity of the middle vertical map

$$0 \to H^1(Z_G(H')/H, K^\times) \to H^1(Z_G(H'), K^\times) \to H^1(H, K^\times)^{Z_G(H')/H} \to 0$$

Here, the top row comes from the Hochschild-Serre spectral sequence. In a symbol, we have $\beta$ generating the left-hand group in the bottom row, while $\gamma$ is a sequence of characters of $H'$, whose images generate $H^\vee$. Consequently, $\beta$ and $\gamma$ together generate $H^\vee$. Thus we have a homomorphism

$$\psi_I : \text{Burn}_{n,I}(G) \to \text{Burn}_n(G),$$
sending \((H \subseteq H', Z' \simeq K, \beta, \gamma)\) to
\[(H', Z' \simeq K, \beta \cup \gamma)\]
when \(\gamma\) is a sequence of nontrivial characters, otherwise to 0.

In order to explain the relevance of this definition, we introduce a map which converts some of the characters in \(\gamma\) to a transcendental extension of the Galois algebra. Let
\[J \subseteq I\]
be a subset. Given a symbol \((H \subseteq H', Z' \simeq K, \beta, \gamma)\), we use \(J\) to define subgroups
\[\overline{H}' := \bigcap_{i \in I \setminus J} \ker(c_i) \subseteq H',\]
\[\overline{H} := H \cap \overline{H}' \subseteq H.\]
Then we define
\[\omega_{I,J} : \text{Burn}_{n,I}(G) \to \text{Burn}_{n,J}(G),\]
by applying Construction (A) when possible:
\[(H \subseteq H', Z' \simeq K, \beta, \gamma) \mapsto (\overline{H} \subseteq \overline{H}', Z_G(H')/\overline{H}' \simeq K((t_i)_{i \in I \setminus J}), \overline{\beta}, \overline{\gamma}),\]
where \(\overline{\gamma} = (\overline{c}_j)_{j \in J}\), when all of the characters of \(\overline{\beta}\) are nonzero, and
\[(H \subseteq H', Z' \simeq K, \beta, \gamma) \mapsto 0,\]
otherwise.

This is compatible with relations: the only one that is nontrivial to check is (B2), where \(\Theta_1\) maps to \(\overline{\Theta}_1\), as we see by dividing into cases according to the vanishing of \(\overline{b}_1\) or \(\overline{b}_2\), or their equality, and \(\Theta_2\) maps to \(\overline{\Theta}_2\), as we see using Remark 2.3.

We recall the setting of [7, Defn. 5.4]: Let \(X\) be a smooth projective variety of dimension \(n\), with a generically free action of \(G\), satisfying Assumption 2. Let \(D_1, \ldots, D_\ell\) be \(G\)-stable divisors, with
\[D_I := \bigcap_{i \in I} D_i, \quad \text{for } I \subseteq \mathcal{I} := \{1, \ldots, \ell\}, \quad D_\emptyset = X.\]
We suppose, for notational simplicity, that for every \(I\) the generic stabilizers of the components of \(D_I\) belong to a single conjugacy class of subgroups, and take \(H_I\) to be a representative. Then to \(I \subseteq M \subseteq \mathcal{I}\) we attach the following class in \(\text{Burn}_{n,I}(G)\):
\[\chi_{I,M}(X \ltimes G, (D_i)_{i \in I}) := \sum_{H' \supseteq H_I} \sum_{\substack{W \subseteq D_I \\text{generic stabilizer} \ H' \ \text{at} \ M}} (H_I \subseteq H', N' \simeq k(W), \beta, \gamma),\]
where
• the first sum is over conjugacy class representatives $H'$ of abelian subgroups of $N_G(H_I)$, containing $H_I$,
• the second sum is over $N_{N_G(H_I)}(H')$-orbits of components $W$ with generic stabilizer $H'$, contained in components of $D_I$ with generic stabilizer $H_I$ and satisfying $\{i \in I \mid W \subset D_i\} = M$,
• $\beta = \beta_W(D_I)$ encodes the normal bundle to $W$ in $D_I$, and
• $\gamma = (c_i)_{i \in I}$, the characters coming from $D_i$ with $i \in I$.

Then
\[
[N_{D_I/X} \otimes G]^{\text{naive}} = \sum_{I \subseteq M \subseteq I \setminus J \subseteq M} \psi_{I \cap J}(\omega_{I,J}(X \otimes G, (D_i)_{i \in I}))),
\]
where the terms with $J = \emptyset$ contribute $[N_{D_I/X} \otimes G]^{\text{naive}}$. This provides some insight to [7, Lemma 5.7].

5. Fibrations

In this section, we define a projectivized version of the equivariant indexed Burnside group and use it to give a formula for the class in $\text{Burn}_n(G)$ of the projectivization of a sum of line bundles.

Let $G$ be a finite group and $I \subset \mathbb{N}$ a nonempty finite index set. The equivariant projectively indexed Burnside group

\[
\text{Burn}_{n,F(I)}(G)
\]

is defined with generators and relations as in Section 4 where

• $\beta$ consists of $n - d - |I| + 1$ characters (so $d \leq n - |I| + 1$),
• the differences of pairs of characters of $\gamma$ should generate $H^\vee$,
• and there is an additional relation:

(P): If $\gamma' - \gamma$ is a constant sequence then

\[
(H \subseteq H', N' \subset K, \beta, \gamma) = (H \subseteq H', N' \subset K, \beta, \gamma').
\]

We define

\[
\omega_{F(I),J} : \text{Burn}_{n,F(I)}(G) \to \text{Burn}_{n,J}(G),
\]

for a proper subset

\[
J \subsetneq I,
\]

by

• choosing $i_0 \in I \setminus J$,
• applying (P) to get a representative symbol

\[
(H \subseteq H', N' \subset K, \beta, \gamma)
\]

with $\gamma_{i_0} = 0$, and
• applying \( \omega_{I \setminus \{i_0\}, J} \) to the class of \((H \subseteq H', N' \subset K, \beta, (c_i)_{i \in I \setminus \{i_0\}}) \) in \( \operatorname{Burn}_{n, I \setminus \{i_0\}}(G) \).

Let \( X \) be a smooth projective variety over \( k \). Assume that \( X \) carries a \( G \)-action, and let \( L_0, \ldots, L_r \) be \( G \)-linearized line bundles on \( X \). The next statement examines the condition, for \( G \) to act generically freely on \( \mathbb{P}(L_0 \oplus \cdots \oplus L_r) \), so that Assumption 2 satisfied.

**Lemma 5.1.** Let \( X \) be a smooth projective variety over \( k \) with a \( G \)-action and \( G \)-linearized line bundles \( L_0, \ldots, L_r \). Let \( H \) be the stabilizer at the generic point of a component of \( X \), and let us denote the \( N_G(H) \)-orbit of the component by \( X' \). The following are equivalent.

(i) The \( N \)-action on \( X' \) satisfies Assumption 2, and \( H \) is abelian with \( H' \) spanned by the differences of characters determined by \( L_0, \ldots, L_r \).

(ii) The \( G \)-action on \( \mathbb{P}(L_0 \oplus \cdots \oplus L_r) \) is generically free and satisfies Assumption 2.

(iii) The \( G \)-action on \( \mathbb{P}(L_0 \oplus \cdots \oplus L_r) \) is generically free and satisfies Assumption 2 for \( L_0, \ldots, L_r \), together with the \( G \)-linearized line bundles on \( X \) associated with \( N \)-linearized line bundles on \( X' \).

The statement is inspired by [7, Lemma 7.3].

**Proof.** The action of \( G \) on \( \mathbb{P}(L_0 \oplus \cdots \oplus L_r) \) is generically free if and only if the action of \( N_G(H) \) on \( \mathbb{P}(L_0|_{X'}, \ldots, L_r|_{X'}) \) is generically free. The latter has generic stabilizer \( \bigcap_{i=1}^r \ker(b_i - b_0) \). Thus the condition on \( H \) in (i) is equivalent to the condition of generically free action in (ii) and in (iii). We assume this from now on.

An \( N \)-linearized line bundle on \( X' \) determines an \( N_G(H) \)-linearized line bundle on \( X' \), with trivial \( H \)-action. An \( N_G(H) \)-linearized line bundle on \( X' \) determines, and is determined by, \( G \)-linearized line bundle on \( X \); this is the meaning of the associated line bundles in (iii). Conversely, a \( G \)-linearized line bundle on \( X \) which restricts to an \( N_G(H) \)-linearized line bundle on \( X' \) with trivial \( H \)-action, gives rise to an \( N \)-linearized line bundle on \( X' \).

We start by showing (i) implies (iii), using the interpretation of Assumption 2 in terms of the representability of a certain morphism from the quotient stack to a product of copies of \( B \mathbb{G}_m \). Given (i), we have such a representable morphism

\[
[X'/N] \to B \mathbb{G}_m \times \cdots \times B \mathbb{G}_m.
\]

Correspondingly, the fibers of the composite morphism

\[
[X/G] \to [X'/N] \to B \mathbb{G}_m \times \cdots \times B \mathbb{G}_m
\]
all have constant stabilizer group $H$. The condition in (i) implies that the $H$-representation given by $b_0, \ldots, b_r$ is faithful. With $r+1$ additional factors $BG_m$ we get a representable morphism from $[X/G]$, hence also from $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$.

Since trivially (iii) implies (ii), it remains only to show (ii) implies (i). Generally, a line bundle on a projective bundle is isomorphic to the pullback of a line bundle from the base, twisted by a power of the tautological line bundle. Two vector bundles, one obtained from the other by tensoring by a line bundle, have isomorphic projectivizations, the tautological line bundle of one obtained from the other by tensoring by the pullback of the line bundle from the base. A sum of line bundles, after such tensoring, may be brought in a form with trivial $i$th factor, for any $i$, and this way see that any power of the tautological line bundle on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$, restricted to the open $U_i \subset \mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ defined by nonvanishing on the component $L_i$, is identified with a line bundle pulled back from the base; all of these assertions are valid in an equivariant setting. With the notation of Assumption 2 for $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$, we always have $\text{Spec}(k(F)) \subset U_i$ for some $i$. So, (ii) implies that the $G$-action on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ satisfies Assumption 2 for $\text{Pic}^G(X)$. Since $\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \to X$ admits equivariant sections, we deduce that some finite collection of $G$-linearized line bundles on $X$ determines a representable morphism

$$[X/G] \to BG_m \times \cdots \times BG_m.$$ 

Replacing each by a tensor product with combinations of $L_0, \ldots, L_r$, we obtain a $G$-linearized line bundle on $X$ that comes from an $N$-linearized line bundle on $X'$. The corresponding morphism

$$[X'/N] \to BG_m \times \cdots \times BG_m$$

is representable, and thus we have (i). \qed

**Proposition 5.2.** Let $X$ be a smooth projective variety of dimension $n-r$ over $k$ with a $G$-action and $G$-linearized line bundles $L_0, \ldots, L_r$. We assume the conditions and adopt the notation of Lemma 5.1. We define $I := \{0, \ldots, r\}$ and the following class in $\text{Burn}_{n,\mathbb{P}(I)}(G)$:

$$\xi(X \acts G, (L_i)_{i \in I}) := \sum_{H' \supseteq H} \sum_{W \subset X'} (H \subset H', N' \acts k(W), \beta, \gamma),$$

where

- the first sum is over abelian subgroups $H'$ of $G$ that contain $H$, up to conjugacy in $NG(H)$,
• the second sum is over $N_{NC(H)}(H')$-orbits of components $W \subset X'$ where the generic stabilizer is $H'$.
• $\beta = \beta_W(X')$ encodes the normal bundle to $W$ in $X'$, and
• $\gamma = (c_i)_{i \in I}$, the characters coming from $L_i$ with $i \in I$.

Then

$$[\mathcal{P}(L_0 \oplus \cdots \oplus L_r) \otimes G] = \sum_{J \subseteq I} \psi_J(\omega_{\mathcal{P}(I), J}(\xi(X \otimes G, (L_i)_{i \in I})))$$

in $\text{Burn}_n(G)$.

**Proof.** We identify each contribution to $[\mathcal{P}(L_0 \oplus \cdots \oplus L_r) \otimes G]$ as

$$V = \varphi_J^{-1}(W),$$

for some $W$ in the definition of $\xi(X \otimes G, (L_i)_{i \in I})$, where $\varphi_J$ denotes the projection to $X$ in the definition of $\restrict_H$. Then, $$(H \subseteq H', N' \subset k(W), \beta, \gamma) \in \text{Burn}_{n, \mathcal{P}(J)}(G)$$

maps under $\psi_J \circ \omega_{\mathcal{P}(I), J}$ to

$$(\overline{H}, N_{NC(H)}(H') \subset k(V), \beta_V(X)).$$ \qed

**Example 5.3.** Let $G := C_5 \times S_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $S_3$. We take $L_0$ to be trivial and $L_1$ to be the twist of $O_{\mathbb{P}^1}(1)$ by a nontrivial character $\chi$ of $C_5$. Then we have the situation of Lemma 5.1 with $H = C_5$ and $N = S_3$, and the conditions of the lemma are satisfied. We have

$$\xi(X \otimes G, (L_0, L_1)) = (C_5 \subseteq C_5, S_3 \subset k(\mathbb{P}^1), \emptyset, (0, \chi))$$

+ $(C_5 \subseteq C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, (0, 1), (0, (\chi, 0)))$

+ $(C_5 \subseteq C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, (0, 1), (0, (\chi, 1)))$

+ $(C_5 \subseteq C_5 \times A_3, S_3/A_3 \subset k \times k, (0, 1), (0, (\chi, 1)))$.

The outcome of Proposition 5.2 is

$$[\mathcal{P}(L_0 \oplus L_1) \otimes G] = \langle \text{triv}, G \subset k(\mathbb{P}^1)(t), \emptyset \rangle + \langle ((1, 2), k, C_5 \otimes k(t), 1) \rangle$$

+ $(C_5, S_3 \subset k(\mathbb{P}^1), \chi)$ + $(C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, ((0, 1), (\chi, 0)))$

+ $(C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, ((0, 1), (\chi, 1)))$

+ $(C_5 \times A_3, S_3/A_3 \subset k \times k, ((0, 1), (\chi, 1)))$

+ $(C_5, S_3 \subset k(\mathbb{P}^1), -\chi)$ + $(C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, ((0, 1), (-\chi, 0)))$

+ $(C_5 \times \langle(1, 2)\rangle, \text{triv} \subset k, ((0, 1), (-\chi, 1)))$

+ $(C_5 \times A_3, S_3/A_3 \subset k \times k, ((0, 1), (-\chi, 1)))$. 


In the notation of Section 3 we observe that the \( \mathcal{G} \)-prefilter
\[
\mathcal{H} := \{ (C_5, \mathcal{G}_3) \}
\]
satisfies the condition of Proposition 3.3. Upon projection
\[
\text{Burn}_2(G) \rightarrow \text{Burn}_2^\mathcal{H}(G)
\]
(see Definition 3.2), we obtain the class
\[
(C_5, \mathcal{G}_3 \subset k(P^1), \chi) + (C_5, \mathcal{G}_3 \subset k(P^1), -\chi) \in \text{Burn}_2^\mathcal{H}(G).
\]
This class is nonzero. Moreover, it is different for \( \chi \in \{ \pm 1 \} \) as compared to \( \chi \in \{ \pm 2 \} \).

Geometrically, the situation above arises as follows: Consider the 3-dimensional representation \( W_\chi = 1 \oplus (V \otimes \chi) \) of \( G \), sum of a trivial 1-dimensional representation and twist by \( \chi \) of the standard 2-dimensional representation \( V \) of \( \mathcal{G}_3 \). This gives a generically free action of \( G \) on \( P^2 = P(W_\chi) \), with a \( G \)-fixed point \( p \). To bring the \( G \)-action into a form where Assumption 2 is satisfied, we need to blow up \( p \), and
\[
[P(W_\chi) \subset G] = [P(L_0 \oplus L_1) \subset G] \in \text{Burn}_2(G).
\]

6. Products

Let \( G' \) and \( G'' \) be finite groups. Define a product map
\[
\text{Burn}_{n'}(G') \times \text{Burn}_{n''}(G'') \rightarrow \text{Burn}_{n'+n''}(G' \times G'').
\]
On symbols, it is given by
\[
((H', Z' \subset K', \beta'), (H'', Z'' \subset K'', \beta'')) \mapsto (H, Z \subset K, \beta), \quad (6.1)
\]
where
- \( H = H' \times H'' \),
- \( Z = Z' \times Z'' \),
- \( K = K' \otimes_k K'' \), with the natural action of \( Z \),
- \( \beta = \beta' \cup \beta'' \).

**Proposition 6.1.** The product map \( (6.1) \) is well-defined, and satisfies
\[
([X' \subset G'], [X'' \subset G'']) \mapsto [X' \times X'' \subset G' \times G''].
\]

**Proof.** The map clearly respects relations. The only point to remark is that in (B2), the condition for nontriviality of \( \Theta_2 \) holds for \( \beta' \) if and only if it holds for \( \beta = \beta' \cup \beta'' \).\[\square\]
7. Restrictions

Let $G$ be a finite group and $G' \subset G$ a subgroup. A $G$-action on a quasiprojective variety $X$ induces an action of $G'$, and thus it is natural to propose the existence of a restriction homomorphism from $\text{Burn}_n(G)$ to $\text{Burn}_n(G')$, acting by 

$$[X \acts G] \mapsto [X \acts G']. \quad (7.1)$$

In this section we establish the existence and uniqueness of this homomorphism.

**Example 7.1.** Suppose $H$ is an abelian subgroup of $G$, contained in $G'$. Symbols, identified in $\text{Burn}_n(G)$ by relation (C), might no longer be identified in $\text{Burn}_n(G')$. E.g., with $G = D_4$ and $G' = C_4$ the restriction of $(G', G/G' \acts k \times k, 1) \in \text{Burn}_1(G)$ to $\text{Burn}_1(G')$ has to be a sum of two symbols with distinct characters:

$$(G', G/G' \acts k \times k, 1) \mapsto (G', \text{triv} \acts k, 1) + (G', \text{triv} \acts k, 3).$$

**Theorem 7.2.** For all $n \geq 0$, there exists a unique homomorphism of abelian groups

$$\text{res}_{G'}^G : \text{Burn}_n(G) \to \text{Burn}_n(G').$$

compatible with (7.1).

**Proof.** By Lemma 2.1, it suffices to consider symbols of the form

$$s = (H, Z \acts K, \beta).$$

When we act by conjugation by some element of $G$, we obtain an equivalent symbol, where $H$ is replaced by a conjugate, the corresponding centralizer quotient replaces $Z$, and conjugation is used to form from $\beta$ a sequence of characters of the conjugate of $H$. By conjugation we have a transitive action of $G$ on a set $\mathcal{S}$ of symbols, where $s \in \mathcal{S}$ has stabilizer $Z_G(H)$. The restriction of the action to $G'$ consists of finitely many orbits; in the formula below the sum is over orbit representatives

$$s' = (H', Z' \acts K, \beta'),$$

such that the restriction $\beta'|_{H' \cap G'}$ of $\beta'$ to $H' \cap G'$ has trivial space of invariants; here, $Z'$ denotes $Z_G(H')/H'$. Then we define the restriction to $G'$ by

$$s \mapsto \sum_{s'} (H' \cap G', (Z_G(H') \cap G')/(H' \cap G') \acts K, \beta'|_{H' \cap G'});$$

this map respects relations. Uniqueness follows from [7, Rmk. 5.16]. \qed
As an application of the restriction construction, we obtain a map
\[ \text{Burn}_{n'}(G) \times \text{Burn}_{n''}(G) \to \text{Burn}_{n'+n''}(G), \]
using the product construction in Section 6 with \( G' = G'' = G \), followed restriction to the diagonal
\[ G \subset G \times G. \]
This map on Burnside groups sends
\[ ([X' \otimes G], [X'' \otimes G]) \mapsto [X' \times X'' \otimes G]. \]

8. Combinatorial analogs

Here we define and study a quotient
\[ \text{Burn}_n(G) \to \mathcal{B}_n(G) \]
to a combinatorial version of the equivariant Burnside group, by forgetting the information about the Galois algebra.

**Definition 8.1.** The combinatorial symbols group
\[ \mathcal{B}_n(G) \]
is the \( \mathbb{Z} \)-module, generated by symbols
\[ (H, Y, \beta) \]
with \( H \) abelian, \( Y \subseteq Z_G(H)/H \), and \( \beta \) a sequence of nonzero elements generating \( H' \), of length at most \( n \), modulo relations:
- **(O):** \( (H, Y, \beta) = (H, Y, \beta') \) if \( \beta' \) is a reordering of \( \beta \).
- **(C):** \( (H, Y, \beta) = (gHg^{-1}, gYg^{-1}, \beta') \) for \( g \in G \), with \( \beta \) and \( \beta' \) related by conjugation by \( g \).
- **(B1):** \( (H, Y, \beta) = 0 \) when \( b_1 + b_2 = 0 \).
- **(B2):** \( (H, Y, \beta) = \Theta_1 + \Theta_2 \), where \( \Theta_1 \) and \( \Theta_2 \) are as in Section 2 i.e.,
\[
\Theta_1 = \begin{cases} 
0, & \text{if } b_1 = b_2, \\
(H, Y, \beta_1) + (H, Y, \beta_2), & \text{otherwise},
\end{cases}
\]
with \( \beta_1 \) and \( \beta_2 \) as in \( (2.3) \), and
\[
\Theta_2 = \begin{cases} 
0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\
(H, \overline{Y}, \bar{\beta}), & \text{otherwise},
\end{cases}
\]
where \( \overline{H} = \ker(b_1 - b_2) \), \( \overline{Y} \) is the pre-image of \( Y \) in \( Z_G(H)/\overline{H} \), and \( \bar{\beta} \) consists of the restrictions to \( \overline{H} \) of the characters of \( \beta \).
Proposition 8.2. The map sending the class of a triple
\((H,Y \subset K, \beta) \in \text{Burn}_n(G)\),
for fields \(K \in \text{Alg}_Y(K_0)\), \(K_0 \in \text{Bir}_d(k)\), with \(d \leq n\), to
\([k': k](H,Y, \beta) \in \mathcal{BC}_n(G)\),
where \(k'\) is the algebraic closure of \(k\) in \(K_0\), gives a surjective homomorphism
\(\text{Burn}_n(G) \to \mathcal{BC}_n(G)\).

Proof. This is clear from the description of the relations in \(\text{Burn}_n(G)\) from Section 2. \(\square\)

Definition 8.3. Given a \(G\)-prefilter \(H\), we let
\(\mathcal{BC}^H_n(G)\)
be the quotient of \(\mathcal{BC}_n(G)\) by the subgroup generated by classes \((H,Y, \beta)\)
with \((H,Y) \not\in H\).

Exactly as in Section 3 we have

Proposition 8.4. Let \(H\) be a \(G\)-prefilter, satisfying the hypothesis of
Proposition 3.3. Then \(\mathcal{BC}^H_n(G)\) is generated by symbols \((H,Y, \beta)\) for
\((H,Y) \in H\), subject to relations (O), (C), (B1), and (B2) applied
to these symbols.

Additionally, upon passage to the combinatorial analogue we also have
the other structures developed in this paper:
- equivariant (projectively) indexed combinatorial Burnside group;
- product map;
- restriction homomorphisms.

Example 8.5. Suppose that \(G\) is abelian.
- We have (cf. [7, §8])
  \(\mathcal{B}_n(G) = \mathcal{BC}^{(G, \text{triv})}_n(G)\),
  where \(\mathcal{B}_n(G)\) is the symbols group from [5].
- There is a commutative diagram
  \[
  \begin{array}{ccc}
  \text{Burn}_n(G) & \longrightarrow & \mathcal{BC}_n(G) \\
  \downarrow & & \downarrow \\
  \text{Burn}^G_n(G) & \longrightarrow & \mathcal{B}_n(G)
  \end{array}
  \]
  (The factor factor \([k': k]\) in Proposition 8.2 matches the similar
  factor in [7, Prop. 8.1].)
9. Applications

As a first application of the formalism in [7] for nonabelian groups, we gave in [3] an example of $G = C_2 \times S_3$-actions on $\mathbb{P}^2$ and a quadric surface $Q \subset \mathbb{P}^3$, which were distinguished by the respective classes in $\text{Burn}_2(G)$. The actions are stably $G$-equivariantly rational [9].

Here we give a further application, for $G = S_4$, acting on $\mathbb{P}^2$ and a del Pezzo surface of degree 6. This example was treated in [1], via birational rigidity techniques (Noether inequality).

We recall basic facts about the subgroup lattice of $G = S_4$:

- Conjugacy classes of nonabelian subgroups: $S_3, D_4, A_4$,
- Conjugacy classes of abelian subgroups: trivial, even $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$, even $K_4 \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$, odd $\mathfrak{K}_4$.

**First action:** On $\mathbb{P}^2$, we consider the projectivization of the standard 3-dimensional representation $V_3$, with respect to basis 

$$(-1,1,1,-1), \quad (1,-1,1,-1), \quad (1,1,-1,-1),$$

given by the 4 matrices below.

$$\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \lambda_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\sigma$ and $\tau$ generate $S_3$ and $\lambda_1, \lambda_2$ the even $\mathfrak{K}_4$.

The restriction of $V_3$ to $D_4$ decomposes as 1-dimensional plus irreducible 2-dimensional representation. Each $D_4 \subset G$, gives a distinguished line and a point; together these form a triangle. We blow up the 3 points to get a hexagon of lines, which form two triples of disjoint lines, each line has faithful Klein 4-group action and generic stabilizer even $\mathbb{Z}/2$. The intersection points of the lines have stabilizer even $\mathfrak{K}_4$. We blow up these intersection points. The result is a wheel of 12 rational curves:

$$D_1 \quad R_1 \quad D'_1 \quad R_2 \quad D_2 \quad R_3 \quad R_6 \quad D'_3 \quad R_5 \quad D_3 \quad R_4 \quad D'_2$$

Each rational curve has generic stabilizer even $\mathbb{Z}/2$, and their intersection points have stabilizer even $\mathfrak{K}_4$. The 12 curves form 3 orbits: $\{D_1, D_2, D_3\}$
and \{D_1', D_2', D_3'\} consist of lines with generic stabilizer even \(\mathbb{Z}/2\) and faithful \(\mathfrak{A}_4\)-action, and the lines in the \(G\)-orbit \{\(R_1, \ldots, R_6\)\} have generic stabilizer even \(\mathbb{Z}/2\) and a nontrivial \(\mathbb{Z}/2\)-action.

The restriction of \(V_3\) to \(\mathfrak{S}_3\) also decomposes into a 1-dimensional and an irreducible 2-dimensional representation. Looking at \(G\)-orbits, we find a \(G\)-orbit of 4 distinguished \(\mathfrak{S}_3\)-lines, which intersect in 6 points with odd \(\mathfrak{A}_4\) stabilizer. We also have a \(G\)-orbit of 4 distinguished points with \(\mathfrak{S}_3\)-stabilizer. We blow up the points with odd \(\mathfrak{A}_4\)-stabilizer and also those with \(\mathfrak{S}_3\)-stabilizer. We get a \(G\)-orbit of 6 lines, each with \(\mathbb{Z}/2\)-action and generic stabilizer odd \(\mathbb{Z}/2\), as well as an orbit of 4 exceptional curves with \(\mathfrak{S}_3\)-action.

**Second action:** Let \(X\) be a del Pezzo surface of degree 6 given by

\[
x_0y_0z_0 = x_1y_1z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.
\]

We write the action of \(G\) in coordinates

\[
x := x_0/x_1, \quad y := y_0/y_1, \quad z := z_0/z_1.
\]

Then, \(\mathfrak{S}_3 = \langle \sigma, \tau \rangle\) acts by permuting \(x, y, z\); \(\lambda_1\) changes signs on \(x\) and \(z\), and \(\lambda_2\) changes signs on \(x\) and \(y\).

There are three orbits of points, of length 4, with stabilizer \(\mathfrak{S}_3\) (see [1, Lemma 1.3]). Blowing these up, we obtain 3 \(G\)-orbits of \(\mathfrak{S}_3\)-lines. These do not contribute to \([X \acts G]\). There are also two \(G\)-orbits of points with \(D_4\)-stabilizers, these points are precisely the intersection points of the 6 lines at infinity, i.e., in the locus

\[
\{x_0 = 0\} \cup \{y_0 = 0\} \cup \{z_0 = 0\}.
\]

These lines have generic stabilizer even \(\mathbb{Z}/2\) and a nontrivial \(\mathbb{Z}/2\)-action, and they form a single \(G\)-orbit. After we blow up the two orbits of 3 points points, we obtain precisely the wheel configuration we described above.

To summarize, the difference

\[
[X \acts G] - [\mathbb{P}^2 \acts G]
\]

is a symbol

\[
(\text{odd } \mathbb{Z}/2, \mathbb{Z}/2 \acts k(t), (1))
\]

(9.1)

corresponding to a \(G\)-orbit of 6 lines with generic stabilizer odd \(\mathbb{Z}/2\) and nontrivial \(\mathbb{Z}/2\)-action. By a computation, analogous to the determination of \(\text{Burn}_2(\mathbb{Z}/2 \oplus \mathbb{Z}/2)\) in [3 §5.4], the class (9.1) is nontrivial in \(\text{Burn}_2(G)\).
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