Lagrangian symmetries and supersymmetries depending on derivatives. Conservation laws and cohomology

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Abstract. Motivated by BRST theory, we study generalized symmetries and supersymmetries depending on derivatives of dynamic variables in a most general setting. We state the first variational formula and conservation laws for higher order Lagrangian systems on fiber bundles and graded manifolds under generalized symmetries and supersymmetries of any order. Cohomology of nilpotent generalized supersymmetries are considered.

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1 Introduction

Symmetries of differential equations under transformations of dynamic variables depending on their derivatives have been intensively investigated (see [3, 19, 23] for a survey). Following [3, 23], we agree to call them the generalized symmetries in contrast with classical (point) symmetries. In mechanics, conservation laws corresponding to generalized symmetries are well known [23]. In field theory, BRST transformations provide the most interesting example of generalized symmetries [11, 12], but they involve odd ghost fields. Therefore, we aim to consider both generalized symmetries of classical Lagrangian systems on smooth fiber bundles and generalized supersymmetries of Lagrangian systems on graded manifolds.

Generalized symmetries of Lagrangian systems on a local coordinate domain of a trivial fiber bundle have been described in detail [23]. In the recent work [11], a global analysis of first order Lagrangian systems and conservation laws under generalized symmetries depending on first order derivatives has been provided. We aim studying the conservation laws in higher order Lagrangian systems on fiber bundles and graded manifolds under generalized symmetries and supersymmetries of any order. Let us emphasize the following.

(i) An \(r\)-order differential equation on a fiber bundle \(Y \to X\) is conventionally defined as a closed subbundle of the \(r\)-order jet bundle \(J^rY \to X\) of sections of \(Y \to X\) [8, 19]. Euler–Lagrange equations need not satisfy this condition, unless an Euler–Lagrange operator is of constant rank. Therefore, we regard infinitesimal symmetry transformations of
Lagrangians and Euler–Lagrange operators as differential operators on a graded differential algebra (henceforth GDA) of exterior forms, but not as manifold maps. For instance, we are not concerned with dynamic symmetries. This approach is straightforwardly extended to Lagrangian systems on graded manifolds.

(ii) We use the first variational formula in order to obtain Lagrangian conservation laws. Recall that an \( r \)-order Lagrangian of a Lagrangian system on a fiber bundle \( Y \to X \) is defined as a horizontal density \( L : J^rY \to \wedge^r T^*X \), \( n = \dim X \), on the \( r \)-order jet manifold \( J^rY \). Let \( u \) be a projectable vector field on \( Y \to X \) seen as an infinitesimal generator of a local one-parameter group of bundle automorphisms of \( Y \to X \). Let \( L_{J^r u}L \) be the Lie derivative of \( L \) along the jet prolongation \( J^r u \) of \( u \) onto \( J^r Y \). The first variational formula provides its canonical decomposition

\[
L_{J^r u}L = u_V \delta L + d_H (h_0 (J^{2r-1} u | \Xi_L)),
\]

where \( \delta L \) is the Euler–Lagrange operator, \( \Xi_L \) is a Lepagean equivalent of \( L \) (e.g., a Poincaré–Cartan form), \( u_V \) is a vertical part of \( u \), \( d_H \) is the total differential, and \( h_0 \) is the horizontal projection (see their definitions below) [13, 21, 28]. Let \( u \) be a divergence symmetry of \( L \), i.e., \( L_{J^r u}L \) is a total differential \( d_H \sigma \). Then, the first variational formula (1) on the kernel of the Euler–Lagrange operator \( \delta L \) leads to the conservation law

\[
0 \approx d_H (h_0 (J^{2r-1} u | \Xi_L) - \sigma).
\]

If \( u \) is a (variational) symmetry of \( L \), i.e., \( L_{J^r u}L = 0 \), the conservation law (2) comes to the familiar Noether one. Our goal is to extend the first variational formula (1) to generalized symmetries and supersymmetries, and to obtain the corresponding Lagrangian conservation laws.

(iii) A vector field \( u \) in the first variational formula (1) is a derivation of the \( \mathbb{R} \)-ring \( C^\infty (Y) \) of smooth real functions on \( Y \). Let \( \vartheta \) be a derivation of \( C^\infty (Y) \) with values in the ring \( C^\infty (J^rY) \) of smooth real functions on the jet manifold \( J^rY \). It is called a \( k \)-order generalized vector field. A generalized symmetry can be defined as the prolongation \( J^r \vartheta \) of \( \vartheta \) onto any finite order jet manifold \( J^rY \). This definition recovers both the notion of a local generalized symmetry in [23] and the definition of a generalized vector field as a section of the pull-back bundle \( TY \times J^kY \to J^kY \) in [11]. The key point is that, in general, \( L_{J^r \vartheta} \) is an exterior form on the jet manifold \( J^{r+k}Y \). By virtue of the well-known Bäcklund theorem, \( L_{J^r \vartheta} \) preserves the GDA \( \mathcal{O}_r^* \) of exterior forms on \( J^rY \) iff either \( \vartheta \) is a vector field on \( Y \) or \( Y \to X \) is a one-dimensional bundle and \( \vartheta \) is a generalized vector field at most of first order. Thus, considering generalized symmetries, we deal with Lagrangian systems of unspecified finite order. Infinite order jet formalism [2, 19, 21, 28, 30] provides a convenient tool for studying these systems both on fiber bundles and graded manifolds. In the framework of this formalism, the first variational formula issues from the variational bicomplex, whose cohomology provides some topological obstruction to generalized symmetries and supersymmetries. For instance, if \( \vartheta \) is a divergence symmetry of a Lagrangian \( L \), the equality

\[
\delta (L_{J^r \vartheta} L) = 0
\]

(3)
holds, but the converse is not true. There is a topological obstruction to \( \vartheta \) in (3) to be a divergence symmetry. At the same time, one can think of the equality (3) as being at least locally the characteristic equation for divergence symmetries of a given Lagrangian \( L \). Recall that, by virtue of the master identity
\[
L_{J^2u} \delta L = \delta (L_{J^2u} L),
\]
any classical divergence symmetry of a Lagrangian is also a symmetry of its Euler–Lagrange operator. However, this identity is not extended to generalized symmetries [23].

The following peculiarities of generalized supersymmetries should be additionally noted. They are expressed into jets of odd variables, and they can be nilpotent.

We do not concern particular geometric models of ghost fields in gauge theory, but consider Lagrangian systems of odd variables in a general setting. For this purpose, one calls into play fiber bundles over graded manifolds or supermanifolds [9, 10, 22]. However, Lagrangian BRST theory on \( X = \mathbb{R}^n \) [4, 6, 7] involves jets of odd fields only with respect to space-time coordinates. Therefore, we describe odd variables on a smooth manifold \( X \) as generating elements of the structure ring of a graded manifold whose body is \( X \). By the well-known Batchelor theorem [5], any graded manifold is isomorphic to the one whose structure sheaf is the sheaf \( \mathfrak{A}_Q \) of germs of sections of the exterior product
\[
\wedge Q^* = \mathbb{R} \bigoplus_{X} Q^* \bigoplus_{X} \wedge^2 Q^* \bigoplus_{X} \cdots,
\]
where \( Q^* \) is the dual of some real vector bundle \( Q \to X \). In physical models, a vector bundle \( Q \) is usually given from the beginning. Therefore, we restrict our consideration to so called simple graded manifolds \( (X, \mathfrak{A}_Q) \) where the Batchelor isomorphism holds fixed. We agree to say that \( (X, \mathfrak{A}_Q) \) is constructed from \( Q \). Accordingly, \( r \)-jets of odd variables are defined as generating elements of the structure ring of the simple graded manifold \( (X, \mathfrak{A}_{J^rQ}) \) constructed from the jet bundle \( J^rQ \) of \( Q \) [21, 26]. This definition of jets differs from that of jets of a graded fiber bundle in [18], but reproduces the heuristic notion of jets of ghosts in Lagrangian BRST theory on \( \mathbb{R}^n \) in [4, 6, 7]. Moreover, this definition enables one to study Lagrangian systems on a graded manifold similarly to those on a fiber bundle.

The BRST transformation in gauge theory on a principal bundle exemplifies a generalized supersymmetry (see \( \psi \) (40) below). The BRST operator is defined as the Lie derivative \( L_\psi \) along this generalized symmetry. The fact that it is nilpotent on horizontal (local in the terminology of [4, 6]) forms motivates us to study nilpotent generalized supersymmetries. They are necessarily odd, i.e., there is no nilpotent generalized symmetry. The key point is that the Lie derivative \( L_\psi \) along a generalized supersymmetry and the total differential \( d_H \) mutually commute. If \( L_\psi \) is nilpotent, we obtain a bicomplex whose iterated cohomology classifies Lagrangians with a given nilpotent divergence symmetry.
2 Lagrangian systems of unspecified finite order on fiber bundles

Finite order jet manifolds make up the inverse system

\[ X \leftarrow Y \leftarrow J^1 Y \leftarrow \cdots \cdots J^{r-1} Y \leftarrow J^r Y \leftarrow \cdots. \tag{5} \]

Its projective limit $J^\infty Y$, called the infinite order jet space, is endowed with the weakest topology such that surjections $\pi^r_r : J^\infty Y \rightarrow J^r Y$ are continuous. This topology makes $J^\infty Y$ into a paracompact Fréchet manifold [30]. Any bundle coordinate atlas \( \{ U_Y, (x^\lambda, y^i) \} \) of $Y \rightarrow X$ yields the manifold coordinate atlas

\[ \{(\pi^r_0)^{-1}(U_Y), (x^\lambda, y^i_{\Lambda})\}, \quad y^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x^{\lambda}} d_\mu y^i_{\lambda}, \quad 0 \leq |\Lambda|, \]

of $J^r Y$ where $\Lambda = (\lambda_1, \ldots, \lambda_l)$ is a symmetric multi-index of length $k$, $\lambda + \Lambda = (\lambda \lambda_k, \ldots, \lambda_1)$, and

\[ d_\lambda = \partial_\lambda + \sum_{|\Lambda| \geq 0} y^i_{\lambda+\Lambda} \partial^A_i, \quad d_\lambda = d_{\lambda_k} \circ \cdots \circ d_{\lambda_1}, \quad \Lambda = (\lambda_k, \ldots, \lambda_1), \tag{7} \]

are the total derivatives. Hereafter, we fix an atlas of $Y$ and, consequently, that of $J^\infty Y$ containing a finite number of charts, though their branches $U_Y$ need not be domains [17].

With the inverse system (5), we have the direct system

\[ \mathcal{O}^r(X) \xrightarrow{\pi^r_r} \mathcal{O}^r(Y) \xrightarrow{\pi^r_1} \mathcal{O}^0 \cdots \cdots \cdots \cdots \quad \mathcal{O}^r \rightarrow \cdots \]

of the GDAs of exterior forms on finite order jet manifolds with respect to the pull-back monomorphisms $\pi^r_{r-1}$. Its direct limit $\mathcal{O}^*\infty$ is a GDA, whose de Rham cohomology equals that of the fiber bundle $Y$ [2]. Though $J^\infty Y$ is not a smooth manifold, one can think of elements of $\mathcal{O}^*\infty$ as being objects on $J^\infty Y$ as follows. Let $\mathfrak{D}_r$ be the sheaf of germs of exterior forms on the $r$-order jet manifold $J^r Y$, and let $\overline{\mathfrak{D}}_r$ be its canonical presheaf. There is the direct system of presheaves

\[ \overline{\mathfrak{D}}_X \xrightarrow{\pi^r_r} \overline{\mathfrak{D}}_0 \xrightarrow{\pi^1_0} \overline{\mathfrak{D}}_1 \cdots \cdots \cdots \cdots \quad \overline{\mathfrak{D}}_r \rightarrow \cdots. \]

Its direct limit $\overline{\mathfrak{D}}_\infty$ is a presheaf of GDAs on $J^\infty Y$. Let $\mathfrak{X}^*\infty = \Gamma(\overline{\mathfrak{D}}_\infty)$ of sections of $\overline{\mathfrak{D}}_\infty$ is a GDA whose elements $\phi$ possess the following property. For any point $z \in J^\infty Y$, there exist its open neighbourhood $U$ and an exterior form $\phi^{(k)}$ on some jet manifold $J^k Y$ such that $\phi|_U = \phi^{(k)} \circ \pi^\infty|_U$. There is the monomorphism $\mathcal{O}^*\infty \rightarrow \mathfrak{X}^*\infty$ whose image consists of all exterior forms on finite order jet manifolds.

Restricted to a coordinate chart (6), elements of $\mathcal{O}^*\infty$ can be written in a coordinate form, where horizontal forms $\{ dx^\lambda \}$ and contact 1-forms $\{ \theta^i_\lambda = dy^i_\lambda - y^i_{\lambda + \Lambda} dx^\lambda \}$ make up local generators of the $\mathcal{O}^0\infty$-algebra $\mathcal{O}^*\infty$. There is the canonical decomposition

\[ \mathcal{O}^*\infty = \bigoplus_{k,m} \mathcal{O}^k_m, \quad 0 \leq k, \quad 0 \leq m \leq n, \]
of $O_\infty$ into $O_\infty^0$-modules $O_\infty^{k,m}$ of $k$-contact and $m$-horizontal forms together with the corresponding projections $h_k : O_\infty^* \rightarrow O_\infty^{k,*}$ and $h^m : O_\infty^* \rightarrow O_\infty^{*,m}$. Accordingly, the exterior differential on $O_\infty^*$ is split into the sum $d = d_H + d_V$ of the total and vertical differentials of $O_\infty^*$ such that $\varrho \circ d_H = 0$, and the nilpotent variational operator $\delta = \varrho \circ d$ on $O_\infty^{*,n}$. Then, $O_\infty^*$ is split into the well-known variational bicomplex. If $Y$ is contractible, this bicomplex at terms except $\mathbb{R}$ is exact. This fact is known as the algebraic Poincaré lemma (e.g., [23]).

Here, we consider only the variational complex

$$0 \rightarrow \mathbb{R} \rightarrow O_\infty^0 \xrightarrow{d_H} O_\infty^{0,1} \rightarrow O_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots, \quad E_k = \varrho(O_\infty^{k,n}).$$

One can think of

$$L = L\omega \in O_\infty^{0,n}, \quad \omega = dx \wedge \cdots \wedge dx, \quad \omega_\mu = \partial_\mu \omega,$$

as being a finite order Lagrangian, while $\delta L$ is its Euler–Lagrange operator

$$\delta L = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} d_\Lambda (\partial^\Lambda L) \theta^i \wedge \omega.$$ 

**THEOREM 1.** Cohomology of the variational complex (10) is isomorphic to the de Rham cohomology of the fiber bundle $Y$, i.e., $H^{k\leq n}(d_H) = H^{k\leq n}(Y)$, $H^{k-n}(\delta) = H^{k\geq n}(Y)$.

**Outline of proof.** [14, 29] (see also [1]). We have the complex of sheaves of $O_\infty^*$-modules

$$0 \rightarrow \mathbb{R} \rightarrow \mathfrak{g}_\infty^0 \xrightarrow{d_H} \mathfrak{g}_\infty^{0,1} \rightarrow \mathfrak{g}_\infty^{0,n} \xrightarrow{\delta} \mathfrak{e}_1 \xrightarrow{\delta} \mathfrak{e}_2 \rightarrow \cdots$$

on $J^\infty Y$ and the complex of their structure modules

$$0 \rightarrow \mathbb{R} \rightarrow Q_\infty^0 \xrightarrow{d_H} Q_\infty^{0,1} \rightarrow Q_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots.$$ 

Since the paracompact space $J^\infty Y$ admits a partition of unity by elements of $Q_\infty^0$ [30], the sheaves of $Q_\infty^0$-modules on $J^\infty Y$ are acyclic. Then, by virtue of the above mentioned algebraic Poincaré lemma, the complex (13) is a resolution of the constant sheaf $\mathbb{R}$. In accordance with the abstract de Rham theorem, cohomology of the complex (14) equals the cohomology of $J^\infty Y$ with coefficients in $\mathbb{R}$. This cohomology, in turn, is isomorphic to the de Rham cohomology of $Y$, which is a strong deformation retract of $J^\infty Y$ [2, 30]. Finally, the $d_H$- and $\delta$-cohomology of $Q_\infty^*$ is proved to equal that of its subalgebra $O_\infty^*$ [14, 29].
A corollary of Theorem 1 is that any $\delta$-closed form $L \in \mathcal{O}^{0,n}$ is split into the sum
\[ L = h_0 \varphi + d_H \xi, \quad \xi \in \mathcal{O}^{0,n-1}_\infty, \]
where $\varphi$ is a closed $n$-form on $Y$. In other words, a finite order Lagrangian $L$ is variationally trivial iff it takes the form (15).

**PROPOSITION 2.** For any Lagrangian $L \in \mathcal{O}^{0,n}_\infty$, there is the decomposition
\[ dL = \delta L - d_H(\Xi), \quad \Xi \in \mathcal{O}^{1,n-1}_\infty. \]

**Proof.** Let us consider another subcomplex
\[ 0 \to \mathcal{O}^{1,0}_\infty \xrightarrow{d_H} \mathcal{O}^{1,1}_\infty \to \cdots \to \mathcal{O}^{1,n}_\infty \xrightarrow{\delta} E_1 \to 0 \]
(17)
of the variational bicomplex. Similarly to the proof of Theorem 1, one can show that it is exact [14, 29]. Its exactness at the term $\mathcal{O}^{1,n}_\infty$ implies the $\mathbb{R}$-module decomposition
\[ \mathcal{O}^{1,n}_\infty = E_1 \oplus d_H(\mathcal{O}^{1,n-1}_\infty) \]
with respect to the projector $\varrho$. Applied to $dL \in \mathcal{O}^{1,n}_\infty$, it gives the decomposition (16). $\square$

The form $\Xi$ in the decomposition (16) is not unique. It reads
\[ \Xi = \sum_{s=0}^{s} F_{i}^{\lambda_{\nu_{s}}...\nu_{1}} \theta_{\nu_{s}}...\nu_{1} \wedge \omega_{\lambda}, \quad F_{i}^{\nu_{k}...\nu_{1}} = \partial_{i}^{\nu_{k}...\nu_{1}} L - d_{a} F_{i}^{\lambda_{\nu_{k}...\nu_{1}}} + h_{i}^{\nu_{k}...\nu_{1}}, \]
where functions $h$ obey the relations $h_{i}^{\nu} = 0, h_{i}^{(\nu_{k}...\nu_{k-1})...\nu_{1}} = 0$. It follows that $\Xi L = \Xi + L$ is a Lepagean equivalent, e.g., a Poincaré–Cartan form of a finite order Lagrangian $L$ [16].

The decomposition (16) leads to the desired first variational formula.

### 3 Generalized Lagrangian symmetries

Let $\mathfrak{o} \mathcal{O}^{0}_\infty$ be the Lie algebra of derivations of the $\mathbb{R}$-ring $\mathcal{O}^{0}_\infty$ of smooth real functions of finite jet order on $J^\infty Y$. A derivation $v \in \mathfrak{o} \mathcal{O}^{0}_\infty$ is said to be a generalized symmetry if the Lie derivative $L_v \phi$ of any contact one-form $\phi \in \mathcal{O}^{1,0}_\infty$ is also a contact form. Forthcoming Propositions 3 – 5 confirm the contentedness of this definition.

**PROPOSITION 3.** The derivation module $\mathfrak{o} \mathcal{O}^{0}_\infty$ is isomorphic to the $\mathcal{O}^{0}_\infty$-dual $(\mathcal{O}^{1}_\infty)^*$ of the module of one-forms $\mathcal{O}^{1}_\infty$.

**Proof.** At first, let us show that $\mathcal{O}^{1}_\infty$ is generated by elements $df, f \in \mathcal{O}^{0}_\infty$. It suffices to justify that any element of $\mathcal{O}^{1}_\infty$ is a finite $\mathcal{O}^{0}_\infty$-linear combination of elements $df, f \in \mathcal{O}^{0}_\infty$. Indeed, every $\phi \in \mathcal{O}^{1}_\infty$ is an exterior form on some finite order jet manifold $J^k Y$ and, by virtue of the Serre–Swan theorem (extended to non-compact manifolds [24, 27]), it is represented by a finite sum of elements $df, f \in C^\infty(J^k Y) \subset \mathcal{O}^{0}_\infty$. Any element $\Phi \in (\mathcal{O}^{1}_\infty)^*$ yields a derivation $f \to \Phi(df)$ of the ring $\mathcal{O}^{0}_\infty$. Since the module $\mathcal{O}^{0}_\infty$ is generated by elements $df, f \in \mathcal{O}^{0}_\infty$, the
different elements of \((O^1_\infty)^*\) provide different derivations of \(O^0_\infty\), i.e., there is a monomorphism \((O^1_\infty)^* \to \mathfrak{d}O^0_\infty\). By the same formula, any derivation \(\upsilon \in \mathfrak{d}O^0_\infty\) sends \(df \mapsto \upsilon(f)\) and, since \(O^0_\infty\) is generated by elements \(df\), it defines a morphism \(\Phi_\upsilon : O^1_\infty \to O^0_\infty\). Moreover, different derivations \(\upsilon\) provide different morphisms \(\Phi_\upsilon\). Thus, we have a monomorphism and, consequently, an isomorphism \(\mathfrak{d}O^0_\infty \cong (O^1_\infty)^*\).

\[\square\]

**Proposition 4.** With respect to the atlas \((6)\), any derivation \(\upsilon \in \mathfrak{d}O^0_\infty\) is given by the coordinate expression

\[\upsilon = \upsilon^\lambda \partial_\lambda + \upsilon^i \partial_i + \sum_{|\Lambda|>0} \upsilon^i_\Lambda \partial^\Lambda_i,\]  

(19)

where \(\upsilon^\lambda, \upsilon^i, \upsilon^i_\Lambda\) are smooth functions of finite jet order obeying the transformation law

\[\upsilon'^\lambda_i = \frac{\partial x^\lambda}{\partial x'^\mu} \upsilon^\mu_i, \quad \upsilon'^i = \frac{\partial y^i}{\partial y'^j} \upsilon^j_i + \frac{\partial y^i}{\partial x'^\mu} \upsilon^\mu_i, \quad \upsilon'^i_\Lambda = \sum_{|\Sigma| \leq |\Lambda|} \frac{\partial y^i_\Sigma}{\partial x'^\mu} \upsilon^\mu_\Sigma + \frac{\partial y^i_\Lambda}{\partial x'^\mu} \upsilon^\mu_\Lambda.\]  

(20)

**Proof.** Restricted to a coordinate chart \((6)\), \(O^1_\infty\) is a free \(O^0_\infty\)-module countably generated by the exterior forms \(dx^\lambda, \theta^i_\Lambda\). Then, \(\mathfrak{d}O^0_\infty = (O^1_\infty)^*\) restricted to this chart consists of elements (19), where \(\partial_\lambda, \partial^\lambda_i\) are the duals of \(dx^\lambda, \theta^i_\Lambda\). The transformation rule (20) results from the transition functions (6). Since the atlas \((6)\) is finite, a derivation \(\upsilon\) preserves \(O^*_\infty\). \(\square\)

The contraction \(\upsilon \rfloor \phi\) and the Lie derivative \(L_\upsilon \phi, \phi \in O^*_\infty\), obey the standard formulas.

**Proposition 5.** A derivation \(\upsilon\) (19) is a generalized symmetry iff

\[\upsilon^i_\Lambda = d_\Lambda(\upsilon^i - y^i_\mu \upsilon^\mu), \quad 0 < |\Lambda|.\]  

(21)

**Proof.** The expression (21) results from a direct computation similarly to the first part of the above mentioned Bäcklund theorem. Then, one can justify that local functions (21) fulfill the transformation law (20). \(\square\)

Thus, we recover the notion of a generalized symmetry in item (iii) in Introduction.

Any generalized symmetry admits the horizontal splitting

\[\upsilon = \upsilon_H + \upsilon_V = \upsilon^\lambda \partial_\lambda + (\upsilon^i \partial_i + \sum_{|\Lambda|>0} d_\Lambda \upsilon^i_\Lambda), \quad \upsilon^i = \upsilon^i - y^i_\mu \upsilon^\mu,\]  

(22)

relative to the canonical connection \(\nabla = dx^\lambda \otimes d_\lambda\) on the \(C^\infty(X)\)-ring \(O^0_\infty[21]\). For instance, let \(\tau\) be a vector field on \(X\). Then, the derivation \(\tau \rfloor (d_H f), f \in O^0_\infty\), is a horizontal generalized symmetry \(\upsilon = \tau^\mu d_\mu\). It is easily justified that any vertical generalized symmetry \(\upsilon = \upsilon_V\) obeys the relations

\[\upsilon \rfloor d_H \phi = -d_H (\upsilon \rfloor \phi), \quad L_\upsilon (d_H \phi) = d_H (L_\upsilon \phi), \quad \phi \in O^*_\infty.\]  

(23)
PROPOSITION 6. Given a Lagrangian $L \in \mathcal{O}_{\infty}^{0,n}$, its Lie derivative $\mathbf{L}_u L$ along a generalized symmetry $u$ (22) obeys the first variational formula

$$\mathbf{L}_u L = v^*_V dL + dH(h_0(v^*_L)) + \mathcal{L}_{v^*_H}(v_H|\omega),$$

(24)

where $\Xi_L$ is a Poincaré–Cartan form of $L$.

Proof. The formula (24) comes from the splitting (16) and the first equality (23):

$$\mathbf{L}_u L = v^*_V dL + d(v^*_L) = v^*_V dL + dH(v_H|L) + \mathcal{L}_{v^*_H}(v_H|\omega) =$$

$$v^*_V \delta L - v^*_V dH \Xi + dH(v_H|L) + \mathcal{L}_{v^*_H}(v_H|\omega) =$$

$$v^*_V \delta L + dH(v^*_V \Xi + v_H|L) + \mathcal{L}_{v^*_H}(v_H|\omega), \quad \Xi_L = \Xi + L.$$

Let $u$ be a divergence symmetry of $L$, i.e., $\mathbf{L}_u L = dH\sigma$, $\sigma \in \mathcal{O}_{\infty}^{0,n-1}$. By virtue of the expression (25), this condition implies that a generalized symmetry $u$ is projected onto $X$, i.e., its components $u^\lambda$ depend only on coordinates on $X$. Then, the first variational formula (24) takes the form

$$dH\sigma = v^*_V \delta L + dH(h_0(v^*_L)).$$

(26)

Restricted to $\ker \delta L$, it leads to the generalized Noether conservation law

$$0 \approx dH(h_0(v^*_L) - \sigma).$$

(27)

A glance at the expression (25) shows that a generalized symmetry $u$ (22) projected onto $X$ is a divergence symmetry of a Lagrangian $L$ iff its vertical part $v^*_V$ is so. Moreover, $u$ and $v^*_V$ lead to the same conservation law (27).

Finally, let us obtain the characteristic equation for divergence symmetries of a Lagrangian $L$. Let a generalized symmetry $u$ (22) be projected onto $X$. Then, the Lie derivative $\mathbf{L}_u L$ (25) is a horizontal density. Let us require that it is a $\delta$-closed form, i.e., $\delta(\mathbf{L}_u L) = 0$. In accordance with the equality (15), this condition is fulfilled iff

$$\mathbf{L}_u L = h_0 \varphi + dH\sigma,$$

(28)

where $\varphi$ is a closed $n$-form on $Y$, i.e., $u$ is a divergence symmetry of $L$ at least locally. Note that the topological obstruction $h_0 \varphi$ (28) to $u$ to be a global divergence symmetry is at most of first order. If $Y \to X$ is an affine bundle, its de Rham cohomology equals that of $X$ and, consequently, the topological obstruction $h_0 \varphi = \varphi$ (28) reduces to a non-exact $n$-form on $X$.

4 Lagrangian systems on graded manifolds

In order to describe Lagrangian systems on a graded manifold, we start from constructing the corresponding GDA $\mathcal{S}_{\infty}^*$. 

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Let \((X, \mathcal{A}_Q)\) be the simple graded manifold constructed from a vector bundle \(Q \to X\). Its structure ring \(\mathcal{A}_Q\) consists of sections of the exterior bundle (4) called graded functions. Given bundle coordinates \((x^\lambda, q^a)\) on \(Q\) with transition functions \(q^a = \rho^a_b q^b\), let \(\{c^a\}\) be the corresponding fiber bases for \(Q^* \to X\), together with the transition functions \(c^a_{\alpha} = \rho^a_b c^b\). Then, \((x^\lambda, c^a)\) is called the local basis for the graded manifold \((X, \mathcal{A}_Q)\) [5, 21]. With respect to this basis, graded functions read

\[ f = \sum_{k=0}^{\infty} \frac{1}{k!} f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}, \]

where \(f_{a_1 \cdots a_k}\) are smooth local functions on \(X\).

Let \(\mathfrak{d}\mathcal{A}_Q\) be the Lie superalgebra of graded derivations of the \(\mathbb{R}\)-ring \(\mathcal{A}_Q\), i.e.,

\[ u(f f') = u(f) f' + (-1)^{|u||f|} f u(f'), \quad f, f' \in \mathcal{A}_Q, \]

where \([\cdot]\) denotes the Grassmann parity. Its elements are called graded vector fields on \((X, \mathcal{A}_Q)\). Due to the canonical splitting \(VQ = Q \times Q\), the vertical tangent bundle \(VQ \to Q\) of \(Q \to X\) can be provided with the fiber bases \(\{\partial_a\}\), dual of \(\{c^a\}\). Then, a graded vector field takes the local form \(u = u^\lambda \partial_\lambda + u^a \partial_a\), where \(u^\lambda, u^a\) are local graded functions, and \(u\) acts on \(\mathcal{A}_Q\) by the rule

\[ u(f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}) = u^\lambda \partial_\lambda (f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}) + u^a f_{a_1 \cdots a_k} \partial_a (c^{a_1} \cdots c^{a_k}). \]

This rule implies the corresponding transformation law

\[ u^\lambda = u^\lambda, \quad u^a = \rho^a_{\beta} u^\beta + u^\lambda \partial_\lambda (\rho^a_{\beta} c^\beta). \]

Then, one can show that graded vector fields on a simple graded manifold are sections of a certain vector bundle \(V_Q \to X\) which is locally isomorphic to \(\wedge Q^* \otimes (Q \oplus TX)\) [21, 25].

Using this fact, one can introduce graded exterior forms on the graded manifold \((X, \mathcal{A}_Q)\) as sections of the exterior bundle \(\wedge^k V_Q\), where \(V_Q^* \to X\) is the pointwise \(\wedge Q^*\)-dual of \(V_Q\). Relative to the dual bases \(\{dx^\lambda\}\) for \(T^* X\) and \(\{db^a\}\) for \(Q^*\), graded one-forms read

\[ \phi = \phi_\lambda dx^\lambda + \phi_a db^a, \quad \phi'_a = \rho^{-1b}_a \phi_b, \quad \phi'_\lambda = \phi_\lambda + \rho^{-1b}_a \partial_\lambda (\rho^a_{\beta} \phi_b) c^\beta. \]

Graded exterior forms constitute the GDA \(\mathcal{C}_Q^*\) with respect to the graded exterior product \(\wedge\) and the even exterior differential \(d\). Recall the standard formulas

\[ d(\phi \wedge \sigma) = d\phi \wedge \sigma + (-1)^{|\phi||\sigma|} \phi \wedge d\sigma, \]
\[ u \cdot (\phi \wedge \sigma) = (u \cdot \phi) \wedge \sigma + (-1)^{|\phi||u|} \phi \wedge (u \cdot \sigma), \]
\[ L_u \phi = u \cdot d\phi + d(u \cdot \phi), \quad L_u (\phi \wedge \sigma) = L_u (\phi) \wedge \sigma + (-1)^{|\phi||u|} \phi \wedge L_u (\sigma). \]

Since the jet bundle \(J^r Q \to X\) of the vector bundle \(Q \to X\) is a vector bundle, let us consider the simple graded manifold \((X, \mathcal{A}_{J^r Q})\) constructed from \(J^r Q \to X\). Its local basis is \(\{x^\lambda, c^a_\Lambda\}, 0 \leq |\Lambda| \leq r\), together with the transition functions

\[ c^a_{\lambda+\Lambda} = d_\Lambda (\rho^a_\beta c^\beta_\Lambda), \quad d_\Lambda = \partial_\lambda + \sum_{|\Lambda|<r} c^a_{\lambda+\Lambda} \partial_a^\lambda, \quad \text{if } 0 \leq |\Lambda| \leq r, \]

This concludes the formulation of the graded differential calculus on graded manifolds.
where \( \partial^X_a \) are the duals of \( c^a_A \). Let \( \mathcal{C}^*_{r,Q} \) be the GDA of graded exterior forms on the graded manifold \((X, \mathfrak{A}_{r,Q})\). Since \( \pi^{r-1}_r : J^r Q \to J^{r-1} Q \) is a linear bundle morphism over \( X \), it yields the morphism of graded manifolds \((X, \mathfrak{A}_{r,Q}) \to (X, \mathfrak{A}_{r-1,Q})\) and the monomorphism of the GDAs \( \mathcal{C}^*_{r-1,Q} \to \mathcal{C}^*_{r,Q} \) [21]. Hence, there is the direct system of the GDAs

\[
\mathcal{C}^*_0 \to \mathcal{C}^*_1 \to \cdots \to \mathcal{C}^*_{r,Q} \to \cdots.
\]

Its direct limit \( \mathcal{C}^*_\infty \) consists of graded exterior forms on graded manifolds \((X, \mathfrak{A}_{r,Q}), 0 \leq r\), modulo the pull-back identification. It is a locally free \( C^\infty(X)\)-algebra generated by the elements \((1, c^a_A, dx^\lambda, \theta^a_A = dc^a_A - c^a_{A+} dx^\lambda)\).

This construction of odd jets enables one to describe odd and even variables (e.g., ghosts, ghosts-for-ghosts and antifields in BRST theory) on the same footing. Let us assume that a of the GDA \( \mathcal{O} \) functions (6). It is readily observed that \( \mathcal{C} \) is a \( \mathfrak{A} \)-algebra generated by the elements \((1, y^\lambda_A, dx^\lambda), \theta^i_A\). Let us consider the \( C^\infty(X)\)-product of graded algebras \( \mathcal{C}^*_\infty \) and \( \mathcal{P}^*_\infty \) over their common subalgebra \( \mathcal{O}^*(X) \). It is a graded algebra \( \mathcal{S}^*_\infty(Q,Y) \) (or, simply, \( \mathcal{S}^*_\infty \) if there is no danger of confusion) with respect to the exterior product \( \wedge \) such that

\[
f(\psi \wedge \phi) = (f\psi) \wedge \phi = \psi \wedge (f\phi), \quad \psi \wedge \phi = (-1)^{|\phi||\psi|} \phi \wedge \psi, \quad |\psi \wedge \phi| = |\psi| + |\phi|
\]

for all \( \psi \in \mathcal{C}^*_\infty, \phi \in \mathcal{P}^*_\infty \) and \( f \in C^\infty(X) \). Elements of \( \mathcal{S}^*_\infty \) are also endowed with the Grassmann parity such that \([\phi] = 0\) for all \( \phi \in \mathcal{P}^*_\infty \). Therefore, we continue to call elements of the ring \( \mathcal{S}^0_\infty \) the graded functions. They are polynomials of \( c^a_A \) and \( y^i_A \) with coefficients in \( C^\infty(X) \). The sum of exterior differentials on \( \mathcal{C}^*_\infty \) and \( \mathcal{P}^*_\infty \) makes \( \mathcal{S}^*_\infty \) into a GDA generated locally by the elements \((1, c^a_A, dx^\lambda, \theta^a_A, \theta^i_A)\). One can think of \( \mathcal{S}^*_\infty \) as being the algebra of even and odd variables on a smooth manifold \( X \). In particular, this is the case of the above mentioned Lagrangian BRST theory on \( X = \mathbb{R}^n \) [4, 6, 7]. Let the collective symbol \( s^a_A \) further stand both for its even and odd generating elements \( c^a_A \) and \( y^i_A \).

The algebra \( \mathcal{S}^*_\infty \) is decomposed into \( \mathcal{S}^0_\infty \)-modules \( \mathcal{S}^{k,r}_\infty \) of \( k \)-contact and \( r \)-horizontal graded forms. Accordingly, the graded exterior differential \( d \) on \( \mathcal{S}^*_\infty \) is split into the sum \( d = d_H + d_V \) of the total differential \( d_H(\phi) = dx^\lambda \wedge d\lambda(\phi), \phi \in \mathcal{S}^*_\infty \), and the vertical one. Provided with the projection endomorphism \( \varrho \) given by the expression similar to (9) and the graded variational operator \( \delta = \varrho \circ d \), the algebra \( \mathcal{S}^*_\infty \) is split into the variational bicomplex.

Here, we are concerned only with the following three complexes:

\[
0 \to \mathbb{R} \to \mathcal{S}^0_\infty \xrightarrow{d} \mathcal{S}^1_\infty \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}^k_\infty \xrightarrow{d} \cdots, \quad (31)
\]

\[
0 \to \mathbb{R} \to \mathcal{S}^0_\infty \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty \xrightarrow{d_V} \cdots \xrightarrow{d_V} \mathcal{S}^{0,n}_\infty \xrightarrow{\delta} 0, \quad (32)
\]

\[
0 \to \mathcal{S}^{1,0}_\infty \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty \xrightarrow{d_V} \cdots \xrightarrow{d_V} \mathcal{S}^{1,n}_\infty \xrightarrow{\varrho} \mathcal{E}_1 \to 0, \quad \mathcal{E}_1 = \varrho(\mathcal{S}^{1,n}_\infty). \quad (33)
\]
The first of them is the graded de Rham complex. The second one is the short variational complex, where $L = L_\omega \in S_{\infty}^{0,n}$ is a graded Lagrangian and

$$
\delta(L) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^a \Lambda \Lambda^a + d_\Lambda (\partial^\Lambda L) \tag{34}
$$

is its Euler–Lagrange operator. The third complex leads us to the first variational formula.

**THEOREM 7.** The cohomology of the complexes (31) – (32) equals the de Rham cohomology of $X$. The complex (33) is exact.

**Proof.** The proof follows the scheme of the proof of Theorem 1. It is given in Appendix. \hfill \blacksquare

**COROLLARY 8.** Every $d_H$-closed form $\phi \in S_{\infty}^{0,m<n}$ falls into the sum $\phi = \varphi + d_H \xi$, where $\varphi$ is a closed $m$-form on $X$. Every $\delta$-closed form $L \in S_{\infty}^{0,n}$ (a variationally trivial graded Lagrangian) is the sum $\phi = \varphi + d_H \xi$, where $\varphi$ is a non-exact $n$-form on $X$.

The exactness of the complex (33) at the term $S_{\infty}^{1,n}$ results in the following.

**PROPOSITION 9.** Given a graded Lagrangian $L = L_\omega$, there is the decomposition

$$
dL = \delta L - d_H(\Xi), \quad \Xi \in S_{\infty}^{1,n-1}, \quad \Xi = \sum_{s=0} \theta^a_{\nu_1...\nu_s} \wedge F^{\nu_1...\nu_s}_a (\partial_\lambda \omega), \quad F^{\nu_1...\nu_s}_a = \partial^\nu_a F^{\nu_1...\nu_s}_a - \partial^\nu_a F^{\nu_1...\nu_s}_a + h^\nu_{\nu_1...\nu_s}, \tag{35}
$$

where graded functions $h$ obey the relations $h^\nu = 0$, $h^\nu_{\nu_1...\nu_{k-1}} = 0$.

**Proof.** The proof repeats that of Proposition 2. \hfill \blacksquare

Proposition 9 shows the existence of a Lepagean equivalent $\Xi_L = \Xi + L$ of a graded Lagrangian $L$. Locally, one can always choose $\Xi$ (36) where all functions $h$ vanish.

## 5 Generalized Lagrangian supersymmetries

Generalized supersymmetries are defined as graded derivations $\nu \in \mathfrak{d} S_{\infty}^0$ of the $\mathbb{R}$-ring $S_{\infty}^0$ such that the Lie derivative $L_\nu \phi$ of any contact graded one-form $\phi \in S_{\infty}^{1,0}$ is also a contact form. Similarly to the case of generalized symmetries (Propositions 3 – 5), on can show that any generalized supersymmetry takes the local form

$$
v = v_H + v_V = v_\lambda d_\lambda + (v^a \partial_a + \sum_{|\Lambda| > 0} d_\Lambda v^a \partial^a_\lambda), \tag{37}
$$

where $v^\lambda$, $v^a$ are local graded functions. Then, it is easily justified that any vertical generalized supersymmetry $\nu$ (37) obey the relations (23) where $\phi \in S_{\infty}^*.$

**PROPOSITION 10.** Given a graded Lagrangian $L \in S_{\infty}^{0,n}$, its Lie derivative $L_\nu L$ along a generalized supersymmetry $\nu$ (37) obeys the first variational formula

$$
L_\nu L = v_V [\delta L] + d_H(h_0(\nu \Xi_L)) + d_V(v_H \omega L), \tag{38}
$$
where $\Xi_L$ is a Lepagean equivalent of $L$.

The proof is similar to that of Proposition 6. In particular, let $v$ be a divergence symmetry of $L$, i.e., $L_v L = d_H \sigma$, $\sigma \in S_{\infty}^{0,n-1}$. Then, the first variational formula (38) restricted to $\text{Ker} \delta L$ leads to the conservation law

$$0 \approx d_H (h_0 (v \Xi_L) - \sigma). \quad (39)$$

The BRST transformation in gauge theory on a principal bundle $P \to X$ with a structure Lie group $G$ gives an example of a vertical generalized supersymmetry as follows. Principal connections on $P$ are represented by sections of the affine bundle $C = J^1 P/G \to X$ coordinated by $(x^\lambda, a^r_\lambda)$ [13, 21, 28]. Infinitesimal generators of one-parameter groups of vertical automorphism (gauge transformations) of $P \to X$ are associated to sections of the vector bundle $V_G P = VP/G$ of right Lie algebras of the group $G$. Let us consider the simple graded manifold $(X, A_{V_G Y})$ constructed from this vector bundle. Its local basis is $(x^\lambda, a^r_\lambda)$. Let $S_{\infty}^*(C, V_G P)$ be the above algebra of even and odd variables $(a^r_\lambda, C^r)$ on $X$. Then, the generalized symmetry

$$v = v^r_\lambda \frac{\partial}{\partial a^r_\lambda} + v^r C^r + \sum_{|\Lambda| > 0} \left( d_\Lambda v^r_\lambda \frac{\partial}{\partial a^r_{\Lambda,\lambda}} + d_\Lambda v^r \frac{\partial}{\partial C^r_\Lambda} \right), \quad (40)$$

$$v^r_\lambda = C^r_\lambda + c^r_{pq} a^p_\lambda C^q, \quad v^r = \frac{1}{2} c^r_{pq} C^p C^q,$$

is the BRST transformation. The BRST operator is defined as the Lie derivative $s = L_v$ acting on $S_{\infty}^*(C, V_G P)$. It is readily observed that it is nilpotent on the module $S_{\infty}^{0,*}(C, V_G P)$ of horizontal forms.

Therefore, let us focus on nilpotent generalized supersymmetries. We say that a vertical generalized supersymmetry $v$ (37) on a GDA $S_{\infty}^*$ is nilpotent if

$$L_v (L_v \phi) = \sum_{|\Sigma| \geq 0, |\Lambda| \geq 0} (v^b_\Sigma \partial_b^\Sigma (v^a_\Lambda) \partial_a^\Lambda + (-1)^{|\Sigma|} v^b_\Sigma \partial_b^\Sigma (v^a_\Lambda) \partial_a^\Lambda) \phi = 0 \quad (41)$$

for any horizontal form $\phi \in S_{\infty}^{0,*}$. A glance at the second term in the expression (41) shows that a nilpotent generalized supersymmetry is necessarily odd. Furthermore, if the equality

$$L_v (v^a) = \sum_{|\Sigma| \geq 0} v^b_\Sigma \partial_b^\Sigma (v^a) = 0$$

holds for all $v^a$, a generalized supersymmetry $v$ is nilpotent. A useful example of a nilpotent generalized supersymmetry is an odd supersymmetry

$$v = v^a (x) \partial_a + \sum_{|\Lambda| > 0} \partial_\Lambda v^a \partial_a^\Lambda, \quad (42)$$

where all $v^a$ are real smooth functions on $X$, but all $s^a$ are odd.

Since the Lie derivative $L_v$ and the total differential $d_H$ mutually commute, let us suppose that the module of horizontal forms $S_{\infty}^{0,*}$ is split into a complex of complexes $\{S_k^{m}\}$ with
respect to $d_H$ and the Lie derivative $L_v$. In order to make it into a bicomplex, let us introduce the nilpotent operator $s_v \phi = (-1)^{\phi} L_v \phi$, $\phi \in S_\infty^{0,*}$, such that $d_H \circ s_v = -s_v \circ d_H$. This bicomplex

$$d_H : S^{k,m} \to S^{k,m+1}, \quad s_v : S^{k,m} \to S^{k+1,m}$$

is graded by the form degree $0 \leq m \leq n$ and an integer $k \in \mathbb{Z}$, though it may happen that $S^{k,*} = 0$ starting from some $k = k_0$. For short, let us call $k$ the charge number. For instance, the BRST bicomplex $S_\infty^{0,*}(C,V_G \mathcal{P})$ is graded by the charge number $k$ which is the polynomial degree of its elements in odd variables $C_\Lambda$. The bicomplex defined by the supersymmetry (42) has the similar gradation, but its nilpotent operator decreases the odd polynomial degree.

Let us consider horizontal forms $\phi \in S_\infty^{0,*}$ such that a nilpotent generalized supersymmetry $v$ is their divergence symmetry, i.e., $s_v \phi = d_H \sigma$. We come to the relative and iterated cohomology of the nilpotent operator $s_v$ with respect to the total differential $d_H$. Recall that a horizontal form $\phi \in S^{*,*}$ is said to be a relative $(s_v/d_H)$-closed form if $s_v \phi$ is a $d_H$-exact form. This form is called exact if it is a sum of an $s_v$-exact form and a $d_H$-exact form. Accordingly, we have the relative cohomology $H^{*,*}(s_v/d_H)$. In BRST theory, it is known as the local BRST cohomology [4, 6]. If a $(s_v/d_H)$-closed form $\phi$ is also $d_H$-closed, it is called an iterated $(s_v|d_H)$-closed form. This form $\phi$ is said to be exact if $\phi = s_v \xi + d_H \sigma$, where $\xi$ is a $d_H$-closed form. Note that the iterated cohomology $H^{*,*}(L_v|d_H)$ of a $(s_v,d_H)$-bicomplex $S^{*,*}$ is exactly the term $E_2^{s,*}$ of its spectral sequence [20]. There is an obvious isomorphism $H^{*,n}(s_v/d_H) = H^{*,n}(s_v|d_H)$ of relative and iterated cohomology groups on horizontal densities. This cohomology naturally characterizes Lagrangians $L$, for which $v$ is a divergence symmetry, modulo the Lie derivatives $L_v \xi$, $\xi \in S_\infty^{0,*}$, and the $d_H$-exact forms. One can apply Theorem 1 in [14] in order to state the relations between the iterated cohomology and the total $(s_v + d_H)$-cohomology of the bicomplex $S^{*,*}$ under the assumptions that all exterior forms on $X$ are of the same charge number (since $v$ is vertical, they are $s_v$-closed) and they are not $s_v$-exact. This is the case of the BRST transformation (40), but not the supersymmetry (42).

6 Appendix. Proof of Theorem 7

We start from the exactness of the complexes (31) – (33), except the terms $\mathbb{R}$, on $X = \mathbb{R}^n$. The Poincaré lemma and the algebraic Poincaré lemma have been extended to the complexes (31) and (32) [4, 5, 6]. The algebraic Poincaré lemma is applied to the complex (33) as follows.

The fact that a $d_H$-closed graded exterior form $\phi \in S^{1,m\leq n}$ is $d_H$-exact results from the algebraic Poincaré lemma for horizontal graded exterior forms $\phi \in S^{0,m\leq n}$. Indeed, let us formally associate to an $(m + 1)$-form $\phi = \sum \phi^a \Lambda_1 \wedge ds_\alpha$ the horizontal $m$-form $\bar{\phi} = \sum \phi^a s_\alpha$ depending on additional variables $s_\alpha$ of the same Grassmann parity as $s_\alpha$. It is easily justified that $d_H \bar{\phi} = d_H \phi$. If $d_H \phi = 0$, then $d_H \bar{\phi} = 0$ and, consequently, $\bar{\phi} = d_H \bar{\psi}$ where $\bar{\psi} = \sum \phi^a s_\alpha$. 

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is linear in $S^*_\Lambda$. Then, $\phi = d_H \psi$ where $\psi = \sum \psi^A_\alpha \wedge ds^A_\alpha$. It remains to show that, if

$$\varphi(\phi) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^\alpha \wedge [d_\Lambda(\partial^\Lambda_\alpha)](\phi) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^\alpha \wedge [d_\Lambda \phi^\Lambda_\alpha] = 0, \quad \phi \in S^1_{\infty},$$

then $\phi$ is $d_H$-exact. A direct computation gives

$$\phi = d_H \psi, \quad \psi = - \sum_{|\Lambda| \geq 0} \sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|} \theta_\Sigma^\alpha \wedge d_{\Sigma} \phi^{\Lambda+\mu}_\alpha \omega_\mu.$$ 

Let us associate to each open subset $U \subset X$ the $\mathbb{R}$-module $\mathcal{S}^*_U$ of elements of $S^*_\infty$ restricted to $U$. It is readily observed that these make up a presheaf on $X$. Let $\mathcal{S}^*_\infty$ be the sheaf constructed from this presheaf and $\Gamma(\mathcal{S}^*_\infty)$ its structure module of sections. One can show that $\mathcal{S}^*_\infty$ inherits the bicomplex operations, and $\Gamma(\mathcal{S}^*_\infty)$ does so. For short, one can say that $\Gamma(\mathcal{S}^*_\infty)$ consists of polynomials in $s^\Lambda_\alpha$, $ds^\Lambda_\alpha$ of locally bounded jet order $|\Lambda|$. There is the monomorphism $S^*_\infty \rightarrow \Gamma(\mathcal{S}^*_\infty)$.

Let us consider the complexes of sheaves of $C^\infty(X)$-modules

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{G}^0 \xrightarrow{d} \mathcal{G}^1 \xrightarrow{d} \mathcal{G}^k \rightarrow \cdots, \quad \mathcal{G}^k \xrightarrow{\delta} \mathcal{G}^k \rightarrow \cdots, \quad (43)$$

$$0 \rightarrow \mathcal{G}^0 \xrightarrow{d_H} \mathcal{G}^0,1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{G}^n \xrightarrow{\delta} 0, \quad (44)$$

$$0 \rightarrow \mathcal{G}^{1,0} \xrightarrow{d_H} \mathcal{G}^{1,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{G}^{1,n} \xrightarrow{\delta} \mathcal{E}_1 \rightarrow 0, \quad (45)$$

on $X$ and the complexes of their structure modules

$$0 \rightarrow \mathbb{R} \rightarrow \Gamma(\mathcal{G}^0) \xrightarrow{d} \Gamma(\mathcal{G}^1) \xrightarrow{d} \Gamma(\mathcal{G}^k) \rightarrow \cdots, \quad (46)$$

$$0 \rightarrow \mathbb{R} \rightarrow \Gamma(\mathcal{G}^0) \xrightarrow{d_H} \Gamma(\mathcal{G}^{0,1}) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Gamma(\mathcal{G}^{0,n}) \xrightarrow{\delta} 0, \quad (47)$$

$$0 \rightarrow \Gamma(\mathcal{G}^{1,0}) \xrightarrow{d_H} \Gamma(\mathcal{G}^{1,1}) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Gamma(\mathcal{G}^{1,n}) \xrightarrow{\delta} \Gamma(\mathcal{E}_1) \rightarrow 0. \quad (48)$$

The complexes (43) – (44) are resolutions of the constant sheaf $\mathbb{R}$, while the complex (45) is exact. By virtue of the abstract de Rham theorem, the cohomology of the complexes (46) – (47) equals the de Rham cohomology $H^*(X)$ of $X$, whereas the complex (48) is exact. It remains to prove that cohomology of the complexes (31) – (33) equals that of the complexes (46) – (48). The proof follows that of Theorem 9 in [14] and Theorem 5.1 in [29].

Let the common symbols $\Gamma^*_\infty$ and $D$ stand for all modules and operators in the complexes (46) – (48), respectively. With this notation, one can say that any $D$-closed element $\phi \in \Gamma^*_\infty$ takes the form $\phi = \varphi + D\xi$, where $\varphi$ is an exterior form on $X$. Then, it suffices to show that, if an element $\phi \in S^*_\infty$ is $D$-exact in the module $\Gamma^*_\infty$, then it is so in $S^*_\infty$. By virtue of the above mentioned Poincaré lemmas, if $X$ is contractible and a $D$-exact element $\phi$ is of finite jet order $[\phi]$ (i.e., $\phi \in S^*_\infty$), there exists an element $\varphi \in S^*_\infty$ such that $\phi = D\varphi$. Moreover, a glance at the corresponding homotopy operators shows that the jet order $[\varphi]$ of $\varphi$ is bounded by an integer $N([\phi])$, depending only on $[\phi]$. We agree to call this fact the finite exactness of the operator $D$. Given an arbitrary manifold $X$, the finite exactness takes place on any domain $U \subset X$. The following statements are proved similarly to those in [14, 29].
(i) Given a family \( \{ U_\alpha \} \) of disjoint open subsets of \( X \), let us suppose that the finite exactness takes place on every subset \( U_\alpha \). Then, it is true on the union \( \bigcup \alpha U_\alpha \).

(ii) Suppose that the finite exactness of the operator \( D \) takes place on open subsets \( U, V \) of \( X \) and their non-empty overlap \( U \cap V \). Then, it is also true on \( U \cup V \).

It remains to choose an appropriate cover of \( X \). It admits a countable cover \( \{ U_\xi \} \) by domains \( U_\xi, \xi \in \mathbb{N} \), and its refinement \( \{ U_{ij} \} \), where \( j \in \mathbb{N} \) and \( i \) runs through a finite set, such that \( U_{ij} \cap U_{ik} = \emptyset, j \neq k \) \[17\]. Then, \( X \) has the finite cover \( \{ U_i = \bigcup_j U_{ij} \} \). Since the finite exactness of \( D \) takes place on any domain \( U_\xi \), it also holds on any member \( U_{ij} \) of the refinement \( \{ U_{ij} \} \) of \( \{ U_\xi \} \) and, in accordance with the assertion (i), on any member of the finite cover \( \{ U_i \} \) of \( X \). Then, the assertion (ii) states the finite exactness of \( D \) on \( X \).

References

[1] Anderson, I and Duchamp, T.: On the existence of global variational principles, Amer. J. Math. 102 (1980), 781-868.

[2] Anderson, I.: Introduction to the variational bicomplex, Contemp. Math. 132 (1992), 51-73.

[3] Anderson, I., Kamran, N. and Olver, P: Internal, external and generalized symmetries, Adv. Math. 100 (1993), 53-100.

[4] Barnich, G., Brandt, F. and Henneaux, M.: Local BRST cohomology in gauge theories, Phys. Rept. 338 (2000), 439-569.

[5] Bartocci, C., Bruzzo, U. and Hernández Ruipérez, D.: The Geometry of Supermanifolds, Kluwer Academic Publ., Dordrecht, 1991.

[6] Brandt, F.: Locally BRST cohomology and covariance, Commun. Math. Phys. 190 (1997) 459-489.

[7] Brandt, F.: Jet coordinates for local BRST cohomology, Lett. Math. Phys. 55 (2001), 149-159.

[8] Bryint, R., Chern, S., Gardner, R., Goldschmidt, H. and Griffiths, P.: Exterior Differential Systems, Springer, Berlin, 1991.

[9] Cariñena, J. and Figueroa, H.: Hamiltonian versus Lagrangian formulations of supermechanics, J. Phys. A 30 (1997), 2705-2724.

[10] Cianchi, R., Francaviglia, M. and Volovich, I.: Variational calculus and Poincaré–Cartan formalism in supermanifolds, J. Phys. A. 28 (1995), 723-734.

[11] Fatibene, L., Ferraris, M., Francaviglia, M. and McLenaghan, R.: Generalized symmetries in mechanics and field theories, J. Math. Phys. 43 (2002), 3147-3161.
[12] Fulp, R., Lada, T. and Stasheff, J.: Noether variational Theorem II and the BV formalism, E-print arXiv: math.QA/0204079

[13] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: New Lagrangian and Hamiltonian Methods in Field Theory, World Scientific, Singapore, 1997.

[14] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Iterated BRST cohomology, Lett. Math. Phys. 53 (2000), 143-156.

[15] Giachetta, G., Mangiarotti, L. and Sardanashvily, G.: Cohomology of the infinite-order jet space and the inverse problem, J. Math. Phys. 42 (2001), 4272-4282.

[16] Gotay, M.: A multisymplectic framework for classical field theory and the calculus of variations, In: Mechanics, Analysis and Geometry: 200 Years after Lagrange, North Holland, Amsterdam, 1991, pp. 203-235.

[17] Greub, W., Halperin, S. and Vanstone, R.: Connections, Curvature, and Cohomology, Vol. 1, Academic Press, New York, 1972.

[18] Hernández Ruipérez, D. and Muñoz Masqué, J.: Global variational calculus on graded manifolds, J. Math. Pures Appl. 63 (1984), 283-309.

[19] Krasil’shchik, I., Lychagin, V. and Vinogradov, A.: Geometry of Jet Spaces and Non-linear Partial Differential Equations, Gordon and Breach, New York, 1985.

[20] Mac Lane, S.: Homology, Springer, Berlin, 1967.

[21] Mangiarotti, L. and Sardanashvily, G.: Connections in Classical and Quantum Field Theory, World Scientific, Singapore, 2000.

[22] Monterde, J. and Vallejo, J.: The symplectic structure of Euler–Lagrange superequations and Batalin–Vilkoviski formalism, J. Phys. A 36 (2003), 4993-5009.

[23] Olver, P.: Applications of Lie Groups to Differential Equations, Springer, Berlin, 1998.

[24] Rennie, A.: Poincaré duality of Spin^c structures for non-commutative manifolds, E-print arXiv: math-ph/0107013.

[25] Sardanashvily, G.: SUSY-extended field theory, Int. J. Mod. Phys. A 15 (2000), 3095-3112; E-print arXiv: hep-th/9911108.

[26] Sardanashvily, G.: Cohomology of the variational complex in field-antifield BRST theory, Mod. Phys. Lett. A 16 (2001) 1531-1541; E-print arXiv: hep-th/0102175.

[27] Sardanashvily, G.: Remark on the Serre–Swan theorem for non-compact manifolds, E-print arXiv: math-ph/0102016.
[28] Sardanashvily, G.: Ten lectures on jet manifolds in classical and quantum field theory, 
*E-print arXiv*: math-ph/0203040.

[29] Sardanashvily, G.: Cohomology of the variational complex in the class of exterior forms 
of finite jet order, *Int. J. Math. and Math. Sci.* 30 (2002), 39-48.

[30] Takens, F.: A global version of the inverse problem of the calculus of variations, *J. Diff. 
Geom.* 14 (1979), 543-562.