FRÉCHET GLOBALISATIONS OF HARISH-CHANDRA SUPERMODULES

ALEXANDER ALDDRIDGE

Abstract. For any Lie supergroup whose underlying Lie group is reductive, we prove an extension of the Casselman–Wallach globalisation theorem: There is an equivalence between the category of Harish-Chandra modules and the category of $SF$-representations (smooth Fréchet representations of moderate growth) whose module of finite vectors is Harish-Chandra. As an application, we extend to Lie supergroups a general general form of the Gel’fand–Kazhdan criterion due to Sun–Zhu.

Introduction

In the study of continuous representations of non-compact real-reductive Lie groups $G_0$, a basic obstacle is that almost all representations of interest are infinite-dimensional. A basic tool, which reduces many analytic questions to algebraic ones, is the passage to the module of $K_0$-finite vectors. The fundamental Casselman–Wallach theorem [13, 56] guarantees that every Harish-Chandra $(g_0, K_0)$-module occurs in this way. This is essential, in particular in applications to the classification problem for irreducible unitary representations.

Lie supergroups were introduced by Berezin, Kostant, and Leites in the 1970s as a mathematical framework for the study of the supersymmetries occurring in quantum field theory. Meanwhile, the subject of representations of Lie superalgebras is well-established in mathematics and physics, with a literature far too extensive to cite; compare the monographs [18, 41]. On the level of Lie supergroups, there is a sizeable literature in physics, but the subject has been hardly studied from a mathematical perspective.

Most mathematical works (e.g. Refs. [25, 27, 30, 44, 45]) consider the unitarisable Harish-Chandra modules, without exploring the issue whether they arise as the space of finite vectors of some ‘global’ representation. The first to take a ‘global’ perspective in the mathematical community were Carmeli–Cassinelli–Toigo–Varadarajan [16], who introduce a notion of unitary Lie supergroup representations. This has spawned a flurry of further investigation [40, 42, 43, 48].

Meanwhile, beyond the obvious fact that non-unitary representations may occur as intermediates in the study of unitary ones, it has become clear that unitary representations alone are insufficient for the purpose of Fourier–Plancherel decomposition, even in simple cases [5]. This is confirmed by applications of supersymmetry to number theory and random matrices [19, 31], as well as in physics, for instance in the study of the Chalker–Coddington model with point contacts [10].

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Finally, it has become increasingly clear, in the recent investigation of the Gel'fand and Gel'fand–Kazhdan properties [1–3, 52] beyond the setting of Riemannian symmetric pairs, that Casselman–Wallach theory is eminently useful for the study of branching multiplicities. Here, we argue that similar statements hold true also for the setting of Lie supergroups.

Therefore, it seems paramount to study the globalisations of ‘algebraic’ representations, irrespective of unitary, for Lie supergroups. In this paper, we generalise the Casselman–Wallach theorem to Lie supergroups, as follows.

**Theorem A.** Let $G$ be a Lie supergroup whose underlying Lie group $G_0$ is almost connected and real reductive, $\mathfrak{g}$ its Lie superalgebra, and let $K_0 \subseteq G_0$ be maximal compact. Then any Harish-Chandra $(\mathfrak{g}, K_0)$-module has a unique $\mathfrak{sl}$-globalisation.

This defines an additive equivalence between the category $\mathcal{HC}(\mathfrak{g}, K_0)$ of Harish-Chandra $(\mathfrak{g}, K_0)$-modules and the category $\mathcal{CW}(G)$ of $\mathfrak{sl}$-representations of $G$ whose module of $K_0$-finite vectors is Harish-Chandra.

Here, we follow Ref. [8] in using the term ‘$\mathfrak{sl}$-representation’ (resp. ‘$\mathfrak{f}$-representation’) instead of ‘smooth Fréchet representation of moderate growth’ (resp. ‘Fréchet representation of moderate growth’).

As an application of our results on globalisation, we study the Gel'fand–Kazhdan property for pairs of supergroups, to arrive by the following version of the Gel'fand–Kazhdan criterion, which generalises that given recently by Sun–Zhu [51].

**Theorem B.** Let $H_1, H_2 \subseteq G$ be closed subsupergroups, $\chi_i, i = 1, 2$, characters of $H_i$, $i = 1, 2$, and $\sigma$ an antiautomorphism of $G$. Assume that any even relatively $(\chi_1^{-1} \otimes \chi_2^{-1})$-invariant tempered superfunction $G$-that is a joint eigenvector of all even $G$-invariant $D \in \mathfrak{u}(\mathfrak{g})$ is fixed by $\sigma$.

Then, for any contragredient pair $(E, F)$ of $\mathfrak{f}$-representations of $G$ such that $E_\infty$ and $F_\infty$ are irreducible $G$-representations whose modules of $K_0$-finite vectors are Harish-Chandra, we have

$$\dim \text{Hom}_{H_1}(E_\infty, \chi_1) \dim \text{Hom}_{H_2}(F_\infty, \chi_2) \leq 1.$$ 

Theorem A (Theorem 4.6) is derived in the framework of convolution algebras of Schwartz functions, introduced by Bernstein–Krötz [8] in their proof of a Casselman–Wallach theorem for holomorphic families of Harish-Chandra modules.

As it turns out, the framework of convolution superalgebras of superdistributions and Berezinian densities is well-adapted for the study of the classes of continuous and weakly smooth representations, introduced here. In fact, a version of the Dixmier–Malliavin theorem holds (Proposition 2.15).

Moreover, the convolution algebra of Berezin–Schwartz densities is equally well suited for the study of $\mathfrak{f}$- and $\mathfrak{sl}$-representations (or moderate growth representations) of Lie supergroups. Indeed, we prove a Schwartzian Dixmier–Malliavin theorem for $\mathfrak{f}$-representations (Proposition 3.8), generalising the corresponding result of Bernstein–Krötz [8].

What makes the proof of our main results tick is the fact that all of the convolution superalgebras in question can be presented as coinduced modules (Proposition 2.2, Corollary 2.9, Proposition 3.2, and Proposition 3.3), allowing for a passage from Lie supergroups to supergroup pairs. We can thus reduce most analytic questions to the underlying Lie group and use Hopf algebraic methods of computations to arrive by our conclusions.

The expression of the convolution superalgebras via coinduced modules whilst preserving the convolution product, is, however, a non-trivial fact. It is based on an extension of Bruhat’s regularity theorem for left-invariant distributions (Proposition 2.4). Together with dualising module techniques, this implies an expression
of the invariant Berezin density in terms of the Haar density on the underlying Lie group (Proposition 2.8). Such an expression was previously only known in very special cases, where, in particular, the ‘odd modular function’ is trivial [20].

In the final Section 5, we apply our results to the generalisation of the multiplicity-one Theorem B (Theorem 5.7). The setting of Sun–Zhu [51] goes over more or less verbatim, due to our extension of the Casselman–Wallach theory.

We do not yet view these last results as definitive statements on multiplicity freeness for supergroups. Indeed, there are many issues special to the super case that need to be addressed, such as \( Q \) type modules and lack of semi-simplicity at the level of finite-dimensional modules. Moreover, non-trivial examples that verify this question in future work. However, the ease with which at least the purely even results transfer to the super case is to our mind a strong indication to the utility of the super Casselman–Wallach Theorem A.

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1. Supergroup representations

In this section, we collect some preliminary material on supergroups and their representations.

1.1. Supergroups and supergroup pairs. Concerning supermanifolds, we will use standard notions, as are to be found in Refs. [14,21,36,37]. We give some basic definitions to fix our terminology.

Let \( K \) be the field \( \mathbb{R} \) of real or the field \( \mathbb{C} \) of complex numbers. Consider the category of \( K \)-superspaces: Its objects are pairs \( X = (X_0, \mathcal{O}_X) \) comprised of a topological space \( X_0 \) and a sheaf \( \mathcal{O}_X \) on \( X_0 \) of supercommutative \( K \)-superalgebras with local stalks; its morphisms \( \varphi : X \rightarrow Y \) are pairs \((\varphi_0, \varphi^2)\) consisting of a continuous map \( \varphi_0 : X_0 \rightarrow Y_0 \) and an even unital morphism of \( K \)-superalgebra sheaves \( \varphi^2 : \varphi_0^* \mathcal{O}_Y \rightarrow \mathcal{O}_X \). Given some finite-dimensional super-vector space \( V = V_0 \oplus V_1 \) over \( K \), together with a compatible \( K \)-structure on the odd part \( V_1 \), we define the affine superspace \( \mathcal{A}(V) \) by

\[
\mathcal{A}(V)_0 := V_0, \quad \mathcal{O}_{\mathcal{A}(V)} := C_0^\infty(\mathbb{R}) \Lambda(V_1^*).
\]

Given a \( K \)-superspace \( X \), an open subspace is one of the form \( X|_{U'} := (U, \mathcal{O}_X|_{U'}) \) for some open subset \( U \subseteq X_0 \). A \( K \)-superspace \( X \) is called a supermanifold if \( X_0 \) admits an open cover \( (U_i) \) such that for every index \( i \), \( X|_{U_i} \) is isomorphic to an open subspace of some affine superspace \( \mathcal{A}(V) \). In this case, for \( x \in U_i \), the tuple \( \dim_K V_0| \dim_{K} V_1 \) is denoted \( \dim_x X \) and called the superdimension of \( X \) at \( x \).

For \( K = \mathbb{R} \), one customarily calls supermanifolds as defined above real supermanifolds; in the case \( K = \mathbb{C} \), they are called cs manifolds [21].

It is known that the category of supermanifolds admits finite products [21,36]. Thus, group objects and their morphisms in this category are well-defined [39]. A group object in the category of supermanifolds will be called a Lie supergroup or simply a supergroup. For \( K = \mathbb{R} \), these are real Lie supergroups, while for \( K = \mathbb{C} \), they are cs Lie supergroups.

For applications to linear representations, the following definition proves useful.

Definition 1.1 (Supergroup pairs). Let \( G_0 \) be a real Lie group with Lie algebra \( \mathfrak{g}_0 \), \( g \) be a Lie superalgebra over \( K \) such that \( \mathfrak{g}_0 = \mathfrak{g}_0 \otimes_K \mathbb{K} \), and \( \text{Ad} : G_0 \rightarrow \text{Aut}(\mathfrak{g}) \) a smooth action of \( G_0 \) by Lie \( K \)-superalgebra automorphisms. We say that \( (\mathfrak{g}, G_0) \)
(where the action is understood) is a supergroup pair if the differential $d\text{Ad}$ of $\text{Ad}$ is the restriction of the bracket $[\cdot, \cdot]$ of $\mathfrak{g}$ to $\mathfrak{g}_0 \times \mathfrak{g}$.

A morphism of supergroup pairs $(\mathfrak{g}, G_0) \to (\mathfrak{h}, H_0)$ consists by definition of a morphism $\varphi_0 : G_0 \to H_0$ of real Lie groups and a $\varphi_0$-equivariant Lie $\mathbb{K}$-superalgebra morphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $d\varphi_0 = d\varphi|_{\mathfrak{g}_0}$.

In the literature, supergroup pairs are referred to as Harish-Chandra pairs. Since to our knowledge, Harish-Chandra never worked on supergroups, we prefer to use a less colourful nomenclature.

The following proposition is due to Kostant [32] and Koszul [34] in the case $\mathbb{K} = \mathbb{R}$; see Ref. [14] for a detailed exposition. The extension to the case of $\mathbb{K} = \mathbb{C}$ presents no difficulty.

**Proposition 1.2.** Consider the functor that associates to a Lie supergroup $G$ the supergroup pair $(\mathfrak{g}, G_0)$, where $G_0$ is the underlying Lie group of $G$, $\mathfrak{g}$ is its Lie superalgebra, and $G_0$ acts on $\mathfrak{g}$ by the natural adjoint action.

This functor defines an equivalence of the category of Lie supergroups and their morphisms with the category of supergroup pairs and their morphisms.

**Remark 1.3.** In particular, we may associate with any real Lie supergroup $G$ the cs Lie supergroup whose supergroup pair is $(G_0, \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. On the level of superspaces, this sends $G$ to the complex superspace $(G_0, O_G \otimes_{\mathbb{R}} \mathbb{C})$.

We are mainly interested in complex representations, so we consider the case of cs Lie supergroups to be more relevant than the case of real Lie supergroups. Compare Ref. [21, § 4.9] for a list of five exemplary situations where it is more natural or even required to consider cs manifolds instead of real supermanifolds. In particular, Example 4.9.3 (op. cit.) describes a cs Lie supergroup which does not admit a real form. By contrast, any complex Lie supergroup has a cs form.

Most aspects of real supermanifolds carry over to the cs case. A notable exception is that for a cs manifold $X$, the induced almost complex structure on the odd part of the tangent bundle of the associated real supermanifold $X_{\mathbb{R}}$ is non-integrable [9].

### 1.2. Smooth and continuous supergroup representations

In what follows, let $G_0$ be a Lie group with Lie algebra $\mathfrak{g}_0$. To fix our terminology, we recall the following somewhat standard definitions.

**Definition 1.4** (Continuous and smooth representations). Let $G_0$ be a Lie group, $E$ a topological vector space over $\mathbb{K}$ and $G_0 \times E \to E$ a linear left action of $G_0$ on $E$. If the action is a continuous map, then we say that the induced map $\pi_0 : G \to \text{GL}(E)$ is a continuous representation of $G_0$ on $E$.

Let the topology on $E$ be locally convex. A vector $v \in E$ is called smooth if the orbit map $\gamma_v : G_0 \to E : g \mapsto \pi_0(g)v$ is a smooth map. For $x \in \mathfrak{g}_0$, one defines

$$d\pi_0(x)v := \frac{d}{dt} |_{t=0} \pi_0(\exp(tx))v.$$ 

This defines an action of $\mathfrak{g}_0$ on the space $E_\infty$ of all smooth vectors. One endows $E_\infty$ with the coarsest locally convex topology such that for all $u \in \mathfrak{U}(\mathfrak{g}_0)$, the linear map

$$d\pi_0(u) : E_\infty \to E$$

is continuous. Compare Refs. [8,53,57] for various defining sets of seminorms for this topology. The representation $\pi_0$ is called weakly smooth if the canonical inclusion $E_\infty \to E$ is an isomorphism of topological vector spaces. (For the reasons explained in Ref. [8], we reserve the term smooth for $\mathcal{F}$-representations, to be defined below.)
In what follows, let $G$ be a Lie supergroup with underlying Lie group $G_0$ and Lie superalgebra $\mathfrak{g}$. We continue to denote the Lie algebra of $G_0$ by $\mathfrak{g}_0$; in particular, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, that is $\mathfrak{g}_0 = \mathfrak{g}_0 \otimes_R \mathbb{K}$.

The following proposition is well-known and can be easily derived from standard facts on supergroup actions given ample exposition in the literature, see e.g. Ref. [14].

**Proposition 1.5.** Let $E$ be a finite-dimensional super-vector space over $\mathbb{K}$. Then the following data are in one-to-one correspondence:

(i) Pairs $(d\pi, \pi_0)$ of graded linear representations $\pi_0$ of $G_0$ on $E$ and $G_0$-equivariant Lie superalgebra actions $\pi$ of $\mathfrak{g}$ on $E$ with $d\pi_0 = d\pi|_{\mathfrak{g}_0}$;
(ii) left actions $G \times E \to E$ that are linear over $G$, in the sense that

$$g(u + \lambda v) = g(u) + \lambda g(v)$$

for all $S$ and all $S$-valued points $g \in S$, $u, v \in S E$, and $\lambda \in S \mathbb{K}$.

Here, we write $x \in S X$ if $x : S \to X$ is a morphism, and for any morphism $f : X \to Y$ and $x \in S X$, we define $f(x) \in S Y$ by $f(x) := f \circ x$.

Observe that for $\mathbb{K} = \mathbb{C}$ and $E \neq E_0$, no Lie supergroup with Lie superalgebra $\mathfrak{gl}(E, \mathbb{K})$ and underlying Lie group $\text{GL}(E_0, \mathbb{K}) \times \text{GL}(E_1, \mathbb{K})$ exists, so the proposition does not admit a statement in terms of supergroup homomorphisms.

On grounds of the above equivalence, we adopt the following terminology.

**Definition 1.6** (Continuous and smooth supergroup representations). Let $E$ be a locally convex super-vector space (i.e. $E$ is a locally convex vector space with a grading that exhibits $E$ as a locally convex direct sum). Assume given a continuous representation $\pi_0$ of $G_0$ on $E_0$ and a Lie superalgebra representation $d\pi$ of $\mathfrak{g}$ on $E_\infty$ such that the map $\mathfrak{g} \times E_\infty \to E_\infty : (x, v) \mapsto d\pi(x)v$ is continuous.

We say that $(d\pi, \pi_0)$ is a continuos $G$-representation if $d\pi$ is $G_0$-equivariant, i.e.

$$d\pi(\text{Ad}(g))(x) = \pi_0(g)d\pi(x)\pi_0(g^{-1})$$

for all $x \in \mathfrak{g}$ and $g \in G_0$, and $d\pi|_{\mathfrak{g}_0} = d\pi_0$. If in addition, $E$ is weakly smooth as a $G_0$-representation, then we call $E$ a weakly smooth $G$-representation.

The definition given above for continuous supergroup representations is compatible with the corresponding ones given in the literature for the case of unitary representations [16, 40].

2. **Convolution superalgebras and representations**

In what follows, let $G$ be a Lie supergroup, where $G_0$ is assumed to be $\sigma$-compact. Let $\mathfrak{g}$ be its Lie superalgebra. In this section, we introduce a convolution superalgebra of compactly supported Berezinian densities on $G$ and show that there is a one-to-one between its (non-degenerate) representations and the smooth representations of $G$.

To that end, we will identify the sheaf of Berezinian densities of $G$ within the sheaf of superdistributions as the $\mathfrak{g}$-module induced from the sheaf of densities on the underlying Lie group $G_0$. We begin by discussing superdistributions.

2.1. **Superdistributions.** In this section, we introduce superdistributions on $G$, and show how to express them in terms of the underlying Lie group.

For any open $U \subseteq G_0$, we endow $\mathcal{O}_G(U)$ with the locally convex topology generated by the seminorms

$$p_{u, v, K}(f) := \sup_{x \in K} |(L_u R_v f)(x)|$$
where $K \subseteq U$ is compact and $u,v \in \mathcal{U}(\mathfrak{g})$. Here, $L$ and $R$, respectively, denote the left and right regular representation. It is known [14, 34] that there is an isomorphism

$$\mathcal{O}_G(U) \xrightarrow{\phi} \text{Hom}_{\mathfrak{g}_0}(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(U, \mathbb{K}))$$

given by

$$\phi(f)(u;x) := (-1)^{|f||u|}(R_u f)(x)$$

for all $f \in \mathcal{O}_G(U)$, $u \in \mathcal{U}(\mathfrak{g})$, and $x \in U$. Here, the action of $\mathfrak{g}_0$ on $\mathcal{C}^\infty(U, \mathbb{K})$ is by left-invariant differential operators \textit{i.e.} infinitesimal right translations, and the algebra product is expressed on the right-hand side by the rule

$$f h = m \circ (f \otimes h) \circ \Delta,$$

where $m$ denotes multiplication in $\mathcal{C}^\infty$ and $\Delta$ denotes comultiplication in $\mathcal{U}(\mathfrak{g})$. For future use, we note that the multiplication morphism $m$ is given in terms of the isomorphism $\phi$ by

$$(2.1) \quad \phi(m^I(f))(u \otimes v; g, h) = \phi(f)(\Lambda d(h^{-1})(u)v; gh) = (-1)^{|u||v|+|f|}(L_{s(\Lambda d(h^{-1})(u))}R_u f)(gh).$$

Since $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_0) \wedge \mathfrak{g}_1$ as graded $\mathfrak{g}_0$-modules [49], we have

$$\mathcal{O}_G(U) \cong \mathcal{C}^\infty(U, \mathbb{K}) \otimes_{\mathbb{K}} (\mathfrak{g}_1)^*.$$

Since the Grassmann factor is finite-dimensional, one readily checks that is an isomorphism of locally convex super-vector spaces, where $\mathcal{C}^\infty(U, \mathbb{K})$ is given the usual topology of uniform convergence with all derivatives on compact subsets. In particular, $\mathcal{O}_G(U)$ is an $m$-convex Fréchet algebra [38].

Similarly, we give $\Gamma_c(\mathcal{O}_G)$ the locally convex inductive limit topology for the embeddings of the subspaces $\Gamma_k(\mathcal{O}_G)$ of sections $f$ with support $\text{supp} f \subseteq K$, where $K \subseteq G_0$ is compact. The latter are given the relative topology induced by $\Gamma(\mathcal{O}_G)$. Then $\Gamma_c(\mathcal{O}_G)$ is an LF space, and the multiplication is jointly continuous.

\textbf{Definition 2.1} (Superdistributions). For any open $U \subseteq G_0$, define

$$\mathcal{D}b_G(U) := \Gamma_c(\mathcal{O}_G|_U)'$$

the strong continuous dual space. Since $\mathcal{O}_G$ is a $c$-soft sheaf, we have by [11, Chapter V, §1, Proposition 1.6] that $U \mapsto \Gamma_c(\mathcal{O}_G|_U)$ is a flabby cosheaf. The corestriction maps are continuous by the definition of the topology. Thus, $\mathcal{D}b_G$ is a sheaf of locally convex super-vector spaces, called the \textit{sheaf of superdistributions}. In particular, we let $\mathcal{D}'(G) := \Gamma(\mathcal{D}b_G) = \Gamma_c(\mathcal{O}_G)'$.

The sheaf $\mathcal{D}b_G$ is naturally a right $\mathcal{O}_G$-module by

$$\langle \mu f, \varphi \rangle := \langle \mu, f \varphi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathcal{D}b_G(U)$ and $\Gamma_c(\mathcal{O}_G|_U)$.

The Lie supergroup $G$ acts from the left on $\mathcal{D}b_G$, where the $G_0$- and $\mathfrak{g}$-action are given respectively by

$$\langle L_g \mu, \varphi \rangle := \langle \mu, L_g^{-1} \varphi \rangle, \quad \langle L_x \mu, \varphi \rangle := -\langle \mu, L_x \varphi \rangle.$$

Here, in terms of the isomorphism $\phi$, we have

$$\phi(L_g \varphi)(u;h) = \phi(\varphi)(u; g^{-1}h), \quad \phi(L_x \varphi)(u;h) = -\phi(\varphi)(\text{Ad}(h^{-1})(x)u;h).$$

In what follows, if $H$ is a subsupergroup of $G$ and $\mathcal{A}$ is a subalgebra of $\mathcal{O}_G$, we will call a sheaf on $G_0$ with a left $H$-action commuting with a right $\mathcal{A}$-action a $(H, \mathcal{A})$-module. Thus, $\mathcal{D}b_G$ is a $(G, \mathcal{O}_G)$-module.

\textbf{Proposition 2.2}. Let $\mathcal{D}b_G_0$ be the sheaf of superdistributions on $G_0$. There is an isomorphism of $(G, \mathcal{O}_G_0)$-modules
\[ \mathcal{U}(g) \otimes \mathcal{U}(g_0) \xrightarrow{\text{Db}_{G_0}} \text{Db}_G, \]

given by
\[ \langle u \otimes \mu, \varphi \rangle = \langle \mu, j^* (L_{S(u)} \varphi) \rangle = (-1)^{|u|} \langle \mu_g, \varphi(Ad(g^{-1})(u); g) \rangle \]
for all open \( U \subseteq G_0, u \in \mathcal{U}(g), \mu \in \text{Db}_{G_0}(U), \) and \( \varphi \in \Gamma_c(\mathcal{O}_G|_U) \).

**Proof.** Firstly, we check that the map is well-defined. Indeed, we compute for \( x \in g_0 \):
\[ \langle ux \otimes \mu, \varphi \rangle = \langle \mu, j^* (L_{S(u)} x \varphi) \rangle = \langle \mu, j^* (L_x L_{S(u)} \varphi) \rangle = \langle u \otimes L_x \mu, \varphi \rangle, \]
where we abbreviate \( j = j_{G_0} \). Similarly, one verifies that the map is \( G \)-equivariant. Since it is manifestly right \( \mathcal{O}_{G_0} \)-linear, it is a morphism of \( (G, \mathcal{O}_{G_0}) \)-modules.

To see that it is an isomorphism, we define an involutive anti-automorphism \((-)^\vee = i^*_0 : i_0^{-1} \mathcal{O}_G \to \mathcal{O}_G \) where \( i_0(g) = g^{-1} \) by
\[ f^\vee(u; g) := f(Ad(g)(S(u); g^{-1})). \]
(This just the inversion morphism \( i : G \to G \).) Then we compute
\[ (-1)^{|u|} (S(u) \otimes \mu, \varphi) = \langle \mu_g, \varphi(Ad(g^{-1})(u); g) \rangle = \langle \mu_g^{-1}, \varphi(u; g) \rangle. \]
We recall again that there is an isomorphism of right \( g_0 \)-modules
\[ \bigwedge g_1 \otimes \mathcal{U}(g_0) \to \mathcal{U}(g) : \eta \otimes u \mapsto \beta(\eta) u \]
where \( \beta \) is supersymmetrisation. Applying this decomposition in Equation (2.3) readily implies our claim. \( \square \)

The proof shows that
\[ \mathcal{U}(g) \otimes \mathcal{U}(g_0) \xrightarrow{\text{Db}_{G_0}} \bigwedge g_1 \otimes_k \text{Db}_{G_0}. \]

If we consider on this sheaf the obvious tensor product locally convex topology (there is no choice which one to take, since \( \bigwedge g_1 \) is finite-dimensional), then it is easy to check that the above isomorphism is in fact one of sheaves of locally convex super-vector spaces.

### 2.2. Left-invariant superdistributions

In this section, we show that left-invariant superdistributions are smooth and hence proportional to the invariant Berezian density. To state this precisely, we recall the definition of Berezinian densities.

**Definition 2.3** (Berezinian densities). Let \( \text{Ber}_G \) denote the Berezia sheaf of \( G \) and \( |\Omega|_G := \text{or}_{G_0} \otimes \mathbb{Z} \text{Ber}_G \). Its set of global sections is denoted by \( |\Omega|_G(G) \); elements thereof are called \( \text{Berezian densities} \). The set of compactly supported sections is denoted by \( |\Omega|_{c(G)} \).

Recall \([21, 36, 37]\) that \( |\text{Ber}|_G \) is naturally a \( (G, \mathcal{O}_G) \)-module. Moreover, if \( U \subseteq G_0 \) is open and \( \omega \in \Gamma_c(|\Omega|_G|_U) \), then \( \int_{G|U} \omega \in \mathbb{K} \), the **Berezin integral** of \( \omega \), is well-defined. In particular, there is an embedding \( |\Omega|_G \to \text{Db}_G \), given by
\[ \langle \omega, \varphi \rangle := \int_{G|U} \omega \varphi \quad \text{for all} \quad \omega \in |\Omega|_G(U), \varphi \in \Gamma_c(\mathcal{O}_G|_U). \]

By Ref. [4], \( |\Omega|_G \) has a nowhere vanishing \( G \)-invariant section \( |Dg| \), which is unique up to constant multiples. It furnishes a module basis of \( |\Omega|_G \).

The following generalises a result due to Bruhat [12, Chapitre I, Proposition 3.1].

**Proposition 2.4** (Super Bruhat regularity theorem). Let \( \mu \in \mathcal{D}'(G) \) be left-invariant under \( G \). Then for some constant \( c \), we have \( \mu = c|Dg| \).

The proof of the proposition uses the following definition and basic lemmas.
Definition 2.5 (Convolution of superdistributions). Let $\mu, \nu \in \Gamma(D\theta_G)$. We say that $(\mu, \nu)$ is a proper pair if $m_0 : \operatorname{supp} \mu \times \operatorname{supp} \nu \to G_0$ is a proper map.

If $\varphi \in \Gamma_c(O_G)$, then $K := (\operatorname{supp} \mu \times \operatorname{supp} \nu) \cap m_0^{-1}(\operatorname{supp} \varphi)$ is compact. Let $\chi \in \Gamma_c(O_{G \times G})$ such that $\chi|_U = 1$ for some open neighbourhood of $K$. The quantity

$$\langle \mu \ast \nu, \varphi \rangle := \langle \mu \otimes \nu, \chi m_0^2(\varphi) \rangle$$

is independent of $\chi$. Moreover, it depends continuously on $\varphi$, thus defining an element $\mu \ast \nu \in \Gamma(O_G)$, the convolution of $\mu$ and $\nu$. Clearly, if either $\mu$ or $\nu$ is compactly supported, then $(\mu, \nu)$ is a proper pair.

Lemma 2.6. Let $\mu \in \Gamma(D_G)$ and $\omega \in |\Omega|^c(G)$. Then $\mu \ast \omega \in |\Omega|(G)$.

Proof. Let $\varphi \in \Gamma_c(O_G)$ and set

$$\psi(g) := \int_G \omega(h)m_0^2(\varphi)(g, h)$$

for any $T$ and any $h \in T \setminus G$. In the integral, $h$ denotes the generic point $h = \operatorname{id}_G \in G$.

Then by Yoneda’s lemma, we have $\psi \in \Gamma(O_G)$, and this superfunction has compact support $\subseteq (\operatorname{supp} \varphi)(\operatorname{supp} \omega)^{-1}$. Hence, we find that

$$\langle \mu \ast \omega, \varphi \rangle = \langle \mu, \psi \rangle = \left\langle \mu_g, \int_G |Dh| f(h) \varphi(gh) \right\rangle.$$

Writing $\omega = |Dg| f$, we have

$$\varrho := (\mu \otimes \operatorname{id})(m \circ (i \times \operatorname{id}))^2(f) \in \Gamma(O_G),$$

since $\Gamma(O_{G \times G}) = \Gamma(O_G) \otimes_{\pi} \Gamma(O_G)$, the completion of the projective tensor product [6, Corollary C.9], and $(\mu \otimes \operatorname{id})$ extends continuously this space. We thus compute

$$\langle \mu \ast \omega, \varphi \rangle = \left\langle \mu_g, \int_G |Dh| f(g^{-1}h) \varphi(h) \right\rangle = \langle |Dg| \varrho, \varphi \rangle,$$

so that $\mu \ast \omega = |Dg| \varrho$, proving the claim.

\[ \square \]

Lemma 2.7. Let $U$ be the filter of open neighbourhoods of $1 \in G_0$. There exist Berezinian densities $\chi_U = \chi_U \in |\Omega|_c(G)$, $\operatorname{supp} \chi_U \subseteq U \in U$, such that

$$\lim_{U \in U} \chi_U \ast \mu = \lim_{U \in U} \mu \ast \chi_U = \mu$$

in $D'(G)$, for any $\mu \in D'(G)$. If $\mu \in \Gamma_c(O_G)$, then the convergence is in $|\Omega|_c(G)$.

Proof. For $U$ sufficiently small, we may choose local coordinates $(u, \xi)$ and define

$$\chi_U := |D(u, \xi)| \xi_1 \cdots \xi_q g_U,$$

where $\int_U |du| g_U = 1$ and $\dim G = p/q$. Then for $\varphi \in \Gamma_c(O_G)$, we have

$$\int_G \chi_U \varphi = \int_U |du| g_U \varphi_0 \to \varphi_0(1) = \varphi(1),$$

where the convergence is uniform for $\varphi$ in compact subsets of $\Gamma_c(O_G)$. Indeed, Ref. [26, Proposition 2.42] gives uniform convergence, and compactness is preserved when passing to a coarser topology.

Now, the computation in the proof of Lemma 2.6 shows that

$$\langle \mu \ast \chi_U, \varphi \rangle = \langle \mu, \chi_U \ast \varphi \rangle$$

where we set

$$\varphi_U(h) := \int_G \chi_U(g) \varphi(g^{-1}h).$$
for any $T$ and any $h \in T G$. Then for $h \in T G$

$$(\chi_U \ast \varphi)(h) - \varphi(h) = \int_G \chi_U(g)(\varphi(g^{-1}h) - \varphi(h)) \to 0,$$

the convergence being in $\Gamma_c(O_T)$. Taking $T = G$ and $h = \text{id}_G \in G$, the assertion follows for right convolutions, and the case of left convolutions is similar.

\textbf{Proof of Proposition 2.4.} The proof is the same as Bruhat’s, based on the super-extensions of classical facts stated as the lemmas above. Let $(\chi_U)$ be as in the statement of Lemma 2.7. We have, for any $|g| \in \overline{\text{closure of the line spanned by}} \chi$,

$L_g \mathcal{L}_u(\mu \ast \chi_U) = (L_g \mathcal{L}_u \mu) \ast \chi_U,$

so the superdistribution $\mu \ast \chi_U$, which is a Berezin density by Lemma 2.6, is left-invariant under $G$ and thus equals $c_U |Dg|$ for some constant $c_U$.

But by Lemma 2.7, we have $\mu = \lim_{T \to G} \mu \ast \chi_U = \lim_{T \to G} c_U |Dg|$, so that $\mu$ is contained in the closure of the line spanned by $|Dg|$. But this line is finite-dimensional, and hence a closed subspace of $\Gamma(Db_G)$, since the unique Hausdorff vector space topology on $K$ is complete. This shows the assertion. \hfill $\Box$

2.3. \textbf{Berezinian densities via ordinary densities.} In this subsection, we show how Berezinian densities can be expressed in terms of ordinary densities on the underlying Lie groups.

To that end, let $|\Omega|_{G_0} := \text{ord}_{G_0} \otimes_{\mathbb{K}} \Omega^p_{G_0}$, where $p/q = \dim G$, denote the sheaf of $K$-valued smooth densities on $G_0$. Its global sections will be denoted by $|\Omega|(G_0)$, and the subspace of compactly supported sections by $|\Omega|_c(G_0)$. As above, there is an embedding $|\Omega|_{G_0} \to Db_{G_0}$, given by

$$(\omega, \varphi) := \int_U \omega \varphi \quad \text{for all} \quad \omega \in |\Omega|_{G_0}(U), \varphi \in \Gamma_c(O_{G_0}|U) = C_c^\infty(U).$$

The isomorphism in Proposition 2.2 suggests that we can identify $|\Omega|_G$ and $\mathcal{U}(g) \otimes_{\mathcal{U}(g_0)} |\Omega|_{G_0}$ within $Db_G$. Although this is not completely straightforward, it turns out to be quite generally true, as we now proceed to explain.

Let $\delta_1$ be the character by which $g_0$ acts on $\text{Ber}(g_0)$, \textit{i.e.}

$$\delta_1(x) = - \text{tr}_{\delta_1} \text{ad}(x) \quad \text{for all} \quad x \in g_0.$$

This character extends naturally to $\mathcal{U}(g_0)$. It is the differential of the character $\Delta_1$ of $G_0$, given by

$$\Delta_1(g) = \text{Ber}(g_0)^\ast(\text{Ad}(g)) = (\det_{\delta_1} \text{Ad}(g))^{-1}.$$

For any $g_0$-module $N$ (say), there is a well-known $[7,15,17,23,28]$ isomorphism of graded $g$-modules

$$\mathcal{U}(g) \otimes_{\mathcal{U}(g_0)} N \xrightarrow{\Phi} \text{Hom}_{g_0}(\mathcal{U}(g), \text{Ber}(g_0)^\ast) \otimes K N.$$
A notable special case occurs when $N = \text{Ber}(g/g_0)$. In this case, we may consider the action of $G_0$ on $N$, and $\text{Ber}(g/g_0)^* \otimes_K N \cong \mathbb{K}$ as $G$-modules. Moreover, if $g \in G_0$, then we have the equation

$$\iota(\text{Ad}(g)(u\beta(\eta))) = \text{Ad}(g)(u) \int_{B_1} \omega_1 \text{Ad}(g)(\eta) = \Delta_1(g) \cdot \text{Ad}(g)(\iota(u\beta(\eta)))$$

by the change of variables formula for the Berezin integral. Combining these facts with the definition of $\Phi$, one arrives by the formula

$$(2.5) \quad \Phi(\text{Ad}(g)(u \otimes n)(v)) = \Delta_1(g) \Phi(u \otimes n)(\text{Ad}(g^{-1})(v))$$

for $u, v \in \mathcal{U}(g)$ and $n \in \text{Ber}(g/g_0)$.

Let $I_{\delta_1}$ be the left ideal of $\mathcal{U}(g)$ generated by the set

$$\{x \in g_0 \mid x - \delta_1(x)\}.$$

By [23, Proposition 3.5], the space of $g$-invariants in

$$\mathcal{U}(g) \otimes_{\mathcal{U}(g_0)} \text{Ber}(g/g_0) = \text{Ber}(g/g_0)/I_{\delta_1}$$

is one-dimensional. Let $\gamma \in \mathcal{U}(g)$ be a representative of a basis.

**Proposition 2.8.** For a suitable normalisation of $|Dg|$ and $|dg|$, we have

$$(2.6) \quad |Dg| = L_\gamma(|dg| \Delta_1).$$

**Proof.** Let us consider the isomorphism $\Phi$ for $N = \text{Ber}(g/g_0)$. Since there is a canonical isomorphism

$$(2.7) \quad \text{Ber}((g/g_0)^*) \otimes_K \text{Ber}(g/g_0) \rightarrow \mathbb{K}$$

of $g_0$-modules [23, Lemma 1.4], we may view $\Phi$ as an isomorphism

$$\Phi : \mathcal{U}(g) \otimes_{\mathcal{U}(g_0)} \text{Ber}(g/g_0) \rightarrow \text{Hom}_{g_0}(\mathcal{U}(g), \mathbb{K}).$$

Moreover, by Ref. [23, p. 150], the coset of $\gamma$ corresponds under the canonical isomorphism $\Phi$ to the element $\varepsilon : \mathcal{U}(g) \rightarrow \mathbb{K}$, which is the extension of $0 : g \rightarrow \mathbb{K}$ to a superalgebra morphism. Hence, by Equation (2.5), for any $g \in G_0$, we have

$$(2.8) \quad \text{Ad}(g)(\gamma) \equiv \Delta_1(g)\gamma \pmod{I_{\delta_1}}.$$

Since $|dg| \Delta_1$ is relatively $g_0$-invariant for the character $\delta_1$, this quantity is annihilated by $I_{\delta_1}$. In particular, the superdistribution

$$\Omega := L_\gamma(|dg| \Delta_1) \in \mathcal{D}'(G)$$

depends only on the coset of $\gamma$.

By Proposition 2.4, it will be sufficient to show that $\Omega$ is a $g$- and $G_0$-invariant functional. Firstly, let $x \in g$ be homogeneous. Then we compute

$$\langle \Omega, L_x f \rangle = \langle |dg| \Delta_1, L_{S(\gamma)x}(f) \rangle = (-1)^{|x||\gamma|} \langle L_{x\gamma}(|dg| \Delta_1), f \rangle = 0$$

since by the choice of $\gamma$, we have $x\gamma \in I_{\delta_1}$ for any $x \in g$.

Secondly, we compute

$$\langle \Omega, L_h f \rangle = \langle L_{\text{Ad}(h^{-1})(\gamma)}(L_{h^{-1}}(|dg| \Delta_1)), f \rangle = \langle \Omega, f \rangle,$$

by the use of the relation $L_{h^{-1}}(|dg| \Delta_1) = \Delta_1(h)|dg| \Delta_1$ and Equation (2.8). Thus, we reach our conclusion. \qed

**Corollary 2.9.** As $(G, \mathcal{O}_{G_0})$-submodules of $\mathcal{D}b_G$, we have

$$|\Omega|_G = \mathcal{U}(g) \otimes_{\mathcal{U}(g_0)} |\Omega|_{G_0}.$$ 

Writing $\Delta(\gamma) = \sum_i \gamma_i' \otimes \gamma_i''$, the absolute Berezianian $|Dg|f$ corresponds to

$$(2.9) \quad \sum_i \gamma_i' \otimes |dg| \Delta_1 j_{G_0}(L_{S(\gamma_i'', \gamma_i')}(f)).$$
Conversely, the element $1 \otimes |dg|$ is mapped to $|Dg|\psi$, where $\psi \in \Gamma(O_G)$ is defined by

$$
\psi(u; g) := (R_{(u)}\Delta^{-1})(g)
$$

for all $u \in \mathfrak{U}(g)$, $g \in G_0$.

Proof. Consider the isomorphism

$$
\mathfrak{U}(g) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} \mathcal{D}b_{G_0} \longrightarrow \mathcal{D}b_G
$$

from Proposition 2.2. For $f, \varphi \in O_G(U)$, we compute

$$
(L_{S(\gamma)}(f\varphi))(g) = \sum_i (-1)^{\gamma_i(\varphi)+\gamma''_i}(L_{S(\gamma_i)}(f))(g)(L_{S(\gamma_i)}(\varphi))(g).
$$

For the non-zero summands, we have $|\gamma''_i| + |f| \equiv 0 \pmod{2}$. Hence, under the isomorphism, the expression in Equation (2.10) is mapped to $|Dg|f$. Thus, $|\Omega|_G$ is contained in the image of the subsheaf

$$
\mathfrak{U}(g) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} |\Omega|_{G_0} \subseteq \mathfrak{U}(g) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} \mathcal{D}b_{G_0}.
$$

For the converse, i.e. that $\mathfrak{U}(g) \otimes_{\mathfrak{g}_0} |\Omega|_{G_0}$ is mapped to $|\Omega|_G$, we need only show that this is the case for the $\mathfrak{U}(g) \otimes_{O_G}$-generator $1 \otimes |dg|$. To that end, consider the superfunction $\psi \in \Gamma(O_G)$, defined by Equation (2.10). It is well-defined, because the map $\iota$ is by definition left $\mathfrak{g}_0$-linear.

By [23, Theorem 3.1, Equation (65)], we have $\gamma \equiv \beta(x_1 \cdots x_q J)$ ($I_{j_1}$), where $J \in \bigwedge_0 \mathfrak{g}_1$ is the Jacobian of the exponential map (compare loc. cit.). Set $\tilde{\gamma} := x_1 \cdots x_q J \in S(\mathfrak{g})$ and consider the grading with components

$$
S^{\bullet,k} := S^{\bullet,k}(\mathfrak{g}) := S(\mathfrak{g}_0) \otimes \bigwedge^k \mathfrak{g}_1.
$$

Observe

$$
\Delta(S^{\bullet,k}) \subseteq \bigoplus_{a+b=k} S^{\bullet,a} \otimes S^{\bullet,b}.
$$

In particular, we have

$$
\Delta(\tilde{\gamma}) \equiv \gamma \otimes 1 \pmod{\bigoplus_{a<q} S^{\bullet,q} \otimes S(\mathfrak{g})}.
$$

On the other hand, by the definition of $\iota$, we have $\iota(u\gamma(\eta)) = 0$ for $u \in \mathfrak{U}(\mathfrak{g}_0)$ and $\eta \in \bigwedge \mathfrak{g}_1$, unless $\gamma$ has a non-zero component in top degree. Since $\beta : \mathfrak{U}(\mathfrak{g}) \rightarrow S(\mathfrak{g})$ is an isomorphism of coalgebras [46], we find

$$
\Delta(\gamma) \equiv \gamma \otimes 1 \pmod{\ker \iota \otimes \mathfrak{U}(\mathfrak{g})}.
$$

As observed in the proof of Proposition 2.8, we have for all $v \in \mathfrak{U}(g)$:

$$
\varepsilon(v) = \Phi(\tilde{\gamma})(v) = \iota(\varepsilon\gamma),
$$

where $\tilde{\gamma} \in \mathfrak{U}(\mathfrak{g})/I_{\tilde{\gamma}}$ denotes the coset of $\gamma$ and we have used Equation (2.4). No signs occur, since the left-hand side of the equation is independent of the odd part of $v$. In particular, $\iota(\gamma) = 1$.

Hence, we compute for any $\varphi, \psi \in \Gamma(O_G)$ that

$$
L_{S(\gamma)}(\psi\varphi)(1; g) = \Delta_1(g)^{-1}\varphi(1; g).
$$

For compactly supported $\varphi$, this implies that

$$
\langle |Dg|\psi, \varphi \rangle = \langle |Dg|, \Delta_1^{-1}j_{G_0}^*(L_{S(\gamma)}(\psi\varphi)) \rangle = \langle |Dg|, j_{G_0}^*(\varphi) \rangle.
$$

Thus, we find that $1 \otimes |dg|$ is mapped to $|\Omega|(G)$; this proves the claim. \qed
2.4. Convolution of superdistributions and Berezinian densities.

**Definition 2.10** (Compactly supported superdistributions). We let $E'(G)$ be the strong dual space of $\Gamma(O_G) = O_G(G_0)$ and call its elements *compactly supported superdistributions*. For $\mu, \nu \in E'(G)$, the convolution $\mu * \nu \in E'(G)$ from Definition 2.5 takes the form

$$\langle \mu * \nu, f \rangle := \langle \mu \otimes \nu, m^I(f) \rangle$$

for all $f \in \Gamma(O_G)$. Here, $m : G \times G \to G$ is the multiplication of $G$.

If $A$ is a topological $K$-vector space with an algebra structure, then we call $A$ a *topological algebra* if multiplication is separately continuous. We allow non-unital algebras, but unless called ‘non-unital’ expressly, they are assumed to have a unit.

In the following, let $E'(G_0)$ be the strong dual of $\Gamma(O_{G_0})$. It carries a natural convolution, see Ref. [35, 53]. Recall that $\mathfrak{U}(g_0) \subseteq E'(G_0)$ is a subalgebra via $u \mapsto L_u \delta$, where $\delta$ denotes the Dirac delta distribution supported at the neutral element of $G_0$.

**Proposition 2.11.** The convolution product on $E'(G)$ is well-defined and turns it into an associative and unital topological superalgebra. We have $E'(G) = \Gamma_c(\mathcal{D}G)$ and there is an isomorphism

$$\mathfrak{U}(g) \otimes \mathfrak{U}(g) E'(G_0) \to E'(G)$$

of locally convex super-vector spaces. In terms of this isomorphism, the superalgebra structure is uniquely determined by the following facts:

(i) The following are graded subalgebras:

$$\mathfrak{U}(g) = \mathfrak{U}(g) \otimes \mathfrak{U}(g), \quad E'(G_0) = \mathfrak{U}(g_0) \otimes \mathfrak{U}(g_0) E'(G_0).$$

(ii) For $\mu \in E'(G_0)$ and $u \in \mathfrak{U}(g)$, the products $u * \mu$ and $\mu * u$ are given by

$$(2.11) \quad \langle u * \mu, \varphi \rangle = \langle \mu, j^G_{\mathfrak{g}_0}(L_{S(u)} \varphi) \rangle, \quad \langle \mu * u, \varphi \rangle = \langle \mu, j^G_{\mathfrak{g}_0}(R_u \varphi) \rangle$$

for all superfunctions $\varphi \in \Gamma(O_G)$.

Proof. Since $\Gamma_c(O_G)$ is dense in $\Gamma(O_G)$, $E'(G)$ may be identified with a subspace of $\mathcal{D}'(G)$. On the other hand, one knows that $E'(G_0) = \Gamma_c(\mathcal{D}G_0)$. Therefore, Proposition 2.2 gives an isomorphism of super-vector spaces as stated and $E'(G) = \Gamma_c(\mathcal{D}G)$. Moreover, it is straightforward to prove that it is indeed a homeomorphism for the topology on $E'(G)$ and the natural topology on $\mathfrak{U}(g) \otimes \mathfrak{U}(g_0) E'(G_0) = \wedge \mathfrak{g}_0 \otimes \mathfrak{g} E'(G_0)$.

It is clear that there is at most one algebra structure on $E'(G)$ determined by the information stated in (i) and (ii). Conversely, we compute for $\mu, \nu \in E'(G_0)$ and $u, v \in \mathfrak{U}(g)$, by the use of Equations (2.1) and (2.2):

$$\langle (u \otimes \mu) * (v \otimes \nu), \varphi \rangle = (-1)^{|\varphi||u| + |\nu|} \langle \mu_0 \otimes v_0, \nu_h, (m^h \varphi)(\text{Ad}((g,h)^{-1})(u \otimes v); g, h) \rangle$$

$$= (-1)^{|\varphi||u| + |v|} \langle \mu_0 \otimes v_0, \nu, \varphi(\text{Ad}(h^{-1})((g^{-1})(u)v); gh) \rangle.$$

For $u = \delta$ and $v = 1$, we obtain

$$\langle u * \nu, \varphi \rangle = (-1)^{|\varphi||u|} \langle \nu_h, \varphi(\text{Ad}(h^{-1})(u); h) \rangle = \langle \nu, L_{S(u)} \varphi \rangle,$$

and for $u = 1$ and $\nu = \delta$, we get

$$\langle \mu * v, \varphi \rangle = (-1)^{|\varphi||v|} \langle \mu_0, \varphi(v; g) \rangle = \langle \mu, R_v \varphi \rangle.$$

This shows Equation (2.11).

The convolution on $E'(G)$ is an even bilinear map by definition. That it is associative operation follow either from $m \circ (m \times \text{id}) = m \circ (\text{id} \times m)$, or also easily from Equation (2.11), together with the fact that $\mathfrak{U}(g)$ and $E'(G_0)$ are algebras and that the actions $L$ and $R$ commute. \qed
The convolution algebra structure on $\mathcal{E}'(G)$ admits a natural $\mathbb{K}$-linear antilinear involution, defined by
\begin{equation}
\langle \mu, \varphi \rangle := \langle \mu, \check{\varphi} \rangle = \langle \mu, i^2 \varphi \rangle,
\end{equation}
where $i : G \to G$ is the inversion morphism, and $\check{\varphi} = i^2 \varphi$ was employed above in the proof of Proposition 2.2. Since Berezinian densities pull back under isomorphisms, the involution leaves $[\Omega]_c(G) \subseteq \mathcal{E}'(G)$ stable.

**Corollary 2.12.** The dense subspace $[\Omega]_c(G) \subseteq \mathcal{E}'(G)$ is a graded ideal and a non-unital Fréchet algebra with the topology induced from $\Gamma_c(O_G)$. In terms of the isomorphism $\mathcal{E}'(G) = \mathfrak{g}(g) \otimes \Omega_c(g_0) \otimes \mathcal{E}'(G_0)$, its $\mathcal{E}'(G)$-bimodule structure is determined uniquely by the following facts:

1. The following is a non-unital graded subalgebra bi-invariant under $\mathcal{E}'(G_0)$:
   \[ [\Omega]_c(G_0) = \mathfrak{g}(g_0) \otimes \Omega_c(g_0) \otimes [\Omega]_c(G_0). \]
2. For $u, v \in \mathfrak{g}(g)$ and $\omega \in [\Omega]_c(G_0)$, we have
   \[ u \ast (v \otimes \omega) = (u \otimes 1) \ast (v \otimes \omega) = uv \otimes \omega. \]
3. For $\omega \in [\Omega]_c(G_0)$ and $u \in \mathfrak{g}(g)$, the products $u \ast \omega$ and $\omega \ast u$ are given by
   \begin{equation}
   \int_{\mathcal{G}_0} \big( u \ast \omega \big) \phi = \int_{\mathcal{G}_0} \omega \mathcal{E}_0(L_S(u)) \phi = \langle u \otimes \omega, \phi \rangle,
   \end{equation}
   \begin{equation}
   \int_{\mathcal{G}_0} \big( \omega \ast u \big) \phi = \int_{\mathcal{G}_0} \omega \mathcal{E}_0(R_u) \phi,
   \end{equation}
   for all superfunctions $\phi \in \Gamma(O_G)$.

**Proof.** Let us verify that $[\Omega]_c(G)$ is indeed a convolution ideal in $\mathcal{E}'(G)$. Indeed, this follows from Lemma 2.6. Alternatively, one may proceed as follows.

Certainly, $[\Omega]_c(G_0)$ is an ideal of $\mathcal{E}'(G_0)$. Let $\omega$ and $u \in \mathfrak{g}(g)$. Since $u \ast \omega$ corresponds to $u \otimes \omega$, it is obvious that $u \ast \omega \in [\Omega]_c(G_0)$. On the other hand, we have
\begin{align*}
\int_{\mathcal{G}_0} (\omega \ast u) \phi &= \int_{\mathcal{G}_0} \omega(g) \phi(u; g) = \int_{\mathcal{G}_0} \omega(g) \phi(S(u); g) = \int_{\mathcal{G}_0} \tilde{\omega}(g) \check{\phi}(u) = \int_{\mathcal{G}_0} \tilde{\omega}(g) \check{\phi}(u) = \int_{\mathcal{G}_0} \tilde{\omega}(g) \check{\phi}(u),
\end{align*}
where $\Omega \in [\Omega]_c(G)$ corresponds to $S(u) \otimes \tilde{\omega}$ and $\tilde{\omega}$ was defined in Equation (2.12). This shows that $\omega \ast u \in [\Omega]_c(G_0)$.

Thus, $[\Omega]_c(G)$ is indeed a graded ideal of $\mathcal{E}'(G)$, and the remaining statements follow readily from Proposition 2.11. \(\square\)

### 2.5. Convolution action on representations

We now show how supergroup representations on Fréchet spaces can be characterised in terms of the action of convolution superalgebras. We use the following terminology.

**Definition 2.13.** Let $A$ be a topological algebra. A left $A$-module will be called a **continuous module** if the action map is separately continuous. An $A$-module $E$ is called **non-degenerate** if
\[ E = AE := \langle av \mid a \in A, v \in E \rangle. \]

**Lemma 2.14.** Let $(E, \pi)$ be a weakly smooth Fréchet $G$-representation. Then the $\mathcal{E}'(G_0)$-module structure inherited from $E|G_0$ combines with the $\mathfrak{g}(g)$-action on $E$ to a unique continuous $\mathcal{E}'(G)$-module structure on $E$, denoted by $\Pi$.

**Proof.** Let $\pi_0$ denote the action of $\mathfrak{g}$ and $G_0$ on $E$, respectively. Take $v \in E$. Then $\pi_0((-)v) : G \to E : g \mapsto \pi_0(g)v$ is a smooth map and there is an element $\pi(-)v \in \text{Hom}_{\pi_0}(\mathfrak{g}(g), C^\infty(G_0, E))$, defined by
\[ \pi(u; g)v := (\pi(-)v)(u)(g) := \pi_0(g)d\pi(u)v. \]
Since $\Gamma(O_G, E)$ is a nuclear Fréchet space [54, Corollary to Theorem 51.4], we have $C^\infty(G_0, E) = \Gamma(O_G, E)$, where $\hat{\otimes}_\pi$ denotes the completed (projective) tensor product. We may thus define for $u \in \frak{U}(\frak{g})$ and $\mu \in \mathcal{E}'(G_0)$:

$$
\Pi(u \otimes \mu)v := d\pi(u)\Pi_0(\mu)v = d\pi(u)\langle \mu_g, \pi_0(g)v \rangle,
$$

where $\Pi_0$ is the integrated version of the $G_0$-representation $\pi_0$ on $E$, see Refs. [35, 53]. We compute

$$
\Pi(u \otimes \omega)v = \langle \mu_g, \pi_0(g)\text{Ad}(g^{-1})(u)\rangle v = \langle \mu, L_{S(u)}\pi(-v) \rangle.
$$

In particular, for $x \in \frak{g}_0$, we obtain

$$
\Pi(ux \otimes \mu)v = \langle \mu, L_xL_{S(\omega)}\pi(-v) \rangle = \langle L_x\mu, L_{S(\omega)}\pi(-v) \rangle = \Pi(u \otimes L_x\mu)v.
$$

This shows that the action $\Pi$ is well-defined on $\frak{U}(\frak{g}) \otimes \mathcal{E}'(G_0)$, and hence, the assertion follows from Proposition 2.11. \hfill $\square$

We call $\Pi$ the integrated action of $\pi$. Restricting it to densities, we obtain the following proposition, which generalises a theorem of Dixmier–Malliavin [22].

**Proposition 2.15** (Super Dixmier–Malliavin theorem). Let $E$ be a Fréchet super-vector space over $\mathbb{K}$. Then we have the following facts:

(i) If $E$ carries the structure of a continuous $G$-representation, then the action of $[\Omega]_c(G)$ on $E_\infty$ extends continuously to $E$. The induced action of $[\Omega]_c(G)$ on $E_\infty$ is non-degenerate. More precisely, we have the equality

$$
E_\infty = \Pi([\Omega]_c(G))E = \Pi([\Omega]_c(G))E_\infty.
$$

(ii) Conversely, let $\Pi$ be a non-degenerate continuous action of $[\Omega]_c(G)$ on $E$. Then $\Pi$ is integrated from a unique weakly smooth $G$-representation.

In particular, the category of weakly smooth Fréchet $G$-representations and the category of non-degenerate continuous Fréchet $[\Omega]_c(G)$-modules are equivalent.

**Proof.** Assume that $\pi$ is a continuous $G$-representation on $E$, so that we have by Lemma 2.14 the integrated representation $\Pi$ of $\mathcal{E}'(G_0)$ on $E_\infty$. Let $\Pi_0$ be the integrated version of the $G_0$-representation $\pi_0$ on $E$. By a theorem of Gårding [53], we have $\Pi_0([\Omega]_c(G_0))E \subseteq E_\infty$.

Thus, it makes sense to define, for $u \in \frak{U}(\frak{g})$, $\omega \in [\Omega]_c(G_0)$, and $v \in E$:

$$
\Pi(u \otimes \omega)v := d\pi(u)\Pi_0(\omega)v.
$$

Indeed, this coincides with the definition of $\Pi$ on $E_\infty$ given in Equation (2.14). In addition, for $x \in \frak{g}_0$, we have

$$
\Pi(ux \otimes \omega)v = d\pi(u)d\pi_0(x)\Pi_0(\omega)v = d\pi(u)\Pi_0(L_x\omega)v = \Pi(u \otimes x\omega)v,
$$

so that $\Pi$ defines a continuous representation of $[\Omega]_c(G)$ by Corollary 2.9.

By the Dixmier–Malliavin theorem [22, Theorem 3.3], we have

$$
E_\infty = \Pi_0([\Omega]_c(G_0))E_\infty.
$$

Applying the definition of $\Pi$ in Equation (2.14), we obtain

$$
E_\infty = d\pi(\frak{U}(\frak{g})))E_\infty = d\pi(\frak{U}(\frak{g})\Pi_0([\Omega]_c(G_0)))E_\infty = \Pi([\Omega]_c(G))E_\infty.
$$

But $\Pi([\Omega]_c(G))E \subseteq E_\infty$, so we have proved part (i) of the proposition.

Conversely, assume that $E$ is a non-degenerate continuous $[\Omega]_c(G)$-module. If $\nu \in E$ is a vector, then we may express it as $\nu = \sum_{j \in J} \Pi(\omega_j)v_j$ where $J$ is finite and $\omega_j \in [\Omega]_c(G)$, $v_j \in V$. We wish to define $\pi_0$ and $d\pi$ for $g \in G_0$ and $u \in \frak{U}(\frak{g})$ by

$$
\pi_0(g)v := \sum_{j \in J} \Pi(L_g\omega_j)v_j, \quad d\pi(u)v := \sum_{j \in J} \Pi(L_u\omega_j)v_j.
$$

The first task is to show that these quantities are independent of all choices.
To that end, let $\sum_{j \in J} \Pi(\omega_j) v_j = 0$ in $E$. Choose $(\chi_U)$ as in Lemma 2.7. Then $L_g \omega_j = \delta_g \ast \omega_j$, where $\delta_g$ is the Dirac distribution supported at $g$, and

$$\sum_{j \in J} \Pi(L_g \omega_j) v_j = \lim_{U \subseteq U} \sum_{j \in J} \Pi(\delta_g \ast \chi_U \ast \omega_j) v_j = \lim_{U \subseteq U} \Pi(L_g \chi_U) v = 0.$$ 

A similar argument applies for $u \in \mathfrak{U}(g)$, and so Equation (2.16) indeed defines actions $\pi_0$ of $G_0$ and $d\pi$ of $g$.

Moreover, in case $\Pi$ is already integrated from a weakly smooth $G$-representation $\pi'$, then analogously

$$\pi'_0(g) v = \lim_{U \subseteq U} \sum_j \pi'_0(g) \Pi(\chi_U \ast \omega_j) v_j = \lim_{U \subseteq U} \Pi(L_g \chi_U) v = \pi_0(g) v.$$ 

Similarly, one shows that $d\pi' = d\pi$, so $\Pi$ is integrated from at most one weakly smooth $G$-representation, and if it is, then the corresponding actions of $G_0$ and $\mathfrak{g}$ are given by $\pi_0$ and $d\pi$, respectively. It therefore remains to be shown that $\pi_0$ and $d\pi$ combine to a weakly smooth $G$-representation.

For this, we observe that the action map $\Pi : [\Omega|^c_c(G) \otimes_i E \to E$ is continuous and surjective, $\otimes_i$ denoting injective tensor product. It extends to a continuous surjective map $\Pi$ on the completed tensor product $\hat{\otimes}$. Since $[\Omega|^c_c(G)$ is nuclear [6, Proposition C.7], we have $\hat{\otimes} = \otimes [54, Theorem 50.1]$, the latter denoting the completed projective tensor product. Since $G$ acts weakly smoothly on $[\Omega|^c_c(G)$, it acts weakly smoothly on $[\Omega|^c_c(G) \otimes_E E$. Since $E$ is as a Fréchet $G_0$- and $g$-representation a quotient of this space, it follows that is a weakly smooth $G$-representation. □

**Remark 2.16.** The graded version of the Dixmier–Malliavin theorem offered above (part (i) of Proposition 2.15) admits an independent proof, which does not appeal to Corollary 2.9, but rather follows a similar path as Dixmier and Malliavin in their original proof, reducing the statement to low-dimensional cases. To simplify the exposition, we restrict ourselves to the case of a weakly smooth $G$-representation.

We need to show that $E \subseteq \Pi([\Omega|^c_c(G)] E$. To that end, we introduce the following terminology: A closed Lie subsupergroup $H$ of $G$ is called 

$singly generated$ if its Lie superalgebra $\mathfrak{h}$ is generated by a single homogeneous element.

Then the following is straightforward: Any singly generated Lie subsupergroup is locally isomorphic to one of the Abelian supergroups $\mathfrak{a}$ and $\mathfrak{a}^{[0]}$, or to $\mathfrak{a}^{[1]}$, where the Lie superalgebra has the unique non-zero homogeneous relation $x = [y, g]$. Moreover, there exist singly generated closed Lie subsupergroups $H_1$, ..., $H_n$ such that the $n$-fold multiplication morphism $m : H_1 \times \cdots \times H_n \to G$ is an isomorphism in a neighbourhood $U$ of the identity.

Now, fix $v \in E$. We claim the following: For any singly generated sub-supergroup $H$ and any neighbourhood $V \subseteq U$ of 1, there exist $\omega_0, \omega_1 \in [\Omega|^c_c(H) \subseteq E'(H) \subseteq E'(G)$ and $w \in E$ with $\operatorname{supp} \omega_j \subseteq H \cap V$, such that $v = \Pi(\omega_0)v + \Pi(\omega_1)w$. Since $\mathfrak{h} \subseteq \mathfrak{a}^{[1]}$ as a closed Lie subsupergroup, this follows from Dixmier–Malliavin [22, Theorem 3.3] in case $H_0$ is locally isomorphic $\mathfrak{h}$.

In case $H$ is isomorphic to $\mathfrak{a}^{[0]}$, we have $\Gamma(\mathcal{O}_H) = \mathbb{K}[\tau]$ where $\tau$ is odd. It follows that $|\delta\tau|$, defined by $\int_H \delta\tau f = \frac{\delta f}{\delta \tau}$, is a smooth density, and $\int_H |\delta\tau| (\tau f) = f(0)$. Thus, $d\delta = |\delta\tau| \tau \in [\Omega|^c_c(\mathcal{O}_H)$ and hence, the statement is obvious in this case.

Applying the statement inductively, we find $f_0, f_1 \in [\Omega|^c_c(H_j) \subseteq E'(G)$ and $\omega_1, ..., \omega_n \in E$, $i_j = 0, 1$, such that

$$v = \sum_{i_1, ..., i_n = 0, 1} \Pi(f_{i_1} \ast \cdots \ast f_{i_n}) \omega_{i_1, ..., i_n}.$$ 

Now, for $\omega_j \in [\Omega|^c_c(H_j)$ and $\varphi \in \Gamma(\mathcal{O}_G)$, we have

$$\langle \omega_1 \ast \cdots \ast \omega_n, \varphi \rangle = \langle \omega_1 \otimes \cdots \otimes \omega_n, m^\sharp \varphi \rangle.$$
Since $\omega_1 \otimes \cdots \otimes \omega_n$ is in $|\Omega|_c(H_1 \times \cdots \times H_n)$, we find $\omega_1 \ast \cdots \ast \omega_n \in |\Omega|_c(G)$, provided that the sup $\omega_j$ are small enough. This finally proves the claim.

3. SF-representations

In this section, we extend the notion of smooth representations of moderate growth, or SF-representations, to the case of Lie supergroups. We construct a superalgebra of Schwartz–Berezin densities and show that its representations are in one-to-one correspondence with SF-representations of $G$.

3.1. Schwartz–Berezin densities. Following Ref. [8], we will call a measurable function $s : G_0 \to (0, \infty)$ a scale if $s$ and $1/s$ are essentially bounded and

$$s(gh) \leq s(g)s(h).$$

We write $s \preceq s'$ for scales $s, s'$ if is a constant $C > 0$ and an integer $N \geq 0$ with

$$s(g) \leq Cs(g)^N$$

for all $g \in G_0$. This defines a preorder. The equivalence classes for the largest equivalence relation contained in $\preceq$ are denoted by $[s]$ and called scale structures.

In what follows, we fix a scale $s$ on $G_0$. We will always make the assumption that $s$ dominates the $\mathfrak{g}$-adjoint scale, i.e. $s \geq s_{\mathfrak{g}}$ where

$$s_{\mathfrak{g}}(g) := \max(\|\text{Ad}(g)\|_{\mathfrak{g}}, \|\text{Ad}(g)\|_{\mathfrak{g}})$$

where we fix some norm on $\mathfrak{g}$. Observe that there is a constant $C > 0$ such that

$$\max(\|\Delta_1(g)\|, \|\Delta_1(g)^{-1}\|) \leq Cs_{\mathfrak{g}}(g)^N,$$

where $N = \dim \mathfrak{g}_1$.

Definition 3.1 (Schwartz–Berezin densities). We define the space of Schwartz–Berezin densities to be

$$\mathcal{S}(G, [s]) := \left\{ |Dg| f \bigg| \forall u, v \in \Omega(\mathfrak{g}), N \geq 0 : \int_{G_0} |dg| s(g)^N |(L_u R_v f)(g)| < \infty \right\},$$

where $|Dg|$ and $|dg|$ are some choices of left invariant Berezin density on $G$ resp. left invariant density on $G_0$. This space is endowed with the locally convex topology generated by the seminorms

$$p_{u,v,N}(|Dg| f) := \int_{G_0} |dg| s(g)^N |(L_u R_v f)(g)|.$$

Clearly, the locally convex super-vector space $\mathcal{S}(G, [s])$ is independent of the choice of $|Dg|$, $|dg|$, and the representative $s$ of the scale structure $[s]$.

Similarly, there is a space of Schwartz densities $\mathcal{S}(G_0, [s]) \subseteq |\Omega|(G_0)$. According to Ref. [8], it is defined as the space of smooth vectors for the bi-regular representation $L_0 \times R_0$ of $G_0 \times G_0$ on the space $\mathcal{R}(G_0, [s])$, the set of continuous densities $\omega$ that are rapidly decreasing in the sense that

$$\forall N \in \mathbb{N} : \int_{G_0} |\omega| s^N < \infty.$$

We have the following description of $\mathcal{S}(G, [s])$ in terms of $\mathcal{S}(G_0, [s])$.

Proposition 3.2. The isomorphism from Corollary 2.9 induces an isomorphism

$$\mathcal{S}(G, [s]) \cong \Omega(\mathfrak{g}) \otimes_{\Omega(\mathfrak{g}_0)} \mathcal{S}(G_0, [s]).$$

In particular, $\mathcal{S}(G, [s])$ is nuclear space and $G$-invariant for the left regular representation $L$, as a subspace of $|\Omega|(G)$. 

Proof. By the above definitions, we have $|Dg| f \in \mathcal{F}(G, [s])$ if and only if for any $u,v \in \mathfrak{U}(g)$, we have
\[ \omega := |dg| j^2(L_u R_v f) \in \mathcal{R}(G_0, [s]), \]
where we abbreviate $j := j_{G_0}$. Such a density $\omega$ is smooth, and for $x \in g_0$, we have
\[ L_x \omega = |dg| j^2(L_{xu} R_v f) \in \mathcal{R}(G_0, [s]). \]
One argues similarly for $R_x$, so that $\omega \in \mathcal{F}(G_0, [s])$.

Now, let $|Dg| f \in \mathcal{O}([G])$. We may assume w.l.o.g. that $|Dg|$ and $|dg|$ are related by Equation (2.6). Then Equation (2.9) implies that $|Dg| f$ corresponds to
\[ \sum_i \gamma_i' \otimes |dg| j^2(L_{S(\gamma_i')}(f)) \]
where $\Delta(\gamma) = \sum_i \gamma_i' \otimes \gamma_i''$. For any $u \in \mathfrak{U}(g)$, we therefore have
\[ |dg| \Delta_1 j^2(L_u(f)) \in \mathcal{F}(G_0, [s]), \]
since $s$ dominates the $g$-adjoint scale by assumption.

Conversely, let the Berezinian density $|Dg| f$ correspond to $u \otimes \omega$, where we assume $\omega = |dg|h \in \mathcal{F}(G_0, [s])$ and $u \in \mathfrak{U}(g)$. By Corollary 2.9, we have $f = L_u(\psi h)$, with $\psi$ defined in Equation (2.10).

For $v,w \in \mathfrak{U}(g)$, $g \in G_0$, we expand
\[ (-1)^{|w||u|} \Delta(vu) = \sum_i \nu_i' \otimes \nu_i'', \quad \Delta(w) = \sum_j \nu_j' \otimes \nu_j''. \]
Then we compute for $\chi_{ij} := (-1)^{|\psi||\nu_i'|+|\nu_i''|+|\nu_j'|+|\nu_j''|}$ that
\[ j^2(L_v R_w(\psi f))(g) = \sum_{i,j} \chi_{ij}(L_{\nu_i'} R_{\nu_j'}(\psi))(g)(L_{\nu_j'} R_{\nu_i'}(h))(g) \]
with
\[ (L_{\nu_i'} R_{\nu_j'}(\psi))(g) = (R_i(\text{Ad}(g^{-1})(S(\nu_j'))\Delta_i^{-1})(g) \]
\[ = \delta_1(S(\text{Ad}(g^{-1})(S(\nu_j'))\Delta_i^{-1}))(g)^{-1}. \]
We have
\[ \iota(xay) = \iota(a)(y - \delta_1(y)) \]
for all $x,y \in g_0$ and $a \in \mathfrak{U}(g)$ [28, Equation (3)]. Moreover, $\delta_1$ is a character of $\mathfrak{U}(g_0)$ and in particular $\text{Ad}(G_0)$-invariant. Finally, there is a constant $C > 0$ such that
\[ |\iota(\beta(\text{Ad}(g)(\xi)\eta))| = \int_{g_1} \text{Ad}(g)(\xi)\eta \leq C\|\text{Ad}(g)\|_{g_1}\|\xi\|\|\eta\| \]
for all $\xi \in \Lambda^k g_1$, $\eta \in \Lambda g_1$. (Here, $\|\|$ denotes some submultiplicative norm on $\Lambda g_1$.) It follows that there exist a constant $C > 0$ and an integer $N > 0$ such that
\[ |\delta_1(S(\iota(\text{Ad}(g^{-1})(S(\nu_j'))\nu_j')))| \leq C s_0(g)^N. \]
for all $g \in G_0$. The sum in Equation (3.2) is finite, so we may conclude that $|dg| j^2(L_u R_w(f))$ is a Schwartz density on $(G_0, [s])$ once so is $|dg|(L_{\nu_i'} R_{\nu_j'}(h))$.

To that end, similarly as above, we note that
\[ j^2(L_{xa} R_{yb}(h)) = L_{x} R_{y} j^2(L_{a} R_{b} h) \]
for all $x,y \in g_0$, $a,b \in \mathfrak{U}(g)$, that
\[ (R_{\beta(\text{Ad}(g)(\xi)\eta)} h)(g) = \epsilon(\beta(\text{Ad}(g)(\xi)\eta)), \]
and that there is a constant $C > 0$ such that
\[ |\epsilon(\beta(\text{Ad}(g)(\xi)\eta))| \leq C\|\text{Ad}(g)\|_{g_1}\|\xi\|\|\eta\|. \]
Thus, there is an integer $N > 0$ such that for all $g \in G_0$, we have
\[ |(L_{\nu_i'} R_{\nu_j'}(h))(g)| \leq s_0(g)^N |H(g)| \]
where $H = \sum_{\ell} (L_{a_{\ell}} R_{b_{\ell}} h)$ for some $a_{\ell}, b_{\ell} \in \mathfrak{u}(g_0)$.

In summary, we have shown the isomorphism in Equation (3.1), in particular, $\mathcal{S}(G,[s])$ is a $G$-invariant subspace of $\mathfrak{U}(\mathfrak{g})$. Inspecting the above formulae, it is evident that it is an isomorphism of topological vector spaces, if $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} \mathcal{S}(G,[s])$ is endowed with the natural topology on $\bigwedge \mathfrak{g}_1 \otimes \mathcal{S}(G,[s])$. The nuclearity now follows from that of $\mathcal{S}(G,[s])$ [8, Corollary 5.6].

**Proposition 3.3.** The subspace $\mathcal{S}(G,[s]) \subseteq \mathcal{D}(G)$ is bi-invariant under the regular representation of $G$. Via the isomorphism in Equation (3.1), it inherits a non-unital Fréchet superalgebra structure with continuous multiplication, determined uniquely by the following facts:

(i) The following is a non-unital graded subalgebra bi-invariant under $G$:

\[ \mathcal{S}(G,[s]) = \mathfrak{U}(\mathfrak{g}_0) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} \mathcal{S}(G,[s]). \]

(ii) For $u, v \in \mathfrak{U}(\mathfrak{g})$ and $\omega \in \mathcal{S}(G,[s])$, we have

\[ u \ast (v \otimes \omega) = (u \otimes 1) \ast (v \otimes \omega) = uv \otimes \omega. \]

(iii) For $\omega \in \mathcal{S}(G,[s])$ and $u \in \mathfrak{U}(\mathfrak{g})$, the product $u \ast \omega$ is given by

\[ \int_{G_0} (u \ast \omega) \varphi = \int_{G_0} \omega^* \int_{G_0} \omega (L_S(\omega) \varphi) = \langle u \otimes \omega, \varphi \rangle, \]

for all compactly supported superfunctions $\varphi \in \Gamma_c(\mathcal{O}_G)$.

**Proof.** We already know that $\mathcal{S}(G,[s])$ is invariant under $L_u$ and $L_g$ for any $u \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$. To see that $\mathcal{S}(G,[s])$ is also invariant under the right regular action $R$, it will be sufficient to show that $\mathcal{S}(G,[s])$ is stable under $(-)^\gamma$, defined in Equation (2.12).

Choose a basis $x_1, \ldots, x_q$ of $\mathfrak{g}_1$, and let $x^1, \ldots, x^q$ be the dual basis of $\mathfrak{g}_1^\ast$. We write $x_I := x_{i_1} \cdots x_{i_k} \in \bigwedge \mathfrak{g}_1$ and $x'^I := x^{i_1} \cdots x^{i_k} \in \bigwedge \mathfrak{g}_1^\ast$ for $I = (1 \leq i_1 < \cdots < i_k \leq q)$. Then we compute by Equation (2.2)

\[ \langle (\beta(x_I) \otimes \omega)^\gamma, \varphi \rangle = (-1)^{|I||\gamma|} \int_{G_0} \omega(g) \varphi(S(\beta(x_I)); g^{-1}) \]

\[ = \sum_J \int_{G_0} (-1)^{|J||\gamma|} \omega(g) \langle x_J', \text{Ad}(g)(x_I) \rangle \varphi(\text{Ad}(g^{-1}))(S(x_J)); g \rangle \]

\[ = \langle \sum_J \beta(x_I) \otimes \omega(x_J', \text{Ad}(\cdot)(x_I)), \varphi \rangle \]

for all $\varphi \in \Gamma_c(\mathcal{O}_G)$. Here, observe that $|J| = |I|$, because the adjoint action by $G_0$ on $\bigwedge \mathfrak{g}_1$ respects the $\mathbb{Z}$-grading.

By the assumption on the scale $s$, we have $\omega \in \mathcal{S}(G_0,[s])$ and thus

\[ |dg| \sum_J \omega(x_J', \text{Ad}(\cdot)(x_I)) \in \mathcal{S}(G_0,[s]). \]

In view of Proposition 3.2, this shows that $(\beta(x_I) \otimes \omega)^\gamma \in \mathcal{S}(G,[s])$. Therefore, $\mathcal{S}(G,[s])$ is invariant under $(-)^\gamma$ and bi-invariant under $G$.

In follows that there is a well-defined operation $\ast$ on $\mathcal{S}(G,[s])$, defined by

\[ (u \otimes \omega) \ast (v \otimes \varphi) := \sum_J uv_J \otimes (\omega_j \ast \varphi), \]

for arbitrary $u, v \in \mathfrak{U}(\mathfrak{g})$ and $\omega, \varphi$, where we decompose $R_{S(\omega)}\omega = \sum_j v_j \otimes \omega_j$. 
If $\omega, \pi$ are compactly supported, then by Equation (2.2), we have

$$
\langle (u \otimes \omega) * (v \otimes \pi), \varphi \rangle = \sum_j (-1)^{|uv_j||\omega|} \int_{G_0} (\omega_j * \pi)(g) \varphi(\Ad(g^{-1}) (uv_j); g) \\
= \sum_j (-1)^{|uv_j||\omega|} \int_{G_0 \times G_0} \omega_j(g) \pi(h)(\Ad((gh)^{-1})(uv_j); gh) \\
= \sum_j (-1)^{|uv_j||\omega|} \int_{G_0 \times G_0} \omega_j(g) \pi(h) m^2(\varphi)(\Ad(g^{-1})(uv_j) \otimes 1; g, h) \\
= \langle \sum_j (uv_j \otimes \omega_j) \otimes \pi, m^2(\varphi) \rangle \\
= \langle L_u R_{S(v)} \omega \otimes \pi, m^2(\varphi) \rangle \\
= (-1)^{|uv||\omega|} \int_{G_0 \times G_0} \omega(g) \pi(h) m^2(\varphi)(\Ad(g^{-1})(u) \otimes 1; g, h) \\
= (-1)^{|uv||\omega|} \int_{G_0 \times G_0} \omega(g) \pi(h) m^2(\varphi)(\Ad(g^{-1})(u \otimes \Ad(h^{-1})(v); g, h) \\
= \langle (u \otimes \omega) \otimes (v \otimes \pi), m^2(\varphi) \rangle,
$$

so that $\ast$ on $\mathcal{F}(G, [s])$ extends the convolution on $[\Omega]_{c}(G)$. By Proposition 3.2, $[\Omega]_{c}(G) \subseteq \mathcal{F}(G, [s])$ is a dense subspace and $\mathcal{F}(G, [s])$ is nuclear. To finish the proof of our assertion, it is by [54, Theorem 50.1] sufficient to show that the convolution $\ast$ on $[\Omega]_{c}(G)$ is separately continuous in the topology induced from $\mathcal{F}(G, [s])$.

Since $(-)^\ast$ is continuous on $\mathcal{F}(G, [s])$, it will be sufficient to show continuity in the second argument. In view of Corollary 2.12 i–iii, we have the identity

$$
L_u R_v [\omega \ast \pi] = (-1)^{|v||\omega|} (L_u \omega) \ast (R_v \pi)
$$

for all $u, v \in \mathfrak{U}(g)$ and $\omega, \pi \in [\Omega]_{c}(G)$. Together with the fact that for any $v \in \mathfrak{U}(g)$, $R_v$ is continuous on $\mathcal{F}(G, [s])$, it follows that it is sufficient to show that

$$
\pi \mapsto p_{1,1,N}((L_u R_{S(v)} \omega) \ast \pi) : \mathcal{F}(G_0, [s]) \to \mathbb{R}
$$

is a continuous seminorm for any integer $N \geq 0$. But this follows from the continuity of convolution on $\mathcal{F}(G_0, [s])$ [55, Theorem 7.1].

### 3.2. SF-representations of supergroups.

Fix a scale $s$ on $G_0$ dominating the $g$-adjoint scale. Recall [8, Definition 2.6, Lemma 2.10] that a continuous Fréchet $G$-representation $\pi_0$ on $E$ is called an $F$-representation or a Fréchet representation of moderate growth of $(G_0, [s])$ if the topology of $E$ is generated by a countable collection $(p_j)$ of seminorms such that for any $j$, there exist an index $k$, a constant $C > 0$, and an integer $N \geq 0$ with

$$
p_j(\pi_0(g)v) \leq C s(g)^N p_k(v)
$$

for all $v \in E$ and $g \in G_0$. It is called an SF-representation or smooth if it is in addition weakly smooth.

In view of this terminology, we make the following definition.

**Definition 3.4 (SF-representations).** Let $\pi$ be a continuous representation $G$ on a Fréchet super-vector space $E$. Then $\pi$ is called an $F$-representation of $(G_0, [s])$ if the topology of $E$ is generated by a countable collection $(p_j)$ of seminorms such that for some norm $|| \cdot ||$ on $\bigwedge g_1$ and for any index $j$, there is an index $k$, a constant $C > 0$, and an integer $N \geq 0$ with

$$
p_j(\pi(g)v) \leq C ||\eta|| s(g)^N p_k(v)
$$

for all $v \in E_\infty$, $g \in G_0$, and $\eta \in \bigwedge g_1$. If in addition, $\pi$ is a weakly smooth $G$-representation, then it is called an SF-representation of $(G_0, [s])$. 

Remark 3.5. If $\pi$ is an $F$-representation (resp. an $SF$-representation) of $(G, [s])$, then $\pi_0$ is an $F$-representation (resp. an $SF$-representation) of $(G_0, [s])$. Indeed, $E_\infty$ is dense in $E$, and taking $\eta = 1$ in Equation (3.6), we obtain Equation (3.5). Also by definition, if $\pi$ is an $F$-representation of $(G, [s])$ on $E$, then the subrepresentation on the space $E_\infty$ of smooth vectors is an $SF$-representation [8, Corollary 2.16].

In particular, using Ref. [8, (2.2)], Proposition 3.2, and Proposition 3.3, we find that the left and right regular representations $L$ and $R$ on $\mathcal{F}(G, [s])$ are $SF$-representations of $(G, [s])$.

In fact, the $F$-representations are characterised among the continuous representations of $G$ by the growth of the underlying $G_0$-representation.

Lemma 3.6. Let $\pi$ be a continuous (resp. weakly smooth) representation of $G$ on a Fréchet super vector-space $E$. Then $\pi$ is an $F$-representation (resp. an $SF$-representation) of $(G, [s])$ if and only if $\pi_0$ is an $F$-representation (resp. an $SF$-representation) of $(G_0, [s])$.

Proof. It is sufficient to consider the case of $F$-representations. As noted above, if $\pi$ is an $F$-representation of $(G, [s])$, then $\pi_0$ is an $F$-representation of $(G_0, [s])$. Conversely, assume that $\pi_0$ is an $F$-representation of $(G_0, [s])$. Since $s$ dominates the $\theta$-adjoint scale, the adjoint representation of $G_0$ on $\bigwedge \mathfrak{g}_\theta$ is an $F$-representation. Hence, so is $\bigwedge \mathfrak{g}_\theta \otimes E$. Manifestly, this gives the condition in Equation (3.6). □

Remark 3.7. From Lemma 3.6, we obtain the following: Let $\pi$ be a continuous $G$-representation on a Banach super vector-space $E$. Then $\pi$ is an $F$-representation of $(G, [s])$ if and only if $\pi_0$ is $s$-bounded in the sense that $s \geq s_{\pi_0}$ where

$$s_{\pi_0}(g) := \max(\|\pi_0(g)\|, \|\pi_0(g^{-1})\|).$$

In particular, in this case, the $G$-representation on $E_\infty$ is an $SF$-representation [8, Corollary 2.16].

For $F$-representations of $G$, we obtain the following variant of the Dixmier–Malliavin theorem, generalising Ref. [8, Proposition 2.20]. Compare Ref. [24, Example 2.3.3].

Proposition 3.8. Let $E$ be a Fréchet super-vector space over $\mathbb{K}$. Then we have the following facts:

(i) If $E$ carries the structure of an $F$-representation $\pi$ of $(G, [s])$, then the integrated action $\Pi$ extends continuously to an action of $\mathcal{F}(G, [s])$, also called the integrated action of $\pi$. We have the equality

$$E_\infty = \Pi(\mathcal{F}(G, [s]))E = \Pi(\mathcal{F}(G, [s]))E_\infty.$$

(ii) Conversely, let $\mathcal{F}(G, [s])$ act continuously and non-degenerately via $\Pi$ on $E$. Then $\Pi$ is integrated from a unique $SF$-representation of $(G, [s])$.

In particular, we obtain an equivalence of the category of $SF$-representations of $(G, [s])$ with the category of non-degenerate continuous Fréchet $\mathcal{F}(G, [s])$-modules.

Proof. If $E$ is an $F$-representation of $(G_0, [s])$, then $\mathcal{F}(G_0, [s])$ acts continuously on $E$, and $E_\infty = \mathcal{F}(G_0, [s])E = \mathcal{F}(G_0, [s])E_\infty$ [8, Proposition 2.20]. Conversely, if $E$ carries a continuous non-degenerate action of $\mathcal{F}(G_0, [s])$, then this action is integrated from a unique $SF$-representation of $(G_0, [s])$ (loc. cit.). Using these facts, together with Proposition 3.2 and Proposition 3.3, the proof of the claim is the same as that of Proposition 2.15. We therefore leave the details to the reader. □

4. Harish-Chandra Supermodules

In this section, we come to our main result, a generalisation of the Casselman–Wallach theorem to supergroups.
4.1. Basic facts and definitions. In what follows, we assume that the underlying Lie group $G_0$ of $G$ is almost connected and real reductive [55] and let $K_0 \subseteq G_0$ be a maximal compact subgroup. We fix on $G_0$ the maximal scale structure [8, 2.1.1] and omit the mention of $[s]$ in our notation. In particular, any Banach representation of $G$ is an $F$-representation.

**Definition 4.1** (Harish-Chandra supermodules). A $(\mathfrak{g}, K_0)$-module is by definition a complex, $\mathbb{Z}/2\mathbb{Z}$ graded, locally finite $K_0$-representation $V$, endowed with a $K_0$-equivariant $\mathfrak{g}$-module structure, which extends the derived $\mathfrak{t}_0$-action on $V$. A morphism of $(\mathfrak{g}, K_0)$-modules $\phi : U \to V$ is an even $\mathbb{C}$-linear map that is equivariant for the actions of $\mathfrak{g}$ and $K_0$.

A $(\mathfrak{g}, K_0)$-module is called Harish-Chandra or a Harish-Chandra supermodule if it is $K_0$- multiplicity finite and finitely generated over $\mathfrak{U}(\mathfrak{g})$. The full subcategory of the category of $(\mathfrak{g}, K_0)$-modules whose objects are the Harish-Chandra supermodules is denoted by $\text{HC}(\mathfrak{g}, K_0)$.

The following observation is elementary, but effective.

**Lemma 4.2.** Let $V$ be a $(\mathfrak{g}, K_0)$-module. Then $V \in \text{HC}(\mathfrak{g}, K_0)$ if and only if its restriction $V|_{(\mathfrak{g}_0, K_0)}$ to a $(\mathfrak{g}_0, K_0)$-module lies in $\text{HC}(\mathfrak{g}_0, K_0)$.

**Proof.** We need only observe that $\mathfrak{U}(\mathfrak{g})$ is finitely generated as a $\mathfrak{U}(\mathfrak{g}_0)$-module. □

**Lemma 4.3.** Let $E$ be an $F$-representation of $G$ (for instance, a Banach representation). Then the space $E^{(K_0)}$ of $K_0$-finite and smooth vectors is a $(\mathfrak{g}, K_0)$-module.

**Proof.** Since the action of $K_0$ on $\mathfrak{U}(\mathfrak{g})$ is locally finite, we see that the $\mathfrak{g}$-action on $E^{(K_0)}$ is invariant. □

**Remark 4.4.** Let $\pi_0$ be a continuous $G$-representation on a complex Banach super-vector space $E$. Denoting by $C$ the Casimir element of $\mathfrak{g}_0$, assume that either

(i) $d\pi_0(C) \in \text{End}(E^{(K_0)})$ extends continuously to $E$, or

(ii) $P(d\pi_0(C)) = 0$ on $E^{(K_0)}$ for some polynomial $P$.

Then it is known that the space $E^{(K_0)}$ of $K_0$-finite vectors is contained in $E^{(K_0)}$ [8, Corollary 3.10]. Hence, if $\pi_0$ is the $G_0$ part of a continuous $G$-representation, then $E^{(K_0)}$ is a $(\mathfrak{g}, K_0)$-module, by Lemma 4.3.

4.2. Globalisation of Harish-Chandra supermodules.

**Definition 4.5** (Casselman–Wallach representations). An SF-representation $(E, \pi)$ of $G$ is called Casselman–Wallach or a CW representation if the space $E^{(K_0)}$ of $K_0$-finite and smooth vectors is in $\text{HC}(\mathfrak{g}, K_0)$.

If $V \in \text{HC}(\mathfrak{g}, K_0)$, then an isomorphism $\phi : V \to E^{(K_0)}$ of $(\mathfrak{g}, K_0)$-modules, where $(E, \pi)$ is an SF-representation, is called an $SF$-globalisation of $V$. Any $SF$-globalisation of a Harish-Chandra supermodule is a CW representation of $G$.

A CW globalisation $\phi : V \to E$ is called minimal if for any CW globalisation $\psi : V \to H$, there exists an even continuous $G$-equivariant map $\hat{\psi} : E \to H$ such that $\psi \circ \varphi = \varphi$. Since the $K_0$-finite vectors are dense in $E$, such a map $\hat{\psi}$ is unique. Thus, minimal globalisations (if they exist) are unique up to canonical isomorphism.

Dually, a CW globalisations $\phi : V \to E$ is called maximal if for any CW globalisation $\psi : V \to H$, there exists an even continuous $G$-equivariant map $\hat{\psi} : H \to E$ such that $\psi \circ \varphi = \varphi$. Again, maximal globalisations (if they exist) are unique up to canonical isomorphism.

We are now ready to state our main theorem.

**Theorem 4.6** (Super Casselman–Wallach theorem). Let $V \in \text{HC}(\mathfrak{g}, K_0)$. Up to isomorphism, there is a unique CW globalisation of $V$. 

We postpone the proof to Subsection 4.3 and give a number of corollaries. The derivation of these follows the same procedures as in the Lie group case [56].

**Corollary 4.7.** The functor mapping \((E, \pi)\) to \(E^{(K_0)}_\infty\) sets up an additive equivalence between the category \(\text{CW}(G)\) of CW representations of \(G\) and the category \(\text{HC}(\mathfrak{g}, K_0)\) of Harish-Chandra supermodules. In particular, the category \(\text{CW}(G)\) is Abelian.

**Corollary 4.8.** Let \(f : E \to F\) be a morphism of \(CW\ G\)-representations. Then \(f\) is a topological morphism with closed image.

Here, \(f : E \to F\) is called a topological morphism if the induced map

\[
E/\ker f \to \operatorname{im} f
\]

is an isomorphism of topological vector spaces.

As a corollary to the proof of Theorem 4.6, we obtain the following.

**Corollary 4.9.** Any \(E \in \text{CW}(G)\) is the space of smooth vectors of a continuous Hilbert \(G\)-representation.

### 4.3. Proof of Theorem 4.6.

Having stated our main result, together with some immediate corollaries, let us come to its proof.

**Proof of Theorem 4.6.** Firstly, we show that \(V\) has a minimal SF-globalisation \(V \subseteq V_1\). We mimic the construction detailed in Ref. [8, §6].

By Lemma 4.2, we have \(U := V|_{(\mathfrak{g}_0, K_0)} \in \text{HC}(\mathfrak{g}_0, K_0)\). Thus, there is a finite set \(v_1, \ldots, v_n\) of homogeneous vectors generating the \(\mathfrak{u}(\mathfrak{g}_0)\)-module \(V\) and a continuous Hilbert representation \((E, \pi_0)\) of \(G_0\) such that \(E^{(K_0)}_\infty = U\) [8, §5.1].

Since \(\mathcal{J}(G)\) is invariant under \((-)^\vee\), Proposition 3.2 shows that the map

\[
\mathcal{J}(G_0) \otimes \mathfrak{u}(\mathfrak{g}) \to \mathcal{J}(G) : \omega \otimes u \mapsto R_{S(u)}(\omega)
\]

is an isomorphism of right \(\mathfrak{u}(\mathfrak{g})\)-modules. Here, \(\mathfrak{u}(\mathfrak{g})\) acts from the right on \(\mathcal{J}(G)\) by \(\omega u := (-1)^{|\omega||u|} R_{S(u)}(\omega)\). We define, for \(\omega \in \mathcal{J}(G_0)\) and \(v \in V\)

\[
\Pi(\omega)v := \sum_i \Pi_0(\omega_i)u_i v,
\]

where

\[
\omega = \sum_i R_{S(u_i)}(\omega_i)
\]

is any decomposition with \(\omega_i \in \mathcal{J}(G_0)\) and \(u_i \in \mathfrak{u}(\mathfrak{g})\). To see that this is well-defined, we need only remark that

\[
\Pi_0(R_{-x}\omega) = \Pi_0(\omega)d\pi_0(x)
\]

for all \(\omega \in \mathcal{J}(G_0)\) and \(x \in \mathfrak{g}_0\).

Now, consider the graded subspace \(\mathcal{N} \subseteq \mathcal{J}(G)^n\), defined by

\[
\mathcal{N} := \{ (\omega_1, \ldots, \omega_n) \in \mathcal{J}(G)^n \mid \sum_j \Pi(\omega_j)v_j = 0 \}.
\]

We claim that it is closed and invariant under the action of \(\mathcal{J}(G)\) by left convolution. To prepare the proof of this claim, we briefly suspend our argument and establish some ancillary lemmas.

Let \(\tilde{V} \in \text{HC}(\mathfrak{g}, K_0)\) be the dual Harish-Chandra module of \(V\), defined as the set of \(K_0\)-finite vectors in the algebraic dual \(V^*\). Then \(\tilde{V}\) is also the dual of \(V|_{(\mathfrak{g}_0, K_0)}\) [8, §4], and in particular \(\tilde{V} \subseteq E^*\) [8, Lemma 5.3].

**Lemma 4.10.** Let \(v \in V\), \(u \in \mathfrak{u}(\mathfrak{g})\) and \(g \in G_0\). We have the identity

\[
\langle \xi, \pi_0(g^{-1}) \text{Ad}(g)(u)v \rangle = (-1)^{|u||\xi|} \langle S(u)\xi, \pi_0(g^{-1})v \rangle.
\]
Thus, we have

$$\exp$$ is a local diffeomorphism and $$G$$

The equality is obvious for $$g \in K_0$$. Since $$G_0 K_0 = G_0$$, where $$G_0$$ is the connected component of the identity of $$G_0$$, we may assume that $$G_0$$ is connected.

To prove the assertion in that case, assume first that $$u \in \beta(\wedge g_1)$$. The image $$F$$ of $$\beta(\wedge g_1)$$ in $$\text{End}(V)$$ is finite-dimensional, so the linear map

$$F \to V \subseteq E : u \mapsto uv$$

is continuous. For $$x \in g_0$$, we may hence exchange limits and compute

$$\frac{d}{dt} \bigg|_{t=0} \text{Ad}(\exp(tx))(u)v = [x,u]v = d\pi_0(x)uv - uxv.$$ 

Thus, we have

$$\frac{d}{dt} \bigg|_{t=0} \pi_0(\exp(-tx)) \text{Ad}(\exp(tx))(u)v = -d\pi_0(x)uv + [x,u]v = -uxv,$$

by the smoothness of the $$G_0$$-representation $$E_\infty$$. Hence

$$\frac{d}{dt} \bigg|_{t=0} \langle \xi, \pi_0(e^{-tx}) \text{Ad}(e^{tx})(u)v \rangle = (-1)^{|\xi||u|} \langle S(u)\xi, d\pi_0(x)v \rangle$$

$$= (-1)^{|\xi||u|} \frac{d}{dt} \bigg|_{t=0} \langle S(u)\xi, \pi_0(e^{-tx})v \rangle.$$ 

By the uniqueness of initial value problems, the equality follows for $$g = e^x$$. Since $$\exp$$ is a local diffeomorphism and $$G_0$$, being connected, is generated by a neighbourhood of the identity, the equality holds for arbitrary $$g \in G_0$$.

To remove the restriction on $$u$$, recall that $$\U(\mathfrak{g}) = \U(\mathfrak{g}_0)\beta(\wedge g_1)$$. By linearity in $$u$$, it is sufficient to consider $$u = u^u$$ for $$u^u \in \U(\mathfrak{g}_0)$$ and $$u^v \in \beta(\wedge g_1)$$. Then

$$\langle \xi, \pi_0(g^{-1}) \text{Ad}(g)(u)v \rangle = \langle \xi, \pi_0(g^{-1}) \text{Ad}(g)(u^u)v \rangle$$

$$= \langle S(u^u)\xi, \pi_0(g^{-1}) \text{Ad}(g)(u^v)v \rangle$$

$$= (-1)^{|\xi||u|} \langle S(u^u)S(u^v)\xi, \pi_0(g^{-1})v \rangle$$

$$= (-1)^{|\xi||u|} \langle S(u)\xi, \pi_0(g^{-1})v \rangle.$$ 

This proves the claim in general.

$$\square$$

**Lemma 4.11.** For $$u \in \U(\mathfrak{g})$$, $$\omega \in \mathcal{S}(G)$$, $$v \in V$$, and $$\xi \in \tilde{V}$$, we have

$$\langle \xi, \Pi(L_u(\omega))v \rangle = (-1)^{|\xi||v|} \langle S(u)\xi, \Pi(\omega)v \rangle.$$ 

**Proof.** For $$v \in V$$ and $$\xi \in \tilde{V}$$, we define $$M_{\xi,v} \in \Gamma(\mathcal{O}_G)$$ by

$$M_{\xi,v}(u; g) := (-1)^{|\xi||v|} \langle \xi, \pi_0(g)v \rangle.$$ 

Clearly, this is well-defined.

For $$u \in \U(\mathfrak{g})$$ and $$\omega \in \mathcal{S}(G_0)$$, we compute

$$\langle \xi, \Pi(R_{S(u)}(\omega))v \rangle = (-1)^{|\xi||v|} \langle \xi, \Pi_0(\omega)uv \rangle = (-1)^{|\xi||v|} \int_{G_0} \omega(g) \langle \xi, \pi_0(g)uv \rangle$$

$$= (-1)^{|\xi||v|} \langle R_{S(u)}(\omega), M_{\xi,v} \rangle.$$ 

By Equation (4.1), it follows that

$$\langle \xi, \Pi(\omega)v \rangle = (-1)^{|\xi||v|} \langle \omega, M_{\xi,v} \rangle$$

for any $$\omega \in \mathcal{S}(G)$$. In particular, if $$u \in \U(\mathfrak{g})$$, we have

$$\langle \xi, \Pi(L_u(\omega))v \rangle = (-1)^{|\xi||u|+|\omega|+|u||v|+|\xi||w|} \langle \omega, L_{S(u)}(M_{\xi,v}) \rangle$$

$$= (-1)^{|\xi||u|+|\omega|+|u||w|} \langle \omega, M_{S(u)\xi,v} \rangle = (-1)^{|\xi||u|} \langle S(u)\xi, \Pi(\omega)v \rangle.$$
since
\[ L_u(M_{\xi,v})(u'; g) = (-1)^{|u||\xi|+|u'||v|}\langle \xi, \pi_0(g) \Ad(g^{-1})(S(u))u'v \rangle \]
\[ = (-1)^{|u'||v|}\langle uv, \pi_0(g)u'v \rangle = M_{u\xi,v}(u'; g), \]
by Lemma 4.10. This proves the assertion. \[ \square \]

We now again take up the proof of our main theorem.

**Proof of Theorem 4.6 (continued).** For \( v' \in E \), we have
\[ v' = 0 \iff \forall \xi \in \tilde{V} : \langle \xi, v' \rangle = 0. \]
Hence, by Lemma 4.11, the subspace \( \mathcal{N} \) is invariant under \( L^n \), where \( L \) is the regular \( G \)-representation. That it is invariant under left convolution by \( \mathcal{S}(G) \) now follows from the identity
\[ R_{S(u)}(\omega) * \varpi = \omega * (L_u(\varpi)) \]
valid for \( u \in \mathfrak{U}(\mathfrak{g}), \omega \in \mathcal{S}(G_0) \), and \( \varpi \in \mathcal{S}(G) \), together with Equation (4.1).

Since \( \mathcal{S}(G) \cong \mathcal{S}(G_0) \otimes \bigwedge \mathfrak{g}_1 \) is the locally convex direct sum of finitely many copies of \( \mathcal{S}(G_0) \), it follows directly from the definition in Equation (4.2) that
\[ \phi : \mathcal{S}(G)^n \to E : (\omega_1, \ldots, \omega_n) \mapsto \sum_j \Pi(\omega_j) v_j \]
is continuous, so that \( \mathcal{N} \) is also closed, as claimed.

Hence, if we define
\[ V_+ := \mathcal{S}(G)^n / \mathcal{N}, \]
then this is a continuous non-degenerate Fréchet \( \mathcal{S}(G) \)-module. By Proposition 3.8, the \( \mathcal{S}(G) \)-action is integrated from a unique \( SF \)-representation \( \pi \) of \( G \).

The map induced by \( \phi \) identifies \( V_+ \) (as a super-vector space) with the subspace
\[ U_+ = \Pi(\mathcal{S}(G)) V = \Pi(\mathcal{S}(G_0)) V \]
of \( E \). By construction [8, § 6], \( U_+ \) is, with the quotient topology defined by the natural map \( \mathcal{S}(G_0)^n \to U_+ \) induced by \( \phi \), the minimal globalisation of the module \( U \in \mathcal{H}C(\mathfrak{g}_0, K_0) \). But by the Casselman–Wallach theorem [8, Theorem 10.6], it holds that \( U_+ = E_\infty \) as locally convex spaces.

Since \( \mathfrak{U}(\mathfrak{g}) \) is \( \Ad(K_0) \)-locally finite, the space of \( K_0 \times K_0 \)-finite vectors is
\[ \mathcal{S}(G)^{(K_0 \times K_0)} = \mathcal{S}(G_0)^{(K_0 \times K_0)} \otimes_{\mathfrak{U}(\mathfrak{g}_0)} \mathfrak{U}(\mathfrak{g}). \]
From this, it is easy to deduce that \( V_+ \) is an \( SF \)-globalisation of \( V \). In particular, \( (V_+)_{|G_0} \) is an \( SF \)-globalisation of \( U \). From the Casselman–Wallach theorem [8, Theorem 10.6] again, it follows that the map \( V_+ \to U_+ \) induced by \( \phi \) is an isomorphism of locally convex vector spaces. In particular, \( V_+ \) is the space of smooth vectors of a continuous Hilbert \( G \)-representation.

Now, let \( F \) be any \( SF \)-globalisation of \( V \), so that we are given an isomorphism \( \psi : V \to F^{(K_0)} \) of \( (\mathfrak{g}, K_0) \)-modules. Invoking the Casselman–Wallach theorem (loc. cit.), there is a unique isomorphism \( \tilde{\psi} : V_+ \to F \) of \( SF \)-representations of \( G_0 \) extending \( \psi \). For any \( u \in \mathfrak{U}(\mathfrak{g}) \), the action by \( u \) on \( V_+ \) and \( F \) is continuous. Hence, by the density of \( V \) in \( V_+ \), it follows that \( \tilde{\psi} \) is \( \mathfrak{g} \)-equivariant. This shows that \( V_+ \) is a minimal \( SF \)-globalisation. The same argument shows that it is maximal, and hence follows the claim. \[ \square \]
5. Application: Gel’fand–Kazhdan representations

In this section, we show, by way of application of our results in Section 4, that the Gel’fand–Kazhdan criterion for multiplicity freeness carries over to the case of Lie supergroups. Therein, we build on the work of Sun–Zhu [51] who have shown how to present this within the framework of Lie group Casselman–Wallach theory. Antecedents are the classical results of Gel’fand–Kazhdan [29] and Shalika [50], as well as theorems of Kostant [33], Yamashita [58], and Prasad [47].

We retain our assumptions on the Lie supergroup $G$ from Section 4.

**Definition 5.1** (Contragredient pairs). A pair $(E, F)$ of continuous $G$-representations is called *contragredient* if there exists a $G_0$-invariant continuous bilinear map

$$\langle \cdot, \cdot \rangle : E \times F \to \mathbb{K}$$

that is a perfect pairing whose restriction to $E_\infty \times F_\infty$ is $G$-invariant.

Here, by a *perfect pairing* we mean that the canonical maps

$$E \to F', \quad F \to E'$$

are isomorphisms of topological vector spaces.

**Remark 5.2.** Assume $(E_\infty, F_\infty)$ is a pair of $SF$-representations of $G$ and

$$\langle \cdot, \cdot \rangle : E_\infty \times F_\infty \to \mathbb{K}$$

is a non-degenerate continuous bilinear form that is $G$-invariant. If $U$ is a Hilbert globalisation of $E_\infty$ (which exists if $E_\infty$ is CW), then the space of $G_0$-smooth vectors in $F := E'$ coincides with $F_\infty$. Thus, $(E, F)$ a contragredient pair with underlying $SF$-representations $E_\infty$ and $F_\infty$.

As we shall presently see, contragredient pairs of representations allow for an abstract matrix coefficient map. To state this precisely, we introduce the following definition.

**Definition 5.3** (Tempered superfunctions). A superfunction $f \in \Gamma(O_G)$ is called *tempered* if for all $u, v \in \mathfrak{U}(g)$

$$t_{u,v,N}(f) := \sup_{g \in G_0} s(g)^{-N} \left| (L_u R_v f)(g) \right| < \infty$$

for some $N \geq 0$. Here, $s$ denotes the maximal gauge.

The space of tempered superfunctions is denoted by $\mathcal{T}(G)$. It is topologised as the locally convex inductive limit of the spaces $\mathcal{T}_N(G) := \bigcap_{u,v} \{t_{u,v,N} < \infty \}$, endowed with the locally convex topology generated by the seminorms $t_{u,v,N}$, $u, v \in \mathfrak{U}(g)$.

For any $\omega \in \mathcal{S}(G)$, the Berezin integral

$$\varphi \mapsto \langle \omega, \varphi \rangle := \int_G \omega \varphi$$

extends uniquely to a continuous functional on $\mathcal{T}(G)$. This is easy to deduce from Proposition 2.8 and the corresponding classical facts.

Define $\mathcal{S}'(G)$, the space of tempered generalised functions, to be the strong dual of $\mathcal{S}(G)$. There is a continuous linear injection

$$\mathcal{T}(G) \hookrightarrow \mathcal{S}'(G).$$

The following proposition generalises Ref. [51, Theorem 2.1].
Theorem 2.3 (i) to Lie supergroups. Lemma 3.5, we see that it takes values in $\omega$ since for spaces, and $M$ (5.1) $-\infty$ in view of Proposition 3.8 (i). Thus, $\Phi$ is an immediate consequence of Proposition 3.2 and [51, Lemma 3.3]. □

Proof. Let $\Pi$. Then $\Pi$ is a topological morphism with closed image.

We now generalise Sun–Zhu’s version of the Gel’fand–Kazhdan criterion [51, Theorem 2.3 (i)] to Lie supergroups.

Proposition 5.4. Let $(E, F)$ be a contragredient pair of continuous $F$-representations of $G$. Then the map

$$M : E_\infty \times F_\infty \to \mathcal{T}(G), \quad M_{\omega,v}(u, g) := (-1)^{|u||v|+|v'|}\langle \pi_{E,0}(g)d\pi_E(u)v, v' \rangle$$

extends continuously to a $G \times G$-equivariant separately continuous bilinear map

$$M^-\infty : E^-\infty \times F^-\infty \to \mathcal{S}(G),$$

where $E^-\infty := (F_\infty)'$, $F^-\infty := (E_\infty)'$, and $(d\pi_E, \pi_{E,0})$ is the $G$-action on $E$.

If, moreover, $E_\infty$ and $F_\infty$ are CW $G$-representations, then $M^-\infty$ is continuous and the induced $(G \times G)$-equivariant continuous linear map

$$E^-\infty \hat{\otimes}_\pi F^-\infty \to \mathcal{S}(G)$$
is a topological morphism with closed image.

The structure of the proof is manifestly the same as the one given by Sun–Zhu [51], so we shall be brief. We begin with the following lemma.

Lemma 5.5. Let $E$ be an $F$-representation of $G$. Then the bilinear map

$$\Phi_E : \mathcal{S}(G) \times E \to E_\infty : (\omega, v) \mapsto \Pi_E(\omega)v$$
is well-defined and continuous.

Proof. That the map is well-defined follows from Proposition 3.8 (i). The continuity is an immediate consequence of Proposition 3.2 and [51, Lemma 3.3]. □

Proof of Proposition 5.4. For $\omega \in \mathcal{S}(G)$, we may define $\Pi_E^-\infty(\omega) : E^-\infty \to E$ by

$$\langle \Pi_E^-\infty(\omega)v, v' \rangle := (-1)^{|\omega||v|}\langle \omega, \Phi_E(\omega)v' \rangle, \quad v \in E^-\infty, v' \in F.$$Then $\Pi_E^-\infty(\omega)$ is continuous, and the bilinear map

$$\Phi_E^-\infty : \mathcal{S}(G) \times E^-\infty \to E : (\omega, v) \mapsto \Pi_E^-\infty(\omega)v$$
is separately continuous, both by Lemma 5.5. Applying Proposition 3.2 and [51, Lemma 3.5], we see that it takes values in $E_\infty$ and is separately continuous with respect to the natural topology on this space.

We compute for $v \in E$ and $v' \in F$:

$$\langle \Phi_E^-\infty(\omega, v), v' \rangle = (-1)^{|\omega||v|}\langle \omega, (\Phi_E(\omega)v) v' \rangle = \langle \Pi_E(\omega)v, v' \rangle = \langle \Phi_E(\omega, v), v' \rangle,$$
since for $\omega = u \otimes \omega, u \in \mathfrak{u}(\mathfrak{g}), \omega \in \mathcal{S}(G_u)$, we have

$$\langle v, \Phi_E(\omega)v' \rangle = \langle v, \Pi_{E,0}(\omega)\langle S(u)v, v' \rangle$$
$$= (-1)^{|\omega||v|}\langle d\pi_E(u)\Pi_{E,0}(\omega)v, v' \rangle = (-1)^{|\omega||v|}\langle \Pi_E(\omega)v, v' \rangle,$$in view of Proposition 3.8 (i). Thus, $\Phi_E^-\infty$ extends $\Phi_E$.

Altogether, the map $M^-\infty : E^-\infty \times F^-\infty \to \mathcal{S}(G)$,

$$(5.1) \quad \langle \omega, M^-\infty(v, v') \rangle := \langle \Phi_E^-\infty(\omega, v), v' \rangle = (-1)^{|\omega||v|}\langle \omega, \Phi_E^-\infty(\omega, v') \rangle,$$
is well-defined, separately continuous, and extends $M$.

Now, assume that $E_\infty$ and $F_\infty$ are CW $G$-representations. As such, they are nuclear Fréchet spaces [8, Corollary 5.6] and hence reflexive [54, Corollary 3 to Proposition 50.2, Corollary to Proposition 36.9]. The same holds for $\mathcal{S}(G)$, by Proposition 3.2. Thus, $E^-\infty$, $F^-\infty$, and $\mathcal{S}(G)$ are strong duals of reflexive Fréchet spaces, and $M^-\infty$ is automatically continuous (op. cit., Theorem 41.1). The final statement now follows from Corollary 4.8. □

We now generalise Sun–Zhu’s version of the Gel’fand–Kazhdan criterion [51, Theorem 2.3 (i)] to Lie supergroups.
Definition 5.6 (Irreducible representations). Let $U$ be an \textit{SF}-representation of $G$. We say that $U$ is \textit{irreducible} if there is no non-zero proper closed subspace of $U$ that is $G$-invariant.

Theorem 5.7 (Super Gel'fand–Kazhdan criterion). Let $H_1, H_2$ be closed subsupergroups of $G$, $\chi_i : H_i \to \mathbb{K}^\times$ characters of $H_i$, and $\sigma : G \to G$ an anti-automorphism. Assume any $T \in \mathcal{A}(G)_0$, which is at once $(H_1 \times H_2)$-relatively invariant for the character $\chi_1^{-1} \otimes \chi_2^{-1}$ and a joint eigenvector of all $D \in \mathcal{U}(g)^G_0$ is fixed by $\sigma$.

Then, for any $(E, F)$ contragredient pair of $F$-representations of $G$ such that $E_{\infty}, F_{\infty}$ are irreducible CW $G$-representations, we have

$$\dim \text{Hom}_{H_1}(E_{\infty}, \chi_1) \dim \text{Hom}_{H_2}(F_{\infty}, \chi_2) \leq 1.$$ 

Here, $\text{Hom}_H$ denotes continuous even linear maps that are equivariant with respect to the supergroup $H$.

Proof. Again, our argument is largely that of Sun–Zhu [51], with appropriate modification and references to our results. Let

$$0 \neq v \in \text{Hom}_{H_1}(E_{\infty}, \chi_1) \subseteq F_{-\infty}, \quad 0 \neq u \in \text{Hom}_{H_2}(F_{\infty}, \chi_2) \subseteq E_{-\infty},$$

and set $T := M_{u \otimes v}^{-\infty} \in \mathcal{A}(G)$, appealing to Proposition 5.4.

For $D \in \mathcal{U}(g)$ and $\omega \in \mathcal{A}(G)$, we compute

$$\langle \omega, DT \rangle = (-1)^{|D||\omega|} \langle R_D \omega, M_{u \otimes v}^{-\infty} \rangle = (-1)^{|D||\omega|} \langle \Pi_{E_{\infty}}^{-\infty}(R_{D} \omega) u, v \rangle = \langle \Pi_{E_{\infty}}^{-\infty}(\omega) d_{E_{\infty}}^{-\infty}(D) u, v \rangle,$$

by the use of Equation (5.1) and Equation (3.4). If now $D$ is even and $G$-invariant, then $D$ commutes with the $G$-action on $E_{-\infty}$.

The Harish-Chandra $(g, K_0)$-module $E_{\infty}^{(K_0)}$ is countable-dimensional, and $d_{E_{\infty}}(\mathcal{U}(g))$ acts irreducibly, hence Dixmier’s Lemma [55, Lemma 0.5.2] applies, and $S(D)$ acts by a scalar. Since $E_{\infty}^{(K_0)} \subseteq E_{\infty}$ is dense, it follows that $D$ acts by a scalar on $E_{-\infty}$. Thus, by the computation above, $T$ is an eigenvector of $D$.

On the other hand, as a similar computation shows, $T$ is also relatively $(\chi_1^{-1} \otimes \chi_2^{-1})$-invariant under $(H_1 \times H_2)$. By assumption, $T$ is fixed by $\sigma$.

Let $\omega \in \mathcal{A}(G)$ and $g \in G_0$. We compute

$$\langle \Pi_{E_{\infty}}^{-\infty}(\omega) u, \pi_F(\sigma(g))^{-\infty} v \rangle = \langle \omega, R_{\sigma(g)} T \rangle = \langle \omega, R_{\sigma(T)} \sigma(T) \rangle = \langle \omega, \sigma(L_g T) \rangle = (-1)^{|u||\omega|} \langle \pi_{E_{\infty}}(g) u, \Pi_{E_{\infty}}^{-\infty}(\omega) v \rangle.$$

By the irreducibility of $E_{\infty}$ and $F_{\infty}$, we conclude that

$$\Pi_{E_{\infty}}^{-\infty}(\omega) u = 0 \Leftrightarrow \Pi_{E_{\infty}}^{-\infty}(\omega) v = 0.$$

Hence, for any other $0 \neq u' \in \text{Hom}_{H_2}(F_{\infty}, \chi_2)$, the continuous linear maps

$$\mathcal{A}(G) \to E_{\infty} : \omega \mapsto \Pi_{E_{\infty}}^{-\infty}(\omega) u, \quad \mathcal{A}(G) \to E_{\infty} : \omega \mapsto \Pi_{E_{\infty}}^{-\infty}(\omega) u',$$

have the same kernel $W$ (say), and induce continuous linear maps

$$\varphi, \varphi' : \mathcal{A}(G)/W \to E_{\infty}.$$ 

These are $G$-equivariant by their definition, so they are isomorphisms with closed image, by the token of Corollary 4.8. They are non-zero, and therefore surjective, by the assumption of irreducibility.

Hence, $\psi := \varphi'^{-1} \circ \varphi$ is a well-defined continuous even linear and $G$-equivariant automorphism of $E_{\infty}$. Restricted to $E_{\infty}^{(K_0)}$, it is a constant, by Dixmier’s Lemma (loc. cit.) again. This shows that $u' \in \mathbb{K} u$, by applying Lemma 2.7. A similar argument applies to $v$, proving the assertion. $\square$
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\textsc{Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany}

\textit{E-mail address:} alldridge@math.uni-koeln.de