Extreme amenability and nice enumerations

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Abstract. We consider automorphism groups of countably categorical structures which satisfy some properties related to nice enumerability. We study (extreme) amenability of these groups. We analyse a particular $\omega$-categorical structure which is a candidate for an example of an $\omega$-categorical structure with amenable automorphism group and without $\omega$-categorical precompact expansions with extremely amenable automorphism groups.

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0 Introduction

A group $G$ is called amenable if every $G$-flow (i.e. a compact Hausdorff space along with a continuous $G$-action) supports an invariant Borel probability measure. If every $G$-flow has a fixed point then we say that $G$ is extremely amenable. Let $M$ be a relational countably categorical structure which is a Fraïssé limit of a Fraïssé class $\mathcal{K}$. In particular $\mathcal{K}$ coincides with $Age(M)$, the class of all finite substructures of $M$. By Theorem 4.8 of [15] the group $Aut(M)$ is extremely amenable if and only if the class $\mathcal{K}$ has the Ramsey property and consists of rigid elements. Here the class $\mathcal{K}$ is said

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to have the Ramsey property if for any \( k \) and a pair \( A < B \) from \( \mathcal{K} \) there exists \( C \in \mathcal{K} \) so that each \( k \)-coloring

\[
\xi : \binom{C}{A} \rightarrow k
\]

is monochromatic on some \( \binom{B'}{A'} \) from \( C \) which is a copy of \( \binom{B}{A} \), i.e.

\[
C \rightarrow (B)^A_k.
\]

This result has become a basic tool to amenability of automorphism groups. To see whether \( Aut(M) \) is amenable one usually looks for an expansion \( M^* \) of \( M \) so that \( M^* \) is a Fraïssé structure with extremely amenable \( Aut(M^*) \). Moreover it is usually assumed that \( M^* \) is a precompact expansion of \( M \), i.e. every member of \( \mathcal{K} \) has finitely many expansions in \( Age(M^*) \), see [15], [16], [23], [3] and [25]. Theorem 9.2 from [3] and Theorem 2.1 from [25] describe amenability of \( Aut(M) \) in this situation. The question if there is a countably categorical structure \( M \) with amenable automorphism group which does not have expansions as above was formulated by several people, for example see Problems 27, 28 in [4].

In our paper having in mind these respects, we consider automorphism groups of countably categorical structures which satisfy some properties related to nice enumerability, [1]. We will see in Section 2 that properties of this kind are connected with extreme amenability of automorphism groups and the generic point property.

This suggests that the \( \omega \)-categorical structure found by the author in [12] is a natural candidate for an example of an \( \omega \)-categorical structure with amenable automorphism group but without \( \omega \)-categorical precompact expansions with extremely amenable automorphism groups. The theory of this structure is not G-compact and it does not have AZ-enumerations (see Appendix). Analysing the automorphism group of this structure we obtain in Section 3 some partial results supporting this conjecture.

Since our example from [12] does not have elimination of quantifiers we slightly modify the approach from [15] and [23] to extreme amenability so that it works for expansions of structures obtained by Hrushovski’s amalgamation method. This is presented in Section 1 and uses the approach from [11] (and [17]).

1 Generic expansions of \( \omega \)-categorical structures and extreme amenability

We fix a countable structure \( M \) in a language \( L \). We assume that \( M \) is \( \omega \)-categorical (most of the terms below make sense under the assumption that \( M \) is atomic). Let \( T \) be an extension of \( Th(M) \) in the language with additional relational and functional symbols \( \bar{r} = (r_1, ..., r_t) \). We assume that \( T \) is axiomatizable by sentences of the following form:

\[
(\forall \bar{x})(\bigvee_i (\phi_i(\bar{x}) \land \psi_i(\bar{x}))),
\]
where $\phi_i$ is a quantifier-free formula in the language $L \cup \bar{r}$, and $\psi_i$ is a first-order formula of the language $L$. Consider the set $X$ of all possible expansions of $M$ to models of $T$.

Following [11] we define for a tuple $\bar{a} \subset M$ a diagram $\phi(\bar{a})$ of $\bar{r}$ on $\bar{a}$. To every functional symbol from $\bar{r}$ we associate a partial function from $\bar{a}$ to $\bar{a}$. Choose a formula from every pair $\{r_i(\bar{a}'), \neg r_i(\bar{a}')\}$, where $r_i$ is a relational symbol from $\bar{r}$ and $\bar{a}'$ is a tuple from $\bar{a}$ of the corresponding length. Then $\phi(\bar{a})$ consists of the conjunction of the chosen formulas and the definition of the chosen functions (so, in the functional case we look at $\phi(\bar{a})$ as a tuple of partial maps).

Consider the class $B_T$ of all theories $D(\bar{a}), \bar{a} \subset M$, such that each of them consists of $Th(M, \bar{a})$ and a diagram of $\bar{r}$ on $\bar{a}$ satisfied in some $(M, \bar{r}) \models T$. We order $B_T$ by extension: $D(\bar{a}) \leq D'(\bar{b})$ if $\bar{a} \subset \bar{b}$ and $D'(\bar{b})$ implies $D(\bar{a})$ under $T$ (in particular, the partial functions defined in $D'$ extend the corresponding partial functions defined in $D$). Since $M$ is an atomic model, each element of $B_T$ is determined by a formula of the form $\phi(\bar{a}) \land \psi(\bar{a})$, where $\psi$ is a complete formula for $M$ and $\phi$ is a diagram of $\bar{r}$ on $\bar{a}$. The corresponding formula $\phi(\bar{x}) \land \psi(\bar{x})$ will be called basic.

On the set $X = \{(M, \bar{r}') : (M, \bar{r}') \models T\}$ of all $\bar{r}$-expansions of the structure $M$ we consider the topology generated by basic open sets of the form

$$[D(\bar{a})] = \{(M, \bar{r}') : (M, \bar{r}') \models D(\bar{a})\}, \bar{a} \subset M.$$  

It is easily seen that any $[D(\bar{a})]$ is clopen. The topology is metrizable: fix an enumeration $\bar{a}_0, \bar{a}_1, ...$ of $M^{<\omega}$ and define

$$d((M, \bar{r}'), (M, \bar{r}'')) = \sum 2^{-n} : \text{there is a symbol } r \in \bar{r} \text{ such that its interpretations on } \bar{a}_n \text{ in the structures } (M, \bar{r}') \text{ and } (M, \bar{r}'') \text{ are not the same (if } r \text{ is a functional symbol then } r'(\bar{b}) \neq r''(\bar{b}) \text{ for some } \bar{b} \subseteq \bar{a}_n \text{ ).}$$

It is easily shown that the metric $d$ defines the topology determined by the sets of the form $[D(\bar{a})]$.

By the assumptions on $T$ ($T$ is axiomatizable by sentences which are universal with respect to symbols from $\bar{r}$) the space $X$ forms a closed subset of the complete metric space of all $\bar{r}$-expansions of $M$. Thus $X$ is complete and the Baire Category Theorem holds for $X$. We say that $(M, \bar{r}) \in X$ is generic if the class of its images under $Aut(M)$ is comeagre in $X$ [11].

**Remark 1.1** All our arguments also work for the case when $\bar{r} = (r_1, ..., r_t, ...)$ is an infinite sequence, but for every tuple $\bar{b}$ from $M$ the family $B_T$ has finitely many diagrams defined on $\bar{b}$. In this case we say that $X$ consists of precompact expansions.

Notice that the space $Aut(M)$ under the conjugacy action, and generic automorphisms (introduced in [24]) provide a particular example of this construction. Indeed, identify each $\alpha \in Aut(M)$ with the expansion $(M, \alpha, \alpha^{-1})$. The class of structures of this form is axiomatized in the language of $M$ with the functional symbols $\{\alpha, \beta\}$ by $Th(M)$, the sentence $\alpha(\beta(x)) = \beta(\alpha(x)) = x$ and universal sentences asserting that $\alpha$ preserves the relations of $M$. Also, any partial isomorphism $\bar{a} \rightarrow \bar{a}'$ can be viewed as
the diagram corresponding to the maps $\bar{a} \to \bar{a}'$ and $\bar{a}' \to \bar{a}$. It is clear that a generic automorphism $\alpha$ (see [24]) defines generic expansion $(M, \alpha, \alpha^{-1})$.

Similar considerations can be applied in the following general situation. Let $M$ be an $\omega$-categorical structure in a language $L$. Let $\bar{r}$ be a tuple of relations on $M$ and $T$ be $Th(M)$ extended by all the sentences from $Th(M, \bar{r})$ of the form $\forall \bar{x} \exists \bar{x}(D(\bar{x}))$, where $D(\bar{x})$ is basic for $(M, \bar{r})$. It is clear that $T$ satisfies the conditions from the beginning of the section. Note that $B_T$ consists of all diagrams $D(\bar{b})$ such that the corresponding formula $D(\bar{x})$ is realizable in $(M, \bar{r})$. The expansion $(M, \bar{r})$ is ubiquitous in category if $(M, \bar{r})$ is generic with respect to $B_T$. It is clear that generic expansions (with respect to some theory $T'$) are always ubiquitous in category.

Theorem 1.5 of [11] states that a structure $(M, \bar{r})$ is ubiquitous in category if and only if every complete type over $\emptyset$ realizable in $(M, \bar{r})$ is determined in $(M, \bar{r})$ by a formula of the form $\exists \bar{y} D(\bar{x}\bar{y})$ where $D(\bar{x}\bar{y})$ is basic.

We now give several important definitions from [11]. We say that $B_T$ has the joint embedding property if for any $D_1(\bar{a}), D_2(\bar{b}) \in B_T$ there exist $D(\bar{c}) \in B_T$ and $M$-elementary maps $\delta : \bar{a} \to \bar{c}$ and $\sigma : \bar{b} \to \bar{c}$ such that $D(\bar{c})$ extends $D_1(\delta(\bar{a})) \cup D_2(\sigma(\bar{b}))$. The class $B_T$ has the weak amalgamation property (see [17], in the original paper [11] it is called the almost amalgamation property) if for every $D(\bar{a}) \in B_T$ there is an extension $D'(\bar{ab}) \in B_T$ such that for any $D_1(\bar{ac}_1), D_2(\bar{ac}_2) \in B_T$, where $D'(\bar{ab}) \leq D_i(\bar{ac}_i), i = 1, 2$, there exists a common extension $D''(\bar{ac}) \in B_T$ under some $(M, \bar{a})$-elementary maps $\bar{c}_i \to \bar{c}, i = 1, 2$ (which may move $\bar{b}$).

**Theorem A.** ([11], Theorem 1.2 and Corollary 1.4) (a) The set $X$ has a generic structure if and only if $B_T$ has the joint embedding property and the weak amalgamation property.

(b) If there are no continuum many pairwise non-isomorphic elements of $X$, then $B_T$ has the weak amalgamation property.

It is worth noting that in (a) for any member of $B_T$ the corresponding basic formula is realized in a generic structure.

An element $D(\bar{b}) \in B_T$ is called an amalgamation base if any two of its extensions have a common extension in $B_T$ under some automorphism of $M$ fixing $\bar{b}$. We say that $B_T$ satisfies Truss’ condition if any element of $B_T$ extends to an amalgamation base. If it holds then the set of amalgamation bases is a cofinal subset of $B_T$ which has the amalgamation property. It is clear that Truss’ condition implies the weak amalgamation property. In particular it together with the joint embedding property implies the existence of a generic expansion $(M, \bar{r})$. The proof of Theorem 1.5 of [11] shows that when $B_T$ satisfies Truss’ condition and $D(\bar{b}) \in B_T$ is an amalgamation base, then the type of $\bar{b}$ in this $(M, \bar{r})$ is determined by the basic formula $D(\bar{x})$. We will use this fact later.

The group $Aut(M)$ has a natural action on $B_T$. Moreover if $\bar{a}$ and $\bar{b}$ have the same type with respect to $Th(M)$, then after replacement $\bar{a}$ by $\bar{b}$ in $D(\bar{a})$ we obtain an image
of $D(\bar{a})$ under an automorphism of $M$. This is a consequence of an $\omega$-homogeneity of $M$. It is also clear that $\text{Aut}(M)$ acts continuously on $X$ with respect to the topology defined above.

Let $D(\bar{a}) \in B_T$ have an extension $D'(\bar{b}) \in B_T$. Let $(D'(\bar{b}))_{D(\bar{a})}$ be the set of all images of $D(\bar{a})$ in $D'(\bar{b})$ under elementary maps of $M$.

**Definition 1.2** An $\text{Aut}(M)$-invariant subfamily $C \subseteq B_T$ satisfies the Ramsey property if for any $k$ and a pair $D(\bar{a}) < D'(\bar{b})$ from $C$ there exists $D''(\bar{c}) \in C$ so that each $k$-coloring

$$\xi : \left( \frac{D''(\bar{c})}{D(\bar{a})} \right) \to k$$

is monochromatic on some $(D'(\bar{b}))_{D(\bar{a})}$ from $D''(\bar{c})$ which is a copy of $(D'(\bar{b}))_{D(\bar{a})}$, i.e.

$$D''(\bar{c}) \to (D'(\bar{b}))_{D(\bar{a})}^k.$$

**Remark 1.3** By the proof of Theorem 1.2 (1 $\to$ 2) of [10] the weak amalgamation property is a consequence of the following version of the Ramsey property:

For any $D(\bar{a}) \in B_T$ there is an extension $D'(\bar{a}\bar{b}) \in B_T$ such that for any $D_1(\bar{a}\bar{c}_1) \in B_T$, where $D'(\bar{a}\bar{b}) \leq D_1(\bar{a}\bar{c}_1)$, there exists an extension $D_1(\bar{a}\bar{c}_1) < D_2(\bar{a}\bar{c}_2) \in B_T$ such that

$$D_2(\bar{a}\bar{c}_2) \to (D_1(\bar{a}\bar{c}_1))_{D(\bar{a})}^2.$$

under some $(M, \bar{a})$-elementary maps.

The following theorem is a slightly generalized version of Theorem 4.5 from [15].

**Theorem 1.4** Let $M$ be an $\omega$-categorical structure and $B_T$ satisfy Truss’ condition. Let $C \subseteq B_T$ be an invariant cofinal subset of amalgamation bases with the joint embedding property and the amalgamation property.

Then the automorphism group $\text{Aut}(M, \bar{\bar{r}})$ of a generic expansion corresponding to $C$ is extremely amenable if and only if the class $C$ has the Ramsey property and consists of rigid elements, i.e. no $D(\bar{a}) \in C$ can be taken onto itself by a non-trivial elementary map $\bar{a} \to \bar{a}$ with respect to $\text{Th}(M, \bar{\bar{r}})$.

**Proof.** We adapt the proof of Theorem 4.5 from [15] as follows. Firstly we remind the reader that for a closed subgroup $G < \text{Aut}(M)$ a $G$-type of a tuple $\bar{a}$ is just the orbit $G\bar{a}$. It is a consequence of the proof of Theorem 1.5 from [11] that for $G = \text{Aut}(M, \bar{\bar{r}})$ if $D(\bar{a}) \in C$, then the $G$-type of $\bar{a}$ is determined by $D(\bar{x})$, i.e. coincides with the set of all realizations of this formula in $(M, \bar{\bar{r}})$. Then the ordering of $G$-types $G\bar{a} \leq G\bar{b}$ introduced in [15] just corresponds to the relation $D(\bar{y}) \models D(\bar{x})$ for some embedding of $\bar{x}$ (corresponding to $\bar{a}$) into $\bar{y}$ (corresponding to $\bar{b}$). The Ramsey property introduced in [15] in the case of $\text{Aut}(M, \bar{\bar{r}})$-types of tuples $\bar{a}$ with $D(\bar{a}) \in C$ coincides with Definition 1.2 for $C$. The rest follows from Proposition 4.3, Theorem 4.5 and Remark 4.6 of [15]. □
Remark 1.5 Under the assumption of that theorem assume that \( I \subseteq M \) is an order indiscernible sequence of type \( \omega \) (say with respect to the order \( < \)). Then the subclass of all members of \( \mathcal{C} \) of the form \( D(\vec{a}) \) with \( \vec{a} \subseteq I \) has the Ramsey property if and only if so does the class of all ordered finite \( \mathfrak{r} \)-structures realisable on \( (I, <) \). Under the additional assumption of monotonicity such classes are characterised by Theorem 1.4 of [10].

It is worth noting that the majority of basic statements of [15] can be adapted to the situation of Theorem 1.4. In particular Theorem 7.5 of [15] and Theorem 5 of [23] (a generalisation of the former one) can be stated as follows.

**Theorem 1.6** Let \( M \) be an \( \omega \)-categorical structure, \( G = \text{Aut}(M) \) and \( B_T \) satisfy Truss’ condition. Let \( \mathcal{C} \subset B_T \) be an invariant cofinal subset of rigid amalgamation bases with the joint embedding property and the amalgamation property. Assume that \( (M, \mathfrak{r}) \) is a generic expansion corresponding to \( \mathcal{C} \).

Then the space \( X \) is the universal minimal flow of \( G \) if and only if the class \( \mathcal{C} \) has the Ramsey property and the following expansion property (relative to \( M \)):

any tuple \( \vec{a} \) from \( M \) extends to a tuple \( \vec{b} \in M \) so that for any \( D(\vec{a}) \) and \( D'(\vec{b}) \in \mathcal{C} \) there is an \( M \)-elementary map \( \vec{a} \rightarrow \vec{b} \) which embeds \( D(\vec{a}) \) into \( D'(\vec{b}) \).

A proof of this theorem can be obtained by a straightforward adaptation of Sections 4 and 5 of [23]. Extending the standard terminology from [3] it is natural to call \((M, \mathcal{C})\) from Theorem 1.6 an excellent pair. Then Proposition 9.2 from [3] can be generalised to a description of amenability of \( \text{Aut}(M) \). Moreover repeating Proposition 14.3 from [3] we can show that in the situation of Theorem 1.6 the group \( G \) has the generic point property, i.e. every minimal \( G \)-flow has a comeager orbit.

### 2 Nice enumerations

**Definition 2.1** A linear ordering \( < \) of a countable structure \( M \) is called an AZ-enumeration of \( M \) if it has order-type \( \omega \) and for any \( n \geq 1 \) it satisfies the following property:

whenever \( \vec{b}_i, \; i < \omega \), is a sequence of \( n \)-tuples from \( M \), there exist some \( i < j < \omega \) and a \( < \)-preserving elementary map \( f : M \rightarrow M \) such that \( f(\vec{b}_i) = \vec{b}_j \).

It is known (and easily seen) that any structure having an AZ-enumeration is countably categorical. It is also clear that any AZ-enumeration is nice, i.e. for any infinite \( < \)-increasing sequence \( a_i \in M \) there are \( i \) and \( j \) so that the initial segment \( \{ c : c < a_i \} \) is isomorphic to a substructure of the initial segment \( \{ c : c < a_j \} \) by an automorphism mapping \( a_i \) to \( a_j \).
There are countably categorical structures without AZ-enumerations [14]. On the other hand it is an open question if there are ω-categorical structures without nice enumerations [5]. Since by Theorem 2.4 of [5] any permutation module of a structure having a nice enumeration, has the ascending chain condition for submodules it is also open if there are countably categorical structures so that there is a permutation module of this structure which does not have the ascending chain condition.

Permutation modules which we consider appear as follows. For a field \( F \) let \( FM \) be the \( F \)-vector space where \( M \) is a basis. Then the group algebra \( FAut(M) \) naturally acts on \( FM \), i.e. \( FM \) becomes a module over \( FAut(M) \). We usually consider right modules. When \( v \in FM \), then \( \text{supp}(v) \) is the set of all elements of \( M \) which appear in \( v \) with non-zero coefficients. When we have some ordering of \( \text{supp}(v) \) we denote by \( \text{Head}(v) \) the maximal element of \( \text{supp}(v) \) under this ordering.

2.1 Ramsey property and nicely extended orderings

Let \( M \) be an ω-categorical structure. Consider \( M \) as a relational structure admitting elimination of quantifiers. Let \( (I,<) \) be an ordered subset of \( M \) representing all 1-types of \( \text{Th}(M) \).

**Definition 2.2** We say that \( (I,<) \) nicely extends to \( \omega \) with respect to an initial segment \( \bar{a} \subset I \) if for any 1-type \( p \) of \( \text{Th}(M) \) there is a finite sequence \( a^p_1, ..., a^p_s \) of its realisations from \( \bar{a} \) with the following property. The set \( (I,<) \) extends to an ordered set \( (C,<) \subset M \) of type \( \omega \) so that for any 1-type \( p \) and any \( a_i \in C \) of the type \( p \) with \( i \) greater than the indices of \( a^p_1, ..., a^p_s \) in \( C \), the initial segment of \( \bar{a} \) determined by some \( a^p_j \) embeds into the initial segment of \( (C,<) \) determined by \( a_i \) so that \( a_i \) is the image of \( a^p_j \).

It is clear that if \( (I,<) \) is a nice enumeration of \( M \) then any initial segment of \( (I,<) \) is contained in an initial segment \( \bar{a} \subset (I,<) \) so that \( (I,<) \) nicely extends to itself with respect to \( \bar{a} \).

Note that any ω-enumeration of a transitive ω-categorical structure \( M \) has this property with respect to \( \bar{a} \) which is a singleton. On the other hand we do not see any reason why this holds with respect to arbitrary long segments (as in the case of nice enumerations).

Let \( S \) be a family of tuples from \( M \) which includes all singletons and has the property that any finite set occurs in some tuple from \( S \). Any tuple of \( M \) may be viewed as an ordered substructure of \( M \).

We say that \( S \) has the **Ramsey property** if the Ramsy property holds for the corresponding class of ordered substructures. The following lemma and discussions after it suggest that nice extendibility to \( \omega \) may be viewed as a variant of the Ramsey property.

From now on we fix an ordered subset \( (I,<) \) of \( M \) so that any finite subset of \( I \) belongs to \( S \) as an ordered structure.
Lemma 2.3 Assume that $S$ has the Ramsey property. Let $\bar{a} \subset (I, <)$ be a finite substructure. For any 1-type $p(x)$ realizable by an element of $\bar{a}$ fix such a realisation, say $a^p$.

Then $\bar{a}$ embeds into a finite substructure $\bar{c} \in S$ so that for any type $p$ and any $a_i \in \bar{a}$ of the type $p$, the initial segment of $\bar{a}$ determined by $a^p$ embeds into the initial segment of $\bar{c}$ determined by $a_i$ so that $a_i$ is the image of $a^p$.

Proof. We may assume that any 1-type of $Th(M)$ is realised in $\bar{a}$. Let $p_1, \ldots, p_k$ be the list of these types. By the Ramsey property there is a tuple $\bar{b}_1$ so that for any 2-colouring of singletons from $\bar{b}_1$ of the type $p_1$ there is a monochromatic subtuple of the type $\bar{a}$. Then applying the Ramsey property again we find $\bar{b}_2$ so that for any 2-colouring of singletons from $\bar{b}_2$ of the type $p_2$ there is a monochromatic subtuple of the type $\bar{b}_1$. After $k$ steps of this procedure we find the corresponding sequence $\bar{b}_1, \ldots, \bar{b}_k$. Let $\bar{c} = \bar{b}_k$.

To check the statement of the lemma let us colour black all $p_k$-realisations $d \in \bar{b}_k$ so that $d$ is the last element of a subtuple of $\bar{b}_k$ of the type of the initial segment of $\bar{a}$ finished by $a^{p_k}$. Otherwise we colour $d$ white. We may assume that $\bar{b}_{k-1}$ is monochromatic for this colouring. Since it contains the initial segment of $\bar{a}$ determined by $a^{p_k}$, all $p_k$-realisations from $\bar{b}_{k-1}$ are black. Now applying this argument consequently to all pairs $(p_i, \bar{b}_i)$ with $i = k - 1, \ldots, 2, 1$, we obtain a copy of $\bar{a}$ as in the formulation. $\square$

Applying the argument of Lemma 2.3 we obtain a sequence $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m, \ldots$ so that $\bar{c}_1 = \bar{c}$ and for any $m$, any type $p$ and any $a_i \in \bar{c}_m$ of the type $p$, the initial segment of $\bar{a}$ determined by $a^p$ embeds into the initial segment of $\bar{c}_{m+1}$ determined by $a_i$ so that $a_i$ is the image of $a^p$. Let $(C, <) = \bigcup_{m>0} \bar{c}_m$. We may arrange that $(I, <)$ is a subordering of $(C, <)$. Taking $\bar{a}$ long enough and extending $(I, <)$ if necessary, we may assume that all 1-types of $Th(M)$ are realisable in $\bar{a}$. The main point of Definition 2.2 is that such $(C, <)$ can be found of type $\omega$. The following theorem shows if this is possible then the corresponding permutation modules have the ascending chain condition for submodules.

It is a generalization of Theorem 2.4 of [5].

Theorem 2.4 Let $M$ be an $\omega$-categorical structure and $F$ be a field. Assume that
(i) $\mathcal{V} = \{v_1, v_2, \ldots\}$ is an independent subset of the $FAut(M)$-module $FM$, i.e. for every $i$
$\langle \mathcal{V} \rangle_{FAut(M)} \neq \langle v_1, \ldots, v_{i-1} \rangle_{FAut(M)}$,
(ii) the union $I$ of the supports of all elements from $\mathcal{V}$ realises all 1-types of $Th(M)$.

Then in any elementary extension of $M$ the set $I$ cannot be enumerated so that $(I, <)$ nicely extends to an $\omega$-ordered set $(C, <)$ so that for any 1-type $p$ the corresponding sequence $a^p_0, \ldots, a^p_k \in I$ consists of elements of the form $Head(v), \, v \in \mathcal{V}$, under the ordering $<$.\n
Proof. Let $(I, <)$ be an enumeration of the union of supports from $\mathcal{V}$. Assume that $(C, <)$ is an extension of $(I, <)$ which together with some initial segment $\bar{a} \subset (I, <)$ witnesses that the statement of the theorem does not hold in $M$ (or in some
elementary extension of $M$). Well-order $F$ in such a way that $0$ and $1$ are the first two elements. Let $<$ also denote the resulting lexicographic ordering on $FC$. We define a sequence of elements of $⟨V⟩_{F, Aut(M)} \cap FC$ as follows. Let $u_1$ be the minimal element of $⟨V⟩_{F, Aut(M)} \cap FC \setminus \{0\}$. For each $i$ let $u_i$ be the minimal element of $\langle u_1, \ldots, u_{i-1} \rangle_{F, Aut(M)}$. Clearly $u_1 < u_2 < \ldots$ and $Head(u_i)$ occurs in $u_i$ with coefficient $1$. Moreover $Head(u_i) < Head(u_{i+1})$. Indeed, if $Head(u_i) = Head(u_{i+1})$, then the element $u_{i+1} - u_i$ contradicts the choice of $u_{i+1}$ as the minimal element outside $\langle u_1, \ldots, u_{i-1} \rangle_{F, Aut(M)}$.

So $\{Head(u_i) : i \in \omega\}$ is an infinite subset of $C$. By the choice of $C$ there is a number $i_0$ so that for any $i > i_0$ there is $j < i_0$ with $a_j \in I$ of the form $Head(v_i)$, $v_i \in V$, and an automorphism of $M$ which takes the initial segment of $(I, <)$ determined by $a_j$ to the initial segment of $(C, <)$ determined by $u_i$. In particular for sufficiently large $i$ we find $v_j$ with $j < i_0$ and an automorphism $\gamma$ so that the heads of $u_i$ and $v_j\gamma$ are the same. Then $u_i - v_j\gamma < u_i$ which contradicts the choice of $u_i$. □

2.2 AZ-enumerations and generic expansions

The following theorem shows that AZ-enumerations provide generic expansions by orderings.

Theorem 2.5 Let $M$ be a Fraisse limit of a relational amalgamation class $\mathcal{K}$. Let $<$ be a linear ordering of $M$ of type $\omega$. Let $T$ be $Th(M)$ extended by all the sentences from $Th(M, <)$ of the form $\forall \bar{x} \lnot D(\bar{x})$, where $D(\bar{x})$ is basic for $(M, <)$.

If the linear ordering $<$ is an AZ-enumeration of $M$ then the space of all $T$-expansions of $M$ contains a generic structure which is $\omega$-categorical.

Proof. Assume that $<$ is an AZ-enumeration of $M$. The definition of $T$ implies that $B_T$ has JEP. Let us show that $B_T$ satisfies WAP, the weak amalgamation property. If WAP does not hold for some $D(\bar{a}) \in B_T$, then we build a tree in $(B_T, <)$ with the root $D(\bar{a})$. At every step we split the already constructed extensions of $D(\bar{a})$ as follows. If $D'(\bar{a}\bar{b})$ is an extension corresponding to the vertex $\epsilon_1 \epsilon_2 \ldots \epsilon_n$ of the tree (with $\epsilon_i \in \{0, 1\}$), find $D_1(\bar{a}c_1)$ and $D_2(\bar{a}c_2)$ extending $D'(\bar{a}\bar{b})$ which cannot be amalgamated over $D(\bar{a})$. These extensions correspond to vertices $\epsilon_1 \epsilon_2 \ldots \epsilon_n 0$ and $\epsilon_1 \epsilon_2 \ldots \epsilon_n 1$.

Now choose tuples $\bar{a}_i$ from $(M, <)$ which correspond to $\bar{a}$ in all extensions $D'(\bar{a}\bar{b})$ with numbers $0, 10, \ldots, 111\ldots 10, \ldots$ in the tree. In other words $\bar{a}_i$ extends to a tuple $\bar{a}_i \bar{b}^j$ realizing $D'(\bar{x}\bar{y})$ corresponding to the number $1\ldots 10$ with $i$ units. It is clear that no $\bar{a}_i$ can be taken to $\bar{a}_j$ with $i < j$ by a $<$-preserving elementary map.

By Theorem A find a generic structure $(M, \prec^*)$. If this structure is not $\omega$-categorical, then for some natural $k > 0$ there infinitely many $k$-types over $\emptyset$. By Theorem 1.5 of [11] each type over $\emptyset$ realizable in $(M, \prec^*)$ is determined by a formula of the form $\exists \bar{y} D(\bar{x}\bar{y})$, where $D(\bar{x}\bar{y})$ is basic. Thus there is an infinite family $\Phi$ of basic formulas of the form $D(\bar{x}\bar{y})$ with $|\bar{x}| = k$ so that

(a) each element of $\Phi$ can be realized both in $(M, \prec^*)$ and $(M, <)$,
(b) for any pair \(D_1(\bar{x}\bar{y}_1)\) and \(D_2(\bar{x}\bar{y}_2)\) from the family the conjunction \(D_1(\bar{x}\bar{y}_1) \land D_2(\bar{x}\bar{y}_2)\) cannot be realized neither in \((M, \prec)\) nor in \((M, \prec^*)\).

Thus the set of \(\bar{x}\)-parts of realizations in \((M, \prec)\) of formulas from \(\Phi\) has the property contradicting to AZ-enumeration by \(\prec\). □

**Remark 2.6** Let us assume that \(M\) has an \(\omega\)-categorical expansion by a linear order, say \(\prec^*\) which is ubiquitous in category. Assume that \(I \subset M\) is an order indiscernible sequence of type \(\omega\) with respect to \(Th(M)\). Let \(\prec\) be the corresponding order of \(I\). Let \(T = Th(M, \prec^*)\) and let \(C\) be the subclass of all members of \(B_T\) of the form \(D(\bar{a})\) with \(\bar{a} \subset I\). When does \(C\) have an infinite anti-chain with respect to the natural embedding defined in \(B_T\)? If all members of \(C\) are \(\prec\)-rigid but all possible linear orderings \(\prec^*\) can be realised on \(\prec\)-increasing tuples from \(I\), then the answer is positive by Section 3 of [2]. On the other hand it is impossible if \(I\) is an indiscernible set with respect to \(Th(M)\). In this case \(\prec\) is a nice enumeration of the structure \((I, \prec^*)\). Moreover by Kruskal’s tree theorem [18] in this case it suffices to assume that \(\prec^*\) is a tree ordering.

We suspect that the question of this remark may have some connections with the version of the Ramsey property appearing in Remark 1.5.

We now prove a proposition which connects the ascending chain condition for submodules of permutation modules with the existence of infinite anti-chains of substructures.

**Proposition 2.7** Let \(M\) be an \(\omega\)-categorical structure. Considering \(M\) as a relational structure assume that the family \(K\) of all finite substructures of \(M\) does not have an infinite antichain with respect to embedding induced by automorphisms of \(M\).

Then for any finite field \(GF(q)\) the permutation module \(GF(q)M\) over \(GF(q)Aut(M)\) has the ascending chain condition for submodules.

It is worth also noting that the proposition in fact reduces the case of the ascending chain condition of permutation modules over finite fields to the case of permutation modules over \(GF(2)\) (the permutation module \(GF(2)M\) over \(GF(2)Aut(M)\) can be identified with \(K\)).

**Proof of Proposition 2.7.** Suppose that there is a sequence \(p_i \in GF(q)M, i \in \omega\), which generates a non-finitely generated \(GF(q)\)-permutation submodule of \(GF(q)M\). We express each \(p_i\) as a sum

\[
 f_1(\sum_{j \in D_1} a_j) + f_2(\sum_{j \in D_2} a_j) + ... + f_{q-1}(\sum_{j \in D_{q-1}} a_j),
\]

where \(GF(q) \setminus \{0\}\) is enumerated as \(\{f_1, ..., f_{q-1}\}\) and sets \(D_k\) are pairwise disjoint. We may assume that for sufficiently large \(i\) the element \(p_i\) always has the minimal number of non-zero sums \(\sum_{j \in D_k} a_j\) among all elements from the difference

\[
 \langle p_0, ..., p_{i-1} \rangle GF(q) \setminus \langle p_0, ..., p_{i-1} \rangle GF(q).
\]
Then there is an infinite subsequence of $p_i$ consisting of elements which have non-zero members $f_k(\sum_{j \in D_k} a_j)$ for the same $k$. Moreover we may assume that any $p_i$ of this subsequence satisfies the condition

$$1 \leq |D_1| \leq \min(|D_1|, |D_2|, ..., |D_{l-1}|)$$

with $D_l = ... = D_{q-1} = \emptyset$.

We may also assume that the size $|D_1|$ in these $p_i$ is minimal among all elements from the difference

$$\langle p_0, ..., p_t \rangle_{GF(q)} \setminus \langle p_0, ..., p_{t-1} \rangle_{GF(q)}$$

with $D_t = ... = D_{q-1} = \emptyset$.

We now identify each $D_1$ with a substructure from $\mathcal{K}$. Using the assumption of the theorem we choose two members $p_i = f_1(\sum_{j \in D_1} a_j) + ...$ and $p_l = f_1(\sum_{j \in D_1} a_j) + ...$ of the subsequence described above so that $D_1$ embeds into $D_1$ by a map extending to an automorphism of $M$, say $\alpha$. Then the element $p_l - \alpha p_i$ contradicts the choice of $p_i$. □

**Remark 2.8** Let us notice that if a countable structure $M$ is a model of an $\omega$-categorical universal theory, then any enumeration of $M$ is nice. This follows from Theorem 1 of [19] stating the existence of a function $s : \omega \to \omega$, so that for any substructure $B < M$ of size $\geq s(n)$ any 1-type over an $n$-element subset of $B$ is realized in $B$ (i.e. in particular any $n$-type over any $b \in B$ is realized in $B$). Now let $p(x)$ be a non-algebraic type over $\emptyset$. If $a_1, ..., a_n, ...$ is an infinite sequence of realizations of $p(x)$ in order of the enumeration of $M$, then choosing $n$ sufficiently big, we find a realization of the initial segment defined by $a_1$ in the initial segment of $a_n$, where $a_n$ realizes the place of $a_1$. Now the condition of nice enumeration can be easily verified.

It is an old open question if there is an $\omega$-categorical universal theory which is not $\omega_1$-categorical [19].

### 3 Examples

The countably categorical structure with non-G-compact theory, found by the author in [12] does not have AZ-enumerations. The results of the previous section suggest that its automorphism group can have interesting properties from the point of view of topological dynamics. We will in particular show that this structure has the following properties:

- it does not satisfy Hrushovski’s extension property;
- it does not have an order expansion with the Ramsey property;
- the automorphism group is amenable.
3.1 Equivalence relations

We start with a very interesting reduct of the structure from [12]. It has already deserved some attention in model-theoretic community, see [6].

Let \( L_0 = \{ E_n : 0 < n < \omega \} \) be a first-order language, where each \( E_n \) is a relational symbol of arity \( 2n \). Let \( \mathcal{K}_0 \) be the class of all finite \( L_0 \)-structures \( C \) where each relation \( E_n(\bar{x}, \bar{y}) \) determines an equivalence relation on the set (denoted by \( (C)_n \)) of unordered \( n \)-element subsets of \( C \). In particular we have that \( \mathcal{K} \) satisfies the sentence

\[
\forall \bar{x}\bar{y}(E_n(x_1, \ldots, x_n, y_1, \ldots, y_n) \to \bigwedge\{E_n(y_1, \ldots, y_n, x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \sigma \in \text{Sym}(n)\}).
\]

For \( C \in \mathcal{K}_0 \) and \( n > |C| \) we put that no \( 2n \)-tuple from \( C \) satisfies \( E_n(\bar{x}, \bar{y}) \). It is easy to see that \( \mathcal{K}_0 \) is closed under taking substructures and the number of isomorphism types of \( \mathcal{K}_0 \)-structures of any size is finite.

Let us verify the amalgamation property for \( \mathcal{K}_0 \). Given \( A, B_1, B_2 \in \mathcal{K}_0 \) with \( B_1 \cap B_2 = A \), define \( C \in \mathcal{K}_0 \) as \( B_1 \cup B_2 \). We only obey the following rules. When \( n \leq |B_1 \cup B_2| \) and \( \bar{a} \in (B_1)_n \cup (B_2)_n \) we put that the \( E_n \)-class of \( \bar{a} \) in \( C \) is contained in \( (B_1)_n \cup (B_2)_n \). We also assume that all \( n \)-tuples meeting both \( B_1 \setminus B_2 \) and \( B_2 \setminus B_1 \) are pairwise equivalent with respect to \( E_n \). In particular if \( n \geq \max(|B_1|, |B_2|) \) we put that all \( n \)-element \( n \)-tuples from \( C \) are pairwise \( E_n \)-equivalent.

It is easy to see that this amalgamation also works for the joint embedding property.

Let \( M_0 \) be the countable universal homogeneous structure for \( \mathcal{K}_0 \). It is clear that in \( M_0 \) each \( E_n \) defines infinitely many classes and each \( E_n \)-class is infinite.

Theorem 3.1 which we prove below, shows that \( M_0 \) cannot be treated by the methods of [13]. It states that \( M_0 \) does not have a linear ordering so that the corresponding age has the order property and the Ramsey property. On the other hand it is worth noting that the class \( \mathcal{K}_0 \) of all linearly ordered members of \( \mathcal{K}_0 \) has JEP and AP, i.e. there is a generic expansion of \( M_0 \) by a linear ordering. To see AP we just apply the amalgamation described above together with the standard amalgamation of orderings.

**Theorem 3.1** The structure \( M_0 \) does not have any expansion by a linear order so that \( \text{Th}(M_0, <) \) admits elimination of quantifiers and the age of \( (M_0, <) \) satisfies the Ramsey property.

**Proof.** Consider a linearly ordered expansion \( (M_0, <) \) together with the corresponding age, say \( \mathcal{K}^< \). Assume that \( \mathcal{K}^< \) has the Ramsey property.

Note that \( \mathcal{K}^< \) does not contain any three-element structure of the form \( a < b < c \), where \( a \) and \( c \) belong to the same \( E_1 \)-class which is distinct from the \( E_1 \)-class of \( b \). Indeed, otherwise repeating the argument of Theorem 6.4 from [13], we see that in any larger structure from \( \mathcal{K}^< \) we can colour two-elements structures \( a < b \) with \( \neg E_1(a, b) \), so that there is no monochromatic three-element structure of the form above.

As a result we see that any \( E_1 \)-class of \( (M_0, <) \) is convex. We now claim that the following structure \( B \) can be embedded into \( (M_0, <) \).
Let $B = \{ a_1 < a_2 < a_3 < a_4 < b_1 < b_2 \}$, where the $E_1$-classes of all elements are pairwise distinct, but the pairs $\{ a_1, a_2 \}$ and $\{ b_1, b_2 \}$ are $E_2$-equivalent. We assume that in all other cases any two distinct pairs from $B$ belong to distinct $E_2$-classes. Moreover we assume that for each $k = 3, 4, 5$ any two distinct $k$-subsets from $B$ belong to distinct $E_k$-classes. In particular the ordered structures defined on $\{ a_1, a_2, a_3, a_4 \}$ and $\{ a_3, a_4, b_1, b_2 \}$ are isomorphic. Let $A$ be a representative of this isomorphism class.

Since $M_0$ is the universal homogeneous structure with respect to $K_0$, taking any tuple $a_1' < a_2' < a_3' < a_4' < b_1' < b_2'$ with pairwise distinct $E_1$-classes we can find $B$ in $M_0$ as a half of a copy of a structure from $K_0$ consisting of 12 elements where each $E_1$-class is represented by a pair.

To show that the Ramsey property does not hold for the age of $(M_0, <)$ take any finite substructure $C$ of this age which extends $B$. Fix any enumeration of $E_2$-classes occurring in $C$. Then colour a copy of $A$ red if the class of the first two elements is enumerated before the class of the last pair. Otherwise colour such a copy green. It is clear that $C$ does not contain a structure isomorphic to $B$ so that all substructures of type $A$ are of the same colour. □

3.2 Adding circular orders

The structure found in [12] is build by a generalized Fra\'issé’s construction, appealing to Theorem 2.10 of [7], p.44. We now recall that material.

Let $L$ be a relational language and let $C$ be a class of finite $L$-structures. Let $E$ be a class of embeddings $\alpha : A \to B$ (where $A, B \in C$) such that any isomorphism $\delta$ between $C$-structures (from $\text{Dom}(\delta)$ onto $\text{Range}(\delta)$) is in $E$, the class $E$ is closed under composition and the following property holds:

if $\alpha : A \to B$ is in $E$ and $C \subseteq B$ is a substructure in $C$ such that $\alpha(A) \subseteq C$, then the map obtained by restricting the range of $\alpha$ to $C$ is also in $E$.

We say that a structure $A \in C$ is a strong substructure of an $L$-structure $M$ if $A \subseteq M$ and any inclusion $A \subseteq B$ with $B \in C$ and $B \subseteq M$ is an $E$-embedding. We call an embeddings $\rho : C \to M$ strong if $C \in C$ and $\rho(C)$ is a strong substructure of $M$.

Theorem 2.10 of [7] states that if

(a) the number of isomorphism types of $C$-structures of any finite size is finite;
(b) the class $E$ satisfies the joint embedding property and the amalgamation property and
(c) there is a function $\theta$ on the natural numbers such that any finite $L$-structure $C$ embeds into some $A \in C$ of size $\leq \theta(|C|)$ such that any embedding from $A$ to a $C$-structure is strong;

then there exists a countably categorical $L$-structure $M$ such that $M$ is generic, i.e.

(a’) $C$ is the class (up to isomorphism) of all strong substructures of $M$;
(b’) $M$ is a union of a chain of $E$-embeddings and
Moreover, any isomorphism between strong finite substructures of $M$ is strongly embeddable into $M$ over $A$.

For such a structure $M$ we say that $M$ satisfies Hrushovski’s extension property if for any finite family of isomorphisms between finite strong substructures of $M$, $\phi_i : B_i \to C_i$, $i \leq k$, there is a finite strong substructure $B < M$ containing $\bigcup_{i \leq k}(B_i \cup C_i)$ so that each $\phi_i$ extends to an automorphism of $B$.

We now describe our example, say $M$. Let $L = \{E_n, K_n, R_n : 2 < n \in \omega\}$ be a first-order language, where each $E_n$ and $R_n$ is a relational symbol of arity $2n$ and each $K_n$ has arity $3n$. The $L$-structure $M$ is built by the version of Fraïssé’s construction presented above. Let us specify a class $\mathcal{K}$ of finite $L$-structures, which will become the class of all finite substructures of $M$.

Assume that in each $C \in \mathcal{K}$ each relation $E_n(\bar{x}, \bar{y})$ determines an equivalence relation on the set (denoted by $\binom{C}{n}$) of unordered $n$-element subsets of $C$. As before for $C \in \mathcal{K}$ and $n > |C|$ we put that no $2n$-tuple from $C$ satisfies $E_n(\bar{x}, \bar{y})$.

The relations $R_n$ are irreflexive. The $R_n$-arrows respect $E_n$,

$$\forall \bar{x}, \bar{y}, \bar{u}, \bar{w}(E_n(\bar{x}, \bar{y}) \land E_n(\bar{u}, \bar{w}) \land R_n(\bar{x}, \bar{u}) \to R_n(\bar{y}, \bar{w})),$$

and define a partial 1-1-function on $\binom{C}{n}/E_n$.

Every $K_n$ is interpreted by a circular order $\ll$ on the set of $E_n$-classes. Therefore we take the axiom

$$\forall \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}(E_n(\bar{x}, \bar{y}) \land E_n(\bar{u}, \bar{w}) \land E_n(\bar{z}, \bar{v}) \land K_n(\bar{x}, \bar{z}, \bar{u}) \to K_n(\bar{y}, \bar{v}, \bar{w})).$$

and the corresponding axioms of circular orders. We also take some axioms connecting $K_n$ and $R_n$:

$$R_n(\bar{x}, \bar{y}) \land R_n(\bar{y}, \bar{z}) \to K_n(\bar{x}, \bar{y}, \bar{z});$$

$$\forall \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3(\bigwedge_{i \leq 3} R_n(\bar{v}_i, \bar{w}_i) \to (K_n(\bar{v}_1, \bar{v}_2, \bar{v}_3) \leftrightarrow K_n(\bar{w}_1, \bar{w}_2, \bar{w}_3))).$$

These axioms say that $R_n$ defines a partial automorphism of the circular order induced by $K_n$ on $\binom{C}{n}/E_n$. Our final axioms state that this partial automorphism admits an extension to a 1-1-function $f$ (on some larger domain) such that $f^m$ is the identity on its domain, but for each $V \in \binom{C}{n}/E_n$ and $m \neq 0$ with $m < n$ we have $f^m(V) \neq V$. These conditions can be written by an infinite set of universal first-order formulas (which forbid all inconsistent situations).

We say that a structure $A \in \mathcal{K}$ is strong, if for every $n \in \omega$ all elements of $\binom{A}{n}$ are pairwise equivalent with respect to $E_n$ or for any $\bar{a} \in \binom{A}{n}$ there is a sequence $\bar{a}_1 (\equiv \bar{a}), \ldots, \bar{a}_n$ of pairwise non-$E_n$-equivalent tuples from $\binom{A}{n}$ such that $(\bar{a}_i, \bar{a}_{i+1}) \in R_n$, $1 \leq i \leq n - 1$, and $(\bar{a}_n, \bar{a}_1) \in R_n$. Let $\mathcal{C}$ be the class of all strong structures from $\mathcal{K}$.

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(c') if $A$ is a strong substructure of $M$ and $\alpha : A \to B$ is in $E$ then $B$ is strongly embeddable into $M$ over $A$. 

1 A twisted around total order with the natural ternary relation induced by the relation $x < y < z$
It is proved in [12] that \( C \) is cofinal in \( K \) and satisfies the joint embedding property and the amalgamation property.

Let \( \mathcal{E} \) be the class of all embeddings between (strong) structures from \( C \). It is proved in [12] that the pair \(( C, \mathcal{E})\) satisfies the conditions of Theorem 2.10 of [7]. The generic structure of this class is \( \omega \)-categorical and non-G-compact.

For every \( n > 2 \) let \( M_n = (M_n)/E_n \). The construction of [12] (and a general theorem from [8]) implies that the structure induced on \( M_n \) by the relations of \( M \) coincides with \((M_n, C^3, R_n)\), where \( C^3 \) is the ternary relation of a dense circular ordering defined by \( K_n \) and \( R_n \) is a binary relation inducing an \( n \)-cycle as described above. This in particular means that any automorphism of \((M_n, C^3, R_n)\) can be realized by an automorphism of \( M \).

Let \( \mathcal{L}' \) be an extension of \( \mathcal{L} \) and \( M' = (M, \bar{r}) \) be an \( \mathcal{L}' \)-expansion of \( M \) which is ubiquitous in category. We do not demand that \( \bar{r} \) is finite we only assume that \( M' \) is a precompact expansion. Let \( \mathcal{B}_T \) be the corresponding set of \( \bar{r} \)-diagrams. Let \( \mathcal{C}' \subset \mathcal{B}_T \) be a class of diagrams \( D(\bar{b}) \) so that \( \bar{b} \) is an enumeration of a strong substructure from \( C \). In the following theorem we assume that \( \mathcal{C}' \) satisfies Truss’ condition.

**Theorem 3.2** Let \( M \) be the generic structure of \((C, \mathcal{E})\). Then the group \( G = \text{Aut}(M) \) is amenable and \( M \) does not satisfy Hrushovski’s extension property.

Let \( M' \) be a precompact expansion of \( M \) which is ubiquitous in category and is defined with respect to a family \( \mathcal{C}' \subset \mathcal{B}_T \) satisfying Truss’ condition. If \( \text{Aut}(M') \) is extremely amenable, then for any \( n \) realized as \( |\bar{b}| \) with some \( D(\bar{b}) \in \mathcal{C}' \) there is \( \alpha \in \text{Aut}(M_n, C^3, R_n) \) which cannot be realized by an automorphism of \( M' \).

The main point of this theorem is that although in different arities the structures induced by \( M \) are completely independent, any expansion \( M' \) as in the formulation simultaneously destroys \( M \) in all reasonable arities. The proof uses some material from [9]. We now describe it.

Let \( L \) be a finite relational language. An \( L \)-structure \( F \) is called a link structure if \( F \) is a singleton or \( F \) can be enumerated \( \{a_1, ..., a_n\} \) so that \((a_1, ..., a_n)\) satisfies an irreflexive relation from \( L \).

Let \( \mathcal{S} \) be a finite set of link structures. Then an \( L \)-structure \( N \) is of link type \( \mathcal{S} \) if any substructure of \( N \) which is a link structure is isomorphic to a structure from \( \mathcal{S} \).

An \( L \)-structure \( F \) is packed if any pair from \( F \) belongs to a link structure which is a substructure of \( F \).

If \( \mathcal{R} \) is a finite family of packed irreflexive \( L \)-structures, then an \( L \)-structure \( F \) is called \( \mathcal{R} \)-free if there does not exist a weak homomorphism (a map preserving the predicates) from a structure from \( \mathcal{R} \) to \( F \).

Proposition 4 and Theorem 5 of [9] state that for any family of irreflexive link structures \( \mathcal{S} \) and any finite family of irreflexive packed \( L \)-structures \( \mathcal{R} \) the class of all irreflexive finite \( L \)-structures of link type \( \mathcal{S} \) which are \( \mathcal{R} \)-free, has the free amalgamation property and Hrushovski’s extension property for partial isomorphisms (i.e. Hrushovski’s extension property holds in the Fraïssé limit of this class).
Proof of Theorem 3.2. For each $n > 2$ enumerate all $E_n$-classes. Consider the expansion of $M$ by distinguishing each $E_n$-class by a predicate $P_{n,i}$ according the enumeration. Let $L^*$ be the language of all predicates $P_{n,i}$ and let $M^*$ be the $L^*$-structure defined on $M$. For every finite sublanguage $L' \subseteq L^*$ let $M^*(L')$ be the $L'$-reduct of $M^*$ defined by these interpretations.

We denote by $\mathcal{K}(L')$ the class of all finite $L'$-structures with the properties that for any arity $l$ represented by $L'$:

- any $l$-relation is irreflexive and invariant with respect to all permutations of variables,
- any two relations of $L'$ of arity $l$ have empty intersection.

Let $\mathcal{S}(L')$ be the set of all link structures of $\mathcal{K}(L')$ satisfying these two properties. Thus $\mathcal{K}(L')$ is of link type $\mathcal{S}(L')$.

**Claim 1.** For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is a universal structure with respect to the class $\mathcal{K}(L')$.

It is easy to see that any structure from $\mathcal{K}(L')$ can be expanded to a structure from $\mathcal{K}$ so that $L'$-predicates become classes of appropriate $E_n$’s.

**Claim 2.** For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is an ultrahomogeneous structure.

Let $f$ be an isomorphism between finite substructures of $M^*(L')$. We may assume that $\text{Dom}(f)$ contains tuples representing all $M^*(L')$-predicates of $L'$. Moreover some tuples can be added so that $\text{Dom}(f)$ as a substructure of $M$ belongs to $\mathcal{C}$. Then $f$ extends to an automorphism of $M$ fixing the classes of appropriate $E_n$’s which appear in $L'$. Thus this automorphism is an automorphism of $M^*(L')$ and of the $L'$-reduct of $M^*(L')$ too.

**Claim 3.** For each finite sublanguage $L' \subseteq L^*$ let $\mathcal{R}(L')$ be the family of all packed $L'$-structures of the form $\langle \{a_1, \ldots, a_n\}, P_{n,i}, P_{n,j} \rangle$, where $i \neq j$, $P_{n,i} = \{(a_1, \ldots, a_n)\}$ and $P_{n,j} = \{(a_{\sigma(1)}, \ldots, a_{\sigma(n)})\}$ for some permutation $\sigma$. Then the class $\mathcal{K}(L')$ is the class of all irreflexive finite $L'$-structures of link type $\mathcal{S}(L')$, which are $\mathcal{R}(L')$-free.

The claim is obvious. By Proposition 4 and Theorem 5 of [9] we now see that $\mathcal{K}(L')$ is closed under substructures, has the joint embedding property, the free amalgamation property and Hrushovski’s extension property.

By Claim 1 and Claim 2 the structure $M^*(L')$ is the universal homogeneous structure of $\mathcal{K}(L')$. In particular any tuple of finite partial isomorphisms of $M^*(L')$ can be extended to a tuple of automorphisms of a finite substructure of $M^*(L')$.

Note that the same statement holds for the structure $M^*$. To see this take any tuple $f_1, \ldots, f_k$ of finite partial isomorphisms of $M^*$. Let $r$ be the size of the union $\bigcup_{i \leq k} \text{Dom}(f_i)$ and $L'$ be the sublanguage of $L^*$ of arity $r$ consisting of all relations of $M^*$ which meet any tuple from $\bigcup_{i \leq k} \text{Dom}(f_i)$. Then there is a finite substructure $A$ of $M^*(L')$ containing $\bigcup_{i \leq k} \text{Dom}(f_i)$ so that each $f_i$ extends to an automorphism of $A$.

Let $r'$ be the size of $A$. Let $L''$ be a sublanguage of $L^*$ so that $L' \subseteq L''$ and for each arity $l \leq r'$ the sublanguage $L'' \setminus L'$ contains exactly exactly one $l$-relation, say
Since $M^*$ is the universal homogeneous structure of $\mathcal{K}(L'^{n})$ the substructure $A$ can be chosen so that any $l$-subset of $A$ which does not satisfy any relation from $L'$, does satisfy $P_{l,n_i}$.

As a result any automorphism of $A$ extends to an automorphism of $M^*(L'^{n''})$ for any $L'^{n''} \subset L'^{n}$ containing $L'^{n}$. In particular it extends to an automorphism of $M^*$.

As in Proposition 6.4 of [17] we see that $\text{Aut}(M^*)$ has a dense subgroup which is the union of a countable chain of compact subgroups. In particular $\text{Aut}(M^*)$ is amenable.

Let us consider the structure $(M_n, C^3, R_n)$, where $n > 2$. Treating $R_n$ as an automorphism of it we easily see that the quotient of $\text{Aut}(M_n, C^3, R_n)$ by the closed central subgroup $(R_n)$ is isomorphic to the induced group of permutations on the set of $R_n$-orbits (where each orbit consists of $n$ elements). It is also clear that the structure of these $R_n$-orbits is isomorphic to the circular order $(\mathbb{Q}, C^3)$ (where $\mathbb{Q}$ is twisted around). By Theorem 449C of [3] the group $\text{Aut}(M_n, C^3, R_n)$ is amenable if so is $\text{Aut}(\mathbb{Q}, C^3)$.

**Claim 4.** $\text{Aut}(\mathbb{Q}, C^3)$ is amenable.

Let $\mathcal{K}_C$ be the class of all finite substructures of $(\mathbb{Q}, C^3)$. It is an easy exercise that the latter is the Fraïssé limit of the former (in particular $\mathcal{K}_C$ satisfies JEP and AP). Let $\mathcal{K}_<$ be the class of all expansions of $\mathcal{K}_C$ by linear orderings which after twisting become the given circular order. It also has JEP and AP and the structure $(\mathbb{Q}, C^3, <)$ is the Fraïssé limit of $\mathcal{K}_<$.

It can be easily verified that $(\mathcal{K}_C, \mathcal{K}_<)$ is an excellent pair in the sense of Section 9 of [3]. For example the ordering property and the Ramsey property follow from the fact that they hold for linear orderings [15]. Thus the space $X$ of all $C$-admissible order expansions of $(\mathbb{Q}, C^3)$ forms the universal minimal flow of $\text{Aut}(\mathbb{Q}, C^3)$.

For every finite substructure $A \subset (\mathbb{Q}, C^3)$ and an ordering $< \in A$ let $N_{A,<}$ be the set of all expansions of $(\mathbb{Q}, C^3)$ by a $C$-admissible linear orders which agree with $<$. The class of sets of this form generates the Borel $\sigma$-algebra of $X$. If $|A| = n$, put

$$\mu(N_{A,<}) = \frac{1}{n}.$$  

This uniquely defines an $\text{Aut}(\mathbb{Q}, C^3)$-invariant measure on $X$ (the discussion in the end of Section 9 of [3] can be helpful at this point). This proves Claim 4.

Since each automorphism of $M$ preserves all $R_i$, $i > 2$, it is easy to see that there is a natural homomorphism from $\text{Aut}(M)$ to the product $\prod_{i>2} \text{Aut}(\mathbb{Q}_i/C^3, R_i)$ and $\text{Aut}(M^*)$ is the kernel of it. By Theorem 449C of [3] the group $\text{Aut}(M)$ is amenable.

To see that $M$ does not satisfy Hrushovski’s extension property let us consider any triple of $n$-tuples $\bar{a}, \bar{b}, \bar{c}$ representing pairwise distinct elements of $M_n$ so that for some $\bar{a}'$ with $R_n(\bar{a}, \bar{a}')$ we have

$$K_n(\bar{a}, \bar{b}, \bar{a}') \land K_n(\bar{a}, \bar{c}, \bar{a}') \land K_n(\bar{a}, \bar{b}, \bar{c}).$$

Then the map $\phi$ fixing $\bar{a}$ and taking $\bar{b}$ to $\bar{c}$ cannot be extended to an automorphism of a finite substructure of $M$. Moreover we may arrange that $\text{Dom}\phi$ is a strong substructure of $M$. 

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Let us prove the second statement of the theorem. Let \(n\) be as in the formulation, \(A\) be a strong substructure of size \(n\) and let \(A'\) be an expansion of \(A\) to the language of \(M'\) realizing an element from \(C'\). Since \(\text{Aut}(M')\) is extremely amenable, the class \(C'\) has the Ramsey property. We assign black to all \(\bar{b}\) of the type \(tp^{M'}(A')\) (and to the corresponding diagrams on these \(\bar{b}\)) so that

\[
M' \models \exists \bar{y}(R_n(A', \bar{y}) \land K_n(A', \bar{b}, \bar{y})).
\]

We color \(\bar{b} \models tp^{M'}(A')\) white if this condition does not hold. Let \(B\) be a strong substructure of \(M\) containing an \(R_n\)-cycle starting with \(A\) so that its type defines some \(D(B) \in C'\). Assuming that any \(\alpha \in \text{Aut}(M_n, C^3, R_n)\) can be realized by an automorphism of \(M'\), we may choose \(B\) so that each vertex of our \(R_n\)-cycle is represented in \(B\) by a tuple realizing \(tp^{M'}(A')\). By the proof of Theorem 1.5 of [11], Truss’ condition on \(C'\) implies that the type of any \(\bar{b}\) with \(D(\bar{b}) \in C'\) is determined in \(M'\) by the basic formula corresponding to \(D(\bar{b})\). Thus any copy of \(D(B)\) in \(C'\) contains and \(R_n\)-cycle consisting of tuples realizing \(tp^{M'}(A')\). This contradicts to the Ramsey property. \(\square\)

**Remark 3.3** The structure \(M\) does not have generic expansions by a linear order as in Theorem [14]. This can be proved by an appropriate version of Theorem [3.1].

### 4 Appendix: Enumerations

For convenience of the reader in this appendix we repeat an argument from [12] which shows that the \(L_0\)-structure \(M_0\) introduced above does not have a nice enumeration.

**Proposition 4.1** The structure \(M_0\) is \(K_0\)-categorical and does not have any AZ-enumeration.

**Proof.** Since for each \(n\) the number of finite structures of \(K_0\) of size \(n\) is finite, the structure \(M_0\) is \(K_0\)-categorical and admits elimination of quantifiers (by Fraïssé’s theorem). Now for a contradiction suppose that there is an ordering \(\prec\) defining an AZ-enumeration of \(M_0\). We will define an infinite sequence of pairs \(a_n \prec b_n, n \in \omega\), satisfying the following conditions. For \(n > 2\) the elements \(a_n\) and \(b_n\) are chosen so that for any \((n - 1)\)-tuple of the form \(x_1 \prec x_2 \prec \ldots \prec x_{n-1}\) with \(x_{n-1} \prec a_n\) the tuple \((x_1, x_2, \ldots, x_{n-1}, b_n)\) is \(E_n\)-equivalent with some \(n\)-tuple \(\bar{y}\) satisfying \(y_1 \prec \ldots \prec y_n \prec a_n\). On the other hand we also demand that for each \(j < n\), any \(j\)-tuple of the form \(d_1 \prec d_2 \prec \ldots \prec d_{j-1} \prec b_n\) with \(d_{j-1} \prec a_n\) is not \(E_j\)-equivalent with any \(j\)-tuple \(y_1 \prec \ldots \prec y_j\) with \(y_j \prec a_n\).

The pairs \((a_n, b_n)\) can be defined by induction. Let \(a_0 \prec a_1 = b_0 \prec b_1 \prec a_2 \prec b_2\) be the initial 5-element \(\prec\)-segment of \(M_0\). At step \(n > 2\) we just take \(a_n\) as the next element enumerated after \(b_{n-1}\). To define \(b_n\) consider the substructure of \(M_0\) defined on \(D = \{x : x \prec a_n\}\). We embed \(D\) into some \(K_0\)-structure \(D \cup \{b\}\) such that for each \(j < n\) all tuples \((y_1, \ldots, y_{j-1}, b)\) with \(y_1 \prec y_2 \prec \ldots \prec y_{j-1} \prec a_n\) form an \(E_j\)-class which does not meet any \(j\)-tuple from \(D\). We also demand that each \(n\)-tuple of \(D \cup \{b\}\) is
$E_n$-equivalent with an $n$-tuple of $D$. Since $M_0$ is universal homogeneous, the element $b$ can be found in $M_0$. Let $b_n$ be the element of $M_0$ with $D \cup \{b_n\}$ isomorphic with $D \cup \{b\}$ over $D$ and having the minimal number with respect to $\prec$.

If $f : M_0 \to M_0$ is a $\prec$-preserving elementary map taking $(a_i, b_i)$ to $(a_j, b_j)$, then by the definition of $b_j$ any $i$-tuple of $\{x : x \prec a_j\} \cup \{b_j\}$ with $b_j$ is not $E_i$-equivalent with any tuple of $\{x : x \prec a_j\}$. By the definition of $b_i$ this is impossible. Therefore we have a contradiction with the definition of an AZ-enumeration. □

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