RIGIDITY AND REGULARITY OF CO-DIMENSION ONE
SOBOLEV ISOMETRIC IMMERSIONS

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Abstract. We prove the developability and $C^{1,1/2}_{\text{loc}}$ regularity of $W^{2,2}$ isometric immersions of $n$-dimensional domains into $\mathbb{R}^{n+1}$. As a conclusion we show that any such Sobolev isometry can be approximated by smooth isometries in the $W^{2,2}$ strong norm, provided the domain is $C^4$ and convex. Both results fail to be true if the Sobolev regularity is weaker than $W^{2,2}$.

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1. Introduction

It has been known since at least the 19th century that any smooth surface with zero Gaussian curvature is locally ruled, i.e. passing through any point of the surface is a straight segment lying on the surface. Such surfaces were called developable surfaces. This terminology was used as an indication that any such surface is in isometric equivalence with the plane, i.e. any piece of it can be developed on the flat plane without any stretching or compressing. Meanwhile, it was already suspected that there exist somewhat regular surfaces applicable to the plane, but yet not developable (See [4] for a review of this question). Nevertheless, it was not until the work of John Nash at the zenith of the last century that the existence of such unintuitive phenomena was rigorously established.

In his pioneering work, Nash settled several questions. He established that any Riemannian manifold can be isometrically embedded in an Euclidean space [19]. Moreover, if the dimension of the space is large enough, this embedding can be done in a manner so that the diameter of the image is as small as one wishes. As for the lower dimensional embeddings, Nash [20] and Kuiper [17], established the existence of a $C^1$ isometric embedding of any Riemannian manifold into another manifold of one higher dimension. Their method, which is now famously re-cast in 1991 Mathematica Subject Classification. 58D10, 46T10, 30L05, 53C24.

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the framework of convex integration [8], involved iterated perturbations of a given short mapping of the manifold towards realizing an isometry.

A surprising corollary of these results is the existence of a $C^1$ flat torus in $\mathbb{R}^3$ [3]. Another one is that there are $C^1$ isometric embeddings of the two dimensional unit sphere into three dimensional space with arbitrarily small diameter. By contrast, it was established by Hartman and Nirenberg that any flat $C^2$ surface in $\mathbb{R}^3$ must be developable [9], while Hilbert had already shown that any $C^2$ isometric immersion of the sphere must be a rigid motion. This latter result is a special case of a similar statement for any closed convex surface in $\mathbb{R}^3$, see [24 Chapter 12]. On the other hand, the former result was generalized by Pogorelov’s for $C^1$ isometries with total zero curvature in [23, Chapter II] and [24, Chapter IX].

A natural question arises in this context for the analyst: What about isometric immersions of intermediate regularity, say of Hölder or Sobolev type? Regarding Hölder regularity, rigidity of $C^{1,\alpha}$ isometries of 2 dimensional flat domains has been established for $\alpha \geq 2/3$ [11,2], while their flexibility in the sense of Nash and Kuiper is known for $\alpha < 1/7$ [2,6]. The critical value for $\alpha$ is conjectured to be $1/2$ in this case. As for the regularity of Sobolev isometries, following the results of Kirchheim in [10] on $W^{2,\infty}$ solutions to degenerate Monge-Ampère equations (see Proposition 1.3), the rigidity of $W^{2,2}$ isometries of a flat domain was established in [22]. More precisely, it was established that such mappings are developable in the classical sense, i.e.

**Theorem 1.1** (Pakzad [22]). Let $v \in W^{2,2} (\Sigma, \mathbb{R}^3)$ be an isometric immersion, where $\Sigma$ is a bounded Lipschitz domain in $\mathbb{R}^2$. Then $v \in C^{1,1/2}_{\text{loc}} (\Sigma, \mathbb{R}^2)$. Furthermore, for every point of $x$, either there exists a neighborhood of $x$, or a unique segment passing through $x$ and joining $\partial \Sigma$ at both ends, on which $\nabla v$ is constant.

**Remark 1.2.** It can be shown that this statement is actually valid for all bounded open sets $\Sigma \subset \mathbb{R}^2$, i.e. without any assumption on the regularity of the boundary. All one must prove is that the constancy segments, whose existence are locally established, can be extended all the way to the boundary one step at a time. Assuming the existence of any supposedly maximal constancy segment which does not reach the boundary, a contradiction could be achieved by creating a Lipschitz domain $\Sigma' \subset \Sigma$ including the closure of that segment and applying Theorem 1.1 to $\Sigma'$. In the same manner, the regularity assumption on $\partial \Omega$ in Theorem 1.4 can be removed.

To put this result in context, it is noteworthy that a $W^{2,2}$ function on a two dimensional domain fails barely to be $C^1$, but there is information available about second weak derivatives, and e.g. the Gaussian curvature of the image of a $W^{2,2}$ isometric immersion of a flat domain is identically zero as an $L^1$ function. This indicates that these isometries are far from the highly oscillatory solutions of Nash and Kuiper and hence possibly should behave in a rigid manner. Note that only the $C^1$ regularity result was stated in [22] and was a major ingredient of the proof, but the higher Hölder regularity announced here is an immediate consequence of the developability. In [18] it was established that the $C^1$ regularity can be extended up to the boundary if the domain is of class $C^{1,\alpha}$. This does not hold true anymore for merely $C^1$ regular domains. Finally, the following proposition is a key step in establishing the above rigidity result and will be instrumental in proving Theorem 1.4.
**Proposition 1.3** (Kirchheim [16], Pakzad [22]). Let $\Sigma$ be as above and let $f \in W^{1,2}(\Sigma, \mathbb{R}^3)$ be a map with almost everywhere symmetric and singular (i.e. of zero determinant) gradient. Then $f \in C^0(\Sigma)$ and for every point $x \in \Sigma$, there exists either a neighborhood $U$ of $x$, or a segment passing through it and joining $\partial \Sigma$ at its both ends, on which $f$ is constant.

It was proved furthermore in [22] that any $W^{2,2}$ isometry on a convex 2d domain can be approximated in strong norm by smooth isometries. This is a nontrivial result, since the usual regularization techniques fail due to the non-linearity of the isometry constraint. The idea was to make use of the developability structure of these mappings and reduce the approximation problem to the one about mollifying the expressions $R^T R'$ for the Darboux moving frames $R(t)$ along the curves orthogonal to the rulings. The convexity assumption is a technical one, and as shown by Hornung [10, 12], can be replaced by e.g. piece-wise $C^1$ regularity of the boundary, see also [11].

It is natural to ask whether these results can be generalized to higher dimensions. In [26], the authors showed the generalized developability of smooth isometric immersions of Euclidean domains into Euclidean spaces. We would like to pose the same problem for the same class of isometric immersions but only considered under sufficient Sobolev regularity assumptions. A first main result in this direction, presented in this paper, is the developability of $W^{2,2}$ co-dimension one isometries of flat domains in $\mathbb{R}^n$:

**Theorem 1.4.** Let $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ be an isometric immersion, where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Then $u \in C^{1,1/2}_{\text{loc}}(\Omega, \mathbb{R}^{n+1})$. Moreover, for every $x \in \Omega$, either $\nabla u$ is constant in a neighborhood of $x$, or there exists a unique $(n-1)$-dimensional hyperplane $P \ni x$ of $\mathbb{R}^n$ such that $\nabla u$ is constant on the connected component of $x$ in $P \cap \Omega$.

The interesting feature of this new result is that the Sobolev regularity $W^{2,2}$ is much below the required $W^{2,n+\varepsilon}$ for obtaining $C^1$ regularity. An extra difficulty which comes in the way of the proof in dimensions higher than 2 is that the argument used in Lemma 2.1 of [22] to show the continuity of the derivatives of the given Sobolev isometry is no more generalizable to our case. Indeed, in [22], a very important first step of the proof of developability is to show the $C^1$ regularity. Here, on the other hand, we first show the developability of the mapping without having the $C^1$ regularity at hand. Our proof is based an induction on the dimension of slices of the domain and careful and detailed geometric arguments. Having established developability, the $C^1$ regularity (and better) follows in a straightforward manner.

The problem of regularity and developability of Sobolev isometric immersions of co-dimension $k > 1$ is more involved and could not be tackled through the methods discussed in this paper. In a forthcoming paper by Jerrard and the second author [15], another approach, more analytical in nature, is adapted to study this problem. It is based on the fact that the Hessian rank inequality

$$\text{rank}(\nabla^2 v) \leq k \quad \text{a.e. in } \Omega$$

is satisfied by the components $v = v^j$ of such isometry. Note that this equation becomes the degenerate Monge-Ampère equation when $k = n - 1$. Similar as in [22], regularity and developability of the Sobolev solutions to (1.1) directly implies the same results for the corresponding isometries. However, one loses some natural
advantages when working with (1.1) rather than with the isometries themselves as done in the present paper: the solution \( v \) is no more Lipschitz and being just a scalar function, one loses the extra information derived from the length preserving properties of isometries. Methods of geometric measure theory applied to the class of Monge-Ampère functions developed by Jerrard in [13, 14] are used to overcome these obstacles.

The second main result of this paper concerns approximation of \( W^{2,2} \) isometries by smooth ones:

**Theorem 1.5.** Assume \( \Omega \subset \mathbb{R}^n \) is a \( C^1 \) bounded convex domain and that \( u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \) is an isometric immersion. Then there is a sequence of isometric immersions \( u_m \in C^\infty(\Omega, \mathbb{R}^{n+1}) \) converging to \( u \) in \( W^{2,2} \) norm.

The main idea of the proof, similar as in the 2d case, is to mollify the curves which pass orthogonally through the constancy hyperplanes of Theorem 1.4 both in the domain and on the image. This latter problem, framed within the general isometry mollification problem, is still nonlinear. However, identifying these curves with suitable orthonormal moving Darboux frames \( R(t) \in SO(n) \) and \( \tilde{R}(t) = [(\nabla u)R(t), n(t)] \in SO(n+1) \), where \( n \) is the unit normal to the image of the isometry in \( \mathbb{R}^{n+1} \), we could linearize the problem by considering the curvature matrices \( R^T R'(t) \in so(n) \) and \( \tilde{R}^T \tilde{R}'(t) \in so(n+1) \) and recover an approximating sequence of moving frames through their regularization. Many technical details must nevertheless be taken care of in this process; in particular one must make sure that the mollified curves can be used to define new smooth isometries. Also, the mapping as a whole cannot be described by one single couple of such curves and the domain must be partitioned into suitable subdomains.

**Remark 1.6.** Neither the \( C^1 \) regularity nor the convexity of the boundary seems to be absolutely necessary for the density result to hold true (see e.g. [12] for finer results in 2d), but omitting these assumptions goes beyond the scope of our paper. However, both of the results in Theorems 1.4 and 1.5 are sharp in the sense that they fail to be true if the isometric immersion is only of class \( W^{2,p} \) for \( p < 2 \). An immediate counterexample is the following isometric immersion \( u : B^2 \times (0,1)^{n-2} \to \mathbb{R}^{n+1} \), whose image can be visualized as a family of cones over a hyperplane of dimension \( n-2 \):

\[
u(r \cos \theta, r \sin \theta, x_3, \cdots, x_n) := \left( \frac{r}{2} \cos(2\theta), \frac{r}{2} \sin(2\theta), \frac{\sqrt{3}}{2} r, x_3, \cdots, x_n \right).
\]

The paper is organized as follows: In Section 2 we will review some basic analytic properties of isometric immersions with second order derivatives. Section 3 is dedicated to the proof of Theorem 1.4. In Section 4 we will show that smooth isometric immersions are strongly dense in the space of \( W^{2,2} \) isometric immersions from a domain of \( \mathbb{R}^n \) into \( \mathbb{R}^{n+1} \). The proof of Lemma 4.13, which is a crucial and difficult step in establishing the density result is postponed to the Appendix for the convenience of the reader.

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Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^n, n \geq 2$. We define the class of Sobolev isometric immersions from $\Omega$ to $\mathbb{R}^{n+1}$ as,

$$I^{2,2}(\Omega, \mathbb{R}^{n+1}) := \{ u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) : (\nabla u)^T \nabla u = I \text{ a.e.} \}. $$

Note that the condition $(\nabla u)^T \nabla u = I$ implies that $u$ is Lipschitz continuous, thus,

$$I^{2,2}(\Omega, \mathbb{R}^{n+1}) \subset W^{2,2}(\Omega, \mathbb{R}^{n+1}) \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1}).$$

Given $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, let $u^j, 1 \leq j \leq n + 1$, be the $j$-th component of $u$ and let $u_i = \partial u / \partial x_i, 1 \leq i \leq n$, be the partial derivative of $u$ in the $e_i$ direction.

Throughout the paper we will use the same notation for all functions.

For a.e. $x \in \Omega$, consider the cross product

$$\mathbf{n}(x) = u_1(x) \times \cdots \times u_n(x).$$

That is, $\mathbf{n}(x)$ is the unique unit vector orthogonal to $u_i(x)$ for all $1 \leq i \leq n$ such that

$$u_1(x), \ldots, u_n(x), \mathbf{n}(x)$$

form a positive basis of $\mathbb{R}^{n+1}$.

Note that $\mathbf{n}$ can also be identified as differential forms: consider the 1-form,

$$\omega_i = \sum_{j=1}^{n+1} u_i^j dx_j.$$ 

Then

$$\mathbf{n} = *(\omega_1 \wedge \cdots \wedge \omega_n).$$

because for any $\xi \in \bigwedge^1(\mathbb{R}^{n+1}),$

$$\langle \xi, \mathbf{n} \rangle = \langle \xi, *(\omega_1 \wedge \cdots \wedge \omega_n) \rangle = (-1)^n \xi \wedge \omega_1 \wedge \cdots \wedge \omega_n = (-1)^n \det[\xi, u_1, \ldots, u_n].$$

Since $u \in W^{2,2} \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1})$, it follows from (2.3) that $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{R}^n)$.

Since $u$ is isometric immersion, $\langle u_i, u_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, we can differentiate using the product rule to obtain,

$$\langle u_{ik}, u_j \rangle + \langle u_i, u_{jk} \rangle = 0 \text{ a.e.}$$

Permutation of indices $i, j, k$ yields,

$$\langle u_{ij}, u_k \rangle + \langle u_i, u_{kj} \rangle = 0 \text{ a.e.}$$

$$\langle u_{ki}, u_j \rangle + \langle u_k, u_{ji} \rangle = 0 \text{ a.e.}$$

Using the fact that $u_{ij} = u_{ji}$ for all $i, j$, we add (2.5) and (2.6), then subtract (2.6) to obtain,

$$\langle u_i, u_{jk} \rangle = 0 \text{ a.e. for all } 1 \leq i, j, k \leq n.$$ 

Since for a.e. points in the domain, $\mathbf{n}, u_1, \ldots, u_j$ form an orthonormal basis for $\mathbb{R}^{n+1}$, we can write,

$$u_{jk} = \sum_{i=1}^{n} \langle u_{jk}, u_i \rangle u_i + \langle u_{jk}, \mathbf{n} \rangle \mathbf{n}. $$
\[ u_{jk} = \langle u_{jk}, n \rangle \quad \text{a.e. for all } \quad 1 \leq j, k \leq n. \]  

Note that \( A_{jk} := \langle u_{jk}, n \rangle \) is the element in row \( j \) and column \( k \) of the second fundamental form \( A \), which is a symmetric \( n \times n \) matrix. In particular, (2.8) holds for each component of \( u_{jk} \) and \( n \), i.e.,

\[ u_{jk} = A_{jk} n^\ell \quad \text{for all } \quad 1 \leq \ell \leq n + 1, \quad 1 \leq j, k \leq n. \]

Thus, the Hessian of \( u^\ell \) satisfies,

\[ \nabla^2 u^\ell = n^\ell A, \quad 1 \leq \ell \leq n + 1. \]

**Lemma 2.1.** The second fundamental form \( A \in M^{n \times n} \) has the following properties,

\[ \frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} = 0 \quad \text{in distributional sense for all } \quad 1 \leq i, j, k \leq n, \]

and

\[ A_{ij} A_{kl} - A_{il} A_{kj} = 0 \quad \text{for all } \quad 1 \leq i, j, k, l \leq n. \]

**Proof.** For a smooth immersion \( v : \Omega \to \mathbb{R}^{n+1} \), not necessarily isometric, let \( g_{ij} = \langle v_{,i}, v_{,j} \rangle \) be the first fundamental forms, then by differentiating \( g_{ij} \) twice,

\[ g_{ijkl} = \langle v_{,ijkl}, v_{,j} \rangle + \langle v_{,ikl}, v_{,ji} \rangle + \langle v_{,il}, v_{,jk} \rangle + \langle v_{,i}, v_{,jkl} \rangle. \]

The summation over the proper permutations of \( i, j, k, l \) yields

\[ g_{ijkl} + g_{klij} - g_{iljk} - g_{ijkl} = -2 \langle v_{,ij}, v_{,kl} \rangle + 2 \langle v_{,il}, v_{,kj} \rangle. \]

Given any other smooth immersion \( w : \Omega \to \mathbb{R}^{n+1} \), the following identity is also obvious,

\[ \langle v_{,ij}, w \rangle_{,k} - \langle v_{,ik}, w \rangle_{,j} = \langle v_{,ij}, w_{,k} \rangle - \langle v_{,ik}, w_{,j} \rangle. \]

Now we let a sequence of smooth immersions \( u_m \to u \) in \( W^{2,2}(\Omega, \mathbb{R}^{n+1}) \) and \( n_m \to n \) in \( W^{1,2}(\Omega, \mathbb{R}^{n+1}) \). Writing the left hand sides of (2.12) and (2.13) as distributional derivatives and passing to the limit we get,

\[ 0 = -2 \langle u_{,ij}, u_{,kl} \rangle + 2 \langle u_{,il}, u_{,kj} \rangle. \]

because \( \langle u_{,i}, u_{,j} \rangle = \delta_{ij} \) for all \( i, j \). In addition, since \( n \) is a unit vector, \( \langle n_{,k}, n \rangle = 0 \). Then by (2.8), \( \langle u_{,ij}, n_{,k} \rangle = 0 \) for all \( i, j, k \), thus,

\[ \langle u_{,ij}, n \rangle_{,k} - \langle u_{,ik}, n \rangle_{,j} = 0 \]

The two identities in the lemma follow easily from \( A_{ij} = \langle u_{,ij}, n \rangle \), (2.14), and (2.15). The proof is complete. \( \square \)

**Corollary 2.2.** The second fundamental form \( A \) satisfies \( \text{rank } A \leq 1 \) and \( A \) is symmetric a.e. in \( \Omega \). Moreover, the Hessian of each component of \( u \) satisfies \( \text{rank } \nabla^2 u^\ell \leq 1 \) for all \( 1 \leq \ell \leq n + 1 \) a.e. on \( \Omega \).

**Proof.** By identity (2.11), all \( 2 \times 2 \) minors of \( A \) vanish, hence the rank of \( A \) is less than or equal to 1. By (2.9), \( \text{rank } \nabla^2 u^\ell \leq \text{rank } A \leq 1 \) and \( A \) is symmetric a.e. since \( \nabla^2 u^\ell \) is symmetric a.e. The proof is complete. \( \square \)
3. Developability and Regularity

Our first main result- Theorem 1.4- follows from the following proposition:

**Proposition 3.1.** Let \( u \in H^{2,2}(\Omega, \mathbb{R}^{n+1}) \), where \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \). Let \( A \) be the second fundamental form of \( u \). Let \( P_k \) be a \( k \)-dimensional plane of \( \mathbb{R}^n \), \( k \leq n \). Suppose on \( P_k \cap \Omega \) we have the following properties,

1. There exists a sequence of smooth functions \( u^\varepsilon \) defined in the domain \( \Omega \) such that
   \[
   \int_{P_k \cap \Omega} |u^\varepsilon - u|^2 + |\nabla u^\varepsilon - \nabla u|^2 + |\nabla^2 u^\varepsilon - \nabla^2 u|^2 \, d\mathcal{H}^{n-k} \to 0.
   \]
   Here \( \nabla u^\varepsilon \), \( \nabla u \), \( \nabla^2 u^\varepsilon \) and \( \nabla^2 u \) denote the first and second full gradients with respect to the domain \( \Omega \).
2. The full gradient \( \nabla u \) satisfies \( \nabla u^\varepsilon \nabla u = 1 \, d\mathcal{H}^{k-1} \)-a.e. on \( P_k \cap \Omega \).
3. \( \nabla^2 u^\varepsilon = \mathbf{n}^k A \) for each \( 1 \leq \ell \leq n \) \( \mathcal{H}^{k-1} \)-a.e. on \( P_k \cap \Omega \).
4. \( \text{rank } A \leq 1 \) and \( A \) is symmetric \( \mathcal{H}^{k-1} \)-a.e. on \( P_k \cap \Omega \).

Then \( u \in C^{1,1/2}_{\text{loc}}(P_k, \mathbb{R}^{n+1}) \). Moreover, for every \( x \in P_k \cap \Omega \), either \( \nabla u \) is constant in a neighborhood in \( P_k \cap \Omega \) of \( x \), or there exists a unique \( (k-1) \)-dimensional hyperplane \( P_{k-1}^x \) such that \( \nabla u \) is constant on the connected component of \( x \) in \( P_{k-1}^x \cap \Omega \).

The proof of this proposition is based on induction on lower dimensional slices. Before we prove Proposition 3.1 we will show that it implies Theorem 1.4.

**Proof of Theorem 1.4.** We simply take \( k = n \) in Proposition 3.1, in which case \( P_n \cap \Omega = \Omega \). Since \( u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \), the convolution of \( u \) with the standard mollifier \( u^\varepsilon \) apparently satisfies assumption (1). By the fact that \( u \in H^{2,2}(\Omega, \mathbb{R}^{n+1}) \), \( \nabla u^\varepsilon \nabla u = 1 \) a.e. in \( \Omega \), which is property (2). Property (3) follows from equation (2.9) and property (4) follows from Corollary 2.2. Therefore, all the assumptions of Proposition 3.1 are satisfied, and hence the conclusion of Theorem 1.4 follows from the conclusion of Proposition 3.1. The proof is complete.

**Corollary 3.2.** Let \( u \in H^{2,2}(\Omega, \mathbb{R}^{n+1}) \), where \( \Omega \) is a Lipschitz domain in \( \mathbb{R}^n \). Then for every \( k \) dimensional slice \( P_k \cap \Omega \), \( \nabla u \) is constant either on \( k \) dimensional neighborhoods of \( P_k \cap \Omega \), or constant on \( (k-1) \)-dimensional slice of \( P_k \cap \Omega \).

**Proof.** Since assumptions (1)-(4) of Proposition 3.1 are satisfied a.e. in \( \Omega \). By Fubini Theorem, assumptions (1)-(4) also holds in a.e. \( k \)-dimensional slice. Thus the conclusion of Proposition 3.1 holds for a.e. \( k \)-dimensional slices. Since \( \nabla u \) is continuous, by a simple approximation argument, it holds on every \( k \)-dimensional slices. The proof is complete.

Assumptions (2) (3) and (4) regard the properties of isometric immersions, while (1) can be formulated for any general Sobolev function. This latter assumption is necessary for allowing the use of the chain rule which involves the full gradient even in lower dimensional slices. To be precise, we prove the following lemma which will play an important role everywhere in the proof of Proposition 3.1.

**Lemma 3.3** (Chain Rule). Let \( \Psi \in W^{1,2}(\Omega, \mathbb{R}^N) \), \( N \geq 1 \). Let \( \Sigma \subset \Omega \) be a \( k \)-dimensional flat domain. Suppose that there exist a sequence of smooth functions \( \Psi^\varepsilon \in C^\infty(\Omega, \mathbb{R}^N) \) such that

\[
\int_\Sigma |\Psi^\varepsilon - \Psi|^2 + |\nabla \Psi^\varepsilon - \nabla \Psi|^2 \, d\mathcal{H}^k \to 0,
\]
where \( \nabla \Psi \) denotes the full gradient with respect to the domain \( \Omega \). Let \( \mathbf{v} \) be any directional vector tangent to \( \Sigma \), then the chain rule,

\[
\frac{d}{dt}
\big|_{t=0} \Psi(\cdot + t \mathbf{v}) = \nabla \Psi \mathbf{v}
\]

holds in the weak sense over the domain \( \Sigma \). In particular,

\( \Psi \in W^{1,2}(\Sigma, \mathbb{R}^N) \).

**Proof.** Let \( \phi \in C_0^\infty(\Sigma) \), then,

\[
\int_{\Sigma} \frac{d}{dt}
\big|_{t=0} \Psi^t(x + t \mathbf{v})\phi(x) d\mathcal{H}^k = -\int_{\Sigma} \Psi^t(x) \frac{d}{dt}
\big|_{t=0} \phi(x + t \mathbf{v}) d\mathcal{H}^k.
\]

Since \( \Psi^t \) is smooth in \( \Omega \), we have,

\[
\int_{\Sigma} \frac{d}{dt}
\big|_{t=0} \Psi^t(x + t \mathbf{v})\phi(x) d\mathcal{H}^k = \int_{\Sigma} \nabla \Psi^t(x) \mathbf{v}\phi(x) d\mathcal{H}^k.
\]

By \( (3.1) \) we pass to the limit to conclude that,

\[
\int_{\Sigma} \nabla \Psi^t(x) \mathbf{v}\phi(x) d\mathcal{H}^k = -\int_{\Sigma} \Psi^t(x) \frac{d}{dt}
\big|_{t=0} \phi(x + t \mathbf{v}) d\mathcal{H}^k.
\]

Thus the chain rule as stated in the Lemma hold in the weak sense over the domain \( \Sigma \). The proof is complete. \( \Box \)

**Remark 3.4.** Note that the above lemma involves the full gradient of \( \Psi \). The assumption \( \Psi \in W^{1,2}(\Sigma, \mathbb{R}^N) \) by itself is not enough to conclude the chain rule.

### 3.1. Base case- 2-dimensional slices.

Suppose for a 2-dimensional plane \( P_2 \) all the assumptions (1)-(4) in Proposition \( 3.1 \) are satisfied. Without loss of generality, we can assume \( P_2 \) is parallel to the space spanned by \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \). Indeed, it is easy to see that assumption (1)-(4) in Proposition \( 3.1 \) are invariant under rotating the coordinate system. We denote \( P_2 \) by \( P_{\mathbf{e}_1, \mathbf{e}_2} \) to remind ourselves of this fact.

Let \( f = \nabla u^\ell \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \) for some arbitrary \( 1 \leq \ell \leq n + 1 \). Define,

\( g := (f_1, f_2)|_{P_{\mathbf{e}_1, \mathbf{e}_2} \cap \Omega} \in W^{1,2}(P_{\mathbf{e}_1, \mathbf{e}_2} \cap \Omega, \mathbb{R}^2) \).

**Lemma 3.5.** Let \( f^\ell : \Omega \to \mathbb{R}^n \) be a smooth sequence converging strongly to \( f \) in \( W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \). Let \( C \) be a line segment in \( P_{\mathbf{e}_1, \mathbf{e}_2} \cap \Omega \) such that

\[
\int_C |f^\ell - f|^2 + |\nabla f^\ell - \nabla f|^2 d\mathcal{H}^1 \to 0,
\]

\( \text{rank} \nabla f \leq 1 \) and \( \nabla f \) is symmetric for \( \mathcal{H}^1 \)-a.e. points on \( C \). Then if \( g \) is constant on \( C \), so is \( f \).

**Proof.** Let \( \mathbf{v} \) be the unit directional vector of \( C \). Since \( \mathbf{v} \) is a linear combination of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \),

\( \mathbf{v} = (v_1, v_2, 0, \cdots, 0) \).

Let \( \tilde{\mathbf{v}} = (v_1, v_2) \), then the first two components of \( f \) satisfy \( \nabla f^1 \cdot \mathbf{v} = \nabla g^1 \cdot \tilde{\mathbf{v}} \) a.e. on \( C \) and \( \nabla f^2 \cdot \mathbf{v} = \nabla g^2 \cdot \tilde{\mathbf{v}} \) a.e. on \( C \).

Since \( f \) satisfies the assumption of Lemma \( 3.3 \), the chain rule,

\[
\frac{d}{dt}
\big|_{t=0} f(\cdot + t \mathbf{v}) = (\nabla f)\mathbf{v}
\]

holds in the weak sense on \( C \). In particular, it holds for it first two component \( f^1 \) and \( f^2 \) and, of course, \( g \).
As \( g \) is constant on \( C \),

\[
0 = \frac{d}{dt}(t = 0) g(\cdot + t\tilde{v})
\]
in the weak sense. Hence,

\[
(\nabla g) \tilde{v} = 0 \quad \text{a.e. on } C.
\]

This implies,

\[
\nabla f^1 \cdot v = 0 \quad \text{and} \quad \nabla f^2 \cdot v = 0 \quad \text{a.e. on } C.
\]

For \( z \in C \) such that \( \nabla f^1(z) \cdot v = 0 \) and \( \nabla f^2(z) \cdot v = 0 \), rank \( \nabla f(z) \leq 1 \) and \( \nabla f(z) \) is symmetric, we have two cases: 1) \( \nabla f^1(z) \neq 0 \) or \( \nabla f^2(z) \neq 0 \), 2) \( \nabla f^1(z) = \nabla f^2(z) = 0 \) In the first case, we can assume with loss of generality that \( \nabla f^1(z) \neq 0 \).

Therefore, rank \( \nabla f(z) = 1 \) and

\[
\nabla f^i(z) = a_z^i \nabla f^1(z) \quad \text{for all } i \geq 1.
\]

It then follows that

\[
\nabla f^i(z) \cdot v = a_z^i (\nabla f^1(z) \cdot v) = 0 \quad \text{for all } i \geq 1.
\]

In the second case, by symmetry,

\[
f^i_j(z) = f^j_i(z) = 0, \quad \text{for } j = 1, 2, \text{ and } i = 1, \cdots, n.
\]

As \( v = (v_1, v_2, 0, \cdots, 0) \),

\[
\nabla f^i(z) \cdot v = 0 \quad \text{for all } i = 1, \cdots, n.
\]

Therefore, in either cases, we have proved

\[
\nabla f^i \cdot v = 0 \quad \text{a.e. on } C \quad \text{for all } i = 1, \cdots, n.
\]

Therefore, \( f \) is constant on \( C \) by the chain rule in (3.3). The proof is complete.

**Corollary 3.6.** If \( g \) is constant on a 2-dimensional region \( U \) in \( P_{e_1 e_2} \cap \Omega \), \( f \) is constant on \( U \) as well.

**Proof.** Observe that if \( U \) is a 2-dimensional region of \( P_{e_1 e_2} \cap \Omega \), which has strictly positive 2-dimensional Hausdorff measure, then the assumptions (1) and (4) of Proposition 3.1 imply,

\[
\int_U |f^\ell - f|^2 + |\nabla f^\ell - \nabla f|^2 d\mathcal{H}^2 \to 0,
\]

rank \( \nabla f \leq 1 \) and \( \nabla f \) is symmetric for \( \mathcal{H}^2 \) a.e. points on \( U \). Thus the same argument for line segments in Lemma 3.5 gives for any directional vector \( v \) of \( U \), \( \nabla f^i \cdot v = 0 \) a.e. on \( U \) for all \( i = 1, \cdots, n \), hence the chain rule implies \( f \) is constant on \( U \). The proof is complete.

**Lemma 3.7.** Suppose assumptions (1)-(4) of Proposition 3.1 are satisfied on the two dimensional region \( P_{e_1 e_2} \cap \Omega \). Let \( f = \nabla u^\ell \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n) \) for some arbitrary \( 1 \leq \ell \leq n + 1 \). Then the restriction \( f \in C^{0,1/2}_{\text{loc}}(P_{e_1 e_2} \cap \Omega, \mathbb{R}^n) \). Moreover, for every point \( x \in P_{e_1 e_2} \cap \Omega \), either there exists a neighborhood in \( P_{e_1 e_2} \cap \Omega \) of \( x \), or a unique line segment in \( P_{e_1 e_2} \cap \Omega \) passing through \( x \) and joining \( \partial \Omega \) at both ends, on which \( f \) is constant.
Proof. The proof is divided into seven steps.

**Step 0.** Preliminary set up: by assumption (4) of Proposition 3.1, \( \nabla f \) satisfies rank \( \nabla f \leq 1 \) and \( \nabla f = \nabla^2 u \) is symmetric a.e. on \( P_{e_1 e_2} \cap \Omega \). Therefore, \( g := (f^1, f^2)|_{P_{e_1 e_2} \cap \Omega} \in W^{1,2}(P_{e_1 e_2} \cap \Omega, \mathbb{R}^2) \) also satisfies rank \( \nabla g \leq 1 \) and \( \nabla g \) is symmetric a.e. on \( P_{e_1 e_2} \cap \Omega \). We employ [22, Prop.1], which is cited above as Proposition 1.3. The function \( g \) satisfies the assumption of this proposition on the domain \( P_{e_1 e_2} \cap \Omega \) and hence the conclusions holds true for \( g \). Suppose \( g \) is constant on some maximal connected neighborhood \( U \subset P_{e_1 e_2} \cap \Omega \), by continuity of \( g \), it is also constant on its closure \( U \cap \Omega \).

**Step 1.** We claim that the boundary of \( U \) only consists of line segments joining the boundary and none of these line segments intersect inside \( \Omega \). Indeed, if \( x \in \partial U \cap \Omega \), then \( x \) is not contained in a constancy neighborhood of \( g \), therefore by Proposition 1.3 there exists a unique line segment \( C^U_x \subset P_{e_1 e_2} \cap \Omega \) passing through \( x \) and joining \( \partial \Omega \) at both ends on which \( g \) is constant, which implies \( \partial U \cap \Omega \subset \bigcup_{x \in \partial U \cap \Omega} C^U_x \). Moreover, for \( x, z \in \partial U \cap \Omega \), \( C^U_x = C^U_z \) if \( z \in C^U_x \) and \( C^U_x \cap C^U_z \cap \Omega = \emptyset \) if \( z \notin C^U_x \) (Figure 1). This follows from the fact that if \( g \) is constant on two such intersecting segments, it must be constant on their convex hull inside \( \Omega \) too. On the other hand, suppose \( g \) is constant on some line segment \( C^U_z \) passing through \( x \in \partial U \cap \Omega \) and joining \( \partial \Omega \) at both end, since \( g \) is constant on \( \overline{U} \) and \( C^U_x \), which intersect at \( x \), it must be constant on the convex hull of \( \overline{U} \) and \( C^U_x \) inside \( \Omega \). But \( U \) is maximal, hence \( \bigcup_{x \in \partial U \cap \Omega} C^U_x \subset \partial U \cap \Omega \). Therefore,

\[
\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} C^U_x.
\]

**Step 2.** We claim that we can choose small enough \( \delta > 0 \) so that for any region \( U \) on which \( g \) is constant, the 2-dimensional ball \( B^2(x_0, \delta) \subset P_{e_1 e_2} \cap \Omega \) intersects \( \partial U \) at no more than two line segments belonging to \( \partial U \). Indeed, let \( x_0 \in P_{e_1 e_2} \cap \Omega \) be such that \( g \) is not constant in a neighborhood of \( x_0 \). We use the fact that for any maximal constant region \( U \), line segments in \( \partial U \) do not intersect inside \( \Omega \). If \( x_0 \) is at a positive distance of all constancy regions, the conclusion is trivial. The
same is true if it lies on the boundary of one of the constancy regions and yet is positively distant from all others. Suppose therefore that there is a sequence of maximal constancy regions \( U_m \) converging to \( x_0 \) in distance, in which case there are two line segments \( C_{x_1}^U \) and \( C_{x_2}^U \) in \( \partial U_m \) whose angle (if they are nonparallel) or distance (if they are parallel) converges to zero, since both of these sequences of segments must converge to the same constancy segment passing through \( x_0 \). Then since all the other line segments in \( \partial U_m \) must be arbitrarily close to \( \partial \Omega \), we can again choose \( \delta \) small enough so that \( B^2(x_0, \delta) \) is away from \( \partial \Omega \) and hence it does not intersect a third line segment in \( \partial U_m \) (Figure 2).

![Figure 2. \( B^2(x_0, \delta) \) intersects \( \partial U_m \) at two line segments.](image)

**Step 3.** We construct a foliation of the ball \( B^2(x_0, \delta) \subset P_{e_1, e_2} \cap \Omega \). Firstly, for any \( x \in B^2(x_0, \delta) \), we will construct a line segment \( C_x \) in \( B^2(x_0, \delta) \) passing through \( x \) and joining \( \partial B^2(x_0, \delta) \) at both ends on which \( g \) is constant and \( C_x \cap C_z \cap B^2(x_0, \delta) = \emptyset \) if \( z \notin C_x \). The construction is as follows: for those \( x \) not contained in a constant region of \( g \), this line segment is given automatically by Proposition 1.3. If \( x \) is contained in a constant maximal region \( U \) of \( g \), then it is constant on every line segment in \( U \) that passes through it so we have to choose the appropriate one: 1) If \( B^2(x_0, \delta) \) intersect only one line segment \( C^U \) in \( \Omega \) that belongs to \( \partial U \), then we define \( C_x \) to be the line segments inside \( B^2(x_0, \delta) \) passing through \( x \) and parallel to \( C^U \); 2) If \( B^2(x_0, \delta) \) intersects two line segments \( C_{x_1}^U, C_{x_2}^U \) in \( \Omega \) that belongs to \( \partial U \), let \( L_1 \) and \( L_2 \) be the two lines that contain \( C_{x_1}^U \) and \( C_{x_2}^U \). If \( L_1 \) and \( L_2 \) are not parallel, let \( O := L_1 \cap L_2 \) and let \( C_x \) be the segment given by the intersection with \( B^2(x_0, \delta) \) of the line passing through \( O \) and \( x \). If \( L_1 \) and \( L_2 \) are parallel, then we let \( C_x \) be the line segment inside \( B^2(x_0, \delta) \) passing through \( x \) and parallel to \( L_1 \). (Figure 3). In this way, we have constructed a family of line segments \( \{ C_x \}_{x \in B^2(x_0, \delta)} \) in \( B^2(x_0, \delta) \) on which \( g \) is constant and \( C_x \cap C_z \cap B^2(x_0, \delta) = \emptyset \) if \( z \notin C_x \).

Secondly, for every \( x \in B^2(x_0, \delta) \), let \( N(x) \) be the vector field orthogonal to \( C_x \). By making \( \delta \) smaller we can make sure that none of the \( C_x \)'s intersect inside \( B^k(x_0, 2\delta) \), and therefore we can choose an orientation such that \( N \) is a Lipschitz
Figure 3. Construction of foliations.

vector field inside the ball of radius \( \delta \). The ODE,

\[
\gamma'(t) = N(\gamma(t)) \quad \gamma(0) = x_0,
\]

then has a unique solution \( \gamma : (a, b) \to B^2(x_0, \delta) \) for some interval \( (a, b) \subset \mathbb{R} \) containing 0. Moreover, if necessary by making \( \delta \) smaller, \( \cup \{C_{\gamma(t)}\}_{t \in (a, b)} = B^2(x_0, \delta) \). Therefore, \( \{C_{\gamma(t)}\}_{t \in (a, b)} \) is a foliation of \( B^2(x_0, \delta) \) (Figure 4).

Figure 4. Foliations of \( B^2(x_0, \delta) \).

**Step 4.** We now want to show the assumptions of Lemma 3.5 are satisfied along \( C_{\gamma(t)} \) for a.e. \( t \in (a, b) \). We define the function \( h : B^2(x_0, \delta) \to B^2(x_0, \delta) \) as

\[
h(x) = \gamma(t) \quad \text{if} \quad x \in C_{\gamma(t)}.
\]

Since none of the \( C_{\gamma(t)} \) intersect inside \( B^2(x_0, \delta) \), \( h \) is well defined and \( h \) is constant along each \( C_{\gamma(t)} \), i.e. \( h^{-1}(\gamma(t)) = C_{\gamma(t)} \). Since \( \gamma \) is Lipschitz, \( h \) is Lipschitz as
well. Moreover, since $|\gamma''(t)|$ is uniformly bounded, we have the one dimensional Jacobian $J_h > C > 0$. (For the definition of general $k$-dimensional Jacobian see [2] p. 88.)

Let $E_0$ be the set of all $x \in B^2(x_0, \delta)$ such that rank $\nabla f(x) > 1$ or $\nabla f(x)$ is not symmetric. By assumption (4) of Proposition 3.1 on $f$, $|E_0| = 0$. As $h$ is Lipschitz, we can apply the general co-area formula [7, p. 112] to $h$ to obtain,

$$0 = \int_{E_0} J_h(x) dx = \int_{h^{-1}(w)} H^1(E_0 \cap h^{-1}(w)) dH^1(w)$$

$$= \int_{a}^{b} H^1(E_0 \cap h^{-1}(\gamma(t))) |\gamma'(t)| dt$$

Therefore, for a.e. $t \in (a, b)$, $H^1(E_0 \cap C_{\gamma(t)}) = 0$ since $|\gamma'| = 1$. Moreover, by change of variable formula, if $f^\epsilon$ is a smooth approximation sequence,

$$\int_{B^2(x_0, \delta)} (|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2) J_h$$

$$= \int_{\gamma} \int_{h^{-1}(w)} (|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2) dH^1 dH^1(w)$$

$$= \int_{a}^{b} \int_{h^{-1}(\gamma(t))} (|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2) dH^1 |\gamma'(t)| dt$$

$$= \int_{a}^{b} \int_{C_{\gamma(t)}} (|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2) dH^1 dt.$$

Since $J_h$ is bounded, together with assumption (1) in Proposition 3.1, we then have for a.e. $t \in (a, b)$,

$$\int_{C_{\gamma(t)}} (|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2) dH^1 \to 0.$$

Therefore, the assumptions of Lemma 5.5 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a, b)$.

**Step 5.** We now prove the Lemma for the ball $B^2(x_0, \delta)$. Step 4 and Lemma 5.5 imply that $f$ is constant on $C_{\gamma(t)}$ for a.e. $t \in (a, b)$. By choosing an initial value for $\gamma$ arbitrary close to $x_0$ and applying the general co-area formula in a similar manner we can make sure that $f$ is of class $W^{1,2}$ on $\gamma$. Hence we conclude that $f$ is $C^{0,1/2}$ on $\gamma$ by the Sobolev embedding theorem. Let $F$ be the set of $t \in (a, b)$ such that $f$ is not constant along $C_{\gamma(t)}$, then $H^1(F) = 0$. We modify $f$ to be constant along $C_{\gamma(t)}$ for each $t \in F$. Note that,

$$H^2(\bigcup\{C_{\gamma(t)} : t \in F\}) \leq 2\delta \sup J_h^{-1}H^1(\{\gamma(t) : t \in F\}) = 2\delta \sup J_h^{-1}H^1(F) = 0.$$

Hence $f$ is $C^{0,1/2}$ up to modification of a set of measure zero in $B^2(x_0, \delta)$. Moreover, $f$ is constant on $C_{\gamma(t)}$ for all $t$, which foliates $B^2(x_0, \delta)$. In addition, by Corollary 3.6, $f$ is constant on every 2-dimensional region in $B^2(x_0, \delta)$ on which $g$ is constant. Therefore, $f$ is either constant on a line segment joining $\partial B^2(x_0, \delta)$ at both ends, or constant on a 2-dimensional region in $B^2(x_0, \delta)$. This proves Lemma 5.7 for the ball $B^2(x_0, \delta)$.

**Step 6.** Finally, we prove the lemma for the entire domain $P_{e_1, e_2} \cap \Omega$. Suppose there is some $x \in P_{e_1, e_2} \cap \Omega$ that is not contained in a constant region of $f$. Then by what we have proved, $f$ is constant on a line segment passing through $x$ and joining
the boundary of \( B^2(x, \delta_x) \subset P_{e_1, e_2} \cap \Omega \) for some \( \delta_x > 0 \). Let \( \overline{y_1 y_2} \) be the largest line segment containing this segment on which \( f \) is constant. Suppose \( y_1 \in P_{e_1, e_2} \cap \Omega \), then from what we have proved, \( f \) is either constant on 2-dimensional regions or line segments passing through \( y_1 \) and joining the boundary of \( B^2(y_1, \delta_{y_1}) \subset P_{e_1, e_2} \cap \Omega \) for some \( \delta_{y_1} > 0 \). Firstly, \( y_1 \) cannot be contained in a constant region of \( f \), otherwise we can prolong the segment \( [y_1, y_2] \). Thus, there must be a line segment \( \overline{z_1 z_2} \) passing through \( y_1 \) and joining the boundary of \( B^2(y_1, \delta_{y_1}) \) at both end on which \( f \) is constant. Secondly, \( \overline{z_1 z_2} \) cannot have the same direction as \( \overline{y_1 y_2} \), otherwise, we can again prolong the segment \( [y_1, y_2] \). Then we consider the region \( \Delta \) bounded by \( y_1 y_2, \overline{z_1 z_2} \), and \( y_2 \). Since \( g \) is constant on \( \overline{y_1 y_2} \) and \( \overline{z_1 z_2} \), by Proposition 1.3 \( g \) must be constant on \( \Delta \) because no line segment can join the boundary of \( P_{e_1, e_2} \cap \Omega \) passing through a point inside \( \Delta \) without intersecting either \( \overline{y_1 y_2} \) or \( \overline{z_1 z_2} \) (Figure 5). Hence \( \Delta \) cannot be contained in a constant region of \( f \). The proof is complete.

![Figure 5](image-url)
for $t' \neq t$. Moreover, $\nabla^2 u^1$ is constant on $C_\gamma(t)$ for every $t \in (a, b)$. Assumption (1) and (3) in Proposition 3.1 together with the same argument using co-area and change of variable formulas as in the proof of Lemma 3.7 yield for a.e. $t \in (a, b)$

$$\int_{C_\gamma(t)} |\nabla u^e - \nabla u|^2 + |\nabla^2 u^e - \nabla^2 u|^2 d\mathcal{H}^1 \to 0$$

and $\nabla^2 u^\ell = (n^\ell / n^1)\nabla^2 u^1$, $2 \leq \ell \leq n + 1$, $\mathcal{H}^1$ a.e on $C_\gamma(t)$.

Let $v$ be the directional vector of one such $C_\gamma(t)$, then the chain rule in Lemma 3.3 and the fact that $\nabla u$ is constant on $C_\gamma(t)$ imply

$$0 = \frac{d}{dt} \bigg|_{t=0} \nabla^1 (\cdot + tv) = (\nabla^2 u^1)v$$

in the weak sense in $C_\gamma(t)$. Therefore,

$$(\nabla^2 u^\ell)v = \frac{n^\ell}{n}(\nabla^2 u^1)v = 0, \quad 2 \leq \ell \leq n + 1 \text{ a.e. on } C_\gamma(t).$$

Hence again by the chain rule in Lemma 3.3 $\nabla^\ell, 2 \leq \ell \leq n + 1$, is constant on $C_\gamma(t)$. Therefore, each $\nabla^\ell$ is constant on $C_\gamma(t)$ for a.e. $t \in (a, b)$. Furthermore, since for each $1 \leq \ell \leq n + 1$, $\nabla^\ell$ is continuous on $P_{e_\ell e_\ell} \cap \Omega$, we conclude that $\nabla^\ell$ for all $1 \leq \ell \leq n + 1$ are constant on all $C_\gamma(t)$ that foliates $B^2(x_0, \delta)$. On the other hand, each 2-dimensional region $U$ of $B^2(x_0, \delta)$ automatically satisfies all the assumptions (1) and (3) in Proposition 3.1 hence the same argument for each $C_\gamma(t)$ gives that $\nabla^\ell$ for all $2 \leq \ell \leq n + 1$ is constant on the same region on which $\nabla^1$ is constant. This proves $\nabla u$ is either constant on 2-dimensional regions or constant on line segments in $B^2(x_0, \delta)$ joining the boundary. The proof of Proposition 3.1 for the domain $P_{e_\ell e_\ell} \cap \Omega$ follows from exactly the same argument as the last step of the proof of Lemma 3.7. The proof for the base case is complete. \hfill \Box

3.2. Inductive step- $k$-dimensional slices. In this subsection, we will prove that Proposition 3.1 holds true for $k$ if it holds true for $k - 1$ when $2 < k \leq n$. This, combined with the base case $k = 2$ established in the previous step, completes the proof of Proposition 3.1

3.2.1. Developability. Based on the induction hypothesis for $k - 1$, we first prove a weaker result in $k$-dimensional slices of $\Omega$ than Proposition 3.1. That is, we prove that $u$ is developable on all $k$-dimensional slices satisfying assumptions (1)-(4) of Proposition 3.1 in the following sense:

**Proposition 3.8.** Suppose Proposition 3.1 is true for any $(k-1)$-dimensional slice of $\Omega$ on which assumptions (1)-(4) are satisfied. Let $P_k$ be any $k$-dimensional plane such that assumptions (1)-(4) for $u$ holds on $P_k \cap \Omega$, then for every $x \in \Omega$, either $u$ is affine in a neighborhood in $P_k \cap \Omega$ of $x$, or there exists a unique $(k-1)$-dimensional hyperplane $P_{x-1}^x$ such that $u$ is affine on the connected component of $x$ in $P_{x-1}^x \cap \Omega$.

**Proof.** We first need to define a terminology that is the higher dimensional version of “line segments joining the boundary of some domain at both ends”.

**Definition 3.9.** By a $k$-plane $P$ in $\Sigma$, we mean a connected component of a $k$-dimensional plane $P \cap \Sigma$, where $\Sigma$ is any $N$-dimensional region with $N \geq k \geq 1$. 

Remark 3.10. We emphasize here that such \( k \)-plane \( P \) in \( \Sigma \) refers to not the entire plane, but just the part inside a region. On the other hand, it refers to the entire connected part inside this region.

Let \( v \) be any unit directional vector of \( P_k \), let \( v_1, \cdots, v_{k-1} \) be a set of linearly independent unit vectors of \( P_k \) perpendicular to \( v \). We parametrize the family of \((k-1)\)-dimensional planes parallel to the space spanned by these vectors as follows:

\[
P_{v_1 \cdots v_{k-1}}^y = \left\{ z : z = y + \sum_{i=1}^{k-1} s_i v_i, \ s_i \in \mathbb{R} \right\}, \quad y \in \text{span}(v).
\]

Lemma 3.11. Given the direction \( v \), for a.e. \( y \in \text{span}(v) \), \( u \) is \( C_{\text{loc}}^{1,1/2} \) and is an isometry on \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \). Moreover for every \( x \in P_{v_1 \cdots v_{k-1}}^y \cap \Omega \), \( u \) is either affine on a \((k-1)\)-dimensional region in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) containing \( x \), or affine on a \((k-2)\)-plane in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) passing through \( x \).

Proof. Since \( u \) satisfies assumptions (1)-(4) on \( P_k \cap \Omega \), by Fubini Theorem, for a.e. \( y \in \text{span}(v) \), assumptions (1)-(4) are also satisfied on \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \). Hence by our induction hypothesis on \((k-1)\) slices of \( \Omega \), \( \nabla u \) is \( C_{\text{loc}}^{0,1/2} \) on \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \). By assumption (2) \( \nabla u^T \nabla u = I \) a.e., and hence everywhere in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) by continuity. Therefore, by assumption (1) and the chain rule in Lemma 3.3 \( u \) is an isometry on \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \).

Moreover by our induction hypothesis, for every \( x \in P_{v_1 \cdots v_{k-1}}^y \cap \Omega \), \( \nabla u \) is either constant on a \((k-1)\)-dimensional region in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) containing \( x \), or constant on an \((k-2)\)-plane in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) passing through \( x \). Hence by the chain rule in Lemma 3.3 \( u \) is either affine on a \((k-1)\)-dimensional regions in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \), or affine on \((k-2)\)-planes in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \). The proof is complete.

Now we want to show that a substantial part of Lemma 3.11 is true for every rather than a.e. \((k-1)\)-dimensional planes in \( \Omega \).

Lemma 3.12. Given direction \( v \), for all \( y \in \text{span}(v) \) and for all \( x \in P_{v_1 \cdots v_{k-1}}^y \cap \Omega \), \( u \) is either an affine isometry on a \((k-1)\)-dimensional region in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) containing \( x \), or an affine isometry on a \((k-2)\)-plane in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) passing through \( x \).

Remark 3.13. We obtain from the proof of Lemma 3.11 that \( u \) is \( C^1 \) on a.e. planes. However, Lemma 3.11 does not imply \( u \) is \( C^1 \) on every plane because even though \( \nabla u \) is continuous on a.e. planes, we cannot conclude from here that \( \nabla u \) is continuous in \( \Omega \), so we cannot pass to the limit to conclude as Lemma 3.11.

Proof. Given \( y \in \text{span}(v) \), Lemma 3.11 guarantees a sequence \( y_m \in \text{span}(v) \), \( y_m \to y \) such that Lemma 3.11 is true on \( P_{v_1 \cdots v_{k-1}}^{y_m} \cap \Omega \) for every \( m \).

Let \( x \in P_{v_1 \cdots v_{k-1}}^y \cap \Omega \), we divide the proof into the following two cases:

1. There is a sequence of \((k-2)\)-planes \( P_m \) in \( P_{v_1 \cdots v_{k-1}}^{y_m} \cap \Omega \) on which \( u \) is an affine isometry and \( P_m \) converges to \( x \) in distance.

2. There does not exist such a sequence of \((k-2)\)-planes.

Suppose we are in case (1), then the limit of \( P_m \) must also be a \((k-2)\)-plane \( P \) in \( P_{v_1 \cdots v_{k-1}}^y \cap \Omega \) passing through \( x \). Also since \( u \) is Lipschitz continuous, \( u \) must also be an affine isometry on \( P \), which proves the Lemma in this case (Figure 6).

Suppose now we are in case (2). If we cannot find such a sequence of \((k-2)\)-planes, then we can find \( x_m \in P_{v_1 \cdots v_{k-1}}^{y_m} \cap \Omega \), \( x_m \to x \) with the property that there is
Suppose $u$ is an affine isometry on two line segments $C_1$ and $C_2$ in $P_k \cap \Omega$ intersecting at a point $x$ in the interior of both $C_1$ and $C_2$. Let $H$ be the convex hull of the line segments $C_1$ and $C_2$, then $u$ is an affine isometry on $H \cap \Omega$.

**Proof.** We parametrize $C_1$ and $C_2$ by \( \{x + t\mathbf{v}_1, t \in [-a, b]\} \) and \( \{x + s\mathbf{v}_2, s \in [-c, d]\} \), respectively, with both \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) unit vectors. We can assume \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent, otherwise, the conclusion of the lemma is obvious. Since $u$ is affine on both $C_1$ and $C_2$, $u(C_1)$ and $u(C_2)$ are both line segments in $\mathbb{R}^{n+1}$. We can again parametrize the lines that contains the line segments $u(C_1)$ and $u(C_2)$ by $u(x) + t\mathbf{v}_1$ and $u(x) + s\mathbf{v}_2$, where both $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors due to the isometry assumption.
Let $y \in H \cap \Omega$, we can of course assume that $y$ is neither in $C_1$ nor $C_2$, otherwise, there is nothing to prove. In this way, we can find a line $L_3$ passing through $y$ and intersecting $C_1$ at only one point, denoted $x_{13}$; and $C_2$ at only one point, denoted $x_{23}$, where the segment $\overline{x_{13}x_{23}}$ lies inside $\Omega$. Since $x_{13} \in C_1$, $x_{13} = x + t_0v_1$ for some $t_0 \in [-a, b]$. Similarly $x_{23} = x + s_0v_2$ for some $s_0 \in [-c, d]$. Then since

$$y = wx_{13} + (1 - w)x_{23} \quad \text{for some } w \in [0, 1],$$

it follows

$$y = x + wt_0v_1 + (1 - w)s_0v_2.$$

To prove that $u$ is an affine isometry on $H$, we need to prove

$$u(y) = u(x) + wt_0\tilde{v}_1 + (1 - w)s_0\tilde{v}_2. \quad (3.5)$$

We first claim that the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between $C_1$ and $C_2$. Since $x$ is in the interior of $C_1$ and $C_2$, we can construct a parallelogram $ABCD$ centered at $x$, with $A, C \in C_1$ and $B, D \in C_2$. Since $u$ is an affine isometry on $C_1$ and $C_2$, $|u(A) - u(x)| = |A - x|$, $|u(B) - u(x)| = |B - x|$, $|u(C) - u(x)| = |C - x|$ and $|u(D) - u(x)| = |D - x|$. On the other hand, $|u(A) - u(B)| \leq |A - B|$ and $|u(B) - u(C)| \leq |B - C|$ since $u$ is 1-Lipschitz (Figure 8).

![Figure 8](image_url)

This implies the angle $\alpha_2$ between the line segments $\overline{u(x)u(A)}$ and $\overline{u(x)u(B)}$ must be smaller than or equal to the angle $\alpha_1$ between $\overline{xA}$ and $\overline{xB}$, and the angle $\beta_2$ between the line segments $\overline{u(x)u(B)}$ and $\overline{u(x)u(C)}$ must be smaller than or equal to the angle $\beta_1$ between $\overline{xB}$ and $\overline{xC}$. Hence $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. This proves our claim.

Since by assumption, $u$ is an affine isometry on $\overline{x_{13}x}$ and $\overline{x_{23}x}$, we have

$$u(x_{13}) - u(x) = t_0\tilde{v}_1 \quad \text{and} \quad u(x_{23}) - u(x) = s_0\tilde{v}_2$$

for the same $t_0, s_0$ and unit vector $\tilde{v}_1, \tilde{v}_2$ as defined before. In particular, $|u(x_{13}) - u(x)| = |x_{13} - x|$ and $|u(x_{23}) - u(x)| = |x_{23} - x|$. Moreover, since the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between $C_1$ and $C_2$, we have $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$. 
On the other hand, $u(x_{13}x_{23})$ is a 1-Lipschitz curve, hence the length the the curve $u(x_{13}x_{23})$, denoted by $|u(x_{13}x_{23})|$, satisfies $|u(x_{13}x_{23})| \leq |x_{13} - x_{23}|$. Altogether we have

$$|u(x_{13}) - u(x_{23})| \leq |u(x_{13}x_{23})| \leq |x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|.$$ 

This implies

$$|u(x_{13}x_{23})| = |u(x_{13}) - u(x_{23})|.$$ 

Hence the curve $u(x_{13}x_{23})$ must coincide with line segment $u(x_{13}x_{23})$. Therefore, $u$ also maps the line segment $x_{13}x_{23}$ onto a line segment $u(x_{13})u(x_{23})$, which means $u$ is affine on $x_{13}x_{23}$.

Finally, since $u$ is 1-Lipschitz, $|u(x_{13}) - u(y)| \leq |x_{13} - y|$ and $|u(x_{23}) - u(y)| \leq |x_{23} - y|$. However, since $u$ is affine on $x_{13}x_{23}$, $|u(x_{13}) - u(x_{23})| = |u(x_{13}) - u(y)| + |u(y) - u(x_{23})| \leq |x_{13} - y| + |y - x_{23}| = |x_{13} - x_{23}|.$

But we already showed that $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$. Hence $|u(x_{13}) - u(y)| = |x_{13} - y|$ and $|u(x_{23}) - u(y)| = |x_{23} - y|$. Therefore,

$$u(y) = wu(x_{13}) + (1 - w)u(x_{23})$$

for the same $w$ as (3.13), which yields (3.15). The proof is complete.

**Corollary 3.15.** Given a $\ell$-dimensional ($\ell \leq k$) region $U$ in $P_k \cap \Omega$, and a line segment $C$ in $P_k \cap \Omega$ for which there exists $x \in C \cap U$ that lies in the interior of both $U$ and $C$, if $u$ is an affine isometry on both $U$ and $C$, then $u$ is an affine isometry on the convex hull $H$ of $U$ and $C$ inside $\Omega$.

**Proof.** Let $y \in H \cap \Omega$. We need to show that $u(y) = u(x) + t\tilde{v}$ for some $\tilde{v}$ given by a linear combination of directional vectors in $u(U)$ and $u(C)$ and $|t\tilde{v}| = |y - x|$. Let $P_y$ be a 2-dimensional plane that contains $y$ and $C$. Then $P_y$ intersects $U$ at some line segment $C_y$. Since $u$ is an affine isometry on both $C$ and $C_y$, by Lemma 3.14 $u$ is an affine isometry on the convex hull of $C$ and $C_y$ (Figure 9). Since this convex hull contains both $y$ and $x$, this implies $u(y) = u(x) + t\tilde{v}$ for some vector $\tilde{v}$, $|t\tilde{v}| = |y - x|$, and $\tilde{v}$ is a linear combination of directional vectors of $u(C)$ and $u(C_y)$. Our claim then follows because $C_y \subset U$ and $u$ is an affine isometry on $U$, so any vectors of $u(C_y)$ is a linear combination of vectors in $u(U)$. The proof is complete.

By obvious induction we then have

![Figure 9](image-url)
Corollary 3.16. Suppose $U_1$ and $U_2$ are $k_1$ and $k_2$-dimensional regions ($k_1, k_2 \leq k$) in $P_k \cap \Omega$ with nonempty intersections. Moreover, there exists a point $x \in U_1 \cap U_2$ belonging to the interior of both $U_1$ and $U_2$. If $u$ is an affine isometry on both $U_1$ and $U_2$, then $u$ is an affine isometry on the convex hull of $U_1$ and $U_2$ inside $\Omega$.

Now we are ready to prove Proposition 3.8. Given $x \in P_k \cap \Omega$, we first claim that there is a $(k - 1)$-dimensional hyperplane $P^x_0$ in $P_k$ and a $(k - 1)$-dimensional neighborhood $U^x_0 \subset P^x_0 \cap \Omega$ containing $x$ on which $u$ is an affine isometry. Otherwise, for all $(k - 1)$-dimensional hyperplanes in $P_k \cap \Omega$ that pass through $x$, $x$ is not contained in any $(k - 1)$-dimensional neighborhood on which $u$ is an affine isometry. In particular, let $v_1, \ldots, v_k$ be linearly independent vectors of $P_k$ and let $P^x_{v_1, v_2, \ldots, v_k}$, $i = 1, \ldots, k$ be the $(k - 1)$-dimensional hyperplanes in $\Omega$ passing through $x$ and parallel to the space spanned by $v_1, \ldots, v_{i - 1}, v_{i + 1}, \ldots, v_k$. Since $x$ is not contained in any $(k - 1)$-dimensional neighborhood in $P^x_{v_1, v_2, \ldots, v_k} \cap \Omega$ on which $u$ is an affine isometry, by Lemma 3.12 there exists $(k - 2)$-planes $P_i \ni x$ in $P^x_{v_1, v_2, \ldots, v_k} \cap \Omega$ such that $u$ is an affine isometry on $P_i$. By Corollary 3.16 $u$ is an affine isometry on the convex hull of $P_i$ for all $1 \leq i \leq k$ (Figure 11 Case 1). Let $v_i$ be a directional vector of $P_i$. Since $P_i \subset P^x_{v_1, v_2, \ldots, v_k}$, which is orthogonal to $v_i$, at least $k - 1$ out of these $k$ vectors are linearly independent. This convex hull has $k - 1$ linearly independent directional vectors, hence it must contain a $(k - 1)$-dimensional neighborhood of $x$, contradiction to our assumption, which proves our claim.

![Figure 10. Case 1 (left) and Case 2 (right).](image)

Therefore, we have proved that $x$ must be contained in a $(k - 1)$-dimensional neighborhood $U^x_0 \subset P^x_0 \cap \Omega$ for some $(k - 1)$-dimensional hyperplane $P^x_0$ and $u$ is an affine isometry on $U^x_0$. If $U^x_0$ is the entire connected component containing $x$ in $P^x_0 \cap \Omega$, then the conclusion of the proposition is achieved. Otherwise, we can find a maximal $(k - 2)$-plane $P_x$ in $U^x_0$, which is not a $(k - 2)$-plane in $P^x_0 \cap \Omega$, i.e., it is away from $\partial \Omega$, on which $u$ is an affine isometry. Let $P^x_1$ be any other $(k - 1)$-dimensional hyperplane containing the region $P_x$. We have $P_x = U^x_0 \cap P^x_1$ and since the maximal affine region $P_x \subset P^x_1 \cap \Omega$ is not a $(k - 2)$-plane in $P^x_1 \cap \Omega$, by Lemma 3.12 $x$ must be contained in a $(k - 1)$-dimensional neighborhood $U^x_1 \subset P^x_1 \cap \Omega$ on
Lemma 3.17. We will first prove the following lemma. The arguments are very similar to what we used in the proofs of Lemmas 3.5-3.7. Let \( u \) be an affine isometry on the convex hull of \( U_0^k \) and \( U_1^k \), whose interior is a \( k \)-dimensional region, which also achieves the conclusion of Proposition 3.8. The proof is complete. \( \square \)

3.2.2. Regularity and the conclusion of the inductive step. In our last step, we will essentially show that Proposition 3.8 combined with assumptions (1)-(4) of Proposition 3.1 hold on \( P_k \). Let \( \lambda \) be any \( n \)-dimensional region, implies the conclusion of the latter proposition. This will hence conclude the inductive step. The key point is to show that if \( u \) is affine on a \((k-1)\)-plane, then its full gradient must be constant on the same region. The arguments are very similar to what we used in the proofs of Lemmas 3.5, 3.6, 3.7.

We will first prove the following lemma.

**Lemma 3.17.** Suppose on a \( k \)-plane \( P \) (\( 1 \leq k \leq n \)) in \( \Omega \) we have the following:

1. There is a sequence of smooth functions \( u^\ell \in C^\infty(\Omega, \mathbb{R}^{n+1}) \) such that
   \[
   \int_P |u^\ell - u|^2 + |\nabla u^\ell - \nabla u|^2 + |\nabla^2 u^\ell - \nabla^2 u|^2 dH^k \to 0.
   \]

2. rank \( \nabla^2 u^\ell \leq 1 \) and \( \nabla^2 u^\ell \) is symmetric a.e. on \( P \) for all \( 1 \leq \ell \leq n+1 \).

Then if \( u \) is affine on \( P \), \( \nabla u \) is constant on \( P \).

**Proof.** Let \( v \) be any unit directional vector in \( P \). By assumption (1) and the chain rule in Lemma 3.3, \( u \) is affine on \( P \) implies
\[
\nabla u(x)v = \text{constant} \quad \text{for a.e. } x \in P.
\]

Take the directional derivative one more time, together with assumption (1) we obtain,
\[
(3.6) \quad (v)^T \nabla^2 u^\ell v = 0 \quad \text{for a.e. } x \in P
\]
for all \( 1 \leq \ell \leq n+1 \). However, to show that \( \nabla u \) is constant on \( P \), we need a conclusion stronger than (3.6), i.e.,
\[
(3.7) \quad \nabla^2 u^\ell v = 0 \quad \text{for a.e. } x \in P
\]
for all \( 1 \leq \ell \leq n+1 \). Indeed, by assumption (2), we can write \( \nabla^2 u^\ell \) as
\[
\nabla^2 u^\ell(x) = \lambda(x)b(x) \otimes b(x) \quad \text{a.e.}
\]
for some scalar function \( \lambda \) and \( b \in S^{n-1} \). Then (3.6) implies,
\[
(3.7) \quad (v)^T \lambda(x)b(x) \otimes b(x)v = \lambda(x)(v, b(x))^2 = 0 \quad \text{a.e.}
\]
This then implies
\[
\lambda(x)(v, b(x))^2 = 0 \quad \text{a.e.}
\]
Therefore,
\[
\nabla^2 u^\ell v = \lambda(x)(v, b(x))b(x) = 0 \quad \text{a.e.}
\]
which is exactly (3.7). The proof of the lemma is complete. \( \square \)

Let \( P_k \) be any \( k \)-dimensional plane such that assumptions (1)-(4) in Proposition 3.1 hold on \( P_k \cap \Omega \).

By means of Lemma 3.17 and making use of Proposition 3.8 and Corollary 3.10 similar as before we get
\[
\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U
\]
where \( P_x^U \) is some \((n-1)\)-plane in \( \Omega \) containing \( x \) with the property that for \( x, z \in \partial U \cap \Omega \), \( P_x^U = P_z^U \) if \( z \in P_x^U \) and \( P_x^U \cap P_z^U \cap \Omega = \emptyset \) if \( z \notin P_x^U \).
Similarly as in the proof of Lemma 3.7 (Figure 5), it suffices to show that the conclusions hold locally true. If \( x_0 \in P_k \cap \Omega \) is a point lying in an affine neighborhood for \( u \) in \( P_k \), then Lemma 3.7 and the assumptions of Proposition 3.1 immediately imply that \( \nabla u \) must be constant in the same neighborhood, which is the desired conclusion. Otherwise, we may and do choose a small \( \delta > 0 \) so that for any region \( U \) on which \( u \) is affine, the \( k \)-dimensional ball \( B^k(x_0, \delta) \subset P_k \cap \Omega \) intersects \( \partial U \) at no more than two \((k-1)\)-planes belonging to \( \partial U \).

We now focus on \( B^k(x_0, \delta) \subset P_k \cap \Omega \). For any \( x \in B^k(x_0, \delta) \), as in Lemma 3.7 we construct a \((k-1)\)-plane \( P_x \) in \( B^k(x_0, \delta) \) passing through \( x \) on which \( u \) is affine and \( P_x \cap P_z \cap B^k(x_0, \delta) = \emptyset \) if \( z \notin P_x \), see Figure 3. We then construct a foliation of \( B^k(x_0, \delta) \), see Figure 4 and obtain that the assumptions of Lemma 3.7 are satisfied along \( P_{\gamma(t)} \) for a.e. \( t \in (a, b) \) by the same argument as Step 4 and Step 5 of Lemma 3.7. It then follows that \( \nabla u \) is constant on \( P_{\gamma(t)} \) for a.e. \( t \in (a, b) \).

By choosing an initial value for \( \gamma \) arbitrary close to \( x_0 \) and applying the co-area formula in a similar manner we can make sure that \( \nabla u \) is of class \( W^{1,2} \) on \( \gamma \). Hence we conclude that \( \nabla u \) is \( C^{0,1/2} \) on \( \gamma \) by the Sobolev embedding theorem. Let \( F \) be the set of \( t \in (a, b) \) such that \( \nabla u \) is not constant along \( P_{\gamma(t)} \), then \( \mathcal{H}^1(F) = 0 \). We modify \( \nabla u \) to be constant along \( P_{\gamma(t)} \) for each \( t \in F \). Note that

\[
\mathcal{H}^k(\{P_{\gamma(t)} : t \in F\}) \leq c(2\delta)^{k-1} \mathcal{H}^1(\{\gamma(t) : t \in F\}) = c(2\delta)^{k-1} \mathcal{H}^1(F) = 0
\]

for some constant \( c \). Hence \( \nabla u \) is \( C^{0,1/2} \) up to modification of a set of measure zero in \( B^k(x_0, \delta) \). Moreover, \( \nabla u \) is constant on \( P_{\gamma(t)} \) for all \( t \), which foliates \( B^k(x_0, \delta) \). Thus \( \nabla u \) is constant on any region on which \( u \) is affine. Therefore, \( \nabla u \) is constant either on a \((k-1)\)-plane or \( k \)-dimensional region in \( B^k(x_0, \delta) \). This implies that the conclusions of Proposition 3.1 under the induction hypothesis are true and hence the inductive step is established. As a conclusion the proofs of Proposition 3.1 and Theorem 1.4 are complete. \( \square \)

4. Density: Proof of Theorem 1.5

In this section we show that isometric immersions smooth up to the boundary are strongly dense in \( I^{2,2}(\Omega, \mathbb{R}^{n+1}) \) if \( \Omega \subset \mathbb{R}^n \) is a convex \( C^1 \) domain. Note that it is sufficient to prove that \( I^{2,2} \cap C^{\infty}(\Omega, \mathbb{R}^{n+1}) \) is strongly dense in \( I^{2,2}(\Omega, \mathbb{R}^{n+1}) \). Having this result at hand, and since \( \Omega \) is assumed convex, the approximating sequence can be easily rescaled to be smooth up to the boundary.

4.1. Foliations of the domain. We have argued in the proof of Theorem 1.4 in section 3.2.2 that for every maximal region \( U \subset \Omega \) on which \( u \) is affine, \( \partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P^U_x \), where \( P^U_x \) is some \((n-1)\)-plane in \( \Omega \) containing \( x \) with the property that for \( x_1, x_2 \in \partial U \cap \Omega \), \( P^U_{x_1} = P^U_{x_2} \) if \( x_2 \in P^U_{x_1} \) and \( P^U_{x_1} \cap P^U_{x_2} \cap \Omega = \emptyset \) if \( x_2 \notin P^U_{x_1} \).

We say a maximal region on which \( u \) is affine is a body if its boundary contains more than two different \((n-1)\)-planes in \( \Omega \).

**Lemma 4.1.** It is sufficient to prove Theorem 1.3 for a function in \( I^{2,2}(\Omega, \mathbb{R}^{n+1}) \) with a finite number of bodies.

**Proof.** The proof is similar to the proof of [22, Lemma 3.8] and is omitted for brevity. \( \square \)

Now we can just assume \( u \in I^{2,2}(\Omega, \mathbb{R}^{n+1}) \) has finite number of bodies. Each body is closed and so is therefore their union, whose complement we denote by
Note that now for every \( n \)-dimensional maximal-affine region \( U \subset \tilde{\Omega} \), \( \partial U \cap \tilde{\Omega} \) consists of at most two \((n-1)\)-planes.

Similarly as in the proof of Lemma 3.7, for every \( x \in \tilde{\Omega} \), we will construct an \((n-1)\)-plane \( P_x \) in \( \tilde{\Omega} \) passing through it on which \( \nabla u \) is constant and \( P_x \cap P_z \cap \tilde{\Omega} = \emptyset \) if \( z \notin P_x \). To apply the same construction in Lemma 3.7, we make use of Theorem 1.4 and the fact that \( \partial U \cap \tilde{\Omega} \) consists of at most two \((n-1)\)-planes, see in Figure 11.

For every \( x \in \tilde{\Omega} \), we define the normal vector field \( N(x) \) as the unit vector orthogonal to the family \( P_x \) constructed above. Since none of the \( P_x \)'s intersect inside \( \tilde{\Omega} \) we can choose an orientation such that \( N \) is a Lipschitz vector fields. The ODE,

\[
\gamma'(t) = N(\gamma(t)) \quad \gamma(0) = x_0
\]

has a unique solution \( \gamma : (a, b) \to \tilde{\Omega} \) for some interval \((a, b) \subset \mathbb{R} \) containing 0. Note that \( P_x = P_{\gamma(t)} \) if \( x \in P_{\gamma(t)} \), therefore, \( \{P_{\gamma(t)}\}_{t \in (a, b)} \) is a local foliation of \( \tilde{\Omega} \) such that \( \nabla u \) is constant on \( P_{\gamma(t)} \) for all \( t \in (a, b) \) (Figure 12).

4.2. Leading curves in the domain.

**Definition 4.2.** Let \( \{P_x\}_{x \in \tilde{\Omega}} \) be a family of \((n-1)\)-planes in \( \tilde{\Omega} \) passing through \( x \) on which \( \nabla u \) is constant, satisfying \( P_x \cap P_z \cap \tilde{\Omega} = \emptyset \) if \( z \notin P_x \) and \( P_x = P_z \) if \( z \in P_x \).

We say that a curve \( \gamma \in C^{1,1}([0, \ell], \tilde{\Omega}) \) parametrized by arclength is a leading curve if it is orthogonal at any possible point of intersection \( z \in \gamma([0, \ell]) \cap P_x \) to \( P_x = P_z \) for all \( x \in \tilde{\Omega} \) (Figure 13).

It is easy to see that \( \gamma \) constructed in Subsection 4.1 when restricted to the interval \([0, \ell]\) is a leading curve, since by the ODE (4.1), \( |\gamma'| = 1 \) and \( |\gamma''| \) is bounded as \( N \) is Lipschitz.

**Definition 4.3.** The \((n-1)\)-dimensional hyperplane \( F_x(t) \) orthogonal to \( \gamma(t) \) at \( t \in [0, \ell] \) is called the leading front of \( \gamma \) at \( t \in [0, \ell] \) (Figure 13).

---

**Figure 11.** Construction of global foliations in \( \tilde{\Omega} \).
Remark 4.4. It then follows from the definition of the leading curve that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \bar{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0, \ell]$ such that $t \neq \tilde{t}$. Moreover, $F_\gamma(t) \subset \bar{\Omega}$, otherwise, $F_\gamma(t) \cap B \neq \emptyset$ where $B$ is one of the bodies in $\Omega \setminus \bar{\Omega}$. Since $\nabla u$, being continuous, is constant on $F_\gamma(t) \cap \bar{\Omega}$ and $B$, it must be constant on their convex hull, which is again a body, contradiction to that a body is a maximal region.

We say that a curve $\gamma$ covers the domain $A \subset \Omega$ if

$$A \subset \bigcup \{F_\gamma(t) : t \in [0, \ell]\}.$$ 

By $\Omega(\gamma)$ we refer to the biggest set covered by $\gamma$ in $\Omega$. We now restrict our attention to the covered domain $\Omega(\gamma)$. It is obvious that $\Omega(\gamma)$ is convex since it is bounded by $F_\gamma(0)$, $F_\gamma(\ell)$ and $\partial \Omega$. 

Figure 12.

Figure 13. Leading curve and leading fronts.
From the construction in subsection 4.1, the \((n-1)\)-planes \(P_{\gamma(t)}\) in \(\tilde{\Omega}, t \in [0, \ell]\) which constitute a local foliation of \(\tilde{\Omega}\) are global foliations of \(\Omega(\gamma)\). Moreover, \(P_{\gamma(t)} = F_{\gamma}(t) \cap \Omega(\gamma) = F_{\gamma}(t) \cap \Omega\) for all \(t \in [0, \ell]\). We relabel them \(P_{\gamma}(t)\) to be in consistence of notation and we name them:

**Definition 4.5.** The component \(P_{\gamma}(t) := F_{\gamma}(t) \cap \Omega\) is called the leading \((n-1)\)-planes in \(\Omega\) of \(\gamma\) at \(t \in [0, \ell]\).

Let \(\{N_i(t)\}_{i=1}^{n-1}\) be an orthonormal basis for the leading front \(F_{\gamma}(t)\) (Figure 13) such that \(N_i\) is Lipschitz for all \(1 \leq i \leq n-1\) and \(\text{det}[\gamma'(t), N_1(t), \ldots, N_{n-1}(t)] = 1\). It is obvious such orthonormal basis exists because we can pick \(\{N_i(0)\}_{i=1}^{n-1}\) as an orthonormal basis for \(F_{\gamma}(0)\) that form a positive orientation with \(\gamma'(0)\) and then move this frame along \(\gamma\) in an orientation preserving way (note that \(\gamma\) is not a closed curve so this is possible). Let \(\Phi : [0, \ell] \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n\) be defined as,

\[
\Phi(t, s) := \gamma(t) + \sum_{i=1}^{n-1} s_i N_i(t),
\]

where \(s = (s_1, \ldots, s_{n-1})\). Then we can represent the leading front at \(t \in [0, \ell]\) as,

\[
F_{\gamma}(t) = \{\Phi(t, s), s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1}\}. \tag{4.3}
\]

For each \(t \in [0, \ell]\), define the open set,

\[
\Sigma^\gamma(t) = \{s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1} : \Phi(t, s) \in \Omega\}. \tag{4.4}
\]

It is obvious that \(0 \in \Sigma^\gamma(t)\), hence it is non-empty. Then we can also parametrize the leading planes as

\[
P_{\gamma}(t) = \{\Phi(t, s), s = (s_1, \ldots, s_{n-1}) \in \Sigma^\gamma(t)\}. \tag{4.5}
\]

Now define,

\[
\Sigma^\gamma := \{(t, s), \Phi(t, s) \in \Omega\}. \tag{4.6}
\]

Of course we can also write,

\[
\Sigma^\gamma = \{(t, s), t \in [0, \ell], s = (s_1, \ldots, s_{n-1}) \in \Sigma^\gamma(t)\}.
\]

We will focus on the restriction of \(\Phi\) in \(\Sigma^\gamma\). However, if no confusion is caused, we still denote such restriction \(\Phi\). It is easy to see that \(\Phi\) maps \(\Sigma^\gamma\) into \(\Omega(\gamma)\). Indeed, if \(x = \Phi(t, s)\) for some \((t, s) \in \Sigma^\gamma\), by definition of \(\Sigma^\gamma\), \(\Phi(t, s) \in \Omega\). On the other hand, \(\Phi(t, s) \in F_{\gamma}(t)\), thus, \(x = \Phi(t, s) \in F_{\gamma}(t) \cap \Omega \subset \Omega(\gamma)\).

**Lemma 4.6.** \(\Phi : \Sigma^\gamma \rightarrow \Omega(\gamma)\) is one-to-one and onto. In particular,

\[
\Omega(\gamma) = \{\Phi(t, s), (t, s) \in \Sigma^\gamma\} = \bigcup \{P_{\gamma}(t) : t \in [0, \ell]\}.
\]

**Proof.** We first show one-to-one. Suppose \(\Phi(t_1, s_1) = \Phi(t_2, s_2)\) while \((t_1, s_1) \neq (t_2, s_2)\). Since \(s \rightarrow \Phi(t, s)\) is obviously one-to-one by the definition of \(\Phi\), it must be \(t_1 \neq t_2\). We have argued in Remark 4.3 that \(F_{\gamma}(t_1) \cap F_{\gamma}(t_2) \cap \Omega = \emptyset\). Therefore, \(F_{\gamma}(t_1) \cap F_{\gamma}(t_2) \cap \Omega(\gamma) = \emptyset\) since \(\Omega(\gamma) \subset \Omega\). However, \(\Phi(t_1, s_1) \in F_{\gamma}(t_1)\) and \(\Phi(t_2, s_2) \in F_{\gamma}(t_2)\), contradiction to \(\Phi(t_1, s_1) = \Phi(t_2, s_2)\).

We will now show onto. Let \(x \in \Omega(\gamma)\), then \(x = \Phi(t, s)\) for some \(t \in [0, \ell]\) and \(s \in \mathbb{R}^{n-1}\). Since \(x \in \Omega(\gamma)\), \(\Phi(t, s) \in \Omega(\gamma) \subset \Omega\), hence \((t, s) \in \Sigma^\gamma\). The proof is complete. \(\square\)
Similarly we can also write
\[ \Phi(t, S \cdot s) = \gamma(t) + S \left( \sum_{i=1}^{n-1} s_i N_i(t) \right), \quad t \in [0, \ell], s \in \mathbb{R}^{n-1}, \]
as
\[ \Phi(t, S \cdot s) = \gamma(t) + S \left( \sum_{i=1}^{n-1} s_i N_i(t) \right), \quad t \in [0, \ell], s \in \mathbb{S}^{n-2}, S \geq 0. \]

We then rewrite the representation of leading front in (4.3) in an equivalent way:
\[ F_\gamma(t) = \{ \Phi(t, S \cdot s), S \geq 0, s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2} \}. \]
For each \( t \in [0, \ell] \) and \( s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2} \), define the scalar function,
\[ S^\gamma_*(t) := \sup \{ S \geq 0 : \Phi(t, S \cdot s) \in \Omega \}. \]
That is, \( S^\gamma_*(t) \) is the distance from \( \gamma(t) \) to \( \partial \Omega \) in the direction \( \sum_{i=1}^{n-1} s_i N_i(t) \). From the definition of \( \Sigma^\gamma(t) \) and \( \Sigma^\gamma \),
\[ \Sigma^\gamma(t) = \{ (S \cdot s) : s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S^\gamma_*(t) \}, \]
and
\[ \Sigma^\gamma = \{ (t, S \cdot s), t \in [0, \ell], s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S^\gamma_*(t) \}. \]
Since \( |\gamma'(t)| = 1, \gamma''(t) \cdot \gamma'(t) = 0 \), we can then write \( \gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t) N_i(t) \).

Similarly we can also write
\[ N'_i = \kappa_{i_0} \gamma' + \sum_{j=1}^{n-1} \kappa_{i_j} N_j. \]

It is easy to see that \( \kappa_{i_0} = -\kappa_i, \kappa_{i_1} = 0 \) and \( \kappa_{i_j} = -\kappa_j \). These equations can be written as the matrix equation
\[
\begin{pmatrix}
N_1 \\
N_2 \\
\vdots \\
N_{n-1}
\end{pmatrix}' =
\begin{pmatrix}
0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} \\
-\kappa_1 & 0 & \kappa_{12} & \cdots & \kappa_{1n-1} \\
-\kappa_2 & -\kappa_{12} & 0 & \cdots & \kappa_{2n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\kappa_{n-1} & -\kappa_{1n-1} & -\kappa_{2n-1} & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\gamma' \\
N_1 \\
N_2 \\
\vdots \\
N_{n-1}
\end{pmatrix}
\]

Given two non-parallel leading fronts \( F_\gamma(t) \) and \( F_\gamma(\tilde{t}) \), denote their \( (n-2) \)-plane intersection by \( F(t, \tilde{t}) \). Given \( s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2} \), define \( L_s(t, \tilde{t}) \) as the distance from \( \gamma(t) \) to \( F(t, \tilde{t}) \) along the direction \( \sum_{i=1}^{n-1} s_i N_i(t) \) (we set \( L_s(t, \tilde{t}) = +\infty \) if it does not hit \( F(t, \tilde{t}) \) along this direction) (Figure 4). We then define,
\[ L^\gamma_*(t) := \inf \{ L_s(t, \tilde{t}) : \tilde{t} \neq t \}. \]
Since all \( F(t, \tilde{t}) \) are outside \( \Omega \), \( L^\gamma_*(t) \geq S^\gamma_*(t) \) for all \( s \in \mathbb{S}^{n-2} \) and \( t \in [0, \ell] \).

**Lemma 4.7.** \( L^\gamma_*(t) \left( \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \leq 1 \) for all \( t \in [0, \ell] \) and \( s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2} \).
**Proof.** Suppose $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel. Solving for their intersection yields
\[
\gamma(t) + \sum_{i=1}^{n-1} s_i N_i(t) = \gamma(\tilde{t}) + \sum_{i=1}^{n-1} r_i N_i(\tilde{t}).
\]
This is a linear system of $n$ equations and $2n-2$ unknowns $(s_i)_{i=1}^{n-1}$ and $(r_i)_{i=1}^{n-1}$. A solution for this system of equations exists because the two leading fronts are not parallel. Then direct computation using Cramer’s rule gives the formula for $F(t, \tilde{t})$ explicitly,
\[
F(t, \tilde{t}) = \{ x \in F_\gamma(t) : (x - \gamma(t)) \cdot \left( -\sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} N_i(t) \right) = 1 \},
\]
where
\[
h_i(t, \tilde{t}) := \det[N_1(\tilde{t}), \ldots, N_{n-1}(\tilde{t}), N_i(t)]
\]
for $1 \leq i \leq n-1$, and
\[
H(t, \tilde{t}) = \det[N_1(\tilde{t}), \ldots, N_{n-1}(\tilde{t}), \gamma(t) - \gamma(\tilde{t})].
\]
Note that $H(t, \tilde{t}) \neq 0$ since $\gamma(t) - \gamma(\tilde{t})$ is not parallel to $F_\gamma(\tilde{t})$.

We firstly claim that
\[ L_s(t, \tilde{t}) \left( -\sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} N_i(t) \right) \leq 1. \]
Indeed, we divide the situation into two cases. In the first case, suppose we travel from $\gamma(t)$ along a given direction $\sum_{i=1}^{n-1} s_i N_i(t)$ and hit $F(t, \tilde{t})$, then for $x \in F(t, \tilde{t})$,
\[
x - \gamma(t) = L_s(t, \tilde{t}) \left( \sum_{i=1}^{n-1} s_i N_i(t) \right).
\]
Therefore,
\begin{equation}
L_s(t, \tilde{t}) \left( \sum_{i=1}^{n-1} s_i N_i(t) \right) \cdot \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} N_i(t) \right) \\
= L_s(t, \tilde{t}) \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} s_i \right) = 1.
\end{equation}

(4.15)

Suppose for a certain direction \( \sum_{i=1}^{n-1} s_i N_i(t) \) we do not hit \( F(t, \tilde{t}) \), in which case we set \( L_s(t, \tilde{t}) = +\infty \), then we must hit \( F(t, \tilde{t}) \) through the direction \( - \sum_{i=1}^{n-1} s_i N_i(t) \), therefore, by (4.15),
\begin{equation}
L_{-s}(t, \tilde{t}) \left( \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \right) = 1.
\end{equation}

(4.15)

In particular, since \( L_{-s}(t, \tilde{t}) > 0 \),
\begin{equation}
\sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i > 0.
\end{equation}

We then must have,
\begin{equation}
L_s(t, \tilde{t}) \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \right) < 0.
\end{equation}

(4.16)

(4.15) and (4.16) together gives that in either case (4.14) holds true, which proves our claim.

We secondly claim that
\begin{equation}
L_\gamma^s(t) \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \right) \leq 1
\end{equation}

for all \( t, \tilde{t} \in [0, \ell] \) and \( s \in S^{n-2} \). Indeed, if for a given \( t, \tilde{t} \) and \( s \in S^{n-2} \), \( F_\gamma(t) \) and \( F_\gamma(\tilde{t}) \) are not parallel, and
\begin{equation}
- \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \geq 0,
\end{equation}
then,
\begin{equation}
L_\gamma^s(t) \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \right) \leq L_s(t, \tilde{t}) \left( - \sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, t)} s_i \right) = 1
\end{equation}

which gives (4.17) for this case. If for a certain \( t, \tilde{t} \) and \( s \in S^{n-2} \), (4.18) fails to hold, then (4.17) is obviously satisfied. Finally, if \( F_\gamma(t) \) and \( F_\gamma(\tilde{t}) \) are parallel, then \( h_i(t, \tilde{t}) = 0 \) for all \( 1 \leq i \leq n - 1 \), hence the (4.17) is again satisfied.

We thirdly claim that
\begin{equation}
- \frac{h_i(t, \tilde{t})}{H(t, t)} \to \kappa_i(t), \quad 1 \leq i \leq n - 1.
\end{equation}

(4.19)
as $\tilde{t} \to t$. Indeed, since $\det[\gamma'(t), N_1(t), \cdots, N_{n-1}(t)] = 1$ for all $t \in [0, \ell]$, 

$$H(t, \tilde{t}) \approx \det[N_1(\tilde{t}), \cdots, N_{n-1}(\tilde{t}), \gamma'(\tilde{t})(t - \tilde{t})] = (-1)^{n-1}(t - \tilde{t})$$

as $\tilde{t} \to t$. Moreover, 

$$h_i(t, t) = \det[N_1(t), \cdots, N_{n-1}(t), N_i(t)] = 0.$$

Then, 

$$\frac{d}{dt} \left( \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} \right) \approx \frac{h_i(t, \tilde{t}) - h_i(t, t)}{(-1)^{n-1}(t - \tilde{t})} \to (-1)^{n-1} \left( \det[N'_1(t), \cdots, N'_{n-1}(t), N_i(t)] + \cdots + \det[N_1(t), \cdots, N'_{n-1}(t), N_i(t)] \right).$$

Recalling (4.11) and plugging this expression into (4.20) and it is easy to see that all other terms vanish except 

$$\det[N_1(t), \cdots, N'_1(t), \cdots, N_{n-1}(t), N_i(t)]$$

$$= -\kappa_i \det[N_1(t), \cdots, \gamma'(t), \cdots, N_{n-1}(t), N_i(t)]$$

$$= \kappa_i \det[N_1(t), \cdots, N_{n-1}(t), \gamma'(t)] = (-1)^{n-1}\kappa_i$$

because $\det[\gamma'(t), N_1(t), \cdots, N_{n-1}(t)] = 1$. This proves (4.19).

Passing in (4.17) to the limit $\tilde{t} \to t$ we obtain the lemma. The proof is complete. □

Recall that $S^\gamma(t)$ as defined in (4.8) satisfies $0 \leq S^\gamma(t) \leq L^\gamma(t)$ for all $s \in \Sigma^n$ due to the fact that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = 0$ for all $t, \tilde{t} \in [0, \ell], \tilde{t} \neq t$. We then have,

**Corollary 4.8.** $S^\gamma(t) \left( \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \leq 1$ for all $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \Sigma^n$.

**Proof.** If $\sum_{i=1}^{n-1} s_i \kappa_i(t) \geq 0$, then $S^\gamma(t) \left( \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \leq L^\gamma(t) \left( \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \leq 1$. If $\sum_{i=1}^{n-1} s_i \kappa_i(t) < 0$, then the result is obviously true. □

From the definition of $\Phi$ in (4.2), $\Phi$ is Lipschitz, hence its Jacobian $J_\Phi = \det D\Phi$ exists a.e. on $\Sigma^\gamma$, where $\Sigma^\gamma$ has two equivalent representations (4.6) and (4.10). We will show the Corollary 4.8 implies $J_\Phi > 0$ a.e. on $\Sigma^\gamma$, namely,

**Lemma 4.9.** $J_\Phi(t, s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_i(t) > 0$ for all $(t, s) \in \Sigma^\gamma$.

**Proof.** Differentiating $\Phi(t, s)$ with respect to $(t, s_1, \cdots, s_{n-1})$ gives,

$$J_\Phi(t, s) = \det[\gamma'(t) + \sum_{i=1}^{n-1} s_i N'_i(t), N_1(t), \cdots, N_{n-1}(t)].$$

Substituting (4.11) into (4.21), we obtain, after Gaussian elimination, that,

$$J_\Phi(t, s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_i(t).$$
If $\sum_{i=1}^{n-1} s_i \kappa_i(t) \leq 0$, then obviously $J_\Phi(t,s) > 0$. Suppose now $\sum_{i=1}^{n-1} s_i \kappa_i(t) > 0$. By (4.8) and (4.10) we have

$$\sum_{i=1}^{n-1} s_i \kappa_i(t) = |s| \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_i(t) \right) < S_n^\gamma(t) \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_i(t) \right) \leq 1$$

by Corollary 4.8 Therefore, $J_\Phi(t,s) > 0$ for all $(t,s) \in \Sigma^\gamma$. The proof is complete. \qed

4.3. Moving Frames in the target space. We are now in a position to define the moving frame in the target space $\mathbb{R}^{n+1}$. Let $N_i(t), 1 \leq i \leq n-1$ be as in subsection 4.2. Define the leading curve corresponding to $\gamma$ in $u(\Omega(\gamma))$ to be

$$\tilde{\gamma} := u \circ \gamma.$$

We also recall from subsection 4.1 the definitions (4.2), (4.5), and that $\nabla u$ is constant on $P_\gamma(t)$ for each $t \in [0, \ell]$. Hence for each $t \in [0, \ell]$, $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$.

Consider the Darboux frame $(\tilde{\gamma}', v_1, \ldots, v_{n-1}, n)$ where $v_i(t) = \nabla u(\gamma(t)) N_i(t)$, $i = 1, \ldots, n-1$ and $u(t) = \tilde{\gamma}'(t) \times v_1(t) \times \cdots \times v_{n-1}(t)$. Since $u$ is an isometric affine map along $P_\gamma(t)$ for each $t \in [0, \ell]$ we obtain

$$u(\Phi(t,s)) = \tilde{\gamma}(t) + \sum_{i=1}^{n-1} s_i v_i(t) \tag{4.23}$$

for all $t \in [0, \ell]$ and $s \in \Sigma^\gamma(t)$. Differentiating with respect to $t$, by (4.2) we get

$$\nabla u(\Phi(t,s)) \left( \gamma'(t) + \sum_{i=1}^{n-1} s_i N_i'(t) \right) = \tilde{\gamma}'(t) + \sum_{i=1}^{n-1} s_i v_i'(t), \tag{4.24}$$

and differentiating with respect to $s_i$, $1 \leq i \leq n-1$ we obtain for each $i$,

$$\nabla u(\Phi(t,s)) N_i(t) = v_i(t). \tag{4.25}$$

By the linear expansion of $N_i'$ in (4.11) and (4.12), together with (4.24) and (4.25) we get

$$\tilde{\gamma}'(t) + \sum_{i=1}^{n-1} s_i v_i'(t) \tag{4.26}$$

$$= \nabla u(\Phi(t,s)) \left( 1 - \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \gamma'(t) + \sum_{i=1}^{n-1} s_i \left( \sum_{j=1}^{n-1} \kappa_{ij}(t) v_j(t) \right)$$

with $\kappa_i = 0$ and $\kappa_{ij} = -\kappa_{ji}$. Also, by (4.24), for $s = 0$ we have

$$\nabla u(\Phi(t,0)) \gamma'(t) = \tilde{\gamma}'(t).$$

Since $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$ for each $t \in [0, \ell]$, we obtain

$$\nabla u(\Phi(t,s)) \gamma'(t) = \nabla u(\Phi(t,0)) \gamma'(t) = \tilde{\gamma}'(t) \tag{4.27}$$

for all $s \in \Sigma^\gamma(t)$.

Alongside (4.25), this shows that at each point in $\Omega(\gamma)$, $\nabla u$ maps an orthonormal frame to another orthonormal frame and this orthonormal frame only depends on
Finally, using \( (4.26) \) and matching coefficients yields for all \( 1 \leq i \leq n - 1, \)

\[
v'_i = -\kappa_i \gamma'_i + \sum_{j=1}^{n-1} \kappa_{ij} v_j, \quad \kappa_{ii} = 0 \text{ and } \kappa_{ij} = -\kappa_{ji}.
\]

In other words, the following system of ODEs is satisfied by the Darboux frame of \( \tilde{\gamma} \):

\[
\begin{pmatrix}
\gamma' \\
v_1 \\
v_2 \\
\vdots \\
v_{n-1} \\
n
\end{pmatrix}' = \mathcal{K}
\begin{pmatrix}
\gamma' \\
v_1 \\
v_2 \\
\vdots \\
v_{n-1} \\
n
\end{pmatrix},
\]

where the skew-symmetric curvature matrix \( \mathcal{K} \) is given by

\[
\mathcal{K} = \\
\begin{pmatrix}
0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} & \kappa_n \\
-\kappa_1 & 0 & \kappa_{12} & \cdots & \kappa_{1,n-1} & 0 \\
-\kappa_2 & -\kappa_{12} & 0 & \cdots & \kappa_{2,n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\kappa_{n-1} & -\kappa_{1,n-1} & -\kappa_{2,n-1} & \cdots & 0 & 0 \\
-\kappa_n & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

4.4. Change of variable formula. Recall that \( \Phi : \Sigma^\gamma \rightarrow \Omega(\gamma) \) is one-to-one and onto, where \( \Sigma^\gamma \) was defined in \( (4.10) \). For \((t, s) \in \Sigma^\gamma\), let \( u_i(t, s) := (\frac{\partial}{\partial x_i} u) \circ \Phi(t, s) \), note that \( u_i \) is the \( i \)th column of \( \nabla u \circ \Phi \). The following holds for all \((t, s) \in \Sigma^\gamma\): since \( \nabla^T u \cdot \gamma' = n \cdot \nabla u \gamma' = n \cdot \gamma' = 0 \) and \( \nabla^T u n \cdot N_j = n \cdot \nabla u N_j = n \cdot v_j = 0 \) for all \( 1 \leq j \leq n - 1 \), we have \( \nabla^T u n = 0 \), i.e. \( u_i \cdot n = 0 \) for all \( 1 \leq i \leq n \). Thus,

\[
u_i = (u_i \cdot \gamma') \gamma' + \sum_j (u_i \cdot v_j) v_j + (u_i \cdot n) n
\]

\[
= (u_i \cdot \gamma') \gamma' + \sum_j (u_i \cdot v_j) v_j
\]

\[
= (u_i \cdot \nabla u \gamma') \gamma' + \sum_j (u_i \cdot \nabla u N_j) v_j
\]

\[
= (\nabla^T u_i \cdot \gamma') \gamma' + \sum_j (\nabla^T u_i \cdot N_j) v_j
\]

\[
= (e_i \cdot \gamma') \gamma' + \sum_j (e_i \cdot N_j) v_j,
\]

where \( e_i = (0, \ldots, 1, \ldots, 0) \). Note that the right hand side of \( (4.30) \) is independent of \( s \). Differentiating with respect to \( s_j, 0 \leq j \leq n - 1 \), by \( (4.2) \) we get for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \),

\[
(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t, s)) N_j(t) = 0.
\]
Differentiating $u_i$ with respect to $t$ we obtain,

\[
(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t, s))(\gamma'(t) + \sum_{j=1}^{n-1} s_j N_j(t))
= (e_i \cdot \gamma''(t))\tilde{\gamma}(t) + (e_i \cdot \gamma'(t))\tilde{\gamma}''(t)
+ \sum_{j=1}^{n-1} (e_i \cdot N_j(t))v_j(t) + \sum_{j=1}^{n-1} (e_i \cdot N_j(t))v'_j(t).
\]

(4.32)

If we write out $N_j'$ as a linear combination of $\gamma'$ and $N_j$, $j = 1, \cdots, n-1$ as in (4.11) and (4.12), the left hand side of (4.32) becomes

\[
(1 - \sum_{j=1}^{N-1} s_j \kappa_j(t)) (\nabla \frac{\partial}{\partial x_i} u)(\Phi(t, s))\gamma'(t).
\]

For the right hand side of (4.32), if we write out $\gamma''$, $\tilde{\gamma}''$, $N_j'$ and $v_j'$ as linear combinations of $\gamma'$, $\tilde{\gamma}'$, $N_i$ and $v_\ell$, $\ell = 1, \cdots, n-1$ and $n$ as in (4.11) and (4.28), we obtain,

\[
(e_i \cdot \gamma'')(\tilde{\gamma}' + \sum_j (e_i \cdot N_j')v_j + \sum_j (e_i \cdot N_j)v'_j
= (e_i \cdot \sum_j \kappa_j N_j)\tilde{\gamma}' + (e_i \cdot \gamma')(\sum_j \kappa_j v_j + \kappa_n n) + \sum_j (e_i \cdot (\gamma_j' + \sum_\ell \kappa_\ell N_\ell))v_j
+ \sum_j (e_i \cdot N_j)(-\kappa_j \tilde{\gamma}' + \sum_\ell \kappa_j \gamma' + \sum_\ell \kappa_j v_\ell)
= \sum_j (e_i \cdot \kappa_j N_j)\tilde{\gamma}' + (e_i \cdot \gamma')(\sum_j \kappa_j v_j + \sum_j (e_i \cdot N_j)\tilde{\gamma}' + \sum_\ell \sum_j (e_i \cdot \kappa_j N_j)\gamma'_\ell + \sum_\ell \sum_j (e_i \cdot N_j)\kappa_j v_j = (e_i \cdot \gamma')\kappa_n n
\]

where we used the fact that $\kappa_{ij} = -\kappa_{ji}$. By Lemma 4.9, $1 - \sum_{j=1}^{n-1} s_j \kappa_j(t) > 0$ for all $(t, s) \in \Sigma'$. Therefore,

\[
(\nabla \frac{\partial}{\partial x_i} u)(\Phi(t, s))\gamma'(t) = \frac{(e_i \cdot \gamma'(t))\kappa_n(t)n(t)}{1 - \sum_{j=1}^{n-1} s_j \kappa_j(t)}.
\]

(4.33)

Since $\Phi$ is Lipschitz with $J_\Phi(t, s) = 1 - \sum_{j=1}^{n-1} s_j \kappa_j(t) > 0$, the change of variable $x = \Phi(t, s)$ with (4.23) and (4.33) yields,

\[
\int_{\Omega(\gamma)} |u(x)|^2 dx
= \int_0^\ell \int_{\Sigma'\gamma(t)} |\tilde{\gamma}(t)|^2 + \sum_{i=1}^{n-1} s_i |v_i(t)|^2 \cdot (1 - \sum_{j=1}^{n-1} s_j \kappa_j(t)) d\mathcal{H}^{n-1}(s) dt,
\]

(4.34)

\[
\int_{\Omega(\gamma)} |\nabla u(x)|^2 dx = n|\Omega(\gamma)|,
\]

(4.35)
Lemma 4.10. There exists a sequence of isometries\( u \) converging strongly to\( u \) and (4.8) respectively. Since all leading fronts meet outside\( \Omega \), we must have\( u \) is a leading curve. We then define\( \gamma \) such that satisfies\( \gamma \). Suppose\( \Lambda \) is well-defined.\( \gamma \) is a sequence of smooth maps in\( I \) and our final goal is to glue all\( \gamma \) together smoothly. By flattening\( \gamma \) around the end points 0 and\( \ell \), so we cannot call the (n 2)-dimensional hyperplane orthogonal to\( \gamma \) at\( t \) leading fronts. Instead we call them orthogonal fronts and denote them by\( F_{\gamma}(t) \). If we manage to show all such orthogonal fronts meet outside\( \Omega(\gamma_m) \),\( \gamma_m \) becomes a leading curve for\( u_m \) and\( F_{\gamma_m}(t) \) are actually the leading fronts. We then define\( u_m \) to be isometric affine mapping along each leading front\( F_{\gamma_m}(t) \). Since all the leading fronts intersect outside\( \Omega \),\( u_m \) is well-defined.

We first need the following lemma,

Lemma 4.13. There exists smooth curve\( \gamma_m \) such that\( \gamma_m(t) \to \gamma(t) \) strongly in\( W^{2,p}([0,\ell],[\mathbb{R}^n]) \) for all\( 1 \leq p < \infty \) and satisfies\( F_{\gamma_m}(t) \cap F_{\gamma_m}(\ell) \cap \Omega = \emptyset \) for all\( t, t' \in [0,\ell] \).

Proof. The construction is long and technical so we postpone the proof to Appendix A.\( \square \)

We also need to define the curves\( \gamma_m \) in the target space\( u(\Omega(\gamma)) \) corresponding to\( \gamma_m \). Recall that the normal curvature\( \kappa_n \) defined in (4.29) is bounded. We choose a sequence of uniformly bounded smooth function\( \tilde{\kappa}_{n,m} \) such that\( \tilde{\kappa}_{n,m} \to \kappa_n \) a.e. in\( [0,\ell] \), (and hence in\( L^p \) for all\( 1 \leq p < \infty \)).

We need to flatten\( \tilde{\kappa}_{n,m} \) around the end points 0 and\( \ell \) for two reasons: first, it might happen that\( \Omega(\gamma) \not\subseteq \Omega(\gamma_m) \) so we need to extend the isometric immersion defined on\( \Omega(\gamma_m) \) smoothly to the region of\( \Omega(\gamma) \) outside\( \Omega(\gamma_m) \). Second, so far all the construction is on one covered domain\( \Omega(\gamma) \) and our final goal is to glue all the different covered domains together smoothly. By flattening\( \tilde{\kappa}_{n,m} \) around the
end point 0 and \( \ell \), \( u_m \) constructed later is affine near the leading planes \( P_i(0) \) and \( P_i(\ell) \) (for definition of leading planes see Definition (4.5)) so that we can join all the pieces smoothly. The modification goes as follows: by (4.33), the second derivative of \( u \) vanishes whenever \( \kappa_n = 0 \). Put

\[
\ell^*_m = \begin{cases} \ell & \text{if } \Omega(\gamma) \subset \Omega(\gamma_m) \text{ and}, \\
\sup\{t \in [0, \ell] : F_{\gamma_m}(t) \cap F_i(\ell) \cap \Omega(\gamma) = \emptyset \} \text{ otherwise.}
\end{cases}
\]

By step 1 of Lemma (4.13) in the Appendix, \( F_{\gamma_i}(t) \rightarrow F_i(t) \) uniformly, hence \( \ell^*_m \rightarrow \ell \) as \( m \rightarrow \infty \).

Let \( \psi_1 \) be any smooth non-negative function which is 0 on \([-1, \infty)\) and 1 on \((-\infty, -2)\). Let \( \psi_2 \) be any smooth positive function which is 0 on \((-\infty, 1]\) and 1 on \((2, \infty)\). We put,

\[
\kappa_{n,m}(t) := \psi_1(m(t - \ell^*_m)) \psi_2(mt) \kappa_{n,m}(t), \quad t \in [0, \ell]
\]

and we solve the following linear system for initial values \( \tilde{\gamma}_m'(0) = \tilde{\gamma}'(0), v_{i,m}(0) = v_i(0), \) and \( n_m(0) = n(0) \):

\[
\begin{pmatrix}
\tilde{z}'_m \\
v_{1,m} \\
v_{2,m} \\
\vdots \\
v_{n-1,m} \\
n_m
\end{pmatrix}' = \mathcal{K}_m
\begin{pmatrix}
\tilde{z}'_m \\
v_{1,m} \\
v_{2,m} \\
\vdots \\
v_{n-1,m} \\
n_m
\end{pmatrix},
\]

where the matrix \( \mathcal{K}_m \) is given by

\[
\mathcal{K}_m = \begin{pmatrix}
0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} & \kappa_{n,m} \\
-\kappa_{1,m} & 0 & \kappa_{12,m} & \cdots & \kappa_{1n-1,m} & 0 \\
-\kappa_{2,m} & -\kappa_{12,m} & 0 & \cdots & \kappa_{2n-1,m} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\kappa_{n-1,m} & -\kappa_{1n-1,m} & -\kappa_{2n-1,m} & \cdots & 0 & 0 \\
-\kappa_{n,m} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

We define

\[
\tilde{\gamma}_m(t) = \tilde{\gamma}(0) + \int_0^t \tilde{z}'_m(\tau)d\tau.
\]

By the same argument as in step 1 in the proof of Lemma (4.13), \( \tilde{\gamma}_m \rightarrow \tilde{\gamma} \) in \( W^{2,p}([0, \ell], \mathbb{R}^{n+1}) \) and the moving frame \((\tilde{\gamma}_m', v_{1,m}, \cdots, v_{n-1,m}, n_m)\) converges to \((\gamma', v_1, \cdots, v_{n-1}, n)\) uniformly.

Eventually, we define our approximating sequence \( u_m \) on \( \Omega(\gamma_m) \):

\[
(4.37) \quad u_m(\gamma_m(t) + \sum_{i=1}^{n-1} s_i N_{i,m}(t)) = \tilde{\gamma}_m(t) + \sum_{i=1}^{n-1} s_i v_{i,m}(t),
\]

where \( \gamma_m \) is defined in Lemma (4.13). Such \( \gamma_m \) assures that all its leading fronts intersect outside \( \Omega \), hence \( u_m \) is well-defined and smooth over \( \Omega(\gamma) \cap \Omega(\gamma_m) \).

As before, let \( \Phi_m : [0, \ell] \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n \) be defined as

\[
\Phi_m(t, s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i N_{i,m}(t),
\]
and let $\Delta^{\gamma} = \{(t, s) : \Phi_m(t, s) \in \Omega(\gamma)\}$. Same argument as Step 6 in Lemma 4.13 gives that $\Phi_m(t, s)$ is a bi-Lipschitz mapping of $\Delta^{\gamma}$ onto $\Omega(\gamma) \cap \Omega(\gamma_m)$. By differentiating with respect to $t, s, \cdots, s_{n-1}$, as in (4.27) and (4.25), we see that at each point of $x$, $\nabla u_m(x)$ maps an orthonormal frame to an orthonormal frame. Hence $\nabla u_m(x)^{\top} \nabla u_m(x) = 1$. Moreover, $u_m$ is affine near $P_{\gamma_m}(\ell)$ and can be extended by an affine isometry over $\Omega(\gamma)$. Therefore, $u_m \in I^{2,2}(\Omega(\gamma), \mathbb{R}^n)$. Everything we have proved for isometric immersions of course applies, in particular, by (4.30), (4.33), and (4.31) we have for all $1 \leq i \leq n - 1$,

\begin{equation}
(4.38) \quad \frac{\partial}{\partial x_i} u_m \circ \Phi_m(t, s) = (e_i \cdot \gamma_m(t)) \gamma'_m(t) + \sum_{j=1}^{n-1} (e_i \cdot N_{j,m}(t))v_{j,m}(t),
\end{equation}

\begin{equation}
(4.39) \quad (\nabla \frac{\partial}{\partial x_i} u_m)(\Phi_m(t, s)) \gamma'_m(t) = \frac{(e_i \cdot \gamma_m(t))\kappa_{n,m}(t)\kappa(t)}{1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)}, \quad \text{and}
\end{equation}

\begin{equation}
(4.40) \quad (\nabla \frac{\partial}{\partial x_i} u_m)(\Phi_m(t, s))N_{j,m}(t) = 0, \quad \text{for all } 1 \leq j \leq n - 1.
\end{equation}

for all $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \Delta^{\gamma_m}(t)$. Moreover, by (4.34), (4.35), and (4.36) we compute,

\begin{equation}
(4.41) \quad \int_{\Omega(\gamma)} |u_m(x)|^2 dx = \int_{\Omega(\gamma) \cap \Omega(\gamma_m)} |u_m(x)|^2 dx + \int_{\Omega(\gamma) \setminus \Omega(\gamma_m)} |u_m(x)|^2 dx,
\end{equation}

\begin{equation}
= \int_0^{\ell} \int_{\Delta^{\gamma_m}(t)} |\gamma_m(t)| + \sum_{i=1}^{n-1} s_i v_{i,m}(t)^2 \cdot \left(1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\right) d\mathcal{H}^{n-1}(s) dt
+ \int_{\Omega(\gamma) \setminus \Omega(\gamma_m)} |u_m(\ell) + \nabla u_m(\ell)(x - \gamma_m(\ell))|^2 dx,
\end{equation}

\begin{equation}
(4.42) \quad \int_{\Omega(\gamma)} |\nabla u_m(x)|^2 dx = n|\Omega(\gamma)|,
\end{equation}

\begin{equation}
(4.43) \quad \int_{\Omega(\gamma)} |\nabla^2 u_m(x)|^2 dx
= \int_{\Omega(\gamma) \cap \Omega(\gamma_m)} |\nabla^2 u_m(x)|^2 dx + \int_{\Omega(\gamma) \setminus \Omega(\gamma_m)} |\nabla^2 u_m(x)|^2 dx
= \int_0^{\ell} \int_{\Delta^{\gamma_m}(t)} \left(1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\right) d\mathcal{H}^{n-1}(s) dt + 0.
\end{equation}

It is easy to see that $u_m \to u$ in $W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ because $(\gamma'_m, N_1, \cdots, N_{n-1})$ uniformly, $(\gamma'_m, v_1, \cdots, v_{n-1}, N_m)$ converges to $(\gamma'_m, v_1, \cdots, v_{n-1}, N)$ uniformly, $\kappa_{n,m} \to \kappa_n$, $\kappa_{i,m} \to \kappa_i$, $1 \leq i \leq n - 1$ in $L^p([0, \ell])$ for all $1 \leq p < \infty$, $1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \geq \min\{p/16d, 1/2\}$, $\Delta^{\gamma_m}(t) \to \Sigma^{\gamma}(t)$ for all $t \in [0, \ell]$ and $|\Omega(\gamma) \setminus \Omega(\gamma_m)| \to 0$. The proof is complete. \qed
Combining Lemmas 4.10 and 4.12, we get a smooth approximation sequence for any isometry \( u \) in \( \Omega(\gamma) \).

4.6. **Approximation for** \( u \) **in** \( \Omega \). The proof is exactly the same as the proof in section 3.3 in [22]. Since it is the final part of the argument, we briefly review it for the convenience of the reader.

Recall that we defined a maximal region on which \( u \) is affine a **body** if its boundary contains more than two different \( (n-1) \)-planes in \( \Omega \) (recall Definition 3.9 for the definition of \( (n-1) \)-planes in \( \Omega \)) and we have shown that we can assume \( \Omega \) has only a finite number of bodies and is partitioned into bodies and covered domains. We call the maximal subdomain covered by some leading curve \( \gamma \) an **arm**. Similarly to Lemma 4.1, we also have,

**Lemma 4.14.** It is sufficient to prove Theorem 1.5 for a function in \( I^{2,2}(\Omega, \mathbb{R}^{n+1}) \) with finite number of arms.

**Proof.** The proof is the same as the two dimensional case in [22, Lemma 3.9] and is omitted for brevity.

Now since \( \Omega \) is convex and simply-connected, we claim that two bodies are connected through one chain of bodies and arms: it suffices to consider the graph obtained by retracting bodies to vertices and arms to edges. This graph is simply connected because it is a deformation retract of \( \Omega \). Therefore every two vertices are connected through only one chain of edges, which proves the claim (Figure 15).

![Graph of retraction of \( \Omega \).](image)

We begin by a central body \( B_1 \) and define our approximating sequence on each arm as in subsection 4.5. Note that for this final purpose, we have constructed our approximating smooth isometric immersion to be affine near both ends, this allows us to apply an affine transformation to the target space of each arm so that the affine regions near its ends join together smoothly all the way till we reach \( B_2 \). Meanwhile, we also apply an affine transformation to \( u(B_2) \) so that it joins the last arm smoothly. It is easy to see from the uniform convergence of each term in representation (4.38) that such affine transformation converges to identity as \( m \to 0 \). Now we continue our construction using \( B_2 \) as a new starting point. Note that we will never come back to \( B_1 \) because they are connected through only one chain of arms. The construction of the approximating sequence on the entire domain \( \Omega \) is complete. \( \square \)
APPENDIX A. PROOF OF LEMMA 4.13

Step 1. Recall from the matrix of moving frame defined in Subsection 4.2 that \( \gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t)N_i(t) \), with \( \kappa_i \) bounded. We can choose uniformly bounded smooth functions \( \tilde{\kappa}_{i,m} \to \kappa_i \) a.e. on \([0, \ell]\), and hence in measure due to the fact that \([0, \ell]\) is bounded. Since the sequence \( \tilde{\kappa}_{i,m} \) are uniformly bounded, it follows \( \tilde{\kappa}_{i,m} \to \kappa_i \) in \( L^p \) for all \( 1 \leq p < \infty \). Similarly we can find uniformly bounded smooth functions \( \kappa_{ij,m} \to \kappa_{ij} \) a.e. on \([0, \ell]\) (hence in \( L^p \) for all \( 1 \leq p < \infty \)) for \( \kappa_{ij}, 1 \leq i, j \leq n-1 \). By solving the following system of ODEs:

\[
\begin{pmatrix}
\Gamma' \\
N_{1,m} \\
N_{2,m} \\
\vdots \\
N_{n-1,m}
\end{pmatrix}' = \kappa_m
\begin{pmatrix}
0 & \tilde{\kappa}_{1,m} & \tilde{\kappa}_{2,m} & \cdots & \tilde{\kappa}_{n-1,m} \\
-\tilde{\kappa}_{1,m} & 0 & \kappa_{1,2,m} & \cdots & \kappa_{1,n-1,m} \\
-\tilde{\kappa}_{2,m} & -\kappa_{1,2,m} & 0 & \cdots & \kappa_{2,n-1,m} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
-\tilde{\kappa}_{n-1,m} & -\kappa_{1,n-1,m} & -\kappa_{2,n-1,m} & \cdots & 0
\end{pmatrix},
\]

where the matrix \( \kappa_m \) is given by

\[
\kappa_m = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2}
\end{pmatrix}
\]

we obtain a unique orthogonal frame \((\Gamma'_m(t), N_{1,m}(t), \cdots, N_{n-1,m}(t))\) with initial condition \( \Gamma'_m(0) = \gamma'(0) \), and \( N_{i,m}(0) = N_i(0) \). We can then define

\[
\Gamma_m(t) = \Gamma(0) + \int_0^t \Gamma'_m(\tau)d\tau.
\]

We want to show that \((\Gamma'_m, N_{1,m}, \cdots, N_{n-1,m}) \to (\gamma', N_1, \cdots, N_{n-1})\) uniformly. This result is given by the following theorem due to Opial, [21], Theorem 1.

Lemma A.1 (Opial). Suppose the linear system of differential equations,

\[
(A.1) \quad x'(t) = A_k(t)x(t), \quad x(0) = a_k, \quad k = 0, 1, 2, \cdots
\]

admits a solution \( x_k(t) \) in \([0, \ell]\) for all \( k \). Suppose \( a_k \to a_0 \),

\[
\int_0^t A_k(s)ds \to \int_0^t A_0(s)ds
\]

uniformly for all \( t \in [0, \ell] \) and \( A_k \) is a bounded sequence in \( L^1 \), i.e.

\[
\sup_k \|A_k\|_{L^1([0, \ell])} < \infty,
\]

then the solutions \( x_k(t) \) converge to \( x_0(t) \) uniformly.

Since \( \tilde{\kappa}_{i,m} \to \kappa_i \) and \( \kappa_{ij,m} \to \kappa_{ij} \) in \( L^p \) for all \( 1 \leq p < \infty \), in particular for \( p = 1 \), the conditions in Lemma A.1 are satisfied, hence \((\Gamma'_m, N_{1,m}, \cdots, N_{n-1,m})\) converges to \((\gamma', N_1, \cdots, N_{n-1})\) uniformly. Since \( \Gamma''_m = \sum_{i=1}^{n-1} \tilde{\kappa}_{i,m}N_{i,m} \), \( \Gamma''_m \) are uniformly bounded, and \( \Gamma''_m \to \gamma'' \) a.e. (and hence in \( L^p \) for all \( 1 \leq p < \infty \),
Poincaré inequality for intervals implies that \( \Gamma_m \to \gamma \) in \( W^{2,p}(\Omega) \) for all \( 1 \leq p < \infty \).

However \( \Gamma_m \) is not our desired curve since we cannot guarantee that all its leading fronts intersect outside \( \Omega \). This happens if \( \Gamma_m \) is too “curvy”. We need to “flatten” its curvature continuously. This needs to be done in several steps:

**Step 2.** We construct \( \tilde{k}_m = (\tilde{k}_{1,m}, \ldots, \tilde{k}_{n-1,m}) \) continuous on \( t \in [0, \ell] \) and for each \( t \in [0, \ell] \) and \( s = (s_1, \ldots, s_{n-1}) \in \mathbb{S}^{n-2} \),

\[
(S^m_s(t) + \frac{\rho}{2})(\sum_{i=1}^{n-1} s_i \tilde{k}_{i,m}(t)) \leq 1
\]

where

\[
S^m_s(t) = \sup \{ S \geq 0 : \Gamma_m(t) + S \sum_{i=1}^{n-1} s_i N_i(t) \in \Omega \}.
\]

We first need the following lemma using implicit function theorem for \( C^1 \) functions.

**Lemma A.2.** \( S^m_s(t) \) is uniformly continuous on \( (s,t) \in \mathbb{S}^{n-2} \times [0, \ell] \) and \( S^m_s(t) \) converges to \( S_s(t) \) uniformly on \( (s,t) \in \mathbb{S}^{n-2} \times [0, \ell] \).

**Proof.** Let \( t_0 \in [0, \ell] \) and \( s^0 = (s_1^0, \ldots, s_{n-1}^0) \in \mathbb{S}^{n-1} \) be arbitrary. We parametrize locally \( \mathbb{S}^{n-2} \) by the polar coordinates: \( s_i = s_i(\theta) \) where \( \theta = (\theta_1, \ldots, \theta_{n-2}) \in U_1 \subset [0, \pi)^{n-3} \times [0, 2\pi) \). Let \( \theta^0 \in U_1 \) be such that \( s_i^0 = s_i(\theta^0) \).

Let \( \gamma^0 = \gamma(t_0) \) and \( N^0_i = N_i(t_0) \). Let \( x_0 \) be the intersection of the line segment \( L = \{ \gamma^0 + S(\sum_{i=1}^{n-1} s_i^0 N_i^0), 0 \leq S \} \) and \( \partial \Omega \). Then \( x_0 = \gamma^0 + S_0(\sum_{i=1}^{n-1} s_i^0 N_i^0) \) for some \( S_0 > 0 \).

Since \( \Omega \) is a \( C^1 \) domain, there exits an open subset of \( U_2 \subset \mathbb{R}^{n-1} \) and a \( C^1 \) function \( \alpha : U_2 \to \partial \Omega \) and \( \alpha(\eta_1^0, \ldots, \eta_{n-1}^0) = x_0 \) for some \( (\eta_1^0, \ldots, \eta_{n-1}^0) \in U_2 \).

Consider \( F : \mathbb{R}^n \times \mathbb{R}^{n \times (n-1)} \times U_1 \times \mathbb{R} \times U_2 \to \mathbb{R}^n \)

\[
F(\gamma, \eta_1, \ldots, \eta_{n-1}, \theta, S, \eta_1, \ldots, \eta_{n-1}) = \gamma + S(\sum_{i=1}^{n-1} s_i(\theta) N_i) - \alpha(\eta_1, \ldots, \eta_{n-1}).
\]

Since \( x_0 \in \partial \Omega \cap L \),

\[
F(\gamma^0, \eta_1^0, \ldots, \eta_{n-1}^0, \theta^0, S^0, \eta_1^0, \ldots, \eta_{n-1}^0) = 0.
\]

Let

\[
x = (\gamma, \eta_1, \ldots, \eta_{n-1}, \theta), \quad y = (S, \eta_1, \ldots, \eta_{n-1})
\]

and

\[
\alpha_k := \frac{\partial \alpha}{\partial \eta_k} \quad 1 \leq k \leq n-1.
\]

Then

\[
\det \left[ \frac{\partial F}{\partial y} (\gamma^0, \eta_1^0, \ldots, \eta_{n-1}^0, \theta^0, S^0, \eta_1^0, \ldots, \eta_{n-1}^0) \right] = \det \left[ \sum_{i=1}^{n-1} s_i(\theta) N_i, \alpha_1(\eta_1^0, \ldots, \eta_{n-1}^0), \ldots, \alpha_{n-1}(\eta_1^0, \ldots, \eta_{n-1}^0) \right] \neq 0.
\]

Otherwise, the line segment \( L \) would be parallel to the tangent plane of \( \partial \Omega \) at \( x_0 \), which is not possible since \( \Omega \) is convex.
By implicit function theorem, there is an open neighborhood
\[ V_1 \subset \mathbb{R}^n \times \mathbb{R}^{n \times (n-1)} \times U_1 \]
of \( x_0 = (\gamma^0, N^0_1, \cdots, N^0_{n-1}, \theta^0) \), \( V_2 \subset \mathbb{R} \times U_2 \) of \( y_0 = (S^0, \eta^0_1, \cdots, \eta^0_{n-1}) \), and a \( C^1 \)
map \( y : V_1 \to V_2 \) such that
\[ F(x, y(x)) = F(x, S(x), \eta_1(x), \cdots, \eta_{n-1}(x)) = 0. \]
for all \( x \in V_1 \).

Since \( \gamma, N_i, 1 \leq i \leq n-1 \) are Lipschitz on \([0, \ell]\) and \( \Gamma_m \to \gamma \) uniformly and
\( N_{i,m} \to N_i \) uniformly on \([0, \ell]\) for all \( 1 \leq i \leq n-1 \), there exists an open interval
\( O \subset \mathbb{R} \) containing \( t_0 \), an open subset \( \Delta \subset U_1 \) containing \( \theta_0 \) and an integer \( M \) such
that for all \( t \in [0, \ell] \cap O, \theta \in \Delta \) and \( m \geq M \),
\[ x(t, \theta) = (\gamma(t), N_1(t), \cdots, N_{n-1}(t), \theta) \in V_1 \quad \text{and,} \]
\[ x_m(t, \theta) = (\Gamma_m(t), N_{1,m}(t), \cdots, N_{n-1,m}(t), \theta) \in V_1. \]

Apparently \( x_m(t, \theta) \to x(t, \theta) \) uniformly for all \( t \in [0, \ell] \cap O \) and \( \theta \in \Delta \). Since \( S \)
is \( C^1 \) on \( x \in V_1 \),
\[ S(x_m(t, \theta)) \to S(x(t, \theta)) \quad \text{uniformly on } t \in [0, \ell] \cap O \quad \text{and} \quad \theta \in \Delta. \]

Moreover, since \( S \) is \( C^1 \) on \( x \in V_1 \) and \( x \) is uniformly continuous on \( t \in [0, \ell] \cap O \)
and \( s \in s(\Delta) \), \( S \) is uniformly continuous on \( t \in [0, \ell] \cap O \) and \( s \in s(\Delta) \).

Now note that since \( F(x(t, \theta), y(x(t, \theta))) = 0 \) and \( F(x_m(t, \theta), y(x_m(t, \theta))) = 0 \),
for each \( s = s(\theta) \in s(\Delta) \subset \mathbb{S}^{n-2} \), we have \( S^*_{s}(t) = S(x(t, \theta)) \) and \( S^*_{s,m}(t) = S(x_m(t, \theta)) \). Thus by \([A.3]\),
\[ S^*_{s,m}(t) \to S^*_{s}(t) \quad \text{uniformly on } t \in [0, \ell] \cap O \quad \text{and} \quad s \in s(\Delta), \]
and \( S^*_{s,m}(t) \) is uniformly continuous on \( t \in [0, \ell] \cap O \) and \( s \in s(\Delta) \).

It remains to observe that since \([0, \ell]\) and \( \mathbb{S}^{n-2} \) are both compact they can be covered by a finite union of neighborhoods on which \([A.3]\) holds. The proof is complete. \( \Box \)

Define
\[ \lambda_m(t) := \min \left\{ 1, 1 / (\sup |s|=1 \{ (S^*_{s,m}(t) + \rho/2) / \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \}) \} \]
where \( \tilde{\kappa}_{i,m}, 1 \leq i \leq n-1 \) are those found in Step 1. A first observation is that
\( 0 < \lambda_m \leq 1 \). Indeed, there must exist \( s \in \mathbb{S}^{n-2} \) such that \( \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0 \) so the
supreme over all \( s \in \mathbb{S}^{n-2} \) must be nonnegative. On the other hand, \( S^*_{s,m} \) as well
as all \( \tilde{\kappa}_{i,m} \) are bounded so \( \lambda_m \) is bounded below by a positive number.

Second, we observe that \( \lambda_m \) is continuous. Indeed, by Lemma \([A.2]\)
\[ (S^*_{s,m}(t) + \rho/2) / \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \]
is uniformly continuous on \((s,t) \in \mathbb{S}^{n-2} \times [0, \ell] \). Hence the supreme over \( \mathbb{S}^{n-2} \) is
attained and a simple argument gives
\[ h(t) := \sup |s|=1 \{ (S^*_{s,m}(t) + \rho/2) / \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \} \]
is continuous.

We then define a vector valued function \( \tilde{\kappa}_m = (\tilde{\kappa}_{1,m}, \ldots, \tilde{\kappa}_{n-1,m}) \) as
\[
(\tilde{\kappa}_{1,m}(t), \ldots, \tilde{\kappa}_{n-1,m}(t)) := \lambda_m(t) (\kappa_{1,m}(t), \ldots, \kappa_{n-1,m}(t))
\]
\( \tilde{\kappa}_m \) is obviously continuous. It remains to show \( \tilde{\kappa}_m \) satisfies (A.2). Indeed, for any \( s = (s_1, \ldots, s_{n-1}) \in \mathbb{S}^{n-2} \),
\[
(S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) = \lambda_m(S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)).
\]
If \( \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0 \), then by the definition of \( \lambda_m \),
\[
\lambda_m(t)(S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) \leq \min \{(S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)), 1\} \leq 1.
\]
If \( \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) < 0 \), then
\[
\lambda_m(t)(S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) < 0 \leq 1.
\]
Thus (A.2) is satisfied.

**Step 3.** We want to show that \( (\tilde{\kappa}_{1,m}, \ldots, \tilde{\kappa}_{n-1,m}) \rightarrow (\kappa_1, \ldots, \kappa_{n-1}) \) a.e. Indeed, we know that \( \tilde{\kappa}_m = (\tilde{\kappa}_{1,m}, \ldots, \tilde{\kappa}_{n-1,m}) \rightarrow (\kappa_1, \ldots, \kappa_{n-1}) \) a.e.. Therefore, all we need to show is \( \lambda_m \rightarrow 1 \) a.e..

By possibly replacing \( \lambda_m \) by a subsequence, it suffices to prove \( \lambda_m \rightarrow 1 \) in measure. From the definition of \( \lambda_m \), it is enough to show the Lebesgue measure of the set:
\[
E_m = \{ t \in [0, l] \mid \exists s \in \mathbb{S}^{n-2}, (S_s \Gamma^m(t) + \frac{\rho}{2} \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)) > 1 \}
\]
goes to zero. First by assumption, \( L^\gamma(t) - S_s^\gamma(t) > \rho > 0 \) and by Lemma 4.7
\[
L^\gamma(t)(\sum_{i=1}^{n-1} \kappa_i(t)s_i) \leq 1,
\]
thus,
\[
(S_s^\gamma(t) + \rho)(\sum_{i=1}^{n-1} s_i \kappa_i(t)) \leq 1,
\]
for all \( t \in [0, l] \) and \( s \in \mathbb{S}^{n-2} \). Indeed, if \( \sum_{i=1}^{n-1} s_i \kappa_i(t) \geq 0 \),
\[
(S_s^\gamma(t) + \rho)(\sum_{i=1}^{n-1} s_i \kappa_i(t)) \leq L^\gamma(t)(\sum_{i=1}^{n-1} \kappa_i(t)s_i) \leq 1
\]
which gives (A.4).
If \( t \in E_m \), there is \( s \in S^{n-2} \) such that
\[
\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) > \frac{1}{S^\Gamma_m(t) + \rho/2}.
\]
Therefore all \( t \in E_m \) and our choice of \( s = s(t) \) as above, we have,
\[
|\tilde{\kappa}_m(t) - \kappa(t)| \geq \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) - (\sum_{i=1}^{n-1} s_i \kappa_i(t))
\]
\[
> \frac{\rho/2 + S^\gamma_s(t) - S^\Gamma_m(t)}{(S^\Gamma_m(t) + \rho/2)(S^\gamma_s(t) + \rho)} \geq \frac{\rho/2 - |S^\gamma_s(t) - S^\Gamma_m(t)|}{\rho^2/2}.
\]
By Lemma A2 we have
\[
S^\Gamma_m(t) \rightarrow S^\gamma_s(t) \text{ uniformly on } s \in S^{n-2} \text{ and } t \in [0, \ell],
\]
then we can find \( m \) sufficiently large so that \( |S^\gamma_s(t) - S^\Gamma_m(t)| < \rho/4 \) for all \( s \in S^{n-2} \) and \( t \in [0, \ell] \). Since \( \tilde{\kappa}_m \rightarrow \kappa \text{ a.e.} \),
\[
\lim_{m \to \infty} |E_m| \leq \lim_{m \to \infty} |\{t : |\tilde{\kappa}_m(t) - \kappa(t)| \geq \frac{1}{2\rho}\}| = 0
\]
which is what we wanted to show.

**Step 4.** Since \( \tilde{\kappa}_m = (\tilde{\kappa}_{1,m}^1, \cdots, \tilde{\kappa}_{n-1,m}^1) \) are continuous, for each \( m \) we can find \( \kappa_m \) smooth and \( |\tilde{\kappa}_m - \kappa_m| \to 0 \) uniformly on \( t \in [0, \ell] \). Hence for \( m \) sufficiently large,
\[
(S^\Gamma_p(t) + \rho^2/4) \Big(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\Big) \leq 1
\]
\[(A.5)\]

**Step 5.** We now define our desired curve \( \gamma_m \). Given \( \kappa_m = (\kappa_{1,m}, \cdots, \kappa_{n-1,m}) \) smooth as found in Step 4, and \( \kappa_{ij,m} \to \kappa_{ij} \) found in step 1, we again solve the system of ODEs
\[
\begin{pmatrix}
\gamma'_m \\
N_{1,m} \\
N_{2,m} \\
\vdots \\
N_{n-1,m}
\end{pmatrix} = \mathcal{K}^{n \times n}_m
\begin{pmatrix}
\gamma'_m \\
N_{1,m} \\
N_{2,m} \\
\vdots \\
N_{n-1,m}
\end{pmatrix},
\]
where
\[
\mathcal{K}^{n \times n}_m = \begin{pmatrix}
0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} \\
-\kappa_{1,m} & 0 & \kappa_{2,m} & \cdots & \kappa_{n-1,m} \\
-\kappa_{2,m} & -\kappa_{1,m} & 0 & \cdots & \kappa_{n-2,m} \\
& \vdots & \vdots & \ddots & \vdots \\
-\kappa_{n-1,m} & -\kappa_{n-2,m} & \cdots & -\kappa_{1,m} & 0
\end{pmatrix},
\]
and denote by the orthogonal frame \( (\gamma'_m(t), N_{1,m}(t), \cdots N_{n-1,m}(t)) \) the unique solution with initial conditions \( \gamma'_m(0) = \gamma'(0) \) and \( N_{1,m}(0) = N_1(0) \). Moreover, by Lemma A1 \( (\gamma'_m(t), N_{1,m}(t), \cdots N_{n-1,m}(t)) \to (\gamma'(t), N_1(t), \cdots N_{n-1}(t)) \) uniformly. Let
\[
\gamma_m(t) = \gamma(0) + \int_0^t \gamma'_m(\tau) d\tau.
\]
We claim \( \gamma_m \) satisfies for \( m \) sufficiently large,

\[
(S_\gamma^m(t) + \frac{\rho}{8} \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)) \leq 1
\]

Indeed, by the same argument of Lemma \([A,2]\) using implicit function theorem, \( S_\gamma^m \) also converges to \( S_\gamma \) uniformly. Together with Lemma \([A,2]\) we obtain that \( |S_\gamma^m - S_\gamma^m| \) converges to 0 uniformly. Thus the claim follows from \([A,5]\).

**Step 6.** Finally, we claim that orthogonal fronts satisfy \( F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \Omega = \emptyset \) for all \( t, \tilde{t} \in [0, \ell] \).

For \( \gamma_m \) and its moving frame \((\gamma'_m, N_{1,m}, \ldots, N_{n-1,m})\) found in Step 5, let \( \Phi_m : [0, \ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n \) be defined as

\[
\Phi_m(t, s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i N_{i,m}(t).
\]

Let \( \Sigma_{\gamma_m} = \{(t, s) : \Phi_m(t, s) \in \Omega\} \). By the same argument as Lemma \([4.6]\), \( \Phi_m \) maps \( \Sigma_{\gamma_m} \) onto \( \Omega(\gamma_m) \) where \( \Omega(\gamma_m) \) is the subset of \( \Omega \) covered by all orthogonal fronts \( F_{\gamma_m}(t), t \in [0, \ell] \). By the same computation as in \([1.22]\),

\[
J_{\Phi_m}(t, s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t).
\]

Let \( d := \text{diam}(\Omega) \), we claim that

\[
1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \geq \min\{\rho/16d, 1/2\},
\]

for all \( (t, s) \in \Sigma_{\gamma_m} \). Indeed, if \( \sum_{i=1}^{n-1} (s_i/|s|) \kappa_{i,m}(t) \geq 1/2d \), then by \([A,6]\),

\[
1 - |s| \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) \geq 1 - S_\gamma^m(t) \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) \geq \frac{\rho}{8} \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) \geq \frac{\rho}{8} \frac{1}{2d}.
\]

If \( \sum_{i=1}^{n-1} (s_i/|s|) \kappa_{i,m}(t) < 1/2d \), then

\[
1 - |s| \left( \sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) > 1 - \frac{|s|}{2d} \geq \frac{1}{2}.
\]

Hence, the claim follows. By Inverse function theorem due to Clarke \([5]\), \( \Phi_m \) admits a local Lipschitz inverse, actually a **global** Lipschitz inverse \( \Phi_m^{-1} : \Omega(\gamma_m) \to \Sigma_{\gamma_m} \) since the Jacobian is everywhere bounded below by a positive constant in \( \Sigma_{\gamma_m} \). In particular, \( \Phi_m \) is one-to-one on \( \Sigma_{\gamma_m} \). This implies all orthogonal fronts \( F_{\gamma_m}(t), t \in [0, \ell] \) meet outside \( \Omega \). The proof of Lemma \([4.13]\) is complete. \( \square \)
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