On the Edge-Length Ratio of Planar Graphs

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Introduction

The *edge-length ratio* of a drawing is a natural metric to guarantee the readability of a graph drawing.
The edge-length ratio $\rho(\Gamma)$ of a straight-line drawing $\Gamma$ of a graph $G = (V, E)$ is the ratio between the lengths of the longest and of the shortest edge in the drawing.

$$\rho(\Gamma) = \max_{e_1, e_2 \in E(G)} \frac{\ell_\Gamma(e_1)}{\ell_\Gamma(e_2)},$$

where $\ell_\Gamma(e)$ denotes the length of the segment representing an edge $e$ in $\Gamma$. 
Planar edge-length ratio

Definition

The *planar edge-length ratio* $\rho(G)$ of a graph $G$ is the minimum edge-length ratio of any planar straight-line drawing $\Gamma$ of $G$.

$$\rho(G) = \min(\rho(\Gamma))$$
Examples of graphs admitting a good edge-length ratio

Example 1: The nested-triangle graph has planar edge-length ratio less than $1 + \varepsilon$. 
Examples of graphs admitting a good edge-length ratio

Example 2: The plane 3-tree obtained as the join of a path with an edge has planar edge-length ratio less than 3.
Deciding whether a graph has planar edge-length ratio equal to 1 is an **NP-hard** problem.

- Eades et al.\(^1\) for biconnected planar graphs;
- Cabello et al.\(^2\) for triconnected planar graphs.

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\(^1\) “Fixed edge-length graph drawing is NP-hard”, Discrete Applied Mathematics 28(2), (1990)

\(^2\) “Planar embeddings of graphs with specified edge lengths”, J. Graph Algorithms Appl. 11(1), (2007)
The study of combinatorial bounds for the planar edge-length ratio of planar graphs started with Lazard et al.\textsuperscript{3}.

1. Outerplanar graphs have planar edge-length ratio smaller than 2.
2. There exist outerplanar graphs whose planar edge-length ratio is larger then $2 - \epsilon$.

\textsuperscript{3}“On the edge-length ratio of outerplanar graphs”, Theor. Comput. Sci. 770, (2019)
The questions we look at

1. What is the edge-length ratio for planar graphs?
2. What is the edge-length ratio for notable classes of graphs like series-parallel or bipartite graphs?
Our results

1. **Theorem 1**: planar graphs have planar edge-length ratio in $\Theta(n)$

2. **Theorem 2**: planar 3-trees with depth $k$ have planar edge-length ratio in $O(k)$

3. **Theorem 3**: 2-trees have planar edge-length ratio in $O(n^{0.695})$

4. **Theorem 4**: for any fixed $\epsilon > 0$, bipartite planar graphs have planar edge-length ratio smaller than $1 + \epsilon$
Theorem 1: edge-length ratio of planar graphs (1)

For arbitrarily large values of $n$, there exists an $n$-vertex planar graph whose planar edge-length ratio is in $\Omega(n)$.

Proof:
- Consider any planar straight-line drawing $\Gamma$ of $G$
- Assume that the length of the shortest edge of $G$ in $\Gamma$ is 1
- Let $T_k = a_k b_k c_k$ and $T_{k-1} = a_{k-1} b_{k-1} c_{k-1}$. We prove that: $P(T_k) \geq P(T_{k-1}) + c$, for a constant $c$

This implies that the edge-length ratio of $\Gamma$ is $\Omega(n)$. 

\[ a_k \]
\[ G_k \]
\[ a_{k-1} \]
\[ c_{k-1} \]
\[ b_{k-1} \]
\[ b_k \]
Theorem 1: edge-length ratio of planar graphs (2)

**Lemma**

Let $T$ and $T'$ be triangles such that $T'$ is contained into $T$, then $P(T) > P(T')$

![Diagram of triangles $T$ and $T'$]

**Lemma**

If $||ad|| \geq 1$ and $\hat{b}ac \leq 90^\circ$, then $P(T) > P(T') + 1$

![Diagram of triangles $T$ and $T'$ with angles and side lengths]
Theorem 1: edge-length ratio of planar graphs (3)

If $b_{k-1} a_{k-1} c_{k-1} \leq 90^\circ$, then $P(T_k) > P(T_{k-1}) + 1$
Theorem 1: edge-length ratio of planar graphs (4)

If $b_{k-1}a_{k-1}c_{k-1} > 90^\circ$ and $c_{k-1}b_{k-1}a_k \leq 90^\circ$, then $P(T_k) > P(T_{k-1}) + 1$
Theorem 1: edge-length ratio of planar graphs (5)

Let $p_i$ be the intersection point between the straight line $a_{k-1}b_{k-1}$ with $c_{k-1}a_k$. 

\[ q_i \]
Let $q_i$ be the intersection point between the straight line $a_{k-1}c_{k-1}$ with $b_{k-1}a_k$.

We distinguish two cases:

1. $|a_kq_i| \geq 0.4$
2. $|a_kq_i| \leq 0.4$
If $|a_k q_i| \geq 0.4$, then $P(b_{k-1} c_{k-1} q_i) > P(T_{k-1})$ and since $c_{k-1} \hat{q}_i a_k > 90^\circ$ we have $|c_{k-1} a_k| > |c_{k-1} q_i|$, and hence $P(T_k) > P(T_{k-1}) + 0.4$.
If $|a_k q_i| \leq 0.4$, then $|a_k p_i| \geq 0.4$, and hence $P(T_k) - P(T_{k-1})$ will assume its minimum value when $|b_{k-1} a_k| = 1$ and $|a_k p_i| = 0.4$, then $P(T_k) > P(T_{k-1}) + 0.32$.
Theorem 2: edge-length ratio of plane 3-trees

Every plane 3-tree with depth $k$ has planar edge-length ratio in $O(k)$. 

A plane 3-tree $G$ is naturally associated with a rooted ternary tree $T_G$, whose internal nodes represent the internal vertices of $G$ and whose leaves represent the internal faces of $G$.

The proof is by induction. Let $\text{depth}(G) := \text{depth}(T_G) = k$, then the planar edge-length ratio of $G$ is in $O(k)$. 

![Diagram of a plane 3-tree and its associated rooted ternary tree](image-url)
Theorem 3: edge-length ratio of 2-trees (1)

Every $n$-vertex 2-tree has planar edge-length ratio in $O(n \log_2 \phi) \subseteq O(n^{0.695})$, where $\phi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.

Lazard et al.\textsuperscript{4} asked whether the planar edge-length ratio of 2-trees is bounded by a constant; recently, at the 14\textsuperscript{th} Bertinoro Workshop on Graph Drawing, Fiala announced a negative answer to the above question.

\textsuperscript{4} “On the edge-length ratio of outerplanar graphs”, Theor. Comput. Sci., (2019)
Theorem 3: edge-length ratio of 2-trees (2)

Definition
An *apex vertex* of the edge \((u, v)\) is a vertex that is connected to \(u\) and \(v\).

Definition
The *side edges* of \((u, v)\) are all the edges with a vertex \(u\) or \(v\) and apex vertex of \((u, v)\).

Definition
An edge \((u, v)\) is *trivial* if it has no apex, otherwise it is *non-trivial*. 
Theorem 3: L2T-drawer algorithm (3)

Definition

A *linear 2-tree* is a 2-tree such that every edge has at most one non-trivial side edge.

Our $L2T$-drawer algorithm constructs a planar straight-line drawing $\Gamma$ of a linear 2-tree $H$. 
Theorem 3: edge-length ratio of 2-trees (4)

Proof:

1. Find a subgraph $H$ of $G$ that is a linear 2-tree, and such that every $H$-component of $G$ has ”few” internal vertices.
2. Construct a planar straight-line drawing $\Gamma$ of $H$ by the algorithm $L2T$-drawer.
3. Recursively draw each $H$-component independently, plugging such drawings into $\Gamma$, thus obtaining a drawing of $G$. 

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Theorem 4: edge-length ratio of bipartite planar graphs

Theorem

For every $\epsilon > 0$, every $n$-vertex bipartite planar graph has planar edge-length ratio smaller than $1 + \epsilon$.

Proof:
The proof is based on the work of Brinkman et al.\textsuperscript{[5]} and is by induction on $n$. The figure shows the expansion and contraction operations we use in order to perform induction.

\textsuperscript{5} “Generation of simple quadrangulations of the sphere”, Discrete Mathematics 305(1 – 3), (2005)
Open problems

- What is the asymptotic behavior of the planar edge-length ratio of 2-trees?
- Is the planar edge-length ratio of cubic planar graphs sub-linear?
- Is the planar edge-length ratio of $k$-outerplanar graphs bounded by some function of $k$?
Thank you for your attention!