HIGHER THEORIES OF ALGEBRAIC STRUCTURES

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Abstract. The notion of (symmetric) coloured operad or “multicategory” can be obtained from the notion of commutative algebra through a certain general process which we call “theorization” (where our term comes from an analogy with William Lawvere’s notion of algebraic theory). By exploiting the inductivity in the structure of higher associativity, we obtain the notion of “n-theory” for every integer n ≥ 0, which inductively theorizes n times, the notion of commutative algebra. As a result, (coloured) morphism between n-theories is a “graded” and “enriched” generalization of (n − 1)-theory. The inductive hierarchy of those higher theories extends in particular, the hierarchy of higher categories. Indeed, theorization turns out to produce more general kinds of structure than the process of categorification in the sense of Louis Crane does. In a part of low “theoretic” order of this hierarchy, graded and enriched 1- and 0-theories vastly generalize symmetric, braided, and many other kinds of enriched multicategories and their algebras in various places.

We make various constructions of/with higher theories, and obtain some fundamental notions and facts. We also find iterated theorizations of more general kinds of algebraic structure including (coloured) properad of Bruno Vallette and various kinds of topological field theory (TFT). We show that a “TFT” in the extended context can reflect a datum of a very different type from a TFT in the conventional sense, despite close formal similarity of the notions.

This work is intended to illustrate use of simple understanding of higher coherence for associativity.

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0. Introduction

0.0. Higher theories.

0.0.0. Among variations of the notion of operad, the symmetric, the planar and the braided (see Fiedorowicz [9]) versions are particularly simple to describe, and are very commonly worked with. Over an operad of one of these types, an algebra can be considered respectively in a symmetric, associative and braided (see Joyal and Street [12]) monoidal category.

In general, each operad governs algebras over it, and this role is important. In other words, the notion of operad is important since it gives a way to do universal algebra in (e.g., symmetric) monoidal categories. For this role, we consider an operad as analogous to Lawvere’s algebraic theory [14].

By visiting the conceptual origin of the notion of operad, one finds that the notion of symmetric operad arises naturally through a certain process, which we call “theorization” (inspired by Lawvere’s notion), from the notion of commutative algebra. Moreover, the same process can be started from the notion of $\mathcal{U}$-algebra for any operad $\mathcal{U}$ in sets or groupoids, to produce a new kind of algebraic structure which we call “$\mathcal{U}$-graded” operad (Section 0.2.0). It turns out that planar and braided operads are $\mathcal{U}$-graded operads where $\mathcal{U} = E_1, E_2$ respectively.
The notion of $\mathcal{U}$-graded operad in groupoids has a deceptively simple description, namely, a $\mathcal{U}$-graded operad in groupoids can be described as an operad $\mathcal{X}$ in groupoids equipped with a morphism $P: \mathcal{X} \to \mathcal{U}$. (This $\mathcal{X}$ equipped with $P$ corresponds to a $\mathcal{U}$-graded operad in sets if the maps induced by $P$ on the groupoids of operations have everywhere, homotopy fibre with a discrete homotopy type. The details will be discussed in Remark 0.13.) This might hide the notion of theorization from a non-obsessed mind. However, we have come to think that theorization is an important notion.

One purpose of this long introductory section is to introduce the notion of theorization, which will be a generalization with its own mathematical content, of the notion of categorification in the sense of Crane [7, 6] (about which the most influential pioneer may have been Grothendieck), but will be at least as informal a notion as categorification. Even though a very precise understanding of the notion of theorization is not technically necessary for the body of the article, at least a rough understanding will be essential for understanding the ideas of our work. In the body, we shall use the language of theorization to navigate the reader through ideas.

Let us, however, start with a sketch of what we actually do in this work. After that, the main purpose of this introduction will be to introduce the idea of theorization, and describe more topics to be covered in the body.

0.0.1. In this work, we introduce and study ‘higher order’ generalizations of (coloured) operads, which we call “higher theories” of algebras. Higher theories will be obtained by iterating the process of theorization starting from the notion of coloured operad. Introduction of these objects leads to (among other things) a framework for a natural explanation and a vast generalization of the fact which we formulate below as Proposition 0.0. Let us describe it now.

The proposition will be about places where the notion of $\mathcal{U}$-graded operad can be enriched, but the reader may assume that $\mathcal{U} = E_1$ or $E_2$ (see above). As another notice, colours in operads will not play an essential role for a while, so we shall consider just uncoloured operads everywhere till we start taking colours explicitly into consideration.

For a symmetric monoidal category $\mathcal{A}$, let us denote by $\mathbf{Op}_A(\mathcal{A})$ the category of $\mathcal{U}$-graded operads in $\mathcal{A}$. $\mathbf{Op}_A$ is a category-valued functor on the 2-category $\mathbf{Alg}_{\mathbf{Com}}(\mathbf{Cat})$ of symmetric monoidal categories, where $\mathbf{Cat}$ denotes the 2-category of categories (with a fixed limit for size), equipped with the symmetric monoidal structure given by the direct product operations.

For an operad $\mathcal{U}$, let us mean by an $(\mathcal{U} \otimes E_1)$-monoidal category, an associative monoidal object in the 2-category of $\mathcal{U}$-monoidal categories, or equivalently, a $\mathcal{U}$-monoidal object in the 2-category of associative monoidal categories. Note that there is a forgetful functor $\mathbf{Alg}_{\mathbf{Com}}(\mathbf{Cat}) \to \mathbf{Alg}_{\mathbf{U}}(\mathbf{Alg}_{\mathbf{Com}}(\mathbf{Cat})) \to \mathbf{Alg}_{\mathbf{U}}(\mathbf{Alg}_{E_1}(\mathbf{Cat}))$ from the symmetric monoidal categories to $(\mathcal{U} \otimes E_1)$-monoidal categories, where we have used the canonical functor $\mathbf{C} \to \mathbf{Alg}_{\mathbf{U}}(\mathbf{C})$ existing for every coCartesian symmetric monoidal 2-category $\mathbf{C}$ (namely, a 2-category $\mathbf{C}$ closed under the finite coproducts, made symmetric monoidal by the finite coproduct operations) obtained by letting every operation on a given object, say $X$, of $\mathbf{C}$, be the codiagonal map of $X$.

The formulation of the proposition is as follows. (See Remark 0.1 for a technical point.)

**Proposition 0.0.** For every symmetric operad $\mathcal{U}$, the functor $\mathbf{Op}_A$ on $\mathbf{Alg}_{\mathbf{Com}}(\mathbf{Cat})$ has an extension to $\mathbf{Alg}_{\mathbf{U}}(\mathbf{Alg}_{E_1}(\mathbf{Cat}))$ (in a manner which is functorial in $\mathcal{U}$).
The meaning of Proposition is that there is a natural notion of $\mathcal{U}$-graded operad in every $(\mathcal{U} \otimes E_1)$-monoidal category, such that the notion of $\mathcal{U}$-graded operad in a symmetric monoidal category $\mathcal{A}$ coincides naturally with the notion of $\mathcal{U}$-graded operad in the $(\mathcal{U} \otimes E_1)$-monoidal category underlying $\mathcal{A}$. Our definition will generalize the familiar notions of

- associative algebra in a associative monoidal category,
- planar operad in a braided monoidal category,
- braided operad in a $E_3$-monoidal infinity 1-category

(in addition to vacuously, the notion of symmetric operad in a symmetric monoidal category).

Remark 0.1. In order to actually have these examples, Proposition needs to be interpreted in the framework of sufficiently high dimensional category theory. (Infinity 1-category theory is sufficient.) However, let us not emphasize this technical point in this introduction, even though our work will eventually be about higher category theory.

Remark 0.2. It seems to be a reasonable guess that the notion of $\mathcal{U}$-graded multicategory had been known before our work even in the form enriched in a symmetric monoidal category, and if it had in the enriched form, then we expect Proposition 0.0 to have also been known. We simply have failed to confirm this from the literature.

In fact, we introduce in this work much more general notion of “grading”, generally for higher theories, and find quite general but natural places where the notions of graded higher theory can be enriched. An explanation of Proposition 0.0 from the general perspective to be so acquired, will be given in Section 0.0.4. In fact, all of these will result from extremely simple ideas, which we would like to describe with their main consequences.

0.0.2. Our starting point is the idea that an operad and more generally, a coloured operad or “multicategory” (see Lambek [15] or Section 1.1.2) is analogous to an algebraic theory for its role of governing algebras over it. We extend this by defining, for every integer $n \geq 0$, a notion of “$n$-theory”, where a 0-theory is an algebra (commutative etc.), a 1-theory will be a multicategory (symmetric etc.), and, for $n \geq 2$, each $n$-theory will come with a natural notion of algebra over it, in such a way that an $(n-1)$-theory coincides precisely with an algebra over the terminal $n$-theory, generalizing from the case $n = 1$, the fact that, e.g., a commutative algebra is an algebra over the terminal symmetric multicategory $\text{Com}$.

Remark 0.3. While algebraic theory in Lawvere’s sense is about a kind of algebraic structure which makes sense in any Cartesian symmetric monoidal category, the notion of algebra over an $n$-theory can be enriched (in particular) in any, e.g., symmetric, monoidal category (Definition 3.0). In this work, we only consider kinds of algebraic structure which are definable at least in every symmetric monoidal category.

The stated objective will be achieved by defining an $n$-theory inductively as a theorized form of an $(n-1)$-theory, where the theorization we consider is a “coloured” version of it. Theorization in the coloured sense, of commutative algebra, will be symmetric multicategory.

Let us describe what about the notion of multicategory we would like to generalize.

We first note that the notion of symmetric monoidal category is a categorification of the notion of commutative algebra, and the process of categorification plays here
the role of enabling us to define the notion of commutative algebra in an arbitrary symmetric monoidal category. A commutative algebra in a symmetric monoidal category $A$ is simply a lax symmetric monoidal functor $1 \to A$, where $1$ denotes the unit symmetric monoidal category.

Next, the notion of multicategory generalizes the notion of symmetric monoidal category in the sense that there is a functor which associates to a given symmetric monoidal category $A$, a multicategory $\Theta A$ from the structure of which $A$ can be recovered (up to canonical equivalence). Indeed, we can define $\Theta A$ as the multicategory in which

- an object is an object of $A$,
- a multimap $X \to Y$, where $X$ is a family of objects indexed by a finite set, say $S$, is a map $\bigotimes_S X \to Y$ in $A$,
- multimaps compose by composition in $A$ in the obvious manner.

Moreover, the notion of commutative algebra makes sense also in a multicategory $M$ so that, in the case $M = \Theta A$, we get back the notion of commutative algebra in $A$. Indeed, a commutative algebra in $M$ is a functor $\text{Com} \to M$ of multicategories.

We note that the notion of functor of multicategories is ‘primitive’ in that it is not a lax notion of morphism between categorified structures, as the notion of lax symmetric monoidal functor is.

Now, a standard definition is that, for a multicategory $U$, a $U$-algebra in $M$ is a functor $U \to M$ of multicategories, generalizing the notion of commutative algebra in $M$. Even in the special case where $M = \Theta A$, this gives a vast generalization of the notion of commutative algebra in $A$. The notion for a general $M$, is a version which is enriched in a more general manner.

Now we would like to get back to the general context, and sketch the relation between the notions, of $n$-theory and of $(n-1)$-theory. (The generalization of what has been discussed so far, will be as summarized by the diagrams (0.6) and (0.7) below, of which the case already discussed is the case $n = 1$.)

Firstly, there is an appropriate categorification of the notion of $(n-1)$-theory, which enables us to generalize the notion of $(n-1)$-theory to the notion of $(n-1)$-theory in a categorified $(n-1)$-theory. An $(n-1)$-theory in a categorified $(n-1)$-theory $C$ will be a lax functor $1_{\text{Com}}^{n-1} \to C$ (possibly with “colours” in a suitable sense, which is not important in this introduction, but is to be commented on after Theorem 0.4 below) of categorified $(n-1)$-theories, where $1_{\text{Com}}^{n-1}$ denotes the terminal $(n-1)$-theory.

Next, the notion of $n$-theory will generalize the notion of categorified $(n-1)$-theory in the similar manner as the manner how the notion of multicategory generalized the notion of symmetric monoidal category. (General discussions of this will be given in Sections 0.3, 0.4.) Moreover, an $(n-1)$-theory in a categorified $(n-1)$-theory $C$ will be equivalent as a datum to a (possibly coloured) functor $1_{\text{Com}}^n \to \Theta C$, where the target here is the $n$-theory corresponding to $C$.

In fact, $1_{\text{Com}}^n$ is equivalent to $\Theta 1_{\text{Com}}^{n-1}$ in a suitable sense, and we shall have the following.

**Theorem 0.4** (Theorem 2.14). Let $B$ and $C$ be categorified $(n-1)$-theories. Then a functor $\Theta B \to \Theta C$ of uncategorified $n$-theories, is equivalent as a datum to a lax functor $B \to C$.

The mentioned “colouring” of a (lax) functor (which the reader may ignore in this introduction) means the replacement of the source $1_{\text{Com}}^{n-1}$ by a suitable $B$ (Section 2.6.2).

The notion of coloured functor $1_{\text{Com}}^n \to \mathcal{V}$ of $n$-theories thus generalizes the notion of $(n-1)$-theory, to the similar notion in an $n$-theory $\mathcal{V}$. In general, given
a specific kind of algebraic structure such as “\((n-1)\)-theory” here, a “theorized” instance of that kind of structure, such as \(\mathcal{V}\) here, will be a general and natural place along which we hope to enrich the kind of structure. 

Functor of higher theories will again be a “primitive” notion, unlike lax functor of categorified higher theories.

Since the notion of \(n\)-theory generalizes the notion of categorified \((n-1)\)-theory, we inductively find within the hierarchy of \(n\)-theories as \(n\) varies, the hierarchy of \(n\)-categories or iterated categorifications of category, as in fact a very small part of it. Quite a variety of other hierarchies of iterated categorifications, such as the hierarchy of symmetric monoidal \(n\)-categories, of operads in \(n\)-categories, and so on, also form very small parts of the same hierarchy.

**Remark 0.5.** It has not been clear whether there is an analogous hierarchy starting from Lawvere theory. Possible iterated categorifications of Lawvere’s notion are algebraic theories enriched in \(n\)-categories, but we are looking for a larger hierarchy than iterated categorifications.

0.0.3. Let us sketch the basic structures which we obtain further.

Let \(n \geq 1\), and let \(\mathcal{U}\) and \(\mathcal{V}\) be unenriched \(n\)-theories, i.e., those enriched in sets (or in groupoids). Then it is appropriate to refer to a (coloured) functor \(\mathcal{U} \to \mathcal{V}\) as a \(\mathcal{U}\)-algebra in \(\mathcal{V}\). (We have already mentioned this terminology for \(n = 1\), where we did not want “colours”. For a general \(n\), we shall treat an algebra over an enriched \(n\)-theory using a kind of Day convolution, to handle the colours in a simple manner. See Section 0.4.) In the case where \(\mathcal{U}\) is terminal, this notion of \(\mathcal{U}\)-algebra was the (enriched) notion of \((n-1)\)-theory. Thus, we have obtained as various kinds of structure which are analogous to the structure of an \((n-1)\)-theory, as the choices there are of an \(n\)-theory \(\mathcal{U}\).

The situation can be summarized by the diagram

\[
\begin{array}{ccc}
\text{n-theory} & \overset{\text{theorize}}{\nearrow} & \mathcal{U} \\
\text{(n-1)-theory \overset{\text{“grade” by } \mathcal{U}}{\nearrow}} & \searrow & \mathcal{V} \\
\mathcal{U}\text{-algebra \overset{\text{control}}{\nearrow} (n-1)\text{-theory}} & \Downarrow & \mathcal{V}\text{-algebra}
\end{array}
\]

To be detailed, various enriched notions are related as depicted in the following diagram, where \(\mathcal{U}, \mathcal{V}\) vary through \(n\)-theories, and \(\mathcal{C}\) varies through categorified \((n-1)\)-theories. (For example, the “unenriched” \((n-1)\)-theory is where \(\mathcal{C} = \mathbb{B}^{n-1}\text{Set}\) for a certain construction \(\mathbb{B}\) (Section 2.5, where Set denotes the Cartesian symmetric monoidal category of sets.)

\[
\begin{array}{ccc}
\text{Functor} & \overset{\text{forbid colours}}{\Downarrow} & \text{Lax functor} \\
\mathcal{U} \to \mathcal{V} & \mathcal{U}\text{-algebra in } \mathcal{V} & 1^n_{\text{Com}} \to \mathcal{C} \\
\mathcal{V} = \mathcal{C} & \mathcal{U}\text{-algebra in } \mathcal{C} & \mathcal{V} = \mathcal{C} \\
\mathcal{U}\text{-algebra in } \mathcal{C} & \overset{\text{forbid colours}}{\Downarrow} & \text{Lax functor} \\
(n-1)\text{-theory} & \overset{\text{control}}{\Downarrow} & 1^n_{\text{Com}} \to \mathcal{C}.
\end{array}
\]

On the other hand, there is a notion of \(\mathcal{U}\)-graded \(n\)-theory, which theorizes the notion of \(\mathcal{U}\)-algebra, so the relation between the former and the latter notions is analogous to (and in this particular case, generalizes in fact; see below) the relation between the notions, of \(n\)-theory and of \((n-1)\)-theory, described in Section 0.0.2. The term “graded \(n\)-theory” comes from the following proposition. The previous, un-“graded” version of an \(n\)-theory will be called a symmetric \(n\)-theory.
Proposition 0.8 (Proposition 3.3). An unenriched $\mathcal{U}$-graded $n$-theory is equivalent as a datum to an unenriched symmetric $n$-theory $\mathcal{X}$ equipped with a functor $\mathcal{X} \to \mathcal{U}$ of symmetric $n$-theories.

In the case where $\mathcal{U}$ is terminal, the notion of $\mathcal{U}$-graded $n$-theory, which theorizes the notion of $1^n_{\text{Com}}$-algebra $=(n-1)$-theory, coincides indeed with the notion of symmetric $n$-theory.

One might think that, for a general $\mathcal{U}$, it makes sense to refer to a $\mathcal{U}$-algebra also as a $\mathcal{U}$-graded $(n-1)$-theory. In fact, there are even “lower” order notions of graded theory as follows.

If $n \geq 2$, then as there is a notion of algebra over an $(n-1)$-theory, there also is a notion of algebra over a monoid $\mathcal{X}$ over an $n$-theory $\mathcal{U}$, where by a monoid, we mean an algebra enriched in sets. In the case where $\mathcal{X}$ is the terminal monoid $1^{n-1}_\mathcal{U}$, the notion of $\mathcal{U}$-algebra theorizes the notion of $\mathcal{X}$-algebra, generalizing the manner how the notion of $(n-1)$-theory theorized the notion of $(n-2)$-theory. It is appropriate to refer to $1^{n-1}_\mathcal{U}$-algebra also as a $\mathcal{U}$-graded $(n-2)$-theory.

We can further iterate a similar construction of a new notion, and obtain for every integer $m$ such that $0 \leq m \leq n-2$, the notion of $\mathcal{U}$-graded $m$-theory, which is theorized by the notion of $\mathcal{U}$-graded $(m+1)$-theory.

There is also a notion of $\mathcal{U}$-graded $(n+1)$-theory, which theorizes the notion of $\mathcal{U}$-graded $n$-theory. We in fact obtain the following fundamental results.

Theorem 0.9 (Theorems 3.17, 3.19). A $\mathcal{U}$-graded $n$-theory is equivalent as a datum to an algebra over the $(n+1)$-theory $\Theta \mathcal{U}$. A $\mathcal{U}$-graded $(n+1)$-theory is equivalent as a datum to a $\Theta \mathcal{U}$-graded $(n+1)$-theory.

The notion of $\mathcal{U}$-graded $(n+1)$-theory can further be theorized iteratively, but the resulting notion of $\mathcal{U}$-graded $m$-theory for $m \geq n+2$ coincides with the notion of $\Theta^m_{\mathcal{U}}$-$\mathcal{U}$-graded $m$-theory, where $\Theta^m_{\mathcal{U}}$ denotes the $m$-theory obtained by applying the construction $\Theta m - n$ times on $\mathcal{U}$ (Section 2.4). (Theorem also suggests that $\mathcal{U}$-graded $m$-theory in the case $n=0$ should mean $\Theta \mathcal{U}$-graded $m$-theory for $m \geq 0$.)

In the case where $\mathcal{U}$ is terminal, the hierarchy of $\mathcal{U}$-graded higher theories coincides with the original hierarchy of symmetric higher theories.

We further consider higher theories graded by a graded theory, and so on, and shall find out that this does not add much to what have already been described. See Section 0.7.1 for some more on this.

In this work, we shall investigate relationship among various mathematical structures related to these objects, do and investigate various fundamental constructions, as well as consider a generalization and its relation to other subjects of mathematics. See Section 0.7.

0.4. The notion of $\mathcal{U}$-graded 1-theory in a $\mathcal{U}$-graded 2-theory gives a generalization of Proposition 0.0. Let us see this. (The discussion below will be slightly imprecise even though essentially correct. A more technical account can be found in Section 2.1)

Given a $(\mathcal{U} \otimes E_1)$-monoidal category $\mathcal{A}$, one can ‘categorically deloop’ $\mathcal{A}$ using the $E_1$-monoidal structure (see Bénabou [1, 2.2] or the review in Section 1.1.2) to obtain a $\mathcal{U}$-monoidal 2-category $\mathcal{B} \mathcal{A}$, and hence a $\mathcal{U}$-graded 1-theory $\Theta \mathcal{B} \mathcal{A}$ enriched in categories. Since this is a categorified $\mathcal{U}$-graded 1-theory, the notion of $\mathcal{U}$-graded operad in $\Theta \mathcal{B} \mathcal{A}$ makes sense, which we declare to be the notion of $\mathcal{U}$-graded operad in $\mathcal{A}$. This naturally generalizes the notion with the same name from the case where $\mathcal{A}$ is symmetric monoidal. Moreover, it is easy to check when $\mathcal{U}$ is one of the most familiar operads, that this coincides with the usual notion. However, this notion of operad, including the coloured version of it, is nothing but the notion of 1-theory in the $\mathcal{U}$-graded 2-theory $\Theta (\Theta \mathcal{B} \mathcal{A})$. 
0.0.5. Our method for iterative theorization will be by a technology of producing from a given kind of associative operation, a new kind of associative operation, which is based on fundamental understanding of the higher structure of associativity. See Section 1.1.3. The author indeed expects a hierarchy of iterated theorizations to exist starting from a kind of algebraic structure which can be expressed as defined by an “associative” operation, much more generally than has been discussed so far. Our construction of the hierarchy of \( n \)-theories, and its graded generalization, may be showing the meaningfulness of a general question of the existence of iterated theorizations for kinds of “algebraic” structure, and this may be our deepest contribution at the conceptual level.

Indeed, we shall consider in Section 5, a modest generalization of symmetric higher theories, which are obtained by iteratively theorizing some algebraic structures in which the operations may have multiple inputs and multiple outputs, such as various versions of topological field theories. See Section 0.7.4 for brief descriptions of the examples to be discussed in this work.

0.0.6. For the rest of this introduction, we shall describe the details of the idea of theorization, and more results of this work, and then give the outline of the body of the paper, and some notes for the reader for proceeding.

0.1. The conceptual origin of the notion of operad.

0.1.0. Theorization will be a process which produces a new kind of algebraic structure from a given kind. In order to start a discussion of the idea of theorization, we would like to be able to talk about kinds of algebraic structure.

An example of a kind of structure definable in a symmetric monoidal category, is a symmetric monoidal functor from a fixed symmetric monoidal category, say \( \mathcal{B} \). Thus, let us mean by a \( \mathcal{B} \)-algebra in a symmetric monoidal category \( \mathcal{A} \), simply a symmetric monoidal functor \( F: \mathcal{B} \to \mathcal{A} \). It is a ‘representation’ of \( \mathcal{B} \) in \( \mathcal{A} \) (or an \( \mathcal{A} \)-valued point of the ‘affine scheme’ \( \text{Spec} \mathcal{B} \)).

Concretely, by describing the structure of \( \mathcal{B} \) using a collection of generating objects and generating maps (as well as decomposition of each of the source and the target of every generating map into a monoidal product of generating objects), one may obtain a presentation of the form of the structure of a \( \mathcal{B} \)-algebra in terms of structure maps and equations satisfied by the structure maps. For example, the coCartesian symmetric monoidal category \( \mathcal{B} = \text{Fin} \) of finite sets, is generated under the symmetric monoidal multiplication operations \( \otimes = \coprod \) by the terminal object \( * \), and one map \( \nabla_S: * \otimes S \to * \) for each finite set \( S \). It follows that the datum of a symmetric monoidal functor \( F: \text{Fin} \to \mathcal{A} \) can be described as the object \( A = F(*) \) of \( \mathcal{A} \) equipped with one operation \( F(\nabla_S): A \otimes S = F(*) \otimes S \to A \) for each finite set \( S \), satisfying suitable equations resulting from the relations one has in Fin. Thus, we have obtained a presentation of the form of the structure of a Fin-algebra, as the form of datum for defining a commutative algebra.

For a general \( \mathcal{B} \), if objects of \( \mathcal{B} \) are generated under the monoidal multiplication by a family \( b = (b_\lambda)_{\lambda \in \Lambda} \) of objects, then a \( \mathcal{B} \)-algebra defined by a symmetric monoidal functor \( F: \mathcal{B} \to \mathcal{A} \), can be considered similarly as structured on the family \( Fb \) of objects of \( \mathcal{A} \), by structure maps satisfying equations imposed by the structure of \( \mathcal{B} \). For example, the universal \( \mathcal{B} \)-algebra defined by id: \( \mathcal{B} \to \mathcal{B} \), is structured on the family \( b \) of objects of \( \mathcal{B} \), where the structure maps will be the chosen generating maps of \( \mathcal{B} \).

Algebra over a symmetric monoidal category in our sense, is simply the most obvious formalization of kind of structure which can be presented as defined by structure maps satisfying some specific equations. Lawvere’s theory is based on this idea. Indeed, a multi-sorted, i.e., “coloured”, Lawvere theory is essentially
a Cartesian symmetric monoidal category $\mathcal{B}$ which is given a nice collection of generating objects. A PROP [16] or “category of operators” [3], with colours [4], is similar.

An algebra over a multicategory is also covered. Indeed, given a multicategory $\mathcal{U}$, one can freely generate from it a symmetric monoidal category, say $L\mathcal{U}$, so a $\mathcal{U}$-algebra in a symmetric monoidal category $\mathcal{A}$ will be equivalent as a datum to an $L\mathcal{U}$-algebra in $\mathcal{A}$.

Even though we are not particularly interested in the notion of algebra over a general symmetric monoidal category, this can be the starting point for our purpose of finding kinds of algebraic structure which generalize nicely. For example, recall that the basic idea of categorification is that a categorification of a certain kind of algebraic structure, is a kind of structure on category obtained by replacing structure maps by functors, and structural equations by suitably coherent isomorphisms, forming a part of the structure. We have a canonical categorification of $B$-algebra, which we shall call a $B$-monoidal category, and it is simply a symmetric monoidal functor $B \to \text{Cat}$, where $\text{Cat}$ denotes the Cartesian symmetric monoidal category enriched in groupoids, of categories (with a fix limit for size), where for $\mathcal{X}, \mathcal{Y} \in \text{Cat}$, we let $\text{Map}_{\text{Cat}}(\mathcal{X}, \mathcal{Y})$ be the groupoid formed by functors $\mathcal{X} \to \mathcal{Y}$ and isomorphisms between them.

Remark 0.10. This is a technical remark.

Here and everywhere else in this introduction, a functor which we consider to a category enriched in groupoids (or in categories) is a functor in the usual “weakened” sense (which is sometimes called a pseudo-functor, see Grothendieck [11], Section 8], [10]). Even though we shall not need to look into the details of this till we enter the body, a symmetric monoidal structure on such a functor can also be defined in an appropriate manner.

In fact, it should be understood that every categorical term in this introduction is used in the similarly appropriate sense when there is enrichment of the relevant categorical structures in groupoids or categories, where enrichment itself should be understood to be done in the standard “weakened” manner. See Bénabou [1]. (The reader who is comfortable with homotopy theory may instead replace all sets/groupoids with infinity groupoids, and understand everything as enriched in infinity groupoids, and this will trivialize the process of categorification since infinity 1-categories (of size up to a fixed limit) are already forming a (larger) infinity 1-category.)

However, we notify the reader of a circularity here. Namely, we have used one particular categorification of the notion of commutative algebra, i.e., the notion of symmetric monoidal category, to categorify kinds of structure which are similar to commutative algebra. Even though the result obtained is not bad, one may not be able to expect that the same framework would also be the most useful for categorifying very different kinds of structure.

For example, for categorification of the notion of multicategory, a method which takes account of the categorical dimensionality appears to lead to a cleaner and less redundant presentation of the result than the method of reformulating the notion of multicategory as algebra over a symmetric monoidal category, and then applying the previous definition. (It appears simpler to treat a multicategory as an algebra over a categorically 2-dimensional algebraic structure, e.g., the terminal “2-theory”, than to treat it as an algebra over a multicategory or a symmetric monoidal category, which can naturally be seen as 1-dimensional structures.)

Therefore, the general idea which we have described of a kind of algebraic structure and its categorification, seems more important for practical purposes, than
precise but limited formulations of the notions in particular contexts, such as $B$-algebra and $B$-monoidal category. Nevertheless, we hope that the examples above was clarifying on our view on algebraic structures.

Finally, we remark that, in order to consider a categorification of the notion of $B$-algebra, the target of the symmetric monoidal functor on $B$ to define a categorified structure, did not need to be $\text{Cat}$. Namely, the notion of symmetric monoidal functor $B \to A$, where $A$ is any symmetric monoidal category enriched in groupoids, formalizes the idea of replacing structural equations for the structure of a $B$-algebra (in some presentation of the notion) by coherent isomorphisms.

Therefore, it seems reasonable to expect in general, that a meaningful categorification of a kind of structure definable in a symmetric monoidal category, should be a kind of structure definable in any symmetric monoidal category $A$ enriched in groupoids. In fact, the right way to consider categorification is perhaps as about enrichment in groupoids, and the resulting weakening in a coherent manner, of the structure.

While the expectation above is indeed fulfilled in the concrete cases which we consider in this work, we shall keep concentrating on the case $A = \text{Cat}$ of categorized structures for the time being, since this will keep things simpler. One relation between the mentioned general form of categorification and the idea of theorization, will be seen in Section 1.1.1. On the other hand, for the kinds of structure which we theorize in this work, the general form of categorification can be understood in any case, as a specific kind of structure residing in a suitably associated theorized structure, leaving us no need to consider more general situation than $A = \text{Cat}$ in this early stage. See e.g., Corollary 2.16 (or Theorem 0.4). For example, the case “$n = 0$” of this result applies to the categorized form of the notion of $B$-algebra.

0.1.1. In order to get to the idea of theorization, we first recall that a basic feature expected of the categorified structure is that, if a category $C$ is equipped with a categorized form of a certain kind of algebraic structure, then the original, uncategorized form of the same structure should naturally make sense in $C$. For example, if $C$ is given a monoidal structure over a symmetric monoidal category $B$, then a “$B$-algebra” in $C$ means a lax $B$-monoidal functor $1 \to C$, where $1$ denotes the unit $B$-monoidal category.

However, for some kinds of algebraic structure, we have more general instances of this phenomenon. In a symmetric monoidal category for example, the notion of algebra makes sense over any symmetric operad or multicategory, and the same moreover makes sense also in any symmetric multicategory. Indeed, an algebra over a symmetric multicategory $U$ in a symmetric multicategory $V$, is simply a morphism $U \to V$. Similarly, the notion of algebra over any planar multicategory makes sense in any associative monoidal category, and more generally, in any planar multicategory in the same manner.

While the notion of associative monoidal category categorifies the notion of associative algebra, the process of theorization, which will be more general than the process of categorification, will produce the notion of planar multicategory from the notion of associative algebra, and symmetric multicategory from commutative algebra. In general, theorization will produce from a given kind of algebraic structure, a new kind of algebraic structure generalizing its categorification, in such a manner that the original notion of algebra reduces to the notion of algebra over the terminal one among the theorized objects (meaning symmetric multicategories, for commutative algebras, so generalizing the simple fact that an commutative algebra is an algebra over the terminal symmetric multicategory).

Let us thus recall how one may naturally arrive at the notion of symmetric operad, starting from the notion of commutative algebra (and we are suggesting
that the same procedure will produce the notion of planar operad from the notion of associative algebra, for example). Specifically, let us try to find the notion of symmetric operad (in sets) out of the desire of generalizing the notion of commutative algebra to the notions of certain other kinds of algebra which makes sense in a symmetric monoidal category. Indeed, one of the most important roles of a multicategory is definitely the role of governing algebras over it.

The way how we generalized the notion of commutative algebra is as follows. Recall, as we have already seen, that the structure of a commutative algebra on an object \( A \) of a symmetric monoidal category, was given by a single \( S \)-ary operation \( A^{\otimes S} \to A \) for every finite set \( S \) which, collected over all \( S \), had appropriate consistency. We get a generalization of this by allowing not just a single \( S \)-ary operation, but a family of \( S \)-ary operations parametrized by a set prescribed for \( S \). This “set of \( S \)-ary operations” for each \( S \), is the first bit of the datum defining an operad in sets. Having this, we next would like to compose these operations just as we can compose multiplication operations of a commutative algebra, and the composition should have appropriate consistency. A symmetric multicategory is simply a more general version of this, with many objects, or “colours”. (We shall take a look at colours in a theorized structure in Section 0.3.1.)

Remark 0.11. In this formulation of an operad, part of the composition structure makes the set of \( S \)-ary operations functorial with respect to bijections of \( “S” \). This gives the “action of the symmetric group” in another common formulation of an operad.

A similar procedure can be imagined once a kind of “algebraic” structure in a broad sense is specified as a specific kind of system of operations, in place of commutative or associative algebra. Inspired by Lawvere’s notion of algebraic theory, we call a multicategory also a (symmetric) \( 1 \)-theory, and then generally call theorization, a process similar to the process above through which we have obtained \( 1 \)-theories from the notion of commutative algebra. The result of such a process will also be called a theorization. Thus, the notion of \( 1 \)-theory is a theorization of the notion of commutative algebra. We shall see in Sections 0.3 and 0.4 how the process of theorization indeed generalizes the process of categorification.

Remark 0.12. Recall that operad was a kind of structure which made sense in any symmetric monoidal category, the meaning of which for us was that the form of datum to define an operad, could be presented in terms of structure maps and structural equations. In general, it seems reasonable to expect that a meaningful theorization of a given kind of algebraic structure should have a similar presentation, and in particular, should make sense in any symmetric monoidal category \( A \). This ability of presentation will be important when we would like to theorize a theorized kind of structure once again, even though we shall till that time, stick normally to the case where \( A \) is the Cartesian symmetric monoidal category Set of sets, in order to keep our exposition simpler.

0.2. Theorization of algebra.

0.2.0. As the simplest example of a theorization process next to the one which we have seen in the previous section, let us consider theorization of the notion of \( U \)-algebra for a symmetric operad \( U \) in sets (see Remark 0.13 below for the case of an operad in groupoids). By using the same method as in the previous section, we shall obtain a theorization of the notion of \( U \)-algebra, which we call \( U \)-graded operad (in the uncoloured version). Let us assume for simplicity, that \( U \) is an uncoloured operad.

Recall that the structure of a \( U \)-algebra on an object \( A \) of a symmetric monoidal category, is defined by an associative action on \( A \), of operations in \( U \). If \( u \) is an
S-ary operation in $U$ for a finite set $S$, then it should act as an $S$-ary operation $A^\otimes S \to A$. Now, to theorize the notion of $U$-algebra given by an action of the operators of $U$, means to modify the definition of this structure by replacing an action of every operator $u$ in $U$, by a choice of the set “of operations of shape (so to speak) $u$”. We call an element of this set an operation of degree $u$.

Thus the datum of a $U$-graded operad $X$ in sets should include, for every operation $u$ in $U$, a set whose element we shall call an operation in $X$ of degree $u$. If the operation $u$ is $S$-ary in $U$, then we shall say that any operation of degree $u$ in $X$ has arity $S$.

There should further be given a consistent way to compose operations in $X$, according to the way how operations in $U$ compose, namely, in a manner which respects the degrees of the operations. These will be a complete set of data for a $U$-graded operad $X$ in sets.

There is also a coloured version of this, which we call $U$-graded 1-theory, multicategory or coloured operad, and this is a theorization of $U$-algebra in a more general sense. From the general discussions of theorization in Sections 0.3 and 0.4, $U$-graded multicategory will turn out to be also a generalization of $U$-monoidal category, generalizing the way how symmetric multicategory generalizes symmetric monoidal category.

0.2.1. By reflecting on what we have done above, we immediately find that a $U$-graded operad in sets is in fact exactly a symmetric operad $Y$ in sets equipped with a morphism $P: Y \to U$. The relation between $X$ above and $Y$ here is that an $S$-ary operation in $Y$ is an $S$-ary operation in $X$ of any degree. The map $P$ maps an operation in $Y$ to the degree which the operation had when it was in $X$.

Conversely, given an $S$-ary operation $u$ in $U$, an operation in $X$ of degree $u$ is an $S$-ary operation in $Y$ which lies over $u$. For example, $U$, lying terminally over itself, indeed corresponds in this manner, to the terminal $U$-graded operad, which has exactly one operation of each degree.

Remark 0.13. In the case where $U$ is an operad in groupoids, the set of $S$-ary operations in $X$ of degree $u$ should be functorial in $u$ on the groupoid of $S$-ary operations in $U$. The $S$-ary operations in $Y$ will then be the corresponding groupoid projecting to the groupoid of $S$-ary operations in $U$ (obtained by the Grothendieck construction [11] Section 8) or the (homotopy) colimit in groupoids). For $U = E_2$, the mentioned functoriality corresponds to the action of the pure braid groups in a braided operad. See Fiedorowicz [9].

In general, a symmetric operad $Y$ in groupoids equipped with $P: Y \to U$, corresponds to a $U$-graded operad $X$ in groupoids. $X$ is in sets if for every operation $u$ in $U$, the groupoid of operations in $X$ of degree $u$, obtained as the (homotopy) fibre over $u$ in $Y$, is a homotopy 0-type, namely, a groupoid in which every pair of maps $f, g: x \Rightarrow y$ between the same pair of objects are equal.

Remark 0.14. Inclusion of operads in groupoids in the discussion leads to a subtle situation. For example, if $U = E_2$, then a $U$-algebra in a symmetric monoidal category is simply a commutative algebra. Therefore, we are considering both $E_2$-graded multicategory and symmetric multicategory as theorizations of commutative algebra. The crucial difference between the two theorizations is between the categorifications being generalized, namely, braided monoidal category and symmetric monoidal category.

Similarly, a $U$-graded multicategory enriched in sets is a symmetric multicategory in which multimaps are graded by multimaps of $U$. In other words, it is just a symmetric multicategory (enriched in sets) equipped with a morphism to $U$. Now,
given a $\mathcal{U}$-graded multicategory $\mathcal{X}$, an $\mathcal{X}$-algebra in a $\mathcal{U}$-graded 1-theory $\mathcal{Y}$ will be just a functor $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{U}$-graded 1-theories.

**Example 0.15.** A multicategory graded by the initial operad Init, is a multicategory with only unary multimaps, which is equivalent as a datum to a category.

A similar theorization of the notion of $\mathcal{U}$-algebra, can be defined for a coloured symmetric operad $\mathcal{U}$ enriched in sets, and a $\mathcal{U}$-graded 1-theory enriched in sets will be again a symmetric multicategory $\mathcal{X}$ (enriched in sets) equipped with a functor to $\mathcal{U}$. In other words, $\mathcal{X}$ will be such that not only multimaps in it are graded, but objects are also graded by objects of $\mathcal{U}$. For an object $u$ of $\mathcal{U}$, an object of $\mathcal{X}$ of degree $u$, will be just an object of $\mathcal{X}$ lying over $u$.

**Example 0.16.** Recall, as noted in Example 0.15, that a category $\mathcal{C}$ can be considered as a symmetric multicategory having only unary multimaps. If we consider $\mathcal{C}$ as a multicategory in this way, then a $\mathcal{C}$-graded 1-theory enriched in sets is a category equipped with a functor to $\mathcal{C}$, and this theorizes $\mathcal{C}$-algebra, or functor on $\mathcal{C}$ (which one might also call a left $\mathcal{C}$-module). On the other hand, a categorification of $\mathcal{C}$-algebra is a category valued functor $\mathcal{C} \to \text{Cat}$, and among the theorizations, categorifications correspond to op-fibrations over $\mathcal{C}$.

Suppose given a category $\mathcal{C}$ and two functors $F, G: \mathcal{C} \to \text{Cat}$, corresponding respectively to categories $\mathcal{X}, \mathcal{Y}$ lying over $\mathcal{C}$, mapping down to $\mathcal{C}$ by op-fibrations. Note that, by Example 0.16, $F$ and $G$ are categorified $\mathcal{C}$-modules, and $\mathcal{X}$ and $\mathcal{Y}$ as categories over $\mathcal{C}$, are the corresponding $\mathcal{C}$-graded 1-theories.

In this situation, the relation between maps $F \to G$ and maps $\mathcal{X} \to \mathcal{Y}$, is as follows. Namely, a functor $\phi: \mathcal{X} \to \mathcal{Y}$ of categories over $\mathcal{C}$ (see Remark 0.10, to be technical), corresponds to a map $F \to G$ if and only if $\phi$ preserves coCartesian maps, and an arbitrary functor $\phi$ over $\mathcal{C}$ only corresponds to a lax map $F \to G$ (defined with respect to the standard 2-category structure on Cat).

This is an instance of Theorem 0.4.

**0.3. Theorization in general.**

0.3.0. For the idea for theorization of a general algebraic structure, the notion of profunctor/distributor/bimodule is useful. For categories $\mathcal{C}, \mathcal{D}$, a $\mathcal{D}$–$\mathcal{C}$-bimodule (in the category Set) is a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Set}$. The category of $\mathcal{D}$–$\mathcal{C}$-bimodules contains the opposite of the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$ as a full subcategory, where a functor $F: \mathcal{C} \to \mathcal{D}$ is identified with the bimodule $\text{Map}_{\mathcal{D}}(F-, -)$. Let us say that this bimodule is corepresented by $F$. By symmetry, the category of bimodules also contains $\text{Fun}(\mathcal{D}, \mathcal{C})$. However, for the purpose of theorization, we treat $\mathcal{C}$ and $\mathcal{D}$ asymmetrically, and mostly consider only corepresentation of bimodules. Bimodules compose by tensor product, to make categories form a 2-category, extending the 2-category formed with (opposite) functors as 1-morphisms, by the identification of a functor with the bimodule corepresented by it.

0.3.1. We would like to consider the general idea of theorization, while having in mind as an example, the case of the notion of $\mathcal{B}$-algebra for a symmetric monoidal category $\mathcal{B}$.

We shall find that theorization is in fact more general than lax (or “op-lax”; see below) categorification, where, by a lax categorification of a kind of structure, we mean a relaxation of a specific categorification of the same kind of structure in the sense that it is a specific generalization of the specified categorification such that, in a lax categorified structure, a non-invertible map (going in a specified direction) is allowed in place of every one of some specified structure isomorphisms in some presentation of the categorified form of structure of the kind. Let us denote by
Cat, the 2-category of categories and functors. Then the notion of B-algebra has a canonical lax categorification, which we shall call lax B-monoidal category, where a **lax B-monoidal category** is by definition, a lax functor $B \to \text{Cat}$ which is given a datum of (not lax) commutation with the symmetric monoidal structures. In the case where $B$ is Fin, this coincides with the standard notion of lax symmetric monoidal category.

There is another lax categorification of the notion of B-algebra, which we shall call **op-lax B-monoidal category**. An op-lax B-monoidal category is similar to a lax B-monoidal category except that the laxness of the functor $B \to \text{Cat}$ should be opposite, namely, the non-invertible structure maps should go in the opposite direction. In other words, it should be an symmetric monoidal op-lax functor in one of the common conventions, in which a lax functor $1 \to \text{Cat}$ from the unit category $1$, is a category equipped with a monad, rather than a comonad, on it (see Bénabou [1]).

The idea of theorization which we have described in Sections 0.1, 0.2, can now be expressed as that it is a virtualization of an op-lax categorification, where by a virtualization of op-lax B-monoidal structure, or any kind of structure as far as the following makes sense, we mean a specific generalization of the structure such that, in a virtualized structure, non-corepresentable bimodules are allowed in place of some specified structure functors. (Note here also that the structure maps on bimodules should be understood to be in the opposite direction to the structure maps on functors, owing to the contravariance of the corepresentation of bimodules by functors.)

To be cautious, specification of a theorization of B-algebra for example, includes specification of the notion of “structure functor” for B-monoidal structure, which should at least include specification of a collection of generating objects of $B$. Specifically, given a family $b = (b_\lambda)_{\lambda \in \Lambda}$ of objects of $B$ which generates all objects under the monoidal multiplication, we consider for a family $X = (X_\lambda)$ of categories, a symmetric monoidal functor $F: B \to \text{Cat}$ with $Fb = X$ as a structure on $X$, and then see a theorization as a more general kind of structure which we can consider on $X$ (although a slightly more precise understanding will be that the structure is on the collections of colours to be described shortly, as would be concluded from the discussions of Section 0.3.3 below).

For example, theorization of algebra over a multicategory $U$, can be considered as the case where $B$ is freely generated by $U$, and the generating objects which we choose will be the indecomposable objects, i.e., the objects which come from $U$. See Section 0.3.2 below.

Note in particular, that the idea above does not determine the theorization from a categorification uniquely, so there is a question on which theorization if any, we would like. In the case of algebra over a multicategory $U$, we achieve the following through theorization. Firstly, our categorification is $U$-monoidal category, and this already allows us to define a $U$-algebra internal in a $U$-monoidal category $A$ as a lax $U$-monoidal functor $1 \to A$, where $1$ denotes the unit $U$-monoidal category. Now, through the process of theorization, this notion of $U$-algebra becomes generalized to the notion of functor of $U$-graded multicategories. See Example 0.22.

In general, given a specific kind of structure in a specific context, and some reasonable categorification of it, we would like a similar extension of the notion through theorization. It is a theorization which allows this that we would like, and existence of such a theorization appears to be usually a non-trivial question.

Another thing to note is that the notion of theorization which we have formulated is the “coloured” version which we did not discuss in detail in Section 0.1 or 0.2. A **colour** in the theorized structure is an object of a category in the underlying family...
of categories, e.g., \((X_\lambda)_{\lambda \in \Lambda}\) above. This generalizes the colours in a multicategory. See Section 0.3.2 below.

To be more detailed, in the case discussed above, one can consider an object of \(X_\lambda\) as a colour having degree \(\lambda\).

Remark 0.17. As mentioned in Remark 0.12, it is better to have a presentation of the form of datum for a theorized structure, so in particular, we have a definition of a theorized structure enriched in a symmetric monoidal category. In practice, it is perhaps not difficult usually, to write down a presentation by looking at the process of the theorization carefully. Essentially, one simply needs to run the virtualization process using the enriched version of bimodules, even though, to be rigorous, there is a minor issue here that bimodules in a general symmetric monoidal category do not necessarily compose, so we need to work actually in a 2-theory formed by enriched categories and bimodules. However, we shall not worry about this in this introduction, and shall mainly consider only unenriched theorized structures.

In the concrete situations which we treat in the body, another, simpler method for theorization (which demands more concrete data as an input) will in fact give a simpler solution for enriching the theorized structure in a symmetric monoidal category, as will be seen in Section 0.3.4.

0.3.2. For a multicategory \(U\), let us try to interpret \(U\)-graded multicategory as a “theorization” of \(U\)-algebra in the defined sense.

Firstly, we consider the structure of a \(U\)-algebra as a structure on family of objects (of a symmetric monoidal category) indexed by the objects of \(U\). Namely, we consider a \(U\)-algebra \(A\) in a symmetric monoidal category \(A\), as consisting of

- for every object \(u \in \text{Ob} U\), an object \(A(u)\) of \(A\),
- for every finite set \(S\) and an \(S\)-ary operation \(f: u \to u'\) in \(U\), where \(u = (u_s)_{s \in S}\) is a family of objects of \(U\) indexed by \(S\), a map \(Af: A(u) \to A(u')\) in \(A\), where \(A(u) := \bigotimes_{s \in S} A(u_s)\),

and then consider the latter as a structure on the family \(\text{Ob} A := \{A(u)\}_{u \in \text{Ob} U}\) of objects of \(A\). The structure is thus an action of every multimap \(f\) in \(U\) on the relevant members of the family \(\text{Ob} A\).

We would like next to obtain from a \(U\)-graded multicategory \(X\), a family similar to \(\text{Ob} A\) above, of categories, to underlie \(X\). For this, we take the family \(\text{Ob} \mathcal{X} := (X_u)_{u \in \text{Ob} U}\), where \(X_u\) denotes the category formed by the objects of \(X\) of degree \(u\), and maps (i.e., unary multimaps) between them of degree \(\text{id}_u\).

The rest of the structure of \(X\) can then be considered as a lax associative action of the rest of the multimaps \(f\) in \(U\), on these categories \(X_u\), each \(f\) acting as the bimodule formed by the multimaps in \(X\) of degree \(f\), so \(X\) can be interpreted as obtained by putting a theorized \(U\)-algebra structure on the family \(\text{Ob} \mathcal{X}\). (See Section 0.3.3 below to be more precise.)

Remark 0.18. Lax associativity of an action through bimodules generalizes \(op\)-lax associativity of an action through functors.

If a \(U\)-graded multicategory \(X\) is seen as a theorized structure in this manner, then a colour in this theorized structure is an object of a category \(X_u\), where \(u\) is any object of \(U\). In other words, it is an object of the multicategory \(X\).

0.3.3. We would like to give a minor and technical remark.

For a multicategory \(U\), a natural theorization which our definition expects of \(U\)-algebra would appear to be lax \(U\)-algebra in the 2-category mentioned above formed by categories and bimodules between them. (Note Remark 0.18.) This does not coincide with our desired theorization, which is \(U\)-graded multicategory.
Indeed, for an object $u$ of $\mathcal{U}$, if a category, say $\mathcal{X}_u$, is associated to $u$, and the identity map of $u$ acts on $\mathcal{X}_u$ in our 2-category of bimodules, then this action gives another category, say $\mathcal{Y}_u$, with objects the objects of $\mathcal{X}_u$, and a map, say $F: \mathcal{X}_u \to \mathcal{Y}_u$, of the structures of categories on the same collection of objects. However, the theorization in the idea described in the previous sections, is not where $\text{id}_u$ acts on an already existing category $\mathcal{X}_u$, but where the structure of $\mathcal{X}_u$ itself as a category, is the action of $\text{id}_u$.

In other words, we usually do not just want to consider a relaxed structure in the 2-category of categories and bimodules, but we would further like to require that the resulting map corresponding to $F$ in the example above, associated to each member of the underlying family of categories, to be an isomorphism.

Remark 0.19. In the example of Section 0.3.2, there is another category structure on the objects of $\mathcal{X}_u$, in which a map is a (unary) map in $\mathcal{X}$ of degree an arbitrary endomorphism of $u$ in $\mathcal{U}$, rather than just the identity. This clearly does not interfere with the remark here.

0.3.4. For a final remark, for the kinds of structure which we know how to theorize, we actually have a more economical description of the theorizations than we have given above. This uses the construction of a 2-category by ‘categorically delooping’ an associative monoidal category $\mathcal{A}$. See Bénabou [1, 2.2] or the review in Section 1.1.2. Note that, if $\mathcal{A}$ is a symmetric monoidal category, then the resulting 2-category, which we shall denote by $\mathcal{B}\mathcal{A}$, inherits a symmetric monoidal structure.

To turn to the description of the theorization, for the case of the notion of algebra over a multicategory $\mathcal{U}$ for example, a $\mathcal{U}$-graded multicategory enriched in a symmetric monoidal category $\mathcal{A}$, can be described as a coloured version of a lax $\mathcal{U}$-algebra in the symmetric monoidal 2-category $\mathcal{B}\mathcal{A}$. We refer the reader to Section 1.1.2 for the case $\mathcal{U} = \text{Com}$ of this. In general, it will be seen in Section 3.2.3 that, for an $n$-theory $\mathcal{U}$, our theorization of $\mathcal{U}$-algebra will, in the $\mathcal{A}$-enriched form, be an (“$n$-tuply”) coloured lax $\mathcal{U}$-algebra enriched in $\mathcal{B}\mathcal{A}$.

This idea of coloured lax structure enriched in $\mathcal{B}\mathcal{A}$, is actually less redundant than the idea of theorization which we have expressed above for a general situation, and does not produce the issue discussed in Section 0.3.3 either. This idea is the one along which we actually theorize kinds of algebraic structure which we can theorize so far. See the definitions in the body. In particular, the simplest case will be observed explicitly in Proposition 2.13.

However, compared to our previous formulation of the idea of theorization, the formulation of the notion of “coloured lax structure” enriched in a symmetric monoidal 2-category, would rely more on the manner how we generate the relevant structure (e.g., the symmetric monoidal category $\mathcal{B}$ for the case of “$\mathcal{B}$-algebra”). Since the author does not know what exact data are needed for theorizing a kind of “algebraic structure” in general, he does not know a general definition of a coloured lax structure. To formulate this notion for a given kind of structure, seems essentially equivalent to theorizing the kind of structure.

Remark 0.20. On the other hand, we have an ‘uncoloured theorization’ as soon as we have a lax version of the kind of structure. However, unless we can further find a reasonable common generalization of this uncoloured “theorization” and the categorification, there might not be a reasonable notion of algebra over an uncoloured “theorized” object of the kind.

0.4. A basic construction. The idea of theorization was such that a categorified structure was an instance of the theorized form of the same structure. Let us suppose given a kind of structure and a theorization of it. Then, for a categorified structure $\mathcal{X}$, let us denote by $\Theta \mathcal{X}$, the theorized structure corresponding to $\mathcal{X}$.
Concretely, $\Theta X$ is obtained by replacing as needed, structure functors of $X$ with bimodules copresented by them. The construction $\Theta$ generalizes the usual way to construct a multicategory from a monoidal category. Let us say that $\Theta X$ is represented by $X$.

**Remark 0.21.** Recall that a theorized structure in general could have colours. This flexibility is playing an important role here. Indeed, if the categorified structure $X$ is structured on a family, say $\text{Ob}_X = (\text{Ob}_X^\lambda)_{\lambda \in \Lambda}$, of categories, where $\Lambda$ denotes a collection which is suitably specified in the chosen presentation of the form of the structure in question, then any object of $\text{Ob}_X^\lambda$ for any $\lambda \in \Lambda$, is being a colour in the theorized structure $\Theta X$.

Thus the coloured version of the notion of theorization is indeed necessary in order for every categorified structure to be an instance of a theorized structure.

As in the case of monoidal structure, $\Theta$ is usually only faithful, but not full. Indeed, for (families of) categories $X, Y$ each equipped with a categorified structure, a morphism $\Theta X \to \Theta Y$ is equivalent as a datum to a lax morphism $X \to Y$. See Section 2.6.3.

**Example 0.22.** Let $U$ be a symmetric multicategory, and let $A$ be a $U$-monoidal category. Then a $U$-algebra in $A$, namely, a lax $U$-monoidal functor $\Theta 1 \to A$, where $1$ denotes the unit $U$-monoidal category, is equivalent as a datum to a functor $\Theta 1 \to \Theta A$ of $U$-graded multicategories, which is by definition, a $U$-algebra in $\Theta A$.

**Remark 0.23.** The functor $\Theta$ has a left adjoint (which in fact can be described in a concrete manner). In the example above, $U$-algebra in $A$ is thus equivalent to a $U$-monoidal functor to $A$ from the $U$-monoidal category freely generated from the terminal $U$-graded 1-theory $\Theta 1$ (which thus has a concrete description).

### 0.5. Theorization of category.

**0.5.0.** For illustration of the general definition, let us describe a natural theorization of the notion of category, which we shall call “categorical theory” here.

In order to define this, we first note that category can be understood as a kind of structure on family of sets. Indeed, for a category $X$, there is the family $\text{Map}_X := (\text{Map}_X(x, y))_{x, y}$ of sets of morphisms, parametrized by pairs $x, y$ of objects of $X$, so the structure of $X$ can be understood as defined on this family $\text{Map}_X$ of sets, by the composition operations. Moreover, this presentation immediately leads to a generalization of the notion to the notion of category enriched in a symmetric monoidal category.

Now, if we choose and fix a collection as the collection of objects for our (enriched) categories, then 2-category with the same collection of objects can be considered as a categorification of category with that collection of objects. Categorical theory will be a theorization of category whose associated categorification is 2-category.

The description of a categorical theory (enriched in sets) is as follows. Firstly, it, like a 2-category (our categorification) has objects, 1-morphisms, and sets of 2-morphisms. 1-morphisms do not compose, however. Instead, for every nerve $f : x_0 \to_{f_1} \cdots \to_{f_n} x_n$ of 1-morphisms and a 1-morphism $g : x_0 \to_{g} x_n$, one has the notion of $(n$-ary) 2-multimap $f \to g$. The 2-morphisms, which were already mentioned, are just unary 2-multimaps. There are given unit 2-morphisms and associative composition for 2-multimaps, analogously to the similar operations for multimaps in a planar multicategory.

A 2-category in particular represents a categorical theory, in which a 2-multimap $f \to g$ is a 2-morphism $f_n \circ \cdots \circ f_1 \to g$ in the 2-category. Between two 2-categories, a natural map of the represented categorical theories is not precisely a functor, but is a lax functor of the 2-categories.
As we have also suggested, a categorical theory is also a generalization of a planar multicategory. Indeed, planar multicategory was a theorization of associative algebra. The relation between the notions of planar multicategory and of categorical theory, is parallel to the relation between the notions of associative monoid and of category. Namely, categorical theory is a ‘many objects’ (or ‘coloured’) version of planar multicategory, where the word “object” here refers to one at a deeper level than the many objects which a multicategory (as a ‘coloured’ operad) may already have are at.

One thing one should note then, is that, while the 2-multimaps in a categorical theory is generalizing the multimaps in a planar multicategory here, the 1-morphisms in a categorical theory is generalizing the objects of a planar multicategory, and no longer have the characteristic of operators like the 1-morphisms in a category. Indeed, a 1-morphism in a categorical theory and an object of a planar multicategory are both ‘colours’, and we have also mentioned earlier that there is no operations of composition given for 1-morphisms in a categorical theory.

What we said above in comparison of the structures of a categorical theory and of a planar multicategory, is that a categorical theory \( \mathcal{X} \) has one more layer of ‘colouring’ under the 1-morphisms, given by the collection of the objects of \( \mathcal{X} \).

**Example 0.24.** Let \( \mathcal{A} \) be an associative monoidal category. Then the categorical theory corresponding to the planar multicategory \( \Theta \mathcal{A} \), is represented by the 2-category \( BA \). (To be technical, the former categorical theory is “simply coloured” in the sense that it has only one layer of colours, and it is equivalent to the ‘simply coloured part’ at the base object, of the categorical theory \( \Theta BA \). See Section 2.5.4 for an explanation in a similar situation.)

**Example 0.25.** The “Morita” 2-category due to Bénabou [1] of associative algebras and bimodules in a monoidal category \( \mathcal{A} \) with nicely behaving colimits, is well-defined as a categorical theory when \( \mathcal{A} \) is more generally, an arbitrary planar multicategory. There is a forgetful functor from this “Morita” categorical theory to the categorical theory of Example 0.24.

0.5.1. Following the general pattern about theorization, there is a notion of category in a categorical theory, and, as an uncoloured version of it, monoid in a categorical theory, which generalizes monad in a 2-category. A monad in a 2-category \( \mathcal{X} \) was a lax functor to \( \mathcal{X} \) from the unit 2-category (see Bénabou [1]), which can also be considered as a map between the categorical theories represented by these 2-categories. See Section 0.3 However, the latter is a monoid in the target categorical theory \( \Theta \mathcal{X} \) by definition.

**Example 0.26.** Let \( \mathcal{C} \) be a category enriched in groupoids, and let \( M \) be a monad on \( \mathcal{C} \). Then there is a categorical theory as follows.

- An object is an object of \( \mathcal{C} \).
- A map \( x \to y \) is a map \( Mx \to y \) in \( \mathcal{C} \).
- Given a sequence of objects \( x_0, \ldots, x_n \) and maps \( f_i : Mx_{i-1} \to x_i \) in \( \mathcal{C} \) and \( g : Mx_0 \to x_n \), the set \( \text{Mul}(f; g) \) of 2-multimaps \( f \to g \), is the set of commutative diagrams

\[
\begin{array}{ccc}
M^n x_0 & \xrightarrow{M^{n-1} f_1} & \cdots & \xrightarrow{f_n} & x_n \\
\downarrow m & & & & \downarrow = \\
M x_0 & \xrightarrow{g} & x_n
\end{array}
\]

in \( \mathcal{C} \) (i.e., the set of isomorphisms filling the rectangle), where \( m \) denotes the multiplication operation on \( M \).
• Composition is done in the obvious manner.

A monoid in this categorical theory is exactly an $M$-algebra in $C$.

More generally, a category in a categorical theory can be described as a coloured version of a map of categorical theories. A basic example is a category in the categorical theory obtained by considering a planar multicategory $U$ as a categorical theory. This is equivalent as a datum to a category enriched in the planar multicategory $U$.

0.6. Iterating theorization.

0.6.0. In Section 0.5, we have theorized the notion of category by considering a category as a structure on the family of sets consisting of the sets of maps. Recall from Example 0.15 that a category was an ‘initially graded’ 1-theory. Section 0.3.2 shows thus that category is a theorization of Init-algebra, or bare, i.e., unstructured, object. (It is not difficult to see in the similar manner, that the versions enriched in a symmetric monoidal category, of the relevant notions also coincide.) For these reasons, we shall call a categorical theory also an “Init-graded 2-theory”.

One can similarly consider the structure of a categorical theory as a structure on the sets of its 2-multimaps, and then try to theorize the notion of categorical theory after fixing the collections of objects and of 1-morphisms. It turns out that there is indeed an interesting theorization in this case, which in particular generalizes the notion of 3-category. One might call the resulting theorized object a categorical 2-theory or an initially graded 3-theory.

One might ask whether it is possible to iterate theorization in a similar manner here, or starting from some specific additional structure, rather than ‘no structure at all’. We should of course ask possibility of interesting theorizations. One condition for this has been mentioned in Section 0.3.1. Other desirable things may include abundance of natural examples, and reasonable properties.

In the case where the answer to the question is affirmative, just as the original structure could be expressed as the structure of an algebra over the terminal object among the (unenriched) theorized objects of the same kind, the theorization similarly becomes the structure of an algebra over the terminal object among the twice theorized objects, and so on, so all the structures can be described using their iterated theorizations. Moreover, by considering an algebra over a non-terminal theory, an algebra over it, and so on, one obtains various general structures, which can all be treated in a unified manner.

The question asked above is non-trivial. However, we introduce the notion of $n$-theory in this work, which will inductively be an interesting theorization of $(n-1)$-theory. The hierarchy as $n$ varies, of $n$-theories will be an infinite hierarchy of iterated theorizations which extends the various standard hierarchies of iterated categorifications, in particular, the hierarchy of $n$-categories in the “initially graded” case.

While our higher theories will be in general a completely new mathematical objects, we have already found very classical objects of mathematics among 2-theories. Namely, while we have seen that a categorical theory was an “initially graded” 2-theory, we have also noted in Section 0.5 that planar multicategories were among categorical theories.

We have also seen non-classical objects among 2-theories in Section 0.5. Less exotic examples of higher theories comes from the construction of Section 0.4. Namely, a higher categorified instance of a lower theorized structure leads to a higher theorized structure through the iterated application of the construction $\Theta$ (details of which can be found in Section 2.4). As an object, this is less interesting among the
general higher theories for the very reason that it is represented by a lower theory. However, the functors between these theories are interesting in that it is much more general than functors which we consider between the original lower theories. Namely, a functor between such higher theories amounts to highly relaxed functor between the higher categorified lower theories, as follows from an iteration of the remark in Section 0.4.

In Section 2.5 (as will be previewed in Section 0.7.2), we discuss a general construction which we call “delooping”, through which we obtain an \((n + 1)\)-theory which normally fails to be representable by a categorified \(n\)-theory. Another construction, which is closely related to the classical Day convolution, will also be discussed in Section 4.4, and this also produces similar examples.

0.7. Further developments.

0.7.0. Let us preview a few more highlights of our work.

0.7.1. In Section 0.0.3 we have hinted at a notion of algebra over a monoid (i.e., algebra enriched in sets) over an \(n\)-theory enriched in sets (if \(n \geq 2\)), a notion of algebra over unenriched such (if \(n \geq 3\)), and so on. For a \(U\)-graded \(m\)-theory \(X\) enriched in sets, we indeed define the notion of \(X\)-algebra, and more generally, of \(X\)-graded \(\ell\)-theory, as well as the notion of higher theory graded by unenriched such, and so on. We can actually give simple definitions of all these, using Theorem 0.9 as a general principle (Section 3.3). At the end, every structure (which is enriched in sets or groupoids) will come with a hierarchy of higher theories “graded” by it. We obtain a generalization of Proposition 0.8 with these new notions (Proposition 3.37). We also treat some basic questions concern the general enriched versions of all the notions in Section 4.

0.7.2. An \(n\)-theory enriched in a symmetric monoidal category \(\mathcal{A}\), can also be described as a (suitable) \(n\)-theory in the \((n + 1)\)-theory \(\Theta^{n+1}B^n\mathcal{A}\) obtained by applying the construction \(\Theta\) \(n + 1\) times (Section 2.6) to the symmetric monoidal \((n + 1)\)-category \(B^n\mathcal{A}\) (the \(n\)-th iterated categorial deloop of \(\mathcal{A}\)). In Section 2.6 we generalize the categorical delooping construction for symmetric monoidal higher category, to a certain construction \(\mathcal{B}\) which produces a symmetric \(n\)-theory from a symmetric \((n - 1)\)-theory. This is a generalization of the categorical delooping in such a manner that, for a symmetric monoidal category \(\mathcal{A}\), there is a natural ‘equivalence’ \(\Theta^{n+1}B^n\mathcal{A} \simeq \mathcal{B}^n\Theta\mathcal{A}\) (we refer the reader to Section 2.5.4 for the precise relation). It follows that a natural notion of \(n\)-theory enriched in a symmetric multicategory \(\mathcal{M}\), which is not necessarily of the form \(\Theta\mathcal{A}\), is \(n\)-theory in the \((n + 1)\)-theory \(\mathcal{B}^n\mathcal{M}\), and, as \(n\) increases, the notion iteratively ‘theorizes’ the previous notions in a suitable sense (see Remark 2.14 and Proposition 2.13).

Incidentally, if \(\mathcal{M}\) is not of the form \(\Theta\mathcal{A}\), then \(\mathcal{B}^n\mathcal{M}\) is usually not representable by a categorified \(n\)-theory.

0.7.3. For a symmetric monoidal \(n\)-category \(\mathcal{C}\), we construct a certain \((n + 1)\)-theory \(A_n\mathcal{C}\) and a functor \(A_n\mathcal{C} \rightarrow \Theta^{n+1}B^n\text{Set}\) of \((n + 1)\)-theories. The use of this is the following. Namely, while we have already mentioned that an \(n\)-theory \(\mathcal{X}\) enriched in sets can be considered as an \(n\)-theory in \(\Theta^{n+1}B^n\text{Set}\), the construction above allows us to understand an \(\mathcal{X}\)-graded \(m\)-theory enriched in a symmetric monoidal category \(\mathcal{A}\), where \(0 \leq m \leq n - 1\), as an appropriate lift of the theory to \(A_nB^n\mathcal{A}\) (a more general statement being as Corollary 4.10 where \(A_n = \Theta_n\mathcal{A}, \Theta^n_0\)
in the notation there). There is also a version of this for \( m = n \) (Corollary 4.12), which will have an application in our work.

0.7.4. We also touch on more topics, such as the following.

- Pull-back and push-forward constructions which changes gradings, and their properties (Sections 3.3, 4.3). The result Corollary 4.12 mentioned in Section 0.7.3 will be used for the construction of the push-forward ‘on the right side’.

- Some other basic constructions such as a construction for higher theories related to Day’s convolution [8]. The Day type construction leads very easily to a notion of algebra over an enriched higher theory (Section 4.4).

- Hierarchies of iterated theorizations associated to more general systems of operations, with multiple inputs and multiple outputs, such as operations of ‘shapes’ of bordisms as in various versions of a topological field theory (Section 5). Examples also include iterated theorizations of the notion due to Vallette of (coloured) properad [17]. See Example 5.5.

The last topic leads to vast generalizations of the relevant versions of the notion of topological field theory. We obtain a simple but in a way exotic example (Section 5.2) in addition to examples of a more expected type.

0.7.5. We also enrich everything we consider in this work, in the Cartesian symmetric monoidal infinity 1-category of infinity groupoids, instead of in sets. Fortunately, this does not add any difficulty to the discussions. However, we invite the reader who does not wish to deal with homotopy theory, to Section 0.9.0.

0.8. Outline. We shall give the definition of an ungraded higher theory with a minimal degree of enrichment, in Section 1. We shall follow up the definition in Section 2 with discussions of simple subjects such as a planar variant, a less colored variant, the construction \( \Theta \), and the generalized “delooping” construction. We shall then discuss algebras and graded higher (and “lower”) theories over a higher theory in Section 3. We shall then discuss in Section 4 various topics about enrichment of the notion of higher theory, including a construction for higher theories which is related to the Day convolution. Finally, we shall discuss iterated theorizations of more general algebraic structure in Section 5. Appendix is for comparison of this work with a related important work [7] of Baez and Dolan.

0.9. Notes for the reader.

0.9.0. For practical purposes, it seems best to build our theory in the framework of homotopy theory. Thus, we let the infinity 1-category of infinity groupoids be the default place where we enrich any categorical or algebraic structure.

For the reader who does not wish to deal with homotopy theory, our terminology in the body will be such that it could be read as if we are working in the framework of the classical, discrete category theory. For example, we shall say “category” to in fact mean “infinity 1-category”. So such a reader would be comfortable with interpreting what we write in the normal, classical manner, and then ignoring what is redundant in such an interpretation.

Remark 0.27. Here are two cautions. One is that, when we say “groupoid” (while in fact meaning infinity groupoid), this can often be interpreted as set in the classical context, but sometimes, it will be better to interpret it honestly as “groupoid” (i.e., 1-groupoid). The other is that there is fear that some of the examples we give may degenerate to trivialities in the non-homotopical interpretation.
While we also welcome the reader who does this, our method for theorization is by dealing with the structure of the coherence for higher associativity, which is also the key for higher category theory, so our expectation is that the reader who interprets our work in the classical context, would eventually find the “homotopical” interpretation more natural. (The subject of the present work does not select any particular model for higher category theory. In order to communicate to as wide audience as we can, we shall try to make clear which data are being used from the theory of higher categories when we use them, so the hope is that even the reader who has no more than basic ideas on the essence of the higher category theory, would find our exposition largely accessible.)

0.9.1. As we have mentioned, we adopt the convention that all terms should be interpreted in homotopical/infinity 1-categorical sense. Namely, categorical terms are used in the sense enriched in the infinity 1-category of infinity groupoids, and algebraic terms are used freely in the sense generalized in accordance with the enriched categorical structures.

However, we do welcome the reader who prefers to work in the classical setting, to interpret the terms in the usual, non-homotopical manner. In this case, categorical terms (e.g., multicategory) should be understood in the sense enriched in the category of sets (or sometimes better groupoids) unless otherwise specified, and Remark 0.10 would continue in effect.

For example, by a 1-category, we officially mean an infinity 1-category, while also welcoming the classical interpretation. We often call a 1-category (namely an infinity 1-category) simply a category. More generally, for an integer $n \geq 0$, by an $n$-category, we mean an infinity $n$-category.

0.9.2. Unless otherwise noted, we do not consider non-unital associative algebraic structures. Moreover, we normally treat unitality as part of associativity. (Specification of the unit will be a nullary operation.)

0.9.3. In our notations, we shall freely put a non-negative integer (or a variable, such as “$n$”, for a non-negative integer) as a superscript to a letter in order to avoid excessive use of multiple subscripts. Other than the exceptions listed below, and unless otherwise noted, such a superscript will be a label just like a subscript, put on the right upper corner in order to preserve rooms for subscripts. In particular, there will be only few occasions where we need to take a power of a thing, in which case, we shall indicate so.

Major exceptions are as follows.

- “$\Delta^n$”, “$d_i^n$”, “$R^n$” will respectively denote the $n$-dimensional symplex, the $i$-th simplicial coface operator, $n$-dimensional Euclidean space.
- “$f^{-1}$” for a map or a morphism $f$, will denote the inverse of $f$, or the inverse image by the map $f$.
- “$B^n$” and “$\mathbb{B}^n$” will denote the $n$-fold applications of the (“delooping”) constructions “$B$” and “$\mathbb{B}$” respectively (which will have been defined).
- “$Q_n$” will denote the instance of a certain construction (defined in this work) which applies to an “$n$-theory”, and produces an “$n$-theory”.
- “$1^n_U$” and “$\mathcal{U}^n$” will respectively denote the terminal ($\mathcal{U}$-graded, unenriched uncoloured) $n$-theory and the $n$-dimensional “universal” monoid.

All other exceptions will be noted at the relevant places.
1. Symmetric higher theories

1.0. Introduction. After giving a small number of preliminary definitions, we shall give in this section, the definition of a higher theory with least amount of structure.

1.1. The definition.

1.1.0. Let
- \( \text{Ord} \) denote the category (in the classical, discrete sense) of finite ordinals (including the empty set \( \emptyset \)),
- \( \Delta \) denote the category (again in the discrete sense) of combinatorial simplices, in other words, non-empty finite ordinals.

For example, we have objects \([0] = \{0\}\) and \([1] = \{0 < 1\}\) of \( \Delta \), and the maps in \( \Delta \) called the coface operators \( d^i: [0] \to [1] \), for \( i = 0, 1 \), where \( d^0(0) = 1, d^0(0) = 0 \).

The following is about all of the combinatorics which we need for this work. Namely, there is a functor \([-]: \text{Ord} \to \Delta^{\text{op}} \) defined as follows.

For an object \( I \in \text{Ord} \), we define
\[
[I] := [1] \cup_{[0]} d_0^{I} \cdots d_0^{I} [1] \quad (I\text{-fold; e.g., } [\emptyset] = [0]).
\]

In other words, \([I] = \bigcup_{i \in I} [1] \) is obtained by gluing for every pair \( i < i+1 \) of adjacent elements of \( I \), 1 in the \( i \)-th component \([1]\), with 0 in the \((i+1)\)-th component \([1]\).

For a map \( \phi: I \to J \) in \( \text{Ord} \), we note that \([I] = \bigcup_{j \in J} [\phi^{-1} j] \), where adjacent components are glued (similarly to before) at the respective maximal and minimal elements. We define \( [\phi]: [I] \to [J] \) in \( \Delta \) to be the map obtained by gluing over \( j \in J \), the maps \([1] \to [\phi^{-1} j] \) in \( \Delta \) preserving the minimum and the maximum.

Remark 1.0. Let
- \( \text{Fin} \) denote the category (in the discrete sense) of finite sets,
- \( \text{Fin}_{*} \) denote the category (discrete sense) of pointed finite sets.

Namely, an object of \( \text{Fin}_{*} \) is a finite set \( S \) equipped with a “base point” \(* \in S\), and a morphism is a map \( f \) for which we have \( f(*) = * \).

Then there is a commutative square
\[
\begin{array}{ccc}
\text{Ord} & \xrightarrow{[-]} & \Delta^{\text{op}} \\
\text{forget} \downarrow & & \downarrow \Delta^1/\partial \Delta^1 \\
\text{Fin} & \xrightarrow{(\_)+} & \text{Fin}_{*},
\end{array}
\]
where \( \Delta^1 \) denotes the simplicial 1-simplex \( \Delta^{\text{op}} \to \text{Fin} \), with its boundary \( \partial \Delta^1 \), and \((\_)+\) is the functor which externally adds a base point to every finite set.

Remark 1.1. The functor \([-]: \text{Ord} \to \Delta^{\text{op}} \) has a right adjoint \( \tilde{\Delta}^1: \Delta^{\text{op}} \to \text{Ord} \), which, as a functor, is the lift of \( \Delta^1: \Delta^{\text{op}} \to \text{Fin} \) obtained by putting the natural total order on the set of all faces of \( \Delta^1 \) of each fixed dimension. In particular,
\[
[I] = \text{Hom}_{\Delta}([0], [I]) \simeq \text{Hom}_{\text{Ord}}(I, \tilde{\Delta}^1_0)
\]
as a functor of \( I \in \text{Ord} \).

Notation 1.2. In the following, we usually write the elements of an ordinal \( I \) as \( 1 < 2 < \cdots \) in the ascending order, and then the elements of \([I]\) as \( 0 < 1 < 2 < \cdots \).
1.1.1. Next, we explain our terminology and notations concerning families, nerves and operations on them. Here we introduce only a minimal amount of it; more will be introduced later during various other definitions.

For \( S \in \text{Fin} \), we mean by an \( S \)-family a family of mathematical objects indexed by the elements of \( S \).

Let \( \phi : T \to S \) be a map in \( \text{Fin} \). Then from an \( S \)-family \( x = (x_s)_{s \in S} \), we obtain a \( T \)-family \( \phi^*x := (x_{\phi t})_{t \in T} \).

For \( I \in \text{Ord} \), we mean by an \( I \)-nerve in a category, a pair consisting of

- a \([I]\)-family of objects \( x = (x_i)_{i \in |I|} \), and
- an \( I \)-family of maps \( f = (f_i)_{i \in I} \), where \( f_i : x_{i-1} \to x_i \).

Such an \( f \) is also called an \( I \)-nerve connecting the \([I]\)-family \( x \).

Let \( \phi : I \to J \) be a map in \( \text{Ord} \). Then from an \([I]\)-family \( x \), we obtain an \([J]\)-family \( \phi^*x \), and from an \( I \)-nerve \( f \) as above connecting \( x \), we obtain an \( J \)-nerve \( \phi \circ f \) connecting \( \phi^*x \), defined by

\[
(\phi \circ f)_j = f_1(\phi(j)) \cdots f_{|J|+1}(\phi(j)).
\]

Note that \( \{\phi(j) = 1 < \cdots < |\phi|\} = \phi^{-1}(1) \subset I \).

Definition 1.3. Let \( I \in \text{Ord} \). Then an \([I]\)-family \( J \) in either \( \text{Ord} \) or \( \text{Fin} \), is said to be elemental if \( J_{|\pi(1)} = \ast \), the terminal object, where \( \pi \) denotes the unique map \( I \to \{1\} \), so \( |\pi|(1) \) is the maximum of \( |I| \).

1.1.2. The notion of multicategory is a theorization of the notion of commutative algebra. Indeed, a multicategory (enriched in groupoids) is a virtualized form of an op-lax symmetric monoidal category.

We can also formulate the notion of coloured lax commutative algebra as follows.

Definition 1.4. A coloured lax commutative algebra \( U \) in a symmetric monoidal 2-category \( \mathcal{A} \) consists of the following data.

(i) A collection \( \text{Ob} \, U \), whose member is called an object of \( U \), and, for every object \( u \in \text{Ob} \, U \), an object \( U(u) \in \text{Ob} \, \mathcal{A} \).

(ii) For every finite set \( S \), every \( S \)-family \( u_0 = (u_{0_s})_{s \in S} \) of objects of \( U \), and every object \( u_1 \), a map \( m^U_{1} (u_0; u_1) : U(u_0) \to U(u_1) \), where \( U(u_0) := \bigotimes_{s \in S} U(u_{0_s}) \).

Suppose given

- a finite ordinal \( I \),
- an elemental \([I]\)-family \( S = (S_i)_{i \in |I|} \) of finite sets, and an \( I \)-nerve \( \phi = (\phi_i)_{i \in I} \) in \( \text{Fin} \) connecting \( S \),
- for every \( i \in [I] \), an \( S_i \)-family \( u_i = (u_{is})_{s \in S_i} \) in \( \text{Ob} \, U \).

Then a 2-morphism \( m^U_{2} : \pi_1 m^U_{1}[u] \to m^U_{1}(u_0; u_{|\pi|}) \) in \( \mathcal{A} \), where

\[
m^U_{1}[u] := (m^U_{1}(u_{i-1}; u_i))_{i \in I}, \quad m^U_{1}(u_{i-1}; u_i) := \bigotimes_{s \in S_i} m^U_{1}(u_{i-1}; u_{is}), \quad m^U_{1}(u; u_{|\pi|}) := \bigotimes_{s \in \pi} m^U_{1}(u_s; u_{|\pi|})
\]

of objects,

- \( \pi \) denotes the unique map \( I \to \{1\} \).

(∞) A datum of coherence for the structure.

We shall not write down the details of (∞) since the explicit form of it is not so important here. A more general case is treated in Definition 1.5 in particular, Sections 1.8, 1.9 below.

Now, given a monoidal category \( \mathcal{A} \), by its categorical delooping, we mean the 2-category \( \mathcal{B} \mathcal{A} \) with a chosen “base” object, in which all objects are equivalent, and the endomorphism monoidal category of the base object is given an equivalence with \( \mathcal{A} \). (Note that this determines \( \mathcal{B} \mathcal{A} \) uniquely.)
For a symmetric monoidal category $A$, we can consider a multicategory enriched in $A$ as, by definition, a coloured lax commutative algebra $U$ in the symmetric monoidal 2-category $BA$ (with the induced symmetric monoidal structure) such that, for every object $u \in \text{Ob} U$, $U(u)$ is the unit (i.e., the base) object of $BA$.

For a multicategory $U$, we denote the object $m^U_{1}(u_0; u_1) \in A$ by $\text{Mul}_U(u_0; u_1)$. In the case where $A$ is the Cartesian symmetric monoidal category $\text{Gpd}$ of groupoids, $\text{Mul}_U(u_0; u_1)$ is the groupoid of multimaps $u_0 \to u_1$ in $U$. For a general $A$, the object $\text{Mul}_U(u_0; u_1)$ of $A$ is made to behave as if it were formed by (generally fantastical) multimaps $u_0 \to u_1$.

For example, in the case where $A$ is the Cartesian symmetric monoidal category of categories (with a fixed limit for the size), a multicategory enriched in $A$ is the obvious categorified form of a multicategory. This is not a useless notion, and the notion of coloured lax commutative algebra can more generally be defined in a categorified multicategory, for example. A more general notion of enrichment will be the subject of Section 4.

1.1.3. Starting from commutative algebra and multicategory, there is an infinite hierarchy of iterated theorizations. The idea for its construction is simple. We consider the structure of a multicategory as given by an associative system of “composition” operations for multimaps. In general, we would like to produce from one kind of associative system of operations, another kind of associative system of operations. This can be done using the following simple observation on the inducitivity of the structure of coherent associativity.

Suppose that we have a collection $m$ of operations (e.g., $m^U_{1}$ of Definition 1.4) which, if made coherently associative, would define (in perhaps a special case) $n$-th theorized version of a multicategory (e.g., a multicategory if $n = 0$). Suppose further that we actually have a collection $m'$ of (at least lax) associativity maps for the operations $m$, but that we still do not have a coherence datum for these maps $m'$, so $m$ is not yet coherently (lax) associative. An example of $m'$ is $m^U_{2}$ for $m = m^U_{1}$ in Definition 1.4.

In this situation, datum of coherence for the lax associativity has the following interpretation. Consider $m'$ as itself a new collection of operations. Then a coherence datum we are looking for amounts precisely to a datum of coherent associativity for the collection $m'$ of operations.

A $(n+1)$-theorization of multicategory is then obtained by formalizing the structure given by the collection $m'$ of operations and its coherent associativity. See Definition 1.5 below for the details.

**Definition 1.5.** Let $n \geq 2$ be an integer. A symmetric $n$-theory $U$ (which will often be called simply an “$n$-theory”) enriched in a symmetric monoidal category $A$, consists of data of the forms specified below as $(0), (1), (2)$ (or just $(0), (1)$ if $n = 2$), “$(k)$” for every integer $k$ such that $3 \leq k \leq n - 1$, $(n), (n + 1), (n + 2)$, and “$(n + \ell)$” for every integer $\ell \geq 3$.

We refer to a multicategory enriched in $A$ also as a (symmetric) 1-theory enriched in $A$. We refer to a commutative algebra in $A$ also as a (symmetric) 0-theory (“enriched”) in $A$.

The case where $A$ is the Cartesian symmetric monoidal category $\text{Gpd}$ of groupoids, of $n$-theories, will play important roles, so we let $\text{Gpd}$ be the default place where an $n$-theory is to be enriched, and consider an $n$-theory enriched in groupoids as an unenriched $n$-theory. An $n$-theory in the narrower sense will mean an “unenriched” $n$-theory.
An \( n \)-theory in the broader sense will mean an enriched (or unenriched) \( n \)-theory. Later, enrichment of an \( n \)-theory will be considered in a more general place than a symmetric monoidal category.

Specification of the forms of data will occupy the rest of this section.

1.2. Objects of a higher theory. The form of datum (0) for Definition 1.5 is as follows.

(0) A collection \( \text{Ob} \mathcal{U} \), whose member will be called an object of \( \mathcal{U} \).

1.3. Multimaps in a higher theory. The form of datum (1) for Definition 1.5 is as follows.

(1) Suppose given

\((0')\) an elemental \([1]\)-family \( I = (S, \ast) \) of finite sets, and a \([1]\)-nerve \( (\pi : S \to \ast) \) in \( \text{Fin} \) connecting \( I \), whole of which is determined by a free choice of \( S \),

\((0'')\) an \( S \)-family \( u_0 = (u_{0s})_{s \in S} \) of objects of \( \mathcal{U} \), and an object \( u_1 \).

Then a collection \( \text{Mul}^{[0]}_{\mathcal{U}}(u_0; u_1) \) or \( \text{Mul}^{[0]}_{\mathcal{U}}[u] \) for short, whose member will be called an \( (S \text{-ary}) (1\text{-})\text{multimap} \ u_0 \to u_1 \) in \( \mathcal{U} \).

Observe that a datum of this form extends as follows. Suppose given

- \( \psi : S \to T \) in \( \text{Fin} \),
- \( u_0 \) as above,
- a \( T \)-family \( u_1 \) of objects of \( \mathcal{U} \).

Then we let \( \text{Mul}^{[0]}_{\mathcal{U}}[u] \) denote the collection of all \( T \)-families \( v = (v_t)_{t \in T} \), where \( v_t \) is a multimap \( u_0|_{t} \to u_1|_{t} \) in \( \mathcal{U} \), where the source here is the restriction of the \( S \)-family \( u_0 \) to \( \psi^{-1}t \subseteq S \), so \( v_t \) is \( \psi^{-1}t \)-ary.

1.4. 2-multimaps in an \( n \)-theory, in the case \( n \geq 3 \).

1.4.0. The form of datum (2) for Definition 1.5 is as follows.

(2) Suppose given

\((1')\) an elemental \([1]\)-family \( I^1 = (I^1_0, \{1\}) \) in \( \text{Ord} \), and an \([1]\)-nerve \( (\pi : I^1_0 \to \{1\}) \),

\((0')\) an elemental \([1]^2\)-family \( I^0 = (I^0_s)_{s \in [1]^2} \) in \( \text{Fin} \), and a \( I^0 \)-nerve \( \phi^0 \)

connecting \( I^0 \), namely, \( \phi^0 = (\phi^0_{i_0})_{i_0 \in I^1_0} \), where \( \phi^0_{i_0} : I^0_{i_0-1} \to I^0_{i_0} \),

\((0'')\) an \( I^0 \)-family \( u^0 \) in \( \text{Ob} \mathcal{U} \), namely, \( u^0 = (u^0_{i_0})_{i_0 \in I^1_0} \), where \( u^0_{i_0} \) is an \( I^0 \)-family in \( \text{Ob} \mathcal{U} \).

\((1'')\) * a \( \phi^0 \)-nerve \( u^0_0 \) of multimaps in \( \mathcal{U} \), connecting \( u^0 \), which by definition means that \( u^0_0 = (u^0_{i_0})_{i_0 \in I^1_0} \), where \( u^0_{i_0} \in \text{Mul}^{[0]}_{\mathcal{U}}(u^0_{i_0-1}; u^0_{i_0}) \),

Then a collection \( \text{Mul}^{[0]}_{\mathcal{U}}[u^0](u^0_0; u^0_1) \) or \( \text{Mul}^{[0]}_{\mathcal{U}}[u] \) for short, whose member will be called a \( 2\text{-multimap} \ u^0_0 \to u^0_1 \) in \( \mathcal{U} \).

1.4.1. We can extend a datum of this form from above for a similar input datum with the requirement that the \([1]^2\)-family \( I^0 \) be elemental discarded. The idea is to treat such an input datum as an \( I^0_{[\pi(1)]} \)-family of elemental data, to obtain an \( I^0_{[\pi(1)]} \)-family of outputs.

Thus, for input data similar to \((1')\) through \((1'')\) above with \( I^0 \) non-elemental, we let \( \text{Mul}^{[0]}_{\mathcal{U}}[u] \) denote the collection whose member is an \( I^0_{[\pi(1)]} \)-family \( (v_s)_{s \in I^0_{[\pi(1)]}} \) of 2-multimaps in \( \mathcal{U} \), where \( v_s \in \text{Mul}^{[0]}_{\mathcal{U}}[u^0_s](u^0_{i_0}; u^0_{i_0+1}) \), where

- \( u^0_{i_0} : (u^0_{i_0-1})_{i_0 \in I^0_{[\pi(1)]}} \) is the restriction of \( u^0_{i_0} \) to \( I^0_{s} := (\phi^0_{i_0})^{-1} s \subseteq I^0_{i_0} \), where \( \phi^0_{i_0} := \phi^0_{i_0} \cdots \phi^0_{i_0+1} \).
1.4.2. We can extend the datum (2) further by discarding the requirement that I^1 be elemental, in the similar way as above.

To do this, suppose given

- a map \( \psi: I^1_0 \to I^1_1 \) in Ord,
- not necessarily elemental data of the form (0') and (0'') of (2),
- a \( \phi^0 \)-nerve \( u^0 \) and a \( \psi_! \phi^0 \)-nerve \( u^1 \) respectively, of multimaps in \( U \) (which we mean to be interpreted following the previous step).

Then we let \( \text{Mul}^0_U[u] \) denote the collection whose member is an \( I^1_1 \)-family \( v = (u_i)_{i \in I^1_1} \) of 2-multimaps in \( U \), where \( u_i \in \text{Mul}^0_U[u^0_i]) \), where

- \( \pi_i \) denotes the unique map \( I^1_0 \to \), so \( \psi = \sum \pi_i \) (where \( \sum \pi_i \) denotes the functor which takes the disjoint union equipped with the lexicographical order),
- \( u^0_i \) denotes the restriction of \( u^0 \) to \( I^1_0 \subset I^1_1 \),
- \( u^1_i \) denotes the restriction of \( u^1 \) to \( I^1_0 \).

so \( u^1_i \) is a \( \phi^0 \)-nerve-connecting \( u^0_i \), where \( \phi^0 \) denotes the restriction of \( \phi \) to \( I^1_0 \).

1.5. \( k \)-multimaps in a higher theory.

1.5.0. The form of datum \( (k) \) for \( 3 \leq k \leq n - 1 \) for Definition 1.5.0 is specified inductively as follows.

- (k) Suppose given
  - \( (k' - 1) \) an elemental \([1]\)-family \( I^k-1 = (I^k_0, [1]) \) in Ord, and an \([1]\)-nerve \( \pi: I^k_0 \to [1] \),
  - \( (k' - 2) \) an elemental \([1]^{-1}\)-family \( I^k-2 = (I^k_1, [i]) \) in Ord, and an \( I^k_1 \)-nerve \( \phi^k-2 = (\phi^k-2_i)_{i \in I^k_1} \) connecting \( I^k-2 \),
  - \( (k' - 3) \) through \( (k' - 3') \) of \( (k - 1), \)
  - \( (k' - 2) \) a \( (k' - 2) \)-family \( u^k-2 \) of \( I^k_1 \) of \( (k - 2) \)-multimaps in \( U \), where
    - \( I^k_1 \)-family \( u^k-2 \) is in fact a \( \phi^k-2 \)- \( \phi^k-3 \)-nerve (see \( (k - 1') \)) below of \( \phi^k-3 \)-multimaps in \( U \) (where \( \phi : I^k_1 \to \phi^k-3 \),
  - \( (k' - 1) \) * a \( \phi^k-2 \)-nerve \( u^k-1 \) of \( (k - 1) \)-multimaps connecting \( u^k-2 \) in \( U \), which by definition means that \( u^k-1 \) consists of \( u^k-1 \in \text{Mul}^0_U^{\leq k-2} u^k-2 \) (see below), where \( u^k-2 \) consists of
    - \( u^k-4 \) \( u^k-2 \) \( u^k-2 \) \( u^k-2 \),

    where \( u^k-2 \) denotes the restriction of \( u^k-2 \) to \( [i - 1, i] \subset I^k_0 \),
    - \( u^k-1 \) \( u^k-1 \) \( u^k-1 \) \( u^k-1 \),

Then a collection \( \text{Mul}_{\mathcal{U}}^\infty(u^k-2)(u^k-1) \) or \( \text{Mul}_{\mathcal{U}}^\infty[u] \) for short, whose member will be called a \( k \)-multimap \( u^k-1 \to u^k-1 \) in \( U \).
Definition 1.6. We refer to a datum of the form \((I; \pi, \phi)\), where \(I := (I^\nu)_{0 \leq \nu < k - 1}\), \(\phi := (\phi^\nu)_{0 \leq \nu < k - 2}\), specified by \((k - 1')\) through \((0')\) above, as the \textbf{arity of a} \(k\)-\textbf{multimap} in a symmetric higher theory.

We refer to a datum of the form \(u := (u^\nu)_{0 \leq \nu < k - 1}\) specified by \((0')\) through \((k - 1')\) above (by induction in \(k\)), as the \textbf{type of a} \(k\)-\textbf{multimap} in \(U\), of \textbf{arity} \(I(1; \nu, \phi)\).

Remark 1.7. Even though we have not yet specified the form of the rest of data for \(U\), note that the notion of the type of a \(k\)-multimap “in \(U\)” makes sense as soon as data of the forms \((0)\) through \((k - 1)\) are given “for \(U\)”.

A datum of the form \((k)\) above extends for a similar input datum with the elementality requirements discarded. This can be done by induction, starting from the elemental case above, as follows.

1.5.1. Fix an integer \(\nu\) such that \(1 \leq \nu \leq k - 1\), and suppose as an inductive hypothesis, that we have extended the datum \((k)\) for input data similar to \((k - 1')\) through \((k - 1'')\) above, where the families \(I^0\) through \(I^{\nu - 2}\) are allowed to be non-elemental. Then we extend the datum \((k)\) for input data with the families up to \(I^{\nu - 1}\) allowed to be non-elemental, as follows.

Suppose given data similar to \((k - 1')\) through \((k - 1'')\) above, where the families \(I^0\) through \(I^{\nu - 1}\) are allowed to be non-elemental (which we mean to be interpreted following the previous inductive step). Then we let \(\text{Mul}_{\nu}^U[u]\) denote the collection whose member is an \(I^{\nu - 1}_{\pi[i]}\)-family \((v_i)_{i \in I_{\nu[i]}^{\nu - 1}}\), where \(\pi^\nu\) denotes the unique map \(I^\nu \to \{1\}\), and \(v_i \in \text{Mul}_{\nu}^U[u_i]\), where \(u_i\) consists of

\[u^{\nu - 4}_i, (\phi^\nu, u^{\nu - 3}_{-1}), u^{\nu - 2}_i := (u^\nu_i)_{i \geq \nu - 2},\]

where if \(\nu \leq k - 2\),

- \(u^{\nu - 2, \leq k - 3}_i = (u^\nu_i)_{\nu - 2 \leq \kappa \leq k - 3}\) are as already defined by the previous step of the induction on \(k\) (see the case \(\nu = k - 1\) below for \(u^{\nu - 2}_{-1}\) and \(u^{\nu - 1}_{-1}\) and the next point for \(u^{\nu - 2}_{\leq k - 3}(i)\)),

and if \(\nu = k - 1\),

- \(u^{\nu - 2}_{-1}\) denotes the restriction of \(u^{\nu - 2}\) to \([(\pi^\nu, \phi^{\nu - 1})^{-1}i] \in I^{\nu - 1}\),

- \(u^{\nu - 1}_{-1} := (u^{\nu - 1}_j)_{j \in I^{\nu - 1}_j}\), where \(u^{\nu - 1}_j\) denotes the \((\phi^\nu_{-1})_{-1}\)-nerve connecting \((\phi^{\nu - 1}_{-1})_{-1}\), obtained by restricting \(u^{\nu - 1}\) to \(I^{\nu - 1}_j := (\phi^\nu_{-1})_{-1} \subseteq I^{\nu - 1}\), where

- \(\phi^\nu_{-1}\) denotes the \(I^\nu_{-1}\)-nerve in Ord connecting the \textit{elemental} \([I^\nu_{-1}]\)-family \(I^{\nu - 1}_j\), obtained by restricting \(\phi^\nu\),

- \(\phi^{\nu - 2}_{-1}\) is the \(I^{\nu - 1}_{-1}\)-nerve obtained by restricting \(\phi^{\nu - 2}\), connecting the \([I^{\nu - 1}_{-1}]\)-family \(I^{\nu - 2}_{-1}\) obtained by restricting \(I^{\nu - 2}\),

so

\[\left((\phi^{\nu - 1}_{-1})_{-1}\right)_j, \phi^{\nu - 2}_{-1} = (\phi^{\nu - 1}_{-1}, \phi^{\nu - 2})_{I^{\nu - 1}_j},\]

\[\left((\phi^{\nu - 1}_{j})_{-1}\right)_j, (u^{\nu - 2}_{-1})_j = (\phi^{\nu - 1}_{j}, u^{\nu - 2})_{I^{\nu - 1}_j},\]

and for any \(\nu, u^{\nu - 1}_j \in (u^{\nu - 1}_j)_{j \in I^{\nu - 1}_j}\), and \(u^{\nu - 1}_{-1} = (u^{\nu - 1}_i)\) are as already specified by the previous step of the induction on \(\nu\). Note (see below) that, by induction on \(k\), \(u^{\nu - 1}_j\) for \(j \in I^{\nu - 1}_j\) is an \(I^{\nu - 2}_{\pi[i]}\)-family \((u^{\nu - 1}_{ij})_{i \in I^{\nu - 2}_{\pi[i]}}\), where \(u^{\nu - 1}_{ij} \in \text{Mul}_{\nu - 1}^{I^{\nu - 2}_{\pi[i]}[u_{\leq k - 2}[i,j]]}\), similarly for \(u^{\nu - 1}_1\).
1.5.2. Finally, the datum \((k)\) extends for input data with \(I^{k-1}\) non-elemental, as follows. Suppose given
\[\begin{itemize}
\item a map \(\psi: I_0^{k-1} \rightarrow I_1^{k-1}\) in \(\text{Ord}\),
\item not necessarily elemental data of the form \((k-2')\) through \((k-2'')\) above,
\item a \(\phi^{k-2}\)-nerve \(u_0^{k-1}\) connecting \(u^{k-2}\), and a \(\psi|\phi^{k-2}\)-nerve \(u_1^{k-1}\) connecting \(\psi|u^{k-2}\) respectively, of \((k-1)\)-multimaps in \(\mathcal{U}\).
\end{itemize}\]
Then we let \(\text{Mul}_U^n[u]\) denote the collection whose member is an \(I_1^{k-1}\)-family \(v = (v_i)_{i \in I_1^{k-1}}\) of \(k\)-multimaps in \(\mathcal{U}\), where \(v_i \in \text{Mul}^\pi[u_i]\), where
\[\begin{itemize}
\item \(\pi_i\) denotes the unique map \(I_0^{k-1} = \psi^{-1}i \rightarrow *\), so \(\psi = \sum_i I_1^{k-1} \pi_i\),
\item \(u_i\) consists of
\[u_{<k-4}, (\phi^{<2} \rightarrow [\psi|\pi_i]_{i-1})_{\pi_i} u^{<k-3}, u^{<k-2}_{i}, u^{<k-1}_{i}, \]
where
\[-u^{<k-4}_{i} \text{ denotes the restriction of } u^{<k-2} \text{ to } [I_0^{k-1}] \subset [I_0^{k-1}],
\[-u^{<k-1}_{i} = (u_j^{<k-1}_{i})_{j \in [1]}, \text{ where } u^{<k-1}_{i} \text{ denotes the restriction of } u_{0}^{<k-1}_{i} \text{ to } [I_0^{k-1}], \text{ and } u^{<k-1}_{i} = (u^{<k-1}_{i})_{i} \text{ where } u^{<k-1}_{i} \text{ denotes the restriction of } \phi^{k-2}_{i} \text{ to } [I_0^{k-1}].\]
\end{itemize}\]
so \(u^{<k-1}_{i}\) is a \((\phi^{k-2}_{i})\)-nerve connecting \(u^{<k-2}_{i}\), where \(\phi^{k-2}_{i}\) denotes the restriction of \(\phi^{k-2}\) to \(I_0^{k-1}\).

1.5.3. This allows us to go to the next inductive step.

1.6. The object “formed by \(n\)-multimaps in an \(n\)-theory”. The form of datum \((n)\) for Definition \ref{def:n-multimaps} is as follows.
\[(n) \text{ Suppose given the type } u \text{ of an } (n)\text{-multimap in } \mathcal{U} \text{ of arity given as } (I; \pi, \phi). \]
Namely, suppose given a set of data similar to \((k-1)\) through \((k-1')\) of “\((k)\)” in Section \ref{sec:composition} but with \(k\) substituted by \(n\) (where \(I^{k-2}\) should be a family in Fin if \(n = 2\)), so these will be \((n-1)\) through \((n-1')\) here. Then an object \(\text{Mul}_U^n[u^{<n-2}](u_0^{n-1}; u_1^{n-1})\) of \(\mathcal{A}\), or \(\text{Mul}_U^n[u]\) for short. In the case where \(\mathcal{A}\) is Gpd or some other category so that \(\text{Mul}_U^n[u]\) can have its objects, then those objects will be called \(n\)-multimaps \(u_{0}^{n-1} \rightarrow u_1^{n-1}\) in \(\mathcal{U}\). For a general \(\mathcal{A}\), we shall call \(\text{Mul}_U^n[u]\) the object “of \(n\)-multimaps”.

A datum of this form extends for a non-elemental input datum just as the datum “\((k)\)” did in Section \ref{sec:composition}.

To see this, for the purpose of induction starting from the elemental case above, fix an integer \(\nu\) such that \(1 \leq \nu \leq n-1\), and suppose given data similar to \((n-1')\) through \((n-1)\) above, where the families \(I^0\) through \(I^{n-1}\) are allowed to be non-elemental. Then we define \(\text{Mul}_U^n[u] := \bigotimes_{i \in I_1^{n-1}} \text{Mul}_U^n[u_i]\), which makes sense by induction on \(\nu\).

Suppose given next, instead of \(\pi: I_0^{n-1} \rightarrow \{1\}\) above, a map \(\psi: I_0^{n-1} \rightarrow I_1^{n-1}\) in \(\text{Ord}\), and suppose \(u_i^{n-1}\) is now an \(\psi|n^{-2}\)-nerve of \((n-1)\)-multimaps connecting \(\psi|u^{n-2}\). Then we let \(\pi_i\) for \(i \in I_1^{n-1}\) denote the unique map \(\psi^{-1}i \rightarrow *\) (so \(\psi = \sum_i \pi_i\)), and define \(\text{Mul}_U^n[u] := \bigotimes_{i \in I_1^{n-1}} \text{Mul}^{\pi_i}[u_i]\).

1.7. Composition of \(n\)-multimaps. The form of datum \((n+1)\) for Definition \ref{def:n-multimaps} is as follows.
\[(n+1) \text{ Suppose given the arity } (I; \pi, \phi) \text{ of an } (n+1)\text{-multimap in a symmetric higher theory, namely}
\[-(k-1') \text{ and } (k-2') \text{ of “}(k)\text{” in Section } 1.5 \text{ but with } k \text{ substituted by } n+1, \text{ so these will be } (n') \text{ and } (n-1') \text{ here.}
\[-(n-2') \text{ through } (0') \text{ of } (n) \text{ in Section } 1.6 \text{ as well as}
\]
Definition 1.8. We refer to a datum of the form $(n_0)$ through $(n + 1)$ above, as the \textbf{type of a $\phi^{n-1}$-nerve of n-multimaps} in $\mathcal{U}$.

A datum of the form $(n + 1)$ above extends for a non-elemental input datum as follows.

To begin with, fix an integer $\nu$ such that $1 \leq \nu \leq n - 1$, and suppose given data similar to $(n')$ through $(n - 1')$ above, where the families $I^0$ through $I^{n-1}$ are allowed to be non-elemental. Then we define $m^I_1(\pi): \text{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u] \to \text{Mul}_{\mathcal{U}}^{\phi^{n-1}}[\pi u]$ as the monoidal product over $i \in I_{[\nu+1]}$ of the maps $m^I_1(\pi): \text{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u_i] \to \text{Mul}^{\phi^{n-1}}[\pi u_i]$, which makes sense by induction on $\nu$.

Next, suppose given data similar to $(n')$ through $(n - 1')$ above, where the families $I^0$ through $I^{n-1}$ are allowed to be non-elemental. Then we define $m^I_2(\pi): \text{Mul}_{\mathcal{U}}^{\phi^{n-1}}[u] \to \text{Mul}_{\mathcal{U}}^{\phi^{n-1}}[\pi u]$ as the monoidal product over $i \in I_{[\nu+1]}$ of the maps $m_1(\pi_i): \text{Mul}^{\phi_{n-1}}[u_i] \to \text{Mul}^{\phi_{n-1}}[\pi u_i]$, where $u_i$ consists of

\[ u^{n-3}, \ (\phi^{n-1}_{\text{def}}[\nu i]), u^{n-2}, u^{n-1}_i, \]

so $u_i$ consists of

\[ u^{n-3}, \ ([\nu n-1]_{\text{def}}), u^{n-2}, ([\nu n-1])_i. \]

1.8. \textbf{The associativity isomorphism for the composition.} The form of datum $(n+2)$ for Definition 1.8 is as follows.

$(n+2)$ Suppose given the arity $(I; \pi, \phi)$ of an $(n+2)$-multimap in a symmetric higher theory, and the type $u$ of a $\phi^{n-1}$-nerve of n-multimaps in $\mathcal{U}$. Then a 2-isomorphism $m^I_2(\pi): \pi m^I_1(\phi^n) \to m^I_1(\pi \phi^n)$ in $\mathcal{A}$, where $m^I_1(\phi^n)$ denotes the $I^0_{n+1}$-nerve obtained by indexing with $i \in I^0_{n+1}$, the maps

\[ m_1(\phi^n_i): \text{Mul}^{\phi_{n-1},\phi^{n-1}}[\phi^{n-1}_{n-1}][u] \to \text{Mul}^{\phi_{n-1},\phi^{n-1}}[\phi^{n-1}_{n-1}][u]. \]

A datum of this form extends for a non-elemental input datum as follows.

To begin with, fix an integer $\nu$ such that $1 \leq \nu \leq n - 1$, and suppose given data similar to $(n + 1)$ through $(n - 1')$ above, where the families $I^0$ through $I^{n-1}$ are allowed to be non-elemental. Then we define $m^I_2(\pi): \pi m^I_1(\phi^n) \to m^I_1(\pi \phi^n)$ as the monoidal product over $i \in I_{[\nu+1]}$ of the 2-isomorphisms

\[ m^I_2(\pi): \pi m_1(\phi^n) \to m_1(\phi^n): \text{Mul}^{\phi^{n-1}}[u_i] \to \text{Mul}^{\phi^{n-1}}[\pi u_i], \]
which makes sense by induction on $\nu$.

Next, suppose given data similar to $(n+1')$ through $(n-1')$ above, where the families $I^0$ through $I^{n-1}$ are allowed to be non-elemental. Then we define $m^2_{I^0}(\pi) : \pi_1 m_1 (\phi^n) \simto m_1 (\pi_1 \phi^n)$ as the monoidal product over $i \in I^{n-1}_{[\pi_1 \phi^n]}(1)$ of the 2-isomorphisms

\[
m_2(\pi) : \pi_1 m_1 (\phi^n) \simto m_1 (\pi_1 \phi^n) : \text{Mul}^\phi_{\pi_1 \phi^n} [u_{\pi_1}] \to \text{Mul}^\phi_{\pi_1 \phi^n} [u_{\pi_1}].
\]

Next, suppose given data similar to $(n+1')$ through $(n-1')$ above, where the families $I^0$ through $I^n$ are allowed to be non-elemental. Then we write for $i \in I^{n}_{[\pi_1]}(1)$,

\[

\pi_i^n := \pi_1 (\phi^n_{i_1}) \quad (\text{so } \pi_i \phi^n = \sum \pi_i^n),
\]

and define $m_2(\pi) : \pi_1 m_1 (\phi^n) \simto m_1 (\pi_1 \phi^n)$ as the monoidal product over $i \in I^{n}_{[\pi_1]}(1)$ of the 2-isomorphisms

\[
m_2(\pi) : \pi_1 m_1 (\phi^n_{i_1}) \simto m_1 (\pi_1 \phi^n) : \text{Mul}^\phi_{\pi_1 \phi^n} [u_{\pi_1}] \to \text{Mul}^\phi_{\pi_1 \phi^n} [u_{\pi_1}].
\]

Finally, suppose given, instead of $\pi : I^{n+1} \to I^{n+1}$ in Ord. Then we let $\pi_i$ for $i \in I^{n+1}_0$ denote the unique map $\psi : I^{n+1} \to \pi$ (so $\pi = \sum_i \pi_i$), and define the isomorphism $m_2(\psi) : \psi \pi_1 m_1 (\phi^n) \simto m_1 (\psi \phi^n)$ of $I^{n+1}_1$-nerves in $\mathcal{A}$ as the family indexed by $i \in I^{n+1}_1$ of the isomorphisms

\[
m_2(\pi_i) : \pi_1 m_1 (\phi^n_{i_1}) \simto m_1 (\pi_1 \phi^n_{i_1}).
\]

1.9. Coherence for the associativity.

1.9.0. The form of datum $(n+\ell)$ for $\ell \geq 3$ for Definition [123] is specified inductively as follows.

$(n+\ell)$ Suppose given the arity $(I; \pi, \phi)$ of an $(n+\ell)$-multimap in a symmetric higher theory, and the type $u$ of a $\phi^{n+\ell-1}$-nerve of $n$-multimaps in $\mathcal{U}$. Then an $\ell$-isomorphism

\[
m_{\ell I}^U(\pi) : \pi_1 m_{\ell-1} (\phi^{n+\ell-2}) \simto m_{\ell-1} (\pi \phi^{n+\ell-2})
\]

in $\mathcal{A}$, where $m_{\ell-1} (\phi^{n+\ell-2})$ denotes the $I^{n+\ell-1}$-nerve of $(\ell-1)$-isomorphisms obtained by indexing with $i \in I^{n+\ell-1}_0$ the isomorphisms

\[
m_{\ell-1} (\phi^{n+\ell-2}) : m_{\ell-2} (\phi^{n+\ell-3}) \simto m_{\ell-2} (\phi^{n+\ell-3})
\]

of $(\ell-2)$-isomorphisms

\[
\pi_i^{n+\ell-3} m_{\ell-3} (\phi^{n+\ell-4}) \simto m_{\ell-3} (\phi^{n+\ell-4})
\]

(where $\pi^{n+\ell-3} := (\pi \phi^{n+\ell-2})$) in $\mathcal{A}$, or $\text{Mul}^{\phi^{n-1}} [u] \to \text{Mul}^{\phi^{n-1}} [u]$ if $\ell = 3$.

A datum of this form extends for a non-elemental input datum as follows.

The initial step is a similar induction as before. Fix an integer $\nu$ such that $1 \leq \nu \leq n + \ell - 1$, and suppose given data similar to $(n+\ell-1')$ through $(n-1')$ above, where the families $I^0$ through $I^{n-1}$ are allowed to be non-elemental. Then we define $m_{\ell I}^U(\pi) : \pi_1 m_{\ell-1} (\phi^{n+\ell-2}) \simto m_{\ell-1} (\pi \phi^{n+\ell-2})$ as

- if $\nu \leq n + 1$, the monoidal product over $I^{n-1}_{[\pi \phi^{n+\ell-2}]}(1)$,
- if $\nu \geq n + 2$, the $(n+\ell+1-\nu)$-isomorphism of $I^{n+\ell-1}_{[\pi \phi^{n+\ell-2}]}$-nerves of $(\nu-n-1)$-isomorphisms (or 1-morphisms if $\nu = n + 2$) in $\mathcal{A}$, given by the family indexed by $I^{n+\ell-1}_{[\pi \phi^{n+\ell-2}]}(1)$,

of $m_{\ell I}^U(\pi)$ defined for each $i \in I^{n-1}_{[\pi \phi^{n+\ell-2}]}(1)$ as an instance of the previous inductive step.

For instance, as the case $\nu = n + \ell - 1$, suppose given data similar to $(n+\ell-1')$ through $(n-1')$ above, where the families $I^0$ through $I^{n+\ell-2}$ are allowed to be non-elemental. Then we write for $i \in I^{n+\ell-2}_{[\pi \phi^{n+\ell-2}]}(1)$, $\pi_i^{n+\ell-2} \equiv \pi_i \phi^{n+\ell-2}$ (so
\[ \pi_i \phi^{n+\ell-2} = \sum_i \pi_i^{n+\ell-2}, \]
and define the isomorphism
\[ m_\ell(\pi): \pi m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\pi_i^{n+\ell-2}) \]
of isomorphisms
\[ (\pi_i^{n+\ell-2}, m_{\ell-2}(\phi^{n+\ell-3}) \xrightarrow{\sim} m_{\ell-2}(\pi_i^{n+\ell-2} \phi^{n+\ell-3}) \]
of \( I_{[n]}^{n+\ell-2} \)-nerves of \((\ell - 2)\)-isomorphisms (or 1-morphisms if \( \ell = 3 \)) in \( \mathcal{A} \), as given by the family indexed by \( i \in I_{[n]}^{n+\ell-2} \) of
\[ m_\ell(\pi): \pi m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\pi_i^{n+\ell-2}) :\]
\[ \pi_i^{n+\ell-2} m_{\ell-2}(\phi^{n+\ell-3}) \xrightarrow{\sim} m_{\ell-2}(\pi_i^{n+\ell-2} \phi^{n+\ell-3}). \]

Finally, suppose given, instead of \( \pi: I_0^{n+\ell-1} \rightarrow \{1\} \) above, a map \( \psi: I_0^{n+\ell-1} \rightarrow I_1^{n+\ell-1} \) in \( \text{Ord} \). Then we let \( \pi_i \) for \( i \in I_1^{n+\ell-1} \) denote the unique map \( \psi^{-1} i \rightarrow * \) (so \( \psi = \sum_i \pi_i \) ), and define the isomorphism
\[ m_\ell(\psi): \psi m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\psi \phi^{n+\ell-2}) \]
of \( I_1^{n+\ell-1} \)-nerves of \((\ell - 1)\)-isomorphisms in \( \mathcal{A} \), as given by the family indexed by \( i \in I_1^{n+\ell-1} \) of the isomorphisms
\[ m_\ell(\pi_i): \pi_i m_{\ell-1}(\phi^{n+\ell-2}) \xrightarrow{\sim} m_{\ell-1}(\psi \phi^{n+\ell-2}). \]

1.9.1. We can thus proceed to the next inductive step, and this completes Definition 1.9.0.

2. Simple variants and basic constructions

2.0. Introduction. In this section, we shall first discuss a planar variant of higher theories, which iteratively theorize associative algebra. In particular, we shall find the structure of a planar \((n-1)\)-theory formed by endomorphisms in a symmetric \( n \)-theory. This will lead to a discussion of other theorized structures similarly residing in a symmetric higher theory. We shall further discuss less coloured variants of higher theory, relation of higher theory to higher categorified structure, and a construction for a higher theory which generalizes the “delooping” construction for a symmetric monoidal category.

2.1. Planar theories.

2.1.0. The notion of symmetric multicategory had variants such as planar and braided. While a generalization of these will appear in Section 3, the notion of planar \( n \)-theory is particularly simple to describe, and turns out to be also fundamental, so we shall discuss it here.

The definition of a planar \( n \)-theory is the same as the definition of a symmetric \( n \)-theory except that one uses the category \( \text{Ord} \) instead of \( \text{Fin} \). Namely, \( I_0^0 \) (and \( S \)) appearing in the definition 1.3.5 of an \( n \)-theory should now be in \( \text{Ord} \), and everything else is as it makes sense under this modification.

In particular, one obtains a planar \( n \)-theory from a symmetric \( n \)-theory \( \mathcal{U} \) by restricting the data defining \( \mathcal{U} \), through the forgetful functor \( \text{Ord} \rightarrow \text{Fin} \).

2.1.1. The notion of planar \( n \)-theory is fundamental for the following reason. Given a symmetric \( n \)-theory \( \mathcal{U} \), and its object \( x \), the structure of a planar \((n-1)\)-theory underlies the structure formed by endomorphisms (i.e., unary endomultimaps) of \( x \).

The idea is that if one intends to take as the part \((0')\) of an input datum for \( \mathcal{U} \), the constant elemental \( I_0^0 \)-nerve \(* \rightarrow \cdots \rightarrow * \), and as the part \((0'')\), the constant family at \( x \), then the rest of the required input data is of the same form as the form for an input datum for a planar \((n-1)\)-theory.
Thus, a planar \((n-1)\)-theory \(V = \text{Map}_U(x,x)\) is obtained as follows. Suppose inductively in \(k\), that an input for the datum \((k)\) for the planar \((n-1)\)-theory \(V\) is given by \(J^\nu, \psi^\nu, \nu^\nu\) for all integers \(\nu\) in the suitable range. Then one obtains an input for the datum \((k+1)\) for \(U\) by letting

- \(I^\nu := J^{\nu-1}, \phi^\nu := \psi^{\nu-1}\) for \(\nu \geq 1\),
- \(I^0\) and \(\phi^0\) constant as above, as well as \(u^0\) constant at \(x\),
- \(u^\nu := \nu^{\nu-1}\) for \(\nu \geq 1\),

so we use the output for this by \(U\) as the datum \((k)\) for \(V\) for the original input.

For example, the collection of the objects of \(V\) is \(\text{Mul}_U^\pi(x;x)\), where \(\pi\) is the identity map of \(I^\nu_0 = \{1\}\).

### 2.2. More related higher theorizations

In Section 2.1 we have found the structure of a planar \((n-1)\)-theory within the structure of every \(n\)-theory. The case \(n = 1\) of this is the associative algebra of endomorphisms within a multicategory, and by no accident, a planar \(n\)-theory is an \(n\)-th theorization of an associative algebra.

Since we can also find other structures within the structure of a multicategory, we can find higher theorized forms of them within the structure of an \(n\)-theory.

For example, if we focus on, instead of endomorphisms on a selected object, all unary multimaps between arbitrary objects within an \(n\)-theory, then we find that they naturally form a structure which is an \((n-1)\)-th theorization of the structure of a category. We have seen examples of theorized categories in Section 0.5. We have also noted there that theorized category was a ‘more coloured’ version of planar multicategory. Higher theorizations of category relates to planar higher theories in a similar manner.

For another example, if we fix one object of an \(n\)-theory to look at, but allow all endomultimaps of arbitrary arities (and arbitrary higher multimaps between them), then we find the structure of an \((n-1)\)-th theorization of an uncoloured operad.

The theorizations can have colours at ‘shallower’ levels, and these will be precisely defined as \((n-1)\)-tuply coloured \(n\)-theories in Section 2.3.

### 2.3. Restricting strata for colours

#### 2.3.0. The datum of a multicategory, or a coloured operad, enriched in groupoids, say, consists of collection of objects or “colours”, and operations or “multimaps” which compose. In an \(n\)-theory, only the multimaps of dimension \(n\) are required to compose, so only these are really considered as operations, while the collections of lower dimensional multimaps are then considered as forming strata of colours whose role is to specify the types of the operations in the top dimension.

As we could consider uncoloured operad, we sometimes want to consider uncoloured, and only partially coloured, higher theories. Specifically, we would like to consider situations in which the data below some dimension are all fixed to be ‘trivial’. We actually do this by simulating such a situation, instead of defining what we mean by the “trivial” data.

**Definition 2.0.** Let \(n \geq 2\) be an integer. Then a (symmetric) uncoloured \(n\)-theory \(U\) enriched in a symmetric monoidal category \(A\), consists of data of the forms specified below as \((n)\), \((n+1)\) and \((\infty)\).

- \((n)\) (The object “formed by objects”.) Suppose given the arity \((I; \pi, \phi)\) of an \(n\)-multimap in a symmetric higher theory, namely, \((n-1)\) through \((\nu)\) of \((n)\) in Section 1.6. Then an object \(\text{Ob}^nU\) of \(A\), to be called the object of objects of \(U\) of the specified arity.
This extends for a non-elemental input datum just as the datum \((n)\) of Section 1.2 did, and we use the resulting extended datum in the specification of the next datum form.

\((n + 1)\) \((Composition of objects.)\) Suppose given the arity \((I; \pi, \phi)\) of an \((n + 1)\)-multimap in a symmetric higher theory. Then a map \(m_I(\pi): \text{Ob}^{\phi_{n+1}} \to \text{Ob}^{\phi_{n}} \cdot \text{Ob}^{\phi_{m-1}} \cdot \text{A}\), where the source here is the object \(\bigotimes_{i \in I_0} \text{Ob}^{\phi_{m-1}} \cdot \text{A}\). The map \(m_I(\pi)\) will be called the \textbf{composition} operation for objects of \(\mathcal{U}\).

\((\infty)\) A datum of coherent associativity for the composition operations, corresponding to that for an \(n\)-theory described in Sections 1.8 and 1.9.

This completes Definition 2.0.

**Definition 2.1.** Let \(n \geq 2\) be an integer. We say that an \(n\)-theory as defined in Definition 1.5 as \textbf{\(n\)-tuply coloured}.

Let \(m\) be an integer such that \(1 \leq m \leq n - 1\). Then a \((symmetric)\) \textbf{\(m\)-tuply coloured \(n\)-theory} \(\mathcal{U}\) enriched in a symmetric monoidal category \(\mathcal{A}\), consists of data of the forms specified below as \((n - m)\), “\((n)\)” for every integer \(k\) such that \(n - m + 1 \leq k \leq n - 1\), \((n), (n + 1)\) and \((\infty)\).

\((n - m)\) \((Object.)\) Suppose given the arity \((I; \pi, \phi)\) of an \((n - m)\)-multimap in a symmetric higher theory (or a finite set \(\mathcal{S}\) with unique map \(\pi: \mathcal{S} \to \ast\), if \(n - m = 1\)). Then a collection \(\text{Ob}^{\phi} \mathcal{U}\), whose member will be called an \textbf{object} of \(\mathcal{U}\) of the specified arity.

This extends for a non-elemental input datum just as the datum \((k)\) of Section 1.2 did for \(k = n - m\), and we use the resulting extended datum in the specification of the next datum form.

\((k)\) \((\(k + n + m\)\)-multimap, inductively for \(n - m + 1 \leq k \leq n - 1\).\) Suppose given

- the arity \((I; \pi, \phi)\) of a \(k\)-multimap in a symmetric higher theory,
- if \(k - 3 \geq n - m\), then \((n - m')\) through \((k - 3')\) of \((k - 1)\) here,

and

\((k - 2)\) if \(k - 2 \geq n - m\), then an \(I_{k - 2}\)-family \(u^{k-2-n+m} = (u^{k-2-n+m}_{i \in I_{k-1}})\) of \((k - n + m - 2)\)-multimaps (or objects if \(k - n + m - 2 = 0\)) in \(\mathcal{U}\),

where, if \(k - n + m + 2 \geq 1\), then the \(I_{k - 2}\)-family \(u^{k-2-n+m}_{i \in I_{k-1}}\) is in fact a \(\phi^{k-2}_{i \in I_{k-1}} \phi^{k-3}\)-nerve (see \((k - 1')\) below) of \((k - n + m - 2)\)-multimaps in \(\mathcal{U}\), connecting \(\phi^{k-2}_{i \in I_{k-1}} \phi^{k-3}_{i \in I_{k-1}}\).

\((k - 1')\) if \(k - 1 = n - m\), then

* a \(\phi^{k-2}_{i \in I_{k-1}} \text{-nerve} u^{0}_{0}\) of objects in \(\mathcal{U}\), which \textit{by definition} means that 

\[ u^{0}_{0} = (u^{0}_{0})_{i \in I_{k-1}}, \quad \text{where} \quad u^{0}_{0} \in \text{Ob}^{\phi_{k-2}} \mathcal{U}, \]

* an object \(u^{0}_{0} \in \text{Ob}^{\phi_{k-2}} \mathcal{U};\)

if \(k - 1 \geq n - m + 1\), then

* a \(\phi^{k-2}_{i \in I_{k-1}} \text{-nerve} u^{k-1-n+m}_{0}\) of \((k - n + m - 1)\)-multimaps in \(\mathcal{U}\) connecting \(u^{k-1-n+m}_{0} = (u^{k-1-n+m}_{0})_{i \in I_{k-1}}\), where \(u^{k-1-n+m}_{0} \in \text{Mul}^{\phi_{k-2}}_{\text{u} \in u^{k-1-n+m-1}_{0}}\)

(see below),

* \(u^{k-1-n+m-1}_{0} \in \text{Mul}^{\phi_{k-2}}_{\text{u} \in u^{k-1-n+m-1}_{0}}\).

Then a collection \(\text{Mul}^{\phi_{k-2}}_{\text{u} \in u^{k-1-n+m-1}_{0}}\) or \(\text{Mul}^{\phi_{k-2}}_{\text{u} \in u^{k-1-n+m-1}_{0}}\) for short, whose member will be called a \((k - n + m)\)-\textbf{multimap} \(u^{0}_{0} \rightarrow u^{k-1-n+m-1}_{0}\) in \(\mathcal{U}\).
This extends for a non-elemental input datum just as the datum \((k)\) of Section 1.8 did, and we use the resulting extended datum in the specification of the next datum form.

\((n)\) (The object “formed by \(m\)-multimaps.”) Suppose given the arity \((I; \pi, \phi)\) of an \(n\)-multimap in a symmetric higher theory, and a set of data similar to \((0')\) through \((k-1')\) of \((k)\) above, but with \(k\) substituted by \(n\), so these will be \((0')\) through \((n-1')\) here. Then an object \(\text{Mul}^n_U[u_0^{m-2}][u_0^{m-1}; u_1^{m-1}]\) of \(\mathcal{A}\), or \(\text{Mul}^n_U[u]\) for short, to be called the object of \(m\)-multimaps \(u_0^{m-1} \rightarrow u_1^{m-1}\) in \(\mathcal{U}\).

This extends for a non-elemental input datum just as the datum \((n)\) of Section 1.8 did, and we use the resulting extended datum in the specification of the next datum form.

\((n+1)\) (Composition of \(m\)-multimaps.) Suppose given

- the arity \((I; \pi, \phi)\) of an \((n+1)\)-multimap in a symmetric higher theory,
- \((n-m')\) through \((n-2')\) of \((n)\) above,
- \((k-2')\) of \((k)\) above, but with \(k\) substituted by \(n+1\), so this will be \((n-1')\) here.

Then a map \(m_1U(\pi): \text{Mul}^{n-1}_U[u] \rightarrow \text{Mul}^{n-1}_U[u; \pi, u]\) in \(\mathcal{A}\), where the source here is the object \(\bigotimes_{i \in I^n} \text{Mul}^{n-1}_U[u_i|\pi|]\) of \(\phi^{n-1}\)-nerve of \(m\)-multimaps connecting \(u^{m-1}\) in \(\mathcal{U}\). The map \(m_1(\pi)\) will be called the composition operation for \(m\)-multimaps in \(\mathcal{U}\).

\((\infty)\) A datum of (coherent) associativity for the composition operations, corresponding to that for an \(n\)-theory described in Sections 1.8 and 1.9.

This completes Definition 2.7.

2.3.1. Any notion which makes sense for a general \(n\)-theory also makes sense for an \(n\)-theory with restricted strata of colours as above. Indeed, given the definition of a notion concerning an \(n\)-theory, one obtains the definition of the corresponding notion for a less coloured \(n\)-theory just by suppressing from the definition, every specification involving colours in the lower dimensions in the \(n\)-theory. For example, at a place where one needs to choose an object of an \(n\)-theory, one just does not need to make any choice with an \(m\)-tuply coloured \(n\)-theory if \(m \leq n-1\). Similarly, at a place where one needs to make some choice for every object of the \(n\)-theory, one just make one choice with a not fully coloured \(n\)-theory.

2.4. Forgetting categorifications to theorizations.

2.4.0. For an integer \(m \geq 0\), let us denote by \(\text{Cat}_m\) the Cartesian symmetric monoidal category of \(m\)-categories with a fix limit for the size, where we let a \(0\)-category mean a groupoid by convention.

While an \(n\)-theory enriched in \(\text{Cat}_m\) is an instance of an enriched \(n\)-theory, it is also unenriched in the sense that we can more generally consider \(m\)-categories enriched in a symmetric monoidal category. Thus it can be considered both as an “unenriched” instance of an \(m\)-categorified \(n\)-theory, and as an enriched “uncategorified” \(n\)-theory. Interpolating these two views, it can also be considered as an \(\ell\)-categorified \(n\)-theory enriched in \((m-\ell)\)-categories, for every integer \(\ell\) such that \(1 \leq \ell \leq m-1\).

2.4.1. Since \((n+1)\)-theory is a theorization of \(n\)-theory, there are in particular, \(n\)-theories enriched in \(\text{Cat}_{m+1}\) among \((n+1)\)-theories enriched in \(\text{Cat}_m\). Indeed, a op-lax \(n\)-theory enriched in \(\text{Cat}_{m+1}\) can be characterized among \((n+1)\)-theories enriched in \(\text{Cat}_m\), as one in which every profunctor/distributor/bimodule (enriched in \(\text{Cat}_m\)) virtually giving the composition of \(n\)-multimaps, is corepresentable, and
an \( n \)-theory can be characterized among \( \text{op-lax} \ n \)-theories as one in which every associativity map is an isomorphism.

Given an \( n \)-theory \( \mathcal{U} \) enriched in \((m+1)\)-categories, let us denote by \( \Theta_n \mathcal{U} \), \((n+1)\)-theory enriched in \( m \)-categories obtained by replacing each functor of composition operation for \( n \)-multimaps in \( \mathcal{U} \), by the bimodule corepresented by it. We shall say that \( \Theta_n \mathcal{U} \) is \textbf{represented} by \( \mathcal{U} \).

**Definition 2.2.** Let \( n \geq 0 \), \( m \geq 2 \) be integers, and let \( \mathcal{U} \) be an \( n \)-theory enriched in \( m \)-categories. Given an integer \( \ell \) such that \( 0 \leq \ell \leq m \), we define an \((n+\ell)\)-theory \( \Theta_{n+\ell} \mathcal{U} \) enriched in \((m-\ell)\)-categories, by the inductive relations

\[
\Theta_{n+\ell} \mathcal{U} = \begin{cases} 
\mathcal{U} & \text{if } \ell = 0, \\
\Theta_{n+\ell-1} \Theta_n^{n+\ell-1} \mathcal{U} & \text{if } \ell \geq 1.
\end{cases}
\]

For example, we obtain from a symmetric monoidal \( n \)-category \( \mathcal{A} \) an uncategorified \( n \)-theory \( \Theta_n^{[1]} \mathcal{A} \). Even though this is an \( n \)-theory, it is true that this is not really a new mathematical object since it is essentially just a symmetric monoidal \( n \)-category \( \mathcal{A} \). An \( n \)-theory which \textit{fails} to be represented by an \((n-1)\)-theory arises, for example, through the “delooping”, as well as the “convolution”, constructions, which we shall discuss in Sections 2.5 and 4.4 respectively.

However, considering symmetric monoidal \( n \)-categories as \( n \)-theories means considering very different \textit{morphisms} between symmetric monoidal \( n \)-categories, since a functor of these \( n \)-theories turns out to be an \( n \)-lax version of a symmetric monoidal functor of the original symmetric monoidal \( n \)-categories. Here and everywhere, by “\( n \)-lax” we mean “relaxed \( n \) times”. The way in which the structure is relaxed each time is actually quite interesting here, and the author also finds the structure resulting from iteration of these processes of relaxation fascinating.

Even though \( n \)-lax symmetric monoidal functor may still not be a \textit{very} new notion, the construction above is certainly giving a new meaning to this notion, in a richer environment where many new and natural mathematical structures interact, as we shall show through this work.

In general, for \( m \)-categorified \( n \)-theories \( \mathcal{U}, \mathcal{V} \), a functor \( \Theta_{n+\ell} \mathcal{U} \rightarrow \Theta_{n+\ell} \mathcal{V} \) of uncategorified \((n+m)\)-theories is equivalent as a datum to an \( m \)-lax functor \( \mathcal{U} \rightarrow \mathcal{V} \), as follows similarly to Theorem 2.4.4 below.

2.4.2. There are of course, less coloured versions of all the above. In particular, in the situation of Definition 2.2 if \( \mathcal{U} \) is \( k \)-tuply coloured, where \( 0 \leq k \leq n-1 \), then \( \Theta_{n+\ell} \mathcal{U} \) is obtained as \((k+\ell)\)-tuply coloured.

2.5. Delooping a higher theory.

2.5.0. There is a construction, which we shall call the \textit{delooping}, of a symmetric \( n \)-tuply coloured \((n+1)\)-theory from a symmetric \( n \)-theory. We shall describe this construction, and then discuss its relation to the “categorical” delooping.

An \((n+1)\)-theory which is obtained through this construction normally fails to be representable by an \( n \)-theory. Delooped theories will be conveniently used throughout this work as the targets of (possibly “coloured”) functors of higher theories.

2.5.1. The delooping construction relies on the following construction.

Suppose first that the arity of a \( 2 \)-multimap is specified as in (1′) and (0′) of (2) in Section 1.3 Then since our notation 1.2 chooses an embedding \( I_0^1 \rightarrow [I_0^1] \), it makes sense to take the coproduct \( \coprod_{I_0^1} I_0^0 \). Then note that a \( \phi^0 \)-nerve \( v_0^0 \) as in (2) of Definition 2.1 in the case \( m = n = 1 \), of objects of an \((n-1)\)-tuply coloured
n-theory, can be considered as a \((\coprod I^n_0)\)-family, while an \((I^n_0\)-ary\) object \(u^n_0\) is a \((\coprod I^n_1)\)-family, where \(I^n_1 := \{1\} \).

Suppose next given the arity \((I^n_1, \pi, \phi)\) of a 3-multimap in a symmetric higher theory. Then for every \(i \in I^n_2\) and \(j \in I^n_1\), we have a map \(\coprod (\phi^n_{i,j})^{-1} (\phi^n_{i,j-1}) I^0 \to I^n_{j-1}(I^0)\) whose component for \(k \in (\phi^n_{i,j})^{-1} j\) is the composite

\[
\phi^n_{i,j}(k) \cdot \phi^n_{i,j-1}(k+1) : I^n_{j-1}(I^0) \to I^n_{j-1}(I^0)
\]

Taking the coproduct of these over \(j\), we obtain a map \(\coprod I^n_{i-1} (\phi^n_{i,j-1}) I^0 \to I^n_i (\phi^n_{i,j+1}) I^0\), which together form an \(I^n_2\)-family.

Let us denote this nerve by \(f_1, \phi^n_0\), and the \(|I^n_2|\)-family of finite sets connected by it by \(f_1, I^0\). Then the datum \(u^n_0\) of \((1^n)\) of (3) for Definition 2.1 in the case \(m = n - 1\), can be considered as a \((f_1, I^0)_i\)-family of objects in \(U\), so \(u^n_0\) can be considered as a \((f_1, I^0)\)-family.

2.5.2. Suppose now given an \(n\)-theory \(V\). Then we wish to construct a new \(n\)-tuply coloured \((n + 1)\)-theory \(U = BV\) by precomposing the above constructions to the data for \(V\).

The construction of \(U\) is as follows.

1. Given a finite set \(S\), we let \(\text{Ob}^S U = \text{Ob} V\), where \(\pi\) denotes the unique map \(S \to \ast\).

2. Given the arity of a 2-multimap as in the preliminary construction above, if \(n = 0\), then we let \(m^n_{I^0}(\pi) : \text{Ob}\mathcal{V}_n \to \text{Ob}\mathcal{V}_n\) be the multiplication map \((\text{Ob}\mathcal{V}_n)^{\otimes \coprod I^n_0} \to \text{Ob}\mathcal{V}_n\). If \(n \geq 1\), and further given a \(\phi^n_0\)-nerve \(u^n_0\) of objects of \(\mathcal{U}\) and an \(I^n_0\)-ary object \(u^n_0\) as in (2) of Definition 2.1 in the case \(m = n - 1\), then we let \(\text{Mul}\mathcal{I}^n_0[u^n_0] = \text{Mul}\mathcal{V}_n[u^n_0]\), where \(\pi\) denotes the unique map \(\coprod I^n_0 \to \ast\), and \(u^n_0\) is considered as a \((f_1, I^0)\)-family of objects in \(V\), by the preliminary construction above.

Next, if \(n \geq 2\), suppose given an input datum for (3) for Definition 2.1 in the case \(m = n - 1\). Then we can construct a set of data of the form required of an input to (2) in Section 1.4 for \(V\), as follows. If we define

\[
J^n_0 := I^n_0, \quad J^n := \int J^n_1, \quad \psi^n := \int \phi^n, \quad \psi^n := (\pi : J^n_0 \to \{1\}),
\]

then \(J, \psi, u\) give the required form of datum. Using this, we let \(\text{Mul}\mathcal{I}^n_0[u^n_0] = \text{Mul}\mathcal{V}_n[u^n_0]\).

If \(n \geq 1\), we let \(m^n_{I^0}(\pi) = m^n_{V}(\pi)\) for \(\nu = 1\) or \(2\), in the similar manner, where the form of \(u\) is different in the case \(n = 1\), and we are not given \(u\) in the case \(n = 0\).

For every \(k \geq 4\), the datum \((k)\) for \(U\) is constructed in the similar manner from the datum \((k - 1)\) for \(V\).

2.5.3. It is clear that if \(U\) is an \(m\)-tuply coloured \(n\)-theory, then its deloop \(BU\) is obtained as an \(m\)-tuply coloured \((n + 1)\)-theory.

2.5.4. Our delooping construction for higher theories relates to the categorical delooping to be recalled now. The categorical delooping construction associates to a monoid \(A\) (or more generally, a monoidal \(n\)-category), a category (or \((n + 1)\)-category in the general case) \(BA\) with a chosen “base” object, in which

- all objects are equivalent,
- the endomorphism monoid (or monoidal \(n\)-category) of the base object is given an equivalence with \(A\).
Note that, if $\mathcal{A}$ is a symmetric monoidal $n$-category, then $B\mathcal{A}$ is canonically a symmetric monoidal $(n+1)$-category (with unit the base object) since the functor $B$ preserves direct products.

Let us describe one relation which will be convenient for us.

Specifically, let $n \geq 0$ be an integer, and $\mathcal{A}$ be a symmetric monoidal $n$-category. Then for an integer $m$ such that $0 \leq m \leq n$, we would like to relate the $(m+1)$-theory $\mathbb{B}\Theta_m^0\mathcal{A}$ to the $(m+1)$-theory $\Theta_m^{n+1}B\mathcal{A}$, both enriched in $(n-m)$-categories.

An obvious issue for this is that $\Theta_m^0\mathcal{A}$ is fully coloured whereas $\mathbb{B}\Theta_m^0\mathcal{A}$ is only $m$-tuply coloured. However, if we restrict the data for $\Theta_m^{n+1}B\mathcal{A}$ so the only object we consider is the base object of $B\mathcal{A}$, then the rest of the data for $\Theta_m^{n+1}B\mathcal{A}$ is of the form of datum for an $m$-tuply coloured $(m+1)$-theory enriched in $(n-m)$-categories.

**Proposition 2.3.** Let $n \geq 0$ be an integer, and $\mathcal{A}$ be a symmetric monoidal $n$-category. Then for every integer $m$ such that $0 \leq m \leq n$, $\mathbb{B}\Theta_m^0\mathcal{A}$ is equivalent to the $(n-m)$-categorified $m$-tuply coloured $(m+1)$-theory obtained as above from $\Theta_m^{n+1}B\mathcal{A}$ by restricting objects to just the base object of $B\mathcal{A}$.

**Proof.** The case $m = 0$ is immediate from the constructions, and the general case follows by induction, from Lemma 2.4 below. ∎

The lemma is as follows, and follows immediately from the definitions.

**Lemma 2.4.** For an arbitrary times categorified $n$-theory $\mathcal{U}$, we have

$$\mathbb{B}\Theta_n^0\mathcal{U} \simeq \Theta_{n+1}^0\mathcal{U}.$$ 

2.6. Theorization and lax functors.

2.6.0. The obvious notion of functor of $n$-theories has a reasonable lax version. Let us start with recording the definition of a functor.

**Definition 2.5.** Let $n \geq 2$ be an integer, and let $\mathcal{U}$ and $\mathcal{V}$ be $n$-theories enriched in a symmetric monoidal category $\mathcal{A}$. Then a **functor** $F : \mathcal{U} \to \mathcal{V}$ of $n$-theories consists of data of the forms specified below as (0), “(k)” for every integer $k$ such that $1 \leq k \leq n-1$, (n), (n + 1), (n + 2), and “(n + $\ell$)” for every integer $\ell \geq 3$.

Similarly to the data for an $n$-theory, data for a functor will be associated to input data satisfying the same elementality requirements as before. As before, each datum form specified needs to be extended for a non-elemental input datum before the next datum form is specified. The way how we do this is more or less the same as before, and we shall do this implicitly.

The forms of data are as follows.

(0) **(Action on objects.)** For every object $u$ of $\mathcal{U}$, an object $Fu$ of $\mathcal{V}$.

(k) **(Action on $k$-multimaps, inductively for $1 \leq k \leq n-1$.)** Suppose given the type $u$ of a $k$-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$. Then for every $k$-multimap $x \in \text{Mul}_k^{I}(u)$, a $k$-multimap $Fx \in \text{Mul}_k^{I}(Fu)$.

(n) **(Action on $n$-multimaps.)** Suppose given the type $u$ of an $n$-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$. Then a map $\tilde{m}_k^I(\pi) : \text{Mul}_k^{I}[u] \to \text{Mul}_k^{I}[Fu]$ in $\mathcal{A}$.

(n + 1) **(The isomorphism of compatibility with the composition.)** Suppose given the arity $(I; \pi, \phi)$ of an $(n+1)$-multimap in a symmetric higher theory, the type $u$ of a $\phi^{n-1}$-nerve of $n$-multimaps in $\mathcal{U}$.

Then a 2-isomorphism $\tilde{m}_k^I(\pi): \tilde{m}_k^I(\pi_0) \circ \tilde{m}_I^I(\pi) \cong \tilde{m}_k^I(\pi) \circ \tilde{m}_k^I(\phi^{n-1})$. 

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in $\mathcal{A}$, filling the square

$$
\begin{array}{ccc}
\text{Mul}^{\phi^{-1}}_U[u] & \xrightarrow{m^U_0(\pi)} & \text{Mul}^{\phi^{-1}}_U[\pi u] \\
\downarrow \phi^U_0 & & \downarrow \phi^U_0 \\
\text{Mul}^{\phi^{-1}}_V[Fu] & \xrightarrow{m^V_0(\pi)} & \text{Mul}^{\phi^{-1}}_V[\pi Fu],
\end{array}
$$

where $\phi^U_0$ denotes the monoidal product over $i \in I^n_0$ of the maps

$$
\phi^U_0(\phi^{-1}) : \text{Mul}^{\phi^{-1}}_U[u_i] \longrightarrow \text{Mul}^{\phi^{-1}}_U[\pi Fu].
$$

$(n+2)$ (The isomorphism of coherence for the compatibility with the composition.) Suppose given

- the arity $(I; \pi, \phi)$ of an $(n+2)$-multimap in a symmetric higher theory,
- the type $u$ of a $\phi^{n-1}$-nerve of $n$-multimaps in $\mathcal{U}$.

Then an 3-isomorphism

$$
\tilde{m}^F_2(\pi) : m^F_1(\pi; \phi^n) \circ m^U_2(\pi) \cong m^V_2(\pi) \circ \pi \tilde{m}^F_1(\phi^n)
$$

in $\mathcal{A}$, filling the square

$$
\begin{array}{ccc}
\text{Mul}^{\phi^{-1}}_U[u] & \xrightarrow{m^U_0(\pi)} & \text{Mul}^{\phi^{-1}}_U[\pi u] \\
\downarrow \phi^U_0 & & \downarrow \phi^U_0 \\
\text{Mul}^{\phi^{-1}}_V[Fu] & \xrightarrow{m^V_0(\pi)} & \text{Mul}^{\phi^{-1}}_V[\pi Fu],
\end{array}
$$

where $\phi^U_0$ denotes the $(I^{n+1})^{op}$-nerve of $\phi^n$-nerve of $n$-multimaps in $\mathcal{U}$.

$(n+\ell)$ (Higher coherence, inductively for $\ell \geq 3$.) Suppose given

- the arity $(I; \pi, \phi)$ of an $(n+\ell)$-multimap in a symmetric higher theory,
- the type $u$ of a $\phi^{n-1}$-nerve of $n$-multimaps in $\mathcal{U}$.

Then an $(\ell+1)$-isomorphism

$$
\tilde{m}^F_\ell(\pi) : \tilde{m}^F_{\ell-1}(\pi; \phi^{n+\ell-2}) \circ m^U_\ell(\pi) \cong m^V_\ell(\pi) \circ \pi \tilde{m}^F_{\ell-1}(\phi^{n+\ell-2})
$$

in $\mathcal{A}$, filling the square

$$
\begin{array}{ccc}
\text{Mul}^{\phi^{-1}}_U[u] & \xrightarrow{m^U_0(\pi)} & \text{Mul}^{\phi^{-1}}_U[\pi u] \\
\downarrow \phi^U_0 & & \downarrow \phi^U_0 \\
\text{Mul}^{\phi^{-1}}_V[Fu] & \xrightarrow{m^V_0(\pi)} & \text{Mul}^{\phi^{-1}}_V[\pi Fu],
\end{array}
$$

where $\phi^U_0$ denotes the $(I^{n+\ell})^{op}$-nerve of $\phi^n$-nerve of $n$-multimaps in $\mathcal{U}$.

$(n+\ell)$ (Higher coherence, inductively for $\ell \geq 3$.) Suppose given

- the arity $(I; \pi, \phi)$ of an $(n+\ell)$-multimap in a symmetric higher theory,
- the type $u$ of a $\phi^{n-1}$-nerve of $n$-multimaps in $\mathcal{U}$.

Then an $(\ell+1)$-isomorphism

$$
\tilde{m}^F_\ell(\pi) : \tilde{m}^F_{\ell-1}(\pi; \phi^{n+\ell-2}) \circ m^U_\ell(\pi) \cong m^V_\ell(\pi) \circ \pi \tilde{m}^F_{\ell-1}(\phi^{n+\ell-2})
$$

in $\mathcal{A}$, filling the square

$$
\begin{array}{ccc}
\text{Mul}^{\phi^{-1}}_U[u] & \xrightarrow{m^U_0(\pi)} & \text{Mul}^{\phi^{-1}}_U[\pi u] \\
\downarrow \phi^U_0 & & \downarrow \phi^U_0 \\
\text{Mul}^{\phi^{-1}}_V[Fu] & \xrightarrow{m^V_0(\pi)} & \text{Mul}^{\phi^{-1}}_V[\pi Fu],
\end{array}
$$

where $\phi^U_0$ denotes the $(I^{n+\ell})^{op}$-nerve of $\phi^n$-nerve of $n$-multimaps in $\mathcal{U}$.
the \((\ell - 1)\)-isomorphisms
\[
m^V_{\ell - 1}(\phi^{n+\ell-2}) \circ m^H_{\ell - 1}(\phi^{n+\ell-2}) \circ \cdots \circ m^H_{-1}(\phi^{n+\ell-2}) \circ m^U_{\ell - 1}(\phi^{n+\ell-2}) \circ \cdots \circ m^U_{-1}(\phi^{n+\ell-2}) \circ m^F_{\ell - 3}(\pi^{n+\ell-3}) \circ \cdots \circ m^F_{-1}(\pi^{n+\ell-3}) \circ m^F_0(\phi^{n-1}) \circ m^F_1(\pi) \circ m^F_2(\pi) \circ m^F_3(\pi) \circ \cdots \circ m^F_{n+1}(\phi)
\]
in \(\mathcal{A}\).

2.6.1. Now a lax functor can be defined as follows.

**Definition 2.6.** Let \(n \geq 2\) be an integer, and let \(\mathcal{U}\) and \(\mathcal{V}\) be \(n\)-theories enriched in a symmetric monoidal 2-category \(\mathcal{A}\) (i.e., in the symmetric monoidal category underlying \(\mathcal{A}\)). Then a **lax functor** \(F: \mathcal{U} \to \mathcal{V}\) consists of data of the forms

- (0) through \((n)\) specified above for Definition 2.5 for the same value of \(n\),
- \((n + 1)\) and \((\infty)\) below.

\((n + 1)\) (The map of *compatibility* with the composition.) Suppose given
- the arity \((I: \pi, \phi)\) of an \((n + 1)\)-multimap in a symmetric higher theory,
- the type \(u\) of a \(\phi^{n-1}\)-nerve of \(n\)-multimaps in \(\mathcal{U}\).

Then a 2-map
\[
m^F_1(\pi): m^V_1(\pi) \circ m^H_1(\phi^{n-1}) \to m^F_2(\pi, \phi^{n-1}) \circ m^F_1(\pi)
\]
in \(\mathcal{A}\). See \((n + 1)\) for Definition 2.5 above.

\((\infty)\) A datum of coherence, similar to that for a functor.

This completes Definition 2.5.

2.6.2. The notion of lax functor can be used to define the notion of \("(n + 1)\)-tuple\) **coloured lax \(n\)-theory**, which will allow us to describe an \((n + 1)\)-theory along the line discussed in Section 0.3.4, as to be done in Proposition 2.13. In order to do this, let us first understand an \(n\)-theory as a functor of higher theories.

In order to describe the source of the functor, we need the following definitions. Recall that an \(n\)-theory \(\mathcal{U}\) consists by definition, of data \((k)\) for all integers \(k \geq 0\), of the forms specified in Section 1. Of these, the part \(k \leq n - 1\) does not involve the information of where the theory is enriched.

**Definition 2.7.** Let \(n \geq 0\) be an integer. Then we refer to the system consisting of data of the forms \((0)\) through \((n - 1)\) as specified for Definition 1.5 of an \(n\)-theory, as a system of **colours up to dimension** \(n - 1\) for a higher theory, or a system of colours for an *\(n\)-theory*.

In particular, for every \(n\)-theory \(\mathcal{U}\) and every integer \(m\) such that \(0 \leq m \leq n - 1\), we have a system of colours up to dimension \(m\) **underlying** \(\mathcal{U}\), consisting of the data \((0)\) through \((m)\) for \(\mathcal{U}\), so \(\mathcal{U}\) consists of this system of colours and a structure on it.

**Definition 2.8.** Let \(m \geq 0\) and \(n \geq m + 1\) be integers. Then for the system of colours up to dimension \(m\) consisting of data of the forms \((0)\) through \((m)\) as specified for Definition 1.5 of an \(n\)-theory, we refer to the rest of data for an \(n\)-theory, consisting of data of the forms \((k)\) for \(k \geq m + 1\), as the **structure of an \(n\)-theory** on the system of colours.

Now the following gives an interpretation of a higher theory as a functor of higher theories.
Example 2.9. Choose and fix a system of colours up to dimension \( n - 1 \), and denote by \( \mathcal{T} \) the terminal object among unenriched \( n \)-theories extending this system of lower colours. Explicitly, \( \mathcal{T} \) is such that every groupoid of \( n \)-multimaps in it is contractible.

Then the structure on the chosen system of colours, of an \( n \)-theory enriched in a symmetric monoidal category \( \mathcal{A} \), is equivalent as a datum to a lax functor \( \mathcal{T} \to \mathcal{B}^n \mathcal{A} \) of categorified \( n \)-theories. This can also be described as a functor \( \Theta_n \mathcal{T} \to \Theta_n \mathcal{B}^n \mathcal{A} \) of \((n + 1)\)-theories.

The equivalence of these two descriptions will be generalized by Theorem 2.14 below. Note that \( \Theta_n \mathcal{T} \) is terminal among unenriched \((n + 1)\)-theories extending our system of lower colours.

Remark 2.10. Lemma 2.4 implies an equivalence \( \Theta_n \mathcal{B} \mathcal{A} \simeq \mathcal{B}^n \Theta_0 \mathcal{A} \).

Let us say that an \( n \)-theory \( \mathcal{T} \) as in Example 2.9 is terminal on the chosen system of colours up to dimension \( n - 1 \). Considering \( \mathcal{T} \) as a coloured variant of the terminal unenriched uncoloured \( n \)-theory \( \mathcal{1}^n \mathcal{C} \) (where \( \mathcal{C} = \mathcal{E}_\infty \) denotes the commutative operad), one might say that an \( n \)-theory enriched in \( \mathcal{A} \) is a lax functor \( \mathcal{1}^n \mathcal{C} \to \mathcal{B}^n \mathcal{A} \) with strata of colours for an \( n \)-theory, and similarly, also a functor \( \mathcal{1}^{n+1} \mathcal{C} \to \mathcal{B}^n \Theta_0 \mathcal{A} \) with similar strata of colours.

Definition 2.11. Let \( n \geq 0 \) be an integer. Then, an \( n \)-theory enriched in a multicategory \( \mathcal{M} \), is a functor \( \mathcal{1}^{n+1} \mathcal{C} \to \mathcal{B}^n \mathcal{M} \) with strata of colours for an \( n \)-theory.

Concretely,
- the colours of the functor will be the colours of the \( n \)-theory, say \( \mathcal{U} \), enriched in \( \mathcal{M} \), defined by the coloured functor,
- \( n \)-multimaps in such \( \mathcal{U} \) are similar to those in the case enriched in a symmetric monoidal category, but they ‘form’ objects of \( \mathcal{M} \),
- the composition operations for \( n \)-multimaps in \( \mathcal{U} \) are multimaps in \( \mathcal{M} \).

Accordingly, the notion of lax functor generalizes in an obvious manner, to that between \( n \)-theories enriched in a multicategory \( \mathcal{M} \) enriched in categories. See Definition 2.6.

Definition 2.12. Let \( n \geq 0 \) be an integer, and \( \mathcal{M} \) be a multicategory enriched in categories. Then,
- a lax \( n \)-theory enriched in \( \mathcal{M} \) is a lax functor \( \mathcal{1}^{n+1} \mathcal{C} \to \mathcal{B}^n \mathcal{M} \) with strata of colours for an \( n \)-theory,
- an \((n + 1)\)-tuply coloured lax \( n \)-theory enriched in \( \mathcal{M} \) is a lax functor \( \mathcal{1}^{n+1} \mathcal{C} \to \mathcal{B}^n \mathcal{M} \) with strata of colours up to dimension \( n \), namely, a lax functor \( F: \mathcal{T} \to \mathcal{B}^n \mathcal{M} \), where \( \mathcal{T} \) is an unenriched \((n + 1)\)-theory, which is terminal on a system of colours for an \((n + 1)\)-theory.

We record Example 2.9 using the term just introduced.

Proposition 2.13. Let \( \mathcal{A} \) be a symmetric monoidal category. Then, for an integer \( n \geq 1 \), an \( n \)-theory enriched in \( \mathcal{A} \) is equivalent as a datum to an \((n + 1)\)-theory enriched in \( \mathcal{B} \mathcal{A} \).

2.6.3. The following is a key fact on the relation between theorization and categorification.

Theorem 2.14. Let \( n \geq 0 \) be an integer, and \( \mathcal{U} \) and \( \mathcal{V} \) be categorified \( n \)-theories. Then a functor \( \Theta_n \mathcal{U} \to \Theta_n \mathcal{V} \) of uncategorified \((n + 1)\)-theories, is equivalent as a datum to a lax functor \( \mathcal{U} \to \mathcal{V} \).
Conversely, suppose given a functor \( 3 \).

\[ \text{data} \] (which will turn out in the next step to have been actually redundant).

\[ V \] by the bimodules/distributors/profunctors represented by the \( m \), and similarly for \( n \) actions up to dimension \( n \)-theories) naturally formed respectively by the structures on these same data, of functors (of \( (n + 1) \)-theories), and of lax functors (of categorified \( n \)-theories), are equivalent.

1. The structure of a categorified \( n \)-theory is given on the categories of \( n \)-multimaps, by the (coherently) associative composition functors. A lax functor \( F : U \to V \) extending the chosen action, is then seen to be given by

- \( 0 \) the action of \( F \) on the categories of \( n \)-multimaps through functors specified by the datum \( m^F \), and
- \( 1 \) the (lax) compatibility specified by the data \( m^F \) and \( n^F \) for \( k \geq 3 \), of the mentioned action with the associative composition functors for \( n \)-multimaps in \( U \) and in \( V \).

2. Recall that we have obtained \( \Theta_n U \) by replacing the composition functors of \( U \) by the bimodules/distributors/profunctors represented by them, and similarly for \( V \).

Given a datum as \( 1 \) above for a lax functor \( F \), of an action on \( n \)-multimaps, a datum of compatibility as \( 1 \) above, of this action with the composition functors for \( n \)-multimaps, can equivalently be described as an action on the (associative) composition bimodules for \( n \)-multimaps (which were represented by the composition functors) in \( \Theta_n U \) and in \( \Theta_n V \). Note that these bimodules (in \( \Theta_n U \) or in \( \Theta_n V \)) are formed by \( (n + 1) \)-multimaps, and the (coherent) associativity of the composition (for \( n \)-multimaps) can equivalently be described as the (coherently associative) composition operations for \( (n + 1) \)-multimaps.

In this manner, we consider \( F \) as a structure between \( \Theta_n U \) and \( \Theta_n V \).

The difference of this structure between \( \Theta_n U \) and \( \Theta_n V \), from a functor \( G : \Theta_n U \to \Theta_n V \) of \( (n + 1) \)-theories, is that

- the action of \( G \) on \( n \)-multimaps does not (explicitly) include the datum of functoriality included in \( m^F \), and
- the action of \( G \) on \( (n + 1) \)-multimaps (given by \( m^G \)) and its compatibility (given by \( m^G \) for \( k \geq 2 \)) with the composition operations for \( (n + 1) \)-multimaps in \( \Theta_n U \) and in \( \Theta_n V \), do not explicitly include the data of compatibility with the bimodules structures, included in \( m^F \) and \( n^F \).

Never the less, we obtain a functor \( \Theta_n F : \Theta_n U \to \Theta_n V \) by forgetting these extra data (which will turn out in the next step to have been actually redundant).

3. Conversely, suppose given a functor \( G : \Theta_n U \to \Theta_n V \) which extends the chosen actions up to dimension \( n - 1 \). Then, from the discussions above, a lax functor \( H : U \to V \) extending the same lower actions, such that \( \Theta_n H = G \), is constructed if we equip the higher data for \( G \) with the following.

- A functoriality in the \( n \)-multimap \( x \in \text{Mul}_n[U] \), on the \( n \)-multimap \( Gx \in \text{Mul}_n[V][Hu] \).
- A compatibility of this functoriality with the data \( n^G \), \( i \geq 1 \), of the action of \( G \) (in a manner compatible with the (associative) composition operations for \( (n + 1) \)-multimaps in \( \Theta_n U \) and in \( \Theta_n V \)) on \( (n + 1) \)-multimaps.

In order to obtain a functoriality of the \( n \)-multimap \( Gx \), one notes that a map in the category \( \text{Mul}_n[U] \) is suitably ‘unary’ \( (n + 1) \)-multimaps in \( \Theta_n U \). The desired functoriality is given by the restriction of the associative action of \( G \) to unary \( (n + 1) \)-multimaps in \( \Theta_n U \).

A compatibility of this functoriality with the action of \( G \) means...
position structures of $\Theta$

functoriality of $G$

4.

We have thus constructed a desired lax functor $G$.

In order to complete the proof, we would like to verify in the case of $\Theta_n U$ and $\Theta_n V$, that the lax functor $G$ constructed from $G$ on the category of $n$-multimaps is as constructed above, we see that the compatibility of the action of $G$ on $(n+1)$-multimaps with the composition structures of $\Theta_n U$ and of $\Theta_n V$, also induces a desired datum.

4. We have thus constructed a desired lax functor $H : U \to V$ naturally from $G$.

In order to complete the proof, we would like to verify in the case $G = \Theta_n U$ for a lax functor $F : U \to V$, that the lax functor $\Theta_n U$ constructed from $G$ will be naturally equivalent to $F$. For this, it suffices to observe that the set of data which we have added to the datum of $G$ to construct $H$ in the step 3 is naturally equivalent to the set of data which was “forgotten” in the construction $F \mapsto \Theta_n U$ in the step 2 but this is clear.

Remark 2.15. We have thus described for the $(n+1)$-theory $W = \Theta_n V$, a functor $G : \Theta_n U \to W$ as a lax functor $U \to V$. Similar arguments lead to a description of $G$ for an arbitrary $(n+1)$-theory $W$, as a structure generalizing a lax functor $U \to V$. One might call this structure between $U$ and $V$ a functor $U \to V$. Thus, a functor $\Theta_n U \to W$ will be equivalent as a datum to a functor $U \to W$.

For example, an $n$-theory enriched in a multicategory $\mathcal{M}$ (Definition 2.11) is equivalent as a datum to a functor $1^n_{\text{Com}} \to \mathbb{B}^n \mathcal{M}$ with strata of colours up to dimension $n - 1$ for a higher theory.

Note that $n$-theories enriched in groupoids are also among $n$-theories enriched in categories since a groupoid can be considered as a category in which every morphism is invertible. This kind of unenriched categorified $n$-theories is special as a target of a functor in that there is no difference between an ordinary functor and a lax functor from an unenriched categorified $n$-theory to such a target.

Corollary 2.16. The functor $\Theta_n$ is fully faithful on $n$-theories enriched in groupoids.

3. Graded higher theories

3.0. Introduction. The purpose of this section is to discuss grading of higher theory by a higher theory (enriched in groupoids). It turns out that this notion naturally arises by considering theorization of algebra over a higher theory. We shall also see interrelations between the notions, of “graded” lower theory over a higher theory, and of iterated monoid, i.e., monoid (i.e., algebra in the category of
groupoids) over a monoid over ... over a monoid, over a higher theory enriched in groupoids. Some results obtained along the way will be basic.

3.1. Algebras over a higher theory.

3.1.0. We have sought for the notion of \( n \)-theory as an interesting \( n \)-th theorization of the notion of commutative algebra. As have been discussed in Section 0, we wished to get from this was for each \( n \)-theory to govern algebras over it, which naturally generalize \((n-1)\)-theories.

**Definition 3.0.** Let \( n \geq 2 \) be an integer, and let \( \mathcal{U} \) be an unenriched \( n \)-theory. Then a \( \mathcal{U} \)-algebra \( \mathcal{X} \) enriched in a multicategory \( \mathcal{M} \) consists of data of the forms specified below as \((0)\) and \((1)\) (or just \((0)\) if \( n = 2 \)), “\((k)\)” for every integer \( k \) such that \( 2 \leq k \leq n-2 \), \((n-1)\), \((n)\) and \((\infty)\).

Let us specify the forms of data for Definition.

**Remark 3.1.** As before, we implicitly extend each datum form specified below for a non-elemental input datum before proceeding to specifying the next datum form, in a more or less similar manner as before.

\((0)\) (Object.) For every object \( u \) of \( \mathcal{U} \), a collection \( \text{Ob}_u \mathcal{X} \), whose member will be called an object of \( \mathcal{X} \) of degree \( u \).

\((1)\) (Multimap, in the case \( n \geq 3 \).) Suppose given \((0')\) and \((0'')\) of \((1)\) in Section 1.3 and \((0')\) an \( S \)-family \( x_0 \) of objects of \( \mathcal{X} \) of degree \( u_0 \) (namely, \( x_0 \in \text{Ob}_{u_0} \mathcal{X} \) for every \( s \in S \), and an object \( x_1 \) of degree \( u_1 \).

Then for every multimap \( v \in \text{Mul}_{u_0}^v \), a collection \( \text{Mul}_{X,v}^X(x_0;x_1) \) or \( \text{Mul}_{X,v}^X[x] \) for short, whose member will be called an \((S\text{-arity})\) \((1)\)-multimap \( x_0 \rightarrow x_1 \) in \( \mathcal{X} \) of degree \( v \).

\((k)\) \((k\text{-multimap})\), inductively for \( 2 \leq k \leq n-2 \).) Suppose given \((1'), (0''), (1'')\), \((k-2)\) (or just \((1'')\), “\((0'')\)” if \( k = 2 \), and \((k-1)\) below:

(a) the type \( u \) of a \( k \)-multimap in \( \mathcal{U} \) of arity given as \((I;\pi,\phi)\).

\((0')\) an \( I \)-family \( x^0 \) of objects of \( \mathcal{X} \), of degree \( u^0 \), namely, \( x^0 = (x^0)^i \in [u^0] \), where \( x^0_i \) is an \( I^0 \)-family of objects of \( \mathcal{X} \), of degree \( u^0_i \),

(b) if \( k \geq 4 \), then \((1')\) through \((k-3)\) of \((k-1)\) here,

\((k-2)\) (in the case \( k \geq 3 \)) an \( I^k \)-family \( x^{k-2} = (x^{k-2})^i \in [u^{k-1}] \) of \((k-2)\)-multimaps in \( \mathcal{X} \), where \( x^{k-2} \) is an \( \phi^{k-2} \)-nerve of \((k-2)\)-multimaps in \( \mathcal{X} \), connecting \( \phi^{k-2}(\mathcal{X}) \) of degree \( u^{k-2}_i \).

\((k-1)\) a \( \phi^{k-2} \)-nerve \( x^{k-1}_0 \) of \((k-1)\)-multimaps connecting \( x^{k-2} \), of degree \( u^{k-1}_0 \) in \( \mathcal{X} \), namely, \( x^{k-1}_0 \in \text{Mul}_{v_0}^{\phi^{k-2}}[x^{k-2}] \), for every \( i \in [u^{k-1}] \),

\[ x^{k-1}_0 \in \text{Mul}_{v_0}^{\pi,v_1} \phi^{k-2} \pi[x^{k-2}]. \]

Then for every \( k \)-multimap \( v \in \text{Mul}_{u_0}^v \), a collection \( \text{Mul}_{X,v}^{X,v} [x^{k-2}] (x^{k-1}_0;x^{k-1}_1) \) or \( \text{Mul}_{X,v}^X[x] \) for short, whose member will be called a \((k\text{-multimap})\) \( x^{k-1}_0 \rightarrow x^{k-1}_1 \) in \( \mathcal{X} \) of degree \( v \).

**Definition 3.2.** We refer to a datum of the form \( x = (x^i)_{0 \leq i \leq k-1} \) specified by \((0')\) through \((k-1)\) above (by induction in \( k \)), as the type of a \( k \)-multimap in \( \mathcal{X} \) of arity \((I;\pi,\phi)\) and of degree \( u \).

**Remark 3.3.** Even though we have not yet specified the form of the rest of data for \( \mathcal{X} \), note that the notion of the type of a \( k \)-multimap “in \( \mathcal{X} \)” makes sense as soon as data of the forms \((0')\) through \((k-1)\) are given “for \( \mathcal{X} \)”.
(n − 1) (The object “formed by (n − 1)-multimaps”.) Suppose given the type u of an (n − 1)-multimap in \( \mathcal{U} \) of arity given as \((I; \pi, \phi)\), and the type x of an \((n-1)\)-multimap in \(X\) of the same arity of degree u, namely, a set of data similar to \((0^\circ)\) through \((k-1^\circ)\), of \("(k)\"\) above, but with k substituted by \(n-1\), so these will be \((0^\circ)\) through \((n-2^\circ)\) here. Then for every \((n-1)\)-multimap \(v \in \text{Mul}_U^n[u]\), an object \(\text{Mul}_X^\pi^n[x^{\leq n-3}][(x_0^{n-2}, x_1^{n-2})]\) of \(\mathcal{M}\), or \(\text{Mul}_X^\pi^n[x]\) for short. In the case where \(\mathcal{M}\) is \(\Theta_0\text{Gpd}\) or some other multicategory so that \(\text{Mul}_X^\pi^n[x]\) can have its objects, then those objects will be called \((n-1)\)-multimaps \(x_0^{n-2} \to x_1^{n-2} \in \mathcal{X}\) of degree \(v\). For a general \(\mathcal{M}\), we shall call \(\text{Mul}_X^\pi^n[x]\) the object “of \((n-1)\)-multimaps in \(X\) of degree \(v\)”.

(n) (Action of the \(n\)-multimaps of \(\mathcal{U}\).) Suppose given
- the type u of an \(n\)-multimap in \(\mathcal{U}\) of arity given as \((I; \pi, \phi)\),
- \((0^0)\) through \((n-3^\circ)\) of \((n-1)\) above,
- \((k-2^\circ)\) of \("(k)\"\) above, but with k substituted by \(n\), so this will be \((n-2^\circ)\) here.

Then a map
\[
\bar{m}_0^n(\pi): \text{Mul}_U^n[u] \to \text{Mul}_M^n\left(\text{Mul}_X^{\phi^{n-2}}_{X,u_0^{n-1}}[x]; \text{Mul}_X^{\pi^{n-2}}_{X,u_1^{n-1}}[\pi x]\right)
\]
of groupoids, where \(\text{Mul}_X^{\phi^{n-2}}_{X,u_0^{n-1}}[x]\) denotes the \((\coprod_{I_0^{n-1}} I^{n-2})\)-family \(\coprod_{I_1^{n-1}} \text{Mul}_X^{\phi^{n-2}}_{X,u_0^{n-1}}[x]\) of objects of \(\mathcal{M}\), and \(\pi x\) consists of \(x^{\leq n-3}\), \(\pi x^{n-2}\). For \(v\) in the source of this map, the multimap
\[
\bar{m}_0^n(\pi)(v): \text{Mul}_X^{\phi^{n-2}}_{u_0^{n-1}}[x] \to \text{Mul}_X^{\pi^{n-2}}_{u_1^{n-1}}[\pi x]
\]
in \(\mathcal{M}\) will be called the composition operation along \(v\) for \((n-1)\)-multimaps in \(X\).

**Definition 3.4.** We refer to a datum of the form \(x = (x^v)_{0 \leq v \leq n-2}\) specified by \((0^0)\) through \((n-2^\circ)\) above, as the **type of a \(\phi^{n-2}\)-nerve of \((n-1)\)-multimaps** in \(X\) of degree \(u \leq n-2\).

\((\infty)\) A datum of (coherent) associativity for the action of \(n\)-multimaps, similar to a datum for a functor, of compatibility of the action with the composition (Definition 2.3).

This completes Definition 3.0.

3.1.1. Similarly, one can define the notion of algebra over a less coloured \(n\)-theory. See Section 2.3.

**Example 3.5.** A \(1^n_{\text{Con}}\)-algebra enriched in a multicategory \(\mathcal{M}\) is equivalent as a datum to an \((n-1)\)-theory enriched in \(\mathcal{M}\) (Definition 2.11).

This can be generalized over an arbitrary \(n\)-theory \(\mathcal{U}\) after we theorize the notion of \(\mathcal{U}\)-algebra, which we shall do next.

3.2. **Iterated theorizations of algebra.**

3.2.0. We would like to theorize the notion of algebra. Let us first relax the notion.

**Definition 3.6.** Let \(n \geq 2\) be an integer, and let \(\mathcal{U}\) be an unenriched \(n\)-theory. Then a *lax \(\mathcal{U}\)-algebra* \(X\) enriched in a categorified multicategory \(\mathcal{M}\) consists of data of the forms \((0^0)\) through \((n)\), specified above for Definition 3.0 for the same value of “\(n\)”, and a datum of coherent lax associativity for the action of \(n\)-multimaps, similar to a datum for a lax functor, of coherent compatibility of the action with the composition (Definition 2.0).
This essentially contains at least the unenriched version of a virtualization of the notion of op-lax categorified $U$-algebra, namely, a *theoryization* of the notion of $U$-algebra. We shall write down an enriched version of the definition explicitly, in a form which will be convenient shortly. (The notion will be generalized in Section 4) A natural name for the kind of thing will turn out to be “graded $n$-theory”.

**Definition 3.7.** Let $n \geq 1$ be an integer, and let $U$ be an $n$-theory enriched in groupoids. Then a $U$-graded $n$-theory $\mathcal{X}$ enriched in a multicategory $\mathcal{M}$ consists of data of the forms specified below as (0) and (1) (or just (0) if $n=2$), “(k)” for every integer $k$ such that $2 \leq k \leq n-2$, $(n-1)$, $(n)$, $(n+1)$, and $(\infty)$.

The forms of data are as follows. Remark 3.1 applies here again.

(0) (Object.) For every object $u$ of $\mathcal{U}$, a collection $\text{Ob}_u \mathcal{X}$, whose member will be called an object of $\mathcal{X}$ of degree $u$.

(1) (Multimap, in the case $n \geq 2$.) Suppose given $(0^0)$ and $(0^1)$ of (1) in Section 1.3 and

(0$^0$) an $S$-family $x_0$ of objects of $\mathcal{X}$ of degree $u_0$, and an object $x_1$ of degree $u_1$.

Then for every multimap $v \in \text{Mul}^U_\pi[u]$, a collection $\text{Mul}^\mathcal{X}_{X,v}(x_0; x_1)$ or $\text{Mul}^\mathcal{X}_{v}(x)$ for short, whose member will be called an $(S\text{-ary})$ $(1)$-multimap $x_0 \to x_1$ in $\mathcal{X}$ of degree $v$.

(k) (k-multimap, inductively for $2 \leq k \leq n-1$.) Suppose given the type $u$ of a $k$-multimap in $\mathcal{U}$ of arity given as $(I; \pi, \phi)$, and the type $x$ of a $k$-multimap in $\mathcal{X}$ of the same arity of degree $u$, namely, $(0^0)$, $(0^1)$, $(k-2^0)$ (or just “(0$^0$)” if $k = 2$), and $(k-1^0)$ below:

(0$^0$) an $I^0$-family $x^0$ of objects of $\mathcal{X}$ of degree $u^0$, namely, $x^0_i \in \text{Ob}_{u^0} \mathcal{X}$ for every $i \in [I^0]$,

(b) if $k \geq 4$, then $(1^0)$ through $(k-3^0)$ of $(k-1)$ here,

(k-2$^0$) (in the case $k \geq 3$) an $I^{k-2}$-family $x^{k-2} = (x_i^{k-2})_{i \in [I^{k-1}_i]}$, where $x_i^{k-2}$ is a $\phi^{k-2}$-family of $(k-2)$-multimaps connecting $\phi^{k-2}$-$x^{k-3}$ in $\mathcal{X}$, of degree $u_i^{k-2}$,

(k-1$^0$) a $\phi^{k-1}$-family $x^{k-1}$ of $(k-1)$-multimaps connecting $\phi^{k-1}$-$x^{k-2}$ of degree $u_i^{k-1}$ in $\mathcal{X}$, and $x_i^{k-1} \in \text{Mul}^{\pi_i \phi^{k-2}}_i[x; x^{k-2}]$.

Then for every $k$-multimap $v \in \text{Mul}^U_{\pi}[u]$, a collection $\text{Mul}^\mathcal{X}_v[x; x^{k-2}; x^{k-1}]$ or $\text{Mul}^\mathcal{X}_v[x]$ for short, whose member will be called a $k$-multimap $x^{k-1} \to x^{k-1}$ in $\mathcal{X}$ of degree $v$.

(n) (Action of the $n$-multimaps of $\mathcal{U}$.) Suppose given the type $x$ of an $n$-multimap in $\mathcal{X}$ of arity and degree given respectively as $(I; \pi, \phi)$ and $u$. Then a functor (to the underlying category of $\mathcal{M}$)

$$M^\mathcal{X}_0(\pi)[x^{\leq n-2}; x^{n-1}]: \text{Mul}^U_{\pi}[u] \to \mathcal{M}$$

which will also be denoted by $M^\mathcal{X}_0(\pi)[x]$ for short. For $v$ in the source of this functor, we write

$$\text{Mul}^\mathcal{X}_v[x] := \text{Mul}^\mathcal{X}_v[x^{\leq n-2}; x^{n-1}]: = M^\mathcal{X}_0(\pi)[x](v).$$

In the case where $\mathcal{M}$ is $\Theta_0$Gpd or some other multicategory so that $\text{Mul}^\mathcal{X}_v[x]$ can have its objects, then those objects will be called $n$-multimaps $x^{n-1}_0 \to x^{n-1}_1$ in $\mathcal{X}$ of degree $v$. For a general $\mathcal{M}$, we shall call $\text{Mul}^\mathcal{X}_v[x]$ the object “of $n$-multimaps in $\mathcal{X}$ of degree $v$”.

(n+1) (Associativity map.) Suppose given

- the arity $(I; \pi, \phi)$ of an $(n+1)$-multimap in a symmetric higher theory,

- the type $u$ of a $\phi^{n-1}$-nerve of $n$-multimaps in $\mathcal{U}$,
– the type \( x \) of a \( \phi^{n-1} \)-nerve of \( n \)-multimaps in \( \mathcal{X} \) of degree \( u \leq n-1 \),
namely, \( (0^u) \) through \((k-2^u)\) of \((k)\) above, but with \( k \) substituted
by \( n+1 \), so this will be \((0^u)\) through \((n-1^u)\) here.

Then a multimap
\[
M^1(X)(\pi): M^0(\phi^{n-1})[x] \rightarrow M^0(\pi\phi^{n-1})[\pi x] \circ M^1(\pi)
\]
of functors \( \text{Mul}^{n-1}_U[u] \rightarrow \mathcal{M} \), where the source of this multimap is the
\( \coprod_{i \in I} I \rightarrow \mathcal{F} - \text{family} \coprod_{i \in I} \text{pr}_i^* M(\phi^{n-1})[x_i] \),
where \( \text{pr}_i \) denotes the projection \( \text{Mul}^{n-1}_U[u] \rightarrow \text{Mul}^{n-1}_U[u_i] \).
For \( v \) in the source of this functor, we write
\[
m^1_\pi(\pi)_v := M^1_X(\pi)(v): \text{Mul}^{\phi^{n-1}}_U[x] \rightarrow \text{Mul}^{\phi^{n-1}}_U[\pi x],
\]
where \( \pi_\pi := m^1_\pi(\pi)(v) \), and call it the composition operation along \( v \)
for \( n \)-multimaps.

\((\infty)\) A datum of coherence for the associativity, corresponding to that for a lax
\( \mathcal{U} \)-algebra (Definition 3.6).

This completes Definition 3.7.

3.2.1. Let us consider the unenriched case where \( \mathcal{M} = \Theta_0 \text{Gpd.} \).

Note that in this case, \( M_0(\pi)[x] \) above can be identified with the datum of the
canonical projection map
\[
\text{colim}_{v \in \text{Mul}_U[u]} \text{Mul}_U^\infty([x]) \rightarrow \text{Mul}_U^1[u]
\]
of groupoids. The groupoid \( \text{colim}_{\text{Mul}_U[u]} \text{Mul}_U^\infty([x]) \) of \( n \)-multimaps of arbitrary degrees,
will further be the groupoid of \( n \)-multimaps in a symmetric \( n \)-theory.

Indeed, the datum \( M_1 \) induces “composition” operations \( m^1_\pi(\pi)_v := \text{colim}_{v \in \text{Mul}_U[u]} \text{Mul}_U^\infty([x]) \)
on these groupoids, covering the composition operations \( m^1_\pi(\pi)_v \) in \( \mathcal{U} \). Writing down
the form of datum for the coherence for the associativity in \( \mathcal{X} \) (which is straightforward),
we obtain the case \( m = 0 \) of Proposition 3.8 below (hence our term for
the notion).

The construction simply remains valid in the unenriched (higher) categorified
case \( \mathcal{M} = \Theta_0 \text{Cat}_m \).

**Proposition 3.8.** Let \( n \geq 1 \) be an integer, and let \( \mathcal{U} \) be an \( n \)-theory enriched in
groupoids. Then for every integer \( m \geq 0 \), an unenriched \( m \)-categorified \( \mathcal{U} \)-graded \( n \)-theory
(i.e., \( \mathcal{U} \)-graded \( n \)-theory enriched in \( \Theta_0 \text{Cat}_m \)) is equivalent as a datum to an
unenriched \( m \)-categorified symmetric \( n \)-theory \( \mathcal{Y} \) equipped with a functor \( \mathcal{Y} \rightarrow \mathcal{U} \).

In order to see this more precisely, let us introduce the following terminology,
which will be justified shortly.

**Definition 3.9.** Let \( n \geq 1 \) be an integer, and \( \mathcal{U} \) be an (unenriched) \( n \)-theory.
Then, for an integer \( m \) such that \( 0 \leq m \leq n \), we refer to data of the forms \( (0) \)
through \((m-1)\) specified for Definition 3.7 as a system of colours up to dimension \( m-1 \)
for a \( \mathcal{U} \)-graded higher theory, or a system of colours for a \( \mathcal{U} \)-graded
\( m \)-theory.

Suppose given a system \( \mathcal{X} \) of colours up to dimension \( n-2 \) for a \( \mathcal{U} \)-graded
higher theory. Then we obtain a system \( \Delta_n \mathcal{X} \) (where \( \Delta: \mathcal{U} \rightarrow 1_{\text{Sym}} \)) of colours up
to dimension \( n-2 \) for a symmetric higher theory, by inductively defining as follows.

We first define \( \text{Ob} \Delta_n \mathcal{X} \) as the collection whose member is a pair \((u,x)\), where
\( u \in \text{Ob} \mathcal{U} \) and \( x \in \text{Ob}_u \mathcal{X} \). If \( n \geq 3 \), then inductively for an integer \( k \) such
that \( 1 \leq k \leq n-2 \), the type of a \( k \)-multimap in \( \mathcal{Y} \) will be an identical form of
datum as a pair \((u,x)\), where \( u \) is the type of a \( k \)-multimap in \( \mathcal{U} \), and \( x \) is the
type of a \( k \)-multimap in \( \mathcal{X} \) of degree \( u \). We then inductively define \( \text{Mul}^n_{\Delta_n \mathcal{X}}([u,x]) \)
as the collection whose member is a pair \((v, y)\) consisting of \(v \in \text{Mul}_U^*[u]\) and \(y \in \text{Mul}_V^*[x]\). Moreover, one can associate to every member \((v, y)\) the member \(v \in \text{Mul}_U^*[u]\).

Proposition 3.3 now follows since the category (or \((m+2)\)-category) of extensions of the datum \(\Delta_X\) to an unenriched \(m\)-categorified symmetric \(n\)-theory equipped with a functor to \(U\), gets equated by the described construction, to the category (or \((m+2)\)-category) of extensions of the datum \(X\) to an unenriched \(m\)-categorified \(U\)-graded \(n\)-theory.

**Example 3.10.** Let \(\text{Init}\) denote the initial uncoloured operad in groupoids.
- A category is equivalent as a datum to an \(\text{Init}\)-graded 1-theory.
- A planar multicategory is equivalent as a datum to an \(E_1\)-graded 1-theory.
- A braided multicategory (see Fiedorowicz [9]) is equivalent as a datum to an \(E_2\)-graded 1-theory.

**Example 3.11.** Since \(U\)-graded \(n\)-theory is a theorization of \(U\)-algebra, one obtains from a \(U\)-monoidal category \(X\), a \(U\)-graded \(n\)-theory by replacing the functors giving the composition operations by the bimodules/distributors/profunctors corepresented by them. We shall say that this \(n\)-theory is **represented** by \(X\), and shall denote it by \(\Theta_{n-1}X\), where the subscript comes from the fact that \(U\)-algebra is a generalization of \((n-1)\)-theory from the case \(U = \text{Id}_{\text{Com}}\).

**Example 3.12.** Recall that we called a plain \(n\)-theory, i.e., an \(n\)-theory which is not considered with any grading, also a symmetric \(n\)-theory. Every symmetric \(n\)-theory is canonically graded by the terminal unenriched uncoloured \(n\)-theory \(1^n_{\text{Com}}\), and there is no difference between a symmetric \(n\)-theory and a \(1^n_{\text{Com}}\)-graded \(n\)-theory.

3.2.2. For an unenriched \(U\)-graded \(n\)-theory \(X\), let us denote the symmetric \(n\)-theory underlying \(X\) (which maps to \(U\); see Proposition 3.8) by \(\Delta X\), where \(\Delta\) denotes the unique functor \(U \to 1^n_{\text{Com}}\). For example, for the terminal unenriched uncoloured \(U\)-graded \(n\)-theory \(1^n_U\), we have that the canonical projection functor \(\Delta: 1^n_U \to U\) is an equivalence.

Using this, we can obtain a compact reformulation of the notion of algebra, as will be given now.

Suppose given a system of colours up to dimension \(n-2\) for \(U\)-graded higher theory, and let \(T\) denote the terminal unenriched \(U\)-graded \(n\)-theory on this system of colours. Note that one can consider the structure of a \(U\)-algebra on this system of colours. Indeed, the structure of a \(U\)-algebra equipped with strata of colours up to dimension \(n-2\), or colours for a \(U\)-algebra.

For example, a \(U\)-monoid, i.e., a \(U\)-algebra enriched in groupoids, is naturally equivalent as a datum to a coloured functor \(U \to B^{n-1}\Theta_0\text{Gpd}\). In this sense, the symmetric \((n+1)\)-theory \(B^{n-1}\Theta_0\text{Gpd}\) classifies (uncoloured) monoids over unenriched symmetric \(n\)-theories, where the universal monoid is the uncoloured \(B^{n-1}\Theta_0\text{Gpd}\)-monoid \(\bigvee^{n-1}\) “classified” by the identity functor of \(B^{n-1}\Theta_0\text{Gpd}\).

**Proposition 3.13.** For the universal monoid \(U^{n-1}\), the projection functor of the \(B^{n-1}\Theta_0\text{Gpd}\)-graded \(n\)-theory \(\Theta_{n-1}U^{n-1}\) is equivalent to \(B^{n-1}\Theta_0\text{Gpd}\).

The proof is straightforward from the definitions.
3.2.3. Now the notion of algebra generalizes immediately as follows.

**Definition 3.14.** Let $n \geq 2$ be an integer, and $\mathcal{U}$ be an unenriched $n$-theory. Then a **$\mathcal{U}$-algebra in a symmetric $n$-theory** $\mathcal{V}$ is a functor $\mathcal{U} \to \mathcal{V}$ with strata of colours for a $\mathcal{U}$-algebra, namely, a functor $\Delta_1 \mathcal{T} \to \mathcal{V}$, where $\mathcal{T}$ is the terminal unenriched $\mathcal{U}$-graded $n$-theory on a system of colours up to dimension $n - 2$ for a $\mathcal{U}$-graded higher theory.

We generalize this as follows.

**Definition 3.15.** Let $n \geq 1$ be an integer, and $\mathcal{U}$ be an unenriched $n$-theory. Then an $n$-tuply coloured lax $\mathcal{U}$-algebra in a categorified symmetric $n$-theory $\mathcal{V}$, is a lax functor $\mathcal{U} \to \mathcal{V}$ with strata of colours up to dimension $n - 1$, namely, a lax functor $\Delta_1 \mathcal{T} \to \mathcal{V}$, where $\mathcal{T}$ is the terminal unenriched $\mathcal{U}$-graded $n$-theory on a system of colours up to dimension $n - 1$ for a $\mathcal{U}$-graded higher theory.

We obtain from Definition 3.7 that a $\mathcal{U}$-algebra enriched on a symmetric monoidal category $\mathcal{A}$ for a symmetric monoidal category $\mathcal{B}$, is a theory $\mathcal{V}$, is equivalent as a datum to a $\mathcal{U}$-graded $n$-theory on the chosen system of colours, of a $\mathcal{U}$-algebra enriched in a symmetric monoidal category $\mathcal{A}$, which follows from Lemma 2.4.

We obtain the following fundamental result.

**Theorem 3.17.** Let $n \geq 1$ be an integer, and let $\mathcal{U}$ be an $n$-theory enriched in groupoids. Then a $\mathcal{U}$-graded $n$-theory enriched in a multicategory $\mathcal{M}$ is equivalent as a datum to a $\Theta_0 \mathcal{U}$-algebra enriched in $\mathcal{M}$.

**Proof.** We shall prove the case where $\mathcal{M}$ is represented by a symmetric monoidal category $\mathcal{A}$. The general case follows from essentially the same (and simpler) argument, but one needs to use Remarks 3.16 and 2.15 of which we have omitted the details.

Let us first note that systems of colours for a $\mathcal{U}$-graded $n$-theory and for a $\Theta_0 \mathcal{U}$-algebra are identical forms of datum. Choose and fix a datum of this form. Then we would like to show that the categories of the structures on this same system of colours, of a $\mathcal{U}$-graded $n$-theory and of $\Theta_0 \mathcal{U}$-algebras, are equivalent. Let us prove this.

Let $\mathcal{T}$ denote the terminal unenriched $\mathcal{U}$-graded $n$-theory on the chosen system of colours. Then the structure of a $\mathcal{U}$-graded $n$-theory on those strata of colours could be described as a lax functor $\Delta_1 \mathcal{T} \to \mathcal{B}^n \mathcal{A}$, which Theorem 2.14 and Lemma 2.3 equates with the datum of a functor $\Theta_n \Delta_1 \mathcal{T} \to \mathcal{B}^n \Theta_0 \mathcal{A}$.

Let next $\mathcal{J}$ denote the terminal unenriched $\Theta_n \mathcal{U}$-graded $(n + 1)$-theory on the same system of colours. Then the structure of a $\Theta_n \mathcal{U}$-algebra on those strata of colours can be described as a functor $\Delta_1 \mathcal{J} \to \mathcal{B}^n \Theta_0 \mathcal{A}$, where $\Delta_1 : \Theta_n \mathcal{U} \to \mathcal{I}^n_{\text{Com}}$.

However, it is immediate that the strata of colours up to dimension $n - 1$, of $\Theta_n \Delta_1 \mathcal{T}$, and of $\Delta_1 \mathcal{J}$, are identical, and these two $(n + 1)$-theories on the same strata of colours are in fact equivalent. The result follows. □

**Definition 3.18.** Let $\mathcal{A}$ be a 0-theory in groupoids, i.e., a commutative monoid. Then an $\mathcal{A}$-graded 0-theory is a $\Theta_0 \mathcal{A}$-algebra.

In particular, an unenriched $\mathcal{A}$-graded 0-theory is equivalent as a datum to a commutative monoid $X$ equipped with a morphism $X \to \mathcal{A}$. 

**Remark 3.16.** Using the definition mentioned in Remark 2.15 of a “functor”, one can define an $n$-tuply coloured $\mathcal{U}$-algebra in a symmetric $(n + 1)$-theory $\mathcal{W}$ as a functor $\mathcal{U} \to \mathcal{W}$ with strata of colours up to dimension $n - 1$. In particular, a $\mathcal{U}$-graded $n$-theory enriched in a multicategory $\mathcal{M}$ will be equivalent as a datum to an $n$-tuply coloured $\mathcal{U}$-algebra in $\mathcal{B}^n \mathcal{M}$. Note the equivalence $\Theta_n \mathcal{B}^n \mathcal{A} = \mathcal{B}^n \Theta_0 \mathcal{A}$ for a symmetric monoidal category $\mathcal{A}$, which follows from Lemma 2.3.
3.2.4. The notion of \( U \)-graded \( n \)-theory can be theorized in almost the same way as how we have theorized the notion of algebra over an \( n \)-theory. We might call the resulting object a \( U \)-\textit{graded} \( (n+1) \)-theory.

However, this is not really a new notion as should be expected from Theorem 3.17.

\[ \text{Theorem 3.19.} \]
\[ \text{Let } n \geq 0 \text{ be an integer, and let } U \text{ be an } n \text{-theory enriched in groupoids. Then a } U \text{-graded } (n+1) \text{-theory is equivalent as a datum to a } \Theta_n U \text{-graded } (n+1) \text{-theory.} \]

The proof is also similar to the proof of Theorem 3.17. We leave the details to the interested reader.

\[ \text{Definition 3.20.} \]
\[ \text{Let } n \geq 0 \text{ be an integer, and let } U \text{ be an } n \text{-theory enriched in groupoids. For an integer } m \geq n+2, \text{ a } U \text{-graded } m \text{-theory is a } \Theta_m U \text{-graded } m \text{-theory, or equivalently, a } \Theta_m^{n+1}U \text{-algebra.} \]

Thus, for every \( U \), \( U \)-graded \( m \)-theories are iterated theorizations of \( U \)-algebra for \( m \geq n \).

\[ \text{Example 3.21.} \]
An \( E_1 \)-graded \( n \)-theory is equivalent as a datum to a planar \( n \)-theory defined in Section 2.1.

More generally, for every multicategory \( U \) enriched in groupoids, there is a similar description of a \( U \)-graded \( n \)-theory. Recall that the notion of planar \( n \)-theory was defined by replacing the category Fin in the definition of a symmetric \( n \)-theory, with the category Ord. For a similar description of the notion of \( U \)-graded \( n \)-theory, we would like to replace Fin by a category which is an analogue for \( U \), Ord.

In order to construct this category, note that the forgetful functor \( \Theta_0 \) from \( U \)-monoidal categories to \( U \)-graded multicategories, has a left adjoint \( L_U \), and that the unique functor \( \Delta: U \rightarrow \text{Com} \) of symmetric multicategories, where \( \text{Com} = E_\infty \) denotes the commutative operad (i.e., the terminal unenriched symmetric operad), induces a functor \( \Delta_*: L_U 1_U \rightarrow \Delta^* \text{Com} 1_{\text{Com}} = \Delta^* \text{Fin} \) of \( U \)-monoidal categories, where \( 1_U \) denotes the terminal unenriched \( U \)-graded multicategory.

For simplicity, suppose first that \( U \) is uncoloured. Then we obtain a description of a \( U \)-graded \( n \)-theory by replacing the category Fin in the definition of a symmetric \( n \)-theory, with \( \text{Com} := L_U 1_U \), where

- by a family \textit{indexed} by \( J \in \mathcal{L} \), we mean a family indexed by \( \Delta_* J \in \text{Fin} \), and
- for \( I \in \text{Ord} \), we say that a \( [I] \)-family \( J \) of objects of \( \mathcal{L} \) is \textit{elemental} if \( \Delta_* J \) is an elemental \( [I] \)-family in Fin.

If \( U \) is instead a coloured multicategory, then the description is similar except that we would need to have objects of the “theory” to have “degrees” in \( U \). We leave the details to the reader.

3.3. Iterated monoids.

3.3.0. Let \( U \) be an \( n \)-theory enriched in groupoids, and \( X \) be a \( U \)-monoid. Then since the structure of \( X \) is a generalization of the structure of an \( (n-1) \)-theory, it is natural to consider the notion of \( X \)-algebra if \( n \geq 2 \).

The notion can actually be reduced to the same notion in the special case where \( X \) is the unit (or terminal uncoloured) \( U \)-monoid. Indeed, if we denote by \( \Delta \) the unique functor \( U \rightarrow 1_{\text{Com}} \), then an \( X \)-algebra will simply be an algebra over the unit \( \Delta_0 \Theta_{n-1}X \)-monoid. (In particular, the dependence of the notion on \( U \) will be only through the presentation of the symmetric \( n \)-theory \( Y = \Delta_0 \Theta_{n-1}X \) as underlying \( \Theta_{n-1}X \).)
3.3.1. Let now \( \mathcal{X} \) be the unit \( \mathcal{U} \)-monoid (so \( \mathcal{U} = \Delta\Theta_{n-1}\mathcal{X} \)). Then the notion of \( \mathcal{X} \)-algebra will be such that the notion of \( \mathcal{U} \)-algebra theorizes it. In fact, the notion of \( \mathcal{U} \)-algebra ‘dethorizes’ more times, and the most dethorized notion will at the end be equivalent to the notion of algebra over the unit monoid over ... over the unit monoid over \( \mathcal{U} \), where we should have \( n - 1 \) unit monoids in the expression.

We can directly define all iteratively dethorized notions as follows.

**Definition 3.22.** Let \( n \geq 1 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids. Then for an integer \( m \) such that \( 0 \leq m \leq n - 1 \), a **\( \mathcal{U} \)-graded \( m \)-theory** enriched in a multicategory \( \mathcal{M} \), is a functor \( \mathcal{U} \to \Theta_{m+1}^n \mathcal{B}^m \mathcal{M} \) with strata of colours for a \( \mathcal{U} \)-graded \( m \)-theory.

To be explicit, in the case \( m = 0 \), there is no colours added. In the case \( m \geq 1 \), given a system of colours for a \( \mathcal{U} \)-graded \( m \)-theory \( \mathcal{X} \), the structure on it, of a \( \mathcal{U} \)-graded \( m \)-theory, is a functor \( \mathcal{T} \to \Theta_{m+1}^n \mathcal{B}^m \mathcal{M} \), where \( \mathcal{T} \) is the symmetric \( n \)-theory described as follows.

(0) \( \text{Ob } \mathcal{T} \) is the collection whose member is a pair \((u, x)\), where \( u \in \text{Ob } \mathcal{U} \) and \( x \in \text{Ob}_n \mathcal{X} \).

By induction, for \( k \) such that \( 1 \leq k \leq m \), the type of a \( k \)-multimap in \( \mathcal{T} \) of a given arity \( (I, \pi, \phi) \), is specified by

- the type \( u \) of a \( k \)-multimap in \( \mathcal{U} \) of arity \( (I, \pi, \phi) \),
- the type \( x \) of a \( k \)-multimap in \( \mathcal{X} \) of the same arity of degree \( u \).

(k) (Inductively for \( k \) such that \( 1 \leq k \leq m - 1 \).) Suppose given the arity \( (I, \pi, \phi) \) of a \( k \)-multimap, and the type \( (u, x) \) of a \( k \)-multimap in \( \mathcal{T} \) of arity \( (I, \pi, \phi) \). Then \( \text{Mul}_T^k[(u, x)] \) is the collection whose member is a pair \((v, y)\), where \( v \in \text{Mul}_{\mathcal{U}}^k[u] \) and \( y \in \text{Mul}_{\mathcal{X}}^k[v] \).

(m) Suppose given a datum similar to an input datum for “(k)” above, but with \( k \) substituted by \( m \). Then \( \text{Mul}_T^m[(u, x)] = \text{Mul}_{\mathcal{U}}^m[u] \).

(ℓ) (Inductively for \( \ell \) such that \( m + 1 \leq \ell \leq n \).) Suppose given the arity \( (I, \pi, \phi) \) of an \( \ell \)-multimap, and the type of an \( \ell \)-multimap in \( \mathcal{T} \) of arity \( (I, \pi, \phi) \), which by induction, will be specified by

- the type \( u \) of a \( \ell \)-multimap in \( \mathcal{U} \) of arity \( (I, \pi, \phi) \),
- the type \( x \) of a \( \phi^m-1 \)-nerve of \( m \)-multimaps in \( \mathcal{X} \) of degree \( u \geq m \).

Then \( \text{Mul}_T^\ell[(u, x)] = \text{Mul}_{\mathcal{U}}^\ell[u] \).

The composition is given by the composition in \( \mathcal{U} \).

Thus, the notion of \( \mathcal{U} \)-graded \((n - 1)\)-theory coincides with the notion of \( \mathcal{U} \)-algebra, and for all \( m \geq 0 \) (which may be \( \geq n \)), the notion of \( \mathcal{U} \)-graded \( m \)-theory is iteratively an \( m \)-th theorization of the notion of \( \mathcal{U} \)-graded 0-theory.

**Lemma 3.23.** A **\( \mathcal{U} \)-graded \( m \)-theory** is equivalent as a datum to a \( \Theta_m \mathcal{U} \)-graded \( m \)-theory.

The proof of this is direct from Corollary 2.16.

**Definition 3.24.** Let \( n \geq 0 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids. Then we refer to a \( \mathcal{U} \)-graded 1-theory also as a **\( \mathcal{U} \)-graded multicategory**.

**Definition 3.25.** Let \( n \geq 1 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids. Let \( m \geq 0 \) and \( \ell \geq 1 \) be integers. Then, for an \( \ell \)-categorified \( \mathcal{U} \)-graded \( m \)-theory \( \mathcal{X} \), we denote by \( \Theta_m \mathcal{X} \), the \((\ell - 1)\)-categorified \( \mathcal{U} \)-graded \((m + 1)\)-theory represented by \( \mathcal{X} \), namely, obtained by replacing the structure functors of \( \mathcal{X} \) by the bimodules/distributors/profunctors corepresented by them.

If \( \mathcal{X} \) is enriched in \( \ell \)-categories, then it can be regarded as \( k \)-categorified for any \( k \geq \ell \). However, resulting \( \Theta_m \mathcal{X} \) is independent of \( k \). In particular, \( \Theta_m \mathcal{X} \) is also
defined for an $U$-graded $m$-theory $X$ enriched in groupoids, and it is an uncategoryed $U$-graded $(m+1)$-theory, which can be considered as as highly categorified (trivially) as one wishes to.

**Definition 3.26.** Let $n \geq 1$ be an integer, and $U$ be an $n$-theory enriched in groupoids. Let $m \geq 0$ be an integer, and $X$ be a $U$-graded $m$-theory which is possibly higher categorified. Then, for an integer $\ell \geq 0$, we define a (less or uncategoryed) $U$-graded $(m+\ell)$-theory $\Theta^{m+\ell}_m X$ by the inductive relations

\[
\Theta^{m+\ell}_m X = \begin{cases} X & \text{if } \ell = 0, \\ \Theta_{m+\ell-1}^{m+\ell-1} \Theta^{m+\ell-1}_m X & \text{if } \ell \geq 1. \end{cases}
\]

3.3.2. The following definition essentially achieves (more than) our initial goal of the discussions here.

**Definition 3.27.** Let $n \geq 0$ be an integer, and $U$ be an $n$-theory enriched in groupoids. Let $m \geq 0$ be an integer, and $X$ be a $U$-graded $m$-theory enriched in groupoids. Then, for an integer $\ell \geq 0$, an $X$-graded $\ell$-theory is a $\Delta \Theta^n_m X$-graded $\ell$-theory, where $N := \max\{m, n\}$, and $\Delta: \Theta^n_m U \rightarrow 1_N^\Com$. In the case where $m = n$, one has $\Delta \Theta^n_n X = \Theta^n_n U$, and for an $X$-graded $\ell$-theory is equivalent to a datum to a $U$-graded $\ell$-theory, as had been predicted.

3.3.3. In order to analyse the notions further, we shall next consider how gradings can be altered.

Given a functor $F: V \rightarrow U$ of $n$-theories enriched in groupoids, it is clear that a $U$-graded $m$-theory $X$ gets pulled back by $F$. Let us denote the resulting $V$-graded $m$-theory by $F^* X$. In the case where $m = n$, and $X$ is unenriched, the projection $(\Delta_V) F^* X \rightarrow V$ (where $\Delta_V: V \rightarrow 1^\Com$) is the base change of the projection $(\Delta_U): X \rightarrow U$ by $F$ in a suitable sense. In the case where $m \leq n - 1$, one has $V \Theta^n_m (F^* X) = F^* (\Theta^n_m X)$, where the superscripts to $\Theta^n_m$ indicate the gradings considered. If $m \geq n + 1$, then $F^*$ is the pull-back by $\Theta^n_m F: \Theta^n_m V \rightarrow \Theta^n_m U$. In the case where $m = n$, $P \mathcal{Y}$ has the same underlying symmetric $n$-theory as that of $\mathcal{Y}$.

**Example 3.28.** Let $X$ be the terminal uncoloured $U$-graded $m$-theory enriched in groupoids. Then we have $\Delta \Theta^n_m X = \Theta^n_n U$, so an $X$-graded $\ell$-theory is equivalent to a datum to a $U$-graded $\ell$-theory, as had been predicted.

**Example 3.29.** For an $n$-theory $V$ enriched in groupoids, and an uncoloured $V$-monoid $X$ defined by a functor $F: V \rightarrow B^n \Gpd$, we have an equivalence $\Theta_{n-1} V = F^* (\Theta_{n-1} B^n)$ of (simply coloured) $V$-graded $n$-theories, which has been described in Proposition 3.13.

Let again $U$ be an $n$-theory enriched in groupoids. For an integer $m \geq 0$, suppose that $V$ is an $U$-graded $m$-theory enriched in groupoids, and denote by $P$, the projection $\Delta \Theta^n_m V \rightarrow \Theta^n_m U$, where $N \geq m, n$, and $\Delta: U \rightarrow 1_N^\Com$. Then also clearly, one obtains from an unenriched (possibly higher categorified) $V$-graded $m$-theory $\mathcal{Y}$, its push forward $P \mathcal{Y}$ as a $U$-graded $m$-theory, generalizing the construction $\Delta$.

In the case $m = n$, $P \mathcal{Y}$ has the same underlying symmetric $n$-theory as that of $\mathcal{Y}$, with projection to $U$ given by the projection to $\Delta V$ composed with $P: \Delta V \rightarrow U$.

In the case $m \leq n - 1$, the description of $P \mathcal{Y}$ is as follows.

(i) For an object $u \in \text{Ob} U$, $\text{Ob}_u P \mathcal{Y}$ is the collection whose member is a pair $(v, y)$, where $v \in \text{Ob}_u \mathcal{V}$ and $y \in \text{Ob}_u \mathcal{Y}$.

Let $k$ be an integer such that $1 \leq k \leq m$, and suppose given the type $u$ of a $k$-multimap in $U$ whose arity is given as $(I, \pi, \phi)$. Then by induction, the type of a $k$-multimap in $P \mathcal{X}$ of the same arity of degree $u$, is specified by

- the type $v$ of a $k$-multimap in $\mathcal{V}$ of the same arity of degree $u$,
- the type $y$ of a $k$-multimap in $\mathcal{Y}$ of the same arity of degree $v$.

(k) (Inductively for $1 \leq k \leq m - 1$.) Suppose given
– a $k$-multimap $w^k$ in $U$ whose arity and type are given respectively as
$(I; \pi, \phi)$ and $u^k_{\leq k-1} = (u^\nu)_{0 \leq \nu \leq k-1}$,
– the type $(v, y)$ of a $k$-multimap in $P; \mathcal{X}$ of the same arity of degree
$u^k_{\leq k-1}$.
Then $\text{Mul}_{P; Y, w^k}^m[(v, y)]$ is the collection whose member is a pair $(w, z)$, where
\[ w \in \text{Mul}_{w^k}^m[v] \quad \text{and} \quad z \in \text{Mul}_{Y, w}^m[y]. \]

$(m)$ Suppose given a datum similar to an input datum for “$(k)$” above, but with $k$
substituted by $m$. Then
\[ \text{Mul}_{P; Y, w^m}^n[(v, y)] = \colim_{w \in \text{Mul}_{w^m}^m[v]} \text{Mul}_{Y, w}^m[y]. \]
The composition is given by the composition in $Y$.

It is immediate to verify that we have an equivalence $\Theta^m_{\pi n}(P; \mathcal{Y}) \simeq P; \Theta^m_{\pi n} \mathcal{Y}$.
In the case $m \geq n+1$, the description above applies after $U$ is replaced by $\Theta^m_{\pi} U$.
In general, between suitable categories, the construction $P$ gives a left adjoint
to the functor $P^*$.

Remark 3.30. A construction $P_\ast$, which suitably gives a right adjoint of $P^*$, will
be considered later in Section 3.3. This construction is not as obvious, and will in
general, only produce an $(m+1)$-theory from an $m$-theory.

Example 3.31. $P 1^0_V = V$ for the terminal unenriched uncoloured $V$-graded $m$-
theory $1^m_V$.

Example 3.32. In the case where $V$ is the terminal unenriched $U$-graded $m$-theory
enriched in groupoids, $P$ is an equivalence, and $P^*$ and $P$ are the mutually inverse
equivalences giving the identification of Example 3.28.

Example 3.33. Consider the case where $U$ is an initially graded 1-theory enriched
in groupoids, and $m = 0$. In this case, a $U$-monoid $V$ is a groupoid-valued functor on
$U$, and the projection $P: (\Delta_U)_0 \mathcal{V} \rightarrow U$ is the corresponding op-fibration (fibred
in groupoids). For a $V$-graded 0-theory $\mathcal{X}$, the op-fibration

\[ (\Delta_U)_0: P; \Theta^0_{0} \mathcal{X} \rightarrow U \]
describing the $U$-monoid $P; \mathcal{X}$ is the composite of the op-fibrations $(\Delta_V)_0 \mathcal{V} \rightarrow
\Theta^0_{0} \mathcal{V}$ and $P$, agreeing with the alternative description of the $U$-module $P; \mathcal{X}$ as
the left Kan extension along $P$ of the $(\Delta_U)_0 \mathcal{V}$-module $\mathcal{X}$.

The version of this connection for $U = 1^m_{\text{Com}}$, has been concretely expressed in
Proposition 3.3.

Proposition 3.3 generalizes as follows.

Proposition 3.34. Let $n \geq 0$ be an integer, $U$ be an $n$-theory enriched in groupoids,
m $\geq 0$ be an integer, and $V$ be a $U$-graded $m$-theory enriched in groupoids. Then an
unenriched $V$-graded $m$-theory is equivalent as a datum to an unenriched $U$-graded
$m$-theory $\mathcal{X}$ equipped with a “projection” functor $\mathcal{X} \rightarrow V$.

Proof. We may assume $m \leq n$ without loss of generality, and the case $m = n$ may
not be tautologous, but is obviously true.

For the case $m \leq n-1$, denote by $P$, the projection $(\Delta_U)_0 \Theta^m_{m} V \rightarrow U$, where
$\Delta_U: U \rightarrow 1^m_{\text{Com}}$. Then, from an unenriched $V$-graded $m$-theory $\mathcal{Y}$, we obtain a
$U$-graded $m$-theory $P; \mathcal{Y}$ equipped with the functor

\[ P; \Delta_\mathcal{Y}: P; \mathcal{Y} \rightarrow P 1^m_V = V, \]

where $\Delta_\mathcal{Y}$ denotes the unique functor $\mathcal{Y} \rightarrow 1^m_V$.

Conversely, given an unenriched $U$-graded $m$-theory $\mathcal{X}$ with projection $Q: \mathcal{X} \rightarrow
V$, one obtains a $V$-graded $m$-theory $((\Delta_U)_0 \Theta^m_{m} V), 1^m_{\mathcal{X}}$.
We would like to show that these constructions are inverse to each other. The functor $P\Delta : P((\Delta_U, \Theta^*_m P), 1^n_V \to P1^n_V$ is equivalent to $Q : X \to V$.

It therefore, suffices to show a natural equivalence $(\Delta_U, \Theta^*_m P, \Delta_{Y}), 1^n_{P, Y} \simeq Y$.

However, the functor $(\Delta_U, \Theta_m P, \Delta_{Y})$ can be identified with the functor $(\Delta_U \circ \Theta^n_m P), Y \Theta^n_m Y \longrightarrow (\Delta_U \circ \Theta^n_m P), \Theta_m 1^n_V$,

so we obtain

$$Y\Theta^n_m (\Delta_U, \Theta^n_m P, \Delta_{Y}), 1^n_{P, Y} = ((\Delta_U, \Theta^n_m P, \Delta_{Y}), \Theta_m 1^n_V, 1^n_{P, Y}),$$

$$= ((\Delta_U \circ \Theta^n_m P), \Theta^n_m \Delta_{Y}, 1^n_{P, Y}),$$

$$= Y\Theta^n_m Y,$$

from which the result follows. \hfill \Box

4. Enrichment of higher theories

4.0. Introduction. The subject of this section will be a general notion of enrichment for graded higher theories. We shall show how some previous notions such as grading by a graded higher theory, can be compactly understood using the new notions introduced in this section. We shall also show how the new framework helps with considering push-forward construction ‘on the right side’, along some functors of higher theories. We shall also discuss a construction for higher theories which is related to the Day convolution for symmetric monoidal categories, and leads to a notion of algebra over an enriched higher theory.

4.1. Enriched theories.

4.1.0. Given a kind of algebraic structure, one motivation for theorizing it was to generalize it to similar structure definable in a theorized form of the same kind of algebraic structure. Specifically, we would like to define the kind in question of algebraic structure in a theorized structure $V$, as a (coloured) morphism to $V$ from the terminal unenriched theorized structure.

For example, for an $n$-theory $U$ enriched in groupoids, we would like to define the notion of $U$-algebra in a $U$-graded $n$-theory, since $U$-graded $n$-theory is the kind of structure which theorize the structure of a $U$-algebra. More generally, the following definition seems reasonable, which can be considered as giving a quite general manner of enrichment, of the notion of graded higher theory.

Definition 4.0. Let $n \geq 0$ be an integer, and $U$ be an $n$-theory enriched in groupoids. Let $m \geq 0$ and $M \geq m + 1$ be integers, and $V$ be a $U$-graded $M$-theory. Then an $m$-theory in $V$ is a functor $1^M_U \to V$ of $U$-graded $M$-theories with strata of colours for a $U$-graded $m$-theory. Namely, it consists of

- a system of colours up to dimension $m - 1$ for a $U$-graded higher theory,
- a functor $\Theta^M_{m+1} T \to V$, where $T$ denotes the terminal unenriched $U$-graded $(m + 1)$-theory on the chosen system of colours.

The system of colours will be called the system of colours of the $m$-theory defined.

In the case $m = n - 1$, $m$-theory will also be called an algebra.

Remark 4.1. In order to avoid redundancy, we call the defined kind of object simply an “$m$-theory” (or “$U$-algebra”) instead of a “$U$-graded $m$-theory” (or $U$-algebra) when it is understood that a $U$-grading is contained is the datum $V$. We may explicitly refer to the $m$-theory as a $U$-graded $m$-theory when $(M \geq n)$ and $V$ is a priori, just a symmetric $M$-theory, and different higher theories may be being considered over which we would like to grade $V$, perhaps including the commutative operad $\text{Com}$. 
This would clarify that a $U$-grading is considered for $V$. Note however, that this would be still ambiguous if more than one $U$-gradings are being considered for $V$.

Similarly to the definitions in Section 2.3, there are also less coloured versions of the notion of enriched graded theory. These are the cases where the $U$-graded theory $T$ which were considered in Definition 4.0, are less coloured.

There is also a lax generalization.

**Definition 4.2.** Let $n \geq 0$ be an integer, and $U$ be an $n$-theory enriched in groupoids. Let $m \geq 0$ and $M \geq m + 1$ be integers, and let $V$ be a possibly higher categorified $U$-graded $M$-theory. Then for an integer $\ell \geq 0$, an $\ell$-lax $m$-theory in $V$ is an $m$-theory in $\Theta_{M+\ell}^{M}V$.

$V$ needs to be at least $\ell$-categorified in order for this to be properly more general than the $(\ell - 1)$-lax notion.

If $V$ is represented by a once more categorified $U$-graded $(M - 1)$-theory $W$, then an $\ell$-lax $m$-theory in $V = \Theta_{M-1}W$, is also an “$m$-tuply coloured” $(\ell + 1)$-lax $(m - 1)$-theory in $W$.

4.1.1. As we have discussed in Section 0, we obtain the following in a low “theoretic” level of algebra. For a functorial formulation of the following, see Proposition 4.3.

**Proposition 4.3.** For every coloured symmetric operad $U$ in groupoids, the notion of coloured $U$-graded operad in a symmetric monoidal category has a generalization in a $(\mathcal{U} \otimes E_{1})$-monoidal category. Namely, there is a notion of coloured $U$-graded operad in a $(\mathcal{U} \otimes E_{1})$-monoidal category, such that the notion of coloured $U$-graded operad in a symmetric monoidal category coincides with the notion of coloured $U$-graded operad in its underlying $(\mathcal{U} \otimes E_{1})$-monoidal category.

Indeed, for a $(\mathcal{U} \otimes E_{1})$-monoidal category $A$, it sufficed to define a coloured $U$-graded operad in $A$ as a 1-theory in the $U$-graded 2-theory $\Theta_{0} B A$, where $B A$ denotes the $U$-monoidal 2-category obtained by categorically delooping $A$ using the $E_{1}$-monoidal structure. To be more precise, the 1-theory in $\Theta_{0} B A$ should satisfy the following condition as a functor

$$F: T \to \Theta_{0} B A,$$

where $T$ denotes the unenriched $U$-graded 2-theory which is terminal on the colours of the 1-theory. The condition is that $F$ should send any object of $T$ (i.e., colour of the 1-theory) to the base object of $B A$.

In the case where $U = \text{Init}, E_{1}, E_{2}$, this coincides with the familiar notion. See Example 3.10.

**Remark 4.4.** It seems natural that the notion of $U$-graded multicategory can further be generalized to be enriched in a $(\mathcal{U} \otimes E_{1})$-graded multicategory. We hope to come back to this in a sequel to this work.

4.1.2. We can more generally consider the similarly general enrichment of the notion of $U$-graded higher theory in the case where $U$ is graded by an $N$-theory, where $N \geq n + 1$. In this case, a $U$-graded higher theory in the previous sense was simply a $\Delta \Theta_{n}^{N} U$-graded higher theory.

The notion is therefore a special case of the notion defined in Definition 4.0. To be explicit, we have the following. (There will be changes in the notation.)

Let

- $n \geq 0$ be an integer, and $U$ be an $n$-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $X$ be a $U$-graded $m$-theory enriched in groupoids,
- $\ell \geq 0$ and $L \geq \ell + 1$ be integers, and $Y$ be an $X$-graded $L$-theory.
Then $\mathcal{Y}$ is a $\Delta\Theta^N_m \mathcal{X}$-graded $L$-theory, where $N := \max\{m, n\}$, and $\Delta : \mathcal{U} \rightarrow 1^n_{\text{Com}}$, so an $\ell$-theory in $\mathcal{Y}$ makes sense according to Definition 4.0.

**Definition 4.5.** Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an $n$-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{X}$ be a $\mathcal{U}$-graded $m$-theory enriched in groupoids,
- $\ell \geq 0$ and $L \geq \ell + 1$ be integers, and $\mathcal{V}$ be a $\mathcal{U}$-graded $L$-theory.

Then an $\mathcal{X}$-graded $\ell$-theory in $\mathcal{V}$ is an $\ell$-theory in the $\mathcal{X}$-graded $L$-theory $P^*\mathcal{V}$, where $P$ denotes the projection $\Delta : \Theta^N_m \mathcal{X} \rightarrow \Theta^N_m \mathcal{U}$, where $N := \max\{m, n\}$, and $\Delta : \mathcal{U} \rightarrow 1^n_{\text{Com}}$.

In the case $\ell = m - 1$, an $\mathcal{X}$-graded $\ell$-theory will also be called an $\mathcal{X}$-algebra.

Thus, a $\mathcal{U}$-graded $m$-theory in $\mathcal{V}$ is an uncoloured $\mathcal{T}$-algebra in $\mathcal{V}$ for an unenriched $\mathcal{U}$-graded $(m + 1)$-theory $\mathcal{T}$ which is terminal on a system of colours up to dimension $m - 1$.

**Remark 4.6.** For the notion of Definition 4.5, we are refraining from saying “$\mathcal{U}$-graded $\mathcal{X}$-graded” theory when we do not intend to emphasize the $\mathcal{U}$-grading, and this is part of our convention on the terminology. See Remark 4.4.

**Example 4.7.** For $\mathcal{U}$ and $\mathcal{X}$ as in Definition 4.5, a $\mathcal{X}$-graded $\ell$-theory enriched in a multicategory $\mathcal{M}$, is equivalent as a datum to an $\mathcal{X}$-graded $\ell$-theory in the $\mathcal{U}$-graded $(\ell + 1)$-theory $\Delta^* \mathcal{B}^m \mathcal{M}$, where $\Delta : \mathcal{U} \rightarrow 1^n_{\text{Com}}$.

**Example 4.8.** In Definition 4.5 if $\mathcal{X}$ is the terminal uncoloured $\mathcal{U}$-graded $m$-theory enriched in groupoids, then $P$ is an equivalence, and an $\mathcal{X}$-graded $\ell$-theory in $\mathcal{V}$ is equivalent as a datum to an $\ell$-theory in $\mathcal{V}$.

### 4.2. Graded theories as lifts of an algebra

**4.2.0.** Using Definition 4.5, an $\mathcal{X}$-graded $\ell$-theory in $\mathcal{V}$ in the notation there, can be written as an appropriate coloured functor of $\mathcal{U}$-graded higher theories. On the other hand, $\mathcal{X}$ itself may be defined by a coloured functor $F : 1^m_{\mathcal{U}} \rightarrow \Delta^* \mathcal{B}^m \Theta_0 \text{Gpd}$ of $\mathcal{U}$-graded theories. In this situation, one might wish to describe a higher theory graded by $\mathcal{X}$, directly in terms of $F$. We shall show that it can indeed be described as a coloured lift of $F$.

Note that we may assume $m = L(\geq \ell + 1)$ without loss of generality. We shall first discuss a result in this situation (Proposition 4.9). For application in Section 4.3, we shall also give an analogous result Proposition 4.11 for the case “$\ell = m$”.

**4.2.1.** Let us first recall common notation.

If $\mathcal{C}$ is a category and $x$ is an object of $\mathcal{C}$, then we denote by $\mathcal{C}_{/x}$, the category of objects of $\mathcal{C}$ lying over $x$, i.e., equipped with a map to $x$. More generally, if a category $\mathcal{D}$ is equipped with a functor to $\mathcal{C}$, then we define $\mathcal{D}_{/x} := \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/x}$. Note here that $\mathcal{C}_{/x}$ is mapping to $\mathcal{C}$ by the functor which forgets the structure map to $x$.

Note that the notation is abusive in that the name of the functor $\mathcal{D} \rightarrow \mathcal{C}$ is dropped from it. In order to avoid this abuse from causing any confusion, we shall use this notation only when the considered functor $\mathcal{D} \rightarrow \mathcal{C}$ is clear from the context.

**4.2.2.** To get on the task now, let us denote by CAT the very large category of large categories. Consider this as symmetric monoidal by the Cartesian structure, and give the functor category Fun(Gpd^{op}, CAT) the structure of a multicategory by the Day convolution. Namely, we consider the structure of a multicategory underlying (or “represented” by) the symmetric monoidal structure given by the Day convolution. Then the Grothendieck construction defines a functor $G : \text{Fun}(\text{Gpd}^{op}, \text{CAT}) \rightarrow \Theta_0(\text{CAT}_{/\text{Gpd}})$ of multicategories, where CAT_{/Gpd} is
made symmetric monoidal by the structure induced from the (Cartesian) symmetric monoidal structure of Gpd.

Furthermore, considering Gpd as a symmetric monoidal (full) subcategory of CAT, we obtain the composite

\[
\Theta_0 \text{Gpd} \xrightarrow{\text{Yoneda}} \text{Fun}(\text{Gpd}^{\text{op}}, \text{Gpd}) \xrightarrow{\text{inclusion}} \text{Fun}(\text{Gpd}^{\text{op}}, \text{CAT}) \overset{G}{\longrightarrow} \Theta_0(\text{CAT}_{/\text{Gpd}}),
\]

where the structures of multicategories on the functor categories are by the Day convolution. We denote this functor by \(T: \Theta_0 \text{Gpd} \to \Theta_0(\text{CAT}_{/\text{Gpd}})\), \(X \to T_X\), so \(T_X\) is defined by the forgetful functor \(AX := \text{Gpd}_{/X} \to \text{Gpd}\).

Now let \(n \geq 0\) be an integer, and \(U\) be an \(n\)-theory. Then for an integer \(m \geq 0\), we obtain from a \(U\)-graded \(m\)-theory \(V\) enriched in Gpd, a \(U\)-graded \(m\)-theory \(T_{\ast}; V\) enriched in CAT, by postcomposition with \(\text{B}^n T; \text{B}^n \Theta_0 \text{Gpd} \to \text{B}^n \Theta_0(\text{CAT}_{/\text{Gpd}})\).

Proposition 4.4.9. Let \(n \geq 0\) be an integer, and \(U\) be an \(n\)-theory in groupoids. Suppose that an uncoloured \(T\)-monoid \(X\) is defined by a functor \(F: T \to \Delta^\ast \text{B}^n \Theta_0 \text{Gpd}\), where \(\Delta: U \to \text{1}_{\text{Com}}^n\), of \(U\)-graded \((m + 1)\)-theories (e.g., these data may be specifying a \(U\)-graded \(m\)-theory).

Then, for an integer \(\ell\) such that \(0 \leq \ell \leq m - 1\), an \(X\)-graded \(\ell\)-theory in \(U\)-graded \(m\)-theory \(V\) enriched in groupoids, is equivalent as a datum to a lift with strata of colours up to dimension \(\ell - 1\), of \(F\) to \(\Theta_m A_{\ast} V\).

Proof. Since \(X\) is uncoloured, a system of colours for an \(X\)-graded \(\ell\)-theory (up to dimension \(\ell - 1\)) is equivalent as a datum to a system of colours for an \(T\)-graded \(\ell\)-theory. If \(J\) is the terminal unenriched \(T\)-thgraded \((\ell + 1)\)-theory on a system of colours up to dimension \(\ell - 1\), then, from the definitions, a correspondence is immediate between the structures on this system of colours, of \(X\)-graded \(\ell\)-theories in \(V\), and lifts of \(F\) to functors \(P\Theta_\ell^{\ell + 1} J \to \Theta_m A_{\ast} V\), where \(P\) denotes the projection \(\Delta^N\Theta_\ell^{\ell + 1} T \to \Theta_m N U\), \(N \geq m + 1, n\).

Corollary 4.10. Let \(F, X\) be as in Proposition. Then, for an integer \(\ell\) such that \(0 \leq \ell \leq m - 1\), an \(X\)-graded \(\ell\)-theory enriched in a symmetric multicategory \(M\) is equivalent as a datum to a lift with strata of colours up to dimension \(\ell - 1\), of \(F\) to \(\Delta^\ast \theta_\ell^{n+1} A_{\ast}^\ell M\).

Proof. An \(X\)-graded \(\ell\)-theory enriched in a symmetric multicategory \(M\) is an \(X\)-graded \(\ell\)-theory in \(\Delta^\ast W\), where \(W\) denotes the symmetric \((\ell + 1)\)-theory \(\text{B}^\ell M\). Proposition identifies this with a coloured lift of \(F\) to \(\Delta^\ast \Theta_m A_{\ast} \Theta_\ell^{n+1} W\). Moreover, there is an equivalence

\[
\Theta_m A_{\ast} \Theta_\ell^{n+1} W = \text{B}^\ell \Theta_m A_{\ast} \Theta_\ell^{n-\ell} M,
\]

of unenriched \((m + 1)\)-theories lying over \(\text{B}^n \Theta_0 \text{Gpd} = \text{B}^\ell \text{B}^m \Theta_0 \text{Gpd}\).}

4.2.3. In Section 4.3, we shall use a natural analogue of Proposition 4.9 for \(\ell = m\). The way how we obtain it will be by restricting the context.

Let us denote by \(\text{CAT}\) the very large 2-category of large categories extending the 1-category CAT. In order to formulate the result, we first extend \(T\) to the composite

\[
T: \Theta_0 \text{CAT} \xrightarrow{\text{Yoneda}} \text{Fun}(\text{CAT}^{\text{op}}, \text{CAT}) \xrightarrow{\text{restriction}} \text{Fun}(\text{Gpd}^{\text{op}}, \text{CAT}) \overset{G}{\longrightarrow} \Theta_0(\text{CAT}_{/\text{Gpd}})
\]
of functors of categorified multicategories, where \( \text{Fun} \) indicates the 2-categories of functors extended to categorified multicategories by the Day convolution, and \( \text{Gpd} \) is considered as a symmetric monoidal subcategory of \( \text{CAT} \). Thus, for \( C \in \text{CAT} \), the object \( T_C \in \text{CAT}/\text{Gpd} \) is defined by the forgetful functor \( AC := \text{Gpd}/C \to \text{Gpd} \).

Now let \( n \geq 0 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory. Then as before, for an integer \( m \geq 0 \), we obtain from a \( \mathcal{U} \)-graded \( m \)-theory \( V \) enriched in \( \text{CAT} \),

- a \( \mathcal{U} \)-graded \( m \)-theory \( A, V \) enriched in \( \text{CAT} \),
- a functor \( \Theta_m A, V \to \Delta^* \mathbb{B}^m \Theta_0 \text{Gpd} \),

by considering \( T_* \).

**Proposition 4.11.** For \( \mathcal{U}, F, \mathcal{X} \) as in Proposition 4.9, an \( \mathcal{X} \)-graded \( m \)-theory in the \((m + 1)\)-theory represented by a \( \mathcal{U} \)-graded \( m \)-theory \( V \) enriched in \( \text{CAT} \), is equivalent as a datum to a lift with strata of colours up to dimension \( m - 1 \), of \( F \) to \( \Theta_m A, V \).

The proof is similar to the proof of Proposition 4.9.

**Corollary 4.12.** For \( F, \mathcal{X} \) as in Proposition 4.9, an \( \mathcal{X} \)-graded \( m \)-theory enriched in a symmetric monoidal category \( A \), is equivalent as a datum to a lift with strata of colours up to dimension \( m - 1 \), of \( F \) to \( \Delta^* \mathbb{B}^m \Theta_0 A, A \).

**Proof.** This is the case \( V = \Delta^* \mathbb{B}^m A \) of Proposition since we have an equivalence \( A, \mathbb{B}^m A = \mathbb{B}^m A, A \) of \((m + 1)\)-theories enriched in \( \text{CAT} \), lying over \( \mathbb{B}^m \text{Gpd} \). \( \square \)

### 4.3. The right adjoint of the restriction of degrees.

**4.3.0.** In Section 3.3 we have considered the left adjoint to the functor ‘restricting’ the degrees for graded theories. We would like to consider the right adjoint in the following situation.

Let

- \( n \geq 0 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids,
- \( m \geq 0 \) be an integer, and \( \mathcal{V} \) be an \( \mathcal{U} \)-graded \( m \)-theory enriched in groupoids,
- \( \mathcal{X} \) be an unenriched \( \mathcal{V} \)-graded \( m \)-theory.

Denote by \( P \) the projection functor \( \Delta : \Theta_m^N \mathcal{V} \to \Theta_n^N \mathcal{U} \), where \( N \geq m, n \), and \( \Delta : \mathcal{U} \to 1_{\text{Con}}^{\text{n}} \). Then it turns out that we can construct a \( \mathcal{U} \)-graded \((m + 1)\)-theory \( P, \mathcal{X} \) having an appropriate universal property.

We shall do this construction in two steps. The key observation is that we can reduce the situation above into two simpler situations according to the factorization of \( P \) as

\[
\Delta : \Theta_m^N \mathcal{V} \xrightarrow{B} \Delta : \Theta_m^V \mathcal{T} \xrightarrow{Q} \Theta_m^N \mathcal{U},
\]

where \( \mathcal{T} \) denotes the \( \mathcal{U} \)-graded \( m \)-theory enriched in groupoids which is terminal on the system of colours of \( \mathcal{V} \), so \( \mathcal{V} \) can be identified with an uncoloured \( \mathcal{T} \)-graded \( m \)-theory.

The two cases which we shall treat separately below, will imply respectively that we obtain a \( \mathcal{T} \)-graded \((m + 1)\)-theory \( R, \mathcal{X} \), and that we obtain from this, a \( \mathcal{U} \)-graded \((m + 1)\)-theory \( Q, R, \mathcal{X} \). Moreover, it will follow that the construction \( P_* := Q, R_* \) has a universal property desired of the right push-forward.

**Remark 4.13.** As we shall see in the construction, \( Q_* \) will produce an \((m + 1)\)-theory in which the groupoids of \((m + 1)\)-multimaps may not necessarily be small, even if the groupoids of \( m \)-multimaps are all small in \( \mathcal{X} \).

Let us see the constructions \( Q_* \) and \( R_* \).
4.3.1. In order to construct $R$, above, it suffices to construct a $U$-graded $(m+1)$-
theory $P, X$ in the spacial case of the original situation where $V$ is an uncoloured $U$-
graded $m$-theory. This can be done with the following construction at the universal
level.

Recall from Section 3.2 and Corollary 4.12 that $V$ corresponds to a functor $F_V: \Theta^N U \to \Theta_m^N B^m \Theta_0 Gpd$, where $N \geq n, m + 1$, and $X$ corresponds then to a
functor $F_X: J \to F_V^* B^m \Theta_0 A_* Gpd$ of unenriched $U$-graded $(m+1)$-theories, where $J$
is terminal on the system of colours of $X$.

Let $\text{Cocorr}$ denote the symmetric monoidal 2-category of groupoids and corco-
respondences. Thus, its object is a groupoid, and for groupoids $X, Y$, the category $\text{Map}_{\text{Cocorr}}(X, Y)$ is the category formed naturally by the diagrams of the form

$$X \to M \leftarrow Y$$
in $\text{Gpd}$, where the groupoid $M$ is allowed to vary arbitrarily. Composition is done
by the obvious push-out operation. The symmetric monoidal structure is induced from
the Cartesian product in $\text{Gpd}$.

Then there is a symmetric monoidal lax functor $\Gamma: A_* \text{Gpd} \to \text{Cocorr}$ which sends

- an object of $A_* \text{Gpd}$ given by $X \in \text{Gpd}$ and a functor $F: X \to \text{Gpd}$, to the
groupoid $\lim_X F$,
- a map $(X, F) \to (Y, G)$ in $A_* \text{Gpd}$ given by a map $f: X \to Y$ and a map $\tilde{f}: F \to \tilde{f}^* G$, to the map in $\text{Cocorr}$ corresponding to the diagram

$$\lim_X F \xrightarrow{\tilde{f}} \lim_X \tilde{f}^* G \xleftarrow{\lim_Y G}$$
in $\text{Gpd}$,

and extends these data naturally. From this, we obtain the induced functor $\Gamma: \Theta^2_0 A_* \text{Gpd} \to \Theta^2_0 \text{Cocorr}$ of 2-theories, and hence

$$\mathbb{B}^m \Gamma: \Theta_{m+1}^N B^m \Theta_0 A_* \text{Gpd} = \mathbb{B}^m \Theta^2_0 A_* \text{Gpd} \to \mathbb{B}^m \Theta^2_0 \text{Cocorr}.$$  

Denote by $\text{Cocorr}$, the symmetric monoidal 2-category of co-correspondences
in the category $\text{Gpd}_*$ of pointed groupoids. The forgetful functor $\text{Gpd}_* \to \text{Gpd}$
(which preserves push-outs and direct products) induces a symmetric monoidal
functor $\text{Cocorr} \to \text{Cocorr}$. In particular, we can consider the $(m+2)$-theory
$\Theta_{m+1}^N B^m \Theta_0 \text{Cocorr}$, as graded by $\Theta_{m+1}^N B^m \Theta_0 \text{Cocorr} = \mathbb{B}^m \Theta^2_0 \text{Cocorr}$. This
$\Theta_{m+1}^N B^m \Theta_0 \text{Cocorr}$-graded $(m+2)$-theory is in fact representable by a $(m+1)$-
theory. This will be an instance of the following.

**Lemma 4.14.** Let $C$ and $D$ be $n$-theories enriched in CAT, and suppose given a
functor $P: D \to C$ of categorified $n$-theories. Then the $\Theta_n C$-graded $(n+1)$-theory
corresponding to the induced functor $P: \Theta_n D \to \Theta_n C$ of symmetric $(n+1)$-theories,
is representable by a categorified $\Theta_n C$-graded $n$-theory if the functors induced by $P$
on the categories of $n$-multimaps, are all op-fibrations.

Moreover, the representing $\Theta_n C$-graded $n$-theory in this situation, is enriched in fact
in groupoids if and only if the op-fibrations are all fibred in groupoids.

**Proof.** Let $y$ be the type of an $n$-multimap in $D$, and $x$ be an $n$-multimap in $C$
of type $P y$. Then the category of $n$-multimaps in the representing $n$-theory, of
type $y$ of degree $x$, will be the fibre of the op-fibration over $x$ in the category of
$n$-multimaps of type $y$ in $D$. We would like to let $(n+1)$-multimaps in $\Theta_n C$ act on
these categories, but an $(n+1)$-multimap in $\Theta_n C$ is a morphism in the category of
appropriate $n$-multimaps in $C$, which acts on the fibres of the op-fibrations. It is
straightforward to check that this extends to the structure of a categorified $\Theta_n C$-
graded $n$-theory which represents the $\Theta_n C$-graded $(n+1)$-theory $\Theta_n D$. 

Remark 4.15. The condition is also necessary.

To see this, note that the category of $n$-multimaps in $C$ of a given arity can be recovered from $\Theta_n C$, as the category formed by $n$-multimaps in $\Theta_n C$ of the same arity, under the unary $(n + 1)$-multimaps between them in $\Theta_n C$. Moreover, we obtain a similar description of the category of $n$-multimaps in $D$ of a given arity, as a category lying over the corresponding category for $C$, in view of the noted description of the latter category. On the other hand, for a categorified $\Theta_n C$-graded $n$-theory $\mathcal{E}$, the category formed by $n$-multimaps in $\Delta_! \Theta_n \mathcal{E}$ of a given arity (where $\Delta : \Theta_n C \to 1_{\text{Comp}}^{n+1}$) under the unary $(n + 1)$-multimaps between them in $\Delta_! \Theta_n \mathcal{E}$, can be seen to be lying over the corresponding category in $F := \Theta_n C$, as the op-fibration corresponding to the action of the unary $(n + 1)$-multimaps in $F$ between those $n$-multimaps, on the categories of $n$-multimaps in $\mathcal{E}$ of the same arity of appropriate degrees. Therefore, if $\Theta_n D$ corresponds to the $F$-graded $(n + 1)$-theory $\Theta_n \mathcal{E}$, then we conclude that the assumption of Proposition is satisfied by $D$.

We can indeed apply Lemma to the forgetful functor $\mathbb{B}^m \Theta_0 \text{Cocorr}_* \to \mathbb{B}^m \Theta_0 \text{Cocorr}$ of categorified $(m + 1)$-theories, since the forgetful functor $\text{Cocorr}_* \to \text{Cocorr}$ is such that the functors induced on the categories of 1-morphisms are easily seen to be op-fibrations fibred in groupoids.

Let us identify $\mathbb{B}^m \Theta_0^2 \text{Cocorr}_*$ with the $\mathbb{B}^m \Theta_0^2 \text{Cocorr}$-graded $(m + 1)$-theory enriched in groupoids representing it. From this, we obtain a simply coloured $F$-graded $(m + 1)$-theory $F' (\mathbb{R}^m \Gamma)^*(\mathbb{B}^m \Theta_0^2 \text{Cocorr}_*)$.

Definition 4.16. Let $U, X, P, J, F_X$ be as above. Denote by $Q$ the projection $\Delta_k \Theta_m^{N+1} U \to \Theta_m^N U$, where $N := \max\{m + 1, n\}$, and $\Delta : U \to 1_{\text{Comp}}^\infty$.

Then we define a $U$-graded $(m + 1)$-theory $P, X$ as $Q, F_X*(\mathbb{B}^m \Gamma)^*(\mathbb{B}^m \Theta_0^2 \text{Cocorr}_*)$.

Example 4.17. In the case where $V$ is the terminal unenriched uncoloured $U$-graded $m$-theory $1_U^m, X$ can be identified with a $U$-graded $m$-theory, and it follows from Proposition 5.17 that $P, X = \Theta_m X$.

4.3.2. In order to do the other construction, let us introduce a notation.

Suppose given a collection $\Lambda$, and a family $X = (X_\lambda)_{\lambda \in \Lambda}$ of groupoids parametrized by $\Lambda$. Then by $\prod_{\Lambda} X = \prod_{\lambda \in \Lambda} X_\lambda$, we denote the not necessarily small groupoid whose truncated $n$-type is $\prod_{\Lambda} X^{\leq n}$ naturally formed by associations $\sigma$ to every member $\lambda \in \Lambda$, of an object $\sigma(\lambda) \in X_\lambda^{\leq n}$, where $X_\lambda^{\leq n}$ denotes the truncated $n$-type of $X_\lambda$.

Note that $\prod_{\Lambda} X^{\leq n}$ may not be small, but is well-defined as a homotopy $n$-type by induction on $n$. For example, we define $\sigma, \tau \in \prod_{\Lambda} X^{\leq 0}$ to be equal if and only if $\sigma(\lambda) = \tau(\lambda)$ for every $\lambda \in \Lambda$, and then this equality relation is an equivalence relation, so the members of $\prod_{\Lambda} X^{\leq 0}$ form a possibly large $0$-type under this relation of equality.

4.3.3. In order to construct $Q_*$ of Section 4.3.1 it suffices to construct a $U$-graded $(m + 1)$-theory $P, X$ in the original situation modified as follows.

- $V$ is terminal on the system of colours for a $U$-graded $m$-theory.
- $X$ is a $V$-graded $(m + 1)$-theory.

The construction is as follows. If necessary, Theorem 3.17 allows us to replace $U$ by $\Theta_n U$ for any $N > n$, so we shall assume without loss of generality, that the dimension of $U$ is sufficiently high.

For an object $u \in \text{Ob} U$, we let an object $\sigma \in \text{Ob}_u P, X$ of $P, X$ of degree $u$ be an association to every $v \in \text{Ob}_u V$ of an object $\sigma(v) \in \text{Ob}_v X$. 

Let $k$ be an integer such that $1 \leq k \leq m - 1$. Then the collections of $k$-multimaps in $P_\mathcal{X}$ will inductively be as follows. Suppose given a k-multimap $u^k$ in $\mathcal{U}$ of arity and type given respectively as $(I; \pi, \phi)$ and $u^{\leq k-1} = (u^r)_0 \leq \nu \leq k-1$, and the type $\sigma = (\sigma^r)_0 \leq \nu \leq k-1$ of a k-multimap in $P_\mathcal{X}$ of the same arity of degree $u^{\leq k-1}$ (see Definition 5.7). Then, we let a k-multimap $\tau \in \text{Mul}^k_{P_\mathcal{X}, u^k}[\sigma]$ be an association to every k-multimap $u^k$ in $\mathcal{V}$ of arity $(I; \pi, \phi)$ and degree $u^k$, of a k-multimap $\tau(v^k) \in \text{Mul}^\pi_{P_\mathcal{X}, \sigma}[\sigma(v^{\leq k-1})]$, where $v^{\leq k-1} = (v^r)_0 \leq \nu \leq k-1$ is the type of $u^k$ (of degree $u^{\leq k-1}$), and $\sigma(v^{\leq k-1}) := (\sigma^r(v^r))_0 \leq \nu \leq k-1$ (where $\sigma^r := (\sigma_i^r(v^r))_{i \in [k]}$ if $\nu \leq k - 2$, etc.) by induction, the type of a k-multimap in $\mathcal{X}$ of the same arity of degree $v^{\leq k-1}$.

The collections of m-multimaps in $P_\mathcal{X}$ will be as follows. Suppose given an input datum similar to above, but with $k$ replaced by $m$. Then we let an m-multimap $\tau \in \text{Mul}^m_{P_\mathcal{X}, u^m}[\sigma]$ be an association to every one of the types $v$ of m-multimaps in $\mathcal{V}$ of arity $(I; \pi, \phi)$ and degree $u^{\leq m-1}$, of an m-multimap $\tau(v) \in \text{Mul}^\pi_{P_\mathcal{X}, u^m}[\sigma(v)]$.

The groupoids of $(m + 1)$-multimaps in $P_\mathcal{X}$ will be as follows. Suppose given a $(m + 1)$-multimap $u^{m+1}$ in $\mathcal{U}$ of arity and type given respectively as $(I; \pi, \phi)$ and $u^{\leq m}$, and the type $\sigma$ of a $(m + 1)$-multimap in $P_\mathcal{X}$ of the same arity of degree $u^{\leq m}$. Then we let

$$
\text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma] = \prod_{v} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma(v^{\leq m-1})(v)[\sigma_0^{m}[v]; \sigma_1^{m}[\pi v]],
$$

where $v$ runs through all the types of $\phi^{m-1}$-nerves of m-multimaps in $\mathcal{V}$ of degree $u^{\leq m-1}$.

The action of $(m + 2)$-multimaps of $\mathcal{U}$ on the groupoids of $(m + 1)$-multimaps in $P_\mathcal{X}$, will be as follows. Suppose given a $(m + 2)$-multimap $u^{m+2}$ in $\mathcal{U}$ of arity and type given respectively as $(I; \pi, \phi)$ and $u^{\leq m+1}$, and the type $\sigma$ of a $\phi^m$-nerve of $(m + 1)$-multimaps in $P_\mathcal{X}$ of degree $u^{\leq m}$. Then we let the action $\text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma] \rightarrow \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\pi \sigma]$ of $u^{m+2}$ be given by composing the following two maps, namely,

- the map

$$
\prod_{i \in I_{m+1}} \prod_{w_i} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma^{\leq m-2}(w_i^{\leq m-2})] \quad \left[ ((\phi^{m-1})_i^{m-1})(w_i^{m-1}) \right] \left[ \sigma^{m-1}[w_i]; \sigma_i^{m}[\phi^m(w_i)] \right]
$$

$$
\rightarrow \prod_{v} \prod_{i \in I_{m+1}} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma^{\leq m-2}(v^{\leq m-2})] \left[ ((\phi^{m-1})_i^{m-1})(v^{m-1}) \right] \quad \left[ \sigma^{m-1}[v^{m-1}]; \sigma_i^{m}[\phi^m(v^{m-1})] \right],
$$

where

- the source is simply the result of expanding the factors of the product $\text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma] = \prod_{i \in I_{m+1}} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma_i]$, $v$ runs through all the types of $\phi^{m-1}$-nerves of m-multimaps in $\mathcal{V}$ of degree $u^{\leq m-1}$,
- the map is induced from the correspondence $v \mapsto w_i = (\phi^{m-1})_i^{m-1}(v)$, and

- the map

$$
\prod_{v} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma^{\leq m-1}(v)] \left[ \sigma^{m-1}[v^{m-1}]; \sigma_1^{m}[\pi v] \right]
$$

$$
\rightarrow \prod_{v} \text{Mul}^\pi_{P_\mathcal{X}, u^{m+1}}[\sigma^{\leq m-1}(v)] \left[ \sigma^{m-1}[v]; \sigma_0^{m}[v]; \sigma_1^{m}[\pi v] \right],
$$

where $-v$ runs through all the types of $\phi^{m-1}$-nerves of m-multimaps in $\mathcal{V}$ of degree $u^{\leq m-1}$.
where \( \nu \) runs through the same range, and the map is given by the action of \( u^{n+2} \) in \( P_{\varepsilon}X \).

It is straightforward to extend these data naturally to a full datum for a \( \mathcal{U} \)-graded \((m+1)\)-theory \( P_{\varepsilon}X \).

4.3.4. From these constructions, we in particular have obtained the constructions \( R_* \) and \( Q_* \) in the situation of Section 4.3.0. Therefore, we can extend the previous constructions to this situation by defining \( P_* := Q_* R_* \). Note Example 4.17. \( P_* \) will have the following universal property.

**Proposition 4.18.** Let

- \( n \geq 0 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids,
- \( m \geq 0 \) be an integer, and \( \mathcal{V} \) be a \( \mathcal{U} \)-graded \( m \)-theory enriched in groupoids,
- \( \mathcal{X} \) be an unenriched \( \mathcal{V} \)-graded \( m \)-theory.

Denote by \( P \) the projection functor \( \Delta : \Theta^N_m \mathcal{V} \to \Theta^N_m \mathcal{U} \), where \( N \geq m, n, \) and \( \Delta : \mathcal{U} \to 1^N_{\text{con}} \).

Then, for a \( \mathcal{U} \)-graded \((m+1)\)-theory \( \mathcal{Z} \), a functor \( \mathcal{Z} \to P_* \mathcal{X} \) of \( \mathcal{U} \)-graded \((m+1)\)-theories is naturally equivalent as a datum to a functor \( P^* \mathcal{Z} \to \Theta^N_m \mathcal{X} \) of \( \mathcal{V} \)-graded \((m+1)\)-theories.

Indeed, this is an immediate consequence of the similar universal properties of the constructions \( R_* \) and \( Q_* \), which can be verified easily from the constructions.

**Example 4.19.** Let \( \mathcal{U}, \mathcal{V} \) be as in Proposition, but suppose moreover that \( \mathcal{V} \) is uncoloured. Then a system of colours for a \( \mathcal{V} \)-graded higher theory is the same as a system of colours for a \( \mathcal{U} \)-graded higher theory. Let \( \ell \geq 0 \) be an integer, and suppose given such a system of colours up to dimension \( \ell - 1 \). Let \( \mathcal{T} \) denote the terminal unenriched \( \mathcal{V} \)-graded \((\ell+1)\)-theory on this system of colours. We would like to consider for an integer \( L \geq \ell + 1, m \), Proposition for an unenriched \( \mathcal{V} \)-graded \( L \)-theory \( \mathcal{X} \) and the \( \mathcal{U} \)-graded \((L+1)\)-theory \( \mathcal{Z} := P \Theta^L_{\ell+1} \mathcal{T} \), where \( P \) is as in Proposition.

We obtain that a functor \( \mathcal{Z} \to P_* \mathcal{X} \) is equivalent as a datum to a functor \( \Theta^L_{\ell+1} \mathcal{T} = P^* P \Theta^L_{\ell+1} \mathcal{T} \to \mathcal{X} \), which simply defines the structure of an \( \ell \)-theory in \( \mathcal{X} \), on the considered system of colours.

**Corollary 4.20.** In addition to \( \mathcal{U}, \mathcal{V}, \mathcal{X} \) of Proposition, suppose given a \( \mathcal{U} \)-graded \( m \)-theory \( \mathcal{W} \) enriched in groupoids. Let \( \tilde{P} : \tilde{P}^* \mathcal{W} \to \mathcal{W} \) be the counit for the adjunction, and \( Q : Q^* \mathcal{W} \to \Theta^N_m \mathcal{U} \) and \( \tilde{Q} : \tilde{Q}^* \mathcal{W} \to \mathcal{V} \) be respective projections.

Then there is a natural equivalence \( Q^* \mathcal{P} \mathcal{X} \simeq \tilde{P} \tilde{Q}^* \mathcal{X} \) of \( \mathcal{W} \)-graded \((m+1)\)-theories.

**Proof.** This follows immediately from Proposition and the following lemma. \( \square \)

**Lemma 4.21.** Let

- \( n \geq 0 \) be an integer, and \( \mathcal{U} \) be an \( n \)-theory enriched in groupoids,
- \( m \geq 0 \) be an integer, and \( \mathcal{V} \) and \( \mathcal{W} \) be \( \mathcal{U} \)-graded \( m \)-theories enriched in groupoids.

Let \( P, \tilde{P}, Q, \tilde{Q} \) be similar to those in Proposition and Corollary 4.20. Then, for a \( \mathcal{W} \)-graded \( m \)-theory \( \mathcal{Y} \), there is a natural equivalence \( P^* Q_\mathcal{Y} \simeq \tilde{Q}_\mathcal{Y} \tilde{P}^* \mathcal{Y} \) of \( \mathcal{V} \)-graded \( m \)-theories.

**Proof.** Straightforward from the definitions. \( \square \)
4.4. Convolution for higher theories. In Example 4.19 if $\mathcal{X}$ is of the form $P^\ast \mathcal{Y}$ for a $\mathcal{U}$-graded $L$-theory $\mathcal{Y}$, then we obtain that a $\mathcal{V}$-graded $\ell$-theory in $\mathcal{Y}$ can equivalently be written as an $\ell$-theory in the $\mathcal{U}$-graded $(L+1)$-theory $P_\ast P^\ast \mathcal{Y}$.

There is also a coloured generalization of this. We change the notations, and consider the following situation.

Let
- $n \geq 0$ be an integer, and $\mathcal{U}$ be an $n$-theory enriched in groupoids,
- $m \geq 0$ be an integer, and $\mathcal{T}$ be a $\mathcal{U}$-graded $(m+1)$-theory enriched in groupoids, which is terminal on a system of colours up to dimension $m-1$,
- $\mathcal{X}$ be an uncoloured $\mathcal{T}$-monoid, so together with $\mathcal{T}$, this is defining a $\mathcal{U}$-graded $m$-theory,
- $\ell \geq 0$ and $L \geq \ell + 1$, $m$ be integers, and $\mathcal{Y}$ be an unenriched $\mathcal{U}$-graded $L$-theory.

For these, let $P : \Delta \Theta^N_{m+1} \mathcal{T} \rightarrow \Theta^N_m \mathcal{U}$ (where $N \geq m+1, n$, and $\Delta : \mathcal{U} \rightarrow 1^n_{\text{Com}}$) and $R : P \Theta_m \mathcal{X} \rightarrow \mathcal{T}$ be the respective projections.

Then the $\mathcal{T}$-graded $(L+1)$-theory $R_\ast R^\ast P^\ast \mathcal{Y}$ is such that an $\mathcal{X}$-graded $\ell$-theory in $\mathcal{Y}$ is equivalent as a datum to an $\ell$-theory in $R_\ast R^\ast P^\ast \mathcal{Y}$. Therefore, $R_\ast R^\ast P^\ast \mathcal{Y}$, considered as a construction between the $\mathcal{U}$-graded $m$-theories $\mathcal{X}$ and $\mathcal{Y}$, is in a way analogous to the Day convolution for monoidal categories [8].

Remark 4.22. In the case $\ell = 0$, we obtain that the datum of an $\mathcal{X}$-graded 0-theory in $\mathcal{Y}$ is further equivalent to the datum of a 0-theory in $(PR)_\ast (PR)^\ast \mathcal{Y}$. This may be closer to the conventional contexts for the Day convolution.

The purpose of this section is to obtain an enriched generalization of this for the case $\ell = m - 1$, where enrichment is in a symmetric monoidal higher category.

Let us start with the following.

Definition 4.23. Suppose given
- an integer $d \geq 0$, and an $d$-theory $\mathcal{T}$ enriched in groupoids,
- an integer $k$ such that $0 \leq k \leq d$, and a symmetric monoidal $k$-category $\mathcal{A}$,
- functors $F, G : \mathcal{T} \rightarrow \mathcal{B}^{d-k} \Theta^k_0 \mathcal{A}$, or equivalently, $(F, G) : \mathcal{T} \rightarrow \mathcal{B}^{d-k} \Theta^k_0 (\mathcal{A} \times \mathcal{A})$.

Then we define a $\mathcal{T}$-graded $d$-theory $\text{Fun}(F, G)$ as $(F, G)^\ast \mathcal{B}^{d-k} \text{Fun}_{\mathcal{A}}$, where $\text{Fun}_{\mathcal{A}}$ denotes $\Theta^k_0 \text{Fun}([1], \mathcal{A})$ (where the $k$-category $\text{Fun}([1], \mathcal{A})$ is given the object-wise symmetric monoidal structure) considered as a $\Theta^k_0(\mathcal{A} \times \mathcal{A})$-graded $k$-theory via the symmetric monoidal functor

$$(d_1, d_0) : \text{Fun}([1], \mathcal{A}) \rightarrow \mathcal{A} \times \mathcal{A}.$$ 

induced from the simplicial coface operators $d^1, d^0 : [0] \rightarrow [1]$.

We would like to generalize the construction “Fun” for coloured theories. The following definition includes this.

Definition 4.24. Let
- $n \geq 0$ be an integer, and $\mathcal{U}$ be an $n$-theory enriched in groupoids with the unique functor $\Delta : \mathcal{U} \rightarrow 1^n_{\text{Com}}$,
- $m \geq 0$ be an integer, and $\mathcal{T}, \mathcal{J}$ be $\mathcal{U}$-graded $(m+1)$-theory enriched in groupoids,
- $k$ be an integer such that $0 \leq k \leq m + 1$, and $\mathcal{A}$ be a symmetric monoidal $k$-category,
- $F : \mathcal{T} \rightarrow \Delta^\ast \mathcal{B}^{m+1-k} \Theta^k_0 \mathcal{A}$ and $G : \mathcal{J} \rightarrow \Delta^\ast \mathcal{B}^{m+1-k} \Theta^k_0 \mathcal{A}$ be functors of $\mathcal{U}$-graded $(m + 1)$-theories.
For these, let $P : \Delta \Theta^N_{m+1} \mathcal{T} \to \Theta^N_n \mathcal{U}$ (where $N := \max\{m+1, n\}$) and $\tilde{Q} : R^* P^* \mathcal{J} \to \mathcal{T}$ be the respective projections, and $\tilde{P} : R^* P^* \mathcal{J} \to \mathcal{J}$ be the counit for the adjunction.

Then we define the $\mathcal{T}$-graded $(m+1)$-theory $\text{Fun}((\mathcal{T}, F), (\mathcal{J}, G))$ as $\tilde{Q}_! \text{Fun}(F \tilde{Q}, G \tilde{P})$, where

$$F \tilde{Q}, G \tilde{P} : \Delta \Theta^N_{m+1} R^* P^* \mathcal{J} \to \Theta^N_{m-k+1} B^{m-k+1} \mathcal{A}$$

are the indicated functors of symmetric $N$-theories.

This indeed gives the desired enriched generalization of the previous construction done using the right push-forward construction. Namely, we obtain the following in the special case where $k = 1$ and $\mathcal{A} = \text{Gpd}$ of the construction here.

**Proposition 4.25.** Let

- $n \geq 0$ be an integer, and $\mathcal{U}$ be an $n$-theory enriched in groupoids with the unique functor $\Delta : \mathcal{U} \to \mathbf{1}^n_{\text{set}}$.
- $m \geq 0$ be an integer, and $\mathcal{T}$ and $\mathcal{J}$ be $\mathcal{U}$-graded $(m+1)$-theories enriched in groupoids, each of which is terminal on a system of colours up to dimension $m-1$.
- $F : \mathcal{T} \to \Delta^* B^m \Theta_0 \text{Gpd}$ and $G : \mathcal{J} \to \Delta^* B^m \Theta_0 \text{Gpd}$ be functors of $\mathcal{U}$-graded $(m+1)$-theories.

Denote by $\mathcal{X}$ and $\mathcal{Y}$, the $\mathcal{U}$-graded $m$-theories defined by $F$ and $G$.

Then $\text{Fun}(\mathcal{X}, \mathcal{Y}) := \text{Fun}((\mathcal{T}, F), (\mathcal{J}, G))$ is equivalent to $R_! R^* P^* \mathcal{Y}$, where $R : \Theta_m \mathcal{X} \to \mathcal{T}$ and $P : \Delta \Theta^N_{m+1} \mathcal{T} \to \Theta^N_n \mathcal{U}$ (where $N \geq m + 1, n$) are the respective projections.

The proof is straightforward by direct inspection of the constructions.

It follows that, if $m \geq 1$, an $\mathcal{X}$-algebra in $\mathcal{Y}$ is equivalent as a datum to an $(m-1)$-theory in $\text{Fun}(\mathcal{X}, \mathcal{Y})$. Since the latter notion makes sense for any symmetric monoidal $k$-category $\mathcal{A}$, where $0 \leq k \leq m + 1$, we can think of it as the definition in such an enriched context, of an $\mathcal{X}$-algebra in $\mathcal{Y}$.

**5. Higher theorization of symmetric monoidal functor**

**5.0. The definition.**

5.0.0. We have so far considered iterated theorizations of algebra over a multicategory. One might wonder whether there are iterative theorizations of symmetric monoidal functor on a fixed symmetric monoidal category $\mathcal{B}$, to a varying target symmetric monoidal category.

This is actually a generalization of the previous case. Indeed, the functor $\Theta_0$ from symmetric monoidal categories to multicategories, has a left adjoint $L$, so, for a multicategory $\mathcal{U}$, a symmetric monoidal functor on $L \mathcal{U}$ is the same as a $\mathcal{U}$-algebra, which we have already theorized.

We would like to show here that, if a symmetric monoidal category $\mathcal{B}$ admits a certain concrete form of description, then we indeed obtain iterative theorizations of symmetric monoidal functor on $\mathcal{B}$, by replacing in Definition 1.5 of a symmetric higher theory, the coCartesian symmetric monoidal category Fin by $\mathcal{B}$.

It turns out that $\mathcal{B}$ may more generally be a symmetric monoidal higher category satisfying certain conditions.

**Remark 5.0.** When the dimension of the category $\mathcal{B}$ is at least 2, then a source of difficulty for having interesting symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ is a symmetric monoidal category, is that such a functor must invert all maps in $\mathcal{B}$ in dimensions $\geq 2$. However, this restriction will be discarded in one dimension at a time as theorization (in particular, relaxation) of the notion is iterated. Indeed, in the $n$-dimensional theory which we define below, the inversion of maps of $\mathcal{B}$ will be forced only in dimensions $\geq n + 2$. 

Let \( \mathcal{B} \) be a symmetric monoidal higher category, and denote the underlying higher category of \( \mathcal{B} \) by \( \mathcal{C} \), so \( \mathcal{B} \) is \( \mathcal{C} \) equipped with a symmetric monoidal structure. The conditions we would like to impose on \( \mathcal{B} \) are (0) and “(k)” below for all integers \( k \geq 1 \).

(0) The groupoid \( \text{Ob}\mathcal{C} \) of objects of \( \mathcal{C} \) is free as a commutative monoid on a groupoid of generators.

(k) Suppose given data of the forms \( (k-1) \) through \( (1) \) of \( (k) \) in Section 1.5 or equivalently, the arity of a \( k \)-multimap in an Init-graded higher theory. Then the groupoid formed by all \( k \)-multimap of the specified arity in the Init-graded \( k \)-theory \( \Theta_k^1\mathcal{C} \), is free on a groupoid as a commutative monoid under the symmetric monoidal structure of \( \mathcal{B} \).

Note that the groupoid of free generators is then the full subgroupoid consisting of indecomposable objects (from which we exclude the unit(s) by definition).

Remark 5.1. If \( \mathcal{B} \) is in fact a symmetric monoidal \( d \)-category for a finite value of \( d \), then the groupoid of \( (d+1) \)-multimaps, and more generally, of \( \phi^{d+1} \)-nerves of \( (d+1) \)-multimaps in \( \Theta_{d+1}^1\mathcal{C} \) (see Definition 1.5), will be equivalent to the groupoid of \( \phi^{d-1} \)-nerves of \( d \)-morphisms in \( \Theta_d^1\mathcal{C} \). In particular, the conditions \( (k) \) for all integers \( k \geq 0 \) hold in this case if the conditions hold for all \( k \leq d \), and the groupoid of \( \phi^{d-1} \)-nerves of \( d \)-morphisms is free as a commutative monoid for every specification of the arity.

In addition to \( \mathcal{B} = \text{Fin} \), the following are examples of \( \mathcal{B} \) satisfying these conditions. The symmetric monoidal structures are all given by the “disjoint union”.

- The 2-category \( \text{Corr}(\text{Fin}) \) of correspondences of finite sets, and its underlying 1-category \( \text{Corr}(\text{Fin}) \).
- The category \( \text{Bord}_d \) of compact 0-dimensional manifolds and (the groupoids of) 1-dimensional bordisms between them. Any choice of tangential structure on manifolds.
- The category \( \text{End}_{\text{Bord}_d}(\emptyset) \) of closed \( (d-1) \)-manifolds and (the groupoids of) \( d \)-dimensional bordisms. Any choice of tangential structure.
- The fully extended cobordism \( d \)-category \( \text{Bord}_d \) of bordisms up to dimension \( d \). Any choice of tangential structure.
- For an integer \( k \) such that \( 2 \leq k \leq d-1 \), the \( k \)-category \( \text{End}_{\text{Bord}_d}(\emptyset) \) of endomorphisms in \( \text{Bord}_d \) of the empty \( (d-k-1) \)-dimensional cobordism.
- The \( (d+1) \)-category of \( d \)-th iterated cocorrespondences in \( \text{Fin} \), and its underlying \( d \)-category. See e.g., Lurie [15, Section 3.2] for the idea, and Ben-Zvi and Nadler [2, Remark 1.17] for an explicit discussion of a definition which applies readily here. See also Calaque [5].

Remark 5.2. One can also define versions of the fully extended cobordism category where each bordism is given a codimension \( n \) embedding into the Euclidean space. These bordisms will form an \( E_n \)-monoidal \( d \)-category.

While our technique so far does not seem to apply directly for theorizing the notion of topological field theory on such a category, this kind of category seems close to satisfying an \( E_n \) analogue of the assumption required for our technique. We hope to treat theorization of these topological field theories in a sequel to this work.

Remark 5.3. Some other non-embedded (symmetric monoidal) variants of the cobordism category, such as discussed by Lurie in [15, Section 4], also satisfy the conditions of Remark 5.1.

5.0.1. Let now \( \mathcal{B} \) be a symmetric monoidal higher category satisfying the conditions \( (k) \) above for every integer \( k \geq 0 \). We shall obtain an \( n \)-th theorization of
symmetric monoidal functor (to a variable symmetric monoidal 1-category) on $B$. (See Remark 5.0.) Our $n$-theorized objects will be called “$B$-graded $n$-theories”.

A $B$-graded $n$-theory will consist of data similar to the data $(k)$ for $k \geq 0$, for a symmetric (i.e., “Fin-graded”) $n$-theory (see Section 1), but with appropriate modifications applied as follows.

Firstly, the form of datum $(0)$ (in the case $n \geq 1$) will be as follows.

$(0)$ For every indecomposable object $b$ of $B$, a collection $\text{Ob}_b U$, whose member will be called an object of $U$ of degree $b$.

We extend this for an arbitrary object $b \in B$ as follows. Namely, we let $\text{Ob}_b U$ be the collection of $b$-families of objects of $U$, defined as follows.

**Definition 5.4.** Let $b$ be an object of $B$. Then a $b$-family of objects of a $B$-graded higher theory $U$, is a pair consisting of

- a decomposition $b \simeq \bigotimes_{s \in S} c_s$, where $S$ is a finite set, and $c = (c_s)_{s \in S}$ is an $S$-family of indecomposable objects of $B$, and
- a $c$-family $u \in \text{Ob}_b U$, by which we mean that $u$ is an $S$-family $(u_s)_{s \in S}$, where $u_s \in \text{Ob}_{c_s} U$.

Let us next describe the type of a 1-multimap. This is where the true difference of a general $B$-graded theory from the case $B = \text{Fin}$ is seen. Namely, for a general $B$, a multimap in a $B$-graded theory will in general, not only accept multiple inputs, but also emit multiple outputs.

Thus, the type of a multimap in a $B$-graded higher theory $U$ consists of

$(0')$ a map $b^1 : b^0 \to b^0$ in $B$ which is indecomposable with respect to the commutative monoid structure,

$(0'')$ a $b^0$-family $u$ of objects of $U$, namely, $u = (u_i)_{i=0,1}$, where $u_i$ is a $b^0_i$-family of objects of $U$.

In general, we modify Definition 1.5 of a symmetric $n$-theory in the following two respects.

- We modify the form of datum $(k)$ for every $k \geq 1$ as follows. We
  - keep the forms of input data $(k - 1')$ through $(1')$ unchanged,
  - replace the nerve in $\text{Fin}$ in $(0')$ with a $k$-multimap in $\Theta_1^k \mathcal{C}$ (where $C$ denotes the higher category underlying $B$ as before) of the specified arity, which is moreover, indecomposable with respect to the commutative monoid structure,
  - modify the form of the rest of input data accordingly,
  - let the similar datum as associated in $(k)$ of Definition 1.5 be associated to the input data of these modified forms.

- We generalize the process of extension of the datum $(k)$ to a process of extension from indecomposable to arbitrary $k$-multimaps, using the free decomposition of $k$-multimaps just similarly to before.

**Example 5.5.** For the symmetric monoidal 1-category $B = \text{Cocorr(Fin)}$ underlying the symmetric monoidal 2-category of cocorrespondences in $\text{Fin}$, the notion of $B$-graded 1-theory enriched in a symmetric monoidal category $A$, coincides with the notion of coloured properad of Vallette in $A$ [17]. Thus, the notion of (coloured) properad is a theorization of the notion of $B$-algebra, and $B$-graded higher theories give further theorizations.

5.1. Symmetric monoidal functors as algebras in a theory.

5.1.0. We would like to see that, if $B$ is a symmetric monoidal $d$-category which satisfies our assumptions, then symmetric monoidal functors $B \to A$ with $A$ any symmetric monoidal $d$-category, are indeed included in our framework.
5.1.1. Let us start with the following. Let $\mathcal{B}$ be a symmetric monoidal $d$-category which satisfies our assumptions ($k$) for all integers $k \geq 0$. Then we would like to construct a $\mathcal{B}$-graded $d$-theory from a symmetric monoidal $d$-category $\mathcal{E}$ equipped with a symmetric monoidal functor $P: \mathcal{E} \to \mathcal{B}$. The $d$-theory, which we shall denote by $\mathcal{E}^\theta$, is as follows.

For a indecomposable object $b \in \mathcal{B}$, an object of $\mathcal{E}^\theta$ of degree $b$ is an object of $\mathcal{E}$ in the fibre over $b$.

For every integer $k$ such that $1 \leq k \leq d - 1$, the datum $(k)$ which we specify for $\mathcal{E}^\theta$ is inductively as follows. Denote by $C$ the $d$-category, i.e., $(d - 1)$-categorified Init-graded 1-theory, underlying $\mathcal{B}$. Suppose given

- an indecomposable $k$-multimap $b^k$ in $\Theta^\circ_c C$ of arity specified by data of the forms $(k - 1')$ through $(1')$ of $(k)$ in Section 5.5 and of type given as $b^{\leq k - 1} = (b^{p_0})_{0 \leq p \leq k - 1}$,

- a type $e = (e^{p_1})_{0 \leq p \leq k - 1}$ of a $k$-multimap in the $\mathcal{B}$-graded theory $\mathcal{E}^\theta$ of the same arity of degree $b^{\leq k - 1}$.

By induction, $e^{k - 1}_i$ (where $i = 0, 1$) will be a family consisting of lifts in $\mathcal{E}$ of the factors/components (in the unique decomposition in $\mathcal{B}$) of the (nerve of) $(k - 1)$-multimaps (or objects if $k = 1$) $b^{k - 1}_i$, so $\otimes e^{k - 1}_i$ in $\mathcal{E}$ lifts $b^{k - 1}_i$, where $\otimes$ indicates taking the monoidal product of the members of the family (which is $e^{k - 1}_i$ here) in $\mathcal{E}$. Moreover, if $k \geq 2$, then the $(k - 1)$-morphisms $\pi_i(\otimes e^{k - 1}_0)$ and $\otimes e^{k - 1}_1$ in $\mathcal{E}$ have common source and target by induction.

Given these data, we define a $k$-multimap $e^{k - 1}_0 \to e^{k - 1}_1$ in $\mathcal{E}^\theta$ of degree $b^k$, to be a lift of $b^k$ to a $k$-morphism $\pi_i(\otimes e^{k - 1}_0) \to \otimes e^{k - 1}_1$ (or $\otimes e^{k - 1}_0 \to \otimes e^{k - 1}_1$ if $k = 1$) in $\mathcal{E}$, completing the induction.

Similarly, the groupoids of $n$-multimaps in $\mathcal{E}^\theta$ will be the groupoids of similar lifts, and $n$-multimaps in $\mathcal{E}^\theta$ compose by the composition of $n$-multimaps in $\Theta^\circ_0 \mathcal{E}$.

Thus we have constructed a $\mathcal{B}$-graded $d$-theory $\mathcal{E}^\theta$.

Remark 5.6. This construction is not faithful in $\mathcal{E}$ (equipped with $P: \mathcal{E} \to \mathcal{B}$). Instead, the construction $(\ )^\theta$ gives (non-trivial) right localization functors of suitable categories.

5.1.2. Let us denote by $1^d_{\mathcal{B}}$, the terminal unenriched uncoloured $\mathcal{B}$-graded $d$-theory. Inspecting the construction above, it is easy to see that a $0$-theory in $\mathcal{E}^\theta$, or a functor $1^d_{\mathcal{B}} \to \mathcal{E}^\theta$, is equivalent as a datum to a section to the symmetric monoidal functor $P: \mathcal{E} \to \mathcal{B}$, which commutes with the symmetric monoidal structures, but is $(d - 1)$-lax as a functor.

Let now $\mathcal{A}$ be a symmetric monoidal $d$-category. Then we have the projection functor $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$, which is symmetric monoidal. It follows that a $0$-theory in $(\mathcal{A} \times \mathcal{B})^\theta$ is a symmetric monoidal $(d - 1)$-lax functor $\mathcal{B} \to \mathcal{A}$.

Remark 5.7. Even though the construction above has thus captured symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$ for every symmetric monoidal $1$-category $\mathcal{A}$, this is not the most interesting target if $d \geq 2$. However, in the case where $d \geq 2$ and the dimension of $\mathcal{A}$ is also $d$, the reader may be unsatisfied for the laxness which has crept in. (In relation to Remark 5.6, this laxness is due to the way how $\mathcal{B}$ as the terminal one among symmetric monoidal $d$-categories lying over $\mathcal{B}$, can fail to be local with respect to the right localization if $d \geq 2$.) Compared with $d$-lax symmetric monoidal functors $\mathcal{B} \to \mathcal{A}$, which could be captured in the framework of $\Theta^\circ_0 \mathcal{B}$-graded $d$-theories in the previous approach, our new approach here has only eliminated the laxness with respect to the symmetric monoidal structure. However, in order to deal with the remaining laxness with a similar technique, we would need to assume a finer version of the unique decomposition, which would be more difficult to be satisfied.
5.2. An example of different nature. In the case $\mathcal{B} = \text{Bord}_1$, there is an example of a 1-theory which is associated to a category rather than a symmetric monoidal category. An algebra in it will appear very different from a 1-dimensional field theory in the usual sense. Let us sketch these. The tangential structure we consider is framing, or equivalently, orientation.

Let $\mathcal{C}$ be a category. Then we can use it to construct a $\text{Bord}_1$-graded 1-theory as follows.

Firstly, we need to associate to every indecomposable object of $\text{Bord}_1$, a collection to be the collection of objects of that degree. To every 0-dimensional manifold consisting of one point $\text{pt}$ with any framing of $\text{pt} \times \mathbb{R}^1$, we associate the collection $\text{Ob}\mathcal{C}$.

Next, we need to associate to every indecomposable map in $\text{Bord}_1$, a groupoid to be the groupoid of 1-multimaps of that degree.

- If the bordism is diffeomorphic to the interval as a manifold, then to every object $x \in \text{Ob}\mathcal{C}$ at the incoming (relatively to the orientation) end point, and every object $y \in \text{Ob}\mathcal{C}$ at the outgoing end point, we associate the groupoid $\text{Map}_\mathcal{C}(x, y)$.
- If the bordism is diffeomorphic to the circle, then, for simplicity, we associate the terminal groupoid (but note Remark 5.8 below).

Finally, we need to define the composition operations. This can be given by the composition of maps in $\mathcal{C}$ (and its associativity).

Thus, we have sketched a construction of a $\text{Bord}_1$-graded 1-theory. Let us denote this theory by $\mathcal{Z}_\mathcal{C}$.

Note that, in the case where $\mathcal{C}$ is the unit category $\mathbf{1}$, we obtain $\mathcal{Z}_\mathbf{1} = \mathbf{1}_{\text{Bord}_1}$, the terminal $\text{Bord}_1$-graded 1-theory. It follows that, in general, any object $x \in \mathcal{C}$, or equivalently, a functor $\mathbf{1} \to \mathcal{C}$, induces a functor $\mathcal{Z}_x: \mathbf{1}_{\text{Bord}_1} \to \mathcal{Z}_\mathcal{C}$, which, by definition, is a field theory in $\mathcal{Z}_\mathcal{C}$.

The contractibility of the diffeomorphism group of the framed interval implies that $\mathcal{Z}_x$ for $x \in \mathcal{C}$ exhaust all field theories in $\mathcal{Z}_\mathcal{C}$.

Remark 5.8. There is another version of $\mathcal{Z}_\mathcal{C}$, in which the groupoid which we associate to the circle is the Hochschild homology $\text{HH}_n\mathcal{C} \simeq \int \mathcal{C} \text{Map}_\mathcal{C}(x, x)$. All the claims above also hold for this version of $\mathcal{Z}_\mathcal{C}$, as a result of the following observation (and simple computations).

The observation is as follows. Let $\uparrow, \downarrow$ denote the two 1-framed points of opposite framings, and let $\mathcal{I}$ denote the terminal category (i.e., $\text{Init}$-graded 1-theory) on the two colours “$\uparrow$” and “$\downarrow$” (so, as a category, $\mathcal{I}$ is a contractible groupoid). Let $\mathcal{T}_\mathcal{I}$ and $\mathcal{Th}_\text{Bord}$ respectively denote the categories of $\mathcal{I}$-graded and of $\text{Bord}_1$-graded 1-theories. There is an obvious adjunction $\Delta: \mathcal{T}_\mathcal{I} \rightleftarrows \text{Cat}: \Delta^*$, where $\Delta: \{\uparrow, \downarrow\} \to *$. The contractibility of the diffeomorphism group of the framed interval implies that there is also a functor $\mathcal{Th}_\text{Bord} \to \mathcal{T}_\mathcal{I}$ with left adjoint $\overline{\mathcal{Z}}$ satisfying $\mathcal{Z} = \overline{\mathcal{Z}} \circ \Delta^*$.

Appendix A. A comparison to the work of Baez and Dolan

A.0. We have been informed of resemblance between our work and the beautiful pioneering work as early as about two decades ago, of Baez and Dolan [0]. Since some of our purposes overlap theirs, and the methods also has great similarity, we think that a comparison of two works would be worthwhile.

Specifically, Baez and Dolan introduce what they call the “slice operad” construction for the purpose of defining the notion of “opetopic set”, which they use to give a definition of an $n$-category. The slice operad construction is not just interesting and powerful, but some of the ideas which they have developed for this construction, are quite close to some of the ideas which we have used for our work.
There seems to be no doubt therefore, that our work was shaped by the great influences from some ideas which go back to their work (or were at least popularized quite possibly as we imagine, through their work).

The “slice” construction constructs from a multicategory $O$, a new multicategory $O^+$. This looks close to our $\Theta O$, even though $\Theta O$ is a 2-theory, rather than a 1-theory. Since Baez and Dolan constructed $O^+$ as a multicategory, they did not need to introduce a new concept like our concept of higher theory. Moreover, iteration of their construction is automatic, unlike iteration of the process of theorization, which was the first main theme of our work. We recognize this as a great advantage of their construction.

A.1. This does not mean, however, that staying in the world of multicategories is necessary or desirable in all respects. We have succeeded after all, in defining all the higher notions of theory, and the new framework accommodates simpler approaches to some issues. For example, the case $n = 1$ of our Theorem 3.17 implies that the 2-theory $\Theta O$ is such that an uncoloured $\Theta O$-algebra is precisely the same as an $O^+$-algebra (described in the quotation below). Moreover, the construction of $\Theta O$ from $O$ was direct and immediate, while the construction of $O^+$ is one of the major steps in Baez and Dolan’s work.

Example A.0. In the case where $O$ is the associative operad, an $O^+$-algebra is precisely an uncoloured planar operad, while a $\Theta O$-algebra (which naturally has “colours” in general) is precisely a planar multicategory, i.e., a coloured planar operad.

The notion of theorization also clarify the work of Baez and Dolan conceptually. Let us first hear the description of the slice operad in the inventors’ own words. We shall quote from [0]. In the context at hand, their term “operad” means coloured operad (in sets, over which “algebras” are also considered in sets), and “type” means colour in our terminology.

“We define the “slice operad” $O^+$ of an operad $O$ in such a way that an algebra of $O^+$ is precisely an operad over $O$, i.e., an operad with the same set of types as $O$, equipped with an operad homomorphism to $O$. Syntactically, it turns out that:

1. The types of $O^+$ are the operations of $O$.
2. The operations of $O^+$ are the reduction laws of $O$.
3. The reduction laws of $O^+$ are the ways of combining reduction laws of $O$ to give other reduction laws.

This gets at the heart of the process of “categorification,” in which laws are promoted to operations and these operations satisfy new coherence laws of their own. Here the coherence laws arise simply from the ways of combining the the old laws.” [sic]

(John C. Baez and James Dolan [0 Section 1])

In their work, they observe the points (1), (2), (3) from the actual construction of $O^+$, but do not explicitly discuss the conceptual reason for why $O^+$ had to be related to categorification. The notion of theorization sheds light on this. Indeed, “an operad over $O^+$ in their definition, is precisely an (uncoloured) theorized O-monoid (or algebra “enriched” in sets), as those authors may have known in some formulation.

A.2. There are also other advantages in employing higher theories. For example, recall that the ultimate goal of Baez and Dolan was to give a definition of an $n$-category. For this, they needed a few more steps after defining the slice operad. On the other hand, a version of $n$-categories are already among the $n$-theories. To
examine the difference closely, we generally consider an \( n \)-theory formed not just by the \( n \)-multimaps, but with strata of colours consisting of objects to \((n - 1)\)-multimaps (which is the usual “colour” in an operad in the case \( n = 1 \)), and in a special case, the structure of an \( n \)-category is formed by these objects as objects, and the unary higher multimaps as higher morphisms. Contrary to this, Baez and Dolan do not consider an algebraic structure having more than one layer of colours since they consider only multicategories. This is the reason why they needed to find another route which might appear like a detour from the point of view of higher theories. (However, some opetopic sets appear to be modeling a version of initially graded higher theories, so their method merely does not appear as direct as one can wish to make it.)

The flexibility coming from the rooms for strata of colours, is also important for considering enrichment, since possibility for more interesting enrichment requires more strata of colours. Even though the purposes of Baez and Dolan did not motivate them to consider a very general notion of enrichment, their framework as built may not support a very interesting notion of enrichment, either. Note also that, even if one intends to work only with multicategories, enrichment of multicategories is most generally done along 2-theories.

A.3. From the quotation of Baez and Dolan’s words above, the idea expressed in the final two sentences is remarkable. Indeed, it is exactly the idea which we have described in Section 1.1 and used in our definition of an \( n \)-theory, except for two differences.

One difference is that we see the same, more generally at the heart of theorization. The other is that we have a simpler understanding of the “new coherence law”, in terms of the theorized form of the structure.

Now, our version of their idea has led to a process which keeps the complexity of structures from increasing rapidly by instead raising the theoretic order, and the resulting simplicity helped us enormously with various constructions concerning higher theories. (In those constructions, roles were also played by the flexibility from the rooms for strata of colours.)

Our version of their idea also helped us with treating some systems of operations with multiple inputs and multiple outputs.

A.4. To summarize, our work has benefited from the fruits of the developments which were initiated by such prominent works as Baez and Dolan’s.

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