Some unity results on entire functions and their difference operators related to 4 CM theorem

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Abstract
This paper is to consider the unity results on entire functions sharing two values with their difference operators and to prove some results related to 4 CM theorem. The main result reads as follows: Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a_1, a_2 \) be two distinct finite complex constants. If \( f(z) \) and \( \Delta_n f(z) \) share \( a_1 \) and \( a_2 \) "CM", then \( f(z) \equiv \Delta_n f(z) \), and hence \( f(z) \) and \( \Delta_n f(z) \) share \( a_1 \) and \( a_2 \) CM.

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1 Introduction and main results
It is well known that a monic polynomial is uniquely determined by its zeros and a rational function by its zeros and poles ignoring a constant factor. But it becomes much more complicated to deal with the transcendental meromorphic function case. In 1929, Nevanlinna proved his famous 5 IM theorem and 4 CM theorem (see e.g. [20, 23]): if meromorphic functions \( f(z) \) and \( g(z) \) share five (respectively, four) distinct values in the extended complex plane IM (respectively, CM), then \( f(z) \equiv g(z) \) (respectively, \( f(z) = T(g(z)) \), where \( T \) is a Möbius transformation). Here and in what follows, we say that \( f(z) \) and \( g(z) \) share the finite value \( a \) CM(IM) if \( f(z) - a \) and \( g(z) - a \) have the same zeros with the same multiplicities (ignoring multiplicities), and we say that \( f(z) \) and \( g(z) \) share the \( \infty \) CM(IM) if \( f(z) \) and \( g(z) \) have the same poles with the same multiplicities (ignoring multiplicities).

To relax those shared conditions in Nevanlinna’s 4 CM theorem, Gundersen provided an example to show that 4 CM shared values cannot be replaced with 4 IM shared values, but with 3 CM shared values and 1 IM shared value in [5]. That is, “4 IM ≠ 4 CM” and “3 CM + 1 IM = 4 CM”. In addition, he showed that “2 CM + 1 IM = 4 CM” in [6] (see correction in [8]), as well as by Mues in [17]. The problem that “1 CM + 3 IM = 4 CM” is still open. We recall the following result by Mues in [17], which mainly inspired us to write this paper.

Theorem A ([17]) Let \( f \) and \( g \) be nonconstant meromorphic functions sharing four distinct values \( a_j \) (\( j = 1, 2, 3, 4 \)) “CM”. If \( f \not\equiv g \), then \( f \) and \( g \) share \( a_j \) (\( j = 1, 2, 3, 4 \)) CM.
In Theorem A, \( f \) and \( g \) share the value \( a \) “CM” means that
\[
\overline{N}(r, \frac{1}{f-a}) - N_E(r, a) = S(r, f), \quad \text{and} \quad \overline{N}(r, \frac{1}{g-a}) - N_E(r, a) = S(r, g),
\]
where \( N_E(r, a) \) is defined to be the reduced counting function of common zeros of \( f(z) - a \) and \( g(z) - a \) with the same multiplicities. Similarly, \( N_E^{(1)}(r, a) \) used later is defined to be the reduced counting function of common simple zeros of \( f(z) - a \) and \( g(z) - a \).

Applying Theorem A, one can get (see Theorem 4.8 in [23]) the following.

**Theorem B** ([23]) Let \( f \) and \( g \) be nonconstant meromorphic functions and \( a_j \), \( j = 1, 2, 3, 4 \) be distinct values. If \( f \not\equiv g \) share \( a_j \), \( j = 1, 2, 3, 4 \) IM and if \( \overline{N}(r, \frac{1}{f-a}) = S(r, f) \) \( (j = 1, 2) \), then \( f \) and \( g \) share \( a_j \), \( j = 1, 2, 3, 4 \) CM.

**Remark 1.1** Let \( \delta(a, f) \) denote the deficiency of \( a \) with respect to \( f(z) \), which is defined as
\[
\delta(a, f) = \lim_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} + 1 - \lim_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.
\]
Then we see that the condition \( \overline{N}(r, \frac{1}{f-a}) = S(r, f) \) \( (j = 1, 2) \) in Theorem B means that \( \delta(a, f) = 1 \) \( (j = 1, 2) \). And we say that \( a \) is a Nevanlinna exceptional value of \( f(z) \), provided that \( \delta(a, f) > 0 \).

To reduce the number of shared values, Rubel and Yang appear to be the first to consider the unity of the entire function sharing two values with its first derivative in [21]. They proved that, for a nonconstant entire function \( f \), if \( f \) and \( f' \) share values \( a, b \) CM, then \( f \equiv f' \). Mues and Steinmetz [18] improved Rubel and Yang’s result by replacing “2 CM” with “2 IM” in 1979, and then by replacing “entire function” with “meromorphic function” in [19] (see also Gundersen [7]). In 2013, Li [15] improved these results by adding some condition on the poles of the meromorphic function \( f \).

This paper is to consider replacing the “derivative” with “difference operator”, which is defined as follows:
\[
\Delta_nf(z) = f(z + \eta) - f(z) \quad \text{and} \quad \Delta_n^{n+1}f(z) = \Delta_n^n f(z + \eta) - \Delta_n^nf(z), \quad n \in \mathbb{N}^*,
\]
where \( \eta \) is always a nonzero complex constant. This idea is partly due to the work by Heittokangas et al. in [12]. They were the first to consider a nonconstant meromorphic function \( f(z) \) sharing values with its shift \( f(z + \eta) \) and to prove the following.

**Theorem C** ([12]) Let \( f(z) \) be a meromorphic function of finite order, and let \( \eta \in \mathbb{C} \). If \( f(z) \) and \( f(z + \eta) \) share three distinct periodic functions \( a_1, a_2, a_3 \in \mathcal{S}(f) \) with period \( \eta \) CM, then \( f(z) = f(z + \eta) \) for all \( z \in \mathbb{C} \).

In Theorem C, \( \mathcal{S}(f) = S(f) \cup \{\infty\} \), where \( S(f) \) is the set containing all meromorphic functions \( a(z) \) satisfying
\[
T(r, a) = S(r, f), \quad \text{as} \ r \to \infty, r \notin E,
\]
where $E$ is an exceptional set of finite logarithmic measure. Theorem C can be read as a “3 CM” theorem and it has been improved to “2 CM + 1 IM” by Heittokangas et al. [13]. The key theory used in their research consists of the difference counterparts of Nevanlinna theory of meromorphic functions (see e.g. [3, 10, 11]).

In 2013, Chen and Yi [2] proved the following Theorem D, which was then extended to Theorem E by Cui and Chen in [4], and to Theorem F by Zhang and Liao in [24].

**Theorem D** ([2]) Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f)$ is not integer or infinite, and let $\eta$ be a constant such that $f(z + \eta) \neq f(z)$. If $f(z)$ and $\Delta_s f(z)$ share three distinct values $a, b, \infty$ CM, then $f(z) \equiv \Delta_s f(z)$.

**Theorem E** ([4]) Let $f(z)$ be a nonconstant meromorphic function of finite order, and let $\eta$ be a nonzero finite complex constant. Let $a, b$ be two distinct finite complex constants and $n$ be a positive integer. If $f(z)$ and $\Delta^n_s f(z)$ share $a, b, \infty$ CM, then $f(z) \equiv \Delta^n_s f(z)$.

**Theorem F** ([24]) Let $f(z)$ be a nonconstant entire function of finite order and $\eta$ be a nonzero finite complex constant. Let $a, b$ be two distinct finite complex constants. If $f(z)$ and $\Delta_s f(z)$ share $a, b$, CM, then $f(z) \equiv \Delta_s f(z)$.

**Remark 1.2** We will improve Theorems D–F by the following Theorem 1.1, whose proof is given with a different method from those in [2, 4, 24].

**Theorem 1.1** Let $f(z)$ be a nonconstant entire function of finite order, and let $a_1, a_2$ be two distinct finite complex constants. If $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ “CM”, then $f(z) \equiv \Delta^n_s f(z)$, and hence $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ CM.

**Theorem 1.2** Let $f(z)$ be a nonconstant entire function of finite order, and let $a_1, a_2$ be two distinct finite complex constants. If $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ IM, and

$$\overline{N}(r, \frac{1}{f - a_1}) = S(r,f) \quad (1.1)$$

holds, then $f(z) \equiv \Delta^n_s f(z)$, and hence $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ CM.

As a continuation of Theorem B and Theorem 1.2, we prove the following.

**Theorem 1.3** Let $f(z)$ be a nonconstant entire function of finite order, and let $a_1, a_2$ be two distinct finite complex constants. If $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ IM, and there exists some constant $\lambda > \frac{1}{2}$ such that $\delta(a_1 f) + \delta(a_2 f) \geq \lambda$, then $f(z) \equiv \Delta^n_s f(z)$, and hence $f(z)$ and $\Delta^n_s f(z)$ share $a_1$ and $a_2$ CM.

Other basic concepts and fundamental results of the Nevanlinna theory of meromorphic functions (see e.g. [14, 23]) may be used directly in what follows.

### 2 Lemmas

Now we recall two lemmas which are important in the proofs of our theorems. The first lemma has been used frequently in dealing with value sharing problems related to difference operators.
Lemma 2.1 ([11]) Let $\eta \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then, for any small periodic function $a(z)$ with period $\eta$, with respect to $f(z)$,

$$m\left(r, \frac{\Delta_n^a f}{f-a}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof Suppose that $f$ havethe following.

According to the second fundamental theorem and Lemma 2.1, we can easily derivethat

Let $f(z)$ be a nonconstant meromorphic function, $a$ be an arbitrary complex number, and $k$ be a positive integer. Then

(i) $\overline{N}(r, \frac{1}{f-a}) \leq \frac{k}{k+1} \overline{N}_{\delta}(r, \frac{1}{f-a}) + \frac{1}{k+1} N(r, \frac{1}{f-a})$;

(ii) $\overline{N}(r, \frac{1}{f-a}) \leq \frac{k}{k+1} \overline{N}_{\delta}(r, \frac{1}{f-a}) + \frac{1}{k+1} T(r,f) + O(1)$.

Lemma 2.3 Suppose that $a_1, a_2 \in \mathbb{C}$ satisfying $a_1 \neq a_2$, $f(z)$ is a nonconstant entire function of finite order sharing $a_1$ and $a_2$ "CM" with $\Delta_n^a f(z)$. If $f(z) \neq \Delta_n^a f(z)$, then

$$T(r,f) + S(r,f) = \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{f-a_j}\right) = \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_n^a f-a_j}\right).$$

What is more, if $f(z) \neq \Delta_n^a f(z)$ and (1.1) holds, then

(i) $T(r, \Delta_n^a f) = T(r,f) + S(r,f)$;

(ii) $\forall b \in \mathbb{C} \setminus \{a_1, a_2\}, \overline{N}(r, \frac{1}{f-b}) = T(r,f) + S(r,f), \overline{N}(r, \frac{1}{f-a_1 b}) = T(r,f) + S(r,f)$;

(iii) $\overline{N}(r, \frac{1}{f}) = S(r,f), \overline{N}(r, \frac{1}{f-a_1}) = S(r,f)$;

(iv) $\overline{N}(r, a_1) + \overline{N}(r, a_2) = S(r,f)$, where $\overline{N}(r, a_i)$ is the reduced counting function of the multiple common zeros of $f-a_i$ and $\Delta_n^a f-a_i$ ($i = 1, 2$).

Proof Suppose that $f(z) \neq \Delta_n^a f(z)$. Since $f(z)$ and $\Delta_n^a f(z)$ share two values $a_1$ and $a_2$ "CM", according to the second fundamental theorem and Lemma 2.1, we can easily derive that

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r,f)$$

$$= \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_n^a f-a_j}\right) + \sum_{j=1}^{2} \left[ \overline{N}\left(r, \frac{1}{f-a_j}\right) - N_E(r,a_j) \right] + S(r,f)$$

$$\leq \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_n^a f-a_j}\right) + S(r,f) \leq \overline{N}\left(r, \frac{1}{f-\Delta_n^a f}\right) + S(r,f)$$

$$\leq T(r,f-\Delta_n^a f) + S(r,f) = m(r,f-\Delta_n^a f) + S(r,f)$$

$$\leq m\left(r, \frac{\Delta_n^a f}{f}\right) + m(r,f) + S(r,f) \leq T(r,f) + S(r,f).$$

Hence we prove the first conclusion.
Suppose that \( f(z) \not\equiv \Delta^n f(z) \) and (1.1) holds, and we prove conclusions (i)–(iv) step by step.

**Step 1.** Notice that \( f(z) \) and \( \Delta^n f(z) \) share the value \( a_1 \) “CM” and (1.1) imply that

\[
\overline{N} \left( r, \frac{1}{\Delta^n f - a_1} \right) = \overline{N} \left( r, \frac{1}{f - a_1} \right) + \left[ \overline{N} \left( r, \frac{1}{\Delta^n f - a_j} \right) - N_F(r, a_j) \right] = S(r,f).
\]  

(2.1)

Then, applying the second fundamental theorem again, we have

\[
T(r, \Delta^n f) \leq \overline{N} \left( r, \frac{1}{\Delta^n f - a_1} \right) + \overline{N} \left( r, \frac{1}{\Delta^n f - a_2} \right) + S(r, \Delta^n f) 
\]

\[
\leq \overline{N} \left( r, \frac{1}{\Delta^n f - a_2} \right) + S(r,f) \leq T(r, \Delta^n f) + S(r,f).
\]

From this and the second equality in the first conclusion, we can see that

\[
T(r, \Delta^n f) = T(r,f) + S(r,f).
\]

**Step 2.** For all \( b \in \mathbb{C} \setminus \{a_1, a_2\} \), from the second fundamental theorem, the second equality in the first conclusion, and conclusion (i), we can derive that

\[
2T(r,f) + S(r,f) = 2T(r, \Delta^n f) 
\]

\[
\leq \overline{N} \left( r, \frac{1}{\Delta^n f - a_1} \right) + \overline{N} \left( r, \frac{1}{\Delta^n f - a_2} \right) + S(r, \Delta^n f) 
\]

\[
\leq T(r,f) + \overline{N} \left( r, \frac{1}{\Delta^n f - b} \right) + S(r,f) 
\]

\[
\leq T(r,f) + T(r, \Delta^n f) + S(r,f) = 2T(r,f) + S(r,f),
\]

which leads to

\[
\overline{N} \left( r, \frac{1}{\Delta^n f - b} \right) = T(r,f) + S(r,f).
\]

Similarly, we can prove that the following equality holds:

\[
\overline{N} \left( r, \frac{1}{f - b} \right) = T(r,f) + S(r,f).
\]

**Step 3.** Set

\[
h(z) = \frac{(\Delta^n f')}{\Delta^n f - a_1}.
\]  

(2.2)

Then we get from (2.1) and the lemma of logarithmic derivatives that

\[
T(r,h) = m(r,h) + N(r,h) 
\]

\[
= m \left( r, \frac{(\Delta^n f')}{\Delta^n f - a_1} \right) + \overline{N} \left( r, \frac{1}{\Delta^n f - a_1} \right) = S(r,f). 
\]  

(2.3)
It is obvious that \( h(z) \neq 0 \) since \( \Delta^nf(z) \) is not a constant. Hence from (2.1)–(2.3) we can deduce that

\[
\mathcal{N}(r, \frac{1}{(\Delta^nf)'}) \leq \mathcal{N}(r, \frac{1}{\Delta^nf - a_1}) + \mathcal{N}(r, \frac{1}{\Delta^nf - a_2}) \\
\leq \mathcal{N}(r, \frac{1}{\Delta^nf - a_1}) + T(r, h) = S(r, f).
\]

Similarly, we can prove that \( \mathcal{N}(r, \frac{1}{g}) = S(r, f) \).

Step 4. Consider the following function:

\[
g(z) = \frac{f'(f - \Delta^nf)}{(f - a_1)(f - a_2)}. \tag{2.4}
\]

The condition that \( f(z) \) and \( \Delta^nf(z) \) share two values \( a_1 \) and \( a_2 \) “CM” ensures that \( g(z) \) is a meromorphic function such that all poles of \( g(z) \) consist of zeros of \( f(z) - a_1 \) and \( f(z) - a_2 \).

We obtain from Lemma 2.1 and the lemma of logarithmic derivatives that

\[
T(r, g) = m(r, g) + N(r, g) = m(r, \frac{f'(f - \Delta^nf)}{(f - a_1)(f - a_2)}) + N(r, \frac{f'(f - \Delta^nf)}{(f - a_1)(f - a_2)}) \\
\leq m\left(r, \frac{f'}{(f - a_1)(f - a_2)}\right) + m\left(r, \frac{f'}{f - a_2}\right) \\
+ \sum_{j=1}^2 \left[ \mathcal{N}(r, \frac{1}{f - a_j}) - N_E(r, a_j) \right] \\
\leq m\left(r, \frac{a_1}{a_1 - a_2}, \frac{f'}{f - a_1}\right) + m\left(r, \frac{a_2}{a_1 - a_2}, \frac{f'}{f - a_2}\right) + S(r, f) \\
= S(r, f).
\]

Let \( z_j \ (j = 1, 2, \ldots) \) be the multiple common zeros of \( f - a_i \) and \( \Delta^nf - a_i \) \((i = 1, 2)\), and let \( m_{ij} \) and \( n_{ij} \) be the multiplicities of the zero \( z_j \) of \( f - a_i \) and \( \Delta^nf - a_i \), respectively. Note that \( m_{ij}, n_{ij} \geq 2 \). It follows from expression (2.4) of \( g(z) \) that \( z_j \ (j = 1, 2, \ldots) \) are zeros of \( g(z) \) with multiplicity at least \( \min\{m_{ij}, n_{ij}\} - 1 \geq 1 \). This and (2.5) show that

\[
\mathcal{N}(r, a_1) + \mathcal{N}(r, a_2) \leq \mathcal{N}(r, \frac{1}{g}) \leq T(r, g) = S(r, f). \tag{2.5}
\]

Remark 2.1 Checking the proof of Lemma 2.3 carefully, we can see that all the conclusions of it still hold when 2 “CM” is replaced with 2 IM.

Lemma 2.4 Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a_1, a_2 \) be two distinct finite complex constants. If \( f(z) \) and \( \Delta^nf(z) \) share \( a_1 \) and \( a_2 \) IM and (1.1) holds, then \( f(z) \) and \( \Delta^nf(z) \) share \( a_1 \) and \( a_2 \) “CM”.

Proof It is easy to find that \( f(z) \) and \( \Delta^nf(z) \) share \( a_1 \) “CM”, since \( f(z) \) and \( \Delta^nf(z) \) share the value \( a_1 \) IM and (1.1) holds.
From the second fundamental theorem and (1.1), we have

\[ T(r,f) \leq \overline{N}(r, \frac{1}{f-a_1}) + \overline{N}(r, \frac{1}{f-a_2}) + S(r,f) \]

\[ = \overline{N}(r, \frac{1}{f-a_2}) + S(r,f). \]  \hspace{1cm} (2.6)

Let \( k = 1 \). Then (ii) in Lemma 2.2 can be rewritten as

\[ \overline{N}(r, \frac{1}{f-a_2}) \leq \frac{1}{2} \overline{N}_1(r, \frac{1}{f-a_2}) + \frac{1}{2} T(r,f) + O(1). \]  \hspace{1cm} (2.7)

(2.6) and (2.7) give

\[ T(r,f) \leq \overline{N}_1(r, \frac{1}{f-a_2}) + S(r,f). \]

And due to

\[ \overline{N}_1(r, \frac{1}{f-a_2}) \leq \overline{N}(r, \frac{1}{f-a_2}) \leq T(r,f) + S(r,f), \]

the above inequality implies

\[ T(r,f) = \overline{N}_1(r, \frac{1}{f-a_2}) + S(r,f) = \overline{N}(r, \frac{1}{f-a_2}) + S(r,f), \]  \hspace{1cm} (2.8)

and thus

\[ \overline{N}_2(r, \frac{1}{f-a_2}) = S(r,f). \]

Similarly, the following equality holds:

\[ T(r, \Delta^n f) = \overline{N}_1(r, \frac{1}{\Delta^n f - a_2}) + S(r, \Delta^n f) = \overline{N}(r, \frac{1}{\Delta^n f - a_2}) + S(r, \Delta^n f). \]

Then, by (i) in Lemma 2.3, we can derive that

\[ T(r,f) = \overline{N}_1(r, \frac{1}{\Delta^n f - a_2}) + S(r,f) = \overline{N}(r, \frac{1}{\Delta^n f - a_2}) + S(r,f), \]  \hspace{1cm} (2.9)

and thus

\[ \overline{N}_2(r, \frac{1}{\Delta^n f - a_2}) = S(r,f). \]  \hspace{1cm} (2.10)

By (iv) in Lemma 2.3, one can easily see that

\[ N_E(r, a_2) - N_E^{(1)}(r, a_2) \leq \overline{N}(r, a_2) = S(r,f). \]  \hspace{1cm} (2.11)
It follows from (2.8), (2.10), and (2.11) that

\[
\overline{N}\left(r, \frac{1}{f-a_2}\right) - N_E(r,a_2) \\
= \overline{N}\left(r, \frac{1}{f-a_2}\right) - N_E^{(1)}(r,a_2) + S(r,f) \\
\leq \overline{N}\left(r, \frac{1}{f-a_2}\right) - \left(\overline{N}_1\left(r, \frac{1}{f-a_2}\right) - \overline{N}_2\left(r, \frac{1}{\Delta^nf-a_2}\right)\right) + S(r,f) \\
= S(r,f).
\]

(2.12)

Since \(f(z)\) and \(\Delta^nf(z)\) share \(a_2\) IM, from (2.12) we obtain that

\[
\overline{N}\left(r, \frac{1}{\Delta^nf-a_2}\right) - N_E(r,a_2) = S(r,f).
\]

(2.13)

Thus, \(f(z)\) and \(\Delta^nf(z)\) share the value \(a_2\) “CM”. \(\square\)

**Remark 2.2** We can find that (2.8), (2.12), and (2.13) used in the proof of Theorem 1.1 still hold when 2 IM is replaced with 2 “CM”.

**Lemma 2.5** ([3]) Let \(f(z)\) be a meromorphic function of finite order \(\rho\), \(\varepsilon\) be a positive constant, \(\eta_1\) and \(\eta_2\) be two distinct nonzero complex constants. Then there exists a subset \(E \subset (1, +\infty)\) of finite logarithmic measure such that, for all \(z\) satisfying \(|z| = r \in [0,1] \cup E\) and as \(r \to \infty\) sufficiently large,

\[
\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}.
\]

**Lemma 2.6** ([1, 9]) Let \(f(z)\) be a meromorphic function with finite order \(\rho\). Then, for any given \(\varepsilon > 0\), there exists a set \(E \subset (1, +\infty)\) of finite linear measure such that, for all \(z\) satisfying \(|z| = r \in [0,1] \cup E\) and \(r\) sufficiently large,

\[
\exp\{-r^{\rho+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\rho+\varepsilon}\}.
\]

**Lemma 2.7** ([23]) Suppose that \(f(z)\) is a nonconstant meromorphic function in \(|z| < R\) and \(a_j\) \((j = 1, 2, \ldots, q)\) are \(q\) distinct finite complex numbers. Then

\[
m\left(r, \sum_{j=1}^{q} \frac{1}{f-a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + O(1)
\]

holds for \(0 < r < R\).

### 3 Proof of Theorem 1.1

Suppose that \(f(z) \not\equiv \Delta^nf(z)\). Since \(f(z)\) is a nonconstant entire function sharing \(a_1\) and \(a_2\) “CM” with \(\Delta^nf(z)\),

\[
\frac{\Delta^nf(z) - a_1}{f(z) - a_1} = p_1(z)e^{q_1(z)}
\]

(3.1)
where \( p_j(z) \) are meromorphic functions such that \( \rho(p_j) < \rho(f) \) \((j = 1, 2)\), and \( q_1(z), q_2(z) \) are polynomials such that \( \deg p_1(z) \leq \rho(f) \), \( \deg q_2(z) \leq \rho(f) \).

If \( p_1(z)e^{\eta_1(z)} \equiv p_2(z)e^{\eta_2(z)} \), then we get from (3.1) and (3.2) that \( f(z) \equiv \Delta^n f(z) \). Next, we suppose that \( p_1(z)e^{\eta_1(z)} \not\equiv p_2(z)e^{\eta_2(z)} \). From (3.1) and (3.2), we have

\[
\frac{\Delta^n f(z) - a_2}{f(z) - a_2} = \frac{p_2(z)e^{\eta_2(z)}}{p_1(z)e^{\eta_1(z)} - p_2(z)e^{\eta_2(z)}}. \tag{3.3}
\]

Since \( \rho(p_j) < \rho(f) \) \((j = 1, 2)\), we can deduce from (3.3) that almost all (except at most \( S(r, f) \)) zeros of \( f(z) - a_2 \) are zeros of \( g(z) := 1 - p_2(z)e^{\eta_2(z)} \). Hence

\[
N\left(r, \frac{1}{f - a_1}\right) \leq N\left(r, \frac{1}{1 - p_2e^{\eta_2}}\right) + S(r, f) \leq T(r, e^{\eta_2}) + S(r, f). \tag{3.4}
\]

Next, we discuss two cases.

**Case 1:** \( \deg q_2(z) < \rho(f) \). It follows from (3.4) that

\[
N\left(r, \frac{1}{f - a_1}\right) = S(r, f).
\]

Therefore, Lemma 2.3 is valid now. Let us consider the following two functions:

\[
F(z) = \frac{\Delta^n f - a_1}{\Delta^n f - a_2}, \quad G(z) = \frac{f - a_1}{f - a_2}. \tag{3.5}
\]

Notice that \( f(z) \) and \( \Delta^n f(z) \) share two values \( a_1 \) and \( a_2 \) “CM”, and we see that \( F(z) \) and \( G(z) \) are meromorphic functions sharing 0 and \( \infty \) “CM”. By (3.5), (ii) in Lemma 2.3, and the Valiron–Mokhon’ko theorem (see e.g. [16, 22]), we have

\[
T(r, F) = T(r, \Delta^n f) + S(r, \Delta^n f) = T(r, f) + S(r, f), \tag{3.6}
\]

\[
T(r, G) = T(r, f) + S(r, f).
\]

Let

\[
\wp(z) = \frac{F''}{F'} - \frac{G''}{G'}, \tag{3.7}
\]

and we get, by applying the lemma of logarithmic derivatives,

\[
m(r, \wp) \leq m\left(r, \frac{F''}{F'}\right) + m\left(r, \frac{G''}{G'}\right) + O(1)
= S(r, F') + S(r, G') = S(r, f).
\tag{3.8}
\]

Clearly, (3.7) shows that the poles of \( \wp(z) \) are simple, and they can only come from the zeros of \( F'(z) \) and \( G'(z) \) as well as the poles of \( F(z) \) and \( G(z) \).
In the following, suppose that \( z_2 \) is a pole of \( F(z) \) and \( G(z) \) with the same multiplicity \( k \), which comes from the zero \( z_2 \) of \( f - a_2 \) and \( \Delta_0^nf - a_2 \) with the same multiplicity \( k \). And suppose that the following two expansions hold in the neighborhood of \( z - z_2 \):

\[
F(z) = \frac{A_{-k}}{(z-z_2)^k} + \frac{A_{-k+1}}{(z-z_2)^{k-1}} + \cdots,
\]

\[
G(z) = \frac{B_{-k}}{(z-z_2)^k} + \frac{B_{-k+1}}{(z-z_2)^{k-1}} + \cdots,
\]

a simple computation shows that

\[
\phi = \frac{F''}{F} - \frac{G''}{G} = \left( -\frac{k+1}{z-z_2} + \frac{(k-1)A_{-k+1}}{kA_{-k}} + O(z-z_2) \right) - \left( -\frac{k+1}{z-z_2} + \frac{(k-1)B_{-k+1}}{kB_{-k}} + O(z-z_2) \right) = \frac{k-1}{k} \left( \frac{A_{-k+1}}{A_{-k}} - \frac{B_{-k+1}}{B_{-k}} \right) + O(z-z_2),
\]

which implies that \( z_2 \) is not the pole of \( \psi(z) \).

To consider the zeros of \( F''(z) \) and \( G''(z) \), we derive from (3.5) that

\[
F'' = \frac{(a_1 - a_2)(\Delta_0^nf')}{(\Delta_0^nf - a_2)^2}, \quad G'' = \frac{(a_1 - a_2)f'}{(f - a_2)^2}. \tag{3.10}
\]

Now (iv) in Lemma 2.3 and (3.10) imply that

\[
N \left( r, \frac{1}{F''} \right) \leq N \left( r, \frac{1}{(\Delta_0^nf')}, \frac{1}{(\Delta_0^nf - a_2)^2} \right) = S(r, f), \quad N \left( r, \frac{1}{G''} \right) \leq N \left( r, \frac{1}{f'} \right) = S(r, f). \tag{3.11}
\]

Then, by (2.12),(2.13),(3.5),(3.7), and (3.11), we can deduce that

\[
N(r, \psi)
\leq N \left( r, \frac{1}{F''} \right) + N \left( r, \frac{1}{G''} \right) + N(r, F) + N(r, G) - 2N_E(r, a_2) + S(r, f) \tag{3.12}
\]

\[
= N \left( r, \frac{1}{\Delta_0^nf - a_2} \right) + N \left( r, \frac{1}{f - a_2} \right) - 2N_E(r, a_2) + S(r, f) = S(r, f).
\]

Thus (3.8) and (3.12) give immediately

\[
T(r, \psi) = m(r, \psi) + N(r, \psi) = S(r, f). \tag{3.13}
\]

If \( \psi(z) \neq 0 \), suppose that \( z^*_2 \) is a simple common pole of \( F(z) \) and \( G(z) \), which comes from the simple common zero \( z_2 \) of \( f - a_2 \) and \( \Delta_0^nf - a_2 \). Then (3.9) implies that \( z^*_2 \) is a zero of \( \psi(z) \) with the multiplicity at least 1, which means that

\[
N_E^1(r, a_2) \leq N \left( r, \frac{1}{\psi} \right) \leq T(r, \psi) = S(r, f). \tag{3.14}
\]
Combining (2.8), (2.11), and (2.12) shows that

\[
N^{(1)}_{E}(r, a_2) = N_{E}(r, a_2) + S(r, f) = \mathcal{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f) = T(r, f) + S(r, f).
\]  

(3.15)

Thus clearly \(T(r, f) \leq S(r, f)\) follows immediately from (3.14) and (3.15). That is impossible.

Now, we have proved that \(\varphi(z) \equiv 0\), that is,

\[
\frac{F''}{F'} = \frac{G''}{G'}.
\]

Taking integration of this identity twice, we can derive that

\[
F \equiv \alpha G + \beta,
\]

(3.16)

where \(\alpha \neq 0\) and \(\beta\) are constants.

Next, we discuss two subcases.

Case 1.1: \(a_1\) is not a Nevanlinna exceptional value of \(f(z)\). The condition that \(f(z)\) and \(\Delta^n f(z)\) share the value \(a_1\) “CM” ensures that there exists \(z_3\) such that \(f(z_3) = \Delta^n f(z_3) = a_1\), and then from (3.5), \(F(z_3) = G(z_3) = 0\). Therefore \(\beta = 0\), and (3.16) becomes

\[
F \equiv \alpha G.
\]

(3.17)

Since \(f(z) \neq \Delta^n f(z)\), we see that \(\alpha \neq 1\). Thus, 1 must be a Picard exceptional value of \(F(z)\) and \(G(z)\), we can deduce easily from (3.17) that 1, \(\alpha\) are Picard exceptional values of \(F(z)\), and 1, \(\frac{1}{\alpha}\) are Picard exceptional values of \(G(z)\). This fact and (3.5) show that

\[
G = f - a_1 \neq \frac{1}{\alpha}.
\]

That is,

\[
f \neq \frac{a_1 \alpha - a_2}{\alpha - 1},
\]

(3.18)

which means that \(\frac{a_1 \alpha - a_2}{\alpha - 1}\) is a Picard exceptional value of \(f(z)\). Obviously, \(\frac{a_1 \alpha - a_2}{\alpha - 1} \neq a_1, a_2\).

On the other hand, by (ii) in Lemma 2.3, the following equality holds:

\[
\mathcal{N}\left(r, \frac{1}{f - \frac{a_1 \alpha - a_2}{\alpha - 1}}\right) = T(r, f) + S(r, f).
\]

This contradicts (3.18).

Case 1.2: \(a_1\) is a Nevanlinna exceptional value of \(f(z)\). Since \(f(z)\) and \(\Delta^n f(z)\) share \(a_1\) “CM”, \(a_1\) is also a Nevanlinna exceptional value of \(\Delta^n f\), and thus 0 is a Nevanlinna exceptional value of \(F(z)\) and \(G(z)\). From (3.5) and (3.16), we see that 0, 1, \(\beta, \alpha + \beta\) and 0, 1, \(-\frac{\beta}{\alpha}\),
\(1-\beta\) are Nevanlinna exceptional values of \(F(z)\) and \(G(z)\), respectively. As \(f(z) \neq \Delta^n f(z)\), we get \(\alpha \neq 1\). Hence \(\beta = 1\), \(\alpha + \beta = 0\). And now (3.16) is of the form

\[
F \equiv -G + 1. \tag{3.19}
\]

From (3.19), we know that \(F(z)\) and \(G(z)\) share \(\frac{1}{2}\) CM, which implies that \(f(z)\) and \(\Delta^n f(z)\) share another value \(2a_1 - a_2\) \((\neq a_1, a_2)\) CM. Then

\[
\overline{N}(r, \frac{1}{f - a_2}) + \overline{N}(r, \frac{1}{f - (2a_1 - a_2)}) \leq \overline{N}(r, \frac{1}{f - \Delta^n f}) \leq T(r, f - \Delta^n f) = m(r, f - \Delta^n f) \leq m\left(\frac{\Delta^n f}{f}\right) + m(r, f) \leq T(r, f) + S(r, f).
\]

This and (3.15) yield \(\overline{N}(r, \frac{1}{f - (2a_1 - a_2)}) = S(r, f)\). On the other hand, by (i) in Theorem 2.3, the following equality holds:

\[
\overline{N}(r, \frac{1}{f - (2a_1 - a_2)}) = T(r, f) + S(r, f).
\]

This is a contradiction.

Case 2: \(\deg q_2(z) = \rho(f)\). If \(\deg q_1(z) < \rho(f)\), then we can deduce similar contradictions as in Case 1. Thus, \(\deg q_1(z) = \rho(f)\). Suppose that \(\deg q_1(z) = \deg q_2(z) = \rho(f) = d\). Obviously, \(d \geq 1\). Otherwise, we get a contradiction from (3.3) that

\[
\rho(f) \leq \max\{\rho(p_1), \rho(p_2)\}.
\]

Set

\[
q_1(z) = A_d z^d + A_{d-1} z^{d-1} + \cdots + A_0
\]

and

\[
q_2(z) = B_d z^d + B_{d-1} z^{d-1} + \cdots + B_0,
\]

then \(A_d B_d \neq 0\). Denote \(A_d = r_1 e^{i\theta_1}, B_d = r_2 e^{i\theta_2}, A_d + B_d = r_3 e^{i\theta_3}\), where \(\theta_j \in [-\pi, \pi], j = 1, 2, 3\).

From (3.1) and (3.2), we have

\[
\Delta^n f = \frac{\Delta^n f}{f - a_1} = \frac{a_2 p_1 e^{i\theta_1} - a_1 p_2 e^{i\theta_2} + (a_1 - a_2)p_1 p_2 e^{i\theta_1 + \theta_2}}{(a_2 - a_1)(1 - p_2 e^{i\theta_2})}. \tag{3.20}
\]

Notice that

\[
\Delta^n f = \Delta^n (f - a_j) = \sum_{j=0}^{n} (-1)^j C^n_j (f(z + (n-j)\eta) - a_j), \quad j = 1, 2.
\]
Set \( q = \max\{\rho(p_1), \rho(p_2)\} \) and \( \epsilon = \min\{\frac{2q}{3}, \frac{1}{2}\} \), then applying Lemma 2.5 we see that there exists a subset \( E_1 \subset (1, +\infty) \) of finite logarithmic measure such that, for all \( z \) satisfying \( |z| = r \notin [0, 1] \cup E_1 \) and as \( r \to \infty \) sufficiently large,

\[
\exp\left[-r^{d-1+\epsilon}\right] \leq \left| \frac{\Delta^2 f}{f - a_1} \right| \leq \exp\left[r^{d-1+\epsilon}\right], \quad j = 1, 2. \tag{3.21}
\]

By Lemma 2.6, for \( \epsilon \) given above, there exists a set \( E_2 \subset (1, +\infty) \) of finite linear measure such that, for all \( z \) satisfying \( |z| = r \notin [0, 1] \cup E_2 \) and \( r \) sufficiently large,

\[
\exp\left[-r^{d+\epsilon}\right] \leq |p_j(z)| \leq \exp\left[r^{d+\epsilon}\right], \quad j = 1, 2. \tag{3.22}
\]

**Case 2.1:** \( r_1 > \max\{r_2, r_3\} := r_4 \). For the point \( \theta_1/d \in [-\pi, \pi) \), we see that, for all \( z = |z|e^{i\theta_1} = r e^{i\theta_1} \),

\[
A_d z^d = r_4 z^d > r_3 z^d = \max\{r_2 z^d, r_3 z^d\} \geq \max\{\Re A_d z^d, \Re (A_d + B_d) z^d\}. \tag{3.23}
\]

Then we deduce from (3.20)–(3.22) that, for all \( z = r e^{i\theta_1} \) satisfying \( |z| = r \notin [0, 1] \cup E_1 \cup E_2 \) and \( r \) sufficiently large,

\[
|a_2| \exp\left[r_1 z^d (1 + o(1)) - r^{d+\epsilon}\right] \\
< |a_2 p_1 e^{i\theta_1}| = \left| (a_2 - a_1)(1 - p_2 e^{i\theta_2}) \right| \frac{\Delta^2 f}{f - a_1} + |a_1 p_2 e^{i\theta_2} - (a_1 - a_2) p_1 p_2 e^{i\theta_1 + \epsilon q_2}| \leq \left| (a_2 - a_1)(1 - p_2 e^{i\theta_2}) \right| \frac{\Delta^2 f}{f - a_1} + |a_1 p_2 e^{i\theta_2}| + |(a_1 - a_2) p_1 p_2 e^{i\theta_1 + \epsilon q_2}| \leq (|a_1| + |a_2|)(1 + \exp|\exp r^{d+\epsilon}|) \exp|\exp r^{d-1+\epsilon}| + |a_1| \exp\left|r_3 z^d (1 + o(1)) + r^{d+\epsilon}\right| \\
< \exp\left|r_4 z^d (1 + o(1))\right|. \]

However, from (3.23) we see that this is impossible.

**Case 2.2:** \( r_2 > \max\{r_1, r_3\} \). Rewrite (3.20) as the form

\[
\frac{\Delta^2 f}{f - a_2} = \frac{a_2 p_1 e^{i\theta_1} - a_2 p_1 e^{i\theta_2} + (a_1 - a_2) p_1 p_2 e^{i\theta_1 + \epsilon q_2}}{(a_2 - a_1)(1 - p_1 e^{i\theta_1})}.
\]

With this and reasoning as in Case 2.1, we can deduce a similar contradiction.

**Case 2.3:** \( r_3 > \max\{r_1, r_2\} \). Reasoning as in Case 2.1, we can deduce a similar contradiction again.

**Case 2.4:** \( r_1 = r_2 = r_3 \). Since \( A_d = r_1 e^{i\theta_1}, B_d = r_2 e^{i\theta_2}, A_d + B_d = r_3 e^{i\theta_3}, \theta_1, \theta_2, \) and \( \theta_3 \) must be distinct and satisfy

\[
|\theta_j - \theta_k| \notin [0, 2\pi], \quad 1 \leq j < k \leq 3.
\]

Thus, for \( \varphi_1 = -\theta_1/d \) and \( z = re^{i\varphi_1} \), we have

\[
A_d z^d = r_1 z^d > \max\{r_1 \cos(\theta_2 + d\varphi_2) r^d, r_1 \cos(\theta_2 + d\varphi_2) r^d\} \\
= \max\{\Re B_d z^d, \Re (A_d + B_d) z^d\}.
\]

With this and arguing as in Case 2.1, we can also deduce a similar contradiction.
Thus, we finally prove that $f(z) \equiv \Delta_0 f(z)$.

**Remark 3.1** From Lemma 2.4 and Theorem 1.1, we can get Theorem 1.2 immediately. And we omit it.

### 4 Proof of Theorem 1.3

We begin our proof by supposing that $f(z) \not\equiv \Delta_0 f(z)$. We get immediately from Lemma 2.1 that

$$T(r, \Delta_0^n f) = m(r, \Delta_0^n f) \leq m\left(r, \frac{\Delta_0^n f}{f}\right) + m(r, f) = T(r, f) + S(r, f),$$

which also gives $S(r, \Delta_0^n f) \leq S(r, f)$.

Since $f(z)$ and $\Delta_0^n f(z)$ share two values $a_1$ and $a_2$ IM, we get by applying the second fundamental theorem that

$$T(r, f) \leq N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) + S(r, f)$$

$$= N\left(r, \frac{1}{\Delta_0^n f - a_1}\right) + N\left(r, \frac{1}{\Delta_0^n f - a_2}\right) + S(r, f) \leq 2T(r, \Delta_0^n f) + S(r, f).$$

And hence

$$T(r, \Delta_0^n f) \geq \frac{1}{2} T(r, f) + S(r, f). \quad (4.1)$$

Using Lemma 2.1 and Lemma 2.7, one can easily prove that

$$m\left(r, \frac{1}{f-a_1}\right) + m\left(r, \frac{1}{f-a_2}\right)$$

$$= m\left(r, \frac{1}{f-a_1} + \frac{1}{f-a_2}\right) + O(1) \quad (4.2)$$

$$\leq m\left(r, \frac{1}{\Delta_0^n f}\right) + m\left(r, \frac{\Delta_0^n f}{f-a_1} + \frac{\Delta_0^n f}{f-a_2}\right) + O(1) \leq m\left(r, \frac{1}{\Delta_0^n f}\right) + S(r, f).$$

The assumption $\delta(a_1, f) + \delta(a_2, f) \geq \lambda$ means that

$$N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) \leq (2-\lambda)T(r, f) + S(r, f). \quad (4.3)$$

We get by combining (4.2) and (4.3) that

$$m\left(r, \frac{1}{\Delta_0^n f}\right) \geq 2T(r, f) - N\left(r, \frac{1}{f-a_1}\right) - N\left(r, \frac{1}{f-a_2}\right) + O(1)$$

$$\geq \lambda T(r, f) + S(r, f). \quad (4.4)$$
On the other hand, we can derive by using Lemma 2.7 and the lemma of logarithmic derivatives that

\[
m(r, \frac{1}{\Delta_{\alpha_1}^n f}) + m(r, \frac{1}{\Delta_{\alpha_2}^n f - a_1}) + m(r, \frac{1}{\Delta_{\alpha_2}^n f - a_2})
\]

\[
= m\left(r, \frac{1}{\Delta_{\alpha_1}^n f - a_1} + \frac{1}{\Delta_{\alpha_2}^n f - a_2}\right) + O(1)
\]

\[
\leq m\left(r, \frac{(\Delta_{\alpha_1}^n f)'}{\Delta_{\alpha_1}^n f} + \frac{(\Delta_{\alpha_2}^n f)'}{\Delta_{\alpha_2}^n f - a_1} + \frac{(\Delta_{\alpha_2}^n f)'}{\Delta_{\alpha_2}^n f - a_2}\right) + m\left(r, \frac{1}{(\Delta_{\alpha_2}^n f)'}\right) + O(1)
\]

\[
\leq m\left(r, \frac{\Delta_{\alpha_2}^n f}{\Delta_{\alpha_2}^n f'}\right) + S(r, f) \tag{4.5}
\]

Noting that \(f(z)\) shares two values \(a_1\) and \(a_2\) IM with \(\Delta_{\alpha_2}^n f(z)\), we can derive that

\[
N\left(r, \frac{1}{\Delta_{\alpha_1}^n f - a_1}\right) + N\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_2}\right)
\]

\[
\leq N\left(r, \frac{1}{f - \Delta_{\alpha_2}^n f}\right) \leq T(r, f - \Delta_{\alpha_2}^n f) = m(r, f - \Delta_{\alpha_2}^n f) \quad \tag{4.6}
\]

\[
\leq m\left(r, \frac{\Delta_{\alpha_2}^n f}{f}\right) + m(r, f) \leq T(r, f) + S(r, f).
\]

Furthermore, from (4.6) we know

\[
N\left(r, \frac{1}{\Delta_{\alpha_1}^n f - a_1}\right) + N\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_2}\right)
\]

\[
\leq N\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_1}\right) + N\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_2}\right) + N\left(r, \frac{1}{(\Delta_{\alpha_2}^n f)'}\right) \tag{4.7}
\]

\[
\leq T(r, f) + N\left(r, \frac{1}{(\Delta_{\alpha_2}^n f)'}\right) + S(r, f).
\]

Hence (4.5) and (4.7) lead to

\[
m\left(r, \frac{1}{\Delta_{\alpha_2}^n f}\right) + 2T(r, \Delta_{\alpha_2}^n f) + O(1)
\]

\[
= m\left(r, \frac{1}{\Delta_{\alpha_2}^n f}\right) + T\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_1}\right) + T\left(r, \frac{1}{\Delta_{\alpha_2}^n f - a_2}\right)
\]

\[
\leq T(r, f) + T\left(r, \frac{1}{(\Delta_{\alpha_2}^n f)'}\right) + S(r, f)
\]

\[
\leq T(r, f) + T(r, \Delta_{\alpha_2}^n f) + S(r, f).
\]

Since \(\lambda > \frac{1}{2}\), we can conclude easily from the above inequality and (4.4) that

\[
T(r, \Delta_{\alpha_2}^n f) \leq T(r, f) - m\left(r, \frac{1}{\Delta_{\alpha_2}^n f}\right) + S(r, f)
\]

\[
\leq (1 - \lambda)T(r, f) + S(r, f) < \frac{1}{2}T(r, f) + S(r, f),
\]

which contradicts (4.1). Hence \(f(z) \equiv \Delta_{\alpha_2}^n f(z)\).
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References
1. Chen, Z.: The growth of solutions of a class of second-order differential equations with entire coefficients. Chin. Ann. Math., Ser. A 20(1), 7–14 (1999) (in Chinese)
2. Chen, Z., Yi, H.: On sharing values of meromorphic functions and their differences. Results Math. 63, 557–565 (2013)
3. Chiang, Y., Feng, S.: On the Nevanlinna characteristic of f(α + η) and difference equations in the complex plane. Ramanujan J. 16(1), 105–129 (2008)
4. Cui, N., Chen, Z.: The conjecture on unity of meromorphic functions concerning their differences. J. Differ. Equ. Appl. 22(10), 1452–1471 (2016)
5. Gundersen, G.G.: Meromorphic functions that share three or four values. J. Lond. Math. Soc. (2) 203(3), 457–466 (1979)
6. Gundersen, G.G.: Meromorphic functions that share four values. Trans. Am. Math. Soc. 227, 457–466 (1983)
7. Gundersen, G.G.: Meromorphic functions that share two finite values with their derivative. Pac. J. Math. 105(2), 299–309 (1983)
8. Gundersen, G.G.: Correction to ‘Meromorphic functions that share four values’. Trans. Am. Math. Soc. 304, 847–850 (1987)
9. Gundersen, G.G.: Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. J. Lond. Math. Soc. 37(2), 88–104 (1988)
10. Halburd, R., Korhonen, R.: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. 314(2), 477–487 (2006)
11. Halburd, R., Korhonen, R.: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. 31, 463–478 (2006)
12. Heittokangas, J., Korhonen, R., Laine, I., Rieppo, J.: Uniqueness of meromorphic functions sharing values with their shifts. Complex Var. Elliptic Equ. 56(1–4), 81–92 (2011)
13. Heittokangas, J., Korhonen, R., Laine, I., Rieppo, J., Zhang, J.: Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity. J. Math. Anal. Appl. 355, 352–363 (2009)
14. Laine, I.: Nevanlinna Theory and Complex Differential Equations. de Gruyter Studies in Mathematics, vol. 15. de Gruyter, Berlin (1993)
15. Li, S.: Meromorphic functions sharing two values IM with their derivatives. Results Math. 63(3–4), 965–971 (2013)
16. Mohon’ko, A.Z.: The Nevanlinna characteristics of certain meromorphic functions. Teor. Funkc. Funkc. Anal. Ih Prilozh. 14, 83–87 (1971) (in Russian)
17. Mues, E.: Meromorphic functions sharing four values. Complex Var. Elliptic Equ. 12, 167–179 (1989)
18. Mues, E., Steinmetz, S.: Meromorphe funktionen, die mir oher ableitung zwei werte teilen. Manuscr. Math. 6(29), 195–206 (1979)
19. Mues, E., Steinmetz, S.: Meromorphe funktionen, die mit ihrer ableitung zwei werte teilen. Results Math. 6(1–2), 48–53 (1983)
20. Nevanlinna, R.: Le théorème de Picard–Borel et l’itérée des fonctions méromorphes. Gauthier-Villars, Paris (1929)
21. Rubel, L.A., Yang, C.: Values shared by an entire function and its derivative. Lect. Notes Math. 599, 101–103 (1977)
22. Valiron, G.: Sur la dérivée des fonctions algébroides. Bull. Soc. Math. Fr. 59, 17–39 (1931)
23. Yi, H., Yang, C.: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003)
24. Zhang, J., Liao, L.: Entire functions sharing some values with their difference operators. Sci. China Math. 57(10), 2143–2152 (2014)