Slow dynamics of the contact processes on complex networks

Géza Ódor

RESEARCH INSTITUTE FOR NATURAL SCIENCES (MFA) BUDAPEST

- Exploration of complex networks is flourishing since ~2000 (Barabási & Albert)

- Dynamical systems living on networks are of current interest

- Origin of slow (dynamic) scaling behavior in internet, brain, quantum systems,… etc.

- Open question : Complex networks + quenched disorder ?

Diffusion spectrum imaging
Observed slow dynamics in networks

- **Brain**: Size distribution of neural avalanches
  
  *G. Werner*: *Biosystems*, 90 (2007) 496,

  Correlation length (\(\xi\)) diverges
  
  *Tagliazucchi & Chialvo* (2012): Brain complexity born out of criticality.

- **Internet**: worm recovery time is slow:

  Small world networks  \(\rightarrow\) fast dynamics

  What is the cause?
Scaling and universality classes appear in complex system due to: $\xi \to \infty$

i.e: near critical points

Basic models classified by universal scaling behavior

G. Ódor: *Universality in nonequilibrium system* (World Scientific 2008), Rev. Mod. Phys. 2004

- Why don't we see universal behavior in networks?
- Tuning to critical point is needed!

I'll show a possible way to understand this
Modelling dynamics on fundamental (nonequilibrium) models

Prototype: Contact Process describing “epidemic/info” propagation (1d):

- In regular, Euclidean lattices:
  - Order parameter: $\rho$ the density of active sites
  - Phase transition between active and inactive (absorbing)
  - Critical point: $\lambda_c > 0$

- Exhibits scaling behavior belonging to the DP universality class, still rarely observed in nature

- Sensitivity to spatially/temporal (quenched) disorder → The scaling behavior is slow, non-universal
**Rare Region argument for Q-disordered CP**

- **Fixed (quenched) disorder/impurity changes the local birth rate** ⇒ $\lambda_c > \lambda_c^0$

- **Locally active**, arbitrarily large Rare Regions in the inactive phase due to the *inhomogeneities*

- Probability or RR of size $L_R$:
  
  $w(L_R) \sim \exp (-c L_R^d)$

  Contribute to the density:
  
  $\rho(t) \sim \int dL_R \ L_R^d \ w(L_R) \ \exp [-t / \tau(L_R^d)]$

- For $\lambda < \lambda_c^0$ : conventional (exponentially fast) decay

- At $\lambda_c^0$ the characteristic time scales as: $\tau(L_R) \sim L_R^{-Z}$ ⇒ saddle point analysis:
  
  $\ln \rho(t) \sim t^{d / (d + Z)}$

  *(stretched exponential)*

- For $\lambda_c^0 < \lambda < \lambda_c$ : $\tau(L_R) \sim \exp(b L_R^d)$:
  
  Griffiths Phase

  *(continuously changing exponents)*

- At $\lambda_c$ Ultra slow time dependences : $\rho(t) \sim \ln(t)^{-\alpha}$
Basic network models

From regular to random networks:

Erdős-Rényi \((p = 1)\)

Degree \((k)\) distribution in \(N \to \infty\) node limit:
\[
P(k) = e^{<k>} \frac{k^k}{k!}
\]

Topological dimension: \(N(r) \sim r^d\)
Above perc. thresh.: \(d = \infty\)
Below percolation \(d = 0\)

Scale free networks:

Degree distribution:
\[
P(k) = k^{-\gamma} \ (2 < \gamma < 3)
\]

Topological dimension: \(d = \infty\)
Example: Barabási-Albert lin. prefetential attachment

Focus on dynamical systems living on networks: Fast dynamics is expected
Networks: fast dynamics, mean-field behavior expected

Effect of disorder:
Rare active regions in the absorbing phase: \( \tau(A) \sim e^A \)
\( \rightarrow \) slow dynamics (Griffiths Phase)?

M. A. Munoz, R. Juhász, C. Castellano and G. Ódor, PRL 105, 128701 (2010)

1. Inherent disorder in couplings
2. Disorder induced by topology

Optimal fluctuation theory + simulations:

- In Erdős-Rényi networks below the percolation threshold
- In generalized small-world networks for finite topological dimension
CP + Topological disorder results

Generalized Small World networks: \( P(l) \sim \beta l^{-2} \)
(link length probability)

- **Top. dim: \( N(r) \sim r^d \) \( d(\beta) \) finite:**

  \( \lambda_c(\beta) \) decreases monotonically from 
  \( \lambda_c(0) = 3.29785 \) (1d CP) to:
  \[
  \lim_{\beta \to \infty} \lambda_c(\beta) = 1
  \]
  towards mean-field CP value

  \( \lambda < \lambda_c(\beta) \) inactive, there can be
  **locally ordered, rare regions** due to more
  than average, active, incoming links

- **Griffiths phase:** \( \lambda \) -- dep. continuously changing
dynamical power laws:
  for example: \( \rho(t) \sim t^{-\alpha(\lambda)} \)

  Logarithmic corrections!

- **Ultra-slow** (“activated”) scaling: \( \rho \propto \ln(t)^{-\alpha} \) at \( \lambda_c \)

- As \( \beta \to 1 \) Griffiths phase shrinks/disappears

- **Same results for** cubic, regular random networks,
  higher dimensions

---

**FIG. 3:** Density decay in Benjamini-Berger networks with \( s = 2 \) and \( \beta = 0.2 \) for different values of \( \lambda \) (from top to bottom: 2.81, 2.795, 2.782, 2.77, 2.75, 2.73, 2.71, 2.70, 2.69, 2.67, 2.65, 2.6). Straight lines lie in the Griffiths phase. 
Inset: Corresponding effective exponents, illustrating the presence of corrections to scaling.
Contact process on Barabási-Albert (BA) network

- Heterogeneous mean-field theory: conventional critical point, with linear density decay:

\[ \rho(t) \sim [t \ln(t)]^{-1}. \]

with logarithmic correction

- Extensive simulations confirm this

- No Griffiths phase observed

- Steady state density vanishes at \( \lambda_c \approx 1 \)

  linearly, \( \text{HMF: } \beta = 1 \)

FIG. 1. Density decay \( t\rho(t) \) as a function of \( \ln(t) \) for the CP on unweighted looped BA networks with \( m = 3 \) of size \( N = 8 \times 10^7 \). The different curves correspond to \( \lambda = 1.2068, \ldots, 1.24 \) (bottom to top). Inset: Steady state density, showing agreement with HMF theory scaling. The full line shows a power-law fitting to the data points in the form \[ -0.36(5)x^{0.98(2)}. \]
CP on Barabási-Albert trees
hunt for GP-s, by slowing the propagation

- Lack of loops slows propagation
- For $\langle k \rangle = 3 : \lambda_c > 0$

Weighted networks:

$$\omega_{ij} = \omega_0 (k_i k_j)^{-v}$$
$$\omega_{ij} = \frac{|i-j|^x}{N}$$

Strong size corrections

Non mean-field transition:

Power-laws for: $x = 2, 3$

Heterogeneous mean-field theory: critical point, with linear density decay: $\rho \mu 1/t$
can't describe frozen disorder!
Do power-laws survive the thermodynamic limit?

- Finite size analysis shows the disappearance of a power-law scaling:

- Smeared phase transition: power-law → saturation:
  - Rare sub-spaces, but infinite dimensional?

---

**FIG. 5.** Density decay as a function of time $\rho(t)$ for the CP on weighted BA trees with a multiplicative weighting scheme (WBAT-I) with exponent $\nu = 1.5$. Plots correspond to two sets of $\lambda$ (upper branch: $\lambda = 144$, lower branch $\lambda = 140$) at different network sizes $N$. Dashed lines represent PL fittings. Inset: Initial time region of the same data, showing an stretched exponential behavior.

**FIG. 8.** Finite-size scaling analysis of the density decay exponent for $\lambda = 6.75$ (triangles), $\lambda = 6.8$ (boxes), $\lambda = 6.82$ (triangles), $\lambda = 6.85$ (bullets), $\lambda = 6.9$ (rhombes) in the CP on weighted BA trees with a age-dependent weighting scheme (WBAT-II) with exponent $x = 2$. Top inset: $\rho(t)$ for $\lambda = 6.82$ ($N = 10^6$, $N = 4 \times 10^5$, $N = 10^5$ top to bottom). Bottom inset: Initial time density.
Percolation analysis of the weighted BA tree

We consider a network of a given size $N$, and delete all the edges with a weight smaller than a threshold $\omega_{th}$.

For small values of $\omega_{th}$, many edges remain in the system, and they form a connected network with a single cluster encompassing almost all the vertices in the network.

When increasing the value of $\omega_{th}$, the network breaks down into smaller subnetworks of connected edges, joined by weights larger than $\omega_{th}$.

The size of the largest ones grows linearly with the network size $N$ $\leftrightarrow$ standard percolation transition.

These clusters, which can become arbitrarily large in the thermodynamic limit, play the role of correlated RR$^*$s, sustaining independently activity and smearing down the phase transition.
Summary

- Quenched disorder in complex networks can cause slow dynamics: Rare-regions $\rightarrow$ (Griffiths) phases $\rightarrow$ no tuning or self-organization needed!

- In **finite dim.** (for CP) GP can occur due to topological disorder

- In **infinite dim.**, scale-free, BA network mean-field transition of CP with logarithmic corrections (HMF+simulations)

- In **BA trees** non mean-field transition observed

- In **weighted BA trees** non-universal, slow, power-law dynamics can occur for finite $N$, but in the $N \rightarrow \infty$ limit saturation is observed

- Smeared transition can describe this, percolation analysis confirms the existence of arbitrarily large dimensional sub-spaces with (correlated) large weights

- Acknowledgements to: HPC-Europa2, OTKA, Osiris FP7

[1] M. A. Munoz, R. Juhasz, C. Castellano, and G. Ódor, Phys. Rev. Lett. 105, 128701 (2010)
[2] G. Ódor, R. Juhasz, C. Castellano, M. A. Munoz, AIP Conf. Proc. 1332, Melville, New York (2011) p. 172-178.
[3] R. Juhasz, G. Ódor, C. Castellano, M. A. Munoz, Phys. Rev. E 85, 066125 (2012)
[4] G. Ódor and Romualdo Pastor-Satorras, Phys. Rev. E 86, 026117 (2012)