Moduli Spaces of Framed $G$–Higgs Bundles and Symplectic Geometry

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Abstract: Let $X$ be a compact connected Riemann surface, $D \subset X$ a reduced effective divisor, $G$ a connected complex reductive affine algebraic group and $H_x \subsetneq G$ a Zariski closed subgroup for every $x \in D$. A framed principal $G$–bundle on $X$ is a pair $(E_G, \phi)$, where $E_G$ is a holomorphic principal $G$–bundle on $X$ and $\phi$ assigns to each $x \in D$ a point of the quotient space $(E_G)_x/H_x$. A framed $G$–Higgs bundle is a framed principal $G$–bundle $(E_G, \phi)$ together with a holomorphic section $\theta \in H^0(X, \text{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D))$ such that $\theta(x)$ is compatible with the framing $\phi$ at $x$ for every $x \in D$. We construct a holomorphic symplectic structure on the moduli space $\mathcal{M}_{FH}(G)$ of stable framed $G$–Higgs bundles on $X$. Moreover, we prove that the natural morphism from $\mathcal{M}_{FH}(G)$ to the moduli space $\mathcal{M}_H(G)$ of $D$-twisted $G$–Higgs bundles $(E_G, \theta)$ that forgets the framing, is Poisson. These results generalize (Biswas et al. in Int Math Res Not, 2019. https://doi.org/10.1093/imrn/rnz016, arXiv:1805.07265) where $(G, \{H_x\}_{x \in D})$ is taken to be $(\text{GL}(r, \mathbb{C}), \{I_{r \times r}\}_{x \in D})$. We also investigate the Hitchin system for the moduli space $\mathcal{M}_{FH}(G)$ and its relationship with that for $\mathcal{M}_H(G)$.

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1. Introduction

Higgs bundles on Riemann surfaces were introduced by Hitchin in [Hi1] and the Higgs bundles on higher dimensional complex manifolds were introduced by Simpson in [Si1]. The moduli spaces of Higgs bundles on Riemann surfaces have been extensively studied because of their rich symplectic geometric, differential geometric as well as algebraic geometric structures; they also play an important role in geometric representation theory [Ngo]. In particular, in his foundational papers [Hi1,Hi2], Hitchin showed that such a moduli space is a holomorphically symplectic manifold which contains the total space of the cotangent bundle of a moduli space of vector bundles as a Zariski dense open subset such that the restriction of the symplectic form to this Zariski open subset coincides with the standard Liouville symplectic form on the total space of the cotangent bundle. Moreover, he constructed a fibration of the moduli space of Higgs bundles over an affine space which he went on to prove to be an algebraically completely integrable system; this completely integrable system nowadays is known as the Hitchin system.

Over time, moduli spaces of Higgs bundles have undergone diverse generalizations. Here we will consider $D$–twisted $G$–Higgs bundles to which we shall add an extra structure which is called a framing. Similar objects were considered earlier in [Si2,Si3, Ma] and [Ni].

Take a compact connected Riemann surface $X$, and fix a reduced effective divisor $D$ on it. Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. A $D$–twisted $G$–Higgs bundle $(E_G, \theta)$ on $X$ consists of a holomorphic principal $G$–bundle $E_G \rightarrow X$ together with a $D$-twisted Higgs field $\theta \in H^0(X, \text{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D))$, where $\text{ad}(E_G)$ is the adjoint vector bundle for principal $G$–bundle $E_G$ while $K_X$ denotes the holomorphic cotangent bundle of $X$.

The isomorphism classes of all topological principal $G$–bundles on $X$ are parametrized by the fundamental group $\pi_1(G)$. Once we fix a topological isomorphism class $\nu \in \pi_1(G)$, the moduli space of stable $D$–twisted $G$–Higgs bundles is a smooth connected orbifold [Ni,Hi3]; such a moduli space will be denoted by $\mathcal{M}_H(G)$.

It is known that $\mathcal{M}_H(G)$ is equipped with a natural holomorphic Poisson structure [Bot, Ma, BR]. It should be mentioned that this Poisson structure is never symplectic unless $D$ is actually the zero divisor.

Fix a nondegenerate symmetric $G$–invariant bilinear form $\sigma$ on the Lie algebra $\mathfrak{g} := \text{Lie}(G)$. For each point $x \in D$, fix a Zariski closed subgroup $H_x \subset G$. A framing on a holomorphic principal $G$–bundle $E_G$ on $X$ is a map $\phi : D \rightarrow \bigcup_{x \in D} (E_G)_x / H_x$ such that $\phi(x) \in (E_G)_x / H_x$ for every $x \in D$. Using the bilinear form $\sigma$ in (2.9) and
the framing $\phi$ on $E_G$, we construct a subspace $\mathcal{H}_x^+ \subset \text{ad}(E_G)_x$ for each $x \in D$; see Sect. 2.1 for the construction. Let 

$$\text{ad}_\phi^n(E_G) \subset \text{ad}(E_G)$$

be the subsheaf uniquely identified by the condition that a locally defined holomorphic section $s$ of $\text{ad}(E_G)$ lies in $\text{ad}_\phi^n(E_G)$ if and only if $s(x) \in \mathcal{H}_x^+ \subset \text{ad}(E_G)_x$ for every $x \in D$ that lies in the domain of the locally defined section $s$.

A Higgs field on a framed principal $G$–bundle $(E_G, \phi)$ is a holomorphic section $\theta$ of the holomorphic vector bundle $\text{ad}_\phi^n(E_G) \otimes K_X \otimes \mathcal{O}_X(D)$, where $\text{ad}_\phi^n(E_G)$ is described above. Such a triple $(E_G, \phi, \theta)$ will be called a framed $G$–Higgs bundle. In particular, the pair $(E_G, \theta)$ is a $D$–twisted $G$–Higgs bundle. If $H_x$ is the trivial subgroup $e \in G$ for all $x \in D$, then $\mathcal{H}_x^+ = \text{ad}(E_G)_x$ for all $x$. Hence in that case a Higgs field on $(E_G, \phi)$ is simply an element of $H^0(X, \text{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D))$ (a $D$–twisted $G$–Higgs field on $E_G$).

We prove the following:

1. A moduli space of framed $G$–Higgs bundles has a natural holomorphic symplectic structure. (See Theorem 5.4.)
2. The forgetful map from a moduli space of framed $G$–Higgs bundles to a moduli space of $D$–twisted $G$–Higgs bundles, defined by $(E_G, \phi, \theta) \mapsto (E_G, \theta)$, is Poisson. (See Theorem 5.5.)

In particular, the holomorphic Poisson manifold given by a moduli space of $D$–twisted $G$–Higgs bundles $\mathcal{M}_H(G)$ can be enhanced to a symplectic manifold by augmenting the $D$–twisted $G$–Higgs bundle with a framing for the trivial sub group $e \in G$ for all points of $D$.

The Hitchin system

$$h : \mathcal{M}_H(G) \longrightarrow \mathcal{B}$$

is defined by evaluating the Chevalley morphism $\chi : g \longrightarrow g \sslash G$ on the Higgs field. This is again, despite $\mathcal{M}_H(G)$ being only Poisson and not symplectic unless $D$ is the zero divisor, an algebraically completely integrable system ([Ma, Remark 8.6], [DM, Section 5]).

Hitchin systems constitute a very large family of algebraically completely integrable systems. Moreover, it is known that for suitable choices of the Riemann surface $X$, the group $G$, and the twisting, many classical integrable systems are embedded in them as symplectic leaves (see [Ma, Section 9]). In [BLP] we showed that the Hitchin systems provided by the moduli spaces of framed $G$–Higgs bundles, when $G = \text{GL}(r, \mathbb{C})$ and $H_x = \text{I}_{r \times r}$ for all $x \in D$, are no longer algebraically completely integrable systems. Firstly, the number of Poisson commuting functions given by the Hitchin map falls short of the dimension of the moduli space of framed principal $G$–bundles (which is half the dimension of the moduli spaces of framed $G$–Higgs bundles). Secondly, its fibers are not abelian varieties, but torsors over the fibers of the non-framed Hitchin system. We also investigate two subsystems which come with the correct number of Poisson commuting functions. We also show that these results generalize for any connected complex reductive affine algebraic group $G$.

Let $\mathcal{M}_{FH}(G)$ denote the moduli space of stable framed $G$–Higgs bundles with a fixed topological class $\nu$, and let

$$h_{FH} : \mathcal{M}_{FH}(G) \longrightarrow \mathcal{B}$$
be the corresponding Hitchin system.

We prove the following:

(3) The Hitchin map \( h_{FH} \) in (1.2) produces a set of \( N := \dim B \) Poisson commuting functions on \( \mathcal{M}_{FH}(G) \), i.e., \( h_{FH} = (h_1, \ldots, h_N) \) with \( \{h_i, h_j\}_p = 0 \) for all \( i, j = 1, \ldots, N \). (See Corollary 7.4.)

(4) The generic fibers of the map \( h_{FH} \) are torsors over the abelian varieties \( J_b = h^{-1}(b) \) where \( h : \mathcal{M}_H(G) \to B \) is the Hitchin map in (1.1). (See Corollary 7.6.)

(5) There is a moduli space \( \mathcal{M}_{FH}^\Delta(G) \) which is a subsystem of \( \mathcal{M}_{FH}(G) \) and it is maximally abelianizable. (See Corollary 7.10 and Remark 7.12.)

The above results specialize to the results in [BLP] when \( G = \text{GL}(r, \mathbb{C}) \) and \( H_x = I_{r \times r} \) for every \( x \in D \).

In Sect. 2 we introduce \( D \)-twisted \( G \)-Higgs bundles as well as framed structures for principal bundles and their juxtaposition, namely framed \( G \)-Higgs bundles. In Sect. 3 we study the infinitesimal deformations of the \( D \)-twisted \( G \)-Higgs bundles and framed \( G \)-Higgs bundles. In Sect. 5, we construct a symplectic structure on the moduli space \( \mathcal{M}_{FH}(G) \) of stable framed \( G \)-Higgs bundles, as well as a Poisson structure on the moduli space \( \mathcal{M}_H(G) \) of stable \( D \)-twisted \( G \)-Higgs bundles.

In Sect. 7 we investigate the integrability properties of the Hitchin system in (1.1). For the sake of clarity, we focus on the case \( H_x = e \) for all \( x \), nevertheless discussing the general case in Remark 7.12.

We also describe a subsystem of (1.2) which is maximally abelianizable. This is done using the cameral cover approach of Donagi–Gaitsgory [DG] (see also [Ngo]), which identifies the generic fiber of the Hitchin map in (1.1) with a subvariety of the Jacobian of the cameral cover. We find that the generic fibers (corresponding to smooth cameral covers unramified over \( D \)) are \( G^n/Z(G) \)-torsors over the fibers of the map in (1.1), where \( n = \#D \) and \( Z(G) \) is the center of \( G \). It turns out that we may naturally identify \( T^n/Z(G) \)-sub-torsors therein (where \( T \subset G \) is a maximal torus) with moduli spaces of framed Higgs bundles. More precisely, they correspond to the fibers of the restriction of the Hitchin map to the locus of relatively framed Higgs bundles defined in (7.16). This parametrizes Higgs bundles together with a framing of both the bundle and the Higgs field (see Proposition 7.8, Theorem 7.9 and Remark 7.11).

2. Framed \( G \)-Higgs Bundles and Stability

2.1. Framings and \( G \)-Higgs bundles. Let \( X \) be a compact connected Riemann surface. Denote by \( K_X \) the holomorphic cotangent bundle of \( X \). Let

\[
D = \{x_1, \ldots, x_n\} \subset X
\]

be a reduced effective divisor on \( X \) consisting of \( n \geq 1 \) distinct points.

To clarify, we shall always assume that \( D \neq \emptyset \).

The holomorphic line bundle \( K_X \otimes \mathcal{O}_X(D) \) on \( X \) will be denoted by \( K_X(D) \). For any \( x \in D \), the fiber \( K_X(D)_x \) of \( K_X(D) \) over \( x \) is identified with \( \mathbb{C} \). Indeed, for any holomorphic coordinate function \( z \) on \( X \) defined around the point \( x \) such that \( z(x) = 0 \), consider the homomorphism

\[
\mathbb{C} \to K_X(D)_x, \quad c \mapsto c \cdot \left. \frac{dz}{z} \right|_{z=x}.
\]
The homomorphism in (2.2) is in fact independent of the choice of the above holomorphic coordinate function \( z \), and thus \( K_X(D)_x \) is canonically identified with \( \mathbb{C} \).

Let \( G \) be a connected complex Lie group. Let

\[
p : E_G \rightarrow X
\]

be a holomorphic principal \( G \)–bundle over \( X \); we recall that this means that \( E_G \) is a holomorphic fiber bundle over \( X \) equipped with a holomorphic right-action of the group \( G \)

\[
q' : E_G \times G \rightarrow E_G
\]

such that

\[
p(q'(z, g)) = p(z)
\]

for all \((z, g) \in E_G \times G\), where \( p \) is the projection in (2.3) and, furthermore, the resulting map to the fiber product

\[
E_G \times G \rightarrow E_G \times_X E_G, \quad (z, g) \mapsto (z, q'(z, g))
\]

is a biholomorphism. For notational convenience, the point \( q'(z, g) \in E_G \), where \((z, g) \in E_G \times G\), will be denoted by \( zg \). For any \( x \in X \), the fiber \( p^{-1}(x) \subset E_G \) will be denoted by \((E_G)_x\).

For each point \( x \in D \), fix a complex Lie proper subgroup \( H_x \subset G \).

A framing of \( E_G \) over the divisor \( D \) in (2.1) is a map

\[
\phi : D \rightarrow \bigcup_{x \in D} (E_G)_x / H_x,
\]

where \( H_x \) is the subgroup in (2.5), such that \( \phi(x) \in (E_G)_x / H_x \) for every \( x \in D \). So the space of all framings of \( E_G \) over \( D \) is the Cartesian product

\[
\mathcal{F}(E_G) := \prod_{x \in D} (E_G)_x / H_x.
\]

Let

\[
\hat{p}_x : \mathcal{F}(E_G) \rightarrow (E_G)_x / H_x
\]

be the natural projection.

A framed principal \( G \)–bundle on \( X \) is a holomorphic principal \( G \)–bundle \( E_G \) on \( X \) equipped with a framing over \( D \).

The first remark below is due to the referee.

**Remark 2.1.** Take \( G \) to be a reductive algebraic group. A parabolic subgroup of \( G \) is a Zariski closed connected subgroup \( P \subset G \) such that the quotient variety \( G/P \) is projective. Set each \( H_x \) to be some parabolic subgroup of \( G \). Then a framed principal \( G \)–bundle is a quasiparabolic \( G \)–bundle with parabolic divisor \( D \) and quasiparabolic type \( H_x \) for \( x \in D \). In particular, when \( G = \text{GL}(r, \mathbb{C}) \) and \( H_x \subset \text{GL}(r, \mathbb{C}) \) is a parabolic subgroup for every \( x \in D \), a framed principal \( G \)–bundle corresponds to a holomorphic vector bundle \( E \) on \( X \) of rank \( r \) equipped with a strictly decreasing filtration, by linear subspaces, of the fiber \( E_x \) for all \( x \in D \). The dimensions of the subspaces in the filtration of \( E_x \) are determined by the conjugacy class of the subgroup \( H_x \). Conversely, these dimensions determine the conjugacy class of \( H_x \).
Remark 2.2. If $H_x$ is a normal subgroup of $G$, then the action of $G$ on $(E_G)_x$ produces an action of the quotient group $G/H_x$ on the quotient manifold $(E_G)_x/H_x$. This action of $G/H_x$ on $(E_G)_x/H_x$ is evidently free and transitive. In other words, $(E_G)_x/H_x$ is a torsor for the group $G/H_x$. Therefore, if $H_x$ is a normal subgroup of $G$ for all $x \in D$, then $\mathcal{F}(E_G)$ in (2.6) is a torsor for the group $\prod_{x \in D} G/H_x$. The special case where $(G, \{H_x\}_{x \in D}) = (\text{GL}(r, \mathbb{C}), \{\text{I}_{r \times r}\}_{x \in D})$ is treated in [BLP].

Let $\mathfrak{g}$ denote the Lie algebra of $G$. For any $x \in D$, the Lie algebra of the subgroup $H_x$ in (2.5) will be denoted by $\mathfrak{h}_x$. Since the adjoint action of $H_x$ on $\mathfrak{g}$ preserves the sub-algebra $\mathfrak{h}_x$, the quotient space $\mathfrak{g}/\mathfrak{h}_x$ is equipped with an action of $H_x$ induced by the adjoint action of $H_x$.

The quotient map $(E_G)_x \longrightarrow (E_G)_x/H_x$ defines a holomorphic principal $H_x$--bundle over the complex manifold $(E_G)_x/H_x$. Let

$$V_x^0 := (E_G)_x \times^{H_x} (\mathfrak{g}/\mathfrak{h}_x) \longrightarrow (E_G)_x/H_x$$

be the holomorphic vector bundle over $(E_G)_x/H_x$ associated to this holomorphic principal $H_x$--bundle $(E_G)_x \longrightarrow (E_G)_x/H_x$ for the above $H_x$--module $\mathfrak{g}/\mathfrak{h}_x$. Then the holomorphic tangent bundle of the space of all framings $\mathcal{F}(E_G)$ defined in (2.6) has the expression

$$T \mathcal{F}(E_G) = \bigoplus_{x \in D} \hat{p}_x^* V_x^0,$$

where $\hat{p}_x$ is the projection in (2.7).

Henceforth, $G$ will always be assumed to be a connected complex reductive affine algebraic group. The subgroup $H_x \subset G$ in (2.5) is assumed to be Zariski closed for every $x \in D$.

Since the group $G$ is reductive, its Lie algebra $\mathfrak{g}$ admits $G$--invariant nondegenerate symmetric bilinear forms. To construct such a form, consider the decomposition $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Take the Killing form $\kappa$ on $[\mathfrak{g}, \mathfrak{g}]$ and take any nondegenerate symmetric bilinear form $\sigma'$ on $Z(\mathfrak{g})$; the direct sum $\sigma' \oplus \kappa$ is a $G$--invariant nondegenerate symmetric bilinear form on $Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Fix a $G$--invariant nondegenerate symmetric bilinear form

$$\sigma : \text{Sym}^2(\mathfrak{g}) \longrightarrow \mathbb{C}$$

on $\mathfrak{g}$.

Take a holomorphic principal $G$--bundle $E_G$ on $X$. Let $\text{ad}(E_G)$ be the adjoint vector bundle over $X$ associated to $E_G$ for the adjoint action of $G$ on $\mathfrak{g}$. Therefore, each fiber of $\text{ad}(E_G)$ is a Lie algebra isomorphic to $\mathfrak{g}$. More precisely, for any $y \in X$, there is an isomorphism of Lie algebras $\mathfrak{g} \sim \text{ad}(E_G)_y$ which is unique up to automorphisms of $\mathfrak{g}$ given by the adjoint action of the elements of $G$. Using such an isomorphism $\mathfrak{g} \sim \text{ad}(E_G)_y$, the $G$--invariant form $\sigma$ in (2.9) produces a symmetric nondegenerate bilinear form on the fiber $\text{ad}(E_G)_y$; note that this bilinear form on $\text{ad}(E_G)_y$ does not depend on the choice of the above isomorphism $\mathfrak{g} \sim \text{ad}(E_G)_y$ because $\sigma$ is $G$--invariant. Let

$$\hat{\sigma} : \text{Sym}^2(\text{ad}(E_G)) \longrightarrow \mathcal{O}_X$$

be the bilinear form constructed as above using $\sigma$. Let

$$T_{rel}^p \subset TE_G$$
be the relative tangent bundle for the projection \( p \) in \((2.3)\). The action of \( G \) on \( E_G \) produces an action of \( G \) on \( TE_G \). This action preserves the subbundle \( T^p_{rel} \) because of the condition in \((2.4)\). The trivial holomorphic vector bundle \( E_G \times g \rightarrow E_G \) equipped with the action of \( G \), given by the action of \( G \) on \( E_G \) and the adjoint action of \( G \) on \( g \), is identified with \( T^p_{rel} \); this identification between \( T^p_{rel} \) and \( E_G \times g \) is evidently \( G \)–equivariant. The quotient \( T^p_{rel} / G \) is a holomorphic vector bundle over \( E_G / G = X \). This holomorphic vector bundle \( T^p_{rel} / G \) over \( X \) is holomorphically identified with the adjoint vector bundle \( \text{ad}(E_G) \).

Let \( \phi : D \rightarrow \bigcup_{x \in D}(E_G)_x / H_x \) be a framing on \( E_G \). For each \( x \in D \), let
\[
q_x : (E_G)_x \rightarrow (E_G)_x / H_x
\]
be the natural quotient map.

Using the framing \( \phi \) we shall construct a subspace \( \mathcal{H}_x \subset \text{ad}(E_G)_x \) for every \( x \in D \). For that purpose, first recall that the elements of \( \text{ad}(E_G)_x \) are the \( G \)–invariant sections of the vector bundle \( T^p_{rel}\mid_{p^{-1}(x)} \rightarrow p^{-1}(x) \), where \( p \) is the projection in \((2.3)\). Consider all \( G \)–invariant sections
\[
v \in H^0(p^{-1}(x), T^p_{rel}\mid_{p^{-1}(x)})^G
\]
such that the restriction \( v\mid_{q_x^{-1}(\phi(x))} \) satisfies the condition that
\[
v\mid_{q_x^{-1}(\phi(x))} \subset q_x^{-1}(\phi(x)) \times h_x \subset q_x^{-1}(\phi(x)) \times g = T^p_{rel}\mid_{q_x^{-1}(\phi(x))},
\]
where \( q_x \) is the projection in \((2.11)\), and \( h_x \) as before is the Lie algebra of \( H_x \); here we are using the earlier observation that \( T^p_{rel} = E_G \times g \), and we also have identified the section \( v\mid_{q_x^{-1}(\phi(x))} \) with the subset of \( T^p_{rel}\mid_{q_x^{-1}(\phi(x))} \) given by its image. Let
\[
\mathcal{H}_x \subset \text{ad}(E_G)_x
\]
be the subspace defined by all such \( v \). Note that \( \mathcal{H}_x \) is a Lie subalgebra of \( \text{ad}(E_G)_x \) which is identified with \( h_x \) by an isomorphism that is unique up to automorphisms of \( h_x \) given by the adjoint action of the elements of the group \( H_x \).

The following construction of \( \mathcal{H}_x \) was suggested by the referee.

Remark 2.3. The framing \( \phi \) produces a reduction of structure group of the principal \( G \)–bundle \( (E_G)_x \rightarrow x \), defined just over the point \( x \), to the subgroup \( H_x \subset G \) for each \( x \in D \). Indeed,
\[
E^x_{H_x} := q_x^{-1}(\phi(x)) \subset (E_G)_x
\]
is a principal \( H_x \)–bundle, where \( q_x \) and \( \phi \) are the maps in \((2.11)\) and \((2.3)\) respectively. So we have
\[
\text{ad}(E^x_{H_x}) \subset \text{ad}((E_G)_x) = \text{ad}(E_G)_x.
\]
The subspace \( \mathcal{H}_x \) in \((2.12)\) coincides with \( \text{ad}(E^x_{H_x}) \).
For every $x \in D$, let

$$\mathcal{H}_x^\perp \subset \text{ad}(E_G)_x$$

be the annihilator of $\mathcal{H}_x$ with respect to the bilinear form $\hat{\sigma}(x)$ constructed in (2.10).

A **Higgs field** on the framed principal $G$–bundle $(E_G, \phi)$ is a holomorphic section

$$\theta \in H^0(X, \text{ad}(E_G) \otimes K_X(D))$$

such that

$$\theta(x) \in \mathcal{H}_x^\perp \subset \text{ad}(E_G)_x$$

for every $x \in D$; recall from (2.2) that $K_X(D)_x = \mathbb{C}$, so we have $(\text{ad}(E_G) \otimes K_X(D))_x = \text{ad}(E_G)_x$, and hence we have $\theta(x) \in \text{ad}(E_G)_x$.

Notice that in [BLP] the interlinking between the framing and the Higgs field was not explicit due to the assumption that $H_x = e$ for all $x \in D$.

**Definition 2.4.** A **framed $G$–Higgs bundle** is a triple of the form $(E_G, \phi, \theta)$, where $(E_G, \phi)$ is a framed principal $G$–bundle on $X$, and $\theta$ is a Higgs field on $(E_G, \phi)$.

The following remark is due to the referee.

**Remark 2.5.** As in Remark 2.1, assume that each $H_x$ is a parabolic subgroup of $G$. Therefore, a framed principal $G$–bundle $(E_G, \phi)$ is also a quasiparabolic $G$–bundle. Then a Higgs field on $(E_G, \phi)$ is a logarithmic Higgs field $\theta$ on $E_G$, with polar part on $D$, such that the residue of $\theta$ at every $x \in D$ is nilpotent with respect to the quasi-parabolic structure at $x$. Recall from Remark 2.1 that when $G = \text{GL}(r, \mathbb{C})$ and $H_x$ is a parabolic subgroup for all $x \in D$, a framed principal $G$–bundle $(E_G, \phi)$ corresponds to a holomorphic vector bundle $E$ on $X$ of rank $r$ equipped with a filtration of subspaces of $E_x$ for every $x \in D$. In that case, a Higgs field on $(E_G, \phi)$ is a strongly parabolic Higgs field on the quasiparabolic bundle $E$; see [LM] for strongly versus non-strongly parabolic Higgs fields.

### 2.2. Stability of framed $G$–Higgs bundles.

Recall that a parabolic subgroup of $G$ is a Zariski closed connected subgroup $P$ such that the quotient variety $G/P$ is projective. Let $Z_0(G)$ denote the (unique) maximal connected subgroup of the center of $G$. A character

$$\hat{\chi} : P \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

of a parabolic subgroup $P \subset G$ is called **strictly anti-dominant** if

- $\hat{\chi}$ is trivial on $Z_0(G)$ (note that $Z_0(G) \subset P$), and
- the holomorphic line bundle over $G/P$ associated to the holomorphic principal $P$–bundle $G \longrightarrow G/P$ for the character $\hat{\chi}$ of $P$ is ample.

The unipotent radical of a parabolic subgroup $P \subset G$ is denoted by $R_u(P)$. The quotient $P/R_u(P)$ is a reductive affine complex algebraic group. A Zariski closed connected reductive complex algebraic subgroup $L(P) \subset P$ is called a **Levi factor** of $P$ if the composition of maps

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$
is an isomorphism [Bor, p. 158, § 11.22]. There are Levi factors of $P$, moreover, any two Levi factors of $P$ differ by the inner automorphism of $P$ produced by an element of the unipotent radical $R_u(P)$ [Bor, p. 158, § 11.23], [Hum, § 30.2, p. 184].

Let $E_G$ be a holomorphic principal $G$–bundle over $X$. Let

$$
\theta \in H^0(X, \text{ad}(E_G) \otimes K_X(D))
$$

be a holomorphic section. The $D$–twisted $G$–Higgs bundle $(E_G, \theta)$ is called stable (respectively, semistable) if for all triples of the form $(P, E_P, \hat{\chi})$, where

- $P \subset G$ is a proper (not necessarily maximal) parabolic subgroup,
- $E_P \subset E_G$ is a holomorphic reduction of structure group of $E_G$ to $P$ over $X$ such that

$$
\theta \in H^0(X, \text{ad}(E_P) \otimes K_X(D)) \subset H^0(X, \text{ad}(E_G) \otimes K_X(D)),
$$

and

- $\hat{\chi}$ is a strictly anti-dominant character of $P$,

the inequality

$$\text{degree}(E_P(\hat{\chi})) > 0$$

(respectively, degree$(E_P(\hat{\chi})) \geq 0$) holds, where $E_P(\hat{\chi})$ is the holomorphic line bundle over $X$ associated to the holomorphic principal $P$–bundle $E_P$ for the character $\hat{\chi}$ of $P$. (See [Hi2, Si2, Si3, Ra2, Ra1, RS, AnBi, BG].)

Let $P$ be a parabolic subgroup of $G$ and $E_P \subset E_G$ a holomorphic reduction of structure group of $E_G$ over $X$ to the subgroup $P$. Such a reduction of structure group is called admissible if for every character $\hat{\chi}$ of $P$ trivial on $Z_0(G)$, the associated holomorphic line bundle $E_P(\hat{\chi})$ on $X$ is of degree zero.

A $D$–twisted $G$–Higgs bundle $(E_G, \theta)$ is called polystable if either $E_G$ is stable, or there is a parabolic subgroup $P \subset G$ and a holomorphic reduction of structure group $E_{L(P)} \subset E_G$ over $X$ to a Levi factor $L(P)$ of $P$, such that

- $\theta \in H^0(X, \text{ad}(E_{L(P)}) \otimes K_X(D)) \subset H^0(X, \text{ad}(E_G) \otimes K_X(D))$,
- the holomorphic $L(P)$–Higgs bundle $(E_{L(P)}, \theta)$ is stable, and
- the reduction of structure group of $E_G$ to $P$ given by the extension of the structure group of $E_{L(P)}$ to $P$, corresponding to the inclusion of $L(P)$ in $P$, is admissible.

(See [Hi2, Si2, Si3, RS, AnBi, BG].) In particular, a polystable $D$–twisted $G$–Higgs bundle is semistable.

**Definition 2.6.** A framed $G$–Higgs bundle $(E_G, \phi, \theta)$ over $X$ is called stable if the $D$–twisted $G$–Higgs bundle $(E_G, \theta)$ is stable. Similarly, $(E_G, \phi, \theta)$ is called semistable (respectively, polystable) if $(E_G, \theta)$ is semistable (respectively, polystable).

It should be mentioned that there are other definitions of (semi)stability of a framed $G$–Higgs bundle. The one given in Definition 2.6 is in fact a special case.

**Remark 2.7.** When $G = \text{GL}(r, \mathbb{C})$ and $H_x = I_{rxr}$ for all $x \in D$, Definition 2.6 reduces to the definition of (semi)stable framed Higgs bundles given in [BLP, Definition 2.2] (see also [BLP, Remark 2.6]).
3. Infinitesimal Deformations

3.1. Infinitesimal deformations of a framed principal bundle. The infinitesimal deformations of a holomorphic principal $G$–bundle $E_G$ over $X$ are parametrized by $H^1(X, \text{ad}(E_G))$ (see [Do], [Se, Appendix III]).

To describe the space of all infinitesimal deformations of a framed holomorphic principal $G$–bundle, first consider the special case where $H_x = \{e\}$ for every $x \in D$ (see (2.5)); as before the identity element of $G$ is denoted by $e$. In this case, the infinitesimal deformations of a framed holomorphic principal $G$–bundle $(E_G, \phi)$ are parametrized by $H^1(X, \text{ad}(E_G) \otimes O_X(-D))$; in the special case where $G = \text{GL}(r, \mathbb{C})$ and $H_x = \{I_{r \times r}\}$ for all $x \in D$, this is Lemma 2.5 of [BLP]. For notational convenience, the tensor product $\text{ad}(E_G) \otimes O_X(-D)$ will be denoted by $\text{ad}(E_G)(-D)$. Consider the following short exact sequence of coherent analytic sheaves on $X$:

$$0 \rightarrow \text{ad}(E_G)(-D) \rightarrow \text{ad}(E_G) \rightarrow \text{ad}(E_G)|_D \rightarrow 0.$$ 

Let

$$\rightarrow H^0(X, \text{ad}(E_G)) \rightarrow H^0(X, \text{ad}(E_G)|_D) \overset{\alpha_1}{\rightarrow} H^1(X, \text{ad}(E_G)(-D))$$

$$\overset{\alpha_2}{\rightarrow} H^1(X, \text{ad}(E_G)) \rightarrow H^1(X, \text{ad}(E_G)|_D) = 0$$

be the long exact sequence of cohomologies associated to it; we have $H^1(X, \text{ad}(E_G)|_D) = 0$ in (3.1) because $\text{ad}(E_G)|_D$ is a torsion sheaf supported on points. The homomorphism $\alpha_2$ in (3.1) sends an infinitesimal deformation of $(E_G, \phi)$ to the infinitesimal deformation of $E_G$ obtained from it by simply forgetting the framing. Consider the space of framings $\mathcal{F}(E_G)$ in (2.6) (at present $H_x = \{e\}$ for every $x \in D$). Note that

$$T_{\phi} \mathcal{F}(E_G) = H^0(X, \text{ad}(E_G)|_D) = \mathfrak{g}^D := \bigoplus_{x \in D} \mathfrak{g}.$$ 

Indeed, $\phi(x) \in (E_G)_x$ identifies the fiber $(E_G)_x$ with $G$ by sending any $g \in G$ to $\phi(x)g \in (E_G)_x$. This trivialization of $(E_G)_x$ produces an identification of the Lie algebra $\text{ad}(E_G)_x$ with $\mathfrak{g}$; indeed, both $\text{ad}(E_G)_x$ and $\mathfrak{g}$ are identified with the right $G$–invariant vector fields on $(E_G)_x$ and $G$ respectively. The homomorphism $\alpha_1$ in (3.1) gives the infinitesimal deformations of the framed principal $G$–bundle $(E_G, \phi)$ obtained by deforming the framing while keeping the holomorphic principal $G$–bundle $E_G$ fixed.

Now we consider the general case of framings. The subgroups $H_x \subsetneq G$, $x \in D$, in (2.5) are no longer assumed to be trivial.

Consider the subspaces $\mathcal{H}_x \subset \text{ad}(E_G)_x$, $x \in D$, constructed in (2.12). Let $\text{ad}_\phi(E_G)$ be the holomorphic vector bundle on $X$ defined by the following short exact sequence of coherent analytic sheaves:

$$0 \rightarrow \text{ad}_\phi(E_G) \rightarrow \text{ad}(E_G) \rightarrow \bigoplus_{x \in D} \text{ad}(E_G)_x/\mathcal{H}_x \rightarrow 0,$$

where $\text{ad}(E_G)_x/\mathcal{H}_x$ is supported at $x$. Let

$$\rightarrow H^0(X, \text{ad}(E_G)) \rightarrow H^0(X, \bigoplus_{x \in D} \text{ad}(E_G)_x/\mathcal{H}_x) \overset{\tilde{\alpha}_1}{\rightarrow} H^1(X, \text{ad}_\phi(E_G))$$

$$\overset{\tilde{\alpha}_2}{\rightarrow} H^1(X, \text{ad}(E_G)) \rightarrow H^1(X, \bigoplus_{x \in D} \text{ad}(E_G)_x/\mathcal{H}_x) = 0$$

(3.3)
be the long exact sequence of cohomologies associated to the short exact sequence of coherent analytic sheaves in (3.2); we have $H^1(X, \bigoplus_{x \in D \text{ad}(E_G)_x/\mathcal{H}_x}) = 0$ in (3.3) because $\text{ad}(E_G)_x/\mathcal{H}_x$ is a torsion sheaf supported on points.

**Lemma 3.1.** (1) The infinitesimal deformations of any framed holomorphic principal $G$–bundle $(E_G, \phi)$ on $X$ are parametrized by $H^1(X, \text{ad}_\phi(E_G))$, where $\text{ad}_\phi(E_G)$ is constructed in (3.2).

(2) The homomorphism $\widehat{\alpha}_2$ in (3.3) sends an infinitesimal deformation of $(E_G, \phi)$ to the infinitesimal deformation of $E_G$ obtained from it by simply forgetting the framing.

(3) Consider the space of framings $\mathcal{F}(E_G)$ on $E_G$ in (2.6). The tangent space of it at $\phi$ is

$$T_{\phi}\mathcal{F}(E_G) = \bigoplus_{x \in D} \text{ad}(E_G)_x/\mathcal{H}_x$$

(see (2.8)). The homomorphism $\widehat{\alpha}_1$ in (3.3) gives all the infinitesimal deformations of the framed principal $G$–bundle $(E_G, \phi)$ obtained by deforming the framing while keeping the holomorphic principal $G$–bundle $E_G$ fixed.

**Proof.** First note that for any open subset $U \subset X$, the space of all holomorphic sections of $\text{ad}(E_G)|_U$ is the space of all holomorphic vector fields $v$ on $p^{-1}(U) \subset E_G$, where $p$ is the projection in (2.3), satisfying the following two conditions:

- $v$ is invariant under the action of $G$ on $E_G$, and
- $v$ is vertical for the projection $p$.

The subsheaf $\text{ad}_\phi(E_G) \subset \text{ad}(E_G)$ coincides with the subsheaf that also preserves the framing $\phi$. The lemma follows from this; we omit the details. \qed

For any two subgroups $H' \subset H \subset G$, and any holomorphic principal $G$–bundle $F_G$ on $X$, there is a natural projection $(F_G)_y/H' \rightarrow (F_G)_x/H$ for any point $y \in X$. Therefore, if we have $H'_x \subset H_x \subset G$ for every $x \in D$, then a framing of $E_G$ for $\{H'_x\}_{x \in D}$ produces a framing of $E_G$ for $\{H_x\}_{x \in D}$. In particular, a framing of $E_G$ for the trivial groups $\{e\}_{x \in D}$ produces a framing of $E_G$ for $\{H_x\}_{x \in D}$.

From (3.2) we conclude that $\text{ad}_\phi(E_G)$ fits in the following short exact sequence of sheaves on $X$:

$$0 \rightarrow \text{ad}(E_G)(-D) := \text{ad}(E_G) \otimes \mathcal{O}_X(-D) \xrightarrow{\zeta} \text{ad}_\phi(E_G) \rightarrow \bigoplus_{x \in D} \mathcal{H}_x \rightarrow 0.$$  

(3.4)

Let

$$\zeta_* : H^1(X, \text{ad}(E_G)(-D)) \rightarrow H^1(X, \text{ad}_\phi(E_G))$$

be the homomorphism of cohomologies induced by the homomorphism $\zeta$ in (3.4). This homomorphism $\zeta_*$ coincides with the homomorphism of infinitesimal deformations corresponding to the above map from the framings of a holomorphic principal $G$–bundle $F_G$ for $\{e\}_{x \in D}$ to the framings of $F_G$ for $\{H_x\}_{x \in D}$. 
3.2. Infinitesimal deformations of a framed G–Higgs bundle. Take a holomorphic principal G–bundle $E_G$ on $X$, and also take a holomorphic section

$$\theta \in H^0(X, \text{ad}(E_G) \otimes K_X(D)).$$

Let

$$f_\theta : \text{ad}(E_G) \longrightarrow \text{ad}(E_G) \otimes K_X(D)$$

be the $O_X$–linear homomorphism defined by $t \mapsto [\theta, t]$. Now we have the 2-term complex

$$C'_\bullet : C'_0 = \text{ad}(E_G) \xrightarrow{f_\theta} C'_1 = \text{ad}(E_G) \otimes K_X(D), \quad (3.5)$$

where $C'_i$ is at the $i$-th position.

The following lemma is proved in [BR, p. 220, Theorem 2.3], [Bot, p. 399, Proposition 3.1.2], [Ma, p. 271, Proposition 7.1] (see also [Bi]).

**Lemma 3.2.** The infinitesimal deformations of the $D$–twisted $G$–Higgs bundle $(E_G, \theta)$ are parametrized by elements of the first hypercohomology $\mathbb{H}^1(C'_\bullet)$, where $C'_\bullet$ is the complex in (3.5).

The following lemma gives the dimension of the infinitesimal deformations.

**Lemma 3.3.** Assume that genus$(X) \geq 1$. Let $(E_G, \theta)$ be a stable $D$–twisted $G$–Higgs bundle. Then

$$\mathbb{H}^0(C'_\bullet) = \{ v \in H^0(X, \text{ad}(E_G)) \mid [\theta, v] = 0 \} = Z(g),$$

where $Z(g) \subset g$ as before is the center, and

$$\mathbb{H}^2(C'_\bullet) = 0.$$

Moreover,

$$\dim \mathbb{H}^1(C'_\bullet) = \dim G \cdot (2 \cdot (\text{genus}(X) - 1) + n) + \dim Z(g),$$

where $n = \#D$.

**Proof.** Consider the short exact sequence of complexes of sheaves

$$0 \longrightarrow 0 \longrightarrow \text{ad}(E_G) \otimes K_X(D) \longrightarrow C'_0 \xrightarrow{f_\theta} C'_1 = \text{ad}(E_G) \otimes K_X(D) \longrightarrow 0 \longrightarrow 0.$$
on $X$. Let

$$
\begin{align*}
0 & \rightarrow \mathbb{H}^0(C'_\bullet) \rightarrow H^0(X, \text{ad}(E_G)) \rightarrow H^0(X, \text{ad}(E_G) \otimes K_X(D)) \rightarrow \mathbb{H}^1(C'_\bullet) \\
& \rightarrow H^1(X, \text{ad}(E_G)) \xrightarrow{\varpi} H^1(X, \text{ad}(E_G) \otimes K_X(D)) \rightarrow \mathbb{H}^2(C'_\bullet) \rightarrow 0
\end{align*}
$$

(3.6)

be the long exact sequence of hypercohomologies associated it. First note that the trivial holomorphic vector bundle $X \times Z(\mathfrak{g})$ over $X$ is a holomorphic subbundle of $\text{ad}(E_G)$, because the adjoint action of $G$ on $\mathfrak{g}$ fixes $Z(\mathfrak{g})$ pointwise. The stability condition of $(E_G, \theta)$ implies that

$$[v \in H^0(X, \text{ad}(E_G)) | [\theta, v] = 0] = H^0(X, X \times Z(\mathfrak{g})) = Z(\mathfrak{g}).$$

(3.7)

On the other hand, from (3.6) it follows that

$$\mathbb{H}^0(C'_\bullet) = \{v \in H^0(X, \text{ad}(E_G)) | [\theta, v] = 0\}.$$

Hence, we have that $\mathbb{H}^0(C'_\bullet) = Z(\mathfrak{g})$.

Next note that from (3.7) it follows that

$$\{v \in H^0(X, \text{ad}(E_G) \otimes \mathcal{O}_X(-D)) | [\theta, v] = 0\} = 0.$$

(3.8)

The nondegenerate symmetric bilinear form $\tilde{\sigma}$ in (2.10) identifies the holomorphic vector bundle $\text{ad}(E_G)$ with its dual $\text{ad}(E_G)^*$. Hence Serre duality gives that

$$H^1(X, \text{ad}(E_G)) = H^0(X, \text{ad}(E_G) \otimes K_X)^*$$

and $H^1(X, \text{ad}(E_G) \otimes K_X(D)) = H^0(X, \text{ad}(E_G) \otimes \mathcal{O}_X(-D))^*$. Using these isomorphisms, the homomorphism $\varpi$ in (3.6) coincides with the dual of the homomorphism

$$H^0(X, \text{ad}(E_G) \otimes \mathcal{O}_X(-D)) \rightarrow H^0(X, \text{ad}(E_G) \otimes K_X), \ v \mapsto [\theta, v].$$

(3.9)

Therefore, the homomorphism in (3.9) will be denoted by $\varpi^*$. From (3.8) it now follows that $\varpi^*$ in (3.9) is injective. Hence its dual $\varpi$ is surjective. Consequently, from (3.6) we now conclude that $\mathbb{H}^2(C'_\bullet) = 0$.

From (3.6) it follows immediately that

$$\dim \mathbb{H}^1(C'_\bullet) = \chi(\text{ad}(E_G) \otimes K_X(D)) - \chi(\text{ad}(E_G)) + \dim \mathbb{H}^0(C'_\bullet) + \dim \mathbb{H}^2(C'_\bullet),$$

where $\chi(F) := \dim H^0(X, F) - \dim H^1(X, F)$ is the Euler characteristic. By Riemann–Roch, we have $\chi(\text{ad}(E_G)) = \dim G \cdot (1 - \text{genus}(X))$, and

$$\chi(\text{ad}(E_G) \otimes K_X(D)) = -\chi(\text{ad}(E_G) \otimes \mathcal{O}_X(-D)) = \dim G \cdot (\text{genus}(X) - 1 + n).$$

Consequently, from the above computations of $\mathbb{H}^0(C'_\bullet)$ and $\mathbb{H}^2(C'_\bullet)$ it follows that $\dim \mathbb{H}^1(C'_\bullet) = \dim G \cdot (2 \cdot (\text{genus}(X) - 1 + n) + \dim Z(\mathfrak{g})$. □

Remark 3.4. For a stable $D$–twisted $G$–Higgs bundle $(E_G, \theta)$, since $\mathbb{H}^2(C'_\bullet) = 0$, the deformations of $(E_G, \theta)$ are unobstructed.
We shall describe the space of all infinitesimal deformations of a framed \( G \text{–Higgs} \) bundle. For that, we first consider the special case where \( H_x = \{ e \} \) for every \( x \in D \).

Consider the following 2-term sub-complex of the complex \( \mathcal{C}'_\bullet \) in (3.5):

\[
\mathcal{C}_\bullet : \quad \mathcal{C}_0 = \text{ad}(E_G)(-D) := \text{ad}(E_G) \otimes \mathcal{O}_X(-D) \xrightarrow{f_0} \mathcal{C}_1 = \text{ad}(E_G) \otimes K_X(D)
\]

(3.10)

(here the restriction of the homomorphism \( f_0 \) to \( \text{ad}(E_G)(-D) \subset \text{ad}(E_G) \) is also denoted by \( f_0 \)). Let \( \phi \) be a framing on \( E_G \). Since \( H_x = e \) for all \( x \in D \), we have that \( \mathcal{H}_x = 0 \), which implies that \( \text{ad}_\phi(E_G) = \text{ad}(E_G) \). Consequently, the triple \( (E_G, \phi, \theta) \) is a framed \( G \text{-Higgs} \) bundle.

**Lemma 3.5.** Assume that \( H_x = \{ e \} \) for every \( x \in D \). The infinitesimal deformations of the framed \( G \text{-Higgs} \) bundle \( (E_G, \phi, \theta) \) are parametrized by elements of the first hypercohomology \( \mathbb{H}^1(\mathcal{C}_\bullet) \), where \( \mathcal{C}_\bullet \) is the complex in (3.10).

**Proof.** The proof of Lemma 3.2 also works for this lemma after some very minor and straightforward modifications. (In the special case where \( G = \text{GL}(r, \mathbb{C}) \), this lemma reduces to Lemma 2.7 of [BLP].) \( \square \)

We have the following short exact sequence of complexes of sheaves on \( X \)

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{C}_\bullet : & \text{ad}(E_G)(-D) & \xrightarrow{f_0} & \text{ad}(E_G) \otimes K_X(D) & = \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{C}'_\bullet : & \text{ad}(E_G) & \xrightarrow{f_0} & \text{ad}(E_G) \otimes K_X(D) & = \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{C}'_0 : & \text{ad}(E_G)|_D & \to & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

where \( \mathcal{C}'_0 \) is a 1-term complex, and \( \mathcal{C}'_\bullet \) is defined in (3.5). Let

\[
\begin{align*}
\to \mathbb{H}^0(X, \mathcal{C}'_0) &= \mathcal{H}^0(X, \text{ad}(E_G)|_D) \xrightarrow{\beta^1_1} \mathbb{H}^1(X, \mathcal{C}_\bullet) \xrightarrow{\beta^1_2} \mathbb{H}^1(X, \mathcal{C}'_\bullet) \\
\to \mathbb{H}^1(X, \mathcal{C}'_0) &= 0
\end{align*}
\]

(3.11)

be the long exact sequence of hypercohomologies associated to this short exact sequence of complexes; note that we have \( \mathbb{H}^1(X, \mathcal{C}'_0) = 0 \) because \( \mathcal{C}'_0 \) is a torsion sheaf supported on points. The homomorphism \( \beta^1_2 \) in (3.11) coincides with the homomorphism of infinitesimal deformations corresponding to the forgetful map that sends any framed \( G \text{-Higgs} \) bundle \( (E'_G, \phi', \theta') \) to the \( D \)-twisted \( G \text{-Higgs} \) bundle \( (E'_G, \theta') \) by forgetting the framing; see Lemmas 3.2 and 3.5 (we have \( H_x = \{ e \} \) for every \( x \in D \)). The homomorphism \( \beta^1_1 \) in (3.11) corresponds to moving just the framing while keeping the \( D \)-twisted \( G \text{-Higgs} \) bundle \( (E_G, \theta) \) fixed.

Now we consider framings of general type. The subgroups \( H_x \subset G, x \in D \), in (2.5) are no longer assumed to be trivial.
Let \((E_G, \phi, \theta)\) be a framed \(G\)–Higgs bundle. Consider the subspace \(\mathcal{H}_x^\perp \subset \text{ad}(E_G)_x\) in (2.13). Let \(\text{ad}_\phi^n(E_G)\) be the holomorphic vector bundle on \(X\) defined by the following short exact sequence of coherent analytic sheaves on \(X\):

\[
0 \rightarrow \text{ad}_\phi^n(E_G) \rightarrow \text{ad}(E_G) \rightarrow \bigoplus_{x \in D} \text{ad}(E_G)_x / \mathcal{H}_x^\perp \rightarrow 0 ,
\]

(3.12)

where \(\text{ad}(E_G)_x / \mathcal{H}_x^\perp\) is supported at \(x\). From (3.12) it follows immediately that the holomorphic sections of \(\text{ad}_\phi^n(E_G)\) are precisely the sections \(s \in H^0(X, \text{ad}(E_G))\) such that \(s(x) \in \mathcal{H}_x^\perp\) for every \(x \in D\). Hence from the definition of Higgs fields on \((E_G, \phi)\) it follows that Higgs fields on \((E_G, \phi)\) are precisely the holomorphic sections of the holomorphic vector bundle \(\text{ad}_\phi^n(E_G) \otimes K_X(D)\).

**Lemma 3.6.** The homomorphism \(f_\theta\) in (3.5) sends the subsheaf \(\text{ad}_\phi(E_G) \subset \text{ad}(E_G)\) constructed in (3.2) to the subsheaf \(\text{ad}_\phi^n(E_G) \otimes K_X(D) \subset \text{ad}(E_G) \otimes K_X(D)\), where \(\text{ad}_\phi^n(E_G)\) is constructed in (3.12).

**Proof.** Let \(S\) be a Lie subalgebra of \(\mathfrak{g}\). Let \(S^\perp\) be the annihilator of it for the symmetric bilinear form \(\sigma\) in (2.9). Then it can be shown that

\[
[S, S^\perp] \subset S^\perp .
\]

(3.13)

Indeed, the \(G\)–invariance condition on \(\sigma\) implies that

\[
\sigma([a, b] \otimes c) + \sigma(c \otimes [a, b]) = 0
\]

(3.14)

for all \(a, b, c \in \mathfrak{g}\). In particular, for any \(a, b \in S\) and \(c \in S^\perp\),

\[
\sigma([a, b] \otimes c) = -\sigma(c \otimes [a, b]) = 0,
\]

because \([b, a] \in S\) and \(c \in S^\perp\).

For any \(x \in D\), the image of the homomorphism \(\text{ad}_\phi(E_G)_x \rightarrow \text{ad}(E_G)_x\) in (3.2) is \(\mathcal{H}_x\), while the image of the homomorphism \(\text{ad}_\phi^n(E_G)_x \rightarrow \text{ad}(E_G)_x\) in (3.12) is \(\mathcal{H}_x^\perp\). From (3.13) we know that

\[
[\mathcal{H}_x, \mathcal{H}_x^\perp] \subset \mathcal{H}_x^\perp .
\]

(3.15)

Since \(\theta\) is a holomorphic section of \(\text{ad}_\phi^n(E_G) \otimes K_X(D)\), the lemma follows from (3.15). \(\Box\)

The restriction of \(f_\theta\) (defined in (3.5)) to \(\text{ad}_\phi(E_G) \subset \text{ad}(E_G)\) will be denoted by \(f_0^\theta\). Let \(\mathcal{D}_\bullet\) be the following 2-term sub-complex of \(\mathcal{C}_\bullet\) constructed in (3.5):

\[
\mathcal{D}_\bullet : \mathcal{D}_0 = \text{ad}_\phi(E_G) \xrightarrow{f_0^\theta} \mathcal{D}_1 = \text{ad}_\phi^n(E_G) \otimes K_X(D)
\]

(3.16)

(note that Lemma 3.6 shows that \(f_\theta(\mathcal{D}_0) \subset \mathcal{D}_1\)).

**Lemma 3.7.** All the infinitesimal deformations of the given framed \(G\)–Higgs bundle \((E_G, \phi, \theta)\) are parametrized by the elements of the first hypercohomology \(\mathbb{H}_1(\mathcal{D}_\bullet)\), where \(\mathcal{D}_\bullet\) is constructed in (3.16).
Proof. Just as the proof of Lemma 3.2 also works for Lemma 3.5, it works even for this lemma after the framing is suitably taken into account. We omit the details. It should be mentioned that this lemma can also be proved using the framework of Sect. 6. □

We have the following short exact sequence of complexes of sheaves on $X$

$$
\begin{align*}
0 & \longrightarrow 0 & 0 \\
\mathcal{D}' : & \longrightarrow 0 & \mathcal{D}'_1 = \text{ad}^n_\phi(E_G) \otimes K_X(D) \\
& \downarrow & \downarrow \\
\mathcal{D} : & \text{ad}_\phi(E_G) \longrightarrow f^0_\phi \text{ad}^n_\phi(E_G) \otimes K_X(D) \\
& \downarrow & \downarrow \\
\mathcal{D}'' : & \text{ad}_\phi(E_G) \longrightarrow 0 \\
& \downarrow & \downarrow \\
0 & 0 & 0
\end{align*}
$$

(both $\mathcal{D}'$ and $\mathcal{D}''$ are 1-term complexes concentrated at the first position and the 0-th position respectively). Let

$$
\begin{align*}
\mathbb{H}^0(\mathcal{D}'') = H^0(X, \text{ad}_\phi(E_G)) & \longrightarrow \mathbb{H}^1(\mathcal{D}') = H^0(X, \text{ad}^n_\phi(E_G) \otimes K_X(D)) \\
\beta_3' & \longrightarrow \mathbb{H}^1(\mathcal{D}) \longrightarrow \mathbb{H}^1(\mathcal{D}'') = H^1(X, \text{ad}_\phi(E_G))
\end{align*}
$$

be the long exact sequence of hypercohomologies associated to (3.17). The homomorphism $\beta_3'$ in (3.18) corresponds to deforming the Higgs field keeping the framed principal bundle $(E_G, \phi)$ fixed; recall that the Higgs fields on $(E_G, \phi)$ are the holomorphic sections of $\text{ad}^n_\phi(E_G) \otimes K_X(D)$. The homomorphism $\beta_4'$ in (3.18) corresponds to the forgetful map that sends an infinitesimal deformation of $(E_G, \phi, \theta)$ to the infinitesimal deformation of $(E_G, \phi)$ it gives by simply forgetting the Higgs field (see Lemma 3.1(1)).

The hypercohomologies of $\mathcal{D}$ will be computed in Sect. 5.1.

4. Framed $G$–Higgs Bundles and Symplectic Geometry

4.1. Construction of a symplectic structure.

Proposition 4.1. The dual $\text{ad}_\phi(E_G)^*$ of the vector bundle $\text{ad}_\phi(E_G)$ in (3.2) is identified with $\text{ad}^n_\phi(E_G) \otimes \mathcal{O}_X(D)$, where $\text{ad}^n_\phi(E_G)$ is constructed in (3.12). This identification is canonical in the sense that it depends only on $\sigma$ in (2.9).

The dual vector bundle $\text{ad}^n_\phi(E_G)^*$ is identified with $\text{ad}_\phi(E_G) \otimes \mathcal{O}_X(D)$; this identification is canonical in the above sense.

Proof. Consider the fiberwise nodegenerate symmetric bilinear form

$$
\hat{\sigma} : \text{ad}(E_G)^{\otimes 2} \longrightarrow \mathcal{O}_X
$$

in (2.10). Tensoring it with $\mathcal{O}_X(D)$ We get the homomorphism

$$
\hat{\sigma} \otimes \text{Id}_{\mathcal{O}_X(D)} : \text{ad}(E_G) \otimes \text{ad}(E_G) \otimes \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D).
$$

(4.1)
Recall that both \( \text{ad}_\phi(E_G) \) and \( \text{ad}_\phi^0(E_G) \) are contained in \( \text{ad}(E_G) \) (see (3.2) and (3.12)). It can be shown that the image of the restriction of the homomorphism \( \hat{\sigma} \otimes \text{Id}_{\mathcal{O}_X(D)} \) in (4.1) to the subsheaf

\[
\text{ad}_\phi(E_G) \otimes \text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D) \subset \text{ad}(E_G) \otimes \text{ad}(E_G) \otimes \mathcal{O}_X(D)
\]
is contained in \( \mathcal{O}_X \subset \mathcal{O}_X(D) \). Indeed, this follows from the facts that for any \( x \in D \), the image of \( \text{ad}_\phi(E_G)_x \) (respectively, \( \text{ad}_\phi^0(E_G)_x \)) in \( \text{ad}(E_G)_x \) is \( \mathcal{H}_x \) (respectively, \( \mathcal{H}_x^\perp \)), and \( \mathcal{H}_x \) annihilates \( \mathcal{H}_x^\perp \) for the form \( \hat{\sigma}(x) \).

The above restricted homomorphism

\[
\hat{\sigma} \otimes \text{Id}_{\mathcal{O}_X(D)} : \text{ad}_\phi(E_G) \otimes \text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X
\]
produces a homomorphism

\[
S : \text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D) \longrightarrow \text{ad}_\phi(E_G)^* . \tag{4.2}
\]

This homomorphism \( S \) is an isomorphism over the complement \( X \setminus D \), because

- the pairing \( \hat{\sigma} \) in (2.10) is fiberwise nondegenerate, and
- \( \text{ad}_\phi^0(E_G)|_{X \setminus D} = \text{ad}(E_G)|_{X \setminus D} = \text{ad}_\phi(E_G)|_{X \setminus D} \).

Denote the torsion sheaf \( \text{ad}_\phi(E_G)^*/\text{Image}(S) \) by \( Q \), where \( S \) is the homomorphism in (4.2). So we have

\[
\text{degree}(Q) = \text{degree}(\text{ad}_\phi(E_G)^*) - \text{degree}(\text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D)) . \tag{4.3}
\]

Since \( \hat{\sigma} \) in (2.10) produces an isomorphism of \( \text{ad}(E_G) \) with the dual vector bundle \( \text{ad}(E_G)^* \), it follows that \( \text{degree}(\text{ad}(E_G)) = 0 \). Hence from (3.2) it follows immediately that

\[
\text{degree}(\text{ad}_\phi(E_G)) = \sum_{x \in D} (\dim \mathcal{H}_x - \dim \mathfrak{g}) ,
\]
while from (3.12) it follows that

\[
\text{degree}(\text{ad}_\phi^0(E_G)) = \sum_{x \in D} (\dim \mathcal{H}_x^\perp - \dim \mathfrak{g}) .
\]

Consequently, we have 
\( \text{degree}(\text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D)) = \sum_{x \in D} \dim \mathcal{H}_x^\perp \). As \( \dim \mathcal{H}_x + \dim \mathcal{H}_x^\perp = \dim \mathfrak{g} \), from (4.3) it now follows that \( \text{degree}(Q) = 0 \). Since \( Q \) is a torsion sheaf with \( \text{degree}(Q) = 0 \), we conclude that \( Q = 0 \). Consequently, the homomorphism \( S \) in (4.2) is an isomorphism. This proves the first statement of the proposition.

The isomorphism in the second statement of the proposition is given by \( S^* \otimes \text{Id}_{\mathcal{O}_X(D)} \).

\[\square\]

**Remark 4.2.** Consider the dual of the homomorphism \( \text{ad}_\phi(E_G) \hookrightarrow \text{ad}(E_G) \) in (3.2).

From the first statement in Proposition 4.1 we have

\[
\text{ad}(E_G) = \text{ad}(E_G)^* \hookrightarrow \text{ad}_\phi(E_G)^* \longrightarrow \text{ad}_\phi^0(E_G) \otimes \mathcal{O}_X(D) ;
\]
as in the proof of Proposition 4.1, the two vector bundles \( \text{ad}(E_G) \) and \( \text{ad}(E_G)^* \) are identified using \( \hat{\sigma} \).
Consider the complex $\mathcal{D}_\bullet$ in (3.16). Its Serre dual complex, which we shall denote by $\mathcal{D}_\bullet^\vee$, is the following:

$$
\mathcal{D}_\bullet^\vee : \mathcal{D}_0^\vee = (\text{ad}_\phi^n(EG) \otimes K_X(D))^* \otimes K_X \overset{(f_0^0)^* \otimes \text{Id}_{K_X}}{\longrightarrow} \mathcal{D}_1^\vee = \text{ad}_\phi(EG)^* \otimes K_X;
$$

(4.4)

to clarify, $\mathcal{D}_0^\vee$ and $\mathcal{D}_1^\vee$ are at the 0-th position and 1-st position respectively. From Proposition 4.1 it follows that

- $\mathcal{D}_0^\vee = \text{ad}_\phi(EG)$, and
- $\mathcal{D}_1^\vee = \text{ad}_\phi^n(EG) \otimes K_X(D)$.

Moreover, using these two identifications, the homomorphism $(f_0^0)^* \otimes \text{Id}_{K_X}$ in (4.4) coincides with $f_0^0$. In other words, the dual complex $\mathcal{D}_\bullet^\vee$ is canonically identified with $\mathcal{D}_\bullet$; this isomorphism of course depends on the bilinear form $\sigma$ in (2.9). Let

$$
\xi : \mathcal{D}_\bullet \sim \mathcal{D}_\bullet^\vee
$$

be this isomorphism. This isomorphism $\xi$ produces an isomorphism

$$
\Phi := \xi_* : H^1(\mathcal{D}_\bullet) \sim H^1(\mathcal{D}_\bullet^\vee)
$$

(4.6)

go of hypercohomologies. On the other hand, Serre duality gives that

$$
H^1(\mathcal{D}_\bullet^\vee) = H^1(\mathcal{D}_\bullet)^*
$$

(cf. [Huy, p. 67, Theorem 3.12]). Using this, the isomorphism $\Phi$ in (4.6) produces an isomorphism

$$
\Phi_{(EG, \phi, \theta)} : H^1(\mathcal{D}_\bullet) \sim H^1(\mathcal{D}_\bullet^\vee)^*.
$$

(4.7)

This homomorphism $\Phi_{(EG, \phi, \theta)}$ is clearly skew-symmetric.

We shall now describe an alternative construction of the homomorphism $\Phi_{(EG, \phi, \theta)}$ in (4.7).

Consider the tensor product of complexes $\mathcal{D}_\bullet := \mathcal{D}_\bullet \otimes \mathcal{D}_\bullet$. So

$$
\mathcal{D}_\bullet : \mathcal{D}_0 = \text{ad}_\phi(EG) \otimes \text{ad}_\phi(EG) \overset{f_0^0 \otimes \text{Id} + \text{Id} \otimes f_0^0}{\longrightarrow} \mathcal{D}_1 = ((\text{ad}_\phi^n(EG) \otimes K_X(D)) \otimes \text{ad}_\phi(EG)) \oplus (\text{ad}_\phi(EG) \otimes (\text{ad}_\phi^n(EG) \otimes K_X(D)))
$$

$$
\overset{\text{Id} \otimes f_0^0 - f_0^0 \otimes \text{Id}}{\longrightarrow} \mathcal{D}_2 = (\text{ad}_\phi^n(EG) \otimes K_X(D)) \otimes (\text{ad}_\phi^n(EG) \otimes K_X(D));
$$

to clarify, $\mathcal{D}_i$ is at the $i$-th position. Consider the homomorphism

$$
\gamma : \gamma : \mathcal{D}_1 = ((\text{ad}_\phi^n(EG) \otimes K_X(D)) \otimes \text{ad}_\phi(EG)) \oplus (\text{ad}_\phi(EG) \otimes (\text{ad}_\phi^n(EG) \otimes K_X(D))) \rightarrow K_X, \quad (a \otimes b + (c \otimes d)) \mapsto \hat{\sigma}(a \otimes b) + \hat{\sigma}(c \otimes d),
$$

where $\hat{\sigma}$ is the pairing in (2.10). Using (3.14) it is straight-forward to deduce that

$$
\gamma \circ (f_0^0 \otimes \text{Id} + \text{Id} \otimes f_0^0) = 0.
$$
Consequently, \( \gamma \) produces a homomorphism of complexes
\[
\Gamma : \mathcal{D}_\bullet \longrightarrow K_X[-1],
\]
(4.8)
where \( K_X[-1] \) is the complex \( 0 \longrightarrow K_X \), with \( K_X \) being at the 1-st position. More precisely, \( \Gamma \) is the following homomorphism of complexes:
\[
\begin{array}{ccc}
\mathcal{D}_\bullet : & \mathcal{D}_0 & \longrightarrow \mathcal{D}_1 & \longrightarrow \mathcal{D}_2 \\
\Gamma & \downarrow & \downarrow & \downarrow \\
K_X[-1] : & 0 & \longrightarrow K_X & \longrightarrow 0
\end{array}
\]

Now we have the homomorphisms of hypercohomologies
\[
H^1(D_\bullet) \otimes H^1(D_\bullet) \longrightarrow H^2(D_\bullet \otimes D_\bullet) = H^2(\mathcal{D}_\bullet) \xrightarrow{\Gamma_*} H^2(K_X[-1])
\]
\[
= H^1(X, K_X) = H^1(X, O_X)^* = \mathbb{C},
\]
(4.9)
where \( \Gamma_* \) is the homomorphism of hypercohomologies induced by the homomorphism \( \Gamma \) in (4.8).

The bilinear form on \( H^1(D_\bullet) \) constructed in (4.9) coincides with the one given by the isomorphism \( \Phi_{(E_G, \phi, \theta)} \) in (4.7).

Recall from Lemma 3.7 that the space of infinitesimal deformations of \((E_G, \phi, \theta)\) is identified with \( H^1(D_\bullet) \).

The above constructions are summarized in the following lemma:

**Lemma 4.3.** The space of infinitesimal deformations of any given framed \( G \)-Higgs bundle \((E_G, \phi, \theta)\), namely \( H^1(D_\bullet) \), is equipped with a natural symplectic structure \( \Phi_{(E_G, \phi, \theta)} \) that is constructed in (4.7) (and also in (4.9)).

### 4.2. A Poisson structure.

Take a \( D \)-twisted \( G \)-Higgs bundle \((E_G, \theta)\) as in Lemma 3.2. Consider the hypercohomology \( H^1(C'_\bullet) \), where \( C'_\bullet \) is constructed in (3.5). Following [Bot, BR, Ma], we shall show that there is a natural homomorphism to it from its dual \( H^1(C'_\bullet)^* \).

**Proposition 4.4.** There is a natural homomorphism
\[
P_{(E_G, \theta)} : H^1(C'_\bullet)^* \longrightarrow H^1(C'_\bullet).
\]

**Proof.** Let \( (C'_\bullet)^\vee \) denote the Serre dual complex of \( C'_\bullet \) in (3.5). So we have
\[
(C'_\bullet)^\vee : (C'_\bullet)^\vee_0 = (\text{ad}(E_G) \otimes K_X(D))^* \otimes K_X \xrightarrow{f_0^* \otimes \text{Id}_{K_X}} (C'_\bullet)^\vee_1 = \text{ad}(E_G)^* \otimes K_X.
\]
(4.10)
where \( (C'_\bullet)^\vee \) is at the \( i \)-th position. As done before, the form \( \hat{\sigma} \) in (2.10) identifies \( \text{ad}(E_G)^* \) with \( \text{ad}(E_G) \). So
\[
(C'_\bullet)^\vee_0 = \text{ad}(E_G) \otimes O_X(-D) \quad \text{and} \quad (C'_\bullet)^\vee_1 = \text{ad}(E_G) \otimes K_X.
\]
Using these two identifications, the homomorphism \( f_0^* \otimes \text{Id}_{K_X} \) in (4.10) coincides with the restriction of \( f_0 \) to the subsheaf \( \text{ad}(E_G) \otimes O_X(-D) \subset \text{ad}(E_G) \); this restriction of
\( f_\theta \) to \( \text{ad}(E_G) \otimes \mathcal{O}_X(-D) \) will also be denoted by \( f_\theta \). Hence the complex \((C')^\vee \) in (4.10) becomes
\[
(C')^\vee : (C')^\vee_0 = \text{ad}(E_G) \otimes \mathcal{O}_X(-D) \xrightarrow{f_\theta} (C')^\vee_1 = \text{ad}(E_G) \otimes K_X. \tag{4.11}
\]
Consequently, we have a homomorphism of complexes \( R : (C')^\vee \to C' \) defined by
\[
\begin{array}{ccc}
(C')^\vee : \text{ad}(E_G) \otimes \mathcal{O}_X(-D) & \to & \text{ad}(E_G) \otimes K_X \\
\downarrow R & & \downarrow y \\
C' : \text{ad}(E_G) & \to & \text{ad}(E_G) \otimes K_X(D)
\end{array}
\]
where the homomorphisms
\[
\text{ad}(E_G) \otimes \mathcal{O}_X(-D) \to \text{ad}(E_G) \quad \text{and} \quad \text{ad}(E_G) \otimes K_X \to \text{ad}(E_G) \otimes K_X(D)
\]
are the natural inclusions (recall that the divisor \( D \) is effective, so \( \mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \)).

Serre duality gives that
\[
H^1((C')^\vee) = H^1(C'_*) \tag{4.12}
\]
Hence the above homomorphism \( R \) of complexes produces the following homomorphism of hypercohomologies
\[
H^1(C'_*) = H^1((C')^\vee) \xrightarrow{R_*} H^1(C'_*), \tag{4.13}
\]
where \( R_* \) is the homomorphism of hypercohomologies induced by \( R \); the above isomorphism \( H^1(C'_*) = H^1((C')^\vee) \) is the one in (4.12). The homomorphism \( H^1(C'_*) \to H^1(C'_*) \) in (4.13) is the homomorphism \( P_{(E_G, \theta)} \) in the proposition that we are seeking. \( \square \)

5. Symplectic Geometry of Moduli of Framed \( G \)--Higgs Bundles

5.1. Moduli space of framed \( G \)--Higgs bundles. As before, for each point \( x \in D \), fix a Zariski closed complex algebraic proper subgroup \( H_x \) of the complex reductive affine algebraic group \( G \).

The topologically isomorphism classes of principal \( G \)--bundles on \( X \) are parametrized by the elements of the fundamental group \( \pi_1(G) \) \cite{St}, \cite[BLS, p. 186, Proposition 1.3(a)]. Fix an element
\[
\nu \in \pi_1(G).
\]

Let \( \mathcal{M}_H(G) \) denote the moduli space of stable \( D \)--twisted \( G \)--Higgs bundle on \( X \) of the form \((E_G, \theta)\), where
\begin{itemize}
\item \( E_G \) is a holomorphic principal \( G \)--bundle on \( X \) of topological type \( \nu \), and
\item \( \theta \in H^0(X, \text{ad}(E_G) \otimes K_X(D)) \).
\end{itemize}

See \cite{Si2, Si3, Ni} for the construction of the moduli space.

Lemmas 3.2 and 3.3 combine together to give the following (see also Remark 3.4):
Corollary 5.1. Assume that \( \text{genus}(X) \geq 1 \). For any point \( (E_G, \theta) \in \mathcal{M}_H(G) \),
\[
T_{(E_G, \theta)} \mathcal{M}_H(G) = \mathbb{H}^1(C'_\bullet),
\]
where \( C'_\bullet \) is the complex in (3.5).

The moduli space \( \mathcal{M}_H(G) \) is a smooth orbifold of dimension \( \dim G \cdot (2 \cdot (\text{genus}(X) - 1) + n) + \dim Z(\mathfrak{g}) \), where \( n = \#D \), and \( Z(\mathfrak{g}) \subseteq \mathfrak{g} := \text{Lie}(\mathfrak{g}) \) is the center of the Lie algebra.

Let \( \mathcal{M}_{FH}(G) \) denote the moduli space of stable framed \( G \)–Higgs bundles of topological type \( \nu([S_i^2, S_i^3, N_i, M, D, M, D, G]) \). Let
\[
\varphi : \mathcal{M}_{FH}(G) \longrightarrow \mathcal{M}_H(G) \quad (5.1)
\]
be the forgetful morphism that sends any triple \((E_G, \phi, \theta)\) to \((E_G, \theta)\).

Define
\[
Z_h = (\bigcap_{x \in D} h_x) \cap Z(\mathfrak{g}), \quad (5.2)
\]
where \( h_x \) as before denotes the Lie algebra of the subgroup \( H_x \) of \( G \).

Proposition 5.2. Assume that \( \text{genus}(X) \geq 1 \). Let \((E_G, \phi, \theta)\) be a stable framed \( G \)–Higgs bundle. Let \( D_\bullet \) be the complex in (3.16) associated to \((E_G, \phi, \theta)\). Then the following three hold:
\begin{enumerate}
  \item \( \mathbb{H}^0(D_\bullet) = Z_h \), where \( Z_h \) is defined in (5.2),
  \item \( \mathbb{H}^2(D_\bullet) = Z_h^* \),
  \item \( \dim \mathbb{H}^1(D_\bullet) = 2(\dim Z_h + \dim G \cdot (\text{genus}(X) - 1 + n) - \sum_{x \in D} \dim h_x) \), where \( n = \#D \).
\end{enumerate}

Proof. Since \( D_\bullet \) is a sub-complex of \( C'_\bullet \) constructed in (3.5), it follows that \( \mathbb{H}^0(D_\bullet) \subset \mathbb{H}^0(C'_\bullet) \). More precisely, from (3.2) we know that an element
\[
v \in \mathbb{H}^0(C'_\bullet) \subset H^0(C'_0) = H^0(X, \text{ad}(E_G))
\]
lies in \( \mathbb{H}^0(D_\bullet) \) if and only if \( v(x) \in \mathcal{H}_x \) for every \( x \in D \). Now, from Lemma 3.3 we know that \( \mathbb{H}^0(C'_\bullet) = Z(\mathfrak{g}) \). Combining these it yields that \( \mathbb{H}^0(D_\bullet) = Z_h \). This proves (1) in the proposition.

Using Serre duality and the isomorphism \( \xi \) in (4.5), we have that
\[
\mathbb{H}^2(D_\bullet) = \mathbb{H}^0(D_\bullet)^* = \mathbb{H}^0(C'_\bullet)^* = Z_h^*.
\]
This proves (2) in the proposition.

To prove (3), first note that from the long exact sequence of hypercohomologies associated to the short exact sequence of complexes in (3.17) it follows immediately that
\[
\dim \mathbb{H}^1(D_\bullet) = \dim \mathbb{H}^0(D_\bullet) + \dim \mathbb{H}^2(D_\bullet) - \chi(\text{ad}_\phi(E_G)) + \chi(\text{ad}_\phi^0(E_G) \otimes K_X(D));
\]
\[
(5.3)
\]
as before, \( \chi \) denotes the Euler characteristic. Now, from (3.4) we know that
\[
\chi(\text{ad}_\phi(E_G)) = \chi(\text{ad}(E_G)(-D)) + \sum_{x \in D} \dim \mathcal{H}_x.
\]
Hence $\chi(\text{ad}_\phi(E_G)) = (\sum_{x \in D} \dim h_x) - \dim G \cdot (\text{genus}(X) - 1 + n)$.

Since $\text{ad}_\phi(E_G)^* \otimes K_X = \text{ad}_\phi^n(E_G) \otimes K_X(D)$ (see Proposition 4.1(1)), using Serre duality, we have that

$$\chi(\text{ad}_\phi^n(E_G) \otimes K_X(D)) = \chi(\text{ad}_\phi(E_G)^* \otimes K_X) = -\chi(\text{ad}_\phi(E_G))$$

$$= \dim G \cdot (\text{genus}(X) - 1 + n) - \sum_{x \in D} \dim h_x.$$

On the other hand, it was shown above that

$$\dim \mathbb{H}^2(D_\bullet) = \dim \mathbb{H}^0(D_\bullet) = \dim Z_{\mathfrak{h}}.$$

Combining these with (5.3), the third statement in the proposition follows. $\square$

Lemma 3.7 and Proposition 5.2 combine together to give the following:

**Corollary 5.3.** Assume that genus$(X) \geq 1$. For any point $(E_G, \phi, \theta) \in M_{FH}(G)$,

$$T_{(E_G, \phi, \theta)} M_{FH}(G) = H^1(D_\bullet),$$

where $D_\bullet$ is the complex in (3.16).

The moduli space $M_{FH}(G)$ is a smooth orbifold of dimension $2(\dim Z_{\mathfrak{h}} + \dim G \cdot (\text{genus}(X) - 1 + n) - \sum_{x \in D} \dim h_x)$.

Henceforth, we would always assume that genus$(X) \geq 1$.

5.2. Symplectic form on the moduli space. Consider the symplectic form $\Phi_{(E_G, \phi, \theta)}$ in Lemma 4.3. In view of Corollary 5.3, this pointwise construction defines a holomorphic two-form on the moduli space $M_{FH}(G)$. This holomorphic two-form on $M_{FH}(G)$ will be denoted by $\Phi$.

**Theorem 5.4.** The above holomorphic form $\Phi$ on $M_{FH}(G)$ is symplectic.

**Proof.** The form $\Phi$ is fiberwise nondegenerate by Lemma 4.3. So it suffices to show that $\Phi$ is closed.

Take any point $(E_G, \phi, \theta) \in M_{FH}(G)$. Corollary 5.3 says that

$$T_{(E_G, \phi, \theta)} M_{FH}(G) = H^1(D_\bullet).$$

Now consider the homomorphism

$$\beta'_4 : H^1(D_\bullet) \longrightarrow H^1(X, \text{ad}_\phi(E_G))$$

in (3.18). In view of the first statement in Proposition 4.1, Serre duality gives that

$$H^1(X, \text{ad}_\phi(E_G))^* = H^0(X, \text{ad}_\phi(E_G)^* \otimes K_X) = H^0(X, \text{ad}_\phi^n(E_G) \otimes K_X(D)).$$

Now, since $\theta \in H^0(X, \text{ad}_\phi^n(E_G) \otimes K_X(D))$, we have the homomorphism

$$\Psi_{(E_G, \phi, \theta)} : T_{(E_G, \phi, \theta)} M_{FH}(G) = H^1(D_\bullet) \longrightarrow \mathbb{C}, \ w \mapsto \theta(\beta'_4(w)).$$

This pointwise construction of $\Psi_{(E_G, \phi, \theta)}$ produces a holomorphic 1-form on the moduli space $M_{FH}(G)$. This holomorphic 1-form on $M_{FH}(G)$ will be denoted by $\Psi$.

The holomorphic 2-form $d\Psi$ coincides with $\Phi$. Hence the form $\Phi$ is closed. $\square$
5.3. A Poisson map. Take any \((E_G, \theta) \in \mathcal{M}_H(G)\). From Corollary 5.1 and (4.12) we know that

\[
T_{(E_G, \theta)}\mathcal{M}_H(G) = \mathbb{H}^1(C'_\bullet) \quad \text{and} \quad T^*_{(E_G, \theta)}\mathcal{M}_H(G) = \mathbb{H}^1((C'_\bullet)^\vee).
\]

The pointwise construction of the homomorphism \(P_{(E_G, \theta)}\) in Proposition 4.4 produces a homomorphism

\[
P : T^*\mathcal{M}_H(G) \longrightarrow T\mathcal{M}_H(G).
\]

(5.4)

This \(P\) is a Poisson form on the moduli space \(\mathcal{M}_H(G)\) [Bot, p. 417, Theorem 4.6.3].

**Theorem 5.5.** The forgetful function \(\varphi\) in (5.1) is a Poisson map.

**Proof.** Take any \(z := (E_G, \phi, \theta) \in \mathcal{M}_{FH}(G)\). Let

\[
y := \varphi(z) = (E_G, \theta) \in \mathcal{M}_H(G)
\]

be its image under \(\varphi\). Consider the differential of the map \(\varphi\)

\[
d\varphi(z) : T_z\mathcal{M}_{FH}(G) \longrightarrow T_y\mathcal{M}_H(G)
\]

(5.5)

at the point \(z \in \mathcal{M}_{FH}(G)\). Let

\[
d\varphi(z)^* : T_y^*\mathcal{M}_H(G) \longrightarrow T_z^*\mathcal{M}_{FH}(G)
\]

(5.6)

be the dual homomorphism.

In view of Corollary 5.3, the isomorphism \((\Phi_{(E_G, \phi, \theta)})^{-1}\) in (4.7) is a homomorphism

\[
(\Phi_{(E_G, \phi, \theta)})^{-1} : T_y^*\mathcal{M}_{FH}(G) \sim T_z^*\mathcal{M}_{FH}(G).
\]

(5.7)

Note that the homomorphism \((\Phi_{(E_G, \phi, \theta)})^{-1}\) in (5.7) defines the Poisson structure on \(\mathcal{M}_{FH}(G)\) associated to the symplectic form \(\Phi\) (see Theorem 5.4).

To prove the theorem, we need to show the following: For every \(w \in T_y^*\mathcal{M}_H(G)\),

\[
d\varphi(z) \circ (\Phi_{(E_G, \phi, \theta)})^{-1} \circ d\varphi(z)^*(w) = P(w),
\]

(5.8)

where \(P\), \((\Phi_{(E_G, \phi, \theta)})^{-1}\), \(d\varphi(z)\) and \(d\varphi(z)^*\) are the homomorphisms constructed in (5.4), (5.7), (5.5) and (5.6) respectively, or in other words, the following diagram of homomorphisms is commutative

\[
\begin{array}{ccc}
T_y^*\mathcal{M}_H(G) & \xrightarrow{P} & T_y\mathcal{M}_H(G) \\
d\varphi(z)^* \downarrow & & \downarrow d\varphi(z) \\
T_z^*\mathcal{M}_{FH}(G) & \xrightarrow{(\Phi_{(E_G, \phi, \theta)})^{-1}} & T_z\mathcal{M}_{FH}(G)
\end{array}
\]

(see [BLP, Section 4]).

First consider the homomorphism \(d\varphi(z)\) in (5.5). Recall from Corollary 5.3 and Corollary 5.1 respectively that \(T_z\mathcal{M}_{FH}(G) = \mathbb{H}^1(D_\bullet)\) and \(T_y\mathcal{M}_H(G) = \mathbb{H}^1(C'_\bullet)\). Now from the definition of the forgetful map \(\varphi\) in (5.1) it follows immediately that
$d\varphi(z)$ coincides with the homomorphism of hypercohomologies $H^1(D_\bullet) \longrightarrow H^1(C'_\bullet)$ corresponding to the following homomorphism of complexes:

$$
\begin{array}{ccc}
D_\bullet : \text{ad}_{}(E_G) & \longrightarrow & \text{ad}_{}^n(E_G) \otimes K_X(D) \\
\downarrow & & \downarrow \\
C'_\bullet : \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G) \otimes K_X(D)
\end{array}
$$

where the homomorphisms

$$
\text{ad}_{}(E_G) \longrightarrow \text{ad}(E_G) \quad \text{and} \quad \text{ad}_{}^n(E_G) \otimes K_X(D) \longrightarrow \text{ad}(E_G) \otimes K_X(D)
$$

are the natural inclusions (see (3.2) and (3.12)).

Next consider the homomorphism $d\varphi(z)^\ast$ in (5.6). Using Corollary 5.3 and the isomorphism $\Phi_{(E_G,\phi,\theta)}$ in (4.7) it follows that $T^*_xM_{FH}(G) = H^1(D_\bullet)$. On the other hand, we have $T^*_xM_H(G) = H^1((C')_\bullet^\vee)$ (see Corollary 5.1 and (4.12)); also, the complex $(C')_\bullet^\vee$ is realized as the complex in (4.11). Using these, the homomorphism $d\varphi(z)^\ast$ coincides with the homomorphism of hypercohomologies $H^1((C')_\bullet^\vee) \longrightarrow H^1(D_\bullet)$ corresponding to the following homomorphism of complexes:

$$
\begin{array}{ccc}
(C')_\bullet^\vee : \text{ad}(E_G) \otimes O_X(-D) & \longrightarrow & \text{ad}(E_G) \otimes K_X \\
\downarrow & & \downarrow \\
D_\bullet : \text{ad}_{}(E_G) & \longrightarrow & \text{ad}_{}^n(E_G) \otimes K_X(D)
\end{array}
$$

where the homomorphisms

$$
\text{ad}(E_G) \otimes O_X(-D) \longrightarrow \text{ad}_{}(E_G) \quad \text{and} \quad \text{ad}(E_G) \otimes K_X \longrightarrow \text{ad}_{}^n(E_G) \otimes K_X(D)
$$

are the natural inclusions; see (3.4) and Remark 4.2.

Consequently, the homomorphism $d\varphi(z)^\ast \circ (\Phi_{(E_G,\phi,\theta)})^{-1} \circ d\varphi(z)^\ast$ in (5.8) coincides with the homomorphism of hypercohomologies

$$
\eta : H^1((C')_\bullet^\vee) \longrightarrow H^1(C'_\bullet) \quad (5.9)
$$

corresponding to the following homomorphism of complexes:

$$
\begin{array}{ccc}
(C')_\bullet^\vee : \text{ad}(E_G) \otimes O_X(-D) & \longrightarrow & \text{ad}(E_G) \otimes K_X \\
\downarrow & & \downarrow \\
C'_\bullet : \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G) \otimes K_X(D)
\end{array}
$$

where the homomorphisms

$$
\text{ad}(E_G) \otimes O_X(-D) \longrightarrow \text{ad}(E_G) \quad \text{and} \quad \text{ad}(E_G) \otimes K_X \longrightarrow \text{ad}(E_G) \otimes K_X(D)
$$

are the natural inclusions. But the homomorphism $\eta$ in (5.9) evidently coincides with the homomorphism $P_{(E_G,\theta)}$ constructed in Proposition 4.4. Hence (5.8) is proved. As noted before, this completes the proof of the theorem.  \(\Box\)
6. The Framework of Atiyah–Bott

In this section we sketch an alternative construction of the symplectic form $\Phi$ in Theorem 5.4 using the framework developed by Atiyah and Bott in [AtBo]. This framework was also employed by Hitchin in [Hi1].

Take any element $v \in \pi_1(G)$. Fix a $C^\infty$ principal $G$–bundle $E^0_G$ on $X$ of topological type $v$. Fix Zariski closed subgroups $H_x \subset G$ for all $x \in D$.

1. Fix a framing $\phi$ on $E^0_G$ of type $\{H_x\}_{x \in D}$, so $\phi_0(x)$ is an element of the quotient space $(E^0_G)_x / H_x$ for every $x \in D$.

2. The space of all holomorphic structures on the principal $G$–bundle $E^0_G$ is an affine space for the vector space $C^\infty(X, \text{ad}(E^0_G) \otimes \Omega^0_X)$. Fix a holomorphic structure on the $C^\infty$ principal $G$–bundle $E^0_G$; the resulting holomorphic principal $G$–bundle will be denoted by $E^0_G$.

3. Fix a Higgs field $\theta_0$ on the framed holomorphic principal $G$–bundle $(E^0_G, \phi_0)$.

As done in (2.10), let

$$\tilde{\sigma}_0 \in C^\infty(X, \text{Sym}^2(\text{ad}(E^0_G)))$$

be the fiberwise nondegenerate symmetric bilinear form defined by $\sigma$ in (2.9); the subscript “0” in “$\tilde{\sigma}_0$” is to emphasize the fact that this pairing is on a fixed vector bundle $\text{ad}(E^0_G)$.

Recall the constructions of $\text{ad}_\phi(E_G)$ and $\text{ad}_\phi^n(E_G)$, done in (3.2) and (3.12) respectively, for a framed principal $G$–bundle $(E_G, \phi)$. Substituting the above framed principal $G$–bundle $(E^0_G, \phi_0)$ in place of $(E_G, \phi)$ in the constructions done in (3.2) and (3.12), we get holomorphic vector bundles $\text{ad}_{\phi_0}(E^0_G)$ and $\text{ad}_{\phi_0}^n(E^0_G)$ respectively.

Let $\mathcal{V}^{0.1}$ denote the space of all $C^\infty$ sections of the vector bundle

$$\text{ad}_{\phi_0}(E^0_G) \otimes \overline{K_X} = \text{ad}_{\phi_0}(E^0_G) \otimes \Omega^0_X.$$ 

The space of all $C^\infty$ sections of the vector bundle $\text{ad}_{\phi_0}^n(E^0_G) \otimes K_X(D)$ will be denoted by $\mathcal{V}^{1.0}$. Now construct the direct sum of vector spaces

$$\mathcal{W} := \mathcal{V}^{0.1} \oplus \mathcal{V}^{1.0}.$$ (6.2)

Given any $v \in \mathcal{V}^{0.1}$, we get a framed holomorphic principal $G$–bundle $(E^0_G, \phi_v)$ on $X$. To clarify, the underlying $C^\infty$ framed principal $G$–bundle for $(E^0_G, \phi_v)$ is $(E^0_G, \phi_0)$, and the almost complex structures of $E^0_G$ and $E^0_G$ differ by $v$; as mentioned before, the space of all holomorphic structures on $E^0_G$ is an affine space for $C^\infty(X, \text{ad}(E^0_G) \otimes \Omega^0_X)$. It may be mentioned that these conditions uniquely determine $E^u_G$. Also, note that the framing $\phi_v$ coincides with $\phi_0$ using the $C^\infty$ identification between $E^0_G$ and $E^u_G$. Now consider the Dolbeault operator for the holomorphic vector bundle $\text{ad}(E^u_G)$; we shall denote it by $\overline{\partial}^u$. This Dolbeault operator $\overline{\partial}^u$ and the Dolbeault operator for the holomorphic line bundle $K_X(D)$ together define the Dolbeault operator for the holomorphic vector bundle $\text{ad}(E^u_G) \otimes K_X(D)$. This Dolbeault operator for $\text{ad}(E^u_G) \otimes K_X(D)$ will be denoted by $\overline{\partial}^u$.

Let

$$\mathcal{W}^0 \subset \mathcal{W}$$ (6.3)
be the subset of the direct sum in (6.2) consisting of all \((v, w) \in \mathcal{Y}^{0,1} \oplus \mathcal{Y}^{1,0}\) such that
\[
\overline{\partial}^v (w) = 0.
\] (6.4)

Therefore, for any \((v, w) \in \mathcal{W}^0\), the section \(w\) is a (holomorphic) Higgs field on the framed holomorphic principal \(G\)–bundle \((E_0^G, \phi_0)\).

We shall now construct a complex 1-form on \(\mathcal{W}\). For any \((v, w) \in \mathcal{W}^0\), we have \(\hat{\sigma}_0(v, w) \in C^\infty(X, \Omega^{0,1}_X)\), where \(\hat{\sigma}_0\) is constructed in (6.1). Note that while \(w\) may have a pole over \(D\) as a section of \(\text{ad}(E_0^G)\), the pairing \(\hat{\sigma}_0(v, w)\) does not have a pole as a section of \(\Omega^{1,1}_X\), because the image of \(\text{ad}_{\phi_0}(E_0^G)_x\) in \(\text{ad}(E_0^G)_x\) annihilates the image of \(\text{ad}_{\phi_0}(E_0^G)_x\) for the nondegenerate bilinear form \(\hat{\sigma}_0(x)\) on \(\text{ad}(E_0^G)_x\) for all \(x \in D\). (To see this, recall from (3.2) that the image of \(\text{ad}_{\phi}(E_0 G)_x\) in \(\text{ad}(E_0 G)_x\) is \(\mathcal{H}_x\), while from (3.12) we know that the image of \(\text{ad}_{\phi}(E_0 G)_x\) in \(\text{ad}(E_0 G)_x\) is \(\mathcal{H}_x^\perp\).)

Let \(\Psi'_0 \in H^0(\mathcal{W}, \Omega^1_{\mathcal{W}})\) be the holomorphic 1-form on \(\mathcal{W}\) defined by
\[
\Psi'_0(v, w)(v_1, w_1) \mapsto \int_X \hat{\sigma}_0(v_1, w) \in \mathbb{C},
\]
for all \((v, w) \in \mathcal{W}\) and \((v_1, w_1) \in T_{(v, w)} \mathcal{W} = \mathcal{W}\); here we are using the fact that the tangent space \(T_{(v, w)} \mathcal{W}\) is canonically identified with \(\mathcal{W}\) itself as \(\mathcal{W}\) is a complex vector space. Note that using the element of \(\mathcal{W}^*\) defined by
\[
(v, w) \mapsto \int_X \hat{\sigma}_0(v, w) \in \mathbb{C},
\]
the vector space \(\mathcal{Y}^{1,0}\) is embedded into the dual vector space \((\mathcal{Y}^{0,1})^*\). This embedding produces a holomorphic embedding of \(\mathcal{W}\) inside the holomorphic cotangent bundle \((T \mathcal{W})^*\). Using this embedding, the form \(\Psi'_0\) in (6.5) is the restriction of the Liouville 1-form on the holomorphic cotangent bundle \((T \mathcal{Y}^{0,1})^*\).

The de Rham differential
\[
d\Psi'_0 =: \Phi'_0
\] (6.6)
has the following expression: For any
\[
(v, w) \in \mathcal{W}
\]
and any two tangent vectors \((v_1, w_1), (v_2, w_2) \in T_{(v, w)} \mathcal{W}\),
\[
\Phi'_0(v, w)((v_1, w_1), (v_2, w_2)) = \int_X (\hat{\sigma}_0(v_2, w_1) - \hat{\sigma}_0(v_1, w_2)).
\]
Let \(\Psi_0\) and \(\Phi_0\) be the restrictions to \(\mathcal{W}^0\) (see (6.3)) of the above defined differential forms \(\Psi'_0\) and \(\Phi'_0\) respectively.

Let \(\mathcal{G}\) denote the group of all \(C^\infty\) automorphisms of the principal \(G\)–bundle \(E_0^G\) preserving the framing \(\phi_0\). It is straight-forward to check that the Lie algebra of \(\mathcal{G}\) is
$C^\infty(X, \text{ad}_{\phi_0}(E_G^0))$. This group $G$ has a natural action on $\mathcal{W}$; this action of $G$ on $\mathcal{W}$ evidently preserves the subset $\mathcal{W}^0$ defined in (6.3). The 1-form $\Psi_0$ on $\mathcal{W}$ is evidently preserved by the action of $G$ on $\mathcal{W}$, because $\hat{\sigma}_0$ is preserved under the action of $G$ on $\text{ad}(E_G^0)$ induced by the action of $G$ on the principal $G$-bundle $E_G^0$. Consequently, the action of the group $G$ on $\mathcal{W}^0$ preserves the form $\Psi_0$. The de Rham differential $d\Psi_0'$ is preserved by the action of $G$ on $\mathcal{W}$, because $\Psi_0'$ is preserved by the action of $G$ on $\mathcal{W}$. Therefore, the 2-form $\Phi_0 = (d\Psi_0')|_{\mathcal{W}^0}$ is also preserved by the action of $G$ on $\mathcal{W}^0$.

Take any element $(v, w) \in \mathcal{W}^0$. As before, $\hat{\sigma}_1^v$ and $\hat{\sigma}_v^w$ denote the Dolbeault operators for $\text{ad}(E_G^v)$ and $\text{ad}(E_G^v) \otimes K_X(D)$ respectively. Take any section $\beta \in C^\infty(X, \text{ad}_{\phi_0}(E_G^0))$. Now we have

$$\int_X \hat{\sigma}_0(\hat{\sigma}_1^v(\beta), w) = -\int_X \hat{\sigma}_0(\beta, \hat{\sigma}_v^w(w)) = 0,$$

because $\hat{\sigma}_v^w(w) = 0$ (see (6.4)). As a consequence of it, the 1-form $\Psi_0$ on $\mathcal{W}^0$ descends under the action of $G$ on $\mathcal{W}^0$. Hence $\Phi_0 = d\Psi_0$ also descends under the action of $G$ on $\mathcal{W}^0$. The descent of $\Psi_0$ corresponds to the form $\Psi$ in the proof of Theorem 5.4, while the descent of $\Phi_0$ corresponds to the form $\Phi$ in Theorem 5.4. From (6.6) it follows that $\Phi = d\Psi$.

7. The Hitchin System: Cameral Data for Framed $G$–Higgs Bundles

In this section we shall describe the Hitchin integrable system for framed $G$–Higgs bundles. We will assume that $H_x = e$ for all $x \in D$ as it is quite similar to the general case while being simpler to present; some remarks on the general case are included for the sake of completeness.

For any holomorphic Poisson manifold $(M, \mathfrak{P})$, we denote by $\{\cdot, \cdot\}_z$ the associated Poisson bracket on $\mathcal{O}_M$, i.e., $\{f, g\}_z = \pi^z(df, dg)$ where $\pi^z \in \Gamma(\wedge^2(TM))$ is the Poisson bi-vector.

A symplectic structure $\omega$ on $M$ also defines a Poisson bracket on $\mathcal{O}_M$ by assigning to $(f, g) \in \mathcal{O}_M \times \mathcal{O}_M$ the function $\{f, g\}_\omega = \omega(X_g, X_f)$, where $X_f$ and $X_g$ are the Hamiltonian vector fields defined by $f$ and $g$ with respect to $\omega$.

Two functions $f, g \in \mathcal{O}_M$ are said to Poisson commute if

$$\{f, g\} = 0.$$

An algebraically completely integrable system on $M$ consists of functions $f_1, \ldots, f_d \in \mathcal{O}_M$ with $d = \frac{1}{2} \dim M$, such that

- $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq d$,
- the corresponding Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_d}$ are linearly independent at the general point, and
- the generic fiber of the map $(f_1, \ldots, f_d) : M \rightarrow \mathbb{C}^d$ is a open set in an abelian variety such that the vector fields $X_{f_1}, \ldots, X_{f_d}$ are linear on it.

7.1. Recollection: the Hitchin system for Higgs bundles. Fix a Borel subgroup $B \subset G$ and a Cartan subgroup $T \subset B$. Let $t \subset b$ be the Lie algebras of $T$ and $B$. The Weyl group $N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$, will be denoted by $W$. 

Consider the Chevalley morphism
\[ \chi : g \longrightarrow t/\mathbb{C}^x \] (7.1)
constructed using the isomorphism \( t/\mathbb{C}^x \cong \mathfrak{g}/G := \text{Spec}(\mathbb{C}[\mathfrak{g}]^G) \). Since \( \mathbb{C}[\mathfrak{g}]^G \) is generated by homogeneous polynomials of degrees \( d_1, \ldots, d_r \), where \( r = \text{rank}(G) \), it admits a graded \( \mathbb{C}^x \)-action. The induced action on \( \mathfrak{g}/G \) turns \( \chi \) into a \( \mathbb{C}^x \)-equivariant morphism. This, using the \( G \)-invariance property of the morphism (7.1), induces a map:
\[ h : \mathcal{M}_H(G) \longrightarrow B := H^0(X, t \otimes K_X(D)/W) \] (7.2)
given by
\[ h(E, \theta)(x) = \chi(\theta(x)) . \]
Alternatively, the choice of \( r \) generators \( p_1, \ldots, p_r \) of \( \mathbb{C}[\mathfrak{g}]^G \) of degrees \( \deg p_i = d_i \), \( i = 1, \ldots, r \), induces an isomorphism
\[ H^0(X, t \otimes K_X(D)/W) \cong \bigoplus_{i=1}^r H^0(X, K_X^{d_i}(d_i D)) \]
under which \( h \) can be described as
\[ h(E, \theta)(x) = (p_1(\theta(x)), \ldots, p_r(\theta(x))) , \quad p_i(\theta) \in H^0(X, K_X^{d_i}(d_i D)) . \] (7.3)
The dimension of the vector space \( B \) thus is
\[ N := \dim B = \sum_{i=1}^r (d_i(2g(X) - 2 + n) - g(X) + 1) \]
\[ = (g(X) - 1) \sum_{i=1}^r (2d_i - 1) + n \sum_{i=1}^r d_i \]
\[ = (g(X) - 1) \dim G + n \cdot \dim B , \] (7.4)
where \( g(X) \) is the genus of \( X \).

Given any \( b \in B \), we define the corresponding \textit{cameral cover} as the curve \( X_b \) given by the commutative diagram:
\[
\begin{array}{ccc}
X_b & \longrightarrow & t \otimes K_X(D) \\
\downarrow{\pi_b} & & \downarrow \\
X & \longrightarrow & t \otimes K_X(D)/W
\end{array}
\] (7.5)
Consider the generic locus \( B^\text{sm} \) corresponding to sections whose associated cameral cover in (7.5) is smooth. Then, by [Ngo, Proposition 4.7.7], the inverse image \( h^{-1}(b) \) is contained in the locus of \( \mathcal{M}_H(G) \) consisting of Higgs bundles for which the Higgs field \( \theta \) is regular at every point, meaning that the orbit of \( \theta(x) \) is maximal dimensional. Moreover, by [DG, Corollary 17.8], the choice of a point in the fiber induces an isomorphism
\[ h^{-1}(b) \cong H^1(X_b, T)^W , \] (7.6)
where the action of $W$ on a principal $T$-bundle $P 	o X_b$ is given by
\[ w \cdot P = (w^* P \times_w T) \otimes \mathcal{R}_w . \]

In the above, $\mathcal{R}_w$ is a principal $T$-bundle naturally associated to the ramification divisor of $w$ (cf. [DG, § 5]). Moreover, there exists a group scheme $J \to X \times B$ such that $h^{-1}(b) \cong H^1(X, J_b)$, where $J_b = (J \mid_{X \times \{b\}}$). In other words, the automorphism group of elements of the Hitchin fiber $h^{-1}(b)$ (seen as torsors over $X_b$) descends to $J_b \to X$.

In the language of stacks, let $\mathcal{M}_H(G)$ be the stack of $G$–Higgs bundles. In a similar way as done in (7.2) we may define a stacky Hitchin map by:
\[ h : \mathcal{M}_H(G) \to B \]
\[ (E, \phi) \mapsto \chi(\phi), \]

where $\chi$ is the Chevalley morphism in (7.1).

Consider the Picard stack $\mathcal{P} \to B$ of principal $J$–bundles. Then, $\mathcal{M}_H(G) \mid_{B^{sm}}$ is a torsor over $\mathcal{P} \mid_{B^{sm}}$ relative to $B^{sm}$. In particular, if $b \in B^{sm}$, we have an isomorphism
\[ h^{-1}(b) \cong \mathcal{P}_b, \]
determined by a choice of an element of the fiber.

Lemma 7.1.
\[ \dim h^{-1}(b) = (g(X) - 1) \dim G + n(\dim B - \dim T) + \dim Z(G). \]

Proof. By Lemma 3.3 we have that $\mathcal{M}_H(G) = \mathcal{M}_H(G) \parallel Z(G)$ (where the symbol $\parallel$ denotes rigidification [AOV, Appendix A]). So it follows that
\[ \dim h^{-1}(b) = \dim \mathcal{P}_b + \dim Z(G) \]
\[ = (g(X) - 1) \dim G + n(\dim B - \dim T) + \dim Z(G), \]

where the second equality is [Ngo, Corollary 4.13.3]. □

The above facts about abelianization of generic fibers (7.6) and (7.8), the dimensions in Lemma 7.1 and Corollary 5.1, together with the following proposition prove that the Hitchin map is an algebraically completely integrable system on the Poisson variety $\mathcal{M}_H(G)$.

Proposition 7.2. Let $P$ be the Poisson structure on $\mathcal{M}_H(G)$ described in (5.4) and $\{\cdot, \cdot\}_P$ its associated Poisson bracket. The $N$ functions on $\mathcal{M}_H(G)$ provided by the Hitchin system $h$ in (7.2) Poisson-commute with respect to $\{\cdot, \cdot\}_P$.

Proof. This follows from the results in [Ma, Theorem 8.5, Remark 8.6] and [DM, Section 5]. □
7.2. The Hitchin morphism for framed $G$–Higgs bundles. Now consider the morphism

$$h_{FH} : \mathcal{M}_{FH}(G) \longrightarrow B$$

(7.10)

defined by the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_{FH}(G) & \xrightarrow{\varphi} & \mathcal{M}_H(G) \\
\downarrow & & \downarrow h \\
& & B
\end{array}$$

(7.11)

where $\varphi$ is defined in (5.1) and $h$ is in (7.2).

Remark 7.3. By commutativity of (7.11), it turns out that $h_{FM}$ can also be expressed in terms of invariant polynomials as in (7.3).

Note that Proposition 7.2 and Theorem 5.5 together give the following.

Corollary 7.4. Let $\Phi$ be the holomorphic symplectic form on $\mathcal{M}_{FH}(G)$ and $\{\cdot, \cdot\}_\Phi$ the associated Poisson bracket. Then the $N$ functions in $h_{FH}$ Poisson commute with respect to $\{\cdot, \cdot\}_\Phi$.

Let $Z(G)$ denote the center of $G$.

Proposition 7.5. The forgetful map $\varphi$ in (5.1) makes $\mathcal{M}_{FH}(G)$ a torsor over the orbifold $\mathcal{M}_H(G)$ for the group $(\prod_{x \in D} G)/Z(G) = G^n/Z(G)$, where $n = \#D$ and $Z(G)$ is embedded diagonally in the Cartesian product $G^n$.

Proof: take any $(E_G, \theta) \in \mathcal{M}_H(G)$. The group $Z(G)$ is a subgroup of the group $\text{Aut}(E_G, \theta)$ parametrizing all holomorphic automorphisms of the $D$-twisted $G$–Higgs bundle $(E_G, \theta)$. In fact $Z(G)$ is a normal subgroup of $\text{Aut}(E_G, \theta)$ such that quotient $\text{Aut}(E_G, \theta)/Z(G)$ coincides with the inertia group of the orbifold point $(E_G, \theta) \in \mathcal{M}_H(G)$.

Now consider

$$\mathcal{F}(E_G) = \prod_{x \in D} (E_G)_x/H_x = \prod_{x \in D} (E_G)_x$$

constructed in (2.6). From the action of $G$ on $(E_G)_x, x \in D$, we get an action of $G^n = \prod_{x \in D} G$ on $\prod_{x \in D} (E_G)_x$. Consider $Z(G)$ embedded diagonally in $\prod_{x \in D} G$. The action of this subgroup $Z(G) \subset \prod_{x \in D} G$ on $\prod_{x \in D} (E_G)_x$ factors through the tautological action of $\text{Aut}(E_G, \theta)$ on $\prod_{x \in D} (E_G)_x$.

On the other hand, the inverse image $\varphi^{-1}(E_G, \theta) \subset \mathcal{M}_{FH}(G)$ is evidently identified with $\mathcal{F}(E_G)/\text{Aut}(E_G, \theta)$. This proves that the orbifold $\mathcal{M}_{FH}(G)$ is a torsor over $\mathcal{M}_H(G)$ for the group $(\prod_{x \in D} G)/Z(G) = G^n/Z(G)$. $\square$

From Proposition 7.5 a description of the Hitchin fibers is obtained.

Corollary 7.6. The forgetful morphism that forgets the framing induces a $G^n/Z(G)$-torsor structure

$$h_{FH}^{-1}(b) \longrightarrow h^{-1}(b).$$
In particular, the Hitchin system is not abelianizable, thus neither is it algebraically completely integrable. Note also that the number of Poisson commuting functions provided by $h_{FH}$ is less than half of the dimension of $\mathcal{M}_{FH}(G)$. We next define a maximally abelianizable subsystem such that its dimension doubles the number of Poisson commuting functions. In order to do that, we need to introduce some more notation.

Consider the stack of stable framed Higgs bundles $\mathcal{M}_{FH}(G)$. Forgetting the frame induces a $G^n$-torsor $\Phi : \mathcal{M}_{FH}(G) \to \mathcal{M}_H(G)$ by Proposition 7.5. Now, the Hitchin map in (7.10) also admits a stacky version $h_{FH}$ defined by the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}_{FH}(G) & \to & \mathcal{M}_H(G) \\
\downarrow & & \downarrow \\
\mathcal{M}_{FH}(G)/\Phi & \to & \mathcal{M}_H(G),
\end{array}
$$

where $\Phi$ is the forgetful morphism and $h$ is defined in (7.7). Note that by Proposition 5.2 we have $\mathcal{M}_{FH}(G) = \mathcal{M}_{FH}(G)/Z(G)$, (see the first sentence in the proof of Lemma 7.1 for notation), so the following commutative diagram is obtained

$$
\begin{array}{ccc}
\mathcal{M}_{FH}(G) & \xrightarrow{h_{FH}} & \mathcal{M}_{FH}(G) \\
\downarrow & & \downarrow \\
\mathcal{M}_H(G) & \xrightarrow{h} & \mathcal{M}_H(G),
\end{array}
$$

where the horizontal arrows are $Z(G)$–torsors defined via rigidification.

**Lemma 7.7.** The forgetful morphism $h_{FH}^{-1}(b) \to h^{-1}(b)$ that forgets the framing induces a $G^n$–torsor structure.

**Proof.** Commutativity of (7.12) implies that $\Phi$ takes fibers of $h_{FH}$ to fibers of $h$. The rest follows as in the proof of Proposition 7.5, after incorporating the observation that quotienting by automorphisms of the base is not necessary when working with stacks.

\[\square\]

### 7.3. Relatively framed Higgs bundles

In this section we produce a subsystem of the Hitchin system (7.10) which is an algebraically completely integrable system.

Consider $\mathcal{B}_{sm}^{nr} \subset \mathcal{B}_{sm}$, the subset of smooth cameral covers unramified over $D$. Over this we consider the stack $\mathcal{P}_{FH}$ of $J$ principal bundles with a $W$ and $T$–equivariant framing over $D_B = (D \times B) \times T / \otimes K_X(D)/ \otimes K_D$. If $D_b = D_B|_{D_b}$, then equivariance of $\delta : P|_{D_b} \cong D_b \times T$ is given by

$$
\delta_{w^{-1}x} = w^{-1} \circ \delta_x
$$

where

$$
\delta_x : P_x \xrightarrow{\sim} T
$$

is the frame at a point $x \in D_b$ and $w^{-1} : T \to T$ is the usual action.

By the following proposition, $\mathcal{P}_{FH}$ is an abelian group stack relative to $\mathcal{B}_{sm}^{nr}$.
Proposition 7.8. The forgetful morphism
\[ \mathcal{P}_{FH} \longrightarrow \mathcal{P} \] (7.15)
induces a $T^n$ torsor structure.

Proof. Let $(E, \theta, \delta_i) \in \mathcal{P}_{F,b}(X), i = 1, 2.$ Then, the equivariance condition (7.14) implies that $\delta_i$ commutes with all the automorphisms of $(E, \theta)$ inside $h^{-1}(b).$ Hence one obtains a $J|_D$ torsor. But since by assumption $D_b \longrightarrow D$ is unramified, this is a $T^n$–torsor. See [Ngo, § 2.5]. \hfill \Box

Theorem 7.9. The equivalence $h^{-1}(b) \cong \mathcal{P}_b$ induces a faithful morphism
\[ \mathcal{P}_{F,b} \hookrightarrow h^{-1}_{FH}(b). \]

Proof. Let $(E_G, \theta, \delta) \in h^{-1}_{FH}(b),$ and let $P \in \mathcal{P}_b(X)$ be the object corresponding to $(E_G, \theta)$ via the equivalence $h^{-1}_{FH}(b) \cong \mathcal{P}_b(X).$ Since $X_b$ is not ramified over $D,$ the equivariance conditions on $P$ and $\delta,$ together with [LP, Proposition 7.5] imply that $P|_{D_b}$ and $\delta$ descend to $E|_D$ and a trivialization $E|_D \cong D \times N,$ where $N$ is the normalizer of $T$ in $G.$

Since all the steps are functorial, this defines a morphism of stacks. Faithfulness follows from the fact that these are categories fibered in groupoids and that the action of $\mathcal{P}_{FH}$ on $h^{-1}_{FH}(b)$ is compatible with the torsor structures over $\mathcal{P}$ and $h^{-1}(b)$ respectively. \hfill \Box

We define the sub-stack of relatively framed Higgs bundles as
\[ M^{\Delta}_{FH}(G) := \text{Im} \left( \mathcal{P}_{F,b} \hookrightarrow h^{-1}_{FH}(b) \right). \] (7.16)

Let $M^{\Delta}_{FH}(G) := M^{\Delta}_{FH}(G) \big/ Z(G),$ (see the first sentence in the proof of Lemma 7.1 for notation). Consider the restriction of the Hitchin map
\[ h^{\Delta}_{FH} : M^{\Delta}_{FH}(G) \longrightarrow B_{nr}^\text{sm}. \] (7.17)

Corollary 7.10. The fibers of $h^{\Delta}_{FH}$ are $N$-dimensional semiabelian varieties. Therefore the moduli space $M^{\Delta}_{FH}(G)$ is maximally abelianizable. Moreover, the $N$-functions $(h_1, \ldots, h_N)$ obtained by identifying $B \cong \mathbb{C}^N$ and $h_{FH} = (h_1, \ldots, h_N)$ are in involution.

Proof. We have a commutative diagram
\[ \begin{array}{ccc} h^{-1}_{FH}(b) & \longrightarrow & h^{-1}_{FH}(b) \\ \downarrow & & \downarrow \\ h^{-1}(b) & \longrightarrow & h^{-1}_H(b), \end{array} \]

which by Theorem 7.9 implies that there is a short exact sequence
\[ 0 \longrightarrow T^n / Z(G) \longrightarrow (h^{\Delta})^{-1}_{FH}(b) \longrightarrow h_H(b) \longrightarrow 0. \]

By [BSU, Proposition 7.2.1] these are semiabelian varieties. The dimensional count follows from Lemma 7.1 and the above exact sequence.
Poisson commutativity and linearity of the vectors \( X_{hi}, i = 1, \ldots, N \) follows as in [BLP, Proposition 5.12].

The Hitchin system (7.17) is a maximally abelianizable subsystem as the dimension of the fibers justifies. \( \square \)

**Remark 7.11.** Given a framed camera datum, the corresponding Higgs bundle is naturally endowed with a framing of the principal bundle and of the Higgs field.

**Remark 7.12.** For general groups \( H_x \) one may produce the following maximally abelianizable subsystem. Given \( x \in D \), let \( T \subset G \) be a maximal torus, and let \( T_x := T \cap H_x \). Then, one may consider the stack of camera data together with a framing, that is, a \( T \)-equivariant morphism \( P|_{D_x} \rightarrow \prod_{x \in D_x} T/T_x \) which is \( T \)-equivariant and \( W \)-equivariant, in the same sense as (7.14). The same reasoning as done for \( H_x = e \) produces a \( \prod_{x \in D} T/T_x \)-torsor \( (\mathfrak{h}^\Delta_{FH})^{-1}(b) \subset \mathfrak{h}^{-1}(b) \), that we call the stack of framed camera data (over \( X_b \)). On the level of the moduli space, one obtains a torsor for the group

\[
\left( \prod_{x \in D} T/T_x \right) / (Z(G)/Z_{H_x}(G))
\]

which is maximal (of dimension \( N - \sum_{x \in D} \dim T_x + \dim Z_{H_x}(G) \)). The fibers are thus semiabelian varieties of the same dimension as \( B \) if and only if \( \dim T_x = \dim Z_{H_x}(G) \).

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**References**

[AOV] Abramovich, D., Olsson, M., Vistoli, A.: Tame stacks in positive characteristic. Annales de l’Institut Fourier 58, 1057–1091 (2008)

[AnBi] Anchouche, B., Biswas, I.: Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. Am. J. Math. 123, 207–228 (2001)

[AtBo] Atiyah, M.F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond. Ser. A 308, 523–615 (1983)

[BLS] Beauville, A., Laszlo, Y., Sorger, C.: The Picard group of the moduli of G-bundles on a curve. Compos. Math. 112, 183–216 (1998)

[BG] Biswas, I., Gómez, T.L.: Connections and Higgs fields on a principal bundle. Ann. Glob. Anal. Geom. 33, 19–46 (2008)

[BR] Biswas, I., Ramanan, S.: An infinitesimal study of the moduli of Hitchin pairs. J. Lond. Math. Soc. 49, 219–231 (1994)

[Bi] Biswas, I.: A remark on a deformation theory of Green and Lazarsfeld. J. Reine Angew. Math. 449, 103–124 (1994)

[BLP] Biswas, I., Logares, M., Peón-Nieto, A.: Symplectic geometry of a moduli space of framed Higgs bundles. Int. Math. Res. Not. (2019). https://doi.org/10.1093/imrn/rnz016, arXiv:1805.07265

[Bor] Borel, A.: Linear Algebraic Groups, Second Edition, Graduate Texts in Mathematics, 126. Springer, New York (1991)

[Bot] Bottacin, F.: Symplectic geometry on moduli spaces of stable pairs. Ann. Sci. École Norm. Sup. 28, 391–433 (1995)

[BSU] Brion, M., Samuel, P., Uma, V.: Lectures on the structure of algebraic groups and geometric applications. CMI Lecture Series in Mathematics, vol. 1. Hindustan Book Agency, Chennai (2013)
[Do] Donin, I.F.: Construction of a versal family of deformations for holomorphic bundles over a compact complex space. Math. USSR Sb. 23, 405–416 (1974)

[DG] Donagi, R.Y., Gaitsgory, D.: The gerbe of Higgs bundles. Transform. Groups 7, 109–153 (2001)

[DM] Donagi, R.Y., Markman, E.: Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, Integrable systems and quantum groups (Montecatini Terme, 1993), 1–119, Lecture Notes in Mathematics, 1620, Fond. CIME/CIME Found. Subser., Springer, Berlin (1996)

[Hi1] Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. 55, 59–126 (1987)

[Hi2] Hitchin, N.J.: Stable bundles and integrable systems. Duke Math. J. 54, 91–114 (1987)

[Hi3] Hitchin, N.J.: Lie groups and Teichmüller space. Topology 31, 449–473 (1992)

[Hum] Humphreys, J.E.: Linear Algebraic Groups. Graduate Texts in Mathematics, vol. 21. Springer, New York (1975)

[Huy] Huybrechts, D.: Fourier-Mukai Transforms in Algebraic Geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford (2006)

[LP] Lange, H., Pauly, C.: Polarizations of Prym varieties for Weyl groups via abelianization. J. Eur. Math. Soc. 11, 315–349 (2009)

[LM] Logares, M., Martens, J.: Moduli of parabolic Higgs bundles and Atiyah algebroids. J. Reine Angew. Math. 649, 89–116 (2010)

[Ma] Markman, E.: Spectral curves and integrable systems. Compos. Math. 93, 255–290 (1994)

[Ngo] Ngô, B.C.: Le lemme fondamental pour les algèbres de Lie. Publ. Math. Inst. Hautes Études Sci. 111, 1–169 (2010)

[Ni] Nitsure, N.: Moduli space of semistable pairs on a curve. Proc. Lond. Math. Soc. 62, 275–300 (1991)

[Ra1] Ramanathan, A.: Stable principal bundles on a compact Riemann surface. Math. Ann. 213, 129–152 (1975)

[Ra2] Ramanathan, A.: Moduli of principal bundles over algebraic curves. Proc. Indian Acad. Sci. Math. Sci. 106, 301–328 (1996)

[RS] Ramanathan, A.: Subramanian: Einstein-Hermitian connections on principal bundles and stability. J. Reine Angew. Math. 390, 21–31 (1988)

[Se] Seshadri, C.S.: Fibrés vectoriels sur les courbes algébriques, Notes written by J.-M. Drézet, Astérisque, vol. 96. Société Mathématique de France, Paris (1982)

[Si1] Simpson, C.T.: Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Am. Math. Soc. 1, 867–918 (1988)

[Si2] Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety I. Inst. Hautes Études Sci. Publ. Math. 79, 47–129 (1994)

[Si3] Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety II. Inst. Hautes Études Sci. Publ. Math. 80, 5–79 (1995)

[St] Steenrod, N.: The Topology of Fibre Bundles. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton (1951)