Poset structures on \((m + 2)\)-angulations and polynomial bases of the quotient by \(G^m\)-quasisymmetric functions

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Abstract

For integers \(m, n \geq 1\), we describe a bijection sending dissections of the \((mn + 2)\)-regular polygon into \((m + 2)\)-sided polygons to a new basis of the quotient of the polynomial algebra in \(mn\) variables by an ideal generated by some kind of higher quasi-symmetric functions. We show that divisibility of the basis elements corresponds to a new partial order on dissections, which is studied in some detail.

1 Introduction

Let \(m \geq 1\) be an integer. For every integer \(n \geq 1\), we define a simple poset structure \(P_{m,n}\) on the set of \((m + 2)\)-angulations of a \((mn + 2)\)-gon. This generalizes the construction by Pallo on triangulations [Pal03], which is closely related to the Tamari lattice.

Quasisymmetric functions, a generalisation of symmetric functions, were introduced in [Ges84] and are now classical in algebraic combinatorics. Some higher analogues were introduced in [Poi98] and further studied in [BH08]. We recall some results about quotients of polynomial rings by higher quasi-symmetric functions, first obtained for quasisymmetric functions in [AB03, ABB04] and extended to the higher case in [Ava07].

We then show that the poset \(P_{m,n}\) is isomorphic to the divisibility poset of a new particular basis of the quotient of the polynomial ring in \(m\) sets of \(n\) variables by the ideal generated by \(G^m\)-quasisymmetric functions without constant term. Our description of a new basis builds upon the basis indexed by \(m\)-Dyck paths that was introduced for general \(m\) in [Ava07].

For \(m = 1\), the posets \(P_{1,n}\) were introduced by Pallo in [Pal03] and further studied in [CSS12, CSS14], and the basis indexed by triangulations was defined in [Cha05], but the connection between them is new.

The last two sections of the article are devoted to some enumerative results on the posets (enumeration of intervals, rank generating function) and to a recursive description of the intervals as distributive lattices of orders ideals of forests.
2 A poset structure on $M$-angulations

Let $m \geq 1$ be an integer. For the sake of readability, we will use the expression $M$-angulation instead of $(m + 2)$-angulation. An $M$-angulation of a regular convex polygon is a set of diagonals that cut the polygon into $(m + 2)$-sided polygonal regions.

We consider the set $Q_{m,n}$ of $M$-angulations of a $(mn+2)$-gon. Every element of $Q_{m,n}$ contains $n$ regions, separated by $n - 1$ diagonal edges. The cardinality of $Q_{m,n}$ is given by $\frac{1}{mn+1} \binom{(m+1)n}{n}$, the number of $(m+1)$-ary planar rooted trees with $n$ inner vertices (often called a Fuss-Catalan number). A simple bijection between these two classes of objects is given by planar duality. Some elements of $Q_{2,7}$ are shown in Figures 1 and 2.

![Figure 1: A quadrangulation of the 16-gon](image)

To define a poset structure on $Q_{m,n}$, we fix a particular element $Q_0$, which is a fan (every diagonal edge involves a fixed vertex, denoted 0 and called the apex, see Figure 2).

![Figure 2: The fan $Q_0$ for $m = 2$ and $n = 7$](image)

We consider the following order relation. An element $Q$ of $Q_{m,n}$ is covered in $P_{m,n}$ by the $M$-angulations obtained by flipping one of its diagonal edges included in $Q_0$. Here flipping means removing this diagonal edge and replacing it by another diagonal edge cutting again the $(2m + 2)$-gon created by the removal into two $(m + 2)$-gons. Note that any diagonal edge may be flipped in
exactly $m$ different ways. As an example the poset $\mathcal{P}_{2,3}$ is shown on Figure 3.

![Diagram of the poset $\mathcal{P}_{2,3}$](image)

This poset is clearly graded, the rank of an element $Q$ being $n - 1$ minus the number of diagonal edges shared by $Q_0$ and $Q$. This is also the number of diagonal edges in $Q$ that are not in $Q_0$.

It follows from the description of the coverings that an element of rank $r$ is covered by exactly $m(n - 1 - r)$ elements. This implies the formula $m^{n-1}(n-1)!$ for the number of maximal chains in $\mathcal{P}_{m,n}$.

This poset has $Q_0$ as unique minimum. The fact that $Q_0$ is smaller than all $M$-angulations follows from the next lemma by induction on the number of diagonals that are both in $Q$ and $Q_0$. This lemma implies that unless $Q$ is $Q_0$, there is a cover relation $Q' \triangleleft Q$ where $Q'$ has one more diagonal in common with $Q_0$.

**Lemma 1.** Let $Q$ be distinct from $Q_0$. Then there is always at least one diagonal $d_0$ in $Q_0$ that cuts exactly one diagonal of $Q$.

**Proof.** One can assume without restriction that $Q$ and $Q_0$ have no common diagonal, otherwise one can find $d_0$ (by induction) inside one of the parts cut by the common diagonals. In at least one of these parts the restriction of $Q$ must differ from the restriction of $Q_0$, because $Q$ is not $Q_0$.

Let us label all the vertices of the regular polygon counter-clockwise by integers, starting from the vertex 0 which is the apex of $Q_0$.

Because $Q$ and $Q_0$ have no common diagonal, there exists a unique region $R_0$ of $Q$ that contains the vertex 0 in its boundary. Removing $R_0$ from the ambient polygon, one gets one or more convex polygons, all of them with $km + 2$ vertices for some $k$. Let us choose one of these polygons, and let $Q'$ be its $M$-angulation obtained from $Q$ by restriction. Let $R'$ be the unique region of $Q'$ which is adjacent to $R_0$. 

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Excluding minimal and maximal indices, vertices of $Q'$ form a sequence of $mk$ vertices numbered consecutively, in which every residue class modulo $m$ is represented exactly $k$ times.

Because of the tree-like structure of $M$-angulations, one can build the $M$-angulation $Q'$ by successive additions of $M$-angles, starting from the region $R'$. Going backwards, one can go from $Q'$ to $R'$ by removal of $M$-angles on the boundary.

Using this leaf-removal induction, one can then prove that the boundary of $R'$ (excluding minimal and maximal indices) contains exactly one representative of every residue class modulo $m$.

Note now that in $Q_0$, the vertices linked by a diagonal to the apex 0 form a residue class modulo $m$, when numbered in the same way as the vertices of $Q$.

It follows that exactly one of the vertices of $R'$ (excluding minimal and maximal indices) is the end of a diagonal $d_0$ in $Q_0$. The diagonal $d_0$ does only cross one diagonal of $Q$, namely the diagonal separating the regions $R_0$ and $R'$.

The maximal elements in the poset $P_{m,n}$ are the $M$-angulations that have no common diagonal with $Q_0$. We will call them final $M$-angulations.

Remark: There are several interesting existing families of posets on objects in bijection with $(m+2)$-angulations, including $m$-Tamari lattices [BPR12] and $m$-Cambrian lattices of type $A$ [STW15]. The posets introduced here seem to be new.

### 3 $m$-analogues of $B$-quasisymmetric functions

In [BH08], Baumann and Hohlweg introduced the ring of $B$-quasisymmetric functions as the graded dual Hopf algebra of the analog in type $B$ of Solomon’s descent algebra. This ring is contained in the polynomial ring in two sets of $n$ variables.

In [Ava07], a quotient ring of this polynomial ring by an ideal of $B$-quasisymmetric functions was studied. Moreover, for every integer $m \geq 1$, an analog of the ring of $B$-quasisymmetric functions and an analog of the quotient ring were also defined and studied, involving $m$ sets of $n$ variables.

We refer to [Ava07] for the original motivations of the study of these rings and for the proof of the results that we will use. Let us now summarize the results of [Ava07] in their most general form.

Let us denote by $M = \{x, y, z, \ldots, \omega\}$ a set of $m$ distinct letters, endowed with a total order $x < y < z < \cdots < \omega$. We will mostly illustrate our constructions with the cases $m = 2$ and $m = 3$, therefore using only the letters $x, y, z$.

Let us start with polynomials in the union of $m$ alphabets of each $n$ variables: $X_n = x_1, \ldots, x_n, Y_n = y_1, \ldots, y_n$, up to $\Omega_n = \omega_1, \ldots, \omega_n$. Denote this polynomial ring by $\mathbb{Q}[X_n, Y_n, \ldots, \Omega_n]$. Inside this polynomial ring, one can define a space of $G^m$-quasisymmetric functions, which reduces when $m = 1$ to the classical quasisymmetric functions.

In [Ava07], a Gröbner basis for the ideal $J_{m,n}$ generated by constant-term-free $G^m$-quasisymmetric functions was described, and from that was deduced a
monomial basis for the quotient $\mathcal{R}_{m,n}$ of the polynomial ring $\mathbb{Q}[X_n, Y_n, \ldots, \Omega_n]$ by $\mathcal{J}_{m,n}$.

This monomial basis for the quotient $\mathcal{R}_{m,n}$ is indexed by $m$-Dyck paths, which gives the dimension formula $\dim \mathcal{R}_{m,n} = \frac{1}{mn+1} (m+1)n$.

### 3.1 Definitions

For these definitions, we follow [BH08], with some minor differences, for the sake of simplicity of the computations we will have to make. The main change is to describe directly the general case for any $m \geq 1$ and not the special case $m = 2$.

An $m$-vector of size $n$ is a vector $v = (v_1, v_2, \ldots, v_{mn})$ of length $mn$ with entries in $\mathbb{N}$. One must think of $m$-vectors as the concatenation of $n$ sequences of length $m$. An $m$-composition is an $m$-vector in which there is no sequence of $m$ consecutive zeros.

The integer $n$ is called the size of $v$. The weight of $v$ is by definition the $m$-tuple $(w_1, \ldots, w_m)$ where $w_j = \sum_{i=0}^{n-1} v_{mi+j}$. We also set $|v| = \sum_{i=0}^{mn} v_i$. For example $(1, 0, 2, 1, 0, 2, 3, 0)$ is a $2$-composition of size $4$, and of weight $(6, 3)$.

To make notations lighter, we shall sometimes write $m$-vectors or $m$-compositions as words with bars, where the bars separates the word into $n$ blocks of length $m$. For example, $10|21|02|30$ stands for the $2$-vector $(1, 0, 2, 1, 0, 2, 3, 0)$ (see also the following definition).

Let us now define the fundamental $G^m$-quasisymmetric polynomials, indexed by $m$-compositions.

Let $c = (c_1, \ldots, c_{mn})$ be an $m$-composition. One can decompose $c$ as a concatenation of $n$ blocks of $m$ integers. Let us first associate to $c$ a word $w_c$ in the alphabet $\{x, y, z, \ldots, \omega\}$, defined as the concatenation, over all blocks, of $c$, of the word $x^{b_1} y^{b_2} \ldots \omega^{b_m}$. For example, for the $3$-composition $010|201$, one obtains $w_c = yxxz$ (powers are written as repeated letters).

Then the fundamental $G^m$-quasisymmetric polynomial of index $c$ is

$$F_c = \sum_{t \in w_c} \prod_{i \in w_c} t_{i(t)},$$

where the sum is taken over all maps $i$ from the sequence of letters of the word $w_c$ to the set $\{1, \ldots, n\}$ such that $i$ is weakly increasing inside every block of $c$ and strictly increasing between two blocks.

Let us give some examples for $m = 2$:

$$F_{12} = \sum_{i \leq j \leq k} x_i y_j y_k.$$  \hspace{1cm} (1)

$$F_{02|10} = \sum_{i \leq j < k} y_i y_j x_k.$$  \hspace{1cm} (2)

It is clear from the definition that the multidegree (i.e. the $m$-tuple (degree in $x$, degree in $y$, ..., degree in $\omega$)) of $F_c$ in $\mathbb{Q}[X_n, Y_n, \ldots, \Omega_n]$ is the weight of $c$. If the size of $c$ is greater than $n$, we set $F_c = 0$.

The space of $G^m$-quasisymmetric polynomials, denoted by $QSym_n(G^m)$ is the vector subspace of the ring $\mathbb{Q}[X_n, Y_n, \ldots, \Omega_n]$ generated by the $F_c$, for all $m$-compositions $c$. 

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Let us denote by $\mathcal{J}_{m,n}$ the ideal $\langle Q\text{Sym}_n(G^m) \rangle$ generated by $G^m$-quasisymmetric polynomials with zero constant term.

3.2 A monomial basis for $\mathcal{R}_{m,n}$

Let $v = (v_1, v_2, \ldots, v_{mn-1}, v_{mn})$ be an $m$-vector of size $n$. We associate to $v$ a path $\pi(v)$ in the plane $\mathbb{N} \times \mathbb{N}$, with steps $(0,1)$ (up step) or $(m,0)$ (right step). We start from $(0,0)$ and for each entry $v_i$ (read from left to right) add $v_i$ right steps $(m,0)$ followed by one up step $(0,1)$. This clearly defines a bijection between $m$-vectors of size $n$ and such paths of height $mn$.

As an example, the path associated to the 2-vector $(1,0,1,2,0,0,1,1)$ is

![Path Illustration](image)

If the path $\pi(v)$ associated with an $m$-vector $v$ always remains above the diagonal $x = y$, we call this path an $m$-Dyck path, and say that the corresponding $m$-vector $v$ is an $m$-Dyck vector.

Being an $m$-Dyck vector is equivalent to the condition that, for any $1 \leq \ell \leq mn$, one has

$$m(v_1 + v_2 + \cdots + v_\ell) < \ell.$$ 

For $v$ an $m$-vector (of length $mn$), we denote by $\mathcal{A}_v$ the monomial

$$\mathcal{A}_v = (x_1^{v_1}y_1^{v_2} \cdots \omega_1^{v_m})(x_2^{v_{m+1}}y_2^{v_{m+2}} \cdots \omega_2^{v_{2m}}) \cdots (x_n^{v_{(n-1)m+1}} \cdots \omega_n^{v_{mn}}).$$

This clearly defines a bijection between $m$-vectors and all monomials in the polynomial ring $\mathbb{Q}[X_n,Y_n,\ldots,\Omega_n]$.

For example, the monomial associated to the 2-vector $(1,0,1,2,0,0,1,1)$ is $x_1x_2y_2x_4y_4$.

The following result was proved in [Ava07, Th. 5.1].

**Proposition 2.** The set $\mathcal{B}_{m,n}$ of monomials $\mathcal{A}_v$ for $v$ varying over $m$-Dyck vectors of size $n$ is a basis for the space $\mathcal{R}_{m,n} = \mathbb{Q}[X_n,Y_n,\ldots,\Omega_n]/\mathcal{J}_{m,n}$.

4 A bijection between $M$-angulations and $m$-Dyck paths

We assign to each vertex of the $mn + 2$-gon (except the vertex 0) a letter in $M$ as follows. The vertices are labelled by repeating the sequence $x, y, z, \ldots, \omega$ in counter-clockwise order around the polygon, in such a way that the final vertex just before the vertex 0 receives the last letter $\omega$ of $M$. It follows that the first vertex just after the vertex 0 also receives the letter $\omega$.

See Figure 4 for an illustration of this labelling when $m = 2$ with the ordered set of letters $x < y$. 

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Figure 4: The labelling of vertices by $x$ and $y$ letters ($m = 2$)

Let us also label the inner diagonals of $Q_0$ from 1 to $n-1$ in counter-clockwise order.

Then we consider an $M$-angulation $Q$. To any diagonal $d$ of $Q$, we associate a polynomial $m_d$. If $d$ coincides with a diagonal of $Q_0$, we set $m_d = 1$. Otherwise, and reading counter-clockwise, $d$ starts from a vertex labelled by the letter $u$, intersects consecutive diagonals of $Q_0$ labelled from $i$ to $j$ and ends at a vertex labelled by the letter $v$. Then we set $m_d = v_{j+1} - u_i$. We then associate to $Q$ the polynomial $P_Q$ defined as the product of $m_d$ over its diagonals.

As an example, Figure 5 shows the polynomials associated to the quadrangulations of Figure 3, in the corresponding positions.

$$
\begin{align*}
(x_3 - y_2) & (x_1 - y_2) & (y_3 - x_2) & (y_3 - x_2) & (x_2 - y_1) & (y_2 - x_1) & (y_2 - x_1) \\
(y_3 - x_1) & (x_3 - y_1) & (x_3 - y_1) & (x_2 - y_1) & (x_3 - y_1) & (x_3 - y_1) & (y_3 - x_1)
\end{align*}
$$

Figure 5: The polynomials associated to quadrangulations of Figure 3

To deal with leading terms of polynomials, we will use the lexicographic order induced by the ordering of the variables:

$$
x_1 < y_1 < \cdots < \omega_1 < x_2 < y_2 < \cdots < x_n < y_n < \cdots < \omega_n. \quad (3)
$$

The lexicographic order is defined on monomials as follows: $A_v <_{\text{lex}} A_w$ if and only if the last non-zero entry of $v - w$ (componentwise) is negative.
Note that the leading monomial of the polynomial $P_Q$ attached to an $M$-angulation is easily described: in every binomial factor written $v_{j+1} - u_i$ with $j + 1 > i$, keep only the monomial $v_{j+1}$.

**Proposition 3.** The set of leading monomials of the polynomials $P_Q$ when $Q$ varies over the set of $M$-angulations coincides with the monomial basis $B_{m,n}$.

**Proof.** We need a bijection $\Phi$ between $M$-angulations $Q$ and $m$-Dyck paths (or rather $m$-Dyck vectors $v$) such that the leading monomial of $P_Q$ is equal to $A_{\Phi(Q)}$.

The idea to define $\Phi$ is to compose the leading-monomial application $Q \mapsto \text{LM}(P_Q)$ with the bijection between monomials and $m$-Dyck paths described in section 3.2.

Let us instead define the reverse bijection $\Psi$.

Let $v$ be an $m$-Dyck vector of size $n$. We start from the empty set of diagonals on the $mn + 2$-gon. We shall add iteratively diagonals. Let $D$ denote the current set of diagonals, under construction. We read the $m$-vector $v$ from left to right. To any non-zero entry $c$ associated to variable $z_k$ we add to $D$ a fan with $c$ diagonals as follows. Let $t$ be the letter before $z$ in the cyclic order $x < y < \cdots < \omega < x$. The (common) ending point of the added fan is the first vertex labelled $z$ that comes counter-clockwise after the diagonal $k - 1$ of $Q_0$, and the starting points are the $c$ last vertices labelled $t$ that are available before the ending point. Here being available means not being separated from the ending point by a diagonal already in $D$.

See Figure 6 for an example of this construction.

![Figure 6: The quadrangulation $Q$ associated to the monomial $x_5 y_6^3 y_7$ with $P_Q = (x_5 - y_4)(y_5 - x_3)(y_5 - x_2)(y_5 - x_1)(y_7 - x_6)$.](image-url)

Now the key point is that, when adding such a fan, the starting points are always strictly between the apex 0 and the ending point in counter-clockwise
order. This is because of the \( m \)-Dyck word property of \( v \): for any \( 1 \leq \ell \leq mn \), one has

\[
m(v_1 + v_2 + \cdots + v_\ell) < \ell.
\]

Indeed, when numbering the vertices of the polygon counter-clockwise from the apex 0 to \( mn + 1 \), the index of the first vertex \( \ell \) in counter-clockwise order for the fan added at step \( \ell \) is exactly \((1 + \ell) - m(v_1 + v_2 + \cdots + v_\ell) - 1\).

The leading term of the product of binomials \( m_d \) attached to the diagonals in the fan just added is therefore the monomial \( z_\ell^k \), by definition of the lexicographic order.

At the end of the process, there remains only to add some diagonals of \( Q_0 \) to obtain an \( M \)-angulation. There is a unique way to do that. This does not change the product of \( m_d \) over all diagonals.

One has therefore defined a map \( \Psi \) from \( m \)-Dyck words to \( M \)-angulations. This has the property that the leading monomial of \( P_{\Psi(v)} \) is equal to \( A_v \). Because one can recover the \( m \)-Dyck word \( v \) from \( A_v \), this map is injective.

This proves that \( \Psi \) is an injection between sets of equal cardinalities, hence a bijection.

The reverse bijection \( \Phi \) can be described as first removing from \( Q \) all diagonals in \( Q_0 \), then proceeding by successive removals of fans.

**Proposition 4.** The set of polynomials \( P_Q \) (when \( Q \) varies over the set of \( M \)-angulations) endowed with the poset structure given by divisibility is isomorphic to \( P_{m,n} \).

**Proof.** The map \( Q \mapsto P_Q \) is a bijection, with inverse obtained from the factorisation of the polynomial. The flip in \( Q \) of a diagonal of \( Q_0 \) corresponds to the multiplication of \( P_Q \) by a linear binomial.

Remark: the fan \( Q_0 \) has an obvious involutive symmetry that fixes the apex 0 and flips the ambient polygon. Given the labelling of the polygon by \( M = \{x, y, \ldots, \omega\} \), this involution acts on variables by simultaneous substitution of letters

\[
\omega, x, y, z, \ldots \leftrightarrow \omega', \ldots, z, y, x.
\]

and renumbering of indices \( i \leftrightarrow n+1-i \). By construction, the set of polynomials \( P_Q \) is sent to itself (up to signs) by this involution acting on variables.

**5 Enumerative aspects**

**5.1 Recursive description**

Let \( T_m = \sum_{n \geq 1} Q_{m,n} x^n \) be the generating series for \( M \)-angulations according to their size \( n \). Note that the power of \( x \) is chosen to correspond to the number of regions.

There is a classical recursive decomposition of \( M \)-angulations, which describes them as an \( M \)-angle with an \( M \)-angulation (or nothing) grafted on all but one sides. This implies that

\[
T_m = x (1 + T_m)^{m+1}.
\]

(4)
Recall that the final $M$-angulations are those which do not contain any diagonal of $Q_0$. They are the maximal elements of the posets $P_{m,n}$.

Let $F_m$ be the generating series for final $M$-angulations according to the number of regions. These objects can be decomposed as one $M$-angle (the unique region having 0 in its boundary), on which one can graft any $M$-angulation on all sides that do not contain the vertex 0. One obtains that

$$F_m = x(1 + T_m)^m = T_m/(1 + T_m).$$  \hfill (5)

Let us now describe another simple decomposition of $M$-angulations, corresponding to the rightmost expression in the equation (5).

To every $M$-angulation $Q$, one can associate a list $L(Q)$ of final $M$-angulations, obtained by cutting $Q$ along its initial diagonals. Let us assume that these pieces are listed in counter-clockwise order.

**Proposition 5.** Sending $Q$ to $L(Q)$ defines a bijection from $M$-angulations to lists of final $M$-angulations.

The inverse bijection is very simple: given a list of final $M$-angulations, one can glue them back along their sides into one single $M$-angulation.

This inverse bijection from $L(Q)$ to $Q$ can also be interpreted in the following way, that will be useful later. Given a list of $k$ final $M$-angulations, one considers the $M$-angulation $Q_0$ with $k$ regions. One then replaces the regions of $Q_0$ by the final $M$-angulations, in the counter-clockwise order. In the resulting $M$-angulation $Q$, the initial $Q_0$ can be identified with the union of all regions that are adjacent to the vertex 0.

![Figure 7: Illustration of construction $L$](image)

### 5.2 Rank generating function

Let $G_m$ be the generating function for the elements of all posets $P_{m,n}$, according to their size $n$ and their rank, namely

$$G_m = \sum_{n \geq 1} x^n \sum_{Q \in P_{m,n}} z^{rkQ}. \hfill (6)$$

The generating function $G_m$ satisfies

$$G_m = \frac{F_m(zx)/z}{1 - F_m(zx)/z} \hfill (7)$$

where $F_m$ is the generating series for the final elements. Indeed, any $M$-angulation can be written uniquely as a list of final $M$-angulations by cutting
along the diagonal edges shared with \( Q_0 \). The rank parameter is multiplicative along this decomposition. And the rank of final \( M \)-angulations is just their size minus 1, so that the generating series of final \( M \)-angulations according to size and rank is just \( F_m(zx)/z \).

Using Lagrange inversion followed by a simple summation of binomial coefficients, one can deduce from (4), (5) and (7) that the rank generating function of \( P_{m,n} \) is given by

\[
\sum_{k=0}^{n-1} \frac{n-k}{n} \binom{mn+k-1}{k} z^k.
\]  

(8)

For \( m = 1 \), this enumeration according to rank was already done in [CSS14, §3].

5.3 Decomposition of intervals

Let us now study the intervals in the posets \( P_{m,n} \).

An interval \( A \) is a pair of elements \((A^-, A^+)\) in \( P_{m,n} \) that satisfies \( A^- \leq A^+ \). In every finite poset, the number of intervals is also the dimension of the incidence algebra.

Let us call an interval \( A \) initial if its lower bound \( A^- \) is \( Q_0 \). The set of initial intervals can be identified with the set \( Q_{m,n} \) of \( M \)-angulations in \( P_{m,n} \). Indeed, \( Q_0 \) is smaller than all \( M \)-angulations by Lemma 1.

Let \( I_{m,n} \) be the set of intervals in \( P_{m,n} \). Let \( I_m = \sum_{n \geq 1} \#I_{m,n} x^n \) be the generating series for intervals.

One can get a recursive decomposition for intervals, similar to the previous decomposition for \( M \)-angulations. For this, one needs the following construction.

Suppose that one has an \( M \)-angulation \( B_0 \) with \( k \) regions, and a list of \( k \) final \( M \)-angulations \( B_1, \ldots, B_k \). From this data, one can build an \( M \)-angulation \( G(B_0; B_1, \ldots, B_k) \) as follows. First build the \( M \)-angulation \( L^{-1}(B_1, \ldots, B_k) \). Removing its \( k-1 \) initial diagonals creates a region with \( mk+2 \) sides. Place \( B_0 \) inside this region to define the \( M \)-angulation \( G(B_0; B_1, \ldots, B_k) \). See Figure 8 for an example.

If \( B_0 \) is \( Q_0 \), the construction \( G \) is just the inverse of the \( L \) map.

Figure 8: \( B_0, B_1, B_2 \) and \( G(B_0; B_1, B_2) \)

Given an interval \( A \), one can apply the map \( L \) to its bottom element \( A^- \). This gives a list of final \( M \)-angulations \( A_1^-, \ldots, A_k^- \), such that

\[
A^- = G(Q_0; A_1^-, \ldots, A_k^-).
\]  

(9)
Proposition 6. There exists a unique $M$-angulation $A'_0^+$ such that
\[ A' = G(A'_0^+; A_1^-, \ldots, A_k^-). \] (10)

Proof. Uniqueness is clear by the definition of the construction $G$.
Existence is proved by induction on the difference of initial diagonals in $A^-$ and $A^+$. If $A^+ = A^-$, then the only possible choice is $A'_0^+ = Q_0$.
Otherwise, let us consider a cover relation $A^- \leq Q' \prec Q''$. Assume that $Q' = G(Q'_0; A'_1^-, \ldots, A'_k^-)$ by induction. Because the cover relations flips an initial diagonal $d$ and does not change the other ones, the diagonal $d$ must in fact belong to $Q'_0$. Therefore one can flip it in $Q'_0$ to get $Q'' = G(Q''_0; A'_1^-, \ldots, A'_k^-)$. \hfill $\square$

Keeping the same notations, one also has the following result.

Proposition 7. Every element of the interval $[A^-, A^+]$ can be uniquely written as $G(Q; A_1^-, \ldots, A_k^-)$ for some $Q$ in $[Q_0, A'_0^+]$. The interval $A$ is isomorphic to the interval $[Q_0, A'_0^+]$.

Proof. This result follows from the proof of Prop. 6. In fact, more is true: all elements greater than $A^-$ can be uniquely written $G(Q; A_1^-, \ldots, A_k^-)$ for some $Q$ and this bijection identifies the upper ideal of $A^-$ with a smaller poset of type $\mathcal{P}$. \hfill $\square$

Proposition 6 implies the following decomposition.

Proposition 8. The map sending an interval $A$ to the pair $(A'_0^+, \mathcal{L}(A^-))$ defines a bijection between intervals and pairs $(B_0, (B_1, \ldots, B_k))$ where $B_0$ is an $M$-angulation with $k$ regions and $B_1, \ldots, B_k$ are final $M$-angulations.

The inverse bijection is given by
\[ A^- = G(Q_0; B_1, \ldots, B_k) \quad \text{and} \quad A^+ = G(B_0; B_1, \ldots, B_k). \] (11)

Corollary 9. The generating series of intervals can be expressed using those of $M$-angulations and final $M$-angulations as
\[ I_m = T_m(F_m) = T_m(x(1 + T_m)^m). \] (12)

The second equality follows from (5).

6 Isomorphism types of intervals

The aim of this section is to give a description of the intervals as posets, and to prove in particular that they are distributive lattices with Möbius numbers in $\{-1, 0, 1\}$.

By Proposition 7, every interval is isomorphic to an initial interval. It is therefore enough to study initial intervals.

Let us call an interval initial-final if its minimum is $Q_0$ and its maximum is a final quadrangulation.

Proposition 10. Every initial interval is isomorphic to a product of initial-final intervals.
Proof. Let $A = [Q_0, A^+]$ be an initial interval. Consider the set of diagonals of $Q_0$ that belong to $A^+$. Then one can cut both $Q_0$ and $A^+$ along these diagonals. Every piece $A^-_i$ of $Q_0$ is a smaller $Q_0$. Every piece $A^+_i$ of $A^+$ is a final quadrangulation. By definition of the partial order, the same diagonals belong to every element of $A$. Cutting along them gives an isomorphism with the product of the intervals $[A^-_i, A^+_i]$, which are initial-final intervals.

Let us now proceed to a more subtle decomposition.

Let $Q$ be a final triangulation. Let $R_0$ be the unique region in $Q$ with 0 in its boundary. Removing $R_0$ from $Q$ leaves a certain number $k$ of polygons, that will be called the blocks. Let us call $k$ the width of $Q$. The width can be 0 only if $Q$ is reduced to $R_0$. Otherwise it lies between 1 and $m$.

**Proposition 11.** Every initial-final interval $A$ is isomorphic to a product of initial-final intervals of width 1.

Proof. The main idea is that the flips downwards from $Q$ happen completely independently in distinct blocks. Let now us give a detailed argument for this independence.

Let us consider the set $D$ of diagonals that go from vertex 0 to one of the vertices of the region $R_0$, except the two vertices that are neighbors of 0. These diagonals are inside the region $R_0$ and therefore do not belong to $Q$. The extremities of these diagonals receive exactly once every label from 0 to $m - 2$ in clockwise order, and therefore never get the label $m - 1$. Note that the extremities of the initial diagonal are labeled by $m - 1$.

Let us show by induction downward from $Q$ that for every $Q' \in [Q_0, Q]$, the diagonals in $D$ do not belong to $Q'$ and do not cross any diagonal of $Q'$. This is clear for $Q$. The effect of a down flip is to replace a diagonal by an initial diagonal. This initial diagonal is not in $D$ and does not cross any diagonal in $Q$, because of the labeling of its extremity is $m - 1$.

One can therefore cut along the diagonals in $D$. Replacing every block but a fixed one by a trivial block gives a map to an initial-final interval of width 1. Taking the product of all these maps gives the desired isomorphism.

**Proposition 12.** The number of elements covered by the maximum in a initial-final interval $[Q_0, Q]$ is the width of $Q$.

Proof. In the proof of Lemma 1, it was shown that in every polygon of $Q$ minus $R_0$, there is exactly one initial diagonal $d$ that cross just one diagonal $d'$ of $Q$. The set of diagonals of $Q$ that can be flipped down in the poset is exactly the collection of these $d'$, and their number is therefore the width of $Q$.

In particular, initial-final intervals of width 1 have a maximum that covers a unique element.

**Corollary 13.** The intervals in $P_{m,n}$ can be build iteratively by either adding a maximum to a smaller one or by taking a product of several smaller ones.

Proof. By induction on both the height (difference of ranks) of the intervals and the size of the ambient polygon. Every interval is a product of initial-final ones. Every initial-final interval is a product of initial-final intervals of width 1.
In both cases, if the product is not reduced to one element, the factors live in strictly smaller polygons.

Every initial-final interval of width 1 is obtained by adding a maximum to an interval, which has smaller height.

Let us define a forest poset as a poset where every element is covered by at most one element. These posets can be build iteratively from the empty poset by two operations, namely adding a maximum element and taking the disjoint union. Their Hasse diagrams are forests of rooted trees, where the roots are the maximal elements.

**Proposition 14.** Every interval in $P_{m,n}$ is a distributive lattice, isomorphic to the lattice of order ideals of a forest poset.

**Proof.** By the results above, the intervals in $P_{m,n}$ can be obtained from smaller intervals using two operations, namely adding a maximum and taking a product. These two operations correspond to adding a maximum or taking the disjoint union, on the poset of join-irreducible elements. The statement follows by induction.

For $m = 1$, this property of intervals was already obtained in [CSS14, §2].

**Corollary 15.** Möbius numbers of intervals in posets $P_{m,n}$ belong to $\{-1, 0, 1\}$.

**Proof.** This property is preserved by adding a maximum or taking a product.

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