What do Abelian categories form?

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Abstract. Given two finitely presentable Abelian categories $A$ and $B$, we outline a construction of an Abelian category of functors from $A$ to $B$, which has nice 2-categorical properties and provides an explicit model for a stable category of stable functors between the derived categories of $A$ and $B$. The construction is absolute, so it makes it possible to recover not only Hochschild cohomology but also Mac Lane cohomology.

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Introduction

A long time ago when mathematics was a ‘science about numbers’, which by the way it still is, at least nominally, in East Asian languages, the objects of mathematical study were typically elements in a set. These days they usually form a category. However, what happens when these objects are themselves categories? Small categories by themselves, without any adornment, of course form a 2-category, and this is all there is: we have the category of functors $\text{Fun}(I, I')$ for any small categories $I$ and $I'$. These are equipped with composition functors, identity objects, and associativity and unitality isomorphisms, and this is the end of the story. But things are much less clear when we are talking about categories with additional structures and/or categories of a special type.

The case in point is modern ‘non-commutative’ or ‘categorical’ algebraic geometry as formulated, for example, by Kontsevich and Soibelman [21]. Stated briefly, this is the ‘geometry of derived categories’: one studies an algebraic variety $X$ by looking at its derived category $D(X)$ of coherent sheaves. But $D(X)$ is more than just a category. At the very least it carries a triangulated structure in the sense of Verdier [28], but it is well known that this is not enough. In particular, triangulated functors between triangulated categories do not form a triangulated category, and worse then that, while it is more-or-less clear what this ‘triangulated category of functors’ from $D(X)$ to $D(X')$ should be, it is not possible to recover it just from $D(X)$ and $D(X')$. The correct object of study is a triangulated category ‘with an enhancement’, and the meaning of ‘enhancement’ is a matter of choice.

One rather radical choice, which is becoming more popular, is to take a step back and say that, in fact, all categories should be equipped with an enhancement. Triangulated categories then correspond to stable enhanced categories, and stability is a condition and not a structure. Stable functors between stable categories do form a stable category, so the theory looks reasonably complete and natural. However, in practice, in all the existing formalisms such as quasi-categories or complete Segal spaces, an enhanced category is something rather large and dependent on arbitrary choices, and it only makes sense to consider it up to a ‘weak equivalence’ of some sort. Thus, to work with enhanced categories one has to use the cumbersome machinery of abstract homotopy theory, that is, model categories, simplicial homotopy theory, and so on. This is fine in topology, where this machinery is needed in any case, but feels excessive in more algebraic applications, where people are used to the simple and powerful homological algebra of [10], and strongly prefer chain complexes to simplicial sets.

Perhaps for this reason, the most common technique of enhancements used in categorical geometry is based on the notion of a differential-graded or DG-category; we refer the reader to [20] for a very good overview. In this context the question posed in the title of this paper was addressed in the pioneering paper [26]. Among other things, Tamarkin constructs the correct DG-category of DG-functors between
two DG-categories over a fixed field $k$, and then studies in detail the composition functors and all sorts of higher structures that arise in the theory. In particular, if we restrict our attention to a single DG-category $A_*$, then its Hochschild cohomology $HH^q(A_*)$ is defined as the algebra $R\text{Hom}^*(\text{Id}, \text{Id})$, where $\text{Id}$ is the identity endofunctor of $A_*$, and it carries an additional structure of an $E_2$-algebra or, equivalently, of a $B_{\infty}$-algebra; see [20], §5.4. This is crucial for developing deformation theory of $A_*$ with first-order deformations described by classes in $HH^2(A_*)$.

Our goal in this paper is to some extent complementary to what was done in [26]. Namely, we observe that if not all, then at least many triangulated categories that appear in geometry not only come from DG-categories, but are also derived categories of Abelian ones. So, assume that we consider derived categories $D(C)$, $D(C')$ of some Abelian categories $C$, $C'$. Can we recover the ‘correct’ category of functors from $D(C)$ to $D(C')$ if we remember not only $D(C)$, $D(C')$, but also the Abelian categories $C$, $C'$? If so, do we have a convenient model for this category of functors in terms of $C$ and $C'$?

In a sense, this looks like a toy model for the whole theory since derived categories are somewhat special, and remembering the Abelian category $C$ is even more restrictive. One advantage of this toy model is that the resulting theory is absolute: while Abelian categories linear over a fixed field $k$ can be considered, one can also work without fixing the ground field. In fact, if one does fix $k$, then the answer to our question has been known for a long time: if $C$ is small and the target category is large enough (say, the ind-completion $\text{Ind}(C')$ of a small Abelian category $C'$), then $k$-linear left-exact functors $C \to \text{Ind}(C')$ form an Abelian category. (This, essentially, goes back to the famous Gabriel–Popescu Theorem from the 1960s; see Example 5.2 below for a precise statement.) Its derived category is exactly what we want. In particular, it gives the correct Hochschild cohomology $HH^q(C) = R\text{Hom}^*(\text{Id}, \text{Id})$, which can be identified with the DG version, and deformation theory in this context was successfully developed in [22] and subsequent work.

Alternatively, instead of functors $C \to \text{Ind}(C')$, one can consider functors $\text{Ind}(C) \to \text{Ind}(C')$ that are continuous, that is, commute with filtered colimits. This gives the same category, but it can be defined in greater generality: instead of ind-completions of small Abelian categories, one can consider arbitrary finitely presentable Abelian categories (see §1.2 for more details).

In the absolute case these things are less well-studied, or at least, not so easy to find in the literature. If we look at deformation theory, then the basic example of a first-order deformation that is not linear over a field is the square-zero extension $\mathbb{Z}/p^2\mathbb{Z}$ of the prime field $\mathbb{Z}/p\mathbb{Z}$. In this case, to get a deformation class one has to replace Hochschild cohomology with the so-called Mac Lane cohomology $HM^*(\mathbb{Z}/p\mathbb{Z})$. One attempt to extend this to general Abelian categories was [18], where a bunch of functor categories were constructed together with associated versions of Hochschild cohomology, and comparison theorems were proved that show how to recover Mac Lane cohomology and some other generalizations of Hochschild cohomology. However, the emphasis in [18] was on these cohomology theories and comparison maps. What we want to do in the present paper is to concentrate on the 2-categorical structure. For any finitely presentable Abelian categories $C$ and $C'$, we construct one particular functor category $\text{Mor}(C,C')$, with its derived
version $\text{DMor}_{\text{st}}(C,C')$, and then we construct the composition functors and show that objects in $\text{Mor}(C,C')$ and $\text{DMor}_{\text{st}}(C,C')$ indeed act naturally on $C$ and $D(C)$, respectively.

The underlying idea of the construction is somewhat surprising but very old. Again, it essentially goes back to the Gabriel–Popescu Theorem. Given an additive functor $E: A \to C'$ between Abelian categories, one reinterprets the condition that $E$ is left-exact as a condition of $E$ being a sheaf for an appropriate Grothendieck topology on $C^0$ (sometimes called the ‘single-epi topology’). Then one can drop additivity: being a sheaf and being additive are independent conditions, either of which makes perfect sense without the other. We then take finitely presentable Abelian categories $C$ and $C'$, and consider the category $\text{Fun}_e(C,C')$ of all continuous functors $C \to C'$. The category $\text{Fun}_e(C,C')$ is Abelian, and our category $\text{Mor}(C,C') \subset \text{Fun}_e(C,C')$ is the full subcategory formed by sheaves. This category is also Abelian, we have the left-exact fully faithful embedding $e: \text{Mor}(C,C') \to \text{Fun}_e(C,C')$, and its left-adjoint associated sheaf functor $a: \text{Fun}_e(C,C') \to \text{Mor}(C,C')$ is exact.

To extend this to derived categories we start with the positive part $D^{\geq 0}(C)$ of the derived category $D(C)$ and use the classic extension technology due to Dold [6]. Namely, for any Abelian $A$, the Dold–Kan equivalence identifies the category $\text{Fun}(A,A)$ of cosimplicial objects in $A$ with the category $C^{\geq 0}(A)$ of chain complexes in $A$ concentrated in non-negative cohomological degrees. A functor $E: A \to A'$ to some Abelian $A'$ then extends to a functor $D(E): C^{\geq 0}(A) \to C^{\geq 0}(A')$ by passing to cosimplicial objects and applying $E$ pointwise. More generally, if we have a functor $E: A \to C^{\geq 0}(A') \cong \text{Fun}(A,A')$, we can define its Dold extension $D(E): C^{\geq 0}(A) \to C^{\geq 0}(A')$ by applying $E$ pointwise, followed by restricting the resulting bisimplicial object to $\Delta \subset \Delta \times \Delta$. If $A = C$ and $A' = C'$ are finitely presentable, then this construction sends continuous functors to continuous functors, and it always sends pointwise quasi-isomorphisms to pointwise quasi-isomorphisms, thus descends to a functor $D: D^{\geq 0}(\text{Fun}_e(C,C')) \to D^{\geq 0}(\text{Fun}_e(C^{\geq 0}(C),C'))$,

where we identify $C^{\geq 0}(\text{Fun}(-,-)) \cong \text{Fun}(-,C^{\geq 0}(-))$ and localize with respect to pointwise quasi-isomorphisms.

Now, we denote by $\text{DMor}(C,C')$ the derived category of our functor category $\text{Mor}(C,C')$, with $\text{DMor}^{\geq 0}(C,C')$ standing for its positive part, and we observe that the derived functor $R^e$ of the embedding $e$ provides a full embedding $R^e: \text{DMor}^{\geq 0}(C,C') \to D^{\geq 0}(\text{Fun}_e(C,C'))$. On the other hand, let us say that a functor $C^{\geq 0}(C) \to C^{\geq 0}(C')$ is homotopical if it sends quasi-isomorphisms to quasi-isomorphisms, and let $D\mathcal{H}^{\geq 0}(C,C') \subset D^{\geq 0}(\text{Fun}_e(C^{\geq 0}(C),C'))$ be the full subcategory formed by homotopical continuous functors. With this notation, we prove the following assertion (for precise statements, see Theorem 5.5 and Corollary 5.6):

- An object $E \in D^{\geq 0}(\text{Fun}_e(C,C'))$ lies in the essential image of the embedding $R^e$ if and only if its Dold extension $D(E)$ is homotopical. Moreover, $D \circ R^e: \text{DMor}^{\geq 0}(C,C') \to D\mathcal{H}^{\geq 0}(C,C')$ is an equivalence.

It is interesting to note that Dold himself used his extension procedure in a slightly different manner. He discovered that for any $E: A \to A'$ whatsoever, $D(E)$ sends
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chain-homotopic maps between complexes to chain-homotopic ones (in our language this is Lemma 4.6), and then defined the derived functors $D^i(E)$, $i \geq 0$, by applying $D(E)$ to an injective resolution of some $A \in \mathcal{A}$ and taking homology of the resulting complex. This actually fits together quite nicely with the sheaf-theoretic approach: for any $E$, $D^0(E)$ is a sheaf, and $E \cong D^0(E)$ if and only if $E$ was a sheaf. The higher derived functors $D^i(E)$ are the homology objects of $R^i(e(a(E)))$ (this is Proposition 4.8).

To extend our functors to the full derived category $D(C)$ we need to know that the Dold extension $D(E)$ commutes with homological shifts. However, in general, this is not true and should not be true. The reason for this is the additivity condition that we have dropped, and we now reinstate it at the derived level. This can be done in several equivalent ways, see Proposition 6.3, but the simplest one is just to require that $E: C \to C'(C')$ becomes additive when we project to $D(C')$. This distinguishes a full triangulated subcategory $DMor_{st}(C,C') \subset DMor(C,C')$ of stable objects, and these, as soon as they are bounded below for technical reasons, act naturally by functors $D(C) \to D(C')$. Note that $DMor_{st}(C,C')$ inherits a t-structure from $DMor(C,C')$, and its heart $Mor_{st}(C,C')$ is simply the category of additive left-exact continuous functors $C \to C'$, just like in the $k$-linear situation. However, the whole category is not the derived category of its heart. The difference appears already in degree two, and includes Mac Lane cohomology classes responsible for non-linear deformations.

To complete the introduction, let us give a section-by-section overview of what we do throughout the paper. But before that, let us mention things that we do not do:

- We do not prove that our category $DMor(C,C')$ is indeed the category of stable enhanced functors $D(C) \to D(C')$ in the homotopically enhanced world. Indeed, doing this would require us to pick a model for this enhanced world, and none are too appealing. However, given what we do prove, it should be a trivial exercise to take this last step in any particular model.
- We consistently restrict ourselves to finitely presentable Abelian categories, such as inductive completions $Ind(C)$ of small Abelian categories, and do not explore weaker finiteness conditions. It should be possible to do something for more general Abelian categories, but for illustration purposes we stick to the simplest possible case.
- We do not explore at all the higher structures on Hochschild cohomology or its generalizations, which was the main content of [26]. We believe that there is a very interesting story to explore here, and that looking at things at the level of Abelian categories might clarify the general theory, but this should be the subject of further research.
- We do not touch deformation theory. We shall return to this elsewhere.

When one reads this list of omissions, one realizes that very little, possibly nothing in what remains is new. Thus, the paper should be treated as an overview, with the main goal of presenting and maybe reassembling known issues in a slightly different way, and highlighting the main ideas. The ideas themselves are definitely not new either, and not due to us (in particular, the idea of using sheaves is borrowed from the exposition of the Gabriel–Popescu Theorem in [4], and the importance of
dropping additivity is inspired by the seminal paper [13] and subsequent work of Pirashvili and others). All the proofs are presented for the sake of completeness and for the convenience of the reader, and much earlier alternative proofs are probably available in the literature. Having said that, we now give an overview of the overview.

**Overview of the paper.** Section 1 contains various preliminaries. It should not be understood as a self-contained introduction to category theory and homological algebra. Our goal is to fix notation, explain precisely the non-standard terminology that we use, and emphasize useful facts that are not usually emphasized (such as the fact that being additive is a condition on a category and not an additional structure). Subsection 1.1 is devoted to general category theory. Here the non-standard terms are *left-closed subcategories* and *left-pointed categories*. Subsection 1.2 is concerned with presentability and ind- and pro-completions in the spirit of [19]. Subsection 1.3 deals with Abelian categories, and §1.4 is devoted to derived ones. We assume known all the standard material which can be found in any textbook on homological algebra, but we do discuss less standard items such as homotopy limits and colimits, and the relationship between short exact sequences and bi-Cartesian squares (Remark 1.9), and then between distinguished triangles and homotopy bi-Cartesian squares (Example 1.11). The latter is of course inspired by the notion of stability in the homotopy enhanced world, but it is useful even in the unenhanced setting.

Section 2 is devoted to Grothendieck topologies. We give a very brief overview in §2.1 with some illuminating examples such as topologies on finite partially ordered sets. Among other things, we recall that for any small category $I$ equipped with a topology and any finitely presentable Abelian category $E$, the category $\text{Shv}(I,E)$ of $E$-valued sheaves on $I$ is Abelian, and the embedding $e: \text{Shv}(I,E) \rightarrow \text{Fun}(I^{op},E)$ is left-exact with an exact left-adjoint associated sheaf functor $a$. Then, in §2.2 we turn to a particular class of topologies, namely those generated by coverings formed by a single morphism. To axiomatize the situation we introduce the notion of a *covering class* $F$ and prove one general result on the existence of certain special coverings (Lemma 2.11).

In §3 we turn to hypercoverings. These are usually understood as augmented simplicial sets of a certain type, but, in fact, a large part of the theory exists in much greater generality (namely, with $\Delta$ replaced by a more general small category $I$), and proofs actually become easier when unencumbered by simplicial combinatorics. Thus we take the liberty of spending some time on hypercoverings of various types for various categories $I$. Subsection 3.1 deals with finite partially ordered sets, and §3.2 extends this further to a class of categories that includes both $\Delta$ and the category $\text{Pos}$ of finite partially ordered sets. Then, in §3.3 we return to the standard simplicial story. This is the only part of the paper where some of the results might be new.

Both §§2 and 3 are completely categorical. Homological algebra first appears in §4. We start by recalling basic facts about the Dold–Kan equivalence (we skip the proofs). We then introduce our main character, the single-epi topology on an Abelian category $A$ (in the terminology of §2.2 it corresponds to the covering class of epimorphisms). We show that, by virtue of the Dold–Kan equivalence,
hypercoverings in the single-epi topology can be identified with left resolutions in the sense of homological algebra. We then prove a general result, Proposition 4.8, which computes the derived functor $R^e$ of the embedding $e:\text{Shv}(I,\mathcal{E})\to\text{Fun}(I^o,\mathcal{E})$ in terms of hypercoverings in $I$ for any small category $I$ with a covering class $F$ and any finitely presentable Abelian category $\mathcal{E}$. The result is in fact standard, and it also holds for more general topologies (‘local cohomology can be computed by hypercoverings’), but the proof in our case is easy and the end result is nice: if $I = \mathcal{A}$ with the single-epi topology, then the homology objects of $R^e(\text{a}(E))$ for some functor $E: \mathcal{A}^o \to \mathcal{E}$ are simply the Dold-derived functors of $E$ since hypercoverings are left resolutions.

Having finished with the preliminaries, we then turn to the main subject of the paper, namely functor categories. In §5 we define the category of functors $\text{Mor}(\mathcal{C},\mathcal{C}')$ for any finitely presentable Abelian categories $\mathcal{C}, \mathcal{C}'$ and its derived category $\text{DMor}(\mathcal{C},\mathcal{C}')$, and we prove our main extension results, Theorem 5.5 and Corollary 5.6. These provide an action of the positive part $\text{DMor}^{>0}(\mathcal{C},\mathcal{C}')$ by functors $\mathcal{D}^{>0}(\mathcal{C}) \to \mathcal{D}^{>0}(\mathcal{C}')$. Then, in §6 we introduce the stability condition on functors in several equivalent forms given in Proposition 6.3, and show that the full subcategory $\text{DMor}^+(\mathcal{C},\mathcal{C}') \subset \text{DMor}(\mathcal{C},\mathcal{C}')$ spanned by stable objects bounded below acts by functors $\mathcal{D}^+(\mathcal{C}) \to \mathcal{D}^+(\mathcal{C}')$.

1. Generalities

1.1. Categories and functors. We denote by $\text{pt}$ the point category (one object, one morphism). For any category $I$, we denote by $I^o$ the opposite category. For any functor $\gamma: I \to \mathcal{E}$, we denote by $\gamma^o: I^o \to \mathcal{E}^o$ the opposite functor. For any object $E \in \mathcal{E}$, we denote by $E_I: I \to \mathcal{E}$ the constant functor with value $E$. Somewhat non-standardly, we shall say that a full subcategory $I' \subset I$ is left-closed if for any morphism $f: i' \to i$ in $I$ with $i \in I'$, we also have $i' \in I'$ and $I' \subset I$ is right-closed if $I'^o \subset I^o$ is left-closed. A category $I$ is connected if any two objects are connected by a chain of morphisms. For any object $e \in \mathcal{E}$, we denote by $I/e \subset I$ the left comma-fibre of the functor $\gamma$, that is, the category of pairs $\langle i, \alpha \rangle$, where $i \in I$ and $\alpha: \gamma(i) \to e$ is a morphism, we let $\sigma(e): I/\gamma e \to I$ be the forgetful functor, and we drop $\gamma$ from the notation when it is clear from the context (in particular, for any $i \in I$, we shorten $I/\sigma_i I$ to $I/i$). Dually, the right comma-fibre is $e \backslash I = (I^o/e)^o$, and the fibre $I_e$ is the full subcategory $I_e \subset I/e$ spanned by $\langle i, \alpha \rangle$ such that $\alpha$ is an isomorphism.

We treat a partially ordered set $J$ as a small category in the usual way (the objects are elements $j \in J$, and there is a single morphism from $j$ to $j'$ if and only if $j' \geq j$). A small category $I$ is equivalent to a partially ordered set if and only if there is at most one morphism between any two objects. Admitting some abuse of terminology, we shall simply say that $I$ is a partially ordered set. Note that in this case, for any $i \in I$, the comma-fibre $I/i$ is a left-closed full subcategory $I/i \subset I$ and, being full, it is also a partially ordered set.

As another non-standard bit of terminology, which will prove useful, we say that a category $I$ is left-pointed if it has an initial object $o \in I$ and $\{o\} \subset I$ is left-closed. Any category $I$ can be turned into a left-pointed one by formally adding an initial object $o$. We denote the resulting category by $I^\leq$. Any left-pointed category $I$ is of this type (namely, we have $I \cong (I \setminus \{o\})^\leq$). The product $I_0 \times I_1$ of left-pointed
categories $I_0$, $I_1$ is left-pointed, and for any two categories $I_0$, $I_1$, we define their extended product by

$$I_0 \ast I_1 = (I_0^< \times I_1^<) \setminus \{ o \times o \},$$

so that $I_0^< \times I_1^< \cong (I_0 \ast I_1)^<$. 

**Example 1.1.** The category pt*pt can be naturally identified with the opposite $\mathcal{V}$ to the partially ordered set $\mathcal{V} = \{0, 1\}^<$ with three elements $o$, 0, 1 and order relations $0, 1 \geq o$.

A functor between left-pointed categories is left-pointed if it sends $o$ to $o$, and for any categories $I_0$, $I_1$, a functor $\varphi: I_0 \to I_1^<$ canonically extends to a left-pointed functor $\varphi^<: I_0^< \to I_1^<$. An augmented functor from a category $I$ to some category $\mathcal{E}$ is a functor $E^<: I^< \to \mathcal{E}$. It is $e$-augmented for some object $e \in \mathcal{E}$ if $E^<(o) = e$, and an augmentation of a given functor $E: I \to \mathcal{E}$ is an augmented functor $E^<: I^< \to \mathcal{E}$ equipped with an isomorphism $E^<|_I \cong E$. Augmentations of a given functor $E$ form a category, which, by definition, has a terminal object $E^<$ if and only if the limit $\lim_I E$ exists and we have $E^<(o) \cong \lim_I E$. In this case we say that the augmentation is universal. Specifying an $e$-augmentation of a functor $E: I \to \mathcal{E}$ is equivalent to prescribing a map $e_I \to E$ from the constant functor with value $e$. Dually, we write $I^> = (I^<)^o$, and a coaugmented functor $I \to \mathcal{E}$ is a functor $I^> \to \mathcal{E}$. The universal coaugmentation is given by the colimit colim$_I$ (if it exists). A category $\mathcal{E}$ is complete (finitely complete) if $\lim_I E$ exists for any small (finite, respectively) category $I$ and functor $E: I \to \mathcal{E}$, and cocomplete (finitely cocomplete) if $\mathcal{E}^o$ is complete (finitely complete, respectively). For example, the category Sets of all sets is complete and cocomplete.

A retract of an object $i \in I$ in a category $I$ is an object $i' \in I$ equipped with maps $a: i' \to i$ and $b: i \to i'$ such that $b \circ a = \text{id}$. The composition $p = a \circ b: i \to i$ is then idempotent, $p^2 = p$, and $i'$ is the image of the idempotent endomorphism $p$. The image is unique if it exists, and is automatically preserved by any functor. A category is Karoubi-closed if every idempotent endomorphism of any object admits an image. If a category $I$ is Karoubi-closed, then so is the opposite category $I^o$.

A category that is complete or cocomplete is Karoubi-closed. For any small category $I$ and an arbitrary category $\mathcal{E}$, we let $\text{Fun}(I, \mathcal{E})$ be the category of functors $E: I \to \mathcal{E}$. For any functor $\gamma: I' \to I$ from a small $I'$, we let $\gamma^*E = E \circ \gamma \in \text{Fun}(I', \mathcal{E})$. We have $\text{Fun}(\text{pt}, \mathcal{E}) = \mathcal{E}$. For any $E_0, E_1: I \to \mathcal{E}$, we denote by $\text{Hom}_I(E_0, E_1)$ the set of maps from $E_0$ to $E_1$, and we drop the index $I$ when it is clear from the context. We also define a functor $\mathcal{H}\text{om}_I(E_0, E_1): I \to \text{Sets}$ by

$$\mathcal{H}\text{om}_I(E_0, E_1)(i) = \text{Hom}_{I\setminus I}(\sigma(i)^oE_0, \sigma(i)^oE_1),$$

and note that $\mathcal{H}\text{om}_I(E_0, E_1) = \lim_I \mathcal{H}\text{om}_I(E_0, E_1)$. For any $E \in \text{Fun}(I', \mathcal{E})$, the left Kan extension $\gamma_!E$ is a functor $\gamma_!E: I \to \mathcal{E}$ equipped with a map $E \to \gamma^*\gamma_!E$ satisfying the usual universal property. If $\text{colim}_{I'/I} E$ exists for any $i \in I$, then the left Kan extension $\gamma_!E$ also exists and is given by

$$\gamma_!E(i) = \text{colim}_{I'/I} E^i, \quad i \in I.$$

If this happens for any $E$ (for instance, if the target category $\mathcal{E}$ is cocomplete), then $\gamma_!: \text{Fun}(I', \mathcal{E}) \to \text{Fun}(I, \mathcal{E})$ is left-adjoint to $\gamma^*$. If $\gamma = \tau: I \to \text{pt}$ is the tautological
projection to a point, then \( \gamma = \text{colim}_I \) is the colimit itself. Dually, the right Kan extension \( f_* E \) is \((\gamma E)^o)\), and there is a dual version of (1.3) expressing \( \gamma_* E \) in terms of limits over right comma-fibres.

**Remark 1.2.** The left Kan extension \( \gamma E \) may exist even if some of the colimits in (1.3) do not. Namely, for any functor \( E': I \to \mathcal{E} \) and object \( i \in I \), a map \( E \to \gamma_* E' \) induces an \( E'(i) \)-coaugmentation of the functor \( \sigma(i)*E: I'/i \to \mathcal{E} \). We shall say that \( E' \) is a universal left Kan extension if all these coaugmentations are universal. Then a universal left Kan extension exists if and only if so do all the colimits in (1.3). A universal left Kan extension is, in particular, a left Kan extension. Under certain assumptions (for instance, if \( I = \text{pt} \) or if \( \mathcal{E} \) has arbitrary products), any left Kan extension is universal, but this is not true in general. For example, if \( \mathcal{E} \) is a discrete category with more than one object, then \( \text{Fun}(I, \mathcal{E}) \cong \mathcal{E} \) if and only if the small category \( I \) is connected, but since \( \mathcal{E} \) has no initial object, \( \text{colim}_I E \) never exists for a functor \( E: I \to \mathcal{E} \) from an empty category \( I \). Therefore, for any functor \( \gamma: I' \to I \) between connected small categories and any \( E: I' \to \mathcal{E} \), the left Kan extension \( \gamma E \) exists tautologically, but if at least one comma-fibre \( I'/i \) is empty, it is not universal.

**Example 1.3.** For any small category \( I \) and functor \( X: I^o \to \text{Sets} \), define the category of elements \( IX \) as the category of pairs \( \langle i, x \rangle \), where \( i \in I \) and \( x \in X(i) \), with morphisms \( \langle i, x \rangle \to \langle i', x' \rangle \) given by morphisms \( f: i \to i' \) such that \( f(x') = x \). We then have the forgetful functor \( \pi: IX \to I, \langle i, x \rangle \mapsto i \) and, for any functor \( E: I^o \to \mathcal{E} \) to a complete category \( \mathcal{E} \), we can define

\[
\text{Hom}(X, E) = \text{lim}_{IX^o} \pi^{o*} E.
\]  

Moreover, we can define a functor \( \text{Hom}(X, E): I^o \to \mathcal{E} \) by

\[
\text{Hom}(X, E) = \pi^{o*} \pi^{o*} E,
\]

and then \( \text{Hom}(E', \text{Hom}(X, E)) \cong \text{Hom}(X, \text{Hom}(E', E)) \) for any \( E' \in \text{Fun}(I^o, \mathcal{E}) \), where \( \text{Hom}(E', E) \) is given by (1.2). If \( \mathcal{E} = \text{Sets} \), then (1.4) simply computes the set of morphisms \( X \to E \), and (1.5) reduces to (1.2) by the dual version of (1.3).

For any integer \( n \geq 0 \), we denote by \( [n] \) the ordinal \( \{0, \ldots, n\} \) with the usual order. When needed, we treat it as a partially ordered set or a category. We have \( [n]^o \cong [n] \) and \( [n]^c \cong [n]^e \cong [n+1] \). For example, \( [0] = \text{pt} \) is a point, and \( [1] \) is the ‘single arrow category’ with two objects 0, 1 and a single non-trivial arrow 0 \( \to \) 1. Functors \( [1] \to \mathcal{E} \) to some \( \mathcal{E} \) correspond to arrows in \( \mathcal{E} \). Functors \( [2] \to \mathcal{E} \) correspond to composable pairs of arrows \( f, f' \). We have the embeddings \( s, t: [1] \to [2] \) onto the initial (respectively, terminal) segment of the ordinal \([2]\), and a functor \( [2] \to \mathcal{E} \) produces \( f \) (respectively, \( f' \)) by restriction via \( s \) (respectively, \( t \)). We also have the embedding \( m: [1] \to [2] \) onto \( \{0, 2\} \subset [2] \), and restricting via \( m \) produces the composition \( f' \circ f \). More generally, for any category \( I \) and two functors \( E_0, E_1: I \to \mathcal{E} \), specifying a map \( f: E_0 \to E_1 \) is equivalent to specifying a functor \( \iota(f): [1] \times I \to \mathcal{E} \) whose restriction to \( \{l\} \times I, l = 0, 1 \) is identified with \( E_l \). A composable pair of morphisms \( f, f' \) gives a functor \( \iota(f, f'): [2] \times I \to \mathcal{E} \) equipped with the isomorphisms

\[
(s \times \text{id})^* \iota(f, f') \cong \iota(f) \quad \text{and} \quad (t \times \text{id})^* \iota(f, f') \cong \iota(f'),
\]
and we have a canonical identification
\[ \iota(f' \circ f) \cong (m \times \text{id})^* \iota(f, f'). \]

Analogously, commutative squares in a category \( \mathcal{E} \) correspond to functors \([1]^2 \to \mathcal{E} \), where \([1]^2 = [1] \times [1] \) is the Cartesian square of the single-arrow category \([1] \). The single-arrow category \([1] \cong \text{pt} \leq \) is left-pointed, and \( \text{pt} \ast \text{pt} \cong [1]^2 \setminus \{0 \times 0\} \) is the partially ordered set \( V^0 \) from Example 1.1, so that a commutative square \( \gamma: [1]^2 \to \mathcal{E} \) defines an augmented functor from \( V^0 \) to \( \mathcal{E} \). The square is Cartesian if and only if the augmentation is universal (that is, \( \lim_{V^0} \gamma \) exists and the map \( \gamma(0 \times 0) \to \lim_{V^0} \gamma \) is an isomorphism). Dually, we have \([1]^o \cong [1] \), so that we also have \([1]^2 \cong V^> \), and a commutative square is a coaugmented functor from \( V \). The square is co-Cartesian if and only if the coaugmentation is universal.

### 1.2. Inductive completions.

Let us recall some more advanced results on limits and inductive completions that we shall need (a good recent general reference for this is [19], Chap. 6). For a connected small category \( I \) with the tautological projection \( \tau: I \to \text{pt} \), the functor \( \tau^*: \mathcal{E} \to \text{Fun}(I, \mathcal{E}) \) is fully faithful, so that, by adjunction, \( \text{colim}_I E_i \cong \lim_I E_i \cong E \) for any \( E \in \mathcal{E} \) (and both the limit and colimit exist). A functor \( \gamma: I' \to I \) is cofinal if \( i \setminus I' \) is connected for any \( i \in I \). In this case the dual version of (1.3) shows that \( \gamma_* E_{I'} \cong E_I \) with \( E \in \mathcal{E} \), and that for any \( E: I \to \mathcal{E} \), we have the adjunction isomorphism
\[ \text{colim}_{I'} \gamma^* E \cong \text{colim}_I E, \tag{1.6} \]
where both sides exist at the same time. A category that has an initial object is trivially connected, so that any functor that admits a left-adjoint is cofinal. A useful example of such a situation occurs when \( \gamma^o \) is a ‘Grothendieck fibration’ of \([11] \) (see, for instance, [17], §1.3, for a recent overview with the same notation as here). In this case the embedding \( I'_i \to I'/i \) admits a left-adjoint for any \( i \in I \), so that one can replace the left comma-fibres \( I'/i \) in (1.3) with the usual fibres \( I'_i \).

**Definition 1.4.** A non-empty category \( I \) is **directed** if

(i) for any two objects \( i, i' \in I \), there is an object \( i'' \in I \) with morphisms \( i \to i'' \), \( i' \to i'' \),

and **filtered** if, moreover,

(ii) for any two maps \( f, f': i \to i' \), there is a map \( g: i' \to i'' \) such that \( g \circ f = g \circ f' \).

A directed category is obviously connected, so that if every right comma-fibre \( i \setminus I' \) of a functor \( \gamma: I' \to I \) is non-empty and directed for any \( i \in I \), then \( \gamma \) is cofinal. If \( I' \) is filtered, then all these comma-fibres satisfy condition (ii) of Definition 1.4 automatically, so they are also filtered. If \( I \) is a partially ordered set, then again condition (ii) of Definition 1.4 is automatic, and \( I \) is filtered if and only if it is directed. A finitely cocomplete category \( I \) is trivially filtered. For any filtered \( I \), the colimit functor \( \text{colim}_I: \text{Fun}(I, \text{Sets}) \to \text{Sets} \) preserves finite limits, and this property is the main reason why the notion of a filtered category is useful.

**Remark 1.5.** It is a pleasant exercise to check that the converse is also true. Namely, if \( \text{colim}_I: \text{Fun}(I, \text{Sets}) \to \text{Sets} \) preserves finite limits, then \( I \) is filtered.

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For any category $C$, the objects of the inductive completion $\text{Ind}(C)$ are pairs $\langle I, c \rangle$ consisting of a small filtered category $I$ and a functor $c: I \to C$, and the morphisms are

$$\text{Hom}(\langle I, c \rangle, \langle I', c' \rangle) = \lim_{i \in I} \text{colim}_{i' \in I'} \text{Hom}(c(i), c'(i')).$$

We have the tautological full embedding $\iota: C \to \text{Ind}(C)$, $c \mapsto \langle \text{pt}, c \rangle$. The category $\text{Ind}(C)$ has filtered colimits and is universal among categories with this property. Namely, we say that a functor is continuous if it preserves filtered colimits. Then, for any target category $E$ that has filtered colimits and for any functor $E: C \to E$, the left Kan extension $\iota_! E: \text{Ind}(C) \to E$ exists, it is continuous, and is the unique (up to a unique isomorphism) continuous extension of $E$ to $\text{Ind}(C)$. For any $C \in \text{Ind}(C)$ represented by a pair $\langle I, c \rangle$, the natural projection $I \to C/I$ is cofinal, so that $\iota_! (E)(C) \cong \text{colim}_{i \in I} E(c(i))$ by (1.3) and (1.6). Dually, the projective completion $\text{Pro}(C)$ is given by $\text{Pro}(C) = \text{Ind}(C^\circ)$. Objects in $\text{Ind}(C)$ ($\text{Pro}(C)$) are also called ind-objects (pro-objects, respectively) in $C$. If $C = I$ is a small category, then we have the Yoneda full embedding

$$Y: I \to \text{Fun}(I^\circ, \text{Sets}), \quad Y(i)(i') = \text{Hom}(i', i),$$

and $\iota_! Y: \text{Ind}(I) \to \text{Fun}(I^\circ, \text{Sets})$ is also a full embedding that identifies $\text{Ind}(I)$ with the full subcategory in $\text{Fun}(I^\circ, \text{Sets})$ spanned by filtered colimits of representable functors. Note that the Yoneda embedding (1.7), hence also the embedding $I \to \text{Ind}(I)$, reflects monomorphisms (that is, a map $f$ is a monomorphism in $I$ if and only if it is a monomorphism in $\text{Ind}(I)$).

An object $c \in C$ in a category $C$ is finitely presentable or compact if the corepresentable functor $\text{Hom}(c, -)$ preserves filtered colimits. Let $C_c \subset C$ be the full subcategory spanned by compact objects in a cocomplete category $C$. Then, for any full subcategory $I \subset C_c$, the full embedding $I \to C_c \to C$ canonically extends to a fully faithful functor

$$\text{Ind}(I) \to \text{Ind}(C_c) \to C.$$  \hspace{1cm} (1.8)

A category $C$ is finitely presentable if it is cocomplete and there is a small full subcategory $I \subset C_c$ such that the functor (1.8) is essentially surjective. Since the functor is fully faithful, it is then automatically an equivalence. Moreover, the functor $Y_I: C \to \text{Fun}(I^\circ, \text{Sets})$ induced by the Yoneda embedding (1.7) is a full embedding that preserves limits and filtered colimits. In particular, filtered colimits in $C$ commute with finite limits, just as in the case when $C = \text{Sets}$. For any two finitely presentable categories $C$ and $C'$, continuous functors from $C$ to $C'$ form a well-defined category $\text{Fun}_c(C, C')$, and we have

$$\text{Fun}_c(C, C') \cong \text{Fun}(I, C'),$$

where $I \subset C_c \subset C$ is a small subcategory such that $C \cong \text{Ind}(I)$.

**Example 1.6.** Any colimit in a category $C$ can be represented as a filtered colimit of finite colimits. Therefore, for any small finitely cocomplete category $I$, the ind-completion $\text{Ind}(I)$ is finitely presentable. The Yoneda embedding $Y_I$ induced by (1.7) identifies $\text{Ind}(I)$ with the full subcategory $\text{Fun}_{ex}(I^\circ, \text{Sets}) \subset \text{Fun}(I^\circ, \text{Sets})$ of
functors \( X : I^o \to \text{Sets} \) that preserve finite limits. Indeed, \( \text{Fun}_{\text{ex}}(I^o, \text{Sets}) \) contains the representable functors \( Y(i) \) and is closed under filtered colimits, so that \( \text{Ind}(I) \subset \text{Fun}_{\text{ex}}(I^o, \text{Sets}) \). On the other hand, for any \( X \in \text{Fun}_{\text{ex}}(I^o, \text{Sets}) \), the category of elements \( IX \) from Example 1.3 is finitely cocomplete, thus filtered, so that \( X \cong \text{colim}_{(i,x) \in IX} Y(i) \) is in \( \text{Ind}(I) \).

**Example 1.7.** The situation of Example 1.6 is in fact general. Namely, for any small category \( I \), an object \( c \in \text{Ind}(I) \) is compact if and only if it is a retract of an object \( i \in I \subset \text{Ind}(I) \). (By definition, \( c \cong \text{colim}_J i \) for some filtered \( J \) and functor \( i : J \to I \), and if \( c \) is compact, the isomorphism \( c \to \text{colim}_J i \) must factor through \( i(j) \in I \) for some \( j \in J \).) Therefore, for any finitely presentable category \( C \cong \text{Ind}(I) \) with small \( I \), the subcategory \( C_c \subset C \) is essentially small, and then we also have \( C \cong \text{Ind}(C_c) \cong \text{Fun}_{\text{ex}}(C_c^o, \text{Sets}) \) (in particular, \( C \) is automatically complete). Since filtered colimits of sets commute with finite limits, \( C_c \subset C \) is closed under finite colimits, thus finitely cocomplete.

### 1.3. Abelian categories

A category \( C \) is **pointed** if it has an initial object \( 0 \) and a terminal object \( 1 \), and the unique map \( 0 \to 1 \) is an isomorphism (so that \( 0 \) is both an initial and a terminal object, unique up to a unique isomorphism). For any two objects \( A, B \in C \) in a pointed category \( C \), we have a unique map \( A \to B \) that factors through \( 0 \), so that the Hom-sets \( \text{Hom}(-,-) \) are naturally pointed. If a pointed category \( C \) admits finite products and coproducts, then we have a natural map

\[
A \sqcup B \xrightarrow{(\text{id} \times 0) \cup (0 \times \text{id})} A \times B \tag{1.10}
\]

for any \( A, B \in C \). We say that a category is **preadditive** if it is pointed, has finite products and coproducts, and all the maps (1.10) are isomorphisms. Thus we have \( A \sqcup B \cong A \times B \) canonically. This object is denoted by \( A \oplus B \) and called the **sum** of \( A \) and \( B \) (all coproducts that exist in \( C \) are then also called ‘sums’). For any preadditive category, Hom-sets carry a commutative monoid structure with \( 0 \) as the unity element, and compositions are monoid maps. A category is **additive** if it is preadditive, and the monoids \( \text{Hom}(A,B), A,B \in C \), are Abelian groups (that is, admit inverses). The category \( \text{Ab} \) of all Abelian groups is additive, and any additive category is automatically enriched over \( \text{Ab} \) (that is, compositions are compatible with the Abelian group structures on Hom-sets). The original reference for the notion of an additive category is [10]. However, it is useful to remember that the Ab-enrichment required in [10], §1.3, is actually automatic and unique, so that being additive is a condition on a category and not an additional structure.

We also note that the property of being additive is self-dual, that is, the opposite \( C^o \) to an additive category \( C \) is also additive.

**Remark 1.8.** The term ‘preadditive’ used above is non-standard (and sometimes appears in the literature with a different meaning). It seems that there is no standard term.

The notion of an Abelian category also goes back to [10], and it is also self-dual. Namely, for any map \( f : A \to B \) in a pointed category \( A \), the **kernel** is given by \( \text{Ker} f = A \times_B 0 \) and, dually, the **cokernel** is \( \text{Coker} f = (\text{Ker} f^o)^o \). Both need not exist in general. An additive category is Abelian if it has all kernels and cokernels.
What do Abelian categories form?

(‘axiom AB1’), and for any morphism $f: A \to B$ with kernel $k: \text{Ker } f \to A$ and cokernel $c: B \to \text{Coker } f$, the natural map $\text{Coker } k \to \text{Ker } c$ is an isomorphism (AB2). An additive category satisfying AB1 is finitely complete and cocomplete, and if it satisfies AB2, it is also Karoubi-closed. An Abelian subcategory $A' \subset A$ in an Abelian category $A$ is a full subcategory closed under finite sums, kernels and cokernels (so that, in particular, $A'$ is also Abelian). Alternatively, a short exact sequence in a pointed category $A$ is a sequence

$$A \xrightarrow{i} B \xrightarrow{p} C$$

(1.11)
such that $p \circ i = 0$, $\text{Ker } p$ and $\text{Coker } i$ exist, and the maps $A \to \text{Ker } p$ and $\text{Coker } i \to C$ are isomorphisms. Then, another way to phrase AB2 is to say that every map $f: A \to B$ admits a decomposition

$$C_0 \to A \to C_1 \to B \to C_2$$

(1.12)
such that $C_0 \to A \to C_1$ and $C_1 \to B \to C_2$ are short exact sequences. (Note that such a decomposition is necessarily unique.)

**Remark 1.9.** The following repackaging of the notion of a short exact sequence is sometimes useful. Specifying a sequence (1.11) with $p \circ i = 0$ is equivalent to specifying a commutative square

$$\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow & & \downarrow p \\
0 & \rightarrow & C
\end{array}$$

(1.13)

Then a sequence (1.11) is exact on the left (on the right) if and only if the corresponding square (1.13) is Cartesian, that is, $\text{Ker } p$ exists and $A \to \text{Ker } p$ is an isomorphism (respectively, it is co-Cartesian, that is, $\text{Coker } i$ exists and $\text{Coker } i \to C$ is an isomorphism). The sequence is exact if the square is bi-Cartesian, that is, Cartesian and co-Cartesian at the same time.

Grothendieck lists further conditions of increasing strength that one can impose on an Abelian category $C$: it can have arbitrary coproducts (AB3), coproducts of short exact sequences can be exact (AB4), and the same can hold for filtered colimits (or equivalently, filtered colimits can commute with finite limits; this is AB5). There is also a further property AB6 whose real importance has begun to emerge only recently, so we skip it. The additional properties are not self-dual. An Abelian category $C$ is said to satisfy $ABN^*, N = 3, 4, 5, 6$, if $C^0$ satisfies $ABN$. The category $\text{Ab}$ satisfies $AB5$ and $AB4^*$. (There is a theorem that an Abelian category satisfying $AB5$ and $AB5^*$ is trivial.)

A generator of an Abelian category $C$ is an object $U \in C$ such that $\text{Hom}(U, -)$ is faithful (equivalently, for any morphism $f: M \to M'$ in $C$, $\text{Hom}(U, f) = 0$ implies $f = 0$). A Grothendieck Abelian category is an Abelian category $C$ satisfying $AB5$ that admits a generator. One of the main results of [10] is that a Grothendieck Abelian category has enough injectives (that is, any $A \in C$ admits a monomorphism $A \to I$ with injective $I$).
One usually defines additive functors for additive categories, but it is useful to do it slightly more generally. We say that a functor $E : \mathcal{A} \to \mathcal{B}$ between categories with finite products is additive if it commutes with finite products (that is, for any $A, B \in \mathcal{A}$, the natural map $E(A \times B) \to E(A) \times E(B)$ is an isomorphism). This is again a condition and not a structure. However, if $\mathcal{A}$ and $\mathcal{B}$ are additive, then an additive functor $E$ is automatically enriched over $\text{Ab}$. On the other hand, if $\mathcal{A}$ is additive and $\mathcal{B} = \text{Sets}$, then $E$ automatically and uniquely factors through the forgetful functor $\text{Ab} \to \text{Sets}$ (more precisely, the forgetful functor $\text{Fun}(\mathcal{A}, \text{Ab}) \to \text{Fun}(\mathcal{A}, \text{Sets})$ induces an equivalence between the full subcategories spanned by additive functors). In particular, since filtered colimits of sets commute with finite products, all objects $E \in \text{Ind}(\mathcal{A}) \subset \text{Fun}(\mathcal{A}^{\circ}, \text{Sets})$ in the inductive completion of a small additive category $\mathcal{A}$ are additive, $\text{Ind}(\mathcal{A})$ is an additive category, and the full embedding $\text{Ind}(\mathcal{A}) \subset \text{Fun}(\mathcal{A}^{\circ}, \text{Sets})$ factors through a full embedding $\text{Ind}(\mathcal{A}) \subset \text{Fun}(\mathcal{A}^{\circ}, \text{Ab})$.

A functor between Abelian categories is left-exact (right-exact) if it commutes with finite limits (finite colimits, respectively). In particular, a left- or right-exact functor is automatically additive. Alternatively, an additive functor is left-exact (right-exact) if and only if it sends short exact sequences to sequences exact on the left (exact on the right, respectively). The simplest way to see that the two notions are equivalent is to use the description of short exact sequences in terms of squares (1.13). A functor is exact if it is both left- and right-exact.

If a small category $\mathcal{A}$ is Abelian, then $\text{Ind}(\mathcal{A})$ is finitely presentable by Example 1.6, and it is a Grothendieck Abelian category (this is well known but rather non-trivial, see Example 5.3 below). Conversely, any finitely presentable Abelian category $\mathcal{E}$ satisfies AB5 almost by definition (filtered colimits commute with finite limits and colimits), and for any small full subcategory $I \subset \mathcal{E}_{c} \subset \mathcal{E}$ such that $\mathcal{E} \cong \text{Ind}(I)$, the coproduct of all objects $i \in I \subset \mathcal{E}$ is a generator, so that $\mathcal{E}$ is a Grothendieck Abelian category. The full subcategory $\mathcal{E}_{c} \subset \mathcal{E}$ of compact objects is additive and, by Example 1.7, it is essentially small and has cokernels (and all finite colimits). It is also closed under extensions and thus is exact in the sense of [24], §2.

**Definition 1.10.** A small additive category $\mathcal{A}$ with cokernels is pre-Abelian if $\text{Ind}(\mathcal{A})$ is Abelian.

Any small Abelian category is pre-Abelian, but the converse is not true, so the notion is not vacuous. For example, the category $R$-mod of left modules over a ring $R$ is Abelian, $M \in R$-mod is compact if and only if it is the cokernel of a map $f : R^{m} \to R^{n}$ for some integers $m, n \geq 0$, so that $R$-mod is obviously finitely presentable, but $(R$-mod)$_{c}$ is Abelian only if the ring $R$ is left-coherent.

As in Remark 1.8, the term ‘pre-Abelian’ is non-standard, but it seems that there is no standard term. An abstract characterization of pre-Abelian categories can be found in [25] (under the name ‘ind-Abelian’). By Example 1.7, a pre-Abelian category $\mathcal{A}$ is of the form $\mathcal{E}_{c}$ for an Abelian finitely presentable $\mathcal{E}$ if and only if $\mathcal{A}$ is Karoubi-closed.

**1.4. Derived categories.** We denote by $C^{\bullet}(\mathcal{E})$ the category of chain complexes $M^{\bullet} = \langle M^{\bullet}, d \rangle$ in an additive category $\mathcal{E}$, and we let $C_{\ast}(\mathcal{E})$ be the same category but with complexes indexed by homological rather than cohomological degrees, with
the convention that $M^i = M_{-i}$. The homological shift $M^*[n]$ of a complex $M^*$ by an integer $n$ is given by $(M^*[n])^i = M^{i+n}$. The cone of a morphism $f : M^* \to N^*$ in $C^*(E)$ is given by $C(f)^i = N^i \oplus M^{i+1}$, with the usual upper-triangular differential. If $E$ is Abelian, we assume known the standard notions of a homology object, an acyclic complex, a quasi-isomorphism, and so on. We denote by $C^+(E), C^-(E) \subset C^*(E)$ the full subcategories spanned by complexes bounded below, that is, $M^i = 0$ for $i \ll 0$ (respectively, bounded above, that is, $M^i = 0$ for $i \gg 0$), and we let $C^*_E(E) = C^+(E) \cap C^-(E) \subset C^*(E)$ be the full subcategory of bounded complexes. If $E$ is finitely presentable, with the subcategory $E_c \subset E$ of compact objects, then $C^*(E)$ is also finitely presentable, and $C^*_b(E_c) \subset C^*(E)$ is the full subcategory of compact objects. In particular, for a small Abelian category $A$, we have $C^*(\text{Ind}(A)) \cong \text{Ind}(C^*_b(A))$.

The localization $h(C, W)$ of a category $C$ with respect to a class of morphisms $W$ is a category $h(C, W)$ equipped with a functor $h : C \to h(C, W)$ that inverts all morphisms in $W$ and is universal with this property: any functor $C \to E$ to some $E$ that inverts all morphisms in $W$ factors through $h$ uniquely up to a unique isomorphism. For any small category $I$, we denote by $W^I$ the class of morphisms in the functor category $\text{Fun}(I, C)$ that are pointwise in $W$. If both localizations $h(C, W)$ and $h(\text{Fun}(I, C), W^I)$ exist, we have the tautological functor

$$h(C, W) \to h(\text{Fun}(I, C), W^I), \quad c \mapsto c_I,$$

and the homotopy limit and homotopy colimit are by definition its left- and right-adjoint functors

$$\text{hocolim}_I, \text{holim}_I : h(\text{Fun}(I, C), W^I) \to h(C, W) \quad (1.14)$$

if they exist. If $\text{holim}_I$ exists, then an augmented functor $c : I^\prec \to C$ gives rise to a comparison map $c(o) \to \text{holim}_I c$, and we say that the augmentation is homotopy universal if the map is an isomorphism in $h(C, W)$. In particular, a commutative square $[1]^2 \to C$ is homotopy Cartesian if it is homotopy universal when considered as an augmented functor (just as in the non-homotopical case). Dually, if $\text{hocolim}_I$ exists, we have the notion of a homotopy universal coaugmentation and that of a homotopy co-Cartesian square.

Localization does not always exist (if $C$ is large, there could be set-theoretical issues) and is notoriously difficult to construct explicitly and to describe. (In particular, even if relevant localizations exist, constructing homotopy limits and colimits is a highly non-trivial task.) One additional structure that helps to control localization is that of Quillen’s model category; see, for example, [23], [8], and [12] (although [12] has to be used carefully since the author takes the liberty of redefining standard notions according to his needs). Since we shall only use it tangentially (for example, in Lemma 4.4 below), we do not give any details. Let us just mention that if $C$ is a model category and $W$ is the class of weak equivalences, then by [7], $h(\text{Fun}(I, C), W^I)$ exists for any finite $I$, and so do the homotopy limit and colimit functors of (1.14). In particular, homotopy Cartesian and homotopy co-Cartesian squares are well-defined in any model category $C$.

In the Abelian context, the most common example of a localization is the category $C^*(E)$ of chain complexes in an Abelian category $C$ whose localization with
respect to the class of quasi-isomorphisms produces the derived category $\mathcal{D}(\mathcal{C})$. If $\mathcal{C}$ has enough injectives, this localization can be constructed by model category techniques, but there is a simpler and earlier alternative, which works under much milder restrictions: one first constructs the homotopy category $\text{Ho}(\mathcal{C})$ of chain complexes and chain-homotopy classes of maps between them, and then applies a general localization theorem of Verdier [28] (this relies on the structure of a triangulated category on $\text{Ho}(\mathcal{C})$, which was introduced by Verdier specifically for this purpose). We refer the reader to any standard textbook on homological algebra such as [29] or [9] for basic facts on derived categories. In particular, we assume known the fact that the derived category $\mathcal{D}(\mathcal{C})$ is additive, and the dual notions of a total right-derived functor $R^E: \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{E})$ (left-derived functor $L^E$) of a left-exact (right-exact, respectively) functor $E: \mathcal{C} \to \mathcal{E}$ between Abelian categories.

For any Abelian category $\mathcal{C}$ and small category $I$, the functor category $\text{Fun}(I, \mathcal{C})$ is Abelian, and we simplify notation by writing $\mathcal{D}(I, \mathcal{C}) = \mathcal{D}(\text{Fun}(I, \mathcal{C}))$. If the Abelian category $\mathcal{C}$ is finitely presentable, then for any finitely presentable $I$, the continuous functor category $\text{Fun}_c(I, \mathcal{C})$ is also Abelian by (1.9), and we write $\mathcal{D}_c(I, \mathcal{C}) = \mathcal{D}(\text{Fun}_c(I, \mathcal{C})) \cong \mathcal{D}(I_c, \mathcal{C})$. Alternatively, $\mathcal{D}(I, \mathcal{C})$ for a small $I$ and an Abelian $\mathcal{C}$ can by obtained by localizing the functor category $\text{Fun}(I, \mathcal{C}^\ast(\mathcal{C})) \cong C^\ast(\text{Fun}(I, \mathcal{C}))$ with respect to the class of pointwise quasi-isomorphisms, and if $I$ is finite, the homotopy limit and colimit (1.14) both exists and are given by $\text{hocolim}_I = L^c \text{colim}_I$ and $\text{holim}_I = R^c \text{lim}_I$. An object $E \in \mathcal{D}(I, \mathcal{C})$ tautologically defines a functor $\mathcal{D}(E): I \to \mathcal{D}(\mathcal{C})$, so that we have a comparison functor

$$\mathcal{D}: \mathcal{D}(I, \mathcal{C}) \to \text{Fun}(I, \mathcal{D}(\mathcal{C})).$$

(1.15)

This functor is not an equivalence unless $I = \text{pt}$.

Example 1.11. Take $I = [1]$, the single-arrow category. For any Abelian category $\mathcal{C}$, the objects in $\text{Fun}([1], \mathcal{C})$ are arrows in $\mathcal{C}$, and taking the cokernel of an arrow provides a right-exact functor $\text{Coker}: \text{Fun}([1], \mathcal{C}) \to \mathcal{C}$ with derived functor $L^c \text{Coker}: \mathcal{D}([1], \mathcal{C}) \to \mathcal{D}(\mathcal{C})$. For any $E \in \mathcal{D}([1], \mathcal{C})$, the functor $\mathcal{D}(E): [1] \to \mathcal{D}(\mathcal{C})$ is an arrow in the derived category $\mathcal{D}(\mathcal{C})$, and $L^c \text{Coker}(E)$ gives its cone in the sense of the triangulated structure on $\mathcal{D}(\mathcal{C})$. However, this version of the cone is functorial. The necessary rigidity is added exactly by lifting $\mathcal{D}(E)$ to an object $E \in \mathcal{D}([1], \mathcal{C})$. Analogously, distinguished triangles in $\mathcal{D}(\mathcal{C})$ can be naturally rigidified by considering squares (1.13) in $\mathcal{D}([1]^2, \mathcal{C})$ that are homotopy bi-Cartesian. Every such square produces a distinguished triangle after applying (1.15) and, conversely, any distinguished triangle lifts to such a square. The lifting is unique but only up to a non-unique isomorphism.

One can also consider the category $C_{\geq 0}(\mathcal{C}) = C^{\leq 0} \subset C_{\ast}(\mathcal{C}) = C^\ast(\mathcal{C})$ of complexes concentrated in non-negative homological (that is, non-positive cohomological) degrees. Its localization produces the full subcategory $D^{\leq 0}(\mathcal{C}) \subset \mathcal{D}(\mathcal{C})$ of connective objects, a part of a standard $t$-structure on $\mathcal{D}(\mathcal{C})$. Dually, localizing the category $C_{\geq 0}(\mathcal{C})$ of complexes concentrated in non-negative cohomological degrees produces the full subcategory $D^{\geq 0}(\mathcal{C}) \subset \mathcal{D}(\mathcal{C})$ of coconnective objects, which is another part of the standard $t$-structure. For general facts on $t$-structures, see [2].
Let us just recall that the embedding $\mathcal{D}^{\leq 0}(\mathcal{C}) \subset \mathcal{D}(\mathcal{C})$ admits a right-adjoint canonical truncation functor $\tau^{\leq 0}: \mathcal{D}(\mathcal{C}) \to \mathcal{D}^{\leq 0}(\mathcal{C})$ with $\mathcal{D}^{\leq 0}(\mathcal{C}) \cap \mathcal{D}^{\geq 0}(\mathcal{C}) \cong \mathcal{C}$, so that $\tau^{\geq 0}$ induces a functor $\mathcal{D}^{\geq 0}(\mathcal{C}) \to \mathcal{C}$.

2. Topologies and coverings

2.1. Revision on Grothendieck topologies. The original reference for Grothendieck topologies and topos theory is [1], but a very concise and useful overview can be found in [14], §0.3. Let us recall the basic issues.

By definition, a sieve on an object $i \in I$ in a small category $I$ is a subfunctor in the representable functor $\mathcal{Y}(i) = \text{Hom}(-, i): I^o \to \text{Sets}$. The collection of all sieves on an object $i$ is denoted by $\Omega(i)$ and, for any map $f: i' \to i$ and sieve $s \in \Omega(i)$,

$$f^*s = s \times_{\mathcal{Y}(i)} \mathcal{Y}(i')$$

is a sieve on $i'$, so that $\Omega$ is itself a contravariant functor $I^o \to \text{Sets}$. A Grothendieck topology on $I$ is given by collections of sieves $T(i)$, one for each object $i \in I$, satisfying the following axioms:

(i) for any $i \in I$, the maximal sieve $\mathcal{Y}(i)$ is in $T(i)$,

(ii) for any morphism $f: i \to i'$ and $s \in T(i')$, we have $f^*s \in T(i)$ (in other words, $T \subset \Omega$ is a subfunctor), and

(iii) if for some $s \in \Omega(i)$ and $s' \in T(i)$ we have $f^*s \in T(i')$ for any $f: i' \to i$ in $s'(i') \subset \text{Hom}(i', i)$, then $s \in T(i)$.

For any $i \in I$, the set $\Omega(i)$ is partially ordered with respect to the inclusion, and the axioms (i)–(iii) imply that $T(i) \subset \Omega(i)$ is right-closed and closed under intersections, so that $T(i)^o$ is a directed partially ordered set.

A functor $E: I^o \to \text{Sets}$ is a separable presheaf (sheaf) with respect to a topology $T$ if the map $E(i) \to \text{Hom}(s, E)$ is injective (bijective, respectively) for any $i \in I$ and $s \in T(i)$. If we compute $\text{Hom}(s, E)$ by (1.4), then these conditions make sense for a functor $E: I^o \to \mathcal{E}$ to any complete target category $\mathcal{E}$, so that the notion of being a sheaf or a separable presheaf is also defined for $\mathcal{E}$-valued functors. Explicitly, the category of elements $\mathcal{I}s$ of Example 1.3 is equivalent to the full subcategory $I/s_i \subset I/i$ spanned by arrows $f \in s(i) \subset \text{Hom}(i', i)$, and we have

$$\text{Hom}(s, E) = \lim_{\gamma \in (I/s_i)^o} E(i').$$

We denote by $\text{Shv}(I, \mathcal{E}) \subset \text{Fun}(I^o, \mathcal{E})$ the full subcategory spanned by sheaves. Then, if the target category $\mathcal{E}$ is finitely presentable (thus, in particular, complete and cocomplete), the embedding $\text{Shv}(I, \mathcal{E}) \to \text{Fun}(I^o, \mathcal{E})$ admits a left-adjoint associated sheaf functor $a: \text{Fun}(I^o, \mathcal{E}) \to \text{Shv}(I, \mathcal{E})$. To construct it we define a functor $a_0: \text{Fun}(I^o, \mathcal{E}) \to \text{Fun}(I^o, \mathcal{E})$ by

$$a_0(E)(i) = \text{colim}_{s \in T(i)^o} \text{Hom}(s, E), \quad i \in I, \quad E \in \text{Fun}(I^o, \mathcal{E}).$$

This is functorial in $i$ since so is $T(i)$. (Slightly more precisely, the partially ordered sets $T(i)$, $i \in I$, fit together into a Grothendieck fibration $\gamma: T \to I$ whose fibres are partially ordered sets $T(i)$ with the inclusion order, and $a_0(E) = \gamma^o_0 \text{Hom}(-, E)$ is the left Kan extension along the opposite functor $\gamma^o: T^o \to I^o$.) Then the natural maps $E(i) \to \text{Hom}(s, E)$ provide a functorial map $E \to a_0(E)$, and one can check
that \( a_0(E) \) is a separated presheaf for any \( E \), and a sheaf if \( E \) is separated. (For \( \mathcal{E} = \text{Sets} \), this is Proposition 3.2 in [27], and the general case reduces to this by the Yoneda embedding (1.7).) Therefore \( a_0^2(E) = a_0(a_0(E)) \) is a sheaf for any \( E \), and we have an isomorphism \( a_0^2 \cong e \circ a \) for a unique functor \( a : \text{Fun}(\mathcal{I}, \mathcal{E}) \to \text{Shv}(\mathcal{I}, \mathcal{E}) \), while \( E \to a_0(E) \) provides the adjunction map \( E \to e(a(E)) \). By adjunction, \( a \) commutes with arbitrary colimits, and since the colimits in (2.2) are filtered, it also commutes with finite limits. A functor \( E : \mathcal{I}^o \to \mathcal{E} \) is a sheaf if and only if the adjunction map \( f : E \to e(a(E)) \) is an isomorphism, and in fact it suffices to require that it admits a splitting \( g : e(a(E)) \to E \), \( g \circ f = \text{id} \) (because then \( a(g) \circ a(f) = \text{id} \), and since \( a(f) \) is invertible, \( a(g) \circ a(f) = \text{id} : a(E) \to a(E) \), so that \( g \circ f = \text{id} \) since \( e \) is fully faithful).

**Example 2.1.** If the target category \( \mathcal{E} \) is Abelian, then \( \text{Shv}(\mathcal{I}, \mathcal{E}) \) is also Abelian, with kernels taken pointwise and cokernels created by the associated sheaf functor \( a \). (That is, for any morphism \( f : E_0 \to E_1 \) between \( E_0, E_1 \in \text{Fun}(\mathcal{I}^o, \mathcal{E}) \) that are actually sheaves, \( \text{Ker} f \) is a sheaf, and \( a(\text{Coker} f) \) is a cokernel in \( \text{Shv}(\mathcal{I}, \mathcal{E}) \).) To check AB2 note that \( a \) preserves cokernels by definition, but also commutes with finite limits, hence preserves kernels and short exact sequences (1.11). Then a decomposition (1.12) in \( \text{Shv}(\mathcal{I}, \mathcal{E}) \) can be obtained by applying \( a \) to the corresponding decomposition in \( \text{Fun}(\mathcal{I}^o, \mathcal{E}) \). Moreover, \( \text{Shv}(\mathcal{I}, \mathcal{E}) \) is a Grothendieck Abelian category that satisfies AB3*. (We recall that \( \mathcal{E} \) is finitely presentable by our standing assumption to ensure the existence of \( a \).) Indeed, products in \( \text{Shv}(\mathcal{I}, \mathcal{E}) \) are products in \( \text{Fun}(\mathcal{I}^o, \mathcal{E}) \), and \( a \) commutes with colimits and finite limits, so that AB5 and AB3* are inherited from \( \text{Fun}(\mathcal{I}^o, \mathcal{E}) \). To obtain a generator it suffices to take the sum of all objects \( a(\mathcal{Y}(E)), i \in \mathcal{I} \), where \( E \) is a fixed generator of \( \mathcal{E} \) and \( \mathcal{Y}(E) \) is the corepresentable functor given by

\[
\mathcal{Y}(E)(i') = E[\text{Hom}(i', i)], \quad i' \in \mathcal{I},
\]

(2.3)

where the right-hand side is shorthand for the ‘sum of copies of \( E \) numbered by elements in the set \( \text{Hom}(i', i) \).’ If \( \mathcal{I} \) has finite coproducts, one can also consider the full subcategory \( \text{Shv}_{\text{add}}(\mathcal{I}, \mathcal{E}) \) spanned by sheaves that are additive. Since filtered colimits in \( \mathcal{E} \) commute with finite products, the functor \( a \) preserves additivity and \( \text{Shv}_{\text{add}}(\mathcal{I}, \mathcal{E}) \subset \text{Shv}(\mathcal{I}, \mathcal{E}) \) is an Abelian subcategory.

**Remark 2.2.** In fact, *any* Grothendieck Abelian category satisfies AB3*, but this is a rather non-trivial theorem. For the categories \( \text{Shv}(\mathcal{I}, \mathcal{E}) \) of Example 2.1, the claim is obvious.

**Example 2.3.** For any small \( I \), the minimal topology \( T_{\text{min}} \) consists of the maximal sieves \( \mathcal{Y}(i), \ i \in I \). We have \( T_{\text{min}} \cong \text{pt}_I \), the functor sending everything to the one-point set \( \text{pt} \), and the corresponding embedding

\[
1 : \text{pt}_I \to \Omega
\]

(2.4)

is given by \( 1(i)(\text{pt}) = \mathcal{Y}(i) \in \Omega(i), \ i \in I \). (Note that \( \text{pt}_I \) is the terminal object in \( \text{Fun}(\mathcal{I}^o, \text{Sets}) \).) Sheaves for the minimal topology are all functors \( \mathcal{I}^o \to \mathcal{E} \). The maximal topology \( T_{\text{max}} \) consists of all sieves including the empty one, \( T_{\text{max}} \cong \Omega \), but it is not very interesting since the only sheaf in this topology is the constant functor \( \mathcal{I}^o \to \mathcal{E} \) sending everything to the terminal object in \( \mathcal{E} \). However, if \( I \) is left-pointed,
then all non-empty sieves also form a topology, which we call submaximal. For this topology, we have Shv(I^o, E) ∼= E with the equivalence given by evaluation at the initial object o ∈ I.

**Example 2.4.** Let J be a partially ordered set. Then prescribing a sieve on an object j ∈ J is the same as prescribing a left-closed subset in the comma-set J/j ⊂ J. For any subset J′ ⊂ J and element j ∈ J, let T_{J'}(j) ⊂ Ω(j) consist of sieves J_0 ⊂ J/j that contain J′ ∩ (J/j). Then T_{J'} is a Grothendieck topology on J, and if J is finite, then every Grothendieck topology is of this form. (To recover J′ from a topology T we take the subset J′ ⊂ J of elements j ∈ J such that T(j) ⊂ Ω(j) consists of the maximal sieve Y(i).) We have Shv(J, E) ∼= Fun(J^o, E) for any finitely presentable E, and the associated sheaf functor a is given by restriction to J′ ⊂ J. If J is left-pointed (that is, has the smallest element o ∈ J), then the topology corresponding to {o} ⊂ J is the submaximal topology of Example 2.3.

**Remark 2.5.** One does not need to specify the whole topology T ⊂ Ω to define sheaves and compute the associated sheaf functor. Indeed, we say that a base C of a topology T on a small category I is a subfunctor C ⊂ T such that C(i)^o ⊂ T(i)^o is a cofinal partially ordered subset for any i ∈ I. Then, for any functor E: I^o → E to some finitely presentable E, one can replace the colimit over T(i)^o in (2.2) with the colimit over C(i)^o, so that a map E → a(E) is an isomorphism (that is, E is a sheaf) if and only if Hom(s, E) ∼= E(i) for any i ∈ I and s ∈ C(i). Note that T(i) ⊂ Ω(i) can be recovered as the right closure of C(i) ⊂ Ω(i), that is, the subset of sieves s ∈ Ω(i) with non-empty C(i)/s.

**Remark 2.6.** For any small I, the functor Ω: I^o → Sets has the following universal property: for any monomorphism X → Y in Fun(I^o, Sets), there is a unique map Y → Ω that fits into a Cartesian square

\[
\begin{array}{ccc}
X & \longrightarrow & pt_I \\
\downarrow & & \downarrow 1 \\
Y & \longrightarrow & Ω
\end{array}
\]

where 1 is the embedding (2.4) (for a representable Y, this is simply the definition of Ω). Then, in particular, a subfunctor T ⊂ Ω defines a map j: Ω → Ω, and one can show that T is a topology if and only if

(i) \quad j \circ 1 = 1, \quad (ii) \quad \land \circ (j \times j) = j \circ \land, \quad \text{and} \quad (iii) \quad j \circ j = j,

where \land: Ω × Ω → Ω corresponds to the square (2.5) for the embedding

\[
(1 \times 1): pt_I = pt_J \times pt_I → Ω \times Ω.
\]

The functor Ω is known as the ‘subobject classifier’ in Fun(I^o, Sets), and the image Ω_j of the idempotent endomorphism j: Ω → Ω is a subobject classifier in the sheaf category Shv(I, Sets). In general, sheaf categories Shv(I, Sets) are known as toposes, and it turns out that it is the existence of a subobject classifier that characterizes them in the most natural way. This is the subject of abstract topos theory, for which we refer the reader to the wonderful book [14] (although, when
can define a pair of adjoint functors extends uniquely to a continuous functor of the category $Y_i$ by elements in the set where the right-hand side is shorthand for the 'product of copies of $I$ is Cartesian. Up to an equivalence, the category $C$ is Cartesian. Up to an equivalence, the category $Y_i$ indexed by some set of indices $\alpha$, with the sieve $s(\{i_{\alpha} \to i\})$ consisting of all morphisms $i' \to i$ that factor through one of the morphisms $i_{\alpha} \to i$. A Grothendieck pretopology on $I$ is given by a bunch of covering families for every $i \in I$, again satisfying certain axioms (which ensure, in particular, that the resulting collection of sieves is a base for a Grothendieck topology in the sense of Remark 2.5).

We shall not need the full definition of a pretopology (see, for instance, [14], §0.3), but we do need one somewhat degenerate example when all the covering families consist of one object. Namely, we say that a class $F$ of morphisms in $I$ is a covering class if it is closed under compositions, contains all identity maps, and such that any morphism $f: i' \to i$ in $F$ admits a pullback with respect to any morphism $i'' \to i$ (‘morphisms in $F$ admit pullbacks’) and the pullback is also in $F$ (‘$F$ is stable under pullbacks’). Then, for any covering class $F$, putting $C(i) = \{s(\{f: i' \to i\}) | f \in F\}$ defines a base $C$ of a Grothendieck topology $T$ on $I$, which we call the $F$-topology. A functor $E: I^o \to E$ to some finitely presentable $E$ is separable with respect to the $F$-topology if and only if $E(i) \to E(i')$ is injective for any $f: i' \to i$ in $F$, and it is a sheaf if and only if the square

$$
\begin{array}{ccc}
E(i) & \longrightarrow & E(i') \\
\downarrow & & \downarrow \\
E(i') & \longrightarrow & E(i' \times_i i')
\end{array}
$$

is Cartesian. Up to an equivalence, the category $C(i)$, $i \in I$, can be obtained by considering the full subcategory $I/_{\neq} i \subset I/i$ spanned by morphisms in $F$ and identifying all the morphisms between any two objects (so that, in the end, there is at most one map between any two objects, and the category is canonically equivalent to a partially ordered set). In particular, if an arrow $f: i' \to i$ splits, that is, admits an inverse $g: i \to i'$, $f \circ g = \text{id}$, then the corresponding object $s(f) \in C(i)$ is isomorphic to the maximal sieve.

For any object $i \in I$, we denote by $\text{ev}_i : \text{Fun}(I^o, E) \to E$ the evaluation funcator sending $E$ to $E(i)$, and note that since $E$ is complete, $\text{ev}_i$ has a right-adjoint $Y_i : E \to \text{Fun}(I^o, E)$ given by

$$
Y_i(E)(i') = E(\text{Hom}(i, i')) , \quad i' \in I,
$$

where the right-hand side is shorthand for the ‘product of copies of $E$ numbered by elements in the set $\text{Hom}(i, i')$’. Equivalently, we have $Y_i(E) \cong Y^i(E)^o$, where $Y^i(E)$ is as in (2.3). More generally, we consider the projective completion $\text{Pro}(I)$ of the category $I$. Then, since $E$ has filtered colimits, any functor $E: I^o \to E$ extends uniquely to a continuous functor $\hat{E} = \iota_I : \text{Pro}(I)^o = \text{Ind}(I^o) \to E$, and we can define a pair of adjoint functors

$$
\text{ev}_{\hat{i}} : \text{Fun}(I^o, E) \to E \quad \text{and} \quad Y_{\hat{i}} : E \to \text{Fun}(I^o, E)
$$

2.2. Coverings. A practical way to describe a sieve is by specifying a covering family $\{i_\alpha \to i\}$ indexed by some set of indices $\alpha$, with the sieve $s(\{i_\alpha \to i\})$ consisting of all morphisms $i' \to i$ that factor through one of the morphisms $i_\alpha \to i$.
for any proobject \( \tilde{i} \in \text{Pro}(I) \) by

\[
ev_{\tilde{i}}(E) = E(\tilde{i}) = \colim_{\tilde{i} \to i} E(i), \quad E \in \text{Fun}(I^o, \mathcal{E}),
\]

\[
\mathcal{Y}_{\tilde{i}}(E')(i') = E'(\text{Hom}(\tilde{i}, i')) = \lim_{\tilde{i} \to i} E'(\text{Hom}(i, i')), \quad E' \in \mathcal{E}, \tag{2.9}
\]

where the colimit (the limit) actually reduces to a cofinal filtered diagram representing \( \tilde{i} \) (to the opposite diagram, respectively).

**Definition 2.7.** A proobject \( \tilde{i} \in \tilde{I} \) is said to be \( F \)-liftable if \( \text{Hom}(\tilde{i}, -) \) sends maps in \( F \) to surjective maps.

**Lemma 2.8.** For any \( F \)-liftable proobject \( \tilde{i} \in \tilde{I} \), the functor \( \mathcal{Y}_{\tilde{i}} \) of (2.8) takes values in \( \text{Shv}(I, \mathcal{E}) \subset \text{Fun}(I^o, \mathcal{E}) \), and the adjoint evaluation functor \( \ev_{\tilde{i}} \) factors through the associated sheaf functor \( a : \text{Fun}(I^o, \mathcal{E}) \to \text{Shv}(I, \mathcal{E}) \).

**Proof.** For the second claim, it suffices to check that \( \ev_{\tilde{i}} \) inverts the map \( E \to a_0(E) \) for any \( E \). By (2.9) and (2.2), we have

\[
\ev_{\tilde{i}}(a_0(E)) \cong \colim_{\tilde{i} \to i} \colim_{s \in C(i)} \text{Hom}(s, E) = \colim_{(\tilde{i} \setminus C)^o} E(s), \tag{2.10}
\]

where \( \tilde{i} \setminus C = (\tilde{i} \setminus I) \times_I C \) is the category of triples \( \langle s, i, \tilde{i} \to i \rangle \) consisting of an object \( i \in I \), an arrow \( \tilde{i} \to i \), and a sieve \( s \in C(i) \). However, the \( F \)-liftable of \( \tilde{i} \) ensures that for any such \( \langle s, i, \tilde{i} \to i \rangle \) with \( s \) represented by an arrow \( f: i' \to i \) in \( F \), the arrow \( \tilde{i} \to i \) factors through \( f \), and then \( f^*(s) \) is split. Therefore the full subcategory in \( (\tilde{i} \setminus C)^o \) spanned by triples with split \( s \) is cofinal, so that we can reduce the colimit in (2.10) to a colimit over this subcategory. But the smaller colimit is exactly \( \ev_{\tilde{i}}(E) \), so we have proved the second claim. By adjunction, this means that, for any \( E' \in \text{Fun}(I^o, \mathcal{E}) \) with \( E \in \mathcal{E} \), any map \( g: E' \to \mathcal{Y}_{\tilde{i}}(E) \) factors uniquely through the adjunction map \( E' \to e(a(E')) \). Taking \( E' = \mathcal{Y}_{\tilde{i}}(E) \) and \( g = \text{id} \), we can deduce that the adjunction map \( E' \to e(a(E')) \) splits, so that \( E' \) is a sheaf. \( \square \)

**Remark 2.9.** Note that since the colimit in (2.9) reduces to a filtered colimit, the functor \( \ev_{\tilde{i}} \) commutes with finite limits. If \( \mathcal{E} = \text{Sets} \), then this means, by definition, that the adjoint pair \( \langle \ev_{\tilde{i}}, \mathcal{Y}_{\tilde{i}} \rangle \) determines a point of the topos \( \text{Shv}(I, \text{Sets}) \).

To construct \( F \)-liftable proobjects in \( I \) we take an object \( i \in I \) and let \( \text{Cov}(i) \subset I / i \) be the full subcategory spanned by arrows \( i' \to i \) in \( F \) with the induced projection \( \sigma(i) : \text{Cov}(i) \to I / i \to I \). The class \( \sigma(i)^* F \) of morphisms \( f \) such that \( \sigma(i)^{0}(f) \in F \) is then a covering class of morphisms in \( \text{Cov}(i) \), and we have the following result.

**Definition 2.10.** An \( F \)-hull of an object \( i \in I \) is a \( \sigma(i)^* F \)-liftable proobject \( \tilde{i} \) in \( \text{Cov}(i) \).

**Lemma 2.11.** (i) For any \( F \)-hull \( \tilde{i} \in \text{Pro}(\text{Cov}(i)) \) of an object \( i \in I \), the proobject \( \sigma(i)(\tilde{i}) \in \text{Pro}(I) \) is \( F \)-liftable.

(ii) For any \( i \in I \), there is an \( F \)-hull \( \tilde{i} \in \text{Pro}(\text{Cov}(i)) \).
Proof. For (i), assume given a morphism \( f : i'' \to i' \) in \( F \). We need to show that any map \( g : \sigma(i)(i) \to i' \) factors through \( i'' \). But, by definition, \( \tilde{i} \) comes from a projective system in \( \text{Cov}(i) \), so \( g \) factors through \( \sigma(g') \) for some morphism \( g' : \tilde{i} \to i_0 \) in \( \text{Cov}(i) \). Then it suffices to check that \( \sigma(g') \) factors through the map \( i'' \times_{i'} i_0 \to i_0 \). But this map is in \( F \), so it is comes from \( \text{Cov}(i) \), and \( \tilde{i} \) is \( \sigma(i)^*F \)-liftable in \( \text{Cov}(i) \).

For (ii), the argument is completely standard and goes back at least to [10] (where it is already called ‘standard’). Note that since \( I \) is small, so is \( \text{Cov}(i) \), and then for any proobject \( \tilde{i} \in \text{Pro}([\text{Cov}(i)]) \), there is a set \( S \) whose elements \( s \in S \) enumerate, up to an isomorphism, all diagrams

\[
\tilde{i} \to i'_s \leftarrow i''_s
\]

in \( \text{Cov}(i) \) with \( f \in \sigma(i)^*F \). Moreover, \( \text{Cov}(i) \) has finite products, and finite products of maps in \( F \) are in \( F \), so for any finite subset \( S_0 \subset S \), we can define a proobject \( \tilde{i}[S_0] \) as the fibred product

\[
\tilde{i}[S_0] \to \prod_{s \in S_0} i''_s
\]

\[
\tilde{i} \to \prod_{s \in S_0} i'_s
\]

We can then let \( H(\tilde{i}) = \lim_{S_0 \subset S} \tilde{i}[S_0] \), where the limit is taken over the directed partially ordered set of finite subsets in \( S \). This is a proobject that comes equipped with a map \( H(\tilde{i}) \to \tilde{i} \) and, by construction, for any diagram (2.11), the composition map \( H(\tilde{i}) \to i'_s \) factors through \( f_s \). To complete the proof, let \( i_0 = i \) with the identity map \( i \to i \), define inductively \( i_{n+1} = H(i_n), n \geq 0 \), and take \( \tilde{i} = \lim_n i_n \). □

3. Hypercoverings

3.1. Partially ordered sets. Fix a category \( I \) equipped with a covering class \( F \). Recall that \( \text{pt}^< \cong [0]^< \cong [1] \) is the single arrow category and \( [1]^o \cong [1] \), so that the coverings of an object \( i \in I \) can be understood as \( i \)-coaugmented functors from the point category \( \text{pt} \) to \( I \). The notion of a hypercovering extends this to categories other than \( \text{pt} \). The usual application is to the simplex category \( \Delta \) (see §3.3 below), but it is instructive to develop the theory in a more general context. We start with finite partially ordered sets.

Definition 3.1. For any left-pointed finite partially ordered set \( J \), a functor \( E : J^o \to I \) is called a hypercovering if \( \lim_{J^o} E \) exists for any non-empty left-closed subset \( J_0 \subset J \), and the natural map \( \lim_{J^o} E \to \lim_{J_0} E \) is in \( F \) for any two non-empty left-closed subsets \( J_0 \subset J_1 \subset J \).

In the situation of Definition 3.1 we denote by \( \text{HCov}(J) \subset \text{Fun}(J^o, I) \) the full subcategory spanned by hypercoverings. If \( \varphi : J' \to J \) is a left-pointed functor between left-pointed finite partially ordered sets, then (1.3) shows immediately that for any \( E \in \text{HCov}(J') \), the right Kan extension \( \varphi_* E \) exists and is a hypercovering, and if \( \varphi \) is a left-closed embedding, then \( \varphi^{**} \) also tautologically sends hypercoverings
to hypercoverings. For any finite partially ordered set $J$, we have the left-pointed partially ordered set $J^<$, and we define a $J$-hypercovering of an object $i \in I$ as an $i$-coaugmented functor $E: J^{<o} = J^o \to I$ that is a hypercovering in the sense of Definition 3.1. We denote the category of $J$-hypercoverings of an object $i$ by $\text{HCov}(J, i) \subset \text{HCov}(J^<)$. Note that non-empty left-closed subsets in $J^<$ correspond bijectively to all left-closed subsets in $J$. If $J = \text{pt}$, then the only left-closed subsets in $J$ are the empty set and $J$ itself, so that a $J$-hypercovering is a covering and $\text{HCov}(\text{pt}, i) \cong \text{Cov}(i)$. In the general case, $J/j \subset J$ is left-closed for any $j \in J$, and since it has the largest element, (1.6) provides an isomorphism

$$\lim_{(J/j)^{<o}} E \cong E(j) \quad (3.1)$$

for any $E \in \text{HCov}(J, i)$. Thus, for any $j \leq j'$, the map $E(j') \to E(j)$ is in $F$, so that a $J$-hypercovering factors through the subcategory $I_F \subset I$ with the same objects as $I$ and morphisms that are in $F$.

Example 3.2. For any $n \geq 0$, every non-empty left-closed subset $J_0 \subset [n]$ is of the form $[m] = \{0, \ldots, m\} \subset [n]$, $0 \leq m \leq n$, so that $J^o_0$ has the smallest element and $\lim_{J^{<o}_0}$ reduces to evaluation by (3.1). Thus $E: [n]^o \to I$ is a hypercovering if and only if it factors through $I_F$.

Remark 3.3. A non-empty left-closed subset $J_0 \subset J$ in a left-pointed partially ordered set $J$ is the preimage of $\{0\} \subset [1]$ under a unique left-pointed map $J \to [1]$ and, similarly, pairs $J_0 \subset J_1 \subset J$ of non-empty left-closed subsets correspond to left-pointed maps $J \to [2]$. Then, by virtue of (1.3) and Example 3.2, Definition 3.1 can be rephrased as follows: $E: J^o \to I$ is a hypercovering if and only if for any left-pointed map $\varphi: J \to [2]$, the Kan extension $\varphi_\ast E$ exists, is universal in the sense of Remark 1.2, and factors through $I_F$.

Example 3.4. Let $J = \{0, \ldots, n\}$ be the set of integers $0, \ldots, n$ with discrete order. Then any subset $J_0 \subset J$ is left-closed, and if $J_0$ has at most one element, then $\lim_{J^{<o}_0}$ again reduces to evaluation. For larger $J_0$ this is not true. However, since the morphisms in a covering class admit pullbacks and are stable under pullbacks, induction on the cardinality of $J_0$ shows that any functor $E: J^{<o} \to I_F \subset I$ is a hypercovering once again.

Example 3.5. Consider the square $[1]^2$ of the single arrow category $[1]$. Then $[1]^{2o} \cong [1]^2$, the functors $[1]^{2o} \to I$ correspond to commutative squares

$$
\begin{array}{ccc}
i_{11} & \longrightarrow & i_{10} \\
\downarrow & & \downarrow \\
i_{01} & \longrightarrow & i_{00}
\end{array}
\quad (3.2)
$$

and such a square is a hypercovering if and only if the morphisms $i_{01}, i_{10} \to i_{00}$ are in $F$, and so is $i_{11} \to i_{10} \times_{i_{00}} i_{01}$. In particular, it is not sufficient to require that the functor factors through $I_F$.

In the general case, testing whether or not a given functor $E: J^{<o} \to I$ is a $J$-hypercovering requires checking a lot of conditions. However, this can be reduced to one condition for each element $j \in J$ by induction and the following result.
Lemma 3.6. Let \( j \in J \) be a maximal element in a finite partially ordered set \( J \), and let
\[
J' = J \setminus \{ j \} \subset J \quad \text{and} \quad L(j) = J/j \cap J' \subset J.
\]
Then \( E: J^\circ \twoheadrightarrow I \) is a \( J \)-hypercovering if and only if
(i) its restriction to \( J' \subset J \) is a \( J' \)-hypercovering, and
(ii) the natural map \( E(j) = \lim_{(J/j)^{\circ}} E \to \lim_{L(j)} E \) is in \( F \).

Proof. The ‘only if’ part is clear. For the ‘if’ part, by induction on cardinality, it suffices to check that \( \lim_{J^\circ} E \) exists and the map \( \lim_{J^\circ} E \to \lim_{J_0^\circ} E \) is in \( F \) for any left-closed \( J_0 \subset J \). For the first claim, let \( V = \{ 0, 1 \}^< \) be as in Example 1.1, and consider the map \( \varphi: J \to V \) sending \( L(j) \) to \( o, j \) to 1, and the rest to 0. Then all the comma-fibres \( J^</\varphi^<V \) either have cardinality smaller than \( J \) or have a largest element or both, so that \( \varphi_*^<E \) exists by induction and satisfies the assumptions of the Lemma for \( J = V \) and \( j = 1 \). Hence all is reduced to this case, which immediately follows from Example 3.4.

For the second claim, consider the map \( \varphi: J \to V \) sending \( J_0 \cap (J \setminus \{ j \}) \) to \( o, j \) to 1, and the rest to 0. Then again \( \varphi_*^<E \) exists by induction, and all is once again reduced to the obvious case \( J = V, j = 1 \). □

Example 3.7. The class \( F \) of all monomorphisms is a covering class in any complete category \( E \). In this case, by Lemma 3.6, a functor \( J^o \to E \) is a hypercovering in the sense of Definition 3.1 if and only if it is a separable presheaf for the submaximal topology of Example 2.3. More generally, for any \( I \) and \( F \), a functor \( E: J^o \to I \) is a hypercovering if and only if \( \text{Hom}(s, E) \) defined as the limit (2.1) exists for any sieve \( s \subset Y(i) \) in the submaximal topology, and the map \( E(i) \to \text{Hom}(s, E) \) is in \( F \).

It is useful to extend the class \( F \) to a covering class in the hypercovering categories \( \text{HCov}(J, i) \). Namely, note that the category \([n] \) is left-pointed for any \( n \geq 0 \), so that for any finite partially ordered set \( J \), the product \([n] \times J^< \) is a left-pointed finite partially ordered set. Recall also that \([n]^o \cong [n] \).

Definition 3.8. A map \( f: E_0 \to E_1 \) in \( \text{HCov}(J^<) \) is in the class \( F(J) \) if the corresponding functor \( i(f): [1] \times J^{<o} \cong ([1] \times J^<)^o \to I \) is a hypercovering.

Lemma 3.9. For any finite partially ordered set \( J \), the class \( F(J) \) is a covering class in \( \text{HCov}(J^<) \) and it restricts to a covering class in \( \text{HCov}(J, i) \subset \text{HCov}(J^<) \) for any \( i \in I \). For any map \( \varphi: J' \to J^< \) from a finite partially ordered set \( J' \), the right Kan extension functor \( \varphi_\circ^< \) sends morphisms in \( F(J') \) to morphisms in \( F(J) \). Moreover, for any left-closed subset \( J' \subset J \) with embedding functor \( \varphi: J' \to J \) and any \( E \in \text{HCov}(J^<) \), the adjunction map \( a: E \to \varphi_\circ^<E \) is in \( F(J) \).

Proof. For the first claim, put \( J_n = [n] \ast J, \) \( n = 0, 1, \) where \(- \ast -\) is the extended product (1.1), and note that for any composable pair \( f, f' \) of morphisms in \( F(J) \), the functor \( i(f, f') \) satisfies condition (ii) of Lemma 3.6 at any \( l \times i \in J_1^{<o} \subset ([2] \times J^<)^o \). Indeed, for \( l = 0, 1 \), this is precisely the same condition at \( l \times i \in J_0^{<o} \) for \( i(f') = (t \times \text{id})^*i(f, f') \) (where we have \( s^o = t \) and \( t^o = s \) under the identifications \([1]^o \cong [1] \) and \([2]^o \cong [2] \)). For \( l = 2 \), the functor \( s: [1] \to [2] \) has a right-adjoint \( s^\dagger: [2] \to [1] \), and the product \( s^\dagger \times \text{id} \) restricts to a functor \( s^\dagger \times \text{id}: L(2 \times j)^{<o} \to \)
\(L(1 \times j)^{<o} \subset [1] \times J^{<o}\) right-adjoint to \(s \times \text{id}\). We then have

\[
\lim_{L(2 \times j)^{<o}} \iota(f, f') \cong \lim_{L(1 \times j)^{<o}} s_{*}^{\dagger} \iota(f, f') \\
\cong \lim_{L(1 \times j)^{<o}} s^{*} \iota(f, f') \cong \lim_{L(1 \times j)^{<o}} \iota(f),
\]

so everything is reduced to condition (ii) of Lemma 3.6 for \(\iota(f)\). Thus \(\iota(f, f')\) is a \(J_{1}\)-hypercovering, and since \(m : [1] \to [2]\) also has a right-adjoint \(m^{\dagger}\), it follows that \(\iota(f \circ f') \cong m^{*} \iota(f, f') \cong m_{*}^{\dagger} \iota(f, f')\) is a \(J_{0}\)-hypercovering. This shows that \(F(J)\) is closed under compositions. Now the identification (3.1) implies immediately that the maps in \(F(J)\) are pointwise in \(F\), so that they admit pullbacks, and they are stable under pullbacks again by Lemma 3.6. For the second claim, it suffices to observe that \(\iota(\varphi_{*}^{<o}(f)) \cong (\text{id} \times \varphi)^{<o} \iota(f)\), and for the third claim, let \(J'' = J_{0}' \cup (\{1\} \times J) \subset J_{0}\) with embedding map \(\varphi_{1} : J'' \to J_{0}\), and note that \(\iota(a) \cong \varphi_{1*}^{<o} \varphi_{1}^{<o} \iota(\text{id}_{E})\), where \(\text{id}_{E} : E \to E\) is the identity map. \(\square\)

### 3.2. Thin \(ML\)-categories.

To go beyond partially ordered sets recall that a factorization system \(\langle L, R \rangle\) in a category \(C\) is given by two classes of morphisms \(L\) and \(R\) in \(C\), both closed under compositions and containing all isomorphisms, such that any map \(f : c' \to c\) in \(C\) factors as

\[
c' \xrightarrow{l} c'' \xrightarrow{r} c,
\]

with \(l \in L\) and \(r \in R\), and the factorization is unique up to a unique isomorphism. This very useful notion goes back to [3], and we refer to [3], §2, for further details. Given a factorization system \(\langle L, R \rangle\), we denote by \(C_{L}, C_{R} \subset C\) the subcategories with the same objects as \(C\) and maps that are in \(L\) and \(R\). We note that, as a consequence of the uniqueness of the factorization (3.3), the obvious functor \(C_{R}/c \to C/c, c \in C\) is a fully faithful embedding that admits a left adjoint (sending \(f : c' \to c\) to the second term \(r : c'' \to c\) in (3.3)).

**Definition 3.10.** A thin \(ML\)-category is a small category \(X\) equipped with a factorization system \(\langle M, L \rangle\) such that \(X_{L}/x\) is a finite partially ordered set for any \(x \in X\). A functor \(\varphi : X \to X'\) between thin \(ML\)-categories \(\langle X, M, L \rangle\) and \(\langle X', M', L' \rangle\) is called an \(ML\)-functor if it sends maps in \(M\) (\(L\)) to maps in \(M'\) (\(L'\), respectively).

For any thin \(ML\)-category \(X\), we can turn the augmented category \(X^{<}\) into a thin \(ML\)-category by placing all the maps \(o \to x, x \in X\) in the class \(L\), so that \(X_{L}^{<} \cong (X_{L})^{<}\). For any \(x \in X\), the partially ordered set \(X_{L}/x\) has the largest element \(\text{id} : x \to x\), and we write \(L(x) = (X_{L}/x) \setminus \text{id}\).

**Definition 3.11.** For any thin \(ML\)-category \(X\), an \(X\)-hypercovering of an object \(i \in I\) is an \(i\)-coaugmented functor \(E : X^{<o} = X^{>o} \to I\) such that for any \(x \in X\), the limit \(\lim_{X^{<o}} \sigma(x)^{<o} E\) exists and the map

\[
E(x) \cong \lim_{X_{L}^{<o}} \sigma(x)^{<o} E \to \lim_{L(x)^{<o}} \sigma(x)^{<o} E
\]

is in the class \(F\).
Example 3.12. Any finite partially ordered set $J$ is trivially a thin $ML$-category, with $M$ ($L$) consisting of the identity morphisms (all morphisms, respectively). In this case Lemma 3.6 immediately shows that Definition 3.11 reduces to Definition 3.1.

Remark 3.13. One can also describe hypercoverings in terms of Grothendieck topologies, as in Example 3.7. Namely, for any thin $ML$-category $X$, a sieve $s$ on an object $x \in X^<_L$ defines an ‘induced’ sieve on the same object in $X^<_L$ consisting of all maps $x' \to x$ whose component $x'' \to x$ of the factorization (3.3) is in $s$. Then all induced sieves $s \in Y(x)$ for the submaximal topology on $X^<_L$ form a topology on $X^<_L$, and $E: X^<_L \to I$ is a hypercovering if and only if $\text{Hom}(s, E)$ exists for any $s$ and the map $E(x) \to \text{Hom}(s, E)$ is in $F$. In good cases this induced topology can be described explicitly. For example, one can show that if all the maps $l \in L$ are monomorphisms and any map $m \in X$ admits a one-sided inverse $l$, $m \circ l = id$, then the induced topology on $X^<_L$ is submaximal (that is, consists of all non-empty sieves). This happens, for example, for the category $\Delta$ considered below in §3.3.

For an arbitrary thin $ML$-category $X$, we denote the category of $X$-hypercoverings of an object $i \in I$ by $HCov(X, i)$. By virtue of Example 3.12, this is consistent with our earlier notation. For any map $l: x' \to x$ in the class $L$, we have $(X_L / x) / l \cong X_L / x'$, so that $\sigma(x)^{<o}$ sends $X$-hypercoverings to $(X_L / x)$-hypercoverings. In particular, this implies that any $X$-hypercovering $E: X^<_L \to I$ sends $X^{<o}_L \subset X^{<o}$ into $I_F \subset I$, so that we have a full embedding $HCov(X, i) \subset \text{Fun}(X^{<o}, \text{Cov}(i))$.

Example 3.14. We say that a full subcategory $X \subset X'$ in a thin $ML$-category $(X, M, L)$ is an $ML$-subcategory if the middle term $x''$ of the decomposition (3.3) is in $X'$ for any morphism $f: x' \to x$ in $X'$. Then $X'$ with the classes $M$, $L$ is a thin $ML$-category, and the embedding $X' \to X$ is an $ML$-functor.

Example 3.15. For any two thin $ML$-categories $(X, M, L)$ and $(X', M', L')$, the product $(X \times X', M \times M', L \times L')$ is a thin $ML$-category, and so is the extended product $X \star X'$ defined in (1.1).

Combining Examples 3.12 and 3.15, we can see that the extended product $J \star X$ of a finite partially ordered set $J$ and a thin $ML$-category $X$ is naturally a thin $ML$-category. In particular, as in Definition 3.8, we can take $J = [0]$, and say that a map $f$ in $HCov(X, i)$ is in the class $F(X)$ if $\iota(f)$ is a hypercovering. Then Lemma 3.9 shows immediately that $F(X)$ is a covering class.

Example 3.16. Let $\text{Pos}$ be the category of all non-empty finite partially ordered sets, let $M$ be the class of all surjective maps, and let $L$ be the class of all injective maps $J' \to J$ that are full when considered as functors (that is, $J'$ can be identified with its image in $J$ equipped with the induced partial order). Then $\text{Pos}$ is a thin $ML$-category in the sense of Definition 3.11.

Lemma 3.17. Assume given an object $x \in X$ in a thin $\text{Hom}$-finite $ML$-category $(X, M, L)$, and let $\phi: \text{pt} \to X$ be the embedding onto $x$. Then, for any object $i \in I$, $\phi^{<o}_x$ and $\phi^{<o}_x$ define an adjoint pair of functors between $HCov(X, i)$ and $\text{Cov}(i)$ sending morphisms in $F(X)$, $F$ to morphisms in $F$, $F(X)$.

Proof. Since the restriction of an $X$-hypercovering to $X^{<o}_L$ factors through $I_F$, it follows that $\phi^{<o}_x$ sends hypercoverings to coverings. Moreover, by the definition of
a covering class, $\text{Cov}(i)$ has finite products. Since $X$ is $\text{Hom}$-finite, the right Kan extension $\varphi_*^{<o}E$ exists by (1.3) for any $E \in \text{Cov}(i)$ and is given by

$$\varphi_*^{<o}E(x') \cong \prod_{f : x \to x'} E,$$

with the product of copies of $E$ on the right numbered by maps $f : x \to x'$. But the set of maps $x \to x'$ splits into a disjoint union according to the isomorphism class of the middle term $x''$ of the decomposition (3.3), and $\lim_{L(x')}\varphi_*^{<o}E$ is then given by the same product but over the subset of maps such that $x'' \to x'$ is not an isomorphism. Since maps in $F$ are stable under pullbacks, the map (3.4) for $\varphi_*^{<o}E$ is in $F$ for any $x' \in X$. □

Lemma 3.18. Assume given a thin ML-category $\langle X, M, L \rangle$ and a full ML-subcategory $X' \subset X$. Let $\varphi : X' \to X$ be the embedding functor, and assume that $X'_M \subset X_M$ is right-closed. Then $\varphi_*^{<o}E$ exists and is an $X$-hypercovering for any $X'$-hypercovering $E$.

Proof. For any $x \in X$, let $\varphi_x : X'_L/x \to X_L/x$ be the embedding induced by $\varphi$, and consider the base change map

$$\sigma(x)^{<o} \varphi_*^{<o}E \to \varphi_x^{<o} \sigma(x)^{<o}E. \quad (3.5)$$

Since $\sigma(x)^{<o}E$ is a hypercovering, we already know that the target of (3.5) exists and is a hypercovering, so it suffices to show that the map is an isomorphism. By (1.3), this amounts to checking that $X'_L/x/\varphi^{<o}x$ is cofinal in $X'/^{<o}/\varphi^{<o}x$ for any $x \in X$. But since $X'_M \subset X_M$ is right-closed, the functor $X^{<o}/x \to X'_L/x$ left-adjoint to the embedding $X'_L/x \subset X^{<o}/x$ sends $X'/^{<o}/\varphi^{<o}x$ into $X'_L/^{<o}/\varphi^{<o}x$, so the assertion follows from (1.6). □

Lemma 3.19. Assume given thin ML-categories $\langle X, M, L \rangle$, $\langle X', M', L' \rangle$ and an ML-functor $\varphi : X \to X'$ that admits a left-adjoint $\psi : X' \to X$. Moreover, assume that $\psi$ is also an ML-functor. Then $\varphi^{<o}$ sends $X'$-hypercoverings to $X$-hypercoverings.

Proof. For any $x \in X$, the ML-functor $\varphi$ induces a functor $\varphi_x : X'_L/x \to X_L/x(\varphi(x))$. Since its adjoint $\psi_x$ is also an ML-functor, it induces a functor $\psi_x : X'_L/x(\varphi(x)) \to X_L/\psi(\varphi(x)) \to X_L/x$ left-adjoint to $\varphi_x$. Then $\varphi_x^{<o} \cong \psi_x^{<o}$ sends hypercoverings to hypercoverings for any $x$, hence so does $\varphi^{<o}$. □

Lemma 3.20. For any two thin ML-categories $X_0$ and $X_1$, a functor $X_0^{<o} \times X_1^{<o} \to I$ is a hypercovering if and only if the corresponding functor $X_0^{<o} \to \text{Fun}(X_1^{<o}, I)$ factors through $\text{HCov}(X_1)$ and is a hypercovering with respect to the class $\text{F}(X_1)$.

Proof. Since

$$(X_0^{<} \times X_1^{<})_L/(x_0 \times x_1) \cong (X_0^{<}_L/x_0) \times (X_1^{<}_L/x_1)$$

for any $x_0 \times x_1 \in X_0^{<} \times X_1^{<}$, it suffices to prove the claim when $X_0$ and $X_1$ are finite partially ordered sets. Then Remark 3.3 and (1.3) immediately reduce everything to the case $X_0 = [1]$, and the argument in this case is the same as in the proof of Lemma 3.9. □
3.3. Simplicial objects. Now, as usual, denote by $\Delta \subset \text{Pos}$ the full subcategory spanned by the ordinals $[n], n \geq 0$. A simplicial object in a category $I$ is by definition a functor $i_*: \Delta^o \to I$ with $i_n = i_*([n]), n \geq 0$. One traditionally writes

$$\Delta^o I = \text{Fun}(\Delta^o, I).$$

An $i$-augmented simplicial object for some $i \in I$ is an $i$-coaugmented functor $i_*: \Delta^{<o} \cong \Delta^{\geq o} \to I$. It is explicitly given by a triple $\langle i_*, i, a \rangle$, where $i_*$ is a simplicial object, and $a: i_* \to i$ is the augmentation map to the constant simplicial object with value $i$.

The full subcategory $\Delta \subset \text{Pos}$ inherits the structure of a thin ML-category of Example 3.16, so it is possible to speak of $\Delta$-hypercoverings in the sense of Definition 3.11. These are usually just called hypercoverings and appear in the literature in many places and forms. Here are some of them.

(i) For any $n \geq 0$, let $\Delta^{\leq n} \subset \Delta^< \mathbf{c}$ be the full subcategory spanned by $[m]$ with $m < n$, and let $j_n: \Delta^{\leq n} \to \Delta^<$ be the embedding functor. The $n$-th coskeleton $\cosk_n i_*$ of an augmented simplicial object $i_*: \Delta^{<o} \to I$ is given by the right Kan extension $j_n^o \circ j_n^{*o} i_*$, if it exists, and if it does, it comes equipped with the adjunction map $E \to \cosk_n E$. The 0-th coskeleton $\cosk_0 i_*$ always exists. By (1.3), this is just the constant functor with value $i = i_*(o)$, and the adjunction map $i_* \to \cosk_0 i_*$ is $\text{id}$ over $o$ and the augmentation map $a: i_* \to i$ over $\Delta^o$. Then Lemma 3.18 applies to the embeddings $\Delta^{\leq n} \subset \Delta^<$ and shows that an augmented simplicial object $i_*: \Delta^{<o} \to I$ is a $\Delta$-hypercovering if and only if $\cosk_n i_*$ exists for any $n \geq 0$ and the map $i_n \to (\cosk_n i_*)_n$ is in $F$. If $I$ is the category of schemes and $F$ is the covering class of proper maps, then these are the original hypercoverings of [5].

(ii) Alternatively, one can describe $\Delta$-hypercoverings in terms of sieves as in Remark 3.13. Namely, let $\Delta_n = \text{Hom}(-, [n]): \Delta^o \to \text{Sets}$ be the elementary simplex, and let $S_{n-1} \subset \Delta_n$ be the standard simplicial sphere. We augment both by the one-point set pt. Then $i_*: \Delta^{<o} \to I$ is a hypercovering if and only if $\text{Hom}(S_{n-1}, i_*)$ exists for any $n \geq 1$, and the map $i_n = \text{Hom}(\Delta_n, i_*) \to \text{Hom}(S_{n-1}, i_*)$ is in $F$. If this holds, then the same holds for any non-empty sieve $S \subset \Delta_n$. For example, if $I = \text{Sets}$ is the category of sets and $F$ is the class of all surjective maps, then an augmented simplicial set $X_* = \langle X_*, X, a \rangle: \Delta^{<o} \to \text{Sets}$ is a $\Delta$-hypercovering if and only if $a: X_* \to X$ is a trivial fibration with respect to the Kan-Quillen model structure on $\Delta^o \text{Sets}$. Explicitly, the hypercovering condition at $[n] \in \Delta$ says that any augmented map $S_n \to X_*$ extends to an augmented map $\Delta_n \to X_*$. If this holds, then the same is true for any non-empty sieve $S \subset \Delta_n$. We also note that the coaugmentation given by a $\Delta$-hypercovering $X_*: \Delta^{<o} \to \text{Sets}$ is universal: if we put

$$\pi_0(X_*) = \text{colim}_{\Delta^o} X_*, \quad X_* \in \Delta^o \text{Sets},$$

then for any $\Delta$-hypercovering $\langle X_*, X, a \rangle$, we have $X \cong \pi_0(X_*)$ and $a: X_* \to X$ is the canonical map.

(iii) Now assume that $I$ is a model category and $F$ is the class of fibrations (respectively, trivial fibrations). Then $\Delta$ has the structure of a Reedy category (see [12], §5.2), with our classes $M$ and $L$ corresponding to matching and latching maps (and this is why we use this notation). The category of simplicial objects in $I$ carries a Reedy model structure, and an augmented object $\langle i_*, i, a \rangle$ in $I$ is
a $\Delta$-hypercovering if and only if $a: i_* \to i$ is a fibration (respectively, trivial fibration). The partially ordered set $L([n])$ of Definition 3.11 is the latching category for the Reedy structure on $\Delta$ (it becomes the matching category for the opposite category $\Delta^o$).

From now on, we shall also shorten ‘$\Delta$-hypercovering’ to ‘hypercovering’, and write $\text{HCov}(i) = \text{HCov}(\Delta, i)$. We recall that $\text{HCov}(i)$ comes equipped with a covering class $F(\Delta)$. Recall also that for any two simplicial objects $E_0, E_1 \in \text{Fun}(\Delta^o, \mathcal{E})$ in a category $\mathcal{E}$, we also have the simplicial set $\text{Hom}(E_0, E_1)$ given by (1.2), and since $[0] \in \Delta^o$ is the initial object, we have

$$\text{Hom}(E_0, E_1) \cong \lim_{\Delta^o} \text{Hom}(E_0, E_1) \cong \text{Hom}(E_0, E_1)_0.$$ 

In particular, this applies to hypercoverings $i_* \in \text{HCov}(i) \subset \text{Fun}(\Delta^o, \text{Cov}(i)).$

**Definition 3.21.** For any object $i \in I$, the objects in the category $HC(i)$ are the hypercoverings $i_* \in \text{HCov}(i)$, and the morphisms are given by

$$\text{Hom}(i_*, i'_*) = \pi_0(\text{Hom}(i_*, i'_*)),$$

where $\pi_0(\cdot)$ is as in (3.6).

To see the set $\pi_0(X_*)$ more explicitly, note that since any object $[n] \in \Delta$ admits a map $[0] \to [n]$, the natural map $X_0 \to \pi_0(X)$ is surjective. Then define an *elementary homotopy* between two elements $x, x' \in X_0$ as an element $\tilde{x} \in X_1$ with $X(s)(\tilde{x}) = x$ and $X(t)(\tilde{x}) = x'$, and say that two elements $x, x' \in X_0$ are *chain-homotopic* if they can be connected by a finite chain of elementary homotopies (going in either direction). Then, being chain-homotopic is an equivalence relation on $X_0$, and $\pi_0(X)$ is the corresponding set of equivalence classes.

**Definition 3.22.** For any two simplicial objects $E_0, E_1 \in \text{Fun}(\Delta^o, \mathcal{E})$ in a category $\mathcal{E}$, two maps $E_0 \to E_1$ are *chain-homotopic* if so are the corresponding elements in $\text{Hom}(E_0, E_1)_0$.

In this language morphisms in the category $HC(i)$ of Definition 3.21 are maps in $\text{HCov}(i)$ considered up to a chain homotopy.

**Lemma 3.23.** Let $\delta^{<}: \Delta^{<} \to \Delta^{<} \times \Delta^{<}$ be the diagonal embedding. Then $\delta_{\leq o}^*: E \leq o$ exists and is a $(\Delta \times \Delta)$-hypercovering for any hypercovering $E$.

**Proof.** The embedding $\varphi: \Delta \to \text{Pos}$ satisfies the assumptions of Lemma 3.18, and so does the embedding $\varphi \times \varphi: \Delta \times \Delta \to \text{Pos} \times \text{Pos}$. Therefore it suffices to prove the statement for the diagonal embedding $\delta: \text{Pos}^{\leq} \to \text{Pos}^{\leq} \times \text{Pos}^{\leq}$, and then apply it to $\varphi_{\leq o}^*: E$, which exists by Lemma 3.18. But if we reinterpret $\text{Pos}^{\leq}$ as the category of all finite partially ordered sets by treating the initial object $o$ as the empty set, then $\delta$ has a right-adjoint $\mu: \text{Pos}^{\leq} \times \text{Pos}^{\leq} \to \text{Pos}^{\leq}$ given by the Cartesian product, which completes the proof by Lemma 3.19. □

**Corollary 3.24.** For any $i \in I$, the category $HC(i)^o$ opposite to $HC(i)$ of Definition 3.21 is filtered.

**Proof.** Condition (i) of Definition 1.4 is obvious ($\text{HCov}(i)$ has finite products). For (ii), note that for any $i_*, i'_* \in \text{HCov}(i)$ and $n \geq 0$, we have

$$\text{Hom}(i'_*, i_n) \cong \text{Hom}(i'_*, \text{Hom}(\Delta_n, i_*)) \cong \text{Hom}(i'_*, e_n^* \delta_* i'_*),$$
where \( \mathcal{H}om(\Delta_n, -) \) is as in (1.5), and \( \varepsilon_n : \Delta^< \to \Delta^< \times \Delta^< \) is the embedding onto \([n] \times \Delta^<\). Then, by Lemmas 3.23, 3.20, and 3.17, the object \( \mathcal{H}om(\Delta_n, i) \) is a hypercovering, and the map \( \mathcal{H}om(\Delta_n, i) \to \mathcal{H}om(S_{n-1}, i) \) is in \( F(\Delta) \). Thus, if two maps \( f, f' : i' \to i \) are given in \( \text{H Cov}(i) \), we can construct the fibred product

\[
\begin{array}{ccc}
i'' & \longrightarrow & \mathcal{H}om(\Delta_1, i) \\
g \downarrow & & \downarrow \\
i' & \longrightarrow & \mathcal{H}om(S_0, i) \cong i \times_i i
\end{array}
\]

in \( \text{H Cov}(i) \), and then \( f \circ g \) and \( f' \circ g \) are connected by an elementary homotopy, thus give the same map in \( HC(i) \). \( \square \)

**Remark 3.25.** We say that a hypercovering \( i, \in \text{H Cov}(i) \) is \textit{n-truncated} for some \( n \geq 0 \) if the map \( i, \to \cosk_n(i) \) is an isomorphism. Then a 1-truncated hypercovering of an object \( i \) is the same as a covering. Moreover, any two maps between 1-truncated hypercoverings are related by an elementary homotopy. Thus, in fact, the full subcategory \( HC(i)_1 \subset HC(i) \) spanned by 1-truncated hypercoverings can be canonically identified with \( C(i) \).

### 4. Dold–Kan equivalence and derived functors

#### 4.1. Dold–Kan equivalence.

Now, let \( \mathcal{E} \) be an additive Karoubi-closed category (for example, an Abelian one). Then we have the \textit{Dold–Kan equivalence}

\[
\text{Fun}(\Delta^o, \mathcal{E}) \cong C_{\geq 0}(E), \quad E \mapsto C_*(E),
\]

where \( C_{\geq 0}(\mathcal{E}) \) is the category of chain complexes in \( \mathcal{E} \) concentrated in non-negative homological degrees and \( C_*(E) \) is the normalized chain complex of the simplicial object \( E \). The original reference for (4.1) is [6], but there are many expositions in the literature. A constant functor \( \Delta^o \to \mathcal{E} \) with some value \( E \) corresponds to the complex that consists of \( E \) placed in degree 0. If \( \mathcal{E} \) is cocomplete, then, by adjunction, \( \text{colim}_{\Delta^o} E \) for some \( E \in \text{Fun}(\Delta^o, \mathcal{E}) \) is the cokernel of the differential \( C_1(E) \to C_0(E) \) in the corresponding chain complex. Augmented simplicial objects correspond to augmented complexes, that is, triples \( \langle M, \alpha, M_\alpha \rangle \) of a chain complex \( M_\alpha \), an object \( M \), and a morphism \( \alpha : M_0 \to M \) such that \( \alpha \circ d = 0 : M_1 \to M_0 \).

For any \( n \geq 0 \), (4.1) also identifies \( \text{Fun}(\Delta^o_{\leq n}, \mathcal{E}) \) with chain complexes \( C_{[0, n]}(\mathcal{E}) \) concentrated in degrees \( 0 \leq i < n \), and the restriction \( j^o_n \) sends a complex \( M_\alpha \) to its \( n \)-th stupid truncation, that is, to the complex \( M_n \to \cdots \to M_0 \). If \( \mathcal{E} \) has kernels, then the right Kan extension \( j^o_n \) exists and sends a complex \( M_n \to \cdots \to M_0 \) to

\[
\text{Ker} d \to M_n \xrightarrow{d} \cdots \xrightarrow{d} M_0.
\]

One can also iterate (4.1) and obtain the equivalence

\[
\text{Fun}(\Delta^o \times \Delta^o, \mathcal{E}) \cong \text{Fun}(\Delta^o, C_{\geq 0}(\mathcal{E})) \cong C_{\geq 0, \geq 0}(\mathcal{E}),
\]

whose target is the category of bicomplexes in \( \mathcal{E} \) concentrated in non-negative homological bidegrees. By an abuse of notation, for any bicomplex \( M_{\bullet \bullet} \), corresponding to
a bisimplicial object $M$ under (4.3), we denote by $\delta^* M_{\cdot \cdot}$ the complex corresponding to $\delta^* M$, where $\delta: \Delta^o \to \Delta^o \times \Delta^o$ is the diagonal embedding. The complex $\delta^* M_{\cdot \cdot}$ differs from the totalization $\text{Tot}(M_{\cdot \cdot})$ of the bicomplex $M_{\cdot \cdot}$, but they are canonically quasi-isomorphic. Namely, there are functorial shuffle maps

$$\text{Tot}(M_{\cdot \cdot}) \to \delta^* M_{\cdot \cdot} \to \text{Tot}(M_{\cdot \cdot}) \quad (4.4)$$

whose composition is the identity map, and both are quasi-isomorphisms if $\mathcal{E}$ is Abelian. (Again, there are many expositions of this in the literature; for example, see [15], §3.4, for a coordinate-free construction.) The individual terms $M_n$ of the complex $M = \delta^* M_{\cdot \cdot}$ are finite sums of the terms of the bicomplex $M_{\cdot \cdot}$. In particular, if all the $M_{\cdot \cdot}$ are projective in $\mathcal{E}$, then so are all the $M_n$.

**Definition 4.1.** Assume that $\mathcal{E}$ is Abelian. Then a map $M_{\cdot \cdot} \to N_{\cdot \cdot}$ in $C_{\geq 0, \geq 0}(\mathcal{E})$ is a left (right) quasi-isomorphism if $M_{n, \cdot} \to N_{n, \cdot}$ ($M_{\cdot, n} \to N_{\cdot, n}$, respectively) is a quasi-isomorphism for any $n \geq 0$.

**Example 4.2.** The totalization functor $\text{Tot}: C_{\geq 0, \geq 0}(\mathcal{E}) \to C_{\geq 0}(\mathcal{E})$ admits two obvious one-sided inverses $L, R: C_{\geq 0}(\mathcal{E}) \to C_{\geq 0, \geq 0}(\mathcal{E})$ given by $L(M_{i,j}) = M_i$ if $j = 0$ and 0 otherwise, and, dually, $R(M_{i,j}) = M_j$ if $j = 0$ and 0 otherwise. (In terms of the Dold–Kan equivalence, we have $L \cong \pi_0^*$ and $R \cong \pi_1^*$, where $\pi_0, \pi_1: \Delta^o \times \Delta^o \to \Delta^o$ are the projections onto the two factors.) There are no functorial maps between $L$ and $R$. However, the totalization functor $\text{Tot}$ also admits a left-adjoint $l: C_{\geq 0}(\mathcal{E}) \to C_{\leq 0, \geq 0}(\mathcal{E})$ given by $l(M_{i,j}) = M_{i,j} \oplus M_{i,j+1}$, with both differentials equal to $d + \text{id}$. Then the isomorphisms $\text{Tot} \circ L \cong L \circ \text{Tot} \cong \text{id}$ provide maps $l \to L$ and $l \to R$ by adjunction, and if $\mathcal{E}$ is Abelian, then these two adjoint maps are, respectively, left and right quasi-isomorphisms in the sense of Definition 4.1.

**Lemma 4.3.** Assume that $\mathcal{E}$ is Abelian and we have bicomplexes $M_{\cdot \cdot}, M'_{\cdot \cdot}$ in $\mathcal{E}$ and a map $f: M_{\cdot \cdot} \to M'_{\cdot \cdot}$ that is a left or right quasi-isomorphism in the sense of Definition 4.1. Then $\delta^*(f): \delta^* M_{\cdot \cdot} \to \delta^* M'_{\cdot \cdot}$ is also a quasi-isomorphism.

**Proof.** The totalization $\text{Tot}(f)$ is obviously a quasi-isomorphism, and so are the shuffle maps (4.4) both for $M_{\cdot \cdot}$ and $M'_{\cdot \cdot}$. □

Dually, one can replace $\Delta^o$ in (4.1) with $\Delta$, and then $C_{\geq 0}(\cdot)$ gets replaced with the category $C_{\geq 0}(\cdot)$ of complexes concentrated in non-negative cohomological degrees. (To deduce the corresponding statement for $\mathcal{E}$ we apply (4.1) to $\mathcal{E}^o$, which is additive and Karoubi-closed.) The rest of the material up to and including Example 4.2 and Lemma 4.3 also has an obvious dual counterpart.

### 4.2. Covering classes.

Next, we want to equip our additive Karoubi-closed category $\mathcal{E}$ with a covering class $F$. There are two possible choices. Firstly, assume that $\mathcal{E}$ has kernels. Then $\mathcal{E}$ is finitely complete, and we can take the class of all maps. The hypercovering condition for this class is empty ($\text{H Cov}(M)$ is simply the category of all chain complexes equipped with an augmentation $M_{\cdot} \to M$), but Definition 3.21 is still non-trivial. To describe what it says we note that any additive category $\mathcal{E}$ is trivially a module category over the monoidal category $\mathbb{Z}$-$\text{mod}^{ff}$ of finitely generated free $\mathbb{Z}$-modules. While the Dold–Kan equivalence is not tensorial, for any $M: \Delta^o \to \mathcal{E}$ and $V: \Delta^o \to \mathbb{Z}$-$\text{mod}^{ff}$, we have shuffle maps

$$C_*(M) \otimes C_*(V) \to C_*(M \otimes V) \to C_*(M) \otimes C_*(V) \quad (4.5)$$
induced by (4.4). Indeed, if one considers the bicomplex $C_\ast(M) \boxtimes C_\ast(V)$ with terms $C_n(M) \otimes C_n(V)$, then

$$C_\ast(M) \otimes C_\ast(V) \cong \operatorname{Tot}(C_\ast(M) \boxtimes C_\ast(V)) \quad \text{and} \quad C_\ast(M \otimes V) \cong \delta^*(C_\ast(M) \boxtimes C_\ast(V)).$$

If we let $\mathbb{Z}[\Delta_1]: \Delta^\circ \to \mathbb{Z}-\text{mod}$ be the simplicial $\mathbb{Z}$-module generated pointwise by the elementary 1-simplex, then specifying an elementary homotopy in the sense of Definition 3.21 between two maps $f, f': E_\ast \to E'_\ast$ is equivalent to specifying a map

$$h: C_\ast(E_\ast \otimes \mathbb{Z}[^\Delta^1_1]) \to C_\ast(E'_\ast) \quad (4.6)$$

with a given restriction to $C_\ast(E_\ast \otimes \mathbb{Z}[S_0]) \cong C_\ast(E_\ast) \oplus C_\ast(E_\ast)$. However, $C_\ast(\mathbb{Z}[\Delta_1])$ is the length-2 complex $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, so that specifying a chain homotopy between $f$ and $f'$ in the usual sense is equivalent to specifying a map

$$h': C_\ast(E_\ast) \otimes C_\ast(\mathbb{Z}[\Delta_1]) \to C_\ast(E'_\ast). \quad (4.7)$$

By virtue of (4.5), the domain of (4.7) is a retract of the domain of (4.6), so that an elementary homotopy exists if and only if the maps are chain-homotopic in the standard sense. This relation is already transitive, so that being chain-homotopic in the usual sense and in the sense of Definition 3.21 amounts to the same thing.

A more interesting alternative is to assume further that $\mathcal{E}^\circ$ is either Abelian, or small and pre-Abelian in the sense of Definition 1.10, so that $\operatorname{Pro}(\mathcal{E}) \cong \operatorname{Ind}(\mathcal{E}^\circ)^\circ$ is Abelian, and to take as $F$ the class of all epimorphisms in $\mathcal{E}$. Since epimorphisms in an Abelian category are obviously stable under pullbacks, and the full embedding $\mathcal{E} \subset \operatorname{Pro}(\mathcal{E}) = \operatorname{Ind}(\mathcal{E}^\circ)^\circ$ reflects epimorphisms, $F$ is again a covering class. Then (4.2) shows immediately that an augmented complex $\langle M_\ast, M, a \rangle$ is an $F$-hypercovering if and only if $a: M_\ast \to M$ is a quasi-isomorphism, that is, $M_\ast$ is a left resolution of $M$ in $\operatorname{Pro}(\mathcal{E}) \supset \mathcal{E}$, and $HC(M)$ is the category of left resolutions of the object $M \in \mathcal{E}$ and chain-homotopy classes of maps between them. Dually, if $\mathcal{E}$ is pre-Abelian, then hypercoverings in the opposite category $\mathcal{E}^\circ$ correspond to right resolutions in $\operatorname{Ind}(\mathcal{E}^\circ)$.

If we have categories $I, I'$ with covering classes $F, F'$ and a functor $\gamma: I \to I'$ that commutes with finite limits and sends morphisms in $F$ to morphisms in $F'$, then it tautologically sends $F$-hypercoverings to $F'$-hypercoverings. For example, this happens if $I = \mathcal{E}$ is an Abelian category with some projective object $E$, $E' = \{\text{Sets, } F \text{ and } F' \text{ consist of epimorphisms, and } \gamma \text{ is the functor } \operatorname{Hom}(E, -)\}$. If $\mathcal{E}$ has arbitrary sums, then this particular functor has a left-adjoint $\operatorname{Sets} \dashv \mathcal{E}$, $\mathcal{S} \mapsto E[S]$, where, as in Example 2.1, $E[S]$ stands for the sum of copies of $E$ numbered by elements $s \in S$. This adjoint still sends epimorphisms to epimorphisms even if $E$ is not projective, but it certainly never commutes with limits. However, the following is still true.

**Lemma 4.4.** Assume given a set $S$, a collection $E_s \in \mathcal{E}, s \in S$, of objects in an Abelian category $\mathcal{E}$ with arbitrary sums, and a collection of augmented simplicial sets $X_s: \Delta^{\leq \circ} \to \{\text{Sets, } s \in S, \text{ that are hypercoverings with respect to the class of epimorphisms. Then the augmented simplicial object } \bigoplus_s E_s[X_s]: \Delta^{\leq \circ} \to \mathcal{E} \text{ is a hypercovering with respect to the class of epimorphisms.}**
Sketch of a proof. Since $\bigoplus_s E_s[X_s]$ is a retract of $E[X]$, $E = \bigoplus_s E_s$, and $X = \bigprod_s X_s$, it suffices to consider the case when $S = \text{pt}$ has a single element. Moreover, it suffices to prove that for any $E' \in \mathcal{E}$, the augmented functor $\text{Hom}(E(X), E') : \Delta^< \rightarrow \text{Ab}^o$ is a hypercovering in the category $\text{Ab}^o$ opposite to the category of Abelian groups, again with respect to epimorphisms, and this amounts to our statement with $\mathcal{E}$ replaced by $\text{Ab}$ and $E$ by $\text{Hom}(E, E')$. In other words, we can assume right away that $\mathcal{E} = \text{Ab}$. Then $E[X] \cong E \otimes \mathbb{Z} [X]$, so we can assume that $E = \mathbb{Z}$, and we can further apply the Dold–Kan equivalence to replace $E[X]$ with the normalized chain complex $C_s(X, \mathbb{Z})$. The statement is then well known: a hypercovering is, in particular, a weak equivalence, and $C_\ast(-, \mathbb{Z})$ sends weak equivalences to quasi-isomorphisms. Here is one argument for this: since all simplicial sets are cofibrant, any weak equivalence is a composition of trivial cofibrations and their one-sided inverses, so it suffices to treat trivial cofibrations; since filtered colimits in $\text{Ab}$ preserve quasi-isomorphisms, it further suffices to consider elementary trivial cofibrations, that is, pushouts of horn extensions; but a pushout of an injective quasi-isomorphism is a quasi-isomorphism, so it further suffices to consider horn embeddings. These give quasi-isomorphisms because of how the Dold–Kan equivalence works.

Remark 4.5. The statement of Lemma 4.4 is essentially a fact from topology (‘no homotopy implies no homology’, which is, roughly speaking, the Hurewicz theorem). The proof sketched above uses Quillen model structures. There are many alternative proofs, but none are elementary or purely categorical.

4.3. Dold-derived functors. Now assume given a functor $E : \mathcal{C} \rightarrow \mathcal{E}$ between complete categories $\mathcal{C}$ and $\mathcal{E}$. We extend $E$ to a functor $E : \text{Fun}(\Delta^o, \mathcal{C}) \rightarrow \text{Fun}(\Delta^o, \mathcal{E})$ by applying it termwise. The following is an essentially classical observation due to Dold.

Lemma 4.6. If two maps $f$ and $f'$ in $\text{Fun}(\Delta^o, \mathcal{C})$ are chain-homotopic in the sense of Definition 3.22, then so are the maps $E(f)$ and $E(f')$ in $\text{Fun}(\Delta^o, \mathcal{E})$.

Proof. By induction on the length of the chain, we can assume that the maps $f, f' : c \rightarrow c'$ are connected by a homotopy

$$\tilde{f} : c \rightarrow \text{Hom}(\Delta_1, c') = \pi_\ast \pi^\ast c',$$

where $\pi = \sigma([1])^o : (\Delta/[1])^o \rightarrow \Delta^o$ is the projection. But then we tautologically have $E \circ \pi^\ast \cong \pi^\ast \circ E$, and the adjunction map $a : \pi^\ast \pi_\ast \pi^\ast c' \rightarrow \pi^\ast c'$ gives rise to a map

$$E(a) : \pi^\ast E(\pi^\ast \pi^\ast c') \cong E(\pi^\ast \pi_\ast \pi^\ast c') \rightarrow E(\pi^\ast c') \cong \pi^\ast E(c').$$

This is in turn adjoint to the map $a' : E(\pi_\ast \pi^\ast c') \rightarrow \pi_\ast \pi^\ast E(c')$, and $a' \circ E(\tilde{f})$ is an elementary homotopy connecting $E(f)$ and $E(f')$. \(\square\)

If the source category $\mathcal{C}$ is not complete, Lemma 4.6 still holds, with the same proof, if we restrict our attention to a full subcategory in $\text{Fun}(\Delta^o, \mathcal{C})$ where $\text{Hom}(\Delta_1, -)$ and $\text{Hom}(S_0, -)$ exist. In particular, assume given a small category $I$ equipped with a covering class $F$. Then one can take $\mathcal{C} = \text{Cov}(i)$, $i \in I$, and consider the subcategory $\text{HCOv}(i) \subset \text{Fun}(\Delta^o, \mathcal{C})$. As a target category $\mathcal{E}$ let us
take a finitely presentable Abelian category. Then, for any functor $E: I^o \to \mathcal{E}$ and any object $i \in I$, we can consider the functor $E_i^o = E^o \circ \sigma(i): \text{Cov}(i) \to \mathcal{E}^o$ and apply Lemma 4.6 to its extension $E_i^o(\Delta): \text{HCov}(i) \to \text{Fun}(\Delta^o, \mathcal{E}^o)$. Thus, if we compose $E_i^o(\Delta)$ with the Dold–Kan equivalence, we obtain a functor $\text{HCov}(i) \to C_{\geq 0}(\mathcal{E}^o)$ that sends chain-homotopic maps to chain-homotopic maps. Dually, $E_i: \text{HCov}(i)^o \to \text{Fun}(\Delta, \mathcal{E}) \cong C^{\geq 0}(\mathcal{E})$ also sends chain-homotopic maps to chain-homotopic maps, so that for any $n$, the $n$-th homology group $H^n(E_i(i'))$ of the complex $C(E_i(i'))$ defines a functor $\text{HCov}(i)^o \to \mathcal{E}$ that factors through the filtered category $HC(i)$.

**Definition 4.7.** The $n$-th Dold-derived functor $D^n(E): I^o \to \mathcal{E}$ of a functor $E: I^o \to \mathcal{E}$ is given by

$$D^n(E)(i) = \text{colim}_{i \in HC(i)^o} H^n(C_*(E_i(i))).$$

This is functorial in $i$ since any map $i' \to i$ in $I$ induces a functor $\text{HCov}(i) \to \text{HCov}(i')$ that sends chain-homotopic maps to chain-homotopic maps, so we again have a Grothendieck fibration $HC \to I$ with fibres $HC(i)$ and can apply (1.3).

**Proposition 4.8.** For any $n \geq 0$ and $E \in \text{Fun}(I^o, \mathcal{E})$, we have a functorial isomorphism

$$D^n(E) \cong R^n e a(E),$$

where $a: \text{Fun}(I^o, \mathcal{E}) \to \text{Shv}(I, \mathcal{E})$ is the associated sheaf functor and $R^* e$ are the derived functors of the embedding functor $e: \text{Shv}(I, \mathcal{E}) \to \text{Fun}(I^o, \mathcal{E})$.

**Proof.** Take an object $i \in I$, consider the category $\text{HCov}(i)$ with the covering class $F_\Delta$, and choose an $F_\Delta$-hull $\tilde{i} \in \text{HCov}(i)$, which exists by Lemma 2.11, (ii). By definition, $\tilde{i}$ is a proobject in $\text{HCov}(i)$ represented by some filtered diagram $\gamma: J \to \text{HCov}(i)^o$. Since $\tilde{i}$ is $F_\Delta$-liftable, $i' \setminus J$ is non-empty and directed for any $i' \in \text{HCov}(i)$, hence the same is true for the right comma-fibres of the composition functor $J \to \text{HCov}(i)^o \to HC(i)^o$. Since $J$ is filtered, the composition functor is cofinal, and we have

$$D^n(E)(i) \cong \text{colim}_{j \in J} H^n(E_i(\gamma(j))) \cong H^n(E(\sigma(i)(\tilde{i}))),$$

(4.8)

where the second isomorphism holds since filtered colimits in $\mathcal{E}$ are exact. Moreover, by Lemma 3.17 and adjunction, $ev_n(\tilde{i})$ is $\sigma(i)^* F$-liftable in $\text{Cov}(i)$, so $\sigma(i)(\tilde{i})$ is $F$-liftable in $I$ by Lemma 2.11, (i). By Lemma 2.8, (4.8) then immediately implies that $D^n(E) \cong D^n(a(E))$, so that $D^n(\cdot)$ factors through the associated sheaf functor. Moreover, if $n = 0$, then Dold–Kan equivalence provides a Cartesian square

$$
\begin{array}{ccc}
H^0(E_i(i')) & \longrightarrow & E_i(i'_0) \\
\downarrow & & \downarrow \\
E_i(i'_0) & \longrightarrow & E_i(i'_1)
\end{array}
$$

(4.9)

for any hypercovering $i' \in \text{HCov}(i)$, and if $E$ is a sheaf, then the map $E_i(i'_0 \times_i i'_0) \to E_i(i'_1)$ is injective, so we can replace $E_i(i'_1)$ in (4.9) with $E_i(i'_0 \times_i i'_0)$ and conclude
from the sheaf condition that \( H^0(E_i(i')) \cong E(i) \) for any \( i' \in \text{HCov}(i') \). Therefore \( D^0(E) \cong e(a(E)) \). Finally, by (4.8), the collection \( D^n(\cdot) \) defines a \( \delta \)-functor on \( \text{Shv}(I, \mathcal{E}) \) in the sense of [10], so if we say that a sheaf \( E \in \text{Shv}(I, \mathcal{E}) \) is exact when \( H^n(E_i(i')) = 0 \) for \( n \geq 1 \) and any \( i \in I \) and \( i' \in \text{HCov}(i) \), then it suffices to show that any sheaf \( E \) admits an injective map \( E \to E' \) to an exact sheaf \( E' \). For any object \( i \in I \), choose an \( F \)-hull \( \tilde{i} \) provided by Lemma 2.11,(ii), and consider the product

\[
E' = \prod_{i \in I} Y_{\tilde{i}}(E(\tilde{i})) ,
\]

where \( Y_{\tilde{i}}(E(\tilde{i})) \), \( i \in I \), are as in Lemma 2.8. Then, on the one hand, \( E' \) is exact by Lemma 4.4, and on the other hand, the maps \( \tilde{i} \to i \) induce a map \( a: E \to E' \), and since \( E \) is a sheaf, all the maps \( E(i) \to E(\tilde{i}) \) are injective. Thus the map \( e\nu_i(a) \) is injective for every \( i \in I \), so that \( a \) itself is injective. □

### 5. Categories of morphisms

We can now define and study the main subject of the paper, namely, the category of morphisms between Abelian categories. Assume given finitely presentable Abelian categories \( A \) and \( B \). Recall that the subcategory \( \mathcal{A}_c \subset A \) of compact objects in \( A \) is pre-Abelian in the sense of Definition 1.10, and the class \( \text{Epi} \) of epimorphisms is a covering class in the opposite category \( A^o \).

**Definition 5.1.** The category of morphisms \( \text{Mor}(A, B) \) is the full subcategory \( \text{Mor}(A, B) = \text{Shv}(\mathcal{A}_c^o, B) \subset \text{Fun}(A^c, B) \cong \text{Fun}_c(A, B) \) spanned by sheaves with respect to the \( \text{Epi} \)-topology on \( A^o \).

By Example 2.1, the category \( \text{Mor}(A, B) \) is a Grothendieck Abelian category, and we have a tautological action functor

\[
A \times \text{Mor}(A, B) \to B .
\]  

(5.1)

Conversely, a functor \( E: A \to B \) comes from a morphism if and only if

(i) \( E \) is continuous, and

(ii) for any injective morphism \( a: A \to B \) in \( \mathcal{A}_c \), the morphism \( E(a): E(A) \to E(B) \) is injective in \( B \), and so is the map

\[
E(B) \oplus_{E(A)} E(B) \to E(B \oplus_A B) ,
\]  

(5.2)

where we denote by \( B \oplus_A B \) the cokernel of the map \( a \oplus (-a): A \to B \oplus B \), and similarly for \( E(B) \oplus_{E(A)} E(B) \).

Indeed, (5.2) is just (2.6) for \( \langle \mathcal{A}_c^o, \text{Epi} \rangle \), and we note that epimorphisms in \( \mathcal{A}_c^o \) are monomorphisms in \( A_c \). Since both (i) and (ii) are obviously closed under compositions, we have natural composition functors

\[
\text{Mor}(A, B) \times \text{Mor}(B, C) \to \text{Mor}(A, C)
\]

for any three finitely presentable Abelian categories \( A, B, C \).
Example 5.2. The sheaf conditions (i) and (ii) imply immediately that an object $E \in \text{Fun}(A_c, B)$ is an additive sheaf if and only if its continuous extension $E : A \to B$ is left-exact in the usual sense (that is, it commutes with finite limits). Thus the full subcategory $\text{Mor}_\text{add}(A, B) \subset \text{Mor}(A, B)$ spanned by additive functors is the category of left-exact continuous functors $A \to B$. In particular, this category of functors is a Grothendieck Abelian category. However, $\text{Mor}(A, B)$ is strictly bigger: not all morphisms are additive. For example, if $A = B = \text{Ab}$, then $\text{Mor}(A, B) \subset \text{Fun}_c(\text{Ab}, \text{Ab})$ contains the functor sending an Abelian group $A$ to the free Abelian group $\mathbb{Z}[A]$ spanned by the underlying set of $A$.

Example 5.3. Note that if $A$ is a small Abelian category, then to define the category $\text{Mor}(\text{Ind}(A), \text{Ab}) \cong \text{Shv}(A^o, \text{Ab})$ with its full subcategory $\text{Mor}_\text{add}(\text{Ind}(A), \text{Ab}) \subset \text{Mor}(\text{Ind}(A), \text{Ab})$ and to prove that both are Grothendieck Abelian categories, we do not need to know that $\text{Ind}(A)$ is Abelian. Conversely, the simplest way to prove that $\text{Ind}(A)$ is Abelian is to show that the Yoneda embedding (1.7) identifies it with $\text{Mor}_\text{add}(\text{Ind}(A), \text{Ab})$, which is a version of the fundamental Gabriel–Popescu Theorem (see [4], Chap. 5, §10) and in our setting it immediately follows from Example 1.6.

On the derived level, we denote by $\text{DMor}(A, B)$ the derived category of the Abelian category $\text{Mor}(A, B)$, and we let $\text{DMor}^{\geq 0}(A, B) \subset \text{DMor}(A, B)$ be the full subcategory spanned by coconnective objects. We have a fully faithful embedding $R^e : \text{DMor}^{\geq 0}(A, B) \to \mathcal{D}^{\geq 0}(A, B) \cong \mathcal{D}_c^{\geq 0}(A, B)$ constructed in Proposition 4.8. We recall that the target category can be obtained by localizing the category of complexes

$$C^{\geq 0}(\text{Fun}(A_c, B)) \cong \text{Fun}(A_c, C^{\geq 0}(B)) \cong \text{Fun}_c(A, C^{\geq 0}(B))$$

with respect to quasi-isomorphisms, and one can also use the Dold–Kan equivalence to replace $C^{\geq 0}(-)$ with $\text{Fun}(\Delta, -)$. This makes it possible to extend the functor interpretation of $\text{Mor}(A, B)$ to $\text{DMor}^{\geq 0}(A, B)$. Namely, for any functor $E : A_c \to \text{Fun}(\Delta, B)$, we have its canonical continuous extension $E : A \to \text{Fun}(\Delta, B)$, and we can further define the Dold extension

$$\text{D}(E) \text{Fun}(\Delta, A) \to \text{Fun}(\Delta, B) \quad (5.3)$$

as the composition

$$\text{Fun}(\Delta, A) \xrightarrow{E} \text{Fun}(\Delta \times \Delta, B) \xrightarrow{\delta^*} \text{Fun}(\Delta, B),$$

where the first functor is $E$ applied termwise, and $\delta : \Delta \to \Delta \times \Delta$ is the diagonal embedding.

Definition 5.4. A continuous functor $C^{\geq 0}(A) \to C^{\geq 0}(B)$ is said to be homotopical if it sends quasi-isomorphisms to quasi-isomorphisms.

Theorem 5.5. A functor $E : A_c \to C^{\geq 0}(B) \cong \text{Fun}(\Delta, B)$ represents an object in $\text{DMor}^{\geq 0}(A, B) \subset \mathcal{D}^{\geq 0}(A_c, B)$ if and only if its Dold extension (5.3) is homotopical in the sense of Definition 5.4.
Proof. Let us begin with the ‘only if’ part. We say that $E: A_c \to C^{\geq 0}(B)$ is good up to degree $n \geq 0$ if for any quasi-isomorphism $f$ in $C^{\geq 0}(A)$, $D(E)(f)$ is a quasi-isomorphism in degrees $\leq n$. By Lemma 4.3, a pointwise quasi-isomorphism $E_0 \to E_1$ induces a pointwise quasi-isomorphism $D(E_0) \to D(E_1)$, so that being good only depends on the object in $D^{\geq 0}(A_c, B)$ represented by $E$. Moreover, for any complex $M^\bullet \in C^{\geq 0}(\text{Mor}(A, B))$, note that (i) $R^e(M^\bullet)$ is good up to degree $n$ if and only if so is $R^e(M^{\leq n+1})$, where $M^{n+1}$ is the stupid truncation $M^0 \to \cdots \to M^{n+1}$, and (ii) the property of being good up to degree $n$ is stable under extensions, and the property of being good up to any degree is also stable under homological shifts. Hence to prove that $R^e(M^\bullet)$ is good in all degrees for any complex $M^\bullet$ it suffices to consider complexes concentrated in degree 0.

Moreover, we say that a sheaf $M \in \text{Mor}(A, B)$ is ind-exact if $H^n(M(\tilde{A})) = 0$ for any $n \geq 1$ and any hypercovering $\tilde{A}$ of an object $A \in A^\circ$. (In particular, this applies to hypercoverings in $A^\circ_c \subset A^\circ$, so that an ind-exact sheaf is exact in the sense used in the proof of Proposition 4.8.) Then, since $A$ is finitely presentable, it is a Grothendieck Abelian category, thus has enough injectives, and any injective $I \in A$ is Epi-liftable when considered as an object of $A^\circ \cong \text{Pro}(A^\circ_c)$. Then, by Lemma 2.8, $Y_{A}(B) \in \text{Fun}(A_c, B)$ is a sheaf for any $B \in B$ and $A \in A$, and by exactly the same argument as in the proof of Proposition 4.8, any $M \in \text{Mor}(A, B)$ admits an embedding into an ind-exact sheaf of the form

$$M' = \prod_{A \in A_c} Y_A(M(\tilde{A})),$$

where we choose an embedding $A \to \tilde{A}$ into some injective $\tilde{A} \in A$ for any $A \in A_c$. We conclude that every sheaf in $\text{Mor}(A, B)$ admits a resolution by ind-exact sheaves, so it suffices to prove that $R^e(M)$ is good for an ind-exact sheaf $M \in \text{Mor}(A, B)$. In this case the Dold-derived functors $D^n(M)$ vanish for $n \geq 1$, so we have $R^e(M) \cong M$ by Proposition 4.8, and we need to show that $D(M)(f)$ is a quasi-isomorphism for any quasi-isomorphism $f: A^\bullet \to B^\bullet$ in $C^{\geq 0}(A)$.

In the simple case when $A^\bullet = A \in A \subset C^{\geq 0}(A)$ is an object in $A$ considered as a complex concentrated in degree 0, the claim follows from the definition since $B^\bullet$ is a hypercovering of $A$.

In the general case note that the Abelian category $C^{\geq 0}(A)$ is also a Grothendieck Abelian category, thus has enough injectives. Moreover, for any injective $I^\bullet \in C^{\geq 0}(A)$, $I^n \in A$ is injective for any $n \geq 0$. Therefore $A^\bullet$ admits an injective resolution in $C^{\geq 0}(A)$, and this is a bicomplex $I^{\bullet, \bullet} \in C^{\geq 0, \geq 0}(A)$ with injective terms equipped with a map $A^\bullet \to I^{\bullet, \bullet}$ such that $A^n \to I^{n, n}$ is a quasi-isomorphism for any $n \geq 0$. Then, on the one hand, $f_A: A^\bullet \to I^\bullet = \delta^* I^{\bullet, \bullet}$ is a quasi-isomorphism by Lemma 4.3 and $I^\bullet$ is a complex of injectives, and on the other hand, the simple case of the claim, which we have already proved, together with Lemma 4.3 show that $D(M)(f_A)$ is also a quasi-isomorphism. Applying the same argument to $B^\bullet$, we can reduce the claim to the case when $f: A^\bullet \to B^\bullet$ is a map between complexes of injectives. But in this case $f$ is invertible up to a chain-homotopy equivalence, and everything follows from Lemma 4.6.

Finally, for the ‘if’ part, we say that $E \in D^{\geq 0}(A_c, B)$ is good if $D(E)$ is homotopical. Then, in particular, $E$ inverts all maps between 1-truncated coverings
of objects in $A$ in the sense of Remark 3.25, so that the canonical truncation $\tau^{\leq 0} E \in \text{Fun}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}^{\geq 0}(\mathcal{A}, \mathcal{B})$ is a sheaf. Thus, if $a(E) = 0$, then $\tau^{\leq 0}(E) = 0$, so that $E \in \mathcal{D}^{\geq 1}(\mathcal{A}, \mathcal{B})$. Then the same argument applies to the homological shift $E[1]$, so $E \in \mathcal{D}^{\geq n}(\mathcal{A}, \mathcal{B})$ for any $n \geq 0$ by induction. We conclude that if $a(E) = 0$ for a good $E$, then $E = 0$. If not, let $E'$ be the cone of the adjunction map $E \to a(R^e(E))$. Since we have already proved that $a(R^e(E))$ is good, so is $E'$, and since $a(E') = 0$, we have $E' = 0$ and $E \cong R^e(a(E))$. □

By Theorem 5.5, any continuous functor $E : \mathcal{A} \to \mathcal{C}^*(\mathcal{B})$ that represents an object in $\text{DMor}^{\geq 0}(\mathcal{A}, \mathcal{B})$ extends to a homotopical continuous functor $\mathcal{C}^{\geq 0}(\mathcal{A}) \to \mathcal{C}^{\geq 0}(\mathcal{B})$. Let us complement this result by showing that any continuous homotopical functor appears in this way. Namely, recall that $\mathcal{C}^{\geq 0}(\mathcal{A}) \cong \text{Ind}(\mathcal{D}_h^{\geq 0}(\mathcal{A}))$ is finitely presentable, and let $\mathcal{D}_h^{\geq 0}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}(\mathcal{C}^{\geq 0}(\mathcal{A}), \mathcal{B})$ be the full subcategory spanned by homotopical functors. We then have a natural projection

$$
\tau^* : \mathcal{D}h^{\geq 0}(\mathcal{A}, \mathcal{B}) \to \mathcal{D}^{\geq 0}(\mathcal{A}, \mathcal{B}) \cong \mathcal{D}_c^{\geq 0}(\mathcal{A}, \mathcal{B}),
$$

(5.4)

where $\tau : \mathcal{A} \to \mathcal{C}^{\geq 0}(\mathcal{A})$ sends an object $A \in \mathcal{A}$ to the same object, but considered as a complex concentrated in degree 0.

**Corollary 5.6.** The projection (5.4) factors through an equivalence of categories $\mathcal{D}h^{\geq 0}(\mathcal{A}, \mathcal{B}) \cong \text{DMor}^{\geq 0}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}^{\geq 0}(\mathcal{A}, \mathcal{B}) \cong \mathcal{D}_c^{\geq 0}(\mathcal{A}, \mathcal{B})$.

**Proof.** The fact that (5.4) factors through $\text{DMor}^{\geq 0}(\mathcal{A}, \mathcal{B})$ follows immediately from Theorem 5.5. Moreover, we obviously have $\tau^*(\mathcal{D}(E)) \cong E$, so that the correspondence $E \mapsto \mathcal{D}(E)$ provides a one-sided inverse to the functor $\tau^* : \mathcal{D}h^{\geq 0}(\mathcal{A}, \mathcal{B}) \to \text{DMor}^{\geq 0}(\mathcal{A}, \mathcal{B})$. To prove the claim it then suffices to construct a functorial isomorphism $E \cong \mathcal{D}(\tau^* E)$ for any $E \in \mathcal{D}h^{\geq 0}(\mathcal{A}, \mathcal{B})$. To do this we represent $E$ by a homotopical continuous functor $E : \mathcal{C}^{\geq 0}(\mathcal{A}) \to \mathcal{C}^{\geq 0}(\mathcal{B})$ and note that $\mathcal{C}^{\geq 0}(\mathcal{A})$ is also an Abelian category, so that $E$ admits a Dold extension to a continuous functor $\mathcal{D}(E) : \mathcal{C}^{\geq 0, \geq 0}(\mathcal{A}) \to \mathcal{C}^{\geq 0}(\mathcal{B})$. We then have $E \cong \mathcal{D}(E) \circ \mathcal{L}$ and $\mathcal{D}((\tau^* E)) \cong \mathcal{D}(E) \circ \mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are the embeddings of Example 4.2. However, we also have the embedding $\mathcal{I}$ and maps $\mathcal{L}, \mathcal{R} \to \mathcal{I}$ which give rise to functorial maps

$$
E \cong \mathcal{D}(E) \circ \mathcal{L} \to \mathcal{D}(E) \circ \mathcal{I} \leftarrow \mathcal{D}(E) \circ \mathcal{R} \cong \mathcal{D}(\tau^* E),
$$

(5.5)

so it suffices to prove that both maps in (5.5) are quasi-isomorphisms. By Example 4.2, it then suffices to prove that $\mathcal{D}(E)$ sends both left and right quasi-isomorphisms of Definition 4.1 to quasi-isomorphisms in $\mathcal{C}^{\geq 0}(\mathcal{B})$. For right quasi-isomorphisms, this follows immediately from the fact that $E$ is homotopical, and for left quasi-isomorphisms it follows from Theorem 5.5. □

**Remark 5.7.** As we saw in the proof of Theorem 5.5, any object $E$ in $\text{DMor}^{\geq 0}(\mathcal{A}, \mathcal{B})$ can be represented by a complex $M'$ of ind-exact sheaves, and then $E \cong e(M') : \mathcal{A} \to \mathcal{C}^{\geq 0}(\mathcal{B})$ is also a complex of sheaves, so that the Dold extension $\mathcal{D}(E)$ can be represented by a continuous homotopical functor $\mathcal{C}^{\geq 0}(\mathcal{A}) \to \mathcal{C}^{\geq 0}(\mathcal{B})$ that sends monomorphisms to monomorphisms. This makes it a left-derivable functor with respect to the injective model structures on $\mathcal{C}^{\geq 0}(\mathcal{A})$ and $\mathcal{C}^{\geq 0}(\mathcal{B})$ in the sense of Definition 3.1 in [16]. By Corollary 5.6, any continuous homotopical
functor is then quasi-isomorphic to a left-derivable one. This has its uses. In particular, left-derivable functors can be applied in gluing constructions such as in [16], §3.

6. Stability and extensions

6.1. Stability. Under the assumptions of Theorem 5.5, any homotopical functor $E: C^{>0}(A) \to C^{>0}(B)$ descends, by definition, to a functor

$$D(E): D^{>0}(A) \to D^{>0}(B),$$

and Theorem 5.5 then shows that (5.1) extends to an action functor

$$D^{>0}(A) \times \text{DMor}^{>0}(A, B) \to D^{>0}(B).$$

However, there is nothing in Theorem 5.5 to ensure that the functor $D(E)$ is triangulated in any sense of the word. In particular, it is not necessarily additive and does not commute with homological shifts, so that there is no easy way to extend (6.2) to full derived categories. The latter can actually be made more precise in the following way.

For any cosimplicial object $c: \Delta \to \mathcal{C}$ in a complete category $\mathcal{C}$ and any simplicial set $X$, we have the cosimplicial object $c(X): \Delta \to \mathcal{C}$ with terms $c(X)([n]) = c([n])(X([n]))$. Equivalently, $c(X) \cong \pi_* \pi^* c$, where $\pi: \Delta X \to \Delta$ is the forgetful functor from the category of elements of $X: \Delta^o \to \text{Sets}$. If we have a functor $E: \mathcal{C} \to \mathcal{E}$ to another complete category $\mathcal{E}$ and extend it to functors $E: \text{Fun}(\Delta, \mathcal{C}) \to \text{Fun}(\Delta, \mathcal{E})$ and $\tilde{E}: \text{Fun}(\Delta X, \mathcal{C}) \to \text{Fun}(\Delta X, \mathcal{E})$ by applying it pointwise, then by the same argument as in Lemma 4.6, the tautological isomorphism $E \circ \pi^* \cong \pi^* \circ E$ induces by adjunction a map $E \circ \pi_* \to \pi_* \circ \tilde{E}$, and taken together the two give a map

$$E(c(X)) \to E(c)(X),$$

functorial in $E$, $c$, and $X$. Now take $\mathcal{C} = A$ and $\mathcal{E} = B$, and let $S^1: \Delta^o \to \text{Sets}$ be the simplicial circle obtained by gluing together both ends of the simplicial interval $\Delta_1$. (This is different from the standard simplicial circle $S_1$, which we define to be the boundary of the 2-simplex $\Delta_2$.) Then the projection $S^1 \to \Delta_0 = \text{pt} \Delta^o$ admits a unique splitting $\Delta_0 \to S^1$, so that for any $A: \Delta \to A$, we have a splitting

$$A(S^1) \cong A \oplus \Omega(A)$$

for some object $\Omega(A): \Delta \to A$ functorial in $A$. Up to a quasi-isomorphism, $\Omega(A)$ corresponds to the homological shift $A[-1]$ under the Dold–Kan equivalence (4.1). The morphism (6.3) provides a functorial map

$$E(\Omega(A)) \to \Omega(E(A)),$$

and one can ask whether or not this map is a quasi-isomorphism for any $A$. This turns out to be a non-trivial condition on $E$, and it can be stated in several equivalent ways.
Definition 6.1. For any two Abelian categories $A$ and $B$, a functor $E: A \to C^*(B)$ is stable if, for any short exact sequence (1.11) in $A$ with the corresponding bi-Cartesian square $\gamma: [1]^2 \to A$ as in (1.13), the induced square $\gamma^*E \in \mathcal{D}([1]^2, B)$ is homotopy bi-Cartesian.

Stability in the sense of Definition 6.1 is obviously invariant under taking cones and quasi-isomorphisms, so if $A$ and $B$ are finitely presentable and $E$ is continuous, it only depends on the object in $\mathcal{D}_c(A, B)$ represented by $E$. In particular, it makes sense to say that an object in $\text{DMor}(A, B) \subset \mathcal{D}_c(A, B)$ is stable. Moreover, since $C^{\geq 0}(A)$ is also Abelian and finitely presentable, stability also makes sense for objects in $\mathcal{D}H^{\geq 0}(A, B) \subset \mathcal{D}_c(C^{\geq 0}(A), B)$.

Remark 6.2. In practice, stability means two things: (i) the functor $\mathcal{D}(E)$ of (1.15) is pointed, so that $\mathcal{D}(E)(p) \circ \mathcal{D}(E)(i) = \mathcal{D}(E)(p \circ i)$ is 0, and (ii) the induced map from a cone of $\mathcal{D}(E)(i)$ to $\mathcal{D}(E)(C)$ is an isomorphism, so that $\mathcal{D}(E)$ sends short exact sequences to distinguished triangles. However, as we saw in Example 1.11, this ‘induced map’ only becomes uniquely defined once we lift $\mathcal{D}(E)$ to an object $E \in \mathcal{D}(A, B)$. Knowing $\mathcal{D}(E)$ alone is not enough.

Proposition 6.3. Assume given an object $E \in \mathcal{D}^{\geq 0}_c(A, B)$ represented by a continuous homotopical functor $E: C^{\geq 0}(A) \to C^{\geq 0}(B)$ with the corresponding functor $\mathcal{D}(E)$ of (6.1). Then the following conditions are equivalent:

(i) The functor $\mathcal{D}(E)$ is additive;
(ii) The object $E$ is stable;
(iii) The object $\tau^*E \in \text{DMor}^{\geq 0}(A, B)$ is stable;
(iv) The functor $\mathcal{D}(\tau^*E): A \to \mathcal{D}(B)$ is additive;
(v) The morphism (6.5) is a quasi-isomorphism for any $A \in C^{\geq 0}(A)$.

Proof. Since stability for split short exact sequences is equivalent to additivity, it follows that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv). We obviously have (iv) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) by restriction to $A \subset C^{\geq 0}(A)$. Conversely, since $E$ is quasi-isomorphic to the Dold extension $D(\tau^*E)$ by Corollary 5.6, we have (iv) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). Moreover, (iv) in fact implies stability for $E$ and termwise-split short exact sequences in $C^{\geq 0}(A) \cong \text{Fun}(\Delta, A)$, and every short exact sequence (1.11) in $A \subset C^{\geq 0}(A)$ is quasi-isomorphic to the sequence

\[ 0 \to C(p)[-1] \to C(i \oplus i)[-1] \to C \to 0 \tag{6.6} \]

in $C^{\geq 0}(A)$, where $i \oplus i: B \oplus_A B \to C$ is the natural map. Since (6.6) is termwise-split, it follows that (iv) $\Rightarrow$ (iii), hence (i), (ii), (iii), and (iv) are all equivalent. Analogously, (iv) applied pointwise shows that the map (6.3) is a quasi-isomorphism for any finite $X$, which implies (v), and to finish the proof it remains to show that (v) $\Rightarrow$ (iv).

To do this assume that (v) holds, and note that since (6.5) is a quasi-isomorphism for $A = 0$, the functor $\mathcal{D}(E)$ is pointed. Thus, for any two objects $A_0, A_1 \in A \subset C^{\geq 0}(A)$ with embeddings $i_l: A_l \to A_0 \oplus A_1$, $l = 0, 1$, and projections $p_l: A_0 \oplus A_0 \to A_l$, $l = 0, 1$, the composition

\[ E(A_0) \oplus E(A_1) \xrightarrow{E(i_0) \oplus E(i_1)} E(A_0 \oplus A_1) \xrightarrow{E(p_0) \oplus E(p_1)} E(A_0) \oplus E(A_1) \]
is an identity map in \( \mathcal{D}(\mathcal{B}) \). Therefore \( E \in \mathcal{D}_{\omega}(A \times A, \mathcal{B}) \) admits a functorial splitting \( E(A_0 \oplus A_1) \cong E(A_0) \oplus E(A_1) \oplus E'(A_0, A_1) \) for a certain object \( E' \in \mathcal{D}(A \times A, \text{Ind}(\mathcal{B})) \), and it suffices to invoke the following assertion.

**Lemma 6.4.** Let \( \mathcal{A}, \mathcal{B} \) be Abelian categories, and assume given a functor \( E: \mathcal{A} \times \mathcal{A} \to C^{>0}(\mathcal{B}) \) with the Dold extension \( D(E): C^{>0}(\mathcal{A} \times \mathcal{A}) \to C^{>0}(\mathcal{B}) \) and the corresponding functor \( \mathcal{D}(E): \mathcal{A} \times \mathcal{A} \to \mathcal{D}^{>0}(\mathcal{B}) \) of (1.15). Moreover, assume that for any \( A \in \mathcal{A} \), \( \mathcal{D}(E)(0 \times A) = \mathcal{D}(E)(A \times 0) = 0 \) and for any \( A' \in C^{>0}(\mathcal{A} \times \mathcal{A}) \), the map

\[
\mathcal{D}(E)(\Omega(A')) \to \Omega(\mathcal{D}(E)(A'))
\]

given by (6.5) is a quasi-isomorphism. Then \( \mathcal{D}(E) = 0 \).

**Proof.** Assume that \( \mathcal{D}(E) \neq 0 \), and let \( n \) be the largest integer such that \( \mathcal{D}(E): \mathcal{A} \times \mathcal{A} \to \mathcal{D}^{>0}(\mathcal{B}) \) lands in \( \mathcal{D}^{\geq n}(\mathcal{B}) \). Replacing \( E \) with the canonical truncation of its homological shift \( E[n] \), we can reduce the task to the case \( n = 0 \). To obtain a contradiction it suffices to show that \( \mathcal{D}(E) \) lands in \( \mathcal{D}^{\geq 1}(\mathcal{B}) \). Since (6.7) is a quasi-isomorphism, it further suffices to show that for any \( A_0, A_1 \in C^{>0}(\mathcal{A}) \), the complex \( \mathcal{D}(E)(\Omega(A_0') \times \Omega(A_1')) \in C^{>0}(\mathcal{B}) \) projects into \( \mathcal{D}^{>2}(\mathcal{B}) \). However, for any \( B_0, B_1 \in \text{Fun}(\Delta, \mathcal{A}) \), we have \( B_0 \times B_1 \cong \delta^*(B_0 \boxtimes B_1) \), where the box product \( B_0 \boxtimes B_1 \in \text{Fun}(\Delta \times \Delta, \mathcal{A} \times \mathcal{A}) \) sends \( [m] \times [m] \in \Delta \times \Delta \) to \( B_0([m]) \times B_1([m]) \). Therefore \( \mathcal{D}(E)(B_0 \times B_1) \) can be computed by taking \( B_0 \times B_1 \in \text{Fun}(\Delta, \mathcal{A}) \times \text{Fun}(\Delta, \mathcal{A}) \) and applying the composition

\[
\text{Fun}(\Delta, \mathcal{A}) \times \text{Fun}(\Delta, \mathcal{A}) \xrightarrow{\Box} \text{Fun}(\Delta \times \Delta, \mathcal{A} \times \mathcal{A}) \xrightarrow{E} \text{Fun}(\Delta \times \Delta, C^{>0}(\mathcal{B})) \xrightarrow{\delta^*} \text{Fun}(\Delta, C^{>0}(\mathcal{B})) \cong \text{Fun}(\Delta \times \Delta, \mathcal{B}) \xrightarrow{\delta^*} \text{Fun}(\Delta, \mathcal{B}) \cong C^{>0}(\mathcal{B}).
\]

If we now let \( B_0 \) and \( B_1 \) be the images of the complexes \( \Omega(A_0') \) and \( \Omega(A_1') \) under the Dold–Kan equivalence, then \( B_0([0]) = B_1([0]) = 0 \) by the definition of the functor \( \Omega \), and since \( E(-, 0) \) and \( E(0, -) \) are acyclic, the functor \( E(B_0 \boxtimes B_1): \Delta \times \Delta \to C^{>0}(\mathcal{B}) \) lands in acyclic complexes after restriction to \( \Delta \times [0] \) and \([0] \times \Delta \). In other words, if we apply the Dold–Kan equivalence (4.3) and let \( M^{**} \in C^{>0,>0}(C^{>0}(\mathcal{B})) \) be the corresponding bicomplex with values in \( C^{>0}(\mathcal{B}) \), then \( M^{n,0} \) and \( M^{0,m} \) are acyclic for any \( n, m \geq 0 \). This implies that the double totalization \( \text{Tot}(\text{Tot}(M^{**})) \in C^{>0}(\mathcal{B}) \) projects into \( \mathcal{D}^{>2}(\mathcal{B}) \), and by virtue of the shuffle quasi-isomorphism (4.4), the same holds for \( \mathcal{D}(E)(\Omega(A_0') \times \Omega(A_1')) \cong \delta^*(\delta^* M^{**}) \). \( \square \)

**6.2. Extensions.** Since stability in the sense of Definition 6.1 is closed under taking cones, stable objects form a full triangulated subcategory \( \text{DMor}_{st}(\mathcal{A}, \mathcal{B}) \subset \text{DMor}(\mathcal{A}, \mathcal{B}) \). The standard \( t \)-structure on \( \text{DMor}(\mathcal{A}, \mathcal{B}) \) induces a \( t \)-structure on \( \text{DMor}(\mathcal{A}, \mathcal{B}) \), and by Proposition 6.3, (ii), and Example 5.2, its heart is the category \( \text{Mor}_{add}(\mathcal{A}, \mathcal{B}) \) of continuous left-exact functors \( \mathcal{A} \to \mathcal{B} \). However, the whole \( \text{DMor}_{st}(\mathcal{A}, \mathcal{B}) \) is not the derived category of this heart: it is bigger. Indeed, Proposition 6.3 imposes additivity on the derived category level only. It is not required, and it is in general not true that an object \( E \) with additive
$\mathcal{D}(E)$ can be represented by a complex of additive sheaves. It is only the homology objects of the complex that are required to be additive.

**Example 6.5.** Let $\mathcal{A} \cong \mathcal{B} \cong k$-mod be the category of vector spaces over a perfect field $k$. Then, since $k$-mod is semisimple, every left-exact functor $k$-mod $\to k$-mod is exact, and if it is also continuous, then it is completely determined by its value on the one-dimensional vector space $k$, so that $\text{Mor}_{\text{add}}(\mathcal{A}, \mathcal{B}) \cong k$-mod, with $k$ corresponding to the identity functor. However, $\text{RHom}^*(k, k)$ computed in the category $\text{DMor}_{\text{st}}(\mathcal{A}, \mathcal{B})$ is the Mac Lane cohomology $HM^*(k)$, and it is highly non-trivial when $k$ has positive characteristic (in particular, there is a non-trivial class in $\text{Ext}^2(k, k)$).

The simplest way to extend (6.2) to full derived categories, or at least to the derived category $\mathcal{D}^+(\cdot)$ of complexes bounded below, is to use Proposition 6.3, (v). Namely, for every stable $E \in \text{DMor}^{>0}(\mathcal{A}, \mathcal{B})$, the quasi-isomorphism (6.5) induces a functorial isomorphism

$$\mathcal{D}(E)(A[-1])[1] \cong \mathcal{D}(E)(A),$$  \hspace{1cm} (6.8)

and then $\mathcal{D}(E)$ extends immediately to a triangulated functor

$$\mathcal{D}(E): \mathcal{D}^+(\mathcal{A}) = \bigcup_{n \geq 0} \mathcal{D}^{\leq -n}(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$$  \hspace{1cm} (6.9)

with the aid of the limit with respect to the maps (6.8). (The limit exists since for any individual $A \in \mathcal{D}^+(\text{Ind}(\mathcal{A}))$, the inverse system stabilizes at a finite step.) Since we also have the obvious identification $E(A)[-1] \cong (E[-1])(A)$, it follows that (6.2) extends to a functor

$$\mathcal{D}^+(\mathcal{A}) \times \text{DMor}^+_{\text{st}}(\mathcal{A}, \mathcal{B}) \to \mathcal{D}^+(\mathcal{B})$$  \hspace{1cm} (6.10)

triangulated separately in each of the two variables. Let us finish the paper by showing how to lift (6.9) to the chain level. This is not quite trivial: to use (6.5) directly would require taking limits over chain-level liftings of the maps (6.8), and infinite limits do not behave nicely unless $\mathcal{B}$ satisfies AB4*. Therefore we use an alternative approach based on ‘chain-cochain complexes’ as, for example, in [15].

For any Abelian category $\mathcal{E}$, we denote by $C^\geq 0(\mathcal{E})$ the category of second-quadrant bicomplexes in $\mathcal{E}$ (they are called ‘chain-cochain complexes’ in [15], § 3.1). We say that a map $f: E^*_i \to F^*_i$ in $C^\geq 0(\mathcal{E})$ is a vertical quasi-isomorphism if $f: E^*_i \to F^*_i$ is a quasi-isomorphism for any $n \geq 0$. We have the sum-totalization functor $\text{Tot}: C^\geq 0(\mathcal{E}) \to C^*(\mathcal{E})$ given by $\text{Tot}(E^*_i)^n = \bigoplus_{i-j=n} E^j_i$. It sends vertical quasi-isomorphisms to quasi-isomorphisms and, as in Example 4.2, it has a right-adjoint $1: C^*(\mathcal{E}) \to C^\leq 0(\mathcal{E})$ given by $1(E_i^*)^j = E^{-i-j} \oplus E^{-i-j-1}$. Again as in Example 4.2, we also have full embeddings $L: C^\geq 0(\mathcal{E}) \to C^\geq 0(\mathcal{E})$ and $R: C^\geq 0(\mathcal{E}) \to C^\geq 0(\mathcal{E})$ onto the full subcategories of chain-cochain complexes concentrated in cohomological and, respectively, homological degree 0, and the isomorphism $\text{Tot} \circ R \cong \text{id}$ induces a vertical quasi-isomorphism $R \to 1 \circ L$, where $\iota: C^\geq 0(\mathcal{E}) \to C^*(\mathcal{E})$ is the tautological embedding. The Dold–Kan equivalence (4.1) identifies $C^\geq 0(\mathcal{E})$ with the category $\text{Fun}(\Delta^o \times \Delta, \mathcal{E})$ of simplicial-cosimplicial
objects in \( \mathcal{E} \), and a morphism is a vertical quasi-isomorphism if and only if it becomes a quasi-isomorphism after restriction to \([n] \times \Delta \subset \Delta^0 \times \Delta\) for any \([n] \in \Delta^0\).

Now assume given a stable object in \( \text{DMor}^{>0}(\mathcal{A}, \mathcal{B}) \) represented by a continuous functor \( E: \mathcal{A} \to C^{>0}(\mathcal{B}) \). We consider its Dold extension \( \text{D}(E) \) given by (5.3), and extend it further to a functor

\[
\text{D}(E): \text{Fun}(\Delta^0 \times \Delta, \mathcal{A}) \to \text{Fun}(\Delta^0 \times \Delta, \mathcal{B})
\]

(6.11) by applying it pointwise along \( \Delta^0 \). Then we can consider a continuous functor

\[
\tilde{E} = \text{Tot} \circ \text{D}(E) \circ \iota: C^+(\mathcal{A}) \to C^+(\mathcal{B}),
\]

(6.12) where we restrict our attention to complexes bounded below, and the map \( R \to \iota \circ \iota \) provides a map

\[
\iota \circ \text{D}(E) \to \tilde{E} \circ \iota,
\]

(6.13) where \( \text{D}(E) \) on the left is the Dold extension (5.3).

**Proposition 6.6.** For any continuous functor \( E: \mathcal{A} \to C^{>0}(\mathcal{B}) \) representing a stable object in \( \text{DMor}^{>0}(\mathcal{A}, \mathcal{B}) \), the functor (6.12) sends quasi-isomorphisms to quasi-isomorphisms, and the map (6.13) is a quasi-isomorphism.

**Proof.** To prove that (6.13) is a quasi-isomorphism it suffices to recall that \( R \to \iota \circ \iota \) is a vertical quasi-isomorphism and observe that (6.11) sends vertical quasi-isomorphisms to vertical quasi-isomorphisms by Theorem 5.5. For the first claim, we note that every quasi-isomorphism \( f: A^* \to B^* \) factors as

\[
A^* \xrightarrow{f \oplus i} B^* \oplus C(\text{id}_{A^*}) \xrightarrow{\text{id} \oplus 0} B^*,
\]

(6.14) where \( C(\text{id}_{A^*}) \) is the cone of \( \text{id}: A^* \to A^* \) and \( i: A^* \to C(A^*) \) is the natural embedding. Then \( f \oplus i \) in (6.14) is an injective quasi-isomorphism, and \( \text{id} \oplus 0 \) admits an injective left-inverse. Therefore it suffices to consider injective quasi-isomorphisms \( f: A^* \to B^* \). Moreover, since \( E \) is stable, it suffices to check that \( \tilde{E} \) sends any acyclic complex, for example \( \text{Coker} f \), to an acyclic complex. We then observe that for any \( n \geq 0 \) and any acyclic complex \( A^* \) in \( C^*(\text{Ind}(\mathcal{A})) \), the complex \( \text{I}(A^*)^n_\tau \) has homology in degree 0 only, and we have a natural vertical quasi-isomorphism \( L(\tau^{>0} A^*) \to \text{I}(A^*) \), where \( \tau^{>0} A^* \) is the canonical truncation. Therefore it suffices to check that for any acyclic complex \( A^* \) concentrated in cohomological degrees \( \leq 0 \) and bounded below, \( \tilde{E}(A^*) \) is acyclic. However, since \( A^* \) is bounded below, it has a finite filtration with contractible associated graded quotients. Since \( E \) is stable, it further suffices to consider the case when \( A^* \) is contractible. But the functor \( \tilde{E} \circ \text{I}: C^{\leq 0}(\mathcal{A}) \cong C_{>0}(\mathcal{A}) \to C^*(\text{Ind}(\mathcal{B})) \) is just the simplicial Dold extension of the functor \( E \), so everything follows from Lemma 4.6. \( \square \)

**Remark 6.7.** As in Remark 5.7, if a stable object \( E \in \text{DMor}^{>0}(\mathcal{A}, \mathcal{B}) \) is represented by a complex of ind-exact sheaves, then the resulting functor (6.12) is also left-derivable with respect to the injective model structure (that is, sends monomorphisms to monomorphisms).
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