Vacuum Brane and the Bulk Dynamics in Dilatonic Brane World

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We investigate the dynamics of vacuum brane and the bulk in dilatonic brane world. We present exact dynamical solutions which describe the vacuum dilatonic brane world. We find that the solution has initial singularity and singularity at spatial infinity.

I. INTRODUCTION

In modern theories of unified physical interactions, spacetime has more than four dimensions. It is well-known that Kaluza-Klein theory \cite{1,2} tells us that the spacetime which has more than four dimensions. Superstring theories (e.g. Ref. \cite{3} ) are at the moment the most promising candidates for a unified description of the basic physical interactions. There are five anomaly-free, perturbative superstring theories. The critical dimensions of spacetime are ten for these theories. There is now evidence that the five superstring theories are related with each other by dualities. It is conjectured that these theories can be regarded as the limits of an unique theory, so called M-theory, in which spacetime has eleven dimensions. M-theory seems to be related to $N = 1, D = 11$ supergravity theory at low energy.

Conventionally, it is thought that the extra dimension is quite small, say the Planck scale, and then it has not been observed yet.

How does compactification of the extra dimension take place? The explanation is through multidimensional cosmology \cite{4,5}. In general relativity, the geometry of spacetime is dynamical. The three-dimensional space we observe was once as small as the internal space, and expanded during evolution of the universe, while internal space contracted or has remained small during evolution of the universe. Therefore, internal space is microscopic and is not observable. This explanation is called dynamical compactification.

One of alternative approaches by branes in string theory has been suggested \cite{6,7}. That is the brane world scenario, where the Standard Model gauge and matter fields exist inside branes while gravitons propagate in the bulk
of spacetime. If this is true, the situation in which gravity exists in the bulk of the 11-dimensional spacetime, while the Standard Model particles exist on a 3-brane is possible. Phenomenologically, the brane world scenario was proposed to solve the hierarchy problem (e.g., Refs. [7,8]). This type of the brane world scenario may be motivated by Hořava and Witten [10,11]. Hořava-Witten scenario is reduced to five-dimensional effective theory at low energy. Indeed, Lukas and et al. [12] have shown that this five-dimensional theory admits a supersymmetric solution describing a pair of thin domain walls. One of the four-dimensional domain walls in five-dimensional spacetime is considered as our universe.

Randall and Sundrum (RS1) [8] proposed a toy model to solve the hierarchy problem. This model is the system with two branes like the Hořava-Witten scenario. In addition, Randall and Sundrum (RS2) [9] presented the scenario with noncompact extra space. This model is the system with a single positive tension brane. In this scenario, although the hierarchy problem is not solved, it is very interesting that gravity is localized on 3-brane due to the exponentially decreasing warp factor of the bulk metric. In the basis of the scenario, the brane world cosmology has been studied so far [13–23].

The low-energy effective theories of superstring theories and M-theory are supergravity theories, including dilaton field and antisymmetric tensor field etc. [3]. The dilaton field exists naturally in string theories. Since we might be able to assume that the antisymmetric field behaves as a cosmological constant effectively, it is interesting to investigate dilatonic brane world scenario with a cosmological constant. The related works have been done so far [24–37].

We investigate the dilatonic brane world in this paper. For simplicity, we concentrate on vacuum brane cases. The static solutions in this model have been already investigated in Ref. [24]. On the ground of the static solutions, we find the dynamical solutions which may be able to describe an era of the early universe. The related studies can be seen in [38].

The plan of the paper is as follows. In §II, we introduce the model of dilatonic brane world. In §III, we briefly review the static solutions of the model. In §IV, we find the dynamical solutions of the model. In the last section, we give the conclusions and remarks.
II. THE MODEL

We consider a brane world scenario in dilatonic gravity with a cosmological constant and interpret the \( (D-1) \)-dimensional thin domain wall of the \( D \)-dimensional spacetime as a \( (D-2) \)-brane world. For simplicity, we consider the cases that the brane is vacuum. The action of the model in the Einstein frame is given by

\[
S = S_{\text{bulk}} + S_{\text{brane}}.
\]

\[
S_{\text{bulk}} = \frac{1}{2} \int d^D x \sqrt{-g} \left[ R - \frac{4}{D-2} (\partial \phi)^2 - \Lambda e^{\lambda \phi/(D-2)} \right],
\]

\[
S_{\text{brane}} = -\frac{1}{2} \int_{\text{brane}} d^{D-1} x \sqrt{-q} V e^{\lambda \phi/2(D-2)} + S_{\text{YGH}},
\]

where \( q \) is the determinant of the induced metric of the brane, \( \lambda \) is the dilaton coupling, \( \Lambda \) is the cosmological constant in the bulk, \( V \) is the cosmological constant in the brane. The last term \( S_{\text{YGH}} \) is the York-Gibbons-Hawking boundary term \[39,40\]. From now on, we assume that a brane is located at \( y = 0 \) and that the metric is the form

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = q_{\mu\nu} dx^\mu dx^\nu + b(x, y)^2 dy^2,
\]

where \( \alpha, \beta = 0, 1, 2, \cdots, (D-1) \) and \( \mu, \nu = 0, 1, 2, \cdots, (D-2) \). The variation of the metric leads us the \( D \)-dimensional Einstein equations

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta} - \frac{V}{2b} e^{\lambda \phi/2(D-2)} g_{\alpha\beta} \delta(y),
\]

where the energy-momentum tensor \( T_{\alpha\beta} \) in the bulk is given by

\[
T_{\alpha\beta} = \frac{4}{D-2} \left[ -\frac{1}{2} (\partial \phi)^2 g_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi \right] - \frac{1}{2} \Lambda g_{\alpha\beta} e^{\lambda \phi/(D-2)}.
\]

The \( D \)-dimensional dilaton field equation is also obtained as

\[
8 \Box \phi - \lambda \Lambda e^{\lambda \phi/(D-2)} - \frac{\lambda V}{2b} e^{\lambda \phi/2} \delta(y) = 0.
\]

III. STATIC SOLUTIONS

Before presenting our dynamical solutions, we briefly review the static solution \[24\]. We assume a metric of the form
\[ ds^2 = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu + b(y)^2 dy^2, \]  

(8)

where \( \mu, \nu = 0, 1, 2, \cdots, (D-2) \) and \( \eta_{\mu\nu} \) denotes the metric of the \((D-1)\)-dimensional Minkowski space. The variables \( a \) and \( b \) are the scale factors of \((D-1)\)-dimensional spacetime and the orbifold, respectively. The dilatonic field \( \phi \) is a function of the coordinate \( y \) of the extra space i.e.

\[ \phi = \phi(y). \]  

(9)

The \( D \)-dimensional Einstein equations are given by

\[
(D - 2) \frac{a''}{a} + \frac{(D - 2)(D - 3)}{2} \left( \frac{a'}{a} \right)^2 - (D - 2) \frac{a'b'}{ab} \\
+ \frac{2}{(D - 2)} (\phi')^2 + \frac{\Lambda b^2 e^{\lambda \phi/(D-2)}}{2b} + \frac{V}{2b} e^{\lambda \phi/(D-2)} \delta(y) = 0, \\
\frac{1}{2} (D - 1)(D - 2) \left( \frac{a'}{a} \right)^2 - \frac{2}{(D - 2)} (\phi')^2 + \frac{1}{2} \frac{\Lambda b^2 e^{\lambda \phi/(D-2)}}{2b} = 0,
\]

(10)

where primes denote the derivatives with respect to the coordinate \( y \) of the orbifold. The \( D \)-dimensional dilatonic field equation is described as

\[
\phi'' + (D - 1) \frac{a'}{a} \phi' - \frac{b'}{b} \phi' - \frac{\lambda \Lambda}{8} b^2 e^{\lambda \phi/(D-2)} - \frac{\Lambda V}{2b} e^{\lambda \phi/(D-2)} \delta(y) = 0.
\]

(12)

When the orbifold has the \( S^1/Z_2 \) symmetry, the junction conditions on the brane \([41]\) are given by

\[
a'(+0) = \frac{V}{4(D - 2)}, \\
\phi'(+0) = \frac{\lambda V}{16},
\]

(13)

(14)

The solution satisfying the junction conditions has the following form:

\[
a(y) = H^{\frac{2}{(D-2)\Delta}}, \\
b(y) = H^{\frac{2(D-1)}{2(D-2)\Delta}}, \\
\Phi(y) = e^{-\frac{\lambda \phi}{2(D-2)}} = H^{\frac{\lambda^2}{2(D-2)\Delta}},
\]

(15)

(16)

(17)

where

\[
H = 1 + Q|y|, \\
\Delta = \frac{\lambda^2 - 16(D - 1)}{8(D - 2)}.
\]

(18)

(19)
The parameter $Q$ is related to the bulk cosmological constant $\Lambda$ as

$$\Lambda = \frac{2Q^2}{\Delta}. \quad (20)$$

From the junction conditions on the brane, we obtain the relation between the bulk cosmological constant $\Lambda$ and the brane tension $V$

$$\Lambda = 2^{-5}\Delta V^2. \quad (21)$$

When $\lambda = 0$, this solution corresponds to the Randall-Sundrum non-dilatonic solution and when $\lambda = 4\sqrt{6}$, this solution does to the Hořava-Witten scenario \[10\]. As shown in Ref. \[24\], when $\Delta$ is lower than $-2$, the graviton is trapped within the brane and then the Newton gravity is reproduced. This is expected to the extension of the Randall-Sundrum scenario \[9\].

In the dilatonic case ($\lambda \neq 0$) with positive $Q$, the dilaton field becomes singular at $|y| \to \infty$. When the parameter $Q$ is negative, the singularity of the dilaton field appears at $|y| = |Q|^{-1}$. We prefer $Q > 0$ because we are interested in the noncompact extra dimension like the RS2. Then, when $-2(D-1)/(D-2) < \Delta < 0$, the spacetime has the curvature singularity at $|y| \to \infty$. By solving the null geodesics equations, we find that the $|y| \to \infty$ singularity occurs at the spatial infinity. Otherwise, when $\Delta > 0$, the curvature singularity does not appear at the singular point of the dilatonic field ($|y| \to \infty$).

**IV. DYNAMICAL SOLUTIONS**

In this section we obtain the dynamical solutions by following the procedure in \[12\]. We assume that the metric was the form:

$$ds^2 = -N(\tau, y)^2d\tau^2 + a(\tau, y)^2\delta_{mn}dx^m dx^n + b(\tau, y)^2dy^2, \quad (22)$$

where $m, n = 1, 2, \cdots, (D-2)$. The dilaton field is a function of time coordinate $\tau$ and the coordinate $y$ of internal space, i.e.

$$\phi = \phi(\tau, y). \quad (23)$$

We assume that all of dynamical variables can be separable as
In the above, $a(y)$, $b(y)$ and $\phi(y)$ are the static solutions in the previous section. As a result, the Einstein equations are given by

\[
\frac{a^2}{b^2} \left[ -(D-2) \frac{a''}{a} - \frac{(D-2)(D-3)}{2} \left( \frac{a'}{a} \right)^2 + (D-2) \frac{a'b'}{ab} \right] - \frac{2}{D-2} (\phi')^2 - \frac{1}{2} \Lambda \epsilon^{\lambda \phi/(D-2)} \beta^2 b^2 e^{\lambda \phi_1/(D-2)}
\]

\[
- \frac{V}{2b} e^{\lambda \phi/(D-2)} \beta^2 b^2 e^{\lambda \phi_1/(D-2)} \delta(y) + \frac{\beta^2}{n^2} \left[ (D-2)(D-3) \left( \frac{\alpha'}{\alpha} \right)^2 + (D-2) \frac{\dot{\alpha}^2}{\alpha} - 2 \frac{\dot{\phi}^2}{D-2} \right] = 0,
\]

\[
\frac{a^2}{b^2} \left[ \frac{1}{2} (D-1)(D-2) \left( \frac{a'}{a} \right)^2 - \frac{2}{D-2} (\phi')^2 \right] + \frac{1}{2} \Lambda \epsilon^{\lambda \phi/(D-2)} \beta^2 b^2 e^{\lambda \phi_1/(D-2)}
\]

\[
\frac{\beta^2}{n^2} \left[ -(D-2) \frac{\ddot{\alpha}}{\alpha} - \frac{\ddot{\beta}}{\beta} - \frac{(D-2)(D-3)}{2} \left( \frac{\dot{\alpha}}{\alpha} \right)^2 - (D-2) \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} + (D-3) \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} - 2 \frac{\dot{\phi} \dot{\phi}}{D-2} \right] = 0,
\]

\[
\frac{a^2}{b^2} \left[ \phi'' + (D-1) \frac{a'}{a} \phi' - \frac{b'}{b} \phi' - \frac{\lambda \Lambda}{8} e^{\lambda \phi/(D-2)} e^{\lambda \phi_1/(D-2)}
\]

\[
- \frac{\lambda V}{2b} e^{\lambda \phi/(D-2)} \beta^2 b^2 e^{\lambda \phi_1/(D-2)} \delta(y) \right] = 0.
\]

The $D$-dimensional dilatonic field equation is
\[-\frac{\beta^2}{n^2} \left[ \ddot{\phi}_1 + (D-2) \frac{\dot{\alpha}}{\alpha} \dot{\phi}_1 + \frac{\dot{\beta}}{\beta} \dot{\phi}_1 - \dot{n} \dot{\phi}_1 \right] = 0.\]

From Eq. (31), we obtain \( \beta \propto e^{-\frac{\lambda \phi_1}{2(D-2)}} \) and set

\[\beta e^{-\frac{\lambda \phi_1}{2(D-2)}} = 1, \quad (33)\]

and then the equations of motion (Eq. (28-32)) are separated by variables. The first \([\ ]\) of the left hand sides of the Einstein equations (28)-(30) and the dilaton field equation (32) vanishes by using the \(D\)-dimensional Einstein equations and the dilaton field equation in static case.

From now on, we solve the time-dependent parts of equations of motion. We choose the gauge condition as

\[n(\tau) = \text{const.} \quad (34)\]

The solution is given by

\[\alpha = \alpha_0 |\tau - \tau_0|^p, \quad (35)\]

\[\beta = \beta_0 |\tau - \tau_0|^q, \quad \Phi \equiv e^{-\frac{\lambda \phi_1}{2(D-2)}} = \beta = \beta_0 |\tau - \tau_0|^q, \]

where

\[p = p_\pm \quad (36)\]

\[:= \frac{1 + (D-2)(4/\lambda)^2 \pm \sqrt{1 + (D-2)(4/\lambda)^2}}{D - 1 + (4(D-2)/\lambda)^2}, \]

\[q = q_\pm := -(D-2)p_\pm + 1. \quad (37)\]

The parameter dependence of the power indices \(p_\pm, q_\pm\) are shown in the Figs. 1 and 2. When \(\tau - \tau_0\) is negative, the first solution ((+) solution) is that the worldvolume contracts and the orbifold expands. The second solution ((−) solution) is that both the worldvolume and the orbifold contracts. When \(\tau - \tau_0\) is positive, the first solution ((+) solution) is that the worldvolume expands and the orbifold contracts. The second solution ((−) solution) is that both the worldvolume and the orbifold expand. Since the powers \(p\) or \(q\) of the scale factors is smaller than one, the expansion is subluminal. This solution may be interpreted as Friedmann-Robertson-Walker universe. As shown in the Fig. 1, as the absolute value of the dilaton coupling parameter is getting larger, the power indices \(p_+, q_+\) of the scale
factors is getting larger in the (+) solutions. The dilaton coupling makes the absolute values of Hubble parameters large. Otherwise, in the (−) solutions, as the absolute value of the dilaton coupling parameter is getting larger, the index \( p_− \) of the scale factor of the external space is getting larger and that the index \( q_− \) of the internal space is getting smaller.

FIG. 1. The indices \( p_+, q_+ \) of the dynamical solutions for \( D = 5 \). In the non-dilatonic case (\( \lambda = 0 \)), the values of the indices \( p_+, q_+ \) are 1/3 and 0, respectively. In the Hořava-Witten model (\( \lambda = 4\sqrt{6} \)), the values of the indices \( p_+, q_+ \) are \( \frac{2+4\sqrt{3}}{33} \) and \( \frac{-4\sqrt{3}+2}{11} \), respectively. The values of the indices \( p_+, q_+ \) are asymptotic to 1/2, and −1/2, respectively, as the absolute value of the dilatonic coupling constant is getting larger.
FIG. 2. The indices $p_-$, $q_-$ of the dynamical solutions for $D = 5$. In the non-dilatonic case ($\lambda = 0$), the values of the index $p_-$, $q_-$ equal the values of the indices $p_+$, $q_+$ and are $1/3$ and 0, respectively. In the Hořava-Witten model ($\lambda = 4\sqrt{5}$), the values of the indices $p_-$, $q_-$ are $\frac{2-4\sqrt{3}}{11}$ and $\frac{4\sqrt{3}+2}{11}$, respectively. The values of the indices $p_+$, $q_+$ are asymptotic to 0, and 1, respectively, as the absolute value of the dilatonic coupling constant is getting larger.

It is illustrative to figure out the Friedmann equation on the brane. The equation is derived from the $(0-0)$ component of the $D$-dimensional Einstein equations (Eq. (28):

$$
\frac{(D-2)(D-3)}{2} \left( \frac{\dot{\alpha}}{\alpha} \right)^2 + (D-2) \frac{\ddot{\alpha}}{\alpha^3} - \frac{2}{D-2} \dot{\phi}^2 = 0.
$$

(38)

In Appendix, we give the sketch of the derivation of the $(D-1)$-dimensional effective gravitational equations on the brane \cite{34,35} to obtain to physical meaning of each terms in the left-hand side of Eq. (38). The effective gravitational equation is given by:

$$(D-1)G_{\mu\nu} = \frac{4(D-3)}{(D-2)^2} \hat{T}_{\mu\nu} - (D-1)\Lambda q_{\mu\nu} - E_{\mu\nu},$$

(39)

where

$$
\hat{T}_{\mu\nu} = D_\mu \phi D_\nu \phi - \frac{D}{2(D-1)} (D_\mu \phi)^2 q_{\mu\nu},
$$

(40)

$$(D-1)\Lambda = \frac{1}{2} \left[ \Lambda - \frac{\Delta}{25} V^2 e^{\lambda \phi/(D-2)} \right] = 0.
$$

(41)

Note that the $(D-1)$-dimensional effective cosmological constant vanishes in this model. In the $(D-1)$-dimensional
effective gravitational equations on the brane, the term $E_{\mu\nu}$ is the contribution of the bulk to the brane. In appendix, we derive the expression $E_{\mu\nu}$ in the case with the radion $b$. For our solutions we evaluate each terms as follows:

\begin{align}
(\mathcal{D}_{-1})G_{00} &= \frac{(D - 2)(D - 3)}{2} \left( \frac{\dot{\alpha}}{\alpha} \right)^2, \\
\frac{4(D - 3)}{D - 2} \tilde{T}_{\mu\nu} &= \frac{2(D - 3)}{(D - 1)(D - 2)} (\dot{\phi})^2, \\
E_{00} &= \frac{D - 3}{D - 1} \left[ \frac{\dot{\alpha} - \dot{\beta}}{\alpha - \beta} - \left( \frac{\dot{\alpha}}{\alpha} \right)^2 + \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} \right] \\
&= \frac{(D - 2)(D - 3)}{D - 1} \left[ - \left( \frac{\dot{\alpha}}{\alpha} \right)^2 + \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} \right].
\end{align}

Using the spatial components of the $D$-dimensional Einstein equations (Eqs. (29-30)), we can eliminate the term $\ddot{\alpha} - \ddot{\beta}$ in Eq. (44) and obtain Eq. (45). The first term on the left-hand side of Eq. (38) is obtained from the $(D - 1)$-dimensional Einstein tensor $(\mathcal{D}_{-1})G_{00}$ and $E_{00}$. The second term on the left-hand side is obtained from $E_{00}$. The third term on the left-hand side corresponds to the $(D - 1)$-dimensional energy-momentum tensor of the dilaton field. In the work [13], it is shown that $E_{\mu\nu}$ fall off and cannot carry away the energy momentum from a system to infinity in the perturbation theory around the Randal-Sundrum solution. In our results, $E_{00}$ affect on the Hubble parameter $\frac{\dot{\alpha}}{\alpha}$ and is not negligible. This is because the energy-flow of the dilatonic field from the brane to the bulk has an effect on $E_{\mu\nu}$.

Finally, we comment the global structure of the bulk geometry. From the trace of the $D$-dimensional Einstein equation, the scalar curvature in the bulk is given by

\begin{equation}
R = - \frac{4}{D - 2} (\nabla \phi)^2 + \frac{D}{D - 2} \Lambda e^{\lambda \phi/(D - 2)}.
\end{equation}

Substituting our solutions to the above, we obtain

\begin{equation}
R = - \frac{16(D - 2)q^2}{\lambda^2} H^{-\frac{4}{(D - 2)\Delta}} |\tau - \tau_0|^{-2} \\
+ \frac{1}{(D - 2)(\beta_0)^2} \left( D + \frac{\lambda^2}{8\Delta} \right) \Lambda H^{-\frac{\lambda^2}{4(D - 2)\Delta}} |\tau - \tau_0|^{-2q}.
\end{equation}

Note that the $|y| \to \infty$ singularity appears. As discussed in §III, the $|y| \to \infty$ singularity is the curvature singularity when $-2(D - 1)/(D - 2) < \Delta < 0$, and is not so when $\Delta > 0$. In addition, the initial singularity exist at $\tau = \tau_0$, the big crunch singularity also appears at $|\tau| \to \infty$ when $q$ is negative.
V. DISCUSSION AND SUMMARY

In this paper we have investigated the bulk dynamics in a dilatonic vacuum brane world with a cosmological constant. We presented the dynamical bulk-solutions in §IV. The behavior of the solutions depend on the dilaton coupling constant \( \lambda \). It is the next issue to investigate the dilatonic brane world with ordinary matter.

The Randall-Sundrum toy model with noncompact extra space is given by the limit when the dilaton coupling parameter vanishes. In the Randall-Sundrum toy model, the orbifold is stabilized i.e. \( q = 0 \). This scenario can be considered as the valid model for the universe after the stabilization of the orbifold. As shown in the case of the static solution, the nearby solutions of the Randall-Sundrum solution are extended to the Randall-Sundrum scenario.

Even if the solutions do not correspond to the extended Randall-Sundrum scenario, when the fifth dimension is compact, the solutions are not necessarily excluded physically. When \( \tau - \tau_0 \) is positive, there are the solutions in which the four dimensional worldvolume expands and internal space contracts. If the fifth dimension is compact, this solution is approximately reduced to the normal Kaluza-Klein scenario with the dynamical compactification. This solution may be a candidate for a expanding phase of the early universe.

For the explanation of homogeneity and density perturbation, it is believed that the early universe has been in the inflationary paradigm. Though the dynamical solutions which we have presented in this paper correspond not to the inflation but to the subluminal expansion, the inflationary solutions can exist in more realistic model. Two types of inflation is considered in the brane-world scenario [33]. The first type is bulk inflation. If the moduli field \( \phi \) in the bulk has the appropriate potential \( V(\phi) \), it can cause the modular inflation [44]. The second type is brane inflation. The matter fields in the brane cause the inflation. In general, the inflation take place as a mixture of both types of the inflation [33].

We obtained two types of the dynamical solutions (\( \pm \)) in §IV. It is interesting to estimate which solution is most probable by the framework of quantum cosmology.

There are the initial singularity at \( \tau - \tau_0 = 0 \) and the singularity at \( |y| \rightarrow \infty \) in the solutions which we have presented. By solving the null geodesics equations, we find that the \( |y| = \infty \) singularity occurs at the spatial infinity. The 5-dimensional effective theory breaks down near the singularities. It is thought that the initial singularity may be removed by quantum gravity and quantum cosmology. The singularity at \( |y| \rightarrow \infty \) may be removed in the model
with the dimensions more than five \((D > 5)\) \([12,43]\). It is interesting issue to construct such a model. From the view of AdS/CFT correspondence, the curvature singularity at \(|y| \to \infty\) may have physical meaning. For example, in Ref. \([43]\), it is conjectured that large curvatures in geometries with the Minkowski brane are physical only if the scalar potential is bounded above in the asymptotically AdS solution. But the Gubser’s conjecture can not be applied to our solutions and is necessary to be generalized.

While this work was being completed we became aware of related work by Maeda and Wands, \([34]\) Mennim and Battye \([35]\) and Barcelo and Visser \([36]\). We added the comment about this in §IV and appendix.

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**APPENDIX:**

We sketch the derivation of the effective equation on the brane following Refs. \([14,34–36]\) and give the useful expression of the \(E\) in the case with the radion \(b\). In the brane world scenario, the brane world is described by a thin domain wall \(((D-2)\text{-brane}) (M, q_{\mu\nu})\) in \(D\)-dimensional spacetime \((V, g_{\mu\nu})\). We denote the vector unit normal to \(M\) by \(n^\alpha\) and the induced metric on \(M\) by \(q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu\). We begin with the Gauss equation,

\[
(D-1) R^\alpha_{\beta\gamma\delta} = (D) R^\alpha_{\nu\rho\sigma} q^\nu_{\mu} q^\rho_{\beta} q^\sigma_{\gamma} + K^\alpha_{\beta} K_{\gamma\delta} - K^\alpha_{\delta} K_{\beta\gamma},
\]

(A1)

where the extrinsic curvature of \(M\) is denoted by \(K_{\mu\nu} = q^\alpha_{\mu} q^\beta_{\nu} \nabla_\alpha n_\beta\). Contracting the Gauss equation (A1) on \(\alpha\) and \(\gamma\), we find

\[
(D-1) R_{\mu\nu} = (D) R_{\rho\sigma} q^\rho_{\mu} q^\sigma_{\nu} - (D) R^\alpha_{\beta\gamma\delta} n_\alpha q^\beta_{\mu} n_\gamma q^\delta_{\nu} + KK_{\mu\nu} - K^\alpha_{\mu} K_{\nu\alpha},
\]

(A2)

where \(K = K^\mu_{\mu}\) is the trace of the extrinsic curvature. This readily gives

\[
(D-1) G_{\mu\nu} = \left[(D) R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} (D) R\right] q^\rho_{\mu} q^\sigma_{\nu} + (D) R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu}
\]

\[+ KK_{\mu\nu} - K^\beta_{\mu} K_{\nu\beta} - \frac{1}{2} q_{\mu\nu} (K^2 - K^\alpha_{\beta} K_{\alpha\beta}) - \tilde{E}_{\mu\nu},\]

(A3)

where
Using the $D$-dimensional Einstein equations,

$$
(D) R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta} (D) R = T_{\alpha\beta},
$$

(A5)

where $T_{\mu\nu}$ is the $D$-dimensional energy-momentum tensor, together with the decomposition of the Riemann tensor into the Weyl curvature, the Ricci tensor and the scalar curvature;

$$
(D) R_{\mu\alpha\nu\beta} = (D-3) (D-2) (g_{\mu}[\nu (D) R_{\beta\alpha} - g_{\alpha[\nu} (D) R_{\beta]\mu])
$$

(A6)

$$
- \frac{1}{2(D-3)} g_{\mu[\nu} g_{\beta]\alpha} (D) R + (D) C_{\mu\alpha\nu\beta},
$$

we obtain the $(D - 1)$-dimensional equations as

$$
(D-1) G_{\mu\nu} = \frac{D-3}{D-2} \left( T_{\rho\sigma} q_{\rho}^{\mu} q_{\sigma}^{\nu} + \left( T_{\rho\sigma} n^{\rho} n^{\sigma} - \frac{1}{D-1} T_{\rho}^{\rho} \right) q_{\mu\nu} \right)
$$

$$
+ K K_{\mu\nu} - K_{\mu}^{\sigma} K_{\nu}^{\sigma} - \frac{1}{2} q_{\mu\nu} (K^2 - K^{\alpha\beta} K_{\alpha\beta}) - E_{\mu\nu},
$$

(A7)

where

$$
E_{\mu\nu} = (D) C_{\beta\rho\sigma} n_{\alpha} n^{\rho} q_{\mu}^{\beta} q_{\nu}^{\sigma}.
$$

(A8)

Note that $E_{\mu\nu}$ is traceless.

The dilaton field equation in the bulk is reduced to

$$
\Box \phi - n^{\nu} (\nabla_{\nu} n^{\mu}) \nabla_{\mu} \phi + K \nabla_{n} \phi + L_{n}^{2} \phi - \frac{\lambda \Lambda}{8} e^{\lambda \phi/(D-2)} = 0.
$$

(A9)

The junction condition for the extrinsic curvature and $Z_{2}$ symmetry tells us that the extrinsic curvature on the brane is uniquely written in the terms of the energy-momentum tensor $S_{\mu\nu}$ on the brane [1];

$$
K_{\mu\nu}|_{brane} = - \frac{1}{2} (S_{\mu\nu} - \frac{1}{D-2} q_{\mu\nu} S).
$$

(A10)

In our system the energy-momentum tensor $S_{\mu\nu}$ on the brane is given by

$$
S_{\mu\nu} = - \frac{V}{2} e^{\lambda \phi/2(D-2)} q_{\mu\nu}.
$$

(A11)

The junction condition for the dilatonic field and $Z_{2}$ symmetry imply us that the derivative of the dilaton field with the coordinate $y$ of the fifth dimension on the brane is written as
\[ \phi'|_{\text{brane}} = \frac{D-2}{16} V e^{\lambda \phi/(D-2)}, \]  
\[ (A12) \]

where \( \phi'|_{\text{brane}} = \lim_{\epsilon \to +0} \phi'(\epsilon) \).

Substituting (A10) into (A7) and using (A12), we obtain the \((D-1)\)-dimensional effective equations on the brane (Eq. (39)). The effective equation for the dilaton field is given by

\[ \Box \phi = \frac{4(D-2)\Lambda + V^2}{32(D-2)} e^{\lambda \phi/(D-2)} - \phi''|_{\text{brane}}. \]  
\[ (A13) \]

We have assumed the \(D\)-dimensional metric to have the form,

\[ ds^2 = b^2 dy^2 + q_{\mu\nu} dx^\mu dx^\nu. \]  
\[ (A14) \]

It is useful to write down the following formula for \(E_{\mu\nu}\):

\[ E_{\mu\nu} = \tilde{E}_{\mu\nu} - \frac{1}{D-2} q_{\mu\nu}^{(D)} R_{\alpha\beta} n^\alpha n^\beta - \frac{1}{D-2} q_{\mu}^{\alpha} q_{\nu}^{\beta}^{(D)} R_{\alpha\beta} + \frac{1}{(D-1)(D-2)} q_{\mu\nu}^{(D)} R, \]  
\[ (A15) \]

where

\[ \tilde{E}_{\mu\nu} = (D) R_{\mu\nu\rho\beta} n^\alpha n^\beta \]

\[ = -\mathcal{L}_n K_{\mu\nu} + K_{\mu\alpha} K_n^\alpha - D_\nu D_\mu \log b - D_\mu \log b D_\nu \log b, \]  
\[ (A16) \]

and \(D_\mu\) is the covariant differentiation with respect to \(q_{\mu\nu}\).

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