On color isomorphic subdivisions

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Abstract

Given a graph $H$ and an integer $k \geq 2$, let $f_k(n, H)$ be the smallest number of colors $C$ such that there exists a proper edge-coloring of the complete graph $K_n$ with $C$ colors containing no $k$ vertex-disjoint color isomorphic copies of $H$. In this paper, we prove that $f_2(n, H_t) = \Omega(n^{1+\frac{1}{2t-3}})$ where $H_t$ is the 1-subdivision of the complete graph $K_t$. This answers a question of Conlon and Tyomkyn (arXiv: 2002.00921).

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1 Introduction

Recently, Conlon and Tyomkyn \cite{4} initiated the study of a new problem on extremal graph theory, which aims to find two or more vertex-disjoint color isomorphic copies of some given graph in proper edge-colorings of complete graphs. Formally, we say that two vertex-disjoint copies of a graph $H$ in a coloring of $K_n$ are color isomorphic if there exists an isomorphism between them preserving the colors. For an integer $k \geq 2$ and a graph $H$, let $f_k(n, H)$ be the smallest number of colors $C$ such that there exists a proper edge-coloring of the complete graph $K_n$ with $C$ colors containing no $k$ vertex-disjoint color isomorphic copies of $H$. Obviously we have $n - 1 \leq f_k(n, H) \leq \binom{n}{2}$ since the coloring of $K_n$ is proper. Conlon and Tyomkyn \cite{4} first verified that finding $f_k(n, H)$ is indeed an extremal problem. Hence one may ask the following question.

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Problem 1.1. Given a graph $H$ and an integer $k \geq 2$, determine the order of growth of $f_k(n, H)$ as $n \to \infty$.

In [4], Conlon and Tyomkyn showed various general results about the function $f_k(n, H)$, such as the following upper bounds.

**Theorem 1.2** ([4]). The followings hold.

(i) For any graph $H$ with $v$ vertices and $e$ edges,

$$f_k(n, H) = O\left(\max\{n, n^{\frac{k+2}{k-1}}\}\right).$$

(ii) For every graph $H$ containing a cycle, there exists $k = k(H)$ such that

$$f_k(n, H) = \Theta(n).$$

There were also some known results on this function. For example, Theorem 1.2 (ii) is from the random algebraic method of Bukh [1]. When $H$ is an even cycle, the above constant $k = k(H)$ obtained by the random algebraic method is likely very large due to the Lang-Weil bound [14]. Recently, Ge, Jing, Xu and Zhang [6] improved the constant $k = k(C_4)$ by showing that $f_3(n, C_4) = \Omega(n^{\frac{4}{3}})$. Very recently, Janzer [8] developed a new method for finding suitable cycles of given length and then obtained a general lower bound as follows.

**Theorem 1.3** ([8]). Let $k, \ell$ be fixed integers. Then we have

$$f_k(n, C_{2\ell}) = \Omega(n^{\frac{k}{k-1} \cdot \frac{\ell-1}{\ell}}).$$

As a corollary of Theorem 1.2 (i), one can see that if $e(H) \geq 2v(H) - 2$, then $f_2(n, H) = \Theta(n)$. Conlon and Tyomkyn [4] asked how sharp this bound is and they also proved that $f_2(n, \theta_3, \ell) = \Omega(n^{\frac{4}{3}})$. Since $e(\theta_3, \ell) = \frac{3}{2}v(\theta_3, \ell) - 3$, the above corollary cannot be improved, to say that, $e(H) \geq \frac{3}{2}v(H) - 3$ implies that $f_2(n, H) = \Theta(n)$. They also suggested that an interesting test case for deciding whether this lower bound can be pushed closer to $2v(H)$ might be to study $f_2(n, H_t)$, where $H_t$ is the 1-subdivision of the complete graph $K_t$. The main result of this paper answers their question as follows.

**Theorem 1.4.** Let $t \geq 3$ be a fixed integer. Then we have

$$f_2(n, H_t) = \Omega(n^{1+\frac{1}{t-3}}).$$

The proof of our main result is mainly based on the ideas in [7]. We modify them at some points and add some new ideas such as the deletion method. Theorem 1.4 indicates that $f_2(n, H_t)$ grows superlinearly with $n$. Since $e(H_t) = 2v(H_t) - 2t$, our result tells that, $e(H) \geq 2v(H) - 2t$ with $t \geq 3$ does not imply that $f_2(n, H) = \Theta(n)$.

**Notation:** The notations $o$, $O$, $\Omega$, $\Theta$ have their usual asymptotic meanings. For a graph $G$ and subset $X \subseteq V(G)$, we denote $G[X]$ as the induced subgraph of $G$. Usually we denote $N_G(v)$ as the set of neighbors of $v$ in $G$ and denote $\deg(v) := |N_G(v)|$ as the degree of $v$ in $G$. 

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2 Proof of Theorem 1.4

Suppose that $C = \gamma n^{1+\frac{1}{16}}$, where $\gamma$ is a sufficiently small constant. Suppose also that $n$ is taken sufficiently large. For convenience in our proof, we will also assume that $n$ is divided by 4. First we need the following lemma, which will help us construct the auxiliary graph.

Lemma 2.1. Given a proper $C$-coloring $\chi$ of $G = K_n$, take a random equipartition of $V(G)$ into four parts $X_1$, $X_2$, $X_3$ and $X_4$. Then the expected number of monochromatic matchings of the form $\{x_1x_3, x_2x_4\}$ in $G$ is $\frac{1}{256} \sum_{c \in \chi} \binom{n}{2}$, where $e_c$ is the number of edges with color $c$ in $G$.

Proof of Lemma 2.1. Given a proper $C$-coloring $\chi$ of $G = K_n$, take a random equipartition of $V(G)$ into four parts $X_1$, $X_2$, $X_3$ and $X_4$. Since a monochromatic matching of size two in $G$ contains 4 vertices, and the probability of each of such vertex in $X_i$ is $\frac{1}{4}$. Hence, the expected number of such monochromatic matchings of the form $\{x_1x_3, x_2x_4\}$ is $\frac{1}{4}$, where $x_i \in X_i$ for $i = 1, 2, 3, 4$. Then by the linearity of expectation, the result is proved. \qed

Next we construct the auxiliary graph $G$ as follows. Given a proper $C$-coloring $\chi$ of $G = K_n$, we choose an equipartition of $V(G)$ into four parts $X_1$, $X_2$, $X_3$ and $X_4$, such that the number of monochromatic matchings of the form $\{x_1x_3, x_2x_4\}$ in $G$ is at least $\frac{1}{256} \sum_{c \in \chi} \binom{n}{2}$, where $e_c$ is the number of edges with color $c$ in $G$ and $x_i \in X_i$ for $i = 1, 2, 3, 4$. Let $G$ be a bipartite graph, the vertex set $V(G) = (X_1 \times X_2) \cup (X_3 \times X_4)$ and $(x_1, x_2) \in X_1 \times X_2$ is adjacent to $(x_3, x_4) \in X_3 \times X_4$ if and only if $\{x_1x_3, x_2x_4\}$ is a monochromatic matching in $G$. It is not hard to show that $|V(G)| = \frac{n^2}{8}$ and the number of edges in the auxiliary graph $G$ is equal to the number of monochromatic matchings of the form $\{x_1x_3, x_2x_4\}$ in $K_n$. Hence, we have

$$|E(G)| \geq \frac{1}{256} \sum_{c \in \chi} \binom{e_c}{2} \geq \frac{n^4}{1024C} > \frac{|V(G)|^2}{1024}\frac{1}{\gamma},$$

where the second inequality uses the convexity of $\binom{n}{2}$ and the formula $\sum_{c \in \chi} e_c = \binom{n}{2}$.

For the rest of the proof, we will show that $G$ contains a copy of $H_t$ with the property that the vertices are pairwise disjoint sets. The next observation is inspired by Lemma 4.3 in [S], which is useful for making sure that the vertices are disjoint sets.

Lemma 2.2. For any vertex $S$ in $G$ and any vertex $v$ in $G$, there is at most one vertex $T$ in $G$ such that $ST$ is an edge in $G$ and $v$ is in $T$.

Proof of Lemma 2.2. Without loss of generality, assume that $S = \{s_1, s_2\} \in X_1 \times X_2$ and $v \in X_3$. If there are two distinct vertices $T_1 = \{v, t_1\}$ and $T_2 = \{v, t_2\}$ such that both of $T_1$ and $T_2$ are adjacent to $S$, then the colors of edges $s_2t_1$ and $s_2t_2$ are the same, a contradiction. \qed
We say a graph $F$ is $K$-almost-regular if $\min_{v \in V(F)} \deg(v) \leq K \cdot \max_{v \in V(F)} \deg(v)$. Moreover, we say $F$ is a bipartite balanced graph with $V(F) = A \cup B$ if $\frac{1}{2}|B| \leq |A| \leq 2|B|$. We shall use the following lemma, which has been used in many problems [2, 3, 7, 12, 15].

**Lemma 2.3.** For any positive constant $\alpha < 1$, there exists $n_0$ such that if $n > n_0$, $c \geq 1$ and $F$ is an $n$-vertex graph with at least $cn^{1+\alpha}$ edges, then $F$ has a $K$-almost-regular balanced bipartite subgraph $F'$ with $m$ vertices such that $m \geq n^{\frac{\alpha(1-\alpha)}{2(1+\alpha)}}$, $|E(F')| \geq \frac{c}{10}m^{1+\alpha}$ and $K = 60 \cdot 2^{1+\frac{1}{\alpha}}$.

Let $G$ be the auxiliary graph defined as above. By Lemma 2.3, we can find a $K$-almost-regular balanced bipartite subgraph $G_0$ with $|V(G_0)| = n_1 \geq |V(G)|^{\frac{a(1-\alpha)}{2(1+\alpha)}}$, $|E(G_0)| \geq \frac{c_0}{10}n_1^{1+\alpha}$, where $\alpha = \frac{\delta - 2}{2\delta - 3}$, $K = 60 \cdot 2^{1+\frac{1}{\alpha}}$ and $c_0 = \frac{1}{1024\gamma} \geq 1$. Since the constant $\gamma$ is chosen to be sufficiently small, $c_1 = \frac{c_0}{10}$ is a sufficiently large constant. To prove our main result, it suffices to show that in $G_0$, there exists a copy of $H_t$ in which the vertices are pairwise disjoint, because if we can find a copy of $H_t$ in $G_0$ such that the vertices are pairwise disjoint, then we can find two vertex-disjoint color isomorphic copies of $H_t$ in $G$.

**Theorem 2.4.** Let $G_0$ be the subgraph of $G$ defined as above. $G_0$ contains a copy of $H_t$ in which the vertices are pairwise disjoint.

Before we prove the above theorem, we collect a few results that will be useful to us. By the definition of $K$-almost-regular balanced bipartite graph, let $G_0 = A \cup B$ with $|B| = m$ and the degree of every vertex of $G_0$ be between $\delta$ and $K\delta$, where $\delta \geq c_1 m^{\frac{\delta - 2}{2\delta - 3}}$ for some sufficiently large constant $c_1$. Then we define the neighborhood graph $W_A$ on vertex set $A$, where the weight of the pair $uv$ in $W_A$ is $d_{G_0}(u,v) = |N_{G_0}(u) \cap N_{G_0}(v)|$. Sometimes we also write the weight of the pair $uv$ as $W(u,v)$. Moreover, for a subset $U$ of $A$, write $W(U) = \sum_{uv \in \binom{U}{2}} d_{G_0}(u,v)$.

The following lemma on weighted graph $W_A$ of $G_0$ has been shown in [3].

**Lemma 2.5 ([3]).** Let $G_0 = A \cup B$, be the bipartite graph with $|B| = m$ and minimum degree being at least $\delta$ in $A$. Then for any subset $U \subseteq A$ with $\delta|U| \geq 2m$, we have

$$W(U) = \sum_{uv \in \binom{U}{2}} d_{G_0}(u,v) \geq \frac{\delta^2}{2m} \binom{|U|}{2}.$$

We further consider the weighted graph $W_A$ of $G_0$. For distinct vertices $u, v \in A$, we say that the edge $uv$ is light if $1 \leq W(u, v) < 2\binom{\delta}{2}$, and that is heavy if $W(u, v) \geq 2\binom{\delta}{2}$. Observe that if there is a copy of $K_t$ in $W_A$ formed by heavy edges, then there is a copy of $H_t$ in $G_0$, in which the vertices of $H_t$ are pairwise disjoint. Based on the above observation, we can obtain the following lemma.
Lemma 2.6. If $G_0$ does not contain a copy of $H_t$ in which the vertices are pairwise disjoint, then for any subset $U \subseteq A$ with $|U| \geq \frac{8tm}{\delta}$ and $|U| \geq 2$, the number of light edges in $W_A[U]$ is at least $\frac{\delta^2}{16t^3m}\binom{|U|}{2}$.

Proof of Lemma 2.6. By Lemma 2.5, for any subset $U \subseteq A$ with $|U| \geq \frac{8tm}{\delta}$, we have

$$W(U) \geq \frac{\delta^2}{2m}\binom{|U|}{2} \geq \frac{\delta^2}{8m}|U|^2 \geq 8t^2m.$$  

Let $B = \{b_1, b_2, \ldots, b_m\}$ and $h_i := |N_{G'}(b_i)|$. Let $G'$ be the induced subgraph $G_0[U, B]$ of $G_0$. Now suppose that for some $i$, $h_i \geq 2(t-1)$. Since $G_0$ does not contain a copy of $H_t$ in which the vertices are pairwise disjoint, there is no $K_t$ in the weighted graph $W_A[N_{G'}(b_i)]$ formed by heavy edges. Hence by Turán theorem of $K_t$-free graph, the number of light edges in $W_A[N_{G'}(b_i)]$ is at least

$$\frac{1}{t-1}\binom{h_i}{2} \geq \frac{h_i^2}{4(t-1)}.$$  

Moreover, note that

$$\sum_{i: h_i < 2(t-1)} \frac{h_i}{2} < 4t^2m \leq \frac{W(U)}{2},$$  

which implies that

$$\sum_{i: h_i \geq 2(t-1)} \frac{h_i}{2} \geq \frac{W(U)}{2}.$$  

By the definition of light edge, every light edge is presented in at most $2\binom{t}{2}$ of the set $N_{G'}(b_i)$. Thus, the total number of edges in $W_A[U]$ is at least

$$\frac{1}{2\binom{t}{2}} \sum_{i: h_i \geq 2(t-1)} \frac{h_i^2}{4(t-1)} \geq \frac{W(U)}{8t^3} \geq \frac{\delta^2}{16t^3m}\binom{|U|}{2}.$$  

The proof is finished. $\square$

With the above tools in hand, now we are ready to prove Theorem 1.4. Actually, it suffices to prove Theorem 2.4. In order to avoid ambiguity, we need to clarify the specific meaning of some expressions. When we say $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$ in $G_0$, we mean that there is no pair of vertices $(x, y) \subseteq G$ as a vertex $S \in G_0$ such that $S \in N_{G_0}(u) \cap N_{G_0}(v)$ in graph $G_0$. Moreover, we say two vertices $S, T \in V(G_0)$ do not share a vertex in $G$, we mean that if $S = (s_1, s_2) \subseteq V(G)$ and $T = (t_1, t_2) \subseteq V(G)$, then $s_i \neq t_j$ with $i, j \in \{1, 2\}$.  

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Proof of Theorem 2.4. If we can find a copy of \( H_t \) in \( G_0 \) with vertices \( u_1, u_2, \ldots, u_t \) on one side and vertex \( v_{i,j} \) joined to \( u_i \) and \( u_j \) for each \( 1 \leq i < j \leq t \). By Lemma 2.2, for any \( 1 \leq i < j \leq t \), \( u_i \) and \( u_j \) cannot share a vertex in \( G \), since they have the common neighbor \( v_{i,j} \). Similarly, \( v_{i,j} \) and \( v_{i,k} \) cannot share a vertex in \( G \). Hence we only need to show that for any distinct \( i, j, k, \ell \), \( v_{i,j} \) and \( v_{k,\ell} \) cannot share a vertex in \( G \).

We shall define \( u_1, u_2, \ldots, u_{t-1} \in A \) recursively with the following properties.

(i) For any distinct \( i, j \), \( j \leq t-1 \), \( u_i \) and \( u_j \) form a light edge in \( W_A \).

(ii) For any distinct \( i, j, k \), \( k \leq t-1 \), \( N_{G_0}(u_i) \cap N_{G_0}(u_j) \cap N_{G_0}(u_k) = \emptyset \) in \( G_0 \).

(iii) For any distinct \( i, j, k, \ell \), \( \ell \leq t-1 \), the vertices in \( N_{G_0}(u_i) \cap N_{G_0}(u_j) \) and the vertices in \( N_{G_0}(u_k) \cap N_{G_0}(u_\ell) \) do not share an element in \( G \).

(iv) For each \( 1 \leq i \leq t-1 \), the number of \( u \in A \) with the property that for every \( j \leq i \), \( u_j u \) is light is at least \((\frac{\delta^2}{64tm})^{i} \cdot |A| \).

As we have discussed above, combining the properties (i), (ii) and (iv) together helps us to find a copy of \( H_t \) in \( G_0 \) and the property (iii) helps us to show the vertex-disjoint property of such \( H_t \) in \( G_0 \).

Since \( G_0 \) is balanced, \( |A| \geq \frac{m}{2} \geq \frac{8tm}{\delta} \) as \( m \) is sufficiently large. By Lemma 2.6, there are at least \( \frac{\delta^2}{16t^3m} \binom{|A|}{2} \) light edges in \( A \). So by pigeonhole principle, we can pick some vertex \( u_1 \in A \) such that the number of light edges \( u_1 u \) is at least \( \frac{\delta^2}{16t^3m} (|A| - 1) \geq \frac{\delta^2}{64tm} |A| \). Suppose that for \( 2 \leq \ell \leq t-1 \), \( u_1, u_2, \ldots, u_{\ell-1} \) have been chosen with properties (i), (ii), (iii) and (iv). Let \( U_0 \) be the set of vertices \( u \in A \) such that \( u_j u \) is a light edge for every \( j \leq \ell - 1 \). By the property (iv), we have that \( |U_0| \geq (\frac{\delta^2}{64tm})^{\ell-1} |A| \). Let \( U \) be consisted of those \( u \in U_0 \) with the following properties.

- For all \( 1 < i < j < \ell - 1 \), \( N_{G_0}(u_i) \cap N_{G_0}(u_j) \cap N_{G_0}(u) = \emptyset \) in \( G_0 \).
- For all \( 1 < i < j < k < \ell - 1 \), \( N_{G_0}(u_i) \cap N_{G_0}(u_j) \) and \( N_{G_0}(u_k) \cap N_{G_0}(u) \) do not share a vertex in \( G \).

Next we show that the cardinality of \( |U_0 \setminus U| \) cannot be large, that means we can always guarantee \( |U| \) is large enough. First, since the edge \( u_i u_j \) in \( W_A \) is light, \( W(u_i, u_j) < 2\binom{t}{2} \). On the other hand, since \( G_0 \) is \( K \)-almost regular, the degree of every vertex in \( B \) is at most \( K \delta \), which implies that the number of vertices \( u \in A \) such that \( N_{G_0}(u_i) \cap N_{G_0}(u_j) \cap N_{G_0}(u_k) \neq \emptyset \) in \( G_0 \) is at most \( \binom{\ell-1}{2} \cdot 2\binom{t}{2} \cdot K \delta \).

Second, for any \( i, j, k \leq \ell - 1 \), consider those two elements \( x, y \) in \( G \) which are in \( v_{i,j} = (x, y) \in N_{G_0}(u_i) \cap N_{G_0}(u_j) \). Observe that, each of \( x \) and \( y \) is contained in at most one vertex of \( N_{G_0}(u_k) \) by Lemma 2.2. Hence, for any fixed \( i, j, k \leq \ell - 1 \), there are at most \( 4\binom{t}{2} \) bad vertices.
in $B$ that we will not pick as $v_{k,\ell}$ of $H_t$. Next we delete all neighbors of such bad vertices in $A$ from $U_0$. Note that we need to do this for all distinct $i, j, k \leq \ell - 1$, hence we will delete at most $\binom{t-1}{3} \cdot 4\binom{t}{2} \cdot K\delta$ vertices from $U_0$. Therefore, $|U_0 \setminus U| \leq \binom{t-1}{2} \cdot 2\binom{t}{2} \cdot K\delta + \binom{t-1}{3} \cdot 4\binom{t}{2} \cdot K\delta$. Since $m$ is sufficiently large, we have

$$\left(\frac{\delta^2}{64t^3m}\right)^{\ell-1}|A| \geq 2\binom{\ell-1}{3} \cdot 4\binom{t}{2} \cdot K\delta + 2\binom{t-1}{2} \cdot 2\binom{t}{2} \cdot K\delta.$$ 

which implies that

$$|U| \geq |U_0| \geq \frac{1}{2}\left(\frac{\delta^2}{64t^3m}\right)^{\ell-1}|A|.$$ 

Moreover, note that $\delta \geq c_1 m^{\frac{1-2}{2\ell-3}}$ for some sufficiently large constant $c_1$. Hence for any $2 \leq \ell \leq t - 1$, we have

$$\frac{1}{2}\left(\frac{\delta^2}{64t^3m}\right)^{\ell-1}|A| \geq \frac{8tm}{\delta}.$$ 

By Lemma 2.6 and pigeonhole principle, there exists some $u_{t} \in U$ such that there are at least $\frac{\delta^2}{64t^3m}(|U| - 1) \geq \left(\frac{\delta^2}{64t^3m}\right)^{\ell-1}|A|$ light edges adjacent to $u_t$ in $U$. Now we have chosen the suitable $u_t$ with $2 \leq \ell \leq t - 1$. This completes the recursive construction of $u_1, u_2, \ldots, u_{t-1}$.

Now we set $\ell = t - 1$ and then there is a set $V \subseteq A$ with $|V| \geq \left(\frac{\delta^2}{64t^3m}\right)^{t-1}|A|$, such that for every $i \leq t - 1$ and $v \in V$, $u_i$ is a light edge. Finally, we need to prove that there is a vertex $u_t \in V$ such that for any $i < j < t$, $N_{G_0}(u_i) \cap N_{G_0}(u_j) \cap N_{G_0}(u_t) = \emptyset$ in $G_0$ and for any distinct $i, j, k < t$, $N_{G_0}(u_i) \cap N_{G_0}(u_j)$ and $N_{G_0}(u_k) \cap N_{G_0}(u_t)$ do not share a vertex in $G$. Using the similar argument and deletion method as above, we will delete at most $\binom{t-1}{3} \cdot 4\binom{t}{2} \cdot K\delta$ vertices from $V$. It is easy to see that such $u_t$ exists because $|V| \geq \left(\frac{\delta^2}{64t^3m}\right)^{t-1}|A| \geq \binom{t-1}{3} \cdot 4\binom{t}{2} \cdot K\delta + \binom{t-1}{2} \cdot 2\binom{t}{2} \cdot K\delta$. Hence there exists a copy of $H_t$ in which the vertices are pairwise disjoint, the proof is finished.

3 Conclusions and some open problems

Regarding the question about the function $f_k(n, H)$, there have been several interesting results and methods shown in [4, 6, 8]. In this paper, we mainly focus on the case of $H = H_t$ is the 1-subdivision of the complete graph $K_t$ and we prove that $f_2(n, H_t) = \Omega(n^{1+\frac{1}{2t}})$. Note that Theorem 1.2 (i) gives that $f_2(n, H_t) = O(n^{1+\frac{1}{2t}})$. Hence it will be interesting to determine the exponent $\beta$, if exists, such that $f_2(n, H_t) = \Theta(n^{\beta})$.

Theorem 1.2 (i) also indicates that if $H$ is a bipartite graph with $e(H) \geq \frac{k}{k-1}|H| - \frac{2}{k}$, then $f_k(n, H) = \Theta(n)$. Our main result shows that, when $k = 2$, this bound can be pushed to $2|H| - 2t$, which answers a question of Conlon and Tyomkyn. It will be interesting to further decide whether this bound can be pushed closer to $\frac{k}{k-1}|H|$, with $k \geq 3$.

In the classical Turán problem, there is a famous conjecture called rational exponent conjecture [5, Conjecture 1.6], which states that for every rational number $r \in (1, 2)$, there exists
a single bipartite graph $H$ such that $ex(n, H) = \Theta(n^r)$. This conjecture is still open, and the current progress of this conjecture can be seen in [2] [9] [10] [11] [13] and the references therein. The known results show that something broadly similar holds for $f_2(n, H)$. Hence we think the following conjecture may be of interest.

**Conjecture 3.1.** For every rational number $r \in (1, 2)$, there exists a single bipartite graph $H$ such that

$$f_2(n, H) = \Theta(n^r).$$

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