Recursive quantum gauge theories

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Abstract
Quantum gauge theories with finite-dimensional representation spaces are constructed that can have canonical gauge field theories as singular limits. They describe nature as a recursive quantum assembly by iterating Fermi-Dirac quantification. Six iterations are necessary and sufficient for present physics. The gauge structure, the spin-statistics correlation, the space-time metric, and the Higgs field are modeled.

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1 Finite quantum theories

Canonical gauge theories like general relativity and the Standard Model are triply singular, due to (1) the canonical commutation relation between the differentiator $\partial_\mu$ and the coordinate $x^\mu$, (2) the canonical commutation relations between the gauge connection $\Gamma_\mu$ and its canonical-conjugate gauge field, and (3) the vanishing Hessian determinant of the gauge-invariant action with respect to time derivatives of the gauge vector potentials. The current gauge theories are often optimistically supposed to be singular limits of some regular theory to come. A kinematics for such a regular theory is developed here to the point where it provides finite correspondents for the operators in the usual action principle for a general gauge group. Dirac often emphasized the need for a finite theory; Bopp and Haag (1950) posed a regularity principle for spin models, and Segal (1951) posed one for quantum field theory in general.

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The proposed quantum theory, termed recursive, represents the system as a recursive quantum assembly. Its modules have Fermi-Dirac statistics, and are modularizations, or unitizations, of like assemblies of a lower level, or rank. Each assembly is also interpreted as a quantum topological simplex with its constituent modules as its vertices. Let us call such a recursive assembly or simplex a plexus for short. Multiple quantification was proposed by Weizsäcker (1955), John Baez (2005), and others.

A recursive assembly has a natural gauge theory, discrete for classical systems, continuous for quantum ones. For Weyl the gauge was a machinist’s gauge block or a carpenter’s gauge, a metaphor for a precise standard of measurement that undergoes a non-integrable change during its transport around a tube of magnetic flux. The gauge block for gravity is a vector in a Minkowski tangent space of the space-time manifold as a standard of direction. In Dirac’s revision of Weyl, and in the Standard Model, the gauge block is the quantum particle under study, undergoing a semi-simple group of its representation space when it is carried around flux of the gauge field. The gauge block of a quantum plexus is the module of a certain stage of assembly. Let us call it the cell of the plexus. The gauge Lie algebra of the first kind describes the statistics of the vertices of the cell. It induces transformations of the gauge Lie algebra of the second kind in assemblies representing events, perhaps two ranks higher than the cell. Its non-integrability, as expressed by the gauge curvature, results from dislocations in the organization of the plexus.

Stages of assembly are linearly ordered by rank (as in the von Neumann universe of sets). For example, since dynamical variables are functions of time, they have higher rank than time. In the kinematics considered here, the cell has some rank \( C \approx 4 \), and the space-time event and field have a rank \( E \approx 6 \), which is large enough for present physics.

### 1.1 Quantization as quantification

It is well known that “second quantization” is a misnomer for a process that does not quantize but quantifies, passes from one quantum to an assembly of many. Its commutation relations express statistics, not just complementarity.

The same can be said of canonical quantization, however. In general, canonical quantization can be broken down schematically into two steps:

1. Choose a vector space of canonically conjugate dynamical variables.
2. Form polynomials in the chosen variables subject to graded canonical commutation relations.

In retrospect, the first step specifies the representation space of a certain sub-quantum, and the second step is a quantification, assembling replicas of that
sub-quantum into a quantum system with Bose-Einstein or Fermi-Dirac statistics.

In recursive quantification, the initial vector space is $\mathbb{R}$, and the single step is quantification with Fermi-Dirac statistics, iterated recursively as often as necessary. Six recursions are necessary and perhaps sufficient for present physics.

Thus one functor Grass is used whenever we quantify, quantize, or gauge, to convert a one quantum theory to a many-quantum one, and one gauge block to many.

While “second quantization” is a misnomer, “second quantification” may be accurate. It is unreasonable to assume that some quantum non-commutativity comes from statistics and some from some more mysterious complementarity, with no experimental evidence for such a division. Let us suppose here that all complementarity expresses statistics on some level.

1.2 The laminar phase

The Standard Model groups can all be represented faithfully on about 16 vertices. If there are $N$ vertices in quantum history, a random simplex would have $N/2 \gg 16$ vertices on the average. Clearly the ambient plexus is not random but highly organized as a lamina, in that the spectral multiplicity of its basic variables are either $\leq 10^4$ or $\gg 10^{15}$. Let us call them the short variables and the long. Short variables may be further classified as longitudinal if they transform under transformations of the long variables, or else transverse. The space-time coordinates $x^\mu$ and momentum-energy coordinates $p_\mu$, called orbital, and gauge boson field variables, are long; spins are longitudinal short variables; the hypercharge $y$, isospin $\tau^k$, color charges $\chi^c$, and generation $\Gamma$ are transverse short variables. The Higgs field at each event is long.

Let us provide a regular kinematics for the laminar variables; that is, one based on regular Lie algebras, which are those with regular Killing form. A regular theory must be quantum, since commutative Lie algebras are singular, and must have a finite-dimensional representation space for its operators, like a finite aggregate of spins. Let us suppose that their laminar structure originates much as a graphene does, from organization of the plexus out of cellular elements.

A regular Lie algebra is semi-simple, a direct sum of simple ones. But in a quantum theory such a direct sum represents a disjunction of possible theories, one of which can be selected by a single measurement. Let us therefore specialize from regular Lie algebras to simple with no loss of generality.
1.3 Quantum space-times

In order to bring the spirit of quantum physics to bear on space-time, we must reconsider the experimental meaning of the event construct. Einstein represented an event by a smallest possible occurrence, such as a collision of two small bodies. We may accept this. But then he took as its defining variables only space-time coordinates defined originally by a lattice of clocks and later by light signals. Today this would appear as a gratuitous inconsistency, since it neglects the influence of observation on the observed. If an event is a collision then the 10 channels of the collision already describe it. If these do not carry photons, they will carry other kinds of quantum that are just as good. A supernumerary lattice or light signal would participate in the event, resulting in a quite different event.

The insistence on space-time coordinates seems to be a vestige of the Cartesian hypothesis that position in space and time are the necessary and sufficient variables for a true model. This was already displaced by Newtonian mechanics with its forces and masses. The test of a quantum theory is not whether it permits a space-time picture but whether it represents experimental processes with their observed spectra, transition probabilities, and assembly relations, including our macroscopic physical experience as a correspondence limit of large quantum numbers. For this we require all the coordinates of \( \sigma \) at least initially..

1.4 Spinorial quantum spaces

Feynman, Yang, and Penrose made helpful attempts at quantizing the orbital space-time variables. Let us compare them in natural units. Here \( \hat{x} \) designates a quantized \( x \); \( k \in \{1, 2, 3\} \); \( m \in \{1, 2, 3, 4\} \); and \( \delta x \) indicates an “atom” of \( x \): a finite-difference element of \( x \), to be summed later. The elements of these Lie algebras are dynamical transformations, some of which are symmetries as well.

\[
\begin{align*}
\text{Feynman (1941)} \quad & \delta \hat{x}^m \sim \gamma^m, \\
\text{Yang (1947)} \quad & \hat{x}^m \sim i\eta^{[5}\partial^m], \quad \hat{p}_m \sim i\eta^{[6}\partial^m], \\
\text{Penrose (1971)} \quad & \delta \hat{x}^k \sim \sigma^k, \\
\text{Present} \quad & \delta \hat{x}^m \sim \gamma^{m5}, \quad \delta \hat{p}_m \sim \gamma_{m6}.
\end{align*}
\]

The Penrose (1971) and Feynman (1941) quantum spaces still assume an absolute space or space-time points, with coordinates \( x^\mu \) but not \( p_\mu \). Guided by group simplicity, Yang (1947) relativized space-time within a larger quantum phase space, Yang space, whose quantum coordinates include positions, momenta, boosts, angular momenta, and a quantized imaginary \( L_i = \hat{i} \) that we will interpret as electric generator and Higgs field; all on an equal footing. A
quantum space today must fit in all the variables of \[\Box\] and their commutation relations. This can be done in several ways for the transverse variables, corresponding to possible GUT theories, so let us consider only the orbital variables at first.

Yang, following Hartland Snyder (1947), represented his algebra by differential operators on an infinite-dimensional function space, as shown in (1). Let us retain the Yang Lie algebra for its regularity but replace the classical \(\mathbb{R}^6\) infrastructure for its singularity, in favor of a quantum space that is regular like Feynman’s and Penrose’s, with odd statistics to go with its odd spin parity. This is the bottom line of (1). This finite-dimensional representation space must have an indefinite metric. The unit cell of a lamina may have the Yang dynamical group, but not the lamina itself, whose organization rearranges the Yang group in the sense of Umezawa (1993). A quantum space-time lamina of Connes (1994) has the singular long dimensions of a manifold and regular short dimensions; here both are regular.

2 The recursive Grassmann algebra

Fermi-Dirac quantification preserves regularity; Bose-Einstein does not. Let us use Fermi-Dirac statistics for each stage of recursive assembly, therefore.

The classical prototype recursive assembly is the recursively finite set. The space \(C\) of recursively finite sets is made from \(\emptyset C := \{\emptyset\}\) in a finite number of stages of power-set formation \(r+1C = 2^r C =: \exp r C\); prefixes are ranks.

We form the power set \(2^X\) by unitizing the sets in \(X\), forming all possible disjoint unions of the resulting unit sets, and collecting them into \(2^C\). all their disjoint unions. Let us represent the disjoint union as a multiplication operation \(\lor\), often left unwritten, and unitization as Peano’s monadic operation \(\iota: x \mapsto \{x\}\). Multiplication assembles modules, unitization modularizes assemblies. Let us also interpret any recursive finite set as a recursive simplicial complex, or plexus, whose vertices are unit sets, monads. The classical algebra \(C\) of recursively finite sets with the operations \(\emptyset, \iota, \lor\) is a recursive binary exterior algebra (over itself). Its identity is \(1 = \emptyset\). Every other element has square 0.

2.1 Recursive quantification

Let us use \(C\) as armature on which to construct an analogous recursive quantum algebra \(Q\). The necessary minimum operations for quantum recursive assembly seem to be \(1, \iota, \lor, \text{ and } +\). \(\lor\) is now a Grassmann product, sometimes unwritten, used to assemble quantum modules. \(\iota\) is now an upgrade of Peano’s \(\iota\) to a linear
operator that maps $Q$ onto its first-degree sector $Q^1 \subset Q$, by identification modulo the equivalences
\[
\forall a, a' \in \mathbb{R}, \, Q, Q' \in Q : \iota(a'a'Q' + aQ) \equiv a'aQ' + a\iota Q.
\] (2)

Addition $+$ in $Q$ is quantum superposition. These operations are commonplace in quantum physics, though $\iota$ is usually tacit. When we attach indices to a scalar field to build a tensor field, we implicitly form a tensor product and then unitize it so that its factors are not confused with those of other similar tensors in a product.

$Q$ is a recursive Grassmann algebra, over itself; $C$ corresponds to the rays of a classical basis for $Q$, made without quantum superposition.

There is a standard way to propagate a metric from a one-quantum representation space $\mathcal{V}$ to the many-quantum space $2^\mathcal{V} := \text{Grass} \mathcal{V}$. Applied recursively to the natural metric $\|r\| = r^2$ on $\mathbb{R} = _0Q$, which is stage 0 of $Q$, it defines a Hilbert metric $H : Q \to \text{Dual} Q$ that will serve as a reference metric on the representation space, and a Hilbert norm $\|Q\|_H := HQ \circ Q$.

Also useful is the neutral duplex norm $D$ on the duplex space $\text{Dup} Q := Q \oplus \text{Dual} Q$, whose value $\|Q \oplus Q'\|_D$ for any $Q \in Q, Q' \in \text{Dual} Q$ is the valuation $Q' \circ Q$. The duplex space occurs in the spinor constructions of Cartan (1913) and Chevalley (1954) and in the theory of Saller (2006), who calls it the quantum space of the system.

Chevalley (1954) and Bohm (1962) represented one-level simplicial complexes in the Grassmann algebra on its vertices. The plexus iterates the functor Grass recursively.

To bring out the intended parallel to the classical power set $2^X$, let us also write the Grassmann algebra as
\[
\text{Grass} \mathcal{V} =: 2^\mathcal{V}.
\] (3)
The boldfaced $2$ stands for the quantum binary decision, to be or not to be in a product, which has to be made for each basis element in $\mathcal{V}$, and their quantum superpositions. The many-quantum representation space is the binary exponential of the one-quantum space as the many-body phase space is the binary exponential of the one-body. Conversely, we recognize a system as composite if its representation space $\mathcal{X}$ has a useful logarithm $\mathcal{V} = \log_2 \mathcal{X}$, in the sense that $\mathcal{X}' = 2^\mathcal{V}$.

Let us also use the standard apostrophe notation
\[
f'^X := \{ f(x) | x \in X \}
\] (4)
for “the $f$’s of the $X$’s”. Then $Q = \text{Grass} Q$ is recursively generated from $\mathbb{R}$ by iterating Grass $\iota'$:
\[
_0Q := \mathbb{R}.
\] (5)
\( r+1 \mathcal{Q} := \text{Grass}_R \iota' (r \mathcal{Q}) \sim \text{Grass}_R \mathcal{Q}. \) 

(6)

\( \mathcal{Q} \) is doubly graded, by polynomial degree \( g \) and by von Neumann rank \( r \):

\[
\mathcal{Q} = \bigoplus_{g,r} r \mathcal{Q}^g. 
\]

(7)

Degree counts monadic factors \( \iota \mathcal{Q} \). Rank counts nested \( \iota \)'s. A polyadic in \( \mathcal{Q} \) that is homogeneous of degree \( g \) is called a \( g \)-adic.

The serial number \( q \) for the classical basis element \( e_q \) gives the order of creation, when first priority of operation is given to 1, second to \( \vee \), and third to \( \iota \), and factors in a product are ordered by their serial number. Then

\[
\forall q, q' \in \mathbb{N} : e_{(2q)} = \iota e_q, \quad e_{q+q'} = e_q \vee e_q 
\]

(8)

Table 1 lists some basic polyads \( \mathcal{C} \subset \mathcal{Q} \), by stratum, with their serial number and statistics (exchange parity). The basis tabulated is generated from 1 by \( \iota \) and \( \vee \), without quantum superposition, and so is called a classical basis. Its symbols may be regarded as hyperbinary numbers, two-dimensional positional notations for their serial numbers \( q = \sum q_n \exp n \). The positional values \( \exp n \) grow hyperexponentially with position \( n \) where binary values \( 2^n \) merely grow exponentially. The hyperbinary coefficients \( q_n \) have growing ranges \( 0 \leq q_n < \exp n \) instead of the fixed range \( q_n = 0, 1 \) of binary coefficients.

Stratum 4 would overflow the page; Table 2 lists its monadics. Stratum 5 in Planck-size characters would fill the known universe many times. In Table 1 the rank of a polyadic is its height in bars. The degree is the number of stacks, as seen from above. The statistics is +, − for even and odd.

Linear operators \( \text{Deg}, \text{Rank} : \mathcal{Q} \to \mathcal{Q} \) multiply each element of \( r \mathcal{Q}^g \) by \( g \) or \( r \) respectively. Let us define the hyper-exponential function \( \exp r \) by setting \( \exp 0 := 1 \) and \( \exp (r+1) := 2^\exp r \). The dimension of stage \( r \mathcal{Q} = \text{Grass}_R \mathcal{Q} \) is \( \exp r \).

Table 2 gives the 16 basic monadics \( e_q \) of rank 4, where \( q \) is the serial number of a basis element and \( L = \log_2 q \).

### 2.2 Interpretation

Rays in \( \mathcal{Q} \) and its dual space \( \text{Dual} \mathcal{Q} \) represent quantum io (input-output) channels for the system under study.

1 in \( \mathcal{Q} \) represents the io channel for an empty plexus. 0 \( \in \mathcal{Q} \) represents the empty channel.

Mathematical objects and classical plexuses, by definition, do not change under perception. Quantum ones on the contrary change unpredictably, in
quantities not being perceived. Thus the quantum simplex or plexus is not a mathematical object as a classical one might be. Its channels, however, due to their macroscopic statistical aspect, can be represented by mathematical objects, the elements of the representation space $Q$ or its dual.

$Q$ has no significant relativity group; every ray in $Q$ is intrinsically different from every other. This holds also for $\mathbb{R}^4$, the representation space for space-time, and for the Hilbert space $S^2(\mathbb{N})$ of complex sequences of summable square, the representation space for canonical quantum theory. Let us take this categorical describability as a prerequisite for a representation space. The relativity groups of the physical system preserve selected structural elements of the representation space and violate others. Here as in canonical theories the linear algebraic operations $+, \vee$ on the representation space are supposed to have physical meaning, and the relativity group of the plexus respects them, but violates $\iota$. There is such an $\iota$ violation in Hilbert’s $S^2(\mathbb{N})$ too:

Whatever the statistics of $Q$, $\iota Q$ has odd statistics. This curious violation of statistics seems acceptable, since we see conservation of statistics only on the particle rank. It is already implanted in quantum field physics. All the components of a spinor, regardless of their degree as Grassmann products of semivectors, are given a fresh degree of 1; and the Dirac vector $\gamma^\mu$, of even spin parity, obeys odd commutation relations with itself. Nevertheless $\iota$ preserves the spin-statistics correlation, by violating spin parity exactly as much as exchange parity.

## 3 The spinor tree

$Q$ is constructed on the basis of Fermi statistics, spinor theory, and classical set theory. The spin-statistics equality $W = X$ is a major clue to its interpretation.

Since monadics in $Q^1$ anti-commute, vertices obey the exclusion principle, and have $X = 1$. Therefore, since $W = X$, monadics must be spinors, with $W = 1$. Vertices must have spin 1/2, and $Q$ must consist of multispinors of various degrees and ranks and their superpositions. The theory of the algebra $Q$ must then include the quantum theory of Fermi statistics, and the classical theory of spinors.

The theory of spinors is commonly based, following Cartan and Dirac, on a classical space-time background. Presumably there is no classical space-time. Then we need another interpretation for spinor theory. The spin-statistics equality $W = X$ suggests one. When Cartan (1913) constructed double-valued representations for rotations, Schur (1911) had already constructed them for permutations, a more primitive construct, using much the same matrices, among others.
The spin-statistics correlation can then be made into an identity. The classical angular-momentum Poisson Bracket relations are a singular classical limit of a regular quantum statistics of the Palev (1977) kind, reviewed in §4.5, which is taken as more fundamental. This statistics in turn follow from Fermi statistics for quanta that are exchanged by spin operations.

3.1 **The quadratic space** \( \mathcal{W} \)

3.2 **Indefinite metric**

Cartan constructed his spinor space as the exterior algebra over a “semivector” space. This step in the construction uses no metric. Let us identify it with the construction of each stage of \( Q \) from the preceding. Thus each stage serves as semivectors for the next and spinors for the preceding; the constructs of semivector and spinor are relativized. The spinor space of any rank is the exterior algebra over that of the previous rank. \( Q \) can be interpreted either as a spinor space, a modularization of the infinite-dimensional spinor space of Dirac (1974), or as a semivector space.

In the Cartan theory, a quadratic vector space is prior to the spinor space and defines its orthogonal group. It is the duplex space of the Cartan semivector space. Here therefore the quadratic space of stage \( r \) is

\[
\mathcal{W}_r := \text{Dup}_r Q, \quad \| v \oplus v' \|_D := v'(v) = v' \circ v.
\]

The quadratic space \( \mathcal{W} \) for spinors of any stage \( r+1 \) is the duplex space of the spinors of the previous stage. Its orthogonal group \( \text{SO}(\text{Exp}^r, \text{Exp}^r) \) includes the linear group \( \text{SL}(\text{Exp}^r) \) of the spinors of the previous stage. The duplex space is a representation space for a pair of a simplex and a dual simplex, which will be called a duplex for short. It carries a neutral quadratic form that can serve as the origin of the Minkowski metric of space-time tangent spaces. It is encoded in Dirac operators \( \gamma_w \) for the duplex. This Clifford algebra is isomorphic for all duplexes of a given rank, as for all tangent spaces of Minkowski space-time. It varies at the event level because the cell assembly is a variable.

The spinor space is also duplicated in practice. If \( Q \) is is used for kets then \( \text{Dual} Q \) is used for bras, and \( \text{Dup} Q \) contains superpositions of bras and kets. These do not arise in canonical quantum theories. This can be regarded as a superselection rule resulting from a singular limit of large numbers at the level of the observer.

The statistical norm \( \| Q \| \) for any vector \( Q \) of the representation space can be normalized to be the average number of systems flowing into the experiment.
in the channel represented by \( Q \). Then the norm \( \| Q \| \) is positive for kets, negative for bras; it is no longer a probability but a probability flux. Perhaps Dirac (1974) intended this interpretation of indefinite metrics. Transition probabilities are homogeneous of degree 0 in the norm, so that no physical conclusion is changed if a norm is replaced by its negative everywhere. Processes associated with bras are defined to lower the energy; kets raise the energy. The energy distinction does not depend on the sign of the norm and therefore can be used to define it.

Null vectors in \( \text{Dup} \, Q \) are inaccessible in the singular canonical limit, like null vectors of space-time in the Galilean limit.

### 3.3 Dirac spin operators

The quadratic space and the spinor space are linked by an algebra \( rA \), which is both the Clifford algebra of the quadratic space and the operator algebra of the spinor space. Here therefore the Clifford algebra of stage \( r \) is

\[ rA := \text{Cliff} \, r \; W = \text{Cliff} \; \text{Dup} \, i \; r \; Q. \tag{10} \]

*The operator algebra of rank \( r \) is a Clifford algebra over the duplex space of the previous rank, taken with the duplex metric:*

\[ \text{Alg} \, 2^V = 2^V \otimes 2^\text{Dual} \; V = 2^\text{Dup} \; V = \text{Cliff} \; \text{Dup} \; V. \tag{11} \]

Since the elements of this Clifford algebra are the linear operators on Grass \( V \), Grass \( V \) is indeed a spinor space for this Clifford algebra and its orthogonal group.

In particular, each basis vector \( e_w \in r_{-1} \; W \) may be identified with a Dirac operator, a generator of a Clifford algebra of the next rank, when it is usually written as \( \gamma_w \). Dropping the rank prescript for now, we define the operator \( \gamma_v \) to be the left exterior multiplication by \( e_v \), and \( \gamma^u \) to be the left exterior differentiator \( \partial^u \) with respect to \( e_u \). These indeed generate \( \text{Cliff}(r \; W) \):

\[ \{ \gamma_u, \gamma^v \} = \delta_u^v, \quad \{ \gamma^u, \gamma^v \} = 0 = \{ \gamma'_u, \gamma_u \}. \tag{12} \]

The usual spin operators generating the spin group \( \text{Spin} \; (W) \) are the semi-commutators

\[ \gamma_{w'w} := \frac{1}{2} [\gamma_{w'}, \gamma_w]. \tag{13} \]

A rotation of a simplex through \( 2\pi \) is represented by

\[ e^{iW} = e^{i\pi \gamma_{w'w}}. \tag{14} \]
where the sum is over the vertices. The spinor tree, the central column of Table 3, is constructed inductively, by iterating Grass, beginning with the trivial exterior algebra $\mathbb{R}$ of degree 0 and rank 0. Orthogonal groups and Clifford-Fermi algebras perch on each level of the resulting spinor tree of Table 3. The symmetry Lie algebra of $r\mathbb{Q}$ as vector space is sl($r\mathbb{Q}$). The symmetry Lie algebra of $r\mathbb{Q}$ as exterior algebra is only sl($r-1\mathbb{Q}$), exponentially smaller.

### 3.4 Pauli metrics

When monadics in $r\mathbb{Q}$ serve as spinors, let us endow them with a recursive Pauli metric form $r\beta$ that is invariant under the spin group Spin($r\mathcal{W}$). The usual Pauli form is stage $r = 2$. Dropping the prescript $r$ for clarity, $\beta$ is constructed to define a pseudo-expectation value

$$\text{Av}_Q A := Q^\beta AQ = \beta Q A Q = \beta_{q q'} Q^{q'} A^{q'} Q^q$$

(16)

giving the flux of any variable $A \in \text{Alg}_r \mathbb{Q}$ for any $Q \in r\mathbb{Q}$. This transforms in the same way as $A$ under the spin group of stage $r$.

This requires that the Pauli metric $\beta$ of each Clifford algebra Cliff $\mathcal{W}$ skew-symmetrize all the $\Gamma = \gamma_{w w'} \in \text{Cliff}_r \mathcal{W}$, in the sense that (designating the transpose operation by $T$)

$$\forall \Gamma \in \text{Cliff}^2 \mathcal{W} : T(\beta \Gamma) = -\beta \Gamma.$$ 

(17)

The construction of $\beta$ for SO($n, n$) then follows the usual one for SO($3, 1$) closely. To skew-symmetrize the second grade it suffices to skew-symmetrize the first grade. Call a basis for $\mathcal{W}$ orthonormal if the duplex metric $D$ is diagonal in that basis and has diagonal elements $\pm 1$. One matrix representing a Pauli metric $\beta$ in an orthonormal basis is the product (in any order) of those $\gamma_{w}$ whose squares are $-1$. Metrics and operators transform differently; hence the frame restriction.

The standard Dirac case $2\beta$ is skew-symmetric, but $r\beta$ is symmetric for $r > 2$. The Pauli metrics $r\beta$ do not have a well-defined limit as $r \to \infty$. Since the stages nest, however, so do their groups, and the $\beta$ of any stage serves as well for all lower stages.

### 4 Chiral spinors

#### 4.1 Infinites

In classical mechanics the center of the coordinate Lie algebra is the whole algebra. Canonical quantization reduces the center to the one-dimensional
Lie algebra $\mathbb{C}$ of complex numbers, generated by the right-hand sides of the canonical commutation relation

$$[ip, iq] = i; \quad [i, ip] = 0; \quad [iq, i] = 0;$$

(18)
in which $ip, iq, i$ are skew-hermitian infinitesimal isometries of the quantum theory. Representing the $i$ in (18) by a scalar matrix is well known to make the Lie algebra singular; in a regular theory in a complex $n$-dimensional representation space the trace of the first equation would be $0 = ni$.

These canonical commutation relations, in their many appearances, from the differential calculus to the Bose statistics of gaugeons, seem to be the only obstacle to regularity in all of quantum field theory. They account for all the infinities of present-day quantum physics. Therefore their reform is a critical precondition for a regular theory. Yang (1947) reformed the canonical commutation relations between $x^\mu$ and $p^\mu$, and Palev (1977) those between bosonic creators and annihilators.

The Yang (1947) Lie algebra of (1) replaces the $i$ in (18) by a generator $L_{i}^{65} =: L_{i}$ of the Yang Lie algebra, here taken with the neutral signature of $\text{so}(3, 3) \subset \text{sl}(4\mathbb{C}) \subset \text{sl}(8\mathbb{R})$. Its spinor representation space is a subspace of cellular stage $C\mathcal{Q}$, with rank $C$. It seems that an 8-dimensional spinor space with $C = 4$ suffices for this. This cell will be adequate for the gauge theories of gravity and electricity only. The Standard Model requires a cell of perhaps 16 vertices, which will still fit in stage 4.

4.2 Higgs field and $i$

Suitably renormalized, one generator $L_{i}$ becomes a quantized imaginary $\hat{i}$, which becomes $i$ in a suitable singular limit with organization. Let us therefore adopt the real field $\mathbb{R}$ for the coefficients of the representation space $\mathcal{Q}$. This makes it necessary to replace the Schrödinger first-order differential equation for the time-development of input vectors, with its prominent $i$, by a real one, that does not require an explicit $i$, like Dirac’s or Maxwell’s equation; though it is understood that the canonical action principles for both Dirac’s and Maxwell’s equations presently include essential $i$’s that must now be replaced by an operator like $\hat{i}$. In various limits, stages, and normalizations, Yang’s $L_{i}$ is supposed to underlie the $i$ of quantum probability amplitudes, the electric charge gauge generator $iq$, and the Higgs field.

The dynamical symmetry rearrangement that centralizes the imaginary $i$ reduces general spinors of the Yang group to chiral spinors of the Lorentz group.

We see this as follows. The chirality of a fermion in the standard quantum theory is an operator at the top of its 16-dimensional Dirac Clifford algebra,

$$i\gamma^{421} := i\gamma^{4}\gamma^{3}\gamma^{2}\gamma^{1} \doteq \pm 1 =: i\gamma^\top \doteq \pm 1.$$  

(19)
\(i\gamma^{4321} = \pm 1\) for a left-handed electron, with isospin \(1/2\), and \(-1\) for a right-handed electron, with isospin 0.

The Yang SO(3,3) group has a spinor space \(\cong 8\mathbb{R}\), the square root of its Clifford algebra. Its pseudoscalar volume element \(\gamma^T := \gamma^{654321} = \gamma^{65}\gamma^{4321}\) commutes with so(3,3) transformations \(\gamma^y \gamma^{y'}\) \((y, y' = 1, \ldots, 6)\) and has eigenvalues \(\gamma^T = \pm 1\). \(\gamma^T\) reduces the spinor space to two eigenspaces \(\cong 4\mathbb{R}\) with

\[\gamma^{4321} = \mp\gamma^{65} = \mp\gamma\hat{1}\]  \hspace{1cm} (20)

These are therefore chiral spinors.

The atoms of the quantized orbital operators may then be represented by \(\delta\hat{x}^m \sim \gamma^m\), \(\delta\hat{p}_m = \gamma^{4321}\gamma_m\), as was suggested also by Marks (2008).

The canonical relations work in a part of the spectrum of \(|\hat{\gamma}| = +\sqrt{-\hat{\gamma}^2}\) that is so near to the maximum value \(N\) as to be indistinguishable from it, yet has passed for infinite until now. For example the band

\[1 - N^{-1/2} < |\hat{\gamma}| \leq 1\]  \hspace{1cm} (21)

is sufficiently narrow and crowded, with width \(N^{-1/2} \to 0\) and multiplicity \(O(\sqrt{N}) \to \infty\).

Stückelberg (1960) showed how to extract the complex quantum theory from a real one like \(\mathcal{Q}\) given an operator \(\hat{\gamma}\) whose square is \(-1\). The main point is that when the superselection rule for \(\hat{\gamma}\) is in force, physical \(i\)o channels are not represented by vectors in \(\mathcal{Q}\) but by planes in \(\mathcal{Q}\) invariant under \(\hat{\gamma}\). For physical operators must then commute with \(\hat{\gamma}\) and no one-dimensional projector can do this, while a two-dimensional one can. Two vectors \(Q_1, Q_2 \in \mathcal{Q}\) with \(Q_2 = \hat{\gamma}Q_1\) define such a plane, and combine into one ray in a singular limit where \(\hat{\gamma} \rightarrow i\) is adjoined to the real field. The origin of this superselection rule was unspecified. In the case of a plexus, it arises from the quantum law of large numbers, as for all the coordinates of classical mechanics. The imaginary \(i\) is the singular limit \(\hat{\gamma}\), the sum of \(N\) cell operators \(\delta\hat{\gamma}\), all much smaller than unity, and commuting with one another. The other basic variables are similar sums. The commutator of a variable with \(i\) then has only \(n\) terms, while the product has \(\sim n^2\). For large enough \(n\) the commutator is negligible compared to the product.

### 4.3 Cumulation

A vertex monadic of any stage is a superposition of unitized products of monadics of the previous stage. Thus low-level operators induce high-level ones. The cumulation operator \(\Sigma\) converts any property \(x\) of a module of low rank into a property \(\Sigma x\) of a module one rank higher, the cumulant of \(x\), by summing over all replicas of the lower-rank module in the higher. Kostant
(1961) calls it $\theta$. In particular, in a quantum plexus the quantum spin
operators of a single-cell stage induce orbital operators on the event stage by $\Sigma$, 
often written in terms of annihilation and creation operators $\psi$ and $\psi^H$ as
\[ \Sigma x := \psi^H x \psi. \] 
$\Sigma$ is a Lie homomorphism. It injects the Lie algebra of each rank into that of
the next. $\Sigma^n x$ represents any operator $x \in \text{Alg}_r(Q)$ in the algebra $\text{Alg}_{r+1}(Q)$. 
Let us also write $x = \delta y$ to mean that $y = \Sigma x$, and $x = \delta^n y$ to mean that $y = \Sigma^n x$.

4.4 Single-cell symmetries

Not all of the generators of Yang $\text{so}(6−n,n)$ are symmetries of the lamina, evidently. Our ambient space-time lamina is not six-dimensional on our 
macroscopic scale. But its simplicial cells might be six-dimensional on their 
sub-microscopic scale. We can use the Yang group generators as symmetries 
of the single-cell stage, when we write them as $\delta L_{n'm'}$. Their macroscopic cu-
mulants are $L_{n'n} := \Sigma_{E-C}^\delta L_{n'n}$. Then
\[ [\delta L_{n'm}, \delta L_{m'm}] = g_{nm'} \delta L_{n'm'} - g_{n'm} \delta L_{nm} + g_{n'm} \delta L_{nm'} - g_{nm} \delta L_{n'm'}, \] 
with $m, m', n, n' \in \{1, \ldots, 6\}$. The six-dimensional quadratic Yang space re-
quires a three-dimensional semi-vector space $3\mathbb{R} \subset cQ$ of the cellular rank, and
its eight-dimensional spinors are elements of the eight-dimensional Grass $3\mathbb{R}$.

Relative to any frame, these carry atomistic elements of familiar variables 
according to
\[ \delta L = \begin{bmatrix}
0 & \ldots & \delta L^{14} & -\delta x^1/X & -\delta p^1/E \\
\delta L^{21} & \ldots & \delta L^{24} & : & : \\
: & \ldots & : & : & : \\
\delta L^{41} & \ldots & 0 & -\delta x^4/X & -\delta p^4/E \\
\delta x^1/4X & \ldots & \delta x^4/X & 0 & \delta L^{56} \\
\delta p^1/E & \ldots & \delta p^4/E & \delta L^{65} & 0
\end{bmatrix}, \] 
whose matrix elements are elements of $\text{so}(6−n,n)$. Here $X$ and $E$ are fundamental 
units of time and energy, the chronon and the ergon; it is not excluded that these have the order of magnitude of Planck units. $\delta L_i = \delta L^{56}$ is one 
atom in the cumulant $\Sigma \delta L^{56} = N\tilde{i}$, $\tilde{i}$ is a quantized $i$, and $N$ is the number of 
terms in the sum.
On the other hand, the commutation relations of the Heisenberg-Poincaré Lie algebra of space-time position \(x^\mu\), momentum-energy \(p^\mu\), and dimensionless Lorentz generator \(L_{\mu'\mu}\), in \(c = 1\) units, are

\[
\begin{align*}
[L^\nu, L^\mu] &= g^{\nu \mu'} L^\nu_{\mu'} - g^{\nu \mu'} L^{\nu'}_{\mu} + g^{\nu \mu} L^{\nu'}_{\mu'} - g^{\nu \mu} L_{\nu' \mu'}, \\
[L^\mu, x^\mu] &= g^{\mu \mu'} x_{\mu'} - g^{\mu \mu'} x^\mu, \\
[L^\mu, p^\mu] &= g^{\mu \mu'} p_{\mu'} - g^{\mu \mu'} p^\mu, \\
[x^\mu, p^\mu] &= ihg^{\mu \mu'}, \\
[L^\mu, \hat{i}] &= [x^\mu, x^\mu] = [x^\mu, i] = 0
\end{align*}
\]

(25)

with \(\mu, \mu', \nu, \nu' \in \{1, \ldots, 4\}\). To contract (23) \(\rightarrow\) (25), we must polarize and centralize one of its generators, which we may take to be \(L_{56}\) in an adapted frame, and turn off the gauge fields. Let us invoke a dynamical self-organization akin to magnetization. It is sufficient if (23) acts on an io space \(\mathcal{V} = 2(2N + 1)\mathbb{R} \subset \mathcal{Q}\) with large dimension \(2(2N + 1) \gg 1\), so that the spectra of the \(L_{m'm}\) are quasi-continuous; and if in a polarized sector \(\mathcal{V}_{\text{pol}} \subset \mathcal{V}\), supposed to represent the usual environment, \(|L_{65}|\) is close to its maximum eigenvalue \(N\), and the other components of \(L_{\mu'\mu}\) are much smaller, though still quasi-continuous:

\[
|L_{65}| \approx N \gg |L_{\mu'\mu}|
\]

(26)

The matrices of the single-cell so(\(W[C]\)) are not orbital variables but their “atoms”, spin variables. The orbital variables are their cumulants on a later stage. A Yang so(3,3) of level \(C\) is faithfully represented on level \(E\) by second cumulants of its generators \(L^C\). The \(L^C\) in one \(\mathcal{Q}\) frame, up to constant multipliers, approach the operators

\[
\begin{align*}
\delta \hat{x}^m &= X \gamma^m 5, \quad m, n = 1, 2, 3, 4, \\
\delta \hat{p}_m &= E \gamma^m 6, \\
\delta \hat{i} &= \gamma^{65}, \\
\delta \hat{L}_{nm} &= h \gamma_{nm},
\end{align*}
\]

(27)

as \(h(4) \leftarrow so(3,3) \leftarrow sl(6)\). The quantized cumulant imaginary \(\hat{i}\) is normalized to unit magnitude with a small factor \(N^{-1}\). To form macroscopic event coordinates, we must cumulate these single-cell variables at least twice, to reach at least level 6, with \(\text{Exp} 6 = 2^{(2^6)}\) points, enough for a quasi-continuum. Let us set

\[
\begin{align*}
x^\mu &= XL_{\mu}^5, \\
p^\mu &= N^{-1}EL_{\mu}^6, \\
\hat{i} &= N^{-1}L_{65}^5,
\end{align*}
\]

(28)

following Yang. \(\hat{i}^2 \approx -1\) in \(\mathcal{V}_{\text{pol}}\). The factor \(N^{-1}\) results from the duality between position and momentum, which requires us to average one when we sum the other.
In the limit
\[ N \to \infty, \quad \text{with } EX =\hbar, \] (29)
the Heisenberg-Poincaré relations (25) follow.

The electric generator \( \delta L_{56} \) commutes with the Lorentz generators \( \delta L_{\mu'\mu} \) (in an adapted frame) at the cellular level but not with the atoms of momentum \( E\delta L_{\mu6} \) or position \( X\delta L_{\mu5} \). That commutativity comes at the event level in the lamina, due to the large value of \( N \).

### 4.5 Palev statistics

The commutation relations for Bose statistics define a canonical Lie algebra, and so they too require regularization. Palev (1977), pursuing simplicity, reformed the canonical algebra of Bose statistics to a simple Lie algebra, such as \( \text{so}(N) \).

Pairs of \( Q \) fermions do not exactly obey Bose statistics but a Palev statistics defining an \( \text{so}(N_+, N_-) \) Lie algebra. The canonical Lie algebra of Bose statistics is a singular organized limit of this \( \text{so}(N_+, N_-) \) with \( N_\pm \to \infty \). In a recursive quantum theory it is natural to regard all empirical bosons as approximations to palevons with even-degree fermionic cores. In a canonical quantum theory a fermionic core would show up as a hard core in high-energy collisions. In the recursive quantum theory this does not follow, because at the cellular level the canonical indeterminacy relations have not yet set in, and because elementary energy transfers are local in both position space and momentum space.

### 4.6 Vertex statistics

To be sure, these event spaces are too small for the lamina variables of §1.2. A recursive quantum theory must fit all these variables and their commutation relations into the operator algebra of \( Q \). All these new groups, however, are simple and fit readily into low ranks of \( Q \). The infinities come from the orbital and bosonic field variables, which generate singular Lie algebras. This study limits itself to this core problem.

The first models in (1) do not impose odd statistics on their space-time atoms or chronons for obvious reasons. If the chronon has spin 1/2 and no other variables, odd statistics would exclude histories with more than four events. In the recursive quantum theory, unitization allows us to clone a spin and build histories with arbitrarily many spins 1/2 with odd statistics.
5 Recursive gauge

The quantum plexus must have a regular gauge group that can be approximated by the singular ones of gravity and the Standard Model. Historically, gauge theories were made by gauging un-gauged theories. Recursive quantum theories, however, are born gauged. The event stage contains enormous numbers of clones of the single cell. The plexus gauge group of the first (Dirac) kind is the group $\text{SL}(n)$ of a single cell, a simplex some two ranks below the space-time event-vertex. The gauge group of the second kind is that generated by all the single-cell groups of the event rank. In the laminar phase, it is to reduce to that of the Standard Model and gravity, first by dynamical symmetry re-arrangement by the organization of the lamina, which results in a semi-simple group, and then by a singular approximation, including the limit of classical space-time, that results in the usual singular gauge group.

Evidently quantization and gauging are both forms of quantification, acting at different stages. *The single cell is the gauge block in the Weyl sense.* It defines a fundamental quantum of time-interval as well as Planck’s quantum of action.

This unanticipated unification of the quantum and the gauge fits into a long-standing conjecture: that gauge theory, including gravity, is an extension of dislocation theory to a quantum crystalline medium; that every gauge current seen close up is a Burgers-Volterra vector describing dislocations. The medium is the quantum plexus, organized into a crystalline polarized lamina, and the gauge fluxes, interpreted as vertex permutations rather than rotations, characterize its dislocations.

To see this unification in more detail let us return to classical mechanics. There one has three sets of algebraic commutation relations, schematically $[x, x] = 0$, $[x, p] = 0$, and $[p, p] = 0$. This trivial Lie algebra is maximally singular, in that its Killing form is not merely singular but vanishes identically.

A gauge theory reforms the commutators $[p, p]$, making them into a gauge curvature field. A canonical quantum theory reforms the commutators $[x, p]$ making them $i\hbar$. Let us define the power constant $W := E/NX$. In a recursive quantum theory, the lowest-order corrections to the commutators have the form

$$[x, x] \sim \frac{\hbar}{W}, \quad [x, p] \sim \hbar, \quad [p, p] \sim \hbar W.$$ (30)

Thus classical gauge theory handles the $o(W)$ correction, canonical quantum theory deals with the $o(1)$ correction, and the $o(W^{-1})$ correction, presumably the smallest in ordinary experiments, is not yet taken up.

This symmetry is masked at first because the $[p, p]$ commutators are field variables while the $[x, p]$ commutators have passed for constants. In a quantum
plexus the \([x, p]\) commutator is also a field, the Higgs field, a singular normalized limit of \(L_i\). This passes for constant in ordinary circumstances because it is polarized near its maximum value by the dynamical symmetry-rearrangement that forms the lamina.

A gauge connection associates a Lie algebra element \(a\) with a direction \(p\) (or its dual) at a point \(x\). A direction in Yang space (24) is a Lie algebra element. This permits us to assume that a singular quantum gauge field might actually be an organized singular limit of a regular sea of events, taken from a quantum space like a Yang space with an appropriate orthogonal group.

\(Q\) degree counts vertices of the simplex, \(Q\) rank counts nested iotas, and the basic \(Q\) dynamical operators \(L^C \in \mathfrak{so}(\mathcal{W}[C])\) of the cell rank \(C\) count components of generalized angular momentum in units of the roots of this Lie algebra. The eight spin-like atoms of orbital angular momentum \(\hat{x}^m\) and \(\hat{p}_m\) \((m \in 6)\), and the quantized imaginary \(\hat{i}\), are among the 15 generators \(\mathcal{C}^C\) of a Yang \(\mathfrak{so}(3,3)\) Lie algebra, forming a Lie subalgebra of the 120-dimensional Lie algebra \(\mathfrak{so}(\mathcal{W}[C] \subset \mathfrak{alg} Q^1 \cong \text{Cliff } \mathcal{W}[C]\) of rank \(C\).

The elementary momenta do not commute. Neither do the cumulative momenta, which correspond to infinitesimal translations. This quantum non-commutativity survives into general relativity as part of the curvature, perhaps including a cosmologically constant part.

The Higgs field is usually posited ad hoc. The quantized \(i\) of Yang provides it with a theoretical foundation, as in Tavel (1965) (which flowed from a suggestion of Yang at a Rochester Conference). It seems consistent with the technicolor hypothesis of Susskind (1979) and Weinberg (1979) that the Higgs field, like the BCS pair \(i\) vector, be an order parameter of a self-organization, namely that of the crystalline laminar condensation whose dislocations are responsible for gauge interactions.

The Yang group and the Heisenberg-Poincaré group include orthogonal transformations that interchange space-time position and momentum, a symmetry called reciprocity by Born (1949). Reciprocity is badly broken by

1. The canonical locality principle. This requires elementary particles in interaction to have approximately the same position, but allows them to have widely different momenta; especially by its sharpest form,

2. The gauge principle. This requires the local gauge group to be replicated at every position, not every momentum.

3. The vacuum. This has no natural zero for position in space-time at the particle level of experiments, but has a natural zero for momentum-energy.

The recursive gauge principle merely requires covariance under the group of
every simplicial cell, which includes changes in momentum as well as in position.

Locality in momentum and asymptotic freedom both limit high-momentum transfers in elementary particle interactions, so they may be related. One rather tenuous relation arises at once. Simplicial locality leads in a singular limit to a non-abelian gauge. When this is adjusted to fit the Standard Model, it will permit the Gross-Wilczek deduction of asymptotic freedom, which seems to be a mild form of locality in momentum, from the non-abelian gauge of the Standard Model.

In a quantum plexus, the structures treated as elementary particles in the canonical theory are edges of extended dislocations and their interactions consist of many simplicial processes. Then the observed large momentum transfers \( p^\mu \) may occur in many small steps \( \delta p^\mu \) consistent with simplicial locality. These violations of reciprocity are trying to tell us something important about the fine structure of the lamina, but we have not made out what they are saying yet.

5.1 Dynamics

Let us consider other problems that a recursive quantum electrodynamics must still solve. The Yang group suffices for its cellular gauge group, with \( L_{56} \to N_i \) as the electric axis in Yang space, the Higgs field to be frozen, and the electric gauge group generator. The \( i \) of the event level is to be treated as a constant. While \( N_i \) rotates position into momentum, the limit \( N \to \infty \) permits us to take \([i, x^\mu] = 0\).

The simplicial operators for the orbital variables have already been given. Since the momentum-energy variables \( \hat{p}_\mu \) are covariant under the simplicial gauge group, they correspond to the kinetic momentum-energy \( p_\mu - \Gamma_\mu \) of the canonical gauge theory rather than to the total momentum-energy \( p_\mu \) or the potential momentum \( \Gamma_\mu \), which includes both the electric and the gravitational vector potential. No plexus expression has been found for either \( p_\mu \) or \( \Gamma_\mu \) separately. Perhaps the search was misguided. Since only \( p_\mu - \Gamma_\mu \) has invariant physical meaning, the absence of separate \( p \) and \( \Gamma \) makes the recursive theory more physical, not less, than the canonical one. To recover pure quantum electrodynamics the gravitational vector potential can be suppressed by setting \([\gamma^\mu, p'^\mu] \to 0\).

The Lorentz group corresponds to the centralizer of the electric group, with generators \( L_{\mu'\mu} \) in an adapted frame. Thus dynamical symmetry rearrangement and a singular limit have to account for the vacuum averages of (1) \( p^\mu \to 0 \), (2) \( \hat{\tau} \to i \), the imaginary constant and the Higgs field and (3) \( g' \to g_{\mu'\mu} \), the gravitational field, as order parameters. This brings in rather
new considerations, appropriate for a sequel.

6 Discussion

If we take both the quantum and gravity theories seriously, space-time is more likely to be composed of spins, as Feynman and Penrose have suggested, than conversely. The geometric meaning of spinors is mysterious, as Atiyah (1998) points out. The quantum meaning is simple: They represent the most elementary quantum actions, the creation and annihilation of vertex elements, which are of the Fermi-Dirac kind. Their Clifford algebra is a Fermi-Dirac algebra. Spinors cannot be explained in simpler terms if there are none. They can still be understood by describe everything else in terms of them, including geometry, as attempted here. In physics as in archeology, the foundations are the last thing discovered. Classical geometry and mechanics are explained by quantum geometry and mechanics, not conversely.

Iterated, the classic spinor construction of Cartan leads to a recursive hierarchy of spinors that can be interpreted as representations of recursive quantum simplicial complexes with symmetries that are useful for quantum theories. The semivectors underlying the Cartan construction of spinors are also spinors, but they are of the previous stage, and are given a different transformation law under the orthogonal group, yet to be explained physically. The neutral metric of the Cartan construction is now a relativistic probability flux metric that distinguishes bras from kets in each frame by the signs of their norms, and reduces to the classical Minkowski metric in an organized singular limit.

Canonical quantum theories imagine a manifold of events without spins. Spinless events do not occur in the Standard Model. The events of a recursive quantum theory are vertices of a simplex and carry the spin of its group. A complex of such vertices, as opposed to a lattice, is finite in that its dynamical variables all have discrete finite spectra.

Manifold theories represent the event space, including event coordinates that suffer gauge transformations, as a manifold, preferably without boundaries, after Kaluza. Then they need a high energy-density and stress to curve and compactify the circular transverse dimensions of the lamina. Humbler laminas like bubbles, snowflakes, and graphenes do not compactify but organize. Their quantum elements organized themselves in only some of the possible directions, typically because their interactions have short range and saturate, like covalent bonds; and so they have boundaries in the transverse dimension. Let us assume the same for the ambient laminar plexus. This replaces the compactification problem by the organizational problem: How do the available modules assemble themselves into the ambient lamina? Can one
not write down a vector in $\mathbb{Q}$ describing this organization?

It is proposed that the imaginary $i$ of quantum theory is like the $i$ of AC circuit theory: Both represent a contingent symmetry of the environment, an AC generator in one case and a quantum simplicial plexus in the other. When the generator hunts or the lamina is dislocated, the constant $i$ gives way to a dynamical one. $\hat{i}$ hopefully also serves as Higgs field, inflicting mass on any gaugeon that does not respect it.

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Table 1: Polyadics and their statistics, by rank and serial number

| r | e_q | q |
|---|-----|---|
| 0 | .   | 0 |
| 1 |     | 1 |
| 2 |     | 2 3|
| 3 |     | 4 5 6 7 8 9 10 11 12 13 14 15 |
| 4 |     | 16 17 18 19 20 21 22 23 24 25 26 27 ... |
| 5 |     | Exp 5 ... |
| 6 |     | Exp 6 ... |

Table 2: Monadics of rank 4

| log_2(q) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| e_q      | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  | .  |
| $r$ | Level | Alg $r\mathcal{Q}$ | $r\mathcal{Q}$ | $r\mathcal{W}$ | $SO(r\mathcal{W})$ |
|-----|-------|-------------------|-------------|-------------|----------------|
| 3   | Fermi 4$\mathbb{R}$ | $4\mathbb{R}$ | $16\mathbb{R}$ | $32\mathbb{R}$ | $SO(16, 16)$ |
| 2   | Fermi 2$\mathbb{R}$ | $2\mathbb{R}$ | $4\mathbb{R}$ | $8\mathbb{R}$ | $SO(4, 4)$ |
| 1   | Fermi $\mathbb{R}$ | $\mathbb{R}$ | $2\mathbb{R}$ | $4\mathbb{R}$ | $SO(2, 2)$ |
| 0   | Fermi 0 | $\mathbb{R}$ | $\mathbb{R}$ | 0 | 1 |

Table 3: **Stages of the spinor tree $\mathcal{Q}$**.