PARTIAL COMBINATORY ALGEBRA AND
GENERALIZED NUMBERINGS

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Abstract. Generalized numberings are an extension of Ershov’s
notion of numbering, based on partial combinatory algebra (pca)
instead of the natural numbers. We study various algebraic prop-
erties of generalized numberings, relating properties of the num-
bering to properties of the pca. As in the lambda calculus, exten-
sionality is a key notion here.

1. Introduction

A numbering is a surjective mapping \( \gamma : \omega \to S \) from the natural
numbers \( \omega \) to a set \( S \). The theory of numberings was started by Ershov
in a series of papers, beginning with [8] and [9]. Ershov studied the
computability-theoretic properties of numberings, as generalizations of
numberings of the partial computable functions. In particular, he called
a numbering precomplete if for every partial computable unary function
\( \psi \) there exists a computable unary \( f \) such that for every \( n \)
\[
\psi(n) \downarrow \implies \gamma(f(n)) = \gamma(\psi(n)).
\] (1)
Following Visser, we say that \( f \) totalizes \( \psi \) modulo \( \gamma \). Ershov showed
that Kleene’s recursion theorem holds for arbitrary precomplete num-
berings. Visser [26] extended this to his so-called “anti diagonal nor-
malization theorem” (ADN theorem). Another generalization of the re-
cursion theorem is the famous Arslanov completeness criterion [2], that
extends the recursion theorem from computable functions to all func-
tions bounded by an incomplete computably enumerable (c.e.) Turing
degree. Barendregt and Terwijn [5] showed that Arslanov’s result also
holds for any precomplete numbering. In Terwijn [25], a joint general-
ization of Arslanov’s completeness criterion and Visser’s ADN theorem
was proved. It is currently open whether this joint generalization also
holds for every precomplete numbering.

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A classic example of numberings are numberings of the partial computable functions. Such a numbering is acceptable if it can be effectively translated back and forth into the standard numbering of the p.c. functions \( \varphi_e \). Rogers [22] proved that a numbering is acceptable if and only if it satisfies both the enumeration theorem and parametrization (also known as the S-m-n–theorem). It follows from this that every acceptable numbering is precomplete. On the other hand, Friedberg [11] showed that there exist numberings of the p.c. functions without repetitions. Friedberg’s 1-1 numbering is not precomplete, as can be seen as follows. Suppose that \( \gamma : \omega \to \mathcal{P} \) is a 1-1 numbering of the (unary) p.c. functions that is precomplete. By (1), we then have for every p.c. function \( \psi \) a computable function \( f \) such that

\[
\psi(n) \downarrow \implies \gamma(f(n)) = \gamma(\psi(n)) \implies f(n) = \psi(n).
\]

The second implication follows because \( \gamma \) is 1-1. So we see that in fact \( f \) is a total extension of \( \psi \). But it is well-known that there exist p.c. functions that do not have total computable extensions. So we see that 1-1 numberings of the p.c. functions are never precomplete. For more about 1-1 numberings see Kummer [14].

A topic closely related to numberings is that of computably enumerable equivalence relations (ceers). For every numbering \( \gamma \) we have the equivalence relation defined by \( n \sim_\gamma m \) if \( \gamma(n) = \gamma(m) \). Conversely, for every countable equivalence relation, we have the numbering of its equivalence classes. Hence the above terminology about numberings also applies to ceers. Lachlan [15], following work of Bernardi and Sorbi [6], proved that all precomplete ceers are computably isomorphic. For a recent survey about ceers, see Andrews, Badaev, and Sorbi [1].

In the examples of numberings given above, the set \( \omega \) is not merely a set used to number the elements of a set \( S \), but it carries extra structure as the domain of the partial computable functions, making it into a so-called partial combinatory algebra (pca). In section 2 below we review the basic definitions of pca. We can extend the notion of numbering from \( \omega \) to arbitrary pca’s as follows. A generalized numbering is a surjective mapping \( \gamma : \mathcal{A} \to S \), where \( \mathcal{A} \) is a pca and \( S \) is a set. This notion was introduced in Barendregt and Terwijn [5]. Below we study generalized numberings in relation to the algebraic structure of the pca \( \mathcal{A} \). Just as in the lambda-calculus, the notion of extensionality is central here. A pca is called extensional if \( f = g \) whenever \( fx \simeq gx \) for every \( x \). We have similar notions of extensionality based on generalized numberings (Definitions 5.1 and 6.1). We also have a notion of precompleteness for generalized numberings, analogous to Ershov’s notion (Definition 5.1). In section 5 we show that there is a relation
between extensionality (an algebraic property) and precompleteness (a computability theoretic property). Precompleteness of generalized numberings is related to the topic of complete extensions. For example, the identity on a pca $A$ is precomplete if and only if every element of $A$ (seen as a function on $A$) has a total extension in $A$.

In section 4 we show that the numbering of functions of a pca is precomplete, which is the analog of the precompleteness of the standard numbering of the p.c. functions. In general the functions modulo extensional equivalence do not form a pca. This prompts the definition of the notion of algebraic numbering, which is a generalized numbering that preserves the algebraic structure of the pca.

Combinatory completeness is the characteristic property that makes a structure with an application operator a pca. This is the analog of the S-m-n–theorem (parametrization) for the p.c. functions. In section 5 we study the relation between combinatory completeness, extensionality, and precompleteness.

In section 6 we introduce strong extensionality, and in section 7 we introduce some auxiliary equivalence relations. In section 8 we investigate the relations between various notions of extensionality and algebraic numberings. We will see that neither notion implies the other, and that they are in a sense complementary.

The Friedberg numbering of the p.c. functions quoted above also exists for the class of c.e. sets. In section 9 we discuss the existence of 1-1 numberings of classes of sets that are uniformly c.e. by considering the complexity of equality on those classes.

Our notation is mostly standard. In the following, $\omega$ denotes the natural numbers. $\varphi_e$ denotes the $e$-th partial computable (p.c.) function, in the standard numbering of the p.c. functions. We write $\varphi_e(n) \downarrow$ if this computation is defined, and $\varphi_e(n) \uparrow$ otherwise. $W_e = \text{dom}(\varphi_e)$ denotes the $e$-th computably enumerable (c.e.) set. For unexplained notions from computability theory we refer to Odifreddi [18] or Soare [24]. For background on lambda-calculus we refer to Barendregt [4].

2. Partial combinatory algebra

Combinatory algebra predates the lambda-calculus, and was introduced by Schönfinkel [23]. It has close connections with the lambda-calculus, and played an important role in its development. Partial combinatory algebra (pca) was first studied in Feferman [10]. To fix notation and terminology, we will briefly recall the definition of a pca, and for a more elaborate treatment refer to van Oosten [20].
Definition 2.1. A partial applicative structure (pas) is a set \( A \) together with a partial map from \( A \times A \) to \( A \). We denote the image of \((a, b)\), if it is defined, by \( ab \), and think of this as ‘a applied to b’. If this is defined we denote this by \( ab \downarrow \). By convention, application associates to the left. We write \( abc \) instead of \((ab)c\).

Terms over \( A \) are built from elements of \( A \), variables, and application. If \( t_1 \) and \( t_2 \) are terms then so is \( t_1 t_2 \). If \( t(x_1, \ldots, x_n) \) is a term with variables \( x_1, \ldots, x_n \), and \( a_1, \ldots, a_n \in A \), then \( t(a_1, \ldots, a_n) \) is the term obtained by substituting the \( a_i \) for the \( x_i \).

For closed terms (i.e. terms without variables) \( t \) and \( s \), we write \( t \simeq s \) if either both are undefined, or both are defined and equal. Here application is strict in the sense that for \( t_1 t_2 \) to be defined, it is required that both \( t_1, t_2 \) are defined. We say that an element \( f \in A \) is total if \( fa \downarrow \) for every \( a \in A \).

A pas \( A \) is combinatory complete if for any term \( t(x_1, \ldots, x_n, x) \), \( 0 \leq n \), with free variables among \( x_1, \ldots, x_n, x \), there exists a \( b \in A \) such that for all \( a_1, \ldots, a_n, a \in A \),

\begin{enumerate}
  \item \( ba_1 \cdots a_n \downarrow \)
  \item \( ba_1 \cdots a_n a \simeq t(a_1, \ldots, a_n, a) \).
\end{enumerate}

A pas \( A \) is a partial combinatory algebra (pca) if it is combinatory complete.

Theorem 2.2. (Feferman [10]) A pas \( A \) is a pca if and only if it has elements \( k \) and \( s \) with the following properties for all \( a, b, c \in A \):

\begin{itemize}
  \item \( k \) is total and \( kab = a \),
  \item \( sab \downarrow \) and \( sabc \simeq ac(bc) \).
\end{itemize}

Note that \( k \) and \( s \) are nothing but partial versions of the familiar combinators from combinatory algebra. As noted in [10, p95], Theorem 2.2 has the consequence that in any pca we can define lambda-terms in the usual way (cf. Barendregt [4, p152]). For every term \( t(x_1, \ldots, x_n, x) \), \( 0 \leq n \), with free variables among \( x_1, \ldots, x_n, x \), there exists a term \( \lambda^*x.t \) with variables among \( x_1, \ldots, x_n \), with the property that for all \( a_1, \ldots, a_n, a \in A \),

\begin{itemize}
  \item \((\lambda^*x.t)(a_1, \ldots, a_n) \downarrow \)
  \item \((\lambda^*x.t)(a_1, \ldots, a_n)a \simeq t(a_1, \ldots, a_n, a) \).
\end{itemize}

The most important example of a pca is Kleene’s first model \( K_1 \), consisting of \( \omega \) with application defined as \( nm = \varphi_n(m) \). Kleene’s second model \( K_2 \) [12] consists of the reals (or more conveniently Baire space \( \omega^\omega \)), with application \( \alpha \beta \) defined as applying the continuous functional

\[ \begin{align*}
  \text{(1) Because the lambda-terms in combinatory algebra do not have the same substitution properties as in the lambda calculus, we use the notation } \lambda^* \text{ rather than } \lambda. \\
  \text{Curry used the notation } [x] \text{ to distinguish the two (cf. [7]).}
\end{align*} \]
with code $\alpha$ to the real $\beta$. See Longley and Normann [16] for a more detailed definition.

In the axiomatic approach to the theory of computation, there is the notion of a basic recursive function theory (BRFT). Since this is supposed to model basic computability theory, it will come as no surprise that every BRFT gives rise to a pca. In case the domain of the BRFT is $\omega$, it actually contains a copy of the p.c. functions. See Odifreddi [18] for a discussion of this, and references to the literature, including the work of Wagner, Strong, and Moschovakis.

Other pca’s can be obtained by relativizing $\mathcal{K}_1$, or by generalizing $\mathcal{K}_2$ to larger cardinals. Also, every model of Peano arithmetic gives rise to a pca by considering $\mathcal{K}_1$ inside the model. Further constructions of pca’s are discussed in van Oosten and Voorneveld [21], and in van Oosten [20] even more examples of pca’s are listed. Finally, it is possible to define a combination of Kleene’s models $\mathcal{K}_1$ and $\mathcal{K}_2$ (due to Plotkin and Scott) using enumeration operators, cf. Odifreddi [19, p857 ff].

3. Generalized numberings

A generalized numbering is a surjective mapping $\gamma: \mathcal{A} \to S$, where $\mathcal{A}$ is a pca and $S$ is a set. As in the case of ordinary numberings, we have an equivalence relation on $\mathcal{A}$ defined by $a \sim \gamma b$ if $\gamma(a) = \gamma(b)$.

As for ordinary numberings, in principle every generalized numbering corresponds to an equivalence relation on $\mathcal{A}$, and conversely. However, below we will mainly be interested in numberings that also preserve the algebraic structure of the pca, making this correspondence less relevant.

The notion of precompleteness for generalized numberings was defined in [5]. By [5, Lemma 6.4], the following definition is equivalent to it.

**Definition 3.1.** A generalized numbering $\gamma: \mathcal{A} \to S$ is precomplete if for every $b \in \mathcal{A}$ there exists a total element $f \in \mathcal{A}$ such that for all $a \in \mathcal{A}$,

$$ba \Downarrow \implies fa \sim \gamma ba.$$  \hspace{1cm} (2)

In this case, we say that $f$ totalizes $b$ modulo $\sim \gamma$.

A precomplete generalized numbering $\gamma$ is complete if there is a special element $s \in S$ (not depending on $b$) such that in addition to (2), $\gamma(fa) = s$ for every $a$ with $ba \uparrow$.

In [5], generalized numberings were used to prove a combination of a fixed point theorem for pca’s (due to Feferman [10]), and Ershov’s recursion theorem [9] for precomplete numberings on $\omega$. 
Every pca $\mathcal{A}$ has an associated generalized numbering, namely the identity $\gamma_\mathcal{A} : \mathcal{A} \to \mathcal{A}$. In section 5 we will see examples of when the numbering $\gamma_\mathcal{A}$ is or is not precomplete.

4. ALGEBRAIC NUMBERINGS

Definition 4.1. Let $\mathcal{A}$ be a pca. Define an equivalence on $\mathcal{A}$ by $a \sim_e b$ if
$$\forall x \in \mathcal{A} (ax = bx).$$

The following result generalizes the precompleteness of the numbering $n \mapsto \varphi_n$ of the partial computable functions.

Proposition 4.2. The natural map $\gamma_e : \mathcal{A} \to \mathcal{A}/\sim_e$ is precomplete. Note that $\gamma_e$ is a generalized numbering of the equivalence classes.

Proof. Let $b \in \mathcal{A}$. We have to prove that there is a total $f \in \mathcal{A}$ such that when $ba \downarrow$ then $fa \sim_e ba$, i.e. $\forall c \in \mathcal{A} (fac = bac)$. This follows from the combinatory completeness of $\mathcal{A}$: Consider the term $bxy$. By combinatory completeness there exists $f \in \mathcal{A}$ such that for all $a,c \in \mathcal{A}$, $fa \downarrow$ and $fac = bac$. $\square$

Remark 4.3. Note that $\mathcal{A}/\sim_e$ is in general not a pca, at least not with the natural application defined by $\overline{a \cdot b} = \overline{a} \cdot \overline{b}$. For example, in Kleene's first model $K_1$ we have for $n,m \in \omega$ that $n \sim_e m$ if $\varphi_n = \varphi_m$. Now we can certainly have that $m \sim_e m'$, but $\varphi_n(m) \neq \varphi_n(m')$, so we see that the natural definition of application $\overline{a \cdot m} = \overline{a} \cdot \overline{m}$ in $\omega/\sim_e$ is not independent of the choice of representative $m$.

The previous considerations prompt the following definition. First we extend the definition of $\sim_\gamma$ from $\mathcal{A}$ to the set of closed terms over $\mathcal{A}$ as follows:

Definition 4.4. $a \sim_\gamma b$ if either $a$ and $b$ are terms that are both undefined, or $a,b \in \mathcal{A}$ and $\gamma(a) = \gamma(b)$.

This extended notion $\sim_\gamma$ is the analog of the Kleene equality $\simeq$.

Definition 4.5. Call a generalized numbering $\gamma : \mathcal{A} \to S$ algebraic if $\sim_\gamma$ is a congruence, i.e.
$$a \sim_\gamma a' \land b \sim_\gamma b' \implies ab \sim_\gamma a'b'.$$

In this case, we also call the pca $\mathcal{A}$ $\gamma$-algebraic.

If $\gamma$ is algebraic, we can factor out by $\sim_\gamma$, as in algebra:

Proposition 4.6. Suppose that $\mathcal{A}$ is $\gamma$-algebraic. Then $\mathcal{A}/\sim_\gamma$ is again a pca.
Proof. Define application in $A/\sim_{\gamma}$ by $\bar{a} \cdot \bar{b} = a \cdot b$. By algebraicity this is well-defined. Combinatory completeness follows because we have the combinator symbols $\bar{\pi}$ and $\bar{K}$.

We note that for a generalized numbering $\gamma : A \to S$ there is in general no relation between the notion of precompleteness and $\gamma$-algebraicity. This follows from results in the following sections.

- $\gamma$-algebraic does not imply precomplete. Namely, let $\gamma_{K_2}$ be the identity on Kleene’s second model $K_2$. Then $K_2$ is trivially $\gamma$-algebraic. However, by [5], the numbering $\gamma_{K_2}$ is not precomplete.
- precomplete does not imply $\gamma$-algebraic. Otherwise, by Proposition 5.2 we would have that $\gamma$-extensional implies $\gamma$-algebraic, contradicting Proposition 8.2. Alternatively: The canonical map $\gamma_e : A \to A/\sim_e$ is precomplete by Proposition 4.2. However, it is not algebraic, since otherwise we would have by Proposition 4.6 that $A/\sim_e$ is a pca, which in general it is not by Remark 4.3.

5. Precompleteness and Extensionality

We can think of combinatory completeness of a pca as an analog of the S-m-n–theorem (also called the parametrization theorem) from computability theory [18]. Suppose $A$ is a pca, and $\gamma : A \to S$ is a generalized numbering. We have the following analogies of concepts:

| Kleene’s first model $K_1$ | pca $A$ |
|-----------------------------|---------|
| S-m-n–theorem               | $\iff$ combinatory completeness $A$ |
| $\downarrow$                | $\downarrow$? |
| precompleteness of $n \mapsto \varphi_n$ | $\iff$ precompleteness of $\gamma : A \to S$ |

Since the precompleteness of the numbering $n \mapsto \varphi_n$ of the partial computable functions follows from the S-m-n–theorem, and the notions of combinatory completeness and precompleteness are analogous to these, one might guess that the precompleteness of $\gamma$ could be a consequence of the combinatory completeness of $A$. However, this is not the case: The identity on $K_1$ is not precomplete, as there exist p.c. functions that do not have a total computable extension. In general, every pca $A$ has the identity $\gamma_A : A \to A$ as an associated generalized numbering, and $\gamma_A$ is precomplete if and only if every element $b \in A$ has a total extension $f \in A$. In [5] it was shown that the the identity $\gamma_{K_2}$ on Kleene’s second model $K_2$ is also not precomplete.
Definition 5.1. Let $\mathcal{A}$ be a pca, and $\gamma : \mathcal{A} \to S$ a generalized numbering. We say that $\mathcal{A}$ is $\gamma$-extensional if

$$\forall a \in \mathcal{A} (fa \simeq ga) \implies f \sim_{\gamma} g$$

for all $f, g \in \mathcal{A}$.

In other words, $\mathcal{A}$ is $\gamma$-extensional if the relation $\sim_{\gamma}$ extends the relation $\sim_e$ from Definition 4.1. For the special case where $\gamma : \mathcal{A} \to \mathcal{A}$ is the identity, this is called extensionality of $\mathcal{A}$, cf. Barendregt [3, p1094].

Proposition 5.2. Suppose $\mathcal{A}$ is $\gamma$-extensional. Then $\gamma$ is precomplete.

Proof. This is similar to Proposition 4.2. Given $b \in \mathcal{A}$, we have to prove that there exists a total $f \in \mathcal{A}$ such that for every $a \in \mathcal{A}$, $fa \sim_{\gamma} ba$ whenever $ba \downarrow$. Consider the term $bxy$. By combinatory completeness of $\mathcal{A}$ there exists a total $f \in \mathcal{A}$ such that $fac \simeq bac$ for all $a, c \in \mathcal{A}$. Now suppose $ba \downarrow$. It follows from $\gamma$-extensionality of $\mathcal{A}$ that $fa \sim_{\gamma} ba$. Hence $\gamma$ is precomplete. \(\square\)

In particular, we see from Proposition 5.2 that the identity $\gamma_\mathcal{A}$ on $\mathcal{A}$ is precomplete if $\mathcal{A}$ is extensional.

It is possible that a generalized numbering $\gamma : \mathcal{A} \to S$ is precomplete for some other reason than $\mathcal{A}$ being $\gamma$-extensional. For example, suppose that $\mathcal{A}$ is a total pca. Then $\gamma$ is trivially precomplete (this is immediate from Definition 3.1), but a total pca $\mathcal{A}$ need not be extensional, as the next proposition shows.

Proposition 5.3. There exists a generalized numbering $\gamma : \mathcal{A} \to S$ that is precomplete, but such that $\mathcal{A}$ is not $\gamma$-extensional.

Proof. This follows from the fact that there exists a total pca that is not extensional. For example, let $\mathcal{A}$ be a model of the lambda calculus. This certainly does not have to be extensional, for example the graph model $P\omega$ is not extensional, cf. [1], p474]. (An example of a model of the lambda calculus that is extensional is Scott’s model $D_\infty$.)

Another example for is the set of terms $M(\beta\eta)$ in the lambda calculus. This combinatory algebra is extensional, by inclusion of the $\eta$-rule. However, the set of closed terms $M^0(\beta\eta)$ is not extensional by Plotkin [17]. \(\square\)

2 Alternatively, we could derive Proposition 5.2 from Proposition 4.2 by noticing that if $\gamma, \gamma' : \mathcal{A} \to S$ are generalized numberings such that $\sim_{\gamma}$ extends $\sim_{\gamma'}$, and $\gamma'$ is precomplete, then also $\gamma$ is precomplete. By Proposition 4.2 we have that $\gamma' = \gamma_\mathcal{A}$ is precomplete, and if $\mathcal{A}$ is $\gamma$-extensional then $\sim_{\gamma}$ extends $\sim_e$, so it follows that $\gamma$ is precomplete.
Proposition 5.3 shows that the converse of Proposition 5.2 does not hold. By the results of [5], combinatory completeness of $A$ does not imply precompleteness of $\gamma_A$. Conversely, precompleteness of $\gamma_A$ also does not imply combinatory completeness of $A$, as we now show. To prove this, we have to extend the definition of generalized numbering from pca’s to pas’s.

**Proposition 5.4.** There exists a pas $A$ that is not combinatory complete, and such that $\gamma_A$ is precomplete.

**Proof.** Consider the set $A = 2^{<\omega}$ of finite binary strings, with concatenation of strings as application. This is a pure term model, in which application of terms can only increase the length of terms. This implies that this pas is not combinatory complete, as there cannot be a combinator $k$, which does reduce the length of terms. However, the identity $\gamma_A : A \rightarrow A$ is precomplete, as the pas $A$ is total. □

Finally, we note that for a generalized numbering $\gamma : A \rightarrow S$, there is also no relation between combinatory completeness of the pca $A$ and the property of $A$ being $\gamma$-extensional:

- There exists a pas $A$ that is $\gamma$-extensional but not combinatory complete. (This is stronger statement than Proposition 5.4 by Proposition 5.2.) Namely, the counterexample from Proposition 5.4 is extensional.

- Conversely, there exists a pca $A$ and a generalized numbering $\gamma : A \rightarrow S$ such that $A$ is not $\gamma$-extensional. Namely, otherwise combinatory completeness of $A$ would imply precompleteness of $\gamma$ by Proposition 5.2 contradicting [5, Proposition 7.2]. Alternatively, we can just take any pca $A$ that is not extensional.

6. **Strong extensionality**

Given a generalized numbering $\gamma : A \rightarrow S$, we have two kinds of equality on $A$: $a \simeq b$ and $a \sim_{\gamma} b$. The notion of $\gamma$-extensionality is based on the former. We obtain a stronger notion if we use the latter, where we use the extended notion of $\sim_{\gamma}$ for closed terms from Definition 4.4.

**Definition 6.1.** We call $A$ **strongly $\gamma$-extensional** if
\[
\forall x (fx \sim_{\gamma} gx) \implies f \sim_{\gamma} g
\]

This counterexample is also strongly $\gamma$-extensional (in the sense of Definition 6.1 below). So we also have that strong $\gamma$-extensional does not imply combinatory complete.
Theorem 6.2. Strong $\gamma$-extensionality implies $\gamma$-extensionality, but not conversely.

Proof. First, strong $\gamma$-extensionality implies $\gamma$-extensionality because $\forall x \, fx \simeq gx$ implies $\forall x \, fx \sim_\gamma gx$, so the premiss of the first notion is weaker than that of the second.

To see that the implication is strict, we exhibit a pca $\mathcal{A}$ that is $\gamma$-extensional but not strongly $\gamma$-extensional. Take $\mathcal{A}$ to be Kleene’s first model $\mathcal{K}_1$, and let $\gamma = \gamma_e$, the numbering of equivalence classes from Proposition 4.2. That $\mathcal{A}$ is $\gamma_e$-extensional is trivial, since $f \sim_e g$ means precisely $\forall x \, fx \simeq gx$.

To see that $\mathcal{A}$ is not strongly $\gamma_e$-extensional, let $d, e \in \mathcal{A}$ be such that $d \neq e$ and $\forall x \, dx \simeq ex$, and define $f = kd$ and $g = ke$, with $k$ the combinator. Then $fx = d$ and $gx = e$, hence $\forall x \, fx \sim_e gx$ because $d \sim_e e$. But not $\forall x \, fx \simeq gx$ because $d \neq e$, so $f \not\sim_e g$. □

7. LEFT AND RIGHT EQUivalENCES

For the discussion below (and also to aid our thinking), we introduce a number of equivalence relations. Let $\mathcal{A}$ be a pca, and $\gamma : \mathcal{A} \to S$ a generalized numbering.

Definition 7.1. We define two kinds of equivalence relations on $\mathcal{A}$, corresponding to right and left application:

- $f \sim_R g$ if $\forall x \, fx \simeq gx$.
- $f \sim_L g$ if $\forall z \, zf \simeq zg$.
- $f \sim_{R, \gamma} g$ if $\forall x \, fx \sim_\gamma gx$.
- $f \sim_{L, \gamma} g$ if $\forall z \, zf \sim_\gamma zg$.

Note that $\sim_R$ is the same as the relation $\sim_e$ from section 4. Also note that $\sim_R$ is equal to $\sim_{R, \gamma}$ for $\gamma = \gamma_e$ from Proposition 4.2.

Note that with these equivalences we can succinctly express extensionality as follows:

$\mathcal{A}$ is $\gamma$-extensional if $f \sim_R g \Rightarrow f \sim_\gamma g$;

$\mathcal{A}$ is strongly $\gamma$-extensional if $f \sim_{R, \gamma} g \Rightarrow f \sim_\gamma g$.

Proposition 7.2. For all $f, g \in \mathcal{A}$ we have:

(i) $f \sim_L g \Rightarrow f \sim_R g$,

(ii) $f \sim_{L, \gamma} g \Rightarrow f \sim_{R, \gamma} g$.

Proof. (i) is a special case of (ii), so it suffices to prove the latter. For every $x$, define $z_x = \lambda^* h. hx$. (Note that in every pca we can define...
such lambda terms, cf. section 2. Then

\[ f \sim_L g \implies \forall x. z x f \sim \gamma z x g \]
\[ \implies \forall x. f x \sim \gamma g x \]
\[ \implies f \sim_R g. \]

8. Algebraic versus extensional

For a given pca \( A \) and a generalized numbering \( \gamma \) on \( A \), note that the following hold: If \( A \) is \( \gamma \)-algebraic, then for every \( f, g \in A \),

\[ f \sim \gamma g \implies f \sim_R \gamma g, \quad (4) \]
\[ f \sim \gamma g \implies f \sim_L \gamma g. \quad (5) \]

This holds because in Definition 4.5, we can either take the right sides equal, obtaining (4), or take the left sides equal, obtaining (5). Also note that (5) actually implies (4) by Proposition 7.2. We could call (4) right-algebraic and (5) left-algebraic.

Proposition 8.1. \( \gamma \)-algebraic is equivalent with (5).

Proof. That (5) follows from \( \gamma \)-algebraicity was noted above. Conversely, assume (5) and suppose that \( a \sim_\gamma a' \) and \( b \sim_\gamma b' \). We have to prove that \( ab \sim_\gamma a'b' \). Indeed we have

\[ ab \sim_\gamma a'b' \quad \text{by (4)} \]
\[ \sim_\gamma a'b' \quad \text{by (5)}. \]

On the other hand, if \( A \) is strongly \( \gamma \)-extensional, we have

\[ f \sim_R \gamma g \implies f \sim_\gamma g \]

which is the converse of (4). So we see that in a sense, the notions of algebraicity and extensionality are complementary. We now show that neither of them implies the other.

Proposition 8.2. \( \gamma \)-extensional does not imply \( \gamma \)-algebraic.

Proof. Consider Kleene's first model \( K_1 \), and let \( \gamma = \gamma_e \) be the numbering from Proposition 4.2. Every pca is trivially \( \gamma_e \)-extensional, as \( f \sim_e g \Rightarrow f \sim_e g \). However, \( K_1 \) is not \( \gamma_e \)-algebraic. Namely, (5) above does not hold: There are \( n, m \in K_1 \) such that \( n \sim_e m \), i.e. \( n \) and \( m \) are codes of the same partial computable function, but \( n \neq m \), so that \( \forall z. z n \sim_e z m \) does not hold: There is a p.c. function \( \varphi \) such that \( \varphi(n) \neq \varphi(m) \).

We can strengthen Proposition 8.2 to the following.

Theorem 8.3. Strong \( \gamma \)-extensional does not imply \( \gamma \)-algebraic.
Proof. We show that (6) does not imply (5). As a pca we take Kleene’s first model $K_1$, and we define a generalized numbering $\gamma$ on it as follows. We start with the equivalence $\sim_e$ on $K_1$, and we let $\sim_\gamma$ be the smallest extension of $\sim_e$ such that
\[ f \sim_\gamma g \iff \forall x \ f x \sim_\gamma g x. \] (7)
The equivalence relation $\sim_\gamma$ is the smallest fixed point of the monotone operator that, given an equivalence $\sim$ on $K_1$ that extends $\sim_e$, defines a new equivalence $\approx$ by
\[ f \approx g \iff \forall x \ f x \sim g x. \] (8)
The existence of $\sim_\gamma$ is then guaranteed by the Knaster-Tarski theorem on fixed points of monotone operators [13]. Note that by Remark 4.3, $K_1/\sim_e$ is not a pca, and neither are the extensions $K_1/\approx$, but this is not a problem for the construction (8), since the application $fx$ keeps taking place in the pca $K_1$. Note that by (7) we have that $K_1$ is strongly $\gamma$-extensional.

We claim that (5) fails for $\gamma$, and hence that $K_1$ is not $\gamma$-algebraic. First we observe that $\gamma$ is not trivial, i.e. does not consist of only one equivalence class. Namely, let $ax \uparrow$ for every $x$, and let $b \in K_1$ be total. Then obviously $\forall x \ ax \sim_\gamma bx$ does not hold, hence by (7) we have $a \not\sim_\gamma b$.

For the failure of (5) we further need the existence of $f \sim_\gamma g$ such that $f \neq g$. Such $f$ and $g$ exist, since they already exist for $\sim_e$, and $\sim_\gamma$ extends $\sim_e$. Now let $a \not\sim_\gamma b$ (the existence of which we noted above), and let $z$ be a code of a partial computable function such that $zf = a$ and $zg = b$. Then $zf \not\sim_\gamma zg$, hence $\forall z \ zf \sim_\gamma zg$ does not hold, and thus (5) fails.

Corollary 8.4. (5) implies (4), but not conversely.

Proof. (5) implies (4) by Proposition 7.2. In the counterexample of Theorem 8.3 the equivalence (7) holds, so both (4) and (6) hold, but (5) does not.

Proposition 8.5. $\gamma$-algebraic does not imply $\gamma$-extensional.

Proof. Let $A$ be a pca that is not extensional (such as Kleene’s $K_1$), and let $\gamma$ be the identity on $A$. Every pca is always $\gamma$-algebraic, so this provides a counterexample to the implication.}

The proof of Theorem 8.3 shows that (6) does not imply (5). Since (5) strictly implies (4) by Corollary 8.4, a stronger statement would be to show that (6) does not imply (4). We can obtain this with a
variation of the earlier proof (though this construction is less natural, and only serves a technical purpose).

**Theorem 8.6.** (6) does not imply (4).

**Proof.** As before, we take Kleene’s first model \( K_1 \), and we define a generalized numbering \( \gamma \) on it. Now we do not want the equivalence (7) to hold, so we do not start with the equivalence relation \( \sim_e \).

Let \( f, g \in K_1 \) and \( x \in \omega \) be such that \( fx \downarrow \) is total, and \( gx \uparrow \). Start with \( f \sim_\gamma g \), so that \( \gamma \) equates \( f \) and \( g \) and nothing else. Let \( \sim_\gamma \) be the smallest extension of this equivalence such that

\[
\forall x (fx \sim_\gamma gx) \implies f \sim_\gamma g.
\]

As before, the equivalence relation \( \sim_\gamma \) exists by the Knaster-Tarski theorem \([13]\). This ensures that \( \gamma \) satisfies (6).

We claim that \( \forall x fx \sim_\gamma gx \) does not hold, and hence that (4) fails. Note that we would only have \( \forall x fx \sim_\gamma gx \) if at some stage \( \forall x, y fxy \sim_\gamma gxy \) would hold. However, since \( fx \) is total, \( fxy \) is always defined, whereas \( gxy \) is never defined by choice of \( x \). Hence, by Definition 4.4, \( fxy \not\sim_\gamma gxy \), whatever \( \gamma \) may be. \( \square \)

Consider the following properties, which are the converses of the implications from Proposition 7.2:

\[
\begin{align*}
f \sim_R g & \implies f \sim_L g, \quad (9) \\
f \sim_{R_\gamma} g & \implies f \sim_{L_\gamma} g. \quad (10)
\end{align*}
\]

These properties express that whenever \( f \) and \( g \) denote the same function, they are *inseparable* in the pca (compare Barendregt \([4\text{, p48}]\)). Combining algebraicity and extensionality, we obtain the following relations.

**Proposition 8.7.**

(i) \( \gamma \)-algebraic + \( \gamma \)-extensional \( \implies \) (9).

(ii) \( \gamma \)-algebraic + strongly \( \gamma \)-extensional \( \implies \) (10).

**Proof.** We only prove (ii), since (i) is similar. By \( \gamma \)-algebraicity we have (5), hence

\[
f \sim_{R_\gamma} g \implies f \sim_\gamma g \quad \text{by strong } \gamma \text{-extensionality}
\]

\[
\implies f \sim_{L_\gamma} g \quad \text{by (5)}. \quad \square
\]

Note that the identity \( \gamma_A \) is algebraic for any pca \( A \). So any extensional \( A \) (meaning \( \gamma_A \)-extensional, which in this case coincides with strongly \( \gamma_A \)-extensional) is an example where the conditions of Proposition 8.7 hold.
9. A NOTE ON 1-1 NUMBERINGS OF UNIFORMLY C.E. CLASSES

A class of c.e. sets $\mathcal{L}$ is called \textit{uniformly c.e.} if it admits a computable numbering of its indices, i.e. if it is of the form

$$\mathcal{L} = \{W_{f(n)} : n \in \omega\}$$

with $f$ a computable function. Such a numbering $f$ is called 1-1 if the numbering does not have any repetitions, i.e. if the sets $W_{f(n)}$ are all different. A classic question is which uniformly c.e. classes admit a 1-1 numbering. For a list of references about this topic see Odifreddi \[18\] and Kummer \[14\].

Friedberg \[11\] showed that the class of all c.e. sets has a 1-1 numbering. The difficulty of course lies in the fact that equality for c.e. sets is hard to decide: The set $\{(n,m) \mid W_n = W_m\}$ is $\Pi^0_2$-complete, cf. Soare \[24\]. Here we prove two results relating the problem above to the complexity of the equality relation.

**Proposition 9.1.** If for a uniformly c.e. class $\mathcal{L}$ the equality relation is $\Pi^0_1$ then $\mathcal{L}$ has a 1-1 numbering.

**Proof.** Let $f$ be computable such that $\mathcal{L} = \{W_{f(n)} : n \in \omega\}$. The statement that the equality relation of $\mathcal{L}$ is $\Pi^0_1$ means that the set

$$U = \{(n,m) \mid W_{f(n)} \neq W_{f(m)}\}$$

is c.e. Enumerate $\mathcal{L}$ as follows. Enumerate $f(n)$ in $\mathcal{L}$ if and only if $(\forall m < n) [(n,m) \in U]$. This is clearly a computable enumeration because $U$ is c.e. Also, it is easy to see that for every set in $\mathcal{L}$, $f(n)$ is enumerated into $\mathcal{L}$ for the minimal $n$ such that $f(n)$ is a code for this set. This proves that the enumeration is a 1-1 numbering. \qed

**Proposition 9.2.** A uniformly c.e. class $\mathcal{L}$ for which the equality relation is $\Sigma^0_1$ does not necessarily have a 1-1 numbering.

**Proof.** Define a uniformly c.e. class $\mathcal{L}$ as follows. We want to ensure that $\varphi_e$ is not a 1-1 enumeration of $\mathcal{L}$. To this end, for every $e$ we put two codes $x_e$ and $y_e$ of c.e. sets into $\mathcal{L}$, with $W_{y_e} = \{2e, 2e + 1\}$. The code $x_e$ is defined by the following enumeration procedure for $W_{x_e}$. Enumerate $2e$ into $W_{x_e}$. Search for two different codes $a$, $b$ in the range of $\varphi_e$ such that $2e \in W_a$ and $2e \in W_b$. If such $a$ and $b$ are found enumerate $2e + 1$ in $W_{x_e}$. The class $\mathcal{L}$ thus defined is uniformly c.e. and has a $\Sigma^0_1$ equality relation, because to find out whether two sets in $\mathcal{L}$ are equal, enumerate them and compare the first even elements enumerated (every set in $\mathcal{L}$ contains exactly one even element). If these are equal, say they both equal $2e$, then we know that the two sets are
$W_{xe}$ and $W_{ye}$, and these are equal if and only if the $\Sigma^0_1$-event from the enumeration procedure of $W_{xe}$ occurs.

To prove that no computable function 1-1 enumerates $L$ fix $e$ such that $\varphi_e$ is total. Now if two different codes $a$ and $b$ occur in the range of $\varphi_e$ such that $2e \in W_a \cap W_b$, then $W_{xe} = W_{ye}$ by definition of $x_e$, hence $\varphi_e$ does not one-one enumerate $L$. In the case that such $a$ and $b$ do not appear in range($\varphi_e$) it holds that $W_{xe} \neq W_{ye}$, hence $\varphi_e$ does not enumerate all the elements of $L$. \qed

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