COUNTING CONJUGACY CLASSES IN THE UNIPOTENT RADICAL OF PARABOLIC SUBGROUPS OF $GL_n(q)$

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Abstract. Let $q$ be a prime power. Let $P$ be a parabolic subgroup of the general linear group $GL_n(q)$ that is the stabilizer of a flag in $F_n^q$ of length at most 5, and let $U = O_p(P)$. In this note we prove that, as a function of $q$, the number $k(U)$ of conjugacy classes of $U$ is a polynomial in $q$ with integer coefficients.

1. Introduction

Let $GL_n(q)$ be the finite general linear group defined over the field $F_q$ of $q$ elements, where $q$ is a prime power. A longstanding conjecture attributed to G. Higman (cf. [12]) asserts that the number of conjugacy classes of a Sylow $p$-subgroup of $GL_n(q)$ is given by a polynomial in $q$ with integer coefficients. This has been verified by computer calculation by A. Vera-López and J. M. Arregi for $n \leq 13$, see [17]. There has been much interest in this conjecture, for example from G. R. Robinson [15] and J. Thompson [16]. The reader is referred to [1], [6], [9], [8], [10], and [11] for recent related results.

In this note we consider the following question, which is precisely Higman’s conjecture when $P = B$ is a Borel subgroup of $GL_n(q)$.

Question 1.1. Let $P$ be a parabolic subgroup of $GL_n(q)$ and let $U = O_p(P)$. As a function of $q$, is the number $k(U)$ of conjugacy classes of $U$ a polynomial in $q$?

In this paper, we give an affirmative answer to Question 1.1 in the following cases.

Theorem 1.2. Let $P$ be a parabolic subgroup of $GL_n(q)$ that is the stabilizer of a flag in $F_n^q$ of length at most 5, and let $U = O_p(P)$. Then, as a function of $q$, the number $k(U)$ of conjugacy classes of $U$ is a polynomial in $q$ with integer coefficients.

We now explain the significance of the hypothesis imposed in Theorem 1.2. Let $P$ be a parabolic subgroup of $GL_n(F_q)$ and let $U$ be the unipotent radical of $P$; where $F_q$ denotes the algebraic closure of $F_q$. All instances when $P$ acts on $U$ with a finite number of orbits were determined in [13]; this is precisely the case when $P$ is the stabilizer of a flag in $F_n^q$ of length at most 5. So Theorem 1.2 deals with parabolic subgroups $P$ of $GL_n(q)$ that correspond to parabolic subgroups $P$ of $GL_n(F_q)$ with a finite number of conjugacy classes in $U$. In such cases, it is observed in [13, §4, Rem. 4.13] that the parameterization of the $P$-conjugacy classes in $U$ is independent of $q$: this is the crucial point that we require for our proof of Theorem 1.2.

The proof of Theorem 1.2 involves a translation of the problem to a representation theoretic setting. More precisely, we recall from [13, §4] that the $P$-conjugacy classes in $U$ correspond bijectively to the so called $\Delta$-filtered modules of a certain quasi-hereditary algebra $A_q$. This

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allows us to see that the parameterization of the $P$-orbits in $U$ is independent of $q$ and that we can choose a set $\mathcal{R}$ of representatives that are matrices with entries equal to 0 or 1. The other key point is that the structures of the centralizers $C_P(x)$ and $C_U(x)$ for $x \in \mathcal{R}$ do not depend on $q$; this is covered in Propositions 2.5 and 2.7.

We now discuss some natural generalizations of Theorem 1.2. First consider the case of a normal subgroup $N$ of $P$ with $N \subseteq U$. Still assuming that there is only a finite number of $P$-orbits in $U$, we readily derive from the proof of Theorem 1.2 that $k(U, N)$, the number of $U$-conjugacy classes in $N = N \cap U$, is given by a polynomial in $q$ with integer coefficients. It should also be possible to prove that the number $k(U, N)$ is a polynomial in $q$ with just the assumption that there are finitely many $P$-orbits in $N$. For example, for $N = U^{(l)}$ the $l$th member of the descending central series of $U$, there is a classification of all instances when $P$ acts on $U^{(l)}$ with a finite number of orbits, see [2]. In such situations a generalization of the proof of Theorem 1.2 would require detailed knowledge of the $P$-conjugacy classes in $N$.

Further, it is natural to consider the generalization of Question 1.1, where $GL_n(q)$ is replaced by any finite reductive group $G$; and also to consider the number $k(P, U)$ of $P$-conjugacy classes in $U$ rather than $k(U)$. (In order to avoid degeneracies in the Chevalley commutator relations, it is sensible to only consider these generalizations when $q$ is a power of a good prime for $G$.)

At present there are no known examples, where $k(U)$ is not given by a polynomial in $q$, and there are many cases not covered by Theorem 1.2, where $k(U)$ is given by a polynomial in $q$, see for example [10] and [17]. However, it is not necessarily the case that $k(P, U)$ is a polynomial in $q$: indeed in [7, Exmp. 4.6], it shown that in case $G$ is of type $G_2$, and $P = B$ is a Borel subgroup of $G$, the number $k(B, U)$ is given by two different polynomials depending on the residue of $q$ modulo 3.

Let $P$ be a parabolic subgroup of a reductive algebraic group $G$ defined over $\mathbb{F}_q$, and suppose that $P$ has finitely many conjugacy classes in $U$; let $P$ and $U$ be the groups of $\mathbb{F}_q$-rational points of $P$ and $U$ respectively. Given the discussion after Theorem 1.2 a natural generalization to consider is whether the number $k(U)$ of conjugacy classes of $U$ is a polynomial in $q$. Our proof of Theorem 1.2 is dependent on the detailed information about the $P$-conjugacy classes in $U$. For this reason the argument does not adapt to the case, where $G$ is any finite reductive group. The main difficulty is that is it is not clear whether the parameterization of $P$-orbits in $U$ and the structure of centralizers depends on the characteristic of the underlying ground field. Another problem is that centralizers $C_P(u)$ for $u \in U$ need not be connected, so determining the $P$-classes in $U$ from the $P$-classes in $U$ may be non-trivial.

2. Translation to representation theory

In this section we recall the relationship between adjoint orbits of parabolic subgroups and modules for a certain quasi-hereditary algebra that was established in [13, §4]. This relationship is central to our proof of Theorem 1.2. In particular, it is crucial for Propositions 2.5 and 2.7, which describe the structure of certain centralizers. Throughout this section we work in generality over any field, before specializing to finite fields for the proof of Theorem 1.2 in Section 3.

Let $K$ be any field, and let $n, t \in \mathbb{Z}_{\geq 1}$. Let $d = (d_1, \ldots, d_t) \in \mathbb{Z}_{\geq 0}^t$ with $d_i \leq d_{i+1}$ and $d_t = n$. We define the parabolic subgroup $P(d) = P_K(d)$ of $GL_n(K)$ to be the stabilizer of
the flag $0 \subseteq K^{d_1} \subseteq K^{d_2} \subseteq \ldots \subseteq K^{d_t}$ in $K^n$; any parabolic subgroup of $GL_n(K)$ is conjugate to $P(d)$ for some $d$. We write $U(d) = U_K(d) = \{ u \in GL_n(K) \mid (u - 1)V_i \subseteq V_{i-1} \text{ for each } i \}$ for the unipotent radical of $P(d)$, and $u(d) = u_K(d) = \{ x \in M_n(K) \mid xV_i \subseteq V_{i-1} \text{ for each } i \}$ for the Lie algebra of $U(d)$. Then $P(d)$ acts on $u(d)$ via the adjoint action: $g \cdot x = g x g^{-1}$ for $g \in P(d)$ and $x \in u(d)$. For $x \in u(d)$, we write $P \cdot x$ for the adjoint $P$-orbit of $x$ and $C_P(x)$ for the centralizer of $x$ in $P$; we define $U \cdot x$ and $C_U(x)$ analogously.

Though we are primarily interested in the conjugacy classes of $U(d)$ and the $P(d)$-conjugacy classes in $U(d)$, it is more convenient to consider the adjoint $P(d)$-orbits in $u(d)$. The map $x \mapsto 1 + x$ is a $P(d)$-equivariant isomorphism between $u(d)$ and $U(d)$, which means that the adjoint $P(d)$-orbits in $u(d)$ are in bijective correspondence with the $P(d)$-conjugacy classes in $U(d)$; this allows us to work with the adjoint orbits.

The quiver $Q_t$ is defined to have vertex set $\{1, \ldots, t\}$; there are arrows $\alpha_i : i \to i + 1$ and $\beta_i : i + 1 \to i$ for $i = 1, \ldots, t - 1$. Below, in Figure 1, we give an example of a quiver $Q_5$.

![Figure 1. The quiver $Q_5$.](image)

Let $I_t = I_{t,K}$ be the ideal of the path algebra $KQ_t$ of $Q_t$ generated by the relations:

\[ \beta_i \alpha_i = 0 \text{ and } \alpha_i \beta_i = \beta_{i+1} \alpha_{i+1} \text{ for } i = 1, \ldots, t - 2. \]

The algebra $\mathcal{A}_t = \mathcal{A}_{t,K}$ is defined to be the quotient $KQ_t/I_t$.

Recall that an $\mathcal{A}_t$-module $M$ is determined by a family of vector spaces $M(i)$ over $K$ for $i = 1, \ldots, t$ such that $M = \bigoplus_{i=1}^t M(i)$, and linear maps $M(\alpha_i) : M(i) \to M(i + 1)$ and $M(\beta_i) : M(i + 1) \to M(i)$ for $i = 1, \ldots, t - 1$ that satisfy the relations (2.1). The dimension vector $\text{dim } M \in \mathbb{Z}_{\geq 0}^t$ of an $\mathcal{A}_t$-module is defined by $\text{dim } M = (\text{dim } M(1), \ldots, \text{dim } M(t))$.

Let $\mathcal{M}_t = \mathcal{M}_{t,K}$ be the category of $\mathcal{A}_t$-modules $M$ such that $M(\alpha_i)$ is injective for all $i$. Write $\mathcal{M}_t(d) = \mathcal{M}_{t,K}(d)$ for the class of modules in $\mathcal{M}_t$ with dimension vector $d$. It is shown in [13, §4] that the orbits of $P(d)$ in $u(d)$ are in bijective correspondence with the isoclasses in $\mathcal{M}_t(d)$. Moreover, there is a unique structure of a quasi-hereditary algebra on $\mathcal{A}_t$ such that $\mathcal{M}_t$ is the category of $\Delta$-filtered modules, see [13, §4] and [5, §§6–7].

Suppose for this paragraph that $K$ is infinite. Using the above correspondence from [13, §4] and the results from [5], it was proved in [13, Thm. 4.1] that there is a finite number of $P(d)$-orbits in $u(d)$ if and only if $t \leq 5$. More precisely, this is deduced from the fact that $\mathcal{A}_t$ has finite $\Delta$-representation type if and only if $t \leq 5$, see [5, Prop. 7.2].

Let $t \leq 5$. Because the results in [13, §4] are proved for an arbitrary field, see [13, Rem. 4.13], the parametrization of indecomposable $\Delta$-filtered $\mathcal{A}_t$-modules does not depend on the field $K$; we explain this more explicitly below. Let $\{I_1, \ldots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable $\Delta$-filtered $\mathcal{A}_t$-modules, and write $d_i$ for the dimension vector of $I_i$. Let $x_i \in u(d_i)$ be such that the $P(d_i)$-orbit of $x_i$ corresponds to the isoclass of $I_i$. As discussed in [13, §7], see also [3, Fig. 10], one can choose $x_i$ to be a matrix with entries 0 and 1; and these matrices do not depend on $K$. In particular, this implies that the modules $I_i$ are absolutely indecomposable.

Another important consequence for us is the following lemma.
Lemma 2.2. Assume $t \leq 5$. We may choose a set $\mathcal{R}$ of representatives of the adjoint $P(d)$-orbits in $u(d)$ such that each element of $\mathcal{R}$ is a matrix with all entries equal to 0 or 1. Moreover, the elements of $\mathcal{R}$ do not depend on the field $K$, i.e. the positions of entries equal to 1 do not depend on $K$.

We continue to assume that $t \leq 5$, and let $d \in \mathbb{Z}_{\geq 0}$. Let $P = P(d)$, $U = U(d)$ and $x \in u = u(d)$. For the proof of Theorem 1.2 we require information about the structure of the centralizers $C_P(x)$ and $C_U(x)$, which is given by Propositions 2.5 and 2.7.

Let $M$ be a $\Delta$-filtered $A_t$-module (with dimension vector $d$) whose isoclass corresponds to the $P$-orbit of $x$. Extending the arguments of [13, §4], one can show that the automorphism group $\text{Aut}_{A_t}(M)$ of $M$ is isomorphic to $C_F(x)$. Below we explain the structure of $\text{End}_{A_t}(M)$ and $\text{Aut}_{A_t}(M)$, this uses standard arguments that we outline here for convenience. We proceed to explain how $C_F(x)$ is related to $\text{End}_{A_t}(M)$.

As above, let $\{I_1, \ldots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable $\Delta$-filtered $A_t$-modules. We may decompose $M$ as a direct sum of indecomposable modules

\begin{equation}
M \cong \bigoplus_{i=1}^m n_i I_i,
\end{equation}

where $n_i \in \mathbb{Z}_{\geq 0}$. Then

\[ \text{End}_{A_t}(M) \cong \bigoplus_{i,j=1}^m n_i n_j \text{Hom}_{A_t}(I_i, I_j) \]

as a vector space and composition is defined in the obvious way.

We observed above that $I_i$ is absolutely indecomposable, which means that $\text{End}_{A_t}(I_i)$ is a local ring, and that we have the decomposition $\text{End}_{A_t}(I_i) = K \oplus m_i$, where $K$ is acting by scalars and $m_i$ is the maximal ideal. Therefore,

\[ n_i^2 \text{End}_{A_t}(I_i) \cong M_{n_i}(K) \oplus M_{n_i}(m_i), \]

where $M_{n_i}(K)$ is a subalgebra and $M_{n_i}(m_i)$ is an ideal. In fact, we have that $M_{n_i}(m)$ is the Jacobson radical of $n_i^2 \text{End}_{A_t}(I_i)$.

Now one can see that the Jacobson radical of $\text{End}_{A_t}(M)$ is

\[ J(\text{End}_{A_t}(M)) \cong \bigoplus_{i=1}^m M_{n_i}(m_i) \oplus \bigoplus_{i \neq j} n_i n_j \text{Hom}_{A_t}(I_i, I_j). \]

Further, there is a complement to $J(\text{End}_{A_t}(M))$ in $\text{End}_{A_t}(M)$ denoted by $C(\text{End}_{A_t}(M))$ with

\[ C(\text{End}_{A_t}(M)) \cong \bigoplus_{i=1}^m M_{n_i}(K). \]

We are now in a position to describe the automorphism group $\text{Aut}_{A_t}(M)$. We have

\[ \text{Aut}_{A_t}(M) \cong U(C(\text{End}_{A_t}(M))) \rtimes (1_M + J(\text{End}_{A_t}(M))). \]

where $U(C(\text{End}_{A_t}(M)))$ denotes the group of units of $C(\text{End}_{A_t}(M))$ and $1_M + J(\text{End}_{A_t}(M))$ is the unipotent group $\{1_M + \phi \mid \phi \in J(\text{End}_{A_t}(M))\}$. We have $U(C(\text{End}_{A_t}(M))) \cong \ldots \ldots$
\[ \prod_{i=1}^{m} \text{GL}_{n_i}(K), \] and therefore
\[ \text{Aut}_{A_i}(M) \cong \prod_{i=1}^{m} \text{GL}_{n_i}(K) \rtimes N, \]
where \( N \) is a split unipotent group over \( K \). By saying \( N \) is a split unipotent group we mean that \( N \) has a normal series with all quotients isomorphic to the additive group \( K \). The dimension of \( N \) is
\[ (2.4) \quad \delta := \sum_{i=1}^{m} n^2_i (\text{dim } \text{End}_{A_i}(I_i) - 1) + \sum_{i \neq j} n_i n_j \text{dim } \text{Hom}_{A_i}(I_i, I_j). \]
One can compute all \( \text{Hom} \)-groups \( \text{Hom}_{A_i}(I_i, I_j) \) from the underlying Auslander-Reiten quivers of \( A_i \) in [5, p221–222], see also [3, App. A]; the dimensions \( \text{dim } \text{Hom}_{A_i}(I_i, I_j) \) are independent of \( K \). Therefore, the positive integer \( \delta \) is also independent of \( K \).

We said above that \( \text{Aut}_{A_i}(M) \) of \( M \) is isomorphic to \( C_p(x) \), so we have the following proposition.

**Proposition 2.5.** The Levi decomposition of \( C_p(x) \) is given by
\[ C_p(x) \cong \prod_{i=1}^{m} \text{GL}_{n_i}(K) \rtimes N, \]
where \( N \), the unipotent radical of \( C_p(x) \), is a split unipotent group over \( K \) of dimension \( \delta \).

We now wish to give the structure of the centralizer \( C_U(x) \). By a further extension of the arguments in [13, §4], one sees that there is an isomorphism
\[ C_U(x) \cong 1_M + \text{End}'_{A_i}(M), \]
where
\[ \text{End}'_{A_i}(M) := \{ \phi \in \text{End}_{A_i}(M) \mid \phi M(i) \subseteq M(i-1) \text{ for all } i \}, \]
here we are identifying \( M(i-1) \) with its image in \( M(i) \) under \( M(\alpha_i-1) \). We have that \( \text{End}'_{A_i}(M) \) is a nilpotent ideal of \( \text{End}_{A_i}(M) \). We define
\[ \text{Hom}'_{A_i}(I_i, I_j) := \{ \phi \in \text{Hom}_{A_i}(I_i, I_j) \mid \phi I_i(l) \subseteq I_j(l-1) \text{ for all } l \}. \]
Then we have the isomorphism
\[ \text{End}'_{A_i}(M) \cong \bigoplus_{i,j=1}^{m} n_i n_j \text{Hom}'_{A_i}(I_i, I_j). \]
We write
\[ (2.6) \quad \delta' := \text{dim } \text{End}'_{A_i}(M) = \sum_{i,j=1}^{m} n_i n_j \text{dim } \text{Hom}'_{A_i}(I_i, I_j). \]
From the Auslander-Reiten quivers of \( A_i \) exhibited in [5, p221–222], one can compute the dimensions \( \text{dim } \text{Hom}'_{A_i}(I_i, I_j) \). These integers are independent of \( K \), so that \( \delta' \) is also independent of \( K \). The above discussion proves the following proposition, which concludes this section.

**Proposition 2.7.** The centralizer \( C_U(x) \) is a split unipotent group over \( K \) of dimension \( \delta' \).
3. Proof of Theorem 1.2

Let \( q \) be a prime power and let \( K = \mathbb{F}_q \) be the field of \( q \) elements. Let \( t \leq 5 \) and let \( d \in \mathbb{Z}_{\geq 0}^t \). Let \( P = P(d), U = U(d) \) and \( u = u(d) \) be as in the previous section, so \( P \) is a parabolic subgroup of \( \text{GL}_n(q) \).

The following corollary of Propositions 2.5 and 2.7 is a key step in our proof of Theorem 1.2. It follows immediately from these Propositions 2.5 and 2.7 along with the elementary fact that the order of a general linear group over \( \mathbb{F}_q \) is given by a polynomial in \( q \). The positive integers in the statement are determined in (2.3), (2.4) and (2.6).

**Corollary 3.1.** Let \( x \in u \). Then there are positive integers \( n_1, \ldots, n_m, \delta \) and \( \delta' \) independent of \( q \) such that

\[
|C_P(x)| = \prod_{i=1}^{m} |\text{GL}_{n_i}(q)| \cdot q^\delta,
\]

and

\[
|C_U(x)| = q^{\delta'}.
\]

In particular, both \( |C_P(x)| \) and \( |C_U(x)| \) are polynomials in \( q \) with integer coefficients.

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** We have to prove that \( k(U) \) is given by a polynomial in \( q \). As discussed in the previous section \( k(U) = k(U,u) \), the number of adjoint \( U \)-orbits in \( u \). We will prove that \( k(U,u) \) is a polynomial in \( q \) with integer coefficients.

We may choose a set of representatives \( \mathcal{R} \) of the adjoint \( P \)-orbits in \( u \), as in Lemma 2.2 and consider \( \mathcal{R} \) to be independent of \( q \). We have

\[
k(U,u) = \sum_{x \in \mathcal{R}} k(U, P \cdot x),
\]

where \( k(U, P \cdot x) \) is the number of \( U \)-orbits contained in \( P \cdot x \). For \( x \in u \) and \( g \in P \), we have \( C_U(g \cdot x) = gC_U(x)g^{-1} \). Therefore, we get \( |U \cdot x| = |U \cdot (g \cdot x)| \) and \( k(U, P \cdot x) = |P \cdot x|/|U \cdot x| \).

It follows that

\[
k(U,u) = \sum_{x \in \mathcal{R}} k(U, P \cdot x) = \sum_{x \in \mathcal{R}} \frac{|P \cdot x|}{|U \cdot x|} = \frac{|P|}{|U|} \sum_{x \in \mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|} = |L| \sum_{x \in \mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|},
\]

where \( L \) is a Levi subgroup of \( P \). Since \( |L| \) is a polynomial in \( q \), it follows from Corollary 3.1 and the fact that \( \mathcal{R} \) is independent of \( q \) that \( k(U,u) = k(U) \) is a rational function in \( q \).

Since \( k(U) \) takes integer values for all prime powers, standard arguments show that \( k(U) \) is in fact a polynomial in \( q \) with rational coefficients, see for example [9, Lem. 2.11].

Let \( P \) be the subgroup of \( \text{GL}_n(\mathbb{F}_q) \) corresponding to \( P \) and let \( U \) be the unipotent radical of \( P \). The **commuting variety of** \( U \) is the closed subvariety of \( U \times U \) defined by

\[
\mathcal{C}(U) = \{(u, u') \in U \times U \mid uu' = u'u\}.
\]

Setting \( \mathcal{C}(U) = \mathcal{C}(U) \cap (U \times U) \) and using the Burnside counting formula we get

\[
|\mathcal{C}(U)| = \sum_{x \in U} |C_U(x)| = |U| \cdot k(U).
\]

Since \( |U| = q^{\dim U} \) and \( k(U) \) is a polynomial in \( q \) with rational coefficients, so is \( |\mathcal{C}(U)| \).

Now using the Grothendieck trace formula applied to \( \mathcal{C}(U) \) (see [4, Thm. 10.4]), standard
arguments prove that the coefficients of this polynomial are integers, see for example [14, Prop. 6.1]. Thus, it follows that $k(U)$ is a polynomial function in $q$ with integer coefficients, as claimed.

**Remark 3.2.** Let $t \leq 5$ and $d, d' \in \mathbb{Z}_{>0}^t$ with $d_i = d'_i = n$. Suppose that $P = P(d)$ and $Q = P(d')$ are associated parabolic subgroups of $GL_n(\mathbb{F}_q)$, i.e. $P$ and $Q$ have Levi subgroups that are conjugate in $GL_n(q)$. This means that there is a $\sigma \in S_n$ such that $d_i - d_{i-1} = d_{\sigma(i)} - d_{\sigma(i-1)}$ for all $i = 1, \ldots, t$, with the convention that $d_0 = d'_0 = 0$. Let $U = U(d)$ and $V = U(d')$. A consequence of [13, Cor. 4.7] is that the number $k(P(U))$ of $P$-conjugacy classes in $U$ is the same as $k(Q,V)$; the reader is referred to [9, Cor. 4.8]. However, it is not always the case that the number of conjugacy classes of $U$ is the same as the number of conjugacy classes of $V$. For example, take $t = 3$ and consider the dimension vectors $d = (2,3,4)$ and $d' = (1,3,4)$. Then $P(d)$ and $P(d')$ are associated parabolic subgroups of $GL_4(q)$. Let $U = U(d)$ and $V = U(d')$. Then by direct calculation one can check that

$$k(U) = (q - 1)^3 + 6(q - 1)^2 + 5(q - 1) + 1$$

$$\neq (q - 1)^4 + 4(q - 1)^3 + 6(q - 1)^2 + 5(q - 1) + 1 = k(V).$$

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