Deriving exact prepotential for $N = 2$ supersymmetric Yang-Mills theories from superconformal anomaly with rank two gauge groups

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September 18, 2018

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Abstract

The exact prepotential for $N = 2$ supersymmetric Yang-Mills theory is derived from the superconformal anomalous Ward identity for the gauge group $SU(2)$ and $SU(3)$ which can be generalized to any other rank two gauge group.
1. Introduction

There is a spectacular progress in understanding the non-perturbative behaviour of supersymmetric Yang-Mills theories in recent past \[1\]. In their pioneering work Seiberg and Witten obtained an exact low energy Wilsonian effective action for \( N = 2 \) supersymmetric Yang-Mills theory for the \( SU(2) \) gauge group. Due to \( N = 2 \) SUSY the effective action has to be holomorphic and its perturbative one loop \( \beta \) function is exact. The quantum moduli space of this theory is described as a one parameter family of elliptic curves. The period of this curve fixes the metric on the quantum moduli space as well as the spectrum of BPS states. Then making use of the electric-magnetic duality i.e. strong coupling behaviour is systematically related to the weak coupling behaviour of the dual fields, the exact prepotential is obtained. This was immediately generalized to theory with arbitrary \( SU(N) \) gauge group \[2, 3\] and with matter multiplet \[4\].

Later on Boneli et. al. and Matone \[5\] have shown that the moduli space is equivalent to genus zero Riemann surface and \( u = < tr\phi^2 > \) (where \( \phi \) is the scalar component of the \( N = 2 \) SUSY) is the uniformizing coordinate of the moduli space. In this way without using the arguments of duality, they obtain the effective potential which is same as that of Seiberg and Witten. Recently in two consecutive works Magro et al.\[6\] and Flume et al. \[7\] have obtained the same result without making use of duality. Their main argument is that the moduli space is the space of all inequivalent couplings \( \tau \), which is the multiple covering of the fundamental domain of the \( \tau \) plane. Then using the fact that the BPS mass spectrum is finite and the superconformal anomaly is a \( U(1) \) section, they uniquely fixed the form of the effective potential. So far their procedure is not generalized to higher rank groups.

For \( N = 2 \) SUSY Yang-Mills theory with gauge group \( G \) spontaneously broken to \( U(1)^r \), where \( r \) is the rank of the group, the super conformal Ward identity implies that \[8\],

\[
\sum_{i=1}^{N_c-1} a_i \frac{\partial F}{\partial a_i} - 2F = 8 \pi i \beta \ u, \tag{1}
\]

where \( F \) is the effective action and \( a_i = < \phi_i > \), \( \phi \) is the scalar component of the \( N = 2 \) superfield and \( \beta \) is the one loop beta function given as \( \beta = \frac{2N_c - N_f}{16\pi^2} \). Starting from the Seiberg - Witten solution which are described by a family of hyper elliptic curves, and then using Whitham dynamics this relation is also obtained by Eguchi
For the time being let us pretend to forget about the rich structure of the SUSY theories due to duality unearthed by Seiberg and Witten. We only know from the conventional quantum field theory the super conformal Ward identity for the $N = 2$ SUSY Yang Mills theory in its coulomb phase and only its weak coupling perturbative behaviour. The main objective of our work is to derive the exact form of the prepotential starting from this anomaly equation. We are illustrating it for the $SU(2)$ gauge group in section 3, and for $SU(3)$ in section 4. First of all we derive a set of second order partial differential equation in terms of the gauge invariant variables starting from the Ward identity. The $Z_N$ symmetry, remnant of the $U(1)_R$ symmetry fixes the form of the differential equation. The weak coupling result which correspond to the asymptotic limit is used as the boundary condition which completely fixes the form of the differential equation with some unknown coefficients. Then demanding the differential equation to be regularly singular all the coefficients are fixed.

2. Basic formalism

We will consider here the $N = 2$ supersymmetric $SU(N_c)$ gauge theories. For the sake of simplicity we neglect here flavours. The field content of this theory consists of gauge fields $A_\mu$, two Weyl fermions $\lambda$ and $\psi$ and a complex scalar field $\phi$ all in their adjoint representation of the gauge group $SU(N_c)$. The theory has $N_c - 1$ complex dimensional moduli space of vacua which are parametrized by the gauge invariant order parameters

$$u_k = \frac{1}{k} Tr \langle \phi^k \rangle, \quad k = 2, ..., N_c,$$

(2)

where $\phi$ is the scalar component of the $N = 2$ chiral super multiplet. The moduli space of vacua corresponds to the flatness condition $[\phi, \phi^\dagger] = 0$. We can always rotate $\phi$ in to the caratan subalgebra $\phi = \sum_{k=1}^{N_c} a_k H_k$ with $H_k = E_{k,k} - E_{k+1,k+1}$, and $(E_{k,l})_{ij} = \delta_{ik}\delta_{jl}$. At a generic point the vacuum expectation value of $\phi$ breaks the $SU(N_c)$ gauge symmetry to $U(1)^{N_c-1}$. The low energy $N = 2$ effective Lagrangian is written in terms of two chiral multiplets $(A_i, W_i)$ as

$$\mathcal{L}_{eff} = Im \frac{1}{4\pi} \left[ \int d^4\theta \partial_i \mathcal{F}(A) \bar{A}^i + \frac{1}{2} \int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j \right],$$

(3)

where $\mathcal{F}(A)$ is the holomorphic prepotential and $\partial_i = \frac{\partial}{\partial A_i}$. Classically
\[ F_{cl}(A) = \tau \sum_{i=1}^{N_c} A_i - \frac{1}{N_c} \sum_{j=1}^{N_c} A_j \]  

(4)

where \( \tau = N_c \left\{ \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \right\} \). The one loop perturbative part of the prepotential is given by

\[ F_1 = i \frac{2N_c - N_f}{8\pi} \sum_{i<j} (A_i - A_j)^2 \log \frac{(A_i - A_j)^2}{\Lambda^2} \]  

(5)

Due to \( N = 2 \) supersymmetry there are no further perturbative correction, however there are non-perturbative instanton correction to \( F \).

3. \( SU(2) \) gauge group

For the sake of clarity of our method we are illustrating the case for \( SU(2) \) here. However the differential equation approach to this problem is presented by Bilal [12] and also pursued by many others. For \( SU(2) \) gauge group the anomalous Ward identity implies

\[ a \frac{\partial F}{\partial a} - 2F = 2 \frac{i u}{\pi} \]  

(6)

We define \( \frac{\partial F}{\partial a_i} = b_i \) which will be interpreted as the dual of \( a_i \). Taking further derivatives in \( u \) we get

\[ \frac{a''}{a} = \frac{b''}{b} = -V(u) \]  

(7)

which implies

\[ a'' + V(u)a = 0 \]

\[ b'' + V(u)b = 0. \]  

(8)

Let \( \eta_1 \) and \( \eta_2 \) be two linearly independent solutions of the differential equations. Let \( Z = \frac{\eta_1}{\eta_2} \) be the quotient of these two solutions. Very easily it can be shown that

\[ V(u) = \frac{1}{2} \{ \frac{3}{2} \left( \frac{Z''}{Z'} \right)^2 - \frac{Z'''}{Z'} \} \]

\[ = \frac{1}{2} \{ Z, u \} \]  

(9)

Where \( \{ Z, u \} \) is the Schwartzian. Let \( V(u) \) be analytic in the neighbourhood of \( u = u_0 \) except at \( u_0 \) itself, then if \( Z \) is continued analytically around a closed curve
$C$ encircling $u_0$ but no other singularity, we get

$$
\begin{align*}
\eta'_1 &= \alpha \eta_1 + \beta \eta_2 \\
\eta'_2 &= \gamma \eta_1 + \delta \eta_2 \\
Z' &= \frac{\alpha Z + \beta}{\gamma Z + \delta}.
\end{align*}
\tag{10}
$$

This implies that $Z$ transforms linearly under this monodromy. There is a theorem which states that if $u(Z)$ is an automorphic function, then in order that the inverse function $u(Z)$ be single valued in the neighbourhood of $u_0$, the transformation be an elliptic transformation of angle $\frac{2\pi}{p}$ where $p$ is an integer. Without going into details we refer to the reader to Ref. [9, 10], which elaborates how to get $V(u)$.

$$
V(u) = \frac{1}{4} \sum_{i=1}^{n} \left[ \frac{1 - \alpha_i^2}{(u - u_i)^2} + \frac{2\beta_i}{u - u_i} + \gamma(u) \right]
\tag{11}
$$

where $\alpha_i \pi$ is the internal angle of the $Z$ polygon at the corner homologous with $u_i$. If $\infty$ is also a singular value of $u$ which has an angular point of polygon as its homologue with angle $\kappa \pi$, then

$$
\sum_i \beta_i = 0, \quad \sum_i u_i \beta_i = -\frac{1}{2} \sum_i (1 - \alpha_i^2) + \frac{1}{2} (1 - \kappa^2).
\tag{12}
$$

The classical action possesses $U(1)_R$ symmetry. However the anomaly and the instantons break this symmetry to $Z_8$ discrete symmetry. Under this $\phi \to e^{i\frac{2\pi}{n}\phi},$ so for odd $n$, $\phi^2 \to -\phi^2$. The non-vanishing vacuum expectation value $u = \langle tr\phi^2 \rangle$ breaks this symmetry further to $Z_4$. This also implies that in the moduli space $u \to -u$ is a symmetry. So besides $\infty$ we have even number of singularities. It is convenient to fix $u_i = u_{i+1}, \alpha_i = \alpha_{i+1}$ and $\beta_i = -\beta_{i+1}$. Thus $V(u)$ has this specific form

$$
V(u) = \frac{1}{2} \sum_{i=1}^{n} \left[ \frac{(1 - \alpha_i^2)(u^2 + u_i^2)}{(u^2 - u_i^2)} + \frac{2u_i \beta_i}{(u^2 - u_i^2)} \right].
\tag{13}
$$

Around $u \to \infty$

$$
V(u) = \frac{1}{2u^2} \sum_{i=1}^{n} [(1 - \alpha_i^2) + 2u_i \beta_i]
\tag{14}
$$

which is $\frac{(1-\kappa^2)}{4u^2}$ c.f. eq.(12 ). This has the solution $\eta = c \sqrt[4]{1 \pm \kappa}$. However for $u \to \infty$ theory is asymptotically free and the perturbative result is exact which gives $\eta_1 = c \sqrt{u}$ and $\eta_2 = \sqrt{u} \log u$. This fixes $\kappa$ to be 0.
To get the solution around an arbitrary singular point $u_i$ we make a transformation

$$ a = (u - u_i)^{\frac{1}{2}}(1 - \alpha_i) f(u). $$

This gives rise to

$$ f'' + \left(\frac{1 - \alpha_i}{u - u_i}\right)f' + \frac{1}{2} \frac{\beta_i}{(u - u_i)} f = 0. \quad (15) $$

To solve the above equation as a power series we take

$$ f = (u - u_i) s F(u - u_i), $$

where $F(u - u_i)$ is the power series and the indicial equation is

$$ s(s - 1) + (1 - \alpha_i) s = 0 \quad (16) $$

which gives $s = \alpha_i$. For $\beta_i$ non-zero one of the consistent solution will be for $\alpha \geq 1$. Since maximum value of $\alpha_i$ is one so we set $\alpha_i$ to be one. So the two solutions are $f_1 = (u - u_i)$ and $f_2 = \frac{\beta_i}{\alpha_i}(u - u_i) \log(u - u_i)$. Here $f_1$ and $f_2$ can correspond either to $a$ or $b$ or a linear combination of them. We take here a most general form of the solution like $f_1 = pb + qa$ and $f_2 = rb + sa$ such that $ps - rq = 1$. They correspond to $(p, q)$ and $(r, s)$ dyons becoming massless at these singularities. The monodromy around $u_i$ is given by $(u - u_i) \to (u - u_i)e^{2\pi i}$. Under this

$$ pb' + qa' = pb + qa $$

$$ rb' + sa' = rb + sa - 2(pb + qa) \quad (17) $$

This gives the monodromy matrix

$$ M(p, q) = \begin{pmatrix} 1 + 2pq & 2q^2 \\ -2p^2 & 1 - 2pq \end{pmatrix}. \quad (18) $$

If $u_i$ is a singularity also $-u_i$ is a singularity. The monodromy around $\infty$ has to be the composition of all the monodromies in the $u$ plane.

$$ M_\infty = M_{-n} \cdots M_{-1} M_1 \cdots M_n \quad (19) $$

If we chose $M_1$ to be monodromy matrix of a $(1, 0)$ dyon which is a monopole and $M_{-1}$ to be due to the $(1, 1)$ dyon we can very easily see that

$$ M_\infty = M_{-1} M_1 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \quad (20) $$

This gives

$$ M_\infty = M_{-n} \cdots M_{-2} M_\infty M_2 \cdots M_n. \quad (21) $$

In order to satisfy equation (21), $(M_2 \cdots M_n)^2 = I = (M_{-n} \cdots M_{-2})^2$ has to be satisfied. Here $M_2, \cdots, M_n$ are all $SL(2, z)$ group elements. Then $G_1 = M_2 \cdots M_n$
is also an $SL(2,z)$ element. If $G_1^2 = I$ then either $G_1 = I$ or $G_1 = -I$. Similarly also $G_{-1} = M_{-n} \cdots M_{-2}$ is either $I$ or $-I$. So the consistency condition requires that $G_1 = G_{-1} = I$ or $-I$. Only contractible loop has monodromy which is $I$. This shows that there is no other singularity besides two strong coupling singularities due to (1,0) and (1,1) dyon and $\infty$. However from this we are not able to algebraically prove that these three singularities are unique. Since we are assuming from the beginning that first two singularities are due to (1,0) and (1,1) dyon. If we don’t assume these singularities from the beginning then we cannot show from the first principle that these three are the only singularities. There can be some arbitrary finite number of monodromy matrices whose product will be $M_{\infty}$. Only consistency requirement does not give the uniqueness of the three singularities. Thus one needs here more input to show the uniqueness. This is argued by Magro et al. [6] in a different way taking the anomaly to be the $U(1)$ section and the finiteness of the dyon masses.

4. **SU(3) case**

For the $SU(3)$ gauge group the anomalous Ward identity gives

$$\sum_i \left( \frac{1}{2} a_i b_i - F \right) = \frac{3}{2} i \pi u.$$  \hspace{1cm} (22)

As mentioned earlier the Cartan subalgebra variables are not gauge invariant and we use $u_k(a_k)$ as our working variables. We define $u = u_2$ and $v = u_3$. For $SU(3)$ we have

$$\phi = a_1 H_1 + a_2 H_2$$  \hspace{1cm} (23)

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$  \hspace{1cm} (24)

This implies classically $u = a_1^2 + a_2^2 - a_1 a_2$ and $v = a_1 a_2 (a_1 - a_2)$.

We take derivatives of eq.(22) with respect to $u$ and $v$ up to second order and get a set of second order differential equations as

$$\sum_i b_i \partial_{uu}^2 a_i - a_i \partial_{uu}^2 b_i = 0$$  \hspace{1cm} (25)

$$\sum_i b_i \partial_{uv}^2 a_i - a_i \partial_{uv}^2 b_i = 0$$  \hspace{1cm} (26)
\[
\sum_i b_i \partial_{uv}^2 a_i = 0 \quad (27)
\]
\[
\sum_i b_i \partial_{uv} a_i = 0 \quad (28)
\]
\[
\sum_i \partial_v b_i \partial_u a_i = 0 \quad (29)
\]

It is possible to derive differential equations for each \(a_i\) and \(b_i\) separately from equations (25)- (29). More explicitly taking the sum we can rewrite e.g. (25) and (26) as

\[
a_1 b_1 \left[ \frac{1}{b_1} \partial_{uu}^2 b_1 - \frac{1}{a_1} \partial_{uu}^2 a_1 \right] = a_2 b_2 \left[ \frac{1}{a_2} \partial_{uu}^2 a_2 - \frac{1}{b_2} \partial_{uu}^2 b_2 \right], \quad (30)
\]

and

\[
a_1 b_1 \left[ \frac{1}{b_1} \partial_{vv}^2 b_1 - \frac{1}{a_1} \partial_{vv}^2 a_1 \right] = a_2 b_2 \left[ \frac{1}{a_2} \partial_{vv}^2 a_2 - \frac{1}{b_2} \partial_{vv}^2 b_2 \right]. \quad (31)
\]

Let us compare the ratio \(\frac{a_1 b_1}{a_2 b_2}\) from equation (30) and (31) to get

\[
\frac{\left[ \frac{1}{b_1} \partial_{uu}^2 b_1 - \frac{1}{a_1} \partial_{uu}^2 a_1 \right]}{\left[ \frac{1}{b_1} \partial_{vv}^2 b_1 - \frac{1}{a_1} \partial_{vv}^2 a_1 \right]} = \frac{\left[ \frac{1}{a_2} \partial_{uu}^2 a_2 - \frac{1}{b_2} \partial_{uu}^2 b_2 \right]}{\left[ \frac{1}{a_2} \partial_{vv}^2 a_2 - \frac{1}{b_2} \partial_{vv}^2 b_2 \right]} = f_1(u, v), \quad (32)
\]

where \(f_1(u, v)\) is an arbitrary function of \(u\) and \(v\). Further we can write

\[
\frac{1}{b_1} \left[ \partial_{uu}^2 b_1 - f_1 \partial_{vv}^2 b_1 \right] = \frac{1}{a_1} \left[ \partial_{uu}^2 a_1 - f_1 \partial_{vv}^2 a_1 \right] = f_2(u, v), \quad (33)
\]

where \(f_2(u, v)\) is also an arbitrary function of \(u\) and \(v\). This gives

\[
\partial_{uu}^2 a_1 - f_1(u, v) \partial_{vv}^2 a_1 - f_2(u, v) a_1 = 0. \quad (34)
\]

and

\[
\partial_{uu}^2 b_1 - f_1(u, v) \partial_{vv}^2 b_1 - f_2(u, v) b_1 = 0. \quad (35)
\]

Similarly we can get separate equations for \(a_2\) and \(b_2\) as

\[
\partial_{uu}^2 a_2 - f_1(u, v) \partial_{vv}^2 a_2 - f_3(u, v) a_2 = 0 \quad (36)
\]

and

\[
\partial_{uu}^2 b_2 - f_1(u, v) \partial_{vv}^2 b_2 - f_3(u, v) b_2 = 0. \quad (37)
\]
Also from equations (25), (27) and (28) we get

\[ \frac{1}{a_1} \partial_{u_2}^2 b_1 - \frac{1}{a_1} \partial_{u_2}^2 a_1 = \frac{1}{a_2} \partial_{u_2}^2 b_2 - \frac{1}{a_2} \partial_{u_2}^2 b_2 \]

Now using simple rules of ratios (i.e. \( \frac{p}{q} = \frac{p+tq}{q+ts} \)) and some simple algebraic manipulations we get another set of separate equations for \( a_i \) and \( b_i. \)

\[
\begin{align*}
\partial_{u_1}^2 a_1 - g_1(u,v) \partial_{u_1}^2 a_1 - g_2(u,v) \partial_v a_1 - g_3(u,v) a_1 &= 0, \\
\partial_{u_2}^2 b_1 - g_1(u,v) \partial_{u_2}^2 b_1 - g_2(u,v) \partial_v b_1 - g_3(u,v) b_1 &= 0, \\
\partial_{u_2}^2 a_2 - g_1(u,v) \partial_{u_2}^2 a_2 - g_2(u,v) \partial_v a_2 - g_4(u,v) a_2 &= 0, \\
\partial_{u_2}^2 b_2 - g_1(u,v) \partial_{u_2}^2 b_2 - g_2(u,v) \partial_v b_2 - g_4(u,v) b_2 &= 0.
\end{align*}
\]

We note here that there exists various ways of framing equations in various forms from the Ward identity keeping up to the second derivative terms. Here \( f_1, f_2, f_3, g_1, g_2, g_3 \) and \( g_4 \) are all unknown functions which will be determined by the argument of symmetry and the existing known solutions in the semiclassical regime as the boundary condition. All these equations are self-consistent and integrable since all these are derived from the anomalous Ward identity keeping up to the second derivative terms. So there are no further constraints on \( f_i \) and \( g_i \) for the equations to be integrable. These unknowns are like \( V(u) \) of the previous section which in principle can be algebraically determined.

We observe that \( a_1 \) and \( b_1 \) obey the same differential equation and so also \( a_2 \) and \( b_2. \) In the semiclassical limit \( u \rightarrow \infty \) and \( v \rightarrow \infty \) (c.f. eq. (23)), we have \( a_1' = a_1 \) and \( a_2' = a_1 - a_2 \) and \( a_1' = -a_2 \) and \( a_2' = a_1 - a_2 \) respectively. This implies that \( a_1 \) and \( a_2 \) obey the same differential equation which suggests \( f_2 = f_3 \) and \( g_3 = g_4. \) So we have

\[ \partial_{u_1}^2 a - f_1(u,v) \partial_{u_2}^2 a - f_2(u,v) a = 0 \]

and

\[ \partial_{u_2}^2 a - g_1(u,v) \partial_{u_2}^2 a - g_2(u,v) \partial_v a - g_3(u,v) a = 0 \]

Finally we have a pair of second order differential equation for \( a \) which has four independent solutions \( a_1, a_2, b_1 \) and \( b_2. \) For the case of \( SU(2) \) we have only two solutions which ratio gives \( Z \) and from there a Schartzian is emerged which happens to be the potential. Here also one can form an analogue of \( Z \) which will be the period matrix, say \( T_{ij} \) in terms of \( \frac{b_i}{a_1}, \frac{b_i}{a_2}, \frac{b_i}{a_1} \) and \( \frac{b_i}{a_2}. \) Any one of the solutions can be expressed as a linear combination of four linearly independent solutions; namely a solution \( \psi = \alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3 + \alpha_4 \psi_4. \) If \( \omega \) is a singular point in the
two dimensional $U \times V$ complex space, then a monodromy around $\omega$ will change $\psi_i \rightarrow S_{ij} \psi_j$. $S_{ij}$ can be shown to be $Sp(4, Z)$ elements. Under this the period matrix $T_{ij}$ is also transformed under $Sp(4, Z)$. An analogue of Schwartzian in higher complex dimensional space can be found however the inverse map $\omega(T_{ij})$ is unknown. This way we cannot proceed to fix all these unknown function as like as $SU(2)$. So we will emphasize on discrete symmetry and the semiclassical boundary condition to recognize the form of these unknown functions.

Here $Z_6$ is the remnant of $U(1)_R$ symmetry due to anomaly and instatons on the $SU(3)$ moduli space. This implies $\phi \rightarrow e^{\frac{2\pi}{3}} \phi$. In otherwords $u \rightarrow e^{\frac{2\pi}{3}} u$ and $v \rightarrow e^{i\pi} v$. So $u^3$ and $v^2$ are the invariant quantities. Of course any constant of mass dimension six is also an invariant quantity which we denote it as $\Lambda^6$ which may coresspond to the scale parameter of the Wilsonian effective action. This dictates the form the differential equation to be

$$\partial_{uu}^2 a - f_1(u^3, v^2, \Lambda^6) u \partial_{uv}^2 a - f_2(u^3, v^2, \Lambda^6) u a = 0$$

(45)

and

$$\partial_{uu}^2 a - g_1(u^3, v^2, \Lambda^6) u^2 v \partial_{uv}^2 a - g_2(u^3, v^2, \Lambda^6) uv \partial_v a - g_3(u^3, v^2, \Lambda^6) u a = 0.$$  (46)

For $u \rightarrow \infty$ and $v \rightarrow \infty$ we have

$$\partial_{uu}^2 a \rightarrow \frac{a}{4\pi^2}$$

$$\partial_{uv}^2 a \rightarrow \frac{a}{4\pi^2}$$

$$\partial_{uv}^2 a \rightarrow \frac{a}{4uv}$$

$$\partial_v a \rightarrow \frac{a}{2v}.$$  (47)

In the plane when $v = \text{const.}$ we shall have $\partial_{uu}^2 a = V(u) a$ where $V(u)$ is given in eqn.(13). Similarly this is the case for the plane where $u = \text{const.}$ Inorder that all these conditions be satisfied by equations ((45)-(46)), we make an ansatz for

$$f_1 = \text{const.} = \varphi_1$$

$$f_2 = \frac{\varphi_2}{u^3 + c_1 v^2 + c_2 \Lambda^6}$$

$$g_1 = \frac{\xi_1}{u^4 + d_1 v^2 + e_1 \Lambda^6}$$

$$g_2 = \frac{\xi_2}{u^4 + d_2 v^2 + e_2 \Lambda^6}$$

$$g_3 = \frac{\xi_3}{u^4 + d_3 v^2 + e_3 \Lambda^6},$$  (48)

where $\varphi_1$, $\varphi_2$, $\xi_1$, $\xi_2$, $\xi_3$, $c_1$, $c_2$, $d_1$, $d_2$, $d_3$, $e_1$, $e_2$ and $e_3$ are just complex numbers. We note here that these may not be the unique form of these functions. This is merely
our guess since they satisfy the asymptotic conditions in a very simple way. So many unknown parameters correspond to a large number of singularities. These numbers are fixed by demanding that the equations are regularly singular otherwise solutions around an essential singularity can be isolated and we cannot recover from that the known results in the semiclassical regime. The singularity in the semiclassical regime is known which fixes some of these numbers. We show in detail how this is done in the sequel.

In the semiclassical limit \( u \to \infty \) and \( v \to \infty \), we can neglect \( \Lambda \). Besides separate singularities due to \( u \to \infty \) and \( v \to \infty \) there is a plane of singularity when \( a_1 = -a_2 \), then \( u = 3a^2 \) and \( v = 2a^3 \) and \( u^3 = cv^2 \) where \( c = \frac{27}{4} \). All \( f_i \)'s and \( g_i \)'s are singular for \( c_1 = d_1 = d_2 = d_3 = -c \). Since this is the only possible singularity in this limit which implies that \( c_1 = d_1 = d_2 = d_3 = -c \).

For convenience let us change the variable \( u^3 = \alpha \) and \( v^2 = \beta \). Now we get

\[
9\alpha \partial_{\alpha\alpha}^2 \mathbf{a} + 6 \partial_{\alpha} \mathbf{a} - 2 \varphi_1(2\beta \partial_{\beta\beta}^2 \mathbf{a} + \partial_{\beta} \mathbf{a}) - \frac{\varphi_2}{\alpha - c\beta} \mathbf{a} = 0
\]

and

\[
9\alpha \partial_{\alpha\alpha}^2 \mathbf{a} + 6 \partial_{\alpha} \mathbf{a} - \frac{6 \xi_1 \alpha \beta}{\alpha - c\beta} \partial_{\alpha\beta}^2 \mathbf{a} - \frac{2\xi_2 \beta}{\alpha - c\beta} \partial_{\beta} \mathbf{a} - \frac{\xi_3}{\alpha - c\beta} \mathbf{a} = 0.
\]

However these equations ought to be regularly singular for \( \alpha = c\beta \). By demanding that we get the following conditions

\[
9\alpha^2 - 4\varphi_1 c^3 \beta^2 - 6\xi_1 c\alpha \beta = 9(\alpha - c\beta)^2,
\]

and

\[
6\alpha + 2\varphi_1 c^2 \beta + 2\xi_2 c\beta = 6(\alpha - c\beta),
\]

and \( \varphi_2 = 0 \). This gives \( \varphi_1 = -\frac{9}{4c} \), \( \xi_1 = 3 \) and \( \xi_2 = -\frac{3}{4} \). When \( u \to \infty \) and \( u \geq v \) the asymptotic condition will be satisfied for \( \xi_3 = \frac{1}{4} \) (c.f. eq.(46)). Since \( \varphi_2 = 0 \), from eq. (48) we get \( f_2 = 0 \) If we don’t take the equation to be regularly singular then isolated singularities can be easily separated from the equation. Then we can never recover the known semiclassical result and monodromy around \( \infty \). Thus we get

\[
\partial_{uu}^2 \mathbf{a} + \frac{9}{4c} u \partial_{uv}^2 \mathbf{a} = 0
\]

and

\[
\partial_{uu}^2 \mathbf{a} - \frac{3u^2 v}{u^3 - cv^2 + e_1 \Lambda^6} \partial_{uu}^2 \mathbf{a} + \frac{3}{4 \frac{u^3}{u^3 - cv^2 + e_2 \Lambda^6}} \partial_{uv} \mathbf{a} - \frac{u}{4(u^3 - cv^2 + e_3 \Lambda^6)} \mathbf{a} = 0.
\]
Still we have to fix $e_1$, $e_2$ and $e_3$. For different values of $e_i$'s we get different singularities for every term. If really they are different then at any such singularity that term can be separated since all other terms will be finite. This will correspond to an essential singularity for which in the limit $\Lambda \to 0$, then we cannot recover the semiclassical result. The only legitimate singularities will correspond to the same singularity for all the terms. This implies that $e_1 = e_2 = e_3$.

To compare our results with the standard Picard-Fuchs equation of Ref. [2] we have to substitute $u = -u$ and take $e_1 = e_2$, then we will get the exact form. Now by changing the variables as $x = c\frac{\sqrt{u}}{w}$ and $y = c\frac{\sqrt{v}}{w}$, equation (53) and (54) are written as the hypergeometric differential equation for the Appel system of type $F_4 (a, b, c, d, x, y)$ [13]. This equation has been thoroughly studied in Ref. [2] in three different regimes of $u$, $v$ and $\Lambda$ which we don’t want to repeat. To compare the results of our prepotentials with the explicit instaton calculations [14] we take the regime where $u$ is large and $v$ and $\Lambda$ are small which correspond to small $x$ and $y$. Four independent solutions are

$$a_1 = \sqrt{u}F_4\left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{27\Lambda^6}{4u^3}, \frac{27v^2}{4u^3}\right) + \frac{v}{2u}F_4\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{27\Lambda^6}{4u^3}, \frac{27v^2}{4u^3}\right)$$

$$b_1 = \frac{1}{2\pi i}\sqrt{u} \sum_{m,n} \left(\frac{1}{6}\right)_{m+n} \left(\frac{1}{2}\right)_{m-n} \left(\frac{27\Lambda^6}{4u^3}\right)^m \left(\frac{27v^2}{4u^3}\right)^n$$

$$\times \left[-2\psi(m+1) + \psi\left(-\frac{1}{6} + m + n\right) + \psi\left(\frac{1}{6} + m + n\right) + \log \left(-\frac{27\Lambda^6}{4u^3}\right)\right]$$

$$+ \frac{1}{2\pi i} \sum_{m,n} \left(\frac{1}{6}\right)_{m+n} \left(\frac{1}{2}\right)_{m-n} \left(\frac{27\Lambda^6}{4u^3}\right)^m \left(\frac{27v^2}{4u^3}\right)^n$$

$$\times \left[-2\psi(m+1) + \psi\left(-\frac{1}{6} + m + n\right) + \psi\left(\frac{1}{6} + m + n\right) + \log \left(-\frac{27\Lambda^6}{4u^3}\right)\right]$$

and similarly

$$a_2 = -\sqrt{u}F_4\left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{27\Lambda^6}{4u^3}, \frac{27v^2}{4u^3}\right) - \frac{v}{2u}F_4\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{27\Lambda^6}{4u^3}, \frac{27v^2}{4u^3}\right),$$

$$b_2 = -\frac{1}{2\pi i}\sqrt{u} \sum_{m,n} \left(\frac{1}{6}\right)_{m+n} \left(\frac{1}{2}\right)_{m-n} \left(\frac{27\Lambda^6}{4u^3}\right)^m \left(\frac{27v^2}{4u^3}\right)^n$$

$$\times \left[-2\psi(m+1) + \psi\left(-\frac{1}{6} + m + n\right) + \psi\left(\frac{1}{6} + m + n\right) + \log \left(-\frac{27\Lambda^6}{4u^3}\right)\right]$$

$$+ \frac{1}{2\pi i} \sum_{m,n} \left(\frac{1}{6}\right)_{m+n} \left(\frac{1}{2}\right)_{m-n} \left(\frac{27\Lambda^6}{4u^3}\right)^m \left(\frac{27v^2}{4u^3}\right)^n$$

$$\times \left[-2\psi(m+1) + \psi\left(-\frac{1}{6} + m + n\right) + \psi\left(\frac{1}{6} + m + n\right) + \log \left(-\frac{27\Lambda^6}{4u^3}\right)\right]$$
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) and

\[
F_4(a, b, c, d; x, y) = \sum_{m,n} \frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(d)_n} x^m y^n.
\]

where \( (a)_m = \Gamma(a + m)/\Gamma(a) \). In order to calculate the prepotential \( \mathcal{F} \) we have to first express \( u(a_1, a_2) \) and \( v(a_1, a_2) \) as power series in \( \frac{a_1}{\Lambda} \) and \( \frac{a_2}{\Lambda} \) by inverting eqn.(54) and (56). Then these are substituted in \( b_1 \) and \( b_2 \). The integration over \( a_1 \) and \( a_2 \) gives the prepotential.

We obtained equations (53) and (54) on the basis of \( Z_6 \) symmetry and asymptotic conditions. The most general form of the equations will be

\[
\partial^2_{u} a - \sum_i \left[ \frac{3u^2 v}{u^3 - cv^2 + e_i \Lambda^6} \partial^2_{u} a - \frac{3u v}{4u^3 - cv^2 + e_i \Lambda^6} \partial_e a + \frac{u}{4(u^3 - cv^2 + e_i \Lambda^6)} a \right] = 0.
\]

(60)

Here again we have multiple strong coupling singularities like the case of \( SU(2) \) theory. Again we may use the argument of monodromy to conjecture that there are only six singularities. However we cannot prove that this is the unique solution. In this case ordering of monodromy matrices are quite complicated unlike \( SU(2) \) case where singularities were paired on the real line.

5. Conclusion

We are able to derive here an exact prepotential starting from anomalous Ward identity of the \( N = 2 \) SUSY Yang-Mills theory in its broken phase (coulomb phase) for \( SU(2) \) gauge group which agrees with Seiberg-Witten result [1, 13] and for \( SU(3) \) gauge group which agrees with the results of Ref.[2, 3, 14]. We can use this procedure for any rank two gauge group. Extensions to higher rank gauge group is possible however it will be quite lengthy and tedious. Inclusion of flavours is quite easy for \( SU(2) \) but it will be quite complicated for the rank two gauge groups.

Although we have not assumed electric-magnetic duality at all however the physical interpretation of these singularities will be vanishing condition of the \( SU(N) \) dyon masses. We are unable to algebraically prove here that the Seiberg-Witten solutions [1, 2, 3] are the unique solutions. The fact that the monodromy around \( \infty \) has to be the composition of all the monodromies around all the singularities, in principle there exists a possibility of finite number of singularities other than these Seiberg-Witten singularities around which the monodromies may combine to satisfy this condition. We fail to rule out this possibility by imposing only consistency condition. Somehow it needs another constraint stemming from physical condition to show the uniqueness of the Seiberg-Witten singularities.
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