Induced topological pressure for topological dynamical systems

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Abstract. In this paper, inspired by the article [5], we introduce the induced topological pressure for a topological dynamical system. In particular, we prove a variational principle for the induced topological pressure.

Keywords and phrases: Induced pressure, dynamical system, variational principle.

1 INTRODUCTION AND MAIN RESULT

The present paper is devoted to the study of the induced topological pressure for topological dynamical systems. Before stating our main result, we first give some notation and background about the induced topological pressure. By a topological dynamical system (TDS) $(X, f)$, we mean a compact metric space $(X, d)$ together with a continuous map $f : X \to X$. Recall that $C(X, \mathbb{R})$ is the Banach algebra of real-valued continuous functions of $X$ equipped with the supremum norm. For $\varphi \in C(X, \mathbb{R}), n \geq 1$, let $(S_n\varphi)(x) := \sum_{i=0}^{n-1} \varphi(f^i x)$ and for $\psi \in C(X, \mathbb{R})$ with $\psi > 0$, let $m := \min\{\psi(x) : x \in X\}$. Mathematics Subject Classification: 37D25, 37D35
We denote by $M(X, f)$ all $f$-invariant Borel probability measures on $X$ endowed with the weak-star topology.

Topological pressure is a basic notion of the thermodynamic formalism. It first introduced by Ruelle [11] for expansive topological dynamical systems, and later by Walters [1,9,10] for the general case. The variational principle established by Walters can be stated as follows: Let $(X, f)$ be a TDS, and let $\varphi \in C(X, \mathbb{R})$, $P(\varphi)$ denote the topological pressure of $\varphi$. Then

$$P(\varphi) = \sup \{h_\mu(f) + \int \varphi d\mu : \mu \in M(X, f)\}.$$ (1.1)

where $h_\mu(f)$ denotes the measure-theoretical entropy of $\mu$. The theory of topological pressure and its variational principle plays a fundamental role in statistics, ergodic theory, and the theory of dynamical systems [3,9,13]. Since the works of Bowen [4] and Ruelle [12], the topological pressure has become a basic tool in the dimension theory of dynamical systems [8,14].

Recently Jaerish, Kesseböhmer and Lamei [5] introduced the notion of the induced topological pressure of a countable Markov shift, and established a variational principle for it. One important feature of this pressure is the freedom in choosing a scaling function, and this is applied to large deviation theory and fractal geometry. In this paper we present the induced topological pressure for a topological dynamical system and consider the relation between it and the topological pressure. We set up a variational principle for the induced topological pressure. As an application, we will point out that the BS dimension is a special case of the induced topological pressure.

Let $(X, f)$ be a TDS. For $n \in \mathbb{N}$, the $n$th Bowen metric $d_n$ on $X$ is defined by

$$d_n(x, y) = \max \{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n - 1\}.$$ 

For every $\epsilon > 0$, we denote by $B_n(x, \epsilon)$, $\overline{B}_n(x, \epsilon)$ the open (resp. closed) ball of radius $\epsilon$ and order $n$ in the metric $d_n$ around $x$, i.e.,

$$B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\} \quad \text{and} \quad \overline{B}_n(x, \epsilon) = \{y \in X : d_n(x, y) \leq \epsilon\}.$$ 

Let $Z \subseteq X$ be a non-empty set. A subset $F_n \subset X$ is called an $(n, \epsilon)$-spanning set of $Z$ if for any $y \in Z$, there exists $x \in F_n$ with $d_n(x, y) \leq \epsilon$. A subset $E_n \subset Z$ is called an $(n, \epsilon)$-separated set of $Z$ if $x, y \in E_n, x \neq y$ implies $d_n(x, y) > \epsilon$.

Now we define a new notion, the induced topological pressure which extends the definition in [5] for topological Markov shifts if the Markov shift is compact, as follows.

**Definition 1.1.** Let $(X, f)$ be a TDS and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. For $T > 0$, define

$$S_T = \{n \in \mathbb{N} : \exists x \in X \text{ such that } S_n \psi(x) \leq T \text{ and } S_{n+1} \psi(x) > T\}.$$
For \( n \in S_T \), define
\[
X_n = \{ x \in X : S_n \psi(x) \leq T \text{ and } S_{n+1} \psi(x) > T \}.
\]

Let
\[
Q_{\psi,T}(f, \varphi, \epsilon) = \inf \left\{ \sum_{n \in S_T} \sum_{x \in F_n} \exp(S_n \varphi)(x) : F_n \text{ is an } (n, \epsilon)\text{-spanning set of } X_n, n \in S_T \right\}.
\]

We define the \( \psi \)-induced topological pressure of \( \varphi \) (with respect to \( f \)) by
\[
P_\psi(\varphi) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log Q_{\psi,T}(f, \varphi, \epsilon) \quad (1.2)
\]

Remarks.

(i) Let \([T_m]\) denote the integer part of \( \frac{T}{m} \). Then for \( n \in S_T \), \( n \leq [\frac{T}{m}] + 1 \), i.e., \( S_T \) is a finite set.

(ii) If \( 0 < \epsilon_1 < \epsilon_2 \), then \( Q_{\psi,T}(f, \varphi, \epsilon_1) \geq Q_{\psi,T}(f, \varphi, \epsilon_2) \), which implies the existence of the limit in (1.2) and \( P_\psi(\varphi) > -\infty \).

(iii) \( P_1(\varphi) = P(\varphi) \).

The variational principle for induced topological pressure is stated as follows.

**Theorem 1.1.** Let \((X, f)\) be a TDS and \( \varphi, \psi \in C(X, \mathbb{R}) \) with \( \psi > 0 \). Then
\[
P_\psi(\varphi) = \sup \left\{ \frac{h_\nu(f)}{\int \psi d\nu} + \frac{\int \varphi d\nu}{\int \psi d\nu} : \nu \in M(X, f) \right\} \quad (1.3)
\]

This paper is organized as follows. In Section 2, we provide an equivalent definition of induced topological pressure. We prove Theorem 1.1 in Section 3. We point out that the BS dimension is a special case of the induced topological pressure in Section 4. In Section 5, we study the equilibrium measures for the induced topological pressure.

## 2 AN EQUIVALENT DEFINITION

In this section, we obtain an equivalent definition of the induced topological pressure by using separated sets (from now on, we omit the word ‘topological’ if no confusion can arise).

**Proposition 2.1.** Let \((X, f)\) be a TDS and \( \varphi, \psi \in C(X, \mathbb{R}) \) with \( \psi > 0 \). For \( T > 0 \), define
\[
P_{\psi,T}(f, \varphi, \epsilon) = \sup \left\{ \sum_{n \in S_T} \sum_{x \in E_n} \exp(S_n \varphi)(x) : E_n \text{ is an } (n, \epsilon)\text{-separated set of } X_n, n \in S_T \right\}.
\]

Then
\[
P_\psi(\varphi) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \epsilon) \quad (2.4)
\]
Proof. We note that since the map $\epsilon \mapsto \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \epsilon)$ is nondecreasing, the limit in (2.4) is well defined when $\epsilon \to 0$. For $n \in S_T$, let $E_n$ be an $(n, \epsilon)$-separated set of $X_n$ which fails to be $(n, \epsilon)$-separated when any point of $X_n$ is added. Then $E_n$ is an $(n, \epsilon)$-spanning set of $X_n$. Therefore

$$Q_{\psi,T}(f, \varphi, \epsilon) \leq P_{\psi,T}(f, \varphi, \epsilon)$$

and

$$P_{\psi}(\varphi) \leq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \epsilon).$$

To show the reverse inequality, for any $\epsilon > 0$, we choose $\delta > 0$ small enough so that

$$d(x, y) \leq \frac{\delta}{2} \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon. \quad (2.5)$$

For $n \in S_T$, let $E_n$ be an $(n, \delta)$-separated set of $X_n$ and $F_n$ an $(n, \frac{\delta}{2})$-spanning set of $X_n$. Define $\phi : E_n \to F_n$ by choosing, for each $x \in E_n$, some point $\phi(x) \in F_n$ with $d_{n}(\phi(x), x) \leq \frac{\delta}{2}$. Then $\phi$ is injective. Therefore,

$$\sum_{n \in S_T} \sum_{y \in F_n} \exp(S_n \varphi)(y) \geq \sum_{n \in S_T} \sum_{y \in \phi E_n} \exp(S_n \varphi)(y) \geq \sum_{n \in S_T} \left( \min_{x \in E_n} \exp((S_n \varphi)(\phi x) - (S_n \varphi)(x)) \right) \sum_{x \in E_n} \exp(S_n \varphi)(x) \geq \exp\left(-\left(\frac{T}{m} + 1\right)\epsilon\right) \sum_{n \in S_T} \sum_{x \in E_n} \exp(S_n \varphi)(x).$$

We conclude that

$$\lim_{\delta \to 0} \limsup_{T \to \infty} \frac{1}{T} \log Q_{\psi,T}(f, \varphi, \frac{\delta}{2}) \geq -\frac{1}{m} \epsilon + \lim_{\delta \to 0} \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \delta).$$

As $\epsilon \to 0$, we have

$$P_{\psi}(\varphi) \geq \lim_{\delta \to 0} \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \delta).$$

3 THE PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1. Firstly, we study the relation between $P_{\psi}(\varphi)$ and $P(\varphi)$, which will be needed for the proof of Theorem 1.1. The following Theorem 3.1 is very similar to Theorem 2.1 of [5], and it is a generalization of this theorem in the case of a compact topological Markov shift.
Theorem 3.1. Let \((X, f)\) be a TDS and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). For \(T > 0\), define
\[
G_T = \{ n \in \mathbb{N} : \exists x \in X \text{ such that } S_n \psi(x) > T \}.
\]
For \(n \in G_T\), define
\[
Y_n = \{ x \in X : S_n \psi(x) > T \}.
\]
Let
\[
R_{\psi, T}(f, \varphi, \epsilon) = \sup \left\{ \sum_{n \in G_T} \sum_{x \in E'_n} \exp(S_n \varphi)(x) : E'_n \text{ is an } (n, \epsilon)-\text{separated set of } Y_n, n \in G_T \right\}.
\]
We have
\[
P_{\psi}(\varphi) = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(f, \varphi - \beta \psi, \epsilon) < \infty \}. \tag{3.6}
\]
Here we make the convention that \(\inf \emptyset = \infty\).

**Proof.** For \(n \in \mathbb{N}, x \in X\), we define \(m_n(x)\) to be the unique positive integer such that
\[
(m_n(x) - 1)\|\psi\| < S_n \psi(x) \leq m_n(x)\|\psi\|.
\]
Observing that
\[
\exp(-\beta\|\psi\| m_n(x)) \exp(-|\beta|\|\psi\|) \leq \exp(-\beta S_n \psi(x)) \leq \exp(-\beta\|\psi\| m_n(x)) \exp(|\beta|\|\psi\|)
\]
for all \(x \in X\). For \(\xi_T = \{ \xi_n : X \to \mathbb{R} \}_{n \in G_T}\), we define
\[
R_{\psi, T}(f, \varphi, \xi_T, \epsilon)
= \sup \left\{ \sum_{n \in G_T} \sum_{x \in E'_n} \exp((S_n \varphi)(x) - \xi_n(x)) : E'_n \text{ is an } (n, \epsilon)-\text{separated set of } Y_n, n \in G_T \right\}.
\]
We conclude that
\[
\limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(f, \varphi - \beta \psi, \epsilon) < \infty
\]
if and only if
\[
\limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(f, \varphi, \{ -\beta\|\psi\| m_n \}_{n \in G_T}, \epsilon) < \infty.
\]
Hence, it will be sufficient to verify that
\[
P_{\psi}(\varphi) = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(f, \varphi, \{ -\beta\|\psi\| m_n \}_{n \in G_T}, \epsilon) < \infty \}.
\]
By the equivalent definition of \(P_{\psi}(\varphi)\), for every \(\delta > 0, \beta \in \mathbb{R}\) with \(\beta < P_{\psi}(\varphi) - \delta\), there exists an \(\epsilon_0 > 0\) with
\[
\beta + \delta < \limsup_{T \to \infty} \frac{1}{T} \log P_{\psi, T}(f, \varphi, \epsilon) \leq P_{\psi}(\varphi), \quad \forall \epsilon \in (0, \epsilon_0),
\]
and we can find a sequence \( \{T_j\}_{j \in \mathbb{N}} \) such that for every \( j \in \mathbb{N} \), \( T_{j+1} - T_j > 2\|\psi\| \) and for each \( j \in \mathbb{N} \), there exists an \( E_{T_j} = \bigcup_{n \in S_{T_j}} E_n \) with
\[
\sum_{n \in S_{T_j}} \sum_{x \in E_n} \exp(S_n \varphi)(x) \geq \exp(T_j (\beta + \frac{\delta}{2})).
\]
Since for \( j \in \mathbb{N}, n \in S_{T_j}, x \in E_n, T_j - \|\psi\| < S_n \psi(x) \leq T_j \), we have
\[
S_{T_i} \cap S_{T_j} = \emptyset, i \neq j
\]
and
\[
\|\psi\| m_n(x) - T_j < 2\|\psi\|.
\]
It follows that
\[
R_{\psi,T}(f, \varphi, \{-\beta \|\psi\| m_n\}_{n \in G_T}, \epsilon)
\]
\[
\geq \sum_{j \in \mathbb{N}, T_j - \|\psi\| > T} \sum_{n \in S_{T_j}} \sum_{x \in E_n} \exp((S_n \varphi)(x) - \beta \|\psi\| m_n(x))
\]
\[
\geq \exp(-2\|\psi\|) \sum_{j \in \mathbb{N}, T_j - \|\psi\| > T} \sum_{n \in S_{T_j}} \sum_{x \in E_n} \exp((S_n \varphi)(x) - \beta T_j)
\]
\[
\geq \exp(-2\|\psi\|) \sum_{j \in \mathbb{N}, T_j - \|\psi\| > T} \exp((\beta + \frac{\delta}{2}) T_j - \beta T_j)
\]
\[
= \infty.
\]
Therefore, for all \( \beta < P_{\psi}(\varphi) - \delta \),
\[
\lim_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}(f, \varphi, \{-\beta \|\psi\| m_n\}_{n \in G_T}, \epsilon) = \infty. \tag{3.7}
\]
This argument is not only valid for \( P_{\psi}(\varphi) \in \mathbb{R} \), but also for \( P_{\psi}(\varphi) = \infty \), in which case (3.7) holds for every \( \beta \in \mathbb{R} \). Then
\[
P_{\psi}(\varphi) \leq \inf\{\beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \lim_{T \to \infty} R_{\psi,T}(f, \varphi, \{-\beta \|\psi\| m_n\}_{n \in G_T}, \epsilon) < \infty\}. \tag{3.8}
\]
Next, we establish the reverse inequality. We consider the case \( P_{\psi}(\varphi) \in \mathbb{R} \) and show that for any \( \delta > 0 \),
\[
\limsup_{\epsilon \to 0} \lim_{T \to \infty} R_{\psi,T}(f, \varphi, \{-(P_{\psi}(\varphi) + \delta) \|\psi\| m_n\}_{n \in G_T}, \epsilon) < \infty.
\]
Again, by the equivalent definition of \( P_{\psi}(\varphi) \), we have, for any \( \epsilon > 0 \),
\[
\limsup_{T \to \infty} \frac{1}{T} \log P_{\psi,T}(f, \varphi, \epsilon) < P_{\psi}(\varphi) + \frac{\delta}{2}.
\]
and we can find an \( l_0 \in \mathbb{N} \) such that for all \( l \in \mathbb{N} \) with \( l \geq l_0 \),

\[
P_{\psi,lm}(f, \varphi, \epsilon) \leq \exp(lm(P_\psi(\varphi) + \frac{2\delta}{3})�.
\]

Note that for \( n \in S_{lm}, x \in E_n \), we have

\[
\|\psi\|_n(x) - lm < 2\|\psi\|
\]

and

\[-(P_\psi(\varphi) + \delta)\|\psi\|_n(x) \leq -lm(P_\psi(\varphi) + \delta) + 2|P_\psi(\varphi) + \delta|\psi|.
\]

Moreover, for sufficiently large \( T > 0, n \in G_T, x \in E'_n \subset Y_n \), there exists a unique \( l \in \mathbb{N} \) such that \((l - 1)m < S_n\psi(x) \leq lm \). Obviously \( S_{n+1}\psi(x) > lm \). Hence, we obtain

\[
R_{\psi,T}(f, \varphi, \{- (P_\psi(\varphi) + \delta)\|\psi\|_n\}_{n \in G_T}, \epsilon)
\]

\[
\leq \sum_{l \geq l_0} \sup \left\{ \sum_{n \in S_{lm}} \sum_{x \in E_n} \exp((S_n\varphi)(x) - (P_\psi(\varphi) + \delta)\|\psi\|_n(x)) : E_n \text{ is an } (n, \epsilon)-\text{separated set of } X_n, n \in S_{lm} \right\}
\]

\[
\leq \exp(2\|\psi\|_nP_\psi(\varphi) + \delta) \sum_{l \geq l_0} \exp(-(P_\psi(\varphi) + \delta)lm)P_{\psi,lm}(f, \varphi, \epsilon)
\]

\[
\leq \exp(2\|\psi\|_nP_\psi(\varphi) + \delta) \sum_{l \geq l_0} \exp(-\frac{\delta}{3}lm)
\]

\[
< \exp(2\|\psi\|_nP_\psi(\varphi) + \delta) \frac{1}{1 - \exp(-\frac{\delta m}{3})}.
\]

This implies

\[
\lim_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}(f, \varphi, \{- (P_\psi(\varphi)) + \delta)\|\psi\|_n\}_{n \in G_T}, \epsilon) < \infty,
\]

and hence,

\[
P_\psi(\varphi) \geq \inf \{ \beta \in \mathbb{R} : \lim_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}(f, \varphi, \{- \beta\|\psi\|_n\}_{n \in G_T}, \epsilon) < \infty \}.
\] (3.9)

Combining (3.8) and (3.9) we obtain (3.6).

**Corollary 3.1.** Let \((X, f)\) be a TDS, and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). We have

\[
P_\psi(\varphi) \geq \inf \{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \leq 0 \}.
\] (3.10)

**Proof.** Let \(\beta \in \{ \beta \in \mathbb{R} : \lim_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < \infty \}\) and

\[
\lim_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) = a.
\]

Combining (3.8) and (3.9) we obtain (3.6).
Then for any \( \epsilon > 0 \),
\[
\limsup_{T \to \infty} R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < a + 1.
\]
We can find a \( T_0 > 0 \) such that for all \( T > T_0 \),
\[
R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < a + 2.
\]
Now, for sufficiently large \( n \in \mathbb{N} \),
\[
S_n \psi(x) > T, \quad \forall x \in X,
\]
and hence, for such \( n \in G_T \), \( E_n \) is an \( (n, \epsilon) \)-separated set of \( X \) and
\[
\sum_{x \in E_n} \exp(S_n(\varphi - \beta \psi))(x) < a + 2.
\]
It follows from this that
\[
P(\varphi - \beta \psi) \leq 0.
\]
Since
\[
\inf\{ \beta \in \mathbb{R} : \lim \limsup_{\epsilon \to 0} R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < \infty \}
\geq \inf\{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \leq 0 \},
\]
the inequality (3.10) follows by Theorem 3.1.

**Corollary 3.2.** Let \((X, f)\) be a TDS, and \( \varphi, \psi \in C(X, \mathbb{R}) \) with \( \psi > 0 \). We have
\[
P_{\psi}(\varphi) = \inf\{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \leq 0 \} = \sup\{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \geq 0 \}.
\]

**Proof.** If there exists a \( \beta \in \mathbb{R} \) such that \( P(\varphi - \beta \psi) = \infty \), then \( P(\varphi - \beta \psi) = \infty \) for all \( \beta \in \mathbb{R} \). By Corollary 3.1, we have
\[
P_{\psi}(\varphi) = \inf\{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \leq 0 \} = \sup\{ \beta \in \mathbb{R} : P(\varphi - \beta \psi) \geq 0 \}.
\]
Suppose for any \( \beta \in \mathbb{R} \), \( P(\varphi - \beta \psi) < \infty \). By (1.1) we have
\[
P(\varphi - \beta \psi) = \sup\{ h_\nu(f) + \int \varphi d\nu - \beta \int \psi d\nu : \nu \in M(X, f) \}.
\]
Then for each \( \beta_1, \beta_2 \in \mathbb{R}, \beta_1 < \beta_2 \) and \( 0 < \epsilon < \frac{m(\beta_2 - \beta_1)}{2} \), there exists a \( \mu \in M(X, f) \) such that
\[
\begin{align*}
\sup\{ h_\nu(f) + \int \varphi d\nu - \beta_2 \int \psi d\nu : \nu \in M(X, f) \} &< h_\mu(f) + \int \varphi d\mu - \beta_2 \int \psi d\mu + \epsilon \\
= h_\mu(f) + \int \varphi d\mu - \beta_1 \int \psi d\mu + \epsilon - (\beta_2 - \beta_1) \int \psi d\mu \\
< h_\mu(f) + \int \varphi d\mu - \beta_1 \int \psi d\mu - (\beta_2 - \beta_1)(\int \psi d\mu - \frac{m}{2}) \\
\leq \sup\{ h_\nu(f) + \int \varphi d\nu - \beta_1 \int \psi d\nu : \nu \in M(X, f) \} - (\beta_2 - \beta_1)(\int \psi d\mu - \frac{m}{2}).
\end{align*}
\]
Thus, the map $\beta \mapsto P(\varphi - \beta \psi)$ is strictly decreasing.

Next, we prove that

$$P(\varphi - \beta \psi) < 0 \implies R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < \infty.$$  

Let $P(\varphi - \beta \psi) = 2a < 0$. For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$,

$$\sup_{E_n} \sum_{x \in E_n} \exp(S_n(\varphi - \beta \psi))(x) \leq \exp(na),$$

where the supremum is taken over all $(n, \epsilon)$-separated sets of $X$. Consequently, for sufficiently large $T > 0$, we have

$$R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) \leq \sum_{n \geq N} \sup_{E_n} \sum_{x \in E_n} \exp(S_n(\varphi - \beta \psi))(x) \leq \frac{1}{1 - \exp(a)} < \infty,$$

and the conclusion holds.

Since

$$\inf\{\beta \in \mathbb{R} : P(\varphi - \beta \psi) < 0\} \geq \inf\{\beta \in \mathbb{R} : \lim_{\epsilon \to 0} \lim_{T \to \infty} R_{\psi,T}(f, \varphi - \beta \psi, \epsilon) < \infty\},$$

by Theorem 3.1 and Corollary 3.1, we conclude that

$$\inf\{\beta \in \mathbb{R} : P(\varphi - \beta \psi) \leq 0\} = \inf\{\beta \in \mathbb{R} : P(\varphi - \beta \psi) < 0\} = \sup\{\beta \in \mathbb{R} : P(\varphi - \beta \psi) \geq 0\}.$$

**Corollary 3.3.** Let $(X, f)$ be a TDS and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. Suppose that for each $\beta \in \mathbb{R}$ we have $P(\varphi - \beta \psi) \in \mathbb{R}$. Then $P(\varphi - P_{\psi}(\varphi)\psi) = 0$.

**Proof.** By the proof of Corollary 3.2 the map $\beta \mapsto P(\varphi - \beta \psi)$ is a strictly decreasing, continuous map on $\mathbb{R}$. Hence $P(\varphi - P_{\psi}(\varphi)\psi) = 0$.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Firstly, we show

$$P_{\psi}(\varphi) \geq \sup \left\{ \frac{h_{\nu}(f)}{\psi d\nu} + \frac{\varphi d\nu}{\int \psi d\nu} : \nu \in M(X, f) \right\}. \quad (3.11)$$

By Corollary 3.1 we have $0 \geq P(\varphi - \beta \psi)$ for $\beta > P_{\psi}(\varphi)$. It follows from (1.1) that

$$0 \geq P(\varphi - \beta \psi)$$

$$= \sup \left\{ h_{\nu}(f) + \int \varphi d\nu - \beta \int \psi d\nu : \nu \in M(X, f) \right\}$$

$$= \sup \left\{ \int \psi d\nu \left( \frac{h_{\nu}(f)}{\psi d\nu} + \frac{\varphi d\nu}{\int \psi d\nu} - \beta \right) : \nu \in M(X, f) \right\},$$
and hence (3.11) holds.

Next, we establish the reverse inequality. Similarly by Corollary 3.2 we have $P(\varphi - \beta \psi) \geq 0$ for $\beta < P(\varphi)$. Then

$$P(\varphi - \beta \psi) = \sup \{ h_\nu(f) + \int \varphi d\nu - \beta \int \psi d\nu : \nu \in M(X, f) \}$$

$$= \sup \{ \int \psi d\nu(\frac{h_\nu(f)}{\psi} + \frac{\varphi d\nu}{\psi} - \beta) : \nu \in M(X, f) \}$$

$$\geq 0.$$ It is easy to see that

$$P(\varphi) \leq \sup \{ \frac{h_\nu(f)}{\psi} + \frac{\varphi d\nu}{\psi} : \nu \in M(X, f) \}.$$ (3.12)

Combining (3.11) and (3.12), we obtain (1.3).

4 A SPECIAL CASE (BS-DIMENSION)

In this section we will show that the BS dimension with Carathéodory structure is a special case of the induced pressure. The BS dimension was first defined by Barreira and Schmeling [2] as follows.

For $n \geq 1, \epsilon > 0$, we put

$$W_n(\epsilon) = \{ B_n(x, \epsilon) : x \in X \}.$$ For any $B_n(x, \epsilon) \in W_n(\epsilon), \psi \in C(X, \mathbb{R})$ with $\psi > 0$, the function $\psi$ can induce a function by

$$\psi(B) = \sup_{x \in B} (S_n \psi)(x).$$ We call $\mathcal{G} \subset \cup_{j \geq N} W_j(\epsilon)$ covers $X$, if $\bigcup_{B \in \mathcal{G}} B = X$.

Definition 4.1. Let $(X, f)$ be a TDS. For any $\alpha > 0, N \in \mathbb{N}$ and $\epsilon > 0$, we define

$$M(\alpha, \epsilon, N) = \inf_{\mathcal{G}} \{ \sum_{B \in \mathcal{G}} \exp(-\alpha \psi(B)) \},$$

where the infimum is taken over all finite $\mathcal{G} \subset \cup_{j \geq N} W_j(\epsilon)$ that cover $X$. Obviously $M(\alpha, \epsilon, N)$ is a finite outer measure on $X$ and increases as $N$ increases. Define

$$m(\alpha, \epsilon) = \lim_{N \to \infty} M(\alpha, \epsilon, N)$$
and
\[\dim_{BS}(X, \epsilon) = \inf\{\alpha : m(\alpha, \epsilon) = 0\} = \sup\{\alpha : m(\alpha, \epsilon) = \infty\}.\]

The BS dimension is \(\dim_{BS} X = \lim_{\epsilon \to 0} \dim_{BS}(X, \epsilon)\): this limit exists because given \(\epsilon_1 < \epsilon_2\), we have \(m(\alpha, \epsilon_1) \geq m(\alpha, \epsilon_2)\), so \(\dim_{BS}(X, \epsilon_1) \geq \dim_{BS}(X, \epsilon_2)\).

**Proposition 4.1.** For a TDS, we have \(P_\psi(0) = \dim_{BS} X\).

**Proof.** By [2, Proposition 6.4], we have \(P(-\psi \dim_{BS} X) = 0\). Now it follows from Corollary 3.3 that \(P_\psi(0) = \dim_{BS} X\).

## 5 EQUILIBRIUM MEASURES AND GIBBS MEASURES

In this section we consider the problem of the existence of equilibrium measures for the induced pressure. We also study the relation between Gibbs measures and equilibrium measures for the induced pressure in the particular case of symbolic dynamics.

**Definition 5.1.** Let \((X, f)\) be a TDS and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). A member \(\mu\) of \(M(X, f)\) is called an equilibrium measure for \(\psi\) and \(\varphi\) if \(P_\psi(\varphi) = h_\mu(f) + \int \varphi d\mu - \int \psi d\mu\). We will write \(M_{\psi, \varphi}(X, f)\) for the collection of all equilibrium measures for \(\psi\) and \(\varphi\).

**Definition 5.2.** Let \((X, f)\) be a TDS. Then \(f\) is said to be positively expansive if there exists \(\epsilon > 0\) such that \(x = y\) whenever \(d(f^n(x), f^n(y)) < \epsilon\) for every \(n \in \mathbb{N} \cup \{0\}\).

The entropy map of a TDS is the map \(\mu \mapsto h_\mu(f)\), which is defined on \(M(X, f)\) and has values in \([0, \infty]\). The entropy map \(\mu \mapsto h_\mu(f)\) is called upper semi-continuous if given a measure \(\mu \in M(X, f)\) and \(\delta > 0\), we have \(h_\nu(f) < h_\mu(f) + \delta\) for any measure \(\nu \in M(X, f)\) in some open neighborhood of \(\mu\). Now we show that any expansive map has equilibrium measures.

**Proposition 5.1.** Let \((X, f)\) be a TDS and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). Then

(i) If \(f\) is a positively expansive map, then \(M_{\psi, \varphi}(X, f)\) is compact and non-empty.

(ii) If \(\varphi, \phi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\) and if there exists a \(c \in \mathbb{R}\) such that
\[\varphi - \phi - c \int \psi d\mu \in \{\tau \circ f - \tau : \tau \in C(X, \mathbb{R})\}\]
for each \(\mu \in M(X, f)\), then \(M_{\psi, \varphi}(X, f) = M_{\psi, \phi}(X, f)\).

**Proof.** (i) For a positively expansive map \(f\), it follows from the proof in [1,9] that the map \(\mu \mapsto h_\mu(f)\) is upper semi-continuous. Then \(\mu \mapsto \frac{h_\mu(f)}{\int \psi d\mu}\) is upper semi-continuous. Since the map
\[\mu \mapsto \int \frac{\varphi}{\psi d\mu} d\mu\]
is continuous for each \( \varphi \in C(X, f) \), then

\[
\mu \mapsto \frac{h_\mu(f) + \int \varphi \, d\mu}{\int \psi \, d\mu}
\]

is upper semi-continuous. Since an upper semi-continuous map has a maximum on any compact set, it follows from Theorem 1.1 that \( M_{\psi, \varphi}(X, f) \neq \emptyset \). The upper semi-continuity also implies \( M_{\psi, \varphi}(X, f) \) is compact because if \( \mu_n \in M_{\psi, \varphi}(X, f) \) and \( \mu_n \to \mu \in M(X, f) \), then

\[
\frac{h_\mu(f) + \int \varphi \, d\mu}{\int \psi \, d\mu} \geq \limsup_{n \to \infty} \frac{h_{\mu_n}(f) + \int \varphi \, d\mu_n}{\int \psi \, d\mu_n} = P_\psi(\varphi),
\]

so \( \mu \in M_{\psi, \varphi}(X, f) \).

(ii) Note that for each \( \mu \in M(X, f) \)

\[
\frac{h_\mu(f) + \int \varphi \, d\mu}{\int \psi \, d\mu} = \frac{h_\mu(f) + \int \phi \, d\mu}{\int \psi \, d\mu},
\]

therefore \( P_\psi(\varphi) = P_\psi(\phi) + c \), hence \( M_{\psi, \varphi}(X, f) = M_{\psi, \phi}(X, f) \).

Next, we consider symbolic dynamics. Let \((\Sigma_A, \sigma)\) be a one-sided topological Markov shift (TMS, for short) over a finite set \( S = \{1, 2, \ldots, k\} \). This means that there exists a matrix \( A = (t_{ij})_{k \times k} \) of zeros and ones (with no row or column made entirely of zeros) such that

\[
\Sigma_A = \{ \omega = (i_1, i_2, \ldots) \in S^\mathbb{N} : t_{i_j i_{j+1}} = 1 \text{ for every } j \in \mathbb{N} \}.
\]

The shift map \( \sigma : \Sigma_A \to \Sigma_A \) is defined by \((i_1, i_2, i_3 \ldots) \mapsto (i_2, i_3, \ldots) \). We call \( C_{i_1 \ldots i_n} = \{(j_1, j_2, \ldots) \in \Sigma_A : j_l = i_l \text{ for } l = 1, \ldots, n\} \) the cylindrical set of \( \omega \). We equip \( \Sigma_A \) with the topology generated by the cylindrical sets. The topology of a TMS is metrizable and may be given by the metric \( d_\alpha(\omega, \omega') = e^{-n|\omega \wedge \omega'|} \), \( \alpha > 0 \), where \( \omega \wedge \omega' \) denotes the longest common initial block of \( \omega, \omega' \in \Sigma_A \). The shift map \( \sigma \) is continuous with respect to this metric. A TMS \((\Sigma_A, \sigma)\) is called a topologically mixing TMS if for every \( a, b \in S \), there exists an \( N_{ab} \in \mathbb{N} \) such that for every \( n > N_{ab} \), we have \( C_a \cap \sigma^{-n}C_b \).

**Definition 5.3.** Let \((\Sigma_A, \sigma)\) be a TMS and \( \varphi, \psi \in C(\Sigma_A, \mathbb{R}) \) with \( \psi > 0 \). We say that a probability measure \( \mu \) in \( \Sigma_A \) is a Gibbs measure for \( \psi \) and \( \varphi \) if there exists a \( K > 1 \) such that

\[
K^{-1} \leq \frac{\mu(C_{i_1 \ldots i_n})}{\exp[-(S_n \psi)(\omega)P_\psi(\varphi) + (S_n \varphi)(\omega)]} \leq K
\]

for each \((i_1, i_2, \ldots) \in \Sigma_A, n \in \mathbb{N} \) and \( \omega \in C_{i_1 \ldots i_n} \).

We show that \( \sigma \)-invariant Gibbs measures are equilibrium measures. Making a similar proof as in [1, Theorem 3.4.2], we can obtain the following statement:
Proposition 5.2. If a probability measure \( \mu \) in \((\Sigma_A, \sigma)\) is a \( \sigma \)-invariant Gibbs measure for \( \varphi \) and \( \psi \), then it is also an equilibrium measure for \( \psi \) and \( \varphi \).

Now we establish the existence of Gibbs measures.

Proposition 5.3. Let \((\Sigma_A, \sigma)\) be a topologically mixing TMS. Suppose that \( \varphi \) and \( \psi \) are Hölder continuous functions and \( \psi > 0 \). Then there exists at least one \( \sigma \)-invariant Gibbs measure for \( \psi \) and \( \varphi \).

Proof. By Corollary 3.3 we have

\[ P(\varphi - P_\psi(\varphi)\psi) = 0. \]

As \( \varphi - P_\psi(\varphi)\psi \) is Hölder continuous, it follows from [1, Theorem 3.4.4] that there exists a \( K > 1 \) such that

\[ K^{-1} \leq \exp[\frac{\mu(C_{i_1...i_n})}{n} - (S_n\psi)(\omega) + (S_n\varphi)(\omega)] \leq K. \]

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