Well-posedness of initial-boundary value problem for
time-fractional diffusion-wave equation
with time-dependent coefficients

Xinchu HUANG\textsuperscript{1,*}, Masahiro YAMAMOTO\textsuperscript{2}

\textbf{Abstract} We consider the well-posedness of the initial-boundary value problem for a time-fractional
partial differential equation with the fractional order lying in (1,2]. First, we show some results on the
unique existence of solution to time-fractional ordinary differential equations. Then the unique existence
of weak and strong solutions to a time-fractional partial differential equation is derived by using the
Galerkin method.

\textbf{Keywords} time-fractional diffusion/wave equation, well-posedness, Fredholm
alternative, Galerkin approximation, regularity estimate

1 Introduction and main results

1.1 Settings and governing equation

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary (e.g., of $C^2$-class). Put $Q := \Omega \times (0, T)$ and
$\Sigma := \partial \Omega \times (0, T)$, where $T > 0$ is arbitrarily fixed. In what follows, $A$ is defined by

$$A(x,t)u := - \sum_{i,j=1}^{n} \partial_i (a_{ij}(x,t)\partial_j u) + \sum_{j=1}^{n} b_j(x,t)\partial_j u + c(x,t)u, \quad (x,t) \in Q$$

where $b_j, c \in W^{1,\infty}(0, T; L^\infty(\Omega))$, $1 \leq j \leq n$, $a_{ij} = a_{ji} \in C^1(\overline{Q})$, $1 \leq i, j \leq n$, satisfy

$$\sigma_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i \xi_j \leq \sigma_1 |\xi|^2, \quad (x,t) \in \overline{Q}, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \quad \text{(1)}$$

\begin{flushleft}
\footnotesize
\textsuperscript{1}Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
JSPS Postdoctoral Fellowships for research in Japan. E-mail: huangxc@ms.u-tokyo.ac.jp

\textsuperscript{2}Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan;
Honorary Member of Academy of Romanian Scientists, Ilfov, nr. 3, Bucuresti, Romania;
Correspondence member of Accademia Peloritana dei Pericolanti, Palazzo Universit`a, Piazza S. Pugliatti 1 98122 Messina, Italy. E-mail: myama@ms.u-tokyo.ac.jp

\textsuperscript{*}Corresponding author: huangxc@ms.u-tokyo.ac.jp
\end{flushleft}
with some constants $\sigma_0, \sigma_1 > 0$.

An initial-boundary value problem is fundamental for describing a related phenomenon and the unique existence of solution to such an problem is definitely a primary mathematical subject. More precisely, in this article, for $\alpha \in (0, 2]$, we consider the following initial-boundary value problem for the time-fractional diffusion/wave equation:

$$
\begin{aligned}
\partial^\alpha_t u(x, t) + A(x, t)u(x, t) &= F(x, t), \quad (x, t) \in Q, \\
\partial_t u(x, 0) &= u_1(x), \quad x \in \Omega, \\
u(x, t) &= 0, \quad (x, t) \in \Sigma, \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
$$

where $\partial^\alpha_t$ denotes the (Caputo) fractional derivative formally given by

$$
\partial^\alpha_t g(t) := \int_0^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \partial^m_\tau g(\tau)d\tau
$$

for $m - 1 < \alpha < m, m \in \mathbb{N}$. Throughout this paper, $\Gamma(\cdot)$ denotes the gamma function. In the following section, the fractional derivative will be defined in some functional space so that one can consider a weak and a strong solutions in some fractional Sobolev space.

From the mathematical viewpoint, the time-fractional diffusion/wave equation (2) is a generalization of the classical diffusion equation or the wave equation since one considers an integro-differential operator of real order $\alpha \in (0, 2]$ and the solution to (2) corresponds to the solutions of the diffusion equation as $\alpha \to 1$ and the wave equation as $\alpha \to 2$, respectively. From the practical viewpoint, the time-fractional diffusion/wave equation is regarded as a feasible candidate of modeling the anomalous diffusion in porous media (see e.g., Nigmatullin [13] for $0 < \alpha < 1$) or the mechanical diffusive waves in viscoelastic media which admit a power-law creep (see e.g., Mainardi [12] for $1 < \alpha < 2$). Actually with the time-fractional derivatives, one can expect some memory effects of power-law type in the diffusion or the wave phenomena.

Owing to the applications in engineering and other applied sciences, initial-boundary value problems for the time-fractional partial differential equations have been intensively investigated in the last few decades. For special cases of time-independent coefficients, there are many existing results. For example, one can readily derive the solution formula to the time-fractional diffusion/wave equation (2) in terms of the Fourier method and the so-called Mittag-Leffler functions (see [15]). For some generalizations of the equation (2) including the discussions on the multi-term and the distributed-order time-fractional diffusion equations with time-independent coefficients, we refer to e.g., [3, 8, 9, 10, 11].

On the other hand, there are very few works on the time-fractional partial differential equations with time-dependent coefficients, where the available methods are limited. For example, one can no longer apply the methods of the Laplace transform and the eigenfunction expansion to obtain an explicit representation of the
solution. If the principal coefficients $a_{ij}$ in the elliptic part $A$ do not depend on the time variable, then one can regard the lower order terms with time-dependent coefficients as a new source term and apply the fixed-point theorem to overcome the difficulty (e.g., [4]). However, this is impossible for general $a_{ij} = a_{ij}(x, t)$. In order to deal with the case of time-dependent coefficients, Zacher [17] and Kubica, Ryszewska and Yamamoto [7] applied the Galerkin approximation method to prove the unique existence of solution to the initial-boundary value problem (2) for $0 < \alpha < 1$.

However, to the authors' best knowledge, there are no publications for the well-posedness for the initial-boundary value problem (2) with $1 < \alpha < 2$. Thus the main purpose of this article is to fill the lack of such a fundamental result. More precisely, as the main results of this article, we prove the unique existence and regularity estimates of solution to the initial-boundary value problem for the time-fractional diffusion-wave equation in the case of $1 < \alpha < 2$ with time-dependent coefficients, which are stated in Subsection 1.4.

Before the statement, we formulate the fractional derivatives in Subsections 1.2 and 1.3.

1.2 Fractional Sobolev spaces

For the statement of our main results, following [7], we introduce notations, operators, and function spaces. Here and henceforth, by $L^p(X)$ and $W^{k,p}(X)$, $k \in \mathbb{N}$, $p \in \mathbb{N} \cup \{\infty\}$ we mean the usual Lebesgue space and the $k$-th order Sobolev space of $L^p$ functions on $X \subset (0, T)$ (or $X \subset \Omega$), respectively, and in particular, for $p = 2$ we denote $W^{k,2}(X)$ by $H^k(X)$ (see e.g., [1, 5]).

Moreover, by $H^\gamma(0, T)$, $0 < \gamma < 1$, we denote the fractional Sobolev-type spaces called Sobolev-Slobodecki spaces (e.g., [1, 5]) and we define the norm in $H^\gamma(0, T)$ by

$$
\|u\|_{H^\gamma(0,T)} := \left( \|u\|_{L^2(0,T)}^2 + \int_0^T \int_0^T \frac{|u(t) - u(s)|^2}{|t-s|^{1+2\gamma}} \, dt \, ds \right)^{\frac{1}{2}}.
$$

For $\gamma > 1$ and $\gamma \notin \mathbb{N}$, we write $\gamma = \ell + \theta$ where $\ell \in \mathbb{N}$ and $\theta \in (0, 1)$. Then the fractional Sobolev-type space is defined by

$$
H^\gamma(0,T) := \left\{ u \in H^\ell(0,T); \frac{d^\ell}{dt^\ell} u \in H^\theta(0,T) \right\}
$$

and it is known that this is a Banach space with respect to the norm

$$
\|u\|_{H^\gamma(0,T)} := \left( \|u\|_{H^\ell(0,T)}^2 + \left\| \frac{d^\ell}{dt^\ell} u \right\|_{H^\theta(0,T)}^2 \right)^{\frac{1}{2}}.
$$

We set $H^0(0,T) := L^2(0,T)$ and $\frac{du}{dt} := u$.

Next we introduce a subspace of $H^\gamma(0,T)$, which was considered in [4, 7]. For $0 < \gamma \leq 1$, we define the
Banach space:

\[ H_\gamma(0,T) := \begin{cases} 
\{ u \in H^\gamma(0,T); u(0) = 0 \}, & \frac{1}{2} < \gamma \leq 1, \\
\{ u \in H^{\frac{1}{2}}(0,T); \int_0^T t^{-1} |u(t)|^2 \, dt < \infty \}, & \gamma = \frac{1}{2}, \\
H^\gamma(0,T), & 0 < \gamma < \frac{1}{2}, 
\end{cases} \]

with the following norm

\[ \|u\|_{H_\gamma(0,T)} := \begin{cases} 
\|u\|_{H^\gamma(0,T)}, & 0 < \gamma \leq 1, \gamma \neq \frac{1}{2}, \\
\left( \|u\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T t^{-1} |u(t)|^2 \, dt \right)^{\frac{1}{2}}, & \gamma = \frac{1}{2}. 
\end{cases} \]

For any \( \ell \in \mathbb{N} \), we define

\[ H_\ell(0,T) := \left\{ u \in H^\ell(0,T); \frac{d^k}{dt^k} u(0) = 0, k = 0, 1, \ldots, \ell - 1 \right\}. \]

Then for \( \gamma := \ell + \theta \) with \( \ell \in \mathbb{N} \) and \( 0 \leq \theta < 1 \), we define the Banach space:

\[ H_{\ell+\theta}(0,T) := \left\{ u \in H_\ell(0,T); \frac{d^\ell}{dt^\ell} u \in H_\theta(0,T) \right\} \]

with the following norm

\[ \|u\|_{H_{\ell+\theta}(0,T)} := \left( \|u\|^2_{H^\ell(0,T)} + \left\| \frac{d^\ell}{dt^\ell} u \right\|^2_{H_\theta(0,T)} \right)^{\frac{1}{2}}. \]

By setting \( H_0(0,T) := H^0(0,T) = L^2(0,T) \), we defined the Banach space \( H_\gamma(0,T) \) for any \( \gamma \geq 0 \).

### 1.3 Definition of time-fractional derivative on \( H_\gamma(0,T) \)

For any function \( v \in L^2(0,T) \), we introduce the Riemann-Liouville fractional integral operator \( J^\gamma \) given by

\[ J^\gamma v(t) := \begin{cases} 
\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} v(s) \, ds, & \gamma > 0, \\
v(t), & \gamma = 0. 
\end{cases} \]

Then we have the following theorem showing that \( J^\gamma L^2(0,T) = H_\gamma(0,T) \).

**Theorem 1.1.** Let \( \gamma > 0 \).

(i) \( J^\gamma : L^2(0,T) \rightarrow H_\gamma(0,T) \) is injective and surjective.

(ii) For all \( v \in L^2(0,T) \), there exist constants \( C_1, C_2 > 0 \), depending on \( \gamma, T \), such that

\[ C_2 \|J^\gamma v\|_{H_\gamma(0,T)} \leq \|v\|_{L^2(0,T)} \leq C_1 \|J^\gamma v\|_{H_\gamma(0,T)}. \]

For \( 0 < \gamma < 1 \), one can refer to e.g., [7], and then Theorem 1.1 can be easily verified by the definition of \( H_\gamma(0,T) \) for \( \gamma \geq 1 \). The proof is given as a special case of Theorem 2.2 in Section 2.
In terms of Theorem 1.1, we can define
\[ \partial_t^\gamma := J^{-\gamma} \equiv (J^\gamma)^{-1} : H_\gamma(0,T) \to L^2(0,T) \]
and we have the following corollary:

**Corollary 1.1.** Let \( \gamma > 0 \).

(i) \( \partial_t^\gamma J^\gamma v = v, \quad v \in L^2(0,T) \) and \( J^\gamma \partial_t^\gamma u = u, \quad u \in H_\gamma(0,T) \).

(ii) For all \( u \in H_\gamma(0,T) \), there exist constants \( C_1, C_2 > 0 \), depending on \( \gamma, T \) such that
\[ C_2 \| \partial_t^\gamma u \|_{L^2(0,T)} \leq \| u \|_{H_\gamma(0,T)} \leq C_1 \| \partial_t^\gamma u \|_{L^2(0,T)}. \]

**1.4 Statement of main results**

Now we can formulate an initial-boundary value problem for a time-fractional diffusion-wave equation (1 < \( \alpha \) ≤ 2):
\[
\begin{align*}
\partial_t^\alpha (u - a_0 - ta_1) &= -A(x,t)u(x,t) + F(x,t), \quad (x,t) \in \Omega \times (0,T), \\
u(\cdot, t) &\in H^1_0(\Omega), \quad t \in (0,T), \\
u(x,0) &= a_0 \quad \text{and} \quad \partial_t u(\cdot,0) = a_1 \quad \text{in} \quad \Omega,
\end{align*}
\]
where \( H_\alpha(0,T) \) and \( \partial_t^\alpha \) denotes the fractional Sobolev space and the fractional derivative respectively which are introduced in the above subsections. Here we impose the initial conditions by \( u - a_0 - ta_1 \in H_\alpha(0,T; H^{-1}(\Omega)) \) instead of \( u(\cdot,0) = a_0 \) and \( \partial_t u(\cdot,0) = a_1 \) in \( \Omega \), but one will find by the definition that they are equivalent as long as the solution \( u \) is sufficiently regular.

Now we are ready to state the first of the main theorems, which are concerned with a weak solution.

**Theorem 1.2.** Let \( F \in L^2(0,T; L^2(\Omega)) \), \( a_0 \in H^1_0(\Omega) \) and \( a_1 \in L^2(\Omega) \). Then there exists a unique solution \( u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1_0(\Omega)) \) to (3) satisfying \( u - a_0 - ta_1 \in H_\alpha(0,T; H^{-1}(\Omega)) \). Moreover, there exists a constant \( C = C(\Omega,T,\alpha,\sigma_0,\sigma_1) > 0 \) such that
\[ \| u - a_0 - ta_1 \|_{H_\alpha(0,T; H^{-1}(\Omega))} + \| u \|_{L^\infty(0,T; H^1_0(\Omega))} + \| u \|_{H^1(0,T; L^2(\Omega))} \leq C \left( \| a_0 \|_{H^1_0(\Omega)} + \| a_1 \|_{L^2(\Omega)} + \| F \|_{L^2(0,T; L^2(\Omega))} \right). \]

If the regularity of \( a_0, a_1 \) and \( F \) is improved, then we can gain a strong solution as follows.

**Theorem 1.3.** Let \( F - Aa_0 \in H^1_0(0,T; L^2(\Omega)) \), \( a_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( a_1 \in H^1_0(\Omega) \). Then the solution to (3) satisfies \( u \in H^1(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \) and \( \partial_t^\alpha (u - a_0 - ta_1) \in L^\infty(0,T; L^2(\Omega)) \) and \( \partial_t^\alpha (u - a_0 - ta_1) \in L^\infty(0,T; L^2(\Omega)) \). Moreover, there exists a constant \( C = C(\Omega,T,\alpha,\sigma_0,\sigma_1) > 0 \) such that
\[ \| \partial_t^\alpha (u - a_0 - ta_1) \|_{L^\infty(0,T; L^2(\Omega))} + \| u \|_{H^1(0,T; H^1_0(\Omega))} + \| u \|_{L^\infty(0,T; H^2(\Omega))} \leq C \left( \| a_0 \|_{H^2(\Omega)} + \| a_1 \|_{H^1_0(\Omega)} + \| F \|_{H^1(0,T; L^2(\Omega))} \right). \]
Remark 1.1. In Theorem 1.3, we need the assumption $F - Aa_0 \in H^1_1(0,T;L^2(\Omega))$ for deriving the strong solution. According to the definition of $H^1_1(0,T)$, this assumption is equivalent to the following two assumptions:

(i) $F \in H^1_1(0,T;L^2(\Omega))$, (ii) $F(x,0) = A(x,0)a_0(x)$, for $x \in \Omega$.

The former one (i) is a regularity assumption while the latter one (ii) can be regarded as a compatible condition between the source term $F$ and the initial condition $a_0$.

The remaining part of this paper is organized as follows. In Section 2, we present some properties of the fractional derivative which are very useful in the following sections. In order to prove the main results, we discuss the unique existence of solution to a system of time-fractional ordinary differential equations in Section 3. After that, Sections 4 and 5 are devoted to the proofs of Theorems 1.2 and 1.3 respectively. For the completeness, Appendix provides two important inequalities which are needed for the proofs of the main results.

2 Properties of fractional derivatives

In view of the formulation of $\partial_t^\alpha$, we can state several properties.

First we state a more detailed characterization of the Riemann-Liouville fractional integral operator with the domain $H^\beta(0,T) \subset L^2(0,T), \beta \geq 0$.

Theorem 2.1. Let $\alpha > 0, \beta \geq 0$.

(i) $J_\alpha : H^\beta(0,T) \longrightarrow H^{\alpha + \beta}(0,T)$ is injective and surjective and

\[ \| J_\alpha u \|_{H^{\alpha + \beta}(0,T)} \sim \| u \|_{H^\beta(0,T)}, \quad u \in H^\beta(0,T). \]

(ii) $\partial_t^\alpha : H^{\alpha + \beta}(0,T) \longrightarrow H^\beta(0,T)$ is injective and surjective and

\[ \| \partial_t^\alpha u \|_{H^\beta(0,T)} \sim \| u \|_{H^{\alpha + \beta}(0,T)}, \quad u \in H^{\alpha + \beta}(0,T). \]

Here and henceforth, we denote a norm equivalence by $\sim$.

Proof. Step 1: We prove a special case of $\beta = 0$, that is equivalent to Theorem 1.1.

The case of $0 < \alpha < 1$ has already been proved in Theorem 2.1 of [7] and the case of $\alpha = 1$ is trivial by the definition of $H_1(0,T)$. Here we prove the case of $\alpha > 1$ by the result of $0 < \alpha \leq 1$. For $\alpha > 1$, we let $\alpha = m + \theta$ where $m \in \mathbb{N}$ and $\theta \in (0,1]$. In other words, we take $m = \lfloor \alpha \rfloor - 1 \geq 1$ where $\lfloor \cdot \rfloor$ denotes the ceiling function and $\theta = \alpha - m$.

First we prove $J_\alpha u \in H_\alpha(0,T)$ for all $u \in L^2(0,T)$. Since $J_\alpha u = J^m J^\theta u$ for $u \in L^2(0,T)$, we have

\[ \frac{d^j}{dt^j} u = J^{m-j} J^\theta u = J^{m-j+\theta} u, \quad j = 0, \ldots, m. \]
Then by Young’s convolution inequality we see that \( \frac{d^j}{dt^j} u \in L^2(0,T) \) for \( j = 0, \ldots, m \). In addition, for \( j = 0, \ldots, m - 1 \), \( \frac{d^j}{dt^j} u = J^j \cdot J^{m-1-j+\theta} u \), which indicates \( \frac{d^j}{dt^j} u \in H_1(0,T) \) since \( J^{m-1-j+\theta} u \in L^2(0,T) \). Hence we have \( \frac{d^j}{dt^j} u(0) = 0 \) for \( j = 0, \ldots, m - 1 \). This implies \( u \in H_m(0,T) \).

Moreover, we take \( j = m \) and then we obtain \( \frac{d^m}{dt^m} u = J^\theta u \in H_\theta(0,T) \) since \( 0 < \theta \leq 1 \). Together with \( u \in H_m(0,T) \), we proved \( J^\alpha u \in H_\alpha(0,T) \).

Then we prove that \( J^\alpha \) is injective and surjective. Assume \( u_1, u_2 \in L^2(0,T) \) such that \( J^\alpha u_1 = J^\alpha u_2 \). By taking \( m \) times derivative, we have \( J^\theta u_1 = J^\theta u_2 \), which implies \( u_1 = u_2 \).

On the other hand, for any \( v \in H_\alpha(0,T) \), we have \( v \in H_m(0,T) \) and \( \frac{d^m}{dt^m} v \in H_\theta(0,T) \), which yields \( \frac{d^j}{dt^j} v(0) = 0 \) for \( j = 0, \ldots, m - 1 \) and \( \frac{d^m}{dt^m} v = J^\theta u \) for some \( u \in L^2(0,T) \). Thus, we apply \( J^m \) to \( \frac{d^m}{dt^m} v = J^\theta u \) and obtain \( v = J^\alpha u \). This proves the surjectivity.

Finally, we prove the equivalence of the norms. By definition, we have

\[
||J^\alpha u||_{H_\alpha(0,T)}^2 = ||J^\alpha u||_{H^m(0,T)}^2 + \left|\left| \frac{d^m}{dt^m} J^\alpha u \right|\right|_{H_\theta(0,T)}^2 = \sum_{j=0}^m \left|\left| J^{m-j+\theta} u \right|\right|_{L^2(0,T)}^2 + ||J^\theta u||_{H_\theta(0,T)}^2.
\]

By Young’s convolution inequality and the norm equivalence of \( ||J^\theta u||_{H_\theta(0,T)} \) and \( ||u||_{L^2(0,T)} \) for \( 0 < \theta \leq 1 \), we obtain

\[
||J^\alpha u||_{H_\alpha(0,T)}^2 \leq C ||u||_{L^2(0,T)}^2.
\]

On the other hand, by the norm equivalence of \( ||J^\theta u||_{H_\theta(0,T)} \) and \( ||u||_{L^2(0,T)} \) for \( 0 < \theta \leq 1 \), we have

\[
||J^\alpha u||_{H_\alpha(0,T)}^2 \geq ||J^\theta u||_{H_\theta(0,T)}^2 \geq \tilde{C} ||u||_{L^2(0,T)}^2.
\]

Step 2: We prove the general case of \( \beta > 0 \).

Since the second statement (ii) of Theorem 2.1 is equivalent to the first one, we only need to prove (i).

The injectivity follows immediately from the injectivity of \( J^\alpha \) proved above. Now for any \( v \in H_{\alpha+\beta}(0,T) \), we need to find \( u \in H_\beta(0,T) \) such that \( J^\alpha u = v \). Indeed, by step 1, there exists \( w \in L^2(0,T) \) such that \( v = J^{\alpha+\beta} w \). Since \( J^{\alpha+\beta} w = J^\alpha J^\beta w \), we choose \( u = J^\beta w \in H_\beta(0,T) \).

For the norm equivalence, by Corollary 2.1 we have

\[
||J^\alpha u||_{H_{\alpha+\beta}(0,T)} \sim ||\partial_t^{\alpha+\beta} J^\alpha u||_{L^2(0,T)}, \quad ||u||_{H_\beta(0,T)} \sim ||\partial_t^\beta u||_{L^2(0,T)}.
\]

Then we complete the proof by noting the following equalities:

\[
\partial_t^{\alpha+\beta} J^\alpha u = \partial_t^{\alpha+\beta} J^\alpha J^\beta \partial_t^\beta u = \partial_t^{\alpha+\beta} J^{\alpha+\beta} \partial_t^\beta u = \partial_t^\beta u, \quad u \in H_\beta(0,T).
\]

Then we have the following properties as a corollary.
Corollary 2.1. Let $\alpha, \beta > 0$.

(i) $\partial_t^{\alpha+\beta} u = \partial_t^\alpha \partial_t^\beta u = \partial_t^\beta \partial_t^\alpha u, \quad u \in H_{\alpha+\beta}(0, T)$.

(ii) Let $\alpha \geq \beta$. Then

\[
J^{\alpha-\beta} v = \partial_t^\beta J^\alpha v, \quad v \in L^2(0, T) \quad \text{and} \quad J^{\alpha-\beta} v = \partial_t^\beta J^\alpha v = J^\alpha \partial_t^\beta v, \quad v \in H_\beta(0, T).
\]

(iii) Let $\alpha < \beta$. Then

\[
\partial_t^{\beta-\alpha} v = \partial_t^\beta J^\alpha v, \quad v \in H_{\beta-\alpha}(0, T) \quad \text{and} \quad \partial_t^{\beta-\alpha} v = \partial_t^\beta J^\alpha v = J^\alpha \partial_t^\beta v, \quad v \in H_\beta(0, T).
\]

Moreover, we have the following proposition.

Proposition 2.1. Let $\gamma = m - 1 + \theta$ with $m \in \mathbb{N}$ and $0 < \theta \leq 1$. Then

\[
\partial_t^\gamma u = \frac{d^m}{dt^m} J^{1-\theta} u, \quad u \in H_\gamma(0, T).
\]

In particular, let $\gamma \in \mathbb{N}$. Then

\[
\partial_t^\gamma u = \frac{d^\gamma}{dt^\gamma} u, \quad u \in H_\gamma(0, T).
\]

Proof. By Theorem 1.1 for any $u \in H_\gamma(0, T)$, there exists $w \in L^2(0, T)$ such that $u = J^\gamma w$. Then by the facts that $J^\alpha J^\beta = J^{\alpha+\beta}$, $J^\alpha = J^{\alpha+\beta}$ for any $\alpha, \beta \geq 0$ and $\frac{d}{dt} J^1 v = v$, we obtain

\[
\frac{d^m}{dt^m} J^{1-\theta} u = \frac{d^m}{dt^m} J^{m-\gamma} u = \frac{d^m}{dt^m} J^{m-\gamma} J^\gamma w = \frac{d^m}{dt^m} J^m w = \frac{d^{m-1}}{dt^{m-1}} J^1 J^{m-1} w = \ldots = w = \partial_t^\gamma u.
\]

\[\square\]

3 System of time-fractional ordinary differential equations

In this section, for the sake of the proofs of Theorems 1.2 and 1.3 we establish the well-posedness of the initial value problem for a system of time-fractional ordinary differential equations with the fractional order $\alpha \in (1, 2]$.

Let $N \in \mathbb{N}$. We discuss the solution $u = (u_1, \ldots, u_N)^T$ to the following initial value problem for a linear system of time-fractional ordinary differential equations:

\[
\begin{aligned}
\partial_t^\alpha (u - a_0 - ta_1) &= P(t)u(t) + F(t), \quad 0 < t < T, \\
u - a_0 - ta_1 &\in (H_\alpha(0, T))^N,
\end{aligned}
\]

where $a_j = (a_{j,1}, \ldots, a_{j,N})^T$, $j = 0, 1$, $F = (f_1, \ldots, f_N)^T$ is given and $P = (p_{ij})_{i,j=1}^N$ is a given $N \times N$ matrix function. We have the following theorem.

Theorem 3.1. Let $a_0, a_1 \in \mathbb{R}^N$, $F \in L^2(0, T)^N$ and $P \in L^\infty(0, T)^{N \times N}$. Then \( (4) \) admits a unique solution $u \in (H^\alpha(0, T))^N$ satisfying

\[
\|u - a_0 - ta_1\|_{(H_\alpha(0, T))^N} + \|u\|_{(H_\alpha(0, T))^N} \leq C \left(\|a_0\|_{\mathbb{R}^N} + \|a_1\|_{\mathbb{R}^N} + \|F\|_{(L^2(0, T))^N} \right)
\]

for some positive constant $C = C(\|P\|_{L^\infty(0, T)^{N \times N}}, N, T, \alpha) > 0$.  

8
Remark 3.1. We can generalize Theorem 3.1 for all $\alpha > 0$. Indeed we consider
\[\partial_t^\alpha \left( u - \sum_{j=0}^{[\alpha]-1} \frac{t^j}{\Gamma(j+1)} a_j \right) = P(t)u(t) + F(t), \quad 0 < t < T,\]
for $a_j \in \mathbb{R}^N$, $j = 0, \ldots, [\alpha] - 1$. Then we can prove the unique existence of solution and a similar estimate as \(\text{(5)}\) in the same way.

Proof. Let
\[v(t) = u(t) - a_0 - t a_1, \quad \tilde{F}(t) = F(t) + P(t)a_0 + tP(t)a_1.\]
Then we find that $\tilde{F} \in (L^2(0, T))^N$ and $v \in (H_\alpha(0, T))^N$ satisfies
\[\partial_t^\alpha v(t) = P(t)v(t) + \tilde{F}(t), \quad 0 < t < T.\]
By Corollary 1.1, we apply $J^\alpha$ on both sides of the above equation and it is sufficient to prove there exists a unique solution to the integral equation:
\[v(t) = J^\alpha(Pv)(t) + J^\alpha \tilde{F}(t), \quad 0 < t < T.\] (6)
Let $K : (L^2(0, T))^N \rightarrow (L^2(0, T))^N$ be defined by
\[Kw := J^\alpha(Pw), \quad w \in (L^2(0, T))^N.\]
According to the assumption that $P \in (L^\infty(0, T))^{N \times N}$, it is clear that $Pw \in (L^2(0, T))^N$. Thus, by Theorem 1.1 we have $J^\alpha(Pw) \in (H_\alpha(0, T))^N$. Since the embedding $(H_\alpha(0, T))^N \rightarrow (L^2(0, T))^N$ is compact, we find that $K$ is a compact operator from $(L^2(0, T))^N$ to itself. By Fredholm alternative, it left to prove that
\[Kw = w \quad \text{implies} \quad w = 0.\]
In fact, we have
\[w(t) = Kw(t) = J^\alpha(Pw)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} P(s)w(s)ds.\]
With the assumption $P \in (L^\infty(0, T))^{N \times N}$, we estimate
\[|w(t)|_{\mathbb{R}^N} \leq \sqrt{N} \frac{\|P\|_{(L^\infty(0, T))^{N \times N}}} {\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |w(s)|_{\mathbb{R}^N} ds \leq C \int_0^t (t-s)^{\alpha-1} |w(s)|_{\mathbb{R}^N} ds.\]
Then we apply the generalized Grönwall’s inequality (e.g., [7] Appendix A) and obtain
\[|w(t)|_{\mathbb{R}^N} = 0, \quad 0 < t < T.\] (7)
Again from Theorem 1.1 we note that $J^\alpha \tilde{F} \in (H_\alpha(0, T))^N \subset (L^2(0, T))^N$. Therefore, by Fredholm alternative, we reach that there exists a unique solution $v \in (L^2(0, T))^N$ to \(\text{(6)}\). Moreover, we have further
\[v = J^\alpha(Pv + \tilde{F}) \in (H_\alpha(0, T))^N.\]
This proves the unique existence of the solution to (4).

Next we show the estimate $\|v\|_{(L^2(0,T))^N}$. According to (3) and the estimation

$$(t-s)^{\alpha-1} = (t-s)(t-s)^{\alpha-2} \leq T(t-s)^{\alpha-2}, \quad 0 \leq s < t \leq T,$$

we have

$$|v(t)|_{\mathbb{R}^N} \leq C \int_0^t (t-s)^{\alpha-1} |v(s)|_{\mathbb{R}^N} ds + C \int_0^t (t-s)^{\alpha-2} \int_0^s (s-\tau)^{\alpha-1} |\tilde{F}(\tau)|_{\mathbb{R}^N} d\tau ds$$

$$\leq C \int_0^t (t-s)^{\alpha-1} |\tilde{F}(s)|_{\mathbb{R}^N} ds + C \int_0^t (t-s)^{\alpha-2} |\tilde{F}(s)|_{\mathbb{R}^N} ds.$$

By the generalized Grönwall’s inequality and Fubini’s theorem, we obtain

$$|v(t)|_{\mathbb{R}^N} \leq C \int_0^t (t-s)^{\alpha-1} |\tilde{F}(s)|_{\mathbb{R}^N} ds + C \int_0^t (t-s)^{\alpha-2} \int_0^t (s-\tau)^{\alpha-1} |\tilde{F}(\tau)|_{\mathbb{R}^N} d\tau ds$$

$$\leq C \int_0^t (t-s)^{\alpha-1} |\tilde{F}(s)|_{\mathbb{R}^N} ds + C \int_0^t (t-\tau)^{2\alpha-2} |\tilde{F}(\tau)|_{\mathbb{R}^N} d\tau.$$

Since $\alpha - 1 > 0$, we take $L^2(0,T)$-norm of the above inequality and by Young’s convolution inequality (e.g., Appendix A), we obtain

$$\|v\|_{(L^2(0,T))^N} \leq C \|\tilde{F}\|_{(L^2(0,T))^N}.$$

Hence together with Corollary 1.1(ii) and (6), this inequality yields

$$\|v\|_{(H^s(0,T))^N} \leq C \|\partial_t^s v\|_{(L^2(0,T))^N} \leq C (\|Pv\|_{(L^2(0,T))^N} + \|\tilde{F}\|_{(L^2(0,T))^N}) \leq C \|\tilde{F}\|_{(L^2(0,T))^N}.$$

Therefore, we prove (5) by combining the above inequality and the following two estimates:

$$\|u\|_{(H^s(0,T))^N} \leq \|v\|_{(H^s(0,T))^N} + \|a_0 + t\beta a_1\|_{(H^s(0,T))^N} \leq C (\|v\|_{(H^s(0,T))^N} + |a_0|_{\mathbb{R}^N} + |a_1|_{\mathbb{R}^N}),$$

$$\|\tilde{F}\|_{(L^2(0,T))^N} \leq \|F\|_{(L^2(0,T))^N} + \|P a_0 + tP a_1\|_{(L^2(0,T))^N} \leq C (\|F\|_{(L^2(0,T))^N} + |a_0|_{\mathbb{R}^N} + |a_1|_{\mathbb{R}^N}).$$

We end up this section by the following result of an improved regularity.

**Theorem 3.2.** For arbitrarily fixed $\beta > 0$. Let $a_0, a_1 \in \mathbb{R}^N$, and $P \in (W^{[\beta],\infty}(0,T))^N \times N$. Moreover, we assume

$$F + P a_0 + tP a_1 \in (H^s(0,T))^N.$$

Then the solution $u$ to (4) satisfies

$$u - a_0 - ta_1 \in (H^s+\beta(0,T))^N.$$

10
Proof. By the proof of Theorem 3.1 the solution to (4) admits

\[ u - a_0 - ta_1 = J^\alpha (Pu + F). \]  

By setting

\[ v(t) = u(t) - a_0 - ta_1, \quad t \in [0, T], \]

we rewrite (5) by

\[ v = J^\alpha (Pv) + J^\alpha (F + Pa_0 + tPa_1). \]

Since \( v \in (H_\alpha(0,T))^N \) and \( F + Pa_0 + tPa_1 \in (H_\beta(0,T))^N \), this equation yields \( v \in (H_{\min(2\alpha,\alpha+\beta)}(0,T))^N \).

Then again by the equation, we have further \( v \in (H_{\min(3\alpha,\alpha+\beta)}(0,T))^N \). We end up with \( v \in (H_{\alpha+\beta}(0,T))^N \) by iterating for finite times.

We finish this section with a special case \( \beta = 1 \) of Theorem 3.2 which will be used in Section 5.

Corollary 3.1. Let \( a_0, a_1 \in \mathbb{R}^N \), and \( P \in (W^{1,\infty}(0,T))^{N \times N} \). Moreover, we assume

\[ F + Pa_0 \in (H_1(0,T))^N \]

Then the solution \( u \) to (4) satisfies

\[ u - a_0 - ta_1 \in (H_{\alpha+1}(0,T))^N. \]

This corollary follows immediately from Theorem 3.2 by noting that \( F + Pa_0 \in (H_1(0,T))^N \) is equivalent to \( F + Pa_0 + tPa_1 \in (H_1(0,T))^N \).

4 Proof of Theorem 1.2

In this section and the next section, we apply the Galerkin approximation to prove Theorems 1.2 and 1.3 respectively.

Proof of Theorem 1.2. We divide the proof into four steps.

Step 1. Construction of the approximate solutions

We introduce \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) as the eigenvalues of \( -\Delta \) with the homogeneous Dirichlet boundary condition. By \( \varphi_k \in H^2(\Omega) \cap H^1_0(\Omega), \quad k \in \mathbb{N}, \) we denote the corresponding eigenfunction of \( -\Delta \) for \( \lambda_k \).

Without loss of generality, we can choose \( \varphi_k \) such that \( \{\varphi_k\}_{k=1}^{\infty} \) forms an orthogonal basis in \( L^2(\Omega) \), that is, we have

\[ (\varphi_k,\varphi_\ell)_{L^2(\Omega)} = \delta_{k\ell}, \quad H^{-1}(\Omega)(\varphi_k,\varphi_\ell)_{H^1_0(\Omega)} = \delta_{k\ell} \]  

(9)
where
\[
\delta_{k\ell} := \begin{cases} 
1, & k = \ell, \\
0, & k \neq \ell 
\end{cases}
\]
denotes the Kronecker delta.

For arbitrarily fixed \(N \in \mathbb{N}\), we seek for the approximate solution:
\[
u_N(x,t) = \sum_{k=1}^{N} p_k^N(t) \varphi_k(x)
\]
satisfying
\[
\begin{align*}
\partial^\alpha_t (u_N - a_{0,N} - ta_{1,N}) &= -A(x,t)u_N(x,t) + F_N(x,t), \quad (x,t) \in \Omega \times (0,T), \\
u_N - a_{0,N} - ta_{1,N} &\in H_\alpha(0,T; L^2(\Omega)),
\end{align*}
\]
where
\[
a_{j,N} = \sum_{k=1}^{N} a^j_k \varphi_k, \quad a^j_k := (a_j, \varphi_k)_{L^2(\Omega)}, \quad j = 0, 1,
\]
\[
F_N(t) = \sum_{k=1}^{N} f_k(t) \varphi_k, \quad f_k(t) := H^{-1}(\Omega) \langle F(t), \varphi_k \rangle_{H^0_\alpha(\Omega)}, \quad 0 < t < T.
\]

For each \(\ell = 1, \ldots, N\), we multiply \(\varphi_\ell\) on both sides of (10) and take integral over \(\Omega\). Then we obtain
\[
\begin{align*}
\partial^\alpha_t (p_\ell^N - a_\ell^0 - ta_\ell^1) &= \sum_{k=1}^{N} p_k^N (-A(t)\varphi_k, \varphi_\ell)_{L^2(\Omega)} + f_\ell(t), \quad t \in (0,T), \\
p_\ell^N - a_\ell^0 - ta_\ell^1 &\in H_\alpha(0,T), \\
\ell &= 1, \ldots, N.
\end{align*}
\]
We denote \(Q(t) = (q_{\ell k}(t))_{\ell,k=1}^{N}\) with
\[
q_{\ell k}(t) := (-A(t)\varphi_k, \varphi_\ell)_{L^2(\Omega)}, \quad 0 < t < T.
\]

By the regularity assumptions on the coefficients of \(A\) and \(F \in L^2(0,T; L^2(\Omega))\), we have
\[
Q \in (L^\infty(0,T))^{N \times N}, \quad \text{and} \quad f = (f_1, \ldots, f_N)^T \in (L^2(0,T))^N.
\]

Then we rewrite the above equations by setting \(a^j = (a^j_1, \ldots, a^j_N)^T\) for \(j = 0, 1\), \(p^N = (p^N_1, \ldots, p^N_N)^T\):
\[
\begin{align*}
\partial^\alpha_t (p^N - a^0 - ta^1) &= Q(t)p^N(t) + f(t), \quad t \in (0,T), \\
p^N - a^0 - ta^1 &\in (H_\alpha(0,T))^N.
\end{align*}
\]

Therefore, by Theorem 4.1 we find a unique solution \(p^N \in (H^\alpha(0,T))^N\) satisfying \(p^N - a^0 - ta^1 \in (H_\alpha(0,T))^N\) and we construct the approximate solution \(u_N\) which is the solution to (10) satisfying
\[
u_N \in H^\alpha(0,T; H^2 \cap H^0_0(\Omega)), \quad u_N - a_{0,N} - ta_{1,N} \in H_\alpha(0,T; H^2 \cap H^0_0(\Omega)).
\]

By varying \(N\) in \(\mathbb{N}\), we obtain the approximate solutions \(\{u_N\}_{N=1}^\infty\).
Step 2. A priori estimate of the approximate solutions

We multiply $\partial_t u_N$ on both sides of the equation \([\text{13}]\) and then integrate over $\Omega$. Moreover, we apply the fractional integral operator $J^{\alpha - 1}$ and obtain

$$
J^{\alpha - 1} \int_{\Omega} \partial_t^\alpha (u_N - a_0, N - ta_1, N) \partial_t (u_N - a_0, N - ta_1, N) dx + J^{\alpha - 1} \int_{\Omega} \partial_t^\alpha (u_N - a_0, N - ta_1, N) a_{1, N} dx
$$

$$
= J^{\alpha - 1} \int_{\Omega} -A u_N \partial_t u_N dx + J^{\alpha - 1} \int_{\Omega} F_N \partial_t u_N dx.
$$

(13)

As we find $u_N - a_0, N - ta_1, N \in H_{\alpha}(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \subset H_1(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, we have by Corollary \([2.1\text{i}]\) and Proposition \([2.1\text{j}]\) that

$$
\partial_t (u_N - a_0, N - ta_1, N) = \partial_t^1 (u_N - a_0, N - ta_1, N)
$$

and

$$
\partial_t^\alpha (u_N - a_0, N - ta_1, N) = \partial_t^{\alpha - 1} \partial_t^1 (u_N - a_0, N - ta_1, N) = \partial_t^{\alpha - 1} \partial_t (u_N - a_0, N - ta_1, N).
$$

Then by setting

$$
v(t) := \partial_t^1 (u_N - a_0, N - ta_1, N) = \partial_t (u_N - a_0, N - ta_1, N) = \partial_t u_N - a_1, N,
$$

which belongs to $H_{\alpha - 1}(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ by Theorem \([2.1\text{j}]\) we employ the coercivity inequality (Lemma \([2.1\text{i}]\)) to estimate the first term on the left-hand side of \([13]\):

$$
J^{\alpha - 1} \int_{\Omega} \partial_t^\alpha (u_N - a_0, N - ta_1, N) \partial_t (u_N - a_0, N - ta_1, N) dx = J^{\alpha - 1} (\partial_t^{\alpha - 1} v(\cdot, t), v(\cdot, t))_{L^2(\Omega)}
$$

$$
\geq \frac{1}{2} \|v(\cdot, t)\|^2_{L^2(\Omega)} = \frac{1}{2} \|\partial_t u_N(\cdot, t) - a_{1, N}\|^2_{L^2(\Omega)}
$$

$$
\geq \frac{1}{4} \|\partial_t u_N(\cdot, t)\|^2_{L^2(\Omega)} - \frac{1}{2} \|a_{1, N}\|^2_{L^2(\Omega)}.
$$

Here in the last line we used the estimate $(A - B)^2 \geq \frac{1}{2} A^2 - B^2$. For the second term on the left-hand side of \([13]\), by Fubini’s theorem and Corollary \([1.1\text{i}]\), we have

$$
J^{\alpha - 1} \int_{\Omega} \partial_t^\alpha (u_N - a_0, N - ta_1, N) a_{1, N} dx = \int_{\Omega} J^{\alpha - 1} \partial_t^{\alpha - 1} v(x, t) a_{1, N} dx
$$

$$
= (\partial_t u_N(\cdot, t), a_{1, N})_{L^2(\Omega)} - \|a_{1, N}\|^2_{L^2(\Omega)}
$$

$$
\geq -\frac{1}{8} \|\partial_t u_N(\cdot, t)\|^2_{L^2(\Omega)} - 3\|a_{1, N}\|^2_{L^2(\Omega)}.
$$

Here in the last line we used the estimate $AB = (\frac{1}{2} A)(2B) \geq -\frac{1}{8} A^2 - 2B^2$.

On the other hand, we estimate the first term on the right-hand side of \([13]\) by integration by parts and we have

$$
J^{\alpha - 1} \int_{\Omega} -A u_N \partial_t u_N dx
$$

$$
= -\frac{1}{4} \int_{\Omega} \sum_{i, j=1}^n a_{ij} \partial_i u_N \partial_j u_N dx + J^{\alpha - 1} \int_{\Omega} \left( \sum_{j=1}^n b_j \partial_j u_N + cu_N \right) \partial_t u_N dx
$$

13
\[-\frac{1}{2}J^{\alpha-1}\partial_t \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t)\partial_t u_N \partial_j u_N \, dx + J^{\alpha-1} \int_{\Omega} \left( \frac{1}{2} \sum_{i,j=1}^{n} \partial_t a_{ij}(t) \partial_i u_N \partial_j u_N + \frac{1}{2} \sum_{j=1}^{n} b_j \partial_t u_N \partial_t u_N + cu_N \partial_t u_N \right) \, dx \]
\[
\leq -\frac{1}{2}J^{\alpha-1}\partial_t \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t)\partial_t u_N \partial_j u_N \, dx + CJ^{\alpha-1}\|\nabla u_N\|_{L^2(\Omega)}^2 + \frac{1}{2}J^{\alpha-1}\|\partial_t u_N\|_{L^2(\Omega)}^2.
\]

Moreover, we have
\[
J^{\alpha-1} \int_{\Omega} F_N \partial_t u_N \, dx \leq \frac{1}{2}J^{\alpha-1}\|F_N\|_{L^2(\Omega)}^2 + \frac{1}{2}J^{\alpha-1}\|\partial_t u_N\|_{L^2(\Omega)}^2.
\]

Therefore, combining the above four estimates and \[14\] yields
\[
\frac{1}{8}\|\partial_t u_N(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2}J^{\alpha-1}\partial_t \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t)\partial_t u_N(t) \partial_j u_N(t) \, dx + \frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(0)\partial_t u_N(0) \partial_j u_N(0) \, dx
\]
\[
\leq \frac{7}{2}a_{1,N}\|\partial_t u_N\|_{L^2(\Omega)}^2 + \frac{1}{2}J^{\alpha-1}\|F_N\|_{L^2(\Omega)}^2 + J^{\alpha-1}\|\partial_t u_N\|_{L^2(\Omega)}^2 + CJ^{\alpha-1}\|\nabla u_N\|_{L^2(\Omega)}^2
\]
for \(0 < t < T\). Next we apply the fractional integral operator \(J^{2-\alpha}\) on the both sides above. By noting
\[
J^{2-\alpha}J^{\alpha-1}v = J^{\alpha-1}J^{2-\alpha}v = J^1v = \int_{0}^{t} v(s) \, ds \quad \text{for} \quad v \in L^2(0, T),
\]
we obtain
\[
\frac{1}{8}J^{2-\alpha}\|\partial_t u_N\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t)\partial_t u_N(t) \partial_j u_N(t) \, dx + \frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(0)\partial_t u_N(0) \partial_j u_N(0) \, dx
\]
\[
\leq \frac{7}{2}\frac{s^{2-\alpha}}{\Gamma(3-\alpha)}\|a_{1,N}\|_{L^2(\Omega)}^2 + \frac{1}{2}J^1\|F_N\|_{L^2(\Omega)}^2 + J^1\|\partial_t u_N\|_{L^2(\Omega)}^2 + CJ^1\|\nabla u_N\|_{L^2(\Omega)}^2
\]
(14)\\nfor \(0 < t < T\). By the assumption on \(a_{ij}\), we have
\[
\frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t)\partial_t u_N(t) \partial_j u_N(t) \, dx \geq \frac{\sigma_0}{2}\|\nabla u_N(\cdot, t)\|_{L^2(\Omega)}^2 \quad \text{and}\\n\frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(0)\partial_t u_N(0) \partial_j u_N(0) \, dx \leq \frac{\sigma_1}{2}\|\nabla u_N(\cdot, 0)\|_{L^2(\Omega)}^2.
\]

Moreover, \(u_N - a_{0,N} - ta_{1,N} \in H_0(\Omega); H^2(\Omega) \cap H_0^1(\Omega) \subset H_1(0, T; H^0_0(\Omega))\) implies \(u_N(0) = a_{0,N}\) and hence
\[
\|\nabla u_N(\cdot, 0)\|_{L^2(\Omega)}^2 \sim \|u_N(\cdot, 0)\|_{H^1_0(\Omega)}^2 = \|a_{0,N}\|_{H^1_0(\Omega)}^2.
\]

Thus, we put the above estimates into (14) and we obtain
\[
J^{2-\alpha}\|\partial_t u_N\|_{L^2(\Omega)}^2 + \|\nabla u_N(\cdot, t)\|_{L^2(\Omega)}^2
\]
\[
\leq C \left( \|a_{1,N}\|_{L^2(\Omega)}^2 + \|a_{0,N}\|_{H^1_0(\Omega)}^2 + J^1\|F_N\|_{L^2(\Omega)}^2 \right) + CJ^1\|\partial_t u_N\|_{L^2(\Omega)}^2 + CJ^1\|\nabla u_N\|_{L^2(\Omega)}^2
\]
(15)\\nfor \(0 < t < T\). Furthermore, we note that \(J^1\|\partial_t u_N\|_{L^2(\Omega)}^2 = J^{\alpha-1}J^{2-\alpha}\|\partial_t u_N\|_{L^2(\Omega)}^2\) and
\[
J^1\|\nabla u_N\|_{L^2(\Omega)}^2 = \int_{0}^{t} \|\nabla u_N(\cdot, s)\|_{L^2(\Omega)}^2 \, ds = \int_{0}^{t} \|\partial_t u_N(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
14
Then by setting 
\[ w(t) := J^{2-\alpha} \|\partial_t u_N \|_{L^2(\Omega)}^2 + \|\nabla u_N(\cdot, t)\|_{L^2(\Omega)}^2 \geq 0, \]
we rewrite (15) by

\[ w(t) \leq C \left( \|a_0, N\|_{H^1_0(\Omega)}^2 + \|a_1, N\|_{L^2(\Omega)}^2 + \|F_N\|_{L^2(0, t, L^2(\Omega))}^2 \right) + C J^{\alpha-1} w(t), \quad 0 < t < T. \]

Therefore, by the generalized Grönwall’s inequality (e.g., [7, Appendix A]), we obtain

\[ w(t) \leq C \left( \|a_0, N\|_{H^1_0(\Omega)}^2 + \|a_1, N\|_{L^2(\Omega)}^2 + \|F_N\|_{L^2(0, t, L^2(\Omega))}^2 \right), \quad 0 < t < T \]
with a new constant \( C > 0 \). By the definition of \( a_j, N, j = 0, 1 \) and \( F_N \), we see that

\[ \|a_0, N\|_{H^1_0(\Omega)}^2 \leq \|a_0\|_{H^1_0(\Omega)}^2, \quad \|a_1, N\|_{L^2(\Omega)}^2 \leq \|a_1\|_{L^2(\Omega)}^2, \quad \|F_N(\cdot, t)\|_{L^2(\Omega)}^2 \leq \|F(\cdot, t)\|_{L^2(\Omega)}, \quad 0 < t < T, \]
and hence we have

\[ J^{2-\alpha} \|\partial_t u_N \|_{L^2(\Omega)}^2 + \|\nabla u_N(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \left( \|a_0\|_{H^1_0(\Omega)}^2 + \|a_1\|_{L^2(\Omega)}^2 + \|F\|_{L^2(0, t, L^2(\Omega))}^2 \right) \tag{16} \]
for \( 0 < t < T \). By applying \( J^{\alpha-1} \) and taking \( t = T \), we use Young’s convolution inequality (e.g., [7, Appendix A]) to obtain

\[ \|\partial_t u_N\|_{L^2(0, T, L^2(\Omega))} \leq C \left( \|a_0\|_{H^1_0(\Omega)}^2 + \|a_1\|_{L^2(\Omega)}^2 + \|F\|_{L^2(0, T, L^2(\Omega))}^2 \right). \tag{17} \]

Next we estimate \( u_N - a_0, N - ta_1, N \|_{H^\alpha(0, T; H^{-\alpha}(\Omega))} \). For any \( \psi \in H^1_0(\Omega) \), by the equation (11), we have

\[ \|H^{-\alpha}(\partial_t^\alpha (u_N - a_0, N - ta_1, N), \psi)_{H^1_0(\Omega)}\|_{H^{-\alpha}(\Omega)} \leq \|H^{-\alpha}(\partial_{\alpha}^\alpha (u_N - a_0, N - ta_1, N), \psi)_{H^1_0(\Omega)}\|_{H^{-\alpha}(\Omega)} \leq C \left( \|\nabla u_N(\cdot, t)\|_{L^2(\Omega)} + \|F(\cdot, t)\|_{H^{-\alpha}(\Omega)} \right) \|\psi\|_{H^1_0(\Omega)} \]

Therefore, by \( \|F(\cdot, t)\|_{H^{-\alpha}(\Omega)} \leq C \|F(\cdot, t)\|_{L^2(\Omega)} \leq C \|F(\cdot, t)\|_{L^2(\Omega)} \) for \( 0 < t < T \) and the above inequality, we find that \( u_N - a_0, N - ta_1, N \in H^\alpha(0, T; H^{-\alpha}(\Omega)) \) with

\[ \|u_N - a_0, N - ta_1, N\|_{H^\alpha(0, T; H^{-\alpha}(\Omega))} \leq C \|\partial_t^\alpha (u_N - a_0, N - ta_1, N)\|_{L^2(0, T, H^{-\alpha}(\Omega))} \leq C \left( \|\nabla u_N\|_{L^2(0, T, L^2(\Omega))} + \|F\|_{L^2(0, T, L^2(\Omega))} \right). \tag{18} \]

To sum up, we combine the estimates (16)–(18) and obtain the a priori estimate:

\[ \|u_N\|_{L^\infty(0, T, H^1_0(\Omega))} + \|\partial_t u_N\|_{L^2(0, T, L^2(\Omega))} + \|u_N - a_0, N - ta_1, N\|_{H^\alpha(0, T; H^{-\alpha}(\Omega))} \leq C \left( \|a_0\|_{H^1_0(\Omega)} + \|a_1\|_{L^2(\Omega)} + \|F\|_{L^2(0, T, L^2(\Omega))} \right). \tag{19} \]
Step 3. Existence of solution by taking a limit of a weakly convergent subsequence

In the last step, since the constant $C > 0$ is independent of $N$, we derived a uniform estimate of some norms of the approximate solutions. By (19), in particular, we see that

$$
\|u_N\|_{L^2(0,T;H^1_0(\Omega))} + \|u_N\|_{H^1(0,T;L^2(\Omega))} + \|u_N - a_0,N - ta_1,N\|_{H_0(0,T;H^{-1}(\Omega))} 
\leq C \left( \|a_0\|_{H^1_0(\Omega)} + \|a_1\|_{L^2(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))} \right).
$$

Therefore, $\{u_N\}_{N=1}^\infty$ is bounded in $L^2(0,T;H^1_0(\Omega))$ and $H^1(0,T;L^2(\Omega))$, $\{u_N - a_0,N - ta_1,N\}_{N=1}^\infty$ is bounded in $H_0(0,T;H^{-1}(\Omega))$. Then there exist functions $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $\tilde{u} \in \cap H_0(0,T;H^{-1}(\Omega))$ and a subsequence $\{u_{N'}\}$ of $\{u_N\}_{N=1}^\infty$ such that

$$
\begin{align*}
\begin{cases}
  u_{N'} \to u & \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\
  u_{N'} - a_0,N - ta_1,N' \to \tilde{u} & \text{ weakly in } H_0(0,T;H^{-1}(\Omega))
\end{cases}
\end{align*}
$$

as $N' \to \infty$. Since $a_0,N' \to a_0$ and $a_1,N' \to a_1$ strongly in $L^2(\Omega)$ as $N' \to \infty$, we have $u_{N'} - a_0,N' - ta_1,N' \to u - a_0 - ta_1$ weakly in $L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$. By noting that the weak convergence is unique, we obtain $\tilde{u} = u - a_0 - ta_1$. Therefore, we derive

$$
\|u\|_{L^2(0,T;H^1_0(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))} + \|u - a_0 - ta_1\|_{H_0(0,T;H^{-1}(\Omega))} 
\leq \liminf_{N' \to \infty} \left( \|u_{N'}\|_{L^2(0,T;H^1_0(\Omega))} + \|u_{N'}\|_{H^1(0,T;L^2(\Omega))} + \|u_{N'} - a_0,N' - ta_1,N\|_{H_0(0,T;H^{-1}(\Omega))} \right) 
\leq C \left( \|a_0\|_{H^1_0(\Omega)} + \|a_1\|_{L^2(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))} \right).
$$

Similarly, we have $u \in L^p(0,T;H^1_0(\Omega))$ for any $1 < p < \infty$ and

$$
\|u\|_{L^p(0,T;H^1_0(\Omega))} \leq C \left( \|a_0\|_{H^1_0(\Omega)} + \|a_1\|_{L^2(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))} \right).
$$

Since the right-hand side above is independent of $p$, we can prove by contradiction that $u \in L^\infty(0,T;H^1_0(\Omega))$ satisfies

$$
\|u\|_{L^\infty(0,T;H^1_0(\Omega))} \leq C \left( \|a_0\|_{H^1_0(\Omega)} + \|a_1\|_{L^2(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))} \right).
$$

Next we show that $u$ is a weak solution to (3).

Recall that $u$ is called a weak solution to (3) if $u - a_0 - ta_1 \in H_0(0,T;H^{-1}(\Omega))$, $u \in L^2(0,T;H^1_0(\Omega))$ and for any $\psi \in H^1_0(\Omega)$,

$$
H^{-1}(\Omega) (\partial_t^p (u - a_0 - ta_1), \psi)_{H^1_0(\Omega)} = B[u(t),\psi;t] + H^{-1}(\Omega) (F(v,t),\psi)_{H^1_0(\Omega)}
$$

(21)

holds true for a.e. $t \in [0,T]$. Here $B[u,v;t]$ is a bilinear form defined by

$$
B[u,v;t] := \int_{\Omega} \left( - \sum_{i,j=1}^n a_{ij}(t) \partial_i u \partial_j v + \sum_{j=1}^n b_j(t) \partial_j u v + c(t) u v - \sum_{i,j=1}^n a_{ij}(t) \partial_i \psi \partial_j v + \sum_{j=1}^n b_j(t) \partial_j \psi u + c(t) \psi u \right) \, dx.
$$
Recall that $u_N$ solves (10) in $L^2$ sense. Then for any $\tilde{\psi} \in L^2(0,T; H^1_0(\Omega))$, we multiply (10) by $\tilde{\psi}$ and then integrate over $\Omega \times (0,T)$, which gives

$$
\int_0^T H^{-1}(\Omega) \langle \partial_t^\alpha (u_N - a_0,N - ta_{1,N}), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt = \int_0^T B[u_N(t), \tilde{\psi}(t); t] dt + \int_0^T H^{-1}(\Omega) \langle F_N(\cdot, t), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt
$$

where we used integration by parts:

$$
\int_0^T (-Au_N(t), \tilde{\psi}(t))_{L^2(\Omega)} dt = \int_0^T \int_\Omega \left( \sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u_N) + \sum_{j=1}^n b_j \partial_j u_N + cu_N \right) \tilde{\psi} dx dt = \int_0^T B[u_N(t), \tilde{\psi}(t); t] dt.
$$

By taking the subsequence \{u_{N'}\}, we have

$$
\int_0^T H^{-1}(\Omega) \langle \partial_t^\alpha (u_{N'} - a_{0,N'} - ta_{1,N'}), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt = \int_0^T B[u_{N'}(t), \tilde{\psi}(t); t] dt + \int_0^T H^{-1}(\Omega) \langle F_{N'}(\cdot, t), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt.
$$

(22)

According to the weak convergence (20), we have $u_{N'} \to u$ weakly in $L^2(0,T; H^1_0(\Omega))$ and $\partial_t^\alpha (u_{N'} - a_{0,N'} - ta_{1,N'}) \to \partial_t^\alpha (u - a_0 - ta_1)$ weakly in $L^2(0,T; H^{-1}(\Omega))$ as $N' \to \infty$. Moreover, by Lebesgue’s dominated convergence theorem, we have $F_{N'} \to F$ strongly in $L^2(0,T; L^2(\Omega))$ which implies $F_{N'} \to F$ weakly in $L^2(0,T; H^{-1}(\Omega))$. Since $\int_0^T B[\cdot, \tilde{\psi}; t] dt$ is a bounded linear functional on $L^2(0,T; H^1_0(\Omega))$, we let $N' \to \infty$ in (22) and obtain

$$
\int_0^T H^{-1}(\Omega) \langle \partial_t^\alpha (u - a_0 - ta_1), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt = \int_0^T B[u(t), \tilde{\psi}(t); t] dt + \int_0^T H^{-1}(\Omega) \langle F(\cdot, t), \tilde{\psi}(t) \rangle_{H^1_0(\Omega)} dt.
$$

Note that in the above equality $\tilde{\psi}$ can be taken arbitrarily in $L^2(0,T; H^1_0(\Omega))$. Therefore, (21) holds true for any $\psi \in H^1_0(\Omega)$ and a.e. $t \in [0,T]$, and $u$ is a solution to (3).

Step 4. Uniqueness of solution

In the case of $\alpha \leq 1$, as one can see from the proof of Theorem 4.1 in [1], the uniqueness of solution can be easily proved by taking $\psi = u(t) \in H^1_0(\Omega)$ in (24) and applying the generalized Grönwall’s inequality. However, in our case of $1 < \alpha \leq 2$, we cannot do in the same way because $\partial_t u$ does not belong to $L^2(0,T; H^1_0(\Omega))$. Here we modify the idea used in [2, Chapter 7] to show the uniqueness of the solution by proving first the uniqueness of the integral of the solution.

By taking difference of the two possible solutions, it is sufficient to show that the weak solution $\bar{u} \in L^2(0,T; H^1_0(\Omega))$ to

$$
\begin{cases}
\partial_t^\alpha \bar{u}(x,t) = -A(x,t)\bar{u}(x,t), & (x,t) \in \Omega \times (0,T), \\
\bar{u} \in H^\alpha(0,T; H^{-1}(\Omega))
\end{cases}
$$

can only be zero. By the definition of the weak solution, we have $H^{-1}(\Omega) \langle \partial_t^\alpha \bar{u}(s), \psi \rangle_{H^1_0(\Omega)} = B[\bar{u}(s), \psi; s]$ for any $\psi \in H^1_0(\Omega)$ and a.e. $s \in [0,T]$. We integrate with respect to $s$ over $(0,\tau)$ and obtain

$$
\int_0^\tau H^{-1}(\Omega) \langle \partial_t^\alpha \bar{u}(s), \psi \rangle_{H^1_0(\Omega)} ds = \int_0^\tau B[\bar{u}(s), \psi; s] ds
$$

17
for any $\psi \in H^1_0(\Omega)$ and a.e. $\tau \in [0,T]$. Now we take $\psi = \bar{u}(\tau) \in H^1_0(\Omega)$ and for arbitrarily fixed $t \in [0,T]$ we integrate the above equality over $\tau \in (0,t)$, which gives us

\[
\int_0^t \int_0^\tau H^{-1}(\Omega) (\partial^{\alpha}_t \bar{u}(s), \bar{u}(\tau)) H^1_0(\Omega) ds d\tau = \int_0^t \int_0^\tau B[\bar{u}(s), \bar{u}(\tau); s] ds d\tau
\]  

(23)

for any $t \in [0,T]$. Next we estimate the left-hand side and the right-hand side of (23), respectively.

By employing Corollary 2.1 (iii) and Lemma 3.1, we have

\[
\int_0^t \int_0^\tau H^{-1}(\Omega) (\partial^{\alpha}_t \bar{u}(s), \bar{u}(\tau)) H^1_0(\Omega) ds d\tau = \int_0^t \int_0^\tau H^{-1}(\Omega) (J^1 \partial^{\alpha}_t \bar{u}(\tau), \bar{u}(\tau)) H^1_0(\Omega) d\tau
\]

\[
= \int_0^t \int_0^\tau H^{-1}(\Omega) (\partial^{\alpha-1}_t \bar{u}(\tau), \bar{u}(\tau)) H^1_0(\Omega) d\tau
\]

\[
= J^{2-\alpha} J^{\alpha-1} H^{-1}(\Omega) (\bar{u}(t), \bar{u}(t)) H^1_0(\Omega)
\]

\[
\geq \frac{1}{2} J^{2-\alpha} \|\bar{u}(\cdot, t)\|_{L^2(\Omega)}^2.
\]  

By letting $\bar{v} = J^1 \bar{u}$, which implies $\bar{u} = \partial_t \bar{v}$, the above inequality yields

\[
\int_0^t \int_0^\tau H^{-1}(\Omega) (\partial^{\alpha}_t \bar{u}(s), \bar{u}(\tau)) H^1_0(\Omega) ds d\tau \geq \frac{1}{2} J^{2-\alpha} \|\partial_t \bar{v}(\cdot, t)\|_{L^2(\Omega)}^2.
\]  

(24)

On the other hand, we let

\[
\int_0^t \int_0^\tau B[\bar{u}(s), \bar{u}(\tau); s] ds d\tau = - \int_0^t \int_0^\tau \sum_{i,j=1}^n a_{ij}(s) \partial_i \bar{u}(s) \partial_j \bar{u}(\tau) dx ds d\tau
\]

\[
+ \int_0^t \int_0^\tau \left( \sum_{j=1}^n b_j(s) \partial_j \bar{u}(s) + c(s) \bar{u}(s) \right) \bar{u}(\tau) dx ds d\tau
\]

=: I_1(t) + I_2(t).

By Fubini’s theorem and integration by parts, we estimate

\[
I_1(t) = - \int_\Omega \sum_{i,j=1}^n \int_0^t a_{ij}(s) \partial_i \bar{u}(s) \int_s^t \partial_j \bar{u}(\tau) d\tau ds dx
\]

\[
= - \int_\Omega \sum_{i,j=1}^n \int_0^t a_{ij}(s) \partial_i \bar{u}(s) \partial_j \bar{v}(t) dx ds d\tau + \int_\Omega \sum_{i,j=1}^n \int_0^t a_{ij}(s) \partial_i \bar{u}(s) \partial_j \bar{v}(s) ds d\tau dx
\]

\[
= - \int_\Omega \sum_{i,j=1}^n (a_{ij}(t) \partial_i \bar{v}(t) - a_{ij}(0) \partial_i \bar{v}(0)) \partial_j \bar{v}(t) dx d\tau + \int_\Omega \sum_{i,j=1}^n \int_0^t \partial_s a_{ij}(s) \partial_i \bar{v}(s) \partial_j \bar{v}(t) ds d\tau dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^n (a_{ij}(t) \partial_i \bar{v}(t) \partial_j \bar{v}(t) - a_{ij}(0) \partial_i \bar{v}(0) \partial_j \bar{v}(0)) dx - \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \int_0^t \partial_s a_{ij}(s) \partial_i \bar{v}(s) \partial_j \bar{v}(s) ds d\tau dx
\]

\[
= \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(t) \partial_i \bar{v}(t) \partial_j \bar{v}(t) dx + \int_\Omega \sum_{i,j=1}^n \int_0^t \partial_s a_{ij}(s) \partial_i \bar{v}(s) \partial_j \bar{v}(t) ds d\tau dx
\]

\[
- \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \partial_s a_{ij}(s) \partial_i \bar{v}(s) \partial_j \bar{v}(s) ds dx.
\]
Then by the assumption on \( a_{ij} \) and Hölder’s inequality, we have

\[
- \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(t) \partial_{i} \bar{v}(t) \partial_{j} \bar{v}(t) dx \leq -\frac{\sigma_{0}}{2} \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2},
\]

\[
\int_{0}^{t} \left( \sum_{i,j=1}^{n} \int_{\Omega} \partial_{i} a_{ij}(s) \partial_{i} \bar{v}(s) \partial_{j} \bar{v}(t) ds dx \right) \leq \varepsilon \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \frac{C}{\varepsilon} \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds,
\]

\[
- \frac{1}{2} \int_{0}^{t} \sum_{i,j=1}^{n} \int_{0}^{t} \partial_{i} a_{ij}(s) \partial_{i} \bar{v}(s) \partial_{j} \bar{v}(s) ds dx \leq C \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds.
\]

Thus, we obtain

\[
I_{1}(t) \leq -\left( \frac{\sigma_{0}}{2} - \varepsilon \right) \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \left( C + \frac{C}{\varepsilon} \right) \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds.
\]

Similarly, we estimate

\[
I_{2}(t) = \int_{0}^{t} \int_{0}^{t} \left( \sum_{j=1}^{n} b_{j}(s) \partial_{j} u(s) + c(s) \bar{u}(s) \right) \int_{s}^{t} \bar{v}(\tau) d\tau ds dx
\]

\[
= \int_{0}^{t} \int_{0}^{t} \left( \sum_{j=1}^{n} b_{j}(s) \partial_{j} u(s) + c(s) \bar{u}(s) \right) \bar{v}(t) ds dx - \int_{0}^{t} \int_{0}^{t} \left( \sum_{j=1}^{n} b_{j}(s) \partial_{j} u(s) + c(s) \bar{u}(s) \right) \bar{v}(s) ds dx
\]

\[
+ \int_{0}^{t} \int_{0}^{t} \left( \sum_{j=1}^{n} b_{j}(s) \partial_{j} \bar{v}(s) + c(s) \bar{v}(s) \right) \partial_{s} \bar{v}(s) ds dx.
\]

By the assumptions on \( b_{j}, c \), Hölder’s inequality and the Poincaré inequality, we have

\[
I_{2}(t) \leq \varepsilon \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \left( C + \frac{C}{\varepsilon} \right) \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds + C \int_{0}^{t} \| \partial_{s} \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds.
\]

Hence by taking \( \varepsilon = \frac{\sigma_{0}}{8} \), we obtain

\[
\int_{0}^{t} \int_{0}^{T} B[u(s), \bar{u}(\tau); s] ds d\tau \leq -\frac{\sigma_{0}}{4} \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + C \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds + C \int_{0}^{t} \| \partial_{s} \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds. \quad (25)
\]

Inserting (24) and (26) into (25) yields

\[
J^{2-\alpha} \| \partial_{t} \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} \leq C \int_{0}^{t} \| \partial_{t} \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds + C \int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds.
\]

In the same way as we did in Step 3, we have

\[
\int_{0}^{t} \| \partial_{s} \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds = J^{\alpha-1} J^{2-\alpha} \| \partial_{t} \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2}, \quad \text{and}
\]

\[
\int_{0}^{t} \| \nabla \bar{v}(\cdot, s) \|_{L^{2}(\Omega)}^{2} ds \leq \Gamma(\alpha - 1) T^{2-\alpha} J^{\alpha-1} \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2},
\]

and thus, with a new constant \( C > 0 \), we obtain

\[
J^{2-\alpha} \| \partial_{t} \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} \leq C J^{\alpha-1} \left( J^{2-\alpha} \| \partial_{t} \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \| \nabla \bar{v}(\cdot, t) \|_{L^{2}(\Omega)}^{2} \right).
\]

Therefore, we apply the generalized Grönwall’s inequality (e.g., [7, Appendix A]) and the Poincaré inequality and we reach \( \bar{v}(\cdot, t) = 0 \) for a.e. \( t \in [0, T] \). Noting that \( \bar{u} = \partial_{t} \bar{v} \), we have also \( \bar{u}(\cdot, t) = 0 \) for a.e. \( t \in [0, T] \).
5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. If $a_1 \in H^2(\Omega) \cap H^1_0(\Omega)$ and we allow $\|a_1\|_{H^2(\Omega)}$ on the right-hand side of the regularity estimate, then the proof would be easy since one could denote $v = u - a_0 - ta_1$ and apply $L^2$ estimate to $v$ in space. In Theorem 1.3, we have only $\|a_1\|_{H^2_0(\Omega)}$ on the right-hand side and thus we need some technical treatment as follows.

Proof of Theorem 1.3. If we prove the a priori estimate

$$
\| \partial_t^2 (u_N - a_0, N - ta_1, N) \|_{L^2(0, T; L^2(\Omega))} + \| u_N \|_{H^1(0, T; H^2_0(\Omega))} + \| u_N \|_{L^2(0, T; H^2(\Omega))} 
\leq C \left( \| a_0 \|_{H^2(\Omega)} + \| a_1 \|_{H^1_0(\Omega)} + \| F \|_{H^1(0, T; L^2(\Omega))} \right),
$$

then we can derive the desired regularity and estimate for $u$ by a similar argument used in Step 3 of the proof in Section 4. Then it is sufficient to establish the uniform estimate.

Here we consider slightly different approximate solutions due to some technical reasons. For simplicity, we still use the same notation $u_N$. Precisely, we construct the approximate solutions:

$$
u_N(x, t) = \sum_{k=1}^{N} p_N^k(t) \varphi_k(x), \quad (x, t) \in \Omega \times (0, T)
$$
satisfying

$$
\begin{cases}
\partial_t^2 (u_N - a_0, N - ta_1, N) = -A(x, t)(u_N - a_0, N)(x, t) + G_N(x, t), & (x, t) \in \Omega \times (0, T), \\
u_N - a_0, N - ta_1, N \in H_\alpha(0, T; L^2(\Omega)),
\end{cases}
$$

where

$$
a_{j, N} = \sum_{k=1}^{N} a_{j, k} \varphi_k, \quad a_{j, k} := (a_j, \varphi_k)_{L^2(\Omega)}, \quad j = 0, 1,
$$

$$
G_N(t) = \sum_{k=1}^{N} g_k(t) \varphi_k, \quad g_k(t) := (F(t) - A(t)a_0, \varphi_k)_{L^2(\Omega)}, \quad 0 < t < T.
$$

Then $p^N = (p_1^N, \ldots, p_N^N)^T$ solves

$$
\begin{cases}
\partial_t^4 (p^N - a_0 - ta_1) = Q(t)(p^N(t) - a_0) + g(t) = Q(t)p^N(t) + (g(t) - Q(t)a_0), & t \in (0, T), \\
p^N - a_0 - ta_1 \in (H_\alpha(0, T))^N
\end{cases}
$$

where $Q(t) = (q_{k, l}(t))_{k, l=1}^{N}$ is defined by (11) and $g = (g_1, \ldots, g_N)^T$. By the assumption $F - Aa_0 \in H_1(0, T; L^2(\Omega))$, we find $g \in (H_1(0, T))^N$, and thus $(g - Qa_0) + Qa_0 = g \in (H_1(0, T))^N$. Now we employ Corollary 3.1 and we obtain $p^N - a_0 - ta_1 \in (H_{\alpha+1}(0, T))^N$, which implies

$$
u_N - a_0, N - ta_1, N \in H_{\alpha+1}(0, T; H^2(\Omega) \cap H^1_0(\Omega)).$$
Moreover, since $ta_{1,N} \in H_1(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, we have $u_N - a_{0,N} \in H_1(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. Then we take time derivative in (28) and we obtain
\[
\partial_s \partial_t^\alpha (u_N - a_{0,N} - ta_{1,N}) = -A\partial_t (u_N - a_{0,N}) + \partial_t G_N - (\partial_t A)(u_N - a_{0,N}).
\]

Now by setting $v := \partial_t (u_N - a_{0,N} - ta_{1,N}) \in H_\alpha(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, we rewrite the above equation:
\[
\partial_t^\alpha v(t) = -Av(t) - A(t) a_{1,N} + \partial_t G_N(t) - (\partial_t A)(u_N - a_{0,N}), \quad 0 < t < T. \tag{28}
\]

We multiply (28) by $\partial_t^{\alpha - 1} v$ and integrate over $\Omega$. By integration by parts and substituting $t$ by $s$, we obtain
\[
\frac{1}{2} \partial_s \| \partial_t^{\alpha - 1} v(s) \|^2_{L^2(\Omega)} + \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j v(s) \partial_t^{\alpha - 1} \partial_i v(s) \, dx
\]
\[
= \int_\Omega \left( \sum_{i,j=1}^n b_{ij}(s) \partial_j v(s) + c(s)v(s) \right) \partial_t^{\alpha - 1} v(s) \, dx - \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} \partial_t^{\alpha - 1} \partial_i v(s) \, dx
\]
\[
+ \int_\Omega \left( \sum_{i,j=1}^n b_{ij}(s) \partial_j a_{1,N} + c(s)a_{1,N} \right) \partial_t^{\alpha - 1} v(s) \, dx + \int_\Omega \partial_s G_N(s) \partial_t^{\alpha - 1} v(s) \, dx
\]
\[
- \int_\Omega \partial_s A(s) (u_N(s) - a_{0,N}) \partial_t^{\alpha - 1} v(s) \, dx =: \sum_{k=1}^5 I_k(s). \tag{29}
\]

We estimate the right-hand side of (29) term by term. Recalling that the coefficients $a_{ij}, b_j, c$ are bounded from above, by Hölder’s inequality and the Poincaré inequality, we have
\[
I_1(s) \leq \varepsilon \| \nabla v(\cdot, s) \|^2_{L^2(\Omega)} + \frac{C}{\varepsilon} \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} \quad \text{for any } \varepsilon > 0,
\]
\[
I_2(s) \leq C \left( \| a_{1,N} \|^2_{H_0^1(\Omega)} + \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} \right),
\]
\[
I_4(s) \leq C \left( \| \partial_s G_N(s) \|^2_{L^2(\Omega)} + \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} \right),
\]
\[
I_5(s) \leq C \left( \| u_N(s) - a_{0,N} \|^2_{H^2(\Omega)} + \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} \right).
\]

Moreover, by the equation (28) and the elliptic regularity (e.g., [2] Theorem 4, Chapter VI), we obtain
\[
\| u_N(s) - a_{0,N} \|^2_{H^2(\Omega)} \leq C \| -A(s)(u_N(s) - a_{0,N}) \|^2_{L^2(\Omega)} + C \| u_N(s) - a_{0,N} \|^2_{L^2(\Omega)}
\]
\[
\leq C \| \partial_t^\alpha (u_N(s) - a_{0,N} - sa_{1,N}) \|^2_{L^2(\Omega)} + C \| G_N(s) \|^2_{L^2(\Omega)} + C \| u_N(s) - a_{0,N} \|^2_{L^2(\Omega)}
\]
\[
\leq C \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} + C \| G_N(s) \|^2_{L^2(\Omega)} + C \| u_N(s) - a_{0,N} \|^2_{L^2(\Omega)}, \tag{30}
\]
and hence
\[
I_5(s) \leq C \left( \| \partial_t^{\alpha - 1} v(\cdot, s) \|^2_{L^2(\Omega)} + \| G_N(s) \|^2_{L^2(\Omega)} + \| u_N(s) - a_{0,N} \|^2_{L^2(\Omega)} \right).
\]

Furthermore, by Proposition [24] we estimate $I_2$ as follows:
\[
I_2(s) = - \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} \partial_s J^{2-\alpha} \partial_i v(s) \, dx.
\]
Also we employ the coercivity inequality (Lemma A.2) to obtain
\[ \|\partial^\alpha v(0)\|^2_{L^2(\Omega)} \leq \sigma_0 J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)} - C J^{3-\alpha}\|\nabla v\|^2_{L^2(\Omega)}. \]

Moreover, we have
\[ J^1\|\nabla v\|^2_{L^2(\Omega)} = \int_0^t \|\nabla v(\cdot, s)\|^2_{L^2(\Omega)} ds \leq \int_0^t \Gamma(2-\alpha)(t-s)^{\alpha-1}\|\nabla v(\cdot, s)\|^2_{L^2(\Omega)} ds \]
\[ \leq \Gamma(2-\alpha)T^{\alpha-1} J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)}. \]

Therefore, by taking \( \varepsilon > 0 \) small enough, the above four estimates yields
\[ \|\partial^\alpha v(\cdot, t)\|^2_{L^2(\Omega)} + J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)} \]
\[ \leq C \left( \|a_{1,N}\|^2_{H^1_0(\Omega)} + \|G_N\|^2_{L^2(0,T;L^2(\Omega))} + \|\partial_t G_N\|^2_{L^2(0,T;L^2(\Omega))} + J^1\|u_N - a_{0,N}\|^2_{L^2(\Omega)} \right) \]
\[ + C J^{1} \left( \|\partial^\alpha v\|^2_{L^2(\Omega)} + J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)} \right) \]
\[ \tag{31} \]
for all \( 0 \leq t \leq T \). It is readily to check that
\[ \|a_{1,N}\|^2_{H^1_0(\Omega)} \leq \|a_1\|^2_{H^1_0(\Omega)}, \]
\[ \|G_N\|^2_{L^2(0,T;L^2(\Omega))} \leq C \left( \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|a_0\|^2_{H^2(\Omega)} \right), \]

Then we apply the operator \( J^1 \) to (29), that is, we integrate (29) over \( s \in (0, t) \). Noting that \( \partial^\alpha v(0) = 0 \) and \( J^1 = J^{2-\alpha} J^{\alpha-1} \), we derive
\[ \frac{1}{2}\|\partial^\alpha v(\cdot, t)\|^2_{L^2(\Omega)} + J^{2-\alpha} J^{\alpha-1} \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} J^{2-\alpha} \partial_s a_{ij}(s) dx \]
\[ \leq - \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} J^{2-\alpha} \partial_s a_{ij}(s) dx + C \left( \|a_{1,N}\|^2_{H^1_0(\Omega)} + \|G_N\|^2_{L^2(0,T;L^2(\Omega))} + \|\partial_t G_N\|^2_{L^2(0,T;L^2(\Omega))} \right) \]
\[ + \varepsilon J^1\|\nabla v\|^2_{L^2(\Omega)} + C J^{3-\alpha}\|\nabla v\|^2_{L^2(\Omega)} + \left( C + \frac{C}{\varepsilon} \right) J^1\|\partial^\alpha v\|^2_{L^2(\Omega)} + C J^1\|u_N - a_{0,N}\|^2_{L^2(\Omega)}. \]
By Hölder’s inequality, for any \( \varepsilon > 0 \), we have
\[ - \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} J^{2-\alpha} \partial_s a_{ij}(s) dx \leq C J^{2-\alpha} \int_\Omega \sum_{i,j=1}^n |\partial_j a_{1,N}|^2 \|\partial_s a_{ij}(s)\| dx \leq \frac{C}{\varepsilon} \|a_{1,N}\|^2_{H^1_0(\Omega)} + \varepsilon J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)}. \]

Also we employ the coercivity inequality (Lemma A.2) to obtain
\[ J^{2-\alpha} J^{\alpha-1} \int_\Omega \sum_{i,j=1}^n a_{ij}(s) \partial_j a_{1,N} J^{2-\alpha} \partial_s a_{ij}(s) dx \geq \frac{\sigma_0}{2} J^{2-\alpha}\|\nabla v\|^2_{L^2(\Omega)} - C J^{3-\alpha}\|\nabla v\|^2_{L^2(\Omega)}. \]
\[ \| \partial_t G_N \|^2_{L^2(0,T;L^2(\Omega))} \leq C \left( \| F \|^2_{H^1(0,T;L^2(\Omega))} + \| a_0 \|^2_{H^1(\Omega)} \right). \]

According to the estimate of Theorem 1.2 we have
\[ J^1 \| a_N - a_0, N \|^2_{L^2(\Omega)} \leq C \| a_N \|^2_{L^2(0,T;L^2(\Omega))} + C \| a_0, N \|^2_{L^2(\Omega)} \leq C \left( \| F \|^2_{L^2(0,T;L^2(\Omega))} + \| a_0 \|^2_{H^1(\Omega)} + \| a_1 \|^2_{L^2(\Omega)} \right). \]

Combining the above estimates with (31), we reach
\[ J^1 \| u_N - a_0, N \|^2_{L^2(\Omega)} \leq C \| u_N \|^2_{L^2(0,T;L^2(\Omega))} + C \| a_N \|^2_{L^2(\Omega)} \leq C \left( \| F \|^2_{L^2(0,T;L^2(\Omega))} + \| a_0 \|^2_{H^1(\Omega)} + \| a_1 \|^2_{L^2(\Omega)} \right). \]

for all \( 0 \leq t \leq T \). By the generalized Grönwall’s inequality (e.g., [7, Appendix A]), the above inequality implies
\[ \| \partial_t^{\alpha - 1} v(\cdot, t) \|^2_{L^2(\Omega)} + J^{2-\alpha} \| \nabla v \|^2_{L^2(\Omega)} \leq C \left( \| a_0 \|^2_{H^2(\Omega)} + \| a_1 \|^2_{H^1(\Omega)} + \| F \|^2_{H^1(0,T;L^2(\Omega))} \right) \]
for all \( 0 \leq t \leq T \). Then we return to (30) and see that
\[ \| u_N(t) \|^2_{H^2(\Omega)} \leq C \| u_N(t) - a_0, N \|^2_{H^2(\Omega)} + C \| a_0, N \|^2_{H^2(\Omega)} \leq C \left( \| a_0 \|^2_{H^2(\Omega)} + \| a_1 \|^2_{H^1(\Omega)} + \| F \|^2_{H^1(0,T;L^2(\Omega))} \right). \]

Finally, substituting \( v = \partial_t (u_N - u_0, N - ta_1, N) \) into (32) yields
\[ \| \partial_t^{\alpha} (u_N(t) - u_0, N - ta_1, N) \|^2_{L^2(\Omega)} + J^{2-\alpha} \| \nabla (\partial_t u_N - a_1, N) \|^2_{L^2(\Omega)} + \| u_N(t) \|^2_{H^2(\Omega)} \leq C \left( \| a_0 \|^2_{H^2(\Omega)} + \| a_1 \|^2_{H^1(\Omega)} + \| F \|^2_{H^1(0,T;L^2(\Omega))} \right) \]

We apply the operator \( J^{\alpha - 1} \) to the second term on the left-hand side and take \( t = T \), which implies
\[ \| \nabla \partial_t u_N \|^2_{L^2(0,T;L^2(\Omega))} \leq C \left( \| a_0 \|^2_{H^2(\Omega)} + \| a_1 \|^2_{H^1(\Omega)} + \| F \|^2_{H^1(0,T;L^2(\Omega))} \right). \]

This completes the proof of Theorem 1.3. \( \square \)

A Appendix

In the Appendix, we give the statements of two coercivity inequalities which are used several times in the proofs of the main results.

**Lemma A.1** (A coercivity inequality). Let \( 0 < \gamma \leq 1 \). For each \( u \in L^2(0,T;H^0_\gamma(\Omega)) \cap H_\gamma(0,T;H^{-1}(\Omega)) \),
\[ \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} H^{-1}(\partial_t^\gamma u(\cdot,s), t(\cdot,s))H^0_\gamma(\Omega)ds \geq \frac{1}{2} \| u(\cdot,t) \|^2_{L^2(\Omega)}. \]

For this lemma, we refer to Theorem 3.4(ii) of [7].
Lemma A.2 (Second coercivity inequality). Let $0 < \gamma \leq 1$ and $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega \times (0, T))$, $i, j = 1, \ldots, n$ satisfy the assumption \[\text{(1)}.\] Then for each $v = (v_1, \ldots, v_n)^T \in H_1(0, T; L^2(\Omega))$,

$$
\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,s) v_i(x,s) \partial^\gamma_{x} v_j(x,s) dxdx s \geq \frac{\sigma_0}{2} \|v(v,t)\|^2_{L^2(\Omega)} - C \int_0^t \|v(s,t)\|^2_{L^2(\Omega)} ds
$$

for some constant $C = C(\gamma, \|a_{ij}\|_{W^{1,\infty}}) > 0$.

**Proof.** For completeness, we give the detailed proof but we also refer to [7, Proof of Theorem 4.2] for a similar idea. For the case of $\gamma = 1$, the left-hand side of our desired inequality reads

$$
\int_0^t \int_\Omega \sum_{i,j=1}^n a_{ij}(x,s) v_i(x,s) \partial_s v_j(x,s) dx ds.
$$

Then by the assumption \[\text{(1)}.\] and integration by parts, we can immediately prove the desired inequality. Now we consider the case of $0 < \gamma < 1$. Let

$$
I_0(x,s) := \sum_{i,j=1}^n a_{ij}(x,s) v_i(x,s) \partial^\gamma_{x} v_j(x,s), \quad (x,s) \in \Omega \times (0, T).
$$

In the following estimate of $I_0$, we may omit $x$ for simplicity. Since we assume $v \in H_1(0, T; L^2(\Omega))$, the fractional derivative $\partial^\gamma_{x}$ coincides with the Caputo fractional derivative $C\partial^\gamma_{x}$ and by $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$, we have

$$
I_0(s) = \sum_{i,j=1}^n a_{ij}(s) v_i(s) \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \partial_\tau v_j(\tau) d\tau
$$

$$
= \sum_{i,j=1}^n a_{ij}(s) \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \partial_\tau v_j(\tau) v_i(\tau) d\tau + \sum_{i,j=1}^n a_{ij}(s) \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \partial_\tau v_j(\tau) (v_i(s) - v_i(\tau)) d\tau
$$

$$
=: I_1(s) + I_2(s).
$$

By noting $v_i(0) = 0$ for $i = 1, \ldots, n$ and

$$
\lim_{\tau \to s} \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} (v_i(s) - v_i(\tau))(v_j(s) - v_j(\tau)) \leq \lim_{\tau \to s} \frac{(s-\tau)^{-1-\gamma}}{\Gamma(1-\gamma)} \|v_i\|_{H^1(0,T)} \|v_j\|_{H^1(0,T)} = 0,
$$

we use integration by parts to estimate $I_2$ as follows:

$$
I_2(s) = -\frac{1}{2} \sum_{i,j=1}^n a_{ij}(s) \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \partial_\tau ((v_j(s) - v_j(\tau))(v_i(s) - v_i(\tau))) d\tau
$$

$$
= \frac{1}{2} \sum_{i,j=1}^n a_{ij}(s) \frac{s^{-\gamma}}{\Gamma(1-\gamma)} (v_j(s)v_i(s)) + \frac{\gamma}{2} \sum_{i,j=1}^n a_{ij}(s) \int_0^s \frac{(s-\tau)^{-\gamma-1}}{\Gamma(1-\gamma)} ((v_j(s) - v_j(\tau))(v_i(s) - v_i(\tau))) d\tau
$$

$$
\geq \frac{\sigma_0}{2} \frac{s^{-\gamma}}{\Gamma(1-\gamma)} \|v(s)\|^2_{L^2(\Omega)} + \frac{\sigma_0}{2} \frac{\gamma}{\Gamma(1-\gamma)} \int_0^s \frac{|v(s) - v(\tau)|^2}{(s-\tau)^{\gamma+1}} d\tau \geq 0.
$$

Here in the last line we used the assumption \[\text{(1)}.\] Similarly we estimate $I_1$ as follows:

$$
I_1(s) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(s) \int_0^s \frac{(s-\tau)^{-\gamma}}{\Gamma(1-\gamma)} \partial_\tau (v_i(\tau)v_j(\tau)) d\tau
$$

\[\text{(2)}.\]
\[
\begin{align*}
&= \frac{1}{2} \int_0^s (s-\tau)^{-\gamma} \frac{\partial}{\partial \tau} \left( \sum_{i,j=1}^n (a_{ij}(s) - a_{ij}(\tau))v_i(\tau)v_j(\tau) \right) d\tau + \frac{1}{2} \int_0^s (s-\tau)^{-\gamma} \frac{\partial}{\partial \tau} \left( \sum_{i,j=1}^n a_{ij}(\tau)v_i(\tau)v_j(\tau) \right) d\tau \\
&= -\frac{\gamma}{2} \int_0^s \sum_{i,j=1}^n \frac{(s-\tau)^{\gamma}}{s-\tau} a_{ij}(s) - a_{ij}(\tau) \frac{v_i(\tau)v_j(\tau)}{s-\tau} d\tau + \frac{1}{2} J^{1-\gamma} \partial_s \left( \sum_{i,j=1}^n a_{ij}v_i v_j \right).
\end{align*}
\]

Since \( a_{ij} \in W^{1,\infty}(\Omega \times (0,T)) \), there exists a generic constant \( C > 0 \) depending on some norm of \( a_{ij} \) such that
\[
\left| \frac{a_{ij}(s) - a_{ij}(\tau)}{s-\tau} \right| \leq C.
\]
Thus, by the triangle inequality \(|AB| \leq \frac{1}{2}(A^2 + B^2)\), we have the lower bound of \( I_1 \):
\[
I_1(s) \geq \frac{1}{2} J^{1-\gamma} \partial_s \left( \sum_{i,j=1}^n a_{ij}v_i v_j \right) - C J^{1-\gamma} |v|_{L^2(\Omega)}^2(s).
\]

Finally, by Fubini’s theorem, we obtain
\[
\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \int_\Omega I_0(x,s) dx ds = J^\gamma \int_\Omega I_0 dx \geq \frac{1}{2} J^{1-\gamma} \partial_s \left( \sum_{i,j=1}^n a_{ij}v_i v_j \right) - C J^{1-\gamma} |v|_{L^2(\Omega)}^2
\]
\[
\geq \frac{\sigma_0}{2} \|v(\cdot,t)\|_{L^2(\Omega)}^2 - C \int_0^t \|v(\cdot,s)\|_{L^2(\Omega)}^2 ds.
\]

\[\square\]

**Acknowledgments**

The first author was supported by Grant-in-Aid for JSPS (Japan Society for the Promotion of Science) Fellows 20F20319. The second author was supported by Grant-in-Aid for Scientific Research (A) 20H00117, JSPS.

**References**

[1] R. A. Adams and J. J.F. Fournier. *Sobolev Spaces*, Vol. 140, 2nd edition, Academic Press, 2003.

[2] L. Evans. *Partial Differential Equations*, American Mathematical Society, 1998.

[3] R. Gorenflo, A. A. Kilbas, F. Mainardi and S.V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer, Berlin, 2014.

[4] R. Gorenflo, Y. Luchko and M. Yamamoto. Time-fractional diffusion equation in the fractional Sobolev spaces, Fract. Calc. Appl. Anal. 18 (2015): 799-820.

[5] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.
[6] A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion, J. Math. Anal. Appl. 340 (2008): 252-281.

[7] A. Kubica, K. Ryszewska and M. Yamamoto. *Time-Fractional Differential Equations: A Theoretical Introduction*, Springer, Singapore, 2020.

[8] Z. Li, Y. Liu and M. Yamamoto. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. Applied Mathematics and Computation 257 (2015): 381-397.

[9] Y. Luchko. Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation. J. Math. Anal. Appl. 374 (2011): 538-548.

[10] Y. Luchko and M. Yamamoto. General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary-value problems, Fract. Calc. Appl. Anal. 19 (2016): 676-695.

[11] F. Mainardi, A. Mura, G. Pagnini and et al. Time-fractional diffusion of distributed order. Journal of Vibration and Control 14 (2008): 1267-1290.

[12] F. Mainardi. Fractional diffusive waves in viscoelastic solids, In *Nonlinear Waves in Solids* (J. L. Wegner and F. R. Norwood eds.), ASME/AMR, Fairfield (1995): 93-97.

[13] R. R. Nigmatullin. The realization of the generalized transfer equation in a medium with fractal geometry, Physica Status Solidi 133 (1986): 425-430.

[14] I. Podlubny. *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[15] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. Journal of Mathematical Analysis and Applications 382 (2011): 426-447.

[16] L. N. Slobodeckij. Generalized Sobolev spaces and their applications to boundary value problems of partial differential equations, Leningrad. Gos. Ped. Inst. Ucep. Zap. 197 (1958): 54-112.

[17] R. Zacher. Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, Funkcial. Ekvac. 52 (2009): 1-18.