Pitfalls of Zero Voltage Values in Optimal Power Flow Problems

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Abstract—The existence of strictly positive lower bounds on voltage magnitude is taken for granted in optimal power flow problems. Nevertheless, it is not possible to rely on such bounds for a variety of real-world network optimization problems. This paper discusses a few issues related to 0V assumptions made during the process of deriving optimization formulations in the current-voltage, power-voltage and power-lifted-voltage variable spaces. The differences between the assumptions are illustrated for a 2-bus 2-wire test case, where the feasible sets are visualized. A nonzero relaxation gap is observed for the canonical multiconductor nonlinear power-voltage formulation. A zero gap can be obtained for the branch flow model semi-definite relaxation, using newly proposed valid equalities.

Index Terms—Optimal power flow, mathematical optimization, nonlinear programming.

I. INTRODUCTION

In the context of ac transmission system optimization, it has generally been established [1] that the nonlinear nonconvex power-voltage formulation of the circuit physics (in polar coordinates, ‘SVP’) is the most scalable, when used with nonlinear programming solvers such as IPOPT [2], MIPS [3] or BELTISTOS [4]. Other exact formulations, e.g. the rectangular current-voltage (IVR) and the rectangular power-voltage formulation (SVR) result in overall slower computations [1].

The canonical optimal power flow (OPF) problem specification furthermore assumes the existence of a number of bounds, and postulates a generation cost minimization objective. In general authors assume that voltage magnitude has a strictly positive lower bound, thereby excluding the 0 V from the feasible set. Nevertheless, many real-world network optimization problems may not have this property of strictly positive lower bounds, as 1) the values may be missing entirely; or 2) 0 V solutions can not be excluded a-priori. In the author’s experience, such contexts are not far-fetched:

- Short circuit situations, e.g. in the context of (security-constrained) OPF with explicit representation of short circuit conditions [5], [6].
- Polynomial chaos OPF [7], [8] has no magnitude lower bound for all \( k > 1 \) (i.e. there is a valid lower bound on the mean, but the 0 V solution cannot be excluded from the variance).
- Harmonic OPF [9] has no magnitude lower bound for the true harmonics \( h > 1 \) (e.g. voltage magnitude at \( h = 3 \), i.e., the triplen harmonic, can perfectly be 0).
- Four-wire OPF [10], [11] has no voltage magnitude lower bound for the neutral terminal\(^1\).
- OPF with HVDC lines has no voltage lower bound on the nodes for a metallic return [12].

In grounded three-phase four-wire networks, the neutral current may be (close to) zero due to balanced loading, so excluding the 0 V neutral solution may be impossible. Allowing 0 V values causes real algorithmic problems for SVR [11]:

- 0 V solutions were naturally found by IPOPT;
- the solution obtained was very sensitive to variable initialization, i.e. different nodes ended up at 0 V;
- it led to convergence problems, requiring much larger computation time than the IVR form;
- solutions were likely to be strictly locally optimal.

Similarly, polynomial chaos OPF paper [8] observed a significant computational speedup using IVR over SVR. Therefore, this paper showcases a number of pitfalls related to 0 V assumptions and solutions in the feasible sets of OPF in different variable spaces. The 0 V solutions lead to problems in the lifting of Kirchhoff’s current law (KCL) to power variables. Furthermore, loads and generators are often assumed to be connected between a phase and the neutral at 0 V, and dropping this assumption requires attention to detail.

The paper is structured as follows. First we derive a variety of formulations in a consistent notation, which are compared numerically next on a 2-bus 2-wire (explicit neutral) power flow case study\(^2\). Finally we present some conclusions and directions of future work.

II. DERIVATION OF FORMULATIONS

Fig. 1 illustrates the definitions of the current, voltage and power variables and indices used in this paper. We will explore the following three-phase four-wire a.k.a. unbalanced rectangular formulations:

- NLP unbalanced IVR form, presented in [11];
- NLP SVR form, with extensions to avoid violation of KCL in loads/generators, as discussed in [11];
- Branch flow model (BFM) SDP relaxation [13], extended to avoid violation of KCL in load/generators.

\(^1\)Even in multi-earthed networks, a behind-the-meter network may have neutral voltage rise, making Kron’s reduction of the neutral problematic.

\(^2\)Extensive numerical results with 0 V issues available in [8], [11].
We derive the n-conductor expressions in a consistent notation in vector-matrix variables, but illustrate for a 2-wire system with phase a and an explicit neutral n. We ignore branch shunt terms for brevity, and refer to [14] for a treatment of 4-wire OPF with asymmetric II-model branches, and bus shunts.

### A. Current-Voltage (IVR)

The complex voltage vector \( U_i \) at bus \( i \), a.k.a. phasor, stacks the values of the nodes at buses \( i \),

\[
U_i \equiv \begin{bmatrix} U_{i,a} & U_{i,n} \end{bmatrix}^T.
\]  

(1)

For a two-wire set of electromagnetically coupled conductors, we end up with a \( 2 \times 2 \) impedance matrix, capturing the self-impedance (diagonal) and mutual impedance (off-diagonal).

By taking the matrix inverse, we obtain the admittance,

\[
Z_l \equiv R_l + jX_l \equiv \begin{bmatrix} Z_{l,aa} & Z_{l,an} \\ Z_{l,na} & Z_{l,nn} \end{bmatrix},
\]  

\[
Y_l \equiv (Z_l)^{-1}.
\]  

(2)

Note that, based on the physics (Carson/Pollaczek eq.), we expect that \( R_l \) and \( X_l \) are symmetric and positive definite. We define the complex current vector in branch \( l \) in the direction of \( i \) to \( j \) to stack the individual values for each of the wires,

\[
I_{lj} \equiv \begin{bmatrix} I_{lij,a} & I_{lij,n} \end{bmatrix}^T,
\]  

\[
I_{ji} \equiv \begin{bmatrix} I_{lij,a} & I_{lij,n} \end{bmatrix}^T.
\]  

(3)

The branch currents satisfy,

\[
I_{ij} + I_{ji} = 0.
\]  

(4)

The multiconductor ac Ohm’s law is,

\[
U_j = U_i - Z_l I_{lj}.
\]  

(5)

The power flow into the branch is,

\[
S_{lj} \equiv \begin{bmatrix} S_{lij,aa} \\ S_{lij,nn} \end{bmatrix} \equiv U_i \circ (I_{lj})^* = \begin{bmatrix} U_{i,a} I_{lij,a}^* \\ U_{i,n} I_{lij,n}^* \end{bmatrix},
\]  

(6)

where ‘\( \circ \)’ indicates the element-wise product. The current vector for a generator is,

\[
I_g \equiv \begin{bmatrix} I_{g,a} \\ I_{g,n} \end{bmatrix}^T,
\]  

(7)

and by convention flows from the generator to the bus it is connected to, in each of its wires. Note that conservation of current applies through the generator,

\[
1^T I_g = 0.
\]  

(8)

The power from the generator through each of the wires is,

\[
S_g \equiv \begin{bmatrix} S_{g,aa} \\ S_{g,nn} \end{bmatrix} \equiv U_i \circ I_g^* = \begin{bmatrix} U_{i,a} I_{g,a}^* \\ U_{i,n} I_{g,n}^* \end{bmatrix}.
\]  

(9)

For loads, the current flows from the bus to the load,

\[
I_d \equiv \begin{bmatrix} I_{d,a} \\ I_{d,n} \end{bmatrix}^T.
\]  

(10)

Conservation of current through the load is,

\[
1^T I_d = 0.
\]  

(11)

The power flowing through each of the wires to the load is,

\[
S_d \equiv \begin{bmatrix} S_{d,aa} \\ S_{d,nn} \end{bmatrix} \equiv U_i \circ I_d^*.
\]  

(12)

To force the load value (i.e. constant-power set point), we account for the voltage drop over the load, i.e. \( U_{i,a} - U_{i,n} \),

\[
S_{d}^\text{ref} = (U_{i,a} - U_{i,n}) I_{d,a}^*.
\]  

(13)

We now continue to expand the RHS, and use (11) and (12),

\[
U_{i,a} I_{d,a}^* - U_{i,n} I_{d,a}^* = U_{i,a} I_{d,a}^* + U_{i,n} I_{d,n}^* = S_{d,aa} + S_{d,nn} = 1^T S_d
\]  

(14)

and finally obtain,

\[
S_{d}^\text{ref} = 1^T S_d.
\]  

(15)

Note that this does not imply conservation of current,

\[
\sum_{ij} I_{ij} + \sum_{di} I_d - \sum_{gi} I_g = 0.
\]  

(16)

The effective dispatch value for a generator is,

\[
S_g^\text{disp} \equiv P_g^\text{disp} + jQ_g^\text{disp} = 1^T S_g.
\]  

(17)

Finally, KCL at bus \( i \) is,

\[
\sum_{li} I_{lij} + \sum_{di} I_d - \sum_{gi} I_g = 0.
\]  

(18)

### B. Canonical Power-Voltage (SVR-1)

We state Ohm’s law (5) in admittance form,

\[
I_{ij} = Y_l (U_i - U_j)
\]  

(19)

and substitute this into (6) to obtain power flow vector into the branch \( l \) in the direction of \( i \) to \( j \),

\[
S_{lij} = U_i \circ (Y_l (U_i - U_j))^*.
\]  

(20)

We lift KCL (18) to power variables (vectors with the same sizes as the original current vectors in (18)), by taking the conjugate and multiplying element-wise with \( U_i \),

\[
\sum_{lij} S_{lij} + \sum_{di} S_d - \sum_{gi} S_g = 0.
\]  

(21)

When we allow 0V entries in \( U_i \), the lifted KCL is a relaxation, as it does not imply current-based KCL (18), due to (possible) multiplication with 0V. This is equivalent to adding a slack variable to the original KCL, e.g. when \( U_{i,n} = 0 \):

\[
\sum_{lij} I_{lij,n} + \sum_{di} I_d,n - \sum_{gi} I_{g,n} + I_{i,n}^\text{slack} = 0.
\]  

(22)

As we are working with complex values, \( I_{n}^\text{slack} \) actually represents two degrees of freedom, which can be exploited by optimization solvers.
C. Power-Voltage Extension (SVR-2)

Even though SVR-1 has proven to work well to optimize three-phase Kron-reduced four-wire networks with only wye loads\textsuperscript{3}, we need to strengthen the feasible set when we model loads/generators between phases (‘delta’) or phase-to-neutral\textsuperscript{4}. Constraints (15)-(17) are insufficient in general, as they do not imply (8) or (11). To resolve this, we use the variables $I_d, I_g$ and add the constraints (8), (9), (11), (12).

D. Canonical Power-Lifted-Voltage BFM (SWR-1)

We define lifted variables for voltage, $W_i$, current, $L_i$, and power, $\bar{S}_{ij}$,

$$ W_i \equiv U_i(U_i)^H, \quad L_i \equiv I_{ij}(I_{ij})^H, \quad \bar{S}_{ij} \equiv U_i(I_{ij})^H, \quad (23) $$

where superscript ‘H’ indicates the conjugate transpose. Note that the substitutions don’t destroy any information\textsuperscript{5}, as long as $U_i, I_{ij}$ are not the zero vector - individual entries being zero is fine. The power variable $\bar{S}_{ij}$ is a square matrix, and therefore distinct from the vector $\bar{S}_{ij}$, however they are related,

$$ \text{diag}(\bar{S}_{ij}) = S_{ij}. \quad (24) $$

We can construct a block matrix based on the lifted variables,

$$ M_{ij} \equiv \begin{bmatrix} U_i & U_i \end{bmatrix}^H \begin{bmatrix} I_{ij} \end{bmatrix}^H = \begin{bmatrix} W_i & \bar{S}_{ij} \end{bmatrix}. \quad (25) $$

In convex relaxation schemes, this constraint gets cast as,

$$ \text{rank}(M_{ij}) = 1, M_{ij} \succeq 0, \quad (26) $$

and the rank constraint is dropped to obtain a semi-definite programming formulation. Ohm’s law (5) is multiplied with its own conjugate transpose to obtain,

$$ W_j = W_i - \bar{S}_{ij}(Z_{ij})^H - Z_i(\bar{S}_{ij})^H + Z_i L_i (Z_i)^H. \quad (27) $$

Furthermore, by multiplying Ohm’s law (5) on the right with $\bar{I}_{ij}^H$ and using (4), we obtain,

$$ \bar{S}_{ij} + \bar{S}_{ji} = Z_i L_i. \quad (28) $$

Combining (21) with (24), we have the BFM formulation with a diagonal KCL expression, as proposed by Gan and Low [13].

E. Power-Lifted-Voltage BFM Extension (SWR-2)

As we now generalize the definitions of the load and generator power to matrix variables as well\textsuperscript{6},

$$ \bar{S}_d \equiv U_d^H \equiv \begin{bmatrix} S_{d,a} & S_{d,an} \\ S_{d,na} & S_{d,n} \end{bmatrix} = \begin{bmatrix} U_{i,a}I_{d,a} & U_{i,a}I_{d,n} \\ U_{i,n}I_{d,a} & U_{i,n}I_{d,n} \end{bmatrix}, \quad (29) $$

We can partition based on the rows,

$$ \bar{S}_d \equiv \begin{bmatrix} S_{d,a} \\ S_{d,nn} \end{bmatrix}. \quad (30) $$

\text{and recognize that (11) implies,}

$$ 1^T S_{d,a} = 0, \quad 1^T S_{d,n} = 0. \quad (31) $$

Similarly, for generators we obtain,

$$ \bar{S}_g \equiv U_g^H \equiv \begin{bmatrix} S_{g,a} \\ S_{g,n} \end{bmatrix}, \quad 1^T S_{g,a} = 0, \quad 1^T S_{g,n} = 0. \quad (32) $$

We note valid equalities of the matrix variables w.r.t. the vector-style ones previously defined,\textsuperscript{7}

$$ \text{diag}(\bar{S}_d) = S_d, \quad \text{diag}(\bar{S}_g) = S_g. \quad (33) $$

Note that we do not project KCL on the rows of $\bar{S}_{ij}$ or $L_i$ in a similar fashion to (31)-(32). Due capacitive coupling to earth as well as grounding of the neutral, the current through a branch does not in general sum to zero. Finally we generalize KCL, using the matrix variables $\bar{S}$,

$$ \sum_{i,j} \bar{S}_{ij} + \sum_i \bar{S}_d - \sum_j \bar{S}_g = 0. \quad (34) $$

F. Voltage Bounds

For visualizing the feasible sets we enforce voltage bounds,

$$ \begin{bmatrix} U_{\text{min}}^2 \\ U_{\text{min}}^2 \end{bmatrix} \leq \begin{bmatrix} U_{i,a}^2 \\ U_{i,n}^2 \end{bmatrix} \leq \begin{bmatrix} U_{\text{max}}^2 \\ U_{\text{max}}^2 \end{bmatrix} \quad (35) $$

By setting $U_{\text{min}}^2$ for instance to 0.9 pu, we typically exclude the low-voltage solution to quadratic equality (13).

G. Feasible sets and generalizations

Table I summarizes the variables and constraints for five optimization formulation variants, using the previously-defined constraints. Implementation of these equations in nonlinear optimization frameworks typically requires translating the complex-valued expressions to sets of equivalent real-valued ones. Furthermore, lines should be modelled with shunt admittances, typically used to represent capacitive behavior w.r.t earth. Note that in general grid codes specify phase-to-neutral voltage magnitude limits, not phase-to-ground. Geth and Ergun [14] present the corresponding real-value expressions, and derive expressions for a variety of bounds. Claeyss et al. [17] propose a framework to represent unbalanced transformers across variable spaces.

III. OPF FOR A 2-BUS 2-WIRE CASE

Note that extensive numerical results showcasing 0 V issues (\geq 128 networks, up to 2156 nodes) have been published in papers [8], [11] co-authored by this paper’s author. Therefore, in this work, the focus is to provide insight, using a 2-bus test case (data in Table II). The problems are implemented in JUMP [18]. The nonconvex formulations are solved with IPOPT [2]; the semidefinite relaxation is solved with CLARABEL [16]. We solve repeatedly for different values $0 \leq \theta \leq 2\pi$ in the objective, to sample the edge of the solution space.

Note that this is actually a power flow simulation, i.e. feasibility, problem, as with no physical degree of freedom,
as the generator needs to produce a specific amount of power to satisfy the demand, taking into account the branch losses. As will be shown though, only IVR has a single (high-voltage) solution, all other forms have multiple solutions. The solutions sets are visualized in Fig. 2 and are discussed next.

A. Numerical results: amount of solutions

1) IVR has a single solution: The solution to this problem is uniquely specified by the voltage phasor at bus $j$, i.e.,

$$U_j = \begin{bmatrix} 0.937066 + j0.002500 \\ 0.062934 - j0.002500 \end{bmatrix},$$

and a phase-to-neutral voltage magnitude of 0.874146 pu. The generator output is, $S_g^\text{disp} = 1.147226 + j0.565434$. This solution is that of the circuit in Fig. 1.

2) SVR-2 has two solutions: The IVR solution is feasible w.r.t. SVR-2, however SVR-2 has a second spurious solution,

$$U_j = \begin{bmatrix} 0.924699 - j0.007591 \\ 0 + j0 \end{bmatrix},$$

associated with generator dispatch, $S_g^\text{disp} = 1.076921 + j0.549558$. The relaxation gap of this solution is 6.1282%. The voltage drop across the load differs, with the phase-to-neutral voltage magnitude at 0.924 where it was 0.874 pu for IVR. With the neutral being hard grounded at both buses, this solution is equivalent to that of a transformed circuit with Kron’s reduction of the neutral ($Z_N^\text{disp} = 0.052622 + j0.033902$).

This spurious solution is that of the circuit in Fig. 3a, where the neutral at bus $j$ is grounded. The existence of two distinct solutions suggests the problem for bigger networks has combinatorial (discontinuous) features, which may limit the performance of many algorithms, including the NLP interior-point algorithm in IPOPT.

3) SVR-1 has infinite solutions: A two-dimensional infinite set of solutions is feasible w.r.t. SVR-1. Note that the IVR solution is on the edge of this set, whereas the spurious SVR-2 solution is on the interior. Despite the IVR solution being on the edge, it is not in an ideal location, as it does not align with the minimum active power search direction, so likely leads to nonzero relaxation gap under a broad range of objectives.

For the 2-bus network, the combination of dropping the load KCL and allowing the 0 neutral voltage solution is equivalent to that of a transformed circuit with Kron’s reduction of the neutral ($Z_N^\text{disp} = 0.052622 + j0.033902$).

Fig. 2. The top figure illustrates the solution sets for the formulations, the bottom one zooms into the solutions with minimum active power. The middle figure indicates the corresponding value of the neutral voltage magnitude at bus $j$ for the solutions on the edges of the solution sets.
to the solutions of the circuit presented in Fig. 3b, where a variable complex current source is added to the bus $j$. The $\min P_g^{disp}$ solution is,

$$U_j = \begin{bmatrix} 0.926241 - j0.013080 \\ -0.007515 - j0.016886 \end{bmatrix}, U_j^{mag} = \begin{bmatrix} 0.926333 \\ 0.018483 \end{bmatrix} \tag{38}$$

with phase-to-neutral voltage magnitude at $0.933764$ pu. The generator output is $\delta^*_y = 1.071996 + j0.552133$, i.e. a gap of 6.5575\%

4) **SWR-2 has infinite solutions:** The solution space is a 1-dimensional infinite set. The IVR solution is an extreme point on this line, in the direction of minimum apparent power, therefore leading to a zero relaxation gap across a broad range of real-world objectives. The 0 V SVR-2 solution is *not* in the feasible set of SWR-2.

5) **SWR-1 has infinite solutions:** The solution space is a 2-dimensional infinite set, with a nonzero gap. We do not establish if it is comparable with SVR-1. The $\min P_g^{disp}$ solution is the same as that of SVR-1.

**B. Discussion**

The exclusion of the 0 V solution in SWR-2 is due to the combination of 1) KCL as a matrix equality 2) conservation of current through loads/generators. Adding either constraint individually to SWR-1 is not sufficient to obtain a 0 gap. That suggests the spurious solution in SWR-2 can be excluded by also using the matrix KCL (34) and generalizing (20) to the outer product (instead of element-wise). Alternatively, as voltages are inherently relative, we can also choose to redefine the ground voltage value, at for instance 1 V.

Modelling loads with a nonzero voltage at both terminals requires additional constraints. In SVR-2, the current variables could still be eliminated in favor of power variables (i.e. as $(S/U)^*$), following the approach in [11], but this may result in higher order polynomial expressions. Furthermore, constant-power three-phase loads have multiple solutions, with the amount depending on the configuration and neutral grounding impedance [19], [20]. Deriving valid bounds to exclude the ‘low-voltage solution’ may speed up the computation.

**IV. Conclusions**

In the absence of voltage magnitude lower bounds, the ‘exact’ canonical unbalanced power-voltage formulation is a relaxation of the circuit laws with a nonzero gap. Conversely, the unbalanced SDP BFM formulation - often trivialized for its theoretical inexactness, has a zero gap under the same conditions. The nonzero gap solution permitted by the power-voltage formulation, itself characterized by a zero voltage on one of the bus terminals, can furthermore be proven infeasible by the SDP relaxation. In what amounts to a reversal of fortune w.r.t. transmission-style (positive-sequence) OPF in terms of reliability, convergence and gap, with SDP now performing better than NLP, bound tightening based on a convex relaxation could be performed to generate valid lower bounds for the power-voltage form, thereby making the solution unique again.

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