The ratio of domination and independent domination numbers on trees

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Abstract

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Let $\gamma(G)$ and $i(G)$ be the domination number and the independent domination number of $G$, respectively. In 1977, Hedetniemi and Mitchell began with the comparison of $i(G)$ and $\gamma(G)$ and recently Rad and Volkmann posted a conjecture that $i(G)/\gamma(G) \leq \Delta(G)/2$, where $\Delta(G)$ is the maximum degree of $G$. In this work, we prove the conjecture for trees and provide the graph achieved the sharp bound.

Keywords: Extremal graphs; Domination number; Independent domination number; Comparison.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $N_G(v) = \{w \in V(G) : vw \in E(G)\}$ is the open neighborhood of $v$ and $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of $v$ in $G$. If $N_G(v) = \emptyset$, $v$ is called an isolated vertex. For $S \subseteq V(G)$, $N_G(S)$ is the open neighborhood of $S$, $N_G[S] = N_G(S) \cup S$ is the closed neighborhood of $S$ and $G - S$ is a subgraph induced by $V(G) - S$. A graph $F$ is a forest if it has no cycles. Specially, $F$ is a tree if it contains only one component. A double star is a tree with exactly two vertices of degree greater than 1. In particular, if the two vertices have same degree,
It is known that a vertex set \( D \subset V(G) \) is a dominating set if every vertex of \( V(G) - D \) is adjacent to some vertices of \( D \). The minimum cardinality of a dominating set is called the domination number, denoted by \( \gamma(G) \). Similarly, a vertex set \( I \subset V(G) \) is an independent dominating set if \( I \) is both an independent set and a dominating set in \( G \), where an independent set is a set of vertices in a graph such that no two of which are adjacent. The minimum cardinality of an independent dominating set is called the independent domination number, denoted by \( i(G) \). Currently, lots of work relating domination number and independent domination number have been studied, referred to surveys \([3, 5]\).

In 1977, S. Hedetniemi and S. Mitchell \([6]\) showed that for any tree \( T \), \( i(L(T))/\gamma(L(T)) = 1 \), where \( L(T) \) is the line graph of \( T \). Because any line graph is a \( K_{1,3} \)-free graph, R. B. Allan and R. Laskar \([1]\) extended the previous result in 1978 and obtained that if a graph does not have an induced subgraph isomorphic to \( K_{1,3} \), then \( i(G)/\gamma(G) = 1 \). Recently, Goddard et al.\([4]\) considered the ratio \( i(G)/\gamma(G) \) for regular graphs and proved that \( i(G)/\gamma(G) \leq 3/2 \) for cubic graphs. In 2013, Southey and Henning \([8]\) improved the previous result to \( i(G)/\gamma(G) \leq 4/3 \) for connected cubic graphs except for \( K_{3,3} \). During the same year, Rad and Volkmann \([7]\) got an upper bound of \( i(G)/\gamma(G) \) for a graph \( G \) and proposed the conjecture.

**Theorem 1** (Rad and Volkmann \([7]\)) Let \( G \) be a graph, then

\[
\frac{i(G)}{\gamma(G)} \leq \begin{cases} 
\frac{\Delta(G)}{2}, & \text{if } 3 \leq \Delta(G) \leq 5, \\
\Delta(G) - 3 + \frac{2}{\Delta(G)-1}, & \text{if } \Delta(G) \geq 6.
\end{cases}
\]

**Conjecture 2** (Rad and Volkmann \([7]\)) Let \( G \) be a graph with \( \Delta(G) \geq 3 \), then \( i(G)/\gamma(G) \leq \Delta(G)/2 \).

In 2014, Furuta et al.\([2]\) showed that \( i(G)/\gamma(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2 \) for a graph \( G \) and gave the graph achieved the new bound. However, when \( \Delta(G) \) is big enough, then \( \Delta(G) - 2\sqrt{\Delta(G)} + 2 > \Delta(G)/2 \). Now there is a natural question that

Q: Is there other class of graphs, which has an affirmative answer for Conjecture 2?

Motivated by Conjecture 2 and the above question, we prove that Conjecture 2 is true for the tree and provide the graph \( G \), which attains the sharp bound \( \Delta(G)/2 \).
Theorem 3 Let $G$ be a forest, then

$$\frac{i(G)}{\gamma(G)} \leq \begin{cases} 1, & \text{if } \Delta(G) \leq 2, \\ \frac{\Delta(G)}{2}, & \text{if } \Delta(G) \geq 3, \end{cases}$$

and the equalities hold if either $\Delta(G) \leq 2$ or each component of $G$ is a balanced double star (see figure 1).

As an immediate consequence of Theorem 3, we obtain that

Theorem 4 Let $G$ be a tree, then

$$\frac{i(G)}{\gamma(G)} \leq \begin{cases} 1, & \text{if } \Delta(G) \leq 2, \\ \frac{\Delta(G)}{2}, & \text{if } \Delta(G) \geq 3, \end{cases}$$

and the equalities hold if either $\Delta(G) \leq 2$ or $G$ is a balanced double star (see figure 1).

2 Proof of Theorem 3

In this section, we will prove Theorem 3 and start with an interesting lemma.

Lemma 1 Let $r_1, r_2, r_3, r_4, t$ be positive numbers with $\frac{r_1}{r_2} \leq t$ and $\frac{r_3}{r_4} \leq t$. Then $\frac{r_1 + r_3}{r_2 + r_4} \leq t$.

Since $r_1 \leq r_2 t, r_3 \leq r_4 t$, we replace $r_1, r_3$ and obtain that Lemma 1 is true. Next we will give the main proof of this note.

Proof of Theorem 3. For $\Delta(G) \leq 1$, $G$ contains only isolated vertices or edges and $i(G) = \gamma(G)$, that is, $i(G)/\gamma(G) = 1$. Next, we will consider the case of $\Delta(G) \geq 2$ and begin with the case that the forest $G$ contains only one component, that is, $G$ is a tree.

Let $D$ be a minimum dominating set of $G$. Then $G[D]$ is also a forest. We build $\{G_i\}, \{x_i\}$ with $i \geq 1$ as follows: Let $G_1 = G[D]$ and $x_1 \in V(G_1)$
with \( d_{G_i}(x_1) = 0 \) or 1; For \( i \geq 2 \), if \( V(G_i - N_{G_i}[x_i]) = \phi \), then stop and set \( i = k \). Otherwise, let \( G_i = G_{i-1} - N_{G_{i-1}}[x_{i-1}] \) and \( x_i \in V(G_i) \) with \( d_{G_i}(x_i) = 0 \) or 1.

Set \( X = \{ x_1, x_2, ..., x_k \} \). Then \( X \) is an independent dominating set of \( G[D] \) and \( \{ N_{G_i}[x_i], 1 \leq i \leq k \} \) is a partition of \( D \), that is, \( \sum_{1 \leq i \leq k} (d_{G_i}(x_i) + 1) = |D| = \gamma(G) \). Choose \( I \subset V(G) - D \) such that \( X \cup I \) is an independent dominating set of \( G \), that is, \( i(G) \leq |X| + |I| = k + |I| \). Since \( D \) is a dominating set of \( G \), then \( I = \bigcup_{v \in D - X} (N_G(v) \cap I) = \bigcup_{1 \leq i \leq k} (\bigcup_{v \in N_{G_i}(x_i)} (N_G(v) \cap I)) \). By the choice of \( x_i \), for \( 1 \leq i \leq k \) and \( v \in N_{G_i}(x_i) \), we have \( d_{G_i}(x_i) \leq d_{G_i}(v) \). Thus, \( |N_G(v) \cap I| \leq d_G(v) - d_{G_i}(v) \leq \Delta(G) - d_{G_i}(x_i) \) and

\[
\begin{align*}
|I| &\leq \sum_{1 \leq i \leq k} \left( \sum_{v \in N_{G_i}(x_i)} |N_G(v) \cap I| \right) \\
&\leq \sum_{1 \leq i \leq k} \left( \sum_{v \in N_{G_i}(x_i)} (\Delta(G) - d_{G_i}(x_i)) \right) \\
&= \sum_{1 \leq i \leq k} \left( \sum_{v \in N_{G_i}(x_i)} \Delta(G) - \sum_{1 \leq i \leq k} \left( \sum_{v \in N_{G_i}(x_i)} d_{G_i}(v) \right) \right) \\
&= (|D| - k)\Delta(G) - \sum_{1 \leq i \leq k} d_{G_i}(x_i)^2. \tag{1}
\end{align*}
\]

By (1) and \( |D| = \gamma(G) \), we can obtain that

\[
\begin{align*}
i(G) &\leq k + |I| \\
&\leq k + (|D| - k)\Delta(G) - \sum_{1 \leq i \leq k} d_{G_i}(x_i)^2 \\
&= \Delta(G)\gamma(G) - \sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2),
\end{align*}
\]

that is,

\[
\frac{i(G)}{\gamma(G)} \leq \Delta(G) - \frac{\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2)}{\gamma(G)}.
\]

Now, it suffices to show that \( -\frac{\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2)}{\gamma(G)} \leq -\frac{\Delta(G)}{2} \), that is,

\[
\begin{align*}
\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2) &\geq \frac{1}{2} \Delta(G)\gamma(G) \\
&= \frac{1}{2} \Delta(G) \left( \sum_{1 \leq i \leq k} (d_{G_i}(x_i) + 1) \right) \tag{2}
\end{align*}
\]

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By the construction of $G_i, x_i, d_{G_i}(x_i) = d_{G_i}(x_i)^2 = 0$ or 1. Thus, (2) is the same as (3) below.

$$\Leftrightarrow k\Delta(G) - k + \sum_{1 \leq i \leq k} d_{G_i}(x_i) - \frac{1}{2}\Delta(G)(\sum_{1 \leq i \leq k} d_{G_i}(x_i))$$

$$-\frac{1}{2}\Delta(G)k \geq 0$$

$$\Leftrightarrow (1 - \frac{1}{2}\Delta(G))(\sum_{1 \leq i \leq k} d_{G_i}(x_i)) - k \geq 0 \quad \text{(3)}$$

Furthermore, $d_{G_i}(x_i) = 0$ or 1 yields that $(\sum_{1 \leq i \leq k} d_{G_i}(x_i)) - k \leq 0$. Since $\Delta(G) \geq 2$, then $1 - \frac{1}{2}\Delta(G) \leq 0$. Thus, (3) is true, that is, Theorem 3 is true for the tree.

Next we will consider the case that $G$ has more than one component. In this case, each component of $G$ is either an isolated vertex or a tree, say $G_1, G_2, \ldots, G_s$ with an integer $s \geq 2$. For $1 \leq j \leq s$, if $G_j$ is an isolated vertex, then $i(G_j)/\gamma(G_j) = 1/1 \leq \Delta(G)/2$; If $G_j$ is a tree, by the above proof, $i(G_j)/\gamma(G_j) \leq \Delta(G)/2$. Finally, using Lemma 1, $i(G)/\gamma(G) \leq \Delta(G)/2$ holds for the forest. Furthermore, if $\Delta(G) \leq 2$, all forests achieve the bound; if $\Delta(G) \geq 3$, the union of balanced double stars attain the bound. Thus, Theorem 3 is true.

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