Bidirectional teleportation is a fundamental protocol for exchanging quantum information between two parties by means of a shared resource state and local operations and classical communication (LOCC). In this paper, we develop two seemingly different ways of quantifying the simulation error of unideal bidirectional teleportation by means of the normalized diamond distance and the channel infidelity, and we prove that they are equivalent. By relaxing the set of operations allowed from LOCC to those that completely preserve the positivity of the partial transpose, we obtain semi-definite programming lower bounds on the simulation error of unideal bidirectional teleportation. We evaluate these bounds for several key examples: when there is no resource state at all and for isotropic and Werner states, in each case finding an analytical solution. The first aforementioned example establishes a benchmark for classical versus quantum bidirectional teleportation. Another example consists of a resource state resulting from the action of a generalized amplitude damping channel on two Bell states, for which we find an analytical expression for the simulation error that is in agreement with numerical estimates (up to numerical precision). We then evaluate the performance of some schemes for bidirectional teleportation due to [Kiktenko et al., Phys. Rev. A 93, 062305 (2016)] and find that they are suboptimal and do not go beyond the aforementioned classical limit for bidirectional teleportation. We offer a scheme alternative to theirs that is provably optimal. Finally, we generalize the whole development to the setting of bidirectional controlled teleportation, in which there is an additional assisting party who helps with the exchange of quantum information, and we establish semi-definite programming lower bounds on the simulation error for this task. More generally, we provide semi-definite programming lower bounds on the performance of bipartite and multipartite channel simulation using a shared resource state and LOCC.

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I. INTRODUCTION

Quantum teleportation is one of the most remarkable protocols in quantum information [BBC+93], and it is one of the earliest examples of the fascinating possibilities that local operations and classical communication processing [BDSW96] on entangled states offers. Due to the fragile nature of qubits, the protocol was established as an alternative to direct transmission of a quantum state between two parties, as depicted in Figure 1. Teleportation is now used routinely as a basic primitive in quantum information science, with applications in quantum communication, quantum error correction, quantum networking, etc. See Figure 2 for a depiction of the teleportation protocol.

Teleportation has been extended in various ways, and one interesting way of doing so in a basic quantum network is via the method of bidirectional teleportation. In the ideal version of this protocol, Alice and Bob share two ebits of entanglement and teleport qubits to each other in opposite directions. The ideal protocol realizes a perfect swap channel, as shown in Figure 3. This possibility was observed early on [Vai94], and it was subsequently considered in [HVCP01, HPV02]. More recently, there has been a flurry of research on the topic, with various proposals for bidirectional teleportation [FTH14, HH16]. There has been even more interest recently in a variation called bidirectional controlled teleportation, using five-qubit [ZZQS13, SBP13, LN13, LLS+13, Che14], six-qubit [Yan13, SZ13, DZ14, LNLS16, ZXL19], seven-qubit [DZSX14, Hon16, San16], eight-qubit [ZZLY15, SZHA17], and nine-qubit [LJ16] entangled resource states (see also [TVP15]). Bidirectional controlled teleportation is a three-party protocol in which three parties, typically called Alice, Bob, and Charlie, share an entangled resource state, and they use local operations and classical communication to exchange qubits between Alice and Bob. See also [GSWL17] for other variations of bidirectional teleportation.

The applications of bidirectional teleportation align with those of standard, unidirectional teleportation, but they apply in a basic quantum network setting in which two parties would like to exchange quantum information. Although the ideal version of bidirectional teleportation is manifestly a trivial extension of the original protocol in which it is simply conducted twice (but in opposite directions), the situation becomes less trivial and more relevant to experimental practice when the resource state shared by the two parties deviates from the ideal resource of two maximally entangled states. Indeed, much of the prior work cited above focuses on precisely this kind of case, when the resource state is different from two maximally entangled states, either by being a different pure state, a mixed state, or a state with insufficient entanglement to accomplish the task. These kinds of investigations are important for understanding ways to simulate the ideal protocol approximately in an experimental setting. More generally, a way of framing bidirectional teleportation is that Alice and Bob share a resource state $\rho_{AB}$ and they are allowed local operations and classical communication (LOCC) for free, with the goal of simulating an ideal swap channel, as depicted in Figure 4.

In spite of the many works listed above on the topic of bidirectional teleportation, what appears to be missing is a systematic method for quantifying its performance in the case that it does not operate perfectly. There is a need for this because any experimental implementation of bidirectional teleportation will necessarily be imperfect. Indeed, entangled states generated in experimental pro-
FIG. 3. Ideal swap channel between two parties, Alice and Bob, realized by ideal bidirectional teleportation. This is a two-party generalization of the ideal unidirectional channel depicted in Figure 1.

FIG. 4. The figure depicts a general framework for understanding the simulation of bipartite quantum channels, realized by combining an LOCC protocol and a quantum resource state $\rho_{AB}$. In experimental implementations, the resource state $\rho_{AB}$ is imperfect. An example of a target bipartite channel to simulate is the ideal bidirectional teleportation, which is equivalent to a swap channel, as depicted in Figure 3.

tocols such as spontaneous parametric down-conversion are only approximations to ideal maximally entangled states [Cou18]. Our aim here is to fill this void.

The contributions of our paper are as follows:

1. After reviewing ideal bidirectional teleportation in Section III and recognizing that it implements a unitary swap channel between Alice and Bob (see Figure 5), we define two seemingly different ways to quantify the performance of unideal bidirectional teleportation by means of the normalized diamond distance and the channel infidelity (Sections IV A and IV B). We provide definitions of channel infidelity and diamond distance here, but they can also be found in Sections 3.5.2 and 3.5.3 of [KW20b], respectively. Even though these performance measures are generally different, we prove that they lead to the same values for the case of simulating the swap channel (Section IV C). More generally, we also discuss how these measures can be employed for quantifying the performance of bipartite channel simulation by means of a shared quantum state and local operations and classical communication (LOCC) (Section IV D).

2. Optimizing these performance measures is in general challenging because such optimizations are conducted over the set of LOCC channels, and it is known that optimizing over LOCC channels is difficult. We then relax the optimization problem such that it is conducted over the larger set of channels that completely preserve the positivity of the partial transpose (C-PPT-P), as shown in Figure 6 (Section VA). For both error measures (normalized diamond distance and channel infidelity), we show how the relaxed optimization problems can be evaluated by means of semi-definite programs (SDPs). See Sections VA and VC. SDPs are optimization problems in which the cost or objective function is linear, along with constraints and optimization variables that are semi-definite. This optimization technique is widely used in quantum information theory due to the semi-definite constraints that apply to the basic constituents of quantum mechanics (including states and channels). More information on semi-definite programs and C-PPT-P channels can be found in Sections 2.4 and 3.2.12 of [KW20b], respectively.

3. For the specific case of bidirectional teleportation, we show how symmetries of the unitary swap channel, some of which are depicted in Figure 7, lead to a much simpler semi-definite program for quantifying performance (Section VB). The resulting semi-
We then consider some specific examples of resource states for bidirectional teleportation (Section VI). These include the case when no resource state is available, as well as isotropic and Werner states (Sections VIA, VIB, and VIC, respectively). We also consider a resource state resulting from the action of a generalized amplitude damping channel on two Bell states (Section VID). In the first three cases, we reduce the semi-definite program for quantifying performance to a linear program, which we then solve analytically. The case when no resource state is available provides a benchmark for classical versus quantum bidirectional teleportation, and it is thus important for assessing any experimental implementation of bidirectional teleportation. Specifically, we prove that, when no resource state is available, the simulation error when simulating bidirectional teleportation of $d$-dimensional systems cannot be smaller than $1 - \frac{1}{2^d}$. For some particular values of the parameters for isotropic and Werner states, we also prove that the simulation error when using LOCC channels is equal to the simulation error when using channels that completely preserve the positivity of the partial transpose. For the generalized amplitude damping channel example, we find an analytical expression for the simulation error, which is correct up to numerical precision (thus we expect there to be an analytical proof).

5. Next, in Section VII, we use our previous results to assess the performance of previous proposals for bidirectional teleportation from [KPF16], which are for the case when the resource state available is a single ebit, instead of the required two ebits that are necessary for a perfect implementation of bidirectional teleportation. We find that the proposals from [KPF16] are suboptimal and do not go beyond the classical limit for bidirectional teleportation. We also provide a simple protocol that is provably optimal.

6. We finally generalize the whole development to quantify the performance of multipartite channel simulation, when using LOCC channels and a shared resource state for channel simulation (Section VIII A). We then analyze the specific case of bidirectional controlled teleportation and provide a semi-definite program that can be used to assess the performance of this latter protocol.

We note here that our general approach is similar in spirit to the approach taken in [IP05, H15], but the applications we consider here are different. We begin in the next section with some preliminary material and set some notation. The rest of our paper proceeds in the order presented above, and we finally conclude in Section IX with a summary and a list of open questions for future work.

II. PRELIMINARIES

In this section, we review some preliminary concepts in quantum information and set some notation used throughout the rest of our paper. More background on quantum information is available in [Hay17, Hol19, Wat18, Wil17, KW20b].

A quantum state is a positive semi-definite operator with trace equal to one. A bipartite quantum state $\rho_{AB}$ acts on a tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and we denote the dimension of system $A$ by $d_A$ and that of $B$ by $d_B$. A bipartite state $\rho_{AB}$ is entangled if it cannot be written in the following form:

$$\sum_x p(x) \sigma_x^A \otimes \tau_x^B,$$

where $\{p(x)\}_x$ is a probability distribution and $\{\sigma_x^A\}_x$ and $\{\tau_x^B\}_x$ are sets of states. For many discussions and applications of entanglement, see the review [HHHH09].

A quantum channel is a completely positive and trace-preserving map. We denote the unnormalized maximally entangled operator by

$$\Gamma_{RC} := |\Gamma\rangle \langle \Gamma |_{RC},$$

and we define the following operators:

$$|\Gamma\rangle_{RC} := \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_C,$$

where $R \simeq C$ with dimension $d$ and $\{|i\rangle_R\}_{i=0}^{d-1}$ and $\{|i\rangle_C\}_{i=0}^{d-1}$ are orthonormal bases. The notation $R \simeq C$...
means that the systems $R$ and $C$ are isomorphic. The maximally entangled state is denoted by

$$\Phi_{RC} := \frac{1}{d} \Gamma_{RC},$$

(4)

and the maximally mixed state by

$$\pi_C := \frac{1}{d} I_C.$$  

(5)

The Choi operator of a quantum channel $\mathcal{N}_{C \rightarrow D}$ (and more generally a linear map) is defined as

$$\Gamma^N_{RD} := \mathcal{N}_{C \rightarrow D}(\Gamma_{RC}).$$

(6)

In the notation above, there is an implicit identity channel acting on the reference system $R$, so that

$$\mathcal{N}_{C \rightarrow D}(\Gamma_{RC}) \equiv (id_R \otimes \mathcal{N}_{C \rightarrow D})(\Gamma_{RC}),$$

(7)

and we employ this convention throughout our paper. A linear map $\mathcal{M}_{C \rightarrow D}$ is completely positive if and only if its Choi operator $\Gamma^M_{RD}$ is positive semi-definite, and $\mathcal{M}_{C \rightarrow D}$ is trace preserving if and only if its Choi operator satisfies $\text{Tr}[\Gamma^M_{RD}] = I_R$.

An LOCC channel $\mathcal{L}_{A'B' \rightarrow AB}$ is a bipartite channel that can be written in the following form:

$$\mathcal{L}_{A'B' \rightarrow AB} = \sum_y E^y_{A' \rightarrow A} \otimes F^y_{B' \rightarrow B},$$

(8)

where $\{E^y_{A' \rightarrow A}\}_y$ and $\{F^y_{B' \rightarrow B}\}_y$ are sets of completely positive, trace-non-increasing maps, such that the sum map $\sum_y E^y_{A' \rightarrow A} \otimes F^y_{B' \rightarrow B}$ is a quantum channel (completely positive and trace preserving) [CLM+14]. However, not every channel of the form in (8) is an LOCC channel (there are separable channels of the form in (8) that are not implementable by LOCC [BDF+99]).

We make extensive use of the following bilateral unitary twirl channel in our paper:

$$\widetilde{T}_{CD}(X_{CD}) := \int dU \mathcal{U}_C \otimes \mathcal{U}_D(X_{CD}),$$

(9)

where $U(\cdot) = U(\cdot)U^\dagger$, $\mathcal{U}(\cdot) = \mathcal{U}(\cdot)U^T$, the overline indicates the complex conjugate, and $dU$ denotes the Haar measure (uniform distribution on unitary operators). This channel is an LOCC channel, in the sense that Alice can pick a unitary at random according to the Haar measure, apply it to her system, report to Bob which one she selected, who can then apply the complex conjugate unitary to his system. In order to make this process feasible in practice, note that the channel in (9) can be simulated by a unitary two-design [Mat14], in which only a finite amount of classical data is required to communicate to Bob when implementing (9) via LOCC. The following identity from [Wer89, HH99, Wat18] simplifies the calculation of the action of $\widetilde{T}_{CD}$ on an arbitrary input operator $X_{CD}$:

$$\widetilde{T}_{CD}(X_{CD}) = \Phi_{CD} \text{Tr}_{CD}[\Phi_{CD} X_{CD}] + \frac{I_{CD} - \Phi_{CD}}{d^2 - 1} \text{Tr}_{CD}[(I_{CD} - \Phi_{CD}) X_{CD}].$$

(10)

We denote the transpose map acting on the quantum system $C$ by

$$T_C(\cdot) := \sum_{i,j=0}^{d-1} |i\rangle_C \langle j| C(\cdot) |i\rangle_C |j\rangle_C.$$  

(11)

A state $\rho_{CD}$ is a positive partial transpose (PPT) state if $T_D(\rho_{CD})$ is positive semi-definite. The partial transpose is its own adjoint, in the sense that

$$\text{Tr}[Y_{CD} T_C(X_{CD})] = \text{Tr}[T_C(Y_{CD}) X_{CD}],$$

(12)

for all linear operators $X_{CD}$ and $Y_{CD}$.

The following post-selected teleportation identity [Ben05] plays a role in our analysis:

$$\mathcal{N}_{C \rightarrow D}(\rho_{SC}) = (\Gamma_C \otimes \rho_{SC} \otimes \Gamma^N_{RD}) \Gamma_{CR}.$$  

(13)

We also make frequent use of the identities

$$\text{Tr}_{C}[X_{CD}] = (\Gamma_{RC}(I_R \otimes X_{CD})) \Gamma_{RC},$$  

(14)

$$X_{CD}[\Gamma]_{CR} = T_R(X_{RD}) \Gamma_{CR}.$$  

(15)

Given channels $\mathcal{N}_{C \rightarrow D}$ and $\mathcal{M}_{D \rightarrow E}$, the Choi operator $\Gamma^\mathcal{M}_{SE}$ of the serial composition $\mathcal{M}_{D \rightarrow E} \circ \mathcal{N}_{C \rightarrow D}$ is given by

$$\Gamma^\mathcal{M}\mathcal{N}_{SE} = \langle \Gamma \mid D \Gamma \otimes \Gamma^\mathcal{N}_{SE} \rangle = \text{Tr}_D[\Gamma^\mathcal{N}_{RD}] T_D(\Gamma^\mathcal{M}_{DE}),$$

(16)

where $D \equiv S$, the operator $\Gamma^\mathcal{N}_{RD}$ is the Choi operator of $\mathcal{N}_{C \rightarrow D}$, and $\Gamma^\mathcal{M}_{SE}$ is the Choi operator of $\mathcal{M}_{D \rightarrow E}$.

### III. IDEAL BIDIRECTIONAL TELEPORTATION

Let us examine the case of ideal bidirectional teleportation on two qudits in detail [Vai94]. Doing so is helpful for us in establishing a basic metric for the performance of unideal bidirectional teleportation. Put simply, ideal bidirectional teleportation consists of an ideal unidirectional teleportation [BBC+93] from Alice to Bob and an ideal unidirectional teleportation from Bob to Alice, where Alice and Bob are two spatially separated parties. As such, the protocol uses entanglement and classical communication to simulate the following unitary swap channel:

$$S_{AB}^d(\rho_{AB}) := F_{AB}(\rho_{AB}) F_{AB}^\dagger,$$

(18)

where the unitary swap or flip operator $F$ is defined as

$$F_{AB} := \sum_{i,j=0}^{d-1} |i\rangle_A \otimes |j\rangle_B.$$  

(19)
In the above, $A \simeq B$, and $\{|i\rangle_A\}_{i=0}^{d-1}$ and $\{|i\rangle_B\}_{i=0}^{d-1}$ are orthonormal bases. Denoting the identity operator from Alice to Bob by $I_{A \rightarrow B}$

$$I_{A \rightarrow B} := \sum_{i=0}^{d-1} |i\rangle_B \langle i|_A, \quad (20)$$

and the identity operator from Bob to Alice by $I_{B \rightarrow A}$:

$$I_{B \rightarrow A} := \sum_{i=0}^{d-1} |i\rangle_A \langle i|_B, \quad (21)$$

we see that

$$F_{AB} = \sum_{i,j=0}^{d-1} |i\rangle_A \langle j|_A \otimes |j\rangle_B \langle i|_B \quad (22)$$

$$= \sum_{i,j=0}^{d-1} |j\rangle_B \langle j|_A \otimes |i\rangle_A \langle i|_B \quad (23)$$

$$= \sum_{j=0}^{d-1} |j\rangle_B \langle j|_A \otimes \sum_{i=0}^{d-1} |i\rangle_A \langle i|_B \quad (24)$$

$$= I_{A \rightarrow B} \otimes I_{B \rightarrow A}. \quad (25)$$

The identity operators realize the following identity channels

$$\text{id}_{A \rightarrow B}(\cdot) := I_{A \rightarrow B}(\cdot)(I_{A \rightarrow B})^{\dagger},$$

$$\text{id}_{B \rightarrow A}(\cdot) := I_{B \rightarrow A}(\cdot)(I_{B \rightarrow A})^{\dagger},$$

and we see that the ideal swap channel is equivalent to

$$S_{AB}^d = \text{id}_{B \rightarrow A} \otimes \text{id}_{A \rightarrow B}. \quad (28)$$

Even though our choice of notation might suggest that the swap channel is a tensor product of local identity channels, we should note that this is not the case: the swap channel is a global channel that cannot be realized by local actions alone. Our notation $\text{id}_{A \rightarrow B}$ indicates that Alice’s input system $A$ is placed at Bob’s output $B$ and the notation $\text{id}_{B \rightarrow A}$ indicates that Bob’s input system $B$ is placed at Alice’s output $A$.

In more detail, recall that the standard, ideal unidirectional teleportation protocol [BBC+93] begins with Alice and Bob sharing the following maximally entangled resource state:

$$\Phi_{AB}^d := \frac{1}{d} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B, \quad (29)$$

where $\hat{A} \simeq B$ and $\{|i\rangle_A\}_{i=0}^{d-1}$ and $\{|i\rangle_B\}_{i=0}^{d-1}$ are orthonormal bases. The amount of entanglement in this state is $\log_2 d$ [BBPS96], and so the state above is said to be equivalent to $\log_2 d$ ebits. Alice then prepares the system $A$ in the state $\rho_A$, where $A \simeq \hat{A}$, so that the overall state is

$$\rho_A \otimes \Phi_{AB}^d. \quad (30)$$

Alice performs a Bell measurement on systems $A\hat{A}$, which is specified in terms of the following measurement operators:

$$\{\Phi_{AA}^x \}_{z,x \in \{0, \ldots, d-1\}}, \quad (31)$$

where

$$\Phi_{AA}^x := (W^z_{\hat{A}} \otimes I_A)\Phi_{\hat{A}A}^d (W^z_{\hat{A}} \otimes I_A)^{\dagger}, \quad (32)$$

$$W^z_{\hat{A}} := Z(z)X(z), \quad (33)$$

$$Z(z) := \sum_{k=0}^{d-1} e^{\frac{2\pi i k z}{d}} |k\rangle\langle k|, \quad (34)$$

$$X(z) := \sum_{k=0}^{d-1} |k \oplus_d z\rangle\langle k|, \quad (35)$$

and $\oplus_d$ denotes addition modulo $d$. Defining the Bell measurement in terms of the following quantum instrument:

$$\mathcal{B}_{\hat{A}A \rightarrow A\hat{A}CA} (\omega_{\hat{A}A}) := \sum_{z,x=0}^{d-1} \Phi_{AA}^x \omega_{\hat{A}A} \Phi_{\hat{A}A}^x \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C, \quad (36)$$

the following identity holds

$$\mathcal{B}_{\hat{A}A \rightarrow A\hat{A}CA} (\rho_A \otimes \Phi_{AB}^d)$$

$$= \sum_{z,x=0}^{d-1} \Phi_{AA}^x (\rho_A \otimes \Phi_{AB}^d) \Phi_{\hat{A}A}^x \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C, \quad (37)$$

$$= \frac{1}{d^2} \sum_{z,x=0}^{d-1} \Phi_{AA}^x \otimes (W^z_B)^{\dagger} \rho_B W^z_B \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C. \quad (38)$$

Alice traces out the systems $A\hat{A}$, leaving the state

$$(\text{Tr}_{\hat{A}A} \circ \mathcal{B}_{\hat{A}A \rightarrow A\hat{A}CA})(\rho_A \otimes \Phi_{AB}^d)$$

$$= \frac{1}{d^2} \sum_{z,x=0}^{d-1} (W^z_B)^{\dagger} \rho_B W^z_B \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C. \quad (39)$$

Alice then communicates the classical register $C_A$ to Bob over a $d^2$-dimensional classical channel, defined by

$$\Delta_{A \rightarrow C_B} (\cdot) := \sum_{z,x=0}^{d-1} |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C. \quad (40)$$

The amount of classical information that can be communicated by this channel is $2 \log_2 d$ bits. Bob finally performs the following correction channel on systems $BC_B$:

$$\mathcal{C}_{BC_B \rightarrow B} (\omega_{BZX}) := \sum_{z,x} W^z_B \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C \otimes |z\rangle\langle z|_C \otimes |x\rangle\langle x|_C. \quad (41)$$

so that
\[ C_{BC_B \rightarrow B} \left( \frac{1}{d^2} \sum_{z,x=0}^{d-1} (W_B^{z,x})^\dagger \rho_B W_B^{z,x} \otimes |z, x\rangle\langle z, x|_{CB} \right) = \rho_B. \] (42)

Since the state \( \rho_A \) on Alice’s system is perfectly reconstructed on Bob’s system \( B \) at the end of this process, we conclude that the whole process simulates the identity channel \( \text{id}_{A \rightarrow B} \). In more detail, let us denote the channel realized by the whole protocol as

\[ T_{A \rightarrow B} := C_{BC_B \rightarrow B} \circ \Sigma_{C_A \rightarrow C_B} \circ \text{Tr}_{A\tilde{A}} \circ B_{A\tilde{A} \rightarrow A\tilde{A}C_A} \circ P_{\Phi}. \] (43)

The discussion above then argues that

\[ T_{A \rightarrow B} = \text{id}_{A \rightarrow B}. \] (44)

See, e.g., [Wil17, Section 6.5.3] for a more detailed argument.

The unidirectional teleportation protocol can be run in the opposite direction, from Bob to Alice, and this process realizes the channel \( T_{B \rightarrow A} \). Following the same argument above, but swapping the roles of Alice and Bob, the equality \( T_{B \rightarrow A} = \text{id}_{B \rightarrow A} \) holds. Thus, by consuming \( 2 \log_2 d \) ebits, \( 2 \log_2 d \) bits of classical communication from Alice to Bob, and \( 2 \log_2 d \) bits of classical communication from Bob to Alice, they can realize the ideal swap channel in (18):

\[ S_{AB}^d = T_{A \rightarrow B} \otimes T_{B \rightarrow A}. \] (45)

IV. QUANTIFYING THE PERFORMANCE OF UNIDEAL BIDIRECTIONAL TELEPORTATION

Having established that ideal bidirectional teleportation realizes an ideal swap channel, let us now discuss unideal bidirectional teleportation. Succinctly, the goal of unideal bidirectional teleportation is to use a resource state \( \rho_{AB} \) and local operations and classical communication (LOCC) to simulate a \( d \)-dimensional swap channel of the form in (18), where

\[ d = d_A = d_B. \] (46)

The goal is to make the error between the simulation and the ideal swap channel as small as possible.

In more detail, we assume that Alice and Bob share a quantum state \( \rho_{AB} \), instead of two maximally entangled states of the form in (29). Such a state could be generated by an unideal experimental process, such as spontaneous parametric down-conversion [Cou18]. We make no assumption about the dimensions of systems \( \tilde{A} \) and \( \tilde{B} \), other than that they are finite dimensional. In particular, it need not be the case that \( d_{\tilde{A}} \) is equal to \( d_{\tilde{B}} \). There are two other systems \( A \) and \( B \), that serve as inputs for Alice and Bob, respectively, to the unideal bidirectional teleportation. They then act with an LOCC channel \( \mathcal{L}_{AB\tilde{A}B \rightarrow AB} \), on their input systems \( A \) and \( B \) and their shares \( \tilde{A} \) and \( \tilde{B} \) of the resource state \( \rho_{\tilde{A}B} \), to produce the output systems \( A \) and \( B \). As mentioned in Section II, the LOCC channel \( \mathcal{L}_{AB\tilde{A}B \rightarrow A\tilde{A}B} \) can be written as

\[ \mathcal{L}_{AB\tilde{A}B \rightarrow AB} = \sum_y \mathcal{E}_{A \rightarrow A}^y \otimes \mathcal{F}_{BB \rightarrow B}^y, \] (47)

where \( \{ \mathcal{E}_{A \rightarrow A}^y \}_y \) and \( \{ \mathcal{F}_{BB \rightarrow B}^y \}_y \) are sets of completely positive maps such that \( \mathcal{L}_{AB\tilde{A}B \rightarrow AB} \) is a quantum channel. Thus, the overall channel realized by the simulation is as follows:

\[ \tilde{S}_{AB}(\omega_{AB}) := \mathcal{L}_{AB\tilde{A}B \rightarrow AB}(\omega_{AB} \otimes \rho_{AB}), \] (48)

which is depicted in Figure 4. In the simulation, we allow for classical communication between Alice and Bob for free, so that \( \mathcal{L}_{AB\tilde{A}B \rightarrow AB} \) can be considered a free channel, as is common in the resource theory of entanglement [BDSW96, CG19].

A. Quantifying error with normalized diamond distance

Let us now discuss how to quantify the simulation error. The standard metric for doing so is the normalized diamond distance [Kit97], having been used in both quantum computation [Kit97] and quantum information [Wat18, Wil17]. In short, this metric quantifies the maximum absolute deviation between the probabilities of observing the same outcome when each quantum channel is applied to the same input state and the same measurement is made.

Let us elaborate upon the explanation of diamond distance given above, in a similar way to the motivation presented in [WW19b, Wil20]. Suppose that the ideal channel to be implemented is \( \mathcal{N}_{C \rightarrow D} \), shown in Figure 8. Suppose further that \( \tilde{\mathcal{N}}_{C \rightarrow D} \) is the simulation of \( \mathcal{N}_{C \rightarrow D} \), shown in Figure 9. To interface with these channels and obtain classical data for the purpose of distinguishing them, the most general way for doing so is to prepare a
where the diamond distance calculation of the diamond distance simplifies as follows:
\[ \sum_{x \in \mathcal{X}} x^x = I_{RD} \text{ and } x^x_{RD} \geq 0 \text{ for all } x \in \mathcal{X}. \]

The result of this procedure (preparation, channel evolution, and measurement) is a classical outcome \( N \) that occurs with probability \( \text{Tr}[\Lambda^x_{RD} N_{C \rightarrow D}(\rho_{RC})] \) if the channel \( N_{C \rightarrow D} \) is applied, while the outcome \( x \in \mathcal{X} \) occurs with probability \( \text{Tr}[\Lambda^x_{RD} N_{C \rightarrow D}(\rho_{RC})] \) if the channel \( N_{C \rightarrow D} \) is applied. The error or difference between these probabilities is naturally quantified by the absolute deviation
\[
\left| \text{Tr}[\Lambda^x_{RD} N_{C \rightarrow D}(\rho_{RC})] - \text{Tr}[\Lambda^x_{RD} N_{C \rightarrow D}(\rho_{RC})] \right|. \tag{49}
\]

We can then quantify the maximum possible error between the channels \( N_{C \rightarrow D} \) and \( \tilde{N}_{C \rightarrow D} \) by optimizing (49) with respect to all preparations and measurements:
\[
\sup_{\rho_{RC}, \Lambda_{RD}} \left| \text{Tr}[\Lambda^x_{RD} N(\rho_{RC})] - \text{Tr}[\Lambda^x_{RD} \tilde{N}(\rho_{RC})] \right| = \sup_{\rho_{RC}, \Lambda_{RD}} \left| \text{Tr}[\Lambda^x_{RD} N(\rho_{RC})] - \text{Tr}[\Lambda^x_{RD} \tilde{N}(\rho_{RC})] \right|, \tag{50}
\]

where it is implicit that the channels \( N_{C \rightarrow D} \) and \( \tilde{N}_{C \rightarrow D} \) above have input system \( C \) and output system \( D \). Mathematically, this optimization has the effect of removing the dependence on the preparation and measurement such that the error is a function solely of the two channels \( N_{C \rightarrow D} \) and \( \tilde{N}_{C \rightarrow D} \). It is a fundamental and well known result in quantum information theory [Kit97, AKN98] that the error in (50) is equal to the normalized diamond distance:
\[
\text{Eq. (50)} = \frac{1}{2} \left\| N - \tilde{N} \right\|_\diamond, \tag{51}
\]
where the diamond distance \( \left\| N - \tilde{N} \right\|_\diamond \) is defined as
\[
\left\| N - \tilde{N} \right\|_\diamond := \sup_{\rho_{RC}} \left\| N_{C \rightarrow D}(\rho_{RC}) - \tilde{N}_{C \rightarrow D}(\rho_{RC}) \right\|_1, \tag{52}
\]
and the trace norm of an operator \( X \) is given by \( \left\| X \right\|_1 = \text{Tr}[|X|^*X] \), where \( |X| := \sqrt{X^*X} \). It is well known that the calculation of the diamond distance simplifies as follows:
\[
\left\| N - \tilde{N} \right\|_\diamond = \sup_{\psi_{RC}} \left\| N_{C \rightarrow D}(\psi_{RC}) - \tilde{N}_{C \rightarrow D}(\psi_{RC}) \right\|_1, \tag{53}
\]
where the optimization is with respect to every pure bipartite state \( \psi_{RC} \) with system \( R \) isomorphic to the channel input system \( C \).

Intuitively, the diamond distance can be thought of as a metric characterizing the distinguishability of two quantum channels. As indicated in the previous paragraph, this metric quantifies the maximum absolute deviation between the probabilities of observing the same outcome when each quantum channel is applied to the same input state and the same measurement is made. It is used as a way to quantify the distance between two quantum channels and is a standard metric used in quantum computation and quantum information.

The normalized diamond distance can be computed by means of the following semi-definite program [Wat09]:
\[
\frac{1}{2} \left\| N - \tilde{N} \right\|_\diamond = \inf_{z_{RD} \geq 0} \left\{ \left\| \text{Tr}_D [Z_{RD}] \right\|_\infty : Z_{RD} \geq \Gamma^N_{RD} - \Gamma^\tilde{N}_{RD} \right\}, \tag{54}
\]

where \( \Gamma^N_{RD} \) and \( \Gamma^\tilde{N}_{RD} \) are the Choi operators of \( N \) and \( \tilde{N} \), respectively. This will be helpful for us later on in Sections V A and V B.

Returning to our case of interest, the simulation error when employing a specific LOCC channel \( \mathcal{L}_{AB \rightarrow AB} \) is quantified as follows:
\[
e_{\text{LOCC}}(S^d_{AB}, \rho_{\tilde{A}B}, \mathcal{L}_{AB \rightarrow AB}) := \frac{1}{2} \left\| S^d - \tilde{S} \right\|_\infty, \tag{55}
\]
where \( d \) is the dimension of the swap channel \( S^d \) (see (46)), \( S^d \) is defined in (18), and \( \tilde{S} \) in (48). Since we are interested in the minimum possible simulation error, taken over all possible LOCC channels, we define the simulation error of bidirectional teleportation, when employing the resource state \( \rho_{\tilde{A}B} \), as follows:
\[
e_{\text{LOCC}}(S^d_{AB}, \rho_{\tilde{A}B}) := \inf_{\mathcal{L} \in \text{LOCC}} e_{\text{LOCC}}(S^d_{AB}, \rho_{\tilde{A}B}, \mathcal{L}_{AB \rightarrow AB}). \tag{56}
\]

This simulation error is difficult to compute as \( d, d_A \), and \( d_B \) become larger. This computational difficulty is related to how it is difficult to optimize functions over the set of separable states [Gur04, Gha10, HM13]. In Section V B, we determine a lower bound on the simulation error \( e_{\text{LOCC}}(S^d_{AB}, \rho_{\tilde{A}B}) \) that can be computed by means of semi-definite programming [VB96] and is thus efficiently computable. For some states \( \rho_{\tilde{A}B} \) of interest, the lower bound is achievable, so that we can determine the error of unideal bidirectional teleportation precisely in these cases.

### B. Quantifying error with channel infidelity

Another way to quantify error between channels is by using the fidelity. Recall that the fidelity of quantum states \( \omega \) and \( \tau \) is defined as follows [Uhl76]:
\[
F(\omega, \tau) := \left\| \sqrt{\omega} \sqrt{\tau} \right\|_1^2. \tag{57}
\]
This quantity is equal to one if and only if the states $\omega$ and $\tau$ are the same, and it is equal to zero if and only if the states are orthogonal. If $\omega$ is a pure state, i.e., equal to $|\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle$, then the fidelity reduces to the following expression:

$$F(|\psi\rangle\langle\psi|, \tau) = \langle\psi|\tau|\psi\rangle. \quad (58)$$

In this case, it has the operational meaning that $F(|\psi\rangle\langle\psi|, \tau)$ is the probability with which the state $\tau$ passes a test for being the state $|\psi\rangle\langle\psi|$. The test in this case is given by the binary measurement $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$, and the first outcome corresponds to the decision “pass.” So the probability of passing is equal to $F(|\psi\rangle\langle\psi|, \tau)$.

We can then lift this to a measure of similarity for quantum channels $N_{C \rightarrow D}$ and $\tilde{N}_{C \rightarrow D}$ as follows:

$$F(N, \tilde{N}) := \inf_{\rho_{RC}} F(N_{C \rightarrow D}(\rho_{RC}), \tilde{N}_{C \rightarrow D}(\rho_{RC})), \quad (59)$$

which can be viewed as the fidelity counterpart of the diamond distance in (52). Just like (53), the following simplification holds

$$F(N, \tilde{N}) = \inf_{\psi_{RC}} F(N_{C \rightarrow D}(\psi_{RC}), \tilde{N}_{C \rightarrow D}(\psi_{RC})), \quad (60)$$

where the optimization is with respect to all pure bipartite states $\psi_{RC}$ with system $R$ isomorphic to the channel input system $C$. We note here that the channel infidelity is defined as

$$1 - F(N, \tilde{N}). \quad (61)$$

Since the channel fidelity is a measure of similarity, the channel infidelity is a measure of distinguishability and thus can be understood as an error measure in our context. Thus, in what shortly follows, we employ it as a simulation error, with the goal of minimizing it.

The root fidelity of channels can be computed by means of the following semi-definite program [YF17, KW20a]:

$$\sqrt{F}(N, \tilde{N}) = \frac{1}{2} \inf_{\rho_{WRD, ZRD}} \text{Tr}[W_{RD} \Gamma_{RD}^N + Z_{RD} \Gamma_{RD}^\tilde{N}],$ \quad (62)$$

subject to

$$\rho_R \geq 0, \quad \text{Tr}[\rho_R] = 1, \quad \begin{bmatrix} W_{RD} & \rho_R \otimes I_D & Z_{RD} \end{bmatrix} \geq 0. \quad (63)$$

In the above, the optimization is over all linear operators $W_{RD}$ and $Z_{RD}$, and $\Gamma_{RD}^N$ and $\Gamma_{RD}^\tilde{N}$ are the Choi operators of $N$ and $\tilde{N}$, respectively. The dual of this semi-definite program is given by

$$\sup_{\lambda \geq 0, Q_{RD}} \lambda \quad (64)$$

subject to

$$\lambda I_R \leq \text{Re}[\text{Tr}[Q_{RD}]], \quad (65)$$

Using the infidelity of channels, we can define an alternate notion of simulation error as follows:

$$e^{F}_{\text{LOCC}}(S_{AB}^d, \rho_{AB}, L_{AB\tilde{A}B\rightarrow AB}) := 1 - F(S_{AB}^d, \tilde{S}), \quad (67)$$

where $d$ is the dimension of the swap channel $S_{AB}^d$ (see (46)), $S_{AB}^d$ is defined in (18), and $\tilde{S}$ in (48). Minimizing this error with respect to all LOCC channels, we arrive at the following:

$$e^{F}_{\text{LOCC}}(S_{AB}^d, \rho_{AB}) := \inf_{L \in \text{LOCC}} e^{F}_{\text{LOCC}}(S_{AB}^d, \rho_{AB}, L_{AB\tilde{A}B\rightarrow AB}). \quad (68)$$

For the same reasons given previously, this quantity is difficult to compute, and so we seek alternative ways to estimate it.

### C. Equality of simulation errors when simulating the swap channel

Even though we have defined two different notions of LOCC simulation error of bidirectional teleportation based on the normalized diamond distance and channel infidelity, it turns out that they are equal. This result follows as a consequence of the swap channel $S_{AB}^d$ in (18) having the following symmetry:

$$S_{AB}^d(\mathcal{U}_A \otimes \mathcal{V}_B) = (\mathcal{V}_A \otimes \mathcal{U}_B) S_{AB}^d, \quad (69)$$

holding for all unitary channels $\mathcal{U}_A$ and $\mathcal{V}_B$. An additional symmetry of the swap channel $S_{AB}^d$ is that it commutes with itself:

$$S_{AB}^d \circ S_{AB}^d = S_{AB}^d \circ S_{AB}^d. \quad (70)$$

Although at first glance this latter symmetry might seem trivial, it is actually helpful in further simplifying the optimization problem for bidirectional teleportation. More generally, a unitary $U$ is a symmetry of a channel $N$ if it commutes with the action of the channel: $N \circ U = U \circ N$. Clearly, the equalities in (69) and (70) represent symmetries of the swap channel $S_{AB}^d$.

By exploiting the symmetries in (69) and (70), we prove in Appendix A that it suffices to optimize both the normalized diamond distance and the channel infidelity with respect to LOCC channels $L_{AB\tilde{A}B\rightarrow AB}$ having the following form:

$$L_{AB\tilde{A}B\rightarrow AB}(\omega_{AB} \otimes \rho_{AB}) = S_{AB}^d(\omega_{AB}) \text{Tr}[K_{\tilde{A}B} \rho_{AB}] + \frac{1}{2} \text{Re}[\text{Tr}[Q_{RD}]], \quad (71)$$

$$+ (D_{A \rightarrow B} \otimes D_{B \rightarrow A} + (\mathcal{D}_{A \rightarrow B} \otimes \mathcal{D}_{B \rightarrow A}) (\omega_{AB}) \text{Tr}[L_{\tilde{A}B} \rho_{AB}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A} (\omega_{AB}) \text{Tr}[N_{AB} \rho_{AB}]. \quad (71)$$
where $D$ denotes the following generalized Pauli channel:

$$
D(\sigma) := \frac{1}{d^2-1} \sum_{(x,z)\neq (0,0)} W^{z,x} \sigma (W^{z,x})^\dagger. \quad (72)
$$

Thus, the interpretation of the simulating channel is that it measures the resource state $\rho_{\tilde{A}\tilde{B}}$ according to the POVM $\{K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}}, N_{\tilde{A}\tilde{B}}\}$, which is subject to the constraint that the overall channel $L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}$ is LOCC.

After that, it takes the following action:

1. If the first outcome $K_{\tilde{A}\tilde{B}}$ occurs, then apply the ideal swap channel to the input state $\omega_{AB}$.

2. If the second outcome $L_{\tilde{A}\tilde{B}}$ occurs, then with probability $1/2$, apply the identity channel $\text{id}_{A\rightarrow B}$ to transfer Alice’s input system $A$ to Bob, but then garble Bob’s input system $B$ by applying the channel $D$ and transfer the resulting system to Alice; with probability $1/2$, apply the identity channel $\text{id}_{B\rightarrow A}$ to transfer Bob’s input system $B$ to Alice, but then garble Alice’s input system $A$ by applying the channel $D$ and transfer the resulting system to Bob.

3. If the third outcome $N_{\tilde{A}\tilde{B}}$ occurs, then apply the garbling channel $D$ to both Alice and Bob’s systems individually and exchange them.

We again stress that the constraint on the set $\{K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}}, N_{\tilde{A}\tilde{B}}\}$ is that the overall channel $L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}$ is LOCC.

We state the equality of the simulation errors as follows and prove this result in Appendix A:

**Proposition 1** The optimization problems in (56) and (68), for the error in simulating the unitary SWAP channel $S_d$ in (18), simplify as follows:

$$
\epsilon_{\text{LOCC}}(S_d^{AB}, \rho_{\tilde{A}\tilde{B}}) = \epsilon_{\text{LOCC}}(S_d^{AB}, \rho_{\tilde{A}\tilde{B}}) = 1 - \sup_{K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}}, N_{\tilde{A}\tilde{B}}} \text{Tr}[\rho_{AB} K_{\tilde{A}\tilde{B}}], \quad (73)
$$

subject to $K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}} + N_{\tilde{A}\tilde{B}} = I_{\tilde{A}\tilde{B}}$ and the following channel $L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}$ being LOCC:

$$
L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}(\omega_{AB}) = S_d^{\dagger}(\text{Tr}_{\tilde{A}\tilde{B}}[K_{\tilde{A}\tilde{B}} \omega_{AB\tilde{A}\tilde{B}}]) + \frac{1}{2} \left( \text{id}_{A\rightarrow B} \otimes D_{B\rightarrow A} + D_{A\rightarrow B} \otimes \text{id}_{B\rightarrow A} \right) (\text{Tr}_{\tilde{A}\tilde{B}}[L_{\tilde{A}\tilde{B}} \omega_{AB\tilde{A}\tilde{B}}]) + (D_{A\rightarrow B} \otimes D_{B\rightarrow A}) (\text{Tr}_{\tilde{A}\tilde{B}}[N_{\tilde{A}\tilde{B}} \omega_{AB\tilde{A}\tilde{B}}]). \quad (74)
$$

As a consequence of Proposition 1, there is no need for two different notions of simulation error when considering the simulation of the swap channel.

**Remark 2** If one had to pick one error metric over the other, we think the diamond distance is preferable for comparing general channels. It captures a notion of error that makes physical sense as the largest deviation in outcome probabilities that could be observed by performing the most general physical procedure to distinguish an ideal channel from its simulation. Related to this, it has an operational interpretation in terms of hypothesis testing of channels. It also has nice properties like the triangle inequality, data processing under the action of a superchannel, and stability under tensoring with the identity. For these reasons, it is the standard theoretical tool used in the study of fault tolerant quantum computation, and one can consult [Wat18] and find it used to define quantum channel capacities. Thus we are using it here also.

The channel fidelity has a sensible operational interpretation if the target channel is a unitary channel, as the probability with which the simulation channel can pass a test for being the unitary channel. It also possesses the properties of stability and data processing mentioned above, and if one takes the square root of the infidelity (often called sine distance), then it also obeys the triangle inequality.

So we view both of these error metrics as being important and thus we have considered them both here. In light of the fact that these error metrics are generally different, we find it an interesting conclusion that the normalized diamond distance and the infidelity give the same value when considering simulation of the SWAP channel.

### D. LOCC simulation of general bipartite channels

In the previous sections, we discussed how to quantify the simulation error for ideal bidirectional teleportation. In this section, we generalize the task to the LOCC simulation of an arbitrary bipartite channel.

A bipartite channel $N_{AB\rightarrow A'B'}$, depicted in Figure 10, is a quantum channel with input systems $A$ and $B$ and output systems $A'$ and $B'$ [BHLS03, CLL06]. Alice has control of the systems $A$ and $A'$, and Bob has control of the output systems $B$ and $B'$. The swap channel in (18) is a particular example of a bipartite channel, but of course there are many other interesting examples, such as the controlled-NOT gate.

We can thus generalize the simulation task in the previous section to be about simulating a general bipartite channel $N_{AB\rightarrow A'B'}$. This was considered for point-to-point channels in [BDSW96, HHH99] and for bipartite channels in [BHLS03, BDWW19, GS19]. In this case, the simulating channel is defined similarly to (48):

$$
N_{AB\rightarrow A'B'}(\omega_{AB}) := L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}(\omega_{AB} \otimes \rho_{\tilde{A}\tilde{B}}), \quad (75)
$$

where $\rho_{\tilde{A}\tilde{B}}$ is a resource state and $L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}$ is an LOCC channel. The simulation error when employing a specific LOCC channel $L_{A\tilde{A}B\tilde{B}A\rightarrow A'B'}$ is quantified as
\[ T_B \cdot T_{B'} \cdot (N_{AB \rightarrow A'B'} \cdot \rho_{AB}) = \sum_{i,j} |i\rangle \langle j| B \omega_B |i\rangle \langle j| B, \] 

where \( T_B \) denotes the transpose map, defined by 

\[
T_B(\omega_B) = \sum_{i,j} |i\rangle \langle j| B \omega_B |i\rangle \langle j| B, \tag{82}
\]

and with \( T_{B'} \) defined similarly on the system \( B' \). It is well known that every LOCC channel is a C-PPT-P channel [Rai99, Rai01], but the reverse containment does not hold. Thus,

\[
\text{LOCC} \subset \text{C-PPT-P}, \tag{83}
\]

as depicted in Figure 6.

Having defined this set of channels, we define the simulation error under C-PPT-P channels as follows:

\[
e_{\text{PPPT}}(N_{AB \rightarrow A'B'}, \rho_{AB}) := \inf_{P \in \text{C-PPT-P}} \frac{1}{2} \sum_{i,j} \| N_{AB \rightarrow A'B'} - \tilde{N}_{AB \rightarrow A'B'} \|_\diamond , \tag{84}
\]

where the optimization is with respect to C-PPT-P channels \( P_{AB \rightarrow A'B'} \)

\[
\tilde{N}_{AB \rightarrow A'B'}(\omega_{AB}) := P_{AB \rightarrow A'B'}(\omega_{AB} \otimes \rho_{AB}). \tag{85}
\]

We note here that both of the optimization problems given in (78) and (84) are special cases of the optimization discussed in [FWTB20, Section II].

Furthermore, the following bound holds due to the containment in (83):

\[
e_{\text{PPPT}}(N_{AB \rightarrow A'B'}, \rho_{AB}) \leq e_{\text{LOCC}}(N_{AB \rightarrow A'B'}, \rho_{AB}). \tag{86}
\]

We now discuss how the error in (84) can be computed by means of a semi-definite program. To do so, we need to review a few key concepts. First, recall from [Wat09] that the normalized diamond distance between channels \( N_{C \rightarrow D} \) and \( M_{C \rightarrow D} \) can be computed by means of the following semi-definite program:

\[
\frac{1}{2} \| N_{C \rightarrow D} - M_{C \rightarrow D} \|_\diamond = \inf_{\mu \geq 0, Z_{RD} \geq 0} \left\{ \mu : \mu R \geq Z_R, Z_{RD} \geq \Gamma_{RD}^N - \Gamma_{RD}^M \right\}, \tag{87}
\]

where \( \Gamma_{RD}^N \) is the Choi operator of the channel \( N_{C \rightarrow D} \), defined as

\[
\Gamma_{RD}^N := N_{C \rightarrow D}(\Gamma_{RC}), \tag{88}
\]

\[
\Gamma_{RC} := \Gamma \Gamma_{RC} \tag{89}
\]

and system \( R \) is isomorphic to the channel input system \( A \). The Choi operator \( \Gamma_{RD}^M \) is defined similarly. Thus, when calculating the error in (84), we can employ this semi-definite program, as well as the semi-definite constraints corresponding to the optimization over C-PPT-P channels.
To arrive at the desired conclusion, let us recall some facts about quantum channels and their Choi operators. A quantum channel has two properties: it should be completely positive and trace preserving. The relation of these properties to the Choi operator is given by the following:

1. a linear map \( N_{AB\rightarrow CD} \) is completely positive if and only if its Choi operator \( \Gamma^N_{RD} \) is positive semi-definite, and

2. it is trace preserving if and only if \( \text{Tr}_D[\Gamma^N_{RD}] = I_R \).

Each of these conditions is semi-definite and can be incorporated into an optimization over C-PPT-P channels. The condition in (81) corresponds to the Choi operator being PPT [Rai99, Rai01]; that is,

1. A bipartite channel satisfies (81) if and only if its Choi operator \( \Gamma^N_{AB\rightarrow AB} \) satisfies \( T_{BB'}(\Gamma^N_{AB\rightarrow AB}) \geq 0 \).

Finally, suppose that we have a bipartite channel \( N_{AB\rightarrow CD} \) and a channel \( M_{E\rightarrow A} \). Then the serial composition of them leads to the channel

\[
R_{EB\rightarrow CD} = N_{AB\rightarrow CD} \circ M_{E\rightarrow A}.
\]

It is natural then to express the Choi operator of \( R_{EB\rightarrow CD} \) in terms of those for \( M_{E\rightarrow A} \) and \( N_{AB\rightarrow CD} \). It is known that

\[
\Gamma^R_{EB\rightarrow CD} = \text{Tr}_A[T_A(\Gamma^M_{EA})\Gamma^N_{ABCD}],
\]

where \( T_A \) is the transpose map from (82).

Combining all of the above, we arrive at the following:

**Proposition 3** The simulation error in (84) can be computed by means of the following semi-definite program:

\[
\begin{align*}
\mathcal{C}_{\text{PPT}}(N_{AB\rightarrow AB'}, \rho_{\hat{A}B}) &= \inf_{\mu \geq 0, Z_{AB'}, X_{AB'A'B'} \geq 0, \mu,} \\
& \quad \text{subject to} \\
& \quad \mu I_{AB} \geq Z_{AB}, \\
& \quad T_{BB'}(P_{AB\hat{A}B'\hat{A}}) \geq 0, \\
& \quad \text{Tr}_{A'B'}(P_{AB\hat{A}B'\hat{A}}) = I_{AB\hat{A}B}, \\
& \quad Z_{AB'A'B'} \geq \Gamma^N_{AB'A'B'} - \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})P_{AB\hat{A}B'\hat{A}}].
\end{align*}
\]

The objective function and the first two constraints come from the semi-definite program for the normalized diamond distance in (87). The quantity \( \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})P_{AB\hat{A}B'\hat{A}}] \) comes about from the Choi operator of the composition of two channels in (92), while noting that a state \( \rho_{\hat{A}B} \) is a particular kind of channel from a trivial system to the systems \( \hat{A}B \). The third constraint comes from the fact that \( P_{AB\hat{A}B'\hat{A}} \) should be the Choi operator for a C-PPT-P map, and the fourth constraint comes from the fact that \( P_{AB\hat{A}B'\hat{A}} \) should be the Choi operator for a trace-preserving map.

As we show in Appendix B, the SDP dual to (93) is as follows:

\[
\begin{align*}
& \sup_{X_{AB'A'B'}, X_{ABA'B'} \geq 0, M_{AB}} \text{Tr}[\Gamma^N_{AB'A'B'}X_{AB'A'B'}^2 - \text{Tr}[W_{ABAB}]], \\
& \text{subject to} \\
& \quad \text{Tr}[X_{AB}^0] \leq 1, \\
& \quad X_{AB'A'B'}^2 \leq X_{AB} \otimes I_{A'B'}, \\
& \quad X_{AB'A'B'}^2 \otimes T_{BB}(\rho_{\hat{A}B}) + T_{BB'}(X_{AB'B'A'B'}^3) \leq W_{ABAB} \otimes I_{A'B'}.
\end{align*}
\]

**B. Semi-definite programming lower bound on the simulation error of unideal bidirectional teleportation**

The semi-definite program in Proposition 3 can be evaluated for our bipartite channel of interest, i.e., the unitary swap channel \( S_{AB}^d \) in (18). Even though the semi-definite program is efficiently computable with respect to the dimensions of systems \( A, B, A', B', \hat{A}, \hat{B} \), one finds that it can take some time in practice when evaluated for states of interest. Thus, we are interested in ways of reducing its complexity.

To reduce its complexity, we recall that the unitary swap channel \( S_{AB}^d \) in (18) has the following symmetry:

\[
S_{AB}^d(U_A \otimes V_B) = (V_A \otimes U_B)S_{AB}^d,
\]

where \( U_A \) and \( V_B \) are arbitrary unitary channels. Furthermore, it commutes with itself, i.e.,

\[
S_{AB}^d \circ S_{AB}^d = S_{AB}^d \circ S_{AB}^d.
\]

By exploiting these symmetries, we arrive at a semi-definite program for evaluating \( \mathcal{C}_{\text{PPT}}(S_{AB}^d, \rho_{\hat{A}B}) \), which has significantly lower complexity than the generic semi-definite program in Proposition 3. In particular, its complexity is polynomial in the dimension of the systems \( \hat{A} \) and \( \hat{B} \). We provide a proof of Proposition 4 in Appendix C.

**Proposition 4** The semi-definite program in Proposition 3, for the error in simulating the unitary SWAP channel \( S_{AB}^d \) in (18), simplifies as follows:

\[
\begin{align*}
\mathcal{C}_{\text{PPT}}(S_{AB}^d, \rho_{\hat{A}B}) &= 1 - \sup_{K_{AB}, L_{AB} \geq 0} \text{Tr}[\rho_{\hat{A}B}K_{\hat{A}B}], \\
& \quad \text{subject to} \\
& \quad T_{\hat{B}}(K_{\hat{A}B} + \frac{L_{\hat{A}B}}{d+1} + \frac{N_{\hat{A}B}}{(d+1)^2}) \geq 0,
\end{align*}
\]
that it measures the resource state. Thus, the interpretation of the simulating channel is

\[ \{ \text{POVM} \} \]

Remark 5 The proof of Proposition 4 demonstrates that an optimal C-PPT-P channel for simulating the unitary swap channel has the following structure:

\[ \mathcal{P}_{AB\bar{A}\bar{B} \rightarrow AB}(\omega_{AB} \otimes \rho_{\bar{A}\bar{B}}) = S_{AB}^d(\omega_{AB}) \text{Tr}[K_{\bar{A}B} \rho_{\bar{A}B}] + \frac{1}{2} \left( \text{id}_{A \rightarrow B} \otimes D_{B \rightarrow A} + D_{A \rightarrow B} \otimes \text{id}_{B \rightarrow A} \right) (\omega_{AB}) \text{Tr}[L_{\bar{A}B} \rho_{\bar{A}B}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A})(\omega_{AB}) \text{Tr}[N_{\bar{A}B} \rho_{\bar{A}B}], \]

(105)

where \( D \) denotes the following generalised Pauli channel:

\[ D(\sigma) := \frac{1}{d^2 - 1} \sum_{(x,z) \neq (0,0)} W^{z,x} \sigma(W^{z,x})^\dagger. \]

Thus, the interpretation of the simulating channel is that it measures the resource state \( \rho_{\bar{A}B} \) according to the POVM \( \{ K_{\bar{A}B}, L_{\bar{A}B}, N_{\bar{A}B} \} \), which is subject to the inequality constraints in Proposition 4. After that, it takes the following action:

1. If the first outcome \( K_{\bar{A}B} \) occurs, then apply the ideal swap channel to the input state \( \omega_{AB} \).

2. If the second outcome \( L_{\bar{A}B} \) occurs, then with probability \( 1/2 \), apply the identity channel \( \text{id}_{A \rightarrow B} \) to transfer Alice’s input system \( A \) to Bob, but then garble Bob’s input system \( B \) by applying the channel \( D \) and transfer the resulting system to Alice; with probability \( 1/2 \), apply the identity channel \( \text{id}_{B \rightarrow A} \) to transfer Bob’s input system \( B \) to Alice, but then garble Alice’s input system \( A \) by applying the channel \( D \) and transfer the resulting system to Bob.

3. If the third outcome \( N_{\bar{A}B} \) occurs, then apply the garbling channel \( D \) to both Alice and Bob’s systems individually and exchange them.

The fact that the measurement operators obey the inequality constraints in Proposition 4 implies that the quantum channel \( \mathcal{P}_{AB\bar{A}\bar{B} \rightarrow AB} \) is C-PPT-P.

C. Semi-definite programming lower bounds when using channel infidelity

As mentioned at the end of Section IV D, we can also employ infidelity to quantify the simulation error. In this section, we briefly detail the semi-definite programming lower bound that results when using the infidelity to measure simulation error.

First, consider that the simulation error is defined as follows, when optimizing over C-PPT-P channels and using the infidelity measure:

\[ \epsilon^F_{\text{PPT}}(N_{AB \rightarrow A'B'}, \rho_{\bar{A}B}) := \inf_{\mathcal{P} \in \text{C-PPT-P}} 1 - F(N_{AB \rightarrow A'B'}, \tilde{N}_{AB \rightarrow A'B'}), \]

(107)

where the optimization is with respect to C-PPT-P channels \( \mathcal{P}_{AB\bar{A}\bar{B} \rightarrow A'B'} \) and

\[ \tilde{N}_{AB \rightarrow A'B'}(\omega_{AB}) := \mathcal{P}_{AB\bar{A}\bar{B} \rightarrow A'B'}(\omega_{AB} \otimes \rho_{\bar{A}B}). \]

(108)

It is clear that

\[ \epsilon^F_{\text{PPT}}(N_{AB \rightarrow A'B'}, \rho_{\bar{A}B}) \leq \epsilon_{\text{LOCC}}^F(N_{AB \rightarrow A'B'}, \rho_{\bar{A}B}), \]

(109)

for the same reason that (86) holds.

Then we find the following result, from combining the semi-definite program for channel fidelity in (64–66), along with reasoning similar to that used to justify Proposition 3:

**Proposition 6** The simulation error in (107) can be computed by means of the following semi-definite program:

\[ \epsilon_{\text{PPT}}^F(N_{AB \rightarrow A'B'}, \rho_{\bar{A}B}) = 1 - \left[ \sup_{\lambda \geq 0, P_{AB\bar{A}BA'B'}} \lambda \right]^2, \]

(110)

subject to

\[ \lambda_{AB} \leq \text{Re}[\text{Tr}_{A'B'}(Q_{AB\bar{A}B'})], \]

(111)

\[ T_{BB'}(P_{AB\bar{A}B'A'B'}) \geq 0, \]

(112)

\[ \text{Tr}_{A'B'}(P_{AB\bar{A}B'A'B'}) = I_{AB\bar{A}B'}, \]

(113)

\[ \left[ \Gamma_{AB\bar{A}B'A'B'}^{N_{AB\bar{A}B'A'B'}}, Q_{AB\bar{A}B'A'B'}^{N_{AB\bar{A}B'A'B'}}, \text{Tr}_{AB} (P_{AB\bar{A}B'} \rho_{\bar{A}B}) P_{AB\bar{A}B'A'B'}^{N_{AB\bar{A}B'A'B'}} \right] \geq 0. \]

(114)

**Remark 7** There are important implications of Proposition 6 beyond the problems considered in this paper and which are relevant for quantum resource theories of channels (see, e.g., [LW19, LW19, WW19b]). In prior work, the normalized diamond distance was used to measure simulation error [FWTB20, LW19, WW19b], and if the class of channels being considered for the simulation obeys semi-definite constraints, then it follows that the simulation error can be computed by means of a semi-definite program. What Proposition 6 demonstrates is that the same is true when using channel infidelity for the simulation error. As a key example, Proposition 6 demonstrates that the channel box transformation optimization problem from [WW19b] can be computed by means of a semi-definite program when using channel infidelity for approximation error. We show this explicitly in Appendix D.
Applying similar reasoning used to arrive at Propositions 1 and 4, we find that the PPT simulation error of the unitary swap channel is equal to the expression from Proposition 4.

**Proposition 8** The semi-definite program in Proposition 6, for the error in simulating the unitary SWAP channel $S_{AB}^d$ in (18), simplifies to the expression from Proposition 4. That is,

$$e_{\text{PPT}}(S_{AB}^d, \rho_{AB}) = e_{\text{PPT}}(S_{AB}^d, \rho_{AB}) = e_{\text{PPT}}(S_{AB}^d, \rho_{AB}) = 1 - \frac{1}{d^2}, \quad (115)$$

As a consequence of Proposition 8, there is no need for different notions of simulation error when considering the simulation of the unitary swap channel using C-PPT-P channels.

**VI. EXAMPLES**

In this section, we consider several examples of resource states that can be used for bidirectional teleportation, and we evaluate their performance. For several cases of interest, we establish an exact evaluation not only for the error when using a PPT simulation, but also when using an LOCC simulation.

One key example resource state, considered in Section VI A below, is when there is in fact no resource state at all. In such a situation, Alice and Bob can only employ a PPT or LOCC simulation of bidirectional teleportation. In doing so, they could prepare a PPT or separable resource state for free, respectively, by means of a C-PPT-P or LOCC channel, and so this situation also captures the case in which the resource state that they share is a PPT or separable state, respectively. We find that the simulation error in both cases is equal to $1 - \frac{1}{d^2}$. If there is no resource state, then the error in simulating the SWAP channel $S_{AB}^d$ is equal to $1 - \frac{1}{d^2}$ when LOCC is free, a separable state. The quantity $d$ is the dimension of the SWAP channel.

Other resource states that we consider are isotropic [HH99] and Werner [Wer89] states, which are states that are simply characterized by a single parameter, due to the large amount of symmetry that they possess. We consider these states in Sections VI B and VI C, respectively.

We note here that we arrived at the analytical conclusions in Sections VI A, VI B, and VI C by employing Matlab and Mathematica. We used Matlab to implement the linear programs numerically, whose numerical solutions were subsequently used to infer analytical solutions through the use of Mathematica. We have included all of these files with the arXiv posting of our paper. The Mathematica files are especially useful for verifying that the values of primal and dual variables are feasible for the linear programs given.

**A. No resource state: Benchmark for classical versus quantum bidirectional teleportation**

We begin by considering the scenario in which there is no resource state at all. As mentioned above, the utility of this scenario is that it provides a dividing line between a classical and quantum implementation of bidirectional teleportation. The proof is sufficiently short that we provide it below. Figure 11 plots the expression in (116) for the simulation error.

**Proposition 9** If there is no resource state, then the error in simulating the unitary SWAP channel $S_{AB}^d$ in (18) is equal to $1 - 1/d^2$:

$$e_{\text{PPT}}(S_{AB}^d, \emptyset) = e_{\text{LOCC}}(S_{AB}^d, \emptyset) = 1 - \frac{1}{d^2}, \quad (116)$$

where the notation $\emptyset$ indicates the absence of a resource state.

**Proof.** If there is no resource state, this is equivalent to the state $\rho_{AB}$ simply being equal to the number one, and the three operators $K_{AB}, L_{AB},$ and $N_{AB}$ from Proposition 4, reduce to real numbers $p_1, p_2,$ and $p_3$. Then the semi-definite program from Proposition 4 simplifies to the following linear program:

$$1 - \sup_{p_1, p_2, p_3 \geq 0} p_1, \quad (117)$$
subject to
\[
\begin{align*}
p_1 + \frac{p_2}{d+1} + \frac{p_3}{(d+1)^2} &\geq 0, \\
\frac{1}{d^2-1}(p_2 + p_3) &\geq p_1, \\
p_1 + \frac{p_3}{(d-1)^2} &\geq \frac{p_2}{d-1}, \\
p_1 + p_2 + p_3 &\geq 1.
\end{align*}
\] (118)

Note that the first inequality constraint is redundant (a trivial consequence of \(p_1, p_2, p_3 \geq 0\)), and so it can be eliminated. A feasible point of the linear program above is given by
\[
p_1 = \frac{1}{d^2}, \quad p_2 = 0, \quad p_3 = 1 - \frac{1}{d^2},
\] (122)

implying that
\[
e_{\text{PPT}}(S_{AB}^d, \emptyset) \leq 1 - \frac{1}{d^2}.
\] (123)

To find a matching lower bound, we determine the linear program dual to that in (117). Consider that (117)–(121) can be written in the standard form of a linear program as follows [BV04]:
\[
1 - \sup_{x \geq 0} \left\{ c^T x : Ax \leq b \right\},
\]
(124)

where
\[
x^T = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix},
\]
\[
c^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\]
\[
A = \begin{bmatrix} 1 - \frac{1}{d^2-1} & -\frac{1}{d^2-1} \\ -1 & \frac{1}{d^2-1} & -\frac{1}{(d-1)^2} \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix},
\]
\[
b^T = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}.
\]
(125) \quad (126) \quad (127) \quad (128)

Note that, in deriving the matrix \(A\), we have eliminated the redundant constraint in (118). The dual of this linear program is given by [BV04]
\[
1 - \inf_{y \geq 0} \left\{ b^T y : A^T y \geq c \right\},
\] (129)

Due to weak duality [BV04], the optimal value of (129) does not exceed the optimal value of (124). Writing (129) out in detail using the definitions in (125)–(128) gives
\[
1 - \inf_{y_1, \ldots, y_d \geq 0} y_3 - y_4,
\] (130)

subject to
\[
y_1 - y_2 + y_3 - y_4 \geq 1,
\] (131)
\[
-\frac{y_1}{d^2-1} + \frac{y_2}{d-1} + y_3 - y_4 \geq 0,
\] (132)

A solution is given by
\[
y_1 = 1 - \frac{1}{d^2}, \quad y_2 = y_4 = 0, \quad y_3 = \frac{1}{d^2},
\] (135)

implying that
\[
e_{\text{PPT}}(S_{AB}^d, \emptyset) \geq 1 - \frac{1}{d^2}.
\] (136)

Putting together (123) and (136), we conclude the equality
\[
e_{\text{PPT}}(S_{AB}^d, \emptyset) = 1 - \frac{1}{d^2}.
\] (137)

By applying the inequality in (86), we conclude that
\[
e_{\text{PPT}}(S_{AB}^d, \emptyset) \leq e_{\text{LOCC}}(S_{AB}^d, \emptyset).
\] (138)

Thus, it remains to establish an upper bound on \(e_{\text{LOCC}}(S_{AB}^d, \emptyset)\), by demonstrating a scheme for bidirectional teleportation that achieves the simulation error \(1 - \frac{1}{d^2}\).

A scheme to achieve this error using LOCC consists of Alice and Bob preparing the \(d^2\)-dimensional states \(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B\), applying the bilateral twirl in (9) to get the state
\[
\omega_{AB} := T_{\text{bil}} (|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B)
\]
(139)

\[
= \frac{1}{d^2} \Phi^d_{AB} + \left(1 - \frac{1}{d^2}\right) \left(I_{\hat{A}} - \Phi^d_{AB} \right)
\]
(140)

separating out \(\Phi^d_{AB}\) to two e-dits \(\Phi^d_{A_1B_1} \otimes \Phi^d_{A_2B_2}\) via local isometries, and then performing teleportation in opposite directions.

Let us now prove that this simulation achieves the simulation error \(1 - \frac{1}{d^2}\). Consider that the action of the bidirectional teleportation operations is equivalent to an LOCC channel \(\mathcal{L}_{AB\hat{A}B\hat{A}}\). Furthermore, we apply the same LOCC channel \(\mathcal{L}_{AB\hat{A}B\hat{A}}\) regardless of whether we are conducting ideal or unideal bidirectional teleportation. So it follows that ideal bidirectional teleportation is given by
\[
S_{\hat{A}B}^d(\cdot) = \mathcal{L}_{AB\hat{A}B\hat{A}}(\cdot) \otimes \Phi^d_{AB},
\]
(141)

and the channel realized by unideal bidirectional teleportation is
\[
\tilde{S}_{\hat{A}B}^d(\cdot) = \mathcal{L}_{AB\hat{A}B\hat{A}}(\cdot) \otimes \omega_{\hat{A}B}.
\]
(142)

Let \(\psi_{AB}\) be an arbitrary input state for these channels. Then we find that
\[
\frac{1}{2} \left\| S_{\hat{A}B}^d(\psi_{AB}) - \tilde{S}_{\hat{A}B}^d(\psi_{AB}) \right\|_1
\] (143)
establishing an upper bound on the LOCC simulation

The final equality follows because \( \Phi_{\hat{A}\hat{B}} \) by applying the equality in (53). Continuing, we find

so that the trace norm is equal to the sum of the traces

The inequality follows from the data-processing inequal-

The following proposition establishes a simple expres-

for bidirectional teleportation. It is given exclusively in
terms of the dimension \( d \) of the swap channel that is being
simulated and the two parameters \( F \) and \( d_A \) that char-
acterize the isotropic resource state. A proof is available in
Appendix E. The proof exploits the symmetries of an
isotropic state to reduce a variation of the semi-definite
program in Proposition 4 to a linear program, which we
then solve analytically.

**Proposition 10** The simulation error for the unitary
swap channel when using an isotropic resource state \( \rho_{\hat{A}\hat{B}}^{(F,d_A)} \) is as follows:

\[
\epsilon_{\text{PPT}}(S_{\hat{A}\hat{B}}, \rho_{\hat{A}\hat{B}}^{(F,d_A)}) =
\begin{cases}
1 - \frac{1}{d_A} & \text{if } F \leq \frac{1}{d_A} \\
1 - \frac{Fd_A}{d^2} & \text{if } F > \frac{1}{d_A} \text{ and } d_A \leq d^2 \\
\frac{(1 + F d_A)(1 - F)}{1 - \frac{1}{d_A}} & \text{if } F > \frac{1}{d_A} \text{ and } d_A > d^2
\end{cases}
\]

(151)

We also have that

\[
\epsilon_{\text{PPT}}(S_{\hat{A}\hat{B}}, \rho_{\hat{A}\hat{B}}^{(F,d_A)}) = \epsilon_{\text{LOCC}}(S_{\hat{A}\hat{B}}, \rho_{\hat{A}\hat{B}}^{(F,d_A)})
\]

(152)

if \( F \leq \frac{1}{d_A} \) or if \( F > \frac{1}{d_A} \) and \( d_A \leq d^2 \).

It remains open to determine if the equality in (152) holds when \( F > \frac{1}{d_A} \) and \( d_A > d^2 \). The main question is to design an LOCC-assisted scheme that achieves a simulation error of \( (1 - \frac{1}{d_A})(1 - F)/(1 - \frac{1}{d_A}) \) when \( F > \frac{1}{d_A} \) and \( d_A > d^2 \).

Figure 12 plots the expression given in (151) for the simulation error.

**Remark 11 (Optimal strategy with a single e-dit)**

A special case of Proposition 10 above occurs when Alice
and Bob share a single e-dit, so that \( F = 1 \) and \( d_A = d \). In this case, the following equality holds

\[
e_{\text{PPT}}(\mathcal{S}_{AB}^d, \rho_{AB}^{(1,d)}) = e_{\text{LOCC}}(\mathcal{S}_{AB}^d, \rho_{AB}^{(1,d)}) = 1 - \frac{1}{d}.
\] (153)

An optimal protocol to achieve this simulation error is for Alice and Bob to embed the resource state \( \rho_{AB}^{(1,d)} = \Phi_{AB}^d \) in larger Hilbert spaces, each of dimension \( d^2 \) and labeled by \( \hat{A} \) and \( \hat{B} \) without loss of generality, twirl it according to (9) and (10), which results in the following resource state:

\[
\frac{1}{d} \hat{\Phi}_{AB}^{d^2} + \left( 1 - \frac{1}{d} \right) \frac{(I_{\hat{A}\hat{B}} - \hat{\Phi}_{AB}^{d^2})}{d^4 - 1}.
\] (154)

From there, Alice and Bob can locally separate out \( \hat{\Phi}_{AB}^{d^2} \) to two e-dits \( \hat{\Phi}_{AB}^{d} \otimes \hat{\Phi}_{AB}^{d} \) via local isometries and each perform a teleportation in opposite directions. By following an error analysis similar to that in (143)–(149), we conclude that this scheme achieves a simulation error equal to \( 1 - \frac{1}{d} \).

### C. Werner states

Another general class of bipartite states of interest in quantum information is the class of Werner states [Wer89]. A Werner state of parameter \( p \in [0,1] \) and dimension \( d_A \in \{2, 3, 4, \ldots\} \) is defined as follows:

\[
W_{AB}^{(p,d)} := \frac{2}{d_A (d_A + 1)} \Pi^S_{AB} + \frac{2}{d_A (d_A - 1)} \Pi^A_{AB},
\] (155)

where \( \Pi^S_{AB} := (I_{\hat{A}\hat{B}} + F_{AB})/2 \), \( \Pi^A_{AB} := (I_{\hat{A}\hat{B}} - F_{AB})/2 \), and \( F_{AB} \) is the unitary swap operator defined in (19).

The general interest in Werner states stems from the fact that an arbitrary state of systems \( AB \) can be twirled in a different way to a Werner state. That is, a Werner state results from the following bilateral twirl:

\[
\hat{W}_{CD}(X_{CD}) := \int dU \ (U_C \otimes U_D)(X_{CD}),
\] (156)

where the notation is defined similarly to that in (9). It is well known that the action of a Werner twirl as above, on an arbitrary input operator \( X_{CD} \), is as follows [Wat18]:

\[
\hat{W}_{CD}(X_{CD}) = \text{Tr}[\Pi^S_{AB} X_{CD}] \frac{2}{d_A (d_A + 1)} \Pi^S_{AB} + \text{Tr}[\Pi^A_{AB} X_{CD}] \frac{2}{d_A (d_A - 1)} \Pi^A_{AB}.
\] (157)

Just as with isotropic states, the general interest in them stems from the fact that they result from a twirl of an arbitrary bipartite state according to \( \hat{W}_{CD} \). Also, the states are easily characterized in terms of just two parameters.

The following proposition establishes a simple expression for the simulation error when using a Werner state for bidirectional teleportation. It is given exclusively in terms of the dimension \( d \) of the swap channel that is being simulated and the two parameters \( p \) and \( d_A \) that characterize the Werner resource state. A proof is available in Appendix F. The proof exploits the symmetries of a Werner state to reduce a variation of the semi-definite program in Proposition 4 to a linear program, which we then solve analytically.

**Proposition 12** The simulation error for the unitary swap channel when using a Werner resource state \( W_{AB}^{(p,d)} \) is as follows:

\[
e_{\text{PPT}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)}) = e_{\text{LOCC}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)}) = \left\{ \begin{array}{ll} 1 - \frac{1}{d} & \text{if } p \leq 1/2, \\ 1 - \frac{1}{d} & \text{if } p > 1/2. \end{array} \right.
\] (158)

If \( p \leq 1/2 \), then

\[
e_{\text{PPT}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)}) = e_{\text{LOCC}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)})
\] (159)

It is an open question to determine if

\[
e_{\text{PPT}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)}) = e_{\text{LOCC}}(\mathcal{S}_{AB}^d, W_{AB}^{(p,d)})
\] (160)

for \( p > 1/2 \). The main question is to design an LOCC-assisted scheme that achieves a simulation error of \( 1 - \frac{1}{4p - 2 + d_A} \) when \( p > 1/2 \).

Figure 13 plots the expression given in (158) for the simulation error.
D. Resource state resulting from generalized amplitude damping channel

In this section, we consider a numerical example in which we can apply the semi-definite program from Proposition 4. This example involves a resource state resulting from two Bell states affected by noise from a generalized amplitude damping channel (GADC). The GADC can be understood as a qubit thermal channel, in which the input qubit interacts with a thermal qubit environment according to a beamsplitter-like interaction, after which the environment qubit is discarded [KSW20]. In more detail, recall that the GADC has the following in (9), uses this resource state. If Alice and Bob perform a bilateral teleportation, we obtain an upper bound by demonstrating a protocol that consists of four qubits in total, each of which is acted upon by a GADC with the same parameters $\gamma$ and $N$. When $\gamma$ and $N$ are both equal to zero, the resource state is equivalent to two ebits and perfect bidirectional teleportation is possible. As the noise parameters increase, the bidirectional teleportation is imperfect and occurs with some error.

The resource state we consider is then

$$\mathcal{A}_{\gamma,N}(\rho) := \sum_{i=1}^{4} A_i \rho A_i^\dagger,$$  \hspace{1cm} (161)

where $\gamma \in [0,1]$ is the damping parameter, $N \in [0,1]$ is the noise parameter, and

$$A_1 := \sqrt{1-N} \left( |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1| \right),$$  \hspace{1cm} (162)

$$A_2 := \sqrt{\gamma} (1-N) |0\rangle\langle 1|,$$  \hspace{1cm} (163)

$$A_3 := \sqrt{N} \left( \sqrt{1-\gamma} |0\rangle\langle 0| + |1\rangle\langle 1| \right),$$  \hspace{1cm} (164)

$$A_4 := \sqrt{\gamma} N |1\rangle\langle 0|.$$  \hspace{1cm} (165)

The resource state in (166) is equivalent to two ebits, consisting of four qubits in total, each of which is acted upon by a GADC with the same parameters $\gamma$ and $N$. When $\gamma$ and $N$ are both equal to zero, the resource state is equivalent to two ebits and perfect bidirectional teleportation is possible. As the noise parameters increase, the bidirectional teleportation is imperfect and occurs with some error.

By evaluating the semi-definite program in Proposition 4 for this resource state, we obtain a lower bound on the simulation error of bidirectional teleportation. We obtain an upper bound by demonstrating a protocol that uses this resource state. If Alice and Bob perform a bilateral twirl on their state, specifically, the channel in (9), where $U$ is a unitary that acts on two qubits, then the resulting state is an isotropic state of the following form:

$$F(\gamma, N) \Phi \otimes 2 + (1 - F(\gamma, N)) \frac{I \otimes 4 - \Phi \otimes 2}{15},$$  \hspace{1cm} (168)

where

$$F(\gamma, N) := \text{Tr}[\Phi \otimes 2 \mathcal{A}_{\gamma,N}(\Phi \otimes 2)].$$  \hspace{1cm} (169)

By applying Proposition 10 and noting that $d_A = d^2 = 4$ for this example, we find that the simulation error, when using this protocol, is given by

$$1 - \max \left\{ F(\gamma, N), \frac{1}{16} \right\}.$$  \hspace{1cm} (171)

Up to numerical precision, we find that the upper bound in (171) and the SDP lower bound from Proposition 4 match, so that (171) should in fact be an exact analytical expression for the simulation error when using this resource state.

Figure 14 plots the expression in (171) for the simulation error. The simulation error tends to zero as the damping parameter $\gamma$ approaches zero (so that the channel $\mathcal{A}_{\gamma,N}$ is converging to an identity channel and thus the resource state to two ebits). For fixed $\gamma$ and the noise parameter $N$ converging to 1/2, the simulation error increases.

VII. On the performance of the KPF16 protocols for bidirectional teleportation

We now use our framework to evaluate the performance of some proposals for bidirectional teleportation from [KPF16]. Let us call these proposals KPF16, after the authors and year of publication of [KPF16]. Our main conclusion is that the schemes from [KPF16] are suboptimal according to the performance metric in (78), whereas our simple strategy proposed in Remark 11 is optimal.
FIG. 15. Shown above is the first proposal for bidirectional teleportation from Kitkenko et al. The scheme utilizes ten qubits with subindices A and B to denote Alice and Bob, respectively. $T_A$ and $T_B$ denote the trigger qubits of Alice and Bob, respectively, whose states dictate the actions of both parties. If either party’s trigger qubit is in the state 1, that party performs unidirectional quantum teleportation on their end. There are four auxiliary qubits, labeled by $M^1_A$, $M^2_A$, $M^1_B$, and $M^2_B$, which are each initialized to the state $|0\rangle$ and are used to store projective measurement outcomes. The states $|\psi_Q_A\rangle$ and $|\psi_Q_B\rangle$ are those that Alice and Bob wish to swap (more generally, the state can be a joint state $|\psi\rangle_{AB}$ of both systems). The resource state shared by Alice and Bob to help with the task is a single Bell state $|\Phi\rangle_{C_AC_B} := (|0\rangle_C A C_B+|1\rangle_C A C_B)/\sqrt{2}$.  

### A. Bipartite channel for the first KPF16 protocol

To see this, let us recall the first proposal for bidirectional teleportation from [KPF16] shown in Figure 15. The scheme presented in Figure 2(a) of [KPF16] utilizes ten qubits with subindices A and B to denote Alice and Bob, respectively. The system label $T_A$ denotes a “trigger qubit” of Alice, and the system label $T_B$ denotes a trigger qubit of Bob (the notation here overlaps with our notation for partial transpose, but it should be clear from the context). There are four auxiliary qubits, labeled by $M^1_A$, $M^2_A$, $M^1_B$, and $M^2_B$, that are each initialized to the state $|0\rangle$ and are used to store projective measurement outcomes. The states $|\psi_Q_A\rangle$ and $|\psi_Q_B\rangle$ are those that Alice and Bob wish to swap (more generally, the state can be a joint state $|\phi\rangle_{AB}$ of both systems). The resource state shared by Alice and Bob to help with the task is a single Bell state $|\Phi\rangle_{C_AC_B} := (|0\rangle_C A C_B+|1\rangle_C A C_B)/\sqrt{2}$. The trigger qubits of Alice and Bob are initialized to the states $|\psi_0\rangle$ and $|\psi_2\rangle$, respectively, where

$$|\psi_0\rangle := \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle.$$  

The trigger qubits act as controls for local Bell measurements. That is, if the trigger qubit of Alice is in the state $|0\rangle$, then a Bell measurement is not performed on her systems $Q_A$ and $C_A$. If the trigger qubit of Alice is in the state $|1\rangle$, then a Bell measurement is performed on her systems $Q_A$ and $C_A$. The quantum circuit in Figure 2(a) of [KPF16] indicates that these actions happen coherently. However, the trigger qubits are discarded at the end of the circuit, and so they really just end up playing the role of random control bits. A similar description applies to Bob’s side.

Thus, the following actions are taken, depending on the value of the trigger qubits. If both are zero, then no action is taken and the output of the circuit is a Bell state on systems $C_A$ and $C_B$. If Alice’s trigger qubit is equal to one and Bob’s is equal to zero, then unidirectional teleportation is performed from Alice to Bob. The result is that an identity channel takes system $C_A$ to system $C_B$, while a completely depolarizing channel is performed on system $C_B$ and takes it to system $C_A$. If Alice’s trigger qubit is equal to zero and Bob’s is equal to one, then the actions are exactly opposite of the previous setting. Finally, if both Alice and Bob’s trigger qubits are equal to one, then completely depolarizing channels act on both systems $C_A$ and $C_B$, so that the output in this case is the maximally mixed state of two qubits. In this last case, the protocol is such that the output states become “jammed” due to Alice and Bob both trying to teleport at the same time. The various actions are summarized in Table I.

The bipartite channel realized by the first KPF16 protocol is thus as follows:

$$K_{Q_A Q_B \rightarrow C_A C_B} = (1-p_1)(1-p_2) \text{Tr}[\rho_{Q_A Q_B} |\Phi\rangle_{C_AC_B}$$  

$$+ (1-p_1)p_2(\mathcal{R}_Q \rightarrow C_B \otimes \text{id}_{Q_B \rightarrow C_A})(\rho_{Q_A Q_B})$$  

$$+ p_1(1-p_2)(\text{id}_{Q_A \rightarrow C_B} \otimes \mathcal{R}_Q \rightarrow C_A)(\rho_{Q_A Q_B})$$  

$$+ p_1 p_2 \mathcal{R}_Q \rightarrow C_B \otimes \mathcal{R}_Q \rightarrow C_A(\rho_{Q_A Q_B}),$$  

where $p_i = \sin^2(\theta_i/2)$ for $i \in \{1, 2\}$, $\mathcal{R}^\pi$ is a replacer channel that traces out its input and replaces it with the maximally mixed qubit state $\pi = I/2$, and $\mathcal{R}^\pi \otimes \pi$ is a replacer channel that traces out its two-qubit input and replaces with the maximally mixed state $\pi \otimes \pi$ of two qubits.

One of the first observations that we can make about the bipartite channel in (173) is that none of the terms contain the ideal swap channel. That is, with the first KPF16 protocol, the ideal swap channel does not occur.
even probabilistically. Furthermore, the bipartite channel in (173) does not obey the symmetry of the swap channel in (101), due to the presence of the first term in (173), whereas we know from the analysis in (A5)–(A10) that it should if it is to be an optimal simulation. As such, these are strong indicators that this protocol will not perform well as an approximation of bidirectional teleportation, according to the metric defined in (77). In what follows, we prove that this is the case.

Let us now calculate the Choi operator of the channel $K_{QACBQB} := K_{QACBQB}(\Gamma_{QAC} \otimes \Gamma_{QB})$, where

$$\Gamma_{QAC} := \sum_{i,j \in \{0,1\}} |i⟩⟨j| \otimes |i⟩⟨j|_{QAC},$$

and $\Gamma_{QB}$ is similarly defined. To do so, let us consider the four terms in (173) separately. We find that the first term is

$$\text{Tr}_{QACBQB} [\Gamma_{QAC} \otimes \Gamma_{QB}] \Phi_{CAB} = I_{Q} \otimes \Phi_{CAB} \otimes I_{B}.$$  

The second term is

$$(R_{QCQB} \otimes id_{QAC})(\Gamma_{QAC} \otimes \Gamma_{QB}) = I_{QAC} \otimes \Gamma_{CAB} \otimes \pi_{B}.$$  

The third term is

$$(id_{QAC} \otimes R_{QCB})(\Gamma_{QAC} \otimes \Gamma_{QB}) = I_{QAC} \otimes \Gamma_{CAB} \otimes \pi_{C} \otimes I_{B}.$$  

The fourth term is

$$R_{QACQB} \otimes \pi_{C} \otimes \Gamma_{QB} = I_{QAC} \otimes \pi_{C} \otimes \pi_{C} \otimes I_{B}.$$  

Putting everything together, we conclude that the Choi operator $K_{QACBQB}$ of the channel $K_{QACBQB}$ is as follows:

$$K_{QACBQB} = (1 - p_1)(1 - p_2)I_{Q} \otimes \Phi_{CAB} \otimes I_{B} + (1 - p_1)p_2I_{Q} \otimes \Gamma_{CAB} \otimes \pi_{B} + p_1(1 - p_2)I_{Q} \otimes \pi_{C} \otimes \Gamma_{B} + p_1p_2I_{Q} \otimes \pi_{C} \otimes \pi_{C} \otimes I_{B}.$$  

This is helpful not only for calculating the normalized diamond distance between the ideal swap channel and the first KPF16 protocol, but also for estimating the fidelity when sending in states of maximally entangled states.

**B. Fidelity of the first KPF16 protocol to ideal bidirectional teleportation**

We now exploit the Choi operator in (180) in order to compare the first KPF16 protocol with ideal birectional teleportation via fidelity. As recalled in Section III, ideal bidirectional teleportation realizes the following state when acting on maximally entangled inputs:

$$S_{QACBQB \rightarrow CACB}(\Phi_{QAC} \otimes \Phi_{QB}) = \Phi_{CAB} \otimes \Phi_{CAB}.$$  

Since the state above is a pure state, we can calculate the fidelity by means of the following formula:

$$\text{Tr}[(\Phi_{CAB} \otimes \Phi_{CAB})K_{QACBQB}/4],$$

where we have normalized the Choi operator to become $K_{QACBQB} = 1$. Although the fidelity formula in (182) can be readily calculated by hand, it is helpful to employ a diagrammatic calculus to do so. Scalable ZX (SZX) calculus is a low-level graphical language with a written set of rules utilized for the verification of quantum computations [CD11]. This technique, in which each wire in a diagram represents a qubit, allows for one diagram to be transformed into another if both represent the same quantum process. This makes SZX-calculus useful for a variety of applications ranging from error correction to circuit optimization. The basic rules of the SZX calculus that we need are summarized in Figure 16. In what follows, we utilize the language to calculate the fidelity of the ideal versus unideal state for each individual scenario in Table I. To be clear, each diagram in Figures 17–20 is a visual representation of one term in the following formula:

$$(\Phi_{CAB} \otimes \Phi_{CAB}) \frac{1}{4} \ K \ (\Phi_{CAB} \otimes \Phi_{CAB})$$

$$= (1 - p_1)(1 - p_2)F_1 + p_2(1 - p_1)F_2 + p_1(1 - p_2)F_3 + p_1p_2F_4,$$

where we have omitted the system labels $QACBQB$ of $K$ for brevity. That is, our goal is to calculate $F_1$, $F_2$, $F_3$, and $F_4$. Figures 17–20 accomplish this goal, from which we conclude that $F_1 = F_4 = \frac{1}{16}$ and $F_2 = F_3 = \frac{1}{4}$. See the captions of Figures 17–20 for explanations of these calculations. Then we find that the fidelity is equal to

$$(1 - p_1)(1 - p_2)\frac{1}{16} + p_2(1 - p_1)\frac{1}{4} + p_1(1 - p_2)\frac{1}{4} + p_1p_2\frac{1}{16} = \frac{1}{16}(1 + 3(p_1 + p_2) - 6p_1p_2).$$  

(185)
FIG. 16. (a) Tracing out one share of the maximally entangled state and replacing with a maximally mixed state leads to two maximally mixed states. The diagram depicts this transformation and the diagrammatic representations. The right-hand side of the bottom diagram indicates the normalization factor needed for two maximally mixed states. (b) The overlap of the maximally entangled state $|\Phi\rangle$ with itself is 1. (c) Postselected teleportation transformation, in which the overlap of maximally entangled states is reduced to an identity channel with normalization factor $1/2$.

This function has a maximum at either $p_1 = 1$ and $p_2 = 0$ or $p_1 = 0$ and $p_2 = 1$, with both cases leading to a maximum fidelity of $\frac{1}{2}$ for the KPF16 protocol. These optimal values of $p_1$ and $p_2$ correspond to a unidirectional teleportation strategy for one party and a replacer channel for the other and vice versa, which clearly do not suffice for bidirectional teleportation.

In contrast, our simple approach from Remark 11 achieves a fidelity of $\frac{1}{4}$, which follows from a straightforward calculation. Thus, our approach provides a significant improvement over the first KPF16 protocol when only a single ebit is available for bidirectional teleportation.

We now consider the normalized diamond distance between the KPF16 protocol and ideal bidirectional teleportation, i.e.,

$$\frac{1}{2} \left\| K_{Q_1 Q_2 \rightarrow C_A C_B}^{p_1, p_2} - S_{Q_1 Q_2 \rightarrow C_A C_B} \right\|_1,$$  \hspace{1cm} (186)

where we have included the dependence of $K_{Q_1 Q_2 \rightarrow C_A C_B}^{p_1, p_2}$ on $p_1, p_2 \in [0, 1]$ for clarity. We have numerically implemented the semi-definite program to calculate the simulation error above for all values of $p_1, p_2 \in [0, 1]$. The results are displayed in Figure 21. We observe that the simulation error is never below $\frac{3}{4}$ and this lowest value is achieved at either $p_1 = 1$ and $p_2 = 0$ or $p_1 = 0$ and $p_2 = 1$.

We now prove that this is the case. That is, we prove...
FIG. 19. The analysis here is the same as that in Figure 18, leading to a value of \( \frac{1}{4} \).

FIG. 20. The four horizontal lines in the middle section are each weighted by \( \frac{1}{2} \), due the presence of maximally mixed states. Then we apply the second rule from Figure 16 to collapse the remaining two loops, leading to a value of \( \frac{1}{16} \).

FIG. 21. Numerical calculation of normalized diamond distance in (186) for all \( p_1, p_2 \in [0, 1] \). The lowest simulation error is achieved at either \( p_1 = 1, p_2 = 0 \) or \( p_1 = 0, p_2 = 1 \).

that

\[
\min_{p_1, p_2 \in [0, 1]} \frac{1}{2} \left\| K^{p_1, p_2}_{Q_A Q_B \rightarrow C_A C_B} - S_{Q_A Q_B \rightarrow C_A C_B} \right\|_\infty = \frac{3}{4}
\]  

(187)

As such, the first KPF16 protocol does not perform better for bidirectional teleportation than the classical limit from Proposition 9, which is contrary to the claim from [KPF16]. This discrepancy is due to the different performance metrics being used in [KPF16] and in our paper to quantify the performance of bidirectional teleportation. One of the main themes of our paper is that (78) or (80) with \( N \) set to the unitary swap channel is the correct way of quantifying the performance of bidirectional teleportation.

To establish (187), recall from (53) that the simulation error involves an optimization over all input states to the channels. Let us pick the input state to be a tensor product of maximally entangled states \( \Phi_{Q_A Q_B} \otimes \Phi_{Q_B Q_B} \). Then we find that

\[
\begin{align*}
\frac{1}{2} \left\| K^{p_1, p_2}_{Q_A Q_B \rightarrow C_A C_B} - S_{Q_A Q_B \rightarrow C_A C_B} \right\|_\infty \\
\geq \frac{1}{2} \left\| (K^{p_1, p_2} - S)(\Phi_{Q_A Q_B} \otimes \Phi_{Q_B Q_B}) \right\|_1 \\
= \frac{1}{2} \left\| \frac{1}{4} K^{p_1, p_2}_{Q_A Q_B \otimes C_A C_B Q_B} - \Phi_{Q_A C_B} \otimes \Phi_{C_A Q_B} \right\|_1,
\end{align*}
\]  

(188)

(189)

where we have omitted system labels in the second line. We can now apply the following measurement channel:

\[
\omega_{Q_A C_A C_B Q_B} \rightarrow \\
\text{Tr}[\omega_{Q_A C_A C_B Q_B} (\Phi_{Q_A C_B} \otimes \Phi_{C_A Q_B})][1][1]+ \\
\text{Tr}[\omega_{Q_A C_A C_B Q_B} (I_{Q_A C_B C_A Q_B} - \Phi_{Q_A C_B} \otimes \Phi_{C_A Q_B})][0][0].
\]  

(190)

For the state \( \Phi_{Q_A C_B} \otimes \Phi_{C_A Q_B} \), the output is \( |1\rangle[1] \), while for \( \frac{1}{4} K^{p_1, p_2}_{Q_A Q_B \otimes C_A C_B Q_B} \), the output is \( F|1\rangle[1] + (1 - F) |0\rangle[0] \), where \( F \) is the fidelity formula in (185). Since the trace distance does not increase under the action of a measurement channel, we conclude that

\[
\begin{align*}
\frac{1}{2} \left\| \frac{1}{4} K^{p_1, p_2}_{Q_A Q_B \otimes C_A C_B Q_B} - \Phi_{Q_A C_B} \otimes \Phi_{C_A Q_B} \right\|_1 \\
\geq \frac{1}{2} \left\| F|1\rangle[1] + (1 - F) |0\rangle[0] - |1\rangle[1] \right\|_1 \\
= 1 - F \\
= 1 - \frac{1}{16} (1 + 3 (p_1 + p_2) - 6p_1 p_2).
\end{align*}
\]  

(191)

(192)

(193)

Since we already argued that the function in (185) takes its maximum value of \( 1/4 \) at either \( p_1 = 1, p_2 = 0 \) or \( p_1 = 0, p_2 = 1 \), we conclude that

\[
\min_{p_1, p_2 \in [0, 1]} \frac{1}{2} \left\| K^{p_1, p_2}_{Q_A Q_B \rightarrow C_A C_B} - S_{Q_A Q_B \rightarrow C_A C_B} \right\|_\infty \geq \frac{3}{4}.
\]  

(194)

This value is actually achieved because the optimal input state for \( p_1 = 1, p_2 = 0 \) or \( p_1 = 0, p_2 = 1 \) is a tensor
product of maximally entangled states. This is due to the covariance of the bipartite channel at these special points and from an application of [LKDW18, Corollary II.5].

In contrast, we know from Proposition 10 and Remark 11 that the optimal simulation error for bidirecional teleportation of qubits, when only a single ebit is available, is equal to $\frac{3}{4}$. Furthermore, this simulation error is achievable using the simple strategy outlined in Remark 11.

D. Second protocol of KPF16

We now analyze a second protocol from [KPF16], shown in Figure 22, in which there is a single trigger qubit that controls whether Alice teleports to Bob or Bob teleports to Alice. As discussed in [KPF16], this scheme is equivalent to one in which a classical random bit selects whether Alice teleports or Bob does, and it can be communicated via a classical channel. The scheme thus realized a bipartite channel that is a probabilistic mixture of the two middle terms in (173). The bipartite channel realized by this second protocol is as follows:

$$K_{Q_AQ_B\rightarrow C_AC_B}(\rho_{Q_AQ_B}) := p(R_{Q_A\rightarrow C_B} \otimes \text{id}_{Q_B\rightarrow C_A})(\rho_{Q_AQ_B}) + (1 - p) (\text{id}_{Q_A\rightarrow C_B} \otimes R_{Q_B\rightarrow C_A})(\rho_{Q_AQ_B}),$$

where $p \in [0, 1]$. This channel is covariant and so [LKDW18, Corollary II.5] applies; we thus conclude that the tensor product of maximally entangled states is an optimal input for either the normalized diamond distance or the channel infidelity to the ideal swap channel. By applying the analysis in Figures 18 and 19, we conclude that both terms above have a fidelity of $\frac{1}{4}$. Thus, the overall channel infidelity is equal to $\frac{3}{4}$ for all $p \in [0, 1]$. One can also check that the normalized diamond distance is equal to $\frac{1}{4}$. Thus, this second protocol of [KPF16] does not go beyond the classical limit from Proposition 9.

Another protocol suggested in [KPF16] is to mix the two previous protocols (i.e., to use one or the other probabilistically). However, such a strategy never achieves a fidelity higher than $\frac{3}{4}$, and by previous analyses that we have given, such a strategy does not go beyond the classical limit from Proposition 9.

As indicated previously, an optimal strategy in this scenario is to employ that given in Remark 11.

VIII. GENERALIZATION TO MULTIPARTITE CHANNEL SIMULATION AND BIDIRECTIONAL CONTROLLED TELEPORTATION

In this section, we generalize the development in Section IV D to the case of multipartite channel simulation, and then we consider the specific case of bidirectional controlled teleportation. The latter has been considered extensively in the literature [ZZQS13, SBP13, LNS16, LLS13, Che14, Yan13, SZ13, DZ14, DZSL15, ZXL19, DZSX14, Hon16, San16, ZZLY15, SJHA17, LJ16].

A. Multipartite channel simulation

Let us begin by recalling that a multipartite LOCC channel can be written in the following form [CLM+14]:

$$\mathcal{L}_{A_1 \cdots A_M \rightarrow A_1' \cdots A_M'} = \sum_{y} \bigotimes_{i=1}^{M} \mathcal{E}_{A_i \rightarrow A_i'}^y,$$

where $M$ is the number of parties and the set $\{\mathcal{E}_{A_i \rightarrow A_i'}^y\}_y$, for each $i \in \{1, \ldots, M\}$, is a set of completely positive maps such that the sum map in (196) is trace preserving. However, just as in the bipartite case, there exist channels of the form in (196) that are not implementable by means of LOCC.

Let us now define LOCC simulation of a multipartite channel. Let $\mathcal{N}_{A_1 \cdots A_M \rightarrow A_1' \cdots A_M'}$ be a multipartite channel, in the sense that there are $M$ input systems $A_1 \cdots A_M$ and $M$ output systems $A_1' \cdots A_M'$. Furthermore, the $i$th party controls the $i$th input $A_i$ to the channel, as well as the $i$th output $A_i'$, for $i \in \{1, \ldots, M\}$. Suppose now that the $M$ parties share a resource state $\rho_{A_1 \cdots A_M}$. Then an LOCC simulation of $\mathcal{N}_{A_1 \cdots A_M \rightarrow A_1' \cdots A_M'}$ consists of an LOCC channel
\[ \mathcal{L}_{A_1 \cdots A_M \hat{A}_1 \cdots \hat{A}_M \to A'_1 \cdots A'_M} \] such that
\[ \tilde{N}_{A'_M \to A'_1} (\omega_{A'_M}) := \mathcal{L}_{A'_M \hat{A}_1 \cdots \hat{A}_M \to A'_1} (\omega_{A'_M} \otimes \rho_{\hat{A}_M}), \] (197)
where we have employed the shorthand
\begin{align*}
A_M^M & \equiv A_1 \cdots A_M, \\
A_M^{M'} & \equiv A_1 \cdots A_{M'}, \\
\hat{A}_M^M & \equiv \hat{A}_1 \cdots \hat{A}_M.
\end{align*}
\[ (198) \]
\[ (199) \]
\[ (200) \]
Note that \[ \mathcal{L}_{A_M^M \hat{A}_M^M \to A'_M} \] takes the following form:
\[ \mathcal{L}_{A_M^M \hat{A}_M^M \to A'_M} = \sum_{y} \bigotimes_{i=1}^{M} \mathcal{E}_{A_i \hat{A}_i \to A'_i}. \] (201)

The simulation error when using the LOCC channel \[ \mathcal{L}_{A_M^M \hat{A}_M^M \to A'_M} \] is then defined as follows:
\[ e_{\text{LOCC}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M}, \mathcal{L}_{A_M^M \hat{A}_M^M \to A'_M}) := \frac{1}{2} \left\| \mathcal{N}_{A'_M \to A'_1} - \tilde{N}_{A'_M \to A'_1} \right\|_1. \] (202)
The simulation error minimized over all LOCC channels is defined as follows:
\[ e_{\text{LOCC}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M}) := \inf_{\mathcal{L} \in \text{LOCC}} e_{\text{LOCC}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M}, \mathcal{L}_{A_M^M \hat{A}_M^M \to A'_M}). \] (203)

Just as in the bipartite case, the LOCC simulation error in (203) is generally hard to calculate. So we instead seek ways of estimating or bounding it, and we can make use of an idea from [IP05], which is helpful for developing a multipartite extension of the bipartite case presented in Section IVD. Let us define a multipartite quantum channel \[ \mathcal{P}_{A'_M^M} \to A_M^M \] to be completely PPT preserving if the following maps are completely positive:
\[ T_{S'} \circ \mathcal{P}_{A'_M^M} \to A_M^M \circ T_{S}, \] (204)
for all \[ S \in \mathbb{P}(\{A_1, \ldots, A_M\}) \], where \( \mathbb{P} \) denotes the power set. In the above, \( T_S \) and \( T_{S'} \) denote partial transpose maps acting on all subsystems in \( S \) and \( S' \), respectively, and the subset \( S' \) is chosen to correspond to the same systems in \( S \) but for the channel output. Note that there is some redundancy in this specification. There is no need to include the null set or full set from \( \mathbb{P}(\{A_1, \ldots, A_M\}) \) because the map \( \mathcal{P}_{A'_M^M} \to A_M^M \) is completely positive. That is, the null set corresponds to no transposes being taken, and the full set corresponds to a full transpose on both the input and output, and it is known that the resulting map is completely positive if and only if the original map is completely positive. There is further redundancy in the sense that \( T_{S'} \circ \mathcal{P}_{A'_M^M} \to A_M^M \circ T_S \) is completely positive if and only if \( T_{S''} \circ \mathcal{P}_{A'_M^M} \to A_M^M \circ T_{S''} \) is.

Let us illustrate this concept with an example. Suppose that \( M = 4 \) and the four parties are labeled as \[ ABCD. \] Then a four-partite channel \[ \mathcal{P}_{A'_M^M} \to A'_1 \to B'_1 \to C'_1 \to D'_1 \] is completely PPT preserving if the maps
\[ T_{S'} \circ \mathcal{P}_{A'_M^M} \to A'_1 \to B'_1 \to C'_1 \to D'_1 \circ T_{S} \] (205)
are completely positive, where \( S \in \{A, B, C, D, AD, AC, DC\} \) and \( S' \) is the corresponding subset on the output systems. As stated above, there is no need to include all of the elements of the power set.

Extending the bipartite case, we can consider this concept from the perspective of the Choi operator. The Choi operator of a multipartite linear map \[ \mathcal{P}_{A'_M^M} \to A_M^M \] is defined as follows:
\[ \mathcal{P}_{A'_M^M} \to A_M^M := \mathcal{P}_{A'_M^M} \to A_M^M \left( \bigotimes_{i=1}^{M} \Gamma_{A_i, A_i} \right). \] (206)
The multipartite linear map \[ \mathcal{P}_{A'_M^M} \to A_M^M \] is completely positive if and only if its Choi operator \[ \mathcal{P}_{A'_M^M} \to A_M^M \] is positive semi-definite
\[ \mathcal{P}_{A'_M^M} \to A_M^M \geq 0, \] (207)
and \[ \mathcal{P}_{A'_M^M} \to A_M^M \] is trace preserving if and only if \[ \mathcal{P}_{A'_M^M} \to A_M^M \] satisfies
\[ \text{Tr}_{A'_M} [\mathcal{P}_{A'_M^M} \to A_M^M] = I_{A'_M}. \] (208)
Furthermore, a multipartite channel \[ \mathcal{P}_{A'_M^M} \to A_M^M \] is completely PPT-preserving if and only if its Choi operator satisfies
\[ T_{S'''} (\mathcal{P}_{A'_M^M} \to A_M^M) \geq 0 \] (209)
for all \( S \in \mathbb{P}(\{A_1, \ldots, A_M\}) \) and with \( S' \) corresponding to \( S \).

We then define a PPT simulation when using a multipartite C-PPT-P channel \[ \mathcal{P}_{A'_M^M} \to A_M^M \] along with a resource state \( \rho_{\hat{A}_M^M} \):
\[ \tilde{N}_{A'_M \to A'_1} (\omega_{A'_M}) := \mathcal{P}_{A'_M^M} \to A_M^M (\omega_{A'_M} \otimes \rho_{\hat{A}_M^M}), \] (210)
the simulation error as
\[ e_{\text{PPT}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M^M}, \mathcal{P}_{A'_M^M} \to A_M^M) := \frac{1}{2} \left\| \mathcal{N}_{A'_M \to A'_1} - \tilde{N}_{A'_M \to A'_1} \right\|_1, \] (211)
and the simulation error minimized over all C-PPT-P channels \[ \mathcal{P}_{A'_M^M} \to A_M^M \] as
\[ e_{\text{PPT}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M^M}) := \inf_{\mathcal{P} \in \text{C-PPT-P}} e_{\text{PPT}} (\tilde{N}_{A'_M \to A'_1}, \rho_{\hat{A}_M^M}, \mathcal{P}_{A'_M^M} \to A_M^M). \] (212)

One of the main observations of [IP05] is that the set of multipartite LOCC channels is contained in the set of multipartite C-PPT-P channels:
\[ \text{LOCC} \subset \text{C-PPT-P}, \] (213)
which is the main reason that this concept is useful for providing bounds on what is achievable using LOCC. As a direct consequence of (213), we conclude that

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) \leq e^{\text{LOCC}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}).$$

(214)

The main advantage of this approach is due to the following generalization of Proposition 3:

Proposition 13 The simulation error in (212) can be computed by means of the following semi-definite program:

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) = \inf_{\mu, Z, A^M \rightarrow A'^M, A^M, \lambda I_{A^M}} \mu, Z \geq 0,$$

subject to

$$\mu I_{A^M} \geq Z_{A^M},$$

$$\text{Tr}_{A'^M} [P_{A^M A^M} \rho_{A^M}] = I_{A^M A^M},$$

(215)

$$Z_{A^M A'^M} \geq \Gamma_{A^M A'^M} - \text{Tr}_{A^M} [T_{A^M} (\rho_{A^M}) P_{A^M A^M A'^M}],$$

and

$$T_{S\hat{S}S'} (P_{A^M A^M A'^M}) \geq 0,$$

(216)

for all $S \in \mathbb{P}(\{A_1, \ldots, A_M\})$, with $\hat{S}$ and $S'$ corresponding to $S$.

The proof of Proposition 13 is very similar to the proof of Proposition 3. It just combines the semi-definite program for the diamond distance in (87) along with the constraints in (207)–(209).

As we did previously, we can also define simulation error in terms of infidelity. For this case, we define

$$e^{\text{LOCC}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}, \mathcal{L}_{A^M A'^M} - A^M, A^M) :=$$

$$1 - F(\mathcal{N}_{A^M \rightarrow A'^M}, \hat{\mathcal{N}}_{A^M A'^M}),$$

(217)

where $\hat{\mathcal{N}}_{A^M A'^M}$ is defined in (197), and the simulation error minimized over all LOCC channels as

$$e^{\text{LOCC}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) :=$$

$$\inf_{\mathcal{L} \in \text{LOCC}} e^{\text{LOCC}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}, \mathcal{L}_{A^M A'^M} - A^M).$$

(218)

We also define the following simulation error:

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}, \mathcal{P}_{A^M A^M A'^M}) :=$$

$$1 - F(\mathcal{N}_{A^M \rightarrow A'^M}, \hat{\mathcal{N}}_{A^M A'^M}),$$

(219)

where $\hat{\mathcal{N}}_{A^M A'^M}$ is defined in (201), and the simulation error minimized over all C-PPT-P channels $\mathcal{P}_{A^M A^M A'^M}$ as

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) :=$$

$$\inf_{\mathcal{P} \in \text{PPT}} e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}, \mathcal{P}_{A^M A^M A'^M}).$$

(220)

FIG. 23. Shown above is a diagram for bidirectional controlled teleportation. The goal of this protocol is the same as the usual BQT protocol, but there is a third party, Charlie, who helps Alice and Bob with the task. The three parties share a resource state $\rho_{A'B'C}$ and perform an LOCC protocol to approximate a swap of the input systems $A$ and $B$.

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) :=$$

$$\inf_{\mathcal{P} \in \text{PPT}} e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}, \mathcal{P}_{A^M A^M A'^M}).$$

(221)

By reasoning similar to that used to conclude Proposition 6, we conclude the following:

Proposition 14 The simulation error in (223) can be computed by means of the following semi-definite program:

$$e^{\text{PPT}}(\mathcal{N}_{A^M \rightarrow A'^M}, \rho_{A^M}) =$$

$$1 - \left[ \sup_{\lambda \geq 0} \| \lambda I_{A^M A'^M} \rho_{A^M} \|^2 \right],$$

(222)

subject to

$$\lambda I_{A'^M} \leq \text{Re} \left[ \text{Tr}_{A'^M} [Q_{A^M A'^M}] \right],$$

$$\text{Tr}_{A'^M} [P_{A^M A'^M A'^M}] = I_{A^M A'^M},$$

(223)

$$\left[ \Gamma_{A^M A'^M} - Q_{A^M A'^M} - \text{Tr}_{A^M} [T_{A^M} (\rho_{A^M}) P_{A^M A'^M A'^M}] \right] \geq 0,$$

and

$$T_{S\hat{S}S'} (P_{A^M A^M A'^M}) \geq 0,$$

for all $S \in \mathbb{P}(\{A_1, \ldots, A_M\})$, with $\hat{S}$ and $S'$ corresponding to $S$.

B. Bidirectional controlled teleportation

An important special case of LOCC simulation of a multipartite channel is bidirectional controlled teleportation, depicted in Figure 23. For this problem, the goal is
Note that the LOCC channel $\mathcal{N}_{\hat{A}B\hat{B}C\rightarrow AB}$ has trivial output system for Charlie. In more detail, it can be written in the following form:

$$\mathcal{L}_{AB\hat{A}\hat{B}\hat{C}\rightarrow AB}(\tau_{A\hat{A}\hat{B}\hat{C}}) = \sum_y (\mathcal{E}_{A\hat{A}}^y \otimes \mathcal{F}_{BB\rightarrow B}^y) \text{Tr}_C [\Lambda_C^y \tau_{A\hat{A}\hat{B}\hat{C}}],$$

(230)

where $\{\mathcal{E}_{A\hat{A}}^y\}_y$ and $\{\mathcal{F}_{BB\rightarrow B}^y\}_y$ are sets of completely positive maps and $\{\Lambda_C^y\}_y$ is a set of positive semi-definite operators such that the sum map in (230) is a quantum channel.

The simulation error for bidirectional controlled teleportation, when using a particular LOCC channel $\mathcal{L}_{A\hat{A}\hat{B}\hat{C}\rightarrow AB}$, is given by

$$\epsilon_{\text{LOCC}}(S_{AB}^d, \rho_{A\hat{B}\hat{C}}; \mathcal{L}_{AB\hat{A}\hat{B}\hat{C}\rightarrow AB}) := \frac{1}{2} \left\| S_{AB} - \tilde{S}_{AB} \right\|_1,$$

(231)

and the simulation error minimized over all LOCC channels $\mathcal{L}_{A\hat{A}\hat{B}\hat{C}\rightarrow AB}$ is given by

$$\epsilon_{\text{LOCC}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}}) := \inf_{\mathcal{L} \in \text{LOCC}} \epsilon_{\text{LOCC}}(S_{AB}^d, \rho_{A\hat{B}\hat{C}}; \mathcal{L}_{AB\hat{A}\hat{B}\hat{C}\rightarrow AB}).$$

(232)

We can also define simulation errors in terms of channel infidelity:

$$\epsilon_{\text{LOCC}}^F(S_{AB}^d; \rho_{A\hat{B}\hat{C}}; \mathcal{L}_{AB\hat{A}\hat{B}\hat{C}\rightarrow AB}) := 1 - F(S_{AB}^d, \tilde{S}_{AB}^d),$$

(233)

$$\epsilon_{\text{LOCC}}^F(S_{AB}^d; \rho_{A\hat{B}\hat{C}}) := \inf_{\mathcal{L} \in \text{LOCC}} \epsilon_{\text{LOCC}}^F(S_{AB}^d, \rho_{A\hat{B}\hat{C}}; \mathcal{L}_{AB\hat{A}\hat{B}\hat{C}\rightarrow AB}).$$

(234)

As before, the simulation error in (232) is difficult to compute, and so we bound it from below by the PPT simulation error, which is defined from (212) and denoted by $\epsilon_{\text{PPT}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}})$. As a generalization of Proposition 4, we show that the semi-definite program for calculating the PPT simulation error $\epsilon_{\text{PPT}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}})$ simplifies as given in Proposition 15 below. A proof is available in Appendix G and is similar to the proof of Proposition 4. The interpretation of an optimal simulating channel is the same as given in Remark 5, except with respect to the measurement operators $K_{A\hat{B}C}$, $L_{A\hat{B}C}$, $M_{A\hat{B}C}$, and $N_{A\hat{B}C}$ being subject to the conditions in Proposition 15 below. Furthermore, the two different notions of simulation error based on $\epsilon_{\text{PPT}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}})$ and $\epsilon_{\text{PPT}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}})$ coincide again.

**Proposition 15** The semi-definite programs in Propositions 13 and 14, for the error in simulating the unitary SWAP channel $S_{AB}^d$, using a resource state $\rho_{A\hat{B}\hat{C}}$ simplify as follows:

$$\epsilon_{\text{PPT}}(S_{AB}^d; \rho_{A\hat{B}\hat{C}}) = \epsilon_{\text{PPT}}^F(S_{AB}^d; \rho_{A\hat{B}\hat{C}}) = 1 - \inf_{\mathcal{L}_{AB\hat{A}\hat{B}\hat{C}} \subset \mathbb{R}} \text{Tr}[\rho_{A\hat{B}\hat{C}} K_{A\hat{B}C}],$$

(235)

subject to

$$T_S \left( K_{A\hat{B}C} + \frac{L_{A\hat{B}C}}{d+1} + \frac{N_{A\hat{B}C}}{(d+1)^2} \right) \geq 0,$$

(236)

where $S_{AB}^d \{ \hat{A}, \hat{B} \}$ and

$$T_C(K_{A\hat{B}C}) = T_C(L_{A\hat{B}C}), T_C(N_{A\hat{B}C}) \geq 0.$$ (237)

(238)

**Proposition 16** If there is no resource state, then the error in (212) and (232) for simulating the unitary SWAP channel $S_{AB}^d$ in (18) is equal to $1 - 1 / d^2$:

$$\epsilon_{\text{PPT}}(S_{AB}^d; \emptyset) = \epsilon_{\text{LOCC}}(S_{AB}^d; \emptyset) = 1 - \frac{1}{d^2},$$

(239)

where the notation $\emptyset$ indicates the absence of a resource state.

**Proof.** This follows from the same proof given for Proposition 9. When there is no resource state, the state $\rho_{A\hat{B}\hat{C}}$ collapses to the number one, and the operators $K_{A\hat{B}C}$, $L_{A\hat{B}C}$, and $N_{A\hat{B}C}$ collapse to real numbers as well. So the optimization in Proposition 15 collapses to the linear program in (117)–(121), and we conclude the statement above from the rest of the proof of Proposition 9.

**IX. CONCLUSION**

In this paper, we have provided a systematic approach for quantifying the performance of bidirectional teleportation and bidirectional controlled teleportation. We have established a benchmark for classical versus quantum bidirectional teleportation, and we have evaluated
semi-definite programming lower bounds on the simulation error for some key examples of resource states. More generally, we have demonstrated that semi-definite programs are possible when using the channel infidelity as an error measure, which addresses an open question from [WW19a, WW19b] and should have applications more generally in quantum resource theories.

Going forward from here, there are several avenues for future work. First, we can consider other unitary channels besides the swap channel, and the line of thinking developed here could be useful for related scenarios considered in [STM11, WSM19]. We can also consider applying the framework used here to analyze multidirectional teleportation between more than two parties. We also wonder whether there is an LOCC simulation that achieves a performance matching the lower bound found here, for all parameter values for isotropic and Werner states. As mentioned in the introduction of our paper, there have been many proposals for bidirectional (controlled) teleportation. One could also evaluate the semi-definite programming bounds from this paper for imperfect versions of the resource states in those works in order to determine how robust those entangled resource states are to noise.

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[AKN98] Dorit Aharonov, Alexei Kitaev, and Noam Nisan. Quantum circuits with mixed states. In Proceedings of the thirtieth annual ACM Symposium on Theory of Computing, pages 20–30, New York, NY, USA, May 1998. ACM. arXiv:quant-ph/9806029.

[BBC93] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Physical Review Letters, 70(13):1895–1899, March 1993.

[BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. Physical Review A, 53(4):2046–2052, April 1996. arXiv:quant-ph/9511030.

[BDF99] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters. Quantum nonlocality without entanglement. Physical Review A, 59(2):1070–1091, February 1999. arXiv:quant-ph/9804053.

[BDSW96] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. Physical Review A, 54(5):3824–3851, November 1996. arXiv:quant-ph/9604024.

[BDWW19] Stefan Bäuml, Siddhartha Das, Xin Wang, and Mark M. Wilde. Resource theory of entanglement for bipartite quantum channels. July 2019. arXiv:1907.04181.

[Ben05] Charles H. Bennett. Simulated time travel, teleportation without communication, and how to conduct a romance with someone who has fallen into a black hole. https://www.research.ibm.com/people/b/bennetc/QUPONshort.pdf, May 2005.

[BHLS03] Charles H. Bennett, Aram W. Harrow, Debbie W. Leung, and John A. Smolin. On the capacities of bipartite Hamiltonians and unitary gates. IEEE Transactions on Information Theory, 49(8):1895–1911, August 2003. arXiv:quant-ph/0205057.

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, The Edinburgh Building, Cambridge, CB2 8RU, UK, 2004.

[CD11] Bob Coecke and Ross Duncan. Interacting quantum observables: categorical algebra and diagrammat-
via a maximally seven-qubit entangled state. *International Journal of Theoretical Physics*, 53(8):2697–2707, August 2014.

[FTH14] Hong-Zi Fu, Xiu-Lao Tian, and Yang Hu. A general method of selecting quantum channel for bidirectional quantum teleportation. *International Journal of Theoretical Physics*, 53(6):1840–1847, June 2014.

[FWTB20] Kun Fang, Xin Wang, Marco Tomamichel, and Mario Berta. Quantum channel simulation and the channel’s smooth max-information. *IEEE Transactions on Information Theory*, 66(4):2129–2140, April 2020. arXiv:1807.05354.

[GH10] Sevag Gharibian. Strong NP-hardness of the quantum separability problem. *Quantum Information and Computation*, 10(3):343–360, March 2010. arXiv:0810.4507.

[GS19] Gilad Gour and Carlo Maria Scandolo. The entanglement of a bipartite channel. July 2019. arXiv:1907.02552.

[GSWL17] Yi-Tao Guo, Hai-Long Shi, Xiao-Hui Wang, and Si-Yuan Liu. Probabilistic resumable bidirectional quantum teleportation. *Quantum Information Processing*, 16(11):278, 2017.

[Gur04] Leonid Gurvits. Classical complexity and quantum entanglement. *Journal of Computer and System Sciences*, 69(3):448–484, 2004. arXiv:quant-ph/0003055.

[Hay17] Masahito Hayashi. *Quantum Information Theory: A Mathematical Foundation*. Springer, second edition, 2017.

[HH99] Michał Horodecki and Paweł Horodecki. Reduction criterion of separability and limits for a class of distillation protocols. *Physical Review A*, 59(6):4206–4216, June 1999. arXiv:quant-ph/9708015.

[HH16] Shima Hassanpour and Monireh Houshmand. Bidirectional teleportation of a pure EPR state by using GHZ states. *Quantum Information Processing*, 15(2):905–912, February 2016. arXiv:1411.0207.

[HHNH99] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. General teleportation channel, singlet fraction, and quasidistillation. *Physical Review A*, 60(3):1888–1898, September 1999. arXiv:quant-ph/9807091.

[HHNH09] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81(2):865–942, June 2009. arXiv:quant-ph/0702225.

[HM13] Aram W. Harrow and Ashley Montanaro. Testing product states, quantum Merlin-Arthur games and tensor optimization. *Journal of the ACM*, 60(1):3:1–3:43, February 2013. arXiv:1001.0017.

[Hol19] Alexander S. Holevo. *Quantum Systems, Channels, Information: A Mathematical Introduction*. Walter de Gruyter, second edition, 2019.

[Hon16] Wen-quin Hong. Asymmetric bidirectional controlled teleportation by using a seven-qubit entangled state. *International Journal of Theoretical Physics*, 55(1):384–387, January 2016.

[HPV02] Susana F. Huelga, Martin B. Plenio, and Joan A. Vaccaro. Remote control of restricted sets of operations: Teleportation of angles. *Physical Review A*, 65(4):042316, April 2002. arXiv:quant-ph/0107110.

[HVCPC01] Susana F. Huelga, Joan A. Vaccaro, Anthony Cheffes, and Martin B. Plenio. Quantum remote control: Teleportation of unitary operations. *Physical Review A*, 63(4):042303, March 2001. arXiv:quant-ph/0005061.

[I115] Takaya Ikuto and Satoshi Ishizaka. Entanglement and swap of quantum states in two qubits. *Quantum Information and Computation*, 15:0923–0931, 2015.

[IP05] Satoshi Ishizaka and Martin B. Plenio. Multipartite entanglement manipulation under positive partial transpose preserving operations. *Physical Review A*, 71(5):052303, May 2005. arXiv:quant-ph/0412193.

[KLi17] Alexei Kitaev. Quantum computations: algorithms and error correction. *Russian Mathematical Surveys*, 52(6):1191–1249, 1997.

[KPF16] Evgeniy O. Kiktenko, A. A. Popov, and Aleksy K. Fedorov. Bidirectional imperfect quantum teleportation with a single Bell state. *Physical Review A*, 93(6):062305, June 2016. arXiv:1602.01420.

[KSW20] Sumeet Khatri, Kunal Sharma, and Mark M. Wilde. Information-theoretic aspects of the generalized amplitude-damping channel. *Physical Review A*, 102(1):012401, July 2020. arXiv:1903.07747.

[KW20a] Vishal Katariya and Mark M. Wilde. Geometric distinguishability measures limit quantum channel estimation and discrimination. April 2020. arXiv:2004.10708.

[KW20b] Sumeet Khatri and Mark M. Wilde. *Principles of Quantum Communication Theory: A Modern Approach*. November 2020. arXiv:2011.04672v1.

[LJ16] Yuan-hua Li and Xian-min Jin. Bidirectional controlled teleportation by using nine-qubit entangled state in noisy environments. *Quantum Information Processing*, 15(2):929–945, 2016.

[LKD18] Felix Leditzky, Eneet Kaur, Nilanjan Datta, and Mark M. Wilde. Approaches for approximate additivity of the Holevo information of quantum channels. *Physical Review A*, 97(1):012332, January 2018. arXiv:1709.01111.

[LLS’13] Yuan-hua Li, Xiao-lan Li, Ming-huang Sang, Yiyou Nie, and Zhi-sheng Wang. Bidirectional controlled quantum teleportation and secure direct communication using five-qubit entangled state. *Quantum Information Processing*, 12(12):3835–3844, December 2013.

[LN13] Yuan-hua Li and Li-ping Nie. Bidirectional controlled teleportation by using a five-qubit composite GHZ-Bell state. *International Journal of Theoretical Physics*, 52(5):1630–1634, May 2013.

[LNL16] Yuan-hua Li, Li-ping Nie, Xiao-lan Li, and Ming-huang Sang. Asymmetric bidirectional controlled teleportation by using six-qubit cluster state. *International Journal of Theoretical Physics*, 55(6):3008–3016, June 2016.

[LW19] Zi-Wen Liu and Andreas Winter. Resource theories of quantum channels and the universal role of resource erasure. April 2019. arXiv:1904.04201.

[LY20] Yunchao Liu and Xiao Yuan. Operational resource theory of quantum channels. *Physical Review Research*, 2(1):012035(R), February 2020. arXiv:1904.02680.

[Mat14] Olivia Di Matteo. A short introduction to unitary 2-designs. https://glassnotes.github.io/OliviaDiMatteo_Unitary2Designs.pdf, November 2014.

[Rai99] Eric M. Rains. Bound on distillable entanglement. *Physical Review A*, 60(1):179–184, July 1999. arXiv:quant-ph/9809082.

[Rai01] Eric M. Rains. A semidefinite program for distillable entanglement. *IEEE Transactions on Information Theory*, 47(7):2921–2933, November 2001. arXiv:quant-ph/0008047.
Appendix A: Proof of Proposition 1

1. Exploiting symmetries of the unitary swap channel

The main idea of the proof is to simplify the optimization problems in (56) and (68) by exploiting the symmetries of the unitary swap channel, as given in (69) and (70). To begin with, let us note that the Choi operator of the SWAP channel can be written as

$$\Gamma_{ABA'B'}^d = \sum_{i,j,k,\ell} |i\rangle\langle j|_A \otimes |k\rangle\langle \ell|_B \otimes |k\rangle\langle \ell|_{A'} \otimes |i\rangle\langle j|_{B'}$$

(A1)

and

$$\Gamma_{ABA'} \otimes \Gamma_{BA'},$$

(A2)

for orthonormal bases \{\{i\}_A\}, \{\{k\}_B\}_k, \{\{k\}_{A'}\}_k, and \{\{i\}_{B'}\}_i. Let us define the unitary channel \(U(\cdot) = U(\cdot)U^\dagger\). To exploit symmetries of the unitary SWAP channel, recall from (69) that the channel \(S_{AB}^d\) is covariant in the following way:

$$S_{AB}^d = (U_A^\dagger \otimes U_B^\dagger) \circ S_{AB}^d \circ (U_A \otimes V_B).$$

(A3)
Let $\mathcal{A}_{AB}^\rho$ denote the channel that appends the bipartite state $\rho_{AB}$ to its input:

$$\mathcal{A}_{AB}^\rho(\omega_{AB}) = \omega_{AB} \otimes \rho_{AB}. \tag{A4}$$

Let us start with the diamond distance, but note that the reasoning employed in the first part of the proof (just below) applies equally well to channel infidelity. Let $L_{ABAB\to AB}$ be an arbitrary LOCC channel to consider for the optimization problem in (56). Exploiting the unitary invariance of the diamond distance with respect to input and output unitaries [Wat18, Proposition 3.44], we find the following:

$$\left\| L_{ABAB\to AB} \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right\|_\diamond = \left\| (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ \left[ L_{ABAB\to AB} \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right] \circ (\mathcal{U}_A \otimes \mathcal{V}_B) \right\|_\diamond \tag{A5}$$

$$= \left\| (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ L_{ABAB\to AB} \circ (\mathcal{U}_A \otimes \mathcal{V}_B) \circ \mathcal{A}_{AB}^\rho - (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ S_{AB}^d \circ (\mathcal{U}_A \otimes \mathcal{V}_B) \right\|_\diamond \tag{A6}$$

$$= \left\| (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ L_{ABAB\to AB} \circ (\mathcal{U}_A \otimes \mathcal{V}_B) \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right\|_\diamond. \tag{A7}$$

Thus, the channels $L_{ABAB\to AB}$ and $(\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ L_{ABAB\to AB} \circ (\mathcal{U}_A \otimes \mathcal{V}_B)$ perform equally well for the optimization. Now we exploit the convexity of the diamond distance with respect to one of the channels [Wat18], as well as the Haar probability measure over the unitary group, to conclude that

$$\left\| L_{ABAB\to AB} \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right\|_\diamond = \int \int dU \, dV \, \left\| (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ L_{ABAB\to AB} \circ (\mathcal{U}_A \otimes \mathcal{V}_B) \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right\|_\diamond \tag{A8}$$

$$\geq \left\| \tilde{L}_{ABAB\to AB} \circ \mathcal{A}_{AB}^\rho - S_{AB}^d \right\|_\diamond, \tag{A9}$$

where

$$\tilde{L}_{ABAB\to AB} := \int \int dU \, dV \, (\mathcal{V}_A^\dagger \otimes \mathcal{U}_B^\dagger) \circ L_{ABAB\to AB} \circ (\mathcal{U}_A \otimes \mathcal{V}_B). \tag{A10}$$

Thus, we conclude that it suffices to optimize (84) over LOCC channels that possess this symmetry. Critical to this argument is the observation that the channel twirl in (A10) can be realized by LOCC, so that $\tilde{L}_{ABAB\to AB}$ is an LOCC channel if $L_{ABAB\to AB}$ is.

What is the form of LOCC channels possessing this symmetry? Let $L_{ABAB\to AB}$, denote the Choi operator of the channel $L_{ABAB\to AB}$, where $A'$ and $B'$ denote the output systems. Then consider that the Choi operator $\tilde{L}_{ABAB\to AB}$ of the double twirled channel $\tilde{L}_{ABAB\to AB}$ is as follows:

$$\tilde{L}_{ABAB\to AB} := \int \int dU \, dV \, (\mathcal{U}_A \otimes \mathcal{V}_B \otimes \mathcal{U}_{A'} \otimes \mathcal{U}_{B'})(L_{ABAB\to AB}), \tag{A11}$$

where $\overline{U(\cdot)} = U(\cdot)^T$ (with the overbar denoting complex conjugate). Now recall the following identity from [Wer89, HH99, Wat18]:

$$\tilde{T}_{CD}(X_{CD}) := \int dU \, (\mathcal{U}_C \otimes \mathcal{U}_D)(X_{CD}) = \Phi_{CD} \operatorname{Tr}_{CD}[\Phi_{CD} X_{CD}] + \frac{I_{CD} - \Phi_{CD}}{d^2 - 1} \operatorname{Tr}_{CD}[(I_{CD} - \Phi_{CD}) X_{CD}]. \tag{A12}$$

We apply this to (A11) to write it as

$$\tilde{L}_{ABAB\to AB} = (\tilde{T}_{AB'}) \otimes (\tilde{T}_{BA'})(L_{ABAB\to AB})$$

$$= (\Phi_{AB'} \otimes \Phi_{BA'}) \operatorname{Tr}_{ABAB'}[(\Phi_{AB'} \otimes \Phi_{BA'}) L_{ABAB\to AB}]$$

$$+ \left( \Phi_{AB'} \otimes \frac{I_{BA'} - \Phi_{BA'}}{d^2 - 1} \right) \operatorname{Tr}_{ABAB'}[(\Phi_{AB'} \otimes \Phi_{BA'}) L_{ABAB\to AB}]$$

$$+ \left( \frac{I_{AB'} - \Phi_{AB'}}{d^2 - 1} \Phi_{BA'} \right) \operatorname{Tr}_{ABAB'}[(\Phi_{AB'} \otimes \Phi_{BA'}) L_{ABAB\to AB}].$$
\[
+ \left( \frac{I_{AB'} - \Phi_{AB'}}{d^2 - 1} \otimes \frac{I_{BA'} - \Phi_{BA'}}{d^2 - 1} \right) \text{Tr}_{ABA'B'}[(I_{AB'} - \Phi_{AB'}) \otimes (I_{BA'} - \Phi_{BA'})] L_{AB\hat{A}B'}.
\]

Now defining
\[
K_{\hat{A}B}' := \text{Tr}_{ABA'B'}[(\Phi_{AB'} \otimes \Phi_{BA'}) L_{AB\hat{A}B'}] \\
L_{\hat{A}B}' := \text{Tr}_{ABA'B'}[(\Phi_{AB'} \otimes [I_{BA'} - \Phi_{BA'}]) L_{AB\hat{A}B'}] \\
M_{\hat{A}B}' := \text{Tr}_{ABA'B'}[[I_{AB'} - \Phi_{AB'}] \otimes \Phi_{BA'}) L_{AB\hat{A}B'}] \\
N_{\hat{A}B}' := \text{Tr}_{ABA'B'}[([I_{AB'} - \Phi_{AB'}] \otimes [I_{BA'} - \Phi_{BA'}]) L_{AB\hat{A}B'}],
\]

we can write
\[
\tilde{L}_{AB\hat{A}B'A'B'} = \Phi_{AB'} \otimes \Phi_{BA'} \otimes K_{\hat{A}B}' + \Phi_{AB'} \otimes \frac{I_{BA'} - \Phi_{BA'}}{d^2 - 1} \otimes L_{\hat{A}B}' + \frac{I_{AB'} - \Phi_{AB'}}{d^2 - 1} \otimes \Phi_{BA'} \otimes M_{\hat{A}B}' + \frac{I_{AB'} - \Phi_{AB'}}{d^2 - 1} \otimes \frac{I_{BA'} - \Phi_{BA'}}{d^2 - 1} \otimes N_{\hat{A}B}'.
\]

What are the conditions on the operators \(K_{\hat{A}B}', L_{\hat{A}B}', M_{\hat{A}B}', N_{\hat{A}B}'\)? In order for \(\tilde{L}_{AB\hat{A}B'A'B'}\) to be the Choi operator of a channel, the following conditions should hold, thus imposing conditions on the operators \(K_{\hat{A}B}', L_{\hat{A}B}', M_{\hat{A}B}', N_{\hat{A}B}'\):

\[
\tilde{L}_{AB\hat{A}B'A'B'} \geq 0, \\
\text{Tr}_{ABA'B'}[\tilde{L}_{AB\hat{A}B'A'B'}] = I_{AB\hat{A}B}. \tag{A18}
\]

The first condition in (A18) imposes that
\[
K_{\hat{A}B}', L_{\hat{A}B}', M_{\hat{A}B}', N_{\hat{A}B}' \geq 0. \tag{A20}
\]

The second condition in (A19) imposes that
\[
\pi_{AB} \otimes K_{\hat{A}B}' + \pi_{AB} \otimes L_{\hat{A}B}' + \pi_{AB} \otimes M_{\hat{A}B}' + \pi_{AB} \otimes N_{\hat{A}B}' = I_{AB\hat{A}B}, \tag{A21}
\]

which is the same as
\[
K_{\hat{A}B}' + L_{\hat{A}B}' + M_{\hat{A}B}' + N_{\hat{A}B}' = d^2 I_{AB\hat{A}B}. \tag{A22}
\]

Thus, we can think of the operators \(K_{\hat{A}B}', L_{\hat{A}B}', M_{\hat{A}B}', N_{\hat{A}B}'\) normalized by \(d^2\) as measurement operators, which gives an interesting physical interpretation to them. Let us then define
\[
K_{\hat{A}B} := \frac{1}{d^2} K_{\hat{A}B}', \quad L_{\hat{A}B} := \frac{1}{d^2} L_{\hat{A}B}', \quad M_{\hat{A}B} := \frac{1}{d^2} M_{\hat{A}B}', \quad N_{\hat{A}B} := \frac{1}{d^2} N_{\hat{A}B}'.
\]

and note that (A22) is equivalent to
\[
K_{\hat{A}B} + L_{\hat{A}B} + M_{\hat{A}B} + N_{\hat{A}B} = I_{\hat{A}B}. \tag{A24}
\]

(A20) is equivalent to
\[
K_{\hat{A}B}, L_{\hat{A}B}, M_{\hat{A}B}, N_{\hat{A}B} \geq 0, \tag{A25}
\]

and (A17) is equivalent to
\[
\tilde{L}_{AB\hat{A}B'A'B'} = \Gamma_{AB'} \otimes \Gamma_{BA'} \otimes K_{\hat{A}B} + \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes L_{\hat{A}B} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \otimes M_{\hat{A}B} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes N_{\hat{A}B}'. \tag{A26}
\]

Now consider that the Choi operator of the composite channel \(\mathcal{L}_{AB\hat{A}B} : A_{\hat{A}B}' \rightarrow A_{\hat{A}B} \otimes A_{\hat{A}B}'\) is given by
\[ \text{Tr}_{AB}[T_{AB}(\rho_{AB})\tilde{L}_{AB\hat{A}B'\hat{A}'B'}] = \]

\[ = \Gamma_{AB'} \otimes \Gamma_{BA'} \text{Tr}[T_{AB}(\rho_{AB})K_{AB}] + \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[T_{AB}(\rho_{AB})L_{AB}] + \]

\[ + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \text{Tr}[T_{AB}(\rho_{AB})M_{AB}] + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[T_{AB}(\rho_{AB})N_{AB}]. \] (A27)

By making the substitutions \( K_{\hat{A}B} \rightarrow T_{\hat{A}B}(K_{\hat{A}B}), L_{\hat{A}B} \rightarrow T_{\hat{A}B}(L_{\hat{A}B}), M_{\hat{A}B} \rightarrow T_{\hat{A}B}(M_{\hat{A}B}), \) and \( N_{\hat{A}B} \rightarrow T_{\hat{A}B}(N_{\hat{A}B}), \)
and using the facts that

\[ K_{\hat{A}B} \geq 0 \iff T_{\hat{A}B}(K_{\hat{A}B}) \geq 0, \] (A28)

and the same for \( L_{\hat{A}B}, M_{\hat{A}B}, \) and \( N_{\hat{A}B}, \) as well as

\[ K_{\hat{A}B} + L_{\hat{A}B} + M_{\hat{A}B} + N_{\hat{A}B} = I_{\hat{A}B} \iff T_{\hat{A}B}(K_{\hat{A}B}) + T_{\hat{A}B}(L_{\hat{A}B}) + T_{\hat{A}B}(M_{\hat{A}B}) + T_{\hat{A}B}(N_{\hat{A}B}) = T_{\hat{A}B}(I_{\hat{A}B}) = I_{\hat{A}B}, \] (A29)

and the fact that the channel \( \tilde{L}_{AB\hat{A}B'\rightarrow AB} \) remains LOCC under these changes, we conclude that the optimization problem does not change if we make these substitutions. Thus, we can take the Choi operator of \( \tilde{L}_{AB\hat{A}B'\rightarrow AB} \circ \mathcal{A}_{\hat{A}B}^0 \) to be

\[ \Gamma_{AB'} \otimes \Gamma_{BA'} \text{Tr}[\rho_{AB}K_{AB}] + \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[\rho_{AB}L_{AB}] + \]

\[ + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \text{Tr}[\rho_{AB}M_{AB}] + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[\rho_{AB}N_{AB}]. \] (A30)

Regarding the observation in (71), consider from (A30) that the Choi operator of an optimal LOCC simulating channel acting on the resource state \( \rho_{AB} \) is as follows:

\[ \Gamma_{AB'} \otimes \Gamma_{BA'} \text{Tr}[\rho_{AB}K_{AB}] + \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[\rho_{AB}L_{AB}] + \]

\[ + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \text{Tr}[\rho_{AB}M_{AB}] + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \text{Tr}[\rho_{AB}N_{AB}], \] (A31)

where the operators \( K_{AB}, L_{AB}, M_{AB}, \) and \( N_{AB} \) obey the constraints

\[ K_{AB}, L_{AB}, M_{AB}, N_{AB} \geq 0, \] (A32)

\[ K_{\hat{A}B} + L_{\hat{A}B} + M_{\hat{A}B} + N_{\hat{A}B} = I_{\hat{A}B}, \] (A33)

such that

\[ \Gamma_{AB'} \otimes \Gamma_{BA'} \otimes T_{\hat{A}B}(K_{\hat{A}B}) + \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes T_{\hat{A}B}(L_{\hat{A}B}) + \]

\[ + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \otimes T_{\hat{A}B}(M_{\hat{A}B}) + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes T_{\hat{A}B}(N_{\hat{A}B}) \] (A34)

is the Choi operator of an LOCC channel. Thus, the set \( \{K_{\hat{A}B}, L_{\hat{A}B}, M_{\hat{A}B}, N_{\hat{A}B}\} \) constitutes a POVM. Given that \( \Gamma_{AB'} \) is the Choi operator of an identity channel from \( A \) to \( B', \) \( \Gamma_{BA'} \) is the Choi operator of an identity channel from \( B \) to \( A', \) and \( \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \) is the Choi operator of the generalized Pauli channel in (106), the interpretation in Remark 5 follows. The last observation about \( \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \) follows because

\[ \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} = \frac{d}{d^2 - 1} \left( I_{BA'} - \frac{1}{d} \Gamma_{BA'} \right) \] (A35)

\[ = \frac{d}{d^2 - 1} (I_{BA'} - \Phi_{BA'}) \] (A36)

\[ = \frac{d}{d^2 - 1} \sum_{(z,z) \neq (0,0)} W_{A'}^{z,x} \Phi_{BA'}(W_{A'}^{z,x})^\dagger \] (A37)
Applying this to (A34), we conclude that it has the form

$$\mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} = \frac{1}{2} \left( \mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} + S^d_{AB} \circ \mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} \circ S^d_{AB} \right).$$  \hfill (A39)

Equivalently, its Choi operator $\mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB}$ should satisfy

$$\tilde{L}_{A\bar{B}\bar{A}\rightarrow AB} = \frac{1}{2} \left( \tilde{L}_{A\bar{B}\bar{A}\rightarrow AB} + (S^d_{AB} \otimes S^d_{A'B'}) \left( \tilde{L}_{A\bar{B}\bar{A}\rightarrow AB} \right) \right).$$  \hfill (A40)

Applying this to (A34), we conclude that it has the form

$$\Gamma_{AB'} \otimes \Gamma_{BA'} \otimes T_{\bar{A} \bar{B}}(K_{\bar{A} \bar{B}}) + \frac{1}{2} \left( \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \right) \otimes T_{\bar{A} \bar{B}}(L_{\bar{A} \bar{B}} + M_{\bar{A} \bar{B}}) + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes T_{\bar{A} \bar{B}}(N_{\bar{A} \bar{B}}).$$  \hfill (A41)

After defining $L_{\bar{A} \bar{B}} + M_{\bar{A} \bar{B}}$ as $L_{\bar{A} \bar{B}}$, we obtain the form stated in Proposition 1.

Note that the swap itself cannot be implemented by LOCC. However, once we have the reduction of an optimal LOCC to the form in (A34), this corresponds to a channel of the following form

$$S^d_{AB}(\text{Tr}_{\bar{A}}[K_{\bar{A} \bar{B}} \omega_{A\bar{B} \bar{A}}]) + (\text{id}_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[L_{\bar{A} \bar{B}} \omega_{A\bar{B} \bar{A}}] + (D_{A \rightarrow B} \otimes \text{id}_{B \rightarrow A}) \text{Tr}_{\bar{A}}[M_{A\bar{B}} \omega_{A\bar{B} \bar{A}}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[N_{A\bar{B}} \omega_{A\bar{B} \bar{A}}].$$  \hfill (A42)

Thus, the variant of the channel corresponding to $S^d_{AB} \circ \mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} \circ S^d_{AB}$ is as follows:

$$S^d_{AB}(\text{Tr}_{\bar{A}}[K_{\bar{A} \bar{B}} \omega_{A\bar{B} \bar{A}}]) + (D_{A \rightarrow B} \otimes \text{id}_{B \rightarrow A}) \text{Tr}_{\bar{A}}[L_{\bar{A} \bar{B}} \omega_{A\bar{B} \bar{A}}] + (\text{id}_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[M_{A\bar{B}} \omega_{A\bar{B} \bar{A}}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[N_{A\bar{B}} \omega_{A\bar{B} \bar{A}}],$$  \hfill (A43)

which is still LOCC if the original channel is, because it is related to the original merely by Alice and Bob flipping their local actions in the case of the $L_{\bar{A} \bar{B}}$ and $M_{\bar{A} \bar{B}}$ measurement outcomes. When we randomly apply either of these channels, the random mixture is LOCC and corresponds to

$$S^d_{AB}(\text{Tr}_{\bar{A}}[K_{\bar{A} \bar{B}} \omega_{A\bar{B} \bar{A}}]) + \frac{1}{2} (D_{A \rightarrow B} \otimes \text{id}_{B \rightarrow A} + \text{id}_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[(L_{\bar{A} \bar{B}} + M_{\bar{A} \bar{B}}) \omega_{A\bar{B} \bar{A}}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A}) \text{Tr}_{\bar{A}}[N_{A\bar{B}} \omega_{A\bar{B} \bar{A}}]),$$  \hfill (A44)

which allows us to lump together the measurement outcomes for $L_{\bar{A} \bar{B}}$ and $M_{\bar{A} \bar{B}}$ into a single measurement outcome.

2. Evaluating normalized diamond distance

We have now reduced the optimization problem in (56), for the swap channel, to the following one:

$$\epsilon_{\text{LOCC}}(S^d_{AB}, \rho_{\bar{A} \bar{B}}) = \inf_{\mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} \in \text{LOCC}} \frac{1}{2} \left\| \mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB} \circ A^\rho_{\bar{A} \bar{B}} - S^d_{AB} \right\|_1,$$  \hfill (A45)

subject to

$$\mathcal{L}_{A\bar{B}\bar{A}\rightarrow AB}(\omega_{AB} \otimes \rho_{\bar{A} \bar{B}}) = S^d_{AB}(\omega_{AB}) \text{Tr}[K_{\bar{A} \bar{B}} \rho_{\bar{A} \bar{B}}] + \frac{1}{2} (\text{id}_{A \rightarrow B} \otimes D_{B \rightarrow A} + D_{A \rightarrow B} \otimes \text{id}_{B \rightarrow A}) (\omega_{AB}) \text{Tr}[L_{\bar{A} \bar{B}} \rho_{\bar{A} \bar{B}}] + (D_{A \rightarrow B} \otimes D_{B \rightarrow A}) (\omega_{AB}) \text{Tr}[N_{\bar{A} \bar{B}} \rho_{\bar{A} \bar{B}}].$$  \hfill (A46)
We then conclude that because $\mu$ for which the smallest $\mu$ subject to the channel $\tilde{L}_{A\hat{A}B\rightarrow AB}$ is satisfied. This is the positive part of the operator on the right-hand side of the inequality. Since the operator on the right-hand side has the following Jordan–Hahn decomposition (A51) is satisfied. This is the positive part of the operator on the right-hand side of the inequality. Since the operator on the right-hand side has the following Jordan–Hahn decomposition

$$K_{A\hat{B}}, L_{A\hat{B}}, N_{A\hat{B}} \geq 0,$$

(A47)

$$K_{A\hat{B}} + L_{A\hat{B}} + N_{A\hat{B}} = I_{A\hat{B}}.$$  

(A48)

Keep in mind that the operators $K_{A\hat{B}}, L_{A\hat{B}},$ and $N_{A\hat{B}}$ are further constrained so that $\tilde{L}_{A\hat{A}B\rightarrow AB} \in \text{LOCC}$, as indicated in (A45). We can exploit the form of the optimization of the diamond distance from (87) to rewrite the optimization for the simulation error of a unitary swap channel as follows:

$$\inf_{\mu, Z_{A\hat{A}B'}, K_{A\hat{B}}, L_{A\hat{B}}, N_{A\hat{B}} \geq 0} \mu,$$

(A49)

subject to

$$\mu I_{AB} \geq Z_{AB},$$

(A50)

$$Z_{A\hat{A}B'} \geq \Gamma_{A\hat{B}} \otimes \Gamma_{B\hat{B}} (1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}])$$

$$- \frac{1}{2} \left( \begin{array}{c} d_{BA'} \frac{d_{A\hat{B}} - d_{BA'}}{d^2 - 1} + d_{I_{AB}} - \Gamma_{BA'} \otimes \Gamma_{B\hat{B}} \end{array} \right) \text{Tr}[\rho_{A\hat{B}} L_{A\hat{B}}]$$

$$- \frac{d_{I_{AB}} - \Gamma_{A\hat{B}}}{d^2 - 1} \otimes \frac{d_{I_{BA'}} - \Gamma_{B\hat{B}}}{d^2 - 1} \text{Tr}[\rho_{A\hat{B}} N_{A\hat{B}}],$$

(A51)

subject to the channel $\tilde{L}_{A\hat{A}B\rightarrow AB}$ in (A46) being LOCC. Finally, since we are trying to minimize with respect to $\mu$ and $Z_{A\hat{A}B'}$, we can choose $Z_{A\hat{A}B'}$ to be the smallest positive semi-definite operator such that the inequality in (A51) is satisfied. This is the positive part of the operator on the right-hand side of the inequality. Since the operator on the right-hand side has the following Jordan–Hahn decomposition

$$\Gamma_{A\hat{B}} \otimes \Gamma_{B\hat{B}} (1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}]) - \left[ \frac{1}{2} \left( \begin{array}{c} d_{BA'} \frac{d_{A\hat{B}} - d_{BA'}}{d^2 - 1} + d_{I_{AB}} - \Gamma_{BA'} \otimes \Gamma_{B\hat{B}} \end{array} \right) \text{Tr}[\rho_{A\hat{B}} L_{A\hat{B}}] \right],$$

(A53)

it follows that its positive part is given by

$$(1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}]) \Gamma_{A\hat{B}} \otimes \Gamma_{B\hat{B}}.$$  

(A54)

Thus, an optimal solution is given by

$$Z_{A\hat{A}B'} = \Gamma_{A\hat{B}} \otimes \Gamma_{B\hat{B}} (1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}]),$$  

(A55)

for which the smallest $\mu$ possible is

$$\mu = 1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}],$$  

(A56)

because

$$Z_{AB} = \text{Tr}_{A'B'}[Z_{A\hat{A}B'B}] = I_{AB} (1 - \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}]).$$  

(A57)

We then conclude that

$$\epsilon_{\text{LOCC}}(S_{AB}^d, \rho_{A\hat{B}}) = 1 - \sup_{K_{A\hat{B}}, L_{A\hat{B}}, N_{A\hat{B}} \geq 0} \text{Tr}[\rho_{A\hat{B}} K_{A\hat{B}}],$$  

(A58)

subject to

$$K_{A\hat{B}} + L_{A\hat{B}} + N_{A\hat{B}} = I_{A\hat{B}}$$  

(A59)

and the following channel is LOCC:

$$\tilde{L}_{A\hat{A}B\rightarrow AB}(\omega_{A\hat{A}B\hat{B}}) = S_{AB}^d(\text{Tr}_{A\hat{B}}[K_{A\hat{B}} \omega_{A\hat{A}B\hat{B}}]) + \frac{1}{2} \left( \text{id}_{A\rightarrow B} \otimes D_{B\rightarrow A} + D_{A\rightarrow B} \otimes \text{id}_{B\rightarrow A} \right) \left( \text{Tr}_{A\hat{B}}[L_{A\hat{B}}^\tau A\hat{B}] \right)$$

$$+ (D_{A\rightarrow B} \otimes D_{B\rightarrow A}) \left( \text{Tr}_{A\hat{B}}[N_{A\hat{B}} \omega_{A\hat{A}B\hat{B}}] \right).$$  

(A60)
3. Evaluating channel infidelity

Let us start from (A3) and recall the symmetries of the unitary swap channel. This implies the following symmetry for the Choi operator in (A1)–(A2) for the swap channel:

\[
\Gamma_{ABA'B'}^{\mathcal{G}} = \Gamma_{AB'} \otimes \Gamma_{BA'}
\]

(A61)

for all unitary channels \( \mathcal{U} \) and \( \mathcal{V} \). This implies that

\[
\Gamma_{ABA'B'}^{\mathcal{G}} = \int \int d\mathcal{U} \ d\mathcal{V} \ (\mathcal{U}_A \otimes \mathcal{V}_B \otimes \mathcal{V}_A' \otimes \mathcal{U}_B') (\Gamma_{ABA'B'}^{\mathcal{G}}).
\]

(A63)

By exploiting the semi-definite program in (64)–(66) for channel infidelity, we find that

\[
\epsilon_{\text{LOCC}}(S_{ABA'B'}^{\mathcal{G}}, \rho_{\hat{A}B}) = 1 - \left[ \sup_{\lambda \geq 0, L_{ABAB'))^{\mathcal{G}} \geq 0, Q_{ABA'B'}} \lambda \right]^2,
\]

(A64)

subject to

\[
\lambda I_{AB} \leq \text{Re}[\text{Tr}_{A'B'}(Q_{ABA'B'})]
\]

(A65)

\[
\text{Tr}_{A'B'}[L_{ABAB'})^{\mathcal{G}}] = I_{ABAB'}
\]

(A66)

\[
\left[ \Gamma_{ABA'B'}^{\mathcal{G}}, Q_{ABA'B'}^{\mathcal{G}}, \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})L_{ABAB'})^{\mathcal{G}}] \right] \geq 0,
\]

(A67)

and \( L_{ABAB')}^{\mathcal{G}} \) is the Choi operator for an LOCC channel. Note that we can write the last constraint as

\[
|0\rangle\langle 0| \otimes \Gamma_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 1| \otimes Q_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 0| \otimes Q_{ABA'B'}^{\mathcal{G}} + |0\rangle\langle 1| \otimes \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})P_{ABA'B'}] \geq 0.
\]

(A68)

Suppose that \( \lambda, L_{ABAB')}^{\mathcal{G}} \), and \( Q_{ABA'B'}^{\mathcal{G}} \) is an optimal solution. Let

\[
\mathcal{W}_{ABA'B'} := \mathcal{U}_A \otimes \mathcal{V}_B \otimes \mathcal{V}_A' \otimes \mathcal{U}_B'.
\]

(A69)

Then it follows that \( \lambda, \mathcal{W}_{ABA'B'}(L_{ABAB')}^{\mathcal{G}} \), and \( Q_{ABA'B'}^{\mathcal{G}}(L_{ABAB')}^{\mathcal{G}} \) is an optimal solution. This is because all of the constraints are satisfied for these choices while still achieving the same optimal value. Indeed, consider that

\[
\lambda I_{AB} \leq \text{Re}[\text{Tr}_{A'B'}(Q_{ABA'B'})]
\]

(A70)

\[
\lambda(\mathcal{U}_A \otimes \mathcal{V}_B)(I_{AB}) \leq (\mathcal{U}_A \otimes \mathcal{V}_B)(\text{Re}[\text{Tr}_{A'B'}(Q_{ABA'B'})])
\]

(A71)

\[
\lambda I_{AB} \leq \text{Re}[\text{Tr}_{A'B'}(\mathcal{W}_{ABA'B'}(Q_{ABA'B'}))]
\]

(A72)

\[
\lambda I_{AB} \leq \text{Re}[\text{Tr}_{A'B'}(\mathcal{W}_{ABA'B'}(Q_{ABA'B'}))]
\]

(A73)

\[
\text{Tr}_{A'B'}[L_{ABAB')}^{\mathcal{G}}] = I_{ABAB'} \iff \text{Tr}_{A'B'}[\mathcal{W}_{ABA'B'}(L_{ABAB')}^{\mathcal{G}}] = I_{ABAB'}
\]

(A74)

and that

\[
|0\rangle\langle 0| \otimes \Gamma_{ABA'B'}^{\mathcal{G}} + |0\rangle\langle 1| \otimes Q_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 0| \otimes Q_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 1| \otimes \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})L_{ABAB')}^{\mathcal{G}}] \geq 0
\]

(A75)

\[
|0\rangle\langle 0| \otimes \Gamma_{ABA'B'}^{\mathcal{G}} + |0\rangle\langle 1| \otimes Q_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 0| \otimes Q_{ABA'B'}^{\mathcal{G}} + |1\rangle\langle 1| \otimes \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})L_{ABAB')}^{\mathcal{G}}] \geq 0
\]

(A76)

\[
|0\rangle\langle 0| \otimes \mathcal{W}(\Gamma_{ABA'B'}^{\mathcal{G}}) + |0\rangle\langle 1| \otimes \mathcal{W}(Q_{ABA'B'}^{\mathcal{G}}) + |1\rangle\langle 0| \otimes \mathcal{W}(Q_{ABA'B'}^{\mathcal{G}}) + |1\rangle\langle 1| \otimes \text{Tr}_{AB}[T_{AB}(\rho_{\hat{A}B})L_{ABAB')}^{\mathcal{G}}] \geq 0
\]

(A77)

Also, \( \mathcal{W}_{ABA'B'}(L_{ABAB')}^{\mathcal{G}} \) is the Choi operator for an LOCC channel if \( L_{ABAB')}^{\mathcal{G}} \) is. Furthermore, due to the fact that the objective function is linear and the constraints are linear operator inequalities, it follows that convex
combinations of solutions are solutions as well. So this implies that if \( \lambda, L_{AB\tilde{A}B'} \), and \( Q_{ABA'B'} \) is an optimal solution, then so is \( \lambda \),

\[
\tilde{L}_{AB\tilde{A}B'B'} := \int \int dU \ dV \ (\mathcal{U}_A \otimes \mathcal{V}_B \otimes \mathcal{V}_A' \otimes \mathcal{U}_B')(L_{AB\tilde{A}B'B'}),
\]

(A79)

\[
\tilde{Q}_{ABA'B'} := \int \int dU \ dV \ (\mathcal{U}_A \otimes \mathcal{V}_B \otimes \mathcal{V}_A' \otimes \mathcal{U}_B')Q_{ABA'B'}. \tag{A80}
\]

Furthermore, \( \tilde{L}_{AB\tilde{A}B'B'} \) is the Choi operator for an LOCC channel if \( L_{AB\tilde{A}B'} \) is. As argued in Appendix C, \( \tilde{L}_{AB\tilde{A}B'B'} \) has a simpler form as follows:

\[
\tilde{L}_{AB\tilde{A}B'B'} = \Gamma_{AB'} \otimes \Gamma_{BA'} \otimes K_{\tilde{A}B} + \Gamma_{AB'} \otimes \frac{dI_{AB'} - \Gamma_{BA'}}{d^2 - 1} \otimes L_{\tilde{A}B} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \otimes M_{\tilde{A}B} + \frac{dI_{AB'} - \Gamma_{BA'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes N_{\tilde{A}B}, \tag{A81}
\]

and the constraints in (A64)–(A66) on \( \tilde{L}_{AB\tilde{A}B'B'} \) simplify to

\[
K_{\tilde{A}B}, L_{\tilde{A}B}, M_{\tilde{A}B}, N_{\tilde{A}B} \geq 0, \tag{A82}
\]

\[
K_{\tilde{A}B} + L_{\tilde{A}B} + M_{\tilde{A}B} + N_{\tilde{A}B} = I_{\tilde{A}B}. \tag{A83}
\]

We then find that \( \tilde{Q}_{ABA'B'} \) simplifies to

\[
\tilde{Q}_{ABA'B'} = q_1 \Gamma_{AB'} \otimes \Gamma_{BA'} + q_2 \Gamma_{ABA'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} + q_3 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} + q_4 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \tag{A84}
\]

where \( q_1, q_2, q_3, q_4 \in \mathbb{C} \). The constraint in (A65) reduces to the following:

\[
\lambda \leq \text{Re}[q_1 + q_2 + q_3 + q_4], \tag{A85}
\]

because

\[
\text{Tr}_{A'B'}[\tilde{Q}_{ABA'B'}] = (q_1 + q_2 + q_3 + q_4)I_{AB}. \tag{A86}
\]

Furthermore, the constraint in (A68) then reduces to the following:

\[
|0\rangle\langle 0| \otimes \Gamma_{AB'} \otimes \Gamma_{BA'} + |0\rangle\langle 1| \otimes \left[ \begin{array}{c}
q_1 \Gamma_{AB'} \otimes \Gamma_{BA'} + q_2 \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} + q_3 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} + q_4 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \\
+ q_1 \Gamma_{AB'} \otimes \Gamma_{BA'} + q_2 \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} + q_3 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} + q_4 \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1}
\end{array} \right]
\]

\[
+ |1\rangle\langle 0| \otimes \left[ \begin{array}{c}
\frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \Gamma_{BA'}
\end{array} \right] + |1\rangle\langle 1| \otimes \left[ \begin{array}{c}
\frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \Gamma_{BA'}
\end{array} \right] \geq 0. \tag{A87}
\]

Now exploiting the orthogonality of the operators \( \Gamma_{AB'} \otimes \Gamma_{BA'}, \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1}, \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'}, \) and \( \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \), we conclude that the single constraint above is equivalent to the following four constraints:

\[
|0\rangle\langle 0| + q_1 |0\rangle\langle 1| + q_2 |1\rangle\langle 0| + \text{Tr}[T_{AB}(\rho_{AB})K_{\tilde{A}B}]|1\rangle\langle 1| \geq 0, \tag{A88}
\]

\[
q_2 |0\rangle\langle 1| + q_1 |1\rangle\langle 0| + \text{Tr}[T_{AB}(\rho_{AB})L_{\tilde{A}B}]|1\rangle\langle 1| \geq 0, \tag{A89}
\]

\[
q_1 |0\rangle\langle 0| + q_3 |0\rangle\langle 1| + \text{Tr}[T_{AB}(\rho_{AB})M_{\tilde{A}B}]|1\rangle\langle 1| \geq 0, \tag{A90}
\]

\[
q_4 |0\rangle\langle 0| + q_4 |1\rangle\langle 1| + \text{Tr}[T_{AB}(\rho_{AB})N_{\tilde{A}B}]|1\rangle\langle 1| \geq 0. \tag{A91}
\]
These constraints can in turn be expressed as the following four matrix inequalities:

\[
\begin{bmatrix}
1 & q_1^* \\
q_1 & \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})K_{\hat{A}\hat{B}}]
\end{bmatrix} \geq 0, \quad (A92)
\]

\[
\begin{bmatrix}
0 & q_2^* \\
q_2 & \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})L_{\hat{A}\hat{B}}]
\end{bmatrix} \geq 0, \quad (A93)
\]

\[
\begin{bmatrix}
0 & q_3^* \\
q_3 & \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})M_{\hat{A}\hat{B}}]
\end{bmatrix} \geq 0, \quad (A94)
\]

\[
\begin{bmatrix}
0 & q_4^* \\
q_4 & \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})N_{\hat{A}\hat{B}}]
\end{bmatrix} \geq 0. \quad (A95)
\]

Since \( \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})K_{\hat{A}\hat{B}}], \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})L_{\hat{A}\hat{B}}], \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})M_{\hat{A}\hat{B}}], \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})N_{\hat{A}\hat{B}}] \geq 0 \), the inequalities above hold if and only if

\[
\text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})K_{\hat{A}\hat{B}}] \geq |q_1|^2, \quad q_2 = q_3 = q_4 = 0. \quad (A96)
\]

Note that we can perform the following substitutions as we did previously: \( K_{\hat{A}\hat{B}} \rightarrow T_{\hat{A}\hat{B}}(K_{\hat{A}\hat{B}}), L_{\hat{A}\hat{B}} \rightarrow T_{\hat{A}\hat{B}}(L_{\hat{A}\hat{B}}), M_{\hat{A}\hat{B}} \rightarrow T_{\hat{A}\hat{B}}(M_{\hat{A}\hat{B}}), \) and \( N_{\hat{A}\hat{B}} \rightarrow T_{\hat{A}\hat{B}}(N_{\hat{A}\hat{B}}) \). The objective function does not change under these substitutions. Thus, the optimization problem in (A64) reduces to the following:

\[
1 - \sup_{\lambda \geq 0, K_{\hat{A}\hat{B}}, L_{\hat{A}\hat{B}}, M_{\hat{A}\hat{B}}, N_{\hat{A}\hat{B}} \geq 0, q_i \in \mathbb{C}} \left( \lambda^2 \right), \quad (A97)
\]

subject to

\[
\text{Tr}[\rho_{\hat{A}\hat{B}}K_{\hat{A}\hat{B}}] \geq |q_1|^2, \quad \lambda \leq \text{Re}[q_1], \quad (A98)
\]

\[
K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} + M_{\hat{A}\hat{B}} + N_{\hat{A}\hat{B}} = I_{\hat{A}\hat{B}}, \quad (A100)
\]

and the following channel is LOCC:

\[
\tilde{\mathcal{E}}_{ABAB\rightarrow AB}(\omega_{ABAB}) = \mathcal{S}_{AB}^d(\text{Tr}_{\hat{A}\hat{B}}[K_{\hat{A}\hat{B}}\omega_{ABAB}]) + (\text{id}_{A\rightarrow B} \otimes \mathcal{D}_{B\rightarrow A})(\text{Tr}_{\hat{A}\hat{B}}[L_{\hat{A}\hat{B}}\tau_{\hat{A}\hat{B}}])
+ (\mathcal{D}_{A\rightarrow B} \otimes \text{id}_{B\rightarrow A})(\text{Tr}_{\hat{A}\hat{B}}[M_{\hat{A}\hat{B}}\omega_{ABAB}]) + (\mathcal{D}_{A\rightarrow B} \otimes \mathcal{D}_{B\rightarrow A})(\text{Tr}_{\hat{A}\hat{B}}[N_{\hat{A}\hat{B}}\omega_{ABAB}]). \quad (A101)
\]

Since we are trying to maximize the value of \( \lambda \) subject to these constraints, it is clear that we should pick \( \lambda = q_1 = \sqrt{\text{Tr}[\rho_{\hat{A}\hat{B}}K_{\hat{A}\hat{B}}]} \). We can furthermore apply the other symmetry in (70) to get a similar reduced form for the optimization problem. This concludes the proof.

**Appendix B: Proof of Eq. (98)**

Let \( A \) and \( B \) be Hermitian operators, and let \( \Phi \) be a Hermiticity-preserving map. Recall from [Wat18] that if a primal semi-definite program (SDP) is given by

\[
\inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi(Y) \geq A \}, \quad (B1)
\]

then its dual is given by

\[
\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi^\dagger(X) \leq B \}, \quad (B2)
\]

where \( \Phi^\dagger \) is the Hilbert–Schmidt adjoint of \( \Phi \). For our semi-definite program of interest in (93), we find that (93) can be written in the standard form in (B1) with

\[
Y = \text{diag}(\mu, Z_{ABA'B'}, P_{ABA'B'}) , \quad (B3)
\]
\[ B = \text{diag}(1, 0, 0), \]  
\[ A = \text{diag}(0, \Gamma_{A'B'B'}, 0, I_{ABAB'}, -I_{ABAB'}), \]  
\[ \Phi(Y) = \text{diag}(\mu I_{AB} - Z_{AB}), \]  
\[ Z_{ABA'B'} + \text{Tr}_{\bar{A}B}[T_{\bar{A}B}(\rho_{\bar{A}B} P_{AB\bar{A}A'B'})], \]  
\[ T_{B\bar{B}B'}(P_{AB\bar{A}A'B'}), \]  
\[ \text{Tr}_{A'B'}[P_{AB\bar{A}A'B'}]). \]  

So we need to compute the Hilbert–Schmidt adjoint of \( \Phi \), which is defined for a Hermiticity-preserving map by the equation

\[ \text{Tr}[X \Phi(Y)] = \text{Tr}[\Phi^\dagger(X)Y], \]

holding for all Hermitian \( X \) and \( Y \). By defining

\[ X = \text{diag}(X^1_{AB}, X^2_{ABA'B'}, X^3_{AB\bar{A}A'B'}, X^4_{AB\bar{A}A'B'}, X^5_{AB\bar{A}A'B'}), \]

consider that

\[ \text{Tr}[X \Phi(Y)] = \text{Tr}[X^1_{\bar{A}B}(\mu I_{AB} - Z_{AB})] + \text{Tr}[X^2_{ABA'B'}(Z_{ABA'B'} + \text{Tr}_{\bar{A}B}[T_{\bar{A}B}(\rho_{\bar{A}B} P_{AB\bar{A}A'B'})]) + \text{Tr}[X^3_{AB\bar{A}A'B'} T_{B\bar{B}B'}(P_{AB\bar{A}A'B'})] + \text{Tr}[X^4_{AB\bar{A}A'B'} T_{A'B'}[P_{AB\bar{A}A'B'}]] - \text{Tr}[X^5_{AB\bar{A}A'B'} T_{A'B'}[P_{AB\bar{A}A'B'}]]. \]

So this means that

\[ \Phi^\dagger(X) = \text{diag}(\text{Tr}[X^1_{AB}], -X^1_{AB} \otimes I_{A'B'} + X^2_{ABA'B'}, \text{Tr}[X^3_{AB\bar{A}A'B'} T_{B\bar{B}B'}(X^3_{AB\bar{A}A'B'}) + (X^4_{AB\bar{A}A'B'} - X^5_{AB\bar{A}A'}) \otimes I_{A'B'}]. \]

Then the dual SDP is found by plugging into (B2):

\[ \sup_{X^1, X^2, X^3 \geq 0} \text{Tr}[\Gamma_{A'B'B'}^N X^2_{ABA'B'}] + \text{Tr}[X^4_{AB\bar{A}A'B'} - X^5_{AB\bar{A}A'B'}], \]

subject to

\[ \text{Tr}[X^1_{AB}] \leq 1, \]  
\[ X^2_{ABA'B'} \leq X^1_{AB} \otimes I_{A'B'}, \]  
\[ X^2_{ABA'B'} \otimes T_{\bar{A}B}(\rho_{\bar{A}B}) + T_{B\bar{B}B'}(X^3_{AB\bar{A}A'B'}) \leq -(X^4_{AB\bar{A}A'B'} - X^5_{AB\bar{A}A'B'}) \otimes I_{A'B'}. \]

This can then be rewritten as

\[ \sup_{X^1, X^2, X^3 \geq 0, W \in \text{Herm}} \text{Tr}[\Gamma_{A'B'B'}^N X^2_{ABA'B'}] - \text{Tr}[W_{AB\bar{A}A'B'}], \]

subject to

\[ \text{Tr}[X^1_{AB}] \leq 1, \]  
\[ X^2_{ABA'B'} \leq X^1_{AB} \otimes I_{A'B'}, \]  
\[ X^2_{ABA'B'} \otimes T_{\bar{A}B}(\rho_{\bar{A}B}) + T_{B\bar{B}B'}(X^3_{AB\bar{A}A'B'}) \leq W_{AB\bar{A}A'B'} \otimes I_{A'B'}. \]

concluding the proof.
Appendix C: Proof of Proposition 4

By following precisely the same reasoning given in Appendix A, we conclude that the desired optimization can be written as
\[
e_{\text{PPT}}(S_{AB}^d, \rho_{AB}) = 1 - \sup_{K_{AB}, L_{AB}, N_{AB} \geq 0} \text{Tr}[\rho_{AB} K_{AB}],
\] (C1)

subject to
\[
K_{AB} + L_{AB} + N_{AB} = I_{AB}
\] (C2)

and the following channel is C-PPT-P:
\[
\tilde{\mathcal{P}}_{ABAB\rightarrow AB}(\omega_{ABAB}) = S_{AB}^d(\text{Tr}_{AB}[K_{AB} \omega_{ABAB}]) + \frac{1}{2} (\text{id}_{A\rightarrow B} \otimes \mathcal{D}_{B\rightarrow A} + \mathcal{D}_{A\rightarrow B} \otimes \text{id}_{B\rightarrow A}) (\text{Tr}_{AB}[L_{AB} \tau_{AB}]) + (\mathcal{D}_{A\rightarrow B} \otimes \mathcal{D}_{B\rightarrow A}) (\text{Tr}_{AB}[N_{AB} \omega_{ABAB}]).
\] (C3)

Recall that the Choi operator of \(\tilde{\mathcal{P}}_{ABAB\rightarrow AB}\) is given by
\[
\tilde{\mathcal{P}}_{ABAB\rightarrow AB} = \Gamma_{AB'} \otimes \Gamma_{BA'} \otimes T_{AB}(K_{AB}) + \frac{1}{2} \left( \Gamma_{AB'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \Gamma_{BA'} \right) \otimes T_{AB}(L_{AB}) + \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes T_{AB}(N_{AB}).
\] (C4)

The conditions \(K_{AB}, L_{AB}, N_{AB} \geq 0\) and \(K_{AB} + L_{AB} + N_{AB} = I_{AB}\) guarantee that \(\tilde{\mathcal{P}}_{ABABAB'}\) is the Choi operator of a channel. In order for \(\tilde{\mathcal{P}}_{ABABAB'}\) to be the Choi operator of a C-PPT-P channel, the following condition should hold
\[
T_{BBB'}(\tilde{\mathcal{P}}_{ABABAB'}) \geq 0.
\] (C5)

In what follows, we explore what this condition imposes on the operators \(K_{AB}, L_{AB}, N_{AB}\). Consider that
\[
\begin{align*}
T_{BBB'}(\tilde{\mathcal{P}}_{ABABAB'}) & = T_B(\Gamma_{AB'}) \otimes T_B(\Gamma_{BA'}) \otimes T_A(K_{AB}) \\
& + \frac{1}{2} \left( T_B(\Gamma_{AB'}) \otimes T_B \left( \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \right) + T_B \left( \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \right) \otimes T_B(\Gamma_{BA'}) \right) \otimes T_A(L_{AB}) \\
& + T_B \left( \frac{dI_{AB'} - \Gamma_{AB'}}{d^2 - 1} \right) \otimes T_B \left( \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \right) \otimes T_A(N_{AB}).
\end{align*}
\] (C6)

\[
= F_{AB'} \otimes F_{BA'} \otimes T_A(K_{AB}) \\
+ \frac{1}{2} \left( F_{AB'} \otimes \frac{dI_{BA'} - F_{BA'}}{d^2 - 1} + \frac{dI_{AB'} - F_{AB'}}{d^2 - 1} \otimes F_{BA'} \right) \otimes T_A(L_{AB}) \\
+ \frac{dI_{AB'} - F_{AB'}}{d^2 - 1} \otimes \frac{dI_{BA'} - F_{BA'}}{d^2 - 1} \otimes T_A(N_{AB}).
\] (C7)

Now consider that the SWAP operator \(F_{CD}\) and the identity \(I_{CD}\) can be written in terms of the projections \(\Pi_{CD}^S\) and \(\Pi_{CD}^A\) onto the respective symmetric and antisymmetric subspaces as
\[
F_{CD} = \Pi_{CD}^S - \Pi_{CD}^A,
\] (C8)
\[
I_{CD} = \Pi_{CD}^S + \Pi_{CD}^A,
\] (C9)

which implies that
\[
\begin{align*}
\frac{dI_{CD} - F_{CD}}{d^2 - 1} & = \frac{1}{d^2 - 1} \left( d\Pi_{CD}^S + d\Pi_{CD}^A - (\Pi_{CD}^S - \Pi_{CD}^A) \right) \\
& = \frac{d - 1}{d^2 - 1} \Pi_{CD}^S + \frac{d + 1}{d^2 - 1} \Pi_{CD}^A.
\end{align*}
\] (C10)
Continuing, we find that

\[
\text{Eq. (C7)} = (\Pi^S_{AB'} - \Pi^A_{AB'}) \otimes (\Pi^S_{BA'} - \Pi^A_{BA'}) \otimes T_A(K_{AB})
\]

\[
= \frac{1}{d+1} \Pi^S_{CD} + \frac{1}{d-1} \Pi^A_{CD}.
\]  

(C12)

Since the operators in the first two factors of the tensor product are orthogonal projectors, we then see that the condition in (C5) is equivalent to the following four conditions:

\[
T_B(K_{AB}) + \frac{T_B(L_{AB})}{d+1} + \frac{T_B(N_{AB})}{(d+1)^2} \geq 0,
\]  

(C15)

\[
- T_B(K_{AB}) + \frac{T_B(L_{AB})}{2(d-1)} - \frac{T_B(L_{AB})}{2(d+1)} + \frac{T_B(N_{AB})}{(d+1)(d-1)} \geq 0,
\]  

(C16)

\[
- T_B(K_{AB}) - \frac{T_B(L_{AB})}{2(d-1)} + \frac{T_B(L_{AB})}{2(d+1)} + \frac{T_B(N_{AB})}{(d+1)(d-1)} \geq 0,
\]  

(C17)

\[
T_B(K_{AB}) + \frac{T_B(L_{AB})}{d-1} + \frac{T_B(N_{AB})}{(d-1)^2} \geq 0.
\]  

(C18)

It is clear that the third one is redundant. These in turn are equivalent to the following three conditions:

\[
T_A(K_{AB}) + \frac{T_A(L_{AB})}{d+1} + \frac{T_A(N_{AB})}{(d+1)^2} \geq 0,
\]  

(C19)

\[
\frac{1}{d-1} \left( \frac{T_A(L_{AB})}{2} + \frac{T_A(N_{AB})}{d+1} \right) \geq T_A(K_{AB}) + \frac{T_A(L_{AB})}{2(d+1)},
\]  

(C20)

\[
T_A(K_{AB}) + \frac{T_A(N_{AB})}{(d-1)^2} \geq \frac{1}{d-1} \left[ T_A(L_{AB}) \right].
\]  

(C21)

Using the linearity of the transpose operation, we can rewrite these conditions a final time as

\[
T_A \left( K_{AB} + \frac{L_{AB}}{d+1} + \frac{N_{AB}}{(d+1)^2} \right) \geq 0,
\]  

(C22)

\[
\frac{1}{d-1} T_A \left( \frac{L_{AB}}{2} + \frac{N_{AB}}{d+1} \right) \geq T_A \left( K_{AB} + \frac{L_{AB}}{2(d+1)} \right),
\]  

(C23)

\[
T_A \left( K_{AB} + \frac{N_{AB}}{(d-1)^2} \right) \geq \frac{1}{d-1} T_A (L_{AB}).
\]  

(C24)

Since \( T_A(G_{AB}) \geq 0 \) if and only \( T_B(G_{AB}) \geq 0 \), we can equivalently write the partial transpose constraints with respect to \( T_B \). Some simple algebra reduces the inequality in (C23) to

\[
\frac{1}{d^2-1} T_A (L_{AB} + N_{AB}) \geq T_A (K_{AB}).
\]  

(C25)
By combining with (C1)–(C3), we conclude the proof of Proposition 4.

We note here that, if we do not employ the symmetry in (70), then we arrive at the following SDP:

\[ e_{\text{PPT}}(S_{AB}^A, \rho_{AB}) = 1 - \sup_{K_{AB}, L_{AB}, M_{AB} : N_{AB} \geq 0} \text{Tr}[\rho_{AB} K_{AB}], \tag{C26} \]

subject to

\[ T_B \left( K_{AB} + \frac{L_{AB}}{d+1} + \frac{M_{AB}}{(d+1)^2} + \frac{N_{AB}}{(d+1)^2} \right) \geq 0, \tag{C27} \]

\[ \frac{1}{d-1} T_B \left( L_{AB} + \frac{N_{AB}}{d+1} \right) \geq T_B \left( K_{AB} + \frac{M_{AB}}{d+1} \right), \tag{C28} \]

\[ \frac{1}{d-1} T_B \left( M_{AB} + \frac{N_{AB}}{d+1} \right) \geq T_B \left( K_{AB} + \frac{L_{AB}}{d+1} \right), \tag{C29} \]

\[ T_B \left( K_{AB} + \frac{N_{AB}}{(d-1)^2} \right) \geq \frac{1}{d-1} T_B \left( L_{AB} + M_{AB} \right), \tag{C30} \]

\[ K_{AB} + L_{AB} + M_{AB} + N_{AB} = I_{AB}, \tag{C31} \]

where \( K_{AB}, L_{AB}, M_{AB}, \) and \( N_{AB} \) are positive semi-definite Hermitian matrices and elements of a POVM and \( d \) is the dimension of the SWAP channel. This SDP was derived in an earlier version of our paper, available at [SW20], before we noticed the extra symmetry in (70). We have used this latter SDP to evaluate some examples in our paper. The main difference between the SDP in (103) and that in (C26) is that, in the latter, by exploiting the symmetry in (70), we can set \( M_{AB} = L_{AB} \) and eliminate the constraint in (C29). Then we set \( L_{AB} := 2L_{AB} \).

**Appendix D: Channel box transformation using infidelity**

Although the topic of this appendix is tangential to the main theme of this paper, we think it is nevertheless important to go through a different example in which the channel infidelity as a measure of error leads to a semi-definite program. We suspect that this observation will have wide application in the context of quantum resource theories [CG19].

Recall from [WW19b] that the channel box transformation problem refers to the task of converting a pair \((N_{A \rightarrow B}, M_{A \rightarrow B})\) of channels to another pair \((K_{C \rightarrow D}, L_{C \rightarrow D})\) by means of a superchannel \( \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)} \). In particular, the goal is to minimize the error \( \varepsilon \) in the following:

\[ \Theta(N_{A \rightarrow B}) \approx_\varepsilon K_{C \rightarrow D}, \tag{D1} \]

\[ \Theta(M_{A \rightarrow B}) = L_{C \rightarrow D}. \tag{D2} \]

See [WW19b] for a full exposition of this problem. The problem can be phrased in a more mathematical way as follows:

\[ \varepsilon((N, \mathcal{M}) \rightarrow (K, \mathcal{L})) := \inf_{\Theta \in \text{SC}} \{ \varepsilon \in [0,1] : \Theta(N_{A \rightarrow B}) \approx_\varepsilon K_{C \rightarrow D}, \Theta(M_{A \rightarrow B}) = L_{C \rightarrow D} \}, \tag{D3} \]

where SC is the set of superchannels.

In [WW19b], the approximation error in (D1) was taken to be with respect to normalized diamond distance. In addition to various operational motivations for doing so, an additional implicit motivation was that doing so led to a semi-definite program.

What we show here is that if we take the approximation error in (D1) to be with respect to channel infidelity, so that

\[ \Theta(N_{A \rightarrow B}) \approx_\varepsilon K_{C \rightarrow D} \iff 1 - F(\Theta(N_{A \rightarrow B}), K_{C \rightarrow D}) \leq \varepsilon, \tag{D4} \]

then we still arrive at a semi-definite program to perform the optimization. The semi-definite program is as follows:

\[ \varepsilon((N, \mathcal{M}) \rightarrow (K, \mathcal{L})) = 1 - \sup_{\lambda \geq 0, \, \mathcal{E}_{BAD} \geq 0, \, Q_{CD}} \lambda^2 \tag{D5} \]
subject to
\[
\begin{align*}
\Gamma_{CB}^\Theta &= I_{CB}, & (D6) \\
\Gamma_{CBA}^\Theta &= \Gamma_{CA}^\Theta \otimes \frac{I_B}{d_B}, & (D7) \\
\Gamma_{CD}^\epsilon &= \text{Tr}_{AB}[T_{AB}(\rho^M_{AB})\Gamma_{CBA}^{\Theta}], & (D8) \\
\lambda I_C &\leq \text{Re}[\text{Tr}_D[Q_{CD}]], & (D9) \\
\left[\begin{array}{c}
\Gamma_{CD}^\epsilon \\
Q_{CD} \\
\text{Tr}_{AB}[T_{AB}(\rho^M_{AB})\Gamma_{CBA}^{\Theta}]
\end{array}\right] &\geq 0. & (D10)
\end{align*}
\]

The constraints $\Gamma_{CBAD}^\Theta \geq 0$, $\Gamma_{CB}^\Theta = I_{CB}$, and $\Gamma_{CBA}^\Theta = \Gamma_{CA}^\Theta \otimes \frac{I_B}{d_B}$ correspond to $\Gamma_{CBAD}^\Theta$ being a Choi operator for a superchannel. The constraint $\Gamma_{CD}^\epsilon = \text{Tr}_{AB}[T_{AB}(\rho^M_{AB})\Gamma_{CBA}^{\Theta}]$ ensures that the superchannel $\Theta$ transforms $\mathcal{M}_{A\rightarrow B}$ exactly to $\mathcal{L}_{C\rightarrow D}$. The other constraints ensure that the superchannel $\Theta$ transforms $\mathcal{N}_{A\rightarrow B}$ approximately to $\mathcal{K}_{C\rightarrow D}$ with channel infidelity as the error measure, making use of the semi-definite program for root channel fidelity from (64–66) (see also [KW20a]).

**Appendix E: Proof of Proposition 10**

In this appendix, we establish how the semi-definite program in (C26) simplifies for isotropic states. Recall from (150) that an isotropic state has the following form:

\[
\rho^{(F,d_A)}_{AB} := F\Phi_{AB} + (1 - F)\frac{I_{AB} - \Phi_{AB}}{d_A^2 - 1}.
\]

We prove that

\[
\epsilon_{\text{PPT}}(S^d_{AB}, \rho^{(F,d_A)}_{AB}) = \begin{cases} 
1 - \frac{1}{d^2} & \text{if } F \leq \frac{1}{d_A} \\
1 - \frac{dA}{d^2} & \text{if } F > \frac{1}{d_A} \text{ and } d_{A} \leq d^2 \\
\frac{(1 - \frac{1}{d_A})(1 - F)}{1 - \frac{1}{d^2}} & \text{if } F > \frac{1}{d_A} \text{ and } d_{A} > d^2
\end{cases},
\]

as stated in Proposition 10.

To begin with, if $F \leq 1/d_A$, then the resource state is separable [HHHH09, Wat18]. So the resource state can be prepared by LOCC, and then our previous result from Proposition 9 applies, so that we can conclude that the simulation error is

\[
1 - \frac{1}{d^2}
\]

in this case. So in what follows, we focus exclusively on the case when $F > 1/d_A$.

Due to the objective function in (C26) being linear and the constraints being linear inequalities, it follows that a convex combination of optimal solutions is also optimal. Since the resource state is isotropic, then this means that if $K_{AB}$, $L_{AB}$, $M_{AB}$, and $N_{AB}$ is an optimal solution, then $(U_A \otimes \overline{U}_B)(K_{AB})$, $(U_A \otimes \overline{U}_B)(L_{AB})$, $(U_A \otimes \overline{U}_B)(M_{AB})$, and $(U_A \otimes \overline{U}_B)(N_{AB})$ is too, where $U_A$ and $U_B$ unitary channels corresponding to an arbitrary unitary operator $U$. Following the reasoning above, it follows that the twirled versions of these operators is also optimal, so that it suffices for each of $K_{AB}$, $L_{AB}$, $M_{AB}$, and $N_{AB}$ to have an isotropic form:

\[
\begin{align*}
K_{AB} &= k_1\Phi_{AB} + k_2(I_{AB} - \Phi_{AB}), & (E4) \\
L_{AB} &= l_1\Phi_{AB} + l_2(I_{AB} - \Phi_{AB}), & (E5) \\
M_{AB} &= m_1\Phi_{AB} + m_2(I_{AB} - \Phi_{AB}), & (E6) \\
N_{AB} &= n_1\Phi_{AB} + n_2(I_{AB} - \Phi_{AB}). & (E7)
\end{align*}
\]

The objective function then evaluates to

\[
\text{Tr}[K_{AB}\rho^{(F,d_A)}_{AB}] = k_1F + k_2(1 - F).
\]

(E8)
Consider that the equality constraint

\[ K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} + M_{\hat{A}\hat{B}} + N_{\hat{A}\hat{B}} = I_{\hat{A}\hat{B}} \]  

implies that

\[ k_1 + l_1 + m_1 + n_1 = 1, \]  

\[ k_2 + l_2 + m_2 + n_2 = 1. \]

We also have that all coefficients are non-negative:

\[ k_1, l_1, m_1, n_1, k_2, l_2, m_2, n_2 \geq 0, \]

which follows from the constraints \( K_{\hat{A}\hat{B}}, L_{\hat{A}\hat{B}}, M_{\hat{A}\hat{B}}, N_{\hat{A}\hat{B}} \geq 0. \) All of the inequality constraints involve the terms \( T_B(K_{\hat{A}\hat{B}}), T_B(L_{\hat{A}\hat{B}}), T_B(M_{\hat{A}\hat{B}}), \) and \( T_B(N_{\hat{A}\hat{B}}). \) Let us evaluate the first of these and the rest follow similarly:

\[ T_B(K_{\hat{A}\hat{B}}) = \frac{k_1}{d_A} F_{\hat{A}\hat{B}} + k_2 (I_{\hat{A}\hat{B}} - F_{\hat{A}\hat{B}}/d_A) \]  

\[ = \frac{k_1}{d_A} (\Pi_{\hat{A}\hat{B}}^S - \Pi_{\hat{A}\hat{B}}^A) + k_2 (\Pi_{\hat{A}\hat{B}}^S + \Pi_{\hat{A}\hat{B}}^A) - \frac{k_2}{d_A} (\Pi_{\hat{A}\hat{B}}^S - \Pi_{\hat{A}\hat{B}}^A) \]  

\[ = \frac{k_1}{d_A} (k_2 + k_2 (d_A - 1)) \Pi_{\hat{A}\hat{B}}^S + (k_2 (d_A + 1) - k_1) \Pi_{\hat{A}\hat{B}}^A \]

where we used the facts that

\[ T_B(\Phi_{\hat{A}\hat{B}}) = \frac{1}{d_A} F_{\hat{A}\hat{B}}, \]  

\[ F_{\hat{A}\hat{B}} = \Pi_{\hat{A}\hat{B}}^S - \Pi_{\hat{A}\hat{B}}^A, \]  

\[ I_{\hat{A}\hat{B}} = \Pi_{\hat{A}\hat{B}}^S + \Pi_{\hat{A}\hat{B}}^A. \]

Similarly,

\[ T_B(L_{\hat{A}\hat{B}}) = \frac{1}{d_A} [(l_1 + l_2 (d_A - 1)) \Pi_{\hat{A}\hat{B}}^S + (l_2 (d_A + 1) - l_1) \Pi_{\hat{A}\hat{B}}^A], \]

\[ T_B(M_{\hat{A}\hat{B}}) = \frac{1}{d_A} [(m_1 + m_2 (d_A - 1)) \Pi_{\hat{A}\hat{B}}^S + (m_2 (d_A + 1) - m_1) \Pi_{\hat{A}\hat{B}}^A], \]

\[ T_B(N_{\hat{A}\hat{B}}) = \frac{1}{d_A} [(n_1 + n_2 (d_A - 1)) \Pi_{\hat{A}\hat{B}}^S + (n_2 (d_A + 1) - n_1) \Pi_{\hat{A}\hat{B}}^A]. \]

Let us consider how each of the constraints simplify. Consider that the first inequality constraint

\[ T_B \left( \frac{K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} + M_{\hat{A}\hat{B}} + N_{\hat{A}\hat{B}}}{d + 1} \right) \geq 0, \]

simplifies to the following two inequalities:

\[ (k_1 + k_2 (d_A - 1)) + \frac{l_1 + l_2 (d_A - 1) + m_1 + m_2 (d_A - 1)}{d + 1} + \frac{n_1 + n_2 (d_A - 1)}{(d + 1)^2} \geq 0, \]

\[ k_2 (d_A + 1) - k_1 + \frac{l_2 (d_A + 1) - l_1 + m_2 (d_A + 1) - m_1}{(d + 1)} + \frac{n_2 (d_A + 1) - n_1}{(d + 1)^2} \geq 0. \]

The second inequality constraint

\[ \frac{1}{d - 1} T_B \left( \frac{L_{\hat{A}\hat{B}} + N_{\hat{A}\hat{B}}}{d + 1} \right) \geq T_B \left( \frac{K_{\hat{A}\hat{B}} + M_{\hat{A}\hat{B}}}{d + 1} \right), \]
simplifies to the following two inequalities:

\[
\frac{1}{d-1} \left( l_1 + l_2 (d_A - 1) + \frac{n_1 + n_2 (d_A - 1)}{d+1} \right) \geq k_1 + k_2 (d_A - 1) + \frac{m_1 + m_2 (d_A - 1)}{d+1}, \\
\frac{1}{d-1} \left( l_2 (d_A + 1) - l_1 + \frac{n_2 (d_A + 1) - n_1}{d+1} \right) \geq k_2 (d_A + 1) - k_1 + \frac{m_2 (d_A + 1) - m_1}{d+1}.
\] (E25, E26)

The third inequality constraint

\[
\frac{1}{d-1} T_B \left( M_{\bar{A}\bar{B}} + \frac{N_{\bar{A}\bar{B}}}{d+1} \right) \geq T_B \left( K_{\bar{A}\bar{B}} + \frac{L_{\bar{A}\bar{B}}}{d+1} \right),
\] (E27)

simplifies to the following two inequalities:

\[
\frac{1}{d-1} \left( m_1 + m_2 (d_A - 1) + \frac{n_1 + n_2 (d_A - 1)}{d+1} \right) \geq k_1 + k_2 (d_A - 1) + \frac{l_1 + l_2 (d_A - 1)}{d+1}, \\
\frac{1}{d-1} \left( m_2 (d_A + 1) - m_1 + \frac{n_2 (d_A + 1) - n_1}{d+1} \right) \geq k_2 (d_A + 1) - k_1 + \frac{l_2 (d_A + 1) - l_1}{d+1}.
\] (E28, E29)

The final inequality constraint

\[
T_B \left( K_{\bar{A}\bar{B}} + \frac{N_{\bar{A}\bar{B}}}{(d-1)^2} \right) \geq \frac{1}{d-1} T_B \left( L_{\bar{A}\bar{B}} + M_{\bar{A}\bar{B}} \right),
\] (E30)

simplifies to the following two inequalities:

\[
k_1 + k_2 (d_A - 1) + \frac{n_1 + n_2 (d_A - 1)}{(d-1)^2} \geq \frac{1}{d-1} \left( l_1 + l_2 (d_A - 1) + m_1 + m_2 (d_A - 1) \right), \\
k_2 (d_A + 1) - k_1 + \frac{n_2 (d_A + 1) - n_1}{(d-1)^2} \geq \frac{1}{d-1} \left( l_2 (d_A + 1) - l_1 + m_2 (d_A + 1) - m_1 \right).
\] (E31, E32)

Consider that the first inequality constraint in (E23) is redundant, following from (E12). So it can be eliminated. Summarizing, we have reduced the semi-definite program in (C26) to the following linear program:

\[
1 - \sup_{k_1,l_1,m_1,n_1, k_2,l_2,m_2,n_2 \geq 0} [k_1 F + k_2 (1 - F)],
\] (E33)

subject to

\[
k_1 + l_1 + m_1 + n_1 = 1, \\
k_2 + l_2 + m_2 + n_2 = 1,
\] (E34, E35)

\[
k_2 (d_A + 1) - k_1 + \frac{l_2 (d_A + 1) - l_1 + m_2 (d_A + 1) - m_1}{(d+1)} + \frac{n_2 (d_A + 1) - n_1}{(d+1)^2} \geq 0,
\] (E36)

\[
\frac{1}{d-1} \left( l_1 + l_2 (d_A - 1) + \frac{n_1 + n_2 (d_A - 1)}{d+1} \right) \geq k_1 + k_2 (d_A - 1) + \frac{m_1 + m_2 (d_A - 1)}{d+1},
\] (E37)

\[
\frac{1}{d-1} \left( l_2 (d_A + 1) - l_1 + \frac{n_2 (d_A + 1) - n_1}{d+1} \right) \geq k_2 (d_A + 1) - k_1 + \frac{m_2 (d_A + 1) - m_1}{d+1},
\] (E38)

\[
\frac{1}{d-1} \left( m_1 + m_2 (d_A - 1) + \frac{n_1 + n_2 (d_A - 1)}{d+1} \right) \geq k_1 + k_2 (d_A - 1) + \frac{l_1 + l_2 (d_A - 1)}{d+1},
\] (E39)
\[
\frac{1}{d-1} \left( m_2 (d_\hat{A} + 1) - m_1 + \frac{n_2 (d_\hat{A} + 1) - n_1}{d + 1} \right) \geq k_2 (d_\hat{A} + 1) - k_1 + \frac{l_2 (d_\hat{A} + 1) - l_1}{d + 1}. \tag{E40}
\]

\[
k_1 + k_2 (d_\hat{A} - 1) + \frac{n_1 + n_2 (d_\hat{A} - 1)}{(d - 1)^2} \geq \frac{1}{d - 1} \left( l_1 + l_2 (d_\hat{A} - 1) + m_1 + m_2 (d_\hat{A} - 1) \right), \tag{E41}
\]

\[
k_2 (d_\hat{A} + 1) - k_1 + \frac{n_2 (d_\hat{A} + 1) - n_1}{(d - 1)^2} \geq \frac{1}{d - 1} \left( l_2 (d_\hat{A} + 1) - l_1 + m_2 (d_\hat{A} + 1) - m_1 \right). \tag{E42}
\]

The standard form of a linear program is as follows [BV04]:

\[
1 - \sup_{x \geq 0} \{ c^T x : Ax \leq b \}. \tag{E43}
\]

We can write (E33)–(E42) in this way, with

\[
x^T = \begin{bmatrix} k_1 & l_1 & m_1 & k_2 & l_2 & m_2 & n_2 \end{bmatrix}, \tag{E44}
\]

\[
c^T = \begin{bmatrix} F & 0 & 0 & 0 & 1 - F & 0 & 0 \end{bmatrix}, \tag{E45}
\]

\[
b^T = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{E46}
\]

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\
1 & \frac{1}{d+1} & \frac{1}{d+1} & -\frac{1}{(d+1)^2} & - (d_\hat{A} + 1) & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 \\
1 & -\frac{1}{d-1} & \frac{1}{d-1} & \frac{1}{(d-1)^2} & d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 \\
-1 & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{(d+1)^2} & d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 \\
1 & \frac{1}{d+1} & \frac{1}{d+1} & -\frac{1}{(d+1)^2} & d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 \\
-1 & \frac{1}{d-1} & \frac{1}{d-1} & -\frac{1}{(d-1)^2} & - (d_\hat{A} - 1) & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 & -d_\hat{A} + 1 \\
1 & -\frac{1}{d-1} & \frac{1}{d-1} & -\frac{1}{(d-1)^2} & - (d_\hat{A} + 1) & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1 & -d_\hat{A} - 1
\end{bmatrix}. \tag{E47}
\]

If \( F > \frac{1}{d_\hat{A}} \) and \( d_\hat{A} \leq d^2 \), then the following choices are feasible for the primal linear program in (E43)

\[
k_1 = \frac{d_\hat{A}}{d^2}, \quad l_1 = m_1 = 0, \quad n_1 = 1 - \frac{d_\hat{A}}{d^2}, \quad k_2 = 0, \tag{E48}
\]

\[
l_2 = m_2 = \frac{(d + 1)^2 k_1 + n_1 - (d_\hat{A} + 1)}{2d (d_\hat{A} + 1)} = \frac{d_\hat{A}}{d^2 (d_\hat{A} + 1)}, \tag{E49}
\]

\[
n_2 = 1 - \frac{2d_\hat{A}}{d^2 (d_\hat{A} + 1)}. \tag{E50}
\]

Thus, it follows that, in the case that \( F > \frac{1}{d_\hat{A}} \) and \( d_\hat{A} \leq d^2 \),

\[
e_{\text{PPT}}(S^d_{AB}, \rho^{(F,d_\hat{A})}_{AB}) \leq 1 - \frac{Fd_\hat{A}}{d^2}. \tag{E51}
\]

Now we turn to the dual linear program. It is given by

\[
1 - \inf_{y \geq 0} \{ b^T y : A^T y \geq c \}. \tag{E52}
\]

A feasible choice for the dual linear program is as follows:

\[
y_1 \in \left[ \frac{1}{d_\hat{A} d^2}, \frac{F}{d^2} \right], \quad y_2 = 0, \quad y_3 = \frac{Fd_\hat{A}}{d^2} - y_1. \tag{E53}
\]
Thus, we find for the case
\begin{equation}
y_4 = 0, \quad y_5 = \frac{1}{4d^2} (d + 1)^2 \left(F - d^2 y_1\right),
\end{equation}
\begin{equation}
y_6 = \frac{1}{4d^2} (d^2 - 1) \left(F + d^2 y_1\right), \quad y_7 = 0,
\end{equation}
\begin{equation}
y_8 = y_6, \quad y_9 = 0, \quad y_{10} = 0, \quad y_{11} = \frac{1}{4d^2} (d - 1)^2 \left(F - d^2 y_1\right).
\end{equation}

This implies that
\begin{equation}
\epsilon_{\text{PPT}}(S_{AB}^d, \rho_{AB}^{(F,d_A)}) \geq 1 - \frac{Fd_A}{d^2}
\end{equation}

For the final case of $F > \frac{1}{d_A}$ and $d_A > d^2$, for the primal, one can check that the following choices are feasible
\begin{equation}
k_1 = 1, \quad l_1 = 0, \quad m_1 = 0, \quad n_1 = 0,
\end{equation}
\begin{equation}
k_2 = \frac{d_A - d^2}{d^2(d_A - 1)},
\end{equation}
\begin{equation}
l_2 = \begin{cases}
\frac{(1+d)(d^2+d-|d_A+1|d_A)}{d^2(d_A-1)}, & \text{if } d_A + 1 < d + d^2 \\
0, & \text{if } d_A + 1 \geq d + d^2
\end{cases}
\end{equation}
\begin{equation}
m_2 = l_2,
\end{equation}
\begin{equation}
n_2 = 1 - k_2 - l_2 - n_2.
\end{equation}

This implies that
\begin{equation}
\epsilon_{\text{PPT}}(S_{AB}^d, \rho_{AB}^{(F,d_A)}) \leq 1 - k_1 F - k_2 (1 - F)
\end{equation}
\begin{equation}
= 1 - F - \frac{d_A - d^2}{d^2(d_A - 1)} (1 - F)
\end{equation}
\begin{equation}
= \frac{\left(1 - \frac{1}{d^2}\right)(1 - F)}{1 - \frac{1}{d_A}}.
\end{equation}

Now we consider the dual program for the case $F > \frac{1}{d_A}$ and $d_A > d^2$. A feasible solution is given by
\begin{equation}
y_1 = \frac{1 - F + d^2(d_A F - 1)}{d^2(d_A - 1)}, \quad y_2 = 0, \quad y_3 = \frac{(d_A - 1)(F - y_1)}{d^2 - 1},
\end{equation}
\begin{equation}
y_4 = 0, \quad y_5 = 0, \quad y_6 = \frac{F - y_1}{2}, \quad y_7 = 0, \quad y_8 = y_6,
\end{equation}
\begin{equation}
y_9 = y_{10} = y_{11} = 0.
\end{equation}

Note that
\begin{equation}
F - y_1 = \frac{(d^2 - 1)(1 - F)}{d^2(d_A - 1)} \geq 0.
\end{equation}

Thus, we find for the case $F > \frac{1}{d_A}$ and $d_A > d^2$ that
\begin{equation}
\epsilon_{\text{PPT}}(S_{AB}^d, \rho_{AB}^{(F,d_A)}) \geq 1 - (y_1 + y_3)
\end{equation}
\begin{equation}
= \frac{\left(1 - \frac{1}{d^2}\right)(1 - F)}{1 - \frac{1}{d_A}}.
\end{equation}

This completes the proof of the equality in (E2).
the resource state $\rho_{A\hat{B}}^{(F,d,\hat{A})}$ and then perform teleportation in opposite directions. Consider that the bilateral twirl in (9) realizes the following evolution:

$$X \to \text{Tr}[\Phi_{AB}^{d^2} X] \Phi_{\hat{A}\hat{B}}^{d^2} + \text{Tr}[(I_{AB} - \Phi_{AB}^{d^2}) X] \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1}.$$  \hspace{1cm} (E72)

Consider that

$$\text{Tr}[\Phi_{AB}^{d^2} \rho_{A\hat{B}}^{(F,d,\hat{A})}] = F \text{Tr}[\Phi_{AB}^{d^2} \Phi_{\hat{A}\hat{B}}] + (1 - F) \frac{\text{Tr}[\Phi_{AB}^{d^2} (I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}})]}{d^2 - 1}$$  \hspace{1cm} (E73)

$$= F \frac{d_{\hat{A}}}{d^2} + (1 - F) \left( \frac{d_{\hat{A}}}{d^2} \right) \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1}$$  \hspace{1cm} (E74)

$$= F \frac{d_{\hat{A}}}{d^2},$$  \hspace{1cm} (E75)

where we are embedding the space $\hat{A}\hat{B}$ in the larger space $AB$ and we used the facts that

$$\text{Tr}[\Phi_{AB}^{d^2} \Phi_{\hat{A}\hat{B}}] = \frac{d_{\hat{A}}}{d^2},$$  \hspace{1cm} (E76)

$$\text{Tr}[\Phi_{AB}^{d^2} I_{\hat{A}\hat{B}}] = \frac{d_{\hat{A}}}{d^2}.$$  \hspace{1cm} (E77)

Thus,

$$\rho_{A\hat{B}}^{(F,d,\hat{A})} \to \text{Tr}[\Phi_{AB}^{d^2} \rho_{A\hat{B}}^{(F,d,\hat{A})}] \Phi_{AB}^{d^2} + \text{Tr}[(I_{AB} - \Phi_{AB}^{d^2}) \rho_{A\hat{B}}^{(F,d,\hat{A})}] \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1}$$  \hspace{1cm} (E78)

$$= F \frac{d_{\hat{A}}}{d^2} \Phi_{AB}^{d^2} + \left( 1 - F \frac{d_{\hat{A}}}{d^2} \right) \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1}.$$  \hspace{1cm} (E79)

Following a similar analysis as given in (143)–(149), we bound the normalized trace distance between the ideal state $\Phi_{AB}^{d^2}$ and this one as follows:

$$\frac{1}{2} \left\| \Phi_{AB}^{d^2} - \left( F \frac{d_{\hat{A}}}{d^2} \Phi_{AB}^{d^2} + \left( 1 - F \frac{d_{\hat{A}}}{d^2} \right) \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1} \right) \right\|_1$$

$$= \frac{1}{2} \left\| \left( 1 - F \frac{d_{\hat{A}}}{d^2} \right) \Phi_{AB}^{d^2} - \left( 1 - F \frac{d_{\hat{A}}}{d^2} \right) \frac{I_{AB} - \Phi_{AB}^{d^2}}{d^2 - 1} \right\|_1$$  \hspace{1cm} (E80)

$$= 1 - F \frac{d_{\hat{A}}}{d^2}.$$  \hspace{1cm} (E81)

This implies that

$$\epsilon_{\text{LOCC}}(S_{AB}^d, \rho_{A\hat{B}}^{(F,d,\hat{A})}) \leq 1 - F \frac{d_{\hat{A}}}{d^2},$$  \hspace{1cm} (E82)

and combined with the general inequality in (86), we conclude that

$$\epsilon_{\text{LOCC}}(S_{AB}^d, \rho_{A\hat{B}}^{(F,d,\hat{A})}) = 1 - F \frac{d_{\hat{A}}}{d^2},$$  \hspace{1cm} (E83)

for the case $F > \frac{1}{d_{\hat{A}}}$ and $d_{\hat{A}} \leq d^2$.

For the case $F > \frac{1}{d_{\hat{A}}}$ and $d_{\hat{A}} > d^2$, it is unclear to us at the moment how to achieve the PPT simulation error by means of an LOCC-assisted scheme.
Appendix F: Proof of Proposition 12

In this appendix, we establish how the semi-definite program in (C26) simplifies for Werner states. The analysis bears some structural similarities to that in Appendix E. Recall from (155) that a Werner state has the following form:

\[ W_{AB}^{(p,d_A)} := (1-p) \frac{2}{d_A (d_A+1)} \Pi^S_{AB} + p \frac{2}{d_A (d_A-1)} \Pi^A_{AB}, \]  

where \( p \in [0,1] \) and \( \Pi^S_{AB} := (I_{AB} + F_{AB})/2 \) and \( \Pi^A_{AB} := (I_{AB} - F_{AB})/2 \).

We prove that

\[ \epsilon_{\text{PPT}}(\mathcal{S}_{AB}, W_{AB}^{(p,d_A)}) = \begin{cases} 1 - \frac{1}{d^2} & \text{if } p \leq \frac{1}{2} \\ 1 - \frac{4p-2+d_A}{d^2} & \text{if } p > \frac{1}{2}. \end{cases} \]  

(F2)

If \( p \leq \frac{1}{2} \), the state is separable [HHHH09, Wat18]. So the resource state can be prepared by LOCC, and then our previous result from Proposition 9 applies, so that we can conclude that the simulation error is

\[ 1 - \frac{1}{d^2} \]  

(F3)

in this case. So in what follows, we focus exclusively on the case when \( p > \frac{1}{2} \).

By applying a similar argument as given above (E4) but using (156)–(157) instead, it suffices for each of \( K_{AB}, L_{AB}, M_{AB}, \) and \( N_{AB} \) to have the Werner form:

\[ K_{AB} = k_1 \Pi^S_{AB} + k_2 \Pi^A_{AB}, \]  

(F4)

\[ L_{AB} = l_1 \Pi^S_{AB} + l_2 \Pi^A_{AB}, \]  

(F5)

\[ M_{AB} = m_1 \Pi^S_{AB} + m_2 \Pi^A_{AB}, \]  

(F6)

\[ N_{AB} = n_1 \Pi^S_{AB} + n_2 \Pi^A_{AB}. \]  

(F7)

The objective function evaluates to

\[ \text{Tr}[K_{AB} W_{AB}^{(p,d_A)}] = k_1 (1-p) + k_2 p. \]  

(F8)

Consider that the equality constraint

\[ K_{AB} + L_{AB} + M_{AB} + N_{AB} = I_{AB} \]  

(F9)

implies that

\[ k_1 + l_1 + m_1 + n_1 = 1, \]  

(F10)

\[ k_2 + l_2 + m_2 + n_2 = 1. \]  

(F11)

We also have that all coefficients are non-negative:

\[ k_1, l_1, m_1, n_1, k_2, l_2, m_2, n_2 \geq 0. \]  

(F12)

All of the inequality constraints involve the terms \( T_B(K_{AB}), T_B(L_{AB}), T_B(M_{AB}), \) and \( T_B(N_{AB}) \). Let us evaluate the first of these and the rest follow similarly:

\[ T_B(K_{AB}) = T_B(k_1 \Pi^S_{AB} + k_2 \Pi^A_{AB}) \]  

(F13)

\[ = k_1 T_B(\Pi^S_{AB}) + k_2 T_B(\Pi^A_{AB}) \]  

(F14)

\[ = k_1 T_B \left( \frac{I_{AB} + F_{AB}}{2} \right) + k_2 T_B \left( \frac{I_{AB} - F_{AB}}{2} \right) \]  

(F15)

\[ = \frac{k_1}{2} I_{AB} + \frac{k_1 d_A}{2} \Phi_{AB} + \frac{k_2}{2} I_{AB} - \frac{k_2 d_A}{2} \Phi_{AB} \]  

(F16)

\[ = \frac{d_A}{2} (k_1 - k_2) \Phi_{AB} + \frac{1}{2} (k_1 + k_2) I_{AB}. \]  

(F17)
The first one simplifies as follows:

\[ \frac{d_A}{2} (k_1 - k_2) \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (k_1 + k_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} + \Phi_{\hat{A}\hat{B}} \right) \]  

\[ = \left[ \frac{d_A}{2} (k_1 - k_2) + \frac{1}{2} (k_1 + k_2) \right] \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (k_1 + k_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} \right) \]  

\[ = \left[ \frac{d_A + 1}{2} k_1 - \frac{d_A - 1}{2} k_2 \right] \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (k_1 + k_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} \right). \]  

For the other operators \( L_{\hat{A}\hat{B}}, M_{\hat{A}\hat{B}}, \) and \( N_{\hat{A}\hat{B}}, \) we have that

\[ T_{\hat{B}}(L_{\hat{A}\hat{B}}) = \left[ \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 \right] \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (l_1 + l_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} \right), \]  

\[ T_{\hat{B}}(M_{\hat{A}\hat{B}}) = \left[ \frac{d_A + 1}{2} m_1 - \frac{d_A - 1}{2} m_2 \right] \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (m_1 + m_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} \right), \]  

\[ T_{\hat{B}}(N_{\hat{A}\hat{B}}) = \left[ \frac{d_A + 1}{2} n_1 - \frac{d_A - 1}{2} n_2 \right] \Phi_{\hat{A}\hat{B}} + \frac{1}{2} (n_1 + n_2) \left( I_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}} \right). \]  

Now we evaluate the inequalities from (C26). We begin with the following one

\[ T_{\hat{B}} \left( K_{\hat{A}\hat{B}} + \frac{L_{\hat{A}\hat{B}}}{d + 1} + \frac{M_{\hat{A}\hat{B}}}{d + 1} + \frac{N_{\hat{A}\hat{B}}}{(d + 1)^2} \right) \geq 0, \]

and observe that it reduces to the following two inequalities:

\[ \frac{d_A + 1}{2} k_1 - \frac{d_A - 1}{2} k_2 + \frac{1}{(d + 1)^2} \left( \frac{d_A + 1}{2} n_1 - \frac{d_A - 1}{2} n_2 \right) \]

\[ + \frac{1}{d + 1} \left( \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 + \frac{d_A + 1}{2} m_1 - \frac{d_A - 1}{2} m_2 \right) \geq 0 \]  

\[ \frac{1}{2} \left( k_1 + k_2 + \frac{l_1 + l_2 + m_1 + m_2}{d + 1} + \frac{n_1 + n_2}{(d + 1)^2} \right) \geq 0. \]  

The first one simplifies as follows:

\[ (d_A + 1) \left( k_1 + \frac{l_1 + m_1}{d + 1} + \frac{n_1}{(d + 1)^2} \right) \geq (d_A - 1) \left( k_2 + \frac{l_2 + m_2}{d + 1} + \frac{n_2}{(d + 1)^2} \right). \]  

The second inequality is redundant, as a consequence of (F12). The following inequality

\[ \frac{1}{d - 1} T_{\hat{B}} \left( L_{\hat{A}\hat{B}} + \frac{N_{\hat{A}\hat{B}}}{d + 1} \right) \geq T_{\hat{B}} \left( K_{\hat{A}\hat{B}} + \frac{M_{\hat{A}\hat{B}}}{d + 1} \right), \]

reduces to the following two inequalities:

\[ \frac{1}{d - 1} \left( \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 + \frac{d_A + 1}{2} n_1 - \frac{d_A - 1}{2} n_2 \right) \geq \frac{d_A + 1}{2} k_1 - \frac{d_A - 1}{2} k_2 \]

\[ + \frac{d_A + 1}{2} m_1 - \frac{d_A - 1}{2} m_2 \]  

\[ \frac{1}{2(d - 1)} \left( l_1 + l_2 + \frac{n_1 + n_2}{d + 1} \right) \geq \frac{1}{2} \left( k_1 + k_2 + \frac{m_1 + m_2}{d + 1} \right). \]  

The first inequality simplifies as follows:

\[ \frac{d_A + 1}{d - 1} \left( l_1 + \frac{n_1}{d + 1} \right) + (d_A - 1) \left( k_2 + \frac{m_2}{d + 1} \right) \geq (d_A + 1) \left( k_1 + \frac{m_1}{d + 1} \right) + \left( \frac{d_A - 1}{d - 1} \right) \left( l_2 + \frac{n_2}{d + 1} \right). \]
and the second one as
\[
\left(\frac{1}{d-1}\right) \left( l_1 + l_2 + \frac{n_1 + n_2}{d+1} \right) \geq k_1 + k_2 + \frac{m_1 + m_2}{d+1} \tag{F30}
\]

The following inequality
\[
\frac{1}{d-1} T_B \left( M_{AB} + \frac{N_{AB}}{d+1} \right) \geq T_B \left( K_{AB} + \frac{L_{AB}}{d+1} \right)
\]

reduces to the following two inequalities:
\[
\frac{1}{d-1} \left( \frac{d_A + 1}{d} m_1 - \frac{d_A - 1}{d} m_2 + \frac{d_A + 1}{2} n_1 - \frac{d_A - 1}{2} n_2 \right) \geq \frac{d_A + 1}{2} k_1 - \frac{d_A - 1}{2} k_2 + \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 
\]
\[
\frac{d_A + 1}{2} \left( m_1 + m_2 + \frac{n_1 + n_2}{d+1} \right) \geq \frac{1}{2} \left( k_1 + k_2 + \frac{l_1 + l_2}{d+1} \right) \tag{F31}
\]

The first simplifies as follows:
\[
\frac{d_A + 1}{d-1} \left( m_1 + \frac{n_1}{d+1} \right) + (d_A - 1) \left( k_2 + \frac{l_2}{d+1} \right) \geq \frac{d_A - 1}{d-1} \left( m_2 + \frac{n_2}{d+1} \right) + (d_A + 1) \left( k_1 + \frac{l_1}{d+1} \right) \tag{F32}
\]
and the second as
\[
\frac{1}{d-1} \left( m_1 + m_2 + \frac{n_1 + n_2}{d+1} \right) \geq k_1 + k_2 + \frac{l_1 + l_2}{d+1} \tag{F33}
\]

The final inequality
\[
T_B \left( K_{AB} + \frac{N_{AB}}{(d-1)^2} \right) \geq \frac{1}{d-1} T_B \left( L_{AB} + M_{AB} \right)
\]

reduces to the following two inequalities:
\[
\frac{d_A + 1}{2} k_1 - \frac{d_A - 1}{2} k_2 + \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 + \frac{d_A + 1}{2} m_1 - \frac{d_A - 1}{2} m_2 \geq \frac{1}{d-1} \left( \frac{d_A + 1}{2} l_1 - \frac{d_A - 1}{2} l_2 + \frac{d_A + 1}{2} m_1 - \frac{d_A - 1}{2} m_2 \right) \tag{F34}
\]
\[
\frac{1}{2} \left( k_1 + k_2 + \frac{n_1 + n_2}{(d-1)^2} \right) \geq \frac{1}{2} \left( l_1 + l_2 + m_1 + m_2 \right). \tag{F35}
\]

The first inequality simplifies as follows:
\[
(d_A + 1) \left( k_1 + \frac{n_1}{(d-1)^2} \right) + \frac{d_A - 1}{d-1} \left( l_2 + m_2 \right) \geq (d_A - 1) \left( k_2 + \frac{n_2}{(d-1)^2} \right) + \frac{d_A + 1}{d-1} \left( l_1 + m_1 \right) \tag{F36}
\]
and the second as
\[
k_1 + k_2 + \frac{n_1 + n_2}{(d-1)^2} \geq \frac{1}{d-1} \left( l_1 + l_2 + m_1 + m_2 \right). \tag{F37}
\]

Summarizing, we have reduced the optimization problem to the following linear program:
\[
1 - \sup_{k_1, l_1, m_1, n_1, k_2, l_2, m_2, n_2 \geq 0} \left[ k_1 \left( 1 - p \right) + k_2 p \right] \tag{F38}
\]
subject to
\[
k_1 + l_1 + m_1 + n_1 = 1, \tag{F39}
\]
Let us refer to this as the primal linear program. A feasible solution for the primal linear program is as follows:

\[
\begin{align*}
(k_1 + \frac{l_1 + m_1}{d+1} + \frac{n_1}{(d+1)^2}) & \geq (d\hat{A} - 1) \left( k_2 + \frac{l_2 + m_2}{d+1} + \frac{n_2}{(d+1)^2} \right), \\
(d\hat{A} + 1) \left( k_1 + \frac{l_1 + m_1}{d+1} + \frac{n_1}{(d+1)^2} \right) & \geq (d\hat{A} - 1) \left( k_2 + \frac{l_2 + m_2}{d+1} + \frac{n_2}{(d+1)^2} \right),
\end{align*}
\]

We can write this in the standard form of a linear program as follows:

\[
1 - \sup_{x \geq 0} \{ c^T x : Ax \leq b \},
\]

where

\[
\begin{align*}
A &= \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
-\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} \\
\frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} & \frac{1}{d+1} \\
-\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} \\
-\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}, \\
k_1 &= \frac{d\hat{A} - 2}{d^2 d\hat{A}}, \\
l_1 &= \frac{2 + d(d\hat{A} - 1) - d\hat{A}}{d^2 (d\hat{A} + 1)}, \\
m_1 &= l_1, \quad n_1 = 1 - k_1 - l_1 - m_1, \\
k_2 &= \frac{d\hat{A} + 2}{d^2 d\hat{A}}.
\end{align*}
\]
\[ l_2 = \frac{d^2(d_A + 1)l_1 - 2}{d^2(d_A - 1)}, \quad m_2 = l_2, \quad n_2 = 1 - k_2 - m_2 - n_2. \]  

(F60)

Then we conclude that

\[ e_{\text{PPT}}(S_{AB}, W_{AB}^{(p,d_A)}) \leq 1 - [(1 - p) k_1 + pk_2] \]

\[ = 1 - \frac{d_A - 2 + 4p}{d^2d_A}. \]

(F61)  

(F62)

The dual linear program is given by

\[ 1 - \inf_{y \geq 0} \{ b^T y : A^T y \geq c \}. \]

(F63)

A feasible choice for the dual variables is

\[ y_1 = \frac{(d_A + 2)p - 1}{d^2d_A}, \quad y_2 = 0, \]

(F64)

\[ y_3 = \frac{d_A (1 - p) + 2p - 1}{d^2d_A}, \quad y_4 = 0, \]

(F65)

\[ y_5 = \frac{(d + 1)^2(2p - 1)}{4d^2d_A}, \quad y_6 = 0, \]

(F66)

\[ y_7 = \frac{(d^2 - 1)(2p - 1 + d_A)}{4d^2d_A}, \]

(F67)

\[ y_8 = 0, \quad y_9 = y_7, \]

(F68)

\[ y_{10} = \frac{(d - 1)^2(2p - 1)}{4d^2d_A}, \quad y_{11} = 0. \]

(F69)

Then we find that

\[ e_{\text{PPT}}(S_{AB}, W_{AB}^{(p,d_A)}) \geq 1 - (y_1 + y_3) \]

\[ = 1 - \frac{4p - 2 + d_A}{d^2d_A}, \]

(F70)  

(F71)

and finally conclude that

\[ e_{\text{PPT}}(S_{AB}, W_{AB}^{(p,d_A)}) = 1 - \frac{4p - 2 + d_A}{d^2d_A}. \]

(F72)

Appendix G: Proof of Proposition 15

A proof for Proposition 15 proceeds similarly to the proof for Proposition 4, as given in Appendix C. The main goal is to simplify the semi-definite program from Proposition 13.

Let \( P_{ABAB\hat{C} \rightarrow A'B'} \) be an arbitrary multipartite C-PPT-P channel, to be considered for bidirectional controlled teleportation. Its Choi operator \( P_{ABAB\hat{C} \rightarrow A'B'} \) obeys the following conditions:

\[ P_{ABAB\hat{C} \rightarrow A'B'} \geq 0, \]  

(G1)

\[ \text{Tr}_{A'B'}[P_{ABAB\hat{C} \rightarrow A'B'}] = I_{ABAB}, \]  

(G2)

\[ T_{A\hat{A}A'}(P_{ABAB\hat{C} \rightarrow A'B'}) \geq 0, \]  

(G3)

\[ T_{B\hat{B}B'}(P_{ABAB\hat{C} \rightarrow A'B'}) \geq 0, \]  

(G4)

\[ T_{C}(P_{ABAB\hat{C} \rightarrow A'B'}) \geq 0. \]  

(G5)

The simulation error for a particular channel \( P_{ABAB\hat{C} \rightarrow A'B'} \) is as follows:

\[ \frac{1}{2} \left\| S_{AB \rightarrow A'B'}^\dagger - P_{ABAB\hat{C} \rightarrow A'B'} \circ A_{AB\hat{C}}^\dagger \right\|. \]  

(G6)
By following the same reasoning in (A3)–(A26), it suffices to optimize over symmetrized channels $\tilde{P}_{\hat{A}\hat{B}\hat{A}'\hat{B}'}$, with Choi operator given by

$$
\begin{align*}
\tilde{P}_{\hat{A}\hat{B}\hat{A}'\hat{B}'} &= \Gamma_{\hat{A}B'} \otimes \Gamma_{BA'} \otimes K_{\hat{A}\hat{B}C} + \Gamma_{\hat{A}B'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes L_{\hat{A}\hat{B}C} \\
&\quad + \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \Gamma_{BA'} \otimes M_{\hat{A}\hat{B}C} + \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes N_{\hat{A}\hat{B}C},
\end{align*}
$$

where

$$(G7)$$

This Choi operator satisfies the conditions in (G1) and (G2). Then we apply the conditions in (G3)–(G5) to determine further conditions on $K_{\hat{A}\hat{B}C}$, $L_{\hat{A}\hat{B}C}$, $M_{\hat{A}\hat{B}C}$, and $N_{\hat{A}\hat{B}C}$. Following the same reasoning given in (C6)–(C24), we conclude that the following conditions hold

$$
T_{\hat{S}} \left( K_{\hat{A}\hat{B}C} + \frac{L_{\hat{A}\hat{B}C}}{d+1} + \frac{M_{\hat{A}\hat{B}C}}{d+1} + \frac{N_{\hat{A}\hat{B}C}}{(d+1)^2} \right) \geq 0,
$$

$$
\frac{1}{d-1} T_{\hat{S}} \left( L_{\hat{A}\hat{B}C} + \frac{N_{\hat{A}\hat{B}C}}{d+1} \right) \geq T_{\hat{S}} \left( K_{\hat{A}\hat{B}C} + \frac{M_{\hat{A}\hat{B}C}}{d+1} \right),
$$

$$
\frac{1}{d-1} T_{\hat{S}} \left( M_{\hat{A}\hat{B}C} + \frac{N_{\hat{A}\hat{B}C}}{d+1} \right) \geq T_{\hat{S}} \left( K_{\hat{A}\hat{B}C} + \frac{L_{\hat{A}\hat{B}C}}{d+1} \right),
$$

$$
T_{\hat{S}} \left( K_{\hat{A}\hat{B}C} + \frac{N_{\hat{A}\hat{B}C}}{(d-1)^2} \right) \geq \frac{1}{d-1} T_{\hat{S}} \left( L_{\hat{A}\hat{B}C} + M_{\hat{A}\hat{B}C} \right),
$$

for $\hat{S} \in \{\hat{A}, \hat{B}\}$. Finally imposing the condition in (G5) and using the orthogonality of $\Gamma_{\hat{A}B'} \otimes \Gamma_{BA'}$, $\Gamma_{\hat{A}B'} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1}$, $\frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \Gamma_{BA'}$, and $\frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1} \otimes \frac{dI_{BA'} - \Gamma_{BA'}}{d^2 - 1}$, we conclude that

$$
T_{\hat{C}}(K_{\hat{A}\hat{B}C}), T_{\hat{C}}(L_{\hat{A}\hat{B}C}), T_{\hat{C}}(M_{\hat{A}\hat{B}C}), T_{\hat{C}}(N_{\hat{A}\hat{B}C}) \geq 0.
$$

(G10)

The rest of the proof then proceeds as in (A49)–(A57). We can furthermore exploit the extra symmetry in (70) to simplify the SDP, as stated in Proposition 15. The proof that $e_{\hat{A}\hat{B}C}^{\text{PPT}}(S_{\hat{A}B}, \rho_{\hat{A}\hat{B}C})$ can be calculated by the same semi-definite program follows from reasoning similar to that given in Appendix A3.