ABSTRACT. With the existence and uniqueness of a vector value solution for the full non-linear homogeneous Boltzmann system of equations describing multi-component monatomic gas mixtures for binary interactions proved [8], we present in this manuscript several properties for such a solution. We start by proving the gain of integrability of the gain term of the multispecies collision operator, extending the work done previously for the single species case [3]. In addition, we study the integrability properties of the multispecies collision operator as a bilinear form, revisiting and expanding the work done for a single gas [2]. With these estimates, together with a control by below for the loss term of the collision operator as in [8], we develop the propagation for the polynomially and exponentially $\beta$-weighted $L_p^\beta$-norms for the vector value solution. Finally, we extend such $L_p^\beta$-norms propagation property to $p = \infty$.

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1. Introduction

We consider a mixture of $I$ monatomic gases in a space homogeneous setting. Each component of the mixture can be statistically described by its distribution function $f_i := f_i(t, v)$, depending on time $t \geq 0$ and velocity of molecules $v \in \mathbb{R}^N$, that changes due to binary interactions with other particles of the same or different species. For an $i$ fixed, each $f_i$ solves a Boltzmann type equation where the collision operator takes into account not only the influence of particles of the same species, but all other species. Since we are considering all species simultaneously, we introduce a vector valued set of distribution functions $\mathbb{F} := \{f_i\}_{1 \leq i \leq I}$, whose change due to binary collisions of particles is expressed by a vector of collision operators, with its $i$–th component given by $[Q(\mathbb{F})]_i := \sum_{j=1}^{I} Q_{ij}(f_i, f_j)$. Then, the evolution of a mixture is leaded by a system of Boltzmann equations.

The existence and uniqueness of the solution for the non-linear system of spatially homogeneous Boltzmann equations for multi-species mixtures with binary interactions has been proven recently in [8], in a vector valued Banach space with a norm depending on the species mass fractions, to be defined in the next sections. That result is obtained by following general ODE theory in Banach spaces that studies the evolution of such vector valued solutions with their suitable norm, without requesting entropy boundedness. Such normed spaces provided estimates to rigorously prove the generation and propagation of scalar polynomial and exponential moments of the vector valued $\mathbb{F}$.

The innovative tools and results at the core of this manuscript are the following: first the introduction of suitable $L^p_\beta$ vector valued spaces whose norms have dependence on specific weights depending on the species mass fractions and a new explicit Carleman integral representation for the positive part of the collisional operator, associated to a binary interaction of two different species with different masses. Next, by means of the new Carleman representation and the vector valued $L^p_\beta$-spaces associated to the multi-species system, we obtain a gain of integrability estimate for the positive contribution of the collisional form that provides explicit constants rates. Last, we show that gain of integrability estimates prove the propagation of the $L^p$ norms with polynomial and exponential weights of the vector valued solution of the Boltzmann system of equations for mixtures.

The techniques used in this manuscript are extensions or adaptations of results developed for the scalar Boltzmann equations. But since in the mixture framework
each component of the mixture is characterized by the molecular mass $m_i$, the symmetry properties of the binary collisions are no longer valid, which yields to changes in the mathematical treatment.

It is noteworthy that the gain of integrability estimates essentially follows, after the generalization to the multi-species problem, the strategy devised in [3], which consists in first showing the control of a weighted $L^2$ norm of the positive part of the collision operator of each species by a lower order weighted $L^2$ norm of the input vector value function in a suitable Banach space. After the $L^2$ control, one can obtain a Young’s type inequality for the gain term of the collision operator for mixture of gases by means of a weak form of the positive term of the collisional operator as an extension of the original strategy developed in [2] and a subsequent interpolation argument for the system.

This extends the work in [2] and [3], both proven for a single species Boltzmann equation. With this two estimates, we are able to state and prove the gain of $L^p$ integrability of the positive part of the collision operator $1 < p < \infty$. In addition, we prove the propagation of $L^\infty$ norms, following the same approach as in [5] applied to the single component gas.

The paper is organized as follows. In section 2 we describe the kinetic model along with the notation used through the manuscript and state the main results of this work. In the same section we define the suitable Banach spaces needed to get our results and their notation and we state different ways to write the gain part of the collisional operator, both in the classical definition and the weak formulation. Then, in Section 3 and 4 we prove the gain of integrability for $L^2$ and $L^p$ polynomially weighted norms respectively. In section 5 we prove the main results: the propagation of $L^p$ norms, both with polynomial and exponential weights, and in Section 6 we extend this results for the case $p = \infty$. Finally, there is an Appendix with some calculations and statements needed.

2. Preliminaries and Main results

2.1. The kinetic model for monatomic gas mixtures. Each component of the mixture, namely $A_i$ with $1 \leq i \leq I$, is described with its own distribution function $f_i := f_i(t, v) \geq 0$, that depends on time $t > 0$ and particle velocity $v \in \mathbb{R}^N$. In order to describe its evolution, we first model the binary interaction of two colliding molecules.

2.1.1. Collision process. We fix two molecules of species $A_i$ and $A_j$ that are going to collide. Let molecule of species $A_i$ have mass $m_i$ and velocity $v'$; and molecule of species $A_j$, mass $m_j$ and velocity $v'_s$ before the collision. After the collision, they belong to the same species, have the same mass, but their velocities changed to $v$ and $v'_s$, respectively. During the elastic collision, conservation of momentum and kinetic energy hold, that is

\begin{align}
    m_i v' + m_j v'_s &= m_i v + m_j v'_s, \\
    m_i |v'|^2 + m_j |v'_s|^2 &= m_i |v|^2 + m_j |v'_s|^2. \tag{1}
\end{align}
Let \( r_{ij} \in (0, 1) \) be the mass contribution of the molecule of species \( A_i \) to the sum of masses of two colliding molecules of species \( A_i \) and \( A_j \), i.e. denote
\[
 r_{ij} := \frac{m_i}{m_i + m_j} \quad \Rightarrow \quad r_{ji} := 1 - r_{ij} = \frac{m_j}{m_i + m_j}.
\] 

Then equations (1) can be parametrized with a parameter \( \sigma \in \mathbb{S}^{N-1} \), so that pre-collisional quantities can be written in terms of post-collisional ones as
\[
v' = v + (1 - r_{ij})(|u| - u), \quad v'_* = v_* - r_{ij}(|u| - u),
\]
where \( u := v - v_* \) is the relative velocity. In other words, \( \sigma \) is in the direction of the pre-collisional relative velocity \( u' = v' - v'_* \).
\[
u' = |u| \sigma.
\]

2.1.2. The system of Boltzmann equations. Since the whole mixture is considered simultaneously, we are led to introduce a vector of distribution functions,
\[
 F = [f_{i1 \leq t \leq i}].
\]

The vector-value function \( F \) satisfies the system of Boltzmann equations,
\[
 \partial_t F(t, v) = Q(F, F)(t, v), \quad t > 0, \quad v \in \mathbb{R}^N,
\]
where \( Q(F, F) \) is a vector of multispecies collision operators whose \( i \)-th component is
\[
 [Q(F, F)]_i = \sum_{j=1}^t Q_{ij}(f_i, f_j)(v),
\]
and \( Q_{ij} \) is defined below.

2.2. Pairwise collision operator \( Q_{ij} \). Let \( f \) be the distribution function for the species \( A_i \) and let the distribution function \( g \) be associated to the species \( A_j \). The pairwise collision operator describing the collisions of molecules of species \( A_i \) with the molecules of species \( A_j \) is defined as
\[
 Q_{ij}(f, g)(v) = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} (f(w') - f(v)) g(v'_*) \mathcal{B}_{ij}(|u|, \hat{u} \cdot \sigma) \, d\sigma \, dv_* \quad \text{and} \quad \hat{u} := u/|u|,
\]
where the pre-collisional velocities are given by (3), and \( \hat{u} := u/|u| \).

The other way around, we define the pairwise collision operator that describes collision of molecules of species \( A_j \) with the ones of species \( A_i \)
\[
 Q_{ji}(g, f)(v) = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} (g(w'_*) - g(v_*)) f(v) \mathcal{B}_{ji}(|u|, \hat{u} \cdot \sigma) \, d\sigma \, dv_*,
\]
where now velocities \( w' \) and \( w'_* \) differ from (3) by mass interchange \( m_i \leftrightarrow m_j \),
\[
 w' = v + (1 - r_{ji})(|u| - u), \quad v'_* = v_* - r_{ji}(|u| - u),
\]
after noting that \( r_{ji} = 1 - r_{ij} \) from definition (2).

The collision kernel associated to transition probabilities of exchanging states at an interaction, \( \mathcal{B}_{ij} \), is a positive a.e. measure that satisfies the micro-reversibility assumption given by the invariance between the switching of post- and pre-collisional velocities for a pair of interacting particles. That is, \( \mathcal{B}_{ij} \) remains invariant under
Figure 1. Illustration of the collision transformation, with notation $V_{ij} := \frac{m_i v + m_j v^*}{m_i + m_j}$, $u := v - v_*$, $u' := v' - v'_*$. The displacement of the center of mass with respect to a single component elastic binary interaction is given by $(r_{ij} - \frac{1}{2})u = \frac{m_i - m_j}{2(m_i + m_j)}u$, if $m_i > m_j$. Solid lines denote vectors after collision, or given data. Dash-dotted vectors represent primed (pre-collisional) quantities that can be calculated from the given data, and compared to the case $m_i = m_j$, represented by dotted vectors. Dashed vector direction is the displacement along the direction of the relative velocity $u$ proportional to the half difference of relative masses, (which clearly vanishes for $m_i = m_j$, reducing the model to a classical collision). Note that the scattering direction $\sigma$ is preserved as the pre-collisional relative velocity $u'$ keeps the same magnitude as the post-collisional $u$, $u'$ is parallel the reference elastic pre-collisional relative velocity $|u|\sigma$. This figure is reproduced from [8].

The following exchange $(v, v_*, \sigma) \leftrightarrow (v', v'_*, \sigma')$, with $\sigma' = u/|u|$, and $(v, v_*, \sigma) \leftrightarrow (v_*, v, -\sigma)$, resulting in the following identity

$$B_{ij}(|u|, \hat{u} \cdot \sigma) = B_{ij}(|u'|, \hat{u}' \cdot \sigma') = B_{ji}(|u|, \hat{u} \cdot \sigma).$$

(9)
In our scope, we only deal with the hard potential and integrable angular transition probability case for each particle pair $A_i$ and $A_j$, $1 \leq i, j \leq I$. That means, the terms $B_{ij}$, $i, j = 1, \ldots, I$ are assumed to take the following form

$$B_{ij}(|u|, \hat{u} \cdot \sigma) = |u|^{\gamma_{ij}} b_{ij}(\hat{u} \cdot \sigma), \quad \gamma_{ij} \in (0, 1], \quad \text{and} \quad b_{ij}(\hat{u} \cdot \sigma) \in L^1(S^{N-1}),$$  \hspace{1cm} (10)

where $b_{ij}(\hat{u} \cdot \sigma)$ is the angular transition rate. We assume that each $b_{ij}$ has been symmetrized with respect to the polar angle $\theta$, and therefore its support lays in $[0, 1]$.

We note that this introductory part does not need the split in (10). The general form of cross section $B_{ij}$ with the symmetries from (9) is enough to develop the new Carleman representation, to be shown in next sections. However, existence and uniqueness theory is built in [8] for the cross section (10).

2.2.1. Bilinear form of collision operator in the vector form notation. Once we defined the pairwise collision operator $Q_{ij}(f, g)$, we can introduce the vector value bilinear form of multispecies collision operator $Q(F, G)$. Let $F$ and $G$ be vectors of distribution functions $F = [f_i]_{i=1}^I$ and $G = [g_i]_{i=1}^I$. Then the collision operator associated to the distribution functions $F$ and $G$ is defined through its $i$–th component by

$$[Q(F, G)]_i(v) := \sum_{j=1}^I Q_{ij}(f_i, g_j)(v), \quad i = 1, \ldots, I.$$  

Clearly, the Boltzmann operator (5) is obtained for $F = G$.

2.2.2. Gain and loss terms. When it is possible to separate the collision operator (6) into the sum of two operators (typical situation is the cut-off regime when the angular part of the cross section (10) is integrable), we are led to define the gain and the loss term. Namely, the first part of the collision operator (6) is called the gain term,

$$Q_{ij}^+(f, g)(v) = \int_{\mathbb{R}^N} \int_{S^{N-1}} f(v') g(v'_*) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_*,$$

while the second part is called the loss term,

$$Q_{ij}^-(f, g)(v) = f(v) \int_{\mathbb{R}^N} \int_{S^{N-1}} g(v'_*) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_*,$$

so that (6) can be rewritten as the difference of this two operators

$$Q_{ij}(f, g)(v) = Q_{ij}^+(f, g)(v) - Q_{ij}^-(f, g)(v).$$

Passing to the vector notation, we define the vector value gain $Q^+(F, G)$ and loss $Q^-(F, G)$ terms. Namely, with the pairwise operators $Q_{ij}^+$ and $Q_{ij}^-$ defined above, the gain term $Q^+(F, G)$ is defined as

$$[Q^+(F, G)]_i = \sum_{j=1}^I Q_{ij}^+(f_i, g_j) = \sum_{j=1}^I \int_{\mathbb{R}^N} \int_{S^{N-1}} f_i(v') g_j(v'_*) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_*,$$

while the loss term $Q^-(F, G)$ is defined with

$$[Q^-(F, G)]_i = \sum_{j=1}^I Q_{ij}^-(f_i, g_j) = f_i(v) \sum_{j=1}^I \int_{\mathbb{R}^N} \int_{S^{N-1}} g_j(v'_*) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_*,$$
with \( u = v - v_\ast, \hat{u} = u/|u| \).

2.3. Functional spaces and integral representations for binary interactions of mixtures of monatomic gases. We will study the gain operator and introduce some essential concepts and notation for later studies.

2.3.1. Functional spaces. We are working in Banach spaces associated to the mixture, as well as to its components. Therefore we need to define the associated vector valued \( L^p \) weighted spaces (\( L^p_\beta \)), where \( \beta \) is either a polynomial weight of order \( k \) (where we will denote \( \beta = k \)), i.e. \( \beta \equiv \langle v \rangle^k_i \), or an exponential weight with rate \( \alpha > 0 \) and order \( s \), for \( 0 < s \leq 1 \), i.e. \( \beta \equiv \exp(\alpha \langle v \rangle^s_i) \), and their respective norms, where both weights depend on a renormalized mass \( m_i \) for each specie mass density \( f_i \) with \( 1 \leq i \leq I \).

For the case where \( \beta \) is a polynomial weight of order \( k \), the notation is drawn from the previous related work defined for the \( L^1_k \) vector value functional space case for the system of Boltzmann equation for mixtures [8], that we extend here to \( 1 < p \leq \infty \). We point out that in the case of \( L^1_k \)-spaces these polynomial weighted norms depend on time and can be viewed describing the time evolution of observables (or \( L^1_k \)-moments) associated to the vector valued probability densities. In addition, such evolution of norms was crucial to obtain a set of ordinary differential inequalities that enabled to show that \( L^1_k \)-moments of each species were generated and propagated uniformly in time depending on the initial data, as much as to show these \( L^1_k \)-moments were summable in \( k \), for \( k > k^* \), with \( k^* \) a constant given in (72) depending on \( b_{ij} \) and \( r_{ij} \), to obtain uniform in time estimates of exponential moments.

Such time dependent norms are obtained from the weak formulation associated to the system, which will be defined as follows.

Following the introduction of \( L^1_k \) norms in [8], we recall their definition of the Lebesgue weight \( \langle v \rangle_i := \sqrt{1 + \frac{m_i}{\sum_{j=1}^I m_j} |v|^2} \). We remark that the renormalization of the weight is a sufficient condition to obtain the energy identity [8, Lemma 4], which is essential to obtain Povzner estimates and propagation of \( L^1_k \) norms.

The associated norm is then

\[
\|F\|_{L^p_k} := \left( \sum_{i=1}^I \int_{\mathbb{R}^N} \left( |f_i(v)| \langle v \rangle^k_i \right)^p \, dv \right)^{1/p}.
\]  

(13)

For \( p = \infty \), we define

\[
L^\infty_k := \left\{ F = [f_i]_{1 \leq i \leq I} : \sum_{i=1}^I \sup_{v \in \mathbb{R}^N} \left( |f_i(v)| \langle v \rangle^k_i \right) < \infty, \, k \geq 0 \right\}.
\]
with its associated norm
\[ \| \mathcal{F} \|_{L_k^\infty} := \sum_{i=1}^I \text{ess sup}_{v \in \mathbb{R}^N} \left( |f_i(v)| \langle v \rangle_i^k \right). \]

We will also work in the components framework. We define the space of each mixture component as
\[ L^p_{k,i} := \left\{ g : \int_{\mathbb{R}^N} \left( |g(v)| \langle v \rangle_i^k \right)^p dv < \infty, k \geq 0, 1 \leq p < \infty \right\}, \]

with its norm
\[ \| g \|_{L^p_{k,i}} := \left( \int_{\mathbb{R}^N} \left( |g(v)| \langle v \rangle_i^k \right)^p dv \right)^{1/p}. \]

When \( p = \infty \) the space related to the specie \( A_i \) is
\[ L^\infty_{k,i} := \left\{ g : \text{ess sup}_{v \in \mathbb{R}^N} \left( |g(v)| \langle v \rangle_i^k \right) < \infty, k \geq 0 \right\}, \]

with its norm
\[ \| g \|_{L^\infty_{k,i}} := \text{ess sup}_{v \in \mathbb{R}^N} \left( |g(v)| \langle v \rangle_i^k \right). \]

Note that the norm of \( \mathcal{F} \) in \( L^p_k \) is related to the norm of its components \( f_i \) in the space \( L^p_{k,i} \) via
\[ \| \mathcal{F} \|_{L^p_k} = \sum_{i=1}^I \| f_i \|_{L^p_{k,i}}, \quad \| \mathcal{F} \|_{L^\infty_k} = \sum_{i=1}^I \| f_i \|_{L^\infty_{k,i}}. \]

In the special case when \( k = 0 \), we introduce the following notation for the norm
\[ \| g \|_p := \| g \|_{L^p_{0,i}}, \quad 1 \leq p \leq \infty, \]

for any \( i = 1, \ldots, I \).

Now, when \( \beta \) is an exponential weight with rate \( \alpha > 0 \) and order \( s \), for \( 0 < s \leq 1 \) we define the norm as
\[ \left\| \mathcal{F} e^{\alpha(\cdot)} \right\|_{L^p_0} := \left( \sum_{i=1}^I \left\| f_i(t, \cdot) e^{\alpha(\cdot)} \right\|_{L^p_0}^p \right)^{1/p}. \] (14)

2.3.2. Weak form of the gain term. Weak formulation of the pairwise collision operator plays a central role in all further calculations. In this Section, we define it for the gain part of the collision operator.

We integrate the gain operator (11) over the velocity space against some suitable test function \( \psi(v) \). After performing the change of variables \((v, v_*, \sigma) \leftrightarrow (v', v'_*, \sigma')\), with \( v, v' \) given in (3) and \( \sigma' = |u| \hat{u} \cdot \sigma \), we obtain the following weak form of the pairwise collision operator \( Q_{ij} \) that corresponds to an interaction of species \( A_i \) with the species \( A_j \)
\[ \int_{\mathbb{R}^N} Q_{ij}^+(f,g)(v) \psi(v) dv = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} f(v) g(v_*) B_{ij}(|u|, \hat{u} \cdot \sigma) \psi(v') d\sigma dv_* dv. \] (15)
Moreover, by changing \((v, v_*, \sigma) \leftrightarrow (v_*, v, -\sigma)\) in the last integral we get another representation

\[
\int_{\mathbb{R}^N} Q^+_i(f, g)(v) \psi(v) \, dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} f(v_*) \, g(v) \, B_{ij}(|u|, \hat{u} \cdot \sigma) \, \psi(u_*)^* \, d\sigma \, dv_*, \quad (16)
\]

where \(u_*^*\) is

\[
u_*^* = v_* - (1 - r_{ij})(|u| \sigma - u),
\]
as introduced in \((8)\).

A different weak form representation for the gain term, associated to binary interactions via angular integration. Following the representation introduced in \([2]\), we need to write the integration on the sphere of any test function for a binary interaction of two particles with different masses as follows. Let \(\varphi\) and \(\chi\) be bounded and continuous functions. We define the collision weight angular integral operator acting on the test functions \(\varphi\) and \(\chi\),

\[
\mathcal{P}_{ij}(\varphi, \chi)(u) := \int_{\mathbb{S}^{N-1}} \varphi(u_{ij}^-) \chi(u_{ij}^+) \, b_{ij}(\hat{u} \cdot \sigma) \, d\sigma,
\]

where \(u^+\) and \(u^-\) are defined by

\[
u_{ij}^- := (1 - r_{ij})(u - |u|\sigma) \quad \text{and} \quad u_{ij}^+ := u - u_{ij}^- + r_{ij}u + (1 - r_{ij})|u|\sigma.
\]

Moreover, let \(\tau\) and \(\mathcal{R}\) denote the translation and reflection operators,

\[
\tau_v \psi(x) := \psi(x - v) \quad \text{and} \quad \mathcal{R} \psi(x) := \psi(-x), \quad v, x \in \mathbb{R}^N.
\]

Then the weak formulation \((15)\) of the gain part of the collision operator for the cross section \((10)\) can be rewritten as

\[
\int_{\mathbb{R}^N} Q^+_i(f, g)(v) \psi(v) \, dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(v)g(v - u)|u|^\gamma \, \mathcal{P}_{ij}(\tau_v(\mathcal{R} \psi), 1)(u) \, du \, dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(v - u)g(v)|u|^\gamma \, \mathcal{P}_{ij}(1, \tau_v(\mathcal{R} \psi))(u) \, du \, dv. \quad (18)
\]

2.3.3. Carleman representation. The Carleman integral representation in the study of the Boltzmann equation for elastic interactions \([7]\, \text{Appendix}\) plays an important role in the analysis as much as in approximation theory to the studies of the initial value problem associated to this model. It provides a strong form alternate representation of the gain operator where the angular integration is performed in the orthogonal direction to the one corresponding to the difference of a molecular velocity \(v\) and its post collisional one \(v'\). Such framework has been useful, both, in the study of propagation of \(L^\infty\)-estimates of classical solutions for elastic interactions with hard potentials and integrable cross sections \([7]\, \text{or for hard potentials with non-integrable cross sections in} \quad [9]\, \text{. It was also used in the gain of integrability properties for the elastic Boltzmann equation for hard potentials}\quad \text{[1, 3]} \quad \text{to obtain explicit estimates in}\quad L^p, \quad \text{that are sharp in some cases.}

In the case of a system of Boltzmann equations for binary interactions for mixture of gases, the analogue to the Carleman representation, has only been recently addressed in \([6]\) where some constant parameters are undetermined.
We present here a new formulation for the Carleman representation for the strong collisional form associated to the gain operator for a binary interaction, that is compatible with the Banach spaces and \(L^1\) norms that allowed us to construct vector valued solution in [8]. All parameters in our new representation are determined by functions of the corresponding two different masses in the interaction.

**Theorem 2.1** (Carleman representation of the gain term). Let

\[
P_{r_{ij}}(v, v') = \frac{(v - (2r_{ij} - 1)v')}{2(1 - r_{ij})}, \quad r_{ij} \in (0, 1).
\]

Denote with \(E_{v-v'}\) the hyperplane orthogonal to the vector \(v - v'\), that is

\[
E_{v-v'} = \{ y \in \mathbb{R}^N : (v - v') \cdot y = 0 \} \subset \mathbb{R}^{N-1}.
\]

Then the gain operator \(G_{v-v'}\) follows as follows

\[
Q^+_{ij}(f, g)(v) = (1 - r_{ij})^{-N+1} \int_{x \in \mathbb{R}^N} \frac{f(x)}{|x - v|} \int_{z \in E_{v-x}} g(z + P_{ij}(v, x))
\]

\[
\times \left| \frac{(v - x)}{2(1 - r_{ij})} + z \right|^{2-N} \mathcal{B}_{ij} \left( \left| \frac{(v - x)}{2(1 - r_{ij})} + z \right|, 1 - \frac{|x - v|^2}{2(1 - r_{ij})^2} \right) d\mu.
\]

The proof of this Theorem is given in Appendix A.

**2.3.4. Kernel form.** By means of the Carleman representation (20), the operator \(Q^+(F, G)\) can be written in a kernel form as follows.

**Lemma 2.2.** (Kernel form of the gain operator) Let \(F = [f_i]_{1 \leq i \leq I}\) and \(G = [g_i]_{1 \leq i \leq I}\), where \(f_i(v) \geq 0\) and \(g_i(v) \geq 0\) for all \(v \in \mathbb{R}^N\) and all \(1 \leq i \leq I\). Then the gain operator \(Q^+(F, G) (12)\) can be written in the following kernel form

\[
\left[Q^+(F, G)\right]_i(v) = \int_{\mathbb{R}^N} f_i(x) K_i[G](v, x) dx,
\]

where the kernel is

\[
K_i[G](v, x) = \sum_{j=1}^I \tau_x Q^+_{ij}(\delta_0, \tau_{-x}g_j)(v),
\]

with the translation operator \(\tau_w\) defined by

\[
\tau_w g(v) = g(v - w), \quad \text{for any } v, w \in \mathbb{R}^N.
\]

**Proof.** The proof of this lemma uses the Carleman representation for the gain operator for a binary interaction in species mixture (20), proven in Appendix A. Indeed, it follows

\[
Q^+_{ij}(\delta_0, g)(v) = (1 - r_{ij})^{-N+1} \int_{v \in \{v\}} \frac{1}{|v|} g \left( \frac{v}{2(1 - r_{ij})} + z \right) \left| \frac{v}{2(1 - r_{ij})} + z \right|^{2-N}
\]

\[
\times \mathcal{B}_{ij} \left( \left| \frac{v}{2(1 - r_{ij})} + z \right|, 1 - \frac{|v|^2}{2(1 - r_{ij})^2} \right) d\mu.
\]

Then the kernel representation is obtain by using the translation operator. \(\square\)
2.4. Propagation of polynomial and exponential weighted $L^p$ norms. We start by recalling that in [8] the authors proved existence and uniqueness of the vector value solution of the Cauchy problem for the homogeneous Boltzmann system of equations and generation and propagation of polynomial and exponential weighted $L^1$ norms by means of an existence theorem for ODE systems in suitable Banach spaces. We stress that norms are generated in $L^1_k$, $k \geq 0$ (observables) because of the use of Jensen’s inequality for probability. In the present manuscript, we cannot have an analogue of such an inequality for $L^p$, $p > 1$; therefore, we will prove the propagation of polynomial and exponential weighted $L^p$ norms. In order to do so, we need to obtain a lower bound for the negative contribution of the loss term and upper bounds for the gain part of the collision operator that produces a gain of integrability, by meaning that $L^p_k$ norm of the positive part of the collision operator is controlled sublinearly by the $L^p_k$ norms of the input functions.

**Theorem 2.3.** (Propagation of polynomially weighted $L^p$ norms.) If $\mathbb{F}$ is a solution of the Boltzmann system

\[
\begin{align*}
\partial_t \mathbb{F}(t, v) &= Q(\mathbb{F}, \mathbb{F})(t, v), \quad t > 0, \ v \in \mathbb{R}^N, \\
\mathbb{F}(0, v) &= \mathbb{F}_0(v),
\end{align*}
\]

with the cross section (10) where $b_{ij} \in L^1(\mathbb{S}^{n-1})$, an initial data $\mathbb{F}_0 := \mathbb{F}(0, \cdot) \in \Omega$, with $\Omega$ defined in (86), and $\|\mathbb{F}_0\|_{L^p_k} < \infty$ for $k > k^*$, with $k^*$ as given in (72), then there is a constant $D_k = \left( \frac{B_k}{A_k} \right)^{\frac{1}{k}}$, where $B_k$ and $A_k$ are defined as

\[
A_k = \min_{1 \leq i, j \leq I} \frac{\|b_{ij}\|_{L^1(\mathbb{S}^{n-1})}}{\max_{1 \leq i \leq I} m_i} - e^{-\gamma} I^{2^+ - \frac{2}{p} + \frac{2\gamma}{p}} \|\mathbb{F}\|_{L^1_k} C^Q_{p, 1, p} \\
&\quad - \left( \frac{\sum_{k=1}^I m_k}{\min_{1 \leq i \leq I} m_i} \right)^{\frac{1}{\gamma}} \|\mathbb{F}\|_{L^1_k}^{\frac{1}{\gamma}},
\]

\[
B_k = I^{2 - \frac{2}{p} + \frac{2\gamma}{p}} e^{2\gamma - N} \hat{C}_N \|\mathbb{F}\|_{L^1_k} \|\mathbb{F}\|_{L^1_k}^{2 - \theta} \|\mathbb{F}\|_{L^1_k}^{2 + \theta} \|\mathbb{F}\|_{L^1_k},
\]

such that

\[
\|\mathbb{F}\|_{L^1_k}^p \leq \max\{D_k, \|\mathbb{F}_0\|_{L^p_k}^p\}.
\]

We can also state the theorem for propagation of $L^p$ norms with exponential weights, for $p \in (1, \infty)$. The idea behind the proof will follow from what is proven for polynomial weights.

**Theorem 2.4.** (Propagation of exponentially weighted $L^p$ norms.) Let $\mathbb{F}$ be the solution of the Boltzmann system (21) with the cross section (10) where $b_{ij} \in L^1(\mathbb{S}^{n-1})$ and let

\[
\bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij} \in (0, 1).
\]

Assume that the initial data $\mathbb{F}_0 := \mathbb{F}(0, \cdot)$ satisfies the assumptions of Theorem 2.3. If additionally

\[
\|\mathbb{F}_0 e^{\gamma_{ij}v} \|_{(L^1_{c_{\gamma}n} \cap L^p_{c_{\gamma}})} = C_0 < \infty,
\]


for some $s \in (0, 1]$, $p \in (1, \infty)$ and positive constants $\alpha_0$ and $C_0^\gamma$, then there exist a positive constant $\alpha$ such that

$$
\| F e^{\alpha(x,y)} \|_{L^p_0} \leq \max \left\{ \left( \frac{\hat{B}_0}{\hat{A}_0} \right)^{-\frac{\gamma}{p^*}}, \| F_0 e^{\alpha(x,y)} \|_{L^p_0} \right\}, \quad t \geq 0.
$$

with $\hat{B}_0$ and $\hat{A}_0$ given as

$$
\hat{A}_0 = \min_{1 \leq i,j \leq l} \| b_{ij} \|_{L^1(\mathbb{S}^{N-1}, d\sigma)} \frac{1}{\max_{1 \leq i,j \leq l} m_{ij}} C_{ib} - \varepsilon^\gamma l^2 - \frac{1}{2} \sum_{i,j} \mathcal{C}_{ij} \| e^{\gamma(x,y)} \|_{L^p_0}^\gamma,
$$

$$
\hat{B}_0 = l^2 - \frac{1}{2} \sum_{i,j} \mathcal{C}_{ij} \| e^{\gamma(x,y)} \|_{L^p_0}^\gamma.
$$

and $\gamma$ as given in (68).

Moreover, we can extend this results for the case $p = \infty$.

**Theorem 2.5.** (Propagation of polynomially weighted $L^\infty$ norms.) Let $\gamma_{ij} \in (0, 1]$, $b_{ij} \in L^1(\mathbb{S}^{N-1})$, and an initial data satisfying the hypothesis of Theorem 2.3 and such that

$$
\| F_0 \|_{L^\infty} = C_0,
$$

for $k > k^*$ with $k^*$ as given in (72), and for some positive constant $C_0$. Then there exists a constant $C(F_0)$ depending on $\bar{\gamma}$, $m_{ij}$, $b_{ij}$, $k$ such that

$$
\| F(t, \cdot) \|_{L^\infty} \leq C(F_0), \quad t \geq 0,
$$

for $F$ the solution of the Boltzmann system (21), with $\bar{\gamma}$ defined as in (24).

**Theorem 2.6.** (Propagation of exponentially weighted $L^\infty$ norms.) Let $F$ be the solution of the Boltzmann system (21) with the cross section (10) where $b_{ij} \in L^1(\mathbb{S}^{N-1})$ and let $\bar{\gamma}$ defined as in (31). Assume that the initial data $F_0 := F(0, \cdot)$ satisfies the assumptions of Theorem 2.3 and that

$$
\| F_0 \|_{L^\infty} = C_0.
$$

If additionally,

$$
\| F_0 e^{\alpha(x,y)} \|_{(L^\infty_0 \cap L^p_0)} = C_0^\gamma < \infty,
$$

for some $s \in (0, 1]$, $p \in (1, \infty)$ and positive constants $\alpha_0$ and $C_0^\gamma$, then, there exist positive constants $\alpha$ such that

$$
\| F e^{\alpha(x,y)} \|_{L^\infty} \leq \max \left\{ \left( \frac{\hat{B}_0}{\hat{A}_0} \right)^{-\frac{\gamma}{p^*}}, \| F_0 e^{\alpha(x,y)} \|_{L^\infty} \right\}, \quad t \geq 0.
$$

with $\hat{B}_0$ and $\hat{A}_0$ as given in (26).

### 3. Estimate of the gain operator in $L^2$ framework

In this section we provide $L^2$ estimates for the gain term of the collision operator $Q^+(F, G)$ assuming that the angular part $b_{ij}(\cdot, \cdot, \sigma) \in L^\infty(\mathbb{S}^{N-1})$. First we work in a space without weight, and then we add polynomial weight, following the strategy developed in [3] for the single species case.
3.1. Estimate of the gain operator in a plain $L^2$ space. As done in [3] for the single specie case, we will start by stating and proving the $L^2_0$ estimate for the gain part of the collision operator.

**Proposition 3.1.** Let $N \geq 3$ and let $F$ and $G$ be distribution functions, such that $F \in L^1_0$ and $G \in L^{1-\theta}_{1,N}$ and $L^2_0$. We will consider the transition probability terms (10) with the angular part satisfying additionally that

$$b_{ij}(\hat{u} \cdot \sigma) \in L^\infty(S^{N-1}) \text{ for any } 1 \leq i, j \leq I.$$  

(29)

Then the following estimate holds

$$\|Q^+(F, G)\|_{L^2_0} \leq \left( \frac{\sqrt{2I} k_0 \varepsilon^\gamma \|G\|_{L^2_0} \sqrt{2I} k_N \varepsilon^{2-N+2\theta} \|G\|_{L^1_0} \|G\|_{L^2_0}}{\sqrt{2I} k_N \varepsilon^{1-\theta} \|G\|_{L^1_0}} \right) \|F\|_{L^1_0},$$  

(30)

with $\theta = 1/N$, $\varepsilon > 0$, $\bar{\gamma}$ as in (24), and

$$\bar{\gamma} = \min_{1 \leq i, j \leq I} \gamma_{ij},$$  

(31)

and the constants $k_0$ and $k_N$ are from Lemmas 3.2 and 3.1 respectively.

**Proof.** Once the operator $Q^+(F, G)$ is written in a kernel form, from Minkowski’s integral inequality we get an estimate

$$\|Q^+(F, G)\|_{L^2_0} \leq \left( \frac{\sqrt{2I} k_0 \varepsilon^\gamma \|G\|_{L^2_0} \sqrt{2I} k_N \varepsilon^{2-N+2\theta} \|G\|_{L^1_0} \|G\|_{L^2_0}}{\sqrt{2I} k_N \varepsilon^{1-\theta} \|G\|_{L^1_0}} \right) \|F\|_{L^1_0},$$  

(30)

The following step is to estimate

$$\int_{v \in \mathbb{R}^N} K_i[G](v, x)^2 dv$$  

for the specific choice of the cross section (10).  

(32)

Whenever explicitly needed, we will use an extra subindex to clarify the cross section we use, i.e.

$$K_{\gamma_{ij},i}[G](v, x) := K_i[G](v, x), \text{ and } Q^+_{\gamma_{ij},ij}(f, g)(v) := Q^+_{ij}(f, g)(v) \text{ with } B_{ij} \text{ from (10)}.$$  

Note that notation $K_{\gamma_{ij},i}[G](v, x)$ is ambiguous, since the kernel $K_i$ does not depend on $j$. Then, since

$$|u|^{\gamma_{ij}} \leq \varepsilon^{\gamma_{ij}} 1_{|u| \leq \varepsilon} + |u|^{\gamma_{ij}} 1_{|u| > \varepsilon},$$
(32) becomes

\[
\int_{v \in \mathbb{R}^N} K_{\gamma, i}[G](v, x)^2 \, dv \leq 2 \varepsilon^2 \int_{v \in \mathbb{R}^N} K_{0, i}[G](v, x)^2 \, dv \\
+ 2 \varepsilon^{2(2-N+\gamma)} \int_{v \in \mathbb{R}^N} K_{N-2, i}[G](v, x)^2 \, dv \\
\leq 2 I \varepsilon^2 \gamma \sum_{j=1}^I \int_{v \in \mathbb{R}^N} (\tau x, Q_{0, i j} (\delta_0, \tau x, g_j)(v))^2 \, dv \\
+ 2 I \varepsilon^{2(2-N+\gamma)} \sum_{j=1}^I \int_{v \in \mathbb{R}^N} (\tau x, Q_{N-2, i j} (\delta_0, \tau x, g_j)(v))^2 \, dv,
\]

since \( 2 - N + \gamma_{ij} \leq 0 \) for \( N \geq 3 \). The final estimate (30) follows from the following two Lemmas. \( \Box \)

**Lemma 3.1.** Let \( N \geq 3 \) and denote \( G = [g_j]_{1 \leq j \leq I} \), with \( g_j(v) \geq 0 \) for all \( v \in \mathbb{R}^N \) and all \( 1 \leq j \leq I \). Assume that the cross section takes the form (10) with the angular part satisfying the extra assumption (29). Then the following estimate holds

\[
\sum_{j=1}^I \int_{v \in \mathbb{R}^N} Q_{N-2, i j}^+(\delta_0, g_j)(v)^2 \, dv \leq k_N \| G \|_{L_{\gamma}^1}^{2(1-\theta)} \| G \|_{L_{\gamma}^1}^{2\theta}, \tag{33}
\]

with \( \theta = \frac{1}{N} \),

\[
k_N = I^{1-\theta} \tilde{C}_N 2^{N-2} |S^{N-2}| \max_{1 \leq i, j \leq I} \left( \| b_{ij} \|_{L_{\infty}(S^{N-1})}^2 (1 - r_{ij})^{-N} \right)^{N-3} \left( \sqrt{\sum_{i=1}^I m_i} \right)^{N-3},
\]

\[
\tilde{C}_N = \pi^{1/2} \frac{\Gamma(N/2 - 1/2)}{\Gamma(N - 1/2)} \left( \frac{\Gamma(N/2)}{\Gamma(N)} \right)^{-1+1/N},
\]

and \( r_{ij} \) as in (2).

**Proof.** Since the angular part \( b_{ij} \) of the cross section is assumed bounded, we can write

\[
\int_{v \in \mathbb{R}^N} Q_{N-2, i j}^+(\delta_0, g_j)(v)^2 \, dv \\
\leq \| b_{ij} \|_{L_{\infty}(S^{N-1})}^2 (1 - r_{ij})^{2(1-N)} \int_{v \in \mathbb{R}^N} \frac{1}{|v|^2} \left( \int_{z \in \{v\}} g_j \left( \frac{1}{2(1 - r_{ij})} v + z \right) \, dz \right)^2 \, dv.
\]
For $v \in \mathbb{R}^N$ we pass to its spherical coordinates $r \omega$, with $r = |v| \in \mathbb{R}$ and $\omega \in S^{N-1}$, and then change $r \mapsto s = \frac{1}{2(1-r_{ij})} r$, so that the integral becomes

$$
\int_{v \in \mathbb{R}^N} Q^+_{N-2,i,j}(\delta_0, g_j)(v)^2 dv \\
\leq \|b_{ij}\|_{L^\infty(S^{N-1})}^2 \frac{2^N - 2}{N-2} (1 - r_{ij})^{-N} \int_{v \in \mathbb{R}^N} \int_{\omega \in S^{N-1}} (y \cdot \omega)^{N-3} g_j(y) \\
\times \left( \int_{\omega \in S^{N-1}} g_j((y \cdot \omega + z_2) d\omega \right) dy d\omega.
$$

For fixed $\omega$, we combine integration with respect to $z_1 \in \{\omega\}^\perp$ and in the direction of $\omega$ with magnitude $s$ to form an integration in $\mathbb{R}^N$ with respect to the new variable $y = z_1 + s\omega$. Calculating $y \cdot \omega = s$, we have

$$
\int_{v \in \mathbb{R}^N} Q^+_{N-2,i,j}(\delta_0, g_j)(v)^2 dv \\
\leq \|b_{ij}\|_{L^\infty(S^{N-1})}^2 \frac{2^N - 2}{N-2} (1 - r_{ij})^{-N} \int_{v \in \mathbb{R}^N} \int_{\omega \in S^{N-1}} (y \cdot \omega)^{N-3} g_j(y) \\
\times \left( \int_{z_2 \in \{\omega\}^\perp} g_j((y + z_2 d\omega \right) dy d\omega.
$$

Moreover, in the last integral, we change $z_2 \mapsto z = z_2 + (y \cdot \omega)\omega - y$, and noting that $z$ still belongs to the same space as $z_2$ because $z \cdot \omega = 0$, we have

$$
\int_{v \in \mathbb{R}^N} Q^+_{N-2,i,j}(\delta_0, g_j)(v)^2 dv \\
\leq \|b_{ij}\|_{L^\infty(S^{N-1})}^2 \frac{2^N - 2}{N-2} (1 - r_{ij})^{-N} \int_{v \in \mathbb{R}^N} \int_{\omega \in S^{N-1}} (y \cdot \omega)^{N-3} g_j(y) \\
\times \left( \int_{z \in \{\omega\}^\perp} g_j(y + z d\omega \right) dy d\omega.
$$

Then bounding the term $y \cdot \omega \leq |y|$, for $N \geq 3$, and using the representation with the Dirac delta function, we have

$$
\int_{v \in \mathbb{R}^N} Q^+_{N-2,i,j}(\delta_0, g_j)(v)^2 dv \leq \|b_{ij}\|_{L^\infty(S^{N-1})}^2 \frac{2^N - 2}{N-2} (1 - r_{ij})^{-N} \\
\times \int_{\omega \in S^{N-1}} \int_{y \in \mathbb{R}^N} \int_{z \in \mathbb{R}^N} |y|^{N-3} g_j(y) g_j(y + z) \delta_0(\omega \cdot z) dz dy d\omega.
$$

Calculating

$$
\int_{\omega \in S^{N-1}} \delta_0(\omega \cdot z) d\omega = \frac{|S^{N-2}|}{|z|},
$$

we get

$$
\int_{v \in \mathbb{R}^N} Q^+_{N-2,i,j}(\delta_0, g_j)(v)^2 dv \\
\leq \|b_{ij}\|_{L^\infty(S^{N-1})}^2 \frac{2^N - 2}{N-2} (1 - r_{ij})^{-N} |S^{N-2}| \int_{y \in \mathbb{R}^N} \int_{z \in \mathbb{R}^N} |y|^{N-3} g_j(y) g_j(y + z) dz dy.
$$
We can bound \( |y| \) in terms of its \( j \)-th Lebesgue weight,

\[
|y| \leq (y)_j \left( \sqrt{\sum_{i=1}^{l} \frac{m_i}{m_j}} \right),
\]

(34)

and then denote

\[
\tilde{g}_j(y) = g_j(y) (y)_j^{N-3},
\]

so that the integral becomes, after translation in \( z \) variable

\[
\int_{v \in \mathbb{R}^N} Q_{N-2,ij}^{+}(\delta_0, g_j)(v)^2 dv \leq C_{N,ij} \int_{y \in \mathbb{R}^N} \int_{z \in \mathbb{R}^N} \frac{\tilde{g}_j(y) g_j(z)}{|z-y|} dz dy,
\]

with

\[
C_{N,ij} = \|b_{ij}\|_{L^\infty(S^{N-1})}^2 2^{N-2} |\mathbb{S}^{N-2}| (1 - r_{ij})^{-N} \left( \sqrt{\sum_{i=1}^{l} \frac{m_i}{m_j}} \right)^{N-3}.
\]

(35)

Applying Hardy-Littlewood-Sobolev inequality, we obtain

\[
\int_{y \in \mathbb{R}^N} \int_{z \in \mathbb{R}^N} \frac{\tilde{g}_j(y) g_j(z)}{|z-y|} dz dy
\]

\[
\leq \tilde{C}_N \left( \int_{y \in \mathbb{R}^N} \tilde{g}_j(y) \frac{2^{N}}{2^N - 1} dy \int_{z \in \mathbb{R}^N} g_j(z) \frac{2^{N}}{2^N - 1} dz \right)^{\frac{2^N - 1}{2^N}}
\]

\[
\leq \tilde{C}_N \left( \int_{y \in \mathbb{R}^N} \tilde{g}_j(y) \frac{2^{N}}{2^N - 1} dy \right)^{\frac{2^N - 1}{2^N}},
\]

with the exact constant, as proved in [10, Theorem 4.3], given by

\[
\tilde{C}_N = \pi^{1/2} \frac{\Gamma(N/2 - 1/2)}{\Gamma(N - 1/2)} \left( \frac{\Gamma(N/2)}{\Gamma(N)} \right)^{-1+1/N}.
\]

(36)

Then using log-convexity of \( L^p \) norms,

\[
\int_{y \in \mathbb{R}^N} \left( g_j(y) \langle v \rangle_j^{N-3} \right) \frac{2^N}{2^N - 1} dy
\]

\[
\leq \left( \int_{y \in \mathbb{R}^N} g_j(y) \langle v \rangle_j^{N-3} dy \right)^{\frac{2^N (1 - \theta)}{2^N - 1}} \left( \int_{x \in \mathbb{R}^N} g_j(x)^2 dx \right)^{\frac{N^2}{2^N}}, \quad \theta := \frac{1}{N}.
\]

Therefore,

\[
\int_{v \in \mathbb{R}^N} Q_{N-2,ij}^{+}(\delta_0, g_j)(v)^2 dv
\]

\[
\leq C_{N,ij} \tilde{C}_N \left( \int_{y \in \mathbb{R}^N} g_j(y) \langle v \rangle_j^{N-3} dy \right)^{2(1-\theta)} \left( \int_{x \in \mathbb{R}^N} g_j(x)^2 dx \right)^{\theta}, \quad \theta = \frac{1}{N}.
\]
In order to represent this estimate in norm notation, we sum the last inequality over \( j = 1, \ldots, I \), obtaining

\[
\sum_{j=1}^{I} \int_{v \in \mathbb{R}^N} Q_{N-2,i,j}^+(\delta_0, g_j)(v)^2 \, dv \leq I^{1-\theta} \hat{C}_N \max_{1 \leq i,j \leq I} (C_{N,ij})
\]

\[
\times \left( \sum_{j=1}^{I} \int_{y \in \mathbb{R}^N} g_j(y) \frac{\chi^{-2}}{\|y\|^{N-2}} \, dy \right)^{2(1-\theta)} \left( \sum_{j=1}^{I} \int_{x \in \mathbb{R}^N} g_j(x)^2 \, dx \right)^\theta,
\]

since \( 2(1-\theta) \geq 1 \) and \( \theta < 1 \) for \( \theta = \frac{1}{N} \) and \( N \geq 2 \). Using the norm notation (13), we finally get estimate (33). \( \square \)

**Lemma 3.2.** Assume \( N \geq 3 \) and the cross section in the form (10) with the angular part satisfying (29). Let \( G = [g_j]_{1 \leq j \leq I} \), with \( g_j(v) \geq 0 \) for all \( v \in \mathbb{R}^N \) and all \( 1 \leq j \leq I \). Then the following estimate holds

\[
\sum_{j=1}^{I} \int_{v \in \mathbb{R}^N} Q_{0,ij}^+(\delta_0, g_j)(v)^2 \, dv \leq k_0 \|G\|_{L_2^2},
\]

with a constant \( k_0 = \max_{1 \leq i,j \leq I} \left( \|b_{ij}\|_{L^\infty(S^{N-1})} \left( C_{ij}^N \right)^2 \right) \) and

\[
C_{ij}^N = \left| S^{N-2} \right| \int_{-1}^{1} \left( \sqrt{2}(1 - r_{ij})(1 + \mu) \right)^{-N/2} \left( 1 - \mu^2 \right)^{N-3} \, d\mu.
\]

**Proof.** The proof starts with the weak form of the gain term (16), after a rotation \( \sigma \mapsto -\sigma \),

\[
\int_{\mathbb{R}^N} Q_{0,ij}^+(\delta_0, g_j)(v) \psi(v) \, dv
\]

\[
\leq \|b_{ij}\|_{L^\infty(S^{N-1})} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} \delta_0(v_*) \, g_j(v_*) \psi(v - (1 - r_{ij})(|u| \sigma - u)) \, d\sigma \, dv \, du
\]

\[
= \|b_{ij}\|_{L^\infty(S^{N-1})} \int_{\mathbb{R}^N} g_j(v) \int_{S^{N-1}} \psi((1 - r_{ij})(|v| \sigma + v)) \, d\sigma \, dv.
\]

Applying Hölder inequality,

\[
\int_{\mathbb{R}^N} Q_{0,ij}^+(\delta_0, g_j)(v) \psi(v) \, dv \leq \|b_{ij}\|_{L^\infty(S^{N-1})} \left( \int_{\mathbb{R}^N} g_j(v)^2 \, dv \right)^{1/2}
\]

\[
\times \left( \int_{\mathbb{R}^N} \left( \int_{S^{N-1}} \psi((1 - r_{ij})(|v| \sigma + v)) \, d\sigma \right)^2 \, dv \right)^{1/2}.
\]

Assuming \( \psi \) is a radial function, that is

\[
\psi((1 - r_{ij})(|v| \sigma + v)) = \psi \left( \sqrt{2}(1 - r_{ij}) |v| (1 + \sigma \cdot \hat{v}) \right),
\]

the integration with respect to \( \sigma \) can be simplified

\[
\int_{S^{N-1}} \psi((1 - r_{ij})(|v| \sigma + v)) \, d\sigma
\]

\[
= \left| S^{N-2} \right| \int_{-1}^{1} \psi \left( \sqrt{2}(1 - r_{ij}) |v| (1 + \mu) \right) \left( 1 - \mu^2 \right)^{N-3} \, d\mu.
\]
Then, by the Minkowski inequality,

\[ |S^N| \left( \int_{\mathbb{R}^N} \psi \left( \frac{1}{\sqrt{2(1 - r_{ij})}} |v| (1 + \mu) \left( 1 - \mu^2 \right)^{N-3} d\mu \right)^2 dv \right)^{1/2} \]

\[ \leq |S^N| \left( \int_{\mathbb{R}^N} \psi \left( \frac{1}{\sqrt{2(1 - r_{ij})}} |v| (1 + \mu) \left( 1 - \mu^2 \right)^{N-3} d\mu \right)^2 dv \right)^{1/2} \]

\[ = C_{ij}^N \left( \int_{\mathbb{R}^N} \psi(|v|^2) dv \right)^{1/2} \]

with

\[ C_{ij}^N = |S^N| \int_{-1}^{1} \left( \frac{1}{\sqrt{2(1 - r_{ij})}} (1 + \mu) \left( 1 - \mu^2 \right)^{N-3} d\mu \right). \]

Therefore, (37) becomes

\[ \int_{\mathbb{R}^N} Q_{0,ij}^+ (\delta_0, g_j)(v) \psi(v) dv \]

\[ \leq \|b_{ij}\|_{L^\infty(S^N)} C_{ij}^N \left( \int_{\mathbb{R}^N} g_j(v)^2 dv \right)^{1/2} \left( \int_{\mathbb{R}^N} \psi(|v|^2) dv \right)^{1/2}. \]

We finally get

\[ \sum_{j=1}^{I} \int_{\mathbb{R}^N} Q_{0,ij}^+ (\delta_0, g_j)(v)^2 dv \]

\[ \leq \max_{1 \leq i, j \leq I} \left( \|b_{ij}\|_{L^\infty(S^N)}^2 \left( C_{ij}^N \right)^2 \right) \sum_{j=1}^{I} \int_{\mathbb{R}^N} g_j(v)^2 dv \]

\[ = \max_{1 \leq i, j \leq I} \left( \|b_{ij}\|_{L^\infty(S^N)}^2 \left( C_{ij}^N \right)^2 \right) \|G\|_{L^4_0}, \]

which concludes the proof.

Therefore, with the completion of these two lemmas 3.1 and 3.2, the proof of (30) from proposition 3.1 is completed.

3.2. Estimate of the gain operator in a polynomially weighted \( L^2 \) space.

The estimate of the gain operator \( Q^+ (F, G) \) in \( L^2_0 \) norm makes use of the estimate in \( L^2_0 \) and the following inequality

\[ \langle v \rangle_i \leq \langle v' \rangle_i \langle v'_j \rangle_j, \]

which immediately follows from the conservation law of kinetic energy (1). Indeed, it holds

\[ \|Q^+ (F, G)\|_{L^6_0} \leq \left\| Q^+ (\bar{F}, \bar{G}) \right\|_{L^6}, \]

where

\[ [\bar{F}]_i(v) = f_i(v) \langle v \rangle_i^k, \quad i = 1, \ldots, I, \]

having in mind \( [F]_i(v) = f_i(v) \). Then we apply the result of Proposition 3.1 to \( Q^+ (\bar{F}, \bar{G}) \). Therefore, the following Proposition holds.
Theorem 4.1 (L" control of the collision weight angular integral operator). Let $1 \leq p,q,r \leq \infty$ with $1/p + 1/q = 1/r$. Then if $r \neq \infty$, for the operator $\mathcal{P}_{ij}(\varphi,\chi)$ (17) with $b_{ij} \in L^1(S^{N-1})$, the following estimate holds,

$$
\|\mathcal{P}_{ij}(\varphi,\chi)\|_r \leq C_{p,q,r} \|\varphi\|_p \|\chi\|_q,
$$

for any $\varphi \in L^p(\mathbb{R}^N)$ and $\chi \in L^q(\mathbb{R}^N)$, where the constant is

$$
C_{p,q,r} = |S^{N-2}| \int_{-1}^{1} \left( (1 - r_{ij})^2 (2 - 2s) \right)^{-\frac{N}{2p}} \times \left( r_{ij} (1 - r_{ij}) \left( \frac{1}{r_{ij}(1 - r_{ij})} - 2 + 2s \right) \right)^{-\frac{N}{2r}} d\xi^{b_{ij}}_N(s),
$$

with the measure $\xi^{b_{ij}}_N$ defined on $[-1,1]$ as

$$
d\xi^{b_{ij}}_N(s) = b_{ij}(s)(1 - s^2)^{N-3} ds.
$$

For $r = \infty$ the estimate (38) still holds but the constant simplifies to

$$
C_{\infty} := C_{\infty,\infty,\infty} = \|b_{ij}\|_{L^1(S^{N-1};\mathcal{G})}.
$$

Proof. The proof consists in multiple steps.

Step 1. (Radial symmetrization) Following the strategy in [2], we will consider $G = SO(N)$ the group of rotations of $\mathbb{R}^N$ (orthonormal transformations of determinant 1) and we will denote $R$ a generic rotation. Moreover, we assume the Haar measure $d\mu$ of this compact topological group normalized, i.e., $\int_G d\mu(R) = 1$.

For $p \geq 1$, let $f \in L^p(\mathbb{R}^N)$ and we will define the radial symmetrization $f^*_p$ of $f$ by

$$
f^*_p(x) = \left( \int_G |f(Rx)|^p d\mu(R) \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty.
$$
\[ f^* = \text{ess sup}_{|y|=|x|} |f(y)| \]

This radial rearrangement can be seen as an \( L^p \)-average of \( f \) over all the rotations \( R \in G \) and its properties can be found in [2]. It is not hard to prove that

\[ \|f\|_p = \|f^*\|_p. \]

**Step 2. (Passage from \( \mathcal{P}_{ij} \) to an one-dimensional operator \( \mathcal{B}_{ij} \))** First we use the following lemma, whose proof only uses the structure of \( \mathcal{P}_{ij} \) but not the definition of \( u^+_{ij} \) and \( u^{-}_{ij} \).

**Lemma 4.2** (Lemma 3 in [2]). Let \( \varphi, \chi, \psi \in C_c(\mathbb{R}^N) \) and \( 1/p + 1/q + 1/r' = 1 \), with \( 1 \leq p, q, r' \leq \infty \). Then

\[
\left| \int_{\mathbb{R}^N} \mathcal{P}_{ij}(\varphi, \chi)(u)\psi(u) du \right| \leq \int_{\mathbb{R}^N} \mathcal{P}_{ij}(\varphi^*_p, \chi^*_q)(u)\psi^*_r(u) du.
\]

From this Lemma, \( L^p \)-estimates for the operator \( \mathcal{P}_{ij} \) will follow by considering radial functions. As in [2] for any radial function \( f : \mathbb{R}^N \to \mathbb{R} \), we can define \( \tilde{f} := \mathbb{R}^+ \to \mathbb{R} \) by

\[ f(x) = \tilde{f}(|x|). \]

In addition, for any \( 1 \leq p < \infty \),

\[ \int_{\mathbb{R}^N} f(x)^p dx = |\mathbb{S}^{N-1}| \int_0^\infty \tilde{f}(t)^p t^{N-1} dt, \quad \text{(42)} \]

and for \( p = \infty \),

\[ \|f\|_\infty = \|\tilde{f}\|_{\infty,+} := \text{ess sup}_{y \in \mathbb{R}^+} |\tilde{f}(y)|. \quad \text{(43)} \]

Let us now show how the operator \( \mathcal{P}_{ij} \) simplifies to a 1-dimensional operator when applied to radial functions. Indeed, if \( \varphi \) and \( \chi \) are radial

\[
\mathcal{P}_{ij}(\varphi, \chi)(u) = \int_{\mathbb{S}^{N-1}} \hat{\varphi}(|u|) \hat{\chi}(|u|) b_{ij}(\hat{u} \cdot \sigma) d\sigma
\]

\[ = \int_{\mathbb{S}^{N-1}} \hat{\varphi}(a^1_{ij}(|u|, \hat{u} \cdot \sigma)) \hat{\chi}(a^2_{ij}(|u|, \hat{u} \cdot \sigma)) b_{ij}(\hat{u} \cdot \sigma) d\sigma
\]

\[ = |\mathbb{S}^{N-2}| \int_{-1}^1 \hat{\varphi}(a^1_{ij}(|u|, s)) \hat{\chi}(a^2_{ij}(|u|, s)) b_{ij}(s)(1 - s^2)^{N-2} ds,
\]

where \( a^1_{ij}, a^2_{ij} : \mathbb{R}^+ \times [-1, 1] \to \mathbb{R}^+ \) are defined by

\[ a^1_{ij}(x, s) = (1-r_{ij})x(2-2s)^{1/2}, \quad a^2_{ij}(x, s) = (r_{ij}(1-r_{ij}))^{1/2} \left( \frac{1}{r_{ij}(1-r_{ij})} - 2 + 2s \right)^{1/2}. \quad \text{(44)} \]

Considering the measure \( \xi^{b_{ij}}_N \) on \([-1, 1]\) as \( d\xi^{b_{ij}}_N(s) = b_{ij}(s)(1 - s^2)^{N-3} ds \), as given in (40), we conclude that

\[
\mathcal{B}_{ij}(\varphi, \chi)(x) = |\mathbb{S}^{N-2}| \int_{-1}^1 \hat{\varphi}(a^1_1(x, s)) \hat{\chi}(a^2_1(x, s)) d\xi^{b_{ij}}_N(s).
\]

(45)
Therefore, we are led to introduce the following bilinear operator for any two bounded and continuous functions \( \varphi, \chi : \mathbb{R}^+ \rightarrow \mathbb{R} \) as

\[
\mathcal{B}_{ij}(\varphi, \chi)(x) := \int_{-1}^{1} \varphi(a_{ij}^1(x, s)) \chi(a_{ij}^2(x, s)) d\xi_N^{b_{ij}}(s),
\]

with mappings \( a_{ij}^1 \) and \( a_{ij}^2 \) defined in (44).

**Step 3. (Study of the operator \( \mathcal{B}_{ij} \))**

**Lemma 4.3.** Let \( 1 \leq p, q, r \leq \infty \) with \( 1/p + 1/q = 1/r \), \( \varphi \in L^p(\mathbb{R}^+, x^{N-1}dx) \) and \( \chi \in L^q(\mathbb{R}^+, x^{N-1}dx) \). For \( r \neq \infty \) we have

\[
||\mathcal{B}_{ij}(\varphi, \chi)||_{r, +} \leq C_{p,q,r} ||\varphi||_{p, +} ||\chi||_{q, +},
\]

where the constant is

\[
C_{p,q,r} = \int_{-1}^{1} \left( (1 - r_{ij})(2 - 2s)^{1/2} \right)^{-\frac{N}{r}}
\]

\[
\times \left( r_{ij}(1 - r_{ij}) \left( \frac{1}{r_{ij}(1 - r_{ij})} - 2 + 2s \right) \right)^{-\frac{N}{r}} d\xi_N^{b_{ij}}(s).
\]

For \( r = \infty \), we have

\[
||\mathcal{B}_{ij}(\varphi, \chi)||_{\infty, +} \leq C_{\infty} ||\varphi||_{\infty, +} ||\chi||_{\infty, +},
\]

with the constant that simplifies to

\[
C_{\infty} := C_{\infty, \infty} = \frac{1}{|S^{N-2}|} ||b_{ij}||_{L^1(S^{N-1}; d\sigma)}.
\]

**Proof.** Using Minkowski’s and Hölder’s inequalities with exponents \( p/r \) and \( q/r \) we obtain

\[
\left( \int_0^\infty \left| \int_{-1}^{1} \varphi(a_{ij}^1(x, s)) \chi(a_{ij}^2(x, s)) d\xi_N^{b_{ij}}(s) \right|^r x^{N-1} dx \right)^{1/r}
\]

\[
\leq \int_{-1}^{1} \left( \int_0^\infty |\varphi(a_{ij}^1(x, s))|^r |\chi(a_{ij}^2(x, s))|^r x^{N-1} dx \right)^{\frac{1}{r}} d\xi_N^{b_{ij}}(s)
\]

\[
\leq \int_{-1}^{1} \left( \int_0^\infty |\varphi(a_{ij}^1(x, s))|^p x^{N-1} dx \right)^{\frac{1}{p}} \left( \int_0^\infty |\chi(a_{ij}^2(x, s))|^q x^{N-1} dx \right)^{\frac{1}{q}} d\xi_N^{b_{ij}}(s).
\]

Then for \( \varphi \) we can consider the change of variables \( y = a_{ij}^1(x, s) \), for any fixed \( s \in [-1, 1] \),

\[
\left( \int_0^\infty |\varphi(a_{ij}^1(x, s))|^p x^{N-1} dx \right)^{\frac{1}{p}} = \left( (1 - r_{ij})(2 - 2s)^{1/2} \right)^{-\frac{N}{p}} \left( \int_0^\infty |\varphi(y)|^p y^{N-1} dy \right)^{\frac{1}{p}}.
\]

And we can repeat the same for \( \chi \),

\[
\left( \int_0^\infty |\chi(a_{ij}^2(x, s))|^q x^{N-1} dx \right)^{\frac{1}{q}} = \left( r_{ij}(1 - r_{ij}) \left( \frac{1}{r_{ij}(1 - r_{ij})} - 2 + 2s \right) \right)^{-\frac{N}{q}} \left( \int_0^\infty |\chi(y)|^q y^{N-1} dy \right)^{\frac{1}{q}},
\]

which yields (47).
For \( r = \infty \), from the definition of the operator \( B_{ij} \) and pulling out the \( L^\infty \) norms of \( \varphi \) and \( \chi \), we obtain

\[
B_{ij}(\varphi, \chi)(x) \leq \frac{1}{|S_{N-2}|} \|\varphi\|_{\infty, +} \|\chi\|_{\infty, +} \|b_{ij}\|_{L^1(S^{N-1}; d\sigma)},
\]

and taking the supremum we obtain (48).

**Step 4. (Conclusion of the proof for Theorem 4.1)** From the Lemma 4.2, for \( 1/p + 1/q = 1/r \) and \( r \neq \infty \), by duality we have

\[
\left( \int_{\mathbb{R}^N} \left| \mathcal{P}_{ij}(\varphi, \chi) \right|^r \right)^{1/r} \leq \left( \int_{\mathbb{R}^N} \left| \mathcal{P}_{ij}(\varphi^*_p, \chi^*_q) \right|^r \right)^{1/r}
\]

Now, using equations (42), (45) and Lemma 4.3

\[
\left( \int_{\mathbb{R}^N} \left| \mathcal{P}_{ij}(\varphi^*_p, \chi^*_q) \right|^r \right)^{1/r} = |S_{N-1}|^{1/r} \left( \int_0^\infty \left| \mathcal{P}_{ij}(\varphi^*_p, \chi^*_q)(x) \right|^r x^{N-1} dx \right)^{1/r}
\]

\[
\leq C_{p,q,r} |S_{N-1}|^{1/r} \left( \int_0^\infty \varphi^*_p(x)^r x^{N-1} dx \right)^{1/r/2} \left( \int_0^\infty \chi^*_q(x)^r x^{N-1} dx \right)^{1/r/2}
\]

which yields (38). If \( r = \infty \), by duality and Lemma 4.2, we have

\[
\|\mathcal{P}_{ij} (\varphi, \chi)\|_\infty \leq \|\mathcal{P}_{ij} (\varphi^*_\infty, \chi^*_\infty)\|_\infty = \|\mathcal{P}_{ij} (\varphi^*_\infty, \chi^*_\infty)\|_\infty
\]

\[
\leq \|\varphi^*_\infty\|_{\infty, +} + \|\chi^*_\infty\|_{\infty, +} + \|b_{ij}\|_{L^1(S^{N-1}; d\sigma)} = \|b_{ij}\|_{L^1(S^{N-1}; d\sigma)} \|\varphi\|_\infty \|\chi\|_\infty,
\]

with the norm on radially symmetrized functions as defined in (43). Now the proof of the Theorem 4.1 is completed.

**Theorem 4.4.** Let \( p, q, r \in [1, \infty] \) with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \). Assume that \( B_{ij} \) takes the form

\[
B_{ij}(x, y) = x^{\lambda_{ij}} b_{ij}(y), \quad \lambda_{ij} \geq 0, \quad \text{and} \quad b_{ij} \in L^1([0, 1]; d\xi_N(s)),
\]

for any \( i, j = 1, \ldots, I \), with the measure from (40). Then we have the following estimate for \( F \in L^p(\mathbb{R}^N) \) and \( G \in L^q(\mathbb{R}^N) \)

\[
\|Q^+(F, G)\|_{L^0_\epsilon} \leq C_{p,q,r}^Q \|F\|_{L^p_\lambda} \|G\|_{L^q_\lambda},
\]

with

\[
\lambda = \max_{1 \leq i, j \leq I} \lambda_{ij},
\]

and where the constant \( C_{p,q,r}^Q \) for \( r \neq 1 \) and \( r \neq \infty \) is given by

\[
C_{p,q,r}^Q = I^{\frac{r-1}{r}} \max_{1 \leq i, j \leq I} (2^{\lambda_{ij}} M_{ij} C_{p,q,r}^Q),
\]
contains three steps. The first step aims at estimating

\[ C_{p,q,r} := \left( C_{r'} \right)^{r'/q'} \left( C_{r''} \right)^{r''/p'} \]

and

\[ M_{ij} := \left( \sum_{k=1}^{r} m_k \right)^{\frac{\lambda_{ij}}{m_i}} + \left( \sum_{k=1}^{r} m_k \right)^{\frac{\lambda_{ij}}{m_j}}, \quad (51) \]

\[ C_{ij}^{p,q,r} := \left( C_{r'} \right)^{r'/q'} \left( C_{r''} \right)^{r''/p'} \]

\[ = |S|^{-2} \left( \int_{-1}^{1} \left( (1 - r_{ij})^2 (2 - 2s) \right)^{-\frac{N}{2}} d \xi_N^1(s) \right)^{r'/q'} \]

\[ \times \left( \int_{-1}^{1} \left( r_{ij} (1 - r_{ij}) \left( \frac{1}{r_{ij} (1 - r_{ij})} - 2 + 2s \right) \right)^{-\frac{N}{2}} d \xi_N^1(s) \right)^{r'/p'}. \quad (52) \]

When \( r = 1 \) and \( r = \infty \) the constants change to

\[ C_{1}^{Q} := C_{1,1,1}^{Q} = \max_{1 \leq i,j \leq L} \left( 2^{\lambda_{ij}} M_{ij} \| b_{ij} \|_{L^1(S^{N-1})} \right), \]

\[ C_{\infty}^{Q} = \max_{1 \leq i,j \leq L} \left( 2^{\lambda_{ij}} M_{ij} C_{ij}^{p,q,r} \right). \quad (53) \]

**Proof.** This proof is inspired of the one written in [2], Theorem 1. We rewrite it here in complete detail as several changes are needed for the adaptation to gas mixtures models.

If \( \lambda_{ij} = 0 \) we can consider the gain operator in a weak form given in (18),

\[ J_0 := \int_{\mathbb{R}^N} Q_{ij}^+(f_i, g_j)(v) \psi(v) dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v) g_j(v-u) \mathcal{P}_{ij}(\mathcal{R}(\psi), 1)(u) du dv. \quad (54) \]

When \( \lambda_{ij} > 0 \) we use the following additional inequality,

\[ |u|^{\lambda_{ij}} \leq (|v-u| + |v|)^{\lambda_{ij}} \leq 2^{\lambda_{ij}} (|v-u|^{\lambda_{ij}} + |v|^{\lambda_{ij}}), \]

so that the weak form of the Gain operator for the pair \( \{ij\} \) is estimated by

\[ J_{\lambda_{ij}} := \int_{\mathbb{R}^N} Q_{ij}^+(f_i, g_j)(v) \psi(v) dv \]

\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v) g_j(v-u) \mathcal{P}_{ij}(\mathcal{R}(\psi), 1)(u)|u|^{\lambda_{ij}} du dv \]

\[ \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v) g_j(v-u) \mathcal{P}_{ij}(\mathcal{R}(\psi), 1)(u)(2^{\lambda_{ij}} (|v-u|^{\lambda_{ij}} + |v|^{\lambda_{ij}})) du dv \]

\[ \leq 2^{\lambda_{ij}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v) g_j(v-u) |v-u|^{\lambda_{ij}} \mathcal{P}_{ij}(\mathcal{R}(\psi), 1)(u) du dv \right) \]

\[ + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v) |v|^{\lambda_{ij}} g_j(v-u) \mathcal{P}_{ij}(\mathcal{R}(\psi), 1)(u) du dv. \quad (55) \]

The proof is separated into three subcases, depending whether \( i \) \((p,q,r) \notin \{(1,1,1), (p,p',\infty)\}, ii) (p,q,r) = (1,1,1) \) or \( iii) (p,q,r) = (p,p',\infty) \). Each subcase contains three steps. The first step aims at estimating \( J_0 \) from (56). Then, in the second step, by applying the very same estimate on \( J_0 \), but now on functions \( f_i \) and \( g_j \) (respectively on \( f_i \) and \( g_j \)) we estimate \( J_{\lambda_{ij}} \) given in (57). Finally, in step three, by duality arguments, we obtain an estimate for \( \| Q_{ij}^+(f_i, g_j) \|_r \) that will yield
Then, we can conclude that
\[
\|f\|_p \leq \|f\|_{L^{p,q}_{\lambda,i}} \quad \text{and} \quad \|f\|^{\lambda_{ij}}_p \leq \sqrt{\sum_{k=1}^l \frac{m_k}{m_i}} \|f\|_{L^{p,q}_{\lambda,i}},
\]
the first one following from the monotonicity of norms, and the second one from (34).

Subcase 1. \((p, q, r) \notin \{(1, 1, 1), (p, p', \infty)\}\).

Step 1. (Estimate of \(J_0\)). Since the exponents \(p, q, r\) in the Theorem satisfy
\(1/p' + 1/q' + 1/r = 1\) we can regroup the terms conveniently.

\[
J_0 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (f_i(v))^\frac{q}{r} g_j(v-u)^\frac{p}{r} \left( f_i(v) \frac{r}{q} \mathcal{P}_{ij}(\tau-v(R\psi), 1)(u) \right) du dv.
\]

Then using Hölder’s inequality
\[
J_0 \leq I_1I_2I_3.
\]

where
\[
I_1 := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v)^p g_j(v-u)^q du dv \right)^{\frac{1}{p}} = \|f_i\|_p^{p/r} \|g_j\|_q^{q/r},
\]
\[
I_2 := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v)^p \mathcal{P}_{ij}(\tau-v(R\psi), 1)(u)^{q'} du dv \right)^{\frac{1}{q'}} \leq \left( C_{r', \infty, r'}^{\mathcal{P}_{ij}} \right)^{r'/q'} \|f_i\|_p^{p/r'} \|\psi\|^{r'/q'},
\]
\[
I_3 := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_j(v-u)^q \mathcal{P}_{ij}(\tau-v(R\psi), 1)(u)^{r'} du dv \right)^{\frac{1}{r'}} \leq \left( C_{r', \infty, r'}^{\mathcal{P}_{ij}} \right)^{r'/p'} \|g_j\|_q^{q/p'} \|\psi\|^{r'/p'}.
\]

Then, we can conclude that
\[
J_0 \leq C_{p,q,r}^{ij} \|f_i\|_p \|g_j\|_q \|\psi\|_r',
\]
where \(C_{p,q,r}^{ij}\) is (54).

Step 2. (Estimate of \(J_{\lambda_{ij}}\)). Using the estimate (59) for the functions \(f_i, g_j \cdot |\lambda_{ij}\)
and for \(f_i |\cdot |^{\lambda_{ij}}, g_j\) then (57) becomes
\[
J_{\lambda_{ij}} \leq 2^{\lambda_{ij}} C_{p,q,r}^{ij} \left( \|f_i\| |\lambda_{ij}|_p \|g_j\|_q + \|f_i\|_p \|g_j\| |\lambda_{ij}|_q \|\psi\|_r' \right).
\]

Then, using (58), we obtain the final estimate for \(J_{\lambda_{ij}}\),
\[
J_{\lambda_{ij}} \leq 2^{\lambda_{ij}} C_{p,q,r}^{ij} M_{ij} \|f_i\|_{L^{p}_{\lambda_{ij}}} \|g_j\|_{L^{q}_{\lambda_{ij}}} \|\psi\|_r',
\]
where \(M_{ij}\) is from (51).

Step 3. (Estimate of \(\|Q^+_i(f_i, g_j)\|_r\)). Thanks to the last estimate on \(J_{\lambda_{ij}}\), by
duality we obtain
\[
\|Q^+_i(f_i, g_j)\|_r \leq 2^{\lambda_{ij}} C_{p,q,r}^{ij} M_{ij} \|f_i\|_{L^{p}_{\lambda_{ij}}} \|g_j\|_{L^{q}_{\lambda_{ij}}}.
\]
Furthermore, we can estimate the norm of the vector form collision operator in terms of its components,
\[
\|Q^+(F,G)\|_{L^1_\rho} = \left( \sum_{i=1}^{l} \int_{\mathbb{R}^N} \left| \sum_{j=1}^{l} Q^+_{ij}(f_i, g_j)(v) \right|^r \, dv \right)^{1/r} \leq \left( \sum_{i=1}^{l} \int_{\mathbb{R}^N} \|Q^+_{ij}(f_i, g_j)\|_r \, dv \right)^{1/r},
\]
which yields, using (60),
\[
\|Q^+(F,G)\|_{L^1_\rho} \leq C_{p,q,r} \sum_{i=1}^{l} \sum_{j=1}^{l} \|f_i\|_{L^p_{\lambda_{ij}}} \|g_j\|_{L^q_{\lambda_{ij}}} \leq C_{p,q,r} \sum_{i=1}^{l} \sum_{j=1}^{l} \|f_i\|_{L^p_{\lambda_{ij}}} \|g_j\|_{L^q_{\lambda_{ij}}},
\]
by monotonicity of norms.

**Subcase 2.** \((p,q,r) = (1,1,1)\)

**Step 1. (Estimate of \(J_0\)).** For this choice of indexes, \(J_0\) can be estimated with
\[
J_0 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v)g_j(v-u)\mathcal{P}_{ij}(\tau_v(\mathcal{R}^\psi),1)(u)\,dv \, du \\
\leq \|\mathcal{P}_{ij}(\tau_v(\mathcal{R}^\psi),1)\|_\infty \|f_i\|_1 \|g_j\|_1 \leq C_{\infty} \|f_i\|_1 \|g_j\|_1 \|\psi\|_\infty,
\]
where the constant \(C_{\infty}\) is from (41).

**Step 2. (Estimate of \(J_{\lambda_{ij}}\)).** Using the same idea as in the previous subcase, we apply the inequality above to the functions \(f_i, g_j\cdot|\lambda_{ij}\) and to \(f_i\cdot|\lambda_{ij}\), \(g_j\), and then using (58) we get the estimate for \(J_{\lambda_{ij}}\),
\[
J_{\lambda_{ij}} \leq 2^\lambda_{ij} C_{\infty} M_{ij} \|f_i\|_{L^1_{\lambda_{ij}}} \|g_j\|_{L^1_{\lambda_{ij}}} \|\psi\|_\infty.
\]

**Step 3. (Estimate of \(\|Q^+(F,G)\|_{L^1_\rho}\)).** Writing the norm of the vector value collision operator in terms of norms of its components and exploiting duality arguments, we have
\[
\|Q^+(F,G)\|_{L^1_\rho} = \sum_{i=1}^{l} \int_{\mathbb{R}^N} \left| \sum_{j=1}^{l} Q^+_{ij}(f_i, g_j)(v) \right| \, dv \\
\leq C_{1}^{Q^+} \sum_{i=1}^{l} \sum_{j=1}^{l} \|f_i\|_{L^p_{\lambda_{ij}}} \|g_j\|_{L^q_{\lambda_{ij}}} \leq C_{1}^{Q^+} \sum_{i=1}^{l} \sum_{j=1}^{l} \|f_i\|_{L^p_{\lambda_{ij}}} \|g_j\|_{L^q_{\lambda_{ij}}},
\]
where the constant \(C_{1}^{Q^+}\) from (55), and the last inequality follows from the monotonicity of norms.

**Subcase 3.** \((p,q,r) = (p,p',\infty)\).

**Step 1. (Estimate of \(J_0\)).** If additionally \((p,p') = (\infty,1)\), then \(J_0\) from (56) can be rewritten and estimated using the same ideas as in the Subcase 1,
\[
J_0 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i(v)g_j(v-u)\mathcal{P}_{ij}(\tau_v(\mathcal{R}^\psi),1)(u)\,dv \, du \\
\leq \|f_i\|_\infty \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_j(v)\mathcal{P}_{ij}(1,\tau_v(\mathcal{R}^\psi))(u)\,dv \, du \right) \leq C_{\infty,1} \|f_i\|_1 \|g_j\|_1 \|\psi\|_1.
\]
Similarly, for \((p,p') = (1,\infty)\), we have
\[
J_0 \leq C_{1,\infty} \|f_i\|_1 \|g_j\|_\infty \|\psi\|_1.
\]
If \((p, p') \notin \{(1, \infty), (\infty, 1)\}\), we rewrite \(J\) as
\[
J_0 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( f_i(v) \mathcal{P}_{ij}(\tau_v(\mathcal{R} \psi), 1)(u) \right) \times \left( g_j(v - u) \mathcal{P}_{ij}(\tau_v(\mathcal{R} \psi), 1)(u) \right) \, dv.
\]
Then by the Hölder inequality, we obtain
\[
J_0 \leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( f_i(v) \mathcal{P}_{ij}(\tau_v(\mathcal{R} \psi), 1)(u) \right)^{1/p} \times \left( g_j(v - u) \mathcal{P}_{ij}(\tau_v(\mathcal{R} \psi), 1)(u) \right)^{1/p'} \, dv \right)^{1/p}.
\]
Repeating the same procedure as in the Subcase 1, we obtain
\[
J_0 \leq C_{p, p', \infty}^{ij} \| f_i \|_p \| g_j \|_{p'} \| \psi \|_1,
\]
with a constant
\[
C_{p, p', \infty}^{ij} = \left( C^{\mathcal{P}_{ij}}_{1, \infty, 1} \right)^{1/p} \left( C^{\mathcal{P}_{ij}}_{\infty, 1, 1} \right)^{1/p'}.
\]
This notation merges all possible pairs \((p, p')\).

**Step 2. (Estimate of \(J_{\lambda_{ij}}\)).** Applying the inequality above to the functions \(f_i, g_j \cdot |^\lambda_{ij}\) and to \(f_i \cdot |^\lambda_{ij}, g_j\), and then using (58) we get the estimate for \(J_{\lambda_{ij}}\),
\[
J_{\lambda_{ij}} \leq 2^{\lambda_{ij}} C_{p, p', \infty}^{ij} M_{ij} \| f_i \|_{L^\infty_{\lambda_{ij}, i}} \| g_j \|_{L^\infty_{\lambda_{ij}, j}} \| \psi \|_1.
\]

**Step 3. (Estimate of \(\| Q^+(F, G) \|_{L^p_0}\)).** Finally, from the inequality above by duality, we obtain
\[
\| Q^+(f_i, g_j) \|_{\infty} \leq 2^{\lambda_{ij}} C_{p, p', \infty}^{ij} M_{ij} \| f_i \|_{L^\infty_{\lambda_{ij}, i}} \| g_j \|_{L^\infty_{\lambda_{ij}, j}},
\]
which yields
\[
\| Q^+(F, G) \|_{L^p_0} = \sum_{i=1}^{l} \left\| \sum_{j=1}^{l} Q^+(f_i, g_j) \right\|_{\infty} \\
\leq C^{Q^+}_{\infty} \sum_{i=1}^{l} \sum_{j=1}^{l} \| f_i \|_{L^\infty_{\lambda_{ij}, i}} \| g_j \|_{L^\infty_{\lambda_{ij}, j}} \leq C^{Q^+}_{\infty} \sum_{i=1}^{l} \sum_{j=1}^{l} \| f_i \|_{L^p_{\lambda_{ij}, i}} \| g_j \|_{L^p_{\lambda_{ij}, j}},
\]
with the constant \(C^{Q^+}_{\infty}\) from (55). \(\square\)

### 4.2. Estimate of the gain operator in a polynomially weighted \(L^p\) space.

To find the estimates for the positive part of the collision operator for any \(r \in (1, \infty)\) we will use the estimates proved in Theorem 4.4, the one from Lemma 3.2 and Riesz-Thorin interpolation theorem.

**Theorem 4.5. (Gain of integrability)** For any \(\epsilon > 0\), \(p \in (1, \infty)\) and \(k \geq 0\) the collision operator can be estimated in the following way
\[
\| Q^+(F, G) \|_{L^p_0} \leq I^{1 - \frac{1}{p}} \frac{1}{r} \| F \|_{L^p} \left( \epsilon^{\frac{1}{2}} C^{Q^+}_{p, 1, p} \| G \|_{L^p} \right)^{1 - \theta} \| G \|_{L^p}^{1 - \theta} \| G \|_{L^p}^{k} + \tilde{C}_N \epsilon^{2 + 2 - N} \| G \|_{L^p_{\frac{\lambda_{ij}}{2}}}^{1 - \theta} \| G \|_{L^p_{\frac{\lambda_{ij}}{2}}}^{k}
\]

with $C_{p,1,p}^{G^+}$ as given in (50) and $\hat{C}_N$ will be given in (69).

**Proof.**

**Step 1. Estimate of $\|Q_{N-2}^+(\delta_0, G)\|_{L_0^1}$ by Riesz Thorin Interpolation.**

We will separate first the interpolation between $L^1$ and $L^2$, and then $L^2$ and $L^\infty$.

**Case 1:** $p \in (1, 2)$. The first estimate we need can be proven by the Theorem 4.4, and is given by

$$\|Q_{N-2}^+(\delta_0, G)\|_{L_0^1} \leq C_{1,p}^{G^+} \|\hat{G}\|_{L_0^1},$$

Now, from Lemma 3.1 and the fact that $L^{\frac{2N}{N-3}} \hookrightarrow L^{\frac{2N}{N-2}}$ we can prove

$$\|Q_{N-2}^+(\delta_0, G)\|_{L_0^2} \leq I^{\frac{2N+1}{N}} C_{1}^{G^+} \hat{C}_{2N} \|\hat{G}\|_{L^{\frac{2N}{N-2}}},$$

with $C_N = \max_{1 \leq i,j \leq I}(C_{N,ij})$ and $C_{N,ij}$ and $\hat{C}_N$ as given in (35) and (36) respectively.

Then, by the Riesz-Thorin theorem, the interpolation of $L^1$ and $L^2$ by $L^1$ and $L^{\frac{2N}{N-1}}$ yields

$$\|Q_{N-2}^+(\delta_0, G)\|_{L_0^p} \leq 2 \left( I^{\frac{2N+1}{N}} C_{1}^{G^+} \hat{C}_{2N} \right) \|\hat{G}\|_{L^{\frac{2N}{N-1}}},$$

with the relation

$$\frac{1 - 1/N}{1} + \frac{1/N}{p} = \frac{1}{r}, \text{ that is, } r = \frac{Np}{p(N-1) + 1},$$

and $\vartheta = 2 - 2/p$.

**Case 2:** $p \in [2, \infty)$. For the interpolation in this case we will use the following estimate from Theorem 4.4:

$$\|Q_{N-2}^+(\delta_0, G)\|_{L_0^\infty} \leq C_{\infty}^{G^+} \|\hat{G}\|_{L^{\frac{2N}{N-2}}},$$

We use again Riesz-Thorin theorem, for the interpolation of $L^2$ and $L^\infty$ by $L^{\frac{2N}{N-1}}$ and $L^{\infty}$, to get

$$\|Q_{N-2}^+(\delta_0, G)\|_{L_0^p} \leq 2 \left( I^{\frac{2N+1}{N}} C_{1}^{G^+} \hat{C}_{2N} \right)^{1-\vartheta} \|\hat{G}\|_{L^{\frac{2N}{N-1}}},$$

where

$$r = \frac{Np}{2N-1},$$

and $\vartheta = 1 - 2/p$.

**Step 2. Estimate of $\|Q_{N-2}^+(\delta_0, G)\|_{L_0^p}$ by Hölder’s inequality.** Consider $\theta \in (0, 1)$ that relates to $r$ and $p$ in the following way

$$\frac{1 - \theta}{1} + \frac{\theta}{p} = \frac{1}{r},$$
Then Hölder’s inequality implies that
\[
\|G\|_{L_{N-2}^p} = \left( \sum_{i=1}^{I} \int_{R^N} \left( g_i(v) \left( v_i \right)^{N-2} \right) dv \right)^{\frac{1}{p}} \\
\leq \sum_{i=1}^{I} \left( \int_{R^N} \left( g_i(v) \left( v_i \right)^{N-2} \right)^{\frac{N-2}{N}} g_i(v) dv \right)^{\frac{1}{p}} \\
\leq \sum_{i=1}^{I} \left( \int_{R^N} g_i(v) \left( v_i \right)^{\frac{N-2}{N}} \right)^{1-\theta} \left( \left( \int_{R^N} g_i(v)^p \right)^{\frac{1}{p}} \right)^{\theta} \\
\leq I \|G\|_{L_{N}^\theta}^{\frac{1}{\theta}} \|G\|_{L_{N}^\theta}.
\]
Therefore, we conclude that for any \( r \) that satisfies (64) or (65), there exists \( \theta \in (0, 1) \) such that
\[
\|Q_{N-2}^+(\delta_0, G)\|_{L_0^p} \leq \tilde{C}_N \|G\|_{L_{N}^\theta} \|G\|_{L_{N}^\theta}, \tag{67}
\]
where the parameter \( \theta \) is defined as
\[
\theta := \theta_{p,N} = \begin{cases} \frac{r}{N}, & \text{if } p \in (1, 2] \text{ and } r = \frac{Np}{p(N-1)+1}, \\ \frac{r}{N(p-2)+1}, & \text{if } p \in [2, \infty) \text{ and } r = \frac{Np}{2N-1}, \end{cases}
\tag{68}
\]
and the constant \( \tilde{C}_N \) is given by
\[
\tilde{C}_N = \begin{cases} 2I \left( C_{1}^{\frac{N}{p}} \right)^{1-\theta} \left( I \left( \frac{Np}{p(N-1)+1} \right) \right)^{\theta}, & \text{if } p \in (1, 2], \ r = \frac{Np}{p(N-1)+1}, \ \text{and } \theta = 2 - 2/p, \\ 2I \left( I \left( \frac{Np}{2N-1} \right) \right)^{1-\theta} \left( C_{1}^{\frac{N}{p}} \right)^{\theta}, & \text{if } p \in [2, \infty), \ r = \frac{Np}{2N-1}, \ \text{and } \theta = 1 - 2/p. \end{cases}
\tag{69}
\]

**Step 3. Estimate for \( \|Q_{\gamma_{ij}}^+(\delta_0, G)\|_{L_0^p} \):** Now, we will proceed as we did for the case of the \( L^2 \) norms. Following the idea of Proposition 3.1 we can show that
\[
\|Q_{\gamma_{ij}}^+(\delta_0, G)\|_{L_0^p} \leq 2^{\frac{p+1}{p}} \epsilon^{1+\frac{p}{2}} \|Q_{0}^+(\delta_0, G)\|_{L_p^p} + 2^{\frac{p+1}{p}} \epsilon^{1+\frac{p}{2}} \|Q_{N-2}^+(\delta_0, G)\|_{L_0^p},
\]
then using Theorem 4.4 and (67) we get
\[
\|Q_{\gamma_{ij}}^+(\delta_0, G)\|_{L_0^p} \leq 2^{\frac{p+1}{p}} \epsilon^{1+\frac{p}{2}} C_{p,1,p} \|G\|_{L_0^p} + 2^{\frac{p+1}{p}} \epsilon^{1+\frac{p}{2}} \tilde{C}_N \|G\|_{L_{N}^\theta} \|G\|_{L_{N}^\theta}, \tag{70}
\]
with \( \theta \) defined as in (68) and \( \tilde{C}_N \) as in (69).

**Step 4. Estimate for \( \|Q^+(F, G)\|_{L_0^p} \):** Now, from Minkowski’s integral inequality we can derive the following estimate
\[
\|Q^+(F, G)\|_{L_0^p} \leq \sum_{i=1}^{I} \int_{R^N} f_i(x) \left( \int_{R^N} \left( \sum_{j=1}^{J} \tau_a Q_{ij}^+(\delta_0, \tau_a g_j)(v) \right) dv \right)^{1/p} dx \\
\leq I^{1-\frac{1}{p}} \int_{R^N} \sum_{i=1}^{I} f_i(x) \|Q^+(\delta_0, \tau_a G)(\cdot)\|_{L_0^p} dx.
\]
Combining (70) with this last estimate, it is just a matter of adding the weights as in Proposition 3.2 to complete the proof of Theorem 4.5. \( \square \)
5. Propagation of exponentially and polynomially weighted $L^p$ norms

**Proposition 5.1.** For any $\epsilon > 0$, $p \in (1, \infty)$, $\gamma \in (0,1)$ and $k \geq 0$, we have the following estimate,

$$
\sum_{i=1}^{I} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^+ (F,F) \right]_i \langle v \rangle^p_i \, dv 
\leq f^2 \frac{1}{\gamma} 2^{\frac{p+1}{p}} \| F \|_{L_1^1} \left( \epsilon^{\gamma} C_{p,1,p}^{Q^+} \| F \|_{L_k^p}^p + \epsilon^{-2} \| \hat{C}_N \|_{L_1^1}^{1-\theta} \| F \|_{L_k^p}^{p-1+\theta} \right),
$$

where $\hat{C}_N$ depends on $\| b_{ij} \|_\infty$ and it is defined as in (69), and $C_{p,1,p}^{Q^+}$ as given in (50).

**Proof.** The proof immediately follows from Hölder’s inequality and Theorem 4.5. \hfill \square

**Proposition 5.2.** Let $F$ satisfy the assumptions Lemma C.1, that allows to obtain a lower bound for hard potentials transition probabilities i.e. there exists some constant $c_{lb}$ such that

$$
\sum_{i=1}^{I} \int_{\mathbb{R}^N} m_i f_i(t,w) |v - w|^\gamma_i \, dw \geq c_{lb} \langle v \rangle^\gamma_i, \quad \text{for any } j = 1, \ldots, I, \quad (71)
$$

with $\gamma$ defined as in (31).

Then the following estimate holds

$$
\sum_{i=1}^{I} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q(F,F) \right]_i \langle v \rangle^p_i \, dv \leq B_k \| F \|_{L_k^p}^{p-1+\theta} - A_k \| F \|_{L_k^p}^p,
$$

where $A_k$ and $B_k$ are given in (22) and are positive for a small enough $\epsilon > 0$ and $k > k^*$ with

$$
\kappa^* = \max\{k, 2 + 2\gamma\}, \quad (72)
$$

where $\kappa = \max_{1 \leq i,j \leq I} \{k_{ij}^\gamma\}$ and each $k_{ij}^\gamma$ depends on the angular transition rate $b_{ij}$ as well as on the binary mass fraction $m_j/(m_i + m_j)$ as in [8].

**Proof.** Since we work in cut-off framework, the gain and the loss term of the collision operator can be separated

$$
\sum_{i=1}^{I} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q(F,F) \right]_i \langle v \rangle^p_i \, dv
= \sum_{i=1}^{I} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^+ (F,F) \right]_i \langle v \rangle^p_i \, dv - \sum_{i=1}^{I} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^- (F,F) \right]_i \langle v \rangle^p_i \, dv
$$

To operate on the gain part of the collision operator, and since Proposition 5.1 works for $b_{ij} \in L^\infty (\mathbb{S}^{N-1})$, we will split the angular part $b_{ij}$ of the transition probability terms (10) as follows

$$
b_{ij}(y) = b_{ij}^l(y) + b_{ij}^c(y), \quad (73)
$$

where $b_{ij}^c \in L^\infty (\mathbb{S}^{N-1})$ and $\| b_{ij}^l \|_1 \leq \epsilon^\gamma$. Accordingly, in this Section we introduce a notation for the gain operator so that the splitting becomes more visible. Namely, $Q^+_q (F,G)(v)$
will stand for the gain operator (12) and (11) choosing (10), i.e.

\[
Q^+_q(\mathcal{F}, \mathcal{G})(v) = \left[ \sum_{j=1}^I \int_{\mathbb{R}^N} \int_{\mathbb{R}^{N-1}} f_i(v') g_j(v'_*) |u|^{\gamma_{ij}} b^q_{ij}(\hat{u} \cdot \sigma) \, d\sigma \, dv_* \right]_{1 \leq i \leq I}
\]

\[
= \left[ \sum_{j=1}^I Q^+_{\gamma_{ij}, q}(f_i, g_j) \right]_{1 \leq i \leq I},
\]

(74)

where \( q = 1, \infty \). Therefore

\[
\sum_{i=1}^I \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^+_1(\mathcal{F}, \mathcal{F}) \right]_i \langle v \rangle_i^{pk} \, dv
\]

\[
= \sum_{i=1}^I \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^+_1(\mathcal{F}, \mathcal{F}) \right]_i \langle v \rangle_i^{pk} \, dv + \sum_{i=1}^I \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q^+_\infty(\mathcal{F}, \mathcal{F}) \right]_i \langle v \rangle_i^{pk} \, dv
\]

Observe that for the second term we can apply Proposition 5.1, while for the first term we will need an extra computation.

First, we invoke the following point-wise estimate for the relative velocity by the mixture brackets

\[
|u| \leq 2|v - v_*| \leq 2 \frac{\sqrt{m_i}}{\sqrt{m_j}} \langle v \rangle_i \langle v'_* \rangle_j.
\]

(75)

For the first inequality we refer to the proof of Theorem 2.1 in [5], since it is independent from the system, and for the second one just note that

\[
\frac{\sqrt{m_i}}{\sum_{k=1}^I m_k \sum_{k=1}^I m_k} |v - v_*| \leq \min \left\{ \frac{\sqrt{m_i}}{\sum_{k=1}^I m_k}, \frac{\sqrt{m_j}}{\sum_{k=1}^I m_k} \right\} |v - v_*|
\]

\[
\leq \frac{\sqrt{m_i}}{\sum_{k=1}^I m_k} |v| + \frac{\sqrt{m_j}}{\sum_{k=1}^I m_k} |v'_*|
\]

\[
\leq \langle v \rangle_i \langle v'_* \rangle_j.
\]

Hence, considering (75) and the conservation of kinetic energy, we have the following estimate

\[
|u|^{\gamma_{ij}} \langle v \rangle_i^{\frac{\gamma_{ij}}{p}} \leq \left( 2 \frac{\sum_{k=1}^I m_k}{\sqrt{m_i \sqrt{m_j}}} \right)^{\gamma_{ij}} \langle v'_* \rangle_i^{\frac{\gamma_{ij}}{p}} \langle v'_* \rangle_j^{\gamma_{ij}(1 + \frac{1}{p})}.
\]
Therefore, using this estimate, Hölder's inequality and Theorem 4.4,
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q_i \rho (F, F') \right] (v) \, dv 
\leq \left( 2 \sum_{k=1}^{l} m_k \right) \frac{\gamma}{\min_{1 \leq i \leq l} m_i} \sum_{i=1}^{l} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_i (v) v_i^{k+\gamma} f_j (v') v_j^{k+\gamma} \, dv \, dv 
\times b_{ij} (\bar{u} \cdot \sigma) \left( f_i (v) v_i^{k+\gamma} \right)^{p-1} \, dv \, dv 
\leq \left( 2 \sum_{k=1}^{l} m_k \right) \frac{\gamma}{\min_{1 \leq i \leq l} m_i} \sum_{i=1}^{l} \left\| \sum_{j=1}^{l} Q_{ij}^+ \left( \left\langle \gamma_i^k \bar{f}_i, \gamma_j^k \bar{f}_j \right\rangle \right) \right\|_{L^p_{k+\gamma}}^{p-1} 
\leq \left( 2 \sum_{k=1}^{l} m_k \right) \frac{\gamma}{\min_{1 \leq i \leq l} m_i} C_{p,1,p} \left\| F \right\|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1}.
\]
Now observe that from the definition of $C_{p,1,p}^+$ in (50), the fact that $b_{ij}$ is symmetrized, and since $\| b_{ij} \|_1 \leq \bar{c} \gamma$
\[
C_{p,1,p}^+ \leq \frac{2^{\frac{2N-1}{\gamma}} |g|}{S^{N-2}} \times \max_{1 \leq i,j \leq l} \left( 2^{N/\gamma} M_{ij} \max_{0 \leq s \leq 1} \left( r_{ij} (1 - r_{ij}) \left( \frac{1}{r_{ij} (1 - r_{ij})} - 2 + 2s \right) \right)^{-\frac{1}{\gamma}} \right)^{\gamma} 
:= C \bar{c} \gamma
\]
Therefore, and by monotonicity of norms,
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^N} f_i^{p-1} \left[ Q_i (F, F') \right] (v) \, dv 
\leq \left( 2 \sum_{k=1}^{l} m_k \right) \frac{\gamma}{\min_{1 \leq i \leq l} m_i} C \bar{c} \gamma \left\| F \right\|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} + \bar{c} \gamma |g| L^{\frac{p-1}{2}} \frac{2^{\frac{2N-1}{\gamma}} |g|}{S^{N-2}} C_{p,1,p} \left\| F \right\|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} 
+ \epsilon^{2+2^{-N} L^{2-\frac{2}{\gamma}} \frac{1}{2} \frac{1}{\gamma} \frac{1}{\gamma} \bar{c} \gamma \left\| F \right\|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} + \theta \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} \theta
\leq B \left\| F \right\|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} + A \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} \theta
\leq B \| F \|_{L^1_{k+\gamma}} \left\| F \right\|_{L^p_{k+\gamma}}^{p-1} + A \| F \|_{L^p_{k+\gamma}}^{p-1} \theta.
\]
with $B_k$ and $A_k$ as defined in (22) and $\epsilon$ small enough such that $A_k$ is positive. Notice that by propagation of moments, $B_k$ only depends on the initial data.

As a consequence of this Proposition, we are able to state the following result.

**Corollary 1.** If $F$ is a solution of the Boltzmann system (4), then
\[
\frac{1}{p} \partial_t \| F \|_{L^p_k}^{p-1} \leq B_k \| F \|_{L^p_{k+\gamma/p}}^{p-1} - A_k \| F \|_{L^p_{k+\gamma/p}}^{p-1},
\]
for any \( k \geq k^\ast, \ 1 \leq p < \infty, \ k^\ast \) as defined in (72), and \( \theta \) as given in (68).

We are now able to prove the main theorems regarding the propagation of \( L^p \) norms with polynomial and exponential weights, stated in section 2.

**Proof of theorem 2.3.** We start by the inequality (76), and we associate it to the ODE of Bernoulli type

\[
y'(t) = by(t)^{1-c} - ay(t),
\]

with \( a, b, c > 0 \), whose solution will be an upper bound for \( \|F\|_{L^p_k}^p \). Indeed, we can explicitly solve (77) to get to the solution

\[
y(t) = \left( \frac{b}{a} \left( 1 - e^{-act} \right) + y(0)e^{-act} \right)^\frac{1}{p}.
\]

Now, we can follow the proof given in [8], Theorem 2.6, choosing \( y(t) = \|F\|_{L^p_k}^p \), \( b = pB_k, \ a = pA_k, \) and \( c = \frac{1-a}{p} \) to conclude our result.

**Proof of theorem 2.4.** Let’s note that from theorem 2.7 in [8] there exists a constant \( 0 < \tilde{\alpha} \leq \alpha_0 \) and a constant \( \tilde{C}^\ast > 0 \) such that

\[
\|Fe^{\tilde{\alpha}t\gamma}\|_{L^\gamma_k} \leq \tilde{C}^\ast.
\]

On the other hand, for any \( s \in (0,1] \) and from the conservation of kinetic energy,

\[
\langle \nu \rangle^s_i \leq \langle \nu' \rangle^s_i + \langle \nu'' \rangle^s_i.
\]

Therefore

\[
\sum_{j=1}^{l} Q_{ij}(f_i, f_j)(v)e^{\alpha(v)^s_i} \leq \sum_{j=1}^{l} Q_{ij}^+(f_i e^{\alpha(v)^s_i}, f_j e^{\alpha(v)^s_i})(v) - \sum_{j=1}^{l} Q_{ij}(f_i, f_j)(v)e^{\alpha(v)^s_i}.
\]

Now, denoting \( g_i(v) = f_i(v)e^{\alpha(v)^s_i} \), and using the lower bound (71),

\[
\partial_t g_i(v) \leq \sum_{j=1}^{l} Q_{ij}^+(g_i, g_j)(v) - c_{\tilde{B}} \min_{1 \leq i, j \leq l} \|b_{ij}\|_{L^1(G^{\alpha_0})}^{(3)} g_i(v)e^{\alpha(v)^s_i}.
\]

We can continue now, as done in Proposition 5.1 and Proposition 5.2 to conclude as in Theorem 2.3 that

\[
\|G\|_{L^p_k}^p \leq \max \left\{ \left( \frac{\tilde{B}_0}{\tilde{A}_0} \right)^{-\frac{p-1}{p}}, \|G_0\|_{L^p_k}^p \right\},
\]

with \( \tilde{B}_0 \) and \( \tilde{A}_0 \) as given in (26). Let’s note that there exists \( \zeta > 0 \) such that

\[
\langle \nu \rangle^s_i \leq e^{\alpha(v)^s_i^{1/2}}
\]

then,

\[
\|Fe^{\alpha(v)^s}\|_{L^\gamma_k}^{\frac{p}{p-2}} = \|Fe^{\alpha(v)^s}(\nu)^{\frac{p-2}{p}}\|_{L^\gamma_k} \leq \|Fe^{\alpha(v)^s-\zeta/2}\|_{L^\gamma_k} \leq \|Fe^{\alpha(v)^s}\|_{L^\gamma_k}.
\]

Therefore, by (78), this upper estimate will be finite for \( \alpha < \tilde{\alpha} \) and the inequality (25) yields.
6. $L^\infty$ estimates

Our last goal in this work is to extend the propagation of $L^p_\beta$ norms to the case of $p = \infty$. As in [5], equation (3), for technical reasons, we separate the angular part $b$ of the transition probability terms (10) as follows

$$b_{ij}(y) = b_{ij}(y) \left( \mathbb{1}_{y \leq \sqrt{1-\varepsilon_{ij}}} + \mathbb{1}_{y > \sqrt{1-\varepsilon_{ij}}} \right) =: b_{ij}^{\varepsilon_{ij},1}(y) + b_{ij}^{\varepsilon_{ij},2}(y), \quad (80)$$

and, as before, the support of $b_{ij}$ is assumed to be in $[0,1]$, because of the symmetrization assumption. As in (74), we use a notation for the gain operator so that the splitting becomes more visible. Namely,

$$Q_{\varepsilon_{ij},q}^+(\mathbb{F}, \mathbb{G})(v) = \sum_{j=1}^{l} \int_{\mathbb{R}^N} \int_{S^{N-1}} f_i(u') \ g_j(v'_i) \ |u_i| \gamma_{ij}^{\varepsilon_{ij},q} (\hat{u} \cdot \sigma) \ d\sigma \ dv, \quad 1 \leq i \leq l$$

$$= \sum_{j=1}^{l} Q_{\gamma_{ij},ij,q}^+(f_i, g_j), \quad 1 \leq i \leq l,$$

where $q = 1, 2$.

**Lemma 6.1.** Let the transition probability terms $B_{ij}$ be given with (10) and (80). Then, when $\gamma_{ij} = 0$ for all $1 \leq i \leq l$, the following inequalities hold

$$\left\| Q_{\varepsilon_{ij},1}^+(\mathbb{F}, \mathbb{G}) \right\|_{L^\infty_0} \leq C^{b_1} \left\| \mathbb{F} \right\|_{L^2_\beta} \left\| \mathbb{G} \right\|_{L^2_\beta},$$

$$\left\| Q_{\varepsilon_{ij},2}^+(\mathbb{F}, \mathbb{G}) \right\|_{L^\infty_0} \leq C^{b_2} \left\| \mathbb{F} \right\|_{L^\infty_0} \left\| \mathbb{G} \right\|_{L^1_\beta}, \quad (81)$$

where the constant $C^{b_1} \sim \max_{1 \leq i,j \leq l} \left( \varepsilon_{ij}^{\frac{N}{2}} \left\| b_{ij} \right\|_{L^1(S^{N-1}, d\sigma)} \right)$, and

$C^{b_2} \sim \max_{1 \leq i,j \leq l} \left\| \varepsilon_{ij} b_{ij}^{\varepsilon_{ij},2} \right\|_{L^1(S^{N-1}, d\sigma)}$. In particular, $C^{b_2}$ decreases with $\varepsilon_{ij} \searrow 0$. Therefore,

$$C^{b_2} \leq \xi, \quad (82)$$

where $\xi$ can be taken $\xi < 1$ arbitrarily.

**Proof.** For the first estimate, we apply the result of Theorem 4.4 for the case $(p,q,r) = (2,2,\infty)$, which yields the desired estimate with a constant

$$C^{Q^+}_{\infty} = \max_{1 \leq i,j \leq l} M_{ij} \left| S^{N-2} \right| \left( \int_0^1 \left( 2(1-r_{ij})^2(1-s) \right)^{\frac{-N}{2}} b_{ij}^{\varepsilon_{ij},1}(s)(1-s)^{\frac{N-3}{2}} ds \right)^{1/2} \times \left( \int_0^1 \left( r_{ij}(1-r_{ij}) \left( \frac{1}{r_{ij}(1-r_{ij})} - 2 + 2s \right) \right)^{-\frac{N}{2}} b_{ij}^{\varepsilon_{ij},1}(s)(1-s)^{\frac{N-3}{2}} ds \right)^{1/2}.$$
We can estimate this constant using the inequality $(1-s)^{-1} \leq 2\epsilon_{ij}^{-2}$ in the support of $b_{ij}^{\gamma,1}$, and estimate $r_{ij}^{-1}(1-r_{ij}) - 2 + 2s \geq r_{ij}^{-1}(1-r_{ij}) - 2$ for $s \geq 0$,

$$C_{ij}^{\gamma} \leq \max_{1 \leq i,j \leq l} M_{ij} |S^{N-2}| r_{ij}^{-2} \left(1-r_{ij}\right)^{-\frac{2N}{N-1}} \left(1-r_{ij}(1-r_{ij}) - 2\right)^{-\frac{2N}{N-1}} \epsilon_{ij}^{-\frac{2N}{N-1}} \left(\int_0^1 b_{ij}(s)(1-s^{2})^{-\frac{2N}{N-1}} ds\right)$$

$$= \max_{1 \leq i,j \leq l} M_{ij} r_{ij}^{-\frac{2N}{N-1}} \left(1-r_{ij}\right)^{-\frac{2N}{N-1}} \left(1-r_{ij}(1-r_{ij}) - 2\right)^{-\frac{2N}{N-1}} \epsilon_{ij}^{-\frac{2N}{N-1}} \|b_{ij}\|_{L^1(S^{N-1})}$$

Next, for the second estimate we use the same Theorem 4.4 for $(p,q,r) = (\infty,1,\infty)$. The constant then becomes

$$C_{ij}^{\gamma} = \max_{1 \leq i,j \leq l} M_{ij} |S^{N-2}| \int_0^1 \left(r_{ij}(1-r_{ij}) \left(1-r_{ij}(1-r_{ij}) - 2\right)\right)^{-\frac{2N}{N-1}} \epsilon_{ij}^{-\frac{2N}{N-1}} \left(\int_0^1 b_{ij}(s)(1-s^{2})^{-\frac{2N}{N-1}} ds\right)$$

$$\leq \max_{1 \leq i,j \leq l} M_{ij} \left(r_{ij}(1-r_{ij}) \left(1-r_{ij}(1-r_{ij}) - 2\right)\right)^{-\frac{2N}{N-1}} \|b_{ij}^{\gamma,2}\|_{L^1(S^{N-1})}$$

Note that $\|b_{ij}^{\gamma,2}\|_{L^1(\partial\Omega)} \to 0$, as $\epsilon_{ij} \to 0$ by the dominate convergence theorem. Then, inequalities (81) hold. \qed

**Theorem 6.2.** Let $\gamma_{ij} \in (0,1]$, $b_{ij} \in L^1(S^{N-1})$, and an initial data satisfying the hypothesis of Theorem 2.6 in [8] and such that

$$\|F_0\|_{L^\infty} = C_0,$$

for some positive constant $C_0$. Then there exists a constant $C(F_0)$ depending on $\bar{\gamma}$, $m_i$, $b_{ij}$ such that

$$\|F(t,\cdot)\|_{L^\infty} \leq C(F_0), \quad t \geq 0,$$

for $F$ the solution of the Boltzmann system, with $\bar{\gamma}$ defined as in (31).

This proof follows a different approach from the case $1 < p < \infty$, since now we need to focus on point-wise estimates on the collisional integral rather than in computing $L^p$ norms by duality.

**Proof.** Component-wise, we can apply inequality (75) to the gain part of the collisional operator,

$$\sum_{j=1}^{l} Q_{ij}^+(f_i, f_j)(v) \leq \sum_{j=1}^{l} \frac{2 \sum_{k=1}^{l} m_k}{\sqrt{m_i} \sqrt{m_j}} Q_{0,ij}^+(f_i, f_j(\gamma_{ij})) (v) \langle v \rangle_{\gamma_{ij}}$$

$$\leq 2 \frac{\sum_{k=1}^{l} m_k}{\min_{1 \leq i \leq l} \sqrt{m_i} \min_{1 \leq j \leq l} \sqrt{m_j}} \sum_{j=1}^{l} Q_{0,ij}^+(f_i, f_j(\gamma_{ij})) (v) \langle v \rangle_{\gamma_{ij}}$$
For the loss term, from Lemma C.1

$$\sum_{j=1}^{I} Q_{ij}^+(f_i, f_j)(v) = f_i(v) \sum_{j=1}^{I} \left| b_{ij} \right|_{L^1(S^N-1)} \int_{\mathbb{R}^N} f_j(v_*) \left| v - v_* \right|^{\gamma_i} \, d\sigma v_*$$

$$\geq f_i(v) \min_{1 \leq i, j \leq I} \left| b_{ij} \right|_{L^1(S^N-1)} \sum_{j=1}^{I} \int_{\mathbb{R}^N} f_j(v_*) \left| v - v_* \right|^{\gamma_i} \, d\sigma v_*$$

$$\geq \min_{1 \leq i, j \leq I} \left| b_{ij} \right|_{L^1(S^N-1)} \frac{1}{\max_{1 \leq j \leq I} m_j} c_{ib} f_i(v) \langle v \rangle_i^\gamma.$$  

where $c_{ib}$ is given by (88).

Denoting

$$C_G := \frac{\sum_{k=1}^{l} m_k}{\min_{1 \leq i, j \leq I} \sqrt{m_i} \min_{1 \leq i, j \leq I} \sqrt{m_j}}$$

and

$$C_L := \frac{\min_{1 \leq i, j \leq I} \left| b_{ij} \right|_{L^1(S^N-1)}}{\max_{1 \leq j \leq I} m_j}.$$  

Then, we obtain the following control

$$\partial_t f_i = \sum_{j=1}^{I} Q_{ij}(f_i, f_j)(v) \leq C_G \sum_{j=1}^{I} Q_{ij}^+(f_i, f_j)(v) \langle v \rangle_i^\gamma - C_L c_{ib} f_i(v) \langle v \rangle_i^\gamma,$$

or equivalently,

$$\partial_t f_i + C_L c_{ib} f_i(v) \langle v \rangle_i^\gamma \leq C_G \sum_{j=1}^{I} Q_{ij}^+(f_i, f_j)(v) \langle v \rangle_i^\gamma.$$  

Now, we can split the angular function as in (80) and apply Lemma 6.1 component wise, as showed in (63)

$$\partial_t \left( f_i(v) e^{C_L c_{ib}(v) \gamma t} \right) \leq e^{C_L c_{ib}(v) \gamma t} C_G \left( \sum_{j=1}^{I} Q_{0,ij}^+(f_i, f_j)(v) \right) \langle v \rangle_i^\gamma.$$  

$$\leq e^{C_L c_{ib}(v) \gamma t} C_G \left( \sum_{j=1}^{I} Q_{0,ij}^+(f_i, f_j)(v) \right) \langle v \rangle_i^\gamma.$$  

$$\leq e^{C_L c_{ib}(v) \gamma t} C_G \sum_{j=1}^{I} \left( \left| Q_{0,ij}^+(f_i, f_j)(v) \right|_{L^1} + \left| Q_{0,ij}^+(f_i, f_j)(v) \right|_{L^2} + \left| Q_{0,ij}^+(f_i, f_j)(v) \right|_{L^\infty} \right) \langle v \rangle_i^\gamma.$$  

$$\leq e^{C_L c_{ib}(v) \gamma t} C_G \left( C_{1} \sum_{j=1}^{I} \left| f_i \right|_{L^2} \left| f_j \right|_{L^\infty} \langle v \rangle_i^\gamma \right) \langle v \rangle_i^\gamma.$$  

$$\leq e^{C_L c_{ib}(v) \gamma t} C_G \left( C_{1} \sum_{j=1}^{I} \left| f_i \right|_{L^2} \left| f_j \right|_{L^\infty} \langle v \rangle_i^\gamma \right) \langle v \rangle_i^\gamma.$$  

Now, observe that from (72), $\left| F \right|_{L^1_3} \leq \left| F \right|_{L^2_3}$, and since $F_0 \in \Omega$ then by theorem 2.6 in [8], $\left| F \right|_{L^1_3} \leq C(F_0)$, with $C(F_0)$ as given in that theorem. In the same
way, \( \|F\|_{L^2} \leq \|F\|_{L^2} \) and by theorem 2.3 there is propagation of \( L^2 \) norms, so 
\( \|F\|_{L^2} \leq \hat{C}(F_0) \), with \( \hat{C}(F_0) \) as given in (23).
Moreover, we can choose \( \max_{1 \leq i, j \leq t} \epsilon_{ij} \) small enough such that, choosing \( \xi = \frac{C_G C_L}{2C(F_0)C_G T} \) in (82),
\[
C^{b_2} \leq \frac{c_0 C_L}{4C(F_0)C_G T}.
\]
Then, it follows that
\[
\partial_t (f_i(v)) e^{C_L c_{b_2}(v)\gamma t} \left( C_G c_{b_1} \hat{C}(F_0) \|f_i\|_2 + \frac{C_L c_{b_2}}{4T} \|f_i\|_{\infty} \right) < \|f_i\|_{\infty}^\gamma.
\]
Now, doing integration over \( t \) and taking the supremum over \( \|F\|_{L^2} \), \( \|F\|_{L^1} \), \( \|F\|_{L^2} \), we can choose max \( \max_{1 \leq i, j \leq t} \epsilon_{ij} \) small enough such that, choosing \( \xi = \frac{C_G C_L}{2C(F_0)C_G T} \) in (82),
\[
C^{b_2} \leq \frac{c_0 C_L}{4C(F_0)C_G T}.
\]
Then, it follows that
\[
\partial_t (f_i(v)) e^{C_L c_{b_2}(v)\gamma t} \left( C_G c_{b_1} \hat{C}(F_0) \|f_i\|_2 + \frac{C_L c_{b_2}}{4T} \|f_i\|_{\infty} \right) < \|f_i\|_{\infty}^\gamma.
\]
Now, doing integration over \( t \)
\[
f_i(v) \leq f_i(0, v) e^{-C_L c_{b_2}(v)\gamma t} + \int_0^t e^{-C_L c_{b_2}(v)\gamma (t-s)} \left( C_G c_{b_1} \hat{C}(F_0) \|f_i\|_2 + \frac{C_L c_{b_2}}{4T} \|f_i\|_{\infty} \right) ds \|f_i\|_{\infty}^\gamma.
\]
Finally, since the right hand side remains the same, we take the supremum over all \( v \in \mathbb{R}^N \) and sum over \( 1 \leq i \leq I \), to obtain
\[
\|F(t, \cdot)\|_{L^2} \leq \|F_0\|_{L^2} + \frac{C_G C_{b_1} \hat{C}(F_0)}{c_0 C_L} \sum_{i=1}^I \sup_{0 \leq s \leq t} \|f_i(s, \cdot)\|_2 + \frac{1}{4T} \sum_{i=1}^I \sup_{0 \leq s \leq t} \|F(s, \cdot)\|_{L^2}.
\]
Again, by the propagation of polynomially weighted \( L^p \) norms (23),
\[
\|F(s, \cdot)\|_{L^2} \leq \hat{C}(F_0),
\]
and taking the supremum over \( t \in (0, T] \), estimate (83) follows, and the proof of Theorem 6.2 is complete.

**Proof of theorem 2.5** From the conservation law of kinetic energy (1),
\[
\langle v \rangle_i \leq \langle v' \rangle_i \langle v'' \rangle_j.
\]
Then, it holds
\[
\partial_t \hat{f}_i \leq \sum_{j=1}^I Q_{ij} \langle \hat{f}_i, \hat{f}_j \rangle(v),
\]
where
\[ \hat{f}_i(v) = f_i(v) \langle v \rangle_i^k, \quad i = 1, \ldots, I. \]

Note that Lemma 6.1 is also valid for \( \hat{F} \) and \( \hat{G} \) then we can obtain
\[
\partial_t \hat{f}_i \leq \sum_{j=1}^{I} Q_{ij}(\hat{f}_i, \hat{f}_j)(v) \leq C_G \sum_{j=1}^{I} Q_{ij}^+(\hat{f}_i, \hat{f}_j)(v) \langle v \rangle_i^\gamma - C_L c_b \hat{f}_i(v) \langle v \rangle_i^\gamma,
\]
as in the proof of theorem 6.2. So, it is just a matter of repeating the proof for \( \hat{f}_i \).
In fact, we can obtain that
\[
\partial_t \left( \hat{f}_i(v) e^{C_L c_b(v) \gamma I} \right) \leq e^{C_L c_b(v) \gamma} C_G \left( C_{bh} \| \hat{f}_i \|_2 \| \hat{F} \|_{L_2^\gamma} + C_{bt} \| \hat{f}_i \|_{\infty} \| \hat{F} \|_{L_1^\gamma} \right) \langle v \rangle_i^\gamma.
\]
Now, \( \| \hat{F} \|_{L_1^\gamma} = \| F \|_{L_1^{\gamma+k}} \) and since \( F_0 \in \Omega \) then by theorem 2.6 in [8], we can bound \( \| F \|_{L_1^{\gamma+k}} \leq C(F_0) \), with \( C(F_0) \) as given in that theorem for \( k > k^* \). In the same way, \( \| \hat{F} \|_{L_2^\gamma} = \| F \|_{L_2^{\gamma+k}} \) and by theorem 2.3 there is propagation of weighted \( L^2 \) norms, so \( \| F \|_{L_2^{\gamma+k}} \leq \tilde{C}(F_0) \), with \( \tilde{C}(F_0) \) as given in (23) for \( k > k^* \). Then we can redo the computations shown before to obtain
\[
\| \hat{F}(t, \cdot) \|_{L_0^\infty} \leq \| \hat{F}_0 \|_{L_0^\infty} + C_G C_{bh} \tilde{C}(F_0) I \sup_{0 \leq s \leq t} \| \hat{F}(s, \cdot) \|_{L_2^\gamma} + \frac{1}{4} \sup_{0 \leq s \leq t} \| \hat{F}(s, \cdot) \|_{L_0^\infty}.
\]
Again, by the propagation of polynomially weighted \( L^p \) norms (23),
\[
\| \hat{F}(s, \cdot) \|_{L_2^\gamma} \leq \tilde{C}(F_0),
\]
and taking the supremum over \( t \in (0, T] \), estimate (27) follows, and the proof of Theorem 6.2 is complete.

\[ \Box \]

Proof of theorem 2.6 Recall from the proof of Theorem 6.2,
\[
\sum_{j=1}^{I} Q_{ij}^+(f_i, f_j)(v) \leq 2 \sum_{k=1}^{K} \frac{m_k}{\min_{1 \leq i \leq I} \sqrt{m_i} \min_{1 \leq j \leq I} \sqrt{m_j}} \sum_{j=1}^{I} Q_{ij}^+(f_i, f_j\langle \cdot \rangle_j^\gamma)(v) \langle v \rangle_i^\gamma,
\]
then, from (79), yields the following estimate
\[
\partial_t g_i(v) \leq \left( 2 \sum_{k=1}^{K} \frac{m_k}{\min_{1 \leq i \leq I} \sqrt{m_i} \min_{1 \leq j \leq I} \sqrt{m_j}} \sum_{j=1}^{I} Q_{ij}^+(g_i, g_j\langle \cdot \rangle_j^\gamma)(v) \langle v \rangle_i^\gamma \right.
\]
\[ - c_b \frac{\min_{1 \leq i, j \leq I} \| b_{ij} \|_{L^1(G^n)} \| g_i \|_1(G^n)}{\max_{1 \leq j \leq I} \min_{1 \leq i \leq I} m_j}, \]
for \( g_i(\cdot) = f_i(\cdot) e^{\alpha(\cdot) t} \). Then, by splitting the angular function as in (80), we can repeat the same argument as in the proof of Theorem 6.2, to conclude that
\[
\| G(t, \cdot) \|_{L_0^\infty} \leq C(G_0)
\]
Note that we have the boundedness of \( \| G \|_{L_1^1}, \| G \|_{L_2^\gamma} \) and \( \| G \|_{L_2^\gamma} \) by the generation and propagation of exponentially weighted moments and \( L^p \) norms.
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Appendix A. Proof of Theorem 2.1 (Carleman representation of the gain operator)

Proof. The proof follows [7] and [4]. Indeed, using the fact that

\[ \int_{\mathbb{R}^{N}} F(|u|\sigma - u) d\sigma = \frac{2}{|u|^{N-2}} \int_{y \in \mathbb{R}^{N}} \delta(2y \cdot u + |y|^2) F(y) dy, \]

where \( \delta \) is usual Dirac delta function, one has

\[ Q_{ij}^+(f,g)(v) = \int_{v_+ \in \mathbb{R}^N} \frac{2}{|u|^{N-2}} \int_{y \in \mathbb{R}^{N}} \delta(2y \cdot u + |y|^2) f(v + (1 - r_{ij})y) g(v_+ - r_{ij}y) \times B_{ij} \left( |u|, 1 - \frac{|y|^2}{2|u|^2} \right) dy dv_+, \]

where \( u = v - v_+, \) and \( |u| = |v - v_+| = |v - v_+ + y| \). We first perform the change of variable \( y \mapsto v' = v + (1 - r_{ij})y \), with the Jacobian \((1 - r_{ij})^N\), and then for \( v' \) fixed \( v_+ \mapsto v'_+ = v_+ - \frac{r_{ij}}{1 - r_{ij}}(v' - v) \) with the unit Jacobian. After some calculations, this gives

\[ Q_{ij}^+(f,g)(v) = 2(1 - r_{ij})^{-N} \int_{v' \in \mathbb{R}^N} \int_{v'_+ \in \mathbb{R}^N} |u|^{-N+2} f(v') g(v'_+) \times B_{ij} \left( |u|, 1 - \frac{|v' - v|^2}{2(1 - t)^2 |u|^2} \right) dv'_+ dv', \]

where \( u = \frac{1}{1 - r_{ij}} v - \frac{r_{ij}}{1 - r_{ij}} v' - v'_+, \) and \( |u| = |v' - v'_+| \). With this notation, we can precise the argument of delta function,

\[ \frac{2}{1 - r_{ij}} (v' - v) \cdot u + \frac{|v' - v|^2}{(1 - r_{ij})^2} = \frac{2}{1 - r_{ij}} (v - v') \cdot (v'_+ - v) + \frac{1 - 2r_{ij}}{(1 - r_{ij})^2} |v' - v|^2. \]
This leads us to perform the change of variables \( v' \mapsto \bar{v} = v' - v \), with unit Jacobian, which implies

\[
Q^+_ij(f, g)(v) = 2(1 - r_{ij})^{-N} \int_{v' \in \mathbb{R}^N} \int_{\bar{v} \in \mathbb{R}^N} |v' - \bar{v} - v|^{-N+2} f(v') \, g(\bar{v} + v) \times \delta \left( \frac{2}{1 - r_{ij}} (v - v') \cdot \bar{v} + \frac{1 - 2r_{ij}}{(1 - r_{ij})^2} |v' - v|^2 \right) \times B_{ij} \left( |v' - \bar{v} - v|, 1 - \frac{|v' - v|^2}{2(1 - t)^2 |v' - \bar{v} - v|^2} \right) \, d\bar{v} \, dv' .
\]  

(85)

Now, for fixed \( v' \) and \( v \), we decompose the vector \( \bar{v} \) into the component parallel to \( v - v' \) and orthogonal to it. Let

\[
n = \frac{v - v'}{|v - v'|}.
\]

Denote with \( z^\parallel \) the component of \( \bar{v} \) in direction of \( n \), i.e. let \( z^\parallel := n \cdot \bar{v} \). Then its orthogonal vector \( z^\perp \) belong to the hyperplane \( E_{v,v'} \), by definition (19). Thus, one may write

\[
\bar{v} = z^\perp + z^\parallel n, \quad z^\parallel = n \cdot \bar{v}, \quad z^\perp \in E_{v,v'},
\]

from which it follows. Moreover, in (85), we change the variables \( \bar{v} \mapsto z^\perp + z^\parallel n \) with unit Jacobian, and integration is performed via integration with respect to its components \( z^\perp \in E_{v,v'} \) and \( z^\parallel \in \mathbb{R} \). This change of variables simplifies the argument of delta function. Indeed,

\[
\frac{2}{1 - r_{ij}} (v - v') \cdot \bar{v} + \frac{1 - 2r_{ij}}{(1 - r_{ij})^2} |v' - v|^2 = \frac{2|v' - v|}{1 - r_{ij}} \left( \frac{1 - 2r_{ij}}{2(1 - r_{ij})} |v' - v| + z^\parallel \right),
\]

which yields the following representation

\[
Q^+_ij(f, g)(v) = 2(1 - r_{ij})^{-N} \int_{v' \in \mathbb{R}^N} \int_{z^\perp \in E_{v,v'}} \int_{z^\parallel \in \mathbb{R}} |v' - v - z^\perp - z^\parallel n|^{-N+2} \times g \left( v + z^\perp + z^\parallel n \right) \delta \left( -\frac{2|v' - v|}{1 - r_{ij}} \left( \frac{2r_{ij} - 1}{2(1 - r_{ij})} |v' - v| - z^\parallel \right) \right) \times B_{ij} \left( |v' - v - z^\perp - z^\parallel n|, 1 - \frac{|v' - v|^2}{2(1 - r_{ij})^2 |v' - v - z^\perp - z^\parallel n|^2} \right) \, dz^\parallel \, dz^\perp \, dv'.
\]

Then, by the fact

\[
\int_{y \in \mathbb{R}} \delta(a(b - y))F(y)dy = |a|^{-1} F(b),
\]
one obtains
\[
Q_{ij}^+(f,g)(v) = (1 - r_{ij})^{-N+1} \int_{v' \in \mathbb{R}^N} \frac{f(v')}{|v' - v|} \int_{z^+ \in E_{v'}} g \left( z^+ + P_{r_{ij}}(v,v') \right) \times \left( \frac{v - v'}{2(1 - r_{ij})} + z^+ \right)^{2-N} \times B_{ij} \left( \frac{(v - v')}{2(1 - r_{ij})} + z^+ \right)^{1 - \frac{|v' - v|^2}{2(1 - r_{ij})} + z^+} \right) dv' \]

It remains to rename the variables. \hfill \Box

### Appendix B. Existence and Uniqueness

For a closure to this work, we will transcript the theorem from [8] and the idea behind the proof.

**Theorem B.1** (Existence and Uniqueness). Assume that $F(0,v) = F_0(v) \in \Omega$, where

\[
\Omega = \left\{ F(t,\cdot) \in L^1_t : F \geq 0 \text{ in } v, \sum_{i=1}^{I} \int_{\mathbb{R}^N} m_i v f_i(t,v) dv = 0, \right. \\
\left. \exists c_0, c_0, C_2, C_2 + \varepsilon > 0, \text{ and } C_0 < c_2, \text{ such that } \forall t \geq 0, \right. \\
\left. c_0 \leq m_0 F(t) \leq C_0, \right. \\
\left. c_2 \leq m_2 F(t) \leq C_2, \right. \\
\left. m_2 + \varepsilon F(t) \leq C_2 + \varepsilon, \text{ for } \varepsilon > 0, \right. \\
\left. m_k F(t) \leq C_{k_*}, \text{ with } k_* \text{ as in (72), and } C_{k_*} \text{ as in (87)} \right\} \quad (86)
\]

where

\[
m_k F(t) := \|F\|_{L^1} = \sum_{i=1}^{I} \int_{\mathbb{R}^3} |f_i(t,v)| \langle v \rangle^2 dv.
\]

Then the Boltzmann system (21) for the cross section (10) has a unique solution in $C([0,\infty), \Omega) \cap C^1((0,\infty), L^2)$. The proof is based in an abstract framework of ODE theory in Banach spaces, which can be found in [11]. In order to apply that theory, it is crucial to state the invariant region $\Omega \subset L^1_t$ in which the collision operator $Q : \Omega \to L^1_t$ satisfies (i) Hölder continuity, (ii) Sub-tangent and (iii) one-sided Lipschitz conditions.

To that end, the authors studied the map $\mathcal{L}_{\tilde{\gamma}, k_*} : [0, \infty) \to \mathbb{R}$, defined as

\[
\mathcal{L}_{\tilde{\gamma}, k_*}(x) = -Ax^{1+\frac{\tilde{\gamma}}{2}} + Bx,
\]

where $A$ and $B$ are positive constants, $\tilde{\gamma} \in (0, 1]$ and $k_*$ as defined in (72). This map has only one root, denoted with $x_{\tilde{\gamma}, k_*}$, at which $\mathcal{L}_{\tilde{\gamma}, k_*}$ changes from positive to negative. Thus, for any $x \geq 0$, they write

\[
\mathcal{L}_{\tilde{\gamma}, k_*}(x) = \max_{0 \leq z \leq x_{\tilde{\gamma}, k_*}} \mathcal{L}_{\tilde{\gamma}, k_*}(x) =: \mathcal{L}^*_x.
\]

Then, defining

\[
C_{k_*} := x_{\tilde{\gamma}, k_*} + \mathcal{L}^*_x
\]

they were able to write such a region $\Omega$. 

Appendix C. Lower bound of the cross section

We will state the following Lemma, whose proof can be found in [8, Appendix].

Lemma C.1. Let \( \gamma_{ij} \in [0,2] \), for any \( i,j \in \{1,\ldots,I\} \), and assume

\[
0 \leq \left\{ F(t) = [f_1(t) \ldots f_I(t)]^T \right\}_{t \geq 0} \subset L^2_\beta \text{ satisfies}
\]

\[
c \leq \sum_{i=1}^I \int_{\mathbb{R}^3} m_i f_i(t,v) dv \leq C, \quad c \leq \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t,v) |v|^2 dv \leq C,
\]

\[
\sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t,v) m_i v dv = 0,
\]

for some positive constants \( c \) and \( C \). Assume also boundedness of the moment

\[
\sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t,v) m_i |v|^{2+\varepsilon} dv \leq B, \quad \varepsilon > 0.
\]

Then, there exists a constant \( c_{lb} \) defined as

\[
c_{lb} = c \frac{C}{2} \left( \frac{\max\{C,B\}}{c} \right) \left( 1 + \left( \frac{2 C}{\bar{c}} \right)^{\frac{2+\varepsilon}{\varepsilon}} \right)^{-\frac{2+\varepsilon}{\varepsilon}} \times \left( 1 + \frac{\max_{1 \leq i \leq I} m_i}{\sum_{i=1}^I m_i} \left( \frac{2 C}{\bar{c}} \right)^2 \right)^{-\gamma/2}, \quad (88)
\]

such that

\[
\sum_{i=1}^I \int_{\mathbb{R}^3} m_i f_i(t,w) |v-w|^{\gamma_{ij}} dw \geq c_{lb} \langle v \rangle^{\bar{\gamma}_{ij}}, \quad (89)
\]

for any \( j \in \{1,\ldots,I\} \), with \( \bar{\gamma} = \max_{1 \leq i,j \leq I} \gamma_{ij} \).

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