The Complexity of Probabilistic Justification Logic

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August 12, 2018

Abstract

Probabilistic justification logic is a modal logic with two kind of modalities: probability measures and explicit justification terms. We present a tableau procedure that can be used to decide the satisfiability problem for this logic in polynomial space. We show that this upper complexity bound is tight.

1 Introduction

Following [9] we can define a probabilistic version of a base logic by enriching the language of the base logic with probabilistic operators. The probabilistic operators create formulas of the form \( P_{\geq s} A \) which read as “\( A \) holds with probability at least \( s \)”. The models of these probabilistic logics are probability spaces which have models of the base logic as states. In order to obtain a sound and complete axiomatization the usual axioms for probability are combined with the axioms of the base logic [9].

Artemov developed the first justification logic, the Logic of Proofs (LP), to provide intuitionistic logic with a classical provability semantics [1, 2]. In [2] it was proved that any theorem of modal logic \( S4 \) can be translated into a theorem of \( LP \) by replacing any occurrence of the modal operator \( \square \) with an appropriate explicit justification term and that any theorem in \( LP \) can be translated into a theorem in \( S4 \) by replacing any occurrence of a justification term with a \( \square \). In the same way explicit counterparts for several modal logics were found [3]. For example the justification logic \( J \) is the explicit counterpart of the minimal modal logic \( K \).

In [7] a probabilistic justification logic, \( PPJ \), is defined over the basic justification logic \( J \) [3]. In this paper we present a tableau procedure that can be used to decide the satisfiability problem in \( PPJ \). This procedure uses a rule that is applied to all the formulas that appear in the scope of some probabilistic operator in a tableau branch. The rule creates exponentially many branches, however by applying a theorem from the theory of linear systems we show that only polynomially many branches are needed in order to decide the satisfiability of a given formula. This way we can decide the satisfiability problem for \( PPJ \).
in polynomial space. We show that our upper bound is tight via a reduction from modal logic $\mathcal{D}$, which is the modal logic that is complete with respect to serial Kripke structures.

## 2 A Probabilistic Logic over Classical Propositional Logic

Let $\text{Prop}$ be a countable set of atomic propositions. The logic $\mathcal{LPP}_1$ is defined in [9] over the language $\mathcal{L}_{\mathcal{LPP}_1}$:

$$A ::= p \mid P_{\geq s} A \mid \neg A \mid A \land A$$

where $s \in \mathbb{Q} \cap [0,1]$ and $p \in \text{Prop}$. We also use the following abbreviations:

$$P_{< s} A \equiv \neg P_{\geq s} A, \quad P_{\leq s} A \equiv P_{\geq 1-s} A \equiv \neg P_{\leq s} A \quad \text{and} \quad P_{= s} A \equiv P_{\geq s} A \land P_{\leq s} A.$$  

The axiom schemata and the derivation rules of the logic $\mathcal{LPP}_1$ are presented in Table 1.

Axiom (PI) corresponds to the fact that the probability of truthfulness of every formula is at least 0. Axioms (WE) and (LE) describe some properties of inequalities. Axioms (DIS) and (UN) correspond to the additivity of probabilities for disjoint events. The rule (CE) is the probabilistic analogue of the modal necessitation rule and the rule (ST) informally says that if the probability of a formula is arbitrarily close to $s$ then it is at least $s$. (ST) corresponds to the Archimedean property of the real numbers.

### Axiom Schemata:

(P) finitely many axioms schemata for classical propositional logic

(PI) $\vdash P_{\geq 0} A$

(WE) $\vdash P_{\leq r} A \rightarrow P_{< s} A$, where $s > r$

(LE) $\vdash P_{< s} A \rightarrow P_{\leq s} A$

(DIS) $\vdash P_{\geq r} A \land P_{\geq 1} B \land P_{\geq 1-s} (A \land B) \rightarrow P_{\geq \min(1,r+s)} (A \lor B)$

(UN) $\vdash P_{\leq r} A \land P_{\leq s} B \rightarrow P_{< r+s} (A \lor B)$, where $r + s \leq 1$

### Derivation Rules:

(MP) if $T \vdash A$ and $T \vdash A \rightarrow B$ then $T \vdash B$

(CE) if $\vdash A$ then $\vdash P_{\geq 1} A$

(ST) if $T \vdash A \rightarrow P_{\geq s+\frac{1}{2}} B$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \rightarrow P_{\geq s} B$

### Table 1: Axiom Schemata and Derivation Rules of $\mathcal{LPP}_1$

A probability space is a triple $\langle W, H, \mu \rangle$, where $W$ is a non-empty set of states, $H \subseteq \mathcal{P}(W)$ ($\mathcal{P}$ stands for powerset) is closed under finite union and complementation and $\mu : H \rightarrow [0,1]$ such that $\mu(W) = 1$ and for any disjoint $U$ and $V$ in $H$, $\mu(U \cup V) = \mu(U) + \mu(V)$.

$\text{Q}$ denotes the set of rational numbers.
The models for LPP$_1$ are probability spaces where the states contain truth assignments and probability spaces (so that we can deal with iterated probabilities).

**Definition 1 (LPP$_1$-Model).** An LPP$_1$-model is a quintuple $M = \langle U, W, H, \mu, v \rangle$ where:

1. $U$ is a non-empty set of objects called worlds;
2. $W, H, \mu$ and $v$ are functions, which have $U$ as their domain, such that for every $w \in U$:

   \[
   \langle W_w, H_w, \mu_w \rangle \text{ is a probability space with } W_w \subseteq U \text{ and } v_w : \text{Prop} \rightarrow \{ T, F \}, \text{ where } T (F) \text{ stand for true (false).}
   \]

**Definition 2 (Satisfiability in an LPP$_1$-model).** Let $M = \langle U, W, H, \mu, v \rangle$ be an LPP$_1$-model. Satisfiability is defined as follows (the propositional cases are treated classically):

\[
M, w \models p \iff v_w(p) = T \quad \text{for } p \in \text{Prop};
M, w \models P_{\geq s} B \iff \left( \mu_w([A]_{M,w}) \geq s \right), \text{ where } [A]_{M,w} = \{ u \in W_w \mid M, u \models A \}.
\]

Let $M = \langle U, W, H, \mu, v \rangle$ be an LPP$_1$-model. $M$ will be called measurable if for every $w \in U$ and for every $A \in \mathcal{L}_{\text{LPP}_1}$, $[A]_{M,w} \in H_w$. In the rest of the paper we restrict ourselves to measurable models. LPP$_1,\text{Meas}$ denotes the class of LPP$_1$-measurable models.

Soundness and strong completeness for LPP$_1$ with respect to LPP$_1,\text{Meas}$ is proved in [9]. Assume that $A_1, \ldots, A_k$ are the subformulas of some $A \in \mathcal{L}_{\text{LPP}_1}$. A formula of the form $\pm A_1 \land \ldots \land \pm A_k$, where $\pm A_i$ is either $A_i$ or $\neg A_i$, will be called an atom of $A$. In an atom the order of the conjuncts does not matter. So, two atoms are considered the same if they have the same conjuncts. $|A|$ is defined as the number of symbols that are used in order to write $A$ (where all rational numbers are assumed to have size 1). For $A \in \mathcal{L}_{\text{LPP}_1}$, $||A||$ is the biggest size of a rational number that appears in $A$ (where the size of a rational number is equal to the sum of the lengths of the binary representations of its numerator and denominator, when the rational number is written as an irreducible fraction).

As we mentioned in the introduction, a well known theorem from the theory of linear systems is necessary for our results. We present this theorem as Theorem 3. This result is stated (and proved) for the purposes of probabilistic logic as Theorem 5.1.5. in [6]. The interesting part of Theorem 3 is proved in [4, p. 145].

**Theorem 3.** Let $S$ be a linear system of $n$ variables and of $r$ linear equalities and/or inequalities with integer coefficients each of size at most $l$. Assume that the vector $x = x_1, \ldots, x_n$ is a solution of $S$ such that for all $i \in \{ 1, \ldots, n \}$, $x_i \geq 0$. Then there is a vector $x^* = x_1^*, \ldots, x_n^*$ with the following properties:

1. $x^*$ is a solution of $S$ and at most $r$ entries of $x^*$ are positive;
2. for all $i$, $x_i^*$ is a non-negative rational number with size bounded by $2 \cdot (r \cdot l + r \cdot \log_2(r) + 1)$;
3. for all $i \in \{ 1, \ldots, n \}$, if $x_i^* > 0$ then $x_i > 0$.  

Now we can prove the small model property for LPP\(_1\).

**Theorem 4** (Small Model Property for LPP\(_1\)). If \(A\) is LPP\(_{1,\text{Meas}}\)-satisfiable then it is satisfiable in a model \(M = \langle U, W, H, \mu, v \rangle\) that satisfies the following properties:

1. \(|U| \leq 2^{|A|}\) and in every world of \(U\) exactly one atom of \(A\) holds.
2. For every \(w \in U\) the following holds:
   a. \(W_w = U\) and \(H_w\) is the powerset of \(U\).
   b. For every \(u \in W_w\), \(\mu_w(\{u\}) \leq 2 \cdot (|A| \cdot |A| + |A| \cdot \log_2(|A|) + 1)\) and \(\mu_w(\{u\}) \in \mathbb{Q}\).
   c. For every \(V \in H_w\): \(\mu_w(V) = \sum_{u \in V} \mu_w(\{u\})\).
   d. The number of \(u\)’s such that \(\mu_w(\{u\}) > 0\), is at most \(|A|\).

**Proof.** In the proof of Lemma 5.3.6 of [6] a model for \(A\) that satisfies the conditions of the theorem is constructed. The most interesting property of the small model is (2)d, which can be proved by an application of Theorem 3. □

**Theorem 5.** The LPP\(_{1,\text{Meas}}\)-satisfiability problem is PSPACE-hard.

**Proof.** Since probability spaces are non-empty sets it makes sense to draw a reduction from modal logic D, which is complete for serial Kripke structures. Let \(A\) be a modal formula and let \(f(A)\) be the L\(_{LPP_1}\)-formula that is obtained by replacing any occurrence of \(\square\) in \(A\) with \(P \geq 1\). We will prove that \(A\) is satisfiable iff \(f(A)\) is LPP\(_{1,\text{Meas}}\)-satisfiable.

Assume that \(A\) is satisfiable. Then \(A\) is satisfiable in a finite model [5]. We can create an LPP\(_{1,\text{Meas}}\)-model where the probability space of each world \(w\) consists of the worlds accessible to \(w\) and \(w\) assigns a uniform probability to each of these worlds. Then we can prove that \(f(A)\) is satisfied in this LPP\(_{1,\text{Meas}}\)-model.

Assume that \(f(A)\) is satisfiable. Then it is satisfiable in a model that has the properties of Theorem [3]. We define a Kripke model where \(u\) is accessible from \(w\), if \(\mu_w(\{u\}) > 0\). Then we can prove that \(A\) is satisfiable in this Kripke model. □

### 3 Adding Justifications

Justification logics are modal logics that use explicit terms instead of the modality \(\square\). The terms are constructed according to the grammar \(t ::= c \mid x \mid (t \cdot t) \mid !t\) where \(c\) is a constant and \(x\) is a variable. \(\mathbb{T}m\) denotes the set of all terms. For \(t \in \mathbb{T}m\) and any non-negative integer \(n\) we define: \(!^0 t := t\) and \(!^{n+1} t := ! (!^n t)\). The language of justification logic, \(\mathcal{L}_J\), is defined by the grammar \(A ::= p \mid \neg A \mid A \land A \mid t : A\) where \(t \in \mathbb{T}m\) and \(p \in \text{Prop}\). For this presentation we take \((J)\), i.e. \(\vdash u : (A \to B) \to (v : A \to u \cdot v : B)\) as the only axiom of
the logic J. The logic J is defined by taking a system for classical propositional logic, the axiom J and the rule (AN!):

\[ \vdash !^n c : !^{n-1} c : \cdots : ! c : c : A, \]

where \( c \) is a constant, \( A \) is an axiom-instance and \( n \in \mathbb{N} \).

Semantics for J are given by M-models.

**Definition 6 (M-model).** An M-model is a pair \( \langle v, \mathcal{E} \rangle \), where \( v : \text{Prop} \to \{T, F\} \) and \( \mathcal{E} : \text{Tm} \to \mathcal{P}(\mathcal{L}_J) \) such that for every \( u, v \in \text{Tm} \), for a constant \( c \) and \( A \in \mathcal{L}_J \) we have:

1. \( (A \rightarrow B \in \mathcal{E}(u) \land A \in \mathcal{E}(v)) \implies B \in \mathcal{E}(u \cdot v) \);
2. if \( c \) is a constant, \( A \) an axiom and \( n \in \mathbb{N} \) then \( !^n c : !^{n-1} c : \cdots : ! c : c : A \in \mathcal{E}(!^n c) \).

The language of probabilistic justification logic is defined as a combination of \( \mathcal{L}_J \) and \( \mathcal{L}_{LPP_1} \): 

\[ A ::= p \mid \neg A \mid A \land A \mid t : A \mid P_{\geq s} A. \]

So PPJ is defined by taking the axioms and rules of \( \mathcal{L}_{LPP_1} \) together with the axiom and rules of J ((AN!) can now be applied to probabilistic axioms instances too). A measurable model for probabilistic justification logic is defined by replacing the truth assignment in Definition 1 with an M-model. The class of measurable models is \( \text{PPJ}_{\text{Meas}} \). Satisfiability in \( \text{PPJ}_{\text{Meas}} \) is defined by adding the line 

\[ M, w \models t : A \iff A \in \mathcal{E}_w(t) \]

in Definition 2, where \( \mathcal{E}_w \) is the evidence function that corresponds to the M-model assigned to the world \( w \). Soundness and completeness of probabilistic logics with respect to measurable models is proved in [6]. Theorem 4 holds for PPJ as well [6].

**4 The Tableau Procedure**

Our tableaux are trees where the nodes are formulas prefixed with world and truth signs. So, the node \( w T A (w F A) \) intuitively means that formula \( A \) is true (resp. false) at world \( w \). A branch is a path that starts at a result of an application of the rule prob (defined later) or at the root and ends at the premise of an application of the rule prob or at a leaf. A branch is called closed if it contains both \( w T A \) and \( w F A \) for some \( A \). Otherwise it is called open. A branch is called complete if no rule is applicable in this branch. Otherwise it is called incomplete. The only rule that can create new worlds in our tableaux is the rule prob. For this reason we can assign a world to each branch (of course the same world may be assigned to several branches). So, \( b_w \) denotes a branch where all the formulas are prefixed with \( w \). We will use the abbreviation \( "w T A" \) ("\( w F A" \)) to denote that the node \( w T A (w T A) \) appears in the tableau. Our tableau rules are the rules for classical propositional logic and the rule prob:

\[ \text{This paper aims at illustrating the combination of justification logic and probabilistic logic. Therefore, we consider it useful to study the smallest possible framework. As a consequence we present a variant of logic J without the operator + and with the maximal constant specification. Other features of justification logic, like the term operator +, other justification axioms etc. can be added to our framework without complications.}\]
In rule prob the w.'s are new world prefixes and for all i, j, \( w \ T \ P_{≥sij} B_{ij} \) or \( w \ F \ P_{≥sij} B_{ij} \). In our tableaux we treat formulas starting with a justification term as atomic formulas. In other words, no rule can be applied to a formula of the form \( t : A \). The tableau procedure consists of two parts: first we apply the rules and then we mark worlds and applications of prob satisfiable. Assume that A is a formula that we want to test for satisfiability. We take \( w \ T \ A \) as the root of the tableau and then we apply the following steps:

1. Apply the propositional rules for as long as possible. If there exists an open branch that contains \( w \ T \ P_{≥s} B \) or \( w \ F \ P_{≥s} B \) for some s and B then we go to step 2. Otherwise stop.

2. Apply the probabilistic tableau rule to every open branch. Then go to step 1.

The second part of the tableau procedure consists of a method for marking worlds and applications of prob satisfiable. In order to mark worlds satisfiable we traverse the tree from the leaves to the root and we make sure that the justification and the probabilistic restrictions are satisfied. In order to check the probabilistic constraints we have to mark applications of prob as satisfiable as well.

**Marking Worlds Satisfiable.** Let \( b_w \) be one of the branches that correspond to world \( w \). In order to check that “justification constraints” hold in \( b_w \) we have to extend the satisfiability algorithm for justification logic J \[^8\] in the probabilistic context. The algorithm of \[^8\] checks that if \( w \ F \ t : A \) then \( A / \in E(t) \) where \( \langle v, E \rangle \) is the minimum M-model that is defined by the formulas \( u : B \) such that \( w \ T \ u : B' \). This algorithm uses a procedure for unifying axiom schemata of justification logic. In order to extend this algorithm to the probabilistic setting we have to extend the unification to probabilistic axiom schemata. These axioms come with some linear side conditions (see Table 1), so their unification will create a linear system. The unification algorithm then succeeds if this linear system is satisfiable. For more details see Lemma 5.3.3 of \[^6\]. Now \( w \) will be marked satisfiable if there exists an open \( b_w \) such that the extended algorithm for justification satisfiability holds and either it is a complete branch or it ends in an application of prob and this application is marked satisfiable.

**Marking Applications of prob Satisfiable.** Let \( ρ \) be an application of the rule prob on branch \( b_w \). We associate variables \( x_i \) with every world \( w.i \), even if \( w.i \) is marked not satisfiable. The \( x_i \) corresponds to the probabilities that world \( w \) assigns to \( w.i \) in a small model for \( A \) (i.e. \( x_i = μ(w)(\{w.i\}) \)) in the sense of Theorem \[^4\]. We mark \( ρ \) satisfiable if the following linear system is solvable:
\[
\sum_{i=1}^{n} x_i = 1 \\
(\forall 1 \leq i \leq n)[x_i \geq 0]
\]

if \( \text{"} w \ T \ P_{\geq s} C \text{"} \) then \( \sum_{\{i | \text{"} w.i \ T \ C \text{"} \}} x_i \geq s \)

if \( \text{"} w \ F \ P_{\geq s} C \text{"} \) then \( \sum_{\{i | \text{"} w.i \ T \ C \text{"} \}} x_i < s \).

If the initial formula \( A \) belongs to a world that is marked satisfiable then we return satisfiable. After an application of a propositional rule the length of the formula decreases. After an application of the probabilistic rule the nesting depth of probabilistic operators decreases. Hence, our tableau procedure terminates. By the procedure of marking worlds satisfiable and by Theorem 4 we get the following theorem.

**Theorem 7.** Let \( A \) be a PPJ-formula. The tableau method returns \( A \) is satisfiable iff \( A \) is satisfiable in a measurable model.

**Theorem 8.** The satisfiability problem for PPJ is PSPACE-complete.

**Proof.** The lower bound follows from Theorem 5. The upper bound follows by the fact that we can traverse the tableau for probabilistic justification logic in a depth first fashion by reusing space. Whether a world \( w \) will be marked satisfiable depends on the worlds that appear below it in the tableau. We only need a polynomial number of bits that can be reused in order to decide the satisfiability of the linear systems and justification constraints. A complication arises since rule \( \text{prob} \) creates exponentially many worlds. However, because of Theorem 4(2)d in every application of rule \( \text{prob} \) we can guess a linear number of branches to which we will assign non-zero probability. We conclude that the depth first search operates in non deterministic polynomial space.

**Acknowledgements:** The author is grateful to Antonis Achilleos and the anonymous referees for many useful comments and to ERC for financial support (project EPS 313360).

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