Covariant Coordinate Transformations
on Noncommutative Space

R. Jackiw

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge MA 02139-4307, USA

S.-Y. Pi

Physics Department
Boston University
Boston MA 02215, USA

Abstract

We show how to define gauge-covariant coordinate transformations on a noncommuting space. The construction uses the Seiberg-Witten equation and generalizes similar results for commuting coordinates.
Introduction

Coordinate transformations in a gauge theory can be combined with gauge transformations. A “gauge-covariant coordinate transformation” is an especially interesting and useful combination of the two. Identified some years ago [1, 2], this combination has now arisen in discussions of noncommutative gauge theories [3].

In our paper we investigate the action of coordinate transformations on noncommutative vector gauge potentials (connections) and tensor gauge fields (curvatures). We establish that a noncommutative version of the gauge-covariant coordinate transformation is particularly appropriate in this context.

In the next Section, we review the relevant construction for commutative non-Abelian fields. Then the noncommutative story for U(1) fields is explained in Section 2.

1 Coordinate Transformations for Commuting Gauge Fields

Under a coordinate transformation

\[ x^\mu \to x'^\mu = x'^\mu(x) \]  

(1.1)
a covariant vector field \( A \) transforms as

\[ A_\mu \to A'_\mu \]

\[ A'_\mu(x') = \frac{\partial x'^\alpha}{\partial x^\mu} A_\alpha(x) \]  

(1.2)

For infinitesimal transformations

\[ x'^\mu(x) = x^\mu - f^\mu(x) \]

\[ \delta_f x^\mu \equiv x'^\mu - x^\mu = -f^\mu(x) \]  

(1.3)

(1.2) implies

\[ \delta_f A_\mu(x) \equiv A'_\mu(x) - A_\mu(x) = f^\alpha(x) \frac{\partial}{\partial x^\alpha} A_\mu(x) + \left( \frac{\partial}{\partial x^\mu} f^\alpha(x) \right) A_\alpha(x) = L_f A_\mu \]  

(1.4)

where the last equality defines the Lie derivative, \( L_f \), of a covariant vector with respect to the (infinitesimal) transformation \( f \).

The composition law for the transformations is summarized by the commutator algebra

\[ [\delta_f, \delta_g] x^\mu = \delta_h x^\mu \]  

(1.5)

\[ [\delta_f, \delta_g] A_\mu = \delta_h A_\mu \]  

(1.6)
where \( h \) is the Lie bracket of \( f \) and \( g \):

\[
h^\mu = g^\alpha \partial_\alpha f^\mu - f^\alpha \partial_\alpha g^\mu.
\]

(1.7)

Taking \( A \) to be a Hermitian non-Abelian potential/connection and defining the field strength/curvature \( F \) by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]
\]

we find that for any variation \( \delta \) of \( A \)

\[
\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} = L_f F_{\mu\nu}.
\]

(1.8)

When the variation of \( A \) is as in (1.4), \( \delta_f F \) becomes the appropriate Lie derivative

\[
\delta_f F_{\mu\nu} = f^\alpha \partial_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} = L_f F_{\mu\nu}.
\]

(1.11)

Also the algebra (1.5)–(1.7) is maintained:

\[
[\delta_f, \delta_g] F_{\mu\nu} = \delta_h F_{\mu\nu}.
\]

(1.12)

This of course is the entirely familiar geometrical story about coordinate transformations. However, one may point out a defect: gauge covariance is not maintained. This is especially evident in (1.11) where the gauge-covariant \( F \) responds on the right side with an ordinary derivative, which is not gauge covariant. Further, the response (1.4) of the vector potential is noncovariant. This defect leads to further awkwardness. For example, derivation of conserved quantities associated with symmetries of dynamics by Noether’s theorem produces gauge-noninvariant entities that must be “improved” [4]. Also, recognizing symmetric connections and curvatures, i.e., configurations that are invariant under specific coordinate transformations, is obscured by the gauge dependence [4].

The gauge-covariant coordinate transformation is introduced to overcome the above-mentioned defect. Observe that (1.4) may be identically presented as

\[
\delta_f A_\mu = f^\alpha \left( \partial_\alpha A_\mu - \partial_\mu A_\alpha - i[A_\alpha, A_\mu] \right) + f^\alpha \partial_\mu A_\alpha + i f^\alpha [A_\alpha, A_\mu] + \partial_\mu f^\alpha A_\alpha
\]

\[
= f^\alpha F_{\alpha\mu} + D_\mu (f^\alpha A_\alpha).
\]

(1.13)

The last element is recognized as a gauge transformation[1]. Thus if we supplement the diffeomorphism generated by \( f \) with a gauge transformation generated by \( \Lambda_f = -f^\alpha A_\alpha \) (a freedom on which we can rely in a gauge theory) we arrive at the gauge-covariant transformation

\[
\bar{\delta}_f A_\mu = f^\alpha F_{\alpha\mu}.
\]

(1.14)

[1]Formula (1.13) in the Abelian case is an instance of the well-known identity \( L_f = i_f d + d i_f \), where \( i_f \) is the evaluation on the vector \( f \).
Moreover, when the above gauge-covariant response of $A$ is used in (1.9), a gauge-covariant response is experienced by $F$:

$$\delta_f F_{\mu\nu} = f^\alpha D_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha}$$ (1.15)

which again differs from the conventional Lie derivative expression (1.11) by a gauge transformation generated by $\Lambda_f$. We call (1.15) the gauge-covariant Lie derivative.

Adoption of the transformation rules (1.14) and (1.15) simplifies the task of finding gauge-invariant conserved quantities [4], of recognizing symmetric configuration [5], and other advantages ensue as well [3]. However, the composition algebra is modified. One finds

$$[\delta_f, \delta_g] A_\mu = \delta_h A_\mu + D_\mu (g^\alpha f^\beta F_{\alpha\beta})$$ (1.16)

$$[\delta_f, \delta_g] F_{\mu\nu} = \delta_h F_{\mu\nu} + i [g^\alpha f^\beta F_{\alpha\beta}, F_{\mu\nu}] .$$ (1.17)

In both instances the algebra closes up to a gauge transformation generated by $g^\alpha f^\beta F_{\alpha\beta}$. Thus the gauge-covariant coordinate transformation rules provide a representation up to gauge transformations of the coordinate transformations [3].

## 2 Coordinate Transformations for Noncommuting Gauge Fields

When coordinates do not commute, the formulas in the previous Section present questions about how factors should be ordered. We shall answer such questions by adopting simplifying Ansätze, and by some analysis.

To begin, we presume that the relation (1.1) between $x'$ and $x$ is at most linear; i.e., it is affine. Then $\partial x^\alpha / \partial x'^\mu$ is constant and Eq. (1.2) may be taken over for a noncommuting vector field $\hat{A}_\mu$

$$\hat{A}'_\mu(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \hat{A}_\alpha(x) .$$ (2.1)

(Below we shall find a second reason for restricting the the transformations to be affine.)

Next we wish to derive the noncommutative analog to (1.4). We define the ordering of $x$ within $\hat{A}_\mu(x)$ by the Weyl procedure

$$\hat{A}_\mu(x) = \int_p e^{-ip \cdot x} a_\mu(p) .$$ (2.2)

Here $a_\mu(p)$ is a classical, $c$-number function and the ordering of the noncommutative $x$ is
defined by expanding the exponential. It follows that infinitesimally
\[
\hat{A}'_\mu(x') \approx \hat{A}'_\mu(x - f) = \int_p e^{-ip(x - f)} a'_\mu(p) \\
= \int_p e^{ip \frac{x}{2}} e^{-ip \cdot f} e^{ip \frac{x}{2}} a'_\mu(p) \\
\approx \int \left( 1 + i \frac{1}{2} p \cdot f \right) e^{-ip \cdot (x - f)} \left( 1 + i \frac{1}{2} p \cdot f \right) a'_\mu(p) \\
\approx \hat{A}'_\mu(x) + \frac{1}{2} f^\alpha \partial_\alpha \hat{A}_\mu + \frac{1}{2} \partial_\alpha A_\mu f^\alpha .
\](2.3)

That the second integral equals the first follows from the Baker-Hausdorff formula and the linearity of \( f \) in \( x \).

As a consequence of the above, we adopt (1.4) to the noncommutative case as
\[
\delta_f \hat{A}_\mu = \frac{1}{2} \{ f^\alpha, \partial_\alpha \hat{A}_\mu \}_* + \partial_\mu f^\alpha \hat{A}_\alpha .
\](2.4)

Henceforth, we view all quantities as commuting \( c \)-numbers, but all products are “star” products, defined by
\[
(O_1 \star O_2)(x) = e^{\frac{i}{2} \theta_{\alpha \beta} \partial_\alpha \partial'_\beta O_1(x) O_2(x') } \bigg|_{x' \to x} .
\](2.5)

In (2.4), the curly bracket is the “star” anticommutator; later we shall also use a square bracket to denote the “star” commutator: \([ , ]_*\). In the last entry of (2.4), \( \partial_\mu f^\alpha \) is \( x \)-independent, hence no star product is needed. Also we restrict the discussion to noncommutative \( \text{U}(1) \) gauge theories, where the potential responds to a gauge transformation as
\[
\hat{A}_\mu \to \hat{A}_\mu^\lambda = (e^{i\lambda}) \star (\hat{A}_\mu + i\partial_\mu) \star (e^{i\lambda})^{-1} \\
\approx \hat{A}_\mu + \partial_\mu \lambda - i[\hat{A}_\mu, \lambda]_* = \hat{A}_\mu + D_\mu \lambda
\](2.6)
\[
D_\mu = \partial_\mu - i[\hat{A}_\mu, ]_*
\](2.7)

and the field strength is constructed so that its response is covariant:
\[
\hat{F}_{\mu \nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*
\](2.8)
\[
\hat{F}_{\mu \nu} \to (e^{i\lambda}) \star \hat{F}_{\mu \nu} \star (e^{i\lambda})^{-1} \\
\approx \hat{F}_{\mu \nu} - i[\hat{F}_{\mu \nu}, \lambda]_* .
\](2.9)

Formula (2.4) suffers from noncovariance defects in two ways. First, as in the commutative case, the response involves the vector potential and is not gauge covariant. Second, multiplication by \( x \) (which is present in \( f^\alpha \)) is not covariant, in the sense that if \( \Phi \) is a
gauge-covariant quantity, \( x^\mu \ast \Phi \) is not. In order that covariance be preserved \( x^\mu \) should be supplemented by \( \theta^{\mu \nu} \tilde{A}_\nu \); that is, rather than multiplying by \( f^\alpha \) one should multiply by

\[
\tilde{f}^\alpha = f^\alpha + \partial_\beta f^\alpha \theta^{\beta \gamma} \hat{A}_\gamma .
\]  

(2.10)

There does not seem to be a way to remedy these two defects if one remains with \( (2.1) \). But now we adopt the Seiberg-Witten viewpoint \([7]\), and discover a way of constructing a covariant transformation law, which generalizes \( (1.14) \) to the noncommutative situation.

In the Seiberg-Witten framework, we view \( \hat{A} \) to be a function of \( \theta \) and of the commutative \( A \) (and its derivatives). This comes about because the \( \theta \)-dependence of \( \hat{A} \) is governed by the Seiberg-Witten equation

\[
\frac{\partial \hat{A}_\mu}{\partial \theta^{\alpha \beta}} = -\frac{1}{8} \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + \tilde{F}_{\beta \mu} \} \ast - (\alpha \leftrightarrow \beta) .
\]  

(2.11)

Since the equation is of the first order, a solution is determined by specifying an initial condition for \( \hat{A} \) at \( \theta = 0 \), which is just \( A \). Although we do not need to know the explicit solution to \( (2.11) \), for definiteness we record it to lowest order in \( \theta \):

\[
\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha \beta} A_\alpha (\partial_\beta A_\mu + F_{\beta \mu}) + \cdots .
\]  

(2.12)

We now understand that if we accept that \( \hat{A} \) transforms under affine transformations as a covariant vector, so that \( (2.1) \) and \( (2.4) \) are true, then in the Seiberg-Witten expression for \( \hat{A} \) one must transform \( A \) as a covariant vector and \( \theta \) as a contravariant tensor; infinitesimally

\[
\delta f \theta^{\alpha \beta} = f^\gamma \partial_\gamma \theta^{\alpha \beta} - \partial_\gamma f^\alpha \theta^{\gamma \beta} - \partial_\gamma f^\beta \theta^{\alpha \gamma} .
\]  

(2.13)

[This is seen explicitly in the approximate formula \( (2.12) \).] Moreover, since \( \theta \) is constant and should remain constant, the first term on the right in \( (2.13) \) vanishes, and the remaining terms will not generate an \( x \)-dependence, provided \( f \) is affine in \( x \). (This gives the second reason for restricting transformations to be affine.)

We now assert the principle that \( \theta \) should not be transformed, because it is not a dynamical variable; rather it functions as a background. Thus to get the proper infinitesimal transformation for \( \hat{A} \), we must subtract from \( (2.4) \) the infinitesimal transformation of \( \theta \) \( (2.13) \). We assert therefore that the desired infinitesimal coordinate transformation for \( \hat{A} \) is not \( (2.4) \), but rather

\[
\Delta f \hat{A}_\mu = \frac{1}{2} \{ f^\alpha, \partial_\alpha \hat{A}_\mu \}_\ast + \partial_\mu f^\alpha \hat{A}_\alpha - \frac{\partial \hat{A}_\mu}{\partial \theta^{\alpha \beta}} \delta f \theta^{\alpha \beta} .
\]  

(2.14a)

which reads, once \( (2.11) \) and \( (2.13) \) have been used,

\[
\Delta f \hat{A}_\mu = \frac{1}{2} \{ f^\alpha, \partial_\alpha \hat{A}_\mu \}_\ast + \partial_\mu f^\alpha \hat{A}_\alpha - \frac{1}{4} \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + F_{\beta \mu} \}_\ast (\partial_\gamma f^\alpha \theta^{\gamma \beta} + \partial_\gamma f^\beta \theta^{\alpha \gamma}) .
\]  

(2.14b)
Note that just as $\delta f^{\hat{A}}_{\mu}$ in (2.4) this suffers from the same noncovariance defects. A straightforward manipulation shows that the above may also be written as
\[
\Delta f^{\hat{A}}_{\mu} = \frac{1}{2} \{ \hat{f}^\alpha, \hat{F}_{\alpha\mu} \} + D_\mu \frac{1}{2} \{ f^\alpha + \frac{1}{2} \partial_\alpha f^\omega \theta^{\omega\phi} \hat{A}_\phi, \hat{A}_\alpha \}.
\]
(2.15)
This shows that when a coordinate transformation is effected on $\hat{A}$ by only transforming $\hat{A}$ as a vector in the Seiberg-Witten expression for $\hat{A}$, and not transforming $\theta$, then the response is the covariant first term on the right side of (2.15) and a gauge transformation. As in the commutative case, we can supplement the coordinate transformation $\Delta f$ with a gauge transformation generated by
\[
\hat{\delta} f^{\hat{A}}_{\mu} = \frac{1}{2} \{ \hat{f}^\alpha, \hat{F}_{\alpha\mu} \} \cdot
\]
(2.16)
This is the desired generalization of (1.13). It is very satisfying that both defects have been removed in the response (2.16) – the right side is covariant since it involves multiplication by $\hat{f}^\alpha$ (not $f^\alpha$) and there occurs the field strength/curvature $\hat{F}$ (not the potential/connection $\hat{A}$).

Next we look to the transformation law for $\hat{\delta} f^{\hat{A}}$. Formula (1.9) still holds in the noncommutative framework because we do not vary $\theta$ in the $\star$-commutator contributing to $\hat{F}$. Using (2.16) for the variation of $\hat{A}$, we arrive at
\[
\hat{\delta} f^{\hat{A}}_{\mu} = \frac{1}{2} \{ \hat{f}^\alpha, \hat{F}_{\alpha\mu} \} + \partial_\mu f^\alpha \hat{F}_{\alpha\nu} + \partial_\nu f^\alpha \hat{F}_{\nu\alpha} - \frac{1}{2} \partial_\alpha f^\gamma \theta^{\alpha\beta} \{ \hat{F}_{\alpha\beta}, \hat{F}_{\gamma\nu} \}.
\]
(2.17)
The first three terms on the right provide a natural noncommutative generalization of the gauge-covariant Lie derivative of (1.15). But the last term gives an addition. This addition is analyzed further by defining
\[
\theta^{\gamma\beta}_f = \partial_\alpha f^\gamma \theta^{\alpha\beta}
\]
(2.18a)
and recognizing that the part antisymmetric in $(\gamma, \beta)$ is proportional to the variation (2.13) of $\theta$. Therefore
\[
\theta^{\gamma\beta}_f = -\frac{1}{2} \delta f^{\gamma\beta} + \theta^{\gamma\beta}_f
\]
(2.18b)
with $\theta^{(\gamma\beta)}_f$ being the symmetric part. Substituting (2.18a) into (2.17) and noting that last term in parentheses is antisymmetric in $(\gamma, \beta)$, we are left with
\[
\hat{\delta} f^{\hat{A}}_{\mu} = \frac{1}{2} \{ \hat{f}^\alpha, \hat{F}_{\alpha\mu} \} + \partial_\mu f^\alpha \hat{F}_{\alpha\nu} + \partial_\nu f^\alpha \hat{F}_{\nu\alpha} - \frac{1}{2} \delta f^{\alpha\beta} \{ \hat{F}_{\alpha\mu}, \hat{F}_{\beta\nu} \}.
\]
(2.19)
(We emphasize that no transformation of $\theta$ is carried out. Eq. (2.19) arises because a certain combination of $f$ and $\theta$ combines into an expression identical with $\delta f^{\alpha\beta}$.)
Thus we conclude that those affine coordinate transformations generated by \( f \) that also leave \( \theta \) invariant \( (\delta_f \theta = 0) \) transform the curvature by a gauge-invariant, noncommutative Lie derivative

\[
\hat{\delta}_f \hat{F}_{\mu\nu} = \frac{1}{2} \left\{ \hat{f}^\alpha, D_\alpha \hat{F}_{\mu\nu} \right\}_\ast + \partial_\mu f^\alpha \hat{F}_{\alpha\nu} + \partial_\nu f^\alpha \hat{F}_{\mu\alpha} \quad (\delta_f \theta = 0) .
\]  

(2.20)

Now consider the algebra of these transformations. From (2.16) and (2.17) we compute

\[
[\hat{\delta}_f, \hat{\delta}_g] \hat{A}_\mu = \hat{\delta}_h \hat{A}_\mu + D_\mu \left( \frac{1}{8} \left\{ \hat{g}^\alpha, \left\{ \hat{f}^\beta, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast + \frac{1}{8} \left\{ \hat{f}^\beta, \left\{ \hat{g}^\alpha, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast \right) 
+ \frac{i}{2} \partial_\alpha f^\psi \partial_\alpha g^\phi \partial_\gamma \partial_\delta \left( [\hat{F}_{\omega\phi}, \hat{D}_\beta \hat{F}_{\mu\delta} + \hat{D}_\delta \hat{F}_{\mu\beta}]_\ast + [\hat{F}_{\beta\delta}, \hat{D}_\omega \hat{F}_{\mu\phi} + \hat{D}_\phi \hat{F}_{\mu\omega}]_\ast \right. 
+ 2 \left[ \hat{F}_{\omega\delta}, \hat{D}_\beta \hat{F}_{\phi\mu} \right] - 2 \left[ \hat{F}_{\phi\delta}, \hat{D}_\beta \hat{F}_{\omega\mu} \right] .
\]  

(2.21)

In the general case the algebra does not close, not even up to a gauge transformation: The first term on the right, where \( \hat{h}^\alpha = h^\alpha + \partial_\beta h^\alpha \theta^{\beta\gamma} A_\gamma \), with \( h \) given by the Lie bracket (1.17) is needed for closure; the second, gauge transformation term is the noncommutative generalization of the gauge transformation in (1.16); the last term spoils closure. However, when the expressions involving \( \theta \) are written with the help of (2.18), (2.21) becomes

\[
[\hat{\delta}_f, \hat{\delta}_g] \hat{A}_\mu = \hat{\delta}_h \hat{A}_\mu + D_\mu \left( \frac{1}{8} \left\{ \hat{g}^\alpha, \left\{ \hat{f}^\beta, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast + \frac{1}{8} \left\{ \hat{f}^\beta, \left\{ \hat{g}^\alpha, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast \right) 
- \frac{i}{16} \left( \delta_f \theta^{\alpha\beta} \delta_\gamma \theta^{\gamma\delta} + \theta_f^{(\alpha\beta)} \delta_\gamma \theta^{\gamma\delta} - \frac{1}{2} \delta_f \theta^{\alpha\beta} \delta_\gamma \theta^{\gamma\delta} \right) [\hat{F}_{\alpha\delta}, \hat{F}_{\gamma\beta}]_\ast 
- \frac{i}{8} \delta_f \theta^{\alpha\gamma} \delta_\gamma \theta^{\beta\delta} [\hat{F}_{\alpha\beta}, D_\gamma \hat{F}_{\delta\mu} + D_\delta \hat{F}_{\gamma\mu}]_\ast .
\]  

(2.22)

Thus if again we restrict the coordinate transformations to those that leave \( \theta \) invariant, \( \delta_{f,g} \theta = 0 \), the algebra closes up to gauge transformations, in complete analogy to (1.16):

\[
[\hat{\delta}_f, \hat{\delta}_g] \hat{A}_\mu = \hat{\delta}_h \hat{A}_\mu + D_\mu \left( \frac{1}{8} \left\{ \hat{g}^\alpha, \left\{ \hat{f}^\beta, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast + \frac{1}{8} \left\{ \hat{f}^\beta, \left\{ \hat{g}^\alpha, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast \right) 
(\delta_{f,g} \theta = 0) .
\]  

(2.23)

The commutator acting on \( \hat{F}_{\mu\nu} \) behaves similarly. We record only the formula that holds when \( \theta \) is left invariant, that is, when (2.24) is true:

\[
[\hat{\delta}_f, \hat{\delta}_g] \hat{F}_{\mu\nu} = \hat{\delta}_h \hat{F}_{\mu\nu} + \frac{i}{8} \left[ \left\{ \hat{g}^\alpha, \left\{ \hat{f}^\beta, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast + \left\{ \hat{f}^\beta, \left\{ \hat{g}^\alpha, \hat{F}_{\alpha\beta} \right\}_\ast \right\}_\ast, \hat{F}_{\mu\nu} \right]_\ast .
\]  

(2.24)

Evidently this is the noncommutative generalization of (1.17).
Finally, we examine how our transformations look in the context of the Seiberg-Witten map. We begin with the $O(\theta)$ solution in (2.12) and its consequence

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu} + \cdots. \tag{2.25}$$

We transform the left side of (2.12) according to (2.16) and use (2.25) to express everything in terms of commuting variables to $O(\theta)$. In this way we get

$$\tilde{\delta}_f \tilde{A}_\mu = f^\gamma F_{\mu\gamma} + f^\gamma \theta^{\alpha\beta} (F_{\alpha\gamma} F_{\beta\mu} + A_\beta \partial_\alpha F_{\gamma\mu}) + \partial_\alpha f^\gamma \theta^{\alpha\beta} A_\beta F_{\gamma\mu} + \cdots. \tag{2.26a}$$

Correspondingly, the right side of (2.12) is transformed according to (1.14) and (1.15):

$$\tilde{\delta}_f (A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu})) = f^\gamma F_{\gamma\mu} + \frac{1}{2} f^\gamma \theta^{\alpha\beta} (F_{\alpha\gamma} (\partial_\beta A_\mu + F_{\beta\mu}) + A_\beta \partial_\alpha F_{\gamma\mu}) + \theta^{\alpha\beta} A_\beta (\partial_\alpha f^\gamma F_{\gamma\mu} + \frac{1}{2} \partial_\mu f^\gamma F_{\alpha\gamma}) + \cdots. \tag{2.26b}$$

Forming the difference between (2.26a) and (2.26b) we find that it equals $\partial_\mu (\frac{1}{2} f^\gamma \theta^{\alpha\beta} F_{\alpha\gamma} A_\beta)$; that is, the two ways of effecting the coordinate gauge-covariant transformation coincide up to a gauge transformation.

A similar result emerges when transformations of the field strength are compared. We begin with (2.25) and transform the left side according to (2.20) and express everything in terms of commuting variables with the help of (2.12) and (2.25). One finds

$$\tilde{\delta}_f \tilde{F}_{\mu\nu} = f^\gamma \partial_\gamma \tilde{F}_{\mu\nu} + \partial_\mu f^\gamma \tilde{F}_{\gamma\nu} + \partial_\nu f^\gamma \tilde{F}_{\mu\gamma} + f^\gamma \theta^{\alpha\beta} \partial_\alpha A_\gamma \partial_\beta F_{\mu\nu} - \theta^{\alpha\beta} \partial_\beta f^\alpha A_\alpha \partial_\gamma F_{\mu\nu} + \cdots. \tag{2.27a}$$

Correspondingly, the transformation of the right side of (2.25) reads

$$\tilde{\delta}_f (F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu}) = f^\gamma \partial_\gamma \tilde{F}_{\mu\nu} + \partial_\mu f^\gamma \tilde{F}_{\gamma\nu} + \partial_\nu f^\gamma \tilde{F}_{\mu\gamma} + f^\gamma \theta^{\alpha\beta} \partial_\alpha A_\gamma \partial_\beta F_{\mu\nu} - \theta^{\alpha\beta} \partial_\beta f^\alpha A_\alpha \partial_\gamma F_{\mu\nu} + \theta^{\alpha\beta} \partial_\alpha f^\gamma (F_{\gamma\mu} F_{\beta\nu} - F_{\gamma\nu} F_{\beta\mu}) + \cdots. \tag{2.27b}$$

The difference between the two appears in the last entry of (2.27b), which can be rewritten as $\theta^{\gamma\beta} (F_{\gamma\mu} F_{\beta\nu} - F_{\gamma\nu} F_{\beta\mu})$. Since the parenthetical expression is antisymmetric in $(\gamma, \beta)$, the above also equals $\delta_f \theta^{\gamma\beta} F_{\gamma\nu} F_{\beta\mu}$ and vanishes for transformations that preserve $\theta$. Thus the two forms of the gauge-covariant coordinate transformations (2.27) in fact coincide.

### 3 Conclusion

When transformations of noncommuting coordinates are suitably restricted, we can define gauge-covariant rules for the transformation of gauge fields, just as in the commuting case.
To begin, the coordinate transformation is taken to be affine; then the gauge-covariant rule (2.16) for transforming the noncommuting vector potential/connection emerges with the help of the Seiberg-Witten equation. When the coordinate transformation is further restricted so that it leaves the noncommutativity tensor $\theta$ invariant, the gauge-covariant transformation rule extends to the field strength/curvature, (2.20). For these transformations, the commutator algebra (2.23) and (2.24) closes up to a gauge transformation – behavior familiar from the commutative case.

Our investigation concerns the kinematics of coordinate transformations. The dynamical issue of identifying transformations that leave the equations of motion invariant and lead to conserved quantities is addressed in a separate paper [8]. There it is established that in Noether-theorem derivations of conserved charges, $\theta$ should not be transformed, in concert with the position taken in our paper.

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2 Of course with the further restriction that the diffeomorphism leave $\theta$ unchanged, the covariant transformation law (2.16) for the potential/connection is established directly from (2.4) – the Seiberg-Witten equation is not needed in (2.14) because $\delta_f \theta$ is taken to vanish.
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