HIGHER WEAK DERIVATIVES AND COMMUTATORS

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Dedicated to R. V. Kadison on the occasion of his ninetieth birth day.

Abstract. Let $D$ be a self-adjoint operator on a Hilbert space $H$ and $x$ a bounded operator on $H$. We say that $x$ is $n$ times weakly $D$–differentiable, if for any pair of vectors $\xi, \eta$ from $H$ the function $(e^{itD}xe^{-itD}\xi, \eta)$ is $n$ times differentiable. We show that $x$ is $n$–times weakly $D$–differentiable if and only if there exists a core $\mathcal{F}$ for $D$ such that any commutator $[D, [D, \ldots, [D, x] \ldots]]$ of order $k \leq n$ is defined and bounded on $\mathcal{F}$. If this is the case, the $k$th commutator is defined and bounded on $\text{dom}(D^k)$. We give a couple of equivalent characterizations too.

1. Introduction

Let $D$ be a self-adjoint, usually unbounded, operator on a Hilbert space $H$ and $x$ a bounded operator on $H$, then Quantum Mechanics, Operator Algebra and Noncommutative Geometry offer plenty of reasons why we should be interested in operators that are formed as commutators $[D, x] = Dx - xD$. In noncommutative geometry we want to find a set-up such that classical smooth structures may be described in a language based on operators on a Hilbert space. A derivative is described in terms of a commutator $[D, x]$ and a higher derivative via an iterated commutator $[D, [D, \ldots, [D, x] \ldots]]$, so a basic question is to determine the set of operators for which such an iterated commutator makes sense. It is not clear when a commutator such as $[D, x]$ is densely defined and bounded on its domain of definition, and for 2 bounded operators $x, y$ such that $[D, x]$ and $[D, y]$ are bounded and densely defined the sum of the commutators and/or the the commutator $[D, xy]$ may not be densely defined, so the expression $[D, x]$ does not define a derivation on a subalgebra of $B(H)$ in a canonical way. In the article we introduced a subalgebra $\mathcal{A}_w^D$ of $B(H)$ consisting of those bounded operators $x$ on the Hilbert space $H$ which are weakly $D$–differentiable in the sense, that for each pair of vectors $\xi, \eta$ in $H$
the function \( \langle e^{itD}xe^{-itD}\xi, \eta \rangle \) is differentiable. The main result, Theorem 3.8, in [1] consists of a number of operator theoretical characterizations of weak \( D \)-differentiability. In the present article we define a bounded operator \( x \) on \( H \) to be \( n \)-times weakly \( D \)-differentiable if for each pair of vectors \( \xi, \eta \) in \( H \) the function \( \langle e^{itD}xe^{-itD}\xi, \eta \rangle \) is \( n \)-times differentiable. In section 4 we give several characterizations of those operators that are \( n \)-times weakly \( D \)-differentiable, and we would like to mention here, that a bounded operator \( x \) is \( n \)-times weakly \( D \)-differentiable if and only if for any \( k \) in \( \{1, \ldots, n\} \) the \( k \)'th commutator \([D, [D, \ldots, [D, x] \ldots]]\) is defined and bounded on \( \text{dom}(D^k) \). After the article [1] was accepted for publication and proof read, we realized, that the one parameter group of automorphisms of \( B(H) \) given by \( B(H) \ni x \rightarrow e^{itD}xe^{-itD} \in B(H) \) is actually a so-called adjoint semi-group on a dual Banach space. Let \( S_1(H) \) denote the space of trace class operators on \( H \), then we may consider the one parameter group of automorphisms of \( S_1 \) given as \( s \rightarrow e^{-itD}se^{itD} \), and the adjoint semi-group of this group of automorphisms is exactly the one parameter group of automorphisms of \( B(H) \), which we are studying here. Adjoint semigroups were first studied in [6], and [4] contains a survey of the general theory of adjoint semigroups. Our usage of the general theory is limited, but several things could have been presented in an easier way in [1], if we had been able to make references to [4].

2. Weak and higher weak \( D \)-differentiability

In order to avoid confusion we will like to clear up a point which has not been presented in an optimal way in [1]. The Definition 1.1 of [1] defines a bounded operator \( x \) to be weakly \( D \)-differentiable if there exists a bounded operator \( b \) on \( H \) such that for any pair of vectors \( \xi, \eta \) in \( H \) we have

\[
\lim_{t \to 0} \left( \frac{e^{itD}xe^{-itD} - x}{t} - b \right) \xi, \eta \right) = 0.
\]

This definition implies that for any \( \xi, \eta \) the function \( t \rightarrow \langle e^{itD}xe^{-itD}\xi, \eta \rangle \) is differentiable at \( t = 0 \), and it is stated, but not explicitly proven that this latter property implies weak \( D \)-differentiability as defined via a weak derivative \( b \). It is quite easy to see that the two sorts of weak \( D \)-differentiability are equivalent and all the arguments are presented in [1], but the consequences are not made sufficiently clear. The right formal definition of weak \( D \)-differentiability then becomes as follows.
Definition 2.1. A bounded operator $x$ on $H$ is weakly $D-$differentiable if for any pair of vectors $\xi, \eta$ in $H$ the function $t \to \langle e^{itD}xe^{-itD}\xi, \eta \rangle$ is differentiable at $t = 0$.

To see that our present definition of weak $D-$differentiability implies the existence of a weak derivative, i.e. a bounded operator $b$ such that Definition 1.1 of [1] is satisfied, we refer the reader to the proof of (ii) $\Rightarrow$ (iii) in Theorem 3.8 of [1]. That step is the crucial part of the proof, and it is based on the uniform boundedness principle applied to all the operators

$$\left\{ \frac{e^{itD}xe^{-itD} - x}{t} : t \neq 0 \right\}.$$

This set is bounded because any function such as $t \to \langle e^{itD}xe^{-itD}\xi, \eta \rangle$ is differentiable at $t = 0$, and hence the set of values

$$\left\{ \langle \frac{e^{itD}xe^{-itD} - x}{t}\xi, \eta \rangle : t \neq 0 \right\}$$

is bounded and the principle applies. The existence of $b$ then follows from the rest of Theorem 3.8 of [1]. We will quote that theorem below and define the higher weak derivatives, but first we will recall a couple of other forms of $D-$differentiability.

We say that a bounded operator $x$ is uniformly $D-$differentiable if the function $t \to e^{itD}xe^{-itD}$ is differentiable at $t = 0$, with respect to the norm topology on $B(H)$. In analogy with the definition of weak $D-$differentiability we say that $x$ is strongly $D-$differentiable if for each vector $\xi$ in $H$ the function $t \to e^{itD}xe^{-itD}\xi$ is differentiable at $t = 0$ with respect to the norm topology on $H$. It follows from [1] that weak and strong $D-$differentiability are equivalent but uniform $D-$differentiability is in general a stronger property. On the other hand, when speaking of higher derivatives, we quote from [1] the following result, which tells that higher uniform derivatives are closely related to weak derivatives.

Theorem 2.2. Let $x$ be a bounded operator on $H$ and $n \geq 2$. If $x$ is $n-$times weakly $D-$differentiable then $x$ is $n - 1$ times uniformly $D-$differentiable.

Proof. Se Corollary 4.2 of [1].

We will quote Theorem 3.8 from [1] here, without description of all the language used. Not all of the results below may be generalized to higher derivatives and for those properties, which can be extended, we will give the necessary precise definitions, when needed.
Theorem 2.3. Let $x$ be a bounded operator on $H$. The following properties are equivalent:

(i) $x$ is strongly $D-$differentiable.
(ii) $x$ is weakly $D-$differentiable.
(iii) $x$ is $D-$Lipschitz continuous.
(iv) The sesquilinear form $S(i[D,x])$ on the domain of $D$ is bounded.
(v) The infinite matrix $m(i[D,x])$ represents a bounded operator.
(vi) The operator $Dx - xD$ is defined and bounded on a core for $D$.
(vii) The operator $Dx - xD$ is bounded and its domain of definition is $\text{dom}(D)$.

If $x$ is weakly $D-$differentiable then 

$$\forall \xi, \eta \in H \lim_{t \to 0} \frac{\langle (e^{itD}xe^{-itD} - x)\xi, \eta \rangle}{t} = \langle \delta_w^D(x)\xi, \eta \rangle$$

$x \text{ domD} \subseteq \text{domD}$ and $\delta_w^D(x)|\text{dom}(D) = i(Dx - xD)$

$$\forall t \in \mathbb{R} : \|\alpha_t(x) - x\| \leq \|\delta_w^D(x)\||t|.$$

The properties (iii) and (iv) from the theorem just above have no immediate generalizations to higher derivatives and will not be discussed here at all. The remaining 5 properties have natural extensions to the setting of higher weak derivatives and higher commutators as well. First we give the formal and canonical definition of higher weak $D-$differentiability.

Definition 2.4. A bounded operator $x$ on $H$ is said to be $n-$times weakly $D-$differentiable if for any pair $\xi, \eta$ of vectors in $H$ the complex function $t \to \langle e^{itD}xe^{-itD} \xi, \eta \rangle$ is $n-$times differentiable on $\mathbb{R}$.

We start by making some simplifications in the notation. In the rest of this article we will only consider one unbounded self-adjoint operator $D$, with respect to which we will perform our weak $D-$differentiations. We will then leave out the $D$ in weak $D-$differentiability and in the expression $\delta_w^D(x)$, which in [1] was used to denote the weak $D-$ derivative of $x$. There we also met the one parameter unitary group $u_t := e^{itD}$ and the one parameter group of *-automorphisms $\alpha_t(x) := u_t xu_{-t}$, so we will use $u_t$ and $\alpha_t$ below without recalling the definitions all the time. For a bounded operator $x$ on $H$ the function $t \to \alpha_t(x)$ may be uniformly differentiable, which we know happens iff $x$ is in the domain of definition for the generator for $\alpha_t$. In [1] this derivative was denoted $\delta_u^D(x)$. Here we will never look at uniform derivatives, so this notion will not be needed below, and we will let $\delta(x)$ denote the weak $D-$derivative of $\alpha_t(x)$ at $t = 0$, if it exists. The domains of definitions
for $D$ and $\delta$ are written as $\text{dom}(D)$ and $\text{dom}(\delta)$ respectively. We remind the reader, that $\text{dom}(\delta)$ is a self-adjoint subalgebra of $B(H)$ and $\delta$ is a derivation of this algebra into $B(H)$, [1] Theorem 3.9. Before embarking on higher weak derivatives we would like to make the following observation explicit. The reason being, that the result is only indirectly contained in [1].

**Lemma 2.5.** If a bounded operator $x$ on $H$ is weakly differentiable then for any pair of vectors $\xi, \eta$ in $H$: the function $\langle \alpha_t(x)\xi, \eta \rangle$ is differentiable on $\mathbb{R}$ and

$$\frac{d}{dt} \langle \alpha_t(x)\xi, \eta \rangle = \langle \alpha_t(\delta(x))\xi, \eta \rangle.$$

**Proof.** By definition the equality holds for $t = 0$, and arguments similar to the ones given in the proof of Lemma 2.1 of [1] show that the identity may be translated from $t = 0$ to any other real $t$. $\square$

This lemma has an immediate consequence, which we formulate as a proposition, since it is important, although its proof is trivial.

**Proposition 2.6.** A bounded operator $x$ on $H$ is $n-$times weakly differentiable if and only if $x$ is in $\text{dom}(\delta^n)$. If $x$ is $n-$times weakly differentiable then

$$\frac{d^n}{dt^n} \langle \alpha_t(x)\xi, \eta \rangle = \langle \alpha_t(\delta^n(x))\xi, \eta \rangle.$$

**Proof.** If $x$ is $n-$times weakly differentiable then $x$ is weakly differentiable and an element of $\text{dom}(\delta)$. If $n > 1$, then the lemma above shows that $\delta(x)$ is in $\text{dom}(\delta)$ and

$$\frac{d^2}{dt^2} \langle \alpha_t(x)\xi, \eta \rangle = \frac{d}{dt} \langle \alpha_t(\delta(x))\xi, \eta \rangle = \langle \alpha_t(\delta^2(x))\xi, \eta \rangle.$$

If $n > 2$, we may continue in this way in order to show that $x$ is in $\text{dom}(\delta^n)$ and

$$\frac{d^n}{dt^n} \langle \alpha_t(x)\xi, \eta \rangle = \langle \alpha_t(\delta^n(x))\xi, \eta \rangle.$$

Suppose $x$ is in $\text{dom}(\delta^n)$ then $x$ is in $\text{dom}(\delta)$ and then weakly differentiable. By the lemma we have

$$\frac{d}{dt} \langle \alpha_t(x)\xi, \eta \rangle = \langle \alpha_t(\delta(x))\xi, \eta \rangle.$$

If $n > 1$ then by assumption we know that $\delta(x)$ is in $\text{dom}(\delta)$ so

$$\frac{d}{dt} \langle \alpha_t(\delta(x))\xi, \eta \rangle = \langle \alpha_t(\delta^2(x))\xi, \eta \rangle.$$
and \( \langle \alpha_t(x)\xi, \eta \rangle \) is \( 2 \)–times differentiable. Iteration shows that for \( n \geq 2 \) the operator \( x \) is \( n \)–times weakly differentiable.

\[ \square \]

3. Higher weak \( D \)–derivatives and iterated commutators.

Having Proposition 2.6 one might think that our understanding of \( \delta \) and its powers is sufficiently well established for most purposes, but it is not. The problem is that we do not know how to relate higher weak derivatives to expressions involving iterated commutators with \( iD \). If \( x \) is in \( \text{dom}(\delta) \) then it follows from Theorem 2.3 that \( iDx - ixD \) is defined on all of \( \text{dom}(D) \) and \( \delta(x) \) is the closure of \( iDx - ixD \). If \( x \) is in \( \text{dom}(\delta^2) \), then it is obvious to look at the second \( iD \) commutator

\[ iD(iDx - ixD) - (iDx - ixD)(iD), \]

but we know nothing about its domain of definition, possible boundedness and closure. In this section we will show that the properties of the higher commutators are as nice as we can possibly hope for. We will show that for a bounded \( n \)–times weakly differentiable operator \( x \) the \( n \)–times iterated commutator between \( iD \) and \( x \), is defined on \( \text{dom}(D^n) \) and the closure of this operator equals \( \delta^n(x) \). We will base the proof of this on the results of Theorem 2.3. In order to simplify the writings below we define an operator \( d \) on the space of linear operators on \( H \).

**Definition 3.1.**

(i) A linear operator on \( H \) is a linear operator defined on a subspace of \( H \) and with values in \( H \). The space of all linear operators on \( H \) is denoted \( \mathcal{L} \). Sums and products of elements in \( \mathcal{L} \) are defined on the largest possible domain.

(ii) The operator \( d \) on \( \mathcal{L} \) is defined for \( y \) in \( \mathcal{L} \) by \( d(y) := (iD)y - y(iD) \).

We will start our investigation on higher commutators by making the following observation.

**Lemma 3.2.** Let \( x \) be a bounded operator in \( B(H) \) and \( n \) a natural number. If \( x \) is \( n \)–times weakly differentiable then for any \( k \) in \( \{1, \ldots, n\} \):

\[ \delta^{k-1}(x) : \text{dom}(D) \to \text{dom}(D), \]

\[ \delta^k(x)|\text{dom}(D) = i[D, \delta^{k-1}(x)] = d(\delta^{k-1}(x)) \]

**Proof.** If \( x \) is \( n \)–times weakly differentiable, then for any \( k \) in \( \{1, \ldots, n\} \) we have \( \delta^{k-1}(x) \) is in \( \text{dom}(\delta) \). Then Theorem 2.3 item (vii) presents the claimed properties of \( \delta^{k-1}(x) \).
The statements in Lemma 3.2 show that $\delta^k(x)$ is the closure of the commutator $[iD, \delta^{k-1}(x)]$, but if $k > 1$ then $\delta^{k-1}(x)$ is defined as a closure of the commutator $[iD, \delta^{k-2}(x)]$, so we have no direct control over the operator $[iD, \delta^{k-1}(x)]$. This is not sufficient for our purpose, so we want to look at the restriction of such a commutator to $\text{dom}(D^k)$, and then show that on this domain the higher weak derivative may be computed without any closure operations, as a higher commutator, and that the closure of this algebraically defined commutator equals $\delta^k(x)$.

Proposition 3.3. Let $x$ be an $n$--times weakly differentiable bounded operator on $H$, then for $k$ in $\{1, \ldots, n\}$

\[
\begin{align*}
(i) & \quad \delta^{k-1}(x)\text{dom}(D) \subseteq \text{dom}(D), \\
(ii) & \quad x \text{dom}(D^k) \subseteq \text{dom}(D^k), \\
(iii) & \quad \text{dom}(d^k(x)) = \text{dom}(D^k), \\
(iv) & \quad \delta^k(x)|\text{dom}(D^k) = \text{d}^k(x), \\
(v) & \quad \delta^k(x) = \text{closure}(d^k(x)).
\end{align*}
\]

Proof. The item (i) follows from Lemma 3.2. The following 4 items are related and we show them by induction on $k$. For $k = 1$ the results follow again from item (iv) of Theorem 2.3. Then suppose $1 < k \leq n$ and that the statements are true for natural numbers in the set $\{1, \ldots, k-1\}$. We start by proving (iii), so we will choose a vector $\xi$ in $\text{dom}(D^k)$, then $\xi$ is in $\text{dom}(D^{k-1})$ so $d^{k-1}(x)\xi = \delta^{k-1}(x)\xi$ and by item (i) $d^{k-1}(x)\xi$ is in $\text{dom}(D)$, and finally $\xi$ is in $\text{dom}(iDd^{k-1}(x))$. By assumptions (i) $\xi$ is in $\text{dom}(D^{k-1})$ which equals $\text{dom}(d^{k-1}(x))$ so $\xi$ is in $\text{dom}(d^{k-1}(x)/iD))$ too, and $\text{dom}(D^k) \subseteq \text{dom}(d^k(x))$. The opposite inclusion is trivially true since the $d^k(x)$ is a sum of terms, where the last summand is $(-i)^kxD^k$.

With respect to item (iv), we find from the lemma and general properties of domains that

$\delta^k(x)|\text{dom}(D) = d(\delta^{k-1}(x))$, and $D\text{dom}(D^k) \subseteq \text{dom}(D^{k-1})$,

so by the assumptions made and item (iii) proven we get

$\delta^k(x)|\text{dom}D^k = (iD\delta^{k-1}(x) - i\delta^{k-1}(x)D)|\text{dom}(D^k) = \text{d}^k(x)|\text{dom}(D^k),$

and (iv) has been proven.

With respect to (v) we remark, that $\text{dom}(D^k)$ is a core for $D$ since it contains the vectors in the core $\mathcal{E}$, which was introduced in the proof of (v) $\Rightarrow$ (vi) in Theorem 3.8 of [1]. Then $\delta^k(x)$ is the closure of the commutator $d(\delta^{k-1}(x))|\text{dom}(D^k)$, but the latter equals $d^k(x)$ so (v) follows.
To prove (ii) we remark, that from (iv) we can deduce that
\[ d^{k-1}(x) \text{dom}(D^k) \subseteq \text{dom}(D). \]
On the other hand a closer examination of the expression \( d^{k-1}(x)\xi \) for a vector \( \xi \) in \( \text{dom}(D^k) \) shows that
\[ d^{k-1}(x)\xi = (i)^{k-1} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j D^{k-1-j}xD^j\xi \]
For \( j > 0 \) we have \( D^j\xi \) is in \( \text{dom}(D^{k-j}) \) and by assumption \( xD^j\xi \) is in \( \text{dom}(D^k) \) so \( D^{k-1-j}xD^j\xi \) is in \( \text{dom}(D) \). We have just seen that \( d^{k-1}(x)\xi \) is in \( \text{dom}(D) \) so we may conclude that for \( j = 0 \) the element in the sum indexed by 0, i.e. \( D^{k-1}x\xi \) also belongs to \( \text{dom}(D) \), which means that \( x\xi \) is in \( \text{dom}(D^k) \) and item (ii) has been proved. \( \square \)

4. Equivalent Properties

In analogy with the results of Theorem 2.3 we want to show that higher order weak differentiability may be characterized in several different ways. Some of the properties we find are expressed in terms of infinite matrices of operators, so we will include a short description of this set-up here.

In [1] we defined a sequence of pairwise orthogonal projections with sum \( I \) in \( B(H) \) by letting \( e_n \) denote the spectral projection for \( D \) corresponding to the interval \([n-1, n]\). Then we defined \( \mathcal{M} \) to be all matrices \( (y_{rc}) \) with \( r \) and \( c \) integers and \( y_{rc} \) an operator in \( e_re_c \). Any bounded operator \( x \) on \( H \) induces an element \( m(x) \) in \( \mathcal{M} \) which is defined as \( m(x)_{rc} := e_re_ce_c \). The operator \( D \) has a representation \( m(D) \) in \( \mathcal{M} \) too, and it is defined as a diagonal matrix \( m(D)_{rc} = 0 \), if \( r \neq c \) and diagonal elements \( d_r := m(D)_{rr} := De_r \). Then for any element \( y = (y_{rc}) \) in \( \mathcal{M} \), the commutator \( i[m(D), y] \) makes sense in \( \mathcal{M} \) by
\[ i[m(D), y]_{rc} := i(d_{r}y_{rc} - y_{rc}d_{c}) , \]
and we may define a linear mapping \( d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \) by
\[ \forall y = (y_{rc}) \in \mathcal{M} : \ d_{\mathcal{M}}(y)_{rc} := id_{r}y_{rc} - iy_{rc}d_{c} . \]
By the computations above we get that the powers \( d^n_{\mathcal{M}} \) are given as (4.1)
\[ \forall n \in \mathbb{N} \forall y = (y_{rc}) \in \mathcal{M} : \ d^n_{\mathcal{M}}(y)_{rc} = i^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} d_{r}^{k}y_{rc}d_{c}^{n-k} . \]
We can now formulate our main result.
Theorem 4.1. Let $x$ be a bounded operator on $H$ and $n$ a natural number. The following properties are equivalent:

(i) $x$ is in $\text{dom}(\delta^n)$.
(ii) $x$ is $n$-times weakly $D$-differentiable.
(iii) $x$ is $n$-times strongly $D$-differentiable.
(iv) $\forall k \in \{1, \ldots, n\}$
$$x : \text{dom}(D^k) \to \text{dom}(D^k)$$
$d^k(x)$ is defined and bounded on $\text{dom}(D^k)$ with closure $\delta^k(x)$.
(v) For $k \in \{1, \ldots, n\}$ The infinite matrix $d^k_M(m(x)))$ represents a bounded operator.
(vi) There exists a core $\mathcal{F}$ for $D$ such that for any $k \in \{1, \ldots, n\}$ the operator $d^k(x)$ is defined and bounded on $\mathcal{F}$.

Proof. We prove (i) $\iff$ (ii), (ii) $\iff$ (iii), (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii) and (ii) $\iff$ (vi).

(i) $\iff$ (ii):
Follows from Proposition 2.6.

(ii) $\Rightarrow$ (iii):
Follows from the Cauchy-Schwarz inequality.

(iii) $\Rightarrow$ (ii):
Follows by an induction based on the following induction step. Suppose $0 \leq k < n$, $x$ is a bounded $n$-times weakly differentiable operator, which is $k$-times strongly differentiable, then $\delta^k(x)$ is the $k'$th strong derivative by Theorem 2.3, and since this operator is weakly differentiable, the same theorem shows that $\delta^k(x)$ is strongly differentiable with strong derivative $\delta^{k+1}(x)$.

(ii) $\Rightarrow$ (iv):
This follows from Proposition 3.3.

(iv) $\Rightarrow$ (v):
Let $1 \leq k \leq n$, then we are given that $\delta^k(x)$ exists and is a bounded operator such that $\delta^k(x)|\text{dom}(D^k) = d^k(x)$. Let $c$ be an integer then $e_c H \subseteq \text{dom}(D^k)$ so for any integer $r$ we get
$$e_r \delta^k(x) e_c = e_r d^k(x) e_c = i^k \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e_r D^j x D^j e_c = d^k_M(m(x))_{rc},$$

hence $d^k_M(x)$ is the matrix of a bounded operator and (v) follows.

(v) $\Rightarrow$ (ii):
Assume (v), i.e. that for any $k$ in $\{1, \ldots, n\}$ there exists a bounded operator $z_k$ on $H$ such that for any pair of integers $r, c$ we have $e_r z_k e_c = d^k_M(x)_{rc}$. The case $k = 1$ is covered by Theorem 2.3, The proof may
be found in [1], but we recall the main step, because we will use it repeatedly below. For any vector \( \xi \) from \( e_c H \) we showed that \( x \xi \) is in \( \text{dom}(D) \). It then follows that for any integer \( r \) and a vector \( \xi \) in \( e_c H \) we have \( x \xi \) is in \( \text{dom}(D) \) and

\[
e_r z_1 \xi = i(d_r e_r x e_c - e_r x e_c) \xi = i e_r (D x - x D) \xi.
\]

and we concluded that \( x \) is weakly differentiable with \( \delta(x) = z_1 \). We may now assume that \( 1 \leq k \leq n \) and \( x \) is weakly differentiable of order \( k - 1 \) with \( \delta^j(x) = z_j \) for \( 1 \leq j \leq k - 1 \). Then for \( \xi \) in \( e_c H \) we get \( \delta^{k-1}(x) \xi \) is in \( \text{dom}(D) \) so we have

\[
e_r z_k \xi = i(d_r e_r e_{M}^{-1}(x)e_c - e_r e_{M}^{-1}(x)e_c) \xi = i e_r (D \delta^{k-1}(x) - \delta^{k-1}(x) D) \xi.
\]

Hence \( \delta^{k-1}(x) \) is weakly differentiable and \( \delta^k(x) = z_k \), so \( x \) is \( n \)-times weakly differentiable and (ii) follows.

(ii) \( \Rightarrow \) (vi):

For any \( n \in \mathbb{N} \), the space \( \text{dom}(D^n) \) is a core for \( D \), so (vi) follows from (iv), which, in turn, follows from (ii).

(vi) \( \Rightarrow \) (ii):

Now suppose (iv) holds for a bounded operator \( x \) on \( H \). Then for \( k \in \{1, \ldots, n\} \) there exist bounded operators \( y_k = \text{closure}(d^k(x)|\mathcal{F}) \).

Let us look at the case \( k = 1 \) first. Then \( (iD)x - x(iD) \) is defined and bounded on the core \( \mathcal{F} \) for \( D \), so by Theorem 2.3 item (vi) \( x \) is in \( \text{dom}(\delta) \) and \( y_1 = \delta(x) \). Let us now suppose that \( 1 < k \leq n \) and we have shown that \( y_j = \delta^j(x) \), for \( 1 \leq j \leq k - 1 \), then for any \( \xi \) in \( \mathcal{F} \) we can find a sequence of vectors \( \xi_n \) in \( \mathcal{F} \) such that \( \xi_n \to \xi \) and \( D \xi_n \to D \xi \) for \( n \to \infty \). Since \( d^k(x) \) is bounded and defined on \( \mathcal{F} \) we have

\[
y_k \xi = \lim_{n \to \infty} d^k(x) \xi_n = \lim_{n \to \infty} ((iD)d^{k-1}(x)\xi_n - d^{k-1}(x)(iD)\xi_n) = \lim_{n \to \infty} ((iD)\delta^{k-1}(x)\xi_n - \delta^{k-1}(x)(iD)\xi_n).
\]

Since the last part of these equations forms a convergent sequence we find that \( \lim_{n \to \infty} ((iD)\delta^{k-1}(x)\xi_n \) exists and

\[
\lim_{n \to \infty} ((iD)\delta^{k-1}(x)\xi_n = y_k \xi + \delta^{k-1}(x)(iD)\xi.
\]

Hence \( \delta^{k-1}(x) \xi \) is in \( \text{dom}(D) \) and

\[
y_k \xi = (iD)\delta^{k-1}(x)\xi - \delta^{k-1}(x)(iD)\xi.
\]

By Theorem 2.3 we get that \( \delta^{k-1}(x) \) is weakly differentiable and \( \delta^k(x) = y_k \), so \( x \) is \( n \)-times weakly differentiable, and the theorem follows. \( \square \)
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