Estimating Exposure to Information on Social Networks

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ABSTRACT

This paper considers the problem of estimating exposure to information in a social network. Given a piece of information (e.g., a URL of a news article on Facebook, a hashtag on Twitter), our aim is to find the fraction of people on the network who have been exposed to it. The exact value of exposure to a piece of information is determined by two features: the structure of the underlying social network and the set of people who shared the piece of information. Often, both features are not publicly available (i.e., access to the two features is limited only to the internal administrators of the platform) and difficult to be estimated from data. As a solution, we propose two methods to estimate the exposure to a piece of information in an unbiased manner: a vanilla method which is based on sampling the network uniformly and a method which non-uniformly samples the network motivated by the Friendship Paradox. We provide theoretical results which characterize the conditions (in terms of properties of the network and the piece of information) under which one method outperforms the other. Further, we outline extensions of the proposed methods to dynamic information cascades (where the exposure needs to be tracked in real-time). We demonstrate the practical feasibility of the proposed methods via experiments on multiple synthetic and real-world datasets.

CCS CONCEPTS

• Information systems → Social networks; Information retrieval.

KEYWORDS

social networks, exposure to information, friendship paradox, information diffusion, information cascades

1 INTRODUCTION

Online social networks are an important mechanism through which people are exposed to information. Estimating the total number of people who are exposed by their friends to a piece of information on an online social network (e.g., the URL of an article on Facebook, a hashtag on Twitter, etc.) is an important problem with key societal implications. Such estimates of exposure can, for example, help researchers and the public track the prevalence and reach of election misinformation [4, 9] or improve the public health response to the Coronavirus [27, 29].

To measure the exposure to a piece of information on a network, one needs access to two features: the set of people who shared the piece of information, and the structure of the underlying social network. Of these two features, the structure of the underlying social network is often not publicly available and, fully or partially estimating it from data is not a practically feasible task due to the networks’ massive size (e.g., billions of nodes and edges in Facebook), constantly evolving nature [20, 21], and limits placed by corporations on data collection. Similarly, the set of people who shared the piece of information is often also not publicly known and difficult to estimate from data since it evolves as the piece of information spreads through the social network in the form of an information cascade e.g., a URL of a news article that is being shared on Facebook. As a result, calculating exposure to a piece of information on a social network in a data-driven manner remains a challenging task.

This state of affairs is unfortunate as efficient (in terms of computation and resources) and accurate (in terms of statistical properties) estimates of exposure to information can provide two important benefits:

1. Analysis perspective: to identify the pieces of information that are most widely consumed and thus, shed light on how the information consumption patterns affect the outcomes of various high-stake events such as elections, COVID vaccine acceptance, etc. [4, 29].

2. Intervention perspective: to prioritize various pieces of information in the content moderation and fact checking process of a social media platform. Such interventions could be useful to prevent large numbers of users from being exposed to harmful or misleading information.

Surprisingly, the problem of estimating exposure to information has received relatively little attention in the literature despite its importance. Formally, this problem can be stated as follows (in the context of an undirected social network such as Facebook).

Problem of estimating exposure to information: Consider an undirected social network \( G = (V, E) \) and, let \( s(v) = 1 \) if the node \( v \in V \) shared (with the set of their neighbors \( N(v) \subseteq V \)) a piece of information and \( s(v) = 0 \) otherwise. Assuming the graph \( G \) and the sharing function \( s : V \rightarrow \{0, 1\} \) are unknown, estimate the fraction of nodes exposed to the piece of information,

\[
\hat{f} = \frac{\left| \{v \in V : f(v) = 1\} \right|}{|V|} \tag{1}
\]

where, \( f(v) = 1 \) if the node \( v \in V \) has been exposed to the piece of information by one of their neighbors and \( f(v) = 0 \) otherwise i.e., \( f(v) = 1_{\left( \exists u \in V : f(u) = 1 \right)}(v) \).

\footnote{A “piece of information” refers to any uniquely identifiable message (e.g., a URL, a hashtag) that is shared by the users on a social network with their contacts.}
In the above formulation, the value \( s(v) \in \{0, 1\} \) indicates whether the user (i.e., node) \( v \in V \) shared the piece of information in concern and \( f(v) \in \{0, 1\} \) indicates whether the user \( v \in V \) has been exposed to it by one of their neighbours. Thus, the parameter of interest \( \bar{f} \) denotes the average exposure to the piece of information in the social network i.e., \( \bar{f} = \mathbb{E}(f(X)) \) where \( X \) denotes a uniformly sampled node from the set of all nodes \( V \). In this setting, we are tasked with devising a method to estimate the average exposure \( \bar{f} \). Since the sharing function \( s : V \to \{0, 1\} \) and the network \( G = (V, E) \) are both unknown, the exposure function \( f : V \to \{0, 1\} \) is also unknown (as it depends on \( s \) and \( G \)). However, the exposure \( f(v_i) \in \{0, 1\} \) of a small number of sampled nodes \( v_i, i = 1, 2, \ldots, n \) (where \( n \ll |V| \)) can often be found by looking at whether at least one neighbor of \( v_i \) (for each \( i = 1, 2, \ldots, n \)) shared the piece of information.

**Main results:**

1. We propose two methods for estimating the exposure to a piece of information in an undirected social network (i.e., the problem that is formalized above): a vanilla method based on uniform sampling and a friendship paradox-based method. Both methods are intuitive and practically feasible.

2. Via theoretical analysis and numerical experiments, we characterize the conditions (in terms of the properties of the underlying network and the piece of information in concern) under which the vanilla method outperforms the friendship paradox-based method and vice-versa. These conditions depend only on parameters that are typically known apriori (e.g., whether the network is assortative or disassortative). As such, these characterizing conditions help to choose the most accurate method for estimating exposure to information depending on the context of the problem.

3. We extend the two proposed methods to the setting of a dynamic information cascade where the piece of information gradually spreads and more people become exposed to it over time. The resulting algorithms iteratively track the increasing average exposure in real-time. In addition, we show how the proposed methods can be extended to the context of directed networks (e.g., Twitter).

4. We provide detailed numerical simulations (based on synthetic data) as well as empirical experiments (based on real-world data) to illustrate the usefulness and feasibility of the proposed methods under various practical settings.

## 2 RELATED WORK AND PRELIMINARIES

Our problem definition and methods are motivated by recent literature and events, which we expand on below. We then present background preliminaries on the main approach that we use in this work, the friendship paradox.

### 2.1 Motivation

The need for a principled method for estimating exposure to information in social media has intensified recently.

One reason for this increased interest in exposure estimation is the role that exposure to information on social networks can play in affecting the outcomes of high-stake events that define the course of our society. In particular, the question of reach of “fake news” on Facebook has become a major public concern following the 2016 US presidential election [4, 9]. Similarly, large scale exposure to false information on social media has complicated the public health response to the Coronavirus [27, 29]. Both examples highlight the need for tracking and quantifying exposure, for example to prioritize fact-checking of trending coronavirus and election-related information content.

It has also become clear that the platforms cannot be trusted to reliably provide this information, in real-time or retrospectively [10, 14]. For example, Facebook recently acknowledged serious problems in the data provided to academic researchers in 2020 [1, 30], and reportedly shelved “most-viewed pages” reports when those conflicted with the company’s publicity goals [2]. Such incidents have given rise to the need of methods which can accurately estimate the exposure to various pieces of information independently without the involvement of the social media companies.

Researchers looking to estimate exposure to information in social networks are largely limited to survey-based methods. In such methods, a question is presented to a set of respondents to gather information on the frequency and pattern of their social media usage [15]. For example, researchers had used a post-election online survey to assess how exposure to fake news affected the 2016 US election [4]. Other research had used panels of users who provided access to their web traffic to assess such exposure [17]. Closer to our baseline method here, researchers had used a panel of Twitter users to track their exposure to specific “fake news” URLs and domains shared by people they follow, but did not offer network-wide measures [16].

As opposed to the post-event survey-based approach, the social network sampling-based methods proposed in this work can be implemented in real-time (to track the progression of exposure as a piece of information spreads over time). Further, these methods yield unbiased estimates of the exposure to information across the entire social network. In addition, these techniques can be implemented in a practically feasible manner without the full knowledge of the network (e.g., via a random walk) as well as the set of people who shared the piece of information.

### 2.2 The Friendship Paradox

Our work here is motivated by the graph theoretic consequence named **friendship paradox** which states “on average, the number of friends of a random friend is always greater than or equal to the number of friends of a random individual”. Formally:

**Theorem 1 (Friendship Paradox) [12].** Consider an undirected graph \( G = (V, E) \). Let \( X \) be a node sampled uniformly from \( V \) and \( Y \) be a uniformly sampled end-node from a uniformly sampled edge \( e \in E \). Then,

\[
\mathbb{E}(d(Y)) \geq \mathbb{E}(d(X)),
\]

where \( d(X) \) and \( d(Y) \) denote the degrees of \( X \) and \( Y \), respectively.

In Theorem 1, the random variable \( Y \) is called a random friend. This is because it is a random person from a uniformly sampled pair of friends.\(^2\) The intuition behind the friendship paradox (Theorem 1) is that individuals with large number of friends (i.e., high-degree

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\(^2\) A random friend \( Y \) on an undirected graph \( G = (V, E) \) has the distribution \( \mathbb{P}(Y = v) = d(v) \) for all \( v \in V \). In other words, a random friend is a node sampled with a probability proportional to their degrees.
nodes) appear as the friends of many others. Therefore, a random end of a random link (i.e., the random variable $Y$) is more likely to yield a high degree node than a uniformly sampled node (i.e., the random variable $X$). Consequently, sampling random friends (i.e., $Y$) allows us to reach high-degree nodes in the network without the full knowledge of the network.

The friendship paradox has been exploited in many statistical estimation methods e.g., to reduce the variance in survey based polling methods [23], to efficiently estimate power-law degree distributions [11, 24], and to quickly detect the outbreak of a disease [13]. This paper differs from such existing friendship paradox-based methods in several key ways. First, our focus is on the problem of estimating the fraction of people exposed to a piece of information, which is different from the problems studied in prior works (such as polling, estimating degree distributions, detecting disease outbreaks) that are based on the friendship paradox. Second, we do not claim that the friendship paradox-based method is the better approach for any given setting. Instead, we provide an exact characterization of the conditions where the friendship paradox-based method outperforms the vanilla method based on uniform sampling. Finally, the methods that we propose are provably unbiased whereas most previously proposed estimators based on the friendship paradox attempt to trade-off the bias and variance in order to achieve a smaller mean squared error.

3 ALGORITHMIC APPROACH

In this section, we present two methods for estimating the average exposure to information: a vanilla method based on uniform sampling and a friendship paradox-based method.

3.1 Vanilla method based on uniform sampling

The vanilla approach for estimating the average exposure $\bar{f}$ works by obtaining a set of random nodes and checking whether these nodes have been exposed to the piece of information via their contacts.

**Vanilla method for estimating exposure to information**

*Step 1:* Sample $n$ random nodes $X_1, \ldots, X_n$ uniformly and independently from the set of all nodes $V$.

*Step 2:* Use,

$$\hat{f}_{\text{vl}} = \frac{\sum_{i=1}^{n} f(X_i)}{n}$$

as the estimate of the average exposure $\bar{f}$. The vanilla estimator $\hat{f}_{\text{vl}}$ given in Eq. (3) is unbiased i.e., $\mathbb{E}\{\hat{f}_{\text{vl}}\} = \bar{f}$. However, the vanilla estimator $\hat{f}_{\text{vl}}$ would intuitively yield a larger variance since random nodes are not likely exposed to the piece of information when $s(\cdot)$ is a very sparse function (i.e., only very few people shared the information).

3.2 Friendship paradox-based method

In order to reduce the variance in the estimate of average exposure, we can exploit the friendship paradox-based sampling (instead of vanilla uniform sampling) as follows:

**Friendship paradox-based method for estimating exposure to information**

*Step 1:* Sample $n$ random friends $Y_1, \ldots, Y_n$ from the network independently (a random friend $Y_i$ is a random end of a random link i.e., a link is sampled uniformly from the network and one end of that link is taken with an unbiased coin flip).

*Step 2:* Use,

$$\hat{f}_{\text{fp}} = \frac{k}{n} \sum_{i=1}^{n} \frac{f(Y_i)}{d(Y_i)}$$

as the estimate of average exposure $\bar{f}$ where, $d(v)$ denotes the degree of $v \in V$ and $k$ the average degree of the graph $G = (V, E)$.

The friendship paradox-based estimator $\hat{f}_{\text{fp}}$ can be viewed as an application of importance sampling in social networks where the samples are generated from a different distribution (i.e., random friends $Y$) than the distribution that is directly related to the parameter of interest $\hat{f} = \mathbb{E}\{f(X)\}$ (i.e., random nodes $X$). In particular, random friends are more popular than random nodes in expectation (according to Theorem 1) and thus, sampling random friends will lower the variance by accessing individuals who are more likely to be exposed to the piece of information due to their large popularity (even when the sharing function $s(\cdot)$ is sparse).

The average degree $k$ in Eq. 4 is typically a known statistic for most social networks such as Facebook [31], which makes the implementation of the friendship paradox-based estimator $\hat{f}_{\text{fp}}$ practically feasible. Further, in situations where the edges cannot be sampled uniformly from the network, the friendship paradox-based estimator $\hat{f}_{\text{fp}}$ can also be implemented via random walks as described in Appendix C.

To summarize, Sec. 3 presented two methods to estimate the average exposure to information based on uniform (vanilla) and friendship paradox-based sampling. Extensions of the two proposed methods will be discussed in Sec. 5.

4 COMPARISON OF STATISTICAL PROPERTIES OF THE TWO METHODS

In this section, we analyze and compare the statistical properties of the two proposed estimators (the vanilla estimator $\hat{f}_{\text{vl}}$ given in Eq. (3) and the friendship paradox-based estimator $\hat{f}_{\text{fp}}$ given in Eq. (4)). The aim of this analysis is to identify the conditions under which one estimator may be more accurate than the other for estimating the average exposure to information $\bar{f}$.

The following result (see Appendix A.1 for proof) characterizes the bias and variance of the two exposure estimators.

**Theorem 2.** Consider the vanilla estimator $\hat{f}_{\text{vl}}$ given in Eq. (3) and the friendship paradox-based estimator $\hat{f}_{\text{fp}}$ given in Eq. (4).

1. Both the vanilla estimator $\hat{f}_{\text{vl}}$ and the friendship paradox-based estimator $\hat{f}_{\text{fp}}$ are unbiased estimators of the fraction of people exposed to a piece of information $\bar{f}$ (defined in Eq. (1)) i.e.,

$$\mathbb{E}\{\hat{f}_{\text{vl}}\} = \mathbb{E}\{\hat{f}_{\text{fp}}\} = \bar{f}. \quad (5)$$

2. The variance of $\hat{f}_{\text{vl}}$ is larger than that of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse (as is typically the case in social networks).

3. For any setting where $s(\cdot)$ is not extremely sparse, there is a parameter $k$ such that for $k > k^*$, $\mathbb{V}(\hat{f}_{\text{vl}}) > \mathbb{V}(\hat{f}_{\text{fp}})$.

4. For any setting where $s(\cdot)$ is sparse, there is a parameter $k$ such that for $k < k^*$, $\mathbb{V}(\hat{f}_{\text{vl}}) < \mathbb{V}(\hat{f}_{\text{fp}})$.

5. The average degree $k$ can be estimated from the network by counting the number of edges.

6. When $s(\cdot)$ is very sparse, the variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$.

7. When $s(\cdot)$ is not extremely sparse, the variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$.

8. Both estimators are unbiased.

9. The variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is not extremely sparse.

10. The variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse.

11. The variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is not extremely sparse.

12. The variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse.

13. Both estimators are unbiased.

14. The variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is not extremely sparse.

15. The variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse.

16. Both estimators are unbiased.

17. The variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is not extremely sparse.

18. The variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse.

19. Both estimators are unbiased.

20. The variance of $\hat{f}_{\text{vl}}$ is smaller than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is not extremely sparse.

21. The variance of $\hat{f}_{\text{vl}}$ is larger than the variance of $\hat{f}_{\text{fp}}$ when $s(\cdot)$ is very sparse.

22. Both estimators are unbiased.
(2) The variances of vanilla estimator \( \hat{f}_{V1} \) and the friendship paradox-based estimator \( \hat{f}_{FP} \) are,

\[
\begin{align*}
\text{Var}\{ \hat{f}_{V1} \} &= \frac{1}{n} \hat{f} (1 - \hat{f}) \\
\text{Var}\{ \hat{f}_{FP} \} &= \frac{1}{n} \hat{k} E \left[ \frac{f(X)}{d(X)} \right] - \hat{f}^2 \\
\end{align*}
\]

where \( \hat{k} \) is the average degree of the graph.

As stated in the first part of Theorem 2, both the vanilla estimator \( \hat{f}_{V1} \) and the friendship paradox-based estimator \( \hat{f}_{FP} \) are unbiased estimators of the average exposure \( \hat{f} \). Therefore, the method that has the smaller variance in the given setting should be used for estimating the average exposure \( \hat{f} \). In order to do this, the rest of this section aims to identify the conditions under which one method outperforms the other in terms of the variance.

To theoretically compare the variances, we consider the class of undirected Markovian random networks that are completely characterized by their degree distribution \( P(k) \) (which gives the probability that a uniformly sampled node from the network has degree \( k \)) and the conditional degree distribution \( P(k'|k) \) (which gives the conditional probability that an edge from a degree \( k \) node connects to a degree \( k' \) node). The term “Markovian” here refers to the fact that all higher-order correlations can be expressed only in terms of the two functions \( P(k) \) and \( P(k'|k) \). We can derive a joint degree distribution using \( P(k) \) and \( P(k'|k) \) as

\[
P(k, k') = \frac{kP(k)P(k'|k)}{\hat{k}},
\]

which gives the probability that a uniformly sampled link connects two nodes with degrees \( k \) and \( k' \). The correlation coefficient corresponding to this joint degree distribution \( P(k, k') \) is called the assortativity coefficient, and we denote it with \( r \in [-1, 1] \). Networks for which \( r > 0 \) are called assortative networks since high-degree nodes are more likely to be connected to other high-degree nodes and low-degree nodes are more likely to be connected to other low-degree nodes. On the other hand, networks for which \( r < 0 \) are called disassortative networks since high-degree nodes and low-degree nodes are more likely to be connected with each other. A detailed description of Markovian random networks and assortativity can be found in [5, 6].

In addition to \( P(k) \) and \( P(k'|k) \) which characterize the Markovian random network, we also define \( P_s(1|k) \) which is the conditional probability that a node with degree \( k \) shares the piece of information. Consequently, \( P_s(0|k) = 1 - P_s(1|k) \) is the probability that a node with degree \( k \) does not share the piece of information. Intuitively, if the \( P_s(1|k) \) is closer to 1 for larger (resp. smaller) values of \( k \), then high-degree (resp. low-degree) nodes are more likely to share the piece of information. In particular, if the sharing happens independently of the node popularity (i.e., high-degree and low-degree nodes are equally likely to share the piece of information), then \( P_s(1|k) \) would be a constant that does not depend on degree \( k \). Such relations (between sharing and degree) can be captured using the correlation coefficient between the sharing and the degree, which we refer to as the degree-sharing correlation coefficient and denote by \( \rho \in [-1, 1] \).

The following result (see Appendix A.2 for proof) compares the variances of the two estimators \( \hat{f}_{FP}, \hat{f}_{V1} \) (given in Eq. (3) and Eq. (4), respectively) in terms of the degree distribution \( P(k) \), the conditional degree distribution \( P(k'|k) \) and the conditional sharing probability \( P_s(1|k) \) in the context of Markovian random networks.

**Theorem 3.** The variance of the friendship paradox-based estimator \( \hat{f}_{FP} \) given in Eq. (4) is less than or equal to the variance of the vanilla estimator \( \hat{f}_{V1} \) given in Eq. (3) (i.e., \( \text{Var}\{ \hat{f}_{FP} \} \leq \text{Var}\{ \hat{f}_{V1} \} \)) if and only if

\[
\mathbb{E}_{k-P(k)} \left[ \left( 1 - \frac{\hat{k}}{k} \right) P \left\{ f(X) = 1 \mid d(X) = k \right\} \right] \geq 0,
\]

where \( X \) is a uniformly sampled node from the network, \( \mathbb{E}_{k-P(k)} \) denotes the expectation with respect to the degree distribution \( P(k) \), and

\[
P \left\{ f(X) = 1 \mid d(X) = k \right\} = 1 - \left( \sum_k P \left( k'|k \right) P_s \left( 0|k' \right) \right)^k.
\]

**Discussion of Theorem 3:** Theorem 3 yields insights that help identify the settings where one method is more accurate (in terms of variance) compared to the other for estimating the average exposure to information \( \hat{f} \). In particular, these insights relate the variance of the methods to important network parameters such as the degree distribution \( P(k) \), assortativity coefficient \( r \) and the degree-sharing correlation coefficient \( \rho \) as we discuss in detail below.

1. **Choosing the best method based on assortativity coefficient** \( r \) and degree-sharing correlation coefficient \( \rho \): Due to the term \( (1 - \hat{k}/k) \), the condition Eq. (8) is more likely to be satisfied when the value \( p \{ f(X) = 1 \mid d(X) = k \} \) is closer to 1 for larger values of the degree \( k \) and the value \( p \{ f(X) = 1 \mid d(X) = k \} \) is closer to 0 for smaller values of the degree \( k \). According to Eq. (9), this happens when
   i. \( P(k'|k) \) is closer to 1 for \( k' \gg \hat{k} \) and \( P_s(1|k') \) is closer to 0 for \( k' \ll \hat{k} \).
   or,
   ii. \( P(k'|k) \) is closer to 1 for \( k, k' \ll \hat{k} \) and \( P_s(1|k') \) is closer to 0 for \( k' \ll \hat{k} \).

Consequently, friendship paradox based estimator \( \hat{f}_{FP} \) has a smaller variance compared to the vanilla estimator \( \hat{f}_{V1} \) when \( r, \rho > 0 \) (i.e., the network is assortative and the high-degree individuals are more likely to share the piece of information) or \( r, \rho < 0 \) (i.e., the network is disassortative and the low-degree individuals are more likely to share the piece of information). The arithmetic signs of the assortativity coefficient \( r \) and the degree-sharing correlation coefficient \( \rho \) are typically known parameters based on the context. For example, it is known that many social networks are assortative whereas technological networks are disassortative [25]. Similarly, for pieces of information that get originated from highly popular individuals (e.g., political news, updates on government policy, etc.), the degree-sharing correlation coefficient is typically positive. Hence, this first insight allows us to choose the estimator that best suits the given context based on arithmetic signs of the assortativity coefficient \( r \) and the degree-sharing correlation coefficient \( \rho \).

2. **When the sharing is independent from the popularity:** If the node degree and the sharing are statistically independent, Eq. (9) yields that \( p \{ f(X) = 1 \mid d(X) = k \} = 1 - P_s(0)^k \), where \( P_s(0) = 1 - P_s(1) \) is the probability that any node (independent of the degree \( k \)) does not share the piece of information. Consequently, when the node
works (Sec. 5.1) and dynamic information cascades (Sec. 5.2).

Since when the fraction of people who share a piece of information is identifying the correct exposure at this beginning stage is crucial for this situation is the starting phase of an information cascade typically grows over time as it gets reshared and reposted by the users who were exposed it, leading to an information cascades. This subsection extends the vanilla estimator \( \hat{f}_{V1} \) and the friendship paradox-based estimator \( \hat{f}_{FP} \) to track the increasing average exposure to such information cascades in real-time. The key idea is to use a stochastic approximation algorithm with a constant step-size.

To simplify the notation, let us assume that only one sample can be collected at each time instant and there are no samples at time 0, allowing us to use the same variable \( n \) for discrete time and the number of samples available. Further, let the vanilla and friendship paradox-based estimators at time \( n \) be denoted by \( \hat{f}_{V1}^{(n)} \) and \( \hat{f}_{FP}^{(n)} \) respectively. Then, note that,

\[
\begin{align*}
\hat{f}_{V1}^{(n)} &= \hat{f}_{V1}^{(n-1)} + \frac{1}{n} \left( f(X_n) - \hat{f}_{V1}^{(n-1)} \right) \\
\hat{f}_{FP}^{(n)} &= \hat{f}_{FP}^{(n-1)} + \frac{1}{n} \left( \frac{\hat{f}(Y_n)}{\bar{d}(Y_n)} - \hat{f}_{FP}^{(n-1)} \right)
\end{align*}
\]

where, \( X_n, Y_n \) denote a random node and a random friend at time \( n \), respectively. The recursions in Eq. (12) are obtained under the assumption that the average exposure \( f \) is time-invariant and therefore, the update term decays with time (due to the decreasing step-size \( 1/n \)) and converges to zero. Intuitively, this means that one new sample would not make a significant difference to an estimate (of a time-invariant parameter) derived with a relatively large number of samples (i.e., \( n \gg 1 \)). In particular, it can be shown that the recursions in Eq. (12) converge to the average exposure \( f \) with probability 1 under mild conditions.

However, the decreasing step-size \( 1/n \) in Eq. (12) is not suitable when the average exposure is evolving over time (denoted by \( \hat{f}^{(n)} \)). This is because the decreasing step-size \( 1/n \) will stop updating eventually even though the average exposure \( \hat{f}^{(n)} \) will keep changing. As a solution, the decreasing step-size in Eq. (12) can
be replaced with a constant step-size $\epsilon > 0$ for the case of time evolving average exposure $f^{(n)}$. Then, the new recursive methods for tracking the time evolving average exposure $f^{(n)}$ using the vanilla and friendship paradox-based methods will be as follows:

\begin{align}
\hat{V}_n &= \hat{V}_{n-1} + \epsilon \left( f(X_n) - \hat{V}_{n-1} \right) \\
\hat{P}_n &= \hat{P}_{n-1} + \epsilon \left( k \frac{f(Y_n)}{d(Y_n)} - \hat{P}_{n-1} \right).
\end{align}

The above two methods can track the progression of average exposure $f^{(n)}$ when it is evolving on a slower time scale compared to the collection of samples. In other words, $f^{(n)}$ is assumed to remain approximately constant for every $c > 1$ samples being collected; if $c \approx 1$ (resp. $c \gg 1$), we say the piece of information is spreading rapidly (resp. slowly). The value of the step-size parameter $\epsilon > 0$ in Eq. (13) and Eq. (14) determines the effect of the update at each time. In particular, the value of $\epsilon$ should be relatively large (resp. small) to track the the average exposure to a piece of information that is spreading rapidly (resp. slowly) through the social network.

In summary, Sec. 5 extended the vanilla and friendship paradox-based estimators proposed in Sec. 3 to two settings: directed networks and dynamic information cascades. These extensions are numerically and empirically evaluated in the subsequent sections.

### 6 Numerical Experiments

In this section, we numerically compare the two estimators (vanilla estimator $\hat{V}_n$ given in Eq. (3) and the friendship paradox-based estimator $\hat{P}_n$ given in Eq. (4)) using detailed simulation experiments. The aim of this comparison is to verify and complement the theoretical analysis in Sec. 4, and obtain additional insight into the performance of the two methods and how they compare with each other in various settings.

**Simulation setup:** To compare the vanilla estimator $\hat{V}_n$ and the friendship paradox-based estimator $\hat{P}_n$ (given in Eq. (3) and Eq. (4)), we use synthetically generated power-law networks from the configuration model [26] with 10000 nodes and power-law exponents $\alpha = 2.5$ and $\alpha = 2.2$. The edges in each network $G = (V, E)$ are re-wired using the method proposed in [32] to explore the three possible regions of the assortativity coefficient $r \in [0, 1]$ ($r < 0$, $r = 0$, $r > 0$). Next, the value $s(\nu)$, $\nu \in V$ for each node $\nu \in V$ is first assigned as an iid Bernoulli random variable whose parameter determines the fraction of the people that share the piece of information. Then, the values $s(\nu)$, $\nu \in V$ are swapped amongst the nodes using the label-swapping method used in [19] to correlate the sharing and degree to consider the three cases of the degree-label correlation coefficient $\rho \in [0, 1]$ ($\rho < 0$, $\rho = 0$ and $\rho > 0$). The results obtained using this simulation setup are shown in Fig. 1. To compare the recursive algorithms based on the vanilla and friendship paradox-based estimators (given in Eq. (13) and Eq. (14), respectively), we use two well-known information diffusion models: the Independent Cascade model (ICM) and the Linear Threshold Model (LTM). The results for the ICM are shown in Fig. 2 and the results for the LTM are given in Appendix B. Additional details about the simulation setup are given in Appendix B.

![Figure 1](image1.png)

**Figure 1:** The absolute error values of the vanilla estimate $\hat{V}_n$ (given in Eq. (3)) and the friendship paradox-based estimate $\hat{P}_n$ (given in Eq. (4)) for two synthetically generated power-law networks (with power-law exponents $\alpha = 2.5$ and $\alpha = 2.5$) with various values of the assortativity coefficient $r$, degree-sharing correlation coefficient $\rho$ and the unconditional probability of sharing the piece of information $p_s$ (i.e., $p_s = \sum_k P_s (1|k) P (k)$ is the probability that a uniformly chosen node shares the piece of information). The shown values were estimated via a Monte Carlo simulation as explained in Sec. 6. The plots show that the numerical results agree with the conclusions reached in the statistical analysis in Sec. 4. In particular, the $\hat{P}_n$ is the better choice when both $r$ and $\rho$ have the same sign. Further, comparing Fig. 1a with Fig. 1b shows that heavy-tails increase the disparity between the performances of the two estimators.
Figure 2: The performance of the vanilla and friendship paradox-based stochastic approximation algorithms (given in Eq. (13) and Eq. (14), respectively) for tracking the exposure to an information cascade (in real-time) under the Independent Cascade model (ICM) on a synthetic power-law network with the power-law exponent $\alpha = 2.5$. The top row shows the two estimates together with the true parameter $f$ and the middle row shows the absolute errors corresponding to the two estimates at each time instant and the average error over all time instants. The third row shows the variation of the degree-sharing correlation coefficient $\rho$ with time. The friendship paradox-based stochastic approximation algorithm works better for the assortative network (Fig. 2a) while the vanilla stochastic approximation algorithm works better for the disassortative network (Fig. 2b). This observation agrees with the theoretical conclusions reached in Sec. 4.

Discussion of the Numerical Results (Fig. 1 and Fig. 2): The numerical results verify the theoretical conclusions (from Sec. 4) and yield additional insight into the practical usefulness of the two methods as we discuss below.

1. Choosing the method that is best for the context: Fig. 1a shows that the friendship paradox-based estimate $\hat{f}_{FP}$ is more accurate (compared to the vanilla estimate $\hat{f}_{VH}$) when the assortativity coefficient $r$ and the degree-sharing correlation coefficient $\rho$ have the same signs (i.e., Fig. 1a(i) and Fig. 1a(ix)). When the assortativity coefficient $r$ and the degree-sharing correlation coefficient $\rho$ have different signs (i.e., Fig. 1a(iii) and Fig. 1a(vii)), the vanilla estimate $\hat{f}_{VH}$ is more accurate compared to the friendship paradox-based estimate $\hat{f}_{FP}$. In addition, the vanilla estimate $\hat{f}_{VH}$ has a smaller error when the degree and sharing are uncorrelated (i.e., $\rho = 0$ corresponding to middle column of Fig. 1a and Fig. 1b) for when the sharing probability is not too small (so that both methods yield absolute errors smaller than 100% of the true parameter $f$). Further, each subfigure of Fig. 1a and Fig. 1b shows that the difference in the accuracy of the two estimates is larger when the unconditional sharing probability $p_x$ (i.e., $p_x = \sum_k p_x(k) P(k)$) is smaller, highlighting that choosing the best method is crucial when the piece of information has been shared by only a smaller fraction of people. These numerical observations verify the theoretical expectations captured in the first and second points in the discussion related to

Theorem 3 in Sec. 4, and emphasizes the importance of the choice for less widely shared pieces of information, per the third point.

2. Implications of the heavy-tails: Comparing Fig. 1a with Fig. 1b indicates that the difference in the accuracy of the two methods is larger when the tail of the degree distribution is heavier. Since real-world social networks have been empirically shown to have heavy-tails, this observation highlights the importance of utilizing the theoretical insight to pick the best method for the given context.

3. Tracking the exposure to an information cascade in real-time: Fig. 2 shows the performance of the vanilla and the friendship paradox-based stochastic approximation algorithms (Eq. (13) and Eq. (14), respectively) for tracking the exposure to an information cascade simulated from the ICM. In Fig. 2a, it can be clearly seen that the friendship paradox-based stochastic approximation Eq. (14) outperforms the vanilla stochastic approximation Eq. (13), especially after the time-step 40 where the diffusion process speeds up and the degree-sharing correlation coefficient starts increasing rapidly. In particular, the friendship paradox-based method closely detects the sudden phase transition of the diffusion process approximately at time-step 55 where the exposure suddenly jumps to a larger value. This result aligns with the theoretical conclusions in Sec. 4, since both assortativity coefficient $r$ and the degree-sharing correlation coefficient $\rho$ are both positive, leading to the friendship paradox-based method to outperform the vanilla method. On
the other hand, the Fig. 2b corresponds to the disassortative networks. Since the assortativity coefficient $r$ and the degree-sharing correlation coefficient $\rho$ have opposite signs, the vanilla method outperforms the friendship paradox-based method.

4. Implications of assortativity $r$ and degree-sharing correlation $\rho$ on the overall accuracy: It can be seen from Fig. 1 that both estimates ($\hat{f}_{Vl}$ and $\hat{f}_{FP}$) tend to be less accurate when the degree and sharing are negatively correlated (i.e., $\rho < 0$ in first column of Fig. 1). Although the friendship paradox-based estimator performs better in Fig. 1a(i), its accuracy decreases when moving to Fig. 1a(iv,vii). This result is due to the fact that when $\rho < 0$, the nodes who share the piece of information are the less popular nodes, which makes the average exposure $f$ smaller and more difficult to estimate. If $r < 0$ in addition to $\rho < 0$ (e.g., a star graph where outer nodes are sharing), then the friendship paradox-based estimator $\hat{f}_{FP}$ can easily reach the core nodes that are more likely to be exposed due to their popularity, and thus reduce the variance as in Fig. 1a(i). However, when $r > 0$ and $\rho < 0$ (i.e., Fig. 1a(vii)), the less popular fringe nodes who share the piece of information are more likely to be separated from the core of the network so that the friendship paradox-based estimator cannot reach them. As such, both estimators tend to be the least accurate when $\rho < 0, r > 0$. An important example of this is the case where a piece of information originates with the less visible (i.e., fringe) nodes of the network. As such, special attention should be paid to choosing the best method when $\rho < 0$ to get the best possible accuracy.

In summary, Sec. 6 numerically compared the absolute errors of the estimates obtained using the two estimators presented in Sec. 3 (vanilla estimator $\hat{f}_{Vl}$ and the friendship paradox-based estimator $\hat{f}_{FP}$). The numerical results agree with the theoretical conclusions provided in Sec. 4 and shed more light on the conditions under which one method outperforms the other.

7 RESULTS ON REAL-WORLD NETWORKS

Evaluating the accuracy of the two estimators $\hat{f}_{Vl}, \hat{f}_{FP}$ (proposed in Sec. 3) requires the true exposure $f$ (i.e., the ground truth). However, as we stressed in Sec. 1, the exact value of the ground truth $f$ depends on two features (the full network and the set of sharers) which are highly difficult to obtain, and our study was motivated by this difficulty in the first place. As such, comparing the estimates with the ground truth $f$ in real-world networks (e.g., Twitter, Facebook) is not feasible from a resource and computation viewpoint. Therefore, in this section, we first use real-world undirected networks with the sharing function generated synthetically (Sec. 7.1). We then evaluate our estimators on a real-world network with actual sharing data, using the directed ACM citation network (Sec. 7.2).

7.1 Undirected Networks

We tested the vanilla and friendship paradox-based estimators ($\hat{f}_{Vl}, \hat{f}_{FP}$) on four publicly available real-world network datasets in the SNAP database [22]. These networks include: a collaboration network between authors of papers submitted to Astrophysics and General Relativity in the Arxiv website, a network of Facebook pages of athletes, and a Facebook page network of different companies. For these four networks, the sharing function $s : V \rightarrow \{0, 1\}$ was synthetically generated using the methods in Sec. 6. The results obtained using these five real-world networks are shown in Fig. 3.

For the network datasets corresponding to Fig. 3a-Fig. 3c (where $r > 0$), the friendship paradox-based estimator $\hat{f}_{FP}$ outperforms the vanilla estimator $\hat{f}_{Vl}$ when $\rho > 0$ while both methods have generally large error values (above 100%) when $\rho < 0$. This result is as we expected since $\hat{f}_{FP}$ works better when $r, \rho > 0$ (as per the first point in the discussion related to Theorem 3) and both $\hat{f}_{Vl}, \hat{f}_{FP}$ tend to be less accurate when $\rho < 0, r > 0$ (as per the fourth point in the discussion of numerical results). For the network
dataset corresponding to Fig. 3d, \( \hat{f}_{FP} \) (resp. \( \hat{f}_{V} \)) works better when \( \rho < 0 \) (resp. \( \rho > 0 \)) since the network has \( r < 0 \) (as we theoretically expected). Therefore, the empirical findings align with both the theoretical and numerical results (Sec. 4 and Sec. 6, respectively).

### 7.2 ACM Citation Network

In this analysis, we provide a full real-world network (i.e., without subsampling the network) as well as actual sharing data from that network. We use a network of academic papers and their references. We consider a paper that contains a phrase in its title as a "sharer" of that phrase and all papers that cite that paper as the "exposed". We obtain a dataset of 629,814 papers from DBLP, ACM, and MAG (Microsoft Academic Group) [28]. We filtered out papers that did not have references within the original dataset or were not referenced by another paper in the original dataset to create a final dataset of 217,335 papers. We determine phrases of varying popularity to see how our estimates perform depending on the fraction of sharers. To generate the set of phrases, we first filter the papers’ titles for stopwords and determine the frequency of each word to create a numerically sorted dictionary with word frequency pairs. Then, we use NLTK’s bigram association measures to create word pairs (i.e., phrases) using a subset of words from the dictionary’s beginning. We define popular phrases as having more than 400 sharers (e.g., *data mining, information systems*), average phrases with between 200 and 400 sharers (e.g., *computer graphics, embedded systems*), and unpopular phrases with 100 to 200 sharers (e.g., *network design, optimization problems*. We perform this experiment with 25 popular phrases, 25 average phrases, and 25 unpopular phrases.

Based on the detailed results in Section 4 and their extension in Section 5.1, we expect the friendship paradox estimate based on random followers \( \hat{f}_{Fo} \) to have the lowest absolute mean error, and the vanilla estimate \( \hat{f}_{V} \) to outperform the friendship paradox estimate based on random friends \( \hat{f}_{Fr} \). This is due to one version of the friendship paradox that states on average, a random follower has more friends than a random node; therefore, a random follower is more likely to be exposed to a piece of information.

**Discussion of the ACM Citation Results (Fig. 4):** Fig. 4 shows the average vanilla estimate \( \hat{f}_{V} \), friend-based estimate \( \hat{f}_{Fr} \), and follower-based estimate \( \hat{f}_{Fo} \) for popular phrases (a), average phrases (b), and unpopular phrases (c). In each case, the follower-based estimate \( \hat{f}_{Fo} \) outperforms the other two as we expected from the directed versions of the friendship paradox mentioned in Sec. 5.1. In particular, Fig. 4 shows that the absolute error of all three estimates increase as the popularity of the phrases decrease. However, the follower-based estimate \( \hat{f}_{Fo} \) is more accurate compared to the other two estimates (i.e., the difference in the accuracy is larger) for unpopular phrases (Fig. 4(c)). This is because random followers are more likely to be exposed even to an unpopular piece of information due to their larger friend count (according to the directed versions of the friendship paradox) and hence, sampling random friends lowers the variance of the estimate. In comparison, uniform and friend-based sampling are less likely to reach the smaller number of exposed individuals when the popularity of the piece of information is lower and as such, they yield a larger variance.

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A PROOFS OF THEOREMS

A.1 Proof of Theorem 2

Part 1: For the vanilla estimate \( \hat{\mathcal{f}}_{\nu} \), it follows that \( \mathbb{E} \{ \hat{\mathcal{f}}_{\nu} \} = \bar{f} \) since it is the average of \( n \) iid Bernoulli random variables (with parameter \( \bar{f} \)). For the friendship paradox-based estimate \( \hat{\mathcal{f}}_{FP} \),

\[
\mathbb{E} \{ \hat{\mathcal{f}}_{FP} \} = \mathbb{E} \left\{ \frac{k}{n} \sum_{i=1}^{n} f(Y_i) \right\} \\
= k \mathbb{E} \left\{ \frac{f(Y_i)}{d(Y_i)} \right\} (\because Y_1, \ldots, Y_n \text{ are iid samples.}) \\
= k \sum_{v \in V} d(v) \sum_{i \in V} d(v) \left( \begin{array}{c} d(v) \\
\sum_{v \in V} d(v) \end{array} \right) \\
= k \sum_{v \in V} f(n) \sum_{v \in V} f(n) = \bar{f}.
\]

Therefore, both \( \hat{\mathcal{f}}_{\nu}, \hat{\mathcal{f}}_{FP} \) are unbiased estimates of \( \bar{f} \).

Part 2: Consider the variance of the vanilla estimate \( \hat{\mathcal{f}}_{\nu} \). Since the estimate is the average of \( n \) iid Bernoulli random variables (with parameter \( \bar{f} \)), their variance is given by \( \bar{f}(1-\bar{f})/n \). For the friendship paradox-based estimate \( \hat{\mathcal{f}}_{FP} \),

\[
\text{Var} \{ \hat{\mathcal{f}}_{FP} \} = \text{Var} \left\{ \frac{k}{n} \sum_{i=1}^{n} f(Y_i) \right\} \\
= \frac{1}{n} \text{Var} \left\{ \frac{k f(Y_i)}{d(Y_i)} \right\} (\because Y_1, \ldots, Y_n \text{ are iid samples.}) \\
= \frac{1}{n} \mathbb{E} \left\{ \left( \frac{k f(Y_i)}{d(Y_i)} \right)^2 \right\} - \bar{f}^2 (\because \mathbb{E} \hat{\mathcal{f}}_{FP} = \bar{f}) \\
= \frac{k^2}{n} \sum_{v \in V} f^2(d(v)) \times \frac{d(v)}{\sum_{v \in V} d(v)} - \bar{f}^2 \\
= \frac{1}{n} \left( \frac{k^2}{n} \sum_{v \in V} f(d(v)) \times \frac{1}{nk} - \bar{f}^2 \right) = \frac{1}{n} \left( \mathbb{E} \left\{ f(X)/d(X) \right\} - \bar{f}^2 \right).
\]

A.2 Proof of Theorem 3

Note that \( P(k'|k) P_x(0|k') \) is the probability that a degree \( k \) node connects to a degree \( k' \) node that hasn’t shared the piece of information. Therefore, averaging this term over the value \( k' \) (i.e., \( \sum_{k'} P(k'|k) P_x(0|k') \)) yields the probability that a degree \( k \) node having a neighbor that hasn’t shared the piece of information. For a degree \( k \) node to not be exposed to the information, all \( k \) neighbors of that node must not have shared the piece of information. Hence, \( \sum_{k'} P(k'|k) P_x(0|k') \) is the probability that a node with degree \( k \) has not been exposed to the piece of information i.e.,

\[
\mathbb{P} \{ f(X) = 0 | d(X) = k \} = \left( \sum_{k'} P(k'|k) P_x(0|k') \right)^k.
\]

which yields Eq. (9).

Next, using the expressions for the variance in Eq. (6) and the fact that \( \bar{f} = \mathbb{E} \{ f(X) \} \) (where \( X \) is a uniformly sampled node), we
get.

\[
\text{Var}(\hat{f}_1) \geq \text{Var}(\hat{f}_{FP}) \iff \frac{1}{n} \left( \bar{k} \mathbb{E} \left( \frac{f(X)}{d(X)} \right) - \bar{f}^2 \right) \geq \frac{1}{n} \bar{f}(1 - \bar{f})
\]

\[
\iff \bar{f} - \bar{k} \mathbb{E} \left( \frac{f(X)}{d(X)} \right) \geq 0
\]

\[
\iff \mathbb{E} \left( f(X) \left( 1 - \frac{\bar{k}}{d(X)} \right) \right) \geq 0 \quad (\because \mathbb{E} \{ f(X) \} = \bar{f} \text{ from Eq. (5)})
\]

\[
\iff \mathbb{E}_k \{ f(X) \left( 1 - \frac{\bar{k}}{d(X)} \right) \mid d(X) = k \} \geq 0
\]

(by conditioning on \( d(X) = k \) and then averaging over \( k \))

\[
\iff \mathbb{E}_k \{ 1 - \frac{\bar{k}}{k} \mathbb{P} \{ f(X) = 1 \mid d(X) = k \} \} \geq 0
\]

This completes the proof.

**B ADDITIONAL NUMERICAL RESULTS AND DETAILS FOR REPRODUCIBILITY**

The Github repository will be made publicly available for full reproducibility. Below, we provide additional details related to the simulation setup used to generate the numerical results in Sec. 6.

**Detailed Simulation setup for comparing the estimators \( \hat{f}_{FP}, \hat{f}_{FP} \) (Fig. 1):**

To generate the power-law networks, a sequence of 10k random variables from a power-law distribution with the required power-law exponent \( \alpha \) were generated and rounded up to the nearest integer. Then, the first number in the sequence was altered by a value of 1 if needed to make sure that the sum of the numbers is even (to be valid sequence of degrees). Then, the configuration model (configuration_model function in the networkx package) was used to generate the networks with the given degree sequence.

To change the assortativity of the generated power-law networks, we first sample two edges from the network uniformly (without replacement) and rewire them to increase or decrease the assortativity coefficient \( r \) from the initial value. The Lemma 1 of [32], which orders the three possible ways to rewire the two selected edges based on the resulting assortativity coefficient values, is used to get the maximum increase or decrease in the assortativity coefficient when rewiring. This process is repeated until the required assortativity coefficient values \( r \in \{-0.2, 0, 0.2\} \) are reached or the maximum number of iterations (100k) is reached.

To change the degree-sharing correlation coefficient \( \rho \), we follow the attribute swapping procedure used in [19] as follows. We first pick a node \( u \) from the set of nodes who shared the piece of information uniformly (i.e., \( s(u) = 1 \)) and another node \( v \) from the set of the people who has not shared the piece of information uniformly (i.e., \( s(v) = 0 \)). Then, to increase degree-sharing correlation coefficient \( \rho \), we swap \( s(u) \) and \( s(v) \) if \( d(u) < d(v) \) (resp. \( d(u) > d(v) \)). This process is continued until the required degree-sharing correlation coefficient values \( \rho \in \{-0.2, 0, 0.2\} \) are reached or the maximum number of iterations (100k) is reached.

**Simulation setup for comparing the vanilla and the friendship paradox-based stochastic approximations (given in Eq. (13) and Eq. (14)):** Under the ICM used to generate Fig. 2, each neighbor of a node who shared a piece of information at a previous time instant shares in the current time instant with a pre-specified probability named the infection probability. For Fig. 2, the diffusion was initialized with 10 uniformly chosen nodes and the infection probability is set to 0.05. Additionally, the step size \( \epsilon \) of the stochastic approximations Eq. (13), Eq. (14) is set to 0.01. Further, it is assumed that the stochastic approximations Eq. (13), Eq. (14) are updated 100 times for each step of the diffusion process i.e., the samples are collected 100 times faster than the evolution of the diffusion process.

Fig. 5 shows analogous results obtained using LTM where, a node shares a piece of information at the current time instant if the fraction of its neighbors that have already shared it by the previous time instant exceeds a certain threshold. We choose the threshold value to be 5% for the Fig. 5. The step-size of both stochastic approximations Eq. (13), Eq. (14) as well as the number of samples collected at each time instant are the same as the case for the ICM.

**C PRACTICAL IMPLEMENTATION DETAILS**

When edges cannot be sampled uniformly: The implementation of the friendship paradox-based estimate \( \hat{f}_{FP} \) given in Eq. (4) requires the uniform sampling of links from the underlying social network to obtain random friends \( Y_1, Y_2, \ldots, Y_n \). This sampling approach is feasible in situations where links have unique IDs from a range of integers and the link corresponding to a given integer can be accessed. In settings where such uniform edge sampling is not possible (e.g., a fully unknown social network), the friendship paradox-based estimate \( \hat{f}_{FP} \) can be implemented via the use of random walks, since a stationary distribution of a random walk on an undirected, connected, non-bipartite graph samples nodes with probabilities proportional to their degrees (page 298, [8]). Hence, the random variables \( Y_1, Y_2, \ldots, Y_n \) could be replaced with samples from a sufficiently long random walk. Alternatively, one can also use a second version of the friendship paradox which states that "uniformly sampled friend of a uniformly sampled node has more friends than a uniformly sampled node, on average" [7]. Hence, taking \( n \) uniformly sampled nodes and then taking one random friend of each of them would also be an alternative approach for friendship paradox-based sampling in undirected networks.

When the set of sharers is known: The set of sharers \( S = \{ v \in V : s(v) = 1 \} \) maybe publicly known in some contexts (e.g. Twitter users who shared a particular hashtag). In such cases, several improvements can be made to the proposed methods.

First, \( S \) is typically an array that can be ordered (e.g., a set of unique Twitter handles, a set of integer node IDs, etc.). Thus, for each sampled node \( v \in V \), calculating \( f(v) \in \{ 0, 1 \} \) becomes equivalent to the problem of finding out whether two ordered arrays (the node \( v \)'s neighbors \( N(v) \) and the set of sharers \( S \) intersect or not. Hence, it is computationally easier to calculate \( f(v) \) when the set of sharers \( S \subset V \) is known.

Second, when \( S \subset V \) is known, the average degree of the sharers \( \mathbb{E}(d(X) | s(X) = 1) = \frac{\sum_{v \in S} d(v)}{|S|} \) (which is typically hard to estimate if the set \( S \) is small compared to \( V \)) can be calculated. Further, the average degree of the people who have not shared (i.e., \( \mathbb{E}(d(X) | s(X) = 0) \)) can be estimated by sampling. Then, comparing the two values can be used as a heuristic estimate of the sign of the degree-sharing correlation coefficient \( \rho \).
Figure 5: The performance of the two stochastic approximation algorithms based on the vanilla and friendship paradox-based estimates (given in Eq. (13) and Eq. (14), respectively) for tracking the exposure to an information cascade under the Linear Threshold Model (LTM) for a synthetic power-law network with the power-law exponent $\alpha = 2.5$. This result complements the result shown in Fig. 2 for the Independent Cascade Model (ICM). It can be seen that the conclusions reached under the ICM also hold for the LTM as well. Thus, the proposed vanilla and friendship paradox-based stochastic approximation algorithms can be used to track (in real-time) the exposure to information cascades with various dynamical properties.