Product of Periodic Groups

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Article History
Received: 06.07.2019
Accepted: 12.07.2019
Published: 30.07.2019

Abstract: A group G is called radicable if for each element x of G and for each positive integer n there exists an element y of G such that x=y^n. A group G is reduced if it has no non-trivial radicable subgroups. Let the hyper-(locally nilpotent) or finite) group G=AB be the product of two periodic hyper-(abelian or finite) subgroups A and B. Then the following hold: (i) G is periodic. (ii) If the Sylow p-subgroups of A and B are Chernikov (respectively: finite, trivial), then the p-component of every abelian normal section of G is Chernikov (respectively: finite, trivial).

Keywords: Residual, factorize, minimal condition, primary subgroup

2000 Mathematics subject classification: 20B32, 20D10.

INTRODUCTION
In 1968 N. F. Sesekin proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition [1]. N. F. Seskin and B. Amberg independently obtained a similar result for maximal condition around 1972 [1, 2]. Moreover, a little later the later proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the minimal condition, and its Fitting subgroup inherits the factorization. B. Amberg, N. S. Chernikov, S. Franciosi, F. de Gioranni, O. H. Kegel, J. C. Lennox, D. J. S. Robinson, J. F. Roseblade, Y. P. Sysak, J. S. Wilson and D. I. Zaitsev, development the subjects of products and factorizable group for soluble and nilpotent groups [3-12]. In 1989, S. V. Ivanov showed that every countable group may be embedded in a product of two Tarski groups [13]. Now in this paper we show that if A_1, A_n is n periodic subgroups such that satisfying the minimal condition on primary subgroups, and G=A_1...A_n, then G is also satisfies the minimal condition on primary subgroups, and J(G)=J(A_1)...J(A_n), where J(G) is the finite residual of group G.

Elementary Definitions and Theorems
In this section we give the elementary definition and Basic theorems whose used the proof of main theorem.

Definition
A group G is the product of two subgroups A and B if G=AB={ab|aA, bB}. In this case we say that G is factorized by A and B. The factorizer of S in G=AB is denoted by X(S) and defined as following: The intersection X(S) of all factorized subgroup S of G=AB containing the subgroup S is the smallest factorized subgroup of G Containing S [5].

Definition
Let P is the property pertaining to subgroups of a group G. then we define the P-radical ρ(G) of G is the subgroup generated by all normal P-subgroups of G [5].

Lemma [5]
Let A, B and K be subgroups of a group G such that G=AK=HK and AK = BK = 1. If K is normal in G, then ρ(A)K = ρ(B)K, and so this is a normal subgroup of G.
Proof

Since $\varphi(A)K/K = \varphi(G/K) = \varphi(B)K/K$, hence $\varphi(A)K = \varphi(B)K$, and the proof is complete.

Lemma [2]

Let the group $G=AB$ be the product of two subgroups $A$ and $B$. Then:
- If $A$ and $B$ satisfy the maximal condition on subgroups, then $G$ satisfies the maximal condition on normal subgroups.
- If $A$ and $B$ satisfy the minimal condition on subgroups, then $G$ satisfies the minimal condition on normal subgroups.

Proof

Let $(H_n)_{n\in\mathbb{N}}$ be an ascending sequence of normal subgroups of $G$. Then $(A \cap H_n)_{n\in\mathbb{N}}$ and $(B \cap H_n)_{n\in\mathbb{N}}$ are ascending sequence of subgroups of $A$ and $B$, respectively. Hence $A \cap H_n = A \cap H_{n+1}$ and $B \cap AH_n = B \cap AH_{n+1}$, for almost all $n$. It follows that $AH_n = AB \cap AH_n = A(B \cap AH_n) = A(B \cap AH_{n+1}) = AH_{n+1}$, and so $H_n = H_n(A \cap H_{n+1}) = AH_n \cap H_{n+1} = AH_{n+1} \cap H_{n+1} = H_{n+1}$, for almost all $n$. Therefore $G$ satisfies the maximal condition on normal subgroups. The proof of (ii) is similar.

Lemma [5]

Let $N$ be a non-periodic locally nilpotent normal subgroup of a group $G$ such that $G/N$ is locally finite. If $x$ is an element of infinite order in $N$ and $n$ is a positive integer, the coset $x(x^n)^0$ has order exactly $n$ in the factor group $G/(x^n)^0$. Moreover, if $\pi$ is the set of prime divisors of $n$, the group $x^0((x^n)^0)$ is a $\pi$-group.

Proof

Let $m$ be the least positive integer such that $x^m$ belongs to $(x^n)^0$. Then $x^m$ is in $(x^n)^E$ for some finitely generated subgroup $E$ of $G$. Clearly $(x^n)^E$ lies in $N < x^n, E >$. Since $G/N$ is locally finite, $N \cap < x^n, E >$ has finite index in $< x^n, E >$ and so is a finitely generated nilpotent group. It follows that $(x^n)^E = x^{E_{n/m}}$ has index in $E$ and so is a finitely generated infinite nilpotent group, which is generated by the $(n/m)$th powers of its elements. This is possible only if $m=n$. As the locally nilpotent group $x^0((x^n)^0)$ is generated by $\pi$-elements it is clearly a $\pi$-group.

Lemma

Let the group $G=AB=AK=BK$ be the product of two subgroups $A$ and $B$ and a locally finite normal subgroup $K$. If $E$ is a finite subgroup of $A$ such that $\pi(E) \cap \pi(K)$ is empty, then there exists an element $a$ of $A$ such that $E^a$ is contained in $A \cap B$.

Proof: See Lemma 3.2.2. of [5]

Lemma

Let $\pi$ be a set of primes, and let the locally finite group $G=AB=AK=BK$ be the product of two subgroups $A$ and $B$ and a normal $\pi$-subgroup $K$ such that $A \cap K = B \cap K = I$. If $A_0$ and $B_0$ are normal $\pi'$-subgroups of $A$ and $B$, respectively, such that $A_0 \leq B_0 \leq K$, then $N_{A_0}(A_0 \cap B_0) = A_0 \cap B_0$.

Proof: See Lemma 3.2.3. of [5]

Lemma [5]

Let $\pi$ be a set of primes, and let the locally finite group $G=AB=AK=BK$ be the product of two subgroups $A$ and $B$ and a normal $\pi$-subgroup $K$ such that $A \cap K = B \cap K = I$. If $A_0$ is a normal $\pi'$-subgroup of $A$, then $B_0 = O_{\pi'}(B) \cap A_0 K$ is a normal subgroup of $B$ satisfying the following conditions.
- $A_0 K = B_0 K$
- $N_{A_0}(A_0 \cap B_0) = A_0 \cap B_0$
- If $A_0$ is a belian or finite, then $A_0 \leq B_0$ is a normal subgroup of $G$. 

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Proof: (i) By Lemma 2.3 we have that \( O_{\pi'}(A)K = O_{\pi'}(B)K \). Then
\[
A_0K = O_{\pi'}(A)K \cap A_0K = O_{\pi'}(B)K \cap A_0K = (O_{\pi'}(B) \cap A_0K) = B_0K.
\]
- This follows from (i) and lemma 2.7.
- It is clearly sufficient to show that \( A_0 \) it contained in \( B_0 \), since then \( B_0 = A_0K \cap B_0 = A_0 \). If \( A_0 \) is abelian, then \( A_0 \cap B_0 = A_0 \) by (ii), so that \( A_0 \) is contained in \( B_0 \) if \( A_0 \) is finite. By lemma 2.6. there exists an element \( a \) of \( A \) such that \( A_0 = A_0^a \leq A \cap B \leq N_G(B_0) \). Therefore \( A \cup B_0 \) is a \( \pi' \)-group.

Now \( A_0B_0 = A_0B_0 \cap B_0K = B_0(A_0B_0 \cap K) = B_0 \), so that \( A_0 \) is contained in \( B_0 \). The lemma is proved.

Definition
Recall that a group \( G \) is called radicable if for each element \( x \) of \( G \) and for each positive integer \( n \) there exists an element \( y \) of \( G \) such that \( x = y^n \). A group \( G \) is reduced if it has no non-trivial radicable subgroups.

Lemma: Let \( G \) be a locally finite group having no infinite simple sections then:
- If \( G \) satisfies the minimal condition on \( p \)-subgroups for some prime \( p \), then the factor group \( G/O_{\pi'}(G) \) contains a Chernikov \( p \)-subgroup of finite index;
- If \( G \) satisfies the minimal condition on \( p \)-subgroups for every prime \( p \), then the finite residual \( J \) of \( G \) is radicable abelian and the sylow subgroups of \( G/J \) are finite.

Proof [6]
Note: For see definition of locally finite group refered to "Locally Finite Groups" book of Kegel and Wehrfritz [6].

Main Theorem
Let the hyper-(locally nilpotent or finite) group \( G=AB \) be the product of two periodic hyper-(abelian or finite) subgroups \( A \) and \( B \). Then the following hold.
- \( G \) is periodic.
- If the Sylow \( p \)-subgroups of \( A \) and \( B \) are Chernikov (respectively: finite, trivial), then the \( p \)-component of every abelian normal section of \( G \) is Chernikov (respectively: finite, trivial).

Proof: (i) Assume that \( G \) is not periodic. Without loss of generality We may suppose that \( G \) has no non-trivial periodic normal subgroups. Then \( G \) contains a non-trivial torsion-free locally nilpotent normal subgroup \( K \). Clearly the factorizer \( X (K) \) is also a counterexample, so that we may suppose that \( G \) has a triple factorization \( G=AB=AK=KB \), where \( A \cap K = B \cap K = 1 \). Since \( A \) is hyper-(abelian or finite), it contains a non-trivial normal subgroup \( N \) which is either finite or an abelian \( p \)-group. As \( N \) is normal \( G=AK \), the intersection \( NK \cap B \) is normal in \( B \). If \( N \) is contained in \( B \), it is a normal subgroup of \( AB = G \). This contradiction shows that \( N \) is not contained in \( B \). Let \( a \) be an element of \( N \) and write \( a^b=\bar{x} \), with \( b \) in \( B \) and \( x \) in \( K \). Let \( q \) be a prime which is not in the finite set \( \pi (N) \). Consider the normal closures \( L=\langle x \rangle \) and \( V=\langle x^q \rangle \). Since \( x \) has infinite order, by Lemma 2.5 the coset \( \bar{x} = x^V \) has order \( q \) in \( G = \bar{G} = G/N \), and \( \bar{L} = L/V \) is a \( q \)-group. The factorizer \( X \) of \( \bar{L} \) in \( G = \bar{G} = G/N \) has the triple factorization
\[
\bar{X} = \bar{A} * \bar{B} * = \bar{A} * \bar{L} = \bar{B} * \bar{L},
\]
where \( \bar{A} = A \cap BL \) and \( \bar{B} = B \cap AL \). By Lemma 2.8(iii) the subgroup \( \bar{N} \cap \bar{A} \) is contained in \( \bar{B} \) * , so that \( \bar{x} = \bar{b} \bar{a} \) belongs to \( \bar{B} \cap K = 1 \). This contradiction proves.

(ii) Assume that \( G \) contains an abelian normal section \( M \) whose \( p \)-component is not a Chernikov group. Without loss of generality we may suppose that \( M \) is an abelian normal \( p \)-subgroup of \( G \). As the factorizer \( X(M) \) is also a counterexample, it can also be assumed that \( G \) has a triple factorization \( G=AB=AM=BM \). Since \( A \cap M \) and \( B \cap M \) are Chernikov normal \( p \)-subgroups of \( G \), we may replace \( G \) by \( G/(A \cap M) \) and hence suppose that \( A \cap M = B \cap M = 1 \).
By Lemma 2.3 the subgroup $O_{p'}(A)M = O_{p'}(B)M$ is normal in $G$. Clearly $O_{p'}(G)$ is contained in $H=O_{p'}(A)M$. Hence $O_{p'}(G) \leq O_{p'}(H) \leq O_{p'}(A)$, so that $O_{p'}(G)$ is contained in $A \cap B$. Factoring out $O_{p'}(G)$, we may suppose that $G$ has no non-trivial normal $p'$-subgroups. Clearly $A \cap M = B \cap M = 1$. If $O_{p'}(A)$ is not trivial, it contains a non-trivial normal subgroup $A_0$ of $A$ which is either finite or abelian. Then $A_0$ is normal in $G$ by Lemma 2.8(iii). This contradiction shows that $O_{p'}(A) = O_{p'}(B) = 1$. Since $A$ and $B$ satisfy the minimal condition on $p$-subgroups, it follows from Lemma 2.10(i) that $A$ and $B$ contain normal $p$-subgroups of finite index. In particular, $A$ and $B$ are Cherinkov groups. The group $G=AM$ also contains a normal $p$-subgroup $P$ of finite index. Moreover, $G$ satisfies the minimal condition on normal subgroups, by Lemma 2.4(ii). Then $A$ and $B$ contain normal $p$-subgroups of finite index and hence is a Cherinkov group (Robinson 1972, Part 1, Theorem 5.21 and Corollary 2 to Theorem 5.27) [14]. Therefore $G$ is a Cherinkov group. This contradiction completes the proof of the first statement of (ii). The proofs of the other two statements are similar proof of Theorem is complete.

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