DISTRIBUTIVE LATTICE POLYMORPHISM ON REFLEXIVE GRAPHS

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ABSTRACT. In this paper we give two characterisations of the class of reflexive graphs admitting distributive lattice polymorphisms and use these characterisations to attack the problem of finding a polynomial time recognition algorithm for these graphs.

We provide a polynomial time recognition algorithm for R-thin reflexive graphs which admit what we call Type 2 distributive lattice polymorphisms.

1. Introduction and Results

A graph is reflexive if every vertex has a loop. In this paper we give two characterisations of the class of reflexive DL-graphs, that is, of finite reflexive graphs admitting distributive lattice polymorphisms. We then use these characterisations to attack the problem of finding a polynomial time recognition algorithm for these graphs.

A polymorphism on a graph $G$ is a operation $f : V(G)^d \to V(G)$ on its vertices which preserves edges. It is a now basic fact (see for example [3]) in the theory of constraint satisfaction problems (CSPs) that for various CSPs which can be defined for a given graph, the computational complexity of the problems is intimately related to the types of polymorphisms that the graph admits.

A graph admits totally symmetric idempotent polymorphisms of all arities if and only if it has tree duality. The two main sources of totally symmetric idempotent polymorphisms, are near-unanimity (NU) polymorphisms, and semilattice (SL) polymorphisms. While NU polymorphisms have been well studied, and the classes of graphs admitting them have several nice characterisations, no such study had been attempted for SL polymorphisms until [15], where we looked at the family of reflexive graphs admitting SL polymorphisms. We were unable to find a satisfactory characterisation of such graphs, which we called SL-graphs. However; were able to show that the class of reflexive SL-graphs in a very natural way extended the class of chordal graphs, and in fact, the simplest ones, those in which the semilattice is a chain, are exactly the class of interval graphs.

Graphs admitting lattice polymorphisms, are by definition SL-graphs. The current paper, characterising DL-graphs, began as the author’s attempt to gain intuition into SL-graphs. However, they are an interesting class of graphs in their own right. They show up in [4] where it is shown that a structure $H$ is homomorphically equivalent to a structure admitting distributive lattice polymorphisms if and only if
it has so-called caterpillar duality. As a result they get that the list colouring problem for a reflexive graph $H$ is solvable in polynomial time if it is homomorphically equivalent to a DL-graph, and is otherwise NP-complete.

We note here that in [8], the graphs that are Hasse graphs (defined in Section 2) of distributive lattices are characterized. This is somewhat related to the current problem, but Hasse graphs are not the same thing as DL-graphs!

A partial characterisation of DL-graphs can be extracted from known literature. Indeed, it follows from [14] and [16] (see also [10]) that retracts of products of reflexive paths are exactly the reflexive graphs that admit majority, or 3-NU polymorphisms, that is, polymorphisms $f : V(G)^3 \rightarrow V(G)$ satisfying

$$f(x, y, z) = c$$

if at least two of $x, y$ and $z$ are $c$.

If a reflexive graph admits a polymorphic lattice, then it also admits the following majority operation (see, for example, [1])

$$f(x, y, z) = (x \land y) \lor (y \land z) \lor (x \land z).$$

Thus all lattice graphs are retraction products of paths. However, it is easy to find (see, for example, Proposition 3.8) retraction products of paths that are not DL-graphs. This begs the question: which retraction products of paths are DL-graphs? We do not answer this question, though as mentioned later in this introduction, it motivates our results of Section 5. Rather we find that DL-graphs yield a more useful characterisation in terms as subgraphs of products of proper interval graphs.

To state our results on DL graphs, we require several definitions. All graphs considered are finite, and unless otherwise stated, all are reflexive. All statements about lattices stated without reference in this section are basic and can be found in any text on lattices; see for example, [12] (especially Section 7). Moreover, many of these concepts are explained in Section 2.

Recall that a lattice $L$ can be viewed equivalently as a partial ordering $\leq$ on a set $L$, or as a pair of binary operations $\land, \lor : L \times L \rightarrow L$ on $L$, called the meet and join respectively. We sometimes write $L = (L, \leq)$ to designate notation for the order $\leq$ of a lattice $L$. The lattice is distributive if $\land$ and $\lor$ distribute. A graph $G$ on a lattice $L$ is a graph whose vertex set is $V(G) = L$. A lattice $L$, and a graph $G$ on $L$ are compatible if the binary operations $\land$ and $\lor$ of $L$ are polymorphisms of $G$; that is, if the following holds, where ‘∼’ denotes adjacency in $G$:

$$(u \sim u' \text{ and } v \sim v') \Rightarrow (u \land v \sim u' \land v' \text{ and } u \lor v \sim u' \lor v')$$

If $G$ has a compatible lattice, then it is a lattice graph. If it has a compatible distributive lattice, then $G$ is a distributive lattice graph, or DL-graph.

A totally ordered lattice is called a chain. The n-chain denoted $Z_n$ is the chain on the set $[n] = \{0, 1, \ldots, n\}$ with the usual ordering $0 \leq 1 \leq \cdots \leq n$.

The following characterisation of proper interval (PI) graphs was observed in [11]. A graph is a PI-graph if and only if it has a labeling of its vertices with the labels $[n]$ such that the so-called min-max property holds:

$$(i \leq j \leq k \text{ and } i \sim k) \Rightarrow (i \sim j \text{ and } j \sim k)$$

Though this definition is usually made for irreflexive graphs, all our graphs are reflexive, and one will notice that this is, in fact, quite appropriate for the given definitions, as well as for other common definitions of proper interval graphs.
Both of our characterisations start from the simple observation (Lemma 5.3) that a reflexive graph is compatible with a chain if and only if it is a PI-graph.

1.1. Downset Characterisation. For our first characterisation, we recall a well known representation theory of distributive lattices, due to Birkhoff [2].

For a poset $P$, a subset $D$ is a downset if $a \in D$ and $a \geq b$ implies $b \in D$. The family $\mathcal{D}(P)$ of all downsets of $P$ is a distributive lattice under the inclusion ordering. The meet and join operations are $\cap$ and $\cup$, respectively. Birkhoff showed that for any distributive lattices $L$, $L \cong \mathcal{D}(J_L)$, for a unique poset $J_L$, defined in Section 2.

We now give a construction of compatible graphs on $\mathcal{D}(P)$. A poset $P$ can be viewed as an acyclic (except for loops) digraph by setting $u \rightarrow v$ if $u \geq v$. As such we can also talk of a subgraph $A$ of $P$: a subgraph of the digraph $\{u \rightarrow v \mid u \geq v \text{ in } P\}$.

**Definition 1.1** ($G(P, A)$). For a poset $P$ and a subgraph $A$ of $P$, let $G = G(P, A)$ be the graph whose on $\mathcal{D}(P)$, in which two downsets $S, S' \in \mathcal{D}(P)$ are adjacent if all comparable pairs $x \geq y$ in either $S - S'$ or $S' - S$ are edges of $A$.

See Figure 1 for an example. The depiction of posets by their Hasse graph is explained in Section 2. Observe, in the figure, that the pair $\{1, 2, 3, 4, 5\} - \{1, 2, 3, 4\}$ is not an edge of $G$ as the comparable pair $5 \geq 5$ of vertices in $\{5\} = \{1, 2, 3, 4, 5\} - \{1, 2, 3, 4\}$ is not an edge (that is, a loop), of $A$.

It is not too hard to check (Lemma 7.2) that for any poset $P$ and subgraph $A$, the graph $G(P, A)$ is compatible with $\mathcal{D}(P)$, so is a DL-graph. Nor is it hard to show (see Fact 7.7) that edges can be added to $A$ without changing $G(P, A)$, until $A$ satisfies (1).

It is not immediate however that every DL-graph can be constructed in this way. This is our first characterisation.

**Theorem 1.2.** A graph $G$ on a lattice $L$ is compatible with $L$ if and only if $G = G(J_L, A)$ for some subgraph $A$ of $J_L$ satisfying the min-max property (1).
As a distributive lattice $L$ is uniquely defined by the poset $J_L$, this implies that a graph $G$ is a DL-graph if and only if $G = (P,A)$ for some poset $P$ and some subgraph $A$ of $P$ satisfying the min-max property. 

1.2. PI Characterisation. The proof of Theorem 1.2 will come from our second characterisation, which though being not as succinct as the first, is every bit as useful.

Our basic object is a product of chains (the product of lattices is defined in Section 2), which we denote by $P = \prod_{i=1}^{d} C_i$, where $C_i$ is isomorphic to the $n_i$-chain $Z_{n_i}$ for some implicitly defined $n_i$. The elements of $P$ are $d$-tuples in $\prod_{i=1}^{d} [n_i]_0$, and we have $x \leq y$ if $x_i \leq y_i$ for all $i \in [d]$.

It is easily shown, see Lemma 3.1, that the categorical product of graphs is compatible with the lattice product, from which it follows that the product of PI-graphs is a DL-graph. It is well known that any distributive lattice $L$ compatible with the lattice product, from which it follows that the product of $P$ embedded into a product of chains, induced by the vertices of $L$ in each case the compatible lattice $P$ of chains, satisfied by the vertices of $L$. We refer to these sets as vertex and edge bites, let $B$ be the union of all bites in $P$, and $\beta < \alpha$ be empty, and will sometimes be innocuously assumed to be in a family of PI-graphs.

For each $i,j \in [d]$, $\alpha \in [n_i]$ and $\beta \in [n_j-1]_0$ define the following vertex and edge subsets of $\mathcal{G}$. Let $V_i(\alpha,\beta)_j := \{x \mid \alpha \leq x_i \text{ and } x_j \leq \beta\}$, and $E_i(\alpha,\beta)_j = \{uv \in \mathcal{G} \mid u_i \geq \alpha \text{ and } v_j \leq \beta\}$.

We refer to these sets as vertex bites and edge bites respectively. For a family $B$ of vertex or edge bites, let $\cup B$ be the union of all bites in $B$. A family $B$ of vertex bites is closed if $V_i(\alpha,\beta)_j \subseteq \cup B$ implies $V_i(\alpha,\beta)_j \in B$. The closure of a family of edge bites is analogously defined. The bites $V_i(\alpha,\beta)_j$, or $E_i(\alpha,\beta)_j$, when $i = j$ and $\beta < \alpha$ are empty, and will sometimes be innocuously assumed to be in a family of bites, even when it is not closed.

See Figure 2 for examples of vertex and edge bites removed from a product of PI-graphs.

**Theorem 1.3.** Any reflexive distributive lattice graph $G$ is one of the following graphs.

- (i) A product $\mathcal{G}$ of proper interval graphs.
- (ii) An induced subgraph $\mathcal{H}_V = \mathcal{G} - \cup B_V$ of a graph $\mathcal{G}$ for families $B_V$ of vertex bites.
- (iii) A subgraph $\mathcal{H}_E = \mathcal{G}_V - \cup B_E$ of a graph $\mathcal{G}_V$ from item (ii), for families $B_E$ of edge bites such that for all $E_i(\alpha,\beta)_j \in B_E$, $V_i(\alpha,\beta)_j \in \cup(B_V)$.

In each case the compatible lattice $L$ is the sublattice of the corresponding product $P$ of chains, induced by the vertices of $G$.

The three graphs in Figure 3 exemplify the three steps of this description. Notice that $E_i(\alpha,\beta)_j$ contains all edges incident to vertices in $V_i(\alpha,\beta)_j$, including loops, as well as other edges. It is due to reflexivity and to this fact that step (iii) requires the ‘consistency’ between $B_E$ and $B_V$. 

As the embedding of a lattice into a product of chains is sometimes called a factorization, we refer to the representation $G = \mathcal{G} - \bigcup \mathcal{V} - \bigcup \mathcal{E}$ of a DL-graph $G$ in terms of $\mathcal{G}$ as a factorization of $G$ or of the compatible pair $(G, L)$.

A factorization is Type 1, 2, or 3 depending on whether $G$ is described in step (i), (ii), or (iii) of the above theorem. It is called tight if $L$ is a tight (defined in Section 2) sublattice of $\mathcal{P}$. The following example shows that a DL-graph generally has tight factorization of different types. The number of factors is called the dimension of the factorization.

**Example 1.4.** The path $P_3 = 1 \sim 2 \sim 3 \sim 4$ is a proper interval graph so dimension 1 DL-graph. It has a trivial Type 1 factorization of dimension 1. However, it also has the following Type 3 factorization of dimension 3. Let $\mathcal{G}$ be the clique $K_2^2$, (each factor $K_2$ is a PI-graph). Removing the vertex bites $V_1(1,0)_2$, $V_2(1,0)_3$, and $V_1(1,0)_3$ we leave the clique induced by the vertices $(0,0,0)$, $(0,0,1)$, $(0,1,1)$, and $(1,1,1)$. Removing the edge bites $E_1(1,0)_2$, $E_2(1,0)_3$, and $E_1(1,0)_3$, we leave the path $P_3$ on these vertices.

By definition, a Type $i$ factorization is also a Type $i - 1$ factorization. It is shown in Section 2 that there are DL-graphs with Type 2 factorization but not Type 1 factorization, and DL-graphs with Type 3 factorization but not Type 2 factorization.

**Definition 1.5.** A graph $G$ is a DL graph of Type $i$ if $i$ is the minimum $j$ for which it has a tight Type $j$ factorization.

As non-tight factorization are largely overlooked in the literature on representations of finite lattices, every such factorization easily being ‘altered’ to a tight factorization, our original intention was to omit mention of non-tight factorization, and our definition of Type $i$ DL-graph reflects this. However, it turns out that, non-tight factorization are of some interest, and so are considered in Section 6. For Type 1 and 3 the restriction to tight factorization makes no difference: all Type 1 factorization are tight, and from Theorem 1.3 any graph with a Type 3 factorization has a tight Type 3 factorization. However, in Proposition 6.1 we show that any Type 3 DL-graph has a non-tight Type 2 factorization.
This, on its own, is perhaps not sufficient motivation to consider non-tight factorization, but it is clear that to describe DL-graphs as retractions of products of proper interval graphs, it will be necessary to consider non-tight factorization. Indeed, as retractions are necessarily induced subgraphs, a Type 3 DL-graph cannot be described as a retraction of a product of PI-graphs into which it embeds as a tight subgraph.

As a Type 1 DL-graph is a product of proper interval graphs, and these are polynomial time recognizable, the following is immediate from [9], which tells us that strong products of graphs, which is equivalent to the categorical product when we are considering reflexive graphs, are recognizable in polynomial time. (See Chapter 24 of [13] for various algorithms.)

Corollary 1.6. There is a polynomial-time algorithm to decide if a given graph is a Type 1 DL-graph.

Using techniques discussed in [13] we will be able to generalize this somewhat. A graph is $R$-thin if no two vertices have the same closed neighbourhood. Observe that cores are $R$-thin.

Theorem 1.7. For an $R$-thin graph $G$ there is a polynomial time algorithm to decide whether or not $G$ is a Type 2 DL-graph.

The layout of the paper is as follows. In Section 2 we recall necessary results about posets and lattices. In Section 3 we observe some basic properties of DL-graphs and more general lattice graphs. In Section 4 we give some examples of DL-graphs with various properties. In Section 5 we prove Theorem 1.3 giving our more technical PI characterisation of DL-graphs. In Section 6 we augment the results of Section 5 with the consideration of non-tight factorization of DL-graphs. In Section 7 we prove Theorem 1.2 giving the our donwset characterisation of DL-graphs. We further prove Proposition 7.8 which allows us to decide if a graph has a Type 2 factorization with respect to a particular lattice $L$. Finally in Section 8 we prove Theorem 1.7 addressing the problem of recognizing DL-graphs.

2. Basics about Posets and Distributive Lattices

A cover $a \prec b$ of a poset $P = (P, \leq)$ is a pair of elements $a \leq b$ in $P$ such that

$$a \leq x \leq b \Rightarrow a = x \text{ or } x = b.$$  

If $a \prec b$ is a cover, we say $a$ covers $b$. The acyclic digraph $\hat{H}(P)$ of all covers $(a, b)$ of a poset $P$ is called its Hasse diagram, and the transitive closure of the Hasse diagram, is $P$. The graph $H(P)$ we get from the Hasse diagram of $P$ by replacing each arc with a symmetric edge, is called the Hasse graph of $P$. In figures, we represent a poset or lattice by drawing its Hasse graph, in thick grey edges, such that the corresponding edges of the Hasse diagram are directed downwards.

As the operations $\lor$ and $\land$ of a lattice $L = (V, \leq)$ are associative and commutative, and $V$ is finite in all lattices considered, set versions $\lor S$ and $\land S$ are well defined for all subsets $S \subset V$. A lattice $L$ has a minimum, or zero, element $0_L = \land V$ and a maximum, or one, element $1_L = \lor V$. The height of an element $a \in L$ is the distance in $H(L)$ from $a$ to $0_L$. The length $\ell(L)$ of $L$ is the height of $1_L$. 

The product $L = L_1 \times L_2$ of two lattices $L_1 = (L_1, \leq_1)$ and $L_2 = (L_2, \leq_2)$ is the lattice $(L, \leq)$ defined on the set $L = L_1 \times L_2$ by

$$(a_1, a_2) \leq (b_1, b_2) \text{ if } a_i \leq_1 b_i \text{ for } i = 1, 2.$$  

One can show that the operations $\lor$ and $\land$ of a product are defined componentwise from the corresponding operations of the factors. Thus the product of distributive lattices is a distributive lattice.

A sublattice $L' \leq L$ of a lattice is a subset $L' \subset L$ that is closed under the functions $\land$ and $\lor$. A sublattice is itself a lattice, and it is clear that a sublattice of a distributive lattice is distributive. We will be concerned exclusively with the situation where $L$ is a sublattice of a product $\mathcal{P} = \prod_{i=1}^{d} C_i$ of chains. It is well known, see [6], that every distributive lattice can be embedded as a sublattice into a product of chains. A sublattice $L \leq \mathcal{P}$ is full if $0_L = 0_{\mathcal{P}}$ and $1_L = 1_{\mathcal{P}}$. It is tight if $\ell(L) = \ell(\mathcal{P})$, which is equivalent to being full and cover preserving; every cover of $L$ is a cover of $\mathcal{P}$. It was observed in [17] any tight sublattice $L \leq \mathcal{P}$ of a product of chains, is a semi-direct sublattice: the projection maps $\phi_i : \mathcal{P} \to C_i$, restricted to $L$, are surjective. A homomorphism of lattices $L \to L'$ is a map $f : L \to L'$ which commutes with meets and joins. Homomorphisms are necessarily isotone: $x \leq y \Rightarrow f(x) \leq f(y)$. The projection map $\pi_i : \mathcal{P} \to C_i$ is a homomorphism.

For any subset $P \subset L$ of a lattice $L = (V, \leq)$, $L$ induces a poset on $P$ by restriction of $\leq$ to $P$.

An element $j$ of a lattice is join-irreducible if $a \lor b = j$ implies that $j = a$ or $j = b$. Meet-irreducible elements are defined analogously. Let $J_L$ be the set of non-zero join-irreducible elements of a lattice $L = (L, \leq)$, and let $J_L = (J_L, \leq)$ be the poset defined by restricting $\leq$ to $J_L$. For $x \in L$, let $\downarrow x$ be the downset

$$\{y \in L \mid y \leq x\},$$

and let $\uparrow x$ be defined analogously. An irreducible interval of a lattice is the interval $\downarrow j \cap \uparrow m$ for any join-irreducible element $j$ and meet irreducible element $m$. For a product of chains $\mathcal{P}$ these are exactly the intervals

$$\{x \mid x_i \leq x \text{ and } x_j \leq \beta\},$$

for $i, j \in d$, $\alpha \in [n_i]_0$, and $\beta \in [n_j]_0$.

We mentioned before Birkhoff’s result that for any distributive lattice $L, L \cong \mathcal{D}(J_L)$. Dilworth took this further in [5] corresponding every chain decomposition of $J_L$ to an embedding of $L$ into a product of chains.

Recall that $C_i$ denotes a copy of the chain $Z_{n_i}$ on the set $[n_i]_0$ for some implicitly defined $n_i$; let $C_i^*$ denote the copy of $Z_{n_i-1}$ on the set $[n_i]$ we get by removing the element 0 from $C_i$. A chain decomposition of a poset $P$ is a family $\mathcal{C}^*$ of subchains $C_1^*, \ldots, C_d^*$ such that every element of $P$ is in one exactly one chain. Dilworth showed that for every chain decomposition $\mathcal{C}^*$ of $J_L$, the map $e : S \mapsto (|S \cap C_1^*|, \ldots, |S \cap C_d^*|)$ is an embedding of $\mathcal{D}(J_L)$, so of $L$, into a product of chains $\mathcal{P} = \prod_{i=1}^{d} C_i$. Larson [17] showed that these embeddings are tight.

This correspondence between chain decompositions of $J_L$ and tight embeddings of $L$ into products of chains, has a useful interpretation through a result of Rival.

In [18], Rival showed the following, which we have specialized to our purposes. For a family $\mathcal{R}$ of irreducible intervals, let $\cup \mathcal{R} = \cup_{I \in \mathcal{R}} I$. $\mathcal{R}$ is closed if $I \subset \cup \mathcal{R}$ implies $I \in \mathcal{R}$. 
Theorem 2.1. [18] For any sublattice $L$ of a product of chains $\mathcal{P}$ there is a subset $\mathcal{R}$ of irreducible intervals of $\mathcal{P}$ such that

$$L = \mathcal{P} - \cup \mathcal{R}. $$

Further $\mathcal{R}$ may be assumed to be closed.

It is clear that for a full sublattice $L = \mathcal{P} - \cup \mathcal{R}$, $\mathcal{R}$ contains no intervals $i[\alpha, \beta]_j$ with $\alpha = 0$ or $\beta = n_j$. The alert reader will now observe that vertex bites of $\mathcal{G}$ are exactly the irreducible intervals $i[\alpha, \beta]_j$ of the compatible lattice $\mathcal{P}$ for which $\alpha \neq 0$ and $\beta \neq n_j$. We view any poset as a digraph by viewing comparabilities $u \geq v$ as arcs $u \rightarrow v$.

Definition 2.2. For an embedding $L = \mathcal{P} - \cup \mathcal{R}$ of a lattice $L$ as a tight sublattice of a product of chains $\mathcal{P}$, let $\mathcal{C}^*$ denote the disjoint union $\bigcup_{i=1}^{d} C_i^*$. Denote the element $\alpha$ of $C_i^*$ by $\alpha e_i$. Let $R = R_\mathcal{P}(\mathcal{R})$ be the spanning supergraph of $\mathcal{C}^*$ with arcs

$$\{\alpha e_i \rightarrow (\beta + 1)e_j | i[\alpha, \beta]_j \in \mathcal{R}\}. $$

The assumption that $R$ contains $\mathcal{C}^*$ as a spanning subgraph corresponds to the innocuous assumption that the family $\mathcal{R}$ contain the empty intervals $i[\alpha, \beta]_j$ for $i = j$ and $\beta < \alpha$. As a spanning subgraph, $\mathcal{C}^*$ is exactly a chain decomposition of $R$.

In [19] we showed that for a tight sublattice $L = \mathcal{P} - \cup \mathcal{R}$, the digraph $R$ is a poset (that is, symmetric, transitive, and acyclic), and isomorphic to $J_L$. This gives the following restatement of the correspondence due to Dilworth [6] and Larson [17].

Theorem 2.3. For any tight embedding $L = \mathcal{P} - \cup \mathcal{R}$ of a distributive lattice $L$ as sublattice of a product of chains $\mathcal{P}$, $\mathcal{C}^*$ is a chain decomposition of $J_L \cong R_\mathcal{P}(\mathcal{R})$. Further, for any chain decomposition of $J_L$ there is a (unique) embedding of $L$ into a product of chains $\mathcal{P}$ such that the corresponding $\mathcal{C}^*$ is that chain decomposition of $J_L$.

3. Basic Properties of Lattice Graphs

The (categorical) product of two graphs $G_1$ and $G_2$, with vertex sets $V_1$ and $V_2$ respectively, is the graph $G = G_1 \times G_2$ with vertex set $V_1 \times V_2$ and edge set

$$\{(u_1, u_2)(v_1, v_2) | u_i v_i \in G_i \text{ for } i = 1, 2\}. $$

Lemma 3.1. Let $G_1$ be a compatible graph on $L_1$ and $G_2$ be a compatible graph on $L_2$. Then $G_1 \times G_2$ is a compatible graph on $L_1 \times L_2$.

Proof. Let $(u_1, u_2) \sim (v_1, v_2)$ and $(u'_1, u'_2) \sim (v'_1, v'_2)$ in $G_1 \times G_2$. Then

$$(u_1, u_2) \land (u'_1, u'_2) = (u_1 \land u'_1, u_2 \land u'_2) \sim (v_1 \land v'_1, v_2 \land v'_2) = (v_1, v_2) \land (v'_1, v'_2), $$

and similarly $(u_1, u_2) \lor (u'_1, u'_2) = (v_1, v_2) \lor (v'_1, v'_2). \quad \square$

The following is clear from the definition of compatibility.

Fact 3.2. If a graph $G$ is compatible on a lattice $L$, and $L'$ is a sublattice of $L$, then the subgraph $G'$ of $G$ induced by $L'$ is compatible with $L'$. 
A conservative set (or subalgebra) in a reflexive graph $G$ is a subset $S \subset V(G)$ that is the intersection of sets of the form \( \{x \in V(G) \mid d(x, x_0) \leq d\} \) for some vertex $x_0$ and integer $d$. Components and maximal cliques are examples of conservative sets. It is a basic fact, (see [3]), that a conservative set of a graph is closed under any polymorphism.

**Lemma 3.3.** A graph is a (distributive) lattice graph if and only if each component is.

**Proof.** If a graph is disconnected, and each of its components has a compatible lattice $L_i$, then let $L$ be the simple join of the component lattices. That is, let $L$ be the lattice on the set $\bigcup_{i=1}^{d} L_i$ with the ordering defined by $x \leq y$ if $x \leq y$ in some $L_i$ or if $x \in L_i$ and $y \in L_j$ for $i < j$. It is easy to check that this lattice is compatible with $G$, and that it is distributive if the component lattices are.

On the other hand, if a disconnected graph has a compatible lattice, then as each component is a subalgebra, and subalgebras are closed under polymorphisms, each component is closed under the lattice operations. Thus each component induces a sublattice, so is compatible with the component by Fact 3.2. If a lattice is distributive, then so is any sublattice. □

In light of this, we may consider only connected graphs.

**Lemma 3.4.** If a connected reflexive graph $G$ is compatible with a lattice $L$ then $v$ and $v$ are connected for every vertex $v$.

**Proof.** Indeed, let $u_0$ and $u_p$ be in $v$. Then there is a path $u_0 \sim u_1 \sim \cdots \sim u_p$ between them in $G$. So

\[
(v \lor u_0) \sim (v \lor u_1) \sim \cdots \sim (v \lor u_p)
\]

is a walk between them in $v$. □

The main use of this lemma is in the following proposition. The analogous result does not hold for semilattices.

**Proposition 3.5.** For a connected reflexive graph $G$ with a compatible lattice $L$, $H(L)$ is a subgraph of $G$.

**Proof.** It is enough to show for any cover $v \prec u$, that $uv$ is an edge of $G$. But this is clear, as by the previous lemma the subgraph of $G$ induced by $v$ is connected and so is the subgraph of this graph induced by $|v \cap |u$. But this graph contains only $u$ and $v$, so $uv$ is an edge. □

**Corollary 3.6.** For a connected reflexive graph $G$ with a degree one vertex $v$, $v$ must be the minimum or maximum vertex of any compatible lattice $L$.

**Proof.** All other vertices have a cover above them and a cover below. □

**Corollary 3.7.** The only reflexive trees with compatible lattices are paths.

**Proof.** By the previous corollary a reflexive tree with a compatible lattice must be a path. That reflexive paths have compatible lattices is trivial, but also follows from Lemma 5.3. □
Proposition 3.8. Neither the class of graphs admitting compatible lattices, nor the class admitting compatible distributive lattices, are closed under retraction.

Proof. It is easy to see that the reflexive biclique \(K_{1,4}\) is a retract of the product \(P_2^2\) of two reflexive paths. \(P_2^2\) has a distributive lattice by Lemma 3.1 but \(K_{1,4}\) does not, by Corollary 3.7.

4. Some examples

Lemma 4.1. Let \(u\) be a vertex in a graph with a compatible lattice \(L\). Then any two neighbours \(v, w\) of \(u\) with \(u \geq v, w\) (or \(u \leq v, w\)) are adjacent.

Proof. Indeed as \(v \sim u\) and \(u \sim w\), we have that \(v = (v \land u) \sim (u \land w) = w\).

Proposition 4.2. There are graphs that have compatible lattices but have no compatible distributive lattices.

Proof. Let \(G\) be the graph on the left of Figure 3. It is not too hard to verify that the non-distributive lattice shown on the right is compatible. We show that there is no distributive lattice that is compatible with \(G\).

Assume, towards contradiction, that \(G\) has a compatible distributive lattice \((V(G), \leq)\). By Proposition 3.5 \(0\) and \(1\) must be the vertices labelled 0 and 1 in the figure. Further \(0\) must have unique cover \(a\) and \(1\) must cover \(j\). So \(|j| \cap |a|\) is a distributive sublattice with min \(a\) and max \(j\).

As the set \(\{d, e\}\) is the intersection of maximal cliques, it is a conservative set, so induces a sublattice. The only 2 element lattices is the chain, so we may assume, without loss of generality, that \(d \leq e\).

The set \(\{d, e, h\}\) is also an intersection of maximal cliques, so induces a sublattice of three elements, so must also be a chain. If \(h \leq e\) then Lemma 4.1 implies that \(a\) and \(h\) are adjacent, so \(h \geq e\). Similarly \(b \leq d\).

The set \(\{b, d, e, f, h\}\) is a maximal clique, so induces a lattice. As \(b\) is not adjacent to \(a\) or \(j\), it follows from Lemma 4.1 that it can neither be above or below \(d\) or \(e\), so it is incomparable with them. Thus the sublattice induced on \(\{b, d, e, f, h\}\) is as shown in the figure. It is well known that no lattice with this lattice as a sublattice is distributive.

Proposition 4.3. There are Type 2 distributive lattice graphs that are not Type 1.

Proof. The graph \(Z_2 \times Z_2 - V(2, 1)\) is an example of such a graph. It contains an induced \(K_{1,3}\), so is not a PI-graph. As it has a prime number of vertices, specifically 7, it cannot be a product of more than one PI-graphs.

Proposition 4.4. There are Type 3 distributive lattice graphs that are not Type 2.

Proof. For \(i = 1, 2\) let \(C_i\) be the reflexive path \([3]_0\), and let \(G' = C_1 \times C_2 - V(3, 0)z - E(3, 0)\). Let \(G\) be the graph we get by adding a vertex 1 adjacent only to \((3, 3)\) and a vertex 0 adjacent only to \((0, 0)\). Then \(G\) is R-thin and has the obvious Type 3 compatible lattice \(L\) seen in Figure 4. By Corollary 3.6 any compatible lattice \(L\) has \(1_L = 1\) and \(0_L = 0\) (upto isomorphism).
5. PI Characterisation of DL Graphs

We start with some observations relating lattice homomorphisms and graph homomorphisms. The first is clear.

Fact 5.1. Any cover preserving homomorphism $f : L \rightarrow L'$ of lattices induces a homomorphism of Hasse graphs $f : H(L) \rightarrow H(L')$.

For a set map $f : V \rightarrow V'$, and a graph $G$ on the vertex set $V$, let the push $G_f$ of $G$ onto $V'$ be the graph on the vertex set $V'$ with edgeset

$$E(G_f) = \{ f(u)f(v) \mid uv \in E(G) \}.$$ 

This is clearly the minimum graph one can put on $V'$ so that $f$ is a graph homomorphism.

Lemma 5.2. Let $f : L \rightarrow L'$ be a lattice homomorphism, and $G$ be compatible with $L$. Then $G_f$ is compatible with $L'$.
Proof. Let \( f(u)f(v) \) and \( f(u')f(v') \) be edges of \( G_f \). Then by the definition of the push, \( uw \) and \( u'v' \) are edges of \( G \), and as \( G \) is compatible with \( L \), \( (u \land u')v \land v' \) is an edge of \( G \). As \( f \) is a graph homomorphism, \( f(u \land u')f(v \land v') \) is an edge of \( G_f \). And as \( f \) is a lattice homomorphism \( f(u \land u') = f(u') \land f(u) \) and \( f(v \land v') = f(v) \land f(v') \). So \( (f(u') \land f(u))(f(v) \land f(v')) \) is an edge, as needed. □

Lemma 5.3. A graph \( G \) on the chain lattice \( Z_n \) is compatible if and only if it satisfies \([1]\), i.e., if and only if it is a proper interval graph.

Proof. On the one hand, assume that \( G \) is compatible, and that \( i \leq j \leq k \) and \( i \sim k \). As \( G \) is reflexive, we have that \( j \sim j \). So \( i = i \land j \sim k \land j = j \) and \( k = k \lor j \sim i \lor j = j \).

On the other hand, assume that \([1]\) holds. We show that \( \land \) is a polymorphism; the proof that \( \lor \) is a polymorphism is essentially the same. Let \( a \sim b \) and \( c \sim d \). We may assume without loss of generality that \( a \leq b \) and \( c \leq d \). Further we may assume that \( a \leq c \), so \( a \land c = a \). We must show that \( a \sim (b \land d) \). If \( b \leq d \), then \( b \land d = b \) so this is clear. If \( d < b \) then \( b \land d = d \) and by \( a \leq d < b \) and \( a \sim b \), the identity above gives \( a \sim d \).

As any distributive lattice \( L \) is the sublattice of a product of chains \( \mathcal{P} \), any compatible graph is a subgraph of the clique \( K_\mathcal{P} \) on \( \mathcal{P} \). The following simple observation is surprisingly effective in describing \( G \) as a subgraph of \( K_\mathcal{P} \).

Given a graph \( G \) and a lattice \( L \) on its vertices, we define the edge poset \( E_L(G) \) of \( G \) as the poset induced by the product lattice \( L^2 \) on the set \( \{(x, y) \in L^2 \mid xy \in G\} \). As \( G \) is symmetric, \( (x, y) \in E_L(G) \) if and only if \( (y, x) \in E_L(G) \).

Lemma 5.4. A graph \( G \) on a lattice \( L \) is compatible with \( L \) if and only if \( E_L(G) \) is a sublattice of \( L^2 \). If \( G \) is reflexive, then \( E_L(G) \) is a full sublattice.

Proof. That \( G \) is compatible with \( L \) is the statement that \( u \sim v \) and \( u' \sim v' \) implies \( u \land u' \sim v \land v' \) and \( u \lor u' \sim v \lor v' \). But this is also just the statement that if \( (u, v) \) and \( (u', v') \) are in \( E_L(G) \) then \( (u \land u', v \land v') = (u, v) \land (u', v') \) and \( (u \lor u', v \lor v') = (u, v) \lor (u', v') \) are in \( E_L(G) \). But this is the statement that \( E_L(G) \subseteq V(L^2) \) is closed under the operations \( \land \) and \( \lor \). This is enough.

That it is a full sublattice is equivalent to the fact that \( 0_L \) and \( 1_L \) have loops. □

Now let \( G \) be a reflexive graph that is compatible with a full sublattice \( L \) of a product of chains \( \mathcal{P} \). As \( L \leq \mathcal{P} \) is full, and so \( G \) has loops on \( 1_\mathcal{P} \) and \( 0_\mathcal{P} \), we have that \( E_L(G) \leq L^2 \) is a full sublattice of the product of chains \( \mathcal{P}^2 \). By Theorem 2.1 we thus get that \( E_L(G) = \mathcal{P}^2 - \cup \mathcal{R} \) for some closed family \( \mathcal{R} \) of full irreducible intervals of \( \mathcal{P}^2 \). Just as elements of \( E_L(G) \) are pairs, viewed as edges, of \( G \), elements of \( \mathcal{P}^2 \) can be viewed as edges of the complete graph \( K_\mathcal{P} \) on \( \mathcal{P} \). So \( \cup \mathcal{R} \) is the set of edges we must remove from \( K_\mathcal{P} \) to get \( G \).

As \( G \) is reflexive \( E_L(G) \) determines \( G \) exactly: a vertex of \( K_\mathcal{P} \) is in \( G \) if and only if it has a loop. Thus the proof of Theorem 1.3 will come out of a simple analysis of the intervals in \( \mathcal{R} \). The following lemma records this setup, and provides this analysis of \( \mathcal{R} \). It can be viewed as a technical version of Theorem 1.3. The proof of Theorem 1.3 will be given immediately following this.

For the rest of the section, intervals \( i[\alpha, \beta] \) are intervals of \( \mathcal{P}^2 \), unless otherwise stated, so \( i \) and \( j \) are in \([2d] \).
Lemma 5.5. Let \((G, L)\) be a compatible pair where \(G\) is a reflexive graph and \(L\) is a full sublattice of a product of chains \(P = \prod_{i=1}^{n} C_{n_i}\). Let \(R = R_{E}(G, P)\) be the family of irreducible intervals of \(P^2\) such that \(E_L(G) = E(K_P) \cup R\).

Let an interval \(i_{[\alpha, \beta]} \in R\) be in
\[
\begin{align*}
\bullet & \hspace{1em} \mathcal{R}_+ \text{ if } 1 \leq i \neq j \leq d, \\
\bullet & \hspace{1em} \mathcal{R}_- \text{ if } 1 \leq i \leq d < j \leq 2d \text{ and } j \neq i + d, \\
\bullet & \hspace{1em} \mathcal{R}_0 \text{ if } j = i \text{ or } j = i + d, \text{ and } \alpha \leq \beta, \\
\bullet & \hspace{1em} \mathcal{R}_* \text{ if } j = i \text{ or } j = i + d, \text{ and } \alpha > \beta.
\end{align*}
\]

Then the following are true:
\begin{enumerate}
\item \(\mathcal{R} = \bigcup \mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{R}_0\).
\item If \(L\) is a tight sublattice of \(P\) then \(\mathcal{R}_0\) is empty, so \(\bigcup \mathcal{R} = \bigcup \mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{R}_*\).
\item \(K_P - \cup \mathcal{R}_*\) is a product of proper interval graphs.
\item \(i_{[\alpha, \beta]} \subset \bigcup \mathcal{R}_+ \text{ if and only if } V_i(\alpha, \beta) \subset V(K_P) - V(G)\).
\item \(i_{[\alpha, \beta]} \subset \bigcup \mathcal{R}_- \cup \mathcal{R}_* \text{ if and only if } E_i(\alpha, \beta)_{j-d} \subset E(K_P) - E(G)\).
\item If \(i_{[\alpha, \beta]} \in \mathcal{R}_-\) then \(i_{[\alpha, \beta]_{j-d}} \in \mathcal{R}_-\).
\end{enumerate}

Proof. (i) This follows from the fact that as \(P\) is symmetric the sets \(i_{[\alpha, \beta]}\) and \(i_{[\alpha, \beta]_{j+d}}\) consist of the same edges. As do the sets \(i_{[\alpha, \beta]_{j+d}}\) and \(i_{[\alpha, \beta]_{j+d}}\).

(ii) An interval \(i_{[\alpha, \beta]} \in \mathcal{R}_0\) contains the edge from any vertex \(u\) with \(u_i = \alpha\) to any vertex in the case that \(j = i\), or to any vertex \(v\) with \(v_i = \alpha\) in the case that \(j = i + d\). Either way, no vertex \(u\) with \(u_i = \alpha\) has a loop. As \(G\) is reflexive, this means no such vertex is in \(G\). This is impossible if \(L\) is tight. Thus when \(L\) is tight, \(\mathcal{R}_0\) is empty.

(iii) For each \(i\), let \(G_{n_i}\) be the graph we get from the clique \(K_{n_i}\) on the set \([n_i]\) by removing the the edges \(\{xy \mid x_i \leq \beta < \alpha \leq y_i\}\) for each \(i_{[\alpha, \beta]}\).

Then for each \(i\) and all \(x_i \leq y_i \in [n_i]\) we have the following.
\[
\begin{align*}
x_i y_i \notin G_{n_i} & \iff i_{[\alpha, \beta]} \in \mathcal{R}_* \text{ for some } x_i \leq \beta < \alpha \leq y_i \\
& \iff i_{[y_i, x_i]} \in \mathcal{R}_* \text{ for some } x_i \leq \beta < \alpha \leq y_i \\
& \iff uv \notin K_P - \cup \mathcal{R}_* \forall u, v \in K_P \text{ with } u_i = x_i \text{ and } v_i = y_i
\end{align*}
\]

As \(i_{[\alpha, \beta]} \in \mathcal{R}_\ast\) are empty, this gives us that \(K_P - \cup \mathcal{R}_\ast = \prod_{i=1}^{n} G_{n_i} =: \mathcal{G}\). To see that \(G_{n_i}\) is a proper interval graph, observe that \(G_{n_i}\) is clearly the push of \(\mathcal{G}\) by the projection \(\pi_i\) onto \([n_i]\), and that \(K_P - \cup \mathcal{R}_\ast\) is a DL-graph by Lemma 5.4. As \(\pi_i\) is a lattice homomorphism, it follows from Lemma 5.2 that \(G_{n_i}\) is compatible with \(C_{n_i}\). By Lemma 5.3 \(G_{n_i}\) is therefore a proper interval graph.

(iv) Observe that \(i_{[\alpha, \beta]} \in \mathcal{R}_\ast\) is exactly the set of edges, including loops, incident to vertices of \(V_i(\alpha, \beta)\). As \(G\) is reflexive, removing a vertex from \(K_P\) is equivalent to removing its loop, and so all of its edges.

(v) This is just a matter of definitions.

(vi) This follows by observing that the loops in \(i_{[\alpha, \beta]} \in \mathcal{R}_\ast\) are precisely those on vertices of \(V_i(\alpha, \beta)_{j-d}\). As \(G\) is reflexive, if a loop is removed from a vertex, then the vertex is removed. \(\square\)

Theorem 1.3 is now an easy corollary of Lemma 5.5.
Proof of Theorem 1.3. Let $G$ be a reflexive distributive lattice graph. Letting $L$ be any compatible distributive lattice, there is (see Theorem 2.3) a tight embedding of $L$ as a sublattice of a product of chains $\mathcal{P}$. Let $\mathcal{R}$ be $\mathcal{R}_E(G, \mathcal{P})$ of Lemma 5.5. By (i) and (ii) of the Lemma, we get $G$ from $K_{\mathcal{P}}$ by removing the edges of $\bigcup \mathcal{R}_e$, $\bigcup \mathcal{R}_v$ and $\bigcup \mathcal{R}_r$, and then removing any loopless vertices.

Now let
- $\mathcal{B}_V = \{ V_i(\alpha, \beta)_j \mid i[\alpha, \beta]_j \in \mathcal{R}_v \}$,
- $\mathcal{B}_E = \{ E_i(\alpha, \beta)_j \mid i[\alpha, \beta]_j \in \mathcal{R}_e \}$, and
- $\mathcal{B}_K = \{ B_i(\alpha, \beta)_j \mid i[\alpha, \beta]_j \in \mathcal{R}_r \}$.

Then Lemma 5.5(iv) and (v) give us that $G$ is the graph we get from $K_{\mathcal{P}}$ by removing all edge bites $E_i(\alpha, \beta)_j \in \mathcal{B}_E$ and $\mathcal{B}_K$, and all vertex bites $V_i(\alpha, \beta)_j \in \mathcal{B}_V$.

If $\mathcal{B}_V$ and $\mathcal{B}_E$ are empty, then by Lemma 5.5(iii), $G$ is the product $\mathcal{G}$ of interval graphs which we get by removing from $K_{\mathcal{P}}$ all edge bites $E_i(\alpha, \beta)_j \in \mathcal{B}_E$.

If $\mathcal{B}_E$ is empty, then $G$ is the induced subgraph $\mathcal{G}_V$ of $\mathcal{G}$ we get by removing all vertex bites $V_i(\alpha, \beta)_j \in \mathcal{B}_V$.

Otherwise $G$ is the graph $\mathcal{G}_{V,E}$ we get from $\mathcal{G}_V$ by removing all edge bites $E_i(\alpha, \beta)_j$, for $i[\alpha, \beta]_j \in \mathcal{B}_E$. By Lemma 5.5(vi) $\mathcal{B}_E \subset \text{cl}(\mathcal{B}_V)$.

The statement that we may assume that $L$ is a tight sublattice comes from the assumption of tightness made at the beginning of the proof.

6. Non-Tight Factorization

Proposition 6.1. Any Type 3 DL-graph has a semi-direct (not necessarily tight) Type 2 factorization.

Proof. Let $G = \mathcal{G} - \bigcup \mathcal{B}_V - \bigcup \mathcal{B}_E$ be a Type 3 DL-graph with compatible lattice $L$.

By definition, the sublattice $L$ induced on $G$ by the corresponding product of chains $\mathcal{P}$ is a tight, so subdirect sublattice of $\mathcal{P}$, and we may assume that $|\mathcal{B}_E| \geq 1$.

Let $E_0 := E_{(\alpha, \beta)}_0$ be in $\mathcal{B}_E$. Letting $\mathcal{G}' = \mathcal{G} \times P_2$, we show that $G$ can be embedded as $\phi(G) = \mathcal{G} - \bigcup \mathcal{B}_V' - \bigcup \mathcal{B}_E'$, where the sublattice $L'$ induced on $G' = \phi(G)$ by $\mathcal{G}'$ is a subdirect sublattice of $\mathcal{S}'$, and $|\mathcal{B}_E'| < |\mathcal{B}_E|$. The result then follows by induction on $|\mathcal{B}_E|$.

Indeed, the embedding $\phi$ is defined by setting $\phi(v)_i = v_i$ for $i = 1, \ldots, d$, and setting
$$\phi(v)_{d+1} = \begin{cases} 2 & \text{if } v_{i_0} \geq \alpha_0 \\ 0 & \text{if } v_{j_0} \leq \beta_0 \\ 1 & \text{otherwise.} \end{cases}$$

This is well defined, as $E_{(\alpha, \beta)}_0 \in \mathcal{B}_E$ implies that $V_i(\alpha, \beta)_0 \in \mathcal{B}_V$, so the three cases of the definition are a partition of the vertices of $G$. It is clearly injective, and is order preserving on $L$ as $V_2 := \{ v \mid v_{i_0} \geq \alpha_0 \}$ is an upset of $L$, (so no element in $G - V_2$ can be above any element of $V_2$), and $V_0 := \{ v \mid v_{j_0} \leq \beta_0 \}$ is a downset. Thus $\phi : L \rightarrow L'$ is an isomorphism, which means that $G'$ is compatible with $L'$. Moreover $L'$ is subdirect in $\mathcal{P}'$ as the projection $\pi_i$ satisfies $\pi_i(L') = \pi_i(L) = C_i$ for each $i \in [d]$, and $\pi_{d+1}(L') = Z_2$ by the definition of $L' = \phi(L)$.

As $L'$ is a sublattice of $\mathcal{P}'$ we know that $V(G') = V(G) - \bigcup \mathcal{B}_V'$ for some $\mathcal{B}_V'$, but we observe that in fact
$$\mathcal{B}_V' = \mathcal{B}_V \cup \{ V_{i_0}(\beta_0 + 1, 1)_{d+1}, V_{d+1}(1, \alpha_0 - 1)_{i_0} \},$$
where an element $V_i(\alpha, \beta)_j \in \mathcal{B}_V$ contains $d$-tuples, but in $\mathcal{B}_V'$ it contains $(d + 1)$-tuples.
We claim further that \( G' = \mathcal{G} - \cup \mathcal{B}_{V} - \cup \mathcal{B}_{E} \) where \( \mathcal{B}_{E} = \mathcal{B}_{E} - E_{0} \), and again an element \( E_{i}(\alpha, \beta) \) is interpreted differently in \( \mathcal{B}_{E} \) and in \( \mathcal{B}_{E} \). But this also is trivial, as for all edges \( uv \) in \( \cup \mathcal{B}_{E} - E_{0} \) for \( u, v \in G \), \( \phi(u)\phi(v) \) is an edge of \( \mathcal{B}_{E} \), and edges \( uv \) in \( E_{0} \) between \( u \in G \) with \( u_{0} \geq \alpha_{0} \) and \( v_{0} \leq \beta_{0} \) are not in \( G' \) as \( \phi(u)_{d+1} = 2 \) and \( \phi(v)_{d+1} = 0 \).

7. The Downset Characterisation

To prove Theorem 1.2 we must show (a) for all posets \( P \) and subgraphs \( A \) the graph \( G(P, A) \) is a DL-graph; and (b) that for any DL-graph \( G \), there is some poset \( P \) and subgraph \( A \) such that \( G = G(P, A) \). Part (a) is done easily in Lemma 7.1 and 7.2. For part (b) we prove something stronger: that for any factorization of any compatible pair \( (G, L) \) there is a subgraph \( A \) of \( J_{L} \) such that \( G = G(J_{L}, A) \). At the expense of a slightly more complicated proof, this allows us to decide if a given compatible pair \( (G, L) \) is of Type 2, and so to ultimately decide of a graph \( G \) is a Type 2 DL-graph.

As the construction in Definition 2.2 relates \( J_{L} \) to the embedding of \( L \) into \( \mathcal{P} \), and so through Theorem 1.3 to \( \mathcal{B}_{V} \), a similar construction relates the ‘complement’ of the graph \( A \) in \( J_{L} \) to \( \mathcal{B}_{E} \).

In light of this approach, it will be useful to reframe Definition 1.1 in terms of this complement. For a subgraph \( A \) of a poset \( P \), let \( \overline{A} \) be the subgraph of \( P \) with arc set

\[ \{x \to y \mid x \geq y \text{ but } x \not\to y \text{ in } A\} \]

**Lemma 7.1.** Where \( A \) is a subgraph of a poset \( P \), and \( S \) and \( S' \) are downsets, the following are equivalent definitions of the adjacency of \( S \) and \( S' \) in \( G(P, A) \).

(i) All comparable pairs \( x \geq y \) in either \( S - S' \) or \( S' - S \) are edges of \( A \).

(ii) Neither of \( S - S' \) or \( S' - S \) induce an edge of \( \overline{A} \).

(iii) For all \( x \in S \) and edges \( x \to y \) of \( \overline{A} \), \( y \in S' \), and for all \( x \in S' \) and edges \( x \to y \) of \( \overline{A} \), \( y \in S \).

**Proof.** This is clear when noting that all arcs of \( \overline{A} \) are between comparable pairs. \( \square \)

**Lemma 7.2.** If \( P \) is a poset and \( A \subset P \), then \( \mathcal{D}(P) \) is compatible with \( G = G(P, A) \).

**Proof.** Using (iii) of Lemma 7.1 for the definition of adjacency in \( G \), assume that \( S \sim S' \) and \( T \sim T' \). We must show that \( (S \cup T) \sim (S' \cup T') \) and \( (S \cap T) \sim (S' \cap T') \). For the first, let \( x \in S \cup T \) and \( x \to y \) in \( \overline{A} \). Then \( x \in S \) or \( T \), so as \( S \sim S' \) and \( T \sim T' \), we have that \( y \in S' \) or \( T' \). Thus \( y \in S' \cup T' \). That \( x \in S' \cup T' \) implies \( y \in S \cup T \) is the same, so \( (S \cup T) \sim (S' \cup T') \). The proof of \( (S \cap T) \sim (S' \cap T') \) is similar. \( \square \)

Now let \( \mathcal{G} = \cup \mathcal{B}_{V} - \mathcal{B}_{E} \) be a factorization of a compatible pair \( (G, L) \). Clearly we may assume that \( \mathcal{B}_{V} \) is closed. Observing that \( V_{i}(\alpha, \beta) \) in \( \mathcal{B}_{V} \) is exactly the irreducible interval \( \mathcal{I}_{i}[\alpha, \beta] \) of \( \mathcal{P} \) we get by Theorem 2.3 that \( J_{L} \) is the digraph on the set \( \mathcal{C}^{*} \) with arcs \( \alpha e_{i} \to (\beta + 1)e_{j} \) for every vertex \( V_{i}(\alpha, \beta) \) in \( \mathcal{B}_{V} \).

By the condition, in Theorem 1.3 that \( E_{i}(\alpha, \beta) \) in \( \mathcal{B}_{E} \) implies that \( V_{i}(\alpha, \beta) \) is in \( \text{cl}(\mathcal{B}_{V}) = \mathcal{B}_{V} \), we get that the following supergraph of \( \mathcal{C}^{*} \) is a subgraph of \( J_{L} \).

**Definition 7.3.** Let \( \overline{A} \) be the supergraph of \( \mathcal{C}^{*} \) with arcs

\[ \{\alpha e_{i} \to (\beta + 1)e_{j} \mid E_{i}(\alpha, \beta) \in \mathcal{B}_{E} \cup \mathcal{B}_{E}^{*}\} \]
Let $A = J_L - \overline{A}$ be its complement.

(Recall from the proof of Theorem 1.3 that $\mathcal{B}_E^*$ consists of the edge bites $E_i(\alpha, \beta)_j$ with $i = j$ and $\beta < \alpha$ that we remove from the clique on the vertices of $\mathcal{G}$ to make it a product of PI-graphs.)

See Figure 5 for examples of the posets $J_L$ (thick grey edges) and subgraphs $A$ (black edges) corresponding to the compatible graph lattice pairs factored in Figure 2. In the first picture of Figure 2, the first chain has had the arc $3e_1 \rightarrow 0e_1$ removed, so $E_1(3,1)_1$ is in $\mathcal{B}_E^*$. This accounts for the fact that the first component in the first picture of Figure 5 is not a clique.

**Lemma 7.4.** Let $G$ be a graph compatible with the lattice $L$, and $\mathcal{G} - \cup \mathcal{B}_V - \cup \mathcal{B}_E$ be a factorization. Then $G \cong G(J_L, A)$.

**Proof.** As $G(J_L, A)$ is a graph on $D(J_L)$, it is enough to show that Birkhoff’s lattice isomorphism $S : L \rightarrow D(J_L) : x \rightarrow S_x = \{j \in J_L \mid j \leq x\}$ preserves graph edges. That is, we show that $x \sim y$ in $G$ if and only if $S_x \sim S_y$ in $G(J_L, A)$. This is simply an unraveling of definitions.

Recall from the proof of Theorem 1.3 that for $x, y \in V(G)$, $xy$ is an edge of $G$ if and only if it is not in some edge bite in $\mathcal{B}_E \cup \mathcal{B}_E^*$. That is, $x \not\sim y$ if and only if there is some $E_i(\alpha, \beta)_j$ in $\mathcal{B}_E \cup \mathcal{B}_E^*$ with $x_j \leq \beta$ and $\alpha \leq y_i$ or with $y_j \leq \beta$ and $\alpha \leq x_i$. But if $E_i(\alpha, \beta)_j$ is in $\mathcal{B}_E \cup \mathcal{B}_E^*$ and $x_j \leq \beta$ and $\alpha \leq y_i$, then $E_i(y_i, x_j)_j$ is in $\text{cl}(\mathcal{B}_E \cup \mathcal{B}_E^*)$, so this is true if and only if $E_i(y_i, x_j)_j$ or $E_j(x_j, y_i)_i$ are in $\text{cl}(\mathcal{B}_E \cup \mathcal{B}_E^*)$. By the definition of $\overline{A}$ this is true if and only if $y_i e_i \rightarrow (x_j + 1) e_j$ or $x_j e_j \rightarrow (y_i + 1) e_i$ are in $\overline{A}$. Now as $y_i e_i \in S_y$ and $(x_j + 1) e_j \not\in S_x$, this is the negation of (iii) of Lemma 7.1, so is equivalent to $S_x \not\sim S_y$. \qed

Now to finish off the proof of Theorem 1.2 we could show that the assumption of closure on $R_2(G, L)$ gives that $A$, in the previous lemma, satisfies (1). Similar statements are shown in [19]. However, this is a little messy, and we rather appeal to the following observations.

Given a subgraph $\overline{A}$ of a poset $P$, an arc $x' \rightarrow y'$ of $P$ is called extraneous if there is another arc $x \rightarrow y$ in $\overline{A}$ such that $x' \geq x \rightarrow y \geq y'$.
Fact 7.5. The graph $G(J_L, A)$ is unchanged by adding or removing extraneous arcs from $A$.

Proof. For any two downsets $S$ and $S'$ of $J_L$ with $x' \rightarrow y'$ in $S - S'$, we clearly have that $x \rightarrow y$ is in $S - S'$ as well. \hfill \Box

Definition 7.6. For a subgraph $\overline{A}$ of a poset $P$, let $\overline{A}_+$ be the graph we get by adding all extraneous arcs, and let $\overline{A}_-$ be the graph we get by removing all extraneous arcs.

Fact 7.7. It is clear that $\overline{A}_+$ satisfies the following:

$(x' \geq x \geq y \geq y')$ and $(x \sim y) \Rightarrow (x' \sim y')$.

This is equivalent to $A_+ = J_L - \overline{A}_+$ satisfying (1).

Proof of Theorem 1.2. This is an immediate corollary of Lemma 7.2. Lemma 7.4. That $A$ can be taken to satisfy (1) follows from the above two facts by replacing $A$ with $A_+$. \hfill \Box

We finish the section with one useful lemma which we will use in the next section.

While the construction $R = R_p(\mathcal{R})$ for a given embedding $L = \mathcal{P} - \cup \mathcal{R}$ depends on the product $\mathcal{P}$ of chains we embed $L$ into, it is a useful fact that for tight embeddings and closed sets $\mathcal{R}$, the resulting $R$ is the poset $J_L$, regardless of $\mathcal{P}$. Similarly, the subgraphs $A_+$ and $A_-$ are independent of $\mathcal{P}$. Indeed, we can recover them directly from $J_L$ and $G$: an edge $x > y$ of $J_L$ is in $\overline{A}_+$ if and only if there are no two downsets $S$ and $S'$ in $\mathcal{D}(J_L) \cong L$, adjacent in $G$, with $x$ and $y$ in $S - S'$.

Further, $\overline{A}_-$ is determined from $\overline{A}_+$ by removing any extraneous arcs. See Figure 5.

Proposition 7.8. For a compatible pair $(G, L)$, the following are equivalent.

(i) $(G, L)$ has a Type 2 factorization.

(ii) There is a chain decomposition of $J_L$ such that every edge of $\overline{A}_-$ is between elements of a chain in the decomposition.

(iii) No vertex of $v$ of $J_L$ has more than one up-neighbour or down-neighbour in $\overline{A}_-$; that is, for each $v$, $|v \cap N(v)| \leq 1$ or $|v \cap N(v)| \leq 1$, where $N(v)$ is the set of neighbours of $v$ in $\overline{A}_-$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The graphs $\overline{A}_-$ for the graphs $A$ shown in Figure 5.}
\end{figure}
Proof. A Type 2 factorization of \((G, L)\) is a factorization \(G = G - B_V - B_E\), such that \(B_E\) is empty. This is empty if and only if \(R_e\) is empty where \(R = R_T(G, L)\) and \(R_e\) are as in Lemma 5.5. The set \(R_e\) is empty if and only if the only arcs in \(A\), from Definition 7.3, are those defined by \(B^*_E\), which are arcs of the chain decomposition \(C^*\) of \(R\). By Theorem 2.3, \(R = J_L\) and any such chain decomposition corresponds to a factorization. This gives the equivalence of (i) and (ii).

To see the equivalence of (ii) and (iii), observe that if up-neighbours of \(v\) are incomparable, then they cannot both be in a chain; if they are comparable, then they cannot both be in \(A\), as one would be extraneous. \(\square\)

8. Recognition of \(R\)-thin Type 2 DL-graphs

As we observed in Section 1 if \(G\) is a Type 1 DL-graph, then it is a categorical product of PI-graphs. The factorization of a categorical product was shown to be unique (up to certain obviously necessary assumptions) in [7] by Dörfler and Imrich. Feigenbaum and Schäffer [9] showed that a categorical product can be factored in polynomial time. On the other hand, it was shown in [5] that PI-graphs can be recognised in linear time. Thus Corollary 1.6 stating that Type 1 DL graphs can be recognised in polynomial time, is immediate. We suspect the following is true.

Conjecture 8.1. Reflexive DL-graphs can be recognised in polynomial time.

Towards this, we spend the rest of the section proving Theorem 1.7. We prove it by providing a polynomial time algorithm that does the following.

Input: A connected \(R\)-thin graph \(G\) and designated vertices \(1\) and \(0\).
Output: ‘Yes’ or ‘No’ depending on whether or not \(G\) has a tight Type 2 factorization with maximum vertex \(1\) and minimum vertex \(0\). (In the ‘Yes’ case, the factorization is found.)

It is a simple exercise to extend Lemma 3.3 to show that a graph is a Type 2 DL graph if and only if every component is; and as there are at most \(n^2\) choices of the vertices \(1\) and \(0\), this will be enough. The algorithm has three main steps, which we outline now.

8.1. Outline of the Decision Algorithm. We refer to the situation in which for the given graph \(G\) there is a distributive lattice \(L\) with \(0_L = 0\) and \(1_L = 1\) such that \((G, L)\) is a tight Type 2 pair, as the YES case.

(1) Apply Algorithm 1 to \(G\), returning a graph \(G'\). In Lemma 8.4 we show that in the YES case, \(H(L) \leq G' \leq L^*\), where \(L^*\) is the graph we get by viewing \(L\) as a digraph and throwing away orientation.

(2) Apply Algorithm 2 to \(G'\) returning the lattice \(L\), or the response ‘No’, signifying we are not in the YES case. Lemma 8.5 shows that in the YES case we return the lattice \(L\).

(3) Apply Algorithm 3 to decide if \((G, L)\) is a Type 2 pair, and respond ‘Yes’ or ‘No’ accordingly.

For the proof of Theorem 1.7, we must verify that the three algorithms can each be completed in polynomial time, and must prove the lemmas mentioned in the outline.
8.2. Step 1. We begin with a definition which can be found in [13].

**Definition 8.2.** An edge $x \sim y$ of $G$ **dispensable** if it satisfies the following conditions.

(i) $\exists z$ such that $N(x) \subseteq N(z) \subset N(y)$, or
(ii) $\exists z$ such that $N(y) \subseteq N(z) \subset N(x)$, or
(iii) $\exists z$ such that $N(x) \cap N(y) \subseteq N(x) \cap N(z)$ and $N(x) \cap N(y) \subset N(y) \cap N(z)$.

Observe that when $G$ is $R$-thin, we can replace the $\subseteq$ in the first two conditions with $\subset$; they are equivalent.

**Algorithm 1.** Given a graph $G$, let $G'$ be the graph one gets by removing all dispensable edges.

Algorithm [1] can clearly be implemented in polynomial time. In fact, in [13] it is shown that it can be done in time $O(n^4)$. We must now prove Lemma 8.4.

Recall that any lattice graph $G$ is a subgraph of a product $\mathcal{G} = \prod_{i=1}^d G_i$ where each $G_i$ on the vertex set $[n_i]_{10}$ satisfies the min-max identity (1). As the distinction is important in the upcoming arguments, we emphasize the fact that we use $<$ for strict inequality, and $\leq$ for non-strict inequality. As a slight but natural variation on the notation from Definition 2.2, we let $e_i$ denote the element $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathcal{G}$ whose $i^{th}$ coordinate is 1. Thus for an element $v = (v_1, \ldots, v_d)$ of $\mathcal{G}$, the element $v'$ that differs from $v$ only that it is one greater in the $i^{th}$ coordinate, can be represented $v' = v + e_i$.

For a vertex $v_i$ of $G_i$ we let $v_i^+ = \max\{N_{G_i}(v_i)\} \subset G_i$ be the greatest neighbour of $v_i$ in $G_i$ and $v_i^- = \min\{N_{G_i}(v_i)\}$ be its least neighbour. As $G_i$ is proper interval $v_i \leq u_i$ implies that $v_i^+ \leq u_i^+$ and $v_i^- \leq u_i^-$. As $G_i$ is $R$-thin, strict inequality $v_i < u_i$ implies strict inequality in at least one of $v_i^+ \leq u_i^+$ and $v_i^- \leq u_i^-$.  

**Lemma 8.3.** If $G$ is $R$-thin, and $G \subseteq \prod_{i=1}^d G_i$ is a tight Type 2 factorization, then each $G_i$ is $R$-thin.

**Proof.** Towards contradiction, assume that some $G_i$ contains vertices $a$ and $b$ with the same neighbourhoods. As $G_i$ is a proper interval graph, we may assume that $b = a + 1$. Then for any $x \in \mathcal{G}$ with $x_i = a$, the vertices $x$ and $x + e_i$ have the same neighbourhoods in $\mathcal{G}$, and so if both are in the induced subgraph $G$, they have the same neighbourhood in $G$. As $G$ is $R$-thin, this is impossible so for each $x$ with $x_i = a$ only one of $x$ and $x + e_i$ are in $G$. But then $G$ is not tight in $\mathcal{G}$, a contradiction. \qed

**Lemma 8.4.** Let $G$ be an connected $R$-thin graph such that $(G, L)$ has a tight Type 2 factorization, and let $G'$ be the graph we get by removing all dispensable edges. Then

(a) $G'$ contains the Hasse graph $H(L)$ of $L$, and
(b) every edge of $G'$ is between comparable vertices of $L$.

**Proof.** To prove part (a), let $x \leq y$ be a cover of $L$; we show that it is not dispensable. By the $R$-thinness of $G$, we may assume without loss of generality that there is some $v \in N(y) - N(x)$. So immediately, condition (ii) of Definition 8.2 does not hold. We may assume, by permuting factors of the factorization, and possibly reversing the first factor, that $y = x + e_i$.

Assume that (i) holds, that is, that there is some $z$ with $N(x) \subseteq N(z) \subseteq N(y)$. We first observe that $z_1 \geq y_1$. Indeed as $z$ has some neighbour common with
\[ y = x + e_1, \text{ but not with } x, \text{ we must have } z_1 > x_1; \text{ giving } z_1 \geq y_1. \] Now again by 
\[ N(x) \not\subseteq N(z) \not\subseteq N(y), \text{ there is some } w \in N(y) - N(z). \] If \( w_1 < z_1^- \), then because 
\[ y_1^+ \geq x_1^+ \], \( w \) is also a neighbour of \( x \), contradicting \( N(x) \subseteq N(z) \). So because 
\( w \not\in N(z) \), we may assume that \( w_2 < z_2^- \) or \( w_2 > z_2^+ \). Using that \( w \in N(y) \) this
yields the following two possibilities. Either

(i) \( z_2^+ > y_2 \geq y_2 \), and so \( z_2 > y_2 > w_2 \), or

(ii) \( z_2^- < w_2 = z_2^+ + 1 \leq y_2^+ \), and so \( z_2 < y_2 < w_2 \).

Now \( w \in N(y) - N(x) \) so \( w_1 > x_1^+ \), further, the vertex \( w' \) we get from \( w \) by
replacing \( w_1 \) with \( x_1^+ \) is in \( N(x) - N(z) \) in \( \mathcal{G} \), so is not in \( G \) (because \( N(z) \subseteq N(z) \)).

Thus \( w' \) must be in some vertex bite \( V_i(\alpha, \beta)_j \) in \( \mathcal{G} - G \).

This bite cannot contain \( w \), so \( j = 1 \) and \( x_1^+ \leq \beta < w_1 \). Also, it cannot contain \( y \).

In the case that \( y_2 > w_2 = w'_2 \) this means that \( j = 2 \), which is impossible as \( j = 1 \); in the case that \( y_2 < w_2 = w'_2 \) this means that \( i = 2 \) and \( y_2 > \alpha \geq w_2 = z_2^+ + 1 \).

Thus \( V_2(z_2^+ + 1, x_1^+) \) is in \( \mathcal{G} - G \).

Now we claim that the vertex \( x' \) which we get from \( x \) by replacing \( x_2 \) with \( z_2 \) has the same neighbourhood as \( x \) in \( G \), a contradiction. Indeed \( (x') \) it contains
\( N(x) \cap N(z) \), so contains \( N(x) \), and all vertices in \( N_2(x) - N_2(x') \) have been removed from \( \mathcal{G} \) in \( V_2(z_2^+, 1, x_1^+) \). Further, \( x' \) is indeed in \( G \) as a bite removing it would have to leave \( x \) and \( z \). But these are greater than \( x' \) in different coordinates- no vertex bite can remove a vertex but leave vertices that dominate it in two different coordinates. Thus we have our contradiction, so (i) cannot hold.

Finally, assume that (iii) holds. Clearly this implies that both \( N(x) - N(y) \) and \( N(y) - N(x) \) are non-empty, so \( x_1^+ < y_1^- \) and \( x_1^- < y_1^- \). Moreover, \( z \) has a neighbour \( a \in N(Y) - N(X) \), so having \( a_1 > x_1^+ \), and similarly another neighbour \( b \) having \( b_1 < y_1^- \). But then there is no viable value for \( z_1 \).

This completes the proof of (a). Now we prove (b) by showing that any edge \( xy \) between incomparable vertices \( x \) and \( y \) is dispensable. Indeed, as \( x \) and \( y \) are incomparable, we have that \( x \land y \) and \( x \lor y \) are distinct and different from \( x \) and \( y \). Further as \( \land \) and \( \lor \) are polymorphisms, any common neighbour of \( x \) and \( y \) is a neighbour of both of \( x \land y \) and \( x \lor y \), so \( N(x) \cap N(y) \subseteq N(x \land y), N(x \lor y) \). By \( B \)-thinness, \( N(x \land y) \) and \( N(x \lor y) \) are distinct, so one of them properly contains \( N(x) \cap N(y) \). Thus \( xy \) is dispensable.

\[ \square \]

8.3. Step 2. We define an orientation of the edges of \( G' \).

Algorithm 2. Given a graph \( G \) and subgraph \( G' \) with designated vertices \( 0 \) and \( 1 \),
let the sets \( N_j \) and the graphs \( D_j \) for all \( j = 0, \ldots, \text{dist}(1, 0) \) be defined as follows.
\[ N_j = \{ v \in G \mid \text{dist}(1, v) = j \}, \] and \( D_j \) be the subgraph of \( G' \) induced by \( \bigcup_{\alpha=0}^{\text{dist}(1,0)} N_j \).

Let \( G^2 \) be the partial orientation of \( G' \) we get as follows. For \( j = 1, \ldots, \text{dist}(1, 0) \)
do the following. For an edge \( uv \) of \( G' \), let \( u \rightarrow v \) if

(i) \( u \in N_{j-1} \) and \( v \in N_j \), or
(ii) \( u, v \in N_j \) and any one of the following holds
   (a) \( N(u) - N(v) \) has a vertex \( u' \in D_{j-1} \) such that \( u' > v' \) for all \( v' \in N(v) \cap D_{j-1} \). (We consider \( u' > v' \) if there is a directed path in \( D_{j-1} \) from \( u' \) to \( v' \).)
   (b) \( N(v) - N(u) \) has a vertex \( v' \in D_{j-1} \) such that \( u' > v' \) for all \( u' \in N(u) \cap D_{j-1} \).
(c) $N(v) - N(u)$ has a vertex in $N_j \cup N_{j+1}$ but not in $N_{j-1}$.

If the transitive closure of $\bar{G}'$ is a lattice, return it, otherwise, return 'NO'.

This algorithm is clearly polynomial in $n$.

Lemma 8.5. Let $G$ be a connected $R$-thin graph such that $(G, L)$ has a tight Type 2 factorization, and let $G'$ be the graph we get from $G$ by removing all dispensable edges. Then Algorithm 2 applied to $G'$, $1_L$, and $0_L$, returns $L$.

Proof. By Lemma 8.4, it is enough to show that for any (non-loop) edge $uv$ of $G'$ with $u > v$, the above algorithm properly orients $uv$; i.e., sets $u \to v$ and at the same time does not set $v \to u$.

Observe that by construction every edge of $G'$ is either in $D_j$ for some $j$ or is between $D_{j-1}$ and $D_j$ for some $j$. We will prove by induction on $j$ that the $j$th step of the algorithm properly orients such edges, yielding a proper orientation of all the edges of $D_j$. Before we do this though, we first prove that it will never improperly orient an edge.

Claim 8.6. Let $u > v$ then the algorithm will not set $v \to u$.

Proof. We must check that none of the conditions of the algorithm are satisfied when the roles of $u$ and $v$ are reversed.

To see that item (i) is not satisfied observe that if both $u$ and $v$ are in $N_j$, then clearly it is $u$ that is closer to 1. Indeed, if $v = x_1 \sim x_2 \cdots \sim x_k = 1$ is a path in $G$, then so is $u = u \lor x_k \sim u \lor x_{k-1} \cdots \sim u \lor x_1 = 1$. So $u \in N_{j-1}$ and $v \in N_j$. (In fact this shows that the algorithm properly sets $u \to v$ in the case that $u$ and $v$ are not both in $N_j$.

To see that items (ii) and (iib) are not satisfied, it is enough to observe that if $w' \sim u$ and $v' \sim v$ and $v' \geq w'$ then $u' \sim v$ and $u \sim v'$. But this is clear, as the premises imply that

$u' = u' \land v' \sim u \land v = v$

and

$v' = u' \lor v' \sim u \lor v = u$.

To see that item (iiic) is not satisfied, assume that there is some $w \in N(u) - N(v)$. As $N(u)$ is conservative, (recall the definition of conservative sets preceding Lemma 3.3) it induces a sublattice of $L$, so has a maximum element $u'$. This element must also be in $N(u) - N(v)$; as if we had $u' \sim v$, then

$w = w \lor u' \sim u \lor v = v$,

contradicting the fact that $w \notin N(v)$. We now show that $u'$ is in $N_{j-1}$, so item (iiic) is not satisfied. Indeed, some neighbour $x$ of $u$ must be in $N_{j-1}$, as $u \in N_j$. Let $x = x_1 \sim x_2 \cdots \sim x_i = 1$ be a length $i - 1$ walk from $x$ to $1$. Then taking the join of each element in the walk with $u'$ we get a walk $u' = u' \lor x_{i-1} \sim u' \lor x_{i-2} \cdots \sim u' \lor 1 = 1$ from $u'$ to $1$. This shows that $u'$ is in $N_i$ for some $i \leq j - 1$, but being a neighbour of $u$, it must be in $N_{j-1}$.

We now have just to verify that for $u > v$ the algorithm sets $u \to v$.

For the case $j = 1$ let $uv$ be an edge of $G'$ in $D_1$ with $u > v$. Item (i) holds if and only if $u = 1$, and in this case gives $u \to v$, as needed. Assume therefore that $u, v \in N_1$. As all vertices in $N_1$ are adjacent to 1, items (ii) and (iib) are vacuous, so we must show that (iiic) holds. To see this, observe that as $u \geq v$, we
have that \( u_i \geq v_i \) for all \( i \in [d] \). As \( u_i^+ = 1 = v_i^+ \) for all \( i \), we have by \( R \)-thinness that \( N_G(1) \subset N_G(u) \subset N_G(v) \).

The vertex in \( N_G(v) - N_G(u) \) is thus in \( N_2 \) as needed.

Now assume that all edges of \( D_{j-1} \) are properly oriented. We show that the \( j \)th round of the algorithm properly orients the heretofore unoriented edges of \( D_j \).

Let \( uv \) be an edge of \( D_j - D_{j-1} \) with \( u \geq v \). If not both of \( u \) and \( v \) are in \( N_j \), then (as we showed in the claim) \( u \to v \) is properly ordered by step (i) of the algorithm.

So we may assume that both of \( u \) and \( v \) are in \( N_j \). As \( u > v \) we have that for all \( i \), \( u_i \geq v_i \). By \( R \)-thinness there is a vertex \( w \) in either \( N(u) - N(v) \) or in \( N(v) - N(u) \).

We show now that if the first case, (iia) is satisfied, and then that in the second case, (iib) or (iic) are satisfied.

**Claim 8.7.** If \( w \in N(u) - N(v) \), then (iia) is satisfied.

**Proof.** Let \( w \in N(u) - N(v) \). As we showed in the proof that item (iic) is not satisfied in the previous claim, we have that the maximum neighbour \( u' \) of \( u \) is in \( D_{j-1} \cap (N(u) - N(v)) \).

To see that (iia) is satisfied, we must show that \( u' \geq v' \) for any neighbour \( v' \) of \( v \) in \( D_{j-1} \). Indeed, \( v' \sim v \) and \( u' \sim u \) give \( v' \lor u' \sim v \lor u = u \). As \( u' \) is the maximal neighbour of \( u \) this gives us that \( u' \geq v' \lor u' \). This implies however that \( u' = v' \lor u' \), and so \( u' \geq v' \), as needed.

\( \diamond \)

**Claim 8.8.** If \( w \in N(v) - N(u) \), then (iib) or (iic) are satisfied.

**Proof.** We assume that (iic) does not hold, and then show that (iib) must.

Indeed, if (iic) does not hold, then \( w \in D_{j-1} \cap (N(v) - N(u)) \). As \( D_{j-1} \cap N(v) \) is a conservative set it induces a sublattice, so has a minimum element \( v' \). But then for any neighbour \( u' \) of \( u \) in \( D_{j-1} \) we have from \( v' \sim v \) and \( u' \sim u \), that \( v' \lor u' \sim v \lor u = v \). As \( D_{j-1} \) is conservative, \( v' \lor u' \) is in \( D_{j-1} \) so is in \( D_{j-1} \cap N(v) \). Thus \( v' \lor u' \geq v' \) which implies that \( u' \geq v' \). As \( v' \notin N(u) \) we have that \( u' > v' \), as needed.

This completes the proof of the lemma.

\( \square \)

**Corollary 8.9.** If \( G \) is \( R \)-thin, then for a given choice of minimum and maximum vertices \( 0 \) and \( 1 \), there is at most one lattice \( L \) compatible with \( G \) such that \( 0_L = 0 \) and \( 1_L = 1 \) and \( (G, L) \) has a tight Type 2 factorisation.

8.4. Step 3.

**Algorithm 3.** Decide if a given compatible pair \( (G, L) \) is of Type 2 as follows.

(i) Find \( J_L \). An element is in \( J_L \) if and only if it covers a unique element.

(ii) Find \( \bar{A}_+ \) and then \( \bar{A}_- \). How to do this is discussed preceding Proposition 7.8.

(iii) Check if \( J_L \) and \( \bar{A}_- \) satisfy condition (iii) of Proposition 7.8.

Return ‘YES’ or ‘NO’ depending on whether or not \( J_L \) and \( \bar{A}_- \) satisfy condition (iii) of Proposition 7.8.

This algorithm is clearly polynomial in \( n \). Indeed, the first step is \( O(n^2) \) the second is \( O(n^4) \) and the third is \( O(n^2) \).
8.5. **Comments on Recognition.** Compare Lemma 8.4 to similar statements made in [13 Chap 8], where they show that $G'$ is closely related to what they call the *Cartesian skeleton* of a product graph $G$. Our proof is complicated by the fact that $G$ is not a product, but a subgraph of a product.

We have made no effort to optimize our algorithm. In particular, one sees that Step 1 is the same for any choice of $1$ and $0$ so we needn’t really do it $n^2$ times. Furthermore, it is easy to see that one can restrict the choice of $1$ and $0$ to simplicial vertices whose distance is the diameter.

Corollary 8.9 does not hold if we drop the restriction to $R$-thin, or allow Type 3 factorization. This perhaps explains some of the difficulty in extending Theorem 1.7.

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