THE FAMILY INDEX THEOREM AND BIFURCATION OF SOLUTIONS OF NONLINEAR ELLIPTIC BVP

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Abstract. We obtain some new bifurcation criteria for solutions of general boundary value problems for nonlinear elliptic systems of partial differential equations. The results are of different nature from the ones that can be obtained via the traditional Lyapunov-Schmidt reduction. Our sufficient conditions for bifurcation are derived from the Atiyah-Singer family index theorem and therefore they depend only on the coefficients of derivatives of leading order of the linearized differential operators. They are computed explicitly from the coefficients without the need of solving the linearized equations. Moreover, they are stable under lower order perturbations.

1. Introduction

Let us begin with a rough description of the general setting and the background of the problem.

We consider a system of nonlinear partial differential equations of the form

\begin{equation}
\begin{aligned}
F_i(\lambda, x, u, \ldots, D^k u) &= 0 \text{ for } x \in \Omega, 1 \leq i \leq m \\
G_i(\lambda, x, u, \ldots, D^k u) &= 0 \text{ for } x \in \partial \Omega, 1 \leq i \leq r.
\end{aligned}
\end{equation}

Here \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( u : \bar{\Omega} \to \mathbb{R}^m \) is a vector function and \( \lambda \in \mathbb{R}^q \) is a parameter. \( F_i, G_i \) are smooth functions defined on jets of order \( k \) and \( k_i \) respectively such that \( F_i(\lambda, x, 0) = 0 \), \( G_i(\lambda, x, 0) = 0 \). We will not assume that \( k_i \leq k - 1 \) since we won’t need this here.

Because of the last assumption, the function \( u \equiv 0 \) is a solution of problem (1.1) for every \( \lambda \in \mathbb{R}^q \). The set \( \mathbb{R}^q \times \{ 0 \} \) is called a trivial branch of solutions of (1.1). Nontrivial solutions of (1.1) are solutions \((\lambda, u)\) with \( u \neq 0 \). Roughly speaking, a bifurcation point from the trivial branch for solutions of (1.1) is a point \( \lambda \in \mathbb{R}^q \) such that arbitrarily close to \((\lambda, 0)\) there are nontrivial solutions of the equation.

For each \( \lambda \in \mathbb{R}^q \), the functions \( F_i(\lambda, x, 0) \), \( G_i(\lambda, x, 0) \) define a nonlinear differential operator

\begin{equation}
(F_\lambda, G_\lambda) : C^\infty(\bar{\Omega}; \mathbb{R}^m) \to C^\infty(\bar{\Omega}; \mathbb{R}^m) \times C^\infty(\partial \Omega; \mathbb{R}^r).
\end{equation}

Let \((L_\lambda, B_\lambda)\) be the linearization of \((F_\lambda, G_\lambda)\) at \( u \equiv 0 \). It is easy to see that a necessary condition for a point \( \lambda \) to be a bifurcation point is the existence of a non vanishing solution \( v \) of the linearized problem

\begin{equation}
\begin{aligned}
L_\lambda(x, D)v(x) &= 0, \text{ for } x \in \Omega \\
B_\lambda(x, D)v(x) &= 0, \text{ for } x \in \partial \Omega.
\end{aligned}
\end{equation}

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However, the above condition is not sufficient and the goal of the linearized bifurcation theory is to obtain sufficient conditions for the appearance of nontrivial solutions from invariants associated to the linearization \((L_\lambda, B_\lambda)\).

If the operator \(L_\lambda\) is elliptic and the boundary operator \(B_\lambda\) verifies the Shapiro-Lopatinski\j condition with respect to \(L_\lambda\), then for all \(\lambda \in \mathbb{R}^q\), the differential operator \((L_\lambda, B_\lambda)\) induces a Fredholm operator between the Hardy-Sobolev spaces naturally associated to the problem. It follows from this that \(F^i, G^i\) define on a neighborhood of zero a family of nonlinear Fredholm maps. The Fredholm property is essential. It gives the possibility to recast, at least locally, the study of a bifurcation problem to an equivalent problem for a finite number of nonlinear equations in a finite number of indeterminates. This is the essence of the celebrated Lyapunov-Schmidt method. A typical further assumption in this setting is that points where \((1.3)\) holds are isolated. Assuming this, a number of sufficient conditions for bifurcation can be obtained using either analytical or topological methods \[14, 20, 26, 12, 13, 21\].

In the past years we worked out a different approach to bifurcation, based on various homotopy invariants of families of Fredholm operators defined by the linearization along the trivial branch. The invariants under consideration, the index bundle, the spectral flow and others are borrowed from elliptic topology. They arise in bifurcation theory as a tool linking the nontrivial topology of the parameter space with the appearance of nontrivial solutions of the equation.

In \[17\] we introduced an index of bifurcation which counts algebraically the bifurcation points of a family of nonlinear Fredholm maps parametrized by open subsets of a compact manifold or polyhedron \(\Lambda\). The index takes values in a finite abelian group \(J(\Lambda)\) associated to the parameter space. It has similar properties to the well known fixed-point index. Namely, it possesses the existence, additivity, excision and homotopy invariance property. At an isolated singular point of the linearization the index of bifurcation can be computed using the Lyapunov-Schmidt reduction. But, what is most important, the total index is derived from a well known elliptic invariant; the index bundle of the linearization along the trivial branch. The Atiyah-Singer family index theorem allows us to compute the index bundle directly from the principal part of the linearization, i.e., the coefficients of the leading derivatives of \(L_\lambda(x, D)\) and \(B_\lambda(x, D)\).

For families of elliptic boundary value problems parametrized by \(\mathbb{R}^q\), viewed as an open subspace of the sphere \(S^q\), the results are particularly simple. The groups \(J(S^q)\) are finite cyclic groups whose orders have been computed by Adams and others. In Theorem 1.4.1 of \[17\], under the assumption that the principal part of the boundary operator \(B_\lambda(x, D)\) is independent of \(\lambda\) and that the principal part of the interior operator \(L_\lambda(x, D)\) is independent of \(\lambda\) near the boundary, we computed the index of bifurcation obtaining in this way sufficient conditions for bifurcation in terms of a number defined as the integral of a differential form constructed explicitly from the principal symbol of the linearization \(L_\lambda(x, D)\).

The purpose of the present paper is to extend Theorem 1.4.1 of \[17\] to the case in which also the principal symbol of the boundary operator is parameter dependent.

In \[17\] we proved a parametrized version of the Agranovich reduction in order to recast the calculation of the index bundle of the family of linearizations along the trivial branch to that of a family of pseudo-differential operators on \(\mathbb{R}^n\). Then we applied the Atiyah-Singer family index theorem to the latter. In this article, in addition to our previous results, we will need yet another type of reduction. We will show that the index bundle of a family of elliptic boundary value problems whose interior operator is independent from the parameters coincides with that of a family of pseudo-differential operators on the boundary. In the comparison of
two boundary value problems with the same interior operator the latter is known under the name of Agranovich-Dynin reduction [1,22].

Our result will be proved by combining both reductions with the cohomological form of the family index theorem. Consequently, our criteria for the existence of bifurcation points will be formulated in terms of the Bott-Fedosov degree of two maps \( \sigma \) and \( \tau \), with values in \( GL(n;\mathbb{C}) \) and \( GL(r;\mathbb{C}) \) respectively, which are naturally attached to the reductions discussed above.

The map \( \sigma \) is constructed out of the principal symbol of the family of interior operators \( L \) while \( \tau \) is constructed by restricting the principal symbol of the boundary operators to the vector bundle \( M_+ \) of all stable solutions of a family of ordinary differential equations canonically associated to \( L \).

Extending the operators \( L_\lambda(x,D) \) to the double of \( \Omega \) would give us sufficient conditions for bifurcation in a slightly different but equivalent form. We consider the chosen approach more straightforward. Notice that we still have to assume independence from the parameter \( \lambda \) of the coefficients of the operators \( L_\lambda(x,D) \) near to the boundary of \( \Omega \). Taking into account the variation on the boundary would lead us to pseudo-differential operators with operator valued symbols.

The paper is organized as follows. In Section 2 we state our main result. Section 3 is a short review of the background material from [17]. Section 4 contains the proofs and in Section 5 we construct some examples of semi-linear elliptic boundary value problems illustrating our bifurcation result. An appendix is devoted to the comparison of the Fedosov’s approach to Bott-Fedosov degree in [9] with the one used by Atiyah and Singer in [6].

2. Statement of the main theorem

Let \( (\mathcal{F},\mathcal{G}) : \mathbb{R}^q \times C^\infty(\Omega;\mathbb{R}^m) \longrightarrow C^\infty(\bar{\Omega};\mathbb{R}^m) \times C^\infty(\partial\Omega;\mathbb{R}^r) \) be a parametrized family of nonlinear differential operators defined by

\[
\begin{align*}
\mathcal{F}(\lambda, u) &= \{ F^1(\lambda, x, u, \ldots, D^k u), \ldots, F^m(\lambda, x, u, \ldots, D^k u) \} \\
\mathcal{G}(\lambda, u) &= \{ G^1(\lambda, x, u, \ldots, D^k u), \ldots, G^r(\lambda, x, u, \ldots, D^k u) \}.
\end{align*}
\]

Keeping our notations from [17] we will denote with \( (\mathcal{F}_\lambda, \mathcal{G}_\lambda) \) the operator corresponding to the parameter value \( \lambda \in \mathbb{R}^q \). In general families of differential operators will be denoted using calligraphic letters while the families of induced operators on Hardy-Sobolev spaces will be denoted with the corresponding roman capitals. For example, in our notations \( L_\lambda = L_\lambda(x,D) \); the induced operator being \( L_\lambda \).

Denoting by \( v_{\alpha} \) the variable corresponding to \( D^\alpha u_j \), for each fixed \( \lambda \), the linearization of \( (\mathcal{F}_\lambda, \mathcal{G}_\lambda) \) at \( u \equiv 0 \) is the linear differential operator \( (L_\lambda, B_\lambda) \) defined by

\[
\begin{align*}
L_\lambda(x,D) &= \sum_{|\alpha| \leq k} a_\alpha(\lambda,x) D^\alpha, \\
B_\lambda(x,D) &= \sum_{70} b_\alpha(\lambda,x) D^\alpha, \quad 1 \leq i \leq r,
\end{align*}
\]

where the \( ij \)-entries of \( a_\alpha \in C^\infty(\Delta \times \bar{\Omega};\mathbb{R}^{m \times m}) \) and \( b_\alpha \in C^\infty(\Lambda \times \bar{\Omega};\mathbb{R}^{1 \times m}) \) are

\[
a_{ij}(\lambda, x) = \frac{\partial F^i}{\partial v_{\alpha}}(\lambda, x, 0), \quad b_{ij}(\lambda, x) = \frac{\partial G^j}{\partial v_{\alpha}}(\lambda, x, 0),
\]

and \( \gamma_{ij} \) is the operator ”restriction to the boundary”.

We assume:

H1) For all \( \lambda \in \mathbb{R}^q \), the linearization \( (L_\lambda, B_\lambda) \) of \( (\mathcal{F}_\lambda, \mathcal{G}_\lambda) \) at \( u \equiv 0 \), defines an elliptic boundary value problem. Namely, \( L_\lambda \) is elliptic, properly elliptic at the boundary and \( B_\lambda \) verifies the Shapiro-Lopatinskij condition with respect to \( L_\lambda \) (see [17] Definition 5.2.1).
\( H_2 \) The coefficients \( a_{ij}^\alpha, b_{ij}^\alpha \) of the linearized family \((\mathcal{L}, \mathcal{B})\) extend to smooth functions defined on \( S^q \times \bar{\Omega} \), where \( S^q = \mathbb{R}^q \cup \{\infty\} \) is the one point compactification of \( \mathbb{R}^q \). Moreover the problem
\[
\begin{align*}
\mathcal{L}_\infty(x,D)u(x) &= \sum_{|\alpha| \leq k} a_\alpha(\infty, x)D^\alpha u(x) = f(x), \quad x \in \Omega \\
\mathcal{B}_\infty(x,D)u(x) &= \gamma_0 \sum_{|\alpha| \leq k_i} b_\alpha(\infty, x)D^\alpha u(x) = g(x), \quad x \in \partial\Omega, \quad 1 \leq i \leq r,
\end{align*}
\]
is elliptic and has a unique solution for every \( f \in C^\infty(\bar{\Omega} ; \mathbb{R}^m) \) and every \( g \in C^\infty(\partial\Omega; \mathbb{R}^r) \).

\( H_3 \) The restrictions of the coefficients of the leading terms of \( \mathcal{L}_\lambda(x,D) \) to a neighborhood of \( \partial\Omega \) are independent of \( \lambda \). Moreover the principal symbol of the operator \( \mathcal{L}_\infty(x,D) \) commutes with the principal symbol of \( \mathcal{L}_\lambda(x,D) \) for all \( \lambda \in S^q \).

**Remark 2.0.1.** The hypothesis \( H_2, H_3 \) are restrictive. We will discuss elsewhere bifurcation of elliptic boundary value problems under different assumptions, which do not require the extension of the linearized family to \( S^q \). Obviously, the principal symbols of \( \mathcal{L}_\infty \) and \( \mathcal{L}_\lambda \) commute if either the principal symbol of \( \mathcal{L} \) is constant or the principal symbol of \( \mathcal{L}_\infty \) is diagonal. This later condition is not needed if the principal symbol of \( \mathcal{B} \) is independent from the parameter \( \lambda \).

**Definition 2.0.1.** A bifurcation point from the trivial branch for solutions of \((1.1)\) is a point \( \lambda_* \) such that there exists a sequence \((\lambda_n, u_n) \in \mathbb{R}^q \times C^\infty(\bar{\Omega} ; \mathbb{R}^m) \) of nontrivial solutions of \((1.1)\) with \( \lambda_n \to \lambda_* \) and \( u_n \to 0 \) uniformly together with all of their derivatives.

A sufficient condition for the existence of bifurcation points of \((1.1)\) is that index of bifurcation of the family of Fredholm maps between Hardy-Sobolev spaces induced by \((\mathcal{F}, \mathcal{G})\) does not vanish (see [17]). In Section 4 we will compute the index from two integers associated to the linearization \((\mathcal{L}, \mathcal{B})\) at the trivial branch, called the interior and the boundary multiplicity, together with some natural numbers \( n(q) \) related to the order of the group \( J(S^q) \). As we said in the introduction, the interior and boundary multiplicity will be defined as the Bott-Fedosov degree of two maps \( \sigma \) and \( \tau \) with values in \( GL(m; \mathbb{C}) \) and \( GL(r; \mathbb{C}) \) respectively.

The construction of \( \sigma \) is as follows:

Let \( p(\lambda, x, \xi) \equiv \sum_{|\alpha| = k} a_\alpha(\lambda, x)\xi^\alpha \) be the principal symbol of \( \mathcal{L}_\lambda \). Since the principal symbol is obtained substituting the operator \( D_j = -i\frac{\partial}{\partial x_j} \) with the variable \( \xi_j \), \( p(\lambda, x, \xi) \) is a complex matrix verifying the reality condition
\[
p(\lambda, x, -\xi) = \bar{p}(\lambda, x, \xi).
\]

By ellipticity, \( p(\lambda, x, \xi) \in GL(m; \mathbb{C}) \) if \( \xi \neq 0 \). On the other hand, by \( H_3 \), \( p(\lambda, x, \xi) = p(\infty, x, \xi) \) for all \( x \) in a neighborhood of \( \partial\Omega \). Hence putting
\[
\sigma(\lambda, x, \xi) = \begin{cases} 
p(\lambda, x, \xi)p(\infty, x, \xi)^{-1} & \text{if } x \in \Omega, \xi \neq 0 \\
Id & \text{if } x \notin \Omega,
\end{cases}
\]
we get a smooth map
\[
\sigma : S^q \times (\mathbb{R}^{2n} - \Omega \times \{0\}) \longrightarrow GL(m; \mathbb{C}).
\]

Now let us define the matrix function \( \tau \).
We take a neighborhood $\mathcal{N}$ of $\partial\Omega$ of the form $\mathcal{N} \simeq \partial\Omega \times (-1,1)$. At a point $x$ belonging to $\Omega \cap \mathcal{N}$ we will use a coordinate system of the form $(x',t)$, where $x' = (x'_1, \ldots, x'_{n-1})$ is a coordinate system on the manifold $\Gamma = \partial\Omega$ and $-1 < t \leq 0$ is the coordinate in the direction of the inner normal. In particular, points of $\Gamma$ will have coordinates of the form $(x',0)$ which we identify with $x'$. At every point $x' \in \Gamma$ we split the cotangent space $T^*_{x'}(\mathbb{R}^n) \simeq \mathbb{R}^n$ into a direct sum

$$T^*_{x'}(\mathbb{R}^n) = T^*_{x'}(\Gamma) \oplus \mathbb{R}_\eta,$$

where $\eta$ is the conormal at $x'$, and use the coordinates on $T^*_{x'}(\mathbb{R}^n)$ of the form $(\xi'_1, \ldots, \xi'_{n-1}, \nu)$, where $\xi' = (\xi'_1, \ldots, \xi'_{n-1})$ are the coordinates of a vector in $T^*_{x'}(\Gamma)$ and $\nu$ is the coordinate along the conormal.

Since $\mathcal{L}_x(x, D)$ is properly elliptic, $km = 2r$ and, for any $\lambda \in S^q$, $x' \in \Gamma$ and any cotangent vector to the boundary $\xi' \neq 0$ at $x'$, the polynomial $P(\nu) = \det p(\lambda, x', \xi', \nu)$ has exactly $r$ roots in the upper half-plane $3z > 0$ \[17\].

In terms of the ordinary differential operator $p(\lambda, x', \xi', D_1)$, obtained by substituting $\nu$ with $D_1 = i^{-1} \frac{\partial}{\partial x}$ in the principal symbol, the above condition means that the subspaces of stable (resp. unstable) solutions $M^{\pm}(\lambda, x', \xi') \subset L^2(\mathbb{R}^n; \mathbb{C}^m)$, whose elements are solutions of the system $p(\lambda, x', \xi', D_1)v(t) = 0$ exponentially decaying to 0 as $t \to +\infty$ (resp. $t \to -\infty$) are both of dimension $r$. In particular $M^{\pm}(\lambda, x', \xi')$ are the fibers of two vector bundles over $S^q \times [T^*(\Gamma) - \{0\}]$, which will be denoted with $M^{\pm}$.

Let us denote with $\gamma_j u$ the restriction to $\Gamma$ of the function $D^{(j)}_i u(x', t)$.

Using the coordinates on $\mathcal{N}$, we rewrite the boundary operators $B^i_\chi(x, D)$ in the form

$$B^i_\chi(x, D) u = \sum_{j=0}^{k_i} B^i_j(\lambda, x', D) \gamma_j u,$$

where, $B^i_j(\lambda, x', D)$ is a differential operator of order $k_i - j$ acting on vector functions defined on the manifold $\Gamma$. In the new coordinates the principal symbol of the boundary operator $B_\chi(x, D)$ is the matrix function $p_{b}(\lambda, x, \xi)$ whose $i$-th row is

$$p^i_{b}(\lambda, x, \xi) = \sum_{j=0}^{k_i} p^{ij}_{b}(\lambda, x', \xi') \nu^j,$$

where $p^{ij}_{b}(\lambda, x', \xi')$ is the principal symbol of the operator $B^i_j(\lambda, x', D)$.

By $H_1$ and $H_2$, for any $\lambda \in S^q$, the boundary operator $B_\chi(x, D)$ verifies the Shapiro-Lopatinski\'j condition with respect to $\mathcal{L}_x(x, D)$. This means that, for each $x' \in \partial\Omega$ and each $\xi'$ belonging to $T^*_{x'}\Omega$, the subspace $M^{\pm}(\lambda, x', \xi')$ is isomorphic to $\mathbb{C}^r$ via the map $b(\lambda, x', \xi')$ defined by

$$b(\lambda, x', \xi')v = [p_{b}(\lambda, x', \xi', D_1)v](0).$$

In particular both vector bundles $M^{\pm}$ are trivial.

Identifying an endomorphism of $\mathbb{C}^r$ with its matrix in the canonical basis we define our second map $\tau: S^q \times [T^*(\Gamma) - \{0\}] \to GL(r; \mathbb{C})$ by

$$\tau(\lambda, x', \xi') = b(\lambda, x', \xi') b^{-1}(\infty, x', \xi').$$

We will define degree of the matrix functions $\sigma$ and $\tau$ using Fedosov\’s approach in \[9\]. For this we will need matrix-valued differential forms, or equivalently, matrices having differential forms as coefficients. The product of two matrices of this type is defined in the usual way, with the product of coefficients given by the wedge product of forms. The matrix of differentials $(d\sigma_{ij})$ will be denoted by $d\sigma$.

Let us consider a compact oriented manifold $V$ of odd dimension $2v - 1$ and a smooth map $\phi: V \to GL(l; \mathbb{C})$. Taking the trace of the $(2v - 1)$-th power of
the matrix $\phi^{-1}d\phi$ we obtain an ordinary $(2v-1)$-form $tr(\phi^{-1}d\phi)^{2v-1}$ on $V$. The Bott-Fedosov degree of $\phi$ is defined by

$$
(2.10) \quad \deg(\phi) = N \int_V tr(\phi^{-1}d\phi)^{2v-1},
$$

where $N = \frac{(v-1)!}{(2\pi i)^v(2v-1)!}$.

We define now the interior and boundary multiplicity of the linearized boundary value problem (2.2).

Let $q$ be even, we associate to the $GL(m;\mathbb{C})$-valued function $\sigma$ constructed in (2.5) the one form $\sigma^{-1}d\sigma$ defined on $S^q \times (\mathbb{R}^{2n} - K \times \{0\})$. Without loss of generality we can assume that $\Omega \times \{0\}$ is contained in the unit ball $B^{2n} \subset \mathbb{R}^{2n}$ and hence we can consider the restriction of $\sigma^{-1}d\sigma$ to $S^q \times S^{2n-1}$ as a one form on the compact manifold $S^q \times S^{2n-1}$. To be precise, the latter is the pullback of the former by the inclusion $i : S^q \times S^{2n-1} \to S^q \times (\mathbb{R}^{2n} - K \times \{0\})$. Being homogenous, $\sigma$ is uniquely determined by its restriction to $S^q \times S^{2n-1}$. Thus we will not distinguish in the notation the map $\sigma$ from its restriction to this space. On the other hand, the chain rule allows us to denote with $\sigma^{-1}d\sigma$ the pullback form too.

By definition, the interior multiplicity of the family $(\mathcal{L}, \mathcal{B})$ is

$$
(2.11) \quad \mu_i(\mathcal{L}, \mathcal{B}) = \deg(\sigma) = \frac{-(\frac{q}{2}q + n - 1)!}{(2\pi i)^{\frac{q}{2}q+n} (q+2n-1)!} \int_{S^q \times S^{2n-1}} tr(\sigma^{-1}d\sigma)^{q+2n-1},
$$

The boundary multiplicity $\mu_b(\mathcal{L}, \mathcal{B})$ defined in a similar way. Namely:

$$
(2.12) \quad \mu_b(\mathcal{L}, \mathcal{B}) = \deg(\tau) = \frac{-(\frac{q}{2}q + n - 2)!}{(2\pi i)^{\frac{q}{2}q+n-1} (q+2n-3)!} \int_{S^q \times S(\Gamma)} tr(\tau^{-1}d\tau)^{q+2n-3},
$$

where $S(\Gamma)$ is the unit sphere bundle of the cotangent bundle $T^*(\Gamma)$.

It follows from Fedosov’s formula for the Chern character of a family of elliptic pseudo-differential operators on $\mathbb{R}^n$ [5], Corollary 6.5] and Bott’s Integrality Theorem that $\mu_i(\mathcal{L}, \mathcal{B})$ is an integer. We will show in Section 4 that $\mu_b(\mathcal{L}, \mathcal{B}) \in \mathbb{Z}$ as well.

Finally, the multiplicity of $(\mathcal{L}, \mathcal{B})$ is defined as

$$
(2.13) \quad \mu(\mathcal{L}, \mathcal{B}) = \mu_i(\mathcal{L}, \mathcal{B}) + \mu_b(\mathcal{L}, \mathcal{B}).
$$

By construction, the integral number $\mu(\mathcal{L}, \mathcal{B})$ depends only on the principal symbols of the interior and boundary operators and is invariant under homotopies of families of elliptic boundary value problems.

Remark 2.0.2. While the reality condition (2.3) on the principal symbols of the interior and boundary operators may eventually place some restrictions on the possible values of the degree, the above condition was nowhere used in the definition of $\deg(\sigma)$ and $\deg(\tau)$. Hence exactly the same formulas allows to define the multiplicity $\mu(\mathcal{L}, \mathcal{B})$ of any family of linear elliptic boundary value problems with complex coefficients.

Now, let us introduce the natural numbers $n(q)$. Denoting with $\nu_p(n)$ the power of the prime $p$ in the prime decomposition of $n \in \mathbb{N}$, let $m$ be the number-theoretic function constructed as follows: the value $m(s)$ is defined through its prime decomposition, by setting for $p = 2$, $\nu_2(m(s)) = 2 + \nu_2(s)$ if $s \equiv 0 \mod 2$ and $\nu_2(m(s)) = 1$ if the opposite is true. While, if $p$ is an odd prime, then $\nu_p(m(s)) = 1 + \nu_p(s)$ if $s \equiv 0 \mod (p - 1)$ and 0 in the remaining cases. In
particular \( m(s) \) is always even. The function \( m \) was introduced by Adams [2]. It is well known that for \( q = 4s \) the group \( J(S^q) \) is a cyclic group of order \( m(2s) \).

For \( q \equiv 0, 4 \mod 8 \), let

\[
(2.14) \quad n(q) = \begin{cases} 
m(q/2) & \text{if } q \equiv 0 \mod 8 \\
2m(q/2) & \text{if } q \equiv 4 \mod 8 
\end{cases}
\]

and \( m \) is the number theoretic function defined above.

With all of the above said we can state our criteria for bifurcation of solutions of (1.1).

**Theorem 2.0.1.** Let the problem

\[
(2.15) \begin{align*}
F(\lambda, x, u, \ldots, D^k u) &= 0, \quad x \in \Omega \\
G(\lambda, x, u, \ldots, D^k u) &= 0, \quad x \in \partial\Omega,
\end{align*}
\]

verify assumptions \( H_1, H_2 \) and \( H_3 \). If \( q \equiv 0, 4 \mod 8 \), then bifurcation of smooth solutions of (2.15) from some point of the trivial branch arises provided that \( \mu(\lambda, B) \) is not divisible by \( n(q) \).

**Remark 2.0.3.** If the principal part of the boundary operator is independent of \( \lambda \), then \( \mu_b(\lambda, B) = 0 \) and we obtain Theorem 1.4.1 of [17]. If instead the principal part of the operator \( L_\lambda(x, D) \) is independent of \( \lambda \), then bifurcation of solutions is determined by \( \mu_b(\lambda, B) \), i.e., by Bott-Fedosov degree of \( \tau \).

If \((F, G)\) verifies assumptions \( H_1, H_2 \) and \( H_3 \) of Theorem 2.0.1, then for any lower order perturbation

\[
(\mathcal{F}', \mathcal{G}', (\lambda, x, u, \ldots, D^{k-1} u)), \quad 1 \leq i \leq r,
\]

such that \( F(\lambda, x, 0) = 0, G^i(\lambda, x, 0) = 0 \) and such that the coefficients of the linearization of \((\mathcal{F}', \mathcal{G}')\) converge uniformly to 0 as \( \lambda \to \infty \), also the perturbed problem

\[
(2.16) \begin{align*}
\mathcal{F}(\lambda, x, u, \ldots, D^k u) + \mathcal{F}'(\lambda, x, u, \ldots, D^{k-1} u) &= 0, \quad x \in \Omega \\
\mathcal{G}(\lambda, x, u, \ldots, D^k u) + \mathcal{G}'(\lambda, x, u, \ldots, D^{k-1} u) &= 0, \quad x \in \partial\Omega,
\end{align*}
\]

verifies the same assumptions. Taking into account that lower order perturbations do not affect the value of \( \mu(\mathcal{L}, B) \) we have:

**Corollary 2.0.2.** If \((F, G)\) verifies all of the assumptions in Theorem 2.0.1 and if the perturbation \((\mathcal{F}', \mathcal{G}')\) is as above, then there must be some bifurcation point \( \lambda \in \mathbb{R}^q \) for solutions of the perturbed problem (2.16) as well.

The above corollary shows that the bifurcation criterium based on the invariant \( \mu(\mathcal{L}, B) \) is robust. Bifurcation invariants of local type [12] lack of this property. On the other hand Theorem 2.0.1 does not give any information about where the bifurcation points are located.

In many instances \( \mu(L, B) \) vanishes, e.g., when the top order terms of the operator \((\mathcal{L}_\lambda, \mathcal{B}_\lambda)\) is independent of \( \lambda \). However, examples of boundary value problems with non vanishing \( \mu(L, B) \) will be constructed in Section 5 taking into account the topology of \( GL(m; \mathbb{C}) \). In some simple cases the bifurcation index can be computed from the coefficients of the linearization, using rather elementary index theorems, see [18] and [19].

### 3. The index of bifurcation points

Here we will shortly recall the definition of the index of bifurcation. The construction in [17] uses the index bundle of a family of Fredholm operators and the generalized \( J \)-homomorphism.
If \( \Lambda \) is a compact space, the set \( Vect(\Lambda) \) of all isomorphism classes of real vector bundles over \( \Lambda \) is a semigroup under the direct sum. The (real) Grothendieck group \( KO(\Lambda) \) of a compact topological space \( \Lambda \) is the quotient of the semigroup \( Vect(\Lambda) \) by the diagonal sub-semigroup. Elements of \( KO(\Lambda) \) are called virtual bundles. Each virtual bundle can be written as a difference \([E] - [F]\), where \( E, F \) are vector bundles over \( \Lambda \) and \([E]\) denotes the equivalence class of \((E, 0)\). It is easy to see that \([E] - [F] = 0\) if and only if \( E \) and \( F \) become isomorphic after the addition of a trivial vector bundle to both. The complex Grothendieck group \( K(\Lambda) \) is defined by taking complex vector bundles instead of the real ones. In what follows the trivial bundle with fiber \( \Lambda \times V \) will be denoted by \( \Theta(V) \). The trivial bundle \( \Theta(\mathbb{R}^n) \) will be simplified to \( \Theta^n \).

Let \( X, Y \) be real Banach spaces and let \( L: \Lambda \to \Phi(X, Y) \) be a continuous family of Fredholm operators. Since \( \text{coker} L_\lambda \) is finite dimensional, using compactness of \( \Lambda \), one can find a finite dimensional subspace \( V \) of \( Y \) such that
\[
(3.1) \quad \text{Im} L_\lambda + V = Y \quad \text{for all} \quad \lambda \in \Lambda.
\]

Because of the transversality condition \((3.1)\) the family of finite dimensional subspaces \( E_\lambda = L_\lambda^{-1}(V) \) defines a vector bundle \( E \) over \( \Lambda \).

The index bundle of the family \( L \) is the virtual bundle
\[
(3.2) \quad \text{Ind} L = [E] - [\Theta(V)] \in KO(\Lambda).
\]

The index bundle has the same properties of the numerical index. It is homotopy invariant and hence invariant under perturbations by families of compact operators. It is additive under direct sums and the same holds for the index bundle of the family of composed operators (logarithmic property). Clearly \( \text{Ind} L = 0 \) if \( L \) is homotopic to a family of invertible operators. Below we will use the above properties without any further reference.

Notice that the index bundle of a family of Fredholm operators of index 0, belongs to the reduced Grothendieck group \( K\Omega(\Lambda) \) defined as the kernel of the rank homomorphism \( \text{rk}: KO(-) \to \mathbb{Z}, \text{rk}([E] - [F]) = \text{rk} E - \text{rk} F \).

Given a vector bundle \( E \), let \( S[E] \) be the associated unit sphere bundle with respect to some chosen scalar product on \( E \). Two vector bundles \( E, F \) are said to be stably fiberwise homotopy equivalent if, for some \( n, m \), the unit sphere bundle \( S(E \oplus \Theta^n) \) is fiberwise homotopy equivalent to the unit sphere bundle \( S(F \oplus \Theta^m) \). Let \( T(\Lambda) \) be the subgroup of \( K\Omega(\Lambda) \) generated by elements \([E] - [F]\) such that \( E \) and \( F \) are stably fiberwise homotopy equivalent. Put \( J(\Lambda) = K\Omega(\Lambda)/T(\Lambda) \). The projection to the quotient \( J: K\Omega(\Lambda) \to J(\Lambda) \) is the generalized \( j \)-homomorphism.

**Remark 3.0.1.** The group \( J(\Lambda) \) was introduced by Atiyah in \([3]\). He proved that \( J(\Lambda) \) is a finite group if \( \Lambda \) is a finite \( CW \)-complex by showing that \( J(\mathbb{S}^n) \) coincides with the image of the stable \( j \)-homomorphism of G. Whitehead.

Now let us introduce the index of bifurcation points constructed in \([17]\) for families of Fredholm maps of index 0 having as range a Kuiper space \( Y \), i.e., a Banach space whose group of linear automorphisms is contractible.

Let \( U \) be an open subset of a finite \( CW \)-complex \( \Lambda \) and let \( O \) be an open subset of a Banach space \( X \). Let \( f: U \times O \to Y \) be a family of \( C^1 \) Fredholm maps of index 0 such that \( f(\lambda, 0) = 0 \). We will denote by \( L \) the family \( \{Df_\lambda(0); \lambda \in U\} \).

The pair \((f, U)\) is called admissible if the set \( \Sigma(L) = \{\lambda | \text{Ker} L_\lambda \neq 0\} \) is a compact proper subset of \( U \).

If \((f, U)\) is admissible, the index of bifurcation points of \( f \) in \( U \) is defined by
\[
(3.3) \quad \beta(f, U) = J(\text{Ind} L),
\]
where $L: \Lambda \to \Phi_0(X,Y)$ is any family which coincides with $L$ on an open neighborhood of $\Sigma(L)$ with compact closure contained in $U$.

It was shown in [17] that $\beta(f,U)$ verifies the homotopy invariance, additivity and excision property in their usual form. Here we are mainly interested in:

**Existence property:** If $\beta(f,U) \neq 0$, then the family $f$ has at least one bifurcation point $\lambda_*$ in $U$, i.e., a point such that every neighborhood $V$ of $(\lambda_*,0)$ contains a solution of $f(\lambda,x) = 0$ with $x \neq 0$.

**Normalization property:** $\beta(f,\Lambda) = \beta(f) = J(\text{Ind } L)$.

4. Proof of theorem 2.0.1.

Denoting with $H^s(\partial \Omega; \mathbb{R}^r)$ the product $\prod_{i=1}^r H^{k_i+s-k_i-1/2}(\partial \Omega; \mathbb{R})$, the family of nonlinear differential operators

$$(F,G): \mathbb{R}^q \times C^\infty(\Omega; \mathbb{R}^m) \to C^\infty(\Omega; \mathbb{R}^m) \times C^\infty(\partial \Omega; \mathbb{R}^r)$$

induces a smooth map

$$(4.1) \quad h = : \mathbb{R}^q \times H^{k+q}(\Omega; \mathbb{R}^m) \to H^s(\Omega; \mathbb{R}^m) \times H^s(\partial \Omega; \mathbb{R}^r)$$

having $\mathbb{R}^q \times \{0\}$ as a trivial branch (see [17] Section 5.2).

The Frechet derivative $Dh_\lambda(0)$ of the map $h_\lambda$ at $u \equiv 0$ coincides with the operator $(L_\lambda, B_\lambda)$ induced by the linearization $(\mathcal{L}_\lambda, B_\lambda)$. Since, for any $\lambda \in \mathbb{R}^q$, $(\mathcal{L}_\lambda, B_\lambda)$ is elliptic, using proposition 5.2.1 of [17], we can find a neighborhood $O$ of 0 in $H^{k+s}(\Omega; \mathbb{R}^m)$ such that $h: \mathbb{R}^q \times O \to H^s(\Omega; \mathbb{R}^m) \times H^s(\partial \Omega; \mathbb{R}^r)$ is a smooth family of semi-Fredholm maps.

By $H_2$, the family of boundary value problems $(\mathcal{L}, B)$ extends to a smooth family parametrized by $S^q$, which will be denoted in the same way. Let $(L, B)$ be the family of operators induced on Hardy-Sobolev spaces.

Being $(L_\infty, B_\infty)$ invertible, by continuity of the index of semi-Fredholm operators, $(L_\lambda, B_\lambda)$ is Fredholm of index 0 for all $\lambda \in S^q$ which, on its turn, implies that the map $h: \mathbb{R}^q \times O \to H^s(\Omega; \mathbb{R}^m) \times H^s(\partial \Omega; \mathbb{R}^r)$ is a smoothly parametrized family of Fredholm maps of index 0.

Since $(L_\lambda, B_\lambda)$ is invertible in a neighborhood of $\lambda = (\lambda_0, B_0)$, $(L_\lambda, B_\lambda)$ is Fredholm of index 0 for all $\lambda \in S^q$ which, on its turn, implies that the map $h: \mathbb{R}^q \times O \to H^s(\Omega; \mathbb{R}^m) \times H^s(\partial \Omega; \mathbb{R}^r)$ is a smoothly parametrized family of Fredholm maps of index 0.

If, under the hypothesis of Theorem 2.0.1, we can show that $J(\text{Ind } (L, B)) \neq 0$ in $J(S^q)$, then the family $h$ must have a bifurcation point $\lambda \in \mathbb{R}^q$, by the existence property of the bifurcation index. This would complete the proof of the theorem, since by Proposition 5.2.2 of [17] a bifurcation point of the map $h$ is also a bifurcation point for smooth classical solutions of 4.1 in the sense of definition 2.0.1.

We will show that $J(\text{Ind } (L, B)) \neq 0$ in $J(S^q)$ in three steps.

**Step 1** We assume first that the principal part of $B_\lambda$ is independent of $\lambda$ and hence $\mu(L, B) = \text{deg} (\sigma)$. This is precisely the case considered in [17], Theorem 1.4.1]. In that paper, we showed that, if $\text{deg} (\sigma)$ is not divisible by $n(q)$, then $J(\text{Ind } L) \neq 0$.

**Step 2** Let us assume now that the principal part of $\mathcal{L}_\lambda$ is independent of $\lambda$ while there are no restrictions on the boundary condition $B$.

In this case we will use a family version of the Agranovich-Dynin reduction (see [11, 25]). In the case without parameters this reduction computes the difference between the indices of two elliptic boundary value problems for the same interior operator $\mathcal{L}(x, D)$ as the index of a pseudo-differential operator on the boundary $\Gamma$.
whose principal symbol is constructed in terms of the data. The result extends easily to families of elliptic boundary value problems. While discussing the Agranovich reduction in \[17\] we provided full details, we will be slightly more sketchy here.

We will consider only classical pseudo-differential operators acting on complex vector functions on a compact smooth orientable manifold $\Gamma$. This is exactly the same class of pseudo-differential operators as the one introduced in \[5\] but we are dealing here with trivial bundles only. For a detailed exposition see \[22, 25\]. The class of pseudo-differential operators under consideration contains all differential operators on manifolds and moreover is invariant under composition and formation of adjoints. We will need only a few facts about this class.

1) For $s \in \mathbb{N}$, the Hardy Sobolev space $H^s(\Gamma; \mathbb{C}^m)$ can be defined as the space of all vector functions such that $\mathcal{L}(x, D)u \in L^2(\Gamma; \mathbb{C}^m)$ for every differential operator of order less or equal than $s$. This definition extends to all $s \in \mathbb{R}$ in the usual form, using the square root of the Laplacian. Every pseudo-differential operator $\mathcal{R}$ of order $k$ defines a bounded operator $R: H^{k+s}(\Gamma; \mathbb{C}^m) \to H^s(\Gamma; \mathbb{C}^m)$.

2) Each pseudo-differential operator $\mathcal{R}$ of order $k$ on $\Gamma$ has a well defined principal symbol $\rho_k: T^*(\Gamma) - 0 \to \mathbb{C}^{m \times m}$, where $0$ denotes the image of the zero section. The principal symbol $\rho_k$ is a smooth, positively homogeneous function of order $k$. It behaves well under composition of operators. Namely, $\rho_{k+s}(\mathcal{RQ}) = \rho_k(\mathcal{R})\rho_s(\mathcal{Q})$.

3) A pseudo-differential operator $\mathcal{R}$ of order $k$ is elliptic if $\rho_k(\mathcal{R})(x, \xi) \in GL(m; \mathbb{C})$ for all $(x, \xi)$ with $\xi \neq 0$. Every elliptic operator possesses a (rough) parametrix, which is a proper pseudo-differential operator $\mathcal{P}$ of order $-k$ such that both $\mathcal{R} \circ \mathcal{P} - \text{Id}$ and $\mathcal{P} \circ \mathcal{R} - \text{Id}$ are of order $-1$. Since operators of order $-1$ viewed as operators of $H^s(\Gamma; \mathbb{C}^m)$ into itself are compact, by the Riesz characterization, the bounded operator induced by an elliptic pseudo-differential operator on Hardy-Sobolev spaces is Fredholm.

The Agranovich-Dynin reduction can be carried out in the context of continuous families of elliptic boundary value problems. Although here we will consider smooth families only, since we have defined the bifurcation index for continuous families of $C^1$-Fredholm maps we will work out the reduction in the above generality. Continuous families of pseudo-differential operators have been introduced in \[6\] (see also \[17\] for operators on $\mathbb{R}^n$).

Let us denote by $S(\Gamma) \subset T^*(\Gamma)$ the unit sphere bundle with respect to the induced riemannian metric on $\Gamma$. We will need also the following:

**Lemma 4.0.1.** Every continuous map $\rho: \Lambda \times S(\Gamma) \to GL(m, \mathbb{C})$ can be uniformly approximated by the restrictions to $\Lambda \times S(\Gamma)$ the principal symbols $\rho(\mathcal{S})$ of continuous families of pseudo-differential operators $\mathcal{S}$ of any chosen order. Moreover, if $\Lambda$ is a smooth manifold the approximating symbols can be chosen smooth (in the parameter variables as well).

**Proof.** The first assertion is proved in \[6\] Proposition 6.1]. The second is an immediate consequence of the method in proof there.

Let us consider two families of linear elliptic boundary value problems $(\mathcal{L}, \mathcal{B}_+)$ and $(\mathcal{L}, \mathcal{B}_-)$ with the same constant interior operator $\mathcal{L}$, and with parameter dependence only on the boundary operators:

\begin{align}
\mathcal{L}(x, D) &= \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha, \\
\mathcal{B}_\pm(\lambda, x, D) &= \gamma_0 \sum_{|\alpha| \leq k_i} b_\alpha^i(\lambda, x) D^\alpha, \quad 1 \leq i \leq r.
\end{align}
Here the parameter $\lambda$ belongs to a compact connected topological space $\Lambda$, while the matrix functions $a_{\alpha}(x) \in \mathbb{C}^{m \times m}$, $b_{\alpha}^{-1}(\lambda, x) \in \mathbb{C}^{1 \times m}$ are smooth in $x$, and depend continuously on $(\lambda, x)$ together with all their partial derivatives.

Using the coordinates $(x', t)$ on $\mathcal{N}$, we rewrite the boundary operators $B^i_{\pm}(\lambda, x, D)$ in the form described in (2.10), i.e.,

$$B^i_{\pm}(\lambda, x, D)u = \sum_{j=0}^{k_i} B^i_{\pm j}(\lambda, x', D) \gamma_j u.$$  

(4.3)

Here, however, we will have to extend our considerations to pseudo-differential boundary conditions. Namely, the coefficients $B^i_{\pm j}(\lambda, x', D)$ can be pseudo-differential operators on $\Gamma$ of order $k_i-j$. The principal symbol of the boundary operator is defined in the same way as in (2.7) and there are no changes in the formulation of the Shapiro-Lopatinskij condition.

For $(\lambda, x', \xi') \in \Lambda \times (T^*\Gamma - \bar{0})$, let us define

$$\tau(\lambda, x', \xi') = b_+(\lambda, x', \xi')b_-^{-1}(\lambda, x', \xi'),$$  

(4.4)

where

$$b_{\pm}(\lambda, x', \xi') : \mathcal{M}^+(\lambda, x', \xi') \to \mathbb{C}^r$$  

is the isomorphism associated by (2.5) to the boundary operator $B_{\pm}(\lambda, x, D)$.

The following proposition is the Agranovich-Dynin reduction for families:

**Proposition 4.0.2.** Given $(\mathcal{L}, B_{\pm})$ as above, there exists a family $S$ of pseudo-differential operators of order $0$ on $\Gamma$ such that

$$\text{Ind } (L, B_+) - \text{Ind } (L, B_-) = \text{Ind } S.$$  

Moreover the restriction of the principal symbol $\rho_0(S)$ to $S^0 \times S(\Gamma)$ can be taken arbitrarily close in the sup norm to the restriction of $\tau$ to the same subspace.

**Proof.** Using lemma 4.0.1 we can find a smooth family $\mathcal{S} = \{S_\lambda ; \lambda \in \Lambda\}$ of elliptic pseudo-differential operators of order $0$ on $\Gamma$ such that the family of principal symbols restricted to $S^0 \times S(\Gamma)$ is arbitrarily close to the restriction of $\tau$. Since the Shapiro-Lopatinskij condition (2.5) is stable under small perturbations, it follows that the family $(\mathcal{L}, SB_-)$ is a family of elliptic boundary value problems with pseudo differential boundary conditions.

Since $(L, SB_-) = (\text{Id} \times S)(L, B_-)$, by the logarithmic property of the index bundle,

$$\text{Ind } (L, SB_-) = \text{Ind } (\text{Id} \times S) + \text{Ind } (L, B_-) = \text{Ind } S + \text{Ind } (L, B_-).$$

On the other hand, if on $S^0 \times S(\Gamma)$, $\rho_0(S)$ is close enough to $\tau$, the affine homotopy

$$(1-t)(L, SB_-) + t(L, B_+)$$

is a homotopy of families of linear Fredholm operators between $(L, SB_-)$ and $(L, B_+)$. The proposition now follows from the homotopy invariance property of the index bundle.

**Theorem 4.0.3.** Let the problem

$$\begin{cases}
F(\lambda, x, u, \ldots, D^k u) = 0, \ x \in \Omega \\
G_i(\lambda, x, u, \ldots, D^k u) = 0, \ x \in \partial \Omega, 1 \leq i \leq r,
\end{cases}$$

(4.6)

verify the assumptions $H_1, H_2$ and $H_3$. Assume moreover that the family of interior operators $\mathcal{L}(x, D)$ of the linearization at $u \equiv 0$ is independent of $\lambda$. 


If \( q \equiv 0, 4 \mod 8 \), there exists at least one bifurcation point from the trivial branch provided that \( \mu_b(\mathcal{L}, \mathcal{B}) \) is not divisible by \( n(q) \).

**Proof.** We are going to compute \( J(\text{Ind}(\mathcal{L}, \mathcal{B})) \) from \( \mu_b(\mathcal{L}, \mathcal{B}) = \deg(\tau) \) using the complexification \((\mathcal{L}^c, \mathcal{B}^c)\) of the linearized equations at \( u \equiv 0 \).

Since \( \ker(\mathcal{L}^c, \mathcal{B}^c) = \ker(\mathcal{L}, \mathcal{B}) \otimes \mathbb{C} \), from the definition of the index bundle in (4.2) it follows that

\[
\text{Ind}(\mathcal{L}^c, \mathcal{B}^c) = c(\text{Ind}(\mathcal{L}, \mathcal{B})),
\]

where \( c: KO \to \mathcal{K} \) is the complexification homomorphism.

By Bott periodicity, for \( q = 4s \), both \( \mathcal{K}(S^q) \approx \mathbb{Z} \) and \( KO(S^q) \approx \mathbb{Z} \) are infinite cyclic with \( \mathcal{K}(S^q) \) generated by powers \( \xi_q = ([H] - [\Theta^1])^{2s} \), where \( H \) is the tautological line bundle over \( P^1(\mathbb{C}) \). Moreover, by [23] section 13.94 \( c: KO(S^q) \to \mathcal{K}(S^q) \) is an isomorphism for \( q \equiv 0 \mod 8 \) and a monomorphism with image generated by \( 2\xi_q \) for \( q \equiv 4 \mod 8 \).

We choose as generator of \( KO(S^q) \) an element \( \nu_q \) such that

\[
c(\nu_q) = \begin{cases} 
\xi_q & \text{if } q \equiv 0 \mod 8 \\
2\xi_q & \text{if } q \equiv 4 \mod 8.
\end{cases}
\]

Then each element \( \eta \in \tilde{K}(S^q) \) with \( q = 4s \) is uniquely determined by its degree \( d(\eta) \in \mathbb{Z} \) verifying \( \eta = d(\eta)\xi_q \), and each element \( \eta \) of \( KO(S^q) \) has a degree defined in the same way. Clearly,

\[
d(c(\eta)) = \begin{cases} 
d(\eta) & \text{if } q \equiv 0 \mod 8 \\
2d(\eta) & \text{if } q \equiv 4 \mod 8.
\end{cases}
\]

Let us denote with \( H^*(-; \mathbb{C}) \) the de Rham cohomology with coefficients in \( \mathbb{C} \) and compact supports. We will denote with \( H^{\text{even/odd}}(-; \mathbb{C}) \) the cohomology in even degrees and odd degrees respectively. By the uniqueness of the Chern character and Bott’s integrality theorem ([11], Theorem 9.6, Chap.18), \( \text{ch}(\mathcal{K}(S^q)) \to H^{\text{even}}(S^q; \mathbb{C}) \) sends \( \mathcal{K}(S^q) \) isomorphically into \( H^{\text{even}}(S^q; \mathbb{Z}) \subset H^{\text{even}}(S^q; \mathbb{C}) \). Hence the degree of an element \( \eta \in \mathcal{K}(S^q) \) can be computed by evaluating the Chern character on the fundamental class \([S^q]\) of the sphere. Namely,

\[
d(\eta) = <\text{ch}(\eta); [S^q]>.
\]

We will compute the degree of \( \text{Ind}(\mathcal{L}^c, \mathcal{B}^c) \) using (4.10) and the Agranovich-Dynin reduction.

Put \( (\mathcal{L}, \mathcal{B}_+) = (\mathcal{L}^c, \mathcal{B}^c) \) and \( (\mathcal{L}, \mathcal{B}_-) = (\mathcal{L}^c, \mathcal{B}^c_\infty) \) in Proposition 4.0.2. Being \( (\mathcal{L}^c, \mathcal{B}^c_\infty) \) a constant family, \( \text{Ind}(\mathcal{L}^c, \mathcal{B}^c_\infty) = 0 \), and hence by Proposition 4.0.2 we have,

\[
\text{Ind}(\mathcal{L}^c, \mathcal{B}^c) = \text{Ind}(\mathcal{L}, S),
\]

where \( S \) is induced by a family of pseudo-differential operators \( S \) on \( \Gamma \) whose principal symbol \( \rho = \rho_0(S) \) is homotopic to \( \tau \).

Now let us apply the cohomological form of the Atiyah-Singer family index theorem to \( S \).

Since \( \Gamma \) is a boundary, its Todd class vanishes. Thus by the Atiyah-Singer family index theorem ([7], Theorem (3.1))

\[
\text{ch}(\text{Ind}(S)) = p_*\text{ch}[\rho],
\]

where \( p_* : H^*(S^q \times T^*(\Gamma)) \to H^*(S^q) \) is the direct image homomorphism associated to the bundle of tangents along the fiber and \( [\rho] \in K_c(S^q \times T^*(\Gamma)) \) is the symbol class of \( \rho \).
By definition, the symbol class \([\rho] \in K_c(S^q \times T^*(\Gamma))\) is obtained from the map \(\rho: S^q \times (T^*(\Gamma) - 0) \to GL(r; \mathbb{C})\) by means of the clitching construction \([\mathcal{K}]\) described below:

Let \(S^*(\Gamma)\) be the fiberwise compactification of \(T^*(\Gamma)\), obtained by adjoining a point at infinity to each fiber, and let \(\tilde{S}^* = S^q \times S^*(\Gamma)\). Then \(S^*\) is the union of two open sets \(U_0 = S^q \times T^*(\Gamma) = S^q \times (S^*(\Gamma) - \infty)\) and \(U_1 = S^q \times (S^*(\Gamma) - 0)\), where as before \(\infty\) denotes the image of the section at infinity. Let \(D_i \subset U_i\) be the set of points \((\lambda, v) \in S^*\) with the norm of \(|v|\) \(\leq 1\) and \(|v|\) \(\geq 1\) respectively.

We obtain a vector bundle \(E\) over \(S^*\) gluing two trivial bundles \(\theta^*\) with fiber \(\mathbb{C}^r\) over \(D_i\), \(i = 0, 1\) by means of the restriction of \(\rho\) to \(S^q \times S(\Gamma)\). Since the restriction of \(E\) to a neighborhood of \(\infty\) is trivial, \([E] - [\Theta^*]\) defines an element \([\rho]\) belonging to \(\tilde{K}(S^*/\infty) \cong K_c(S^q \times T^*(\Gamma))\). By definition, the above element is the symbol class of \(\rho\).

The symbol class \([\rho]\) is defined in terms of the restriction of \(\rho\) to \(S^q \times S(\Gamma)\) only. Indeed, the above construction associates a (homotopy invariant) symbol class \([\rho]\) \(\in K_c(S^q \times T^*(\Gamma))\) to any continuous map \(\rho: S^q \times S(\Gamma) \to GL(r; \mathbb{C})\).

**Remark 4.0.1.** The formula (4.12) differs from the one in \([7\, \text{Theorem } (3.1)]\) by a factor \((-1)^{n-1}\). This factor, which is irrelevant to our considerations, disappears by substituting the orientation of \(T^*(\Gamma) \otimes \mathbb{C}\) used in the above paper with the one in \([16\, \text{Theorem } 2, \text{Chap } XIX]\).

In \([9\, \text{§3}]\), Fedosov chooses two trivializations of \(E|_{U_i}\), whose transition function over \(U_1 \cap U_2\) coincides with \(\rho\). Using this trivializations he defines a connection on \(E\) and uses its curvature in order to construct a (non homogeneous) differential form representing the Chern character of \([\rho]\). The result in \([9\, \text{§3}, (17)]\) is that \(\text{ch}([\rho])\) is the cohomology class of the differential form:

\[
(4.13) \quad - \sum_{j=1}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} d(h(||v||)tr(\rho^{-1} dp)^{2j-1}),
\]

where \(h(t)\) is a smooth function which vanishes in a neighborhood of 0 and such that \(h(t) = 1\), for \(t \geq 1\). Actually, in \([9\, \text{§3}]\) only the case \(\Lambda = pt\) is considered, but all his arguments hold word for word for families parametrized by compact orientable manifolds.

On the other hand, since the restrictions of \(\rho\) and \(\tau\) to \(S^q \times S(\Gamma)\) are homotopic, the vector bundles obtained by gluing trivial bundles using either \(\rho\) or \(\tau\) are isomorphic and therefore their Chern characters coincide. In conclusion we obtain

\[
(4.14) \quad \text{ch}([\rho]) = \text{ch}([\tau]) = \left\{ - \sum_{j=1}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} d(h(||v||)tr(\tau^{-1} d\tau)^{2j-1}) \right\},
\]

where \(\{\theta\}\) denotes the cohomology class of the form \(\theta\).

The direct image homomorphism \(p_*\) in de Rham cohomology is the homomorphism induced by a cochain homomorphism called integration along the fiber. The latter takes \((d+2n-2)\)-forms with compact support on \(S^q \times T^*(\Gamma)\) into \(d\)-forms on \(S^q\) literally by integrating the fiber variables. (see \([10\, \text{§VII}]\)).

Denoting with \(\mathcal{F}\) the integration along the fiber, from (4.14) we get

\[
(4.15) \quad p_* \text{ch}([\tau]) = \left\{ \sum_{j=n-1}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} \int_{T^*(\Gamma)} d(h(||v||)tr(\tau^{-1} d\tau)^{2j-1}) \right\}.
\]
On the other hand, the evaluation of a cohomology class on the fundamental class of an \(n\)-manifold in de Rham cohomology corresponds, at the cochain level, to the integration of a representing form over the manifold. Therefore, integrating over \(S^n = S^{4s}\) the \(4s\)-homogenous term from (4.15) and using Fubini’s theorem for integration along the fiber [10, § VII], we get

\[
(4.16) \quad < p, \text{ch}([\tau]); [S^q] > = N \int_{S^{4s} \times S^*(\Gamma)} d[h(\|v\|)] tr(\tau^{-1} d\tau)^{4s+2n-3},
\]

where the right hand side is the ordinary integration of the \((4s + 2n - 2)\)-form over a manifold of the same dimension and

\[
(4.17) \quad N = \frac{(2s + n - 2)!}{(2\pi i)^{2s+n-1}(4s + 2n - 3)!}.
\]

It is easy to see that

\[
d[tr(\tau^{-1} d\tau)^{4s+2n-3}] = -tr d(\tau^{-1} d\tau)^{4s+2n-2} = 0,
\]

and since \(h(\|v\|) \equiv 1\) if \(\|v\| \geq 1\), the differential form \(d[h(\|v\|)] tr(\tau^{-1} d\tau)^{4s+2n-3}\) vanishes outside \(D_0\).

Thus (4.16) reduces to an integral over the manifold with boundary \(D_0\). Using Stokes theorem we obtain

\[
(4.18) \quad < p, \text{ch}([\tau]); [S^q] > = N \int_{S^{4s} \times S^*(\Gamma)} tr(\tau^{-1} d\tau)^{4s+2n-3} = \deg(\tau) = \mu_0(\mathcal{L}, \mathcal{B})
\]

From (4.18), (4.11) , (4.12) we have

\[
(4.19) \quad < \text{ch} \text{Ind} (L^s, B^s); [S^q] > = < \text{ch} \text{Ind} S; [S^q] > = \mu_0(\mathcal{L}, \mathcal{B}).
\]

In particular, by (4.19) and Bott’s Integrality Theorem, \(\mu_0(\mathcal{L}, \mathcal{B}) \in \mathbb{Z}\).

Since complexification of the index bundle of \(\text{Ind} (L, B)\) is the index bundle of the family of complexified operators another consequence of (4.19) together with (4.10) is that \(d(c(\text{Ind} (L, B)) = \mu_0(\mathcal{L}, \mathcal{B})\).

From the above observation, using (4.19) we finally obtain

\[
(4.20) \quad d(\text{Ind} (L, B)) = \begin{cases} 
\mu_0(\mathcal{L}, \mathcal{B}) & \text{if } q \equiv 0 \mod 8 \\
\frac{1}{2} \mu_0(\mathcal{L}, \mathcal{B}) & \text{if } q \equiv 4 \mod 8.
\end{cases}
\]

On the other hand, for \(q = 4s\), \(J(S^{4s}) \simeq Z_{m(2s)}\) and \(J(\text{Ind} L) = 0\) if and only if \(d(\text{Ind} (L, B))\) is divisible by \(m(2s)\). Hence Theorem 4.0.2 follows from (4.20) and the definition of \(n(q)\) in (4.14). \(\square\)

**Step 3** We will reduce the general case to the two considered previously.

Together with the family of linearizations along the trivial branch (\(\mathcal{L}, \mathcal{B}\)) we consider \((\mathcal{L}_\infty, \mathcal{B}_\infty)\) as a constant family and compare the following two families of elliptic boundary value problems:

\[
(4.21) \quad (\mathcal{L}^1, \mathcal{B}^1) = (\mathcal{L}_\infty \mathcal{L}, \mathcal{B}_\infty \mathcal{L}, \mathcal{B})
\]

and

\[
(4.22) \quad (\mathcal{L}^2, \mathcal{B}^2) = (\mathcal{L}_\infty, \mathcal{B}_\infty \mathcal{L}_\infty, \mathcal{B}).
\]

Put \(X = H^{2k+s}(\Omega; \mathbb{R}^m), Y = H^{k+s}(\Omega; \mathbb{R}^m), Z = H^{k+s}(\partial\Omega; \mathbb{R}^r), V = H^s(\Omega; \mathbb{R}^m)\) and \(W = H^s(\partial\Omega; \mathbb{R}^r)\).

The operator \((L^1, B^1) \rightarrow V \times W \times Z\) induced by (4.21) is the composition of \((L, B) \rightarrow Y \times Z\) with \((\mathcal{L}_\infty, \mathcal{B}_\infty) \times Id: Y \times Z \rightarrow V \times W \times Z\).
Hence under the assumptions of Theorem 2.0.1, \((L^1, B^1)\) is Fredholm, and the same holds for \((L^2, B^2)\) which is a composition of \((L_\infty, B_\infty)\) with \((L, B_\infty) \times \text{Id}\). In particular, (4.21) and (4.22) are elliptic boundary value problems, being ellipticity equivalent to the Fredholm property of the induced operator.

The above two decompositions give:

\begin{equation}
\text{Ind } (L^1, B^1) = \text{Ind } (L_\infty, B_\infty) + \text{Ind } (L, B)
\end{equation}

\begin{equation}
\text{Ind } (L^2, B^2) = \text{Ind } (L, B_\infty) + \text{Ind } (L_\infty, B).
\end{equation}

Since \(L_\lambda(x, D) = L_\infty(x, D)\) for \(x\) close to \(\Gamma\), we have \(B^1 = B^2\). Moreover, the principal symbols of \(L_\infty\) and \(L_\lambda\) commute and hence the principal symbols of \(L_\infty L\) and \(L_\lambda L\) coincide. It follows that \(L_\infty L - L_\lambda L\) is of order \(-1\). Thus the families \((L^1, B^1)\) and \((L^2, B^2)\) differ by a family of compact operators and therefore

\begin{equation}
\text{Ind } (L^1, B^1) = \text{Ind } (L^2, B^2).
\end{equation}

Since \((L_\infty, B_\infty)\) is a constant family of operators of index 0, from (4.23), (4.24) we obtain:

\begin{equation}
d(\text{Ind } (L, B)) = d(\text{Ind } (L^1, B^1)) = d(\text{Ind } (L, B_\infty)) = d(\text{Ind } (L_\infty, B)).
\end{equation}

The degrees on the right hand side have been computed in [17] and in Step 2. Indeed, \(d(\text{Ind } (L_\infty, B))\) is given by (4.20) and \(d(\text{Ind } (L, B^\infty))\) is given by the same formula involving the interior multiplicity \(\mu(L, B)\) [17, (4.25)].

In conclusion:

\begin{equation}
d(\text{Ind } (L, B)) = \begin{cases} 
\mu(L, B) & \text{if } q \equiv 0 \mod 8 \\
\frac{1}{2}\mu(L, B) & \text{if } q \equiv 4 \mod 8,
\end{cases}
\end{equation}

and the last argument in Step 2 completes the proof of the theorem. \(\square\)

5. AN EXAMPLE

In this section we will show how to construct families of elliptic differential boundary value problems verifying the hypotheses of Theorem 2.0.1. Examples with pseudo-differential boundary conditions are easy to find. However, exhibiting concrete examples with differential boundary conditions is far from being simple. Indeed, very little is known about the set of elliptic systems of differential operators of a given order, and even less about the structure of elliptic boundary value problems.

Following a suggestion of Atiyah in [4], we will take an indirect approach by approximating the principal symbol of a family of elliptic pseudo-differential operators with symbols of families of elliptic differential operators of sufficiently high order. Atiyah’s idea is to consider the set of elliptic symbols \(A(n, r, 2k)\), whose elements are \(r \times r\) matrices with homogeneous polynomial entries of order \(2k\) in variables \((\xi_1, \ldots, \xi_\alpha)\). Then the approximations are constructed using the fact that the set of restrictions to the unit sphere of elements of \(A(n, r) = \bigcup_{k \geq 0} A(n, r, 2k)\) is dense in the set of even continuous functions from the sphere into \(GL(r; \mathbb{C})\) (see also [24] for a related result).

Before going to this point we need some preliminaries. First, let us observe that the reality condition on the principal symbols is irrelevant to the validity of (4.18). Hence (4.18) holds true not only for the map \(\tau\) defined in (2.12) but in general. Thus, for any smooth map \(\phi: S^9 \times S(\Gamma) \to GL(r; \mathbb{C})\) with \(q = 2t\) even, we have:

\begin{equation}
<p_* \text{ch}(\phi); [S^9] > = \text{deg } \phi.
\end{equation}
Secondly, since we are dealing with homotopy invariants of maps with values $GL(r; \mathbb{C})$ and since the unitary group $U(r)$ is a deformation retract of $GL(r; \mathbb{C})$, in our discussion we can safely assume that $\phi$ takes values in the unitary group $U(r)$.

The traces $\theta_1 = tr(u^{-1}du)^{2i - 1}$ of the odd powers of the Maurer-Cartan matrix-differential form $u^{-1}du$ of $U(r)$ are bi-invariant and hence harmonic differential forms. The forms $\theta_1$ define cohomology classes $[\theta_1] \in H^{2i-1}(U(r); \mathbb{C})$ in the de Rham cohomology with coefficients in $\mathbb{C}$ which are known to be generators of the exterior algebra $H^{odd}(U(r); \mathbb{C})$. The pullback of $\theta_{q+2n-3}$ by the map $\phi$ is the pullback of $d\phi$, and hence we can write

\[
\deg(\phi) = N \int_{S^q \times S(\Gamma)} \phi^* \theta_{q+2n-3}.
\]

Here $\phi^*$ denotes the pullback of $\phi$ and $N$ is as in (2.10).

Finally, let us recall that if $\psi: S^{2v-1} \to GL(r; \mathbb{C})$, is any continuous map, the clutching construction associates to $\psi$ a vector bundle $[\psi]$ over $S^{2v}$ obtained by gluing via the map $\psi$ two trivial complex bundles of rank $r$ over the upper and lower hemispheres $D_+ \times \Gamma$ of $S^{2v}$. For $r \geq v$, the above construction induces an isomorphism of $\pi_{2v-1}(U(r))$ with $K(S^{2v})$.

In 3.1 of [9] Fedosov showed that for smooth $\psi$ and an appropriate choice of orientation of $S^{2v-1}$

\[
< \text{ch}([\psi]); [S^{2v}] > = \deg(\psi) = N \int_{S^{2v-1}} tr(\phi^{-1}d\phi)^{2v-1}.
\]

With this said, let us go to the example.

Let $n \geq 3$ be odd. For simplicity, choose $\Omega$ such that the cotangent bundle of $\Gamma$ is trivial, e.g., take as $\Omega$ the region bounded by an $(n-1)$-torus $\Gamma = (S^1)^{n-1}$. Then $S(\Gamma) \equiv \Gamma \times S^{n-2}$. Consider the map $f$ defined as the composition

\[
S^q \times \Gamma \times S^{n-2} \xrightarrow{\text{Id} \times \pi} S^q \times \Gamma \times \mathbb{R}P^{n-2} \xrightarrow{g} S^{q+2n-3},
\]

where $\mathbb{R}P^{n-2}$ is the real projective space, $\pi: S^{n-2} \to \mathbb{R}P^{n-2}$ is the canonical projection and $g: S^q \times T^{n-1} \times \mathbb{R}P^{n-2} \to S^{q+2n-3}$ is a smooth map having Brouwer degree one and sending $\{\infty\} \times \Gamma \times \mathbb{R}P^{n-2}$ into a point.

Notice that Brouwer’s degree $\deg_B g$ is defined because $\mathbb{R}P^n$ is orientable for odd $n$. Moreover, $\deg_B f = 2$ since $\deg_B \pi = 2$, in this case.

Choose an $r \geq q + 2n - 3$ and define $\phi: S^q \times \Gamma \times S^{n-2} \to GL(r; \mathbb{C})$ to be the composition of the map $f$ with a map $\psi: S^{q+2n-3} \to GL(r; \mathbb{C})$ representing a generator of $\pi_{q+2n-3}(U(r)) \simeq K(S^{q+2n-2}) \simeq \mathbb{Z}$. By construction the map $\phi$ is even, in the variable $\xi$, i.e., $\phi(\lambda, x', -\xi') = \phi(\lambda, x', \xi')$.

By the Change of Variables Theorem,

\[
\deg(\phi) = N \int_{S^q \times S(\Gamma)} \phi^* \theta_{q+2n-3} = N \deg_B f \int_{S^{q+2n-3}} \psi^* \theta_{q+2n-3} = 2 \deg(\psi) = 2,
\]

being $\deg(\psi) = 1$ by (5.3).

**Lemma 5.0.1.** The map $\phi: S^q \times \Gamma \times S^{n-2} \to U(r)$ constructed above, considered as a map with values in $GL(r; \mathbb{C})$, can be uniformly approximated by the restriction to $S^q \times \Gamma \times S^{n-2}$ of the symbol of a family of homogeneous elliptic differential operators.
Proof. Let $\mathcal{C} = C(S^{n-2}; \mathbb{C}^r \times r)$, endowed with the sup norm, and let $\mathcal{A} \subset \mathcal{C}$ be the set of all restrictions to $S^{n-2}$ of polynomial maps from $\mathbb{R}^{n-1}$ to $\mathbb{C}^r \times r$. By the Stone-Weierstrass theorem $\mathcal{A}$ is dense in $\mathcal{C}$. Using this and smooth partitions of unity on $S^q \times \Gamma$, for every $\epsilon > 0$ we can find a smooth map
\[
\rho: S^q \times \Gamma \times \mathbb{R}^{n-1} \rightarrow \mathbb{C}^r \times r
\]
such that
\[
\|\rho(\lambda, x', \xi') - \phi(\lambda, x', \xi')\|_\infty < \epsilon \quad \text{for all} \quad (\lambda, x', \xi') \in S^q \times \Gamma \times S^{n-2}
\]
and such that $\rho(\lambda, x', \xi') = \sum_{|\alpha| \leq 2} a_\alpha(\lambda, x')(\xi')^\alpha$.

For simplicity let us assume that $l = 2s$ is even. We rewrite the last expression in the form $\rho(\lambda, x', \xi') = \sum_{i=0}^{2s} h_i(\lambda, x', \xi')$, where $h_i(\lambda, x', \xi') = \sum_{|\alpha| = i} a_\alpha(\lambda, x')(\xi')^\alpha$. Thus the maps $h_i$ are homogeneous polynomials in $\xi$ of degree $i$.

Since $\phi(\lambda, x', \xi') = \frac{1}{2}[\phi(\lambda, x', \xi') + \phi(\lambda, x', -\xi')]$, the restriction to $S^q \times \Gamma \times S^{n-2}$ of the even part $\rho_{ev}(\lambda, x', \xi') = \sum_{i=0}^{2s} h_{2i}(\lambda, x', \xi')$ of $\rho$ also verifies (5.6). Now we can approximate $\phi$ by the restriction of a map $h$ which is a homogeneous polynomial in $\xi$. In fact, at points with $|\xi| = 1$ the values of the even homogeneous polynomial $h(\lambda, x', \xi') = \sum_{i=0}^{2s} |\xi|^{2i} h_{2i}(\lambda, x', \xi')$, coincide with those of $\rho_{ev}$. Thus $h$ is the symbol of a family of homogeneous differential operators which becomes elliptic after choosing an $\epsilon$ small enough. \hfill $\Box$

Remark 5.0.1. In the above lemma the triviality of the cotangent bundle is essential. Indeed, the proof shows that any parametrized family of even maps from the cotangent sphere bundle to $GL(r; \mathbb{C})$ can be uniformly approximated by a family of principal symbols of elliptic differential operators.

Choose $m, l$ such that $ml = r$ and consider the boundary value problem $(\mathcal{L}^0, \mathcal{B}^0)$, where $\mathcal{L}^0 = (\Delta^l + \mu)\text{Id}_m$, acting on $\mathcal{C}^m$ valued functions and $\mathcal{B}^0 = (\gamma_0, \ldots, \gamma_{l-1})$ is the Dirichlet boundary condition of order $l - 1$.

Since $\Delta^l$ is a strongly elliptic operator, taking $\mu$ big enough, we can assume that the operator $(\mathcal{L}^l, \mathcal{B}^l)$ induced by $(\mathcal{L}^0, \mathcal{B}^0)$ on Hardy-Sobolev spaces is an isomorphism. Let us denote with $\mathcal{H} = \{H_\lambda : \lambda \in S^q\}$ the family of homogeneous elliptic differential operators on $\Gamma$ associated to the symbol $h$ constructed in Lemma 5.0.1.

Since the index of an operator depends only on the homotopy class of the restriction of its principal symbol to the unit sphere bundle $S(\Gamma)$, and since the restriction of $h_\infty$ can be taken arbitrarily close to the constant symbol $\phi_\infty$ we have that $\text{ind} H_\infty = 0$. By eventually taking a lower order perturbation, we can also assume that the operator $H_\infty$ induced by $\mathcal{H}_\infty$ is an isomorphism.

Let us consider now the family $(\mathcal{L}, \mathcal{B})$ with $\mathcal{L} = \mathcal{L}^0$ constant and $\mathcal{B} = H \circ \mathcal{B}^0$. The Fredholm property of the induced operator is equivalent to the ellipticity of the boundary value problem. Hence writing $(L, B)$ in the form
\[
(L, B) = (\text{Id} \times H) \circ (\mathcal{L}^0, \mathcal{B}^0)
\]
we see that the family $(\mathcal{L}, \mathcal{B})$ is a family of elliptic differential boundary value problems with complex (matrix) coefficients. Moreover we have that $(L_\infty, B_\infty)$ is an isomorphism. As a family of complex differential operators of index 0, taking $k = 2l$ the induced family $(L, B) : S^q \times H^{k+s}(\Omega; \mathcal{C}^m) \rightarrow H^s(\Omega; \mathcal{C}^m) \times H^{s+\alpha}(\partial \Omega; \mathcal{C}^r)$ has an index bundle $\text{Ind} (L, B) \in \widetilde{K}(S^q)$.

We have
\[
(5.7) \quad \text{Ind} (L, B) = \text{Ind} (\text{Id} \times H) + \text{Ind} (\mathcal{L}^0, \mathcal{B}^0) = \text{Ind} H,
\]
being $(\mathcal{L}^0, \mathcal{B}^0)$ constant.

Since the restriction of $h$ to $S^q \times S(\Gamma)$ homotopic to $\phi$, from (5.7) we obtain
\[
(5.8) \quad \langle \text{ch}(\text{Ind} (L, B)); S^q \rangle = \langle \text{ch}(\text{Ind} H); S^q \rangle = \langle p, \text{ch}(\phi); S^q \rangle = \deg(\phi) = 2.
\]
Now, we identify $\mathbb{C}^n$ and $\mathbb{C}^r$ with $\mathbb{R}^{2n}$ and $\mathbb{R}^{2r}$ respectively and consider $(\mathcal{L},\mathcal{B})$ as a family of real differential operators. In order to avoid confusions we will denote this family with $(\mathcal{L}',\mathcal{B}')$. The ellipticity of this family is a consequence of the ellipticity of the corresponding complex family. By construction the operator $((\mathcal{L}_\infty',\mathcal{B}_\infty'))$ verifies $H_2$, and being $\mathcal{L}'$ constant, the restriction of this family to parameters belonging to $\mathbb{R}^v$ verifies the assumptions $H_1 - H_3$.

Our aim is to apply Corollary 2.0.2 to the (real) nonlinear perturbations of the restricted family. For this we have to evaluate the multiplicity $\mu_0(\mathcal{L}',\mathcal{B}')$.

Much as in the proof of Theorem 4.0.3 we compute it from the degree of the index bundle of the complexification $(\mathcal{L}^{rc},\mathcal{B}^{rc})$ of $(\mathcal{L}',\mathcal{B}')$.

Denoting with $c : \mathcal{KO}(\cdot) \to \mathcal{K}(\cdot)$ and $r : \mathcal{K}(\cdot) \to \mathcal{KO}(\cdot)$ the complexification and the realization homomorphism respectively we have:

$$
\mu_0(\mathcal{L}',\mathcal{B}') = \langle \text{ch}(\text{Ind}(\mathcal{L}^{rc},\mathcal{B}^{rc})); S^q \rangle = \langle c \circ \text{Ind}(L,B)%; S^q \rangle > .
$$

The right hand side of (5.9) can be easily related to the left hand side of (5.8).

Indeed, for $q = 4s$, by Theorem 30, the Chern classes of $c \circ r(\eta) \in \mathcal{K}(S^q)$ verify $c_{2s}(c \circ r(\eta)) = \pm 2c_{2s}(\eta)$. Using $c_{2s}(\eta) = \pm (2s - 1)! \text{ch}(\eta)$ we conclude from (5.8) that

$$
\mu_0(\mathcal{L}',\mathcal{B}') = \pm 2 < \text{ch}(\text{Ind}(L,B)); S^q > = \pm 2 \text{deg}(\phi) = \pm 4.
$$

The function $n(q)$ defined in (4.11) always assumes values greater or equal than 24. Hence, by Corollary 2.0.2, any family $(\mathcal{F}, \mathcal{G}) : \mathbb{R}^q \times C^\infty(\Omega; \mathbb{R}^{2m}) \to C^\infty(\Omega; \mathbb{R}^{2m}) \times C^\infty(\partial \Omega; \mathbb{R}^{2r})$ of the form

$$(\mathcal{L}_\lambda'(x,D)u + \mathcal{F}'(\lambda,x,u,\ldots), \mathcal{B}_\lambda'(x,D)u + \mathcal{G}'(\lambda,x,u,\ldots)),$$

where $(\mathcal{F}', \mathcal{G}')$ is any lower order perturbation with $\mathcal{F}'(\lambda,x,0) = 0, \mathcal{G}'(\lambda,x,0) = 0$ and such that the coefficients of the linearization of $(\mathcal{F}', \mathcal{G}')$ converge uniformly to 0 as $\lambda \to \infty$, will have at least one bifurcation point.

In conclusion, the above construction of linear elliptic boundary value problems with multiplicity smaller in absolute value than any $n(q)$ produces examples to which our bifurcation criteria can be applied for all data $(q,n,m,l)$ with $n$ odd, $q$ divisible by 4 and $ml = r \geq q + 2n - 3$. The same method can be used in order to construct families of linear differential operators with constant boundary conditions and $\mu_1(\mathcal{L},\mathcal{B})$ small but nonzero.

6. Appendix

We are going to discuss an alternative description of $\text{deg}(\phi)$ taken from Atiyah and Singer [3] Section 9] which works also for maps $\phi$ that are only continuous and compare it with Fedosov’s approach used in this paper. This sheds some light on the construction of families of invertible matrices with a given degree.

The left hand side of (5.8) can be computed in a different way. This is done in Section 9 using generators of $H^s(U(r); \mathbb{Z})$ which transgress to the universal Chern classes in $H^*(BU(r))$. Without going into details, which, by the way, are similar to the proof of Theorem 4.0.3 the conclusions are as follows:

As before, let $V = S^q \times S(\Gamma)$ and $v = q/2 + n - 1$. Using $U(r)/U(r-1) = S^{2r-1}$ it is easy to see that, if $r \geq v$, every continuous map $\phi : V \to U(r)$ is homotopic to one of the form $\text{diag}(\phi', \text{Id}_{r-2v+1})$, where $\phi'$ takes its values in $U(v)$. Taking the first column $\phi'_1$ of $\phi'$ we obtain a map $\phi'_1 : V \to S^{2v-1}$ which, being a map
of oriented manifolds of the same dimension, has a well defined Brouwer’s degree 
\( \deg_B \phi'_1 \in \mathbb{Z} \).

It turns out that \( \deg_B \phi'_1 \) is divisible by \((v - 1)!\).

Define
\[
(6.1) \quad \deg' (\phi) = \begin{cases} 
\frac{1}{(v-1)!} \deg_B \phi'_1 & \text{if } r \geq v \\
0 & \text{if } r < v.
\end{cases}
\]

It is shown in [6, corollary(9.5)] that
\[
(6.2) \quad \deg' (\phi) = < \text{ch} ([\phi]); [S^q \times S(\Gamma)] >
\]
(the statement in Corollary (9.5) is in the case without parameters but the proof with parameters is the same).

Since the direct image homomorphism \( f_* \) commutes with composition of maps and since \( cte_* \) coincides with the evaluation on the fundamental class one easily verifies that
\[
(6.3) \quad < \text{ch} ([\phi]); [S^q \times S(\Gamma)] > = < p_* \text{ch} [\phi]; [S^q] >.
\]

The above relation, together with (6.1) and (6.2), allows us to conclude that if \( \phi \) is smooth, then \( \deg' (\phi) = \deg (\phi) \). In particular \( \deg' (\phi) \) extends \( \deg (\phi) \) to all continuous maps from \( S^q \times S(\Gamma) \) to \( GL(r; \mathbb{C}) \).

As a side remark let us observe that the use of to above extension of the degree applied to the map \( \sigma \) allows us relax assumption \( H'_{3} \) in Theorem 2.0.1 to:

\( H'_{\lambda} \) - The restrictions to \( \partial \Omega \) of the coefficients of the leading terms of \( L_{\lambda} (x,D) \) are independent of \( \lambda \).

Indeed, we have only to improve Step 1 of the proof of Theorem 2.0.1. Under assumption \( H'_{3} \), the construction of \( \sigma \) in (2.5) gives only a continuous map. However, using
\[
< \text{ch} \text{Ind} (L^c, B^c); [S^q] > = < p_* \text{ch} [\sigma]; [S^q] > = \deg' (\sigma)
\]
and arguing as in the proof of Theorem 1.4.1 in [17] we can show that \( J (\text{Ind} L) \neq 0 \) whenever \( \deg' (\sigma) \) is not divisible by \( n(q) \).

On the negative side, let us observe that, while \( \deg \) is given explicitly by an integral of a differential form, the definition of \( \deg' \) in (6.1) is far less explicit. Notice also that we could define \( \deg' \) using approximations of continuous maps by the smooth ones, as is often done for the Brouwer degree.

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