Research Article

On the Control of the 2D Navier–Stokes Equations with Kolmogorov Forcing

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This paper is devoted to the control problem of a nonlinear dynamical system obtained by a truncation of the two-dimensional (2D) Navier–Stokes (N-S) equations with periodic boundary conditions and with a sinusoidal external force along the x-direction. This special case of the 2D N-S equations is known as the 2D Kolmogorov flow. Firstly, the dynamics of the 2D Kolmogorov flow which is represented by a nonlinear dynamical system of seven ordinary differential equations (ODEs) of a laminar steady state flow regime and a periodic flow regime are analyzed; numerical simulations are given to illustrate the analysis. Secondly, an adaptive controller is designed for the system of seven ODEs representing the approximation of the dynamics of the 2D Kolmogorov flow to control its dynamics either to a steady-state regime or to a periodic regime; the value of the Reynolds number is determined using an update law. Then, a static sliding mode controller and a dynamic sliding mode controller are designed for the system of seven ODEs representing the approximation of the dynamics of the 2D Kolmogorov flow to control its dynamics either to a steady-state regime or to a periodic regime. Numerical simulations are presented to show the effectiveness of the proposed three control schemes. The simulation results clearly show that the proposed controllers work well.

1. Introduction

In this work, we study the dynamics as well as the adaptive and the sliding mode control problem of seven-mode truncation system of the 2D Navier–Stokes equations with periodic boundary conditions and a sinusoidal external force along the x-direction. This type of forcing is known as the Kolmogorov forcing and the resulting flow is known as the 2D Kolmogorov flow.

In 1958, Kolmogorov [1] introduced the 2D Kolmogorov flow as an example to study transition to turbulence. The model has been successfully used to study 2D turbulent flows in atmospheric, oceanic, and astrophysical flows due to the weak dependence of the velocity field on the third dimension [2, 3]. In the field of magnetohydrodynamics, Kolmogorov flow has been extensively used and was easily reproduced by suitably placed electrical and magnetic fields. Bondarenko et al. [4] observed this flow inside a specific electrically conducting fluid driven by an electromagnetic field [5]. In the literature, linear and nonlinear stability analysis of this flow was investigated for different domain sizes and forcing wave numbers [6–9]. Numerical simulations and investigations helped in the advancement of the understanding of Kolmogorov flow. In particular, it has been shown that the Kolmogorov flow dynamics exhibits complex structures transforming periodic states to chaotic attractors through a sequence of bifurcations including period doubling, period tripling and gluing bifurcations [10–15]. Some of these structures were realized in experimental laboratories [4, 16].

In the last four decades, several reduced order models that approximate the dynamics of the 2D Kolmogorov flow were constructed using the Fourier Galerkin approach [10, 11, 17–25]. Franceschini and Tebaldi [21] constructed a five-mode truncation ODE system of the 2D Kolmogorov flow when the external force acts on the mode (2, −1). In [21], a number of steady states and Hopf bifurcations were observed up to a Reynolds number equal to 50. Later on, in 1987, She [12] investigated the metastability and vortex
pairing of Kolmogorov flow when the external force along the x-direction acts on the mode \((0, 8)\). Also, Nicolaenko and She [14] studied the dynamics of the coherent structures, homoclinic cycles, and vorticity explosion in Kolmogorov flow. In 1997, using the Karhunen–Loève decomposition and symmetry, Smaoui and Armbruster [11] used a computationally effective method to construct a reduced order system of nonlinear ODEs that approximates the dynamics of Kolmogorov flow when the external force acts on the mode \((0, 2)\). One decade later, Chen [23] and Chen et al. [24, 25] obtained a reduced order system of ODEs when the force acts on the mode \((0, 4)\).

Recently, the control problem of nonlinear PDEs with periodic boundary conditions has been the subject of many research studies [26–36]. Because of the infinite dimensional nature of these PDEs, the practical implementation of such controllers is a very difficult task. As a consequence, attempts were made to approximate these PDEs based on ODE approximations. The idea of inertial manifold to obtain such reduced systems of ODEs was introduced by Foias et al. [37]. Other efforts to construct systems of ODEs that capture the dynamics of the original PDEs were made by other researchers [26–29, 38–42]. Smaoui and Zribi [26–28] constructed reduced order ODE systems that approximate the dynamics of the 2D Navier–Stokes equations using the truncated Fourier expansion method when the external force along the x-direction acts on the mode \((0, k)\). Moreover, Smaoui [29] derived a seven-mode truncation system of ODEs and proposed controllers for its dynamics using Lyapunov based controllers. Extensive numerical simulations were presented to show the different behavior of Kolmogorov flow for Reynolds number range \(0 < Re < 26.41\), and Lyapunov-based controllers were designed to control the dynamics of the system of ODEs to different attractors. Although the control problem of parabolic PDEs has been investigated, the control problem of the different finite dimensional approximations of the 2D Kolmogorov flow is not completely investigated.

The main contribution of this paper is the design of an adaptive controller as well as a static and a dynamic sliding mode controllers to control the dynamics of the seven-mode truncation ODEs system of the 2D Navier–Stokes equations. The seven-mode truncation ODEs’ system was completely derived by Smaoui [29]. This ODEs’ system is the lowest dimensional system obtained so far that captures the dynamics of the 2D Navier–Stokes equations with sinusoidal external force \(f = (a^2 \sin ay, 0)\), where \(a = 2\). We should emphasize here that that the design of such controllers for this well-known partial differential equation has not been treated elsewhere in the literature. First, the dynamics of this 2D Navier–Stokes equation described by a laminar steady-state regime and a periodic flow regime is briefly analyzed. Then, an adaptive control law and a static and a dynamic sliding mode control laws are designed and applied to the system of ODEs to control its dynamics either to a steady state or to a periodic state. It should be noted that other types of controllers for nonlinear systems such as observed-based finite-time tracking sliding mode control, output feedback active suspension control, robust \(H^\infty\) sliding mode control, and adaptive dynamic programming-based decentralized sliding mode control were explored by different investigators from various disciplines [43–51].

The paper is organized as follows. In Section 2, the 2D Kolmogorov flow equations and their Fourier Galerkin approximation described by a seventh-order nonlinear ODE system are presented. The dynamics of the reduced order ODE system described by a laminar steady-state regime and a periodic flow regime is described in Section 3. Section 4 presents the design of an adaptive control law which is used to regulate the states of the reduced order ODE system to a desired fixed state or to a periodic state without the knowledge of the Reynolds number. Section 5 presents sliding mode control laws to control the dynamics of the steady state and periodic regimes. The theoretical developments are verified by numerical simulations. Specifically, a static as well as a dynamic sliding mode controllers are proposed for the system of ODEs. Finally, some concluding remarks are given in Section 6.

### Complexity

#### 2. A Seven-Mode Truncation O.D.E System of the 2D Kolmogorov Flow

The “basic 2D Kolmogorov flow” \(\mathbf{u} = (\alpha \sin ay, 0)\) was introduced by Kolmogorov [1] as an example to study transition to turbulence. This flow is the solution of the 2D Navier–Stokes equations:

\[
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

with force \(\mathbf{f} = (\alpha^2 \sin ay, 0)\) and with periodic boundary conditions in two directions \(0 \leq x, y \leq 2\pi\). The kinematic viscosity is \(\nu = (1/Re)\), where \(Re\) is the Reynolds number and the pressure is \(p\).

A system of ODEs can be derived from the Navier–Stokes equations (1) and (2) for \(\alpha = 2\) by expanding, and \(\mathbf{f}\) using the following Fourier expansion forms:

\[
\mathbf{u}^i(x) = \sum_{k \neq 0} \exp(ikx)\gamma_k^i \frac{k^i}{|k|},
\]

\[
p(x) = \sum_{k \neq 0} \exp(ikx)p_k \frac{k^i}{|k|},
\]

\[
\mathbf{f}^i(x) = \sum_{k \neq 0} \exp(ikx)f_k \frac{k^i}{|k|},
\]

where \(k = (k_1, k_2)\) is a wave vector with integer components, \(k^i = (k_2, -k_1)\), and the reality condition \(\gamma_k = -\gamma_{-k}\) must hold.

The equation for \(\gamma_k\) is:

\[
\dot{\gamma}_k = -j \sum_{k_1 + k_2 + k_3 = 0} \frac{(k_1^2 - k_2^2)}{2|k||k_1||k_2||k_3|} \gamma_{k_1} \gamma_{k_2} - \nu |k|^2 \gamma_k + f_k,
\]

(4)
where \( y_\mathbf{k} = -7\mathbf{k} \) and \( f_\mathbf{k} \) is the component of \( f \) with respect to \((\mathbf{k}/|\mathbf{k}|)\exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x})\). In [29], the following system of seven ODEs was constructed by considering the set of vectors, \( \mathbf{k}_1 = (1, 1), \mathbf{k}_2 = (0, 3), \mathbf{k}_3 = (0, 2), \mathbf{k}_4 = (1, 2), \mathbf{k}_5 = (0, 1), \mathbf{k}_6 = (0, 0), \) and \( \mathbf{k}_7 = (1, -2), \) and their negatives in equation (4):

\[
\begin{align*}
\dot{x}_1 &= -2x_1 - 4x_4 x_3 + 4x_2 x_7, \\
\dot{x}_2 &= -9x_2 + 3x_1 x_7, \\
\dot{x}_3 &= -4x_3 - 4\sqrt{2} x_4 x_6 + 4\sqrt{2} x_7 x_7 + 4R_e, \\
\dot{x}_4 &= -5x_4 + x_1 x_5 + 3\sqrt{2} x_3 x_5, \\
\dot{x}_5 &= -x_5 + 3x_1 x_4 - \sqrt{3} x_3 x_6, \\
\dot{x}_6 &= -x_6 + \sqrt{2} x_1 x_5 + \sqrt{2} x_3 x_4 - \sqrt{2} x_3 x_7, \\
\dot{x}_7 &= -5x_7 - 7x_1 x_2 - 3\sqrt{2} x_3 x_6. 
\end{align*}
\]

(5)

**Remark 1.** It can be easily checked that system (5) is invariant under the following symmetries:

\[
\begin{align*}
r_x: \quad & (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \rightarrow (x_1, -x_2, x_3, -x_4, -x_5, x_6, -x_7), \\
r_y: \quad & (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \rightarrow (-x_1, x_2, x_3, x_4, -x_5, x_6, x_7), \\
r_o: \quad & (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \rightarrow (-x_1, x_2, x_3, -x_4, x_5, -x_6, -x_7).
\end{align*}
\]

(6)

where \( r_x, r_y, \) and \( r_o \) are reflection symmetries across the \( x \)-axis, the \( y \)-axis, and the origin, respectively. Therefore, it can be concluded that \( r_x, r_y, r_o \) with the identity transformation \( i \) form an Abelian group: \( G = \{r_x, r_y, r_o, i\} \).

3. The Laminar Regime of the Kolmogorov Flow

In this section, we analyze the dynamics of the seven-mode truncation ODE system given by (5) when the Reynolds number \( R_e = 20 \) and \( R_e = 24 \) corresponding to a steady-state laminar regime and a periodic regime, respectively.

At \( R_e = 20 \), numerical simulations of system (5) shows that the system has seven fixed points. These points are classified as four asymptotically stable points and three unstable points. Figure 1 presents the phase plane of the system for four different initial conditions, and Figure 2 depicts the vorticity \( \omega = \nabla \times \mathbf{u} \) of the corresponding four asymptotically stable fixed points. It can be verified that these fixed points can also be generated by applying the symmetries described in Section 2.

At \( R_e = 24 \), the numerical simulations show the existence of four stable periodic orbits arising from a Hopf bifurcation at \( R_e = 20.845 \) that remain stable up to \( R_e = 26.323 \) (see Figure 3). Figure 4 presents the vorticity corresponding to one of the four periodic regimes at different times. It can be checked that the other stable periodic orbits can also be generated by applying the symmetries described in Section 2.

4. An Adaptive Controller of the 2D Kolmogorov Flow

In this section, we design an adaptive controller to control different dynamics of the seven-mode truncation ODE system given by (5).

4.1. Dynamic Model of the Error System. The model of the first seventh-order ODE system, the master system, is

\[
\begin{align*}
\dot{x}_1 &= -2x_1 - 4x_4 x_3 + 4x_2 x_7, \\
\dot{x}_2 &= -9x_2 + 3x_1 x_7, \\
\dot{x}_3 &= -4x_3 - 4\sqrt{2} x_4 x_6 + 4\sqrt{2} x_7 x_7 + 4R_e, \\
\dot{x}_4 &= -5x_4 + x_1 x_5 + 3\sqrt{2} x_3 x_5, \\
\dot{x}_5 &= -x_5 + 3x_1 x_4 - \sqrt{3} x_3 x_6, \\
\dot{x}_6 &= -x_6 + \sqrt{2} x_1 x_5 + \sqrt{2} x_3 x_4 - \sqrt{2} x_3 x_7, \\
\dot{x}_7 &= -5x_7 - 7x_1 x_2 - 3\sqrt{2} x_3 x_6.
\end{align*}
\]

(7)

The model of the second seventh-order ODEs system, the slave system, is
\[ \begin{align*}
\dot{y}_1 &= -2y_4 - 4y_5y_7 + 4y_6 + u_4, \\
\dot{y}_2 &= -9y_1 + 3y_1y_7, \\
\dot{y}_3 &= 4y_3 - 4\sqrt{2}y_4y_6 + 4\sqrt{2}y_6 + 4R_2 + u_2, \\
\dot{y}_4 &= -5y_4 + y_1y_5 + 3\sqrt{2}y_4y_6, \\
\dot{y}_5 &= -y_5 + 3y_1y_4 - \sqrt{2}y_1y_6 + u_3, \\
\dot{y}_6 &= -y_6 + \sqrt{2}y_1y_5 + \sqrt{2}y_3y_4 - \sqrt{2}y_3y_7 + u_4, \\
\dot{y}_7 &= -5y_2 - 7y_1y_2 - 3\sqrt{2}y_3y_7.
\end{align*} \tag{8} \]

Note the addition of controllers \( u_1, \ldots, u_4 \) in the dynamic model of the slave system. These controllers will be designed to force the states of the slave system to follow the states of the master system.

Define the errors as the subtraction of the states of the master system from the states of the slave system, which can be written as follows:

\[ \begin{align*}
e_1 &= y_1 - x_1, \\
e_2 &= y_2 - x_2, \\
e_3 &= y_3 - x_3, \\
e_4 &= y_4 - x_4, \\
e_5 &= y_5 - x_5, \\
e_6 &= y_6 - x_6, \\
e_7 &= y_7 - x_7.
\end{align*} \tag{9} \]

Using equations (7)–(9), the dynamical model of the error system can be written as follows:

**Figure 2:** The vorticity of each of the four asymptotically stable fixed point corresponding to the laminar regime at \( R_c = 20 \) with different initial conditions: (a) \( x_0^T = (-1, 0.5, 0.7, 2.5, 0.1, 3.5, -0.4) \); (b) \( x_0^2 = (1, -0.5, 0.7, 2.5, -0.1, 3.5, -0.4) \); (c) \( x_0^3 = (-1, -0.5, 0.7, -2.5, -0.1, -3.5, 0.4) \); (d) \( x_0^4 = (1, 0.5, 0.7, -2.5, 0.1, -3.5, 0.4) \).

**Figure 3:** Phase portraits of the four stable periodic orbits at \( R_c = 24 \) arising from a Hopf bifurcation at \( R_c = 20.845 \) that remains stable for up to \( R_c = 26.323 \).

**Figure 4:** Time evolution of the vorticity at different times of one of the four laminar regimes at \( R_c = 24 \).
\[
\begin{align*}
\dot{e}_1 &= -2e_1 - 4e_4 e_5 - 4x_3 e_4 - 4x_4 e_5 + 4e_2 e_7 + 4x_2 e_2 + 4x_2 e_7 + u_1, \\
\dot{e}_2 &= -9e_3 + 3e_1 e_7 + 3x_2 e_1 + 3x_1 e_7, \\
\dot{e}_3 &= -4e_1 - 4\sqrt{2}e_4 e_6 - 4\sqrt{2}x_2 e_4 - 4\sqrt{2}x_1 e_6 + 4\sqrt{2}e_6 e_7 + 4\sqrt{2}x_1 e_6 \\
&\quad\quad + 4\sqrt{2}x_2 e_7 + 4R + u_2, \\
\dot{e}_4 &= -5e_4 + e_1 e_5 + x_2 e_1 + x_1 e_5 + 3\sqrt{2}e_1 e_6 + 3\sqrt{2}x_2 e_3 + 3\sqrt{2}x_3 e_6, \\
\dot{e}_5 &= -e_3 + 3e_4 e_6 + 3x_2 e_1 - 3\sqrt{2}e_1 e_6 - 3\sqrt{2}x_2 e_6 + 3\sqrt{2}e_6 e_7, \\
\dot{e}_6 &= -e_3 + \sqrt{5}e_1 e_6 + \sqrt{5}x_2 e_1 + \sqrt{5}x_1 e_6 + \sqrt{2}e_3 e_4 + \sqrt{2}x_4 e_3 + \sqrt{2}x_3 e_4 \\
&\quad\quad - \sqrt{2}x_2 e_7 - \sqrt{2}x_2 e_3 - \sqrt{2}x_3 e_7 + u_3, \\
\dot{e}_7 &= -5e_2 - 7e_1 e_2 - 7x_2 e_1 - 7x_1 e_2 - 3\sqrt{2}e_3 e_6 - 3\sqrt{2}x_2 e_6 - 3\sqrt{2}x_3 e_6,
\end{align*}
\]

where \( R = R_e - R_c \).

4.2. An Adaptive Controller for the 2D Kolmogorov Flow.

In this section, an adaptive-based controller is designed to drive the states of the system in (8) to asymptotically converge to the states of the system in (7) without knowledge of the value of the Reynolds number.

\[
\begin{align*}
\dot{u}_1 &= 4e_1 e_5 + 4x_3 e_4 + 4x_4 e_5 - 4e_2 e_7 - 4x_2 e_7 - 4x_2 e_3 - \frac{3b_2}{b_1} e_2 e_7 \\
&\quad\quad - 3 \frac{b_2}{b_1} x_2 e_2 - \frac{b_4}{b_1} e_4 e_5 - \frac{b_4}{b_1} x_5 e_4 + 7 \frac{b_2}{b_1} e_3 e_7 + 7 \frac{b_2}{b_1} x_3 e_7 - \alpha_1 e_1, \\
\dot{u}_2 &= 4\sqrt{2}e_4 e_6 + 4\sqrt{2}x_2 e_4 + 4\sqrt{2}x_4 e_6 - 4\sqrt{2}e_6 e_7 - 4\sqrt{2}e_6 e_2 - 4\sqrt{2}x_4 e_6 \\
&\quad\quad - 3\sqrt{2} \frac{b_2}{b_3} e_6 e_6 - 3\sqrt{2} \frac{b_3}{b_3} x_2 e_6 + 3\sqrt{2} \frac{b_3}{b_3} e_6 e_7 + 3\sqrt{2} \frac{b_3}{b_3} x_3 e_7 - 4R - \alpha_2 e_3, \\
\dot{u}_3 &= -3e_1 e_4 - 3x_4 e_1 - 3x_1 e_4 + \sqrt{5}e_1 e_6 + \sqrt{5}x_2 e_1 + \sqrt{5}x_1 e_6 - \frac{b_4}{b_2} x_2 e_4 - \alpha_3 e_5, \\
\dot{u}_4 &= -\sqrt{5}e_1 e_5 - \sqrt{5}x_3 e_1 - \sqrt{5}x_2 e_5 - \sqrt{2}e_5 e_4 - \sqrt{2}x_4 e_5 - \sqrt{2}x_3 e_4 \\
&\quad\quad + \sqrt{2}x_3 e_7 + \sqrt{2}x_2 e_3 + \sqrt{2}x_3 e_7 - 3\sqrt{2} \frac{b_3}{b_6} x_3 e_4 + 3\sqrt{2} \frac{b_3}{b_6} x_3 e_7 - \alpha_4 e_6,
\end{align*}
\]

with

\[
\dot{R} = \frac{d\hat{R}}{dt} = \frac{4b_1}{b_6} e_3, \quad (13)
\]

to the error model given by the set of ODEs (10) guarantees the convergence of the errors \( e_i \) (i = 1, \ldots, 7) to zero as \( t \) tends to infinity.

Proof. Define the parameter error \( e_R \) as \( e_R = \hat{R} - R \); then, \( \dot{e}_R = \dot{\hat{R}} \).

Let the gains \( \alpha_i \) (i = 1, \ldots, 4) be positive scalars. Also, let the control gains \( \beta_i \) (i = 1, \ldots, 7) be positive scalars such that

\[
3\beta_2 - 7\beta_7 = 0. \quad (11)
\]

Also, define the estimate of the parameter \( R \) as \( \hat{R} \).

The control scheme is given by the following theorem.

**Theorem 1.** The application of the adaptive controller,

\[
V_1 = \frac{1}{2} \left[ b_1 e_1^2 + b_2 e_2^2 + b_3 e_3^2 + b_4 e_4^2 + b_5 e_5^2 + b_6 e_6^2 + b_7 e_7^2 \right],
\]

Now, consider the Lyapunov function candidate \( V_1 \) such that

\[
\dot{V}_1 = \frac{1}{2} \left[ b_1 e_1^2 + b_2 e_2^2 + b_3 e_3^2 + b_4 e_4^2 + b_5 e_5^2 + b_6 e_6^2 + b_7 e_7^2 \right].
\]

Using the model of the error system given by (10), the control law given by (11), (12), and the update law of the parameter given by (13), the derivative of \( V_1 \) with respect to time is such that
\[ \dot{V}_1 = b_1 e_1^2 + b_2 e_2^2 + b_3 e_3^2 + b_4 e_4^2 + b_5 e_5^2 + b_6 e_6^2 + b_7 e_7^2 + b_8 e_8^2 \]
\[ = b_1 e_1 \left[-2 e_1 - 4 e_4 e_5 - 4 x_5 e_4 - 4 x_4 e_5 + 4 e_4 e_7 + 4 x_5 e_7 + 4 x_2 e_7 + u_1 \right] \]
\[ + b_2 e_2 \left[-9 e_2 + 3 e_4 e_5 + 3 x_5 e_4 + 3 x_4 e_5 + b_3 e_3 \right] \]
\[ + b_4 e_4 \left[-5 e_4 + e_5 + x_5 e_5 + 3 \sqrt{2} e_5 e_6 + 3 \sqrt{2} x_6 e_5 + 3 \sqrt{2} x_5 e_6 \right] \]
\[ + b_5 e_5 \left[-e_5 + 3 e_4 e_6 + 3 x_6 e_4 + 3 x_4 e_6 - \sqrt{5} e_1 e_6 - \sqrt{5} x_6 e_1 - \sqrt{5} x_1 e_6 + u_1 \right] \]
\[ + b_6 e_6 \left[-e_6 + \sqrt{5} e_4 e_5 + \sqrt{5} x_5 e_4 + \sqrt{5} x_4 e_5 + \sqrt{2} e_5 e_4 + \sqrt{2} x_4 e_5 + \sqrt{2} x_5 e_4 - \sqrt{2} e_7 e_5 - \sqrt{2} x_7 e_5 + u_4 \right] \]
\[ + b_7 e_7 \left[-5 e_7 - 7 e_4 e_7 - 7 e_4 e_7 - 3 \sqrt{2} e_6 e_7 - 3 \sqrt{2} x_6 e_7 - 3 \sqrt{2} x_7 e_6 + 4 b_7 e_3 e_7 \right] \]
\[ (15) \]

or

\[ \dot{V}_1 = -2 b_1 e_1^2 - 9 b_2 e_2^2 - 4 b_3 e_3^2 - 5 b_4 e_4^2 - 6 b_5 e_5^2 - 5 b_6 e_6^2 - 5 b_7 e_7^2 - 5 b_8 e_8^2 - 5 b_9 e_9^2 - a_1 b_1 e_1^2 - a_2 b_2 e_2^2 - a_3 b_3 e_3^2 - a_4 b_4 e_4^2 - a_5 b_5 e_5^2 - a_6 b_6 e_6^2 \]  
\[ (16) \]

Since the design parameters are positive scalars, then it is concluded that the Lyapunov function \( V_1 \) defined by equation (14) is positive definite and its derivative \( \dot{V} \) is negative semidefinite. Also, since \( e_1, \ldots, e_7 \) and \( e_R \) are bounded, then invoking Barbalat’s lemma, the error functions in (9) asymptotically converge to zero as \( t \) tends to infinity.

Therefore, it can be concluded that the states of system (8) asymptotically converge to the states of system (7) as \( t \) tends to infinity.

Numerical simulations were carried out for the proposed adaptive controller. The values of the control gains \( a_1, \ldots, a_4 \) are taken to be \( a_1 = 50, a_2 = 50, a_3 = 50, \) and \( a_4 = 50 \). The values of the control gains \( b_1 = 10, b_2 = 7, b_3 = 10, b_4 = 1, b_5 = 10, b_6 = 10, b_7 = 3, \) and \( b_8 = 10 \). The simulation results for the master system with \( R_s = 20 \) corresponds to an initial condition \( x(0) = [-1, 0.5, 0.7, 2.5, 0.1, 3.5, -0.4]^T \) and for the slave system with \( R_s = 24 \) corresponds to an initial condition \( x(0) = [0.67, -0.32, 0.59, 2.55, 0.165, 4.31, -1.27]^T \). At the beginning of the simulations, the controllers \( u_1, u_2, u_3, \) and \( u_4 \) are set to zero for the first 30 seconds. Then, the control law given by (12) is switched on.

The simulation results of system (8) with the proposed control scheme given by (12) and (13) when the Reynolds number \( R_s = 20 \) and \( R_s = 24 \) are presented in Figures 5 and 6, respectively. Note that the dynamics at \( R_s = 20 \) corresponds to a steady state regime and for \( R_s = 24 \) corresponds to a periodic regime.

Figures 5 and 6 depict the simulation results for the dynamics of the slave system before and after the control is switched on. Figure 5 shows how the control drags the dynamics of a periodic flow regime of the slave system to a steady state fixed point flow regime of the master system. On the contrary, Figure 6 shows how the dynamics of a steady state fixed point flow regime of the slave system is dragged into the dynamics of a periodic flow regime of the master system.

Figure 7 presents the case when both the master system and the slave system are simulated with the same Reynolds number \( R_s = 24 \) but with different initial conditions (i.e., \( x_0 = [0.67, -0.32, 0.59, 2.55, 0.165, 4.31, -1.27]^T \) for the master system, and \( x_0 = [-0.67, -0.32, 0.59, -2.55, 0.165, -4.31, 1.27]^T \) for the slave system). The values of the control gains used in this case are the same used in the previous two cases. Figure 7 shows how a periodic regime can be dragged into another symmetric periodic regime.

Therefore, it can be concluded that the numerical simulations clearly show that the proposed adaptive controller is able to force the states of the slave system to converge to the states of the master system even though the exact value of \( R_s \) is not known.

5. Sliding Mode Controllers for the 2D Kolmogorov Flow

In this section, sliding mode controllers are proposed to control the dynamics of the seven-mode truncation ODE system presented in (5). The choice of this type of controllers is motivated by the fact that sliding mode controllers are known for their robustness and their insensitivity to modelling errors [43–51].

Recall from the previous section that the model of the error system is as follows:
Figure 5: Adaptive control of the periodic regime of the slave system at $R_e = 24$ to a steady state regime. The controller is switched on at time $t = 30$. (a) The $L_2$-norm $\|E\| = \sqrt{\sum_{i=1}^{7} |y_i|^2}$ vs. time. (b) The state $y_7(t)$ of the slave system vs. time. (c) The phase portrait of the states of the slave system $y_7(t)$ vs. $y_1(t)$. (d) Time evolution of $\hat{R}$, the estimate of $R = R_e - R_i$.

Figure 6: Continued.
Figure 6: Adaptive control of the fixed point regime of the slave system at $R_e = 20$ to a periodic state regime. The controller is switched on at time $t = 30$. (a) The $L_2$-norm $\|E\| = \sqrt{\sum_{i=1}^{n} |yi|^2}$ vs. time. (b) The state $y_7(t)$ of the slave system vs. time. (c) The phase portrait of the states of the slave system $y_7(t)$ vs. $y_1(t)$. (d) Time evolution of $\hat{R}$, the estimate of $R = R_e - R_{\nu}$. 

Figure 7: Adaptive control of the periodic regime of the slave system at $R_e = 24$ to a symmetrized periodic state regime at the same Reynolds number. The controller is switched on at time $t = 30$. (a) The $L_2$-norm $\|E\| = \sqrt{\sum_{i=1}^{n} |yi|^2}$ vs. time. (b) The state $y_7(t)$ of the slave system vs. time. (c) The phase portrait of the states of the slave system $y_7(t)$ vs. $y_1(t)$. (d) Time evolution of $\hat{R}$ and the estimate of $R = R_e - R_{\nu}$. 

Complexity
\[ \dot{e}_1 = -2e_1 - 4e_4e_5 - 4x_5e_4 - 4x_4e_5 + 4e_2e_5 + 4x_7e_2 + 4x_2e_7 + u_1, \]
\[ \dot{e}_2 = -9e_2 + 3e_1e_7 + 3x_7e_1 + 3x_1e_7, \]
\[ \dot{e}_3 = -4e_3 - 4\sqrt{2}e_4e_6 - 4\sqrt{2}x_6e_4 - 4\sqrt{2}x_4e_6 + 4\sqrt{2}e_6e_7 + 4\sqrt{2}x_7e_6 + 4\sqrt{2}x_6e_7 + 4R + u_2, \]
\[ \dot{e}_4 = -5e_4 + e_1e_5 + x_5e_1 + x_1e_5 + 3\sqrt{2}e_3e_6 + 3\sqrt{2}x_6e_5 + 3\sqrt{2}x_5e_6, \]
\[ \dot{e}_5 = -e_5 + 3e_1e_4 + 3x_4e_1 + 3x_1e_4 - \sqrt{5}e_1e_6 - \sqrt{5}x_4e_1 - \sqrt{5}x_1e_6 + u_3, \]
\[ \dot{e}_6 = -e_6 + \sqrt{5}e_5e_5 + \sqrt{5}x_5e_5 + \sqrt{5}x_6e_5 + \sqrt{2}e_3e_4 + \sqrt{2}x_3e_3 + \sqrt{2}x_3e_4 \]
\[ - \sqrt{2}e_5e_7 - \sqrt{2}x_5e_7 + \sqrt{2}x_3e_7 + u_4, \]
\[ \dot{e}_7 = -5e_7 - 7x_1e_7 - 7x_7e_7 - 3\sqrt{2}e_3e_6 - 3\sqrt{2}x_5e_3 - 3\sqrt{2}x_5e_6. \]  

where \( R = R_y - R_x \). We will design a static as well as a dynamic sliding mode controllers to force the states of the slave system given by (8) to the states of the master system given by (7).

5.1. A Static Sliding Mode Controller. Define the sliding surfaces \( S_i \) \((i = 1, \ldots, 4)\) such as

\[
S_1 = e_1, \\
S_2 = e_3, \\
S_3 = e_5, \\
S_4 = e_6.
\]  

and define the signum function as follows:

\[
\text{sgn}(S) = \begin{cases} 
1, & \text{if } S > 0, \\
0, & \text{if } S = 0, \\
-1, & \text{if } S < 0.
\end{cases}
\]

Let the control gains \( \alpha_i \) \((i = 1, \ldots, 4)\) and \( \Gamma_i \) \((i = 1, \ldots, 4)\) be positive scalars. The control scheme is introduced by the following theorem.

**Theorem 2.** The sliding mode control law

\[
u_1 = 2e_1 + 4e_4e_5 + 4x_5e_4 + 4x_4e_5 - 4e_2e_7 - 4x_7e_2 - 4x_2e_7 - \alpha_1e_1 - \Gamma_1\text{sgn}(e_1),
\]
\[
u_2 = 4e_3 + 4\sqrt{2}e_4e_6 + 4\sqrt{2}x_6e_4 + 4\sqrt{2}x_4e_6 + 4\sqrt{2}e_6e_7 - 4\sqrt{2}x_7e_6 - 4\sqrt{2}x_6e_7
- 4R - \alpha_3e_3 - \Gamma_3\text{sgn}(e_3),
\]
\[
u_3 = e_5 - 3e_1e_4 - 3x_4e_1 - 3x_1e_4 + \sqrt{5}e_1e_6 + \sqrt{5}x_4e_1 + \sqrt{5}x_1e_6 - \alpha_3e_5 - \Gamma_3\text{sgn}(e_5),
\]
\[
u_4 = e_6 + \sqrt{5}e_5e_5 + \sqrt{5}x_5e_5 + \sqrt{5}x_6e_5 - \sqrt{2}e_3e_4 - \sqrt{2}x_3e_3 - \sqrt{2}x_3e_4
+ \sqrt{2}e_5e_7 + \sqrt{2}x_5e_7 + \sqrt{2}x_3e_7 - \alpha_4e_6 - \Gamma_4\text{sgn}(e_6),
\]

when applied to the error system (17) guarantees the convergence of the errors \( e_i \) \((i = 1, \ldots, 7)\) to zero.

**Proof.** Taking the time derivatives of \( S_1, \ldots, S_4 \) along the trajectories of the errors given by (17) and applying the controllers given by (20), we obtain

\[
\ddot{S}_i = -\alpha_iS_i - \Gamma_i\text{sgn}(S_i), \quad i = 1, \ldots, 4.
\]  

Let the Lyapunov function candidate \( V_2 \) be such that

\[
V_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2 + S_4^2).
\]

Using the control law given by (20) and the error model given in (17), the derivative of \( V_2 \) with respect to time is such that

\[
\dot{V}_2 = S_1\dot{S}_1 + S_2\dot{S}_2 + S_3\dot{S}_3 + S_4\dot{S}_4 = \sum_{i=1}^{4} S_i(-\alpha_iS_i - \Gamma_i\text{sgn}(S_i)) = \sum_{i=1}^{4} (-\alpha_iS_i^2 - \Gamma_i|S_i|).
\]

Therefore, \( \dot{V}_2 < 0 \) for \( S_i \neq 0 \), for \( i = 1, \ldots, 4 \). Hence, the trajectories associated with the discontinuous dynamics given by (21) converge to zero from any initial condition in a finite time given that \( \alpha_i \) and \( \Gamma_i \) \((i = 1, \ldots, 4)\) are chosen to be...
sufficiently large and positive scalars. It should be noted that since
\[ \dot{V}_2 \leq -\sum_{i=1}^{4} \Gamma_i |S_i|, \]  
(24)
where \( \Gamma_i \) (\( i = 1, \ldots, 4 \)) are positive scalars; then, the above inequality is sufficient to ensure the finite time attractiveness of the sliding surfaces \( S_i \) (\( i = 1, \ldots, 4 \)).

The reaching time \( t_{\text{reach}} \) is upper bounded by a function of \( S_i(0) \) (\( i = 1, \ldots, 4 \)) [52].

Therefore, it can be concluded from (18) that the errors \( e_1, e_2, e_3, e_5, \) and \( e_6 \) converge to zero in finite time.

After such a finite time, the errors’ equations of \( e_1, e_2, e_3, e_5, \) and \( e_6 \) can be written as
\[
\begin{align*}
\dot{e}_2 &= -9e_2 + 3x_1e_7, \\
\dot{e}_4 &= -5e_4, \\
\dot{e}_7 &= -5e_7 - 7x_1e_2.
\end{align*}
\]  
(25)

Define the vector of reduced errors such that the system of ODEs given by (25) can be written as
\[
\begin{bmatrix}
\dot{e}_2 \\
\dot{e}_4 \\
\dot{e}_7
\end{bmatrix} = 
\begin{bmatrix}
-9 & 0 & 3x_1 \\
0 & -5 & 0 \\
-7x_1 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
e_2 \\
e_4 \\
e_7
\end{bmatrix},
\]  
(26)
or
\[
\dot{e}_r(t) = A_r e_r(t),
\]  
(27)
where
\[
A_r = 
\begin{bmatrix}
-9 & 0 & 3x_1 \\
0 & -5 & 0 \\
-7x_1 & 0 & -5
\end{bmatrix}.
\]  
(28)

The characteristic polynomial is \( p(\lambda) = \det (A_r - \lambda I) = (\lambda + 5)(\lambda^2 +14\lambda + 45 + 21x_1^2) \). One of the roots of \( p(\lambda) \) is \(-5\). The other two roots are located in the left-half of the complex plane since the coefficients of \( \lambda^2 + 14\lambda + 45 + 21x_1^2 \) are always positive. Therefore, it can be concluded that the characteristic polynomial is Hurwitz and the matrix \( A_r \) is a stable matrix. Therefore, we can conclude that \( \lim_{t \to \infty} e_r(t) = 0 \).

Hence, it can be concluded that the errors \( e_1, e_2, e_3, e_5, \) and \( e_6 \) converge to zero in finite time, while the errors \( e_2, e_4, \) and \( e_7 \) converge to zero asymptotically.

Therefore, the proposed static sliding mode controller forces the states of the slave system given by (8) to converge to the states of the master system given by (7).

The performance of the proposed static sliding mode controller is simulated using the MATLAB software. Numerical simulations were carried out for the proposed sliding mode controller for three cases. The values of the control gains \( \alpha_1, \ldots, \alpha_4 \) are taken to be \( \alpha_1 = 50, \alpha_2 = 50, \alpha_3 = 50, \) and \( \alpha_4 = 50. \) The values of the control gains \( \Gamma_1, \ldots, \Gamma_4 \) are chosen to be \( \Gamma_1 = 30, \Gamma_2 = 40, \Gamma_3 = 30, \) and \( \Gamma_4 = 40. \) Three cases are simulated: for the first case, we choose \( R_e = 24, \) and the initial conditions \( x(0) = [0.67, -0.32, 0.59, 2.55, 0.165, 4.31, -1.27]^T \) for the master system in (7), and \( R_e = 20 \) and the initial condition \( y(0) = [-1, 0.5, 0.7, 2.5, 0.1, 3.5, -0.4]^T \) for the slave system in (8). The first state in this case corresponds to a periodic orbit, while the second state corresponds to an asymptotically stable orbit. For the second case, we choose \( R_e = 24 \) and the initial conditions \( x(0) = [1, 0.5, 0.7, -2.5, 0.1, -3.5, 0.4]^T \) for the master system in (7), and \( R_e = 20 \) and the initial conditions \( y(0) = [-1, 0.5, 0.7, 2.5, 0.1, 3.5, -0.4]^T \) for the slave system in (8) system. The two states in this case correspond to two symmetric asymptotically stable orbits. For the third case, we choose \( R_e = 24 \) and the initial condition \( x(0) = [0.67, -0.32, 0.59, 2.55, 0.165, 4.31, -1.27]^T \) for the master system in (7) and \( R_e = 24 \) and the initial condition \( y(0) = [-0.67, -0.32, 0.59, -2.55, 0.165, -4.31, 1.27]^T \) for the slave system in (8). Note that the two states in this case correspond to two symmetric stable periodic orbits.

Moreover, at the beginning of the simulations in each case, the controllers \( u_1, u_2, u_3, \) and \( u_4 \) are set to zero for the first 10 seconds. Then, the control law given by (20) is switched on to force system (8) to synchronize with system (7).

Figure 8 presents the simulation results for Case 1. Figure 8(a) depicts the \( L_2 \) norm of the error \( \| e \| = \sqrt{\sum_{i=1}^{7} \| x_i - y_i \|^2} \) versus time. Also, the states \( x_1(t) \) and \( y_1(t) \) versus time and \( x_2(t) \) and \( y_2(t) \) versus time are plotted in Figures 8(b) and 8(c), respectively. Figure 8(d) plots the state \( x_2(t) \) and \( y_2(t) \) versus \( x_1(t) \) and \( y_1(t) \); the figure shows the efficacy of the static sliding mode controller to drive the dynamics from one attractor to a different attractor. Figure 9 shows the simulation results for Case 2. Figure 10 shows the results for Case 3. In the three cases, it is shown how the error converges to zero. Hence, it can be concluded that the designed static sliding mode control law in (20) is able to synchronize the ODE systems in (7) and (8) when these systems have the same or different Reynolds numbers, but they start from two different initial conditions. Clearly, the simulation results indicate that the proposed static sliding mode controller works well.

**Remark 2.** Sliding mode controllers were implemented on many systems. However, most of the work on the control of the Navier–Stokes equations has been theoretical; numerical algorithms were developed to verify the theoretical results. The implementation of controllers on the Navier–Stokes equations is generally an open research area. It should be mentioned that a few works discussed some implementation issues related to the control of the Navier–Stokes equations. For example, Yan et al. [53] briefly described a practical control algorithm for these equation; Vazquez and Krstic [54] discussed some implementation issues related to a closed-form feedback controller for stabilization of the linearized 2D Navier–Stokes Poiseuille System.

In addition, it should be mentioned that constrained sliding-mode control is an active research area, and we are not considering it in the proposed work. However, for completeness, we refer the reader to [35, 36, 55, 56].
Figure 8: A static sliding mode control of the asymptotically stable state regime at the Reynolds number $R_e = 20$ in the slave system (8) to periodic regime at $R_e = 24$ in the master system (7). The controller is switched on at time $t = 10$. (a) The $L_2$ norm of the error $e$ versus time. (b) The states $x_1(t)$ and $y_1(t)$ versus time. (c) The states $x_7(t)$ and $y_7(t)$ versus time. (d) The states $x_7(t)$ and $y_7(t)$ versus $x_1(t)$ and $y_1(t)$ showing how the controller drives the states from one attractor to another attractor.

Figure 9: Continued.
Figure 9: A static sliding mode control of an asymptotically stable orbit at the Reynolds number $R_c = 20$ in the slave system (8) to a symmetrized asymptotically stable state regime at $R_c = 20$ in the master system (7). The controller is switched on at time $t = 10$. (a) The $L_2$ norm of the error $e$ versus time. (b) The states $x_1(t)$ and $y_1(t)$ versus time. (c) The states $x_7(t)$ and $y_7(t)$ versus time. (d) The states $x_7(t)$ and $y_7(t)$ versus $x_1(t)$ and $y_1(t)$ showing how the controller drives the states from one attractor to another attractor.

Figure 10: A static sliding mode control of a stable periodic regime at the Reynolds number $R_c = 24$ in the slave system (8) to a symmetrized stable periodic state regime at $R_c = 24$ in the master system (7). The controller is switched on at time $t = 10$. (a) The $L_2$ norm of the error $e$ versus time. (b) The states $x_1(t)$ and $y_1(t)$ versus time. (c) The states $x_7(t)$ and $y_7(t)$ versus time. (d) The states $x_7(t)$ and $y_7(t)$ versus $x_1(t)$ and $y_1(t)$ showing how the controller drives the states from one attractor to another attractor.
5.2. A Dynamic Sliding Mode Controller. In this section, we design a dynamic sliding mode control to a system of seven ODEs derived from the 2D Navier–Stokes equation. Let $c_i (i = 1, \ldots, 4)$ be positive scalars and $\alpha_i (i = 1, \ldots, 4)$ and $W_i (i = 1, \ldots, 4)$ be sufficiently large positive scalars. Define the sliding surfaces $\sigma_i (i = 1, \ldots, 4)$ such as

$$\sigma_1 = \dot{e}_1 + \gamma_1 e_1,$$
$$\sigma_2 = \dot{e}_3 + \gamma_2 e_3,$$
$$\sigma_3 = \dot{e}_5 + \gamma_3 e_5,$$
$$\sigma_4 = \dot{e}_6 + \gamma_4 e_6.$$ (29)

Theorem 3. The dynamic sliding mode control law,

![Figure 11: A dynamic sliding mode control of the asymptotically stable state regime at the Reynolds number $Re = 20$ in the slave system (8) to periodic regime at $Re = 24$ in the master system (7). The controller is switched on at time $t = 10$. (a) The $L_2$ norm of the error $e$ versus time. (b) The states $x_1 (t)$ and $y_1 (t)$ versus time. (c) The states $x_2 (t)$ and $y_2 (t)$ versus time. (d) The states $x_7 (t)$ and $y_7 (t)$ versus time. (d) The states $x_7 (t)$ and $y_7 (t)$ versus $x_1 (t)$ and $y_1 (t)$ showing how the controller drives the states from one attractor to another attractor.](image-url)
Figure 12: A dynamic sliding mode control of an asymptotically stable orbit at the Reynolds number $Re = 20$ in the slave system (8) to a symmetrized asymptotically stable state regime at $Re = 20$ in the master system (7). The controller is switched on at time $t = 10$. (a) The $L_2$ norm of the error $e$ versus time. (b) The states $x_1 (t)$ and $y_1 (t)$ versus time. (c) The states $x_7 (t)$ and $y_7 (t)$ versus time. (d) The states $x_7 (t)$ and $y_7 (t)$ versus $x_1 (t)$ and $y_1 (t)$ showing how the controller drives the states from one attractor to another attractor.

\[
\dot{u}_1 = -[(y_1 - 2)e_1 - (4e_5 + 4x_5)e_4 - (4e_4 + 4x_4)e_5 + (4e_2 + 4x_2)e_7 + (4e_7 + 4x_7)e_2] \\
-4x_4e_2 - 4\dot{x}_4e_4 + 4\dot{x}_2e_7 + 4\dot{x}_7e_2] - W_1 \text{sgn}(\sigma_1) - \sigma_1\sigma_1,
\]

\[
\dot{u}_2 = -[(y_2 - 4)e_3 - (4\sqrt{2} e_6 + 4\sqrt{2} x_6)e_4 - (4\sqrt{2} e_4 + 4\sqrt{2} x_4 - 4\sqrt{2} e_7 - 4\sqrt{2} x_7)e_6] \\
+ (4\sqrt{2} e_6 + 4\sqrt{2} x_6)e_7 - 4\sqrt{2} e_6\dot{x}_4 - (4\sqrt{2} e_4 - 4\sqrt{2} e_7)\dot{x}_6 + 4\sqrt{2} e_6\dot{x}_7] - W_2 \text{sgn}(\sigma_2) - \sigma_2\sigma_2,
\]

\[
\dot{u}_3 = -[(y_3 - 1)e_5 + (3e_4 + 3x_4 - \sqrt{5} e_6 - \sqrt{5} x_6)e_1 + (3e_1 + 3x_1)e_4 - \sqrt{5} e_1 + \sqrt{5} x_1)e_6] \\
+ (3e_4 - \sqrt{5} e_6)\dot{x}_1 + 3e_1\dot{x}_4 - \sqrt{5} e_1\dot{x}_6] - W_3 \text{sgn}(\sigma_3) - \sigma_3\sigma_3,
\]

\[
\dot{u}_4 = -[(y_4 - 1)e_6 + (\sqrt{5} x_5 + \sqrt{5} x_5)e_1 + (\sqrt{5} e_1 + \sqrt{5} x_1)e_5 + (\sqrt{2} e_4 + \sqrt{2} x_4 - \sqrt{2} e_7 - \sqrt{2} x_7) \\
\sqrt{2} e_5 + \sqrt{2} x_5)e_6 - (\sqrt{2} e_5 + \sqrt{2} x_5)e_7 + \sqrt{5} e_5\dot{x}_1 + \sqrt{5} e_4\dot{x}_5 + (\sqrt{2} e_4 - \sqrt{2} e_7) \\
\sqrt{2} e_5\dot{x}_4 - \sqrt{2} e_5\dot{x}_7] - W_4 \text{sgn}(\sigma_4) - \sigma_4\sigma_4.
\]
when applied to the error system (17) guarantees the convergence of the errors \( e_i \) \((i = 1, \ldots, 7)\) to zero in a finite time. Hence, the states of the slave system given by (8) converge to the states of the master system given by (7).

Proof. Taking the time derivatives of the sliding surfaces \( \sigma_i \) \((i = 1, \ldots, 4)\) along the trajectories of the errors given by equation (17), we obtain

\[
\dot{e}_1 + y_1 \dot{e}_1 = (\gamma_1 - 2)e_1 - (4e_5 + 4x_5)e_4 - (4e_4 + 4x_4)e_5 + (4e_2 + 4x_2)e_7 + (4e_7 + 4x_7) \\
\dot{e}_2 - 4\dot{x}_6 e_6 - 4\dot{x}_3 e_4 + 4\dot{x}_2 e_7 + 4\dot{x}_7 e_2 + \dot{u}_1, \\
\dot{e}_3 = \dot{e}_5 + y_3 \dot{e}_3 = (\gamma_2 - 4)e_3 - (4\sqrt{2} e_6 + 4\sqrt{2} x_6)e_4 - (4\sqrt{2} e_4 + 4\sqrt{2} x_4 - 4\sqrt{2} e_7 - 4\sqrt{2} x_7)e_5 \\
+ (4\sqrt{2} e_6 + 4\sqrt{2} x_6)e_7 - 4\sqrt{2} e_6 \dot{x}_6 + (4\sqrt{2} e_4 - 4\sqrt{2} e_4 \dot{x}_4 + 4\sqrt{2} e_6 \dot{x}_6 + 4\sqrt{2} e_6 \dot{x}_7 + \dot{u}_3, \\
\dot{e}_4 = \dot{e}_5 + y_4 \dot{e}_5 = (\gamma_3 - 1)e_5 + (3e_4 + 3x_4 - \sqrt{5} e_6 - \sqrt{5} x_6)e_6 + (3e_1 + 3x_1)e_7 - (\sqrt{5} e_1 + \sqrt{5} x_1)e_6 \\
+ (3e_4 - \sqrt{5} e_6 \dot{x}_1 + 3e_1 \dot{x}_4 - \sqrt{5} e_1 \dot{x}_6 + \dot{u}_5, \\
\dot{e}_5 + \gamma_4 \dot{e}_5 = (\gamma_4 - 1)e_6 + (\sqrt{5} e_5 + \sqrt{5} x_5)e_7 + (\sqrt{5} e_1 + \sqrt{5} x_1)e_5 + (\sqrt{2} e_4 + \sqrt{2} x_4 - \sqrt{2} e_7 - \sqrt{2} x_7) \\
\dot{e}_6 + (\sqrt{2} e_3 + \sqrt{2} x_3)e_4 - (\sqrt{2} e_3 + \sqrt{2} x_3)e_7 + \sqrt{5} e_5 \dot{x}_1 + \sqrt{5} e_1 \dot{x}_5 \\
+ (\sqrt{2} e_4 - \sqrt{2} e_7 \dot{x}_3 + \sqrt{2} e_3 \dot{x}_4 - \sqrt{2} e_3 \dot{x}_7 + \dot{u}_4.
\]

Let the Lyapunov function candidate \( V_3 \) be such that

\[
V_3 = \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2).
\]
6. Concluding remarks

In this paper, the dynamics of a steady state flow regime and a periodic regime flow observed in a dynamical system of a nonlinear dynamical system of seventh-order nonlinear differential equations truncated from the 2D Navier–Stokes equations with periodic boundary conditions and a sinusoidal external force known as 2D Kolmogorov flow is analyzed. Then, an adaptive controller is designed to drag the dynamics of Kolmogorov flow either to a steady state flow regime or to a periodic flow regime. Also, a static and a dynamic sliding mode controllers were designed to stabilize the dynamics of Kolmogorov flow. The presented numerical simulation results illustrate the effectiveness of the proposed adaptive controller.

The dynamics and control of a reduced order system of less than seven ODEs whose dynamics are similar to the dynamics of the 2D Navier–Stokes equations will be the subject of future research work.

Finally, it should be mentioned that the proposed method can be extended to switched system [50, 51, 57].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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