LINE BUNDLES ON THE MODULI SPACE OF LIE ALGEBROID CONNECTIONS OVER A CURVE

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ABSTRACT. We explore algebro-geometric properties of the moduli space of holomorphic Lie algebroid (L) connections on a compact Riemann surface X of genus \( g \geq 3 \). We establish a smooth compactification for the moduli space of L-connections such that underlying vector bundle is stable. The complement of the moduli space into its compactification yields a divisor. We give a criterion for the numerical effectiveness of the divisor. We compute the Picard group of the moduli space, and analyze Lie algebroid Atiyah bundles associated with an ample line bundle. This enables us to conclude that regular functions on the space of certain Lie algebroid connections are constants. Moreover, under some condition, we show that the moduli space of L-connections does not admit non-constant algebraic functions. We also explore rational connectedness of the moduli spaces.

1. INTRODUCTION AND MAIN RESULTS

The notion of Lie algebroid was introduced by J. Pradines [23, 24] to study the differential groupoids. On the other hand, it naturally arises from the properties of space of vector fields on a smooth manifold. The Lie algebroids have been studied in different categories namely smooth (\( \mathcal{C}^\infty \)), holomorphic and algebraic. In the series of papers [20] [8], [11], [13], [34] etc. (of course there are many more) authors have developed the theory of Lie algebroids and Lie algebroid connections over manifolds, and varieties from different perspectives. The notion of relative holomorphic Lie algebroid connections was introduced in [6].

The moduli space of Lie algebroid connections has received great attention over the years because this is a natural generalisation of the moduli space of holomorphic, logarithmic and meromorphic connections or decorated vector bundles (see Remark 3.4).

The Picard group of the moduli space of stable vector bundles over a smooth projective curve has been studied in [5], [25]. Additionally, the Picard group and regular functions have been investigated for the moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface in [10], [28], and [29].

Algebro-geometric invariants play a crucial role when studying the properties of any moduli space. In the present article, our aim is to compute algebro-geometric invariants, such as the Picard group, regular function, rational connectedness, of the moduli space of Lie algebroid connections over a compact Riemann surface. We also

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provide a natural compactification of the moduli space of Lie algebroid connections whose underlying vector bundle is stable.

In [23, 27], Simpson introduced the notion of \( \Lambda \)-modules over a smooth projective variety and constructed the moduli space of semistable \( \Lambda \)-modules. In [31, 32], Tortella proved an equivalence between the category of split almost polynomial sheaves of rings of differential operators and the category of Lie algebroids. Such an equivalence induces a correspondence between the category of integrable Lie algebroid connections and the category of modules over the split almost polynomial sheaves of rings of differential operators associated with the Lie algebroid. And this equivalence preserves the semistability, and hence one has the moduli space of Lie algebroid connections. Moreover, Krizka [18, 19] used analytic tools to construct moduli space of Lie algebroid connections. In [1], authors have studied the motives of the Hodge moduli space of Lie algebroid connections, where \( \lambda \in \mathbb{C} \).

Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \). We fix a holomorphic Lie algebroid (see Definition 2.1 and Example 2.2) \( L = (L, [,], \#) \) of rank one over \( X \) such that \( \deg L^* > 2g - 2 \), where \( L^* \) is the dual of the line bundle \( L \).

The paper is organised as follows.

In section 2, we recall the notion of holomorphic \( L \)-connection on a holomorphic vector bundle \( E \) over \( X \) (see Definition 2.5). Proposition 2.8 provides a sufficient condition for the existence of holomorphic \( L \)-connections on a semistable vector bundle \( E \) over \( X \).

In section 3, we discuss some basics of the moduli space of \( L \)-connections. Throughout this article, we fix integers \( r \geq 2 \) and \( d \) such that \( r \) and \( d \) are coprime. Let \( M_L(r, d) \) denote the moduli space of stable \( L \)-connections over \( X \) of rank \( r \) and degree \( d \). Then \( M_L(r, d) \) is an irreducible smooth quasi-projective variety of dimension \( 1 + r^2 \deg L^* \) (see Theorem 3.3).

In section 4, we consider the moduli space \( M'_L(r, d) \subset M_L(r, d) \), which is a Zariski open dense subset of \( M_L(r, d) \) consisting of those \( L \)-connections whose underlying vector bundle is stable. Let \( U(r, d) \) be the moduli space of stable vector bundles of rank \( r \) and degree \( d \). Then, \( U(r, d) \) is a smooth projective variety of dimension \( r^2(g - 1) + 1 \). We have a natural projection

\[
p : M'_L(r, d) \longrightarrow U(r, d)
\]

defined by sending \((E, \nabla_L) \mapsto E\), and this is a forgetful map. We recall the notion of torsor (see Definition 4.1) and use it to show that there exists a smooth compactification of the moduli space \( M'_L(r, d) \) (see Theorem 4.5). Now, using this smooth compactification of \( M'_L(r, d) \), we compute the Picard group of the moduli space \( M_L(r, d) \). More precisely, we show that (see Theorem 4.6)

\[
\text{Pic}(M_L(r, d)) \cong \text{Pic}(U(r, d)).
\]

In section 5, we delve into the moduli space of \( L \)-connections with fixed determinant. We fix a holomorphic line bundle \( \xi \) over \( X \) with degree \( d \) and also fix a holomorphic \( L \)-connection \( \nabla^\xi_L \) on \( \xi \). We would like to study the moduli space
\(M_L(r, \xi)\) as described in (5.1). We consider the moduli space \(M'_L(r, \xi) \subset M_L(r, \xi)\) which consists of those \(L\)-connections whose underlying vector bundle is stable and show the following (see Proposition [5.3])

\[\text{Pic}(M_L(r, \xi)) \cong \text{Pic}(M'_L(r, \xi)) \cong \text{Pic}(U(r, \xi)) \cong \mathbb{Z},\]

where \(U(r, \xi)\) denotes the moduli space stable vector bundles of rank \(r\) with fixed determinant \(\xi\).

Let \(P_\xi\) (see (5.5)) denote the moduli space parametrising \(L^*\)-twisted Higgs bundles \((E, \phi)\) on \(X\) of rank \(r\) such that \(E\) is stable, \(\bigwedge^r E \cong \xi\) and \(\text{tr}(\phi) = 0\). Then, \(P_\xi\) is a moduli space of \(L^*\)-connections whose underlying vector bundle is stable and show the following (see Proposition 5.3)

\[\text{Pic}(M_L(r, \xi)) \cong \text{Pic}(M'_L(r, \xi)) \cong \text{Pic}(U(r, \xi)) \cong \mathbb{Z},\]

where \(U(r, \xi)\) denotes the moduli space stable vector bundles of rank \(r\) with fixed determinant \(\xi\).

Let \(P_\xi\) (see (5.5)) denote the moduli space parametrising \(L^*\)-twisted Higgs bundles \((E, \phi)\) on \(X\) of rank \(r\) such that \(E\) is stable, \(\bigwedge^r E \cong \xi\) and \(\text{tr}(\phi) = 0\). Then, \(P_\xi\) is a vector bundle over \(U(r, \xi)\). Now to proceed further, we observe that the dual vector bundle \(P_\xi^*\) is a Lie algebroid over \(U(r, \xi)\) and this construction of Lie algebroid structure on \(P_\xi^*\) has been done in [7, section 4.6]. Let \(\Theta\) be the ample line bundle over \(U(r, \xi)\). Let \(A_{\psi}(\Theta)\) denote the \(P^*_\xi\)-Atiyah bundle associated with \(\Theta\) over \(U(r, \xi)\). Then, using the Hitchin map (see (5.7)) we show the following (see section 5 for the proof).

**Theorem 1.1.** Suppose that genus \(g\) of \(X\) is \(\geq 3\). Then,

\[H^0(U(r, \xi), \text{Sym}^k A_{\psi}(\Theta)) = \mathbb{C}\]

where \(\text{Sym}^k A_{\psi}(\Theta)\) denotes the \(k\)-th symmetric powers of the \(P^*_\xi\)-Atiyah bundle \(A_{\psi}(\Theta)\) associated with \(\Theta\).

We consider the space \(\text{Conn}_{P^*_\xi}(\Theta)\) of all \(P^*_\xi\)-connection on \(\Theta\) and as a consequence of above Theorem 1.1 for \(g \geq 3\), we have (see Corollary 5.4)

\[H^0(\text{Conn}_{P^*_\xi}(\Theta), \mathcal{O}_{\text{Conn}_{P^*_\xi}(\Theta)}) = \mathbb{C},\]

where \(\mathcal{O}_{\text{Conn}_{P^*_\xi}(\Theta)}\) denotes the sheaf of regular functions on \(\text{Conn}_{P^*_\xi}(\Theta)\). Under some conditions we prove that the moduli space \(M_L(r, \xi)\) does not admit any non-constant algebraic function (see Corollary 5.5). Again using the Hitchin map (see (5.7)), we show the following (see section 5 for the proof).

**Theorem 1.2.** Suppose that genus \(g\) of \(X\) is \(\geq 3\). Then, for every \(l < 0\), we have

\[H^0(\text{Conn}_{P^*_\xi}(\Theta), \psi^* \Theta^\otimes l) = 0,\]

where \(\psi : \text{Conn}_{P^*_\xi}(\Theta) \rightarrow U(r, \xi)\) is the natural projection.

In the last section 6, we show that the moduli space \(M_L(r, d)\) is not rational (see Proposition 6.3) and the moduli space \(M_L(r, \xi)\) is rationally connected (see Proposition 6.5).

Let \(P(F)\) be a smooth compactification of the moduli space \(M'_L(r, \xi)\) as in Proposition 5.2 and \(H_0 := P(F) \setminus M'_L(r, \xi)\) the smooth divisor. Then, we characterise the numerical effectiveness of the divisor \(H_0\) (see Proposition 6.7).

A similar result (see Proposition 6.3) can be shown for the divisor \(H := P(F) \setminus M'_L(r, d)\), where \(P(F)\) is the smooth compactification of the moduli space \(M'_L(r, d)\) as in Theorem 4.5.
2. Preliminaries

Let $X$ be a compact Riemann surface of genus $g \geq 2$. Let $T_X$ and $\Omega^1_X$ respectively denote the holomorphic tangent and cotangent bundle on $X$.

**Definition 2.1.** A (holomorphic) Lie algebroid over $X$ is a triple $\mathcal{L} = (L, [\cdot, \cdot], \sharp)$ consisting of

1. a holomorphic vector bundle $L$ on $X$,
2. a $\mathbb{C}$-bilinear and skew-symmetric map $[\cdot, \cdot] : L \otimes \mathbb{C} L \rightarrow L$, called the Lie bracket,
3. a vector bundle homomorphism $\sharp : L \rightarrow T_X$, called the anchor map that induces a homomorphism of Lie algebras from the space of sections of $L$ to the space of sections of $T_X$ satisfying the following properties:
   a. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ (Jacobi identity),
   b. $[u, fv] = f[u, v] + \sharp(u)(f)v$ (Leibniz identity),

for every local holomorphic sections $u, v, w$ of $L$ and every local holomorphic function $f$ on $X$.

From the above definition it is clear that the space of sections of $L$ has a Lie algebra structure. The rank $\text{rk}(\mathcal{L})$ and degree $\deg \mathcal{L}$ of a Lie algebroid $\mathcal{L}$ is by definition rank and degree of the underlying vector bundle $L$ respectively.

By a Lie algebroid we always mean holomorphic Lie algebroid.

**Example 2.2.** Followings are the well-known examples of Lie algebroids.

1. **Tangent Lie algebroid:** The holomorphic tangent bundle $T_X$ is a holomorphic Lie algebroid if we take the Lie bracket to be the usual Lie bracket defined for holomorphic vector fields and anchor map $\sharp$ to be the identity map $1_{T_X}$. We denote it by $T_X = (T_X, [\cdot, \cdot], 1_{T_X})$.

2. **Log Lie algebroid:** Let $S = \{x_1, \ldots, x_m\}$ be a finite subset of compact Riemann surface $X$, and $S := x_1 + \cdots + x_m$ denote the reduced effective divisor. Let $T_X(-\log S) := T_X \otimes \mathcal{O}_X(-S) \subset T_X$ be the line bundle over $X$ consisting of those vector fields which vanish on $S$. Note that the space of sections of $T_X(-\log S)$ form a Lie subalgebra of space of sections of $T_X$ with respect to the bracket on the vector fields. Take anchor map to be the inclusion map $\sharp : T_X(-\log S) \hookrightarrow T_X$. Then $T_X(-\log S)$ form a Lie algebroid called Logarithmic or Log Lie algebroid, and is denoted by $T_X(-\log S) := (T_X(-\log S), [\cdot, \cdot], \sharp)$.

3. **Trivial Lie algebroid:** A holomorphic vector bundle can be equipped with a Lie algebroid structure by considering bracket and anchor map to be zero. Such Lie algebroid is called the trivial Lie algebroid. We denote it by $\mathcal{L}_0 = (L, 0, 0)$, where $L$ is a holomorphic vector bundle over $X$. 
We can define a morphism between Lie algebroids that are not over the same compact Riemann surface. There are actually two different definitions (see, e.g., [20]). We will mostly be dealing with the Lie algebroid over a single compact Riemann surface \( X \). In that case, the two definitions coincide.

**Definition 2.3.** Let \( \mathcal{L} = (L, [\cdot, \cdot], \sharp_L) \) and \( \mathcal{L}' = (L', [\cdot, \cdot], \sharp_{L'}) \) be two Lie algebroids over \( X \). A **Lie algebroid morphism** \( \Phi : \mathcal{L} \rightarrow \mathcal{L}' \) is a vector bundle morphism \( \Phi : L \rightarrow L' \) which is a \( \mathbb{C} \)-Lie algebra homomorphism such that \( \sharp_{L'} \circ \Phi = \sharp_L \).

We say that \( \mathcal{L} \) and \( \mathcal{L}' \) is isomorphic if the vector bundle morphism \( \Phi \) is an isomorphism.

For a Lie algebroid \( \mathcal{L} = (L, [\cdot, \cdot], \sharp) \), the morphism \( \sharp^* : \Omega^1_X \rightarrow L^* \) is the dual of the anchor map, and let \( d_L : \mathcal{O}_X \rightarrow L^* \) be the \( \mathbb{C} \)-derivation defined by

\[
d_L = \sharp^* \circ d,
\]

where \( d : \mathcal{O}_X \rightarrow \Omega^1_X \) is the universal derivation.

**Definition 2.5.** Let \( \mathcal{L} = (L, [\cdot, \cdot], \sharp) \) be a Lie algebroid over \( X \), and \( E \) a holomorphic vector bundle over \( X \). An \( \mathcal{L} \)-connection on \( E \) is a \( \mathbb{C} \)-linear map \( \nabla_\mathcal{L} : E \rightarrow E \otimes L^* \) such that

\[
\nabla_\mathcal{L}(fs) = f \nabla_\mathcal{L}(s) + s \otimes d_L(f),
\]

for every local section \( s \) of \( E \) and \( f \) of \( \mathcal{O}_X \), where \( d_L \) is defined in (2.1). We denote \( \mathcal{L} \)-connection by a pair \( (E, \nabla_\mathcal{L}) \). The rank and degree of an \( \mathcal{L} \)-connection \( (E, \nabla_\mathcal{L}) \) is the rank and degree of underlying vector bundle \( E \).

Now, using the anchor map \( \sharp \), the exterior derivation \( d_L \) can be extended to higher order exterior powers of \( L^* \) such that the composition of successive derivations vanishes (for more details see [14]). Extend \( \mathbb{C} \)-derivation \( d_L : \mathcal{O}_X \rightarrow L^* \) as defined in (2.1) to an operator

\[
d^{p}_L : \bigwedge^p L^* \rightarrow \bigwedge^{p+1} L^*
\]

by setting

\[
d^{p}_L(\alpha)(a_1, \ldots, a_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \sharp a_i \alpha(a_1, \ldots, \hat{a}_i, \ldots, a_{p+1})
\]

\[
+ \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{p+1}),
\]

where \( \alpha \) is a \( p \)-form on \( E \).
where $\alpha$ is a section of $\bigwedge^p L^*$, $a_i$'s are sections of $L$ for $i = 1, \ldots, p + 1$, and $[\cdot, \cdot] : L \otimes L \rightarrow L$ is the Lie bracket of the Lie algebroid $L$. Now, it is easy to check that
\[ d_{p+1} \circ d_p = 0, \]
for every $p \geq 0$, where $d_0 = d$. Thus, we get a complex $(\bigwedge^\bullet L^*, d^\bullet_L)$ called Chevalley–Eilenberg–de Rham complex associated with $L$.

Further, let $\nabla_L$ be an $L$-connection on $E$. As in the classical case, we shall extend $\nabla_L$ to an operator
\[ \nabla_L : E \otimes \bigwedge^p L^* \rightarrow E \otimes \bigwedge^{p+1} L^* \]
by setting
\[ \nabla_L(s \otimes \alpha) = (\nabla_L s) \wedge \alpha + s \otimes d_p^L(\alpha) \]
for every local section $s$ of $E$ and $\alpha$ of $\bigwedge^p L^*$.

Now, we define the $L$-curvature of $\nabla_L$ as follows
\[ R_L := \nabla_L \circ \nabla_L : E \rightarrow E \otimes \bigwedge^2 L^*. \]
As in the classical case, since $R_L$ is an $\mathcal{O}_X$-linear map, it gives a global section of $End(E) \otimes \bigwedge^2 L^*$.

An $L$-connection $\nabla_L$ is called flat or integrable if its $L$-curvature $R_L$ vanishes. The $L$-curvature $R_L$ satisfies analogue of the classical Bianchi identities.

**Example 2.6.** We consider the Lie algebroid connections associated to the Lie algebroids discussed in Example (2.2). Let $E$ be a holomorphic vector bundle over $X$.

(1) **Flat holomorphic connection:** Consider the tangent Lie algebroid $\mathcal{T}_X$ as in Example (2.2)(1). Then, a $\mathcal{T}_X$-connection on $E$ is the usual flat holomorphic connection on $E$. The flatness of the connection follows from the fact that $X$ is a compact Riemann surface which implies $\Omega_X^i = 0$ for $i \geq 2$.

(2) **Flat logarithmic connection:** Consider the logarithmic Lie algebroid $\mathcal{T}_X(-\log S)$ as described in Example (2.2)(2). Then, a $\mathcal{T}_X(-\log S)$-connection on $E$ is a flat logarithmic connection on $E$ singular over $S$ (see [4, 12]).

(3) **Twisted Higgs bundle:** Consider the trivial Lie algebroid $L_0 = (L, 0, 0)$ as in Example (2.2)(3). Then a flat $L_0$-connection on $E$ is a pair $(E, \nabla_{L_0})$, where $\nabla_{L_0} : E \rightarrow E \otimes L^*$ is an $\mathcal{O}_X$-linear morphism, which is nothing but the $L^*$-twisted Higgs bundle.

As in [2], we now describe the first order Lie algebroid jet bundle and corresponding Atiyah exact sequence. Consider the following
\[ J^1_L(E) := E \oplus (E \otimes L^*) \]
as a $\mathbb{C}$-module. We equip $J^1_L(E)$ with an $\mathcal{O}_X$-module structure as follows
\[ f \cdot (s, \sigma) = (fs, f\sigma + s \otimes d_Lf), \]
where $f$, $s$ and $\sigma$ are the local sections of $\mathcal{O}_X$, $E$ and $E \otimes L^*$ respectively. $J^1_L(E)$ is called the first order Lie algebroid jet bundle associated with $E$. 
As in the usual case (see [2]), we have the \( L \)-Atiyah sequence associated with \( E \)
\[ 0 \to E \otimes L^* \to J^1_L(E) \xrightarrow{p_E} E \to 0, \quad (2.2) \]
where \( p_E \) is the natural projection. The above short exact sequence \( (2.2) \) need not split as an \( \mathcal{O}_X \)-module. Let at\(_L(E) \in H^1(X, \mathcal{E}nd(E) \otimes L^*) \) be the extension class of the short exact sequence \( (2.2) \), known as \( L \)-Atiyah class of \( E \). A result similar to the following proposition has been proven in [31, Proposition 17].

**Proposition 2.7.** Let \( E \) be a holomorphic vector bundle over \( X \) and \( L \) a Lie algebroid over \( X \). Then, the followings are equivalent.

1. \( E \) admits an \( L \)-connection.
2. The \( L \)-Atiyah sequence \( (2.2) \) associated with \( E \) splits.
3. The extension class at\(_L(E) \in H^1(X, \mathcal{E}nd(E) \otimes L^*) \) vanishes.

**Proof.** (1) \( \iff \) (2) \( E \) admits an \( L \)-connection \( D : E \to E \otimes L^* \) if and only if the morphism \( \phi : E \to J^1_L(E) \) defined by \( \phi(s) = (s, D(s)) \) is an \( \mathcal{O}_X \)-linear, where \( s \) is a local section of \( E \).

(2) \( \iff \) (3) It follows from the general fact that the short exact sequence splits if and only if the extension class vanishes. \( \Box \)

Now, we give a sufficient condition for the existence of Lie algebroid connection on a semistable vector bundle over \( X \). For any vector bundle \( E \) over \( X \), the slope \( \mu(E) \) of \( E \) is defined by
\[ \mu(E) = \frac{\deg E}{\text{rk}(E)}. \]

A vector bundle \( E \) over \( X \) is said to be semistable if for every non-zero proper subbundle \( F \) of \( E \), we have \( \mu(F) \leq \mu(E) \).

The following proposition is easy to prove (see also [1, Corollary 3.14] ).

**Proposition 2.8.** Let \( E \) be a semistable bundle over \( X \). Let \( \mathcal{L} = (L, [\cdot, \cdot], \sharp) \) be a Lie algebroid such that the vector bundle \( L \) is semistable and \( \mu(L^*) > 2g - 2 \), where \( L^* \) denotes the dual of \( L \). Then, \( E \) admits an \( \mathcal{L} \)-connection. Moreover, if \( \text{rk}(\mathcal{L}) = 1 \), then \( E \) admits an integrable \( \mathcal{L} \)-connection.

**Proof.** Under the conditions on \( E \) and \( L \), it is easy to see that
\[ \mu(\mathcal{E}nd(E) \otimes L \otimes \Omega^1_X) < 0, \]
and therefore
\[ H^0(X, \mathcal{E}nd(E) \otimes L \otimes \Omega^1_X) = 0. \]

From Serre duality, we have
\[ \text{at}_\mathcal{L}(E) \in H^1(X, \mathcal{E}nd(E) \otimes L^*) = H^0(X, \mathcal{E}nd(E) \otimes L \otimes \Omega^1_X)^* = 0. \]

Moreover, if \( \text{rk}(\mathcal{L}) = 1 \), we have \( \bigwedge^2 L^* = 0 \), and hence any \( \mathcal{L} \)-connection on \( E \) is integrable. \( \Box \)
3. Moduli space of $\mathcal{L}$-connections

In this section, we describe the moduli space of Lie algebroid connections over $X$. In what follows, we assume that the rank of Lie algebroid $\mathcal{L} = (L, [\cdot, \cdot], \sharp)$ is one, that is, $L$ is a line bundle.

**Definition 3.1.** An $\mathcal{L}$-connection $(E, \nabla_\mathcal{L})$ is said to semistable (resp. stable) if for every non-zero proper subbundle $F$ of $E$, which is invariant under $\nabla_\mathcal{L}$, that is, $\nabla_\mathcal{L}(F) \subset F \otimes L^*$, we have

$$\mu(F) \leq \mu(E) \text{ (resp. } \mu(F) < \mu(E)),$$

where $\mu(E)$ denotes the slope of $E$ defined above.

A morphism between $\mathcal{L}$-connections $(E, \nabla_\mathcal{L})$ and $(E', \nabla'_\mathcal{L})$ is a morphism

$$\phi : E \to E'$$

of vector bundles such that the following diagram

$$\begin{array}{ccc}
E & \xrightarrow{\nabla_\mathcal{L}} & E \otimes L^* \\
\phi \downarrow & & \phi \otimes 1_{L^*} \downarrow \\
E' & \xrightarrow{\nabla'_\mathcal{L}} & E' \otimes L^*
\end{array}$$

(3.1)

commutes. We say that $(E, \nabla_\mathcal{L})$ and $(E', \nabla'_\mathcal{L})$ are isomorphic if $\phi$ is an isomorphism.

**Lemma 3.2.** Let $\mathcal{L}$ be a Lie algebroid over $X$. Let $(E, \nabla_\mathcal{L})$ and $(E', \nabla'_\mathcal{L})$ be semi-stable $\mathcal{L}$-connections. Then we have

1. Suppose $(E, \nabla_\mathcal{L})$ and $(E', \nabla'_\mathcal{L})$ are stable and $\mu(E) = \mu(E')$. If

$$\phi : (E, \nabla_\mathcal{L}) \to (E', \nabla'_\mathcal{L})$$

is a non-zero morphism of $\mathcal{L}$-connections, then it is an isomorphism.

2. If $(E, \nabla_\mathcal{L})$ is stable, then the only endomorphisms of $(E, \nabla_\mathcal{L})$ are scalars.

**Proof.**

1. Note that $\text{Ker}(\phi) \subset E$ is a $\nabla_\mathcal{L}$-invariant subbundle of $E$. Since $(E, \nabla_\mathcal{L})$ is stable, we have $\mu(\text{Ker}(\phi)) < \mu(E)$. Since $\phi \neq 0$, we have $\text{Im}(\phi) \neq 0$.

Consider the Kernel-Image short exact sequence

$$0 \to \text{Ker}(\phi) \to E \to \text{Im}(\phi) \to 0.$$

Then, $\mu(E) < \mu(\text{Im}(\phi))$. Next, consider the Image-Coimage short exact sequence

$$0 \to \text{Im}(\phi) \to E' \to E'/\text{Im}(\phi) \to 0.$$

Note that $\text{Im}(\Phi)$ is a $\nabla'_\mathcal{L}$-invariant subbundle of $E'$. Since $(E', \nabla'_\mathcal{L})$ is stable, we have $\mu(\text{Im}(\phi)) < \mu(E')$. Thus, we get that $\mu(E) < \mu(E')$, which contradicts the assumption that $\mu(E) = \mu(E')$. Therefore, $\text{Ker}(\phi) = 0$ and $\text{Im}(\phi) = E'$. 


Theorem 3.3. \[1\] Lemma 5.13, Theorem 7.2 Let $X$ be a compact Riemann surface of genus $g \geq 2$. Let $\mathcal{L} = (L, [\cdot , \cdot ], z)$ be a Lie algebroid such that $\text{rk}(\mathcal{L}) = 1$ and $\deg L^* > 2g - 2$. Suppose that the integers $r$ and $d$ are coprime with $r \geq 2$. Then the moduli space $\mathcal{M}_\mathcal{L}(r, d)$ of stable $\mathcal{L}$-connections is an irreducible smooth quasi-projective variety of dimension $1 + r^2 \deg L^*$.

We state the following remarks in order to emphasize the significance of taking into account the overarching perspective of Lie algebroid connections.

Remark 3.4.

(1) Let $D = \sum_{i=1}^{m} a_i x_i$ be an effective divisor on $X$ with $a_i \geq 1$ and $x_i \in X$. Let $\mathcal{M}_\text{conn}(D, r, d)$ be the moduli space of rank $r$ and degree $d$ semistable meromorphic connections with poles of order at most $a_i$ over each $x_i \in D$. If we take Lie algebroid to be $T_X(-D) := (T_X(-D) := T_X \otimes \mathcal{O}_X(-D), [\cdot , \cdot ], z)$,
where \([\cdot, \cdot]\) is the usual Lie bracket on the vector fields and \(\sharp\) is the inclusion morphism \(\sharp : T_X(-D) \hookrightarrow T_X\). Then, \(\mathcal{M}_{T_X(-D)}(r, d) \cong \mathcal{M}_{\text{conn}}(D, r, d)\).

(2) In [1, Corollary 3.20], it has been shown that for any Lie algebroid \(\mathcal{L} = (L, [\cdot, \cdot], \sharp)\) with \(\deg(L^*) > 2g - 2\) and \(\sharp \neq 0\), we have \(\mathcal{M}_{\mathcal{L}}(r, d) \cong \mathcal{M}_{\text{conn}}(D, r, d)\) for a unique effective divisor \(D\) in the linear system \(|L - 1 \otimes T_X|\).

(3) Hence, the formalism of Lie algebroid connections can be effectively applied to uniformly address a wide array of moduli spaces encompassing logarithmic connections, meromorphic connections with poles over distinct divisors. This approach will enable us to uncover meaningful interconnections among their geometries.

Next, let’s assume that the Lie algebroid \(\mathcal{L}\) is trivial, that is, \(\mathcal{L} = (L, 0, 0)\). In this scenario, we observed that \(\mathcal{L}\)-connections correspond to twisted Higgs bundles with a twist by \(L^*\) (as described in the third point of Example 2.6).

Let \(\mathcal{N}_{L^*}(r, d)\) denote the moduli space of semi-stable \(L^*\)-twisted Higgs bundles with rank \(r\) and degree \(d\). The moduli space \(\mathcal{N}_{L^*}(r, d)\) has been constructed in [22]. Under the conditions stated in Theorem 3.3, that is, \(\deg(L^*) > 2g - 2\), \(r\) and \(d\) coprime, and \(r \geq 2\), the moduli space \(\mathcal{N}_{L^*}(r, d)\) is an irreducible smooth quasi-projective variety of dimension \(1 + r^2 \deg(L^*)\), as established in [9, Theorem 1.2, Proposition 3.3].

The moduli space \(\mathcal{N}_{L^*}(r, d)\) of \(L^*\)-twisted Higgs bundles is equipped with the Hitchin map

\[
\mathcal{H} : \mathcal{N}_{L^*}(r, d) \longrightarrow \mathcal{A} := \bigoplus_{i=1}^{r} \mathbb{H}^0(X, (L^*)^i)
\]  

(3.3)

defined by sending a stable pair \((E, \phi)\) to \(\sum_{i=1}^{r} \text{tr}(\wedge^i \phi)\), where \(\phi : E \rightarrow E \otimes L^*\) is an \(\mathcal{O}_X\)-linear morphism. From [22] Theorem 6.1, the Hitchin map \(\mathcal{H}\) defined in (3.3) is proper.

Let

\[
\mathcal{P} := \mathcal{P}_{L^*}(r, d) \subset \mathcal{N}_{L^*}(r, d)
\]

(3.4)

be the moduli space of \(L^*\)-twisted Higgs bundles \((E, \phi)\) such that the underlying vector bundle \(E\) is stable. Then from [21] Theorem 2.8(A)], \(\mathcal{P}\) is a Zariski open dense subset of \(\mathcal{N}_{L^*}(r, d)\). We can restrict the Hitchin map \(\mathcal{H}\) to \(\mathcal{P}\) and we denote the restriction by

\[
\mathcal{H}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{A}.
\]

(3.5)

Using the similar techniques as in [3], one can construct a 1-dimensional scheme \(X_s\) and a finite morphism \(X_s \rightarrow X\), for every point \(s \in \mathcal{A}\). \(X_s\) is called the spectral curve associated to the point \(s\), and \(X_s\) can be singular. Consider the set

\[
U = \{ s \in \mathcal{A} \mid X_s \text{ is integral and smooth} \}.
\]

Then, under the assumption on \(\mathcal{L}\) (i.e. \(\deg(L^*) > 2g - 2\)), \(U\) is a non-empty Zariski open subset of \(\mathcal{A}\).

According to [4] Theorem 2.2.1], for any generic point \(s \in \mathcal{A}\), the fibre \(\mathcal{H}^{-1}(s)\) is an abelian variety denoted as \(J(X_s)\), the Jacobian variety of the spectral curve \(X_s\). Furthermore, as outlined in [4] Remark 2.2.2], for any generic points \(s \in \mathcal{A}\) and with
for \( r \geq 3 \) and \( g \geq 2 \), the fibre \( \mathcal{H}^{-1}_r(s) \) is of the form \( A_{\mathfrak{a}} \setminus F_s \), where \( A_{\mathfrak{a}} \) is an abelian variety and \( F_s \) is a closed subset of \( A_{\mathfrak{a}} \) satisfying \( \text{codim}(F_s, A_{\mathfrak{a}}) \geq 2 \).

Let \( \mathcal{U}(r, d) \) be the moduli space of stable vector bundle of rank \( r \) and degree \( d \), where \( r \) and \( d \) are coprime. Then, \( \mathcal{U}(r, d) \) is a smooth projective variety of dimension \( r^2(g - 1) + 1 \). Let

\[
\pi : \mathcal{P} \longrightarrow \mathcal{U}(r, d)
\]  

be the morphism of varieties sending \((E, \phi) \mapsto E\). In view of [7, Lemma 1.3.1], \( \mathcal{P} \) is a vector bundle over \( \mathcal{U}(r, d) \) with fibre \( \pi^{-1}(E) = H^0(X, \text{End}(E) \otimes L^*) \).

4. Compactification and Picard group

We are using notations and assumptions from previous sections. Let \( \mathcal{M}'_{\mathcal{L}}(r, d) \subset \mathcal{M}_{\mathcal{L}}(r, d) \) denote the moduli space of \( \mathcal{L} \)-connections \((E, \nabla_{\mathcal{L}})\) such that the underlying vector bundle \( E \) is stable. It follows from [21, Theorem 2.8(A)] that \( \mathcal{M}'_{\mathcal{L}}(r, d) \) is a Zariski open dense subset of \( \mathcal{M}_{\mathcal{L}}(r, d) \).

Let \( p : \mathcal{M}'_{\mathcal{L}}(r, d) \longrightarrow \mathcal{U}(r, d) \) be the forgetful map as defined in (1.1).

We recall the definition of a torsor, and show that the moduli space \( \mathcal{M}'_{\mathcal{L}}(r, d) \) is a \( \mathcal{P} \)-torsor over \( \mathcal{U}(r, d) \).

Definition 4.1. Let \( M \) be a connected complex algebraic variety. Let \( \pi : \mathcal{V} \rightarrow M \), be an algebraic vector bundle.

A \( \mathcal{V} \)-torsor on \( M \) is a fibre bundle \( p : Z \rightarrow M \), and an algebraic map from the fibre product

\[
\varphi : Z \times_M \mathcal{V} \rightarrow Z
\]

such that the following conditions are satisfied.

1. \( p \circ \varphi = p \circ p_Z \), where \( p_Z \) is the natural projection of \( Z \times_M \mathcal{V} \) to \( Z \).
2. the map \( Z \times_M \mathcal{V} \rightarrow Z \times_M Z \) defined by \( p_Z \times \varphi \) is an isomorphism.
3. \( \varphi(\varphi(z, v), w) = \varphi(z, v + w) \).

Proposition 4.2. The isomorphic classes of \( \mathcal{V} \)-torsors over \( M \) are parametrized by \( H^1(M, \mathcal{V}) \).

Proof. Let \( p : Z \rightarrow M \) be a \( \mathcal{V} \)-torsor. Let \( \{U_i\}_{i \in I} \) be a covering of \( M \) by open sets, and

\[
\sigma_i : U_i \longrightarrow Z|_{U_i}
\]
a section for every \( i \in I \). Since \( \mathcal{V} \) acts on \( Z \) freely and transitively follows from the definition, \( \sigma_i - \sigma_j \) is a section of \( \mathcal{V}|_{U_i \cap U_j} \). Then \( \{\sigma_i - \sigma_j\}_{i, j \in I} \) forms a 1-cocycle with values in \( \mathcal{V} \), and hence defines a cohomology class in \( H^1(M, \mathcal{V}) \). \( \square \)

Proposition 4.3. Let \( \pi : \mathcal{P} \rightarrow \mathcal{U}(r, d) \) be the algebraic vector bundle defined in (3.6). Then, the fibre bundle \( p : \mathcal{M}'_{\mathcal{L}}(r, d) \rightarrow \mathcal{U}(r, d) \) defined in (1.1) is a \( \mathcal{P} \)-torsor over \( \mathcal{U}(r, d) \).
Proof. Let \((E, \nabla_E), (E, \nabla'_E)\) be two \(\mathcal{L}\)-connections on \(E\). Then
\[
\nabla_E - \nabla'_E \in H^0(X, \text{End}(E) \otimes L^*).
\]
Conversely, given any \(\omega \in H^0(X, \text{End}(E) \otimes L^*)\), \(\nabla_E + \omega\) is again an \(\mathcal{L}\)-connection on \(E\). Thus, \(p^{-1}(E) \subset \mathcal{M}'_{\mathcal{L}}(r, d)\) is an affine space modelled over the vector space \(H^0(X, \text{End}(E) \otimes L^*)\). Note that the fiber of the bundle \(\pi : \mathcal{P} \to \mathcal{U}(r, d)\) at \(E\) is \(H^0(X, \text{End}(E) \otimes L^*)\). Therefore, we get a natural action of \(\pi^{-1}(E)\) on \(p^{-1}(E)\), that is,
\[
\pi^{-1}(E) \times p^{-1}(E) \to p^{-1}(E)
\]
sending \((\omega, \nabla_E)\) to \(\omega + \nabla_E\). This action on the fibre is free and transitive. This action will induce a morphism on the fibre product
\[
\varphi : \mathcal{P} \times_{\mathcal{U}(r,d)} \mathcal{M}'_{\mathcal{L}}(r, d) \to \mathcal{M}'_{\mathcal{L}}(r, d),
\]
which satisfies the above conditions in the definition of the torsor. \(\square\)

Let \(\text{Conn}_{\mathcal{L}}(E)\) denote the space of all \(\mathcal{L}\)-connections \(\nabla_E\) on \(E\) such that \((E, \nabla_E)\) is stable. Notice that \(\text{Conn}_{\mathcal{L}}(E)\) is an affine space modelled over the vector space \(H^0(X, \text{End}(E) \otimes L^*)\).

Given an automorphism \(\Phi\) of \(E\) and an \(\mathcal{L}\)-connection \(\nabla_E\) on \(E\), the \(\mathbb{C}\)-linear morphism \(\Phi \otimes 1_{L^*} \circ \nabla_E \circ \Phi^{-1}\) defines an \(\mathcal{L}\)-connection on \(E\). In fact,
\[
(\nabla_E, \Phi) \mapsto \Phi \otimes 1_{L^*} \circ \nabla_E \circ \Phi^{-1}
\]
defines a natural action of \(\text{Aut}(E)\) on \(\text{Conn}_{\mathcal{L}}(E)\), called gauge transformation. We would like to compute the dimension of the quotient space \(\text{Conn}_{\mathcal{L}}(E)/\text{Aut}(E)\), that parametrizes all isomorphic \(\mathcal{L}\)-connections on \(E\).

The Lie algebra of the holomorphic automorphism group \(\text{Aut}(E)\) is \(H^0(X, \text{End}(E))\). Therefore,
\[
\dim \text{Aut}(E) = \dim H^0(X, \text{End}(E)).
\]
Choose any \(\nabla_E \in \text{Conn}_{\mathcal{L}}(E)\). Then, from Lemma 4.2 (2) the isotropy subgroup
\[
\text{Aut}(E)_{\nabla_E} = \{ \Phi \in \text{Aut}(E) \mid \Phi \otimes 1_{L^*} \circ \nabla_E \circ \Phi^{-1} = \nabla_E \}
\]
of \(\text{Aut}(E)\) is the scalar automorphism of \(E\). Then, the dimension of the space \(\text{Conn}_{\mathcal{L}}(E)/\text{Aut}(E)\) is
\[
\dim H^0(X, \text{End}(E) \otimes L^*) - \dim H^0(X, \text{End}(E)) + 1. \tag{4.1}
\]

**Lemma 4.4.** Let \(E\) be a stable vector bundle of rank \(r\) and degree \(d\). Then
\[
\text{Conn}_{\mathcal{L}}(E)/\text{Aut}(E) \text{ is a } \mathbb{C} \text{-vector space equivalent to } H^0(X, \text{End}(E) \otimes L^*),
\]
and dimension of the space is equal to \(r^2(\deg L^* - g + 1)\).

**Proof.** Note that when \(E\) is stable, \(H^0(X, \text{End}(E)) = \mathbb{C} \cdot 1_E\), that is, automorphism of \(E\) are just non-zero constant multiple of \(1_E\) and in that case the quotient space \(\text{Conn}_{\mathcal{L}}(E)/\text{Aut}(E)\) is equal to the affine space modelled over the vector space \(H^0(X, \text{End}(E) \otimes L^*)\). This is the case with the fibres of the morphism \(p\) defined in (4.1), because the underlying vector bundle is stable.
In view of (4.1), it is enough to compute the dimension of $H^0(X, \text{End}(E) \otimes L^*)$. Given that $E$ is stable, therefore $\text{End}(E) \otimes \Omega_X^1 \otimes L$ is stable. Since $\mu(\text{End}(E) \otimes \Omega_X^1 \otimes L) = 2g - 2 + \text{deg}(L) < 0$, we have $H^0(X, \text{End}(E) \otimes \Omega_X^1 \otimes L) = 0$. Thus, from Serre duality $H^1(X, \text{End}(E) \otimes L^*) = H^0(X, \text{End}(E) \otimes \Omega_X^1 \otimes L)^* = 0$.

From Riemann-Roch theorem for curves, we have $\dim H^0(X, \text{End}(E) \otimes L^*) = \text{deg}(\text{End}(E) \otimes L^*) + \text{rk}(\text{End}(E) \otimes L^*)(1 - g)
\begin{equation} r^2(\text{deg } L^* - g + 1). \end{equation}$

**Theorem 4.5.** There exists an algebraic vector bundle 
\[ \Phi : \mathcal{F} \to \mathcal{U}(r,d) \tag{4.2} \]
of rank $r^2(\text{deg } L^* - g + 1) + 1$ such that $\mathcal{M}_L'(r,d)$ is embedded in $\mathbb{P}(\mathcal{F})$ with $\mathcal{H} := \mathbb{P}(\mathcal{F}) \setminus \mathcal{M}_L'(r,d)$

as the hyperplane at infinity.

**Proof.** Let $\mathcal{G}$ be an affine bundle modeled on a vector bundle $\mathcal{E}$ of rank $n$ over $\mathcal{U}(r,d)$. Now, using the standard inclusion of the affine group in $\text{GL}(n + 1, \mathbb{C})$, we obtain a vector bundle $\mathcal{F}$ of rank $n + 1$ along with an embedding of $\mathcal{G}$ in $\mathbb{P}(\mathcal{F})$ as an open subset with complement being a hyperplane.

From Proposition 4.3 since $\mathcal{M}_L'(r,d)$ is a $\mathcal{P}$-torsor over $\mathcal{U}(r,d)$, the above construction yields an algebraic vector bundle $\mathcal{F}$ over $\mathcal{U}(r,d)$. In this construction, $\mathcal{M}_L'(r,d)$ is embedded in $\mathbb{P}(\mathcal{F})$, and the complement $\mathbb{P}(\mathcal{F}) \setminus \mathcal{M}_L'(r,d)$ forms a hyperplane at infinity denoted as $\mathcal{H}$.

The rank of the vector bundle $\mathcal{F}$ is $r^2(\text{deg } L^* - g + 1) + 1$, a fact that readily follows from Lemma 4.4.

Thus, we get a smooth compactification $\mathbb{P}(\mathcal{F})$ of $\mathcal{M}_L'(r,d)$ by a smooth divisor $\mathcal{H}$ at infinity.

We have the natural inclusion morphism
\[ \iota : \mathcal{M}_L'(r,d) \hookrightarrow \mathcal{M}_L(r,d) \tag{4.4} \]
that induces a homomorphism on the Picard groups
\[ \iota^* : \text{Pic}(\mathcal{M}_L(r,d)) \longrightarrow \text{Pic}(\mathcal{M}_L'(r,d)) \tag{4.5} \]
defined by restriction of the line bundles. Further, the morphism $p$ defined in (1.1) induces a homomorphism of Picard groups
\[ p^* : \text{Pic}(\mathcal{U}(r,d)) \longrightarrow \text{Pic}(\mathcal{M}_L'(r,d)) \tag{4.6} \]
by pullback of line bundles. Then, we prove the following.
Theorem 4.6. Let $g \geq 3$ and $r \geq 2$. Then, the two homomorphisms
\[ \iota^* : \text{Pic}(\mathcal{M}_L(r,d)) \to \text{Pic}(\mathcal{M}'_L(r,d)) \]
and
\[ p^* : \text{Pic}(\mathcal{U}(r,d)) \to \text{Pic}(\mathcal{M}'_L(r,d)) \]
defined in (4.5) and (4.6) respectively, are isomorphisms.

Proof. Let $Z := \mathcal{M}_L(r,d) \setminus \mathcal{M}'_L(r,d)$, that is, $Z$ is the loci of those $\mathcal{L}$-connections such that the underlying vector bundle is not stable. Then, from [1, Lemma 7.1], we have
\[ \text{codim}(Z, \mathcal{M}_L(r,d)) \geq (g-1)(r-1). \]
Thus, for $g \geq 3$ and $r \geq 2$, we have $\text{codim}(Z, \mathcal{M}_L(r,d)) \geq 2$, and hence the morphism $\iota^*$ defined in (4.5) is an isomorphism.

Next, to show that $p^*$ in (4.6) is an isomorphism. First, we show that $p^*$ is injective. Let $\eta \to \mathcal{U}(r,d)$ be an algebraic line bundle such that $p^*\eta$ is a trivial line bundle over $\mathcal{M}'_L(r,d)$. Since $p^*\eta$ is a trivial line bundle, we get a nowhere vanishing section of $p^*\eta$ over $\mathcal{M}'_L(r,d)$. Now, fix a nowhere vanishing section $t \in H^0(\mathcal{M}'_L(r,d), p^*\eta)$, and choose a point $E \in \mathcal{U}(r,d)$. Then, from the following commutative diagram
\[
\begin{array}{ccc}
p^*\eta & \xrightarrow{\tilde{p}} & \eta \\
\downarrow & & \downarrow \\
\mathcal{M}'_L(r,d) & \xrightarrow{p} & \mathcal{U}(r,d)
\end{array}
\]
we get
\[ t|_{p^{-1}(E)} : p^{-1}(E) \to \eta(E) \]
a nowhere vanishing map. Observe that $p^{-1}(E) \cong \mathbb{C}^N$ and $\eta(E) \cong \mathbb{C}$, where $N = r^2(\deg L^* - g + 1)$ (see Lemma 4.4). Further, any nowhere vanishing algebraic function on an affine space $\mathbb{C}^N$ is a constant function, that is, $t|_{p^{-1}(E)}$ is a constant function and therefore it corresponds to a non-zero vector $\alpha_E \in \eta(E)$. Since $t$ is constant on each fiber of $p$, the trivialization $t$ of $p^*\eta$ descends to a trivialization of the line bundle $\eta$ over $\mathcal{U}(r,d)$, and hence giving a nowhere vanishing section of $\eta$ over $\mathcal{U}(r,d)$. Thus, $\eta$ is a trivial line bundle over $\mathcal{U}(r,d)$.

It remains to show that $p^*$ is a surjective morphism. Let $\vartheta \to \mathcal{M}'_L(r,d)$ be an algebraic line bundle. Since $P(\mathcal{F})$ is a compactification of $\mathcal{M}'_L(r,d)$, we can extend $\vartheta$ to a line bundle $\vartheta'$ over $P(\mathcal{F})$. Now, from the morphism
\[ \tilde{\Phi} : P(\mathcal{F}) \to \mathcal{U}(r,d) \]
induced from the morphism in (4.2), we have
\[ \text{Pic}(P(\mathcal{F})) \cong \tilde{\Phi}^*\text{Pic}(\mathcal{U}(r,d)) \oplus \mathcal{O}_{P(\mathcal{F})}(1). \]
Therefore,
\[ \vartheta' = \tilde{\Phi}^*\Lambda \otimes \mathcal{O}_{P(\mathcal{F})}(m) \]
where $\Lambda$ is a line bundle over $U(r,d)$ and $m \in \mathbb{Z}$. Since $H = \mathbb{P}(\mathcal{F}) \setminus \mathcal{M}'(r,d)$ in (4.3) is the hyperplane at infinity, using (4.8) the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(H)$ associated to the divisor $H$ can be expressed as

$$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(H) = \tilde{\Phi}^* \Gamma \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1),$$

(4.10)

where $\Gamma$ is a line bundle over $U(r,d)$. Now, from (4.9) and (4.10), we get

$$\vartheta' = \tilde{\Phi}^* (\Lambda \otimes (\Gamma^*)^\otimes m) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH),$$

where $\Gamma^*$ denotes the dual of $\Gamma$. Since the restriction of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(H)$ to the compliment $\mathbb{P}(\mathcal{F}) \setminus H = \mathcal{M}'(r,d)$ is the trivial line bundle and restriction of $\tilde{\Phi}$ to $\mathcal{M}'(r,d)$ is the map $p$ defined in (1.1), we get

$$\vartheta = p^* (\Lambda \otimes (\Gamma^*)^\otimes m).$$

This completes the proof of the theorem.

\[\square\]

5. REGULAR FUNCTIONS ON THE MODULI SPACE WITH FIXED DETERMINANT

In this section, we explore the moduli space of Lie algebroid connections with a fixed determinant. Under specific assumptions, we show that the moduli space $\mathcal{M}(r,\xi)$ does not possess non-constant regular functions.

Let $\xi$ be a holomorphic line bundle over $X$ of degree $d$, which is coprime to $r$. Consider the fixed $L$-connection

$$\nabla^\xi : \xi \to \xi \otimes L^*$$

on $\xi$. Given an $L$-connection $\nabla_L$ on $E$, we have an $L$-connection $\text{tr}(\nabla_L)$ on $\wedge^r E$.

Consider the moduli space

$$\mathcal{M}(r,\xi) \subset \mathcal{M}(r,d)$$

(5.1)

parametrizing the isomorphic class of pairs $(E, \nabla_L)$ such that

$$(\bigwedge^r E, \text{tr}(\nabla_L)) \cong (\xi, \nabla^\xi).$$

Then, $\mathcal{M}(r,\xi)$ is a smooth quasi-projective variety of dimension $(r^2 - 1) \deg L^*$ (see [1, Proposition 9.7]).

Let

$$\mathcal{M}'(r,\xi) \subset \mathcal{M}(r,\xi)$$

(5.2)

be the subset consisting of those $L$-connections whose underlying vector bundle is stable. Then, $\mathcal{M}'(r,\xi)$ is a Zariski dense open subvariety of $\mathcal{M}(r,\xi)$.

Let $\mathcal{U}(r,\xi)$ be the moduli space of stable vector bundles with fixed determinant $\xi$. Then, $\mathcal{U}(r,\xi)$ is a smooth projective variety of dimension $(r^2 - 1)(g - 1)$. We have a natural projection

$$q : \mathcal{M}'(r,\xi) \longrightarrow \mathcal{U}(r,\xi)$$

(5.3)

defined by sending $(E, \nabla_L) \mapsto E$. 
Next, consider the moduli space $\mathcal{N}_{L^*}(r, d)$ of rank $r$, degree $d$, semi-stable $L^*$-twisted Higgs bundles as described above. Let
\[ \mathcal{N}_{L^*}(r, \xi) \subset \mathcal{N}_{L^*}(r, d) \] (5.4)
be the moduli space parametrising the isomorphic class of pairs $(E, \phi)$ such that $\wedge^r E \cong \xi$ and $\text{tr}(\phi) = 0$.

Further, let
\[ \mathcal{P}_\xi := \mathcal{P}_{L^*}(r, \xi) \subset \mathcal{N}_{L^*}(r, \xi) \] (5.5)
be the moduli space of those $L^*$-twisted Higgs bundles in $\mathcal{N}_{L^*}(r, \xi)$ such that underlying vector bundles are stable.

Let $\pi' : \mathcal{P}_\xi \to \mathcal{U}(r, \xi)$ (5.6)
be the morphism of varieties defined by sending $(E, \phi) \mapsto E$. Then, $\mathcal{P}_\xi$ is a vector bundle over $\mathcal{U}(r, \xi)$ with fibre at $E$ is $\pi'^{-1}(E) = H^0(X, \text{ad}(E) \otimes L^*)$, where $\text{ad}(E) \subset \text{End}(E)$ is a subbundle consisting of those endomorphisms of $E$ whose trace is zero.

We restrict the Hitchin map $\mathcal{H}$ defined in (3.3) to $\mathcal{P}_\xi$ and denote it by $\mathcal{H}_\xi$. Thus, we have
\[ \mathcal{H}_\xi : \mathcal{P}_\xi \to \mathcal{B} := \bigoplus_{i=2}^r H^0(X, (L^*)^i) \] (5.7)
defined by sending
\[ (E, \phi) \mapsto \sum_{i=2}^r \text{tr}(\wedge^i \phi) \]
(for more details see [7]). For $b \in \mathcal{B}$, let $X_b$ denote the spectral curve defined by $b$, which is a ramified $r$-sheeted covering $\epsilon : X_b \to X$ of $X$. Recall that, for a generic $b \in \mathcal{B}$ the kernel of the norm map $\text{Nm} : J(X_b) \to J(X)$ between the Jacobians is called the Prym variety and is denoted by $\text{Prym}(X_b/X)$.

For any generic $b \in \mathcal{B}$, the fibre $\mathcal{H}_\xi^{-1}(b)$ is isomorphic to the open subset $A_b$ of $\text{Prym}(X_b/X)$ consisting of isomorphic class of line bundles $M$ over $X_b$ such that the push forward $\epsilon_* M$ is a stable vector bundle. Let $F_b := \text{Prym}(X_b/X) \setminus A_b$ denote the complement. Then, from [3, Proposition 5.7]
\[ \text{codim}(F_b, \text{Prym}(X_b/X)) \geq 2. \]

Considering again the moduli space $\mathcal{M}'_{L^*}(r, \xi)$ and using the similar statements as in Proposition 4.3, we can show the following.

**Proposition 5.1.** Let $\pi_\xi : \mathcal{P}_\xi \to \mathcal{U}(r, \xi)$ be the algebraic vector bundle defined in (5.6). Then, the fibre bundle
\[ q : \mathcal{M}'_{L^*}(r, \xi) \to \mathcal{U}(r, \xi) \]
defined in (5.3) is a $\mathcal{P}_\xi$-torsor over $\mathcal{U}(r, \xi)$.

Using the same technique as in the proof of the Theorem 4.5, we can compactify the moduli space $\mathcal{M}'_{L^*}(r, \xi)$. More precisely, we have
Proposition 5.2. There exists an algebraic vector bundle
\[ \Psi : \mathcal{F}_\xi \to \mathcal{U}(r, \xi) \]
of rank \((r^2 - 1)(\deg L^* - g + 1) + 1\) such that \(\mathcal{M}'_L(r, \xi)\) is embedded in \(\mathbb{P}(\mathcal{F}_\xi)\) with \(\mathbb{P}(\mathcal{F}_\xi) \setminus \mathcal{M}'_L(r, \xi)\) being the hyperplane at infinity.

The morphism \(q\) in (5.3) induces a morphism of Picard groups
\[ q^* : \text{Pic}(\mathcal{U}(r, \xi)) \to \text{Pic}(\mathcal{M}'_L(r, \xi)) \]
defined by sending a line bundle \(\eta\) over \(\mathcal{U}(r, \xi)\) to a line bundle \(q^*\eta\) over \(\mathcal{M}'_L(r, \xi)\).

Again imitating the similar steps as in the proof of the Theorem 4.6, we have

Proposition 5.3. The morphism \(q^* : \text{Pic}(\mathcal{U}(r, \xi)) \to \text{Pic}(\mathcal{M}'_L(r, \xi))\) of Picard groups in (5.8) is an isomorphism.

The anchor map \(\hat{\flat} : L \to T_X\), induces a morphism
\[ \alpha : T^*\mathcal{U}(r, \xi) \to \mathcal{P}_\xi \]
of vector bundles over \(\mathcal{U}(r, \xi)\), where \(T^*\mathcal{U}(r, \xi)\) denotes the cotangent bundle of \(\mathcal{U}(r, \xi)\).

Note that, according to [7, Section 4.6], the dual vector bundle \(\mathcal{P}_\xi^*\) over \(\mathcal{U}(r, \xi)\) admits a Lie algebroid structure \((\mathcal{P}_\xi^*, [\cdot, \cdot], \hat{\flat})\) such that the dual
\[ \hat{\flat}^* : T^*\mathcal{U}(r, \xi) \to \mathcal{P}_\xi \]
of the anchor map
\[ \hat{\flat} : \mathcal{P}_\xi^* \to TU(r, \xi) \]
coincides with \(\alpha\) in (5.9).

Let \(\Theta\) be the ample generator of the group \(\text{Pic}(\mathcal{U}(r, \xi)) \cong \mathbb{Z}\), where the isomorphism follows from [25, Proposition 3.4, (ii)]. Using the fact that \(\mathcal{P}_\xi^*\) is a Lie algebroid over \(\mathcal{U}(r, \xi)\), we consider the space of \(\mathcal{P}_\xi^*\)-connection on \(\Theta\). Recall that a \(\mathcal{P}_\xi^*\)-connection on \(\Theta\) is a \(C^\infty\)-linear map
\[ \nabla_{\mathcal{P}_\xi^*} : \Theta \to \Theta \otimes \mathcal{P}_\xi \]
which satisfies Leibniz rule
\[ \nabla_{\mathcal{P}_\xi^*}(gt) = g\nabla_{\mathcal{P}_\xi^*}(t) + t \otimes d_{\mathcal{P}_\xi^*}(g), \]
for every local section \(t\) of \(\Theta\) and \(g\) of \(\mathcal{O}_{\mathcal{U}(r, \xi)}\), where \(d_{\mathcal{P}_\xi^*}\) is the following composition
\[ \mathcal{O}_{\mathcal{U}(r, \xi)} \xrightarrow{d} T^*\mathcal{U}(r, \xi) \xrightarrow{\alpha} \mathcal{P}_\xi. \]

Let \(\text{Conn}_{\mathcal{P}_\xi^*}(\Theta)\) be the space of all \(\mathcal{P}_\xi^*\)-connection on \(\Theta\). Then, we have a canonical projection
\[ \psi : \text{Conn}_{\mathcal{P}_\xi^*}(\Theta) \to \mathcal{U}(r, \xi). \]
Moreover, \(\text{Conn}_{\mathcal{P}_\xi^*}(\Theta)\) is a quasi-projective variety and a \(\mathcal{P}_\xi\)-torsor.

Consider the standard first order \(\mathcal{P}_\xi^*\)-jet exact sequence (also called \(\mathcal{P}_\xi^*\)-Atiyah sequence) for the line bundle \(\Theta\) (see [10] for more details) as follows
\[ 0 \to \Theta \otimes \mathcal{O}_{\mathcal{U}(r, \xi)} \mathcal{P}_\xi \to J^1_{\mathcal{P}_\xi^*}(\Theta) \to \Theta \to 0. \]
Applying $\text{Hom}_{\mathcal{O}_{U(r,\xi)}}(-,\Theta)$ to above short exact sequence (5.11), we get

$$0 \rightarrow \mathcal{E}nd_{\mathcal{O}_{U(r,\xi)}}(\Theta) \xrightarrow{\text{Hom}_{\mathcal{O}_{U(r,\xi)}}(J^1_{\xi},\Theta)} \mathcal{E}nd_{\mathcal{O}_{U(r,\xi)}}(\Theta) \otimes P^*_\xi \rightarrow 0$$

which further gives the following short exact sequence

$$0 \rightarrow \mathcal{O}_{U(r,\xi)} \xrightarrow{\iota} \mathcal{A}t_{P^*_\xi(\Theta)} := \text{Hom}_{\mathcal{O}_{U(r,\xi)}}(J^1_{P^*_\xi(\Theta)},\Theta) \xrightarrow{\sigma} \mathcal{E}nd_{\mathcal{O}_{U(r,\xi)}}(\Theta) \otimes P^*_\xi \rightarrow 0$$

which further gives the following short exact sequence

$$0 \rightarrow \mathcal{O}_{U(r,\xi)} \xrightarrow{\iota} \mathcal{A}t_{P^*_\xi(\Theta)} \xrightarrow{\sigma} P^*_\xi \rightarrow 0 \quad (5.12)$$

of vector bundles over $U(r,\xi)$. The vector bundle $\mathcal{A}t_{P^*_\xi(\Theta)}$ is called the $P^*_\xi$-Atiyah bundle associated with $\Theta$ and the morphism $\sigma$ in (5.12) is called the symbol operator.

Now, $\Theta$ admits a holomorphic $P^*_\xi$-connection if and only if the short exact sequence (5.12) splits holomorphically.

We adopt a similar approach as presented in [10, Theorem 4.3] to establish the Theorem 1.1. Furthermore, considering the broader context of Lie algebroid connections, we will make suitable modifications accordingly.

**Proof of Theorem 1.1.** The symbol operator

$$\sigma : \mathcal{A}t_{P^*_\xi(\Theta)} \rightarrow P^*_\xi$$

as described in (5.12), induces a morphism

$$\text{Sym}^k(\sigma) : \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} \rightarrow \text{Sym}^k P^*_\xi$$

on $k$-th symmetric powers of bundles. We also have

$$\text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)} \subset \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} \quad \text{for all } k \geq 1. \quad (5.15)$$

In fact, we have $P^*_\xi$-symbol exact sequence associated with $\Theta$ over $U(r,\xi)$ (for more details see [6]),

$$0 \rightarrow \text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)} \rightarrow \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} \xrightarrow{\text{Sym}^k(\sigma)} \text{Sym}^k P^*_\xi \rightarrow 0. \quad (5.16)$$

In other words, we get a filtration

$$\text{Sym}^0 \mathcal{A}t_{P^*_\xi(\Theta)} \subset \text{Sym}^1 \mathcal{A}t_{P^*_\xi(\Theta)} \subset \ldots \subset \text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)} \subset \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} \subset \ldots \quad (5.17)$$

such that

$$\text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)}/\text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)} \cong \text{Sym}^k P^*_\xi \quad \text{for all } k \geq 1. \quad (5.18)$$

To prove the theorem, it is enough to show that

$$H^0(U(r,\xi),\text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)}) \cong H^0(U(r,\xi),\text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)}) \quad \text{for all } k \geq 1. \quad (5.19)$$

We have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Sym}^{k-1} \mathcal{A}t_{P^*_\xi(\Theta)} & \rightarrow & \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} & \xrightarrow{\text{Sym}^k(\sigma)} & \text{Sym}^k P^*_\xi & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Sym}^{k-1} P^*_\xi & \rightarrow & \text{Sym}^k \mathcal{A}t_{P^*_\xi(\Theta)} & \xrightarrow{\text{Sym}^k(\sigma)} & \text{Sym}^k P^*_\xi & \rightarrow & 0
\end{array}
$$

(5.20)
which gives the following commutative diagram of long exact sequences

\[ \cdots \longrightarrow H^0(U(r, \xi), \text{Sym}^k P^*_\xi) \overset{\delta'_k}{\longrightarrow} H^1(U(r, \xi), \text{Sym}^{k-1} At_{P^*_\xi}(\Theta)) \longrightarrow \cdots \]

\[ \cdots \longrightarrow H^0(U(r, \xi), \text{Sym}^k P^*_\xi) \overset{\delta_k}{\longrightarrow} H^1(U(r, \xi), \text{Sym}^{k-1} P^*_\xi) \longrightarrow \cdots \]

\[ (5.21) \]

In order to show (5.19), it is enough to prove that the coboundary operator \( \delta'_k \) in the above diagram is injective for all \( k \geq 1 \) and which is equivalent to showing that the coboundary operator

\[ \delta_k : H^0(U(r, \xi), \text{Sym}^k P^*_\xi) \rightarrow H^1(U(r, \xi), \text{Sym}^{k-1} P^*_\xi) \]  \[ (5.22) \]

is injective for every \( k \geq 1 \).

Further, a connecting homomorphism can be expressed as the cup product by the extension class of the corresponding short exact sequence. Denote the extension class of the following short exact sequence

\[ 0 \rightarrow \text{Sym}^{k-1} P^*_\xi \rightarrow \text{Sym}^k P^*_\xi \rightarrow \text{Sym}^{k-2} At_{P^*_\xi}(\Theta) \rightarrow 0 \]  \[ (5.23) \]

by \( \gamma_k \).

Let at\(_{P^*_\xi}(\Theta)\) denote the extension class of the \( P^*_\xi \)-Atiyah exact sequence (5.12) (see [6]). Now, the short exact sequence (5.16) is nothing but the \( k \)-th symmetric power of the \( P^*_\xi \)-Atiyah exact sequence (5.12), therefore the extension class \( \gamma_k \) can be expressed in terms of the \( P^*_\xi \)-Atiyah class at\(_{P^*_\xi}(\Theta)\).

Thus, the connecting homomorphism \( \delta_k \) can be described using the \( P^*_\xi \)-Atiyah class at\(_{P^*_\xi}(\Theta)\) \( \in H^1(U(r, \xi), P^*_\xi) \).

The cup product with \( k \) at\(_{P^*_\xi}(\Theta)\) gives rise to a homomorphism

\[ \mu : H^0(U(r, \xi), \text{Sym}^k P^*_\xi) \rightarrow H^1(U(r, \xi), \text{Sym}^k P^*_\xi \otimes P^*_\xi) \]  \[ (5.24) \]

Also, we have a canonical homomorphism of vector bundles

\[ \beta : \text{Sym}^k P^*_\xi \otimes P^*_\xi \rightarrow \text{Sym}^{k-1} P^*_\xi \]

which induces a morphism of \( \mathbb{C} \)-vector spaces

\[ \beta^* : H^1(U(r, \xi), \text{Sym}^k P^*_\xi \otimes P^*_\xi) \rightarrow H^1(U(r, \xi), \text{Sym}^{k-1} P^*_\xi). \]  \[ (5.25) \]

So, we get a morphism

\[ \tilde{\mu} = \beta^* \circ \mu : H^0(U(r, \xi), \text{Sym}^k P^*_\xi) \rightarrow H^1(U(r, \xi), \text{Sym}^{k-1} P^*_\xi). \]  \[ (5.26) \]

Then \( \tilde{\mu} = \delta_k \). Now, it is enough to show that \( \tilde{\mu} \) is injective.

Consider the natural projection in (5.6)

\[ \pi' : P^*_\xi \longrightarrow U(r, \xi). \]

Then, we have

\[ H^0(P^*_\xi, O_{P^*_\xi}) = H^0(U(r, \xi), \pi'_* O_{P^*_\xi}). \]  \[ (5.27) \]
Moreover, since $\pi'$ is an affine fibration, using the projection formula we have

$$\pi'_* \mathcal{O}_\mathcal{P}_\xi = \bigoplus_{k \geq 0} \text{Sym}^k \mathcal{P}_\xi^*.$$  \hfill (5.28)

Using (5.27) and (5.28) we get

$$H^0(\mathcal{P}_\xi, \mathcal{O}_\mathcal{P}_\xi) = \bigoplus_{k \geq 0} H^0(\mathcal{U}(r, \xi), \text{Sym}^k \mathcal{P}_\xi^*)$$  \hfill (5.29)

Now, we use the Hitchin fibration $\mathcal{H}_\xi$ defined in (5.7) to determine the space $H^0(\mathcal{P}_\xi, \mathcal{O}_\mathcal{P}_\xi)$. Further, the fibrewise property of the Hitchin fibration $\mathcal{H}_\xi$ will be used to show that $\tilde{\mu}$ is injective.

First we show that any algebraic function on $\mathcal{P}_\xi$ descends to an algebraic function on $B = \bigoplus_{i=2}^\infty H^0(X, (L^*)^i)$ defined in (5.7). So, let $g : \mathcal{P}_\xi \to \mathbb{C}$ be an algebraic function. Then, for a generic $b \in B$ its restriction $g|_{\mathcal{H}_\xi^{-1}(b)} : \mathcal{H}_\xi^{-1}(b) \to \mathbb{C}$ is an algebraic function. Since $\mathcal{H}_\xi^{-1}(b)$ is an open subset of an abelian variety such that the complement has codimension $\geq 2$, from Hartog's theorem, the algebraic function $g|_{\mathcal{H}_\xi^{-1}(b)}$ is extended to whole abelian variety and hence constant. Thus, $g$ is constant on every generic fibre, and $\mathcal{H}_\xi$ is proper, hence gives an algebraic function on $B$.

Set

$$\mathcal{D} = d(H^0(B, \mathcal{O}_B)) \subset H^0(B, \Omega^1_B)$$

to be the space of all exact algebraic 1-form. Define a map

$$\theta : H^0(\mathcal{P}_\xi, \mathcal{O}_\mathcal{P}_\xi) \to \mathcal{D}$$  \hfill (5.30)

by $g \mapsto dg$, where $\tilde{g}$ is the function on $B$ which is defined by descent of $g$. Then, $\theta$ is a surjection. Note that $\theta$ is not injective, because it sends constant functions to zero.

Therefore, to get an isomorphism, we can restrict $\theta$ to that part of $H^0(\mathcal{P}_\xi, \mathcal{O}_\mathcal{P}_\xi)$ which excludes the constant functions. From (5.29) and (5.30), we have

$$\theta : \bigoplus_{k \geq 1} H^0(\mathcal{U}(r, \xi), \text{Sym}^k \mathcal{P}_\xi^*) \to \mathcal{D}$$  \hfill (5.31)

which is an isomorphism.

Let $T\mathcal{H}_\xi = T\mathcal{P}_\xi/B = \text{Ker}(d\mathcal{H}_\xi)$ be the relative tangent sheaf on $\mathcal{P}_\xi$, where

$$d\mathcal{H}_\xi : TP_\xi \to \mathcal{H}_\xi T\mathbb{B}$$
morphism between tangent bundles.

In view of the canonical symplectic structure on $TP_\xi$, and the fact that $\mathcal{H}_\xi$ defines a completely integrable system [7], we have an isomorphism $\mathcal{H}_\xi^* \Omega^1_B \cong T\mathcal{H}_\xi$ over the general fibre. Thus, we have $H^0(B, \Omega^1_B) \subset H^0(\mathcal{P}_\xi, T\mathcal{H}_\xi)$, and hence from (5.31), we have an injective homomorphism

$$\nu : \mathcal{D} = \bigoplus_{k \geq 1} \theta(H^0(\mathcal{U}(r, \xi), \text{Sym}^k \mathcal{P}_\xi^*)) \to H^0(\mathcal{P}_\xi, T\mathcal{H}_\xi).$$  \hfill (5.32)

Consider the morphism
\[ H^0(\mathcal{P}_\xi, T_{H_\xi}) \to H^1(\mathcal{P}_\xi, T_{H_\xi} \otimes T^*\mathcal{P}_\xi) \]

defined by taking cup product with the first Chern class \( c_1(\pi^*\Theta) \in H^1(\mathcal{P}_\xi, T^*\mathcal{P}_\xi) \).

Using the pairing
\[ T_{H_\xi} \otimes T^*\mathcal{P}_\xi \to \mathcal{O}_{\mathcal{P}_\xi}, \]

we get a homomorphism
\[ \eta : H^0(\mathcal{P}_\xi, T_{H_\xi}) \to H^1(\mathcal{P}_\xi, \mathcal{O}_{\mathcal{P}_\xi}). \]

Since \( c_1(\pi^*\Theta) = \pi^*c_1(\Theta) \), we have
\[ k\eta \circ \nu \circ \theta(\omega_k) = \bar{\mu}(\omega_k), \]

for all \( \omega_k \in H^0(\mathcal{U}(r, \xi), \text{Sym}^k\mathcal{P}_\xi^*) \). Since \( \nu \) and \( \theta \) are injective homomorphisms, it is enough to show that \( \eta\big|_{\nu(\mathcal{D})} \) is injective homomorphism.

Let \( \omega \in \mathcal{D} \setminus \{0\} \) be a non-zero exact 1-form. Choose \( b \in \mathcal{B} \) such that \( \omega(b) \neq 0 \).

From the Hitchin fibration \( H_\xi \) in (5.7) the generic fibre has the form
\[ H_\xi^{-1}(b) = A_b \setminus F_b, \]

where \( A_b \) is the Prym variety and \( F_b \) is the subvariety of \( A_b \) such that \( \text{codim}(F_b, A_b) \geq 2 \). Now, \( \eta(\nu(\omega)) \in H^1(\mathcal{P}_\xi, \mathcal{O}_{\mathcal{P}_\xi}) \) and we have the restriction map
\[ H^1(\mathcal{P}_\xi, \mathcal{O}_{\mathcal{P}_\xi}) \to H^1(H_\xi^{-1}(b), \mathcal{O}_{H_\xi^{-1}(b)}). \]

Since \( \omega(b) \neq 0 \), \( \eta(\nu(\omega)) \in H^1(H_\xi^{-1}(b), \mathcal{O}_{H_\xi^{-1}(b)}) \). Now, because of the following isomorphisms
\[ H^1(H_\xi^{-1}(b), \mathcal{O}_{H_\xi^{-1}(b)}) \cong H^1(A_b, \mathcal{O}_{A_b}) \cong H^0(A_b, TA_b), \]

it follows that \( \eta(\nu(\omega)) \neq 0 \). This completes the proof. \( \square \)

**Corollary 5.4.** Suppose that genus \( g \) of \( X \) is \( \geq 3 \). Then,
\[ H^0(\text{Conn}_{\mathcal{P}_\xi}(\Theta), \mathcal{O}_{\text{Conn}_{\mathcal{P}_\xi}(\Theta)}) = \mathbb{C}, \]

where \( \mathcal{O}_{\text{Conn}_{\mathcal{P}_\xi}(\Theta)} \) denotes the sheaf of regular functions on \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \).

**Proof.** Let \( \mathcal{P}(\mathcal{A}_\mathcal{P}_\xi(\Theta)) \) ( resp. \( \mathcal{P}(\mathcal{P}_\xi^*) \) ) be the projectivization of the \( \mathcal{P}_\xi^* \)-Atiyah bundle \( \mathcal{A}_\mathcal{P}_\xi(\Theta) \) ( resp. \( \mathcal{P}_\xi^* \) ) parametrizing the hyperplanes in the fibres of the vector bundle \( \mathcal{A}_\mathcal{P}_\xi(\Theta) \) ( resp. \( \mathcal{P}_\xi^* \) ). Consider the dual of the short exact sequence (5.12)
\[ 0 \to \mathcal{P}_\xi \to \mathcal{A}_\mathcal{P}_\xi^*(\Theta)^* \to \mathcal{O}_{\mathcal{U}(r, \xi)} \to 0, \]

and using this, it is easy to see that \( \mathcal{P}(\mathcal{P}_\xi^*) \) is a closed subvariety of \( \mathcal{P}(\mathcal{A}_\mathcal{P}_\xi(\Theta)) \) of codimension 1. Since the image of \( \iota^* \) in (5.35) is the trivial line bundle over \( \mathcal{U}(r, \xi) \), the tautological line bundle \( \mathcal{O}_{\mathcal{P}(\mathcal{A}_\mathcal{P}_\xi(\Theta))}(1) \) has a canonical section which vanishes exactly over the divisor \( \mathcal{P}(\mathcal{P}_\xi^*) \) and the order of vanishing is one. Now, from the following description of \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \), it will be clear that the complement \( \mathcal{P}(\mathcal{A}_\mathcal{P}_\xi(\Theta)) \setminus \mathcal{P}(\mathcal{P}_\xi^*) \) is nothing but the space \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \).

Let
\[ u : \mathcal{U}(r, \xi) \to \mathcal{U}(r, \xi) \times \mathbb{C} \]
be the holomorphic map defined by \( E \mapsto (E, 1) \). Let 
\[
S := \text{Im}(u) \subset \mathcal{U}(r, \xi) \times \mathbb{C}
\]
be the image of \( u \). Then, the inverse image \( i^{*-1}(S) \subset \mathcal{A}t_{\mathcal{P}_\xi}(\Theta)^* \) is a vector bundle over \( \mathcal{U}(r, \xi) \) and it coincides with the space \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \), because a holomorphic section of \( i^{*-1}(S) \) gives a holomorphic splitting of \( \mathcal{P}_\xi^* \) and hence a holomorphic \( \mathcal{P}_\xi^* \)-connection on \( \Theta \).

To show that the algebraic variety \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \) does not admit any non-constant global regular function, we use the above description, that is,
\[
\text{Conn}_{\mathcal{P}_\xi}(\Theta) = \mathbf{P}(\mathcal{A}t_{\mathcal{P}_\xi}(\Theta)) \setminus \mathbf{P}(\mathcal{P}_\xi^*).
\]
Thus, we have
\[
H^0(\text{Conn}_{\mathcal{P}_\xi}(\Theta), \mathcal{O}_{\text{Conn}_{\mathcal{P}_\xi}(\Theta)}) = \lim_{k \to 0} H^0(\mathbf{P}(\mathcal{A}t_{\mathcal{P}_\xi}(\Theta)), \mathcal{O}_{\mathbf{P}(\mathcal{A}t_{\mathcal{P}_\xi}(\Theta))}(k)). \tag{5.36}
\]

Since for any finite dimensional vector space \( V \) over \( \mathbb{C} \) and for every \( k \geq 0 \), we have \( H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) = \text{Sym}^k(V) \), where \( \text{Sym}^k(V) \) denote the \( k \)-th symmetric powers of \( V \). We get a natural isomorphism
\[
H^0(\mathbf{P}(\mathcal{A}t_{\mathcal{P}_\xi}(\Theta)), \mathcal{O}_{\mathbf{P}(\mathcal{A}t_{\mathcal{P}_\xi}(\Theta))}(k)) \cong H^0(\mathcal{U}(r, \xi), \text{Sym}^k \mathcal{A}t_{\mathcal{P}_\xi}(\Theta)),
\]
where \( \text{Sym}^k \mathcal{A}t_{\mathcal{P}_\xi}(\Theta) \) denote the \( k \)-the symmetric powers of \( \mathcal{A}t_{\mathcal{P}_\xi}(\Theta) \), and hence from \( \text{(5.36)} \), we get
\[
H^0(\text{Conn}_{\mathcal{P}_\xi}(\Theta), \mathcal{O}_{\text{Conn}_{\mathcal{P}_\xi}(\Theta)}) = \lim_{k \to 0} H^0(\mathcal{U}(r, \xi), \text{Sym}^k \mathcal{A}t_{\mathcal{P}_\xi}(\Theta))).
\]
Now, Corollary follows from Theorem \textbf{11}. \hfill \Box

Let
\[
\gamma \in H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi) \tag{5.37}
\]
denote the cohomology class corresponding to the \( \mathcal{P}_\xi \)-torsor \( \mathcal{M}_\xi^*(r, \xi) \) over \( \mathcal{U}(r, \xi) \) as described in Proposition \textbf{5.1}.

**Corollary 5.5.** Suppose that genus \( g \) of \( X \) is \( \geq 3 \), and \( \gamma \neq 0 \) in \( \text{(5.37)} \). Suppose that the morphism
\[
\alpha^* : H^1(\mathcal{U}(r, \xi), T^* \mathcal{U}(r, \xi)) \to H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi) \tag{5.38}
\]
induced from \( \alpha : T^* \mathcal{U}(r, \xi) \to \mathcal{P}_\xi \) in \( \text{(5.9)} \) is an isomorphism. Then, we have
\[
H^0(\mathcal{M}_\xi^*(r, \xi), \mathcal{O}_{\mathcal{M}_\xi^*(r, \xi)}) = \mathbb{C}. \tag{5.39}
\]

**Proof.** Since the complement of \( \mathcal{M}_\xi^*(r, \xi) \) in \( \mathcal{M}_\xi^*(r, \xi) \) has codimension at least 2, by Hartogs theorem, we can extend any algebraic function from \( \mathcal{M}_\xi^*(r, \xi) \) to \( \mathcal{M}_\xi^*(r, \xi) \). Recall that both \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \) and \( \mathcal{M}_\xi^*(r, \xi) \) are \( \mathcal{P}_\xi \)-torsors over \( \mathcal{U}(r, \xi) \). In view of Corollary \textbf{5.4} it is sufficient to show that \( \text{Conn}_{\mathcal{P}_\xi}(\Theta) \) and \( \mathcal{M}_\xi^*(r, \xi) \) are isomorphic. Note that two non-trivial \( \mathcal{P}_\xi \)-torsors are isomorphic if and only if their corresponding cohomology classes are non-zero constant multiple of each other. Let \( \chi \in \)}
$H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi)$ be the cohomology class corresponding to the $\mathcal{P}_\xi$-torsor $Conn_{\mathcal{P}_\xi}^* (\Theta)$. Since $\Theta$ is an ample line bundle, the first Chern class
\[ c_1(\Theta) \in H^1(\mathcal{U}(r, \xi), T^* \mathcal{U}(r, \xi)) \]
is non-zero. There is a notion of Lie algebroid Chern classes (see [6, Remark 3.6]), so we have first $\mathcal{P}_\xi^*$-Chern class
\[ c_{\mathcal{P}_\xi^*}^1(\Theta) \in H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi) \]
and
\[ \chi = c_{\mathcal{P}_\xi^*}^1(\Theta) = \alpha^*(c_1(\Theta)) \]
where $\alpha^*$ is in (5.38). Therefore $\chi \neq 0$. Since
\[ \dim_\mathbb{C} H^1(\mathcal{U}(r, \xi), T^* \mathcal{U}(r, \xi)) = 1, \]
and $\alpha^*$ in (5.38) is an isomorphism, we have
\[ \dim_\mathbb{C} H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi) = 1. \]
Therefore, any two non-zero cohomology classes in $H^1(\mathcal{U}(r, \xi), \mathcal{P}_\xi)$ are non-zero constant multiple of each other. As we have assumed $\gamma \neq 0$, we get $\gamma = c\chi$, where $c \in \mathbb{C} \setminus \{0\}$. This completes the proof. □

Proof of Theorem 1.2. From (5.36), we have
\[ H^0(Conn_{\mathcal{P}_\xi^*} (\Theta), \psi^* \Theta^\otimes l) = \lim_{k \to \infty} H^0(\mathbb{P}(At_{\mathcal{P}_\xi^*} (\Theta)), \psi^* \Theta^\otimes l \otimes \mathcal{O}_{\mathbb{P}(At_{\mathcal{P}_\xi^*} (\Theta))}(k)). \]
(5.40)
As observed earlier in the proof of Corollary 5.4, using the projection formula, we get a natural isomorphism
\[ H^0(\mathbb{P}(At_{\mathcal{P}_\xi^*} (\Theta)), \psi^* \Theta^\otimes l \otimes \mathcal{O}_{\mathbb{P}(At_{\mathcal{P}_\xi^*} (\Theta))}(k)) \cong H^0(\mathcal{U}(r, \xi), \Theta^\otimes l \otimes Sym^k At_{\mathcal{P}_\xi^*} (\Theta)). \]
Since $\Theta$ is an ample line bundle over $\mathcal{U}(r, \xi)$, we get
\[ H^0(\mathcal{U}(r, \xi), \Theta^\otimes l) = 0 \]
for $l < 0$.
Therefore to prove (1.2), it is enough to show the following
\[ H^0(\mathcal{U}(r, \xi), \Theta^\otimes l \otimes Sym^{k-1} At_{\mathcal{P}_\xi^*} (\Theta)) \cong H^0(\mathcal{U}(r, \xi), \Theta^\otimes l \otimes Sym^k At_{\mathcal{P}_\xi^*} (\Theta)), \]
(5.41)
for all $k \geq 1$ and $l < 0$.
Next, by tensoring $\Theta^\otimes l$ to the commutative diagram in (5.20), we get a commutative diagram, which will produce a commutative diagram of long exact sequences. To show (5.41), it is enough to prove that the coboundary map
\[ \delta^l_k : H^0(\mathcal{U}(r, \xi), \Theta^\otimes l \otimes Sym^k \mathcal{P}_\xi^*) \to H^1(\mathcal{U}(r, \xi), \Theta^\otimes l \otimes Sym^{k-1} \mathcal{P}_\xi^*) \]
(5.42)
is injective for every $k \geq 1$ and $l < 0$. Consider the natural projection in (5.6)
\[ \pi' : \mathcal{P}_\xi \to \mathcal{U}(r, \xi). \]
Since $\pi'$ is an affine fibration, all higher direct images vanish. Therefore, using the projection formula we have

$$\pi'^* \pi'^* \Theta^\otimes l = \bigoplus_{k \geq 0} \Theta^\otimes l \otimes \text{Sym}^k \mathcal{P}^{*}_\xi. \tag{5.43}$$

Using (5.43) we get

$$H^0(\mathcal{P}_\xi, \pi'^* \Theta^\otimes l) = \bigoplus_{k \geq 0} H^0(U(r, \xi), \Theta^\otimes l \otimes \text{Sym}^k \mathcal{P}^{*}_\xi) \tag{5.44}$$

This is the place to use the Hitchin fibration $H^\xi_{5.7}$ again. For a generic $b \in B$, the fibre $H_{\xi}^{-1}(b)$ has the form $H_{\xi}^{-1}(b) = A_b \setminus F_b$, where $A_b$ is the Prym variety and $F_b$ is the subvariety of $A_b$ such that $\text{codim}(F_b, A_b) \geq 2$. From [3, p.177, Theorem 2], it follows that the restriction of the line bundle $\pi'^* \Theta$ to $H_{\xi}^{-1}(b)$ is ample. Therefore, for every $l < 0$ we have

$$H^0(H_{\xi}^{-1}(b), \pi'^* \Theta^\otimes l|_{H_{\xi}^{-1}(b)}) = 0.$$  

Since $H_{\xi}^{-1}(b)$ is a generic fibre, we get

$$H^0(U(r, \xi), \Theta^\otimes l \otimes \text{Sym}^k \mathcal{P}^{*}_\xi) = 0,$$

for every $l < 0$. Thus from (5.44), for every $k \geq 0$ and $l < 0$, we have

$$H^0(U(r, \xi), \Theta^\otimes l \otimes \text{Sym}^k \mathcal{P}^{*}_\xi) = 0,$$

and this shows that the coboundary map $\delta^l_k$ in (5.42) is injective. This completes the proof. 

□

6. Rational connectedness and Divisor at infinity

In [16], rationality of moduli spaces of vector bundles over a smooth projective curve has been studied. Also, the rationality and the rational connectedness of the moduli space of logarithmic connections has been discussed in [30].

Motivated by this, there is a natural question whether the moduli spaces $M_L(r, d)$ and $M_L(r, \xi)$ are rational? For the theory of rational varieties, we refer [17]. In this section, we show that the moduli space $M_L(r, d)$ is not rational and the moduli space $M_L(r, \xi)$ is rationally connected.

Recall that a smooth complex variety $V$ is said to be rationally connected if any two general points on $V$ can be connected by a rational curve in $V$. A Rational variety is always rationally connected. But the converse is not true. The following lemma is an easy consequence of the definition.

**Lemma 6.1.** Let $f : \mathcal{Y} \to \mathcal{X}$ be a dominant rational map of complex algebraic varieties with $\mathcal{Y}$ rationally connected. Then, $\mathcal{X}$ is rationally connected.

**Theorem 6.2** ([16], Theorem 1.1). The moduli space $\mathcal{U}(r, d)$ is birational to $J(X) \times \mathbb{A}^{(n^2-1)(g-1)}$, where $J(X)$ is the Jacobian of $X$.

Note that $J(X)$ is not rationally connected, because it does not contain any rational curve. Therefore, $\mathcal{U}(r, d)$ is not rationally connected.
Proposition 6.3. The moduli space $\mathcal{M}_L(r, d)$ is not rational.

Proof. We show that the moduli space $\mathcal{M}_L(r, d)$ is not rationally connected. Let

$$p : \mathcal{M}_L'(r, d) \to \mathcal{U}(r, d)$$

be the morphism of varieties defined in (4). Suppose that $\mathcal{M}_L'(r, d)$ is rationally connected. Then, from Lemma 6.1, $\mathcal{U}(r, d)$ is rationally connected, which is not true. Since $\mathcal{M}_L'(n, d)$ is an open dense subset of $\mathcal{M}_L(n, d)$, $\mathcal{M}_L(n, d)$ is not rational. □

Lemma 6.4 ([15], Corollary 1.3). Let $f : \mathcal{X} \to \mathcal{Y}$ be any dominant morphism of complex varieties. If $\mathcal{Y}$ and the general fibre of $f$ are rationally connected, then $\mathcal{X}$ is rationally connected.

Proposition 6.5. The moduli space $\mathcal{M}_L(r, \xi)$ is rationally connected.

Proof. Consider the dominant morphism

$$q : \mathcal{M}_L'(r, \xi) \to \mathcal{U}(r, \xi)$$

defined in (5.3). As observed earlier every fibre of $q$ is an affine space and hence rationally connected. Since $\mathcal{U}(r, \xi)$ is rationally connected, $\mathcal{M}_L'(r, \xi)$ is rationally connected follows from Lemma 6.4. Now rational connectedness is a birational invariant, and $\mathcal{M}_L'(r, \xi)$ is a dense open subset of $\mathcal{M}_L(r, \xi)$. □

Therefore, we have a natural question.

Question 6.6. Is the moduli space $\mathcal{M}_L(r, \xi)$ rational?

Next, we discuss the numerical effectiveness of the divisor at infinity. In view of Proposition 5.2, we have a compactification $\mathbf{P}(\mathcal{F}_\xi)$ of the moduli space $\mathcal{M}_L'(r, \xi)$ such that the complement $\mathbf{H}_0 := \mathbf{P}(\mathcal{F}_\xi) \setminus \mathcal{M}_L'(r, \xi)$ is the hyperplane at infinity, and we call $\mathbf{H}_0$ the divisor at infinity.

Proposition 6.7. Let $\mathbf{H}_0 := \mathbf{P}(\mathcal{F}_\xi) \setminus \mathcal{M}_L'(r, \xi)$ be the smooth divisor. Then, the divisor $\mathbf{H}_0$ is numerically effective if and only if the vector bundle bundle $\mathcal{P}_\xi$ over the moduli space $\mathcal{U}(r, \xi)$ is numerically effective.

Proof. Recall that the vector bundle $\mathcal{P}_\xi^*$ is numerically effective if and only if the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{P}_\xi)}(1)$ is numerically effective. Let

$$\Psi : \mathcal{F}_\xi \to \mathcal{U}(r, \xi)$$

be the vector bundle in Proposition 5.2. Then, the divisor $\mathbf{H}_0$ is numerically effective if and only if the restriction of the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{F}_\xi)}(\mathbf{H}_0)$ to $\mathbf{H}_0$ is numerically effective.

First we show that the divisor $\mathbf{H}_0$ is canonically isomorphic to projective bundle $\mathbf{P}(\mathcal{P}_\xi)$. Let

$$\tilde{\Psi} : \mathbf{P}(\mathcal{F}_\xi) \to \mathcal{U}(r, \xi)$$

be the projective bundle. Let $E \in \mathcal{U}(r, \xi)$, and

$$\theta \in \tilde{\Psi}^{-1}(E) \cap \mathbf{H}_0 \subset \mathbf{P}(\mathcal{F}_\xi).$$

(6.1)
Then, $\theta$ represents a hyperplane in the fibre $F_\xi(E) = q^{-1}(E)^{\vee}$ of the vector bundle $F_\xi$, where $q$ is defined in (5.3). Let $H_\theta$ denote this hyperplane represented by $\theta$. Note that $H_\theta \subset q^{-1}(E)^{\vee}$ and $q^{-1}(E)$ is the affine space modelled over the vector space $H^0(X, \text{ad}(E) \otimes L^*)$ which is a fibre of the vector bundle $P_\xi$ over $E$. Therefore for any $f \in H_\theta$, that is, for an affine linear map $f : q^{-1}(E) \to \mathbb{C}$, we have
\[ df \in (P_\xi(E))^* = P_\xi^*(E). \]

Since $\theta \in H_0$, and $H_\theta$ is a hyperplane, the subspace of $P_\xi^*(E)$ generated by \{df\}$_{f \in H_\theta}$ is a hyperplane in $P_\xi^*(E)$. Let $\tilde{\theta} \in P(P_\xi^*)$ denote this hyperplane. Thus, we get a canonical isomorphism
\[ H_0 \cong P(P_\xi^*) \] (6.2)
by sending $\theta$ to $\tilde{\theta}$.

Let $N_{P_\xi(E)/H_0}$ denote the normal bundle of the divisor $H_0 \subset P(F_\xi)$. Note that From Poincaré adjunction formula we have the following
\[ O_{P(F_\xi)}(H_0)|_{H_0} \cong N_{P(F_\xi)/H_0}. \] (6.3)
So, $H_0$ is numerically effective if and only if the normal bundle $N_{P(F_\xi)/H_0}$ is numerically effective.

Thus, to prove the proposition it is enough to show that the normal bundle $N_{P(F_\xi)/H_0}$ is canonically isomorphic to the tautological line bundle $O_{P(P_\xi^*)}(1)$.

Next, note that the fibre $N_{P(F_\xi)/H_0}(\theta)$ of the normal bundle $N_{P(F_\xi)/H_0}$ is canonically isomorphic to the quotient $F_\xi(E)/H_\theta$. Consider the morphism
\[ F_\xi(E) \to P_\xi^*(E) \]
between vector spaces defined by sending $f \mapsto df$. Since image of the hyperplane $H_\theta$ is contained in $\tilde{\theta}$, we have well defined morphism on quotients
\[ F_\xi(E)/H_\theta \to P_\xi^*(E)/\tilde{\theta}, \]
which is an isomorphism. Recall that the fibre of the tautological line bundle $O_{P(P_\xi^*)}(1)$ at $E$ is canonically identified with the quotient $P_\xi^*(E)/\tilde{\theta}$. This completes the proof.

A similar result can be proved for the smooth divisor on the compactification of the moduli space $M'_\xi(r,d)$. Let $H := P(F) \setminus M'_\xi(r,d)$ be the smooth divisor at infinity, where $P(F)$ is the compactification of the moduli space $M'_\xi(r,d)$ as in Theorem 4.5.

Then using the same steps as in proof of Proposition 6.7, we can show the following.

**Proposition 6.8.** The smooth divisor $H$ is numerically effective if and only if the vector bundle $P^*$ is numerically effective, where $P^*$ is the dual of the vector bundle defined in (3.6).

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