NONHOMOGENEOUS QUANTUM MARKOV CHAINS AND A NOTION OF ERGODICITY

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Abstract. Motivated by a model presented by S. Gudder [10] [11], we study a quantum generalization of Markov chains and discuss the relation between these maps and open quantum random walks, a class of quantum channels described by S. Attal et al. [2]. We consider processes which are nonhomogeneous in time, i.e., at each time step, a possibly distinct evolution kernel. Inspired by a spectral technique described by L. Saloff-Coste and J. Zúñiga [22], we define a notion of ergodicity for nonhomogeneous quantum Markov chains and describe a criterion for ergodicity of such objects in terms of singular values. As a consequence we obtain a quantum version of the classical probability result concerning the behavior of the columns (or rows) of the iterates of a stochastic matrix induced by a finite, irreducible, aperiodic Markov chain. We are also able to relate the ergodic property presented here with the notions of weak and uniform ergodicity known in the literature of noncommutative $L^1$-spaces.

1. Introduction

The study of asymptotic behavior of trace-preserving completely positive maps, also known as quantum channels, is a fundamental topic in quantum information theory, see for instance [6, 7, 17, 18, 22, 23]. More recently, an important class of quantum channels, namely Open Quantum Random Walks (OQRWs) has been introduced by S. Attal et al. [2] and its long term behavior studied [3, 15, 16, 24]. OQRWs are such that probability calculations can be expressed in terms of a trace functional (on a noncommutative domain) but their asymptotic limits are seen to be expressed in terms of a trace functional (on a noncommutative domain) but their asymptotic limits are seen to converge to the unique invariant distribution vector for a quantum transition matrix (QTM) and we obtain a natural correspondence between such matrix and stochastic kernel [5]. In our setting this means that we may consider at each time step a distinct quantum channel and study its long term behavior. In terms of classical probability, this corresponds to the problem of studying a time-nonhomogeneous Markov chain: at each (discrete) time step, a possibly distinct evolution kernel. Inspired by a spectral technique described by L. Saloff-Coste and J. Zúñiga [27], we define a notion of ergodicity known in the literature of noncommutative $L^1$-spaces.

Instead of the conditions $0 \leq A_i \leq I$, $\sum_i A_{ij} = I$, one may ask what happens if we allow any matrices $A_{ij}$ satisfying $\sum_i A_{ij}^* A_{ij} = I$, all $j$, and replace the sequential product by the operation $(A, B) \mapsto ABA^*$. The object $A = (A_{ij})$ will then be called a quantum transition matrix (QTM) and we obtain a natural correspondence between such matrix and an OQRW (to be reviewed later). Then we may consider a sequence of QTMs $A = \{A_1, A_2, \ldots\}$ (a nonhomogeneous quantum Markov chain) and study its long term behavior. In terms of classical probability, this corresponds to the problem of studying a time-nonhomogeneous Markov chain: at each (discrete) time step, a (possibly) distinct stochastic kernel [5]. In our setting this means that we may consider at each time step a distinct quantum channel and we would like to describe facts about its asymptotic limit. To achieve this goal, we will define a notion of ergodicity of sequences of QTMs with this we are able to establish an ergodicity criterion in terms of singular values. We will restrict to sequences of unital (unit preserving) QTMs acting on a finite number of sites.

As a consequence of this result, we are able to examine the following problem. We recall from classical probability theory that a finite, irreducible, aperiodic Markov chain is such that the columns (rows) of the iterates of the associated column (row) stochastic matrix $P^r$ converge to the unique invariant distribution vector for $P$, as $r \to \infty$ [21]. In this work we prove an analogous property for QTMs satisfying certain conditions.

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The structure of this work is as follows. In Section 2 we establish basic terminology and notations, review OQRWs, TEMs and define QTMs. In Section 3 following [8,9], we briefly discuss irreducibility and periodicity of channels, as this will serve as a review and as a motivation for our discussion on ergodicity of sequences of QTMs. In Section 4 we discuss two basic examples of QTMs. Section 5 presents a notion of ergodicity for sequences of QTMs (Definition 6.4) and the spectral construction needed for our main result, presented in Section 6. This construction is closely inspired by results presented by L. Saloff-Coste and J. Zúñiga [27] in a classical setting, concerning time-nonhomogeneous Markov chains on a finite state space. In Section 7 we relate the notion of ergodicity presented here with weak ergodicity seen in a noncommutative $L^1$-space setting [20]. We conclude with some open questions.

Understanding ergodicity is a basic goal in the study of Markovian dynamics and similar models but, up to our knowledge, results on ergodicity of sequences of quantum channels besides the case of taking iterations of a single channel are scarce. It is an interesting fact that some classical techniques can be adapted to quantum channels and we hope that this work will stimulate further research on open quantum walks. A different but related work where knowledge, results on ergodicity of sequences of quantum channels are presented in Section 6.

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2. Preliminaries: Quantum Transition Matrices and Open Quantum Random Walks

First we briefly review the setting given by [10,11] and then propose a variation of it. Let $\mathcal{H}$ denote a complex separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. The set of positive definite operators on $\mathcal{H}$ is denoted $B^+(\mathcal{H})$ and the set of positive trace class operators on $\mathcal{H}$ is denoted $\mathcal{T}^+(\mathcal{H})$. Let

$$\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : 0 \leq A \leq I\}$$

$$\mathcal{D}(\mathcal{H}) = \{D \in \mathcal{T}^+(\mathcal{H}) : tr(D) = 1\}$$

denote the set of **effects** and **density operators**, respectively. A **vector state** is a finite or infinite sequence $S = (S_1, S_2, \ldots)$, which can be seen as a distribution vector, where $S_i \in \mathcal{T}^+(\mathcal{H})$ and $\sum_i tr(S_i) = 1$. Denote the convex set of vector states by $\mathcal{S}(\mathcal{H})$.

An **effect matrix** is a finite or infinite square matrix $A = [A_{ij}]$, where $A_{ij} \in \mathcal{E}(\mathcal{H})$, for all $i,j$. A **transition effect matrix** (TEM) is an effect matrix where $\sum_i A_{ij} = I$ for every $j$ and the convergence is in the strong operator topology. The **sequential product** of two effects $A, B \in \mathcal{E}(\mathcal{H})$ is given by

$$A \circ B = A^{1/2}BA^{1/2},$$

where $A^{1/2}$ is the unique positive square root of $A$. For a TEM $A$ and a vector state $S \in \mathcal{S}(\mathcal{H})$, define $A \circ S$ by

$$(A \circ S)_i = \sum_j A_{ij} \circ S_j$$

It is a simple matter to show that $A \circ S$ is a vector state. If $A$ and $B$ are TEMs of the same size on $\mathcal{H}$ we say that $A \circ B$ is **defined** if $\sum_k A_{ik} \circ B_{kj}$ converges in the strong operator topology to an effect $C_{ij}$ for all $i,j$. We define $(A \circ B)_{ij} = C_{ij}$ for all $i,j$ and write $C = A \circ B$. We say a TEM $A$ is **finite** if its effect matrix $A = (A_{ij})$ is such that $i,j$ runs over a set of indices which is finite and every effect matrix $A_{ij}$ is finite dimensional.

As mentioned in the introduction, we propose to study a variation of TEMs: instead of the condition $0 \leq A_i \leq I$, $\sum_i A_{ij} = I$, we will allow any matrices $A_{ij}$ satisfying $\sum_i A^*_{ij}A_{ij} = I$, all $j$, and replace the sequential product by the operation $(A, B) \mapsto ABA^*$. The object $A = (A_{ij})$ will define a **quantum transition matrix** (QTM), and the product of a QTMs by a vector state is given by $(A(S))_i = \sum_j A_{ij}S_jA^*_{ij}$. For a QTMs composed of an order 2 transition matrices, we write

$$A(S) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} := \begin{bmatrix} A_{11}S_1A^*_{11} + A_{12}S_2A^*_{12} \\ A_{21}S_1A^*_{21} + A_{22}S_2A^*_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad tr(S_1) + tr(S_2) = 1,$$

and analogously for higher dimensions. In a similar way as in TEMs, we can define the product of QTMs, and this is defined in analogy with the usual product of matrices. In dimension 2,

$$A^2 = \begin{bmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{bmatrix}. $$
and analogously for $A^r$, $r \geq 3$, and for higher dimensions, taking into consideration the noncommutativity of the terms. A sequence of QTMs will be called a nonhomogeneous quantum Markov chain (QMC). If the sequence consists of the iterates of a fixed QTM then we have a homogeneous quantum Markov chain. That is, the term nonhomogeneous means that the transition kernel employed at each time step $t$ depends on $t$.

In order to describe the correspondence between QTMs and quantum channels, we review basic facts about completely positive maps, more details of which can be seen for instance in [11, 24, 26, 29, 30]. Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be linear. We say $\Phi$ is a positive operator whenever $\rho \geq 0$ implies $\Phi(\rho) \geq 0$. Define for each $k \geq 1$, $\Phi_k : M_k(M_n(\mathbb{C})) \to M_k(M_n(\mathbb{C})), \quad (2.4) \quad \Phi_k(A) = [\Phi(A_{ij})], \quad A \in M_k(M_n(\mathbb{C})), \quad A_{ij} \in M_n(\mathbb{C})$

We say $\Phi$ is $k$-positive if $\Phi_k$ is positive, and we say $\Phi$ is completely positive (CP) if $\Phi_k$ is positive for every $k = 1, 2, 3, \ldots$. It is well-known that CP maps can be written in the Kraus form [26]:

$$(2.5) \quad \Phi(\rho) = \sum_i B_i \rho B_i^*$$

We say $\Phi$ is trace-preserving if $\text{tr}(\Phi(\rho)) = \text{tr}(\rho)$ for all $\rho \in M_d(\mathbb{C})$, which is equivalent to $\sum_i B_i^* B_i = I$. We say $\Phi$ is unital if $\Phi(I) = I$, which is equivalent to $\sum_i B_i^* B_i = I$. Trace-preserving completely positive (CPT) maps are also called quantum channels. Also recall that if $A \in M_d(\mathbb{C})$ there is the corresponding vector representation $\text{vec}(A)$ associated to it, given by stacking together the matrix rows. For instance, if $d = 2$,

$$(2.6) \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \text{vec}(A) = [a_{11} \ a_{12} \ a_{21} \ a_{22}]^T.$$  

The vec mapping satisfies $\text{vec}(AXB^T) = (A \otimes B)\text{vec}(X)$ for any $A, B, X$ square matrices [13] so in particular, $\text{vec}(BX^*B^T) = (B^T \otimes B)\text{vec}(X)$, from which we can obtain the matrix representation $[\Phi]$ for the CP map (2.5):

$$(2.7) \quad [\Phi] = \sum_i B_i \otimes B_i = \sum_{i,j,k,l} \langle E_{kl}, \Phi(E_{ij}) \rangle E_{ki} \otimes E_{lj}$$

We recall the well-known fact that the matrix representation of a CPT map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is independent of the Kraus representation considered. The proof of this result is a simple consequence of the unitary equivalence of Kraus matrices for a given quantum channel [26].

**Remark 2.1. QTMs and OQRWs.** There is a clear correspondence between QTMs and quantum channels describing Open Quantum Random Walks (OQRWs) [2]. For completeness we describe it here. Let $\mathcal{K}$ denote a separable Hilbert space and let $\{|i\rangle\}_{i \in \mathcal{Z}}$ be an orthonormal basis for such space (in case $\mathcal{K}$ is infinite dimensional). The elements of such basis will be called sites (or vertices). Let $\mathcal{H}$ be another Hilbert space, which will describe the degrees of freedom given at each point of $\mathcal{Z}$. Then we will consider the space $\mathcal{H} \otimes \mathcal{K}$. For each pair $i, j$ we associate a bounded operator $B_{ij}$ on $\mathcal{H}$. This operator describes the effect of passing from $|j\rangle$ to $|i\rangle$. We will assume that for each $j, \sum_i B_{ij}^* B_{ij} = I$, where, if infinite, such series is strongly convergent. This constraint means: the sum of all the effects leaving the site $j$ is $I$. We will consider density matrices on $\mathcal{H} \otimes \mathcal{K}$ with the particular form $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$, assuming that $\sum_i \text{tr}(\rho_i) = 1$ (see Remark 2.2 below). For a given initial state of such form, the OQRW induced by the $B_{ij}$ is, by definition,

$$(2.8) \quad \Phi(\rho) = \sum_i \left( \sum_j B_{ij}^* \rho_j B_{ij} \right) \otimes |i\rangle\langle i|.$$  

We say the OQRW is finite if $\text{dim}(\mathcal{H}) < \infty$ and if it acts on a finite number of sites, that is, $\text{dim}(\mathcal{K}) < \infty$. Now let $A = (A_{ij})$ denote a QTM. If $S = (S_i)$ is a vector state then it is clear that the calculation for $A(S)$, defined by

$$(2.9) \quad A(S) = \begin{bmatrix} A_{11} & A_{12} & \cdots & S_i \\ A_{21} & A_{22} & \cdots & S_2 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & S_i \end{bmatrix} := \begin{bmatrix} \sum_i A_{11} S_i A_{11}^* \\ \vdots \\ \vdots \\ \sum_i A_{ii} S_i A_{ii}^* \end{bmatrix}$$

can be identified with the channel iteration

$$(2.10) \quad \Phi_A(S) = \sum_i \left( \sum_j A_{ij} S_j A_{ij}^* \right) \otimes |i\rangle\langle i|.$$  

Comparing (2.9) and (2.10) we realize that the projections $|i\rangle\langle i|$ are being used simply as an index in the vector state space. We conclude that a QTM induces an OQRW in a natural way and conversely.
Proposition 3.1. Let \( \rho \) be a CP map of the form (3.2). Then \( \Phi \) is irreducible if and only if for any \( v \in \mathcal{H} \setminus \{0\} \), the set \( \mathbb{C}[B]v \) is dense in \( \mathcal{H} \).

We can also consider irreducibility conditions for OQRWs. In fact, let \( V = \{1, \ldots, n\} \) be a finite set. For \( i, j \in V \), we call a path from \( i \) to \( j \) any finite sequence \( i_0, \ldots, i_l \) in \( V \) with \( l \geq 1 \) such that \( i_0 = i, i_l = j \). Such path is said to be of length \( l \). We denote by \( \mathcal{P}(i, j) \) the set of paths from \( i \) to \( j \) with arbitrary length. For \( p = (i_0, \ldots, i_j) \in \mathcal{P}(i, j) \), we denote by \( L_p \) the operator from \( \mathcal{H}_i \) to \( \mathcal{H}_j \),

\[
L_p = L_{i_li_{l-1}} \cdots L_{i_1i_0} = L_{ji_{j-1}} \cdots L_{i_1i_0}
\]

Finally, recall that a set \( M \subset W, W \) a topological vector space, is a total set if the linear span of \( M \) is dense in \( W \). The following result is presented in [9], Prop. 3.9.

Proposition 3.2. A CPT map \( \Phi : \mathcal{H} \to \mathcal{H} = \bigoplus_{i \in V} \mathcal{H}_i \)

\[
\Phi(\rho) = \sum_{i,j \in V} B_{ij} \rho B_{ij}^*, \quad B_{ij} = L_{ij} \otimes |i\rangle \langle j|, \quad \sum_{i \in V} L_{ij}^* L_{ij} = I, \quad \forall j
\]

is irreducible if and only if for every \( i, j \in V \) one of the following equivalent conditions holds:

1. For any \( v \in \mathcal{H}_i \setminus \{0\} \), the set \( \{L_pv : p \in \mathcal{P}(i, j)\} \) is total in \( \mathcal{H}_j \).
2. For any \( v \in \mathcal{H}_i \setminus \{0\} \) and \( w \in \mathcal{H}_j \setminus \{0\} \) there exists \( p \in \mathcal{P}(i, j) \) such that \( \langle w, L_pv \rangle \neq 0 \).

Definition 3.3. Let \( \Phi \) be a positive trace-preserving irreducible map and let \( (P_0, \ldots, P_{d-1}) \) be a resolution of identity, i.e., a family of orthogonal projections such that \( \sum_{k=0}^{d-1} P_k = I_d \). Then we say that \( (P_0, \ldots, P_{d-1}) \) is \( \Phi \)-cyclic if \( \Phi^k(P_k) = P_k^\perp \), for \( k = 0, \ldots d - 1 \). The supremum of all \( d \) for which there exists a \( \Phi \)-cyclic resolution of identity \( (P_0, \ldots, P_{d-1}) \) is called the period of \( \Phi \). If \( \Phi \) has period 1 then we call it aperiodic.

With these definitions one may obtain an operator version of the classical Markov chain result [21]. The following has been presented in [9], Theorem 4.18.

Theorem 3.4. Let \( \Phi \) be an irreducible, aperiodic and finite OQRW. For any state \( \rho \) the sequence \( \Phi^r(\rho) \) converges to the invariant state \( \rho_\perp \), which is unique and faithful.

At this point we can state a question. We recall that the classical version of Theorem 3.4 for column stochastic matrices \( P \) [21] implies that the columns of \( P^r \) converge to the unique invariant distribution vector for \( P \), as \( r \to \infty \). One can ask for an analogous property for quantum channels. By considering the matrix representation \( [\Phi] \) of an irreducible, aperiodic, finite quantum channel, does every column converge to a limit distribution? The answer to this question is easily shown to be negative in general. Due to the connection between OQRWs and QTMs, one can ask...
instead the more refined question: given a QTM $A$ and its matrix expression $A = [A_{ij}]$, $\sum_j A_{ij} = I$, all $j$, does the columns of its iterates converge to some kind of limit vector state? In the rest of this work we will present an affirmative answer, under certain conditions, in terms of a notion of ergodicity of sequences of QTMs, to be defined in Section 4.3. We note that Theorem 3.4 is closely related to this question. We aim to present an alternative description which we believe will further clarify this dynamical behavior, possessed by many QTMs and its associated OQRWs.

4. Two examples

Example 4.1. Asymptotic limits for classical QTMs. In this example and the following lemmas we illustrate how classical stochastic matrices may be described in terms of the setting of QTMs. Many aspects of this translation are straightforward, but some details are worth mentioning explicitly. We remark that all QTMs studied in this work are finite (a finite number of finite dimensional matrices $A_{ij}$). For simplicity of notation the following results concern order 2 QTMs $A = (A_{ij})$, $i, j = 1, 2$, where each $A_{ij}$ also has order 2, but generalizations to larger (finite) orders are clear. We say $A$ is a classical bistochastic QTM if $A_{ij} = \sqrt{p_{ij}}$, where $P = (p_{ij})$ is a real bistochastic matrix of finite dimension.

Lemma 4.2. Let $A$ be an order 2 classical bistochastic QTM. a) Let $\rho = (\rho_1, \rho_2)^T$ be a vector state and $v = (v_1, v_2)^T$ probability vector such that $v_i = tr(\rho_i)$, $i = 1, 2$. Then $(Pv)_i = tr(A(\rho))_i$, $i = 1, 2$. b) If $\rho_1 = |x_1\rangle\langle 1|$ and $\rho_2 = |x_2\rangle\langle 2|$, $x_1 + x_2 = 1$, $x_i > 0$, then

\[
A(\rho) = \begin{bmatrix}
p_{11}x_1 & 0 \\
0 & p_{12}x_2 \\
p_{21}x_1 & 0 \\
0 & p_{22}x_2
\end{bmatrix}
\]

Proof. a) Let

\[
\rho_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} f & g \\ h & j \end{bmatrix}, \quad v = \begin{bmatrix} a + d \\ f + j \end{bmatrix}
\]

We have

\[
A(\rho) = \begin{bmatrix}
p_{11}a + p_{12}f & p_{11}b + p_{12}g \\
p_{11}c + p_{12}h & p_{11}d + p_{12}j \\
p_{21}a + p_{22}f & p_{21}b + p_{22}g \\
p_{21}c + p_{22}h & p_{21}d + p_{22}j
\end{bmatrix}, \quad P = \begin{bmatrix} p_{11} + p_{12}f + p_{11}d + p_{12}j \\
p_{21}a + p_{22}f + p_{21}d + p_{22}j
\end{bmatrix}
\]

The proof of b) is immediate from a).

Lemma 4.3. If $P$ is an order 2, aperiodic, irreducible, bistochastic real matrix with stationary vector $\pi = (\pi_1, \pi_2)^T$ then a) for the classical bistochastic QTM $A$ associated to $P$,

\[
A^r \rightarrow \begin{bmatrix} \pi_1I & \pi_1I \\
\pi_2I & \pi_2I \end{bmatrix}, \quad r \rightarrow \infty
\]

b) For any vector state $\rho = (\rho_1, \rho_2)^T$, with entries as in (4.2), if

\[
B = \begin{bmatrix} \pi_1I & \pi_1I \\
\pi_2I & \pi_2I \end{bmatrix}
\]

then

\[
B(\rho) = \begin{bmatrix}
\pi_1 \begin{bmatrix} a + f & b + g \\ c + h & d + j \end{bmatrix} \\
\pi_2 \begin{bmatrix} a + f & b + g \\ c + h & d + j \end{bmatrix}
\end{bmatrix} = \begin{bmatrix} \pi_1(\rho_1 + \rho_2) \\
\pi_2(\rho_1 + \rho_2) \end{bmatrix}
\]
c) For the vector state \( \rho \) with \( \rho_1 = x_1|1\rangle\langle 1 | \) and \( \rho_2 = x_2|2\rangle\langle 2 | \), \( x_1 + x_2 = 1 \), \( x_i > 0 \), and for the classical bistochastic QTM \( A \) associated to \( P \) which is aperiodic and irreducible, \( (A^r)\rho = \) \[
\begin{bmatrix}
p_{11}^{(r)} x_1 & 0 \\
0 & p_{12}^{(r)} x_2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\pi_1 [x_1 & 0 \\
0 & x_2]
\end{bmatrix}, \ r \to \infty
\]

Proof. a) and b) are immediate from the classical calculation. c) Follows from the classical form of \( \rho_{ij}^{(r)} \), the \((i,j)\)-th entry of \( P^r \).

Remark 4.4. The fact that the columns of the iterates of a QTM cannot converge to a vector state should already be clear from the structure we have: on one hand we must have for a QTM \( A = (A_{ij}) \) that \( \sum_p A_{ij}^p A_{ij}^p = I \) for all \( j \), but on the other a vector state \( \rho = (\rho_i) \) must satisfy \( \sum_i \text{tr}(\rho_i) = 1 \). That is, for a column \( (A_{i1}, A_{i2}, \ldots, A_{in})^T \) to be equal to \( \rho \) is impossible in general. This is also indicated by equation (4.7). Nevertheless, if we consider the case in which all matrices \( A_{ij} \) and \( \rho_i \) are one-dimensional then we recover the classical case. As a conclusion, we see that in terms of quantum analogs of the behavior of the columns of the iterates of a finite, aperiodic, irreducible Markov chain, one should take in consideration these computational differences.

Remark 4.5. It is a simple matter to produce an example of a QTM \( A \) which is not classical bistochastic, but still its asymptotic limit \( A^r \) as \( r \to \infty \) equals \((4.7)\). For instance, let \[
A_{11} = A_{22} = \left[ \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{2} \right], \ A_{12} = A_{21} = (I - A_{11}^2)^{1/2}
\]
Then a calculation shows that the QTM \( A = (A_{ij}) \) is such that \( A^r \) converges to \((4.7)\) with \( \pi_1 = \pi_2 = 1/2 \). Motivated by irreducible aperiodic classical bistochastic QTMs and by this example, we would like to determine when this asymptotic behavior occurs in other classes of examples.

We state the following basic fact for later reference.

Lemma 4.6. Let \( P = (p_{ij}) \) be an order 2 bistochastic matrix and \( \Phi(\rho) = \sum_{ij} B_{ij} \rho B_{ij}^* \), \( B_{ij} = \sqrt{p_{ij}} I \otimes E_{ij} \), the OQRW induced by the classical bistochastic QTM \( A = (\sqrt{p_{ij}} I) \), \( i, j = 1, 2 \), \( I \) being the order 2 identity matrix. Then a) The eigenvalues of \( P \) are \( 1, 2p_{11} - 1 \). b) The nonzero eigenvalues for \( \Phi \) are 1 and \( 2p_{11} - 1 \), with multiplicity 4 each.

Example 4.7. Obtaining QTMs from quantum channels. We have seen that every QTM can be seen as a channel acting on one particle on a larger space. This is evident by the definition of \( B_{ij} \) in expression (3.4). In this example we address the converse question: given a quantum channel, can it be described in terms of a channel acting on sites, i.e., an OQRW? We give a simple answer which may be of independent interest: let \( \Phi(\rho) = \sum_{i=1}^n V_i \rho V_i^* \), \( V_i \in M_k(\mathbb{C}) \), and consider the unital QTM
\[
A_{\Phi} = \begin{bmatrix}
V_1 & V_2 & \cdots & V_n \\
V_2 & V_3 & \cdots & V_1 \\
\vdots & \vdots & \ddots & \vdots \\
V_n & V_1 & \cdots & V_{n-1}
\end{bmatrix}
\]
Then, restricted to column operators of the form \( \alpha = [\rho \rho \cdots \rho]^T \) (i.e., equal entries), we obtain
\[
A_{\Phi}(\alpha) = [\Phi(\rho) \Phi(\rho) \cdots \Phi(\rho)]^T.
\]
Note that \( tr(\alpha) = kn \). We regard \((4.9)\) as a codification of the channel \( \Phi \) as an OQRW. Clearly the procedure involved is computationally redundant and makes use of a non-normalized positive column operator. Nevertheless, we will see that our results on ergodicity, stated in terms of QTMs and QMCs, can be applied to sequences of quantum channels due to the method described in this example. Also see Examples \[5.3 \] and \[5.5 \]

Remark 4.8. Despite the fact that many examples in this work will consider QTMs which are induced by a quantum channel, it should be clear that not every QTM arises in such form, even in the particular case of unital channels/QTM. For instance, one can construct a classical bistochastic QTM induced by a nonsymmetric bistochastic real matrix \( P \in M_3(\mathbb{C}) \).
5. A Spectral Technique for QMCs

Recall that the singular values of an operator $T$ are the square roots of the eigenvalues of the map $T^*T$. In this section we consider a quantum channel $\Phi_A = \Phi$ which is unital, induced by an order $n$ QTM $A$ given by order $k$ matrices, that is, $A = (A_{ij})$, $\dim A_{ij} = k$, all $i, j = 1, \ldots, n$. We let $\sigma_i(\Phi)$ be the $i$-th singular value of $\Phi$, $i = 1, 2, \ldots$, arranged in non-increasing order. It is usual to say that $n$ is the number of sites. Note that for any QTM, $\sigma_1(\Phi) = 1$ and $\sigma_i(\Phi) \in [0, 1]$. Define an inner product for columns,

$$\langle \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \rangle_2 := \langle A_1, B_1 \rangle_2 + \langle A_2, B_2 \rangle_2 + \cdots + \langle A_n, B_n \rangle_2,$$

where in the right hand side the inner product appearing is the Hilbert–Schmidt product $\langle X, Y \rangle_2 := tr(X^*Y)$. Denote by $\| \cdot \|_2$ the norm induced by such product. Note that, as a block matrix, a QTM is an order $kn$ matrix (n rows of order k matrices). Inspired by the asymptotic behavior of the unital examples examined, we would like to compare the distance between the columns of a QTM and the column $\pi = \left[ \frac{1}{n} I \cdots \frac{1}{n} I \right]^T$. We have for $A = (A_{ij})$ QTM, for all $j$, 

$$\sum_{i=1}^n \left\| A_{ij}^*A_{ij} - \frac{I}{n} \right\|^2_2 = \sum_{i=1}^n \langle A_{ij}^*A_{ij}, A_{ij}^*A_{ij} \rangle_2 - 2 \sum_{i=1}^n \langle A_{ij}^*A_{ij}, \frac{I}{n} \rangle_2 + n \langle \frac{I}{n}, \frac{I}{n} \rangle_2$$

$$= \sum_{i=1}^n tr((A_{ij}^*A_{ij})^2) - 2 \langle I, \frac{I}{n} \rangle_2 + \frac{1}{n} \langle I, I \rangle_2 = \sum_{i=1}^n tr((A_{ij}^*A_{ij})^2) - \frac{k}{n}$$

(5.2)

**Remark 5.1.** Due to the expression for the matrix representation we see that an OQRW $\Phi$ on $n$ sites, with each transition matrix being an order k matrix, is such that $[\Phi]$ is an order $N_\Phi = (kn)^2$ matrix. To see this, use eq. (2.7) with (3.4). Then, for instance, for a QTM acting on $n$ sites and order 2 transition matrices, the induced OQRW has an order $4n^2$ matrix representation, and a QTM on 2 sites and order 2 transition matrices $A = (A_{ij})$, $A_{ij} \in M_2(\mathbb{C})$, $i, j = 1, 2$, induces an OQRW with an order 16 matrix representation.

With (5.2) in mind, we perform another calculation. Consider an orthonormal basis of eigenstates for $\Phi^*\Phi$ given by $B = \{ \eta_i \}_{i=1}^{N_\Phi}$, such that

$$\eta_1 = \frac{1}{\sqrt{kn}} \left( I \otimes |1\rangle \langle 1| + \cdots + I \otimes |n\rangle \langle n| \right), \quad I = I_k \in M_k(\mathbb{C}),$$

(5.3)

where $I = I_k$ is the order $k$ identity matrix. Note that $\| \eta_1 \| = 1$. Recall that by Remark 2.2 whatever is the initial state $\rho$ on $\mathcal{H} \otimes \mathcal{K}$, the density $\Phi(\rho)$ is of the form $\sum_i \rho_i \otimes \langle i | \langle i |$. This imposes a restriction on the kind of eigenstates present in $B$. Define

$$\rho_1 := \left[ I \ 0 \ \ldots \ 0 \right]^T = \sum_{i=1}^{N_\Phi} d_i \eta_i, \quad d_i \in \mathbb{C},$$

(5.4)

with $N_\Phi$ as in Remark 5.1. We have

$$\langle \Phi^*\Phi \rho_1, \rho_1 \rangle_2 = \langle \Phi^*\Phi \sum_i d_i \eta_i, \sum_j d_j \eta_j \rangle = \sum_{i,j} d_i d_j \langle \Phi^*\Phi \eta_i, \eta_j \rangle = \sum_{i=1}^{N_\Phi} |d_i|^2 \sigma_i^2$$

(5.5)

Also,

$$\Phi(\rho_1) = \begin{bmatrix} A_{11} I A_{11}^* \\ A_{21} I A_{21}^* \\ \vdots \\ A_{n1} I A_{n1}^* \end{bmatrix},$$

(5.6)
so

\[ (5.7) \quad \langle \Phi(\rho_1), \Phi(\rho_1) \rangle_2 = \left\langle \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{bmatrix} \right| \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{bmatrix} \right\rangle_2 = tr((A_{11}^* A_{11})^2) + tr((A_{21}^* A_{21})^2) + \cdots + tr((A_{n1}^* A_{n1})^2). \]

Therefore on one hand we have

\[ (5.8)\quad \sum_{i=1}^n \left\| A_{i1}^* A_{i1} - I/n \right\|^2 = \sum_{i=1}^n \left\| A_{i1}^* A_{i1} - I/n \right\|^2 = \sum_{i=1}^n [d_i]^2 \sigma_i^2, \]

and on the other we obtained, by \( (5.2) \),

\[ (5.9) \quad \sum_{i=1}^n \left\| A_{i1}^* A_{i1} - I/n \right\|^2 = \sum_{i=1}^n \left\| A_{i1}^* A_{i1} - I/n \right\|^2. \]

Finally, note that \( d_1 = \langle \rho_1, \eta_1 \rangle = \frac{1}{\sqrt{n}} tr(I) = \frac{1}{\sqrt{n}} \) and so \( |d_1|^2 = \frac{1}{n}. \) We can repeat an analogous reasoning where we define \( \rho_2, \rho_3, \ldots \) in a similar way as \( \rho_1 \) in \( (5.4) \). We have proved the following:

**Theorem 5.2.** Let \( A = (A_{ij}) \) be a unital QTM (i.e., the induced OQRW preserves the identity) and \( \Phi_A = \Phi : \bigoplus_{i=1}^n M_k(\mathbb{C}) \rightarrow \bigoplus_{i=1}^n M_k(\mathbb{C}) \) be the induced OQRW on \( n \) sites and action given by order \( k \) matrices. Let \( \sigma_i = \sigma_i(\Phi), \) \( i = 1, \ldots, n \) be the singular values of \( \Phi. \) Let \( \{\eta_i\}_{i=1}^{N_k} \) denote an orthonormal basis of eigenstates for \( \Phi^* \Phi, \) associated to eigenvalues arranged in non-increasing order, with \( \eta_i \) given by \( (5.3) \). Then for all \( j = 1, 2, \ldots, n, \)

\[ (5.10) \quad \sum_{i=1}^n \left\| A_{i1}^* A_{i1} - I/n \right\|^2 = \sum_{i=1}^n [d_i]^2 \sigma_i^2, \]

where for each \( j, \) \( [0 \cdots I \cdots 0]^T = \sum_i d_{ij} \eta_i, \) with \( I \) the order \( k \) identity appearing in the \( j \)-th position.

**Example 5.3.** Consider the following unital QTM, induced by the bit-flip channel on 1 qubit \( [23], \)

\[ (5.11) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{11} \end{bmatrix}, \quad A_{11} = \frac{1}{\sqrt{3}} I, \quad A_{12} = \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

In the notation of the theorem, we have in this example \( n = k = 2, \) so the matrix representation of the channel \( \Phi_A = \Phi \) induced by \( A \) has order \( N_k = (kn)^2 = 16. \) The singular values of \( \Phi_A \) are 1 (multiplicity 4), 1/9 (multiplicity 4) and 0 (multiplicity 8). Some eigenstates associated to the nonzero singular values are

\[ (5.12) \quad \eta_{1;1} = \frac{1}{2} \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |0\rangle \langle 0| + \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |1\rangle \langle 1|, \quad \eta_{1;2} = \frac{1}{2} \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |0\rangle \langle 0| + \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |1\rangle \langle 1| \]

\[ (5.13) \quad \eta_{1/9;1} = \frac{1}{\sqrt{2}} \left( \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |0\rangle \langle 0| + \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \otimes |1\rangle \langle 1|, \quad \eta_{1/9;2} = \frac{1}{\sqrt{2}} \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |0\rangle \langle 0| + \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \otimes |1\rangle \langle 1| \]

We note that \( \eta_{1/9;1} \) and \( \eta_{1/9;2} \) have norm 1, and, recalling expression \( (5.3), \) we note that \( \eta_{1;1} + \eta_{1;2} = \eta_1, \) which has norm 1, and we get for \( \rho_1 = [I \ 0]^T = I \otimes |0\rangle \langle 0| \) the decomposition

\[ (5.14) \quad \rho_1 = \eta_1 + \frac{1}{\sqrt{2}} \left( -\eta_{1/9;1} + \eta_{1/9;2} \right) \]

That is, we have \( \rho_1 = \sum_i d_{i1} \eta_i, \) where

\[ (5.15) \quad d_{11} = 1, \quad d_{21} = d_{1/9;1} = -\frac{1}{\sqrt{2}}, \quad d_{31} = d_{1/9;2} = \frac{1}{\sqrt{2}} \]

As for the right hand side of eq. \( (5.10) \) we note that this summation has, in principle, 16 terms, by Remark \( (5.1), \) but in this case several of them are actually zero. We obtain, for \( j = 1, \)

\[ (5.16) \quad \sum_{i=1}^{16} |d_{i1}|^2 \sigma_i^2 = 1 + |d_{1/9;1}|^2 \frac{1}{9} + |d_{1/9;2}|^2 \frac{1}{9} = 1 + \frac{1}{2} \frac{1}{9} + \frac{1}{2} \frac{1}{9} = 1 + \frac{1}{9} \]

On the other side,

\[ (5.17) \quad \sum_{i=1}^2 \left\| A_{i1}^* A_{i1} - I/2 \right\|^2 = \left\| \begin{bmatrix} -\frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\|^2 = \frac{1}{18} + \frac{1}{18} = \frac{1}{9} = \sum_{i=2}^{16} |d_{ij}|^2 \sigma_i^2, \]
as expected by Theorem 5.4, since \( d_{11} = 1 \).

Now let \( \mathcal{A} = \{A_i\}_{i \in \mathbb{N}} \) denote a sequence of QTMs (i.e., a QMC). We use the notation \( A_{i,i} = I \) and
\[
A_{p,q} = A_{p+1} \cdots A_q, \quad p \leq q,
\]
where by the above product we mean the QTM associated to the product of matrix representations of the QTMs \( A_{p+1}, \ldots, A_q \) (equivalently, the product of the associated OQRWs). In the special case of a homogeneous QTM \( \mathcal{A} = \{A^r\}_{r=1}^\infty \) the expression \( A_{p,q} \) equals \( A_{q-p} = A^r \) (product of matrix representations). An advantage on employing matrix representations is that this allows us to make use of well-known results on the singular values of the product of matrices. We make this choice even though this is a matrix which is larger than, say, the Kraus matrices of a channel. Moreover, we denote by \( A_i(l,m) \) the matrix appearing in the \((l,m)\)-th position of \( A_i \) (and not the \((l,m)\)-th numeric entry of the QTM).

**Definition 5.4.** Fix a vector state \( \rho_\pi = [\rho_1 \ldots \rho_n]^T \), \( \sum_i tr(\rho_i) = 1 \), with \( \rho_i \in M_k(\mathbb{C}) \). Let \( \mathcal{Q} = \{Q_1, \ldots, Q_q\} \) be a finite family of QTMs, all admitting \( \rho_\pi \) as an invariant measure. We say that the pair \((\mathcal{Q}, \rho_\pi)\) is **ergodic** if for any QMC defined by the sequence of QTMs \( \mathcal{A} = \{A_i\}_{i \in \mathbb{N}} \) with invariant measure \( \rho_\pi \), such that \( A_i \in \mathcal{Q} \) for infinitely many \( i \)'s, we have for all \( i, j, k \in \{1, \ldots, n\} \),
\[
\lim_{r \to \infty} A_{0,r}(i,j) - A_{0,r}(i,k) = 0.
\]
That is, we verify whether all columns of the resulting product are becoming equal. In [27] the authors remark that if \( \mathcal{Q} \) is an irreducible Markov kernel then the property that the pair \((\mathcal{Q}, \rho_\pi)\) is ergodic in the sense just defined is stronger than the property that
\[
\lim_{n \to \infty} Q^n(x,y) = Q^n(y,z) = 0, \quad \forall \ x, y, z,
\]
which is satisfied if and only if \( \mathcal{Q} \) is aperiodic.

Now we define a notion of distance between columns. In the case of unital channels we will fix the maximally mixed column \( \pi = [I/n \cdots I/n]^T \), \( I \) being the identity matrix of order \( k \), and calculate the distance from a given column to \( \pi \). It is worth noting once again that \( \pi \), being a column of a QTM, is not a vector state. However, we will see that this produces a consistent limit theorems (see equation (4.1) and Remark 4.4). Define
\[
d_2(\mu; \pi) = \left( \sum_{i=1}^{n} \left\| \mu(i) \mu(i)^* - \frac{I}{n} \right\|_2^2 \right)^{1/2},
\]
A typical choice of \( \mu \) will be \( [A_{1j} \cdots A_{nj}]^T \), for every \( j \) (see for instance the calculation made in eq. (5.2)).

We recall the following fact. By exercise 4, p. 182 [13], we have that for any set of matrices \( A_1, \ldots, A_m \in M_k(\mathbb{C}) \), \( m \geq 2 \), \( l = 1, \ldots, k \), \( \sum_{i=1}^{l} \sigma_i(A_1 \cdots A_m) \leq \sum_{i=1}^{l} \sigma_i(A_1) \cdots \sigma_i(A_m) \). In particular, for any set of matrices, and noting that \( \sigma_1(A_1 \cdots A_m) = 1 \),
\[
\sigma_2(A_1 \cdots A_m) \leq \prod_{i=1}^{m} \sigma_2(A_i)
\]
(5.22)

The following is the QTM version of a technical result proved in [27], and will be used in the next section.

**Proposition 5.5.** Let \( \mathcal{A} = (A_i)_{i=1}^\infty \) be a sequence of unital QTMs on \( n \) sites with matrices on \( M_k(\mathbb{C}) \). For each \( j \) let \( \sigma_i(A_j) \), \( i = 1, \ldots, N_{A_j} \) be the singular values of the OQRW induced by \( A_j \). Then for every \( j \geq 1 \), and every \( m \geq 1 \), there is a constant \( C(j,n) \) such that
\[
d_2(\mathcal{A}_{0,m}(\cdot,j); \rho_\pi) \leq C(j,n) \prod_{l=1}^{m} \sigma_2(A_l)
\]
(5.23)

**Proof.** We apply (5.10) with \( \Phi = \mathcal{A}_{0,m} \). By definition, \( \sigma_j(\mathcal{A}_{0,m}) \leq \sigma_2(\mathcal{A}_{0,m}) \), \( j = 2, \ldots, n \) so we get
\[
d_2(\mathcal{A}_{0,m}(\cdot,j); \rho_\pi) = \left( \sum_{i=2}^{N_{A_j}} |d_{ij}|^2 \sigma_i^2 \right)^{1/2} \leq \sigma_2(A_1 \cdots A_m) \left( \sum_{i=2}^{N_{A_j}} |d_{ij}|^2 \right)^{1/2} \leq C(j,n) \prod_{l=1}^{m} \sigma_2(A_l),
\]
(5.24)

where \( C(j,n) = \sum_{i=2}^{N_{A_j}} |d_{ij}|^2 \). Note that the \( d_{ij} \) depend on \( m \) as well, but \( C(j,n) \) does not increase arbitrarily with \( m \), due to the bound \( |d_{ij}| \leq 1 \), for all \( m \).
6. On ergodicity of QMCs

Recall that the geometric multiplicity $γ_Φ(λ)$ of an eigenvalue $λ$ is the dimension of the eigenspace associated with $λ$, i.e., $\dim ker(Φ − λI)$, the maximum number of vectors in any linearly independent set of eigenvectors with that eigenvalue. The algebraic multiplicity $µ_Φ(λ)$ of $λ$ is its multiplicity as a root of the characteristic polynomial and it is well-known that $γ_Φ(λ) ≤ µ_Φ(λ)$. We recall the following important basic fact, the proof of which is described by M. Wolf [20].

**Proposition 6.1.** (Trivial Jordan blocks for peripheral spectrum). Let $Φ$ be a trace-preserving (or unital) positive linear map. If $λ$ is an eigenvalue of $Φ$ with $|λ| = 1$ then its geometric multiplicity equals its algebraic multiplicity, i.e., all Jordan blocks for $λ$ are one-dimensional.

**Remark 6.2.** As a complement to our discussion, by [8], Proposition 3.5, if we have an irreducible quantum channel $Φ$ then for every eigenvalue $λ$ with $|λ| = 1$ we have $\dim ker(Φ − λI) = 1$. By Proposition 6.1 for an irreducible quantum channel we have that 1 is an eigenvalue of algebraic multiplicity 1.

We also need to review the following facts, more details of which can be seen in [6]. By considering the Hilbert-Schmidt inner product on $B(ℋ)$, the adjoint of a unital quantum channel $Φ(ρ) = \sum_i V_i ρ V_i^*$ is the unital channel $Φ^*(ρ) = \sum_i V_i^* ρ V_i$. The square modulus of a unital quantum channel $Φ$ is the unital channel $ΦΦ^*$. It is clear that the square modulus is a self-adjoint non-negative operator, that is, $(A, ΦΦ^*(A)) = (Φ*(A), Φ^*(A)) ≥ 0, ∀ A ∈ B(ℋ)$. This implies that $Φ^*$ can be diagonalized and has only non-negative eigenvalues [6].

The following is the QMC version of the theorem presented in [27].

**Theorem 6.3.** Let $Q = \{Q_1, . . . , Q_q\}$ be a finite family of unital QTM’s. Then the pair $(Q, ρ_τ)$, $ρ_τ$ being the maximally mixed vector state, is ergodic in the sense of Definition 5.4 if and only if $σ_2(Q_j) < 1$ for each $j ∈ \{1, . . . , q\}$.

**Proof.** Assume that $σ = \max_1, . . . , q σ_2(Q_j) < 1$. Let $\{A_i\}_{i=1}^∞$ be a sequence of OQRWs such that

$$N_i = \#\{i ∈ \{1, . . . , l\} : A_i ∈ Q\}$$

tends to infinity with $l$. By (5.28) we have that for every $j$, 

$$\left(\sum_{i=1}^n \|A_{0,i}(i,j)A_{0,i}(i,j)^* − I/\sqrt{n}\|^2\right)^{1/2} ≤ σ^{N_i} C(j, n)$$

which tends to zero as $l → ∞$. Conversely, assume that the pair $(Q, ρ_τ)$ is ergodic. Then equation (5.19) holds for any sequence $(A_i)_{i=1}^∞$ of QTM’s with invariant measure $ρ_τ$ such that $A_i ∈ Q$ for infinitely many $i$’s. That is, the columns of the iterated product are becoming equal to $I/ν$. By contradiction, assume that one of the $Q_i$’s, say $Q_1$ satisfies $σ_2(Q_1) = 1$ and consider the following sequence of QTM’s: $A_{2i+1} = Q_1, A_{2i} = Q_i^*, i = 1, 2, . . .$. Now we consider $Q_1Q_i^*$, for which $σ_2(Q_1) = 1$ is an eigenvalue with algebraic and geometric multiplicity at least 2, i.e., $µ_Φ(1) = γ_Φ(1) ≥ 2$ (see Prop. 6.1).

Now let $Ψ = (Q_1Q_i^*)^r$. It is clear that each $Ψ_r$ is a quantum channel with real spectrum in $[0, 1]$, by the remarks preceding the statement of this theorem. By standard arguments such as the one seen in Novotný et al. [22, 23] (via Jordan blocks), the asymptotic behavior of $Ψ_r$ is determined by the peripheral spectrum, as contributions of eigenspaces associated to eigenvalues with norm less than 1 tend to disappear. Since 1 is the only eigenvalue in the unit circle associated to $Ψ_r$, for all $r$, it is clear that the limit of $Ψ_r$ as $r → ∞$ exists. The QTM $A = \limr→∞ Ψ_r = \limr→∞ (Q_1Q_i^*)^r$ is such that there must be two matrices $A(i,j)$ and $A(l,m)$ which are distinct. Indeed, let $ρ_0$ be an eigenstate of $A$ associated to eigenvalue 1. Suppose that $ρ_0$ is not the maximally mixed column. This assumption is possible since we have that the geometric multiplicity of 1 for $Q_1Q_i^*$ is at least 2. In particular, by writing $ρ_0 = \sum_i η_i |i⟩⟨i|$ we may assume there are $k, l$ such that $η_k ≠ η_l$. Now the fact that $A(ρ_0) = ρ_0$, corresponds to the system of equations

$$A(1, 1)η_1 A(1, 1)^* + · · · + A(1, n)η_n A(1, n)^* = η_1$$

(6.3)

$$A(2, 1)η_1 A(2, 1)^* + · · · + A(2, n)η_n A(2, n)^* = η_2$$

(6.4)

$$\vdots$$

(6.5)

$$A(n, 1)η_1 A(n, 1)^* + · · · + A(n, n)η_n A(n, n)^* = η_n$$

If, on the contrary, all $A(i,j)$ are equal, then by considering the $k$-th and $l$-th equation we conclude that $η_k = η_l$, which is absurd. Therefore the QTM $A = \limr→∞ Ψ_r$ is such that there must be two matrices $A(i,j)$ and $A(l,m)$ which are distinct.
Moreover, there must be a row in $A$ with two different entries, that is, two matrices $A(i, r) \neq A(i, s)$ for some $i, r, s$. In fact, suppose $i_1$ and $i_2$ are rows where we found two distinct elements of $A$, say, $A(i_1, j_1)$ and $A(i_2, j_2)$. If all entries of row $i_1$ are equal then these must be equal to the maximally mixed column. The same conclusion holds for row $i_2$. But then we would have $A(i_1, j_1) = A(i_2, j_2)$, which is absurd. We conclude there must be a row in $A$ with two different entries. Hence we are able to obtain $x, y, z$ such that
\[
\lim_{r \to \infty} A_{0, 2r}(x, y) - A_{0, 2r}(x, z) \neq 0,
\]
as required.

\begin{proof}
Example 6.4. Let
\[
V_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad V_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
Then $\sum_i V_i^* V_i = I$ and we can build an order 3 QTM with the above matrices in the following way. Let
\[
A = \begin{bmatrix} V_1 & V_2 & V_3 \\ V_2 & V_3 & V_1 \\ V_3 & V_1 & V_2 \end{bmatrix}.
\]
In terms of OQRWs this corresponds to define $\rho \mapsto \sum_{i,j} M_{ij} \rho M_{ij}^*$, $M_{ii} = V_i \otimes E_{1i}$, $i = 1, 2, 3$, where $(E_{ij})_{kl} = \delta_{(i,j),(k,l)}$ are the order 3 matrix units, and analogously for the other rows. Then a calculation shows that $\sigma_1(A) = 1$ and $\sigma_2(A) = \frac{2}{3} < 1$. We conclude that the pair $(\{A\}, \rho_\pi)$, $\rho_\pi$ being the maximally mixed state, is ergodic by Theorem 6.3. We note that in this example we need to calculate the singular values of a representation matrix of order $(nk)^2 = (3.2)^2 = 36$, see Remark 5.4. The singular values 1, 2/3, 1/3 and 0 have multiplicities 1, 6, 3 and 26, respectively.
\end{proof}

\begin{proof}
Example 6.5. If a classical bistochastic QTM of order 2 $A = (A_{ij})$, $A_{ij} \in M_2(\mathbb{C})$ belongs to a finite set $Q$ then no pair $(Q, \rho_\pi)$ is ergodic, since $\sigma_2(A) = 1$ by Lemma 4.8. For a different example, let
\[
V_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]
Then $\sum_i V_i^* V_i = I$ and we can build an order 2 QTM with the above matrices by writing
\[
A = \begin{bmatrix} V_1 & V_2 \\ V_2 & V_1 \end{bmatrix}.
\]
We are able to write the corresponding OQRW as in the above example. Then a calculation shows that $\sigma_1(A) = \sigma_2(A) = 1$. We conclude that if $A \in Q$ then no pair $(Q, \rho_\pi)$, will be ergodic ($\rho_\pi$ being the maximally mixed state).

As another observation, consider the 1-qubit quantum channel $\Phi(\rho) = V_1 \rho V_1^* + V_2 \rho V_2^*$ which has 1 as the the unique eigenvalue of modulus 1, and has multiplicity 1. This shows that one can find a quantum channel that has a unique fixed point, which is attractive (a mixing channel, in the terminology of [6]), but such that the induced QTM is not ergodic in our sense.
\end{proof}

7. Ergodicity on noncommutative $L^1$-spaces

As a final motivation we would like to comment on QTM in terms of the setting studied by F. Mukhamedov [20], that is, in a noncommutative $L^1$-space setting. In that context the mentioned author is able to define a Dobrushin coefficient and then analyze ergodicity properties of certain operators. We will discuss some of these results in light of our QTM description and we will see that many facts are easily examined on finite-dimensional Hilbert spaces when proper adaptations are made.

We will consider the algebra $B(H \otimes K)$ of bounded linear operators acting on $H \otimes K$, where $H$ and $K$ finite dimensional (see Remark 2.1). As mentioned before, the vector states described in previous sections form a convex subset of $B(H \otimes K)$. In finite dimensions all norms are equivalent with, for instance, $\| \cdot \|_2 \leq \| \cdot \|_1$, see [29], and $\| \cdot \|_2 \leq \sqrt{n} \| \cdot \|_\infty$, $\| \cdot \|_\infty$ being the usual maximum norm on matrices [3] [12].
As discussed in Section 2, every vector state \( \rho = (\rho_i) \), \( \sum_{i=1}^{n} tr(\rho_i) = 1 \), \( \rho_i \in M_k(\mathbb{C}) \) can be identified with \( \rho = \sum_{i} \rho_i \otimes |i\rangle \langle i| \), and since \( \omega(\rho) := \sum_{i} tr(\rho_i) = tr(\sum_{i} \rho_i \otimes |i\rangle \langle i|) \), we have that \( \hat{\omega} = \frac{1}{k^n} \omega \) is a faithful state functional on the vector states for a QTM. For a given vector state \( \rho \), define \( T_\rho \) acting on vector states as follows,

\[
T_\rho(X) := tr(X)\rho, \quad X = \sum_{i=1}^{n} X_i \otimes |i\rangle \langle i|, \quad X_i \in M_k(\mathbb{C})
\]

Following [20], a QTM \( A \) is uniformly ergodic if there exists an element \( Y \) such that

\[
\lim_{r \to \infty} \|A_{m,r} - T_Y\|_\infty = 0,
\]

for all \( m \geq 0 \). Note that \( Y \) plays the role of an equilibrium state. Also following [20], a QTM \( A \) is weakly ergodic if for every \( k \in \mathbb{N} \cup \{0\} \) we have

\[
\lim_{r \to \infty} \sup_{\rho, \eta \in \mathcal{D}(\mathcal{H})} \|A_{k,r} \rho - A_{k,r} \eta\|_1 = 0,
\]

where \( \|X\|_1 = \hat{\omega}(|X|) \) and \( |X| = \sqrt{X^*X} \). Now we review in our terms an example presented in [20].

**Example 7.1. Sequences of 1-qubit quantum channels.** It is a well known fact that a map \( \Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \) is unital, positive, trace-preserving if and only if

\[
\Phi(w_0 I + w \cdot \sigma) = w_0 I + (Sw) \cdot \sigma,
\]

where \( S \) is an order 3 real matrix with \( \|Sw\| \leq \|w\| \) for all \( w \in \mathbb{C}^3 \). We recall that above we used the notation \( v \cdot \sigma = \sum_{i=1}^{3} v_i \sigma_i \), \( v = (v_1, v_2, v_3) \), \( \sigma_i \) being the Pauli matrices, see [14] [24] [26]. Let \( \{\Phi_k\} \) be a sequence of unital quantum channels with corresponding diagonal real matrices

\[
S^{(k)} = \begin{bmatrix}
\lambda_1^{(k)} & 0 & 0 \\
0 & \lambda_2^{(k)} & 0 \\
0 & 0 & \lambda_3^{(k)}
\end{bmatrix},
\]

where \( |\lambda_i^{(k)}| \leq 1 \), \( i = 1, 2, 3 \). Let \( \nu_k = \max_i\{|\lambda_i^{(k)}|\} \), \( \lambda \in (0, 1) \) and assume that

\[
\nu_k \leq 1 - \lambda, \quad \forall k \in \mathbb{N}.
\]

Then define \( T_k = \Phi_k^* \), \( k \in \mathbb{N} \). Then it is shown in [20] that \( \{T_k\}_{k=1}^{\infty} \) is weakly ergodic. Now consider the pair \( (Q, \rho_\pi) \), where \( Q = \{A_{T_k} \} \) is a finite collection of QTM where \( A_{T_k} \) is induced (following Example 4.7) by the 1-qubit unital channel \( T_k \) above. Then ergodicity may or may not hold, in light of Examples 6.4 and 6.5. As a complement to this fact, we note that one may also be interested in a similar notion of ergodicity for a finite collection of 1-qubit channels. In such case ergodicity will be seen to hold for a collection satisfying (7.6) since the \( \lambda_i \) determine the action of the channel which, in this case, is to transform the Bloch sphere into the ellipsoid

\[
\left( \frac{x_1}{\lambda_1^{(k)}} \right)^2 + \left( \frac{x_2}{\lambda_2^{(k)}} \right)^2 + \left( \frac{x_3}{\lambda_3^{(k)}} \right)^2 = 1, \quad i = 1, \ldots, l,
\]

see for instance [14].

Now we state a result that is also described by [20], with notation adapted for our purpose, considering homogeneous QMCs. Instead of considering the norm induced by the Hilbert-Schmidt norm, as it has been made previously, we will consider \( L^1 \)-spaces, and state results in terms of the norm \( \| \cdot \|_1 \). One of the facts that can be discussed at this point is a relation between weak ergodicity, uniform ergodicity and singular values of a QTM. This gives us an asymptotic notion which is closely related to the one presented in this work. In order to do that, we recall some definitions. Let

\[
Z := \{X = \sum_{i=1}^{n} X_i \otimes |i\rangle \langle i| : tr(X) = 0\}
\]

and for \( T \) a linear operator define

\[
\delta(T) := \sup_{X \in Z, X \neq 0} \frac{\|TX\|_1}{\|X\|_1}
\]

the Dobrushin ergodicity coefficient of \( T \). Basic properties of this coefficient can be seen in [20], where it is proven that

\[
\delta(T) = \sup_{\rho, \eta \in \mathcal{D}(\mathcal{H})} \frac{\|T\rho - T\eta\|_1}{2}.
\]
Theorem 7.2. Let $A$ be a finite dimensional QTM. The following are equivalent: a) The homogeneous QMC is irreducible and aperiodic. b) There exists $s \in [0, 1)$ and $n_0 \in \mathbb{N}$ such that $\delta(A^{n_0}) \leq s$. c) $A = \{A^r\}_{r=1}^{\infty}$ is uniformly ergodic.

In the homogeneous case, Definition 5.4 is reduced to

Definition 7.3. Fix a vector state $\rho_\pi = [\rho_1 \ldots \rho_n]^T$, $\sum_i \rho_i = 1$, with $\rho_i \in M_n(\mathbb{C})$. Let $A$ be a QTM, that admits $\rho_\pi$ as an invariant measure. We say that the pair $(A, \rho_\pi)$ is ergodic if for all $i, j, k \in \{1, \ldots, n\}$,

$$
\lim_{r \to \infty} A^r(i, j) - A^r(i, k) = 0.
$$

The following corollary is immediate from Theorem 7.2 and Definition 7.3:

Corollary 7.4. Suppose one of the conditions of Theorem 7.2 holds for a given homogeneous QMC $A = \{A^r\}_{r=1}^{\infty}$. Then $A$ is ergodic in the sense of Definition 7.3. As a consequence, $\sigma_2(A) < 1$.

On the other hand, assuming that a QTM $A$ is ergodic in the sense of Definition 5.4 then it is a simple matter to show that this does not imply weak ergodicity. In fact, let $A$ be the order 2 maximally mixed QTM and note that for every $\rho$,

$$
A(\rho) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\rho_1 + \rho_2) \\ \frac{1}{2}(\rho_1 + \rho_2) \end{bmatrix}.
$$

Let

$$
\rho = [\rho_1 \rho_2]^T = [I/4 \ I/4]^T, \quad \eta = [\eta_1 \ \eta_2]^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T.
$$

Then

$$
A\rho = \rho, \quad A\eta = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T \implies A\rho - A\eta = \begin{bmatrix} -\frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}.
$$

Hence, we are able to perform a calculation of (7.3) which produces a positive number. This shows that the homogeneous QMC induced by $A$ is ergodic in the sense of Definition 5.4 but is not weakly ergodic.

8. Open questions

In this work we have studied a class of quantum Markov chains induced by transition matrices. We have also described such objects in terms of open quantum random walks and, together with the results on ergodicity, we hope this will encourage further research on dynamical aspects of these operators. Among basic open questions is the problem of establishing results for QTM which are not unital, that is, such that the associated OQRW does not preserve the identity. In this case we note that the property that all entries of a given stationary state are equal (a fact which allows certain calculations performed in our adaptation for the unital case) cannot be used. Many of the questions made here make sense for QTM acting on an infinite dimensional space, but in principal describing and solving these problems would require a treatment which is different than the one used here.

Concerning the theory described in [27] one might be interested in studying quantum adaptations of other properties presented there, particularly aspects appearing in processes that involve groups. Also according to this work, if every stochastic matrix $Q_i$ is reversible with respect to a given distribution, i.e.,

$$
\pi(x)Q_i(x, y) = \pi(y)Q_i(y, x), \quad \forall i,
$$

then the condition $\sigma_2(Q_i) < 1$ for each $i$ is equivalent to the fact that each $Q_i$ is irreducible and aperiodic. Then one may ask for an appropriate notion of reversibility for the setting presented in this work.

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