This paper introduces a class of objects called decision rules that map infinite sequences of alternatives to a decision space. These objects can be used to model situations where a decision maker encounters alternatives in a sequence such as receiving recommendations. Within the class of decision rules, we study two natural subclasses — stopping and uniform-stopping rules. Our main result establishes the equivalence of these two subclasses of decision rules. Next, we introduce the notion of computability of decision rules using Turing machines and show that computable rules can be implemented using a simpler computational device—a finite automaton. We further show that computability of choice rules—an important subclass of decision rules—is implied by their continuity with respect to a natural topology. Finally, we introduce some natural heuristics in this framework and provide their behavioral characterization.

Keywords: decision rules, computability, sequences, revealed preference.

1. INTRODUCTION

1.1. A brief overview

Imagine a decision maker (DM) who faces alternatives sequentially before she decides to “stop” and make a decision. The alternatives keep on being presented and the decision to stop may depend on the history of alternatives examined thus far. A wide range of situations where decision making involves seeking recommendations, receiving bitstreams of information, meeting people, viewing alternatives on websites etc. fall under such a description. There has been substantial evidence that the order in which alternatives are presented to the DM affects the final decision. Incorporating this sequentiality, the literature has introduced models of choice from finite lists i.e. ordered sets (see Rubinstein and Salant (2006) and Horan (2010), among others). However, since the alternatives could possibly be presented “forever” or the DM could potentially go examining the alternatives as long as she “wishes” to in the above mentioned situations, infinite sequences instead of finite lists are a more appropriate modelling device. To that end, we propose a framework of decision making from infinite sequences. We introduce a class of objects called decision rules that generalize choice functions over lists in two ways: (i) It allows the lists to be infinite (and allow for duplication); (ii) It allows the choice to land in an arbitrary decision space making choice from within the set of alternatives a special case.

The present paper focuses on two natural subclasses of decision rules, which we term as stopping rules and uniform-stopping rules. These rules allow us to capture the notion of bounded rationality in our framework and lend their interpretation to commonly studied themes such as shortlisting, screening, limited/bounded attention etc. As the name suggests, for any input i.e. a sequence of alternatives, stopping rules decide within some finite amount of time i.e. after
inspecting a finite segment of alternatives. However, the stopping point may depend on the input. On the other hand, in case of uniform-stopping rules—which are a subclass of stopping rules—there is a fixed upper bound on the stopping point. When the set of alternatives is finite, a fixed upper bound ensures that the set of inputs is finite, thereby allowing one to study various heuristics and provide testable restrictions for their axiomatic characterization. On the other hand, one might suspect that no such guarantee is assured in case of stopping rules even with a finite set of alternatives since the set of infinite sequences is uncountably infinite. Hence, a reasonable question to ask is what restrictions on the class of stopping rules pin down the subclass of uniform-stopping rules? As it turns out, no further restrictions are required i.e. every stopping rule is a uniform-stopping rule. We believe that—due to the generality of our framework—this result might be of independent interest to linguists, computer scientists and game theorists. Our proof relies on the assumption that the set of alternatives is finite and is established via a diagonalization argument.

We discuss the computational aspects of decision making in our framework using two models of computation—a Turing machine and a finite automaton. While much of the literature on computational aspects of choice uses a finite automaton to model a decision maker, a more powerful and faithful model of computation is that of a Turing machine. We define the class of computable decision rules as the ones that can be “implemented” via a Turing machine and automaton-computable rules as the ones which can be implemented via a finite automaton. While it is straightforward to see that automaton-computable rules are computable, we show that the converse is also true. This result is established using the equivalence of stopping and uniform-stopping rules. The connection between computable (automaton-computable) rules and stopping (uniform-stopping) rules allows us to examine the procedural aspects of decision making in our framework.

Next, we turn our attention to an important subclass of decision rules—choice rules. These are the rules in which the decision space is restricted to be the set of alternatives. Choice rules are the analogue of choice functions over lists. We show that continuity of the choice rules with respect to a natural topology—the product topology—on the set of sequences and the discrete topology on the set of alternatives ensures that they are stopping rules, thereby highlighting the link between continuity and computability. Within the class of stopping choice rules, we introduce some natural heuristics and provide their behavioral characterization. In particular, we adapt the idea of satisficing and introduce two variants: ordinal satisficing and cardinal satisficing. In contrast to the result on choice over sets (Rubinstein (2012)), satisficing is behaviorally not equivalent to “rational” choice in our framework. In fact, it turns out that rational behavior is a special case of ordinal satisficing. Next, we introduce a broad class of heuristics called configuration-dependent rules that can explain how position and frequency of the alternatives in a sequence can affect choice. Finally, we provide a topological foundation for our main equivalence result.

1.2. Computability and bounded rationality

Computational aspects of decision making have been an area of interest to economic theorists. As has been pointed out in Richter and Wong (1999)), computability-based economic theories also provide foundations for complexity analysis. The idea of bounded rationality has been closely linked to computational limitations of an economic agent (see Futia (1977)). For instance, in the analysis of repeated games, the strategies of players are implemented using finite automata (Rubinstein (1986)). Further, this implementation of strategies is also assumed to be costly in the state complexity of the automata implementing them (Abreu and Rubinstein (1988)). In the context of finitely repeated games, the imposition of bounds on the complexity of strategies of the players is shown to justify cooperation (see Neyman (1985)).
The discussion of the role of computational constraints in individual decision making goes back to Simon (1955). He remarks “…limits on computational capacity may be important constraints entering into the definition of rational choice under particular circumstances”. To capture the finite informational processing capacity, Kramer (1967) models the DM as a finite automaton and shows the incongruence of rational decision making given this behavioral restriction on the DM. On the other hand, Salant (2003) shows that with the finite automaton model of decision making, implementation of rational choice functions over menus (sets of alternatives) is computationally efficient. He further shows that implementation of other choice functions is much more computationally demanding.

In our framework, decision rules can be implemented by a finite automaton. We call such rules automaton-computable rules. For instance, the decision rule in Example 2 is an automaton-computable rule and can be implemented using a finite automaton with six states if the number of alternatives is 2, both of them having a score of 1 and the threshold score is 2 (see Figure 1).

While modeling DMs using automaton has been a popular approach to model aspects of bounded rationality, to model the computational aspects of decision making, the Turing machine is a more appropriate device. It is a more powerful model of computation than a finite automaton and can be thought of as a precise way of mathematically describing an algorithm. According to the Church-Turing thesis, any realisable computer can be simulated as a Turing machine. Therefore, Turing machines embody the idea of computability in the truest sense. Richter and Wong (1999) use the idea of a Turing machine to define computable preferences and show that computable preferences have computable utility representations. Camara (2021) models the DM as a Turing machine in the environment of decision making under risk. He introduces the notion of computational tractability. A decision problem is intractable if it cannot be implemented by an algorithm in a “reasonable” amount of time. He shows that expected utility maximization is intractable unless the utility function satisfies a strong separable property. Dropping the requirement of tractability and retaining the aspect of computability in our framework, we call a decision rule computable if it can be implemented by a Turing machine. Our main result shows equivalence of computable and simply-computable rules, thereby highlighting the relationship between the notions of computability and bounded rationality.
1.3. Choice rules

An important subclass of decision rules are the rules whose decision space is restricted to be the set of all alternatives itself. These are called choice rules. These are the analogue of choice functions over lists and choice functions over menus in classical abstract choice theory. The idea that a DM may observe alternatives in the form of a list i.e. an ordered set was first introduced in Rubinstein and Salant (2006). Based on their framework, a variety of models incorporating order effects in choice have been introduced in the literature (see for instance Guney (2014) and Dimitrov et al. (2016)).

Satisficing, first introduced by Simon (1955) has been an influential idea in choice theory and many adaptations have been done in the literature. The list setup provides a natural framework to study satisficing behavior. Kovach and Ülkü (2020) introduced one such model. In their model, the DM makes her choice in two stages. In the first stage, she searches through the list till she sees $k$ alternatives. In the second stage, she chooses from the alternatives she has seen. Another adaptation of satisficing was introduced in Manzini et al. (2019). Their model is interpreted as one of approval as against choice. Since our framework is a generalization of lists, satisficing heuristics are a natural choice of study. To behaviorally characterize the satisficing models in our framework, we introduce the informational notions of sufficiency and minimal sufficiency of finite segments of a sequence. The underlying idea is that a DM “makes up” his mind after viewing a certain minimum number of alternatives in a given sequence. Such segments are “sufficient” to implement the choice from the sequence. These notions form the basis of our axioms that characterize the satisficing models in our framework.

As discussed above, an important application of our framework is when alternatives come in the form of streams of recommendations. The idea that recommendations influence choices has been widely accepted. Cheung and Masatlioglu (2021) have introduced a model of decision making under recommendation. However, in their setup, the decision maker observes sets of alternatives and hence is different from our setup. Our object of interest—infinitesimal sequences in the context of choices has been previously studied by Caplin and Dean (2011). Our model differs from their model in terms of incorporating sequences in the domain of choice functions whereas they enrich the observable choice data by incorporating sequences as the output of the choice function and interpret these sequences as provisional choices of the DM with contemplation time.

1.4. Examples

Our framework is general in its scope and provides fruitful avenues to study many situations as the following examples might indicate.

**Example 1** (Language processing). Let $X = \{0, 1\}$. Consider a computer program that receives bitstreams or sequences of symbols from $X$. The bitstreams represent expressions in a natural language (for instance, English) encoded in binary expression i.e. $0’s$ and $1’s$. For every input bitstream, the program declares it as “TRUE” if it contains a grammatically correct sentence. It outputs “FALSE” otherwise.

**Example 2** (Cardinal satisficing). Let $X$ be a set of movies and $\{X_i\}_{i=1}^N$ denote a partition of the movies into $N$ genres for some $N \in \mathbb{N}$. A DM wishes to watch a movie and relies on recommendations. She attaches a “weight” to each genre which indicates the value she attaches to each genre i.e. there exists a function $w : \mathbb{N} \to \mathbb{R}_+$ such that every movie in the $i^{th}$ genre is given the same weight. Her decision procedure involves seeking recommendations sequentially from different sources such as peer groups, websites etc. She has a threshold score in her mind and the first movie in the sequence of recommendations whose cumulative score (due to repetitions) crosses the threshold value is selected by her.
Example 3 (Ordinal satisficing). A DM has to choose a partner based on repeated interactions with a set of potential partners. She has a fixed attention span i.e. finite first-\( k \) interactions. She attaches a “utility” number to each alternative and has some threshold in her mind. She chooses the first potential partner within the first-\( k \) interaction whose utility number exceeds the threshold. Otherwise she chooses the partner with the maximum utility value from the first-\( k \) interactions.

Example 4 (Investment strategy). A company wishes to invest in a fund based on inputs provided by a fund rating agency. Let \( X \) be a finite set of possible performances of the fund (often denoted as \( A_+, B_+ \) etc.). The inputs are in the form of a sequence of performances of the fund i.e. at every time period \( t \in \mathbb{N} \), the agency announces the performance of the fund. Let \( Y = \mathbb{R} \times \mathbb{N} \). The company uses an “algorithm” that —based on the input of performances—decides the amount of money and the time period for which to invest i.e. \( y \in Y \).

Example 6 (Stochastic choice) Let \( X \) be a set of alternatives and \( \sigma \) be a probability distribution over \( \mathbb{N} \). Faced with an input sequence, the DM first draws the “depth” parameter \( k \in \mathbb{N} \) according to \( \sigma \) and then chooses the \( k^{th} \) alternative in the sequence.

Example 7 (Two-stage choice) A DM relies on a shortlisting procedure to make choice and is endowed with two objects: a shortlisting function \( \Gamma \) that selects, for every input sequence, a finite initial segment and an output function, \( O \), that selects an output from the shortlisted segment.

The outline of the paper is as follows. In the next section, we formally introduce stopping and uniform stopping rules and provide the main result. Section 3 introduces the notions of computability of decision making in our framework. Section 4 focuses on choice rules and within the class of choice rules, two variants of satisficing are introduced and behaviorally characterized. Further a broad subclass of choice rules called configuration-dependent rules are introduced and behaviorally characterized. Section 5 discusses the topological approach to our main result and section 6 concludes.

2. MAIN RESULT

Let \( X \) be a finite set of alternatives. The objects of interest are \( X \)-valued infinite sequences i.e. sequences whose terms are the alternatives in \( X \). These are the analogue of “menus” of classical abstract choice theory in our framework. Let \( S \) be the collection of all such sequences i.e. \( S = \{ S \mid S : \mathbb{N} \to X \} \). Denote by \( S(i) \), the \( i^{th} \) element of the sequence \( S \). Let \( S|_k \) denote the segment of \( S \) that comprises of the first-\( k \) elements of the sequence \( S \) and by \( S|_k \cdot T \) a “concatenation” of the segment \( S|_k \), with some sequence \( T \in S \) i.e. \( S|_k \cdot T \in S \) with \( [S|_k \cdot T](i) = S(i) \) for \( i \in \{1, 2, \ldots, k\} \) and \( [S|_k \cdot T](k + i) = T(i) \) for \( i \in \mathbb{N} \). Let \( Y \) be any abstract set which we call the decision space. The DM in our framework is endowed with a decision rule, \( d \), which gives a unique decision in the decision space for any given input sequence.

Definition 1: A decision rule on sequences is a map \( d : S \to Y \).

Consider the following decision rule: Let \( Y = \{0, 1\} \) and \( \{x^*, y, z\} \). For any input \( S \in S \), \( d(S) = 1 \) if \( x^* = S(i) \) for some \( i \in \mathbb{N} \) and \( d(S) = 0 \) otherwise. If the input is examined by the DM sequentially—in discrete time for instance—such a decision rule looks implausible since for the inputs that do not feature \( x^* \), the DM would have to wait “forever” to make a decision. To preclude such decision rules, we restrict our attention to a subclass which we term stopping rules. To define stopping rules formally, we first introduce the following useful object

\[
k_d(S) := \inf \{k \in \mathbb{N} : d(S) = d(S|_k \cdot T) \text{ for all } T \in S\}.
\]
The function $k_d(\cdot)$ captures the smallest truncation of a sequence $S$ beyond which the terms of the sequence do not affect decisions. In other words, for any “tail” appended beyond $k_d(S)$, the decision of the sequence is unaffected. Now, we can formally define stopping rules as follows.

**Definition 2:** A decision rule $d$ is a stopping rule if for every $S \in S$

\[ k_d(S) < \infty \]

Stopping rules capture the idea that for any given input sequence, the DM does not wait indefinitely and “makes up its mind” by a finite amount of time i.e. after viewing a finite initial segment and the subsequent alternatives of the sequence do not affect the decision. It is important to note the stopping time or the “relevant” finite segment can depend on the i.e. input sequence. Since the set of inputs is infinite (uncountably), stopping “times” may not have a finite upper bound. A subclass of stopping rules for which the stopping times have a finite upper bound are called uniform-stopping rules. They are formally defined as

**Definition 3:** A decision rule $d$ is a uniform-stopping rule if

\[ \sup \{ k_d(S) : S \in S \} < \infty \]

Consider a simple uniform-stopping rule: The DM is endowed with a strict preference order $\succ$ over $X$, and for any input sequence, she considers only the first 10 alternatives and picks the $\succ$-maximal alternative from them. While it is clear that every uniform-stopping rule is a stopping rule, we show below the converse is also true.

**Theorem 1:** Every stopping rule is a uniform-stopping rule

**Proof:** Let $d : S \to X$ be a stopping rule. Suppose, for the sake of contradiction, $d$ is not a uniform stopping rule. The proof is organized in steps which are as follows.

**Step 1:** We iteratively define a sequence of pairs $\{(k_j, A_j)\}_{j \in \mathbb{N}}$, where $k_j \in \mathbb{N}$ and $A_j \subseteq S$, as follows:
1. Let $k_1 := \inf \{ k_d(S) : S \in S \}$ and $A_1 := \{ S \in S : k_d(S) = k_1 \}$.
2. For any $j \in \mathbb{N} \setminus \{1\}$, assuming $(k_l, A_l)$ have already been defined for every $l \in \{1, \ldots, j - 1\}$, let

\[ k_j := \inf \{ k_d(S) : S \in S \setminus \bigcup_{l=1}^{j-1} A_l \}, \]

and

\[ A_j := \{ S \in S \setminus \bigcup_{l=1}^{j-1} A_l : k_d(S) = k_j \}. \]

The sets $A_j$ refer to the set of all the sequences for which the stopping time is $k_j$. From our supposition that $d$ is stopping rule and $d$ does not have a uniform stopping time, the following properties are immediate:
1. For each $j \in \mathbb{N}$, $k_j \in \mathbb{N}$ and $A_j \neq \emptyset$.
2. $k_1 < k_2 < \ldots < k_j < \ldots$ and so on.
3. \{ $A_j : j \in \mathbb{N}$\} is a partition of $S$.

These properties shall be referred to in the rest of the argument.

**Step 2:** Pick an arbitrary $S_j \in A_j$ for every $j \in \mathbb{N}$. This generates a sequence of sequences $(S_1, S_2, \ldots)$ such that the stopping time for each $S_j$ is $k_j$. Now, we construct a subsequence
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of this sequence \((\tilde{S}_1, \tilde{S}_2, \ldots)\) that “converges” to some \(S^* \in S\). We do this inductively i.e. we show that for any \(N \in \mathbb{N}\), there exists infinitely many sequences \((S_{N1}, S_{N2}, \ldots)\) in \((S_1, S_2, \ldots)\) that “agree” on the first \(N\) terms. For the base case, note that since \(X\) is finite, there must exist at least one alternative in \(X\) that appears in the first position in infinitely many terms of the sequence \((S_1, S_2, \ldots)\) i.e. in infinitely many sequences. There may be multiple such terms. We pick any one arbitrarily. Consider all such sequences which have the same first term. This forms a subsequence of \((S_1, S_2, \ldots)\). We can write this subsequence as \((S_{11}, S_{12}, \ldots)\) where \(S_{1i} = S_{mi}\) for some \(m_i \geq i\). Note that all sequences in \((S_{11}, S_{12}, \ldots)\) have the same first term. Let \(\tilde{S}_1 = S_{11}\). For the inductive step, suppose we have found out a subsequence \((S_{k1}, S_{k2}, \ldots)\) of \((S_{(k-1)1}, S_{(k-1)2}, \ldots)\) (and consequently of \((S_1, S_2, \ldots)\)) such that all sequences (i.e. terms) in the subsequence “agree” on the first \(k\) terms. Let \(\tilde{S}_k = S_{k1}\). Now, to show that it must be true for \(k+1\) i.e. we can find a subsequence of \((S_{k1}, S_{k2}, \ldots)\) such that all sequences agree on the first \(k+1\) terms, note that since \(X\) is finite, there must exist at least one alternative in \(X\) that occurs at the \((k+1)th\) position in infinitely many sequences i.e. infinitely many terms of \((S_{k1}, S_{k2}, \ldots)\). Let that subsequence be \((S_{(k+1)1}, S_{(k+1)2}, \ldots)\) and \(\tilde{S}_{k+1} = S_{(k+1)1}\). Therefore, we have shown that there exists a subsequence \((\tilde{S}_1, \tilde{S}_2, \ldots)\) of \((S_1, S_2, \ldots)\) such that for any \(k \in \mathbb{N}\), we have

\[
\tilde{S}_{|k} = S_{|k}, \quad \forall \ j \geq k
\]

Therefore \((\tilde{S}_1, \tilde{S}_2, \ldots)\) is a convergent subsequence that converges to some \(S^* \in S\).

**Step 3:** Since \(d\) is a stopping rule, there must exist \(k^* \in \mathbb{N}\) such that

\[
d(S^*) = d(S^*|_{k^*}.T) \quad \forall \ T \in S
\]  

(1)

Note that we can write \((\tilde{S}_1, \tilde{S}_2 \ldots)\) as \((S_{11}, S_{21}, \ldots)\) such that \(S_{1i} = S_{ni}\) for some \(n_i \geq i\) where each \(S_{ni}\) is an element of the initial sequence \((S_1, S_2, \ldots)\) for all \(i \in \mathbb{N}\). Consider \(n_i > k^*\). Note that by construction, \(S_{ni} \in A_{ni}\) i.e. for any \(k < n_i\), there exists some \(T' \in S\) such that \(d(S_{ni}) \neq d(S_{ni}|_{k}.T')\). Let \(k = k^*\). We know that \(S_{ni}|_{k^*} = S^*|_{k^*}\) which implies that \(S_{ni} = S^*|_{k^*}.T''\), where \(T'' \in S\) and \(T''(i) = S_{ni}(k^* + i)\) for all \(i \in \mathbb{N}\). By (1), we know that \(d(S^*) = d(S_{ni})\). However, since \(d(S_{ni}) \neq d(S_{ni}|_{k^*}.T')\) for some \(T' \in S\), we get

\[
d(S_{ni}|_{k^*}.T') = d(S^*|_{k^*}.T') \neq d(S^*)
\]

which contradicts (1). So, there does not exist \(k^* \in \mathbb{N}\) such that (1) holds i.e. there is no finite stopping time for the sequence \(S^*\), a contradiction to \(d\) being a stopping rule. Therefore our supposition is wrong and \(d\) must be a uniform-stopping rule. 

Q.E.D.

Stopping rules —and consequently uniform stopping rules—serve a dual purpose in our framework. First, they encompass various notions of bounded rationality. For instance, they capture the notion of bounded attention. The finite upper bound on the stopping times can be interpreted as the attention span of the DM within which she makes her decision. Alternatively, the “relevant” finite initial segment of a given input sequence can be thought of as a consideration set or a shortlisted set that the DM considers before making the decision. Second, since the set of alternatives is assumed to be finite, the equivalence of stopping and uniform-stopping rules enables us to provide testable restrictions to characterize specific subclasses of stopping decision rules. In section 4, we introduce some natural subclasses of stopping rules and introduce some useful concepts that form the basis of their axiomatic characterization.
3. COMPUTABILITY

A natural constraint on decision making is the limited information processing capability of DMs. While the standard notion of rationality assumes no such constraints, there are a variety of settings where the assumption of infinite processing capabilities is an unrealistic one. The theories of individual decision-making have incorporated such constraints by modelling the decision maker as a finite information-processing device. For instance, Kramer (1967), shows that when the DM suffers from computational constraints, it is impossible to display fully rational behavior. While modelling the decision maker as a finite information-processing device, much of the literature has resorted to the finite automaton model (see Salant (2003), among others). However, a more general model of computation is that of the Turing machine. Richter and Wong (1999) define the computability of preferences using a Turing machine. More recently, Camara (2021) has modelled the DM as a Turing machine and examined the computational tractability of choice under uncertainty.

Our framework also allows us to study computable aspects of decision making. We study the implementation of a decision rule using Turing machines and finite automaton. We first formally define a Turing machine.

**Definition 4:** A Turing machine is a tuple $TM = (Q, \Sigma, \delta)$, where $Q$ is a finite set of states, $\Sigma$ is a finite set of symbols called the alphabet and $\delta : Q \times \Sigma^2 \rightarrow Q \times \Sigma \times \{L, S, R\}^2$ is a transition function.

Our formulation of a Turing machine contains two tapes which are infinite one directional line of “cells”. We denote the two tapes as input and output tapes. Each tape is equipped with a tape head. The tape head of the input tape reads the symbols on the tape one cell at a time whereas the tape head of the output tape can potentially write symbols to the tape one cell at a time. Both the tapes contain $\langle \in \Sigma$ in its first position, the start symbol that initializes the machine. It contains a “register” that holds a single element of $Q$ at a time with the initial state being $q_0$. The transition function maps the current state and the symbol on the current entries of the tapes to a new state and instructions for the heads. The input head can move to the next entry of the tape, or move to the previous entry of the tape, or stay in case a terminal state is reached. The machine stops when a terminal state is reached and the output of the machine, denoted by $TM(I)$ where $I$ is the input sequence, is the entry under head on the output tape. Let $I$ denote a set of possible inputs and $TM_I$ be any Turing machine that stops in finite time for all $I \in I$ i.e. for every $I \in I$, $TM(I)$ is computed in finite number of steps. Using this notion of a Turing machine, a computable rule is defined as follows.

**Definition 5:** A decision rule $d$ is computable, if there exists a Turing machine $TM_S$ such that $d(S) = TM(S)$ for all $S \in S$.

Next, we define the notion of automaton-computability using a finite automaton. A finite automaton is formally defined as follows.

**Definition 6:** A finite automaton is a tuple $A = (Q, \Sigma, \delta)$, where $Q$ is a finite set of states, $\Sigma$ is a finite set of symbols called the input alphabet and $\delta : Q \times \Sigma \rightarrow Q$ is a transition function.

An automaton starts in an initial state $q_0 \in Q$ and reads elements of $\Sigma$ one at a time (in some order). For every input element and the current state, the transition function determines the next state of the automaton. Within the set of states are the terminal or absorbing states, denoted
by $F$. Once the automaton enters one of these states, it remains in that state irrespective of the subsequent inputs. Salant (2011) describes implementation of a choice function on lists using an automaton with an output function. It reads the elements of a list in order and stops either at end of the list or at some intermediate step, depending on the choice function it is implementing. Once it reaches its terminal state i.e. it stops, the output function determines the decision from the decision space (which is the set of alternatives appearing in the list). In our framework, we can denote such an automaton by $A_O$ where for every input sequence, a terminal state is reached in finite number of steps and $O : F \rightarrow Y$ is the output function mapping terminal states to a decision. Using this, we can define implementability of a decision rule on sequences by an automaton in a fairly straightforward manner

**Definition 7:** A decision rule $d$ is automaton-computable if there exists an automaton $A_O$ such that for any input $S \in S$, the output generated by it is $d(S)$.

Our notions of computable and automaton-computable rules are closely linked to stopping and uniform stopping rules. Upon closer examination, one can see that if a decision rule is computable, then it must be a stopping rule. This is because of the fact that since for every input $S$, the Turing machine halts in finite time i.e. after observing a finite initial segment. The sequence beyond this halting point is irrelevant for the output and hence it corresponds to $k_d(S)$ of that input. On the other hand, it is easy to see that uniform stopping rules are implementable using a finite automaton. Our main result therefore establishes the equivalence of computable and automaton-computable rules as well.

**Corollary 1:** A decision rule is computable if and only if it is automaton-computable

The above equivalence collapses when we admit only finite inputs of i.e. finite segments of arbitrary length as against infinite sequences. Consider the following example from: Let $X = \{a,b,\emptyset\}$, $Y = \{\text{yes}, \text{no}\}$ and $S$ be the set of all finite strings that comprise of $a$ and $b$ with the last element being $\emptyset$ (indicating the end of the segment). The decision rule outputs “yes” for any $S \in \{a^n b^n \emptyset : n \in \mathbb{N}\}$ i.e. any segment that comprises of $n$ number of $a$’s followed by $n$ number of $b$’s for any $n \in \mathbb{N}$ and it outputs “no” otherwise. Such a decision rule can be implemented using a Turing machine. However, it cannot be implemented by a finite automaton.

4. **Choice Rules and Satisficing**

An important subclass of decision rules are choice rules. As discussed earlier, since the decision space is restricted to be the set of alternatives, these can be thought of as the analogue of choice functions over lists and choice function over sets. These are formally defined as

**Definition 8:** A decision rule is a choice rule if $d(S) = S(i)$ for some $i \in \mathbb{N}$ and for all $S \in S$

Now, we will characterize the class of stopping choice rules. As it turns out, the only condition we require is continuity with respect to the the discrete topology on $X$ and the induced product topology defined on the set of all sequences$^1$. The product topology here has a behavioral interpretation. It says that a DM considers two sequences “close” or “similar” if their initial segments are the same. Continuity of the choice rule then implies that she cannot display

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$^1$This result goes through even when the decision space is some finite set $Y$
“jumps” in choices for close enough choice problems. This translates into the observation that after a certain finite segment has been observed by the DM, the “tail” of a sequence cannot affect the choice.

**THEOREM 2:** Assume $S$ and $X$ are endowed with the product topology and the discrete topology respectively. Then, a choice rule $d : S \rightarrow X$ is a stopping rule if and only if it is continuous.

**Proof:** First, we show what the product topology on $S$ looks like. We know that $\Pi_S$, the product topology, is the smallest topology with respect to which the projection maps are continuous. Consider any map $M : \{1, \ldots, N\} \rightarrow X$ where $N \in \mathbb{N}$ and define the set $B(M)$ as:

$$B(M) = \left\{ S \in S : \text{ for all } i \in \{1, \ldots, N\}, S(i) = M(i) \right\}$$

Let $B_S$ be the class of all such sets. Note that a for any $N \in \mathbb{N}$, the number of possible maps $M : \{1, \ldots, N\} \rightarrow X$ is $|X|^N$. These sets are what can be interpreted as “open balls” in $S$. Let $\mathcal{T}_S$ be the class of unions of arbitrary subcollections of $B_S$.

**Lemma 1:** $\mathcal{T}_S$ is the product topology on $S$

**Proof:** First, we show that $\mathcal{T}_S$ is indeed a topology over $S$. Notice that $\mathcal{T}_S$ is closed under arbitrary unions by definition. To show that it is closed under finite intersections, let $\bigcap_{i=1}^K B_i$ be a finite intersection such that $B_i \in \mathcal{T}_S$ for all $i \in \{1, \ldots, K\}$. Note that each $B_i$ is a union of some subcollection of $B_S$ and therefore we can write $B_i = \bigcup_{j \in J_i} B_{i}^{j}$, with $J_i$ being some indexed set, where each $B_{i}^{j}$ corresponds to an “open ball” i.e. is a set of the form $B(M)$ for some $M : \{1, \ldots, N\} \rightarrow X$ and $N \in \mathbb{N}$. Using the definition of $B(M)$, we know that there exist sets $A_1, A_2, \ldots$ with $A_j \subseteq X$ for all $j \in \mathbb{N}$ such that

$$B_i = \left\{ S \in S : \text{ for all } j \in \mathbb{N}, S(j) \in A_j \right\}$$

So, we can write $\bigcap_{i=1}^K B_i$ as

$$\bigcap_{i=1}^K B_i = \left\{ S \in S : \text{ for all } j \in \mathbb{N}, S(j) \in \bigcap_{i=1}^K A_j \right\}$$

Clearly, $\bigcap_{i=1}^K B_i = \bigcup_{M \in \mathcal{M}_i} B(M)$ for some collection of maps $\mathcal{M}_i$. Therefore, $\mathcal{T}_S$ is closed under finite intersection. Finally, $\mathcal{T}_S$ contains $S$ and $\varnothing$ as its elements. That $\varnothing \in \mathcal{T}_S$ holds follows from the fact that $\varnothing$ is the union of elements from the empty subcollection of $B_S$. Further, $S$ is the union of elements from the full collection $B_S$. Thus, $\mathcal{T}_S$ is a topology over $S$.

Now, we argue: $\Pi_S \subseteq \mathcal{T}_S$. For this, fix an arbitrary $i_+ \in \mathbb{N}$ and $A \subseteq X$. If $A = \varnothing$, then $\pi_{i_+}^{-1}(A) = \varnothing$. As $\varnothing \in \mathcal{T}_S$, $\pi_{i_+}^{-1}(A) \in \mathcal{T}_S$ if $A = \varnothing$. However, if $A \neq \varnothing$, then observe:

$$\pi_{i_+}^{-1}(A) = \bigcup \left\{ B(M) : M \in X^{\{1, \ldots, i_+\}} ; M(i_+) \in A \right\}$$

Thus, if $A \neq \varnothing$, then $\pi_{i_+}^{-1}(A) \in \mathcal{T}_S$. That is, $\pi_i^{-1}(A) \in \mathcal{T}_S$ for every $A \subseteq X$. Hence, $\{ \pi_i^{-1}(A) : i \in \mathbb{N} ; A \subseteq X \} \subseteq \mathcal{T}_S$ and we have already shown that $\mathcal{T}_S$ is a topology over $S$. Further, by definition, $\Pi_S$ is the smallest topology that satisfies $\{ \pi_i^{-1}(A) : i \in \mathbb{N} ; A \subseteq X \} \in \Pi_S$. Therefore, we obtain: $\Pi_S \subseteq \mathcal{T}_S$. 
Finally, we argue: $\mathcal{T}_S \subseteq \Pi_S$. For this, fix an arbitrary $I \in \mathbb{N}$ and consider an arbitrary map $M : \{1, \ldots, I\} \to X$. For each $i \in \{1, \ldots, I\}$, let $A_i := \{M(i)\}$. Then, we have the following:

$$B(M) = \bigcap \{\pi_i^{-1}(A_i) : i = 1, \ldots, I\}.$$ 

Since $\Pi_S$ is a topology and $\{\pi_i^{-1}(A) : i \in \mathbb{N} ; A \subseteq X\} \subseteq \Pi_S$, it follows that $B(M) \in \Pi_S$. Thus, $\Pi_S$ is a topology over $S$ such that $\mathcal{B}_S \subseteq \Pi_S$. Moreover, $\mathcal{T}_S$ is the smallest topology over $S$ such that $\mathcal{B}_S \subseteq \mathcal{T}_S$. Hence, we conclude: $\mathcal{T}_S \subseteq \Pi_S$.

**Q.E.D.**

Now to show $d$ is a stopping rule if and only if it is continuous, first, assume that $d : S \to X$ is continuous. Fix an arbitrary $S_* \in S$ and let $y_{S_*} := d(S_*)$. Now, we know that $\{y_{S_*}\}$ is open in the discrete topology over $X$. By continuity of the map $d$, the following set:

$$d^{-1}(\{y_{S_*}\}) := \{S \in S : d(S) = y_{S_*}\}$$

satisfies $d^{-1}(\{y_{S_*}\}) \in \Pi_S$. By the lemma above and the definition of $\mathcal{T}_S$, there exists $M : \{1, \ldots, k\} \to X$ such that $S_* 
 B(M) \subseteq d^{-1}(\{y_{S_*}\})$. Now, $S_* \in B(M)$ implies: $M = S_*|_k$ and $B(M) = \{S_*|_k \cdot T : T \in S\}$. Since $B(M) \subseteq d^{-1}(\{y_{S_*}\})$, it follows: $d(S_*|_k \cdot T) = y_{S_*}$ for all $T \in S$. Since $y_{S_*} = d(S_*)$ and $S_*$ was arbitrary, we have established: if the map $d : S \to X$ is continuous, then it is a stopping rule.

Now, assume that $d : S \to X$ is a stopping rule. Since $X$ has the topology $2^X$, we must argue that $d^{-1}(A) := \{S \in S : d(S) \in A\} \in \Pi_S$ for any $A \subseteq X$. Since $d^{-1}$ preserves arbitrary unions and $\Pi_S$ is closed under arbitrary unions, it is enough to argue that $d^{-1}(\{y\}) \in \Pi_S$ for any $y \in X$. So, fix an arbitrary $y_* \in X$. If $d^{-1}(\{y_*\}) = \emptyset$, then we have nothing more to argue as $\emptyset \in \Pi_S$. Hence, assume that $d^{-1}(\{y_*\}) \neq \emptyset$. Consider an arbitrary $S_* \in d^{-1}(\{y_*\})$. Since $d$ is a stopping rule, there exists $k(S_*) \in \mathbb{N}$ such that: $d(S) = y_*$ for every $S \in B(S_*|_{k(S_*)})$. This is because $B(S_*|_{k(S_*)}) = \{S_*|_{k(S_*)} \cdot T : T \in S\}$. Thus, we have:

$$\bigcup \{B(S_*|_{k(S_*)}) : S \in d^{-1}(\{y_*\})\} = d^{-1}(\{y_*\}).$$

Hence, $d^{-1}(\{y_*\}) \in \mathcal{T}_S$ by definition of $\mathcal{T}_S$. By the lemma above, it follows that $d^{-1}(\{y_*\}) \in \Pi_S$. Since $y_* \in Y$ was arbitrary, we have: $d^{-1}(A) \in \Pi_S$ for any $A \in 2^X$. Thus, if the $D : S \to X$ is a stopping rule, then it is continuous.

**Q.E.D.**

### 4.1. Satisficing Rules

Satisficing, first introduced by Herbert Simon (see Simon (1955)), has been a hugely influential model of decision making and has been studied widely in the literature (see Kovach and Ülkü (2020), Aguiar et al. (2016), Tyson (2015) and Papi (2012), among others). Satisficing behavior incorporates search of the DM until a “good enough” alternative is observed. While some existing models endogenize the search order of the DM (see Aguiar et al. (2016)), others treat it as observable in the form of a list and vary the threshold (see Kovach and Ülkü (2020)). In this section, we propose two models similar in spirit and provide their behavioral characterization.

#### 4.1.1. Preliminaries

There is a large literature on limited attention and a variety of modeling approaches have been used (see Masatlıoğlu et al. (2012), Caplin and Dean (2015) and Manzini and Mariotti (2014)).
As mentioned in the above section, our notion of stopping rules also indicate a limited attention span of the DM. From the informational aspect of decision making, we argue that there exists a point by which the DM makes up her mind regarding what to choose. We capture this idea using the concepts of sufficiency and minimal sufficiency of finite segments in a sequence. Let $S_k$ be the set of all segments of length $k$. Then we define a sufficient segment as follows

**Definition 9:** A segment $M \in S_m$ is sufficient if $d(M.T) = d(M.T')$ for all $T, T' \in S$

The intuitive content of the above definition is as follows. As the DM faces a sequence $S \in S$, there comes a point $k \in \mathbb{N}$ when the segment $S|_k$ has enough information for the decision maker to have made up her mind about the choice i.e. $S|_k$ is informationally “sufficient” to enforce a decision. However, the acquired information will not be sufficient until a certain point in time. This motivates the notion of minimal sufficiency. Formally,

**Definition 10:** A segment $M \in S_m$ is minimal sufficient if it is sufficient and for any $k < m$, $M|_k$ is not sufficient

Minimal sufficiency captures the idea of “critical” length of a segment to enforce a decision. By critical, we mean that if the segment is smaller than that length, it can no longer guarantee the same choice irrespective of the tail. Note that the definition of stopping rules indicates that every sequence must have a corresponding minimal sufficient segment that “implements” the choice. Let us denote the class of sufficient and minimal sufficient segments as $S$ and $MS$ respectively. If $M = S|_k$ for some $k \in \mathbb{N}$ and $M \in S \cup MS$, then we will abuse notation and denote the choice of $S$ by $d(M)$ i.e. $d(S) = d(M)$. Also denote by $M(X)$ the set of alternatives that appear in the segment in $M$.

To illustrate the idea of sufficiency and minimal sufficiency, let us consider Example 2 of the introduction. Let $X = \{a, b, c\}$ and $s(a) = s(b) = s(c) = 1$. Suppose the DM has a threshold value of 3 and consider the sequence $S = (a b c a b c..)$ i.e. it consist of “cycles” of alternatives $a, b$ and $c$. Here, the minimal sufficient segment is of length 7 i.e. where $a$ is the first alternative to appear 3 times. Any segment of length less than 7 is not minimal sufficient and any segment of length more than 7 is sufficient.

### 4.1.2. Cardinal Satisficing

We formalize the idea of Example 2 and equip the DM with two objects. The first one is a weight function $w : X \to \mathbb{R}_+$ that assigns a number to every alternative. The weights can be thought of as some scores the DM assigns to the alternatives that are indicative of the relative importance of alternatives. For instance, a DM may give a higher score to “action” movies over the ones belonging to the genre “drama”. The second object that the DM is endowed with is a threshold number $v \in \mathbb{R}_+$. The threshold can be thought of as the satisficing component or the cutoff that the DM uses to make decisions. In what we term as the Cardinal Satisficing Rule (CSR), the DM parses through a sequence and selects the first alternative whose “cumulative” weight crosses the threshold. Since in this heuristic, the “intensity” of the weights can affect the choice, it is termed as “cardinal”. For any given sequence $S \in S$ and a position $N \in \mathbb{N}$, we define the cumulative weight of an alternative $x$ as

$$W^N_S(x) = |\{i \in \{1, \ldots, N\} : S(i) = x\}|.w(x)$$

Now, we can define CSR formally as follows
DEFINITION 11: A stopping rule $d$ is a Cardinal Satisficing Rule if there exists $v \in \mathbb{R}_+$ and $w : X \rightarrow \mathbb{R}_+$ such that for any $S \in \mathcal{S}$,

$$d(S) = \{ x : W^N_S(x) \geq v > W^N_S(y) \}$$

for all $y \neq x$ and some $N \in \mathbb{N}$

Before we state our axioms, we will introduce two concepts: favorable deletion and favorable shift. For any sequence $S \in \mathcal{S}$ and $k \in \mathbb{N}$, let $\hat{S}^k \in \mathcal{S}$ be the sequence which is defined as follows:

$$\hat{S}^k(i) :=
\begin{cases}
S(k+1) & \text{if } i = k; \\
S(k) & \text{if } i = k+1; \\
S(i) & \text{otherwise}.
\end{cases}$$

That is, the sequence $\hat{S}^k$ is obtained from $S$ by interchanging its $k^{th}$ and $k+1^{th}$ elements. We call $\hat{S}^k$ a favorable shift of $S$ with respect to an alternative $x$ if $S(k+1) = x$.

For any $S \in \mathcal{S}$ and $k \in \mathbb{N}$ define $\tilde{S}^k$ as

$$\tilde{S}^k(i) :=
\begin{cases}
S(i) & \text{if } i < k; \\
S(i-1) & \text{if } i > k
\end{cases}$$

The sequence $\tilde{S}^k$ is obtained from $S$ by dropping the alternative located at the $k^{th}$ position. We call $\tilde{S}^k$ a favorable deletion of $S$ with respect to an alternative $x$ if $S(k) \neq x$. The notions of favorable shift and favorable deletion with respect to an alternative capture the idea of bringing it "closer" to the DM. In other words, lowering the position in which an alternative appears in a sequence is considered as "favorable" for it.

Let the class of all favorable shifts of $S$ with respect to $x$ be denoted by $\mathcal{FS}(S, x)$. Further, the class of all favorable deletions of $S$ with respect to $x$ be denoted by $\mathcal{FD}(S, x)$. For any $S \in \mathcal{S}$ and $x \in X$, a favorable transformation of $S$ with respect to $x$ is any favorable shift or favorable deletion. The class of all favorable transformations of $S$ with respect to $x$ shall be denoted by $\mathcal{F}(S, x)$. Therefore, $\mathcal{F}(S, x) = \mathcal{FS}(S, x) \cup \mathcal{FD}(S, x)$ by definition. Now, we are ready to state the axioms that characterize CSR.

4.1.3. Axioms

Our characterization of CSR relies upon two axioms. Our first axiom is an adaptation of the idea of primacy effect to the setting of sequences.

**Axiom 1—Monotonicity:** Let $S \in \mathcal{S}$ and $d(S) = x$. Then $d(S') = x$ for all $S' \in \mathcal{F}(S, x)$.

Intuitively, the axiom captures the idea of primacy effect. It requires the DM to make the same choice if the chosen alternative is brought "closer" to him in the sequence i.e. if a new sequence is more favorable for an alternative that was previously chosen, then it should continue to chosen in the new sequence. Our second axiom relies on the above defined notions of sufficient and minimal sufficient segments.

**Axiom 2—Informational Dominance:** Let $M \in \mathbb{MS}$ and $N \in \mathcal{S}$ such that $d(M) = x$, $d(N) = z$ and $x \notin N(x)$. Then $d([M|_k.N].T) \neq x$ for any $k < m$ and all $T \in \mathcal{S}$. 
This axiom states that if a minimal sufficient segment $M$ “implements” an alternative $x$ and another sufficient segment $N$ that does not contain $x$ implements some other alternative $z$, then concatenating any truncation of $M$ with $N$ prevents $x$ from being chosen. In other words, it asserts that a sufficient segment not containing an alternative can “dominate” a non-minimal sufficient segment in an informational sense. To illustrate, consider the example discussed above. We showed that for a threshold of 3, the minimal sufficient segment for the sequence $S = (a b c a b c a b c \ldots)$ is $M = (a b c a b c a b c \ldots)$. Consider another sequence $S' = (b b c b b c \ldots)$. It is easy to see that the segment $N = (b b c b b)$ is sufficient. Informational dominance says that for any sequence which contains any truncation of $M$ concatenated with the segment $N$ as its initial segment, the choice cannot be equal to $a$. Now, we are ready to state our result.

**Theorem 3:** A stopping choice rule $d$ is a Cardinal Satisficing Rule if and only if it satisfies Monotonicity and Informational Dominance

**Proof:** (Necessity): Given a a choice function is a Cardinal Satisficing Rule, we know that there exists a $v \in \mathbb{R}_+$ and $w : X \rightarrow \mathbb{R}_+$ such that for any $S \in \mathcal{S}$, we have $d(S) = \{x : W_S^N(x) \geq v > W_S^N(y)\}$ for some $N \in \mathbb{N}$ and for all $y \neq x$. To show it satisfies Monotonicity consider any $S$ and its favourable deletion with respect to $d(S)$, say $S'$. Let $N_1 \in \mathbb{N}$ be the position of $S$ where $W_{S}^{N_1}(d(S)) \geq v > W_{S}^{N_1}(y)$ for all $y \neq d(S)$. Note that $S'$ is generated by “deleting” a term of $S$ that is not equal to $d(S)$, for $N_2 = N_1 - 1$ we have $W_{S}^{N_2}(d(S)) \geq v > W_{S}^{N_2}(y)$ for all $y \neq d(S)$ and therefore $d(S') = d(S)$. By similar argument, we can see that $d(S) = d(S')$ where $S'$ is a favorable shift of $S$ with respect to $d(S)$. To show that $d$ satisfies Informational Dominance, consider a minimal sufficient segment $M$ of length $m$ such that $d(M) = x$ and a sufficient segment $N$ of length $n$ such that $d(N) = y$ and $x \notin N(X)$. Assume for contradiction $d([M|k,N],T) = x$ for some $k < m$. Then there exists $N_1 \in \mathbb{N}$ such that $W_{T}^{N_1}(x) \geq v > W_{S}^{N_1}(y)$. Note that since $M|k$ is not minimal sufficient and $x \notin N(X)$, we must have $N_1 > k + n$. But, since the segment $N$ is sufficient, we must have $W_{S}^{N_2}(y) \geq v$ for some $N_2 < N_1$, a contradiction.

(Sufficiency): Let $d$ be a stopping rule that satisfies Monotonicity and Informational Dominance. First, we construct the “revealed” critical frequency of each alternative. Fix $x \in X$. Note that $x$ is chosen from the constant sequence $S^x = (x, x, \ldots)$ i.e. $d(S^x) = x$. Now, by the definition of a stopping rule, there exists $k \in \mathbb{N}$ such that $d(S^x) = d([S^x|k],T)$ for all $T \in \mathcal{S}$. Let $n_x = \inf \{k : d(S^x) = d(S^x|k)\}$. Since $\mathbb{N}$ is well ordered, we know that $n_x \in \mathbb{N}$.

Consider any non-constant sequence $S$ such that $d(S) = x$ (we do not need to prove anything for the case of constant sequences). Denote by $\#x(S|_i) = \{|j \in \{1, \ldots, i\} : [S|_i(j) = x]\}$, i.e. the number of appearances of $x$ in a segment $S|_i$ of $S$. Now, denote by $i(S,x) = \{i \in \mathbb{N} :
she chooses the alternatives considered. This is in contrast with the satisficing model discussed in

tives—her bounded attention span—and picks the first alternative that is ranked above

above the threshold. To illustrate, consider an example where

have

Now, consider finitely many favourable shifts of $S'$ with respect to $x$ to generate $S''$ such that its first $n_x$ terms are all $x$ followed by $n$ terms that are $y$. Again, by Monotonicity, we have $d(S'') = x$.

Denote this segment of $y$'s as $N$ and the segment of $x$'s as $Mx$ where $M$ is $(n_x - 1)$ long segment of $x$'s. So, we can write $S'' = [Mx.N].T$ where $T \in S$ and $T(i) = S(i(S',x) + j)$ for all $j \in \mathbb{N}$. By the definition of $n_x$ we know that there exists some $T \in S$ such that $d(M,T) \neq x$. Also, by the definition of $n_y$, we know that $d(N.T) = y$ for all $T \in S$. In other words, $Mx$ is a minimal sufficient segment and $N$ is a sufficient segment. Using Informational Dominance, we know that $d(MN_TX) \neq x$ for all $T \in S$. It must be that $d(MN_TX) = y$ for all $T \in S$. Suppose not i.e. $d(MN_TX) = z$ for some $z \neq x, y$ and $T \in S$. Then, by Monotonicity, it must be that $d(NxT) = z$, a contradiction since $N$ contains $n_y$ first $y$'s. Therefore $d(MN_TX) = y$ for all $T \in S$. Now, notice that we can generate the earlier sequence $S'$ by successively moving $y$'s to the left i.e. a finitely many favourable shifts with respect to $y$ and by Monotonicity, we must have $d(S') = y$, a contradiction. Therefore $i(S,x) < i(S,y)$.

Let $v = 1$ and $w(x) = \frac{x}{n_x}$ for all $x \in X$. Consider a stopping rule $d^*$ such that $d^*(S) = \{x : W^N_S(x) \geq v > W^N_S(y)\}$ for all $S \in S$. We will show that $d^*$ and $d$ coincide. Consider any arbitrary $S \in S$ and let $d(S) = z$. We know that $i(S,z) < i(S,y)$ for all $y \neq z$. Let $i(S,z) = N$. By construction, we know that $W^N_S(z) > W^N_S(y)$ for all $y \neq z$ and therefore $d^*(S) = z$. Since $S$ was chosen arbitrarily, we have shown that $d^* = d$.

**Q.E.D.**

### 4.1.4. Ordinal satisficing

In this section we introduce another variation of satisficing which we term ordinal satisficing. We use the term “ordinal” because the DM in this model is endowed with a strict preference ordering $\succ$ over the set of alternatives, $X$. She has a threshold alternative, say $a^* \in X$ that reflects the satisficing component. For any sequence, she considers a fixed number of alternatives—her bounded attention span—and picks the first alternative that is ranked above $a^*$. If no such alternative exists in her attention span, she chooses the $\succ$-maximal element among the alternatives considered. This is in contrast with the satisficing model discussed in Rubinstein (2012) where a DM chooses the last alternative from the list if it contains no alternative ranked above the threshold. To illustrate, consider an example where $X = \{a,b,c\}$ with $a \succ b \succ c$, $a^* = b$ and $k = 2$. For the sequence $S = (b\ c\ a\ a\ \ldots)$, the choice is $b$ whereas the choice from the sequence $S' = (a\ b\ c\ a\ \ldots)$ is $a$. Now, we formally define the heuristic.

**Definition 12:** A choice rule $d$ is a **Ordinal Satisficing Rule (OSR)** if there exists $(\succ, a^*, k)$ with $a^* \in X$, $\succ$ a linear order$^2$ over $X$ and $k \in \mathbb{N}$ such that $d(S) = x$ if

(i) If $x \succ a^*$ and $x$ is the first such alternative in the first $k$ positions; or

(ii) $x$ is the $\succ$-maximal element in the first $k$ positions

Before we state the axioms, we need to define the concept of a **decisive** element.

**Definition 13:** A set $D \subseteq X$ is the set of decisive alternatives of $X$ if $D = \{x \in X : (\forall M \in \mathbb{M}(S))[x \in X(M) \implies d(M) = x]\}$. Let $D' = X \setminus D$.

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$^2$A linear order is an asymmetric, complete and transitive binary relation.
The idea behind a “decisive” alternative is that whenever it is present in a minimal sufficient segment, it is chosen. Intuitively, it dominates attention of the DM and enforces its choice. For a given choice function, the set of decisive alternatives may be empty. However, we will show that in the case of OSR, it turns out to be non-empty. Further, in the special case where the DM is an attention-constrained preference maximizer i.e. she chooses the ≻-maximal alternative after viewing a fixed length of alternatives, this set will be a singleton. Based on the notion of a decisive alternative, we can partition the set of minimal sufficient segments into ones that contain at least one decisive alternative and the ones that do not.

**Definition 14:** $\text{MS}_{D'} := \{ M \in \text{MS} : X(M) \subseteq D' \}$ and $\text{MS}_D := \text{MS} \setminus \text{MS}_{D'}$.

4.1.5. Axioms

**Axiom 3—Replacement:** Let $M \in \text{MS}_{D'}$. Consider any $M' = (M \setminus \{x\}) \cup \{y\}$ such that $y \in D'$. Then $M' \in \mathbb{S}$.

The above axiom is related to the informational implications of replacing a non-decisive alternative with another non-decisive alternative. It says that such a replacement does not affect the informational content of a segment. In other words, the new segment generated by replacing one alternative retains its “sufficiency”.

**Axiom 4—Sequential-α:** Consider any $M, M' \in \text{MS}_{D'}$ such that $M(X) \subseteq M'(X)$ and $d(M') \in M(X)$. Then $d(M') = d(M)$.

This axiom is related to the classic condition-α (Sen (1969)) that characterizes rational choice functions on sets. Here, it is restricted to the sequences that contain only non-decisive alternatives. Intuitively, it says that if a non-decisive alternative is revealed “superior” to another non-decisive alternative, then the reverse cannot hold true for those alternatives.

**Axiom 5—Sequential-No Binary Cycles (S-NBC):** Let $x, y, z \in X$ and $M, M', M'' \in \text{MS}_{D'}$ such that $X(M) = \{x, y\}, X(M') = \{y, z\}$ and $X(M'') = \{x, z\}$. If $d(M) = x$ and $d(M') = y$, then $d(M'') \neq z$.

This is a mild condition related to the no binary cycles condition of Manzini and Mariotti (2007). It says that the minimal sufficient segments that contain pairs of alternatives cannot display cycles in choices. Now, we are ready to state our result.

**Theorem 4:** A stopping choice rule $d$ is an OSR if and only if it satisfies Replacement, Sequential-α and Sequential-NBC.

**Proof:** Necessity is easy to establish. So we prove the sufficiency. Suppose $d$ satisfies Replacement, Sequential-α and Sequential NBC. We will use the following two lemmas in establishing that $d$ is an OSR.

**Lemma 2:** $|M| = |M'|$ for any $M, M' \in \text{MS}_{D'}$.

**Proof:** Suppose not. W.L.O.G let $|M| > |M'|$. Since $d$ is a stopping rule, $M$ and $M'$ are finite. Therefore, the restriction of $M$ to $|M'|$ i.e. $M|_{|M'|}$ is also finite. So, we can reach from $M'$ to $M|_{|M'|}$ in finite number of “steps” of replacement i.e. there exists a chain of segments $M_1, \ldots, M_n$ with $M_1 = M'$ and $M_n = M|_{|M'|}$ such that $|\{i : M_j(i) \neq M_{j+1}(i)\}| = 1$ for all
By Theorem 1, we know that sufficient segment makes the DM stop and pick that alternative. Whereas any occurrence of a decisive alternative in a minimal chain has the same length. The DM considers all alternatives in his attention span and then decides. Therefore, \(|M| = i^D\)

**Proof:** Suppose not i.e. there exists a \(M \in \mathbb{M}_{DD}\) such that \(|M| > i^D\). Since \(d\) is a stopping rule, we know that \(M\) is finite. Consider any \(M' \in \mathbb{M}_{DD'}\). Suppose not, then there exists a \(M'' \in \mathbb{M}_{DD''}\) such that \(|M'\) \(\neq i^D\). Therefore, \(|M| \leq i^D\).

By definition of \(D\), we know that \(|M| = i^D\). Assume for contradiction that \(|M| > i^D\) (note that the argument for the case \(|M| < i^D\) is trivial by the definition of \(\mathbb{M}_{DD'}\)). Consider any \((S|_{M|1}, T) \neq d(M) = x\). Let \(\bar{M}\) be the minimal sufficient segment of the sequence \(|M|_{M|-1}, T\). By definition of \(D\), \(|M| < i^D\). Since \(|\bar{M}| < |M|\) and \(M(i) = M(i)\) for all \(i\), this is a contradiction to \(M \in \mathbb{M}_{DD}\). Therefore, \(|M| = i^D\).

**Q.E.D.**

**Remark.** The above lemmas show that all the minimal sufficient segments that do not contain any decisive alternatives have the same length. The DM considers all alternatives in his attention span and then decides. Therefore, any occurrence of a decisive alternative in a minimal sufficient segment makes the DM stop and pick that alternative.

By Theorem 1, we know that \(d\) is a uniform stopping rule and there exists a \(k \in S\), such that for all \(S \in S\), \(d([S]_{k}, T) = d([S]_{k}, T')\) for all \(T, T' \in S\). Now, we consider the following cases:

(i) \(k = 1\). This implies that size of any minimal sufficient segment is 1. Therefore \(D = X\).

Consider any linear order \(\succ\) on \(X\) and \(a^* = \min(X, \succ\)). We can show that \((k, \succ, a^*)\) explain the choice data.

(ii) \(k \geq 2\). Consider any \(x, y \in D'\). Define \(\succ\) as follows: \(x \succ y\) if there exists \(M \in \mathbb{M}_{DD'}\) such that \(\{x, y\} \in X(M)\) and \(c(M) = x\). We first show that \(\succ\) is a linear order over \(D'\). Assume for contradiction that \(\succ\) is not asymmetric. Then there exists \(M, M' \in \mathbb{M}_{DD'}\) such that \(x, y \in X(M) \cap X(M')\), \(d(M) = x\) and \(d(M') = y\). Consider \(M''\) such that \(X(M'') = \{x, y\}\) (such \(M''\) exists due to the assumption that \(k \geq 2\)). By Sequential-\(\alpha\), \(d(M'') = x\) and \(d(M'') = y\), a contradiction. Therefore \(\succ\) is asymmetric. Now, consider any \(x, y \in D'\) and \(M \in \mathbb{M}_{DD'}\) such that \(X(M) = \{x, y\}\). By definition of \(\succ\) we have either \(x \succ y\) or \(y \succ x\). Therefore \(\succ\) is weakly connected. To show \(\succ\) is transitive, we consider two cases:

(a) \(k = 2\). This implies \(|M| = 2\) for all \(M \in \mathbb{M}_{DD'}\). Consider any \(x, y, z \in D'\) and suppose \(x \succ y\) and \(y \succ z\). Therefore, we know that there exists \(M, M' \in \mathbb{M}_{DD'}\) such that \(X(M) = \{x, y\}\), \(d(M) = x\) and \(X(M') = \{y, z\}\), \(d(M') = y\). Consider \(M'' \in \mathbb{M}_{DD'}\) such that \(X(M'') = \{x, z\}\). By Sequential-\(\alpha\), we know that \(d(M'') \neq z\), implying \(d(M') = x\).

(b) \(k > 2\). Consider any \(x, y, z \in D'\). Let \(x \succ y\) and \(y \succ z\). Consider \(M \in \mathbb{M}_{DD'}\) such that \(X(M) = \{x, y, z\}\). By sequential-\(\alpha\), we know that \(d(M) \neq z\) and \(d(M) \neq y\). Therefore \(d(M) = x\) implying \(x \succ z\).
We have shown that $\succ$ is a linear order over $D'$. Now, let $a^* = \max(D', \succ)$ and consider any linear order $\tilde{\succ}$ on $X$ such that $\succ \subseteq \tilde{\succ}$ and $x \tilde{\succ} y$ for all $x \in D$ and $y \in D'$.

Before showing that this linear order and threshold alternative rationalize the choice data, we first show that there exists at least one decisive alternative.

**Lemma 4**: $D$ is non-empty

**Proof**: Assume for contradiction that $D$ is empty i.e. $X = D'$. By Lemma 1, all minimal sufficient segments are of the same length i.e. $M = M'$ for all $M \in \mathcal{MS}$. Since $\succ$ is a linear order over $X$, we have a unique maximal element. W.L.O.G let it be $x$. By Sequential-$\alpha$, we know that $d(M) = x$ for all $M \in \mathcal{MS}$ such that $d \in X(M)$. By definition of $D$, we must have $x \in D$, a contradiction. Q.E.D.

Now, we will show that $(k, \tilde{\succ}, a^*)$ explains the choice data. Let $d(S) = x$. There are two possible cases: (i) The segment of $S_k$ does not contain any alternative from $D$. In this case, by construction, $x$ is the $\succ$-maximal element of $X(S_k)$. (ii) The initial segment $S_k$ contains at least one alternative from $D$. By Lemma 2, $x \in D$ and is the first alternative from $D$ to feature in $S_k$.

Q.E.D.

It is interesting to note that rational behavior within the limited attention span is a special case of our satisficing model i.e. when $D$ is a singleton. This is in contrast with satisficing over sets where satisficing is equivalent to preference maximization (See Rubinstein (2012)). Note that in the case where $|D| > 1$, the identified preference order is not unique i.e. any ordering between the alternatives of $D$ can explain the choice data.

**4.2. Configuration Dependent rules**

In this section, we define a broad class of choice rules that we call configuration dependent rules. The underlying idea is that the decision is made using the “configuration” of the alternatives i.e. the pattern of their occurrence in an input sequence. Configuration dependent rules subsume many possible behaviors. One can think of rules that utilise the information on positioning of alternatives to make choices. Alternatively, one may be interested in rules that focus on frequency of alternatives appearing in the sequence. Rules that utilise any combination of the frequency and positioning of alternatives can also be analysed under the umbrella of configuration dependent rules.

Configuration dependent rules can be described by what we term a bitstream processor. A bitstream is any sequence $b \in \{0,1\}^\mathbb{N}$ i.e. a sequence of 0’s and 1’s. Let $b(i)$ denote the $i^{th}$ component of the bitstream $b$. We call a collection of bitstreams $B$ feasible if it satisfies the following condition

$$|\{b \in B | b(i) = 1\}| = 1 \ \forall i \in \mathbb{N}$$

In other words for any arbitrary position $i \in \mathbb{N}$, there is exactly one bitstream that contains 1 at its $i^{th}$ position. Denote by $\mathbb{B}$ the set of all bitstreams and $\mathcal{B}$ the set of all feasible collections of bitstreams.

**Definition 15**: A bitstream processor is a map $f : \mathcal{B} \rightarrow \mathbb{B}$ such that $f(B) \in B$ for all $B \in \mathcal{B}$.
A bitstream processor selects a bitstream from a feasible collection of bitstreams. One can think of configuration dependent rules "encrypting" any given sequence into a feasible collection sequences of 0’s and 1’s and feeding it into a bitstream processor which then selects one bitstream out of the ones fed into it. This selected bitstream is then “decrypted” into an alternative which is the final choice.

Let \( \{a \in \{0, 1\}^n : a(i) = 1 \text{ if } S(i) = x, \text{ 0 otherwise}\} \). This corresponds to the “configuration” of the alternative \( x \) in sequence \( S \). Note that any sequence \( S \) corresponds to a feasible collection of bitstreams. Denote by \( S \sigma \) the feasible collection of bitstreams generated by \( S \) where \( |B(S)| = |X(S)| \) i.e. the number of bitstreams is equal to the number of alternatives appearing in \( S \). To illustrate, consider the sequence \( S = (a b c a b c...) \) i.e. the sequence of cycles of \( a, b \) and \( c \). This will generate three bitstreams: \( b_1 = (1 0 0 1 0 0...), b_2 = (0 1 0 0 1 0...) \) and \( b_3 = (0 0 1 0 0 1...) \). Note that \( \{b_1, b_2, b_3\} \) form a feasible collection of bitstreams. Now, we can formally define configuration dependent rules.

**Definition 16:** A choice rule \( d \) is a configuration dependent rule if there exists a bitstream processor \( f \) such that \( d(S) = x \) if and only if \( f(B(S)) = x(S) \) for all \( S \in S \).

Intuitively, the choice is made “as if” the input sequence is viewed as a feasible collection of bitstreams and fed into a bitstream processor. The choice of the processor, which is a sequence of 0’s and 1’s, corresponds to one of the alternatives of the sequence. Configuration dependent rules are characterized by a Neutrality axiom that is stated as follows.

**Axiom 6—Neutrality:** Consider any bijection \( \sigma : X \rightarrow X \) and \( S, S' \in S \).

\[ S'(i) = \sigma(S(i)) \forall i \in \mathbb{N} \implies [d(S') = \sigma(d(S))] \]

This axiom states that if a sequence is “transformed” into a new sequence by relabelling the alternatives, then the choice from the new sequence must respect this transformation. In other words, the choice rule is “neutral” with respect to the identity of the alternatives.

**Theorem 5:** A choice rule is a configuration dependent rule if and only if it satisfies Neutrality.

**Proof:** Suppose \( d \) is a configuration dependent rule. Then there exists an \( f \) such that \( d(S) = x \iff f(B(S)) = x(S) \). Consider any \( S, S' \) such that \( d(S) = x \) and for all \( i \in \mathbb{N}, S'(i) = \sigma(S(i)) \). We know that \( B(S) = B(S') \) and \( \sigma(x)(S') = x(S) \). Therefore \( f(B(S')) = \sigma(x)(S') \) which implies \( d(S') = \sigma(d(S)) \).

To show the other direction, consider a choice rule \( d \) that satisfies Neutrality. Define the relation \( \sim_{\sigma} \) as follows: \( S \sim_{\sigma} S' \) if and only if there exists a bijection \( \sigma : X \rightarrow X \) such that \( S'(i) = \sigma(S(i)) \forall i \in \mathbb{N} \). Note that \( \sim_{\sigma} \) is an equivalence relation and hence partitions \( S \). Now, consider any arbitrary \( S \in S \) such that \( d(S) = y \) for some \( y \in X \). Define \( f \) as \( f(B(S)) = y(S) \). Consider any \( S' \) such that \( S \sim_{\sigma} S' \), for some bijection \( \sigma : X \rightarrow X \). By Neutrality, we know that \( d(S') = \sigma(c(S)) \). Also, since \( B(S) = B(S') \) and \( y(S) = \sigma(y)(S') \), we have \( f(B(S')) = \sigma(y)(S') \), by construction. Hence we have defined an \( f \) such that \( d(S) = x \) if and only if \( f(B(S)) = x(S) \). Therefore \( d \) is a configuration dependent rule.

Q.E.D.
4.3. Rational Configuration Dependent Rules

In this section, we provide and characterize a class of rules that are the analogue of rational choice functions over sets. We term them as rational configuration dependent rules. The idea behind rational configuration dependent rules is that the DM has a preference over the patterns or configurations of the alternatives. This is captured by endowing her with a linear order\(^1\) over the set \(\{0, 1\}^N\) which she uses to make choices. The heuristic can be formally defined as follows.

**Definition 17**: A choice rule \(d\) is a rational configuration dependent rule if there exists a linear order \(\triangleright\) over \(\{0, 1\}^N\) such that for all \(S \in S\)

\[
d(S) = \{x : x(S) \triangleright y(S) \text{ for all } y, x \in X(S)\}
\]

As mentioned above, configuration dependent rules can be used to describe behavior where the information about location of alternatives can be used to make choices. Rational configuration-dependent rules, in particular, are useful to this effect. For instance, consider a DM that always picks the first alternative from a sequence. To describe this behavior, let us denote by \(O\) the set of all \(\{0, 1\}^N\) sequences. Let \(O_1\) and \(O_2\) form a partition of \(O\) where \(O_1 = \{a \in O : a(1) = 1\}\) i.e. the set of all 0, 1 sequences that have 1 at its first position and \(O_2 = O \setminus O_1\). Such behavior can be explained as a rational configuration dependent rule by a linear order \(\triangleright\) over \(O\) with \(x \triangleright y\) for any \(x \in O_1\) and \(y \in O_2\).

Rational configuration-dependent rules are characterized using a condition that resembles the well-known Strong Axiom of Revealed Preference (SARP). To state our axiom, we need the notion of an equivalence relation between two sequences with respect to an alternative.

**Definition 18**: For any \(x \in X\), let \(\sim_x \in S \times S\) such that \(S \sim_x S'\) if and only if \(S(i) = x \implies S'(i) = x\) for all \(i \in N\)

The above defined binary relation says that two sequences are related via the relation \(\sim_x\) if the configuration of the alternative \(x\) is the same for both. Now, we are ready to state our axiom.

**Axiom 7—Acyclicity**: For any \(x_1, x_2, \ldots, x_n\) such that \(x_i \in X\) and \(S_1, S_2, \ldots, S_n\) such that \(S_j \sim_{x_{j+1}} S_{j+1}\) for all \(j \in \{1, \ldots, n-1\}\), \(S_n \sim_{x_1} S_1\)

\[
d(S_1) = x_1 \ldots d(S_{n-1}) = x_{n-1} \implies d(S_n) \neq x_n
\]

This axiom is closely related to SARP and it says that if an alternative \(x_1\)'s configuration is directly or indirectly “revealed” preferred to another alternative \(x_n\)'s configuration, then the converse cannot hold. Now, we state our result.

**Theorem 6**: A choice function \(d\) is a rational configuration-dependent rule if and only if it satisfies Neutrality and Acyclicity.

**Proof**: Necessity is easy to establish, so we prove the sufficiency. Define the following “revealed” relation over configurations \(\triangleright^e\) as follows: For any \(a, b \in \{0, 1\}^N\), \(a \triangleright^e b\) if there exists a sequence \(S\) with \(x(S) = a\) and \(y(S) = b\) for some \(x, y \in X\) and \(d(S) = x\). First, we show that \(\triangleright^e\) is asymmetric. Suppose not, then there exist \(a, b \in \{0, 1\}^N\) such that \(a \triangleright^e b\) and \(b \triangleright^e a\), i.e. there exist \(S, S' \in S\) and \(x, y, w, z \in X\) with \(x(S) = w(S') = a\), \(d(S) = x\) and \(y(S) = z(S') = b\), \(d(S') = z\). There are four possible cases:

---

1. A linear order over a set is a complete, transitive and antisymmetric binary relation.
(i) $x = w$ and $y = z$. Note that $S \sim_y S'$ and $S' \sim_x S$. By Acyclicity, $d(S') \neq y$.

(ii) $x \neq w$ and $y = z$. Let $\sigma : X \to X$ be such that $\sigma(x) = w$ and $\sigma(a) = a$ for all $a \neq x$.

Let $S''$ be such that $S''(i) = \sigma(S(i))$ for all $i \in \mathbb{N}$. By Neutrality, $d(S'') = w$. Note that $S'' \sim_y S'$ and $S' \sim_x S''$. By Acyclicity, $d(S') \neq y = z$.

(iii) $x = w$ and $y \neq z$. Let $\sigma : X \to X$ be such that $\sigma(z) = y$ and $\sigma(a) = a$ for all $a \neq x$.

Let $S''$ be such that $S''(i) = \sigma(S'(i))$ for all $i \in \mathbb{N}$. By Neutrality, $d(S'') = y$. Note that $S'' \sim_y S$ and $S \sim_x S''$. By Acyclicity, $d(S) \neq x$.

(iv) $x \neq w$ and $y \neq z$. Let $\sigma : X \to X$ be such that $\sigma(z) = y$, $\sigma(w) = x$ and $\sigma(a) = a$ for all $a \neq x, w$. Let $S''$ be such that $S''(i) = \sigma(S'(i))$ for all $i \in \mathbb{N}$. By Neutrality, $d(S'') = y$.

Note that $S \sim_y S''$ and $S'' \sim_x S$. By Acyclicity, $d(S) \neq x$.

Similarly, by Acyclicity and Neutrality, we can show that the “reaveled” relation $\triangleright^c$ is also acyclic. Now, define an indirect “reaveled” relation $a \triangleright^s b$ as follows: For any $a, b \in \{0, 1\}^n$, $a \triangleright^s b$ if there exists a chain of alternatives $a_1, \ldots, a_n \in \{0, 1\}^n$ with $a = a_1$ and $a_n = b$ such that $a \triangleright c a_1 \triangleright c \ldots a_n \triangleright c b$. Note that by construction $\triangleright^s$ is asymmetric and transitive i.e. a partial order. Consider a exists a linear order $\triangleright$ such that $\triangleright^s \subseteq \triangleright$. Now, define $\tilde{d}(S) = \{x : x(S) \triangleright y(S) \quad \text{for all } y \neq x\}$ for all $S \in S$. Consider any $S$ and W.L.O.G. let $d(S) = x$. Let $x(S) = a$ and now it is easy to see that $d(S) = \tilde{d}$.

Q.E.D.

5. Topological Approach

In this section, we discuss a topological approach to our main result. Having already introduced the product topology on the set of all inputs $S$, we now explicitly describe convergence with respect to the product topology. The following definition is standard.

DEFINITION 3: Suppose, $(Z, \mathcal{T})$ is a topological space. A $Z$–valued sequence $(z_n)$ converges in $\mathcal{T}$ to $z_*$ if, for every $U \in \mathcal{T}$ with $z_* \in U$, there exists $n_U \in \mathbb{N}$ such that $z_n \in U$ for all $n \geq n_U$.

The phrase “$(z_n)$ converges in $\mathcal{T}$ to $z_*$” shall often be abbreviated as “$z_n \to z_*$ in $\mathcal{T}$”. In particular, the meaning of any $X$–valued sequence $(S)$ converges in the product topology $\Pi_S$ to some $S_* \in S$ stands specified. Now, consider the following proposition.

PROPOSITION 1: Let $(S_n)$ be an $X$–valued sequence and $S_* \in S$. Then, $S_n \to S_*$ in $\Pi_S$, if and only if, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $S_n|_k = S_*|_k$ if $n \geq n_k$.

PROOF: First, we assume: $S_n \to S_*$ in $\Pi_S$. Fix an arbitrary $k \in \mathbb{N}$ and define $M : \{1, \ldots, k\} \to X$ as follows: $M(i) := S(i)$ for all $i \in \{1, \ldots, k\}$; that is, $M = S_*|_k$. Now, $\Pi_S = \mathcal{T}_S$ by proposition 1 and recall $B_S \subseteq T_S$ by definition of $\mathcal{T}_S$. Thus, $B(M) \in \Pi_S$. Then, by definition 3, $S_n \to S_*$ in $\Pi_S$ implies: there exists $n_M(B_M) \in \mathbb{N}$ such that $S_n \in B(M)$ if $n \geq n_M(B_M)$. Let $n_k := n_M(B_M)$. Finally, note that $S_n \in B(M)$ and $M = S_*|_k$ implies: $S_n|_k = S_*|_k$.

Now, we prove the converse. For this, consider an arbitrary $U \in \Pi_S$ with $S_* \in U$. By proposition 1, $U \in \mathcal{T}_S$. By definition of $\mathcal{T}_S$, there exists $k \in \mathbb{N}$ and a map $M : \{1, \ldots, k\} \to X$ such that $S_* \in B(M) \subseteq U$. Also, there exists $n_k \in \mathbb{N}$ such that $S_n|_k = S_*|_k$ if $n \geq n_k$. However, $S_* \in B(M)$ implies $S_*|_k = M$. Thus, $S_n \in S_*|_k$ implies $S_n \in B(M)$. Therefore, $S_n \in B(M)$ for all $n \geq n_k$. Since $B(M) \subseteq U$, we have: $S_n \in U$ if $n \geq n_k$. Thus, let $n_U := n_k$ to complete the proof.

Q.E.D.
In addition to the topology over the domain $S$, we must formalize a certain continuity property associated with any stopping rule. For this, we associate to each decision rule $d : S \to Y$ a natural map $k_d : S \to \mathbb{N} \cup \{0, \infty\}$ which was defined as:

$$k_d(S) := \inf \{k \in \mathbb{N} : (\forall T \in S)[d(S|_k \cdot T) = d(S)]\}$$

for all $S \in S$. Assuming that $d : S \to Y$ is not a constant function, the fact that it is a stopping rule is equivalent to asserting that $k_d$ is $\mathbb{N}$–valued. Also, we consider the set $\mathbb{N}$ to be endowed with the discrete topology $2^{\mathbb{N}}$. Note, this topology is the restriction to $\mathbb{N}$ of the standard topology on $\mathbb{R}$. A basic result is as follows.

**Proposition 2**: Let $S$ and $\mathbb{N}$ be endowed with the topologies $\Pi_S$ and $2^{\mathbb{N}}$, respectively. If $d : S \to Y$ is a non–constant stopping rule, then the associated map $k_d : S \to \mathbb{N}$ is continuous.

**Proof**: Let $d : S \to Y$ be a non–constant stopping rule. Fix an arbitrary $A \subseteq \mathbb{N}$. We shall argue: $k_d^{-1}(A) \in \Pi_S$. If $A = \emptyset$, then $k_d^{-1}(A) = \emptyset$. This is because $k_d$ is $\mathbb{N}$–valued since $d$ is a non–constant stopping rule. Thus, $k_d^{-1}(A) \in \Pi_S$ if $A = \emptyset$. Henceforth, we assume that $A \neq \emptyset$. By definition of the map $k_d$, we have:

$$k_d^{-1}(A) = \bigcup \{B(S|_{k_d(S)}) : k_d(S) \in A\}.$$  

Thus, $k_d^{-1}(A) \in \mathcal{I}_S$ by definition of $\mathcal{I}_S$. Since proposition 1 requires that $\mathcal{I}_S = \Pi_S$, we obtain: $k_d^{-1}(A) \in \Pi_S$. Q.E.D.

With this background in place, we are ready to prove Theorem 1 via topological methods. The first of these proofs is as follows.

**Proof 2**: Since $X$ is finite, the topology $2^X$ makes $X$ compact. Thus, by Tychonoff’s theorem, the product topology $\Pi_S$ makes $S$ compact. Since the continuous image of a compact set is compact, it follows that $k_d(S) := \{k_d(S) : S \in S\}$ is a compact subset of $\mathbb{N}$ endowed with the topology $2^{\mathbb{N}}$. Now, $2^{\mathbb{N}}$ is the restriction of the standard topology of $\mathbb{R}$ to $\mathbb{N}$. Thus, $k_d(S)$ is a compact subset of $\mathbb{R}$. Since compact subsets of $\mathbb{R}$ must be bounded, there exists $k_* \in \mathbb{N}$ such that $k_d(S) \leq k_*$ for all $S \in S$. Thus, the decision rule $d : S \to Y$ has a uniform stopping time. Q.E.D.

While the above argument is short, it rests in large measure on the abstract result that continuous maps over compact set have a compact range. The following argument, however, removes the role of this result in furnishing a proof of Theorem 1.

**Proof 3**: Let $d : S \to X$ be a non–constant stopping rule. Thus, the map $k_d$ is $\mathbb{N}$–valued. For each $k \in \mathbb{N}$, let $A_k := \{S \in S : k_d(S) = k\}$. Since $k_d$ is $\mathbb{N}$–valued, we have: $S = \bigcup_{k \in \mathbb{N}} A_k$. Then, $\{A_k : k \in \mathbb{N}\}$ is an open cover of $S$ if we can argue: $A_k \in \Pi_S$ for every $k \in \mathbb{N}$. For this, fix an arbitrary $k \in \mathbb{N}$ and observe:

$$\{S \in S : k_d(S) = k\} = \bigcup \{B(S|_k) : k_d(S) = k\}.$$  

Thus, $A_k \in \mathcal{I}_S$ by definition of $\mathcal{I}_S$. Then, proposition 1 implies that $A_k \in \Pi_S$. That is, $\{A_k : k \in \mathbb{N}\}$ is an open cover of $S$ in the topology $\Pi_S$. However, $\Pi_S$ makes $S$ compact
by Tychonoff’s theorem (see Munkres (2000)). Thus, there exists $L \in \mathbb{N}$ and $k_1 < \ldots < k_L$
such that:

$$S = \bigcup_{l=1}^{L} \{ S \in S : k_d(S) = k_l \}.$$ 

Define $k_* := \max\{k_l : l = 1, \ldots, L\}$. Thus, $k_d(S) \leq k_*$ for all $S \in S$. \textit{Q.E.D.}

A direct comparison of proofs 2 and 3 suggests that the essential ingredient for Theorem 1 to
hold is the compactness of $S$ under the product topology. This was the point of step 2 in the
elementary proof (i.e., proof 1). However, only the sequential compactness of $S$ was established
there which is weaker than compactness. Further, this was done by a direct argument rather
than appealing to the theorem of Tychonoff. The next proof essentially casts the elementary
proof via the compactness of various subsets of $S$ under $\Pi_S$.

PROOF 4: Suppose, $d : S \to Y$ is a stopping rule that does not have a uniform stopping
time. Thus, for each $j \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ such that (1) $k_j < k_{j+1}$ for all $j \in \mathbb{N}$, and (2)
the image of the map $k_d$, which is the set $k_d(S) := \{k_d(S) : S \in S\}$, is $\{k_j : j \in \mathbb{N}\}$. Then, for
each $j \in \mathbb{N}$, define $A_j := \{ S \in S : k_d(S) \geq k_j \}$, is $\{k_j : j \in \mathbb{N}\}$. Then, for
each $j \in \mathbb{N}$, define $A_j := \{ S \in S : k_d(S) \geq k_j \}$. From (1) and (2), we have: $A_j \neq \emptyset$ for every
$j \in \mathbb{N}$. Further, $A_{j+1} \subseteq A_j$ for every $j \in \mathbb{N}$. Thus, $\{A_j : j \in \mathbb{N}\}$ satisfies the finite–intersection property.

We now argue that, if $A_j$ is compact for every $j \in \mathbb{N}$, then the claim of Theorem 1 holds. This is
because the collection $\{A_j : j \in \mathbb{N}\}$ satisfies the finite–intersection property. Thus, by Cantor’s
theorem:

$$\bigcap \{A_j : j \in \mathbb{N}\} \neq \emptyset.$$ 

Thus, there exists $S_* \in S$ such that $S_* \in A_j$ for all $j \in \mathbb{N}$. Then, by definition of $A_j$, we have:

$k_d(S_*) \geq k_j$ for every $j \in \mathbb{N}$. Since $k_j < k_{j+1}$ and $k_j \in \mathbb{N}$ for all $j \in \mathbb{N}$, we have: $k_d(S_*) = \infty$.\n
This contradicts the fact that $d$ is a stopping rule. Thus, our supposition that the stopping rule
d does not have a uniform stopping time must be wrong.

Therefore, it only remains to argue: $A_j$ is compact for each $j \in \mathbb{N}$. For this, fix an arbitrary $j \in \mathbb{N}$. Notice, $A_1 = S$ which is compact under $\Pi_S$ by Tychonoff’s theorem. Hence, we assume $j \geq 2$. For any map $f : \{1, \ldots, L\} \to X$ and $L_* \leq L$, we shall denote the map $l \in \{1, \ldots, L_*\} \mapsto
M(l)$ by $M|_{L_*}$. That is, $M|_{L_*}$ is the truncation of the map $M$ at $L_*$. Now, define:

$$M_j := \{ M \in X^{\{1, \ldots, k_j\}} : \neg(\exists S \in S)[k_d(S) < k_j ; S|_{k_d(S)} = M|_{k_d(S)}] \}.$$ 

Thus, $M_j$ is precisely the collection of maps $M : \{1, \ldots, k_j\} \to X$ such that$^4 k_d(M \cdot T) \geq k_j$
for any $T \in S$. That is:

$$A_j = \bigcup \{B(M) : M \in M_j \}.$$ 

Since $X$ is finite, it follows that $M_j$ is finite. Thus, to show that $A_j$ is compact it is enough to
argue that $B(M_*)$ is compact for any $M_* : \{1, \ldots, K_*\} \to X$.$^5$ However, the map $f_{M_*} : S \to
B(M_*)$, defined by $f_{M_*}(T) := M_* \cdot T$ for all $T \in S$, is a homeomorphism. To see why, it is
equal to argue that the following holds:

$$A \in \Pi_S \iff \{M_* \cdot T : T \in A\} \in \Pi_S \cap B(M_*).$$ 

$^4$Recall, $M \cdot T$ is the concatenation of the map $T$ to the map $M$.

$^5$Here, $B(M_*)$ is endowed with the topology $\Pi_S \cap B(M_*)$. 

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This follows from the fact that \( \{M_\ast \cdot T : T \in B(M)\} = B(M_\ast \cdot M) \) for any map \( M : \{1, \ldots, K\} \to X \) and Lemma 1. Since \( S \) is compact and \( f_{M_\ast} \) is a homeomorphism from \( S \) to \( B(M_\ast) \), it follows that \( B(M_\ast) \) is compact. Thus, the set \( A_j \) is compact for each \( j \in \mathbb{N} \). Q.E.D.

6. CONCLUDING REMARKS

In this paper, we introduced a new framework of decision making that considers infinite sequences. Our framework provides a natural setting to study decision-making situations where the DM faces alternatives sequentially and provides a generalization of the framework on sequences. Our framework provides a natural setting to study decision-making situations where the DM faces alternatives sequentially and provides a generalization of the framework on sequences. Our main result showed that stopping rules are equivalent to its seemingly stricter subclass, uniform-stopping rules. We introduced the notion of computability of a decision rule using Turing machines and showed that any computable decision rule can be implemented by a finite automaton—a less powerful model of computation than the Turing machine.

While we have introduced some natural stopping choice heuristics in this framework, one can think of more such heuristics. The notions of sufficiency and minimal sufficiency of a stochastic choice rules and examining potential applications of our main result.

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6For \( M_\ast : \{1, \ldots, K_\ast\} \to X \) and \( M : \{1, \ldots, K\} \to X \), the map \( M_\ast \cdot M : \{1, \ldots, K_\ast + K\} \to X \) is defined by: \( [M_\ast \cdot M](k) := M_\ast(k) \) if \( k \in \{1, \ldots, K_\ast\} \); otherwise, \( [M_\ast \cdot M](k) := M(k - K_\ast) \).
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