UNIQUENESS OF FOURIER–JACOBI MODELS: THE ARCHIMEDEAN CASE

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ABSTRACT. We prove uniqueness of Fourier–Jacobi models for general linear groups, unitary groups, symplectic groups and metaplectic groups, over an archimedean local field.

1. INTRODUCTION AND THE MAIN RESULT

Uniqueness of Bessel models and Fourier–Jacobi models is the basic starting point to study $L$-functions for classical groups by Rankin–Selberg method ([7, 8]). Breakthroughs have been made towards the proof of the uniqueness in the recent years. Over a non-archimedean local field of characteristic zero, uniqueness of Bessel models and Fourier–Jacobi models is now completely proved, by the works of Aizenbud–Guoerевич–Rallis–Schiffmann [1], Sun [19], Waldspurger [23] and Gan–Gross–Prasad [6]. Over an archimedean local field, uniqueness of Bessel models is proved by Jiang–Sun–Zhu [11]. Only uniqueness of Fourier–Jacobi models in the archimedean case remains open. This article is aimed to prove this remaining case.

Theorem A. Let $G$ be a classical group $\text{GL}_n(\mathbb{R})$, $\text{GL}_n(\mathbb{C})$, $U(p, q)$, $\text{Sp}_{2m}(\mathbb{C})$, or a metaplectic group $\tilde{\text{Sp}}_{2m}(\mathbb{R})$, with an $r$-th Fourier–Jacobi subgroup $S_r$ of it ($r \geq 1$, $n \geq 2r$, $p, q \geq r$, $m \geq r$). Denote by $J_r$ and $N_r$ the Jacobi quotient and the Whittaker quotient of $S_r$, respectively. Then for every irreducible Casselman–Wallach representation $\pi$ of $G$, every nondegenerate irreducible Casselman–Wallach representation $\sigma$ of $J_r$, and every nondegenerate unitary character $\psi$ of $N_r$, one has that

$$\dim \text{Hom}_{S_r}(\pi \hat{\otimes} \sigma, \psi) \leq 1.$$ 

Here $\sigma$ and $\psi$ are viewed as representations of $S_r$ via inflations, and “$\hat{\otimes}$” stands for the completed projective tensor product. By abuse of notation, we do not distinguish representations with their underlying vector spaces. Fourier–Jacobi subgroups as well as their Jacobi quotients and Whittaker quotients are defined in Section 2. The notion of “nondegenerate unitary character on $N_r$” is also explained in Section 2. The notions concerning Casselman–Wallach representations are explained in Section 3.3. Note that Theorem A for $\tilde{\text{Sp}}_{2m}(\mathbb{R})$ implies the analogous result for the symplectic group $\text{Sp}_{2m}(\mathbb{R})$.

When $n = 2r$, or $p = q = r$, or $m = r$, Theorem A asserts uniqueness of Whittaker models for $G$. See [15], [4] for uniqueness of Whittaker models for quasi-split linear
groups over \( \mathbb{R} \) (or [11] for a quick proof). When \( G = U(n, 1) \) and \( \pi \) is unitary, Theorem A is proved in [2].

As in the proof of uniqueness of Bessel models, our idea is to reduce Theorem A to the following basic case, which is called the multiplicity one theorem for Fourier–Jacobi models.

**Theorem B.** Let \( J \) be one of the following Jacobi groups

\[
(1) \quad H_{2n+1}(\mathbb{R}) \rtimes GL_n(\mathbb{R}), \quad H_{2n+1}(\mathbb{C}) \rtimes GL_n(\mathbb{C}), \quad H_{2p+2q+1}(\mathbb{R}) \rtimes U(p,q),
\]

\[
H_{2n+1}(\mathbb{C}) \rtimes Sp_{2n}(\mathbb{C}), \quad H_{2n+1}(\mathbb{R}) \rtimes \tilde{Sp}_{2n}(\mathbb{R}), \quad p, q, n \geq 0,
\]

where "\( H_{2k+1} \)" indicates the appropriate Heisenberg group of dimension \( 2k+1 \). Denote by \( G \) its respective subgroup

\[
GL_n(\mathbb{R}), \quad GL_n(\mathbb{C}), \quad U(p,q), \quad Sp_{2n}(\mathbb{C}), \quad \tilde{Sp}_{2n}(\mathbb{R}).
\]

Then for every nondegenerate irreducible Casselman–Wallach representation \( \rho \) of \( J \), and every irreducible Casselman–Wallach representation of \( \pi \) of \( G \), one has that

\[
\dim \text{Hom}_G(\rho \hat{\otimes} \pi, \mathbb{C}) \leq 1.
\]

In the above inequality, \( \mathbb{C} \) stands for the trivial representation of \( G \). Theorem B is proved in [21], except for the case of \( G = \tilde{Sp}_{2n}(\mathbb{R}) \). But in this case, by the classification of nondegenerate irreducible Casselman–Wallach representations of Jacobi groups (see Section 3.3), Theorem B is obviously equivalent to the analogous result for \( Sp_{2n}(\mathbb{R}) \), which is also proved in [21].

2. **Fourier–Jacobi subgroups**

In order to prove Theorem A uniformly, we introduce the following notation. Let \((\mathbb{K}, \iota)\) be one of the followings five \( \mathbb{R} \)-algebras with involutions:

\[
(2) \quad (\mathbb{R} \times \mathbb{R}, \iota_\mathbb{R}), \quad (\mathbb{C} \times \mathbb{C}, \iota_\mathbb{C}), \quad (\mathbb{C}, \bar{\quad}), \quad (\mathbb{R}, 1_\mathbb{R}), \quad (\mathbb{C}, 1_\mathbb{C}),
\]

where \( \iota_\mathbb{R} \) and \( \iota_\mathbb{C} \) are the maps interchanging the coordinates, \( 1_\mathbb{R} \) and \( 1_\mathbb{C} \) are the identity maps, and "\( \bar{\quad} \)" is the complex conjugation. Let \( E \) be a skew-Hermitian \( \mathbb{K} \)-module; namely, it is a free \( \mathbb{K} \)-module of finite rank, equipped with a nondegenerate \( \mathbb{R} \)-bilinear map

\[
\langle \, , \rangle_E : E \times E \to \mathbb{K}
\]

satisfying

\[
\langle u, v \rangle_E = -\langle v, u \rangle_E, \quad \langle au, v \rangle_E = a\langle u, v \rangle_E, \quad a \in \mathbb{K}, \quad u, v \in E.
\]

Denote by \( U(E) \) the group of all \( \mathbb{K} \)-module automorphisms of \( E \) which preserve the form \( \langle \, , \rangle_E \). According to the five cases of \((\mathbb{K}, \iota)\) in (2), it is respectively a real general linear group, a complex general linear group, a real unitary group, a real symplectic group, or a complex symplectic group. Put

\[
U'(E) := \begin{cases} 
\tilde{Sp}(E) & \text{if } \mathbb{K} = \mathbb{R}; \\
U(E) & \text{otherwise},
\end{cases}
\]
where $\widetilde{\text{Sp}}(E)$ denotes the metaplectic double cover of the symplectic group $\text{Sp}(E)$. Then we have a short exact sequence
\begin{equation}
1 \to \mu_\mathbb{K} \to U'(E) \to U(E) \to 1,
\end{equation}
where
\[
\mu_\mathbb{K} := \begin{cases} 
\{\pm 1\} & \text{if } \mathbb{K} = \mathbb{R}; \\
\{1\} & \text{otherwise.}
\end{cases}
\]

Let $r \geq 1$ and assume that there is a sequence $S\vdash 0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$ of totally isotropic free $\mathbb{K}$-submodules of $E$ so that $\text{rank}_\mathbb{K}(X_i) = i$, $i = 0, 1, \ldots, r$. Put $J_\mathcal{F}(E) := \{g \in U(E) \mid (g - 1)X_i \subset X_{i-1}, i = 1, 2, \ldots, r\}$. Denote by $J_\mathcal{F}'(E)$ the inverse image of $J_\mathcal{F}(E)$ under the covering map $U'(E) \to U(E)$. It is called an $r$-th Fourier–Jacobi subgroup of $U'(E)$.

When $r = 1$, we put $J_X(E) := J_\mathcal{F}(E) = \{g \in U(E) \mid (g - 1)X \subset X\}$ and $J_X'(E) := J_\mathcal{F}'(E)$. Then $J_X'(E)$ is isomorphic to a Jacobi group of Theorem B, and conversely, all Jacobi groups of Theorem B are isomorphic to some $J_X'(E)$ (cf. [19, Section 1]).

For every subset $S$ of $E$, set $S^\perp := \{v \in E \mid \langle v, u \rangle_E = 0 \text{ for all } u \in S\}$. Then $E' := X_r^\perp/X_{r-1}$ is obviously a skew-Hermitian $\mathbb{K}$-module. Put $X' := X_r/X_{r-1} \subset E'$, which is a totally isotropic free $\mathbb{K}$-submodule of $E'$ of rank 1. Restrictions yield a surjective homomorphism
\[
\text{j}_\mathcal{F} : J_\mathcal{F}(E) \to J_X'(E'),
\]
There is a unique surjective homomorphism
\[
\text{j}_\mathcal{F}' : J_\mathcal{F}'(E) \to J_X'(E')
\]
so that the squares in
\[
\begin{array}{cccc}
1 & \to & \mu_\mathbb{K} & \to & J_\mathcal{F}'(E) & \to & J_\mathcal{F}(E) & \to & 1 \\
\downarrow & & \downarrow j_\mathcal{F} & & \downarrow j_\mathcal{F} & & & & \\
1 & \to & \mu_\mathbb{K} & \to & J_X'(E') & \to & J_X'(E') & \to & 1
\end{array}
\]
are commutative. In view of the homomorphism $j_\mathcal{F}'$, we call $J_X'(E')$ the *Jacobi quotient* of $J_\mathcal{F}'(E)$.

Put $N_\mathcal{F}(X) := \{g \in \text{GL}(X) \mid (g - 1)X_i \subset X_{i-1}, i = 1, 2, \ldots, r\}$. 

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It is a maximal unipotent subgroup of the group $GL(X)$ of $K$-linear automorphisms of $X$. Restrictions yield a surjective homomorphism $w_f : J_f(E) \to N_f(X)$. Composing it with the covering map $J'_f(E) \to J_f(E)$, we get a homomorphism

$$w'_f : J'_f(E) \to N_f(X).$$

In view of this homomorphism, we call $N_f(X)$ the Whittaker quotient of the Fourier–Jacobi subgroup $J'_f(E)$.

We review the notion of nondegenerate characters on $N_f(X)$. Define a surjective homomorphism

$$N_f(X) \to A_f(X) := \prod_{i=1}^{r-1} \text{Hom}_K(X_{i+1}/X_i, X_i/X_{i-1}), \quad g \mapsto (a_1, a_2, \ldots, a_{r-1}),$$

where $a_i$ is the map $v + X_i \mapsto (g - 1)v + X_{i-1}$. Then every unitary character $\psi_{N_f(X)}$ on $N_f(X)$ descends to a character $\psi_{A_f(X)}$ on $A_f(X)$ though (4). Note that $A_f(X)$ is a free $K^{r-1}$-module of rank 1. We say that $\psi_{N_f(X)}$ is nondegenerate if the restriction of $\psi_{A_f(X)}$ to every nonzero $K^{r-1}$-submodule of $A_f(X)$ is nontrivial.

3. Preliminaries

3.1. Almost linear Nash groups. We work in the setting of Nash groups. The reader is referred to [16, 17] for details. By a Nash group, we mean a group which is simultaneously a Nash manifold so that all group operations (the multiplication and the inversion) are Nash maps. Every semialgebraic subgroup of a Nash group is automatically closed and is called a Nash subgroup. It is canonically a Nash group. A finite dimensional real representation $V_R$ of a Nash group $G$ is said to be a Nash representation if the action map $G \times V_R \to V_R$ is Nash. A Nash group is said to be almost linear if it admits a Nash representation with finite kernel. For every linear algebraic group $G$ defined over $\mathbb{R}$, every finite fold topological group cover of an open subgroup of $G(\mathbb{R})$ is naturally an almost linear Nash group. On the other hand, every almost linear Nash group is of this form. In particular, all groups which occur in last section are almost linear Nash groups.

A Nash group is said to be unipotent if it admits a faithful Nash representation so that all group elements act as unipotent linear operators. It follows from the corresponding result for linear algebraic groups that every almost linear Nash group has a unipotent radical, namely, the largest unipotent normal Nash subgroup. A reductive Nash group is defined to be an almost linear Nash group with trivial unipotent radical.

Recall that a Nash manifold is said to be affine if it is Nash diffeomorphic to a closed Nash submanifold of some $\mathbb{R}^n$. Since every finite fold topological cover of an affine Nash manifold is an affine Nash manifold, all almost linear Nash groups are affine as Nash manifolds.

3.2. Schwartz inductions. If $M$ is an affine Nash manifold and $V_0$ is a (complex) Fréchet space, then a $V_0$-valued smooth function $f \in C^\infty(M; V_0)$ is said to be Schwartz if

$$|f|_{D,\nu} := \sup_{x \in M} |(Df)(x)|_\nu < \infty$$
for all Nash differential operator $D$ on $M$, and all continuous seminorm $|\cdot|_{\nu}$ on $V_0$. Recall that a differential operator $D$ on $M$ is said to be Nash if $D\varphi$ is a Nash function whenever $\varphi$ is a (complex valued) Nash function on $M$. Denote by $C^s(M; V_0) \subset C^\infty(M; V_0)$ the space of Schwartz functions. Then both $C^s(M; V_0)$ and $C^\infty(M; V_0)$ are naturally Fréchet spaces, and the inclusion map $C^s(M; V_0) \hookrightarrow C^\infty(M; V_0)$ is continuous. Furthermore, we have that (cf. [22, page 533])

$$C^s(M; V_0) = C^s(M) \hat{\otimes} V_0 \quad \text{and} \quad C^\infty(M; V_0) = C^\infty(M) \hat{\otimes} V_0,$$

where $C^s(M) := C^s(M; \mathbb{C})$ and $C^\infty(M) := C^\infty(M; \mathbb{C})$.

Now we recall Schwartz inductions from [5, Section 2]. Let $G$ be an almost linear Nash group. Then it is affine as a Nash manifold. Let $S$ be a Nash subgroup of $G$, and let $V_0$ be a smooth Fréchet representation of $S$ of moderate growth (cf. [5, Definition 1.4.1] or [20, Section 2]). Define a continuous linear map

$$ I_{S,V_0} : C^s(G; V_0) \to C^\infty(G; V_0), \quad f \mapsto \left( g \mapsto \int_S s.f(s^{-1}g) \, ds \right), $$

where $ds$ is a left invariant Haar measure on $S$. We define the unnormalized Schwartz induction $\text{Ind}_G^S V_0$ to be the image of the map (5). Under the quotient topology of $C^s(G; V_0)$ and under right translations, it is a smooth Fréchet representation of $G$ of moderate growth.

A partition of unity argument shows the following

**Lemma 3.1.** Let $f \in C^\infty(G; V_0)$. If

$$ f(sg) = s.f(g), \quad s \in S, \quad g \in G,$$

and $f$ is compactly supported modulo $S$ (that is, the support of $f$ has compact image under the map $G \to S\backslash G$), then $f \in \text{Ind}_G^S V_0$.

If $S'$ is a Nash subgroup of $G$ containing $S$, then we have a canonical isomorphism of representations of $G$ ([5, Lemma 2.1.6]):

$$ \text{Ind}_G^{S'}(\text{Ind}_S^G V_0) \cong \text{Ind}_G^S V_0. $$

We will use the following result.

**Lemma 3.2.** Let $V_0$ and $V$ be smooth moderate growth Fréchet representations of $S$ and $G$, respectively. If $V$ is nuclear, then there is an isomorphism of representations of $G$:

$$ \text{Ind}_S^G(\text{Ind}_S^G(V_0 \hat{\otimes} V|S)) \cong (\text{Ind}_S^G V_0) \hat{\otimes} V. $$

**Proof.** Note that the diagram

$$ \begin{array}{c}
C^s(G; V_0) \hat{\otimes} V \xrightarrow{I_{S,V_0} \hat{\otimes} \text{Id}_V} C^\infty(G; V_0) \hat{\otimes} V \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
C^s(G; V_0 \hat{\otimes} V) \xrightarrow{I_{S,V_0} \hat{\otimes} \text{Id}_{V \hat{\otimes} V}} C^\infty(G; V_0 \hat{\otimes} V)
\end{array} $$


commutes, where the vertical arrows are $G$-representation isomorphisms given by
\[ f \otimes v \mapsto (g \mapsto f(g) \otimes g.v). \]

The image of the bottom horizontal arrow of (8), to be viewed as a representation of $G$ with the quotient topology, equals to the left-hand side of (7). The top horizontal arrow of (8) is the composition of the map
\[(9) \quad \mathcal{C}^c(G; V_0) \hat{\otimes} V \to (\text{Ind}_G^SV_0) \hat{\otimes} V \]
and the map
\[(10) \quad (\text{Ind}_G^SV_0) \hat{\otimes} V \to \mathcal{C}^\infty(G; V_0) \hat{\otimes} V. \]

The continuous linear map (9) is surjective by [22, Proposition 43.9], and is then open by the open mapping theorem for Fréchet spaces. The map (10) is injective since $V$ is nuclear (cf. [22, Proposition 50.4]). Therefore, the image of the top horizontal arrow of (8) equals to the right-hand side of (7). This proves the lemma. \qed

3.3. Casselman–Wallach representations. Let $G$ be an almost linear Nash group as before. Denote by $\mathcal{D}(G)$ the space of (complex valued) Schwartz densities on $G$. Recall that $\mathcal{D}(G) = \mathcal{C}^c(G) dG$, where $dG$ is a left invariant Haar measure on $G$. It is an associative algebra under convolutions.

Let $V$ be a smooth Fréchet representation of $G$ of moderate growth. Then $V$ is a $\mathcal{D}(G)$-module:
\[(f(g) dG).v := \int_G f(g)g.v dG, \quad f \in \mathcal{C}^c(G), v \in V. \]

We say that $V$ is a Casselman–Wallach representation of $G$ if
- every $\mathcal{D}(G)$-submodule of $V$ is closed in $V$, and
- $V$ is of finite length as an abstract $\mathcal{D}(G)$-module.

It is clear that a Casselman–Wallach representation of $G$ is irreducible if and only if it is irreducible as an abstract $\mathcal{D}(G)$-module. Furthermore, by the open mapping theorem for Fréchet spaces, it is easy to see that if
\[0 \to V_1 \to V_2 \to V_3 \to 0\]
is a topologically exact sequence of smooth Fréchet representation of $G$ of moderate growth, then $V_2$ is a Casselman–Wallach representation if and only if both $V_1$ and $V_3$ are so.

When $G$ is reductive, du Cloux proves that $V$ is a Casselman–Wallach representation if and only if its underlying Harish–Chandra module is of finite length (cf. [5, Section 3]), and Casselman and Wallach prove that every finite length Harish–Chandra module has a unique Casselman–Wallach representation as its globalization (cf. [3] and [24, Chapter 11]).

We return to the setup in Section 2. Put
\[\mathfrak{I} := \{a \in \mathbb{K} \mid a' = a\}, \]
which is either \( \mathbb{R} \) or \( \mathbb{C} \). Let \( E \) be a skew-Hermitian \( \mathbb{K} \)-module as before. Put \( E_\mathbb{K} := E \), to be viewed as a symplectic space over \( \mathbb{K} \) under the form

\[
\langle u, v \rangle_{E_\mathbb{K}} := \frac{\langle u, v \rangle_E - \langle v, u \rangle_E}{2}.
\]

The associated Heisenberg group is defined to be

\[
H(E) := \mathbb{K} \times E,
\]

with group multiplication

\[
(a, v) \cdot (a', v') := (a + a' + \langle v', v \rangle_{E_\mathbb{K}}, v + v').
\]

The group \( U(E) \) acts (from left) on \( H(E) \) as group automorphisms through its natural action on \( E \). This defines a semidirect product (the Jacobi group)

\[
\mathbb{J}(E) := H(E) \rtimes U(E),
\]

and its covering

\[
\mathbb{J}'(E) := H(E) \rtimes U'(E).
\]

They are almost linear Nash groups, both having \( H(E) \) as their unipotent radicals. Recall that \( \mathbb{K} \) is the center of \( H(E) \).

Let \( \rho \) be an irreducible Casselman–Wallach representation of \( \mathbb{J}'(E) \). By a version of Schur Lemma ([5, Proposition 5.1.4]), \( \mathbb{K} \) acts through a character \( \psi_\rho \) in \( \rho \). We say that \( \rho \) is nondegenerate if \( \psi_\rho \) is nontrivial. Note that the moderate growth condition implies that \( \psi_\rho \) is unitary. Since all the Jacobi groups which occur in Theorem A and Theorem B are isomorphic (as Nash groups) to some \( \mathbb{J}'(E) \), we also get the notion of “nondegenerate irreducible Casselman–Wallach representations” for these groups.

Fix a nontrivial unitary character \( \psi_\mathbb{K} \) on \( \mathbb{K} \). Let \( \omega \) be a smooth oscillator representation of \( \mathbb{J}'(E) \) associated to it, namely, it is a Casselman–Wallach representation of \( \mathbb{J}'(E) \), and when viewed as a representation of \( H(E) \), it is irreducible with central character \( \psi_\mathbb{K} \). Smooth oscillator representations of \( \mathbb{J}'(E) \) exist by the well known result of splitting metaplectic covers ([14], see also [13, Proposition 4.1]). If \( \pi_0 \) is a Casselman–Wallach representations of \( U'(E) \), to be viewed as a representation of \( \mathbb{J}'(E) \) via inflation, then \( \omega \otimes \pi_0 \) is a Casselman–Wallach representations of \( \mathbb{J}'(E) \) so that \( \mathbb{K} \subset \mathbb{J}'(E) \) acts by the character \( \psi_\mathbb{K} \). Conversely, all such representations are of the form \( \omega \otimes \pi_0 \) for some \( \pi_0 \). Furthermore, \( \omega \otimes \pi_0 \) is irreducible if and only if \( \pi_0 \) is. See [20] for details.

4. Reduction to the basic case

We continue with the notation of Section 2. We reformulate Theorem A more precisely as follows:

**Theorem 4.1.** For every irreducible Casselman–Wallach representation \( \pi \) of \( U'(E) \), every nondegenerate irreducible Casselman–Wallach representation \( \sigma \) of \( J_X'(E') \), and every nondegenerate unitary character \( \psi \) on \( N_\mathbb{F}(X) \), one has that

\[
\dim \text{Hom}_\mathbb{J}'(E) (\pi \otimes \sigma, \psi) \leq 1.
\]
In this section, we explain the strategy of the proof of Theorem 4.1.

Fix two totally isotropic free submodules $Y \supset Y_{r-1}$ of $E$ such that the parings
\[
\langle \cdot, \cdot \rangle_E : X \times Y \to \mathbb{K} \quad \text{and} \quad \langle \cdot, \cdot \rangle_E : X_{r-1} \times Y_{r-1} \to \mathbb{K}
\]
are nondegenerate. Fix $x_r \in X$ and $y_r \in Y$ so that
\[
\langle x_r, Y_{r-1} \rangle_E = 0, \quad \langle X_{r-1}, y_r \rangle_E = 0, \quad \text{and} \quad \langle x_r, y_r \rangle_E = 1.
\]
Identify $E' := X_{r-1}^\perp / X_{r-1}$ with $(X_{r-1} \oplus Y_{r-1})^\perp$, and $E_0 := X^\perp / X$ with $(X \oplus Y)^\perp$. Then we get decompositions
\[
E = X_{r-1} \oplus E' \oplus Y_{r-1} = X \oplus E_0 \oplus Y \quad \text{and} \quad E' = \mathbb{K} x_r \oplus E_0 \oplus \mathbb{K} y_r.
\]
Identify $X' := X / X_{r-1} \subset E'$ with $\mathbb{K} x_r$, and $J_{X'}(E')$ with $\mathbb{J}(E_0)$ via the isomorphism
\[
g \mapsto ((gy_r, y_r)_{E'}, [gy_r - y_r]_{E_0}, [g]_{E_0}).
\]
Here for every $v \in \mathbb{K} x_r \oplus E_0$, denote by $[v]_{E_0} \in E_0$ the image of $v$ under the projection $\mathbb{K} x_r \oplus E_0 \to E_0$, and for every $g \in J_{X'}(E')$, denote by $[g]_{E_0}$ the element of $U(E_0)$ so that the diagram
\[
\begin{array}{ccc}
\mathbb{K} x_r \oplus E_0 & \xrightarrow{g|_{\mathbb{K} x_r \oplus E_0}} & \mathbb{K} x_r \oplus E_0 \\
& \downarrow & \\
E_0 & \xrightarrow{[g]_{E_0}} & E_0
\end{array}
\]
commutes.

Fix the unique identification $J_{X'}(E') = \mathbb{J}'(E_0)$ so that the squares in
\[
\begin{array}{cccc}
1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & J_{X'}(E') & \longrightarrow & J_{X'}(E') & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & \mathbb{J}'(E_0) & \longrightarrow & \mathbb{J}(E_0) & \longrightarrow & 1
\end{array}
\]
are commutative.

Denote by $P'_{X}$ the parabolic subgroup of $U'(E')$ stabilizing $X$, and by $M'_{X}$ the Levi subgroup of $U'(E')$ stabilizing both $X$ and $Y$. Then we have a Levi decomposition
\[
P'_{X} = N_{X} \rtimes M'_{X},
\]
where $N_{X}$ is the unipotent radical of $P'_{X}$. Moreover, we have
\[
M'_{X} = \frac{U'(E_0) \times \text{GL}'(X)}{\Delta \mu_{\mathbb{K}}},
\]
where $\text{GL}'(X)$ is the subgroup of $M'_{X}$ fixing $E_0$ pointwise, and $\Delta \mu_{\mathbb{K}}$ is the group $\mu_{\mathbb{K}}$ diagonally embedded in $U'(E_0) \times \text{GL}'(X)$.

Put $H(X^\perp) := \mathbb{K} \times X^\perp$, which is a subgroup of $H(E)$. Define a homomorphism
\[
p_{X} : H(X^\perp) \rtimes P'_{X} \to H(E_0) \rtimes M'_{X},
\]
where $t \in \mathbb{K}$, $v \in X$, $v_0 \in E_0$, $u \in N_{X}$ and $h \in M'_{X}$. The kernel of $p_{X}$ is $X \rtimes N_{X}$. We always view $H(E_0) \rtimes M'_{X}$ as a quotient of $H(X^\perp) \rtimes P'_{X}$ via the map $p_{X}$.\[\]
Since the covering map $GL'(X) \to GL(X)$ uniquely splits over $N_\mathfrak{f}(X)$, we also view $N_\mathfrak{f}(X)$ as a subgroup of $GL'(X)$.

The following lemma is routine to check and is the key to the proof of Theorem 4.1.

**Lemma 4.2.** The following diagram

\[ J'(E) \xrightarrow{g \mapsto y_r^{-1} g y_r} H(X^\perp) \times P'_X \]

\[ j'_\mathfrak{f} \times \mu'_\mathfrak{f} \]

\[ J'_X'(E') \times N_\mathfrak{f}(X) \xrightarrow{p_X} H(E_0) \times M'_X \]

commutes, where the bottom horizontal arrow is the map

\[ J'_X'(E') \times N_\mathfrak{f}(X) \subset J'_X'(E') \times GL'(X) \]

\[ \frac{\Delta \mu_\mathfrak{f}}{\mu_\mathfrak{f}} = \frac{\Delta \mu_\mathfrak{f}}{\mu_\mathfrak{f}} = \frac{H(E_0) \times (U'(E_0) \times GL'(X))}{\Delta \mu_\mathfrak{f}} = H(E_0) \times M'_X. \]

For the top horizontal arrow of (13), note that $J'_\mathfrak{f}(E) \subset P'_X \subset H(X^\perp) \times P'_X \subset J'(E)$, and $y_r \in E \subset H(E) \subset J'(E)$.

As in Theorem 4.1, let $\sigma$ be a nondegenerate irreducible Casselman–Wallach representation $J'_\mathfrak{f}(E_0) = J'_X'(E')$. Fix a generic irreducible Casselman–Wallach representation $\tau$ of $GL'(X)$ so that $\chi_\tau = \chi_\sigma$, where $\chi_\sigma$ is the character of $\mu_\mathfrak{f}$ through which $\mu_\mathfrak{f} \subset J'_\mathfrak{f}(E_0)$ acts in $\sigma$, and likewise for $\chi_\tau$. Then $\sigma \otimes \tau$ descends to a representation of

\[ \frac{J'(E_0) \times GL'(X)}{\Delta \mu_\mathfrak{f}} = H(E_0) \times M'_X. \]

We recall some notations in [11]. Put

\[ d_\mathfrak{f} := \begin{cases} 1 & \text{if } \mathfrak{f} \text{ is a field;} \\ 2 & \text{otherwise,} \end{cases} \]

and

\[ \mathbb{K}_+^\times := (\mathbb{R}_+^\times)^{d_\mathfrak{f}} \quad (\mathbb{R}_+^\times \text{ is the multiplicative group of positive real numbers}). \]

For all $a \in \mathbb{K}_+^\times$ and $s \in \mathbb{C}^{d_\mathfrak{f}}$, put

\[ a^s := a_1^{s_1} a_2^{s_2} \in \mathbb{C}^\times, \quad \text{if } d_\mathfrak{f} = 2, \quad a = (a_1, a_2), \quad s = (s_1, s_2), \]

if $d_\mathfrak{f} = 1$, $a^s \in \mathbb{C}^\times$ retains the usual meaning.

For every $s \in \mathbb{C}^{d_\mathfrak{f}}$, denote by $\tau_s$ the representation of $GL'(X)$ which has the same underlying space as that of $\tau$, and has the action

\[ \tau_s(g) = |g|^s \tau(g), \quad g \in GL'(X), \]

where $|g|^s$ is the image of $g$ under the composition map

\[ GL'(X) \to GL(X) \xrightarrow{\text{determinant}} \mathbb{K}^\times \xrightarrow{|\cdot|} \mathbb{K}_+^\times \xrightarrow{(\cdot)^s} \mathbb{C}^\times, \]
and

$$| \cdot | : \mathbb{K}^\times \to \mathbb{K}_+^\times$$

is the map of taking componentwise absolute values. Then \(\sigma \otimes \tau_s\) is an irreducible Casselman–Wallach representation of \(H(E_0) \rtimes M'_X\). By inflation through \(p_X\), we view it as an irreducible Casselman–Wallach representation of \(H(X^+) \rtimes P'_X\). For simplicity in notation, put

$$I_s := \text{Ind}_{H(X^+) \rtimes P'_X}^{\mathbb{K}^\times} \sigma \otimes \tau_s.$$

**Proposition 4.3.** Except for a measure zero set of \(s \in \mathbb{C}^{d_k}\), the unnormalized Schwartz induction \(I_s\) is a nondegenerate irreducible Casselman–Wallach representation of \(\mathcal{J}'(E)\).

**Proof.** Assume that \(\mathfrak{k} \subset \mathcal{J}'(E_0)\) acts through the nontrivial unitary character \(\psi_\mathfrak{k}\) in \(\sigma\). Then \(\mathfrak{k} \subset \mathcal{J}'(E)\) also acts through \(\psi_\mathfrak{k}\) in \(I_s\). As in Section 3.3, let \(\omega\) be a smooth oscillator representation of \(\mathcal{J}'(E)\) associated to \(\psi_\mathfrak{k}\).

Denote by \(\omega_X\) the topological \(X\)-coinvariant space of \(\omega\), namely, it is the maximal Hausdorff quotient of \(\omega\) on which \(X \subset H(E)\) acts trivially. This is a representation of \(H(X^+) \rtimes P'_X\). By using the mixed Schrodinger model (cf. [9], [12, Section 5]), we know that \(X \rtimes N_X\) acts trivially on \(\omega_X\), and it descends to a smooth oscillator representation of \(H(E_0) \rtimes M'_X\). By Frobenius reciprocity and using Schrodinger models, we know that the quotient map \(\omega \to \omega_X\) induces an isomorphism of \(H(E) \rtimes P'_X\)-representations:

$$\omega |_{H(E) \rtimes P'_X} \cong \text{Ind}^{H(E) \rtimes P'_X}_{H(X^+) \rtimes P'_X} \omega_X. \quad (14)$$

Let \(s \in \mathbb{C}^{d_k}\). Put

$$\varrho_s := \text{Hom}_{H(E_0)}(\omega_X, \sigma \otimes \tau_s),$$

equipped with the compact open topology. It is an irreducible Casselman–Wallach representation of \(M'_X\), and we have (cf. [20])

$$\sigma \otimes \tau_s \cong \omega_X \otimes \varrho_s \quad (15)$$
as representations of \(H(E_0) \rtimes M'_X\).

Then as \(\mathcal{J}'(E)\)-representation,

$$I_s = \text{Ind}_{H(X^+) \rtimes P'_X}^{\mathbb{K}^\times} \sigma \otimes \tau_s$$

by (6)

$$\cong \text{Ind}_{H(E) \rtimes P'_X}^{\mathbb{K}^\times} \left( \text{Ind}^{H(E) \rtimes P'_X}_{H(X^+) \rtimes P'_X} \sigma \otimes \tau_s \right)$$
by (15)

$$\cong \text{Ind}_{H(E) \rtimes P'_X}^{\mathbb{K}^\times} \left( \text{Ind}^{H(E) \rtimes P'_X}_{H(X^+) \rtimes P'_X} \omega_X \otimes \varrho_s \right)$$
by Lemma 3.2

$$\cong \text{Ind}_{H(E) \rtimes P'_X}^{\mathbb{K}^\times} (\omega |_{H(E) \rtimes P'_X} \otimes \varrho_s)$$
by (14)

$$\cong \omega \otimes \text{Ind}_{P'_X}^{\mathbb{K}^\times} \varrho_s$$
by Lemma 3.2

By using Langlands classification and the result of Speh–Vogan [18, Theorem 1.1], we know that except for a measure zero set of \(s \in \mathbb{C}^{d_k}\), \(\text{Ind}_{P'_X}^{\mathbb{K}^\times} \varrho_s\) is an irreducible
Casselman–Wallach representation of $U'(E)$. This finishes the proof by the argument in the last paragraph of Section 3.3.

Fix a nonzero element $\lambda$ of the one-dimensional space $\text{Hom}_{\mathcal{N}_\mathcal{F}(\mathcal{X})}(\tau, \psi^{-1})$. It induces a continuous linear map

$$\Lambda: \sigma \otimes \tau \to \sigma, \quad u \otimes v \mapsto \lambda(v) u.$$ 

Let $\pi$ be an irreducible Casselman–Wallach representation of $U'(E)$ as in Theorem 4.1, and let

$$(\cdot, \cdot)_\mu: \pi \times \sigma \to \mathbb{C}$$

be a continuous bilinear map which represents an element $\mu \in \text{Hom}_{\mathcal{N}_\mathcal{F}(\mathcal{E})}(\pi \otimes \sigma, \psi)$. As before, let $s \in \mathbb{C}^{d_k}$. For every $f \in I_s$ and $u \in \pi$, consider the following function on $U'(E)$:

$$(16) \quad g \mapsto (g.u, (\Lambda \circ f)(y^{-1}_r g))_\mu.$$ 

Here, both $g \in U'(E)$ and $y_r \in E \subset H(E)$ are viewed as elements in $J'(E) = H(E) \times U'(E)$, and $f$ is viewed as a $\sigma \otimes \tau$-valued function ($\tau_s = \tau$ as vector spaces). It follows from Lemma 4.2 that the function (16) is left $J'_\tau(E)$-invariant.

Put

$$Z_\mu(f, u) := \int_{J'_\tau(E) \setminus U'(E)} (g.u, (\Lambda \circ f)(y^{-1}_r g))_\mu \, dg, \quad f \in I_s, \ u \in \pi,$$

where $dg$ is a fixed right $U'(E)$-invariant positive Borel measure on $J'_\tau(E) \setminus U'(E)$.

We postpone the proof of the following result to Section 5.

**Proposition 4.4.** Assume that $\mu \neq 0$. Then for every $s \in \mathbb{C}^{d_k}$, there are elements $f \in I_s$ and $u \in \pi$ such that the integral $Z_\mu(f, u)$ is absolutely convergent and nonzero.

For every $s \in \mathbb{C}^{d_k}$, denote by $\text{Re} s \in \mathbb{R}^{d_k}$ its componentwise real part. We write $\text{Re} s > c$ for a real number $c$ if every component of $\text{Re} s$ is $> c$. In Section 6, we prove the following

**Proposition 4.5.** There is a real constant $c_\mu$, depending on $\pi, \sigma, \tau$ and $\mu$, such that for every $s \in \mathbb{C}^{d_k}$ with $\text{Re} s > c_\mu$, the integral $Z_\mu(f, u)$ is absolutely convergent for every $f \in I_s$ and $u \in \pi$, and defines a $U'(E)$-invariant continuous linear functional on $I_s \otimes \pi$.

Now we are ready to prove Theorem 4.1, as in the discussion of [11, Section 3.4]. Let $F$ be a finite dimensional subspace of $\text{Hom}_{J'_\sigma(E)}(\pi \otimes \sigma, \psi)$. By Proposition 4.5, we have a linear map

$$F \to \text{Hom}_{U'(E)}(I_s \otimes \pi, \mathbb{C}), \quad \mu \mapsto Z_\mu$$

for $\text{Re} s > c_F$, where $c_F$ is a real constant depending on $\pi, \sigma, \tau$ and $F$. Moreover, by Proposition 4.4, the above map is an injection. In view of Proposition 4.3, choose $s$ such that $\text{Re} s > c_F$ and $I_s$ is irreducible. Then $\dim_\mathbb{C} \text{Hom}_{U'(E)}(I_s \otimes \pi, \mathbb{C}) \leq 1$ by Theorem B. Therefore $\dim_\mathbb{C} F \leq 1$ and Theorem 4.1 is proved.
5. Proof of Proposition 4.4

We continue with the notation of the last section. Denote by $P'_Y$ the parabolic subgroup of $U'(E)$ stabilizing $Y$. It has a Levi decomposition

$$P'_Y = \text{GL}'(Y) \ltimes N_Y,$$

where $N_Y$ is the unipotent radical, and $\text{GL}'(Y) = \text{GL}'(X)$ is the subgroup of $U'(E)$ stabilizing $X$ and $Y$ and fixing $E_0$ pointwise. Denote by $P'_{y_r}(Y)$ the subgroup of $\text{GL}'(Y)$ fixing $y_r$, and by $P'_{Y_{r-1}}(Y)$ the subgroup of $\text{GL}'(Y)$ fixing $Y_{r-1}$ pointwise. Then the multiplication map

$$P'_{y_r}(Y) \times P'_{Y_{r-1}}(Y) \to \frac{\Delta \mu_{\mathbb{K}}}{\text{GL}'(Y)}$$

is an open embedding, and its image has full measure in $\text{GL}'(Y)$. It is also routine to check that the map

$$(17) \quad \frac{(H(X^\perp) \ltimes P'_X) \times (P'_{Y_{r-1}}(Y) \ltimes N_Y)}{\Delta \mu_{\mathbb{K}}} \to J'(E), \quad (g, h) \mapsto gh_r^{-1}h$$

is an open embedding.

Take two vectors $u_\sigma \in \sigma$ and $u_s \in \tau_s$. Take a compactly supported smooth function $\phi$ on $P'_{Y_{r-1}}(Y) \ltimes N_Y$ so that

$$\phi(zh) = \chi(\tau)\phi(h), \quad z \in \mu_{\mathbb{K}}, h \in P'_{Y_{r-1}}(Y) \ltimes N_Y.$$}

Recall that $\sigma \hat{\otimes} \tau_s$ is a representation of $H(E_0) \ltimes M'_X$, and is viewed as a representation of $H(X^\perp) \ltimes P'_X$ by inflation. Put

$$\phi'(g, h) := \phi(h)(g.(u_\sigma \otimes u_s)), \quad g \in H(X^\perp) \ltimes P'_X, h \in P'_{Y_{r-1}}(Y) \ltimes N_Y.$$}

Extension by zero of $\phi'$ through $17$ yields a $\sigma \hat{\otimes} \tau_s$-valued smooth function $f$ on $J'(E)$. By Lemma 3.1, $f \in I_s$.

It is elementary to see that there is a positive smooth function $\gamma_r$ on $(N_{\mathcal{F}}(X) \times \mu_{\mathbb{K}}) \setminus P'_{y_r}(Y)$ so that

$$\int_{J'_E(U'(E))} \varphi(g) \, dg = \int_{((N_{\mathcal{F}}(X) \times \mu_{\mathbb{K}}) \setminus P'_{y_r}(Y)) \times (P'_{Y_{r-1}}(Y) \ltimes N_Y)} \gamma_r(h) \varphi(hk) \, dh \, dk,$$

for all nonnegative continuous functions $\varphi$ on $J'_E(U'(E))$, where $dh$ is a right $P'_{y_r}(Y)$-invariant positive Borel measure on $(N_{\mathcal{F}}(X) \times \mu_{\mathbb{K}}) \setminus P'_{y_r}(Y)$, and $dk$ is a right invariant Haar measure on $P'_{Y_{r-1}}(Y) \ltimes N_Y$. 

For every $u \in \pi$, we have that

$$Z_\mu(f, u) = \int_{J'_r(E) \setminus U'_{Yr}(E)} \{g, u, (\Lambda \circ f)(y_r^{-1}g)\}_\mu \, dg$$

$$= \int_{((N_3(X) \times \mu_K) \setminus P'_{yr}(Y)) \times (P'_{Yr-1}(Y) \times N_Y)} \gamma_r(h) \langle (h k). u, (\Lambda \circ f)(y_r^{-1}h k) \rangle \, dh \, dk$$

$$= \int_{((N_3(X) \times \mu_K) \setminus P'_{yr}(Y)) \times (P'_{Yr-1}(Y) \times N_Y)} \gamma_r(h) \langle (h k). u, \Lambda(h.(f(y_r^{-1}k))) \rangle \, dh \, dk$$

$$= \int_{((N_3(X) \times \mu_K) \setminus P'_{yr}(Y)) \times (P'_{Yr-1}(Y) \times N_Y)} \lambda(\tau_s(h)u_s) \phi(h) \, dh \, dk,$$

where

$$\Phi(h, k) := \gamma_r(h) \phi(k) \langle (h k). u, u_\sigma \rangle_\mu, \quad h \in P'_{Yr-1}(Y), \ k \in P'_{Yr-1}(Y) \times N_Y.$$

Choose $\phi$, $u$ and $u_\sigma$ appropriately so that the function

$$\Psi(h) := \int_{P'_{Yr-1}(Y) \times N_Y} \Phi(h, k) \, dk$$

does not vanish at 1.

Note that the smooth function $\Psi$ on $P'_{yr}(Y)$ satisfies

$$\Psi(bzh) = \psi(b) \chi_{\tau}(z) \Psi(h), \quad b \in N_3(X), \ z \in \mu_K, \ h \in P'_{yr}(Y).$$

Recall from [10, Section 3] that for every smooth function $W$ on $P'_{yr}(Y)$ such that

$$W(bzh) = \psi(b)^{-1} \chi_{\tau}(z) W(h), \quad b \in N_3(X), \ z \in \mu_K, \ h \in P'_{yr}(Y),$$

if $W$ has compact support modulo $N_3(X) \times \mu_K$, then there is a vector $u'_s \in \tau_s$ such that

$$W(h) = \lambda(\tau_s(h)u'_s), \quad h \in P'_{yr}(Y).$$

Therefore we may choose $u_s$ appropriately so that the function $h \mapsto \lambda(\tau_s(h)u_s)$ on $P'_{yr}(Y)$ has compact support modulo $N_3(X) \times \mu_K$, and that

$$\int_{(N_3(X) \times \mu_K) \setminus P'_{yr}(Y)} \lambda(\tau_s(h)u_s) \Psi(h) \, dh \neq 0. \tag{18}$$

Note that the integral $Z_\mu(f, u)$ equals to the left-hand side of (18), and its integrant is smooth and compactly supported. This finishes the proof of Proposition 4.4.
6. Proof of Proposition 4.5

Extend $x_r \in X$ to a $\mathbb{K}$-basis $\{x_1, x_2, \ldots, x_r\}$ of $X$ so that $x_i \in X_i$ for $i = 1, \ldots, r$.

Under this basis, $(\mathbb{K}_+^\times)^r$ embeds in $GL(X)$:

$$(\mathbb{K}_+^\times)^r \subset (\mathbb{K}^\times)^r = \prod_{i=1}^{r} GL(\mathbb{K} x_i) \hookrightarrow GL(X).$$

Since the covering $GL'(X) \to GL(X)$ uniquely splits over $(\mathbb{K}_+^\times)^r$, it also embeds in $GL'(X)$. For every $t = (t_1, t_2, \ldots, t_r) \in (\mathbb{K}_+^\times)^r$, denote by $a_t$ the corresponding element in $GL'(X)$, and put

$$||t|| := \prod_{i=1}^{r} \varpi(t_i + t_i^{-1}) \quad \text{and} \quad \xi(t) := \prod_{i=1}^{r-1} \varpi(1 + \frac{t_i}{t_{i+1}}),$$

where for every $t \in \mathbb{K}_+^\times$,

$$\varpi(t) := \begin{cases} t & \text{if } d_K = 1; \\ t^t & \text{if } d_K = 2 \text{ and } t = (t', t''). \end{cases}$$

Fix a maximal compact subgroup $K$ of $U'(E)$. It is elementary to see that there is a positive character $\delta_r$ on $(\mathbb{K}_+^\times)^r$ such that

$$\int_{J'_r(E) \backslash U'(E)} \varphi(g) \, dg = \int_{(\mathbb{K}_+^\times)^r \times K} \delta_r(t) \varphi(a_t k) \, d^x t \, dk$$

for all nonnegative continuous functions $\varphi$ on $J'_r(E) \backslash U'(E)$, where $dk$ is the normalized Haar measure on $K$, and $d^x t$ is an appropriate Haar measure on $(\mathbb{K}_+^\times)^r$. Pick a positive constant $c_0$ so that

$$\delta_r(t) \leq ||t||^{c_0}, \quad t \in (\mathbb{K}_+^\times)^r.$$  \hspace{1cm} (20)

Recall that

$$\langle \cdot, \cdot \rangle_{\mu} : \pi \times \sigma \to \mathbb{C}$$

is a continuous bilinear map which represents an element $\mu \in \text{Hom}_{J'_r(E)}(\pi \otimes \sigma, \psi)$. Pick a continuous seminorm $| \cdot |_{\pi,1}$ on $\pi$ and a continuous seminorm $| \cdot |_{\sigma}$ on $\sigma$ so that

$$|\langle u, v \rangle_{\mu}| \leq |u|_{\pi,1} |v|_{\sigma}, \quad u \in \pi, \ v \in \sigma.$$  \hspace{1cm} (21)

By the moderate growth condition on $\pi$, there is a constant $c_1 > 0$ and a continuous seminorm $| \cdot |_{\pi,2}$ on $\pi$ such that

$$|(a_t k)_{|\pi,1} \leq ||t||^{c_1} |u|_{\pi,2}, \quad t \in (\mathbb{K}_+^\times)^r, \ k \in K, \ u \in \pi.$$  \hspace{1cm} (22)

Recall that $\lambda \in \text{Hom}_{N_{J'_r}(X)}(\tau, \psi^{-1})$ induces a continuous linear map

$$\Lambda : \sigma \hat{\otimes} \tau \to \sigma, \quad u \otimes v \mapsto \lambda(v) u.$$  \hspace{1cm} (\Lambda)

Still denote by $\tau$ the following continuous linear action of $GL'(X)$ on $\sigma \hat{\otimes} \tau$:

$$\tau(h)(u \otimes v) := u \otimes (\tau(h)v), \quad u \in \sigma, \ v \in \tau.$$
The moderate growth condition on \( \tau \) implies that there is a constant \( c_2 > 0 \) and a continuous seminorm \( | \cdot |_{\sigma \otimes \tau} \) on \( \sigma \otimes \tau \) such that
\[
|\Lambda(\tau(a_t)w)|_\sigma \leq ||t||^{c_2} |w|_{\sigma \otimes \tau}, \quad t \in (\mathbb{K}_+^\times)^r, \ w \in \sigma \otimes \tau.
\]

**Lemma 6.1.** For every positive integer \( N \), there is a continuous seminorm \( | \cdot |_{\sigma \otimes \tau,N} \) on \( \sigma \otimes \tau \) such that
\[
(23) \quad |\Lambda(\tau(a_t)w)|_\sigma \leq \xi(t)^{-N} ||t||^{c_2} |w|_{\sigma \otimes \tau,N}, \quad t \in (\mathbb{K}_+^\times)^r, \ w \in \sigma \otimes \tau.
\]

**Proof.** This is similar to the proof of [11, Lemma 6.2]. We omit the details. \( \square \)

Put \( c_\mu := c_0 + c_1 + c_2 \). Recall that \( s \in \mathbb{C}^{d_K} \).

**Lemma 6.2.** If \( \text{Re } s > c_\mu \), then there is a positive integer \( N \) such that
\[
(24) \quad \int_{(\mathbb{K}_+^\times)^r} ||t||^{c_\mu} \Pi(t)^{\text{Re } s} \xi(t)^{-N} \varpi(1 + t_r)^{-N} d^\times t < \infty,
\]
where
\[
(25) \quad t = (t_1, t_2, \ldots, t_r) \quad \text{and} \quad \Pi(t) := \prod_{i=1}^r t_i \in \mathbb{K}_+^\times.
\]

**Proof.** We assume that \( d_K = 1 \). The other case obviously follows from this one. Write
\[
\alpha_i := \frac{t_i}{t_{i+1}}, \quad i = 1, \ldots, r - 1; \quad \alpha_r := t_r.
\]
Then
\[
\left\{
\begin{aligned}
||t|| &\leq \prod_{i=1}^r (\alpha_i + \alpha_i^{-1})^i, \\
\Pi(t) &\leq \prod_{i=1}^r \alpha_i^i, \\
\xi(t)\varpi(1 + t_r) &\leq \prod_{i=1}^r (1 + \alpha_i).
\end{aligned}
\right.
\]

Therefore the left-hand side of (24) is at most
\[
\prod_{i=1}^r \int_{\mathbb{R}_+^\times} (\alpha_i + \alpha_i^{-1})^{ic_\mu} \alpha_i^{i\text{Re } s} (1 + \alpha_i)^{-N} d^\times \alpha_i,
\]
where \( d^\times \alpha_i \) is an appropriate Haar measure on \( \mathbb{R}_+^\times, i = 1, 2, \ldots, r \). It is elementary to see that if \( N > r(c_\mu + \text{Re } s) \), then
\[
\int_{\mathbb{R}_+^\times} (\alpha_i + \alpha_i^{-1})^{ic_\mu} \alpha_i^{i\text{Re } s} (1 + \alpha_i)^{-N} d^\times \alpha_i < \infty, \quad i = 1, 2, \ldots, r.
\]
This finishes the proof. \( \square \)

Now assume that \( \text{Re } s > c_\mu \). Let \( f \in I_s \), to be viewed as a \( \sigma \otimes \tau \)-valued function on \( \mathbb{J}'(E) \), and let \( u \in \pi \). We want to show that the integral \( Z_\mu(f, u) \) is absolutely convergent and defines a \( U'(E) \)-invariant continuous linear functional of \( I_s \otimes \pi \). The \( U'(E) \)-invariance is obvious as soon as the absolutely convergence is proved.
We have
\[ |Z_\mu(f, u)| \]
\[ \leq \int_{J_\nu(E) \setminus U'(E)} |(g.u, (\Lambda \circ f)(y_r^{-1}g))_\mu| \, dg \]
\[ = \int_{(\mathbb{K}_+^*)^r \times K} \delta_r(t) |((a_t k).u, (\Lambda \circ f)(y_r^{-1}a_t k))_\mu| \, d^x t \, dk \]
\[ \leq \int_{(\mathbb{K}_+^*)^r \times K} |t|^\sigma |(a_t k).u|_{\pi,1} |(\Lambda \circ f)(y_r^{-1}a_t k)|_\sigma \, d^x t \, dk \]
\[ \leq \int_{(\mathbb{K}_+^*)^r \times K} |t|^{\sigma + c_1} |u|_{\pi,2} |(\Lambda \circ f)(y_r^{-1}a_t k)|_\sigma \, d^x t \, dk \]
by \((19)\) and \((20)\) and \((21)\).

Let \(N\) be a positive integer as in Lemma 6.2, and let \(\cdot |_{\sigma \hat{\otimes} \tau, N}\) be a continuous seminorm on \(\sigma \hat{\otimes} \tau\) as in Lemma 6.1. Then for every \(t = (t_1, t_2, \ldots, t_r) \in (\mathbb{K}_+^*)^r\) and \(k \in K\), we have
\[ |(\Lambda \circ f)(y_r^{-1}a_t k)|_\sigma \]
\[ = |\Pi(t)^\ast| |(\Lambda(\tau(a_t) f)(a_t^{-1}y_r^{-1}a_t k))|_\sigma \]
\[ \leq \Pi(t)^{\text{Res}} \xi(t)^{-N} |t|^{\sigma_2} |f(a_t^{-1}y_r^{-1}a_t k)|_{\sigma \hat{\otimes} \tau, N} \]
\[ \leq \Pi(t)^{\text{Res}} \xi(t)^{-N} |t|^{\sigma_2} |f((-t_r y_r) k)|_{\sigma \hat{\otimes} \tau, N} \]
\[ \leq \Pi(t)^{\text{Res}} \xi(t)^{-N} |t|^{\sigma_2} \varpi(1 + t_r)^{-N} |f|_{I_s, N}. \]

Here
\[ |f|_{I_s, N} := \sup \{ \varpi(1 + t)^N |f((-t y_r) k)|_{\sigma \hat{\otimes} \tau, N} \mid t \in \mathbb{K}_+^*, k \in K \}. \]

It is easy to see that \(\cdot |_{I_s, N}\) is a continuous seminorm on \(I_s\).

Therefore
\[ |Z_\mu(f, u)| \leq |f|_{I_s, N} |u|_{\pi,2} \int_{(\mathbb{K}_+^*)^r} |t|^{\sigma_2} \Pi(t)^{\text{Res}} \xi(t)^{-N} \varpi(1 + t_r)^{-N} \, d^x t, \]
and Proposition 4.5 follows by \((24)\).

REFERENCES

[1] A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann, *Multiplicity one theorems*, Ann. of Math. (2) 172 (2010), no. 2, 1407–1434, DOI 10.4007/annals.2010.172.1413. MR2680495 (2011g:22024)

[2] E. M. Baruch and S. Rallis, *On the uniqueness of Fourier Jacobi models for representations of U(n, 1)*, Represent. Theory 11 (2007), 1–15 (electronic), DOI 10.1090/S1088-4165-07-00298-1. MR2276364 (2007m:22006)

[3] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. 41 (1989), no. 3, 385–438, DOI 10.4153/CJM-1989-019-5. MR1013462 (90j:22013)

[4] W. Casselman, H. Hecht, and D. Miličić, *Br Hughes filtrations and Whittaker vectors for real groups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), Proc. Sympos. Pure Math., vol. 68, Amer. Math. Soc., Providence, RI, 2000, pp. 151–190. MR1767896 (2002b:22023)

[5] F. Du Cloux, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. (4) 24 (1991), no. 3, 257–318 (French). MR1100992 (92j:22026)
[6] W. T. Gan, B. H. Gross, and D. Prasad, Symplectic local root numbers, central critical $L$-values, and restriction problems in the representation theory of classical groups, Astérisque 346 (2012), 111–170.

[7] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis, $L$ functions for the orthogonal group, Mem. Amer. Math. Soc. 128 (1997), no. 611, viii+218. MR1357823 (98m:11041)

[8] D. Ginzburg, D. Jiang, S. Rallis, and D. Soudry, $L$-functions for symplectic groups using Fourier-Jacobi models, Arithmetic geometry and automorphic forms, Adv. Lect. Math. (ALM), vol. 19, Int. Press, Somerville, MA, 2011, pp. 183–207. MR2906909

[9] R. Howe, $L^2$ duality for stable reductive dual pairs, preprint, available at http://a4mmcm.googlecode.com/svn/trunk/Lie/L2%20duality.tex.

[10] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic representations, I, Amer. J. Math. 103 (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323 (82m:10050a)

[11] D. Jiang, B. Sun, and C.-B. Zhu, Uniqueness of Bessel models: the Archimedean case, Geom. Funct. Anal. 20 (2010), no. 3, 690–709, DOI 10.1007/s00039-010-0077-4. MR2720228 (2012a:22019)

[12] S. S. Kudla, On the local theta-correspondence, Invent. Math. 83 (1986), no. 2, 229–255, DOI 10.1007/BF01388961. MR818351 (87e:22037)

[13] _____, Splitting metaplectic covers of dual reductive pairs, Israel J. Math. 87 (1994), no. 1-3, 361–401, DOI 10.1007/BF02773003. MR1286835 (95h:22019)

[14] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, Correspondances de Howe sur un corps $p$-adique, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987 (French). MR1041060 (91f:11040)

[15] J. A. Shalika, The multiplicity one theorem for $GL_n$, Ann. of Math. (2) 100 (1974), 171–193. MR0348047 (50 #545)

[16] M. Shiota, Nash manifolds, Lecture Notes in Mathematics, vol. 1269, Springer-Verlag, Berlin, 1987. MR904479 (89b:58011)

[17] _____, Nash functions and manifolds, Lectures in real geometry (Madrid, 1994), de Gruyter Exp. Math., vol. 23, de Gruyter, Berlin, 1996, pp. 69–112. MR1440210 (98k:14081)

[18] B. Speh and D. A. Vogan Jr., Reducibility of generalized principal series representations, Acta Math. 145 (1980), no. 3-4, 227–299, DOI 10.1007/BF02441491. MR590291 (82c:22018)

[19] B. Sun, Multiplicity one theorems for Fourier–Jacobi models, Amer. J. of Math., to appear. arXiv:0903.1417.

[20] _____, On representations of real Jacobi groups, Sci. China Math. 55 (2012), no. 3, 541–555, DOI 10.1007/s11425-011-4333-3.

[21] B. Sun and C.-B. Zhu, Multiplicity one theorems: the Archimedean case, Ann. of Math. (2) 175 (2012), no. 1, 23–44, DOI 10.4007/annals.2012.175.1.2.

[22] F. Trèves, Topological vector spaces, distributions and kernels, Academic Press, New York, 1967. MR0251531 (37 #726)

[23] J.-L. Waldspurger, Une variante d’un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann. arXiv:0911.1618v1.

[24] N. R. Wallach, Real reductive groups. II, Pure and Applied Mathematics, vol. 132, Academic Press Inc., Boston, MA, 1992. MR1170566 (93m:22018)