Global well-posedness and symmetries for dissipative active scalar equations with higher-order couplings

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Abstract  
We are concerned with a family of dissipative active scalar equations recently introduced in [D. Chae, P. Constantin, J. Wu, to appear in Indiana Univ. Math. J. (2013)]. Velocity fields are coupled with the active scalar via a large class of multiplier operators which morally behave as positive derivatives of $(\beta - 1)$-order when $\beta > 1$. We prove global well-posedness and time-decay of solutions, without smallness assumptions, for initial data belonging to the critical Lebesgue space $L^{\frac{n}{2\gamma}}(\mathbb{R}^n)$ which is a class larger than that of the above reference. Symmetry properties of solutions are investigated depending on the symmetry of initial data and coupling operators.

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1 Introduction  
We are concerned with the initial value problem (IVP) for a family of dissipative active scalar equation, which reads as

\[
\begin{cases}
\frac{\partial \theta}{\partial t} + \kappa (-\Delta)^\gamma \theta + u \cdot \nabla_x \theta = 0, & x \in \mathbb{R}^n, \ t > 0, \\
\theta(x, 0) = \theta_0(x), & x \in \mathbb{R}^n,
\end{cases}
\]  

where $n \geq 2$, $\kappa \geq 0$ and $\gamma > 0$. The fractional laplacian operator $(-\Delta)^\gamma$ is defined by

\[ [(-\Delta)^\gamma f](\xi) = |\xi|^{2\gamma} \hat{f}(\xi), \]

where $\hat{f} = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(\xi) d\xi$ stands for the Fourier transform of $f$. The velocity field $u$ is determined by the active scalar $\theta$ by means of the multiplier operators.
Dissipative active scalar equations

\[ u = P[\theta] = (\tilde{P}_1[\theta], ..., \tilde{P}_n[\theta]), \] 

(1.2)
such that \( \nabla \cdot u = 0 \), and

\[ u_j = \tilde{P}_j[\theta] = \sum_{i=1}^{n} a_{ij} \mathcal{R}_i \Lambda^{-1} P_i[\theta], \text{ for } 1 \leq j \leq n, \] 

(1.3)

where \( \Lambda = (-\Delta)^{\frac{1}{2}} \), \( \mathcal{R}_i = -\partial_i(-\Delta)^{-\frac{1}{2}} \) is the \( i \)-th Riesz transform, \( a_{ij} \)'s are constant and

\[ \tilde{P}_i[\theta](\xi) = \hat{P}_i(\xi) \hat{\theta}(\xi). \] 

(1.4)

Denoting \( I = \sqrt{-1} \), it follows that

\[ \hat{\tilde{P}}_j[\theta](\xi) = \hat{P}_j(\xi) \hat{\theta}(\xi) \text{ with } \hat{P}_j(\xi) = \sum_{i=1}^{n} a_{ij} \frac{\xi_i I}{|\xi|^2} P_i(\xi), \] 

and the vector field \( u \) can be expressed in Fourier variables in the shorter form

\[ \hat{u} = \hat{P}[\theta] = P(\xi) \hat{\theta}(\xi) \text{ where } P(\xi) = (\tilde{P}_1(\xi), ..., \tilde{P}_n(\xi)). \] 

(1.5)

Throughout this manuscript the symbol \( P_i(\xi) \) in (1.4) is assumed to belong to \( C^{[\frac{n}{2}] + 1}(\mathbb{R}^n \setminus \{0\}) \) with

\[ \left| \frac{\partial^\alpha P_i}{\partial \xi^\alpha}(\xi) \right| \leq C|\xi|^\beta - |\alpha|, \] 

(1.6)

for all \( \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq \frac{n}{2} + 1 \) and \( \xi \neq 0 \), where \( \beta \geq 0 \). The brackets \([\cdot]\) stands for the greatest integer function.

We could consider an arbitrary \( \kappa > 0 \), nevertheless \( \kappa = 1 \) is assumed for the sake of simplicity. The IVP (1.1)-(1.3) can be converted into the integral equation

\[ \theta(t) = G_\gamma(t)\theta_0 + B(\theta, \theta)(t), \] 

(1.7)

where

\[ B(\theta, \varphi)(t) = - \int_{0}^{t} G_\gamma(t-s)(\nabla_x \cdot (P[\theta] \varphi))(s)ds \] 

(1.8)

and \( G_\gamma(t) \) is the convolution operator with kernel given in Fourier variables by \( \hat{g}_\gamma(\xi, t) = e^{-t|\xi|^{2\gamma}} \).

Solutions of (1.7) are called mild ones for (1.1)-(1.3). Assuming that \( P_i \)'s are homogeneous functions of degree \( \beta \), we have formally that

\[ \theta_\lambda = \lambda^{2\gamma - \beta} \theta(\lambda x, \lambda^{2\gamma} t) \] 

verifies (1.1)-(1.3), for all \( \lambda > 0 \), provided that \( \theta \) does so. It follows that

\[ \theta \to \theta_\lambda = \lambda^{2\gamma - \beta} \theta(\lambda x, \lambda^{2\gamma} t), \text{ for } \lambda > 0, \] 

(1.9)

is the scaling map for (1.1)-(1.3). Also, making \( t \to 0^+ \) in (1.9), one obtains the scaling for the initial data.
In view of (1.6), even when \( P_t \) is not homogeneous, we can consider (1.9) as a kind of intrinsic scaling for (1.1)-(1.3) in the sense that it is useful to identify threshold indexes for functional settings and properties of solutions. One of our aims is to provide a global existence result for (1.1)-(1.3) in a scaling invariant framework.

Active scalar equations like (1.1)-(1.3) arise in a large number of physical models in fluid mechanics and atmospheric science. Examples of those are 2D surface quasi-geostrophic equation (SQG) \( u = \nabla^\perp((-\Delta)^{-1/2} \theta) \) (\( \beta = 1 \)), Burgers equation \( u = \theta \) (\( \beta = 1 \)), 2D vorticity equation \( u = \nabla^\perp(-\Delta)^{-1} \theta \) (\( \beta = 0 \)). SQG is a famous model with a lot of papers concerning existence, uniqueness, regularity and asymptotic behavior of solutions in the inviscid case \( \kappa = 0 \) or in the sub-critical \((1/2 < \gamma < 1)\), critical \((\gamma = 1/2)\) and supercritical \((\gamma \in (0,1/2))\) ranges. Without making a complete list, we would like to mention [1], [2], [10], [11], [12], [13], [15], [19], [21], [24], [25], [26], [32], [34], and their references. In the case \( u = \theta \), see [2] and [17] for results on blow-up, global existence and regularity of solutions. One dimensional active scalar models have also attracted the attention of many authors, see e.g. [4], [14], [16], [18] where the reader can find global existence, finite-time singularity and asymptotic behavior results with velocity coupled via singular integral operators that are zero-order multiplier ones.

In the case of SQG, notice that \( u \) can be written by using Riesz transform as

\[
u = (-R_2 \theta, R_1 \theta)
\]

and then the velocity is coupled to the active scalar via zero-order multiplier operators. The model (1.1)-(1.3) was introduced in [5], [6] and takes into account the effect of higher-order couplings, i.e. \( \beta > 1 \). In this range, the operator \( P[\cdot] \) behaves “morally” like a positive derivative of \((\beta - 1)\)-order and produces more difficulties in comparison with SQG \((\beta = 1)\) and \( \beta < 1 \). Indeed there is an interesting interplay between the field \( u \) and fractional viscosity \((-\Delta)^{\gamma}\) in (1.1)-(1.3), expressed by means of three basic cases: sub-critical \( \beta < 2\gamma \), critical \( \beta = 2\gamma \), and super-critical \( \beta > 2\gamma \).

The paper [5] deals mainly with the inviscid case \( \kappa = 0 \), while [6] with the dissipative one \( \kappa > 0 \). This last work is our main motivation since we also focus in the dissipative model. The authors of [6] showed existence of global solutions in \( L^\infty((0,\infty);Y) \) for (1.1)-(1.3) where \( Y = L^1 \cap L^\infty \cap B^{s,M}_{q,\infty} \) with \( s > 1 \) and \( 2 \leq q \leq \infty \). The index \( M = \{M_j\}_{j \geq 1} \) is a sequence and the space \( B^{s,M}_{q,\infty} \) is an extension of the classical Besov space \( B^s_{q,\infty} \) where the \( B^{s,M}_{q,\infty} \)-norm increases according to the growth of \( M \). The results of [6] consider couplings \( P[\cdot] \) in (1.2) such that \( P_1 \in C^\infty(\mathbb{R}^n \setminus \{0\}) \), \( P_1 \) is radially symmetric, \( P_1 = P_1(|\xi|) \) is nondecreasing with \(|\xi|\), and a technical growth hypothesis involving \( P_t(\xi) \) and the sequence \( M \). Applying their results to the special case

\[
u = \nabla^\perp(\Lambda^{\beta-2}\theta) = \Lambda^{\beta-1}(-R_2 \theta, R_1 \theta)
\]

with \( n = 2 \), \( 0 \leq \beta < 2\gamma < 1 \) (within the sub-critical range), \( \kappa > 0 \) and \( M_j = j + 1 \), they obtained well-posedness of solutions with initial data in \( L^1 \cap L^\infty \cap B^{s,M}_{q,\infty} \). Roughly speaking, the technique employed in [6] for constructing solutions relies on a successive approximation scheme together a priori estimates involving Besov norms. The field (1.12) corresponds to the modified SQG that interpolates 2D vorticity equation and SQG by varying the parameter \( \beta \) from 0 to 1. This model has been studied for instance in [5], [9], [28], [29], [30], [31] where one can find existence and regularity results with data in Sobolev spaces \( H^m \) with \( m \geq 0 \). The conditions \( \kappa > 0 \), \( \beta \in [0,1] \) and \( \beta = 2\gamma \).
Dissipative active scalar equations

were assumed in [9, 26, 29, 30]: \( \kappa > 0 \) and \( 1 \leq \beta < 2 \gamma < 2 \) in [31]; and \( \kappa = 0 \) and \( \beta \in [1, 2] \) in [5]. In this last work, local well-posedness of \( H^m(\mathbb{R}^2) \)-solutions was proved for (1.1)-(1.2) with \( m \geq 4 \).

In this paper we prove the global-in-time well-posedness of (1.1)-(1.3) in the Lebesgue space \( L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \), without smallness conditions (see Theorem 3.1). This is the unique \( L^\prime \)-space whose norm is invariant by the scaling (1.10), that is, \( L^{\frac{n}{2\gamma-\beta}} \) is the critical one in the scale of Lebesgue spaces. In view of the inclusion \( L^1 \cap L^\infty \subset L^{\frac{n}{2\gamma-\beta}} \), our initial data class is larger than that of \([6]\), and we can consider data outside \( L^2 \)-framework. In comparison with \([6]\), some new symbols \( P_t(\xi) \) are considered here, for instance, non-radially symmetric ones. Even for a singular initial data \( \theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \), the global solution \( \theta \in BC((0, \infty); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)) \) is instantaneously \( C^\infty \)-smoothed out and verifies (1.1)-(1.3) classically, for all \( t > 0 \). Here we focus in the range \( \beta \geq 1 \) for the field \( u \) and consider the sub-critical case \( \beta < 2 \gamma \). More precisely, we assume

\[
1 \leq 2\beta - 1 < 2 \gamma < \min\{\frac{2}{\beta}(n+\beta+1), (n+1)\}. \quad (1.13)
\]

The range \( 0 \leq \beta < 2 \gamma \) with \( \beta < 1 \) also can be treated with a slight adaptation on the proofs (see Remark 3.2).

Also, we analyse the asymptotic behavior of solutions by showing some decay properties in \( L^q \)-norms (see Theorem 3.1). Precisely, for \( \frac{n}{2\gamma-\beta} \leq q \leq \infty \) and \( \theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \), we obtain the time-polynomial decay

\[
\|\theta(\cdot, t)\|_{L^q} \leq Ct^{-\left(\frac{2\gamma-\beta}{\gamma} - \frac{n}{2\gamma}\right)}, \text{ for all } t > 0. \quad (1.14)
\]

Assuming further \( \theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \), the corresponding solution \( \theta \) belongs to \( BC((0, \infty); L^1(\mathbb{R}^n)) \) and the estimate (1.14) is improved to

\[
\|\theta(\cdot, t)\|_{L^q} \leq Ct^{-\left(\frac{2\gamma-\beta}{\gamma} - \frac{n}{2\gamma}\right) - \left(\frac{n+\beta}{\beta} - 1\right)}, \text{ for all } t > 0,
\]

where \( 1 \leq q \leq \infty \). Notice that the decay in (1.15) is faster than those of (1.14) due to the condition \( 2\gamma < \frac{2}{\beta}(n+\beta+1) < n+\beta \).

Technically speaking, in order to show existence of solutions we have followed the spirit of [2] by using time-weighted Kato type norms, scaling arguments, \( L^q \)-maximum principles, and arguments of the type parabolic De Giorgi-Nash-Moser. However, due to the higher-order coupling of the field \( u \), the model (1.1)-(1.3) requires involved arguments and is more difficult-to-beating than SQG that was analysed in [2]. For instance, since \( P[\cdot] \) is not continuous in \( L^p \)-spaces, we need to employ an auxiliary Kato-type norm based on homogeneous Sobolev spaces \( \dot{H}^p_\kappa \) with \( p > \frac{n}{2\gamma-\beta} \) in order to control the effects of the nonlinear term in (1.1)-(1.3). So, different from SQG, Sobolev norms play a crucial role for the local existence and extension of solutions for (1.1)-(1.3) with data in Lebesgue spaces.

In view of the dissipative-transport structure of (1.1), it is natural to wonder about symmetry properties of solutions under symmetry conditions for the symbols \( P_t(\xi) \) and data \( \theta_0 \). In Theorem 3.3 we show that the global solution given in Theorem 3.1 is radially symmetric, for all \( t > 0 \), provided that \( \theta_0 \) and \( \text{div}_\xi(P_t(\xi)) \) present this same property. Moreover, results on odd and even symmetry of solutions are obtained under parity conditions for \( \theta_0 \) and \( P_t \)'s. In Remark 3.4, we also comment about conditions for solutions to be non-symmetric.

Let us also comment on \( \log \)-type couplings which are interesting ones covered by (1.1)-(1.3). Ohkitani [33] has presented numerical evidences that, even with \( \kappa = 0 \), (1.1) with \( n = 2 \) and

\[
u = \nabla^\perp (\log(I-\Delta))^{\chi} \theta, \quad \chi > 0,
\]

(1.16)
may be globally well-posed. The authors of [3] have proved local well-posedness of $H^1(\mathbb{R}^2)$-solutions for (1.1)-(1.16) with $\kappa > 0$. As pointed in [5], the field (1.16) is of order higher (at least logarithmically) than derivatives of order 1 and in particular than (1.11). Another examples are

$$P_1(\xi) = |\xi|^\alpha (\log(1 + |\xi|^2))^{\chi}, \quad \chi \geq 0,$$

(1.17)

$$P_1(\xi) = |\xi|^\alpha (\log(1 + \log(1 + |\xi|^2)))^{\chi}, \quad \chi \geq 0,$$

(1.18)

which are indeed of order higher than (1.16) when $\alpha > 1$. These couplings are also treated in [6] with $\alpha = \beta$ and $\chi \geq 0$. When $\alpha = 0$ and $n = 2$, (1.17) and (1.18) correspond to log and log-log Navier-Stokes which are intermediate models between 2D vorticity equation and SQG. See [7] for further details and global existence results in the case $\kappa = 0$, $0 \leq \chi \leq 1$ and data $\theta_0 \in L^1 \cap L^\infty \cap B^s_{q,\infty}$, where $B^s_{q,\infty}$ stands for an inhomogeneous Besov space with $s > 1$ and $q > 2$. An interest in log-type couplings has also arisen in connection with other fluid mechanics models (see [8]).

Finally, it is worthy to remark that our results cover the couplings (1.12), (1.16), (1.17) and (1.18). For instance, the condition (1.6) is clearly satisfied by (1.12), and if $\beta \in [1, 2]$ and $2\beta - 1 < 2\gamma < \min(2 + \frac{2\beta}{\alpha_1}, 3)$ then (1.13) holds true. Also, (1.16) verifies (1.6) with $\beta = 1 + \varepsilon$, for any $\varepsilon > 0$, and we have (1.13) when $\frac{1}{2} < \gamma < \frac{3}{4}$ and $0 < \varepsilon < \gamma - \frac{1}{2}$. Analogously for (1.17) and (1.18) with $\chi > 0$. The case of these two last couplings with $\chi = 0$ follow similarly to (1.12).

This manuscript is organized as follows. In the next section we recall some estimates in $L^q(\mathbb{R}^n)$ and Sobolev homogeneous spaces for Fourier multiplier operators and the semigroup $\{G_\gamma(t)\}_{t \geq 0}$. Our results are stated in section 3 in two theorems, namely Theorems 3.1 and 3.3. Estimates for the bilinear operator (1.8) are obtained in section 4. Local well-posedness and some properties of solutions are proved in subsection 5.1. The proofs of Theorems 3.1 and 3.3 are performed in subsections 5.2 and 5.3, respectively.

## 2 Preliminaries

In this section we recall some estimates for the fundamental solution of the linear part of (1.1) in $L^p(\mathbb{R}^n)$ and $\dot{H}_p^s(\mathbb{R}^n)$, whose norms will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{\dot{H}_p^s}$, respectively.

We remember that given $s \in \mathbb{R}$ and $1 < p < \infty$, the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is the space of all $u \in S'/P$ such that $(-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n)$. In other words, $\dot{H}^s_p = (-\Delta)^{\frac{s}{2}} L^p$ and it is a Banach space with norm

$$\|u\|_{\dot{H}_p^s} = \|(-\Delta)^{\frac{s}{2}} u\|_p.$$  

The following Sobolev type embedding holds true

$$\dot{H}^s_{p_2}(\mathbb{R}^n) \subset \dot{H}^s_{p_1}(\mathbb{R}^n),$$

(2.1)

for $1 < p_2 \leq p_1 < \infty$ and $s_1 - \frac{s}{p_2} = s_2 - \frac{s}{p_2}$. The reader is refereed to [20] chapter 6 for further details on these spaces.

The next lemma gives estimates for certain multiplier operators acting in $\dot{H}^s_p(\mathbb{R}^n)$ (see e.g. [27])

**Lemma 2.1** Let $m, s \in \mathbb{R}$, $1 < p < \infty$ and $F(\xi) \in C_{\alpha}^{\lfloor \frac{m}{2} \rfloor + 1}(\mathbb{R}^n \setminus \{0\})$, where $\lfloor \cdot \rfloor$ stands for the greatest integer function. Assume that there is $L > 0$ such that

$$\left| \frac{\partial^\alpha F}{\partial \xi^\alpha}(\xi) \right| \leq L|\xi|^{m-|\alpha|},$$

(2.2)
Dissipative active scalar equations

for all \( \alpha \in (\mathbb{N} \cup \{0\})^n \), \(|\alpha| \leq \left\lceil \frac{n}{2} \right\rceil + 1 \), and \( \xi \neq 0 \). Then the multiplier operator \( F(D) \) on \( S' / \mathcal{P} \) is bounded from \( \dot{H}^\alpha_p \) to \( \dot{H}^\beta_{p-m} \). Moreover, the following estimate holds true

\[
\| F(D)f \|_{\dot{H}^\beta_{p-m}} \leq C \| f \|_{\dot{H}^\alpha_p},
\]

where \( C > 0 \) is independent of \( f \).

The next lemma gives estimates for \( \{G_\gamma(t)\}_{t \geq 0} \) on spaces \( L^p(\mathbb{R}^n) \) and \( \dot{H}^\beta_p(\mathbb{R}^n) \).

**Lemma 2.2** Let \( n \geq 2 \), \( 0 < \gamma < \infty \), \( 1 \leq p \leq q \leq \infty \), and \( k \in (\mathbb{N} \cup \{0\})^n \). Then

\[
\| \nabla^k \gamma G_\gamma(t) f \|_q \leq C t \left( \frac{1}{2^{\gamma-1}} \right)^{\frac{n}{2}} \left( \frac{1}{p_1} \right)^{\frac{n}{p_1}} \| f \|_{p_1},
\]

for all \( f \in L^p(\mathbb{R}^n) \). Now, let \( s_1 \leq s_2, s_1 \in \mathbb{R} \) and \( 1 < p_1 \leq p_2 < \infty \). There is a constant \( C > 0 \) such that

\[
\| G_\gamma(t) f \|_{\dot{H}^{p_2}_{p_2}} \leq C t \left( \frac{(s_2 - s_1)}{2^{\gamma-1}} \right)^{\frac{n}{2}} \left( \frac{1}{p_1} \right)^{\frac{n}{p_1}} \| f \|_{\dot{H}^{p_1}_{p_1}},
\]

for all \( f \in \dot{H}^{p_1}_{p_1} \). Moreover, given \( f \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \) with \( 1 \leq \beta < 2 \gamma \leq n + \beta \) and \( \frac{n}{2\gamma - \beta} < q < \infty \), then

\[
\sup_{0 < t < T} \| t^{nq} \|_{\dot{H}^{q}_{q}} \leq C \| f \|_{L^{\frac{n}{2\gamma - \beta}}}, \quad \text{and} \quad \lim_{t \to \infty} \| t^{nq} \|_{\dot{H}^{q}_{q}} = 0
\]

where \( \eta_q = \frac{2^\gamma - 1}{2\gamma} - \frac{n}{2\gamma q} \) and \( C \) is independent of \( f \) and \( 0 < t \leq T \).

**Proof.** The estimate (2.4) is well-known (see e.g. [2] for a proof). Also, (2.4) still holds true by replacing \( \nabla^k \gamma \) by \((\Delta)^{\frac{k}{2}}\). In view of the latter comment and \((\Delta)^{\frac{k}{2}} = (-\Delta)^{\frac{k}{2}} = (-\Delta)^{\frac{k}{2}} (-\Delta)^{\frac{k}{2}}\), we obtain (2.5) from (2.4) because \( G_\gamma(t) \) commutes with \((\Delta)^{\frac{k}{2}}\). The estimate in (2.6) comes from (2.5) with \( p_2 = q, s_2 = \beta - 1, p_1 = \frac{n}{2\gamma - \beta} \) and \( s_1 = 0 \). Due to (2.5), it is easy to see that the limit in (2.6) holds true for every \( \theta_0 \in L^{\frac{n}{2\gamma - \beta}} \cap \dot{H}^{\beta-1}_q \). This fact together with \( L^{\frac{n}{2\gamma - \beta}} \cap \dot{H}^{\beta-1}_q = L^{\frac{n}{2\gamma - \beta}} \) and the estimate in (2.6) yield the limit in (2.6), for every \( \theta_0 \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \).

3 Results

This section is devoted to the statements of the results whose proofs will be performed in section 5.

**Theorem 3.1** (Global solutions) Assume the condition (1.13) and let \( \eta_q = \frac{2^\gamma - 1}{2\gamma} - \frac{n}{2\gamma q} \) and \( \tilde{\eta}_q = \frac{2^\gamma - \beta}{2\gamma} - \frac{n}{2\gamma q} \). If \( \theta_0 \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \) then there is a unique global solution \( \theta \in BC((0, \infty); L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n)) \) for (1.1) such that

\[
t^{\eta_q} \theta \in BC((0, \infty); L^q(\mathbb{R}^n)), \quad \text{for all} \quad \frac{n}{2\gamma - \beta} < q \leq \infty,
\]

\[
t^{\tilde{\eta}_q} \theta \in C((0, \infty); \dot{H}^{\beta-1}_q(\mathbb{R}^n)), \quad \text{for all} \quad \frac{n}{2\gamma - \beta} < q < \infty,
\]

where the limits of \( t^{\eta_q} \theta \) in (3.1) and \( t^{\tilde{\eta}_q} \theta \) in (3.2) are zero as \( t \to 0^+ \). Moreover, if \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \) and \( 1 < q \leq \infty \), then \( \theta \in BC((0, \infty); L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n)) \) and

\[
t^{\eta_q + \frac{n}{2\gamma - \beta} - 1} \theta \in BC((0, \infty); L^q(\mathbb{R}^n)).
\]
Remark 3.2 With a slight modification in Theorem 3.1 and an adaptation of the estimates of section 4 (particularly (4.1) and (4.4)), one can treat the range $0 \leq \beta < 1$. For that matter, one should replace $\sup_{0 < t < T} t^{\gamma} \| \theta(t) \|_{H^{\beta-1}_t}$ by $\sup_{0 < t < T} t^{\gamma} \| \theta(t) \|_l$ into those estimates ($l = q, r$). In fact, this case is easier to handling than $\beta > 1$ and it is not necessary to use norms based on homogeneous Sobolev spaces in order to prove existence of solutions.

Before proceeding, we recall the concept of even and odd symmetry. A function $h$ is said to be even (resp. odd) when $h(x) = h(-x)$ (resp. $h(x) = -h(-x)$).

**Theorem 3.3 (Symmetry) Under the hypotheses of Theorem 3.1.**

(i) The solution $\theta(x, t)$ is odd (resp. even) for all $t > 0$, provided that $\theta_0$ and $P_i$’s are odd (resp. even).

(ii) Let $P(\xi)$ be as in (1.2). If $\theta_0$ and $\operatorname{div}_x(P(\xi))$ are radially symmetric then $\theta(x, t)$ is radially symmetric for all $t > 0$.

Remark 3.4 (non-symmetry) Adapting the arguments in the proof of Theorem 3.3, one also can prove the following non-symmetry results: if $\theta_0$ is odd (resp. even) and $P_i$’s are even (resp. odd) then $\theta(x, t)$ is not odd (resp. not even). Also, if $\theta_0$ is nonradial and $\operatorname{div}_x(P(\xi))$ is radial, then $\theta(x, t)$ is not radially symmetric. The detailed verification is left to the reader.

4 Bilinear Estimates

This part of the article is devoted to estimates for the bilinear term (1.8).

**Lemma 4.1** Let $0 < T \leq \infty$, $n \geq 2$, $1 \leq \beta < 2 \gamma < \infty$, and let $1 < q < \infty$ be such that $\frac{\beta-1}{n} < \frac{1}{q} < \frac{\beta-1}{r}$. Denote $\eta_q = \frac{2\gamma-1}{2\gamma} - \frac{n}{2\gamma q}$ and $\eta_r = \frac{2\gamma-1}{2\gamma} - \frac{n}{2\gamma r}$.

(i) If $\frac{2\gamma - (\beta+1)}{n} - \frac{1}{q} < \frac{1}{r} \leq \frac{1}{q'}$ and $q' \leq p \leq \infty$ then there are positive constants $K_1, K_2, K_3$, independent of $\theta, \phi$ and $T$, such that

\[
\sup_{0 < t < T} t^{\eta_q} \| B(\theta, \phi) \|_r \leq K_1 \sup_{0 < t < T} t^{\eta_q} \| \theta \|_{H^{\beta-1}_t} \sup_{0 < t < T} t^{\eta_q} \| \phi \|_r, \tag{4.1}
\]

\[
\sup_{0 < t < T} \| B(\theta, \phi) \|_p \leq K_2 \sup_{0 < t < T} t^{\eta_q} \| \theta \|_{H^{\beta-1}_t} \sup_{0 < t < T} \| \phi \|_p, \tag{4.2}
\]

\[
\sup_{0 < t < T} \| B(\theta, \phi) \|_1 \leq K_3 T^{\frac{2\gamma-1}{n} \eta_q} \sup_{0 < t < T} t^{\eta_q} \| \theta \|_{H^{\beta-1}_t} \sup_{0 < t < T} \| \phi \|_{q'}. \tag{4.3}
\]

(ii) If $\frac{2\gamma - 2}{n} - \frac{1}{q} < \frac{1}{r} < \frac{n+\beta-1}{n} - \frac{1}{q}$ then

\[
\sup_{0 < t < T} t^{\eta_q} \| B(\theta, \phi) \|_{H^{\beta-1}_t} \leq K_4 \sup_{0 < t < T} t^{\eta_q} \| \theta \|_{H^{\beta-1}_t} \sup_{0 < t < T} t^{\eta_q} \| \phi \|_{H^{\beta-1}_t}, \tag{4.4}
\]

where $K_3 > 0$ is a constant independent of $\theta, \phi$ and $T$. 

Dissipative active scalar equations
Proof.

Proof of part (i): Let $p_1 = p$ and $p_2 = r$. Using Lemma 2.2 and Hölder inequality, we estimate

$$\|B(\theta, \phi)\|_{p_i} \leq \int_0^t \|\nabla_x G_\gamma(t-s)(P[\theta]\phi)(s)\|_{p_i} \, ds \leq C \int_0^t (t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma\eta}} \|\|P[\theta]\phi\|(s)\|_{p_i} \, ds$$

where $i = 1, 2$ and in the third line we have used Lemma 2.1 in order to infer

$$\|P[\theta]\|_q \leq C \|\theta\|_{\dot{H}^{\beta-1}}.$$

Therefore

$$\|B(\theta, \phi)\|_p \leq C I_1(t) \sup_{0 < t < T} \|\phi(t)\|_p \sup_{0 < t < T} \|t^{n_\eta} \|\theta(t)\|_{\dot{H}^{\beta-1}} \tag{4.6}$$

$$\|B(\theta, \phi)\|_r \leq C I_2(t) \sup_{0 < t < T} \|t^{n_\eta} \|\phi(t)\|_r \sup_{0 < t < T} \|t^{n_\eta} \|\theta(t)\|_{\dot{H}^{\beta-1}} \tag{4.7}$$

where the integrals $I_1(t)$ and $I_2(t)$ can be computed as

$$I_1(t) = \int_0^t (t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma\eta}} \, ds = \int_0^1 (1-s)^{\eta_0-1}s^{-\eta_\eta} \, ds = C < \infty, \tag{4.8}$$

$$I_2(t) = \int_0^t (t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma\eta}} \, ds = t^{\eta_0-1-\eta_\eta-1} \int_0^1 (1-s)^{\eta_0-1}s^{-\eta_\eta-1} \, ds = C t^{-\eta_\eta}. \tag{4.9}$$

The estimates (4.1) and (4.2) follows from (4.7) with (4.9), and (4.6) with (4.8), respectively.

Moreover, we have that

$$\|B(\theta, \phi)(t)\|_1 \leq \int_0^t \|\nabla_x G_\gamma(t-s)(P[\theta]\phi)(s)\|_1 \, ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2\gamma}} \|P[\theta]\|_q \|\phi\|_r \, ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2\gamma}} \|\|\theta\|_{\dot{H}^{\beta-1}} \|\phi\|_r \, ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-n_\eta} \, ds \sup_{0 < t < T} \|t^{n_\eta} \|\theta\|_{\dot{H}^{\beta-1}} \sup_{0 < t < T} \|\phi\|_r$$

$$\leq CT^{-\frac{1}{2\gamma} - n_\eta} \sup_{0 < t < T} \|t^{n_\eta} \|\theta\|_{\dot{H}^{\beta-1}} \sup_{0 < t < T} \|\phi\|_r,$$

which gives (4.3).
Proof of part (ii): Consider
\[ \frac{1}{h} = \frac{1}{r} + \frac{1}{q} - \frac{\beta - 1}{n} \quad \text{and} \quad \delta = \frac{n}{h} - \frac{n}{r}. \] (4.10)

Note that \( \frac{\beta + \delta}{2} < 1 \) because \( \frac{1}{q} < \frac{2\gamma - 1}{n} \). We employ the continuous inclusion \( H^{\beta-1}_h \subset H^{\beta-1}_r \), and afterwards (2.3) to obtain

\[
\|B(\theta, \phi)\|_{H^{\beta-1}_r} \leq \int_0^t \|G_\gamma(t - s)\|_{H^{\beta-1}_r} ds
\]
\[
\leq \int_0^t \|G_\gamma(t - s)\|_{H^{\beta-1}_h} ds
\]
\[
\leq \int_0^t (t - s)^{-\frac{\beta + \delta}{2n}} ds \|P[B(\theta, \phi)]\|_{H^{\beta-1}_h} ds
\]
\[
\leq C t^{-\frac{\beta + \delta}{2n}} \|\theta\|_{H^{\beta-1}_r} \|\phi\|_{H^{\beta-1}_h}.
\] (4.11)

In view of (4.10), we can choose \( 1 < l < \infty \) in such a way that \( l > q, \frac{1}{h} = \frac{1}{r} + \frac{1}{t} \) and \( \frac{1}{q} = \frac{1}{r} - \frac{\beta - 1}{n} \). Then, Hölder inequality, (2.3), and Sobolev embedding (2.1) imply that

\[
\|P[B(\theta, \phi)]\|_{H^{\beta-1}_r} \leq \|P[\theta]\|_{H^{\beta-1}_r} \|\phi\|_l
\]
\[
\leq \|\theta\|_{H^{\beta-1}_r} \|\phi\|_{H^{\beta-1}_h}.
\] (4.12)

Inserting (4.12) into (4.11), we get

\[
\|B(\theta, \phi)\|_{H^{\beta-1}_r} \leq C t^{-\frac{\beta + \delta}{2n}} \|\theta\|_{H^{\beta-1}_r} \|\phi\|_{H^{\beta-1}_h} ds
\]
\[
\leq C t^{-\frac{\beta + \delta}{2n}} \|\theta\|_{H^{\beta-1}_r} \|\phi\|_{H^{\beta-1}_h} ds
\]
\[
\leq C t^{-\frac{\beta + \delta}{2n}} \|\theta\|_{H^{\beta-1}_r} \|\phi\|_{H^{\beta-1}_h} ds
\]
\[
\leq C t^{-\frac{\beta + \delta}{2n}} \|\theta\|_{H^{\beta-1}_h} \|\phi\|_{H^{\beta-1}_h} ds
\]

which is equivalent to (4.14).

\[
5 \quad \text{Proofs}
\]

5.1 Local in Time Solutions

We start by recalling an abstract lemma in Banach spaces which is useful in order to avoid extensive fixed point computations (see e.g. [28, Theorem 9]).

Lemma 5.1 Let \( X \) be a Banach space with norm \( \|\cdot\|_X \), and \( B : X \times X \to X \) be a continuous bilinear map, i.e., there exists \( K > 0 \) such that

\[
\|B(x_1, x_2)\|_X \leq K \|x_1\|_X \|x_2\|_X,
\]

for all \( x_1, x_2 \in X \).
for all \(x_1, x_2 \in X\). Given \(0 < \varepsilon < \frac{1}{4K_4}\) and \(y \in X\) such that \(\|y\|_X \leq \varepsilon\), there exists a solution \(x \in X\) for the equation \(x = y + B(x, x)\) such that \(\|x\|_X \leq 2\varepsilon\). The solution \(x\) is unique in the closed ball \(\{x \in X : \|x\|_X \leq 2\varepsilon\}\). Moreover, the solution depends continuously on \(y\) in the following sense: if \(\|\tilde{y}\|_X \leq \varepsilon\), \(\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})\), and \(\|\tilde{x}\|_X \leq 2\varepsilon\), then

\[
\|x - \tilde{x}\|_X \leq \frac{1}{1 - 4K_4\varepsilon}\|y - \tilde{y}\|_X.
\]

**Remark 5.2 (Picard sequence)** The solution given by Lemma 5.1 can be obtained as the limit in \(X\) of the Picard sequence \(\{x_k\}_{k \in \mathbb{N}}\) where \(x_1 = y\) and \(x_{k+1} = y + B(x_k, x_k)\), for all \(k \in \mathbb{N}\). Moreover, \(\|x_k\|_X \leq 2\varepsilon\) for all \(k \in \mathbb{N}\).

The following proposition shows that (1.1)-(1.3) is locally in time well-posed for \(L^{\frac{n}{n-\gamma}}\)-data.

**Proposition 5.3 (Local in time solutions)** Assume (1.13) and let \(q\) be such that

\[
\max \left\{ \frac{\beta - 1}{n}, \frac{\gamma - 1}{n} \right\} < \frac{1}{q} < \min \left\{ \frac{2\gamma - \beta}{n}, \frac{n + \beta - 2\gamma}{n}, \frac{n + \beta - 1}{2n} \right\}.
\]

If \(\theta_0 \in L^{\frac{n}{n-\gamma}}(\mathbb{R}^n)\) then there exists \(T > 0\) such that (1.1)-(1.3) has a unique mild solution \(\theta\) in the class

\[
t^\theta \in BC((0, T); \dot{H}_q^{\beta-1}((\mathbb{R}^n))) \text{ and } \lim_{t \to 0^+} t^\theta \|\theta\|_{\dot{H}_q^{\beta-1}} = 0. \tag{5.1}
\]

Moreover, \(\theta \in BC((0, T); L^{\frac{n}{n-\gamma}}(\mathbb{R}^n))\).

**Proof.**

**Step 1:** For \(T > 0\), let us define the Banach space

\[
\mathcal{E}_T = \left\{ \theta \text{ measurable; } t^\theta \in BC((0, T); \dot{H}_q^{\beta-1}((\mathbb{R}^n))) \right\}
\]

with norm given by

\[
\|\theta\|_{\mathcal{E}_T} = \sup_{0 < t < T} t^\theta \|\theta(\cdot, t)\|_{\dot{H}_q^{\beta-1}}. \tag{5.2}
\]

Due to (2.6) in Lemma 2.2 and \(\theta_0 \in L^{\frac{n}{n-\gamma}}(\mathbb{R}^n)\), for any \(\varepsilon > 0\), there exists a \(T > 0\) such that

\[
\sup_{0 < t < T} t^\theta \|G_\gamma(t)\theta_0\|_{\dot{H}_q^{\beta-1}} \leq \varepsilon. \tag{5.3}
\]

Take \(0 < \varepsilon < \frac{1}{4K_4}\) where \(K_4\) is as in (4.4) with \(r = q\). In view of (5.3) and (4.4), we can apply Lemma 5.1 in \(\mathcal{E}_T\) to obtain a unique solution \(\theta(x, t)\) for (1.7) such that

\[
\sup_{0 < t < T} t^\theta \|\theta\|_{\dot{H}_q^{\beta-1}} \leq 2\varepsilon. \tag{5.4}
\]

Using (2.6) and an induction argument, one can shows that every element \(\theta_k\) of the Picard sequence

\[
\theta_1(x, t) = G_\gamma(t)\theta_0(x), \tag{5.5}
\]

\[
\theta_{k+1}(x, t) = \theta_1(x, t) + B(\theta_k, \theta_k), \quad k \in \mathbb{N}, \tag{5.6}
\]

is as in (4.4) with \(r = q\). In view of (5.3) and (4.4), we can apply Lemma 5.1 in \(\mathcal{E}_T\) to obtain a unique solution \(\theta(x, t)\) for (1.7) such that

\[
\sup_{0 < t < T} t^\theta \|\theta\|_{\dot{H}_q^{\beta-1}} \leq 2\varepsilon. \tag{5.4}
\]
satisfies \( \lim_{t \to 0^+} t^{\beta n} \| \theta_k \|_{H^\beta_k} = 0 \). Then the second property in (5.1) follows from the fact that the fixed point \( \tilde{\theta} \) is the limit in \( \mathcal{E}_T \) of \( \{ \theta_k \}_{k \in \mathbb{N}} \) (see Remark 5.2). Further details are left to the reader.

Step 2: In what follows we show that \( \tilde{\theta} \in BC([0, T); L^{2/n - \beta}(\mathbb{R}^n)) \). We have that the recursive sequence (5.5)-(5.6) satisfies (see Remark 5.2)

\[
\sup_{0 < t < T} t^{\beta n} \| \theta_k \|_{H^\beta_k} \leq 2 \varepsilon, \quad \text{for all } k \in \mathbb{N}.
\] (5.7)

Using Lemma 2.2 (4.2) with \( p = \frac{n}{2/n - \beta} \), and (5.7), we get

\[
\sup_{0 < t < T} \| \theta_1(t) \|_{L^{2/n - \beta}} \leq C \| \theta_0 \|_{L^{2/n - \beta}},
\]

and

\[
\sup_{0 < t < T} \| \theta_{k+1}(t) \|_{L^{2/n - \beta}} \leq C \| \theta_0 \|_{L^{2/n - \beta}} + K_2 \sup_{0 < t < T} t^{\beta n} \| \theta_k(t) \|_{H^\beta_k} \sup_{0 < t < T} \| \theta_k(t) \|_{L^{2/n - \beta}} \\
\leq C \| \theta_0 \|_{L^{2/n - \beta}} + 2K_2 \sup_{0 < t < T} \| \theta_k(t) \|_{L^{2/n - \beta}}, \quad \text{for all } k \in \mathbb{N}.
\] (5.8)

By reducing \( T > 0 \) in (5.3) if necessary, we can consider \( 0 < \varepsilon < \min\{ \frac{1}{2K_2}, \frac{1}{2K_2} \} \). Since \( 2K_2 \varepsilon < 1 \), an induction argument in (5.8) shows that \( \{ \theta_k \}_{k \in \mathbb{N}} \) is uniformly bounded in \( L^\infty([0, T); L^{2/n - \beta}(\mathbb{R}^n)) \) and then there exists a subsequence of \( \{ \theta_k \}_{k \in \mathbb{N}} \) (denoted in the same way) that converges toward \( \tilde{\theta} \) weak* in that space and consequently in \( \mathcal{D}'(\mathbb{R}^n \times [0, T)) \). Because \( \theta_k \to \tilde{\theta} \) in \( \mathcal{E}_T \), which implies convergence in \( \mathcal{D}'(\mathbb{R}^n \times [0, T)) \), the uniqueness of the limit in the sense of distributions yields \( \theta = \tilde{\theta} \in L^\infty((0, T); L^{2/n - \beta}(\mathbb{R}^n)) \). The time-continuity of \( \theta \) follows from standard arguments by using that \( \theta \) verifies (1.7), \( \tilde{\theta} \in L^{\infty}((0, T); L^{2/n - \beta}(\mathbb{R}^n)) \cap \mathcal{E}_T \), and the second property in (5.1) (see e.g. [22, 23]).

In the next proposition we investigate the \( L^1 \)-persistence of the solutions obtained in Proposition 5.3.

**Proposition 5.4** Under hypotheses of Proposition 5.3. There exists \( T > 0 \) such that the solution \( \theta \) belongs to \( BC([0, T); L^1(\mathbb{R}^n)) \) provided that \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \).

**Proof.** Let \( q \) be such that \( 1 < q' < \frac{n}{2/n - \beta} \). From interpolation, we have that \( \theta_0 \in L^{q'}(\mathbb{R}^n) \). Employing (5.7) and the estimate (1.2) with \( p = q' \), we get

\[
\sup_{0 < t < T} \| \theta_1(t) \|_{q'} \leq C \| \theta_0 \|_{q'},
\]

and

\[
\sup_{0 < t < T} \| \theta_{k+1}(t) \|_{q'} \leq C \| \theta_0 \|_{q'} + K_2 \sup_{0 < t < T} t^{\beta n} \| \theta_k(t) \|_{H^\beta_k} \sup_{0 < t < T} \| \theta_k(t) \|_{q'} \\
\leq C \| \theta_0 \|_{q'} + 2K_2 \varepsilon \sup_{0 < t < T} \| \theta_k(t) \|_{q'}, \quad \text{for all } k \in \mathbb{N}.
\]

Again reducing \( T > 0 \) if necessary, we can consider \( 2K_2 \varepsilon < 1 \) and proceed similarly to the end of the proof of Proposition 5.3 to obtain that

\[
\theta \in BC([0, T); L^{q'}(\mathbb{R}^n)).
\] (5.9)
Now we use (2.4), (4.3), (5.1), (5.9) to estimate
\[
\sup_{0 < t < T} \|\theta(t)\|_1 \leq C \|\theta_0\|_1 + \sup_{0 < t < T} \|B(\theta, \theta)\|_1
\]
\[
\leq C \|\theta_0\|_1 + K_3 T^{1-\frac{2}{2q} - \eta} \sup_{0 < t < T} t^{\eta_r} \|\theta\|_{\dot{H}^{\beta - 1}_q} \sup_{0 < t < T} \|\theta\|_{q'} < \infty,
\]
as required.

The existence time \(T\) in Propositions 5.3 and 5.4 may depend on index \(q\). In the next proposition we show that indeed one can take a same small time \(T > 0\) for all \(q\).

**Proposition 5.5** Under hypotheses of Proposition 5.3. Let \(\theta\) be the solution given by Proposition 5.3 with data \(\theta_0 \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n)\). There is a \(T > 0\) such that
\[
t^\eta \theta \in BC((0, T); \dot{H}^{\beta - 1}_r(\mathbb{R}^n)), \text{ for all } \frac{n}{2\gamma - \beta} < r < \infty, \tag{5.10}
\]
\[
t^\eta \theta \in BC((0, T); L^r(\mathbb{R}^n)), \text{ for all } \frac{n}{2\gamma - \beta} < r < \infty. \tag{5.11}
\]

**Proof.** Let \(q\) be fixed and as in Proposition 5.3. Given \(\frac{n}{2\gamma - \beta} < r < \infty\) verifying \(\frac{2\gamma - 2}{n} - \frac{1}{q} < \frac{1}{r} < \frac{n + \beta - 1}{n} - \frac{1}{q}\), we can use (4.4) instead of (4.1) and proceed just like as in step 2 of the proof of Proposition 5.3 to obtain (reducing \(T > 0\) if necessary)
\[
t^\eta \theta \in BC((0, T); \dot{H}^{\beta - 1}_r(\mathbb{R}^n)). \tag{5.12}
\]

Now let \(\frac{2\gamma - 2}{n} - \frac{1}{q} < \frac{1}{r} < \frac{2\gamma - 1}{n} - \frac{1}{q}\), and consider \(r < \bar{r} < \infty\). Taking \(\frac{1}{r} = \frac{1}{\bar{r}} + \frac{1}{\tilde{r}}\) and \(\delta = \frac{n}{n - \frac{r}{\bar{r}} - \tilde{r}}\), it follows that \(\delta > 0\), \(\frac{\beta + \delta}{2\gamma} < 1\), and \(\eta_r + \eta_r < 1\). Then, we can estimate
\[
\|\theta\|_{\dot{H}^{\beta - 1}_r} \leq C t^{-\eta_r} \|\theta_0\|_{\dot{H}^{\beta - 1}_r} + \int_0^t \|G_\gamma(t-s) \nabla_x \cdot (P[\theta] \theta)(s)\|_{\dot{H}^{\beta - 1}_r} ds
\]
\[
\leq C t^{-\eta_r} \|\theta_0\|_{\dot{H}^{\beta - 1}_r} + C \int_0^t \|G_\gamma(t-s) \nabla_x \cdot (P[\theta] \theta)(s)\|_{\dot{H}^{\beta - 1 + \delta}_r} ds
\]
\[
\leq C t^{-\eta_r} \|\theta_0\|_{\dot{H}^{\beta - 1}_r} + C \int_0^t (t-s)^{-\frac{\beta + \delta}{2\gamma}} \|\nabla_x \cdot (P[\theta] \theta)(s)\|_{\dot{H}^{\beta - 1}_r} ds, \tag{5.13}
\]
where above we have used Sobolev embedding and afterwards (2.5).

Now we employ (2.3), Hölder inequality and Sobolev embedding in order to estimate
R.H.S. of (5.13)

\[ \leq C t^{-\eta_p} \|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|P[\theta]\theta(s)\|_h \, ds \]

\[ \leq C t^{-\eta_p} \|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|P[\theta]\|_q \|\theta(s)\|_z \, ds \]

\[ \leq C t^{-\eta_p} \|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|\theta\|_{\dot{H}^{\beta-1}_q} \|\theta\|_{H^{\beta-1}_q} \, ds \]

\[ \leq C t^{-\eta_p} \|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_q-\eta_r} \, ds \left( \sup_{0<t<T} t^{\eta_q} \|\theta\|_{\dot{H}^{\beta-1}_q} \right) \left( \sup_{0<t<T} t^{\eta_r} \|\theta\|_{H^{\beta-1}_q} \right) \]

\[ \leq C t^{-\eta_p} \int_0^1 (1-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_q-\eta_r} \, ds \]

\[ \leq Ct^{-\eta_p}, \]

and then we arrive at (5.12) with \( \tilde{r} \) in place of \( r \), and with the same existence time \( T > 0 \). From interpolation, notice that (5.12) also holds true for every \( r = l \) such that \( \frac{n}{2\gamma-\beta} < l < \tilde{r} \). Since \( \tilde{r} > r \) is arbitrary, we obtain (5.12) with \( r = l \) (and the same \( T > 0 \)), for all \( \frac{n}{2\gamma-\beta} < l < \infty \), which gives (5.10).

The proof of (5.11) can be performed in a similar way by using (4.1) instead of (4.4).

\[ \square \]

5.2 Proof of Theorem 3.1

5.2.1 Step 1: Local smoothness and maximum principle

The solutions obtained in Proposition 5.3 are instantaneously \( C^\infty \)-smoothed for any \( t > 0 \) belonging to the existence interval \((0,T)\). This smooth effect holds for several parabolic equations in several frameworks, like e.g. \( L^p \), weak-\( L^p \), Morrey, Besov-Morrey, when mild solutions are constructed by using time-weighted norms of Kato type (see [22]). Precisely, adapting arguments from [22] (see also [23]), one can obtain that the solution verifies

\[ \partial_t^m \nabla_x^k \theta(x,t) \in C^0((0,T); L^\frac{n}{2\gamma-\beta}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) , \] (5.15)

for all \( \frac{n}{2\gamma-\beta} < q < \infty , \ m \in \{0\} \cup \mathbb{N} \) and multi-index \( k \in (\{0\} \cup \mathbb{N})^n \), where \( T > 0 \) is the existence time given in Proposition 5.3. In particular, it follows that \( \theta(x,t) \in C^\infty(\mathbb{R}^n \times (0,T)) \) and \( \theta(t) \in L^\infty(\mathbb{R}^n) \) with

\[ \|\theta(t)\|_\infty \leq C \|\theta(t)\|_{\frac{n+\beta-2\gamma}{n}}^\alpha \|\nabla_x \theta(t)\|_{\frac{2\gamma-\beta}{2\gamma-\beta}}^{1-\alpha} , \] (5.16)

for all \( 0 < t < T \), where \( \alpha = \frac{n+\beta-2\gamma}{n} \). If further \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^\frac{n}{2\gamma-\beta}(\mathbb{R}^n) \) then \( q \) in (5.15) can be taken in the range \( 1 < q < \infty \).
Due to (5.15) we have that \( \theta \) verifies (1.1)-(1.3) in the classical sense and \( \partial_t^p \nabla_x^k \theta(x, t) \to 0 \) when \( |x| \to \infty \), for all \( 0 < t < T \). In view of \( \nabla \cdot P\theta = 0 \), we can integrate by parts to obtain

\[
\frac{\partial}{\partial t} \| \theta(t) \|_p^p = p \int_{\mathbb{R}^n} \theta(t)^{p-1} \frac{\partial}{\partial t} \theta(t) \, dx
\]

\[
= p \int_{\mathbb{R}^n} \theta(t)^{p-1} (-(-\Delta)^\gamma \theta - \nabla_x \cdot (P\theta)) \, dx
\]

\[
= -p \int_{\mathbb{R}^2} \theta(t)^{p-1} (-\Delta)^\gamma \theta \, dx \leq -\int_{\mathbb{R}^2} \left( -\Delta \right)^\gamma (\theta^\frac{p}{2}) \left( \theta^\frac{p}{2} \right) \, dx,
\]

for all \( t \in (0, T) \). The last inequality in (5.17) can be found in [11] [13] (see also [21]).

In view of the estimate (5.17), we have that \( L^p \)-norms of \( \theta(t) \) are non-increasing in \( (0, T) \). If \( \theta_0 \in L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \) and \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \), we obtain respectively

\[
\| \theta(t) \|_{\frac{n}{2\gamma-p}} \leq \| \theta(t_0) \|_{\frac{n}{2\gamma-p}} \quad \text{and} \quad \| \theta(t) \|_1 \leq \| \theta(t_0) \|_1,
\]

for \( 0 < t_0 \leq t < T \). Making \( t_0 \to 0^+ \) in (5.18), it follows that the solution \( \theta(x, t) \) satisfies

\[
\| \theta(t) \|_{\frac{n}{2\gamma-p}} \leq \| \theta_0 \|_{\frac{n}{2\gamma-p}} \quad \text{and} \quad \| \theta(t) \|_1 \leq \| \theta_0 \|_1,
\]

for all \( t \in (0, T) \), when \( \theta_0 \in L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \) and \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \), respectively.

### 5.2.2 Step 2: Extension of the local solution

We start by making the following observation: if \( \frac{n}{2\gamma-p} < q < \infty \) and \( \theta_0 \in L^q(\mathbb{R}^n) \) then

\[
t^n \| G_{\gamma}(t)\theta_0 \|_{H^{\beta-1}} \leq Ct^n \| \theta_0 \|_q \to 0 \quad \text{when} \quad t \to 0^+.
\]

Therefore, for \( \theta_0 \in L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \), the existence time \( T > 0 \) obtained in Proposition 5.3 can be taken depending on the norm \( \| \theta_0 \|_q \). Indeed it can be chosen as

\[
T = \left( \frac{\varepsilon}{\| \theta_0 \|_q} \right)^{\frac{1}{nq}},
\]

where \( 0 < \varepsilon < \frac{1}{4K_4} \) and \( C > 0 \) is as in (5.20).

Now let \( \theta_0 \in L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n) \). From Proposition 5.3, there exist \( M_0 > 0 \), \( T_0 > 0 \) and a unique mild solution for (1.1)-(1.3) in \( (0, T_0) \) such that

\[
\sup_{0 < t < T_0} t^n \| \theta(t) \|_{H^{\beta-1}} \leq M_0 \quad \text{and} \quad \sup_{0 < t < T_0} \| \theta(t) \|_{\frac{n}{2\gamma-p}} \leq \| \theta_0 \|_{\frac{n}{2\gamma-p}},
\]

where (5.19) has been used in the second inequality in (5.22).

Let us now denote

\[
T = \sup \left\{ \tilde{T} > 0; \theta \in C((0, \tilde{T}); H^{\beta-1}_q(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-p}}(\mathbb{R}^n)), \sup_{0 < t < T} t^n \| \theta \|_{H^{\beta-1}_q} < \infty, \sup_{0 < t < T} \| \theta \|_{\frac{n}{2\gamma-p}} \leq \| \theta_0 \|_{\frac{n}{2\gamma-p}} \right\}.
\]
We desire to prove that \( T = \infty \). Suppose by contradiction that \( T < \infty \), and let \( a = \theta(T - \varepsilon) \) where \( 0 < \varepsilon < T \) will be choose later. Proposition 5.3 gives that \( a \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \cap L^{q}(\mathbb{R}^n) \), for all \( \frac{n}{2\gamma - \beta} < q < \infty \). Moreover, if \( \varepsilon < \frac{T}{2} \) we get \( \|\theta(T - \varepsilon)\|_q \leq \|\theta(T)\|_q \). Therefore, taking \( \varepsilon < T/2 \) and \( a \) as initial data, we have that given \( 0 < M_1 < \frac{1}{\theta(T)} \) there exist \( T_1 > 0 \) and a unique mild solution \( \tilde{\theta} \) for (1.1)-(1.3) such that

\[
\tilde{\theta} \in C((T - \varepsilon, T_1 + T - \varepsilon); L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \cap \dot{H}^{\beta-1}_q(\mathbb{R}^n))
\]

and

\[
\sup_{T-\varepsilon < t < T_1 + T - \varepsilon} (t - (T - \varepsilon))^{\eta} \|\tilde{\theta}(t)\|_{\dot{H}^{\beta-1}} \leq 2M_1 \\
\sup_{T-\varepsilon < t < T_1 + T - \varepsilon} \|\tilde{\theta}(t)\|_{\frac{n}{2\gamma - \beta}} \leq \|\theta(T - \varepsilon)\|_{\frac{n}{2\gamma - \beta}}.
\]

From uniqueness part of Proposition 5.3 it follows that \( \theta = \tilde{\theta} \) in \((T - \varepsilon, T)\). In view of (5.21), we can choose \( T_1 = \min \left\{ \left( \frac{M_1}{C\|\theta(T)\|_q} \right)^{\frac{1}{\eta}}, T \right\} \). Taking \( 0 < \varepsilon < \min\{\frac{T}{2}, T_1\} \) and \( T_2 = T_1 + T - \varepsilon \), we have that \( T < T_2 \) and get a solution

\[
\theta \in C((0, T_2); L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \cap \dot{H}^{\beta-1}_q(\mathbb{R}^n))
\]

such that

\[
\sup_{0 < t < \tilde{T}} t^{\eta} \|\theta(t)\|_{\dot{H}^{\beta-1}} < \infty \quad \text{and} \quad \sup_{0 < t < \tilde{T}} \|\theta(t)\|_{\frac{n}{2\gamma - \beta}} \leq \|\theta_0\|_{\frac{n}{2\gamma - \beta}},
\]

for all \( 0 < \tilde{T} < T_2 \), which contradicts the maximality of \( T \) in (5.23). Therefore \( T = \infty \) and we are done.

5.2.3 Step 3: Global \( L^q \)-decay of solutions

We will prove only the part of the statement corresponding to the case \( \theta_0 \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \). The estimate (3.3) for \( \theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n) \) follows similarly to the first one by using \( \|\theta(t)\|_1 \leq \|\theta_0\|_1 \) and the sequence \( q_k = 2^k \) instead of \( \|\theta(t)\|_{\frac{n}{2\gamma - \beta}} \leq \|\theta_0\|_{\frac{n}{2\gamma - \beta}} \) and \( q_k = \frac{n}{2\gamma - \beta} 2^k \).

Since we have extended the solution \( \theta \), it follows that (5.15) and (5.16) hold true for \( T = \infty \). Then

\[
\|\theta(\cdot, t)\|_\infty < \infty, \text{ for all } t > 0. \tag{5.24}
\]

Now we proceed as in [2] and [23]. In view of the Gagliardo-Nirenberg inequality, we have that

\[
\|\phi\|_2 \leq C \|\phi\|_1^\alpha \|(-\Delta)^{\frac{\alpha}{2}} \phi\|_2^{1-\alpha} \quad \text{with } \alpha = \frac{2\gamma}{n + 2\gamma}. \tag{5.25}
\]

Taking \( \phi = \theta^\frac{\alpha}{n} \) in (5.25), it follows that

\[
\|\theta\|_{q}^{\frac{\alpha + 2\gamma}{n}} \leq C \|\theta\|_{\frac{2\gamma}{n}} \|(-\Delta)^{\frac{\alpha}{2}} (\theta^\frac{\alpha}{n})\|_2^2. \tag{5.26}
\]
Denoting $\psi_q(t) = \|\theta(t)\|_q^q$, we obtain from (5.17) and (5.26) that

$$\frac{\partial}{\partial t} \psi_q \leq -C(\psi_q) \frac{n}{\gamma} \psi_{q + \frac{n}{2}}.$$  (5.27)

The differential inequality (5.27) can be solved by an induction procedure. In fact, using the first inequality in (5.19) and considering the sequence $q_k = \frac{n}{2\gamma - \beta} 2^k$ for $k \geq 0$, we arrive at

$$\psi_{q_k}(t) \leq M_{q_k} t^{-\frac{n}{2}(\frac{1}{2^k - 1})},$$  (5.28)

where

$$M_{q_0} = \|\theta_0\|_{\frac{n}{2\gamma - \beta}}$$

and

$$M_{q_k} = \left(\frac{n(2^k - 1)}{2C\gamma}\right)^{\frac{n}{2\gamma - \beta}} M_{q_{k-1}}^{\frac{n}{2\gamma - \beta}},$$

for $k \in \mathbb{N}$.

It follows that

$$M_{q_k}^{-1} = \left(\frac{n(2^k - 1)}{2C\gamma}\right)^{-\frac{n}{2\gamma - \beta}} M_{q_{k-1}}^{-\frac{n}{2\gamma - \beta}},$$

for all $k \in \mathbb{N}$, and then

$$\|\theta(t)\|_{2^q q_0} \leq M_{q_k}^{-1} t^{-\frac{n}{2}(\frac{1}{2^k - 1})} = \left(\prod_{i=1}^k \left(\frac{n(2^i - 1)}{2C\gamma}\right)^{\frac{n}{2\gamma - \beta}} \right) (M_{q_0})^{\frac{n}{2\gamma - \beta}} t^{-\frac{n}{2\gamma - \beta}(\frac{2^k - 1}{2^k - 1})},$$  (5.29)

where $q_0 = \frac{n}{2\gamma - \beta}$. In view of (5.24), we can make $k \to \infty$ in (5.29) to obtain

$$\|\theta(t)\|_{\infty} \leq C\|\theta_0\|_{\frac{2\gamma - \beta}{n}} \left(\frac{2\gamma - \beta}{n}\right)^{-\frac{2\gamma - \beta}{2\gamma}}.$$

(5.30)

Interpolating the first inequality in (5.19) with (5.30), the result is

$$\|\theta(t)\|_q \leq C t^{-\theta},$$

for all $\frac{n}{2\gamma - \beta} \leq q \leq \infty$,

as required.

The uniqueness statement follows from the local uniqueness property in Proposition 5.3.

5.3 Proof of Theorem 3.3

Part (i): We will prove only the odd part of the statement since the even one follows similarly. Let $\theta$ be the solution of Proposition 5.3 with existence time $T > 0$. From step 2 of the proof of Theorem 3.1, $\theta$ can be extended by using Proposition 5.3 and solving (1.1)-(1.3) consecutively with initial data $\theta(T_0), \theta(T_0 + T_1), \theta(T_0 + 2T_1)$ and so on, where $T_1 = \min\left\{\left(\frac{C}{\|\theta(T_0)\|_q}\right)^{\frac{1}{q_0}}, T\right\}$, $\varepsilon = \frac{1}{8\kappa_q}$.
and \( C \) as in (5.20). Because of that, it is sufficient to show the following claim: if \( \theta_0 \in L^{\frac{n}{n-\gamma}} \) is odd then so is the solution \( \theta(x, t) \) given by Proposition 5.3 for all \( t \in (0, T) \). In fact, notice that we can use this claim repeatedly to show that the global solution \( \theta(x, t) \) is odd, for all \( t > 0 \).

Let \( \psi(x, t) = G_\gamma(t)\theta_0 \). We have that \( \theta_0(-x) = -\theta_0(x) \) is equivalent to

\[
-\hat{\theta}_0(\xi) = [\theta_0(-x)](\xi) = \hat{\theta}_0(\xi) \text{ in } S'(\mathbb{R}^n).
\]

It follows from (5.31) that

\[
[\psi(-x, t)](\xi) = e^{-t\|\xi\|^{2\gamma}_\theta} \hat{\theta}_0(-\xi) = -\theta_0(-x)(\xi). \\

\]

which shows that \( G_\gamma(t)\theta_0 \) is odd, for each fixed \( t > 0 \) and \( \theta \) is odd then \( \nabla \theta \) is even, because

\[
\nabla(\theta(x, t)) = \nabla(-\theta(-x, t)) = (\nabla \theta)(-x, t)).
\]

Recall that \( A = (a_{ij}) \) and

\[
P(\xi) = (\tilde{P}_1(\xi), ..., \tilde{P}_n(\xi))
\]

where

\[
\tilde{P}_j(\xi) = \sum_{i=1}^{n} a_{ij} \frac{\xi_i}{|\xi|^2} P_i(\xi).
\]

It follows that

\[
\tilde{u}(-\xi) = (\tilde{u}_1(-\xi), ..., \tilde{u}_n(-\xi)) = P(-\xi)\hat{\theta}(-\xi, t)
\]

with

\[
P(-\xi) = \frac{I}{|\xi|^2} [-\xi_1 P_1(-\xi), ..., -\xi_n P_n(-\xi)] A
\]

\[
= \frac{I}{|\xi|^2} [\xi_1 P_1(\xi), ..., \xi_n P_n(\xi)] A = P(\xi),
\]

because \( P_i \)'s are odd. Therefore \( u = P[\theta] \) is odd when \( \theta \) is odd, and then \( (u \cdot \nabla \theta) = (P[\theta] \cdot \nabla \theta) \) is odd too. Hence if \( \theta \) is odd then so is \( B(\theta, \theta) \).

So, employing an induction argument, one can prove that each element \( \theta_k \) of the Picard sequence (5.5)-(5.6) is odd. Since \( \theta_k \to \theta \) in the norm (5.2), then the sequence (5.5)-(5.6) also converges (up to a subsequence) to \( \theta \) a.e. \( x \in \mathbb{R}^n \), for all \( t \in (0, T) \). It follows that \( \theta(x, t) \) is odd, for each fixed \( t \in (0, T) \), because pointwise convergence preserves odd symmetry. This shows the desired claim.

**Part (ii):** From the same reasons given in part (i), we need only to prove that the local solution of Proposition 5.3 is radially symmetric whenever \( \theta_0 \) and \( \text{div}_\xi(P(\xi)) \) are too. For that matter, we first observe that \( G_\gamma(t)\theta_0 \) is radial because \( \theta_0 \) and the kernel \( \hat{g}_\gamma(\xi, t) = e^{-|\xi|^{2\gamma}_\theta} \) are radial, for all \( t > 0 \). Also, for \( \theta \) radially symmetric, we have that

\[
(u \cdot \nabla \theta) = \sum_{j=1}^{n} u_j \partial_\xi_j \theta = \frac{\theta'(r)}{r} \sum_{j=1}^{n} u_j x_j
\]

\[
= I \frac{\theta'(r)}{r} \sum_{j=1}^{n} (\partial_\xi_j \hat{u}_j)^\vee = I \frac{\theta'(r)}{r} \sum_{j=1}^{n} (\partial_\xi_j \tilde{P}_j(\xi)\hat{\theta})^\vee
\]

\[
= I \frac{\theta'(r)}{r} \left( \hat{\theta}(\xi, t) (\text{div}_\xi(P(\xi)))^\vee \right).
\]

(5.32)
It follows from (5.32) that if \( \theta \) and \((\text{div}_\xi(P(\xi)))\) are radial then so is \((u \cdot \nabla \theta)\). Using that \(G_\gamma(t)\) preserves radiality, we obtain that \(B(\theta, \theta)\) defined in (1.8) is radially symmetric, for each \(t \in (0, T)\), whenever \(\theta\) is too. Analogously to part (i), we now can use induction in order to show that each function \(\theta_k\) defined in (5.5)-(5.6) is also radially symmetric. Since \(\theta_k\) converges (up to a subsequence) to \(\theta\) a.e. \(x \in \mathbb{R}^n\), for each \(t \in (0, T)\), we obtain the required conclusion.

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