Convergence to the Tracy-Widom distribution for longest paths in a directed random graph

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This thesis consists of the following paper, which is referred to in the text by its Roman numeral:

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The aim of this thesis is to study asymptotics of the longest path in a directed random graph which is defined on an \( n \times m \) rectangle with any two points (that are comparable according to the product order) connected with probability \( p \). It is shown that, when the rectangle is taken to be long and thin, the limiting distribution of the longest path, centered/rescaled, is the Tracy-Widom distribution.

1 Directed random graphs

The Erdős-Rényi model is one of the most fundamental models of random graphs. The model consists of graphs with vertices \( \{1, 2, \ldots, n\} \) whose edges are independently present with fixed probability \( p \), \( 0 \leq p \leq 1 \). In other words, among all the graphs on \( n \) vertices, a graph \( G \) is constructed with probability

\[
P(G) = p^m (1 - p)^{\binom{n}{2} - m},
\]

where \( m \) is the number of edges of the graph \( G \). A directed random graph is obtained from an Erdős-Rényi graph by directing all of the edges from the vertex with the smaller label to the vertex with the larger label.

Directed random graphs have applications in computer science, biology and physics. One such example from biology [9] is the modeling of food webs, where vertices \( \{1, 2, \ldots, n\} \) represent different species and the presence of a directed edge \((i, j), i < j\), indicates that species \( i \) is the predator and species \( j \) is the prey. In computer science we can find an example of a model of parallel computation [13] where the directed edge \((i, j)\) indicates that task \( i \) has to be done before task \( j \).
Directed random graphs have also been referred to in the literature as random acyclic directed graphs and random graph orders. The name random acyclic directed graph relates to the fact that for every acyclic directed graph there exists a permutation of the vertices such that all of the edges are directed from the vertex with the smaller label to the vertex with the larger label. The other name, random graph orders, stems from a partial ordered set induced by the edges in the graph, that is, $i \prec j$ whenever $i < j$ and there is an edge between $i$ and $j$. Different properties of random directed graphs have been studied in terms of partial orders, for example width (the longest antichain) [5], height (the longest chain) [1, 7, 11, 10], the number of linear extensions [2, 7] and the number of incomparable pairs [17, 7].

2 The Tracy-Widom distribution

One of the classic random matrix models is the Gaussian Unitary Ensemble (GUE). That is a probability space with the sample space of $n \times n$ Hermitian matrices and a probability measure that is invariant under unitary transformations and the real and imaginary matrix elements are independent and identically distributed random variables. Let $\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n$, with $\lambda_1^n < \lambda_2^n < \cdots < \lambda_n^n$, be the eigenvalues of a random $n \times n$ matrix from the GUE. The empirical distribution function of the eigenvalues, defined by $\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i^n/\sqrt{n}}$, converges weakly, in probability, to the semicircle law with density $\frac{1}{2\pi \sqrt{4-x^2}}$ on $[-2,2]$ [3, Thm. 2.2.1]. Fluctuations of $\lambda_n^n$ around the upper bound of the density support 2 have been quantified by Tracy and Widom [18]. The limiting distribution which they found is today called the Tracy-Widom distribution. This distribution function describes universal limit laws for a wide range of processes arising in mathematical physics and interacting particle systems. A variety of the examples is collected in a survey by Tracy and Widom [19]. Here we present two examples, a directed last-passage percolation and a Brownian directed percolation model, which are connected with a result in Paper I.

A directed last-passage percolation model in $\mathbb{N}_0^2$ is defined as follows. Let $(X(i, j), i, j \in \mathbb{N}_0)$ be independent and identically distributed random variables. With each node $(i, j)$ in $\mathbb{N}_0^2$ we associate the weight $X(i, j)$. Let $\Pi_{m,n}$ be a set of all up/right paths in $\mathbb{N}_0^2$ from $(0,0)$ to $(m,n)$, i.e. sequences $((0,0), (i_1, j_1), \ldots, (i_{m+n-1}, j_{m+n-1}), (m,n))$ such that each successive member of the sequence has a single coordinate increased by 1. The last-passage time from $(0,0)$ to $(m,n)$ is the maximum weight of all directed paths from $(0,0)$ to $(m,n)$, that is,

$$T_{m,n} = \max_{\pi \in \Pi_{m,n}} \sum_{(i,j) \in \pi} X(i,j).$$

Johansson [14] showed that $T_{m,n}$, appropriately rescaled, converges in law to the Tracy-Widom distribution when the weights are geometrically or exponentially distributed and $m \sim n$. Later, Bodineau and Martin [8] proved that $T_{n, \lfloor an \rfloor}$, appropriately rescaled, converges to the Tracy-Widom distribution, whenever the weights have a finite moment greater than two and for $a$ sufficiently small (the threshold depending on the order of the finite moment). Independently, Baik and Suidan [4] obtained the same result for the weights with a finite 4th moment and for $a < 3/14$. 

2
A similar model is a Brownian directed percolation model, defined on $\mathbb{R} \times \{1, 2, \ldots, n\}$ with a standard Brownian motion $(B_t^{(j)}, t \geq 0)$ associated with every line $\mathbb{R} \times \{j\}$. Then a last-passage time is defined as

$$Z_{t,n} := \sup_{0=t_0 < t_1 < \cdots < t_{m-1} < t_n = t} \sum_{j=1}^{n} [B_{t_j}^{(j)} - B_{t_{j-1}}^{(j)}], \quad t \geq 0.$$ 

Baryshnikov [6] showed that $Z_{1,n}$ has the same distribution as the largest eigenvalue of a random $n \times n$ matrix from the GUE, and thus, appropriately scaled, converges in distribution to the Tracy-Widom distribution as $n$ tends to infinity. By Brownian scaling, an analogous result follows for $Z_{t,n}$.

3 Skeleton points

A path in a directed random graph is a sequence of vertices $(i_0, i_1, \ldots, i_\ell)$, $i_0 < i_1 < \cdots < i_\ell$ such that every consecutive pair of vertices in the sequence is connected by an edge. The length of a path is the number of edges traversed along the path. Denote by $L[i,j]$ the length of the longest path between vertices $i$ and $j$, $i < j$, in a random directed graph. The value $L[1,n]$ of a graph with $n$ vertices is also called the height of the graph. In the example with food webs, the maximum path represents the longest food chain, while in parallel computation, if we assume that we have enough processors and each task takes one unit of time, the maximum path length is the total time needed for processing. In [1] and [11] it is shown that there exists a constant $C = C(p)$ such that

$$\lim_{n \to \infty} \frac{L[1,n]}{n} = C \quad \text{a.s.}$$

and also upper and lower bounds on $C$ are provided.

A version of the central limit theorem for $L[1,n]$ is studied in [7] and [10]. The first paper investigates the case that the edge probability $p$ is a function of $n$ such that $p$ tends to 0 slower than $(\log n)^{-1}$, while the second paper considers the case that the edge probability $p$ is constant. Both papers introduce skeleton points as a tool to find an asymptotic distribution of $L[1,n]$. In a natural extension of a random directed graph on vertex set $\mathbb{Z}$, where any two vertices are connected with probability $p$, independently of the other edges, a vertex $i$ is a **skeleton point** if for all $j$, $j < i$, there exists a path from vertex $j$ to vertex $i$ and for all $j$, $j > i$, there exists a path from vertex $i$ to vertex $j$. Alon et al. [2] showed that the probability of vertex $i$ being a skeleton point is given by

$$\lambda = \prod_{k=0}^{\infty} (1 - (1-p)^k)^2.$$ 

They further prove that the sequence of the skeleton points in a directed random graph forms a stationary renewal process with the distances between two successive skeleton points having all moments finite. Denote the skeleton points using the following convention:

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \ldots$$
and associate with it a counting process $(\Phi(n) : n \in \mathbb{Z})$ with the agreement $\Phi(0) = 0$, i.e. $\Phi(n)$ is the index of the last skeleton point before vertex $n$. An important property of the skeleton points is that if $\gamma$ is a skeleton point, and $\gamma \leq n$, then a path with length $L[1, n]$ (a maximum length path) must necessarily contain $\gamma$. Therefore, we can write

$$L[1, n] = L[1, \Gamma_1] + \sum_{i=2}^{\Phi(n)} L[\Gamma_{i-1}, \Gamma_i] + L[\Gamma_{\Phi(n)}, n].$$

The first and last term of the right hand side above are negligible when divided by $\sqrt{n}$ and the middle term is a sum of independent and identically distributed random variables. Thus, the middle term carries the weak limit of $L[1, n]$ [10, Thm. 2]

$$\frac{L[1, n] - Cn}{\lambda \sigma \sqrt{n}} \xrightarrow{d} N(0, 1),$$

where $\sigma^2 = \text{var}(L[\Gamma_1, \Gamma_2] - C(\Gamma_2 - \Gamma_1))$.

4 Summary of Paper I

In Paper I we consider an extension of random directed graphs on $\mathbb{Z}^2$ with vertices ordered according to the product order $\prec$ defined as follows: $((i_1, j_1), (i_2, j_2))$ if the two pairs are distinct and $i_1 \leq i_2$ and $j_1 \leq j_2$. In this setting, a random directed graph with vertex set $\{(i, j) : i \in \{0, 1, \ldots, n\}, j \in \{1, 2, \ldots, m\}\}$ has edges between two comparable vertices present with probability $p$, independently of the other edges, and directed according to the product order. We say that $((i_0, j_0), (i_1, j_1), \ldots, (i_\ell, j_\ell))$, with $(i_0, j_0) \prec (i_1, j_1) \prec \cdots \prec (i_\ell, j_\ell)$, is a path of length $\ell$ if all pairs of consecutive vertices in the sequence are connected via an edge. The random variable of our interest in this graph is, as previously, the maximum length of all paths of the graph, denoted here by $L_{n,m}$. Denisov et al. [10] found a version of a limit theorem for $L_{n,m}$ when $n$ tends to infinity, but $m$ is constant. In Paper I we prove a limit theorem for $L_{n,m}$ when both $m$ and $n$ tend to infinity, so that $m \sim n^a$. Using a slightly modified definition of the skeleton points of a random graph with vertex set $\mathbb{Z}$, we can find upper and lower bounds for $L_{n,m}$ defined in terms of the skeleton points. Further, we make a sequence of transformations in order to establish an estimate $S_{n,m}$ of $L_{n,m} - Cn$ which resembles a last-passage percolation model, that is,

$$\frac{1}{\sigma} S_{n,m} = \sup_{0 = t_0 \prec \cdots \prec t_m = t} \sum_{j=1}^{m} \sum_{k=\Phi^{(j)}(t_j)}^{\Phi^{(j)}(t_{j-1})+1} \chi_k^{(j)},$$

where

$$\chi_k^{(j)} := \frac{1}{\sigma} \left\{ L^{(j)}[\Gamma_k^{(j)}, \Gamma_{k-1}^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}) \right\}.$$ 

The random variables $(\chi_k^{(j)}, j \geq 1, k \geq 2)$ are independent and identically distributed with mean 0 and variance 1.

Similarly as in [8], we couple $\frac{1}{\sigma} S_{n,m}$ with a last-passage time $Z_{n,m}$ and show that, if properly rescaled, they have the same limit distribution. Thus, the asymptotic distribution
of $S_{n,m}$, as well as the the asymptotic distribution of $L_{n,m}$, appropriately rescaled, is the Tracy-Widom distribution when $m/n^a \to 1$ as $n \to \infty$ and $a < 3/14$.

The condition for the constant $a$, $a < 3/14$, follows from the coupling and from a convergence rate of the random indices that appear in the expression for $S_{n,m}$. Intuitively, when $a < 3/14$ and $m = \lfloor n^a \rfloor$, we have a graph on long and thin rectangle and the path with maximum length will consist of long subpaths with vertices of constant second coordinate. Each of these long subpaths goes through enough skeleton points on a line, so that the length of a subpath on each line has approximately a normal distribution, with parameters as in (1).

5  Possible extensions

As mentioned in Section 3, Bollobás and Brightwell [7] studied a central limit theorem for the maximal length path of random directed graphs when the edge probability $p$ is a decreasing function of the number of vertices in the graph. They investigated the rate of convergence to 0 for $p$ as the number of vertices tends to infinity, so that there are enough skeleton points in the graph. One could similarly explore which convergence rate for $p$ is needed in order to have enough skeleton points on every line of a random directed graph on the two-dimensional integer lattice to be able to use coupling with Brownian motion as in Paper I. Then it might be possible to show the convergence of the longest path to the Tracy-Widom distribution.

A result for the maximum length of all paths in an $n \times n$ square, or at least for a bigger constant $a$ in an $n \times n^a$ rectangle, would be more interesting, but using the coupling technique in Paper I is not possible in this case. The first step towards a generalization of the result from Paper I could be to study the behaviour of the function

$$g_L(x, y) = \lim_{n \to \infty} \frac{1}{n} L_{\lfloor nx \rfloor, \lfloor ny \rfloor}$$

for any $(x, y) \in \mathbb{R}^2_+$. That is, one can try to show the law of large numbers for any direction in the positive half-plane.

In the case of a last-passage percolation model, the last-passage time has the superadditivity property: if $(i_1, j_1) \prec (i_2, j_2) \prec (i_3, j_3)$ then

$$T_{(i_1, j_1)} + T_{(i_2, j_2)} \leq T_{(i_3, j_3)}.$$ 

One of the results following from this property and Kingman’s ergodic theorem is that the set

$$B = \{(x, y) \in \mathbb{R}^2_+ : g_T(x, y) \leq 1\}$$

is concave. Denote the last-passage shape by $B(t) = \{(i, j) \in \mathbb{N}^2_0 : T(i, j) \leq t\}$. Martin [15] showed that if the weights in a last-passage percolation model satisfy a certain condition then, for any $\varepsilon > 0$,

$$(1 - \varepsilon)B \subseteq \frac{B(t)}{t} \subseteq (1 + \varepsilon)B,$$
for all sufficiently large $t$, with probability 1. In other words, the asymptotic shape of $B(t)$ is $B$.

The longest path between two skeleton points satisfies the condition of the theorem above. Therefore, an interesting question is if it is possible to prove an analogous shape theorem for a directed random graph. A simulation of last-passage shapes of a random directed graph can be found in Figure 1.

![Figure 1: Simulation of a random directed graph with $p = 1/2$. The sets $B(t)$ are shown for $t = 100$ (darkest), 300, 500 (lightest).]
References

[1] Albert, M. and Frieze, A. (1989) Random graph orders. *Order* **6**, 19–30.

[2] Alon, N., Bollobás, B., Brightwell, G., and Janson, S. (1994) Linear extensions of a random partial order. *Ann. Probab.* **4**, 108–123.

[3] Anderson G. W., Guionnet, A. and Zeitouni, O. (2010) *An Introduction to Random Matrices*. Cambridge Univ. Press, Cambridge

[4] Baik, J. and Suidan, T. M. (2005) A GUE central limit theorem and universality of directed first and last passage site percolation. *Int. Math. Res. Not.* **6**, 325–337.

[5] Barak, A. B. and Erdős, P. (1984) On the maximal number of strongly independent vertices in a random acyclic directed graph. *SIAM J. Algebr. Discr. Methods* **5**, 508–514.

[6] Baryshnikov, Y. (2001). GUEs and queues. *Probab. Theory Related Fields* **119**, 256–274.

[7] Bollobás, B. and Brightwell, G. (1997) The structure of random graph orders. *Siam J. Discrete Math.* **10**, 318–335.

[8] Bodineau, T. and Martin, J. (2005). A universality property for last-passage percolation paths close to the axis. *Electron. Commun. Probab.* **10**, 105–112.

[9] Cohen, J. E. and Newman, C. M. (1991) Community area and food chain length: theoretical predictions. *Amer. Naturalist* **138**, 1542–1554.

[10] Denisov, D., Foss, S. and Konstantopoulos, T. (2012). Limit theorems for a random directed slab graph. *Ann. Appl. Probab.* **22**, 702–733.

[11] Foss, S. and Konstantopoulos, T. (2003) Extended renovation theory and limit theorems for stochastic ordered graphs. *Markov Process. Related Fields* **9**, 413–468.

[12] Glynn, P. W. and Whitt, W. (1991) Departures from many queues in series. *Ann. Appl. Probab.* **1**, 546–572.

[13] Isopi, M. and Newman, C. M. (1994) Speed of parallel processing for random task graphs. *Comm. Pure and Appl. Math* **47**, 361–376

[14] Johansson, K. (2000) Shape fluctuations and random matrices. *Comm. Math. Phys.* **209**, 437–476.

[15] Martin, J. B. (2004) Limiting shape for directed percolation models. *Ann. Probab.* **32**, 2908–2937.

[16] Newman, C. M. (1992) Chain lengths in certain random directed graphs. *Random Structures Algorithms* **3**, 243–253.

[17] Simon, K., Crippa, D. and Collenberg, F. (1993) On the distribution of the transitive closure in a random acyclic digraph. *Lecture Notes in Comput. Sci.* **726**, 345–356.

[18] Tracy, C. A. and Widom, H. (1994) Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159**, 151–174.

[19] Tracy, C. A. and Widom, H. (2002) Distribution functions for largest eigenvalues and their applications. *Proceedings of the International Congress of Mathematicians* **1** (Beijing, 2002), 587–596.
Convergence to the Tracy-Widom distribution for longest paths in a directed random graph

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Abstract. We consider a directed graph on the 2-dimensional integer lattice, placing a directed edge from vertex \((i_1, i_2)\) to \((j_1, j_2)\), whenever \(i_1 \leq j_1, i_2 \leq j_2\), with probability \(p\), independently for each such pair of vertices. Let \(L_{n,m}\) denote the maximum length of all paths contained in an \(n \times m\) rectangle. We show that there is a positive exponent \(a\), such that, if \(m/n^a \to 1\), as \(n \to \infty\), then a properly centered/rescaled version of \(L_{n,m}\) converges weakly to the Tracy-Widom distribution. A generalization to graphs with non-constant probabilities is also discussed.

1. Introduction

Random directed graphs form a class of stochastic models with applications in computer science (Isopi and Newman, 1994), biology (Cohen and Newman, 1991; Newman, 1992; Newman and Cohen, 1986) and physics (Itoh and Krapivsky, 2012). Perhaps the simplest of all such graphs is a directed version of the standard Erdős-Rényi random graph (Barak and Erdős, 1984) on \(n\) vertices, defined as follows: For each pair \(\{i, j\}\) of distinct positive integers less than \(n\), toss a coin with probability of heads equal to \(p\), \(0 < p < 1\), independently from pair to pair; if heads shows up then introduce an edge directed from \(\min(i, j)\) to \(\max(i, j)\). There is a natural extension of this graph to the whole of \(\mathbb{Z}\) studied in detail in Foss and Konstantopoulos (2003). In particular, if we define the asymptotic growth rate \(C = C(p)\), as the a.s. limit of the maximum length of all paths between 1 and \(n\) divided by \(n\), Foss and Konstantopoulos (2003) provide sharp bounds on \(C(p)\) for all values of \(p \in (0, 1)\).

A natural generalization arises when we replace the total order of the vertex set by a partial order, usually implied by the structure of the vertex set. In such
a model, coins are tossed only for pairs of vertices which are comparable in this partial order. The canonical case is to consider, as a vertex set, the 2-dimensional integer lattice \( \mathbb{Z} \times \mathbb{Z} \), equipped with the standard component-wise partial order: \((i_1, i_2) \prec (j_1, j_2)\) if the two pairs are distinct and \(i_1 \leq i_2, j_1 \leq j_2\). Such a graph was considered in Denisov et al. (2012). In that paper, it was shown that if \( L_{n,m} \) denotes the maximum length of all paths of the graph, restricted to \( \{0, \ldots, n\} \times \{1, \ldots, m\} \), then there is a positive \( \kappa \) (depending on \( p \) and the fixed integer \( m \)), such that

\[
\left( \frac{L_{[nt,m]} - Cnt}{\kappa \sqrt{nt}}, t \geq 0 \right) \xrightarrow{\text{d}} \left( Z_{t,m} - \frac{2}{\sqrt{m}}, t \geq 0 \right) \quad \text{as} \quad \sqrt{m} \to \infty.
\]

where \( Z_{t,m} \) is the stochastic process defined in terms of \( m \) independent standard Brownian motions, \( B^{(1)}, \ldots, B^{(m)} \), via the formula

\[
Z_{t,m} := \sup_{0 = t_0 < t_1 \cdots < t_{m-1} < t_m = t} \left\{ \sum_{j=1}^{m} \left[ B_{t_j}^{(i)} - B_{t_{j-1}}^{(i)} \right] : 1 \leq i \leq m \right\}, \quad t \geq 0.
\]

One can speak of \( Z \) as a Brownian directed percolation model, the terminology stemming from the picture of a “weighted graph” on \( \mathbb{R} \times \{1, \ldots, m\} \) where the weight of a segment \([s, t] \times \{j\}\) equals the change \( B_{t_j}^{(i)} - B_{s_j}^{(i)}\) of a Brownian motion. If a path from \((0,0)\) to \((t,m)\) is defined as a union \( \bigcup_{j=1}^{m} [t_{j-1}, t_j] \times \{j\} \) of such segments, then \( Z \) represents the maximum weight of all such paths.

Baryshnikov (2001), answering an open question by Glynn and Whitt (1991), showed that

\[
Z_{1,m} \xrightarrow{\text{d}} \lambda_m,
\]

where \( \lambda_m \) is the largest eigenvalue of a GUE matrix of dimension \( m \). Since \( Z_{*,m} \) is 1/2-self-similar, we see that

\[
Z_{t,m} \xrightarrow{\text{d}} \sqrt{t} \lambda_m.
\]

Now, fluctuations of \( \lambda_m \) around the centering sequence \( 2\sqrt{m} \) have been quantified by Tracy and Widom (1994) who showed the existence of a limiting law, denoted by \( F_{\text{TW}} \):

\[
m^{1/6} \left( \lambda_m - 2\sqrt{m} \right) \xrightarrow{\text{d}} F_{\text{TW}}.
\]

A natural question then, raised in Denisov et al. (2012), is whether one can obtain \( F_{\text{TW}} \) as a weak limit of \( L_{n,m} \) when \( n \) and \( m \) tend to infinity simultaneously. Our paper is concerned with resolving this question. To see what scaling we can expect, rewrite the last display, for arbitrary \( t > 0 \), as

\[
m^{1/6} \left( \frac{Z_{t,m}}{\sqrt{t}} - 2\sqrt{m} \right) \xrightarrow{\text{d}} F_{\text{TW}}.
\]

A statement of the form \( X(t,m) \xrightarrow{\text{d}} X \), where the distribution of \( X(t,m) \) does not depend on the choice of \( t > 0 \), implies the statement \( X(t,m(t)) \xrightarrow{\text{d}} X \), for any function \( m(t) \) such that \( m(t) \to \infty \). Hence, upon setting \( m = \lceil t^a \rceil \), we have

\[
l^{a/6} \left( \frac{Z_{t,[t^a]}}{\sqrt{t}} - 2\sqrt{t^a} \right) \xrightarrow{\text{d}} F_{\text{TW}},
\]

(1.2)
Therefore, it is reasonable to guess that, when \( a \) is small enough, an analogous limit theorem holds for a centered scaled version of the largest length \( L_{n,\lfloor an \rfloor} \), namely that

\[
\frac{n^{a/6}}{c_2 \sqrt{n}} \left( \frac{L_{n,\lfloor an \rfloor}}{c_1 n} - 2 \sqrt{n^2} \right) \xrightarrow{d} F_{\text{TW}},
\]

where \( c_1, c_2 \) are appropriate constants.

A stochastic model, bearing some resemblance to ours, is the so-called directed last passage percolation model on \( \mathbb{Z}^d \) (the case \( d = 2 \) being of interest here). We are given a collection of i.i.d. random variables indexed by elements of \( \mathbb{Z}^d_+ \). A path from the origin to the point \( n \in \mathbb{Z}^d_+ \) is a sequence of elements of \( \mathbb{Z}^d_+ \), starting from the origin and ending at \( n \), such that the difference of successive members of the sequence is equal to the unit vector in the \( i \)th direction, for some \( 1 \leq i \leq d \). The weight of a path is the sum of the random variables associated with its members. Specializing to \( d = 2 \), let \( L_{n,m} \) be the largest weight of all paths from \((0,0)\) to \((n,m)\). Assuming that the random variables have a finite moment of order larger than 2, Bodineau and Martin (2005) showed that (1.3) holds for all sufficiently small positive \( a \) (the threshold depending on the order of the finite moment). Independently, Baik and Suidan (2005) obtained the same result for random variables with a finite 4th moment and for \( a < 3/14 \). In both papers, partial sums of i.i.d. random variables were approximated with Brownian motions, in the first case using the Komlós-Major-Tusnády (KMT) construction, while in the second using Skorokhod embedding.

To show that (1.3) holds for our model, we adopt the technique introduced in Denisov et al. (2012), which involves the existence of skeleton points on each line \( \mathbb{Z} \times \{j\} \). Skeleton points are, by definition, random points which are connected with all the other points on the same line. In Denisov et al. (2012) Denisov, Foss and Konstantopoulos used this fact, together with the fact that, for finite \( m \), one can pick skeleton points common to all \( m \) lines, in order to prove (1.1). However, when \( m \) tends to infinity simultaneously with \( n \), it is not possible to pick skeleton points common to all lines. Modifying the definition of skeleton points enables us to give a new proof of (1.1), as well as to prove (1.3). To achieve the latter, we borrow the idea of KMT coupling from Bodineau and Martin (2005). However, we need to do some work in order to express the random variable \( L_{n,m} \) in a way that resembles a maximum of partial sums.

Although we focus on the case where the edge probability \( p \) is constant, it is possible to consider a more general case, where the probability that a vertex \((i_1, i_2) \in \mathbb{Z} \times \mathbb{Z} \) connects to a vertex \((j_1, j_2) \) depends on the distances \(|j_1 - i_1|\) and \(|j_2 - i_2|\) of the two vertices. This generalization is discussed in the last section of the article.

2. The one-dimensional directed random graph

We summarize below some properties of the directed Erdős-Rényi graph on \( \mathbb{Z} \) with connectivity probability \( p \) taken from Foss and Konstantopoulos (2003). For \( i < j \), let \( L[i, j] \) be the maximum length of all paths with start and end points in the interval \([i, j]\). Then, for \( i < j < k \), we have \( L[i, k] \leq L[i, j] + L[j, k] + 1 \). Since the distribution of the random graph is invariant under translations, and is also ergodic (the natural invariant \( \sigma \)-field is trivial), it follows, from Kingman’s subadditive ergodic theorem, that there is a deterministic constant \( C = C(p) \) such
that
\[
\lim_{n \to \infty} \frac{L[1, n]}{n} = C, \text{ a.s.} \tag{2.1}
\]
In fact, \( C = \inf_{n \geq 1} E L[1, n]/n \). The function \( C(p) \) is not known explicitly; only bounds are known (Foss and Konstantopoulos, 2003, Thm. 10.1). For example, \( 0.5679 \leq C(1/2) \leq 0.5961 \). We also know that there exists, almost surely, a random integer sequence \( \{\Gamma_r, r \in \mathbb{Z}\} \) with the property that for all \( r \), all \( i < \Gamma_r \), and all \( j > \Gamma_r \), there is a path from \( i \) to \( \Gamma_r \) and a path from \( \Gamma_r \) to \( j \). The existence of such points, referred to as skeleton points, is not hard to establish (Denisov et al., 2012). Since the directed Erdős-Rényi graph is invariant under translations, so is the random graph, such that \( \{\Gamma_{\nu r}, r \in \mathbb{Z}\} \) is a stationary point process then
\[
C = E L[\Gamma_{\nu r}, \Gamma_{\nu r+1}]
\]
\[
E(\Gamma_{\nu r+1} - \Gamma_{\nu r}).
\]

Foss and Konstantopoulos, 2012, (2). If we let \( p \) for example, for \( \nu_r \), \( r \in \mathbb{Z} \) such points, referred to as skeleton points, are known (\( C \)). In fact, if \( \{\Gamma_{\nu r}, r \in \mathbb{Z}\} \) has the same law as \( \{n + \Gamma_r, r \in \mathbb{Z}\} \), for all \( n \in \mathbb{Z} \). Moreover, it turns out that the sequence forms a stationary renewal process. If we enumerate the skeleton points according to \( \{\Gamma_{\nu r}, r \in \mathbb{Z}\} \), whereas \( \{\Gamma_{\nu r+1} - \Gamma_{\nu r}, r \neq 0\} \) are i.i.d. Stationarity implies that the law of the omitted difference \( \Gamma_1 - \Gamma_0 \) has a density which is proportional to the tail of the distribution of \( \Gamma_2 - \Gamma_1 \). In Denisov et al. (2012) it is shown that the distance \( \Gamma_2 - \Gamma_1 \) between two successive skeleton points has a finite 2nd moment. One can follow the same steps of the proof, to show that in our case, with constant probability \( p \), this random variable has moments of all orders. Moreover, one can show that for some \( \alpha > 0 \) (the maximal such \( \alpha \) depends on \( p \)) it holds that \( E e^{\alpha(\Gamma_2 - \Gamma_1)} < \infty \).

The rate \( \lambda_0 \) of the sequence of skeleton points can be expressed as an infinite product:
\[
\lambda_0 := \frac{1}{E(\Gamma_2 - \Gamma_1)} = \prod_{k=1}^{\infty} (1 - (1 - p)^k)^2. \tag{2.2}
\]
For example, for \( p = 1/2 \), \( \lambda_0 \approx 1/12 \).

A central limit theorem for \( L[1, n] \) is also available (Denisov et al., 2012, Thm. 2). If we let
\[
\sigma_0^2 := \text{var}(L[\Gamma_1, \Gamma_2] - C(\Gamma_2 - \Gamma_1)), \tag{2.3}
\]
then
\[
\frac{L[1, n] - Cn}{\sqrt{\lambda_0 \sigma_0^2 n}} \xrightarrow{(d)} N(0, 1), \tag{2.4}
\]
where \( N(0, 1) \) is a standard normal random variable. Note that \( \sigma_0^2 \neq \text{var}(L[\Gamma_1, \Gamma_2]) \). Unfortunately, we have no estimates for \( \sigma_0^2 \), but, interestingly, there is a technique for estimating it, based on perfect simulation. This was briefly explained in Foss and Konstantopoulos (2003) in connection with an infinite-dimensional Markov chain which carries most of the information about the law of the directed Erdős-Rényi random graph.

In addition, it is shown in Foss and Konstantopoulos (2003) that \( C \) can also be expressed as
\[
C = \frac{E L[\Gamma_1, \Gamma_2]}{E(\Gamma_2 - \Gamma_1)}. \tag{2.5}
\]
In fact, if \( \{\nu_r, r \in \mathbb{Z}\} \) is a random sequence of integers, defined on the same probability space as the one supporting the random graph, such that \( \{\Gamma_{\nu_r}, r \in \mathbb{Z}\} \) is a stationary point process then
\[
C = \frac{E L[\Gamma_{\nu_r}, \Gamma_{\nu_r+1}]}{E(\Gamma_{\nu_r+1} - \Gamma_{\nu_r})}.
\]
The most important property of the skeleton points is that if \( \gamma \) is a skeleton point, and if \( i \leq \gamma \leq j \), then a path with length \( L[i,j] \) (a maximum length path) must necessarily contain \( \gamma \). This crucial property will be used several times below, especially since, for every \( i < j \), the following equality holds

\[
L[\Gamma_i, \Gamma_j] = L[\Gamma_i, \Gamma_{i+1}] + L[\Gamma_{i+1}, \Gamma_{i+2}] + \cdots + L[\Gamma_{j-1}, \Gamma_j].
\]

Furthermore, the restriction of the graph on the interval between two successive skeleton points is independent of the restriction on the complement of the interval; hence the summands in the right-hand side of the last display are independent random variables.

3. Statement of the main result

It is clear from (2.4) that the constants \( c_1, c_2 \) in (1.3) should be as follows: \( c_1 = C \), \( c_2 = \sqrt{\lambda_0 \sigma^2_0} \). Now we can formulate the main result.

**Theorem 3.1.** Let \( C, \lambda_0, \sigma^2_0 \) be the quantities associated with the directed random graph on \( \mathbb{Z} \) with connectivity probability \( p \), defined by (2.1) (equivalently, (2.5)), (2.2), (2.3), respectively. Consider the directed random graph on \( \mathbb{Z} \times \mathbb{Z} \) and let \( L_{n,m} \) be the maximum length of all paths between two vertices in \([0,n] \times [1,m]\). Then, for all \( 0 < a < 3/14 \),

\[
\frac{n^{a/6} \left( \frac{L_{n,n^a}}{\sqrt{\lambda_0 \sigma^2_0 \sqrt{n}}} - 2 \sqrt{\frac{n^2}{\lambda_0 \sigma^2_0}} \right)}{\sqrt{n}} \xrightarrow{d} \mathcal{N} \rightarrow F_{TW},
\]

where \( F_{TW} \) is the Tracy-Widom distribution.

To prove this theorem, we will first define the notion of skeleton points for the graph on \( \mathbb{Z} \times \mathbb{Z} \) and then prove pathwise upper and lower bounds for \( L_{n,m} \) which depend on paths going through these skeleton points. This will be done in Section 4. In Section 5.1 we show that the difference between these bounds is of the order \( o(n^b) \), where \( b = (1/2) - (a/6) \) is the net exponent in the denominator of (3.1). We will then (Section 5.2) introduce a quantity \( S_{n,m} \) which resembles a last passage percolation problem and show that it differs from \( L_{n,m} \) by a quantity which is of the order \( o(n^b) \), when \( m = [n^a] \). The problem will then be translated to a last passage percolation problem (with the exception of random indices). This will finally, in Section 5.3 be compared to the Brownian directed percolation problem by means of strong coupling.

4. Skeleton points and pathwise bounds

Our model is a directed random graph \( G \) with vertices \( \mathbb{Z} \times \mathbb{Z} \). For each pair of vertices \( i, j \), such that \( i \prec j \), toss an independent coin with probability of heads equal to \( p \); if heads shows up introduce an edge directed from \( i \) to \( j \).

A path of length \( \ell \) in the graph is a sequence \((i_0, i_1, \ldots, i_{\ell})\) of vertices \( i_0 \prec i_1 \prec \cdots \prec i_{\ell} \) such that there is an edge between any consecutive vertices.

We denote by \( G_{n,m} \) the restriction of \( G \) on the set of vertices \([1,\ldots,m] \times [1,\ldots,m]\). The random variable of interest is

\[
L_{n,m} := \text{the maximum length of all paths in } G_{n,m}.
\]

We refer to the set \( Z \times \{j\} \) as “line \( j \)” or “\( j \)th line”, and note that the restriction of \( G \) onto \( Z \times \{j\} \) is a directed Erdős-Rényi random graph. We denote this restriction
by $G^{(j)}$. Typically, a superscript $(j)$ will refer to a quantity associated with this restriction. For example, for $a \leq b$,

$$L^{(j)}[a, b] := \text{the maximum length of all paths in } G^{(j)}$$

with vertices between $(a, j)$ and $(b, j)$ and we agree that $L^{(j)}[a, b] = 0$ if $a \geq b$.

Clearly, the $\{G^{(j)}, j \in \mathbb{Z}\}$ are i.i.d. random graphs, identical in distribution to the directed Erdős-Rényi random graph. Therefore, for each $j \in \mathbb{Z}$,

$$\lim_{n \to \infty} L^{(j)}[1, n]/n = C, \text{ a.s.}$$

To establish upper and lower bounds for $L_{n,m}$, we need to slightly change the definition of a skeleton point in $G$.

**Definition 4.1** (Skeleton points in $G$). A vertex $(i, j)$ of the directed random graph $G$ is called skeleton point if it is a skeleton point for $G^{(j)}$ (for any $i' < i < i''$, there is a path from $(i', j)$ to $(i, j)$ and a path from $(i, j)$ to $(i'', j)$) and if there is an edge from $(i, j)$ to $(i, j+1)$.

Therefore, the skeleton points on line $j$ are obtained from the skeleton point sequence of the directed Erdős-Rényi random graph $G^{(j)}$ by independent thinning with probability $p$. When we refer to skeleton points on line $j$, we shall be speaking of this thinned sequence. The elements of this sequence are denoted by

$$\cdots < \Gamma^{(j)}_{-1} < \Gamma^{(j)}_0 \leq 0 < \Gamma^{(j)}_1 < \Gamma^{(j)}_2 < \cdots$$

and have rate

$$\lambda = \frac{1}{E(\Gamma^{(j)}_2 - \Gamma^{(j)}_1)} = p\lambda_0 = p \prod_{k=1}^{\infty} (1 - (1 - p)^k)^2.$$  

The associated counting process of skeleton points on line $j$ is defined by

$$\Phi^{(j)}(t) - \Phi^{(j)}(s) = \sum_{r \in \mathbb{Z}} 1(s < \Gamma^{(j)}_r \leq t), \quad s, t \in \mathbb{R}, \quad s \leq t,$$

together with the agreement that

$$\Phi^{(j)}(0) = 0.$$  

Note that we insist on having the parameter $t$ in $\Phi^{(j)}(t)$ as an element of $\mathbb{R}$ (and not just $\mathbb{Z}$). We also let

$$X^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)},$$

$$Y^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)+1},$$

be the skeleton points on line $j$ straddling $t$:

$$X^{(j)}(t) \leq t < Y^{(j)}(t). \quad (4.1)$$

Next we prove upper and lower bounds for $L_{n,m}$. The set of dissections of the interval $[0, n] \subset \mathbb{R}$ in $m$ non-overlapping, possibly empty intervals is denoted by

$$\mathcal{T}_{n,m} := \{ t = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : 0 = t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m = n \}.$$
Lemma 4.2. (Upper bound) Define
\[ L_{n,m} := \sup_{t \in T_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_{j})] + m. \]  \hspace{1cm} (4.2)
Then \( L_{n,m} \leq \bar{L}_{n,m} \).

Proof: Let \( \pi \) be a path in \( G_{n,m} \). Consider the lines visited by \( \pi \), denoting their indices by \( 1 \leq \nu_1 < \nu_2 < \cdots < \nu_J \leq m \). Let \((a_j, \nu_j)\) and \((b_j, \nu_j)\) be the first and the last vertex of line \( \nu_j \) in the path \( \pi \). Then the length of \( \pi \) satisfies
\[ |\pi| \leq \sum_{j=1}^{J} L^{(\nu_j)}[a_j, b_j] + J - 1. \]
Since successive vertices in the path should be increasing in the order \( \prec \), we have \( b_{j-1} \leq a_j, 2 \leq j \leq J \). Hence, with \( b_0 := 0 \),
\[ |\pi| \leq \sum_{j=1}^{J} L^{(\nu_j)}[b_{j-1}, b_j] + J - 1 \leq \sum_{j=1}^{J} L^{(\nu_j)}[X^{(\nu_j)}(b_{j-1}), Y^{(\nu_j)}(b_j)] + J - 1, \]
where we used (4.1). Since \( J \leq m \), we can extend \( 0 = b_0 \leq b_1 \leq \cdots \leq b_J \leq n \) to a dissection of \([0, n]\) into \( m \) non-overlapping intervals, showing that the right-hand side of the last display is bounded above by \( \bar{L}_{n,m} \). Taking the maximum over all \( \pi \) in \( G_{n,m} \), we obtain \( L_{n,m} \leq \bar{L}_{n,m} \), as required. \( \square \)

Note that the existence and properties of skeleton points were not used in the proof of the upper bound, other than to ensure that the upper bound is a.s. finite.

Lemma 4.3. (Lower bound) Define
\[ \Delta_n^{(j)} := \max_{0 \leq i \leq \Phi^{(j)}(n)} (\Gamma_{i+1}^{(j)} - \Gamma_{i}^{(j)}), \]
and
\[ L_{n,m} := \sup_{t \in T_{n,m}} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_{j})] - \sum_{j=1}^{m} \Delta_n^{(j)}. \]
Then \( L_{n,m} \geq \bar{L}_{n,m} \).

Proof: We will show that, for all \( t = (t_0, \ldots, t_n) \in T_{n,m} \), there is a path \( \pi \) in \( G_{n,m} \) with length \(|\pi|\) satisfying
\[ \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_{j})] \leq |\pi| + \sum_{j=1}^{m} \Delta_n^{(j)}. \]  \hspace{1cm} (4.3)
Fix \( t \in T_{n,m} \) and use the notation
\[ I_j = [Y^{(j)}(t_{j-1}), X^{(j)}(t_{j})], \quad [a_j, b_j], \quad j = 1, \ldots, m. \]
Note that \( a_j \geq b_j \) if there is one or no skeleton points on the segment \((t_{j-1}, t_j) \times \{j\}\) and then \( L^{(j)}(I_j) = 0 \).

Given two skeleton points \((x, i)\), \((y, j)\) we say that there is a staircase path from \((x, i)\) to \((y, j)\) if there is a sequence of skeleton points
\[ (x, i) = (x_0, i), \quad (x_1, i+1), \ldots, \quad (x_{j-i}, j) = (y, j), \]
such that \( x = x_0 \leq x_1 \leq \cdots \leq x_{j-i} = y \). See Figure 4.1. Clearly then, there is a path from \((x, i)\) to \((y, j)\) which jumps upwards by one step each time it meets a new skeleton point from the sequence. We denote this by \((x, i) \xrightarrow{\omega} (y, j)\).

Among all the staircase paths from \((x, i)\) to \((y, j)\), we will consider the best one, defined by two properties:

- Property 1: A best path from \((x, i)\) to \((y, j)\) jumps from line \(k\) to line \(k+1\), \(k = i, i+1, \ldots, j-1\), at the first next skeleton point on line \(k\), i.e. at the points \(x_0\) and \(x_{k-i+1} = Y^{(k+1)}(x_{k-i})\), \(k = i, \ldots, j-2\).
- Property 2: Every horizontal segment of a best path is a path of maximal length.

If all the intervals \(I_1, \ldots, I_m\) are empty, the left-hand side of (4.3) is zero and the inequality is trivially satisfied for any path \(\pi\).

Otherwise, for a fixed \(t \in G_{n,m}\) we will construct a path \(\pi\) in \(G_{n,m}\) for which (4.3) holds. Define a subsequence \(\nu_1 < \nu_2 < \cdots\) of \(1, \ldots, m\), inductively, as follows:

\[
\nu_1 := \inf\{1 \leq j \leq m : I_j \neq \emptyset\},
\nu_r := \inf\{j > \nu_{r-1} : (b_{\nu_{r-1}}, \nu_{r-1}) \xrightarrow{\omega} (b_j, j)\}, \quad r \geq 2.
\]

See Figure 4.2 for an illustration. The procedure stops if one of the elements of the subsequence exceeds \(m\) or if the condition inside the infimum is not satisfied by a path in \(G_{n,m}\).

Let \(J\) be the last index in the above defined sequence. Let \(\pi_1\) be a path of maximum length from \((a_{\nu_1}, \nu_1)\) to \((b_{\nu_1}, \nu_1)\) and define, for \(r = 2, 3, \ldots, J\), a path \(\pi_r\) as a best staircase path from \((b_{\nu_{r-1}}, \nu_{r-1})\) to \((b_{\nu_r}, \nu_r)\). Note that, for each \(r = 2, 3, \ldots, J, J \geq 2\), the end vertex of \(\pi_{r-1}\) is the start vertex of \(\pi_r\). Therefore we can concatenate the paths \(\pi_1, \ldots, \pi_J\) to obtain a path \(\pi\). This path starts from \((a_{\nu_1}, \nu_1)\) and ends at \((b_{\nu_J}, \nu_J)\).

Let \(\pi^{(j)}\) := the restriction of path \(\pi\) on line \(j\) and \(|\pi^{(j)}|\) its length on line \(j\). Also, for \(j \geq \nu_1\) denote \((c_j, j) := the first vertex on line \(j\) of path \(\pi\).
Illustration of the procedure defined by (4.4)-(4.5). There are four best staircase paths: the path from \((b_2, 2)\) to \((b_3, 3)\), the path from \((b_3, 3)\) to \((b_4, 4)\), the path from \((b_4, 4)\) to \((b_5, 5)\), and the path from \((b_5, 5)\) to \((b_8, 8)\). Observe that \(I_b = (a_j, b_j)\), in the figure, are nonempty only for \(j = 2, 5, 7\) and \(8\) (these are the highlighted intervals), but \(I_7\) is not visited by the constructed path. Moreover, \(I_8\) is only partly visited and the path enters \(I_8\) at a point \(c_8\) between \(b_8\).

Split the sum in the left-hand side of (4.3) along the elements of the subsequence \(\{\nu_1, \ldots, \nu_J\}\):

\[
\sum_{j=1}^m L^{(j)}(I_j) = \sum_{r=1}^{J+1} \sum_{j=\nu_r-1+1}^{\nu_r} L^{(j)}(I_j) =: \sum_{r=1}^{J+1} G_r,
\]

where we have conveniently set

\[
\nu_0 := 0, \; \nu_{J+1} := m,
\]

in order to take care of the first and last terms. By the definition of \(\nu_1\), the intervals \(I_1, I_2, \ldots, I_{\nu_1-1}\) are empty and

\[
G_1 = L^{(\nu_1)}(I_{\nu_1}) = |\pi^{(\nu_1)}|.
\]

Assume now that \(2 \leq r \leq J\), and write

\[
G_r = \sum_{j=\nu_{r-1}+1}^{\nu_r} L^{(j)}(I_j) = \sum_{j=\nu_{r-1}+1}^{\nu_r-1} L^{(j)}(I_j) + L^{(\nu_r)}(I_{\nu_r}).
\]

Since \(\pi_{\nu_r}\) is the path of maximal length from \((c_{\nu_r}, \nu_r)\) to its end-vertex \((b_{\nu_r}, \nu_r)\) (Property 2), if \(c_{\nu_r} < a_{\nu_r}\), then \(L^{(\nu_r)}(I_{\nu_r}) \leq |\pi^{(\nu_r)}|\). Define in this case \(I'_{\nu_r} = \emptyset\). Otherwise, we can write

\[
L^{(\nu_r)}(I_{\nu_r}) \leq L^{(\nu_r)}[a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})] + L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}].
\]

Then, again because of Property 2, \(L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}] < |\pi^{(\nu_r)}|\) and it is left to find a bound on the interval \(I'_{\nu_r} = [a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})]\). Recall that by Property
1, depending whether $j$ is a member of the sequence $\{\nu_r, r = 1, \ldots, J\}$ or not, $c_{j+1} = b_j$ or $c_{j+1} = Y^{(j)}(c_j)$, respectively. Also, because of $I_j \subseteq [t_{j-1}, t_j]$, we know that $L^{(j)}(I_j) \leq t_j - t_{j-1}$. Hence, if $\nu_r - \nu_{r-1} > 1$ it holds
\[
\sum_{j=\nu_{r-1}+1}^{\nu_r-1} L^{(j)}(I_j) + L^{(j)}(I'_{\nu_r}) \leq \sum_{j=\nu_{r-1}+1}^{\nu_r-1} (t_j - t_{j-1}) + (Y^{(\nu_r)}(c_{\nu_r}) - t_{\nu_r-1}) \leq Y^{(\nu_r)}(c_{\nu_r}) - b_{\nu_r-1} = \sum_{j=\nu_{r-1}+1}^{\nu_r} (Y^{(j)}(c_j) - c_j) \leq \sum_{j=\nu_{r-1}+1}^{\nu_r} \Delta_n^{(j)}.
\]
Combining the above, we obtain
\[
G_r \leq \sum_{j=\nu_r+1}^{\nu_r-1} \Delta_n^{(j)} + |\pi^{(\nu_r)}|.
\]
If $\nu_J = m$, then $G_{J+1} = 0$. Otherwise, we can extend the sequence $\{c_j, j = \nu_1, \nu_1+1, \ldots, \nu_J\}$ defining iteratively $c_{\nu_{j+1}} := b_{c_j}$ and $c_{j+1} := Y^{(j)}(c_j)$ until $c_j > n$ for some $j$. Let $K$ be the last index such that $c_K \leq n$. As there was not possible to construct the best staircase path after the line $\nu_j$, $K$ is at most $m$. Similarly as above, for $G_{J+1}$ it holds
\[
G_{J+1} = \sum_{j=\nu_{j+1}}^{m} L^{(j)}(I_j) \leq \sum_{j=\nu_{j+1}}^{m} (t_j - t_{j-1}) \leq n - b_{\nu_j} = \sum_{j=\nu_{j+1}}^{K} (Y^{(j)}(c_j) - c_j) \leq \sum_{j=\nu_{j+1}}^{K} \Delta_n^{(j)}.
\]
Finally, we obtain
\[
\sum_{j=1}^{m} L^{(j)}(I_j) \leq \sum_{j=1}^{m} \Delta_n^{(j)} + \sum_{r=1}^{J} |\pi^{(\nu_r)}| \leq \sum_{j=1}^{m} \Delta_n^{(j)} + |\pi|,
\]
as required. \qed

5. Further estimates in probability and Brownian directed percolation

In the present section we prove Theorem 3.1 as a sequence of lemmas.

5.1. Asymptotic coincidence of the two bounds. Looking at (1.3), we can see that the correct scaling requires exponent
\[
b := \frac{1}{2} - \frac{a}{6}
\]
in the denominator and condition $a < 3/7$, which is equivalent to $a < b$.

In the following two lemmas we will not specifically use the definition of $b$ and condition on $a$. Both lemmas hold for more general $a, b > 0, 0 < b - a < 1$.

Lemma 5.1. With $b = (1/2) - (a/6)$ and $a < 3/7$,
\[
\frac{\mathcal{T}_{n, \lfloor n^a \rfloor}}{n^b} - \frac{L_{n, \lfloor n^a \rfloor}}{n^b} \xrightarrow{(p)} 0, \quad n \to \infty.
\]
Proof: Let \( t \) be such that the maximum in the right-hand side of (4.2) is achieved. Then

\[
\mathcal{T}_{n,m} - L_{n,m} \leq m + \sum_{j=1}^{m} \Delta_{n}^{(j)}
\]

\[
+ \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_{j})] - \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_{j})]
\]

\[
\leq m + \sum_{j=1}^{m} \Delta_{n}^{(j)}
\]

\[
+ \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_{j-1})] + L^{(j)}[X^{(j)}(t_{j}), Y^{(j)}(t_{j})] \right\}
\]

\[
\leq m + \Delta_{n}^{(j)} + 2 \sum_{j=1}^{m} \max_{0 \leq i \leq \Phi_{n}(j)} L^{(j)}[r^{(j)}(i), r^{(j)}(i+1)]
\]

\[
\leq m + 3 \sum_{j=1}^{m} \Delta_{n}^{(j)}.
\]

Hence

\[
\frac{1}{n^{b}} E[\mathcal{T}_{n,[n^{c}]} - L_{n,[n^{c}]}] \leq \frac{n^{a}}{n^{b}} + 3 \frac{n^{a}}{n^{b}} E[\Delta_{n}^{(1)}].
\]

Since \( b > a \) and the random variables \( \{\Gamma_{i+1}^{(1)} - \Gamma_{i}^{(1)}, i \geq 1\} \) have a finite \( 1/(b-a) \)-th moment, the convergence to 0 for the second term above follows by Lemma A.1. \( \square \)

5.2. Centering. We introduce the quantity

\[
S_{n,m} := \sup_{t \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_{j})] - C[X^{(j)}(t_{j}) - X^{(j)}(t_{j-1})] \right\}.
\]

This should be “comparable” to \( L_{m,n} - Cn \) when \( m = [n^{a}] \). Indeed, we have:

Lemma 5.2. With \( b = (1/2) - (a/6) \), and \( a < 3/7 \),

\[
\frac{S_{n,[n^{c}]} - (L_{n,[n^{c}]} - Cn)}{n^{b}} \to 0 \text{ as } n \to \infty.
\]

Proof: We begin by rewriting the numerator above as

\[
S_{n,m} - (L_{n,m} - Cn)
\]

\[
= \sup_{t \in \mathcal{T}_{n,m}} \left\{ \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_{j})] + Cn - C \sum_{j=1}^{m} [X^{(j)}(t_{j}) - X^{(j)}(t_{j-1})] \right\}
\]

\[
- L_{n,m}.
\]
Upon writing $n = \sum_{j=1}^{m} (t_j - t_{j-1})$, for any $t \in T_{n,m}$, we have
\[
\left| n - \sum_{j=1}^{m} [X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right| = \left| \sum_{j=1}^{m} [t_j - X^{(j)}(t_j)] - \sum_{j=1}^{m} [t_{j-1} - X^{(j)}(t_{j-1})] \right| \leq 2 \sum_{j=1}^{m} \Delta^{(j)}_n.
\]

Hence, on the one hand we have
\[
S_{n,m} - (L_{n,m} - Cn) \leq \sup_{t \in T_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} + 2C \sum_{j=1}^{m} \Delta^{(j)}_n.
\]

On the other hand,
\[
S_{n,m} - (L_{n,m} - Cn) \geq \sup_{t \in T_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} - 2C \sum_{j=1}^{m} \Delta^{(j)}_n.
\]

Therefore,
\[
|S_{n,m} - (L_{n,m} - Cn)| \leq L_{n,m} - L_{n,m} + 2C \sum_{j=1}^{m} \Delta^{(j)}_n.
\]

Thus, for $m = [n^a]$, the result follows by applying Lemma A.1 and Lemma 5.1. □

Define now variance $\sigma^2$ as
\[
\sigma^2 := \text{var}(L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}))
\]

and observe that $\sigma^2 = \sigma_0^2/p$. We work with the quantity $\frac{1}{\sigma} S_{n,m}$, which can be rewritten as
\[
\frac{1}{\sigma} S_{n,m} = \sup_{t \in T_{n,m}} \sum_{j=1}^{m} \sum_{k=\Phi^{(j)}(t_{j-1})+1}^{\Phi^{(j)}(t_j)} \lambda^{(j)}_k,
\]

where
\[
\lambda^{(j)}_k := \frac{1}{\sigma} \{L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)})\}.
\]
Note that the random variables \(\{\chi_k^{(j)}\}_{k \geq 1, j \geq 1}\), indexed by both \(k\) and \(j\), are independent and that \(\{\chi_k^{(j)}\}_{k \geq 2, j \geq 1}\) are identically distributed with zero mean and unit variance. The fact that the \(\{\chi_1^{(j)}\}_{j \geq 1}\) do not have the same distribution will not affect the result, so we will not separately take care of it.

5.3. Coupling with Brownian motion. The term \(\frac{1}{n}S_{n,m}\) resembles a centered last passage percolation path weight, except that random indices are involved. Therefore, we start using the idea of strong coupling with Brownian motions, analogously to the proof in Bodineau and Martin [2005]. Let \(B^{(1)}, B^{(2)}, \ldots\) be i.i.d. standard Brownian motions, and recall that

\[
Z_{n,m} := \sup_{t \in \mathbb{T}_{n,m}} \sum_{j=1}^{m} [B_{i,j}^{(j)} - B_{i,j-1}^{(j)}].
\]  

(5.1)

Define the random walks \(R^{(1)}, R^{(2)}, \ldots\) by

\[
R_{i}^{(j)} = \sum_{k=1}^{i} \chi_k^{(j)}, \quad i = 0, 1, 2, \ldots,
\]

(5.2)

with \(R_0^{(j)} = 0\). With this notation, we have

\[
\frac{1}{\sigma}S_{n,m} = \sup_{t \in \mathbb{T}_{n,m}} \sum_{j=1}^{m} [R_{\Phi(t)}^{(j)} - R_{\Phi(t)}^{(j)}].
\]

(5.3)

Taking into account (1.2) and Lemma 5.2, it is evident that to prove Theorem 3.1 it remains to show that

\[
\frac{\sigma^{-1}S_{n,[n^a]} - \sqrt{n}Z_{n,[n^a]}}{n^b} \overset{(p)}{\longrightarrow} 0,
\]

or, using the scaling property of Brownian motion, that:

**Lemma 5.3.** For all \(a < 3/14\),

\[
\frac{\sigma^{-1}S_{n,[n^a]} - Z_{n,[n^a]}}{n^b} \overset{(p)}{\longrightarrow} 0.
\]

To show that the random walks are close enough to the Brownian motion we use the following version of the Komlós-Major-Tusnády strong approximation result (Komlós et al., 1976, Thm. 4):

**Theorem 5.4.** For any \(0 < r < 1\), \(n \in \mathbb{Z}_+\) and \(x \in [c_1(\log n)^{1/r}, c_2(n \log n)^{1/2}]\), starting with a probability space supporting independent Brownian motions \(B^{(j)}\), \(j = 1, 2, \ldots\), we can jointly construct i.i.d. sequences \(\chi^{(j)} = (\chi_1^{(j)}, \chi_2^{(j)}, \ldots), j = 1, 2, \ldots\), with the correct distributions and, moreover, such that, with \(R_{i}^{(j)}\) as in (5.2) above,

\[
P(\max_{1 \leq i \leq n} |B_{i}^{(j)} - R_{i}^{(j)}| > x) \leq Cn \exp\{-\alpha x^r\} \text{ for all } j = 1, 2, \ldots,
\]

where the constants \(C, c_1, c_2\) are depending on \(\alpha\), \(r\) and distributions of \(\chi_1^{(1)}\) and \(\chi_2^{(1)}\).

In addition to this, in order to take care of the random indices appearing in (5.3), we need a convergence rate result for the counting processes \(\{\Phi^{(j)}, j \geq 1\}\) which is proven in the appendix.
**Proof of Lemma 5.3:** From (5.3) and (5.1) we have
\[
|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \leq \sup_{t \in T_{n,m}} \sum_{j=1}^{m} \left\{ \left| (R_{\Phi^{(j)}(t_j)}^{(j)} - R_{\Phi^{(j)}(t_{j-1})}^{(j)}) - (B_{X_{t_j}}^{(j)} - B_{X_{t_{j-1}}}^{(j)}) \right| \right\} \\
= \sup_{t \in T_{n,m}} \sum_{j=1}^{m} \left\{ |R_{\Phi^{(j)}(t_j)}^{(j)} - B_{\Phi^{(j)}(t_j)}^{(j)}| + |R_{\Phi^{(j)}(t_{j-1})}^{(j)} - B_{\Phi^{(j)}(t_{j-1})}^{(j)}| \right. \\
+ \left. |B_{\Phi^{(j)}(t_j)}^{(j)} - B_{X_{t_j}}^{(j)}| + |B_{\Phi^{(j)}(t_{j-1})}^{(j)} - B_{X_{t_{j-1}}}^{(j)}| \right\} \\
\leq 2 \sum_{j=1}^{m} \left\{ \max_{0 \leq i \leq n} |R_i^{(j)} - B_i^{(j)}| + \sup_{0 \leq s \leq n} |B_{\Phi^{(j)}(s)}^{(j)} - B_{\Phi^{(j)}(s)}^{(j)}| \right\}
\]
where
\[
U_n^{(j)} := \max_{0 \leq i \leq n} |R_i^{(j)} - B_i^{(j)}|, \quad V_n^{(j)} := \sup_{0 \leq s \leq n} |B_{\Phi^{(j)}(s)}^{(j)} - B_{\Phi^{(j)}(s)}^{(j)}|.
\]
Therefore, it is enough to show that,
\[
\frac{1}{n^2} \sum_{j=1}^{[n^2]} U_n^{(j)} \xrightarrow{n \to \infty} 0 \quad \text{and} \quad \frac{1}{n^2} \sum_{j=1}^{[n^2]} V_n^{(j)} \xrightarrow{n \to \infty} 0.
\]
For the first convergence we will take into account the coupling estimate as in Theorem 5.4. That is, we will throughout assume that the random walks and Brownian motions have been constructed jointly. The second convergence will be established without this estimate, i.e., we will show that it is true, regardless of the joint construction of the Brownian motions and the random walks. This is because the coupling we use is not detailed enough to give us information about the joint distribution of \(B^{(j)}\) and the counting process \(\Phi^{(j)}\) (the latter is not a function of the random walks used in the coupling).

**Proof of the first convergence.** Let \(\delta > 0\). We need to show that
\[
P\left(\sum_{j=1}^{[n^2]} U_n^{(j)} > \delta n^b\right) \to 0 \quad \text{as} \quad n \to \infty.
\]
Let \(\varepsilon < b - a\). Then
\[
P\left(\sum_{j=1}^{[n^2]} U_n^{(j)} > \delta n^b\right) \leq P\left(\max_{1 \leq j \leq [n^2]} U_n^{(j)} \leq \varepsilon, \sum_{j=1}^{[n^2]} U_n^{(j)} > \delta n^b\right) + P\left(\max_{1 \leq j \leq [n^2]} U_n^{(j)} > \varepsilon\right).
\]
(5.4)
The first term from right-hand side is zero for large \(n\). We are allowed, for \(n\) large enough, to estimate the second term using Theorem 5.4 for an arbitrary \(r \in (0, 1)\) and \(x = n^\varepsilon\) as
\[
P\left(\max_{1 \leq j \leq [n^2]} U_n^{(j)} > n^\varepsilon\right) \leq n^a P\left(\max_{1 \leq i \leq n} |B_i^{(1)} - R_i^{(1)}| > n^\varepsilon\right) \leq n^a C n \exp\{-\alpha n^{cr}\} \to 0, \quad \text{as} \quad n \to \infty.
\]
Proof of the second convergence. Let $1/4 < \varepsilon < b - a$. Replacing $U_n^{(j)}$ by $V_n^{(j)}$ in (5.4), we see that the first term is again zero for large $n$ and it remains to show that second term converge to 0, i.e., it is enough to show the convergence for its upper-bound

$$n^a P(V_n^{(1)} > n^\varepsilon) \to 0,$$

as $n \to \infty$. Let $\gamma > 1$ and $1/2 < q < 2\varepsilon$. Then we can write

$$n^a P(V_n^{(1)} > n^\varepsilon) \leq n^a P\left( \sup_{0 \leq s \leq 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q \right) + n^a P\left( \sup_{0 \leq s \leq 2n} |B_{\Phi^{(1)}(s)}^{(1)} - B_{\lambda s}^{(1)}| > n^\varepsilon, \sup_{0 \leq s \leq 2n} |\Phi^{(1)}(s) - \lambda s| \leq \gamma n^q \right).$$

Corollary A.1 implies the convergence of the first term above

$$n^a P\left( \sup_{0 \leq s \leq 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q \right) \leq n^a \exp\{-\alpha(2n)^{q/2}\} \to 0,$$

as $n \to \infty$, where $0 < r < 2 - 1/q$. Set $\varphi = \gamma n^q/\lambda$. For the second term using the fact that, under our condition, $\lambda s - \gamma n^q \leq \Phi^{(1)}(s) \leq \lambda s + \gamma n^q$ for $s \in [0, 2n]$, we have

$$n^a P\left( \sup_{0 \leq s \leq 2n} |B_{\Phi^{(1)}(s)}^{(1)} - B_{\lambda s}^{(1)}| > n^\varepsilon, \sup_{0 \leq s \leq 2n} |\Phi^{(1)}(s) - \lambda s| \leq \gamma n^q \right) \leq n^a P\left( \max_{0 \leq k \leq \lceil n/2 \rceil} \sup_{0 \leq s \leq \varphi} |B_{\Phi^{(1)}(k\varphi + s)}^{(1)} - B_{\lambda(k\varphi + s)}^{(1)}| > n^\varepsilon \right) \leq \sum_{0 \leq s \leq 2n} |\Phi^{(1)}(s) - \lambda s| \leq \gamma n^q
$$

$$\leq n^a(n/\varphi + 1)P\left( \sup_{0 \leq s \leq 3\gamma n^q} |B_t - B_s| > n^\varepsilon \right) \leq n^a(n/\varphi + 1)P\left( \sup_{0 \leq s \leq 3\gamma n^q} |B_s| > n^\varepsilon/2 \right) \leq 4n^a(n/\varphi + 1)P(B_{3\gamma n^q} > n^\varepsilon/2) \leq 4n^a(n/\varphi + 1)\exp\{-n^2/(24\gamma n^q)\} \to 0, \quad \text{(a)}$$

as $n \to \infty$. The inequality (a) is the consequence of $P(\sup_{0 < s < t} B_s > x) = 2P(B_t > x)$ and the inequality (b) is an estimate for the tail of the normal distribution. \(\Box\)

Remark 5.5. The condition $a < 3/14$ is equivalent to $b - a > 1/4$ and, thus, it ensures existence of an $\varepsilon > 1/4$ and later existence of $1/2 < q < 2\varepsilon$. Therefore, it is necessary for application of Corollary A.1, as well as for the convergence of the last estimate above.

6. Graph with non-constant edge probabilities

In Denisov et al. (2012), the authors consider a one-dimensional model with the connectivity probability depending on the distance between points, i.e. there is an edge between vertices $i$ and $j$ with probability $p_{|i-j|}$. To prove a central limit theorem for the maximal path length in that graph, two conditions are introduced:

- $0 < p_1 < 1$;
- $\sum_{k=1}^{\infty} k(1 - p_1) \cdots (1 - p_k) < \infty$. 

The conditions guarantee the existence of skeleton points and finite variance, defined as in (2.3).

We can extend the idea of the graph on \( \mathbb{Z} \times \mathbb{Z} \) keeping, whenever possible, the notation from the \( \mathbb{Z} \times \mathbb{Z} \) graph with constant probability \( p \).

Let \( \{p_{i,j}, i, j \leq 0\} \) be a sequence of probabilities that satisfies the following:

- \( 0 < p_{1,0} < 1 \);
- \( 0 < p_{0,1} < 1 \);
- \( \sum_{k=1}^{\infty} k^{r-1}(1-p_{1,0})(1-p_{2,0})\cdots(1-p_{k,0}) < \infty \) for some \( r > 2 \).

Then the vertices \((i_1, i_2)\) and \((j_1, j_2)\), \((i_1, i_2) \prec (j_1, j_2)\), are connected with probability \( p_{j_1-i_1, j_2-i_2} \).

The conditions above are needed to ensure the existence of skeleton points, as in Definition 4.1, and a finite \( r \)th moment of the random variables \( \chi_2, \chi_3, \ldots \) for some \( r > 2 \).

From now on, let \( r \) denote the order of the highest finite moment, i.e.

\[
r = \sup\{q > 0 : \sum_{k=1}^{\infty} k^{q-1}(1-p_{1,0})(1-p_{2,0})\cdots(1-p_{k,0}) < \infty\}.
\]

Now we can state the analogue of Theorem 3.1 for this more general setting:

**Theorem 6.1.** For all \( a < \min(3/14, (r-2)/(3r/7 + 1)) \),

\[
n^{a/6}\left(\frac{L_{n,[n^a]} - Cn}{\sqrt{\lambda \sigma^2} \sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow{(d)} F_{TW}.
\]

**Proof:** We follow the lines of the proof of Theorem 3.1, emphasizing the points one should be careful about or which need to be modified.

The construction of the upper and lower bounds for \( L_{n,[n^a]} \) is the same as in Section 4. In Lemma 5.1 and Lemma 5.2, the term \( n^a E|\Delta_1|/n^b \) converges to 0 if \( a + 1/r < b \), which leads to the constraint \( a < 6/7(1/2 - 1/r) \).

Next, we rewrite, as in Lemma 5.3,

\[
|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \leq 2 \sum_{j=1}^{m} U_n^{(j)} + 2 \sum_{j=1}^{m} V_n^{(j)}
\]

and prove convergence of each term separately.

**Proof of the first convergence.** Instead of Theorem 5.4, we use the combination of results from Komlós et al. (1976) and Major (1976), as stated in Proposition 2 of Bodineau and Martin (2005), which allow us to couple Brownian motions \( B^{(j)} \), \( j = 1, 2, \ldots \) and random walks \( R^{(j)} \), \( j = 1, 2, \ldots \) so that

\[
P\left(\max_{1 \leq i \leq n} |B_i^{(j)} - R_i^{(j)}| > x\right) \leq Cn x^{-r}, \text{for all } n \in \mathbb{Z}_+, \text{ and all } x \in [n^{1/r}, n^{1/2}].
\]

(6.1)

Let \( \varepsilon = 1/2 \). We want to establish convergence of the same terms as in (5.4). Applying, on the first term, the Markov inequality and properties of the coupling
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(6.1) yield:

\[ P \left( \max_{1 \leq j \leq [n^a]} U_n^{(j)} \leq n^{1/2} \sum_{j=1}^{[n^a]} U_n^{(j)} > \alpha n^{b} \right) \leq \frac{n^a}{\alpha n^{b}} E[U_n^{(1)} 1(U_n^{(1)} \leq n^{1/2})] \]

\[ \leq \frac{n^a}{\alpha n^{b}} \left( n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} P(U_n^{(1)} > x) dx \right) \leq \frac{n^a}{\alpha n^{b}} \left( n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} Cn x^{-r} dx \right) \]

\[ = \frac{n^a}{\alpha n^{b}} \left( n^{1/r} - \frac{C}{r-1} n^{3/2-r/2} + \frac{C}{r-1} n^{1/r} \right) \leq \left( 1 + \frac{C}{r-1} \right) \frac{n^a n^{1/r}}{\alpha n^{b}} \to 0 \]

as \( n \to \infty \). For the second term we use again the coupling properties (6.1):

\[ P \left( \max_{1 \leq j \leq [n^a]} U_n^{(j)} > n^{1/2} \right) \leq n^a P \left( \max_{1 \leq i \leq n} |B_i^{(1)} - R_i^{(1)}| > \varepsilon n^q \right) \leq n^a C n n^{-r/2} \to 0, \]

as \( n \to \infty \) because \( a + 1 - r/2 < 0 \) due to our new constraint on \( a \).

Proof of the second convergence. Here, the only change is the replacement of Lemma A.2 by part of Theorem 6.12.1 in Gut (2013), in our notation:

If \( 1/2 < q < 1 \), then for all \( \varepsilon > 0 \) it holds that \( n^{ar-2} P(\max_{0 \leq k \leq n} |\Gamma_k - \lambda k| > \varepsilon n^q) \to 0 \) as \( n \to \infty \).

One can, in the same fashion as before, prove the following analogue of Corollary A.1,

\[ n^{ar-2} P \left( \sup_{0 \leq t \leq n} |\Phi(t) - \lambda t| > \varepsilon n^q \right) \to 0 \quad \text{as} \quad n \to \infty. \]

Likewise in Remark 5.5, condition \( a < 3/14 \) is necessary. Another constraint that occurs is \( a \leq q r - 2 \) and it can be shown that this is satisfied if \( a \leq (r - 2)/(3r/7 + 1) \).

Appendix A.

Lemma A.1. Let \( r \geq 1 \) and \( \{X_i, i \geq 1\} \) be a sequence of nonnegative i.i.d. random variables such that \( E|X_1|^r < \infty \). Then,

\[ \frac{1}{n^b} E \left[ \max_{1 \leq i \leq n} X_i \right] \to 0 \quad \text{as} \quad n \to \infty. \]

Proof: It suffices to prove the lemma for \( r = 1 \), since then the result for \( r > 1 \) follows easily by Jensen’s inequality. Let \( M_n = \max_{1 \leq i \leq n} X_i \) and \( S_n = X_1 + X_2 + \cdots + X_n \).

Borel-Cantelli lemma, together with the assumption \( E X < \infty \), give \( X_n/n \to 0 \), almost surely, as \( n \to \infty \). An easy computation shows that then also \( M_n/n \to 0 \), almost surely, as \( n \to \infty \).

On the other hand, we know that \( M_n \leq S_n \) for all \( n \) and that \( \{S_n/n, n \geq 1\} \) is uniformly integrable. Therefore, \( M_n/n \) is also uniformly integrable and \( E M_n/n \to 0 \) as \( n \to \infty \) (Gut, 2013, Thm. 5.4.5, Thm. 5.5.2).

The following lemma is an analogue of a result by Lanzinger (1998, Prop. 2), with slight improvement in the probability bound. Moreover, we specialize the lemma to our case.

Lemma A.2. Let \( 1/2 < q \leq 1 \) and \( r < 2 - 1/q \). Suppose that \( X, X_1, X_2, \ldots \) are i.i.d. random variables such that \( E X = \mu \) and \( E e^{\alpha |X|} < \infty \) for some \( \alpha > 0 \), and set \( S_n = \sum_{i=1}^{n} X_k \). Then, for all \( \varepsilon > 0 \),

\[ \exp \left( \alpha n^q \right) P \left( \max_{1 \leq k \leq n} |S_k - k \mu| > \varepsilon n^q \right) \to 0 \]
as $n \to \infty$.

Proof: Without loss of generality we can assume that $\mu = 0$. We first show that, for all $\varepsilon > 1$, $\exp\{an^{qr}\}P(S_n > \varepsilon n^q) \to 0$ as $n \to \infty$.

For fixed $n \in \mathbb{Z}_+$, define $X'_k = X_k1(X_k \leq \varepsilon n^q)$ and set $S'_n = \sum_{k=1}^n X'_k$. Note that $EX' \leq 0$ and that for $a'$, where $a' < a$ and $a'\varepsilon > a$, it holds that $EX' e^{|X'|} < \infty$. We can write

$$P(S_n > \varepsilon n^q) \leq P(S'_n > \varepsilon n^q) + nP(X > \varepsilon n^q). \quad (A.1)$$

We first find a bound for the moment generating function for $X'$ in $a'(\varepsilon n^q)^{-r}$:

$$Ee^{a'(\varepsilon n^q)^{-r-1}X'} \leq 1 + \frac{1}{2}a'^2(\varepsilon n^q)^{2(r-1)}(EX'^2 + EX'e^{a'(\varepsilon n^q)^{-r-1}}1\{X' > 0\})$$

$$\leq 1 + \frac{1}{2}a'^2(\varepsilon n^q)^{2(r-1)}(EX'^2 + EX'e^{a'(\varepsilon n^q)^{-r}})$$

$$\leq 1 + Cn^{-2q(r-1)} \leq \exp\{Cn^{-2q(1-r)}\},$$

where for the first inequality we used $e^y \leq 1 + \max\{1, e^y\}y^2/2$ and for the last one $1 + y \leq e^y$. Now, using Markov’s inequality and the bound above for the first term in $(A.1)$ yields

$$P(S'_n > \varepsilon n^q) = P(e^{a'(\varepsilon n^q)^{-r-1}S'_n} > e^{a'(\varepsilon n^q)^{-r}}) \leq e^{-a'(\varepsilon n^q)^{-r}}\left(EX' e^{a'(\varepsilon n^q)^{-r-1}X'}\right)^n$$

$$\leq \exp\{-a'(\varepsilon n^q)^{-r} + Cn^{-2q(1-r)}\}.$$

For the second term in $(A.1)$, again using Markov’s inequality, we have

$$nP(X > \varepsilon n^q) = nP(e^{aX'} > e^{a(\varepsilon n^q)^{r}}) \leq ne^{-a(\varepsilon n^q)^{r}}E(e^{a|X'|}).$$

Combining the two estimates finally establishes that

$$\exp\{an^{qr}\}P(S_n > \varepsilon n^q) \leq \exp\{-n^{qr}(a'\varepsilon^r - \alpha - Cn^{1-2q+qr})\}$$

$$+ n \exp\{-n^{qr}\alpha(\varepsilon^r - 1)\}Ee^{a|X'|} \to 0$$

as $n \to \infty$ because of the choice of $r, a'$ and $\varepsilon$.

By symmetry, the same convergence rate holds also for $P(S_n < -\varepsilon n^q)$.

Thus, the statement of the theorem follows using Lévy inequality (Gut, 2013, Thm. 3.7.2) and the fact that for $n$ large enough we can find $\varepsilon'$ such that $1 < \varepsilon' < \varepsilon - \sqrt{2\sigma^2n^{1/2-q}}$:

$$P(\max_{1 \leq k \leq n} |S_k| > \varepsilon n^q) \leq 2P(|S_n| > \varepsilon n^q - \sqrt{2n\sigma^2}) = 2P(|S_n| > n^q(\varepsilon - \sqrt{2\sigma^2n^{1/2-q}}))$$

$$\leq 2P(|S_n| > \varepsilon' n^q). \quad \square$$

Corollary A.1. Let $X, X_1, X_2, \ldots$ be positive, integer-valued i.i.d. random variables and suppose that the assumptions of Lemma A.2 are satisfied. Then, for the counting process $\Phi$, where $\Phi(t) = \max\{n : S_n \leq t\}$, it holds that

$$\exp\{an^{qr}\}P(\sup_{0 \leq t \leq n} |\mu\Phi(t) - t| > \varepsilon n^q) \to 0 \text{ for all } \varepsilon > 1.$$
Proof: For the counting process we have \( t - X_{\Phi(t)+1} \leq S_{\Phi(t)} \leq t \). Therefore, we can write
\[
\{ \sup_{0 \leq t \leq n} |\mu(t) - t| > \varepsilon n^q \} = \\
= \{ \sup_{0 \leq t \leq n} (\mu(t) - t) > \varepsilon n^q \} \cup \{ \inf_{0 \leq t \leq n} (\mu(t) - t) < -\varepsilon n^q \} \\
\subseteq \{ \sup_{0 \leq t \leq n} (\mu(t) - S_{\Phi(t)}) > \varepsilon n^q \} \cup \{ \inf_{0 \leq t \leq n} (\mu(t) - S_{\Phi(t)} - X_{\Phi(t)+1}) < -\varepsilon n^q \} \\
\subseteq \{ \sup_{0 \leq t \leq n} (\mu(t) - S_{\Phi(t)}) > \varepsilon n^q \} \cup \{ \inf_{0 \leq t \leq n} (\mu(t) - S_{\Phi(t)}) < -\varepsilon n^q \} \\
\cup \{ \sup_{0 \leq t \leq n} X_{\Phi(t)+1} > (\varepsilon - \varepsilon') n^q \} \\
\subseteq \{ \max_{1 \leq k \leq n} |S_k - k\mu| > \varepsilon n^q \} \cup \{ \max_{1 \leq k \leq n} |S_k - k\mu| > \varepsilon' n^q \} \\
\cup \{ \max_{1 \leq k \leq n+1} X_k > (\varepsilon - \varepsilon') n^q \},
\]
where \( 0 < \varepsilon' < \varepsilon \). Thus,
\[
\exp\{\alpha n^{qr}\} P \left( \sup_{0 \leq t \leq n} |\mu(t) - t| > \varepsilon n^q \right) \leq \exp\{\alpha n^{qr}\} P \left( \max_{1 \leq k \leq n} |S_k - k\mu| > \varepsilon n^q \right) \\
+ \exp\{\alpha n^{qr}\} P \left( \max_{1 \leq k \leq n} |S_k - k\mu| > \varepsilon' n^q \right) + (n + 1) \exp\{\alpha n^{qr}\} P(X > (\varepsilon - \varepsilon') n^q).
\]
The first and the second term above converge to 0 as \( n \to 0 \) by Lemma A.2. We prove convergence to 0 for the third term using Markov’s inequality,
\[
(n + 1) e^{\alpha n^{qr}} P(X > (\varepsilon - \varepsilon') n^q) \leq (n + 1) e^{\alpha n^{qr} - (\varepsilon - \varepsilon') n^q} E e^{\alpha X} \to 0 \text{ as } n \to \infty.
\]

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References

J. Baik and T. M. Suidan. A GUE central limit theorem and universality of directed first and last passage site percolation. *Int. Math. Res. Not.* 2005 (6), 325–337 (2005). MR2131383.

A. B. Barak and P. Erdős. On the maximal number of strongly independent vertices in a random acyclic directed graph. *SIAM J. Algebr. Discr. Methods* 5 (4), 508–514 (1984). MR0763980.

Y. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields* 119 (2), 256–274 (2001). MR1818248.

T. Bodineau and J. Martin. A universality property for last-passage percolation paths close to the axis. *Electron. Commun. Probab.* 10, 105–112 (2005). MR2150699.

J. E. Cohen and C. M. Newman. Community area and food-chain length: theoretical predictions. *Amer. Naturalist* 138 (6), 1542–1554 (1991). [www.jstor.org/stable/2462559](http://www.jstor.org/stable/2462559).

D. Denisov, S. Foss and T. Konstantopoulos. Limit theorems for a random directed slab graph. *Ann. Appl. Probab.* 22 (2), 702–733 (2012). MR2953567.

S. Foss and T. Konstantopoulos. Extended renovation theory and limit theorems for stochastic ordered graphs. *Markov Process. Related Fields* 9 (3), 413–468 (2003). MR2028221.
P. W. Glynn and W. Whitt. Departures from many queues in series. *Ann. Appl. Probab.* 1 (4), 546–572 (1991). MR1129774.

A. Gut. *Probability: A Graduate Course*. Springer, New York, 2nd edition (2013). MR2977961.

M. Isopi and C. M. Newman. Speed of parallel processing for random task graphs. *Comm. Pure and Appl. Math* 47 (3), 361–376 (1994). MR1266246.

Y. Itoh and P. L. Krapivsky. Continuum cascade model of directed random graphs: traveling wave analysis. *J. Phys. A* 45 (45) (2012). MR2990512.

J. Komlós, P. Major and G. Tusnády. An approximation of partial sums of independent RV’s, and the sample DF. II. *Z. Wahrsch. und Verw. Gebiete* 34 (1), 33–58 (1976). MR0402883.

H. Lanzinger. A Baum-Katz theorem for random variables under exponential moment conditions. *Statist. Probab. Lett.* 39 (2), 89–95 (1998). MR1652524.

P. Major. The approximation of partial sums of independent RV’s. *Z. Wahrsch. und Verw. Gebiete* 35 (3), 213–220 (1976). MR0415743.

C. M. Newman. Chain lengths in certain random directed graphs. *Random Structures Algorithms* 3 (3), 243–253 (1992). MR1164838.

C. M. Newman and J. E. Cohen. A stochastic theory of community food webs IV: Theory of food chains in large webs. *Proc. R. Soc. London Ser. B* 228 (1252), 355–377 (1986). www.jstor.org/stable/36124.

C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* 159 (1), 151–174 (1994). MR1257246.