Chiral Phase Transitions in QED at Finite Temperature: Dyson-Schwinger Equation Analysis in the Real Time Hard-Thermal-Loop Approximation

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(Dated: March 25, 2022)

In order for clarifying what are the essential thermal effects that govern the chiral phase transition at finite temperature, we investigate, in the real-time thermal QED, the consequences of the Hard-Thermal-Loop (HTL) resummed Dyson-Schwinger equation for the physical fermion mass function $\Sigma_R$. Since $\Sigma_R$ is the mass function of an “unstable” quasi-particle in thermal field theories, it necessarily has non-trivial imaginary parts together with non-trivial wave function renormalization constants. The analyses so far completely dismissed this fact, despite it being one of the basic outcome of thermal field theory. The “approximation”, which neglects the imaginary parts of the $\Sigma_R$, gives (non-trivial) constraint equations to be solved simultaneously, thus cannot be consistently carried out except in the trivial case, $\Sigma_R = 0$. In the present analysis we correctly respect this fact, and study, in the ladder approximation, the effect of HTL resummed gauge boson propagator. Our results with the use of numerical analysis, show the two facts; i) The chiral phase transition is of second order, since the fermion mass is dynamically generated at a critical value of the temperature $T_c$, or at the critical coupling constant $\alpha_c$, without any discontinuity, and ii) the critical temperature $T_c$ at fixed value of $\alpha$ is significantly lower than the previous results, namely the restoration of chiral symmetry occurs at lower temperature than previously expected. The second fact shows the importance of correctly taking the essential thermal effect into the analysis of chiral phase transition, which are, in the previous analyses, neglected due to the inappropriate approximations. The procedure how to maximally respect the gauge invariance in the present approximation, is also discussed.

PACS numbers: 11.10.Wx, 11.15.Ex, 11.15.Tk, 11.30.Rd

I. INTRODUCTION

Although lots of efforts have been made to understand the temperature- and/or density-dependent phase transition in thermal QCD/QED, we cannot have yet truly understood even the relation between the chiral transition and the confinement-deconfinement transition. Beginning of the relativistic heavy ion collision experiments at BNL-RHIC, which has been actively producing new results that may prove the generation of the quark-gluon plasma phase, has attracted an increasing interest in studying the physics in thermal QCD, thus has given us an encouraging time to proceed to further investigations of the mechanism of phase transition in hot and dense gauge theories, especially in QCD and QED.

The Dyson-Schwinger (DS) equation is proven to be a powerful tool to investigate with the analytic procedure the phase structure of gauge theories, especially in the vacuum gauge theories [1,2,3]. However, we cannot say that, at finite temperature and/or density, the DS equation analyses of chiral and/or di-quark condensation have been carried out successfully.

In the preceding DS equation analyses [4-8], the lessons from vacuum theories have been so simply applied to thermal theories without close examination. In most analyses the ladder approximation was used by simply neglecting all the hard-thermal-loop (HTL) effects [5,7], or only by taking improperly the HTL effects into the gauge boson propagator [6,8]. As a result they have missed the essential contribution of thermal gauge field theories, especially the important effect from the “dynamically screened” magnetic mode (having in general a momentum-dependent “mass”, though...
being massless in the static limit). Many analyses, by fixing in the Landau gauge, ignored the fermion wave-function renormalization constants (WFRCs) by taking their naive tree values [5,6,7]. Furthermore, most analyses done in the real time formalism did not discuss the physical fermion mass function $\Sigma_R$ itself of the retarded propagator [6,8], without any respect to the existence of imaginary parts, together with the inaccurate use of the instantaneous exchange (IE) approximation to the gauge boson propagation [6,8]. All such improper approximation methods may have caused the neglect of would-be-large contributions to the DS equation otherwise existed.

Then we should seriously ask whether we could rely on the previous results of the DS equation analysis on the chiral phase transition as the real consequences of thermal gauge field theories. Considering the troubles in the previous analyses [4-8] mentioned above, we should make a re-analysis by studying the HTL resummed DS equation in the real time formalism, thus giving a new understanding on the phase structure and the mechanism of phase transition in thermal gauge theories.

Main interest of the present investigation lies in clarifying what are the essential temperature effects that govern the phase transition and also in finding how we can closely take these effects into the “kernel” of the DS equation. Essential procedures of our analysis can be summarised as follows;

i) Firstly we use the real-time closed-time-path (RTCTP) formalism [9], and study the physical mass function $\Sigma_R$ itself, not the $\Sigma_{11}$, of the retarded fermion propagator.

ii) Secondly we correctly respect the fact that the $\Sigma_R$ is the mass function of an “unstable” quasi-particle in thermal field theories, namely that the $\Sigma_R$ has non-trivial imaginary parts as well as non-trivial WFRCs. Neglect of imaginary parts and non-trivial WFRCs actually give (non-trivial) constraint equations to be solved simultaneously. This fact was totally dismissed in the preceding analyses.

iii) Thirdly and most importantly, devoting our attention to closely estimating the dominant temperature-dependent contributions, we focus on studying the DS equation being exact up to HTL approximation: Both the gauge boson propagators and the vertex functions are determined within the HTL resummation [10,11,12,13], with which the gauge invariance of the result at least in the perturbative analysis is guaranteed. With the HTL resummed vertex functions we can explicitly write down the HTL resummed DS equation [12].

iv) Finally in connection with ii) and iii) above, the gauge-parameter dependent contribution must be carefully studied without fixing the gauge into some definite ones, such as the Landau gauge.

The third point listed above is better to be taken step by step into the actual analysis of the DS equation. The advantage of the DS equation analysis lies in the possibility of systematic (step-by-step) improvement of the approximation to the integration kernel through the analytic investigation.

Thus, in the present paper we present the result of our first step investigation in the strong coupling QED: focussing on what happens when we take into account exactly the HTL resummed gauge boson propagators, we investigate the consequences of the ladder (point vertex) DS equation with the use of the fully HTL resummed photon propagator. In this approximation, however, the gauge invariance of the results is spoiled. The procedure how to maximally respect the gauge invariance of the results in the ladder approximation, is discussed and the result is given in a separate paper[14]. Analysis in QCD and effects of fully including the HTL resummed vertices will also be presented in the separate paper [15].

This paper is organized as follows. In the next section II, the DS equation in QED for the fermion self-energy in the HTL approximation is derived, and the improved ladder (point vertex) DS equation with the HTL resummed photon propagator is given. The instantaneous exchange (IE) approximation for the photon propagation is discussed. In section III we give the results of the present investigation with the use of numerical analysis: the order of the chiral phase transition, the critical curve in the $T-\alpha$ plane (where $T$ is the temperature of the environment and $\alpha$ the coupling constant), and values of the critical exponents. Conclusion of the present analysis and discussion on the results are given in the last section IV.

**II. THE DYSON-SCHWINGER EQUATION IN QED FOR THE FERMION SELF-ENERGY FUNCTION**

**A. The DS equation in the Hard-Thermal-Loop approximation**

The DS equation for the physical, i.e., the retarded fermion self-energy function $\Sigma_R$ in the HTL approximation can be obtained by applying the following approximation to the full DS equation;

i) replace the full gauge boson propagator with the HTL resummed propagator, and

ii) approximate the full vertex functions to the HTL resummed vertex functions.

Then in the RTCTP formalism we get in QED the desired DS equation (in QCD, the running coupling should be used inside the loop-momentum integration) [12],
\(-i\Sigma_R(P) = -i\Sigma_{RA}(-P, P) = \frac{e^2}{2} \int \frac{d^4K}{(2\pi)^4} \times [\Gamma_{RRA}^\mu(-P, K, P - K) S_{RA}(K) \ast \Gamma_{RAA}^\nu(-K, P, K - P) \ast G_{RR, \mu\nu}(P - K) \\
+ \Gamma_{RAA}^\mu(-P, K, P - K) S_{RR}(K) \ast \Gamma_{AAR}^\nu(-K, P, K - P) \ast G_{RA, \mu\nu}(P - K)]\),

where \(\ast G^{\rho\sigma}\) is the HTL resummed gauge boson propagator, whose retarded \(R \equiv RA\) component is given by

\[\ast G^{\rho\sigma}_R(K) = \frac{1}{\ast \Pi^R_+(K) - K^2 - i\epsilon k_0} A^{\rho\sigma} + \frac{1}{\ast \Pi^R_L(K) - K^2 - i\epsilon k_0} B^{\rho\sigma} - \frac{\xi}{K^2 + i\epsilon k_0} D^{\rho\sigma},\]

and the \(RR\) component by

\[\ast G^{\rho\sigma}_{RR}(-K, K) = (1 + 2n_B(k_0))[\ast G^{\rho\sigma}_R(K) - \ast G^{\rho\sigma}_A(K)], \quad n_B(k_0) = \frac{1}{\exp(k_0/T) - 1},\]

with \(\ast \Pi^R_+\) and \(\ast \Pi^R_L\) being the HTL contributions to the retarded gauge boson self-energy of the transverse and longitudinal modes [16], respectively. \(A^{\rho\sigma}, B^{\rho\sigma}\) and \(D^{\rho\sigma}\) are the projection tensors [17],

\[A^{\rho\sigma} = g^{\rho\sigma} - B^{\rho\sigma} - D^{\rho\sigma}, \quad B^{\rho\sigma} = -\frac{\tilde{K}^\rho \tilde{K}^\sigma}{K^2}, \quad D^{\rho\sigma} = \frac{K^\rho K^\sigma}{K^2},\]

where \(\tilde{K} = (k, k_0\hat{k}), k = \sqrt{k^2}\) and \(\hat{k} = k/k\) denotes the unit three vector along \(k\).

\(S(-P, P) \equiv \Sigma_{R}(P)\) denotes the full fermion propagator, whose retarded \(RA \equiv R\) component is given by

\[S_R(P) = \frac{1}{P + i\epsilon\gamma_0 - \Sigma_R},\]

and \(S_{RR}\) the \(RR\) component

\[S_{RR}(P) = (1 - 2n_F(p_0))[S_R(P) - S_A(P)], \quad n_F(p_0) = \frac{1}{\exp(p_0/T) + 1}.\]

The fermion self-energy function \(\Sigma_R\) in Eq. (III.4) can be tensor-decomposed as

\[\Sigma_R(P) = (1 - A(P))p_I \gamma^I - B(P) \gamma^0 + C(P),\]

with \(A(P), B(P)\) and \(C(P)\) being the three independent scalar invariants to be determined. At zero temperature, the wave function renormalization constant \(A(P)\) coincides with \(B(P)\) and equals to unity in the Landau gauge, while at finite temperature it is not. \(C(P)/A(P)\) plays the role of mass function, in which we are interested.

\(\ast \Gamma^\mu\) is the HTL resummed 3-point fermion-gauge boson vertex function [12],

\[\ast \Gamma_{\alpha\beta\gamma}^\mu = \gamma_{\alpha\beta\gamma}^\mu + \delta \Gamma_{\alpha\beta\gamma}^\mu, \quad (\alpha, \beta, \gamma = A, R),\]

where \(\delta \Gamma_{\alpha\beta\gamma}^\mu\) represents the HTL contribution to the 3-point fermion-gauge boson vertex function, and \(\gamma_{\alpha\beta\gamma}^\mu\) the tree vertex with \(\gamma_{RRA}^\mu = \gamma_{AAR}^\mu = \gamma^\mu\), otherwise zero. Appearance of the HTL resummed vertex functions together with the HTL resummed gauge boson propagators assures that the HTL approximation is consistently carried out in studying the HTL resummed DS equation [12], and guarantees the result being gauge invariant, at least, in the effective perturbation regime.

### B. The improved ladder DS equation with the HTL resummed gauge boson propagator

Neglecting of \(\delta \Gamma_{\alpha\beta\gamma}^\mu\) in Eq. (III.7), simply brings us to the ladder (point-vertex) DS equation with the HTL resummed gauge boson propagator. It significantly simplifies the structure of the DS equation to be examined, thus reducing the technical difficulty in handling the DS equation itself. The price to pay is to lose the assurance of gauge invariance of the results.

In this paper as already mentioned before, we investigate the consequences of the ladder DS equation with the HTL resummed gauge boson propagator, Eqs. (II.2) and (II.3). In this case three invariants \(A(P), B(P)\) and \(C(P)\) as functions of \(p_0\) and \(p\) satisfy the following coupled integral equations \((P^\mu = (p_0, p))\),
- \rho^2 [1 - A(P)] = -e^2 \int \frac{d^4K}{(2\pi)^4} \left\{ 1 + 2n_B(p_0 - k_0) \right\} \text{Im} \left[ *G_R^\sigma (P - K) \right] \times \\
\left\{ [K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_\rho + K_\rho g_\sigma) - k_0(P_\sigma g_\rho + P_\rho g_\sigma) + pk_\rho g_\sigma] \\
+ 2p_0 k_0 g_\sigma 0 g_\rho \right\} \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ P_\sigma g_\rho + P_\rho g_\sigma \} \\
- 2p_0 g_\sigma 0 g_\rho \right\} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ 1 - 2n_F(k_0) \} \times \right. \\
*G_R^\sigma (P - K) \text{Im} \left[ \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right] \\
+ \{ P_\sigma g_\rho + P_\rho g_\sigma \} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ 1 - 2n_F(k_0) \} \times \right. \\
\left. *G_R^\sigma (P - K) \text{Im} \left[ \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right] \right\}, \quad (II.8)

-B(P) = -e^2 \int \frac{d^4K}{(2\pi)^4} \left\{ 1 + 2n_B(p_0 - k_0) \right\} \text{Im} \left[ *G_R^\sigma (P - K) \right] \times \\
\left\{ [K_\sigma g_\rho + K_\rho g_\sigma - 2k_0 g_\sigma 0 g_\rho] \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \\
+ \{ 2g_\rho 2g_\sigma - g_\sigma \} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ 1 - 2n_F(k_0) \} \times \right. \\
*G_R^\sigma (P - K) \text{Im} \left[ \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right] \\
- 2k_0 g_\sigma 0 g_\rho \right\} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ 2g_\rho 2g_\sigma - g_\sigma \} \right\}, \quad (II.9)

C(P) = -e^2 \int \frac{d^4K}{(2\pi)^4} g_\sigma 0 \{ 1 + 2n_B(p_0 - k_0) \} \text{Im} \left[ *G_R^\sigma (P - K) \right] \times \\
\left\{ \frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{ 1 - 2n_F(k_0) \} \times \right. \\
*G_R^\sigma (P - K) \text{Im} \left[ \frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right] \right\}, \quad (II.10)

where the cutoff scale \Lambda is introduced in order to regularize the integrals over \kappa_0 and \kappa.

C. The ladder DS equation in the improved instantaneous exchange (IE) approximation

The DS equations, Eqs. (II.8)-(II.10), are still quite tough to be attacked, forcing us further approximations for the analysis to be effectively carried out. However, the approximation made use of must be consistent with the HTL approximation, without missing the important thermal effects out of the kernel of the DS equation.

Here it is worth noticing that the instantaneous exchange (IE) approximation to the gauge boson propagation, frequently used in the preceding analyses [6,7,8], is not compatible with the HTL approximation in a strict sense. In the exact IE-limit the HTL resummed transverse mass function, *\Pi_T^\sigma(K), vanishes and the transverse (magnetic) mode becomes totally massless. Namely the IE approximation discards the important thermal effect coming from the Landau damping, thus dismissing the dynamical screening of the magnetic mode. This causes the famous quadratic divergence of the Rutherford scattering cross section.

To see this point more clearly, let us take the IE-limit of the DS equation, Eqs. (II.8)-(II.10), and neglect Im[A(P)] and Im[C(P)], then we obtain the following equation for Im[B(P)],

\text{Im}[B(P)] = \frac{e^2}{4\pi^2} \frac{m_T^2}{E} \int_0^\infty k^2 dk \int_{-1}^1 dz \frac{1}{E} \left[ \frac{1}{E^2 + m_g^2 |z|^2} + \frac{1}{E^4} \right], \quad (II.11)

m_g^2 \equiv \frac{\lambda}{3} e^2 T^2, \quad E \equiv \sqrt{(p - k)^2}. 
showing $\text{Im}[B(P)]$ to be quadratically divergent. In order to ignore the $\text{Im}[B(P)]$, we must disregard the divergence and set $\infty = 0$. This fact means that we cannot naively neglect the imaginary parts of the $\Sigma_R$ without facing the inconsistent constraint equations.

The reason why in the previous analyses this divergence did not appear, is that the imaginary part of $\Sigma_R$ was completely neglected there from the beginning, namely that the equation for $\text{Im}[\Sigma_R]$ was totally disregarded. As is evident from the above, however, the “approximation” made use of in the previous analyses [6,8] is not the consistent approximation.

Taking the above into account, the approximation we further make use of is the improved IE approximation to the longitudinal gauge boson propagator, by keeping the exact HTL resummed transverse propagator. In the IE-limit the HTL resummed longitudinal mass function, $^\star \Pi_{\text{L}}(K)$, has a definite thermal mass $m_\text{th}^2 \sim (\epsilon T)^2$, representing the Debye screening due to thermal fluctuation, thus even in the IE limit the longitudinal mode can take into account the essential thermal effect. In the present analysis the gauge is fixed to the Landau gauge ($\xi = 0$).

Here it is fair to note that in the point vertex ladder approximation, as already mentioned before, the gauge invariance of the results is spoiled. To maximally respect the gauge invariance, we should solve Eqs. (II.8)-(II.10) with the constraint $A(P) = 1$, which guarantees $Z_2 = 1$, being consistent with the Ward identity $Z_1 = Z_2$. This can be done by successively adjusting the gauge-parameter $\xi$ in solving the above equations (II.8)-(II.10). The result of this gauge-parameter dependent analysis will be published in a separate paper [14].

III. RESULTS OF THE NUMERICAL ANALYSIS

A. Numerical solutions

Now we should solve numerically the DS equations Eqs. (II.8)-(II.10) with the IE approximation to the longitudinal mode [18]. As we have already stressed, we respect the existence of the non-trivial imaginary parts, $\text{Im}[A]$, $\text{Im}[B]$ and $\text{Im}[C]$, together with the corresponding real parts, $\text{Re}[A]$, $\text{Re}[B]$ and $\text{Re}[C]$. We use the numerical method consisting of starting with suitable trial functions for the solution and iterating the calculational procedure until stable solutions are obtained. This method is simpler to handle and is useful so long as the convergence of the iteration is guaranteed.

We choose several types of the trial functions: various choices of constants, independent of $p_0$ and $p$.

At each iteration, the three-fold integration is performed, namely over $k$, over $k_0$, which are cut off at the mass scale $\Lambda$, and over $z = \cos \theta$. The integration kernel of the present DS equation shows a little bit singular behavior, and the numerical integration of such a singular integrand needs careful integration prescription, which is properly managed. As a result, in each calculation we performed at most 1000 times iterations, and obtained fairly stable solutions.

Result of the present analysis shows that the wave function renormalization constants receive 10-20 percent corrections, see Figure 1 and Table I, indicating the necessity of gauge-parameter dependent analysis [14]. The generated size of the imaginary part is nearly the same as that of the real part, showing the existence of the non-trivial imaginary parts, which is as expected though completely neglected in the previous analyses.

B. Chiral phase transitions

Let us now show the behavior of the mass function $M(P) \equiv \text{Re}[C(P)/A(P)]$ at some fixed value of $p$ as a function of the parameters $\alpha = e^2/(4\pi)$ and $T$. All the data shown in this section are those calculated at $p_0 = 0$, which is suitable for studying the static mass. The $T$-dependence of the mass function $M(P) \equiv \text{Re}[C(P)/A(P)]$ with $p = 0.1\Lambda$ is shown in Figure 2 for various fixed values of $\alpha$, and in Figure 3 the $\alpha$-dependence of the mass function

| $T$ | $\text{Re}[A(P)]$ | $\text{Im}[A(P)]$ | $\text{Re}[B(P)]$ | $\text{Im}[B(P)]$ | $\text{Re}[C(P)]$ | $\text{Im}[C(P)]$ |
|-----|------------------|------------------|------------------|------------------|------------------|------------------|
| 0.130 | 1.262           | 0.125            | -0.547           | 0.303            | 0.496            | -0.035           |
| 0.135 | 1.227           | 0.067            | -0.551           | 0.311            | 0.466            | -0.057           |
| 0.140 | 1.248           | 0.008            | 0.496            | 0.347            | -0.341           | -0.071           |
| 0.145 | 1.249           | 0.016            | 0.416            | 0.342            | -0.157           | -0.042           |
| 0.150 | 1.217           | 0.059            | 0.378            | 0.315            | 0.000            | 0.000            |
FIG. 1: Typical shape of the function $Re[A(P)]$ at $p_0 = 0$ for $\alpha = 4.0$.

$M(P) \equiv Re[C(P)/A(P)]$ with $p = 0.1\Lambda$ is shown for various fixed values of $T$. The errors resulting from the fluctuations are smaller than the size of the symbol used for each sample point in Figures 2 and 3.

FIG. 2: The $T$-dependence of the mass function $M(P) \equiv Re[C(P)/A(P)]$ with $p = 0.1\Lambda$ for various fixed values of coupling constant $\alpha$.

From these figures we can see the two facts; i) The chiral phase transition is of second order, since a fermion mass is generated at a critical value of the temperature $T$ or at the critical coupling constant $\alpha$ without any discontinuity, and ii) the critical temperature $T_c$ at fixed value of $\alpha$ is significantly lower than the previous results [6,7,8], namely the restoration of chiral symmetry occurs at lower temperature than previously expected. The second fact shows the importance of correctly taking the essential thermal effect into the analysis, which was disregarded in the previous analyses due to the inappropriate approximations.
**C. Critical curve and critical exponents**

Analyzing those data, typically shown in Figures 2 and 3 in the last subsection, we can draw the critical curve that shows the phase-boundary in the \((T, \alpha)\)-plane. The critical curve and the critical exponents can be determined systematically from the data with the use of the following method.

We fit the mass function \(M(P)\) with \(n\) data points, \(M_i, \alpha_i\), and \(T_i\), with \(i = 1, 2, ..., n\) near the critical point by assuming the functional form

\[
M(P) = e^{C_T (T_c - T) \nu},
\]  

(III.1)

for fixed coupling constant \(\alpha\), and

\[
M(P) = e^{C_\alpha (\alpha - \alpha_c)^\eta},
\]  

(III.2)

for fixed temperature \(T\). Here, \(C_T, C_\alpha, T_c, \alpha_c, \nu\) and \(\eta\) are adjustable parameters, with \(T_c\) and \(\alpha_c\) corresponding to the critical temperature and critical coupling constant, and \(\nu\) and \(\eta\) designate the critical exponents. The values \(C_T, C_\alpha, T_c, \alpha_c, \nu\) and \(\eta\) can be determined with the method of the standard least-squares fit.

The critical coupling constant \(\alpha_c\) as a function of \(T\), and the critical temperature \(T_c\) as a function of \(\alpha\), thus determined are shown in Figures 4 and 5. By putting them together we can determine the critical curve in the \((T, \alpha^{-1})\)-plane, Figure 6, which shows two characteristic behaviors: a) As \(T\) becomes smaller, the critical coupling constant \(\alpha_c\) also becomes smaller and seems to consistently decrease from above to the zero temperature result, and b) the critical temperature \(T_c\) increases as a function of the coupling constant \(\alpha\), with the possible saturating behavior in the strong coupling limit, namely the critical curve suggests the existence of the critical temperature, above which the chiral symmetry is always restored.

With the data in the the critical curve we may estimate the critical coupling constant \(\alpha_c\) in the limit \(T \to 0\), \(\alpha_c(T \to 0)\), by assuming the functional form \((\alpha - \alpha_c(T \to 0)) \propto T^\beta\). The fitting curve for \(T \to 0\) is shown in Figure 6, by the dashed curve, with the estimated value \(\alpha_c(T \to 0) = 2.58 (1/\alpha_c(T \to 0) = 0.388)\). The estimated value, \(\alpha_c(T \to 0) = 2.58\), is, however, significantly larger than the value \(\alpha_c(T = 0) = \pi/3\) determined by theoretical analyses [1] of the DS equation for the fermion self-energy part \(\Sigma(P)\) at zero temperature, \(T = 0\), in the ladder approximation in the Landau gauge with the tree level photon propagator.

This result could be expected because of the approximation we have made use of in the present analysis. As we have already noticed, even if the photon propagator is HTL resummed, we adopted the improved IE approximation to the photon propagation. This approximation actually limits the applicability of our analysis to the rather high temperature region. In fact, at lower temperatures \(T < 0.1\Lambda\), the stability of the solution is getting spoiled.
FIG. 4: The critical coupling constant $\alpha_c$ as a function of $T$.

FIG. 5: The critical temperature $T_c$ as a function of $\alpha$.

We may also estimate the value of the critical temperature $T_c$ in the strong coupling limit $\alpha \rightarrow \infty$, $T_c(\alpha \rightarrow \infty)$, by assuming the functional form $(T_c(\alpha \rightarrow \infty) - T) \propto (\alpha^{-1})^\delta$. The fitting curve for $\alpha \rightarrow \infty$ is shown in Figure 6, by the dash-dotted curve, with the estimated value $T_c(\alpha \rightarrow \infty) = 0.155\Lambda$. The existence of the critical temperature, $T_c(\alpha \rightarrow \infty)$, above which the chiral symmetry may always be restored, was also claimed by Kondo and Yoshida [6].

The critical exponents are determined as shown in Table II. As can be seen from the table II, the estimated value of the exponent $\eta$ is fairly stable over the range of temperature $T$. The averaged value of the critical exponent $\eta$ is

$$\eta = 0.44 .$$

(III.3)

Above fact that the critical exponent $\eta$ is cleanly determined in the present analysis, strongly supports the chiral phase transition of QED at finite temperature under the change of the coupling constant to be of second order. As for the exponent $\nu$, the estimated value is not so stable over the range of coupling constant $\alpha$. The averaged value of
FIG. 6: The critical curve in the \((T, \alpha^{-1})\)-plane. The dashed curve shows the fit of the critical curve for \(T \to 0\), and the dash-dotted one the fit for \(\alpha \to \infty\) \((1/\alpha \to 0)\). The estimated values, \(\alpha_c(T \to 0)\) and \(T_c(1/\alpha \to 0)\) are shown with the symbol \(\otimes\), see text.

TABLE II: Critical exponents \(\eta\) and \(\nu\).

| \(T/\Lambda\) | \(\eta\) | \(\alpha\) | \(\nu\) |
|---------------|----------|-----------|--------|
| 0.100         | 0.450    | 2.75      | 0.504  |
| 0.105         | 0.443    | 2.90      | 0.562  |
| 0.110         | 0.452    | 3.00      | 0.535  |
| 0.115         | 0.447    | 3.25      | 0.614  |
| 0.120         | 0.436    | 3.50      | 0.538  |
| 0.125         | 0.442    | 3.75      | 0.460  |
| 0.130         | 0.445    | 4.00      | 0.616  |
| 0.135         | 0.439    | 4.25      | 0.780  |
| 0.140         | 0.425    | 4.50      | 0.498  |
| 0.145         | 0.459    | 4.75      | 0.780  |
| 0.150         | 0.446    | 5.00      | 0.644  |

the exponent \(\nu\) is

\[
\nu = 0.60 .
\]  \hspace{1cm} (III.4)

We must improve our analysis to get stable values of \(\nu\) in order to understand clearly the nature of the temperature-dependent phase transition of QED.

IV. CONCLUSION AND DISCUSSION

In this paper we investigated the consequences of the improved ladder (point vertex) DS equation in QED for the fermion self-energy with the fully HTL resummed photon propagator, where the gauge is fixed to the Landau gauge. We have made use of the improved instantaneous exchange (IE) approximation to the photon exchange, namely, we have kept the exact HTL resummed transverse (magnetic) propagator, while taking the IE limit to the longitudinal one. This approximation can account for the essential temperature effect of the HTL resummation to the photon propagator. The prices we should pay due to the two additional approximations, the ladder approximation and the IE approximation, are the loss of the gauge invariance of the results, and the limitation of the applicability of our
analysis to the rather high temperature region. Several discussion and comments on these points are given at the end of this section.

Before summarizing the results of the present investigation, we should stress the fact that the existence of non-trivial imaginary part in the fermion self-energy function $\Sigma_R$ is essentially important in the analysis of dynamical (fermion) mass generation at finite temperature. As we have shown in section II.3, we cannot naively neglect the imaginary part without facing the inconsistent constraint equations.

With this fact in mind, our results can be summarized as follows.

1. The chiral phase transition is found to proceed through the second order transition. This can be seen with the generation of the dynamical fermion mass at the critical temperature or at the critical coupling constant without any discontinuity, see Figures 2 and 3. The clean determination of the critical exponents also strengthen this fact.

2. The critical temperature $T_c$ at fixed value of $\alpha$ is significantly lower than the previous results [6,7,8], namely, the restoration of chiral symmetry occurs at lower temperature than previously expected. This fact shows that in the previous analyses the important temperature effects were thrown away due to the inappropriate approximations.

3. The critical temperature $T_c$ increases as a function of the coupling constant $\alpha$, with the saturation behavior in the stronger coupling constant region, see Figures 5 and 6. This fact may indicate that the chiral symmetry is always restored at sufficiently high temperature, no matter how large the coupling constant becomes, agreeing with the result of Ref. [6] and contradicting that of Ref. [8].

4. The critical curve determined, Figure 6, shows that the area of the chiral broken (symmetric) phase is significantly smaller (larger) than previously expected [6,7,8]. This fact proves that the effect of thermal fluctuation to liberate the chiral symmetry is stronger than previously considered.

5. The critical exponent $\eta$, that describes the phase transition at finite temperature under the change of the coupling constant, is determined cleanly, $\eta = 0.44$. This fact also supports strongly the phase transition of QED at finite temperature to be of second order.

All the above results show the importance of correctly taking the thermal effects into the analysis of chiral phase transition. Inclusion of only the HTL resummed gauge boson (photon) propagator has significantly changed even the qualitative behavior of the phase transition. This fact proves a posteriori the effectiveness of the DS equation in the HTL approximation, thus indicates the correctness of our research-strategy, namely, the importance of the full HTL resummed DS equation analysis of the chiral phase transition at finite temperature/density. Further investigation along the line of our strategy is needed to answer step by step the question how we can closely take the essential thermal effects into the “kernel” of the DS equation.

As noted above, however, in the present analysis, there are two shortcomings. One is the loss of the gauge invariance of the result due to the ladder approximation, and the other the limitation of the applicability of our analysis to the rather high temperature region due to the IE approximation. The first improvement we should make is to resolve these problems.

As shown in section III.3, see Figure 1 and Table I, all the fermion wave function renormalization constants receive more than 10-20 percent corrections. This fact means $Z_2 \neq 1$, and shows the explicit violation of the gauge invariance of the result, since $Z_1 = 1$ due to the point-vertex approximation, thus indicating the necessity of the gauge-parameter dependent analysis. In order to maximally respect the gauge invariance of the result in the improved ladder approximation, we should keep explicitly the gauge-parameter dependent contributions to the DS equations, Eqs. (II.8)-(II.10), and solve them with the constraint $A(P) = 1$, namely, $Re[A(P)] = 1$ and $Im[A(P)] = 0$, which guarantees $Z_2 = 1$ as required, being consistent with the Ward identity $Z_1 = Z_2$. This can be carried out by successively adjusting the gauge-parameter $\xi$ in solving the above equations (, which may correspond to choose some non-linear gauge). The result of this “gauge invariant” analysis within the improved ladder approximation will be given in a separate paper [14].

The second problem of the limitation of the applicability of our analysis to the rather high temperature region due to the IE approximation, casts the shadow on the reliability of estimated value of $\alpha_c(T \to 0) = 2.576$. Actually, the $T \to 0$ limit of the DS equations with the IE approximation, Eqs. (II.8)-(II.10), do not coincide with the DS equations in the vacuum ($T = 0$) theory. Thus to get more reliable estimate of the critical coupling constant in the zero-temperature limit, we must perform the analysis by getting rid of the IE approximation to the photon propagation, which is now under investigation.

Acknowledgment
We thank the useful discussion at the Workshop on Thermal Quantum Field Theories and their Applications, held at the Yukawa Institute for Theoretical Physics, Kyoto, Japan, 8 – 10 August 2002. This work is partly supported by Grant-in-Aid of Nara University, 2002 (HN). The numerical calculation is mainly done at the Computer Center, Nara University.

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