Can Linear Programs Have Adversarial Examples?
A Causal Perspective

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Abstract

The recent years have been marked by extended research on adversarial attacks, especially on deep neural networks. With this work we intend on posing and investigating the question of whether the phenomenon might be more general in nature, that is, adversarial-style attacks outside classification. Specifically, we investigate optimization problems starting with Linear Programs (LPs). We start off by demonstrating the shortcoming of a naive mapping between the formalism of adversarial examples and LPs, to then reveal how we can provide the missing piece—intriguingly, through the Pearlian notion of Causality. Characteristically, we show the direct influence of the Structural Causal Model (SCM) onto the subsequent LP optimization, which ultimately exposes a notion of confounding in LPs (inherited by said SCM) that allows for adversarial-style attacks. We provide both the general proof formally alongside existential proofs of such intriguing LP-parameterizations based on SCM for three combinatorial problems, namely Linear Assignment, Shortest Path and a real world problem of energy systems.

1 Introduction

Adversarial attacks have gained a lot of traction in recent years [Brendel et al., 2018; Ilyas et al., 2018; Guo et al., 2019] as there has been a lot of focus on safety and robustness of machine learning (ML) systems. An interesting observation, though, is that deep neural networks or rather over-parameterized models are the center of attention for most of such adversarial attacks [Zügner et al., 2018; Akhtar and Mian, 2018; Chen et al., 2018]. We argue that this view is too narrow—the phenomenon around adversarials is more general in nature and actually depends on the problem setup. We conjecture that any differentiable perturbed optimizer (DPO) is prone to a new notion of attack similar to classical adversarials. DPOs are a well studied, pragmatic approach to differentiability of mathematical program (MP) solver by means of perturbation [Papandreou and Yuille, 2011; Berthet et al., 2020; Gumbel, 1954; Bach, 2013]. If our conjecture were to be true, then our new view on adversarial attacks would stand as a very general problem beyond learning to classify. There has been previous works where MPs such as Linear Programs (LP) and Mixed Integer Programs (MIP) [Wu et al., 2020; Tjeng et al., 2019] have been used to compute adversarial attacks but not where such optimization modules (LP, MIP) themselves have been confronted with the attacks. In fact, and also due to the recent interest in tightly integrating MPs and deep learning [Amos and Kolter, 2017; Paulus et al., 2021], this extension of adversarial attacks beyond deep networks already significantly advances our understanding of adversarial attacks i.e., it is not just expressiveness that leads to uninterpretable solutions with counter-intuitive properties.

Interestingly, it turns out that the new type of attack we formalize develops naturally from the Pearlian notion of Causality [Pearl, 2009] when starting from the formalism of classical adversarial attacks. Causality refers to a very general idea, in that understanding causal interactions is even central to human cognition and thereby of high value to science, engineering, business, and law [Penn and Povinelli, 2007]. In the last decade, causality has been thoroughly formalized in various instances [Pearl, 2009; Peters et al., 2017; Hernán and Robins, 2020]. At its core lies a Structural Causal
Model (SCM) which is considered to be the model of reality responsible for data-generation. The SCM implies a graph structure over its modelled variables, can reason about (hidden) confounders, and of course handle both interventions and counterfactuals. The richness of the SCM has been crucial for its successful application for ML in marketing [Hair Jr and Sarstedt, 2021], healthcare [Bica et al., 2020] and education [Hoiles and Schaar, 2016]. While we conjecture applicability to the general class of DPO (MP), the focus of this work will be to motivate, illustrate and finally prove formally that we can exploit an SCM’s hidden confounders to construct a new type of attack based on the classical adversarial attacks in order to attack LPs—the basic sub-class of MPs. We coin this new attack Hidden Confounder Attacks, since exploiting knowledge of hidden confounders is a sufficient condition for construction.

Overall, we make a number of key contributions: (1) We first derive a novel theoretical connection between causality’s SCMs and LPs, by which we then (2) use the hidden confounders of the SCM to devise an adversarial-style attack—which we call Hidden Confounder Attack (HCA)—onto the LP showing that non-classification problems can be prone to adversarial-style attacks; (3) We study and discuss two classical LP families and one real world applied optimization problem to further motivate research on HCA and their potentially worrisome consequences regarding security. For reproduction, we make our code repository publicly available: https://anonymous.4open.science/r/Hidden-Conf-Attacks/.

2 Background and Related Work

Let us briefly review the background on adversarial attacks as defined in their original setting of classification, then the formalism of LPs alongside two famous problem instances, and finally SCMs with their mechanism and hidden confounders.

Adversarial Attacks. By using a simple optimization procedure, Szegedy et al. [2014] were able to find adversarial examples, which they defined to be imperceptibly perturbed input images such that these new images were no longer classified correctly by the predictive neural model. Goodfellow et al. [2015] then proposed the Fast Gradient Sign Method (FGSM) that considers the gradient of the error of the classifier w.r.t to the input image. Mathematically, they investigated perturbations of the form

$$\eta := \epsilon \cdot \text{sign}(\nabla_x J(x; y; \theta)) \in \mathbb{R}^{w \times h \times c}$$

(1)

where \(x \in \mathbb{R}^{w \times h \times c}\) is the input image, \(y \in \mathbb{N}\) a class label, \(\theta\) are the neural function approximator parameter, \(J: \mathbb{R}^{w \times h \times c} \times \mathbb{N} \rightarrow \mathbb{R}\) a scalar-valued objective function, \(\text{sign}: \mathbb{R} \rightarrow [-1, 1]\) an element-wise sign function and \(\epsilon \in \mathbb{R}\) a free-parameter. A perturbation \(\eta\) would then account for mis-classification of the given predictive model \(f(x; \theta)\) i.e.,

$$f(x; \theta) = y \neq f(x + \eta; \theta).$$

(2)

The inequality in Eq.2 represents a possibly strongly significant divergence from the expected semantic meaning (i.e., from a human inspector’s perspective) of the class to be predicted. For example, imagine a photograph of a dog that is being classified by \(f\) as a dog. However, the perturbed image \(x + \eta\) is not classified by \(f\) as dog. What came to a surprise for many is two-fold, (1) the new classification could be something drastically different e.g. not another animal like a cat but for instance a washing machine (2) from a human perspective the perturbed image would still lead to the same classification i.e., still a dog. Naturally, said susceptibility (1-2) led to a significant increase in research interest regarding robustness (to adversarial examples) in neural function approximators evoking the narrative of “attacks” and subsequent “defences” on the inspected classification modules as commonly found in cyber-security [Handa et al., 2019].

Mathematical Programming. Selecting the best candidate from some given set with regard to some criterion is a general description of MPs (or optimization), which arguably lies at the core of machine learning and many applications in science and engineering. Classification, e.g. can be considered as a special instance of mathematical programming. An important optimization family are LPs that are concerned with the optimization of an objective function and constraints that are linear in the respective optimization variables. LPs are being applied widely in the real world, e.g., energy systems [Schaber et al., 2012]. More formally, the optimal solution of an LP is given by

$$x^* = \arg\max_{x} \text{LP}(x; w, A, b)$$

$$= \arg\max_{x \in P_{A,b}} (w, x).$$

(3)

(4)
The set of endogenous variables on which SCM operates first present how a naive mapping of the classical adversarial attack framework fails for LPs. Then, with an informal description first before providing the mathematical formalism. Fig.1 will act as the formalization later will require is the concept of causal sufficiency. Following Spirtes [2010]:

A semi-Markovian thus allowing for shared confounder since it is a common cause of at least two variables. Opposed to a hidden confounder, a “common” confounder would be a common cause from within V. An important concept that the formalization later on will require is the concept of causal sufficiency. Following Spirtes [2010]:

Table 1: Classical Problems formulable as LPs. Linear Assignment (left; abbrev. LA) and Shortest Path (right; SP). In LA one matches “workers” to “jobs” according to their “skills”. In SP one finds the “quickest” path from some node i to node j.

where \( \langle a, b \rangle := a^T b = \sum_i a_i b_i \in \mathbb{R} \) is the inner product (dot product), \( w \in \mathbb{R}^n \) is called the weight/cost vector, \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) are the constraint matrix and vector respectively, and finally \( P_{A,b} \) is the solution polytope (or feasible region) i.e., the convex subset of state space \( X \) s.t. each \( x \) satisfies the constraints. Formally, \( P_{A,b} := \{ x | Ax \leq b \text{ and } x \geq 0 \} \subset \mathbb{R}^n \) (if clear from context, we abbreviate to \( P \)). Table 1 presents two classical problems that can be expressed as linear programs: the Linear Assignment Problem (LA) and the Shortest Path Problem (SP). Both problems formulate the optimization variable \( x \in \mathbb{R}^n \) with either \( n = |A \times B| \) or \( n = |E| \) to be a selector, that is, being worker-job matches for the LA problem and edges part of the selected shortest path for the SP problem respectively. Although the original formulation of the LA and SP problems are actually integer LP formulations, which are generally known to be NP-complete opposed to the less restrictive regular LPs, both problems can be solved in polynomial time. However, extensions of regular SP like the Travelling Salesman or the Canadian Traveller problems are known to be NP- and PSPACE-complete respectively.

Causality. Following the Pearlian notion of Causality [Pearl, 2009], an SCM is defined as a 4-tuple \( M := \langle U, V, F, P(U) \rangle \) where the so-called structural equations

\[
v_i \leftarrow f_i(p_{a_i}, u_i) \in F
\]

assign values (denoted by lowercase letters) to the respective endogenous/system variables \( V_i \in V \)
based on the values of their parents \( Pa_i \subseteq V \setminus V_i \) and the values of their respective exogenous/noise/nature variables \( U_i \subseteq U \), and \( P(U) \) denotes the probability function defined over \( U \). Note that, opposed to the Markovian SCM discussed in for instance [Peters et al., 2017], the definition of \( M \) is semi-Markovian thus allowing for shared \( U \) between the different \( V_i \). Such a \( U \) is also called hidden confounder since it is a common cause of at least two variables. Opposed to a hidden confounder, a “common” confounder would be a common cause from within \( V \). An important concept that the formalization later on will require is the concept of causal sufficiency. Following Spirtes [2010]:

“The set of endogenous variables on which SCM \( M \) acts is called causally sufficient if there exist no hidden confounders.” For completion we mention more interesting properties of any SCM, they induce a causal graph \( G \), they induce an observational/associational distribution denoted \( p^M \), and they can generate infinitely many interventional and counterfactual distributions using the do-operator\(^1\).

3 Generalized Adversarial Perspective

We first present how a naive mapping of the classical adversarial attack framework fails for LPs. Then we present how causality can provide additional semantics to formulate this new adversarial-style attack, that we refer to as Hidden Confounder Attacks. To provide for a better intuition, we start off with an informal description first before providing the mathematical formalism. Fig.1 will act as the lead example.

3.1 Naive Mapping: Adversarial Attacks on LPs

In the past, different classes of MPs (LPs, MIPs) have been used defensively for verifying the robustness of neural learners to adversarial examples [Tjeng et al., 2019] and offensively for generating actual adversarial examples [Zhou et al., 2020]. Here, we are concerned with a fundamentally different research question: “How do adversarial attacks affect MPs themselves?” That is, instead of considering MPs as a service to the system to be attacked, we consider the programs themselves to be under attack.

\(^1\)The do-operation “overwrites” structural equations.
**Figure 1:** Lead Example: Hidden Confounder Attack on LP. A real world inspired matching problem under attack. The new matching shows significant bias for individuals with high wealth value. The adversarial LP cost vector is close to the original both value-wise (left) and cost-wise w.r.t. their optimal solution (middle) which means in words that health-wise people in higher need of vaccination are still guaranteed a vaccine spot. However, w.r.t. hidden confounder (Wealth) the adversarial solution drastically deviates (right) i.e., the distribution of vaccines is being skewed towards people of higher wealth, which can circumvent the originally intended policy. (Best viewed in Color.)

**Hypothesis.** Adversarial attacks refer to a more general concept that affects MPs and thus being a property of the problem specification and not per se a property of the expressiveness of deep models or of the classification task.

To the best of our knowledge, we are the first to ask and investigate this question thoroughly. Therefore, in order to establish an initial connection between adversarial attacks and MPs we will start off with the most basic class of MPs: Linear Programs. Since an adversarial attack depends on gradient information, we also require such first-order information from our LPs. One way to achieve this is to consider the class of ML models which inject some noise that is distributed w.r.t. some differentiable probability distribution into the LP optimizer. These so-called perturbed models have been considered for inference tasks within energy models [Papandreou and Yuille, 2011] and regularization in online settings [Abernethy et al., 2014]. Initial works in this research direction date back to the Gumbel-max [Gumbel, 1954] and were recently generalized to Differentiable Perturbed Optimizers (DPO) featuring end-to-end learnability [Berthet et al., 2020]. To formulate an LP optimizer, \( \mathbf{x}^*(\mathbf{w}) = \arg \max_{\mathbf{x} \in \mathcal{P}_{A,b}} \langle \mathbf{w}, \mathbf{x} \rangle \), as a DPO one requires only the existence of a random noise vector \( \mathbf{z} \in \mathbb{R}^n \) with positive and differentiable density \( p(\mathbf{z}) \) such that for \( \epsilon \in \mathbb{R}_{>0} \),

\[
\mathbf{x}^*(\mathbf{\hat{w}}) = E_{p(\mathbf{z})}[\arg \max_{\mathbf{x} \in \mathcal{P}_{A,b}} \langle \mathbf{w} + \epsilon \mathbf{z}, \mathbf{x} \rangle], \tag{6}
\]

where \( \mathbf{x} \in \mathbb{R}^n \) is the optimization variable living in the solution polytope \( \mathcal{P}_{A,b} \) described by LP constraints \( A, b \), where \( \mathbf{\hat{w}} := \mathbf{w} + \epsilon \mathbf{z} \) is the perturbed LP cost parameterization, \( \langle \cdot, \cdot \rangle \in \mathbb{R} \in \mathbb{R}^n \) the inner product and \( E_{p(\mathbf{X})}[f(\mathbf{X})] \) the expected value of random variable \( \mathbf{X} \) under the predictive model \( f \).

Related work on differentiability of more general MPs like quadratic/cone programs [Agrawal et al., 2019] or linear optimization within predict-and-optimize settings [Mandi and Guns, 2020] generally rely on the Karush-Kuhn-Tucker (KKT) conditions. The general advantage of a perturbation method (as in Eq.6) over the analytical approaches is its “black-box” nature i.e., we don’t require to know what kind of MP we are dealing with, since we simply add stochasticity into the problem. This “invariance” to the underlying MP and the fact that differentiability is a necessary key concept behind adversarial attacks, leads to following (informally stated) conjecture.

**Conjecture 1 (HCAs on MPs, informal).** Differentiability of the MPs is a necessary condition for constructing Hidden Confounder Attacks (to be defined in Sec.3.3).
Let’s start by providing a naive mapping between the classical adversarial attack and the famous class of LPs known as Linear Assignment (LA), both of which have been previously introduced in Sec. 2. Mathematically, the following correspondence can be found,

\[ x + \eta := \hat{w}, \quad f_0 := x^*(), \quad y := x^*(w), \quad J := F(\hat{w}, w), \]

(7)

where \( x \) is the feature vector (e.g. an image), \( y \) the class label, \( f_0 \) the neutral predictive model, and \( J \) the cost function (e.g. mean-squared error)—all of these follow the notation from [Goodfellow et al., 2015]. While from the LP perspective we interpret \( w \in \mathbb{R}^{|A|\times|B|} \) as describing the suitability of worker \( a \in A \) for job \( b \in B \) and the optimal solution \( x^*(w) \in [0, 1]^{|A|\times|B|} \) as indicators highlighting the matched pairs \( (a_i, b_j) \). The only addition we have to make to achieve the naive mapping is some distance measure \( F \) acting on the original \( x^*(w) \) and the expected perturbed solution \( x^*(\hat{w}) \) since we need to allow for a “class” change that should occur through an adversarial attack i.e., in the case of LA, \( F \) could be for instance the Structural Hemming Distance [Hamming, 1950]. Also note, in slight abuse of notation our program solver \( x^*(\cdot) \) denotes \( \arg\max_x \text{LP}(x; w, A, b) \).

Having completed the naive mapping in LA specifically, we could then naively view each optimal matching “code” \( x^*(w) \) as a certain class (or label) and then the gradient \( \nabla_w F \) could be used for performing an “adversarial attack” such that the “class” changes (significantly) while the input remains approximately the same. As one can easily make out, the major problem being faced here is that there exists no “semantic impact” to be observed for the human inspector akin to a neural network wrongly classifying a dog (small animal) as a plane (big travel machine). To make this point more clear:

**Missing Semantic Component.** The human inspector’s invariance to the adversarial example is characteristic of an adversarial attack (e.g. still dog-looking image being classified as depicting a washing machine), while a naive mapping to LPs as in (7) leaves out said human component. In other words, there is no general human expectation towards different optimal solutions to an LP.

### 3.2 Causality Provides Semantic Component

In the previous section we concluded that there is a missing semantic component when naively mapping between adversarials and LPs. Yet, the optimal solutions when considered in terms of codes as in the LA example will actually significantly deviate from each other. This deviation (or difference) seems to suggest that there exists some more “fundamental” difference in solution albeit not for the specific optimization objective at hand since the cost values will remain similar (or even the same). But as suggested by the missing semantic component, a human inspector will have no general expectation towards either of the LP solutions. To put it differently, “they look different but that is that”. Nonetheless, to complete the picture it is worth taking a step back and observing the LP on a “meta” level. On this meta level, we can ask the question of how the LP cost vector \( w \) was provided in the first place. Here causality and its Structural Causal Model come into play. The SCM \( \mathcal{M} \) defines a mechanistic data-generating process which will generate the observational distribution \( p^M \) that the human modeller usually observes an empirical fraction of and denotes as data \( D \). So, to loop back to the meta-question of how the LP parameterization came to be, we observe the following relation for some function \( \phi \):

\[ \phi(D) = w \quad \text{where } D \sim p^M. \]

(8)

According to Eq.8, the human modeller that takes the observed data as a basis for determining the cost vector of the LP and then uses some function-mapping between the SCM and the LP denoted as \( \phi \) to produce said cost vector. To give a concrete example of such a modelling, consider Fig.2 in which the human modeller observes a data set \( D := \{(h_i, p_i)\}_{i}^n \sim p(H, P) = p^M \) but no other information. The human modeller does neither observe the SCM induced distribution \( p^M \) nor any more information about the partial SCM \( \mathcal{M} \) which would include knowledge on the structural equations \( f_H, f_P \) and the hidden confounder \( U_C \) (i.e., \( U_C \) is shared by both equations). Therefore, also no information on the true SCM \( \mathcal{M}^* \) either, where \( U_C \) would be part of the endogenous variables (i.e., there would also be \( f_W \) and all \( U_C \) being replaced by \( W \) standing for Wealth). The only knowledge available is the data set \( D \sim p^M \) which numerically describes the Health \( (H) \) and a
Intriguing LP-Parameterizations Based on SCM. The observed observational distribution \( p(H, P) \) was generated by some unobserved SCM. Even if we had some SCM, it might not be the actual underlying, complete SCM i.e., there could still be hidden confounders in our estimate. The cost vector \( w \) of an LP can be viewed as a function \( \phi \) applied to population individuals \((h, p) \sim p(H, P)\). Intriguingly, \( \phi \) may very well be unaware of confounders.

The general notion of Vaccine Priority \((P)\) of certain individuals. Note that a simple linear regression on the data shown in Fig.2 reveals the existence of a causal relation between \( H \) and \( P \) (but not the direction). The true causal direction is read as “lower health values cause higher priority values” and written \( H \rightarrow P \). However, \( P \) describes other factors as well (for instance the age of an individual) since \( P \) also depends on \( U_P \) and not only on its cause \( H \) and the hidden confounder \( U_C \).

As in the previous section, let’s consider an LA problem as our LP instance. The human modeller is trying to find the optimal matching of individuals (based on the data) to respective, available vaccination spots. The human modeller might choose to find the cost parameterization of the LP that will determine the optimal matching by following some sort of policy (or rule) like “individuals of low health and high priority should be matched to vaccine spots, while others can wait”. In this case, we reason that the human modeller implicitly performed a mapping \( \phi \) as in Eq.8 based on the observed data, and this \( \phi \) essentially captures the policy description. Both interestingly and intriguingly, by construction, \( \phi \) does not consider the hidden confounder \( U_C \) (that might be viewed as something like wealth of an individual) since \( U_C \) is not even defined within the data distribution accessible to the human modeller (since \( M \neq M^* \)). This ignorance is the key to defining a meaningful adversarial-style attack on LPs. Now, our key result might be stated informally as:

**Theorem 1 (HCAs on LPs, informal).** Let \( \phi_M \) denote an LP parameterization based on SCM \( M \) while \( M^* \) denotes the “true” or optimal SCM for the phenomenon of interest. Any \( \phi_M \) that identifies individuals in \( M \) is prone to Hidden Confounder Attacks (to be defined in Sec.3.3) unless \( M = M^* \).

Put loosely, choosing the “wrong” \( \phi_M \), one that does not represent the true, underlying SCM \( M^* \), allows for adversarial-style attacks on LPs. In summary we state:

**HCA Exploits Matching Bias, Fig.1:** The LP-parameterization \( \phi_M \) based on the SCM \( M \), with \( w = \phi(D) \) from Eq.8 where \( w \) is the LP cost vector, follows the previously, informally described policy, so \( \phi_M \) models only \( H, P \) since \( U_C \) is not even defined in \( D \). Therefore, Thm.1 predicts the existence of a HCA for LP\((x; w, A, b)\). Fig.1 reveals such an example HCA: the attack is unnoticeable in visual terms, just as for a classical adversarial example, and so is the difference in cost w.r.t. the optimal matchings—however, w.r.t. to the wealth value of each individual, the new matching shows a significant skew towards individuals (or data points) of higher wealth value.

Next, to complete the discussion, we finally formalize the informal notions presented in this section.

### 3.3 Hidden Confounder Attacks

We now introduce formalism to capture all the previously established ideas mathematically, we start with Eq.8.

**Definition 1.** We call a function \( \phi_M \) LP-parameterization based on SCM \( M \) if for an observational data set \( D \sim p^M \) we can define an LP\((x; \phi(D), A, b)\).
By default (unless clear from context), we might refer to such $\phi_M$ simply as LP-parameterizations. Some LP-parameterizations fulfill a property that “identifies individuals in $M$” which we define next.

**Definition 2.** Let $\phi_M$ be an LP-parameterization based on SCM $M$ with $\phi_M(D) = w = (w_1, ..., w_k)$ and $D = \{d_i\}_{i \in I}$. We call $\phi_M$ integral if it satisfies:

$$\forall i \in \{1, ..., n\}. \exists I \subset \{1, ..., k\}. (d_i = \phi_M^{-1}(w_j)_{j \in I}).$$

In words, the parameterization decomposes on the the data point (or unit) level.

A simple yet important insight that immediately follows.

**Corollary 1.** The LP problems Linear Assignment (LA) and Shortest Path (SP) have integral LP-parameterizations.

**Proof.** For LA, simply map each data point indexed by $i$ to the cost vector slice indexed by indices in the set $I$. For each $i$ refers to the same unique $a$ from the “workers” set $A$ for all the different “jobs” $b \in B$. I.e., one data point sampled from the SCM’s observational distribution will correspond to one worker in the LP cost vector. For SP, there is a more direct one-to-one correspondence where each data point is mapped to a unique edge in the graph. \qed

We now state our two main assumptions:

**Assumption 1.** For some fixed constraints $A, b$ let $X^*(w) := \{x | x = \arg \max_x LP(x; w, A, b)\}$ denote the set of optimal LP solutions under $w$. Further let $B^w_\epsilon$ denote an $\epsilon$-Ball with $\epsilon > 0$ around some LP cost $w$. Then there exists a $w \in B_\epsilon^w$ such that $|X^*(w)| > 1$. In words, we can find an $\epsilon$-close LP instance with multiple optimal solutions.

**Assumption 2.** Like before, let $X^*(w)$ be the set of optimal LP solutions under cost vector $w$. Further, let $x^*(w) \in X^*(w)$ denote the solution returned by our solver and let $\tilde{w} = w + \epsilon \nabla_w F$ be the perturbed cost vector for some function $F$ and $\epsilon > 0$. We assume $x^*(w) \neq x^*(\tilde{w})$. In words, the perturbed LP instance returns a different optimal solution.

Arguably, both assumptions are fairly weak and compare to what can be found in adversarial learning literature; yet, it is crucial to make them explicit both for clarity on the given setting and for proving our theorem of HCA on LPs. Now, we are set to give the technical description of what a Hidden Confounder Attack really is.

**Definition 3.** Let $\phi_M$ be an LP-parameterization based on SCM $M$. We call $\phi_M$ prone to Hidden Confounder Attacks if there exists an injective function $h : X^* \rightarrow \mathbb{R}$ with properties

1. $h(x^*) = f(\bigoplus_{i \in x^*} M'_{C}(i))$ and

2. $\exists \tilde{w}. (h(x^*(w)) \neq h(x^*(\tilde{w})))$

for some function $f : \mathbb{R} \rightarrow \mathbb{R}$, aggregation function $\bigoplus$ over units $i$ active in $x^*$ (e.g. the sum for all matched “workers”), and LP cost vector $w$ where

$$M'_{C} : \mathbb{N} \rightarrow Val(C)$$

is the projection of a unit $i$ to its respective confounder value $M'_{C}(i) \in Val(C)$ where $M'$ is an alternate SCM containing $C$. That is, $C$ is a hidden confounder of $M$.

In simple terms, property 1 in Def.3 refers to the uncountable number of functions that can leverage information on the hidden confounder by using the alternate SCM $M'$ to distinguish between different optimal LP solutions which is required by property 2 (since otherwise, there would be no observed difference, alas no attack). Finally, we can provide our key result in its complete formal version.

**Theorem 1 (HCA on LPs, formal).** Let $\phi_M$ be an integral LP-parameterization based on SCM $M$, then we have:

- $M$ is causally insufficient $\iff$ $\phi_M$ is prone to HCA.
Proof. “implies”: As discussed in Sec.2, for an SCM $\mathcal{M} = (\mathbf{U}, \mathbf{V}, \mathbf{F}, P(\mathbf{U}))$ to be causally sufficient there must exist at least one hidden confounder, denoted $C$, that is not an endogenous variable, $C \notin \mathbf{V}$. Therefore, for any LP-parameterization $\phi_{\mathcal{M}}$ and any LP cost vector $\mathbf{w}$, the latter also doesn’t depend on $C$. Then, take an alternate $\mathcal{M}' = (\mathbf{U}', \mathbf{V}', \mathbf{F}', P'(\mathbf{U}))$ for which $C \in \mathbf{V}'$ and construct $h$ as described by property 1 from Def.3, which is guaranteed to exist since $\phi_{\mathcal{M}}$ is integral. Through Assumption 1 we have guaranteed multiple optimal LP solutions for $\text{LP}(\mathbf{x}; \mathbf{w}, \mathbf{A}, \mathbf{b})$ to choose from. On the other hand, through Assumption 2 we can perturb said LP cost vector $\mathbf{w}$ such that $\mathbf{x}^*(\mathbf{w}) \neq \mathbf{x}^*(\hat{\mathbf{w}})$. Since $h$ is injective, we have that $h(\mathbf{x}^*(\mathbf{w})) \neq h(\mathbf{x}^*(\hat{\mathbf{w}}))$ which is property 2 of Def.3 completing the implication.

“$\Longleftarrow$”: Trivial, since HCA (Def.3) are defined as attacks that exploit hidden confounders. □

This fundamental Theorem of our formalism on HCA guarantees us that the existence of hidden confounders implies the susceptibility of LPs to HCAs. In fact, we can even construct an uncountable number of HCAs based on said confounders. We further argue that the HCAs that follow from Thm.1 are highly non-trivial in the sense that they exploit information “outside of the data” i.e., assuming that the human modeller only uses a causally insufficient $\phi_{\mathcal{M}}$, then an adversary is guaranteed a chance to exploit his/her better knowledge on the true, underlying SCM to perform an attack. Also, a simple corollary of Thm.1 is that LA and SP are always prone to HCAs since they have integral $\phi_{\mathcal{M}}$ and $\mathcal{M} = \mathcal{M}'$ is unlikely.

4 Other Examples

Our discussion throughout most of the main section was alongside our lead example (illustrated in Figs.1,2) that provided an existential proof of what we formally proved by our Thoerem on HCAs on LPs (Thm.1). We constructed a synthetic ground-truth SCM $\mathcal{M}^*$ using all three variables $H, P, W$ from which we then created the sub-SCM $\mathcal{M}$ in which $W$ was a hidden confounder denoted $U_C$. Simulating $\mathcal{M}$ produced the data set in Fig.2, $D \sim p^\mathcal{M}$. As in Cor.1, we chose an integral $\phi_{\mathcal{M}}$ mapping each data point separately. Finally, the HCA was completed with $h$ being defined as

$$h(\mathbf{x}^*) = \sum_{i \in \mathbf{x}^*} \mathcal{M}_C^*(i)$$

meaning that the different optimal solutions (in this case each of the chosen matchings between individuals and vaccine spots) differed in their summed up wealth values.

Energy-Systems LP. We considered an energy model for modelling the energy portfolio of a single-family house based on real world data for demand and commonly used equations from energy systems research Schaber et al. [2012]. The examined model considers photovoltaics (PV), market electricity and heating gas over a year time frame (in hours) and resembles a simplified version of the TIMES model Loulou et al. [2005]. The optimal solution balances the usage of the different technologies for matching the required demand such that overall cost is being minimized. The specific LP template is given in Tab.2. Note that $t \in \{0, \ldots, 8760\}$ with 1 year = 8760 hours rendering the template a very large single LP modelling each hour of the year. However, technologies like PV, in their capacity (C_{ap,PV}), do not depend on $t$ which would correspond to the real world intuition that one does not decide and subsequently build new PV for any given hour as it poses a single, fixed-timeframe investment. We use heuristics for the SCM-part to produce an approximation to HCA, since no notion of SCM is being provided a-priori. We then perform said approximation to produce results in Tab.3. We observe the expected setting of “dominating technologies” where PV is being preferred over market-bought electricity, which can lead to an increased risk in working injury or fire (hidden confounders in this case). The limitations on PV-production and Market-buy of electricity act as discrepancy counter-measures that require the system to balance out different technologies i.e., while there will be dominating technologies under price advantages the maximum skew of the portfolio is naturally being protected from being too drastic as both PV and bought electricity are limited in availability (e.g. solar exposure, roof capacity, regulations etc.) and so cannot be maximized naively.

Shortest Path LP. Due to space restrictions the example is being highlighted in the Appendix.

5 Conclusive Discussion

Ultimately, we believe HCAs to be a fundamental problem of mathematical optimization—to the same extent that hidden confounding is a fundamental problem of the Pearlian notion of causality (or science
\[
\min_{\text{Cap}_p} \quad c_{PV} \times \text{Cap}_{PV} + c_{Bat} \times \text{Cap}_{Bat} + \sum_t c_{Ele} \times p_{Ele}(t) + \sum_t c_{Gas} \times p_{Gas}(t)
\]
\[
s.t. \quad p_{Ele}(t) + p_{PV}(t) + p_{Bat}^{out}(t) - p_{Bat}^{in}(t) + p_{Gas}(t) = D(t), \forall t \quad 0 \leq p_{Ele}
\]
\[
\quad \quad p_{Bat}(t) = p_{Bat}(t-1) + p_{Bat}^{out}(t) - p_{Bat}^{in}(t), t \in \{2, \ldots, T\}, p_{Bat}(0) = 0
\]
\[
0 \leq p_{PV}(t) \leq \text{Cap}_{PV} \times \text{avail}_{PV}(t) \times \delta, \forall t
\]
\[
0 \leq p_{in}(t), p_{out}(t) \leq \text{Cap}_{Bat}, \forall t
\]
\[
0 \leq p_{Gas}(t) \leq U_{Gas}, \forall t
\]

**Table 2:** Real-world Optimization Modelling Example: 1-year Energy Systems LP for an Average Household. A large LP that unrolls for 8760 time steps (8760 hours = 1 year). Model based on [Schaber et al., 2012], the quantities represent: Cost for Photovoltaics \(c_{PV}\) (€/kWh), Battery \(c_{Bat}\) (€/kWh), Market Electricity \(c_{Ele}\) (€/kWh), Gas \(c_{Gas}\) (€/kWh), and the total Demand \(D\) (kWh/Year).

| Dem. (h) | \(\text{Cap}_{PV}\) | \(\text{Cap}_{Bat}\) | Self-Gen. | TOTEX | CAPEX | \(\text{Con}_{Gas}\) | \(\text{Con}_{Ele}\) | \(w_{PV}\) |
|---------|-----------------|------------------|-----------|-------|-------|-----------------|-----------------|--------|
| 3000    | 1.76            | 2.45             | 0.42      | 597.41| 161.64| 1.70            | 1743.06        | .005   |
| 3000    | 7.15            | 4.78             | 0.66      | 468.24| 214.87| 1.95            | 1013.49        | .001   |

**Table 3:** Dominating Technologies. Price perturbations \(w_{PV}\) can boost PV production \(\text{Cap}_{PV}\) (green) which leads to a significant increase in risk of working injury or fire.

in general). It is intriguing that the Pearlian notion of causality allowed for bridging the gap between classical adversarial attacks from deep learning and the first, basic class of mathematical optimization, LPs. Naturally, the question arises on the severity of HCA and we believe that our examples have shown potentially worrisome real-world implications. Especially our lead example in Fig.1 captures the Zeitgeist of the pandemic era. To ask the inverse question on how to defend against HCA is equally important, yet, we believe this question to be ill-posed to begin with. Essentially, *Thm.1 suggests an equivalence on the existence of hidden confounders and such attacks.* Put differently, as long as our model assumptions are imperfect, we are exploitable—again, giving us reason to believe HCA to be of fundamental nature. Therefore, we suggest to study HCA beyond LPs (as initially conjectured, see Conj.1) but also alternate notions of attacks, like HCA as a representative example, that might in fact not be based on causal knowledge (confounders) to begin with. Since we were able to develop HCA from first principles, by starting from classical adversarial attacks and naive mappings to LPs, we may be tempted to believe in the existence of related families of attacks (which HCA is an existential example of). From a theoretical standpoint the question of whether the integral property (Def.2) applied to Thm.1 could be dropped might be interesting for broadening the applicability of HCA, while from a practical standpoint an adequate real-world example might be of great presentation interest since it could increase future research activity. Furthermore, we observe this work to form a triangular relationship to the works of [Ilse et al., 2021] and [Eghbal-zadeh et al., 2020]. The first publication is concerned with Pearlian causality and how it relates to data augmentation, while the second paper bridges augmentation and adversarials—our work then loops back adversarial to causality. Separating our discussion from HCA for the moment, we nonetheless are tempted to believe that HCA (although being the focus and motivation behind this work) are not the most important discovery in this paper. The concept of LP-parameterization based on SCM (Def.1) is an intriguing and original concept that for the first time connects the two seemingly independent notions of Pearlian causality and mathematical optimization—in a non-trivial manner. So, it might turn out to be more fruitful long-term to actually study the properties of mathematical programs which stand in direct relation to the data-generating capabilities of SCM, as we started off in this work on LPs in order to construct sensible adversarial-style attacks. To put it bluntly, our current thinking imagines future research around LP-SCM relations as even more fundamental than HCA.

**Ethical and Societal Concerns.** Our work seems to suggest (1) that we can have a similar, adversarial phenomenon outside classification and (2) that the current state of ML can be viewed “causal” to the extent that the assumptions are (i.e., \(\phi\) summarizes essentially these causal assumptions). Of concern is mostly (1), since adversaries may produce harm, therefore, we feel this work is important to raise awareness. Nonetheless, we think that the further integration of ML with causality on modelling assumptions outside the data can push (2) and subsequently provide guidance for resolving (1).
Acknowledgments

This work was supported by the ICT-48 Network of AI Research Excellence Center “TAILOR” (EU Horizon 2020, GA No 952215), the Nexplore Collaboration Lab “AI in Construction” (AICO) and by the Federal Ministry of Education and Research (BMBF; project “PlexPlain”, FKZ 01IS19081). It benefited from the Hessian research priority programme LOEWE within the project WhiteBox and the HMWK cluster project “The Third Wave of AI” (3AI). We take the opportunity to thank Jonas Hülsmann and Florian Steinke for providing the LP model for the energy system example.

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A Appendix for “Can Linear Programs Have Adversarial Examples? A Causal Perspective”

We make use of this Appendix following the main paper to provide details and additional insights we deem important for the reader and for what has been examined in the main paper. To showcase the theoretical results established, we provided existential proofs of HCAs. We presented a real world inspired example on vaccination scheduling as our lead example (Fig.1) and now also discuss the other two examples.

A.1 Shortest Path LP

The caption of this example might be the following shocking headline: “Travelling from New York City to San Francisco...via Canada?” To further motivate the relevance of adversarial risk outside classification, let us consider a real world inspired example that is being showcased in Fig.3 which is concerned with a Shortest Path (SP) problem. In the corresponding real world setting, we might consider the development of an autonomous car. We let the developmental autonomous car travel within North America from New York City (NY) to San Francisco (SF). Our SP has the intention of reducing overall toll costs for the optimal route, which from experience can be hefty. Our LP cost \( w_{ij} \in \mathbb{R}_{>0} \) represents the toll cost when travelling on any road segment from \( i \) to \( j \). In this example, we know the toll costs for a relevant set of road segments within NA where the Canadian road toll policy is comparably modest. Our LP model subsequently solves any given SP problem instance, fully parameterized by the directed acyclic graph (DAG) \( w \in \mathbb{R}^{n \times n} \) with \( n \) being the total number of different cities we have specified, returning \( x_{US} := x^*(w) \in [0,1]^{n \times n} \) suggesting a route through the mid-US. By minimally perturbing the original DAG, that is \( \tilde{w} \approx w \), our solver now chooses an alternate solution \( x_{CA} := x^*(\tilde{w}) \) suggesting a route across the border via Canada\(^2\). While evidently the alternate route deviates strongly in terms of selected road segments, mathematically \( SHD(x_{US}, x_{CA}) \gg 0 \) where \( SHD(\cdot, \cdot) \in \mathbb{N} \) is the Structural Hemming Distance, our model is in fact trustfully returning the optimal solution as cost-wise the statement \( w^T x_{US} \approx \tilde{w}^T x_{CA} \) holds\(^3\). Nonetheless, the aforementioned deviation in terms of the resulting binary codes lends itself to a severe consequence in terms of the hidden confounder i.e., with respect to \( CO_2 \) emissions. Like in the main text, we construct an HCA with function \( h \) accordingly. The hidden confounder is being exploited by the adversary, the alternate optimal solution performs significantly worse: \( h(x_{CA}) \gg h(x_{US}) \). By this, we have again provided existential proof that a hidden confounder can more generally define adversarial attacks for mathematical programs beyond the original formulation in the classical setting for classification (and deep networks), making the attack a consequence of not the specific methodology being applied to the problem but problem specification itself.

A.2 Energy-Systems LP

Our real world based example considered an energy model for modelling the energy portfolio of a single-family house based on real world data for demand and commonly used equations from energy systems research Schaber et al. [2012]. The examined model considers photovoltaics (PV), market electricity and heating gas over a year time frame (in hours) and resembles a simplified version of the TIMES model Loulou et al. [2005]. The optimal solution balances the usage of the different technologies for matching the required demand such that overall cost is being minimized. The specific LP template is given in Tab.2 of the main paper.

Note that \( t \in \{0, \ldots, 8760\} \) with 1 year = 8760 hours rendering the template a very large single LP modelling each hour of the year. However, technologies like PV, in their capacity \( (Cap_{PV}) \), do not depend on \( t \) which would correspond to the real world intuition that one does not decide and subsequently build new PV per any given hour as it poses a single, fixed-timeframe investment. We use heuristics for the SCM-part to produce an approximation to HCA, since no notion of SCM is being provided a-priori. We then perform said approximation to produce results in Tab.3. We observe the expected setting of “dominating technologies” where PV is being preferred over market-bought electricity, which can lead to an increased risk in working injury or fire (hidden confounders in this case). The limitations on PV-production and Market-buy of electricity act as discrepancy counter-
Another Example: Increased \( CO_2 \) Emissions. The edges in the graph represent tolls to be paid for travelling a given road segment. The hidden confounder are \( CO_2 \) emissions. The HCA reveals that travelling via Canada instead of mid-US will amount to the same total travel toll to be paid but the \( CO_2 \) emissions drastically deviate between the solutions. (Best viewed in Color.)

measures that require the system to balance out different technologies i.e., while there will still be dominating technologies under price advantages the maximum skew of the portfolio is naturally being protected from being too drastic as both PV and bought electricity are limited in their “availability” (e.g. solar exposure, roof capacity, law regulations etc.) and thus cannot be naively maximized.

### A.3 Details for HCA Examples Reproduction

For the LA example, with the vaccination matching bias towards the wealthy, we use \( N = 15 \) sampling iterations with temperature parameter \( \sigma = 0.5 \) for the perturbation in the DPO as defined by Berthet et al. [2020] and an attack step \( \epsilon = 0.01 \) for the final HCA. For the SP example, travelling from NY to SF via Canada, we use more sampling iterations \( (N = 20) \) using a lower temperature \( (\sigma = 0.25) \).

| \( c_{PV} \) (\( €/kWh \)) | \( c_{Bat} \) (\( €/kWh \)) | \( c_{Ele} \) (\( €/kWh \)) | \( D \) (kWh/Year) | \( c_{Gas} \) (\( €/kWh \)) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.005           | 300             | 0.25            | 3000            | 0.25            |
| 0.001           | 300             | 0.25            | 3000            | 0.25            |

Table 4: **Parameterization Energy-System.** Cost for Photovoltaics \( c_{PV} \) (€/kWh), Battery \( c_{Bat} \) (€/kWh), Market Electricity \( c_{Ele} \) (€/kWh), Gas \( c_{Gas} \) (€/kWh), and the total Demand \( D \) (kWh/Year).

Technical Details and Code. All experiments are being performed on a MacBook Pro (13-inch, 2020, Four Thunderbolt 3 ports) laptop running a 2.3 GHz Quad-Core Intel Core i7 CPU with a 16 GB 3733 MHz LPDDR4X RAM on time scales ranging from a few minutes (e.g. evaluating LA/SP examples) up to a few hours (e.g. energy systems real world example). Our code is available at: https://anonymous.4open.science/r/Hidden-Conf-Attacks/README.md.