The automorphism group of the moduli space
of semi stable vector bundles

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0 Introduction

To every smooth curve $C$ one can associate a natural variety $\mathcal{U}(r, d)$ - the moduli space of semi stable vector bundles of rank $r$ and degree $d$ on $C$. This canonical object has a rich geometrical structure which reflects in a beautiful way the geometry of the curve $C$ and its deformations. In the abelian case $r = 1$ this relationship is classical and goes back to the Riemann inversion problem, the theory of Jacobi theta-functions and the Torelli theorem. Understanding the subtleties of the non-abelian case has resulted over the years in a multitude of new ideas and unexpected connections with other branches of mathematics. One of the remarkable similarities between the Jacobian varieties and the higher rank moduli spaces is the existence of theta line bundles on them, see [D-N]. These are ample determinantal line bundles, naturally arising from the interpretation of $\mathcal{U}(r, d)$ as a moduli space and provide valuable information about the projective geometry of $\mathcal{U}(r, d)$.

In this paper we describe the groups of automorphisms and of polarized automorphisms of $\mathcal{U}(r, d)$ for $r > 1$. To understand the problem better, consider first the case $r = 1$. It is well known that for any integer $d$, the group of automorphisms of the Jacobian $J^d(C) = \mathcal{U}(1, d)$ is isomorphic to the semi-direct product

$$ J^0(C) \rtimes \text{Aut}_\text{group}(J^0(C)) \cong \text{Aut}(J^d(C)), $$

where $\text{Aut}_\text{group}(J^0(C))$ is the group of group automorphisms of $J^0$. The above isomorphism depends on the choice of a point $L_0 \in J^d(C)$ and is given explicitly by $(\xi, \phi) \rightarrow T_\xi \circ T_{L_0} \circ \phi \circ T_{L_0}^{-1}$. Moreover,
the subgroup of automorphisms preserving the class of the theta bundle is just

\[ \text{Aut}^\theta(J^d(C)) \cong \begin{cases} J^0(C) \times ((\pm \text{id}) \times \text{Aut}(C)) & C - \text{not hyperelliptic}, \\ J^0(C) \times \text{Aut}(C) & C - \text{hyperelliptic}. \end{cases} \]

Some natural automorphisms of \( \mathcal{U}(r,d) \) can be obtained by analogy with this case. For example, the Jacobian \( J^0(C) \) acts by translations on the moduli space and, as it turns out, it actually coincides with the identity component of the automorphism group. Furthermore, we have a natural morphism

\[ (0.1) \quad \text{det} : \mathcal{U}(r,d) \longrightarrow J^d(C) \]

sending a bundle \( E \) to its determinant \( \text{det}(E) \). The fibers of this morphism are moduli spaces on its own right, e.g. the fiber \( \text{det}^{-1}(L_0) \) over a point \( L_0 \) is just the moduli space \( \mathcal{SU}(r,L_0) \) of semistable bundles of rank \( r \) and fixed determinant \( L_0 \). One can use the fibration \( (0.1) \) to construct some natural automorphisms of \( \mathcal{U}(r,d) \), that is, certain automorphisms of the fiber \( \mathcal{SU}(r,L_0) \) can be glued with certain automorphisms of the base \( J^d(C) \) to produce global automorphisms of the moduli space. After choosing a point \( L_0 \in J^d(C) \) we can trivialize the bundle \( (0.1) \) by pulling it back to an étale cover of \( J^d(C) \):

\[ (0.2) \quad \begin{array}{ccc}
\mathcal{SU}(r,L_0) \times J^0(C) & \xrightarrow{(E,L) \mapsto E \otimes L} & \mathcal{U}(r,d) \\
J^0(C) \xrightarrow{L \mapsto L \otimes \nu \otimes L_0} & J^d(C) \\
\text{det} \end{array} \]

The top and the bottom rows of the diagram \( (0.2) \) are Galois covers with Galois group the group of \( r \)-torsion points \( J^0(C)[r] \). Consider the subgroup \( \text{Aut}_{[r]}(J^0(C)) \subset \text{Aut}(J^0(C)) \) consisting of group automorphisms of \( J^0(C) \) which act trivially on the torsion points \( J^0(C)[r] \). Then given a translation, \( T_\mu \), by a torsion point \( \mu \in J^0(C)[r] \) and \( \varphi \in \text{Aut}_{[r]}(J^0(C)) \), the automorphism \( T_\mu \times \varphi \) of \( \mathcal{SU}(r,L_0) \times J^0(C) \) commutes with the action of the Galois group and hence descends to an automorphism of \( \mathcal{U}(r,d) \). In addition, every symmetry of the curve \( C \) will induce an automorphism of \( \mathcal{U}(r,d) \).

In the special case \( r | 2d \), by choosing \( \nu \in J^{2d}(C) \) with property \( \nu \otimes r = L_0 \otimes 2 \), we can construct an additional automorphism \( \delta \) of order two:

\[ \delta : \mathcal{U}(r,d) \longrightarrow \mathcal{U}(r,d) \]
\[
E \quad \longrightarrow \quad E' \otimes \nu
\]

To combine these, consider the subgroups \( \mathfrak{U} \mathfrak{S}_\left[ r \right] \subset \mathfrak{U} \mathfrak{S}_\left[ r \right] \subset \text{Aut}(J^0(C)) \times \text{Aut}(C) \) defined by
Definition 0.1

\[ UG^\circ_r = \{ (\phi, \sigma) \mid \phi^{-1} \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C)) \}, \]

and

\[ UG_r = \{ (\phi, \sigma) \mid \phi^{-1} \circ (\pm 1) \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C)) \}. \]

The group \( UG^\circ_r \) is of index two in \( UG_r \) and we have a natural homomorphism:

\[ J^0(C) \times UG^\circ_r \rightarrow \text{Aut}(U(r,d)) \]

\[ (\xi, \phi, \sigma) \rightarrow (E \mapsto \sigma^*E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1})) \]

where \( \eta \in J^0(C) \) is an arbitrary line bundle with the property \( \eta^\otimes r = \text{det} E \otimes L_0^{-1} \). If in addition \( r \mid 2d \), then the homomorphism (0.3) can be extended to

\[ J^0(C) \times UG_r \rightarrow \text{Aut}(U(r,d)) \]

\[ (\xi, \phi, \sigma) \rightarrow E \mapsto \begin{cases} \sigma^*E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1}) & \text{if } \phi^{-1} \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C)) \\ \sigma^*E^\vee \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta) & \text{if } \phi^{-1} \circ (\pm 1) \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C)) \end{cases} \]

The commutative diagram (0.2) suggests that the subgroups (0.3) (or (0.4) in the case \( r \mid 2d \)) will not differ too much from the full automorphism group, if the moduli space \( SU(r,L_0) \) doesn’t have excess automorphisms. The variety \( SU(r,L_0) \) has a lot of advantages. It is a Fano variety of Picard number one and by a theorem of Narasimhan and Ramanan, \([N-R 2]\), \( H^0(SU(r,L_0), TSU(r,L_0)) = 0 \) unless \( g = 2, r = 2 \) and \( d \) even. Therefore, in general, \( \text{Aut}(SU(r,L_0)) \) is finite and contains the group \( J^0(C)[r] \). In the case \( r = 2 \), odd degree and \( g = 2 \) the group \( \text{Aut}(SU^s(2,L_0)) \) was described by Newstead \([N]\) as a consequence of his proof of the Torelli theorem for the variety \( SU^s(2,L_0) \).

To generalize his result we adopt a different viewpoint - we use Hitchin’s abelianization to prove our main theorem.

**Theorem A** Let \( C \) be a curve without automorphisms. Then the automorphism group of the moduli space \( SU(r,L_0) \) can be described as follows

1. If \( r \nmid 2d \), then the natural map

\[ J^0(C)[r] \rightarrow \text{Aut}(SU(r,L_0)) \]

\[ \mu \rightarrow (E \mapsto E \otimes \mu) \]

is an isomorphism, and

2. If \( r \mid 2d \), then the natural map

\[ J^0(C)[r] \times \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Aut}(SU(r,L_0)) \]

\[ (\mu, \varepsilon) \rightarrow (E \mapsto \delta^\varepsilon(E) \otimes \mu) \]

is isomorphism for \( r \geq 3 \) and has kernel \( \mathbb{Z}/2\mathbb{Z} \) for \( r = 2 \).
The strategy of the proof is to lift the automorphism \( \Phi \) to a birational automorphism of the moduli space of Higgs bundles and then study the induced automorphism \( \tilde{\Phi}_d \) of the family Prym\(_d\)(\( \mathcal{\tilde{C}}, C \)) of smooth spectral Pryms. To determine the map \( \tilde{\Phi}_d \) explicitly we study the ring of relative correspondences over the universal spectral curve and show that the family of groups Prym\(_d\)(\( \mathcal{\tilde{C}}, C \)) does not have non-trivial global group automorphisms. Furthermore, we show that all the sections of Prym\(_d\)(\( \mathcal{\tilde{C}}, C \)) come from pull-back of line bundles on the base curve \( C \) and conclude that \( \tilde{\Phi}_d \) must be a multiplication by \( \pm 1 \) along the fibers of Prym\(_d\)(\( \mathcal{\tilde{C}}, C \)) followed by a translation by a pull-back bundle. To finish the proof, we use the rational map from a spectral Prym to the moduli space \( SU(r, L_0) \) to recover from \( \tilde{\Phi}_d \) the original automorphism \( \Phi \).

For curves with automorphisms the statement of the main theorem has to be modified appropriately. To avoid some technical complications which occur in the case \( g = 2 \), we assume that \( g \geq 3 \) throughout the paper. Consider the subgroups \( \mathcal{SG}_{[r]} \subseteq \mathcal{SG}_{[r]} \subseteq J^0(C) \times \text{Aut}(C) \) defined as

**Definition 0.2**

\[
\mathcal{SG}_{[r]} = \{ (\xi, \sigma) \mid \xi^r = L_0 \otimes \sigma^*(L_0^{-1}) \},
\]

and

\[
\mathcal{SG}_{[r]} = \{ (\xi, \sigma) \mid \xi^r = L_0 \otimes \sigma^*(L_0^{\pm 1}) \}.
\]

Again \( \mathcal{SG}_{[r]} \) is a normal subgroup of index two in \( \mathcal{SG}_{[r]} \) and we have

**Theorem B** Let \( C \) be any smooth curve of genus \( g \geq 3 \). We have

1. If \( r \nmid 2d \), then the natural map

\[
(\xi, \sigma) \mapsto (E \mapsto \sigma^*E \otimes \xi)
\]

is surjective.

2. If \( r \mid 2d \), then the natural map

\[
(\xi, \sigma) \mapsto E \mapsto \begin{cases} 
\sigma^*E \otimes \xi & \text{if } (\xi, \sigma) \in \mathcal{SG}_{[r]} \subset \mathcal{SG}_{[r]} \\
\sigma^*E^\vee \otimes \xi & \text{if } (\xi, \sigma) \in \mathcal{SG}_{[r]} \setminus \mathcal{SG}_{[r]} \end{cases}
\]

is surjective.

**Remark 0.1** The maps in the above Theorem B are actually isomorphisms. Our proof of the injectivity involves some standard arguments from Prym theory similar to those in [N-R 1]; the proof is not included here because of its length.
The first part of Theorem B gives a positive answer to a question posed by Tyurin in the survey paper [Ty].

The main theorem, together with the observation that every automorphism of \( U(r,d) \) must lift to an automorphism of \( SU(r,L_0) \times J^0(C) \), see Lemma 6.3, gives

**Theorem C** Let \( C \) be a smooth curve of genus \( g \geq 3 \). Then the automorphisms of the full moduli space \( U(r,d) \) can be described as follows

1. If \( r \nmid 2d \), then the map
   \[
   J^0(C) \times \mathfrak{U}^r \hookrightarrow \text{Aut}(U(r,d))
   \]
   \[
   (\xi, \phi, \sigma) \rightarrow (E \mapsto \sigma^* E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1}))
   \]
   is surjective, and

2. If \( r \mid 2d \), then the map
   \[
   J^0(C) \times \mathfrak{U}^r \hookrightarrow \text{Aut}(U(r,d))
   \]
   \[
   (\xi, \phi, \sigma) \rightarrow E \mapsto \begin{cases} 
   \sigma^* E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1}) & \text{if } \phi^{-1} \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C)) \\
   \sigma^* E^\vee \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta) & \text{if } \phi \circ \sigma^* \in \text{Aut}_{[r]}(J^0(C))
   \end{cases}
   \]
   is surjective.

The description of the polarized automorphisms of \( U(r,d) \) follows easily from Theorem C.

**Theorem D** For any curve \( C \) of genus \( g \geq 3 \) the automorphisms of the moduli space \( U(r,d) \) which preserve the class of the theta bundle are those belonging to the image of the subgroup \( J^0(C) \times ((\pm \text{id}) \times \text{Aut}(C)) \subseteq J^0(C) \times \mathfrak{U}^r \) under the map \( \mathfrak{D} \).

As an application of the technics we use in this work, we prove in the Appendix at the end of the paper the Torelli Theorem for the moduli space of vector bundles:

**Theorem E** Let \( C_1, C_2 \) be smooth curves of genus \( g \geq 3 \) and \( L_1, L_2 \) line bundles of degree \( d \) on \( C_1, C_2 \) respectively. If \( SU_{\mathcal{C}_1}(r,L_1) \simeq SU_{\mathcal{C}_2}(r,L_2) \), then \( C_1 \simeq C_2 \).

The paper is organized as follows. In the first section we gather all the facts about the moduli space of Higgs bundles, the linear system of spectral curves and the Hitchin map which we are going to use later on. We have included the proofs of some statements which seem to be well known to the experts in the field but which can not be found in the standard sources, as well as, the proofs of some facts about the discriminant locus in the Hitchin base which help us simplify our main argument. In the second section we construct the induced automorphism \( \Phi_d \) and study its first
properties. The third section contains a discussion of the relative Picard of the universal spectral curve and its sections over a Zariski open set in the Hitchin base. In section four we analyze the ring of relative correspondences on the fibers of the universal spectral curve whose description is the key ingredient in the proof of the main theorem. In the fifth section we give the proof of the main theorem and discuss the case of curves with symmetries. In the sixth section we derive the description of Aut(U(r,d)).

Acknowledgments: We are very grateful to Ron Donagi for his constant support and for many valuable discussions and suggestions. We would like to thank Eyal Markman for explaining to us the proof of Lemma 1.4 and George Pappas for helpful conversations.

Notation and Conventions

$SU(r,L_0)$ : The moduli space of semi stable vector bundles of rank $r$ and fixed determinant $L_0$ of degree $d$.

$SU^s(r,L_0)$ : The moduli space of stable vector bundles of rank $r$ and fixed determinant $L_0$ of degree $d$.

$X(r,L_0)$ : The total space of the cotangent bundle $T^*SU^s(r,L_0)$.

$M(r,L_0)$ : The moduli space of semi stable Higgs pairs.

$H : M(r,L_0) \longrightarrow W$ : The Hitchin map.

$C$ : A smooth curve of genus $g \geq 3$.

$\omega_C$ : The canonical bundle on $C$.

$S^o$ : The total space of the canonical bundle $\omega_C$.

$S$ : The space $\mathbb{P}(\mathcal{O} \oplus \omega_C)$.

$\alpha : S \longrightarrow C$ : The canonical map.

$Y \in H^0(S,\mathcal{O}_S(1))$ : The infinity section of $S$.

$X \in H^0(S,\mathcal{O}_S(1) \oplus \alpha^*\omega_C)$ : The zero section of $S$.

$x = X/Y$ : The tautological section of $S^o$.

$Y_\infty$ : The divisor at infinity $\text{div}(Y) \subset S$.

$X_0$ : The zero divisor $\text{div}(X) \subset S^o$.

$\overline{W} = H^0(C,\mathcal{O}) \oplus H^0(C,\omega_C^2) \oplus \cdots \oplus H^0(C,\omega_C^r)$.

$W_k = H^0(C,\omega_C^k)$.

$W \simeq W_2 \oplus \cdots \oplus W_r$, embedded in $\overline{W}$ as $\{1\} \oplus W_2 \oplus \cdots \oplus W_r$.

$W^\infty \simeq W_2 \oplus \cdots \oplus W_r$, embedded in $\overline{W}$ as $\{0\} \oplus W_2 \oplus \cdots \oplus W_r$.

$D$ : The discriminant divisor.

$W^\text{reg} = W \setminus D$.

$\check{C}_s$ : The spectral curve of genus $\check{g} = r^2(g-1) + 1$ associated to $s \in W$. 

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\(\pi_s : \tilde{C}_s \longrightarrow C\) : The projection to the base curve \(C\).
\(\pi : \tilde{\mathcal{C}} \longrightarrow C\) : The universal spectral curve and its projection to \(C\).
\(\beta : \tilde{\mathcal{C}} \longrightarrow S^0\) : The map to \(S^0\).

\(\text{Prym}(\tilde{C}_s, C)\) : The Prymian of the map \(\pi_s : \tilde{C}_s \longrightarrow C\).
\(\text{Prym}(\tilde{\mathcal{C}}, C)\) : The universal Prymian associated to the map \(\pi : \tilde{\mathcal{C}} \longrightarrow C\).
\(\text{Prym}(\tilde{\mathcal{C}}, C)\) : The universal spectral “prymian” of degree \(\tilde{d}\).

1 Spectral curves

1.1 Linear systems of spectral curves

Let \(C\) be a smooth curve of genus \(g\). Let \(S^0\) be the total space of the line bundle \(\omega_C \rightarrow C\) and let \(S = \mathbb{P}(\omega_C \oplus \mathcal{O}_C)\) be the projective extension of \(\omega_C\). Denote by \(\alpha : S \rightarrow C\) the natural projection and let \(\mathcal{O}_S(1)\) be the relative hyperplane bundle on \(S\) corresponding to the vector bundle \(\omega_C \oplus \mathcal{O}_C\).

**Definition 1.1** An \(r\)-sheeted spectral curve is an element \(\tilde{C}\) of the linear system \(|\alpha^*\omega_C^{\otimes r} \otimes \mathcal{O}_S(r)|\) having the property \(\tilde{C} \subset S^0\)

The adjective spectral refers to another interpretation of the curve \(\tilde{C}\) which we recall next. For any vector bundle \(E\) of rank \(r\) over \(C\) and any \(\omega_C\)-twisted endomorphism \(\theta \in H^0(C, \text{End}E \otimes \omega_C)\) of \(E\) there is a suitable notion of a characteristic polynomial. Define the \(i\)-th characteristic coefficient \(s_i \in H^0(C, \omega_C^{\otimes i})\) of \(\theta\) as \(s_i = (-1)^{i+1} \text{tr} (\wedge^i \theta)\). The characteristic polynomial \(P(x)\) of \(\theta\) can be written formally as \(P(x) = x^r + s_1 x^{r-1} + \ldots + s_r\). Geometrically the polynomial \(P(x)\) corresponds to a subcheme \(\tilde{C}_s \subset S^0\) which via the projection \(\alpha\) is an \(r\)-sheeted branch cover of \(C\). Indeed, let \(X \in H^0(C, \alpha^*\omega_C \otimes \mathcal{O}_S(1))\) be the zero section of the ruled surface \(S\) and let \(Y \in H^0(C, \mathcal{O}_S(1))\) be the infinity section of \(S\). By setting \(x := \frac{X}{Y}\), i.e. \(x\) is the tautological section of \(\alpha^*\omega_C \rightarrow S^0\), we can reinterpret \(Y^{\otimes r} P(x) = X^r + s_1 X^{r-1} Y + \ldots + s_r Y^r\) as a section of \(\alpha^*\omega_C^{\otimes r} \otimes \mathcal{O}_S(r)\) whose divisor is contained in \(S^0\), that is - a spectral curve, for more details see [B-N-R], [Hi 1].

**Remark 1.1** We can view the space \(\tilde{W} := \bigoplus_{r=1}^l H^0(C, \omega_C^{\otimes r})\) as the space of characteristic polynomials of \(\omega_C\)-twisted endomorphisms of rank \(r\) vector bundles. The vector space \(\tilde{W}\) can be identified explicitly as the locus of spectral curves in \(|\alpha^*\omega_C^{\otimes r} \otimes \mathcal{O}_S(r)|\) as follows. The push forward map \(\alpha_*\) induces an isomorphism

\[\alpha_* : H^0(S, \alpha^*\omega_C^{\otimes r} \otimes \mathcal{O}_S(r)) \longrightarrow H^0(C, \alpha_* (\alpha^*\omega_C^{\otimes r} \otimes \mathcal{O}_S(r))).\]
On the other hand,
\[ \alpha_s(\alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r)) = \omega_C^{2r} \otimes \alpha_s(\mathcal{O}_S(r)) = \omega_C^{2r} \otimes (\text{Sym}^r(\mathcal{O}_C \oplus \omega_C^{-1})) = \mathcal{O}_C \oplus \omega_C \oplus \ldots \oplus \omega_C^r. \]

Therefore we get an isomorphism
\[ \alpha_* : H^0(S, \alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r)) \longrightarrow \bigoplus_{i=0}^r H^0(C, \omega_C^{2i}), \]
whose inverse is
\[ \bigoplus_{i=0}^r H^0(C, \omega_C^{2i}) \longrightarrow H^0(S, \alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r)) \quad \text{and} \quad s_0 X^r + s_1 X^{r-1} Y + \ldots + s_r Y^r. \]

Here we have identified the spaces \( H^0(C, \omega_C^{2i}) \) and \( \alpha^*H^0(C, \omega_C^{2i}) \subset H^0(C, \alpha^*\omega_C^{2i}) \) via \( \alpha \).

For any \( s = (s_0, s_1, \ldots, s_r) \) denote by \( \tilde{C}_s \) the divisor \( \text{Div}(s_0X^r + s_1X^{r-1}Y + \ldots + s_rY^r) \) on \( S \). Let \( \text{Div}(X) = X_0 \) and \( \text{Div}(Y) = Y_\infty \). By definition the curve \( \tilde{C}_s \) is spectral if \( \tilde{C}_s \subset S^0 \), or equivalently, if \( \tilde{C}_s \cap Y_\infty = \emptyset \). Since \( X_0 \cap Y_\infty = \emptyset \) it follows that \( \tilde{C}_s \cap Y_\infty \neq \emptyset \) if and only if \( Y_\infty \subset \tilde{C}_s \), i.e. if and only if \( s_0 = 0 \).

Thus the locus of all spectral curves in the projective space \( |\alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r)| \) is the affine open set \( \tilde{W} = \{ \tilde{C}_s \mid s_0 \neq 0 \} \).

For the purposes of this paper we will be mainly concerned with the slightly smaller space
\[ W := H^0(C, \omega_C^{22}) \oplus H^0(C, \omega_C^{23}) \oplus \ldots \oplus H^0(C, \omega_C^{2r}), \]
consisting of the characteristic polynomials of traceless twisted endomorphisms of rank \( r \) vector bundles. Let \( h : \tilde{C} \rightarrow W \) be the family of spectral curves parametrized by \( W \). To construct the variety \( \tilde{C} \) consider the line bundle \( p^*_S(\alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r)) \rightarrow W \times S \). There is a natural tautological section \( \epsilon \in H^0(W \times S, p^*_S(\alpha^*\omega_C^{2r} \otimes \mathcal{O}_S(r))) \) given by
\[ \epsilon(((s_2, \ldots, s_r); p)) = X^r(p) + s_2(p)X^{r-2}(p)Y^2(p) + \ldots + s_r(p)Y^r(p), \]
and \( \tilde{C} = \text{Div}(\epsilon) \) is just the divisor of this section.

The universal family \( \tilde{C} \) is proper over \( W \) and admits a natural compactification to a projective variety \( \overline{\tilde{C}} \) - the universal family for the linear system \( \mathbb{P}(H^0(C, \mathcal{O}_C) \oplus W) \). Set \( \overline{W} := H^0(C, \mathcal{O}_C) \oplus W \).

Next we are going to study the linear system \( \mathbb{P}(\overline{W}) \) and the total spaces of the universal families \( \tilde{C} \) and \( \overline{C} \).

**Lemma 1.1** The linear system \( \mathbb{P}(\overline{W}) \) is base point free. Furthermore, if \( r \geq 3 \) the morphism \( f_{\mathbb{P}(\overline{W})} : S \rightarrow \mathbb{P}(\overline{W})^\vee \) is an inclusion when restricted to \( S^0 \) and contracts the infinity section \( Y_\infty \).
Proof. Let \( p \in S \). If \( p \in Y_\infty \), then the curve \( \text{Div}(X^r) \in \mathbb{P}(W) \) does not pass through \( p \). If \( p \notin Y_\infty \), then choose a section \( s_r \in H^0(C, \omega_C^{\otimes r}) \) not vanishing at \( \alpha(p) \). The curve \( \text{Div}(s_rY^r) \in \mathbb{P}(W) \) does not pass through \( p \).

Consider the morphism

\[ f_{|W|} : S \rightarrow \mathbb{P}(W). \]

Since a curve \( \tilde{\mathcal{C}} \in |\alpha^*\omega_{\tilde{C}}^{\otimes r} \otimes \mathcal{O}_S(r)| \) is either spectral or contains \( Y_\infty \), see Remark 1.1, it follows that \( f_{\mathbb{P}(W)} \) contracts \( Y_\infty \).

If \( p \neq q \in S \) are points in the same fiber \( F \) of \( \alpha \), then they are separated by \( \mathbb{P}(W) \) because by pulling \( \mathbb{P}(W) \) back on \( F \) we get the linear system

\[ \mathbb{P}(\text{Span}(x_0^r, x_0^{r-2}, \ldots, x_1^r)) \subset |\mathcal{O}_F(r)|, \]

which for \( r \geq 3 \) separates the points on \( F \cong \mathbb{P}^1 \).

If \( p \neq q \in S \) are points in two different fibers of \( \alpha \), then by choosing a section \( s_r \in H^0(C, \omega_C^{\otimes r}) \) satisfying \( s_r(\alpha(p)) = 0 \) and \( s_r(\alpha(q)) \neq 0 \), we get a curve \( \text{Div}(s_rY^r) \) passing through \( p \) and not passing through \( q \).

The above considerations and the local triviality of \( S^0 \to C \) show that \( \mathbb{P}(\mathcal{W}) \) separates also the tangent directions because for every point \( p \in S^0 \) the morphism \( f_{\mathbb{P}(W)} \) maps the fiber to through \( p \) and a suitable translate of \( X_0 \) into two transversal curves in \( \mathbb{P}(W) \).

Here are some immediate corollaries from the above lemma which are going to be useful later on.

**Corollary 1.1** The generic spectral curve \( \tilde{\mathcal{C}}_s \) is smooth.

**Proof.** Follows from the fact that \( \mathbb{P}(\mathcal{W}) \) does not have base points and from Bertini’s theorem.

**Corollary 1.2** The total spaces of the universal families \( \tilde{\mathcal{C}} \) and \( \mathcal{C} \) are smooth.

**Proof.** Since \( \tilde{\mathcal{C}} \subset \mathcal{C} \) is a Zariski open set it is enough to show that \( \mathcal{C} \) is smooth.

Let \( \pi : \mathcal{U} \to \mathbb{P}(\mathcal{W}) \) be the universal bundle of linear hypersurfaces in \( \mathbb{P}(\mathcal{W}) \). The total space \( \mathcal{C} \) fits in the fiber product diagram

\[
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \pi \\
S & \longrightarrow & \mathbb{P}(\mathcal{W}) \\
\end{array}
\]
But both $S$ and $\overline{U}$ are smooth varieties and moreover the projection $\pi : \overline{U} \to \mathbb{P}(W)^{\vee}$ is a smooth map. Therefore the fiber product $\mathcal{T} = S_{f|\overline{W}|} \times_{\pi} \mathcal{P}$ is also smooth.

\[ \square \]

Let $W_{r-1,r}$ (resp. $W_{r-1}^\infty$) be the linear subsystem of $W$ consisting of points of the form $(s_0, 0, \ldots, s_{r-1}, s_r)$ (resp. $(0, 0, \ldots, s_{r-1}, s_r)$). According to Remark $\square$, $W_{r-1,r}$ (resp. $W_{r-1}^\infty$) can be embedded naturally in the vector space $H^0(S, \alpha^* \omega_C^{\otimes r} \otimes O_S(r))$ and thus, $|(W_{r-1,r})|$ (resp. $|(W_{r-1}^\infty)|$) can be viewed as a linear subsystem of $|\alpha^* \omega_C^{\otimes r} \otimes O_S(r)|$. The arguments in the proof of Lemma $\square$ yield the following corollary.

**Corollary 1.3** The linear system $W_{r-1,r}$ (resp. $W_{r-1}^\infty$) is base point free. Furthermore if $r \geq 3$, then the morphism $f|_{W_{r-1,r}} : S \to \mathbb{P}(W_{r-1,r})^{\vee}$ (resp. $f|_{W_{r-1}^\infty} : S \to \mathbb{P}(W_{r-1}^\infty)^{\vee}$) is an inclusion when restricted to $S^\circ$ and contracts the infinity section $Y_\infty$.

In the next section we are going to prove the irreducibility of various discriminant loci. The following fact is an essential ingredient in the proof of Corollary $\square$ and is also of independent interest.

**Proposition 1.1** Let $\tilde{C}$ be a spectral curve of degree $r$ and let $\Sigma \subset \tilde{C}$ be an irreducible component of $\tilde{C}$. Then $\Sigma$ is a spectral curve of degree $l \leq r$.

**Proof.** Since $\Sigma \subset \tilde{C} \subset S^\circ$, we only need to show that

\[ O_S(\Sigma) = O_S(l) \otimes \alpha^* \omega_C^{\otimes l}, \]

for some $l \leq r$. But $\text{Pic}(S) = \mathbb{Z} \cdot O_S(1) \oplus \alpha^* \text{Pic}(C)$, and hence

\[ O_S(\Sigma) = O_S(l) \otimes \alpha^* L, \]

for some line bundle $L$ on $C$. Since $\Sigma \subset S^\circ$ we have $\Sigma \cdot Y_\infty = 0$ which yields

\[ 0 = O_S(\Sigma) \cdot O_S(1) = O_S(l) \cdot O_S(1) + \alpha^* L \cdot O_S(1) = -l(2g - 2) + \deg L, \]

i.e. $\deg L = l(2g - 2)$.

On the other hand the line bundle $O_S(l) \otimes \alpha^* L$ has a section $\Sigma$ which does not vanish on the infinity divisor $Y_\infty$. But the sections of $O_S(l) \otimes \alpha^* L$ vanishing on $Y_\infty$ are $H^0(S, O_S(l) \otimes \alpha^* L \otimes O_S(-Y_\infty)) = H^0(S, O_S(l - 1) \otimes \alpha^* L)$. By pushing forward $O_S(l - 1) \otimes \alpha^* L$ on $C$ we get

\[ H^0(S, O_S(l - 1) \otimes \alpha^* L) = H^0(C, \alpha_*(O_S(l - 1)) \otimes L) = H^0(C, \text{Sym}^{l-1}(O_C \otimes \omega_C^{-1}) \otimes L) = H^0(C, L \oplus (L \otimes \omega_C^{-1}) \oplus \ldots \oplus (L \otimes \omega_C^{-(l-1)}) = \oplus_{i=0}^{l-1} H^0(C, L \otimes \omega_C^{-i}). \]
Similarly $H^0(S, \mathcal{O}_S(l) \otimes \alpha^*L) = \bigoplus_{i=0}^l H^0(C, L \otimes \omega_C^{-i})$ and hence

$$H^0(S, \mathcal{O}_S(l) \otimes \alpha^*L) / H^0(S, \mathcal{O}_S(l-1) \otimes \alpha^*L) \cong H^0(C, L \otimes \omega_C^{-i}).$$

The section of the divisor $\Sigma$ gives a non-zero element in $H^0(S, \mathcal{O}_S(l) \otimes \alpha^*L) / H^0(S, \mathcal{O}_S(l-1) \otimes \alpha^*L)$ and therefore $H^0(C, L \otimes \omega_C^{-i}) \neq 0$. Since $\deg(L \otimes \omega_C^{-i}) = 0$ we get that $L \otimes \omega_C^{-i} \cong \mathcal{O}_C$.

\[ \square \]

**Remark 1.2** The meaning of the above proposition becomes transparent if we use the interpretation of the spectral curves as characteristic polynomials of $\omega_C$-twisted endomorphisms.

Let $F \to \widetilde{C}$ be a line bundle. Then $F$ can be viewed as a sheaf on $S$ supported on $\widetilde{C} \subset S^0$. Consider the rank $r$ vector bundle $E = \alpha_* F$ on $C$. The push-forward of the homomorphism

$$F \xrightarrow{\otimes x} F \otimes \alpha^* \omega_C$$

is a $\omega_C$-twisted endomorphism of $E$:

$$\theta : E \to E \otimes \omega_C.$$

In this way the curve $\widetilde{C}$ then can be described as the zero scheme of the section

$$\det(\alpha^* \theta - x \cdot \text{id}_{\alpha^* E}) \in H^0(S, \alpha^* \omega_C^{\otimes r} \otimes \mathcal{O}_S(r)).$$

Let now $\Sigma \subset \widetilde{C}$ be a component of $\widetilde{C}$. Consider the sheaf $F \otimes \mathcal{O}_\Sigma$ on $S$ supported on $\Sigma$. Let $E' := \alpha_* (F \otimes \mathcal{O}_\Sigma)$ and let

$$\theta' : E' \to E' \otimes \omega_C$$

be the push forward of the homomorphism

$$F \otimes \mathcal{O}_\Sigma \xrightarrow{\otimes x} (F \otimes \mathcal{O}_\Sigma) \otimes \alpha^* \omega_C.$$

The commutative diagram of (torsion) sheaves on $S^0$

\[
\begin{array}{ccc}
F & \xrightarrow{\otimes x} & F \otimes \alpha^* \omega_C \\
\uparrow & & \uparrow \\
F \otimes \mathcal{O}_\Sigma & \xrightarrow{\otimes x} & (F \otimes \mathcal{O}_\Sigma) \otimes \alpha^* \omega_C
\end{array}
\]
then pushes down to a commutative diagram of vector bundles on $C$:

$$
\begin{array}{ccc}
E & \xrightarrow{\theta} & E \otimes \omega_C \\
\downarrow & & \downarrow \\
E' & \xrightarrow{\theta'} & E' \otimes \omega_C
\end{array}
$$

In particular $E'$ is $\theta$ invariant and $\theta' = \theta|_{E'}$. Then $\Sigma$ is the zero scheme of the section

$$
det(\alpha^*\theta' - x \cdot \id_{\alpha^*E'}) \in H^0(S, \alpha^*\omega_C^\otimes l \otimes \mathcal{O}_S(l)),
$$

where $l = \text{rk} E'$, i.e. $\Sigma$ is a spectral curve.

**Corollary 1.4** The locus $R \subset W$ consisting of reducible spectral curves has codimension

$$
codim(R, W) \geq g - 1
$$

**Proof.** Set $W_i := H^0(C, \omega_C^\otimes i)$. Let $R_r$ be the image of $R$ under the natural projection $W \to W_r$. It suffices to show that $\text{codim}(R_r, W_r) \geq g - 1$ or equivalently, since $R_r$ is invariant under dilations, that $\text{codim}(\mathbb{P}(R_r), \mathbb{P}(W_r)) \geq g - 1$.

For every $i = 1, \ldots, r - 1$ consider the variety $S_i \subset \mathbb{P}(W_r) \times \mathbb{P}(W_i)$ defined by

$$
S_i = \{(D, G) \mid D \geq G\}.
$$

Since a component of a spectral curve is a spectral curve we have

$$
\mathbb{P}(R_r) \subset p_{\mathbb{P}(W_r)}(S_1) \cup \ldots \cup p_{\mathbb{P}(W_r)}(S_{r-1}),
$$

and therefore $\text{codim}(\mathbb{P}(R_r), \mathbb{P}(W_r)) \geq \min_{1 \leq i \leq r-1} \{\text{codim}(p_{\mathbb{P}(W_r)}(S_i), \mathbb{P}(W_r))\}$. Furthermore $\text{dim}(p_{\mathbb{P}(W_r)}(S_i)) = \text{dim} S_i$ because the map $p_{\mathbb{P}(W_r)} : S_i \to \mathbb{P}(W_r)$ is finite on its image. To compute $\text{dim} S_i$ look at the second projection

$$
(1.5) \quad p_{\mathbb{P}(W_i)} : S_i \to \mathbb{P}(W_i).
$$

The map (1.5) is obviously onto and for any $G \in \mathbb{P}(W_i)$ we get by Riemann-Roch

$$
\text{dim} p_{\mathbb{P}(W_i)}^{-1}(G) = h^0(C, \omega_C^\otimes r(-G)) - 1 = h^0(C, \omega_C^\otimes (r-i)) - 1 \leq (2(r-i) - 1)(g - 1).
$$

Consequently by the fiber-dimension theorem we get $\text{dim} S_i \leq (2r - 2)(g - 1) - 1$ and hence

$$
\text{codim}(p_{\mathbb{P}(W_r)}(S_i), \mathbb{P}(W_r)) = g - 1,
$$

for any $i = 1, \ldots, r - 1$. 

\[\square\]
1.2 Discriminant loci

As we saw in Corollary 1.1 the generic spectral curve is smooth. Therefore the space $W$ of spectral curves contains a natural divisor parametrizing the singular spectral curves:

$$D := \{ s \in W \mid \tilde{C}_s \text{ is not smooth} \},$$

which due to Corollary 1.2 is just the discriminant locus for the map $h : \tilde{C} \to W$.

The divisor $D$ is irreducible. To see this consider the full family $\tilde{W}$ of spectral curves and its discriminant divisor $\tilde{D}$. We have a natural inclusion $W \hookrightarrow \tilde{W}$ and clearly $D = W \cap \tilde{D}$. The first step will be to show that $\tilde{D}$ is irreducible and the second to deduce the irreducibility of $D$ from that. To achieve this we need to introduce some auxiliary objects.

Let $\text{Aff}$ be the additive group of the vector space $H^0(C, \omega_C)$. There is a natural affine action of $\text{Aff}$ on the bundle $\omega_C$ which can be considered as an action on its total space

$$\tau : \text{Aff} \to \text{Aut}(S^o)$$

$$\gamma \to (p \mapsto p + \gamma(\alpha(p))).$$

The action $\tau$ on $S^o$ has a natural lift to an action on the bundle $\alpha^* \omega_C \to S^o$ which gives an affine action of $\text{Aff}$ on the space of its global sections: $\gamma \to (\mu \mapsto \mu + \alpha^* \gamma)$. Consequently we obtain a polynomial action on the space of spectral curves

$$\rho : \text{Aff} \to \text{Aut}(\tilde{W})$$

$$\gamma \to (x^r + s_1 x^{r-1} + \ldots + s_r \overrightarrow{\rho \gamma} (x + \gamma)^r + s_1 (x + \gamma)^{r-1} + \ldots + s_r).$$

For every $\gamma \in \text{Aff}$ we have a commutative diagram

$$\begin{array}{ccc}
\tilde{C} & \xrightarrow{\tau_\gamma} & \tilde{C} \\
\downarrow h & & \downarrow h \\
\tilde{W} & \xrightarrow{\rho_\gamma} & \tilde{W}
\end{array}$$

(1.6)

where $\tau_\gamma$ is the automorphism of the universal spectral curve induced by $\tau_\gamma \in \text{Aut}(S^o)$.

**Proposition 1.2** The discriminant divisor $\tilde{D} \subset \tilde{W}$ is irreducible.

**Proof.** Consider the incidence correspondence $\tilde{\Gamma} \subset \tilde{W} \times S^o$ defined by

$$\tilde{\Gamma} := \{(s, p) \in \tilde{W} \times S^o \mid \text{Ord}_p \tilde{C}_s \geq 2\}.$$

Since $\tilde{D} = p_{\tilde{W}}(\tilde{\Gamma})$ it suffices to show that $\tilde{\Gamma}$ is irreducible. The commutativity of the diagram (1.6) implies that $\tilde{\Gamma}$ is $\rho$-invariant. Furthermore, if $s' = \rho_\gamma(s)$, then $s'_1 = s_1 + r \gamma$ and hence $\rho$ is a free action. Therefore we can form the quotient variety $\tilde{\Gamma} / \text{Aff}$. Since the fibration

$$\tilde{\Gamma} \to \tilde{\Gamma} / \text{Aff}.$$
is a locally trivial affine bundle, the irreducibility of $\tilde{\Gamma}$ is equivalent to the irreducibility of $\tilde{\Gamma}/\text{Aff}$.

The fibers of the projection

$$p_C := \alpha \circ p_S : \tilde{\Gamma} \to C$$

are $\rho$-invariant and by taking the quotient we obtain a morphism

$$p_C : \tilde{\Gamma}/\text{Aff} \to C.$$

Let $\Xi \subset \tilde{W} \times C$ be the incidence correspondence

$$\Xi = \{ (s, a) \in \tilde{W} \times C \mid \tilde{C}_s \text{ is singular at the point } \alpha^{-1}(a) \cap X_0 \}.$$

More explicitly by using the Jacobian criterion for smoothness one gets, see [B-N-R] Remark 3.5,

$$\Xi = \{ (s, a) \in \tilde{W} \times C \mid \text{Div}(s_r) \geq 2a, \text{Div}(s_{r-1}) \geq a \}.$$

The fact that $\omega_C^{\otimes r}$ is very ample for any $r \geq 2$, implies that the fibration $\Xi \to C$ is a vector bundle of rank $\dim \tilde{W} - 3$. On the other hand we have a natural morphism of fibrations

$$\begin{array}{ccc}
\Xi & \xrightarrow{t} & \tilde{\Gamma}/\text{Aff} \\
& & \downarrow p_C \\
& & C
\end{array}$$

where $t((s, a)) = \text{Orb}_\text{Aff}((s, \alpha^{-1}(a) \cap X_0))$. It is easy to see that $t$ is onto. Indeed, if $(s, p) \in \tilde{\Gamma}$, then $\text{Ord}_p(\tilde{C}_s) \geq 2$. Choose $\gamma \in H^0(C, \omega_C)$ with the property $\gamma(\alpha(p)) = p$. Then

$$(\rho - \gamma, \tau - \gamma) \cdot (s, p) = (\rho - \gamma(s), \alpha^{-1}(\alpha(p)) \cap X_0),$$

and hence

$$\text{Orb}_\text{Aff}((s, p)) = \text{Orb}_\text{Aff}((\rho - \gamma(s), \alpha^{-1}(\alpha(p)) \cap X_0)).$$

Consequently $t$ is onto and $\tilde{\Gamma}/\text{Aff}$ is irreducible.

\[\square\]

**Remark 1.3** The variety $\tilde{\Gamma}/\text{Aff}$ (and therefore $\tilde{\Gamma}$) is actually smooth. Indeed, let $\Xi_a$ be the fiber of the bundle $\Xi$ over the point $a \in C$. Let $(s, a), (s', a) \in \Xi_a$ be such that $t((s, a)) = t((s', a))$. Then

$$\text{Orb}_\text{Aff}((s, \alpha^{-1}(a) \cap X_0)) = \text{Orb}_\text{Aff}((s', \alpha^{-1}(a) \cap X_0)),$$

and therefore $s' = \rho_\gamma(s)$ for some $\gamma \in H^0(C, \omega_C)$ satisfying $\gamma(a) = 0$.  

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Let $\Xi_0 \subset \Xi$ be the subbundle 

$$\Xi_0 = \{(s, a) \in \Xi \mid \text{Div}(s_1) \geq a\}.$$ 

The morphism $t$ descends to an isomorphism of fibrations

$$\Xi/\Xi_0 \xrightarrow{t} \tilde{\Gamma}/\text{Aff}$$

and thus $\tilde{\Gamma}/\text{Aff} \cong \Xi/\Xi_0$ is smooth. Furthermore, observe that $p_W^{-1}: \tilde{\Gamma} \to \tilde{D}$ is birational because for the generic $s \in \tilde{D}$ the curve $\tilde{C}_s$ has a unique ordinary double point. Therefore $\tilde{\Gamma}$ can be viewed as a natural desingularization of $\tilde{D}$.

**Corollary 1.5** The discriminant divisor $D \subset W$ is irreducible.

**Proof.** Consider the incidence correspondence

$$\Gamma = \{(s, p) \mid \text{Ord}_p(\tilde{C}_s) \geq 2\}.$$ 

Let $q|_{\Gamma}: \Gamma \to \tilde{\Gamma}/\text{Aff}$ be the restriction to $\Gamma$ of the natural quotient morphism $q: \tilde{\Gamma} \to \tilde{\Gamma}/\text{Aff}$. If $(s, p) \in \tilde{\Gamma}$, then $(\rho_{-\frac{1}{2}-s_1}(s), \tau_{-\frac{1}{2}-s_1}(s)) \in \Gamma$. Combined with the fact that $\rho$ acts freely, this yields

$$\Gamma \cap \text{Orb}_{\text{Aff}}((s, p)) = \{(\rho_{-\frac{1}{2}-s_1}(s), \tau_{-\frac{1}{2}-s_1}(s))\}.$$ 

Consequently $q|_{\Gamma}$ is an isomorphism and in particular $\Gamma$ and $D$ are irreducible.

There are two other discriminant loci whose irreducibility will be useful. Define $W_r := H^0(C, \omega_C^{\otimes r})$ and $W_{r-1,r} := H^0(C, \omega_C^{\otimes (r-1)}) \oplus H^0(C, \omega_C^{\otimes r})$. Let $\mathcal{D}_r = D \cap W_r$ and $\mathcal{D}_{r-1,r} = D \cap W_{r-1,r}$ be the discriminant divisors for the families of spectral curves $W_r$ and $W_{r-1,r}$ respectively.

**Proposition 1.3** The discriminant divisor $\mathcal{D}_r \subset W_r$ is irreducible.

**Proof.** Consider the incidence correspondence $\Gamma_r \subset W_r \times C$ defined by

$$\Gamma_r = \{(s_r, a) \mid \text{Div}(s_r) \geq 2a\}.$$ 

Using the Jacobian criterion it is easy to check that a curve $\tilde{C}_s, s \in W_r$ is singular if and only if $s_r$ has a double zero.
Therefore $D_r = p_{W_r}(\Gamma_r)$ and it suffices to show that $\Gamma_r$ is irreducible. The divisor $\Gamma_r$ is equivariant with respect to the natural $\mathbb{C}^\times$ action on $W_r$ (extended trivially to $W_r \times C$). Thus the problem reduces to showing that $\mathbb{P}(\Gamma_r) \subset \mathbb{P}(W_r) \times C$ is irreducible.

Consider the projection $p_C : \mathbb{P}(\Gamma_r) \to C$. By Riemann-Roch $p_C$ is onto and $\text{codim}(p_C^{-1}(a), \mathbb{P}(\Gamma_r)) = 2$ for every $a \in C$. Moreover, $p_C^{-1}(a) \subset \mathbb{P}(W_r)$ is a linear subspace and hence all the fibers of $p_C$ are equidimensional and irreducible. Therefore $\mathbb{P}(\Gamma_r)$ is irreducible.

\[ \square \]

**Remark 1.4** The divisor $\mathbb{P}(D_r)$ is a divisor in the projective space of the complete linear system $\mathbb{P}(H^0(C, \omega_C^{\otimes r}))$. This allows us to give more geometric description of $\mathbb{P}(D_r)$:

\[ \mathbb{P}(D_r) = \mathbb{P}(\{ H \subset \mathbb{P}(W_r^\vee) \text{-- hyperplane} \mid H \supset \mathbb{P}(T_a f_{|W_r|}(C)) \text{ for some } a \}) \]

that is, $\mathbb{P}(D_r) = f_{|W_r|}(C)^\vee$ is the dual hypersurface of the $r$-canonical model of the curve $C$.

**Proposition 1.4** The divisor $D_{r-1,r} \subset W_{r-1,r}$ is irreducible.

**Proof.** Consider the incidence variety

\[ \Gamma_{r-1,r} = \{(a, b, p) \in W_{r-1,r} \times S^\circ \mid \text{Ord}_p(x^r + ax + b) \geq 2 \}. \]

Again $p_{W_{r-1,r}}(\Gamma_{r-1,r}) = D_{r-1,r}$ so it suffices to show that $\Gamma_{r-1,r}$ is irreducible.

Consider the projection

\[ p_{S^\circ} : \Gamma_{r-1,r} \to S^\circ. \]

Let $(U, z)$ be a coordinate chart on $C$. Then we can choose natural coordinates $(w, z) : \alpha^{-1}(U) \supseteq \mathbb{C} \times U$ on $\alpha^{-1}(U)$, so that the tautological section $x$ is $x = w\alpha^*(dz)$.

Let $(z_0, w_0) \in \alpha^{-1}(U)$. Then for any $(a, b) \in W_{r-1,r}$ one has in a neighborhood of $(z_0, w_0)$:

\[
\begin{align*}
a &= (a_0 + a_1(z - z_0) + \ldots) dz^{\otimes (r-1)} = f(z) dz^{\otimes (r-1)} \\
b &= (b_0 + b_1(z - z_0) + \ldots) dz^{\otimes r} = g(z) dz^{\otimes r}.
\end{align*}
\]

Therefore in the local coordinates $(z, w)$ the equation of the curve $x^r + ax + b$ becomes

\[ (1.7) \quad w^r + f(z)w + g(z) = 0. \]

The conditions for (1.7) to have singularity at $(z_0, w_0)$ are

\[
\begin{align*}
b_0 &= 0 \\
(rw^{r-1} + f(z))|_{(z_0, w_0)} &= 0 \\
(f'(z)w + g'(z))|_{(z_0, w_0)} &= 0.
\end{align*}
\]

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or equivalently
\[
\begin{align*}
    b_0 &= 0 \\
    rw_0^{r-1} + a_0 &= 0 \\
    a_1w_0 + b_1 &= 0.
\end{align*}
\] (1.8)

The equations (1.8) determine a codimension 3 affine subspace of \(W_{r-1,r}\) and hence \(\Gamma_{r-1,r|\alpha^{-1}(U)} \to \alpha^{-1}(U)\) is a trivial affine bundle. Therefore \(\Gamma_{r-1,r}\) is an affine bundle over \(S^0\) and hence is irreducible.

\[\blacksquare\]

**Corollary 1.6** Let \(h : \tilde{\mathcal{C}} \to W\) be the universal spectral curve and let \(h_{r-1,r} : \tilde{\mathcal{C}}_{|W_{r-1,r}} \to W_{r-1,r}\) and \(h_r : \tilde{\mathcal{C}}_{|W_r} \to W_r\) be the subfamilies parametrized by \(W_{r-1,r}\) and \(W_r\) respectively. Then the divisors \(h^{-1}(D), h^{-1}_{r-1,r}(D_{r-1,r})\) and \(h^{-1}_r(D_r)\) are irreducible.

**Proof.** Recall the following standard lemma, see [Sh],

**Lemma 1.2** Let \(f : X \to Y\) be a proper map between algebraic varieties of pure dimension. If \(Y\) is irreducible, the general fiber of \(f\) is irreducible and all the fibers are equidimensional then \(X\) is irreducible.

As we saw above each of the discriminant loci \(D_{r-1,r}, D_r\) and \(D\) is irreducible. According to Lemma 1.2 it suffices to show that the generic fiber of each of the maps \(h, h_{r-1}\) and \(h_{r-1,r}\) is irreducible. But, by Corollary 1.4, there exists a point \(s \in D_r \subset D_{r-1,r} \subset D\) such that \(\tilde{\mathcal{C}}_s\) is irreducible which finishes the proof.

\[\blacksquare\]

### 1.3 The Hitchin map

For a line bundle \(L_0 \in \text{Pic}^d(C)\) denote by \(SU(r, L_0)\) the moduli space of semistable vector bundles of rank \(r\) and determinant \(L_0\). It is well known that \(SU(r, L_0)\) is a normal projective variety whose smooth locus coincides with the locus \(SU^s(r, L_0)\) of stable vector bundles, see [Sh]. Since \(C\) is a curve, the deformations of any vector bundle \(E \to C\) are unobstructed and the Kodaira-Spencer map
\[
(1.9) \quad T_{[E]}SU^s(r, L_0) \longrightarrow H^1(C, \text{End}^0 E),
\]
is an isomorphism; here \(\text{End}^0 E\) is the bundle of traceless endomorphisms of \(E\).

Using the isomorphism (1.9) and Serre’s duality, one gets a canonical identification
\[
T_{[E]}^*SU^s(r, L_0) \cong H^0(C, \text{End}^0 E \otimes \omega_C),
\]

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of the fiber of the cotangent bundle to $SU^s(r,L_0)$ at the point $[E]$ and the space of $\omega_C$-twisted traceless endomorphisms. Therefore the collection of characteristic coefficients defined in Section 1.2 gives rise to a morphism

$$H : T^*SU^s(r,L_0) \rightarrow W$$

$$(E,\theta) \rightarrow (s_2(\theta), s_3(\theta), \ldots, s_r(\theta)).$$

Hitchin [Hi 1] was the first one to study the morphism (1.10) in various situations. He discovered many of its remarkable properties and in particular he showed that $H$ endows the (holomorphic) symplectic manifold $X(r,L_0) := T^*SU^s(r,L_0)$ with a structure of an algebraically completely integrable hamiltonian system. More specifically, he showed that there exists a partial compactification

$$H : M(r,L_0) \rightarrow W$$

with general fiber an abelian variety and such that the fibers of (1.10) embed as Zariski open sets in the fibers of $H : M(r,L_0) \rightarrow W$. The variety $M(r,L_0)$ is again a moduli space which parametrizes the equivalence classes of semistable Higgs pairs - that is, pairs $(E,\theta)$ of a bundle and an $\omega_C$-twisted endomorphism such that for every $\theta$-invariant subbundle $U \subset E$ the usual inequality $\mu(U) \leq \mu(E)$ for the slopes of $U$ and $E$ holds. The characteristic coefficients map (1.10) is a well defined morphism on the whole variety $M(r,L_0)$ and is called the Hitchin map of $M(r,L_0)$. There is a natural $\mathbb{C}^\times$-action on the moduli space $M(r,L_0)$

$$(E,\theta) \mapsto (E,t\theta),$$

which induces via the Hitchin map a weighted $\mathbb{C}^\times$-action on the vector space $W$. A number $t \in \mathbb{C}^\times$ acts on $W$ by multiplying the piece $H^0(C,\omega_C^{2j})$ with $t^j$. The compactification diagram (1.11) and its generalizations have been studied extensively in the last years, see [B-N-R], [Hi 1], [Hi 2], [N], [S]. For our purposes, the most convenient is the approach in [B-N-R] which we proceed to describe.

**Proposition 1.5 (Beauville-Narasimhan-Ramanan)** Assume that $\pi_s : \tilde{C}_s \rightarrow C$ is an integral spectral curve. Then the push-forward map $\pi_{ss}$ induces a canonical bijection between

- Isomorphism classes of rank one torsion-free sheaves $F$ on $\tilde{C}_s$ of Euler characteristic $d - r(g-1)$ satisfying $\det(\pi_{ss}F) = L_0$.
- Isomorphism classes of Higgs pairs $(E,\theta)$ satisfying $\det E = L_0$, $\text{tr}\theta = 0$, $H(\theta) = s$. 

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Remark 1.5 Let $s \in W$ be such that $\tilde{C}_s$ is integral. Then every Higgs pair satisfying $H(\theta) = s$ is stable. Indeed, if we assume that there exists a proper $\theta$-invariant subbundle $U \subset E$, then the spectral curve of $(U, \theta|_U)$ will be contained in $\tilde{C}_s$ which contradicts the integrality of $\tilde{C}_s$. Consequently $E$ does not have $\theta$-invariant subbundles and in particular $(E, \theta)$ is stable.

Under the bijection in Proposition 1.5 this translates into the well known fact that the moduli functor of rank one torsion free sheaves on an integral curve possessing only planar singularities is representable by an irreducible fine moduli space - the compactified Jacobian of the curve, see [A-I-K]. Therefore the push-forward map $\pi_{ss}$ induces an isomorphism

$$\pi_{ss} : \text{Prym}_{d}(\tilde{C}_s, C) \to H^{-1}(s),$$

where $\text{Prym}_{d}(\tilde{C}_s, C) = \{ F \in J_{d-r(g-1)}(\tilde{C}_s) \mid \det(\pi_{ss}F) = L_0 \}$, and $J_{d-r(g-1)}(\tilde{C}_s)$ is the generalized Jacobian parametrizing rank one torsion free sheaves of Euler characteristic $d - r(g - 1)$ on $\tilde{C}_s$.

Remark 1.6 If the curve $\tilde{C}_s$ is smooth, then the Prymian $\text{Prym}_{d}(\tilde{C}_s, C)$ is in a natural way torsor over an abelian variety. Indeed, it is easy to see that

$$\det(\pi_{ss}F) = \text{Nm}_{\pi_{ss}}(F) \otimes \det(\pi_{ss}\mathcal{O}_{\tilde{C}_s}),$$

for any line bundle $F$ on $\tilde{C}_s$. Since $\pi_{ss}\mathcal{O}_{\tilde{C}_s} = \mathcal{O}_{C} \oplus \omega_C^{-1} \oplus \ldots \oplus \omega_C^{-(r-1)}$, see [B-N-R], we get

$$\text{Prym}_{d}(\tilde{C}_s, C) = \{ F \in J^{\tilde{d}}(\tilde{C}_s) \mid \text{Nm}_{\pi_{ss}}(F) = L_0 \otimes \omega_C^{\otimes \frac{r(r-1)}{2}} \},$$

with $\tilde{d} = d + r(r - 1)(g - 1)$.

Remark 1.7 We will need more explicit geometric description for the fiber $H^{-1}(s)$ of the Hitchin map over a generic point $s \in D$ of the discriminant. Observe first that there exists a point $s \in D$ for which $\tilde{C}_s$ is irreducible and has a unique ordinary double point as singularity. Indeed, let $s_r \in H^0(C, \omega_C^{\otimes r})$ be a section having a unique double zero $a \in C$. It follows then by the Jacobian criterion for smoothness that the curve

$$\tilde{C}_s : X^r + s_rY^r = 0,$$

has a unique node lying over $a$. Consequently, by upper semicontinuity there is a non-empty Zariski open set $D^o \subset D$ satisfying

$$D^o = \{ s \in D \mid \tilde{C}_s \text{ is irreducible and has a unique ordinary double point} \}.$$
Let now \( s \in \mathcal{D}^\circ \) and say \( p \in \tilde{C}_s \) is its ordinary double point. Set \( a = \pi_s(p) \). To study the geometric properties of \( \text{Prym}_d(\tilde{C}_s, C) \), we recall the construction of the compactified Jacobian \( J_{d-r(g-1)}(\tilde{C}_s) \) in this simple case. Let

\[
\begin{array}{ccc}
\tilde{C}_s' & \xrightarrow{\nu} & \tilde{C}_s \\
\pi_s' & \downarrow & \downarrow \pi_s \\
C & & 
\end{array}
\]

be the normalization of the curve \( \tilde{C}_s \). Let \( \nu^{-1}(p) = \{p^+, p^-\} \subset \tilde{C}_s' \). For any rank one torsion free sheaf \( F \to \tilde{C}_s \) we have the exact sequence

\[
0 \to F \to \nu_\ast \nu^\ast F \to \mathcal{C}_p \to 0. \tag{1.12}
\]

Therefore \( \chi(F) + 1 = \chi(\nu_\ast \nu^\ast F) = \chi(\nu^\ast F) \) and hence the pull-back induces a morphism

\[
\nu^\ast : J_{d-r(g-1)}(\tilde{C}_s) \to J_{d+1-r(g-1)}(\tilde{C}_s'),
\]

where \( J_{d+1-r(g-1)}(\tilde{C}_s') \) is the component of the Picard group of \( \tilde{C}_s' \) consisting of line bundles of Euler characteristic \( d + 1 - r(g-1) \). One can show that (1.13) is a bundle with structure group \( \mathbb{C}^\times \) and with fibers isomorphic to a \( \mathbb{P}^1 \) glued at two points. To identify the bundle (1.13) explicitly, consider a Poincare bundle

\[
\begin{array}{ccc}
P & \to & J_{d+1-r(g-1)}(\tilde{C}_s') \times \tilde{C}_s' \\
\downarrow & & \\
J_{d+1-r(g-1)}(\tilde{C}_s') \times \tilde{C}_s' & \to & J_{d+1-r(g-1)}(\tilde{C}_s')
\end{array}
\]

and set \( \mathcal{P}_+ = \mathcal{P}|_{J_{d+1-r(g-1)}(\tilde{C}_s') \times \{p^+\}} \) and \( \mathcal{P}_- = \mathcal{P}|_{J_{d+1-r(g-1)}(\tilde{C}_s') \times \{p^-\}} \). The \( \mathbb{P}^1 \)-bundle

\[
\mathbb{P}(\mathcal{P}_+ \oplus \mathcal{P}_-) \to J_{d+1-r(g-1)}(\tilde{C}_s')
\]

does not depend on the choice of the Poincare bundle \( \mathcal{P} \) and is furnished with two sections \( X_+ \) and \( X_- \) corresponding to \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) respectively. Furthermore, there is a bundle isomorphism

\[
\begin{array}{ccc}
J_{d-r(g-1)}(\tilde{C}_s) & \xrightarrow{\cong} & \mathbb{P}(\mathcal{P}_+ \oplus \mathcal{P}_-)/X_+ \sim X_- \\
\nu^\ast & \downarrow & \\
J_{d+1-r(g-1)}(\tilde{C}_s') & & 
\end{array}
\]

To describe the subvariety \( \text{Prym}_d(\tilde{C}_s, C) \subset J_{d+1-r(g-1)}(\tilde{C}_s) \) in these terms, we need the following lemma
Lemma 1.3 If $F_1, F_2 \in \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s)$ are such that
\[
\det(\pi_{ss} F_1) = \det(\pi_{ss} F_2),
\]
then
\[
\det(\pi'_{ss} \nu^* F_1) = \det(\pi'_{ss} \nu^* F_2).
\]

Proof. For any $F \in \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s)$ we have the short exact sequence (1.12). After taking direct images, we get
\[
0 \to \pi_{ss} F \to \pi_{ss} \nu_\ast \nu^* F \to \pi_{ss} \mathcal{C}_p \to R^1 \pi_{ss} F
\]
i.e. we have a short exact sequence of sheaves on $C$:
\[
0 \to \pi_{ss} F \to \pi'_{ss} \nu^* F \to \mathcal{C}_a \to 0.
\]
Therefore the bundle $\pi_{ss} F$ is a Hecke transform of the bundle $\pi'_{ss} \nu^* F$ with center at an $(r-1)$-dimensional subspace of the fiber $(\pi'_{ss} \nu^* F)_a$. Thus
\[
\det(\pi_{ss} F) = \det(\pi'_{ss} \nu^* F) \otimes \mathcal{O}_C(-x),
\]
which yields the statement of the lemma.

According to the above lemma, if a point $F \in \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s)$ belongs to the subvariety $\text{Prym}_{d}(\tilde{C}_s, C)$, then every point in the fiber $(\nu^*)^{-1}(\nu^*(F))$ also belongs to this subvariety.

Define $\text{Prym}(\tilde{C}_s', C) := \{ F \in \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s') \mid \det(\pi'_{ss} F) = L_0(-a) \}$. Then $\text{Prym}_{d}(\tilde{C}_s, C)$ fits in the fiber square
\[
\begin{array}{ccc}
\text{Prym}(\tilde{C}_s', C) & \longrightarrow & \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s) \\
\nu^* & & \nu^* \\
\text{Prym}(\tilde{C}_s', C) & \longrightarrow & \mathcal{J}_{d+1-r(g-1)}(\tilde{C}_s')
\end{array}
\]
In particular, $\text{Prym}_{d}(\tilde{C}_s, C)$ is a bundle over the abelian variety $\text{Prym}(\tilde{C}_s', C)$ whose total space is singular along a divisor and has a smooth normalization which is a $\mathbb{P}^1$-bundle over $\text{Prym}(\tilde{C}_s', C)$.

The total space $X(r, L_0)$ of the cotangent bundle of $SU^e(r, L_0)$ is a Zariski open set in the moduli space of Higgs bundles $M(r, L_0)$. It has the advantage that we have morphisms both to
whose fibers are generically transversal. The natural projection \( \Pi : \mathcal{X}(r, L_0) \to \mathcal{SU}^s(r, L_0) \) is onto by definition and the Hitchin map \( H \) is dominant by an argument of Beauville-Narasimhan-Ramanan, see [B-N-R]. In particular, this implies that for the generic element \( s \in W \) the push-forward map \( \pi_s : H^{-1}(s) \cap \mathcal{X}(r, L_0) \to \mathcal{SU}^s(r, L_0) \) is dominant. In specific geometric situations it is important to know what is the codimension of the locus of those \( s \in W \) for which \( \pi_s \) is not dominant. A general result to this extend can be deduced easily from the following Lemma 1.4 communicated to us by E. Markman.

First we introduce some notation.

**Definition 1.2** A vector bundle \( E \in \mathcal{U}(r, d) \) is called very stable if it does not have non-zero nilpotent \( \omega_C \)-twisted endomorphism.

By a result of Drinfeld and Laumon, see [La], [B-N-R], very stable vector bundles always exist. Let \( U \subset \mathcal{SU}^s(r, L_0) \) be the non-empty Zariski open set consisting of very stable bundles. For an \( E \in \mathcal{X}(r, L_0) \) denote by \( \mathcal{X}_E := T^*_E \mathcal{SU}^s(r, L_0) = H^0(C, \text{End}^o E \otimes \omega_C) \subset \mathcal{X}(r, L_0) \) the fiber of the cotangent bundle at \([E]\).

**Lemma 1.4** For any \( E \in U \) the Hitchin map

\[
H_E = H_{|\mathcal{X}_E} : \mathcal{X}_E \longrightarrow W
\]

is surjective.

**Proof.** Since \( E \) is very stable we have \( H^{-1}_E(0) = 0 \) and hence the dimension of the generic fiber of \( H_E \) is zero. On the other hand, \( \dim \mathcal{X}_E = \dim W \) and therefore \( H_E : \mathcal{X}_E \to W \) is dominant.

Denote by \( W^o \subset W \) the Zariski open subset

\[
W^o = \{ s \in W \mid \dim H^{-1}_E(s) = 0 \}.
\]

Clearly the subvariety \( \mathcal{X}_E \) is preserved by the \( \mathbb{C}^\times \) action on \( \mathcal{M}(r, L_0) \). Furthermore the \( \mathbb{C}^\times \)-equivariance of \( H \) implies that \( W^o \) is a \( \mathbb{C}^\times \)-invariant subset of \( W \). Since \( H^{-1}_E(0) = \{0\} \), we have that

\[
H_E : \mathcal{X}_E \setminus \{0\} \longrightarrow W \setminus \{0\}
\]
is a $\mathbb{C}^\times$-equivariant morphism and thus descends to a morphism
\[ \overline{\Pi}_E : \mathbb{P}(\mathcal{X}_E) \to \mathbb{P}_{\weight}, \]
where $\mathbb{P}_{\weight} = (W \setminus \{0\})/\mathbb{C}^\times$ is the corresponding weighted projective space.

The morphism $\overline{\Pi}_E$ is dominant because its range contains the Zariski open set $(W^0 \setminus \{0\})/\mathbb{C}^\times \subset \mathbb{P}_{\weight}$. But $\mathbb{P}(\mathcal{X}_E)$ is a projective variety and therefore $\overline{\Pi}_E(\mathbb{P}(\mathcal{X}_E))$ is projective and thus $\overline{\Pi}_E$ is onto.

We obtain a commutative diagram of $\mathbb{C}^\times$-bundles over $\mathbb{P}(\mathcal{X}_E)$ and $\mathbb{P}_{\weight}$ respectively

\[
\begin{array}{ccc}
\mathcal{X}_E \setminus \{0\} & \xrightarrow{H_E} & W \setminus \{0\} \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{X}_E) & \xrightarrow{\overline{\Pi}_E} & \mathbb{P}_{\weight}
\end{array}
\]

Since the $\mathbb{C}^\times$ on the fibers of $\mathcal{X}_E \setminus \{0\} \to \mathbb{P}(\mathcal{X}_E)$ and $W \setminus \{0\} \to \mathbb{P}_{\weight}$ is simply transitive, we obtain that the map (1.15) (and consequently the map (1.14)) is surjective.

\[\square\]

**Corollary 1.7** For any $s \in W$ the rational map
\[ \pi_{ss} = \Pi|_{H^{-1}(s)} : H^{-1}(s) \to SU(r,L_0) \]
is dominant.

**Proof.** By Lemma 1.4 the range $\Pi(H^{-1}(s))$ contains the nonempty Zariski open set $U$.

\[\square\]

2 The abelianization of an automorphism

2.1 The map $\varphi_W$

Start with an automorphism $\Phi$ of $SU(r,L_0)$. We would like to abelianize it, i.e. to lift $\Phi$ to the moduli space of Higgs bundles and induce from it an automorphism of the family of spectral Pryms. The hope is that in this way we will get a simpler picture because of the fact that the automorphisms of the abelian varieties are pretty well understood. As we will see in the subsequent sections, this hope turns out to be unsubstantiated in that simple form since the spectral Pryms are not generic abelian varieties and can have non-trivial group automorphisms. This is the reason
why we have to work with the whole family of Pryms rather then the individual Prym varieties and to make essential use of its “rigidity” as a group scheme over $W$.

To construct the abelianized version of $\Phi$, observe first that the codifferential of $\Phi$ provides a natural lift of $\Phi|_{\mathcal{SU}(r,L_0)}$ to the total space of the cotangent bundle

$$d\Phi^*: \mathcal{X}(r,L_0) \to \mathcal{X}(r,L_0).$$

The automorphism $d\Phi^*$ commutes with the projection $\Pi: \mathcal{X}(r,L_0) \to \mathcal{SU}(r,L_0)$ by definition. The behaviour of $d\Phi^*$ with respect to the Hitchin map is described by the following proposition.

**Proposition 2.1** There exists an automorphism $\varphi_W \in \text{Aut}(W)$ making the following diagram commutative

$$\begin{array}{ccc}
\mathcal{X}(r,L_0) & \xrightarrow{d\Phi^*} & \mathcal{X}(r,L_0) \\
H & \downarrow & H \\
W & \xrightarrow{\varphi_W} & W
\end{array}$$

**Proof.** Let $s \in W \setminus D$. Then the spectral curve $\tilde{C}_s$ is smooth and (see Remark 1.3) $H^{-1}(s) = \text{Prym}_d(\tilde{C}_s, C)$. Consider the fiber

$$H^{-1}_X(s) := H^{-1}(s) \cap \mathcal{X}(r,L_0) = \{ F \in \text{Prym}_d(\tilde{C}_s, C) \mid \alpha_s F \text{ stable} \}.$$ 

The argument in Remark 5.2 of [B-N-R] gives $\text{codim}(H^{-1}_X(s), H^{-1}(s)) \geq 2$. Therefore since $\text{Prym}_d(\tilde{C}_s, C) = H^{-1}(s)$ is smooth, the regular map $H \circ d\Phi^* : H^{-1}_X(s) \to W$ extends by Hartogs theorem to a morphism $\overline{H \circ d\Phi^*} : H^{-1}(s) \to W$. But $\text{Prym}_d(\tilde{C}_s, C)$ is projective and $W$ is an affine space; hence

$$\overline{H \circ d\Phi^*} (\text{Prym}_d(\tilde{C}_s, C)) = \text{point in } W.$$ 

Denote this point by $\varphi_W(s)$. In this way we obtain a rational map $\varphi_W$ on $W$, with possible singularities along $D$ which makes the diagram (2.16) commutative.

This implies that $\varphi_W$ is $\mathbb{C}^\times$-equivariant since $d\Phi^*$ is linear on the fibers of the cotangent bundle and $H$ is $\mathbb{C}^\times$-equivariant. Thus $\varphi_W$ descends to a rational map

$$\varphi_W : \mathbb{P}_{\text{weight}} \dashrightarrow \mathbb{P}_{\text{weight}}.$$ 

But $\mathbb{P}_{\text{weight}}$ is a normal projective variety and therefore $\varphi_W$ is not defined in a locus of codimension $\geq 2$. Finally the $\mathbb{C}^\times$-equivariance of $\varphi_W$ together with the fact that $\mathbb{C}^\times$ acts transitively on the fibers of $W \setminus \{0\} \to \mathbb{P}_{\text{weight}}$ yield that $\varphi_W$ itself is not defined on a locus of codimension $\geq 2$. Since $\varphi_W$ is a map from a vector space to a vector space, it extends by Hartogs to the whole $W$. 

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Since we want to reduce the study of the automorphism $\Phi$ to the study of a suitable fiberwise automorphism of the family of spectral Pryms, the best situation for us would have been if we could say that $\varphi_W = \text{id}$. Unfortunately, at this stage the only information we can extract from the above proposition is that $\varphi_W$ commutes with the $\mathbb{C}^\times$ action on $W$. In the next lemma we use this to determine some further properties of $\varphi_W$.

**Lemma 2.1** The map $\varphi_W$ preserves the subspace $W_{r-1,r} \subset W$ and moreover $\phi_{r-1,r} := \varphi_W|_{W_{r-1,r}}$ is of the form $\phi_{r-1,r} = (\phi_{r-1}, \phi_r)$ where

$$
\phi_{r-1} : H^0(C, \omega_C^\otimes(r-1)) \to H^0(C, \omega_C^\otimes(r-1))
$$

and

$$
\phi_r : H^0(C, \omega_C^\otimes r) \to H^0(C, \omega_C^\otimes r)
$$

are linear maps.

**Proof.** Consider the algebra of regular functions $S := \mathbb{C}[W]$ on $W$. Decomposing $S$ into isotypical components with respect to the $\mathbb{C}^\times$ action, we get a new grading on it:

$$
S = \otimes_{n \geq 0} S_n,
$$

where $S_n$ consists of all functions on $W$ on which a number $t \in \mathbb{C}^\times$ acts by a multiplication by $t^n$. More explicitly, if we choose coordinates $x_{i1}, \ldots, x_{in_i}$ in $W_i = H^0(C, \omega_C^\otimes i)$, then the elements in $S_n$ are just polynomials in $x_{ij}$’s with admissible multidegrees $\{a_{ij}\}$ satisfying

$$
\sum_{i,j} i \cdot a_{ij} = n.
$$

Any automorphism of $W$ commuting with the $\mathbb{C}^\times$ action will preserve this grading and in particular will preserve the subspaces of the form $W_i \oplus W_{i+1} \oplus \ldots \oplus W_r$ for any $i$. In particular, since $\varphi_W$ commutes with the $\mathbb{C}^\times$ action we have

$$
\phi_{r-1,r} : W_{r-1,r} \to W_{r-1,r}.
$$

Moreover, since g.c.d. $(r - 1, r) = 1$, it follows that $\phi_{r-1,r}$ has the required form.

$\square$
2.2 Extension to the family of Pryms

As we have seen in Section 1.3 the fiber of the Hitchin map \( H : \mathcal{M}(r, L_0) \rightarrow W \) over every point \( s \in W^{\text{reg}} := W \setminus D \) is isomorphic to an abelian variety. Moreover, one can show, see [Ni], that \( D \) is exactly the locus of critical values of \( H \). Define \( \text{Prym}_d(\tilde{C}, C) \) to be the preimage of \( W^{\text{reg}} \) under the Hitchin map. Then \( \text{Prym}_d(\tilde{C}, C) \) is a smooth variety, see [Ni], and we have a smooth fibration \( H : \text{Prym}_d(\tilde{C}, C) \rightarrow W^{\text{reg}} \). Our next goal is to extend the automorphism \( d\Phi^* \) to an automorphism of the family \( \text{Prym}_d(\tilde{C}, C) \). The first result in this direction is the following proposition

**Proposition 2.2** The map \( \varphi_W \) preserves the discriminant divisor \( D \subset W \).

**Proof.** Let \( s \in W \). Corollary 1.7 guarantees that \( \text{Prym}_d(\tilde{C}_s, C) \cap \mathcal{X}(r, L_0) \neq \emptyset \) and hence, due to Proposition 2.1, \( d\Phi^* \) induces a birational automorphism between \( H^{-1}(s) \) and \( H^{-1}(\varphi_W(s)) \).

Let \( s \in D^0 \). If we assume that \( \varphi_W(s) \notin D \), then \( \tilde{C}_{\varphi_W(s)} \) is smooth and \( \text{Prym}(\tilde{C}_{\varphi_W(s)}, C) \) is isomorphic to an abelian variety. Therefore \( \text{Prym}_d(\tilde{C}_s, C) \) is birational to an abelian variety. On the other hand \( \text{Prym}_d(\tilde{C}_s, C) \) is birationally isomorphic to a \( \mathbb{P}^1 \) bundle over an abelian variety, see Remark 1.7, which is a contradiction because such a bundle has Kodaira dimension \(-\infty\).

Consequently \( \varphi_W(D^0) \subset D \). But \( \varphi_W \) is an automorphism and hence is a closed map which yields \( \varphi_W(D) = D \) since the discriminant is irreducible, see Corollary 1.8.

\[ \square \]

**Corollary 2.1** Let \( \phi_r \) be the map defined in Lemma 2.1. Then there exists a number \( \lambda \in \mathbb{C}^\times \) and an automorphism \( \sigma \) of the curve \( C \) such that

\[ \phi_r = m_\lambda \circ \sigma^*, \]

where \( m_\lambda \) is the dilation by \( \lambda \).

**Proof.** By Lemma 2.1 the map \( \phi_r \) is a linear map. On the other hand, Proposition 2.2 implies in particular that the map \( \phi_r \) preserves the discriminant divisor \( D_r \). Therefore the induced map \( \overline{\phi}_r \) on the projective space \( \mathbb{P}(W) \) preserves the divisor \( \mathbb{P}(D_r) \). But, by Remark 1.3, the divisor \( \mathbb{P}(D_r) \) is the dual variety of the \( r \)-canonical model of \( C \) and therefore the dual map \( \check{\phi}_r \) preserves the curve \( f_{\mathbb{P}(W_r)}(C) \subset \mathbb{P}(W_r)^V \). Let \( \sigma \) be the induced automorphism of \( C \). Then, since \( f_{\mathbb{P}(W_r)}(C) \) spans \( \mathbb{P}(W_r)^V \) and \( \check{\phi}_r \) is linear, it follows that \( \overline{\phi}_r = \sigma^* \).

\[ \square \]
Remark 2.1 If the curve $C$ does not have automorphisms, then the above corollary implies that $\phi_r$ is just a dilation.

The proposition above yields the commutative diagram

$$
\begin{array}{ccc}
\text{Prym}_d(\tilde{C}, C) & \xrightarrow{d\Phi^*} & \text{Prym}_d(\tilde{C}, C) \\
\phi_W & \quad \downarrow \quad & \text{H} \\
W^{\text{reg}} & \quad \xrightarrow{\phi_W} & W^{\text{reg}}
\end{array}
$$

Furthermore, as we saw in the proof of Proposition 2.2 the map $d\Phi^*$ gives a birational isomorphism between $H^{-1}(s)$ and $H^{-1}(\varphi_W(s))$ for any $s$. In particular, in the case $s \in W^{\text{reg}}$ we get a birational automorphism between abelian varieties and hence $d\Phi^*_{|H^{-1}(s)}$ extends to a biregular isomorphism between $H^{-1}(s)$ and $H^{-1}(\varphi_W(s))$. Therefore $d\Phi^*$ extends as a continuous automorphism $\tilde{\Phi}_d$ of the whole variety Prym$_d(\tilde{C}, C)$ and since Prym$_d(\tilde{C}, C)$ is smooth, we get:

**Proposition 2.3** There exists a biregular automorphism $\tilde{\Phi}_d$ extending $d\Phi^*$ which fits in the commutative diagram

$$
\begin{array}{ccc}
\text{Prym}_d(\tilde{C}, C) & \xrightarrow{\tilde{\Phi}_d} & \text{Prym}_d(\tilde{C}, C) \\
\phi_W & \quad \downarrow \quad & \text{H} \\
W^{\text{reg}} & \quad \xrightarrow{\phi_W} & W^{\text{reg}}
\end{array}
$$

3 The fibration $\tilde{C} \longrightarrow W_{r-1,r}^{\text{reg}}$

3.1 The Picard group of $\tilde{C}$

We start with some notation: For a vector space $V$, the points of the Grassmanian $Gr(k, V)$ correspond to codimension $k$ linear subspaces of $V$. Also, for a linear space of sections $W$ on $S$ and a given point $p$ on $S$, $W_p$ denotes the linear subspace of sections of $W$ vanishing at $p$. We now make the assumption that $r \geq 3$. We are going first to prove our main Theorem A in the case $r \geq 3$ and next, in Section 5.3, we outline the modifications needed for the proof of the rank 2 case. For simplicity denote the space $W_{r-1,r}$ by $B$ and the spaces $\overline{W}_{r-1,r}$ and $W_{r-1,r}^{\text{reg}} = W_{r-1,r} \setminus \mathcal{D}$ by $\overline{B}$ and $B^{\text{reg}}$ respectively. In the following we are going to use repeatedly Proposition 6.5 in [43], which compares the Picard group of a variety $X$ with that of a Zariski open subset $U$ of $X$.

**Notation.** From now on we will denote by $\tilde{C}$ the part of the universal spectral curve sitting over $B^{\text{reg}}$, unless otherwise stated.
We have the diagram:

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{h} & B^{\text{reg}} \\
\downarrow{\beta} & & \downarrow{\pi} \\
S^\circ & \xrightarrow{\alpha} & C
\end{array}
\]

**Proposition 3.1** The Picard group of the variety \( \tilde{C} \) is isomorphic to the pull back of the Picard group of the base curve \( C \) by the map \( \pi \), i.e. \( \text{Pic} \tilde{C} \simeq \pi^* \text{Pic} C \).

**Proof.** The variety \( \tilde{C} \) can be constructed as follows. Consider the diagram:

\[
\begin{array}{ccc}
f^+_B U^{\text{reg}}|_{S^\circ} \simeq \tilde{C} & \xrightarrow{\beta} & U^{\text{reg}} \subset U \\
\downarrow{\beta} & & \downarrow{p_1} \\
S^\circ \subset S & \xrightarrow{f|_B} & \text{Gr}(1, \overline{B}) \\
\end{array}
\]

\begin{equation}
(3.17)
\end{equation}

The map \( f|_B \) is the map associated to the base point free linear system \([B]\) i.e. it sends a point \( p \) to the point \([B_p]\) representing the linear subspace \( B_p \). The bundle \( U \) is the universal bundle over the Grassmanian. The fiber of the bundle \( U \cap p_2^{-1}(B) \) over a point \([H]\) in the Grassmanian is isomorphic to \( H \cap B \). The variety \( U \cap p_2^{-1}(B) \) is an affine bundle over \( f|_B(S^\circ) \) with fiber over the point \( f|_B(p) \) isomorphic to \( B_p \). Define \( U^{\text{reg}} \) as \( U \cap p_2^{-1}(B \setminus D) = U \cap p_2^{-1}(B^{\text{reg}}) \). Then \( \tilde{C} \simeq f^+_B U^{\text{reg}}|_{S^\circ} \). Note that by Corollary 3.1, the divisor \( f^+_B(p_2^{-1}(\Delta) \cap U) \) is irreducible. Also it is linearly equivalent to zero since \( \text{Pic } B \) is trivial.

To complete the proof of the proposition, we have:

\[
\begin{align*}
\text{Pic} \tilde{C} & \simeq \text{Pic} f^+_B U^{\text{reg}}|_{S^\circ} \\
& \simeq \text{Pic} f^+_B (U \cap p_2^{-1}(B \setminus D))|_{S^\circ} \\
& \simeq \text{Pic} f^+_B (U \cap p_2^{-1}(B))|_{S^\circ} \\
& \simeq \beta^* \text{Pic } S^\circ \\
& \simeq \pi^* \text{Pic } C
\end{align*}
\]

since \( U \cap p_2^{-1}D \) is an irreducible divisor lin. equiv. to 0, since \( f^+_B(U \cap p_2^{-1}(B))|_{S^\circ} \) is an affine bundle over \( S^\circ \), since \( S^\circ \) is an affine bundle over \( C \).

\( \square \)
3.2 The Picard group of $\tilde{C}_p$

For a fixed point $p$ in $S^o$ we denote by $\tilde{C}_p$ the subvariety of $\tilde{C}$ consisting of those curves whose image in $S^o$ passes through $p$. $\tilde{C}_p$ sits over the subspace $B^\text{reg}_p$ of $B^\text{reg}$, consisting of those sections vanishing at $p$. Let $h : \tilde{C}_p \to B^\text{reg}_p$ denote again the restriction of the map $h$ on $\tilde{C}_p$. We are going to calculate the Picard group of $\tilde{C}_p$. Observe that the above fibration $h$ has a section corresponding to the point $p$. Take the restriction of the map $\beta$ to $\tilde{C}_p$ i.e. $\beta : \tilde{C}_p \to S^o$. Let $H_p$ be the preimage of $p$ via the map $\beta$. By a construction similar to that for the variety $\tilde{C}$ in the above Section 3.1, one can see that

the fibration $\beta : \tilde{C}_p \setminus H_p \to S^o \setminus \{p\}$ is the complement of an irreducible divisor linearly equivalent to zero in an affine fibration. We thus have $\text{Pic} (\tilde{C}_p \setminus H_p) \simeq \beta^* \text{Pic} (S^o \setminus \{p\}) \simeq \beta^* \text{Pic} S^o \simeq \pi^* \text{Pic} C$. Since $H_p$ is an irreducible divisor in $\tilde{C}_p$, we conclude that $\text{Pic} \tilde{C}_p$ is generated by $\pi^* \text{Pic} C$ and $H_p$.

We proceed by showing that we actually have:

**Proposition 3.2** $\text{Pic} \tilde{C}_p \simeq \pi^* \text{Pic} C \oplus \mathbb{Z}[H_p]$.

**Proof.** Pick a generic pencil $\mathbb{P}^1$ in $\overline{B}_p$ such that all the fibers of the restriction of the compactified universal spectral curve over the pencil are irreducible, except the one over the infinity point $b$ which has exactly two irreducible components, see Corollary [1.4]. We may also assume that none of the singular points of the fibers lies on the $N = 2r^2(g-1)$ base points of the system. The restriction of the compactified universal spectral curve over $\mathbb{P}^1$, is a smooth surface $X$ which is the blow up of the surface $S$ over the $N$ base points $p_1 = p, \ldots, p_N$ of the pencil. We have the following picture:
In the above picture, \( b \) is the point at infinity, \( b_1, \ldots, b_k \) correspond to the singular fibers and \( E_1, \ldots, E_N \) are the exceptional divisors. The curve over \( b \) consist of two components \( \tilde{Y}_\infty \) and \( \tilde{C}_b \), one of which, \( \tilde{Y}_\infty \), is the proper transform of the divisor at infinity on \( S \). The intersection of \( X \) with the variety \( \tilde{C}_b \) is \( X^\circ = X \setminus h^{-1}(b, b_1, \ldots, b_k) \). We define \( E_i^\circ = E_i \cap X^\circ \). Note that \( E_i^\circ = H_p \cap X \). To prove the claim, it is enough to show that on \( X^\circ \) the line bundle \( E_i^\circ \) is independent from \( \pi^* \text{Pic} C \).

We have

\[
\text{Pic} X \simeq \beta^* \text{Pic} S \oplus \mathbb{Z}[E_i]_{i=1,\ldots,N} \simeq \pi^* \text{Pic} C \oplus \mathbb{Z}[\tilde{Y}_\infty] \oplus \mathbb{Z}[\tilde{E}_i].
\]

Now \( \beta^*(rY_\infty + r\alpha^*\omega_C) = h^{-1}(\text{point}) + \sum_{i=1}^N E_i \), where the equality stands for the linear equivalence of divisors. Therefore, \( r\tilde{Y}_\infty + r\pi^*\omega_C = h^{-1}(\text{point}) + \sum_{i=1}^N E_i = \tilde{C}_b^\prime + \tilde{Y}_\infty + \sum_{i=1}^N E_i \). In other words,

\[
\tilde{C}_b^\prime = (r - 1)\tilde{Y}_\infty + r\pi^*\omega_C - \sum_{i=1}^N E_i.
\]

All the fibers of \( h \) define linear equivalent divisors on \( X \) and the restriction on \( X \setminus h^{-1}(b) \) of the line bundles corresponding to the divisors \( \tilde{C}_b^\prime \) and \( \tilde{Y}_\infty \) are trivial. We thus get

\[
\text{Pic} X^\circ \simeq \text{Pic} X \setminus h^{-1}(b) \simeq \text{Pic} X / (\tilde{Y}_\infty = 0, \tilde{C}_b = 0) \\
\simeq \pi^* \text{Pic} C \oplus \mathbb{Z}[E_i]_{i=1,\ldots,N} / (r\pi^*\omega_C = \sum_{i=1}^N E_i^\circ) \quad \text{by (3.18) and (3.19)}.
\]

Assume now that \( \pi^*L + mH_p = 0 \) on \( \tilde{C}_b \). Then \( \pi^*L + mE_i^\circ = 0 \) on \( X^\circ \). By the description of the Pic \( X^\circ \) we conclude that \( \pi^*L = 0 \) and \( m = 0 \) and this completes the proof of the proposition.

\[
\square
\]

### 3.3 The sections of \( H : J^\tilde{a} (\tilde{C}) \longrightarrow W_{r-1}^{\text{reg}} \)

**Proposition 3.3** The only sections of the map \( H : J^\tilde{a} (\tilde{C}) \longrightarrow W_{r-1}^{\text{reg}} = B^{\text{reg}} \) are those coming from a pull back of a fixed line bundle on \( C \). In other words, if \( \sigma : B^{\text{reg}} \longrightarrow J^\tilde{a} (\tilde{C}) \) is a section of \( H \), then \( \sigma(s) = [\pi^*sM] \) where \( M \) is a fixed line bundle on \( C \). In particular, if \( r \) does not divide \( \tilde{a} \), then the map \( H \) has no sections.

We start with two lemmas:

**Lemma 3.1** On \( J^\tilde{a} (\tilde{C}) \times B^{\text{reg}} \tilde{C} \) there exists a line bundle \( \mathcal{P}^\tilde{a} \) such that \( \mathcal{P}^\tilde{a}|_{[L] \times \tilde{C}} \simeq L^\otimes n \) for some integer \( n \).

**Proof.** See [M-R].

\[
\square
\]
Remark 3.1 One can actually prove that the minimum such positive integer \( n \) is equal to \( \gcd(r, d) \).

Lemma 3.2 If \( \sigma \) is a section of the map \( H \) whose image lies in the locus \( \pi^* \text{Pic} C \) i.e. \( \sigma(s) = [\pi^* M_s] \), where \( M_s \) is a line bundle on \( C \), then \( M_s = M \) for all \( s \in B^{\text{reg}} \).

**Proof.** Consider the map \( \gamma : \tilde{C} \longrightarrow J^d(\tilde{C}) \times_{\text{B^{\text{reg}}}} \tilde{C} \) which sends a point \( q \) sitting over \( s \in B^{\text{reg}} \) to \( (\sigma(s), q) \). Then \( [\gamma^* P^d|_L] = n \sigma(s) \). By the description of \( \text{Pic} \tilde{C} \), see Proposition 3.1, we get that \( \gamma^* P^d \simeq \pi^* M_1 \) for a fixed \( M_1 \) in \( \text{Pic} C \). Hence, \( n[\pi^* M_s] = n \sigma(s) = [\pi^* M_1] \) for all \( s \in B^{\text{reg}} \). Since the map \( \pi^* \) is one to one, see Remark 3.10 in [B-N-R], we conclude that the map \( B^{\text{reg}} \rightarrow \text{Pic} C \) that sends \( s \) to \([M_s]\) has finite image. Since \( B^{\text{reg}} \) is connected, the map is constant.

\[\blacksquare\]

**Proof of Proposition 3.3.** Say that \( \sigma \) is not coming from a pull back. By the above Lemma 3.2, we may assume that there exists an \( s_0 \in B^{\text{reg}} \) such that \( \sigma(s_0) \) is not of the form \( \pi^* A \) for some \( A \) in \( \text{Pic} C \). Take a point \( p \) on \( S^0 \) that lies on \( \tilde{C}_{s_0} \subseteq S^0 \). The family of curves \( \tilde{C}_p \) has a section and therefore on \( J^d(\tilde{C}_p) \times_{\text{B^{\text{reg}}}} \tilde{C}_p \) there exists a Poincare bundle \( P^d_p \), see [M-R]. We have

\[ [\gamma^* P^d_p|_{\tilde{C}_s}] = n \sigma(s) \quad \text{and} \quad [\gamma^* P^d_p|_{\tilde{C}_s}] = \sigma(s) \quad \text{for all} \quad s \in B^{\text{reg}}_p. \]

Therefore, \( n \sigma(s) = [\gamma^* P^d_p|_{\tilde{C}_s}] \) for all \( s \in B^{\text{reg}}_p \). Since \( \text{Pic} B^{\text{reg}}_p \) is trivial, we conclude by the see-saw principle, see [Mu], that

\[ (3.20) \quad n \gamma^* P^d_p \simeq \gamma^* P^d_p \quad \text{on} \quad \tilde{C}_p. \]

On the other hand, by the description of the Picard groups of \( \tilde{C}_p \) and \( \tilde{C} \), see Propositions 3.1 and 3.2, we have

\[ (3.21) \quad \gamma^* P^d_p \simeq \pi^* M + mH_p \quad \text{and} \quad \gamma^* P^d \simeq \pi^* L. \]

Hence, by (3.20) and (3.21), we have on \( \tilde{C}_p \) that \( n \pi^* M + n mH_p \simeq \pi^* L \). Proposition 3.2 implies that \( m = 0 \). Hence, \( \gamma^* P^d_p \simeq \pi^* M \). But then, \( \sigma(s_0) = [\gamma^* P^d_p|_{\tilde{C}_{s_0}}] = [\pi^* M] \) which contradicts the assumption on \( \sigma(s_0) \).

\[\blacksquare\]
4 The ring of correspondences

4.1 The map $\beta': \tilde{C}_h \times \phi_h \tilde{C} \rightarrow S^o \times S^o$

Throughout the Sections 4 and 5, we will denote by $\phi$ the map $\phi_{r-1,r}: B \rightarrow B$. According to Lemma 2.1, we have that $\phi_{r-1,r}$ has the form $\phi_{r-1,r} = (\phi_{r-1}, \phi_r)$ where $\phi_i: H^0(C, \omega^i_C) \rightarrow H^0(C, \omega^i_C)$ are linear automorphisms for $i = r - 1, r$. Furthermore, by Proposition 2.2, we have that $\phi(D) = D$. Therefore, the restriction of the map $\phi$ on $B^{\mathrm{reg}}$, which we will denote again by $\phi$, induces an automorphism $\phi: B^{\mathrm{reg}} \rightarrow B^{\mathrm{reg}}$. In this section we study the fiber product $\tilde{C}_h \times \phi_h \tilde{C}$ defined by the diagram

We define $\overline{\phi}: \overline{B} \rightarrow \overline{B}$ to be the map which extends $\phi$ to $\overline{B}$ as $\overline{\phi} = (1, \phi_{r-1}, \phi_r)$. To investigate the map $\beta'$, we define in the product $S \times S$ the following loci $A$ and $\Gamma$.

**Definition 4.1**

$$A \overset{\text{df}}{=} S^o \times S^o \setminus \overline{\text{Im}\beta'}$$

$$\Gamma \overset{\text{df}}{=} \{(p, q) \in S \times S \text{ such that } \overline{B}_p = \overline{\phi}(\overline{B}_q)\}$$

On $S \times S \setminus (A \cup \Gamma)$ we can define a map $f$ to the Grassmanian $Gr(2, \overline{B})$ of codimension 2 linear subspaces of $\overline{B}$, by sending the point $(p, q)$ to the class of the plane $\overline{B}_p \cap \overline{\phi}(\overline{B}_q)$. We have the following diagram:

(4.22) $\beta'$

The notation is in complete analogy with that of the Diagram (3.17). As in the case of $\tilde{C}$, the variety $\tilde{C}_h \times \phi_h \tilde{C}|_{S^o \times S^o \setminus (A \cup \Gamma)}$ is an affine bundle over the image of $S^o \times S^o \setminus (A \cup \Gamma)$ under the map $f$.  

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Lemma 4.1 \( \dim A \leq 2 \)

**Proof.** Let \( B^\infty \) denote the space of sections \( W_{1,1}^\infty \), i.e. the linear subspace of \( W \) consisting of points of the form \((0,0,\ldots,0,s_{r-1},s_r)\). We have

\[
A = \{(p,q) \in S^\circ \times S^\circ \text{ such that } U\vert_{f(p,q)\cap p_2^{-1}B} = \emptyset \} \\
= \{(p,q) \in S^\circ \times S^\circ \text{ such that } B_p \cap \phi(B_q) = \emptyset \} \\
= \{(p,q) \in S^\circ \times S^\circ \text{ such that } B_p^\infty = \emptyset(B_q^\infty) \}
\]

By Corollary 1.3, the linear system defined by \( B^\infty \) separates points on \( S^\circ \). Therefore, given a point \( p \in S^\circ \) there exists at most one \( q \in S^\circ \) with \((p,q) \in A\) and this proves the lemma.

\( \Box \)

We give now a better description of the locus \( \Gamma \). Consider the diagram

\[
\begin{array}{c}
S & \xrightarrow{f_{\vert\mathbb{P}(\mathcal{B})}} & \mathbb{P}(\mathcal{B}) \\
\downarrow{\overline{\phi}^*f_{\vert\mathbb{P}(\mathcal{B})}} & & \downarrow{\overline{\phi}^*} \\
\mathbb{P}(\mathcal{B}) & & \mathbb{P}(\mathcal{B})
\end{array}
\]

In the above diagram the map \( \overline{\phi}^* \) is the induced automorphism of \( \mathbb{P}(\mathcal{B}) \) by the linear automorphism \( \overline{\phi} \). We denote by \( \pi_1 \) and \( \pi_2 \) the two projections of \( \Gamma \) to the surface \( S \). It is easy to see that

\[
\pi_1(\Gamma) = \{ p \in S \text{ such that } f_{\vert\mathbb{P}(\mathcal{B})}(p) \in \overline{\phi}^* f_{\vert\mathbb{P}(\mathcal{B})}(S) \} = f_{\vert\mathbb{P}(\mathcal{B})}^{-1}\left( f_{\vert\mathbb{P}(\mathcal{B})}(S) \cap \overline{\phi}^* f_{\vert\mathbb{P}(\mathcal{B})}(S) \right)
\]

and similar

\[
\pi_2(\Gamma) = (\overline{\phi}^* f_{\vert\mathbb{P}(\mathcal{B})})^{-1}\left( f_{\vert\mathbb{P}(\mathcal{B})}(S) \cap \overline{\phi}^* f_{\vert\mathbb{P}(\mathcal{B})}(S) \right)
\]

We describe the fibers of the map \( \pi_1 \). If \( p \in Y_{\infty} \), then it is easy to see that \( \pi_1^{-1}(p) \) consists of the whole divisor \( Y_{\infty} \). If \( p \in S^\circ \), then \( \pi_1^{-1}(p) \) consists of at most one point since the system \( \overline{\mathcal{B}} \) separates points on \( S^\circ \). By the above discussion we conclude that

Lemma 4.2 \( \dim (\Gamma \cap (S^\circ \times S^\circ)) \leq 2 \).

4.2 The Picard group of \( \tilde{\mathcal{C}}_{h\times_{\phi} \mathcal{C}} \)

We are going to use the following diagram
to calculate the Picard group of $\tilde{C}_h \times h \tilde{C}$.

**Lemma 4.3** Pic$(\tilde{C}_h \times h \tilde{C} | S^0 \times S^0 \setminus \Gamma) \cong \pi^* \text{Pic}(C \times C)$.

**Proof.** We have $\tilde{C}_h \times h \tilde{C} | S^0 \times S^0 \setminus \Gamma \cong f^* U_{\text{reg}} | S^0 \times S^0 \setminus \Gamma$ since the fiber over $A$ is empty. Therefore,

$$\text{Pic}(\tilde{C}_h \times h \tilde{C} | S^0 \times S^0 \setminus \Gamma) \cong \beta^* \text{Pic}(S^0 \times S^0 \setminus (\Gamma \cup A)),$$

since codim$(\Gamma \cup A) \geq 2$,

$$\cong \pi^* \text{Pic}(C \times C),$$

since $S^0 \times S^0$ is a rank two affine bundle over $C \times C$.

The variety $\tilde{C}_h \times h \tilde{C}$ splits as

$$\tilde{C}_h \times h \tilde{C} = \tilde{C}_h \times h \tilde{C} | S^0 \times S^0 \setminus \Gamma \amalg \tilde{C}_h \times h \tilde{C} | \Gamma.$$

To calculate its Picard group, we have to consider the following two cases.

**Case A:** dim$\Gamma \cap (S^0 \times S^0) \leq 1$: Then one can easily see that dim$\tilde{C}_h \times h \tilde{C} | \Gamma \leq \dim(\tilde{C}_h \times h \tilde{C}) - 2$. We thus have:

**Lemma 4.4** In case A, the Pic$(\tilde{C}_h \times h \tilde{C}) \cong \text{Pic}(\tilde{C}_h \times h \tilde{C} | S^0 \times S^0 \setminus \Gamma) \cong \pi^* \text{Pic}(C \times C)$.

**Case B:** dim$\Gamma \cap (S^0 \times S^0) = 2$: Following the notation we used in the description of $\Gamma$ in the above Section 4.1, we have $\pi_1(\Gamma) = S$. This implies that $\overline{\phi}^*$ induces an automorphism of $f_{\overline{B}}|S)$. The following summarizes the basic properties of the induced automorphism $\overline{\phi}^*$.

1. The point $p_{\infty} = f_{\overline{B}}(Y_{\infty})$ remains fixed.

2. $\overline{\phi}^*$ sends a fiber of the map $\alpha$ to a fiber: otherwise we get a $\mathbb{P}^1$ cover of the curve $C$, which contradicts the assumption that genus $g(C) > 0$. 


3. Since \( C \) has no automorphisms, \( \bar{\phi}^* \) sends a fiber of the map \( \alpha \) to the same fiber. Therefore \( \bar{\phi}^* \) induces an automorphism \( \chi : S^\circ \rightarrow S^\circ \) which preserves the fibers of the map \( \alpha : S^\circ \rightarrow C \).

**Lemma 4.5** Let \( s \in B \) and let \( \tilde{C_s} \) denote also the image of the corresponding spectral curve on \( S^\circ \). Then \( \chi(\tilde{C_s}) = \tilde{C_{\phi^{-1}(s)}} \).

**Proof.** Let \( H_s \) denote the hyperplane in \( \mathbb{P}(\mathbb{B}^\vee) \) corresponding to the section \( s \). Then

\[
\tilde{C_s} \simeq f_{|\mathbb{B}|}(\tilde{C_s}) = H_s \cap f_{|\mathbb{B}|}(S^\circ)
\]

and

\[
\tilde{C_{\phi^{-1}(s)}} \simeq f_{|\mathbb{B}|}(\tilde{C_{\phi^{-1}(s)}}) = H_{\phi^{-1}(s)} \cap f_{|\mathbb{B}|}(S^\circ).
\]

It is \( \bar{\phi}^*(H_s) = H_{\phi^{-1}(s)} \) and so,

\[
f_{|\mathbb{B}|}(\tilde{C_{\phi^{-1}(s)}}) = \bar{\phi}^*(H_s) \cap f_{|\mathbb{B}|}(S^\circ) = \bar{\phi}^*(H_s) \cap \bar{\phi}^*(f_{|\mathbb{B}|}(S^\circ)) \quad \text{since} \quad \bar{\phi}^* f_{|\mathbb{B}|}(S^\circ) = f_{|\mathbb{B}|}(S^\circ) \quad \text{in} \quad \mathbb{P}(\mathbb{B}^\vee),
\]

\[
= \bar{\phi}^*(H_s \cap f_{|\mathbb{B}|}(S^\circ)) \quad \text{since} \quad \bar{\phi}^* \text{ is an automorphism},
\]

\[
= \bar{\phi}^* f_{|\mathbb{B}|}(\tilde{C_s}).
\]

\( \square \)

By the above Lemma 4.5, the automorphism \( \chi \) on \( S^\circ \) induces an automorphism \( \psi \) of \( \tilde{C} \) over \( B \) which makes the following diagram commutative

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\psi} & \tilde{C} \\
\downarrow{h} & & \downarrow{h} \\
B & \xrightarrow{\phi^{-1}} & B
\end{array}
\]

Since \( S^\circ \) is the total space of the line bundle \( \omega_C \) on \( C \), the map \( \chi \) acts by a dilation and a translation by a section of \( \omega_C \) on the fibers i.e. it has the form \( \chi = m_\lambda \circ T_s \) where \( \lambda \in \mathbb{C}^* \) and \( s \in H^0(C, \omega_C) \). We actually claim that \( s = 0 \) and that \( \phi^{-1} \) has the form \( \phi^{-1}(s_{r-1}, s_r) = (\lambda^{-1} s_{r-1}, \lambda^{-r} s_r) \).

Indeed,

(4.23) \( \chi \circ h^{-1}(s_{r-1}, s_r) = h^{-1}(\phi^{-1}(s_{r-1}, s_r)) \).

To prove the first claim, we apply (4.23) to \( (s_{r-1}, s_r) = (0, 0) \). Then, \( \chi \circ h^{-1}(0, 0) = h^{-1}(0, 0) \) since \( \phi^{-1} \) is a linear map. But \( h^{-1}(0, 0) \) is the curve \( x^r = 0 \), and so, \( \chi \circ h^{-1}(0, 0) \) is the curve \( (\lambda x + s)^r = 0 \), see beginning of Section 1.2. This implies that \( s = 0 \).
For the second claim: $h^{-1}(s_{r-1}, s_r)$ is the curve $x^r + s_{r-1}x + s_r = 0$ and so, $\chi \circ h^{-1}(s_{r-1}, s_r)$ is the curve $\lambda^r x^r + \lambda s_{r-1}x + s_r = 0$ i.e. the curve $h^{-1}(\lambda^{-(r-1)} s_{r-1}, \lambda^{-r} s_r)$ which completes the proof.

Corollary 4.1 $\phi(s_{r-1}, s_r) = (\lambda^{r-1} s_{r-1}, \lambda^r s_r)$.

We now proceed with our discussion about the Picard group of $\tilde{C}_h \times \phi h \tilde{C}$ in the case of $\dim \Gamma \cap (S^0 \times S^0) = 2$. We have that $\tilde{C}_h \times \phi h \tilde{C}|_{\Gamma} = \Delta_\lambda = \{(p, \psi^{-1}(p)), p \in \tilde{C}\}$. Note that if $\lambda = 1$, then $\Delta_\lambda$ is exactly the diagonal in the fiber product $\tilde{C}_h \times h \tilde{C}$. Combining the latter with Lemma 4.3 we get:

**Proposition 4.1** In case B, the Picard group of $\tilde{C}_h \times \phi h \tilde{C}$ is generated by $\pi'^* \text{Pic} (C \times C)$ and $\Delta_\lambda$.

**Remark 4.1** It is easy to see that $\text{Pic} (\tilde{C}_h \times \phi h \tilde{C}) \simeq \pi'^* \text{Pic} (C \times C) \oplus \mathbb{Z}[\Delta_\lambda]$.

5 Proof of the main Theorem

5.1 The form of $\tilde{\Phi}$

We start with a definition. Let $C$ and $C_1$ are two smooth curves. Given a line bundle $L$ on the product $C \times C_1$, then $L$ induces a map

$$\psi_L : \text{Pic} C \longrightarrow \text{Pic} C_1$$

defined by $\psi_L(\mathcal{O}(p)) = \alpha^*(L|_{C_1^p})$, where $C_1^p$ is the fiber in the product over the point $p \in C$ and $\alpha : C_1 \longrightarrow C_1^p$ the natural isomorphism. The extension of the definition to a point $[L] \in \text{Pic} C$ is given by taking a meromorphic section of the bundle $L$. In the same way, whenever we have a line bundle on a fiber product of two families of curves we get a map between their relative Picard groups.

Consider the diagram

\[
\begin{array}{ccc}
\tilde{C}_h \times \phi h \tilde{C} & \xrightarrow{h'} & B_{\text{reg}} \\
\pi_2 & & \pi_1 \\
\downarrow \pi' & & \\
C \times C & & C
\end{array}
\]

Let $L$ be a line bundle on $C \times C$. Then $\pi'^* L$ is a line bundle on the fiber product $\tilde{C}_h \times \phi h \tilde{C}$. Following the above notation, it is easy to see that
Lemma 5.1 \( \psi_{\pi^\prime * \mathcal{L}} = \pi_2^* \psi_{\mathcal{L}} \text{Nm}_1 \), where \( \text{Nm} \) is the norm map of \( \pi_1 \).

Note that on the level of fibers we have

\[
\psi_{\pi^\prime * \mathcal{L}} : \text{Pic} \tilde{C}_s \to \text{Pic} \tilde{C}_{\phi(s)} \quad \mathcal{O}(p) \mapsto \pi_{\phi(s)}^* \psi_{\mathcal{L}} \text{Nm}_s(O(p))
\]

where the subscripts refer to the restriction on the corresponding fiber over a point of \( B_{\text{reg}} \).

Remark 5.1 We recall the following fact about maps of abelian torsors and induced maps of abelian schemes. Let \( p : \mathcal{G} \to Z \) be an abelian scheme and \( \pi : T \to Z \) a \( \mathcal{G} \)-torsor. Given two elements \( t_1 \) and \( t_2 \) in the same fiber of \( T \) over \( z \in Z \), we denote by \( t_1 - t_2 \) the unique element \( g \in \mathcal{G} \) over \( z \in Z \), with the property: \( T_g t_2 = t_1 \). Let now \( \phi_T : T \to T \) be an automorphism of the abelian torsor \( T \) i.e. an automorphism that sends a fiber of the map \( \pi \) to another fiber and preserves the action of \( \mathcal{G} \). To that, one can associate a group automorphism \( \phi_{\mathcal{G}} \) of the abelian scheme \( \mathcal{G} \) as follows. Given \( g \in \mathcal{G} \) over \( z \in Z \), choose an element \( t \in T_z \) and define

\[
\phi_{\mathcal{G}}(g_z) = \phi_T(T_g t_z) - \phi_T(t_z).
\]

It is easy to check that this is independent from the choice of \( t \) in the fiber \( T_z \) and that it defines a group automorphism. Note that given \( t_1 \) and \( t_2 \) two elements in the same fiber \( T_z \), then

\[
\phi_T(t_1) - \phi_T(t_2) = \phi_{\mathcal{G}}(t_1 - t_2).
\]

Following the notation of Proposition \[2.3\], let \( \tilde{\Phi} : \text{Prym}(\tilde{C}, C) \to \text{Prym}(\tilde{C}, C) \) be the group automorphism associated to the automorphism \( \tilde{\Phi}_d : \text{Prym}_d(\tilde{C}, C) \to \text{Prym}_d(\tilde{C}, C) \) of \( \text{Prym}(\tilde{C}, C) \)-torsors. We determine now the form of the map \( \tilde{\Phi} \). Consider the map

\[
\mu : \tilde{C}_h \times_{\phi_{\tilde{h}}} \tilde{C} \to \text{Prym}(\tilde{C}, C)_{H \times_h \tilde{C}} \\
(p_s, q_{\phi(s)}) \mapsto (\tilde{\Phi}(rp_s - \pi_s^* \text{Nm}_s(p_s)), q_{\phi(s)})
\]

where the notation for a point on a curve, stands also for the line bundle which the point defines. According to Lemma \[3.1\], on the product \( \text{Prym}(\tilde{C}, C)_{H \times_h \tilde{C}} \) we have a line bundle \( \mathcal{P} \) with the property \( \mathcal{P}|_{[L] \times \tilde{C}_{\phi(s)}} \simeq nL \) for some integer \( n \). Hence,

\[
\mu^* \mathcal{P}|_{[p_s] \times \tilde{C}_{\phi(s)}} \simeq n\tilde{\Phi}(rp_s - \pi_s^* \text{Nm}_s(p_s)). \tag{5.24}
\]

By using \[5.24\] and the knowledge of the Picard group of \( \tilde{C}_h \times_{\phi_{\tilde{h}}} \tilde{C} \) we will derive the form of the map \( \tilde{\Phi} \).
We now conclude the proof of the main Theorem A in the case of $r = 3$. By examining the two cases $\Phi = \pm \psi^*$, we start with some notation. We denote by $\psi^*_d : \text{Prym}_d(\tilde{\mathcal{C}}_h, \tilde{\mathcal{C}})$ the pull back of the map $\psi : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$. Note that $\psi^* = \psi^*_0$. We have the diagram:

$$
\begin{array}{ccc}
\text{Prym}_d(\tilde{\mathcal{C}}, C) & \xrightarrow{\psi^*_d} & \text{Prym}_d(\tilde{\mathcal{C}}, C) \\
\downarrow H & & \downarrow H \\
B_{\text{reg}} & \xrightarrow{\phi} & B_{\text{reg}}
\end{array}
$$

**Case A:** $\dim \Gamma \cap (S^0 \times S^0) \leq 2$. Then, by Lemma 4.4, we have that $\text{Pic}(\tilde{\mathcal{C}} \times \phi \tilde{\mathcal{C}}) \simeq \pi^* \text{Pic}(C \times C)$. We thus get

$$
\mu^* \mathcal{P} \simeq \pi^* \mathcal{L} \quad \text{and so,} \quad \mu^* \mathcal{P}|_{[p_s] \times \tilde{\mathcal{C}}_{\phi(s)}} \simeq \pi^* \mathcal{L}|_{[p_s] \times \tilde{\mathcal{C}}_{\phi(s)}},
$$

i.e.

$$
(5.25) \quad \tilde{\Phi}(nr p_s - n \pi_s^* \text{Nm}_s(p_s)) \simeq \pi_s^* \psi_\mathcal{L} \text{Nm}_s(p_s).
$$

Take the map $\tilde{\Phi}_s : \text{Prym}(\tilde{\mathcal{C}}_s, C) \to \text{Prym}(\tilde{\mathcal{C}}_{\phi(s)}, C)$ Now given a line bundle $\tilde{L}_s \in \text{Prym}(\tilde{\mathcal{C}}_s, C)$, choose a line bundle $\tilde{M}_s$ in $\text{Prym}(\tilde{\mathcal{C}}_s, C)$ such that $\tilde{L}_s = nr \tilde{M}_s$. By choosing a meromorphic section, we can write $\tilde{M}_s = O(\sum_i(p_i^1 - p_i^2))$ where $\text{Nm}(\sum_i(p_i^1 - p_i^2)) = 0$. We have

$$
\tilde{\Phi}_s(\mathcal{L}_s) = \tilde{\Phi}_s(nr \mathcal{M}_s) = \tilde{\Phi}_s(\sum_i nr(p_i^1 - p_i^2)) = \sum_i \left( \tilde{\Phi}_s(nr p_i^1 - n \pi_s^* \text{Nm}_s(p_i^1)) - \tilde{\Phi}_s(nr p_i^2 - n \pi_s^* \text{Nm}_s(p_i^2)) \right)
$$

$$
= \sum_i \pi_{\phi(s)}^* \psi_\mathcal{L} \text{Nm}_s(p_i^1) - \sum_i \pi_{\phi(s)}^* \psi_\mathcal{L} \text{Nm}_s(p_i^2)
$$

$$
= \sum_i \pi_{\phi(s)}^* \psi_\mathcal{L} \text{Nm}_s(p_i^1) - \sum_i \pi_{\phi(s)}^* \psi_\mathcal{L} \text{Nm}_s(p_i^2) \quad \text{by (5.25),}
$$

$$
= \sum_i \pi_{\phi(s)}^* \psi_\mathcal{L} \text{Nm}_s(p_i^1 - p_i^2) = 0 \quad \text{since} \quad \text{Nm}(\sum_i(p_i^1 - p_i^2)) = 0.
$$

The later contradicts the fact that $\tilde{\Phi}$ is an isomorphism, which means that case A cannot occur. Therefore $\dim \Gamma \cap (S^0 \times S^0) = 2$, i.e. we are in case B which we examine below:

**Case B:** $\dim \Gamma \cap (S^0 \times S^0) = 2$. By the discussion in Section 4.2, the $\text{Pic}(\tilde{\mathcal{C}}_h \times \phi \tilde{\mathcal{C}})$ is generated by $\pi^* \text{Pic}(C \times C)$ and the divisor $\Delta_\Lambda$. Hence,

$$
\mu^* \mathcal{P} \simeq \pi^* \mathcal{L} + n \Delta_\Lambda.
$$

Working as before and using the definition of $\Delta_\Lambda$, we conclude that $\tilde{\Phi}(\mathcal{L}) = n \psi^*(\mathcal{L})$. Since $\tilde{\Phi}$ and $\psi^*$ are isomorphisms, we get that $n = \pm 1$. To summarize,

**Proposition 5.1** $\tilde{\Phi} = \pm \psi^*$, where $\psi$ is the map defined in Section 4.2.

### 5.2 The conclusion of the proof

We now conclude the proof of the main Theorem A in the case of $r \geq 3$ by examining the two cases $\Phi = \pm \psi^*$. We start with some notation. We denote by $\psi^*_d : \text{Prym}_d(\tilde{\mathcal{C}}_h, \tilde{\mathcal{C}}) \to \text{Prym}_d(\tilde{\mathcal{C}}_h, \tilde{\mathcal{C}})$ the pull back of the map $\psi : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$. Note that $\psi^* = \psi^*_0$. We have the diagram:
Lemma 5.2 Let $x$ denote an element in $\text{Prym}_d(\tilde{C}, C)$. We have that

1. If $\tilde{\Phi} = \psi^*$, then $\psi_d^{-1}\tilde{\Phi}_d(x) - x$ is independent from $x$ on the fibers of the map $H$.

2. If $\tilde{\Phi} = -\psi^*$, then $\psi_d^{-1}\tilde{\Phi}_d(x) + x$ is independent from $x$ on the fibers of the map $H$.

Proof. For the first: Let $y$ be a point in the fiber of $H$ through $x$. Since $\psi^*\tilde{\Phi} = 1$, it is enough to show that $\psi_d^{-1}\tilde{\Phi}_d(x) - \psi_d^{-1}\tilde{\Phi}_d(y) = \psi^*\tilde{\Phi}(x - y)$. Since $\psi^*\tilde{\Phi}$ is the group homomorphism associated to the map $\psi_d^{-1}\tilde{\Phi}_d$ of abelian torsors, the later is true by Remark 5.1. For the second: It is enough to show that $\psi_d^{-1}\tilde{\Phi}_d(x) - \psi_d^{-1}\tilde{\Phi}_d(y) = y - x$. But since $\tilde{\Phi} = -\psi^*$ we have $\psi_d^{-1}\tilde{\Phi}(x - y) = y - x$ and this case follows as well.

Proposition 5.2 Let $-1$ denote the inversion along the fibers of $\text{Prym}(\tilde{C}, C)$. Then

1. If $\tilde{\Phi} = \psi^*$, then $\tilde{\Phi}_d = T_{\pi^*\mu} \circ \psi_d^*$, where $\mu$ is an $r$-torsion line bundle on the base curve $C$.

2. If $\tilde{\Phi} = -\psi^*$, then $r|2\bar{d}$ and $\tilde{\Phi}_d = T_{\pi^*\nu_1} \circ (\psi_d^*) \circ (-1)$, where $\nu$ is a line bundle on the base curve $C$ which satisfies $\nu_1^{\otimes r} = L_0^{\otimes 2} \otimes \omega_C^{\otimes r(r-1)}$.

Proof. For the first: According to Lemma 5.2, on each fiber $H^{-1}(s) = \text{Prym}_d(\tilde{C}_s, C)$ of the map $H$, the maps $\tilde{\Phi}_d$ and $\psi_d^*$ differ by a translation by a unique element $a(s) \in \text{Prym}(\tilde{C}_s, C)$. This defines a section of the map $H : J^0(\tilde{C}) \rightarrow B^{\text{reg}}$ and therefore by Proposition 3.3 it must have the form $\pi^*M$ for some fixed line bundle $\mu$ on $C$. Since the image of the section is in the Prym(\tilde{C}, C) we get that $\mu$ is an $r$-torsion line bundle.

For the second: According to Lemma 5.2, we can construct a section of the map $H : J^{2\bar{d}}(\tilde{C}) \rightarrow B^{\text{reg}}$ by assigning to the point $s \in B^{\text{reg}}$ the line bundle $\psi_d^{-1}\tilde{\Phi}_d(x) + x$ for some point $x \in \text{Prym}_d(\tilde{C}_s, C)$. By Proposition 3.3 we get that $r|2\bar{d}$. Since $\bar{d} = d + r(r-1)(g-1)$ this is equivalent to $r|2\bar{d}$. The same proposition implies that the above section must have the form $\pi^*\nu_1$ for some fixed line bundle $\nu_1$ on $C$. To complete the proof of the second part of the proposition, observe that the map $\psi^*$ commutes with the translations by an element of the form $\pi^*\nu_1$ and that $\text{Nm} \pi^*\nu_1 = L_0^{\otimes 2} \otimes \omega_C^{\otimes r(r-1)}$.

We are ready now to complete the proof of the Theorem A in the case $r \geq 3$. Pick an element element $s \in B^{\text{reg}}$. Then, by Corollary 4.7, the prymian $\text{Prym}_d(\tilde{C}_s, C)$ maps dominantly to
$SU(r, L_0)$. Let $\mathcal{V}$ be its image. It is enough to prove the theorem for the restriction of the map $\Phi$ on $\mathcal{V}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Prym}_d(\tilde{C}_s, C) & \xrightarrow{\tilde{\Phi}_d} & \text{Prym}_d(\tilde{C}_{\phi(s)}, C) \\
\pi_{ss} & \downarrow & \pi_{\phi(s)*} \\
\mathcal{V} & \xrightarrow{\Phi} & \mathcal{V}
\end{array}
$$

where the maps $\pi_{ss}$ and $\pi_{\phi(s)*}$ are rational maps.

Assume first that $\tilde{\Phi} = \psi^*$. Let $E$ a vector bundle in $\mathcal{V}$. Then, there exists a line bundle $\tilde{L}_s$ in $\text{Prym}_d(\tilde{C}_s, C)$ such that $\pi_{ss}(\tilde{L}_s) = E$. By Proposition 5.3, the above diagram and the fact that the map $\psi$ commutes with the projections $\pi_{ss}$ and $\pi_{\phi(s)*}$ to the base curve $C$, we conclude that

$$
\Phi(E) = \Phi \pi_{ss}(\tilde{L}_s) = \pi_{\phi(s)*}\tilde{\Phi}_d(\tilde{L}_s) = \pi_{\phi(s)*}(\psi^*_d(\tilde{L}_s) \otimes \pi_{\phi(s)*}\mu) = \pi_{\phi(s)*}\psi^*_d(\tilde{L}_s) \otimes \mu = E \otimes \mu.
$$

Assume next that $\tilde{\Phi} = -\psi^*$. Then, following the above notation, we have

$$
\Phi(E) = \Phi \pi_{ss}(\tilde{L}_s) = \pi_{\phi(s)*}\tilde{\Phi}_d(\tilde{L}_s) = \pi_{\phi(s)*}(\psi^*_d(\tilde{L}_s^{-1}) \otimes \pi_{\phi(s)*}\nu_1) = \pi_{\phi(s)*}\psi^*_d(\tilde{L}_s^{-1}) \otimes \nu_1 = \pi_{ss}(\tilde{L}_s^{-1}) \otimes \nu_1.
$$

We claim that $\pi_{ss}(\tilde{L}_s^{-1}) = E^\vee \otimes \omega_C^{-(r-1)}$. Indeed, consider the map $\pi_s : \tilde{C}_s \rightarrow C$. By relative duality we have

$$
\mathcal{R}^0 \pi_{ss}(\tilde{L}_s^{-1}) \simeq \mathcal{R}^0 \pi_{ss}(\omega_{\pi_s} \otimes \tilde{L}_s)^\vee.
$$

By the adjunction formula we have, see e.g. [Hi 1], that $\omega_{\pi_s} \simeq \pi_s^* \omega_C^{-(r-1)}$. We thus get

$$
\pi_{ss}(\tilde{L}_s^{-1}) \simeq E^\vee \otimes \omega_C^{-(r-1)}.
$$

By choosing $\nu = \nu_1 \otimes \omega_C^{-(r-1)}$, we get that $\phi(E) = E^\vee \otimes \nu$ where $\nu^{\otimes r} = L_0^{\otimes 2}$.

Thus, we have shown the surjectivity of the maps (1.) and (2.) in the statement of Theorem A. To show the injectivity of the map (1.), pick up a point $\mu \neq 0 \in J^0[r]$ and consider the set of fixed points $SU(r, L_0)^{(T_\mu)}$. A vector bundle $E$ is fixed under the action of $T_\mu$, if we have an isomorphism

$$
\kappa : E \rightarrow E \otimes \mu.
$$

If $p \mid r$ is the order of the torsion point $\mu$, then the $\mu$-twisted Higgs bundle $(E, \kappa)$ gives a $p$-sheeted unramified spectral cover $\pi_\mu : C_\mu \rightarrow C$ and a semistable vector bundle $F_\kappa$ of rank $r/p$ on it with the property $\pi_{\mu*}(F_\kappa) = E$, see [N-R 1] for details. Therefore the locus $SU(r, L_0)^{(T_\mu)}$ can be identified with the image of the moduli space $SU_C(r/p)$ under the pushforward map $\pi_{\mu*}$ and hence is a proper subvariety.
We will sketch the proof for the injectivity of the map (2.) in the case $r \geq 3$ and when $L_0 = \mathcal{O}$. The modifications of the argument for general $L_0$ are minor and are left to the reader. If $L_0 = \mathcal{O}$, then it suffices to check that the map $E \to E^\vee$ is not the identity on $SU(r, \mathcal{O})$. But if $E$ is a stable vector bundle satisfying $E \cong E^\vee$, then the bundle $E^{\otimes 2}$ has a unique (up to scaling) non-zero section $t$. But $H^0(C, E^{\otimes 2}) = H^0(C, \text{Sym}^2 E) \oplus H^0(C, \wedge^2 E)$ and therefore either $H^0(C, E^{\otimes 2}) = 0$ or $H^0(C, \wedge^2 E) = 0$. This implies that the isomorphism $t$ between $E$ and $E^\vee$ is either symmetric or skew-symmetric. Thus, $E$ is either orthogonal or symplectic and hence it lies either in the moduli space of stable $SO(r)$-bundles or in the moduli space of stable $Sp(r)$-bundles. But for $r \geq 3$, those are proper subvarieties of $SU(r, \mathcal{O})$ and this yields that the map (2.) is injective for $r \geq 3$.

**Remark 5.2** For $r = 2$, the moduli space $SU(2, \mathcal{O})$ coincides with the moduli space of $Sp(2)$-bundles; if $E \in SU(2, \mathcal{O})$, then $E \cong E^\vee$.

### 5.3 The rank 2 case

The proof in the rank 2 case is a modification of the proof of the rank $\geq 3$ case. The main difference is that the linear system defined by $B = W_2 = H^0(C, \omega_C^2)$ does not separate the points on the surface $S^\circ$. A section $s \in B$ corresponds to the curve $x^2 + s = 0$ on $S^\circ$. If $m_{-1}$ is the dilation by $-1$ on $S^\circ$, then a curve in the linear system $|B|$ that passes through a point $p$, passes also through the point $m_{-1}(p)$. Therefore the map $f_{|B|}$ sends $S^\circ$ in a $2 : 1$ way to the projective space. We now show briefly the adjustments for the rank 2 case of the argument we used in the rank $\geq 3$ case.

At first one can see e.g. by Corollary 2.1, that the map $\phi = \phi_2$ is a multiplication by a $\lambda \in \mathbb{C}^*$. Following a similar argument with that of Section 3.1, we can prove that the Picard group of $\tilde{C} \to B^{\text{reg}}$ is again

$$\text{Pic } \tilde{C} = \pi^* \text{Pic } C.$$ 

We also have

$$(5.26) \quad \text{Pic } \tilde{C}_p = \text{Pic } \pi^* C \oplus \mathbb{Z}[H_p]$$

but in this case the technicalities of the proof are slightly different: Let $p' = m_{-1}(p)$. On the fiber of $S^\circ$ over $c = \alpha(p) = \alpha(p')$, only the points $p$ and $p'$ belong in the image of the map $\beta$. We write $\tilde{C}_p = \tilde{C}_p \setminus (H_p \cup H_{p'}) \cup (H_p \cup H_{p'})$. Then Pic $\tilde{C}_p$ is generated by Pic $(C \setminus \{c\})$ and $H_p, H_{p'}$. Observe that $\pi^*(c) = H_p + H_{p'}$. We thus have that Pic $\tilde{C}_p$ is generated by Pic $\pi^* C$ and $H_p$. To prove that this is a direct sum, we again take a pencil as in the proof of Proposition 3.2, but now the curve at infinity embedded on $S$, consists of the $Y_\infty$ and a bunch of $N_1$ fibers over the points $c_1, \ldots, c_{N_1}$ on $C$, which pass through the $N = 2N_1$ base points of the pencil. Therefore the fiber over the
infinity point \( b \) on \( X \) consists of the infinity divisor \( \tilde{Y}_\infty \) and a bunch of \( N_1 \) divisors \( A_1, \ldots, A_{N_1} \) which satisfy \( \pi^* (c_i) = A_i + E_i + E'_i \). Using that we get

\[
\text{Pic}\ X^\circ = \pi^* \text{Pic} \ C \oplus_{i=1}^{N_1} (\oplus \mathbb{Z}[E_i]) \oplus_{i=1}^{N_1} (\oplus \mathbb{Z}[E'_i]) \big/ (\pi^* (c_i) = E_i + E'_i)_{i=1, \ldots, N_1}.
\]

Working as in the rank \( \geq 3 \) case, this proves relation (5.26). To prove the analogue of Proposition 3.3, we proceed in the same way as in the rank \( \geq 3 \) case.

5.4 Curves with automorphisms

For the rank \( r \geq 3 \) case, the only modification that has to be done, is in the argument of Section 4.2, about the induced automorphism of the embedded surface \( f_{[B]} \). For the rank 2 case, the only modification is in the use of Corollary 2.1. We leave to the reader to fill up the details for the proof of Theorem B.

6 The automorphisms of \( U(r, d) \)

**Proposition 6.1** Let \( \Phi \) be an automorphism of \( U(r, d) \). Then \( \Phi \) factors through the determinant map i.e. we have the following commutative diagram

\[
\begin{array}{ccc}
U(r, d) & \xrightarrow{\Phi} & U(r, d) \\
\downarrow \text{det} & & \downarrow \text{det} \\
J^d(C) & \xrightarrow{\phi_d} & J^d(C)
\end{array}
\]

**Proof.** We would like to show that the map \( \det \circ \phi : \det^{-1}(L_0) \rightarrow J^d(C) \) is constant for all \( L_0 \in J^d(C) \). By definition, \( \det^{-1}(L_0) = SU(r, L_0) \) and \( \dim SU(r, L_0) = (r^2 - 1)(g - 1) > g, \quad g \geq 2 \). Therefore the map \( \det \circ \phi : SU(r, L_0) \rightarrow J^d(C) \) has positive dimensional fibers and so, no pull back of a line bundle from \( J^d(C) \) can be ample. According to [D-N], \( \text{Pic} SU(r, L_0) = \mathbb{Z}[\Theta] \). Since \( SU(r, L_0) \) is a projective variety, \( \Theta \) is an ample divisor. Let \( \mathcal{L} \) be a line bundle on \( J^d(C) \). We thus have \( (\det \circ \phi)^* \mathcal{L} \simeq n\Theta \) for some integer \( n \). We claim that \( n = 0 \): If not, then say first that \( n > 0 \). Then \( n\Theta \) is ample and so \( (\det \circ \phi)^* \mathcal{L} \) is ample, which is a contradiction. Next, if \( n < 0 \), then the
same argument applied to the line bundle $L^{-1}$ leads to a contradiction. Therefore $(\det \circ \phi)^* L = O$ for all $L \in \text{Pic} J^d(C)$. Since $SU(r, L_0)$ is irreducible, this implies that the map is constant. Indeed, if the image has positive dimension, then there exists a positive divisor on that - the hyperplane section of its embedding in the projective space. But then the pull back of that divisor on $SU(r, L_0)$ by the map $\det \circ \phi$ defines a non-trivial line bundle, which is a contradiction.

\[ \square \]

As in the case of the automorphisms of $SU(r, L_0)$, an automorphism $\Phi$ of $U(r,d)$ induces an automorphism $\tilde{\Phi}_d$ of the Jacobian fibration $H : J^d(\tilde{C}) \to W^{\text{reg}}$ which sends a fiber of the Hitchin map $H$ to a fiber of $H$. Let $N_1 : J^d(\tilde{C}) \to J^d(C)$ be the map defined by $N_1 = \det \circ \pi_*$. Note that $Nm = T_{r(r-1)/2} \omega_C \circ N_1$.

**Lemma 6.1** Let $s$ be a point in $W^{\text{reg}}$. Then the following diagram commutes

\begin{equation}
\begin{array}{ccc}
J^d(\tilde{C}_s) & \xrightarrow{\tilde{\Phi}_d} & J^d(\tilde{C}_{\phi_W(s)}) \\
N_1 \downarrow & & \downarrow N_1 \\
J^d(C) & \xrightarrow{\phi_d} & J^d(C)
\end{array}
\end{equation}

**Proof.** For the proof we are going to use the following commutative diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{X}(r,d) & \xrightarrow{\Phi^*} & \mathcal{X}(r,d) \\
\pi_* \downarrow & & \pi_* \downarrow \\
\mathcal{U}(r,d) & \xrightarrow{\Phi} & \mathcal{U}(r,d) \\
\downarrow \det & & \downarrow \det \\
J^d(C) & \xrightarrow{\phi_d} & J^d(C)
\end{array}
\end{equation}

(6.28)

It is enough to show that $N_1 \tilde{\Phi}_d = \phi_d N_1$ on a Zariski open $U$ in $J^d(\tilde{C}_s)$. Choose $U$ to be the intersection of the cotangent bundle $\mathcal{X}(r,d)$ to $\mathcal{U}(r,d)$ with the Jacobian $J^d(\tilde{C}_s)$. According to Corollary [L.4], this is a non empty Zariski open in $\mathcal{U}(r,d)$. The proof of the Lemma is now a consequence of the commutativity of the above diagram.
Let $\tilde{\Phi}$ and $\phi$ be the group maps associated to $\tilde{\Phi}_d$ and $\phi_d$, see Remark 5.1. By using the above Lemma 6.1, it is easy to see that

**Corollary 6.1** The following diagram is commutative

\[
\begin{array}{ccc}
J^0(\tilde{C}_s) & \xrightarrow{\tilde{\Phi}} & J^0(\tilde{C}_{\phi_W(s)}) \\
\downarrow \text{Nm} & & \downarrow \text{Nm} \\
J^0(C) & \xrightarrow{\phi} & J^0(C)
\end{array}
\]

(6.29)

where Nm is the norm map.

**Lemma 6.2** Following the notation of Corollary 6.1, the diagram below is commutative

\[
\begin{array}{ccc}
J^0(\tilde{C}_s) & \xrightarrow{\tilde{\Phi}} & J^0(\tilde{C}_{\phi_W(s)}) \\
\downarrow \pi^* & & \downarrow \pi^* \\
J^0(C) & \xrightarrow{\phi} & J^0(C)
\end{array}
\]

**Proof.** Note first that

\[
\tilde{\Phi} \pi^*(L) = \pi^*(M)
\]

(6.30)

for some fixed line bundle $M$.

Indeed, $\tilde{\Phi}$ is a global automorphism on $J^0(\tilde{C})$ and so, $\tilde{\Phi} \pi^*(L)$ defines a section of the map $H : J^0(\tilde{C}) \rightarrow W_{\text{reg}}$. Hence, by Proposition 3.3, it must have the form $\pi^*(M)$ for some fixed line bundle $M$ on $C$. By applying the norm map to (6.30), we get $\text{Nm}\tilde{\Phi} \pi^*(L) = \text{Nm} \pi^*(M)$. Corollary 6.1 implies that $r\phi(L) = rM$. Let $n_r$ denote the multiplication by $r$. By composing both sides of (6.30) by $n_r$ and by using the last relation we get that $n_r \tilde{\Phi} \pi^*(L) = n_r \pi^* \phi(L)$. Since the map $n_r$ is onto, this completes the proof.

**Lemma 6.3** For any $E \in \mathcal{U}(r, d)$ and $M \in J^0(C)$ we have $\tilde{\Phi}(E \otimes M) = \Phi(E) \otimes \phi(M)$. 

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Proof. It suffices to prove the relation for all $E$ in a Zariski open $V$ of $U(r,d)$. Take $\tilde{C}$ a smooth spectral curve and let $V$ be the image of $\mathcal{X}(r,d) \cap J^d(\tilde{C})$ on $U(r,d)$. According to Corollary 1.7, this is a Zariski open of $U(r,d)$. Then $E = \pi_*\tilde{L}$. We have

$$
\Phi(E \otimes M) = \Phi(\pi_*\tilde{L} \otimes M) = \Phi(\pi_*(\tilde{L} \otimes \pi^*M)) = \pi_*\tilde{\Phi}(\tilde{L} \otimes \pi^*M) \ 	ext{by diagram (6.28),}
$$

$$
= \pi_*\tilde{\Phi}(\tilde{L} \otimes \tilde{\phi}(L)) \ 	ext{by the definition of the group map } \tilde{\phi},
$$

$$
= \pi_*\tilde{\Phi}(\tilde{L} \otimes \pi^*\tilde{\phi}(M)) \ 	ext{by Lemma 6.2},
$$

$$
= \pi_*\tilde{\phi}(L) \otimes \phi(M) \ 	ext{by the projection formula},
$$

$$
= \Phi(E) \otimes \phi(M).
$$

\[\square\]

Proof of Theorem 3. Let $\Phi : U(r,d) \rightarrow U(r,d)$ be a given automorphism and let $\phi_d$ the induced automorphism on $J^d(C)$ as in Lemma 6.1. Given a vector bundle $E \in U(r,d)$, we can find a vector bundle $E_{L_0} \in SU(r,L_0)$ and a line bundle $\eta \in J^0(C)$ such that $E = E_{L_0} \otimes \eta$. Note that $\eta^{\otimes r} = \text{det}E \otimes L_0^{-1}$. The above decomposition is unique up to a choice of an $r$-torsion point. Let $\xi_1$ be a line bundle such that $\xi_1^{\otimes r} = \phi_d(L_0) \otimes L_0^{-1}$. Then, $T_{\xi_1} \circ \Phi$ induces an automorphism of $SU(r,L_0)$. By the main Theorem 3, we have two cases.

Case 1: $T_{\xi_1} \circ \Phi(E_{L_0}) = \sigma^*E_{L_0} \otimes \mu$, where $\sigma$ is an automorphism of the curve $C$ and $\mu$ a line bundle which satisfies $\mu^{\otimes r} = L_0 \otimes \sigma^*L_0^{-1}$. We choose now $\xi = \xi_1 \otimes \mu$ and so, $\xi^{\otimes r} = \phi_d(L_0) \otimes \sigma^*L_0^{-1}$. We have

$$
\Phi(E) = \Phi(E_{L_0} \otimes \eta) = \Phi(E_{L_0}) \otimes \phi(\eta) = T_{\xi_1}(T_{\xi_1} \Phi(E_{L_0})) \otimes \phi(\eta) = \sigma^*E_{L_0} \otimes \xi \otimes \phi(\eta) = \sigma^*(E \otimes \eta^{-1}) \otimes \xi \otimes \phi(\eta) = \sigma^*E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1}).
$$

Therefore,

$$
\Phi(E) = \sigma^*E \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta^{-1}) \ 	ext{where } \eta^{\otimes r} = \text{det}E \otimes L_0^{-1} \ 	ext{and } \xi^{\otimes r} = \phi_d(L_0) \otimes \sigma^*L_0^{-1}.
$$

Note that, due to Lemma 6.3, the map $\phi^{-1} \circ \sigma^*$ is the identity on the set of $r$-torsion points in $J^0(C)$.

Case 2: $T_{\xi_1} \circ \Phi(E_{L_0}) = \sigma^*E_{L_0} \otimes \nu$, where $\sigma$ is an automorphism of the curve $C$ and $\nu$ is a line bundle which satisfies $\nu^{\otimes r} = L_0 \otimes \sigma^*L_0$. In this case we choose $\xi = \xi_1 \otimes \nu$ and so, $\xi^{\otimes r} = \phi_d(L_0) \otimes \sigma^*L_0$.
We have
\[
\Phi(E) = \Phi(E_{L_0}) \otimes \phi(\eta) \quad \text{by Lemma 6.3},
\]
\[
= T_{\xi_1}(T_{\xi_1}^{-1}\Phi(E_{L_0})) \otimes \phi(\eta)
\]
\[
= \sigma^*E_{L_0}^\vee \otimes \xi \otimes \phi(\eta)
\]
\[
= \sigma^*(E_{L_0}^\vee \otimes \eta) \otimes \xi \otimes \phi(\eta) \quad \text{by the definition of } E_{L_0},
\]
\[
= \sigma^*E_{L}^\vee \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta).
\]
Therefore,
\[
\Phi(E) = \sigma^*E_{L}^\vee \otimes \xi \otimes \phi(\eta) \otimes \sigma^*(\eta) \quad \text{where } \eta^{\otimes r} = \det E \otimes L_0^{-1} \text{ and } \xi^{\otimes r} = \phi_d(L_0) \otimes \sigma^*L_0.
\]

Again, due to Lemma 6.3, the map \( \phi \circ \sigma^* \) has to be the identity map on the set of \( r \)-torsion points in \( J^0(C) \).

A Appendix: A proof of the Torelli Theorem for the moduli space of vector bundles

Using the technics of the paper we sketch in the following the proof of Theorem 2 - the Torelli Theorem for vector bundles.

Remark A.1 In the case \((r, d) = 1\) the theorem has been proven in [M-N], [N-R 2] and [Ty]. The way they prove it is by showing that the intermediate Jacobian of the moduli space is canonically isomorphic to the principally polarized Jacobian of the curve; the theorem then follows from the usual Torelli for Jacobian of curves. In the case \((r, d) \neq 1\), the moduli space of vector bundles is singular and the construction of the intermediate Jacobian does not apply. In the special case \( r = 2 \) and trivial determinant, Balaji has proven a similar result for a desingularization of the space constructed by Seshadri, see [Ba]. Here we prove the theorem for any \( r, d \).

Proof of Theorem 2. The notation we use is in analogy with the one used in the paper. The subscripts 1, 2 in the notation refer to the curves \( C_1, C_2 \) respectively. Let \( \Phi : SU_{C_1}(r, L_1) \to SU_{C_2}(r, L_2) \) be the given isomorphism. After lifting to the cotangent bundle we obtain in a similar way as in the paper the following commutative diagram:

\[
\begin{array}{ccc}
Prym(\tilde{C}_1, C_1) & \xrightarrow{\tilde{\Phi}} & Prym(\tilde{C}_2, C_2) \\
H_1 \downarrow & & \downarrow H_2 \\
B_{1}^{\text{reg}} & \xrightarrow{\phi} & B_{2}^{\text{reg}}
\end{array}
\]
where $\Phi$ is the induced isomorphism, $H_1$, $H_2$ are the Hitchin maps and the map $\phi$ is a linear isomorphism. Consider the following diagram

$$
\begin{array}{ccc}
S_2 & \xrightarrow{i_2} & \mathbb{P}(\overline{B}_2') \\
\downarrow{\Phi^*} & & \downarrow{\Phi^*} \\
S_1 & \xrightarrow{i_1} & \mathbb{P}(\overline{B}_1')
\end{array}
$$

where the maps $i_1, i_2$ are those defined by the linear systems $\overline{B}_1, \overline{B}_2$ respectively and the isomorphism $\overline{\phi}^*$ is the one induced by the extension $\overline{\phi}$ of $\phi$. Using arguments similar to those in Sections 4.2 and 5.1, we deduce that the existence of an isomorphism $\Phi$ in the above diagram, implies that $\overline{\phi}^*$ has to map $i_2(S_2)$ isomorphically to $i_1(S_1)$. Thus the curve $C_2$ maps to $C_1$ and since both are curves of the same genus $g \geq 3$, the map is an isomorphism.

\[\square\]

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