Explicit description of twisted Wakimoto realizations of affine Lie algebras

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Abstract

In a vertex algebraic framework, we present an explicit description of the twisted Wakimoto realizations of the affine Lie algebras in correspondence with an arbitrary finite order automorphism and a compatible integral gradation of a complex simple Lie algebra. This yields generalized free field realizations of the twisted and untwisted affine Lie algebras in any gradation. The free field form of the twisted Sugawara formula and examples are also exhibited.

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1 Introduction

The Wakimoto type free field realizations [1] of the affine Lie algebras [2] proved very useful in many applications, for example in calculations of correlation functions in conformal field theories [3]. After Wakimoto’s study of the $sl_2$ case, the construction was extended to all the untwisted affine Lie algebras by Feigin and Frenkel [4]. Among the large number of further investigations we here mention only the papers [5, 6, 7, 8] dealing with applications to WZNW correlation functions, and the article [9] that contains explicit formulas for the affine currents in terms of the free fields in the general case, which will be used in the present work. Detailed reviews of the Wakimoto construction and its applications can be found in [10, 11, 3].

The aim of the present work is to generalize the explicit formulas of [9] to the twisted affine Lie algebras. The Wakimoto construction has been recently extended to the twisted case by Szczesny [12]. Properly explained in the framework of vertex algebras, the basic idea is to induce the twist of the affine Lie algebra by twisting the free fields that enter the untwisted currents. In this manner Wakimoto modules are constructed in [12] for the twisted affine Lie algebras using their standard realization by means of a diagram automorphism of a simple Lie algebra. The description of the twisted currents in [12] is not quite explicit, only the example $A_2^{(2)}$ is described in a more or less explicit manner. (The $sl_3$ case was considered in [13], too.)

By combining the basic idea of [12] with the explicit formulas of the untwisted Wakimoto currents in [9], we can easily derive explicit formulas that give the twisted affine currents as composites of the twisted free fields in the general case. In fact, we here perform the construction in correspondence with any finite order automorphism of a simple Lie algebra, and present also the realization of the Sugawara formula in terms of the twisted free fields, which was not considered in [12].

Let us recall that there is a natural correspondence between the $\mathbb{Z}$-gradations of the affine Lie algebras and the finite order automorphisms of the simple Lie algebras. Free field realizations compatible with a particular $\mathbb{Z}$-gradation of an affine Lie algebra should be useful, for example, to analyze WZNW orbifolds defined by the corresponding Lie algebra automorphism. They could also yield a convenient tool for the free field realization of the WZNW model under the boundary condition that the group-valued field belongs to a twisted loop group. The construction may be of interest in connection with inner automorphisms of the simple Lie algebras, too. This is illustrated for instance by the papers in [14], where a free field realization of $\hat{sl}_2$ compatible with the principal gradation is investigated. See also [15, 16, 17] for applications of affine Lie algebras twisted by inner automorphisms.

The above remarks motivated us to describe the twisted Wakimoto construction in correspondence with arbitrary finite order automorphisms of the simple Lie algebras. More precisely, in our ‘input data’ the automorphism is complemented by the choice of a compatible integral gradation of the simple Lie algebra. In the untwisted case generalized Wakimoto realizations associated with arbitrary integral gradations (or parabolic subalgebras) of the simple Lie algebras have been considered in [4, 9]. The most important case is that of the principal gradation (Borel subalgebra) studied also in [12].
The paper is organized as follows. In section 2 definitions and a useful lemma are presented. Section 3 contains a recall of the explicit formulas of the generalized ‘Wakimoto homomorphism’ from [9], and a simple proof of its equivariance with respect to the actions of the pertinent Lie algebra automorphism on the affine currents and on the free fields. Section 4 is devoted to the twisted Wakimoto construction. Our main result is the explicit formula of the ‘twisted Wakimoto homomorphism’ given by Proposition 3. We also discuss the realization of the Sugawara formula in terms of the twisted free fields (see eq. (4.30) for the result) and sketch a convenient way to describe the ‘input data’ that are needed in order to obtain examples. Examples are presented in two out of the three appendices to the main text. Appendix B contains a description of the simplest \( \mathfrak{sl}_2 \) case, which the reader may consult first to get a feeling about the twisted Wakimoto construction. In appendix C a generalized (non-principal) free field realization of \( D^{(2)}_3 \) is presented as an illustration.

In addition to deriving new results, our aim also is to provide a self-contained description of the twisted Wakimoto construction, which may be useful for future studies of its applications. Keeping this in mind, we summarized relevant background information on (twisted modules of) vertex algebras briefly in subsection 4.1 as well as in a technical appendix. The content of appendix A is well known to specialists in vertex algebras, but it is apparently less well known in the related physical literature dealing with twisted chiral algebras.

2 Definitions and conventions

Let \( \mathcal{G} \) be a complex simple Lie algebra with invariant scalar product \( \langle \ , \rangle \). We shall use the generalized Wakimoto realizations [4, 9] of the untwisted affine Lie algebra based on \( \mathcal{G} \) that can be associated with any \( \mathbb{Z} \)-gradation of \( \mathcal{G} \). Such a gradation is defined by the eigenvalues of \( \text{ad}_H \) for a diagonalizable element \( H \in \mathcal{G} \) according to

\[
\mathcal{G} = \bigoplus_m \mathcal{G}_m \quad [\mathcal{G}_m, \mathcal{G}_n] \subset \mathcal{G}_{m+n} \quad \text{with} \quad \mathcal{G}_m = \{ X \in \mathcal{G} \mid [H, X] = mX \}. \tag{2.1}
\]

Denoting the subspaces of positive/negative grades by \( \mathcal{G}_\pm \), this yields the decomposition

\[
\mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+. \tag{2.2}
\]

In the general case \( \mathcal{P} = (\mathcal{G}_0 + \mathcal{G}_+) \) is a parabolic subalgebra and \( \mathcal{G}_0 \) is a reductive Lie algebra. That is \( \mathcal{G}_0 \) decomposes into an Abelian factor, say \( \mathcal{G}_0^0 \), and simple factors, say \( \mathcal{G}_0^i \) for \( i > 0 \),

\[
\mathcal{G}_0 = \bigoplus_{i \geq 0} \mathcal{G}_0^i, \tag{2.3}
\]

where the factors are pairwise orthogonal with respect to \( \langle \ , \rangle \). In the ‘principal case’ \( \mathcal{G}_0 \) is a Cartan subalgebra and the parabolic subalgebra is a Borel subalgebra.

For the purposes of this paper we consider a pair \((\tau, H)\), where \( H \) is as above and \( \tau \) is an automorphism of \( \mathcal{G} \) that has finite order, denoted as \( N \). We assume that \( \tau \) and the \( \mathbb{Z} \)-gradation are compatible in the sense that \( \tau(H) = H \), which is clearly equivalent to

\[
\tau(\mathcal{G}_m) = \mathcal{G}_m \quad \forall m \in \mathbb{Z}. \tag{2.4}
\]
2.1 Convenient bases

In our construction we shall use joint eigenbases of \( (\tau, \text{ad}_H) \). We denote by \( \{T_a\} \) and \( \{T^a\} \) dual bases of \( \mathcal{G} \), \( \langle T_a, T^b \rangle = \delta^b_a \), such that the base elements are simultaneous eigenvectors of \( \text{ad}_H \) and \( \tau \) with respective eigenvalues designated as follows:

\[
[ H, T_a ] = h_a T_a, \quad [ H, T^a ] = h^a T^a, \quad \tau(T_a) = \omega_a T_a, \quad \tau(T^a) = \omega^a T^a. \tag{2.5}
\]

We have

\[
\omega^a = \exp \left( \frac{2\pi i}{N} n^a \right), \quad \omega_a = \exp \left( \frac{2\pi i}{N} n_a \right), \quad n^a, n_a \in \{0, 1, \ldots, N - 1\}, \tag{2.6}
\]

\[
h_a + h^a = 0, \quad \omega_a \omega^a = 1 \quad \forall a = 1, \ldots, \dim(\mathcal{G}). \tag{2.7}
\]

We shall also use dual bases \( \{L_\alpha\} \subset \mathcal{G}_- \) and \( \{U^\beta\} \subset \mathcal{G}_+ \) that are assumed to satisfy the relations \( \langle L_\alpha, U^\beta \rangle = \delta^\beta_\alpha \) and

\[
[ H, L_\alpha ] = h_\alpha L_\alpha, \quad [ H, U^\alpha ] = h^\alpha U^\alpha, \quad \tau(L_\alpha) = \omega_\alpha L_\alpha, \quad \tau(U^\alpha) = \omega^\alpha U^\alpha. \tag{2.8}
\]

These notations are consistent by assuming that \( T_a = L_\alpha \) for \( a = \alpha = 1, \ldots, \dim(\mathcal{G}_-) \).

We need two kinds of bases of \( \mathcal{G}_0 \). Firstly, \( \{D^i, \mu_i\} \) and its dual \( \{D^i_\mu\} \) stand for bases compatible with the decomposition \( \mathcal{G}_0 = \bigoplus i \mathcal{G}_0^i \), that is, \( D^i, \mu_i \) and \( D^i_\mu \) belong to \( \mathcal{G}_0^i \). These base elements are not necessarily eigenvectors of \( \tau \). Second, \( D^k \) and \( D_k^k \) \( (k = 1, \ldots, \dim(\mathcal{G}_0)) \) stand for dual bases of \( \mathcal{G}_0 \) consisting of \( \tau \)-eigenvectors. In this case the eigenvalues are denoted according to

\[
\tau(D^k) = \omega_0^k D^k, \quad \tau(D_k) = \omega_{0,k} D_k, \quad \omega_0^k = \exp \left( \frac{2\pi i}{N} n_0^k \right), \quad \omega_{0,k} = \exp \left( \frac{2\pi i}{N} n_{0,k} \right) \tag{2.9}
\]

with \( n_0^k, n_{0,k} \in \{0, 1, \ldots, N - 1\} \). It is not difficult to see [18] that the \( \tau \)-eigenvectors in \( \mathcal{G}_0 \) can be chosen in such a way that each \( D_k \) belongs either to \( \mathcal{G}_0^k \) or to a direct sum of a subset of the factors \( \mathcal{G}_0^i \) (2.3) formed by pairwise isomorphic factors.

2.2 Some polynomials

Consider the simply connected Lie group \( G_- \) whose Lie algebra is \( \mathcal{G}_- \). The general element \( g_- \in G_- \) can be parametrized as\(^2\)

\[
g_- = e^q \quad \text{with} \quad q = q^\alpha L_\alpha \in \mathcal{G}_-, \tag{2.10}
\]

which provides a diffeomorphism between \( G_- \) and \( \mathcal{G}_- \). The group \( G_- \) is naturally represented on the Lie algebra \( \mathcal{G} \), induced by restricting the adjoint representation of a group \( G \) associated with the Lie algebra \( \mathcal{G} \). Using a symbolic notation that pretends that \( G \) is a matrix group, this representation is given by the map \( g_- \mapsto \mathcal{R}(g_-) \in \text{End}(\mathcal{G}) \) for which

\[
\mathcal{R}(g_-) X := g_- X g_-^{-1} \quad \forall X \in \mathcal{G}. \tag{2.11}
\]

\(^2\)Summation over coinciding indices in opposite position is understood throughout the paper.
We now define the polynomials $T^a_b(q)$ in the variables $\{q^a\}$ by
\[ \mathcal{R}(e^{-q})T^a = T^a_b(q)T^b. \] (2.12)
Similarly, we set
\[ \mathcal{R}(e^{-q})U^a = U^a_b(q)T^b, \quad \mathcal{R}(e^{-q})L^a = L_{a,b}(q)T^b = L^a_{\beta}(q)L^b_{\beta}, \] (2.13)
and
\[ \mathcal{R}(e^{-q})D^{i,\mu} = D^{i,\mu}_a(q)T^a, \quad \mathcal{R}(e^{-q})D^k = D^k_a(q)T^a. \] (2.14)
We also need the polynomials $\Phi^\beta_\alpha(q)$ defined by
\[ \frac{\partial e^q}{\partial q^\alpha} e^{-q} = \Phi^\beta_\alpha(q)L^\beta, \] (2.15)
and the polynomials $\Psi^\beta_\alpha(q)$ that form the inverse matrix,
\[ \Phi^\beta_\alpha(q)\Psi^\beta_\alpha(q) = \delta^\beta_\alpha. \] (2.16)
Finally, we introduce the $\mathcal{G}$-valued polynomials $\Lambda^\alpha_{\alpha}(q)$ by
\[ \Lambda^\alpha_{\alpha}(q) = \frac{\partial \Psi^\lambda_\gamma(q)}{\partial q^\alpha} \Phi^\lambda_\gamma(q)\mathcal{R}(e^{-q})[U^\gamma, L^\rho]. \] (2.17)
It can be shown that $\Lambda^\alpha_{\alpha}$ is actually $\mathcal{G}_-$-valued, and hence we can write
\[ \Lambda^\alpha_{\alpha}(q) = \Lambda_{\alpha,b}(q)T^b = \Lambda^\beta_{\alpha}(q)L^\beta. \] (2.18)
We now establish certain homogeneity properties of the above polynomials. For this let us observe that if $\sigma$ is any automorphism of $\mathcal{G}$ and $f(z)$ is any complex power series, then $f(\text{ad}_q)$ ($q \in \mathcal{G}_-$) yields a polynomial, since $\text{ad}_q$ is nilpotent, and this polynomial satisfies
\[ \sigma^{-1} \circ f(\text{ad}_q) \circ \sigma = f(\text{ad}_{\sigma^{-1}(q)}). \] (2.19)
Consider the transformation
\[ q^a \mapsto \omega^a q^a \quad \text{or equivalently} \quad q = q^a L^a \mapsto \tau^{-1}(q). \] (2.20)

**Lemma 1.** The above-defined polynomials in the $\{q^a\}$ obey the following relations:
\[ \mathcal{U}^a_{b}(\tau^{-1}(q)) = \omega^a \omega_b \mathcal{U}^a_{b}(q), \quad \mathcal{L}_{a,b}(\tau^{-1}(q)) = \omega_a \omega_b \mathcal{L}_{a,b}(q), \quad \mathcal{D}^k_a(\tau^{-1}(q)) = \omega^k_0 \omega_a \mathcal{D}^k_a(q), \]
\[ \Phi^\alpha_\beta(\tau^{-1}(q)) = \omega^\alpha \omega_\beta \Phi^\alpha_\beta(q), \quad \Psi^\beta_\alpha(\tau^{-1}(q)) = \omega^\alpha \omega_\beta \Psi^\beta_\alpha(q), \quad \Lambda_{a,b}(\tau^{-1}(q)) = \omega_a \omega_b \Lambda_{a,b}(q). \]

**Proof.** The relations in the first line are special cases of
\[ T^a_{b}(\tau^{-1}(q)) = \langle \exp(-\text{ad}_{\tau^{-1}(q)})(T^a), T^b \rangle = \langle \exp(-\text{ad}_q)(\tau(T^a)), \tau(T^b) \rangle = \omega^a \omega_b T^a_{b}(q), \] (2.21)
where we used (2.19) and the $\tau$-invariance of $\langle \ , \ \rangle$. The other relations follow similarly since
\[
\Phi^\alpha_\beta(q) = \langle U^\alpha, f(\text{ad}q)L_\beta \rangle, \quad \Psi^\alpha_\beta(q) = \langle U^\alpha, f^{-1}(\text{ad}q)L_\beta \rangle \quad \text{with} \quad f(z) = \frac{e^z - 1}{z},
\]
where $f(z)$ and $f^{-1}(z) = \frac{1}{f(z)}$ are expanded into their Taylor series around $z = 0$. Q.E.D.

We shall see that Lemma 1 implies an important equivariance property of the Wakimoto realizations under the assumption in (2.4). For completeness, let us also present another homogeneity property, which is used in the construction of the Wakimoto realizations [9]. Consider now the transformation
\[
q^\alpha \mapsto \lambda^{h^\alpha} q^\alpha \quad \text{or equivalently} \quad q = q^\alpha L_\alpha \mapsto \sigma^{-1}_\lambda(q) \quad \text{with} \quad \sigma_\lambda = e^{(\ln \lambda)\text{ad}H}
\]
for any $\lambda \in \mathbb{C}^\times$. In the same way as Lemma 1, we obtain

**Lemma 2.** The above polynomials, defined by using any basis of $G$ consisting of eigenvectors of $\text{ad}_H$, are homogeneous in the sense that
\[
\mathcal{T}_b^\alpha(\sigma^{-1}_\lambda(q)) = \lambda^{h^\alpha + h^b} \mathcal{T}_b^\alpha(q), \quad \Phi^\alpha_\beta(\sigma^{-1}_\lambda(q)) = \lambda^{h^\alpha + h^\beta} \Phi^\alpha_\beta(q),
\]
\[
\Psi^\alpha_\beta(\sigma^{-1}_\lambda(q)) = \lambda^{h^\alpha + h^\beta} \Psi^\alpha_\beta(q), \quad \Lambda_{\alpha,\beta}(\sigma^{-1}_\lambda(q)) = \lambda^{h^\alpha + h^\beta} \Lambda_{\alpha,\beta}(q).
\]

Notice that $P(\{\lambda^{h^\alpha} q^\alpha\}) = \lambda^n P(\{q^\alpha\})$ with $n < 0$ holds only for the identically zero polynomial $P$, since $h^\alpha > 0$ for any $\alpha$. This implies for example that $\Lambda_{\alpha}(q)$ varies in $G_-$ as stated in (2.18).

## 3 Wakimoto homomorphism and its equivariance

In the first subsection we recall the explicit formula for the generalized Wakimoto realizations of the current algebras derived in [9]. The construction relies only on the data $(G, H)$, the automorphism $\tau$ will be used in the second subsection.

### 3.1 Recall of the generalized Wakimoto homomorphisms

In correspondence with a basis $T_a$ of $G$, let us consider the currents $J_a(z)$ that have the mode expansions
\[
J_a(z) = \sum_{n \in \mathbb{Z}} J_a[n] z^{-n-1}
\]
and are subject to the commutation relations
\[
[J_a(z), J_b(w)] = \langle [T_a, T_b], T_c \rangle J_c(w) \delta(z, w) + K \langle T_a, T_b \rangle \partial_w \delta(z, w).
\]
Here $K$ is a fixed (non-zero) complex number, $z$ and $w$ are formal variables and

$$\delta(z, w) = \sum_{n \in \mathbb{Z}} w^n z^{-n-1}. \quad (3.3)$$

This formula encodes the affine Lie algebra spanned by the modes $J_a[n]$, $[J_a[m], J_b[n]] = \langle [T_a, T_b], T_c \rangle J_c[m + n] + K \langle T_a, T_b \rangle m \delta_{m,-n}. \quad (3.4)$

By considering the vacuum Verma module of the affine Lie algebra in the usual manner, the fields $J_a(z)$ together with their derivatives and normal ordered products generate a vertex algebra (see e.g. [19, 20]), which we denote as $A(\mathcal{G}, K)$.

In association with the triangular decomposition in (2.2), the generalized Wakimoto realization of the current algebra relies on a vertex algebra whose generating fields can be labelled by an $H$-graded basis of $\mathcal{G}$ spanned by

$$L_{\alpha}, \quad U^{\alpha}, \quad D^i_{\mu} \quad (3.5)$$

as introduced above. We denote the corresponding fields as

$$p_\alpha(z), \quad q^{\alpha}(z), \quad j^i_{\mu}(z). \quad (3.6)$$

They are postulated to have the mode expansions

$$p_\alpha(z) = \sum_{n \in \mathbb{Z}} p_\alpha[n] z^{-n-1}, \quad q^{\alpha}(z) = \sum_{n \in \mathbb{Z}} q^{\alpha}[n] z^{-n-1}, \quad j^i_{\mu}(z) = \sum_{n \in \mathbb{Z}} j^i_{\mu}[n] z^{-n-1} \quad (3.7)$$

together with the commutators

$$[q^{\alpha}(z), p_\beta(w)] = \delta^{\alpha}_{\beta} \delta(z, w), \quad (3.8)$$

and

$$[j^i_{\mu}(z), j^j_{\nu}(w)] = \langle [D^i_{\mu}, D^j_{\nu}], D^{i,0}_{\nu} \rangle \delta(z, w) + K^i_0 \langle D^i_{\mu}, D^j_{\nu} \rangle \partial_w \delta(z, w). \quad (3.9)$$

The numbers $K^i_0$ are determined in terms of $K$ by

$$2K^0_0 = 2K + |\psi|^2 h^\vee = 2K^i_{0} + |\psi|^2 h^\vee_i \quad \text{for} \quad i > 0. \quad (3.10)$$

Here $h^\vee$ and $h^\vee_i$ are the dual Coxeter numbers of $\mathcal{G}$ and the simple factors $\mathcal{G}_i^0$ in (2.3), $\psi$ and $\psi_i$ stand for the respective highest roots, whose length is defined by means of $\langle \cdot, \cdot \rangle$ and its restriction to $\mathcal{G}_0^i$. It is understood that all other commutators between the basic fields vanish. The construction of the vertex algebra generated by these fields, which we denote by $W(\mathcal{G}, K, H)$ ($W$ for Wakimoto) is a well known matter [20].

Let us recall that the product of two commuting fields is well-defined in any vertex algebra, and in this case the product is associative. Therefore we can uniquely associate a field $P(z)$ to

\footnote{A conjugate pair $p_\alpha, q^{\alpha}$ subject to (3.8) is often called a $\beta \gamma$-system in the literature.}
any polynomial $P(q)$ in the variables $q^\alpha$, simply by replacing $q^\alpha$ by $q^\alpha(z)$ in the arguments of $P$ (this rule extends obviously to $\mathcal{G}$-valued polynomials as well). If the fields $\phi(z)$ and $\psi(z)$ do not commute, then their normal ordered product is the field $:\phi(z)\psi(z):$ given by

$$ :\phi(z)\psi(z) := \phi_+(z)\psi(z) + \psi(z)\phi_-(z), \quad (3.11) $$

where

$$ \phi(z) = \phi_-(z) + \phi_+(z), \quad \phi_-(z) = \sum_{n \in \mathbb{Z}_+} \phi[n]z^{-n-1} \quad \text{for} \quad \phi(z) = \sum_{n \in \mathbb{Z}} \phi[n]z^{-n-1} \quad (3.12) $$

with $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

After all this preparation, we can now define the ‘Wakimoto currents’ $J_a(z)$ to be the following fields in the vertex algebra $W(\mathcal{G}, K, H)$:

$$ J_a(z) = -:p_\beta(z) (\Psi_\beta^a(z)U_\alpha^a(z)) : + \sum_i j^i_\mu(z)D^{i,\mu}(z) + (K\Phi^\beta_\alpha(z)\mathcal{L}_{\beta,a}(z) + \Lambda_{\alpha,a}(z)) \partial_z q^\alpha(z). \quad (3.13) $$

**Proposition 1 [4, 9].** The currents $J_a(z)$ satisfy

$$ [J_a(z), J_b(w)] = ([T_a, T_b], T^c)\mathcal{J}_c(w)\delta(z, w) + K\langle T_a, T_b \rangle \partial_w \delta(z, w). \quad (3.14) $$

Thus one obtains a vertex algebra homomorphism $W_H : A(\mathcal{G}, K) \to W(\mathcal{G}, K, H)$ by mapping the generating fields $J_a(z)$ according to

$$ W_H : J_a(z) \mapsto J_a(z). \quad (3.15) $$

It is clear that (3.15) extends uniquely to a vertex algebra homomorphism since the currents $J_a(z)$ generate the vertex algebra $A(\mathcal{G}, K)$ in the sense of the reconstruction theorem (Theorem 4.5 in [19] or Theorem 3.6.1 in [20]). We call $W_H$ the (generalized) Wakimoto homomorphism’ associated with the parabolic structure specified by the grading element $H$. This homomorphism was originally established by Feigin and Frenkel [4] using indirect arguments. The explicit formula (3.13) found in [9] allows for a direct verification of (3.14).

It is useful to collect the current components in the ‘$\mathcal{G}$-valued currents’ $J(z) = J_a(z)T^a$, $J(z) = J_a(z)T^a$ and to also introduce the $\mathcal{G}_-$-valued, $\mathcal{G}_+$-valued and $\mathcal{G}_0$-valued fields $q(z), p(z)$ and $j(z)$ by

$$ q(z) = q^\alpha(z)L_\alpha, \quad p(z) = p_\alpha(z)U^\alpha, \quad j(z) = \sum_i j^i_\mu(z)D^{i,\mu}. \quad (3.16) $$

By definition, $J(z)$ is independent of the choice of the basis of $\mathcal{G}$, $q(z)$ is independent of the choice of the basis of $\mathcal{G}_-$ and so on. For example, if $D^k$ is an arbitrary basis of $\mathcal{G}_0$, then the equality

$$ \sum_i j^i_\mu(z)D^{i,\mu} = j(z) = j_k(z)D^k \quad (3.17) $$

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determines the fields \( f_k(z) \) as linear combinations of the \( f^\alpha_k(z) \). With this notation, the Wakimoto current \( \mathcal{J}(z) \) takes the form
\[
\mathcal{J}(z) = - : p_\beta(z) (\Psi^\beta_{\alpha}(z)(e^{-q U^\alpha} e^q)(z)) : + j_k(z)(e^{-q D^k e^q}) + (Ke^{-q \frac{\partial e^q}{\partial q^\alpha}} + \Lambda_\alpha)(z) \partial_z q^\alpha(z),
\]
where \( e^{-q \frac{\partial e^q}{\partial q^\alpha}} \) is of course a \( G_- \)-valued polynomial.

### 3.2 Equivariance of the Wakimoto homomorphism

From now on we assume that the grading element \( H \) and the automorphism \( \tau \) satisfy the compatibility condition (2.4). We can then choose the base elements in \( G \) to be simultaneously \( \text{ad}_H \) and \( \tau \)-eigenvectors with the eigenvalues denoted as in subsection 2.1. The map
\[
\tau_A : J_a(z) \mapsto \omega_a J_a(z)
\]
respects the current algebra (3.2) and thus it extends uniquely to an automorphism of the vertex algebra \( A(G, K) \). This automorphism, also called \( \tau_A \), has the same order \( N \) as \( \tau \). Similarly, we can define an automorphism \( \tau_W \) of the vertex algebra \( W(G, K, H) \) by extension of the map
\[
\tau_W : (p_\alpha(z), q^\alpha(z), j_k(z)) \mapsto (\omega_a p_\alpha(z), \omega^\alpha q^\alpha(z), \omega_0 k j_k(z)).
\]

**Proposition 2.** If the automorphism \( \tau \) and the gradation are compatible in the sense of (2.4), then the Wakimoto homomorphism \( \mathcal{W}_H : A(G, K) \to W(G, K, H) \) is \( \tau \)-equivariant:
\[
\tau_W \circ \mathcal{W}_H = \mathcal{W}_H \circ \tau_A.
\]

**Proof.** Let us denote the ‘Wakimoto current’ defined by the right hand side of (3.13) as
\[
\mathcal{J}_a(\{p_\alpha(z)\}, \{q^\alpha(z)\}, \{j_k(z)\})
\]
to emphasize that it is a composite expression formed out of the fields listed as its arguments. Note that in terms of the homogeneous fields \( j_k(z) \) the second term in (3.13) has the form
\[
j_k(z) D^k_a(\{q^\alpha(z)\}).
\]
The statement of Proposition 2 is obviously equivalent to the relation
\[
\mathcal{J}_a(\{\omega_a p_\alpha(z)\}, \{\omega^\alpha q^\alpha(z)\}, \{\omega_0 k j_k(z)\}) = \omega_a \mathcal{J}_a(\{p_\alpha(z)\}, \{q^\alpha(z)\}, \{j_k(z)\}).
\]
This holds as a result of the homogeneity properties of the polynomials that enter \( \mathcal{J}_a(z) \) with respect to the transformation (2.20), as described in Lemma 1. **Q.E.D.**

The equivariance of the Wakimoto homomorphism has been proved in [12] in the ‘principal special case’ for which \( \tau \) is a diagram automorphism and \( H \) defines the principal grading of \( G \). By taking advantage of the explicit formula (3.13) of [9] we presented a simpler proof, which is valid in the general case for which \( \tau(H) = H \).
4 Wakimoto realizations of twisted current algebras

We below consider the situation for which Proposition 2 holds and explain how the twisted modules of the vertex algebra $W(G, K, H)$ then yield generalized Wakimoto realizations of the $\tau$-twisted current algebra. The idea is the same that has been used by Szczesny in [12], where the special case for which $\tau$ is a diagram automorphism and $H$ defines the principal grading of $G$ was considered. Our result is more general and our formulas are much more explicit.

4.1 Useful identities for twisted modules of vertex algebras

Let $N$ be a positive integer and $M$ a complex vector space. An $N$-twisted field on $M$ is a formal series of the form

$$F(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} F[n]z^{-n-1},$$

(4.1)

where $F[n] \in \text{End}(M)$ and $F(z)v$ contains only finitely many negative powers of the formal variable $z^\frac{1}{N}$ for any $v \in M$.

Let $\sigma$ be an automorphism of order $N$ of a vertex algebra $V$. As a vector space, we identify $V$ with the corresponding space of fields (vertex operators) on $V$. Let us recall (see [21, 22, 23, 24] and appendix A) that a $\sigma$-twisted module of $V$ is a linear map $R_\sigma$ from the fields in $V$ to the $N$-twisted fields on some vector space, here denoted as $R_\sigma : \phi(z) \mapsto \phi^\sigma(z)$,

(4.2)

for which $\text{id}_V$ maps to $\text{id}_M$,

$$\sigma(\phi(z)) = e^{\frac{2\pi i}{N} k}\phi(z) \quad (k \in \mathbb{Z})$$

(4.3)

implies that

$$\phi^\sigma(z) = \sum_{n \in \frac{k}{N} + \mathbb{Z}} \phi^\sigma[n]z^{-n-1},$$

(4.4)

and the so-called twisted Borcherds identity holds.

We shall use the following consequences of the twisted Borcherds identity (see appendix A). Let $V^k$ stand for the homogeneous fields subject to (4.3), $V^k = V^{k+N}$. First, suppose that the fields $\phi(z)$ and $\psi(z)$ in $V$ have the commutator

$$[\phi(z), \psi(w)] = \sum_{s=0}^{s_{max}} \chi_s(w) \partial_w^s \delta(z, w).$$

(4.5)

If $\phi \in V^k$, then this implies

$$[\phi^\sigma(z), \psi^\sigma(w)] = \sum_{s=0}^{s_{max}} \chi_s^\sigma(w) \partial_w^s \delta_k(z, w),$$

(4.6)
Then the twisted field corresponding to $\xi^\alpha$ and
\[
\delta_k(z, w) = w^{\frac{k}{N}} z^{-\frac{k}{N}} \delta(z, w), \quad (\delta_k = \delta_{k+N}).
\]

Of course, if $\phi \in V^k$, $\psi \in V^l$, then $\chi_s \in V^{k+l}$ for any $s$. Second, suppose again that $\phi \in V^k$, $\psi \in V$ and consider the normal ordered product.

\[
\xi(z) =: \phi(z)\psi(z) :
\]

Then the twisted field corresponding to $\xi(z)$ is given by
\[
\xi^\alpha(z) =: \phi^\alpha(z)\psi^\alpha(z) - \sum_{s=0}^{s_{\text{max}}} s! \left( \frac{k}{N} \right)^s \chi_s^\alpha(z) z^{-s-1}
\]
where we use $\left( \begin{array}{c} \alpha \\ s+1 \end{array} \right) = \frac{\alpha(a-1)\ldots(a-s)}{(s+1)!}$ for any $\alpha \in C$, $s \in \mathbb{Z}_+$ and

\[
: \phi^\alpha(z)\psi^\alpha(z) := \phi^\alpha_+(z)\psi^\alpha(z) + \psi^\alpha(z)\phi^\alpha_-(z),
\]

\[
\phi^-_\alpha(z) = \sum_{n \in \mathbb{Z}_+} \phi^\alpha[n+k/N] z^{-k/N-n-1} \quad \text{with} \quad k \in \{0, 1, \ldots, N-1\}.
\]

In fact, $\phi^\alpha(z)\psi^\alpha(z)$ coincides with the ‘mode normal ordering’ and $\xi^\alpha(z)$ with the ‘OPE normal ordering’ of the twisted fields $\phi^\alpha(z)$, $\psi^\alpha(z)$ used in some references (see [17, 25] and references therein). If $[\phi(z), \psi(w)] = 0$, then $\xi^\alpha(z) = \phi^\alpha(z)\psi^\alpha(z)$ as expected. It can also be shown that for any field $\psi(z)$ in $V$ the twisted field corresponding to $\partial_z\psi(z)$ is given by $\partial_z\psi^\alpha(z)$.

### 4.2 Application to twisted current algebras

Let us suppose that we have a twisted module of the affine vertex algebra $A(G, K)$ corresponding to the automorphism $\tau_A$ (3.19) and consider the twisted currents given by

\[
R_{\tau_A} : J_a(z) \mapsto J_{a}^{\tau_A}(z) := J_a^\tau(z).
\]

These admit the mode expansions
\[
J_a^\tau(z) = \sum_{n \in \frac{a}{N} + \mathbb{Z}} J_a^\tau[n] z^{-n-1}
\]
with the $n_a$ defined in (2.6), and possess the commutators
\[
[J_a^\tau(z), J_b^\tau(w)] = \langle [T_a, T_b], T^c \rangle J_c^\tau(z) \delta_{n_a}(z, w) + K \langle T_a, T_b \rangle \partial_w \delta_{n_a}(z, w).
\]

according to (3.2), (4.6). Equivalently, we obtain the ‘$\tau$-twisted current algebra’ in the mode form
\[
[J_a^\tau[m], J_b^\tau[n]] = \langle [T_a, T_b], T^c \rangle J_c^\tau[m+n] + K \langle T_a, T_b \rangle m \delta_{m,-n}.
\]
It is well known [2] that this algebra is isomorphic to its untwisted analogue in (3.4) if $\tau$ is an inner automorphism of $G$, and the isomorphism can be lifted to the corresponding vertex algebras [23]. Nevertheless, it is useful to have different realizations of the same algebra, since they may have advantages from the viewpoint of various applications [26, 14, 15, 16, 17].

Let us now consider a twisted module of the vertex algebra $W(G, K, H)$ with respect to the automorphism $\tau_W$ (3.20). For the corresponding twisted fields now use the notation

$$R_{\tau_W} : (p_\alpha(z), q^\alpha(z), j_k(z)) \mapsto (\tilde{p}_\alpha(z), \tilde{q}^\alpha(z), \tilde{j}_k(z)).$$

(4.16)

The commutators of the basic twisted fields can be written as

$$[\tilde{q}^\alpha(z), \tilde{p}_\beta(w)] = \delta_\beta^\alpha \delta_{n_\alpha}(z, w),$$

(4.17)

$$[\tilde{j}_k(z), \tilde{j}_l(w)] = \langle [D_k, D_l], D^m \rangle \tilde{j}_m(w)\delta_{n_{0,k}}(z, w) + K_k \langle D_k, D_l \rangle \partial_w \delta_{n_{0,k}}(z, w),$$

(4.18)

where the integers $n_\alpha$, $n_\beta$, and $n_{0,k}$ are defined in subsection 2.1. We here use $\tau$-eigenvectors $D_k$ that belong either to $G_0$ or to a direct sum of a subset of the factors $G_0$ (2.3) formed by pairwise isomorphic factors. Correspondingly, $K_k$ is either $K_0$ or one of the $K_0^i$ in (3.10). The mode expansions of these twisted fields can be written as

$$\tilde{p}_\alpha(z) = \sum_{n \in \frac{m_\alpha}{M} + \mathbb{Z}} \tilde{p}_\alpha[n]z^{-n-1}, \quad \tilde{q}^\alpha(z) = \sum_{n \in \frac{m_\alpha}{M} + \mathbb{Z}} \tilde{q}^\alpha[n]z^{-n-1}, \quad \tilde{j}_k(z) = \sum_{n \in \frac{m_{0,k}}{M} + \mathbb{Z}} \tilde{j}_k[n]z^{-n-1}.$$  

(4.19)

One can of course present (4.17), (4.18) as an equivalent Lie algebra of the modes. For any polynomial $P$ in the complex variables $\{q^\alpha\}$, note that

$$\tilde{P}(z) \equiv R_{\tau_W}(P(z))$$

(4.20)

is obtained simply by replacing the $q^\alpha$ with the $\tilde{q}^\alpha(z)$ in the arguments of $P$. We need also the notation

$$\partial_\alpha P \equiv \frac{\partial P}{\partial q^\alpha}.$$  

(4.21)

Now we are ready to state the main result of this paper.

**Proposition 3.** Suppose that the automorphism $\tau$ and the gradation of $G$ are compatible (2.4), and consider a twisted module $R_{\tau_W}$ of $W(G, K, H)$ with respect to $\tau_W$ (3.20). Then one obtains a twisted module of $A(G, K)$ with respect to $\tau_A$ (3.19) by the composition

$$R_{\tau_A} \equiv R_{\tau_W} \circ W_H$$

(4.22)

with $W_H : A(G, K) \rightarrow W(G, K, H)$ being the homomorphism described in Proposition 1. The twisted currents $J^\tau_a(z) = R_{\tau_W}(J_a(z))$ are found from (3.13) explicitly as

$$J^\tau_a(z) = - \tilde{\psi}_a(z) \left( \tilde{U}^a(z) \tilde{U}^a(z) \right) : + \tilde{\Theta}_a(z)z^{-1}$$

$$+ \tilde{j}_k(z) \tilde{D}^k_a(z) + \left( K \tilde{\Phi}^a(z) \tilde{L}^a(z) + \tilde{\Lambda}^a(z) \right) \partial_z \tilde{q}^\alpha(z)$$

(4.23)
where the $\Theta_a$ are polynomials in the $\{q^a\}$ defined by $\Theta_a \equiv -\frac{n_a}{N} \partial_\beta (\Psi^\beta_a U^a_\alpha)$.

Proof. The first part of the statement relies on the $\tau$-equivariance of $\mathcal{W}_H$ (3.21), and is then obvious from the definition of twisted modules of vertex algebras (see appendix A). Formula (4.23) is derived by applying the identities collected in the preceding subsection, and using that for any polynomial $P$ in the variables $\{q^a\}$ one has

$$[p_\beta(z), P(w)] = - (\partial_\beta P)(w) \delta (z, w)$$

(4.24)
as a consequence of the Wick theorem. Q.E.D.

We now wish to describe the image of the ‘Sugawara field’

$$S(z) = \frac{1}{2y} : J_a(z) J^a(z) : = \frac{1}{2y} \eta^{ab} : J_a(z) J_b(z) : \quad \text{with} \quad 2y \equiv 2K + |\psi|^2 h^\vee$$

(4.25)
in a twisted Wakimoto module (4.22) of $A(G, K)$. To do this, we first quote from [4, 9] that under the Wakimoto homomorphism $S(z) \equiv \mathcal{W}_H(S(z))$ has the form

$$S(z) = \frac{1}{2y} : J_a(z) J^a(z) : = - : p_\alpha(z) \partial_z q^a(z) : + \frac{1}{2y} \sum_{i \geq 0} : J_i^\mu(z) J^\mu(z) : + \frac{1}{y} \partial_z Q(z)$$

(4.26)

with

$$J_Q(z) \equiv \eta_0^0(z) \langle D_0^0, Q \rangle \quad \text{for} \quad Q = \frac{1}{2} [U^\alpha, L_\alpha].$$

(4.27)

It is worth remarking that $Q = \rho_G - \sum_{i > 0} \rho_{G_i}$, where the Weyl vector $\rho_G$ (respectively $\rho_{G_i}$) corresponds to half the sum of the positive roots of $G$ (respectively $G_i$) under the scalar product $\langle , \rangle$, which implies that $Q \in G_0^0$. Denoting the inverse of $\eta_{0,kl} \equiv \langle D_k, D_l \rangle$ by $\eta_{0,kl}^0$, we have

$$\sum_{i \geq 0} : J_i^\mu(z) J^\mu(z) : = \eta_0^0 : J_k(z) J_l(z) :$$

(4.28)

As a direct consequence of (4.9), for any twisted module of $A(G, K)$ with respect to $\tau_A$, the image of $S(z)$ under $R_{\tau_A}$ (4.12) is given$^4$ as follows:

$$S^\tau(z) = \frac{1}{2y} \eta^{ab} : J_a^\tau(z) J_b^\tau(z) : = -\frac{z^{-1}}{2yN} \eta^{ab} n_a \langle [T_a, T_b], T^c \rangle J_c^\tau(z) + z^{-2} \frac{K}{4yN^2} n_a n^a$$

(4.29)

with the $n_a$ defined in (2.6). $S^\tau(z)$ obeys the same commutation relations as $S(z)$ since $\tau_A(S(z)) = S(z)$. In the case the twisted module of $A(G, K)$ is given by (4.22), and thus we can determine $S^\tau(z)$ by applying $R_{\tau_W}$ to the right hand side of (4.26). By using (4.9), we find that

$$S^\tau(z) = - : \tilde{p}_\alpha(z) \partial_z q^a(z) : + \frac{1}{2y} : \tilde{j}_k(z) J_k^0(z) : - \frac{z^{-1}}{2yN} \eta_0^0 n_0,k \langle [D_k, D_l], D^m \rangle \tilde{j}_m(z)$$

$$+ \frac{1}{y} \partial_z Q(z) + \frac{z^{-2}}{2N^2} (n_a n^a + \frac{1}{2y} K_n n_0,k n_0,k).$$

(4.30)

$^4$In an equivalent mode form, formula (4.29) was first discovered in [27], see also exercise 12.20 in [2]. The local form (4.29) can be found in [17, 25] under the label ‘twisted affine-Sugawara construction’.
Recall that $K_k$ is defined below (4.18); the $n_{0,k}$ label the eigenvalues of $\tau$ on $G_0$ according to (2.9); the $n_\alpha$ and the $n^\alpha$ similarly label the $\tau$-eigenvalues on $G_-$ and on $G_+$. Note also that $\tilde{j}_Q(z)$ is expanded in integral powers of $z$ since $\tau_W(\tilde{j}_Q(z)) = \tilde{j}_Q(z)$ as a consequence of $\tau(Q) = Q$.

### 4.3 How to construct examples?

Since the automorphisms of finite order are fully under control [2], one can easily exhibit compatible pairs $(\tau,H)$ together with their natural eigenbases. Below we sketch the construction of these ‘input data’ of the twisted Wakimoto realizations in rather general terms.

Let $G$ be a simple Lie algebra with Cartan subalgebra $H$, corresponding set of roots $\Delta$, and simple roots $\alpha_i$ for $i = 1, \ldots, r$. Any inner automorphism of order $N$ can be conjugated to have the form

$$\tau = \exp\left(\frac{2\pi i}{N} \text{ad}_\theta\right)$$

with some $\theta \in H$ for which $\alpha_i(\theta) \in \mathbb{Z}$. Then a compatible integer gradation is obtained by taking $H$ to be any element from $H$ for which $\alpha_i(H) \in \mathbb{Z}$. A joint eigenbasis for $(\tau,H)$ is provided by a Cartan-Weyl basis of $G$, spanned by Cartan elements $H_\alpha$ and root vectors $E_\alpha$ for $\alpha \in \Delta$ normalized according to $\langle E_\alpha, E_{-\alpha}\rangle = 1$. Of course, the corresponding eigenvalues depend on the actual choice of $\tau$ and $H$. A free field construction of the affine Lie algebra $G^{(1)}$ in the principal realization is obtained by taking $N$ to be the Coxeter number of $G$ and setting both $\theta$ and $H$ equal to the element $I_0 \in H$ defined by

$$\alpha_i(I_0) = 1 \quad \forall i = 1, \ldots, r.$$  \hspace{1cm} (4.32)

The simplest case of $G = sl_2$ is described explicitly in appendix B.

Now recall [2] that, up to conjugation, any outer automorphism of order $N$ can be written in the form

$$\tau = \mu \circ \exp\left(\frac{2\pi i}{N} \text{ad}_\theta\right),$$

where $\mu$ is induced by a non-trivial symmetry of the Dynkin diagram of $G$, $\theta$ belongs to the fixed point set of $\mu$ in $H$ and $\alpha_i(\theta) \in \mathbb{Z}$. Then a compatible $\mathbb{Z}$-gradation of $G$ can be specified by choosing any $H \in H$ for which $\mu(H) = H$ and $\alpha_i(H) \in \mathbb{Z}$. For simplicity of writing, let us assume that $\mu$ has order 2, excluding only the cyclic diagram automorphism of $D_4$. Denote by $H^\pm$ and $G^\pm$ the eigensubspaces of $\mu$ in $H$ and in $G$ in correspondence with the eigenvalues $\pm 1$. Recall that $G^+$ is a simple Lie algebra and $G^-$ is an irreducible module of $G^+$ in which the non-zero weights have multiplicity 1. Denote by $\Delta^+$ the roots of $(H^+, G^+)$ and by $\Delta^-$ the non-zero weights of $H^+$ in $G^-$. For $\lambda \in \Delta^\pm$ let $E^\pm_\lambda \in G^\pm$ be an eigenvector of $H^+$ normalized by $\langle E^s_\lambda, E^r_{-\lambda}\rangle = 1$ for $s \in \{\pm\}$. Choose also bases $\{H^+_k\}_{k=1}^{r^+_+}$ of $H^+$ and $\{H^-_k\}_{k=1}^{r^-_-}$ of $H^-$. Now the elements

$$E^\pm_\lambda \quad (\lambda \in \Delta^\pm), \quad H^+_k \quad (1 \leq k \leq r^+_+), \quad H^-_k \quad (1 \leq k \leq r^-_-)$$

form a joint eigenbasis for any pair $(\tau,H)$ described above. The subalgebra $G_0$ equals $H$ if we select $H := I_0$ in (4.32). A free field realization of the twisted affine Lie algebra $G^{(2)}$ in
the standard realization is obtained by taking $\tau := \mu$. This case has been studied in [12].
In our formalism it is equally easy to provide a free field construction of $G^{(2)}$ in the principal realization. It results from our general formulae by setting $H := I_0$ together with $\theta := I_0$ and $N$ twice the Coxeter number of the algebra $G^{(2)}$ (in the convention of [2]).

Our formalism contains also examples for which $G_0$ is non-Abelian. This is illustrated by a simple example displayed in appendix C.

5 Concluding remarks

In this paper we described the generalized Wakimoto realizations of the twisted affine Lie algebras in correspondence with arbitrary finite order automorphisms of the simple Lie algebras. Our main result is the explicit formula given by Proposition 3, which relies on the equivariance of the formula of Proposition 1 valid in the untwisted case. It is worth noting that these formulas admit classical limits characterized by the absence of the terms represented by $\Lambda_0(z)$ in (3.13) and $\tilde{\Theta}_a(z)$ in (4.23). In fact, these terms are quantum corrections arising from the commutators of the basic free fields (the $\beta\gamma$-systems) that contain the Planck constant if one uses suitable normalization.

In the classical WZNW model with twisted boundary condition, the classical twisted current algebra can be viewed as a consequence of a more fundamental quadratic ‘exchange algebra’ of the group-valued twisted chiral WZNW field governed by a (non-unique) monodromy dependent dynamical $r$-matrix (see [28] and references therein). The quantization of these quadratic Poisson algebras in terms of free fields is an interesting problem for the future. It requires the free field realization of twisted chiral primary fields, which should be also useful for further investigations of orbifolds and twisted versions of the WZNW model.

The method used in this paper is applicable in other cases as well, for example to construct free field realizations of twisted $\mathcal{W}$-algebras or twisted current algebras based on reductive Lie algebras that are a direct sum of identical simple factors [29]. Concerning the first case, recall [30] that a $\mathcal{W}$-algebra can be associated with any $sl_2$-embedding into a simple Lie algebra $\mathcal{G}$. If the $sl_2$ generators are invariant with respect to an automorphism $\tau$ of $\mathcal{G}$, then $\tau$ lifts to an automorphism of the $\mathcal{W}$-algebra and the Miura maps underlying the free field realizations of the $\mathcal{W}$-algebra can be shown to be $\tau$-equivariant. In the second case the permutations of the identical factors give rise to automorphisms of the current algebra, and its free field realization obtained by applying the Wakimoto homomorphism (3.15) factorwise is clearly $\tau$-equivariant with respect to the permutations.

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A Vertex algebras and their twisted modules

For convenience, in this appendix we collect some background information on vertex algebras and their twisted modules used in the main text. For more details, see [19, 20, 21, 22, 23, 24] and references therein.

A vertex algebra is a quadruplet $(V, Y, v^0, T)$, where $V$ is a complex vector space and $Y$ is a linear map from $V$ to the fields on $V$,

$$Y : \phi \mapsto Y(\phi, z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-1}, \quad \phi_n \in \text{End}(V), \quad \phi_n \psi = 0 \quad \forall \phi, \psi \in V, \ n \gg 0,$$

with $z$ being a formal variable. By definition, the state-field correspondence $Y$, the vacuum vector $v^0 \in V$, and the translation operator $T \in \text{End}(V)$ obey the basic relations

$$Y(v^0, z) = \text{id}_V, \quad Tv^0 = 0, \quad [T, Y(\phi, z)] = \partial_z Y(\phi, z),$$

$$\phi_n v^0 = \delta_{n,-1} \phi \quad \text{for} \quad n \geq -1, \quad \forall \phi \in V,$$

as well as the Borcherds identity given by

$$\text{Res}_z \left( Y(\phi, z) Y(\psi, w) i_{z,w} F(z, w) - Y(\psi, w) Y(\phi, z) i_{w,z} F(z, w) \right) = \text{Res}_{z-w} \left( Y(Y(\phi, z-w) \psi, w) i_{w,z-w} F(z, w) \right),$$

for any $\phi, \psi \in V$ and $F(z, w) = (z-w)^l z^m$ for any $l, m \in \mathbb{Z}$. Here we have

$$i_{z,w}(z-w)^l = \sum_{n \in \mathbb{Z}_+} \binom{l}{n} z^{l-n} (-w)^n, \quad i_{w,z}(z-w)^l = \sum_{n \in \mathbb{Z}_+} \binom{l}{n} z^n (-w)^{l-n},$$

$$i_{w,z-w} z^m = i_{w,z-w} (w + (z-w))^m = \sum_{n \in \mathbb{Z}_+} \binom{m}{n} w^{m-n} (z-w)^n.$$

Intuitively speaking, the Borcherds identity (A.4) (also called Cauchy-Jacobi identity) means that the ‘usual contour deformation’ [3] is applicable.

Evaluating the Borcherds identity for $F(z, w) = (z-w)^l z^m$ gives

$$\sum_{n \in \mathbb{Z}_+} (-1)^n \binom{l}{n} \phi_{l+m-n} w^n Y(\psi, w) + Y(\psi, w) \sum_{n \in \mathbb{Z}_+} (-1)^{n+l+1} \binom{l}{n} \phi_{m+n} w^{l-n}$$

$$= \sum_{n \in \mathbb{Z}_+} \binom{m}{n} w^{m-n} Y(\phi_{l+n} \psi, w).$$

For $l = 0$ (A.6) simplifies to

$$\phi_m Y(\psi, w) - Y(\psi, w) \phi_m = \sum_{n \in \mathbb{Z}_+} \binom{m}{n} w^{m-n} Y(\phi_n \psi, w),$$

16
and the collection of these relations for all $m \in \mathbb{Z}$ is equivalent to the commutator formula

$$[Y(\phi, z), Y(\psi, w)] = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} Y(\phi_n \psi, w) \partial_w^n \delta(z, w).$$  

(A.8)

The sums on the right-hand-sides of the last two equations are actually finite; $\delta(z, w)$ is defined in (3.3). In the special case $m = 0$, $l = -1$ the Borcherds identity (A.6) can be written as

$$: Y(\phi, w) Y(\psi, w) := Y(\phi_{-1} \psi, w),$$  

(A.9)

and it can also be shown that $Y(T \phi, z) = \partial_z Y(\phi, z)$.

A vertex algebra homomorphism from $(V, Y, v^0, T)$ to $(\tilde{V}, \tilde{Y}, \tilde{v}^0, \tilde{T})$ is defined to be a linear map $f : V \to \tilde{V}$ satisfying

$$fv^0 = \tilde{v}^0, \quad f \circ T = \tilde{T} \circ f, \quad \tilde{Y}(f \phi, z) \circ f = f \circ Y(\phi, z) \quad \forall \phi \in V. \quad (A.10)$$

An isomorphism is an invertible homomorphism. An automorphism $\sigma$ of $(V, Y, v^0, T)$ is thus defined by the properties

$$\sigma v^0 = v^0, \quad [\sigma, T] = 0, \quad Y(\sigma \phi, z) = \sigma \circ Y(\phi, z) \circ \sigma^{-1}. \quad (A.11)$$

Let $\sigma$ be an automorphism of a vertex algebra $(V, Y, v^0, T)$ that has finite order, say $N$. Then

$$V = \bigoplus_{k=0}^{N-1} V^k \quad \text{with} \quad V^k := \{ \phi \in V \mid \sigma(\phi) = e^{2\pi i k/N} \phi \}. \quad (A.12)$$

A twisted $V$-module with respect to $\sigma$ is a vector space $M$ together with a linear map from $V$ to the $N$-twisted fields on $M$,

$$\phi \mapsto Y_M(\phi, z) = \sum_{n \in \mathbb{Z}} \phi_n^M z^{-n-1}, \quad \phi_n^M \in \text{End}(M), \quad \phi_n^M v = 0 \quad \forall v \in M, \ n \gg 0, \quad (A.13)$$

for which one has $Y_M(v^0, z) = \text{id}_M$,

$$Y_M(\phi, z) = \sum_{n \in \frac{1}{N} \mathbb{Z}} \phi_n^M z^{-n-1} \quad \text{if} \quad \phi \in V^k, \quad (A.14)$$

and the following twisted Borcherds identity:

$$\text{Res}_z \left( z^{\frac{k}{N}} Y_M(\phi, z) Y_M(\psi, w) i_{z, w} F(z, w) - Y_M(\psi, w) z^{\frac{k}{N}} Y_M(\phi, z) i_{w, z} F(z, w) \right) = \text{Res}_{z-w} \left( Y_M(Y(\phi, z-w) \psi, \psi) i_{w, z-w} z^{\frac{k}{N}} F(z, w) \right) \quad (A.15)$$

$\forall \phi \in V^k$, $\psi \in V$ and $F(z, w) = (z-w)^l z^m \quad \forall l, m \in \mathbb{Z}$. We here use

$$i_{w, z-w} (z^{\frac{k}{N}} z^m (z-w)^l) = \sum_{n \in \mathbb{Z}_+} \left( \frac{k}{N} + \frac{m}{n} \right) w^{\frac{k}{N} + m-n} (z-w)^{l+n}. \quad (A.16)$$
The insertion of the factor $z^k$ ensures that the integrands in the argument of \( \text{Res} \) contain only integral powers of the respective variables \( z \) and \( (z-w) \), and (A.15) again can be thought of as the contour deformation argument’. If \( \sigma \) is the identity on \( V \), then the twisted modules become just the (ordinary) modules of \( V \).

By evaluating the twisted Borcherds identity (A.15) for \( F(z,w) = (z-w)^l z^m \) one obtains
\[
\sum_{n \in \mathbb{Z}_+} (-1)^n \binom{\frac{l}{n}}{l} \phi_{\frac{k}{n} + l + m - n} M(\psi, w) + Y_M(\psi, w) \sum_{n \in \mathbb{Z}_+} (-1)^{l+n+1} \binom{\frac{m}{n}}{m} \phi_{\frac{k}{n} + m + n} w^{l-n}
= \sum_{n \in \mathbb{Z}_+} \binom{\frac{k}{n}}{n} w^{\frac{k}{n} + m - n} Y_M(\phi_{l+n}, \psi, w).
\] (A.17)

For \( l = 0 \) this simplifies to
\[
\phi_{\frac{k}{n} + m} M(\psi, w) - Y_M(\psi, w) \phi_{\frac{k}{n} + m} = \sum_{n \in \mathbb{Z}_+} \binom{\frac{k}{n}}{n} w^{\frac{k}{n} + m - n} Y_M(\phi_n, \psi, w), \quad \forall m \in \mathbb{Z}, \quad (A.18)
\]
which is equivalent to the commutator formula
\[
[Y_M(\phi, z), Y_M(\psi, w)] = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} Y_M(\phi_n, \psi, w) \partial_w^n \delta_k(z, w) \quad \forall \phi \in V^k, \psi \in V.
\] (A.19)

The sum is finite since \( \phi_n \psi = 0 \) for \( n \gg 0 \); \( \delta_k(z, w) \) appears in (4.7). The \( m = 0, l = -1 \) special case of (A.17) implies that
\[
Y_M(\phi_{-1}, w) =: Y_M(\phi, w) Y_M(\psi, w) := - \sum_{n \in \mathbb{Z}_+} \binom{\frac{k}{n}}{n+1} Y_M(\phi_n, \psi, w) w^{-n-1}.
\] (A.20)

By definition, \( : Y_M(\phi, w) Y_M(\psi, w) : \) equals \( w^{-\frac{k}{n}} \) times the expression on the left-hand-side of (A.17) with \( m = 0, l = -1 \). It can also be shown that \( Y_M(T\phi, z) = \partial_z Y_M(\phi, z) \) for any \( \phi \in V \).

Upon changing the notation by setting
\[
Y(\phi, z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-1} \equiv \phi(z) = \sum_{n \in \mathbb{Z}} \phi[n] z^{-n-1}
Y_M(\phi, z) = \sum_{n \in \mathbb{Z}} \phi_{\frac{k}{n} + m} z^{-n-1} \equiv \phi^\sigma(z) = \sum_{n \in \mathbb{Z}} \phi^\sigma[n] z^{-n-1}, \quad \forall \phi \in V, \quad (A.21)
\]
the above-derived identities reproduce those mentioned in subsection 4.1.

**B The simplest example: \( \mathcal{G} = sl_2 \)**

The general features of the twisted Wakimoto construction can be illustrated on the simplest \( sl_2 \) example as follows. The untwisted current algebra based on \( sl_2 \) is generated by
\[
[J_{\sigma_1}(z), J_{\sigma_2}(w)] = \pm 2J_{\sigma_1}(w) \delta(z, w), \quad [J_{\sigma_1}(z), J_{\sigma_2}(w)] = 2K \partial_w \delta(z, w),
[J_{\sigma_1}(z), J_{\sigma_2}(w)] = J_{\sigma_1}(w) \delta(z, w) + K \partial_z \delta(z, w),
\] (B.1)
where \( J_{\sigma_a}(z) = \sum_{n \in \mathbb{Z}} J_{\sigma_a}[n] z^{-n-1} \) with the Pauli matrices \( \sigma_3 = (E_{1,1} - E_{2,2}) \), \( \sigma_+ = E_{1,2} \), \( \sigma_- = E_{2,1} \). Any automorphism of \( sl_2 \) of order \( N \) is conjugate to 

\[
\tau = \exp(\frac{\pi i}{N} \text{ad}_{\sigma_3}),
\]

(B.2)

and \( \tau \) lifts to the automorphism \( \tau_A \) of (B.1) for which \( \tau_A(J_{\sigma_a}) = J_{\tau(\sigma_a)} \). The corresponding twisted realization of the algebra (B.1) is generated by the twisted currents \( J^r_{\sigma_a} \) subject to the twisted commutation relations

\[
[J^r_{\sigma_3}(z), J^r_{\sigma_\pm}(w)] = \pm 2 J^r_{\sigma_\pm}(w) \delta(z, w), \quad [J^r_{\sigma_3}(z), J^r_{\sigma_3}(w)] = 2 K \partial_w \delta(z, w), \quad [J^r_{\sigma_\pm}(z), J^r_{\sigma_\mp}(w)] = J^r_{\sigma_\pm}(w) W(z, w) + K \partial_w (W(z, w)) \quad (B.3)
\]

and mode expansions

\[
J^r_{\sigma_+}(z) = \sum_{n \in \mathbb{Z}} J^r_{\sigma_+}[n] z^{-n-1}, \quad J^r_{\sigma_-}(z) = \sum_{n \in \mathbb{Z}} J^r_{\sigma_-}[n] z^{-n-1}, \quad J^r_{\sigma_3}(z) = \sum_{n \in \mathbb{Z}} J^r_{\sigma_3}[n] z^{-n-1}.
\]

The free field realization of (B.1) due to [1] is given by the homomorphism \( J_{\sigma_a} \mapsto \mathcal{J}_{\sigma_a} \) with the following ‘Wakimoto currents’:

\[
\mathcal{J}_{\sigma_-}(z) = -p(z), \quad \mathcal{J}_{\sigma_3}(z) = j(z) - 2 : p(z) q(z) :, \quad \mathcal{J}_{\sigma_+}(z) = : p(z) (q^2)(z) : - j(z) q(z) + K \partial_z q(z), \quad (B.4)
\]

where \( q, p \) and \( j \) satisfy the relations \([q(z), p(w)] = \delta(z, w)\) and \([j(z), j(w)] = 2(K + 2) \partial_w \delta(z, w)\). On the free fields \( \tau \) operates by the automorphism \( \tau_W \) that maps \( j \) to itself and \((q, p)\) to \((\omega q, \omega^{-1} p)\) with \( \omega = \exp(\frac{2 \pi i}{N}) \). The corresponding twisted free fields \( \tilde{q} \) and \( \tilde{p} \) satisfy

\[
[q(z), \tilde{p}(w)] = w^{\frac{1}{N}} z^{-\frac{1}{N}} \delta(z, w), \quad \tilde{q}(z) = \sum_{n \in \mathbb{Z}} \tilde{q}[n] z^{-n-1}, \quad \tilde{p}(z) = \sum_{n \in \mathbb{Z}} \tilde{p}[n] z^{-n-1}. \quad (B.5)
\]

It follows from Proposition 3 (eq. (4.23)) that the twisted currents subject to (B.3) can be realized in terms of the twisted free fields \( \tilde{q}, \tilde{p} \) and the (untwisted) \( U(1) \) current \( j \) according to

\[
J^r_{\sigma_-}(z) = -\tilde{p}(z), \quad J^r_{\sigma_3}(z) = j(z) - 2 : \tilde{p}(z) \tilde{q}(z) : -2z^{-1} \frac{N - 1}{N} \tilde{q}(z) + K \partial_z \tilde{q}(z) + 2z^{-1} \frac{N - 1}{N} \tilde{q}(z). \quad (B.6)
\]

The term \( : \tilde{p}(z)(q^2)(z) : = 2z^{-1} \frac{N - 1}{N} \tilde{q}(z) \) is the twisted analogue of \( : p(z)(q^2)(z) : \) as explained in the general case in Section 4. See in particular eq. (4.10) for the definition of the normal ordering of the twisted free fields which is used here.

The Wakimoto realizations of the twisted current algebra (B.3) of \( sl_2 \) described in (B.6) are different from the (similarly named) realization of \( \hat{sl}_2 \) in the principal gradation presented in [14], which is based on different building blocks. It would be interesting to find a connection between the two constructions.
C An example with non-Abelian $G_0$

We below display a simple example for $G := sl_4$ such that $G_0$ is non-Abelian. Since $A_3 = D_3$, this yields a generalized free field realization of the affine Lie algebra $D^{(2)}_3$.

We consider the automorphism $\nu$ induced by transpose with respect to the `secondary diagonal':

$$\nu : E_{i,j} \mapsto -E_{5-j,5-i}$$  \hfill (C.1)

for the usual elementary matrices $E_{i,j} \in gl_4$. A compatible gradation is defined by

$$H := \frac{1}{2} (E_{1,1} + E_{2,2} - E_{3,3} - E_{4,4}).$$  \hfill (C.2)

Let $\mathcal{H}^+$ denote the diagonal matrices in $sl_4$ fixed by $\nu$, and choose a basis of $sl_4$ consisting of weight vectors with respect to $\mathcal{H}^+$. An arbitrary element $D \in \mathcal{H}^+$ has the form

$$D = d_1 E_{1,1} + d_2 E_{2,2} - d_2 E_{3,3} - d_1 E_{4,4}$$  \hfill (C.3)

and we define the functional $e_i$ on $\mathcal{H}^+$ by $e_i(D) = d_i$. A joint eigenbasis of $\nu$ and $ad_H$ in $G_-$ is furnished by the elements

$$E_{-2e_i}^\pm \equiv E_{5-i,i} \quad (i = 1, 2), \quad E_{-e_1-e_2}^\pm \equiv E_{3,1} \mp E_{4,2}. \hfill (C.4)$$

Our notation indicates that $E_{\lambda}^\pm$ has weight $\lambda \neq 0$ with respect to $\mathcal{H}^+$ and $\nu$-eigenvalue $\pm 1$. With respect to $\langle X, Y \rangle \equiv \text{tr}(XY) \quad (\forall X, Y \in G)$, the dual basis of $G_+$ is spanned by

$$E_{2e_i}^- \equiv E_{i,5-i} \quad (i = 1, 2), \quad \frac{1}{2} E_{e_1+e_2}^\pm \quad \text{with} \quad E_{e_1+e_2}^\pm \equiv E_{1,3} \mp E_{2,4}. \hfill (C.5)$$

The base elements of $G_0$ with non-zero weight are listed as

$$E_{e_1-e_2}^\pm \equiv E_{1,2} \mp E_{3,4}, \quad E_{e_2-e_1}^\pm \equiv E_{2,1} \mp E_{4,3}, \hfill (C.6)$$

and this is complemented by the basis of $\mathcal{H} \subset sl_4$ given by

$$H_{e_1-e_2}^\pm \equiv E_{1,1} - E_{2,2} \pm E_{3,3} \mp E_{4,4} \hfill (C.7)$$

together with $H$. Note that $G_0 = sl_2 \oplus sl_2 \oplus \text{span}\{H\}$.

Let us use the expansions

$$q(z) = q^1(z)E_{-2e_1}^- + q^2(z)E_{-2e_2}^- + q^3(z)E_{-e_1-e_2}^+ + q^4(z)E_{-e_1-e_2}^-, \hfill (C.8)$$

and the notation

$$J_X(z) \equiv J_a(z)\langle T^a, X \rangle \quad \forall X \in G, \quad J_X(z) \equiv j_k\langle D^k, X \rangle \quad \forall X \in G_0. \hfill (C.9)$$
Dropping the variable $z$, the current components associated with $X \in \mathcal{G}_-$ are

$$J_{E_{-2e_1}}^X = -p_i \ (i = 1, 2), \quad J_{E_{e_1-e_2}}^+ = -p_3, \quad J_{E_{-e_1-e_2}}^- = -p_4. \quad (C.10)$$

For $X \in \mathcal{G}_0$ we find

$$J_{E_{e_1-e_2}}^+ = J_{E_{e_1-e_2}}^- = 2 : p_2q^4 : - : p_4q^1 :$$

$$J_{E_{-e_1-e_2}}^- = J_{E_{e_1-e_2}}^+ = 2 : p_2q^3 : + : p_3q^1 :$$

$$J_{E_{e_2-e_1}}^+ = J_{E_{e_2-e_1}}^- = 2 : p_1q^4 : - : p_4q^2 :$$

$$J_{H_{e_1-e_2}}^+ = J_{H_{e_1-e_2}}^- = 2 : p_1q^1 : + 2 : p_2q^2 :$$

$$J_{H_{-e_1-e_2}}^- = J_{H_{e_1-e_2}}^+ = 2 : p_4q^3 : - 2 : p_3q^4 :$$

$$J_{H} = J_{H^-} : p_1q^1 : - : p_2q^2 : - : p_3q^3 : - : p_4q^4 : \quad (C.11)$$

For $X \in \mathcal{G}_+$ the explicit formulae are a bit longer. Straightforward calculation gives

$$J_{E_{e_1-e_2}}^- = -\frac{1}{2}q^1 J_{H_{e_1-e_2}}^+ - q^1 J_{H} + q^3 J_{E_{e_1-e_2}}^- - q^4 J_{E_{e_1-e_2}}^+ + : p_1(q^1)^2 :$$

$$+ : p_2((q^1)^2 - (q^3)^2) : + : p_3(q^1q^3) : + : p_4(q^1q^4) : + K \partial q^1, \quad (C.12)$$

$$J_{E_{e_2-e_1}}^+ = \frac{1}{2}q^2 J_{H_{e_1-e_2}}^+ - q^2 J_{H} - q^3 J_{E_{e_2-e_1}}^- - q^4 J_{E_{e_2-e_1}}^+ + : p_2(q^2)^2 :$$

$$+ : p_1((q^1)^2 - (q^3)^2) : + : p_3(q^2q^3) : + : p_4(q^2q^4) : + K \partial q^2, \quad (C.13)$$

$$J_{E_{e_1+e_2}}^+ = q^1 J_{E_{e_1-e_2}}^- - q^2 J_{E_{e_1-e_2}}^- - 2q^3 J_{H} - q^4 J_{H_{e_1-e_2}}^- + 2 : p_1(q^1)^3 :$$

$$+ 2 : p_2(q^2q^3) : + : p_3((q^1)^2 + (q^4)^2 - q^1q^2) : + 2 : p_4(q^3q^4) : + 2K \partial q^3, \quad (C.14)$$

$$J_{E_{e_1+e_2}}^- = -q^1 J_{E_{e_1-e_2}}^+ - q^2 J_{E_{e_1-e_2}}^- - q^3 J_{H_{e_1-e_2}}^- - 2q^4 J_{H} + 2 : p_1(q^1)^4 :$$

$$+ 2 : p_2(q^2q^4) : + 2 : p_3(q^3q^4) : + : p_4((q^1)^2 + (q^4)^2 + q^1q^2) : + 2K \partial q^4. \quad (C.15)$$

The above formulae represent the Wakimoto current (3.13) for $\mathcal{G} = sl_4$ with $H$ in (C.2). The non-zero commutators of the $\mathcal{G}_0$-valued current $j(z)$ in the homogeneous basis of $\mathcal{G}_0$ are

$$[j_H(z), j_H(w)] = (K + 4) \partial_w \delta(z, w),$$

$$[j_{H_{e_1-e_2}}(z), j_{H_{e_1-e_2}}(w)] = 4(K + 2) \partial_w \delta(z, w),$$

$$[j_{E_{e_1-e_2}}(z), j_{E_{e_2-e_1}}(w)] = j_{H_{e_1-e_2}}(w) \delta(z, w) + 2(K + 2) \delta_{e,s} \partial_w \delta(z, w),$$

$$[j_{E_{e_1-e_2}}(z), j_{H_{e_1-e_2}}(w)] = (-2\sigma)J_{E_{e_1+e_2}}^\sigma(w) \delta(z, w), \quad \forall \sigma = \pm 1, \quad (C.16)$$

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where \( s, r \in \{ \pm \} \), \( t = + \) if \( s = r \), and \( t = - \) if \( s \neq r \). Note that the action of \( \nu \) on \( G_0 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \text{span}\{H\} \) is given by the transposition of the two \( \mathfrak{sl}_2 \) factors composed with an inner automorphism of \( G_0 \). On the vertex algebra generated by the components of \( j(z) \), the corresponding automorphism is in fact a special case of the automorphisms studied in relation to permutation orbifolds [29, 31]. Thus one can construct the \( \nu \)-twisted modules of \( W(G, K, H) \) by applying the construction in [31] to the current algebra part and taking the usual twisted representations of the \( \beta\gamma \)-systems. In any such \( \nu \)-twisted module, the twisted Wakimoto currents \( J_\nu^X (X \in G) \) are obtained by substituting the twisted fields \( \tilde{j}_Y (z) (Y \in G_0) \), \( \tilde{q}^\alpha, \tilde{p}_\alpha \) into (C.10) – (C.15), and by also adding the correction terms defined by the \( \Theta_\alpha \) in (4.23).

In our example, the non-zero correction terms associated with the base elements of \( G \) are

\[
\Theta_H = \frac{3}{2}, \quad \Theta_{E_{2e_1}} = -\frac{3}{2}q^1, \quad \Theta_{E_{2e_2}} = -\frac{3}{2}q^2, \quad \Theta_{E_{e_1+e_2}} = -3q^3, \quad \Theta_{E_{e_1-e_2}} = -3q^4, \quad (C.17)
\]

where we use \( \Theta_X = \Theta_\alpha (X, T_\alpha) \) for any \( X \in G \).

Finally, it is easy to describe the Wakimoto realizations of \( D_3^{(2)} \) in correspondence with other automorphisms of \( D_3 = \mathfrak{sl}_4 \), too. For example, the diagram automorphism of \( \mathfrak{sl}_4 \) that leads to the standard realization (gradation) of \( D_3^{(2)} \) can be written as

\[
\mu = \nu \circ \exp(\pi \text{ad} I_0), \quad I_0 = \frac{3}{2}E_{1,1} + \frac{1}{2}E_{2,2} - \frac{1}{2}E_{3,3} - \frac{3}{2}E_{4,4}, \quad (C.18)
\]

and the Coxeter automorphism of \( \mathfrak{sl}_4 \) associated with the principal realization (gradation) of \( D_3^{(2)} \) is given by

\[
\tau = \nu \circ \exp\left(\frac{4\pi i}{3} \text{ad} I_0\right). \quad (C.19)
\]

Since the above introduced basis of \( \mathfrak{sl}_4 \) is an eigenbasis of \( \mu \) and \( \tau \) as well, the twisted currents \( J_\mu^a \) and \( J_\nu^a \) are obtained by correspondingly twisting the constituent fields in (C.10) – (C.15) and adding the appropriate correction terms \( \Theta_\alpha \) according to Proposition 3 (eq. (4.23)). These correction terms depend on the automorphism through the integers \( n_\alpha \) that label the eigenvalues of the automorphism on \( G_- \).

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