Generalizations of Szpilrajn’s Theorem in economic and game theories

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Abstract  Szpilrajn’s Lemma entails that each partial order extends to a linear order. Dushnik and Miller use Szpilrajn’s Lemma to show that each partial order has a relizer. Since then, many authors utilize Szpilrajn’s Theorem and the Well-ordering principle to prove more general existence type theorems on extending binary relations. Nevertheless, we are often interested not only in the existence of extensions of a binary relation $R$ satisfying certain axioms of orderability, but in something more: (A) The conditions of the sets of alternatives and the properties which $R$ satisfies to be inherited when one passes to any member of a subfamily of the family of extensions of $R$ and: (B) The size of a family of ordering extensions of $R$, whose intersection is $R$, to be the smallest one. The key to addressing these kinds of problems is the szpilrajn inherited method. In this paper, we define the notion of $\Lambda(m)$-consistency, where $m$ can reach the first infinite ordinal $\omega$, and we give two general inherited type theorems on extending binary relations, a Szpilrajn type and a Dushnik-Miller type theorem, which generalize all the well known existence and inherited type extension theorems in the literature.

Keywords  Consistent binary relations, Extension theorems, Intersection of binary relations.

JEL Classification  C60, D00, D60, D71.

1 Introduction

One of the most fundamental results on extensions of binary relations is the following theorem proved by E. Szpilrajn in 1930 [30].
**Theorem 1** Let \( \preceq \) be a partial order on a set \( X \) and let \( x \) and \( y \) be two incomparable elements of \( X \) (neither \( x \preceq y \) nor \( y \preceq x \)). Then, there exists a linear order \( \preceq^* \) on \( X \) which contains all pairs of \( \preceq \) and all pairs \((\kappa, \lambda)\) for which \( \kappa \preceq y \) and \( x \preceq \lambda \) holds.

The original proof of the theorem splits into two steps: In the first step, if \( x \) and \( y \) are two incomparable elements of \( X \), then it is constructed a partial order \( \preceq' \) such that every element of \( A = \{\kappa \in X|\kappa \preceq y\} \) must lie below, with respect to \( \preceq' \), to every element of \( B = \{\lambda \in X|x \preceq \lambda\} \). So we take \( \preceq'' = \preceq \cup \preceq' = \preceq \cup (A \times B) \), in other words, we include these obvious consequences of putting \( y \) below \( x \) and no others. Clearly, \( R'' \) is transitive. In the second step, we see that if \( \subseteq \) is a maximal element (under inclusion) in the set of partial orders extending \( \preceq' \), then, \( \subseteq \) must be a total order. Because otherwise, if \( x \) and \( y \) are incomparable in \( \subseteq \), then, by the first step, we have an extension \( \preceq'' \) of \( \subseteq \) such that \( y \preceq' x \), a contradiction to the maximality of \( \subseteq \). (We have first to show that the union of a chain of posets is a poset. This is a standard Zorn’s Lemma argument.)

The crucial point in the original proof of Szpilrajn’s Theorem is the relationship of the pair \((x, y)\) with the pairs \((\kappa, \lambda)\) \(\in A \times B\) which concludes the transitive axiom for the relation \( \preceq' \). In fact: (\(\alpha\)) For every linear extension \( \subseteq \) of \( \preceq \), \( y \subseteq x \) implies \( \kappa \subseteq \lambda \) and: (\(\beta\)) \( x <' \lambda \) implies \( y <' \lambda \) and \( \kappa <' y \) implies \( \kappa <' x \). In case (\(\alpha\)), we say that \((y, x)\) covers \((\kappa, \lambda)\) and in case (\(\beta\)), we say that \((y, x)\) is an uncovered pair of \( \preceq' \) (see [23, Lemma 5]). The true meaning of the Szpilrajn theorem is that, although it is not constructive, it preserves prescribed properties in the extended relation. In fact, by extending a binary relation \( R \), it is interesting to see whether the conditions of the underlying space \( X \) or the properties which \( R \) satisfies should be inherited when one passes to any member of some family of linear extensions of \( R \). Moreover, in extending a binary relation relation \( R \), the problem will often be how to incorporate some additional data depending on the binary relation with a minimum of disruption of the existing structure or how to extend the relation so that some desirable new condition is fulfilled. For example, as shows case (\(\beta\)) above, if we might wish to adjoin the pair \((x, y)\) to a transitive relation \( R \) that does not already relates \( x \) and \( y \), in order to preserve the relation \( R \) the axiom of transitivity, we must also adjoin all other pairs of the form \((\kappa, \lambda)\) where \((\kappa, y) \in R \) and \((x, \lambda) \in R \). Generally speaking, a natural question in an extension process is to ask, when a given binary relation \( R \) defined on a set of alternatives \( X \) will preserve the properties and the characteristics of \( X \), or of \( R \). For instance, if we refer to a property \((P)\) of \( R \), the answer is affirmative if \((P)\) is the property that the chains generated by \( R \) are well-ordered (see [4]) or if \((P)\) is the property that \( x^* \in X \) is a maximal element of \( R \) (see Proposition 4 below). Addressing a slightly different question, we might wish to find conditions under which the properties which \( R \) or \( X \) satisfies to be inherited when one passes to a linear extension of \( R \). For example, Fucks [15, Corollary 13] finds conditions under which homogeneous partial orders can be extended to homogeneous linear orders. Kontolatou and Stabakis [21] give an analogue of
the Szpilrajn Lemma for partially ordered abelian groups. On the other hand, many other papers in the literature deal with the characterization of the set of binary relations which have an ordering extension that satisfies some additional conditions. See, among others, Denuyynck \[8\] for the additional conditions of convexity, monotonicity and homotheticity and Denuyynck and Lauwers \[9\] for the condition of linearity. If \(X\) is endowed with some topology \(\tau\) one is mainly interested in continuous or semicontinuous linear orders or preorders instead of only linear orders or preorders. In this case, two natural problems have to be discussed: (a) Let \(R\) be a continuous binary relation defined on a topological space \((X, \tau)\). Determine necessary and sufficient conditions for \(R\) to have a continuous linear order extension; (b) Determine necessary and sufficient conditions for \(\tau\) on \(X\) to have the Szpilrajn property (every continuous binary relation \(R\) on \(X\) has a continuous linear order extension). In this direction, some authors utilize the method of Szpilrajn to find the conditions under which \(\tau\) is preserved in the extended relation. For example, Jaffray \[20\] and Bossert, Sprumont, and Suzumura \[5\] provide conditions for the existence of upper semicontinuous extensions of strict (or weak) orders and consistent binary relations, respectively and Herden and Pallack \[18\] provide conditions for the existence of continuous extensions.

In conclusion, there are many types of conditions that one may wish to preserve, or to achieve, in an extension process. These include:

(i) Order theoretic conditions (consistency, acyclicity, transitivity, completeness, e.t.c.);

(ii) Topological conditions (continuity, openness or closedness of the preference sets);

(iii) linear-space conditions (convexity, homogeneity, translation-invariance).

In the following, we call as inherited type extensions theorems all these theorems that preserve, or achieve in an extension process the properties of the original space. The main feature of these theorems, is that they don’t use in their proof the Szpilrajn’s Theorem.

On the other hand, many authors give generalizations of the Szpilrajn’s result by utilizing the original theorem. In what follows, we refer to such results as existence type extension theorems. Arrow \[1, page 64\], Hansson \[17\] and Fishburn \[14\] prove on the basis of the original Szpilrajn’s extension theorem that the result remains true if asymmetry is replaced with reflexivity, that is, any quasi-ordering has an ordering extension. While the property of being a quasi-ordering is sufficient for the existence of an ordering extension of a relation, this is not necessary. As shown by Suzumura \[29\], consistency, as it is defined by him, is necessary and sufficient for the existence of an ordering extension. The existence type extension theorems have played an important role in the theory of choice. One way of assessing whether a preference relation is rational is to check whether it can be extended to a transitive and complete

\[1\] It is well known that the economic approach to rational behaviour traditionally begins with a preference relation \(R\) and determines the optimal choice function \(F\) from \(R\). Revealed
relation (see \cite{7} and \cite{24}). In addition, the Szpilrajn’s existence type theorems are applied: (i) By Stehr \cite{27} to characterize the global orientability; (ii) By Sholomov \cite{26} to characterize ordinal relations; (iii) By Nehring and Puppe \cite{22} on a unifying structure of abstract choice theory; (iv) By Blackorby, Bossert and Donaldson \cite{3} in pure population problems e.t.c.

Dushnik and Miller \cite{13} use the Szpilrajn’s Theorem to prove the following result:

**Theorem 2** Let $\leq$ be a partial order on a set $X$. Then, there exists a collection of linear extensions $\mathcal{F}$ of $R$ such that: (\(\alpha\)) The intersection of the members of $\mathcal{F}$ coincides with $\leq$ and: (\(\beta\)) for every pair of elements $x, y \in X$ with $x$ incomparable to $y$, there exists an $Q \in \mathcal{F}$ with $(x, y) \in Q$.

A family $\mathcal{F}$ of linear extensions of $\leq$ which satisfies conditions (\(\alpha\)) and (\(\beta\)) is called a realizer of $R$. By the theorem of Szpilrajn, for every pair $(x, y)$ of incomparable elements of $R$ we choose two linear extensions $\leq_{xy}$ and $\leq_{yx}$ for which there holds $x \leq_{xy} y$ and $y \leq_{yx} x$. Then, the intersection of all linear orderings $\leq_{xy}$ and $\leq_{yx}$, where $(x, y)$ runs through the set of all pairs of incomparable elements of $R$ is the relation $\leq$. But, this set of linear extensions, has many more elements than necessary. As a consequence of what we have said above, by using the notion of uncovered pair, one can obtain a partial order $\leq$ with the intersection of a reduced number of its linear order extensions. The concept of a realizer $\mathcal{F}$ of $R$ leads to the definition of dimension of $\leq$. According to Dushnik and Miller, the *dimension* of a partial order $\leq$ is defined as the minimum size of a realizer of $\leq$. In fact, the Dushnik-Miller’s theorem provides a procedure that represent binary relations as an intersection of a number of linear order extensions equal to its dimension. In what follows, given a binary relation $R$, a Dushnik-Miller *existence type extension theorem* means that there exists a collection of linear extensions $\mathcal{F}$ of $R$ whose intersection is $R$ and a Dushnik-Miller *inherited type extension theorem* means that $R$ has a realizer.

Much of economic and social behavior observed is either group behavior or that of an individual acting for a group. Group preferences may be regarded as derived from individual preferences, by means of some process of aggregation. For example, if all voters agree that some alternative $x$ is preferred to another alternative $y$, then the majority rule will return this ranking. In this case, there is one simple condition that is nearly always assumed called the principle of unanimity or Pareto principle. This declares that the preference relation for a group of individuals should include the intersection of their individual preferences. Another example of the use of intersections is in the description of preference theory provides another axiomatic approach to rational behavior by reversing the above procedure.

In particular Szpilrajn theorem is the main tool for proving a known theorem of Richter that establishes the equivalence between rational and congruous consumers.

Let $(R_1, R_2, \ldots, R_n)$ be a fixed profile of the individual preference relations. A binary relation $Q$, is called Pareto unanimity relation, if

\[ xQy \iff xR_iy \text{ for all } i \in \{1, 2, \ldots, n\} \text{ and all } x, y \in X. \]

If $R_1, R_2, \ldots, R_n$ are transitive then $Q$ is quasi-transitive.
simple games which can be represented as the intersection of weighted majority games [16]. Dushnik-Miller existence type theorems have been given by many authors. For example, the sufficient part of Suzumuras’s extension result, was subsequently used by Donaldson and Weymark [11] in their proof that every quasi-ordering is the intersection of a collection of orderings; this result extends Dushnik and Miller’s fundamental observation on intersections of strict linear orders. Duggan [12] proves a general Dushnik-Miller existence type theorem from which the above results -and several new ones- can be obtained as special cases. On the existence of a social welfare ordering for a fixed profile in the sense of Bergson and Samuelson, Weymark [31] applies Dushnik and Miller extension theorem in order to prove a generalization of Moulin’s Pareto extension theorem.

In this paper, we introduce the notion of A(m)-consistency, where m belongs to the set of all ordinals less than or equal to the first infinite ordinal ω and we characterize: (a) The existence of a general inherited type theorem on extending binary relations and: (b) The existence of a realizer for a binary relation. The results of the two given general inherited type theorems on extending binary relations, namely, the Szpilrajn type extension theorem and the Dushnik-Miller type extension theorem, generalize all the well known existence and inherited type extension theorems in the literature. We also give examples in a general context to highlight the importance of the inherited type extension theorems and to illustrate its difference from the notion of the existence type extension theorems.

2 Notations and definitions

Let X be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary relation on X. We sometimes abbreviate $(x, y) \in R$ as $xRy$. The composition of two binary relations $R_1$ and $R_2$ is given by $R_1 \circ R_2$ where $(x, y) \in R_1 \circ R_2$ if and only if there exists $z \in X$ such that $(x, z) \in R_1$ and $(z, y) \in R_2$. A binary relation $R$ can always be composed with itself, that is $R \circ R = R^2$. This can be generalized to a relation $R^m$ on X where $R^m = R \circ R \circ \ldots \circ R$ (m-times). Let $P(R)$ and $I(R)$ denote, respectively, the asymmetric part of $R$ and the symmetric part of $R$, which are defined, respectively, by $P(R) = \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \notin R\}$ and $I(R) = \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \in R\}$. Let also $\Delta = \{(x, x) | x \in X\}$ denotes the diagonal of X. An element $x \in X$ is called maximal if for all $y \in X$, $yRx$ implies $xRy$. We say that R on X is (i) reflexive if for each $x \in X$ $(x, x) \in R$; (ii) irreflexive if we never have $(x, x) \in R$; (iii) transitive if for all $x, y, z \in X$, $(x, z) \in R$ and $(z, y) \in R \implies (x, y) \in R$; (iv) antisymmetric if for each $x, y \in X$, $(x, y) \in R$ and $(y, x) \in R \implies x = y$; (v) total if for each $x, y \in X$, $x \neq y$ we have $xRy$ or $yRx$. (vi) complete if for each $x, y \in X$, we have $xRy$ or $yRx$. It follows that $R$ is complete if and only if it is reflexive and total. The transitive closure of a relation $R$ is denoted by $\overline{R}$, that is for all $x, y \in X$, $(x, y) \in \overline{R}$
if there exist \( m \in \mathbb{N} \) and \( z_0, \ldots, z_m \in X \) such that \( x = z_0, (z_k, z_{k+1}) \in R \) for all \( k \in \{0, \ldots, m - 1\} \) and \( z_m = y \). Clearly, \( \overline{R} \) is transitive and, because the case \( m = 1 \) is included, it follows that \( R \subseteq \overline{R} \). Acyclicity says that there do not exist \( m \) and \( z_0, z_1, \ldots, z_m \in X \) such that \( x = z_k, (z_k, z_{k+1}) \in R \) for all \( k \in \{0, \ldots, m - 1\} \) and \( z_m = y \). The relation \( R \) is \( S \)-consistent (consistent in the sense of Suzumura [20]), if for all \( x, y \in X \), for all \( m \in \mathbb{N} \), and for all \( z_0, z_1, \ldots, z_m \in X \), if \( x = z_k, (z_k, z_{k+1}) \in R \) for all \( k \in \{0, \ldots, m - 1\} \) and \( z_m = y \), we have that \( (y, x) \notin P(R) \). The following combination of properties are considered in the next theorems. A binary relation \( R \) on \( X \) is (i) quasi-ordering if \( R \) is reflexive and transitive; (ii) ordering if \( R \) is a total quasi-ordering; (iii) partial order if \( R \) is an antisymmetric quasi-ordering; (iv) linear order if \( R \) is a total partial order; (v) strict partial order if \( R \) is irreflexive and transitive. (vi) strict linear order if \( R \) is a total strict partial order. A binary relation \( Q \) is an extension of a binary relation \( R \) if and only if \( R \subseteq Q \) and \( P(R) \subseteq P(Q) \). If an extension \( Q \) of \( R \) is an ordering, we call it an ordering extension of \( R \), and if \( Q \) is an extension of \( R \) that is a linear order, we refer to it as a linear order extension or \( R \). In fact, an extension \( Q \) of \( R \) subsumes all the pairwise information provided by \( R \), and possibly further information.

The following definitions may be seen as natural extensions of classical definitions used in the partial order case. Let \( \text{inc}(R) = \{(x, y) \in X \times X | (x, y) \notin R \text{ and } (y, x) \notin R \} \) be the set of incomparable pairs of \( R \). The set of all of the linear extensions of \( R \) is denoted by \( Q \). For \( (x, y) \) and \( (\kappa, \lambda) \in \text{inc}(\overline{R}) \) we write \( (x, y) \text{ covers } (\kappa, \lambda) \)- in words \( (x, y) \) covers \( (\kappa, \lambda) \)- if for every linear extension \( Q \) of \( R \), \( (x, y) \in Q \) implies \( (\kappa, \lambda) \in Q \). We call a maximal element \((x^*, y^*)\) of \( \text{inc}(\overline{R}), F \), i.e., \((x^*, y^*) \in M(\text{inc}(\overline{R}), F)\), an uncovered pair of \( R \). By \( F(x, y) \) we denote the set \( \{(\kappa, \lambda) \in \text{inc}(\overline{R}) | (x, y), (\kappa, \lambda) \in F \} \). Any subset \( F \subseteq Q \) is a realizer of \( R \) if and only if: (a) The intersection of the members of \( F \) coincides with \( R \) and: (b) for every pair \( (x, y) \in \text{inc}(\overline{R}) \), \( x, y \in X \), there exists an \( Q \in F \) with \( (x, y) \in Q \). The dimension of a binary relation \( R \) (see [13, Page 601]) is the smallest number of linear orderings whose intersection is \( \overline{R} \).

Let \( R \) be a binary relation defined on a topological space \((X, \tau)\). We say that \( R \) is continuous, if it is a closed subset of \( X \times X \). This is the same thing as saying that for every point \( x \in X \), both sets \{\( y \in X | xRy \)\} and \( \{y \in X | yRx\} \) are closed subsets of \( X \) (see [25, Proposition 1]). We say that \( R \) is upper semicontinuous if for all \( y \in X \), the set \( \{x \in X | (x, y) \in P(R)\} \) is open in \( X \). In general, there is no relationship between a binary relation and a topology on a space. However, there is one topology that is inherently connected with a total order \( R \), called the order topology, which is generated by the subbase consisting of all sets of the form \( \{ x \in X | xP(R)a \} \) and \( \{ x \in X | bP(R)x \} \), where \( a \) and \( b \) are points of \( X \). The space \((X, \tau)\) is compact if for each collection of \( \tau \)-open sets which cover \( X \) there exists a finite subcollection that also covers \( X \).
3 The extension theorems

In the context of examining if the individualistic assumptions used in economics can be used in the aggregation of individual preferences ([1, Definition 5, Theorem 2], Arrow proved a key lemma that extends the famous Szpilrajn’s Theorem.

**Arrow’s lemma.** [1, pp. 64-68]. Let \( R \) be a quasi-ordering on \( X, Y \) a subset of \( X \) such that, if \( x \neq y \) and \( x, y \in Y \), then \( (x, y) \notin R \), and \( T \) an ordering on \( Y \). Then, there exists an ordering extension \( Q \) such that \( Q/Y = T \).

In fact, the lemma says that, if \( R \) is a binary relation defined on a set of alternatives \( X \), then given any ordering \( T \), to any subset \( Y \) of incomparable elements of \( R \), there is a way of ordering all the alternatives which will be compatible both with \( R \) and with the given ordering \( T \) in \( Y \). In this case, it is important that the linear extension of \( R \) inherits the relationship we put between the incomparable elements of \( R \).

Arrow’s generalization of the Szpilrajn’s extension theorem as well as all the well known generalizations of this theorem, use in their proof the Szpilrajn theorem itself. This procedure lead us in existence type extension theorems.

In the following \( \omega \) denotes the first infinite ordinal which comes after all natural numbers, that is, the order type of the natural numbers under their usual linear ordering. By \( \Omega_0 \) we denote the set \( \{\omega, 1, 2, 3, \ldots\} \).

A great deal of work in computational economics and Computational social science has been done in an attempt to find a fast algorithm to count the exact number of linear extensions of a partial order, as well as, to find an efficient algorithm to compute the dimension of a partial order. In this direction, we give two general inherited type extension theorems, by reducing the path length of the transitive closure in the definition of \( S \)-consistency to a minimum, without losing information. To be more precise, we give the following definition.

**Definition 1** Let \( R \) be a binary relation on a set \( X \), let \( m \in \Omega_0 \) and let \( x, y \in X \). We say that: (i) \( R \) is \( m \)-consistent if and only if \( P(R) \subseteq P(R^m) \).

**Remark 1** If \( R \) is \( A(m) \)-consistent, then it is \( m' \)-consistent for all \( 1 \leq m' \leq m \). Therefore, if there exist \( x, g_0, g_1, \ldots, g_m \in X \) such that \( x = g_0, (g_k, g_{k+1}) \in R \) for all \( k \in \{0, \ldots, m'-1\} \), and \( g_{m'} = x \), then we have that \( (g_k, g_{k+1}) \in I(R) \).

The following proposition is evident from Definition (i).

**Proposition 1** Let \( X \) be a non empty set and let \( m \in \Omega_0 \). A binary relation \( R \) on \( X \) is \( m \)-consistent if and only if \( P(R) \subseteq P(R^m) \).
If \( m = \omega \), then \( R^o = \bigcup_{k=1}^{\infty} R^k = \overline{R} \). Since \( I(R^o) = I(\overline{R}) = I((R^o)^c) \) holds for all ordinals \( \omega' \geq \omega \), Definition 1 and Proposition 1 imply the following proposition.

**Proposition 2** A binary relation \( R \) is \( A(\omega) \)-consistent if and only if \( R \) is \( S \)-consistent.

As shows in the following example, an \( m \)-consistent binary relation is not an \( S \)-consistent one.

**Example 1** Let \( X = \{ x_1, x_2, x_3, x_4, x_5 \} \) and
\[
G = \{ (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1), (x_2, x_3) \}.
\]

Clearly, \( G \) is a \( 2 \)-consistent binary relation but not an \( S \)-consistent one.

**Theorem 3** Let \( R \) be a binary relation on \( X, m \in \Omega_\nu \) and \( Y \) a subset of \( X \) such that, if \( x \neq y \) and \( x, y \in Y \), then \( (x, y) \notin \overline{R} \), and \( T \) an ordering on \( Y \). Then, there exists an ordering extension \( Q \) of \( R \) such that \( Q/Y = T \) if and only if \( R \) is a \( A(m) \)-consistent binary relation.

**Proof** To prove necessity, let \( R \) be a \( A(m) \)-consistent binary relation on \( X \). Without loss of generality, we can assume that \( R \) is reflexive (see 30, Lemma, Page 387). We put
\[
R^* = R \cup \{ (\kappa, \lambda) | \kappa \overline{R} \lambda \} \text{ and } x \overline{R} \lambda \text{ where } x, y \in Y \text{ and } (y, x) \in T = R \cup \hat{R}.
\]

Since \( R \) is reflexive, we have \( (y, x) \in \hat{R} \) and \( x \neq y \) for all \( x, y \in Y \). It is easy to check that
\[
(R^*)^m = R^m \cup \{ (\kappa, \lambda) | \kappa \overline{R} \lambda \} \text{ and } x \overline{R} \lambda \text{ where } x, y \in Y \text{ and } (y, x) \in T = R^m \cup \hat{R}.
\]

By the definition of \( R^* \), we have \( \kappa \neq \lambda \), because otherwise \( (x, y) \notin \overline{R} \) or \( T = \overline{R} \), a contradiction. We first prove that \( R^* R^* \) is \( A(m) \)-consistent. Indeed, suppose to the contrary that there are alternatives \( \nu, z_0, z_1, z_2, \ldots, z_m \in X \) such that
\[
\nu = z_0 P(R \cup \hat{R}) z_1 (R \cup \hat{R}) z_2 \ldots (R \cup \hat{R}) z_m = \nu.
\]

Since \( R \) is \( A(m) \)-consistent, there must exists \( k = 0, 1, 2, \ldots, m - 1 \) such that \( (z_1, z_{k+1}) \in \hat{R} \) and for all \( i \in \{0, 1, 2, \ldots, m - 1\} \) with \( i \neq k \), \( (z_i, z_{k+1}) \in R^* \) if and only if \( (z_i, z_{k+1}) \in R \). It follows that \( (x, y) \notin \overline{R} \), a contradiction. It remains to prove that for each \( n \geq m \) there holds \( I((R^*)^n) = I((R^*)^m) \). Let \( \kappa, \lambda \in X \) and \( n \in \mathbb{N} \) be such that \( (\kappa, \lambda) \in I((R^*)^m) \). Then, we have four cases to consider:

- **Case 1.** \( (\kappa, \lambda) \in R^m \) and \( (\lambda, \kappa) \in R^m \). It follows that \( (\kappa, \lambda) \in I((R^m) \subseteq I((R^*)^m) \).

- **Case 2.** \( (\kappa, \lambda) \in R^m \), \( (\lambda, \kappa) \in \hat{R} \). It follows that \( (\kappa, \lambda) \in R^m \), \( (x, \kappa) \in \overline{R} \) and \( (\lambda, y) \in \overline{R} \). Therefore, \( (x, y) \in \overline{R} \) which is impossible.
Case 3. It is similar to case 2.

Case 4. In this case we have \((\kappa, y) \in R, (x, \lambda) \in R, (\lambda, y) \in R\) and \((x, \kappa) \in R\). It follows that \((x, y) \in R\) which is impossible. Therefore, \(I((R^*)^n) = I(R^n) = I(R^n) \subseteq I((R^*)^n)\). The last conclusion shows that \(R^*\) is a \(\Lambda(m)\)-consistent binary relation on \(X\) satisfying \(R \cup T \subseteq R^*\). We now prove that \(R^*\) is an extension of \(R\), that is, \(R \subseteq R^*\) and \(P(R) \subseteq P(R^*)\). The first is obvious from the definition of \(R^*\). To prove the second, let \((\kappa, \lambda) \in P(R)\). Then, \((\kappa, \lambda) \in P(R) \subseteq R \subseteq R^*\). Suppose to the contrary that \((\kappa, \lambda) \notin P(R^*)\). It follows that \((\lambda, \kappa) \in R^*\). We have two cases to consider: (a) \((\lambda, \kappa) \in R\); (b) \(xRy\) and \(xR\kappa\).

In case (i), we have a contradiction to \((\kappa, \lambda) \in P(R)\). In case (ii), \(xRy\) and \(xR\kappa\) jointly to and \((\kappa, \lambda) \in P(R)\) implies that \((x, y) \in R \circ P(R) \circ R \subseteq R\) which is impossible. The last contradiction shows that \((\kappa, \lambda) \in P(R^*)\) which implies that \(P(R) \subseteq P(R^*)\).

Suppose that \(\tilde{R} = \{\tilde{R}_i|i \in I\}\) denote the set of \(\Lambda(m)\)-consistent extensions of \(R\) such that \(R \cup T \subseteq \tilde{R}_i\). Since \(R^* \subseteq \tilde{R}_i\) we have that \(\tilde{R}_i \neq \emptyset\). Let \(Q = (Q_i)_{i \in I}\) be a chain in \(\tilde{R}\), and let \(\hat{Q} = \bigcup_{i \in I} Q_i\). We prove that \(\hat{Q} \in \tilde{R}\). Clearly, \(R \cup T \subseteq \hat{Q}\).

To prove that \(\hat{Q}\) is a \(\Lambda(m)\)-consistent extension of \(R\), we first show that \(\hat{Q}\) is \(m\)-consistent. Indeed, suppose to the contrary that there are alternatives \(\mu, \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\) in \(X\) such that

\[\mu = \gamma_0 P(\hat{Q}) \gamma_1 \hat{Q} \gamma_2 \cdots \hat{Q} \gamma_m \hat{Q} \gamma_m = \mu.\]

Consider the largest \(i\) for which there exist such \(\mu, \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{m+1}\). It follows that \(Q_i\) is non \(m\)-consistent, a contradiction. Therefore, \(\hat{Q}\) is \(m\)-consistent. On the other hand, if \(n \geq m\), then \(I(Q_i^n) = I(Q_i^m)\) for all \(i \in I\) implies that \(I(\bigcup_{i \in I} Q_i^n) = I\left(\bigcup_{i \in I} Q_i^m\right)\). Indeed, let \(\kappa, \lambda \in X\) such that \((\kappa, \lambda) \in I(\bigcup_{i \in I} Q_i^n)\) and \((\kappa, \lambda) \notin I\left(\bigcup_{i \in I} Q_i^m\right)\). Since \((Q_i)_{i \in I}\) is a chain, there exists \(i^* \in I\) such that \((\kappa, \lambda) \in I(Q_{i^*}^n)\) and \((\kappa, \lambda) \notin I(Q_{i}^m)\), a contradiction to \(I(Q_{i^*}^n) = I(Q_{i}^m)\). The last conclusion shows that \(\hat{Q}\) is a \(\Lambda(m)\)-consistent. We now prove that \(P(R) \subseteq P(\hat{Q})\). Take any \((\kappa, \lambda) \in P(R)\) and suppose to the contrary that \((\kappa, \lambda) \notin P(\hat{Q})\). Clearly, \(\kappa \neq \lambda\) and for each \(i \in I\), \((\kappa, \lambda) \in Q_i\).

Since \((\kappa, \lambda) \notin P\left(\bigcup_{i \in I} Q_i\right)\) we conclude that \((\lambda, (\kappa, \lambda) \in Q_i\). Hence, \((\lambda, (\kappa, \lambda) \in Q_i\) for some \(i^* \in I\), a contradiction to \((\kappa, \lambda) \in P(R) \subseteq P(Q)\). Therefore, \(\hat{Q}\) is a \(\Lambda(m)\)-consistent extension of \(R\) such that \(R \cup T \subseteq \hat{Q}\). By Zorn's lemma \(\tilde{R}\) possesses an element, say \(Q\), that is maximal with respect to set inclusion.

We prove that \(\tilde{Q}\) is a ordering extension of \(R\) satisfying the requirements of theorem. Since \(\tilde{Q}\) is reflexive and transitive, it remains to prove that: (a) \(\tilde{Q}\) is total and (b) \(P(Q) \subseteq P(\tilde{Q})\). To prove (a), take any \(x, y \in X\) such that \((x, y) \notin \tilde{Q}\) and \((y, x) \notin \tilde{Q}\). Then, we have two subcases to consider: (a) \((x, y) \notin \tilde{Q}\) or \((y, x) \notin \tilde{Q}\). In subcase (a), we have \((x, y) \notin T\). By the
completeness of \( T \) we conclude that \((y, x) \in T \subseteq Q \subseteq \overline{Q}\), a contradiction. In subcase \((\alpha)\), if \((x, y) \notin \overline{Q}\) and \((y, x) \notin Q\), we define

\[
Q^* = Q \cup \{(\kappa, \lambda)|\kappa \overline{Q} y \quad \text{and} \quad x \overline{Q} \lambda\}.
\]

Then, as in the case of \( R^* \) above, where \( Y^* = \{x, y\} \) and \( T^* = \{(y, x)\} \) play the role of \( Y \) and \( T \) respectively, we conclude that \( Q^* \) is a \( A(m)\)-consistent extension of \( R \), a contradiction to the maximality of \( Q \). The last contradiction shows that \( \overline{Q} \) is complete (total and reflexive). To prove \((\beta)\), we first prove that \( \overline{Q} = Q^m \bigcup_{p=m+1}^{\infty} P(Q^p) \). Clearly, \( Q^m \bigcup_{p=m+1}^{\infty} P(Q^p) \subseteq \overline{Q} \).

To prove the converse, suppose to the contrary that \((\kappa, \lambda) \notin \overline{Q} \) and \((\kappa, \lambda) \notin Q^m \bigcup_{p=m+1}^{\infty} P(Q^p) \). Since \((\kappa, \lambda) \in \overline{Q} \setminus Q^m \), there exists \( \rho > m \) such that \((\kappa, \lambda) \in Q^\rho \). On the other hand, \((\kappa, \lambda) \notin \bigcup_{p=m+1}^{\infty} P(Q^p) \) implies that \((\kappa, \lambda) \in I(Q^\rho) \). By \( m\)-rank equivalence we have \( I(Q^\rho) = I(Q^m) \), a contradiction to \((\kappa, \lambda) \notin Q^m \). Therefore, \( \overline{Q} = Q^m \bigcup_{p=m+1}^{\infty} P(Q^p) \). To prove that \( P(Q) \subseteq P(K) \), suppose to the contrary that \((\kappa, \lambda) \in P(Q) \subseteq P(Q^m) \) and \((\kappa, \lambda) \notin P(K) \). It follows that \((\kappa, \lambda) \in Q^m \bigcup_{p=m+1}^{\infty} P(Q^p) \).

Since \((\kappa, \lambda) \in P(Q^m)\), we conclude that \((\lambda, \kappa) \in \bigcup_{p=m+1}^{\infty} P(Q^p) \). It follows that \((\kappa, \lambda) \in I(Q^q) \) for some \( q > m \). But then, \((\kappa, \lambda) \in I(Q^q) = I(Q^m) \), a contradiction to \((\lambda, \kappa) \notin Q^m \). Therefore, \( P(Q) \subseteq P(K) \). To complete the sufficiency part we show that \( \overline{Q}/Y = T \). Evidently, \( T \subseteq \overline{Q}/Y \). To prove the converse, let \((\kappa, \lambda) \in \overline{Q}/Y \). Suppose to the contrary that \((\kappa, \lambda) \notin T \). Since \( T \) is complete \((\lambda, \kappa) \in T \) holds which implies \((\lambda, \kappa) \in R^* \). On the other hand, \((\kappa, \lambda) \notin T \) and \((\kappa, \lambda) \notin \overline{Q} \) imply that \((\kappa, \lambda) \notin R^* \). Since \( Q \) is an ordering extension of \( R^* \), we have that \((\lambda, \kappa) \in P(R^*) \subseteq P(Q) \subseteq P(K) \). It follows that \((\kappa, \lambda) \notin \overline{Q}/Y \), a contradiction. The last contradiction shows that \( \overline{Q}/Y = T \).

In order to prove sufficiency, let us assume that \( R \) has an ordering extension \( Q \) satisfying the requirements of the theorem. Then, \( R \) is \( S\)-consistent and thus \( A(\omega)\)-consistent. Indeed, suppose to the contrary that there are alternatives \( \tau, \pi_0, \pi_1, \pi_2, ..., \pi_n \in X \) such that

\[
\tau = \pi_n P(R) \pi_1 R \pi_2 R ... R \pi_n R \pi_0 = \tau.
\]

Since \( Q \) is an ordering extension of \( R \) we have \( \tau P(Q) \tau \) which is impossible. Therefore, \( R \) is \( S\)-consistent. The last conclusion completes the proof.

**Remark 2** According to Theorem \( m\)-consistency ensures the existence of a reflexive and complete (tournament) extension of \( R \) and it has nothing to do
with the existence of transitivity. As we can see, the relation $G$ of example 1 has a complete extension, the relation

\[
G^* = G^2 \cup \left( \bigcup_{p=3}^{\infty} P(G^p) \right) = G^2 \cup P(G^3) \cup P(G^4) = \\
\{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_3), (x_1, x_1), (x_1, x_4), (x_2, x_2), (x_1, x_5), (x_1, x_3), (x_3, x_1), (x_4, x_1), (x_2, x_4), (x_4, x_2), (x_4, x_5), (x_3, x_5), (x_2, x_5)\}
\]

This means that Theorem 3 guarantees a complete extension $G^*$ without $G$ being $S$-consistent.

The following corollary is an immediate consequence of Theorem 3 for $Y = \{x, y\}$ and $T = \{(y, x)\}$.

**Corollary 1** Let $R$ be a $\Lambda(m)$-consistent binary relation on $X$, $m \in \Omega_0$. Then, for every pair $(x, y) \in \text{inc}(\overline{R})$, $x, y \in X$, there exists an ordering extension $Q_{xy}$ of $R$ such that $(x, y) \in Q_{xy}$.

By interchanging the roles of $x$ and $y$ ($\text{inc}(\overline{R})$ is symmetric), Corollary 1 gives an analogous result.

Since transitivity implies $\Lambda(m)$-consistency, $m \in \Omega_0$, Arrow’s Lemma is an immediate consequence of the sufficient part of Theorem 3 for $m = 1$.

**Definition 2** For each $m \in \mathbb{N}$, a $\Lambda(m)$-consistent binary relation $R$ on $X$ is $\Delta(m)$-consistent if $I(R^m) = \Delta$.

A consequence of Proposition 2 and Theorem 3 for $m = \omega$ is also the Suzumura’s existence type extension theorem in [28, Page 5]. The following corollary shows this fact.

**Corollary 2** Let $R$ be a binary relation on $X$, $Y$ a subset of $X$ such that, if $x \neq y$ and $x, y \in Y$, then $(x, y) \notin \overline{R}$, and $T$ an ordering on $Y$. Then, there exists an ordering extension $Q$ of $R$ such that $Q/Y = T$ if and only if $R$ is $S$-consistent.

As a consequence of the proof of Theorem 3 the following result is also true:

**Corollary 3** Let $R$ be a binary relation on $X$, $m \in \Omega_0$ and $Y$ a subset of $X$ such that, if $x \neq y$ and $x, y \in Y$, then $(x, y) \notin \overline{R}$, and $T$ a linear order on $Y$. Then, there exists a linear order extension $Q$ of $R$ such that $Q/Y = T$ if and only if $R$ is a $\Delta(m)$-consistent.

**Proof** According to Proposition 1 and Theorem 3 there exists an ordering extension $Q$ of $R$ such that $P(R) \subseteq P(R^m) \subseteq P(Q)$ and $Q/Y = T$. Let $\approx$ be the equivalence relation defined by

\[x \approx y \text{ if and only if } (x, y) \in I(Q).\]
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The quotient set by this equivalence relation \( \approx \) will be denoted \( \bar{X} = X^\approx \), and its elements (equivalence classes) by \([x]\). There exists on \( X \) a linear order \( \Omega \) defined by:

\[
(\forall x, y \in X) \ (x \Omega y \iff \exists x', y' \in X, x \in [x'], y \in [y'], [x'] \approx y')
\]

An asymmetric, \( \Lambda(m) \)-consistent binary relation \( \approx \) is defined by:

\[
\forall [x], [y] \in X^\approx ([x] \approx [y] \iff \exists x' \in [x], \exists y' \in [y], x' \approx y').
\]

According to Corollary 4 there exists a strict linear order extension \( \bar{Q}^\approx \) of \( R^\approx \). Therefore, as in the proof of Theorem 1 [20, Pages 399-400], by using a subbase of \( \tau \), we construct a strict linear order extension \( R^* \) of \( R \) such that for each \( y \in X \) the set \{ \( x \in X | xR^* y \) \} belongs to \( \tau \).

We prove that \((x, y) \in I(R^m) = \Delta\), which implies that \( Q \) is antisymmetric and thus it is a linear order. Suppose to the contrary that \((x, y) \notin I(R) = I(R^m)\). Then, since \((x, y) \in P(R^m) \subseteq P(Q)\) and \((y, x) \in P(R^m) \subseteq P(Q)\) is impossible, we conclude that \((x, y) \notin R^m\) and \((y, x) \notin R^m\).

But then, \( x, y \in Y \) and \((x, y) \in T \) or \((y, x) \in T \) which implies that \( xP(Q)y \) or \( yP(Q)x \) which is impossible. The last contradiction shows that \( Q \) is a linear extension of \( R \).

Conversely, if there exists a linear order extension \( Q \) of \( R \), then by Theorem 3 \( R \) is \( \Lambda(\omega) \)-consistent. It remains to show that \( I(R) = \Delta \). Suppose to the contrary that \( I(R) \neq \Delta \). This implies that, there exist \( x, y \in X \), \( x \neq y \), such that

\[
(x, y) \in I(R) \subseteq I(Q) \text{ and } (x, y) \notin \Delta
\]

which contradicts the anti-symmetry of \( Q \).

If \( R \) is a partial order, Corollary 3 implies one of the main results of [19, Theorem 2.2].

As a corollary to Theorem 3 we also obtain the following well known inherited type extension theorem of Szpilrajn [30].

**Corollary 4** Every (strict) partial order \( R \) possesses a (strict) linear order extension \( Q \). Moreover, if \( x \) and \( y \) are any two non-comparable elements of \( R \), then there exists a (strict) linear order extension \( Q' \) in which \( xQ'y \) and a (strict) linear order extension \( Q'' \) in which \( yQ''x \).

**Proof** It is an immediate result of Theorem 3 for \( Y = \{x, y\} \), \( T = \{(x, y)\} \), \( m = 1 \) and \( R \) being (asymmetric and transitive) reflexive, transitive and anti-symmetric.

Since transitivity implies \( \Lambda(m) \)-consistency for all \( m \in \Omega_\alpha \), the following corollary is an immediate consequence of Theorem 3.

**Corollary 5** (Hanson [17] and Fishburn [14]). Every quasi-ordering has an ordering extension.
4 Refinements of Szpilrajn’s type theorems

In this paragraph, we give a general Dushnik-Miller inherited type extension theorem in which all the well known Dushnik-Miller extension theorems are obtained as special cases.

Theorem 4 Let $R$ be a binary relation on $X$. Then, $\overline{R}$ has as realizer the set of ordering extensions of $R$ if and only if $R$ is $\Lambda(m)$-consistent for some $m \in \Omega_n$.

Proof To prove necessity, let $R$ be a $\Lambda(m)$-consistent binary relation on $X$ for some $m \in \Omega_n$ and let $Q$ be the set of all order extensions of $R$. By Theorem $\exists Q$ is non-empty. We show that $\overline{R} = \bigcap Q$. Indeed, since $\overline{R} \subseteq \bigcap Q$, we have to show that $\bigcap Q \subseteq \overline{R}$. Suppose to the contrary that there exists an $(x, y) \in \bigcap Q$ with $(x, y) \notin \overline{R}$. We first prove that $(y, x) \notin \overline{R}$. Suppose to the contrary that $(y, x) \in \overline{R}$. Since $(y, x) \in P(R) \subseteq P(Q)$, this set is non-empty. Let $\overline{T} \subseteq \overline{R}$ be the set of transitive extensions of $R$. Since $\overline{R} \in \overline{T}$, this set is non-empty. Then, as in the proof of Theorem 1, there exists a maximal element $\hat{T}$ of $\overline{T}$. We prove that $\hat{T}$ is complete. Suppose to the contrary that $(x^*, y^*) \notin \hat{T}$ and $(y^*, x^*) \notin \hat{T}$ for some $x^*, y^* \in X$. Then, it is easy to check that the relation

$$\hat{T} = \hat{T} \cup \{ (\kappa, \lambda) | \kappa \overline{T} y^* \text{ and } x^* \overline{T} \lambda \}$$

is transitive, a contradiction to the maximal character of $\hat{T}$. Therefore, $\hat{T} \in Q$. Since $(y, x) \in P(\overline{R}) \subseteq P(\hat{T})$, we have $(x, y) \notin \hat{T}$, again a contradiction to $(x, y) \in \bigcap Q$. Therefore, in any case we have $(y, x) \notin \overline{R}$. We now prove that $(x, y) \notin \overline{R}$ jointly to $(y, x) \notin \overline{R}$ leads again to a contradiction, and thus, $\bigcap Q \subseteq \overline{R}$. Indeed, let

$$S = R \cup \{ (\kappa, \lambda) | \kappa \overline{R} y \text{ and } x \overline{R} \lambda \}.$$
Then, since $R$ is $\Lambda(m)$-consistent, as in the proof of Theorem 1, there exists an ordering extension $\hat{S}$ of $R$ such that $P(R) \subseteq P(S) \subseteq P(\hat{S})$. Since $(y, x) \in P(S)$ ($(x, x) \in R$, $(y, y) \in R$, $(x, y) \notin R$ and $(y, x) \notin R$) we have that $(x, y) \notin \hat{S}$, a contradiction to $(x, y) \in \bigcap Q \in Q$. This contradiction confirms that $\bigcap Q \subseteq \overline{R}$.

To finish the proof of necessity, it remains to show that $Q$ is a realizer. But, this is an immediate consequence of the Corollary 1.

To prove sufficiency, suppose that $\overline{R}$ has as realizer the set of all order extensions of $R$, let $\hat{Q}$. We prove that $R$ is $\Lambda(\omega)$-consistent. Indeed, since $\bigcap Q = \overline{R}$ we have $P(R) \subseteq \bigcap Q \subseteq P(\bigcap Q) = P(\overline{R}) = P(R^-)$. Therefore, $R$ is $\Lambda(\omega)$-consistent. The last conclusion completes the proof.

Theorems 3 and 6 and remark 2 imply the following corollary.

**Corollary 6** Let $R$ be a binary relation on $X$. Then, $R^m$ has as realizer the set of reflexive and complete (tournament) extensions of $R$ if and only if $R$ is $m$-consistent for some $m \in \Omega_0$.

The following result is an immediate consequence of Corollary 3.

**Corollary 7** Let $R$ be a binary relation on $X$ and let $m \in \Omega_0$. Then, $\overline{R}$ has as realizer the set of linear order extensions of $R$ if and only if $R$ is $\Delta(m)$-consistent.

The following corollary is an irreflexive variant of Corollary 7.

**Corollary 8** Let $R$ be a binary relation on $X$ and let $m \in \Omega_0$. Then, $\overline{R}$ has as realizer the set of strict linear order extensions of $R$ if and only if $R$ is asymmetric and $\Lambda(m)$-consistent.

**Proof** Suppose that $R$ is an asymmetric and $\Lambda(m)$-consistent binary relation for some $m \in \Omega_0$. Then, $R \cup \Delta$ is $\Delta(m)$-consistent. Let $Q$ be the class of linear order extensions of $R$. Then, $\overline{R} \cup \Delta = \bigcap Q$. It follows that $\overline{R} = \bigcap Q \setminus \Delta$, where $Q \setminus \Delta$ is a strict linear order extension of $R$. Conversely, suppose that $\overline{R}$ is the intersection of all strict linear order extensions of $R$. Then, $R$ is $\Lambda(\omega)$-consistent and asymmetric since $I(R) \subseteq I(\overline{R}) \subseteq I(Q) = \emptyset$.

Next is a result due to Dushnik and Miller [13, Theorem 2.32].

**Corollary 9** If $R$ is any (strict) partial order on a set $X$, then there exists a collection $Q$ of (strict) linear orders on $X$ which realize $R$.

**Proof** This follows immediately from Theorem 8 by letting $R$ to be (strict) partial order.

The next result, proved by Donaldson and Weymark [14], strengthens Fishburn’s Lemma 15.4 in [14] and Suzumura’s Theorem $\Lambda(4)$ in [29].
Corollary 10 Every quasi-ordering is the intersection of a collection of orderings.

Proof It is an immediate consequence of the sufficiency part of Theorem\ref{thm:extension} by letting $m = 1$.

Definition 3 \cite{12} Definition 6. Given relations $R$ and $R'$, $R'$ is a compatible extension of $R$ if $R \subseteq R'$ and $P(R) \subseteq P(R')$.

In what follows, $\mathcal{R}$ denotes the class of binary relations which are compatible extensions of $R$.

We recall the following definitions from \cite{12}.

Definition 4 The class $\mathcal{R}$ is closed upward if, for all chains $C$ in $\mathcal{R}$,

$$\bigcup \{R' | R' \in C\} \in \mathcal{R}.$$  

Definition 5 The class $\mathcal{R}$ is arc-receptive if, for all distinct $s$ and $t$ and for all transitive $R' \in \mathcal{R}$, $(t, s) \notin R'$ implies $\overline{R \cup \{(s, t)\}} \in \mathcal{R}$.

Proposition 3 Assume $\mathcal{R}$ is closed upward and arc-receptive. If $R$ is $\Lambda(m)$-consistent for some $m \in \mathbb{N}$ and $R \in \mathcal{R}$, then

$$\overline{R} = \bigcap\{R' \mid R' \text{ is a complete, transitive extension of } R\}.$$  

Proof To prove the corollary, let $m \in \mathbb{N}$ and $R$ be a $\Lambda(m)$-consistent binary relation such that $\overline{R} \in \mathcal{R}$. It follows from Theorem\ref{thm:extension} that,

$$\overline{R} = \bigcap\{R' \mid R' \text{ is a complete, transitive extension of } R\}.$$  

It remains to prove that $R' \in \mathcal{R}$. Because $R \subseteq R'$ by transitivity of $R'$, we obtain $\overline{R} \subseteq R'$. If $R' = \overline{R}$, then $R' \in \mathcal{R}$. Otherwise, suppose that $\overline{R} \subset R'$. We first show that there exists a transitive extension of $R$, let $Q$, such that $Q \in \mathcal{R}$ and $\overline{R} \subset Q \subseteq R'$. Indeed, assume that $s, t \in X$ are such that $(s, t) \in R' \setminus \overline{R}$. There are two cases to consider: (i) $(t, s) \in R'$; (ii) $(t, s) \notin R'$.

Case (i). $(t, s) \in R'$. In this case, since $\mathcal{R}$ is arc-receptive, $\overline{R} \in \mathcal{R}$ and $(s, t) \notin \overline{R}$, we conclude that $Q = \overline{R \cup \{(s, t)\}} \in \mathcal{R}$. We now prove that $Q$ is a transitive extension of $R$. Since $Q$ is transitive, it suffices to show that $Q$ is an extension of $\overline{R}$. Clearly, $\overline{R} \subseteq Q$. To verify that $P(\overline{R}) \subset P(Q)$, take any $(p, q) \in P(\overline{R})$ and suppose $(p, q) \notin P(Q)$.

Since $(p, q) \in R \subseteq \overline{R \cup \{(t, s)\}}$, this means that $(q, p) \in \overline{R \cup \{(t, s)\}}$. Hence, there exist $z_0, z_1, z_2, \ldots, z_m \in X$ such that

$$q = z_0 \{(R \cup \{(t, s)\})z_1 \{(R \cup \{(t, s)\})z_2 \ldots \{(R \cup \{(t, s)\})z_m = p.$$  

Thus, there exists at least one $k \in \{0, 1, \ldots, m - 1\}$ such that $(z_k, z_{k+1}) = (t, s)$, for otherwise $(q, p) \in \overline{R}$, a contradiction. Let $z_\lambda$ be the first occurrence of $t$ and let $z_\mu$ the last occurrence of $s$. Then, since $(p, q) \in P(\overline{R}) \subseteq \overline{R}$,
\[ s = z_\mu \mathcal{R}_{z_{\mu+1}} \cdots \mathcal{R}_{z_m} = \mu \mathcal{R}_{q_0} \cdots \mathcal{R}_{z_\lambda} = t. \]

Hence, \((s, t) \in \mathcal{R}\), a contradiction. Since \(\mathcal{R}'\) is transitive, \(\mathcal{R} \subset \mathcal{Q} \subseteq \mathcal{R}'\).

Case (ii). \((t, s) \notin \mathcal{R}'\). In this case, we must have \((t, s) \notin \mathcal{R}\), since otherwise, we must have \((t, s) \in \mathcal{R}'\), a contradiction.

Let \(\mathcal{Q} = \mathcal{R} \cup \{(s, t)\}\). Then, as in the case (i), we obtain \(\mathcal{Q} \in \mathcal{R}\) and \(\mathcal{R} \subset \mathcal{Q} \subseteq \mathcal{R}'\). Let \(\hat{Q} = (\hat{Q}_i)_{i \in I}\) be the set of transitive extensions of \(\mathcal{R}\) such that \(\mathcal{R} \subset \hat{Q}_i \subseteq \mathcal{R}'\) and \(\hat{Q}_i \in \mathcal{R}\). Let \(\mathcal{C}\) be a chain in \(\hat{Q}\), and \(\hat{C} = \bigcup \mathcal{C}\). Clearly, \(\mathcal{R} \subset \hat{C} \subseteq \mathcal{R}'\). Since \(\mathcal{R}\) is closed upward, \(\hat{C} \in \mathcal{R}\). Therefore, by Zorn’s lemma, \(\hat{Q}\) has an element, say \(\hat{Q}\), that is maximal with respect to set inclusion. Then, \(\mathcal{R}' = \hat{Q} \in \mathcal{R}\). Otherwise, there exists \((s, t) \in \mathcal{R}' \setminus \hat{Q}\) such that \(\mathcal{Q}' = \hat{Q} \cup \{(s, t)\}\) or \(\mathcal{Q}' = \hat{Q} \cup \{(t, s)\}\) is a transitive extension of \(\mathcal{R}\) satisfying \(\mathcal{R}^m \subset \hat{Q} \subset \mathcal{Q}' \subseteq \mathcal{R}'\), which is impossible by maximality of \(\hat{Q}\). This completes the proof.

Since \(S\)-consistency is equivalent to \(A(\omega)\)-consistency, the following result is an immediate corollary of the previous proposition.

**Corollary 11** (Duggan’s General Extension Theorem [12]). Assume \(\mathcal{R}\) is closed upward and arc-receptive. If \(\mathcal{R}\) is \(S\)-consistent and \(\mathcal{R} \in \mathcal{R}\), then

\[ \mathcal{R} = \bigcap \{ \mathcal{R}' \in \mathcal{R} | \mathcal{R}' \text{ is a complete, transitive extension of } \mathcal{R} \}. \]

Clearly, Theorem [6] concludes all the extension theorems referred to Duggan [12, pp. 13-14].

### 5 Applications

Actually, it is well known that the notion of maximal element has interesting applications to the study of economic and game theories. In fact, it plays a central role in many economic models, including global maximum of a utility function and Nash equilibrium of a noncooperative game or equilibrium of an economy (Debreu [10]). We prove the following propositions as a general application of the notion of inherited type extension theorems.

**Proposition 4** Let \(\mathcal{R}\) be a \(\Delta(m)\)-consistent binary relation on some nonempty set \(X\), \(m \in \mathbb{N}\), and let \(x^*\) be a maximal element of \(\mathcal{R}\) in \(X\). Then, there exists a linear order extension \(\mathcal{Q}\) of \(\mathcal{R}\) such that \(x^*\) is a maximal element of \(\mathcal{Q}\) in \(X\).

**Proof** We first show that \(x^*\) is a maximal element of \(\mathcal{R}\). Indeed, suppose to the contrary that \((y, x^*) \in P(\mathcal{R})\) for some \(y \in X\). It then follows that there exists \(l \in \mathbb{N}\) and alternatives \(t_1, t_2, \ldots, t_l\) such that \(y R t_1 \cdots t_{a_m} R t_m \cdots t_l x^*\). Since \((t_i, x^*) \notin P(\mathcal{R})\), we conclude that \((t_i, x^*) \in I(\mathcal{R}) \subseteq I(\mathcal{R}^m)\). Hence, because of \(\Delta(m)\)-consistency, we conclude that \(t_i = x^*\). Similarly, \((t_{m-1}, x^*) \in \mathcal{R}\), and
an induction argument based on this logic yields \( y = x^* \), a contradiction to \((y, x^*) \in P(R)\). Hence, \( x^* \) is a maximal element of \( R \). If \( R \) is complete, then it is a linear order extension of \( R \) which has \( x^* \) as maximal element. Otherwise, there are \( x, y \in X \) such that \((x, y) \notin \overline{R} \) and \((y, x) \notin \overline{R} \). Clearly, one of \( x \) and \( y \) is different from \( x^* \). Let \( x \neq x^* \). We define

\[
R^* = R \cup \{(\kappa, \lambda)|\kappa \neq \lambda, x \overline{R} \lambda, (x, y) \in \text{inc}(\overline{R}) \; x, y \in X, \; \text{and} \; x \neq x^* \}.
\]

Then, as in Theorem 3, we conclude that \( R^* \) is a \( \Lambda(m) \)-consistent extension of \( R \). Since \( I((R^*)^m) = I(R^m) = \Delta \), we conclude that \( R^* \) is \( \Delta(m) \)-consistent. To show that \( x^* \) is a maximal element of \( R^* \), suppose to the contrary that \((\kappa, x^*) \in P(R^*) \) for some \( \kappa \in X \). Since \( x^* \) is a maximal element of \( \overline{R} \), we conclude that \( \kappa \neq \overline{R} \lambda \) and \( x \overline{R} x^* \). It follows that \((x, x^*) \in I(\overline{R}) = I(R^m) = \Delta \), a contradiction to \( x \neq x^* \). Hence, \( x^* \) is a maximal element of \( R^* \). Suppose that \( \overline{R} = \{\overline{R}_i|i \in I\} \) denote the set of \( \Delta(m) \)-consistent extensions of \( R \) which has \( x^* \) as maximal element. Since \( R^* \in \overline{R} \) we have that \( \overline{R} \neq \emptyset \). Let \( Q = (Q_i)_{i \in I} \) be a chain in \( \overline{R} \), and let \( \overline{Q} = \bigcup_{i \in I} Q_i \). We show that \( \overline{Q} \in \overline{R} \). As in the proof of Theorem 3 we conclude that \( \overline{Q} \) is a \( \Delta(m) \)-consistent extension of \( R \).

To verify that \( x^* \) is a maximal element of \( \overline{Q} \), take any \( y \in X \) and suppose \((y, x^*) \in P(\overline{Q}) = P(\bigcup_{i \in I} Q_i) \). Clearly, \( y \neq x^* \) and \((y, x^*) \in Q_i \), for some \( i^* \in I \).

Since \((x^*, y) \notin \bigcup_{i \in I} Q_i \), we conclude that \((x^*, y) \notin Q_i \), for each \( i \in I \). Hence, \((y, x^*) \in P(Q_{i^*}) \), a contradiction to \( Q_{i^*} \in \overline{R} \). Therefore, \( \overline{Q} \in \overline{R} \). By Zorn’s lemma \( \overline{R} \) possesses an element, say \( Q \), that is maximal with respect to set inclusion. Therefore, as above we can prove that \( \overline{Q} \) is an extension of \( R \) which has \( x^* \) as maximal element. We prove that \( \overline{Q} \) is complete. Indeed, take any \( x, y \in X \) such that \((x, y) \notin \overline{Q} \) and \((y, x) \notin \overline{Q} \). We define

\[
Q^* = Q \cup \{(\kappa, \lambda)|\kappa \neq \lambda, x \overline{Q} \lambda, (x, y) \notin \overline{Q}, (y, x) \notin \overline{Q} \; \text{and} \; x \neq x^* \}.
\]

Then, as in case of \( R^* \) above, we have that \( Q^* \) is a \( \Delta(m) \)-consistent binary relation which has \( x^* \) as maximal element, a contradiction to the maximality of \( Q \). The last contradiction implies that \( \overline{Q} \) is complete.

As a corollary of the previous result we have a generalization of Sophie Bade’s result in [2] Theorem 1](she uses transitive binary relations) which shows that the set of Nash equilibria of any game [3] with incomplete preferences can be characterized in terms of certain derived games with complete preferences. More general, it is shown a similarity between the theory of games with incomplete preferences and the existing theory of games with complete preferences. I put in mind the following definition:

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4 In this case, \( G = \{(A_i, R_i)|i \in I\} \) is an arbitrary (normal-form) game. Where \( I \) is a set of players, player \( i \)'s nonempty action space is denoted by \( A_i \) and \( R_i \) is player \( i \)'s preference relation on the outcome space \( A = \prod_{i \in I} A_i \).
Equivalence relation \( \approx \) that is, \( x \approx y \) for each \( x, y \in X \).
In what follows, we denote the set of all Nash equilibria of a game \( G \) by \( \Delta(m) \)-consistent for some \( m \in \mathbb{N} \).

### Corollary 12
Let \( G = \{(A_i, R_i)|i \in I\} \) be any game. Then
\[
\Delta(m) \subseteq \bigcup \{ \Delta(m') \}\bigcup \{ R' \} \text{ is a completion of } G.
\]

**Proof**
Clearly, \( \bigcup \{ \Delta(m') \}\bigcup \{ R' \} \subseteq \Delta(m) \). Conversely, let \( a^* \in \Delta(m) \), that is, \( a^* \) is a Nash equilibrium of \( G \). Let us define \( B_i = \{(a_i, a_{-i})|a_i \in A_i\} \) for all players \( i \). Then, for any player \( i \), \( a^* \) is a maximal element of \( R_i \) in \( B_i \). By Proposition [1] there exists a completion \( R'_i \) of \( R_i \) for each player \( i \) such that \( a^* \) is maximal point of \( R'_i \) in \( B_i \). Consequently \( a^* \) is a Nash equilibrium of the completion \( G' = \{(A_i, R'_i)|i \in I\} \). Hence, \( \Delta(m) \subseteq \bigcup \{ \Delta(m') \}\bigcup \{ R' \} \text{ is a completion of } G \).

As we have pointed out in the introduction, we are often interested in particular binary relations which have an ordering extension that satisfies some additional conditions. The following proposition, which generalizes the main result in [20], is an application to this specific case.

### Proposition 5
Let \((X, \tau)\) be a topological space and \( m \in \Omega_\alpha \). If \( R \) is an asymmetric, \( A(m) \)-consistent and upper semicontinuous binary relation on \( X \), then \( R \) has an upper semicontinuous strict linear order extension.

**Proof**
To begin with, we associate to \( R \) the equivalence relation \( \approx \) defined by
\[
x \approx y \text{ if and only if } \forall z \in X, (zRx \Leftrightarrow zRy) \text{ and } (xRz \Leftrightarrow yRz),
\]
that is, \( x \approx y \) if and only if \( x \) covers \( y \) and \( y \) covers \( x \). The quotient set by this equivalence relation \( \approx \) will be denoted \( \tilde{X} = X/\approx \), and its elements (equivalence classes) by \([x]\).

An asymmetric, \( A(m) \)-consistent binary relation \( R^\approx \) is defined by:
\[
\forall [x], [y] \in X/\approx, ([x]R^\approx [y] \Leftrightarrow \exists x' \in [x], \exists y' \in [y], x'Ry').
\]
According to Corollary [3] there exists a strict linear order extension \( \tilde{Q}^\approx \) of \( R^\approx \). Therefore, as in the proof of Theorem 1 [20] Pages 399-400, by using a subbase of \( \tau \), we construct a strict linear order extension \( R^\approx \) of \( R \) such that for each \( y \in X \) the set \( \{ x \in X | xR^\approx y \} \) belongs to \( \tau \).

In direction of the inherited type Szpilrajn extension theorems, Demuynck [8] give results for complete extensions satisfying various additional properties such as convexity, homotheticity and monotonicity. Since Demuynck’s paper

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5 An action profile \( a = (a_1, ..., a_{|i|}) \) is a Nash equilibrium if for no player \( i \) there exists an action \( a'_i \in A_i \) such that \((a'_i, a_{-i})R_i(a_i, a_{-i})\).
generalizes S-consistency by replacing the transitive closure $\overline{R}$ of $R$ with a more general function $F$, I conjecture that these results can be extended to the case of $A(m)$-consistent binary relation for all $m \in \Omega_n$.

The following proposition is given as a general application of the Dushnik-Miller’s inherited type extension theorem.

For each $x \in X$, we define (see [25 Page 20]): $i(x) = \{ y \in X | x \overline{R} y \}$, $d(x) = \{ y \in X | y \overline{R} x \}$ and $I_x = i(x) \cup d(x)$. For any $x \in X$, $x^m$ and $x^M$ denote the minimum and the maximum of $I_x$.

**Definition 6** A binary relation $R$ on $X$ has finite decomposition incomparability, if there exists $n \in \mathbb{N}$ and $(x_\mu, y_\mu) \in \text{inc}(\overline{R})$, $\mu \in \{1,2,...,n\}$, such that:

1. $(x_\mu^m, y_\mu^m) \notin \overline{R}$ and $(y_\mu^m, x_\mu^M) \notin \overline{R}$, and
2. $\text{inc}(\overline{R}) = \{ (\kappa, \lambda) \in I_{x_\mu} \times I_{y_\mu} | (\kappa, \lambda) \in \text{inc}(\overline{R}) \} , 1 \leq \mu \leq n \}.

**Proposition 6** Let $m \in \Omega_n$ and let $R$ be a continuous $A(m)$-consistent binary relation on a topological space $(X, \tau)$ having finite decomposition incomparability. Then, the dimension of $R$ is finite.

**Proof** Without loss of generality, assume that $R$ is reflexive. We first show that for each $x \in X$ the sets $i(x)$ and $d(x)$ are closed. To prove the case of $i(x)$, let $z \notin i(x)$. Then, $(x, z) \notin \overline{R} \supseteq R$. Then, by [25 Proposition 1], there exists an open $R$-increasing neighbourhood $O_x$ of $x$ and an open $R$-decreasing neighbourhood $O_z$ of $z$ such that $O_x \cap O_z = \emptyset$. Since $x \in O_x$ and $O_z$ is $R$-increasing we conclude that $i(x) \subseteq O_z$. It follows that $z \in O_z \subseteq X \setminus i(x)$. Therefore, $i(x)$ is closed. Similarly, we prove that $d(x)$ is closed which implies that $I_x$ is closed as well. Hence, $I_x$ is compact. Then, $I_x$ has a maximum element $x^M$ and a minimum element $x^m$. To see this, note that if $I_x$ has no largest element, then $\{ I_x \setminus d(z) | z \in I_x \}$ is an open cover of $I_x$ in subspace topology with no finite subcover, and if $I_x$ has no smallest element, then $\{ I_x \setminus i(z) | z \in I_x \}$ is an open cover of $I_x$ in subspace topology with no finite subcover.

Since $R$ has finite decomposition incomparability, there exists $n \in \mathbb{N} \cup \{1,2,...,n\}$, such that $(x_\mu^m, y_\mu^m) \notin \overline{R}$ and $(y_\mu^m, x_\mu^M) \notin \overline{R}$, and $\text{inc}(\overline{R}) = \{ (\kappa, \lambda) \in I_{x_\mu} \times I_{y_\mu} | 1 \leq \mu \leq n \}$. On the other hand, by the continuity of $R$ we have $(y_\mu^m, x_\mu^m) \notin \overline{R}$ and $(x_\mu^m, y_\mu^M) \notin \overline{R}$. It follows that $(x_\mu^m, y_\mu^m), (y_\mu^m, x_\mu^M) \in \text{inc}(\overline{R})$. We prove that $\dim(R) \leq n$. We define

$\mathcal{R}_\mu = R \cup \{ (\kappa, \lambda) \in X \times X | \kappa R y_\mu^M, \text{ and } x_\mu^M R \lambda \}$

and

$\mathcal{R}_\mu^D = R \cup \{ (\lambda, \kappa) \in X \times X | \lambda R x_\mu^m, \text{ and } y_\mu^m R \kappa \}$.

By Theorem 3 there exist linear order extensions $Q_\mu$ and $Q_\mu^D$ of $R$ such that $\text{inc}(\overline{R}) \cap (I_{x_\mu} \times I_{y_\mu}) \subseteq Q_\mu$ and $\text{inc}(\overline{R}) \cap (I_{x_\mu} \times I_{y_\mu}) \subseteq Q_\mu^D$. 

We prove that \( \mathcal{R} = \bigcap_{m=1}^{n} (Q_m \cap Q_m^D) \). Clearly, \( \mathcal{R} \subseteq \bigcap_{m=1}^{n} (Q_m \cap Q_m^D) \). To prove the converse, let \( (\alpha, \beta) \in \bigcap_{m=1}^{n} (Q_m \cap Q_m^D) \) and \( (\alpha, \beta) \notin \mathcal{R} \). The proof proceeds in a similar way to Theorem 6 as follows: We first prove that \( (\beta, \alpha) \notin \mathcal{R} \) and by the finite decomposition incomparability property of \( R \), there exists \( \mu^* \in \{1, 2, ..., n\} \) such that \( (\alpha, \beta) \in I_{\mu^*} \times I_{\mu^*} \). Then, we prove that \( (\alpha, \beta) \notin Q_{\mu^*} \), a contradiction to \( (\alpha, \beta) \in \bigcap_{m=1}^{n} (Q_m \cap Q_m^D) \). The last conclusion completes the proof.

Another example is the following: In the games that are compositions of \( m \) individualist games via unanimity, the usual description of the game, by means of minimal winning coalitions, requires \( n_1 \cdot ... \cdot n_m \) coalitions (with \( n_1 = |N_1| \) and if each one of them has \( m \) players, then, \( m \cdot n_1 \cdot ... \cdot n_m \) digits are needed to describe the game. Using [16] Theorem 3.1, \( (n+1) \cdot (m-p) \) \( p < m \) digits are required to describe the game. This latter number is generally much smaller than the former, and so, the description of the game is much shorter.

Many other interesting applications of the dimension of a binary relation are obtained in Economics. For example, Ok [7, Proposition 1] shows that if \( (X, \succ) \) is a preordered set with \( X \) countable and \( \text{dim}(X, \succ) < \infty \), then \( \succ \) is representable by means of a real function \( u \) in such a way that \( x \succ y \) if and only if \( u(x) > u(y) \). From the multicriteria point of view, the classical crisp dimension refers to a minimal representation of crisp partial orders as the intersection of linear orders, in the sense that each of these linear orders is a possible underlying criterion. Brightwell and Scheinerman [6], on the basis of Dushnik-Miller’s original theorem, prove that the fractional dimension of a game with player set \( N = \{1, ..., n\} \) admits a partition \( N_1, ..., N_m \) in such a way that

\[
W = \{ S \subseteq N : |S \cap N_i| \geq 1, \text{ for all } i = 1, ..., m \}
\]

we shall say that this game is a composition of \( m \) individualist games via unanimity.

Let \( (N, W) \) be a composition of \( m \) individualist games \( (N_i, u_i) \) \( i = 1, ..., m \) with \( 1 \leq n_1 \leq ... \leq n_m \) via unanimity and let \( p < m \) such that \( n_p = 1 \), \( n_{p+1} > 1 \) or \( p = 0 \) if \( n_1 > 1 \). Then the dimension of \( (N, W) \) is \( m - p \).

Given a finite set of alternatives \( X = \{x_1, x_2, ..., x_n\} \), a crisp partial order set \( R \subseteq X \times X \) is characterized by a mapping

\[
\mu : X \times X \rightarrow \{0, 1\}
\]

being

(i) irreflexive: \( \mu(x_i, x_i) = 0 \) \( \forall x_i \in X \),
(ii) antisymmetric: \( \mu(x_i, x_j) = 1 \Rightarrow \mu(x_j, x_i) = 0 \),
(iii) transitive: \( \mu(x_i, x_j) = \mu(x_j, x_k) = 1 \Rightarrow \mu(x_i, x_k) = 1 \). It is therefore assumed that \( \mu(x_i, x_j) = 1 \) means that alternative \( x_i \) is strictly better than \( x_j \) \( (\mu(x_i, x_j) = 0 \) otherwise).

Brightwell and Scheinerman [6] introduce the notion of fractional dimension of a poset \( (X, \succ) \). Let \( F = \{L_1, L_2, ..., L_t\} \) be a nonempty multiset of linear extensions of \( (X, \succ) \). The
a partially ordered set \((X, \succ)\) arises naturally when considering a particular two-person game on \((X, \succ)\), e.t.c.

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authors in [6] call \(F\) a \(k\)-fold realizer of \((X, \succ)\) if for each incomparable pair \((x, y)\), there are at least \(k\) linear extensions in \(F\) which reverse the pair \((x, y)\), i.e., \(\{i = 1, ..., t : y < x \text{ in } L_i]\} \geq k\). We call a \(k\)-fold realizer of size \(t\) a \(-t\)-realizer. The fractional dimension of \((X, \succ)\) is then the least rational number \(\text{fdim}(X, \succ) \geq 1\) for which there exists a \(k - t\)-realizer of \((X, \succ)\) so that \(\frac{t}{k} \geq \text{fdim}(X, \succ)\). Using this terminology, the dimension of \((X, \succ)\), is the least \(t\) for which there exists a 1-fold realizer of \((X, \succ)\).
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