A REMARK ON A THEOREM OF ILIADIS CONCERNING
ISOMETRICALLY CONTAINING MAPPINGS

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Abstract. In this paper we give an alternative proof and a refinement of a recent result of S.D.Iliadis, concerning isometrically containing mappings. We address also a recent related result by A.I.Oblakova.

1. Introduction

Following Stavros Iliadis [4], we shall say that a continuous surjection $F : X \to Y$ between separable metric spaces is isometrically containing for a class $\mathcal{F}$ of continuous surjections between compact metric spaces if for each $f \in \mathcal{F}$ there are isometric embeddings $i : dom f \to X$ and $j : ran f \to Y$ of the domain and the range of $f$, respectively, such that $F \circ i = j \circ f$.

Iliadis [4] proved the following theorem.

Theorem 1 (S.D. Iliadis). For any countable ordinals $\alpha, \beta$ there is a continuous surjection $F : X \to Y$ between complete separable metric spaces with small transfinite dimensions $\text{ind} X = \alpha$, $\text{ind} Y = \beta$, which is isometrically containing for the class $\mathcal{F}(\alpha, \beta)$ of continuous surjections $f$ between compact metric spaces with $\text{ind}(\text{dom} f) \leq \alpha$ and $\text{ind}(\text{ran} f) \leq \beta$.

The proof given by Iliadis is based on a rather refined method, developed in his earlier works [1] - [3].

The aim of this remark is to provide another proof of this theorem, using some results from [P1], cf. also [PP] and [P2].

This approach gives also some refinement of the Iliadis theorem to the following effect (recall that the dimension of a mapping is the supremum of dimensions of its fibers):

Theorem 2. Given countable ordinals $\alpha, \beta$ and an $n$ which is a natural number or $\infty$, there is a continuous surjection $F : X \to Y$ between complete separable metric spaces with $\text{ind} X = \alpha$, $\text{ind} Y = \beta$, such that $\text{dim} F \leq n$ and $F$ is isometrically containing for the class $\mathcal{F}(\alpha, \beta, n)$ of the maps $f \in \mathcal{F}(\alpha, \beta)$ with $\text{dim} f \leq n$.

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2. Proof of Theorem 2

In the sequel, we shall adopt the notation from the proof of Proposition 5.1.1 in [PP].

Let $M$ be a complete separable metric space isometrically universal for separable metric spaces, let $\mathcal{K}(M)$ be the hyperspace of compact subsets of $M$ equipped with the Vietoris topology, and for a countable ordinal $\gamma$, let $\mathcal{K}_\gamma(M) = \{ K \in \mathcal{K}(M) : \text{ind}K \leq \gamma \}$.

We shall consider also $M$ as a topological subspace of the Hilbert cube $I^\infty$, and it should be clear from the context, when we refer to the fixed universal metric on $M$ (which does not extend over $I^\infty$) or we just deal with the topology in $M$ (inherited from $I^\infty$).

We denote by $C(I^\infty, I^\infty)$ the space of continuous maps of $I^\infty$ into itself, equipped with the compact - open topology.

Let us fix countable ordinals $\alpha, \beta$ and an $n$ which is a natural number or $\infty$.

Let $\Phi_\alpha : \mathbb{N}^n \to \mathcal{K}_\alpha(M)$, $\Phi_\beta : \mathbb{N}^n \to \mathcal{K}_\beta(M)$ be continuous surjections on the irrationals, considered in [PP], proof of Proposition 5.1.1. Recall that for every $\alpha < \omega_1$, the set $G_\alpha = \{ (t, x) : x \in \Phi_\alpha(t) \}$ is closed in $\mathbb{N}^n \times M$, ind $G_\alpha = \alpha$ and each $K \in \mathcal{K}_\alpha(M)$ is isometric with $\{ t \} \times \Phi_\alpha(t)$, where $\Phi_\alpha(t) = K$.

Let

1. $\mathcal{G} = \{ (s, t, f) \in \mathbb{N}^n \times \mathbb{N}^n \times C(I^\infty, I^\infty) : f(\Phi_\alpha(s)) = \Phi_\beta(t) \text{ and dim}(f^{-1}(y) \cap \Phi_\alpha(s)) \leq n \text{ for each } y \in \Phi_\beta(t) \}$

Using continuity of $\Phi_\alpha$ and $\Phi_\beta$, and the fact that at most $n$-dimensional compacta form a $G_\delta$-set in the hyperspace $\mathcal{K}(I^\infty)$, cf. [K], §45, IV, Theorem 4, we will show that

2. $\mathcal{G}$ is a $G\delta$-set in the product $\mathbb{N}^n \times \mathbb{N}^n \times C(I^\infty, I^\infty)$.

Let $E = \{ K \in \mathcal{K}(I^\infty) : \dim K > n \}$. Then $E = \bigcup_m E_m$, where $E_m$ is closed in the hyperspace. Moreover, the set $E^*_m = \{ L \in \mathcal{K}(I^\infty) : L \supset K \text{ for some } K \in E_m \}$ is also closed in $\mathcal{K}(I^\infty)$, contains $E_m$ and is contained in $E$. Therefore, replacing $E_m$ by $E_m^*$ we can assume that $E_m$ contains all compacta containing some element of $E_m$.

Let us check that, for every $m \in \mathbb{N}$,

3. $\mathcal{F}_m = \{ (s, f, y) \in \mathbb{N}^n \times C(I^\infty, I^\infty) \times I^\infty : f^{-1}(y) \cap \Phi_\alpha(s) \in E_m \}$ is closed in the product $\mathbb{N}^n \times C(I^\infty, I^\infty) \times I^\infty$

To that end, consider $(s_i, f_i, y_i) \to (s, f, y), (s_i, f_i, y_i) \in \mathcal{F}_m$, and let $C_i = f_i^{-1}(y_i) \cap \Phi_\alpha(s_i) \in E_m$. Passing, if necessary, to a subsequence, we can assume that $C_i \to C$ in $\mathcal{K}(I^\infty)$.

We shall check that $C \subset f^{-1}(y) \cap \Phi_\alpha(s)$. Let us pick any $a \in C$, and let $a_i \to a$, $a_i \in C_i$. Since $f_i(a_i) = y_i, f_i \to f, y_i \to y$, we have $f_i(a_i) \to f(a)$, hence $f(a) = y$. Also, since $a_i \in \Phi_\alpha(s_i)$ and $\Phi_\alpha$ is continuous, $a \in \Phi_\alpha(s)$. In effect, $a \in f^{-1}(y) \cap \Phi_\alpha(s)$, and $C \subset f^{-1}(y) \cap \Phi_\alpha(s)$. Since $C \in E_m$ its superset $f^{-1}(y) \cap \Phi_\alpha(s)$ is also in $E_m$, hence $(s, f, y) \in \mathcal{F}_m$, which gives us (3).

Now, denoting by $\pi : \mathbb{N}^n \times C(I^\infty, I^\infty) \times I^\infty \to \mathbb{N}^n \times C(I^\infty, I^\infty)$ the projection parallel to the compact axis $I^\infty$, we conclude that the sets $\pi(\mathcal{F}_m)$ are closed, and hence the sets

4. $\mathcal{H}_m = (\mathbb{N}^n \times C(I^\infty, I^\infty)) \setminus \pi(\mathcal{F}_m)$ are open in $\mathbb{N}^n \times C(I^\infty, I^\infty)$. Therefore, the set
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(5) $\bigcap_m H_m = \{(s, f) \in \mathbb{N}^n \times C(I, I) : \text{for every } y \in I, \dim(f^{-1}(y) \cap \Phi_\alpha(s)) \leq n\}$. This is a $G_\delta$-set in $\mathbb{N}^n \times C(I, I)$.

Since $L = \{(s, t, f) : f(\Phi_\alpha(s)) = \Phi_\beta(t)\} \subset \mathbb{N}^n \times \mathbb{N}^n \times C(I, I)$ is closed, both $\Phi_\alpha$, $\Phi_\beta$ being continuous, the set $G = L \cap \{\{(s, t, f) : (s, f) \in \bigcap_m H_m\} \text{ is a } G_\delta\text{-set in } \mathbb{N}^n \times \mathbb{N}^n \times C(I, I)$, which ends the proof of (2).

By (2), there is a continuous surjection of $\mathbb{N}^n$ onto $G$,

(6) $u \to (s(u), t(u), f_u) \in G, u \in \mathbb{N}^n$,

and we let

(7) $X = \{(u, z) : z \in \Phi_\alpha(s(u))\} \subset \mathbb{N}^n \times M$,

(8) $Y = \{(t, z) : z \in \Phi_\beta(t)\} \subset \mathbb{N}^n \times M$,

(9) $F : X \to Y, F(u, z) = (t(u), f_u(z))$.

The spaces $X, Y$ are considered with the metric inherited from the product $\mathbb{N}^n \times M$, where $M$ is equipped with the universal metric and $\mathbb{N}^n$ has a standard complete metric.

Then (in notation from [PP]), $Y = G_\beta$, hence $\text{ind}Y = \beta$, and since the function $(u, z) \to (s(u), z)$ maps $X$ into $G_\alpha$, using an observation in [PI], §4, Sublemma 3.2, we check that also $\text{ind}X = \alpha$.

The mapping $F$ in (9) is a continuous surjection. For any $(r, w) \in Y, F^{-1}(r, w) = \{(u, z) \in \mathbb{N}^n \times M : t(u) = r, z \in f_u^{-1}(w) \cap \Phi_\alpha(s(u))\}$, hence by (1) the projection of $F^{-1}(r, w)$ onto the first coordinate is a perfect map with at most $n$-dimensional fibers. Therefore, $\dim F \leq n$.

Finally, let $f : K \to L$ be a continuous between compact spaces with $\text{ind}K \leq \alpha, \text{ind}L \leq \beta$ and $\dim f \leq n$. Since $M$ was an isometrically universal space, we can assume that $K$ and $L$ are isometrically embedded in $M$. The function $f : K \to L$ can be extended to a continuous map $\tilde{f} : I^\infty \to I^\infty$. For $s, t \in \mathbb{N}^n$ such that $K = \Phi_\alpha(s), L = \Phi_\beta(t)$, we have $(s, t, \tilde{f}) \in G$, cf. (1), and let $u \in \mathbb{N}^n$ be such that $s = s(u), t = t(u), \tilde{f} = f_u$, cf. (6).

Then, for the isometric identifications $i : K \to \{u\} \times K$ and $j : L \to \{t\} \times L$, we have $F \circ i = j \circ f$, cf. (9).

3. Comment

In [O], Section 4, A.I.Oblakova proved that there exists a Cantor set such that any finite metric space whose diameter does not exceed 1 and the number of points does not exceed $n$ can be isometrically embedded into it. Using a continuous parametrization of some collections of finite metric spaces, one can refine slightly this result, to the following effect:

**Remark.** For each natural number $n$ there are Cantor sets $X$ and $Y$ and a continuous surjection $F : X \to Y$ which is isometrically containing for the class $\mathcal{D}_n$ of non-expanding surjections between at most $n$-element metric spaces of diameter $\leq 1$.

**Proof.** Let us fix a natural number $n$. First we will prove that
(10) there are zero-dimensional compact metric spaces $E$, $H$ and a continuous surjection $f : E \to H$ which is isometrically containing for the class $D_n$.

Indeed, by a result of Gromov [G], sec.6, there is a compact metric space $Z$ which contains isometrically each metric space of diameter $\leq 1$ containing at most $n$ elements (cf. also [O] and [I2], 9.3, for different proofs).

Let $Z \times Z$ be the metric product and $\pi_i : Z \times Z \to Z$, $i = 1, 2$, projections onto the first and the second coordinate, respectively.

Let $\mathcal{F}$ be the collection of at most $n$-element subsets $T$ of $Z \times Z$ which are graphs of non-expanding surjections from $\pi_1(T)$ onto $\pi_2(T)$, cf. [PP], (4).

Then $\mathcal{F}$ is compact in the hyperspace of $Z \times Z$ and let $u : 2^N \to \mathcal{F}$ be a continuous surjection of the Cantor set $2^N$ onto $\mathcal{F}$.

We let

\begin{align*}
(11) & \quad E = \{(t, x) : x \in \pi_1(u(t))\} \subset 2^N \times Z, \\
(12) & \quad H = \{(t, y) : y \in \pi_2(u(t))\} \subset 2^N \times Z, \\
(13) & \quad f : E \to H, \quad f(t, x) = (t, y), \text{ where } (x, y) \in u(t).
\end{align*}

Notice that continuity of $f$ follows from the fact that its graph $\{(t, x), (t, y)\} : (x, y) \in u(t)\}$ is a closed subset of the compact product $(2^N \times Z) \times (2^N \times Z)$.

To see that $f$ is isometrically containing for the class $D_n$, suppose that $g : K \to L$ is a non-expanding surjection between at most $n$-element metric spaces of diameter $\leq 1$. One can assume that $K$ and $L$ are isometrically embedded in $Z$. Since $T = \{(x, g(x)) : x \in K\}$ belongs to $\mathcal{F}$, $T = u(t)$ for some $t \in 2^N$. Then, for the isometric identifications $i : K \to \{t\} \times K$ and $j : L \to \{t\} \times L$, we have $f \circ i = j \circ g$, cf. (13).

To end the proof of Remark, it suffices to embed $E$ and $H$ into Cantor sets $X$ and $Y$ (by a theorem of Hausdorff, one can extend the metrics on $E$ and $H$ to metrics on $X$ and $Y$, respectively), and to define $F : X \to Y$ as $F = f \circ r$, where $r : X \to E$ is a retraction. Finally, to make sure that $F$ is a surjection, we can always add to $X$ a disjoint copy of $Y$ and let $F$ be the identity on this copy. □

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