Kinetic and transport equations for localized excitations in Sine-Gordon model

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Abstract

We analyze the kinetic behavior of localized excitations - solitons, breathers and phonons - in Sine-Gordon model. Collision integrals for all type of localized excitation collision processes are constructed, and the kinetic equations are derived. We prove that the entropy production in the system of localized excitations takes place only in the case of inhomogeneous distribution of these excitations in real and phase spaces. We derive transport equations for soliton and breather densities, temperatures and mean velocities i.e. show that collisions of localized excitations lead to creation of diffusion, thermoconductivity and intrinsic friction processes. The diffusion coefficients for solitons and breathers, describing the diffusion processes in real and phase spaces, are calculated. It is shown that diffusion processes in real space are much faster than the diffusion processes in phase space.
1. INTRODUCTION

The problem of kinetic properties of excitations in integrable models belongs to a class of most nontrivial problems of physical kinetics. First, enormously long relaxation of nonlinear excitation has been found in numerical experiments by Fermi Pasta and Ulam [1]. In Zabusky and Kruskal numerical experiment [2] unexpected behavior of localized nonlinear excitations has been discovered, namely their interaction without changing their forms and velocities. Zabusky and Kruskal named them solitons. Shortly after, the analytical method of solving nonlinear differential equation with partial derivatives - inverse scattering method - was found in the framework of Korteweg - de Vries equation for 1+1 dimensions [3]. Other physically reasonable continuous models, treatable by the inverse scattering method are the Nonlinear Schrödinger (NS) equation, the Sine-Gordon (SG) equation and the Landau-Lifshitz (LL) equation [4,5]. Common excitations in integrable models are: the one parameter localized wave i.e. the soliton and the nonlinear periodic wave for KdV model; and the two parameter localized wave i.e. the breather for SG and LL models; and the envelope soliton for NS equation. All these excitations interact without changing their forms and their velocities and therefore their energies; the only result of their interaction are shifts of their coordinates and for breathers and NS solitons change of their phases as well. It is worthwhile to emphasize that many particle effects are absent in the following meaning: the total shift in collisions involving several excitations is equal to the sum of shifts in each collision (see[4]).

Soon after analytically solving the KdV equation [3], Zakharov considered the possibility of the kinetic decryption of solitons [6]. In [6], kinetic equation for solitons was written but only the effect of renormalization of soliton velocity due to soliton - soliton collisions has been considered. Another approach for the investigation of kinetic properties of kink type solitons was proposed in [7-9]. In [7] numerically and in [8,9] analytically the diffusion of kinks interacting with phonons in the $\phi^4$ model was considered. This model is not exactly integrable, but the potential for kink - phonons interaction has reflectionless form and the result of the interaction is the same as for soliton - soliton collision in integrable models. Taking into account such kind of interaction in [8,9], kink diffusion coefficient was calculated. A bit later the same calculations have been done for SG model [10]. The renormalization of soliton velocity and the diffusion coefficient of the solitons due to soliton - soliton and soliton - magnon collisions have been considered for SG model in [11].

To explain the effect of velocity renormalization it is useful to note that the dependence of soliton coordinate $x$ from time can be written in the following form:

$$x(t) = x_0 + v_1 t + \int_{-\infty}^{t} dt' \int dx_2 dv_2 |v_1 - v_2| \Delta x_s (v_1, v_2) \theta (t - t') f(x_2, v_2, t'), \quad (1.1)$$

when shifts of soliton coordinates $\Delta x_s (v_1, v_2)$ are taken into consideration. In formula (1.1) $f(x, v, t)$ is the distribution function of solitons or phonons (magnons), and the
sample soliton collides with soliton or phonon (magnon) wave packet \((x_2, v_2)\) at the moment \(t'\),

\[ \theta (t) = 1, \quad t > 0, \quad \theta (t) = 0, \quad t < 0. \]  \(1.2\)

From formula (1.1) one can obtain the following expression for the average velocity of the sample soliton:

\[ < v_1 > = v_1 + \int dx_2dv_2|v_1 - v_2|\Delta x_s(v_1, v_2)f(x_2, v_2, t) \]  \(1.3\)

The diffusion coefficient of soliton is given by:

\[ D = \frac{< [x(t) - < v > t - x(0)]^2 >}{2t}, \]  \(1.4\)

(see [8-10]).

So the shift of soliton coordinate \(\Delta x\) leads to its diffusion. Do another kinetic coefficients for solitons exist in the integrable models? This question is closely related with the problem of entropy production. From the fact of conserving of the solitons velocities follows the absence of the entropy production in momentum space only. But in the coordinate space the interaction of solitons is very strong. Therefore the question: “Does entropy increase in integrable systems?” - is not a nonsense. The answer on these questions have been done in [12] on the example of kink-type solitons in SG model. In [12] kinetic equation for the gas of kinks has been formulated, the entropy production has been proved and the existence of the coefficients of intrinsic friction and thermal conductivity have been shown. Obviously that such kinetic coefficients as mobility coefficient appear in the presence of perturbations, which destroy the integrability of the model (see [13] and review [14]).

In this paper the kinetic behavior of solitons, breathers and phonons in the framework of the SG model is considered. We construct collision integrals for all possible collisions and formulate the system of Boltzmann type kinetic equations for solitons, breathers and phonons. Based on the system of kinetic equations thus obtained, we prove that the entropy production takes place as a result of randomization of the distribution of excitations in coordinate and phase space. Thus we are able to derive transport equations for solitons and breathers and to calculate self-diffusion coefficients for soliton-soliton and soliton-breather and breather-breather collisions as an example.

2.ELEMENTARY EXCITATIONS IN SINE-GORDON EQUATION

Sine-Gordon equation in dimensionless variables can be written as:

\[ \varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta} \sin \beta \varphi = 0. \]  \(2.1\)
Here $\varphi$ is the order parameter, $t$ is the dimensionless time, $x$ is the dimensionless space coordinate, $c=1$ is the characteristic velocity, $m$ is the dimensionless mass and $\beta$ is the parameter of nonlinearity.

Eq.(2.1) follows from Hamilton’s equations (see, for example, [5]):

\[ \varphi_t = \{H, \varphi\}, \pi_t = \{H, \pi\} \]  

with Hamiltonian:

\[ H = \frac{1}{2} \int \left[ \pi^2 + \varphi_x^2 + 2 \left( \frac{m}{\beta} \right)^2 (1 - \cos \beta \varphi) \right] dx \equiv \int h(\varphi, \pi). \]  

In eqs. (2.2) -(2.3) $\varphi$ and $\pi$- are canonically conjugated coordinate and momentum:

\[ \{\pi(x), \varphi(y)\} = \delta(x - y), \]  

where the Poisson brackets are defined by usual way:

\[ \{A, B\} = \int \left( \frac{\delta A}{\delta \pi(x)} \frac{\delta B}{\delta \varphi(x)} - \frac{\delta A}{\delta \varphi(x)} \frac{\delta B}{\delta \pi(x)} \right) dx. \]  

It is easy to see that the functionals of total momentum, $P$, of the system:

\[ P = -\int \pi(x, t) \frac{\partial \varphi(x, t)}{\partial x} dx \]  

and total "angular" momentum $K$:

\[ K = \int x h[\pi, \varphi] dx \]  

commute with Hamiltonian (2.3).

It is well known that there are two types of localized excitations (LE) in SG model: solitons and breathers. Soliton is a one parameter solution, breather is a two parameter solution, the breather can be interpreted as bound states of two solitons (see [4,5]). The one parameter solution of the SG model, often named as kink, can be written as:

\[ \varphi_s = -4 \frac{\epsilon}{\beta} \arctg \left[ \exp \left( \delta_s^{-1}(x - v_s t - x_{0s}) \right) \right]. \]

Breather solution has the following form:

\[ \varphi_b = \frac{4}{\beta} \arctg \left( \frac{\omega_2}{\omega_1} \right) \frac{\sin \left[ \omega(v_b t - k(v_b)x - \varphi_{0b}) \right]}{\cosh \left( \delta_b^{-1}(x - v_b t - x_{0b}) \right)}, \]
where:
\[ m(v) = \frac{m}{\sqrt{1 - v^2}}, \quad \omega(v) = m(v)\omega_1, \quad k(v) = vm(v)\omega_1, \]
\[ \delta_s^{-1} = m(v_s), \quad \delta_b^{-1} = \omega_2 m(v_b). \] (2.10)

Here \( \epsilon = \{+1, -1, 0\} \) is the soliton topological charge, +1 corresponds to soliton, -1 to antisoliton, 0 to breather, \( v_s \) and \( v_b \) are the soliton and breather velocities, \( \delta_s \) and \( \delta_b \) are the soliton and breather sizes, \( x_{0s} \) and \( x_{0b} \) are the initial coordinates of soliton and breather, \( \varphi_{0b} \) is the initial breather phase, \( \omega_1 \) and \( \omega_2 \) are the breather parameters satisfying the following condition (see [5]):
\[ \omega_1^2 + \omega_2^2 = 1. \] (2.11)

When \( \omega_2 \to 0 \), the breather solution reduces to the following ones:
\[ \lim_{\omega_2 \to 0} \varphi_b(x, t) = \varphi_p(x, t) = 4\frac{\omega_2}{\beta} \frac{\sin[\omega(v)t - k(v)x - \varphi_{0b}]}{\cosh[m(v)\omega_2(x - kt/\omega - x_{0b})]}, \] (2.12)

and gives a plane wave solution i.e. a phonon.

The phonon frequency is related with wave vector \( k \) by formula:
\[ \omega_{ph}^2 (k) = m^2 + k^2. \] (2.13)

The breather velocity, when \( \omega_2 \to 0 \), reduces to the phonon group velocity:
\[ v_{ph} = \left( k/\omega_{ph} \right) = \left( \partial \omega_{ph} / \partial k \right), \] (2.14)

and \( 4\omega_2 \) becomes the amplitude of phonon oscillations.

From formulas (2.8), (2.9) and (2.3) it is easy to find the soliton and the breather energy:
\[ E_s = M_s/\sqrt{1 - v_s^2}, \quad E_b = M_b/\sqrt{1 - v_b^2}. \] (2.15)

The soliton and the breather rest masses are given by:
\[ M_s = 8m, \quad M_b = 16m\omega_2. \] (2.16)

When \( \omega_2 \to 1 \), the breather reduces to soliton-antisoliton bound state with a mass 16\( m \). The difference between the breather energy on and the sum of energies of isolated soliton and antisoliton gives the binding energy of the breather:
\[ \Delta E = -\frac{16m}{\sqrt{1 - v^2}} (1 - \omega_2) \] (2.17)

If the system consists of \( n_1 \) solitons and \( n_2 \) breather we have:
\[ E = \sum_{s=1}^{n_1} E_s + \sum_{b=1}^{n_2} E_b, \quad P = \sum_{s=1}^{n_1} P_s + \sum_{b=1}^{n_2} P_b, \quad K = \sum_{s=1}^{n_1} K_s + \sum_{b=1}^{n_2} K_b. \] (2.18)
Energy and momentum are related by the following formula:

\[ E^2 = M^2 + P^2. \]  

(2.19)

Formulae (2.18) allow us to consider solitons and breathers as elementary excitations of field \( \varphi \).

3. COLLISIONS OF LOCALIZED EXCITATIONS

Due to the 1D of the problem the collision of solitons and breather take place without dependence from values \( x_{0s} \) and \( x_{0b} \), \( s = 1, \ldots, n_1, b = 1, \ldots, n_2 \) because their velocities do not change. This peculiar property follows from energy and momentum conservation laws in Sine-Gordon equation as well as other exactly integrable models (see [4,5]). Those conservation laws have the form:

\[ v_1 = v_1'; \quad v_2 = v_2'. \]  

(3.1)

Here and later the values after collision will be devoted with a prime.

To analyze collision processes it is necessary to take into account the "angular" momentum conservation law, which can be written as:

\[ x_1 E_1 + x_2 E_2 = x_1' E_1' + x_2' E_2'. \]  

(3.2)

The result of collision is coordinate shifts (changes) and for breathers phase shifts as well. Furthermore only pair collisions exist and there are no many particles effects [4]. Therefore formulae (3.1),(3.2) provide the general framework for studying the effects of collisions. In this section conservation laws and coordinate \( x_{01} \) and phase changes \( \varphi_{01} \) for all type of collisions will be written down explicitly from general formulas [4,5]. For simplicity the index "0" will be omitted.

**Soliton-soliton collisions.** Conservation laws for two soliton collision have the following form:

\[ v_{1s}' = v_{1s}, \quad v_{2s}' = v_{2s}, \quad (x_{1s}' - x_{1s}) E_{1s} + (x_{2s}' - x_{2s}) E_{2s} = 0 \]  

(3.3)

Coordinates shifts are given by:

\[ \Delta x_{1s} = x_{1s}' - x_{1s} = \frac{4}{E_{1s}} \text{sign}(v_0) \ln |Z_{ss}| = \frac{\delta_{1s}}{2} \text{sign}(v_0) \ln |Z_{ss}| \]

\[ \Delta x_{2s} = x_{2s}' - x_{2s} = -\frac{4}{E_{2s}} \text{sign}(v_0) \ln |Z_{ss}| = -\frac{\delta_{2s}}{2} \text{sign}(v_0) \ln |Z_{ss}|, \]  

(3.4)

where

\[ Z_{ss} = \frac{1 + \sqrt{1 - v_0^2}}{1 - \sqrt{1 - v_0^2}}, \quad v_0 = \frac{(v_{1s} - v_{2s})}{(1 - v_{1s} v_{2s})}. \]  

(3.5)
Here $v_0$ is the velocity of relative motion of solitons which in the general case can be written as:

$$v_0 = \frac{(v_{1i} - v_{2k})}{(1 - v_{1i}v_{2k})}, \quad i, k = s, b, ph. \quad (3.6)$$

**Soliton - breather collision.** In this case conservation laws can be written as:

$$v_{1s} = v'_{1s}, \quad v_{2b} = v'_{2b}, \quad \omega_{22} = \omega'_{22},$$

$$(x'_{1s} - x_{1s})E_{1s} + (x'_{2b} - x_{2b})E_{2b} = 0. \quad (3.7)$$

Coordinate shifts $\Delta x$ are described by the following formulae:

$$\Delta x_{1s} = x'_{1s} - x_{1s} = \frac{8}{E_{1s}} \text{sign}(v_0) \ln |Z_{sb}| = \delta_{1s} \text{sign}(v_0) \ln |Z_{sb}|$$

$$\Delta x_{2b} = x'_{2b} - x_{2b} = -\frac{8}{E_{2b}} \text{sign}(v_0) \ln |Z_{ss}| = -\frac{\delta_{2b}}{2} \text{sign}(v_0) \ln |Z_{sb}|, \quad (3.8)$$

where:

$$Z_{sb} = \frac{1 + \omega_{2,b2} \sqrt{1 - v_0^2}}{1 - \omega_{2,b2} \sqrt{1 - v_0^2}} \quad v_0 = \frac{(v_{1s} - v_{2b})}{(1 - v_{1s}v_{2b})}. \quad (3.9)$$

The breather phase shift $(\Delta \varphi)_b$ is determined by the formula:

$$\tan(\Delta \varphi)_b = -\text{sign}(v_0) \sqrt{1 - \frac{v_0^2}{\omega_{1,b2}}} \quad (3.10)$$

Some comments regarding formulae (3.8) and (3.10) are in order. For soliton coordinate shift (formula (3.8)) due to soliton - breather collision one has a factor of "8" instead of "4" as in formula (3.4). This difference can be explained as follows. If $\omega_{2,b2} \rightarrow 1$, one can considered a breather as two solitons with opposite topological charge. Since the coordinate shift does not depend on topological charge, the soliton - breather collision in this case can be considered as a soliton - two solitons collision, and thus the shift is doubled. Let us mention that in the case $\omega_{2,b2} \rightarrow 1$, the quantities $Z_{ss}$ and $Z_{sb}$ are the same.

Formula (3.10) implies that the phase shift is in general large: $(\Delta \varphi)_b \sim \pi$. It is small only in the ultrarelativistic case $|v_s - 1| << 1$. In this case $(\Delta \varphi)_{b1} \approx -\text{sign}(v_0) \sqrt{1 - v_0^2} \omega_{1,b1}$. If $v_s << 1$, then $(\Delta \varphi)_{b1} \approx \pi/2$.

**Soliton-phonons collision.** This process is characterized by the following conservation laws:

$$v'_{1s} = v_{1s}, \quad v'_{2ph} = v_{2ph}, \quad \omega'_{ph} = \omega_{ph},$$

$$(x'_{1s} - x_{1s})E_{1s} + (x'_{2ph} - x_{2ph})E_{2ph} = 0. \quad (3.11)$$
The expressions for the $x$ and $\varphi$ shifts are:

$$\Delta x_{2ph} = x'_{2ph} - x_{2ph} = -\text{sign}(v_0) \frac{m \sqrt{1 - v_0^2}}{\omega_{ph}(\omega_{ph} - kv_s)}$$  (3.12)

$$\Delta x_{1s} = x'_{1s} - x_{1s} = -\frac{E_{2ph}}{E_{1s}} \Delta x_{2ph}$$  (3.13)

$$\tan(\Delta \varphi_{2ph}) = -\text{sign}(v_0) \frac{\sqrt{1 - v_0^2}}{v_0}.$$  (3.14)

Phonon energy $E_{2ph}$ and frequency $\omega_{ph}(k)$ are defined by the formulae:

$$E_{2ph} = 16 \omega_2 \omega_{ph}(k), \quad \omega_{ph}^2(k) = m^2 + k^2.$$  (3.15)

**Breather - breather collisions.** The corresponding conservation law have the form:

$$v_{1b}' = v_{1b}, \quad v_{2b}' = v_{2b}, \quad \omega_{2,b1} = \omega_{2,b1}, \quad \omega_{2,b2} = \omega_{2,b2}$$

$$(x_{1b}' - x_{1b})E_{1b} + (x_{2b}' - x_{2b})E_{2b} = 0.$$  (3.16)

The expressions for the $x_b$ and $\varphi$ shifts are:

$$\Delta x_{1b} = \frac{8}{E_{1b}} \text{sign}(v_0) \ln |Z_{bb}Z_{bb}'| = \frac{\delta_{1b}}{2} \text{sign}(v_0) \ln |Z_{bb}Z_{bb}'|$$  (3.17)

$$\Delta x_{2b} = -\frac{8}{E_{2b}} \text{sign}(v_0) \ln |Z_{bb}Z_{bb}'| = -\frac{\delta_{2b}}{2} \text{sign}(v_0) \ln |Z_{bb}Z_{bb}'|$$

$$\tan(\Delta \varphi_{1b}) = \text{sign}(v_0) \frac{2v_0 \sqrt{1 - v_0^2} \sin \psi_1 \cos \psi_2}{v_0^2 + (1 - v_0^2)(\cos^2 \psi_1 - \cos^2 \psi_2)}.$$  (3.18)

$$\tan(\Delta \varphi_{2b}) = -\text{sign}(v_0) \frac{2v_0 \sqrt{1 - v_0^2} \sin \psi_2 \cos \psi_1}{v_0^2 + (1 - v_0^2)(\cos^2 \psi_1 - \cos^2 \psi_2)}.$$  (3.19)

Here:

$$Z_{bb} = \frac{1 - \sqrt{1 - v_0^2} \cos(\psi_1 + \psi_2)}{1 - \sqrt{1 - v_0^2} \cos(\psi_1 - \psi_2)}, \quad Z_{bb}' = \frac{1 + \sqrt{1 - v_0^2} \cos(\psi_1 - \psi_2)}{1 + \sqrt{1 - v_0^2} \cos(\psi_1 + \psi_2)}.$$  (3.20)

The angles $\psi_1$ and $\psi_2$ are related to the parameters $\omega_{2,b1}$ and $\omega_{2,b2}$ by the formulae:

$$\sin \psi_1 = \omega_{2,b1}, \quad \sin \psi_2 = \omega_{2,b2}.$$  (3.21)

**Breather - phonon collision.** Since phonons are the limiting case of breather when $\omega_2 \to 0$, formulae for breather - phonons collision can be obtained from ones in the previous paragraph by passing to the limit $\omega_{2,b2} \to 0$. It means that the second breather
reduces to phonon wave packet and the notion of group velocity and coordinate of wave packet automatically appears. The breather velocity \( v_b \) reduces to the group velocity of the wave packet and the coordinate \( x_b \) reduces to the coordinate of the center of the wave packet.

The corresponding conservation law has the form:

\[
v'_{1b} = v_{1b}, \quad v'_{2ph} = v_{2ph}, \quad \omega'_{2,b1} = \omega_{2,b1}, \quad \omega'_{2,ph2} = \omega_{2,ph2}
\]

\[
(x'_{1b} - x_{1b})E_{1b} + (x'_{2ph} - x_{2ph})E_{2ph} = 0,
\]

where:

\[
\Delta x_{2ph} = -\text{sign}(v_0) \frac{2m\omega_{2,b1}\sqrt{1 - v_b^2}}{\omega_{ph}[\omega_{ph} - kv_b][1 - (1 - v_0^2)\omega_{2,b1}^2]}, \quad \Delta x_{1b} = -\frac{E_{2ph}}{E_{1b}}\Delta x_{2ph}
\]

\[
\tan(\Delta \varphi_{1b}) = -\text{sign}(v_0) \frac{2v_0\sqrt{1 - v_0^2}\omega_{2,b1}}{2v_0^2 - 1 + (1 - v_0^2)\omega_{1,b1}^2}
\]

\[
\Delta \varphi_{2ph} = -\text{sign}(v_0) \frac{2v_0\sqrt{1 - v_0^2}\omega_{2,ph2}\omega_{1,b1}}{2v_0^2 - 1 + (1 - v_0^2)\omega_{2,b1}^2}.
\]

**Phonon-phonon collisions.** As it is well known, in linear theory phonon-phonon collisions are absent. Here phonon - phonon interaction is the result of nonlinearity of the Sine-Gordon equation. Corresponding relations can be obtained from ones in previous paragraph by passing the limit \( \omega_{2,b1} \to 0 \). By considering \( \omega_{2,b1} \) as a small quantity, \( \omega_{2,b1} << 1 \), one has:

\[
v'_{1ph} = v_{1ph}, \quad v'_{2ph} = v_{2ph}, \quad \Delta x_{1ph}E_{1ph} + \Delta x_{2ph}E_{2ph} = 0.
\]

The shift of phonons coordinate is:

\[
\Delta x_{1ph} = \text{sign}(v_0) \frac{32}{E_{1ph}} \sqrt{1 - v_0^2}\omega_{2,ph2}\omega_{1,ph1} \quad \Delta x_{2ph} = -\frac{E_{1ph}}{E_{2ph}}\Delta x_{1ph}.
\]

where:

\[
E_{1ph} = 16\omega_{21ph}\omega_{1ph}, \quad E_{2ph} = 16\omega_{22ph}\omega_{2ph}.
\]

The change of phonon phase is:

\[
\Delta \varphi_{1ph} = -2\text{sign}(v_0)\omega_{2,ph1} \frac{\sqrt{1 - v_0^2}}{v_0}, \quad \Delta \varphi_{2ph} = -2\text{sign}(v_0)\omega_{2,ph2} \frac{\sqrt{1 - v_0^2}}{v_0}.
\]

The examination of phonons as a limiting case of breathers gives the possibility to consider changes of phonon phase and coordinate simultaneously.
Let us analyze phase shifts of breathers and phonons. It is easy to see from formulae (3.10)-(3.12) and (3.14) that breather and phonon shifts are small for soliton-breather and soliton-phonon collisions in the general case. For breather-phonon collisions breather and phonon shifts are essentially different (see (3.25),(3.29)). Breather phase shift are proportional to $\omega_{2ph}$ and therefore small. Phonon phase shift ($\Delta \varphi_{ph} \approx 1$). For phonon-phonon collisions phase shifts are small (see (3.33)). This agrees with the standard linear theory of phonons. In accordance with the usual theory of phonons shifts of phonon wave packets do not take place.

In conclusion, let us emphasize that soliton topological charge is conserved for each type of collisions. The processes leading to soliton-antisoliton bound state formation and the reverse processes do not take place because of energy conservation. Thus in the framework of the exactly integrable Sine-Gordon model with Hamiltonian (2.3) the numbers of solitons, antisolitons, breathers and phonons are defined by initial conditions at each point of $(x,t)$. Interactions destroying the integrability of the system lead to possibility of energy exchange between the LE and to the formation and decay of breathers. In the present paper only the processes with conservation of energy and topological charge have been considered. The processes with energy exchange will be analyzed in a future publication.

4. PROBABILITY OF SCATTERING PROCESSES

To construct collision integrals for processes considered in the present section let us analyze the probability of the corresponding scattering process. The simplest process is soliton - soliton scattering. By definition the initial state $(i)$ of a soliton pair is described by its velocities and center of gravity coordinates:

$$i \equiv (x_1, v_1, x_2, v_2) \equiv (1,2).$$

The final state $(f)$ is defined by:

$$f \equiv (x'_1, v'_1, x'_2, v'_2) \equiv (1', 2').$$

Current density of solitons per unit density is given by:

$$j_1 = |v_0|.$$

Taking into account formula (4.3) and the fact that two solitons collide in any case, the transition probability per unit time from state $(i)$ to state $(f)$ can be presented as:

$$W_{i \rightarrow f} \equiv W(1', 2'|1, 2) = |v_0| \delta(v'_1 - v_1) \delta(v'_2 - v_2) \delta(E_1/E_2)(x'_1 - x_1) + (x'_2 - x_2)) \delta(x'_1 - x_1 - \Delta x(v_1, v_2)).$$
In this formula the first two delta functions describe energy conservation laws, the third one - "angular" momentum conservation law, and last delta function describes coordinate shift. The coordinate shift of second soliton follows from the "angular" momentum conservation law. The coordinate shift $\Delta x_1$ is defined by expression (3.4), (3.5).

Let us emphasize that the probability defined by formula (4.4) does not satisfy the detailed balance principle in the standard form.

It is convenient to rewrite formula (4.4) in the form:

$$W(1', 2'|1, 2) =$$

$$= R_{ss}(v'_1, v'_2|v_1, v_2)\delta(x'_1 - x_1 - \Delta x_1(v_1, v_2))\delta(x'_2 - x_2 - \Delta x_2(v_1, v_2)). \quad (4.5)$$

where:

$$R_{ss}(v'_1, v'_2|v_1, v_2) = |v_0|\delta(v'_1 - v_1)\delta(v'_2 - v_2). \quad (4.6)$$

Let us note that:

$$R_{ss}(v'_1, v'_2|v_1, v_2) = R_{ss}(v_1, v_2|v'_1, v'_2) = R_{ss}(v'_2, v'_1|v_2, v_1). \quad (4.7)$$

For the probability of the inverse process the following expression can be written:

$$W(1, 2|1', 2') =$$

$$= R_{ss}(v'_1, v'_2|v_1, v_2)\delta(x_1 - x'_1 + \Delta x_1(v_1, v_2))\delta(x_2 - x'_2 + \Delta x_2(v_1, v_2)). \quad (4.8)$$

Let us emphasize that the probability defined by formula (4.4) does not satisfy the detailed balance principle in the standard form:

$$\tilde{W}(1', 2'|1, 2) = \tilde{W}(1^*, 2^*|1'^*, 2'^*). \quad (4.9)$$

In (4.9) the following notation has been used: $(1^*) \equiv (x_1, -v_1)$.

For further analysis it is convenient to present the expression (4.7) for the probability $W$ as a sum of two parts $W^r$ and $W^m$:

$$W_{ss}(1, 2|1', 2') = W^r_{ss}(1, 2|1', 2') + W^m_{ss}(1, 2|1', 2'), \quad (4.10)$$

where:

$$W^r_{ss} = \frac{1}{2} R_{ss}[\delta(x_{1s} - x'_{1s} - \Delta x_{1s})\delta(x_{2s} - x'_{2s} - \Delta x_{2s}) -$$

$$-\delta(x_{1s} - x'_{1s} + \Delta x_{1s})\delta(x_{2s} - x'_{2s} + \Delta x_{2s})]. \quad (4.11)$$

$$W^m_{ss} = \frac{1}{2} R_{ss}[\delta(x_{1s} - x'_{1s} - \Delta x_{1s})\delta(x_{2s} - x'_{2s} - \Delta x_{2s}) +$$

$$+\delta(x_{1s} - x'_{1s} + \Delta x_{1s})\delta(x_{2s} - x'_{2s} + \Delta x_{2s})]. \quad (4.12)$$
The probability $W_{ss}^r$, as it will be shown later, describes the effect of renormalization of soliton velocity, and the probability $W_{ss}^m$ describes the homogenization of solitons phenomena i.e. the effect of mixing of states. The quantity $W_{ss}^m$ is related with entropy production in the soliton gas and the kinetic coefficients (see sections 5,6).

It is easy to convince oneself that:

\[ W_{ss}^m(1',2'|1,2) = W_{ss}^m(1,2|1',2'). \] (4.13)

This equality means that for the dissipative part of the transition probability the usual detailed balance principle is valid.

Let us consider now the general case of LE $i$ and $k$, $(i, k = s, b, ph)$ for soliton, breather and phonon correspondingly. The probability of such type of collisions per unit time $W_{ik}$ can be defined by following formula:

\[ W_{ik} \equiv W_{ik}(1',2'|1,2) = R_{ik}(V_i', V_k'|V_i, V_k) = |v_0| \delta(V_i - V_i') \delta(V_k - V_k'). \] (4.15)

In formulae (4.14),(4.15) the following notations have been used. Numbers 1 and 2 mean the set of variables defining of the colliding LE states, $d_i$, is the shift of LE coordinate or phase. In the case of solitons $1 \equiv (x_s, v_s)$, for breather or phonon $1 \equiv (x_i, \varphi_i, v_i, \omega_2 i)$, $i = b, ph$. For simplicity two component coordinate $X_i$ and two component velocity $V_i$ have been introduced in the following way:

\[ X_s = x_s, V_s = v_s, \quad X_i = (x_i, \varphi_i), V_i = (v_i, \omega_2 i), i = b, ph. \] (4.16)

In (4.15), the delta function, $\delta ( V_i - V_i' )$, describes conservation law of energy and momentum of each LE; in the expression (4.14), the delta function $\delta (X_i' - X_i - d_i(1,2))$ defines the coordinate and phase for breather and phonon shifts after collision.

From the definition of $R_{ik}$ it follows that:

\[ R_{ik} = R_{ik}(V_i', V_k'|V_i, V_k) = R(V_i, V_k|V_i', V_k'). \] (4.17)

Later only the case of weak inhomogeneous gas of LE will be considered, i.e. when the characteristic length $l$ of the distribution function is much larger than the shift $\Delta X_i$:

\[ |l| >> |\Delta X_i| \] (4.18)

The mean free path of LE is:

\[ l \sim \frac{1}{n}, \] (4.20)

because the cross section of scattering is equal to 1.
The coordinate shifts of LE are of the order of \( \delta_i \) (see section 3), where

\[ \delta_i(v, \omega_2) \sim \frac{1}{m(v)\omega_2} \]  

Therefore the condition (4.18) rewritten in the form:

\[ \frac{1}{n_i} >> \delta_i \quad \text{or} \quad m(v)\omega_2 >> \frac{1}{l} \sim n_i \quad i = s, b, ph, \]  

means that average distance between LE much larger their sizes. In other words the condition of small inhomogeneous (4.18) coincides with the condition of small density of LE gas.

Let us present \( W_{ik} \) as a sum of two parts \( W^r_{ik} \) and \( W^m_{ik} \) by the same way as (4.10)-(4.12). Assuming that (4.18) is valid, we can expand the delta functions in the expression (4.14) for \( W_{ik}(1, 2|1, 2) \) in powers series \( \Delta X_i \). Then the expressions for \( W^r_{ik} \) and \( W^m_{ik} \) become:

\[
W^r_{ik}(1', 2'|1, 2) = R_{ik}(1'2', |1, 2)[d_i \frac{\partial}{\partial X_i} + d_k \frac{\partial}{\partial X_k}]\delta(X_i - X'_i)\delta(X_k - X'_k) \]  

\[
W^m_{ik}(1', 2'|1, 2) = R_{ik}(1'2', |1, 2)[2 + d_i ^2 \frac{\partial ^2}{\partial X_i^2} + d_k ^2 \frac{\partial ^2}{\partial X_k^2} + 2d_i d_k \frac{\partial ^2}{\partial X_i \partial X_k}]\delta(X_i - X'_i)\delta(X_k - X'_k). \]

5. COLLISION INTEGRALS

In the present section the expressions for the collision integrals for all type of LE in SG equation will be derived. It is easier to do this starting from the formulae for probabilities of collisions. Let us introduce following notations:

\[ f = f(X, V, t), B = B(X, V, t), N = N(X, V, t). \]  

(5.1)

for the distribution functions of solitons, breathers and phonons as \( f, B \) and \( N \) correspondingly. The probability to find the LE in the state \((1, 1 + d1)\) can be defined the usual way:

\[ dW_i = F_i(1)d1, \]  

(5.2)

where \( i = s, b, ph \) and \( F_s = f, F_b = B, F_{ph} = N \). The total result of collisions of a sample LE with the other LE is the sum of each collision. This is due to the special type of interaction in exactly integrable models. Therefore the general collision integral can
be presented as a sum of partial collision integrals. For example in the case of solitons the collision integral can be written as:

\[ \mathcal{L}_s = \mathcal{L}_{ss}\{f, f\} + \mathcal{L}_{sb}\{f, B\} + \mathcal{L}_{sph}\{f, N\}, \]  

(5.3)

where \( \mathcal{L}_{ss}, \mathcal{L}_{sb}, \mathcal{L}_{sph} \) are soliton-soliton, soliton-breather and soliton-phonon collision integrals respectively, which have the following form:

\[ \mathcal{L}_{ss}\{f, f\} = \int d1'd2'd2\{W_{ss}(1, 2|1', 2')f(1')f(2') - W_{ss}(1', 2'|1, 2)f(1)f(2)\} \]  

(5.4)

\[ \mathcal{L}_{sb}\{f, B\} = \int d1'd2'd2\{W_{sb}(1, 2|1', 2')f(1')B(2') - W_{sb}(1', 2'|1, 2)f(1)B(2)\} \]  

(5.5)

\[ \mathcal{L}_{sph}\{f, N\} = \int d1'd2'd2\{W_{sph}(1, 2|1', 2')f(1')N(2') - W_{ss}(1', 2'|1, 2)f(1)N(2)\}. \]  

(5.6)

The first term in formula (5.4) and in formulae (5.5) and (5.6) describes solitons "arriving" at the state (1) as the result of collisions and second term describes solitons "leaving" this state.

The general expression for the collision integral has the form:

\[ \mathcal{L}_i = \sum_k \mathcal{L}_{ik}\{F_iF_k\}. \]  

(5.7)

Here:

\[ \mathcal{L}_{ik}\{F_iF_k\} = \int d1'd2'd2\{W_{ik}(1, 2|1', 2')F_i(1')F_k(2') - W_{ik}(1', 2'|1, 2)F_i(1)F_k(2)\}. \]  

(5.8)

Let us discuss ones more the detailed balance principle for usual particles in the form:

\[ \tilde{W}(1', 2'|1, 2) = \tilde{W}(1, 2|1', 2') \]  

(5.9)

In this form detailed balance principle describes two particle collision (generalization to three, four, ... particle collision is well known). Formula (5.9) means that "arriving" number in state \((1', 2')\) from state \((1, 2)\) is equal to the "leaving" number from state \((1', 2')\) to state \((1, 2)\). If the total number of "arriving" particles to some fixed state \((1, 2)\) is equal to the total number of "leaving" particles from state \((1, 2)\) then new smoothened local balance principle can be formulated as:
\[ \int W_{ik}(1, 2|1', 2')d1'd2' = \int W_{ik}(1', 2'|1, 2)d1'd2' \]  
\[(5.10)\]

where \( i, k = s, b, ph. \)

It is not difficult to show that the probabilities of collisions defined in section 3 satisfy this condition. Thus the probabilities, \( W_{ik} \), of LE scattering processes in SG model satisfy the smoothened local balance principle smoothened local balance principle (5.10), while the dissipation parts of probabilities \( W_{ik}^m \) satisfy detailed balance principle (5.9).

The distribution functions \( f, B, N \) in thermodynamic equilibrium must satisfy the following condition:

\[ \mathcal{L}_{ik}\{F_i, F_k\} = 0, \]  
\[(5.11)\]
in accordance with the smoothened local equilibrium principle for each \( i - k \) collision integral. It is necessary to emphasize that smoothened local balance principle (5.10) puts limitation on the probabilities of collisions, on the thermodynamic equilibrium condition (5.11) and on the distribution functions \( F_i \).

Having mentioned the general properties of collision integrals of LE in the SG equation, we proceed with the examination of each one. From formulas (4.21)-(4.22) it is easy to obtain the following expression for the collision integral for solitons in the low density case:

\[ \mathcal{L}_s\{f_1, F_2i\} = -\delta v_s \frac{\partial f_1}{\partial x} + D_s(v_s) \frac{\partial^2 f_1}{\partial x^2}, \]  
\[(5.12)\]

where renormalization of soliton velocity \( \delta v \) and local coefficient of self-diffusion \( D_s(v) \) are given by summing the partial contributions:

\[ \delta v_s = \sum_i \delta v_{si}, \quad D_s(v_s) = \sum_i D_{si}(v_s). \]  
\[(5.13)\]

Here

\[ \delta v_{si} = \int |v_0| \Delta x_{si}(v_{1s}, v_{2i}) F_i d2, \]  
\[(5.14)\]

\[ D_{si} = \frac{1}{2} \int |v_0|[\Delta x_{si}(v_{1s}, v_{2i})]^2 F_i d2 \]  
\[(5.15)\]

The collision integrals for breathers can be analyzed by a similar way, but with a very important difference, connected with the conditions (4.20) and (4.21). The simplest case of breather ensemble is a ensemble with distribution function of the form:

\[ B(x, \varphi, v, \omega_2, t) = B(x, \varphi, v, t)\delta(\omega_0 - \omega_2), \]  
\[(5.16)\]

where \( \omega_0 \) and density of particles \( n_i \) satisfy the condition: \( \omega_0 m(v) >> n_i \).

It is possible to consider more general breather distribution functions, e.g.

\[ B = B(x, \varphi, v, t)b(\omega), \]
where $b(\omega)$ has a sharp maximum near the point $\omega = \omega_0$. For simplicity only the case (5.16) will be considered. In this case all integrations under $\omega_2$ are trivial.

For breathers the collision integral has the following form:

$$L_b\{B_1, F_{2i}\} = -\delta v_b \frac{\partial B_1}{\partial x} - \delta \omega_b \frac{\partial B_1}{\partial \varphi} + D_b \frac{\partial^2 B_1}{\partial x^2} + F_b \frac{\partial^2 B_1}{\partial \varphi^2} + 2K_b \frac{\partial^2 B_1}{\partial x \partial \varphi}. \quad (5.17)$$

The first two terms in this formula describe renormalization of breather velocity $\delta v_b$ and its internal oscillation frequency $\delta \omega_b$. The last three terms describe self-diffusion in $(x, \varphi)$ space. As in the soliton case, the quantities $\delta v_b, \delta \omega_b, D_b, F_b$ and $K_b$ are sums of partial contributions:

$$\delta v_b = \sum_i (\delta v_b)_i, \delta \omega_b = \sum_i (\delta \omega_b)_i, D_b = \sum_i (D_b)_i, F_b = \sum_i (F_b)_i, K_b = \sum_i (K_b)_i, \quad (5.18)$$

where:

$$\delta v_{bi} = \int |v_0| \Delta x_{bi}(v_{1b}, v_{2i}) F_{2i} d2 \quad (5.19)$$

$$\delta \omega_{bi} = \int |v_0| \Delta \varphi_{bs}(v_{1b}, v_{2i}) F_{2i} d2 \quad (5.20)$$

$$D_{bi} = \frac{1}{2} \int |v_0| |\Delta x_{bi}(v_{1b}, v_{2i})|^2 F_{2i} d2 \quad (5.21)$$

$$F_{bi} = \frac{1}{2} \int |v_0| |\Delta \varphi_{bi}(v_{1b}, v_{2i})|^2 F_{2i} d2 \quad (5.22)$$

$$K_{bi} = \frac{1}{2} \int |v_0| \Delta x_{bi}(v_{1b}, v_{2i}) \Delta \varphi_{bi}(v_{1b}, v_{2i}) F_{2i} d2. \quad (5.23)$$

6. KINETIC EQUATIONS AND ENTROPY PRODUCTION

The Boltzmann type kinetic equations for LE with collision integrals constructed in previous subsection can be written following the standard way as:

$$\frac{\partial f}{\partial t} + (v + \delta v_s(v)) \frac{\partial f}{\partial x} = D_s(v) \frac{\partial^2 f}{\partial x^2} \quad (6.1)$$

$$\frac{\partial B}{\partial t} + (v + \delta v_b(V)) \frac{\partial B}{\partial x} + (\omega + \delta \omega_b(V)) \frac{\partial B}{\partial \varphi} = D_b(V) \frac{\partial^2 B}{\partial x^2} + 2K_b(V) \frac{\partial^2 B}{\partial x \partial \varphi} + F_b(V) \frac{\partial^2 B}{\partial \varphi^2} \quad (6.2)$$

$$\frac{\partial N}{\partial t} + (v + \delta v_{ph}(V)) \frac{\partial N}{\partial x} + (\omega + \delta \omega_{ph}(V)) \frac{\partial N}{\partial \varphi} = D_{ph}(V) \frac{\partial^2 N}{\partial x^2} + 2K_{ph}(V) \frac{\partial^2 N}{\partial x \partial \varphi} + F_{ph}(V) \frac{\partial^2 N}{\partial \varphi^2} \quad (6.3)$$
Here terms from collision integrals describing velocity renormalization have been written on the left of eqs. (6.1)-(6.3). On the right of these equations there are only those terms describing dissipative processes leading to homogenization of distribution functions of LE.

It is necessary to emphasize that collision integrals in kinetic equations (6.1) - (6.3) are equal to zero in the homogeneous case. Therefore the stationary solution of kinetic equations have following form:

\[ f = f(v), \quad B = B(v, \omega_2). \] (6.4)

Here \( f(v), B(v, \omega_2) \) are arbitrary functions of its arguments.

In other words the kinetic equations (6.1)-(6.2) describe homogenization i.e. mixing of distribution function of LE up to homogeneous state in real space (for breathers - in \((x, \varphi)\) space) and demonstrate that chaotization in momentum space cannot be realized.

For chaotization processes in momentum space it is necessary to exceed the limits of exactly integrable model i.e. to take into account the terms destroying the integrability in the Hamiltonian of the system.

Let us show now that homogenization of distribution function leads to entropy production in the SG localized excitations gas. The entropy of classical soliton gas and boson gases of breathers and phonons can be defined by standard way:

\[ S = \sum_k S_k, \quad k = s, b, ph \] (6.5)

\[ S_k = - \int F_k \ln \left( \frac{F_k}{e} \right) d1. \] (6.6)

The entropy evolution in time is described by the formulae:

\[ \frac{dS_k}{dt} = - \int \frac{\partial F_k(1)}{\partial t} \ln F_k(1) d1. \] (6.7)

Using kinetic equations (6.1)-(6.3), definitions of \( \mathcal{D}, \mathcal{K} \) and \( \mathcal{F} \) one can find that:

\[ \frac{dS_k}{dt} = \int q_k d1, \] (6.8)

where the source \( q \) of the entropy production is:

\[ q_k = \int \sum_i |v_0(1, 2)|_{ki} \frac{F_i}{F_k} \left\{ [\Delta x(1, 2)]_{ki} \frac{\partial F_k(1)}{\partial x} + [\Delta \varphi(1, 2)]_{ki} \frac{\partial F_k(1)}{\partial \varphi} \right\}^2 d2. \] (6.9)

It is obviously that the expression is positive. It means that (6.9) proves the Boltzmann entropy production theorem. We would like to emphasize that the entropy production is connected only with inhomogeneity in real space \((x, \varphi)\). It is easy to see that in homogeneous case \( \partial F/\partial x = \partial F/\partial \varphi = 0 \) and there no entropy production.
7. TRANSPORT EQUATIONS

In the present subsection the consequences from kinetic equations (6.1)-(6.3) will be analyzed. Let us emphasize that the homogenization of LE in real space means the homogenization of LE in temperature, concentration and macroscopic velocity spaces. The local macroscopic temperature \( T(x) \) is defined through the local energy \( E(x) \) averaged over a distance \( d_0 \) around \( x \) satisfying the inequality \( |\Delta X_i| << d_0 \). In other words collisions of LE lead to creation of diffusion, thermoconductivity and intrinsic friction processes. It is easy to note that the transport equations will have the form of local conservation laws for each type of LE separately. In Sine-Gordon system the numbers of solitons, breathers and phonons are conserved separately. Besides, momentum, energy and angular velocity \( \frac{\partial \varphi}{\partial t} \) (in the breather case) of each LE are conserved in each collision. The transport equation can be written in the following general form.

For the solitons:

\[
\frac{\partial}{\partial t} n_s <a_s> + \frac{\partial}{\partial x} [U^r_s + U^m_s] = 0. \tag{7.1}
\]

For the breathers:

\[
\frac{\partial}{\partial t} n_b <a_b> + \frac{\partial}{\partial x} [U^r_b + U^m_b] + \frac{\partial}{\partial \varphi} [W^r_b + W^m_b] = 0. \tag{7.2}
\]

In formulae (7.1)-(7.2) following notation have been used.

For solitons:

\[
n_s <a_s> = \int a(x, v) f(x, v, t) dv \tag{7.3}
\]

\[
U^r_s = \int a(x, v) [v + \delta v] f(x, v, t) dv \tag{7.4}
\]

\[
U^m_s = -\frac{\partial}{\partial x} \int a(x, v) D_s f(x, v, t) dv. \tag{7.5}
\]

For breathers:

\[
n_b <a_b> = \int a(x, v, \varphi, \omega) B(x, v, \varphi, \omega, t) d\omega dv \tag{7.6}
\]

\[
U^r_b = \int a(x, v, \varphi, \omega) [v + \delta v] B(x, v, \varphi, \omega, t) d\omega dv \tag{7.7}
\]

\[
U^m_b = -\frac{\partial}{\partial x} \int a(x, v, \varphi, \omega) D_b B(x, v, \varphi, \omega, t) d\omega dv - \frac{\partial}{\partial \varphi} \int a(x, v, \varphi, \omega) K_b B(x, v, \varphi, \omega, t) d\omega dv. \tag{7.8}
\]
\[ W_b^r = \int a(x, v, \varphi, \omega)[\omega + \delta \omega]B(x, v, \varphi, \omega, t)dv\omega \]  
(7.9)

\[ W_b^m = -\frac{\partial}{\partial \varphi} \int a(x, v, \varphi, \omega)F_bB(x, v, \varphi, \omega, t)dv\omega - \frac{\partial}{\partial x} \int a(x, v, \varphi, \omega)K_bB(x, v, \varphi, \omega, t)dv\omega. \]  
(7.10)

Substituting \( a \), velocity and energy of LE for the variable \( a \) it is easy to obtain following transport equations: continuity equations, hydrodynamics equations and equations for local energy density. Let us write these equations in the explicit form.

*Continuity equations* \((a = 1)\) can be written as:

\[ \frac{\partial n_s}{\partial t} + \frac{\partial}{\partial x}(j^r_s + j^m_s) = 0 \]  
(7.11)

\[ \frac{\partial n_b}{\partial t} + \frac{\partial}{\partial x}(j^r_b + j^m_b) + \frac{\partial}{\partial \varphi}(i^r_b + i^m_b) = 0. \]  
(7.12)

Here the standard notation \( j_i \) and \( i_i \) have been used for \( U_i \) and \( W_i \) with \( a = 1 \) respectively.

*Hydrodynamics equations* can be derived from equations (7.1),(7.2) with \( a = v \) and \((v, \omega_2)\) for solitons and breathers respectively and have the following form:

\[ \frac{\partial}{\partial t} n_su_s + \frac{\partial}{\partial x}(P^r_s + P^m_s) = 0, \]  
(7.13)

\[ \frac{\partial}{\partial t} n buoy_b + \frac{\partial}{\partial x}(P^r_b + P^m_b) + \frac{\partial}{\partial \varphi}(\Pi^r_b + \Pi^m_b) = 0 \]  
(7.14)

\[ \frac{\partial}{\partial t} n_ww_b + \frac{\partial}{\partial x}(Q^r_b + Q^m_b) + \frac{\partial}{\partial \varphi}(R^r_b + R^m_b) = 0. \]  
(7.15)

Here \( u_s \) and \( u_b \) are hydrodynamic velocities of solitons and breathers correspondingly, \( w \) is hydrodynamic velocity in \( \varphi \) space, \( P^r_i \) and \( P^m_i \), \( i = s, b \) are pressures for solitons and breathers, the quantities \( \Pi^r_b \) and \( \Pi^m_b \) are pressures of breathers due to inhomogeneities in \( \varphi \) space, values \( Q \) and \( R \) are defined by formulas (7.7)-(7.10) with \( a = \omega_2 \).

*The energy transport equations* can be derived when \( a = E_i \) for solitons and breathers. These equations can be written as:

\[ \frac{\partial}{\partial t} n_sT_s + \frac{\partial}{\partial x}(U^r_s + U^m_s) = 0, \]  
(7.16)

\[ \frac{\partial}{\partial t} n_bT_b + \frac{\partial}{\partial x}(U^r_b + U^m_b) + \frac{\partial}{\partial \varphi}(W^r_b + W^m_b) = 0. \]  
(7.17)
The quantities $T_i$ means the average energies of corresponding LE. When $u_i = w_i = 0$ the quantities $T_i$ are the average energies of chaotic motion. $U_i$ and $W_i$ are energy density currents; one can conclude that only $U_i^m$ and $W^m_i$ are different from 0 when $u_i = w_i = 0$.

Let us emphasize the important property of transport equations (7.11)-(7.17). It is easy to see that for any homogeneous distribution functions:

$$f = f(v,t), \quad B = B(v,\omega_2, t) \quad (7.18)$$

with constant temperatures, hydrodynamic velocities and chemical potentials $\mu_i$ then all dissipative terms in these equations are equal to zero. In other words there no energy and momentum exchange between homogeneous gases of solitons and breathers.

This special property of equations (7.14) - (7.16) is eliminated by taking into consideration terms in Hamiltonian which destroy the integrability of the model.

As an example of explicit calculation of transport coefficients we shall obtain the expression for the self-diffusion coefficients assuming that the distribution function of solitons and breathers have following form:

$$f = C_s e^{-m_s v_s^2 / 2k_B T_s}, \quad B = C_b e^{-E_b / k_B T_b} \delta(\omega_2 - \omega_0). \quad (7.19)$$

Let us discuss the concrete expressions for solitons and breathers diffusion coefficients due to soliton-solitons, breather - breathers and solitons breathers collisions. For simplicity we will consider the nonrelativistic solitons and breathers only ($v << 1$).

This case corresponds to small temperatures $T << m$.

The diffusion current of solitons can be presented as:

$$j^d_s = \frac{\partial n_s}{\partial x} [n_s D_{ss} + n_b D_{sb}] + \frac{\partial n_b}{\partial x} n_s D_{sb} \quad (7.20)$$

Using formulae for $(\Delta \varphi)_{ik}$ and $(\Delta x)_{ik}$ from paragraph 3 , formulae (5.13), (5.18), (7.5), and (7.19) it is possible to calculate both $D_{ss}$ and $D_{sb}$. We will present here the final results:

$$D_{ss} = (1/4) \delta^2_s (T/\pi M^1_s)^{1/2} \{[\ln(\gamma M^1_s / T)]^2 + C\} \quad (7.21)$$

$$D_{sb} = (1/2) \delta^2_s (2T/\pi \mu_{sb})^{1/2} I_{sb}, \quad (7.22)$$

where:

$$I_{sb} = (\ln \frac{1 + \omega_0}{1 - \omega_0})^2, \quad \text{if} \quad T/\mu_{sb} \ll 1 - \omega_0^2$$

$$I_{sb} = [\ln(2\gamma \mu_{sb} / T)]^2 + C \quad \text{if} \quad 1 \gg T/\mu_{sb} \gg 1 - \omega_0^2. \quad (7.23)$$

Here:
There are two dissipative currents for breathers $j_b^d$ and $i_b^d$, which can be written as:

\[
\begin{align*}
  j_b^d &= (n_b D_{bb} + n_s D_{bs}) \frac{\partial n_b}{\partial x} + n_b D_{bs} \frac{\partial n_s}{\partial x} + (n_b K_{bb} + n_s K_{bs}) \frac{\partial n_b}{\partial \varphi} \\
  i_b^d &= (n_b F_{bb} + n_s F_{bs}) \frac{\partial n_b}{\partial \varphi} + n_b K_{bs} \frac{\partial n_s}{\partial x} + (n_b K_{bb} + n_s K_{bs}) \frac{\partial n_b}{\partial x}
\end{align*}
\]

After routine calculations the coefficients $D_{ik}$, $F_{ik}$ and $K_{ik}$ can be presented in following general form:

\[
\begin{align*}
  D_{bb} &= \frac{\delta_b^2}{4} \left( \frac{T}{\pi M_b} \right)^{\frac{1}{2}} J_{bb}^D, \\
  D_{bs} &= \frac{\delta_b \delta_s}{4\sqrt{2\pi}} \left( \frac{T}{\mu_{bs}} \right)^{\frac{1}{2}} J_{bs}^D, \\
  D_{ik} &= \{D_{ik}, F_{ik}, K_{ik}\}
\end{align*}
\]

Here:

\[
\mu_{ik} = \frac{M_i M_k}{M_i + M_k}, i, k = s, b
\]

where $M_s$ and $M_b$ are defined by formulae (2.16).

The expressions for $J_{ik}^D$ are presented in table1. It is easy to see that the diffusion of breathers in real space (x-space) is much faster then the relaxation on $\varphi$.

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Table 1. Diffusion coefficients for breather

| \( i_k \) | \( T/\mu_{ik} \ll 1 - \omega_0^2 \) | \( 1 - \omega_0^2 \) >> \( T/\mu_{ik} \) |
|----------|----------------|-----------------|
| \( bs \) | \( J_{ik}^D \) \( \ln \left( \frac{1+\omega_0}{1-\omega_0} \right)^2 \) | \( J_{ik}^F \) \( \frac{\pi}{2} \ln \left( \frac{1+\omega_0}{1-\omega_0} \right)^2 + C \) |
| \( bb \) | \( J_{ik}^D \) \( \frac{\pi}{2} \ln \left( \frac{1+\omega_0}{1-\omega_0} \right)^2 \) | \( J_{ik}^F \) \( 4\left( \frac{\ln \left( \frac{\gamma M_b T}{2} \right)}{\pi} \right)^2 + C \) |