On the Universality of Certain Non-Renormalizable Contributions in Two-Dimensional Quantum Field Theory

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Abstract

We consider the question of removing the ultraviolet cutoff in a 2D Quantum Field Theory with an interaction term which is non-renormalizable by power counting. This model arises as the first non-trivial correction beyond the Gaussian approximation of the so called Capillary Wave or Drumhead Model, and is rather important from a physical point of view since it correctly describes the finite size effects of two-dimensional interfaces. Despite the fact that the interaction is non-renormalizable, we prove that for a large class of regularization schemes the finite and divergent parts can be separated in a simple way. Furthermore, the finite part is independent of the choice of cutoff prescription used.
1 Introduction

A model which is widely used to describe fluid interfaces is the so called Capillary Wave Model (CWM) \[1\]. In the literature, this model is also known as Drumhead Model. It is derived from the assumption that the fluctuations of the interface are described by an effective Hamiltonian proportional to the variation of the surface area with respect to the equilibrium solution (which is the solution which minimizes the area). In the case of an interface bounded by a rectangle of size \(L_1 \times L_2\) the Hamiltonian has the following form:

\[
\mathcal{H} = \sigma \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \sqrt{1 + [\nabla \tilde{\phi}(x_1, x_2)]^2},
\]

where \(\sigma\) denotes the interface tension, and the field \(\tilde{\phi}(x_1, x_2)\) describes the interface displacement from the equilibrium position as a function of the two longitudinal coordinates \(x_1\) and \(x_2\). We shall adopt in the following periodic boundary conditions, i.e., \(\tilde{\phi}(x_1, x_2) = \tilde{\phi}(x_1 + L_1, x_2)\) and \(\tilde{\phi}(x_1, x_2) = \tilde{\phi}(x_1, x_2 + L_2)\). It is possible to generalize our results to other boundary conditions.

Due to its non-polynomial nature, until recently the CWM has been studied only in its quadratic approximation. In this limit the Hamiltonian \(\mathcal{H}\) becomes equivalent to a massless 2D Gaussian model, see \[2\] for details. Recently, in \[2, 3\] the first correction beyond the Gaussian approximation was studied in detail. Despite the fact that this corresponds to a scalar field theory with a non-renormalizable interaction term we found strong evidence that the resulting contribution to the interface free energy is universal, i.e. independent of the regularization scheme. We gave some general arguments supporting this assertion and also tested two different regularization schemes, finding the same cutoff independent contribution. In this paper we want to further pursue this analysis, by studying a wider class of regularization schemes.

Let us mention two more reasons to be interested in this model. First, it can be shown by duality, that the Hamiltonian \(\mathcal{H}\) can be used to describe the physics of Wilson loops in the confining regime of Yang-Mills theories. In particular, the model describes Wilson loops beyond the roughening transition, hence in the region (which is the most interesting from a physical point of view) in which the continuum limit can be taken. In this context the Gaussian approximation of the CWM Hamiltonian was discussed for the first time by Lüscher, Symanzik and Weisz in \[4\]. Second, eq. \(\mathcal{H}\) coincides with the Nambu-Goto model for the bosonic string (in a special frame). The corresponding quantum theory is anomalous. Depending on the quantization method one finds either the breaking of rotational invariance or the appearance of interacting longitudinal modes. However, it can be shown \[5\] that in the infrared limit, i.e. when \(L_1\) and \(L_2\) become large, these anomalous features disappear and the string theory becomes a simple Conformal Field Theory, with a well defined quantum behavior. This limit exactly coincides with the Gaussian approximation mentioned above.

This paper is organized as follows: In section 2 we discuss the expansion of the CWM at the first order beyond the Gaussian approximation, to be called 2-loop contribution in the following. We claim that in the ultraviolet limit a finite part of this contribution can be
extracted in a natural way and is independent of the choice of the regularization scheme. This claim is made precise in form of a statement at the end of section 2. In section 3 we explicitly compute the cutoff dependence of the 2-loop contribution for a continuous momentum space cutoff and for the lattice cutoff scheme, thus providing proofs of our statement for these two important regularization schemes.

2 2-Loop Expansion of the CWM

Our aim is to evaluate the partition function

\[ Z = \int D\tilde{\phi} \exp \left\{ -\sigma \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \sqrt{1 + [\nabla\tilde{\phi}(x_1, x_2)]^2} \right\}. \tag{2} \]

We rescale \( \tilde{\phi}(x_1, x_2) = \phi(\xi_1, \xi_2)/\sqrt{\sigma} \), where \( x_i = \xi_i L_i \). Then we expand in the adimensional variable \( (\sigma L_1 L_2)^{-1} \). The Hamiltonian (1) becomes

\[ H[\phi] = \sigma L_1 L_2 + H_g[\phi] - \frac{1}{\sigma L_1 L_2} H_p[\phi] + O((\sigma L_1 L_2)^{-2}), \tag{3} \]

with

\[ H_g[\phi] = \frac{1}{2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 [\nabla_u \phi(\xi_1, \xi_2)]^2 \tag{4} \]

\[ H_p[\phi] = \frac{1}{4} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \left( [\nabla_u \phi(\xi_1, \xi_2)]^2 \right)^2. \tag{5} \]

The ‘asymmetric gradient’ is defined through

\[ \nabla_u = \left( u^{1/2} \frac{\partial}{\partial \xi_1}, u^{-1/2} \frac{\partial}{\partial \xi_2} \right) \equiv (\nabla_{u,1}, \nabla_{u,2}) \text{ with } u \equiv L_2/L_1. \tag{6} \]

The corresponding (formal) expansion of the partition function eq. (2) is

\[ Z = \text{const} \cdot \exp(-\sigma L_1 L_2) \cdot Z_1 \cdot \left( 1 + \frac{1}{8\sigma L_1 L_2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \left( \left( [\nabla_u \phi(\xi_1, \xi_2)]^2 \right)^2 \right) + \ldots \right). \tag{7} \]

Here, we have introduced the Gaussian measure

\[ \langle \mathcal{O} \rangle \equiv \frac{1}{Z_1} \int_a D\phi \exp \left( -\frac{1}{2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 [\nabla_u \phi(\xi_1, \xi_2)]^2 \right) \mathcal{O}(\phi). \tag{8} \]

The first order (1-loop) contribution \( Z_1 \) to eq. (2) is

\[ Z_1 = \int_a D\phi \exp \left( -\frac{1}{2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 [\nabla_u \phi(\xi_1, \xi_2)]^2 \right). \tag{9} \]
The subscript \( a \) indicates that the functional integral (which is divergent in the ultraviolet) is to be regularized by a suitable cutoff. Examples for cutoff schemes will be given below. \( Z_1 \) is a purely Gaussian integral and can be evaluated by using standard techniques. The result turns out to be a function of the adimensional ratio \( u \) alone:

\[
Z_1(u) = \frac{1}{\sqrt{u}} \left| \eta(iu) / \eta(i) \right|^{-2},
\]

where \( \eta \) is the Dedekind eta function

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q \equiv \exp(2\pi i \tau).
\]

This is a well known result in string theory and conformal field theory, and coincides with the partition function of a 2D conformal invariant free boson on a torus of modular parameter \( \tau = iu \).

The calculation of the 2-loop contribution is less simple. One can show that

\[
\left\langle \left( \nabla^2 \phi(\xi_1, \xi_2) \right)^n \right\rangle = (-1)^n \sum_{\mu_1=1}^2 \ldots \sum_{\mu_n=1}^2 \frac{\partial^2}{\partial k_{\mu_1}^2} \ldots \frac{\partial^2}{\partial k_{\mu_n}^2} \exp \left[ -\frac{1}{2} \left( k_1^2 G_1 + k_2^2 G_2 \right) \right] \bigg|_{k=0}.
\]

By using eq. (12) for \( n = 2 \) and translational invariance on the torus, we find for the 2-loop expansion

\[
Z = \text{const} \cdot \exp(-\sigma L_1 L_2) \cdot Z_1 \cdot \left( 1 + \frac{R}{\sigma L_1 L_2} + \ldots \right),
\]

with

\[
R = \frac{1}{8} \left( 3G_1^2 + 3G_2^2 + 2G_1 G_2 \right).
\]

\( R \) can be rewritten as

\[
R = \frac{1}{8} \left( 2(G_1 + G_2)^2 + (G_1 - G_2)^2 \right).
\]

We now write \( R \) as a sum of two parts,

\[
R = R_\infty + R_f,
\]

with

\[
R_\infty = \frac{1}{4}(G_1 + G_2)^2,
\]

\[
R_f = \frac{1}{8}(G_1 - G_2)^2.
\]

\(^1\)The constant \( \eta(i) \) has been introduced to normalize \( Z_1(1) = 1 \).
With these definitions at hand, we now formulate our

**STATEMENT:**

At least for the four regularization schemes

\[ \Lambda_1 : \ \zeta\text{-function regularization}, \]
\[ \Lambda_2 : \ \text{point splitting regularization}, \]
\[ \Lambda_3 : \ \text{continuous momentum space cutoff}, \]
\[ \Lambda_4 : \ \text{lattice cutoff}, \]

the following assertion holds:

The coefficient \( R \) of the 2-loop contribution to the CWM partition function can be naturally split as \( R = R_\infty + R_f \), and

(S1) \( R_\infty \) diverges when the cutoff parameter \( a \) is sent to zero. The singularity depends on the chosen regularization, but in all regularization schemes it is “shape independent”. For any finite value of the cutoff, \( R_\infty \) is proportional to the area \( L_1 L_2 \) of the rectangle, but does not depend on the ratio \( u = L_2/L_1 \).

(S2) \( R_f \) remains finite when the cutoff is removed:

\[
R_f \rightarrow \frac{1}{8} \left[ 2 + \left( \frac{\pi}{3} u E_2(iu) - 1 \right)^2 \right].
\]

(20)

\( E_2 \) denotes the first Eisenstein series:

\[
E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \quad q \equiv \exp(2\pi i \tau).
\]

(21)

The calculation of \( R \) by means of the \( \zeta \)-function regularization can be found in [7], while the analogous calculation with the point-splitting procedure was presented in [2]. We refer to the original papers for the details. Before we turn to the proofs for the two schemes \( \Lambda_3 \) and \( \Lambda_4 \), let us state two immediate consequences of part (S1) of the statement. First, the fact that \( R_\infty \) depends on \( L_1 L_2 \) alone (and not on the shape factor \( u \)) implies that ratios of partition functions of interfaces with different shapes but the same area stay finite when the cutoff is removed. Second, this same fact allows to ‘absorb’ the infinity in a ‘renormalization’ of the interface tension \( \sigma \) (see also [2]).

3 Proofs

In this section we present the proofs of the above statement for the cutoff schemes \( \Lambda_3 \) and \( \Lambda_4 \). We return to dimensionful variables in this section.
3.1 Momentum Space Cutoff

A very general cutoff procedure for fields on a torus is to use the Fourier representation and then cut off large momenta. In such a scheme the regularized version of $G_\mu$ can be written as

$$G_\mu = - \sum_{p \neq 0} \frac{p_\mu^2}{p^2} \chi(a^2 p^2). \quad (22)$$

The momenta are summed over

$$p_\mu = \frac{2\pi}{L_\mu} \cdot \text{integer}, \quad (23)$$

and $p^2 = p_1^2 + p_2^2$. The cutoff function $\chi$ is assumed to have the property $\chi(0) = 1$ and to drop sufficiently fast to zero when the argument goes to infinity. For technical reasons, we shall later assume that there exists a constant $c < 0$ such that

$$|\chi(\rho)| \leq e^{c\rho}. \quad (24)$$

In order to remove the cutoff we have to send $a$ to zero. Let us first study $G_1 + G_2$:

$$G_1 + G_2 = 1 - \sum_p \chi(a^2 p^2), \quad (25)$$

where the sum is over all momenta (including 0). We introduce a Fourier representation of the cutoff function $\chi$:

$$\chi(a^2 p^2) = \int d^2x \, e^{ipx} \tilde{\chi}_a(x). \quad (26)$$

Here,

$$\tilde{\chi}_a(x) = \int \frac{d^2p}{(2\pi)^2} e^{-ipx} \chi(a^2 p^2). \quad (27)$$

Now,

$$\sum_p \chi(a^2 p^2) = \int d^2x \, \tilde{\chi}_a(x) \sum_p e^{ipx}. \quad (28)$$

The sum over the momenta can be performed:

$$\sum_p e^{ipx} = L_1 \sum_{n_1} \delta(x_1 - n_1 L_1) L_2 \sum_{n_2} \delta(x_2 - n_2 L_2). \quad (29)$$

Hence we have

$$\sum_p \chi(a^2 p^2) = L_1 L_2 \sum_{n_1} \sum_{n_2} \tilde{\chi}_a(n_1 L_1, n_2 L_2). \quad (30)$$

The Fourier transform of the cutoff function is

$$\tilde{\chi}_a(x) = \int \frac{d^2p}{(2\pi)^2} e^{-ipx} \chi(a^2 p^2) = \frac{1}{2\pi a^2} \int_0^\infty d\rho \, \rho \chi(\rho^2) J_0(\rho|x|/a). \quad (31)$$
\(J_0\) denotes the Bessel function of order 0. Plugging this into eq. (30), we get
\[
\sum_p \chi(a^2p^2) = \frac{L_1 L_2}{2\pi a^2} \sum_{n_1} \sum_{n_2} \int_0^\infty d\rho \rho \chi(\rho^2) J_0 \left( \rho \sqrt{(n_1 L_1)^2 + (n_2 L_2)^2} / a \right) \tag{32}
\]
In the limit \(a \to 0\) all contributions but the \(n_1 = n_2 = 0\) term vanish. The argument is that \(J_0\) oscillates wildly in the relevant interval, with the consequence that for nonvanishing \(n_i\) the integral vanishes faster than \(a^2\). Since \(J_0(0) = 1\), we end up with
\[
\sum_p \chi(a^2p^2) \to \frac{L_1 L_2}{2\pi a^2} \int_0^\infty d\rho \rho \chi(\rho^2) = \frac{L_1 L_2}{4\pi a^2} \int_0^\infty d\rho \rho \chi(\rho). \tag{33}
\]
This completes the proof of the part \((S1)\) of the statement.

Let us now turn to the discussion of \(G_1 - G_2\). In this case we use the Laplace transform of \(\chi\). For any function with
\[
|f(t)| \leq e^{ct} \tag{34}
\]
there is a representation
\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{ts} \hat{f}(s). \tag{35}
\]
For the cutoff function,
\[
\chi(a^2p^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \hat{\chi}(s) e^{sa^2p^2}. \tag{36}
\]
From the assumption eq. (24) we infer that we can choose \(c < 0\). So the integration is over a contour where the real part of \(s\) is always negative. This allows for a representation of \(G_1 - G_2\) as follows:
\[
G_1 - G_2 = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \hat{\chi}(s) F(s, a), \tag{37}
\]
with
\[
F(s) \equiv -\sum_{p \neq 0} \frac{p_1^2 - p_2^2}{p^2} e^{sa^2p^2}. \tag{38}
\]
Writing out explicitly the momenta, we get
\[
F(s, a) = -\sum_{n_1, n_2} \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \exp \left[ s 4\pi^2 a^2 \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \right) \right]. \tag{39}
\]
Here, the prime on the summation symbol indicates that \(n_1 = n_2 = 0\) is to be excluded. Let us now separate the two terms in the numerator, and let us call them \(P_1\) and \(P_2\).
\[
P_1 = -\sum_{n_1, n_2} \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \exp \left[ s 4\pi^2 a^2 \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \right) \right], \tag{40}
\]
and

\[ P_2 = \sum_{n_1, n_2} \frac{n_2^2}{L_2^2} \exp \left[ s \, 4\pi^2 a^2 \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \right) \right]. \]  \hspace{1cm} (41)

In \( P_1 \) we can safely take the limit \( a \to 0 \) in the 2-direction, because, as a sum over the index \( n_2 \), \( P_1 \) converges absolutely. We find

\[ P_1 \hat{=} - \sum_{n_1, n_2} \frac{n_2^2}{L_1^2} \exp \left[ s \, 4\pi^2 a^2 \frac{n_2^2}{L_1^2} \right]. \]  \hspace{1cm} (42)

Here we have introduced the symbol \( \hat{=} \) which shall signal equality in the limit \( a \to 0 \). Eq. (42) can be rewritten as follows:

\[ P_1 \hat{=} - 2 \sum_{n_1=1}^{\infty} \sum_{n_2=\infty}^{\infty} \frac{u^2 n_2^2}{n_2^2 + u^2 n_1^2} \exp \left[ s \, 4\pi^2 a^2 \frac{n_2^2}{L_1^2} \right]. \]  \hspace{1cm} (43)

The sum over \( n_2 \) can be done explicitly by using the identity (see, e.g., textbooks on quantum statistics)

\[ \frac{b}{\pi} \sum_{n_2=-\infty}^{\infty} \frac{1}{n_2^2 + b^2} = 1 + \frac{2}{e^{2\pi b} - 1} \quad \text{for} \quad b > 0. \]  \hspace{1cm} (44)

We get

\[ P_1 \hat{=} - 2 \pi u \left\{ \sum_{n_1=1}^{\infty} n_1 \left( 1 + \frac{2}{e^{2\pi u n_1} - 1} \right) \exp \left[ s \, 4\pi^2 a^2 \frac{n_1^2}{L_1^2} \right] \right\}. \]  \hspace{1cm} (45)

Let \( q = e^{2\pi i \tau} = e^{-2\pi u} \). Then we have

\[ P_1 \hat{=} - 2 \pi u \left( \sum_{n=1}^{\infty} \left( n \exp \left[ s \, 4\pi^2 a^2 \frac{n_1^2}{L_1^2} \right] \right) - 4 \pi u \sum_{n=1}^{\infty} \left( \frac{nq^n}{1-q^n} \exp \left[ s \, 4\pi^2 a^2 \frac{n_1^2}{L_1^2} \right] \right) \right). \]  \hspace{1cm} (46)

The second term is not singular in the limit \( a \to 0 \) while the first term can be treated with the Euler-McLaurin summation formula:

\[ \sum_{n=0}^{k} f(n) = \int_{0}^{k} f(x) \, dx + \frac{1}{2} \left[ f(0) + f(k) \right] + \sum_{m=1}^{\infty} (-1)^m \left[ f^{(2m-1)}(k) - f^{(2m-1)}(0) \right] \frac{B_m}{2m!}, \]  \hspace{1cm} (47)

where the \( B_m \) are the Bernoulli numbers. Taking \( f(n) = ne^{-bn^2} \) we get \( f^{(2m-1)}(\infty) = 0 \) and

\[ f^{(2m-1)}(0) = -\frac{1}{b} \frac{d^{2m}}{dx^{2m}} e^{-bx^2} \bigg|_{x=0} = bm-1 \frac{(-1)^m}{m!}, \]  \hspace{1cm} (48)

yielding

\[ \sum_{n=0}^{\infty} ne^{-bn^2} = \int_{0}^{\infty} dx \, xe^{-bx^2} - \sum_{m=1}^{\infty} \frac{b^{m-1} \frac{B_m}{m!(2m)!}}{m} \]  \hspace{1cm} (49)
The last sum is an absolutely convergent series which is regular for \( b \to 0 \). The integral is elementary:

\[
\int_0^\infty xe^{-bx^2} \, dx = \frac{1}{2b}.
\]  

(50)

Taking \( b = -s \frac{4\pi^2 a^2}{L_1} \), we get

\[-2\pi u \sum_{n=1}^\infty \left( n \exp \left[ s\frac{4\pi^2 a^2 n^2}{R^2} \right] \right) \sim \frac{L_1 L_2}{s4\pi a^2} + \frac{\pi}{6} u + \cdots,
\]

(51)

where the omitted terms have higher powers of \( a \). The finite part of this sum is exactly what is needed to reconstruct the Eisenstein function (once the limit \( a \to 0 \) is taken in the second part of \( P_1 \)) and we find:

\[ P_1 \equiv \frac{L_1 L_2}{s4\pi a^2} + \frac{\pi}{6} u E_2(iu). \]  

(52)

If we now study \( P_2 \) we see that we only have to exchange \( L_1 \) and \( L_2 \) and change the sign. In this way the divergent parts cancel out:

\[ F(s, a = 0) = \frac{\pi}{6} u E_2(iu) - \frac{\pi}{6} u^{-1} E_2(iu^{-1}) \]

(53)

Using the identity

\[ E_2(-\frac{1}{\tau}) = \tau^2 E_2(\tau) - i \frac{6\tau}{\pi} \]

(54)

we arrive at

\[ F(s, a = 0) = \frac{\pi}{3} E_2(iu) - 1. \]

(55)

Note that when we performed the limit \( a \to 0 \), the function \( F \) became also independent of \( s \). As a consequence, the Laplace integral eq. (37) then becomes just a representation of \( \chi(0) \), which is one, and part \( (S2) \) of the statement is proven.

3.2 Lattice Cutoff

In this case \( G_\mu \) is defined as:

\[ G_\mu = -\sum_{\vec{p} \neq 0} \frac{\vec{p}_\mu^2}{P_1^2 + P_2^2}. \]

(56)

The momenta are summed over

\[ p_\mu = \frac{2\pi}{al_\mu} \cdot n_\mu, \quad n_\mu = 0, \ldots, l_\mu - 1, \quad L_\mu = al_\mu, \]

(57)

and

\[ a^2 \vec{p}_\mu^2 = 4 \sin^2(\alpha p_\mu). \]

(58)
To remove the lattice cutoff one has to send \( l_1 \) and \( l_2 \) to infinity, while keeping the ratio \( \frac{l_2}{l_1} = \frac{L_2}{L_1} \equiv u \) fixed. \( G_1 \) can be written as

\[
G_1 = -\sum_{n_1=1}^{l_1-1} \left\{ \sum_{n_2=0}^{l_2-1} \frac{\sin^2 \left( \frac{\pi n_1}{l_1} \right)}{\sin^2 \left( \frac{\pi n_1}{l_1} \right) + \sin^2 \left( \frac{\pi n_2}{l_2} \right)} \right\}.
\]

The sum over \( n_2 \) can be done explicitly. To this end we employ the identity

\[
\frac{\sinh^2(x/2)}{\sinh^2(x/2) + \sin^2(\omega/2)} = \tanh(x/2) \sum_{n=-\infty}^{\infty} \exp(-x|n| - i\omega n),
\]

which can, e.g., be derived by a series of elementary operations from formula 3.613 of [8]. We make the identifications

\[
\omega = 2\pi n_2/l_2, \\
\sinh^2(x/2) = \sin^2(\pi n_1/l_1).
\]

Using that

\[
\sum_{n_2=0}^{l_2-1} \exp(-i\omega n) = \sum_{n_2=0}^{l_2-1} \exp(-i2\pi n_2/l_2) = l_2 \sum_{k=-\infty}^{\infty} \delta(n - l_2 k),
\]

we find after summing over \( n \),

\[
\sum_{n_2=0}^{l_2-1} \frac{\sin^2 \left( \frac{\pi n_1}{l_1} \right)}{\sin^2 \left( \frac{\pi n_1}{l_1} \right) + \sin^2 \left( \frac{\pi n_2}{l_2} \right)} = l_2 \tanh(x/2) \left( 1 + \frac{2 \exp(-l_2 x)}{1 - \exp(-l_2 x)} \right).
\]

This allows us to identify the convergent and divergent parts in the limit \( l_1, l_2 \to \infty \). We write

\[
G_1 \equiv C_1 + D_1,
\]

where

\[
D_1 = -l_2 \sum_{n_1=1}^{l_1-1} \tanh(x/2),
\]

and

\[
C_1 = -2l_2 \sum_{n_1=1}^{l_1-1} \tanh(x/2) \frac{\exp(-l_2 x)}{1 - \exp(-l_2 x)}.
\]
In all these sums the index $n_1$ is hidden in $x$. $D_1$ can be written as

$$D_1 = -l_2 \sum_{n_1=1}^{l_1-1} \sin \left( \frac{\pi n_1}{l_1} \right) \frac{\sin \left( \frac{\pi n_1}{l_1} \right)}{\sqrt{1 + \sin^2 \left( \frac{\pi n_1}{l_1} \right)}}. \quad (68)$$

This sum can be evaluated by using the Euler-McLaurin formula eq. (47) and gives

$$D_1 = -\frac{l_1 l_2}{2} + \frac{\pi}{6} u + O \left( \frac{u}{l_1^2} \right). \quad (69)$$

In $C_2$ we can safely take the limit $l_1, l_2 \to \infty$. In this limit, we can write $x = 2\pi n_1/l_1 + O(1/l_1^3)$. We can thus approximate $\tanh(x/2) \sim \pi n_1/l_1$, and

$$\exp(-l_2 x) = \exp(-2\pi u n_1) \left( 1 + O \left( \frac{u}{l_1} \right) \right). \quad (70)$$

Let us define as usual $q = e^{-2\pi u}$. Then

$$C_1 = -4\pi u \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}. \quad (71)$$

Here we have taken into account that the two values $n_1$ and $l_1 - n_1$ are identified with the same value of $x$ thus giving a further factor of 2.

Obviously, the corresponding results for $G_2$ can be obtained just by letting $u \to u^{-1}$. Putting together the various parts, we get

$$G_1 = D_1 + C_1 \to -\frac{l_1 l_2}{2} + \frac{\pi}{6} u E_2(iu),$$

$$G_2 = D_2 + C_2 \to -\frac{l_1 l_2}{2} + \frac{\pi}{6} u^{-1} E_2(iu^{-1}). \quad (72)$$

Employing again eq. (54), we now easily verify that

$$G_1 + G_2 \to -(l_1 l_2 - 1),$$

$$G_1 - G_2 \to \frac{\pi}{3} E_2(iu) - 1, \quad (73)$$

which proves the assertions of the statement.

4 Conclusions

It could seem rather surprising that we find a universal result, independent of the regularization scheme, for the finite part of $R$, since it stems from a non-renormalizable operator. We think that the reason, as it was already mentioned in [7] and [2], is that the result is completely fixed by symmetry requirements. In particular, the modular invariance of the
original interaction term (which is a direct consequence of the fact that the model, due to
the periodic boundary conditions on ϕ, is defined on a torus) and its scaling behavior (which
immediately follows from power counting) tells us that \( G_1 - G_2 \) must be a modular form of
weight 2. The normalization coefficients in front of the modular form then follow from the
study of the one-dimensional limit \( u \to \infty \), which was already discussed by Arvis in \[ H \].

The contribution that we have found does not depend on the regularization schemes that
we have discussed because they all preserve the modular symmetry. We are thus led to ask if
there is some physical reason to choose among all the possible schemes those which preserve
this symmetry. We have no definite answer on this point but it is tempting to conjecture that
it is a consequence of the fact that the Hamiltonians \( \mathfrak{H} \) and \( \mathfrak{H} \) are respectively the first
and the second order expansion of a bosonic string action, and that the modular invariance
is actually preserved at the quantum level in the string model.

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