Numerical Solution of Contact Problems with Friction under Dynamic Loads

A A Lukashevich1, N K Lukashevich1

1Department of mechanics, St. Petersburg State University of Architecture and Civil Engineering, 2-nd Krasnoarmeiskaya St. 4, 190005, St. Petersburg, Russia

E-mail: aaluk@bk.ru

Abstract. The article deals with the dynamic problem of deformable solid mechanics with unilateral constraints and Coulomb friction. For the numerical solution of the problem, the finite element method is used, the contact interaction is modeled by means of the contact finite elements (CFE) of the frame-rod type. The statement of dynamic contact problem for the interacting elastic bodies is given. On the basis of the proposed discrete contact model and the method of step-by-step analysis a numerical algorithm has been developed, which allows in one step-by-step (on time) process to perform the integration of the equations of motion and the implementation of the conditions for unilateral constraints with Coulomb friction on the contact simultaneously. Under the conditions of ultimate friction the method of compensating loads has been applied. With the help of the proposed approach, numerical solutions of the problem of contact of the structure with the base at different parameters of the dynamic load have been obtained and analyzed.

1. Introduction

Problems with unilateral constraints and friction between contacting surfaces are often encountered in the calculation of various types of structures. The solution of such problems under the action of static loads and different contact conditions was considered, for example, in works [1–12]. In the meantime, it is not uncommon to take into account the dynamic loads on the structure [13–16]. The constructive nonlinearity here will be manifested in the change of the working schemes of the structure in time – switching on and off unilateral constraints, both in normal and tangential direction. It is assumed that between two consecutive events on the contact, i.e. within each such working scheme, the character of the structure deformation is linear.

The given work is devoted to the development of a numerical model and algorithm for solving contact problems with unilateral constraints and friction under the dynamic load action. The immediate solution of the dynamic contact problem is constructed on the basis of time discretization using direct schemes of integration of the equations of motion [17, 18]. After each time step, the boundary conditions on the contact are checked. If within a certain step $\Delta t$ there is a change of the working scheme, the time point of changing of the contact state (occurrence of the next event) is determined by means of a step-by-step analysis of the contact state with the use of appropriate approximating expressions for displacements, speeds and accelerations on the time interval $\Delta t$. In this case, the integration step size is corrected and the current step is recalculated. As a result, a new state of contact is established at the given time point and, thus, the current working scheme of the structure is changed [7, 18].

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2. Statement and solution of the problem

Let’s consider the dynamic problem of interaction of linearly elastic bodies \( V^+ \) and \( V^- \) (which may be, for example, the structure and base) with contacting surfaces, \( S^c_+ \) and \( S^c_- \) respectively. To calculate this system we use a discrete computational model FEM. Contact interaction is modeled using frame-rod contact finite elements (CFE) [19, 20]. The CFE provide discrete contact between the nodes of the finite element grid located on the boundary surfaces of the contacting bodies.

Let us write down the matrix equation of motion in the form that allows the solution of a constructively nonlinear dynamic contact problem to be reduced to the solution of the sequence of linear dynamic problems on the basis of step-by-step time analysis of the contact state [18, 21]:

\[
M \ddot{U}^{t+\Delta t} + C \dot{U}^{t+\Delta t} + K \Delta U^{t+\Delta t} = P^{t+\Delta t} - KU^t.
\] (1)

Here \( M \), \( C \) and \( K \) are the mass, damping, and stiffness matrices of the finite element system respectively; \( U^{t+\Delta t} \), \( \dot{U}^{t+\Delta t} \), \( \ddot{U}^{t+\Delta t} \) and \( P^{t+\Delta t} \) are the vectors of nodal displacements, speeds, accelerations, and the external nodal load at time point \( t+\Delta t \); \( \Delta U^{t+\Delta t} \) is the increment of displacement at step \( \Delta t \). In addition, on the part of the outer boundaries (for \( V^+ \) and \( V^- \) respectively) boundary conditions on the forces, and on \( S^c_+ \) in the displacements should be given. It is assumed that at the initial time \( t = 0 \) the vectors of displacements, speeds and accelerations are given and it is necessary to find a solution (1) during the time interval from 0 to some value \( T \).

For the numerical integration of the equations of motion (1), an implicit Newmark finite difference scheme is used, which is based on the assumption of a linear change of accelerations in the \( \Delta t \) interval. In this case, the following dependences between the increments of displacements, speeds and accelerations for the time point \( t+\Delta t \) are used:

\[
\hat{U}^{t+\Delta t} = \frac{1}{\alpha(\Delta t)^2} \left[ \Delta U^{t+\Delta t} - \Delta t \dot{U}^t \right] - \left( \frac{1}{2\alpha} - 1 \right) \ddot{U}^t; \quad U^{t+\Delta t} = \dot{U}^t + [(1-\beta)\ddot{U}^t + \beta \ddot{U}^{t+\Delta t}] \Delta t.
\] (2)

Here \( \alpha = 1/4, \beta = 1/2 \), which corresponds to the case of constant average acceleration at each of the intervals \( \Delta t \). In this case, the Newmark integration scheme for linear problems is unconditionally stable, i.e. the solution does not grow indefinitely at large values of the step \( \Delta t \) [17].

Substituting the expression (2) in the equation (1), thereby excluding \( \ddot{U}^{t+\Delta t} \) and \( \dot{U}^{t+\Delta t} \) from the number of unknown ones, after simple transformations, we obtain the following matrix equation to determine \( \Delta U^{t+\Delta t} \):

\[
\hat{K} \Delta U^{t+\Delta t} = \hat{P}^{t+\Delta t},
\] (3)

where \( \hat{K} = K + a_0M + a_4C \); \( \hat{P}^{t+\Delta t} = P^{t+\Delta t} + M \left( a_2 \dot{U}^t + a_3 \ddot{U}^t \right) + C \left( a_4 \dot{U}^t + a_5 \ddot{U}^t \right) - KU^t \). The coefficients \( a_0 - a_5 \) depend on the step \( \Delta t \) and the parameters \( \alpha \) and \( \beta \), their expressions are given in [17].

The system of algebraic equations (3) is solved by the LDLT factorization method, taking into account the sparsity of the symmetric matrix \( \hat{K} \) and its variable profile. After finding \( \Delta U^{t+\Delta t} \) and, accordingly, \( U^{t+\Delta t} \) for the calculation of accelerations \( \ddot{U}^{t+\Delta t} \) and speeds \( \dot{U}^{t+\Delta t} \), equations (2) are used. In their turn, at any time point \( t' \) within the interval \( \Delta t \) \( (t \leq t' \leq t + \Delta t) \), the values of accelerations \( \ddot{U}(t') \), speeds \( \dot{U}(t') \) and displacements \( U(t') \) can be calculated with the following formulas:

\[
\ddot{U}(t') = \ddot{U}^t + \frac{(t' - t)}{\Delta t} \left[ \ddot{U}^{t+\Delta t} - \ddot{U}^t \right]; \quad \dot{U}(t') = \dot{U}^t + \frac{(t' - t)}{2} [\ddot{U}^t + \ddot{U}^t];
\]

\[
U(t') = U^t + (t' - t) \dot{U}^t + \left[ \ddot{U}^t + \ddot{U}^t \right].
\] (4)

The first of the equations (4), according to the Newmark scheme, expresses the linear law of change of acceleration on the interval \( \Delta t \), the second and third ones are obtained from the expressions (2) with the value substitutions \( \alpha = 1/4, \beta = 1/2 \) and the replacement of the value \( t + \Delta t \) by the value \( t' \).
Furthermore, when taking into account unilateral constraint with Coulomb friction in addition to the initial conditions at \( t = 0 \) and boundary conditions on \( S_k^+ \), \( S_k^\frac{z}{c} \), the conditions on the contacting surfaces \( S_k^\frac{c}{z} \) should be satisfied. Let us write these conditions in terms of forces and displacements for each discrete unilateral constraint \( k \) (i.e. the CFE \( k \) [20, 21]), for the time point \( t \):

\[
\begin{align*}
  u'_{nk} & \geq 0; \quad N'_{k} \leq 0; \quad u'_{nk}N'_{k} = 0, \quad k \in S_c. \\
  \left| T'_{k}\right| & \leq \left| T'_{Uk}\right|, \quad T'_{k}u'_{zk} \geq 0; \quad \left( T'_{k} - T'_{Uk}\right) u'_{zk} = 0, \quad k \in S_c.
\end{align*}
\]

(5)

Here \( u'_{nk}, u'_{zk} \) are mutual displacements of opposite nodes for unilateral constraint \( k \) in the normal and tangential direction; \( u'_{zk} = \frac{\partial u'_{zk}}{\partial t} \) is the speeds of mutual tangential displacement on the contact \( k \); \( N'_{k}, T'_{k} \) are contact forces in the normal and tangential direction (forces in CFE \( k \)); \( T'_{Uk} = -f_{k}N'_{k} \) is ultimate Coulomb friction force for the contact \( k \); \( f_{k} \geq 0 \) is the coefficient of friction-sliding.

The last of the conditions (5) means that upon contact \( u'_{nk} = 0, \quad N'_{k} < 0 \); upon separation \( u'_{nk} > 0, \quad N'_{k} = 0 \). The last two conditions (6) establish the correspondence between the speeds of the mutual slippage of the opposite nodes on the contact \( k \) and the magnitude of the force \( T'_{k} \) at the time point \( t \).

Under the conditions \( \dot{u}'_{zk} = 0 \) and \( \left| T'_{k}\right| < \left| T'_{Uk}\right| \) there is a state of clenching (pre-ultimate friction); when \( \dot{u}'_{zk} \neq 0, \left| T'_{k}\right| = \left| T'_{Uk}\right| \) is the state of slippage, while the direction of the slip rate is in line with the direction of the shear force.

Changing the current state, namely the moment of transition from one state to another, is an event – respectively, it will be the events of slippage, clutch, separation (switching off unilateral constraint), or contact (switching on it).

3. Numerical algorithm

Let’s briefly present a sequence of actions implementing a step-by-step algorithm for solving a dynamic problem with unilateral constraints and Coulomb friction. The General case is considered when the normal forces of interaction and, accordingly, the ultimate friction forces on the contact change in the process of dynamic loading, i.e. in time (as it often occurs in practical problems).

It is believed that at the current time point \( t \) the state on the contact is known. The values of mutual displacements \( u'_{nk}, u'_{zk} \), speed \( \dot{u}'_{zk} \) and contact forces \( N'_{k}, T'_{k}, T'_{Uk} \) are determined for each unilateral constraint \( k \). Let part of the constraints \( (k \in S_{1c}) \) be in the state of clenching, the other part \( (k \in S_{2c}) \) – in the state of contact with the slip and, finally, the third part \( (k \in S_{3c}) \) – in the state of separation, \( S_c = S_{1c} \cup S_{2c} \cup S_{3c} \). At the beginning of the calculation, at \( t = 0 \), the displacements and (as well as the speeds and accelerations) are assumed to be zero.

1. The current time step \( \Delta t \) is performed (in the process of calculation its value can be changed in accordance with the established moment of occurrence of the next events on the contact). From the solution (3), the increments \( \Delta u'_{nk}, \Delta u'_{zk} \), then the values of displacements \( u'_{nk}^{t+\Delta t}, u'_{zk}^{t+\Delta t} \), speeds \( \dot{u}'_{zk}^{t+\Delta t} \), and contact forces \( N'_{k}^{t+\Delta t}, u'_{zk}^{t+\Delta t}, T'_{Uk}^{t+\Delta t} \) for the time point \( t+\Delta t \) are determined.

2. The traversal of all discrete constraints is performed, therewith for each constraint \( k \) there is (within the current step \( \Delta t \)) the time point \( T_k \) of occurrence of the next, i.e. the closest in the time event. The expressions for determining the slippage moment, clutch, separation or contact for the constraint \( k \) respectively will have the following form here:
\[ \hat{u}_k = t + \Delta t \left( \frac{T_{u_k}^l - T_{k}^l}{(T_{k}^{l+M} - T_{k}^l)} \right), \quad k \in S_{1c}; \quad \hat{u}_k = t + \Delta t \left( \frac{-u_{nk}^2}{u_{nk}^{l+M} - u_{nk}^2} \right), \quad k \in S_{3c}. \]

Since the change of displacements, speeds (and, consequently, forces) within the step \( \Delta t \) does not follow a linear law, then, additionally, using the expressions (4), an iterative refinement of the time point \( \hat{t}_k \) can be performed, the time consumption at that increases slightly [21].

3. Of all the values \( \hat{t}_k \) having been found using the formulas (7) and situated in the interval \((t, t + \Delta t)\), the smallest one corresponding to the moment of occurrence of the nearest time event on the contact is selected: \( \hat{t} = \min(\hat{t}_k), \quad k \in S_{c} \). In case \( \hat{t} > t + \Delta t \) the next basic integration step \( \Delta t \) is executed (i.e. transition to p.1 is performed).

4. In case \( t < t < t + \Delta t \) – recalculated of updated in such a way step with value \( \Delta \hat{t} = t - t \) is performed. Herewith, to fulfill the conditions of ultimate friction the method of compensating loads is applied [2, 21, 22]. Changing of the ultimate friction forces on the contact is taken into account by the application of compensating forces \( \hat{F}_{nk} = -\Delta \hat{T}_{u_k} = -\Delta t \frac{N_{k}}{(T_{u_k}^{l+M} - T_{u_k}^l)}, \quad k \in S_{2c} \) to the opposite nodes.

The value of the transverse force on the contact \( k \) is corrected by the same quantity: \( T_{k}^l = T_{k}^l + \Delta \hat{T}_{u_k} \).

As a result of the step recalculations, the values of displacements, speeds and forces on the contact in the time point \( \hat{t} \) are determined. The conditions of the expected event are checked; in case of slippage it will be a condition \( T_{k}^l = T_{u_k}^l \); in case of clenching \( -u_{k}^l = 0 \), in case of detachment \( -N_{k}^l = 0 \), in case of contact \( -u_{nk}^l = 0 \). If the corresponding condition does not work, the time point \( \hat{t} \) should be updated again but in the interval \((t, \hat{t})\) or \((\hat{t}, t + \Delta t)\).

5. In case of occurrence of the next on time event on the corresponding support the state of contact changes – thereby the current working scheme of the construction changes too. Therewith the results of the recalculated step are considered final for the time point \( \hat{t} \). Then all the above actions are repeated, but for the next integration step \( \Delta t \).

4. Results and discussion

Next, the problem of interaction of the apron slab of the dam with ground base at hydrodynamic effect of the water flow discharged from the headwater of the dam has been considered. The calculation scheme of the slab (figure 1) corresponds to one of the objects of the Volzhsky hydroelectric complex. The purpose of the calculations was to assess the impact of the pulsating component of water pressure in the discharge flow for the contact interaction of the apron slab of the dam with the base ground. The criterion condition for determining the ultimate values for the slab thickness in this case is to prevent the slippage or separation of the slab from the ground base.

The calculation took into account loads stipulated by own weight of the slab, the hydrodynamic acting from the water flow, the filtration backpressure. The pulsating component of the water pressure in the discharge flow was taken into account as a dynamic impulse load (the amplitude of the pulsating pressure \( q \) and the correlation of its distribution over the slab surface were taken into account according to the recommendations from [23]).

To study the dependence of the solution on the characteristics of the hydrodynamic acting, the behavior of the slab at different positions of the impulse on the apron slab \((x/L)\), as well as at its different directions and duration of the action was calculated.
As the results of the calculations show, the possible separation of the apron slab from the ground base in all cases occurs only at the edge of the slab from the compressed section (left edge of the slab). The moment of separation depends on the position and direction of the impulse of pressure. The most dangerous (from the point of view of the separation slab from the base) is the pressure pulsation with pulse duration of from 0.46 s (at the location of the pulse closer to the edges of the slab) to 0.5 s (for the middle of the slab).

Figure 2 shows the change of contact stresses on the left edge of the slab base in time – to the moment of separation of the sole from the base ($t/T_{imp}$ is the ratio of the current time $t$ to the duration of the impulse of pressure $T_{imp}$, $x/L$ is the ratio of the $x$ coordinate that defines the position of the impulse at the apron slab, to the slab length $L$). The slab thickness in these cases was taken to be less than the ultimate values for separation $h_{pr}$.

The dependences for the ultimate values of the slab thickness on the pulse position have been obtained (figure 3). A solid line shows the envelope relative values of slab thickness, satisfying the condition of non-separation and shear of the slab from the base (here $h_{cr} = 3.48$ m is the critical depth corresponding to the design specific discharge of water). The dotted line corresponds to the ultimate slab thickness when only static loads are applied. Based on the analysis of the obtained results, proposals concerning the constructive solutions of the considered structure, taking into account the nature of the loads acting on the slab, were made.

![Figure 1. Scheme of the apron slab and applied loads.](image1)

![Figure 2. Contact stresses on the left edge of the slab base before the moment of separation.](image2)

![Figure 3. Dependence of the maximum slab thickness $h_{pr}$ on the pulse position $x/L$.](image3)
5. Conclusion
The dynamic contact problems interacting elastic systems with unilateral constraints and Coulomb friction has been considered. To solve the problem a discrete computational model of the FEM was used, upon that the contact interaction was modelled involving using frame-rod CFE. The numerical algorithm combining in one step-by-step process the integration of equations of motion with step-by-time analysis of the contact state is proposed. This approach provides the possibility of analysis of contact interaction of elastic systems at dynamic loading and has the advantage in cases when the solution of the problem depends on the history of loading, in particular, when accounting for friction-sliding in unilateral constraints.

The results of the calculations for the problems having been considered here allow us to conclude about the efficiency and reliability of the proposed algorithm, taking into account the complicated contact conditions and dynamic loading, which is essential for solving applied problems of structural mechanics. In conclusion, based on the example having been considered here, we should note that the account of contact friction forces contributes to the approximation of the calculation scheme to the real picture of the interaction and, thus, allows obtaining more accurate and complete information about the stress-strain state of the structure, and, consequently, its strength and reliability.

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