HADAMARD DIRECTIONAL DIFFERENTIABILITY OF THE OPTIMAL VALUE OF A LINEAR SECOND-ORDER CONIC PROGRAMMING PROBLEM

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ABSTRACT. In this paper, we consider perturbation properties of a linear second-order conic optimization problem and its Lagrange dual in which all parameters in the problem are perturbed. We prove the upper semi-continuity of solution mappings for the perturbed problem and its Lagrange dual problem. We demonstrate that the optimal value function can be expressed as a min-max optimization problem over two compact convex sets, and it is proven as a Lipschitz continuous function and Hadamard directionally differentiable.

1. Introduction. It is well known that stability theory plays an important role in studying the following linear two-stage stochastic optimization problem

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad d^T x + E(\theta(x, \xi)) \\
\text{s.t.} & \quad Ax = b, \, x \geq 0, \\
& \quad \theta(x, \xi) = \min_{y \in \mathbb{R}^m} c^T y \\
& \quad \text{s.t.} \quad Wy + Tx = h, \, y \geq 0,
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$ is the first stage decision variable and $d \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ is the second stage decision variable and $c \in \mathbb{R}^m$, $W \in \mathbb{R}^{l \times m}$, $T \in \mathbb{R}^{l \times n}$, $h \in \mathbb{R}^l$. $\xi$ is a

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random variable which is composed of some elements in \( \{c, W, T, h\} \). The continuity and differential properties of \( \theta(x, \xi) \) are particularly important in the stability analysis for the linear two-stage problem when the probability distribution is perturbed. There are many publications about the stability of two-stage optimization but among them only a few papers consider the case \( \xi = (c, W, T, h) \), namely all parameters in second stage linear program are random. For examples, in Section 3 of [12], Römisch and Wets obtained the Lipschitz continuity of the optimal value \( \theta(x, \xi) \). Han and Chen [8] investigated continuity properties of parametric linear programs.

The literature on perturbation analysis of optimization problems is enormous, and even a short summary about the most important results achieved would be far beyond our reach. For the perturbation analysis of general optimization problem one may refer to [1] and [4]. For structured optimization problems, one may refer to [9] and for stability results about linear complementarity and affine variational inequality problems, see for [5] and [6].

In this paper, instead of linear programming in the second stage, we consider a linear second-order conic optimization problem. There are a variety of engineering applications about the second-order cone optimization problem in [10], such as filter design, antenna array weight design, truss design, and grasping for the optimization in robotics. It’s worth to mention that a classical Portfolio Problem with \( n \) assets or Stocks holding over one period can be transferred into a second-order cone optimization. The problems can be stated as follows. Given a closed convex set \( X \subseteq \mathbb{R}^n \) and a point \( x \in X \), the second-order conic optimization problem is defined by

\[
\min_{y \in \mathbb{R}^n} \quad c^T y \\
\text{s.t.} \quad a_i^T y + q_i^T x - b_i \geq \|B_i y\|, \quad i = 1, \ldots, l,
\]  

where \( \xi = (c; A; Q; B; b) \) is a given parameter and \( \| \cdot \| \) means 2-norm in this paper without special instructions. Here \( c \in \mathbb{R}^n, A = (a_1, \ldots, a_l)^T \in \mathbb{R}^{l \times m}, Q = (q_1, \ldots, q_l)^T \in \mathbb{R}^{l \times n}, b \in \mathbb{R}^l, B = (B^1; \ldots; B^l) \) with \( B^i \in \mathbb{R}^{J \times m} \), \( i = 1, \ldots, l \).

Let \( g^i(y, x, \xi) = (B^i y; a_i^T y + q_i^T x - b_i), \quad i = 1, \ldots, l \) and \( \mathcal{Q}_{J+1} \subseteq \mathbb{R}^{J+1} \) be the second-order cone in \( \mathbb{R}^{J+1} \) defined by

\[
\mathcal{Q}_{J+1} = \{(s, t) \in \mathbb{R}^J \times \mathbb{R} : t \geq \|s\| \}, i = 1, \ldots, l.
\]

Then Problem (2) is expressed as

\[
\min_{y} \quad c^T y \\
\text{s.t.} \quad g^i(y, x; \xi) \in \mathcal{Q}_{J+1}, \quad i = 1, \ldots, l.
\]  

We use \( \theta(x, \xi) \) to denote the optimal value of \( P(x, \xi) \). Let \( u := (x, \xi) \), where \( x \in X \). In this paper we discuss the stability properties of \( P(x, \xi) \) when \( u \) is perturbed to \( \tilde{u} \), and the differentiability property of \( \theta(\cdot, \cdot) \). We denote the perturbed problem by \( P(\tilde{x}, \tilde{\xi}) \).

The remaining parts of this paper are organized as follows. In Section 3, we demonstrate upper continuity of the solution mapping for \( P(\tilde{x}, \tilde{\xi}) \) at some point \( (x, \xi) \). In Section 4, we study the upper continuity of the solution mapping for the Lagrange dual of \( P(\tilde{x}, \tilde{\xi}) \) at some point \( (x, \xi) \). The local Lipschitz continuity of \( \theta \) and its Hadamard directional differentiability at a point \( (x, \xi) \) are established Section 5. We conclude our paper in Section 6.
2. Preliminary.

Definition 2.1. [11] A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous (osc) at $\bar{x}$ if

$$\lim_{x \to \bar{x}} \sup S(x) \subseteq S(\bar{x}),$$

or equivalently $\limsup_{x \to \bar{x}} S(x) = S(\bar{x})$, where $\limsup_{x \to \bar{x}} S(x)$ is the outer limit of $S$ at $\bar{x}$:

$$\limsup_{x \to \bar{x}} S(x) := \{ u : \exists x^k \to \bar{x}, \exists u^k \in S(x^k) \text{ with } u^k \to u \}.$$ 

The mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is inner semicontinuous (isc) at $\bar{x}$ if

$$\liminf_{x \to \bar{x}} S(x) \supseteq S(\bar{x}),$$

where $\liminf_{x \to \bar{x}} S(x)$ is the inner limit of $S$ at $\bar{x}$:

$$\liminf_{x \to \bar{x}} S(x) := \{ u : \forall x^k \to \bar{x}, \exists u^k \in S(x^k) \text{ for large enough } k \text{ with } u^k \to u \}.$$ 

The mapping $S$ is continuous at $\bar{x}$ if

$$\lim_{x \to \bar{x}} S(x) = S(\bar{x}) = \limsup_{x \to \bar{x}} S(x).$$

Definition 2.2. [11] (upper limits and upper semicontinuity) The upper limit of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ is the value in $\mathbb{R}$ defined by

$$\limsup_{x \to \bar{x}} f(x) := \lim_{\delta \searrow 0} \left[ \sup_{x \in \mathbb{B}(\bar{x}, \delta)} f(x) \right] = \inf_{\delta > 0} \left[ \sup_{x \in \mathbb{B}(\bar{x}, \delta)} f(x) \right] = \inf_{V \in \mathbb{N}(\bar{x})} \left[ \sup_{x \in V} f(x) \right].$$

The function $f : \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous at $x$, if

$$\limsup_{x \to \bar{x}} f(x) = f(\bar{x}).$$

Definition 2.3. [11] (lower limits and lower semicontinuity) The lower limit of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ is the value in $\mathbb{R}$ defined by

$$\liminf_{x \to \bar{x}} f(x) := \lim_{\delta \searrow 0} \left[ \inf_{x \in \mathbb{B}(\bar{x}, \delta)} f(x) \right] = \sup_{\delta > 0} \left[ \inf_{x \in \mathbb{B}(\bar{x}, \delta)} f(x) \right] = \sup_{V \in \mathbb{N}(\bar{x})} \left[ \inf_{x \in V} f(x) \right].$$

The function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous at $x$, if

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}), \text{ or equivalently } \liminf_{x \to \bar{x}} f(x) = f(\bar{x}).$$

Lemma 2.4. [11] (characterization of lower limits and upper limits) Due to the definition of lower limits and upper limits, we can obtain that

$$\liminf_{x \to \bar{x}} f(x) = \min \{ \alpha \in \mathbb{R} | \exists x'' \to \bar{x} \text{ with } f(x'') \to \alpha \},$$

$$\limsup_{x \to \bar{x}} f(x) = \max \{ \alpha \in \mathbb{R} | \exists x'' \to \bar{x} \text{ with } f(x'') \to \alpha \}.$$

Proposition 1. [11] (continuity of functions) A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if it is both lower semicontinuous and upper semicontinuous:

$$\lim_{x \to \bar{x}} f(x) = f(\bar{x}) \iff \liminf_{x \to \bar{x}} f(x) = \limsup_{x \to \bar{x}} f(x).$$
Definition 2.5. [11](horizon cones) For a set $C \subset \mathbb{R}^n$, the horizon cone is the closed cone $C^\infty \subset \mathbb{R}^n$ representing the direction set $\text{hzn} C$, so that
\[ \text{hzn} C = \text{dir} C^\infty, \quad \text{csn} C = \text{cl} C \cup \text{dir} C^\infty, \]
this cone being given therefore by
\[ C^\infty = \begin{cases} \{x | \exists x' \in C, \lambda' \searrow 0, \text{ with } \lambda' x' \searrow x\} & \text{when } C \neq \emptyset \\ \{0\} & \text{when } C = \emptyset \end{cases} \]

Proposition 2. [11](horizon criterion for boundedness) A set $C \subset \mathbb{R}^n$ is bounded if and only if its horizon cone is just the zero cone: $C^\infty = \{0\}$.

Proposition 3. [11](horizon cone of a level set) For any function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and any $\alpha \in \mathbb{R}$, one has $(\text{lev} \leq \alpha f)^\infty \subseteq (\text{lev} \leq 0 f)^\infty$, i.e.,
\[ \{x | f(x) \leq \alpha\}^\infty \subseteq \{x | f^\infty(x) \leq 0\}, \]
where $\text{lev} \leq \alpha f(\cdot, \tilde{u}) = \{y \in \mathbb{R}^m : f(y, \tilde{u}) \leq \alpha\}, \alpha \in \mathbb{R}$.

In the following discussions, we need to adopt Proposition 4.4 of Bonnans and Shapiro (2000) [1]. Consider the parameterized optimization problem of the form
\[ (P) \min_{x \in X} f(x) \quad \text{s.t.} \quad G(x) \in K, \]
the perturbed form
\[ (P_u) \min_{x \in X} f(x, u) \quad \text{s.t.} \quad G(x, u) \in K, \]
where $u \in U$, $X$, $Y$ and $U$ are Banach spaces, $K$ is a closed convex subset of $Y$. $f : X \times U \to \mathbb{R}$ and $G : X \times U \to Y$ are continuous. We denote by
\[ \Phi(u) := \{x \in X : G(x, u) \in K\} \]
the feasible set of problem $(P_u)$ and the optimal value function is
\[ \nu(u) := \inf_{x \in \Phi(u)} f(x, u), \]
and the associated solution set
\[ S(u) := \arg\min_{x \in \Phi(u)} f(x, u). \]

Proposition 4. [1, Proposition 4.4] Let $u_0$ be a given point in the parameter space $U$. Suppose that
(i) the function $f(x, u)$ is continuous on $X \times U$,
(ii) the multifunction $\Phi(\cdot)$ is closed,
(iii) there exist $\alpha \in \mathbb{R}$ and a compact set $C \subset X$ such that every $u$ in a neighborhood of $u_0$, the level set
\[ \text{lev} \leq \alpha f(\cdot, u) := \{x \in \Phi(u) : f(x, u) \leq \alpha\} \]
is nonempty and contained in $C$,
(iv) for any neighborhood $\mathcal{V}_X$ of the set $S(u_0)$ there exists a neighborhood $\mathcal{V}_U$ of $u_0$ such that $\mathcal{V}_X \cap \Phi(u)$ is nonempty for all $u \in \mathcal{V}_U$.

Then:
(a) the optimal value function $\nu(u)$ is continuous at $u = u_0$,
(b) the multifunction $S(u)$ is upper semicontinuous at $u_0$. 
For the discussions in the following, we need use the result in Theorem 7.24 of [13], for this we consider the minimax problem

$$\min_{x \in X} \left\{ \phi(x) := \sup_{y \in Y} f(x, y) \right\}$$

(5)

where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are convex and compact and the function $f : X \times Y \to \mathbb{R}$ is continuous. Consider the perturbation of the minimax problem (5):

$$\min_{x \in X} \sup_{y \in Y} \{ f(x, y) + t\eta_t(x, y) \},$$

(6)

where $\eta_t(x, y)$ is continuous in $X \times Y$, $t \geq 0$. Moreover we assume that $f(x, y)$ is convex in $x \in X$ and concave in $y \in Y$. Denoted by $v(t)$ the optimal value of the above problem (6). Clearly $v(0)$ is the optimal value of the unperturbed problem (5). Then the following lemma holds.

**Lemma 2.6.** [13, Theorem 7.24] Suppose that the following conditions hold:

(i) the sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are convex and compact,
(ii) for all $t \geq 0$, the function $\zeta_t := f + t\eta_t$ is continuous on $X \times Y$, convex respects to $x \in X$ and concave respects to $y \in Y$,
(iii) $\eta_t$ converges uniformly as $t \downarrow 0$ to a function $\gamma(x, y) \in C(X, Y)$.

Then we have

$$\lim_{t \downarrow 0} \frac{v(t) - v(0)}{t} = \inf_{x \in X^*} \sup_{y \in Y^*} \gamma(x, y),$$

where $X^*$ and $Y^*$ respectively represent the optimal solution set of problem (6).

3. Upper continuity of primal solution mapping. Let $f(y, \tilde{u}) = \tilde{c}^T y$. We denote by $\Phi(\tilde{u})$ the feasible set of $P(\tilde{x}, \xi)$, namely

$$\Phi(\tilde{u}) = \{ y \in \mathbb{R}^m : g^i(y, \tilde{x}; \xi) \in Q_{i+1}, i = 1, \ldots, l \},$$

(1)

and by $Y^*(\tilde{u})$ the set of optimal solutions for $P(\tilde{x}, \xi)$. For a given parameter $u$, we analyze properties of the optimal value function $\theta(\cdot, \cdot)$ when $u$ is perturbed to $\tilde{u}$. For this purpose we make the following assumptions.

**Assumption 3.1.** The set $X \subseteq \mathbb{R}^n$ is a non-empty compact convex set.

**Assumption 3.2.** For each $x \in X$, the optimal value of $P(x, \xi)$ is finite and the solution set for $P(x, \xi)$ is compact.

**Assumption 3.3.** The slater condition of $P(x, \xi)$ holds for each $x \in X$, namely for each $x \in \mathbb{R}^n$, there exists $y_x$ such that

$$g^i(y_x, x; \xi) \in \text{int} Q_{i+1}, i = 1, \ldots, l,$$

(2)

which can be written as $a_i^T y_x + q_i^T x - b_i > \|B_i y_x\|, i = 1, \ldots, l$.

The Assumption 3.2 can be guaranteed by the boundedness of feasible set to second-order cone optimization problem. Meanwhile, it’s easy to see that if $B^i = 0, i = 1, \cdots, l$ and $(a_1, \cdots, a_l)$ is row full rank, then Assumption 3.3 is satisfied.

**Lemma 3.1.** Let $\xi$ be given such that Assumptions 3.1-3.3 hold, then there exists $\delta_0 > 0$ such that for any $\tilde{x} \in X$, the Slater condition for $P(\tilde{x}, \xi)$ holds when $\|\xi - \xi\| \leq \delta_0$. 
Proof. From Assumption 3.3, for each $\hat{x} \in X$, there exist $y_{\hat{x}} \in \mathbb{R}^m$ and $\varepsilon_{\hat{x}} > 0$ such that

$$a_i^T y_{\hat{x}} + q_i^T \hat{x} - b_i - \|B_i y_{\hat{x}}\| \geq \varepsilon_{\hat{x}}, i = 1, \ldots, l.$$  

Due to Assumptions 3.1 and 3.2, we know that $\hat{x} \in X$ and the solution set of $P(\hat{x}, \hat{\xi})$ are bounded. Define $\Delta a_i = a_i - \hat{a}_i$, $\Delta q_i = q_i - \hat{q}_i$, $\Delta B_i = B_i - \hat{B}_i$ and $\Delta b_i = b_i - \hat{b}_i$, we have that

$$a_i^T y_{\hat{x}} - \hat{a}_i^T y_{\hat{x}} \leq \|a_i^T y_{\hat{x}} - \hat{a}_i^T y_{\hat{x}}\| \leq \|\Delta a_i^T\| \|y_{\hat{x}}\|,$$

$$q_i^T \hat{x} - \hat{q}_i^T \hat{x} \leq \|q_i^T \hat{x} - \hat{q}_i^T \hat{x}\| \leq \|\Delta q_i^T\| \|\hat{x}\|,$$

$$\|B_i y_{\hat{x}}\| - \|\hat{B}_i y_{\hat{x}}\| \leq \|B_i y_{\hat{x}} - \hat{B}_i y_{\hat{x}}\| \leq \|\Delta B_i\| \|y_{\hat{x}}\|,$$

$$b_i - \hat{b}_i \leq \|\Delta b_i\|, i = 1, \ldots, l.$$  

Set $M_{\hat{x}} := \max\{\|y_{\hat{x}}\|, \|\hat{x}\|, 1\}$ and $\max\{\|\Delta a_i\|, \|\Delta q_i\|, \|\Delta b_i\|, \|\Delta B_i\|\} \leq \frac{\varepsilon_{\hat{x}}}{5M_{\hat{x}}}$. Thus we have for $i = 1, \ldots, l$,

$$a_i^T y_{\hat{x}} + q_i^T \hat{x} - b_i - \|B_i y_{\hat{x}}\| \geq a_i^T y_{\hat{x}} + q_i^T \hat{x} - b_i - \|B_i y_{\hat{x}}\| - (\|\Delta a_i^T\| \|y_{\hat{x}}\| + \|\Delta q_i^T\| \|\hat{x}\|)$$

$$\geq \frac{\varepsilon_{\hat{x}}}{5} > 0.$$  

Thus Slater condition for $P(\hat{x}, \hat{\xi})$ holds when $\hat{x} \in X$, $\|(\hat{c}, \hat{A}, \hat{Q}, \hat{B}, \hat{b})-(c, A, Q, B, b)\| \leq \delta_\hat{x}$ for $\delta_\hat{x} = \frac{\varepsilon_{\hat{x}}}{5M_{\hat{x}}}$. For $\omega_{\hat{x}} > 0$ at $\hat{x} \in X$, we have

$$X \subset \bigcup_{\hat{x} \in X} B_{\omega_{\hat{x}}}(\hat{x}),$$

where $B_r(a)$ stands for an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$. From Assumption 3.1, $X$ is compact, from the finite covering theorem we have that there are a finite number of points $x_1, \ldots, x_{n_0}$ and positive numbers $\omega_{x_1}, \ldots, \omega_{x_{n_0}}$ such that

$$X \subset \bigcup_{j=1}^{n_0} B_{\omega_{x_j}}(x_j).$$

Let $\omega_0 = \min\{\omega_{x_j} : j = 1, \ldots, n_0\}$ and $\delta_0 = \min\{\delta_{x_j} : j = 1, \ldots, n_0\}$. Then for any $\hat{x} \in X$, Slater condition for $P(\hat{x}, \hat{\xi})$ holds when $\|\hat{\xi} - \hat{\xi}\| \leq \delta_0$. \hfill $\square$

Given $\xi$, let $\delta_0 > 0$ be the positive number in Lemma 3.1 satisfying that the Slater condition holds for $P(\hat{x}, \hat{\xi})$ when $x \in X$ and $\|\hat{\xi} - \hat{\xi}\| \leq \delta_0$. Let us denote by for $r > 0$,

$$U_r(\xi) = \left\{(\hat{x}, \hat{\xi}) : \hat{x} \in X, \|\hat{\xi} - \hat{\xi}\| \leq r \right\}.$$  

**Lemma 3.2.** Let $\xi$ be given with Assumptions 3.1 - 3.3 being satisfied. Then for any $\hat{u} \in U_{\delta_0}(\xi),$

$$\lim_{\hat{u} \to \hat{u}} \Phi(\hat{u}) = \Phi(\hat{u}).$$

**Proof.** As the following inclusion

$$\limsup_{\hat{u} \to \hat{u}} \Phi(\hat{u}) \subset \Phi(\hat{u})$$

is obvious, we only need to verify that

$$\liminf_{\hat{u} \to \hat{u}} \Phi(\hat{u}) \supset \Phi(\hat{u}).$$

For arbitrary $\hat{y} \in \Phi(\hat{u})$, we now prove $\hat{y} \in \liminf_{\hat{u} \to \hat{u}} \Phi(\hat{u})$. By Lemma 3.1, we have that

$$\exists \hat{y} \text{ such that } a_i^T \hat{y} + q_i^T \hat{x} - \hat{b}_i - \|\hat{B}_i \hat{y}\| \geq \hat{\varepsilon}_i, i = 1, \ldots, l.$$
For any sequence \( \tilde{u}(t) = (\tilde{x} + t(\tilde{x} - \hat{x}), \tilde{c} + t\triangledown c, \tilde{A} + t\triangledown A, \tilde{Q} + t\triangledown Q, \tilde{B} + t\triangledown B, \tilde{b} + t\triangledown b), \)
\( y(t) = \tilde{y} + t(\tilde{y} - \hat{y}), \)
and we obviously have \( \tilde{u}(t) \to \hat{u} \) and \( y(t) \to \hat{y} \) as \( t \downarrow 0 \). Then for \( \Delta u_i = (\Delta x_i, \Delta c_i, \Delta a_i, \Delta q_i, \Delta B_i, \Delta b_i), i = 1, \ldots, l, \)
we have that
\[
\tilde{a}_i(t)^T y(t) + \tilde{q}_i(t)^T \tilde{x}(t) - \tilde{b}_i(t) - \|\tilde{B}_i(t)y(t)\| \geq (\tilde{a}_i + t\triangledown a_i)^T (\tilde{y} + (1 - t)\tilde{y}) + (\tilde{q}_i + t\triangledown q_i)^T (\tilde{x} + t\triangledown x) - (\tilde{b}_i + t\triangledown b_i) - \|\tilde{B}_i + t\triangledown B_i(t)\| (\tilde{y} + (1 - t)\tilde{y}) \geq t(\tilde{a}_i^T (\tilde{y} - \hat{y}) + \tilde{q}_i^T \triangledown x + \triangle a_i^T \tilde{x} - \triangle b_i) + t^2(\triangle a_i^T (\tilde{y} - \hat{y}) + \triangle q_i^T \triangledown x) \\
+ \tilde{a}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}_i\| - t\|\tilde{B}_i\| - t\|\Delta B_i\| - t^2\|\triangle B_i\| (\tilde{y} - \hat{y}) \|t(\tilde{a}_i^T (\tilde{y} - \hat{y}) + \tilde{q}_i^T \triangledown x + \triangle a_i^T \tilde{x} - \triangle b_i) + t^2(\triangle a_i^T (\tilde{y} - \hat{y})) + (1 - t)(\tilde{a}_i^T \tilde{y} + \tilde{q}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}_i\|) \|t(\tilde{a}_i^T (\tilde{y} - \hat{y}) + \tilde{q}_i^T \triangledown x + \triangle a_i^T \tilde{x} - \triangle b_i) + t^2(\triangle a_i^T (\tilde{y} - \hat{y})) + (1 - t)(\tilde{a}_i^T \tilde{y} + \tilde{q}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}_i\|) \|t(\tilde{a}_i^T (\tilde{y} - \hat{y}) + \tilde{q}_i^T \triangledown x + \triangle a_i^T \tilde{x} - \triangle b_i) + t^2(\triangle a_i^T (\tilde{y} - \hat{y})) + (1 - t)(\tilde{a}_i^T \tilde{y} + \tilde{q}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}_i\|) \). 
\]
Therefore for \( \|\Delta u\| \) small enough, there exits \( \tilde{t} > 0 \) such that 
\[
\tilde{a}_i(t)^T y(t) + \tilde{q}_i(t)^T \tilde{x}(t) - \tilde{b}_i(t) - \|\tilde{B}_i(t)y(t)\| \geq 0, i = 1, \ldots, l, \forall t \in [0, \tilde{t}).
\]
Then \( y(t) \in \Phi(\tilde{u}(t)) \), which implies \( \hat{y} \in \liminf_{\tilde{u} \to \hat{u}} \Phi(\tilde{u}) \). The proof is completed.

Lemma 3.3. For given \( \xi \), let Assumptions 3.1-3.3 hold. Then for any \( \alpha \in \mathbb{R}^n \), there exists \( \delta_1 > 0 \) and a bounded set \( B \subset \mathbb{R}^m \) such that 
\[
\Psi(\tilde{u}, \alpha') \subset B, \forall \alpha' \leq \alpha, \forall \tilde{u} \in \mathcal{U}_b(\xi),
\]
where \( \Psi(\tilde{u}, \alpha) = \Phi(\tilde{u}) \cap \text{lev}_{\leq \alpha} f(\cdot, \tilde{u}) \).

Proof. Without loss of generality, we assume that \( \Psi(\hat{u}, \alpha) \neq \emptyset \). Because \( \Psi(\tilde{u}, \alpha') \subset \Psi(\hat{u}, \alpha), \forall \alpha' \leq \alpha \), we only need to prove \( \Psi(\tilde{u}, \alpha) \subset B \). We prove the result by contradiction. Suppose that there is a sequence \( \hat{u}_k = (x_k, \xi_k) \) such that \( x_k \in X \)
and \( \xi_k \to \xi \) and \( y_k \in \Psi(\hat{u}_k, \alpha) \) with \( \|y_k\| \to \infty \) as \( k \to +\infty \). Let \( d_k = y_k/\|y_k\| \), and notice \( X \) is compact, we can find a subsequence \( k_j \) such that \( x_{k_j} \to x^* \) and \( d_{y_j} \to d_y \) for \( x^* \in X \) and \( d_y \in \text{bdry} B \). In view of \( y_{k_j} \in \Psi(\hat{u}_{k_j}, \alpha), \) one has 
\[
\hat{c}_{k_j}^T y_{k_j} \leq \alpha \\
\hat{a}_{k_j} y_{k_j} + \hat{q}_{k_j}^T x_{k_j} - \hat{b}_{k_j} \geq \|\hat{B}_{k_j}\| y_{k_j}, \ i = 1, \ldots, l.
\]
Dividing both sides of the above inequalities by \( \|y_{k_j}\| \), we obtain 
\[
\hat{c}_{k_j}^T d_y \leq \alpha/\|y_{k_j}\| \\
\hat{a}_{k_j}^T d_{y_j} + \hat{q}_{k_j}^T x_{k_j}/\|y_{k_j}\| - \hat{b}_{k_j}/\|y_{k_j}\| \geq \|\hat{B}_{k_j}\| d_{y_j}, \ i = 1, \ldots, l.
\]
Taking the limits by \( j \to \infty \), we have 
\[
c^T d_y \leq 0, \ a_i^T d_y \geq \|B_i d_y\| > 0, i = 1, \ldots, l,
\]
which contradicts with Assumption 3.2. Since the set of solutions to \( P(x, \xi) \) is compact, we have that such \( d_y \) must be zero.

Theorem 3.4. For given \( \xi \), let Assumptions 3.1-3.3 hold. For any \( \tilde{u} \in \mathcal{U}_b(\xi) \) with \( \delta_1 \) defined in Lemma 3.3, one has that \( \theta \) is continuous at \( \hat{u} \) and the solution set
Upper continuity of dual solution mapping. We only need to show conditions (i)-(iv) in Proposition 4 hold. Let
\[ Y^*(\hat{u}) \subset Y^*(\hat{u}) + \epsilon B, \forall \hat{u} \in B_{\delta}(\hat{u}). \]

Proof. We only need to show conditions (i)-(iv) in Proposition 4 hold. Let \( G(y, \hat{u}) := (g^1(y, \hat{u}), \cdots, g^l(y, \hat{u})) \) and \( K := \prod^l_{i=1} Q_{J+1} \). Obviously we have that \( f(y, \hat{u}) \) is continuous in \( \mathbb{R}^m \times U_{d_1}(\xi) \), namely condition (i) of Proposition 4 holds. From Lemma 3.2 and noticing the equivalence between the outer semi-continuity and the closedness for set-value mappings, we have that \( \Phi \) is a closed set-value mapping so that (ii) of Proposition 4 holds. Condition (iii) of Proposition 4 comes from Lemma 3.3. From Lemma 3.1, the Slater condition holds for \( P(\tilde{x}, \tilde{\xi}) \), where \( \|\tilde{\xi} - x\| \leq \delta_0, \tilde{x} \in X \). This implies Robison constraint qualification for \( \Phi(\hat{u}) \) at any point \( \tilde{y} \in Y^*(\hat{u}) \). Then it follows form Theorem 2.87 in [1] that
\[ \text{dist}(\tilde{y}, \Phi(\hat{u})) \leq \kappa(\text{dist}(G(\tilde{y}, \hat{u}), K)) \leq \kappa\|G(\tilde{y}, \hat{u}) - G(\tilde{y}, \hat{u})\| \] (4)
for \( \hat{u} \in V_U \), where \( V_U \) is some neighborhood of \( \hat{u} \) and \( \kappa > 0 \). Since \( G \) is Lipschitz continuous, we have that condition (iv) of Proposition 4 holds.

Therefore, we have from Proposition 4 that the optimal value function \( \theta \) is continuous at \( \hat{u} \) and the solution set \( Y^*(\hat{u}) \) is upper semicontinuous at \( \hat{u} \), namely for \( \epsilon > 0 \) there exists a number \( \delta_2 > 0 \) such that
\[ Y^*(\hat{u}) \subset Y^*(\hat{u}) + \epsilon B, \forall \hat{u} \in B_{\delta}(\hat{u}). \]
The proof is completed. \( \square \)

4. Upper continuity of dual solution mapping. First of all, we derive the Lagrange dual of the \( P(\tilde{x}, \tilde{\xi}) \). Let \( \tilde{\lambda} = (\tilde{\lambda}^1, \cdots, \tilde{\lambda}^l) \in Q := Q^l_{J+1} \), the Lagrange function of problem \( P(\tilde{x}, \tilde{\xi}) \) defined by
\[ L(\tilde{x}, y, \tilde{\lambda}; \tilde{\xi}) := \tilde{c}^T y - \sum^l_{i=1} \langle \tilde{\lambda}^i, g^i(\tilde{x}, y; \tilde{\xi}) \rangle \]
\[ = \tilde{c}^T y - \langle \tilde{\lambda}, \tilde{A} \rangle y - \sum^l_{i=1} \lambda^i_{J+1} (q^T_i \tilde{x} - b_i), \]
where \( \tilde{A} : \mathbb{R}^m \to \mathbb{R}^{(J+1) \times l} \) is a linear operator defined by
\[ \tilde{A} y = \left\{ \begin{array}{c} \tilde{B}_1^T y \\ \tilde{a}_1^T y \\ \vdots \\ \tilde{B}_l^T y \\ \tilde{a}_l^T y \end{array} \right\}. \]
Then the Lagrange dual of \( P(\tilde{x}, \tilde{\xi}) \) becomes
\[ \tilde{D}(\tilde{x}, \tilde{\xi}) \max \sum^l_{i=1} \lambda^i_{J+1} \left( b_i - q^T_i \tilde{x} \right) \]
s.t. \( \tilde{c} - \tilde{A}^* \tilde{\lambda} = 0, \)
\[ \tilde{\lambda} \in Q, \]
where \( \tilde{A}^* \) is the adjoint of \( \tilde{A} \) and \( \tilde{A}^* \tilde{\lambda} \) is calculated by
\[ \tilde{A}^* \tilde{\lambda} = \sum^l_{i=1} (\tilde{B}_i^T y, \tilde{a}_i | \tilde{\lambda}^i). \] (6)
We denote the feasible set and the objective function for $D(\tilde{x}, \tilde{\xi})$ by
\[
\mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) = \{ \tilde{\lambda} = (\tilde{\lambda}^1; \cdots; \tilde{\lambda}^L) \in \mathcal{Q} : \tilde{c} - \tilde{A}^* \tilde{\lambda} = 0 \}\tag{7}
\]
and $\phi(\tilde{\lambda}, \tilde{u}) = \sum_{i=1}^L \tilde{\lambda}^i_j (\tilde{b}_j - \tilde{q}^T \tilde{x})$ respectively. Set $\Lambda^*(\tilde{u})$ be the set of optimal solutions of $D(\tilde{x}, \tilde{\xi})$. Moreover, since the Lemma 3.1 is satisfied, the dual problem $\tilde{D}(\tilde{x}, \tilde{\xi})$ has a nonempty compact solution set and the duality gap between $\tilde{D}(\tilde{x}, \tilde{\xi})$ and $\tilde{D}(\tilde{x}, \tilde{\xi})$ is zero.

**Lemma 4.1.** Let $(c, A, B)$ be given. If Assumption 3.2 holds, then there exists $\delta_2 > 0$ such that the Slater condition for $D(\tilde{x}, \tilde{\xi})$ holds when $\| (\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B) \| \leq \delta_2$, namely there exists $\tilde{\lambda}(\tilde{c}, \tilde{A}, \tilde{B})$ such that
\[
\tilde{c} - \tilde{A}^* \tilde{\lambda}(\tilde{c}, \tilde{A}, \tilde{B}) = 0, \quad \tilde{\lambda}(\tilde{c}, \tilde{A}, \tilde{B}) \in \text{int} \mathcal{Q}
\]
when $\| (\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B) \| \leq \delta_2$.

**Proof.** Since the Theorem 5.81 in [1] and Assumption 3.2, we know that Slater condition for Dual Problem of $P(x, \xi)$ holds, namely there exists a $\lambda$ such that
\[
c - A^* \lambda = 0, \quad \lambda \in \text{int} \mathcal{Q}. \tag{8}
\]
Firstly we prove that the operator $A^*$ is onto. In fact, suppose that there exist $d_y \in \mathbb{R}^m$ such that $Ad_y = 0$, which implies that $B^T d_y = 0, a^T d_y = 0, i = 1, \ldots, l$. From (8) we obtain that $c^T d_y = 0$. Therefore we obtain $d_y \in Y^*(u)$ and this implies $d_y = 0$ because otherwise $Y^*(u)$ is unbounded, a contradiction with Assumption 3.2. Thus we have that $\text{ker} \ A = \{0\}$ and operator $A^*$ is onto. Define $M = [(B^i)^T, a_1], \ldots, [(B^i)^T, a_l]]$, in view of (6) for $A^*$, we have that matrix $M$ is of row full rank.

The validity of Slater condition for $D(\tilde{x}, \tilde{\xi})$ is equivalent to the solvability of the following system in variable $\tilde{\lambda}$:
\[
\tilde{c} - \tilde{A}^* \tilde{\lambda} = 0, \quad \tilde{\lambda} \in \text{int} \mathcal{Q}. \tag{9}
\]
For any $(\tilde{c}, \tilde{A})$ with $(\tilde{c}, \tilde{A}, \tilde{B}) = (c, A, B) + (\Delta c, \Delta A, \Delta B)$ and $\tilde{\lambda} = \lambda + \Delta \lambda$, the first equality in (9) is equivalent to
\[
0 = c + \Delta c - (A^* + \Delta A^*)(\lambda + \Delta \lambda)
= \Delta c - A^* \Delta \lambda - \Delta A^* \lambda - \Delta A^* \Delta \lambda
\]
and thus
\[
- \tilde{M} \Delta \lambda = - \Delta c + \Delta M \lambda. \tag{10}
\]
Since $\text{ker} \ A = \{0\}$ implies that matrix $M$ is full rank in row, we have that $MM^T$ is positive definite. Let $\Delta N = \Delta MM^T + M \Delta M^T + \Delta M \Delta M^T$, when $\Delta M$ is small enough, $\tilde{M} \tilde{M}^T = MM^T + \Delta N$ is nonsingular. We assume that $\delta_3 > 0$ satisfies that $\tilde{M} \tilde{M}^T$ is nonsingular, $\|\Delta M\| \leq \delta_3$. Then we obtain from Sherman-Morrison-Woodbury formula that
\[
\tilde{M}^+ = \tilde{M}^T (MM^T)^{-1}
= (M^T + \Delta M^T)(MM^T + \Delta N)^{-1}
= (M^T + \Delta M^T)[(MM^T)^{-1} - (MM^T)^{-1} \Delta N[I_m + (MM^T)^{-1} \Delta N]^{-1}(MM^T)^{-1}]
= M^+ + \Delta \Sigma,
\]
where $\Delta \Sigma$ satisfies $\|\Delta \Sigma\| = O(\|\Delta M\|)$. Since $\hat{M}\hat{M}^T$ is nonsingular when $\|\Delta M\| \leq \delta_3$, we have that

$$\Delta \lambda^* := -\hat{M}^T(-\Delta c + \Delta M\lambda) = -[M^T + \Delta \Sigma](-\Delta c + \Delta M\lambda) \quad (11)$$

is a particular solution to (10). From the expression for $\Delta \lambda^*$ in (11), we may assume that $\delta_3 > 0$ small enough such that $\|\Delta \mu^* (\Delta M)\| < \min \{\|\lambda\|/2, (\|M^T\| + \delta_3)(1 + \|\lambda\|)\delta_3\}$ when $\|(\hat{c}, \hat{A}, \hat{B}) - (c, A, B)\| \leq \delta_3$. Therefore, for $\Delta \lambda = \Delta \lambda^*$, we have that

$$\hat{\lambda} := \lambda + \Delta \lambda$$

satisfies (9) when $\|(\hat{c}, \hat{A}, \hat{B}) - (c, A, B)\| \leq \delta_3$. The proof is completed. \hfill $\square$

**Lemma 4.2.** Let $(c, A, B)$ be given with Assumption 3.2 being satisfied. Then, for any $(\hat{c}, \hat{A}, \hat{B}) \in \mathbb{B}_{\delta_3}(c, A, B)$,

$$\lim_{(\hat{c}, \hat{A}, \hat{B}) \to (c, A, B)} E(\hat{c}, \hat{A}, \hat{B}) = E(\hat{c}, \hat{A}, \hat{B}).$$

**Proof.** As the following inclusion

$$\lim sup_{(\hat{c}, \hat{A}, \hat{B}) \to (c, A, B)} E(\hat{c}, \hat{A}, \hat{B}) \subset E(\hat{c}, \hat{A}, \hat{B})$$

is obvious, we only need to verify that

$$\lim inf_{(\hat{c}, \hat{A}, \hat{B}) \to (c, A, B)} E(\hat{c}, \hat{A}, \hat{B}) \supset E(\hat{c}, \hat{A}, \hat{B}).$$

For arbitrary $\hat{\lambda} \in E(\hat{c}, \hat{A}, \hat{B})$, we now prove $\hat{\lambda} \in \lim inf_{(\hat{c}, \hat{A}, \hat{B}) \to (c, A, B)} E(\hat{c}, \hat{A}, \hat{B})$. By Lemma 4.1, we have that there exists $\bar{\lambda}$ such that

$$\hat{c} - \hat{A}^*\bar{\lambda} = 0, \quad \bar{\lambda} \in \text{int} \mathcal{Q}.$$  

For an arbitrary sequence $(\hat{c}(t), \hat{A}(t))$ with $(\hat{c}(t), \hat{B}(t), \hat{A}(t)) = (\hat{c} + t\Delta c, \hat{B} + t\Delta B, \hat{A} + t\Delta A)$, we obviously have $(\hat{c}(t), \hat{B}(t), \hat{A}(t)) \to (\hat{c}, \hat{B}, \hat{A})$ as $t \downarrow 0$. Define $\lambda(t) = (\lambda^1(t); \ldots; \lambda^\ell(t))$ by

$$\lambda(t) = (1 - t)\hat{\lambda} + t(\bar{\lambda} + d_\lambda(t)) = \hat{\lambda} + t(\bar{\lambda} - \hat{\lambda}) + td_\lambda(t), \quad (12)$$

where $d_\lambda(t)$ is the unknown vector. We consider the system

$$\dot{\hat{c}}(t) - \hat{A}^*(t)\lambda(t) = 0. \quad (13)$$

Define

$$\dot{\xi}(t) = \hat{\lambda} + t(\bar{\lambda} - \hat{\lambda}).$$

Due to $\hat{\lambda} \in E(\hat{c}, \hat{A}, \hat{B})$, the equation (13) can written as

$$0 = \dot{\hat{c}} + t\Delta c - \hat{A}^*\dot{\xi}(t) - t\Delta \hat{A}^*\dot{\xi}(t) - \hat{A}^*d_\lambda(t)$$

$$\iff 0 = \dot{\hat{c}} - \hat{A}^*((1 - t)\hat{\lambda} + t\bar{\lambda}) + t(\Delta c - \Delta \hat{A}^*\dot{\xi}(t) - t\hat{A}^*d_\lambda(t)$$

$$\iff 0 = (1 - t)(\hat{c} - \hat{A}^*\hat{\lambda}) + t(\hat{c} - \hat{A}^*\bar{\lambda}) + t(\Delta c - \Delta \hat{A}^*\dot{\xi}(t) - t\hat{A}^*d_\lambda(t)$$

$$\iff \hat{A}^*(t)d_\lambda(t) = \Delta c - [\Delta \hat{A}]^*\dot{\xi}(t).$$

Let $d_\lambda(t)$ be the following least square norm solution:

$$d_\lambda(t) = [\hat{A}(t)^*]^\dagger[(\Delta c - [\Delta \hat{A}]^*\dot{\xi}(t))]. \quad (14)$$
Similar to the analysis in the proof of Lemma 4.1, we obtain \( \mathbf{A}(t)^{\dagger} = \mathbf{A}^{\dagger} + O(t\|\Delta A\|) \). We may assume that \( \|\mathbf{A}(t)^{\dagger}\| \leq 2\|\mathbf{A}^{\dagger}\| \) for \((\bar{c}, \bar{A}, \bar{B}) \in \mathbb{B}_{\delta_{3}}(\tilde{c}, \tilde{A}, \tilde{B})\) when \( t > 0 \) is small enough. Let

\[
\kappa = 2\|\mathbf{A}^{\dagger}\| \max\{1, \|\tilde{x}\|, \|\tilde{\lambda}\|\} + 1.
\]

Then for \( \|\Delta c, \Delta B, \Delta A\| \leq \delta_{3} \), we have

\[
\|d_{\tilde{x}}^{\ast}(t)\| \leq \|\mathbf{A}(t)^{\dagger}\|\|\Delta c - \Delta A^{\ast} \tilde{\xi}(t)\| \leq 2\|\mathbf{A}^{\dagger}\| \max\{1, \|\tilde{x}\|, \|\tilde{\lambda}\|\}(\Delta c, \Delta B, \Delta A) \leq \kappa \delta_{3}.
\]

Since \( \tilde{x} \in \text{int} \mathcal{Q} \), we have that \( \tilde{x} + d_{\tilde{x}}^{\ast}(t) \in \text{int} \mathcal{Q} \) when \( \delta_{3} \leq \delta_{2} \) is small enough. Then we obtain for \( t \in [0, 1] \) that

\[
\lambda(t) = (1 - t) \tilde{\lambda} + t(\tilde{x} - d_{\tilde{x}}^{\ast}(t)) \in \mathcal{Q}
\]

and satisfies (13). Therefore, when \((\bar{c}, \bar{A}, \bar{B}) \in \mathbb{B}_{\delta_{3}}(\tilde{c}, \tilde{A}, \tilde{B})\) and for small \( t > 0 \), one has

\[
\lambda(t) \in \mathcal{E}(\bar{c} + t\Delta c, \bar{A} + t\Delta A)
\]

and \( \lambda(t) \to \tilde{\lambda} \). This implies \( \tilde{\lambda} \in \liminf_{(\bar{c}, \bar{A}, \bar{B}) \to (\tilde{c}, \tilde{A}, \tilde{B})} \mathcal{E}(\bar{c}, \bar{A}, \bar{B}) \). This proof is completed. \( \square \)

Define

\[
\Gamma(\tilde{u}, \alpha) = \mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) \cap \text{lev}_{\geq \alpha} \phi(\cdot, \tilde{u})
\]

with

\[
\text{lev}_{\geq \alpha} \phi(\cdot, \tilde{u}) = \{ \tilde{\lambda} \in \mathbb{R}^{J+1} \times \cdots \times \mathbb{R}^{J+1} : \phi(\tilde{\lambda}, \tilde{u}) \geq \alpha \}, \alpha \in \mathbb{R}.
\]

**Lemma 4.3.** For given \( \xi \), let Assumptions 3.1-3.3 hold. Then for any \( \alpha \in \mathbb{R}^{n} \), there exists \( \delta_{3} > 0 \) and a bounded set \( \mathcal{D} \subset \mathbb{R}^{J+1} \times \cdots \times \mathbb{R}^{J+1} \) such that

\[
\Gamma(\tilde{u}, \alpha') \subset \mathcal{D}, \forall \alpha' \geq \alpha, \forall \tilde{u} \in \mathcal{U}_{\delta_{3}}(\xi).
\]

**Proof.** Without loss of generality, we assume that \( \Gamma(\tilde{u}, \alpha) \neq \emptyset \). Because \( \Gamma(\tilde{u}, \alpha') \subset \Gamma(\tilde{u}, \alpha), \forall \alpha' \geq \alpha \), we only need to prove \( \Gamma(\tilde{u}, \alpha) \subset \mathcal{D} \).

We first prove that, for any \( \tilde{\lambda} \in \Gamma(\tilde{u}, \alpha) \), \( \tilde{\lambda} \) is bounded by contradiction. Suppose that there exists a sequence \( \tilde{u}^{k} = (x^{k}, \tilde{x}^{k}) \) such that \( x^{k} \in X, \tilde{x}^{k} \to \xi \) and \( \lambda^{k} \in \Gamma(\tilde{u}^{k}, \alpha) \) with \( \|\lambda^{k}\| \to \infty \). Let \( d_{\lambda}^{k} = \lambda^{k}/\|\lambda^{k}\| \) and notice \( X \) is compact, we can find a subsequence \( k_{j} \) such that \( x^{k_{j}} \to x^{*} \) for \( x^{*} \in X \) and \( d_{\lambda}^{k_{j}} \to d_{\lambda} \) with \( d_{\lambda} \in \text{bdry} \mathbf{B} \). In view of \( \hat{\lambda}^{k_{j}} \in \Gamma(\tilde{u}^{k_{j}}, \alpha) \), we have that

\[
\sum_{i=1}^{l} (\hat{\lambda}_{j_{i}+1}^{k_{j}})^{T} (\hat{b}_{j_{i}}^{k_{j}} - \hat{q}_{j_{i}}^{k_{j}T} x^{k_{j}}) \geq \alpha,
\]

\[
\hat{\lambda}_{j_{i}} - [\mathbf{A}^{k_{j}}]^{\ast} \lambda^{k_{j}} = 0,
\]

\[
\hat{\lambda}^{k_{j}} \in \mathcal{Q}.
\]

Dividing the above inequalities by \( \|\hat{\lambda}^{k_{j}}\| \), we get

\[
\sum_{i=1}^{l} (\hat{d}_{\lambda}^{k_{j}})^{T} (\hat{b}_{j_{i}}^{k_{j}} - \hat{q}_{j_{i}}^{k_{j}T} x^{k_{j}}) \geq \alpha/\|\hat{\lambda}^{k_{j}}\|,
\]

\[
\hat{d}_{\lambda}^{k_{j}}/\|\lambda^{k_{j}}\| - [\mathbf{A}^{k_{j}}]^{\ast} \hat{d}_{\lambda}^{k_{j}} = 0,
\]

\[
\hat{d}_{\lambda}^{k_{j}} \in \mathcal{Q}.
\]
Taking the limits by $j \to \infty$, we have
\[ \sum_{i=1}^{k} \tilde{d}^T_i (b_i - \tilde{q}^T_i x) \geq 0, \quad \tilde{A}^* \tilde{d}_\lambda = 0, \quad \tilde{d}_\lambda \in \mathbb{Q}, \quad \|\tilde{d}_\lambda\| = 1, \]
which contradicts with the compactness of the optimal solution set of $D(\tilde{x}, \tilde{\xi})$. This is implied by Slater condition for the problem $P(\tilde{x}, \tilde{\xi})$ proved in Lemma 3.1.

**Theorem 4.4.** For given $\xi$, let Assumptions 3.1-3.3 hold. For any $\tilde{u} \in \mathcal{U}_{\delta^3}(\xi)$ with $\delta^3$ defined in Lemma 4.3, one has that the solution set mapping $\Lambda^*$ is upper semi-continuous at $\tilde{u}$, namely for $\epsilon > 0$ there exists a number $\delta > 0$ such that
\[ \Lambda^*(\tilde{u}) \subset \Lambda^*(\tilde{u}) + \epsilon \mathcal{B}, \forall \tilde{u} \in \mathcal{B}_{\delta}(\tilde{u}). \]

**Proof.** The results in this theorem can be proved by Lemma 4.2 and Lemma 4.3. The proof is similar to that of Theorem 3.4. We omit it here. \(\square\)

5. **Differentiability of optimal value function.** From Lemma 3.3 and Lemma 4.3, we assume that for some $\delta_4 > 0$, $\alpha \in \mathbb{R}$, and bounded sets $\mathcal{B} \in \mathbb{R}^m$, $\mathcal{D} \subset \mathbb{R}^{J+1} \times \cdots \times \mathbb{R}^{J+1}$,
\[ \Psi(\tilde{u}, \alpha) \subset \mathcal{B}, \Gamma(\tilde{u}, \alpha) \subset \mathcal{D} \]
for any $\tilde{x} \in X$ and $\|\tilde{\xi} - \xi\| \leq \delta_4$. Therefore, by the Lagrange duality theory, the optimal value can be written as
\[ \theta(\tilde{u}) = \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \inf_{y \in \mathcal{B}} L(y, \lambda; \tilde{u}). \quad (17) \]

**Proposition 5.** For given $\xi$ and $x \in X$, let Assumptions 3.1-3.3 hold. Then $\theta(\tilde{u})$ is locally Lipschitz continuous around $u = (x, \xi)$, namely there exists some $\kappa \geq 0$ depending on $\xi$ such that
\[ |\theta(\tilde{u}) - \theta(u')| \leq \kappa \|\tilde{u} - u'\|, \quad (18) \]
when $\tilde{u}, u' \in \mathcal{B}_{\delta_5}(u)$ for some positive constant $\delta_5 > 0$ depending on $u$. Here $\tilde{u} = (\tilde{x}, \tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b})$, $u' = (x', c', A', Q', B', b')$ and
\[ \|\tilde{u} - u'\| = \|\tilde{c} - c'\| + \sum_{j=1}^{l} \|\tilde{B}^j - B^j\| + \|\tilde{A} - A'\| + \|\tilde{Q} - Q'\| + \|\tilde{b} - b'\| + \|\tilde{x} - x'\|. \]

**Proof.** Since $L(y, \lambda, u)$ is a continuous function on bounded set, the max-min values of $L$ at $\tilde{u}$ and $u'$ can be arrived. Let $(\tilde{y}, \tilde{\lambda}), (y', \lambda') \in \mathcal{B} \times [\mathcal{Q} \cap \mathcal{D}]$ satisfy
\[ \theta(\tilde{u}) = L(\tilde{y}, \tilde{\lambda}; \tilde{u}), \quad \theta(u') = L(y', \lambda'; u'). \]
Without loss of generality, we assume that $\theta(\tilde{u}) \leq \theta(u')$. Then we have
\[ |\theta(\tilde{u}) - \theta(u')| = \left| \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \inf_{y \in \mathcal{B}} L(y, \lambda, \tilde{u}) - \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \inf_{y \in \mathcal{B}} L(y, \lambda, u') \right| \]
\[ = \left| L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(y', \lambda'; u') \right| \]
\[ \leq \left| L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(\tilde{y}, \tilde{\lambda}; \tilde{u}) + L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(y', \lambda'; u') \right| \]
\[ \leq \left| L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(y', \lambda'; u') \right| \]
\[ \leq \sup_{y \in \mathcal{B}} \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} |L(y, \lambda; \tilde{u}) - L(y, \lambda; u')|. \]

Since $Q$ is given as one part of $\xi$, we choose $\delta_5 \leq \min\{|\delta_4|, |Q|\}$ and let $\|\tilde{u} - u'\| \leq \delta_5$. Due to the boundedness of $\mathcal{B}, \mathcal{Q} \cap \mathcal{D}$ and $X$, there exist three constants $D_y, D_x, D_x^* > 0$ such that
0 such that for any \( y \in B, \lambda \in Q \cap D \) and \( x \in X, \| y \| \leq D_y, \| \lambda \| \leq D_\lambda \) and \( \| x \| \leq D_x \). Furthermore \( x', Q' \) are around \( x, Q \) respectively, then we obtain \( \| x' \| \leq D_x + 1, \| Q' \| \leq \| Q \| + 1 \). Thus we have that

\[
\begin{align*}
\| L(y, \lambda; \tilde{u}) - L(y, \lambda; u') \| = \| c - c' \| y + \langle \lambda, g(y, \tilde{\xi}) - g(y, x'; \tilde{\xi}') \rangle \leq \| c - c' \| \| y \| \\
+ \| \lambda \| \times \left\{ \sum_{j=1}^{\| \lambda \|} \| \tilde{B}^j - B'^j \| + \| \tilde{A} - A' \| \right\} \| y \| + \| \tilde{Q} \| \| \tilde{x} - x' \| + \| \tilde{Q'} \| \| x' \| + \| \tilde{b} - b' \|
\right\}
\leq \max \{ D_y, D_\lambda, D_x + 1, \| Q \| + 1 \} \times \\
\left\{ \| c - c' \| + \sum_{j=1}^{\| \lambda \|} \| \tilde{B}^j - B'^j \| + \| \tilde{A} - A' \| + \| \tilde{Q} - Q' \| + \| \tilde{b} - b' \| + \| \tilde{x} - x' \| \right\}
\end{align*}
\]

where \( \kappa = \max \{ D_y, D_\lambda, D_x + 1, \| Q \| + 1 \} \). Combining the above inequality with (19), we obtain the inequality (18) when \( \tilde{u}, u' \in B_{\delta_4}(u, x) \).

\[ \square \]

**Theorem 5.1.** For given \( \xi \) and \( x \in X \), let Assumptions 3.1-3.3 hold. Then the optimal value function \( \theta(\tilde{u}) \) is directionally differentiable at \( u \). Moreover, \( \theta(\tilde{u}) \) is Hadamard directionally differentiable at \( u \). Thus we have the following Taylor expansion of \( \theta(\tilde{u}) \) at \( u \)

\[
\begin{equation}
\theta(\tilde{u}) = \theta(u) + \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} (\Delta c - \Delta A^* \lambda)^T y + \sum_{i=1}^{l} \lambda_i b_i + \sum_{i=1}^{l} \lambda_i (\Delta A_i - A_i) y + o(\| D \lambda \|),
\end{equation}
\]

where \( \tilde{u} = (\tilde{x}, \tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}) \) and \( D u = \tilde{u} - u \) satisfying \( \| D u \| \leq \delta_4 \).

**Proof.** Let \( u_t = u + t D u = (x_t, c_t, A_t, Q_t, B_t, b_t) \) and denote

\[
\gamma(y, \lambda) = \lim_{t \downarrow 0} \frac{L(y, \lambda; u_t) - L(y, \lambda; u)}{t}.
\]

From definition, \( L(y, \lambda; u_t) \) is continuous, convex respects to \( y \in B \) and concave respects to \( \lambda \in \Lambda \cap D \). For the convex and compact set of saddle points \( Y^*(u) \times \Lambda^*(u) \), the directional derivative of \( \theta \) at \( u \) in direction \( D u \) can be derived by Lemma 2.6 as follows:

\[
\begin{align*}
\theta'(u; D u) &= \lim_{t \downarrow 0} \frac{\theta(u_t) - \theta(u)}{t} \\
&= \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \lim_{t \downarrow 0} \frac{L(y, \lambda; u_t) - L(y, \lambda; u)}{t} \\
&= \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \lim_{t \downarrow 0} \left( (c - c) y - \langle \lambda, (A - A) y \rangle + \sum_{i=1}^{l} \lambda_i (b_i - b_i) \right) \\
&= \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \lim_{t \downarrow 0} \left( (\Delta c - \Delta A^* \lambda)^T y + \sum_{i=1}^{l} \lambda_i (\Delta A_i - A_i) y + o(\| D u \|) \right)
\end{align*}
\]

Combining with the Lipschitz continuity of \( \theta(\tilde{u}) \) from Proposition 5, we have that \( \theta(\tilde{u}) \) is Hadamard directionally differentiable at \( u \). Therefore Taylor expansion of \( \theta(\tilde{u}) \) is obtained at \( u \):

\[
\theta(\tilde{u}) = \theta(u) + \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} (\Delta c - \Delta A^* \lambda)^T y + \sum_{i=1}^{l} \lambda_i (\Delta A_i - A_i) y + o(\| D u \|).
\]

The proof is completed. \( \square \)
6. **Conclusions.** We consider the stability of a second-order conic optimization problem when all parameters in the problem are perturbed. Under Slater constraint qualification, we prove the upper semi-continuity of the solution set of both the original problem and the dual problem. Furthermore, we show that the optimal value function is locally Lipschitz continuous and Hadamard directionally differentiable. Interestingly, when we express the optimal value function as a min-max optimization problem over two compact convex sets, the asymptotic distribution of the optimal value function can be discussed by Theorem 7.59 of [13] when $\xi$ is a random variable and the sample average approach is adopted.

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