Symmetric monochromatic subsets in colorings of the Lobachevsky plane

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We prove that for each partition of the Lobachevsky plane into finitely many Borel pieces one of the cells of the partition contains an unbounded centrally symmetric subset.

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1 Introduction

It follows from [B1] (see also [BP1, Theorem 1]) that for each partition of the $n$-dimensional space $\mathbb{R}^n$ into $n$ pieces one of the pieces contains an unbounded centrally symmetric subset. On the other hand, $\mathbb{R}^n$ admits a partition $\mathbb{R}^n = B_0 \cup \cdots \cup B_n$ into $(n+1)$ Borel pieces containing no unbounded centrally symmetric subset. For $n = 2$ such a partition is drawn at the picture:
On colorings of the Lobachevsky plane

Taking the same partition of the Lobachevsky plane $\mathbb{H}^2$, we can see that each piece $B_i$ does contain an unbounded centrally symmetric subset (for such a set just take any hyperbolic line lying in $B_i$).

We call a subset $S$ of the hyperbolic plane $\mathbb{H}^2$ centrally symmetric or else symmetric with respect to a point $c \in \mathbb{H}^2$ if $S = f_c(S)$ where $f_c : \mathbb{H}^2 \to \mathbb{H}^2$ is the involutive isometry of $\mathbb{H}^2$ assigning to each point $x \in \mathbb{H}^2$ the unique point $y \in \mathbb{H}^2$ such that $c$ is the midpoint of the segment $[x, y]$. The map $f_c$ is called the central symmetry of $\mathbb{H}^2$ with respect to the point $c$.

By a partition of a set $X$ we understand a decomposition $X = B_1 \cup \cdots \cup B_m$ of $X$ into pairwise disjoint subsets called the pieces of the partition.

The following theorem shows that the Lobachevsky plane differs dramatically from the Euclidean plane from the Ramsey point of view.

**Theorem 1.1** For any partition $\mathbb{H}^2 = B_1 \cup \cdots \cup B_m$ of the Lobachevsky plane into finitely many Borel pieces one of the pieces contains an unbounded centrally symmetric subset.

## 2 Proof of Theorem 1.1

We shall prove a bit more: given a partition $\mathbb{H}^2 = B_1 \cup \cdots \cup B_m$ of the Lobachevsky plane into $m$ Borel pieces we shall find $i \leq m$ and an unbounded subset $S \subset B_i$ symmetric with respect to some point $c$ in an arbitrarily small neighborhood of some finite set $F \subset \mathbb{H}^2$ depending only on $m$.

To define this set $F$ it will be convenient to work in the Poincaré model of the Lobachevsky plane $\mathbb{H}^2$. In this model the hyperbolic plane $\mathbb{H}^2$ is identified with the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane and hyperbolic lines are just segments of circles orthogonal to the boundary of $D$. Let $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the hyperbolic plane $D$ with attached ideal line. For a real number $R > 0$ the set $D_R = \{z \in \mathbb{C} : |z| \leq 1 - 1/R\}$ can be thought as a hyperbolic disk of increasing radius as $R$ tends to $\infty$.

On the boundary of the unit disk $D$ consider the $(m + 1)$-element set

$$A = \{z \in \mathbb{C} : z^{m+1} = 1\}.$$

For two distinct points $x, y \in A$ let $[x|y] \in \overline{D}$ denote the midpoint of the arc in $\overline{D}$ that connects the points $x, y$ and lies on a hyperbolic line in $\mathbb{H}^2 = D$. Then $F = \{|x|y| : x, y \in A, x \neq y\}$ is a finite subset of cardinality $|F| \leq m(m + 1)/2$ in the unit disk $D$.

For $m = 3$ the set $A$ consists of four points $a_1 = 1, a_2 = i, a_3 = -1$ and $a_4 = -i$ while $F$ consists of five points $[a_1|a_2], [a_2|a_3], [a_3|a_4], [a_4|a_1], [a_1|a_3] = [a_2|a_4]$ as shown at the following picture:
We claim that for any open neighborhood $W$ of $F$ in $C$ one of the pieces of a partition $H^2 = B_1 \cup \cdots \cup B_m$ contains an unbounded subset symmetric with respect to some point $c \in W$. To derive a contradiction we assume the converse: for every point $c \in W$ and every $i \leq m$ the set $B_i \cap f_c(B_i)$ is bounded in $H^2$.

For every $n \in \mathbb{N}$ consider the set

$$C_n = \{ c \in W : \bigcup_{i=1}^m B_i \cap f_c(B_i) \subset \mathbb{D}_n \}.$$

We claim that $C_n$ is a coanalytic subset of $W$, that is, the complement $W \setminus C_n$ is analytic, which in its turn means that $W \setminus C_n$ is the continuous image of a Polish space. Observe that

$$W \setminus C_n = \{ c \in W : \text{there are } i \leq m \text{ and } x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n \} = \text{pr}_2(E),$$

where $\text{pr}_2 : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$ is the projection on the second factor and

$$E = \bigcup_{i=1}^m \{ (x, c) \in \mathbb{D} \times W : x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n \}$$

is a Borel subset of $\mathbb{D} \times W$. Being a Borel subset of the Polish space $\mathbb{D} \times W$, the space $E$ is analytic and so is its continuous image $\text{pr}_2(E) = W \setminus C_n$. Then $C_n$ is coanalytic and hence has the Baire property [Ke 21.6], which means that $C_n$ coincides with an open subset $U_n$ of $W$ modulo some meager set. The latter means that the symmetric difference $U_n \Delta C_n$ is meager (i.e., is of the first Baire category) in $W$. Replacing $U_n$ by the interior of the closure $\overline{U}_n$ of $U_n$ in $W$, if necessary, we may additionally assume that $U_n$ is regular open, that is, $U_n$ coincides with the interior of its closure in $W$. 


We claim that $C_n \subset C_{n+1}$ implies $U_n \subset U_{n+1}$. First we check that

$$U_n \setminus U_{n+1} \subset (U_n \triangle C_n) \cup (U_{n+1} \triangle C_{n+1})$$

is meager. Indeed, for every $x \in U_n \setminus U_{n+1}$ we get $x \in U_n \setminus C_n$ if $x \notin C_n$ and $x \in C_{n+1} \setminus U_{n+1} \triangle C_{n+1}$ if $x \in C_n \subset C_{n+1}$. Therefore, the set $U_n \setminus U_{n+1}$ is meager, which implies $U_n \subset U_{n+1}$ and hence $U_n \subset U_{n+1}$ because the set $U_{n+1}$ is regular open.

Let $U = \bigcup_{n=1}^{\infty} U_n$ and $M = \bigcup_{n=1}^{\infty} U_n \triangle C_n$. Taking into account that $W = \bigcup_{n=1}^{\infty} C_n$, we conclude that

$$W \setminus U = \bigcup_{n=1}^{\infty} C_n \setminus \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} C_n \setminus U_n \subset \bigcup_{n=1}^{\infty} C_n \Delta U_n = M$$

which implies that the open set $U$ has meager complement and thus is dense in $W$.

We claim that $F \subset h^{-1}(U)$ for some isometry $h$ of the hyperbolic plane $H^2 = \mathbb{D}$.

For this consider the natural action

$$\mu : \text{Iso}(H^2) \times \mathbb{D} \to \mathbb{D}, \quad \mu : (h, x) \mapsto h(x)$$

of the isometry group $\text{Iso}(H^2)$ of the hyperbolic plane $H^2 = \mathbb{D}$. It is easy to see that for every $x \in \mathbb{D}$ the map $\mu_x : \text{Iso}(H^2) \to \mathbb{D}, \mu_x : h \mapsto h(x)$, is continuous and open (with respect to the compact-open topology on $\text{Iso}(H^2)$). It follows that the set

$$\bigcap_{x \in F} \mu_x^{-1}(W) = \{ h \in \text{Iso}(H^2) : h(F) \subset W \}$$

is an open neighborhood of the neutral element of the group $\text{Iso}(H^2)$.

Taking into account that $U$ is open and dense in $W$, and that for every $x \in F$ the map $\mu_x : \text{Iso}(H^2) \to \mathbb{D}$ is open, we conclude that the preimage $\mu_x^{-1}(U)$ is open and dense in $\mu_x^{-1}(W) \subset \text{Iso}(H^2)$. Then the intersection $\bigcap_{x \in F} \mu_x^{-1}(U)$, being an open dense subset of $\bigcap_{x \in F} \mu_x^{-1}(W)$, is not empty and hence contains some isometry $h$ having the desired property: $F \subset h^{-1}(U)$. Since $F$ is finite, there is $n \in \mathbb{N}$ with $F \subset h^{-1}(U_n)$. For a complex number $r \in \mathbb{D}$ consider the set $rA = \{ rz : z \in A \} \subset \mathbb{D}$ and let

$$F_r = \{ [x|y] : x, y \in rA, x \neq y \} \subset \mathbb{D},$$

where $[x|y]$ stands for the midpoint of the hyperbolic segment connecting $x$ and $y$ in $H^2$. It can be shown that for any distinct points $x, y \in A$ the midpoint $[x|y]$ tends to the midpoint $[x|y] \in F$ as $r$ tends to 1. Such a continuity yields a neighborhood $O_{1}$ of 1 such that $F_r \subset h^{-1}(U_n)$ for all $r \in O_{1} \cap \mathbb{D}$.

It is clear that for any points $x, y \in A$ the map

$$f_{x,y} : \mathbb{D} \to \mathbb{D}, \quad f_{x,y} : r \mapsto [rx|ry]$$

is open and continuous. Consequently, the preimage $f_{x,y}^{-1}(h^{-1}(M))$ is a meager subset of $\mathbb{D}$ and so is the union $M' = \bigcup_{x,y \in A} f_{x,y}^{-1}(h^{-1}(M))$. So, we can find a non-zero point $r \in O_{1} \setminus M'$ so close to 1 that the set $rA$ is disjoint with the hyperbolic disk $h^{-1}(\mathbb{D}_n)$ (observe that for a complex number $r$ close to 1 the set $rA$ is close to the set $A$ lying in the boundary circle of $\mathbb{D}$ and thus $rA$ can be made disjoint with the compact subset $h^{-1}(\mathbb{D}_n)$ of $\mathbb{D}$). For this point $r$ we shall get $F_r \cap h^{-1}(M) = \emptyset$. 
The set $rA$ consists of $m + 1$ points. Consequently, some cell $h^{-1}(B_i)$ of the partition $\mathbb{D} = h^{-1}(B_1) \cup \cdots \cup h^{-1}(B_m)$ contains two distinct points $rx, ry$ of $rA$. Those points are symmetric with respect to the point $\{rx|ry\} \in F_r \subset h^{-1}(U_n) \setminus h^{-1}(M)$.

Then the images $a = h(rx)$ and $b = h(ry)$ belong to $B_i$ and are symmetric with respect to the point $c = h([rx|ry]) \in U_n \setminus M \subset C_n$. It follows from the definition of $C_n$ that $\{a, b\} \subset B_i \cap f_c(B_i) \subset \mathbb{D}_n$, which is not the case because $rx, ry \notin h^{-1}(\mathbb{D}_n)$.

3 Concerning partitions of $H^2$

We do not know if Theorem 1.1 is true for any finite (not necessarily Borel) partition of the Lobachevsky plane $H^2$. For partitions of $H^2$ into two pieces, the Borel assumption is superfluous.

Theorem 3.1 There is a subset $T \subset H^2$ of cardinality $|T| = 3$ such that for any partition $H^2 = A_1 \cup A_2$ of $H^2$ into two pieces either $A_1$ or $A_2$ contains an unbounded subset, symmetric with respect to some point $c \in T$.

Proof: Lemma 3.2 below allows us to find an equilateral triangle $\triangle c_0c_1c_2$ on the Lobachevsky plane $H^2$ such that the composition $f_{c_2} \circ f_{c_1} \circ f_{c_0}$ of the symmetries with respect to the points $c_0, c_1, c_2$ coincides with the rotation by the angle $2\pi/3$ about some point $o \in H^2$. Consequently $(f_{c_2} \circ f_{c_1} \circ f_{c_0})^3$ is the identity isometry of $H^2$.

We claim that for any partition $H^2 = A_1 \cup A_2$ of the Lobachevsky plane into two pieces one of the pieces contains an unbounded subset symmetric with respect to some point in the triangle $T = \{c_0, c_1, c_2\}$. Assuming the converse, we conclude that the set

$$B = \bigcup_{c \in T} \bigcup_{i=1}^2 A_i \cap f_c(A_i)$$

is bounded. It follows that two points $x, y \in H^2 \setminus B$, symmetric with respect to a center $c \in T$ cannot belong to the same cell $A_i$ of the partition.

Let $B_0 = B$ and $B_{i+1} = B_i \cup \bigcup_{j=0}^2 f_{c_j}^{-1}(B_i)$ for $i \geq 0$. By induction it can be shown that each set $B_i$, $i \geq 0$, is bounded in $H^2$.

Fix any point $x_0 \in H^2 \setminus B_0$ and consider the sequence of points $x_1, \ldots, x_9$ defined by the recursive formula: $x_{i+1} = f_{c_{i \mod 3}}(x_i)$. It follows that

$$x_9 = (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3(x_0) = x_0.$$

We claim that for every $i \leq 9$ the point $x_i$ does not belong to the set $B$. Assuming by contradiction that $x_i \in B$, we would conclude that $x_{i-1} \in \bigcup_{j=0}^2 f_{c_j}^{-1}(B) \subset B_1$. Continuing by induction, for every $k \leq i$ we would get $x_{i-k} \in B_k$. In particular, $x_0 \in B_1 \subset B_9$, which contradicts the choice of $x_0$.

The point $x_0$ belongs either to $A_1$ or to $A_2$. We loose no generality assuming that $x_0 \in A_2$. Since the points $x_0, x_1 \notin B$ are symmetric with respect to $c_0$ and $x_0 \in A_2$, we get that $x_1 \in H^2 \setminus A_2 = A_1$. By the same reason $x_1, x_2$ cannot simultaneously belong to $A_1$ and hence $x_2 \in A_2$. Continuing in this fashion we conclude that $x_i$ belongs to $A_1$ for odd $i$ and to $A_2$ for even $i$. In particular, $x_9 \in A_1$, which is not possible because $x_9 = x_0 \in A_2$. \qed
Lemma 3.2 There is an equilateral triangle \( \triangle ABC \) on the Lobachevsky plane such that the composition \( f_C \circ f_B \circ f_A \) of the symmetries with respect to the points \( A, B, C \) coincides with the rotation by the angle \( 2\pi/3 \) about some point \( O \).

Proof: For a positive real number \( t \) consider an equilateral triangle \( \triangle ABC \) with side \( t \) on the Lobachevsky plane. Let \( M \) be the midpoint of the side \( AB \) and \( l \) be the line through \( C \) that is orthogonal to the line \( CM \). Consider also the line \( p \) that is orthogonal to the line \( AB \) and passes through the point \( P \) such that \( A \) is the midpoint between \( P \) and \( M \). Observe that \( |PM| = |AB| = t \) and for sufficiently small \( t \) the lines \( p \) and \( l \) intersect at some point \( O \).

It is easy to see that the composition \( f_B \circ f_A \) is the shift along the line \( AB \) by the distance \( 2t \) and hence the image \( f_B \circ f_A(O) \) of the point \( O \) is the point symmetric to \( O \) with respect to the point \( C \). Consequently, \( f_C \circ f_B \circ f_A(O) = O \), which means that the isometry \( f_C \circ f_B \circ f_A \) is a rotation of the Lobachevsky plane about the point \( O \) by some angle \( \varphi_t \).

To estimate this angle, consider the point \( X \) such that \( P \) is the midpoint between \( X \) and \( M \). Then \( |XM| = 2t \) and consequently, \( f_B \circ f_A(X) = M \) while \( X' = f_C \circ f_B \circ f_A = f_C(M) \) is the point on the line \( CM \) such that \( C \) is the midpoint between \( X' \) and \( M \). It follows that \( |X'X| \leq |XM| + |MX'| < 2t + 2t = 4t \).

Observe that for small \( t \) the point \( X' \) is near to the point, symmetric to \( X \) with respect to \( O \), which means that the angle \( \varphi_t = \angle XOX' \) is close to \( \pi \) for \( t \) close to zero. On the other hand, for very large \( t \) the lines \( p \) and \( l \) on the Lobachevsky plane do not intersect. So we can consider the smallest upper bound \( t_0 \) of numbers \( t \) for which the lines \( l \) and \( p \) meet. For values \( t < t_0 \) near to \( t_0 \) the point \( O \) tends to infinity as \( t \) tends to \( t_0 \). Since the length of the side \( XX' \) of the triangle \( \triangle XOX' \) is bounded by \( 4t_0 \) the angle \( \varphi_t = \angle XOX' \) tends to zero as \( O \) tends to infinity. Since the angle \( \varphi_t \) depends continuously on \( t \) and decreases from \( \pi \) to zero as \( t \) increases from zero to \( t_0 \), there is a value \( t \) such that \( \varphi_t = 2\pi/3 \). For such \( t \) the composition \( f_C \circ f_B \circ f_A \) is the rotation around \( O \) on the angle \( 2\pi/3 \). \( \square \)
4 Some comments and open problems

In contrast with Theorem [1.1], Theorem [3.1] is true for the Euclidean plane $E^2$ even in a stronger form: for any subset $C \subset E^2$ not lying on a line and any partition $E^2 = A_1 \cup A_2$ one of the cells of the partition contains an unbounded subset symmetric with respect to some center $c \in C$, see [B3].

Having in mind this result let us call a subset $C$ of a Lobachevsky or Euclidean space $X$ central for (Borel) $k$-partitions if for any partition $X = A_1 \cup \cdots \cup A_k$ of $X$ into $k$ (Borel) pieces one of the pieces contains an unbounded monochromatic subset $S \subset X$ symmetric with respect to some point $c \in C$. By $c_k(X)$ (resp. $c^B_k(X)$) we shall denote the smallest size of a subset $C \subset X$, central for (Borel) $k$-partitions of $X$. If no such set $C$ exists, then we put $c_k(X) = \infty$ (resp. $c^B_k(X) = \infty$) where $\infty$ is assumed to be greater than any cardinal number. It follows from the definition that $c_k^B(X) \leq c_k(X)$.

We have a lot of information about the numbers $c_k^B(E^n)$ and $c_k(E^n)$ for Euclidean spaces $E^n$, see [E2]. In particular, we know that

1. $c_2(E^n) = c_2^B(E^n) = 3$ for all $n \geq 2$;
2. $c_3(E^3) = c_3^B(E^3) = 6$;
3. $12 \leq c_4^B(E^4) \leq c_4(E^4) \leq 14$;
4. $n(n + 1)/2 \leq c_n^B(E^n) \leq c_n(E^n) \leq 2^n - 2$ for every $n \geq 3$.

Much less is known about the numbers $c_k^B(H^n)$ and $c_k(H^n)$ in the hyperbolic case. Theorem [3.1] yields the upper bound $c_2(H^n) \leq 3$. In fact, 3 is the exact value of $c_2(H^n)$ for all $n \geq 2$.

**Proposition 4.1** $c_2^B(H^n) = c_2(H^n) = 3$ for all $n \geq 2$.

**Proof:** The upper bound $c_2(H^n) \leq c_2(H^3) \leq 3$ follows from Theorem [3.1]. The lower bound $3 \leq c^B_2(H^n)$ will follow as soon as for any two points $c_1, c_2 \in H^n$ we construct a partition $H^n = A_1 \cup A_2$ in two Borel pieces containing no unbounded set, symmetric with respect to a point $c_i$. To construct such a partition, consider the line $l$ containing the points $c_1, c_2$ and decompose $l$ into two half-lines $l = l_1 \cup l_2$. Next, let $H$ be an $(n - 1)$-hyperplane in $H^n$, orthogonal to the line $l$. Let $S$ be the unit sphere in $H$ centered at the intersection point of $l$ and $H$. Let $S = B_1 \cup B_2$ be a partition of $S$ into two Borel pieces such that no antipodal points of $S$ lie in the same cell of the partition. For each point $x \in H^n \setminus l$ consider the hyperbolic plane $P_x$ containing the points $x, c_1, c_2$. The complement $P_x \setminus l$ decomposes into two half-planes $P^+_x \cup P^-_x$ where $P^+_x$ is the half-plane containing the point $x$. The plane $P_x$ intersects the hyperplane $H$ by a hyperbolic line containing two points of the sphere $S$. Finally put

$$A_i = l_i \cup \{ x \in H^n \setminus l : P^+_x \cap B_i \neq \emptyset \}$$

for $i \in \{1, 2\}$. It is easy to check that $A_1 \cup A_2 = H^n$ is the desired partition of the hyperbolic space into two Borel pieces none of which contains an unbounded subset symmetric with respect to one of the points $c_1, c_2$. \qed

The preceding proposition implies that the cardinal numbers $c_2(H^n)$ are finite.

**Problem 4.2** For which numbers $k, n$ are the cardinal numbers $c_k(H^n)$ and $c^B_k(H^n)$ finite? Is it true for all $k \leq n$?
Except for the equality \( c_2(E^n) = 3 \), we have no information on the numbers \( c_k(E^n) \) with \( k < n \).

**Problem 4.3** Calculate (or at least evaluate) the numbers \( c_k(E^n) \) and \( c_k(H^n) \) for \( 2 < k < n \).

In all the cases where we know the exact values of the numbers \( c_k(E^n) \) and \( c_k^B(E^n) \) we see that those numbers are equal.

**Problem 4.4** Are the numbers \( c_k(E^n) \) and \( c_k^B(E^n) \) (resp. \( c_k(H^n) \) and \( c_k^B(H^n) \)) equal for all \( k, n \)?

Having in mind that each subset not lying on a line is central for 2-partitions of the Euclidean plane, we may ask about the same property of the Lobachevsky plane.

**Problem 4.5** Is any subset \( C \subset H^2 \) not lying on a line central for (Borel) 2-partitions of the Lobachevsky plane \( H^2 \)?

Finally, let us ask about the numbers \( c_k^B(H^2) \) and \( c_k(H^2) \). Observe that Theorem 1.1 guarantees that \( c_k^B(H^2) \leq c \) for all \( k \in \mathbb{N} \). Inspecting the proof we can see that this upper bound can be improved to \( c_k^B(H^2) \leq \text{non}(M) \) where \( \text{non}(M) \) is the smallest cardinality of a non-meager subset of the real line. It is clear that \( \aleph_1 \leq \text{non}(M) \leq c \). The exact location of the cardinal \( \text{non}(M) \) on the interval \([\aleph_1, c]\) depends on axioms of Set Theory, see [Bl]. In particular, the inequality \( \aleph_1 = \text{non}(M) < c \) is consistent with ZFC.

**Problem 4.6** Is the inequality \( c_k^B(H^2) \leq \aleph_1 \) provable in ZFC? Are the cardinals \( c_k^B(H^2) \) countable? finite?

The last problem asks if \( H^2 \) contains a countable (or finite) central set for Borel \( k \)-partitions of the Lobachevsky plane. Inspecting the proof of Theorem 1.1 we can see that it gives an “approximate” answer to this problem:

**Proposition 4.7** For any \( k \in \mathbb{N} \) there is a finite subset \( C \subset H^2 \) of cardinality \( |C| \leq k(k+1)/2 \) such that for any partition \( H^2 = B_1 \cup \cdots \cup B_k \) of \( H^2 \) into \( k \) Borel pieces and for any open neighborhood \( O(C) \subset H^2 \) of \( C \) one of the pieces \( B_i \) contains an unbounded subset \( S \subset B_i \) symmetric with respect to some point \( c \in O(C) \).

**Remark 4.8** For further results and open problems related to symmetry and colorings see the surveys [BP2], [BVV] and the list of problems [BBGRZ §4].

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