LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF THE FRACTIONAL BROWNIAN MOTION

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We investigate first and second order fluctuations of additive functionals of a fractional Brownian motion (fBm) of the form

\[
\left\{ \int_0^t f(n^H(B_s - \lambda)) ds ; t \geq 0 \right\}
\]

(0.1)

where \(B = \{B_t; t \geq 0\}\) is a fBm with Hurst parameter \(H \in (0, 1)\), \(f\) is a suitable test function and \(\lambda \in \mathbb{R}\). We develop our study by distinguishing two regimes which exhibit different behaviors. When \(H \in (0, 1/3)\), we show that a suitable renormalization of (0.1), compensated by a multiple of the local time of \(B\), converges towards a constant multiple of the derivative of the local time of \(B\). In contrast, in the case \(H \in [1/3, 1)\) we show that (0.1) converges towards an independent Brownian motion subordinated to the local time of \(B\). Our results refine and complement those from [6], [8], [4], [16] and solve at the same time the critical case \(H = 1/3\), which had remained open until now.

1. Introduction.

1.1. Overview. Let \(B = \{B_t; t \geq 0\}\) be a fractional Brownian motion with Hurst parameter \(H \in (0, 1)\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The purpose of this manuscript is to study the asymptotic behavior, as \(n\) tends to infinity, of the law of the sequence of processes

\[
\left\{ \int_0^t f(n^H(B_s - \lambda)) ds ; t \geq 0 \right\} \overset{\text{law}}{=} \left\{ \frac{1}{n} \int_0^{nt} f(B_s - n^H \lambda) ds ; t \geq 0 \right\},
\]

(1.1)

where \(\lambda \in \mathbb{R}\) is fixed. More specifically, we will describe in full generality the behavior of the first and second order fluctuations of (1.1). An elementary heuristic understanding of the first order asymptotics of (1.1) can be obtained by making use of the local time of \(B\), formally defined by

\[
L_t(\lambda) := \int_0^t \delta_0(B_s - \lambda) ds,
\]

(1.2)

for \(t > 0\) and \(\lambda \in \mathbb{R}\). The variable \(L_t(\lambda)\), which represents the time spent by \(B\) at level \(\lambda\) up to time \(t\), can be rigorously defined by replacing the Dirac delta \(\delta_0\) by a heat kernel of variance

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\( \varepsilon > 0 \) and then taking limit in \( L^2(\Omega) \) as \( \varepsilon \) goes to zero (see Section 2.2 for more details). By standard algebraic simplifications, we observe that

\[
\int_0^t f(n^H(B_s - \lambda)) ds = \int_0^t \int_\mathbb{R} \delta_0(B_s - y)f(n^H(y - \lambda))dy ds = n^{-H} \int_0^t \int_\mathbb{R} \delta_0(B_s - n^{-H}y - \lambda)f(y)dy ds.
\]

Applying a mollification procedure to the relations above, the term \( n^{-H}y \) vanishes after taking the limit in \( L^2(\Omega) \) as \( n \) tends to infinity, leading to

\[
(1.3) \quad n^H \int_0^t f(n^H(B_s - \lambda)) ds \xrightarrow{L^2(\Omega)} \int_0^t \int_\mathbb{R} \delta_0(B_s - \lambda)f(y)dy ds = L_t(\lambda) \int_\mathbb{R} f(y)dy.
\]

Such convergence in \( L^2(\Omega) \) motivates the study of the associated second order fluctuations. In particular, it is a natural problem to determine whether or not we can find a monotone normalization \( \{\beta_n\}_{n \in \mathbb{N}} \) such that the difference between the left and right-hand sides of (1.3), scaled by \( \beta_n \), converges towards a non-trivial limit. Following [19], we refer to the situation where \( \int_\mathbb{R} f(y)dy = 0 \) as the ‘zero energy case’.

Although important advances have been made in this direction, up to this date this question has only been partially answered in three special cases: (i) when \( \lambda = 0 \) and \( B \) is a Brownian motion (see the work by Papanicolaou, Stroock and Varadhan [17]), (ii) when \( \lambda = 0 \), \( \int_\mathbb{R} f(y)dy = 0 \) and \( B \) is a fractional Brownian motion of Hurst parameter \( H > \frac{1}{3} \) (see the work by Hu, Nualart and Xu [4] and by Nualart and Xu [16], as well as [8] for some earlier findings that can be roughly compared to [4, 16]) and (iii) when \( \lambda = 0 = \int_\mathbb{R} f(y)dy \), \( B \) is a fractional Brownian motion of Hurst parameter \( H < \frac{1}{3} \) and the integral in (1.1) is replaced by a sum (see [6]). In particular, in [4] and the subsequent follow-up paper [16], the following result was proved by means of the method of moments.

**Theorem 1.1.** Suppose that \( H > \frac{1}{3} \) and \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( \int_\mathbb{R} f(y)dy = 0 \). Then,

\[
\left\{ n^{\frac{H+1}{2}} \int_0^t f(n^H B_s) ds \mid t \geq 0 \right\} \overset{\text{Law}}{\rightarrow} \{C_f W_{L_t(0)} \mid t \geq 0\},
\]

where \( C_f > 0 \) is a non-zero constant depending on \( f \) and \( W \) is a standard Brownian motion independent of \( B \). The convergence takes place in the topology of uniform convergence over compact sets.

Although the above mentioned results ([4, 6, 8, 16, 17]) seem to address problems which are "similar in spirit", the nature of the process \( X \) as \( H \) ranges over different sub-regions of \( (0, 1) \) demands the application of techniques of different kind. For the Brownian motion case \( H = \frac{1}{2} \), an equality in law of the underlying sequence of additive functionals and an appropriately time changed Brownian motion holds, which, up to the verification of some additional technical conditions, reduces the problem to the study of the associated quadratic variation. The case \( H \neq 1/2 \) exhibits a substantial increase in the complexity of the problem, due to the lack of the martingale or Markov properties that prevent the application of stochastic calculus techniques. The implementation of Malliavin calculus techniques to the problem under consideration has lead in many cases to satisfactory results when the limiting distribution is Gaussian. However, up to our knowledge, with the exception of part of the work presented in [6], the asymptotic distributional properties of the process of interest (1.1) has never been studied with a Malliavin calculus perspective.
In view of Theorem 1.1, one could conjecture that in the general non-zero energy case, the process (1.1), standardized by the local time $L_t(\lambda)$, could exhibit an asymptotic mixed Gaussian as well. Indeed, this will be shown to be true, as illustrated in our Theorem 1.3. More precisely, we will prove that if we allow $f$ to be replaced by a vector-valued function $(f_1, \ldots, f_d): \mathbb{R} \to \mathbb{R}^d$, with $d \in \mathbb{N}$, then the compensation of the vector-valued process

$$\left\{ n^{\frac{H+1}{2}} \int_0^t \left( f_1(n^H B_s), \ldots, f_d(n^H B_s) \right) ds ; \ t \geq 0 \right\}$$

converges towards an explicit linear transformation of a $\mathbb{R}^d$-dimensional Brownian motion $W = (W_1, \ldots, W_d)$ independent of $B$, subordinated to the local time of $B$. Although it is a phenomenon already observed in [8, Theorem 1] for a vector of functionals slightly different from the one we consider here, it is worth noting that such a limit may seem surprising at first glance, because we could have rather expected that no additional sources of randomness should have emerged when we consider vector-valued test functions instead of real-valued ones. Our approach relies on Fourier analysis and martingale techniques, which are adapted to the problem under consideration by means of Malliavin calculus and the Clark-Ocone formula.

The critical case $H = \frac{1}{3}$ is covered by our analysis as well, and is part of the main contributions of our paper. For this regime, in comparison to the case $H > \frac{1}{3}$, we show asymptotic mixed normality after normalizing by an additional logarithmic factor.

It is worth mentioning that the method of moments used in the papers [4] and [16] is replaced in the proof of Theorem 1.3 by a completely different methodology relying on the Clark-Ocone formula from Malliavin calculus. A crucial trick consists in embedding the stochastic integral representation given by the Clark-Ocone formula into a representation where the time parameter in the kernel is considered fixed, so the resulting process is a martingale to which we can apply Knight’s theorem. These techniques play a fundamental role in extending the results of [4, 16] to the “non-zero energy” case $\int_{\mathbb{R}} f(x) dx \neq 0$ and to the critical case $H = \frac{1}{3}$.

Concerning the case $H < \frac{1}{3}$, we will show that the standardizations of (1.1) converge in $L^2(\Omega)$, as $n$ tends to infinity, towards a constant times the spatial derivative of the local time of $B$, formally defined by

$$(1.4) \quad L'_t(\lambda) := \int_0^t \delta'_0(B_s - \lambda) ds.$$ 

As for (1.2), a rigorous definition of (1.4) can be obtained by replacing $\delta'_0(B_s - \lambda)$ by a suitable approximating mollifier (see Section 2.2). The existence of $L'_t(\lambda)$ can be guaranteed only in the case $H < \frac{1}{2}$ and its trajectories in time are Hölder continuous of order $\beta$ for every $\beta < 1 - 2H$, as discussed in [6]. For this problem, the approach we follow consists on directly computing the $L^2(\Omega)$-norm of the associated error by means of a suitable Fourier representation of the local time and its derivative.

1.2. Main results. In this section we present in detail our main results, which are stated in full generality in Theorem 1.3 and Theorem 1.4 below, and require a non-negligible amount of notation. In what follows, $B = \{B_t ; t \geq 0\}$ denotes a fractional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Namely, $B$ is a centered Gaussian process with covariance function $\mathbb{E}[B_s B_t] = R(s, t)$, where

$$(1.5) \quad R(s, t) := \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right).$$

Set
Let \( \beta_{H,1}, \beta_{H,2} \) be the constants
\[
\beta_{H,1} := \begin{cases} 
C_H (H - 1/2)^{-1} & \text{if } H > 1/2 \\
C_H & \text{if } H \leq 1/2
\end{cases}
\quad \text{and} \quad \beta_{H,2} := (2H)^{-1} \beta_{H,1}.
\]
Define the function \( \beta_{H,3} : \mathbb{R}_+^2 \to \mathbb{R}_+ \) by
\[
\beta_{H,3}(s_1, s_2) := C_H |H|^{1/2} - 2 \int_0^\infty ((\theta + s_1)^{H - 1/2} - (\theta + s_2)^{H - 1/2})^2 d\theta.
\]
In the sequel, for every measurable function \( f : \mathbb{R} \to \mathbb{R} \) and any \( w > 0 \) we set
\[
\|f\|_w = \int_\mathbb{R} |f(x)| (1 + |x|^w) dx
\]
and we denote by \( \Xi_w \) the space of functions \( f \) such that \( \|f\|_w < \infty \).

The following lemma, whose proof is postponed at the beginning of Section 5, states that the quantity \( A_H[f, g] \) appearing in one of our main results (the forthcoming Theorem 1.3) is indeed well-defined when \( H > 1/3 \) and \( f, g \in \Xi_1 \).

**LEMMA 1.2.** If \( H > 1/3 \) and \( f, g \in \Xi_1 \), the following integral is absolutely convergent
\[
A_H[f, g] := \frac{\beta_{H,1}^2}{\pi} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} \eta^2 B_\eta[f, g](s_1 s_2)^{H - 1/2} e^{-\frac{1}{2} \left( \beta_{H,3} (s_1^2 + s_2^2) + \beta_{H,3} (s_1, s_2) \right)} n^2 d\eta d\vec{s},
\]
where \( \vec{s} = (s_1, s_2) \), and where \( B_\eta[f, g] \) is defined as
\[
B_\eta[f, g] := \int_{\mathbb{R}_+^2} f(x) g(\vec{x}) (e^{in\vec{x}} - 1) (1 - e^{-in\vec{s}}) d\eta d\vec{x},
\]
with \( \vec{x} = (x, \vec{x}) \).

In the case \( H = 1/3 \), for \( f, g \in \Xi_1 \) we will use the notation
\[
A_{1/3}[f, g] := \frac{6 \beta_{1/3,1}^2}{\pi} \int_{\mathbb{R}_+^2} x f(x) dx \left( \int_{\mathbb{R}_+^2} x g(x) dx \right) \int_0^1 s^{-1/6} (\beta_{1/3,2}(1 + s^{2/3}) + \beta_{1/3,3}(1, s))^{-2} ds,
\]
which is also well-defined as a product of three absolutely convergent integrals.

Finally, we will make use of the normalizing constants
\[
\ell_{n,H} := \mathbb{1}_{\{H > 1/3\}} + (\log n)^{-1/2} \mathbb{1}_{\{H = 1/3\}}, \quad \text{for } H \in \left[\frac{1}{3}, 1\right).
\]
We are now in a position to present our main results. We start with the case \( H \geq 1/3 \).
Theorem 1.3. Fix $H \geq \frac{1}{3}$, and recall the definition (1.12) of $\ell_{n,H}$ and the definition (1.10) of $A_H[g_1,g_2]$, for $g_1, g_2 \in \Xi$. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}^d$ of the form $f = (f_1, \ldots, f_d)$ with $f_i \in \Xi_1$ if $H > \frac{1}{4}$ and $f_i \in \Xi_2$ if $H = \frac{1}{4}$. Define the matrix $C_H[f] = \{c^{(i,j)}_{H}[f] ; 1 \leq i, j \leq d\}$ given by $C_{H}^{(i,j)}[f] = A_H[f_i, f_j]$. Then, as $n$ tends to infinity,

$$
\left\{ \frac{n^{\frac{H+1}{2}}\ell_{n,H}}{n^{\frac{H}{2}}} \left( \int_0^t f(n^H(B_s - \lambda))ds - n^{-H}L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) ; t \geq 0 \right\}
$$

(1.13)

$$
\mathcal{F}_{d,d} \rightarrow \{ \sqrt{C_{H}[f]} \tilde{W}_{L_t(\lambda)} ; t \geq 0 \},
$$

where $\tilde{W} = \{ \tilde{W}_t ; t \geq 0 \}$ is a $d$-dimensional Brownian motion independent of $B$, $\sqrt{C_{H}[f]}$ denotes the square root of $C_{H}[f]$ and $f.d.d.$ means the convergence of the finite-dimensional distributions.

Remark 1.1. It would be desirable to enhance the convergence of the finite-dimensional distributions in (1.13) to the convergence law in the space $C([0,\infty))$. However, although our proof of the convergence of the finite-dimensional distributions is based on Knight’s theorem, and this theorem provides a criterion for convergence law in $C([0,\infty))$, we apply Knight’s theorem to indefinite stochastic integrals where the time is frozen in the integrand and have a completely different time evolution in comparison with the original process. Thus, our method of proof does not lead directly to the convergence in law of the process. A typical approach to establish a functional version of Theorem 1.3 is to prove the tightness property and the key argument for this consists in deriving suitable moment estimates for the increments of the underlying process. This was possible in the zero-energy case considered in [4] and [16] because the proof of the convergence in law in these papers is based precisely on an analysis of moments of arbitrary order. For the process considered in (1.13), moment estimates seem a very hard and challenging task that we do not pursue in this paper.

The $H < \frac{1}{3}$ case is covered by the following result.

Theorem 1.4. Suppose that $H < \frac{1}{3}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function satisfying $\int_{\mathbb{R}} |f(y)|(1 + |y|^{1+\nu})dy$ for some $\nu > 0$. Then, for every $t > 0$ and $\lambda \in \mathbb{R}$, we have that

$$
\int_{\mathbb{R}} \left( n^H \int_0^t f(n^H(B_s - \lambda))ds - n^{-H}L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) ^{L^2(\Omega)} \rightarrow L_0^t(\lambda) \int_{\mathbb{R}} yf(y)dy,
$$

where $L_0^t(\lambda)$ denotes the derivative of the local time of $B$ up to time $t$ at the level $\lambda \in \mathbb{R}$, see Section 2.2.

Remark 1.2. Observe that Theorem 1.4 is stated in terms of a scalar test function $f : \mathbb{R} \rightarrow \mathbb{R}$ only, which may contrast with Theorem 1.3 that considers a vector-valued function $f : \mathbb{R} \rightarrow \mathbb{R}^d$. Actually, it is only for the sake of simplicity, as a vector-valued version of Theorem 1.4 can be immediately obtained by exploiting the fact that the convergence (1.14) holds in $L^2(\Omega)$.

Remark 1.3. Notice also that an external Gaussian source of noise does not appear in Theorem 1.4. Indeed, using the occupation measure formula and the change of variables $n^H(x - \lambda) = y$, we can write

$$
n^H \left( n^H \int_0^t f(n^H(B_s - \lambda))ds - n^{-H}L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right)
= \int_{\mathbb{R}} n^H \left( L_t(n^H y + \lambda) - L_t(y) \right) f(y)dy.
$$
Then, if $H < \frac{1}{3}$, the local time $L_t(\lambda)$ is differentiable and letting $n$ tend to infinity we obtain an heuristic proof of Theorem 1.4. The last step of the proof will be formalized in Section 4 through the Fourier inversion formula and a suitable Fourier representation of the local time of $B$. The fact that the arguments have a path-wise flavor and the convergence holds in $L^2(\Omega)$, are the main reasons for the non-existence of an exogenous Gaussian noise.

**Remark 1.4.** The phase transition at the critical value $H = 1/3$ exhibits the following characteristics: (i) as $H$ varies over $(0, 1)$, the exponent of the normalizing constant multiplying the additive functional changes continuously and non-smoothly from $2H$ to $(H + 1)/2$ around $H = 1/3$, (ii) when $H < 1/3$, the convergence holds in $L^2(\Omega)$, whereas for $H > 1/3$, the convergence holds in distribution (or more generally, in the stable sense) and the limit involves a mixed Gaussian limit induced by an independent noise $W$. This behavior compares closely to the transition phase of the $q$-Hermite weighted variations of the fractional Brownian motion as $H$ crosses the critical point $1 - 1/(2q)$, which is summarized in [13, Section 5.1] and can be regarded as a particular case of the Breuer-Major theorem. A similar phase transition also appears in other limit theorems related to the fractional Brownian motion, such as the asymptotic behavior of the self-intersection local time of the fractional Brownian motion (see [7, Theorem 1.4]). The analogy with these cases extends further to the critical value $H = 1/3$, as the normalization involves as well a logarithmic term and the type of limit is mixed Gaussian. However, it is worth noticing that neither of the results mentioned above have been obtained by means of the Clark-Ocone formula and Knight’s theorem, but on rather different Malliavin calculus techniques.

**Remark 1.5.** Although the current manuscript focuses exclusively on the study of the fractional Brownian motion, we conjecture that the methodology presented in this paper can potentially be adapted to the study of other families of Gaussian processes that possess the local non-determinism property. In particular, our approach can be carried due to the fact that the process $B$ under consideration can be written as

$$B_t = \int_0^t K(s,t)W(ds),$$

where $W$ is a standard Brownian motion. An example is the Riemann-Liouville process. where $K(t,s) = (t - s)^{H - \frac{1}{2}}$. In this case, the Clark Ocone formula can be determined in an explicit way in terms of $K$. However, as the technical computations presented in Section 5, all of which are essential for proving our main results, are intrinsic to the shape of $K$, they would have to be suitably adapted when changing the underlying kernel $K$.

**Remark 1.6.** We would like to mention that we have exclusively focused on $\mathbb{R}$-valued fractional Brownian motions. However, the above problem can be set-up for the $d$-dimensional fractional Brownian motion as well. In this case, the interaction between $H$ and $d$ plays a fundamental role in the behavior of the law of (1.1). The reader is referred to the work by Kallianpur and Robbins [9], Kasahara and Kotani [10], Kôno [11] and Hu, Nualart and Xu ([4] and [16]) for a detailed description of the state of the art of this problem.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results on local times and Malliavin calculus. In Sections 3 and 4 we present the proofs of our main results, Theorems 1.3 and 1.4. Finally, in Section 5 we prove some technical identities that are used in the proof of the main theorems.

2. Preliminaries.
2.1. Malliavin calculus for Gaussian processes. In this section we provide some notation and introduce the basic elements of Malliavin calculus. The reader is referred to [15] and [14] for a comprehensive presentation of this topic. Throughout the paper, \( W = \{ W_t ; t \geq 0 \} \) will denote a standard \( \mathbb{R} \)-valued Brownian motion defined on a probability space \( (\Omega, F, \mathbb{P}) \).

Let \( K_H : \mathbb{R}_+^2 \to \mathbb{R} \) be the kernel given by

\[
K_H(t, s) := C_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du,
\]

if \( H > \frac{1}{2} \), and by

\[
K_H(t, s) := C_H \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^H - \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}} (u - s)^{H - \frac{1}{2}} du \right),
\]

if \( H < \frac{1}{2} \), with the convention that \( K_H(t, s) = 0 \) if \( t \geq s \), and where \( C_H \) is the constant introduced in (1.6). For \( H = \frac{1}{2} \) we simply set \( K_H(t, s) = 1_{\{s < t\}} \).

Let \( B = \{ B_t ; t \geq 0 \} \) be the unique (up to indistinguishability) continuous modification of the process

\[
B_t := \int_0^t K_H(t, s)dW_s,
\]

where \( H \in (0, 1) \). It is well-known that \( B \) is a fractional Brownian motion of Hurst parameter \( H \), namely, \( B \) is a centered Gaussian process with covariance function given by (1.5). A detailed proof of this fact can be found in [12, Proposition 2.5].

The mapping \( \mathbb{1}_{[0,t]} \mapsto W_t \) can be extended to a linear isometry between \( \mathcal{F} := L^2([0, \infty)) \) and the linear Gaussian subspace of \( L^2(\Omega) \) generated by the process \( W \). We denote this isometry by \( h \mapsto W(h) \). Let \( \mathcal{S} \) denote the set of all cylindrical random variables of the form

\[
F = g(W(h_1), \ldots, W(h_n)),
\]

where \( g : \mathbb{R}^n \to \mathbb{R} \) is an infinitely differentiable function with compact support and \( h_1, \ldots, h_n \) are step functions defined over \([0, \infty)\). In the sequel, we refer to the elements of \( \mathcal{S} \) as “smooth random variables”. The derivative operator of a random variable \( F \in \mathcal{S} \) is the \( \mathcal{S} \)-valued random variable \( DF = \{ D_tF; t \geq 0 \} \), defined by

\[
D_tF := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))h_j(t).
\]

For \( p \geq 1 \), the space \( \mathbb{D}^{1,p} \) denotes the closure of \( \mathcal{S} \) with respect to the norm \( \| \cdot \|_{\mathbb{D}^{1,p}} \), defined by

\[
\| F \|_{\mathbb{D}^{1,p}} := \left( \mathbb{E} \left[ \| F \|^p \right] + \mathbb{E} \left[ \| DF \|^p_\mathcal{S} \right] \right)^{\frac{1}{p}}.
\]

The operator \( D \) can be consistently extended to the space \( \mathbb{D}^{1,p} \). One of the key ingredients for proving Theorem 1.3 is the celebrated Clark-Ocone formula, which establishes that every random variable \( F \in \mathbb{D}^{1,2} \) satisfies the stochastic integral representation

\[
F = \mathbb{E}[F] + \int_{\mathbb{R}_+} \mathbb{E}[D_rF \mid \mathcal{F}_t]dW_r,
\]

where \( \mathcal{F}_t \) denotes the natural \( \sigma \)-algebra generated by \( \{ W_s ; s \leq t \} \).
2.2. Local times. We recall that, for $t > 0$ and $\lambda \in \mathbb{R}$, the local time of $B$ up to time $t$ at the level $\lambda$ and its spatial derivative are formally by (1.2) and (1.4). A rigorous definition of these objects can be obtained by considering the approximating random variables

$$L_{t,\varepsilon}(\lambda) := \int_0^t p_\varepsilon(B_s - \lambda) ds, \quad L_{t,\varepsilon}'(\lambda) := \int_0^t p_\varepsilon'(B_s - \lambda) ds,$$

where $p_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} \exp\{-\frac{1}{2\varepsilon} x^2\}$ denotes the heat kernel of variance $\varepsilon > 0$. Then, by [6, Lemma 11], we have that for all $H \in (0, 1)$, as $\varepsilon$ tends to zero,

$$L_{t,\varepsilon}(\lambda) \xrightarrow{L^2(\Omega)} L_t(\lambda).$$

On the other hand, the family $L_{t,\varepsilon}'(\lambda)$ can be shown to be divergent as $\varepsilon$ tends to zero in $L^2(\Omega)$ when $\lambda = 0$ and $H > \frac{1}{3}$, while in the case $H < \frac{1}{3}$, for any $\lambda \in \mathbb{R},$

$$L_{t,\varepsilon}'(\lambda) \xrightarrow{L^2(\Omega)} L_t'(\lambda).$$

The random variable $L_t(\lambda)$ is an ubiquitous object in the theory of stochastic processes, as it naturally emerges in connection with several fundamental topics, such as the extension of Itô’s formula to convex functions, the absolute continuity of the occupation measure of $B$ with respect to the Lebesgue measure, and the study of limit theorems for additive functionals of $B$ — see [1, 2, 3, 5, 18] for some general references on the subject. On the other hand, the study of the variables $L_t'(\lambda)$ has recently gained momentum for the effectiveness of these random variables as a tool for describing the asymptotics of high-frequency statistics (see the work by Jaramillo, Nourdin and Peccati [6]).

A fundamental identity that will be used throughout the paper is the Fourier inversion formula for $p_\varepsilon(x)$. It states that, for all $\varepsilon > 0$ and $x \in \mathbb{R},$

$$p_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{i}{\varepsilon} \xi^2 - i\xi x} d\xi. \tag{2.6}$$

This representation can be replaced in (1.2) and (1.4) to obtain a Fourier representation for the local time and its spatial derivative. Indeed, in [6, Lemma 1.1], it was proved that the local time and its derivative can be represented as

$$L_t(\lambda) = \int_{-\infty}^{\infty} e^{i \xi (B_r - \lambda)} ds d\xi := \lim_{N \to \infty} \int_{-N}^{N} e^{i \xi (B_r - \lambda)} ds d\xi, \tag{2.7}$$

$$L_t'(\lambda) = \int_{-\infty}^{\infty} i \xi e^{i \xi (B_r - \lambda)} ds d\xi := \lim_{N \to \infty} \int_{-N}^{N} i \xi e^{i \xi (B_r - \lambda)} ds d\xi, \tag{2.8}$$

meaning that, as $N \to \infty$, the right-hand sides of (2.7) and (2.8) (if $H < \frac{1}{3}$) converge in $L^2(\Omega)$ to $L_t(\lambda)$ and $L_t'(\lambda)$, respectively.

Along the paper, for any $0 \leq r \leq s$ we will make use of the following notation:

$$B_{r,s} := \int_0^r K_H(s, \theta) dW_\theta \quad \text{and} \quad \mu_{r,s} := \int_r^s K_H^2(s, \theta) d\theta. \tag{2.9}$$

An important ingredient for proving our results is the following stochastic integral representation for $L_t(\lambda)$, which easily follows from the Clark-Ocone formula (2.5).

**Lemma 2.1.** For all $t \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, we have that

$$L_t(\lambda) = \int_0^t p_{2n}(\lambda) ds + \int_0^t \int_r^t p_{\mu_{r,s}}'(B_{r,s} - \lambda) K_H(s, r) ds dW_r, \tag{2.10}$$

where $B_{r,s}$ and $\mu_{r,s}$ are defined in (2.9).
PROOF. Let $n \in \mathbb{N}$ and define
\[ L_t^{(n)}(\lambda) := \int_0^t p_{\frac{1}{n}}(B_s - \lambda) ds. \]
By (2.5), we can write
\[ L_t^{(n)}(\lambda) = \int_0^t \mathbb{E}[p_{\frac{1}{n}}(B_s - \lambda)] ds + \int_0^t \int_0^t \mathbb{E} [D_r p_{\frac{1}{n}}(B_s - \lambda) | \mathcal{F}_r] dsdW_r \]
\[ = \int_0^t p_{\frac{1}{n} + s^2n}(\lambda) ds + \int_0^t \int_0^t \mathbb{1}_{\{r \leq s\}} K_H(s, r) \mathbb{E} [p_{\frac{1}{n}}(B_r + (B_s - B_{r,s}) - \lambda) | \mathcal{F}_r] dsdW_r. \]
Using the fact that $B_s - B_{r,s}$ is independent of $\mathcal{F}_r$, $B_{r,s}$ is $\mathcal{F}_r$ measurable and $\text{Var}[B_s - B_{r,s}] = \mu_{r,s}$, we thus obtain
\[ L_t^{(n)}(\lambda) = \int_0^t p_{\frac{1}{n} + s^2n}(\lambda) ds + \int_0^t \int_0^t K_H(s, r) p_{\frac{1}{n} + \mu_{r,s}}(B_{r,s} - \lambda) dsdW_r. \]
The result is obtained by taking the limit as $n \to \infty$. \qed

2.3. Local nondeterminism. The following property of the local nondeterminism of the fractional Brownian motion will play a fundamental role in our proofs. For a proof, see e.g. Xiao [20] and the references therein.

**Proposition 2.2.** Let $B = \{B_t; t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Then, there exists a constant $\kappa_H$ such that, for any integer $m \geq 1$, any times $t_m > \cdots > t_2 > t_1 > 0$ and $t > 0$, we have
\[ \text{Var}[B_t | B_{t_1}, \ldots, B_{t_m}] \geq \kappa_H (\min\{|t - t_j|, 1 \leq j \leq m\})^{2H}. \]

3. Proof of Theorem 1.3. In what follows, we fix a time $T > 0$, and we assume that the time variable $t$ belongs to the compact interval $[0, T]$. Taking into account that the local time is the density of the occupation measure, for every $g \in \mathcal{C}_c^\infty$, the process
\[ Z_t^{(n)}[g] := n^{\frac{H}{2}} \ell_{n,H} \left( \int_0^t g(n^H (B_s - \lambda)) ds - n^{-H} L_t(\lambda) \int \mathbb{R} g(x) dx \right), \]
can be rewritten in the form
\[ Z_t^{(n)}[g] = n^{\frac{1-H}{2}} \ell_{n,H} \left( n^H \int \mathbb{R} g(n^H (x - \lambda)) L_t(x) dx - \int \mathbb{R} g(x) L_t(\lambda) dx \right) \]
\[ = n^{\frac{1-H}{2}} \ell_{n,H} \int \mathbb{R} g(x) \left( L_t(n^{-H} x + \lambda) - L_t(\lambda) \right) dx, \]
where the second equality follows from the change of variables $n^H (x - \lambda) \to x$. The first step in our proof consists in using (3.1) to write $Z_t^{(n)}[g]$ as the sum of a suitable martingale and a negligible reminder, which will later allow us to show the convergence (1.13) by means of an asymptotic version of Knight’s theorem. To achieve this, we use the stochastic integral representation (2.10) to write the spatial increment of the local time appearing on the right hand side of (3.1), in the form
\[ L_t(\lambda + \frac{x}{n^H}) - L_t(\lambda) = \int_0^t \left( p_{\frac{1}{n^2H}} \left( \frac{x}{n^H} + \lambda \right) - p_{\frac{1}{n^2H}} (\lambda) \right) ds \]
\[ + \int_0^t \int_0^t (p_{\mu_{r,s}}(B_{r,s} - \frac{x}{n^H} - \lambda) - p_{\mu_{r,s}}(B_{r,s} - \lambda)) K_H(s, r) dsdW_r, \]
where we recall that \( B_{r,s} = \int_0^r K_H(s, \theta) dW_\theta \) and \( \mu_{r,s} = \int_r^s K_H^2(s, \theta) d\theta \). This identity, combined with a stochastic Fubini’s theorem yields

\[
Z_t^{(n)}[g] = n^{\frac{1-H}{2}} \ell_{n,H} \int_0^t G_{r,t}^{(g,n)} dW_r + n^{\frac{1-H}{2}} \ell_{n,H} R_t^{(g,n)},
\]

where

\[
G_{r,t}^{(g,n)} := \int_{\mathbb{R}} \int_{\mathbb{R}} g(x) (p_{(r,s),n}(B_{r,s} - \frac{x}{nH} - \lambda) - p_{(r,s),n}(B_{r,s} - \lambda)) K_H(s,r) dsdx,
\]

and

\[
R_t^{(g,n)} := \int_{\mathbb{R}} \int_0^t g(x) (p_{s,n}(\frac{x}{nH} + \lambda) - p_{s,n}(\lambda)) dsdx.
\]

Next we show that the term \( R_t^{(g,n)} \) satisfies

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \ell_{n,H} n^{\frac{1-H}{2}} |R_t^{(g,n)}| = 0.
\]

To this end, using (2.6) and making the change of variables \( \xi s^H \to \xi \) and \( sn \to s \), we can write

\[
\int_0^t (p_{s,n}(\lambda + n^{-H}x) - p_{s,n}(\lambda)) ds = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{-\frac{1}{2} s^2 \xi^2 - i\lambda \xi} (e^{\frac{i}{nH} - 1}) d\xi ds
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{-\frac{1}{2} s^2 \xi^2 - i\lambda \xi} (e^{\frac{i}{nH} - 1}) s^{-H} d\xi ds
\]

\[
= \frac{1}{2\pi} n^{H-1} \int_0^t e^{-\frac{1}{2} s^2 \xi^2 - i\lambda \xi} (e^{\frac{i}{nH} - 1}) s^{-H} d\xi ds.
\]

As a consequence,

\[
\limsup_{n \to \infty} \sup_{0 \leq t \leq T} \ell_{n,H} n^{\frac{1-H}{2}} |R_t^{(n)}| \leq \frac{1}{2\pi} \limsup_{n \to \infty} \ell_{n,H} n^{\frac{H-1}{2}} \int_0^t \int_{\mathbb{R}} e^{-\frac{1}{2} s^2 \xi^2} |e^{\frac{i}{nH} - 1}| s^{-H} |g(x)| dx d\xi ds
\]

\[
\leq 2 \limsup_{n \to \infty} \ell_{n,H} n^{\frac{H-1}{2}} \int_0^1 \int_{\mathbb{R}} e^{-\frac{1}{2} s^2 \xi^2} s^{-H} |g(x)| dx d\xi ds
\]

\[
+ \limsup_{n \to \infty} \ell_{n,H} n^{\frac{H-1}{2}} \int_1^n \int_{\mathbb{R}} e^{-\frac{1}{2} s^2 \xi^2} |x\xi| s^{-2H} |g(x)| dx d\xi ds,
\]

where the last inequality follows by applying the bound

\[
|e^{\frac{i}{nH} - 1}| \leq 1_{\{s \geq 1\}} |x\xi| s^{-H} + 21_{\{s \leq 1\}}.
\]

From here it follows that

\[
\limsup_{n \to \infty} \sup_{0 \leq t \leq T} \ell_{n,H} n^{\frac{1-H}{2}} |R_t^{(n)}| \leq C \limsup_{n \to \infty} \ell_{n,H} n^{\frac{H-1}{2}} \int_1^n s^{-2H} ds,
\]

for some constant \( C > 0 \) depending only depending on \( g \). Identity (3.4) easily follows from (3.5), taking into account that

\[
\int_1^n s^{-2H} ds = \begin{cases} 
\log(nT) & \text{if } H = \frac{1}{2} \\
\frac{1}{1-2H} [(nT)^{1-2H} - 1] & \text{if } H \neq \frac{1}{2}
\end{cases}
\]

and \( \frac{H-1}{2} + 1 - 2H = \frac{3-2H}{2} \leq 0 \) if \( H \geq \frac{1}{2} \).
Fix a vector-valued function $f = (f_1, \ldots, f_d)$ with $f_i \in \Xi_1$ if $H > \frac{1}{3}$ and $f_i \in \Xi_2$ if $H = \frac{1}{3}$. From (3.2) and (3.4), we deduce that to prove Theorem 1.3 it suffices to show that for every $t := (t_1, \ldots, t_Q) \in [0, T]^Q$ and $\rho_{i,j} \in \mathbb{R}$, $1 \leq i \leq Q$, $1 \leq j \leq d$, the martingale $M^{(n)} := \{M_u^{(n)} ; u \in [0, T]\}$, defined by

$$M_u^{(n)} := n^{\frac{1-H}{2}} \ell_{n,H} \sum_{i=1}^{Q} \sum_{j=1}^{d} \rho_{i,j} \int_{0}^{u} F^{(f_i, n)}_{r,t_i} dW_r,$$

with

$$F^{(f_i, n)}_{r,t_i} := \int_{\mathbb{R}} \int_{\mathbb{R}} f_j(x)(p'_{\mu_{r,s}}(B_{r,s} - \frac{x}{H} - \lambda) - p'_{\mu_{r,s}}(B_{r,s} - \lambda)) K_H(s,r)dsdx,$$

with the convention $F^{(f_i, n)}_{r,t_i} = 0$ if $r \geq t_i$, satisfies

$$M_u^{(n)} \overset{\text{Law}}{\to} \sum_{i=1}^{Q} \sum_{j=1}^{d} \rho_{i,j} (C_H[f]^{\frac{1}{2}} \widetilde{W}_{t_1,\wedge u}(\lambda))_j,$$

where $C_H[f]$ is defined as in Theorem 1.3 and $(C_H[f]^{\frac{1}{2}} \widetilde{W}_{t_1,\wedge u}(\lambda))_j$ denotes the $j$-th component of $C_H[f]^{\frac{1}{2}} \widetilde{W}_{t_1,\wedge u}(\lambda)$. By the asymptotic version of Knight’s theorem (as presented in [18, Chapter XIII, Theorem 2.3]), it suffices to show that the following two conditions hold:

(C1) For every $0 \leq u \leq T$, $\langle M^{(n)} \rangle_u$ converges in probability, and

$$\langle M^{(n)} \rangle_u = n^{1-H} \ell_{n,H}^{2} \int_{0}^{u} \left| \sum_{i=1}^{Q} \sum_{j=1}^{d} \rho_{i,j} F^{(f_i, n)}_{r,t_i} \right|^2 dr \overset{\mathbb{P}}{\to} \sum_{1 \leq i_1, i_2 \leq Q} \sum_{1 \leq j_1, j_2 \leq d} \rho_{i_1,j_1} \rho_{i_2,j_2} A_H[f_{i_1}, f_{i_2}](L_{t_{i_1,\wedge u}(\lambda)} \wedge L_{t_{i_2,\wedge u}(\lambda)}).$$

(C2) For every $u \in [0, T]$,

$$\langle M^{(n)}, W \rangle_u = n^{\frac{1-H}{2}} \ell_{n,H} \sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{i,j} \int_{0}^{u} F^{(f_i, n)}_{r,t_i} dr \overset{\mathbb{P}}{\to} 0.$$

By linearity, to show properties (C1) and (C2), it suffices to prove that, for any $t_1, t_2 \in [0, T]$ and any two functions $f, g$ such that $f, g \in \Xi_1$ if $H > \frac{1}{3}$ and $f, g \in \Xi_2$ if $H = \frac{1}{3}$, we have the following two properties:

(A1) For any $t_1, t_2 \in [0, T]$,

$$n^{1-H} \ell_{n,H}^{2} \int_{0}^{u} F^{(f, n)}_{r,t_1} F^{(g, n)}_{r,t_2} dr \overset{\mathbb{P}}{\to} A_H[f, g](L_{t_{1,\wedge u}(\lambda)} \wedge L_{t_{2,\wedge u}(\lambda)}).$$

(A2) For any $t \in [0, T]$,

$$n^{\frac{1-H}{2}} \ell_{n,H} \int_{0}^{u} F^{(f, n)}_{r,t} dr \overset{\mathbb{P}}{\to} 0.$$
3.1. Proof of (A1). Our analysis is divided into several steps. First, we use Fourier transform to find a decomposition for \( F_{r,t_1}^{(f,n)} F_{r,t_2}^{(g,n)} \) as a sum of three processes \( \Lambda^{(n,m)}_{1,r} + \Lambda^{(n,m)}_{2,r} + \Lambda^{(n,m)}_{3,r} \) that will allow us to easily identify the asymptotic behavior of \( F_{r,t_1}^{(f,n)} F_{r,t_2}^{(g,n)} \), for \( r \) given. Then, in a series of subsequent steps we will analyze individually the behavior of \( \int_0^t A^{(n,m)}_{a,r} dr \), for \( a = 1, 2, 3 \), which will ultimately lead to (3.6).

**Step I**

In order to obtain a more convenient expression for \( F_{r,t_1}^{(f,n)} \), where \( 0 \leq r \leq t \leq T \), we use (3.3) together with the change of variables \( s \rightarrow s/n \) to deduce that

\[
F_{r,t_1}^{(f,n)} F_{r,t_2}^{(g,n)} = \int_{\mathbb{R}^4} \int_{0}^{t-r} f(x) \left( p_{\mu_{r,s}} (B_{r,r+s} - \frac{x}{n} - \lambda) - p'_{\mu_{r,s}} (B_{r,r+s} - \lambda) \right) K_H(r + s) ds dx
\]

\[
= \frac{1}{n} \int_{\mathbb{R}^4} \int_{0}^{n(t-r)} f(x) \left( p_{\mu_{r,s} + \tilde{s}} (B_{r,s} - \frac{x}{n} - \lambda) - p'_{\mu_{r,s} + \tilde{s}} (B_{r,s} - \lambda) \right) K_H(r + s) ds dx
\]

\[
= \frac{1}{2\pi n} \int_{\mathbb{R}^4} \int_{0}^{n(t-r)} f(x) (-i\xi) e^{-i\mu_{r,s} + \tilde{s}\xi^2 - \tilde{s}(B_{r,s} - \lambda)} \left( e^{i\frac{x}{n} - \lambda} - 1 \right) K_H(r + s) ds dx.
\]

As a consequence, if \( t_1, t_2 \in [0, T] \), we have

\[
F_{r,t_1}^{(f,n)} F_{r,t_2}^{(g,n)} := \frac{-1}{4\pi^2 n^2} \int_{\mathbb{R}^4} \int_{0}^{n(t_1-t_2)} f(x) g(\tilde{x}) \xi e^{-\frac{1}{2}(\mu_{r,s} + \eta^2 + \mu_{r,s} + \frac{1}{2}\tilde{s}^2)}
\]

\[
\times e^{-\frac{1}{2}(\mu_{r,s} + \eta^2 - \lambda)} \left( e^{i\frac{x}{n}} - 1 \right) \left( e^{i\frac{\tilde{x}}{n}} - 1 \right) K_H(r + s) ds dx.
\]

where \( \tilde{\xi} := (\xi, \tilde{\xi}), \tilde{s} := (s_1, s_2) \) and \( \tilde{x} := (x, \tilde{x}) \). The next step is to decompose the integration on \( \tilde{\xi} \) into the regions \( \{ |\tilde{\xi}| \leq |\xi| \} \) and \( \{ |\tilde{\xi}| > |\xi| \} \). By a symmetry argument, it suffices to treat one of these regions. In fact, we can write

\[
F_{r,t_1}^{(f,n)} F_{r,t_2}^{(g,n)} := \Lambda^{(n)}_{1,r}(f,g,t_1,t_2) + \Lambda^{(n)}_{2,r}(g,f,t_1,t_2),
\]

where

\[
\Lambda^{(n)}_{1,r}(f,g,t_1,t_2) := \frac{-1}{4\pi^2 n^2} \int_{\mathbb{R}^4} \int_{0}^{n(t_1-t_2)} \frac{1}{2} (|\tilde{\xi}| \leq |\xi|) f(x) g(\tilde{x}) \xi e^{-\frac{1}{2}(\mu_{r,s} + \eta^2 + \mu_{r,s} + \frac{1}{2}\tilde{s}^2)}
\]

\[
\times e^{-\frac{1}{2}(\mu_{r,s} + \eta^2 - \lambda)} \left( e^{i\frac{x}{n}} - 1 \right) \left( e^{i\frac{\tilde{x}}{n}} - 1 \right) K_H(r + s) ds dx.
\]

To simplify the presentation we write \( \Lambda^{(n)}_{r} \) for \( \Lambda^{(n)}_{r}(f,g,t_1,t_2) \). Next we apply the change of coordinates \( \eta := n^{-H} \xi \) and \( \tilde{\eta} := \xi + \tilde{\xi} \) and we use the notation

\[
k^{(n)}_{r,s} := n^{2H-1/2} (s_1, s_2) \frac{1}{2} - H K_H(r + s) K_H(r + \frac{s_2}{n}, r)
\]

\[
\alpha^{(n)}_{r,s,\tilde{\eta}} := n^{2H} \mu_{r,s} + \frac{1}{n} \eta^2 + n^{2H} \mu_{r,s} + \frac{1}{n} (n^{-H} \tilde{\eta} - \eta)^2
\]

\[
\beta^{(n)}_{r,s,\tilde{\eta}} := n^H \eta (B_{r,s} - \frac{1}{n} - B_{r,s} + \frac{1}{n})
\]

\[
\psi^{(n)}_{r,s,\tilde{\eta}} := (e^\eta - 1) \left( e^{\frac{1}{n}} - 1 \right)
\]
for $r > 0$, $s := (s_1, s_2) \in \mathbb{R}^2$, $\eta := (\eta, \eta) \in \mathbb{R}^2$ and $n \in \mathbb{N}$, to obtain

$$
\Lambda^{(n)}_r := \frac{-1}{4\pi^2} \int_{\mathbb{R}^4} \int_0^{n(t_1-r)} \int_0^{n(t_2-r)} 1_{\{|n-H\eta-\eta| \leq |\eta|\}} \eta(n-H\eta-\eta)f(x)g(\bar{x}) \\
\times (s_1 s_2)^{1-H} e^{-\frac{1}{2}(\alpha_r,\tilde{\alpha}_r)^T a_r} e^{-i\bar{\eta}(B_{r+\bar{s}} - \lambda)} \psi_r^{(n)}(n) d\bar{s} d\bar{x}.
$$

Now, we fix $m \geq 1$ and we make the decomposition

$$
n^{1-H} \Lambda^{(n)}_r = \Lambda_{1,r}^{(n,m)} + \Lambda_{2,r}^{(n,m)} + \Lambda_{3,r}^{(n,m)},
$$

where

$$
\Lambda_{1,r}^{(n,m)} := \frac{-1}{4\pi^2} \int_{\mathbb{R}^4} \int_0^{n(t_1-r)} \int_0^{n(t_2-r)} 1_{\{|n-H\eta-\eta| \leq |\eta|\}} \eta(n-H\eta-\eta) \mathbb{1}_{\{|\tilde{\eta}| \leq m\}} f(x)g(\bar{x}) \\
\times (s_1 s_2)^{1-H} e^{-\frac{1}{2}(\alpha_r,\tilde{\alpha}_r)^T a_r} (\mu_r^{(n)}(\eta, \tilde{\eta})) e^{-i\bar{\eta}(B_{r+\bar{s}} - \lambda)} \psi_r^{(n)}(n) d\bar{s} d\bar{x},
$$

and

$$
\Lambda_{3,r}^{(n,m)} := \frac{-1}{4\pi^2} \int_{\mathbb{R}^4} \int_0^{n(t_1-r)} \int_0^{n(t_2-r)} 1_{\{|n-H\eta-\eta| \leq |\eta|\}} \eta(n-H\eta-\eta) \mathbb{1}_{\{|\tilde{\eta}| > m\}} f(x)g(\bar{x}) \\
\times (s_1 s_2)^{1-H} e^{-\frac{1}{2}(\alpha_r,\tilde{\alpha}_r)^T a_r} (\mu_r^{(n)}(\eta, \tilde{\eta})) e^{-i\bar{\eta}(B_{r+\bar{s}} - \lambda)} \psi_r^{(n)}(n) d\bar{s} d\bar{x},
$$

The above decomposition follows by first splitting the domain of integration in $\tilde{\eta}$ into $\{|\tilde{\eta}| \leq m\}$ and $\{|\tilde{\eta}| > m\}$, and then expressing $e^{i\bar{\eta}(B_{r+\bar{s}} - \lambda)}$ as

$$
e^{i\bar{\eta}(B_{r+\bar{s}} - \lambda)} = e^{-\frac{1}{2}\mathbb{E}[e^{i\bar{\eta}(\tilde{\eta})}^2]} + (e^{i\bar{\eta}(\tilde{\eta})} - e^{-\frac{1}{2}\mathbb{E}[e^{i\bar{\eta}(\tilde{\eta})}^2]}).
$$

Our goal is now to show the following three convergences:

$$
\lim_{m \to \infty} \lim_{n \to \infty} \ell^2_n \int_0^u \Lambda_{1,r}^{(n,m)} dr = \frac{1}{2} A_H[f, g](L_{t_2 \wedge u}(\lambda) \wedge L_{t_2 \wedge u}(\lambda)) \quad \text{in} \quad L^2(\Omega),
$$

$$
\lim_{n \to \infty} \ell^2_n \int_0^u \Lambda_{2,r}^{(n,m)} dr = 0 \quad \text{for all} \quad m \geq 1
$$

and

$$
\lim_{m \to \infty} \sup_n \ell^2_n \int_0^u \Lambda_{3,r}^{(n,m)} dr \to 0,
$$

where $A_H[f, g]$ is defined in (1.10) for $H > \frac{1}{4}$ and (1.11) for $H = \frac{1}{4}$. Then, (A1) will follow from the convergences (3.20), (3.21) and (3.22), taking into account the decomposition (3.12) and the fact that the limit in (3.20) is symmetric in $(f, g)$ and $(t_1, t_2)$. 
Step II
In this step we will show the convergence (3.20). By applying Fubini’s theorem, we can write
\[
\int_0^u \Lambda_s^{(n,m)}(r) \, dr = \int_0^{nt_1} \int_0^{nt_2} \Psi_s^{(n,m)}(s) \, ds,
\]
where
\[
\Psi_s^{(n,m)} := -\frac{1}{4\pi^2} \int_0^{(t_1-\frac{r}{\sqrt{n}})(t_2-\frac{r}{\sqrt{n}})u} \int_{\mathbb{R}^4} \mathbb{1}_{\{|\tilde{\eta}|\leq |\eta|\}} \mathbb{1}_{\{|\tilde{\eta}|\leq m\}} \eta(n^{-H} \eta - \eta)f(x)g(\tilde{x}) \\
\times (s_1 s_2)^{\frac{1}{2}} \eta e^{-\frac{1}{2}(\alpha(n)_{\tilde{\eta},\eta} + 2(\beta(n)_{\tilde{\eta},\eta})^\alpha)} e^{-i\theta(B_x \cdot u - \lambda)} \mathbf{v}^{(n)}_{\tilde{\eta},\eta}(r) \, drd\tilde{\eta}d\tilde{x}.
\]
Recall that \(\beta_{H,1}\) and \(\beta_{H,2}\) are defined in (1.7) and \(\beta_{H,2}(\tilde{s})\) is given in (1.8). We observe that (3.20) can be obtained by proving the following statements:

(i) The function
\[
\Psi_s^{(\infty,m)} := \frac{\beta_{H,1}^2}{4\pi^2} \int_{t_1 \wedge t_2 \wedge u} \int_{\mathbb{R}^4} \mathbb{1}_{\{|\tilde{\eta}|\leq m\}} \eta^2 (s_1 s_2) H \frac{1}{2} f(x)g(\tilde{x}) \\
\times e^{-\frac{1}{2}(\beta_{H,2}(s_1^{2H} + s_2^{2H}) + \beta_{H,3}(s_1, s_2)) \eta^2} e^{-i\theta(B_x \cdot u - \lambda)} (e^{i\theta n - 1}(1 - e^{-in\theta})) \, drd\tilde{\eta}d\tilde{x},
\]
(3.23)
satisfies
\[
\lim_{n \to \infty} \ell^2_{n,H} \int_{[0,nt_1] \times [0,nt_2]} (\Psi_s^{(n,m)} - \Psi_s^{(\infty,m)}) \, ds = 0.
\]
(3.24)

(ii) The convergence
\[
\lim_{n \to \infty} \ell^2_{n,H} \int_{[0,nt_1] \times [0,nt_2]} \Psi_s^{(\infty,m)} \, ds = \frac{1}{2} A_H[f, g] \frac{1}{2\pi} \int_{t_1 \wedge t_2 \wedge u} \int_{[-m,m]} e^{-i\theta(B_x \cdot u - \lambda)} \, drd\tilde{\eta}
\]
(3.25)
holds in the topology of \(L^2(\Omega)\).

Proof of (3.24): Before giving the details of the proof, and in order to clarify the presentation, we provide first some heuristic arguments. Recall definitions (3.13) to (3.16), and define
\[
\Theta_s^{(n,m)} := (s_1 s_2)^{\frac{1}{2}} - H e^{-\frac{1}{2}(\alpha(n)_{\tilde{\eta},\eta} + 2(\beta(n)_{\tilde{\eta},\eta})^\alpha)} \mathbf{v}^{(n)}_{\tilde{\eta},\eta}(r) \\
\Theta^{(\infty,m)} := \beta_{H,1} e^{-\frac{1}{2}(\beta_{H,2}(s_1^{2H} + s_2^{2H}) + \beta_{H,3}(s_1, s_2)) \eta^2} (e^{i\theta n - 1}(1 - e^{-in\theta})).
\]
and
Then, we have
\[
\lim_{n \to \infty} \Theta_s^{(n,m)} = \Theta^{(\infty,m)}. Indeed, this convergence follows immediately from (5.5), (5.9) and (5.15). As a consequence, the integrand of \(\Psi_s^{(n,m)}\) converges point-wise to that of \(\Psi_s^{(\infty,m)}\). Nevertheless, to show the convergence of the integrals we need some additional work.

We first consider the region of integration \([\varepsilon, t_1 \wedge t_2 \wedge u]\) for the variable \(r\), where \(\varepsilon \in (0, t_1 \wedge t_2 \wedge u)\) is some given positive constant. Define \(\hat{f} := \max\{|f|, |g|\}\). Notice that
\[
\Psi_s^{(n,m)} := -\frac{1}{4\pi^2} \int_0^{(t_1-\frac{r}{\sqrt{n}})(t_2-\frac{r}{\sqrt{n}})u} \int_{\mathbb{R}^4} \mathbb{1}_{\{|\tilde{\eta}|\leq |\eta|\}} \mathbb{1}_{\{|\tilde{\eta}|\leq m\}} \eta(n^{-H} \eta - \eta)f(x)g(\tilde{x}) \\
\times (s_1 s_2)^{\frac{1}{2}} \eta e^{-i\theta(B_x \cdot u - \lambda)} \, drd\tilde{\eta}d\tilde{x}.
\]
Therefore, there exists a constant $C > 0$ such that

$$\| \Psi_s^{(n,m)} - \Psi_s^{(\infty,m)} \|_{L^2(\Omega)} \leq C(T_1(\tilde{s}) + T_2(\tilde{s}) + T_3(\tilde{s}) + T_4(\tilde{s})),$$

where

$$T_1(\tilde{s}) := \int_0^T \int_{\mathbb{R}^4} \frac{1}{2} (\omega - |n - \eta\tilde{\eta}|) \frac{1}{2} (|\eta| \leq m) \eta^2 (s_1 s_2)^{H - \frac{1}{2}}
\times \tilde{f}(x) \tilde{f}(\tilde{x}) \left| \Theta_{\tilde{s},\tilde{s}}^{(n,m)} - \Theta_{\tilde{s},\tilde{s}}^{(\infty,m)} \right| d\tilde{y}d\tilde{y}d\tilde{x}d\tilde{r},$$

$$T_2(\tilde{s}) := \int_0^T \int_{\mathbb{R}^4} \frac{1}{2} (|n - \eta\tilde{\eta}| - |\tilde{\eta} - \eta|) \frac{1}{2} (|\eta| \leq m) \eta^2 (s_1 s_2)^{H - \frac{1}{2}}
\times \tilde{f}(x) \tilde{f}(\tilde{x}) \left| \Theta_{\tilde{s},\tilde{s}}^{(n,m)} \right| + \left| \Theta_{\tilde{s},\tilde{s}}^{(\infty,m)} \right| d\tilde{y}d\tilde{y}d\tilde{x}d\tilde{r},$$

$$T_3(\tilde{s}) := \int_0^T \int_{\mathbb{R}^4} \frac{1}{2} (|n - \eta\tilde{\eta}| - |\tilde{\eta} - \eta|) \frac{1}{2} (|\eta| \leq m) \eta^2 (s_1 s_2)^{H - \frac{1}{2}}
\times \tilde{f}(x) \tilde{f}(\tilde{x}) \left| \Theta_{\tilde{s},\tilde{s}}^{(n,m)} \right| e^{-\tilde{y}B_{r,r} + \frac{\tilde{y}^2}{2}} - e^{-\tilde{y}B_{r,r} + \frac{\tilde{y}^2}{2}} \|_{L^2(\Omega)} d\tilde{y}d\tilde{y}d\tilde{x}d\tilde{r},$$

$$T_4(\tilde{s}) := \int_0^T \int_{\mathbb{R}^4} \frac{1}{2} (|n - \eta\tilde{\eta}| - |\tilde{\eta} - \eta|) \frac{1}{2} (|\eta| \leq m) \eta^2 (s_1 s_2)^{H - \frac{1}{2}}
\times \tilde{f}(x) \tilde{f}(\tilde{x}) \left| \Theta_{\tilde{s},\tilde{s}}^{(n,m)} \right| \left( \left| \Theta_{\tilde{s},\tilde{s}}^{(\infty,m)} \right| d\tilde{y}d\tilde{y}d\tilde{x}d\tilde{r}.\right.$$ 

**Estimation of the term $T_1(\tilde{s})$:** Notice that we are integrating in a domain with the restrictions $|\eta| \leq |\eta - n^{-H} \tilde{\eta}| \leq |\eta|, |\tilde{\eta}| \leq m$ and $r \geq \varepsilon$. In the sequel we will denote by $C$ a constant that may depend on $m, \varepsilon, T$ and $H$. We claim that the following inequalities hold:

$$|\kappa_{r,\tilde{s},\tilde{\eta}}^{(n)}| \leq C \frac{s_1 + s_2}{n}, \quad \frac{n}{2} \leq \frac{s_1 + s_2}{n} H,$$

$$|\psi_{x,\tilde{\eta}}^{(n)} - (e^{x\eta} - 1)(e^{x\tilde{\eta} - 1})| \leq C(1 \land |x|)(1 \land |n^{-H} x|),$$

$$|\alpha_{r,\tilde{s},\tilde{\eta}}^{(n)} - \beta_{H,2}(s_1^{2H} + s_2^{2H})\eta^2| \leq C(s_1^{2H} + s_2^{2H}) \left( \frac{s_1 + s_2}{n} \eta^2 + \frac{\eta}{n^{H}} \right),$$

$$|n^{2H}\eta^2| E[(B_{r,r+\frac{2\eta}{n}} - B_{r,r+\frac{2\eta}{n}}^{2H}) - \beta_{H,3}(s_1, s_2)|\eta|^2| \leq C \eta^2 \left( (s_1 \lor s_2)^2 n^{2H-2} + (s_1 \lor s_2)^{2H+1} n^{-1} \right),$$

and

$$|\psi_{x,\tilde{\eta}}^{(n)}| + |(e^{x\eta} - 1)(e^{x\tilde{\eta} - 1})| \leq C(1 \land |x|)(1 \land |x\tilde{\eta}|),$$

$$|\kappa_{r,\tilde{s},\tilde{\eta}}^{(n)}| \leq C,$$

$$|\kappa_{r,\tilde{s},\tilde{\eta}}^{(n)}| \leq \delta(s_1^{2H} + s_2^{2H})\eta^2,$$

where $\delta > 0$ denotes a constant depending on $H$. Indeed, inequality (3.27) follows fromLemma 5.4 taking into account that $r \geq \varepsilon$. The estimate (3.28) is straightforward. In order to show (3.29), we write, using Lemma 5.5 when $|\eta - n^{-H} \tilde{\eta}| \leq |\eta| \lor |\tilde{\eta}| \leq m$, 

$$|\alpha_{r,\tilde{s},\tilde{\eta}}^{(n)} - \beta_{H,2}(s_1^{2H} + s_2^{2H})\eta^2| \leq \eta^2 |n^{2H} \mu_{r,r+\frac{2\eta}{n}} - \beta_{H,2}s_1^{2H} + \eta^2 |n^{2H} \mu_{r,r+\frac{2\eta}{n}} - \beta_{H,2}s_2^{2H}|$$

$$+ n^{2H} |(n^{-H} \tilde{\eta} - \eta)^2 - \eta^2|$$
where

\[
16
\]

\[
\text{By Lemma 5.9, there exists a constant }
\]

\[
\text{and this implies (3.29). The estimate (3.30) follows from (5.15). The inequality (3.31) is immediate, taking into account that } |\eta - n^{-H} \bar{\eta}| \leq |\eta|, \text{ (3.32) follows from (5.4) and, finally, (3.33) is due to the lower bounds in Lemma 5.5.}
\]

Thus, using the fact that \(|e^{-\alpha} - e^{-\beta}| \leq (1 + |\alpha - \beta|)(e^{-\alpha} + e^{-\beta})\), we obtain

\[
|\Theta_{\eta, s}^{(n,m)} - \Theta_{\eta, s}^{(\infty,m)}| \leq C^{\frac{s + s}{n}}e^{-\delta(\eta^{2H} + s^{2H})n^2} (1 + |x\eta|)(1 + |\bar{x}\eta|)
\]

\[
+ C e^{-\delta(\eta^{2H} + s^{2H})n^2} (1 + |x\eta|)(1 + |n^{-H} \bar{x}|)
\]

\[
+ C e^{-\delta(\eta^{2H} + s^{2H})n^2} (1 + |x\eta|)(1 + |\bar{x}\eta|)
\]

\[
\times \left( s_1 \lor s_2 \right)^{2H} \left( \eta^2 \eta \lor s_2 \frac{n}{2} + \eta^2 \left( \frac{s_1 \lor s_2}{n} \right)^{2-2H} + \frac{|\eta|}{n^H} \right),
\]

where \(\delta\) is a constant depending on \(H\). From here we deduce that

\[
T_1(\bar{s}) \leq C \left( T_1'(\bar{s}) + T_2''(\bar{s}) + T_3''(\bar{s}) \right),
\]

where

\[
T_1'(\bar{s}) := \int_{\mathbb{R}^3} \eta^2 (s_1 s_2)^{H-\frac{1}{2}} \bar{f}(x) \bar{f}(\bar{x}) e^{-\delta(\eta^{2H} + s^{2H})n^2} (1 + |x\eta|)(1 + |\bar{x}\eta|)
\]

\[
\times \left( s_1 \lor s_2 \right)^{2H} \left( \eta^2 \eta \lor s_2 \frac{n}{2} + \eta^2 \left( \frac{s_1 \lor s_2}{n} \right)^{2-2H} + \frac{|\eta|}{n^H} \right) d\eta d\bar{x},
\]

\[
T_2''(\bar{s}) := \int_{\mathbb{R}^3} \eta^2 (s_1 s_2)^{H-\frac{1}{2}} \bar{f}(x) \bar{f}(\bar{x}) e^{-\delta(\eta^{2H} + s^{2H})n^2} n^{-H} |\bar{x}| (1 + |x\eta|) d\eta d\bar{x},
\]

and

\[
T_3''(\bar{s}) := \int_{\mathbb{R}^3} \eta^2 (s_1 s_2)^{H-\frac{1}{2}} \bar{f}(x) \bar{f}(\bar{x}) e^{-\delta(\eta^{2H} + s^{2H})n^2} \left( 1 + |x\eta| \right) \left( 1 + |\bar{x}\eta| \right) \frac{s_1 + s_2}{n} d\eta d\bar{x}.
\]

By Lemma 5.9, there exists a constant \(C > 0\), such that

\[
T_1'(\bar{s}) \leq C \|\bar{f}\|_1^2 \left( 1 + \sqrt{s_1^{2H} + s_2^{2H}} \right)^{-2} \left( s_1^{2H} + s_2^{2H} \right)^{-\frac{3}{2}} (s_1 s_2)^{H-\frac{1}{2}}
\]

\[
\times \left( \frac{(s_1 \lor s_2)^{2H+1}}{n} + \frac{(s_1 \lor s_2)^2}{n^{2-2H}} \right)
\]

\[
+ \frac{C}{nH} \|\bar{f}\|_1^2 \left( 1 + \sqrt{s_1^{2H} + s_2^{2H}} \right)^{-2} \left( s_1^{2H} + s_2^{2H} \right)^{-2} \frac{s \lor s_2}{n^{2H}} (s_1 s_2)^{H-\frac{1}{2}}.
\]

Integrating over the interval \([0, nT]^2\) yields

\[
\int_{[0, nT]^2} T_1'(\bar{s}) d\bar{s} \leq C \|\bar{f}\|_1^2 \int_0^{nT} (1 + s_2^{2H} )^{-2} \left[ n^{-1} s_1^{1-H} + n^{2H-2} s_2^{2-3H} + n^{-H} \right] ds_2
\]

\[
= C \|\bar{f}\|_1^2 \int_0^{nT} \left[ n^{-1} s_1^{1-H} + n^{2H-2} s_2^{2-3H} + n^{-H} \right] ds_2
\]

\[
+ C \|\bar{f}\|_1^2 \int_1^{nT} \left[ n^{-1} s_1^{1-H} + n^{2H-2} s_2^{2-3H} + n^{-H} s_2^{2H} \right] ds_2
\]

\[
\leq C \left( n^{-1} + n^{2H-2} + n^{-H} + n^{1-3H} \right).
\]
By Lemma 5.9, there exists a constant $C > 0$, such that
\[
T''_1(\tilde{s}) \leq C n^{-H} \|\bar{f}\|_1^2 (1 \vee \sqrt{s_1^{2H} + s_2^{2H}})^{-1} (s_1^{2H} + s_2^{2H})^{-\frac{3}{2}} (s_1 s_2)^{H - \frac{1}{2}}.
\]
Integrating over the interval $[0, nT]^2$ yields
\[
\int_{[0,nT]^2} T''_1(\bar{s}) d\bar{s} \leq C n^{-H} \|\bar{f}\|_1^2 \int_0^{nT} (1 \vee s_2^{H})^{-1} s_2^{-H} ds_2
\leq C n^{-H} \|\bar{f}\|_1^2 \left( \int_0^1 s_2^{-H} ds_2 + \int_1^{nT} s_2^{-1-H} ds_2 \right)
\leq C \|\bar{f}\|_1^2 \left( n^{-H} + n^{-2H} \right).
\]
(3.35)

Finally, applying Lemma 5.9 again, there exists a constant $C > 0$, such that
\[
T''_1(\bar{s}) \leq C \|\bar{f}\|_1^2 (1 \vee \sqrt{s_1^{2H} + s_2^{2H}})^{-2} (s_1^{2H} + s_2^{2H})^{-\frac{1}{2}} s_1 H - \frac{1}{2} \bar{s}^{H + \frac{1}{2}}.
\]
Integrating over the interval $[0, nT]^2$ yields
\[
\int_{[0,nT]^2} T''_1(\bar{s}) d\bar{s} \leq \frac{C}{n} \|\bar{f}\|_1^2 \int_0^{nT} (1 \vee s_2^{H})^{-2} s_2^{-1-H} ds_2
= \frac{C}{n} \|\bar{f}\|_1^2 \left( \int_0^1 s_2^{-H} ds_2 + \int_1^{nT} s_2^{-1-3H} ds_2 \right)
\leq C \|\bar{f}\|_1^2 \left( n^{-1} + n^{-1-3H} \right).
\]
(3.36)

From (3.34), (3.35) and (3.36), we obtain
\[
\limsup_{n \to \infty} \ell_{n, H}^2 \int_0^T \int_{[0,nT]^2} T_1(\bar{s}) d\bar{s} dr = 0.
\]

**Estimation of the term $T_2(\bar{s})$:** To bound $T_2(\bar{s})$, we observe that, in view of the estimate (3.33), there exist constants $C, \delta > 0$, such that
\[
|\Theta_{\eta, \bar{s}}^{(m, n)}| + |\Theta_{\eta, \bar{s}}^{(\infty, m)}| \leq C e^{-\delta s_2^{2H} \eta^{-\delta}} (1 \wedge |\eta \bar{x}|)(1 \wedge |\eta \bar{x}|^{-\frac{1}{2}}) \leq C e^{-c_n s_2^{2H} \eta^{-\delta}} |\eta \bar{x}| \eta^2,
\]
which, combined with the inequality
\[
\mathbb{1}_{\{|n^{-H} \eta^{-\frac{1}{2}}| < \frac{1}{2} \}} \leq \mathbb{1}_{\{|n| \leq 2n^{-H} |\eta| \}} \leq \mathbb{1}_{\{|n| \leq 2mn^{-H} \}},
\]
leads to
\[
|T_2(\bar{s})| \leq C \|\bar{f}\|_1^2 n^{-4H} \int_{\mathbb{R}} (s_1 s_2)^{H - \frac{1}{2}} e^{-\delta s_2^{2H} \eta^2} d\eta \leq C \|\bar{f}\|_1^2 n^{-4H} s_2^{H - \frac{1}{2}} s_1^{-\frac{1}{2}}.
\]
Integrating $s_1$ and $s_2$ over $[0, nT]$, we obtain the inequality
\[
\int_{[0,nT]^2} T_2(\bar{s}) d\bar{s} \leq C \|\bar{f}\|_1^2 n^{1-3H}.
\]
From here we conclude that
\[
\lim_{n \to \infty} \ell_{n, H}^2 \int_0^T \int_{[0,nT]^2} T_2(\bar{s}) d\bar{s} dr = 0,
\]
as required.
Consequently, by Lemma 5.9, of constants $T$ as required.

Thus, we have $C > e^{e^9T}r,r$ for some $C > 0$. Since

$$
E[(B_{r,r} + s_2) - B_r)^2] = E[(E(B_{r,r} + s_2) - B_r | F_r)^2] \leq E[(B_{r,r} + s_2) - B_r)^2] = (s_2/n)^{2H},
$$

we have

$$
\|e^{-i\eta B_{r,r} + s_2} - e^{-i\eta B_r}\|_{L^2(\Omega)} \leq C(1 \wedge (s_2/n)^H).
$$

Thus,

$$
|T_3(\bar{s})| \leq C \int_{\mathbb{R}^3} \eta^2 (s_1 s_2)^{H - \frac{1}{2}} \bar{f}(x) \bar{f}(\bar{x}) e^{-\delta(s_1^{2H} + s_2^{2H})\eta^2} (1 \wedge |xe|)(1 \wedge |\eta e|)(1 \wedge (s_2/n)^H) d\eta d\bar{x}.
$$

Applying Lemma 5.9 to the right-hand side, we have that

$$
|T_3(\bar{s})| \leq C \|\bar{f}\|^2_1 (1 \wedge s_1^{2H} + s_2^{2H})^{-2} (s_1^{2H} + s_2^{2H})^{-2} (s_1 s_2)^{H - \frac{1}{2}} \left(\frac{s_2}{n}\right)^H,
$$

and consequently,

$$
\int_{[0,nT]^2} T_3(\bar{s}) d\bar{s} \leq C \|\bar{f}\|^2_1 n^{-H} \int_0^{nT} (1 \wedge s_1^{2H} + s_2^{2H})^{-2} s_2^{2H} ds_2
\leq C \|\bar{f}\|^2_1 n^{-H} \left(\int_1^{nT} s^{-4H} ds_2 + \int_0^1 ds_2\right)
\leq C \|\bar{f}\|^2_1 (n^{1-3H} + n^{-H}).
$$

As a result,

$$
\lim_{n \to \infty} \int_{[0,nT]^2} T_3(\bar{s}) d\bar{s} d\bar{r} = 0,
$$

as required.

Estimation of the term $T_4(\bar{s})$: To estimate $T_4(\bar{s})$, we observe that (3.33) implies the existence of constants $C, \delta > 0$ such that

$$
|\Theta_{\hat{\eta},s}^{(n,m)}| + |\Theta_{\hat{\eta},s}^{(\infty,m)}| \leq C e^{-\delta(s_1^{2H} + s_2^{2H})\eta^2} (1 \wedge |x\eta|)(1 \wedge |\bar{x}\eta|),
$$

which gives

$$
|T_4(\bar{s})| \leq C \int_{\mathbb{R}^3} \eta^2 (s_1 s_2)^{H - \frac{1}{2}} \bar{f}(x) \bar{f}(\bar{x}) e^{-\delta(s_1^{2H} + s_2^{2H})\eta^2} (1 \wedge |x\eta|)(1 \wedge |\eta \bar{e}|) d\eta d\bar{x}.
$$

Consequently, by Lemma 5.9,

$$
|T_4(\bar{s})| \leq C \|\bar{f}\|^2_1 (1 \wedge \sqrt{s_1^{2H} + s_2^{2H}})^{-2} (s_1^{2H} + s_2^{2H})^{-\frac{1}{2}} (s_1 s_2)^{H - \frac{1}{2}}.
$$
Therefore,
\[
\int_{[0,nT]^2} T_4(\vec{s})d\vec{s} \leq \varepsilon C \| \tilde{f} \|^2 \int_0^{nT} (1 \wedge s_2^H)^{-\frac{3}{2}} s_2^{-2H} ds_2
\]
\[
\leq \varepsilon C \| \tilde{f} \|^2 \int_1^{nT} s_2^{-4H} ds_2 + \int_0^1 ds_2
\]
\[
\leq \varepsilon C \| \tilde{f} \|^2 (n^{1-3H} + 1).
\]

From here it easily follows that

\[
\lim_{n \to \infty} \frac{\ell^2}{\ell_n, H} T_4(\vec{s})d\vec{s} dr \leq C \varepsilon.
\]

Relation (3.24) follows from (3.37), (3.38), (3.40) and (3.41) by taking \( \varepsilon \to 0 \).

**Proof of (3.25):** We distinguish the cases \( H > \frac{1}{3} \) and \( H = \frac{1}{3} \). If \( H > \frac{1}{3} \), taking into account the definition of \( A[f, g] \) given in (1.10), it suffices to show that the integral of (3.23) is finite. To this end, notice that the absolute value of the integrand is bounded by

\[
C \eta^2 (s_1 s_2)^{H-\frac{1}{2}} (1 \wedge \sqrt{s_1^{2H} + s_2^{2H}}\eta^2 (1 \wedge |\eta x|)(1 \wedge |\eta \vec{x}|),
\]

for some constants \( \delta, C > 0 \). By Lemma 5.9, the integral over \( \vec{x} \in \mathbb{R}^2 \), \( \eta \in \mathbb{R} \), \( \vec{\eta} \in \mathbb{R}^d \) and \( \vec{s} \in \mathbb{R}^d \), is bounded by

\[
C \| \tilde{f} \|^2 \int_{\mathbb{R}^d_+} (s_1 s_2)^{H-\frac{1}{2}} (1 \wedge \sqrt{s_1^{2H} + s_2^{2H}})^{-\frac{1}{2}} (s_1^{2H} + s_2^{2H})^{-\frac{1}{2}} d\vec{s},
\]

which is finite due to the condition \( H > \frac{1}{3} \).

To handle the case \( H = \frac{1}{3} \), we first prove that

\[
\lim_{n \to \infty} (\log n)^{-1} \int_{[0,nu]^2} \Psi_{\vec{s}}^{(\infty, m)} d\vec{s} = \lim_{n \to \infty} (\log n)^{-1} \int_{[1,nu]^2} \Psi_{\vec{s}}^{(\infty, m)} d\vec{s}.
\]

To show this, we proceed as in the case \( H > \frac{1}{3} \), to deduce the bound

\[
|\Psi_{\vec{s}}^{(\infty, m)}| \leq C \| \tilde{f} \|^2 (s_1 s_2)^{H-\frac{1}{2}} (s_1^{2H} + s_2^{2H})^{-\frac{1}{2}} (1 \wedge \sqrt{s_1^{2H} + s_2^{2H}})^{-1}.
\]

The right-hand side of the above inequality is integrable over \( ([1, \infty) \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times [1, \infty]) \) due to the condition \( H \geq \frac{1}{3} \), and thus,

\[
\lim_{n \to \infty} (\log n)^{-1} \int_{([1, \infty) \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times [1, \infty])} \Psi_{\vec{s}}^{(\infty, m)} d\vec{s}
\]
\[
\leq C \| \tilde{f} \|^2 \lim_{n \to \infty} (\log n)^{-1} \int_{([1, \infty) \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times [1, \infty])} (s_1 s_2)^{H-\frac{1}{2}} (s_1^{2H} + s_2^{2H})^{-\frac{1}{2}} (1 \wedge \sqrt{s_1^{2H} + s_2^{2H}})^{-1} d\vec{s} = 0.
\]

Relation (3.42) thus follows from the fact that

\[
[0, nu]^2 \setminus [1, nu]^2 \subset ([1, \infty) \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times [1, \infty]),
\]
for \( n \geq \frac{1}{a} \). It is therefore sufficient to analyze the right-hand side of (3.42). To do this, we write

\[
\int_{[1,n]^2} \Psi_s^{(\infty,m)} d\vec{s} = \frac{1}{2\pi} \left( \int_{t_1 \wedge t_2 \wedge u} e^{-i\vec{s}(B_r - \lambda)} d\vec{y} dr \right)
\]

(3.43)

\[
\times \left( \int_{[1,n]^2} T_5(\vec{s}) d\vec{s} + \int_{[1,n]^2} T_6(\vec{s}) d\vec{s} \right),
\]

where \( T_5(\vec{s}) \) and \( T_6(\vec{s}) \) are functions satisfying

\[
|T_5(\vec{s})| \leq \frac{\beta^2_{H,1}}{2\pi} \int_{\mathbb{R}^3} \eta^2(s_1s_2)^{H-\frac{1}{2}} f(x) \tilde{f}(x) e^{-\frac{1}{2}((\beta_{H,2}(s_1^2 + s_2^2) + \beta_{H,3}(s_1, s_2))\eta^2}
\]

\[
\times \left| (e^{i\eta} - 1) (1 - e^{-i\eta}) - \eta^2 x_i x_j \right| d\eta d\vec{x}
\]

and

(3.44)

\[
T_6(\vec{s}) := \frac{\beta^2_{H,1}}{2\pi} \left( \int_{\mathbb{R}} x f(x) dx \right) \left( \int_{\mathbb{R}} x g(x) dx \right) \int_{\mathbb{R}} \eta^4(s_1s_2)^{H-\frac{1}{2}} e^{-\frac{1}{2}((\beta_{H,2}(s_1^2 + s_2^2) + \beta_{H,3}(s_1, s_2))\eta^2) d\eta.
\]

Taking into consideration (3.42) and (3.43), it suffices to show that

(3.45)

\[
\lim_{n \to \infty} \frac{1}{\log n} \int_{[1,n]^2} T_5(\vec{s}) d\vec{s} = 0
\]

and

(3.46)

\[
\lim_{n \to \infty} \frac{1}{\log n} \int_{[1,n]^2} T_6(\vec{s}) d\vec{s} = \frac{1}{2} \mathcal{A}_{\frac{1}{3}} [f, g].
\]

To prove (3.45), we proceed as follows. First we use the inequality

\[
|e^{i\eta} - 1 - (1 - e^{-i\eta})| \leq x^2 \eta^2 + (x^2 + \eta^2) |\eta|^3
\]

together with Lemma 5.9 and taking into account that \( f, g \) are in \( \Xi_2 \), to deduce that

\[
|T_5(\vec{s})| \leq C |\vec{f}|^2 \int_{\mathbb{R}^3} (|\eta|^5 + |\eta|^6)(s_1s_2)^{H-\frac{1}{2}} e^{-c\eta^2(s_1^2 + s_2^2)} d\eta
\]

\[
\leq C |\vec{f}|^2 (s_1s_2)^{H-\frac{1}{2}} ((s_1^2 + s_2^2)^{\frac{3}{2}} + (s_1^2 + s_2^2)^{\frac{4}{3}} - \frac{5}{2}),
\]

for some constant \( C > 0 \). We can easily check that the right-hand side is integrable over \( \vec{s} \in \mathbb{R}^2 \), due to the condition \( H = \frac{1}{3} \). Relation (3.45) follows from here.

Next we prove (3.46). Set

\[
T_7(\vec{s}) := \int_{\mathbb{R}} \eta^4(s_1s_2)^{\frac{1}{2}} e^{-\frac{1}{2}((\beta_{\frac{1}{2},2}(\frac{s_1^2}{2} + \frac{s_2^2}{2}) + \beta_{\frac{1}{3},3}(s_1, s_2))\eta^2) d\eta
\]

\[
= 3\sqrt{2\pi}(s_1s_2)^{\frac{1}{2}}((-\beta_{\frac{1}{2},2}(\frac{s_1^2}{2} + \frac{s_2^2}{2}) + \beta_{\frac{1}{3},3}(s_1, s_2))^{\frac{1}{2}}
\]

\[
= 3\sqrt{2\pi}(s_1s_2)^{\frac{1}{2}}((-\beta_{\frac{1}{2},2}(\frac{s_1^2}{2} + \frac{s_2^2}{2}) + \beta_{\frac{1}{3},3}(s_1, s_2))^{\frac{1}{2}}
\]

where in the last equality we used the fact that \( \beta_{\frac{1}{3},3}(s_1, s_2) = \frac{s_2}{2} \beta_{\frac{1}{3},3}(s_1, s_2) \). Using the fact that \( \sup_{x \in [0,1]} \beta_{\frac{1}{4},3}(1, x) < \infty \), we can check that

\[
\limsup_{n \to \infty} \int_{1}^{n} \int_{1}^{s_2} T_7(\vec{s}) d\vec{s} = \infty,
\]
and thus, by L'Hôpital's rule,

\begin{equation}
\lim_{n \to \infty} \frac{1}{\log n} \int_1^{nu} \int_1^{s_2} T_7(\tilde{s}) d\tilde{s} = \lim_{n \to \infty} \frac{nu}{\log n} \int_1^{nu} T_7(s, nu) ds.
\end{equation}

Applying the change of variables \( \frac{s}{n} \to s \), we deduce that

\[ nu \int_1^{nu} T_7(s, nu) ds = 3\sqrt{2\pi} \left( \int_1^{1/nu} s^{-\frac{1}{2}} (\beta_{1/2,2} (1 + s) + \beta_{1,3} (1, s)) d\tilde{s} \right) \]

Therefore,

\[ \lim_{n \to \infty} \frac{1}{\log n} \int_{[1, nu]^2} T_6(\tilde{s}) d\tilde{s} = \frac{6\beta_2^2}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} x f(x) dx \right) \left( \int_{\mathbb{R}} x g(x) dx \right) \times \int_0^1 s^{-\frac{1}{2}} (\beta_{1/2,2} (1 + s) + \beta_{1,3} (1, s)) d\tilde{s}. \]

This finishes the proof of (3.25) and completes the proof of (3.20).

**Step III**

In this step we show the convergence (3.21), that is, for all \( m \geq 1 \), we have

\[ \lim_{n \to \infty} \ell^2_{n, H} \left\| \int_0^{nT} \Lambda_{s, r}^{(n, m)} dr \right\|_{L^2(\Omega)} = 0. \]

From (3.18) and using the estimate (3.31) and Minkowski's inequality, we can write

\[ \left\| \int_0^{nT} \Lambda_{s, r}^{(n, m)} dr \right\|_{L^2(\Omega)} \leq C \int_{\mathbb{R}^4} \int_0^{nT} \int_0^{nT} (s_1 s_2) H^{-\frac{1}{2}} \mathbb{1}_{(|\tilde{s}| \leq m)} \mathbb{1}_{(|n^{-\frac{1}{2}} \tilde{s} - n| \leq \tilde{n}|)} \eta_2 (1 \wedge |\eta x|)(1 \wedge |\eta \tilde{x}|) \]

\[ \times |f(x, g(\tilde{x}))| T_8(\tilde{s}, \tilde{x}) d\tilde{t} d\tilde{x} =: C \Lambda_1^{(n, m)}, \]

where

\[ T_8(\tilde{s}, \tilde{x}) := \sup_{0 \leq t \leq T} \left\| \int_0^t k_{r, \tilde{s}}(e^{-\frac{i}{2} x_0 s} - e^{-\frac{i}{2} \eta (r, \tilde{s}, \eta, x_0)^2}) e^{-i\eta (B_{r, \tilde{s}} + \frac{i}{2} x_0)} dr \right\|_{L^2(\Omega)}. \]

Next, we make the following decomposition

\begin{equation}
\Lambda_{1, n, m} = \Lambda_{2, n, m} + \Lambda_{3, n, m},
\end{equation}

where

\[ \Lambda_{2, n, m} := \int_{\mathbb{R}^4} \int_0^{nT} \int_0^{nT} (s_1 s_2) H^{-\frac{1}{2}} \mathbb{1}_{(|\tilde{s}| \leq m)} \mathbb{1}_{(|\frac{1}{n^{-1/2}} \tilde{s} - n| \leq \tilde{n}|)} \eta_2 (1 \wedge |\eta x|)(1 \wedge |\eta \tilde{x}|) \]

\[ \times |f(x, g(\tilde{x}))| T_8(\tilde{s}, \tilde{x}) d\tilde{t} d\tilde{x} \]

and

\[ \Lambda_{3, n, m} := \int_{\mathbb{R}^4} \int_0^{nT} \int_0^{nT} (s_1 s_2) H^{-\frac{1}{2}} \mathbb{1}_{(|\tilde{s}| \leq m)} \mathbb{1}_{(|\frac{1}{n^{-1/2}} \tilde{s} - n| > \tilde{n}|)} \eta_2 (1 \wedge |\eta x|)(1 \wedge |\eta \tilde{x}|) \]

\[ \times |f(x, g(\tilde{x}))| T_8(\tilde{s}, \tilde{x}) d\tilde{t} d\tilde{x}. \]
Using (3.32) and the fact that \( \Phi(r_1, r_2, \tilde{s}, \tilde{\eta}) = \Phi(r_2, r_1, \tilde{s}, \tilde{\eta}) \), we can write

\[
T_8^2(\tilde{s}, \tilde{\eta}) = \int_{[0,T]^2} e^{-\frac{1}{2} \frac{n}{t_1,t_2} - \frac{1}{2} \frac{\alpha_{\eta_1,\eta_2}}{r_1,r_2,\tilde{s}} \frac{r}{s_1,s_2}} \mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] d\tilde{r},
\]

(3.50)

\[
\leq 2C \int_{[0,T]^2} \mathbb{1}_{\{r_1 \leq r_2\}} e^{-\frac{1}{2} \frac{n}{t_1,t_2} - \frac{1}{2} \frac{\alpha_{\eta_1,\eta_2}}{r_1,r_2,\tilde{s}}} \mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] d\tilde{r},
\]

where

\[
\Phi(\tilde{r}, \tilde{s}, \tilde{\eta}) = e^{-i\tilde{\eta}(B_{r_1,r_1} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} - B_{r_2,r_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1})} (e^{-i\tilde{\eta}(B_{s_1,s_1} - B_{s_2,s_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1})}) (e^{i\tilde{\eta}(B_{r_1,s_1} - B_{r_2,s_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1}))}.\]

On the set \( \{ |\tilde{\eta}| > |n^{-H} \tilde{\eta} - \eta| \} \cap \{|\tilde{\eta}| \leq m \} \) we have

\[
|n^{-H} \tilde{\eta} - \eta| - |\eta| > \frac{|\eta|}{2},
\]

which implies \( |\eta| \leq 2mn^{-H} \). As a consequence, taking into account that \( T_8(\tilde{s}, \tilde{\eta}) \) is bounded by a constant, we obtain

\[
\Lambda_{3}^{(n,m)} \leq C_{n} \|f\|_{1} \|g\|_{1} \int_{0}^{nT} \int_{0}^{nT} \int_{0}^{2mn^{-H}} \mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] d\tilde{r},
\]

(3.51)

\[
\lim_{n \to \infty} \epsilon_{n,H}^{2} \Lambda_{3}^{(n,m)} = 0.
\]

To handle the term \( \Lambda_{2}^{(n,m)} \), we first make the decomposition

\[
T_8^2(\tilde{s}, \tilde{\eta}) = 2C \left( T_{8,1}(\tilde{s}, \tilde{\eta}) + T_{8,2}(\tilde{s}, \tilde{\eta}) \right),
\]

where

\[
T_{8,1}(\tilde{s}, \tilde{\eta}) = \int_{[0,T]^2} \mathbb{1}_{\{2\epsilon \leq r_1 \leq 2\epsilon \leq r_2\}} e^{-\frac{1}{2} \frac{n}{t_1,t_2} - \frac{1}{2} \frac{\alpha_{\eta_1,\eta_2}}{r_1,r_2,\tilde{s}}} \mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] d\tilde{r},
\]

and

\[
T_{8,2}(\tilde{s}, \tilde{\eta}) = \int_{[0,T]^2} \left( \mathbb{1}_{\{r_1 - r_2 \leq \epsilon\}} \mathbb{1}_{\{r_1 \leq 2\epsilon\}} \mathbb{1}_{\{r_1 \leq r_2\}} e^{-\frac{1}{2} \frac{n}{t_1,t_2} - \frac{1}{2} \frac{\alpha_{\eta_1,\eta_2}}{r_1,r_2,\tilde{s}}} \mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] d\tilde{r}.\]

Estimation of \( T_{8,1}(\tilde{s}, \tilde{\eta}) \): We easily check that

\[
\mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})] = \mathbb{E}[e^{-i\tilde{\eta}(B_{r_1,r_1} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} - B_{r_2,r_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1})}] \mathbb{E}[e^{i\tilde{\eta}(B_{s_1,s_1} - B_{s_2,s_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1})}] \mathbb{E}[e^{i\tilde{\eta}(B_{r_1,s_1} - B_{r_2,s_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1}))}].
\]

We can thus write

\[
|\mathbb{E}[\Phi(\tilde{r}, \tilde{s}, \tilde{\eta})]| \leq R_{1}^{n} + R_{2}^{n},
\]

where

\[
R_{1}^{n} = \left| \exp \left( -\frac{1}{2} \text{Var} \left[ \eta B_{r_1,r_1} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} - \eta B_{r_2,r_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} + \beta_{r_1,s_1,\eta} \beta_{r_2,s_2,\eta} \right] \right) \right|,
\]

\[
- \exp \left( -\frac{1}{2} \text{Var} \left[ \eta B_{r_1,r_1} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} - \eta B_{r_2,r_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} + \beta_{r_1,s_1,\eta} \beta_{r_2,s_2,\eta} \right] \right)
\]

and

\[
R_{2}^{n} = \left| \exp \left( -\frac{1}{2} \text{Var} \left[ \eta B_{r_1,r_1} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} - \eta B_{r_2,r_2} + \frac{\alpha_{\eta_1,\eta_2}}{r_{1},r_{2},r_1} + \beta_{r_1,s_1,\eta} \beta_{r_2,s_2,\eta} \right] \right) \right|.
\]
and
\[ R_n^2 = \exp \left( -\frac{1}{2} \E \left[ (\beta_{r_1, \tilde{s}_\eta}^{(n)})^2 \right] \right) \exp \left( -\frac{1}{2} \Var \left[ \tilde{\eta}B_{r_1, r_1+\frac{2}{n}} - \tilde{\eta}B_{r_2, r_2+\frac{2}{n}} - \beta_{r_2, \tilde{s}_\eta}^{(n)} \right] \right) - \exp \left( -\frac{1}{2} \Var \left[ \tilde{\eta}B_{r_1, r_1+\frac{2}{n}} - \tilde{\eta}B_{r_2, r_2+\frac{2}{n}} + \beta_{r_2, \tilde{s}_\eta}^{(n)} \right] \right), \]

where \( \beta_{r_2, \tilde{s}_\eta}^{(n)} \) is an independent copy of \( \beta_{r_2, \tilde{s}_\eta}^{(n)} \). Thus, by the mean value theorem,
\[ R_1^n \leq \left| \Var[\tilde{\eta}B_{r_1, r_1+\frac{2}{n}} - \tilde{\eta}B_{r_2, r_2+\frac{2}{n}} + \beta_{r_1, \tilde{s}_\eta}^{(n)} - \beta_{r_2, \tilde{s}_\eta}^{(n)}] \right| \\
= 2\left| \E[(\tilde{\eta}B_{r_1, r_1+\frac{2}{n}} - \tilde{\eta}B_{r_2, r_2+\frac{2}{n}} + \beta_{r_1, \tilde{s}_\eta}^{(n)} + \beta_{r_2, \tilde{s}_\eta}^{(n)})\beta_{r_1, \tilde{s}_\eta}^{(n)}] \right| \\
= 2n^H|\eta|\E[(\tilde{\eta}B_{r_1, r_1+\frac{2}{n}} - \tilde{\eta}B_{r_2, r_2+\frac{2}{n}} + n^H\eta(B_{r_1, r_1+\frac{2}{n}} - B_{r_2, r_2+\frac{2}{n}}))(B_{r_2, r_2+\frac{2}{n}} - B_{r_2, r_2+\frac{2}{n}})], \]

where the last identity follows from (3.15). Applying Lemma 5.8 and the fact that \( |\tilde{\eta}| \leq m \) and \( 2\varepsilon \leq r_1 + \varepsilon \leq r_2 \), we thus obtain
\[ R_1^n \leq C\varepsilon n^{2H-\frac{2}{7}}(s_1 - s_2)^2 + n^{-1}|\eta||s_1 - s_2| + n^{H-\frac{2}{7}}|\eta|^2(s_1 - s_2)^{H+\frac{2}{7}}). \]

Similarly, we can show that
\[ R_2^n \leq C\varepsilon n^{H-1}|\eta||s_1 - s_2|. \]

From here we conclude that
\[ T_{8,1}(\tilde{\eta}, \tilde{s}) \leq C\varepsilon \int_{[0,T]^2} \exp \left( -\frac{1}{2} (\alpha_{r_1, \tilde{s}_\eta}^{(n)} + \alpha_{r_2, \tilde{s}_\eta}^{(n)}) \right) \\
\times (n^{2H-\frac{2}{7}}(s_1 - s_2)^2 + n^{-1}|\eta||s_1 - s_2| + n^{H-\frac{2}{7}}|\eta|^2(s_1 - s_2)^{H+\frac{2}{7}}) d\tilde{r}, \]

On the set \( \{|\tilde{\eta}| \leq n^{-H}n - \eta| \leq |\tilde{\eta}| \} \), in view of the estimate (3.32), we have
\[ \exp \left( -\frac{1}{2} (\alpha_{r_1, \tilde{s}_\eta}^{(n)} + \alpha_{r_2, \tilde{s}_\eta}^{(n)}) \right) \leq \exp \left( -\delta\varepsilon^2 (s_1^{2H} + s_2^{2H}) \right). \]

Thus, from (3.52) and (3.53), we get
\[ T_{8,1}(\tilde{\eta}, \tilde{s}) \leq C\varepsilon \exp \left( -\delta\varepsilon^2 (s_1^{2H} + s_2^{2H}) \right) \\
\times (n^{2H-\frac{2}{7}}(s_1 - s_2)^2 + n^{-1}|\eta||s_1 - s_2| + n^{H-\frac{2}{7}}|\eta|^2(s_1 - s_2)^{H+\frac{2}{7}}). \]

**Estimation of** \( T_{8,2}(\tilde{\eta}, \tilde{s}) \). In view of (3.53), we have
\[ T_{8,2}(\tilde{\eta}, \tilde{s}) \leq C\varepsilon \exp \left( -\delta\varepsilon^2 (s_1^{2H} + s_2^{2H}) \right). \]

Thus, by (3.54) and (3.55), we can write
\[ T_6(\tilde{\eta}, \tilde{s}) \leq e^{-\delta(s_1^{2H} + s_2^{2H})}\varepsilon^2 \left( C\varepsilon^2 + C\varepsilon (n^{H-1}|\eta||s_2 - s| + n^{\frac{H+2}{2}}|\eta|^2|s - s|^{1/2}) \right. \\
\left. + n^{-\frac{2}{7}}|\eta||s_2 - s|^{1/2} + n^{-\frac{2}{7}}|\eta||s - s|^{1/2}) \right). \]
Therefore
\[ A_s^{(n,m)} \leq \int_{\mathbb{R}^3} \int_{[0,nT]^2} (s_1 s_2)^{H - \frac{1}{2}} \eta^2 e^{-\delta(s_1^2 + s_2^2)} \eta^2 (1 \wedge |\eta x|)(1 \wedge |\eta \bar{x}|) |\bar{f}(x)g(\bar{x})| \]
\[ \times \left( C \epsilon \frac{1}{\eta} + C \epsilon (n^{H-1} |\eta||s_2 - s| + n^{\frac{H-1}{2}} |\eta|^{\frac{1}{2}} |s - s_2|^{\frac{1}{2}} \right) d\bar{s} \eta d\bar{x}. \]

By Lemma 5.9, the previous quantity is bounded by
\[ \limsup_{n \to \infty} \frac{\ell^2}{n} A_s^{(n,m)} \leq C \epsilon \frac{1}{\eta}. \]

Relation (3.21) follows from (3.48), (3.49), (3.51) and (3.58).

The quantity (3.57) can be bounded by distinguishing the cases \( s_1 \leq s_2 \) and \( s_1 \geq s_2 \), leading to
\[ \limsup_{n \to \infty} \frac{\ell^2}{n} A_2^{(n,m)} \leq C \epsilon \frac{1}{\eta}. \]

Step IV
Next we prove that
\[ \lim_{m \to \infty} \sup_n \frac{\ell^2}{n} \left\| \int_0^u A_s^{(n,m)} \, dr \right\|_{L^2(\Omega)} \to 0. \]

From (3.19) and the estimates (3.32) and (3.33), we can write, using Minkowski’s inequality,
\[ \left\| \int_0^u A_s^{(n,m)} \, dr \right\|_{L^2(\Omega)} \leq C \int_{[0,nT]^2} \int_{\mathbb{R}^3} \left( s_1 s_2 \right)^{H - \frac{1}{2}} |\bar{f}(x)g(\bar{x})|(1 \wedge |\eta x|)(1 \wedge |\eta \bar{x}|) \]
\[ \times \left\| \lambda_s^{(n,m)} (\eta, \bar{s}) \right\|_{L^2(\Omega)} d\eta d\bar{x} d\bar{s}, \]
where
\[ \lambda_s^{(n,m)} (\eta, \bar{s}) := \int_0^u \int_{\mathbb{R} \setminus [-m,m]} 1_{\{|n - \eta \bar{s} - |\eta| \leq |\eta|\}} (n^{-H} \eta - \eta) \eta \lambda_r^{(n)} e^{-2(n - \eta \bar{s}) - 4(n - \eta \bar{s})^{-\ell}} d\eta dr. \]
Recall that we are assuming 
\[ r = (3.64) \]
\[ r = (3.65) \]
\[ r = (3.63) \]

Let us introduce the following notation \[ \Phi_1(r, s, \eta) := \mathbb{E}[e^{-\eta B_{r_1} - \eta B_{r_2} + \eta B_{s_1} + \eta B_{s_2}} | \beta_{r_1}, \beta_{r_2}, \beta_{s_1}, \beta_{s_2}, \eta] \]

Recall that we are assuming \( r_1 \leq r_2 \). We distinguish the following three cases:

(i) \( r_1 + \frac{s_1 + s_2}{n} < r_2 + \frac{s_1^\Lambda/s_2}{n} \),
(ii) \( r_1 + \frac{s_1 + s_2}{n} \geq r_2 + \frac{s_1^\Lambda/s_2}{n} \) and \( \frac{|s_1 - s_2|}{n} \leq 2(r_2 - r_1) \),
(iii) \( r_1 + \frac{s_1 + s_2}{n} \geq r_2 + \frac{s_1^\Lambda/s_2}{n} \) and \( \frac{|s_1 - s_2|}{n} > 2(r_2 - r_1) \).

We begin with the case (i). By equation (5.28) in Lemma 5.7 and Lemma 5.5,

\[ \Phi_1(r, s, \eta) \leq e^{-\delta^2 \eta^2} \exp \left( -\delta \mathbb{E}[\overline{\eta} B_{r_1} + \overline{\eta} B_{r_2} - \overline{\eta} B_{s_1} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \times \exp \left( -\delta^2 \eta^2 \mathbb{E}[\overline{\eta} B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \right) \]

\[ \times \exp \left( -\delta^2 \eta^2 \mathbb{E}[\overline{\eta} B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \right) \]

By the local non-determinism property of \( B \) (see (2.11)), when \( r_1 + \frac{s_1 + s_2}{n} < r_2 + \frac{s_1^\Lambda/s_2}{n} \) we have the inequalities

\[ n^{2H} \mathbb{E}[B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \geq \delta |s_2 - s_1|^2H. \]

and

\[ n^{2H} \mathbb{E}[B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \geq \delta |s_2 - s_1|^2H. \]

Now we handle the case (ii), namely, when \( r_1 + \frac{s_1 + s_2}{n} \geq r_2 + \frac{s_1^\Lambda/s_2}{n} \) and \( \frac{|s_1 - s_2|}{n} \leq 2(r_2 - r_1) \). By equation (5.30) in Lemma 5.7 and Lemma 5.5,

\[ \Phi_1(r, s, \eta) \leq e^{-\delta^2 \eta^2} \times \exp \left( -\delta \mathbb{E}[\overline{\eta} B_{r_1} + \overline{\eta} B_{r_2} - \overline{\eta} B_{s_1} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \times \exp \left( -\delta^2 \eta^2 \mathbb{E}[\overline{\eta} B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \right) \]

Applying the local non-determinism of \( B \) (see (2.11)) and taking into account that \( r_1 + \frac{s_1 + s_2}{n} \geq r_2 + \frac{s_1^\Lambda/s_2}{n} \) and \( \frac{|s_1 - s_2|}{n} \geq 2(r_2 - r_1) \), we obtain

\[ \mathbb{E}[B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \geq \delta (r_2 - r_1)^2H \geq \delta^2 |s_1 - s_2|^2H n^{-2H}. \]

Finally, we handle the case (iii), namely, when \( r_1 + \frac{s_1 + s_2}{n} \geq r_2 + \frac{s_1^\Lambda/s_2}{n} \) and \( \frac{|s_1 - s_2|}{n} > 2(r_2 - r_1) \). Notice that, by the local non-determinism property of the process \( B \) (see (2.11)),

\[ \mathbb{E}[B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \mid B_{r_1} + \overline{\eta} B_{r_2} - B_{s_2} \right) \]
\[
\var{B_{r_2 + \frac{s_2}{n}} - B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}}}
\geq \delta |r_2 - r_1| \frac{|s_2 - s_1|}{n} 2^H,
\]

which leads to the estimate

\[ (3.66) \quad \var{B_{r_2 + \frac{s_2}{n}} - B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}}} \geq \frac{\delta}{2} |s_1 - s_2|^2 n^{-2^H}. \]

From (3.62), (3.63), (3.64), (3.65) and (3.66), we obtain

\[
\Phi_1(r^*, s^*, \tilde{\eta}) \leq e^{-\delta (s_1^2 H + |s_1 - s_2|^2 |H^2)} \eta^2 \times \exp \left( -\delta \var{\tilde{\eta} B_{r_1 + \frac{s_1}{n}} - \tilde{\eta} B_{r_2 + \frac{s_2}{n}} | B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}} - B_{r_2 + \frac{s_2}{n}}} \right). \]

Next we bound from below the variance appearing in the right-hand side of the above expression. Let \( \Sigma \) denote the covariance matrix of \( B_{r_1 + \frac{s_1}{n}} - B_{r_1 + \frac{s_1}{n}} \) and \( B_{r_2 + \frac{s_2}{n}} - B_{r_2 + \frac{s_2}{n}} \). Define also the matrix

\[
\tilde{\Sigma} := \Sigma^{-1} = \begin{pmatrix} \Sigma_{1,2} & -\Sigma_{1,2} \\ -\Sigma_{1,2} & \Sigma_{1,1} \end{pmatrix}.
\]

Notice that

\[
e^{-\delta \var{\tilde{\eta} B_{r_1 + \frac{s_1}{n}} - \tilde{\eta} B_{r_2 + \frac{s_2}{n}} | B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}} - B_{r_2 + \frac{s_2}{n}}} = (\det \Sigma)^{-\frac{1}{2}} \var{\tilde{\eta} B_{r_1 + \frac{s_1}{n}} - \tilde{\eta} B_{r_2 + \frac{s_2}{n}} | B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}} - B_{r_2 + \frac{s_2}{n}}},
\]

where \( p_{\tilde{\Sigma}} \) denotes the probability density of a centered Gaussian random vector \( (N_1, N_2) \) in \( \mathbb{R}^2 \), with covariance \( \tilde{\Sigma} \). From here it follows that

\[
\int_{\mathbb{R}^2 \setminus [-m, m]^2} e^{-\delta \var{\tilde{\eta} B_{r_1 + \frac{s_1}{n}} - \tilde{\eta} B_{r_2 + \frac{s_2}{n}} | B_{r_1 + \frac{s_1}{n}}, B_{r_1 + \frac{s_1}{n}}, B_{r_2 + \frac{s_2}{n}} - B_{r_2 + \frac{s_2}{n}}} \, d\tilde{\eta} d\tilde{\eta}
\leq \frac{1}{2\delta} (\det \Sigma)^{-\frac{1}{2}} \mathbb{P} \{ N_1 \geq \sqrt{2\delta m}, N_2 \geq \sqrt{2\delta m} \}
\leq \frac{1}{2\delta} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{2\delta |\Sigma|}{2^H}},
\]

where the last inequality follows from Chebyshev inequality. Notice that

\[
\Sigma_{1,2} \leq (2T)^{2^H}.
\]

Therefore, for any \( \gamma > 0 \) such that \( (\frac{1}{2} + \gamma) H < 1 \), we can write

\[ (3.67) \quad \mathbb{E} \left[ |\lambda_{3, \gamma}^{(n, m)}(\eta, s)|^2 \right] \leq C m^{-2^H} e^{-\delta (s_1^2 H + |s_1 - s_2|^2 |H^2)} \eta^2 \int_0^u \int_0^{2^H} (\det \Sigma)^{-\gamma} \, d\eta \, dr_1 \, dr_2.
\]

We claim that

\[ (3.68) \quad \sup_{n \geq 1} \sup_{s_1, s_2 \in [0, nT]} \int_0^u \int_0^{2^H} (\det \Sigma)^{-\gamma} \, d\eta \, dr_1 \, dr_2 < \infty.
\]

This follows easily from Lemma 5.2, taking \( a = r_1 + \frac{s_1}{n}, b = r_1 + \frac{s_2}{n} \) and \( h = \frac{s_1 - s_2}{n} \). Indeed we get the upper bound

\[
\det \Sigma \leq \delta \left( r_1 + \frac{s_2}{n} \right)^{2^H} \left( r_2 - r_1 - \frac{s_1 - s_2}{n} \right)^{2^H} 1_{\{ \frac{s_1 - s_2}{n} < r_2 - r_1 \}}
\]

\[
+ \delta \left( r_1 + \frac{s_2}{n} \right)^{2^H} \left( r_2 - r_1 \right) \wedge \left( \frac{s_1 - s_2}{n} - (r_2 - r_1) \right)^{2^H} 1_{\{ \frac{s_1 - s_2}{n} > r_2 - r_1 \}}.
\]
Applying Lemma 5.9 as in the previous steps, we can deduce that
\[ E \]
Relation (3.59) is a consequence of Lemma 5.10.

Notice that by Jensen’s inequality,
\[
(3.69)
\]
\[
(3.70)
\]
and
\[
\]
Making the change of variables \( r_1 = x \) and \( r_2 - r_1 = y \) the claim (3.68) follows.

Therefore, by (3.60), (3.67) and (3.68), we obtain
\[
\left\| \int_0^t \Lambda_{3,r}^{(n,m)} \, dr \right\|_{L^2(\Omega)} \leq C m^{-2\gamma} \int_{[0,n]^2} \int_{\mathbb{R}^3} (s_1 s_2)^{H - \frac{1}{2}} e^{-\delta(s_1^H + |s_2 - s_1|^{2H})} |\hat{f}(x)\hat{f}(\bar{x})| \
\times (1 \land |\eta x|)(1 \land |\eta \bar{x}|) \, dy \, d\bar{y}.
\]
Applying Lemma 5.9 as in the previous steps, we can deduce that
\[
\left\| \int_0^t \Lambda_{3,r}^{(n,m)} \, dr \right\|_{L^2(\Omega)} \leq C m^{-2\gamma} \|f\|_1^2 \int_{[0,n]^2} \int_{\mathbb{R}^3} (s_1 s_2)^{H - \frac{1}{2}} (s_1^{2H} + |s_2 - s_1|^{2H})^{-\frac{3}{2}} \
\times (1 \lor \sqrt{s_1^{2H} + |s_2 - s_1|^{2H}})^{-1} \, ds \, dr.
\]
Relation (3.59) is a consequence of Lemma 5.10.

3.2. Proof of (A2). We can easily prove that
\[
\mathbb{E} \left[ n^{1-H} \left( \int_0^u F_{r,t}^{(n)} \, dr \right)^2 \right] = -\frac{n^{-3H}}{4\pi^2} \int_{[0,u]^2} \int_0^{n(t-r_1)} \int_0^{n(t-r_2)} \int_{\mathbb{R}^3} (s_1 s_2)^{H - \frac{1}{2}} \times f(x) f(\bar{x}) \xi e^{-\frac{1}{2} \hat{\alpha}_{r,x}^{(n)}} e^{-\frac{1}{2} \hat{\alpha}_{r,\xi}^{(n)}} \mathbb{E} \left[ e^{-\xi \left( \hat{B}_{r_1} + \frac{\xi^2}{n} \right) - \xi \left( \hat{B}_{r_2} + \frac{\xi^2}{n} \right) - \left( \xi \hat{N}_1 + \hat{\xi} \hat{N}_2 \right)} \right] \
\times \left( e^{\frac{1}{2} \hat{\alpha}_{r,x}^{(n)}} - 1 \right) \left( e^{\frac{1}{2} \hat{\alpha}_{r,\xi}^{(n)}} - 1 \right) \hat{K}_{r,x}^{(n)} d\xi \, d\xi d\hat{d}r,
\]
where
\[
\hat{K}_{r,x}^{(n)} := n^{2H - 1} (s_1 s_2)^{\frac{1}{2} - H} K_H \left( r_1 + \frac{s_1}{n}, r_1 \right) K_H \left( r_2 + \frac{s_2}{n}, r_2 \right)
\]
and
\[
(3.69)
\]
\[
(3.70)
\]
To estimate the expectation in the right-hand side of (3.69), we define the random variables
\[
N_1 := B_{r_1} + \frac{\xi^2}{n}, \quad N_2 := B_{r_2} + \frac{\xi^2}{n}, \quad N_3 := B_{r_1} - B_{r_2} + \frac{\xi^2}{n}, \quad N_4 := B_{r_2} - B_{r_2} + \frac{\xi^2}{n}.
\]
Using (2.9) and (3.70), we can write
\[
\mathbb{E} \left[ e^{-\xi \left( \hat{B}_{r_1} + \frac{\xi^2}{n} \right) - \xi \left( \hat{B}_{r_2} + \frac{\xi^2}{n} \right) - \left( \xi \hat{N}_1 + \hat{\xi} \hat{N}_2 \right)} \right] = e^{-\frac{1}{2} \text{Var}[\xi N_1 + \hat{\xi} N_2]},
\]
and
\[
e^{-\frac{1}{2} \hat{\alpha}_{r,x}^{(n)}} = e^{-\frac{1}{2} \text{Var}[\xi N_3 + \hat{\xi} N_4 + \xi N_1 + \hat{\xi} N_2]},
\]
Notice that by Jensen’s inequality,
\[
\text{Var}[\xi B_{r_1} + \frac{\xi^2}{n} + \hat{\xi} B_{r_2} + \frac{\xi^2}{n}] = \text{Var}[\xi N_3 + \hat{\xi} N_4 + \xi N_1 + \hat{\xi} N_2]
\]
\[
\leq 3 (\text{Var}[\xi N_3] + \text{Var}[\xi N_4] + \text{Var}[\xi N_1 + \hat{\xi} N_2])
\]
\[
= 3 (\hat{\alpha}_{r,x}^{(n)} + \text{Var}[\xi N_1 + \hat{\xi} N_2]),
\]
and consequently, 
\[
\xi^2 \text{Var}[B_{r_2 + \frac{\hat{s}_2}{n}} \mid B_{r_1 + \frac{\hat{s}_1}{n}}] + \xi^2 \text{Var}[B_{r_1 + \frac{\hat{s}_1}{n}} \mid B_{r_2 + \frac{\hat{s}_2}{n}}] \\
\leq 6(\alpha^*(n)_{r,s,\xi} + \text{Var}[\xi N_1 + \xi N_2]).
\]

The above inequality combined with Lemma 5.5 implies the existence of a constant \(\delta > 0\) such that
\[
\mathbb{E}\left[\int_0^u F_{r,t}^{(f,n)} dr\right]^2 \leq C \int_{[0,t]^2} \int_{[0,n]^2} |\xi||\xi|(s_1 s_2)^{H-\frac{1}{2}} |f(x)f(\bar{x})| \\
\times (1 \wedge |n^{-H}\xi x|) (1 \wedge |n^{-H}\bar{\xi} \bar{x}|) e^{-\delta n^{-2H}(s^2 + \hat{s}_1^2 + \hat{s}_2^2)} \\
\times e^{-\delta \xi^2 (r_1 + \frac{1}{n})^2 |r_2 - r_1 + \frac{2s_1}{n}|^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H} |r_2 - r_1 + \frac{2s_1}{n}|^{2H}} d\xi d\bar{\xi} d\bar{x} d\bar{x}.
\]

By the local non-determinism property of \(B\), we thus conclude that
\[
\mathbb{E}\left[\int_0^u F_{r,t}^{(f,n)} dr\right]^2 \\
\leq C \int_{[0,t]^2} \int_{[0,n]^2} (s_1 s_2)^{H-\frac{1}{2}} (n^{-2H} s_1^{2H} + (r_1 + \frac{s_1}{n})^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H})^{-1} \\
\times (1 + n^{2H} (s_1^{2H} + (r_1 + \frac{s_1}{n})^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H})^{-\frac{1}{2}} \\
\times (1 + n^{2H} (s_1^{2H} + (r_1 + \frac{s_1}{n})^{2H} |r_2 - r_1 + \frac{2s_2}{n}|^{2H})^{-\frac{1}{2}} d\bar{s} d\bar{r}.
\]

Thus, by changing variables \((r_1, r_2)\) by \((r_1 + \frac{s_1}{n}, r_2 + \frac{s_2}{n})\), we obtain
\[
\mathbb{E}\left[\int_0^u F_{r,t}^{(f,n)} dr\right]^2 \\
\leq C \int_{[0,t]^2} \int_{[0,n]^2} (s_1 s_2)^{H-\frac{1}{2}} (n^{-2H} s_1^{2H} + r_1^{2H} |r_2 - r_1|^{2H})^{-1} \\
\times (1 + n^{2H} (s_1^{2H} + r_1^{2H} |r_2 - r_1|^{2H})^{-\frac{1}{2}} \\
\times (1 + n^{2H} (s_1^{2H} + r_1^{2H} |r_2 - r_1|^{2H})^{-\frac{1}{2}} d\bar{s} d\bar{r}.
\]

By the symmetry on \(r_1\) and \(r_2\), we can assume without loss of generality that \(r_1 \leq r_2\). Thus, by changing the coordinates \((r_1, r_2)\) by \((r := r_1, z := r_2 - r_1)\), we obtain the inequality
\[
\mathbb{E}\left[\int_0^u F_{r,t}^{(f,n)} dr\right]^2 \\
\leq C \int_{[0,t]^2} \int_{[0,n]^2} (s_1 s_2)^{H-\frac{1}{2}}
\]

The right-hand side converges to zero due to the condition

\[ H \leq \frac{1}{2} \]

Using the geometric mean/arithmetic mean inequality we deduce the existence of a constant \( C > 0 \) such that

\[
T_n^1 := C n^{-3H} \int_{[0,2t]^2} \int_{[0,nt]^2} (s_1 s_2)^{H - \frac{1}{2}} \left( n^{-2H} s_1^{2H} + r^{2H} \right)^{-1} (1 + n^{-2H} s_1^{2H})^{-1/2} \\
\times (n^{-2H} s_2^{2H} + z^{2H})^{-1} (1 + n^{-2H} s_2^{2H})^{-1/2} d\tilde{s} dr dz 
\]

(3.72)

\[
T_n^2 := C n^{-3H} \int_{[0,2t]^2} \int_{[0,nt]^2} (s_1 s_2)^{H - \frac{1}{2}} \left( n^{-2H} s_1^{2H} + z^{2H} \right)^{-1} (1 + n^{-2H} s_1^{2H})^{-1/2} \\
\times (n^{-2H} s_2^{2H} + z^{2H})^{-1} (1 + n^{-2H} s_2^{2H})^{-1/2} d\tilde{s} dr dz. 
\]

(3.73)

By distinguishing the cases \( r \leq z \) and \( z \leq r \), we get the bound

\[
\mathbb{E} \left[ n^{1-H} \left( \int_0^n F_{r,t}^{(f,n)} \right)^2 \right] \leq C (T_1^n + T_2^n),
\]

(3.71)

where

\[
T_1^n \leq C n^{-H} \int_{[0,2t]^2} \int_{[0,nt]^2} (s_1 s_2)^{-\frac{1}{2}} r^{-H} (1 + n^{-2H} s_1^{2H})^{-1/2} \\
\times z^{-H} (1 + n^{-2H} s_2^{2H})^{-1/2} d\tilde{s} dr dz 
\]

\[ = C (1 - H)^{-2} (2t)^{-2} n^{-H} \left( \int_{[0,nt]} s^{-\frac{1}{2}} (1 + n^{-2H} s^{2H})^{-1/2} ds \right)^2. \]

The convergence towards zero of the term \( T_1^n \) is easy to get from here. In order to handle the term \( T_2^n \), we set \( \alpha := \frac{5}{6} - \frac{1}{12H} \). Notice that \( \alpha \) lies in the interval \((0, 1)\) due to the condition \( H \geq \frac{1}{3} \). Consequently, by the generalized geometric mean/arithmetic mean inequality, we deduce that for every \( a, b > 0 \),

\[
(a^{2H} + b^{2H})^{-1} \leq Ca^{-2H} \alpha b^{2H(1-\alpha)} = Ca^{-\frac{2H}{3} + \frac{1}{6} - \frac{1}{3}}. 
\]

(3.74)

By first choosing \( a = n^{-2H} s_1 \), \( b = z \) and then \( a = n^{-2H} s_2 \), \( b = z \) in (3.74) and substituting the resulting inequalities in (3.73), we obtain the following bound for \( T_2^n \):

\[
T_2^n \leq C n^{\frac{1}{3}(H-1)} \int_{[0,2t]^2} \int_{[0,nt]^2} (z s_1 s_2)^{-\frac{2H}{3} - \frac{1}{3}} (1 + n^{-2H} s_1^{2H})^{-1/2} (1 + n^{-2H} s_2^{2H})^{-1/2} d\tilde{s} dr dz 
\]

\[ = \frac{3}{2} (1 - H)^{-1} (2t)^{\frac{1}{3}(0-2H)} C n^{\frac{1}{3}(H-1)} \left( \int_{[0,nt]} s^{-\frac{2H}{3} - \frac{1}{3}} (1 + n^{-2H} s^{2H})^{-1/2} ds \right)^2. \]

The right-hand side converges to zero due to the condition \( H \geq \frac{1}{3} \). Condition (A2) follows then from (3.71). The proof is now complete.

4. Proof of Theorem 1.4. For any fixed \( \lambda \in \mathbb{R} \), we put

\[
D_n(t) := n^H \int_0^t f(n^H (B_s - \lambda)) ds - L_t(\lambda) \int_\mathbb{R} f(x) dx - n^{-H} L_0(\lambda) \left( \int_\mathbb{R} y f(y) dy \right).
\]
Using the occupation measure formula and the change of variables \( n^H(x - \lambda) = y \), we can write
\[
n^H \int_0^t f(n^H(B_s - \lambda))ds = n^H \int \frac{1}{nH} f(n^H(x - \lambda)L_t(x))dx
\]
\[
= \int f(y)L_t(\frac{y}{nH} + \lambda)dy.
\]
As a consequence, we obtain
\[
D_n(t) = \int f(y)\Psi_n(y)dy,
\]
where
\[
\Psi_n(y) := L_t(\frac{y}{nH} + \lambda) - L_t(\frac{y}{nH}) - \frac{y}{nH}L_t(\lambda).
\]
Therefore,
\[
(4.1) \quad \mathbb{E}[n^{2H}D_n^2(t)] = n^{2H} \int \mathbb{E}[\Psi_n(y)\Psi_n(\tilde{y})]dyd\tilde{y}.
\]
Then, we can compute the expectation \( \mathbb{E}[\Psi_n(y)\Psi_n(\tilde{y})] \) using the Fourier representation of the local time and its derivative given in (2.7) and (2.8):
\[
\Psi_n(s,y) = \frac{1}{2\pi} \int_0^t \int e^{i\xi(B_s - \lambda)} \left[ e^{-i\frac{\xi y}{nH}} - 1 + i\frac{y}{nH} \right] d\xi ds
\]
and in this way, we obtain
\[
(4.2) \quad \mathbb{E}[\Psi_n(y)\Psi_n(\tilde{y})] = \frac{1}{4\pi^2} \int_\mathbb{R}^2 \int_{[0,1]^2} e^{-\frac{1}{2}(\tilde{\xi},\Sigma(\tilde{s})\tilde{\xi})} \left[ e^{-i\frac{\tilde{\xi} y}{nH}} - 1 + i\frac{y}{nH} \right] d\tilde{s}d\tilde{\xi},
\]
where \( \tilde{\xi} = (\xi,\tilde{\xi}), \tilde{y} = (y,\tilde{y}) \) and \( \tilde{s} = (s_1, s_2) \) and where \( \Sigma(\tilde{s}) \) denotes the covariance matrix of \( (B_{s_1}, B_{s_2}) \). Since \( H < \frac{1}{2} \), there exists \( 0 < \alpha < 1 \wedge \nu \), such that \( 3H + 2H\alpha < 1 \). For such \( \alpha \), we have that for all \( z \in \mathbb{R} \)
\[
|e^{iz} - 1 - iz| \leq |z|^{1+\alpha},
\]
which, combined with (4.1) and (4.2), gives
\[
(4.3) \quad \mathbb{E}[n^{2H}D_n^2(t)] \leq n^{-2\alpha H} \int_{[0,T]^2} |\xi\tilde{\xi}|^{1+\alpha}e^{-\frac{1}{2}(\tilde{\xi},\Sigma(\tilde{s})\tilde{\xi})}|y\tilde{y}|^{1+\alpha}|f(y)f(\tilde{y})|d\tilde{s}d\tilde{y}d\xi.
\]
By applying Lemma 5.1 in the Appendix of [6] to the right-hand side, we get that
\[
\mathbb{E}[n^{2H}D_n^2(t)] \leq n^{-2\alpha H} \int_{[0,T]^2} (s_1 \wedge s_2)^{-H(s_1 \wedge s_2 \wedge |s - s_2|)} -3H - 2\alpha H |y\tilde{y}|^{1+\alpha}|f(y)f(\tilde{y})|d\tilde{s}d\tilde{y}.
\]
Since \( 1 - 3H - 2H\alpha > 0 \), from here we conclude that
\[
(\mathbb{E}[n^{2H}D_n^2(t)])^{1/2} \leq Cn^{-\alpha H} \int |x|^{1+\alpha}|f(x)|dx.
\]
The result easily follows from here.
5. Technical lemmas. We start with the proof of Lemma 1.2, stated in Section 1.

**Proof.** (of Lemma 1.2) We use the inequality $|e^{ax} - 1| \leq 2(1 + |a|)$ for all $a \in \mathbb{R}$, to deduce that

$$|B_\eta[f, g]| \leq 4 \int_{\mathbb{R}^2} |f(x)g(\overline{x})|(1 \wedge |\eta x|)(1 \wedge |\eta \overline{x}|)d\overline{x}.$$  

Consequently,

$$\int_{\mathbb{R}_+^2} \int_{\mathbb{R}} |B_\eta[f, g]|D(s_1, s_2)^{H-\frac{1}{2}}e^{-\frac{1}{2}(\beta_{H, 2}(s_1^2H + s_2^2H) + \beta_{H, 3}(s_1, s_2))|\eta|^2}d\eta d\overline{s}$$

$$\leq 4 \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^3} |f(x)g(\overline{x})|(1 \wedge |\eta x|)(1 \wedge |\eta \overline{x}|)D(s_1, s_2)^{H-\frac{1}{2}}e^{-\frac{1}{2}(\beta_{H, 2}(s_1^2H + s_2^2H))|\eta|^2}d\overline{x}d\eta d\overline{s}.$$  

By Lemma 5.9 and taking into account that $f$ and $g$ belong to $\Xi_1$, the integral in the right-hand side is bounded by a constant multiple of

$$\|f\|_1\|g\|_1 \int_{\mathbb{R}_+^2} (s_1s_2)^{H-\frac{1}{2}}(s_1^{2H} + s_2^{2H})^{-\frac{1}{2}}(1 \wedge (s_1^{2H} + s_2^{2H}))^{-1}d\overline{s}$$

$$\leq 2\|f\|_1\|g\|_1 \int_{\mathbb{R}_+} \int_0^{s_2} (s_1s_2)^{H-\frac{1}{2}}s_2^{-3H}(1 \wedge s_2^{-2H})d\overline{s}$$

$$= \frac{4}{2H + 1}\|f\|_1\|g\|_1 \int_{\mathbb{R}_+} s_2^{-H}(1 \wedge s_2^{-2H})ds_2.$$  

The integral in the right-hand side is finite due to the condition $H > \frac{1}{3}$. The result follows from here. 

We now present a lemma containing a useful relationship between conditional variances with respect to different $\sigma$-algebras generated by a Gaussian vector.

**Lemma 5.1.** Let $(N_1, \ldots, N_r, A, B)$ be a non-degenerate Gaussian vector and denote by $\mathcal{F}_N, \mathcal{F}_A, \mathcal{F}_B$ the $\sigma$-algebras generated by $(N_1, \ldots, N_r), A$ and $B$ respectively. Then,

$$\text{Var}[B \mid \mathcal{F}_N \vee \mathcal{F}_A] = \frac{\text{Var}[A \mid \mathcal{F}_N \vee \mathcal{F}_B] \text{Var}[B \mid \mathcal{F}_N]}{\text{Var}[A \mid \mathcal{F}_N]}$$

**Proof.** The determinant of the covariance matrix of $(N_1, \ldots, N_r, A, B)$ can be written as

$$\text{Var}[N_1]\text{Var}[N_2|N_1] \cdots \text{Var}[N_r|N_1, \ldots, N_{r-1}]\text{Var}[A|\mathcal{F}_N]\text{Var}[B|\mathcal{F}_A \vee \mathcal{F}_N]$$

and, also as

$$\text{Var}[N_1]\text{Var}[N_2|N_1] \cdots \text{Var}[N_r|N_1, \ldots, N_{r-1}]\text{Var}[B|\mathcal{F}_N]\text{Var}[A|\mathcal{F}_B \vee \mathcal{F}_N].$$

This equality implies the result. 

The following lemma gives an upper bound for the determinant of the covariance matrix of a bidimensional vector built from the increments of the fractional Brownian motion $B$. 

LEMMA 5.2. Fix $0 < a < b$ and $h > a$. Let $\Sigma$ denote the covariance matrix of $(\Delta a B, \Delta b B)$, where $\Delta a B := B_{a+h} - B_a$ and $\Delta b B := B_{b+h} - B_b$. Then, there exists a constant $\delta > 0$ such that

$$\det \Sigma \geq \delta \begin{cases} a^{2H} (b - a - h)^{2H} & \text{if } 0 < h < b - a, \\ a^{2H} [(h - (b - a)) \wedge (b - a)]^{2H} & \text{if } b - a < h, \\ (a - |h|)^{2H} [(b - a - |h|) \wedge |h|]^{2H} & \text{if } h < 0 \text{ and } |h| < b - a, \\ (a - |h|)^{2H} [(|h| - (b - a)) \wedge (b - a)]^{2H} & \text{if } h < 0 \text{ and } |h| > b - a. \end{cases}$$

PROOF. We have the following formula for the determinant of $\Sigma$:

$$(5.1) \quad \det \Sigma = \text{Var}[B_a | \Delta a B, \Delta b B] \text{Var}[B_b | B_a, \Delta a B, \Delta b B].$$

Let $\Sigma_1$ denotes the covariance matrix of the random vector $(B_a, \Delta a B, \Delta b B)$ and let $\Sigma_2$ be the covariance matrix of the random vector $(\Delta a B, \Delta b B)$. Then,

$$\text{Var}[B_a | \Delta a B, \Delta b B] = \frac{\det \Sigma_1}{\det \Sigma_2} = \frac{\text{Var}[B_a] \text{Var}[\Delta a B] \text{Var}[B_a | \Delta a B] \text{Var}[\Delta b B] \text{Var}[B_a | \Delta a B, \Delta b B]}{\text{Var}[\Delta a B] \text{Var}[B_a | \Delta a B] \text{Var}[\Delta b B] \text{Var}[B_a | \Delta a B, \Delta b B]}.$$

We distinguish several cases:

Case (i) Suppose $a < a + h < b < b + h$. Consider the random variables $A_1 = \Delta a B$, $A_2 = B_b - B_{a+h}$ and $A_3 = \Delta b B$. Then, by the local nondeterminism property of $B$, we can write

$$\text{Var}[B_a | \Delta a B, \Delta b B] \geq \text{Var}[B_a | A_1, A_2, A_3] \geq \frac{\text{Var}[B_a] \text{Var}[A_1] \text{Var}[A_2] \text{Var}[A_3 | A_1, A_2]}{\text{Var}[A_1] \text{Var}[A_2] \text{Var}[A_3 | A_1, A_2]} \geq \delta a^{2H}.$$ 

Consider now the second factor in the right-hand side of (5.1). Using the local nondeterminism property (2.11) of $B$, we obtain

$$\text{Var}[B_b | B_a, \Delta a B, \Delta b B] = \frac{\text{Var}[\Delta b B | B_a, \Delta a B, B_b] \text{Var}[B_b | B_a, \Delta a B]}{\text{Var}[\Delta b B | B_a, \Delta a B]} \geq \delta \frac{h^{2H} (b - a - h)^{2H}}{h^{2H}} = \delta (b - a - h)^{2H}.$$ 

Case (ii) Suppose $a < b < a + h < b + h$. Proceeding as in case (i), but with the random variables $A_1 = B_b - B_a$, $A_2 = B_{a+h} - B_b$ and $A_3 = B_{b+h} - B_{a+h}$ we obtain

$$\text{Var}[B_a | \Delta a B, \Delta b B] \geq \text{Var}[B_a | A_1, A_2, A_3] \geq \delta a^{2H}.$$ 

Consider now the second factor in the right-hand side of (5.1). Using (2.11), we obtain

$$\text{Var}[B_b | B_a, \Delta a B, \Delta b B] = \frac{\text{Var}[\Delta b B | B_a, \Delta a B, B_b] \text{Var}[B_b | B_a, \Delta a B]}{\text{Var}[\Delta b B | B_a, \Delta a B]} \geq \delta \frac{(b - a)^{2H} ((a + h - b) \wedge (b - a))^{2H}}{\text{Var}[\Delta b B | B_a, \Delta a B]}.$$ 

We can get the following upper bound for the denominator of the above expression:

$$\text{Var}[\Delta b B | B_a, \Delta a B] \leq 2 \text{Var}[B_{b+h} - B_{a+h} | B_a, B_{a+h}] + 2 \text{Var}[B_{a+h} - B_b | B_a, B_{a+h}] \leq 2 (b - a)^{2H} + 2 \text{Var}[B_b - B_a | B_a, B_{a+h}] \leq 4 (b - a)^{2H}.$$
As a consequence,
\[ \text{Var}[B_b \mid B_a, \Delta_a B, \Delta_b B] \geq \delta ((a + h - b) \land (b - a))^{2H}. \]

Case (iii) Suppose \( a + h < a < b + h < b \). Proceeding as in case (i), but with the random variables \( A_1 = B_a - B_{a+h}, A_2 = B_{b+h} - B_a \) and \( A_3 = B_b - B_{b+h} \) we obtain
\[ \text{Var}[B_a \mid \Delta_a B, \Delta_b B] = \text{Var}[B_{a+h} \mid \Delta_a B, \Delta_b B] \geq \text{Var}[B_{a+h} \mid A_1, A_2, A_3] \geq \delta (a + h)^{2H}. \]

Consider now the second factor in the right-hand side of (5.1). Using (2.11) we obtain
\[ \text{Var}[B_a \mid B_a, \Delta_a B, \Delta_b B] = \frac{\text{Var}[\Delta_a B \mid B_a, \Delta_a B, B_b] \text{Var}[B_b \mid B_a, \Delta_a B]}{\text{Var}[\Delta_a B \mid B_a, \Delta_a B]} \geq \delta \frac{(b - |h| - a) \land |h|^{2H} (b - a)^{2H}}{\text{Var}[\Delta_a B \mid B_a, \Delta_a B]}. \]

We can get the following upper bound for the denominator of the above expression:
\[ \text{Var}[\Delta_a B \mid B_a, \Delta_a B] \leq |h|^{2H} \leq (b - a)^{2H}. \]

As a consequence,
\[ \text{Var}[B_b \mid B_a, \Delta_a B, \Delta_b B] \geq \delta ((b - |h| - a) \land |h|)^{2H}. \]

Case (iv) Suppose \( a + h < b + h < a < b \). Proceeding as in case (i), but with the random variables \( A_1 = B_{b+h} - B_{a+h}, A_2 = B_a - B_{b+h} \) and \( A_3 = B_b - B_a \) we obtain
\[ \text{Var}[B_a \mid \Delta_a B, \Delta_b B] = \text{Var}[B_{a+h} \mid \Delta_a B, \Delta_b B] \geq \text{Var}[B_{a+h} \mid A_1, A_2, A_3] \geq \delta (a + h)^{2H}. \]

Consider now the second factor in the right-hand side of (5.1). Using (2.11) we obtain
\[ \text{Var}[B_a \mid B_a, \Delta_a B, \Delta_b B] = \frac{\text{Var}[\Delta_a B \mid B_a, \Delta_a B, B_b] \text{Var}[B_b \mid B_a, \Delta_a B]}{\text{Var}[\Delta_a B \mid B_a, \Delta_a B]} \geq \delta \frac{(a - b - |h|) \land (b - a)^{2H}}{\text{Var}[\Delta_a B \mid B_a, \Delta_a B]}. \]

We can get the following upper bound for the denominator of the above expression:
\[ \text{Var}[\Delta_a B \mid B_a, \Delta_a B] \leq 2\text{Var}[B_{b+h} - B_{a+h} \mid B_a, \Delta_a B] + 2\text{Var}[B_b - B_a \mid B_a, \Delta_a B] \leq 4(b - a)^{2H}. \]

As a consequence,
\[ \text{Var}[B_b \mid B_a, \Delta_a B, \Delta_b B] \geq \delta ((a - b - |h|) \land (b - a))^{2H}. \]

The next lemma gives lower and upper bounds for the kernel \( K_H(t, s) \):

**Lemma 5.3.** For all \( 0 < H < 1 \) and \( 0 \leq s \leq t \), the following bounds hold:

(i) If \( H > \frac{1}{2} \),
\[ C_H(H - 1/2)^{-1}(t - s)^{H - \frac{1}{2}} \leq K_H(t, s) \leq \left( \frac{t}{s} \right)^{H - \frac{1}{2}} C_H(H - 1/2)^{-1}(t - s)^{H - \frac{1}{2}}. \]
(ii) If $H \leq \frac{1}{2}$,

\begin{equation}
C_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \frac{1/2 - H}{1/2 + H} t^{-1} \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H+\frac{1}{2}} \right] \leq K_H(t, s) \leq C_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \frac{1/2 - H}{1/2 + H} s^{-1} (t-s)^{H+\frac{1}{2}} \right].
\end{equation}

**Proof.** The inequalities are trivial in the case $H = \frac{1}{2}$, so we only consider the cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

**Case $H > \frac{1}{2}$:** In this case, by (2.1) we have

\[ K_H(t, s) \geq C_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} s^{\frac{1}{2}} du = C_H (H - 1/2)^{-1} (t-s)^{H-\frac{1}{2}}, \]

as required. For the upper bound, we observe that

\[ K_H(t, s) = C_H s^{\frac{1}{2}-H} \int_0^{l-s} u^{H-\frac{3}{2}} (s+u)^{H-\frac{1}{2}} du \leq C_H s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} \int_0^{l-s} u^{H-\frac{3}{2}} du = \left( \frac{t}{s} \right)^{H-\frac{1}{2}} C_H (H - 1/2)^{-1} (t-s)^{H-\frac{1}{2}}. \]

Relation (5.2) follows from here.

**Case $H < \frac{1}{2}$:** We use (2.2) to write

\[ K_H(t, s) = C_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \left( \frac{1}{2} - H \right) s^{\frac{1}{2}-H} \int_0^{l-s} (s+u)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right]. \]

The integral of the second term in parenthesis in the right-hand side can be bounded as follows

\[ (H + \frac{1}{2})^{-1} t^{H-\frac{3}{2}} (t-s)^{H+\frac{1}{2}} = \int_0^{l-s} t^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \leq \int_0^{l-s} (s+u)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \leq \int_0^{l-s} s^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du = (H + \frac{1}{2})^{-1} s^{H-\frac{3}{2}} (t-s)^{H+\frac{1}{2}} \]

Relation (5.3) thus follows by adding $\left( \frac{1}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}}$ in both sides of the previous inequality, and then multiplying by $C_H$.

The following estimates for the kernel $K_H$ plays a role in our proof of (3.24).

**Lemma 5.4.** Recall the constant $\beta_{H,1}$ introduced in (1.7). Then, for any $r, s > 0$ and $n \in \mathbb{N}$ we have

\begin{equation}
n^{H-\frac{1}{2}} \left( \frac{1}{s} \right)^{H-\frac{1}{2}} K_H \left( r + \frac{s}{n}, r \right) \leq C \begin{cases} 
\left( 1 + \frac{s}{nr} \right)^{H-\frac{1}{2}} & \text{if } H > \frac{1}{2} \\
1 + \frac{1}{r} \left( \frac{s}{n} \right)^{H+\frac{1}{2}} & \text{if } H \leq \frac{1}{2},
\end{cases}
\end{equation}
and

\[(5.5) \quad |n^{H-\frac{1}{2}} s^{\frac{1}{2}-H} K_H(r + \frac{s}{n}, r) - \beta_{H,1}| \leq C \frac{s}{rn}, \]

for some constant $C > 0$ only depending on $H$.

\[\text{Proof.} \quad \text{The inequality (5.4) is a consequence of the estimates (5.2) and (5.3). It remains to show (5.5). In the case } H > \frac{1}{2}, \text{ we use (5.2), to deduce that} \]

\[\beta_{H,1} \leq n^{H-\frac{1}{2}} s^{\frac{1}{2}-H} K_H(r + \frac{s}{n}, r) \leq (1 + \frac{s}{rn})^{H-\frac{1}{2}} \beta_{H,1}. \]

Inequality (5.5) in the case $H > \frac{1}{2}$ then follows from the fact that

\[(5.6) \quad \left| (1 + \frac{s}{rn})^{H-\frac{1}{2}} - 1 \right| \leq \frac{s}{nr}.\]

To handle the case $H < \frac{1}{2}$, we observe that by (5.3),

\[\mathcal{C}_H \left[ \left( 1 + \frac{s}{rn} \right)^{H-\frac{1}{2}} + \frac{1/2 - H}{1/2 + H} \frac{1}{n} \right] \left( r + \frac{s}{n} \right) \leq C \left[ \left( 1 + \frac{s}{rn} \right)^{H-\frac{1}{2}} + \frac{1/2 - H}{1/2 + H} \frac{s}{nr} \right].\]

Thus, by (5.6), we obtain

\[\left| n^{H-\frac{1}{2}} s^{\frac{1}{2}-H} K_H(r + \frac{s}{n}, r) - \beta_{H,1} \right| \leq C \left[ 1 - \left( 1 + \frac{s}{rn} \right)^{H-\frac{1}{2}} \right] + \frac{1/2 - H}{1/2 + H} \frac{s}{nr}. \]

This finishes the proof of (5.5) in the case $H \leq \frac{1}{2}$. \qed

For $\mu_{r,s}$ defined by (2.9), we prove the following useful bounds.

**Lemma 5.5.** Suppose that $s \geq 0$ and $r > 0$ and recall the definition of $\mu_{r,r+z}$ from (2.9). Then, if $H > \frac{1}{2}$,

\[(5.7) \quad \frac{C_H s^{2H}}{2H (H - 1/2)^2} \leq n^{2H} \mu_{r,r+z} \leq \frac{C_H s^{2H}}{2H (H - 1/2)} \left( 1 + \frac{s}{rn} \right)^{2H-1}, \]

while if $H < \frac{1}{2}$,

\[(5.8) \quad \frac{C_H s^{2H}}{2H} \leq n^{2H} \mu_{r,r+z} \leq \frac{C_H s^{2H}}{2H} \left( 1 + \frac{1/2 - H}{1/2 + H} \frac{s}{rn} \right)^2. \]

In particular, there exists a constant $C > 0$, only depending on $H$, such that

\[(5.9) \quad \left| n^{2H} \mu_{r,r+z} - \beta_{H,2} s^{2H} \right| \leq \frac{C s^{2H+1}}{rn} \left( 1 + \frac{s}{rn} \right), \]

where $\beta_{H,2}$ is defined in (1.7).
PROOF. The case $H = \frac{1}{2}$ is clear, as in this instance $K_H(t, s) = 1$ for all $0 \leq s \leq t$. Thus, we will assume without loss of generality that $H \neq \frac{1}{2}$.

First we prove (5.7) in the case $H > \frac{1}{2}$. Recall that $\mu_{r,s} = \int_r^s K_H^2(r + \frac{s}{n}, \theta) d\theta$, so that
\begin{align*}
\mu_{r,r+s} &= \int_r^{r+s} K_H^2(r + \frac{s}{n}, \theta) d\theta = \int_0^{\frac{s}{n}} K_H^2(r + \frac{s}{n}, r + \theta) d\theta = \int_0^{\frac{s}{n}} K_H^2(r + \frac{s}{n}, r + \frac{s}{n} - \theta) d\theta.
\end{align*}

We thus deduce from (5.2) that
\begin{align*}
\mu_{r,r+s} &\leq C_H^2 \left( \frac{r + \frac{s}{n}}{r + \frac{s}{n} - \theta} \right)^{2H-1} \theta^{2H-1} d\theta \\
&\leq C_H^2 \left( \frac{r + \frac{s}{n}}{r} \right)^{2H-1} \theta^{2H-1} d\theta,
\end{align*}
which gives
\begin{equation}
(5.11) \quad \mu_{r,r+s} \leq \frac{C_H^2}{2H(H - 1/2)^2} \left( 1 + \frac{s}{r} \right)^{2H} \left( \frac{s}{n} \right)^{2H}.
\end{equation}

To lower bound $\mu_{r,r+s}$, we apply (5.10) and (5.2) to get
\begin{equation}
(5.12) \quad \mu_{r,r+s} \geq \frac{C_H^2}{2H(H - 1/2)^2} \left( \frac{r + \frac{s}{n}}{r} \right)^{2H} \theta^{2H-1} d\theta = \frac{C_H^2}{2H(H - 1/2)^2} \left( \frac{s}{n} \right)^{2H}.
\end{equation}

Relation (5.7) follows from (5.11) and (5.12).

To handle the case $H < \frac{1}{2}$, we use (5.3) to deduce that
\begin{equation}
(5.13) \quad K_H(t, s) \leq C_H(t - s)^{H - \frac{1}{2}} \left( 1 + \frac{1/2 - H}{1/2 + H} s^{-1}(t - s) \right).
\end{equation}

The above inequality can be combined with (5.10), and we can write
\begin{align*}
\mu_{r,r+s} &\leq C_H^2 \int_0^{\frac{s}{n}} \left( 1 + \frac{1/2 - H}{1/2 + H} \frac{s}{r n} \right)^{2H} \theta^{2H-1} d\theta \\
&= \frac{C_H^2}{2H} \left( \frac{s}{n} \right)^{2H} \left( 1 + \frac{1/2 - H}{1/2 + H} \frac{s}{r n} \right)^{2H}.
\end{align*}

On the other hand, by (5.3), for all $0 \leq s \leq t$,
\begin{align*}
C_H \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} &\leq K_H(t, s),
\end{align*}
which by (5.10) leads to
\begin{equation}
(5.14) \quad \mu_{r,r+s} \geq \frac{C_H^2}{2H} \int_0^{\frac{s}{n}} \theta^{2H-1} d\theta \geq \frac{C_H^2}{2H} \left( \frac{s}{n} \right)^{2H}.
\end{equation}

Inequality (5.8) thus follows from (5.13) and (5.14).

Finally, relation (5.9) in the case $H > \frac{1}{2}$ follows by applying the mean value theorem in (5.7), and in the case $H \leq \frac{1}{2}$, it follows by expanding the square in the right-hand side of (5.8).
The conclusion of the following lemma is needed in the proof of (3.24).

**Lemma 5.6.** Let $T, \varepsilon > 0$ be fixed. Then, there exists a constant $C > 0$ only depending on $H, T$ and $\varepsilon$, such that for all $1 < s_1 < s_2 \leq nT$ and $\varepsilon < r \leq T$,

\[(5.15) \quad |n^{2H}E[(B_{r,r+\frac{r}{n}} - B_{r,r+\frac{s}{n}})^2] - \beta_{H,3}(s_1, s_2)| \leq C \left(s_2^2n^{2H-2} + s_2^{2H+1}n^{-1}\right),\]

where $\beta_{H,3}(s_1, s_2)$ is defined in (1.8).

**Proof.** We first prove (5.15). To this end, we write

\[E[(B_{r,r+\frac{r}{n}} - B_{r,r+\frac{s}{n}})^2] = \int_0^T (K_H(r + \frac{r}{n}, \theta) - K_H(r + \frac{s}{n}, \theta))^2 d\theta\]

\[= \frac{1}{n} \int_0^{nr} (K_H(r + \frac{r}{n}, r - \frac{\theta}{n}) - K_H(r + \frac{s}{n}, r - \frac{\theta}{n}))^2 d\theta.\]

By defining $\Delta_n(\theta) := K_H(r + \frac{r}{n}, r - \frac{\theta}{n}) - K_H(r + \frac{s}{n}, r - \frac{\theta}{n})$, we can write

\[(5.16) \quad n^{2H}E[(B_{r,r+\frac{r}{n}} - B_{r,r+\frac{s}{n}})^2] = \int_0^{nr} (n^{H-\frac{1}{2}} \Delta_n(\theta))^2 d\theta.\]

From (2.1) and (2.2), one can easily check that

\[(5.17) \quad \frac{\partial}{\partial t}K_H(t, s) = C_H s^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}t^{H-\frac{1}{2}},\]

and thus,

\[\Delta_n(\theta) = C_H(r - \frac{\theta}{n})^{\frac{1}{2}-H} \int_r^{r+\frac{r}{n}} (u + \frac{\theta}{n} - r)^{H-\frac{3}{2}}u^{H-\frac{1}{2}} du\]

\[= C_H(r - \frac{\theta}{n})^{\frac{1}{2}-H} \int_0^{\frac{s}{n}} \left(\frac{s}{n} + u + \frac{\theta}{n}\right)^{H-\frac{3}{2}}(r + \frac{s}{n} + u)^{H-\frac{1}{2}} du,\]

so that

\[(5.18) \quad n^{H-\frac{1}{2}} \Delta_n(\theta) = C_H(r - \frac{\theta}{n})^{\frac{1}{2}-H} \int_0^{s_2-s_1} (s_1 + u + \theta)^{H-\frac{3}{2}}(r + \frac{s_1+u}{n})^{H-\frac{1}{2}} du.\]

Suppose first that $\theta \leq \frac{nr}{2}$. In this case, because $T \geq r - \frac{\theta}{n} \geq \frac{r}{2} \geq \frac{\delta}{2}$, by the mean value theorem, for any $x > 0$ we have

\[(5.19) \quad \left| \left( r - \frac{\theta}{n} \right)^{\frac{1}{2}-H} \left( r - \frac{\theta}{n} + x \right)^{H-\frac{1}{2}} - 1 \right| \leq Cx,\]

for some constant $C$ depending on $T$ and $\delta$. From (5.18) and the estimate (5.19), we can easily check that there exists a constant $C > 0$, such that if $\theta \leq \frac{nr}{2}$, then

\[(5.20) \quad \left|n^{H-\frac{1}{2}}\Delta_n(\theta) - C_H \int_0^{s_2-s_1} (s_1 + u + \theta)^{H-\frac{3}{2}} du\right| \leq C \left(\frac{\theta + s_2}{n}\right) \int_0^{s_2-s_1} (s_1 + u + \theta)^{H-\frac{3}{2}} du.\]

Moreover, it is easy to see that

\[(5.21) \quad n^{H-\frac{1}{2}}|\Delta_n(\theta)| + C_H \int_0^{s_2-s_1} (s_1 + u + \theta)^{H-\frac{3}{2}} du \leq C \int_0^{s_2-s_1} (s_1 + u + \theta)^{H-\frac{3}{2}} du.\]
Thus, from (5.20) and (5.21), we deduce

\[
(5.22) \quad \left| \left( n^{H - \frac{1}{2}} \Delta_n(\theta) \right)^2 - \frac{C^2_H ((s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}})^2}{(H - 1/2)^2} \right| \leq C \frac{\left( \theta + s_2 \right)^2 (s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}}}{n}.
\]

We therefore conclude that

\[
\int_0^{\frac{\pi}{2}} \left| \left( n^{H - \frac{1}{2}} \Delta_n(\theta) \right)^2 - \frac{C^2_H ((s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}})^2}{(H - 1/2)^2} \right| d\theta \leq C (A_n^1 + A_n^2),
\]

where

\[
A_n^1 := \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leq s_2\}} \frac{S_2}{n} \left| (s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}} \right|^2 d\theta
\]

\[
A_n^2 := \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta > s_2\}} \frac{\theta}{n} \left| (s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}} \right|^2 d\theta.
\]

The term \( A_n^2 \) can be bounded by means of the inequality

\[
\left| (s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}} \right|^2 \leq \left( H - \frac{1}{2} \right)^2 \theta^{2H - 3} |s_2 - s_1|^2,
\]

which is valid for all \( s_2 \leq \theta \). This gives

\[
(5.23) \quad A_n^2 \leq C \frac{1}{n} \int_0^{\frac{\pi}{2}} s_2^2 \theta^{2H - 2} d\theta \leq C r^{2H - 1} s_2^2 n^{2H - 2}.
\]

For handling \( A_n^1 \), we observe that by the change of variables \( \theta \to s_2 \theta \), we can write

\[
A_n^1 = \frac{s_2^{H + 1}}{n} \int_0^1 \left| (1 + \theta)^{H - \frac{1}{2}} - \left( \frac{s_1}{s_2} + \theta \right)^{H - \frac{1}{2}} \right|^2 d\theta.
\]

Then, we use the inequality

\[
\left| (1 + \theta)^{H - \frac{1}{2}} - \left( \frac{s_1}{s_2} + \theta \right)^{H - \frac{1}{2}} \right| \leq \mathbb{1}_{\{H \geq \frac{1}{2}\}} 2^{2H - \frac{1}{2}} + \mathbb{1}_{\{H < \frac{1}{2}\}} \theta^{H - \frac{1}{2}},
\]

in order to deduce that

\[
(5.24) \quad A_n^1 \leq C \frac{s_2^{H + 1}}{n} \int_0^1 (1 + \theta^{2H - 1}) d\theta \leq C s_2^{2H} \frac{s_2}{n}.
\]

From (5.23) and (5.24) here we conclude that

\[
(5.25) \quad \int_0^{\frac{\pi}{2}} \left| \left( n^{H - \frac{1}{2}} \Delta_n(\theta) \right)^2 - \frac{C^2_H ((s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}})^2}{(H - 1/2)^2} \right| d\theta \leq C \left( s_2^{2n^{2H} + s_2^{2H + 1}} n^{-1} \right).
\]

For handling the case \( \theta \geq \frac{mr}{2} \), we use the inequality

\[
\left| \left( r + \frac{s + u}{n} \right)^{H - \frac{1}{2}} - r^{H - \frac{1}{2}} \right| \leq \left( H - \frac{1}{2} \right) r^{H - \frac{1}{2}} \frac{s_2}{n},
\]

as well as (5.18), to conclude that

\[
(5.26) \quad \int_{\frac{mr}{2}}^{\pi} \left| \left( n^{H - \frac{1}{2}} \Delta_n(\theta) \right)^2 - \frac{C^2_H ((s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}})^2}{(H - 1/2)^2} \right| d\theta \leq C A_n^3,
\]
where $C$ is some constant and $A_n^3$ given by
\[
A_n^3 := \int_0^{nr} \left( r^{H - \frac{1}{2}} \left| (r - \frac{\theta}{n})^\frac{1}{2} - H - r^\frac{1}{2} - H \right| + r^{H - \frac{3}{2}} \frac{s_0}{n} (r - \frac{\theta}{n})^{\frac{1}{2}} \right) \times \left| (s_2 + \theta)^{H - \frac{1}{2}} - (s + \theta)^{H - \frac{1}{2}} \right|^2 d\theta.
\]
By first using the inequality
\[
\left| (s_2 + \theta)^{H - \frac{1}{2}} - (s_1 + \theta)^{H - \frac{1}{2}} \right| \leq C |s_2 - s_1| (nr/2)^{H - \frac{1}{2}},
\]
for $\theta \geq \frac{nr}{2}$ and then the fact that
\[
\int_0^{nr} \left( r^{H - \frac{1}{2}} \left| (r - \frac{\theta}{n})^\frac{1}{2} - H - r^\frac{1}{2} - H \right| + r^{H - \frac{3}{2}} \frac{s_0}{n} (r - \frac{\theta}{n})^{\frac{1}{2}} \right) \leq C n,
\]
for some constant $C > 0$, we obtain
\begin{equation}
A_n^3 \leq C n^{2H - 2} |s_2 - s_1|^2 \leq C s_2^2 n^{2H - 2}.
\end{equation}
Inequality (5.15) follows from (5.25), (5.26), (5.27) and the fact that
\[
\int_{nt}^{\infty} |(\theta + s_1)^{H - \frac{1}{2}} - (\theta + s_2)^{H - \frac{1}{2}}|^2 d\theta \leq C s_2^2 n^{2H - 2}.
\]

The following lemma is used in the proof of (3.25).

**Lemma 5.7.** For $r_1, r_2, s_2, s_1 \geq 0$, we define the random variables $\beta^{(n)}_{r_1, s, \eta}, \beta^{(n)}_{r_2, s, \eta}, \alpha^{(n)}_{r_1, s, \eta, \bar{\eta}}, \alpha^{(n)}_{r_2, s, \eta, \bar{\eta}}$ as in (3.14) and (3.15). Then, for every $\eta, \bar{\eta} \in \mathbb{R}$,
\[
\left| \mathbb{E} \left[ e^{-i(\beta^{(n)}_{r_1, s, \eta} - \beta^{(n)}_{r_2, s, \eta} + \bar{\eta}(B_{r_1, r_1} + \frac{s_2}{n} - \lambda) - \bar{\eta}(B_{r_2, r_2} + \frac{s_2}{n} - \lambda))} \right] \exp \left( -\frac{1}{2} \left( \alpha^{(n)}_{r_1, s, \eta, \bar{\eta}} + \alpha^{(n)}_{r_2, s, \eta, \bar{\eta}} \right) \right) \right| \leq \exp \left( -\frac{1}{4} \left( \alpha^{(n)}_{r_1, s, \eta, \bar{\eta}} + \alpha^{(n)}_{r_2, s, \eta, \bar{\eta}} \right) \right)
\]
\[
\times \exp \left( -\frac{1}{60} \text{Var} \left[ \bar{\eta} B_{r_1, r_1} + \frac{s_2}{n} - \bar{\eta} B_{r_2, r_2} + \frac{s_2}{n} \right] | B_{r_1, r_1} + \frac{s_2}{n} - B_{r_1, r_1} + \frac{s_2}{n} - B_{r_2, r_2} + \frac{s_2}{n} \right) \right)
\]
\[
\times \exp \left( -\frac{1}{60} \text{Var} \left[ n^H \eta (B_{r_1, r_1} + \frac{s_2}{n}) - B_{r_1, r_1} + \frac{s_2}{n} \right] | B_{r_1, r_1} + \frac{s_2}{n}, B_{r_1, r_1} + \frac{s_2}{n}, B_{r_2, r_2} + \frac{s_2}{n} \right) \right)
\]
\[
\times \exp \left( -\frac{1}{60} \text{Var} \left[ n^H \eta (B_{r_1, r_1} + \frac{s_2}{n} - B_{r_1, r_1} + \frac{s_2}{n}) \right] | B_{r_1, r_1} + \frac{s_2}{n}, B_{r_2, r_2} + \frac{s_2}{n}, B_{r_1, r_1} + \frac{s_2}{n} \right) \right).
\]
and
\[
\left| \mathbb{E} \left[ e^{-i(\beta^{(n)}_{r_1, s, \eta} - \beta^{(n)}_{r_2, s, \eta} + \bar{\eta}(B_{r_1, r_1} + \frac{s_2}{n} - \lambda) - \bar{\eta}(B_{r_2, r_2} + \frac{s_2}{n} - \lambda))} \right] \exp \left( -\frac{1}{2} \left( \alpha^{(n)}_{r_1, s, \eta, \bar{\eta}} + \alpha^{(n)}_{r_2, s, \eta, \bar{\eta}} \right) \right) \right| \leq \exp \left( -\frac{1}{4} \left( \alpha^{(n)}_{r_1, s, \eta, \bar{\eta}} + \alpha^{(n)}_{r_2, s, \eta, \bar{\eta}} \right) \right)
\]
\[
\times \exp \left( -\frac{1}{28} \text{Var} \left[ \bar{\eta} B_{r_1, r_1} + \frac{s_2}{n} - \bar{\eta} B_{r_2, r_2} + \frac{s_2}{n} \right] | B_{r_1, r_1} + \frac{s_2}{n} - B_{r_1, r_1} + \frac{s_2}{n}, B_{r_2, r_2} + \frac{s_2}{n} - B_{r_2, r_2} + \frac{s_2}{n} \right) \right).
\]
\begin{align}
&\times \exp \left( -\frac{1}{28} \frac{1 - \beta(n)}{\beta(n)} \right) \frac{1}{\beta(n)}
&+ \exp \left( \frac{1}{2} \alpha_{r_1, s, \eta, \bar{\eta}} \right) \left( \eta(N_1 - N_2) + \bar{\eta}n - N_3 - \bar{\eta}m - H N_4 \right)
&+ \exp \left( \frac{1}{2} \alpha_{r_1, s, \eta, \bar{\eta}} \right)
\end{align}

PROOF. First we show (5.28). Define the random variables
\begin{align}
N_1 &:= n^H B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n} - B_{r_2, r_2 + 2a_n}
N_3 &:= n^H B_{r_1, r_1 + 2a_n} - B_{r_2, r_2 + 2a_n}
\bar{N}_1 &:= n^H (B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n}) - \bar{\eta} \left( B_{r_2, r_2 + 2a_n} - \bar{\eta} \right)
\bar{N}_3 &:= n^H (B_{r_2, r_2 + 2a_n} - B_{r_2, r_2 + 2a_n}) - \bar{\eta} \left( B_{r_2, r_2 + 2a_n} - \bar{\eta} \right).
\end{align}

Using (3.14) and (3.15), we can write
\begin{align}
\exp \left[ e^{-i \beta(n)} \eta(N_1 - N_2) + \bar{\eta}n - N_3 - \bar{\eta}m - H N_4 \right]
&= \exp \left( \frac{1}{2} \Var[\eta(N_1 - N_2) + \bar{\eta}n - N_3 - \bar{\eta}m - H N_4] \right)
\end{align}

If we add all the random variables whose variances appear in the expressions (5.31) and (5.32), we obtain
\begin{align}
\eta(N_1 - N_2) + \bar{\eta}n - N_3 - \bar{\eta}m - H N_4 &= \eta \left( B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n} - B_{r_2, r_2 + 2a_n} \right) + \bar{\eta} \left( B_{r_2, r_2 + 2a_n} - B_{r_2, r_2 + 2a_n} \right) - \bar{\eta}B_{r_2, r_2 + 2a_n} =: Z.
\end{align}

As a consequence,
\begin{align}
\Var(Z) &\leq 5 \left( \Var[\eta \bar{N}_1] + \Var[(\eta^H \bar{\eta} - \eta) \bar{N}_2] + \Var[\eta \bar{N}_3] \right)
&+ \Var[(\eta^H \bar{\eta} - \eta) \bar{N}_4] + \Var[\eta(N_1 - N_2) + n^{-H}(\eta N_3 - \bar{\eta} N_4)]
&= 5 \left( \alpha_{r_1, s, \eta, \bar{\eta}} \right) + \alpha_{r_1, s, \eta, \bar{\eta}} + \Var[\eta(N_1 - N_2) + n^{-H}(\eta N_3 - \bar{\eta} N_4)]
\end{align}

The next step is to make use of the following inequalities, which are an immediate consequence of (5.33) and (5.34):
\begin{align}
\Var[\eta B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n} \mid B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n} + B_{r_2, r_2 + 2a_n} - B_{r_2, r_2 + 2a_n}]
&\leq \Var[\eta(n^H(B_{r_1, r_1 + 2a_n} - B_{r_1, r_1 + 2a_n}) + n^H(B_{r_2, r_2 + 2a_n} - B_{r_2, r_2 + 2a_n})) + \eta B_{r_1, r_1 + 2a_n} - \bar{\eta} B_{r_2, r_2 + 2a_n}]
\end{align}

\begin{align}
\Var(Z) &\leq 5 \left( \alpha_{r_1, s, \eta, \bar{\eta}} \right) + \alpha_{r_1, s, \eta, \bar{\eta}} + \Var[\eta(N_1 - N_2) + \bar{\eta} N_3 - \bar{\eta} N_4],
\end{align}
as well as
\[
\text{Var}[n^H \eta(B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}})] \leq \text{Var}[(n^H (B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}}) + n^H (B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}}))] + \eta B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}
\]
(5.36)
\[
\leq 5(\alpha_{r_1, \sigma_1, \eta}^{(n)} + \alpha_{r_2, \sigma_2, \eta}^{(n)} + \text{Var}[\eta(N_1 - N_2) + \eta N_3 - \eta N_4]),
\]
and
\[
\text{Var}[n^H \eta(B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}})] \leq \text{Var}[(n^H (B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}}) + n^H (B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}}))] + \eta B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}
\]
(5.37)
\[
\leq 5(\alpha_{r_1, \sigma_1, \eta}^{(n)} + \alpha_{r_2, \sigma_2, \eta}^{(n)} + \text{Var}[\eta(N_1 - N_2) + \eta N_3 - \eta N_4]).
\]
Putting together (5.35), (5.36) and (5.37), yields
(5.38)
\[
15(\alpha_{r_1, \sigma_1, \eta}^{(n)} + \alpha_{r_2, \sigma_2, \eta}^{(n)} + \text{Var}[\eta(N_1 - N_2) + \eta N_3 - \eta N_4])
\]
\[
\geq \text{Var}[\eta B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}] + \text{Var}[n^H \eta(B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}}) + \eta B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}]
\]
\[
+ \text{Var}[n^H \eta(B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}})]
\]
Relation (5.28) follows from (5.31) and (5.38).
In order to show (5.30), we observe that
\[
\text{Var}[n^H \eta(B_{r_2 + \frac{\sigma_2}{n}} - B_{r_1 + \frac{\sigma_1}{n}})] \leq \text{Var}[\eta^H(B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}} + B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}}) + \eta(B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}]
\]
\[
\leq 5(\alpha_{r_1, \sigma_1, \eta}^{(n)} + \alpha_{r_2, \sigma_2, \eta}^{(n)} + \text{Var}[\eta(N_1 - N_2) + \eta N_3 - \eta N_4]),
\]
Combining the previous inequality with (5.35), we obtain
(5.39)
\[
7(\alpha_{r_1, \sigma_1, \eta}^{(n)} + \alpha_{r_2, \sigma_2, \eta}^{(n)} + \text{Var}[\eta(N_1 - N_2) + \eta N_3 - \eta N_4])
\]
\[
\geq \text{Var}[\eta B_{r_1 + \frac{\sigma_1}{n}} - \eta B_{r_2 + \frac{\sigma_2}{n}}] + \text{Var}[n^H \eta(B_{r_2 + \frac{\sigma_2}{n}} - B_{r_1 + \frac{\sigma_1}{n}})]
\]
Relation (5.28) follows from (5.31) and (5.39).

The following lemma is also used in the proof of (3.25).

\textbf{Lemma 5.8.} Suppose that \(H \neq \frac{1}{2}\) and let \(T, t, \epsilon > 0\) be fixed. Then, for every \(nT \geq s_2 \geq s_1 > 0, nT \geq v_2 \geq v_1 > 0\) and \(T \geq r_2 \geq r_1 + \epsilon \geq 2\epsilon,\) we have
(5.40)
\[
n^{2H} \mathbb{E}[(B_{r_2 + \frac{\sigma_2}{n}} - B_{r_2 + \frac{\sigma_2}{n}})(B_{r_1 + \frac{\sigma_1}{n}} - B_{r_1 + \frac{\sigma_1}{n}})]
\]
\[
\leq C_6(n^{2H-2}(s_2 - s_1)(v_2 - v_1) + n^{H-\frac{3}{2}}(v_2 - v_1)^{H+\frac{1}{2}}(s_2 - s_1)),
\]
\[ n^H \left| E[(B_{r_2, r_2 + \frac{s_2}{n}} - B_{r_2, r_2 + \frac{s_1}{n}})B_{r_1, r_1 + \frac{s_2}{n}}] \right| \leq C_\varepsilon n^{H-1}(s_2 - s_1), \]

for some constant \( C_\varepsilon > 0 \) depending on \( \varepsilon, r_1, r_2, T \) and \( H \).

**Proof.** Recall that \( B_{r, s} = \int_0^r K_H(s, \theta) dW_\theta \). Define

\[ \beta_{H, 3}(\vec{s}, \vec{v}) := E[(B_{r_2, r_2 + \frac{s_2}{n}} - B_{r_2, r_2 + \frac{s_1}{n}})(B_{r_1, r_1 + \frac{s_2}{n}} - B_{r_1, r_1 + \frac{s_1}{n}})]. \]

Notice that, since \( r_1 < r_2 \),

\[
\beta_{H, 3}^{(n)}(\vec{s}, \vec{v}) = \int_0^{r_1} (K_H(r_2 + \frac{s_2}{n}, \theta) - K_H(r_2 + \frac{s_1}{n}, \theta))(K_H(r_1 + \frac{v_2}{n}, \theta) - K_H(r_1 + \frac{v_1}{n}, \theta)) d\theta
\]

\[ = \frac{1}{n} \int_0^{nr_1} (K_H(r_2 + \frac{s_2}{n}, r_1 - \frac{\theta}{n}) - K_H(r_2 + \frac{s_1}{n}, r_1 - \frac{\theta}{n}))
\]

\[ \times (K_H(r_1 + \frac{v_2}{n}, r_1 - \frac{\theta}{n}) - K_H(r_1 + \frac{v_1}{n}, r_1 - \frac{\theta}{n})) d\theta. \]

By defining

\[ \tilde{\Delta}_n(\theta) := K_H(r_2 + \frac{s_2}{n}, r_1 - \frac{\theta}{n}) - K_H(r_2 + \frac{s_1}{n}, r_1 - \frac{\theta}{n}) \]

\[ \tilde{\Delta}_n(\theta) := K_H(r_1 + \frac{v_2}{n}, r_1 - \frac{\theta}{n}) - K_H(r_1 + \frac{v_1}{n}, r_1 - \frac{\theta}{n}), \]

we can write

\[ n^{2H} \beta_{H, 3}^{(n)}(\vec{s}, \vec{v}) = \int_0^{nr_2} (n^{H-\frac{1}{2}} \tilde{\Delta}_n(\theta))(n^{H-\frac{1}{2}} \tilde{\Delta}_n(\theta)) d\theta. \]

From (2.1) and (2.2), one can easily check that

\[ \frac{\partial}{\partial t} K_H(t, s) = C_H s^{\frac{1}{2} - H} (t - s)^{H - \frac{3}{2}} t^{H - \frac{1}{2}}, \]

which implies that the term \( \tilde{\Delta}_n(\theta) \) satisfies

\[ \tilde{\Delta}_n(\theta) = C_H (r_1 - \frac{\theta}{n})^{\frac{1}{2} - H} \int_{r_1 + \frac{ose}{H}}^{r_1 + \frac{un}{H}} (u + \frac{\theta}{n} - r_1)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du 
\]

\[ = C_H (r_1 - \frac{\theta}{n})^{\frac{1}{2} - H} \int_{r_1 + \frac{un}{H}}^{r_1 + \frac{v_2 - v_1}{n}} (v_1 + u + \frac{\theta}{n})^{H - \frac{3}{2}} (r_1 + \frac{v_1 + u + \theta}{n} u)^{H - \frac{1}{2}} du, \]

so that, using the fact that \( \varepsilon \leq r_1 \leq T \), we obtain

\[ n^{H-\frac{1}{2}} \tilde{\Delta}_n(\theta) = C_H (r_1 - \frac{\theta}{n})^{\frac{1}{2} - H} \int_{v_2 - v_1}^{v_2 - v_1} (v_1 + u + \theta)^{H - \frac{3}{2}} (r_1 + \frac{v_1 + u + \theta}{n})^{H - \frac{1}{2}} du \]

\[ \leq C_\varepsilon (r_1 - \frac{\theta}{n})^{\frac{1}{2} - H} \int_{v_2 - v_1}^{v_2 - v_1} (v_1 + u + \theta)^{H - \frac{3}{2}} du, \]

for some constant \( C_\varepsilon > 0 \) depending also on \( T \) and \( H \). In particular, if \( \theta \leq \frac{nr_1}{n} \),

\[ n^{H-\frac{1}{2}} \tilde{\Delta}_n(\theta) \leq C_\varepsilon \int_0^{v_2 - v_1} (v_1 + u + \theta)^{H - \frac{3}{2}} du \]

(5.45)
while if $\theta > \frac{nr_1}{2}$,

\begin{equation}
(5.46) \quad n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta) \leq C_{\varepsilon}(r_1 - \frac{\theta}{n})^{\frac{1}{2}} n^{H-\frac{1}{2}} (v_2 - v_1) n^{H-\frac{1}{2}}.
\end{equation}

On the other hand,

\begin{align*}
\hat{\Delta}_n(\theta) &= C_H(r_1 - \frac{\theta}{n})^{\frac{1}{2}} H \int_{r_1 + \frac{\theta}{n}}^{r_1 + \frac{2\theta}{n}} (u + \frac{\theta}{n} - r_1) H^{-\frac{1}{2}} u^{H-\frac{1}{2}} du \\
&= C_H(r_1 - \frac{\theta}{n})^{\frac{1}{2}} H \int_0^{\frac{2\theta}{n}} (r_2 - r_1 + \frac{s_1}{n} + u + \frac{\theta}{n}) H^{-\frac{1}{2}} (r_2 + \frac{s_1}{n} + u)^{H-\frac{1}{2}} du,
\end{align*}

so that, using the fact that $r_2 - r_1 \geq \varepsilon$, we have

\begin{equation}
(5.47) \quad n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta) \leq C_{\varepsilon}(r_1 - \frac{\theta}{n})^{\frac{1}{2}} n^{H-\frac{1}{2}} (s_2 - s_1),
\end{equation}

for some constant $C_{\varepsilon} > 0$.

From (5.45), (5.46) and (5.47) we conclude that there exists a constant $C_{\varepsilon} > 0$, such that

\begin{align*}
\int_0^{nr_1} (n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta)) (n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta)) d\theta &\leq C_{\varepsilon} n^{H-\frac{1}{2}} (s_2 - s_1) \int_0^{nr_1} (v_1 + u + \theta)^{H-\frac{1}{2}} d\theta \\
&+ C_{\varepsilon} n^{2H-3} (s_2 - s_1) (v_2 - v_1) \int_{nr_1}^{nr_1} (r_1 - \frac{\theta}{n})^{1-2H} d\theta \\
&= C_{\varepsilon} (H - \frac{1}{2})^{-1} n^{H-\frac{1}{2}} (s_2 - s_1) \int_0^{v_2 - v_1} [(v_1 + u + \frac{nr_1}{2})^{H-\frac{1}{2}} - (v_1 + u)^{H-\frac{1}{2}}] d\theta \\
&+ C_{\varepsilon} (H - \frac{1}{2})^{-1} n^{2H-2} (s_2 - s_1) (v_2 - v_1) (r/2)^{2-2H}.
\end{align*}

If $H > \frac{1}{2}$, we use the inequality

\[ |(v_1 + u + \frac{nr_1}{2})^{H-\frac{1}{2}} - (v_1 + u)^{H-\frac{1}{2}}| \leq C h^{H-\frac{1}{2}}, \]

and for $H < \frac{1}{2}$, we use that

\[ (H + \frac{1}{2}) \int_0^{v_2 - v_1} (v_1 + u)^{H-\frac{1}{2}} du = v_2^{H+\frac{1}{2}} - v_1^{H+\frac{1}{2}} \leq (v_2 - v_1)^{H+\frac{1}{2}}. \]

In this way, we obtain

\begin{align*}
\int_0^{nr_1} (n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta)) (n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta)) d\theta &\leq C_{\varepsilon} (n^{2H-2} (s_2 - s_1) (v_2 - v_1) \\
&+ n^{H-\frac{1}{2}} (v_2 - v_1)^{H+\frac{1}{2}} (s_2 - s_1)).
\end{align*}

Relation (5.40) then follows from (5.43).

In order to prove (5.41), we proceed as in the proof of (5.43) to deduce that

\begin{equation}
(5.48) \quad n^{H} |\mathbb{E}[(B_{r_2, r_2 + \frac{\theta}{n}} - B_{r_2, r_2 + \frac{\theta}{n}}) B_{r_1, r_1 + \frac{\theta}{n}}]| = \int_0^{nr_1} (n^{H-\frac{1}{2}} \hat{\Delta}_n(\theta)) (n^{\frac{1}{2}} K_H(r_1 + \frac{v_1}{n}, r_1 - \frac{\theta}{n})) d\theta,
\end{equation}

which by equation (5.47) and Lemma 5.3, gives

\begin{align*}
n^{H} |\mathbb{E}[(B_{r_2, r_2 + \frac{\theta}{n}} - B_{r_2, r_2 + \frac{\theta}{n}}) B_{r_1, r_1 + \frac{\theta}{n}}]| &\leq C_{\varepsilon} n^{-\frac{1}{2}} (s_2 - s_1) \int_0^{nr_1} (r_1 - \frac{\theta}{n})^{-1-2H} |v_1 + \theta|^{H-\frac{1}{2}} d\theta \\
&+ C_{\varepsilon} n^{-\frac{1}{2}} (s_2 - s_1) \mathbb{I}_{(H < \frac{1}{2})} \int_0^{nr_1} (r_1 - \frac{\theta}{n})^{-1-H} |v_1 + \theta|^{H+\frac{1}{2}} d\theta.
\end{align*}
Consequently, for $H > \frac{1}{2}$,
\[
n^H |E[(B_{r_2, r_2 + \frac{\sigma}{n}} - B_{r_2, r_2 + \frac{\sigma}{n}})B_{r_1, r_1 + \frac{\sigma}{n}}]| \leq C_n n^{H-2}(s_2 - s_1) \int_0^{nr_1} (r - \frac{\theta}{n})^{1-2H} d\theta \\
\leq C' n^{H-1}(s_2 - s_1)
\]
In the case $H < \frac{1}{2}$, equation (5.48) gives
\[
n^H |E[(B_{r_2, r_2 + \frac{\sigma}{n}} - B_{r_2, r_2 + \frac{\sigma}{n}})B_{r_1, r_1 + \frac{\sigma}{n}}]| \leq C_n n^{-\frac{1}{2}}(s_2 - s_1) \int_0^{nr_1} |v_1 + \theta|^{H-\frac{1}{2}} d\theta \\
+ C_n n^{-\frac{1}{2}}(s_2 - s_1) \int_0^{nr_1} (r_1 - \frac{\theta}{n})^{-\frac{1}{2}H} |v_1 + \theta|^{H+\frac{1}{2}} d\theta
\]
which leads to
\[
n^H |E[(B_{r_2, r_2 + \frac{\sigma}{n}} - B_{r_2, r_2 + \frac{\sigma}{n}})B_{r_1, r_1 + \frac{\sigma}{n}}]| \leq C_n n^{H-1}(s_2 - s_1),
\]
as required.

The following lemma is used both in the proof of (3.24) and in the proof of Lemma 1.2.

**Lemma 5.9.** For every $a, b, c, \sigma > 0$, there is a constant $C$ such that
(5.49) \[
\int_{\mathbb{R}^2} |f(x)f(\tilde{x})|(1 \wedge |x\eta|)^a(1 \wedge |\eta\tilde{x}|)^b|\eta|^c e^{-\sigma^2 \eta^2} d\eta d\tilde{x} \leq \frac{C}{\sigma + c(1 \vee \sigma)} \|f\|_a \|f\|_b,
\]
where, we recall that $\|f\|_w := \int_{\mathbb{R}} |f(x)|(1 + |x|^w) dx$ for $w > 0$.

In addition,
(5.50) \[
\int_{\mathbb{R}^2} |f(x)(1 \wedge |x\eta|)^a|\eta|^c e^{-\sigma^2 \eta^2} d\eta dx \leq \frac{C}{\sigma + c(1 \vee \sigma)} \|f\|_a,
\]

**Proof.** Define
\[
A := \int_{\mathbb{R}^3} |f(x)f(\tilde{x})|(1 \wedge |x\eta|)^a(1 \wedge |\eta\tilde{x}|)^b|\eta|^c e^{-\sigma^2 \eta^2} d\eta d\tilde{x}.
\]
By first making the change of variables $y = \sigma \eta$, we get
\[
A = \sigma^{-1-c} \int_{\mathbb{R}^3} |f(x)f(\tilde{x})|(1 \wedge |y \tilde{x}/\sigma|)^a(1 \wedge |y x/\sigma|)^b |y|^c e^{-y^2} dy d\tilde{x}.
\]
If $\sigma \geq 1$, we bound the terms $1 \wedge |y x/\sigma|$ and $1 \wedge |y \tilde{x}/\sigma|$ by $|y x/\sigma|$ and $|y \tilde{x}/\sigma|$ respectively, which gives
\[
A \leq \frac{1}{\sigma^{1+a+b+c}} \int_{\mathbb{R}^3} |y|^{a+b+c}|x|^a|\tilde{x}|^b|f(x)f(\tilde{x})| e^{-y^2} dy d\tilde{x} \\
\leq \frac{C}{\sigma^{1+a+b+c}} \|f\|_a \|f\|_b \int_{\mathbb{R}^3} |y|^{a+b+c} e^{-y^2} dy.
\]
If $\sigma \leq 1$, we bound the terms $1 \wedge |y x/\sigma|$ and $1 \wedge |y \tilde{x}/\sigma|$ by one, to obtain
(5.52) \[
A \leq \frac{C}{\sigma^{1+c}} \int_{\mathbb{R}^3} |y|^c |f(x)f(\tilde{x})| e^{-y^2} dy d\tilde{x} \leq \frac{C}{\sigma^{1+c}} \|f\|_a \|f\|_b \int_{\mathbb{R}} |y|^c e^{-y^2} dy.
\]
Relation (5.49) follows from (5.51) and (5.52). The proof of (5.50) is proved in a similar way.
To prove (3.22) we need the following lemma.

**Lemma 5.10.** For any $H > \frac{1}{3}$, we have

$$I := \int_0^\infty \int_0^\infty (s_1 s_2)^{H-\frac{1}{2}} (s_1^{2H} + |s_2 - s_1|^{2H})^{-\frac{3}{2}} \left(1 \lor \sqrt{s_1^{2H} + |s_2 - s_1|^{2H}}\right)^{-2} \, ds_1 \, ds_2 < \infty.$$ 

**Proof.** We can write, making the change of variable $s_2 - s_1 = y$ and $s_1 = s$,

$$I \leq \int_0^\infty \int_0^\infty s^{H-\frac{1}{2}} (s + y)^{H-\frac{1}{2}} (s^{2H} + y^{2H})^{-\frac{3}{2}} \left(1 \lor \sqrt{s^{2H} + y^{2H}}\right)^{-2} \, ds \, dy.$$ 

For $H > \frac{1}{2}$ we use the estimate $(s + y)^{H-\frac{1}{2}} \leq s^{H-\frac{1}{2}} + y^{H-\frac{1}{2}}$ and for $H < \frac{1}{2}$ we write $(s + y)^{H-\frac{1}{2}} \leq \left[\max(s,y)\right]^{H-\frac{1}{2}}$. In this way we obtain

$$I_1 \leq \int_0^\infty \int_0^y s^{2H-1} \left(1 \lor \left[\max(s,y)\right]^{H-\frac{1}{2}}\right) (s^{H-\frac{1}{2}} + y^{H-\frac{1}{2}}) y^{-3H} \left(1 \lor y^{H}\right)^{-2} \, ds \, dy \leq C \int_0^\infty y^{-H} (1 \lor y^{H})^{-2} \, dy < \infty.$$

\[ \square \]

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