An improved phase error tolerance in quantum search algorithm

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Abstract

As the matching condition in Grover search algorithm is transgressed due to inevitable errors in phase inversions, it gives a reduction in maximum probability of success. With a given degree of maximum success, we have derive the generalized and improved criterion for tolerated error and corresponding size of quantum database under the inevitable gate imperfections. The vanished inaccuracy to this condition has also been shown. Besides, a concise formula for evaluating minimum number of iterations is also presented in this work.

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Grover’s quantum search algorithm[1] provides a quadratic speedup over classical counterpart, and it has been proved to be optimal for searching a marked element with minimum oracle calls[2]. It is achieved by applying Grover kernel on an uniform superposition state, which is obtained by applying Walsh-Hadamard transformation on a initial state, in a specific operating steps such that the probability amplitude of marked state is amplified to a desired one. Grover’s kernel is composed of phase rotations and Walsh-Hadamard transformations. The phase rotations include two kinds of operations : π-inversion of the marked state and π-inversion of the initial state. It has shown that the phases, π, can be replaced by two angles, φ and θ, under the phase matching criterion, which is the necessary condition for quantum searching with certainty. In other words, the relation between φ and θ will affect the degree of success of quantum search algorithm. There have been several studies concern with the effect of imperfect phase rotations. In their paper[3], Long et al. have found that the tolerated angle difference between two phase rotations, δ, due to systematic errors in phase inversions, with a given expected degree of success $P_{\text{max}}$, is about $2/\sqrt{NP_{\text{max}}}$, where $N$ is the size of the database. HØyer[4] has shown that after some number of iterations of Grover kernel, depending on $N$ and unperturbed $\theta$, it will give a solution with error probability $O(1/N)$ under a tolerated phase
difference $\delta \sim O(1/\sqrt{N})$. The same result is also derived by Biham et al.[5]. On the other hand, a near conclusion, $\delta \sim O(1/N^{2/3})$, is presented by Pablo-Norman and Ruiz-Altaba[6].

The result of Long et al[3] is based on the approximate Grover kernel and an assumption: large $N$ and small $\delta$ et al. However, we found that the main inaccuracy comes from the approximate Grover kernel. Since all parameters in Grover kernel connect with each other exquisitely, any reduction to the structure of Grover’s kernel would destroy this penetrative relation, so accumulative errors emerge from the iterations to a quantum search. Although this assumption lead their study to a proper result, it cannot be applied to general cases, e.g. any set of two angles in phase rotations satisfies phase matching condition[7][8]. In what follows, we will get rid of the approximation to Grover kernel, then derive an improved criterion for tolerated error in phase rotation and the required number of qubits for preparing a database. Besides, a concise formula for evaluating minimum number of iterations to achieve a maximum probability will also be acquired. By this formula then evaluating the actual maximum probability, one can realize the derived criterion for tolerated error is near exactly.

The Grover kernel is composed of two unitary operators $G_{\tau}$ and $G_{\eta}$, given by

$$
G_{\tau} = I + (e^{i\phi} - 1)|\tau\rangle \langle \tau|,
$$

$$
G_{\eta} = I + (e^{i\theta} - 1)W |\eta\rangle \langle \eta| W^{-1},
$$

where $W$ is Walsh-Hadamard transformation, $|\tau\rangle$ is the marked state, $|\eta\rangle$ is the initial state, and $\phi$ and $\theta$ are two phase angles. It can also be expressed in a matrix form as long as an orthonormal set of basis vectors is chosen. The orthonormal set is

$$
|I\rangle = |\tau\rangle \text{ and } |\tau_\perp\rangle = (W |\eta\rangle - W_{\tau\eta} |\tau\rangle)/l,
$$

where $W_{\tau\eta} = \langle \tau | W |\eta\rangle$ and $l = (1 - |W_{\tau\eta}|^2)^{1/2}$. Letting $W_{\tau\eta} = \sin(\beta)$, we can write, from (2),

$$
|s\rangle = W |\eta\rangle = \sin(\beta) |\tau\rangle + \cos(\beta) |\tau_\perp\rangle,
$$

and Grover kernel can now be written

$$
G = -G_{\eta}G_{\tau} = \begin{bmatrix}
  e^{i\phi}(1 + (e^{i\theta} - 1)\sin^2(\beta)) & (e^{i\theta} - 1)\sin(\beta)\cos(\beta) \\
  e^{i\phi}(e^{i\theta} - 1)\sin(\beta)\cos(\beta) & 1 + (e^{i\theta} - 1)\cos^2(\beta)
\end{bmatrix}.
$$

After $m$ number of iterations, the operator $G^m$ can be expressed as

$$
G^m = (-1)^m e^{im(\phi+\theta)} \begin{bmatrix}
  e^{imw\cos^2(x)} + e^{-imw}\sin^2(x) & e^{-i\tau}i\sin(mw)\sin(2x) \\
  e^{i\tau}i\sin(mw)\sin(2x) & e^{imw}\sin^2(x) + e^{-imw}\cos^2(x)
\end{bmatrix},
$$

where the angle $w$ is defined by

$$
\cos(w) = \cos\left(\frac{\phi - \theta}{2}\right) - 2\sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin^2(\beta),
$$
or

$$\sin(w) = \sqrt{\left(\sin\left(\frac{\theta}{2}\right) \sin(2\beta)\right)^2 + \left(\sin\left(\phi - \frac{\theta}{2}\right) + 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi}{2}\right) \sin(\beta)\right)^2},$$  \hspace{1cm} (7)

the angle $x$ is defined by

$$\sin(x) = \sin\left(\frac{\theta}{2}\right) \sin(2\beta) / \sqrt{l_m},$$  \hspace{1cm} (8)

where

$$l_m = (\sin(w) + \sin\left(\frac{\phi - \theta}{2}\right) + 2 \cos\left(\frac{\phi}{2}\right) \sin\left(\frac{\theta}{2}\right) \sin^2(\beta))^2 + (\sin\left(\frac{\theta}{2}\right) \sin(2\beta))^2$$

$$= 2 \sin(w)(\sin(w) + \sin\left(\frac{\phi - \theta}{2}\right) + 2 \cos\left(\frac{\phi}{2}\right) \sin\left(\frac{\theta}{2}\right) \sin^2(\beta)).$$

More details can be found in the study[8]. Then the probability of finding a marked state is

$$P = 1 - |\langle \tau_\perp | G^m | s \rangle|^2$$
$$= 1 - \left(\cos(mw) \cos(\beta) - \sin(mw) \sin\left(\frac{\phi}{2}\right) \sin(2x) \sin(\beta)\right)^2$$
$$- \sin^2(mw)\left(\cos\left(\frac{\phi}{2}\right) \sin(2x) \sin(\beta) - \cos(2x) \cos(\beta)\right)^2.$$  \hspace{1cm} (9)

Moreover, by the equation $\partial P/\partial (\cos(mw)) = 0$, the minimum number of iterations for obtaining the maximum probability, $P_{\text{max}}(\cos(m_{\text{min}}w))$, is evaluated,

$$m_{\text{min}}(\beta, \phi, \theta) = \frac{\cos^{-1}\left(\sqrt{\frac{b - 2a}{2b}}\right)}{w},$$  \hspace{1cm} (10)

where

$$a = \sin(2x) \cos(2\beta) + \cos(2x) \cos\left(\frac{\phi}{2}\right) \sin(2\beta),$$
$$b = (2 + \sin^2(2x) + (3 \sin^2(2x) - 2) \cos(4\beta) - 2 \sin^2(2x) \cos(\phi) \sin^2(2\beta))$$
$$+ 2 \sin(4x) \cos\left(\frac{\phi}{2}\right) \sin(4\beta).$$

For a sure-success search problem, the phase condition, $\phi = \theta$, provided iterations, $m_{\text{min}} = (\pi/2 - \sin^{-1}(\sin(\phi/2) \sin(\beta))/w$, is required. However, when effects of imperfect phase inversions are considered, the search is not certain, then the new condition to phase error, said $\delta = \phi - \theta$, and the size of database would be rederived in order to accomplish the search with a reduced maximum probability. Now, we suppose the database is large, i.e., if $\sin(\beta) << 1$, and a phase error $\delta$ is small, where $|\delta| << 1$, one will have the following approximation, viz.,
\[
\cos(w) = \cos(\frac{\delta}{2}) - 2\sin(\frac{\theta}{2} + \frac{\delta}{2})\sin(\frac{\theta}{2})\sin^2(\beta)
\]
\[
\approx 1 - (\frac{\delta^2}{8} + 2\beta^2\sin^2(\frac{\theta}{2}))
\]
\[
\sin(w) = (1 - \cos^2(w))^{1/2}
\]
\[
\approx \frac{(\delta^2 + 16\beta^2\sin^2(\frac{\theta}{2}))^{1/2}}{2}
\]
\[
\sin(2x) = \frac{4\beta\sin(\frac{\theta}{2})}{(\delta^2 + 16\beta^2\sin^2(\frac{\theta}{2}))^{1/2}}
\]

The probability \( P \), equation (9), then has the approximation

\[
P \approx 1 - \cos^2(mw)\cos^2(\beta) - \sin^2(mw)\cos^2(2x)
\]
\[
= \sin^2(mw)\sin^2(2x)
\]

with a maximum value, by letting \( \sin^2(mw) = 1 \),

\[
P_{\text{max}} \approx \sin^2(2x) = \frac{16\beta^2\sin^2(\frac{\theta}{2})}{\delta^2 + 16\beta^2\sin^2(\frac{\theta}{2})}
\]

The function (12) for two designations, \( \delta = 0.01 \) and \( \delta = 0.001 \), are depicted in Fig. 1 and Fig. 2 respectively.

Observing Fig. 1 and Fig. 2, one realizes the function (12) depicted by solid line coincides with the exact value, obtained by Eq. (9) and Eq. (10), shown by cross marks. On the contrary, the result of Long et al.,

\[
P_{\text{max}} \approx \frac{4\beta^2\sin^2(\frac{\theta}{2})}{\delta^2 + 4\beta^2\sin^2(\frac{\theta}{2})}
\]

is an underestimation depicted by dash lines.

To summarize, under the inevitable gate imperfections, we have derive the generalized and improved criterion (12), by the exact formulation of Grover kernel after \( m \) numbers of iterations and the approximation to small values of involved parameters, for tolerated error and its corresponding size of quantum database. Moreover, the minimum number of iterations for obtaining the maximum probability, \( m_{\text{min}}(\beta, \phi, \theta) \), is also presented. By observing Fig.1 and Fig.2, one can realize the improved criterion (12) is near an exact one. Besides, utilizing condition (12), one can realize that the value of tolerated error decreases with the growth of database, in other words, it is important to have a good control over tolerated error if we have a large quantum database. Therefore, quantum search machines should avoided gate imperfections as much as possible. If we cannot get rid of these errors, we must limit the size of a quantum database accurately. The result of this study presents a more accurate characterization of the relation between systematic errors and the size of a quantum database.
A nearly exact criterion (12) can be utilized in order to achieve the practical equilibrium between the actual gate imperfection and the size of the quantum database.

- **Figure Caption:**
  
  FIG. 1. Variations of exact value of $P_{\text{max}}(n)$ (cross marks), $16\beta^2 \sin^2(\theta/2) / (\delta^2 + 16\beta^2 \sin^2(\theta/2))$ (solid), and $4\beta^2 / (\delta^2 + 4\beta^2)$ (dash) for $\theta = \pi$, $\delta = 0.01$ where $\beta = \sin^{-1}(2^{-n/2})$.

  FIG. 2. Variations of exact value of $P_{\text{max}}(n)$ (cross marks), $16\beta^2 \sin^2(\theta/2) / (\delta^2 + 16\beta^2 \sin^2(\theta/2))$ (solid), and $4\beta^2 / (\delta^2 + 4\beta^2)$ (dash) for $\theta = \pi$, $\delta = 0.001$ where $\beta = \sin^{-1}(2^{-n/2})$.

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