A constrained proof of the strong version of the Eshelby conjecture for three-dimensional isotropic media

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Eshelby’s seminal work on the ellipsoidal inclusion problem leads to the conjecture that the ellipsoid is the only inclusion possessing the uniformity property that a uniform eigenstrain is transformed into a uniform elastic strain. For the three-dimensional isotropic medium, the weak version of the Eshelby conjecture has been substantiated. The previous work (Ammari et al., 2010) substantiates the strong version of the Eshelby conjecture for the cases when the three eigenvalues of the eigenstress are distinct or all the same, whereas the case where two of the eigenvalues of the eigenstress are identical and the other one is distinct remains a difficult problem. In this work, we study the latter case. To this end, firstly, we present and prove a necessary condition for an inclusion being capable of transforming a uniform eigenstress into a uniform elastic stress field. Since the necessary condition is not enough to determine the shape of the inclusion, secondly, we introduce a constraint that is concerned with the material parameters, and by introducing the concept of dissimilar media we prove that there exist combinations of uniform eigenstresses and the elastic tensors of dissimilar isotropic media such that only an ellipsoid can have the Eshelby uniformity property for these combinations simultaneously. Finally, we provide a more specifically constrained proof of the conjecture by proving that for the uniform strain fields constrained to those induced by an ellipsoid from a set of specified uniform eigenstresses, the strong version of the Eshelby conjecture is true for a set of isotropic elastic tensors which are associated with the specified uniform eigenstresses. This work makes some progress towards the complete solution of the intriguing and longstanding Eshelby conjecture for three-dimensional isotropic media.

Eshelby conjecture, Inclusion problem, Isotropic medium, Eigenstrain, Elasticity

\textbf{1. Introduction}

Eshelby’s work \cite{1, 2} on the inclusion problem is essential to the development of the theories for the mechanical performance of heterogeneous materials. Mura introduced the concept of eigenstrain, and conducted a series of studies via the Eshelby formalism \cite{3}. Many researchers have investigated inclusion problems from diverse aspects. For instance, from the aspect of multi-physical fields, the generalized Eshelby formalisms for piezoelectric inclusions \cite{4, 5} and inclusions governed by more general coupled-fields \cite{6} have been developed; from the aspect of multi-scales, the interface effect of nano-inclusions and nano-inhomogeneities has been studied \cite{7-12}, and the inclusion problem in the context of second gradient elasticity has been investigated \cite{13}. In addition, the classical Eshelby’s inclusion theory for solids has been extended to investigations of liquid inclusions in soft materials \cite{14, 15}. 

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In the field of the inclusion problem, the most significant and fantastic phenomenon is the Eshelby uniformity property of the ellipsoid, which means that the uniform eigenstrain (eigenstress) prescribed in an ellipsoidal inclusion in an infinite medium induces a uniform elastic strain (stress) field inside the inclusion. In 1961, Eshelby [16] conjectured that the ellipsoid uniquely possesses such marvelous uniformity property, which has long been a difficult problem to be proved or disproved. According to previous researches [17-19], the Eshelby conjecture can be more specifically understood in two senses, i.e., the weak and strong versions. The weak version asserts that “an ellipsoid alone transforms all uniform eigenstrains (eigenstresses) into uniform elastic strain (stress) fields in it”, and the strong version asserts that “no inclusion other than an ellipsoid transforms a single uniform eigenstrain (eigenstress) into a uniform elastic strain (stress) field in it” [20]. Note that there are also other ways of statements of the versions [17-19]. It is easily seen that the validity of the strong version leads to the validity of the weak version.

For the isotropic medium in three dimensions, in 2008, the proof of the weak version was fulfilled by Liu [17] with the utilization of the obstacle function on the basis of inclusion problem, and by Kang and Milton [18] with the utilization of the single-layer potential on the basis of the inhomogeneity problem. The conjecture is also valid for the three-dimensional conductivity problem [21]. For the isotropic medium in two dimensions, the proof of the Eshelby conjecture has been fulfilled [22-24]. For anisotropic media, the weak and strong versions of the Eshelby conjecture have already been proved to be valid in two dimensions [19]. For the three-dimensional case, Yuan et al. [20] recently proved that the weak version for cubic, transversely isotropic, orthotropic, and monoclinic symmetries is valid, but there are counterexamples to the strong version.

However, the strong version of the conjecture for the three-dimensional isotropic case has not been fully resolved, though some progress has been made. It was proved by Markenkooff [25] in 1997 that the inclusion that possesses the Eshelby uniformity property cannot have any planus surface. In 1998, Lubarda and Markenkooff [26] further drew the conclusion that the surface of the inclusion that possesses the Eshelby uniformity property needs to satisfy some particular conditions. Then Markenkooff [27] also proved that the only way to assure the Eshelby uniformity property is the infinitesimal perturbation of the ellipsoid into another ellipsoid.

In 2010, a further step towards the proof of the strong version was made by Ammari et al. [28]. By categorizing the induced strain field based on its eigenvalues, Ammari et al. [28] proved that when the three eigenvalues of the elastic strain field corresponding to the remote loading are either all the same or all distinct, the ellipsoidal inhomogeneity uniquely possesses the Eshelby uniformity property.

It is noted that the result given by Ammari et al. [28] for the inhomogeneity problem in the isotropic medium can be extended to prove that the ellipsoidal inclusion uniquely possesses the Eshelby uniformity property, when the eigenvalues of the eigenstress are either all distinct or all identical, that is, the strong version of the Eshelby conjecture is true for these two special cases. In this work, we consider the remaining case where two of the eigenvalues of the eigenstress are identical, and the other one is distinct. In this case, we show and prove three theorems. The first theorem constitutes a necessary condition for an inclusion to possess the Eshelby uniformity property. However, the necessary condition alone cannot help us to determine the shape of the inclusion. Therefore, in the second theorem, we bring in an additional constraint associated with the material parameters, and find that the inclusion can only be an ellipsoid owing to the necessary condition under the constraint that the inclusion possesses the Eshelby uniformity property for two dissimilar isotropic media, where the concept of dissimilar media will be defined in the sequel. Thus, the second theorem provides a constrained proof of the strong version. Further, we evaluate the elastic field induced by an arbitrary inclusion with the Eshelby uniformity property, and find that the uniform strain field induced by an inclusion of any non-ellipsoidal shape, if there is one, cannot be equal to that induced by an ellipsoid for the same uniform eigenstress, which constitutes the third theorem. Thus the third theorem provides an alternative constrained proof of the strong version from another viewpoint when the induced strain field is constrained to be equal to that induced by an ellipsoid for the same uniform eigenstress. All together, this work makes some progress towards the complete resolution of the strong version of the Eshelby conjecture for three-dimensional isotropic media.

2. Basic equations

Let \( \Omega \subset \mathbb{R}^3 \) denote the inclusion domain, which is a one-component connected bounded domain with a Lipschitz boundary. The equilibrium equation for Eshelby’s inclusion problem in the infinite elastic homogenous isotropic medium is

\[
\nabla \cdot \left( \mathbf{C} : (\nabla \otimes \mathbf{u}) - \chi_D \sigma^* \right) = 0 \quad \text{in} \quad \mathbb{R}^3, 
\]

where \( \otimes \) denotes the tensorial product; \( \mathbf{u} \) denotes the displacement field, which is a vector; \( \nabla \) denotes the gradient operator; \( \sigma^* \) denotes a uniform eigenstress, which is a symmetric second-order tensor; \( \chi_D \) denotes the indicator function
with respect to $\Omega$, which satisfies

$$
\chi_{\Omega} = \begin{cases} 
1, & \text{in } \Omega, \\
0, & \text{in } \mathbb{R}^3 \setminus \Omega,
\end{cases} 
$$

(2)

and

$$
\mathbf{C} = [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l
$$

(3)

denotes the fourth-order elastic tensor for the isotropic medium with $\delta_{ij}$ being the Kronecker delta. $\lambda$ and $\mu$ are Lamé parameters that satisfy

$$
\mu > 0, \quad 3\lambda + 2\mu > 0,
$$

(4)

and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the bases of a Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$. Unless otherwise stated, the summation convention on repeated indices will always be stipulated.

By the Fourier analysis [17,20], the solution to Eq. (1) is

$$
\sigma(p)(\mathbf{x}) = -\frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} L_{pq}(\mathbf{q}) \sigma^j_q \mathbf{q} \int_{\Omega} e^{i(x-y) \cdot \mathbf{q}} \, dy, 
$$

(5)

where $i = \sqrt{-1}$ denotes the imaginary unit, and

$$
L_{pq}(\mathbf{q}) = \frac{1}{\mu |\mathbf{q}|^2} \delta_{pq} - \frac{\mu + \lambda}{2(\mu + \lambda)} \frac{\mathbf{q} \cdot \mathbf{q}}{|\mathbf{q}|^4}.
$$

(6)

By substituting Eq. (3) into Eq. (6), we gain

$$
L_{pq}(\mathbf{q}) \xi_p \xi_q = \delta_{pm}.
$$

(7)

Owing to the isotropy of the elastic tensor $\mathbf{C}$, which means that $\mathbf{C}$ possesses the same expression (3) in all Cartesian coordinate systems, we see that Eq. (7) and thus Eq. (5) do not vary with the rotation of the Cartesian coordinate system. Thus to simplify the derivations in the sequel, we let the axes of the Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ coincide with the three mutually orthogonal unit eigenvectors of the eigenstress $\sigma^*$. Then in such a coordinate system, the off-diagonal elements of $\sigma^*$ are zero, and the diagonal elements $\sigma^*_{11}, \sigma^*_{22}, \sigma^*_{33}$ are the three eigenvalues of $\sigma^*$.

In this coordinate system, the inclusion $\Omega$ that possesses the Eshelby uniformity property must yield

$$
\frac{\partial u_p(\mathbf{x})}{\partial x_i} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} L_{pq}(\mathbf{q}) \sigma^*_{pq} \mathbf{q} \int_{\Omega} e^{i(x-y) \cdot \mathbf{q}} \, dy, 
$$

(8)

$$
= \text{constant}.
$$

Note that the left-hand side of Eq. (8) corresponds to the total strain induced by $\Omega$. Since the total strain is the sum of the elastic strain and the uniform eigenstrain, thus the uniformity of the total strain is equivalent to the uniformity of the elastic strain.

Thus, based on Eq. (8), the completion of the proof of the strong version of the Eshelby conjecture for the three-dimensional isotropic medium can be achieved by classifying the three eigenvalues $\sigma^*_{11}, \sigma^*_{22}, \sigma^*_{33}$ of the eigenstress into three cases: they are all identical; they are all distinct; and two of them are identical and the other one is distinct, and then proving $\Omega$ that leads to Eq. (8) must be ellipsoidal for these cases.

As is mentioned before, the strong version for the eigenstress possessing either all identical or all distinct eigenvalues can be proved by extending the result of Ammari et al. [28] to the inclusion problem via the same mathematical manipulations in Sect. 3 of their paper. And we also provide an alternative proof for this case by using the Fourier analysis in 4. Therefore, only the case where two eigenvalues of the eigenstress are identical and the other one is distinct will be studied in detail in the following sections. Note that the classification of the eigenstress is equivalent to the classification of the eigenstrain due to the isotropy of the elastic tensor.

3. A constrained proof of the strong version

We define

$$
g(\mathbf{x}, \mathbf{q}) := \int_{\Omega} e^{i(x-y) \cdot \mathbf{q}} \, dy,
$$

(9)

and let $\sigma^*_{11} = \sigma^*_{22} = k_1, \sigma^*_{33} = k_3$ with $k_1 \neq k_3$.

Since the inclusion $\Omega$ with the Eshelby uniformity property satisfies Eq. (8), then by substituting $\sigma^*_{11} = \sigma^*_{22} = k_1, \sigma^*_{33} = k_3$ along with Eqs. (7) and (9) into Eq. (8), we obtain

$$
\forall j = 1, 2, 3, 
$$

$$
\int_{\mathbb{R}^3} \xi_1 \xi_j \left( \frac{\mu k_1}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{constant}, 
$$

(10)

$$
\int_{\mathbb{R}^3} \xi_2 \xi_j \left( \frac{\mu k_1}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{constant}, 
$$

$$
\int_{\mathbb{R}^3} \xi_3 \xi_j \left( \frac{k_3(\lambda + 2\mu) - k_1(\lambda + \mu)}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{constant}, 
$$

Moreover, it follows from Eq. (10) that

$$
\frac{\partial}{\partial x_1} \int_{\mathbb{R}^3} \xi_1 \xi_j \left( \frac{\mu k_1}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{Linear}, 
$$

(11)

$$
\frac{\partial}{\partial x_2} \int_{\mathbb{R}^3} \xi_2 \xi_j \left( \frac{\mu k_1}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{Linear}, 
$$

$$
\frac{\partial}{\partial x_3} \int_{\mathbb{R}^3} \xi_3 \xi_j \left( \frac{k_3(\lambda + 2\mu) - k_1(\lambda + \mu)}{|\mathbf{q}|^2} + \frac{(\lambda + \mu)(k_1 - k_3)|\xi_j|^2}{|\mathbf{q}|^4} \right) g(\mathbf{x}, \mathbf{q}) = \text{Linear},
$$

for $\mathbf{x} \in \Omega$. 

which finally leads to

$$
\int_{\mathbb{R}^3} \left( \frac{\mu k_1}{|\zeta|^2} + \frac{(\lambda + \mu)(k_1 - k_3) \zeta_3^2}{|\zeta|^4} \right) g(x, \zeta) d\zeta = q_1(x) + \varphi_1(x_3),
$$

$$
\int_{\mathbb{R}^3} \frac{k_3(\lambda + 2\mu) - k_1(\lambda + \mu)}{|\zeta|^4} d\zeta + \frac{(\lambda + \mu)(k_1 - k_3) \zeta_3^2}{|\zeta|^4}
$$

(12)

\cdot g(x, \zeta) d\zeta = q_2(x) + \varphi_2(x_1, x_2), \quad x \in \Omega,

where \( q_1(x) \) and \( q_2(x) \) denote two quadratic functions, and \( \varphi_1(x_3) \) and \( \varphi_2(x_1, x_2) \) denote two unknown functions.

To continue the analysis, we introduce two potentials. The first one is the well-known Newtonian potential \( N_\Omega(x) \) of the inclusion domain \( \Omega \), which can be expressed as

$$
N_\Omega(x) = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{g(x, \zeta)}{|\zeta|^2} d\zeta.
$$

(13)

Note that the Newtonian potential \( N_\Omega(x) \) in Eq. (13) is exactly the first term of the integral on the left-hand side of Eq. (12). Besides, we know that the Newtonian potential \( N_\Omega(x) \) satisfies

$$
\begin{cases}
\Delta N_\Omega(x) = \chi_\Omega, & x \in \mathbb{R}^3, \\
N_\Omega(x) = O \left( \frac{1}{|x|} \right), & \text{as } |x| \to +\infty,
\end{cases}
$$

(14)

where \( \Delta \) denotes the Laplacian operator, and \( O(\cdot) \) denotes the order of magnitude. Eq. (14) admits a unique solution, i.e.,

$$
N_\Omega(x) = -\frac{1}{4\pi} \int_\Omega \frac{1}{|x - y|} dy,
$$

(15)

which provides an explicit expression of \( N_\Omega(x) \).

The second one is a bi-harmonic potential \( H(x) \) expressed as

$$
H(x) = -\frac{1}{8\pi} \int_\Omega \frac{(x_3 - y_3)^2}{|x - y|^3} dy.
$$

(16)

By combining Eq. (16) with Eq. (15), it is straightforward to verify that

$$
\Delta H(x) = N_\Omega(x);
$$

(17)

thus by Eq. (14),

$$
\Delta^2 H(x) = \chi_\Omega, \quad x \in \mathbb{R}^3,
$$

(18)

which indicates \( H(x) \) is bi-harmonic in \( \mathbb{R}^3 \setminus \Omega \).

According to the derivations in 4, we can derive from Eqs. (16) and (17) that

$$
\frac{\partial^2 H(x)}{\partial x_3^2} = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\zeta_3^2}{|\zeta|^4} g(x, \zeta) d\zeta,
$$

(19)

the right-hand side of which is exactly the second term of the integral on the left-hand side of Eq. (12). Therefore, substitution of Eqs. (13) and (19) into Eq. (12) yields

$$
\alpha N_\Omega(x) + \frac{\partial^2 H(x)}{\partial x_3^2} = q_1(x) + \varphi_1(x_3),
$$

$$
\beta N_\Omega(x) + \frac{\partial^2 H(x)}{\partial x_3^2} = q_2(x) + \varphi_2(x_1, x_2), \quad x \in \Omega,
$$

(20)

where \( \alpha \) and \( \beta \) are two real constants defined by

$$
\alpha := \frac{\mu k_1}{(\lambda + \mu)(k_1 - k_3)}, \quad \beta := \frac{k_3(\lambda + 2\mu) - k_1(\lambda + \mu)}{(\lambda + \mu)(k_1 - k_3)}.
$$

(21)

Note that, here, the values of \( q_i \) and \( \varphi_i \) in Eq. (20) are equal to \( \frac{1}{2(2\pi)^3} \) times \( q_i \) and \( \varphi_i \) in Eq. (12). Since \( k_1 \neq k_3 \), it is easy to verify that for any combination of the Lamé parameters \( \lambda, \mu \) that satisfy Eq. (4), Eq. (21) is always valid, and \( \alpha \neq \beta \).

Further, the explicit expression of \( \frac{\partial^2 H(x)}{\partial x_3^2} \) in Eq. (20) can be derived from Eq. (16), i.e.,

$$
\frac{\partial^2 H(x)}{\partial x_3^2} = -\frac{1}{8\pi} \int_\Omega \frac{(x_3 - y_3)^2}{|x - y|^3} dy.
$$

(22)

Given this, we define

$$
\tilde{N}_\Omega(x) := -\frac{1}{4\pi} \int_\Omega \frac{(x_3 - y_3)^2}{|x - y|^3} dy.
$$

(23)

Then by substituting Eqs. (15), (22), and (23) into Eq. (20), we can rewrite Eq. (20) as

$$
g(N_\Omega(x) + \tilde{N}_\Omega(x)) = q_1(x) + \varphi_1(x_3),
$$

$$
\eta N_\Omega(x) + \tilde{N}_\Omega(x) = q_2(x) + \varphi_2(x_1, x_2), \quad x \in \Omega,
$$

(24)

where the values of \( q_i \) and \( \varphi_i \) are now equal to -2 times \( q_i \) and \( \varphi_i \) in Eq. (20); \( N_\Omega(x) \) and \( \tilde{N}_\Omega(x) \) both have explicit expressions, i.e., Eqs. (15) and (23), respectively; and

$$
\gamma := \frac{k_3(\lambda + \mu) - k_1(\lambda + 3\mu)}{(\lambda + \mu)(k_1 - k_3)},
$$

$$
\eta := \frac{k_3(\lambda + 3\mu) - k_1(\lambda + \mu)}{(\lambda + \mu)(k_1 - k_3)}
$$

(25)

are real constants determined by the elastic tensors \( C \) (Lamé parameters \( \lambda, \mu \)) and the eigenvalues \( k_1, k_3 \) of the eigenstress \( \sigma^* \).

Recently, Yuan et al. [20] demonstrate that the material parameters can serve as an important factor in the study of the Eshelby conjecture. Likewise, we also consider the influence of the material parameters in this work. To prove the strong version of the Eshelby conjecture for the eigenstress possessing only two identical eigenvalues is to prove the following proposition:
**Proposition 1.** Let \( \Omega \subset \mathbb{R}^3 \) be a one-component connected bounded open domain with a Lipschitz boundary. Equation (24) holds for \((\sigma^*, C)\), where \(\sigma^*\) is a given uniform eigenstress that possesses two identical eigenvalues \(k_1\) and a distinct eigenvalue \(k_3\), and \(C\) is an isotropic elastic tensor, if and only if \( \Omega \) is of ellipsoidal shape.

To prove or disprove Proposition 1, we investigate Eq. (24). Firstly, we derive a necessary condition for an inclusion possessing the Eshelby uniformity property from Eq. (24), which is stated in the following theorem:

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^3 \) be a one-component connected bounded open domain with a Lipschitz boundary. If Eq. (24) holds for \((\sigma^*, C)\), where \(\sigma^*\) is a given uniform eigenstress that possesses two identical eigenvalues \(k_1\) and a distinct eigenvalue \(k_3\), and \(C\) is an isotropic elastic tensor, then in the Cartesian coordinate system \(x = (x_1, x_2, x_3)\) defined by the eigenvectors of the eigenstress \(\sigma^*\), there must exist an ellipsoid \(E \subseteq \Omega\) that satisfies

\[
N_\Omega(x) - N_E(x) = \varphi(x_1, x_2), \quad x \in E, \tag{26}
\]

where \(\varphi(x_1, x_2)\) is an unknown function that only relies on two spatial coordinates, and \(N_E(x)\) is the Newtonian potential of the ellipsoid \(E\), whose expression is given in Eq. (15) with \(\Omega\) replaced by \(E\).

Theorem 1 implies that the Newtonian potential of an inclusion \(\Omega\) that possesses the Eshelby uniformity property for the eigenstress possessing only two identical eigenvalues is necessarily correlated with the Newtonian potential of an ellipsoid \(E\) via Eq. (26). If \(\Omega\) is ellipsoidal, Eq. (26) is automatically satisfied. Thus, the significance of Theorem 1 is that a non-ellipsoidal inclusion that does not satisfy Eq. (26) is excluded from the set of inclusions that possess the Eshelby uniformity property.

Secondly, we find an extra constraint that can make Eq. (24) the sufficient and necessary condition for \(\Omega\) to be of ellipsoidal shape. The constraint is concerned with the material parameters. To clearly describe the constraint, we introduce the concept of dissimilar media; we define two dissimilar media as follows.

**Definition 1.** Let \(C_1\) and \(C_2\) be two \(6 \times 6\) matrices that express the elastic tensors of two media in the Voigt notation. Let \(a_i\) and \(b_i\) \((i = 1, 2, 3, 4, 5, 6)\) denote the eigenvalues of \(C_1\) and \(C_2\), respectively, and let \(a = (a_1, a_2, a_3, a_4, a_5, a_6)\) and \(b = (b_1, b_2, b_3, b_4, b_5, b_6)\); these two media are dissimilar, if \(a\) and \(b\) are linearly independent. Conversely, the two media are similar, if \(a\) and \(b\) are linearly dependent.

We remark that the above concept of dissimilar media provides a method to classify the materials, which, in addition to facilitating the study of the Eshelby conjecture in this work, may further help reveal the inherent dependence of the general solutions of the linear-elastic problems on the material parameters and hence motivate the development and reformation of classical formalisms of general solutions (e.g., the Stroh formalism) [29, 30].

In this work, we focus on the isotropic media. Let \(v_1\) and \(v_2\) be Poisson’s ratios of two isotropic media. Then, based on the definition of dissimilar media, we find that the Poisson’s ratios of two dissimilar isotropic media satisfy

\[
v_1 \neq v_2, \tag{27}
\]

and the Poisson’s ratios of two similar isotropic media satisfy

\[
v_1 = v_2. \tag{28}
\]

We note that Eq. (27) is equivalent to

\[
\lambda_1\mu_2 - \lambda_3\mu_1 \neq 0, \tag{29}
\]

and Eq. (28) is equivalent to

\[
\lambda_1\mu_2 - \lambda_3\mu_1 = 0, \tag{30}
\]

where \(\lambda_1, \mu_1\) and \(\lambda_2, \mu_2\) are the Lamé parameters of the two isotropic media. Besides, it deserves mentioning that for the two similar isotropic media with \(v_1 = v_2 = v\), the well-known Dunders’ parameters \(t_1, t_2\) for the bi-material system composed of the two similar isotropic media satisfy [31]

\[
t_1 = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}, \quad t_2 = \begin{cases} \frac{1 - 2v}{2 - 2v}, & \text{plane strain}, \\ \frac{1 - v}{2}, & \text{plane stress}. \end{cases} \tag{31}
\]

It has been shown that using linear independence of elastic tensors is helpful in the solutions to the Eshelby conjecture for anisotropic media [20]. Here, we find that the conditions of similarity and dissimilarity of two isotropic media, the latter case of which is a special case of the linear independence of general elastic tensors, are also useful. From Eq. (5), it is straightforward to verify that for an arbitrary eigenstrain field, the solutions to the eigenstrain problem possess the same form for two similar isotropic media, which means that the solution for one of the similar media is a linear function of the solution for the other, whereas the solutions to the eigenstrain problem possess distinct forms for two dissimilar isotropic media, which means that the solution for one of the dissimilar media is not a linear function of the solution for the other. We then propose and prove the following theorem.

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^3 \) be a one-component connected bounded open domain with a Lipschitz boundary. There exist combinations \((\sigma^*, C^{(1)})\) and \((\sigma^*, C^{(2)})\), where \(\sigma^*\) is a given uniform eigenstress that possesses two identical eigenvalues \(k_1\) and a distinct eigenvalue \(k_3\), and \(C^{(1)}\) and \(C^{(2)}\) are elastic tensors of two dissimilar isotropic media, such that Eq. (24) holds for \((\sigma^*, C^{(1)})\) and \((\sigma^*, C^{(2)})\) simultaneously, if and only if \(\Omega\) is of ellipsoidal shape.
By comparing Theorem 2 with Proposition 1, it is straightforward to see that the condition that Eq. (24) holds for \((\sigma^*, C^{(1)})\) and \((\sigma^*, C^{(2)})\) simultaneously is stronger than the condition that Eq. (24) holds for \((\sigma^*, C)\), which reveals extra constraints here. Thus Theorem 2 provides a constrained proof of Proposition 1 and thus the strong version of the Eshelby conjecture for the eigentress possessing only two identical eigenvalues. We cannot prove Theorem 2 if \(C^{(1)}\) and \(C^{(2)}\) are elastic tensors of two similar isotropic media, which will be revealed in the proof of Theorem 2.

It is seen that the above theorems do not exclude the existence of non-ellipsoidal inclusions that can satisfy the Eshelby uniformity property (otherwise the strong version is ultimately proved). Then, if there exist non-ellipsoidal inclusions that satisfy the Eshelby uniformity property, what are the (uniform) strain fields in them like? We explore the answer to this interesting question by the following theorem.

**Theorem 3.** Let \(\Omega \subset \mathbb{R}^3\) be a one-component connected bounded open domain with a Lipschitz boundary. There exists a set of isotropic elastic tensors and an ellipsoid \(E\) with \(E \supset \Omega\) such that the elastic strain field induced by \(\Omega\) in the medium with one of the elastic tensors in this set is equal to that induced by \(E\) within \(\Omega\) for a given uniform eigenstress \(\sigma^*\) that possesses two identical eigenvalues \(k_1\) and a distinct eigenvalue \(k_3\) with \(k_1/k_3 = \sqrt{2} - 1\), if and only if \(\Omega\) is of ellipsoidal shape.

Theorem 3 implies that for the uniform strain fields constrained to be equal to that induced by an ellipsoid from the specified uniform eigenstrains, the strong version of the Eshelby conjecture is true for a set of isotropic elastic tensors \(C\), which, as we will show below, depend on the eigenvalues \(k_1\) and \(k_3\).

Theorem 3 can be expressed mathematically as follows. We define a mapping \(F\) that maps the Cartesian product of the set \(\{\sigma^*\}\) of the uniform eigenstrains, the set \(\{C\}\) of isotropic elastic tensors, and the set \(\{\Omega\}\) of the configurations of the inclusions into the set \(\{\sigma^*\}\) of the induced elastic strains inside \(\Omega\) for the eigenstrain problem, i.e.,

\[
F : \{\sigma^*\} \times \{C\} \times \{\Omega\} \rightarrow \{\sigma^*\},
\]

\[(\sigma^*, C, \Omega) \mapsto F(\sigma^*, C, \Omega) \subset \mathbb{R}^{3 \times 3}.
\]

As is known from the previous researches [1-3],

\[
\forall (\sigma^*, C, E) \in \{\sigma^*\} \times \{C\} \times \{E\},
\]

\[
F(\sigma^*, C, E) = \text{constant},
\]

where \(E\) denotes an ellipsoid, and \(\{E\}\) denotes the set of ellipsoids, which is a subset of \(\{\Omega\}\). In terms of the mapping \(F\) defined by Eq. (32), the strong version of the Eshelby conjecture can be mathematically interpreted as

\[
\forall (\sigma^*, C) \in \{\sigma^*\} \times \{C\}, \quad F(\sigma^*, C, \Omega) = \text{constant},
\]

if and only if \(\Omega \in \{E\}\).

We let \(\sigma^*\) denote a special kind of the uniform eigenstress that possesses two identical eigenvalues \(k_1\) and a distinct eigenvalue \(k_3\) with \(k_1/k_3 = \sqrt{2} - 1\), and we let \(\{\sigma^*\}\) denote the set of \(\sigma^*\), which is a subset of \(\{\sigma^*\}\). Then Theorem 3 can be mathematically interpreted as

\[
\forall \sigma^* \in \{\sigma^*\},
\]

\[
\exists C \in \{C\}, \quad \Omega \in \{\Omega\}, \quad E \in \{E\}\text{ with } E \supset \Omega,
\]

such that \(F(\sigma^*, C, \Omega) = F(\sigma^*, C, E) \equiv \text{constant}\), if and only if \(\Omega \in \{E\}\).

We use \(\{C\}\) to denote the set of \(C\) that makes Eq. (35) valid. Then we can conclude from Eq. (35) that for the uniform strain fields constrained to that induced by an ellipsoid, i.e., \(F(\sigma^*, C, E)\), from the specified uniform eigenstress \(\sigma^*\), the strong version of the Eshelby conjecture is true for a set \(\{C\}\) of isotropic elastic tensors, which is associated with the set \(\{\sigma^*\}\) of the specified uniform eigenstress. In this regard, we will show that the Lamé parameters \(\lambda, \mu\) for the isotropic elastic tensor \(C\) depend on the eigenvalues \(k_1, k_3\) of the specified uniform eigenstress \(\sigma^*\).

Then we turn to prove these three theorems.

### 3.1 Proof of Theorem 1

It can be directly derived from Eq. (24) that

\[
N(\Omega) = \frac{q_1(x) + \varphi_2(x_1, x_2) - q_1(x) - \varphi_1(x_1, x_2)}{\eta - \gamma},
\]

\[
\tilde{N}(\Omega) = \frac{\eta q_1(x) + \eta \varphi_1(x_1, x_2) - \gamma q_2(x) - \gamma \varphi_2(x_1, x_2)}{\eta - \gamma}, \quad x \in \Omega.
\]

(36)

By substituting Eq. (36) into Eq. (14), we gain

\[
0 = \frac{\partial^2 \varphi_1(x_1, x_2)}{\partial x_3^2} + \frac{\partial^2 \varphi_2(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 \varphi_2(x_1, x_2)}{\partial x_2^2} = \text{constant},
\]

which indicates that \(\varphi_1(x_1)\) is a constant, linear or quadratic function, and \(\varphi_2(x_1, x_2)\) must satisfy

\[
\Delta \varphi_2(x_1, x_2) = \text{constant}.
\]

Up to now, it is straightforward to see that Eq. (36) can be rewritten as

\[
N(\Omega) = q'(x) + \varphi'(x_1, x_2), \quad x' \in \Omega,
\]

(39)

where \(q'(x) := \frac{q_1(x) - q_2(x)}{\eta - \gamma}\), and \(\varphi'(x_1, x_2) := \frac{\varphi_2(x_1, x_2)}{\eta - \gamma}\).

Then we are going to derive Eq. (26) from Eq. (39). Since any \(\Omega\) that possesses the Eshelby uniformity property needs to be convex [26], if \(p \in \partial \Omega\), there must be a supporting hyperplane \(S'(p)\) that passes through \(p\) but does not intersect the interior of \(\Omega\), which means \(\Omega\) is entirely on one side of
Then we can expand the right-hand side of Eq. (43) into the Taylor series at $M$ and then take $M$ as the origin of a new Cartesian coordinate system $x' = (x'_1, x'_2, x'_3)$, which yields

$$N_{Q}(x') = - \frac{1}{4\pi|x' - y'|^3} \int_{\Omega} \nabla \cdot \nabla N_{Q}(0) \cdot x' + \phi'(x')$$

(42)

for $x' \in U(0)$, where $y' = (y'_1, y'_2, y'_3)$ are the coordinates of the point within $\Omega$ on the basis of the new coordinate system; $\phi'(x')$ denotes the sum of the terms whose degrees are larger than 2; and $U(0) \subset \Omega$ is the neighborhood of the origin. Note that $\nabla \cdot \nabla N_{Q}(0)$ is positive definite due to the origin being the minimum point of $N_{Q}$.

It can be seen from Eq. (39) that in the new coordinate system $x' = (x'_1, x'_2, x'_3),

$$N_{Q}(x') = q'(x') + \phi'(x'_1, x'_2), \quad x' \in \Omega.$$  

(43)

Then we can expand the right-hand side of Eq. (43) into the Taylor series at the origin, which results in

$$N_{Q}(x') = q'(x') + \phi'(x'_1, x'_2), \quad x' \in U(0),$$

(44)

where $q'(x')$ is a quadratic function, and $\phi'(x'_1, x'_2)$ is the sum of the terms with respect to $x'_1$ and $x'_2$ whose degrees are larger than 2.

Then comparison of Eq. (44) with Eq. (42) leads to

$$\phi'(x'_1, x'_2) = \phi'(x'),$$

(45)

and

$$q'(x') = N_{Q}(0) + \frac{1}{2} x' \cdot \nabla \nabla N_{Q}(0) \cdot x'.$$

(46)

Let

$$q'(x'_1, x'_2) = q'(x') - q'(x'),$$

(47)

thus by substituting Eqs. (46) and (47) into Eq. (43), we obtain

$$N_{Q}(x') = N_{Q}(0) + \frac{1}{2} x' \cdot \nabla \nabla N_{Q}(0) \cdot x'$$

$$+ q'(x'_1, x'_2) + \phi'(x'_1, x'_2), \quad x' \in \Omega.$$  

(48)

Let $\phi''(x'_1, x'_2) := q''(x'_1, x'_2) + \phi'(x'_1, x'_2)$; it follows from Eq. (48) that

$$N_{Q}(x') = N_{Q}(0) + \frac{1}{2} x' \cdot \nabla \nabla N_{Q}(0) \cdot x' + \phi''(x'_1, x'_2), \quad x' \in \Omega.$$  

(49)

Note that the term $x' \cdot \nabla \nabla N_{Q}(0) \cdot x'$ on the right-hand side of Eq. (49) needs to satisfy

$$\Delta(x' \cdot \nabla \nabla N_{Q}(0) \cdot x') = 2,$$

(50)

which is derived by substitution of Eq. (42) into Eq. (14). Besides, we have already proved that $x' \cdot \nabla \nabla N_{Q}(0) \cdot x'$ is positive definite.

Then based on the above property of $x' \cdot \nabla \nabla N_{Q}(0) \cdot x'$, it is known from Ref. [32] that there must be an ellipsoid $E \in \Omega$ satisfying

$$N_{E}(x') = C^{E} + \frac{1}{2} x' \cdot \nabla \nabla N_{Q}(0) \cdot x', \quad x' \in E,$$

(51)

with $C^{E}$ a positive real constant.

Then substituting Eq. (51) into Eq. (49) yields

$$N_{Q}(x') = N_{E}(x') + \phi(x'_1, x'_2), \quad x' \in E,$$

(52)

where $\phi(x'_1, x'_2) := \phi''(x'_1, x'_2) + N_{Q}(0) - C^{E}$. Thus the proof of Theorem 1 is completed.

### 3.2 Proof of Theorem 2

First of all, because we are going to consider two elastic tensors, denoted by $C^{(1)}$ and $C^{(2)}$, all of the parameters, variables, and functions that correspond to $C^{(1)}$ and $C^{(2)}$ will be distinguished by the superscripts (1) and (2), respectively.

Since Eq. (24) holds for $(\sigma^{(1)}, C^{(1)})$ and $(\sigma^{(2)}, C^{(2)})$ simultaneously, it is derived from Eq. (24) that for $x \in \Omega$,

$$\gamma^{(i)} N_{Q}(x) + \tilde{N}_{Q}(x) = q_{1}^{(i)}(x) + \phi_{1}^{(i)}(x_{3}), \quad i = 1, 2,$$

(53)

\[\gamma^{(i)} = \frac{[k_{i}(j^{(i)} + \mu^{(i)}) - k_{i}(j^{(i)} + 3\mu^{(i)})]}{(j^{(i)} + \mu^{(i)}) (k_{1} - k_{3})}, \quad i = 1, 2,\]

(54)

$q_{1}^{(i)}(x)$ ($i = 1, 2$) are quadratic functions; and $\phi_{1}^{(i)}(x_{3})$ ($i = 1, 2$) have been proved to be constant, linear or quadratic functions in the above derivation.

According to Theorem 1, we see that if $\Omega$ has the Eshelby uniformity property, then there exists an ellipsoid $E$ such that Eq. (26) holds. Thus by substituting Eq. (26) into Eq. (53), we gain

$$\phi_{2}(x_{1}, x_{2}) = \frac{q_{1}^{(1)}(x) + \phi_{1}^{(1)}(x_{3}) - [q_{1}^{(2)}(x) + \phi_{1}^{(2)}(x_{3})]}{\gamma^{(1)} - \gamma^{(2)}} - N_{E}(x), \quad x \in \Omega.$$

(55)
The validity of Eq. (55) calls for
\[ \gamma^{(1)} \neq \gamma^{(2)}, \]
which requires Eq. (29). Note that Eq. (29) is satisfied owing to the dissimilarity between the two isotropic media, which is prescribed in Theorem 2. If the two isotropic media are similar, Eq. (29) and thus Eq. (56) no longer hold.

Since the Newtonian potential \( N_E(x) \) of the ellipsoid \( E \) is a quadratic function of \( x \) [33, 34], it is concluded from Eq. (55) that \( \varphi_2(x_1, x_2) \) can only be a constant, linear or quadratic function of \( x_1, x_2 \).

Given that \( \varphi_2(x_1, x_2) \) can only be a constant, linear or quadratic function, we see that \( N_Q(x) \) must be quadratic inside \( \Omega \) due to Eq. (26). To further determine the shape of \( \Omega \), we resort to a powerful theorem, i.e.,

**Theorem 4.** Let \( \Omega \) be a bounded domain with a Lipschitz boundary. The relation
\[ N_\Omega(x) = -\frac{1}{4\pi} \int_{\Omega} \frac{1}{|x - y|} dy = \text{quadratic, } x \in \Omega \]  
holds if and only if \( \Omega \) is an ellipsoid [18].

Since we have proved that \( N_\Omega(x) \) must be quadratic inside \( \Omega \), then according to Theorem 4, we conclude that \( \Omega \) must be ellipsoidal.

We can also verify Theorem 2 in a direct way. It can be derived from Eq. (53) that
\[ N_\Omega(x) = \frac{q_1^{(1)}(x) + \varphi_1^{(1)}(x)-q_1^{(2)}(x) + \varphi_1^{(2)}(x_3)}{\gamma^{(1)} - \gamma^{(2)}}, x \in \Omega. \]  
(58)

Since Eq. (58) still indicates that \( N_\Omega(x) \) must be quadratic inside \( \Omega \), then by Theorem 4, we draw the conclusion that \( \Omega \) is of ellipsoidal shape, which completes the proof of Theorem 2.

### 3.3 Proof of Theorem 3

According to the derivations from Eq. (11) to Eq. (24), Theorem 3 indicates that there exists an ellipsoid \( E \supset \Omega \), such that
\[ \frac{\partial}{\partial x_i} \left[ \gamma N_\Omega(x) + \tilde{N}_E(x) \right] = \frac{\partial}{\partial x_i} \left[ \gamma N_E(x) + \tilde{N}_E(x) \right], i = 1, 2, \]  
(59)

where \( \tilde{N}_E(x) \) is given by Eq. (23) with \( \Omega \) replaced by \( E \).

Let \( E^* = \{ x | x \in E, t > 0 \} \), which is large enough to contain \( \Omega \), and let \( Q = (Q_1, Q_2, Q_3) \) be the coordinates of a point where \( \partial \Omega \) and \( \partial E^* \) intersect each other.

Owing to the quadratic forms of \( N_E(x) \) and \( \tilde{N}_E(x) \), which can be explicitly calculated for any \( E \), it is easy to verify that
\[ N_E(x) - N_E(x) = \text{constant,} \]  
\[ \tilde{N}_E(x) - \tilde{N}_E(x) = \text{constant,} \]  
(60)

substitution of which into Eq. (59) yields
\[ \frac{\partial}{\partial x_i} \left[ \gamma N_{E^*}(x) + \tilde{N}_E(x) \right] = 0, i = 1, 2, \]  
(61)
\[ \frac{\partial}{\partial x_i} \left[ \eta N_{E^*}(x) + \tilde{N}_E(x) \right] = 0, x \in \Omega, \]

where
\[ N_{E^*}(x) := -\frac{1}{4\pi} \int_{E^*} \frac{1}{|x - y|} dy, \]  
(62)
\[ \tilde{N}_E(x) := -\frac{1}{4\pi} \int_{E^*} \frac{(x - y)^2}{|x - y|^3} dy. \]  
(63)

Then we can choose an isotropic elastic tensor \( C \) that leads to
\[ \gamma = 0, \]  
(64)

which requires
\[ k_1(\lambda + 3\mu) = k_3(\lambda + \mu). \]  
(65)

Recall that in Theorem 3, \( k_1/k_3 = \sqrt{2} - 1 \) is prescribed; thus by substituting \( k_1/k_3 = \sqrt{2} - 1 \) into Eq. (64), we gain
\[ \lambda/\mu = \sqrt{2} - 1. \]  
(66)

Hence, the Lamé parameters of the isotropic medium (elastic tensor) we choose shall satisfy Eq. (65). Why \( k_1/k_3 = \sqrt{2} - 1 \) is prescribed in Theorem 3 will be made clear in the sequel.

Then in terms of Eq. (63), it can be derived from Eq. (61) that
\[ N_{E^*}(x) = \frac{1}{\eta} [\varphi_2(x_1, x_2) + C_2 - \varphi_3(x_3) - C_1], \]  
(67)
\[ \tilde{N}_E(x) = C_1 + \varphi_3(x_3), x \in \Omega, \]

where \( C_1, C_2 \) denote two real constants, and \( \varphi_1, \varphi_2 \) still denote two unknown functions.

Let \( n = (n_1, n_2, n_3) \) denote the outward-pointing normal vector of \( \partial E^* \) at the point \( Q \). Then it can be derived from Eq. (62) that
\[ n \cdot \nabla \tilde{N}_{E^*}(x) \bigg|_Q = -2n_3 \int_{E^*} \frac{(Q_3 - y_3)}{4\pi|Q - y|^5} dy \]
\[ + 3 \int_{E^*} \frac{(Q_3 - y_3)^2(Q - y) \cdot n}{4\pi|Q - y|^5} dy \]
\[ = -2n_3 \frac{\partial \tilde{N}_{E^*}(x)}{\partial x_3} \bigg|_Q \]
\[ + 3 \int_{E^*} \frac{(Q_3 - y_3)^2(Q - y) \cdot n}{4\pi|Q - y|^5} dy. \]  
(67)

We define a vector function \( F \),
\[ F(x) := 3 \int_{E^*} \frac{(x_3 - y_3)^2(x - y)}{4\pi|x - y|^5} dy, \]  
(68)
and thus substituting Eq. (68) into Eq. (67) leads to
\[ \mathbf{n} \cdot \mathbf{F}|_{\mathbf{Q}} = \left( \mathbf{n} \cdot \nabla \tilde{N}_{E^*} + 2n_3 \frac{\partial N_{E^*}}{\partial x_3} \right)_{\mathbf{Q}}. \tag{69} \]

Based on Eq. (69), we will make some arguments on the change of the function \( \mathbf{F} \) at the point \( \mathbf{Q} \) near the boundary \( \partial E^* \), just like the arguments on a similar situation for the change of the Newtonian potential made by Kang and Milton [18].

Likewise, we consider two cases concerning the continuity of \( \partial \Omega \).

3.3.1 \( \partial \Omega \) possesses \( C^1 \) continuity

In this case, \( \mathbf{n} \) is also the outward-pointing normal vector of \( \partial \Omega \) at \( \mathbf{Q} \). Thus by substituting Eq. (66) into Eq. (69), we obtain
\[ \mathbf{n} \cdot \mathbf{F}|_{\mathbf{Q}} = \left( \frac{2}{\eta - 1} \right) n_3 \frac{\partial \varphi_1(x_3)}{\partial x_3} \right)_{\mathbf{Q}}. \tag{70} \]

To analyze the change of \( \mathbf{F} \) at \( \mathbf{Q} \) via Eq. (70), we eliminate the unknown function \( \varphi_1(x_3) \) on the right-hand side of Eq. (70) by letting
\[ \eta = 2, \tag{71} \]
which requires
\[ k_1(\lambda + \mu) = k_3(\mu - \lambda). \tag{72} \]

It is straightforward to verify that Eq. (72) is automatically satisfied for \( k_1, k_3 \) that satisfy \( k_1/k_3 = \sqrt{2} - 1 \) and \( \lambda, \mu \) that satisfy Eq. (65). Further, we can substantiate that \( k_1/k_3 = \sqrt{2} - 1 \) prescribed in Theorem 3 actually guarantees the existence of non-trivial \((\lambda, \mu)\) that satisfy both Eqs. (64) and (72). In other words, if \( k_1/k_3 \neq \sqrt{2} - 1 \), we can only find trivial \((\lambda, \mu)\) that satisfy Eqs. (64) and (72), which contradicts Eq. (4) and hence makes the following proof invalid.

Finally, by substituting Eq. (71) into Eq. (70), we obtain
\[ \mathbf{n} \cdot \mathbf{F}|_{\mathbf{Q}} = 0. \tag{73} \]
However, if \( E^* \setminus \Omega \) is not empty, we see \( (\mathbf{Q} - \mathbf{y}) \cdot \mathbf{n} \geq 0 \), substitution of which into Eq. (68) yields
\[ \mathbf{n} \cdot \mathbf{F}|_{\mathbf{Q}} = 3 \int_{E^* \setminus \Omega} \frac{(Q_3 - y_3)^2 (\mathbf{Q} - \mathbf{y}) \cdot \mathbf{n}}{4\pi |\mathbf{Q} - \mathbf{y}|^3} \, dy > 0. \tag{74} \]
Since Eq. (74) contradicts Eq. (73), thus \( E^* \setminus \Omega \) must be empty to avoid the contradiction, which leads to the conclusion \( \Omega = E^* \).

Note that Eq. (70) is valid only when there are line segments \( \{ \mathbf{Q} \pm t \mathbf{n} \mid t \in \mathbb{R} \} \) belonging to \( \Omega \), which is obvious if \( \partial \Omega \) is \( C^1 \) continuous. Here \( \mathbf{Q} \pm t \mathbf{n} = (Q_1 \pm m_1, Q_2 \pm m_2, Q_3 \pm m_3) \) represents the coordinates of the points on the line segments.

3.3.2 \( \partial \Omega \) possesses Lipschitz continuity

In this case, as is stated before, Eq. (70) may not hold since there may not be a line segment \( \{ \mathbf{Q} \pm t \mathbf{n} \mid t \in \mathbb{R} \} \) belonging to \( \Omega \). Hence there is an alternative way to proceed with the argument.

If \( E^* \setminus \Omega \) is not empty, from Eq. (74), we know that the vector function \( \mathbf{F} \) at \( \mathbf{Q} \) must satisfy
\[ \mathbf{F}|_{\mathbf{Q}} \neq 0. \tag{75} \]
Let \( L \) be a line passing through \( \mathbf{Q} \) and satisfying \( L \cap \Omega \subset \Omega \).

The direction of \( L \) is represented by \( \mathbf{v} \). Since \( \partial \Omega \) has Lipschitz continuity, there is a neighborhood \( V \) in \( S^2 \) of \( \mathbf{v} \), where \( S^2 \) denotes the unit sphere in \( \mathbb{R}^3 \); any line \( L^0 \) passing through \( \mathbf{Q} \) and possessing the direction of the vector in \( V \) will lead to \( L^0 \cap \Omega \subset \Omega \). Those lines are contained in a set denoted by \( \{ L^0 \} \). Based on Eq. (75), we know that \( \exists L^* \in \{ L^0 \} \), whose direction is \( \mathbf{v}^* \), such that
\[ \mathbf{v}^* \cdot \mathbf{F}|_{\mathbf{Q}} \neq 0. \tag{76} \]
However, since \( L^* \in \{ L^0 \} \) so that \( L^* \cap \Omega \subset \Omega \), there are line segments \( \{ \mathbf{Q} \pm t \mathbf{v}^* \mid t \in \mathbb{R} \} \) belonging to \( \Omega \), and thus Eq. (70) can still be valid with \( \mathbf{n} \) replaced by \( \mathbf{v}^* \), which leads to
\[ \mathbf{v}^* \cdot \mathbf{F}|_{\mathbf{Q}} = 0, \tag{77} \]
Finally, by similarly letting Eq. (71) and then substituting Eq. (71) into Eq. (77), we obtain
\[ \mathbf{v}^* \cdot \mathbf{F}|_{\mathbf{Q}} = 0, \tag{78} \]
which contradicts Eq. (76). Likewise, \( E^* \setminus \Omega \) must be empty to avoid the contradiction so that \( \Omega = E^* \). Therefore, the proof of Theorem 3 is fulfilled.

4. Conclusions

In this work, we have studied the strong version of the Eschatky conjecture in the context of three-dimensional isotropic elasticity, for the case where two of the eigenvalues of the eigenstress are identical and the other one is distinct. We have made progress towards the proof of the conjecture by presenting and proving three theorems. The first theorem gives a necessary condition for the Newtonian potential of inclusions possessing the Eshelby uniformity property, which can exclude non-ellipsoidal inclusions that do not satisfy the condition. In terms of the necessary condition, and by introducing the effect of the elastic constants of the isotropic medium, the second theorem indicates that there exist combinations of uniform eigenstresses and the elastic tensors of
dissimilar isotropic media such that only an ellipsoid can have the Eshelby uniformity property for these combinations simultaneously. Moreover, the concept of dissimilar media provides a distinctive perspective for the categorization of materials and, more importantly, helps reveal the essence and significance of the material parameters to the solutions of both Eshelby conjectures and other linear-elastic problems. The third theorem provides a more specifically constrained proof of the conjecture. It proves that for the uniform strain fields constrained to that induced by an ellipsoid from a set of the specified uniform eigenstress, the strong version of the Eshelby conjecture is true for a set of isotropic elastic tensors which are associated with the specified uniform eigenstress.

Appendix A: A proof of the strong version for the eigenstress possessing either all distinct or all identical eigenvalues by using Fourier analysis

We are going to prove the strong version for the eigenstress possessing either all distinct or all identical eigenvalues through an alternative method, i.e., the Fourier analysis, which is somehow more concise than the method of Ammari et al. [28].

We then consider the two cases concerning the eigenvalues, separately.

A1 \( \sigma^* \) has three identical eigenvalues

Assume \( \Omega \) possesses the Eshelby uniformity property; thus Eq. (8) holds. Let \( \sigma^*_{11} = \sigma^*_{22} = \sigma^*_{33} = k \). Then by substituting \( \sigma^*_{11} = \sigma^*_{22} = \sigma^*_{33} = k \) along with Eqs. (7) and (9) into Eq. (8) and then substituting Eq. (8) into Eq. (13), we obtain

\[
\nabla \otimes \mathbf{u} = \frac{k}{\lambda + 2\mu} \nabla \otimes \nabla N_{q2}(x) = \text{constant, } x \in \Omega, \tag{A1}
\]

which indicates that \( N_{q2}(x) \) must be quadratic inside \( \Omega \). Then by resorting to Theorem 4, we conclude that \( \Omega \) must be ellipsoidal.

A2 \( \sigma^* \) has three distinct eigenvalues

Likewise, assume \( \Omega \) possesses the Eshelby uniformity property; thus Eq. (8) holds. Let \( \sigma^*_{11} = k_1, \sigma^*_{22} = k_2, \sigma^*_{33} = k_3 \) with \( k_1 \neq k_2, k_2 \neq k_3, k_3 \neq k_1 \). Then by substituting \( \sigma^*_{11} = k_1, \sigma^*_{22} = k_2, \sigma^*_{33} = k_3 \) along with Eqs. (7) and (9) into Eq. (8), we obtain

\[
\nabla \otimes \mathbf{u} = \frac{k_1}{\lambda + 2\mu} \nabla \otimes \nabla N_{q2}(x) = \text{constant, } x \in \Omega, \tag{A2}
\]

The third theorem provides a more specifically constrained proof of the conjecture. It proves that for the uniform strain fields constrained to that induced by an ellipsoid from a set of the specified uniform eigenstress, the strong version of the Eshelby conjecture is true for a set of isotropic elastic tensors which are associated with the specified uniform eigenstress.

Appendix B: The derivation of Eq. (19)

In the main text, we have used the expression of the second derivative \( \frac{\partial^2 H(x)}{\partial x_1^2} \) of the bi-harmonic potential \( H(x) \) in the Fourier space, that is, Eq. (19), to continue the analysis. In this appendix, we present the detailed derivation of Eq. (19). Before the derivation, we state that unless otherwise stipulated, all the integrals below are Lebesgue integrals.
First of all, in terms of the expression (16) of $H(x)$ in the main text, it is straightforward to verify that

\[
\frac{\partial^3 H(x)}{\partial x_i^3 \partial x_j^3 \partial x_k^3} = -\frac{1}{(2\pi)^3} \int_{\Omega} \frac{1}{|x-y|} dy
\]

for $0 \leq \alpha \leq 3$, where $\alpha = i + j + k$ with $i, j, k \in \mathbb{Z}$; thus by Eq. (B1),

\[
\Delta \left( \frac{\partial H}{\partial x_3} \right) = \frac{\partial}{\partial x_3} (\Delta H).
\]  

(B2)

Substitution of the relation (17) between the bi-harmonic potential $H(x)$ and the Newtonian potential $N_2(x)$ in the main text into Eq. (B2) yields

\[
\Delta \left( \frac{\partial H}{\partial x_3} \right) = \frac{\partial}{\partial x_3} N_2(x).
\]

(B3)

Then, by substituting the Fourier form Eq. (13) of $N_2(x)$ in the main text into the right-hand side of Eq. (B3) and further taking the derivative of both sides of Eq. (B3) with respect to $x_3$, we obtain

\[
\frac{\partial}{\partial x_3} \Delta \left( \frac{\partial H}{\partial x_3} \right) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi,
\]

where the expression of $g(x, \xi)$ is given in Eq. (9) in the main text.

It follows from Eq. (B4) that

\[
\Delta \left( \frac{\partial H}{\partial x_3} \right) = \int_{\mathbb{R}^3} \left[ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi \right] dx_3 + \varphi(x_1, x_2)
\]

with $\varphi(x_1, x_2)$ denoting an unknown function. Here, according to the Fourier form (13) of the Newtonian potential $N_2(x)$, we see

\[
\int_{\mathbb{R}^3} \left[ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi \right] dx_3
\]

\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi,
\]

physically denote the first and second derivatives of the Newtonian potential $N_2(x)$ with respect to $x_3$, respectively.

Next, we can derive from Eq. (B5) that

\[
\frac{\partial H}{\partial x_3} = -\int_{\mathbb{R}^3} \left[ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi \right] dx_3 + h_1(x_1, x_2) + h_2(x),
\]

(B6)

where $h_1(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \varphi(y_1, y_2) \ln |x-y| dy$ denotes the Newtonian potential of the function $\varphi(x_1, x_2)$, which satisfies $\Delta h_1(x_1, x_2) = \varphi(x_1, x_2)$, and $h_2(x)$ denotes an unknown harmonic function, which satisfies $\Delta h_2(x) = 0$. Then, by taking the derivative of both sides of Eq. (B6) with respect to $x_3$, we gain

\[
\frac{\partial^2 H}{\partial x_3^3} = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi + \frac{\partial}{\partial x_3} h_2(x).
\]

(B7)

We note that if $\frac{\partial}{\partial x_3} h_2(x) \equiv 0$, then Eq. (B7) will directly give rise to the Fourier form of $\frac{\partial H(x)}{\partial x_3}$, which is exactly Eq. (19). Therefore, to derive Eq. (19), we need to prove $\frac{\partial}{\partial x_3} h_2(x) \equiv 0$. To this end, we then analyze Eq. (B7).

We divide the process of the analysis on Eq. (B7) into two steps. In the first step, we evaluate the magnitude of

\[
\lim_{|\xi| \to \infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi,
\]

as

\[
\left| \int_{\mathbb{R}^3} g(x, \xi) d\xi \right| = \left| \int_{|\xi| = \infty} g(x, \xi) d\xi \right| = \left| \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} g(x, \xi) d\xi \right|. 
\]

(B8)

To make the meaning of right-hand side of Eq. (B8) clear, we bring in new variables $\varepsilon := -x$ and $y := \xi$; thus the right-hand side of Eq. (B8) can be expressed as

\[
\lim_{|\varepsilon| \to \infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} e^{-i\varepsilon \cdot \xi} d\xi,
\]

(B9)

which is exactly the Fourier transformation of $\frac{\partial w(y)}{|y|^4}$ at infinity. Here $\varepsilon$ in Eq. (B9) denotes the variable in the Fourier space.

In terms of the change of variables, we turn to evaluate the magnitude of

\[
\lim_{|\varepsilon| \to \infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_3^2}{|\xi|^4} e^{-i\varepsilon \cdot \xi} d\xi = 0,
\]

(B10)

where $B_r = \{ y \in \mathbb{R}^3 \mid |y| \leq r \}$ denotes a ball with the radius $r$, and $r$ is a real positive constant that is arbitrarily chosen.

We introduce

\[
\mathcal{L}(y) = \left\{ \begin{array}{ll}
\frac{\varepsilon_2 w(y)}{|y|^4}, & y \in B_r, \\
0, & y \in \mathbb{R}^3 \setminus B_r.
\end{array} \right.
\]

(B11)

Since it is straightforward to verify $|w(y)| \leq M$ with $M$ the volume of $\Omega$, thus

\[
\int_{\mathbb{R}^3} \|w_1(y)\| \leq \int_{B_r} \left[ \frac{\varepsilon_2 w(y)}{|y|^4} \right] dy 
\]

\[
\leq \int_{B_r} \frac{M \varepsilon_2}{|y|^4} dy < \int_{B_r} \frac{M}{|y|^2} = 4\pi M r < \infty,
\]

(B12)

which indicates $w_1(y) \in \mathcal{L}^1(\mathbb{R}^3)$, where $\mathcal{L}^1$ denotes the space of absolutely integrable functions. Therefore, according to the well-known Riemann-Lebesgue lemma [35], we have

\[
\lim_{|\varepsilon| \to \infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} w_1(y) e^{-i\varepsilon \cdot \xi} dy = 0,
\]

(B13)
thus, in terms of Eq. (B11), we see Eq. (B13) indicates Eq. (B10).

We remark that Eq. (B10) signifies

\[
\lim_{|x| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\gamma_3 w(y)}{|y|^3} e^{-iyx} dy = \lim_{|x| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \setminus B_0} \frac{\gamma_3 w(y)}{|y|^3} e^{-iyx} dy. \tag{B14}
\]

Owing to Eq. (B14), we turn to evaluate the magnitude of

\[
\lim_{|x| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \setminus B_0} \frac{\gamma_3 w(y)}{|y|^3} e^{-iyx} dy.
\]

Let

\[
w_2(y) = \begin{cases} 0, & y \in B_r, \\ \frac{w(y)}{|y|^3}, & y \in \mathbb{R}^3 \setminus B_r. \end{cases} \tag{B15}
\]

The Fourier transformation \(\widehat{w}_2(\xi)\) of \(w_2(y)\) is

\[
\widehat{w}_2(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} w_2(y) e^{-iy\xi} dy = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \setminus B_0} \frac{\gamma_3 w(y)}{|y|^3} e^{-iy\xi} dy. \tag{B16}
\]

Since

\[
\int_{\mathbb{R}^3} |w_2(y)|^2 dy = \int_{\mathbb{R}^3 \setminus B_0} \frac{\gamma_3^2 |w(y)|^2}{|y|^3} dy \leq \int_{\mathbb{R}^3 \setminus B_0} \frac{M^2 \gamma_3^2}{|y|^3} dy < \int_{\mathbb{R}^3 \setminus B_0} \frac{M^2}{|y|^3} dy = \frac{4\pi M^2}{r} < \infty,
\]

thus by Parseval’s identity [36], that is,

\[
\int_{\mathbb{R}^3} |w_2(y)|^2 dy = \int_{\mathbb{R}^3} \left|\widehat{w}_2(\xi)\right|^2 d\xi, \tag{B18}
\]

we see

\[
\int_{\mathbb{R}^3} \left|\widehat{w}_2(\xi)\right|^2 d\xi < \infty. \tag{B19}
\]

The above equation indicates \(\widehat{w}_2(\xi) \to 0\) as \(|\xi| \to \infty\); thus

\[
\lim_{|\xi| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} w_2(y) e^{-iy\xi} dy = \lim_{|\xi| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \setminus B_0} \frac{w(y)}{|y|^3} e^{-iy\xi} dy = 0, \tag{B20}
\]

substituting which back into Eq. (B14) leads to

\[
\lim_{|x| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\gamma_3 w(y)}{|y|^3} e^{-iyx} dy = 0. \tag{B21}
\]

Finally, by combining Eq. (B21) with Eqs. (B8) and (B9), we obtain

\[
\lim_{|x| \to \infty \omega} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi^2}{|\xi|^3} g(x, \xi) d\xi = 0. \tag{B22}
\]

Therefore, the first step is completed. In the second step, we prove \(\frac{\partial}{\partial \xi_3} h_2(x) = 0\). Based on Eq. (16) in the main text, it is easy to verify

\[
\lim_{|x| \to \infty \omega} \frac{\partial^2 H}{\partial x_3^2} = 0. \tag{B23}
\]

Then, by substituting Eqs. (B23) and (B22) into Eq. (B7), we obtain

\[
\lim_{|x| \to \infty \omega} \frac{\partial}{\partial \xi_3} h_2(x) = 0. \tag{B24}
\]

Recall that \(h_2(x)\) is a harmonic function in \(\mathbb{R}^3\), and hence \(\frac{\partial}{\partial \xi_3} h_2(x)\) is also a harmonic function in \(\mathbb{R}^3\). Owing to the boundary condition (B24) and the analyticity of \(\frac{\partial}{\partial \xi_3} h_2(x)\) in \(\mathbb{R}^3\), we see \(\frac{\partial}{\partial \xi_3} h_2(x)\) is bounded in \(\mathbb{R}^3\). Then by the Liouville theorem, which stipulates that “any harmonic function in \(\mathbb{R}^3\) bounded from above or below is constant” [37], we conclude that \(\frac{\partial}{\partial \xi_3} h_2(x)\) is constant, and the boundary condition (B24) of \(\frac{\partial}{\partial \xi_3} h_2(x)\) further indicates

\[
\frac{\partial}{\partial \xi_3} h_2(x) = 0. \tag{B25}
\]

Hence, the second step is fulfilled. Finally, by substituting Eq. (B25) into Eq. (B7), we gain

\[
\frac{\partial^2 H}{\partial x_3^2} = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi^2}{|\xi|^3} g(x, \xi) d\xi, \tag{B26}
\]

which is exactly Eq. (19) in the main text.

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三维各向同性介质中Eshelby强猜想的受限证明

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摘要  Eshelby在椭球夹杂问题的研究中提出过一个著名的猜想——Eshelby猜想，即椭球是唯一一种能将均匀本征应力转化为均匀弹性的夹杂构型。当该类性质被称为Eshelby均匀特性。在后来的研究中，Eshelby猜想被分为减弱和不变两个版本。在三维各向同性介质中，Eshelby强猜想已得到证明，但Eshelby强猜想仍然悬而未决。其中Ammani等人(2010)仅针对本征应力有两个相同或三个不同特征值的情形证明了强猜想成立。这项工作中，我们考虑尚未被证明的情形，即本征应力仅有两个相同特征值的情形。首先，我们提出了具有Eshelby均匀特性的椭球满足必要的条件。接着，由于该必要条件不足以帮我们判断夹杂构型是否只能为椭球，因此我们额外引入一个与材料常数相关的约束条件，继而基于非线性材料的概念，证明了，在两个非相似各向同性介质中都具有Eshelby均匀特性的夹杂构型只能为椭球。最后，我们提出了一个更具体的强猜想的受限证明，即必须具有Eshelby均匀特性的椭球内部应力场与椭球在相同本征应力下产生的内部应力场相同，继而证明了，对特定本征应力及与本征应力相关的弹性张量的组合，Eshelby强猜想成立。