An Approach towards Quantization of Non-Relativistic Open String Theory

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Abstract. For decades, scientists have worked relentlessly to move forward to see what lies beyond the third dimension and to find out if there is any existence of a unified theory to explain all the workings of the universe from sub-atomic to gigantic inhabitants of cosmos. This insurmountable task has been taken on by many over the last century or so until the emergence of “superstring theory” or “string theory” happened showcasing the fact that an answer seemed possible. From the concepts of a “string”, our ideation on this theory started. In this paper we strived to put forward that the concept of a vibrating one-dimensional microscopic object named “string” can be taken as a fundamental ingredient (in place of point particle) for developing non-relativistic quantum mechanics. Unlike point particle, we take a vibrating string as the quantum object & build a perfectly reasonable quantum mechanical description of the microscopic world. Our main objective in this paper to show that complete development of quantum mechanics is possible based on one dimensional open string.

1. Introduction

At the initial stage of development string theory, it was thought to be a theory for Hadrons. But at later stage of development it was found that one vibrational pattern of string produces graviton. So, people understood that string theory could be the ultimate theory that can provide a solution to Grand Unification, a theory of everything, as said in Einstein’s dream unification project. Related to this, in 2005 TED talk, theoretical physicist Brian Greene, Director of Columbia’s Centre for Theoretical Physics, explains this principle by breaking down our basic knowledge of matter. As per him, the string theory suggests that making up this seemingly indivisible subatomic particle are what appear to be strings, whose vibrating and ever-changing shapes and interactions through space make up all that we know [1]. Though, the above analyses proposed by Prof. Brian and the dream unification project of one of the fathers of modern physics were the shoots of our ideation for this paper but our motivation for this paper is a bit different than most of the string theorists’ dreams. In this paper, our primary focus was to investigate application of string theory at the very basic quantum mechanical (non-relativistic) level & to fortify the fact that a perfectly consistent quantum mechanical dynamics can be achieved by considering a one-dimensional quantum mechanical object like string instead of a point

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particle [9] or extended fuzzy object [10]. Therefore, in this paper we targeted at considering one dimensional string as the fundamental object of our analysis and through this we would put forward our logics to showcase that a perfectly consistent quantum dynamics can be formulated based on that as quantum mechanics developed from classical mechanics always considered point particles as the fundamental object. Hence space time transformation laws are simple Galilean Transformation ignoring (as of now) the space-time pictures (which are as per special theory of relativity). Thus, our quantum mechanical equations of motion should be Schrodinger’s equation (SE) of motion (both time dependent & time independent).

2. The Classical String Equation

We will begin our analysis with the non-relativistic classical string equation of motion which is as follows [3]:

\[ \frac{\mu}{2} \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0 \]  

(2.1)

Where, \( \mu \) = mass of string. \( T \) = string tension and \( y(x,t) \) is the function representing string configuration at any instant of time.

Above equation can be formed once we formulate the Lagrangian (L) for the string action as follows. For a string under vibration, kinetic energy (K.E) for any small length \( dx \) of string can be written as [3]:

\[ K.E = \frac{1}{2} \mu \left( \frac{\partial y}{\partial x} \right)^2 dx \]  

(2.2)

A potential energy (P.E) must arise from this work to stretch the string [3].

So, \( P.E = \frac{1}{2} T \left( \frac{\partial y}{\partial x} \right)^2 dx \)  

(2.3)

So, in general a string is found to have both kinetic & potential energy component intrinsically.

3. Development of Schrodinger Equation for Open String

Let us consider a one-dimensional quantum object of length “L” moving with a velocity \( v \) about its centre of mass. For simplicity we will assume that the string is moving without any rotation about its centre of mass & its vibrational frequency is low, so that we can only concentrate on its translational motion. As per De Broglie’s hypothesis on matter & wave, it’s already established that “to any quantum particle of mass \( \mu \) & velocity \( v \), there exists a corresponding wavelength \( \lambda \)”, such that [2,4,7]:

\[ \lambda = \frac{h}{(\mu v)} = \frac{h}{p} \]  

(3.1)

At this point we ascertain that length of string is same as that of the wavelength associated with the quantum particle as per above mentioned De Broglie’s hypothesis. With this identification we can state that to every quantum point particle of mass \( \mu \) there is a string of length \( L \) and mass \( \mu \) such that,

\[ L = \lambda = \frac{h}{(\mu v)} = \frac{h}{p} \]  

(3.2)

So, this can be regarded as the modified string-wave-particle duality relationship. Therefore, we can develop the dynamical quantum mechanical equations in non-relativistic domain also [6].

Since it was found in section 2 that string has both intrinsic kinetic energy & potential energy hence from energy conservation principle, we can say that,

Total Energy of “free” string = Total energy of “free” particle dual to string = \( E \)

\[ \frac{p^2}{2\mu} + V_y = \frac{p^2}{2\mu} \]  

(3.3)

Now, let’s develop the string operators of motion based on the point particle operators. In standard point particle quantum mechanics momentum operator (in position basis) is written as [2]:

\[ \vec{p}_x = -i\hbar \frac{d}{dx} \]  

(3.4)
Putting this operator representation, in above mentioned equation (3.3), the new operator relationship would be as follows:
\[ \frac{p^2}{2\mu} + \hat{V}_s \] (3.5)
When length of string reduces to zero, then string becomes point particle. So, within limit we can say that \( [3] \) yields \( L = 0 \rightarrow \) string collapses into point particle. Hence potential energy operator for string will be
\[ \hat{V}_s = \frac{1}{2} \mu (L^2) \] (3.6)
Once we get the potential energy for string it can be well understood that kinetic energy will be less than point particle kinetic energy (refer Eq. 3.5). So, in general, string momentum operator can be regarded as some multiple of momentum operator. This multiplication factor should be dimensionless & <1.

Motivated by this argument we can express string momentum operator as:
\[ \frac{p^2}{2\mu} = \frac{p^2}{2\mu} - \hat{V}_s \]
or,
\[ \frac{p^2}{2\mu} = [\gamma(L)]^2 \frac{p^2}{2\mu} \]
where, \( \gamma(L) = a \) dimensionless function of string length \( L \) and \( \gamma(L) < 1 \) (3.7)
So, we define \( \gamma(L) = a \) dimensionless function of string length \( L \) and it has the following property
\( \gamma(L = 0) = 1, \gamma(L \neq 0) < 1 \), Let, \( \gamma(L) = e^{-L} / (L + 1) \)
It is a suitable function which satisfies both the condition as shown below.
\( e^{-L} = 1 - L + L^2 / 2 + \ldots \ldots \) Higher order terms
\( e^{-L} = 1 - L \) (neglecting higher terms as \( L \) is itself very small)
So,
\( e^{-L} / (L + 1) = (1 - L) / (1 + L) < 1 \) for \( L \neq 0 \) and, \( e^{-L} / (L + 1) = (1 - L) / (1 + L) = 1 \) for \( L = 0 \), and the function \( \gamma(L) \) is dimensionless as it is evident from the expansion.

4. Hamiltonian operator, position operator, commutator relationship & Harmonic Oscillator for string
Motivated from the above derivations of momentum operator & potential energy operator we can continue deriving all standard operators & equations as per standard quantum mechanics with the help of string parameter \( \gamma(L) \). Few important operators & equations in “string version” is shown below:

a. Free “string” Hamiltonian operator:
\[ \hat{H}_{se} = \frac{\hat{p}^2}{2\mu} + \hat{V}_s = [\gamma(L)]^2 \frac{\hat{p}^2}{2\mu} + (1/2) \mu L^2 \] (4.1)
So total energy of string in free state will be,
\[ E = \hbar^2 k^2 / 2\mu + (1/2) \mu L^2 = (1/2) \left[ \hbar^2 k^2 / \mu + \mu L^2 \right] \] (4.2)

b. “String” Momentum eigenvalue equation:
\[ \hat{p}_s \psi = p_{sx} \psi \] (where \( p_{sx} \) is the momentum eigenvalue for the string in x direction)
\[ \psi_s(x) = A e^{ikx} \]
Since particle momentum operator is Hermitian [2], hence corresponding string momentum is also Hermitian, as we are multiplying it with a constant term \( \gamma(L) \) with momentum operator from particle. So, both Hamiltonian function & momentum operators are Hermitian as well:
\[ \hat{\rho}_s = \hat{p}_s^* \] and \( \hat{H}_s = \hat{H}_s^* \)
c. “String” position operator for a string according to its point particle counterpart is as follows:
\[ \hat{x}_s = \frac{\hat{x}}{\gamma(L)} = \frac{\hat{x}}{(e^{-L/(L + 1))}) = \hat{a}^L(L + 1)\hat{x} \]

In general, the position operator is expressed by Dirac’s delta function as follows:
\[ \hat{x}_s(x - x') = \frac{\hat{x}}{\gamma(x - x')} = \hat{x}'/\delta(x - x') \]

\[ (4.3) \]

d. “String” commutator relationship:
\[ [\hat{x}_s, \hat{p}_s] = \{ \hat{x}_s \gamma(L), \gamma(L) \hat{p} \} = \{ \hat{x}' \hat{p} - \hat{p} \hat{x}' \} = i\hbar \]

\[ (4.4) \]
e. “String” Time dependent Schrödinger equation for string will be:
\[ \frac{i}{\hbar} \frac{\partial \psi(x, t)}{\partial t} = \hat{H}_s \psi \]

\[ (4.5) \]
f. “String” Harmonic Oscillator:
\[ \hat{H}_s \psi(x, t) = E\psi(x, t) \]

\[ \hat{H}_s \psi(x, t) = \left[ \frac{\hat{x}_s^2}{2\mu} + \frac{(1/2)\rho \omega^2 \hat{x}_s^2}{2} \right] \psi(x, t) + (1/2)TL^2 \psi(x, t) = (E1 + E2)\psi(x, t) \]

In terms of creation & annihilation operator the string harmonic oscillator will be:
\[ \hat{H}_s = [\hbar \omega \hat{a}^\dagger \hat{a} + (1/2)\hbar \omega + (1/2)TL^2] \]

So, we define “0” point energy Eigen value of Harmonic oscillator is \( \frac{1}{2} \hbar \omega + (1/2)TL^2 \).

5. Conclusion:

Physics is such an ocean of possibilities that a new way of looking at a well-known proven system sometimes leads to unexpected dramatic results as it happened when Mr. Richard P. Feynman developed “path integral method” as an alternative form of standard quantum which had huge impact on the development of quantum field theory & Feynman Graphs and similarly another alternative way to his work can be found from the seminal work of Mr. David J. Bohm on the development of “pilot wave theory”. But every analysis has some assumptions for reaching to the targeted goal in a focused fashion. Thus, in ours, they are as listed below:

- A single one-dimensional string has been considered instead of point particle.
- String transverse vibration as well as velocity about its centre of mass is taken much less than velocity of light. Hence relativistic effect has not been considered.

Thus, in the trial of adding few drops of water to an ocean and we have concluded our paper with the following findings with the prevailing assumptions mentioned above:

- The quantum mechanical laws for string got developed and the string-wave-particle duality (as per De-Broglie’s hypothesis) got investigated and proven.
- Schrödinger’s equation for “free” string has been developed and a detailed analysis got carried out along with investigation on both time dependent & time independent equations of motion.
- The quantum mechanical commutator relationship for string got prepared for both of its position & momentum operators. Based on the same, harmonic oscillator problem, creation & annihilation operators also got investigated.

Standard quantum mechanics is fully developed subject and various quantum mechanical phenomena are well established by experimental confirmation. To incorporate all the cases of standard quantum mechanics is almost impossible within this short research document. Therefore, the ocean still remains untouched and several more research possibilities on the subject of this paper still remain open for us and the next first possibility can be carrying on this entire analysis on a closed string [8].
6. References

[1] Article: https://futurism.com/what-is-string-theory-and-why-humanity-absolutely-needs-it
[2] Introductory Quantum Mechanics – Richard L. Liboff Addison Wesley Publication Company
[3] T A First Course in String Theory - Barton Zwiebach, Cambridge University Press
[4] Fundamentals of Quantum Mechanics – V.A. Fock, Mir Publishers, Moscow
[5] Mathematical methods for Physicists – George B. Arfken, Hans J Weber, Elsevier
[6] Classical Mechanics – Goldstein, Poole & Safco – Addison Wesley
[7] Feynman Lectures on Physics – Vol-III
[8] J. Gomis and H. Ooguri, Nonrelativistic closed string theory, Journal of Mathematical Physics 42, 3127 (2001)
[9] C. Batlle, J. Gomis and D. Not, Extended Galilean symmetries of non-relativistic strings, Journal of High Energy Physics 02 (2017) 049 [1611.00026].
[10] Extended objects in quantum systems and soliton solutions; H. Matsumoto, P. Sodano, and H. Umezawa; Phys. Rev. D 19, 511 – Published 15 January 1979