On Neumann problem for the degenerate Monge–Ampère type equations

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Abstract

In this paper, we study the global $C^{1,1}$ regularity for viscosity solution of the degenerate Monge–Ampère type equation $\det[D^2u - A(x, Du)] = B(x, u, Du)$ with the Neumann boundary value condition $D_\nu u = \varphi(x)$, where the matrix $A$ is under the regular condition and some structure conditions, and the right-hand term $B$ is nonnegative.

Keywords: Degenerate equations; Monge–Ampère type equations; Neumann problem; Global second order derivative estimates; Existence

1 Introduction

In this paper, we study the degenerate Monge–Ampère type equation

$$\det[D^2u - A(\cdot, Du)] = B(\cdot, u, Du) \quad \text{in } \Omega,$$

associated with the Neumann boundary value condition

$$D_\nu u = \varphi(x) \quad \text{on } \partial \Omega,$$

where $\Omega$ is a bounded domain, $Du$ and $D^2u$ denote the gradient and Hessian matrix of second order derivatives of the unknown function $u : \Omega \to \mathbb{R}$, respectively, $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a symmetric $n \times n$ matrix-valued function and $A \in C^2(\overline{\Omega} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$, $B : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ is a nonnegative scalar function, $\nu$ denotes the unit inner normal on $\partial \Omega$ and $\varphi$ is a scalar-valued function defined on $\partial \Omega$. We shall use $x$, $z$, and $p$ to denote the points in $\Omega$, $\mathbb{R}$, and $\mathbb{R}^n$, respectively.

We introduce the definitions of nondegenerate and degenerate Monge–Ampère type equation respectively. The Monge–Ampère type equation (1) is called nondegenerate or degenerate if $D^2u - A$ is positive definite or nonnegative definite, namely the right-hand side term $B \geq B_0 > 0$ or $B \geq 0$, respectively, where $B_0$ is a constant. Accordingly, a solution $u$ of the Neumann boundary value problem (1)–(2) is called elliptic (degenerate elliptic) if $D^2u - A > 0$ ($\geq 0$).

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We say that \( A \) is strictly regular in \( \Omega \), if
\[
\sum_{i,j,k,l=1}^{n} A^{kl}_{ij}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2
\] (3)
holds for all \((x,p) \in \Omega \times \mathbb{R}^n, \xi, \eta \in \mathbb{R}^n\) with \( \xi \cdot \eta = 0 \), and some positive constant \( c_0 \), where
\[
A^{kl}_{ij} = \frac{\partial^2 A_{ij}}{\partial p_k \partial p_l}.
\]
If \( c_0 \) on the right-hand side in (3) is replaced with 0, we say that \( A \) is regular in \( \Omega \). As usual, the strictly regular condition and regular condition are also said to be the A3 condition and the A3w condition, respectively, see [1, 2].

For the Dirichlet boundary value problem, the regularity for degenerate Monge–Ampère equation has been extensively studied, see [3–8]. When \( A \equiv 0 \), equation (1) reduces to the classical Monge–Ampère equation. The global \( C^{1,1} \) regularity was obtained by Bao in [4] under the condition \( B^\frac{1}{2}(x) \in C^{1,1}(\tilde{\Omega}) \). Guan, Trudinger, and Wang showed the global \( C^{1,1} \) regularity in [7] with the relaxed condition \( B^\frac{1}{2}(x) \in C^{1,1}(\tilde{\Omega}) \). Using the Pogorelov estimates independent of the lower bound \( B \), the interior regularity was established by Błocki [5] and the authors [8]. When \( A \not\equiv 0 \), Andriyanova [3] proved the second order derivative estimates under the A3w condition with the right-hand term \( B(x, Du) = q(x)\xi(x, Du) \), where \( q^{\frac{1}{n-1}}(x) \in C^{1,1}(\tilde{\Omega}) \) is nonnegative and \( \xi^{\frac{1}{n-1}}(x, Du) \in C^{1,1}(\tilde{\Omega} \times \mathbb{R}^n) \) is positive.

When \( A \not\equiv 0 \), the nondegenerate Monge–Ampère type equation (1) arises in various aspects such as optimal mass transportation problems, geometric optics, and conformal geometry (see, for instance, [1, 9–11]). Ma, Trudinger, and Wang [1] obtained a priori interior second order estimates under the A3 condition. The boundary \( C^{2,\alpha} \) estimate for the Dirichlet problem was established by Huang et al. [12] under the A3 condition. Jiang et al. [13] obtained the global smooth solutions under the A3w condition. Furthermore, Leoper [2] proved that the A3w condition is necessary for \( C^0 \) regularity of the optimal map in the optimal transportation problem.

For the Neumann boundary value problem, the global regularity of solutions for the classical Monge–Ampère equation was established by Lions, Trudinger, and Urbas in [14]. Subsequently, Jiang, Trudinger, and Xiang [15] proved the second order derivative estimates of solutions for the Monge–Ampère type equation (1) under the A3w condition. However, as far as we know, there is no regularity result for the degenerate Monge–Ampère type equation with Neumann boundary value condition, which is the motivation of this work.

Our global second derivative estimate is obtained by mixing the Bernstein type argument and the barrier argument. In order to construct the global barrier function under the regularity assumption of \( A \), we need to assume the existence of a subsolution (strict subsolution) \( \varphi \) to equation (1) satisfying
\[
\det[D^2 \varphi - A(x, Du)] \geq (>) B(x, \varphi, Du) \quad \text{in } \Omega \text{ (on } \tilde{\Omega}),
\] (4)

together with the Neumann boundary condition
\[
D_\nu \varphi = \psi(x) \quad \text{on } \partial \Omega.
\] (5)
In order to treat the degenerate case for equation (1), we assume that $B$ satisfies the following conditions:

$$|B_i|, |B_z|, |B_{p_i}| \leq C_1 B^{1 - \frac{1}{n}}$$ \hspace{1cm} (6)

$$|B_{ij}|, |B_{iz}|, |B_{zp_i}|, |B_{p_ip_j}| \leq C_2 B^{1 - \frac{1}{n}},$$ \hspace{1cm} (7)

and

$$|B_{p_ip_j} - B_{p_i} B_{p_j}|_{n \times n} \geq -C_3 B^2 I,$$ \hspace{1cm} (8)

in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ for $i, j = 1, \ldots, n$, where $C_1$, $C_2$, and $C_3$ are positive constants, $I$ denotes the unit matrix, $B_i$, $B_{ij}$, and $B_{p_i}$ denote the partial derivatives of $B$ with respect to $x_i$, $z$, and $p_i$, respectively. In fact, we can have the following relaxed version of condition (6) if $|B| \leq C$:

$$|B_i|, |B_z|, |B_{p_i}| \leq C_4 B^{1 - \frac{1}{n}},$$ \hspace{1cm} (9)

in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ for $i = 1, \ldots, n$, where $C_4$ depends on $n$, sup $B$, and $C_1$.

We define the domain $\Omega$ to be uniformly $A$-convex with respect to $u \in C^1(\bar{\Omega})$ if

$$\sum_{i,j=1}^{n} (D_i v_j(x) - A_{ij}^1(x, Du)v_k) \tau_i \tau_j \leq -\delta_0 \quad \text{on } \partial \Omega$$ \hspace{1cm} (10)

for all vectors $\tau = \tau(x)$ tangent to $\partial \Omega$ and some positive constant $\delta_0$. For the Neumann problem, the definition of uniform $A$-convexity of $\Omega$ with respect to $u$ is introduced in [15]. We refer the readers to [15] for more descriptions of this definition. Based on (10), we can define a barrier function in the neighborhood of the boundary $\partial \Omega$ by using the defining function of the domain $\Omega$, which will be used in the double normal derivative estimate of $u$ on $\partial \Omega$ in Sect. 3.

We now formulate the main result.

**Theorem 1.1** Let $\Omega$ be a bounded $C^{2,1}$ domain in $\mathbb{R}^n$ and $u \in C^4(\Omega) \cap C^{2,1}(\bar{\Omega})$ be an elliptic solution of Neumann (1)–(2). Assume that $\varphi \in C^{2,1}(\partial \Omega)$, $A \in C^2(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$ satisfies the $A3\omega$ condition and $\Omega$ is uniformly $A$-convex with respect to $u$. Suppose that $B \in C^{2,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying conditions (6)–(8) and $B_z > 0$. Assume that either of the following conditions holds:

(a) $A_{ij} = f_j(x, u_k) \delta_{ij}$, where $f_j$ $(i, j = 1, \ldots, n)$ are given functions and $\delta_{ij}$ is the usual Kronecker delta;

(b) There exists a sufficiently small constant $\delta$ depending on $n, A, B, \text{sup}_\Omega |u|$ and $\text{sup}_\Omega |Du|$ such that if $|D_{p_k} A_{ij}| < \delta$ for all $i, j, k = 1, \ldots, n$.

Assume further that there exists a subsolution $\underline{u} \in C^2(\bar{\Omega})$ of Neumann problem (1)–(2) satisfying (4)–(5). Then we have the estimate

$$\text{sup}_\Omega |D^2 u| \leq C,$$ \hspace{1cm} (11)

where the constant $C$ depends on $n, \Omega, |A|_{2,\bar{\Omega}}, |u|_{1,\bar{\Omega}}, \underline{u}, C_1, C_2, C_3, \text{sup}_\Omega |B|, \|\varphi\|_{C^{2,1}(\partial \Omega)}$, and $\delta_0$. 
We remark that the alternative conditions (a) and (b) for the matrix $A$ are used in the treatment of the third derivative terms in the proof, see the details in Sect. 3.

Since the global second order derivative estimates are independent of the lower bound of $B$, we have the following existence result by approximation. Before stating the theorem, we introduce the definition of viscosity solution of equation (1). An upper (lower)-semicontinuous function $u$ is called a viscosity subsolution (supersolution) of equation (1) if, for any function $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum (minimum) at some point $x_0 \in \Omega$, there holds

$$\det\left[D^2\phi(x_0) - A(x_0, D\phi(x_0))\right] \geq (\leq) B(x_0, u(x_0), D\phi(x_0)).$$

(12)

A continuous function $u$ is a viscosity solution of equation (1) if it is both a viscosity subsolution and a viscosity supersolution of equation (1).

Based on the \textit{a priori} second derivative estimate, we now establish the existence result in the following theorem.

**Theorem 1.2** Under the assumption of Theorem 1.1, assume that “$B$ is a nonnegative function” and “a strict subsolution $u \in C^2(\bar{\Omega})$” in place of “$B$ is a positive function” and “a subsolution $u \in C^2(\bar{\Omega})$”, respectively. Assume further that there exists a supersolution $\bar{u}$ of equation (1) satisfying $D_v \bar{u} \leq \varphi(x)$ on $\partial\Omega$, and $A$ satisfies the quadratic structure condition from below, namely

$$A(x, p) \geq -\mu_0 (1 + |p|^2) I,$$

(13)

where $\mu_0$ is a positive constant depending on $\sup |u|$ and $I$ denotes the $n \times n$ identity matrix. Then there exists a degenerate elliptic viscosity solution $u \in C^{1,1}(\bar{\Omega})$ for the Neumann problem (1)–(2).

**Remark 1** Note that in Theorem 1.2, if $B$ is a positive function and $u \in C^2(\bar{\Omega})$ is merely a subsolution, we can get the existence and uniqueness of the elliptic solution $u \in C^{3,\alpha}(\bar{\Omega})$ of problem (1)–(2) for some $\alpha \in (0, 1)$. We can prove this by following the proof of Theorem 1.2 in [15] and replacing $\bar{u}$ in its original proof with $u$. The reason why we can replace $\bar{u}$ with $u$ is that both $A$ in (1) and $\varphi$ in (2) have no $u$ dependence. Indeed, when $A$ in (1) is independent of $u$, we can obtain Lemma 2.2, which can replace Lemma 2.1 of [15]. Meanwhile, since $\varphi$ in (2) has no $u$ dependence, we are still able to get the key inequality (47) in our proof. Then we can follow the proof of Theorem 1.2 in [15] to establish $u \in C^{3,\alpha}(\bar{\Omega})$.

**Remark 2** It is clear that Theorem 1.2 can be applied to the Monge–Ampère equation with homogenous right-hand side as follows:

$$\det[D^2u - A(x, Du)] = c_0 |x|^\alpha \quad \text{in } \Omega,$$

(14)

where $c_0$ is a nonnegative constant, $\alpha \in [2(n - 1), +\infty)$ is a constant. Thus, there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ of equation (14) under the Neumann boundary value condition (2). For the Dirichlet problem of the totally degenerate Monge–Ampère equation (14) when $c_0 = 0$ and $A \equiv 0$, we refer the reader to [16–18].
Remark 3 Condition (a) in Theorems 1.1 and 1.2 can include $A = \text{diag}(u_2^1, \ldots, u_n^2)$ as an example, which satisfies both the A3w condition and the structure condition (13). Both conditions (a) and (b) in Theorems 1.1 and 1.2 can cover the special case when the matrix $A$ is independent of $p$, so that the standard Monge–Ampère case is already involved. It would be interesting to relax restrictions (a) or (b) of the matrix $A$ in Theorems 1.1 and 1.2.

The paper is organized as follows. In Sect. 2, we introduce two lemmas which will be used to deal with the degenerate term and in the construction of the global barrier function, respectively. In Sect. 3, we give the proof of Theorem 1.1 which constitutes the core of this paper. In this section, we reduce the global estimates to the boundary by using a global auxiliary function, where the computations of the third order derivatives are carried out in detail under the two alternative conditions of Theorem 1.1. For the boundary estimates, we construct a suitable barrier function which can be used to control the degenerate term, originating from the differentiation of equation (1). In this part, we mix the techniques of dealing with the Neumann problems for nondegenerate Monge–Ampère type equations in [14, 15] and the Dirichlet problems for degenerate Monge–Ampère type equations in [3, 5–7]. In Sect. 4, we complete the proof of Theorem 1.2.

2 Preliminaries

In this section, we introduce two lemmas. One lemma is used to deal with the degenerate term which arises in differentiating equation (1). The other lemma is used to construct the global barrier function. In equation (1), we first suppose $B > 0$ in $\Omega$,

$$\{\tilde{u}_ij\} := \{u_{ij} - A_{ij}\} \quad \text{and} \quad \{\tilde{u}^{ij}\} := \{\tilde{u}_ij\}^{-1}.$$ 

Then both the matrices $\{\tilde{u}_ij\}$ and $\{\tilde{u}^{ij}\}$ are positive definite. We can rewrite equation (1) in the form

$$\log[\det(\tilde{u}_ij)] = \tilde{B} \quad \text{in} \ \Omega, \quad (15)$$

where $\tilde{B} := \log B$. By differentiating equation (15) in the direction $\xi \in \mathbb{R}^n$ once and twice respectively, we have

$$\tilde{u}^{ij}(D_\xi u_{ij} - A_{ij} D_\xi \xi_k - A_{ij}^k D_\xi u_k) = D_\xi \tilde{B} \quad (16)$$

and

$$\tilde{u}^{ij}(D_{\xi \xi} u_{ij} - A_{ij} D_{\xi \xi} \xi_k \xi_l - A_{ij}^{kl} D_{\xi \xi} u_k D_{\xi \xi} u_l - A_{ij}^{kl} D_{\xi \xi} u_k D_{\xi \xi} u_l - 2A_{ij}^{kl} D_{\xi \xi} u_k D_{\xi \xi} u_l)$$

$$= \tilde{u}^{ij}\tilde{u}^{jk} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{kl} + D_{\xi \xi} \tilde{B}, \quad (17)$$

where

$$D_\xi \tilde{B} = \frac{B_{ij} + B_{ij} D_\xi u + B_{ii} D_{\xi \xi} u}{B} \quad (18)$$
Assume that $B$ satisfies (2.1) by choosing $\xi$ in (18) and using $\tilde{u}_{ij} = u_{ij} - A_{ij}$ (the relaxed version of (6)), we get (20) and finish the proof of conclusion (i).

Next, we turn to proving (ii). By condition (8), we have

$$\frac{B_{pl}B - B_{pl}B_{pl}u_{ik}u_{ik}}{B^2} \geq -C_{3}\delta_{kl}(\tilde{u}_{jk} + A_{jk})(\tilde{u}_{ik} + A_{ik})$$

$$\geq -C\left[\max_j(|\tilde{u}_{ij}|)\right]^2,$$

for a function $v = v(x,z,p)$. Note that we use the standard summation convention in the context that repeated indices indicate summation from 1 to $n$ unless otherwise specified.

Next, we point out several facts that will be used in the proof of Theorem 1.1.

**Lemma 2.1** Let $u \in C^3(\Omega)$ be a solution of equation (1) in a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that $B$ satisfies (6), $B > 0$, and $\bar{B} = \log B$. Then we have the following properties:

(i) $|D_\xi \bar{B}| \leq C\left[1 + \max_j(|\tilde{u}_{ij}|)\right]B^{\frac{1}{n+1}}$ (20)

holds for $i = 1, \ldots, n$, where the constant $C$ depends on $n$, $C_1$, $A$, and $\|u\|_{C^3(\Omega)}$.

(ii) If $B$ also satisfies conditions (7)–(8), then

$$D_\xi B \geq -C\left[1 + \max_j(|\tilde{u}_{ij}|)\right]B^{\frac{1}{n+1}} - C'\left[1 + \max_j(|\tilde{u}_{ij}|)\right]^2 + \sum_{i=1}^n B_{pl}D_{il}u_{ik}$$

(21)

holds for $i = 1, \ldots, n$, where the constant $C'$ depends on $n$, $C_1$, $C_2$, $A$, and $\|u\|_{C^3(\Omega)}$, and the constant $C'$ depends on $C_3$ and $A$.

Lemma 2.1 can be proved by direct calculations using (6) (or (9)), (7), and (8). We briefly prove it here. A more detailed proof can be found in [19].

**Proof of Lemma 2.1** By choosing $\xi = e_i$ in (18) and using $\tilde{u}_{ij} = u_{ij} - A_{ij}$ and (9) (the relaxed version of (6)), we get (20) and finish the proof of conclusion (i).

Next, we turn to proving (ii). By condition (8), we have

$$\frac{B_{pl}B - B_{pl}B_{pl}u_{ik}u_{ik}}{B^2} \geq -C_{3}\delta_{kl}(\tilde{u}_{jk} + A_{jk})(\tilde{u}_{ik} + A_{ik})$$

$$\geq -C\left[\max_j(|\tilde{u}_{ij}|)\right]^2,$$
where $\delta_{ij}$ denotes the usual Kronecker delta, the constant $C'$ depends on $C_3$ and $A$. It follows from (6) and (7) that the sum of the other terms on the right-hand side of (7) has a lower bound $-C[1 + \max_j(|\tilde{u}_j|)]B^{-\frac{1}{2n-1}}$, where the constant $C$ depends on $n$, $C_1$, $C_2$, $\sup |B|$, $A$, and $\sup_{\Omega_1} |Du|$.

Using the lower bounds $-C[1 + \max_j(|\tilde{u}_j|)]B^{-\frac{1}{2n-1}}$ and (22), and taking $\xi = e_1$ in (19), we get (21) and finish the proof of conclusion (ii).

Remark 4: We remark that $\tilde{B} = \log B$ satisfies condition (8) if it is semi-convex in $p$. The term $\sum_{k=1}^n \tilde{B}_{pk} D_{ni} u_k$ on the right-hand side of (21) can also be dealt with directly. We shall explain the treatment in the later discussion.

We introduce the following linearized operator of equation (1):

$$L = \sum_{i,j,k=1}^n \left[ \tilde{u}^i \left( D_i - A_{ij}^k(x, Du) D_k \right) - \tilde{B}_{pk} D_k \right].$$  \hspace{1cm} (23)

We introduce a fundamental lemma, which is important for constructing the global barrier function in the next section.

Lemma 2.2: Let $u \in C^2(\Omega)$ be an elliptic solution of (1), and let $u \in C^2(\bar{\Omega})$ be an elliptic function of equation (1) satisfying $D^2 u - A(x, Du) \geq \delta I > 0$ in $\bar{\Omega}$ for some positive constant $\delta$, $A$ is regular and $B \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying (6). Then

$$L(e^{K(u - u)}) \geq \epsilon_1 T - C(B^{-\frac{1}{2n-1}} + 1)$$  \hspace{1cm} (24)

holds in $\Omega$ for sufficiently large positive constant $K$ and uniform positive constants $\epsilon_1$, $C$ depending on $\|A\|_{C^2}$, $\|B\|_{C^{1,1}}$, $\Omega$, $\|u\|_{C^1}$, $\|\tilde{u}\|_{C^1}$, $\delta$, and $C_1$ in (6), where $L$ is the operator in (23), and $T = \sum_{i=1}^n \tilde{u}^i$.

Proof: Since $A$ is regular, similar to Lemma 2.1(ii) in [20], we have

$$L(e^{K(u - u)}) \geq \epsilon_1 (T + 1) + C \left\{ \log \frac{\det[D^2 u - A(x, Du)]}{\det[D^2 u - A(x, Du)]} - \sum_{k=1}^n \tilde{B}_{pk}(x, u, Du) D_k(u - u) \right\}$$  \hspace{1cm} (25)

$$\geq \epsilon_1 T + \epsilon_1 + C \left[ \log \frac{\delta^n}{M_1} - B^{-\frac{1}{2n-1}} \right],$$

where $D^2 u - A(x, Du) \geq \delta I > 0$ and condition (6) are used in the second inequality of (25), $M_1 = \sup_{x \in \Omega} B(x, u, Du)$, and the constant $C$ changes from line to line. Then estimate (24) follows by further adjusting the constant $C$. \qed

Remark 5: Note that the barrier inequality in Lemma 2.2 is different from that in [15]. Here the constants $\epsilon_1$ and $C$ in estimate (24) of Lemma 2.2 are independent of the positive lower bound of $B$. Note also that estimate (24) is proved under condition (6), while in [19] a similar estimate is proved under condition (8).
3 Global second order derivative estimates

In this section, by constructing a global auxiliary function, we can reduce the global estimates to the boundary. On the boundary, in order to get the mixed tangential normal second order derivatives of \( u \), we apply the tangential operator to the boundary value condition (2). A bound of double normal derivative of \( u \) on \( \partial \Omega \) is obtained by constructing a suitable auxiliary function. We use the key trick of [14] to establish the global second order derivative estimates of solutions for the degenerate Monge–Ampère type equations with the Neumann boundary value condition. For the argument below, we assume that the functions \( \phi, \nu \) are smoothly extended to \( \bar{\Omega} \). The constant \( C \) in this section changes from line to line.

Mixed tangential normal derivative estimate on \( \partial \Omega \). We introduce the tangential gradient operator \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \), where \( \delta_i = \sum_{j=1}^{n} (\delta_{ij} - \nu_i \nu_j) D_j \) for \( i = 1, \ldots, n \). Applying this tangential operator to boundary condition (2), we have

\[
(D_k u)(\delta_i \nu_k) + \nu_k (D_i D_k u) = \delta_i \phi \quad \text{on} \quad \partial \Omega. \tag{26}
\]

If \( \tau \) is a direction tangential to \( \partial \Omega \) at any point \( y \in \partial \Omega \), we have

\[
D_{\tau \nu} u(y) = \sum_{i,k=1}^{n} \tau_i \nu_k D_{i,k} u = \sum_{i,k=1}^{n} \tau_i \nu_k D_{i,k} u - \sum_{i,j,k=1}^{n} \tau_i \nu_k \nu_j D_{j,k} u \\
= \sum_{i,j,k=1}^{n} \tau_i \nu_k (\delta_{ij} - \nu_i \nu_j) D_{j,k} u = \sum_{i,j,k=1}^{n} \tau_i \nu_k \delta_{ij} D_{j,k} u \\
= \sum_{i,k=1}^{n} \tau_i \delta_i \phi - \tau_i (\delta_i \nu_k) D_k u = \sum_{i,k=1}^{n} \tau_i D_i \phi - \tau_i (D_i \nu_k) D_k u,
\]

where the second equality and the last equality are both valid using the fact that \( \tau \cdot \nu = 0 \), and the fifth equality holds by (26). Hence, from (27) we obtain

\[
|D_{\tau \nu} u| \leq C \quad \text{on} \quad \partial \Omega,
\]

where the constant \( C \) depends on \( \sup_{\bar{\Omega}} |D u| \) and \( \Omega \).

Double normal derivative estimate on \( \partial \Omega \). We may consider any boundary point. Without loss of generality, we may take it to be the origin and the \( x_n \)-axis in the direction of the interior normal. From the uniform \( A \)-convexity of \( \Omega \) and the regularity of \( A \), there exists a \( C^2 \) defining function in \( \bar{\Omega} \) satisfying

\[
\phi = 0 \quad \text{on} \quad \partial \Omega, \quad D_n \phi = -1 \quad \text{on} \quad \partial \Omega, \quad \text{and} \quad \phi < 0 \quad \text{in} \quad \Omega, \tag{29}
\]

together with the inequality

\[
\{D_j \phi - A^j_i(x, Du) D_k \phi\}_{n \times n} \geq \delta_0 I, \tag{30}
\]

in a neighborhood \( N \) of \( \partial \Omega \), whenever \( D_n u \geq \phi(x) \), where \( \delta_0 \) is a positive constant and \( I \) denotes the identity matrix. We employ the auxiliary function

\[
w = \pm (D_n u - \phi(x)) + \beta \phi \quad \text{(31)}
\]
in
\[
\Omega_{d_0} := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) < d_0 \},
\]
where \( \beta \) and \( d_0 \) are two positive constants to be determined. By the continuity of \( D_i \phi \) and \( D_i \phi = -1 \) on \( \partial \Omega \), there exists a small constant \( d_0 \) such that \( D_i \phi \leq -\frac{1}{2} \) in \( \Omega_{d_0} \).

It follows from (16) that
\[
Lw = \pm \tilde{u}^{ij} \left[ D_{ij}(v_k D_k u) - A^{ij}_{ij} D_l(v_k D_k u) \right] \mp L\phi + \beta L\phi
\]
\[
= \pm \tilde{u}^{ij} \left[ (D_{ij} v_k) D_k u + 2D_j v_k D_k u - A^{ij}_{ij} (D_l v_k) D_k u + v_k A_{ij,k} \right]
+ v_l \tilde{B}_{pl} \mp L\phi + \beta L\phi
\geq \beta L\phi - CT - C - CB^{-\frac{1}{n-1}}
\geq \beta L\phi - CT - CB^{-\frac{1}{n-1}},
\]
where \( T = \sum_{i=1}^n \tilde{u}^i \), the first inequality is obtained by using (20) in Lemma 2.1, the second inequality is obtained by using the fact that
\[
\frac{1}{n} T \geq \left( \prod_{i=1}^n \tilde{u}^i \right)^{\frac{1}{n}} \geq \left( \prod_{i=1}^n \tilde{u}_{ii} \right)^{\frac{1}{n}} = B^{-\frac{1}{n}} \geq C. \tag{33}
\]

By a direct calculation, we obtain
\[
L\phi = \tilde{u}^{ij} \left[ D_{ij} \phi - A^{ij}_{ij}(x, Du) D_k \phi \right] - \tilde{B}_{pl} D_k \phi
\geq \delta_0 T - \tilde{B}_{pl} D_k \phi
\geq \delta_0 T - CB^{-\frac{1}{n-1}}, \tag{34}
\]
where the first and second inequalities are established by using (34) and (6), respectively.

Therefore, from (32), (33), and (34), we have
\[
Lw \geq (\beta \delta_0 - C) T - CB^{-\frac{1}{n-1}} - C^2 B^{-\frac{1}{n-1}}. \tag{35}
\]

We decompose \( \Omega_{d_0} = \Omega_{d_0}^1 \cup \Omega_{d_0}^2 \), where
\[
\Omega_{d_0}^1 = \Omega_{d_0} \cap \left\{ \sum_{i=1}^n \tilde{u}_{ii} \geq n \right\}
\]
and
\[
\Omega_{d_0}^2 = \Omega_{d_0} \cap \left\{ \sum_{i=1}^n \tilde{u}_{ii} < n \right\}.
\]

If \( \max_{\Omega_{d_0}} w = w(\tilde{x}) \) for some \( \tilde{x} \in \Omega_{d_0} \). The proof of the double normal derivative estimate of \( u \) on \( \partial \Omega \) splits into three stages according to whether \( \tilde{x} \in \Omega_{d_0}^1 \) or \( \tilde{x} \in \Omega_{d_0}^2 \), or neither.
Case 1. $\bar{x} \in \Omega_{d_0}^1$. Since $\max_{\Omega_{d_0}^1} w = w(\bar{x})$ for $\bar{x} \in \Omega_{d_0}^1$, then we have

$$Lw(\bar{x}) \leq 0. \quad (36)$$

We may assume that $\bar{u}_1(\bar{x})$ is diagonal. Without loss of generality, we may assume that $\bar{u}_{11}(\bar{x}) \geq 1$. At the point $\bar{x}$, we have

$$\mathcal{T} \geq \sum_{i=2}^{n} \bar{u}^{ii}$$
$$\geq (n - 1) \left( \prod_{i=2}^{n} \bar{u}^{ii} \right)^{\frac{1}{n-1}}$$
$$= (n - 1) \left( \prod_{i=1}^{n} \bar{u}^{ii} \right)^{\frac{1}{n-1}} (\bar{u}_{11})^{\frac{1}{n-1}}$$
$$\geq (n - 1) B^{-\frac{1}{n-1}},$$

where $\bar{u}_{11}(\bar{x}) \geq 1$ is used in the last inequality. At $\bar{x}$, we can assume that $\sqrt{\mathcal{T}} \geq \frac{3C}{\sqrt{n-1}}$, otherwise we have already got the global second order derivative estimates. Therefore at $\bar{x}$, by $(35)$, we have

$$Lw \geq \left( \frac{\beta\delta_0}{3} - C \right) \mathcal{T} + \frac{\beta\delta_0}{3} \mathcal{T} - CB^{-\frac{1}{n-1}} + \frac{\beta\delta_0}{3} \mathcal{T} - C B^{-\frac{1}{n-1}}$$
$$\geq \left( \frac{\beta\delta_0}{3} - C \right) \mathcal{T} + \left( \frac{\beta\delta_0}{3} - \frac{C}{n-1} \right) \mathcal{T} + \left( \frac{\beta\delta_0}{3} \sqrt{\mathcal{T}} - \frac{C\beta}{\sqrt{n-1}} \right) \sqrt{\mathcal{T}}$$
$$> 0,$$

where we choose $\beta \geq \frac{3C}{\delta_0} + 1$ such that the last inequality holds. It is a contradiction with $(36)$. Therefore, $\bar{x} \notin \Omega_{d_0}^1$.

Case 2. $\bar{x} \in \Omega_{d_0}^2$. For the unit inner normal vector $\nu = (\nu_1, \ldots, \nu_n)$, we must have

$$0 \leq |\nu_i| \leq 1 \quad \text{for } i = 1, \ldots, n. \quad (39)$$

Since $\max_{\Omega_{d_0}^2} w = w(\bar{x})$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \Omega_{d_0}^2$, we obtain

$$w_\nu(\bar{x}) = 0. \quad (40)$$

Therefore, we have at $\bar{x}$

$$0 = w_\nu = \pm (D_{\nu_1} u - D_{\nu} \varphi) + \beta D_{\nu} \phi$$
$$= \pm \left( \sum_{i \neq \nu} \nu_i \nu_j D_{\nu} u \right) \mp D_{\nu} \varphi + \beta D_{\nu} \phi$$
$$\leq n^2 \sum_{i=1}^{n} \bar{u}_{ii} \mp D_{\nu} \varphi + \beta D_{\nu} \phi. \quad (41)$$
Choose \( \beta \geq 2 \max |D\varphi| + 4n^3 \) such that \( \sum_{i=1}^{n} \tilde{u}_{ii} \geq 2n \), which is a contradiction with \( \sum_{i=1}^{n} \tilde{u}_{ii} < n \). Therefore, \( \tilde{x} \notin \Omega_{d_0}^2 \).

Case 3. \( \tilde{x} \in \partial \Omega_{d_0} \). On \( \partial \Omega_{d_1} \cup \Omega \), we have

\[
 w = \pm(D_{v}u - \varphi) + \beta \varphi \leq 0, \tag{42}
\]

where \( \beta \) is chosen large enough such that the last inequality holds. On \( \partial \Omega \), we have

\[
 w = \pm(D_{v}u - \varphi(x)) + \beta \varphi = 0. \tag{43}
\]

Therefore, for all \( x \in \Omega_{d_0} \), we have

\[
 w(x) \leq 0. \tag{44}
\]

Since \( w(0) = 0 \), we have

\[
 D_{v}w(0) \leq 0, \tag{45}
\]

which implies

\[
 |D_{v,v}u| \leq C \quad \text{on } \partial \Omega. \tag{46}
\]

**Remark 6** In the above proof, we only use (30) in a neighborhood \( N \) of \( \partial \Omega \) whenever \( D_{v}u \geq \varphi(x) \). If \( D_{v}u < \varphi(x) \) in a neighborhood \( N_0 \) of some boundary point \( x_0 \in \partial \Omega \), from the boundary condition, we directly get an upper bound \( D_{v,v}u(x_0) \leq D_{v,v}\varphi(x_0) \). While the lower bound \( D_{v,v}u \geq \sum_{i,j} A_{ij}v_i v_j \) at \( x_0 \) can be derived from the ellipticity.

Based on the mixed tangential normal derivative estimate and the double normal derivative estimate on \( \partial \Omega \), we now use the method in [15] to prove Theorem 1.1, which is a modification of the original method in [14].

**Proof of Theorem 1.1** We modify the elliptic subsolution \( u \) by adding a perturbation function \( a\varphi \), where \( a \) is a small positive constant and \( \varphi \) is the defining function of the domain \( \Omega \) satisfying (29). Note that the function \( u := u - a\varphi \) is still uniformly elliptic in \( \Omega \) if \( a \) is a sufficiently small. By a direct computation, we have

\[
 D_{v}(u - u) = D_{v}u - D_{v}u - aD_{v}\varphi
 = -aD_{v}\varphi
 = a \tag{47}
\]

on \( \partial \Omega \). We define a function with the form \( \Phi = \exp[K(u - u)] \) with a positive constant \( K \). Replacing \( u \) in Lemma 2.2 with \( u \), from (24), we have

\[
 L\Phi \geq c_{1}T - C(B^{-\frac{1}{2(B-1)}} + 1) \quad \text{in } \Omega. \tag{48}
\]
By a direct calculation, we also have
\[
D_i \Phi \geq Ka \inf(e^{K|x-u|}) > 0 \quad \text{on } \partial \Omega. \tag{49}
\]

We employ the auxiliary function
\[
G(x, \xi) = (\tilde{u}_{\xi \xi} - v'(x, \xi)) e^{\frac{\alpha}{2} |Du|^2 + \kappa \Phi} \tag{50}
\]
in $\Omega \times \mathbb{R}^n$, where $|\xi| = 1$, $\alpha$ and $\kappa$ are positive constants to be determined, $v'$ is given by
\[
v'(x, \xi) = 2(\xi \cdot \nu) \xi' j(D_i \phi(x) - D_k u D_i v_k - A_{ij} y_i), \tag{51}
\]
and $\xi'$ is given by
\[
\xi' = \xi - (\xi \cdot \nu) \nu. \tag{52}
\]
Assume that $G$ attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. Let
\[
H(x, \xi) = \log G(x, \xi) = \log (\tilde{u}_{\xi \xi} - v') + \frac{\alpha}{2} |Du|^2 + \kappa \Phi, \tag{53}
\]
then the function $H$ attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. From now on, all the calculations will be made at the point $x = x_0$ and $\xi = \xi_0$ unless otherwise specified. At $x_0$, we obtain
\[
0 = D_i H = \frac{D_i (\tilde{u}_{\xi \xi} - v')}{\tilde{u}_{\xi \xi} - v'} + \alpha D_k u D_i u + \kappa D_i \Phi, \quad \text{for } i = 1, \ldots, n, \tag{54}
\]
\[
0 \geq D_i H
= \frac{D_i (\tilde{u}_{\xi \xi} - v')}{\tilde{u}_{\xi \xi} - v'} - \frac{D_j (\tilde{u}_{\xi \xi} - v') D_j (\tilde{u}_{\xi \xi} - v')}{(\tilde{u}_{\xi \xi} - v')^2}
+ \alpha (D_k u D_i u + D_k D_i u) + \kappa D_i \Phi. \tag{55}
\]
Therefore, at $x_0$ we have
\[
0 \geq LH
= \frac{1}{\tilde{u}_{\xi \xi} - v'} L(\tilde{u}_{\xi \xi} - v') - \frac{1}{(\tilde{u}_{\xi \xi} - v')^2} \tilde{u}_{\xi \xi} D_i (\tilde{u}_{\xi \xi} - v') D_j (\tilde{u}_{\xi \xi} - v')
+ \alpha \tilde{u}_{\xi \xi} D_i u D_{ij} u + \alpha D_k u L u_k + \kappa L \Phi. \tag{56}
\]
By a direct computation and using (17)–(19), we have
\[
Lu_{\xi \xi} \geq \tilde{u}_{\xi \xi} \tilde{u}_{ij} D_i \tilde{u}_{ij} D_k \tilde{u}_{kl} + \tilde{u}_{ij} D_{i,j} A_{ij} D_k u_k D_l u_l
- C[(1 + \tilde{u}_{ii}) T + (\tilde{u}_{ii})^2] - C(1 + \tilde{u}_{ii}) B^{-\frac{1}{p}} \tag{57}
\geq \tilde{u}_{\xi \xi} \tilde{u}_{ij} D_i \tilde{u}_{ij} D_k \tilde{u}_{kl} - C[(1 + \tilde{u}_{ii}) T + (\tilde{u}_{ii})^2] - C(1 + \tilde{u}_{ii}) B^{-\frac{1}{p}},
\]
where the first inequality is established by (21) in Lemma 2.1, and the second inequality is obtained by using the A3w condition. Note that the term $\sum_{k=1}^n \tilde{B}_{pk} D_{li} u_k$ in (21) does not appear on the right-hand side of the first inequality in (57) since it can be subtracted using the definition of the linearized operator $L$ in (23). By a direct computation, we obtain

$$|LA_{\tilde{\xi},\tilde{\xi}}| \leq C\left[(1 + \tilde{u}_{ii})T + \tilde{u}_{ii}\right] + \left|\frac{B_{pi}}{B} A_{\tilde{\xi},\tilde{\xi}}\right|$$

where (20) in Lemma 2.1 is used in the second inequality. It follows from (16) and (20) in Lemma 2.1 that

$$|L\nu| \leq CT + C|L(u_k)|$$

(59)

Note that there is no $1 + \tilde{u}_{ii}$ in the coefficient of $B^{-\frac{1}{n-1}}$ in (59) since the term $\tilde{B}_{pi} D_{li} u_k$ is already subtracted in $L(u_k)$.

Hence, combining (57), (58), and (59), we have

$$L(\tilde{u}_{\tilde{\xi},\tilde{\xi}} - \nu') \geq \tilde{u}^D \tilde{\xi}^D D_{\tilde{\xi}} \tilde{u}_{\tilde{\xi},\tilde{\xi}} - C\left[(1 + \tilde{u}_{ii})T + \tilde{u}_{ii}\right] - C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}}.$$  

(60)

By Cauchy’s inequality, we obtain

$$\tilde{u}^D D_{\tilde{\xi}} (\tilde{u}_{\tilde{\xi},\tilde{\xi}} - \nu') D_{\tilde{\xi}} (\tilde{u}_{\tilde{\xi},\tilde{\xi}} - \nu') \leq (1 + \theta)\tilde{u}^D D_{\tilde{\xi}} \tilde{u}_{\tilde{\xi},\tilde{\xi}} + C(\theta)\tilde{u}^D D_{\nu'} D_{\nu'}$$

(61)

for any $\theta > 0$, where $C(\theta)$ is a positive constant depending on $\theta$. Inserting (48), (60), and (61) into (56), by calculations, we get

$$0 \geq LH$$

$$\geq \frac{1}{\tilde{u}_{\tilde{\xi},\tilde{\xi}} - \nu'} \tilde{u}^D \tilde{\xi}^D D_{\tilde{\xi}} \tilde{u}_{\tilde{\xi},\tilde{\xi}} - \frac{1 + \theta}{(\tilde{u}_{\tilde{\xi},\tilde{\xi}} - \nu')^2}\tilde{u}^D D_{\tilde{\xi}} \tilde{u}_{\tilde{\xi},\tilde{\xi}} + C(\theta)\tilde{u}^D D_{\nu'} D_{\nu'}$$

(62)

Next, we shall deal with the terms on the right-hand side of (62). Without loss of generality, we assume that $\{\tilde{u}_{ij}\}$ is diagonal at $x_0$ with the maximum eigenvalue $\tilde{u}_{11}$. We can always assume that $\tilde{u}_{11} > 1$ and is as large as we want; otherwise we are done. Since $\nu'$ is bounded, $\tilde{u}_{11}$ and $\tilde{u}_{\tilde{\xi},\tilde{\xi}}$ are comparable in the sense that, for any $\theta > 0$, there exists a further constant $C(\theta)$ such that

$$|\tilde{u}_{11} - \tilde{u}_{\tilde{\xi},\tilde{\xi}} + \nu'| < \theta \tilde{u}_{11}$$

(63)
if $\tilde{u}_{11} > C(\theta)$. Therefore, using (63), we have
\[
\begin{align*}
\frac{1}{\tilde{u}_{xx}} - \nu & \left[ (1 + \tilde{u}_{ii})T + (\tilde{u}_{ii})^2 \right] \\
& \leq \frac{1}{(1 - \theta)\tilde{u}_{11}} \left[ (1 + \tilde{u}_{ij})T + (\tilde{u}_{ij})^2 \right] \\
& \leq C \left( T + \sum_{i=1}^{n} \tilde{u}_{ij} \right)
\end{align*}
\]
(64)
for some constant $C$ if $\theta \in (0, 1/2)$.

We shall treat the first two terms on the right-hand side of (62). For this purpose, we will make a more detailed calculation than [15] since the equality $D_{i}A_{\xi \xi} = D_{\xi}A_{ij}$ does not hold in general. Because in (62) the vector $\xi$ is not equal to $e_{1}$, the situation here is different from the formula below (24) in [21]. Next, we divide the discussions into the two cases (a) and (b) assumed in the statement of this theorem.

In case (a), $A_{ij} = f_{ij}(x, u_{i})$. We define
\[
\mathcal{P} := \frac{1}{\tilde{u}_{11}} \tilde{u}^{ik} \tilde{u}^{jl} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} - \frac{1 - 2\theta}{1 - \theta} \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl},
\]
(65)
we shall get a lower bound of the quantity $\mathcal{P}$ in terms of $T$. We define the matrix
\[
\{a_{ik}\} := \{\tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl}\}
\]
(66)
and let $\Lambda$ be its maximum eigenvalue. For the first term of $\mathcal{P}$, we have
\[
\begin{align*}
\frac{1}{\tilde{u}_{11}} \tilde{u}^{ik} \tilde{u}^{jl} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} = \frac{\text{trace}\{\tilde{u}^{ik} a_{ik}\}}{\tilde{u}_{11}} \geq \frac{\tilde{u}^{11} \Lambda}{\tilde{u}_{11}} = \frac{\Lambda}{\tilde{u}_{11}^2}.
\end{align*}
\]
(67)
For the second term of $\mathcal{P}$, using $\{\tilde{u}_{ij}\} := \{u_{ij} - A_{ij}\}$, we have
\[
\begin{align*}
\frac{1 - 2\theta}{1 - \theta} \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} & \leq \frac{1 - 2\theta}{1 - \theta} \frac{2}{\tilde{u}_{11}} \tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} + \frac{1 - 2\theta}{1 - \theta} \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} \\
& \leq \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ik} D_{\xi} \tilde{u}_{ij} D_{\xi} \tilde{u}_{kl} + \frac{1 - 2\theta}{\theta} \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ik} D_{\xi} A_{ij} - D_{\xi} A_{ij} D_{\xi} A_{kl} D_{\xi} A_{kl}.
\end{align*}
\]
(68)
where the last inequality is valid by using Cauchy’s inequality. We now calculate the terms $D_x A_{\xi \xi}$ and $D_{\xi} A_{\xi \xi}$. For general $D(x, Du)$, we have

$$D_x A_{\xi \xi} = D_x u_{\xi \xi} + (D_p A_{\xi \xi}) u_{\xi \xi}, \quad \forall i = 1, \ldots, n. \quad (69)$$

For $A_{ij} = f_{ij}(x, u) \delta_{ij}$, we have

$$D_{\xi} A_{\xi \xi} = (D_p f_{ij}) \xi_{\xi k} + (D_p f_i) \xi_{\xi k} u_{\xi \xi}, \quad \forall i = 1, \ldots, n. \quad (70)$$

Substituting (69) and (70) into the last term of (68), we have

$$\frac{1 - 2\theta}{\tilde{u}_{11}^2} \tilde{u}^\nu (D_x A_{\xi \xi} - D_x A_{\xi \xi})(D_{\xi} A_{\xi \xi} - D_{\xi} A_{\xi \xi})$$
$$= \frac{1 - 2\theta}{\tilde{u}_{11}^2} \tilde{u}^\nu (D_x A_{\xi \xi} D_{\xi} A_{\xi \xi} - D_x A_{\xi \xi} D_{\xi} A_{\xi \xi} - D_x A_{\xi \xi} D_{\xi} A_{\xi \xi} + D_x A_{\xi \xi} D_{\xi} A_{\xi \xi})$$
$$\leq \frac{1 - 2\theta}{\tilde{u}_{11}^2} C(T + \tilde{u}_{11})$$
$$\leq C(T + 1), \quad (71)$$

where $\tilde{u}^\nu \tilde{u}_{ij} = \delta_{ik}$ and $\{\tilde{u}_{ij}\} := \{u_{ij} - A_{ij}\}$ are used in the first inequality, and $\tilde{u}_{11} \geq \frac{1 - 2\theta}{\theta}$ is assumed in the last inequality. Combining (65), (67), (68), and (71), we now get the lower bound of $P$ in terms of $T$,

$$P \geq \frac{1}{\tilde{u}_{11}^2} (\Lambda - a_{ik} \xi_{\xi i} \xi_{\xi j}) - C(T + 1) \geq -C(T + 1), \quad (72)$$

where the definitions of the matrix $\{a_{ik}\}$ and its maximum eigenvalue $\Lambda$ are used successively. On the other hand, it follows from (54) that

$$\tilde{u}^\nu D_{\xi} \tilde{u}_{\xi \xi} D_{\eta} \tilde{u}_{\xi \xi} \leq 2\tilde{u}^\nu \left[|D_{\eta} \nu|^2 + (\tilde{u}_{\xi \xi} - \nu)^2 (a D_{\xi} u D_{\xi} u + \kappa D_{\eta} \Phi)^2 \right]$$
$$\leq 2\tilde{u}^\nu |D_{\eta} \nu|^2 + C(\tilde{u}_{\xi \xi} - \nu)^2 \left(\alpha^2 \sum_{i=1}^n \tilde{u}_{ii} + \kappa^2 T \right). \quad (73)$$

Combining (63), (72), and (73), we obtain

$$\frac{1}{\tilde{u}_{\xi \xi} - \nu} \tilde{u}^\nu \tilde{u}^\mu D_{\xi} \tilde{u}_{\xi \xi} D_{\xi} \tilde{u}_{\xi \xi} \leq \frac{1 + \theta}{(\tilde{u}_{\xi \xi} - \nu)^2} \tilde{u}^\nu D_{\xi} \tilde{u}_{\xi \xi} D_{\xi} \tilde{u}_{\xi \xi}$$
$$\geq \frac{1}{1 - \theta} \left( \frac{1}{\tilde{u}_{11}^2} \tilde{u}^\nu \tilde{u}^\mu D_{\xi} \tilde{u}_{\xi \xi} D_{\xi} \tilde{u}_{\xi \xi} - \frac{1 + \theta}{(1 - \theta)\tilde{u}_{11}^2} \tilde{u}^\nu D_{\xi} \tilde{u}_{\xi \xi} D_{\xi} \tilde{u}_{\xi \xi} \right)$$
$$\geq \frac{1}{1 - \theta} \left( P - \frac{3\theta}{(1 - \theta)\tilde{u}_{11}^2} \tilde{u}^\nu D_{\xi} \tilde{u}_{\xi \xi} D_{\xi} \tilde{u}_{\xi \xi} \right)$$
$$\geq -\frac{C(T + 1)}{1 - \theta} - \frac{3\theta}{[(1 - \theta)\tilde{u}_{11}^2]^2} \left[ 2\tilde{u}^\nu |D_{\eta} \nu|^2 + C(\tilde{u}_{\xi \xi} - \nu)^2 \left(\alpha^2 \sum_{i=1}^n \tilde{u}_{ii} + \kappa^2 T \right) \right]$$
$$\geq -C(T + 1) - C\theta \alpha^2 \sum_{i=1}^n \tilde{u}_{ii} - C\theta \kappa^2 T.$$
for \( \theta \in (0, 1/2) \), where \( C \) becomes a further constant in the last inequality. We can assume \( \tilde{u}_{11} \geq 1 \), otherwise we have already obtained the desired estimate. Similar to (37), we also have

\[
B \frac{1}{\tilde{u}_{11}} \leq \frac{T}{n - 1}.
\]  

(75)

For the last term in (62), using Cauchy’s inequality, we have

\[
\epsilon_1 T - C(B \frac{1}{\tilde{u}_{11}} + 1)
\]

\[
\geq \epsilon_1 T - \epsilon B \frac{1}{\tilde{u}_{11}} - C(\epsilon)
\]

\[
\geq \frac{\epsilon_1}{2} T - C(\epsilon_1),
\]

(76)

where we take \( \epsilon = \frac{n - 1}{2} \epsilon_1 \) and use (75) in the second inequality.

Inserting (63), (64), (74), (75), and (76) into (62), we obtain, for \( \tilde{u}_{11} \geq \max\{C(\theta), 1\},

\[
\alpha \sum_{i=1}^{n} \tilde{u}_{ij} + \frac{\kappa \epsilon_1}{2} T
\]

\[
\leq C \left[ 1 + \kappa C(\epsilon_1) \right] + (1 + \alpha^2 \theta) \sum_{i=1}^{n} \tilde{u}_{ii} + (1 + \alpha + \kappa^2 \theta) T
\]

(77)

By choosing \( \kappa \gg \alpha \gg 1 \) and fixing a small positive \( \theta = 1/\kappa^2 \), we can get from (77) that

\[
\sum_{i=1}^{n} \tilde{u}_{ii}(x_0) \leq C,
\]  

(78)

which implies a corresponding estimate for \( |D^2 u(x_0)| \) in case (a).

In case (b), \( |D_{p_i} A_{ij}| < \delta \) for all \( i, j, k = 1, \ldots, n \), and sufficiently small \( \delta \). We define

\[
\mathcal{P}' := \frac{1}{\tilde{u}_{11}} \tilde{u}_{ij} D_{x_i} \tilde{u}_{ij} D_{x_j} \tilde{u}_{kl} D_{x_l} \tilde{u}_{lk}.
\]

(79)

Similarly to (68), we can get

\[
\frac{1}{\tilde{u}_{11}'^2} \tilde{u}_{ij}' D_{x_i} \tilde{u}_{ij} D_{x_j} \tilde{u}_{kl} D_{x_l} \tilde{u}_{lk}'
\]

\[
= \frac{1}{\tilde{u}_{11}'} \tilde{u}_{ij}' D_{x_i} \tilde{u}_{ij} D_{x_j} \tilde{u}_{kl} (D_{x_j} A_{ij} - D_{x_i} A_{kl})
\]

\[
+ \frac{1}{\tilde{u}_{11}'} \tilde{u}_{ij}' (D_{x_i} A_{ij} - D_{x_j} A_{ij})(D_{x_j} A_{ij} - D_{x_i} A_{kl}).
\]

(80)

For the middle term on the right-hand side of (80), rather than using Cauchy’s inequality, it can be dealt with by using \( D_{x_i} \tilde{u}_{ij} = D_{i} u_{ij} - D_{x} A_{ij}, (69), (70), (54), \) and \( |D_{p_i} A_{ij}| < \delta \), namely

\[
\frac{2}{\tilde{u}_{11}'} \tilde{u}_{ij}' D_{x_i} \tilde{u}_{ij} (D_{x_j} A_{ij} - D_{x_j} A_{ij})
\]

\[
= \frac{2}{\tilde{u}_{11}'} \tilde{u}_{ij}' [D_{i} u_{ij} - (D_{x_j} A_{ij} - D_{x_i} A_{ij})](D_{x_j} A_{ij} - D_{x_j} A_{ij})
\]
\[ \leq \frac{2}{\tilde{u}_{11}} \tilde{u}^a D_\xi \tilde{u}_\xi (D_\zeta A_\zeta - D_j A_\xi) \]  
\[ = \frac{2}{\tilde{u}_{11}} \tilde{u}^a [D_\xi v' - (\tilde{u}_\xi v' - v')(\alpha D_\xi u D_\xi u + \kappa D_j \Phi)] (D_\zeta A_\zeta - D_j A_\xi) \]  
\[ \leq C [1 + \delta (\alpha + \kappa T)], \]  

provided \( \tilde{u}_{11} \geq 1 \). We can analyze the last term on the right-hand side of (80) for general matrix \( A \) to get

\[ \frac{1}{\tilde{u}_{11}} \tilde{u}^a (D_\xi A_\zeta - D_j A_\xi) (D_\zeta A_\zeta - D_j A_\xi) \leq C(T + 1), \]  

provided \( \tilde{u}_{11} \geq 1 \). Combining (79), (80), (81), and (82), we obtain

\[ P' \geq \frac{1}{\tilde{u}_{11}} (\Lambda - a_{ij} \xi_i \xi_j) - C [1 + T + \delta (\alpha + \kappa T)] \]  

\[ \geq -C [1 + T + \delta (\alpha + \kappa T)]. \]  

Combining (63), (83), and (73), we obtain

\[ \frac{1}{\tilde{u}_{11}} \tilde{u}^a D_\xi \tilde{u}_\xi D_\zeta \tilde{u}_\zeta - \frac{1 + \theta}{(\tilde{u}_\xi - v')^2} \tilde{u}^a D_\xi \tilde{u}_\xi D_j \tilde{u}_\xi \]  
\[ \geq \frac{1}{1 - \theta} \left( \frac{1}{\tilde{u}_{11}} \tilde{u}^a D_\xi \tilde{u}_\xi D_\zeta \tilde{u}_\zeta - \frac{1 + \theta}{(1 - \theta)\tilde{u}_{11}^2} \tilde{u}^a D_\xi \tilde{u}_\xi D_j \tilde{u}_\xi \right) \]  
\[ \geq \frac{1}{1 - \theta} \left( P' - \frac{2\theta}{(1 - \theta)\tilde{u}_{11}^2} \tilde{u}^a D_\xi \tilde{u}_\xi D_j \tilde{u}_\xi \right) \]  
\[ \geq -C [1 + T + \delta (\alpha + \kappa T)] - C \theta \alpha^2 \sum_{i=1}^n \tilde{u}_{ii}^2 - C \theta \kappa^2 \alpha T \]  

for \( \theta \in (0, 1/2) \). Using the estimate in (83) and deducing as in case (a), for \( \tilde{u}_{11} \geq \max\{C(\theta), 1\} \), we obtain in place of (77)

\[ \alpha \sum_{i=1}^n \tilde{u}_{ii} + \frac{\kappa \epsilon_1}{2} T \]  
\[ \leq C \left[ 1 + \kappa C(\epsilon_1) + \delta \alpha + (1 + \alpha^2 \theta) \sum_{i=1}^n \tilde{u}_{ii} + (1 + \alpha + \kappa^2 \theta + \kappa \delta) T \right]. \]  

By choosing \( \alpha = 2C + 1 \) and \( \kappa = \frac{2(C(\alpha + 3) + 1)}{\epsilon_1} \) successively and fixing the positive constants \( \theta = 1/\kappa^2 \) and \( \delta = 1/\kappa \), we can get from (85) that (78) holds again, which implies a corresponding estimate for \( |D^2u(x_0)| \) in case (b).

Next, we consider the case \( x_0 \in \partial \Omega \), namely the function \( G \) in (50) attains its maximum over \( \Omega \) at \( x_0 \in \partial \Omega \) and a unit vector \( \xi \). The estimation of the rest of the Hessian \( D^2u \) splits into two stages according to a different direction of \( \xi \).
\textbf{Case (i).} \( \xi \) tangential. Since \( v'(x_0, \xi) = 0 \), at \( x_0 \) we obtain

\[
0 \geq D_v G
= D_v \left[ \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right] e^{\frac{1}{2} |Dv|^2 + \kappa \Phi}
= e^{\frac{1}{2} |Dv|^2 + \kappa \Phi} \left[ \left| \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right| + \frac{\alpha}{2} |Du|^2 + \kappa \Phi \right] + D_v \left( \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right)
= e^{\frac{1}{2} |Dv|^2 + \kappa \Phi} \left[ \left| \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right| + \frac{\alpha}{2} |Du|^2 + \kappa \Phi \right] + D_v \left( \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right)
\]

(86)

\[
\geq e^{\frac{1}{2} |Dv|^2 + \kappa \Phi} \left[ \left| \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right| + \frac{\alpha}{2} |Du|^2 + \kappa \Phi \right] \tilde{u}_{\xi \xi} + D_v \left( \tilde{u}_{\xi \xi} - v'(x_0, \xi) \right)
\]

where \( c_0 = Ka \inf(e^{\frac{1}{2} |Dv|^2 + \kappa \Phi}), M = \max_{\Omega_1} |Du| \). The above inequality gives a relationship between \( \tilde{u}_{\xi \xi}(x_0) \) and \( D_v u_{\xi \xi}(x_0) \) at \( x_0 \), namely

\[
D_v u_{\xi \xi} \leq -(\kappa c_0 - \alpha M) \tilde{u}_{\xi \xi} + D_v \left( A_{\xi \xi} + v' \right).
\]

(87)

On the other hand, by tangential differentiating the boundary condition twice, we obtain

\[
(D_k u) \delta_i \delta_j v_k + (\partial_i D_r u) \delta_j v_k + (\partial_j D_r u) \delta_i v_k + v_k \delta_i \delta_j D_r u = \delta_i \delta_j v \quad \text{on } \partial \Omega.
\]

(88)

Hence, for the tangential direction \( \xi \) at \( x_0 \), we have

\[
D_v u_{\xi \xi} \geq -2(\delta_i v_k) D_{jk} u_{\xi \xi} \xi_j + (\delta_i v_k) \tilde{e}_i \tilde{e}_j D_v u - C
\]

(89)

where the double normal derivative estimate (46) on \( \partial \Omega \) is used in the second inequality. Thus, at \( x_0 \) we have

\[
(\kappa c_0 - \alpha M) \tilde{u}_{\xi \xi} \leq 2(\delta_i v_k) D_{jk} u_{\xi \xi} \xi_j + D_v \left( A_{\xi \xi} + v' \right) + C
\]

(90)

\[
\leq 2(\delta_i v_k) D_{jk} u_{\xi \xi} \xi_j + C |DD_v u| + C
\]

\[
\leq 2(\delta_i v_k) D_{jk} u_{\xi \xi} \xi_j + C,
\]

where the mixed tangential normal derivative estimate (28) and the double normal derivative estimate (46) are used to obtain the last inequality, and the constant \( C \) changes from line to line. Without loss of generality, we can assume the normal at \( x_0 \) to be \( \nu = (0, \ldots, 1) \), and we can assume \( \{\tilde{u}_{ij}(x_0)\}_{i,j=0} \) is diagonal with maximum eigenvalue \( \tilde{u}_{11}(x_0) > 1 \). Then, at \( x_0 \) we obtain

\[
(\kappa c_0 - \alpha M) \tilde{u}_{\xi \xi} \leq C(\tilde{u}_{11} + 1).
\]

(91)

Since \( G(x_0, e_1) \leq G(x_0, \xi) \), we have

\[
\tilde{u}_{11}(x_0) \leq \tilde{u}_{\xi \xi}(x_0) + v'(x_0, e_1) - v'(x_0, \xi).
\]

(92)
Combining (91) and (92) and choosing \( \kappa \geq \frac{2}{C_0}(\alpha M + C) \), we obtain
\[
u_{\xi\xi}(x_0) \leq C. \tag{93}
\]

Case (ii), \( \xi \) non-tangential. We write \( \xi \) as
\[
\xi = \alpha \tau + \beta \nu, \tag{94}
\]
where \( \alpha = \xi \cdot \tau \), \( |\tau| = 1 \), \( \tau \cdot v = 0 \), \( \beta = \xi \cdot v \neq 0 \), and
\[
\alpha^2 + \beta^2 = 1. \tag{95}
\]
Therefore, at \( x_0 \) we have
\[
D_{\xi\xi} u = \alpha^2 \bar{u}_{\tau\tau} + \beta^2 \bar{u}_{\nu\nu} + 2\alpha\beta \bar{u}_{\tau\nu} = \alpha^2 \bar{u}_{\tau\tau} + \beta^2 \bar{u}_{\nu\nu} + v'(x, \xi), \tag{96}
\]
where the definition of \( v' \) in (51) is used. Since \( G(x_0, \tau) \leq G(x_0, \xi) \), we obtain
\[
G(x_0, \xi) = \alpha^2 G(x_0, \tau) + \beta^2 G(x_0, v) \leq \alpha^2 G(x_0, \xi) + \beta^2 G(x_0, v). \tag{97}
\]
Using (95) in (97), we get
\[
G(x_0, \xi) \leq G(x_0, v), \tag{98}
\]
which implies
\[
D_{\xi\xi} u(x_0) \leq C + D_{\nu\nu} u(x_0) \leq C, \tag{99}
\]
where the double normal derivative estimate (46) on \( \partial \Omega \) is used again.

Therefore, we can conclude from the two cases (i) and (ii) that if \( G \) attains its global maximum at \( x_0 \in \partial \Omega \) and a unit vector \( \xi \), then \( D_{\xi\xi} u(x_0) \) is bounded from above. Taking (78) into account, we can derive estimate (11). Hence, Theorem 1.1 is proved. \( \Box \)

4 \( C^0, C^1 \) estimates and the existence
In this section, we derive the lower order a priori derivative estimates (i.e., the \( C^0 \) and \( C^1 \) estimates) for Monge–Ampère type equations (1) with the Neumann boundary value condition (2). Using these estimates together with the second derivative estimate in Theorem 1.1, we give the proof of Theorem 1.2.

Since \( B_2 > 0 \), the comparison principle for the classical solutions is valid. Under the assumptions that \( \underline{u} \) satisfies (4)–(5) and \( \bar{u} \) satisfies (1) and \( D_{\nu} \bar{u} \leq \varphi(x) \) on \( \partial \Omega \), by comparison principle, we have
\[
u \leq u \leq \bar{u}, \tag{100}
\]
in \( \bar{\Omega} \), which gives the \( C^0 \) estimate of the solution \( u \).
The $C^1$ estimate is established in [15, 22] in a general form. For convenience, we repeat it here.

**Theorem 4.1** Let $u \in C^2(\Omega) \cap C^{1,1}(\bar{\Omega})$ satisfy
\begin{equation}
D^2u \geq -\mu_0(1 + |Du|^2)I
\end{equation}

in a $C^2$ domain $\Omega \subset \mathbb{R}^n$, with $D\nu u \geq -\delta$ on $\partial \Omega$, where $\mu_0$ and $\delta$ are nonnegative constants. Then we have the estimate
\begin{equation}
|Du| \leq C,
\end{equation}

where $C$ depends on $\mu_0$, $\delta$, $\Omega$, and $\sup |u|$.

From the ellipticity and the quadratic lower bound for $A$ in (13), it is easy to get inequality (101). Setting $\delta = \max\{-\min_{\partial \Omega} \varphi, 0\}$, $u$ satisfying (2) automatically satisfies $D\nu u \geq -\delta$ on $\partial \Omega$. Hence, the gradient estimate (102) holds for the solution $u$ of problem (1)–(2).

With the preparations of *a priori* estimates (100), (102), and (11), we are able to prove the existence result in Theorem 1.2.

**Proof of Theorem 1.2** We shall prove the existence of a viscosity solution $u$ of the Neumann problem of the degenerate Monge–Ampère equation (1)–(2) by approximations.

We consider the following approximating problem:
\begin{equation}
\begin{align*}
\det[D^2u - A(\cdot, Du)] &= B_\epsilon > 0, \text{ in } \Omega, \\
D_\nu u &= \psi(x), \quad \text{on } \partial \Omega,
\end{align*}
\end{equation}

where
\begin{equation}
B_\epsilon(\cdot, u, Du) := B(\cdot, u, Du) + \epsilon
\end{equation}

for small positive constant $\epsilon$. If $B$ satisfies conditions (6)–(8), it is obvious that $B_\epsilon$ still satisfies the same conditions.

By Remark 1, there exists a unique elliptic solution $u_\epsilon \in C^{3,\alpha}$ of problem (103), where $\alpha \in (0, 1)$. Since $\bar{u}$ is still the supersolution of problem (103), and for sufficiently small $\epsilon > 0$, $\bar{u}$ is still the subsolution of problem (103) as well, we get the uniform estimate
\begin{equation}
\underline{u} \leq u_\epsilon \leq \bar{u}
\end{equation}

for sufficiently small $\epsilon > 0$. Notice that the gradient estimate in Theorem 4.1 is independent of the lower bound of the right-hand side term of the equation. Therefore, we have
\begin{equation}
|D u_\epsilon | \leq C,
\end{equation}

where the constant $C$ is independent of $\epsilon$. By Theorem 1.1, we obtain the uniform second derivative estimate
\begin{equation}
\sup_\Omega |D^2 u_\epsilon | \leq C.
\end{equation}
From the remarks in the paragraph below Theorem 1.1 (or from the proof of Theorem 1.1), we know that the constant $C$ in (106) is independent of $\epsilon$. With the uniform estimates (104), (105), and (106), we can pass the limit to $u_\epsilon$. Now letting $\epsilon \to 0$ (passing to a subsequence if necessary), we get a viscosity solution $u$ of the degenerate Monge–Ampère equation (1) with the Neumann boundary condition (2), which is degenerate elliptic and satisfies the estimate

$$
\|u\|_{C^{1,1}(\bar{\Omega})} \leq C.
$$

(107)

Therefore, the proof of Theorem (1.2) is completed. □

Remark 7 In Theorem 1.2, we proved the existence of a viscosity solution $u \in C^{1,1}(\bar{\Omega})$ by smooth approximations. Such uniqueness of viscosity solutions in our $A_z = 0$, $B_z > 0$, and $\varphi_z = 0$ case is not covered in Sect. 4 of [23]. It would be interesting to investigate the uniqueness of the viscosity solutions of the degenerate Monge–Ampère equation (1) satisfying the Neumann boundary condition (2).

Remark 8 In the current paper, we show that “the existence of a subsolution implies the existence of a solution”, which is the new point of the existence result to the Neumann problem (1)–(2) of degenerate Monge–Ampère type equations. This idea is also new even for nondegenerate Monge–Ampère type equations. For the nondegenerate case in [15], for a more general problem with $A$ and $\varphi$ depending also on $u$,

$$
\det[D^2u - A(\cdot, u, Du)] \equiv B(\cdot, u, Du) \quad \text{in } \Omega,
$$
$$
D_\nu u = \varphi(\cdot, u) \quad \text{on } \partial \Omega,
$$

(108)

the result is that “the existence of a supersolution implies the existence of a solution”. If we consider such a general problem (108) in the degenerate elliptic setting, the supersolution $u$ is only degenerate elliptic and is not enough to construct a barrier function. It needs new ideas to solve the general Neumann problem (108) in the degenerate case.

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