On representing some lattices as lattices of intermediate subfactors of finite index

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Abstract

We prove that the very simple lattices which consist of a largest, a smallest and 2n pairwise incomparable elements where n is a positive integer can be realized as the lattices of intermediate subfactors of finite index and finite depth. Using the same techniques, we give a necessary and sufficient condition for subfactors coming from Loop groups of type A at generic levels to be maximal.

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1 Introduction

Let $M$ be a factor and $N$ a subfactor of $M$ which is irreducible, i.e., $N' \cap M = \mathbb{C}$. Let $K$ be an intermediate von Neumann subalgebra for the inclusion $N \subset M$. Note that $K' \cap K \subset N' \cap M = \mathbb{C}$, $K$ is automatically a factor. Hence the set of all intermediate subfactors for $N \subset M$ forms a lattice under two natural operations $\wedge$ and $\vee$ defined by:

$$K_1 \wedge K_2 = K_1 \cap K_2, K_1 \vee K_2 = (K_1 \cup K_2)''. $$

Let $G_1$ be a group and $G_2$ be a subgroup of $G_1$. An interval sublattice $[G_1/G_2]$ is the lattice formed by all intermediate subgroups $K, G_2 \subseteq K \subseteq G_1$.

By cross product construction and Galois correspondence, every interval sublattice of finite groups can be realized as intermediate subfactor lattice of finite index. The study of intermediate subfactors has been very active in recent years (cf. [9], [18], [21], [29], [39] and [37] for only a partial list). By a result of S. Popa (cf. [34]), if a subfactor $N \subset M$ is irreducible and has finite index, then the set of intermediate subfactors between $N$ and $M$ is finite. This result was also independently proved by Y. Watatani (cf. [39]). In [39], Y. Watatani investigated the question of which finite lattices can be realized as intermediate subfactor lattices. Related questions were further studied by P. Grossman and V. F. R. Jones in [18] under certain conditions. As emphasized in [18], even for a lattice which shapes like a Hexagon and consists of six elements, it is not clear if it can be realized as intermediate subfactor lattice with finite index. This question has been solved recently by M. Aschbach in [1] among other things. In [1], M. Aschbach constructed a finite group $G_1$ with a subgroup $G_2$ such that the interval sublattice $[G_1/G_2]$ is a Hexagon. The lattices that appear in [18, 39, 1] can all be realized as interval sublattice of finite groups.

It turns out that which finite lattice can be realized as an interval sublattice $[G_1/G_2]$ with $G_1$ finite is an old problem in finite group theory. See [32] for an excellent review and a list of references.

Most of the attention has been focused on the very simple lattice $M_n$ consisting of a largest, a smallest and $n$ pairwise incomparable elements. For $n = 1, 2, q + 1$ (where $q$ is a prime power), examples of $M_n$ have been found in the finite solvable groups. After the first interesting examples found by W. Feit (cf. [11]), A. Lucchini (cf. [31]) discovered new series of examples for $n = q + 2$ and for $n = \frac{(q^t+1)}{(q+1)} + 1$ where $t$ is an odd prime.

At the present the only values of $n$ for which $M_n$ occurs as an interval sublattice of a finite group are $n = 1, 2, q + 1, q + 2, \frac{(q^t+1)}{(q+1)} + 1$ where $t$ is an odd prime. The smallest undecided case is $n = 16$. In a major contribution to the problem about subgroup lattices of finite groups in [2], R. Baddeley and A. Lucchini have reduced the problem of realizing $M_n$ as interval sublattice of finite groups to a collection of questions about finite simple groups. These questions are still quite hard, but eventually they might be resolved using the classification of finite simple groups. In this paper, the authors are cautious, but their ultimate goal seems to be to show that the list above is complete. In view of the above results about finite groups, it seems an interesting problem to
ask if $M_{16}$ can be realized as the lattice of intermediate subfactors with finite index. This problem is the main motivation for our paper. One of the main results of this paper, Theorem 2.40, states that all $M_{2n}$ are realized as the lattice of intermediate subfactors of a pair of hyperfinite type $III_1$ factors with finite depth. Note that by [37] this implies that $M_{2n}$ can also be realized as the lattice of intermediate subfactors of a pair of hyperfinite type $II_1$ factors with finite depth. Thus modulo the conjectures of R. Baddeley, A. Lucchini and possibly others we have an infinite series of lattices which can be realized by the lattice of intermediate subfactors with finite index and finite depth but can not be realized by interval sublattices of finite groups.

The subfactors which realize $M_{2n}$ are “orbifold subfactors” of [10, 4, 44], and we are motivated to examine these subfactors by the example of lattice of type $M_6$ in [18] which is closely related to an $\mathbb{Z}_2$ orbifold. To explain their construction, after first two preliminary sections, we will first review the result of A. Wassermann (cf. [23], [40]) about Jones-Wassermann subfactors (cf. Remark (2.27)) coming from representations of Loop groups of type $A$ in section 2.5. Section 2.6 is then devoted to a description of “orbifold subfactors” from an induction point of view. Although it is not too hard to show that the subfactor contains $2n$ incomparable intermediate subfactors, the hard part of the proof of Theorem 2.40 is to show that there are no more intermediate subfactors. Here we give a brief explanation of basic ideas behind our proof and describe how the paper is structured. We will use freely notations and concepts that can be found in preliminary sections. Let $\rho(M) \subset M$ be a subfactor and $M_1$ be an intermediate subfactor. In our examples below all factors are isomorphic to the hyperfinite type $III_1$ factor, and $\rho \bar{\rho}$ are direct sums of sectors from a set $\Delta$ with finitely many irreducible sectors and a non-degenerate braiding. Here we use the endomorphism theory pioneered by R. Longo (cf. [26]). Since $M_1$ is isomorphic to $M$, we can choose an isomorphism $c_1 : M \rightarrow M_1$. Denote by $c_2 = c_1^{-1}\rho$ we have $\rho = c_1c_2$ where $c_1, c_2 \in \text{End}(M)$. Note that $c_1\bar{c}_1 \prec \rho \bar{\rho}$ is in $\Delta$. Our basic idea to investigate the property of $c_1$ is to consider the following set $H_{c_1} := \{[a] | a \prec \lambda c_1, \lambda \in \Delta, a \text{ irreducible}\}$. Since $\Delta$ has finitely many irreducible sectors, $H_{c_1}$ is a finite set. Moreover since $c_1\bar{c}_1 \in \Delta$, an induction method using braiding as in [42] is available. This induction method was used by the author in [42] to study subfactors from conformal inclusions, and developed further by J. Böckenhauer-D. Evans and J. Böckenhauer-D. Evans-Y. Kawahigashi in [3, 4, 5, 6, 7, 8], and leads to strong constraints on the set $H_{c_1}$. Thus by using $\lambda \in \Delta$ to act from the left on $c_1$, one may hope to find what $c_1$ is made of. In the cases of Theorem 2.40 and Cor. 5.23, it turns out that there is a sector $c$ in $H_{c_1}$ with smallest index such that $c_1 \prec \lambda c$, and $c$ is close to be an automorphism (It is an automorphism in the case of Cor. 5.23), and the corresponding subfactors have been well studied as those in [42]. In the simplest case $n = 2$, due to $A-D-E$ classification of graphs with norm less than 2, the above idea can be applied directly to give a rather quick proof of Th. 2.40. We refer the reader to the paragraph after Th. 2.40 which illustrates the above idea.

When $n > 2$, the norms of fusion graphs are in general greater than 2, no $A-D-E$ classification is available, and this is the main problem we must resolve to carry out
the above idea. To explain our method, we note that $S$ matrix as defined in equation (3) has the property

$$|\frac{S_{\lambda\mu}}{S_{1\mu}}| \leq \frac{S_{\lambda1}}{S_{11}}, \forall \mu$$

and

$$|\frac{S_{\lambda\mu}}{S_{1\mu}}| = \frac{S_{\lambda1}}{S_{11}}, \forall \mu$$

iff $\lambda$ is an automorphism, i.e., $\lambda$ has the smallest index 1. Our first observation is that for small index (close to 1) sectors $c$, certain entries of $S$-matrix like quantities (cf. Def. 3.10, Cor. 3.14) called $\psi$-matrix attain their extremum just like $S$-matrices. Hence to detect these small index sectors, we need to have a good estimation of the entries of $\psi$-matrix. In view of the Verlinde formula (cf. equation 4) relating $S$-matrix with fusion rules, it is natural to use the known fusion rules to estimate $\psi$ matrix. However, since the definition of $\psi$ involves sectors which are not braided, the above idea does not work unless one can show that certain intertwining operators are central (cf. Th. 3.8 and Section 5.1 for discussions). Our second observation is that a class of intertwining operators in Definition 3.7 are central (cf. Th. 3.8). Thanks to a number of known results about representations of Loop groups of type $A$, we show that the assumption of Th. 3.8 is verified in our case (cf. Prop. 4.7).

This allows us to show that certain sector with small index does exist (cf. Cor. 3.14), we can indeed find that $c_1$ is made of known subfactors. After a straightforward calculations involving known fusion rules in Prop. 4.10 we are able to finish the proof of Th. 2.40 for general $n$.

In the last section we discuss a few related issues. Conjecture 5.1 is formulated which is equivalent to centrality of certain intertwining operators (cf. Prop. 5.7), and this is motivated by our proof of Th. 2.40. We show in Prop. 5.11 that these intertwining operators are central on a subspace which is a linear span of products of (cf. Def. 5.9) cups, caps and braiding operators only. These motivate us to make Conjecture 5.12 which claims that the subspace is fact the whole space. In view of recent development using category theory (cf. [12]), both conjectures can in fact be stated in categorical terms, and we don’t know any counter examples in the categorical setting. In Prop. 5.17 we prove that a weaker version of Conjecture 5.12 implies Conjecture 5.1 and from this we are able to prove Conjecture 5.1 for modular tensor category from $SU(n)$ at level $k$ (cf. Cor. 5.18).

In section 5.2 we give applications of Cor. 5.18. To explain these applications, recall that a subfactor $N \subset M$ is called maximal if $M_1$ is an intermediate von Neumann algebra between $N$ and $M$ implies that $M_1 = M$ or $M_1 = N$. This notion is due to V. F. R. Jones when he outlined an interesting programme to investigate questions in group theory using subfactors (cf. [22]). In the case when $M$ is the crossed product of $N$ by a finite group $G$, it is easy to see that $N \subset M$ is maximal iff $G$ is an abelian group of prime order. Hence for most of $G$ the corresponding subfactor is not maximal. Cor. 5.23 gives a necessary and sufficient condition for subfactors from representations $\lambda$ of $SU(n)$ at level $k \neq n \pm 2, n$ to be maximal: $\lambda$ is maximal iff $\lambda$
is not fixed by a nontrivial cyclic automorphism of extended Dynkin diagram. Such cyclic automorphisms generate a group isomorphic to $\mathbb{Z}_n$. Hence it follows from Cor. 5.23 that most of such $\lambda$ are maximal. For an example, if $k \neq n \pm 2, n$, $k$ and $n$ are relatively prime, then all $\lambda$ are maximal.

Besides Propositions and Theorems that have been already mentioned, the first two preliminary sections are about sectors, covariant representations, braiding-fusion equations, Yang-Baxter equations, Rehren’s $S,T$ matrices. The third preliminary section summarizes properties of an induction method from [42]. These properties have been extensively studied and applied in subsequent work in [3], [4], [5], [6], [7] and [8] from a different point of view where induction takes place between two different but isomorphic algebras, and we recall a dictionary relating these two as provided in [45]. We think that in this paper it is simpler to take the point of view of [42] when discussing intermediate subfactors, and it is convenient to represent these intermediate subfactors as the range of endomorphisms of one fixed factor, so we do not have to switch between different but isomorphic algebras.

Using the dictionary we translate some properties of relative braidings and local extensions from [6] to our setting (cf. Prop. 2.24). The next two preliminary sections are devoted to subfactors of representations of $SU(n)$ at level $k$ and its extensions. We collect a few properties about fusion rules, $S$ matrices, and we define the subfactor which appears in Th. 2.40. In Prop. 2.41 we show that this subfactor contains $2n$ incomparable proper intermediate subfactors.

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## 2 Preliminaries

For the convenience of the reader we collect here some basic notions that appear in this paper. This is only a guideline and the reader should look at the references such as preliminary sections of [25] for a more complete treatment.

### 2.1 Sectors

Let $M$ be a properly infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of $M$. In this paper $M$ will always be the unique hyperfinite $III_1$ factors. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$. We denote by $[\rho]$ the image of $\rho \in \text{End}(M)$ in $\text{Sect}(M)$.

It follows from [26] and [27] that $\text{Sect}(M)$, with $M$ a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \mapsto \bar{\theta}$; moreover, $\text{Sect}(M)$ is a semiring.

Let $\rho \in \text{End}(M)$ be a normal faithful conditional expectation $\epsilon : M \to \rho(M)$. We define a number $d_{\epsilon}$ (possibly $\infty$) by:

$$d_{\epsilon}^{-2} := \text{Max}\{\lambda \in [0, +\infty)|\epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+\}$$
We define
\[ d = \min \{ d_\epsilon | d_\epsilon < \infty \}. \]

d is called the statistical dimension of \( \rho \) and \( d^2 \) is called the Jones index of \( \rho \). It is clear from the definition that the statistical dimension of \( \rho \) depends only on the unitary equivalence classes of \( \rho \). The properties of the statistical dimension can be found in [26], [27] and [28].

Denote by \( \text{Sect}_0(M) \) those elements of \( \text{Sect}(M) \) with finite statistical dimensions. For \( \lambda, \mu \in \text{Sect}_0(M) \), let \( \text{Hom}(\lambda, \mu) \) denote the space of intertwiners from \( \lambda \) to \( \mu \), i.e. \( a \in \text{Hom}(\lambda, \mu) \) iff \( a \lambda(x) = \mu(x) a \) for any \( x \in M \). \( \text{Hom}(\lambda, \mu) \) is a finite dimensional vector space and we use \( \langle \lambda, \mu \rangle \) to denote the dimension of this space. \( \langle \lambda, \mu \rangle \) depends only on \( [\lambda] \) and \( [\mu] \). Moreover we have \( \langle \nu \lambda, \mu \rangle = \langle \lambda, \nu \mu \rangle \), \( \langle \nu \lambda, \mu \rangle = \langle \nu, \mu \bar{\lambda} \rangle \) which follows from Frobenius duality (See [27]). We will also use the following notation: if \( \mu \) is a subsector of \( \lambda \), we will write as \( \mu \prec \lambda \) or \( \lambda \succ \mu \). A sector is said to be irreducible if it has only one subsector.

For any \( \rho \in \text{End}(M) \) with finite index, there is a unique standard minimal inverse \( \phi_\rho : M \rightarrow M \) which satisfies
\[ \phi_\rho(\rho(m)m'm''\rho(m''')) = m\phi_\rho(m')m''', m,m',m'' \in M. \]
\( \phi_\rho \) is completely positive. If \( t \in \text{Hom}(\rho_1, \rho_2) \) then we have
\[ d_{\rho_1}\phi_{\rho_1}(mt) = d_{\rho_2}\phi_{\rho_2}(tm), m \in M \]

2.2 Sectors from conformal nets and their representations

We refer the reader to §3 of [25] for definitions of conformal nets and their representations. Suppose a conformal net \( \mathcal{A} \) and a representation \( \lambda \) is given. Fix an open interval \( I \) of the circle and Let \( M := \mathcal{A}(I) \) be a fixed type \( III_1 \) factor. Then \( \lambda \) gives rise to an endomorphism still denoted by \( \lambda \) of \( M \). We will recall some of the results of [36] and introduce notations.

Suppose \( \{[\lambda]\} \) is a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net \( \mathcal{A} \). We will use \( \Delta_{\mathcal{A}} \) to denote all finite index representations of net \( \mathcal{A} \) and will use the same notation \( \Delta_{\mathcal{A}} \) to denote the corresponding sectors of \( M \).

We will denote the conjugate of \( [\lambda] \) by \( [\bar{\lambda}] \) and identity sector (corresponding to the vacuum representation) by \([1]\) if no confusion arises, and let \( N_{\lambda\mu}^\nu = \langle [\lambda][\mu], [\nu] \rangle \). Here \( \langle \mu, \nu \rangle \) denotes the dimension of the space of intertwiners from \( \mu \) to \( \nu \) (denoted by \( \text{Hom}(\mu, \nu) \)). We will denote by \( \{T_\epsilon\} \) a basis of isometries in \( \text{Hom}(\nu, \lambda\mu) \). The univalence of \( \lambda \) and the statistical dimension of (cf. §2 of [19]) will be denoted by \( \omega_\lambda \) and \( d(\lambda) \) (or \( d_\lambda \)) respectively. The unitary braiding operator \( \epsilon(\mu, \lambda) \) (cf. [19]) verifies the following

1Many statements in this section and section 2.3 hold true in general case when the the set \( \{[\lambda]\} \) is only braided (cf. [17]) and we hope to consider such cases elsewhere.
Proposition 2.1. (1) Yang-Baxter-Equation (YBE)

\[ \varepsilon(\mu, \gamma)\mu(\varepsilon(\lambda, \gamma))\varepsilon(\lambda, \mu) = \gamma(\varepsilon(\lambda, \mu))\varepsilon(\lambda, \gamma)\lambda(\varepsilon(\mu, \gamma)) \]

(2) Braiding-Fusion-Equation (BFE)

For any \( w \in \text{Hom}(\mu\gamma, \delta) \)

\[ \varepsilon(\lambda, \delta)\lambda(w) = w\mu(\varepsilon(\lambda, \gamma))\varepsilon(\lambda, \mu)(\varepsilon(\lambda, \gamma)) \]

\[ \varepsilon(\delta, \lambda)^*\lambda(w) = w\mu(\varepsilon(\gamma, \lambda)^*\varepsilon(\mu, \lambda)^*) \]

\[ \varepsilon(\lambda, \delta)^*\lambda(w) = w\mu(\varepsilon(\gamma, \lambda)^*\varepsilon(\lambda, \mu)^*) \]

Lemma 2.2. If \( \lambda, \mu \) are irreducible, and \( t_\nu \in \text{Hom}(\nu, \lambda\mu) \) is an isometry, then

\[ t_\nu \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) t_\nu^* = \omega_\nu \omega_\lambda \omega_\mu. \]

By Prop. 2.1, it follows that if \( t_i \in \text{Hom}(\mu_i, \lambda) \) is an isometry, then

\[ \varepsilon(\mu, \mu_i)\varepsilon(\mu_i, \mu) = t_i^* \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) t_i \]

We shall always identify the center of \( M \) with \( \mathbb{C} \). Then we have the following

Lemma 2.3. If

\[ \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) \in \mathbb{C}, \]

then

\[ \varepsilon(\mu, \mu_i)\varepsilon(\mu_i, \mu) \in \mathbb{C}, \forall \mu_i \prec \lambda. \]

Let \( \phi_\lambda \) be the unique minimal left inverse of \( \lambda \), define:

\[ Y_{\lambda\mu} := d(\lambda)d(\mu)\phi_\mu(\varepsilon(\mu, \lambda)^*\varepsilon(\mu, \mu)^*), \]

where \( \varepsilon(\mu, \lambda) \) is the unitary braiding operator (cf. [19]).

We list two properties of \( Y_{\lambda\mu} \) (cf. (5.13), (5.14) of [36]):

Lemma 2.4.

\[ Y_{\lambda\mu} = Y_{\mu\lambda} = Y_{\lambda\mu}^* = Y_{\mu\lambda}. \]

\[ Y_{\lambda\mu} = \sum_k N_{\lambda\mu}^\nu \frac{\omega_{\lambda\mu}}{\omega_\nu} d(\nu). \]

We note that one may take the second equation in the above lemma as the definition of \( Y_{\lambda\mu} \).

Define \( a := \sum_i d_i^2 \omega_{\rho_i}^{-1} \). If the matrix \( (Y_{\mu\nu}) \) is invertible, by Proposition on P.351 of [36] \( a \) satisfies \( |a|^2 = \sum_\lambda d(\lambda)^2 \).

Definition 2.5. Let \( a = |a| \exp(-2\pi i c_0/8) \) where \( c_0 \in \mathbb{R} \) and \( c_0 \) is well defined mod 8\( \mathbb{Z} \).
Define matrices
\[ S := |a|^{-1}Y, \quad T := C \text{Diag}(\omega_\lambda) \] (3)
where
\[ C := \exp(-2\pi i \frac{c_0}{24}). \]
Then these matrices satisfy (cf. [30]):

**Lemma 2.6.**

\[ SS^\dagger = TT^\dagger = \text{id}, \]
\[ STS = T^{-1}ST^{-1}, \]
\[ S^2 = \hat{C}, \]
\[ T\hat{C} = \hat{C}T, \]

where \( \hat{C}_{\lambda\mu} = \delta_{\lambda\bar{\mu}} \) is the conjugation matrix.

Moreover
\[ N_{\lambda\mu}^\nu = \sum_\delta \frac{S_{\lambda\delta}S_{\mu\delta}S_{\nu\delta}^*}{S_{1\delta}}. \] (4)

is known as Verlinde formula. The commutative algebra generated by \( \lambda \)'s with structure constants \( N_{\lambda\mu}^\nu \) is called fusion algebra of \( \mathcal{A} \). If \( Y \) is invertible, it follows from Lemma 2.6 ([1]) that any nontrivial irreducible representation of the fusion algebra is of the form \( \lambda \to S_{\lambda\mu} \frac{S_{1\mu}}{S_{1\nu}} \) for some \( \mu \).

### 2.3 Induced endomorphisms

Suppose that \( \rho \in \text{End}(M) \) has the property that \( \gamma = \rho \bar{\rho} \in \Delta_A \). By §2.7 of [30], we can find two isometries \( v_1 \in \text{Hom}(\gamma, \gamma^2), w_1 \in \text{Hom}(1, \gamma) \) such that \( \bar{\rho}(M) \) and \( v_1 \) generate \( M \) and

\[ v_1^*w_1 = v_1^*\gamma(w_1) = d_{\rho}^{-1} \]
\[ v_1v_1 = \gamma(v_1)v_1 \]

By Thm. 4.9 of [30], we shall say that \( \rho \) is local if

\[ v_1^*w_1 = v_1^*\gamma(w_1) = d_{\rho}^{-1} \] (5)
\[ v_1v_1 = \gamma(v_1)v_1 \] (6)
\[ \bar{\rho}(\epsilon(\gamma, \gamma))v_1 = v_1 \] (7)

Note that if \( \rho \) is local, then
\[ \omega_{\mu} = 1, \forall \mu < \rho \bar{\rho} \] (8)

\[^2\text{We use } v_1, w_1 \text{ instead of } v, w \text{ here since } v, w \text{ are used to denote sectors in Section 2.4.}\]
For each (not necessarily irreducible) $\lambda \in \Delta_A$, let $\varepsilon(\lambda, \gamma) : \lambda \gamma \to \gamma \lambda$ (resp. $\tilde{\varepsilon}(\lambda, \gamma)$), be the positive (resp. negative) braiding operator as defined in Section 1.4 of [42]. Denote by $\lambda_\varepsilon \in \text{End}(M)$ which is defined by

$$
\lambda_\varepsilon(x) := \text{ad}(\varepsilon(\lambda, \gamma))\lambda(x) = \varepsilon(\lambda, \gamma)\lambda(x)\varepsilon(\lambda, \gamma)^* \quad \forall x \in M.
$$

By (1) of Theorem 3.1 of [42], $\lambda_\varepsilon \rho(M) \subset \rho(M), \lambda_\varepsilon \rho(M) \subset \rho(M)$, hence the following definition makes sense.

**Definition 2.7.** If $\lambda \in \Delta_A$ define two elements of $\text{End}(M)$ by

$$
a^\rho_\lambda(m) := \rho^{-1}(\lambda_\varepsilon \rho(m)), \quad \tilde{a}^\rho_\lambda(m) := \rho^{-1}(\lambda_\tilde{\varepsilon} \rho(m)), \forall m \in M.
$$

$a^\rho_\lambda$ (resp. $\tilde{a}^\rho_\lambda$) will be referred to as positive (resp. negative) induction of $\lambda$ with respect to $\rho$.

**Remark 2.8.** For simplicity we will use $a_\lambda, \tilde{a}_\lambda$ to denote $a^\rho_\lambda, \tilde{a}^\rho_\lambda$ when it is clear that inductions are with respect to the same $\rho$.

The endomorphisms $a_\lambda$ are called braided endomorphisms in [42] due to its braiding properties (cf. (2) of Corollary 3.4 in [42]), and enjoy an interesting set of properties (cf. Section 3 of [42]). Though [42] focus on the local case which was clearly the most interesting case in terms of producing subfactors, as observed in [3], [4], [5], [6] that many of the arguments in [42] can be generalized. These properties are also studied in a slightly different context in [3], [4], [5]. In these papers, the induction is between $M$ and a subfactor $N$ of $M$, while the induction above is on the same algebra. A dictionary between our notations here and these papers has been set up in [45] which simply use an isomorphism between $N$ and $M$. Here one has a choice to use this isomorphism to translate all endomorphisms of $N$ to endomorphims of $M$, or equivalently all endomorphims of $M$ to endomorphisms of $N$. In [45] the later choice is made (Hence $M$ in [45] will be our $N$ below). Here we make the first choice which makes the dictionary slightly simpler. Our dictionary here is equivalent to that of [45]. Set $N = \bar{\rho}(M)$. In the following the notations from [3] will be given a subscript BE. The formulas are:

$$
\rho\upharpoonright N = i_{BE}, \quad \bar{\rho}\rho\upharpoonright N = \bar{i}_{BE}i_{BE}, \quad (9)
$$

$$
\gamma = \bar{\rho}^{-1}\theta_{BE}\bar{\rho}, \quad \bar{\rho}\rho = \gamma_{BE}, \quad (10)
$$

$$
\lambda = \bar{\rho}^{-1}\lambda_{BE}\bar{\rho}, \quad \varepsilon(\lambda, \mu) = \bar{\rho}(\varepsilon^+(\lambda_{BE}, \mu_{BE})) \quad (11)
$$

$$
\tilde{\varepsilon}(\lambda, \mu) = \bar{\rho}(\varepsilon^-(\lambda_{BE}, \mu_{BE})) \quad (12)
$$

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3We have changed the notations $a_\lambda, \tilde{a}_\lambda$ of [42] to $a^\rho_\lambda, \tilde{a}^\rho_\lambda$ of this paper to make some of the formulas such as equation (13) simpler.

4 As we will see in Prop. 2.24, the induction with respect to non-local $\rho$ is closely related to induction with respect to certain local $\rho'$ related to $\rho$. 

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The dictionary between $a_\lambda \in \text{End}(M)$ in definition 2.7 and $\alpha_\lambda$ as in Definition 3.3, Definition 3.5 of [3] are given by:

$$a_\lambda = \alpha_\lambda^+, \tilde{a}_\lambda = \alpha_\lambda^-$$

(13)

The above formulas will be referred to as our dictionary between the notations of [42] and that of [3]. The proof is the same as that of [45]. Using this dictionary one can easily translate results of [42] into the settings of [3], [4], [5], [6], [7], [8] and vice versa.

First we summarize a few properties from [42] which will be used in this paper: (cf. Th. 3.1, Co. 3.2 and Th. 3.3 of [42]):

**Proposition 2.9.** (1). The maps $[\lambda] \rightarrow [a_\lambda], [\lambda] \rightarrow [\tilde{a}_\lambda]$ are ring homomorphisms;

(2) $a_\lambda \rho = \tilde{a}_\lambda \rho = \bar{\rho} \lambda$;

(3) When $\rho$ is local, $l a_\lambda, a_\mu \rangle = l \tilde{a}_\lambda, \tilde{a}_\mu \rangle = l \bar{\rho} \lambda, \bar{\rho} \mu \rangle$;

(4) (3) remains valid if $a_\lambda, a_\mu$ (resp. $\tilde{a}_\lambda, \tilde{a}_\mu$) are replaced by their subsectors.

**Definition 2.10.** $H_\rho$ is a finite dimensional vector space over $\mathbb{C}$ with orthonormal basis consisting of irreducible sectors of $[\lambda \rho], \forall \lambda \in \Delta_A$.

$[\lambda]$ acts linearly on $H_\rho$ by $[\lambda] [a] = \sum b l a, b \rangle [b]$ where $[b]$ are elements in the basis of $H_\rho$. By abuse of notation, we use $[\lambda]$ to denote the corresponding matrix relative to the basis of $H_\rho$. By definition these matrices are normal and commuting, so they can be simultaneously diagonalized. Recall the irreducible representations of the fusion algebra of $A$ are given by

$$\lambda \rightarrow \frac{S_{\lambda \mu}}{S_{1 \mu}}$$

**Definition 2.11.** Assume $l a, b \rangle = \sum_{\mu, i \in (\text{Exp})} \phi^{(\mu, i)}_a \phi^{(\mu, i)*}_b$ where $\phi^{(\mu, i)}_a$ are normalized orthogonal eigenvectors of $[\lambda]$ with eigenvalue $\frac{S_{\mu \lambda}}{S_{1 \mu}}$, $\text{Exp}$ is a set of $\mu, i$’s and $i$ is an index indicating the multiplicity of $\mu$. Recall if a representation is denoted by 1, it will always be the vacuum representation.

The following lemma is elementary:

**Lemma 2.12.** (1):

$$\sum_b d^2_b = \frac{1}{S_{11}^2}$$

where the sum is over the basis of $H_\rho$. The vacuum appears once in $\text{Exp}$ and

$$\phi_a^{(1)}(1) = S_{11} d_a$$

(2)

$$\sum_i \frac{\phi_a^{(\lambda, i)} \phi_a^{(\lambda, i)*}}{S_{1 \lambda}^2} = \sum_{\nu} l \nu, b \rangle \frac{S_{\nu \lambda}}{S_{1 \lambda}}$$

where if $\lambda$ does not appear in $\text{Exp}$ then the righthand side is zero.

---

5By abuse of notation, in this paper we use $\sum_b$ to denote the sum over the basis $[b]$ in $H_\rho$. 

---
Proof Ad (1): By definition we have

\[ a \bar{\rho} = \sum_{\lambda} l(\rho, \lambda)\lambda \]  

where in the second = we have used Frobenius reciprocity. Hence

\[ d_a d_\rho = \sum_{\lambda} l(\rho, \lambda) d_\lambda \]  

and we obtain

\[ \sum_{\lambda} d_\lambda^2 = \sum_{\lambda, a} l(\rho, \lambda) d_\lambda d_a / d_\rho = \sum_{a} d_a^2 \]  

(2) follows from definition and orthogality of \( S \) matrix.  

2.4 Relative braidings

In [42], commutativity among subsectors of \( a_\lambda, \tilde{a}_\mu \) were studied. We record these results in the following for later use:

Lemma 2.13. (1) Let \([b]\) (resp. \([b']\)) be any subsector of \( a_\lambda \) (resp. \( \tilde{a}_\lambda \)). Then

\[ [a_\mu b] = [ba_\mu], [\tilde{a}_\mu b'] = [b'\tilde{a}_\mu], [bb'] = [\tilde{a}_\mu \tilde{a}_\mu] \forall \mu, \nu. \]

(2) Let \([b]\) be a subsector of \( a_\mu \tilde{a}_\lambda \), then \([a_\nu b] = [ba_\nu], [\tilde{a}_\nu b] = [b\tilde{a}_\nu], \forall \nu. \]

Proof (1) follows from (1) of Th. 3.6 and Lemma 3.3 of [42]. (2) follows from the proof of Lemma 3.3 of [42]. Also cf. Lemma 3.20 of [5].

In the proof of these commutativity relations in [42], an implicit use of relative braidings was made. These braidings are further studied in [4], [5] and let us recall their properties in our setting by using our dictionary [9], [13].

Let \( \tilde{a}_\lambda \in \text{End}(M) \) be subsectors of \( \tilde{a}_\lambda \) and \( a_\mu \). By Lemma 3.3 of [42], \([\tilde{a}]\) and \([\delta]\) commute. Denote by \( \epsilon_r(\tilde{a}, \delta) \) given by:

\[ \epsilon_r(\tilde{a}_\lambda, \delta) = s^* a_\delta(t^* \bar{\rho}(\delta_\lambda \sigma_\mu) \tilde{a}_\lambda(s) t) \in \text{Hom}(\delta_\lambda \delta_\mu) \]  

\[ \epsilon_r(\delta, \tilde{a}) = : \epsilon_r(\tilde{a}_\lambda, \delta) \]  

with isometries \( t \in \text{Hom}(\tilde{a}_\lambda, \delta) \) and \( s \in \text{Hom}(\delta_\mu, a_\mu) \). Also

\[ \epsilon_r(\tilde{a}_\lambda, a_\mu) = \bar{\rho}(\delta_\lambda \sigma_\mu), \epsilon_r(a_\lambda, \tilde{a}_\lambda) = \bar{\rho}(\bar{\sigma}_\lambda) \]

Lemma 2.14. The operator \( \epsilon_r(\tilde{a}, \delta) \) defined above does not depend on \( \lambda, \mu \) and the isometries \( s, t \) in the sense that, if there are isometries \( x \in \text{Hom}(\tilde{a}_\lambda, a_\mu) \) and \( y \in \text{Hom}(\delta, a_\delta) \), then

\[ \epsilon_r(\tilde{a}, \delta) = s^* a_\delta(t^* \bar{\rho}(\delta_\lambda \sigma_\mu) \tilde{a}_\lambda(s) t) \]
Lemma 2.15. The system of unitaries of Eq. (14) provides a relative braiding between representative endomorphisms of subsectors of \(a_{\lambda}\) and \(a_{\mu}\) in the sense that, if \(\beta, \delta, \omega, \xi\) are subsectors of \([\tilde{a}_{\lambda}],[a_{\mu}],[\tilde{a}_{\mu}],[a_{\xi}]\), respectively, then we have initial conditions

\[ \epsilon_r(\text{id}_M, \delta) = \epsilon_r(\beta, \text{id}_M) = 1, \]

compositions rules

\[ \epsilon_r(\beta \omega, \delta) = \epsilon_r(\beta, \delta) \beta(\epsilon_r(\omega, \delta)), \epsilon_r(\beta, \delta \xi) = \delta(\epsilon_r(\beta, \xi)) \epsilon_r(\beta, \delta), \]

and naturality

\[ \delta(q_+) \epsilon_r(\beta, \delta) = \epsilon_r(\omega, \delta) q_+ q_-, \epsilon_r(\beta, \delta) = \epsilon_r(\beta, \xi) \beta(q_-) \]

whenever \(q_+ \in \text{Hom}(\beta, \omega)\) and \(q_- \in \text{Hom}(\delta, \xi)\).

For the collection of \(\beta, \delta\) such that \(\beta \prec a_{\lambda}, \beta \prec \tilde{a}_{\lambda}\) and \(\delta \prec a_{\mu}, \delta \prec \tilde{a}_{\mu}\) for some (varying) \(\lambda, \mu \in \Delta_{\alpha}\), the unitaries \(\epsilon_r(\beta, \delta)\) defines a braiding: i.e., they verify YBE and BFE in Prop. 2.1.

Lemma 2.16. Let \(r \in \text{Hom}(\lambda_3, \lambda_1 \lambda_2)\). Then

\[ \bar{\rho}(r) \in \text{Hom}(a_{\lambda_3}, a_{\lambda_1} a_{\lambda_2}) \cap \text{Hom}(\tilde{a}_{\lambda_3}, \tilde{a}_{\lambda_1} \tilde{a}_{\lambda_2}). \]

Proof When \(\rho \bar{\rho}\) is local, the lemma follows from Th. 3.3 of [42]. Let us prove the general case. Since \(a_{\lambda} \bar{\rho} = \bar{\rho} \rho\), we have \(\bar{\rho}(r) \in \text{Hom}(a_{\lambda_3} \bar{\rho}, a_{\lambda_1} \lambda_2 \bar{\rho})\). Since \(M\) is generated by \(\bar{\rho}(M), v_1\), to finish the proof we just need to check that

\[ \bar{\rho}(r) a_{\lambda_3}(v_1) = a_{\lambda_1} \lambda_2(v_1) \bar{\rho}(r) \]

Since \(\rho\) is one to one, applying \(\rho\) to the above equation it is sufficient to check that

\[ \gamma(r) \rho a_{\lambda_3}(v_1) = \rho a_{\lambda_1} \lambda_2(v_1) \gamma(r) \]

Using \(\rho a_{\lambda} = \varepsilon(\lambda, \gamma) \rho a(\lambda, \gamma)\), one can check directly that this equation follows from Prop. 2.1. ■

The following is Lemma 3.25 of [3] in our setting:

Lemma 2.17. If \(r \in \text{Hom}(\bar{\rho} \lambda, \bar{\rho} \mu)\), then

\[ r \bar{\rho}(\varepsilon(\mu_1, \lambda)) = \bar{\rho}(\varepsilon(\mu_1, \lambda)) a_{\mu_1}(r), r \bar{\rho}(\tilde{\varepsilon}(\mu_1, \lambda)) = \bar{\rho}(\tilde{\varepsilon}(\mu_1, \lambda)) \tilde{a}_{\mu_1}(r). \]

Following [7] we define

Definition 2.18. For \(\lambda, \mu \in \Delta_{\alpha}\), \(Z_{\lambda \mu} := \langle \lambda a_{\lambda}, \tilde{a}_{\mu} \rangle\).

We can now translate Th. 5.7 and Th. 6.12 of [7] into our setting:

Proposition 2.19. (1) \(\mu\) appears in \(\text{Exp}\) as defined in Definition 2.11 with multiplicity \(Z_{\mu \mu}\);

(2) \(Z_{\lambda \mu}\) as a matrix commute with \(S, T\) matrices as defined in equation (3).
By Lemma 2.12 and Prop. 2.19 we have the following:

**Lemma 2.20.** If 

\[ \sum_{\nu} l_{\nu a, b} \frac{S_{\nu \lambda}}{S_{1 \lambda}} \neq 0, \]

then \( l_{a_\lambda, \tilde{a}_\lambda} \geq 1 \)

The following follows from Prop. 3.1 of [7]:

**Lemma 2.21.** For any \( \lambda \in \Delta_A, b \in H_\rho \) we have \( \varepsilon(\lambda, b\bar{\rho}) \in \text{Hom}(\lambda b, b\lambda), \tilde{\varepsilon}(\lambda, b\bar{\rho}) \in \text{Hom}(\lambda b, \tilde{b}\lambda) \).

Later we will consider the following analogue of S-matrix using relative braidings. Suppose that \( T_\mu \in \text{Hom}(a_\mu, \tilde{a}_\mu), \forall \mu \in \Delta_A \) (\( T_\mu \) can be zero).

**Definition 2.22.** For \( \mu \in \Delta_A, b \in H_\rho \) irreducible, define

\[ \psi_b^{(T_\mu)} := S_{11} d_\mu d_{\tilde{\mu}} (\varepsilon(b\bar{\rho}, \mu)b(T_\mu)\varepsilon(\mu, b\bar{\rho})). \]

**Lemma 2.23.** (1): \( \psi_b^{(T_\mu)} \) depends only on \([b]\);

(2)

\[ \sum_b \psi_b^{(T_\mu)*}[b] \]

is either zero or an eigenvector of \([\lambda]\) with eigenvalue \( \frac{S_{\lambda \mu}}{S_{1 \mu}} \), and \( \sum_b \psi_b^{(T_\mu)}d_b = 0 \) unless \([\mu] = [1]\);

(3) If \( T_\mu, \tilde{T}_\mu \) are unitaries, and for any irreducible \( \lambda \prec \mu \tilde{\mu}, 1 \prec a_\lambda \) iff \([\lambda] = [1]\),

then \( |\sum_b \psi_b^{(T_\mu)}\psi_b^{(\tilde{T}_\mu)}| = 1 \);

(4) If \( T_\mu \) is unitary then \( |\psi_b^{(T_\mu)}| \leq S_{11} d_\mu d_b \).

**Proof** Ad(1): Suppose that \([b_1] = [b]\) and let \( U \in \text{Hom}(b_1, b) \) be a unitary. We have

\[
\psi_b^{(T_\mu)} = S_{11} d_\mu d_{\tilde{\mu}} (\varepsilon(b\bar{\rho}, \mu)b(T_\mu)\varepsilon(\mu, b\bar{\rho})) \\
= S_{11} d_\mu d_{\tilde{\mu}} (\varepsilon(b\bar{\rho}, \mu)bU^*b(T_\mu)\varepsilon(\mu, b\bar{\rho})(U) \\
= S_{11} d_\mu d_{\tilde{\mu}} (\varepsilon(b\bar{\rho}, \mu)b(T_\mu)U\varepsilon(\mu, b\bar{\rho})) \\
= S_{11} d_\mu d_{\tilde{\mu}} (\varepsilon(b\bar{\rho}, \mu)b_1(T_\mu)\varepsilon(\mu, b\bar{\rho})) \\
= \psi_{b_1}^{(T_\mu)}
\]

Where we have used BFE of Prop. 2.1 in the third =.

Ad (2): Let \( t_{b, i} \in \text{Hom}(b, \bar{b}l') \) be isometries such that \( \sum_i t_{b, i}t_{b, i}^* = 1 \). Then

\[
\sum_b \psi_b^{(T_\mu)}lb, \bar{b}l' = \sum_{b, i} S_{11} d_\mu d_{\tilde{\mu}} d_{\bar{\lambda}} \phi_\mu (\mu(t_{b, i})\varepsilon(b\bar{\rho}, \mu)^*b(T_\mu)\varepsilon(\mu, b\bar{\rho})(t_{b, i}^*))
\]
where we have used equation (1). By Prop. 2.1 we have
\[ \sum_{b,i} S_{11} d_{b,i} \phi_\lambda \phi_\mu (\mu(t_{b,i}) \varepsilon(b \rho, \mu) b(T_\mu) \varepsilon(\mu, b \rho) \mu(t_{b,i})) \]
\[ = S_{11} d_{b,i} \phi_\lambda \phi_\mu (\varepsilon(\lambda b' \rho, \mu) b(T_\mu) \varepsilon(\mu, \lambda b' \rho)) \]
\[ = \frac{S_{\lambda \mu}}{S_{1\mu}} \psi_{b'}^{(T_\mu)} \]
Hence
\[ \sum_b [\lambda] \psi_b^{(T_\mu)} [b] = \sum_{b,b'} \psi_b^{(T_\mu)} [b, \lambda b'] [b'] = \frac{S_{\lambda \mu}}{S_{1\mu}} \sum_{b'} \psi_{b'}^{(T_\mu)} [b']. \]
By (1) of Lemma 2.12 we conclude that \( \sum_b \psi_b^{(T_\mu)} d_b = 0 \) unless \( [\mu] = [1] \).

Ad (3): Let \( t_{\lambda,i} \in \text{Hom}(\lambda, \mu \bar{\mu}) \) be isometries such that \( \sum_{\lambda,i} t_{\lambda,i} t_{\lambda,i}^* = 1 \). Then
\[ \psi_b^{(T_\mu)} \psi_{b'}^{(T_{\bar{\mu}})} = S_{11} d_{b,b'} \phi_\lambda \phi_\mu (\varepsilon(b \rho, \mu) b(T_\mu) \varepsilon(\mu, b \rho)) \]
\[ = S_{11} d_{b,b'} \phi_\lambda \phi_\mu (\varepsilon(b \rho, \mu \bar{\mu}) b(T_\mu a_\mu(T_{\bar{\mu}}) \varepsilon(\mu, b \rho)) \]
\[ = S_{11} d_{b,b'} \sum_{\lambda,i} d_{b,b'} \phi_\lambda (\varepsilon(b \rho, \lambda) b(\tilde{\rho}(t_{\lambda,i})^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i})) \varepsilon(\lambda, b \rho)) \]
Where we have used equation (1) and Lemma 2.12 in the second = and BFE of Prop. 2.1 in the third =. By (2) of Lemma 2.23
\[ \sum_b d_{b,b} d_{b,b'} \phi_\lambda (\varepsilon(b \rho, \lambda) b(\tilde{\rho}(t_{\lambda,i})^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i})) \varepsilon(\lambda, b \rho)) = 0 \]
unless \( [\lambda] = [1] \). Denote by \( t_1 \in \text{Hom}(1, \mu \bar{\mu}) \) the unique (up to scalar) isometry. Then we have (Recall we always identify the center of \( M \) with \( \mathbb{C} \))
\[ \sum_b \psi_b^{(T_\mu)} \psi_{b'}^{(T_{\bar{\mu}})} = \tilde{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_1) \]
On the other hand since \( T_\mu, T_{\bar{\mu}} \) are unitaries, we have
\[ \sum_{\lambda,i} \tilde{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i}) \bar{a}_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i}) \]
\[ = \sum_{\lambda,i} \tilde{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i}) \bar{a}_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i}) \]
Since \( \tilde{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i}) \in \text{Hom}(a_\lambda, 1) \), by assumption it is 0 unless \( [\lambda] = [1] \). We conclude that \( [\tilde{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \tilde{\rho}(t_{\lambda,i})] = 1 \) and (3) is proved. (4) follows since \( \phi_\mu \) is completely positive.

Using equation (12), (13), the following is a translation of Prop. 3.2 and Th. 4.7 of [6] into our setting.
Lemma 2.25. in the proof of Lemma 3.2 of [42]):

Proposition 2.24. Suppose that ∀ where ρ

2.5 Jones-Wassermann subfactors from representation of Loop groups

Let G = SU(n). We denote LG the group of smooth maps f : S¹ → G under pointwise multiplication. The diffeomorphism group of the circle DiffS¹ is naturally a subgroup of Aut(LG) with the action given by reparametrization. In particular the group of rotations RotS¹ ≃ U(1) acts on LG. We will be interested in the projective unitary representation π : LG → U(H) that are both irreducible and have positive energy. This means that π should extend to LG × Rot S¹ so that H = ∐ₙ≥₀ H(n), where the H(n) are the eigenspace for the action of RotS¹, i.e., rθξ = exp(i nθ) for θ ∈ H(n) and dim H(n) < ∞ with H(0) ≠ 0. It follows from [35] that for fixed level k which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

\[ P^k_++ = \left\{ \lambda \in P \mid \lambda = \sum_{i=1}^{n-1} \lambda_i \Lambda_i, \lambda_i \geq 0, \sum_{i=1}^{n-1} \lambda_i \leq k \right\} \]

where P is the weight lattice of SU(n) and Λᵢ are the fundamental weights. We will write λ = (λ₁, ..., λₙ₋₁), λ₀ = k - ∑₁≤i≤n⁻¹ λᵢ and refer to λ₀, ..., λₙ₋₁ as components of λ.

We will use Λ₀ or simply 1 to denote the trivial representation of SU(n). For λ, μ, ν ∈ P^k_+, define N^ν_μ = ∑_δ∈P^k_+ S^{(δ)}_λ S^{(δ)}_μ / S^{(δ)}_λ, where S^{(δ)}_λ is given by the Kac-Peterson formula (cf. equation (17) below for an equivalent formula):

\[ S^{(δ)}_\lambda = c \sum_{w \in S_n} \varepsilon_w \exp(i w(δ) \cdot \lambda 2\pi/n) \]

where ε_w = det(w) and c is a normalization constant fixed by the requirement that S^{(δ)}_μ is an orthonormal system. It is shown in [24] P. 288 that N^ν_μ are non-negative
integers. Moreover, define $Gr(C_k)$ to be the ring whose basis are elements of $P_{++}^k$ with structure constants $N_{\lambda \mu}^\nu$. The natural involution $*$ on $P_{++}^k$ is defined by $\lambda \mapsto \lambda^*$, the conjugate of $\lambda$ as representation of $SU(n)$.

We shall also denote $S_{\lambda \mu}^{(L)}$ by $S_{\lambda \mu}^{(L)}$. Define $d_\lambda = \frac{S_{\lambda \mu}^{(L)}}{S_{\lambda \mu}^{(S)}}$. We shall call $(S_{\lambda \mu}^{(L)})$ the $S$-matrix of $LSU(n)$ at level $k$.

We shall encounter the $\mathbb{Z}_n$ group of automorphisms of this set of weights, generated by

$$\sigma : \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) \rightarrow \sigma(\lambda) = (k - 1 - \lambda_1 - \cdots - \lambda_{n-1}, \lambda_1, \cdots, \lambda_{n-2}).$$

Define $\text{col}(\lambda) = \sum_i (\lambda_i - 1)i$. The central element $\exp \frac{2\pi i}{n}$ of $SU(n)$ acts on representation of $SU(n)$ labeled by $\lambda$ as $\exp(\frac{2\pi i \text{col}(\lambda)}{n})$. The irreducible positive energy representations of $LSU(n)$ at level $k$ give rise to an irreducible conformal net $A$ (cf. [25]) and its covariant representations. We will use $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ to denote irreducible representations of $A$ and also the corresponding endomorphism of $M = A(\mathcal{I})$.

All the sectors $[\lambda]$ with $\lambda$ irreducible generate the fusion ring of $A$.

For $\lambda$ irreducible, the univalence $\omega_\lambda$ is given by an explicit formula (cf. 9.4 of [PS]). Let us first define $h_\lambda = \frac{c_2(\lambda)}{k+n}$ where $c_2(\lambda)$ is the value of Casimir operator on representation of $SU(n)$ labeled by dominant weight $\lambda$. $h_\lambda$ is usually called the conformal dimension. Then we have: $\omega_\lambda = \exp(2\pi ih_\lambda)$. The conformal dimension of $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ is given by

$$h_\lambda = \frac{1}{2n(k+n)} \sum_{1 \leq i \leq n-1} i(n-i)\lambda_i^2 + \frac{1}{n(k+n)} \sum_{1 \leq j \leq i \leq n-1} j(n-i)\lambda_j\lambda_i + \frac{1}{2(k+n)} \sum_{1 \leq j \leq n-1} j(n-j)\lambda_j$$

The following form of Kac-Peterson formula for $S$ matrix will be used later:

$$S_{\lambda \mu} = C h_\lambda(x_1, \ldots, x_{n-1}, 1)$$

Where $ch_\lambda$ is the character associated with finite irreducible representation of $SU(n)$ labeled by $\lambda$, and $x_i = \exp(-2\pi i \frac{\mu_i}{k+n}), \mu_i = \sum_{i \leq j \leq n-1} (\mu_j + 1), 1 \leq i \leq n-1$.

It follows that $S$ matrix verifies:

$$S_{\lambda \omega(\mu)} = \exp\left(\frac{2\pi i \text{col}(\lambda)}{n}\right)S_{\lambda \mu}$$

The following result is proved in [40] (See Corollary 1 of Chapter V in [40]).

**Theorem 2.26.** Each $\lambda \in P_{++}^{(k)}$ has finite index with index value $d_\lambda^2$. The fusion ring generated by all $\lambda \in P_{++}^{(k)}$ is isomorphic to $Gr(C_k)$.

**Remark 2.27.** The subfactors in the above theorem are called Jones-Wassermann subfactors after the authors who first studied them (cf. [23], [40]).

**Definition 2.28.** $v := (1, 0, \ldots, 0), v_0 := (1, 0, \ldots, 0, 1), \omega^i = k\Lambda_i, 0 \leq i \leq n-1$.
The following is observed in [16]:

**Lemma 2.29.** Let \((0, ..., 0, 1, 0, ... 0)\) be the \(i\)-th \((1 \leq i \leq n - 1)\) fundamental weight. Then \([\{0, ..., 0, 1, 0, ... 0\} \lambda] \) are determined as follows: \(\mu \prec (0, ..., 0, 1, 0, ... 0)\lambda\) iff when the Young diagram of \(\mu\) can be obtained from Young diagram of \(\lambda\) by adding \(i\) boxes on \(i\) different rows of \(\lambda\), and such \(\mu\) appears in \([\{0, ..., 0, 1, 0, ... 0\} \lambda]\) only once.

**Lemma 2.30.** (1) If \([\lambda] \neq \omega^i\) for some \(0 \leq i \leq n - 1\), then \(v_0 < \lambda \bar{\lambda}\); 
(2) If \(\lambda_1 \lambda_2\) is irreducible, then either \(\lambda_1\) or \(\lambda_2 = \omega^i\) for some \(0 \leq i \leq n - 1\).

**Proof** By Lemma 2.29 we have that
\[
lv \lambda, v \lambda \rangle = 1
\]
iff \(\lambda = \omega^i\) for some \(0 \leq i \leq n - 1\). By Frobenius reciprocity
\[
lv \lambda, v \lambda \rangle = l(1 + v_0, \lambda \bar{\lambda}) = 1 + lv_0, \lambda \bar{\lambda})
\]
Hence
\[
lv_0, \lambda \bar{\lambda}) = 0
\]
iff \(\lambda = \omega^i\) for some \(0 \leq i \leq n - 1\). If \(\lambda_1 \lambda_2\) is irreducible, then by Frobenius reciprocity again we have
\[
lv_0, \lambda \bar{\lambda}) = 1 + lv_0, \lambda_1 \bar{\lambda_1})lv_0, \lambda_2 \bar{\lambda_2})
\]
Hence either
\[
lv_0, \lambda_1 \bar{\lambda_1}) = 0
\]
or
\[
lv_0, \lambda_2 \bar{\lambda_2}) = 0
\]
and the lemma follows.  

**Lemma 2.31.** Suppose \(\lambda \in \Delta_{\mathcal{A}}\) and \(\lambda\) is not necessarily irreducible. Then
\[
\varepsilon(\lambda, v)\varepsilon(v, \lambda) \in \mathbb{C}
\]
iff \([\lambda] = \sum_j [\omega^j]\) where the summation is over a finite set.

**Proof** By Prop. 2.21 we have that
\[
\varepsilon(v^m, \lambda)\varepsilon(\lambda, v^m) \in \mathbb{C}
\]
for all \(m \geq 0\). Since any irreducible \(\mu\) is a subsector of \(v^m\) for some \(m \geq 0\), by Lemma 2.3 we have that \(\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}, \forall \mu, \lambda_1 \prec \lambda\). By definition of \(S\) matrix we have \(|S_{\mu, \lambda_1}|^2 = |S_{\lambda_1, \mu}|^2\). Sum over \(\mu\) we have \(dv_{\lambda_1} = 1\), i.e., \(\lambda_1\) is an automorphism, and this implies that \(v \lambda_1\) is irreducible. The lemma now follows from Lemma 2.30.

**Lemma 2.32.** For any \(m \geq 1\), \(\text{Hom}(v^m, v^m)\) is generated as an algebra by \(1, v^i(\varepsilon(v, v)), 1 \leq i \leq m - 1\).
This is (3) of Lemma 3.1.1 in [45] and is essentially contained in [41]. □

Now let $\rho \bar{\rho} \in \Delta A$ where $A$ is the conformal net associated with $SU(n)$ at level $k$, and consider induction with respect to $\rho$ as defined in Definition 2.7. We have

**Lemma 2.33.** (1) $a\nu, \tilde{a}\nu$ are always irreducible;
(2) $d\nu_0 = 1$ iff $k = n = 2$;
(3) If $k \neq n \pm 2$, then $a\nu_0, \tilde{a}\nu_0$ are irreducible.

**Proof** It is enough to prove the Lemma for positive induction. The negative induction case is similar. Assume that $\rho = \rho' \rho''$ as in Prop. 2.24, since $la_{\lambda,1} = l\rho' \overline{\rho}', \lambda) = la_{\lambda',1}', \forall \lambda$, it is enough to prove the Lemma for induction with respect to $\rho'$. Hence we may assume that $\rho$ is local. By (3) of Prop. 2.9 we have

$$la_{\nu},a_{\nu}) = l\rho \bar{\rho},v\bar{v}) = 1 + l\rho \bar{\rho},v_0)$$

Since $\omega_\nu_0 = \exp(\frac{2\pi i n}{k+n}) \neq 1$, by equation (8) we conclude that $l\rho \bar{\rho},v_0) = 0$ and (1) is proved. (2) follows from equation (17).

Ad (3) By Lemma 2.29 we have

$$[v_0^2] = [1]+2[v_0]+[(2,0,...,0,2)]+[(0,1,0,...,1,0)]+[(0,1,0,...,0,2)]+[(2,0,...,0,1,0)]$$

By computing the conformal dimensions of the descendants of $v_0^2$ using equation (16) we have

$$h_{(2,0,...,0,2)} = \frac{2+2n}{k+n}, h_{(0,1,...,0,2)} = \frac{2n}{k+n}, h_{(0,1,...,1,0)} = \frac{2n-2}{k+n}$$

By equation (8) we conclude that if $k \neq n \pm 2$, then $lv_0^2, \rho \bar{\rho}) = 1$ and (3) is proved.

### 2.6 Induced subfactors from simple current extensions

In this section we assume that the level $k = n'n$ where $n' \geq 3$, and $n'$ is an even integer if $n$ is even. This last condition comes from [44]. For such level it is shown in §3 of [4] that there is $\rho_o \in \text{End}(M)$ such that $[\rho_o \bar{\rho}_o] = \sum_{0 \leq i < n-1} [\omega^i]$ and $\rho_o \bar{\rho}_o$ is local. It also follows from definitions that one can choose $\bar{\rho}_o \rho_o = \sum_{0 \leq i \leq n-1} [g^i]$ where $[g^n] = [1]$ and $[\tilde{a}_o] = [a_o,g]$ (cf. §6.1 of [25]). Also note that $[a_{\omega^i}] = [1], \forall i$. The following is a consequence of Lemma 2.12 and Prop. 2.9

**Lemma 2.34.** There exists an orthonormal basis $\sum_a \phi_a[\mu]$ where $\text{col}(\mu) = 0 \mod n$ and the sum is over all irreducible subsectors of $a_l, \forall \lambda$. The following is a consequence of Lemma 2.12 and Prop. 2.9

$$la_{\lambda,a,b}) = \sum_{\mu,i,\text{col}(\mu) = 0 \mod n} \frac{S_{\lambda\mu}}{S_{1\mu}} \phi_\mu^{a[\mu,i]} \phi_b^{(\mu,i)*}$$

The following follows from Cor. 4.9 of [25]:

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Lemma 2.35. (1) Let $\lambda$ be irreducible and suppose $l$ is the smallest positive integer with $[\omega^l\lambda] = [\lambda]$. Then $[a_\lambda] = \sum_{1 \leq i \leq l} [x_i]$ where $l' = n$ and $[g^i x_1 g^{-i}] = [x_i], 1 \leq i \leq l'$, $[x_i] \neq [x_j]$ if $i \neq j$. Similar statements hold true for $\tilde{a}_\lambda$.

(2) $la_\lambda, a_\mu \neq 0$ iff $[\lambda] = [\omega^j(\mu)]$ for some $1 \leq j \leq n$ iff $[a_\lambda] = [a_\mu]$. Similar statements hold true for $\tilde{a}_\lambda, \tilde{a}_\mu$.

Later we will use the following analogue of Lemma 2.31:

Lemma 2.36. If $\varepsilon(v_0, \lambda) \varepsilon(\lambda, v_0) \in \mathbb{C}$, then $[\lambda] = \sum_j \omega^j$ where the sum is over a finite set of positive integers.

Proof By Prop. 2.1 and Lemma 2.3 we have that $\varepsilon(v_0^m, \lambda_1) \varepsilon(\lambda_1, v_0^m) \in \mathbb{C}$ for all $m \geq 0, \lambda_1 < \lambda$. By Lemma 2.3 again we have $\varepsilon(\mu, \lambda_1) \varepsilon(\lambda_1, \mu) \in \mathbb{C}$ for all $\mu < v_0^m, \lambda_1 < \lambda$. Since by Lemma 2.29 any $\mu$ with $\text{col}(\mu) = 0 \mod n$ is a subsector of $v_0^m$ for some $m \geq 0$, we conclude that $\varepsilon(\mu, \lambda_1) \varepsilon(\lambda_1, \mu) \in \mathbb{C}$ for all $\mu, \text{col}(\mu) = 0 \mod n, \lambda_1 < \lambda$. By the definition of $S$ matrix we have

$$|S_{\mu \lambda_1}| = d_{\lambda_1} |S_{\mu 1}|, \forall \mu, \text{col}(\mu) = 0 \mod n$$

Set $[a] = [b] = [1]$ in Lemma 2.34 we have

$$la_\lambda, a_\lambda \rangle = \sum_{\mu, \text{col}(\mu) = 0 \mod n} d_{\lambda_1}^2 \phi_1^{(\mu, \lambda_1)} \phi_1^{(\mu, \lambda_1)*} = d_{\lambda_1}^2$$

By Lemma 2.35 we have

$$d_{\lambda_1} \geq la_\lambda, a_\lambda \rangle$$

and we conclude that $d_{\lambda_1} = 1$, and in particular $v\lambda_1$ is irreducible. The lemma now follows from Lemma 2.30.

The subfactors $a_\lambda(M) \subset M$ are type III analogue of “orbifold subfactors” studied in [10] and [44].

Lemma 2.37. If $x < a_\lambda, \lambda$ irreducible and $d_x = 1$, then $[\lambda] = [\omega^i], 1 \leq i \leq n$ and $[x] = [1]$.

Proof If $[\lambda] \neq [\omega^i], \forall i$, then by Lemma 2.30 $\lambda \lambda \succ v_0$, and by Lemma 2.33 we have $a_\lambda a_\lambda \succ a_{v_0}$. Since $x < a_\lambda, d_x = 1$, by Lemma 2.35 we conclude that $d_{a_{v_0}} = d_{v_0} = 1$. This is impossible by Lemma 2.33 and our assumption $k = n'n, n' \geq 3$.

Let $(n', n', ..., n')$ be the unique fixed representation under the action of $\mathbb{Z}_n$. By Lemma 2.35

$$[a_{(n', n', ..., n')} ] = \sum_{1 \leq i \leq n} [b_i], [g^i b_1 g^{-i}] = [b_{i+1}], 0 \leq i \leq n - 1$$

Definition 2.38. Denote by $u := (n' + 1, n', n', ..., n')$. 

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Note that by Lemma 2.35 $a_u$ is irreducible.

**Lemma 2.39.** (1) 
$S_{uv} \neq 0$;

(2) Let $\Lambda = (n, 0, \ldots, 0)$. Then $la_{\Lambda}, \tilde{a}_{\Lambda} = 0$, and $S_{u\Lambda} \neq 0$.

**Proof** Ad (1) Since $n[a_u] = [a_v b_i]$, by Lemma 2.34

$$\frac{S_{uv}}{S_{1v}} = \frac{S_{uv} S(n', \ldots, n'v)}{n S_{1v} S_{1v}}$$

Direct computation using equation (17) shows that $S_{vv} S_{1v} \neq 0$. Note that by equation (18) $S(v', \ldots, n'v) S_{1v} = 0$ since $col(v) = 1$, hence

$$S_{1v} = -1$$

and this implies that $S_{(n', \ldots, n'v)} \neq 0$ and (1) is proved.

Ad (2) Since $k = n'n \geq 3n$, it follows that $l \omega^j \Lambda, \tilde{\Lambda} = 0$, $\forall 1 \leq j \leq n$. By Lemma 2.35 $la_{\Lambda}, \tilde{a}_{\Lambda} = 0$. Since $[a_v a_{(n', \ldots, n')}] = n[a_u]$, by Lemma 2.34 we have

$$n S_{u\Lambda} S_{1\Lambda} = S_{v\Lambda} S_{(n', \ldots, n')\Lambda}$$

Hence to finish the proof we just have to check that $S_{v\Lambda} \neq 0, S_{(n', \ldots, n')\Lambda} \neq 0$. Since $Ch_{v}(x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} x_i$, by equation (17) up to a nonzero constant $S_{v\Lambda}$ is equal to

$$\exp(-2\pi i (2n - 1)/(k + n)) + \sum_{0 \leq j \leq n - 2} \exp(-2\pi i j/(k + n))$$

This sum is equal to 0 iff $n = k = 2$. Note that $Ch_{\Lambda}(x_1, \ldots, x_n)$ is a complete symmetric polynomial of degree $n$. $S_{v\Lambda} \neq 0$ now follows directly from equation (17) (cf. (2.7a) of [13] for more general statement).

The main theorem of this section is:

**Theorem 2.40.** The lattice of intermediate subfactors of $a_u(M) \subset M$ is $M_{2n}$.

The proof will be given in section 4. Let us first show that the subfactor in Theorem 2.40 contains $2n$ incomparable intermediate subfactors. By fusion rule with $v$ in Lemma 2.29 we have

$$[a_u] = [a_v b_i] = [b_i a_v], \forall 1 \leq i \leq n.$$ 

Therefore we can assume that

$$a_u = U_i a_v b_i U_i^* = V_i b_i a_v V_i^*, 1 \leq i \leq n$$

where $U_i, V_i$ are unitaries.
Proposition 2.41. (1): As von Neumann algebras

\[ U_i a_v(M) U_i^* = U_j a_v(M) U_j^*, \quad V_i b_i(M) V_i^* = V_j b_j(M) V_j^* \]

iff \( i = j \);

(2) \( U_i a_v(M) U_i^* \) is not an intermediate subfactor in \( V_j b_j(M) V_j^* \subset M \);

(3) \( V_j b_j(M) V_j^* \) is not an intermediate subfactor in \( U_i a_v(M) U_i^* \subset M \).

Proof. Ad(1): If \( U_i a_v(M) U_i^* = U_j a_v(M) U_j^* \), then \( U_i a_v(m) U_i^* = U_j a_v(\theta(m)) U_j^* \), \( \forall m \in M \) where \( \theta \) is an automorphism of \( M \). By Frobenius reciprocity we have \( [\theta] \prec [a_v a_v] \).

By Lemma 2.37 we conclude that \( [\theta] = [1] \) and hence

\[ U_i a_v(m) U_i^* = U_j a_v(U) a_v(m) a_v(U)^* U_j^*, \forall m \in M \]

for some unitary \( U \in M \). Hence

\[ \text{Ad}_a v b_i = \text{Ad}_{U_j a_v(U)} a_v b_i = \text{Ad}_{U_j a_v} b_j \]

and we conclude that \([b_i] = [b_j]\), hence \( i = j \). The second statement in (1) is proved similarly.

Ad (2): If \( U_i a_v(M) U_i^* \) is an intermediate subfactor in \( V_j b_j(M) V_j^* \subset M \), then \( \text{Ad}_{V_j} b_j = \text{Ad}_{U_i} a_v C \) for some \( C \in \text{End}(M) \), and it follows that \([b_j b_j] \supset [a_v a_v] \supset [a_v a_v] \).

Hence

\[ L a_v b_j, a_v b_j \] = \( l b_j b_j, a_v a_v \) \( \geq 2 \)

contradicting the irreducibility of \([a_v a_v] = [a_v b_j]\).

Ad (3): If \( V_j b_j(M) V_j^* \) is an intermediate subfactor in \( U_i a_v(M) U_i^* \subset M \), then there is \( C' \in \text{End}(M) \) such that \([b_j C'] = [a_v] \). Since \([a_v a_v] = [1] + [a_v a_v] \) and \( a_v a_v \) is irreducible by Lemma 2.33, we must have \([b_j b_j] = [a_v a_v] \) and therefore \( d_{C'} = 1 \). By Frobenius reciprocity \( C' \prec [b_j a_v] \), but \([b_j a_v] \) is irreducible since \( a_v \) is irreducible, a contradiction.

Here we give a quick proof of Th. 2.40 for \( n = 2 \) and \( k \neq 10, 28 \) to illustrate some ideas behind the proof. Suppose that \( M_1 \) is an intermediate subfactor of \( a_v(M) \subset M \).

Since all factors in this paper are isomorphic to hyperfinite type \( III_1 \) factor, we can find \( c_1, c_2 \in \text{End}(M) \) such that \( a_v = c_1 c_2 \) and \( c_1(M) = M_1 \). Let \( \rho = c_1 c_1 \), and enumerate the basis of \( H_\rho \) by irreducible sectors \( a \). Note that \( a \) must be of the form \( \rho c \) with \( c \) irreducible, and so \( d_a \geq d_{\rho c} = \sqrt{2} \).

Consider the fusion graph associated with the action of \( \nu \) on \( H_\rho \): the vertices of this graph are irreducible sectors \( a \), and vertices \( a \) and \( b \) are connected by \( l(a, b) \) edges. By Lemma 2.12 this graph is connected and has norm \( 2 \cos(H_{k+2}) \), and hence it must be \( A - D - E \) graph (cf. Chap. 1 of [17]). Since \( k \neq 10, 28 \) it must be \( A \) or \( D \) graph. By Lemma 2.12 we have \( \sum_a d_a^2 = \frac{1}{s_{11}} = \frac{1}{\sin^2(\frac{\pi}{k + 2})} \). Since \( d_a \geq d_{\rho c} = \sqrt{2} \), are the entries of Perron-Frobenius eigenvector for the graph (Such eigenvector is unique up to a positive scalar), compare with the eigenvectors of A-D-E graphs listed for example in Chap. 1 of [17]) we conclude that the graph is \( D \) graph and there is a sector \( c \).
with \( d_\nu = 1 \) and \( c_1 < a_\mu c \) for some \( \mu \in \Delta \). We conclude that either \([c_1] = [a_\mu c]\), or \([c_1] = [b_i c], 1 \leq i \leq 2\). In the former case \([c_2] = [c^{-1} a_\lambda]\) or \([c_2] = [c^{-1} b_j]\), \(1 \leq j \leq 2\). But if \([c_2] = [c^{-1} a_\lambda]\) then \([a_u] = [a_\mu a_\lambda]\) is irreducible, and by Lemma 2.40 \([a_\mu] = [a_u]\) or \([a_\mu] = [1]\), which implies that \(M_1\) is either \(a_u(M)\) or \(M\). If \([c_2] = [c^{-1} b_j], 1 \leq j \leq 2\), then \([a_u] = [a_u b_j]\) and by computing the index and note that the colors of \(u\) and \(b_j\) are \(1 \mod 2, 0 \mod 2\) respectively we have \(a_\mu = a_u\), and we conclude that \(M_1\) must be one of the intermediate subfactors given in Prop. 2.41. The case of \([c_1] = [b_i c], 1 \leq i \leq 2\) is treated similarly. By Prop. 2.41 we have proved Th. 2.40 for \(n = 2, k \neq 10, 28\). The same idea as presented above can be used to give a complete list of all intermediate subfactors of Goodman-Harpe-Jones subfactors. We hope to discuss this and related problems elsewhere.

### 3 Centrality of a class of intertwiners and its consequences

We preserve the setup of section 2.5.

Assume that \(\rho \bar{\rho} \in \Delta_A\). We will investigate a class of inductions which are motivated by finding a proof of Th. 2.40.

In this section we assume that \([a_v] = [h \tilde{a}_v], [h^n] = [1]\), \(a_v\) irreducible, and if \(\mu < v_0^2, 1 < a_\mu\), then \([\mu] = [1]\).

Choose a unitary \(T \in \text{Hom}(a_v, h \tilde{a}_v)\). Such \(T\) is unique up to scalar since \(a_v\) is irreducible. By Lemma 2.13 we have \([h \tilde{a}_v] = [\tilde{a}_v, h]\). Choose a unitary \(T_1 \in \text{Hom}(a_v, h \tilde{a}_v)\). Note that \(T_1\) is unique up to scalar since \(h \tilde{a}_v\) is irreducible.

**Definition 3.1.** Denote by \(U_n := T a_v(T) a_v^2(T) ... a_v^{n-1}(T) \in \text{Hom}(a_v^n, (h \tilde{a}_v)^n)\). Denote by \(T_i := T_i \tilde{a}_v(T_1) ... \tilde{a}_v^{i-1}(T_1) \in \text{Hom}(\tilde{a}_v^i h, h \tilde{a}_v), 1 \leq i \leq n - 1\).

Choose \(T' \in \text{Hom}(h^n, 1)\) (\(T'\) is unique up to scalar).

**Definition 3.2.** Set \(w = v^n\) and define \(u_w := T' h^{n-1}(T_{n-1}) h^{n-2}(T_{n-2}) ... h(T_1) U_n \in \text{Hom}(a_v^n, \tilde{a}_v^n)\).

For example when \(n = 3\), \(u_w = T' h^2(T_1) h^2(\tilde{a}_v(T_1)) h(T_1) T a_v(T) a_v^2(T)\). The reader is encouraged to give a diagrammatic representation of \(u_w\) as in [42].

**Lemma 3.3.** Suppose that \(x, y\) are sectors such that

\[
[x] = \sum_{1 \leq i \leq m} [x_i], [y] = \sum_{1 \leq i \leq m} [y_i], d_{x_i} < d_{x_j}, d_{y_i} < d_{y_j}
\]

if \(i < j\), and \(x_i, y_i\) are irreducible. Let \(T_{x,i} \in \text{Hom}(x_i, x), T_{y,i} \in \text{Hom}(y_i, y), i = 1, ..., m\) be isometries.

If \(U \in \text{Hom}(x, y)\) is unitary then \(U T_{x,i} T_{x,i}^* U^* = T_{y,i} T_{y,i}^*, i = 1, ..., m\).
Proof By assumption $\text{Hom}(x, x), \text{Hom}(y, y)$ are finite dimensional abelian algebras, and so for each $1 \leq i \leq m$ we have $U_{x,i}T_{x,i}^*U_* = T_{y,j}T_{y,j}^*$ for some $j$.

By equation (1) we have

$$d_y\phi_y(U_{x,i}T_{x,i}^*U_*) = d_x\phi_x(T_{x,i}^*T_{x,i}^*) = d_{x_i}$$

Hence $d_{x_i} = d_{y_j}$. By assumption it follows that $i = j, 1 \leq i \leq m$. ■

Lemma 3.4. Let $U \in \text{Hom}(a^2_i h^j, h^l a^2_v), i, j \geq 0$ be a unitary. Then $h^l(\bar{\rho}(\varepsilon(v,v)))U = U\bar{\rho}(\varepsilon(v,v))$.

Proof Since $a_{v_0}$ is irreducible, we have $la_v h_v, a_v h_v) = 2$. We note that $[a_v h_v] = [a_{(2,0,...,0)}] + [a_{(0,1,0,...,0)}]$ and $\frac{d_{a_{(2,0,...,0)}}}{d_{a_{(0,1,0,...,0)}}} = \frac{\sin(\frac{(a+1)\pi}{2})}{\sin(\frac{(a-1)\pi}{2})} > 1$ and so the assumption of Lemma 3.3 is verified. Denote by $P_1, P_2 \in \text{Hom}(v^2, v^2)$ the two different minimal projections corresponding to $(2, 0, ..., 0), (0, 1, ..., 0)$ respectively. Note that $\bar{\rho}(P_1), h^l(\bar{\rho}(P_1)), l = 1, 2$ are minimal projections in $\text{Hom}(a^2_i h^j, a^2_v h^l)$, $\text{Hom}(h^l a^2_v, h^l a^2_v)$ respectively and by Lemma 3.3 we have $U^* h^l(\bar{\rho}(P_1))U = \bar{\rho}(P_1), l = 1, 2$.

Assume that $\varepsilon(v, v) = z_1 P_1 + z_2 P_2$ where $z_1, z_2 \in \mathbb{C}$ (cf. Lemma 3.1.1 [45] for explicit formulas for $z_1, z_2$). Then $h^l(\bar{\rho}(\varepsilon(v,v))) = z_1 h^l(\bar{\rho}(P_1)) + z_2 h^l(\bar{\rho}(P_2))$ and the lemma follows. ■

Lemma 3.5. $\bar{a}_v^i(\bar{\rho}(\varepsilon(v,v)))u_w = u_w a_v^i(\bar{\rho}(\varepsilon(v,v))), 0 \leq i \leq n - 2$.

Proof By Def. 3.2 we can write $u_w = V'_3 V'_2 V'_3$ where

$$V'_3 = a_v^{i+2}(V_3), V'_3 = h^{n-i-3}(T_{n-i-3})h^2(T_2)h(T_1) \in \text{Hom}(a_v^{n-i-2}, h^{n-i-2}a_v^{n-i-2})$$

$$V'_2 = a_v^{(V_2)}, V'_2 = h^{n-i-1}(T_2)h^2(T_2)h(T_1)T_{a_v}(T) \in \text{Hom}(a_v^{n-i-2}, h^{n-i-2}a_v^{n-i-2})$$

and $V'_3 = T^i h^{n-i-1}(T_2)h^2(T_2)h^{i-1}(T_{n-i-2})h(T_1)T_{a_v}(T) \in \text{Hom}(a_v^{i+2}, a_v^{i+2})$.

Although the complicated but explicit formulas of $V'_3, V'_2, V'_3$ are given above, we only use their intertwining properties in what follows.

Hence

$$\bar{a}_v^i(\bar{\rho}(\varepsilon(v,v)))u_w = \bar{a}_v^i(\bar{\rho}(\varepsilon(v,v)))v'_3 a_v^i(V_2)\bar{a}_v^{i+2}(V_3)$$

$$= V'_3 a_v^i(h^{n-i}(\bar{\rho}(\varepsilon(v,v)))V_2)\bar{a}_v^{i+2}(V_3)$$

$$= V'_3 a_v^i(V_2)\bar{\rho}(\varepsilon(v,v))\bar{a}_v^{i+2}(V_3)$$

$$= V'_3 a_v^i(V_2)\bar{a}_v^i(\bar{\rho}(\varepsilon(v,v)))a_v^2(V_3))$$

$$= V'_3 a_v^i(V_2)\bar{a}_v^{i+2}(V_3)\bar{a}_v^i(\bar{\rho}(\varepsilon(v,v)))$$

$$= u_w a_v^i(\bar{\rho}(\varepsilon(v,v)))$$

where in the third = we have used Lemma 3.3. ■

Lemma 3.6. $\bar{a}_v^{n-1}(\bar{\rho}(\varepsilon(v,v)))u_w a_w(u_w) = u_w a_w(u_w)\bar{a}_v^{n-1}(\bar{\rho}(\varepsilon(v,v)))$. 

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Definition 3.7. For each integer \( m \geq 1 \), \( u_w a_w (u_w) \) is a set of isometries such that

\[
\sum_{\mu, i} t_{\mu, i} = 1.
\]

Theorem 3.8. Let \( m \geq 1 \) be any integer and \( R \in \text{Hom}(w^m, w^m) \). Then

\[
\bar{\rho}(R) u_w^m = u_w^m \bar{\rho}(R).
\]

Proof. By Def. 3.2 we can write \( u_w a_w (u_w) = W'_1 W'_2 W'_3 \) where

\[
\begin{align*}
W'_1 & = a_w^{m-1}(W_1) \in \text{Hom}(a_w^{m-1}, a_w^{m-1}) \\
W'_2 & = T(h)^{n-2}(T_n h(T_1) T_a v(T) \cdots a_{n-2}^{n-2}(T) \in \text{Hom}(a_w^{n-1}, a_w^{n-1})
\end{align*}
\]

and

\[
W'_3 = T(h)^{n-2}(T_n h(T_1) T_a v(T) \cdots a_{n-2}^{n-2}(T) \in \text{Hom}(a_w^{n-1}, a_w^{n-1})
\]

As in the proof of Lemma 3.5, even though explicit formulas of \( W_2, W_3, W'_1 \) are given above, what we need in the following is their intertwining properties.

Hence

\[
\begin{align*}
\bar{\rho}(\bar{\varepsilon}(v, v)) u_w a_w (u_w) & = a_w^{n-1}(\bar{\rho}(\bar{\varepsilon}(v, v))) W'_1 a_w^{n-1}(W_2) a_w^{n-1}(W_3) \\
& = W'_1 a_w^{n-1}(h(\bar{\rho}(\bar{\varepsilon}(v, v)))) W_2 a_w^{n-1}(W_3) \\
& = W'_1 a_w^{n-1}(W_2) a_w^{n-1}(\bar{\rho}(\bar{\varepsilon}(v, v))) W_3 a_w^{n-1}(W_1) \\
& = u_w a_w (u_w) a_w^{n-1}(\bar{\rho}(\bar{\varepsilon}(v, v)))
\end{align*}
\]

where in the third \( = \) we have used Lemma 3.4.

Lemma 3.9. Suppose that \( \mu \prec w^m \) are irreducible and Let \( t_{\mu, i} \in \text{Hom}(\mu, w^m) \), \( m \geq 1 \) be a set of isometries such that

\[
\sum_{\mu, i} t_{\mu, i} t_{\mu, i}^* = 1.
\]

(1) For each fixed \( \mu, \hat{\rho}(t_{\mu, i}) u_w^m \hat{\rho}(t_{\mu, i}) \in \text{Hom}(a_{\mu}, \bar{a}_{\mu}) \) is independent of choices of \( t_{\mu, i} \).

(2) \( \hat{\rho}(t_{\mu, i}) u_w^m \hat{\rho}(t_{\mu, i}) \in \text{Hom}(a_{\mu}, \bar{a}_{\mu}) \) is unitary.

Proof. (1) follows immediately from Th. 3.8. To prove (2), note that for each fixed \( \mu, i \)

\[
1 = \sum_{\lambda, j} \hat{\rho}(t_{\mu, i}) u_w^m \hat{\rho}(t_{\lambda, j}) \hat{\rho}(t_{\lambda, j})^* u_w^m \hat{\rho}(t_{\mu, i})^* = \hat{\rho}(t_{\mu, i}) u_w^m \hat{\rho}(t_{\mu, i})^*
\]
where in the second = we have used Th. 3.8. Similarly

\[ 1 = \bar{\rho}(t_{\mu,i})^* u_{w^m}^* \bar{\rho}(t_{\mu,i}) \bar{\rho}(t_{\mu,i})^* u_{w^m} \bar{\rho}(t_{\mu,i}) \]

and the Prop. is proved. \[ \blacksquare \]

The unitary in (2) of Prop. 3.9 will be denoted by \( u_\mu \) (it may depend on \( m \)) in the following.

**Definition 3.10.** Let \( \mu \in \Delta_A \) and \( b \in H_\rho \) be irreducible. Define

\[ \psi_b^{(w)} := S_{11} d_b d_w \phi_w(\varepsilon(b\bar{\rho}, w)b(u_w)\varepsilon(w, b\bar{\rho})), b \in H_\rho. \]

**Lemma 3.11.** Let \( m \geq 1 \) \( t_{\mu,i} \) be as in Prop. 3.9. Then

\[ | \sum_b d_2(b \psi_b^{(w)} d_b S_{11})^m | = \frac{1}{S_{11}^2} lw^m, 1 \rangle, \forall m \geq 1. \]

**Proof**

\[
\begin{align*}
\left( \frac{\psi_b^{(w)}}{d_b S_{11}} \right)^m &= d_w^m \phi_w^m(\varepsilon(b\bar{\rho}, w^m)b(u_{w^m})\varepsilon(w^m, b\bar{\rho})) \\
&= \sum_{\mu,i} d_\mu \phi_\mu(t_{\mu,i}^* \varepsilon(b, w^m)b(u_{w^m})\varepsilon(w^m, b\bar{\rho})t_{\mu,i}) \\
&= \sum_{\mu,i} d_\mu \phi_\mu(\varepsilon(b\bar{\rho}, \mu)b(\rho(t_{\mu,i})^* u_{w^m} \bar{\rho}(t_{\mu,i}))\varepsilon(\mu, b\bar{\rho})) \\
&= \sum_{\mu} l_{\mu, w^m} d_\mu \phi_\mu(\varepsilon(b\bar{\rho}, \mu) u_\mu \varepsilon(\mu, b\bar{\rho})) \\
\end{align*}
\]

where we have used definition of minimal left inverse in the first =, equation (1) in the second =, Prop. 2.1 in the third =, and Lemma 3.9 in the last =.

It follows that

\[
\sum_b d_2^2(b \psi_b^{(w)} d_b S_{11})^m = \sum_{b,\mu} l_{\mu, w^m} d_\mu^2 d_\mu \phi_\mu(\varepsilon(b\bar{\rho}, \mu) u_\mu \varepsilon(\mu, b\bar{\rho})) \\
= \sum_{\mu} l_{\mu, w^m} d_\mu \sum_b d_b d_\mu \phi_\mu(\varepsilon(b\bar{\rho}, \mu) b(u_\mu) \varepsilon(\mu, b\bar{\rho})) \\
= \sum_b d_\mu^2 \phi_1(u_1) l_{1, w^m}
\]

where we have used Lemma 2.23 in the third =. Since \( u_1 \in \text{Hom}(1, 1) \) is unitary by Prop. 3.9, \( |\phi_1(u_1)| = 1 \) and we have proved that

\[ | \sum_b d_2^2(b \psi_b^{(w)} d_b S_{11})^m | = \frac{1}{S_{11}^2} lw^m, 1 \rangle. \]
Proposition 3.12. There is a sector \( c \in H_\rho \) such that \( |\frac{\psi_c(w)}{S_{11}}| = d_c d_w \).

Proof. By Lemma 3.11 we have

\[
|\sum_b d_b^2 \left( \frac{\psi_b(w)}{d_b S_{11}} \right)^m| = \frac{1}{S_{11}^2} l w^m, 1), \forall m \geq 1.
\]

By repeated using Verlinde formula we have

\[
lw^m, 1) = \sum_{\mu} \frac{S_{1\mu}^m}{S_{1\mu}}.
\]

By Lemma 2.31, when \( m \) goes infinity, the leading order of \( |\sum_b d_b^2 \left( \frac{\psi_b(w)}{d_b S_{11}} \right)^m| \)must be \( nd^m \). Note by Lemma 2.23 \( |\frac{\psi_b(w)}{d_b S_{11}}| \leq d_w \). It follows that There is a sector \( c \in H_\rho \) such that \( |\frac{\psi_c(w)}{S_{11}}| = d_c d_w \). ■

Choose \( m = 1 \) and let \( t_{\mu,i} \) be isometries as in Lemma 3.9.

Definition 3.13. Assume that \( \mu \in \Delta_A \) and \([b] \in H_\rho \) is irreducible. Define

\[
\frac{\psi_b^{(\mu)}}{S_{11}} := d_b d_\mu \phi_\mu((\varepsilon(b\bar{\rho}, \mu)b(\bar{\rho}(t_{\mu,i})*u_w \bar{\rho}(t_{\mu,i}))\varepsilon(\mu, b\bar{\rho}))).
\]

Note that by Lemma 3.9 \( \psi_b^{(\mu)} \) is independent of the choice of \( i \).

Corollary 3.14. Assume that \([a_v] = [\bar{h}a_v], [h^n] = [1], a_v \) is irreducible, and if \( \mu < v_0^2, 1 < a_\mu \), then \([\mu] = [1] \). Then there is \([c] \in H_\rho \) such that \( |\frac{\psi_c^{(\lambda)}}{S_{11}}| = d_c d_\lambda, \forall \lambda, \text{col}(\lambda) = 0 \text{mod} n \) and \([c\bar{c}] = \sum_{1 \leq i_1 \leq n_i} \omega^{2i_1} \) where \( i_1 \) is a divisor of \( n \).

Proof. Choose \( m = 1 \) and let \( t_{\mu,i} \) be isometries as in Lemma 3.9. By equation (11) we have

\[
\frac{\psi_c^{(w)}}{S_{11}} = \sum_{\mu} l_{\mu, w} \frac{\psi_c^{(\mu)}}{S_{11}}
\]

By Lemma 2.23 we have

\[
|\frac{\psi_c^{(\mu)}}{S_{11}}| \leq d_c d_\mu
\]

By Prop. 3.12 we conclude that

\[
|\frac{\psi_c^{(\mu)}}{S_{11}}| = d_c d_\mu, \forall \mu < w
\]

In particular \( |\frac{\psi_c^{(v_0)}}{S_{11}}| = d_c d_{v_0} \). By Lemma 2.23 we know that \( \sum_b \psi_b^{(v_0)}[b] \) is a nonzero eigenvector of the action of \([\lambda] \) on \( H_\rho \). Since \( la_{v_0, \bar{a}_{v_0}} = 1 \), by Prop. 2.19 we must have

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\[ \psi_b^{(v_0)} = z \phi_b^{(v_0)}, \] for some constant \( z \) independent of \( b \). Since \([\bar{v}_0] = [v_0]\), \( \sum_b \phi_b^{(v_0)b} \) is also an eigenvector of the action of \([\lambda]\) with eigenvalue \( \frac{S_{\lambda v_0}}{S_{v_0}} \), it follows that \( \phi_b^{(v_0)*} = z' \phi_b^{(v_0)} \), for some constant \( |z'| = 1 \) independent of \( b \). Hence

\[ \sum_b \psi_b^{(v_0)}^2 = \sum_b z^2 z' \phi_b^{(v_0)} \phi_b^{(v_0)*} = z^2 z' \]

By (3) of Lemma 4.2 and our assumption we conclude that \( |z| = 1 \), and so by Lemma 2.12 we have

\[ d_c^2 = \left| \psi_c^{(v_0)} \right|^2 = \left| \frac{\phi_c^{(v_0)}}{S_{v_0}} \right|^2 = \sum_{\mu} l(c \bar{c}, \mu) \frac{S_{\mu v_0}}{S_{v_0}} \]

Since \( \frac{S_{\lambda v_0}}{S_{v_0}} \leq d_\mu \), we must have \( \frac{S_{\lambda v_0}}{S_{v_0}} = d_\mu, \forall \mu \prec c \bar{c} \).

By Lemma 2.36 we conclude that if \( \mu \prec c \bar{c} \), then \( \mu = \omega^i \) for some \( 1 \leq i \leq n \). Let \( 1 \leq i_1 \leq n \) be the smallest positive integer such that \( [\omega^{i_1} c] = [c] \). Then it is clear that

\[ [c \bar{c}] = \sum_{1 \leq i \leq n} [\omega^{i_1} c] \] where \( i_1 \) is a divisor of \( n \).

4 Proof of Th. 2.40

In this section we preserve the setting of section 2.6. Let \( c_1, c_2 \in \text{End}(M) \) such that \( a_u = c_1 c_2, c_1(M) = M_1, M_1 \not= a_u(M), M \). By Prop. 2.41 to prove Th. 2.40 it is enough to show that \( M_1 \) is one of the intermediate subfactors in Prop. 2.41.

4.1 Local consideration

Suppose \( c \) is a sector such that \( c \bar{c} \prec a_\lambda^{\mu} \) where \( \mu \in \Delta_\lambda \) is a direct sum of irreducible sectors with colors divisible by \( n \). Recall from section 2.6 that if \( \lambda = 0 \text{mod } n \), then \( [a_\lambda^{\mu}] = [\tilde{a}_\lambda^{\mu}] \), and we can apply induction of \( a_\lambda^{\mu} \) with respect to \( c \). The following Lemma is proved by a translation of the proof of (3) of Lemma 3.3 in [43] into our setting:

**Lemma 4.1.** If \( \lambda = 0 \text{mod } n \), then \( [a_\lambda^{c_\rho}] = [a_\lambda^{\rho c}] \).

By Prop. 2.24 we have \( c_1 = c_1' c_1'' \). Let \( c_2' = c_1' c_2 \) so that \( a_u = c_1' c_2' \). Consider induction with respect to \( \rho_0 c_1' \).

We have

**Lemma 4.2.** \( [c_1' c_2'] = [1] \).

**Proof** Apply Lemma 2.12 to \( a = \rho_0 c_1', b = \rho_0 c_2' \) we have

\[ \sum_i \frac{\phi_i^{(\lambda, i)}}{S_{1\lambda}^{\lambda}} = \sum_{\nu} \mu \rho_0 c_1' c_2' \rho_0, \nu \frac{S_{\nu \lambda}}{S_{1\lambda}} = \sum_{\nu} \mu \rho_0 \bar{\rho}_0, \nu \frac{S_{\nu \lambda}}{S_{1\lambda}} = \sum_{1 \leq i \leq n} \frac{\exp(\frac{2\pi i s}{n})}{S_{1\lambda}} \]

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Choose \( \lambda = v_0 \) and use Lemma 2.39 we have
\[
\sum_i \phi_i^{(\lambda, i)} \phi_i^{(\lambda, i)*} S_{1\lambda} = 0
\]
Hence by Lemma 2.20 we obtain \( l \alpha_{v_0}^{\rho, c_1, 1} \geq 1 \). For any \( \mu \in \Delta_A \), since \( \rho_0 c_1 \rho_0 < \rho_0 a_{ui} \rho_0 \) and each irreducible sector of \( \rho_0 a_{ui} \rho_0 \) has color divisible by \( n \), it follows that if \( \text{col}(\mu) \neq 0 \text{mod } n \), then \( \lambda, \rho_0 c_1, 1, \rho_0 \) = 0. On the other hand if \( \text{col}(\mu) = 0 \text{mod } n \), by Lemma 4.1 and Prop. 2.9 we have
\[
(\lambda, c, \lambda, \mu, \lambda, \mu, \lambda, n), \mu, \lambda, \mu, \lambda, n)
\]
By (1) of Prop. 2.24 it follows that \( c \) is local.

By Lemma 2.33 we have \( a_{v_0}^{\rho, c_1, 1} = a_{v_0}^{\rho, c_1, 1} \), and by Lemma 2.25 and Lemma 2.36 we conclude that \( \rho_0 c_1 \) is irreducible and \( [\rho_0 c_1] = \sum_{j} [\omega] \) where the sum is over a finite set of positive integers. Since \( \rho_0 c_1 \) is irreducible and \( [\rho_0 c_1] = \sum_{j} [\omega] \) we conclude that \( [\rho_0 c_1, 1, \rho_0] = \sum_{1 \leq j \leq n} [\omega] \) Hence \( d_1 = 1 \) and \( c_1 = 1 \).

By Prop. 2.24 we have proved

**Corollary 4.3.** If \( \lambda \in \Delta_A \) is irreducible, then \( l \alpha, \lambda, \lambda, \mu, \lambda, n, \mu, \lambda, n \geq 1 \) iff \( \lambda = \omega^i, 1 \leq i \leq n \).

### 4.2 Verifying assumptions of Cor. 3.14

Set \( \rho = \rho_0 c_1 \) and all inductions in the rest of this section are with respect to \( \rho \).

**Lemma 4.4.** \( a_\lambda \) is irreducible for all irreducible descendants of \( \nu^2 \nu^2, \nu \nu^3 \).

**Proof** By Lemma 2.29 and Prop. 2.9 we have for \( n \geq 3 \)
\[
[a_{v_0} a_{v_0}] = 2[1] + 4[a_{v_0}] + [a_{2v, 0, 0, 0}, 0] + [a_{2v, 0, 0, 0, 0}, 0] + [a_{2v, 0, 0, 0, 0, 0}, 0] + [a_{4v, 0, 0, 0, 0}, 0]
\]
Note that by Cor. 4.3 we have
\[
la, \mu \rangle = l \langle 1, \mu \rangle \geq 2
\]
iff \( \omega^j(\lambda) = [\mu] \) for some \( 1 \leq j \leq n - 1 \). It is easy to check with the explicit formulas above that \( a_\lambda \) is irreducible for all irreducible descendants of \( \nu^2 \nu^2 \). \( n = 2 \) case is simpler, and similarly one can check directly that \( a_\lambda \) is irreducible for all irreducible descendants of \( \nu \nu^3 \).

**Lemma 4.5.** For all \( \lambda \) with \( \text{col}(\lambda) = 0, [a_\lambda] = [\tilde{a}_\lambda] \).

**Proof** By (2) of Prop. 2.19 and Th. 2.1 of [15] all \( \lambda, \mu \) with \( \mu = 0, 1 \leq i \leq n \) are classified. Using Cor. 4.3, it follows by inspection of Th. 2.1 of [15] that for all \( \lambda \) with \( \text{col}(\lambda) = 0, Z_{\lambda, \mu} = la, \mu, \lambda \rangle \neq 0 \) or \( Z_{\lambda, \mu} = la, \lambda, \mu \rangle \neq 0, \forall \lambda \). In the latter case by Prop. 2.19 we conclude that \( \lambda \) appears in Exp iff \( la, \lambda \rangle \neq 0 \). Choose \( \lambda = (n, 0, ..., 0) = \Lambda \) as in Lemma 2.39. It follows from Lemma 2.39 and Cor. 2.20 that \( \Lambda \in \text{Exp} \), but \( la, \lambda \rangle \neq 0, \forall \lambda, \text{col}(\lambda) = 0 \text{mod } n, \) and by Lemma 2.35 we conclude that for all \( \lambda \) with \( \text{col}(\lambda) = 0, [a_\lambda] = [\tilde{a}_\lambda] \).
Lemma 4.6. Suppose that $x_i \prec a_{\lambda_1, \mu_i}, i = 1, 2$ and $x_1x_2$ is a direct sum of $a_{\nu}$ with $a_{\nu}$ irreducible. Then $[x_1x_2] = [x_2x_1]$.

Proof
By assumption it is enough to check that
$$\langle lx_1x_2, a_{\nu} \rangle = \langle lx_2x_1, a_{\nu} \rangle$$

By Lemma 2.13 we have $[a_{\nu}x_2] = [\bar{x}_2a_{\nu}]$, together with Frobenius reciprocity we obtain
$$\langle lx_1x_2, a_{\nu} \rangle = \langle lx_1, a_{\nu} \rangle + \langle lx_2, a_{\nu} \rangle = \langle lx_1, a_{\nu} \bar{x}_2 \rangle = \langle lx_2x_1, a_{\nu} \rangle$$

Proposition 4.7. There exists $h \in \text{End}(M)$ such that $[\bar{a}_v] = [ha_v], [h^n] = [1]$.

Proof
First suppose that there is no automorphism $h$ such that $[\bar{a}_v] = [ha_v]$ or $[\bar{a}_v] = [ha_v]$. By Lemma 4.5 $[a_v a_0] = [\bar{a}_v a_0] = [1] + [a_{v_0}]$. By Lemma 2.33 $a_{v_0}$ is irreducible, it follows that there are sectors $x_i, y_i$ with $d_{x_i} > 1, d_{y_i} > 1$ such that
$$[a_v a_0] = [x_1] + [x_2], [a_v a_0] = [y_1] + [y_2].$$

We compute
$$[a_v a_0 a_0] = [x_1 a_0] + [x_2 a_0] = [a_v a_0 a_0] = 2[a_v] + [a_{(0,0,...,0,1)}] + [a_{(0,1,0,...,0,1)}]$$

where we have used Lemma 1.5 in the second = . By assumption $d_{x_i} > 1, i = 1, 2$ we have $x_i a_0 \succ a_v$, but $[x_i a_0] \neq [a_v], i = 1, 2$. Hence we can assume that
$$[x_1 a_0] = [a_v] + [a_{(0,0,...,0,1)}], [x_2 a_0] = [a_v] + [a_{(0,1,0,...,0,1)}]$$

Hence
$$la_{v_0} x_i, a_v a_0] = lx_i a_0, x_i a_0] = lx_i, x_i a_0] = lx_i, x_i a_0] = 2$$

where we have used Lemma 2.13 in the first = and Lemma 1.5 in the third =. We can assume that
$$[a_v a_i] = [\bar{a}_v] + [u_i], i = 1, 2$$

where $u_i, i = 1, 2$ is irreducible and we may have $[u_1] = [u_2]$. Note that $[a_v x_1] + [a_v x_2] = [a_v y_1] + [a_v y_2] = [a_v a_0 a_0]$. The same argument applies to $y_i, i = 1, 2$ and we may choose $y_i$ such that
$$[a_v y_i] = [a_v y_i], i = 1, 2$$

Consider now
$$[a_{v_0}^2] = [x_1 x_1] + [x_2 x_2] + [x_1 x_2] + [x_2 x_1]$$

$$= 2[1] + 4[a_{v_0}] + [a_{(2,0,...,0,2)}] + [a_{(0,1,0,...,0,0,2)}] + [a_{(0,0,1,0,...,0,2)}] + [a_{(2,0,...,0,1,0)}]$$
Note that \( x_i \bar{x}_i \sim a_{vi} \), and \([x_i, \bar{x}_j] = [\bar{x}_j, x_i]\) by Lemma 4.4 and Lemma 4.6. Hence
\[
l x_2 \bar{x}_1, x_2 \bar{x}_1) = l x_2 \bar{x}_2, x_1 \bar{x}_1) \geq 2
\]

By computing the index of sectors we conclude that
\[
[x_1 \bar{x}_1] = [a_{vi}] + [a_{(0,0,...,0,2)}], [x_1 \bar{x}_2] = [a_{vi}] + [a_{(0,1,...,0,2)}]
\]
\[
x_2 \bar{x}_2] = [a_{vi}] + [a_{(0,1,0,...,1,0)}], [x_2 \bar{x}_1] = [a_{vi}] + [a_{(2,0,...,0,1,0)}]
\]

Similarly we obtain
\[
y_1 \bar{y}_1] = [a_{vi}] + [a_{(2,0,...,0,2)}], [y_1 \bar{y}_2] = [a_{vi}] + [a_{(0,1,...,0,2)}]
\]
\[
y_2 \bar{y}_2] = [a_{vi}] + [a_{(0,1,0,...,1,0)}], [y_2 \bar{y}_1] = [a_{vi}] + [a_{(2,0,...,0,1,0)}]
\]

Next compute
\[
[a_{vi}a_{vi}] = [a_{vi}a_{vi}a_{vi}] = [y_1 \bar{x}_1] + [y_1 \bar{x}_2] + [y_2 \bar{x}_1] + [y_2 \bar{x}_2].
\]

Note that
\[
l y_2 \bar{y}_1, x_2 \bar{x}_1) = l x_2 y_2, x_1 y_1) = 2
\]
\[
2 = l a_{vi} x_i, a_{vi} y_i) = l a_{vi}^2, y_i \bar{x}_i)
\]

and
\[
l y_2 \bar{y}_1, x_2 \bar{x}_1) = l y_2 y_1, x_2 \bar{x}_1) = 3
\]
\[
l y_1 \bar{x}_2, y_1 \bar{x}_2) = l y_1 y_2, x_2 \bar{x}_2) = 2
\]

where we have also used Lemma 4.6. From these equations we conclude that
\[
y_1 \bar{x}_1] = [a_{i}^2] + [a_{(0,1,...,0,3)}]
\]
or
\[
y_1 \bar{x}_1] = [a_{i}^2] + [a_{(1,0,...,0,0,0)}]
\]

From \([a_{vi}x_1] = [a_{vi}y_1]\) we obtain
\[
[a_{vi}x_1 \bar{x}_1] = [a_{vi}y_1 \bar{x}_1]
\]

Using the formulas for \(x_1 \bar{x}_1, y_1 \bar{x}_1\) we obtain
\[
[a_{vi}a_{(2,0,...,0,2)}] = [a_{vi}a_{(1,0,...,0,1,0,0)}]
\]
or
\[
[a_{vi}a_{(2,0,...,0,2)}] = [a_{vi}a_{(1,0,...,0,3)}]
\]

Both identities are incompatible with Lemma 2.29 and Lemma 4.5.

Therefore there is an automorphism \(h\) such that \([\bar{a}_{vi}] = [ha_{vi}]\) or \([\bar{a}_{vi}] = [ha_{vi}]\). Hence \( h^m \sim [a_{vi}a_{vi}] = [a_{vi}a_{vi}] \) or \( h^m \sim [a_{vi}a_{vi}] = [a_{vi}a_{vi}] \) by Lemma 4.5. Assume that \( h^m \sim a_\mu\) for some \( \mu, \col(\mu) \equiv 0 \mod n \). Since \( \rho = \rho_0 c_1 \), by Lemma 4.11 there is a sector \(x\) of \(a_\mu^n\) such that \([a_\mu^n] = [h^n]\). Since \(d_x = 1\), by Lemma 2.37 we conclude that \([x] = [1]\) and \([h^n]\) = [1].

If \([\bar{a}_{vi}] = [ha_{vi}]\), use \([h^n] = [1]\) we have \([a_{vi}] = [a_{vi}]\). Hence \(\omega^j(n,0,...,0) \sim \bar{v}\) for some \(1 \leq j \leq n\) which is incompatible with fusion rules in Lemma 2.29 since \(k = n' n' \geq 3n\).
4.3 Properties of sectors related to $a_u$

Lemma 4.8. If $\varepsilon(\omega^l, \lambda)\varepsilon(\lambda, \omega^l) = 1$, then $n|\text{col}(\lambda)$.

Proof By monodromy equation $\varepsilon(\omega^l, \lambda)\varepsilon(\lambda, \omega^l) = \exp\left(\frac{2\pi i \text{col}(\lambda)}{n}\right)$ and the lemma follows.

Lemma 4.9. If $[v\lambda] = \sum_{1\leq j\leq k_1-1}[\omega^{j_1}w]$ where $k_1l_1 = n$, $[\omega^{j_1}w] = [\omega^{j_1}w]$ iff $j = j' \mod k_1$, and $\sum_{1\leq i\leq n-1} \lambda_i \leq k - 1$. Then $\lambda = (0, ..., 0, k/k_1, 0, ..., 0, k/k_1, ..., 0)$ where $(0, ..., 0, k/k_1)$ (with $l_1 - 1$ 0’s) appears $k_1 - 1$ times, and the last $l_1 - 1$ entries are 0’s, and $\text{col}(\lambda) = k_1l_1 - 1 \mod n$.

Proof Since $[\omega^{j_1}\lambda] = [\lambda]$, in the components of $\lambda, (\lambda_0, ..., \lambda_{l_1-1})$ appears $k_1$ times. By assumption $v\lambda$ is a sum of $k_1$ distinct irreducible subsectors, it follows from Lemma 2.29 that $\lambda$ has only $k_1$ non-zero components. Since $\lambda_0 \neq 0$, and $\text{col}(\lambda) = \frac{k_1l_1(k_1-1)}{2}$, the lemma follows.

Proposition 4.10. If $[a_u] = [x_1y_1], 1 < d_{x_1} < d_u$ where $x_1 < a_{\lambda_1}, y_1 < a_{\lambda_2}$. Then either $[x_1] = [a_v], [y_1] = [b_1]$ or $[y_1] = [a_v], [x_1] = [b_1], 1 \leq i \leq n$.

Proof By using the action of $\omega$ if necessary, we may assume that the zero-th components of $\lambda_1, \lambda_2$ are positive. By Lemma 2.35 we can assume that

$$[a_{\lambda_1}] = \sum_{1 \leq i \leq k_1} [x_i], [\omega^{j_1}\lambda_1] = [\lambda_1], [g^jx_1g^{-j}] = [x_i], 0 \leq i \leq k_1 - 1, k_1l_1 = n$$

$$[a_{\lambda_2}] = \sum_{1 \leq i \leq k_2} [y_i], [\omega^{j_2}\lambda_2] = [\lambda_2], [g^jx_1g^{-j}] = [x_i], 0 \leq i \leq k_2 - 1, k_2l_2 = n$$

Since $a_u < a_{\lambda_1}, \lambda_2$, $\text{col}(\lambda_1) + \text{col}(\lambda_2) = \text{col}(u) = 1 \mod n$. By Lemma 4.8 $k_i |\text{col}(\lambda_i), i = 1, 2$. Hence $(k_1, k_2) = 1$.

Since $x_1y_1, a_{v0}$ are irreducible, we may assume that $l(x_1, a_{v0}) = 0$, i.e., $a_{v1}x_1$ is irreducible. Let $w \prec v\lambda_1$. Since $\omega^{j_1}[\lambda_1] = [\lambda_1], \omega^{j_1}w \prec v\lambda_1$. Let $t_1|k_1$ be the least positive integer such that $[\omega^{j_1t_1}\lambda] = [\lambda]$. By Lemma 4.8 $n|t_1 \text{col}(w)$. But $\text{col}(w) = 1 + \text{col}(\lambda_1) \mod n$ with $k_1|\text{col}(\lambda_1)$. We conclude that $t_1 = k_1$ and

$$[v\lambda_1] \prec \sum_{0 \leq j \leq k_1-1} [\omega^{j_1j}w]$$

Since $a_{w} \prec a_{v\lambda_1} = \sum_{1 \leq j \leq k_1} [a_vx_j]$ and each $a_vx_j$ is irreducible, $d_{w} = d_w \geq d_vdx_1 = d_vd_{\lambda_1}/n$. Hence

$$[v\lambda_1] = \sum_{0 \leq j \leq k_1-1} [\omega^{j_1j}w]$$

By Lemma 4.9 we have $\text{col}(\lambda_1) = 0 \mod n$. Hence $\text{col}(\lambda_2) = 1 \mod n$ and $k_2 = 1$. If $l_1 = 1$, then $\lambda_1 = (n', ..., n')$, and $d_{\lambda_2} = d_v$. By Proposition on Page 10 of [14] $\lambda_2$ must be in the orbit of $v$ or $\bar{v}$ under the action of $\omega$. But $\text{col}(\lambda_2) = 1 \mod n$, so $[a_{\lambda_2}] = [a_v]$ and Prop. is proved. In the following we assume that $l_1 \geq 2$ to reach a contradiction.
Note that \([a_{\lambda_1,\lambda_2}] = k_1[a_u]\), hence \([\lambda_1\lambda_2] = \sum_{0 \leq i \leq k_1-1} [\omega^{d_i} u]\). By Lemma 2.30, \(k_1 \geq 2\). We have
\[
\langle l\lambda_1, \lambda_2, \lambda_1\lambda_2 \rangle = k_1 \geq 1 + \langle l\lambda_1\lambda_1, v_0 \rangle \langle l\lambda_2\lambda_2, v_0 \rangle = 1 + (k_1 - 1)\langle l\lambda_2\lambda_2, v_0 \rangle
\]
Hence \(\langle l\lambda_2, v\lambda_2 \rangle = 2\).

By Lemma 2.33, Cor. 4.3 and Prop. 4.7, the assumptions of Cor. 3.14 are verified.

We have
\[
\| l\lambda_1\lambda_1, (0, 1, 0, ..., 0) \rangle \langle 0, 0, ..., 1, 0 \rangle \| = \| (0, 1, 0, ..., 0) \rangle \langle 0, 0, ..., 1, 0 \rangle \|
\]
and this implies that the intermediate subfactor \(c_1(M) = adU_{\alpha}(M)\), i.e., it is one of the subfactors in Prop. 2.41. The case when \([c_1] = [b_i], [c_2] = [a_v] 1 \leq i \leq n\) is treated similarly. By Prop. 2.41, Th. 2.40 is proved.
5 Related issues

5.1 Centrality of a class of intertwinners

We preserve the general setup of section 2.3. If \( \rho = \mu c, \mu \in \Delta_A, d_c = 1 \) it follows from definition 2.7 that \([a_\lambda] = [\tilde{a}_\lambda] = [c^{-1} \lambda c], \forall \lambda\), hence \( Z_{\lambda \lambda_1} = \delta_{\lambda, \lambda_1} \). Motivated by our proof of Th. 2.40 we make the following:

**Conjecture 5.1.** If \( Z_{\lambda \lambda_1} = \delta_{\lambda, \lambda_1} \), then \( \rho = \mu c, \mu \in \Delta_A, d_c = 1 \).

We will prove that Conjecture 5.1 is equivalent to the centrality of a class of intertwinners. Assume that \( Z_{\lambda \lambda_1} = \delta_{\lambda, \lambda_1} \). Then for each irreducible \( \lambda \) there is (up to scalar) a unique unitary \( u_\lambda \in \text{Hom}(a_\lambda, \bar{a}_\lambda) \).

Similar to Def. 3.7 we define:

**Definition 5.2.** \( u_{\lambda_1 \lambda_2 ... \lambda_m} := u_\lambda a_{\lambda_1}(u_{\lambda_2})...a_{\lambda_1 \lambda_2 ... \lambda_{m-1}}(u_{\lambda_m}) \in \text{Hom}(a_{\lambda_1 \lambda_2 ... \lambda_m}, \bar{a}_{\lambda_1 \lambda_2 ... \lambda_m}) \)

If \( \rho = \mu c, \mu \in \Delta_A, d_c = 1 \), then it follows from def. (2.7) we can choose \( u_\lambda \) such that \( u_\lambda = c^{-1}(\bar{\varepsilon}(\lambda, \bar{\mu})\varepsilon(\mu, \lambda)) \). Use BFE in Prop. 2.1 we have

\[
u_{\lambda_1 \lambda_2 ... \lambda_m} = c^{-1}(\bar{\varepsilon}(\lambda_1 \lambda_2 ... \lambda_m, \bar{\mu})\varepsilon(\lambda_1 \lambda_2 ... \lambda_m)) \in \text{Hom}(a_{\lambda_1 \lambda_2 ... \lambda_m}, \bar{a}_{\lambda_1 \lambda_2 ... \lambda_m}),
\]

\[	ext{Hom}(a_{\lambda_1 \lambda_2 ... \lambda_m}, \bar{a}_{\lambda_1 \lambda_2 ... \lambda_m}) = c^{-1}(\text{Hom}(\bar{\mu}_{\lambda_1 \lambda_2 ... \lambda_m}, \bar{a}_{\lambda_1 \lambda_2 ... \lambda_m})).
\]

By using BFE in Prop. 2.1 again we have proved the following:

**Lemma 5.3.** If \( \rho = \mu c, \mu \in \Delta_A, d_c = 1 \), then \( u_{\lambda_1 \lambda_2 ... \lambda_m} T u_{\lambda_1 \lambda_2 ... \lambda_m}^* = T, \forall T \in \text{Hom}(a_{\lambda_1 \lambda_2 ... \lambda_m}, a_{\lambda_1 \lambda_2 ... \lambda_m}). \)

Using \( u_\lambda \) we define:

**Definition 5.4.** For any irreducible \([b] \in H_\rho, \lambda \in \Delta_A\),

\[
\psi_b^{(\lambda)} := S_{11} d_b d_\lambda \phi_\lambda(\varepsilon(b \bar{\rho}, \lambda)b(u_\lambda)\varepsilon(\lambda, b \bar{\rho}))
\]

**Lemma 5.5.** For any irreducible \([b] \in H_\rho, \psi_b^{(\lambda)} = c_\lambda \phi_b^{(\lambda)}, |c_\lambda c_\lambda| = 1 \) where \( c_\lambda \) are complex numbers independent of \( b \).

**Proof** Since by Lemma 2.23 \( \sum_b \psi_b^{(\lambda)} \) is an eigenvector of the action of \( \mu \) with eigenvalue \( \frac{s_\lambda}{s_{1\lambda}} \), and by Prop. 2.19 there is up to scalar a unique such eigenvector, it follows that there is a complex number \( c_\lambda \) independent of \( b \) such that \( \psi_b^{(\lambda)} = c_\lambda \phi_b^{(\lambda)}, \forall b \).

Similarly since \( \sum_b \phi_b^{(\lambda)} \) is an orthogonal eigenvector of the action of \( \mu \) with eigenvalue \( \frac{s_\lambda}{s_{1\lambda}} \), we have \( \phi_b^{(\lambda)} = c'\phi_b^{(\lambda)}, |c'| = 1, \forall b \). We have \( \phi_b^{(\lambda)} = c_\lambda c'\phi_b^{(\lambda)}, \forall b, |c'_\lambda| = 1. \)

By Lemma 2.23 \( \sum_b \psi_b^{(\lambda)} \psi_b^{(\lambda)} \) has absolute value 1, and it follows that \( |c_\lambda c_\lambda| = 1. \)

The following Lemma is proved in the same way as Lemma 3.9.

**Lemma 5.6.** If \( u_{\lambda_1 \lambda_2 ... \lambda_m} \) is central, then for fixed \( \mu \), if \( t_\mu \in \text{Hom}(\mu, \lambda_1 \lambda_2 ... \lambda_m) \) is an isometry, then \( \bar{\rho}(t_\mu)^* u_{\lambda_1 \lambda_2 ... \lambda_m} \bar{\rho}(t_\mu) \in \text{Hom}(a_\mu, \bar{a}_\mu) \) is a unitary independent of the choice of \( t_\mu \), and is a scalar multiple of \( u_\mu \).
Proposition 5.7. Conjecture (5.1) is equivalent to the following statement: If $Z_{\lambda_1} = \delta_{\lambda_1 \lambda_1}$, then $u_{\lambda_1 \lambda_2 \ldots \lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m, \forall m$.

Proof Suppose that Conjecture (5.1) is true. Then it follows from Lemma 5.3 that if $Z_{\lambda_1} = \delta_{\lambda_1 \lambda_1}$, then $u_{\lambda_1 \lambda_2 \ldots \lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m, \forall m$. Suppose now that $u_{\lambda_1 \lambda_2 \ldots \lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m, \forall m$. As in the proof of Lemma 3.11 by using centrality $u_{\lambda_1 \lambda_2 \ldots \lambda_m}$ we calculate

$$
\frac{\psi_b^{(\lambda_1)} \psi_b^{(\lambda_2)} \cdots \psi_b^{(\lambda_m)}}{\psi_b^{(1)} \psi_b^{(1)} \cdots \psi_b^{(1)}} = \sum_{\mu} m_{\mu, \lambda_1 \ldots \lambda_m} \phi_{\mu}(\varepsilon(b, \mu)b(\mu)\varepsilon(\mu, b^\varepsilon))c_\mu
$$

where $|c_\mu| = 1$. Hence using Lemma 2.23 as in the proof of Lemma 3.11 we have

$$
\sum_b |d_b^2 \psi_b^{(\lambda_1)} \psi_b^{(\lambda_2)} \cdots \psi_b^{(\lambda_m)}| = \sum_{\mu} \sum_{\lambda} \frac{S_{\lambda_1 \lambda} S_{\lambda_2 \lambda} \cdots S_{\lambda_m \lambda}}{S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_m}} d_b^2
$$

Now choose $m = 2m_1$ and $\lambda_{i+m_1} = \lambda_i, 1 \leq i \leq m_1$, sum over $\lambda_1, \ldots, \lambda_{m_1}$ and use Lemma 5.5 we obtain

$$
\sum_b \frac{1}{d_b^{m-2}} = \sum_{\lambda} \frac{1}{d_{\lambda}^{m-2}}
$$

Let $m = 2m_1$ go to infinity and notice that $d_b \geq 1$ we conclude that there must exist a sector $c$ such that $d_c = 1$ and $\rho = \mu c$ for some $\mu \in \Delta_A$.

For each irreducible $\lambda \in \Delta_A$ we choose $R_{\lambda \lambda}$ so that $R_{\lambda \lambda}^* R_{\lambda \lambda} = c_\lambda \lambda(\lambda R_{\lambda \lambda}^*) R_{\lambda \lambda} = 1$. These operators are unique up to scalars.

Lemma 5.8. (1) We can choose $u_\lambda$ such that

$$
\bar{\rho}(R_{\lambda \lambda}^*) u_{\lambda \lambda} = \bar{\rho}(R_{\lambda \lambda}^*), u_{\lambda \lambda} \bar{\rho}(R_{\lambda \lambda}) = \bar{\rho}(R_{\lambda \lambda}), \forall \lambda;
$$

(2) The relative braiding as defined in Lemma 2.15 among $\alpha'_s$ (resp. $\alpha_\lambda$'s) is a braiding and $\varepsilon(\alpha_\lambda, a_\mu) = \varepsilon(\bar{\alpha}_\lambda, \bar{a}_\mu) = \bar{\rho}(\varepsilon(\lambda, \mu)), \forall \lambda, \mu \in \Delta_A$.

Proof Ad (1): Note that $\bar{\rho}(R_{\lambda \lambda}^*) u_{\lambda \lambda}$ is equal to $\bar{\rho}(R_{\lambda \lambda}^*)$ up to a constant of absolute value 1, hence we can choose multiply $u_{\lambda \lambda}, u_\lambda$ by suitable constants of absolute value 1 so that

$$
\bar{\rho}(R_{\lambda \lambda}^*) u_{\lambda \lambda} = \bar{\rho}(R_{\lambda \lambda}^*)
$$

If

$$
u_{\lambda \lambda} \bar{\rho}(R_{\lambda \lambda}) = c_\lambda \bar{\rho}(R_{\lambda \lambda}), \forall \lambda,
$$
multiply both sides on the left by $\bar{\rho}(R_{\lambda \lambda})^*$ we conclude that $c_\lambda = 1, \forall \lambda$.

Ad (2) The relative braiding are braiding since $[a_\lambda] = [\bar{a}_\lambda]$ by assumption and Lemma 2.15. By definition we have

$$
\varepsilon(\alpha_\lambda, a_\mu) = u_\mu^* \bar{\rho}(\varepsilon(\lambda, \mu)) a_\lambda(u_\mu) = u_\mu^* a_\lambda(u_\mu) = u_\mu^* c_\mu \bar{\rho}(\varepsilon(\lambda, \mu)) = \bar{\rho}(\varepsilon(\lambda, \mu))
$$

where we have used Lemma 2.17 in the second = since $u_\mu \in \text{Hom}(a_\mu, \bar{a}_\mu) \subset \text{Hom}(\bar{\rho}_\mu, \bar{\rho}_\mu)$. The other case is proved similarly. □
Definition 5.9. An operator is a cap (resp. cup) operator if it is \( \mu(R_{\lambda\lambda}) \) (resp. \( \mu(R_{\lambda\lambda})^* \)) for some \( \mu, \lambda \in \Delta_A \). It is a braiding operator if it is \( \mu(\varepsilon(\lambda, \nu)) \) or \( \mu(\overline{\varepsilon(\lambda, \nu)}) \) for some \( \nu, \mu, \lambda \in \Delta_A \).

Definition 5.10. Denote by \( B_{\lambda_1\lambda_2...\lambda_m} \) the subspace of \( \text{Hom}(\lambda_1\lambda_2...\lambda_m, \lambda_1\lambda_2...\lambda_m) \) which is linearly spanned by operators in \( \text{Hom}(\lambda_1\lambda_2...\lambda_m, \lambda_1\lambda_2...\lambda_m) \) consisting of products of only caps, cups and braiding operators.

Proposition 5.11. For any \( T \in \tilde{\rho}(B_{\lambda_1\lambda_2...\lambda_m}), u_{\lambda_1...\lambda_m}T = Tu_{\lambda_1...\lambda_m} \).

Proof It is enough to check for an operator \( T \) which consists of products of only caps, cups and braiding operators. Note that the statement of Prop. is independent of choices of \( u_\lambda \), and we can choose our \( u_\lambda \) so that they verify (1) of Lemma 5.8. It is useful to think of \( T \) as an tangle connecting top \( m \) strings labeled by \( a_{\lambda_1},...a_{\lambda_m} \) to the bottom \( m \) strings labeled by \( a_{\lambda_1},...a_{\lambda_m} \) as in Chapter 2 of [38], where in the tangle only cups, caps and braidings are allowed. Then by Prop. 2.1, \( u\tilde{T}u^* \) will be represented by the same tangle, except the top and bottom \( m \) strings are now labeled by \( \tilde{a}_{\lambda_1},...\tilde{a}_{\lambda_m} \). For each closed string in \( u\tilde{T}u^* \) labeled by \( a_\mu \), by inserting \( u_\mu \) we can change the label \( a_\mu \) to \( \tilde{a}_\mu \) using Prop. 2.1 without changing the operator since we have a closed string. Therefore \( u\tilde{T}u^* \) is represented by the same tangle \( T \) with all labels changed from the original labels \( a_\mu \) of \( T \) to \( \tilde{a}_\mu \). Since \( T \) consists of products of only caps, cups and braiding operators, Prop. follows from Lemma 5.8.

Conjecture 5.12. \( B_{\lambda_1\lambda_2...\lambda_m} = \text{Hom}(\lambda_1\lambda_2...\lambda_m, \lambda_1\lambda_2...\lambda_m), \forall \lambda_1, \lambda_2...\lambda_m, m \geq 1 \).

By Prop. 5.11 and Prop. 5.7 we have proved the following:

Proposition 5.13. Conjecture (5.12) implies Conjecture (5.11).

By examining the proof of Prop. 5.7 we can formulate a weaker version of Conjecture (5.12).

Definition 5.14. We say that \( \lambda \) is a generator for \( \Delta_A \) if for any irreducible \( \mu \in \Delta_A \), there is a positive integer \( m \) such that \( \mu < \lambda^m \).

Conjecture 5.15. For some generator \( \lambda \) of \( \Delta_A \), \( B_{\lambda^1...\lambda^m} = \text{Hom}(\lambda^m, \lambda^m), \forall m \geq 1 \) where \( m \) is the number of \( \lambda \) that appears in the definition of \( B_{\lambda^1...\lambda^m} \).

Lemma 5.16. Assume that \( \lambda \) is a generator for \( \Delta_A \). Then the set \( \{[\mu] | \frac{S_{\mu\lambda}}{S_{1\mu}} = d_\lambda \} \) is a finite abelian group.

Proof Note that by definition \( \frac{|S_{\mu\lambda}|}{S_{1\mu}} = d_\lambda \) implies that \( \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) \in \mathbb{C} \). By Prop. 2.1 this implies that \( \varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C} \) if \( \lambda_1 < \lambda^m, m \geq 1 \). Since \( \lambda \) a generator, it follows that \( \varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}, \forall \lambda_1 \in \Delta_A \). Hence \( \frac{|S_{\mu\lambda_1}|}{S_{1\lambda_1}} = d_\mu, \forall \lambda_1 \in \Delta_A \). By properties of \( S \) matrix this implies that \( d_\mu = 1 \). On the other hand if \( d_\mu = 1 \) then \( |S_{\mu\lambda}| = d_\lambda \) since \( \mu\lambda \) is irreducible. It follows that the set \( \{[\mu] | \frac{|S_{\mu\lambda}|}{S_{1\mu}} = d_\lambda \} \) is a finite abelian group.
Proposition 5.17. Conjecture (5.15) implies Conjecture (5.1).

Proof Assume conjecture (5.15) is true. Then by Prop. 5.11 we know that $u_{\lambda m}$ is central.

As in the proof of Prop. 5.7, replacing $\lambda_i$ by $\lambda$ in the summation we have

$$|\sum_a (\frac{\psi_a(\lambda)}{\psi_a(1)})^m d_a^2| = \sum_{\mu} (\frac{S_{\lambda \mu}}{S_{1 \mu}})^m S_{1 \mu}^2$$

Choose $m$ to be divisible by the order of the finite abelian group in Lemma 5.16 and let $m$ go to infinity, the RHS of the above equation has leading order (up to multiplication by a positive number) $d_{\lambda}^m$. It follows that there is a sector $c$ such that $|\psi_{\lambda}(c)| = d_{\lambda}$. For any $\mu \prec \lambda$, $l \geq 1$. Use the centrality of $u_{\lambda l}$ we have

$$(\frac{\psi_{\lambda}(c)}{\psi_{\lambda}(1)})^l = \sum_{\mu} l_{\mu, \lambda} (\frac{\psi_{\lambda}(\mu)}{\psi_{\lambda}(1)}) c_{\mu}$$

where $|c_{\mu}| = 1$. So we have $\sum_{\mu \prec \lambda} |\psi_{\lambda}(\mu)| \geq d_{\lambda}$. Since $|\psi_{\lambda}(\mu)| \leq d_{\mu}$ and $\sum_{\mu} l_{\mu, \lambda} d_{\mu} = d_{\lambda}$, we conclude that $|\psi_{\lambda}(\mu)| = d_{\mu}, \forall \mu \prec \lambda$. Since $\lambda$ is a generator, we conclude that $|\psi_{\lambda}(\mu)| = d_{\mu}, \forall \mu$. By Lemma 5.3 we conclude that $|\psi_{\lambda}(\mu)|^2 = d_{\mu}^2$. Sum over $\mu$ on both sides we conclude that $d_c = 1$, and the Prop. is proved.

By Prop. 5.17 and Lemma 2.32 we have proved the following:

Corollary 5.18. Conjecture (5.1) is true for $\Delta_A$ where $A$ is the net associated with $SU(n)_k$.

5.2 Maximal subfactors

In this section we give an application of Cor. 5.18.

The following notion is due to V. F. R. Jones:

Definition 5.19. A subfactor $N \subset M$ is called maximal if $M_1$ is a von Neumann algebra such that $N \subset M_1 \subset M$ implies $M_1 = M$ or $M_1 = N$.

We preserve the setting of section 2.5. We will say that $\lambda$ is maximal if $\lambda(M) \subset M$ is a maximal subfactor.

Proposition 5.20. If $S_{c_{\lambda}} \neq 0$, then $\lambda$ is maximal.

Proof Let $M_1$ be an intermediate subfactor between $\lambda(M)$ and $M$. Suppose that $\lambda = c_1 c_2$ and $c_1 = c_1' c_1''$ as in Prop. 2.24. Since $S_{c_{\lambda}} \neq 0$, apply Lemma 2.20 and Lemma 2.25 to induction with respect to $c_1$, we conclude that $\varepsilon(v, c_1' c_1'' \varepsilon(c_1' c_1''), v) \in \mathbb{C}$. By Lemma 2.31 we conclude that $[c_1' c_1''] = [1]$. By Prop. 2.24 we must have $Z_{\lambda_1}^{c_1'} = \delta_{\lambda_1}$.
Since $S_{\lambda^v} \neq 0$, by Lemma 2.20 and §2 of [14] we conclude that $Z_{\mu_1, \mu_2}^{c_1} = \delta_{\mu_1, \mu_2}$. By Prop. 5.18 we conclude that $c_1 = \mu c, \mu \in \Delta_A, d_c = 1$. Replacing $c_1$ by $c_1c^{-1}$ if necessary we may assume that $c_1 = \mu$. It follows that $c_2 = \mu_2$ for some $\mu_2 \in \Delta_A$. By Lemma 2.30 we conclude that $[\mu] = [\lambda]$ or $[\mu] = [\omega^i], 1 \leq \iota \leq n$, hence $M_1 = \lambda(M)$ or $M_1 = M$. 

Corollary 5.21. If $k + n = p'$ where $p$ is a prime number, and $(k, n) \neq (2, 2)$, then $\lambda$ is maximal iff there is no $1 \leq \iota \leq n - 1$ such that $[\omega^i \lambda] = [\lambda]$.

Proof By Th. 5 of [13] when $k + n = p'$ where $p$ is a prime number, $S_{v\alpha} = 0$ iff $[\omega^i \lambda] = [\lambda]$ for some $1 \leq \iota \leq n - 1$. Let $i_1 | i$ be the smallest positive integer such that $[\omega^i \lambda] = [\lambda]$. Then $[\omega^i \lambda] = [\lambda]$ for some $1 \leq \iota \leq n - 1$, then $[\lambda\lambda] \ll \sum_{1 \leq \iota \leq n - 1} [\omega^j\lambda]$ and by [21] and our assumption that $\lambda$ is maximal it follows that $[\lambda\lambda] = \sum_{1 \leq \iota \leq n - 1} [\omega^j\lambda]$. By Lemma 2.30 and Lemma 2.33 this is only possible if $k = n = 2$. The corollary now follows from Prop. 5.20.

Lemma 5.22. Assume that $Z_{\mu}^{c_1} = \delta_{1\mu}, \forall \mu$. Then $\langle c_1c_2, c_1c_2 \rangle = \langle c_1c_2, c_1c_2 \rangle$.

Proof By §2 of [14] we have $Z_{\mu_1, \mu_2}^{c_1} = \delta_{\mu_1, \tau(\mu_2)}$ where $\mu \rightarrow \tau(\mu)$ is an order two automorphism of fusion algebra. It follows that $[\bar{a}_\mu] = [a_{\tau(\mu)}]$, and by [8] irreducible sectors of $\bar{c}_1\nu c_1$ are of the form $a_{\mu}, \forall \mu$. Since

$$\langle c_2c_2, a_\mu \rangle = \langle c_2, a_\mu c_2 \rangle = \langle c_2, c_2 \bar{c}_2 \rangle = \langle c_1, c_1 c_2 \rangle = \langle c_1, c_1 c_2 \rangle = \langle c_1, c_1 c_2 \rangle$$

we conclude that $[c_2c_2] = [a_{c_2c_2}]$, and

$$\langle c_1c_1, \bar{c}_2c_2 \rangle = \langle c_1, \bar{c}_2c_2c_1 \rangle = \langle c_1, c_1 a_{c_2c_2} \rangle = \langle c_1, c_1 c_2c_2 \rangle = \langle c_1, c_1 c_2c_2 \rangle$$

Corollary 5.23. Suppose that $k \neq n - 2, n + 2, n$, then $\lambda$ is maximal iff there is no $1 \leq \iota \leq n - 1$ such that $[\omega^i \lambda] = [\lambda]$.

Proof When $k = 1$ the Cor. is obvious. By Lemma 2.33 we can assume that $k \geq 2$ and $d_{v\alpha} > 1$. As in the proof of Cor. 5.21 $\lambda$ is maximal implies that there is no $1 \leq \iota \leq n - 1$ such that $[\omega^i \lambda] = [\lambda]$. Now suppose that there is no $1 \leq \iota \leq n - 1$ such that $[\omega^i \lambda] = [\lambda]$. If $S_{v\alpha} \neq 0$, then $\lambda$ is maximal by Cor. 5.20. If $k = 2$, the $S$ matrix elements are equal to that of $S$ matrix elements for $SU(2)_n$ up to phase factors, and it follows easily that $S_{v\alpha} = 0$ if there is no $1 \leq \iota \leq n - 1$ such that $[\omega^i \lambda] = [\lambda]$.

Suppose that $k \geq 3, S_{v\alpha} = 0$. Since $[v v] = [1] + [v_0]$ we have $S_{v\alpha} = 0$.

Assume that $M_1$ is an intermediate subfactor between $\lambda(M)$ and $M$, and $\lambda = c_1c_2$ with $c_1(M) = M_1$ and $c_1 = c_1c_2$ as in Prop. 2.24. Apply Lemma 2.20 we have $l(a_{v_0}^c, a_{v_0}^c) \geq 1$. By Lemma 2.33 we must have $[a_{v_0}^c] = [a_{v_0}^c]$ and by Lemma 2.30 $[c_1c_2^j] = \sum_{1 \leq j \leq n/j1} [\omega^j\lambda]$. By Frobenius reciprocity we have $[\omega^j\lambda] = [c_1]$. Since $\lambda = c_1c_2$, $[\omega^j\lambda] = [\lambda]$, and by assumption $j_1 = n$ and $[c_1c_2^j] = [1]$. By Prop. 2.24 we must have $Z_{\mu_1}^{c_1} = \delta_{\mu_1}, \forall \mu$. By §2 of [14] we have $Z_{\mu_1, \mu_2}^{c_1} = \delta_{\mu_1, \tau(\mu_2)}$ where $\tau(\mu) = \omega^{m\text{col}(\mu)}\mu$ or $\tau(\mu) = \omega^{m\text{col}(\mu)}\mu, m \geq 0$. We claim that in fact $[\omega^m] = [1]$ and $\tau(\mu) = \mu$. First we
show that $\tau(\mu) = \omega^{m \col(\mu) \mu}$. If instead $\tau(\mu) = \omega^{m \col(\mu) \bar{\mu}}$, since $k \geq 3$, $\tau((0, 1, 0, \ldots, 0)) \neq (0, 1, 0, \ldots, 0)$, by Lemma 2.20 we must have $S_{\lambda(0,1,0,\ldots,0)} = 0$. From the fusion rule

$$[(0, 1, 0, \ldots, 0)(0, 0, \ldots, 0, 2)] = [(0, 1, 0, \ldots, 0, 2)] + [v_0]$$

we must have $S_{\lambda(0,1,0,\ldots,0,2)} \neq 0$. By Lemma 2.20 we must have $\tau((0, 1, 0, \ldots, 0, 2)) = (0, 1, 0, \ldots, 0, 2) = (2, 0, 0, \ldots, 1, 0)$, a contradiction. So we conclude that $\tau(\mu) = \omega^{m \col(\mu) \mu}, \forall \mu$. It follows that $[\tilde{a}_\mu] = [a_{\omega^{m \col(\mu) \mu}}], a_{\omega^{m \mu}}$, and in particular $[\tilde{a}_v] = [a_{\omega^m a_v}]$. So we have

$$[\omega^m v_c] = [c_1 \tilde{a}_v] = [c_1 a_v] = [v c_1],$$

and similarly $[c_2 \omega^{-m} \bar{v}] = [c_2 \bar{v}]$. If $[\omega^m] \neq [1]$, by our assumption on $\lambda$ we have $\omega^m \neq c_1 \bar{c}_1, \omega^m \neq \bar{c}_2 c_2$. On the other hand we have

$$\langle \bar{v} \omega^m v, c_1 \bar{c}_1 \rangle \geq 1, \langle \bar{v} \omega^m v, \bar{c}_2 c_2 \rangle \geq 1$$

It follows that $\omega^m v_0 \prec c_1 \bar{c}_1, \omega^m v_0 \prec \bar{c}_2 c_2$, and $\langle c_1 \bar{c}_1, \bar{c}_2 c_2 \rangle \geq 2$. By Lemma 5.22 we conclude that $\lambda = c_1 c_2$ is not irreducible, contradicting our assumption. Hence $[\omega^m] = [1]$ and $Z_{\mu_1 \mu_2} = \delta_{\mu_1 \mu_2}$. The rest of the proof now follows in exactly the same way as in the proof of Prop. 5.20. \hfill \blacksquare

**Example 5.24.** When $n = 2$ we have Jones subfactors and their reduced subfactors. In the case $k = n = 2$ there are three irreducible subfactors and they are maximal. Let $n = 2, k \neq 2$. Then $\lambda$ can be labeled by an integer $1 \leq i \leq k$. Cor. 5.23 implies that $i$ is maximal iff $i \neq k/2$ (When $k = 4$ this can be easily checked directly). This can also be proved directly using the same argument at the end of section 2.6.

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