ACTIONS OF HIGHER RANK, IRREDUCIBLE LATTICES ON CAT(0) CUBICAL COMPLEXES

T. TÂM NGUYỄN PHAN

ABSTRACT. Let \( \Gamma \) be an irreducible lattice of \( \mathbb{Q} \)-rank \( \geq 2 \) in a semisimple Lie group of noncompact type. We prove that any action of \( \Gamma \) on a CAT(0) cubical complex has a global fixed point.

1. Introduction

Let \( \Gamma \) be an irreducible lattice of \( \mathbb{Q} \)-rank \( r \geq 2 \) in semisimple Lie groups of noncompact type. It is known that \( \Gamma \) has property (FA), that is, any action of \( \Gamma \) on a tree has a global fixed point. Property (FA) is generalized by Farb \([3]\), who proved that any action of \( \Gamma \) on a \((r - 1)\)-dimensional CAT(0) complex has a global fixed point.

The main theorem of this paper is a generalization of the fact that higher rank lattices \( \Gamma \) have property (FA) in the sense that any action of \( \Gamma \) on a CAT(0) cubical complex (of any dimension) has a global fixed point. This statement has been proved for groups with property (T) \([5]\), which include higher rank, irreducible lattices \( \Gamma \) in semisimple Lie groups \( G \) (of noncompact type) that does not have a \( \text{SO}(n,1) \) or \( \text{SU}(n,1) \) factor.

**Theorem 1** (Main Theorem). Let \( \Gamma \) be an irreducible lattice of \( \mathbb{Q} \)-rank \( \geq 2 \) in semisimple Lie groups of noncompact type. Let \( \Sigma \) be a CAT(0) cubical complex. Suppose that \( \Gamma \) acts on \( \Sigma \) by isometries preserving the cubulation of \( \Sigma \). Then \( \Gamma \) has a global fixed point in \( \Sigma \).

**Remark.** We make the following remarks.

1) The fixed point of \( \Gamma \) does not have to be a vertex of \( \Sigma \), but it is a vertex of the barycentric subdivision of \( \Sigma \).

2) The argument also applies to action without inversions of \( \Gamma \) on CAT(0) polyhedral complexes whose cells are Coxeter polytopes.

This note is a special case of part of a proof of a lemma in the author’s paper on piecewise locally symmetric manifolds \([4]\). Grigori Avramidi has been insisting over more than a year that this part should be written up as a theorem on group actions by higher rank, irreducible lattices on CAT(0) cubical complexes. This paper is dedicated to him for his cube\textsuperscript{cube} birthday.

2. Proof of the main theorem

By passing to the barycentric subdivision of \( \Sigma \) we will assume that action of \( \Gamma \) is without inversion (that is, if an element of \( \Gamma \) preserves a cell of \( \Sigma \), then it fixes that cell pointwise). Also, isometries of \( \Sigma \) are semisimple since the translation distance of an isometry is discrete.

2.1. **Useful theorems on lattices and groups acting semisimply on CAT(0) spaces.** The following theorem on generation of higher rank lattices by nilpotent subgroups is due Farb \([3\text{ Proposition 4.1}]\) and was proved in a more general setting with \( \mathbb{Q} \) replaced by and algebraic number field \( k \).

**Theorem 2** \([3]\). Let \( \Gamma \) be an irreducible lattice of \( \mathbb{Q} \)-rank \( r \geq 2 \), and let \( \Gamma \) act on a CAT(0) space \( Y \). Then exists a collection of subgroups \( \mathcal{C} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_{r+1}\} \) such that
The groups in $C$ generate a finite index subgroup of $\Gamma$.
(2) Any proper subset of $C$ generates a nilpotent subgroup $U$ of $\Gamma$.
(3) There exists $m \in \mathbb{Z}^+$ so that for each $\Gamma_i \in C$, there is a nilpotent group $N < C$ so that $r^m \in [N, N]$ for all $r \in \Gamma_i$.

Since we are dealing with group action on CAT(0) cubical complexes, the following theorem (Proposition 2.3) will prove to be useful.

**Theorem 3.** Let $N$ be a finitely generated, torsion-free, nilpotent group acting on a CAT(0) space $Y$ by semi-simple isometries. Then either $N$ has a fixed point or there is an $N$-invariant flat $L$ on which $N$ acts by translations and hence, factoring through an abelian group.

Given part (3) of Theorem 2 and Theorem 3, the following corollary (Corollary 2.4) of Theorem 2 will prove to be useful.

**Corollary 4.** Let $N$ be a finitely generated, torsion-free, nilpotent group acting on a CAT(0) space $Y$ by semi-simple isometries. Then

1. If $g^m \in [N, N]$ for some $m > 0$, then $g$ has a fixed point.
2. If $N$ is generated by elements each of which has a common fixed point, then $N$ has a global fixed point.

**Remark.** As pointed out in [3], it follows from Corollary 4 above that for each such $U$ in Theorem 2, the set Fix$(U)$ in $Y$ is nonempty.

### 2.2. Proof of the main theorem.

By Theorem 2, the group $\Gamma$ is virtually generated by a collection $C$ of nilpotent subgroups $N_1, N_2, ..., N_{r+1}$. In this proof we only need $\Gamma$ to be virtually generated by 3 nilpotent groups that satisfies the conclusion of Theorem 2. So we write $C = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, for $\Gamma_1 = N_1$, $\Gamma_2 = N_2$, and $\Gamma_3$ is the (nilpotent) group generated by $N_3, N_4, ..., N_{r+1}$. Observe that $\Gamma_i$’s satisfy the first two conclusions of Theorem 2 and the group generated by any two groups $\Gamma_i$ and $\Gamma_j$ fixes a nonempty set by the above remark.

For each $i = 1, 2, 3$, let $F_i$ be the set of points that is fixed by all elements of $\Gamma_i$, and we write $F_i = \text{Fix}(\Gamma_i)$. Let

$$W_{ij} = F_i \cap F_j,$$

for $i, j = 1, 2, 3$. By the remark after Theorem 2, each $W_{ij}$ is nonempty. It is clear that $F_i$’s and thus $W_{ij}$’s are convex. We want to show that $\cap_{i=1,2,3} F_i \neq \emptyset$. Suppose the contrary, that

$$\cap_{i=1,2,3} F_i \neq \emptyset.$$

Let $x_1 \in W_{i3}$, for $i = 1, 2$, be such that the distance $d(x_1, x_2) = d(W_{i3}, W_{23})$. Let $y \in W_{12}$. The geodesic $\gamma_{yx_1}$ (and $\gamma_{yx_2}$, respectively) lies in $F_1$ (and $F_2$, respectively) since $F_1$ and $F_2$ are convex. Without loss of generality, suppose that the angle (see [1]) for the definition of angle)

$$\angle_{x_1}(y, x_2) < \pi/2,$$

**Claim:** there is a simplex $S$ in $W_{i3}$ containing $x_1$ and the angle

$$\angle_{x_1}(S, x_1 x_2) := \min_{s \in S, s \neq x_1} \angle_{x_1}(s, x_1 x_2) < \pi/2.$$

Given the claim, it follows that $S$ contains some point $x'_1$ other than $x_1$ such that $d(x'_1, x_2) < d(x_1, x_2)$, which is a contradiction to the choice of $x_1$ and $x_2$. Therefore,

$$\cap_{i=1,2,3} F_i \neq \emptyset.$$

Therefore $\Gamma$ has a finite index group $\Gamma'$ that fixes a nonempty subset of $\Sigma$. Let $H$ be a finite index subgroup of $\Gamma'$ that is normal in $\Gamma$. Then the set Fix$(H)$ is nonempty. Since $\Gamma$ acts on Fix$(H)$ with bounded orbit, it follows that $\Gamma$ has a global fixed point. We are left to prove the claim.
Before proving the claim, we need the following definitions. A spherical simplex is *all right* if each of its edge lengths is $\pi/2$. A piecewise spherical complex $Z$ is said to be *all right* if all its simplices are all right.

**Proof of Claim.** Let $Q$ be the cell of $\Sigma$ whose interior contains $x_1$. (If $x_1$ is a vertex of $\Sigma$, then $Q = \{x_1\}$.) We consider the following cases.

**Case 1:** If $Q$ is a point, then $Q = \{x_1\}$. Then the link of $\text{Lk}(x_1, \Sigma)$ is a CAT(1) piecewise spherical, all right complex (by [2, Lemma I.5.10]). Apply Lemma 6 below to $\text{Lk}(x_1, \Sigma)$, we get a simplex $S \subset W_{13}$ containing $x_1$ such that the angle $\angle_{x_1}(x_2, S) < \pi/2$. The details of this argument is explain in the next paragraph.

Let $A$ (respectively, $B$) be the cell in $\Sigma$ that intersects nontrivially with $x_1y$ (respectively, $x_1x_2$). Let $A' = \text{Lk}(x_1, A)$ and $B' = \text{Lk}(x_1, B)$. Let $u = L \cap x_1y$ and $v = L \cap x_1x_2$. Then $d(u, v) < \pi/2$ in $L$. Let $U$ (respectively, $V$) be a face of $A'$ (respectively, $B'$) whose interior contains $u$ (respectively, $v$). By Lemma 6 there is a face $C \subset U \cap V$ such that $d_L(C, v) < \pi/2$. Let $\alpha$ (respectively, $\beta$) be the convex hull of $x_1$ and $U$ (respectively, $V$). Then $\alpha \subset F_1$ and $\beta \subset F_3$. Let $S$ be the convex hull of $x_1$ and $C$. Therefore, the edge $S \subset W_{13}$ and has angle $< \pi/2$ with $x_1x_2$.

**Case 2:** If $Q$ is not a point, then $L := \text{Lk}(x_1, \Sigma)$ is the spherical join $\text{Lk}(x_1, Q) \ast \text{Lk}(Q, \Sigma)$.

2a) If $\angle_{x_1}(x_1x_2, Q) < \pi/2$. Then pick $q \in Q$ such that $\angle_{x_1}(x_1x_2, x_1q) < \pi/2$. Since $Q \subset W_{13}$ (because $x_1$ is in the interior of $Q$ and $x_1 \in W_{13}$). We can let $S$ be the edge $x_1q \subset W_{13}$.

2b) Suppose that $\angle_{x_1}(x_1x_2, Q) \geq \pi/2$ (in which case we have equality). Then $x_1x_2$ intersects nontrivially with $\text{Lk}(Q, \Sigma)$ at $v$. 

**Figure 1.** The Fix sets $F_1$, $F_2$ and $F_3$ and their pairwise intersections.
If \(x_1 y\) also intersects nontrivially with \(\text{Lk}(Q, \overline{T})\) at some point \(u\), then argue as in the case \(Q\) is a point using the fact that \(\text{Lk}(Q, \Sigma)\) is \(\text{CAT}(1)\) piecewise spherical, all right complex ([2] Lemma I.5.10) and applying Lemma [6].

Suppose that \(x_1 y\) does not intersect nontrivially with \(\text{Lk}(Q, \Sigma)\). Then there is a point \(q \in Q\) such that \(x_1 y\) intersects with \(H := q \ast \text{Lk}(Q, \Sigma)\) (the spherical joint of \(q\) and \(\text{Lk}(Q, \Sigma)\)) nontrivially. Let \(u = x_1 y \cap H\) and let \(v = x_1 x_2 \cap H\). Then \(d_H(q, v) = \pi/2\) and \(d_H(u, v) < \pi/2\). Therefore, the angle \(\angle_q(u, v) < \pi/2\) by spherical geometry. By applying Lemma [6] to the link \(\text{Lk}(q, H) = \text{Lk}(Q, \Sigma)\), we deduce that there is \(C \subset \text{Lk}(Q, \Sigma)\) such that the convex hull \(S\) of \(Q\) and \(C\) is contained in \(W_{13}\) and \(d_L(C, v) < \pi/2\). Thus \(\angle_{x_1}(x_2, S) < \pi/2\).

2.3. Proof of Lemma [6] First we prove the following lemma.

Lemma 5. Let \(L\) be a \(\text{CAT}(1)\), piecewise spherical, all right complex. Suppose that \(d(x, y) < \pi/2\) for some \(x, y \in L\). Let \(X, Y\) be highest dimensional simplices of \(L\) that contains \(x\) and \(y\) respectively. Then \(X \cap Y \neq \emptyset\), and there is a point \(p \in X \cap Y\) such that \(\angle_p(x, y) < \pi/2\).

Proof. Suppose that \(X \cap Y = \emptyset\). Let \(\gamma\) be a unit speed geodesic connecting \(x\) and \(y\) so that \(\gamma(0) = x\) and \(y \in \gamma([0, \pi/2])\). Let \(a < 0\) and \(b > 0\) so that \(\gamma([a, b])\) is a connected component of the intersection of \(\gamma\) with \(X\). Let \(v\) be a vertex of \(X\), and let \(A_v\) be the union of all cells containing \(v\). Then \(X \subset A_v\). We pick \(v\) so that \(d(v, \gamma(a)) > d(v, \gamma(b))\).

Let \(B_v\) be the closed \(\pi/2\)-ball centered at \(v\). Then \(B_v\) is isometric to the spherical cone \(\text{Cone}(\text{Lk}(v, L))\) on the link of \(v\) in \(Z\). If \(\gamma\) passes through \(v\), then \(d(v, y) < \pi/2\). Hence \(v \in X \cap Y\), and thus \(v \in X \cap Y\), which contradicts the above assumption. So \(\gamma\) does not pass through \(v\).

For each \(t\) such that \(\gamma(t) \in A_v\), let \(s_t\) be the geodesic segment connecting \(v\) passing through \(\gamma(t)\) with length \(\max(d(v, \gamma(t)), \pi/2))\). Let \(S\) be the surface defined as the union of all such \(s_t\). In the same way as in [2] Proof of Lemma I.6.4, the surface \(S\) is a union of triangles with common vertex \(v\) glued together in succession along \(\gamma\). Thus we can develop an \(S\) along \(\gamma\) locally isometric to \(S^2\), i.e. there is a map \(f: S \rightarrow S^2\) that is a local isometry such that \(f(v)\) is the North pole. Hence, \(f(\gamma)\) is a geodesic in \(S^2\) that misses \(f(v)\). Also, \(f(S)\) contains the Northern hemisphere of \(S^2\).

The image \(f(\gamma)\) cuts inside the region \(f(S)\), which contains the Northern hemisphere \(N\) of \(S^2\), so \(f(\gamma)\) has length \(\geq \pi/2\). Let \(d\) be such that \(d(\gamma(t))\) is where \(f(\gamma)\) exits \(f(S)\). It is not hard to see that \(\pi/2 > d > b > 0\). Since \(d(v, \gamma(a)) > d(v, \gamma(b))\), it follows that \(d(f(v), f(\gamma(a))) > d(f(v), f(\gamma(b)))\). Hence \(d(f(\gamma(0)), f(\gamma(d))) > \pi/2\) be spherical geometry, which is a contradiction since \(\gamma\) has unit speed. Therefore, \(X \cap Y \neq \emptyset\). We can pick the point \(p\) to be \(v\). That \(\angle_p(x, y) < \pi/2\) also follows from spherical geometry. 

Lemma 6. Let \(L\) be a \(\text{CAT}(1)\), piecewise spherical, all right complex. Let \(X, Y\) be cells of \(L\), and let \(x \in X, y \in Y\). Let \(X'\) (respectively, \(Y'\)) be a face of \(X\) (respectively, \(y\)) whose interior contains \(x\) (respectively, \(y\)). If \(d(x, y) < \pi/2\), then \(X' \cap Y'\) contains face \(C\) such that \(d(x, C) < \pi/2\).

Proof. We induct on the dimension of \(L\). The base case is when \(L\) has dimension 1, in which case the lemma is obvious. Suppose that \(L\) has dimension \(> 1\). By Lemma [5] there is a point \(p \in X \cap Y\) such that \(\angle_p(x, y) < \pi/2\). Let \(u\) (respectively, \(v\)) be the intersection of the geodesic ray \(px\) (respectively, \(py\)) with the link \(\text{Lk}(p, L)\). Since \(\angle_p(x, y) < \pi/2\), we have \(d_{\text{Lk}(p, L)}(u, v) < \pi/2\). Note that \(\text{Lk}(p, L)\) is also a \(\text{CAT}(1)\) piecewise spherical, all right complex ([2] Lemma I.5.10).

Let \(U'\) (respectively, \(V'\)) be a face in \(\text{Lk}(p, L)\) whose interior contains \(u\) (respectively, \(v\)). Apply the induction hypothesis, there is a point \(q \in \text{Lk}(p, L)\) such that \(U' \cap V'\) contains face \(D\) such that \(d(u, D) < \pi/2\). Let \(C\) be the face in \(L\) that is spanned by \(p\) and \(D\). Then \(C\) satisfies the condition in the lemma. 

□
Acknowledgement. I would like to thank Grigori Avramidi and Benson Farb for useful comments on earlier versions of this paper.

References
1. Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.
2. Michael W. Davis, The geometry and topology of Coxeter groups, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
3. Benson Farb, Group actions and Helly’s theorem, Adv. Math. 222 (2009), no. 5, 1574–1588.
4. T. T. Nguyen Phan, Gluing locally symmetric manifolds, asphericity and rigidity (preprint).
5. Graham Niblo and Lawrence Reeves, Groups acting on CAT(0) cube complexes, Geom. Topol. 1 (1997), approx. 7 pp. (electronic). MR 1432323 (98d:57005)

Department of Mathematics, 5734 S. University Ave., Chicago, IL 60637
E-mail address: ttamp@math.uchicago.edu