The method of calculating the free energy and thermodynamic characteristics of the classical $n$-vector three-dimensional (3D) magnetic model at the microscopic level without any adjustable parameters is proposed. Mathematical description is performed using the collective variables (CV) method in the framework of the $\rho^4$ model approximation. The exponentially decreasing function of the distance between the particles situated at the $N$ sites of a simple cubic lattice is used as the interaction potential. Explicit and rigorous analytical expressions for entropy, internal energy, specific heat near the phase transition point as functions of the temperature are obtained. The dependence of the amplitudes of the thermodynamic characteristics of the system for $T > T_c$ and $T < T_c$ on the microscopic parameters of the interaction potential are studied for the cases $n = 1, 2, 3$ and $n \to \infty$.

I. INTRODUCTION.

Investigating the behaviour of the real three-dimensional systems near the phase transition (PT) point is one of the most important problems of condensed matter physics. In the present paper we propose a theoretical description scheme of the critical behaviour of the classical $n$-vector magnetic model in three dimensions on the microscopic level without any adjustable parameters. The description is based on the original method of calculating the thermodynamic and structural characteristics of 3D model systems near the PT point, which is known as the collective variables (CV) method [1], [2]. Within the frame of an uniform scheme this method allowed to obtain universal characteristics of the system such as critical exponents and critical amplitude ratios for thermodynamic functions above and below the critical point as well as non-universal ones. Starting from the initial principles we calculate the critical temperature and obtain explicit analytical expressions for basic thermodynamic characteristics of the system as functions of the temperature and the $n$ component number of the model. For the first time the dependence of the thermodynamic characteristics of the system on the microscopic parameters of the initial interaction potential is investigated. The investigation of the non-universal properties of the system near the PT point require the use of non-Gaussian distributions of the spin density fluctuations. The suggested calculation of the partition function is based on the use of the simplest non-Gaussian measure density, i.e. the $\rho^4$ model [3].

We have devoted this paper to the widely studied classical $n$-vector magnetic model [4] on three-dimensional simple cubic lattice, which is also known as the Heisenberg classical $O(n)$ spin model or, in field theoretic language, as the lattice $O(n)$ nonlinear $\sigma$ model. The investigation of the critical behaviour of the classical $n$-vector model and its partial cases was conducted in the frame of different methods such as high- and low- temperature series, the field theory, semi-microscopic scaling-field theory and Monte-Carlo calculations. The main attention in these works was given to the investigation of the universal properties of the system such as critical exponents [5]- [12] and some combinations of the critical amplitudes of thermodynamic functions [13]- [20]. Besides, the equation of state of Ising system was obtained to order $\epsilon^2$ by Avdejeva and Migdal [27], Bresin, Wallace and Wilson [28]. Later these results were generalized for the case of the $n$-vector model [28].

Some important results were also obtained for calculating the thermodynamic functions near the critical point. One of the first works in this sphere is a paper by Wegner (the so-called Wegner’s expansion) [29], suggesting the expression for the free energy with “irrelevant” operators in Wilson approach taken into account. The works by Fisher and Aharony [30], Nicoll and Albright [31], Nelson [32] were also devoted to receiving of crossover scaling functions for temperatures $T > T_c$ in zero external field near four dimensions. In 1974, Riedel and Wegner [33] developed a numerical technique, termed the scaling-field method, for obtaining crossover scaling functions for the free energy and susceptibility. Crossover functions rather than power laws [34]- [36] have described the nonasymptotic region between criticality
and the noncritical "background". In the frame of the massive field theory by Bagnuls and Bervillier \[1, 3\] the nonasymptotic critical behaviour for \(d = 3\) in the disordered phase case was analysed. They obtained explicit expressions for the correlation length \(\xi\), the susceptibility \(\chi\) and the specific heat \(C\) as functions of temperature in the disordered phase along the critical isohere for one-, two-, three-component models. The description of nonasymptotical (though still critical) behaviour was obtained as crossover between the Wilson-Fisher (near the critical temperature \(T_c\)) and mean-field (very far from \(T_c\)) behaviours with three adjustable parameters used.

At the present time the actual task of the critical phenomena physics is elaborating the methods giving quantity description of the critical behaviour of the system without using any adjustable parameters. This has been demonstrated, for example, in the works of Dohm and coworkers \[8\] in which calculation of the temperature dependence of the thermodynamic characteristics of the system was performed without using \(\epsilon\) - expansion in the frame of the minimal subtraction scheme for the \(n\)-vector model in three dimensions on the basis of high-order perturbation theory and Borel resummation. In these papers the amplitude functions of the susceptibility, the correlation length, and the specific heat above and below \(T_c\) up to two-loop order within the \(\phi^4\) model for the cases \(n = 1, 2, 3\) were calculated. This approach exploits simultaneously the experimental information and simple relation between the specific heat above and below \(T_c\) for defining the effective renormalized static coupling of the model in the terms of the measured specific heat. The effective renormalized static coupling determined in such a way allows to obtain expressions for other thermodynamic characteristics of the model above and below \(T_c\) without using additional adjustable parameters. Besides, in the recent works by Butera and Comi \[41\] high- and low-temperature expansions for the free energy, the susceptibility and the second correlation moment of the classical \(n\)-vector model on the simple cubic (sc) and the body centered cubic (bcc) lattices were extended to order \(\beta^2\). This research only contains temperature dependence of the thermodynamic characteristics and does not give the possibility to describe the functional dependence of basic thermodynamic functions on the microscopic parameters of the interaction potential and characteristics of the crystal lattice. All this indicates that nonuniversal properties of the 3D system near the phase transition point have not been studied sufficiently yet. However, the precise role and significance of a lattice structure and interaction potential parameters for the approach to asymptotic critical behaviour still seems open to question. We hope that our explicit representations may provide useful benchmarks in studying this question.

The approach to the investigation of the critical properties of the \(n\)-vector magnetic model, suggested in \[4\] provides the necessary conditions for our complex approach to the study of the universal and non-universal phase transition characteristics. The approach suggested in this paper allows us to perform the analysis of the dependence of the thermodynamic characteristics of the \(n\)-vector 3D magnetic model in the vicinity of the phase transition point as functions of temperature and study their dependence on the microscopic parameters of the interaction potential and characteristics of the crystal lattice without any adjustable parameters being used. These results are interesting from the point of view of comparing the theoretical investigations and experimental data.

In Sec.II of the present paper we perform the calculation of the partition function of the \(n\)-vector model using non-Gaussian measure density. The explicit analytical expressions for partial partition functions and general recursion relations (RR) between coefficients of the "effective Hamiltonian blocks" which arise in that case are obtained in the \(\rho^4\)-model approximation. We define the scope of application of these approximate solutions and show that RR as partial solution have a saddle-type fixed point for all \(n\). We perform the calculation of the eigenvectors and eigenvalues of the RG transformation matrix and give the results of the investigation of the dependence of the PT temperature on the microscopic parameters of the interaction potential and the characteristics of the crystal lattice.

Sec.III is devoted to the calculation of the free energy of the \(n\)-vector magnetic model for temperatures above and below the phase transition point. The main idea of such calculation lies in considering separately the contribution from the critical region (CR) where renormalization group symmetry takes place and the region of the long-wavelength fluctuations (LWF) of the spin moment density. It shown that in the case of the temperatures \(T < T_c\) in the region of the LWF the fluctuations are described by non-Gaussian distribution with negative coefficient at square term. The distributions of the spin moment density fluctuations after the selection of the ordering free energy is reduced to the Gaussian distribution. The dependence of the coefficients of the complete expression of the free energy on the microscopic parameters of the initial interaction potential and characteristics of the crystal lattice is investigated for the cases \(n = 1, 2, 3\). The contributions into the expressions for entropy and specific heat from the CR and LWF region are analyzed. It has been shown that considering the contribution of the LWF region satisfies the positiveness of the specific heat of the \(n\) - vector model and system stability.
In Sec.IV. the explicit expressions for thermodynamic functions of the model as functions of the temperature are obtained. The dependence of the critical amplitudes of the thermodynamic functions on the microscopic parameters of the interaction potential and the characteristics of the crystall lattice is investigated for the cases \( n = 1, 2, 3 \). It has been shown that in the case \( n = 3 \) (the Heisenberg model) the specific heat at the \( T = T_c \) has the finite value. The dependence of the maximum of the specific heat on the microscopic parameters of the interaction potential is examined for the case \( n = 3 \). The ratio of the critical amplitudes of the thermodynamic functions at the temperatures \( T > T_c \) and \( T < T_c \) is calculated in order to compare the obtained results with the results of the other methods.

II. CALCULATION OF THE PARTITION FUNCTION AND INVESTIGATION OF THE RECURSION RELATIONS.

The critical behaviour of the different physical systems is characterized by their belonging to a specific class of universality which is defined by the dimensionality of the system \( d \) and the symmetry of the order parameter \( n \). The Stanley model \([4]\) is selected as the object of our investigation of the critical phenomena. This model describes the system of interacting \( n \)-component classical spins localized at the \( N \) sites of the \( d \)-dimensional crystal lattice. The Stanley model is a generalization of the series of different models. In the case \( n = 0 \) it is reduced to the task of a self-avoiding walk and is used for describing the polymerization phenomena. The cases \( n = 1, 2, 3 \) correspond to the Ising model, the \( XY \)-model and the Heisenberg model respectively. The boundary case \( n \to \infty \) is equivalent to the Berlin-Kac spherical model \([43]\) for which the exact result is known. The Stanley model is described by the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \sum_{\mathbf{R} \neq \mathbf{R}'} J \left( |\mathbf{R} - \mathbf{R}'| \right) \hat{S}_\mathbf{R} \hat{S}_{\mathbf{R}'},
\]

(2.1)

where \( \hat{S}_\mathbf{R} = \left( \hat{S}_\mathbf{R}^{(1)}, ..., \hat{S}_\mathbf{R}^{(n)} \right) \) is the \( n \)-component classical spin with the length \( m \left( \sum_{\alpha=1}^{n} |\hat{S}_\mathbf{R}^{(\alpha)}|^2 = m^2 \right) \).

Spins are localized at the \( N \) cites of the \( d \) dimensional simple cubic lattice with coordinates \( \mathbf{R} \). The interaction has an exchange character and can be described by the exponentially decreasing function of distance between the particles.

\[
J \left( |\mathbf{R} - \mathbf{R}'| \right) = A_0 \exp \left( -\frac{|\mathbf{R} - \mathbf{R}'|}{b} \right),
\]

(2.2)

where \( A_0, b \) are the constant. In the CV \( \hat{\rho}_k = \left( \rho_k^{(1)}, ..., \rho_k^{(n)} \right) \) representation the partition function of the model (2.1) is written as [1], [42]

\[
Z = \int \exp \left[ \frac{1}{\beta} \sum_k \beta \Phi(k) \hat{\rho}_k \hat{\rho}_{-k} \right] J[\hat{\rho}] (d\hat{\rho}_k)^N,
\]

(2.3)

where \( \Phi(k) \) is the Fourier-transform of the interaction potential (2.2), the element of the phase space is

\[
(d\hat{\rho}_k)^N = \prod_{a=1}^{n} d\rho_k^{(a)} \prod_{b,c} d\rho_k^{(a,b,c)} d\rho_k^{(a,b,c)},
\]

(2.4)

The prime means that the product over \( k \) is maintained in the upper half-cube of the Brillouin zone and \( k \neq 0 \). The wave vector \( k \) assumes all the values inside the first Brillouin zone

\[
k_i = 2\pi n_i / N^c, \quad i = 1, ..., d,
\]

where \( n_i \) are integers,

\[
-\frac{N^c}{2} \leq n_i < \frac{N^c}{2}, \quad (N^c)^d = N,
\]

and \( N \) is the number of the particles in the periodicity volume \( V = N^d c^d \), \( c \) is a constant of the simple cubic lattice.
and \( J[\rho] \) is the transition Jacobian from the spin variables to the CV. The expression for \( J[\rho] \) is given in Appendix 1. In the coordinate CV representation

\[
\hat{\rho}(R) = \frac{1}{\sqrt{N}} \sum_k \hat{\rho}_k \exp(-i k R)
\]

(2.4)

the expression for \( J[\rho] \) is factored

\[
J[\rho] = \exp(\mu'R') J'[0] \prod_R J[\rho(R)]
\]

(2.5)

where

\[
J[\rho(R)] = \sum_{l \geq 1} a_{2l} (2l)! |\rho(R)|^{2l}.
\]

(2.6)

The expressions for \( a_{2l} \) are shown in Appendix 2. The partition function (2.3) contains the contributions of two different types. The first is energy contributions

\[
\frac{1}{2} \sum_{k \geq B} (a_2 - \beta \Phi(k)) \rho_k \rho_{-k}.
\]

(2.7)

They are diagonal in \( \rho_k \) representation and connected with the interaction potential. The second is entropy contributions

\[
\sum_{l=1}^N \ln J(\rho(R)).
\]

(2.8)

They are diagonal in \( \rho(R) \) coordinate CV representation. There are two possible approaches to calculating (2.3). The first one is using \( \rho_k \) variables in the calculation of the partition function of the system. In such a case the energy contributions are diagonal and the nondiagonality of the entropy contributions leads to the approximate method of calculation. One of such approximations is writing (2.3) in the following form

\[
Z = \prod_R \int \exp\left( -\frac{1}{2} \sum_{k \geq B} (a_2 - \beta \Phi(k)) \hat{\rho}_k \hat{\rho}_{-k} \right) \times [1 + \eta + \frac{1}{2} \eta^2 + \ldots] (d\hat{\rho}_k)^N,
\]

(2.9)

where

\[
\eta = \sum_{l \geq 2} \frac{a_{2l}}{(2l)!} N^{1-l} \sum_{k_1} \ldots \sum_{k_{2m}} \hat{\rho}_{k_1} \ldots \hat{\rho}_{k_{2m}} \delta_{k_1 + \ldots + k_{2m}}.
\]

(2.10)

As rule, such a way of calculation assumes that for the value \( \eta \) it is sufficient to restrict the consideration to one term \( (l=2) \) and in addition it is presupposed that

\[
(a_2 - \beta \Phi(k))^l \gg a_{2l}.
\]

(2.11)

The second way of the calculation of the partition function (2.3) assumes the use of the \( \rho(R) \) CV representation. In that case the entropy contributions are diagonal (2.5), (2.6) and approximation relates to the energy contribution (2.7), as it is nondiagonal in the \( \rho(R) \) representation. The essence of such fitting leads to a certain approximation of \( \Phi(k) \) (see. [1]) which allows to diagonalize the expressions in the exponent under the integrand for the partition function

\[
Z = \prod_R \int \exp\left( -\frac{1}{2} (a_2 - \beta \Phi_{apr}) \tilde{\rho}(R)^2 \right) - \sum_{l \geq 2} \frac{a_{2l}}{(2l)!} (\tilde{\rho}(R)^{2l}) d\tilde{\rho}(R).
\]

(2.12)

We use such approximation for \( \Phi_{apr} \) in which \( \Phi_{apr} \) is a constant for every interval \( k \in (B_l, B_{l-1}) \) and equals the respective mean value \( \Phi(k) \). The first way corresponds to considering the moments of a certain
Gaussian distribution. It allows us to calculate only certain classes of the graphs and does not solve the problem of the description of the critical behaviour as such. Besides, that condition (2.11) is too strong in the vicinity of the phase transition point. Nevertheless, such approach has its advantages at the expense of considering the wave vector dependence of $\Phi(k)$. The second way makes it impossible for us to consider the dependence of $\Phi(k)$ on the wave vector $k$ on the boundary of the Brillouin zone.\[2\] Nevertheless, such a way is not restricted to Gaussian moments and is based on the use of the non-Gaussian measure density. In this case we do not need to perform the summation of various types of infinite series of perturbation theory, the certain terms of which tend to infinity with $T \rightarrow T_c$. Besides, the condition (2.11) becomes optional. In our opinion this condition is the basic barrier in the description of the critical behaviour of the three dimensional systems. The calculation of the nonuniversal characteristics of the phase transition particulary the PT temperature $T_c$ is connected with the choice of the interaction potential. The Fourier - transform of the interaction potential (2.2) takes the form

$$\Phi(k) = \Phi(0) \left(1 + b^2 k^2\right)^2, \quad (2.13)$$

where $\Phi(0) = 8\pi A_0 \left(\frac{b}{c}\right)^3$. The value of $\Phi(k)$ for the wave vectors similar to the boundary of the Brillouin half zone ($B = \frac{\pi}{2}$) is much small than $\Phi(0)$. In this region of the wave vectors a weak dependence of $\Phi(k)$ on the wave vector is observed. This allows to accept the following approximation for $\Phi(k)$

$$\Phi(k) = \begin{cases} 
\Phi(0) \left(1 - 2b^2 k^2\right), & k < B' \\
\hat{\Phi} = \text{const}, & B' \leq k < B.
\end{cases} \quad (2.14)$$

The coordinate $B'$ is obtained from the condition of the applicability of the parabolic approximation for $\Phi(k)$ and equals

$$B' = \left(b \sqrt{2}\right)^{-1}. \quad (2.15)$$

Among the set of the CV $\tilde{\rho}_k$ there are those connected with order parameter. In the case of a model with the exponentially decreasing interaction potential (2.2) it is the $\tilde{\rho}_0$ variable. The investigation of the critical behaviour of this model is largely determined by considering the contribution from the $\tilde{\rho}_0$ variable in calculating the free energy. The mean value $\tilde{\rho}_0$ describes the behaviour of the order parameter. Nevertheless, as seen from the (2.3),(A1.1), all the CV $\tilde{\rho}_k$ are interconnected, and the contribution of the variable $\tilde{\rho}_0$ alone cannot be separated in the partition function (2.3). The given task can be accomplished in case we use the method suggested in [1]. Its essence lies in sequential integrating of the $\tilde{\rho}_k$ variables with $k \neq 0$ and the investigating of the functional from the $\tilde{\rho}_0$ variable. In the case of the $n$-vector model such calculations are made for the first time. The functional representation of the partition function of the $n$ - vector magnetic model considering the interaction $\Phi(k)$ at $k$ close to the boundary of the Brillouin zone (2.14) has the form [4]

$$Z = J'[0] \exp(u'_0 N') \int \exp \left[ - \frac{1}{2} \sum_{k < B'} d(k) \tilde{\rho}_k \tilde{\rho}_{-k} - \right.$$

$$-\frac{a_4}{4! N'} \sum_{k_1,\ldots,k_4, k_i < B'} \tilde{\rho}_{k_1} \cdots \tilde{\rho}_{k_4} \delta_{k_1 + \ldots + k_4} \left( d \tilde{\rho}_k \right)^N. \quad (2.16)$$

Part of the interaction potential is contained in the coefficient

$$d(k) = a_2 - \beta \Phi(k).$$

\[2\] The method considering the dependence on the wave vector $k$ in the frame of the non-Gaussian measure density was proposed in the work [1]. It should be noticed that the $\eta$ correlation function critical exponent in the case of Ising model was calculated in these works.
The value \( \ln \omega_{l+1}(\vec{\rho}) \) corresponds to the block Hamiltonian which depends on the \( N_l \) variables \( \vec{\rho}_k \) connected with the fluctuation of the spin moment density in the blocks. The values which are part of the expressions for the partial partition functions are written in the form

\[
Q(d_1) = (2\pi)^{\frac{3}{2}} \left( \frac{3}{a_4^{(n,l)}} \right)^{\frac{1}{2}} U \left( \frac{n-1}{2}, x_1 \right) \exp \left( \frac{x_1^2}{4} \right),
\]

\[
Q(P_{l-1}) = (2\pi)^{-\frac{3}{2}} \left( \frac{n+2}{3} \right) \frac{a_4^{(n,l-1)}}{\varphi_{n+l-1}} \times U \left( \frac{n-1}{2}, y_{l-1} \right) \exp \left( \frac{y_{l-1}^2}{4} \right),
\]

\[
Q^{N'}(u) = J' [0] \exp \left( u_0^{N'} \right).
\]

For coefficients \( a_{2}^{(n,l)} \) and \( a_{4}^{(n,l)} \) the recursion relations (RR) are valid

\[
a_2^{(n,l+1)} = a_2^{(n,l)} + d^n(B_{l+1}, B_l) M_n(x_l),
\]

\[
a_4^{(n,l+1)} = a_4^{(n,l)} s^{-d} E_n(x_l),
\]

where \( d^n(B_{l+1}, B_l) \) corresponds to the mean value in the \( l \) layer. Specific functions introduced here have the following form

\[
E_n(x_l) = s^{2d} \frac{\varphi_n(y_l)}{\varphi_n(x_l)}, \quad M_n(x_l) = N_n(x_l) - 1, \quad N_n(x_l) = \frac{y_l U_n(y_l)}{x_l U_n(x_l)},
\]

where the functions \( \varphi_n(t) \), \( U_n(t) \) are combinations of the parabolic cylinder functions \( U(a,t) \)

\[
\varphi_n(t) = (n + 2)U_n^2(t) + 2tU_n(t) - 2, \quad U_n(t) = \frac{U^{(n+1)}(t)}{U^{(n)}(t)}.
\]
The variables

\[ x_l = \sqrt{\frac{3}{a_4^{(n,l)}}} d^n(B_{l+1}, B_l), \]
\[ y_l = s^{\frac{d}{2}} U_n(x_l) \left( \frac{n + 2}{\varphi_n(x_l)} \right)^{\frac{1}{2}}. \]  

(2.24)

serve as arguments, where \( N_l = N \cdot s^{-d l} \), \( d \) is the dimension of space, \( l \) is the layer of integration. The obtained expression (2.17) for the partition function enables us to calculate the free energy of the system

\[ F = -kT \sum_{l \geq 1} \ln Q_l. \]  

(2.25)

when the explicit analytical expression for the partial partition function \( Q_l \) (2.19), (2.20) are familiar to us. But, the calculation of the (2.25) can only be done if the explicit solutions of the RR (2.22) are obtained as functions of the phase layer number \( l \). Similar investigation of the RR for the \( n \) - vector model was carried out in series of works [42], [45], [46], [47]. The main attention in these papers was given to calculating the correlation function critical exponent and other universal characteristics of the model for the marginal cases \( s \to 1 \) [42], [45], [47] and \( s \gg 1 \) [46]. In this work we briefly present the results of the investigation of the RR difference form (with \( s > 1 \)) [44] which allow us to perform the calculation of the non-universal system characteristics. We introduce the following definitions

\[ r_{1}^{(n)} = s^{2 l} d^n(0), \quad u_{1}^{(n)} = s^{4 l} a_4^{(n,l)}, \]  

(2.26)

with

\[ d^n(B_{l+1}, B_l) = \frac{r_{1}^{(n)} + q}{s^{2 l}}, \]  

(2.27)

where

\[ q = \beta \Phi(0) \bar{q}, \quad \bar{q} = \frac{d}{(d + 2)} \left( \frac{1 - s^{-(d+2)}}{1 - s^{-d}} \right), \]  

(2.28)

\( \bar{q} \) is the geometric mean value of \( k^2 \) in the interval \( (\frac{1}{s}, 1) \). As a result, the obtained RR (2.22) have the following representation

\[ r_{l+1}^{(n)} = s^{2} \left[ -q + (r_{1}^{(n)} + q) N_n(x_l) \right], \]
\[ u_{l+1}^{(n)} = s^{4 - d} u_{1}^{(n)} E_n(x_l). \]  

(2.29)

The equations (2.29) as a partial solution have a fixed point

\[ r_{n}^* = -f_n \beta \Phi(0), \quad u_{n}^* = \varphi_n(\beta \Phi(0))^2, \]  

(2.30)

where

\[ f_n = \frac{s^2 (N_n(x^*) - 1)}{s^2 N_n(x^*) - 1} \bar{q}, \]
\[ \varphi_n = \frac{3}{(x^*)^2} \bar{q}^2 \left\{ \frac{1 - s^{-2}}{N_n(x^*) - s^2} \right\}^2. \]  

(2.31)

The constants \( f_n \) and \( \varphi_n \) depend on the component number \( n \) of the model and the universal value \( x^* \) which is a solution of the nonlinear equation

\[ s^{4+d} \varphi_n(y(x^*)) = \varphi_n(x^*). \]  

(2.32)

The values \( f_n \) and \( \varphi_n \) for different \( n \) and \( s \) are presented in Table I.

The presence of the fixed point in RR (2.22) allows to write them in the linearized form
\[
\begin{pmatrix}
    r_{i+1}^{(n)} - r_{m}^{*} \\
    u_{i+1}^{(n)} - u_{m}^{*}
\end{pmatrix} = \Re \begin{pmatrix}
    r_{i}^{(n)} - r_{m}^{*} \\
    u_{i}^{(n)} - u_{m}^{*}
\end{pmatrix}.
\]

(2.33)

In calculating the matrix elements of \( \Re \) matrix we restrict our consideration to linear on \((x_i - x^*)\) approximation. As a result the following expressions are obtained for the matrix elements \([44]\)

\[
R_{i1} = s^2 \sqrt{3} \mu_1, \quad R_{i2} = \frac{s^2}{2}(\mu_0 - \mu_1 x^*)(u_{m}^{*})^{-1/2},
\]

\[
R_{21} = s^{4-d} \sqrt{3} u_{m}^{*} \omega_1, \quad R_{22} = s^{4-d} \left( \omega_0 - \frac{\omega_1 x^*}{2} \right).
\]

(2.34)

The eigenvalues \(E_1\) and \(E_2\) of the matrix \( \Re \) are universal values \([44], [48], [49]\)

\[
E_{1,2} = \frac{1}{2} \left\{ R_{11} + R_{22} \pm \left[ (R_{11} - R_{22})^2 + 4R_{12}^0 R_{21}^0 \right]^{1/2} \right\}.
\]

(2.35)

As is obvious from Table I, we have a saddle-type fixed point \(E_1 > 1, E_2 < 1\) for all values \(n\) and \(s\). According to \([48], [49]\) the bigger eigenvalue \(E_1\) defines the critical exponent \(\nu\) for the correlation length

\[
\nu = \frac{\ln s}{\ln E_1}.
\]

(2.36)

Since \(E_1\) is a universal value, the critical exponents are also universal and do not depend on the mesoscopic characteristics of the system. As we can see from (2.34) and (2.35), they only depend on the dimensionality of space \(d\) and the \(n\) spin component number. The results of calculating the critical exponents of the model in the framework of this approach was shown in \([50], [51]\). Thus, in the case \(d = 3, n = 3\) the following values \(\nu = 0.674, \alpha = -0.021, \gamma = 1.347\) were obtained. These values of the critical exponents correspond to the \(\rho^0\) model approximation which gives a good qualitative description of the critical behaviour of the \(n\)-vector model. \([\dagger]\) As is known \([3]\), good quantitative results for the critical exponents can be obtained in the framework of the \(\rho^0\) model approximation. For example, the value of the \(\nu\) critical

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\(\dagger\)It should be pointed out that in (2.34) we introduce the following definition

\[
\mu_1 = \mu_0 \left( a_1 - \frac{q_1}{2} \right); \quad \mu_0 = \frac{s^{d/2} \sqrt{n + 2 u_m(y^*)}}{\sqrt{3} \varphi(x^*)},
\]

\[
\omega_0 = \frac{s^{2d} \varphi(x^*)}{\varphi_n(x^*)}, \quad \omega_1 = \omega_0 (b_1 - q_1),
\]

\[
a_1 = \tilde{P}_1 y^* r_1, \quad r_1 = \partial_t - \frac{q_1}{2}, \quad \tilde{P}_m = \frac{1}{U_m(y^*)} \left( \frac{d^n U_n(y)}{d y^n} \right) y^*,
\]

\[
b_1 = \tilde{Q}_1 y^* r_1, \quad \tilde{Q}_m = \frac{1}{\varphi_n(y^*)} \left( \frac{d^n \varphi_n(y)}{d y^n} \right) y^*,
\]

\[
R_{12}^{(0)} = R_{21}^{(0)} \varphi_n(x^*)^{-1/2}, \quad R_{21}^{(0)} = R_{21} (u_{m}^{*})^{-1/2},
\]

---

\(\dagger\)In the marginal case of the larger \(n \to \infty\), the cumulants \(u_{2l}\) strive for their limit values

\[
\lim_{n \to \infty} u_2 = 1, \quad \lim_{n \to \infty} u_4 = \frac{6}{n + 2} \to 0,
\]

\[
\lim_{n \to \infty} u_2l = 0, \quad l = 3, 4, ...
\]

(in normalizing the spin dimensionality \(m^2 = n\)). Considering only the leading by order of the value \(1/n\) members in the process of integration of the partition functions enables us to employ the method of \(g\)-expansion \([42], [45]\). \([17]\) In this case the values

\[
g_{2l} = \frac{u_{2l}}{(2l)!} \left( \frac{d(B_1, B')}{2} \right)^l, \quad \tilde{g} = (g_4, g_6, ...)
\]

are small. It allows obtaining the next relations for the correlation length critical exponent (in the case \(\eta = 0\)
exponent in the $\rho^6$ model approximation increases and practically remains unchanged with the increase of the $m$ order of the $\rho^{2m}$ model. Obtaining the eigenvectors of the transform matrix $R$ from (2.33) is an essential aspect of the investigation of the RR. They can be represented as follows \[1], [44]

\[ w_1 = w_{11} \left( \frac{1}{R_1} \right), \quad w_2 = w_{22} \left( \frac{R}{1} \right), \]  

(2.37)

where

\[ R_1 = \frac{E_{21}}{E_1 - R_{22}} = \frac{E_1 - R_{11}}{R_{12}}, \]
\[ R = \frac{E_{22}}{E_2 - R_{11}} = \frac{E_2 - R_{22}}{R_{21}}. \]

(2.38, 2.39)

The conjugate vectors $v_1$ and $v_2$ are written in the form

\[ v_1 = v_{11} \left( \frac{1}{R_{12}} \right), \quad v_2 = v_{22} \left( \frac{E_2 - R_{22}}{R_{12}} \right). \]

(2.40)

The normalization conditions $w_1v_1 = 1$, $w_2v_2 = 1$ give the relations for obtaining the coefficients $w_{ii}, v_{ii} (i = 1, 2)$.

Proceeding from (2.33), (2.37) the RR (2.29) can be written in the form

\[ r^{(n)}_i = r^*_n + c_1 E^l_1 + c_2 R E^l_2, \]
\[ u^{(n)}_l = u^*_n + c_1 R E^l_1 + c_2 E^l_2, \]

(2.41)

where $c_1 = c'_1 w_{11}; c_2 = c'_2 w_{22}; c'_1, c'_2 = \text{const}.$

From the initial conditions at $l = 0$

\[ r^{(n)}_0 = a^{(n,0)}_2 - \beta\Phi(0), \quad u^{(n)}_0 = a^{(n,0)}_4, \]

the following expressions for the coefficients $c_1, c_2$ were found

\[ c_1 = \left\{ r^{(n)}_0 - r^*_n + (a^{(n,0)}_4 - u^*_n)(-R) \right\} D^{-1} \]
\[ c_2 = \left\{ a^{(n,0)}_4 - u^*_n + (r^{(n)}_0 - u^*_n)(-R) \right\} D^{-1}, \]

(2.42)

where $D = \frac{E_1 - R_{11}}{R_{12}}, \frac{E_2 - E_1}{E_2 - R_{11}} = \frac{E_1 - E_2}{R_{11} - E_2}$. For temperatures similar to $T_c$ the coefficients $c_1$ and $c_2$ can be represented as follows

\[ g_1^* = 0, \quad \nu = \frac{1}{2}, \quad \text{at} \quad d > 4; \]
\[ g_2^* = \frac{1 - s^{d-4}}{(n + 8)(1 - s^{-d})}, \]
\[ \nu = \left[ 2 + \frac{\ln \left( 1 + \frac{n + 2}{\ln s} (s^{d-4} - 1) \right)}{\ln s} \right]^{-1}, \quad \text{at} \quad d < 4, \]

where $g_1^*$ are the fixed points. But, starting from the type of fixed point we have certain restrictions on the $s$ value ($1 < s < 2$). In the limit $n \to \infty$ for $d < 4$ we obtain $\nu = 1$, which is in agreement with the Berlin-Kac spherical model \[43\]. Analogical expressions for $\nu$ were obtained in a series of the works \[42\], \[16\], \[47\], \[52\], \[53\].

\[ w_{11} v_{11} = \left[ 1 + \frac{R_{12} R_{21}}{E_1 - R_{22}} \right]^{-1} \left[ 1 + \frac{E_1 - R_{11}}{R_{12} R_{21}} \right]^{-1}, \]
\[ w_{22} v_{22} = \left[ 1 + \frac{R_{12} R_{21}}{E_2 - R_{11}} \right]^{-1} \left[ 1 + \frac{E_2 - R_{22}}{R_{12} R_{21}} \right]^{-1}. \]
Having applied these relations we evaluate the critical region of the temperatures \( \tau < \tau^* \) in which the solutions of RR (2.41) are valid. For the critical region to exist it is necessary that the ”exit” from this region at \( l \to 1 \) should not exceed the ”entrance”. It means that the value \( \tau^* \) is equal to the smaller root of the two equations

\[
c_2 RE_2 = c_1 E_1, \quad c_1 R_1 E_1 = c_2 E_2.
\]

Take into account (2.43) we obtain for \( \tau^*_1, \tau^*_2 \)

\[
\tau^*_1 = \left| \frac{c_2 T E_2^{\prime\prime}(0)}{c_1 T E_1} \right|, \quad \tau^*_2 = \left| \frac{c_2 T E_2^{\prime\prime}(0)}{c_1 T E_1 R_1} \right|.
\]

The results of calculating \( \tau^*_1, \tau^*_2 \) in the case \( s = 4 \) are shown in Table II. As \( c_1(T) \sim \tau \), the value \( c_1(T) \) can be represented in the following approximation

\[
c_1 T = c_{1k} + c_{1k1} \tau + O(\tau^2), \tag{2.44}
\]

where

\[
c_{1k} = \left[ c_{11} + \frac{c_{12}}{[\beta_c \Phi(0)]^2} \right] D^{-1}, \tag{2.45}
\]

\[
c_{1k1} = \frac{c_{12}}{[\beta_c \Phi(0)]^2} D^{-1},
\]

with

\[
c_{11} = 1 - f_n - R^* \varphi_n^{1/2},
\]

\[
c_{12} = -a_4^{(n,0)} R^* \varphi_n^{-1/2}, \tag{2.46}
\]

In accordance with (2.43) for \( c_2 T \) we consider only the terms proportional to \( \tau^2 \)

\[
c_2 T = c_{2k} + \tau c_{2k1} + \tau^2 c_{2k2} + O(\tau^3), \tag{2.47}
\]

where the following definitions are introduced

\[
c_{2k} = \left[ c_{23} + \frac{c_{22}}{\beta_c \Phi(0)} + \frac{c_{21}}{[\beta_c \Phi(0)]^2} \right] D^{-1},
\]

\[
c_{2k1} = \left[ \frac{c_{22}}{\beta_c \Phi(0)} + \frac{2c_{21}}{[\beta_c \Phi(0)]^2} \right] D^{-1},
\]

\[
c_{2k2} = \frac{c_{21}}{[\beta_c \Phi(0)]^2} D^{-1}. \tag{2.48}
\]

The calculation of the partial partition functions (2.19), (2.20) are connected with the employment of common RR (2.22). In the vicinity of the fixed point (2.30) they can be replaced by the approximate relations (2.41) which are exact for \( T = T_c \). The question arises under what conditions the relations (2.41) can be made use of instead of common RR (2.22). It is obvious that the system of the relations

\[6\] It should be pointed out that

\[
c_{21} = a_4^{(n,0)}, \quad c_{22} = -a_2^{(n,0)} R^* \varphi_n^{1/2},
\]

\[
c_{23} = R^*_n \varphi_n^{1/2} (1 - f_n) - \varphi_n,
\]

\[
R^*_n = R_1 (u^*_n)^{-1/2}, \quad R^* = R (u^*_n)^{1/2}.
\]
\[ |r^{(n)}_i - r^*_{n_i}| \leq |r^*_i|, \]
\[ |u^{(n)}_i - u^*_{n_i}| \leq u^*_n \]  

is a condition of the applicability of the approximate relations (2.41).

The magnitudes which form (2.49) are connected with the value of the \( n \) spin component number and the value of the interaction number \( l \). The main reason for the deviation of the values \( r^{(n)}_i \) and \( u^{(n)}_i \) from their fixed values is the availability of the terms proportional to \( c_1 E^2_1 \) in the solutions (2.41). For small values \( l \) the contribution of these terms is small compared with \( r^*_n \) and \( u^*_n \), as \( c_1 \sim \tau \). But in the case \( T \neq T_c \) there always exists such a value as \( l = m_\tau \) so that the contribution will be of the order \( r^*_n \) or \( u^*_n \). With \( l > m_\tau \) the deviation will become considerable and the equations (2.41) cannot be used for the description of common RR (2.23). It is necessary to use the first equation in (2.41) for determining the value \( m_\tau \). It is connected with the presence of the \( R_1 \) coefficient near the term \( c_1 E_1 \) in the second equation of (2.41). Proceeding from the assumption that \( R_1 \) is small value as was shown in the case \( n = 1 \) \( \delta \), we see that the deviation \( r^{(n)}_i \) from \( r^*_n \) will be going on faster than the deviation \( u^{(n)}_i \) from \( u^*_n \). Hence it follows that the value \( m_\tau \) may be determined from the equation

\[ r_{m_\tau + 1} - r^*_n = \delta r^*_n, \]  

where \( \delta \) is a certain constant value \( (\delta \leq 1) \). Since later we will compare the results obtained in the case of \( T > T_c \) with those obtained for \( T < T_c \), we assume that \( \delta = 1 \) with \( T < T_c \) and \( \delta = -1 \) with \( T > T_c \). The condition similar to (2.50) was employed in (3). In the first approximation on \( (E_2/E_1)^{m_\tau} \) for \( m_\tau \) we have

\[ m_\tau = -\frac{ln |\tau|}{ln E_1} + m_0 - 1, \]  

where

\[ m_0 = m_c + m_1 \tau. \]

For the coefficients \( m_c \) and \( m_1 \) the following relation take place

\[ m_c = \frac{ln(f_n \delta/c_{1k})}{ln E_1}, m_1 = -\frac{c_{1k1}}{c_{1k} ln E_1}. \]

The obtained value of \( m_\tau \) defines the point of exit of the system from the critical region

\[ B_{m_\tau} = B' s^{-m_\tau}. \]

The analysis of the RR shows, that in the vicinity of the PT point two different fluctuation processes take place. The first one corresponds to the index values \( l \in (0, m_\tau) \) and describes the renormalization group symmetry that takes place in the vicinity of the PT point. It is so called critical regime (CR). The second one corresponds to the index values \( l > m_\tau \) and describes the long-wavelength fluctuations (LWF) of the spin and is valid both near and far from the PT point. It is characterized by Gaussian distribution with dispersion depending on the availability of the CR. It is so called Gaussian regime for \( T > T_c \) and the inverse Gaussian regime for \( T < T_c \).

The calculation of the PT temperature is a significant moment of the investigation of the critical behaviour of the model. As was shown in the works (3), (3), the critical temperature is the point where Gaussian regime is absent and subsequent relations take place

\[ \lim_{l \to \infty} r^{(n)}_l(T_c) = \lim_{l \to \infty} r^{(n)}_l(T)/r^*_n = r^*_n = \text{const}, \]
\[ \lim_{l \to \infty} u^{(n)}_l(T_c) = \lim_{l \to \infty} u^{(n)}_l(T)/u^*_n = u^*_n = \text{const}. \]

In accordance with (2.41), this condition is realized only in the case

\[ c_1(T_c) = 0. \]  

With (2.42) taken into account the explicit equation for critical temperature was obtained

\[ [\beta \Phi(0)]^2 \left( 1 - f_n - \varphi^{1/2} R^* \right) - a_2^{(n,0)} \beta \Phi(0) = -a_4^{(n,0)} R^* \varphi^{1/2}, \]  

(2.54)
where \( R^* = R\sqrt{u_n} \), and \( a_2^{(n,0)}, a_4^{(n,0)} \) are functions of the initial interaction potential \( \frac{2.14}{[54]} \). In this paper we present the results of calculating the PT temperature with the interaction potential of the \( \frac{2.14}{[54]} \) type. As follows from \( \frac{2.14}{[54]} \), the obtaining of the concrete calculation results are connected with the choice of the value \( \Phi \), that is the value \( \Phi(k) \) with \( k \in \Delta \). The correction, considering the influence of the Fourier transform of the potential in the interval \( k \in [B', B] \), makes the results of the calculation more precise.

Let us choose the next form for \( \Phi \)

\[
\Phi = \langle \Phi(k) \rangle + \Phi_{\infty}, \tag{2.55}
\]

where

\[
\langle \Phi(k) \rangle = \frac{\int_{B'}^{B} dk \Phi(k)k^2}{\int_{B'}^{B} dk k^2}. \tag{2.56}
\]

The results obtained for the temperature in the limit \( \frac{b}{c} \to \infty \) must be in agreement with the mean field theory results, i.e.

\[
\beta_c \Phi(0) = \frac{n}{m^2}. \tag{2.57}
\]

Take into account this condition and the equation for the temperature \( T_c \) \( \frac{2.54}{[54]} \), we obtain the equation for defining the \( \Phi_{\infty} \). The solution of this equation is written as

\[
\Phi_{\infty} = -\frac{(n + 2)(f_n + \varphi \Phi^* R^*)}{3n(1 - s_0^{-d})}. \tag{2.57}
\]

The results of calculating the PT temperature of the \( n \)-vector model in the case \( m^2 = n \) are represented in Fig.1. As we can see from the figure above, the PT temperature in the \( \beta_c A_0 \) units (where \( A_0 \) is a constant (see \( \frac{2.2}{[54]} \)) grows with the increase of the \( n \) component number of the model. The PT temperature decreases and strives for the mean field theory results in the case of the increase of the effective range of the interaction potential \( b \). Let us employ the introduced definitions.

### III. CALCULATION OF THE FREE ENERGY OF THE \( N \)-VECTOR MAGNETIC MODEL.

As shown in the previous section, the calculation of the free energy of the system can be performed in accordance with \( \frac{2.23}{[54]} \). The given analysis of the RR and the definition of the region of the availability of their approximate solutions enable us to present the free energy of the system in the vicinity of the PT point in the form

\[
F = F_0 + F_{CR} + F_{LWF}, \tag{3.1}
\]
where $F_0$ is the free energy of the non-interacting spins

$$F_0 = -kT N \ln \left[ \frac{(2\pi)^{n/2} n^{n-1}}{\Gamma(n/2)} \right]. \quad (3.2)$$

$F_{LWF}$ is the LWF free energy, $F_{CR}$ is the CR free energy

$$F_{CR} = -kT \sum_{l=0}^{\nu_r} F_l, \quad (3.3)$$

where

$$F_l = N_l f_l, \quad f_l = \frac{n}{4} \ln \left( \frac{n + 2}{\varphi_n (y_l-1)} \right) + \ln U \left( \frac{n-1}{2}, x_l \right) +$$

$$+ \ln U \left( \frac{n-1}{2}, y_l-1 \right) + \frac{x_l^2}{4} + \frac{y_{l-1}^2}{4}, \quad (3.4)$$

Here $N_l = N' s^{-d l}$. In the case $l = 0$ for $f_l$ the following relation takes place

$$f_0 = \frac{n}{4} \ln \left( \frac{3}{u_0} \right) + u_0' + \frac{3}{4} (u_0')^2 + \ln U \left( \frac{n-1}{2}, z' \right) +$$

$$+ \frac{n}{4} \ln \left( \frac{3}{u_0^{(n)}} \right) + \frac{x_0^2}{4} + \ln U \left( \frac{n-1}{2}, x_0 \right). \quad (3.5)$$

The employment of the RR solutions (2.41) allows us to select in $f_l$ the explicit dependence on the number of the phase layer $l$. Having performed the summation along the layers of the CV phase space to the point $m_r$ of the exit of the system from the CR according to (3.3), for the free energy of the CR we obtain

$$F_{CR}^\pm = -kT N' \left[ \gamma_0' + \gamma_1 \tau + \gamma_2 \tau^2 - \gamma_{10}^\pm |\tau|^{d'} \right]. \quad (3.6)$$

Let note that the signs " + " and " − " correspond to $T > T_c$ and $T < T_c$ respectively. The coefficients $\gamma_0', \gamma_1, \gamma_2, \gamma_{10}^\pm$ are constants and do not depend on the temperature (see Appendix 3). The analytical part of the CR free energy is connected with the coefficients $\gamma_0', \gamma_1, \gamma_2$. It should be mentioned that the expressions of these coefficients coincide at the temperatures above and below the critical temperature. The dependence of $\gamma_1$ on the microscopic parameters of the interaction potential and the $n$ component number of the model are shown in Table II. The nonanalytical part, characterizing the temperature dependence of the specific heat in the vicinity of the PT point, is connected with the term $\gamma_{10}^\pm s^{-d'}$, where

$$\gamma_{10}^\pm = \tilde{\gamma}' s^{-d'}, \quad (3.7)$$

$$\tilde{\gamma}' = \frac{f_{CR}^*}{1 - s^{-d}} - \frac{f_n s d_1}{1 - s^{-d} E_1} + \frac{f_n s^2 d_3}{1 - s^{-d} E_1}. \quad (3.8)$$

For $f_{CR}^*$ we have

$$f_{CR}^* = \frac{n}{2} \ln y^* + \tilde{\alpha}(y^*)^2 + \frac{(x^*)^2}{4} + \ln U \left( \frac{n-1}{2}, x^* \right). \quad (3.9)$$

The values of $d_m$ are given in Appendix 4. In Table 2 the dependence of $\gamma_{10}^\pm$ on the component number $n$ of the model for different ratios $b/c$ is shown. The expression (3.10) describes the contribution of the region of the renormalization group symmetry to the free energy of the system. It allows us to obtain the respective contributions to the specific heat of the system at $T < T_c$ and $T > T_c$

$$C_{CR}^\pm = kN' \left[ c^{(0)} - c_{CR}^\pm |\tau|^{-\alpha} \right], \quad (3.10)$$

where
\[ \alpha = 2 - d \nu, \]
\[ c^{(0)} = 2(\gamma_1 + \gamma_2), \]
\[ c_{CR}^\pm = (1 - \alpha)u_{CR}^\pm, \]
\[ u_{CR}^\pm = d \nu \gamma_1^{\pm}. \]

The curves 1 in Fig. 2 correspond to the contribution of the CR to the specific heat at \( T < T_c \) and \( T > T_c \) respectively. The negative value of the specific heat amplitude, that corresponds to the contribution of the CR, testifies to the nonstability of the contribution of the short-wavelength fluctuations (SWF) of the spin moment density. Considering the contribution of the region of the LWF of the spin moment density in calculating the thermodynamic characteristics of the system is a topical problem of to-day.

![FIG. 2. The temperature dependence of the specific heat. The comparison of the contribution of the CR and the region of the LWF. Notes: 1 - corresponds to the CR, 2 - limit Gaussian region (LGR) (\( \tau > 0 \)) and inverse Gaussian region (IGR) (\( \tau < 0 \)), 3 - contribution to the specific heat due to the rise of the ordering in the system.]

The regions of LGR for \( T > T_c \) and IGR for \( T < T_c \) correspond to the LWF of the spin moment density. The increase of the basic \( x_l \) and the intermediate \( y_l \) variables as functions of \( l \) is a typical peculiarity of the LGR and IGR. In this connection the contribution of these regions to the free energy can be calculated in the Gaussian measure density approximation (as in this region of the wave vectors the term proportional to the fourth power under the exponent of the distribution functions becomes much smaller than the square term). Since, the increase of \( x_l \) is gradual, there exists the so-called transition region (TR) where it is necessary to keep the fourth power of the \( \tilde{c}_k \) variables in the distribution function. The value of the TR is defined by a certain number of the CV phase space layers \( m'' \) following the point \( m_\tau \) of the exit of the system from the CR. The value of the TR is defined by the condition

\[ | x_{m_\tau + m''} | = \frac{\alpha_m}{1 - s^{-d}}, \]

where \( \alpha_m \) is a constant (\( \alpha_m \geq 10 \)). The arguments mentioned above enable us to write the contribution of the LWF at \( T > T_c \) to the free energy of the system in the form

\[ F_{LGR}^+ = -kTN'f_{LGR}^+ r^{3\nu} - \beta \mu_B^2 H^2 N \gamma_4 \tau^{-2\nu}, \]

where

\[ f_{LGR}^+ = f_{TR} + f_*. \]

It should be pointed out that the following definitions are introduced here: \( \mu_B \) is Bor’s magneton, \( f_{TR} \) corresponds to the contribution of the TR and \( f_* \) corresponds to the contribution of the wave vectors region with \( k < B^{b^-(m_\tau + m'' + 1)} \). The explicit analytical expressions of these coefficients was shown in [50]. For \( \gamma_4 \) we have

\[ \gamma_4 = s^{2m''} \left( \frac{f_{TR}}{c_1} \right)^{2\nu} \frac{1}{2\beta \Phi(0) g_0}, \]

where \( g_0 \) is a complex function of \( x_{m_\tau + m'' - 1}^* \), with \( m_\tau = m_\tau + m'' + 2 \) (see. [50]).
The contribution of the region of the LWF of the spin moment density at $T < T_c$ to the free energy of the system is described by the value $F_{LGR}^-$. The value $F_{LGR}^-$ corresponds to the inverse Gaussian region. The calculation of the inverse Gaussian region contribution has its own peculiarities. In the temperature region $T < T_c$ the large-scale fluctuations of the spin moment density are described by the non-Gaussian distribution in which the coefficient near the square term becomes negative. This indicates the appearance of the nonzero order parameter of the system. After selecting the ordering free energy the distribution of the fluctuations becomes Gaussian (see Appendix 5). As a result for $F_{LGR}^-$ we obtain

$$F_{LGR}^- = -kTN'[\tau|3\nu]\gamma_{LGR}^-,$$

where

$$\gamma_{LGR}^- = \gamma_3^{(\mu_+)} + \gamma_3^{<\delta>},$$

$$\gamma_3^{(\mu_+)} = \gamma_g + \gamma_\rho, \quad \gamma_3^{<\delta>} = \epsilon^d\delta\gamma_3^{<\delta>}, \quad \gamma_3^{<\delta>} = s_0^dF_{0}^r.$$

The value $\gamma_3^{<\delta>}$ corresponds to the contribution from the ordering in the system. The coefficients $\gamma_g$ and $\gamma_\rho$ are written in the form

$$\gamma_g = \bar{\gamma}_g^{d\mu_\nu},$$

$$\bar{\gamma}_g = \ln\left[\left(\frac{s^d(n+2)\bar{\mu}_\nu}{3\pi^2\varphi_n(x_{\mu_\nu})}\right)^{n/4} e^{\frac{m}{4}\nu} \left(\frac{n-1}{2n}\right)\right],$$

$$\gamma_\rho = \bar{\gamma}_\rho^{d\mu_\nu},$$

$$\bar{\gamma}_\rho = \frac{5}{12}nL(x) - \frac{n}{2}ln\left(\frac{1+2\bar{\mu}_{\nu,1}}{\pi}\right) + \frac{n}{3} - \frac{\bar{\mu}_{\nu,1}}{8}(\bar{\alpha}_1^2n^2 + 3\bar{\alpha}^2L(x)\bar{\alpha}_2) + \frac{\bar{\mu}_{\nu,1}^2}{48}(\bar{\alpha}_4 + 3\bar{\alpha}_1^2\bar{\alpha}_2) + \frac{9}{4}n^2\bar{\alpha}_2\bar{\rho}_{\nu,1}.$$

It should be pointed out that the following definitions are introduced here

$$\bar{\rho}_{\nu,1} = f_n(1+\delta), \quad \bar{\rho}_{\nu,1} = \varphi_n - f_n\sqrt{\frac{1}{2\pi\rho_{\nu,1}}}.$$

$$L(x) = 3\left(\frac{x - arctg\sqrt{x}}{x^3}\right), \quad x = \frac{1}{\sqrt{2\rho_{\nu,1}}}.$$

besides, $c_\nu = (\frac{\sigma_1}{\sqrt{\frac{1}{2\pi\rho_{\nu,1}}}})$ is a nonuniversal value connected with the microscopic parameters of the initial Hamiltonian. According to (3.11), (3.12), (3.13) and (3.17) the complete expression of the free energy of the three dimensional $n$-vector magnetic model in the absence of the external field can be written as

$$F = -kTN'\left[\gamma_0 - \gamma_1|\tau| + \gamma_2|\tau|^2 + \gamma_3^\pm|\tau|^{3\nu}\right],$$

where

$$\gamma_0 = \gamma_0' + s_0^n\ln\left[\frac{(2\pi)^n/2n^{n-1}}{1/(n/2)}\right],$$

$$\gamma_3^\pm = \gamma_3^{LGR} - \gamma_3^{\pm}.$$

The coefficient $\gamma_3^\pm$ includes the contribution of the CR and the region of the LWF of the spin moment density at the temperatures above and below the critical. It describes the singular behaviour of the specific heat in the vicinity of the PT point. The dependence of the coefficients $\gamma_0, \gamma_3^\pm$ on the microscopic parameters of the interaction potential and the characteristics of the crystal lattice for different $n$ component number of the model is shown in Table III.
IV. THERMODYNAMIC FUNCTIONS OF THE N-VECTOR MAGNETIC MODEL IN THE FRAMEWORK OF THE $\rho^4$ MODEL APPROXIMATION.

The obtained complete expression for the free energy of the $n$-vector magnetic model \((3.21)\) in the $\rho^4$ model approximation allows us to calculate other thermodynamic functions in the vicinity of the phase transition point. As was remarked above, (see \((3.10)\), Fig.2), in calculating the thermodynamic functions of the system especially significant is considering the contribution of the LWF of the spin moment density. So, differentiating the expression of the free energy by the temperature, we obtain the expression for the entropy

$$S = kN \left[ S^0 + S^1 \tau + u^\pm_3 |\tau|^{3\nu-1} \right], \quad (4.1)$$

where

$$S^0 = \gamma_0 + \gamma_1, S^1 = 2(\gamma_1 + \gamma_2), u^\pm_3 = \pm 3\nu\gamma^\pm_3.$$

The values of the coefficients $S^0, S^1, u^\pm_3$ for different $n$ values are shown in Table IV.

It should be mentioned that the correct temperature behaviour of the specific heat curves is ensured by considering the region of the LWF of the spin moment density, i.e. the region of LGR at $T > T_c$ and the region of IGR at $T < T_c$. The significant characteristic of the system is the specific heat for which we obtain

$$C = kN' \left[ C^{(0)} + C^\pm_1 |\tau|^{-\alpha} \right], \quad (4.2)$$

where

$$\alpha = 2 - 3\nu, C^{(0)} = 2(\gamma_1 + \gamma_2), C^\pm_1 = 3\nu(1-\alpha)\gamma^\pm_3.$$

The second term in \((4.2)\) describes the main peculiarity of the specific heat behaviour in the vicinity of the PT point. As we can see from \((4.2)\), the coefficient $C^\pm_1$ includes the contributions of the CR and region of the LWF of the spin moment density. We can see from Fig.2, that considering the influence of the regions of LWF (curves 2) provides the positivity of the specific heat and the system stability respectively. The dependence of the coefficients $C^{(0)}$ and $C^\pm_1$ on the microscopic parameters of the Hamiltonian, i.e. from the $b/c$ ratio for different $n$ is exhibited in Table V.

In Fig.3 the temperature dependence of the specific heat for different $n$ is shown. As it was noticed above, in the case $n = 3$ the critical exponent $\alpha$, which describes the singularity of the specific heat becomes negative $\alpha = -0.021$ \[50\], \[51\]. The analysis of the expression \((4.2)\) and the obtained values of the specific heat amplitudes (see Table V) shows that the specific heat in the case $n = 3$ does not diverge and receives a concrete finite value (see Fig.3).

![FIG. 3. The temperature dependence of the specific heat for different $n$ component number of the model.](image)

The curve of the dependence of the specific heat maximum at $T = T_c$ in the case $n = 3$ on the ratio of the effective range $b$ of the interaction potential to the lattice constant $c$ is shown in Fig.4. One can see from this figure that the value of the specific heat maximum decreases and tends to constant value as

FIG. 4. The temperature dependence of the specific heat maximum at $T = T_c$ in the case $n = 3$ on the ratio of the effective range $b$ of the interaction potential to the lattice constant $c$.
TABLE I. The dependence of the fixed point coordinates and the eigenvalues of the transition matrix on the $s$ parameter of dividing the phase space into the layers and the $n$ component number of the model.

| $s$ | $n$ | $x^*$ | $E_1$ | $E_2$ | $f_n$ | $\varphi_n$ |
|-----|-----|-------|-------|-------|-------|------------|
| 1.1 | 1   | 10.9487 | 1.2077 | 0.9992 | 0.0181 | 0.0201     |
|     | 2   | 11.9534 | 1.2074 | 0.9089 | 0.0202 | 0.0168     |
|     | 3   | 12.8801 | 1.2072 | 0.9086 | 0.0217 | 0.0144     |
| 1.5 | 1   | 3.3645  | 2.1521 | 0.6753 | 0.0998 | 0.1088     |
|     | 2   | 3.5927  | 2.1379 | 0.6710 | 0.1134 | 0.0914     |
|     | 3   | 3.8088  | 2.1273 | 0.6676 | 0.1234 | 0.0787     |
| 2.0 | 1   | 1.5562  | 3.4761 | 0.5347 | 0.2153 | 0.2497     |
|     | 2   | 1.5671  | 3.3901 | 0.5260 | 0.2492 | 0.2105     |
|     | 3   | 1.5848  | 3.3256 | 0.5185 | 0.2746 | 0.1814     |
| 3.0 | 1   | 0.3425  | 6.3085 | 0.4140 | 0.4640 | 0.6263     |
|     | 2   | 0.1684  | 5.9509 | 0.4010 | 0.5498 | 0.5287     |
|     | 3   | 0.0154  | 5.6298 | 0.3880 | 0.6145 | 0.4538     |
| 4.0 | 1   | -0.1789 | 9.6225 | 0.3560 | 0.7167 | 1.0885     |
|     | 2   | -0.4575 | 8.5533 | 0.3402 | 0.8620 | 0.9180     |
|     | 3   | -0.7086 | 7.8304 | 0.3233 | 0.9712 | 0.7841     |

FIG. 4. The dependence of the maximum of the specific heat on the ratio of the effective range $b$ of the interaction potential to the crystal lattice constant $c$ in the case $n = 3$ and $T = T_c$. 
The effective range $b$ of the interaction potential increases. This agrees with the results of the mean field theory.

According to (2.3), the ratio of the critical amplitudes of the leading singular terms of the specific heat at $T > T_c$ and $T < T_c$ can be written in the form

$$A = \frac{\gamma_3^+}{\gamma_3^-}. \tag{4.3}$$

The comparison of the results obtained for the ratio of the critical amplitudes of the specific heat leading singular terms with results obtained by other methods is shown in Table V.

**CONCLUSIONS**

In general, it should be noted that the separate accounting of the contributions of the short- and long-wavelength fluctuations of the spin moment density in the expression of the free energy of the system in the vicinity of the PT temperature allows us to find the explicit analytical expressions for the thermodynamic functions as functions of the temperature. The proposed method enables us to investigate the dependence of the critical amplitudes of the thermodynamic functions on the microscopic parameters of the interaction potential and the characteristics of the crystal lattice. The results obtained for the critical exponents and the ratio of the critical amplitudes agree with those obtained by other methods. The negligible deviation of the obtained results from the experimental data and the results of the numerical calculations is connected with the restriction in the calculation by the $\rho^4$ model approximation. As was seen in [3], [56], the employment of the $\rho^6$ measure density for the investigation of the PT in the case $n = 1$ gives a more precise definition of the calculation results of the universal and nonuniversal characteristics of the system. The extension of the suggested method to the investigation of the critical behaviour of the $n$-vector model in the frame of the $\rho^6$ model approximation does not require any principal changes. To be short, in the present paper we do not give enough attention to calculating the scaling corrections. In the case $n = 1$ such calculations were made in [3]. Besides, in the common case the critical exponent of the correlation function is $\eta \neq 0$. Such values of the correlation function critical exponent can be obtained by the method we propose if we take into account the correction on the averaging of the interaction potential in each layer of the CV. We are going to perform such investigations in our subsequent papers.

**APPENDIX 1**

The $\tilde{\rho}_k$ CV are introduced by means of the functional representation for the operators of the spin density fluctuation $\hat{\rho}$

$$\tilde{\rho}_k = \int \rho_k J \left(\hat{\rho} - \tilde{\rho}\right) (d\tilde{\rho}_k)^N, \quad \tilde{\rho}_k = \frac{1}{\sqrt{N}} \sum_R \tilde{\mathbf{S}}_R \exp (-i\mathbf{kR}) ,$$

where

$$J \left(\hat{\rho} - \tilde{\rho}\right) = \left[ \prod_k \delta \left(\hat{\rho}_k - \tilde{\rho}_k\right) \delta \left(\hat{\rho}_k - \tilde{\rho}_k\right) \right] \delta \left(\hat{\rho}_0 - \tilde{\rho}_0\right)$$

is the transition operator. For the CV $\tilde{\rho}_k$ the following relations come into being

$$\tilde{\rho}_k = \tilde{\rho}_k - i\tilde{\rho}_k, \quad \tilde{\rho}_k = \tilde{\rho}_{-k}, \quad \tilde{\rho}_k = -\tilde{\rho}_{-k}.$$
The Jacobian of the transition from spins to CV has the form \([3], [4]\)
\[
J[\rho] = \tilde{Z}_0 \int \exp(2i\pi \sum_k \tilde{\rho}_k \tilde{\omega}_k + \tilde{D}[\omega]) (d\tilde{\omega}_k)^N, \tag{A1.1}
\]
where for \(\tilde{\omega}_k\) variables conjugate to \(\tilde{\rho}_k\) variables we have
\[
\tilde{\omega}_k = \frac{1}{2} (\tilde{\omega}_k^c + i\tilde{\omega}_k^s), \quad \tilde{\omega}_k^c = \tilde{\omega}_k^c - k, \quad \tilde{\omega}_k^s = -\tilde{\omega}_s - k
\]
and
\[
(d\tilde{\omega}_k)^N = \prod_{a=1}^n d\omega_a^0 \prod_k d\omega_a^c d\omega_a^s,
\]

For the values \(\tilde{D}[\omega]\) the following relation is valid
\[
\tilde{D}[\omega] = \sum_{l \geq 1} \sum_{k_1, ..., k_{2l}} \frac{(2\pi i)^{2l}}{(2l)!} u_{2l} \tilde{\omega}_k^{k_1} ... \tilde{\omega}_k^{k_{2l}} \delta_{k_1 + ... + k_{2l}}. \tag{A1.3}
\]
It should be noted that here and further by the product of the vectors of \(\tilde{\omega}_k^{k_1} ... \tilde{\omega}_k^{k_{2l}}\) type we understand the sum of various possible scalar products of given vectors. The coefficients \(u_{2l}\) have the following expressions \([2]\)
\[
u_2 = \frac{m}{n}, \quad u_4 = -\frac{6m^4}{n^2(n + 2)}, \tag{A1.4}
\]
\[
u_6 = 15m^6 \left[ \frac{1}{n(n + 2)(n + 4)} - \frac{3}{n^2(n + 2)} + \frac{2}{n^3} \right].
\]

**APPENDIX 2**

For the coefficients \(a_{2l}\) the following expressions are available
\[
a_2 = \frac{n}{m^2 s_0} \left[ \frac{1}{1 - \frac{20m^2}{(n+4)}} \sum_{k \in \Delta} \frac{\delta \Phi(k)}{N} \right] \frac{n}{u_0}, \tag{A2.1}
\]
\[
a_4 = \frac{3n^2}{m^4 s_0} \left[ \frac{1}{1 - \frac{20m^2}{(n+4)}} \sum_{k \in \Delta} \frac{\delta \Phi(k)}{N} \right] (1 - z^2 U_n(z' - U_0^2)).
\]

Here subsequent definitions are introduced
\[
U_0 = \sqrt{\frac{n + 2}{2}} U_n \left( z' \right), \quad \Delta = [B', B], \tag{A2.2}
\]
\[
U_n \left( z' \right) = U \left( \frac{n+1}{2}, z' \right) \quad z' = \sqrt{\frac{3}{u_4} u_2}, \tag{A2.3}
\]
where $U(a, x) = D_{-a - \frac{1}{2}}(x)$ are the Weber cylinder parabolic functions. The renormalized $u_{2\ell}$ cumulants considering the availability of $\Phi(k)$ for the large values of $k$ have the form

$$u'_0 = s_0 d u_{2n} \sum_{k \in \Delta} \beta \Phi(k), \quad u'_2 = u_2 - \frac{|u_4| n}{2N} \sum_{k \in \Delta} \beta \Phi(k),$$

$$u'_4 = \left[ |u_4| - \frac{u_6 n}{2N} \sum_{k \in \Delta} \beta \Phi(k) \right] s_0^{-d}.$$

**(APPENDIX 3)**

For the coefficients $d_i$ the following expressions are valid

$$d_1 = B_3 \left( A_3 + \frac{A_1}{E_1} \right), \quad d_2 = B_1 \left( A_3 + \frac{A_1}{E_2} \right), \quad d_3 = B_6 \left( A_3 + \frac{A_1}{E_1} \right) + B_3^2 \left( A_4 + \frac{A_2}{E_1} \right),$$

$$d_4 = B_2 \left( A_3 + \frac{A_1}{E_2} \right) + B_2^2 \left( A_1 + \frac{A_2}{E_2} \right), \quad d_5 = B_4 \left( A_3 + \frac{A_1}{E_1 E_2} \right) + 2B_1 B_3 \left( A_4 + \frac{A_2}{E_1 E_2} \right),$$

$$d_6 = B_7 \left( A_3 + \frac{A_1}{E_1 E_2} \right) + 2(B_1 B_6 + B_3 B_4) \left( A_4 + \frac{A_2}{E_1 E_2} \right),$$

$$d_7 = B_5 \left( A_3 + \frac{A_1}{E_1 E_2^2} \right) + 2(B_1 B_4 + B_2 B_3) \left( A_4 + \frac{A_2}{E_1 E_2^2} \right),$$

$$d_8 = B_8 \left( A_3 + \frac{A_1}{E_1^2 E_2^2} \right) + 2 \left( B_1 B_7 + B_2 B_6 + B_3 B_5 + \frac{B_7^2}{2} \right) \left( A_4 + A_2 E_1^{-2} E_2^{-2} \right),$$

where the coefficients $A_i, B_j$ are universal constants and are defined by the fixed point coordinates. For the $B_j$ coefficients we have

$$B_1 = \varphi_n^{-1} \left( R^* \sqrt{3} - \frac{x^*}{2} \right), \quad B_2 = \varphi_n^{-2} \left( \frac{3}{8} x^* - \frac{\sqrt{3}}{2} R^* \right), \quad B_3 = \varphi_n^{-1} \left( \sqrt{3} - \frac{R^* x^*}{2} \right),$$

$$B_4 = \varphi_n^{-3/2} \left( \frac{3}{2} R^*_1 R^* - \frac{\sqrt{3}}{2} + \frac{3}{4} R^*_1 x^* \right), \quad B_5 = \varphi_n^{-5/2} \frac{3 \sqrt{3}}{4} \left( R^*_1 R^* + \frac{5}{6} R^*_1 x^* \right),$$

$$B_6 = \varphi_n^{-1} \frac{3}{8} x^* R^*_1 - \frac{\sqrt{3}}{2}, \quad B_7 = \varphi_n^{-5/2} \frac{3 \sqrt{3}}{4} \left( 1 + \frac{R^*_1 R^*}{2} - \frac{5 R^*_1 x^*}{4 \sqrt{3}} \right),$$

$$B_8 = \varphi_n^{-3} \frac{15 \sqrt{3}}{16} \frac{R^*_1}{4 \sqrt{3}} \left( 7 x^* R^*_1 - (1 + R^* R^*_1) \right).$$

The coefficients $A_i$ are written in the form

$$A_1 = \frac{n}{2} r_1 - \frac{2 r_1 \tilde{\alpha}}{(y^*)^2}, \quad A_2 = \frac{n}{2} r_2 - \frac{n}{4} r_1^2 + \frac{\tilde{\alpha}(3r_1^2 - 2r_2)}{(y^*)^2}, \quad A_3 = \frac{n}{2} U_n(x^*), \quad A_4 = \frac{n}{4} U'(x^*),$$

where the following definition are introduced

$$\tilde{\alpha} = \frac{(5n + 16)n}{8} - \frac{n/2(n/2 + 1)}{2}, \quad U_n'(x^*) = \left( \frac{dU_n(x_i)}{dx_i} \right)_{x^*},$$

$$r_1 = \tilde{d}_1 \frac{q_1}{2}, \quad r_2 = \frac{1}{2} \tilde{d}_2 - \frac{1}{2} \tilde{d}_1 q_1 + \frac{3}{8} q_1^2 - \frac{1}{4} q_2,$$

$$\tilde{d}_1 = \frac{1}{U_n(x^*)} \left( \frac{d^n U_n(x_i)}{dx_i^n} \right)_{x^*}, \quad q_i = \frac{1}{\varphi_n(x^*)} \left( \frac{d^n \varphi_n(x_i)}{dx_i^n} \right)_{x^*}.$$
APPENDIX 4

The \( \gamma_0 \) coefficients are expressed in the form

\[
\gamma_0 = s^{-d} \left( \frac{f^*}{1 - s^{-d}} + \frac{c_{2k} d_2 E_2}{1 - s^{-d} E_2} + \frac{c_{2k}^2 d_4 E_2^2}{1 - s^{-d} E_2^2} \right),
\]

\[
\gamma_1 = s^{-d} \left( \frac{c_{2k} d_2 E_2}{1 - s^{-d} E_2} + \frac{b_1 d_4 E_2^2}{1 - s^{-d} E_2^2} + \frac{c_{1k} d_2 E_1}{1 - s^{-d} E_1} + \frac{c_{1k} c_{2k} d_2 E_1 E_2}{1 - s^{-d} E_1 E_2} + \frac{c_{1k} b_0 d_7 E_1 E_2^2}{1 - s^{-d} E_1 E_2^2} \right),
\]

\[
\gamma_2 = s^{-d} \left( \frac{c_{2k} d_2 E_2}{1 - s^{-d} E_2} + \frac{c_{1k} c_{2k1} + c_{2k} c_{2k1}) d_4 E_2^2}{1 - s^{-d} E_2^2} + \frac{c_{2k} c_{2k} d_2 E_2 E_1}{1 - s^{-d} E_2 E_1} + \frac{c_{2k} c_{2k} d_2 E_2 E_1^2}{1 - s^{-d} E_2 E_1^2} + \frac{c_{2k} c_{2k} d_2 E_2 E_1^2}{1 - s^{-d} E_2 E_1^2} \right),
\]

where

\[
b_0 = c_{2k}^2, \quad b_1 = 2 c_{2k} c_{2k1}, \quad b_2 = c_{2k1} + 2 c_{2k} c_{2k2}.
\]

For \( \gamma' \) we have

\[
\gamma' = \frac{f^*}{1 - s^{-d}} + \frac{f_n \delta d_1}{1 - s^{-d} E_1} + \frac{f_n^2 \delta^2 d_3}{1 - s^{-d} E_1^3},
\]

where a definition is introduced

\[
f_n = \tau E_1^{m_r+1} c_{1k}.
\]

APPENDIX 5

The contribution of the IGR into the free energy can be written in the form

\[
F_{LGR}^- = -kT N s^{-d(m_r+1)} \ln \left( 2^{n/2} Q(P_{m_r}) \right) - kT \ln Z_{m_r+1},
\]

where

\[
Z_{m_r+1} = \int \exp \left[ -\frac{1}{2} \sum_{k \leq B_{m_r+1}} d^{(n,m_r+1)}(k) \bar{\rho}_k \bar{\rho}_k - \frac{1}{4!} N_{m_r+1} \sum_{k_1 \leq B_{m_r+1}} \bar{\rho}_{k_1} \cdots \bar{\rho}_{k_1} \delta_{k_1 \cdots k_1} \right] (d\bar{\rho})^{N_{m_r+1}},
\]

\[
Q(P_{m_r}) = (2\pi)^{-n/2} \left\{ \frac{n + 2 + \delta}{3} \int d^{(n,m_r-1)} \right\}^{n/4} \times
\]

\[
U \left( \frac{n - 1}{2}, y_{m_r-1} \right) \exp \left( \frac{y_{m_r-1}^2}{4} \right).
\]

After the separation of the ordering free energy in \( Z_{m_r+1} \) the coefficient at the square term becomes positive. It allows us to use under the integration of \( Z_{m_r+1} \) by the variables \( \bar{\rho}_k \) with \( k \neq 0 \) the Gaussian measure density as basic. After the integration for \( Z_{m_r+1} \) we obtain

\[
Z_{m_r+1} = e^{-\beta F_{m_r+1}} \int \exp \left( \beta \sqrt{N} \mu_B \bar{H} \bar{\rho}_0 + B \bar{\rho}_0^2 - \frac{G}{N} \bar{\rho}_0^4 \right) \bar{\rho}_0.
\]

The value \( F_{m_r+1} \) corresponds to the contribution of the region of the LWF of the spin moment density (i.e. from the variables \( \bar{\rho}_k \) with \( k \to 0 \), but \( k \neq 0 \)) into the free energy of the system

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where \( \vec{\rho} \)

It should be mentioned, that

\[ -\beta F_{m,+1} = N_{m,+1} \left\{ \frac{3}{2} | d^{(n,m,+1)}(0) | I_1 - \frac{n}{2} N_{m,+1} \sum_{k \leq B_{m,+1}} \ln \left( \frac{a_{3(k)}}{\pi} \right) - a_4^{(n,m,+1)} I_1^2 + a_4^{2(n,m,+1)} (I_4 + 3I_1^2I_2) + \right\}. \tag{A5.4} \]

For \( I_i (i = 1, \ldots, 4) \) the next relations take place

\[ I_1 = n \bar{\alpha}_1 \frac{s^{2(\mu,+1)}}{\beta \Phi(0)}, \quad \bar{\alpha}_1 = \frac{L(x)}{2F_{\mu,+1}}, \]

\[ I_2 = n^2 \bar{\alpha}_2 \left( \frac{s^{2(\mu,+1)}}{\beta \Phi(0)} \right)^2, \quad \bar{\alpha}_2 = (\bar{\alpha}_1^2 + 6e_1^2 (1 + e_2^2)), \tag{A5.5} \]

\[ I_3 = n^3 \bar{\alpha}_3 \left( \frac{s^{2(\mu,+1)}}{\beta \Phi(0)} \right)^3, \quad \bar{\alpha}_3 = \bar{\alpha}_1^3 + 6e_1^3 \left( 1 + \frac{e_2^3}{\sqrt{2}} \right), \]

\[ I_4 = n^4 \bar{\alpha}_4 \left( \frac{s^{2(\mu,+1)}}{\beta \Phi(0)} \right)^4, \quad \bar{\alpha}_4 = \bar{\alpha}_1^4 + 6e_1^4 \left( 1 + \frac{e_2^4}{\sqrt{2}} \right). \]

It should be mentioned, that

\[ e_1 = \frac{15}{\pi^2 [3 + 10F_{\mu,+1}]}, \tag{A5.6} \]

\[ e_2 = \frac{1}{2\pi} \left\{ \sin(\pi \sqrt{2}) - \pi \sqrt{2} \cos(\pi \sqrt{2}) \right\}. \]

The variable \( \bar{\rho}_0 \) in \( (A5.3) \) is connected with order parameter. Its mean value is proportional to the spin density of the system. For the values \( B \) and \( G \) we obtain

\[ B = | \tau | 2^\nu B_0, B_0 = \beta \Phi(0) B^{(0)}, \tag{A5.7} \]

\[ B^{(0)} = c_0 f_n \frac{1 + \delta}{2} B^{(0)}_1, \]

\[ B^{(1)}_1 = 1 - \alpha_{11} \bar{u}_{\mu,+1} + \alpha_{22} \frac{\bar{u}_{\mu,+1}^2}{2F_{\mu,+1}}, \]

\[ G = \tau | \nu | G_0, G_0 = [\beta \Phi(0)]^2 G^{(0)}, \tag{A5.8} \]

\[ G^{(0)} = c_0 \frac{s_0^3}{24} \bar{u}_{\mu,+1} g^{(0)}, \quad g^{(0)} = 1 - \frac{3}{2} n^2 \bar{\alpha}_2 \bar{u}_{\mu,+1}, \]

where

\[ \alpha_{11} = \frac{n}{2} \bar{\alpha}_1 + \frac{3}{2} n^2 \bar{\alpha}_2, \quad \alpha_{22} = n^3 \left\{ \frac{\bar{\alpha}_1^2}{2} + \frac{\bar{\alpha}_3}{3} \right\}. \tag{A5.9} \]

The variable \( \bar{\rho}_0 \) is a macroscopic value, so we can accept that

\[ \bar{\rho}_0 = \sqrt{N} \bar{\rho}. \tag{A5.10} \]

It makes it possible for us to apply the saddle-point method for the calculation of \( Z_{m,+1} \) in \( (A5.3) \). As a result we find

\[ Z_{m,+1} = \sqrt{\frac{2\pi}{E^m_0(< \bar{\rho} >)}} \exp \left\{-\beta F_{m,+1} - NE_0(< \bar{\rho} >)\right\}, \tag{A5.11} \]

where \( < \bar{\rho} > \) is an extreme point of expression.
\[ E_0(\vec{\rho}) = G\vec{\rho}^4 - B\vec{\rho}^2 - \beta \mu \vec{H} \vec{\rho}, \]  

(A5.12)

which arrises in (A5.3) with the change of variables (A5.10). The variable \( \vec{\rho}_0 \) corresponds to the operator \( \hat{\vec{\rho}}_0 = \frac{1}{\sqrt{N}} \sum_l \hat{\vec{\sigma}}_l \), the mean value of which is connected with the equilibrium value of the order parameter \( \sigma \). In the case of \( \vec{H} = 0 \) we find for \( < \vec{\rho} > \) the following solutions

\[ < \vec{\rho}_{1,2} > = \pm \sqrt{\frac{B}{2G}}, \quad < \vec{\rho}_3 > = 0. \]  

(A5.13)

The solutions \( < \vec{\rho}_{1,2} > \) and \( < \vec{\rho}_3 > \) correspond to extreme value of the functional \( E_0(\vec{\rho}) \) (A5.12). The presence of the nonzero mean spin moment at the temperatures \( T < T_c \) testifies to the appearance of spontaneous magnetization in the system in the absence of the external field. Fig.5 shows the temperature dependence of the order parameter \( \sigma \) for different values of \( n \) component number of the model in the absence of the external magnetic field.

![Fig.5](image)

FIG. 5. The temperature dependence of the order parameter for different values of the \( n \) component number of the model in the case \( \vec{H} = 0 \) (with \( b = c \)).

In accordance with the above mentioned for \( E_0(< \vec{\rho} >) \) we obtain

\[ E_0(< \vec{\rho} >) = -\frac{B^2}{4G} = E'_0 \mid \tau \mid^{3\nu}, \]  

(A5.14)

where

\[ E'_0 = -\frac{3}{2} \frac{\bar{r}_{\mu+1}^2}{u_{\mu+1}} \frac{(B_1^{(0)})^2}{s_{3G}^{(0)}}. \]  

(A5.15)

All this enables us to obtain the explicit analytical expression for the IGR free energy. Summarizing the expression (A5.4) by the variables \( k \leq B_{\mu+1} \) \((k \neq 0)\) and considering the contribution of the \( \vec{\rho}_0 \) variables in accordance with (A5.11) - (A5.14), for \( F_{LGR} \) we obtain the expression (3.15).

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TABLE III. The dependence of the coefficients of the free energy on the microscopic parameters of the interaction potential and the characteristics of the crystal lattice for different $n$ component number of the model.

| $b/c$ | $n$ | $\gamma_{01}$ | $\gamma_1$ | $\gamma_2$ | $\gamma_{10}$ | $\gamma_{11}$ | $\gamma_0$ | $\gamma_1^-$ | $\gamma_1^+$ |
|-------|-----|----------------|------------|------------|--------------|--------------|----------|------------|-------------|
| 0.2887 | 1   | 0.349         | -0.500     | -0.459     | -0.538       | 2.737        | 1.811    | 1.283      | 2.726       |
|       | 2   | 0.727         | -0.976     | -4.427     | 3.315        | 0.368        | 5.335    | 5.795      | 6.294       |
|       | 3   | 1.099         | -1.435     | 24.497     | -25.359      | 30.049       | 9.488    | -22.603    | -22.530     |
| $b=c$ | 1   | 0.297         | -0.521     | -0.122     | -0.448       | 2.276        | 61.085   | 1.066      | 2.266       |
|       | 2   | 0.620         | -1.011     | -3.200     | 2.722        | 0.303        | 192.194  | 4.759      | 5.170       |
|       | 3   | 0.938         | -1.470     | 21.176     | -21.397      | 25.358       | 349.648  | -19.072    | -19.013     |

TABLE IV. The dependence of the amplitudes of the thermodynamic functions on the microscopic parameters of the interaction potential and the characteristics of the crystal lattice for different $n$ component number of the model.

| $b/c$ | $n$ | $S^0$ | $S^1$ | $u^+_3$ | $u^-_3$ | $C^{(0)}$ | $C^+_1$ | $C^-_1$ |
|-------|-----|-------|-------|---------|---------|-----------|----------|---------|
| 0.2887 | 1   | 1.311 | -1.917| 2.355   | -5.007  | -1.917    | 1.972    | 4.190   |
|       | 2   | 4.359 | -10.805| 11.226  | -12.196 | -10.805   | 10.529   | 11.436  |
|       | 3   | 8.053 | 46.124 | -45.677 | 45.530  | 46.124    | -46.629  | -46.478 |
| $b=c$ | 1   | 60.564| -1.286| 1.958   | -4.163  | -1.286    | 1.639    | 3.484   |
|       | 2   | 191.183| -8.421| 9.220   | -10.017 | -8.421    | 8.646    | 9.393   |
|       | 3   | 348.178| 39.412| -38.540 | 38.422  | 39.412    | -39.344  | -39.223 |

TABLE V. The dependence of the ratio of the specific heat critical amplitudes at $T > T_c$ and $T < T_c$ on the $n$ component number of the model.

| $n$ | $A(\text{This study})$ | $\epsilon - \exp$ | $Pade_{\text{apr}}$ | $\frac{1}{\epsilon} - \exp$ | $\text{experiment}$ |
|-----|------------------------|-------------------|---------------------|--------------------------|----------------------|
| 1   | 0.470                  | 0.55              | 0.438               | 0.519                    | 0.538                |
| 2   | 0.921                  | 0.99              | 0.880               | 0.888                    | 1.067                |
| 3   | 1.003                  | 1.36              | 1.326               | 1.152                    | 1.588                |