t-CLASS SEMIGROUPS OF NOETHERIAN DOMAINS

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ABSTRACT. The t-class semigroup of an integral domain R, denoted $S_t(R)$, is the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals with the operation induced by ideal t-multiplication. This paper investigates ring-theoretic properties of a Noetherian domain that reflect reciprocally in the Clifford or Boolean property of its t-class semigroup.

1. INTRODUCTION

Let R be an integral domain. The class semigroup of R, denoted $\mathcal{I}(R)$, is the semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals \cite{3, 19}. We define the t-class semigroup of R, denoted $\mathcal{I}_t(R)$, to be the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals, that is, the semigroup of the isomorphy classes of the t-ideals of R with the operation induced by t-multiplication. Notice that $\mathcal{I}_t(R)$ stands as the t-analogue of $\mathcal{I}(R)$, whereas the class group $\text{Cl}(R)$ is the t-analogue of the Picard group Pic(R). In general, we have

$$\text{Pic}(R) \subseteq \text{Cl}(R) \subseteq \mathcal{I}_t(R) \subseteq \mathcal{I}(R)$$

where the first and third containments turn into equality if R is a Prüfer domain and the second does so if R is a Krull domain.

A commutative semigroup $S$ is said to be Clifford if every element $x$ of $S$ is (von Neumann) regular, i.e., there exists $a \in S$ such that $x = ax^2$. A Clifford semigroup $S$ has the ability to stand as a disjoint union of subgroups $G_e$, where $e$ ranges over the set of idempotent elements of $S$, and $G_e$ is the largest subgroup of $S$ with identity equal to $e$ (cf. \cite{7}). The semigroup $S$ is said to be Boolean if for each $x \in S$, $x = x^2$. A domain $R$ is said to be Clifford (resp., Boole) t-regular if $S_t(R)$ is a Clifford (resp., Boolean) semigroup.

Date: November 28, 2008.

2000 Mathematics Subject Classification. 13C20, 13F05, 11R65, 11R29, 20M14.

Key words and phrases. Class semigroup, t-class semigroup, t-ideal, t-closure, Clifford semigroup, Clifford t-regular, Boole t-regular, t-stable domain, Noetherian domain, strong Mori domain.

This work was funded by KFUPM under Project # MS/t-Class/257.
This paper investigates the \( t \)-class semigroups of Noetherian domains. Precisely, we study conditions under which \( t \)-stability characterizes \( t \)-regularity. Our first result, Theorem 2.2, compares Clifford \( t \)-regularity to various forms of stability. Unlike regularity, Clifford (or even Boole) \( t \)-regularity over Noetherian domains does not force the \( t \)-dimension to be one (Example 2.4). However, Noetherian strong \( t \)-stable domains happen to have \( t \)-dimension 1. Indeed, the main result, Theorem 2.6, asserts that “\( R \) is strongly \( t \)-stable if and only if \( R \) is Boole \( t \)-regular and \( t \)-dim(\( R \)) = 1.” This result is not valid for Clifford \( t \)-regularity as shown by Example 2.9. We however extend this result to the Noetherian-like larger class of strong Mori domains (Theorem 2.10).

All rings considered in this paper are integral domains. Throughout, we shall use \( \text{qf}(R) \) to denote the quotient field of a domain \( R \), \( \mathcal{T} \) to denote the isomorphy class of a \( t \)-ideal \( I \) of \( R \) in \( S_t(R) \), and \( \text{Max}_t(R) \) to denote the set of maximal \( t \)-ideals of \( R \).

## 2. MAIN RESULTS

We recall that for a nonzero fractional ideal \( I \) of \( R \), \( I_v := (I^{-1})^{-1} \), \( I_t := \bigcup J_v \) where \( J \) ranges over the set of finitely generated subideals of \( I \), and \( I_w := \bigcup (J : J) \) where the union is taken over all finitely generated ideals \( J \) of \( R \) with \( J^{-1} = R \). The ideal \( I \) is said to be divisorial or a \( v \)-ideal if \( I = I_v \), a \( t \)-ideal if \( I = I_t \), and a \( w \)-ideal if \( I = I_w \). A domain \( R \) is called strong Mori if \( R \) satisfies the ascending chain condition on \( w \)-ideals [5]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. Suitable background on strong Mori domains is [5]. Finally, recall that the \( t \)-dimension of \( R \), abbreviated \( t \)-dim(\( R \)), is by definition equal to the length of the longest chain of \( t \)-prime ideals of \( R \).

The following lemma displays necessary and sufficient conditions for \( t \)-regularity. We often will be appealing to this lemma without explicit mention.

**Lemma 2.1 ([9] Lemma 2.1).** Let \( R \) be a domain.

1. \( R \) is Clifford \( t \)-regular if and only if, for each \( t \)-ideal \( I \) of \( R \), \( I = (I^2 : I^2) \).
2. \( R \) is Boole \( t \)-regular if and only if, for each \( t \)-ideal \( I \) of \( R \), \( I = c(I^2) \), for some \( c \neq 0 \in \text{qf}(R) \).

An ideal \( I \) of a domain \( R \) is said to be \( L \)-stable (here \( L \) stands for Lipman) if \( R^l := \bigcup_{n \geq 1} (I^n : I^n) = (I : I) \), and \( R \) is called \( L \)-stable if every nonzero ideal is \( L \)-stable. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to
give a characterization of Arf rings; in this context, L-stability coincides with Boole regularity \[12\].

Next, we state our first theorem of this section.

**Theorem 2.2.** Let \( R \) be a Noetherian domain and consider the following statements:

1. \( R \) is Clifford \( t \)-regular;
2. Each \( t \)-ideal \( I \) of \( R \) is \( t \)-invertible in \( (I : I) \);
3. Each \( t \)-ideal is \( L \)-stable.

Then (1) \( \implies \) (2) \( \implies \) (3). Moreover, if \( t\text{-dim}(R) = 1 \), then (3) \( \implies \) (1).

**Proof.** (1) \( \implies \) (2). Let \( I \) be a \( t \)-ideal of a domain \( A \). Then for each ideal \( J \) of \( A \), \( (I : J) = (I : J_t) \). Indeed, since \( J \subseteq J_t \), then \( (I : J_t) \subseteq (I : J) \). Conversely, let \( x \in (I : J) \). Then \( xJ \subseteq I \) implies that \( xJ_t = (xJ)_t \subseteq I_t = I \), as claimed. So \( x \in (I : J_t) \) and therefore \( (I : J) \subseteq (I : J_t) \). Now, let \( I \) be a \( t \)-ideal of \( R \), \( B = (I : I) \) and \( J = (B : B) \). Since \( T \) is regular in \( S \), then \( I = (I^2 : I^2)_1 \). By the claim, \( B = (I : I) = (I : (IJ)_t) = (I : IJ) = ((I : I) : J) = (B : J) \). Since \( B \) is Noetherian, then \( (I(B : I))_{t_1} = J_{t_1} = J_{v_1} = B \), where \( t_1 \)- and \( v_1 \)-denote the \( t \)- and \( v \)-operations with respect to \( B \). Hence \( I \) is \( t \)-invertible as an ideal of \( (I : I) \).

(2) \( \implies \) (3). Let \( n \geq 1 \), and \( x \in (I^n : I^n) \). Then \( xI^n \subseteq I^n \) implies that \( xI^n(B : I) \subseteq I^n(B : I) \). So \( x(I^n(I : I))_{t_1} = x(I^n(I : I))_{t_1} \subseteq (I^n(I : I))_{t_1} = (I^n(I : I))_{t_1} \). Now, we iterate this process by composing the two sides by \( (B : I) \), applying the \( t \)-operation with respect to \( B \) and using the fact that \( I \) is \( t \)-invertible in \( B \), we obtain that \( x \in (I : I) \). Hence \( I \) is \( L \)-stable.

(3) \( \implies \) (1). Assume that \( t\text{-dim}(R) = 1 \). Let \( I \) be a \( t \)-ideal of \( R \) and \( J = (I^2 : I^2)_1 = (I^2 : I^2)_v \) (since \( R \) is Noetherian, and so a TV-domain). We wish to show that \( I = J \). By \[10\] Proposition 2.8.(3), it suffices to show that \( IR_M = JR_M \) for each \( t \)-maximal ideal \( M \) of \( R \). Let \( M \) be a \( t \)-maximal ideal of \( R \). If \( I \not

According to \[2\] Theorem 2.1 or \[8\] Corollary 4.3, a Noetherian domain \( R \) is Clifford regular if and only if \( R \) is stable and \( \text{dim}(R) = 1 \). Unlike Clifford regularity, Clifford (or even Boole) \( t \)-regularity does not force a Noetherian domain \( R \) to be of \( t \)-dimension one. In order to illustrate this fact, we first establish the transfer of Boole \( t \)-regularity to pullbacks issued from local Noetherian domains.
Proposition 2.3. Let \((T, M)\) be a local Noetherian domain with residue field \(K\) and \(\phi : T \rightarrow K\) the canonical surjection. Let \(k\) be a proper subfield of \(K\) and \(R := \phi^{-1}(k)\) the pullback issued from the following diagram of canonical homomorphisms:

\[
\begin{array}{ccc}
R & \rightarrow & k \\
\downarrow & & \downarrow \\
T & \phi & K = T/M
\end{array}
\]

Then \(R\) is Boole \(t\)-regular if and only if so is \(T\).

Proof. By [4, Theorem 4] (or [6, Theorem 4.12]) \(R\) is a Noetherian local domain with maximal ideal \(M\). Assume that \(R\) is Boole \(t\)-regular. Let \(J\) be a \(t\)-ideal of \(T\). If \(J(T : J) = T\), then \(J = aT\) for some \(a \in J\) (since \(T\) is local). Then \(J^2 = aJ\) and so \((J^2)_t = aJ\), where \(t_1\) is the \(t\)-operation with respect to \(T\) (note that \(t_1 = v_1\) since \(T\) is Noetherian), as desired. Assume that \(J(T : J) \subseteq T\). Since \(T\) is local with maximal ideal \(M\), then \(J(T : J) \subseteq M\). Hence \(J^{-1} = (R : J) \subseteq (T : J) \subseteq (M : J) \subseteq J^{-1}\) and therefore \(J^{-1} = (T : J)\). So \((T : J^2) = ((T : J) : J) = ((R : J) : J) = (R : J^2)\). Now, since \(R\) is Boole \(t\)-regular, then there exists \(0 \neq c \in \text{qf}(R)\) such that \((J^2)_t = ((J_t)_t) = cJ_t\). Then \((T : J^2) = (R : J^2) = (R : (J^2)_t) = (R : cJ_t) = c^{-1}(R : J_t) = c^{-1}(R : J) = c^{-1}(T : J)\). Hence \((J^2)_t = (J^2)_{v_1} = cJ_{v_1} = cJ_{t_1} = cJ_t\), as desired. It follows that \(T\) is Boole \(t\)-regular.

Conversely, assume that \(T\) is Boole \(t\)-regular and let \(I\) be a \(t\)-ideal of \(R\). If \(II^{-1} = R\), then \(I = aR\) for some \(a \in I\). So \(I^2 = aI\), as desired. Assume that \(II^{-1} \subseteq R\). Then \(II^{-1} \subseteq M\). So \(T \subseteq (M : M) = M^{-1} = (II^{-1})^{-1} = (I_v : I_v) = (I : I)\). Hence \(I\) is an ideal of \(T\). If \(I(T : I) = T\), then \(I = aT\) for some \(a \in I\) and so \(I^2 = aI\), as desired. Assume that \(I(T : I) \subseteq T\). Then \(I(T : I) \subseteq M\), and so \(I^{-1} \subseteq (T : I) \subseteq (M : I) \subseteq I^{-1}\). Hence \(I^{-1} = (T : I)\). So \((T : I^2) = ((T : I) : I) = ((R : I) : I) = (R : I^2)\). But since \(T\) is Boole \(t\)-regular, then there exists \(0 \neq c \in \text{qf}(T)\) such that \((I^2)_t = ((I_t)_t) = cI_t\). Then \((R : I^2) = (T : I^2) = (T : (I^2)_t) = (T : cI_t) = c^{-1}(T : I_t) = c^{-1}(T : I) = c^{-1}(R : I)\). Hence \((I^2)_t = (I^2)_v = cI_v = cI_t = cI\), as desired. It follows that \(R\) is Boole \(t\)-regular. \(\square\)

Example 2.4. Let \(K\) be a field, \(X\) and \(Y\) two indeterminates over \(K\), and \(k\) a proper subfield of \(K\). Let \(L := K[[X, Y]] = K + M\) and \(R := k + M\) where \(M := (X, Y)\). Since \(T\) is a UFD, then \(T\) is Boole \(t\)-regular [9, Proposition 2.2]. Further, \(R\) is a Boole \(t\)-regular Noetherian domain by Proposition 2.3. Now \(M\) is a \(v\)-ideal of \(R\), so that \(t\)-dim\(R\) = \(\dim(R) = 2\).
Recall that an ideal \( I \) of a domain \( R \) is said to be \( \text{stable} \) (resp., \( \text{strongly stable} \)) if \( I \) is invertible (resp., principal) in its endomorphism ring \((I : I)\), and \( R \) is called a stable (resp., strongly stable) domain provided each nonzero ideal of \( R \) is stable (resp., strongly stable). Sally and Vasconcelos [17] used this concept to settle Bass’ conjecture on one-dimensional Noetherian rings with finite integral closure. Recall that a stable domain is \( L \)-stable [1, Lemma 2.1]. For recent developments on stability, we refer the reader to [1] and [14,15,16]. By analogy, we define the following concepts:

**Definition 2.5.** A domain \( R \) is \( t \)-**stable** if each \( t \)-ideal of \( R \) is stable, and \( R \) is \( t \)-**strongly stable** if each \( t \)-ideal of \( R \) is strongly stable.

Strong \( t \)-stability is a natural stability condition that best suits Boolean \( t \)-regularity. Our next theorem is a satisfactory \( t \)-analogue for Boolean regularity [8, Theorem 4.2].

**Theorem 2.6.** Let \( R \) be a Noetherian domain. The following conditions are equivalent:

1. \( R \) is strongly \( t \)-stable;
2. \( R \) is Boole \( t \)-regular and \( \dim(R) = 1 \).

The proof relies on the following lemmas.

**Lemma 2.7.** Let \( R \) be a \( t \)-stable Noetherian domain. Then \( \dim(R) = 1 \).

**Proof.** Assume \( \dim(R) \geq 2 \). Let \((0) \subseteq P_1 \subseteq P_2 \) be a chain of \( t \)-prime ideals of \( R \) and \( T := (P_2 : P_2) \). Since \( R \) is Noetherian, then so is \( T \) (as \((R : T) \neq 0\)) and \( T \subseteq \overline{R} = R' \), where \( \overline{R} \) and \( R' \) denote respectively the complete integral closure and the integral closure of \( R \). Let \( Q \) be any minimal prime over \( P_2 \) in \( T \) and let \( M \) be a maximal ideal of \( T \) such that \( Q \subseteq M \). Then \( QT_M \) is minimal over \( P_2T_M \) which is principal by \( t \)-stability. By the principal ideal theorem, \( \text{ht}(Q) = \text{ht}(QT_M) = 1 \). By the Going-Up theorem, there is a height-two prime ideal \( Q_2 \) of \( T \) contracting to \( P_2 \) in \( R \). Further, there is a minimal prime ideal \( Q \) of \( P_2 \) such that \( P_2 \subseteq Q \subseteq Q_2 \). Hence \( Q \cap R = Q_2 \cap R = P_2 \), which is absurd since the extension \( R \subseteq T \) is INC. Therefore \( \dim(R) = 1 \). \( \Box \)

**Lemma 2.8.** Let \( R \) be a one-dimensional Noetherian domain. If \( R \) is Boole \( t \)-regular, then \( R \) is strongly \( t \)-stable.

**Proof.** Let \( I \) be a nonzero \( t \)-ideal of \( R \). Set \( T := (I : I) \) and \( J := I(T : I) \). Since \( R \) is Boole \( t \)-regular, then there is \( 0 \neq c \in \text{qf}(R) \) such that \((I^2)_t = cI \). Then \( T : I = (I : I)^2 = (I : I^2) = (I : (I^2)_t) = (I : cI) = c^{-1}I : I = c^{-1}T \). So \( J = I(T : I) = c^{-1}I \). Since \( J \) is a trace ideal of \( T \), then \( (T : J) = (J : J) = (c^{-1}I : c^{-1}I) = (I : I) = T \). Hence \( J_{v_1} = T \), where \( v_1 \) is the \( v \)-operation with respect to \( T \). Since \( R \) is one-dimensional Noetherian domain, then so is \( T \) ([11, Theorem 93]). Now, if \( J \) is a proper ideal of \( T \), then \( J \subseteq N \).
for some maximal ideal \( N \) of \( T \). Hence \( T = J_{v_1} \subseteq N_{v_1} \subseteq T \) and therefore \( N_{v_1} = T \). Since \( \dim(T) = 1 \), then each nonzero prime ideal of \( T \) is \( t \)-prime and since \( T \) is Noetherian, then \( t_1 = v_1 \). So \( N = N_{v_1} = T \), a contradiction. Hence \( J = T \) and therefore \( I = cI = cT \) is strongly \( t \)-stable, as desired. \( \Box \)

**Proof of Theorem 2.6** (1) \( \implies \) (2) Clearly \( R \) is Boole \( t \)-regular and, by Lemma 2.7, \( t \)-dim \( (R) = 1 \).

(2) \( \implies \) (1) Let \( I \) be a nonzero \( t \)-ideal of \( R \). Set \( T := (I : I) \) and \( J := I(T : I) \). Since \( R \) is Boole \( t \)-regular, then there is \( 0 \neq c \in \text{qf}(R) \) such that \( (I^2)_r = cI \). Then \( (T : I) = ((I : I) : I) = (I : I^2) = (I : (I^2)_r) = (I : cI) = c^{-1}(I : I) = c^{-1}T \). So \( J = I(T : I) = c^{-1}I \). It suffices to show that \( J = T \). Since \( T = (I : I) = (I^{-1})^{-1} \), then \( T \) is a divisorial (fractional) ideal of \( R \), and since \( J = c^{-1}I \), then \( J \) is a divisorial (fractional) ideal of \( R \) too. Now, for each \( t \)-maximal ideal \( M \) of \( R \), since \( R_M \) is a one-dimensional Noetherian domain which is Boole \( t \)-regular, by Lemma 2.8 \( R_M \) is strongly \( t \)-stable. If \( I \not\subseteq M \), then \( T_M = (I : I)_M = (IR_M : IR_M) = R_M \) and \( J_M = I(T : I)_M = R_M \). Assume that \( I \subseteq M \). Then \( IR_M \) is a \( t \)-ideal of \( R_M \). Since \( R_M \) is strongly \( t \)-stable, \( IR_M = aR_M \) for some nonzero \( a \in I \). Hence \( T_M = (I : I)_M = (IR_M : IR_M) = R_M \). Then \( J_M = I(M : I)_M = R_M = T_M \). Hence \( J = J_M = \bigcap_{M \in \text{Max}(R)} T_M = \bigcap_{M \in \text{Max}(R)} T = T \). It follows that \( I = cJ = cT \) and therefore \( R \) is strongly \( t \)-stable. \( \Box \)

An analogue of Theorem 2.6 does not hold for Clifford \( t \)-regularity, as shown by the next example.

**Example 2.9.** There exists a Noetherian Clifford \( t \)-regular domain with \( t \)-dim \( (R) = 1 \) such that \( R \) is not \( t \)-stable. Indeed, let us first recall that a domain \( R \) is said to be pseudo-Dedekind if every \( v \)-ideal is invertible \([10]\). In \([18]\), P. Samuel gave an example of a Noetherian UFD domain \( R \) for which \( R[[X]] \) is not a UFD. In \([10]\), Kang noted that \( R[[X]] \) is a Noetherian Krull domain which is not pseudo-Dedekind; otherwise, \( \text{Cl}(R[[X]]) = \text{Cl}(R) = 0 \) forces \( R[[X]] \) to be a UFD, absurd. Moreover, \( R[[X]] \) is a Clifford \( t \)-regular domain by \([9]\) Proposition 2.2] and clearly \( R[[X]] \) has \( t \)-dimension 1 (since Krull). But for \( R[[X]] \) not being a pseudo-Dedekind domain translates into the existence of a \( v \)-ideal of \( R[[X]] \) that is not invertible, as desired.

We recall that a domain \( R \) is called strong Mori if it satisfies the ascending chain condition on \( w \)-ideals. Noetherian domains are strong Mori. Next we wish to extend Theorem 2.6 to the larger class of strong Mori domains.

**Theorem 2.10.** Let \( R \) be a strong Mori domain. Then the following conditions are equivalent:

1. \( R \) is strongly \( t \)-stable;
2. \( R \) is Boole \( t \)-regular and \( t \)-dim \( (R) = 1 \).
**Proof.** We recall first the following useful facts:

**Fact 1** ([10] Lemma 5.11). Let $I$ be a finitely generated ideal of a Mori domain $R$ and $S$ a multiplicatively closed subset of $R$. Then $(I_S)_v = (I_v)_S$. In particular, if $I$ is a $t$-ideal (i.e., $v$-ideal) of $R$, then $I$ is $v$-finite, that is, $I = A_v$ for some finitely generated subideal $A$ of $I$. Hence $(I_S)_v = ((A_v)_S)_v = ((A_S)_v)_v = (A_S)_v = (A_v)_S = I_S$ and therefore $I_S$ is a $v$-ideal of $R_S$.

**Fact 2.** For each $v$-ideal $I$ of $R$ and each multiplicatively closed subset $S$ of $R$, $(I : I)_S = (I_S : I_S)$. Indeed, set $I = A_v$ for some finitely generated subideal $A$ of $I$ and let $x \in (I_S : I_S)$. Then $xA \subseteq xA_v = xI \subseteq xI_S \subseteq I_S$. Since $A$ is finitely generated, then there exists $\mu \in S$ such that $x\mu A \subseteq I$. So $x\mu I = x\mu A \subseteq I_v = I$. Hence $x\mu \in (I : I)$ and then $x \in (I : I)_S$. It follows that $(I : I)_S = (I_S : I_S)$.

(1) $\implies$ (2) Clearly $R$ is Boole $t$-regular. Let $M$ be a maximal $t$-ideal of $R$. Then $R_M$ is a Noetherian domain ([15, Theorem 1.9]) which is strongly $t$-stable. By Theorem $2.6$, $t$-$\dim(R_M) = 1$. Since $MR_M$ is a $t$-maximal ideal of $R_M$ (Fact 1), then $ht(M) = ht(MR_M) = 1$. Therefore $t$-$\dim(R) = 1$.

(2) $\implies$ (1) Let $I$ be a nonzero $t$-ideal of $R$. Set $T := (I : I)$ and $J := I(T : I)$. Since $R$ is Boole $t$-regular, then $(I^2)_T = cI$ for some nonzero $c \in qf(R)$. So $J = c^{-1}I$. Since $J$ and $T$ are (fractional) $t$-ideals of $R$, to show that $J = T$, it suffices to show it $t$-locally. Let $M$ be a $t$-maximal ideal of $R$. Since $R_M$ is one-dimensional Noetherian domain which is Boole $t$-regular, by Theorem $2.6$, $R_M$ is strongly $t$-stable. By Fact 1, $I_M$ is a $t$-ideal of $R_M$. So $I_M = a(I_M : I_M)$. Now, by Fact 2, $T_M = (I : I)_M = (I_M : I_M)$ and then $I_M = aT_M$. Hence $J_M = I_M(T_M : I_M) = T_M$, as desired. 

We close the paper with the following discussion about the limits as well as possible extensions of the above results.

**Remark 2.11.** (1) Unlike Clifford regularity, Clifford (or even Boole) $t$-regularity does not force a strong Mori domain to be Noetherian. Indeed, it suffices to consider a UFD domain which is not Noetherian.

(2) Example [2.4] provides a Noetherian Boole $t$-regular domain of $t$-dimension two. We do not know whether the assumption “$t$-$\dim(R) = 1$” in Theorem $2.2$ can be omitted.

(3) Following [8, Proposition 2.3], the complete integral closure $\overline{R}$ of a Noetherian Boole regular domain $R$ is a PID. We do not know if $\overline{R}$ is a UFD in the case of Boole $t$-regularity. However, it’s the case if the conductor $(R : \overline{R}) \neq 0$. Indeed, it’s clear that $\overline{R}$ is a Krull domain. But $(R : \overline{R}) \neq 0$ forces $\overline{R}$ to be Boole $t$-regular, when $R$ is Boole $t$-regular, and by [9, Proposition 2.2], $\overline{R}$ is a UFD.
(4) The Noetherian domain provided in Example 2.4 is not strongly $t$-discrete since its maximal ideal is $t$-idempotent. We do not know if the assumption “$R$ strongly $t$-discrete, i.e., $R$ has no $t$-idempotent $t$-prime ideals” forces a Clifford $t$-regular Noetherian domain to be of $t$-dimension one.

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