CFAR Based NOMP for Line Spectral Estimation and Detection

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The line spectrum estimation and detection problem are considered in this article. We propose a constant false alarm rate (CFAR)-based Newtonized OMP (NOMP-CFAR) method, which can maintain a desired false alarm rate without the knowledge of the noise variance. The NOMP-CFAR consists of two steps, namely, an initialization step and a detection step. In the initialization step, NOMP is employed to obtain candidate sinusoidal components. In the detection step, CFAR detector is applied to detect each candidate frequency, and provides a relationship between the false alarm rate and the required threshold.

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I. INTRODUCTION

Line spectrum estimation and detection are fundamental problems in signal processing fields, as they arise in various applications such as radar and sonar target estimation and detection [1], [2], [3], direction of arrival estimation [4], [5], situational awareness in automotive millimeter wave radar [6], vital signs identification [7] and so on.

The classical discrete Fourier transform (DFT) based approach is often adopted to obtain the spectrum [8], which can be implemented efficiently via fast Fourier transform (FFT). Then, CFAR detection is implemented to perform target detection. However, DFT is susceptible to “off-grid” effects [9]: the signal from the target leaks into several points in the DFT grid, unless it lies exactly on the DFT grid. Besides, this technique also suffers intertarget interference, i.e., the weak target is hard to detect given that the frequency of the weak target is close to that of a strong target, which limits the resolution capability. Later, subspace methods such as multiple signal classification (MUSIC) [10] and estimation of signal parameters using rotational invariance techniques (ESPRIT) [11] are proposed, which exploit the low-rank structure of the autocorrelation matrix. The capability of resolving multiple closely spaced frequencies is enhanced especially at high signal-to-noise ratio (SNR). However, their performances degrade at medium and low SNRs.

More recent techniques using compressed sensing (CS) based methods exploit the sparse structure of the line spectrum in the frequency domain. By discretizing the frequency onto a finite set of grids, on-grid methods such as orthogonal matching pursuit (OMP) [12], least absolute shrinkage and selection operator (Lasso) [13], [14] are used to solve the line spectrum estimation problems. However, the frequencies are continuous parameters and they cannot lie on the grids exactly. As a consequence, on-grid methods suffer from mismatch issues [9]. To overcome the model mismatch, off-grid and gridless methods are proposed such as relaxation (RELAX) algorithm [15], [16], iterative reweighted approach (IRA) [17], variational line spectra estimation (VALID) [18], atomic norm soft thresholding (AST) [19], NOMP [20], etc. The frequency estimation accuracy of these off-grid and gridless methods is better than the classical methods such as MUSIC and on-grid methods. However, the computational complexity of IRA and AST are high. For VALID, it automatically estimates the noise variance, model order, and other nuisance parameters. However, it is numerically found that when the number of targets is large, VALID tends to output spurious components leading to large false alarms. For NOMP, it
provides the CFAR performance based termination criterion by calculating the threshold with the knowledge of noise variance and a specified probability of false alarm, which is similar to the standard threshold detection. In practice, the interference levels often vary over time. Besides, the standard threshold is sensitive to the changes of the noise variance, and small errors in setting the threshold will have major impacts on radar performance [21].

A classical model arising in adaptive CFAR target detection is to detect the possible presence of a (point-like) coherent target from a given cell under test (CUT) by using the primary and the secondary data, where the noise is colored [22]. Recently, invariance principle and maximal invariance statistics are used to design adaptive radar detectors [23], [24]. In [25], adaptive CFAR radar detection under both steering vector matched and mismatched conditions is designed, which guarantees the CFAR property. Here we study a different setup where the noise is i.i.d., but the secondary data are unavailable and the received data consist of several sinusoidal signals whose frequencies are continuous-valued and whose number is unknown. Therefore, it is of vital importance to incorporate the CFAR detector into the CS based algorithm such that both the advantages of the CS and CFAR are preserved.

A. Related Work

In this article, the CFAR detector is incorporated into the NOMP algorithm named as NOMP-CFAR, so that the integrated detector threshold can be adjusted to maintain the desired false alarm rate. Compared to CFAR method in [20], which is similar to the standard threshold detection with the knowledge of noise variance, NOMP-CFAR estimates the noise variance from data in real time to maintain a CFAR property in the environment with time-varying noise variance. To preserve the superresolution performance of NOMP-CFAR and avoid the target masking effects [21], NOMP-CFAR is divided into two steps: initialization and model order estimation (MOE) or detection step. For the initialization, NOMP is used to obtain a maximum possible number of spectral as a candidate set, which is beneficial for closely spaced weak target detection. For the MOE, the CFAR detector is incorporated and outputs a soft quantity $\Delta (36)$, representing the level in which the amplitude of detected frequency exceeds its corresponding threshold. This is different from the conventional CFAR detector as it outputs a binary decision. The candidate frequency is preserved or removed in the candidate set based on the sign of $\Delta$. In each iteration, the most unlikely frequency corresponding to the maximum negative $\Delta$ is removed. Then, the Newton refinements are conducted to improve the estimation accuracy of the parameters of the candidates.

B. Main Contributions

Motivated by the high estimation accuracy and low computation complexity of NOMP and its excellent performance in the application fields [26], [27], [28], [29], [30], the adaptive CFAR detector is incorporated into the NOMP and NOMP-CFAR is proposed. The main contributions of this work can be summarized in the following three aspects:

1) The NOMP-CFAR algorithm is designed, which inherits both the CFAR and super-resolution advantages of the CFAR and NOMP. Incorporating the CFAR approach into the NOMP is nontrivial as the proposed CFAR detector outputs a soft quantity instead of a simple binary decision. Besides, the output of NOMP is matched to the desired input distribution of the CFAR detector, and an initialization step via the NOMP approach is also provided to improve the detection probability, as shown in Sections III-B and IV-E. Several implement details are also taken into consideration to enhance the performance of NOMP-CFAR. For further details, please refer to Section III.

2) The performances of NOMP-CFAR in terms of false alarm probability and detection probability are analyzed for the cell averaging (CA) CFAR. Specifically, the relationship between the false alarm rate and the required threshold multiplier is established, and insights into the relationship between CFAR and NOMP are also revealed. The detection probability by ignoring inter-sinusoid interference is derived.

3) The NOMP-CFAR is also extended to deal with the compressive measurement scenario and the multiple measurement vector setting (MMV).

4) The performance of NOMP-CFAR algorithm are compared with the NOMP, VALUE, and CFAR detector in numerical simulations and real data experiments. Numerically, it is shown that both NOMP-CFAR and NOMP (noise variance known) provide a CFAR for additive white Gaussian noise (AWGN) scenario. For the detection probability 0.76 and the false alarm probability 0.01, NOMP-CFAR yields 1 dB performance loss, compared to NOMP. For the noise variance which is time-varying, the NOMP algorithm can maintain CFAR performance if the noise variance can be estimated in real time and in high accuracy. However, in practice, the estimated noise variance may differ from the noise variance at the current time, the NOMP algorithm will not preserve its CFAR property by using the estimated noise variance, as the false alarm rate is sensitive to the noise variance [21]. Besides, real experiments are conducted to demonstrate that the detection probability of NOMP-CFAR is higher than that of CFAR and the false alarm is smaller than CFAR.

The rest of this article is organized as follows. Section II introduces the LSE model. In Section III, NOMP-CFAR algorithm is developed. In addition, NOMP-CFAR is also extended to deal with the compressive measurement model and MMV model. In Section IV, numerical experiments are conducted to evaluate the performance of NOMP-CFAR algorithm in terms of estimation accuracy, false alarm probability, and detection probability. The real data experiments
are shown in Section V. Finally, Section VI concludes this article.

**Notation:** Let \( \mathbf{a} \), \( \mathbf{A} \), \( \mathcal{A} \) denote the vector, matrix, and tensor, respectively. For a D-dimension tensor \( \mathcal{Y} \), let \( \mathcal{Y}_k \) denote the \( k \)th element of \( \mathcal{Y} \), where \( k \in \mathbb{N}^D \) and \( \mathbb{N} \) is the set of all natural numbers, and \( \mathcal{Y} \) denotes its spectrum. For a set \( \mathcal{N} \), let \( |\mathcal{N}| \) denote its cardinality. For any two frequencies \( \omega_1 \) and \( \omega_2 \), the wrap-around distance is defined as \( \text{dist}(\omega_1, \omega_2) \triangleq \min_{\omega \in \mathbb{Z}} |\omega_1 - \omega_2 + 2\pi n| \) and \( \mathbb{Z} \) is the set of all integers.

II. PROBLEM SETUP

Consider a multidimensional line spectral \( \mathcal{Z} \in \mathbb{C}^{N_1 \times N_2 \times \cdots \times N_D} \) described as
\[
\mathcal{Z} = \sum_{k=1}^{K} \mathbf{x}_k \mathbf{a}_{N_1}(\omega_{1,k}) \circ \mathbf{a}_{N_2}(\omega_{2,k}) \cdots \mathbf{a}_{N_D}(\omega_{D,k})
\]
where \( D \) denotes the dimension (usually \( D = 1, 2, 3 \)), \( K \) denotes the number of spectral, \( N_d \in \mathbb{N} \) \((d = 1, \ldots, D)\) denotes the number of observations in the \( d \)th dimension, and \( N = \prod_{d=1}^{D} N_d \) is the total number of observations, \( \circ \) denotes the outer product of two vectors, \( \omega_{d,k} \) denotes the frequency of the \( k \)th target in the \( d \)th dimension, \( \mathbf{x}_k \) denotes the \( k \)th target amplitude \((k = 1, \ldots, K)\), \( \mathbf{a}(\omega) \) is an array steering vector defined as
\[
\mathbf{a}(\omega) = \left[1, e^{j\omega}, \ldots, e^{j(P-1)\omega}\right]^T (P \in \mathbb{N}, \omega \in [0, 2\pi]).
\]
(2)
The line spectral \( \mathcal{Z} \) corrupted by additive noise \( \epsilon \) is described as
\[
\mathcal{Y} = \mathcal{Z} + \epsilon
\]
where \( \mathcal{Y} \) is the noisy measurement, \( \epsilon \) denotes the AWGN, and its \( n \)th element follows \( \epsilon_n \sim \mathcal{CN}(0, \sigma^2) \).

By defining \( \mathbf{w}_k = [\omega_{1,k}, \omega_{2,k}, \ldots, \omega_{D,k}]^T \) and \( \mathbf{A}(\mathbf{w}_k) = \mathbf{a}_{N_1}(\omega_{1,k}) \circ \cdots \circ \mathbf{a}_{N_D}(\omega_{D,k}) \), model (1) can be simplified as
\[
\mathcal{Y} = \sum_{k=1}^{K} \mathbf{x}_k \mathbf{A}(\mathbf{w}_k) + \epsilon.
\]
(4)
Vectorizing the tensors, model (3) can be reformulated as a matrix form
\[
\mathbf{y} = \mathbf{Ax} + \epsilon
\]
where \( \mathbf{y} = \text{vec}(\mathcal{Y}) \), \( \mathbf{x} = [x_1, x_2, \ldots, x_K] \) and
\[
\mathbf{a}_N(\mathbf{w}_k) \triangleq \mathbf{a}_{N_1}(\omega_{1,k}) \otimes \cdots \otimes \mathbf{a}_{N_D}(\omega_{D,k})
\]
\( \otimes \) denotes the Kronecker product, \( \epsilon = \text{vec}(\epsilon) \) and \( \epsilon \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_n) \) and \( \mathbf{A} \) is the matrix, which can be shown as
\[
\mathbf{A} = \left[\mathbf{a}_N(\omega_1), \ldots, \mathbf{a}_N(\omega_k), \ldots, \mathbf{a}_N(\omega_K)\right].
\]
(7)
Note that we have provided both the tensor model (4) and the common data model (5). The tensor model is more suitable for practical problems. For example, in radar signal processing, the received data is often represented as a 3-D data cube, where the 3-D correspond to the fast time domain, slow time domain, and spatial domain. Besides, often \( D \) dimensional FFT is adopted to process the data cube and perform CFAR detection. As you see, using the tensor notation is beneficial to design the guard cells and training cells. As for the common data model (5), it is in fact useful for extracting the amplitudes of the frequencies once the frequencies are estimated. Note that the amplitudes can be updated by the least square step \( \hat{x} = \mathbf{A}^T \mathbf{y} \) in NOMP method, where \( \mathbf{A}^T \) denotes the pseudoinverse matrix of \( \mathbf{A} \).

In the following, we will use the two models interchangeably when either one is more suitable.

Consequently, the line spectral estimation and detection problem are to obtain \( \hat{K} \) and the frequencies \( \{\hat{\omega}_{1,k}, \hat{\omega}_{2,k}, \ldots, \hat{\omega}_{D,k}\}_{k=1}^{K} \) without the knowledge of noise variance \( \sigma^2 \).

The SNR in dB of the \( k \)th target is defined as [20]
\[
\text{SNR}_k = 10 \log \frac{N_x}{\sigma^2} = 10 \log \text{SNR}_{\text{sample,}k} = 10 \log \text{SNR}_{\text{sample,}k} - 10 \log(N)
\]
(8)
which is the nominal SNR value in our simulations and is called “integrated SNR.” An alternative definition of SNR is the “per-sample” SNR given by \( \text{SNR}_{\text{sample,}k} = 10 \log \frac{|x_k|^2}{\sigma^2} = 10 \log \text{SNR}_{\text{sample,}k} - 10 \log(N) \). It can be seen that the “integrated SNR” is the “per-sample” SNR plus the coherent integrated gain \( 10 \log(N) \).

III. NOMP-CFAR ALGORITHM

This section develops the NOMP-CFAR algorithm for target detection and estimation. The false alarm probability is often defined clearly for a binary hypothesis testing problem. For the LSE, an intuitively explanation of the false alarm probability is that for the false alarm probability \( \tilde{P}_{FA} \), the NOMP-CFAR algorithm generates about \( N_{\text{MC}} \tilde{P}_{FA} \) false targets whose frequencies are not near the frequencies of the true targets in \( N_{\text{MC}} \) Monte Carlo trials. First, the CFAR criterion is proposed and the threshold with the false alarm probability \( \tilde{P}_{FA} \) is provided. Second, the details of NOMP-CFAR which novelly combines the CFAR and NOMP are presented. Note that we have carefully designed NOMP-CFAR to ensure that the measured false alarm probability is close to the nominal false alarm probability. Finally, NOMP-CFAR is extended to deal with both the compressive measurement model and an MMV model.

A. CFAR Detector Design

The stopping criterion proposed in [20] assumes that the noise variance is known and constant. This stopping criterion sets the threshold accurately to guarantee a specified probability of false alarm. In practice, the noise variance is unknown and can often be variable. To provide predictable detection and false alarm behavior in realistic scenarios, CFAR detection or “adaptive threshold detection” is developed.
The idea of CFAR detector is to estimate the noise variance from the data in real time, so that the detection threshold can be adjusted to maintain the desired $P_{FA}$. Below we present the details of the CFAR design.

We consider a general binary detection problem as

$$
\begin{align*}
\mathcal{H}_0 & : \mathcal{Y} = \varepsilon \\
\mathcal{H}_1 & : \mathcal{Y} = x \mathcal{A}(\omega) + \varepsilon
\end{align*}
$$

(9)

and we choose between the null hypothesis $\mathcal{H}_0$ and the alternative hypothesis $\mathcal{H}_1$, where $\text{vec}(\varepsilon) \sim \mathcal{CN}(0, \sigma^2 I_N)$ is the additive white Gaussian noise (AWGN). For the dimension $D = 1$, the binary detection problem is similar to the sinusoidal signal detection model [2]. The typical detector deciding $\mathcal{H}_1$ can be written as

$$
T(\mathcal{Y}, \sigma^2) = \frac{|\hat{Y}_{\text{peak}}|^2}{\sigma^2} > \alpha' \tag{10}
$$

where $\cdot$ is applied elementwise, $\hat{Y}$ denotes the normalized $D$-dimensional DFT of $\mathcal{Y}$, whose $\hat{n} \triangleq [\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_D]$th element is defined as

$$
\hat{Y}_{\hat{n}} = \frac{1}{N} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_D=1}^{N_D} e^{-j \pi (\hat{n}_1-1)(n_1-1) \cdot } \sum_{n_D=1}^{N_D} e^{-j \pi (\hat{n}_2-1)(n_2-1) \cdot } \sum_{n_D=1}^{N_D} \cdots \sum_{n_D=1}^{N_D} e^{-j \pi (\hat{n}_D-1)(n_D-1) \cdot } Y_n
$$

(11)

and $\hat{n}_{\text{peak}}$ is the peak localization of $|\hat{Y}|^2$ given by

$$
\hat{n}_{\text{peak}} = \arg \max_{\hat{n} \in \hat{N}} |\hat{Y}_{\hat{n}}|^2 \tag{12}
$$

where $\hat{N} = \{ [\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_D] | \hat{n}_1 = 1, 2, \ldots, N_1, \ldots, \hat{n}_D = 1, 2, \ldots, N_D \}$. In particular, for 1-D LSE, i.e., $D = 1$, $\mathcal{Y}$ reduces to $\mathcal{y}$, which are the measurements in time domain and $n$ denotes the index in time domain, $\mathcal{y}$ is the spectrum, and $\mathcal{y}$ denotes the index in frequency domain, and $\mathcal{n}_{\text{peaks}}$ is the peak localization of the spectrum $\mathcal{y}$.

In other words, the detector (10) decides that the signal is present if the peak value of the spectrum exceeds a threshold. It is also worth noting that (10) can be evaluated efficiently through $D$ dimensional DFT. Given the false alarm probability $P_{FA}$, the threshold multiplier $\alpha'$ can be obtained by [20]

$$
\alpha'_{\text{NOMP}} = -\ln \left( 1 - (1 - P_{FA})^{\frac{1}{\sigma^2}} \right). \tag{13}
$$

Motivated by the conventional CFAR approach, the noise variance $\sigma^2$ is estimated with the data samples as

$$
\hat{\sigma}^2 = \frac{1}{N_r} \sum_{\hat{n} \in \hat{T}_{\text{peak}}} |\hat{Y}_{\hat{n}}|^2 \tag{14}
$$

where $\hat{n}_{\text{peak}}$ (12) is the index of the CUT $\hat{Y}_{\text{peak}}$, $\hat{T}_{\text{peak}}$ is the index set of the reference cells, $N_r$ denotes the number of reference cells, and $N_r = |\hat{T}_{\text{peak}}|$. The reference cells are averaged to estimate the noise variance. In practice, we set guard cells between the reference cells and the CUT. The reason is that a frequency not lying on the DFT grid exactly might straddle frequency cells. In this case, the energy in the cell adjacent to $\hat{Y}_{\text{peak}}$ would contain both noise and signal energy. The extra energy from the signal would tend to raise the estimate of the noise variance, resulting in a higher threshold and a lower $P_{FA}$ and $P_{D}$ than intended. The details of selecting reference cells and guard cells can be referred to [21]. However, in the multiple target scenario, the construction of reference cells cannot be directly borrowed.

Fig. 1 gives the CFAR windows with the same reference cells $N_r = 6$ (one dimension) and $N_r = 26$ (two dimension) for the traditional CFAR and the proposed CFAR. Compared to the traditional CFAR window, the detected cells which are closest to the detected frequencies are excluded from the average to estimate the noise variance. The reason is that the detected cells have been fitted with the sinusoidal components. In this case, the energy in the detected cells for the CFAR detection will not be representative of the noise alone and is lower than the energy in the reference cells. If you use the traditional CFAR window, the lower energy due to the detected cells would tend to lower the estimate of the noise variance parameter, which results in a higher $P_{FA}$ and $P_{D}$ than intended. We have also conducted a simulation to show that if we estimate the noise using the traditional approach, the false alarm probability is always higher than the nominal false alarm probability, and the false alarm probability and detection probability are higher than those of the NOMP-CFAR, see Fig. 6 shown later in Section IV-C.

With $\hat{\sigma}^2$ given in (14), the proposed CFAR detector in the case of unknown noise variance is

$$
T(\mathcal{Y}) = \frac{|\hat{Y}_{\text{peak}}|^2}{\hat{\sigma}^2} > \alpha. \tag{15}
$$

Equation (15) indicates that the candidate target is valid when the level target amplitude $|\hat{Y}_{\text{peak}}|^2$ exceeds the noise variance estimation $\hat{\sigma}^2$ multiplied by $\alpha$. For the detector $T(\mathcal{Y})$ in (15), its false alarm probability $P_{FA}$ and detection probability $P_{D}$ are analyzed. First, we need to calculate the probability density function (PDF) of $\hat{\sigma}^2$ (14). Note that the normalized DFT is a unitary matrix, and the DFT of $\mathcal{Y}$ under the null hypothesis $\mathcal{H}_0$ is still the AWGN, i.e.,

$$
p(\mathcal{Y} | \mathcal{H}_0) = \prod_{\hat{n}} \mathcal{CN}(\hat{Y}_{\hat{n}}; 0, \sigma^2). \tag{16}
$$

In general, the exact PDF of $\hat{\sigma}^2$ (14) is hard to obtain as the reference cells depend on the peak localization $\hat{n}_{\text{peak}}$ of the spectrum. Provided that the reference cells are not too near the peak, the PDF of $\hat{Y}_{\hat{n}}$, $\hat{n} \in \hat{T}_{\text{peak}}$ is approximated as a Gaussian distribution, i.e.,

$$
p(\hat{Y}_{\text{peak}} | \mathcal{H}_0) \approx \prod_{\hat{n} \in \hat{T}_{\text{peak}}} \mathcal{CN}(\hat{Y}_{\hat{n}}; 0, \sigma^2). \tag{17}
$$

---

1. In Appendix A, the generalized likelihood ratio test (GLRT) explanation is provided.
Based on (17), we proceed to calculate the average false alarm probability \( \tilde{P}_{FA} \) and the result is summarized as Proposition 1.

**Proposition 1** The average false alarm probability \( \tilde{P}_{FA} \) and the required threshold multiplier \( \alpha \) for the detector (15) is

\[
\tilde{P}_{FA} = 1 - \frac{1}{(N_r - 1)!} \int_0^{+\infty} \left( 1 - e^{-\frac{\alpha}{\sigma}} \right)^N x^{N_r-1} e^{-x} dx
\]

\[
= 1 - \int_0^{+\infty} \exp \left\{ N \ln \left( 1 - e^{-\frac{\alpha}{\sigma}} \right) + (N_r - 1) \ln x - x - \sum_{n=1}^{N_r-1} \ln n \right\} dx
\]

\[
= 1 - \int_0^{+\infty} \exp \{ f(x) \} dx
\]

or

\[
\tilde{P}_{FA} = 1 - \sum_{n=0}^{N} (-1)^n \frac{N \alpha^N}{N_r + 1} \frac{1}{N_r + 1}.
\]

**Proof** The basic idea is to obtain the false alarm probability \( \tilde{P}_{FA} \) with \( \hat{\sigma}^2 \) fixed, and then averaging that with respect to the PDF of \( \hat{\sigma}^2 \) yields the desired false alarm probability \( \tilde{P}_{FA} \).

Define the threshold \( \tilde{T} \) as

\[
\tilde{T} = \alpha \left( \frac{1}{N_r} \sum_{\hat{\mathbf{n}} \notin \hat{\mathbf{Y}}_{peak}} |\hat{\mathbf{Y}}_{\hat{n}}|^2 \right).
\]

Under the assumption (17), the distribution of \( \tilde{T} \) is chi-square distribution with \( 2N_r \) degrees of freedom and variance \( \alpha^2/2 \), i.e.,

\[
p_{\tilde{T}}(\tilde{T}) = \frac{1}{(N_r - 1)!} \left( \frac{N_r}{\alpha^2} \right)^N \tilde{T}^{N_r-1} e^{-\frac{\alpha^2}{N_r}}.
\]

The false alarm event is declared if the peak of the spectrum \( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \) exceeds the threshold \( \tilde{T} \), i.e.,

\[
\tilde{P}_{FA} = E_{\tilde{T}} \left[ P \left( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \leq \tilde{T} \right) \right]
\]

where \( E_{\tilde{T}} \) is taken with respect to the PDF \( p_{\tilde{T}}(\tilde{T}) \) (21).

With \( \tilde{T} \) being fixed, \( P(\hat{\mathbf{Y}}_{\hat{n}}^2 \geq \tilde{T}) \) is

\[
P(\hat{\mathbf{Y}}_{\hat{n}}^2 \geq \tilde{T}) = \max_{\hat{n} \in \hat{\mathbf{T}}} |\hat{\mathbf{Y}}_{\hat{n}}|^2 \geq \tilde{T}
\]

\[
= 1 - P \left( \max_{\hat{n} \in \hat{\mathbf{T}}} |\hat{\mathbf{Y}}_{\hat{n}}|^2 \leq \tilde{T} \right)
\]

\[
= 1 - P \left( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \leq \tilde{T}, \hat{n} \in \hat{\mathbf{T}} \right).
\]

where \( \alpha \) is due to the independence of \( \hat{\mathbf{Y}}_{\hat{n}}, \hat{n} \in \hat{\mathbf{T}} \). The PDF of \( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \) is chi-square distribution with \( 2 \) degrees of freedom and variance \( \sigma^2/2 \),

\[
P \left( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \leq \tilde{T} \right) = 1 - e^{-\tilde{T}/2}. \]

Substituting (23) in (22), the average false alarm probability \( \tilde{P}_{FA} \) is

\[
\tilde{P}_{FA} = 1 - \int_0^{+\infty} \prod_{\hat{n} \in \hat{\mathbf{T}}} P \left( |\hat{\mathbf{Y}}_{\hat{n}}|^2 \leq \tilde{T} \right) \tilde{p}_{\tilde{T}}(\tilde{T}) d\tilde{T}.
\]

Substituting (24) and (21) into (25) yields

\[
\tilde{P}_{FA} = 1 - \int_0^{+\infty} \left( 1 - e^{-\tilde{T}/2} \right)^N \frac{1}{(N_r - 1)!} \left( \frac{N_r}{\alpha^2} \right)^N \tilde{T}^{N_r-1} e^{-\frac{\alpha^2}{N_r}} d\tilde{T}
\]

which can be simplified as (18) by doing a change of variable \( x = \frac{N_r}{\alpha^2} \tilde{T} \). Furthermore, by using the binomial expansion

\[
\left( 1 - e^{-\frac{\alpha^2}{N_r}} \right)^N = \sum_{n=0}^{N} \frac{N!}{n!(N_n)!} \left( -1 \right)^n e^{-\frac{\alpha^2}{N_r}}
\]

(18) can be simplified as

\[
\tilde{P}_{FA} = 1 - \sum_{n=0}^{N} \frac{N!}{n!(N_n)!} \int_0^{+\infty} e^{-\left( \frac{\alpha^2}{N_r} \right) x} x^{N_r-1} dx.
\]

Utilizing

\[
\int_0^{+\infty} e^{-\left( \frac{\alpha^2}{N_r} \right) x} x^{N_r-1} dx = \frac{(N_r-1)!}{(\frac{\alpha^2}{N_r})^N},
\]

(19) is obtained

We have provided two formulas (18) and (19) for \( \tilde{P}_{FA} \) with the required threshold multiplier \( \alpha \). Note that \( \lim_{\alpha \to 0} \exp(f(x)) = \lim_{\alpha \to -\infty} \exp(f(x)) = 0 \), where \( f(x) \) is defined in (18), and the MATLAB built-in integrate function can be used to numerically evaluate the integral \( \int_0^{+\infty} \exp(f(x))dx \). In contrast, (19) is hard to calculate as the dynamic range of the terms in the summation can be very large. Thus, (18) is more robust than (19) in terms of calculating \( \tilde{P}_{FA} \) via numerical method.

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Fig. 1. CFAR windows. (a) and (b) are 1-D and 2-D windows for the traditional CFAR processing, (c) and (d) are 1-D and 2-D windows for the proposed CFAR processing, respectively.

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However, (19) indeed provides some insights into the relationship between the establishing result and the previous results, as shown in the following remarks.

**Remark 1** As $N_r \to \infty$, one has

$$\lim_{N_r \to +\infty} \left( 1 + \frac{n\alpha}{N_r} \right)^{-N_r} = e^{-ma}. \quad (29)$$

Substituting (29) in (19), $\bar{P}_{FA}$ can be simplified as

$$\bar{P}_{FA} = 1 - \sum_{n=0}^{N} (-1)^n \left( \frac{n}{n_r} \right) e^{-ma} = 1 - (1 - e^{-\alpha})^N \quad (30)$$

and $\alpha$ can be calculated as

$$\alpha = -\ln \left( 1 - (1 - \bar{P}_{FA})^{\frac{1}{r}} \right). \quad (31)$$

Similar results, consistent with (31), are obtained in [20], which focuses on the noise variance aware case. This makes sense as $N_r \to \infty$, the noise variance estimate is exact and $\alpha = \alpha'$ with $\alpha'$ given in (13) and $\bar{P}_{FA} = P_{FA}$.

**Remark 2** When the required threshold multiplier $\alpha$ is large such that the false alarm probability $\bar{P}_{FA}$ is small, the first and second terms corresponding to $n = 0$ and $n = 1$ in the summation are much larger than the remaining terms, i.e.,

$$(N - n)!n! \left( \frac{n\alpha + N_r}{\alpha + N_r} \right)^{N_r} \gg 1, n = 2, \ldots, N. \quad (32)$$

Consequently, (19) can be approximated as

$$\bar{P}_{FA} \approx 1 - \sum_{n=0}^{1} (-1)^n \left( \frac{n}{n_r} \right) \left( \frac{n\alpha + N_r}{\alpha + N_r} \right)^{N_r} = N\left( \frac{\alpha}{N_r} + 1 \right)^{-N_r} = N\bar{P}_{FA}(\text{bin}) \quad (33)$$

where $\bar{P}_{FA}(\text{bin}) = (\alpha/N_r + 1)^{-N_r}$ is the average false alarm probability of traditional CA-CFAR method for each cell if we examine one bin. Hence, $\bar{P}_{FA}$ increases approximately linearly with the number of bins examined. Equation (33) demonstrates that when $\bar{P}_{FA}$ is very small, it is equal to the $N$ times of the average false alarm probability of traditional CA-CFAR method. The average number of false alarms among all the $N$ test cells is $N\bar{P}_{FA}(\text{bin}) \ll 1$. Intuitively, when $N\bar{P}_{FA}(\text{bin}) \ll 1$, the false alarm is dominated by the CUT corresponding to the peak of the spectrum with false alarm probability $\bar{P}_{FA}$. This implies the approximation (33). For typical parameters such as $N = 256, N_r = 60$, and $\bar{P}_{FA}$ ranging from 0.001 to 0.02, the required threshold multipliers $\alpha$ calculated through (33) are close to those calculated through (18).

**Remark 3** The two formulas (18) and (19) are obtained with the CA-CFAR approach. Following the similar steps, one could establish the specific $\bar{P}_{FA}$ with the required threshold multiplier $\alpha$ for other CFAR approaches, such as the smallest-of cell-averaging CFAR (SOCA CFAR), greater-of cell-averaging CFAR (GOCA CFAR), order statistic CFAR (OS CFAR) [31], etc. The OS-CFAR rank orders the reference window data samples $\hat{Y}_n$ to form a new sequence in ascending numerical order, $\tilde{n} \in \mathcal{T}_{n_{\text{peak}}}$. The $r$th element of the ordered list is called the $r$th order statistic, denoted by $\hat{Y}_n(r)$. Thus, the new sequence of the reference window data samples $\tilde{Y}_n$ can be denoted as $\{\hat{Y}_n(1), \ldots, \hat{Y}_n(N_r)\}$. In OS-CFAR, the $r$th order statistic is selected as representative of the noise variance and a threshold is set as a multiple of this value

$$\hat{T} = \alpha_{OS}\hat{Y}_n(r). \quad (34)$$

As shown in [31], the PDF of $\hat{T}$ is

$$p_T(\hat{T}) = \frac{r}{\alpha_{OS}} \left( \frac{N}{r} \right) \left[ e^{-\hat{T}/\alpha_{OS}} \right]^{N-r+1} \left[ 1 - e^{-\hat{T}/\alpha_{OS}} \right]^{r-1}. \quad (35)$$

Substituting (35) in (25), the average false alarm rate $\bar{P}_{FA}$ of OS-CFAR can be obtained.

Define

$$\Delta = 10 \log \left( \frac{|\hat{Y}_{n_{\text{peak}}}|^2}{\hat{T}} \right). \quad (36)$$

where $\hat{T}$ is (20). Note that $\Delta = 0$ implies that the integrated amplitude of the detected signal just crosses the corresponding threshold. For $\Delta > 0$, $\Delta$ is the excess in which the amplitude of the detected signal is above the corresponding threshold. While for $\Delta < 0$, $-\Delta$ is the excess in which the amplitude of the detected signal is below the corresponding threshold. For the traditional CFAR detector, it makes a binary decision based on the sign of $\Delta$. In contrast, we provide a “soft” CFAR detector, which outputs $\Delta$. The “soft” CFAR detector is summarized in Algorithm 1.

It is also meaningful to obtain the detection probability $P_D$ for the required threshold multiplier $\alpha$. Under the alternative hypothesis $H_1$, we declare that a frequency $\omega$ exists if the spectrum $\hat{Y}$ at the DFT frequency $\omega$ exceeds its corresponding threshold, where the $d$th element of $\tilde{n}$ is

$$\tilde{n}_d = \arg\min_{n_d} |\omega_d - \tilde{\omega}_{n_d}| \quad (37)$$

where $\tilde{\omega}_{n_d} \triangleq (n_d - 1)2\pi/N_d$ denotes the $d$th dimensional $n_d$th DFT frequency grid. Straightforward calculation

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**Algorithm 1: CFAR Detector.**

1. For a given false alarm probability $P_{FA}$, calculate the required threshold multiplier $\alpha$ for CA-CFAR via (18).
2. Procedure: CFAR ($\mathcal{Y}$, $\alpha$) :
3. Compute $\hat{Y}$ via (11),
4. Find the peak location $\tilde{n}_{\text{peak}}$ via (12),
5. Estimate the noise variance as $\sigma^2$ (14),
6. Compute $\Delta$ (36),
7. Return $\Delta$. 

---
shows that under the alternative hypothesis $H_1$, 
\[
\hat{Y}_h = \sqrt{N_x} \prod_{d=1}^{D} \left( e^{-\frac{1}{2}\sigma^2/N_d} \sin\left( \frac{\omega_d - \hat{\omega}_h}{2} \right) \right) + \bar{e}
\]
\[
= \sqrt{N_x} \prod_{d=1}^{D} \left( e^{-\frac{1}{2}\sigma^2/N_d} \sin\pi \beta_d \right) + \bar{e}
\]
\[
\cong \sqrt{N_x} \beta_h + \bar{e}
\]  
(38)

where
\[
\delta_d = (\omega_d - \hat{\omega}_h)/(2\pi/N_d)
\]
and $-0.5 \leq \delta_d \leq 0.5$ due to $|\omega_d - \hat{\omega}_h| \leq \pi/N_d$, $|\beta_h| \leq 1$. It can be known that $\hat{Y}_h \sim \mathcal{CN}(\sqrt{N_x} \beta_h, \sigma^2)$. Conditioned on $\hat{T}$, the detection probability $P_D(\hat{T})$ is
\[
P_D(\hat{T}) = P\left( |\hat{Y}_h|^2 > \hat{T} \right)
\]
\[
= P\left( \frac{|\hat{Y}_h|^2}{\sigma^2/2} > \alpha \right).
\]  
(40)

As we know, the random variable $|\hat{Y}_h|^2/(\sigma^2/2)$ follows noncentral chi-square distribution with two degrees of freedom, and the random variable $\sigma^2/(\sigma^2/2)$ also follows chi-square distribution with $2N_x$ degrees of freedom. Assume that they are independent, $|\hat{Y}_h|^2/(\sigma^2/2)$ follows noncentral F-distribution, with numerator degrees of 2, denominator degrees of $2N_x$, and positive noncentral parameter $N|x|^2 \beta_h^2$. Define $P_F(x|v_1, v_2, \mu)$ as the noncentral F-distribution CDF at each value in $x$, where $v_1$ denotes the degrees of freedom of the numerator, $v_2$ denotes the degrees of freedom of the denominator, $\mu$ denotes the positive noncentrality parameters. Equation (40) can be simplified as
\[
P_D = 1 - P_F\left( \alpha|2, 2N_x, N|x|^2 \beta_h^2 \right)
\]  
(41)

which is the average detection probability of the CFAR detector (15) for the given target by taking only the noise into account and ignoring the intersinusoidal interference.

B. NOMP-CFAR

Section III-A has proposed the CFAR detector for a single target scenario and provided (18) for calculating the threshold multiplier $\alpha$ for a specified $P_{FA}$. Now we focus on the multiple target detection problem and design NOMP-CFAR algorithm.

For the target detection and estimation problem, one should try to find a minimal representation of the line spectral such that the residue can be well approximated as the AWGN. Note that the conventional CFAR based approach consists of two steps: First, performing the fast Fourier transform (FFT) on $Y$ to obtain the spectrum $\hat{Y}$. Second, implementing the CFAR detection such as CA or OS methods for all the cells by specifying the guard length, training length, and the average false alarm rate $P_{FA}$, the target detection results are obtained. Obviously, the FFT-based approach is suboptimal as it suffers from grid mismatch and intertarget interference. Besides, the resolution of the FFT-based approach is limited by the Rayleigh criterion. Therefore, we propose NOMP-CFAR to overcome the drawbacks of the traditional CFAR based approach and to inherit the advantages of the CFAR behaviour.

The main steps of the NOMP-CFAR can be divided into two steps: Initialization and MOE or detection step. The initialization step provides a good initial point of the NOMP-CFAR. While for the MOE step, it is iterative and it includes an activation step and deactivation step, i.e., by activating the target or deactivating the target. For the initialization step, it is assumed that the number of targets $K$ is upper bounded by a known constant $K_{\text{max}}$, i.e., $K \leq K_{\text{max}}$. This is reasonable as one could choose $K_{\text{max}} = N$ as the number of targets must be less than the number of measurements $N$. Then we run NOMP consisting of coarse detection, single refinement, cyclic refinement, least squares (LS) step to obtain the candidate amplitudes and frequencies set $K_{\text{con}} = \{\hat{\omega}_k, \hat{\omega}_k, k = 1, 2, \ldots, K_{\text{max}}\}$. Note that this step is necessary as it alleviates the intertarget interference and provides fine results for the MOE step. For further details, please see [20]. For the MOE step, the CFAR approach is implemented to perform target detection, i.e., activate and deactivate the components of the frequencies. In addition, all the steps such as coarse detection, single refinement, cyclic refinement, LS in NOMP are also incorporated to enhance the target estimation and detection accuracy. Below we present the details of the MOE step.

Given the candidate set $K = \{\hat{\omega}_k, \hat{\omega}_k, k = 1, 2, \ldots, \hat{K}\}$ and $\hat{K} = |K| \leq K_{\text{max}}$, one could use the CFAR criterion to detect the $k$th frequency. To implement the CFAR detector, we proceed as follows: The residue of the observation $\hat{Y}_r(K)$ after eliminating all the targets $K$ is
\[
\hat{Y}_r(K) = Y - \sum_{k=1}^{\hat{K}} \hat{x}_k A(\hat{\omega}_k).
\]  
(42)

Provided that all the amplitudes and the frequencies are detected and estimated in high accuracy, $\hat{Y}_r(K)$ is approximately AWGN, i.e.,
\[
\hat{Y}_r(K) \approx \varepsilon, \quad (43)
\]
and $\text{vec}(\varepsilon) \sim \mathcal{CN}(0, \sigma^2 I_y)$. To detect the $k$th target, we obtain the pseudomeasurement $\hat{Y}_r(K \setminus (\hat{\omega}_k, \hat{\omega}_k))$ by adding the contribution of the $k$th frequency to the residue $\hat{Y}_r(K)$, i.e.,
\[
\hat{Y}_r(K \setminus (\hat{\omega}_k, \hat{\omega}_k)) = \hat{Y}_r(K) + \hat{x}_k A(\hat{\omega}_k) \overset{\alpha}{\approx} \hat{x}_k A(\hat{\omega}_k) + \varepsilon
\]  
(44)

where $\overset{\alpha}{\approx}$ is due to (43). We now input $\hat{Y}_r(K \setminus (\hat{\omega}_k, \hat{\omega}_k))$ and the required threshold multiplier $\alpha$ to the CFAR detector to provide a soft detection of the $k$th frequency, $k = 1, 2, \ldots, \hat{K}$, i.e., calculate $\Delta_k$ (35). Note that $\Delta_k \geq 0$ means that the

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2For range estimation, range Doppler estimation, the spectrum corresponds to range spectrum and range-Doppler spectrum, respectively.
power of the CUT exceeds the estimated threshold, and this target should be detected if the CFAR is adopted. If we radically implement the CFAR detector to detect all the targets and then stop, miss event may happen in certain scenarios. For example, in the initialization step, a strong target may be detected as two closely spaced frequencies with distinguishable amplitudes. This splitting phenomenon degrades the SNR of the frequency and the CFAR detector may miss the two targets at the same time, which misses the true frequency. To overcome this problem, we only activate or deactivate one frequency in each iteration. In this way, even after splitting, the frequency corresponding to the weaker amplitude may be eliminated and the frequency one with stronger amplitude can be refined, and the true frequency may be detected. To activate or deactivate one frequency in each iteration, the greedy approach is adopted by calculating

$$\hat{k} = \arg \min_{k=1,2,...,K} \Delta_k$$ (45)

and $\Delta_k$. Depending on the sign of $\Delta_k$, we choose two different actions:

A1 $\Delta_k < 0$: In this setting, deactivate the $\hat{k}$th frequency by removing the $\hat{k}$th frequency, i.e., updating $\hat{K}' = \hat{K} \setminus (\hat{x}_k, \hat{\omega}_k)$. Perform the single refinement, cyclic refinement, and LS for the remaining targets and obtain $\hat{K}'$. This refinement step is necessary to ensure the high estimation accuracy. Then for the updated parameter set $\hat{K}'$, perform CFAR detection as done before.

A2 $\Delta_k \geq 0$: In this setting, the frequency set is kept. Then, perform CFAR detection on the residual $\hat{Y}_i(\hat{K})$ to determine whether a new amplitude frequency pair should be added on the list $\hat{K}$ or not. If the CFAR does not detect any target or the number of targets is greater than $K_{\text{max}}$, then the algorithm stops. Otherwise, add the new amplitude frequency pair into the list and iterate.

Note that compared to the traditional CFAR, our procedure can be viewed as a soft CFAR as we proceed very gently. The NOMP-CFAR is summarized as Algorithm 2. In addition, the module of NOMP-CFAR is provided in Fig. 2 to further explain the idea of NOMP-CFAR. The CFAR detector module performs CFAR detection on each candidate frequencies and outputs $\{\Delta_k\}_{k=1}^{K}$. Provided that $\min_{k=1,2,...,K} \Delta_k < 0$, the NOMP module removes the most unlikely frequency component. Otherwise, all the frequencies are kept and a CFAR detection is implemented on the residual $\hat{Y}_i(\hat{K})$ to determine whether a new amplitude frequency pair should be added on the list $\hat{K}$ or not. The CFAR detector determines the validity of the target and provides a “soft” threshold to the NOMP module. NOMP then performs refinement on the remaining components. Besides, we also show that after initialization provided by NOMP, the distribution of the input to the CFAR module is well matched to the desired distribution of the CFAR module. Let $K = 16$ and $K_{\text{max}} = 32$. The quantile-quantile plot (q-q plot) of the residue after eliminating ranging from $\hat{K} = 16$ to $\hat{K} = 31$ targets in a single realization is shown in Fig. 3. The quantile values of the input sample appear along the $y$-axis, and the theoretical values of the specified distribution at the same quantiles appear along the $x$-axis. It can be seen that the resulting plot is approximately linear, demonstrating that the residue after eliminating the effects of all the targets can be well approximated by AWGN, which is the desired distribution of the CFAR detector under the null hypothesis $H_0$.

C. Extension of NOMP-CFAR

The above NOMP-CFAR can be easily extended to deal with the MMV setting and the compressive setting.

1) MMV Setting: The MMV model can be described as

$$\hat{Y}(s) = \sum_{k=1}^{K} \chi_k(s) A(\omega_k(s)) + \epsilon(s), \quad s = 1, 2, \ldots, S$$ (46)
Fig. 3. Quantile-quantile plot (qqplot) of the residue from $K = 16$ to $K = 31$, where $K_{\text{max}} = 32$.

where $S$ denotes the number of snapshots. Details about single refinement, cyclic refinement, and LS step can be referred to [26]. Here we just focus on the difference and provide the details of the CFAR detection.

Define $\tilde{Y}(s)$ as the normalized DFT of $Y(s)$. The CFAR detector deciding the alternative hypothesis $H_1$ is

$$T(Y) = \frac{\frac{1}{S} \sum_{s=1}^{S} |\tilde{Y}_{\text{peak}}(s)|^2}{\hat{\sigma}^2_{\text{mmv}}} \geq \alpha_{\text{mmv}}$$

(47)

where $\tilde{n}_{\text{peak}}$ denotes the peak localization of $\frac{1}{S} \sum_{s=1}^{S} |\tilde{Y}(s)|^2$, which is the average of the spectrum over the snapshots, $\hat{\sigma}^2_{\text{mmv}}$ is

$$\hat{\sigma}^2_{\text{mmv}} = \frac{1}{SN_r} \sum_{s=1}^{S} \sum_{\tilde{n} \in T_{\text{peak}}} |\tilde{Y}_{\tilde{n}}(s)|^2$$

(48)

$\alpha_{\text{mmv}}$ is the required threshold multiplier. Then, given $\alpha_{\text{mmv}}$, the false alarm probability $\tilde{P}_{\text{FA}}$

$$\tilde{P}_{\text{FA}} = 1 - \frac{N_r}{\alpha(SN_r - 1)!} \int_0^{+\infty} \left( 1 - e^{-\frac{N_r}{\alpha}} \sum_{x=0}^{S-1} T_x N \right) \frac{dT}{e^{\frac{N_r}{\alpha}} T^{SN_r-1}}$$

(49)

is established, see Appendix B for the details. Note that for SMV scenario, i.e., $S = 1$, (49) reduces to (18).

2) **Compressive Setting:** For simplicity, we focus on the 1-D compressive line spectrum estimation problem formulated as

$$y = \Phi \left( \sum_{k=1}^{K} a(\omega_k) x_k \right) + \varepsilon$$

(50)

where $\Phi \in \mathbb{C}^{M \times N}$ is the compression matrix and $M/N$ is the compression rate. Define a “generalized” atom as

$$\hat{a}(\omega) = \frac{\Phi a(\omega)}{\|\Phi a(\omega)\|_2}.$$  

(51)

The CS model becomes

$$y = \sum_{k=1}^{K} \tilde{x}_k \hat{a}(\omega) + \varepsilon.$$  

(52)

The steps of the coarse detection, single refinement, cyclic refinement, and LS are the same as that of NOMP. Here we present the CFAR details. Given $y(K) = y - \sum_{|K|} \hat{x}_k \hat{a}(\omega)$, where $K = \{(\hat{x}_k, \hat{\omega}_k), k = 1, 2, \ldots, |K|\}$ and $(\hat{x}_k, \hat{\omega}_k)$ denotes the estimate of the amplitude frequency pair, calculate the “generalized” spectrum as

$$\tilde{y}(K) = \tilde{a}^H(\omega) y(K)$$

(53)

by restricting $\omega$ to the DFT grid. Then the detector (15) is used to perform detection, and the noise variance estimate is given by (14). Besides, the required threshold multiplier for a given $P_{\text{FA}}$ is (18).

IV. NUMERICAL SIMULATION

In this section, the derived theoretical result is verified and the performance of the proposed NOMP-CFAR is investigated. First, the false alarm probability versus the required threshold multiplier is evaluated, which provides a way to calculate the required threshold multiplier for a specified false alarm probability. Second, the effectiveness of the proposed CFAR window is demonstrated by comparing with the traditional CFAR Windows. Third, the CFAR property is validated. Fourth, the benefits of initialization
is demonstrated. Fifth, the performance of the proposed NOMP-CFAR is evaluated, including the SMV, MMV, and compressive settings. Finally, we consider a time-varying noise variance scenario and show the CFAR property of the proposed algorithm. Since the performance of FFT based CFAR detection is poor in multiple target scenarios (here we set the number of frequencies as 16), we do not compare with it. The NOMP achieves the state-of-art performance and we mainly focus on comparing NOMP-CFAR with it. Besides, we also compare with the VALSE [18], which is a Bayesian approach and which automatically estimates the noise variance and model order.

The parameters are set as follows: $N = 256$, the number of frequencies is $K = 16$, the frequencies are randomly drawn and the wrap around distances of any two frequencies are greater than $2.5\Delta_{\text{DFT}}$, where $\Delta_{\text{DFT}} = 2\pi/N$. The noise variance is $\sigma^2 = 1$. The default algorithm parameters are set as follows: The average false alarm probability is $\hat{P}_{\text{FA}} = 10^{-2}$, which corresponds to $\hat{P}_{\text{FA}}(\text{bin}) = 10^{-2}/N \approx 3.9 \times 10^{-5}$, the number of training cells $N_t = 50$ unless stated otherwise. Therefore, the required multiple factor $\alpha$ can be calculated by (28), which is $\alpha = 11.22$. And the threshold of NOMP is $\alpha_{\text{NOMP}} = 0.5\alpha_{\text{NOMP}}\sigma^2 = 10.15$. The over sampling rate is $\gamma = 4$ and the upper bound of the number of targets is $K_{\text{max}} = 2K$. All the results are averaged over 300 Monte Carlo (MC) trials.

A frequency $\omega$ is detected provided that $\min_{k=1,\ldots,K} \text{dist}(\hat{\omega}_k, \omega) \leq 0.5\Delta_{\text{DFT}}$. We compute the detection probability $P_D$ defined as the detection probability of all the true frequencies. An estimated frequency $\hat{\omega}$ is a false alarm provided that $\min_{k=1,\ldots,K} \text{dist}(\hat{\omega}, \omega_k) \geq 0.5\Delta_{\text{DFT}}$, where $\hat{K}$ denotes the number of estimated frequencies. We calculate the false alarm probability $\hat{P}_{\text{FA}}$ defined as the false alarm rate of all the estimated frequencies.

A. Validation of the Independence of Training Cell set

The approximation that samples in training cell set are approximated as independent distributed shown in (17) is verified. For a particular observation tensor $\mathbf{Y}$ or a residual observation tensor $\mathbf{Y}_r$, we calculate the autocorrelation of the training cells set $\hat{Y}_{\text{r,obs}}$ via the MATLAB function “xcorr.” We consider two scenarios: pure noise only and residue after removing the only target via the NOMP approach. The noise variance is 1. The autocorrelations of the two scenarios averaged over 300 Monte Carlo trials are shown in Fig. 4. Clearly, $\hat{Y}_{\text{r,obs}}$ is almost independent, and the noise variance estimate ($-0.09$ dB for noise only and $-0.1$ dB for one target) is close to the true noise variance 0 dB.

B. $\hat{P}_{\text{FA}}$ Versus $\alpha$

In this section, the false alarm probabilities $\hat{P}_{\text{FA}}$ for required threshold multipliers $\alpha$ of NOMP-CFAR and NOMP are calculated and results are shown in Fig. 5. In addition, the required threshold multiplier $\alpha$ of NOMP-CFAR is also approximately calculated through the CFAR approach (33) termed as CFAR approx. for the SMV scenario. For the SMV scenario $S = 1$ and $\hat{P}_{\text{FA}} \leq 0.5$, which is often of interest as $\hat{P}_{\text{FA}}$ is often set as a very small value such as $10^{-2}$ or $10^{-1}$ and $\hat{P}_{\text{FA}}$ is fixed, the required threshold multiplier $\alpha$ of NOMP-CFAR is higher than $\alpha'_{\text{NOMP}}$ of NOMP, i.e., $\alpha'_{\text{NOMP}} < \alpha$. For small $\hat{P}_{\text{FA}}$, i.e., $\hat{P}_{\text{FA}} \leq 0.01$, the required multiplier calculated by CFAR approach approximates that of NOMP-CFAR well. In detail, for $\hat{P}_{\text{FA}} = 10^{-2}$, the required threshold multipliers of NOMP-CFAR, CFAR approx. and NOMP are 11.22, 11.25, and 10.15, respectively, corresponding to 10.50, 10.51, and 10.06 dB. For the MMV scenario, the required threshold multiplier of NOMP-CFAR is close to that of NOMP. In addition, for a specified $\hat{P}_{\text{FA}}$, the required threshold multiplier decreases as the number of snapshots $S$ increases. For example, for $\hat{P}_{\text{FA}} = 10^{-2}$, the required threshold multipliers of NOMP-CFAR for $S = 1$, $S = 10$, and $S = 50$ are 11.22, 2.81, and 1.67, respectively, corresponding to 10.50, 4.49, and 2.23 dB.

C. Comparisons With the Traditional CFAR Windows

The performance of NOMP-CFAR is compared with that of NOMP-CFAR using traditional CFAR windows, and results are shown in Fig. 6. It can be seen that the false alarm rate $\hat{P}_{\text{FA}}$ of NOMP-CFAR using traditional CFAR windows is always higher than that of NOMP-CFAR and the nominal $\hat{P}_{\text{FA}} = 0.01$, and the detection probability $\hat{P}_D$ of NOMP-CFAR using traditional CFAR windows is always higher than that of NOMP-CFAR, and are close to each other for $\text{SNR} \geq 17$ dB. The above result consistent with the analysis in Section III-A demonstrates that the false
alarm can be better controlled by using the proposed CFAR windows which remove the detected cells.

D. Validation of CFAR Property and Detection Probability

The CFAR Property is verified by conducting a simulation in a SMV scenario. The nominal $\bar{P}_{FA}$ is varied from 0.01 to 0.12. The results are shown in Fig. 7. It can be seen that the measured $\bar{P}_{FA}$ is close to that of the nominal $\bar{P}_{FA}$ at three different SNRs SNR = 14 dB, SNR = 15 dB, and SNR = 18 dB, demonstrating the CFAR behavior of both the NOMP and NOMP-CFAR algorithm in constant noise variance scenario.

In the following part, we will show that (41) holds approximately provided that the true frequency is close to the DFT grid through two numerical simulations.

For the first simulation, we consider three scenarios, corresponding to the frequencies, which are closest to the 123th DFT grid with $\delta = 0$, $\delta = 0.25$, $\delta = 0.49$, respectively, where $\delta$ is defined in (39). The detection probabilities of the FFT, which chooses the 123th DFT grid to decide whether the target exists or not and the NOMP-CFAR which chooses the maximum of the DFT grid and may refine the frequency to decide whether the target exists or not by its outputs are shown in Fig. 8. It can be seen that the detection probabilities of the FFT based approach are always close to the corresponding $\bar{P}_D$. For $\delta = 0$ and $\delta = 0.25$ where the frequency is close to the DFT grid, the detection probabilities of the NOMP-CFAR are always close to the corresponding $\bar{P}_D$, whereas for $\delta = 0.49$, the detection probability of the NOMP-CFAR is higher than the corresponding $\bar{P}_D$. The reason is that NOMP-CFAR detects the signal by comparing the peak of the spectrum with the threshold. Besides, a frequency $\omega$ is detected provided that $\min_{k=1,...,\hat{K}} \Delta_{DFT} \leq 0.5\Delta_{DFT}$. These leads NOMP-CFAR to have a higher detection probability than the corresponding $\bar{P}_D$ for the SNRs ranging from 8 dB to 18 dB.

For the second simulation, a multiple targets scenario is considered. We set the number of targets as $K = 16$, where the SNR of the single signal is varied, and its frequency is fixed. Meanwhile, the remaining 15 targets whose frequencies are randomly generated have the same SNRs being 20 dB. We investigate the detection probability of the single target for $\bar{P}_{FA} = 0.01$ and $\bar{P}_{FA} = 0.05$, respectively. An oracle method named FFT (oracle) is also provided, which assumes that the remaining signals are perfectly cancelled and the FFT (oracle) is used to detect the single scenario. Fig. 9 shows that for $\delta = 0.2$, the detection probabilities of both FFT (oracle) and NOMP-CFAR are close to the theoretical detection probability $\bar{P}_D$, demonstrating the excellent detection performance of NOMP-CFAR.

E. Benefits of Initialization

This section conducts a simulation to illustrate the benefits of initialization. We design the NOMP-CFAR (forward)
Besides, for $\hat{P}_D = 0.6$, the SNRs of NOMP, NOMP-CFAR, NOMP-CFAR (forward) are SNR = 14 dB, SNR = 15 dB and SNR = 17 dB, respectively, demonstrating that initialization yields 1 dB performance gain.

F. Performance Versus the SNR in the SMV Setting

The performances of the proposed algorithm is investigated in the SMV scenario. The SNRs of all the targets are identical and are varied from 12 to 22 dB. The false alarm rate $P_{FA}$, the detection probability $P_D$ of all the targets, the correct model order estimation rate $P(\hat{K} = K)$, the MSE of the frequency estimation error $\|\hat{\omega} - \omega\|^2$ averaged over the trials in which $\hat{K} = K$, the normalized reconstruction error $\text{NMSE}(\hat{z}) = \|\hat{z} - z\|^2/\|z\|^2$ are used as the performance metrics. The results are shown in Fig. 11.

Fig. 11(a) shows the false alarm rate $P_{FA}$ versus SNR. At low SNR and SNR \(\leq 16\) dB, the $P_{FA}$ of the NOMP tends to decrease with SNR and the $P_{FA}$ of the NOMP-CFAR increases. For SNR \(\geq 16\) dB, the $P_{FA}$ of the two algorithms are close to the nominal $P_{FA}$, demonstrating the CFAR property of the two algorithms. However, the $P_{FA}$ of VALSE ranges from 0.1 to 0.31, which demonstrates that VALSE violates the CFAR. For $P_D = 0.75$, the corresponding SNRs of NOMP and NOMP-CFAR are 14 dB and 15 dB, respectively, which demonstrates that NOMP-CFAR has 1 dB performance loss due to the lacking of knowledge of noise variance. For SNR \(\leq 14\) dB, the detection probability of VALSE is the highest, followed by NOMP and NOMP-CFAR. As SNR continues to increase, the detection probability of NOMP surpasses that of VALSE. For SNR = 17 dB, the detection probabilities of NOMP and NOMP-CFAR are close to 1 and are higher than that of VALSE.

Fig. 11(c) shows the model order estimation probabilities $P(\hat{K} = K)$ of the three algorithms increase as SNR increases. For SNR = 12 dB, the model order estimation probability $P(\hat{K} = K)$ of VALSE is the highest, followed by NOMP and NOMP-CFAR. For SNR = 13 dB or SNR = 14 dB, the model order estimation probability $P(\hat{K} = K)$ of NOMP surpasses that of VALSE. As SNR continues to increase, the model order estimation probability $P(\hat{K} = K)$ of NOMP-CFAR surpasses that of VALSE and approaches to that of NOMP.

Fig. 11(d) and (e) shows the estimation errors. For the frequency estimation error, the Cramér Rao bound (CRB) is also evaluated for the one target scene. Fig. 11(d) shows that the MSEs of NOMP, NOMP-CFAR, and VALSE closely approach to the CRB computed for the true model order at SNR \(\geq 16\) dB, SNR \(\geq 17\) dB, and SNR \(\geq 19\) dB, respectively. The reason that NOMP-CFAR deviates away from CRB at SNR = 16 dB is that for a frequency lying off the DFT frequency grid, the frequency is not detected. Meanwhile, a wrong frequency is detected which yields a false alarm, causing a large frequency estimation error. For the reconstruction error of the line spectral, NOMP and VALSE perform better than NOMP-CFAR for about SNR \(\leq 16\) dB. NOMP and NOMP-CFAR perform similarly and better than VALSE as SNR increases.

algorithm, which just incrementally detects a new frequency by the CFAR detection algorithm and stops until no frequency is detected. The measured false alarm probability $\hat{P}_{FA}$ and the detection probability $\hat{P}_D$ versus the SNR are shown in Fig. 10. From Fig. 10(a), the measured false alarm rates $\hat{P}_{FA}$ of NOMP, NOMP-CFAR, NOMP-CFAR (forward) are close to the nominal $P_{FA}$ for SNR \(\geq 17\) dB. From Fig. 10(b), the detection probability $\hat{P}_D$ of NOMP is highest, followed by NOMP-CFAR, NOMP-CFAR (forward). For SNR \(\geq 17\) dB, the detection probabilities $\hat{P}_D$ of NOMP and NOMP-CFAR are close to 1, while the detection probability $\hat{P}_D$ of NOMP-CFAR (forward) is near 0.8.
Fig. 11. Performances of NOMP, VALSE, and NOMP-CFAR versus SNR. (a) False alarm rate $P_{FA}$. (b) Detection probability $P_D$ of all the targets. (c) Correct model order estimation rate $P(\hat{K} = K)$. (d) MSE of the frequency estimation error $\|\hat{\omega} - \omega\|^2/K$. (e) Normalized reconstruction error $\text{NMSE}(\hat{z}) = \|\hat{z} - z\|^2/\|z\|^2$.

G. Performance Versus the Number of Snapshots in the MMV Setting

The false alarm rate and the detection probability of the NOMP-CFAR in the MMV setting is investigated. The SNRs of all the targets are set as 10 dB. The false alarm rate and the detection probability are shown in Fig. 12. From Fig. 12(a), the false alarm probability of NOMP [26] is a little lower than that of NOMP-CFAR, and the false alarm probabilities of both algorithms are close to the nominal false alarm probability for $S \geq 3$. Fig. 12(b) shows that the detection probabilities of both algorithms are close to each other and increase as the number of snapshots increases.

H. Performance Versus the Compression Rate in the Compressive Setting

The performance of NOMP-CFAR versus the compression rate $M/N$ is investigated. The elements of the compression matrix are drawn i.i.d. from the complex Bernoulli distribution. The number of frequencies is $K = 8$, the SNRs of all the targets are 22 dB. Results are shown in Fig. 13. From Fig. 13(a), the false alarm probability of NOMP is a little lower than that of NOMP-CFAR, and the false alarm probabilities of both algorithms are close to the nominal false alarm probability for $M/N \geq 0.36$. For the detection probability $P_D$, Fig. 13 shows that NOMP performs better than NOMP-CFAR for $M/N \leq 0.375$, and the $P_D$ of both algorithms approach to 1 as $M/N \geq 0.5$. For $P_D = 0.74$, the compression rates required for NOMP and NOMP-CFAR are 0.24 and 0.32, respectively, meaning that to achieve $P_D = 0.74$, NOMP-CFAR needs $4/3$ times the number of measurements of NOMP.

I. False Alarm Probability Versus Strength of Noise Fluctuation

The nominal SNR of the $k$th target is defined as $\text{SNR}_k \triangleq 10 \log(N|x_k|^2/\sigma_0^2)$, where $\sigma_0^2$ is the nominal noise variance.
Fig. 13. Measured $P_{FA}$ (a) and $P_D$ (b) of the two algorithm in different compressive rate.

Fig. 14. Measured $P_{FA}$ versus the strength of the noise fluctuation.

and is set as $\sigma_0^2 = 1$, i.e., 0 dB. The SNR of each frequency is 22 dB. For each MC trial, the noise variance is drawn uniformly from $[-u, u]$ in dB, where $u$ characterizes the strength of noise fluctuation. The threshold of NOMP is calculated by using $\sigma_0^2$. The results are shown in Fig. 14. Note that the detection probability of all the targets are very close to 1 ($\geq 0.991$) and are omitted here. It can be seen that NOMP violates the CFAR property in varied noise variance scenario, while NOMP-CFAR still preserves the CFAR property.

V. REAL EXPERIMENT

In this section, the performance of proposed NOMP-CFAR is demonstrated using an AWR1642 radar [32]. The AWR1642 radar is an FMCW MIMO radar consisting of two transmitters and four receivers. The radar parameters and wave form specifications are listed in Table I. For the single input multiple output (SIMO) mode, the base band signal model can be formulated as

$$y(n, m, l) = \sum_{k=1}^{K} A_k e^{j[(n-1)\omega_x + (m-1)\omega_y + (l-1)\omega_z]} + \epsilon_{n,m,l}$$

(54)

where $c$ denotes the speed of electromagnetic wave. Note that (54) reduces to the range estimation, range azimuth, and range-Doppler imaging problem, with $M = L = 1$, and $L = 1$, respectively. Thus, one could apply the NOMP-CFAR algorithm to solve the multidimensional line spectrum estimation problem (54), and obtain the estimates of ranges, velocities, and azimuths via (55).

| Parameters | Value |
|------------|-------|
| Number of receivers $L$ | 4 |
| Carrier frequency $f_c$ | 77 GHz |
| Frequency modulation slope $\mu$ | 29.982 MHz/s |
| sweep time $T_s$ | 60 $\mu$s |
| Pulse repeat interval $T_r$ | 160 $\mu$s |
| Bandwidth $B$ | 1798.92 MHz |
| Sampling frequency $f_s = 1/T_r$ | 10 MHz |
| Number of pulses in one CPI $M$ | 128 |
| Number of fast time samples $N$ | 256 |

TABLE I Parameters Setting of the Experiment

$$\omega_x = \frac{4\pi}{c} (f_c v_h + \mu r_k) T_s \approx \frac{4\pi}{c} \mu r_k T_s$$

(55a)

$$\omega_y = \frac{4\pi}{c} f_c v_h T_r$$

(55b)

$$\omega_z = \frac{2\pi}{c} f_c d \sin \theta_k$$

(55c)
The proposed NOMP-CFAR is compared with the traditional CFAR and NOMP. The false alarm rates of NOMP-CFAR and NOMP are set as \( P_{FA} = 10^{-2} \), and the false alarm rate of CFAR is \( P_{FA,CFAR} = 10^{-2}/N_{tot} \) to ensure that false alarms produced by the three algorithms are comparable when \( P_{FA} \) is small, where \( N_{tot} \) denotes the total number of cells under test which equals to the number of measurements. The upper bound of target number for NOMP-CFAR algorithm is \( K_{max} = 32 \) unless stated otherwise.

In the following, three experiments are conducted to evaluate the performance of NOMP-CFAR.

A. Experiment 1

Fig. 15 shows the setup of field experiment 1. The radial distances and the azimuths of the two static people named people 1 and people 2 are about \((4.92 \text{ m}, 0^\circ)\) and \((3.09 \text{ m}, -19.8^\circ)\). First, we conduct range estimation only via the CFAR, NOMP, and NOMP-CFAR approaches. Then, we implement NOMP and NOMP-CFAR for range and azimuth estimation.

The number of the fast domain samples is \( N = 256 \). The number of guard cells and the number of training cells are 8 and 60, respectively. The CA-CFAR is adopted. The required threshold multipliers of the CFAR and NOMP-CFAR are \( \alpha_{CFAR} = 11.06 \) and \( \alpha_{NOMP} = 11.03 \), corresponding to 10.44 dB and 10.43 dB, respectively. Fig. 16(a) shows that the noise variance is about 24 dB, and we set the noise variance \( \sigma^2 = 251 \) for NOMP algorithm, and the corresponding threshold can be calculated as \( 2 \times 10^3 \), i.e., 34.06 dB. The range estimation results are shown in Fig. 16.

Fig. 16(a) shows that CFAR detects people 1 and people 2, which estimates their radial distance as 4.90 m and 3.14 m. From Fig. 16(b), NOMP detects both people 1 and people 2. It is concluded that people 1 and people 2 are extended targets, and each two detected results correspond to people 1 and people 2. The highest amplitudes corresponding to people 1 and people 2 are 48.86 dB and 51.17 dB, respectively. Equivalently, the amplitudes of people 1 and people 2 are 14.80 dB and 17.11 dB above the NOMP thresholds. Fig. 16(c) shows that the detection results of NOMP-CFAR are similar to that of NOMP. In details, the highest amplitudes corresponding to people 1 and people 2 are 48.83 dB and 51.17 dB, respectively. Equivalently, the amplitudes of people 1 and people 2 are 11.53 dB and 13.77 dB above the corresponding thresholds, respectively.

Fig. 18. Field experiment 2 setup.
We then implement the 2-D NOMP and NOMP-CFAR to perform the range and azimuth estimation, the results are shown in Fig. 17. Fig. 17(a) shows that NOMP detects 19 points and the highest 2 points are estimated as \((4.84 \text{ m}, -0.12^\circ)\) and \((3.09 \text{ m}, -22.28^\circ)\), which are close to the truth of people 1 and people 2, respectively and their integrated amplitudes are 54.16 dB, 55.38 dB, respectively. Fig. 17(b) shows that the number of detected points of NOMP-CFAR is 2 and the results are \((4.85 \text{ m}, 0.50^\circ), (5.42 \text{ dB}, 42.30 \text{ dB}), (3.20 \text{ m}, -21.61^\circ), (53.29 \text{ dB}, 51.42 \text{ dB})\), where the first, second, third, and fourth components are the radial distance, the azimuth, reconstructed amplitude, and threshold. The first detected points correspond to people 1 and the second points correspond to people 2. The estimations of radial distances and azimuths of people 1 and people 2 are close to the truth. This demonstrates that NOMP-CFAR suppresses the false alarm significantly, compared to NOMP.

B. Experiment 2

The setup of field experiment 2 is shown in Fig. 18. The radial distances of the two static people named people 3 and people 2 are about \((4.87 \text{ m}, 0^\circ)\) and \((2.63 \text{ m}, 24^\circ)\). A cyclist moves toward the radar with the radial distance starting from 7 to 2 m and the velocity about 2 m/s. The numbers of fast time domain samples and slow time domain samples are \(N = 128\) and \(M = 64\), respectively. Two-dimensional CA-CFAR is adopted. The widths of guard band along the fast time dimension and slow time dimension are 3, the widths of training band along the fast time dimension and slow time dimension are 5. Results are shown in Fig. 19. Fig. 19(a) shows that people 2 and cyclist are detected, and people 1 is missed by CFAR. In addition, CFAR also produces 1 false alarm. From Fig. 19(b), the three targets including people 1, people 2, and cyclist are detected by NOMP. The total number of points detected by NOMP is 47. Fig. 19(c) shows the detection result of NOMP-CFAR algorithm. The total number of points detected by NOMP-CFAR is 24. Note that for people 1, the reconstructed amplitude and the threshold output by NOMP are 61.17 dB and 42.51 dB, respectively and the reconstructed amplitude of people 1 are 18.66 dB above the corresponding thresholds, which is of high confidence of the existence of the people 1.

C. Experiment 3

Fig. 20 shows the setup of field experiment 3. A people and a cyclist move away from the radar at the radial velocity of about 1.2 m/s and 2.1 m/s, and a car moves toward the radar at the radial velocity of about 2.3 m/s. The algorithm’s parameters are the same as that of the experiment 2.

The range azimuth estimation results are shown in Fig. 21. The NOMP algorithm detects 28 targets. Fig. 21(a) shows that the people, cyclist, and car are detected by the NOMP algorithm, whose range and azimuth estimation results are \((4.13 \text{ m}, 29.93^\circ), (5.61 \text{ m}, -29.95^\circ)\), and \((19.90 \text{ m}, -0.27^\circ)\), respectively. For the NOMP-CFAR algorithm, it detects ten targets, including the people, cyclist, and car, as shown in Fig. 21(b). In detail, the range and azimuth estimation results are \((4.12 \text{ m}, 29.73^\circ), (5.62 \text{ m}, -29.91^\circ)\), and \((19.90 \text{ m}, -0.18^\circ)\), respectively. Their corresponding integrated amplitudes and thresholds are \((51.58 \text{ dB}, 40.24 \text{ dB}), (44.44 \text{ dB}, 40.62 \text{ dB}),\) and \((45.52 \text{ dB}, 39.74 \text{ dB})\). The range Doppler estimation results are shown in Fig. 22. The upper bound number of targets for NOMP-CFAR is \(K_{\max} = 48\). Fig. 22(a) shows that the CFAR method detects the people and car but misses the cyclist. NOMP and NOMP-CFAR detect the people, cyclist, and car, and results are shown in Fig. 22(b) and (c), respectively. Meanwhile, the false alarms generated by NOMP-CFAR are smaller than that of NOMP.
VI. CONCLUSION

We have developed NOMP-CFAR algorithm to achieve the high estimation accuracy and maintain the CFAR behavior for line spectrum estimation and detection. The algorithm consists of two steps: Initialization and detection step. For the initialization step, NOMP-CFAR uses the NOMP to provide candidate frequency set in high accuracy, which avoids target masking effects incurred by CFAR. In the detection step, a soft CFAR detector is introduced to output a quantity, which characterizes the confidence of each frequency in the candidate set. A greedy approach is adopted to remove or add one frequency in each iteration, and Newton refinement is then implemented to refine the parameters of the remaining sinusoids. The effectiveness of NOMP-CFAR is verified in substantial numerical experiments and real data.

APPENDIX A

GLRT DERIVATION FOR THE PROPOSED NOMP-CFAR DETECTOR

For simplicity, we consider a general binary detection problem in 1-D model \(^3\)

\[
\begin{cases}
\mathcal{H}_0 : y = \varepsilon \\
\mathcal{H}_1 : y = xa(\omega) + \varepsilon
\end{cases}
\]  

and we choose between the null hypothesis \(\mathcal{H}_0\) and the alternative hypothesis \(\mathcal{H}_1\), where \(\varepsilon \sim CN(0, \sigma^2 I_N)\) is the additive white Gaussian noise (AWGN). However, the noise variance \(\sigma^2\) is unknown, we use \(\sigma^2_1\) and \(\sigma^2_0\) to denote the noise variance under the hypothesis \(\mathcal{H}_1\) and \(\mathcal{H}_0\), respectively. Then, the GLRT detector decides \(\mathcal{H}_1\) if

\[
\frac{\max_{x,\omega,\sigma^2_1} p(y; x, \omega, \sigma^2_1|\mathcal{H}_1)}{\max_{\sigma^2_0} p(y; \sigma^2_0|\mathcal{H}_0)} > \gamma \tag{57}
\]

where \(\gamma\) is a threshold. Under hypothesis \(\mathcal{H}_0\), the pdf of \(y\) is

\[
p(y; \sigma^2_0|\mathcal{H}_0) = \frac{1}{(\pi\sigma^2_0)^{N/2}} \exp \left\{ -\frac{\|y\|^2}{2\sigma^2_0} \right\}. \tag{58}
\]

Maximizing \(p(y; \sigma^2_0|\mathcal{H}_0)\) with respect to \(\sigma^2_0\) yields

\[
\max_{\sigma^2_0} p(y; \sigma^2_0|\mathcal{H}_0) = \left( \frac{N}{\pi e \|y\|^2_2} \right)^{N/2} \tag{59}
\]

where the optimal \(\hat{\sigma}^2_0 = \frac{\|y\|^2_2}{N}\). Similarly, under hypothesis \(\mathcal{H}_1\), maximizing \(p(y; x, \omega, \sigma^2_1|\mathcal{H}_1)\) yields

\[
\max_{x,\omega,\sigma^2_1} p(y; x, \omega, \sigma^2_1|\mathcal{H}_1) = \left( \frac{N}{\pi e \|y - xa(\omega)\|^2_2} \right)^{N} \tag{60}
\]

\(^3\)Extending the 1-D model to multidimensional model is straightforward and is omitted here.
where the optimal \( \hat{\sigma}^2 = \frac{\|y - \hat{x}(\omega)\|^2}{N} \) and \( \hat{x} = a^H(\omega)y/a^H(\omega)a(\omega) = a^H(\omega)y/N \). Substituting (59) and (60) in (57) yields

\[
\min_{\omega} \|y - \frac{1}{N}a(\omega)a^H(\omega)y\|^2 = \min_{\omega} y^H(y - \frac{1}{N}a(\omega)a^H(\omega)y) = \frac{\|y\|^2}{N} - \frac{\|y\|^2}{\max_{\omega}|a^H(\omega)y|^2} > \gamma' (61)
\]

where \( \gamma' = \gamma \frac{\|y\|}{N} \). The GLRT in (61) involves solving an optimization problem with respect to \( \omega \in [0, 2\pi) \). Here we use a suboptimal detector in which we restrict \( \omega \) into the DFT grid, i.e., \( \omega \in \Omega \), where \( \Omega = \{2\pi(n-1)/N, n = 1, 2, \ldots, N\} \). Suppose that \( \hat{n}_{\text{peak}} \) is the peak localization of \( |\hat{y}|^2 \) given by

\[
\hat{n}_{\text{peak}} = \arg\max_{n \in \Omega} |\hat{y}_n|^2 (62)
\]

where \( \hat{y}_n = \frac{1}{\sqrt{N}}a^H(\omega_n)y \), \( \omega_n = 2\pi(n-1)/N \), \( n = 1, 2, \ldots, N \). Note that \( \hat{y} \in C^N \) is the normalized DFT of \( y \), which can be computed efficiently via FFT. Utilizing the fact that \( \|\hat{y}\|^2 = \|y\|^2 / N \) (61) can be further simplified as

\[
\frac{\|\hat{y}\|^2}{\|y\|^2 / N} > \frac{|\hat{y}_{\text{peak}}|^2}{\|\hat{y}\|^2 / N - |\hat{y}_{\text{peak}}|^2} > \gamma' (63)
\]

or

\[
\frac{|\hat{y}_{\text{peak}}|^2}{\|\hat{y}\|^2 / N - |\hat{y}_{\text{peak}}|^2} = \frac{(N-1)}{(N-1)} \sum_{n \in \Omega, n \neq \hat{n}_{\text{peak}}} |\hat{y}_n|^2 > (N-1)(\gamma' - 1). (64)
\]

Note that the detector can be viewed as the ratio of the peak squared magnitude over the noise variance estimate, which is the average of the spectrum excluding the peak component. In practice, a target may not lie on the DFT grid exactly, which might straddle frequency cells. In this case, the energy in the cell adjacent to \( \hat{n}_{\text{peak}} \) would contain both noise and target energy, and the extra energy from the target would tend to raise the estimate of the noise variance, which may result in a threshold that was too high and a lower detection probability \( P_F \). Consequently, the proposed NOMP-CFAR detector (15) is proposed, which can be well explained by the GLRT detector.

**APPENDIX B**

**COMPUTE \( P_{\text{FA}} \) FOR THE MMV SCENARIO**

The CFAR detector deciding \( H_1: T(Y) \geq \alpha_{\text{mmv}} \) (47) can be equivalently written as

\[
\begin{align*}
\sum_{s=1}^{S} |\hat{y}_{\text{peak}}(s)|^2 / \sigma^2 / 2 &\geq \frac{\sum_{s=1}^{S} \sum_{n \in \Omega} |\hat{y}_n(s)|^2 / \sigma^2 / 2} {N_r} \geq \tilde{T} / \alpha_{\text{mmv}} / N_r.
\end{align*}
\]

Under the null hypothesis, \( \sum_{s=1}^{S} |\hat{y}_n(s)|^2 / \sigma^2 / 2 \) follows a chi-squared distribution with a degrees of freedom \( 2S \), i.e.,

\[ P\left( \sum_{s=1}^{S} |\hat{y}_n(s)|^2 / \sigma^2 / 2 \geq \tilde{T} / \alpha_{\text{mmv}} / N_r \right) = 1 - F_{2S} \left( \frac{\tilde{T}}{\alpha_{\text{mmv}} / N_r} \right) \]

where \( N \) denotes the total number of cells and \( F_{2S}(x) \) denotes the cumulative distribution function (CDF) of the chi-squared distribution

\[ F_{2S}(x) = \begin{cases} 1 - e^{-x/2} \sum_{k=0}^{\infty} \frac{(x/2)^k}{k!}, & x > 0 \\ 0, & \text{otherwise.} \end{cases} \]

In addition, \( \tilde{T} \) follows

\[ T \sim \chi^2(2SN_r) = \begin{cases} \frac{1}{2SN_r - 1} e^{-x/2}, & x > 0 \\ 0, & \text{otherwise.} \end{cases} \]

Averaging \( P\left( \sum_{s=1}^{S} |\hat{y}_n(s)|^2 / \sigma^2 / 2 \geq \alpha_{\text{mmv}} / N_r \right) \) over \( T \) yields the false alarm

\[ P_{\text{FA}} = 1 - E_T \sim \chi^2(2SN_r) \left[ F_{2S} \left( \frac{\tilde{T}}{\alpha_{\text{mmv}} / N_r} \right) \right]. \]

Substituting (68) in (69) yields (49).

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