On the Green function for 
an Aharonov–Bohm flux tube

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An earlier contour expression for the Green function of a free complex scalar field in the presence of a conical singularity with localised magnetic flux is shown to yield expressions for the field correlator and defect block expansions that have been more recently found in connection with monodromy defects in conformal field theory. The Green function appears as the Picard integral representation of the Appell $F_1$ function. This is shown to transform into a confluent Horn function, corresponding to a different defect block expansion. Other transformations are discussed.

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1. Introduction and survey

In the extension of conformal field theories to defect conformal field theories, in particular those with a monodromy defect, one soon encounters the bulk two–point field correlator, \( G(x_1, x_2) = \langle \phi(x_1) \overline{\phi}(x_2) \rangle \) in the presence of a conical defect that generates a phase shift on being encircled, (the defect has codimension 2). Of most interest are interacting theories but since explicit expressions for this correlator can be found in the case of free fields there is some value in a further examination of this restricted situation. In an earlier literature, the defect is sometimes referred to as a cosmic string and the correlator as a Green function.

The aim of this technical note is to draw together and extend some existing free–field results and to relate them to more recent investigations. The set–up is the simplest \( i.e. \) a conical defect in flat space–time of \( d \) dimensions with either a Lorentzian or Euclidean signature. The field is a complex scalar conformally invariant one. The phase change is \( 2\pi\delta \) and the cone has the angle, \( \beta \). When \( \beta = 2\pi \), the construction is just an Aharonov–Bohm flux tube.

The essential monodromy expressions were given in [1] for \( d = 2 \), where the Green function appeared in two equivalent forms. The fundamental one, and the most useful, is a Sommerfeld–Carslaw complexified polar angle contour integral which can then be ‘evaluated’, if desired, to give an eigenfunction expression involving Bessel functions. These results are classic in the case of no monodromy, \( \delta = 0 \).

The explicit extension of these results to \( d \) dimensions was made in [2] where the one–point vacuum averaged energy–momentum tensor was calculated in the standard way by applying a differential operator to the Green function. If done directly, the coincidence limit diverges and the infinities have to be removed by subtracting the values corresponding to the usual flat space Green function (\( i.e. \) that with \( \beta = 2\pi, \delta = 0 \), hereafter called \( G_0 \)). Such was the procedure adopted in the earlier [3] (for \( \delta = 0 \)) where exact free Green functions were exhibited. In [2], the subtraction, and also knowledge of the explicit Green function, was avoided by a simple change of contour which eliminated \( G_0 \) from \( G \) at the outset. The coincidence limit is then finite and can be evaluated by contour methods without knowing \( G \). I briefly return to this point in section 5.

In recent works, I cite only [4–6] as being particularly relevant, the Green function is obtained, and displayed, using various methods, eigenfunction expansion being one. Therefore it might be useful to show, again, how these expressions can
be derived from the contour form and so could be considered to be latent in [1,2]. In doing so, certain interesting relations emerge.

2. A contour derivation of the correlator

The normalisation conventions here are those of standard quantum field theory and I make no use of specific CFT notions. The Green function is then normalised as in [4,5] which differs from that in [6]. To aid comparison, my coordinate choices are polar coordinates \(r, \theta\) tangential to the defect and \(y\) along it, i.e. the same as [5]. The Euclidean interval between \(x_1\) and \(x_2\) is

\[
\sigma^2 = r_1^2 + r_2^2 - 2r_1r_2\cos \theta + (y_1 - y_2)^2, \quad \theta \equiv \theta_{12} \equiv \theta_1 - \theta_2.
\]

The standard Green function, \(G_0\), is normalised by the usual textbook expression,

\[
G_0(x_1, x_2) = \frac{\Gamma(\Delta)}{2\pi^{\Delta+1}} \frac{1}{(\sigma^2)^{\Delta}}, \quad \Delta \equiv \frac{d}{2} - 1.
\]

Its dependence on the polar angle difference is written, for short, as \(G_0(\theta)\).

If all that is wanted is an explicit formula for the Green function, equation (4) in [2] is a suitable starting point. This reads,

\[
G(x_2, x_1) = \frac{1}{\beta} \int_A d\alpha G_0(\alpha)P(\alpha + \theta, \delta, \beta),
\]

where the contour \(A\) has two parts, one in the upper half–plane and a reflected one in the lower. \(P\) is a ‘re–periodising’ factor given by, [1],

\[
P(\alpha, \delta, \beta) = \frac{e^{2\pi i \alpha \delta / \beta}}{2i \sin(\pi \alpha / \beta)}, \quad \delta = \delta - 1/2, \quad (0 \leq \delta \leq 1).
\]

It can be written as an image sum on Sommerfeld’s Riemann surface, and related to replicas, but I do not enlarge on this aspect here.

While (2) is perfectly general, it leads to closed forms only in even dimensions or when \(\beta = 2\pi/n, n \in \mathbb{Z}\). Since it is desirable to keep \(d\) arbitrary, I will consider only the case when \(\beta = 2\pi\) i.e. the pure Aharonov-Bohm flux tube which still provides sufficient analytical interest.

The useful hyperbolic angle, \(\alpha_1\), is defined by,

\[
cosh \alpha_1 = \frac{(y_1 - y_2)^2 + r_1^2 + r_2^2}{2r_1r_2},
\]
so that the interval is,

\[ \sigma^2(\alpha) = 2r_1r_2(\cosh \alpha_1 - \cos \alpha), \]

which factorises as,

\[ \sigma^2(u) = r_1r_2u^{-1}(h - u)(u - \frac{1}{h}), \]

where \( h \) and \( u \) are defined by,

\[ h = e^{\alpha_1}, \quad \text{and} \quad u = e^{-i\alpha}. \]

Without loss of generality, \( h \) can be assumed greater than 1. The parameter \( \xi \) used in \([5]\) is related to \( \alpha_1 \) by \( \xi = \sinh^2 \frac{\alpha_1}{2} \).

To proceed with the calculation of (2), the lower contour can be turned into the upper one by reversal of \( \alpha \). Hence we need only compute the upper contribution. Then, changing coordinates from \( \alpha \) to \( u \) the contour becomes one familiar in the theory of Bessel functions \( i.e. \) one running from \(-\infty\), around the origin and back to \(-\infty\).\(^2\)

Convergence requirements at infinity dictate that this contour, \( C \), has to enclose the unit \( u \)-circle including therefore \( u = e^{i\theta} \) and \( u = 1/h \) as well as the origin, 0, while it excludes the points \( h \) and \( \infty \). These are the five singular points of the integrand. (The lower \( \alpha \)-contour gives a singular point at \( u = e^{-i\theta} \). Note that the contour ‘splits’ the light–cone singularity of the \( G_0(\alpha) \) Green function. At the coincidence limit, three of these points come together, pinching the contour.

In terms of \( u \), the periodising function reads,

\[ P_\pm(u) = \frac{e^{i\theta \delta}u^{-\delta+1}}{e^{i\theta} - u} + \frac{e^{-i\theta(1-\delta)}u^\delta}{e^{-i\theta} - u}. \quad (3) \]

The first term is the upper \( \alpha \) contour contribution and the second one the lower contribution. They are related by complex conjugation and the replacement \( \delta \rightarrow 1-\delta \). Hence the first term is computationally sufficient and I denote the corresponding Green function component by \( G_U \).

From (2) with (1) and (3), the Green function component is then given by,

\[ G_U = \frac{\Gamma(\Delta)}{(r_1r_2)^\Delta 2\pi^{\Delta+1}} \frac{e^{i\theta \delta}}{2\pi i} \int_C du u^{-\delta+1+\Delta} (h - u)^{-\Delta} (u - \frac{1}{h})^{\Delta} (u - e^{i\theta})^{-1}. \]

\(^2\) Carslaw makes this manoeuvre which, after expanding the periodising function, yields the Fourier series, eigenfunction expression, at \( \delta = 0 \).
The convergence properties of the integrand allow the contour $C$ to be swung around to the positive real $u$ axis, now running from $+\infty$ and looping around the point $u = h$ (thus enclosing two singular points).

A coordinate change to $v = h/u$, gives the equivalent form, 

$$G_U = \frac{h^{-\Delta} \Gamma(\Delta)}{(r_1 r_2)^{2\Delta+1} 2\pi i} \int_0^{1+} dv \frac{\delta^{-\Delta-1}(v-1)^{-\Delta}}{(1 - \frac{1}{h^2} v)^{-\Delta} (1 - e^{i\theta} h v)^{-1}}.$$ 

The integral around the cut from 0 to 1 converts to a line integral and yields, 

$$G_U = \frac{h^{-\Delta}}{(r_1 r_2)^{2\Delta+1} \Gamma(1-\Delta)} \int_0^1 dv \frac{1}{\delta^{-\Delta-1}(1 - \frac{1}{h^2} v)^{-\Delta} (1 - e^{i\theta} h v)^{-1}},$$

which is, more or less, the final answer except that it can be given a name by noting that Picard's integral formula for the Appell $F_1$ function is, 

$$F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 dt \frac{t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'}},$$

so that,

$$G_U = \frac{h^{-\Delta} (h^{-1} e^{i\theta})^\delta \Gamma(\delta + \Delta)}{(r_1 r_2)^{2\Delta+1} \Gamma(\delta + 1)} F_1(\delta + \Delta; 1, \Delta, \delta + 1; h^{-1} e^{i\theta}, h^{-2}).$$

The total Green function is got by adding the expression obtained by taking the complex conjugate (i.e. $\theta \to -\theta$) and sending $\delta \to 1 - \delta$, i.e. $G(\delta) = G_U(\delta) + \overline{G_U}(1-\delta)$.\(^3\) I will, therefore, consider only $G_U$ in the following and may, occasionally, refer to it as the Green function.

For comparison, in terms of the $x$ and $\overline{x}$ coordinates employed by [6], $h^{-1} e^{i\theta} = x$ and $G_U$ is more neatly expressed as,

$$G_U = \frac{(x \overline{x})^{\Delta/2} x^\delta}{r^\Delta 2\pi^{\Delta+1} \Gamma(\delta + 1)} F_1(\delta + \Delta; 1, \Delta, \delta + 1; x, x \overline{x}).$$

This formula agrees, up to normalisation conventions, with that derived by Gimenez–Grau, \(^4\) using CFT expansions discussed in the next section.

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\(^3\) Incidentally, when physical quantities are being calculated for a complex field, charge conjugation invariance implies that one should use the combination $G(\delta) + G(-\delta)$. Since periodicity in the phase entails $G(-\delta) \equiv G(1-\delta)$, this combination gives a real value.

\(^4\) Private communication.
3. Relation to other formulations

The result (6) has been obtained without any knowledge of eigenfunctions. These are, of course, embedded in the formalism and can be extracted in various ways. The representation of $G$ in terms of ‘defect blocks’ computed in Bianchi et al. [4], employs eigenfunctions and can be construed as just a Fourier series expansion with coefficients related to Bessel functions. Such a form was already given in [1], and related to the contour (2). As mentioned, for no monodromy, this was done by Carslaw many years ago.

In [4], the evaluation of the Fourier coefficients as hypergeometric functions involves the Lipschitz–Hankel integral for the Laplace transform of a Bessel function expression. The original proof of this particular textbook formula involved power series expansion and term by term integration. It gives rise to a ‘radial’ expansion in $\text{sech}^2 \alpha_1$, in the notation here, and is further discussed later.

According to Hankel, there is no contour proof of his formula. However, the same integral is calculated, at some useful length, by Graf and Gubler, [9], in two ways. One is the standard method just mentioned but the other does involve a contour manipulation (the prototype of that used above) and yields a different hypergeometric form. Graf and Gubler then show the equivalence of these two expressions in a very detailed way. It is, actually, a consequence of Kummer’s quadratic hypergeometric relation (see below). The radial expansion is one in $1/h^2 = e^{-2\alpha_1} = x\overline{x}$, and corresponds to the solution here, (6), which can be expanded in hypergeometric functions as given in [10], equation (2), repeated here,

$$F_1(a; b, b'; c; x, y) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} \frac{2F_1(a + n; b', c + n; y)}{2} x^n.$$  

This can be obtained by expansion of one of the brackets in (5) and termwise integration.

Applied to the $F_1$ in (7), this yields the Fourier expansion,

$$F_1(\delta + \Delta; 1, \Delta; \delta + 1; x, x\overline{x}) = \sum_{n=0}^{\infty} \frac{(\delta + \Delta)_n}{n!(\delta + 1)_n} 2F_1(\delta + \Delta + n; \Delta, \delta + 1 + n; x\overline{x}) x^n,$$

which are just the defect blocks written down in [6], equn. (2.3) and so, in the present scheme, have been derived from the contour integral (2).

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5 The mode boundary conditions are built into the contour construction and, by default, give the Friedrich’s extension of the Laplacian. Singular modes, as used in [4], must be treated separately.

6 A slightly more expansive treatment can be found in Kratzer and Franz, [8].
4. Transformations

The fact that the standard Lipschitz–Hankel Bessel integral cannot be derived directly by contours raises the question of the analogue of the Appell formula (6) for this alternative representation. For this, the Fourier series, as given in [5], eqn.(2.12), could simply be taken. However, to keep the present calculation self-contained, I will firstly rederive this and then rewrite it in terms of another named double series.

As already mentioned, Kummer’s quadratic relation can be applied to the hypergeometric function in (8). This relation is, in the form I need it,

\[ h^{-a-c} {}_2F_1\left(a + c, c; a + 1; \frac{1}{h^2}\right) = (2b)^{-a-c} {}_2F_1\left(\frac{a + c}{2}, \frac{a + c + 1}{2}; a + 1; \frac{1}{b^2}\right), \]

where \( h = e^{\alpha_1} \) and \( 2b = 2 \cosh \alpha_1 = h + h^{-1} \).

I have not been able to find the results of this transformation on the Appell \( F_1 \) in the literature and so a little intermediate algebra will be given. Substitution of (9) into (8) produces,

\[ F_1(\delta + \Delta; 1, \Delta; \delta + 1; x, x) = \sum_{n=0}^{\infty} \frac{(\delta + \Delta)_n}{n!(\delta + 1)_n} 2F_1(\delta + \Delta + n; \Delta, \delta + 1 + n; h^{-2}) x^n \]

\[ = \sum_{m,n=0}^{\infty} \frac{(\delta + \Delta)_n}{n!(\delta + 1)_n} x^n b^{-2m} \left( \frac{h}{2b} \right)^{n+\delta+\Delta} \frac{\left( \frac{\delta+\Delta+n+1}{2} \right)_m (\delta+\Delta+n+1)_m}{m!(\delta+n+1)_m}. \]

Use of the \( \Gamma \)-duplication formula allows the Pochhammer combination to be simplified,

\[ \frac{(\delta+\Delta+n)_m (\delta+\Delta+\frac{n+1}{2})_m}{(\delta+n+1)_m} = 2^{-2m} \frac{(\delta+\Delta+n)_{2m}}{(\delta+n+1)_m}, \]

giving,

\[ F_1 = \left( \frac{h}{2b} \right)^{\delta+\Delta} \sum_{m,n=0}^{\infty} \frac{(\delta + \Delta)_n}{m!n!(\delta + 1)_n} \frac{(\delta + \Delta + n)_{2m}}{(\delta + n + 1)_m} e^{i\theta(2b)^{-2m}} \left( \frac{1}{2b} \right)^n \]

\[ = \left( \frac{h}{2b} \right)^{\delta+\Delta} \sum_{m,n=0}^{\infty} \frac{\Gamma(\delta + 1)}{m!n!\Gamma(\delta + \Delta)} \frac{\Gamma(\delta + \Delta + n + 2m)}{\Gamma(\delta + n + m + 1)} e^{i\theta} (2b)^{-2m} \left( \frac{1}{2b} \right)^n \]

\[ = \left( \frac{h}{2b} \right)^{\delta+\Delta} \sum_{m,n=0}^{\infty} \frac{(\delta + \Delta + n+2m)}{m!n!(\delta + 1)_{n+m}} \frac{(e^{i\theta})^n}{(2b)^{-2m}}. \]

7 Graf and Gubler, [9], give a geometrical interpretation of this relation.

8 It has been used in [11] in the context of bulk channel block expansion. The Appell \( F_1 \) function also appears in [11] but only as a hypergeometric surrogate.
after some rearrangement and cancellations.

Defining a new expansion variable, similar to \( x \), by,

\[
\zeta \equiv (2b)^{-1}e^{i\theta},
\]

the sum (11) is a double power series in \( \zeta \) and \( \zeta \xi \) and can be recognised as a confluent Horn series, \( H_6 \), [12] 5.7.1(34). Thus we have shown that the Appell function has the transformation,

\[
x^{\delta+\Delta} F_1(\delta + \Delta; 1, \Delta; \delta + 1; x, x) = \zeta^{\delta+\Delta} H_6(\delta + \Delta, \delta + 1; \zeta \xi, \zeta),
\]

which relates different representations of the expansion of the correlator into defect blocks, for free scalars. I note, in passing, that the convergence criterion, \( \zeta \xi < 1/4 \), is satisfied since \( b \geq 1 \).

There are many other transformations that can be applied to \( F_1 \). Most are to be found in Appell and Kampé de Feriet, [13], and relate \( F_1 \) to itself or to the other Appell functions. For example one has, [13] p.35. equin.(9),

\[
F_1(a; b, b'; c; x, y) = x^{b'} y^{-b'} F_2(b + b', a, b'; c, b + b'; x, 1 - xy^{-1}).
\]

This is actually derived from a double integral representation of \( F_1 \) which yields the more significant expansion,\(^9\), [13], p.34 equin.(8),

\[
F_1(a; b, b'; c; x, y) = \sum_{n=0}^{\infty} \frac{(a)_n (b + b')_n}{n!(c)_n} 2F_1(-n, b', b + b'; 1 - yx^{-1}) x^n,
\]

having, as pointed out in [13], the remarkable property that the coefficients are finite polynomials in \( 1 - yx^{-1} \) of degree \( n \).

Inserting the CFT parameters and putting \( x = x, y = x\bar{x} \), one finds,

\[
F_1(\delta + \Delta; 1, \Delta; \delta + 1; x, x\bar{x}) = \sum_{n=0}^{\infty} \frac{(\delta + \Delta)_n (1 + \Delta)_n}{n!(\delta + 1)_n} 2F_1(-n, \Delta, 1 + \Delta; 1 - \bar{x}) x^n,
\]

which is a Fourier series whose coefficients are finite polynomials in \( 1 - \bar{x} \) of degree \( n \), giving yet another defect block expansion.

As an example of a more out–of–the–way transformation, I cite a formula given by Bailey, [14], which reads,

\[
F_4(\alpha, \beta; \gamma, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}) = (1-x)^\alpha (1-y)^\alpha F_1(\alpha; \gamma, \beta; 1 + \alpha - \gamma; \gamma; x, xy).
\]

\(^9\) Avoid confusing the generic ‘\( x \)’ with the \( x \) variable of [6].
With the parameter choice, $\alpha = \delta + \Delta$, $\beta = \delta$, $\gamma = \delta + 1$ and setting $x = x$, $y = \bar{x}$, it gives another expression for the correlator (7) via the expansion of the Appell $F_4$ function,

$$F_4(a, b, c, c', x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{m!(c)_m} 2F_1(a + m, b + m; c', y) x^m,$$

in terms of another series expansion variable $\kappa \equiv x/|1 - x|^2$.

5. The coincidence limit

One standard computation of one–point vacuum averages consists of applying differential operators to the Green function and then taking the coincidence limit. Divergences appear that have to be taken off, essentially by hand. While not difficult, this process could be avoided if the source of the infinities, the $G_0$ Green function, were to be removed prior to performing the above operation. This can easily be accomplished at the integral level by deforming the contour $A$ in (2) to pass through the pole in the periodising function (this gives $G_0$) and then expunging the circuit around this pole to leave a modified contour which can be taken as two vertical lines in the $\alpha$–plane. In the $u$ plane, this corresponds to moving the Bessel contour, $C$, through the pole at $e^{i\theta}$ which is then discarded to leave a contour, $\overline{C}$, running mostly around the negative real $u$ axis and looping around the point $u = 1/h$. This is finite in the coincidence limit ($h \to 1$, $\theta \to 0$) and the integral becomes an Euler–type one, evaluating to Gamma and Beta functions. This will be considered in a later communication.

6. Comments and conclusion

It has been shown that the free–field bulk correlator for a pure monodromy defect is contained in an old contour integral which evaluates to an Appell $F_1$ function and can be expanded in several different ways into defect blocks.

Knowing that the correlator is an Appell function, gives access to a large number of known transformations. Whether this is physically useful is unclear.

The contour integral applies also if the defect has a conical singularity but the resulting formulae are not so neatly developed.
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