Abstract. We develop a new homotopy method for solving multiparameter eigenvalue problems (MEPs) called the fiber product homotopy method. It requires deformation of exponentially fewer equations compared to existing homotopy methods for MEPs. We show that the fiber product homotopy method theoretically finds all eigenpairs of an MEP with probability one. It is especially well-suited for singular MEPs, a weakness of all other existing methods, as the fiber product homotopy method is provably convergent with probability one for singular problems as well, a fact borne out by numerical experiments. More generally, our numerical experiments indicate that the fiber product homotopy method significantly outperforms the standard Delta method in terms of accuracy, with consistent backward errors on the order of $10^{-16}$ without any use of extended precision. In terms of speed, it significantly outperforms previous homotopy-based methods on all problems and outperforms the Delta method on larger problems, and is also highly parallelizable. We show that the fiber product MEP that we solve in the fiber product homotopy method, although mathematically equivalent to a standard MEP, is typically a much better conditioned problem.

1. Introduction

A multiparameter eigenvalue problem (MEP) is, in an appropriate sense, a system of linear equations

\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k &= b_1, \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k &= b_2, \\
  &\vdots \\
  a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k &= b_k,
\end{align*}

where the coefficients $a_{ij}$'s and $b_i$'s are matrices, and where equality is interpreted to mean on a point in a product of projective spaces (this will be made precise later). These coefficients are square matrices but are of different dimensions in general, so one may not usually regard (1.1) as a linear system over a matrix ring. There is a rich mathematical theory behind MEP [2, 3] that places it at the crossroad of linear and multilinear algebra, ordinary and partial differential equations, spectral theory and Sturm–Liouville theory, among other areas. The problem appeared as early as 1836 in the works of Sturm and Liouville on periodic heat flow in a bar, and was studied over the years by many: Klein, Lamé, Heine, Stieltjes, Pell, Carmichael, Bocher, Hilbert among them (see [2, Preface] and [3, Chapter 1]).

An MEP encompasses many known types of eigenvalue problems: Standard eigenvalue problems $Ax = \lambda x$; generalized eigenvalue problems $Ax = \lambda Bx$; quadratic eigenvalue problems $(\lambda^2 A + \lambda B + C)x = 0$; polynomial eigenvalue problems $(\lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0)x = 0$; quadratic two-parameter eigenvalue problems

\begin{align*}
  (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda \mu A_{11} + \mu^2 A_{02})x_1 &= 0, \\
  (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02})x_2 &= 0;
\end{align*}

may all be reduced to mathematically equivalent MEPs.
Nevertheless MEP remains in the blind spot of most modern mathematicians, whether pure or applied. This is not for its lack of applications; as we pointed out, the problem in fact originated from a study of heat flow, and we will see yet other applications of MEP in Section 7 and that it contains eigenvalue problem and linear system, both ubiquitous in science and engineering, as special cases. We think that a main reason for the obscurity of MEPs is that there are not many effective methods for its computation and there is thus little to be gained from formulating a problem as an MEP. It is with this in mind that we propose a new homotopy method based on what we call fiber product homotopy for computing MEP solutions.

We will now formally define an MEP in more conventional notations. Instead of having a single eigenvalue parameter $\lambda$, an MEP has multiple eigenvalue parameters $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_k)$. We will call

$$H(\mathbf{\lambda}) := A_0 - \lambda_1 A_1 - \cdots - \lambda_k A_k,$$

a linear polynomial matrix in $k$ parameters $\lambda_1, \ldots, \lambda_k$ with matrix coefficients $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$. We will write $\mathbb{P}^n$ for the complex projective $n$-space.

**Definition 1.1.** For a fixed $k \geq 2$ and given matrices $A_{ij} \in \mathbb{C}^{n_i \times n_j}$ with $j = 0, 1, \ldots, k$, $i = 1, \ldots, k$, consider the linear polynomial matrices

$$H_i(\mathbf{\lambda}) := A_{i0} - \lambda_1 A_{i1} - \lambda_2 A_{i2} - \cdots - \lambda_k A_{ik}.$$ 

The multiparameter eigenvalue problem (MEP), or, more precisely, a $k$-parameter eigenvalue problem, is to find $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and corresponding $(x_1, \ldots, x_k) \in \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}$ such that

$$(1.2) \quad H_i(\mathbf{\lambda}) x_i = 0, \quad i = 1, \ldots, k.$$ 

A solution $(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^k \times \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}$ to the MEP is called an eigenpair, the $k$-tuple $(x_1, \ldots, x_k) \in \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}$ an eigenvector, and the $k$-tuple $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ an eigenvalue.

Written out in full, (1.2) takes the form

$$(1.3) \quad (\lambda_1 A_{11} + \lambda_2 A_{12} + \cdots + \lambda_k A_{1k})x_1 = A_{10}x_1,$$

$$(\lambda_1 A_{21} + \lambda_2 A_{22} + \cdots + \lambda_k A_{2k})x_2 = A_{20}x_2,$$

$$(\lambda_1 A_{k1} + \lambda_2 A_{k2} + \cdots + \lambda_k A_{kk})x_k = A_{k0}x_k.$$ 

With $\lambda_i$'s playing the role of $x_i$'s, $A_{ij}$'s and $A_{0i}$'s playing the roles of $a_{ij}$'s and $b_i$'s respectively in (1.1), and interpreting equality of the $i$th equation in (1.1) to mean equality on some $x_i \in \mathbb{P}^{n_i-1}$, we may view (1.3) as an analogue of a linear system that we referred to at the beginning. The analogy is precise when $n_1 = \cdots = n_k = 1 - (1.3)$ is a linear system in the usual sense.

When $k = 1$, (1.3) is a generalized eigenvalue problem. More generally, if $A_{ij} = 0$ for all $i \neq j$, then (1.3) is decoupled into $k$ generalized eigenvalue problems. Hence (1.3) contains both eigenvalue problems and linear systems as special cases. The multiparameter eigenvalue problem is well studied and readers may refer to the books [3][2][23] for a comprehensive treatment.

Since any scalar multiple of $x_i$ is also an eigenvector it is fitting to consider $x_i$ as an element of the projective space $\mathbb{P}^{n_i-1}$ although for practical reason one might prefer to simply normalize $x_i$ to have unit norm.

Our goal is to develop a new homotopy method to solve a multiparameter eigenvalue problem effectively, where effectiveness is measured by the following factors:

- *Speed* as measured by wall time. We record time per path, maximum time over all paths, and total track time of all paths. Our algorithm is highly parallelizable and the per-path times give good speed estimates when there are enough cores to track all paths in parallel.
- *Accuracy* as measured by the backward error. We use the normwise backward error in [13] Theorem 2] for an approximate eigenpair. Our homotopy method tracks several copies of the
eigenvalue $\lambda$; they should all converge to the same value if our method performs correctly and we include the difference between copies of $\lambda$’s as another measure of accuracy.

- *Certificates* of quadratic convergence in terms of Shub–Smale $\alpha$-theory.
- *Number of divergent paths* that fail to converge to the solutions.

The last two measures only apply to methods based on homotopy continuation. We will compare our method to two existing methods:

(i) The *Delta method* [2], which is the de facto standard method for solving MEPs by transforming them into generalized eigenvalue problems; we use the MultiPARÉig package in our experiments with this method.

(ii) The *diagonal coefficient homotopy method* recently proposed in [9] for solving MEPs, where the start system is a random choice of diagonal matrices and the homotopy is a straight-line homotopy that deforms $n_1 \cdots n_k$ of $n_1 \cdots n_k + k$ equations.

The numerical experiments in [9] show that the diagonal coefficient homotopy method outperforms the Delta method in terms of memory usage and is competitive in speed for large values of $n_1, \ldots, n_k$. In both methods, one finds all eigenpairs of an MEP.

Our *fiber product homotopy method* adopts an alternative approach — we solve an MEP (1.2) by solving a mathematically equivalent system that we will call the fiber product multiparameter eigenvalue problem:

\begin{equation}
H_i(\lambda_{i})x_i = 0, \quad \lambda_1 = \lambda_2 = \cdots = \lambda_k, \quad i = 1, \ldots, k,
\end{equation}

where $\lambda_1, \ldots, \lambda_k$ are to be regarded as different copies of $\lambda$. The fiber product homotopy method that we propose has a start system that captures more structure of the MEP and allows us to deform far fewer equations. By introducing $k(k-1)$ auxiliary unknowns, we deform at most $k(k-1)$ of the $n_1 \cdots n_k + k^2$ equations. Compared to the $n_1 \cdots n_k$ equations required by the diagonal coefficient homotopy of [9], our fiber product homotopy is exponentially more efficient as

\[ k(k-1)ln_1 \cdots n_k. \]

For a fixed $k$ and $n_1, \ldots, n_k = O(n)$, fiber product homotopy deforms $O(1)$ equations whereas diagonal coefficient homotopy deforms $O(n^k)$ equations. Furthermore, fiber product homotopy deforms only linear equations whereas diagonal coefficient homotopy deforms nonlinear equations.

An eigenpair $(\lambda, x_1, \ldots, x_k)$ of an MEP is said to be *regular* if the eigenvalue $\lambda$ is isolated and has multiplicity one (see [15] for definitions of algebraic and geometric multiplicity). Since the expected number of regular eigenpairs to an MEP is $n_1 \cdots n_k$, one often considers only relatively small values of $k$ to $n$ when finding all eigenpairs. Our fiber product homotopy method is guaranteed to compute all regular eigenpairs in theory — we show in Theorem 5.4 that our start system in Section 5 is chosen correctly with probability one.

The fiber product homotopy method is motivated by several geometric insights. In Section 3, we will define two algebraic varieties associated with a fiber product MEP: multiparameter eigenvalue variety and multiparameter eigenpair variety. Fiber products, a notion well-known in areas stretching from algebraic geometry to relational database, will be reviewed in Section 4. The name for our method is a result of the fact that a $k$-parameter eigenpair variety is a fiber product of $k$ one-parameter eigenpair varieties. In Section 11, we rely on geometry to define a condition number for the fiber product MEP (1.4), which differs from the condition number for the standard MEP (1.3), these being two different problems, albeit having the same solutions. The geometric nature of our condition number is in that it is the condition number of intersecting an algebraic variety (our multiparameter eigenpair variety) with a varying linear space (defined by our start system).

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1 More precisely, we solve $H_i(\lambda_{i})x_i = 0, G_1 = \cdots = G_k = 0, i = 1, \ldots, k$, where each $G_i : \mathbb{C}^{k^2} \rightarrow \mathbb{C}^{k-1}$ is a linear function of $\lambda_1, \ldots, \lambda_k$, chosen so that the resulting system is equivalent to (1.4). See Section 5.2.
We implemented a purely numerical version (in particular, it does not use multiprecision) of fiber product homotopy in MATLAB and a mixed symbolic-numerical version in BERTINI with MACAULAY2 for comparison. We did extensive numerical experiments with both implementations: randomly generated MEPs in Section 8, the Mathieu two-parameter eigenvalue problem arising from a real-world application — an elliptic membrane vibration problem — in Section 9, and singular MEPs, a challenging class of MEPs with a deficiency in the number of eigenpairs and that breaks most other methods, in Section 10.

We applied a broad range of different measures for speed and accuracy to our numerical experiments in these sections to stress test the robustness of our method. In Section 8, speed is measured via both wall time and iteration count (number of Newton steps); accuracy is measured via both relative backward error of the computed eigenpairs and the deviation in the multiple paths used to track the eigenvalues. In Section 9, we test the effect of reducing the number of Newton steps (by early stopping) on the accuracy of our method and certify the final quadratic convergence speed using Shub–Smale \(\alpha\)-theory. For the singular MEPs in Section 10, what breaks other homotopy methods is the issue of divergent paths and as such we use the number of divergent paths as a measure of effectiveness. The take-away is that our fiber product homotopy has zero divergent path in every case that we tested. In Sections 8 and 10, we also provide the time it takes to track a single path, which gives a good estimate of the speed under parallel execution of our method (since each path can be tracked independently of others).

2. Homotopy methods

We recall the straight-line homotopy and describe the diagonal coefficient homotopy method used in [9] but will defer the description of our fiber product homotopy method to Section 5.

A homotopy deforms solutions of a start system \(Q(z) = 0\) to solutions of a target system \(P(z) = 0\). More precisely, a straight-line homotopy with path parameter \(t\) is defined as

\[
H(z, t) := (1 - t)Q(z) + tP(z), \quad t \in [0, 1].
\]

When \(t = 0\), \(H(z, 0) = Q(z) = 0\) is the start system and when \(t = 1\), \(H(z, 1) = P(z) = 0\) is the target system.

**Definition 2.1.** A start system for the homotopy (2.1) is said to be chosen correctly [16] if the following properties hold:

(i) the solution set of the start system \(Q(z) = 0\) are known or easy to obtain;
(ii) the solution set of \(H(z, t) = 0\) for \(0 \leq t < 1\) consists of a finite number of smooth paths, each parametrized by \(t\) in \([0, 1]\);
(iii) for each isolated solution of the target system \(P(z) = 0\), there is some path originating from a solution of the start system \(Q(z) = 0\) that leads to it.

Let \(D_{ij}\) denote diagonal matrices in \(\mathbb{C}^{n_i \times n_i}\). The diagonal coefficient homotopy method for solving MEP is the straight-line homotopy from \(t = 0\) to \(t = 1\) given by:

\[
H_{dc}(\lambda, x_1, \ldots, x_k, t) := (1 - t)Q_{dc} + tP_{dc},
\]

\[
Q_{dc}(\lambda, x_1, \ldots, x_k) := \begin{bmatrix}
(D_{10} + \lambda_1 D_{11})x_1 \\
\vdots \\
(D_{k0} + \lambda_k D_{kk})x_k
\end{bmatrix},
\]

\[
P_{dc}(\lambda, x_1, \ldots, x_k) := \begin{bmatrix}
H_1(\lambda)x_1 \\
\vdots \\
H_k(\lambda)x_k
\end{bmatrix}.
\]
One should regard
\[ H_{dc} : \mathbb{C}^k \times (\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}) \times \mathbb{C} \to \mathbb{C}^{n_1+\cdots+n_k} \]
as a polynomial map and likewise for
\[ Q_{dc}, P_{dc} : \mathbb{C}^k \times (\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}) \to \mathbb{C}^{n_1+\cdots+n_k}. \]
The homotopy method proposed in [9] is an example of a diagonal coefficient homotopy method.

To implement the homotopy above, one has to account for the scaling of the eigenvectors by introducing \(k\) constraints. One way to do this is by scaling each \(x_i\) so that it has norm one. This is the approach undertaken in [9]. Another way to do this is to place a generic\(^2\) affine constraint on each \(x_i\), which is what we will do in our fiber product homotopy in Section 5.

### 3. Multiparameter eigenvarieties

We will define two algebraic varieties associated with a multiparameter eigenvalue problem.

**Definition 3.1.** In algebraic geometric terms \([8, 11]\), the coordinates of \(H(\lambda)x\) are polynomials that form a subset of \(\mathbb{C}[\lambda, x] = \mathbb{C}[\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n]\) and define an algebraic variety
\[ \mathcal{EP}(H) := \{ (\lambda, x) \in \mathbb{C}^k \times \mathbb{P}^{n-1} : H(\lambda)x = 0 \}. \]

We will call this the eigenpair variety of \(H\). In the context of an MEP, we will call the Cartesian product \(\mathcal{EP}(H_1, \ldots, H_k) := \mathcal{EP}(H_1) \times \cdots \times \mathcal{EP}(H_k)\) the multiparameter eigenpair variety of \(H_1, \ldots, H_k\). Explicitly,
\[ \mathcal{EP}(H_1, \ldots, H_k) = \{ (\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^k \times \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1} : H_1(\lambda_1)x_1 = 0, \ldots, H_k(\lambda_k)x_k = 0 \}. \]

The multiparameter eigenvalue variety of \(H_1, \ldots, H_k\) is the coordinatewise projection of the multiparameter eigenpair variety to \(\mathbb{C}^{k^2}\) and will be denoted by \(\mathcal{EV}(H_1, \ldots, H_k)\). Alternatively, it can be defined explicitly as
\[ \mathcal{EV}(H_1, \ldots, H_k) = \{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k : \det H_1(\lambda_1) = 0, \ldots, \det H_k(\lambda_k) = 0 \}. \]

**Definition 3.2.** An MEP is said to have intrinsic dimension \((d_1, \ldots, d_k)\) if the degree of the polynomial \(\det H_i(\lambda_i)\) is \(d_i\), \(i = 1, \ldots, k\). Such an MEP is said to be generic with respect to intrinsic dimension if the hypersurface defined by \(\det H_i(\lambda_i)\) is generically reduced\(^3\) for \(i = 1, \ldots, k\).

By Bezout’s theorem, the degree of the multiparameter eigenvalue variety \(\mathcal{EV}(H_1, \ldots, H_k) \subseteq \mathbb{C}^{k^2}\) is at most \(\prod_{i=1}^k d_i\). This means that the number of isolated regular points in the intersection of the multiparameter eigenvalue variety with a codimension-\(k(k-1)\) affine linear space in \(\mathbb{C}^{k^2}\) is at most \(\prod_{i=1}^k d_i\). We state this formally below.

**Proposition 3.3.** An MEP with intrinsic dimension \((d_1, \ldots, d_k)\) has at most \(\prod_{i=1}^k d_i\) regular eigenvalues and eigenvectors. This bound is tight if the MEP is generic with respect to intrinsic dimension.

The following simple observation will be a key to our fiber product homotopy method for MEP.

**Lemma 3.4.** Let \(\mathbb{D}\) denote the linear space defined by
\[ \mathbb{D} := \{ (\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^k \times \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1} : \lambda_1 = \cdots = \lambda_k \}. \]
Then the intersection \(\mathbb{D} \cap \mathcal{EP}(H_1, \ldots, H_k)\) corresponds to the set of eigenvpairs of \([1.2]\).
The algebraic variety $\mathcal{D} \cap \mathcal{EP}(H_1, \ldots, H_k)$ is the fiber product of $\mathcal{EP}(H_1), \ldots, \mathcal{EP}(H_k)$ over $\mathbb{C}^k$, a standard notion in algebraic geometry [11]. This is the impetus for the name of our homotopy method — fiber product homotopy — and we will discuss it in the next section.

4. Fiber products

Knowledge of the fiber product’s formal properties at the level of, say, [11] is unnecessary for this article. Instead all we need is the notion of fiber product of sets — an important and well-known concept in relational database theory [18, Section 6.3].

Let $X$ and $Y$ be two sets and $\varphi : X \rightarrow A$ and $\psi : Y \rightarrow A$ be maps to a third set $A$.

$$
\begin{array}{c}
Y \\
\downarrow \psi \\
X \xrightarrow{\varphi} A
\end{array}
$$

The fiber product $X \times_A Y$ of $X$ and $Y$ over $A$ is the subset of the product $X \times Y$ given by

$$
X \times_A Y := \{(x, y) \in X \times Y : \varphi(x) = \psi(y)\}.
$$

The fiber product satisfies the following commutative diagram where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are projection maps:

$$
\begin{array}{ccc}
X \times_A Y & \xrightarrow{\pi_2} & Y \\
\downarrow \pi_1 & & \downarrow \psi \\
X \xrightarrow{\varphi} A
\end{array}
$$

We will illustrate fiber products with a few examples.

Example 4.1 (Relational database). Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and $A = \{-1, +1\}$. Let the maps $\varphi : X \rightarrow A$ and $\psi : Y \rightarrow A$ be given by

$$
\varphi(x) = \begin{cases} 
+1 & x \text{ is odd,} \\
-1 & x \text{ is even,}
\end{cases} \quad \psi(y) = \begin{cases} 
+1 & y \text{ is a vowel,} \\
-1 & y \text{ is a consonant.}
\end{cases}
$$

Then the fiber product of $X$ and $Y$ over $A$ is

$$
X \times_A Y = \{(1, a), (1, e), (2, b), (2, c), (2, d), (3, a), (3, e), (4, b), (4, c), (4, d)\}.
$$

Incidentally, this toy example underlies the JOIN operation in the structured query language (SQL) of a relational database management system (RDBMS). See [18, Section 6.3] for more information.

Example 4.2 (Algebraic geometry). Consider the following cubic curves in $\mathbb{R}^2$:

$$
X = \{(t_1, z_1) \in \mathbb{R}^2 : t_1 = z_1(z_1 - 1)(z_1 - 2) + 1\}, \\
Y = \{(t_2, z_2) \in \mathbb{R}^2 : t_2 = z_2(z_2 - 1)(z_2 - 2) + 1\}.
$$

Let $A = \mathbb{R}$ and consider the maps

$$
\varphi : X \rightarrow A, \quad \varphi(t_1, z_1) = t_1 \quad \text{and} \quad \psi : Y \rightarrow A, \quad \psi(t_2, z_2) = t_2.
$$

Their fiber product,

$$
X \times_A Y = \{(t_1, t_2, z_1, z_2) \in \mathbb{R}^4 : (t_1, z_1) \in X, (t_2, z_2) \in Y, t_1 = t_2\},
$$
is shown in Figure 1. While the Cartesian product $X \times Y$ is an irreducible surface, the fiber product $X \times_A Y$ is a union of two curves — a point $(t_1, t_2, z_1, z_2) \in X \times_A Y$ satisfies

$$z_1 = z_2 \quad \text{or} \quad z_1^2 + z_1 z_2 + z_2^2 = 3z_1 + 3z_2 - 2.$$ 

One of the curves projects onto a line and the other projects onto an ellipse. Whereas the Cartesian product of two irreducible curves is always an irreducible surface, this example shows that the fiber product of two irreducible curves does not need to be irreducible.

**Figure 1.** $X \times_A Y$ and $X \times Y$ are subsets of $\mathbb{R}^4$, but we only plot the coordinates $(z_1, z_2, t_1)$. *Left:* The fiber product $X \times_A Y$ is the union of the blue curve and the magenta curve. The dotted blue line and dotted magenta ellipse are the $(z_1, z_2)$-coordinate projections of the corresponding curve onto the $t_1 = 0$ plane. *Right:* The $(z_1, z_2, t_1)$-coordinate projection of the Cartesian product $X \times Y$.

### Example 4.3 (Multiparameter eigenvalue problem).

Consider a two-parameter eigenvalue problem and the eigenpair varieties of $H_1$ and $H_2$,

$$X = \{(\lambda_1, x_1) \in \mathbb{C}^2 \times \mathbb{P}^{n_1-1} : H_1(\lambda_1)x_1 = 0\},$$

$$Y = \{(\lambda_2, x_2) \in \mathbb{C}^2 \times \mathbb{P}^{n_2-1} : H_2(\lambda_2)x_2 = 0\}.$$

Let $A = \mathbb{C}^2$ and consider the maps

$$\varphi : X \to A, \quad \varphi(\lambda_1, x_1) = \lambda_1 \quad \text{and} \quad \psi : Y \to A, \quad \psi(\lambda_2, x_2) = \lambda_2.$$

Their fiber product is

$$X \times_A Y = \{(\lambda_1, \lambda_2, x_1, x_2) \in \mathbb{C}^4 \times \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} : (\lambda_1, x_1) \in X, (\lambda_2, x_2) \in Y, \lambda_1 = \lambda_2\}$$

$$= \mathcal{E}\mathcal{P}(H_1, H_2),$$

the two-parameter eigenpair variety of $H_1, H_2$.

5. **Fiber product homotopy method**

Fiber products have previously appeared in numerical algebraic geometry in various contexts: study of exceptional sets [20], algorithms to intersect varieties [13], and numerical computations of Galois groups [12 Section 4]. However, the use of fiber products in a homotopy method for solving MEPs is, as far as we know, new. We will now describe this method.
5.1. Start system. We begin our discussion of a start system with the following observation. Let $H(\lambda) = A_0 - \lambda_1 A_1 - \cdots - \lambda_k A_k$ be a linear polynomial matrix and $L(\lambda) = A\lambda - b$ be an affine linear function for some $A \in \mathbb{C}^{(k-1) \times k}$ and $b \in \mathbb{C}^{k-1}$. We claim that the system

$$H(\lambda)x = 0, \quad L(\lambda) = 0$$

is equivalent to a generalized eigenvalue problem, that we will call the associated generalized eigenvalue problem or associated MEP for short.

To see this, let $\lambda = \beta q + p$ where $p, q \in \mathbb{C}^k$ are such that $L(p) = 0$ and $q \in \ker(\nabla L)$, where $\nabla$ denotes the total derivative (also known as total differential). Note that the previous statement says nothing more than $Ap = b$ and $Aq = 0$. Eliminating $q$ and $p$ from

$$(5.1) \quad H(\beta q + p)x = 0, \quad q \in \ker(\nabla L), \quad L(p) = 0,$$

then gives us a GEP with $\beta$ the generalized eigenvalue and $x$ the generalized eigenvector. We will see in Example 5.2 how one may obtain a GEP from (5.1), but expressing the GEP in terms of general $A$, $b$, and $A_0, \ldots, A_k$ is complicated and unilluminating. The associated eigenpairs are

$$(5.2) \quad \{ (\lambda, x) \in \mathbb{C}^k \times \mathbb{P}^{n-1} : \lambda = \beta q + p, \ H(\beta q + p)x = 0 \}.$$

As each coordinate of $H(\beta q + p)x$ is homogeneous in $x$, $H(\beta q + p)x = 0$ is well-defined for $x \in \mathbb{P}^{n-1}$.

For an MEP (1.2), let $L_i : \mathbb{C}^k \to \mathbb{C}^{k-1}, i = 1, \ldots, k$, be affine linear maps. Thus each $L_i(\lambda_i) = 0$ is an affine linear equation in $\lambda_i, i = 1, \ldots, k$. We obtain $k$ associated GEPs:

$$(5.3) \quad H_i(\lambda_i)x_i = 0, \quad L_i(\lambda_i) = 0, \quad i = 1, \ldots, k.$$

The set of solutions to (5.3) will be called start solutions or start points and denoted $S$. If the MEP has intrinsic dimension $(d_1, \ldots, d_k)$, then the $i$th associated GEP has at most $d_i$ generalized eigenvalues. Thus if $S_i$ denotes the set of associated eigenpairs that have distinct generalized eigenvalues, then $S$ is given by the Cartesian product

$$S = S_1 \times \cdots \times S_k$$

and it has cardinality $\prod_{i=1}^k |S_i| \leq \prod_{i=1}^k d_i$.

We define $Q_{FP} : \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1} \to \mathbb{C}^{n_1+\cdots+n_k} \times \mathbb{C}^{k(k-1)}$ by

$$(5.4) \quad Q_{FP}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) := \begin{bmatrix} H_1(\lambda_1)x_1 \\ \vdots \\ H_k(\lambda_k)x_k \\ L_1(\lambda_1) \\ \vdots \\ L_k(\lambda_k) \end{bmatrix}$$

and choose our start system to be the $\sum_{i=1}^k (k-1 + n_i)$ equations in $\sum_{i=1}^k (k + n_i)$ variables

$$(5.5) \quad Q_{FP}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) = 0.$$

Again note that $H_i(\lambda_i)x_i = 0$ is well-defined for $x_i \in \mathbb{P}^{n_i-1}, i = 1, \ldots, k$. We have the following easy observation.

Lemma 5.1. The points in $S$ are regular solutions to the start system (5.5). If the MEP is generic with respect to intrinsic dimension, then $|S| = \prod_{i=1}^k d_i$ for any generic choice of affine linear maps $L_1, \ldots, L_k$.

We now give an illustration of how an MEP can be transformed into a system of GEPs by imposing random affine constraints.
Example 5.2. Consider the two-parameter MEP given by the polynomial matrices below:

$$H_1(\lambda_1) = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} + \lambda_{11} \begin{bmatrix} 11 & 13 \\ 17 & 19 \end{bmatrix} + \lambda_{12} \begin{bmatrix} 23 & 29 \\ 31 & 37 \end{bmatrix},$$

$$H_2(\lambda_2) = \begin{bmatrix} 12 & 31 \\ 15 & 71 \end{bmatrix} + \lambda_{21} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \lambda_{22} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

where we write $\lambda_i = (\lambda_{i1}, \lambda_{i2})$, $i = 1, 2$.

For this MEP, we pick two random (generic) affine linear polynomials $L_1$ and $L_2$, e.g.,

$$L_1(\lambda_1) = (0.6909 + 0.2745i)\lambda_{11} + (0.4277 - 0.1333i)\lambda_{12} - 1,$$

$$L_2(\lambda_2) = (-0.1443 + 0.5711i)\lambda_{21} + (-0.0735 + 1.8085i)\lambda_{22} - 1.$$

Then we have

$$[q_1, q_2, p_1, p_2] = \begin{bmatrix} -0.41 + 0.31i & -0.52 + 0.79i & 1.2501 - 0.4967i & 0 \\ 0.85 + 0.02i & 0.22 - 0.21i & 0 & -0.0224 - 0.5520i \end{bmatrix}.$$

The polynomial matrices for the two associated GEP are

$$H_1(\beta q_1 + p_1) = \begin{bmatrix} -11.75 + 5.46i & -13.25 + 6.45i \\ -16.25 + 8.44i & -16.75 + 9.43i \end{bmatrix} - \beta \begin{bmatrix} 15.17 + 4.02i & 19.48 + 4.79i \\ 19.55 + 6.10i & 23.87 + 6.87i \end{bmatrix},$$

$$H_2(\beta q_2 + p_2) = \begin{bmatrix} 12.04 + 1.10i & 31.04 + 1.10i \\ 15.04 + 1.10i & 71.04 + 1.10i \end{bmatrix} - \beta \begin{bmatrix} -0.08 + 0.35i & -0.08 + 0.35i \\ -0.08 + 0.35i & -0.08 + 0.35i \end{bmatrix}.$$

The first GEP has two finite eigenvalues: $-0.9978 + 1.1933i$ and $-0.5637 + 0.3035i$. The second GEP has only one finite eigenvalue at $-3.6333 - 28.4804i$. Thus the start solutions, i.e., the solutions to our start system, of our homotopy is a set of two points with $(\lambda_1, \lambda_2)$-coordinates below.

$$\begin{array}{c|c}
\lambda_1 & \lambda_2 \\
\hline
(-0.9978 + 1.1933i)q_1 + p_1 & (-3.6333 - 28.4804i)q_2 + p_2 \\
(-0.5637 + 0.3035i)q_1 + p_1 & (-3.6333 - 28.4804i)q_2 + p_2
\end{array}$$

5.2. Target system. For $i = 1, \ldots, k$, let $R_i \in \mathbb{C}^{(k-1)\times k(k-1)}$ be generic and let $G_i: \mathbb{C}^{k^2} \to \mathbb{C}^{k-1}$ be the linear function defined by

$$G_i(\lambda_1, \ldots, \lambda_k) := R_iD_k \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix}, \quad D_k := \begin{bmatrix} I_k & -I_k \\ \vdots & \ddots & \ddots \\ I_k & -I_k \end{bmatrix} \in \mathbb{C}^{k(k-1)\times k^2},$$

where $I_k$ is the $k \times k$ identity matrix. If the matrices $R_1, \ldots, R_k \in \mathbb{C}^{(k-1)\times k(k-1)}$ are generic, then

$$(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^{k^2} : G_1(\lambda_1, \ldots, \lambda_k) = \cdots = G_k(\lambda_1, \ldots, \lambda_k) = 0$$

and therefore,

$$(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^{k^2 \times \mathbb{P}^{n_1-1}} \times \cdots \times \mathbb{P}^{n_k-1} : G_1(\lambda_1, \ldots, \lambda_k) = \cdots = G_k(\lambda_1, \ldots, \lambda_k) = 0$$

$= \{(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^{k^2 \times \mathbb{P}^{n_1-1}} \times \cdots \times \mathbb{P}^{n_k-1} : \lambda_1 = \cdots = \lambda_k \} = \mathbb{D},$$

i.e., the linear space in Lemma 3.4. Hence the system

$$(5.6) \quad H_i(\lambda)x_i = 0, \quad G_1(\lambda_1, \ldots, \lambda_k) = \cdots = G_k(\lambda_1, \ldots, \lambda_k) = 0, \quad i = 1, \ldots, k,$$

is equivalent to the fiber product MEP (1.4), which is equivalent to the original MEP in (1.2).
We define

\[ P_{FP}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) := \begin{bmatrix} H_1(\lambda_1)x_1 \\
\vdots \\
H_k(\lambda_k)x_k \\
G_1(\lambda_1, \ldots, \lambda_k) \\
\vdots \\
G_k(\lambda_1, \ldots, \lambda_k) \end{bmatrix}, \]

and choose our target system to be

\[ P_{FP}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) = 0, \]

which is of course just (5.8). By Proposition 3.4, our target system (5.9) yields the eigenpairs of the MEP.

5.3. Fiber product homotopy. The main objective of this section is to show that our start system is chosen correctly with probability one.

**Definition 5.3.** A fiber product homotopy for the MEP with linear polynomial matrices \( H_1, \ldots, H_k \) is a straight-line homotopy from \( t = 0 \) to \( t = 1 \) given by the polynomial map

\[ H_{FP} : \mathbb{C}^k \times (\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}) \times \mathbb{C} \to \mathbb{C}^{n_1+\ldots+n_k} \times \mathbb{C}^{(k-1)k} \]

where

\[ H_{FP}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k, t) := \begin{bmatrix} H_1(\lambda_1)x_1 \\
\vdots \\
H_k(\lambda_k)x_k \\
(1-t)L_1(\lambda_1) + tG_1(\lambda_1, \ldots, \lambda_k) \\
\vdots \\
(1-t)L_k(\lambda_k) + tG_k(\lambda_1, \ldots, \lambda_k) \end{bmatrix}. \]

Note that \( H_{FP} = (1-t)Q_{FP} + tP_{FP} \).

Throughout the article we assume that \( k \geq 2 \) so that we indeed have a multiparameter eigenvalue problem. If \( k = 1 \), then \( k(k-1) = 0 \), and (5.10) will not involve the path parameter \( t \). In case the reader is wondering whether our homotopy method applies to a standard eigenvalue problem or generalized eigenvalue problem like those discussed in [17, 24], this shows that the answer is no.

**Theorem 5.4.** The fiber product homotopy for an MEP (5.10) with intrinsic dimension \((d_1, \ldots, d_k)\) has a start system chosen correctly with probability one if the start solutions \( S \) has \(|S| = \prod_{i=1}^{k} d_i\).

**Proof.** By Lemma 5.1, the start solutions \( S \) are known after solving \( k \) generalized eigenvalue problems of dimensions \( n_1, \ldots, n_k \) respectively. Consider the variety

\[ \mathcal{C} := \{(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \in \mathbb{C}^k \times \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1} : L(\lambda_1) = 0, \ldots, L(\lambda_k) = 0\}. \]

The intersection \( \mathcal{C} \cap \mathcal{E}\mathcal{P}(H_1, \ldots, H_k) \) consists of \(|S| = \prod_{i=1}^{k} d_i\) points. As each \( G_i \) is a linear function and each \( R_i \in \mathbb{C}^{(k-1)k} \) is generic, \( i = 1, \ldots, k \), it follows from the gamma trick [21, Lemma 7.1.3] that for regular eigenpairs, the homotopy has a start system chosen correctly with probability one.

We will provide more extensive numerical experiments in Section 7 but here we will illustrate our method with a small example: \( k = 3 \) and \( n_1 = n_2 = n_3 = 2 \).
Example 5.5. We generate an MEP by randomly choosing the $2 \times 2$ coefficient matrices $A_{ij}$ for $i = 1, 2, 3$ and $j = 0, 1, 2, 3$. There are eight solutions to the start and target systems. Note that fiber homotopy method requires that we work in $\mathbb{C}^{15}$ since $k^2 + n_1 + n_2 + n_3 = 15$, i.e., $(\lambda_1, \lambda_2, \lambda_3, x_1, x_2, x_3) \in \mathbb{C}^{15}$. The end point will however be of the form $(\lambda, \lambda, \lambda, x_1, x_2, x_3) \in \mathbb{C}^{15}$ where $(\lambda, x_1, x_2, x_3) \in \mathbb{C}^9$ is a multiparameter eigenpair.

With our homotopy (5.10), we deform from $t = 0$ to $t = 1$. Note that $\lambda_i = (\lambda_{i1}, \lambda_{i2}, \lambda_{i3}) \in \mathbb{C}^3$, $i = 1, 2, 3$. In the left plot of Figure 2, we track the $\lambda_{i1}$-coordinate of all eight paths for $t \in [0, 0.9, 1]$, $i = 1, 2, 3$. The horizontal and vertical axes represent the real and imaginary axes. In the plot we see eight sets of three paths (colored red, blue, magenta to represent $i = 1, 2, 3$), each converging to a point that represents the first coordinate of an eigenvalue. We picked the $\lambda_{11}$-coordinate arbitrarily and could have done the same plot for any of the 15 coordinates in $(\lambda_1, \lambda_2, \lambda_3, x_1, x_2, x_3) \in \mathbb{C}^{15}$.

What we are witnessing is a one-dimensional projection of the homotopy path in $\mathbb{C}^{15}$ converging to the eight eigenpairs of the MEP. Note that “one dimension” here means “one complex dimension” which translates to the two real dimensions we see in Figure 2.

The left plot shows only the behavior of the homotopy path towards the end, i.e., only for $t \in [0, 0.9, 1]$. The right plot in Figure 2 shows the full homotopy path, i.e., for all $t \in [0, 1]$, of the $\lambda_{11}$-coordinates for one of the eight solutions of a different MEP of the same dimensions.

Figure 2. Left: The $\lambda_{11}$ coordinates of all eight paths for $t \in [0, 0.9, 1]$, $i = 1, 2, 3$. Right: The $\lambda_{11}$ coordinates of a path for $t \in [0, 1]$, $i = 1, 2, 3$. The horizontal and vertical axes represent the real and imaginary axes.

We would like to emphasize that in Example 5.5, the homotopy path is confined to a six-dimensional subspace of $\mathbb{C}^{15}$ as only $6 = k(k - 1)$ equations involve the path parameter $t$. The following difference between the fiber product homotopy and diagonal coefficient homotopy for MEP cannot be overstated. For the former, at most $k(k - 1)$ equations involve the path parameter, all of which are linear in $\lambda_1, \ldots, \lambda_k$. For the latter, $n_1n_2\ldots n_k$ equations involve the path parameter, all of which are multilinear. As $n$ grows large, as is the case in many applications, $n_1n_2\ldots n_k$ is vastly larger than $k(k - 1)$.

6. Continuation Procedure

Observe that there are $k$ more variables than equations in (5.10) because the eigenvectors are unique only up to scaling, i.e., they are indeed points in projective spaces. To account for this arbitrary scaling, we fix a generic affine chart in each projective space $\mathbb{P}^{n_i-1}$, i.e., by introducing the affine constraints $d_i^j x_i = 1$ with $d_i \in \mathbb{C}^{n_i}$, $i = 1, \ldots, k$.

Let $A_{ij} \in \mathbb{C}^{n_i \times n_j}$, $j = 0, 1, \ldots, k$, $i = 1, \ldots, k$ be the matrices of our MEP as in Definition 1.1 or 1.3. Recall the maps $G_i: \mathbb{C}^{k^2} \to \mathbb{C}^{k-1}$ from (5.6) and $L_i: \mathbb{C}^k \to \mathbb{C}^{k-1}$ from (5.3). For $i = 1, \ldots, k$,
we let \( \mathcal{L}_i : \mathbb{C}^{k^2} \to \mathbb{C}^{k-1} \) be defined by
\[
\mathcal{L}_i(\lambda_1, \ldots, \lambda_k) := L_i(\lambda_i),
\]
i.e., \( \mathcal{L}_i \) extends the domain of \( L_i \) from \( \mathbb{C}^k \) to \( \mathbb{C}^{k^2} \).

In the following, we write \( \mathbb{O}_{m \times n} \in \mathbb{C}^{m \times n} \) and \( \mathbb{O}_n \in \mathbb{C}^n \) for the zero matrix and zero vector, and \( \mathbb{1}_n \in \mathbb{C}^n \) for the all ones vector. To track the homotopy, we use an Euler–Newton predictor–corrector method. The Euler step gives an approximate eigenpair whereas the Newton step refines the approximation:

**Euler step:** This solves the \((k^2 + n_1 + \cdots + n_k) \times (k^2 + n_1 + \cdots + n_k)\) linear system
\[
\begin{bmatrix}
\text{diag}(B_1(x_1), \ldots, B_k(x_k)) & \text{diag}(H_1(\lambda_1), \ldots, H_k(\lambda_k)) \\
0_{k \times k^2} & \text{diag}(d_1^T, \ldots, d_k^T)
\end{bmatrix}
\begin{bmatrix}
\hat{\lambda}_1 \\
\vdots \\
\hat{\lambda}_k \\
\hat{x}_1 \\
\vdots \\
\hat{x}_k
\end{bmatrix}
= \begin{bmatrix}
\mathbb{O}_{n_1 + \cdots + n_k + k} \\
\nabla L_1 - \nabla G_1 - \mathbb{1}_{k-1} \\
\vdots \\
\nabla L_k - \nabla G_k - \mathbb{1}_{k-1}
\end{bmatrix},
\]
where \( B_i(x_i) := -[A_{i1}x_i, \ldots, A_{ik}x_i] \in \mathbb{C}^{n_i \times k}, \ i = 1, \ldots, k \). Here we regard the total derivatives as Jacobian matrices, i.e., \( \nabla L_i, \nabla G_i \in \mathbb{C}^{(k-1) \times k^2}, \ i = 1, \ldots, k \). As \( \mathcal{L}_i \)'s and \( G_i \)'s are affine linear maps, their Jacobians are constant matrices that do not depend on \( \lambda_i \)'s.

**Predictor:** This is given by
\[
p := \begin{bmatrix}
\hat{\lambda}_1 \\
\vdots \\
\hat{\lambda}_k \\
\hat{x}_1 \\
\vdots \\
\hat{x}_k
\end{bmatrix}
= \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_k \\
x_1 \\
\vdots \\
x_k
\end{bmatrix}
+ h \begin{bmatrix}
\hat{\lambda}_1 \\
\vdots \\
\hat{\lambda}_k \\
\hat{x}_1 \\
\vdots \\
\hat{x}_k
\end{bmatrix} \in \mathbb{C}^{n_1 + \cdots + n_k + k^2}
\]
where \( h \) is the step size. The predictor will form the input to the Newton step.

**Newton step:** This solves the \((k^2 + n_1 + \cdots + n_k) \times (k^2 + n_1 + \cdots + n_k)\) linear system
\[
\begin{bmatrix}
\text{diag}(B_1(x_1^{(l)}), \ldots, B_k(x_k^{(l)})) & \text{diag}(H_1(\lambda_1^{(l)}), \ldots, H_k(\lambda_k^{(l)})) \\
0_{k \times k^2} & \text{diag}(d_1^T, \ldots, d_k^T)
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_1 \\
\vdots \\
\Delta \lambda_k \\
\Delta x_1 \\
\vdots \\
\Delta x_k
\end{bmatrix}
= -\begin{bmatrix}
H_1(\lambda_1^{(l)}) \\
\vdots \\
H_k(\lambda_k^{(l)}) \\
d_1^T x_1^{(l)} - 1 \\
\vdots \\
d_k^T x_k^{(l)} - 1
\end{bmatrix}
\]
where \( B_i \) is as defined in the Euler step and
\[
v^{(l)} := \begin{bmatrix}
(1 - t)\nabla L_1 + t\nabla G_1 \\
\vdots \\
(1 - t)\nabla L_k + t\nabla G_k
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{(l)} \\
\vdots \\
\lambda_k^{(l)}
\end{bmatrix}
+ (1 - t) \mathbb{1}_{k-1} \mathbb{1}_{k-1} \in \mathbb{C}^{k(k-1)}.
\]

The initial approximation \([\lambda_1^{(0)}, \ldots, \lambda_k^{(0)}, x_1^{(0)}, \ldots, x_k^{(0)}]^T\) is given by the predictor \( p \).
Corrector: This is given by the solution to the Newton step

\[
\begin{bmatrix}
\Delta \lambda_1 \\
\vdots \\
\Delta \lambda_k \\
\Delta x_1 \\
\vdots \\
\Delta x_k
\end{bmatrix}
\in \mathbb{C}^{n_1 + \cdots + n_k + k^2},
\]

which is then added to refine the approximation,

\[
\begin{bmatrix}
\lambda_1^{(\ell+1)} \\
\vdots \\
\lambda_k^{(\ell+1)} \\
x_1^{(\ell+1)} \\
\vdots \\
x_k^{(\ell+1)}
\end{bmatrix} = \begin{bmatrix}
\lambda_1^{(\ell)} + \Delta \lambda_1 \\
\vdots \\
\lambda_k^{(\ell)} + \Delta \lambda_k \\
x_1^{(\ell)} + \Delta x_1 \\
\vdots \\
x_k^{(\ell)} + \Delta x_k
\end{bmatrix}.
\]

Our Euler–Newton predictor-corrector method uses a predictor \( p \) with size \( h \), followed by Newton steps until \( \|c\|_\infty \) is sufficiently small or when the maximum number of iterations is reached. If \( t + h < 1 \), then we return to the Euler step. If \( t + h \geq 1 \), then we update \( h \) so that \( t + h = 1 \), and do one final Euler step followed by Newton steps.

For a rough idea of the relative costs, tracking one path, i.e., one eigenpair, in our homotopy method typically takes a total of 50 Euler steps and a total of 200 Newton steps. We provide actual implementation details in Section 7.

The matrices in the Euler and Newton steps will in general look like the one depicted in Figure 3 for a four-parameter eigenvalue problem. The \( k \) row vectors \( d_1^T, \ldots, d_k^T \) near the bottom right corner and the \( (k^2 - k) \times k \) block representing \((1 - t)\nabla L_i^i + t \nabla G_i^j \) on the bottom left corner of the matrix are dense — recall from Lemma 5.1, Theorem 5.4, and the first paragraph of this section that we require the entries in these blocks be generic, i.e., there is zero probability that any of these entries is zero. On the other hand, if the matrices \( A_{ij} \in \mathbb{C}^{n_i \times n_i}, j = 0, 1, \ldots, k, i = 1, \ldots, k \), defining the MEP are sufficiently sparse, then the lighter shaded blocks in the top right part of the matrix representing \( H_i(\lambda_i) \) (Euler step) or \( H_i(\lambda_i^{(\ell)}) \) (Newton step), being the sum of \( k + 1 \) sparse matrices, will also be sparse. The darker shaded blocks on the top left part of the matrix representing \( B_i(x_i) \) (Euler step) or \( B_i(x_i^{(\ell)}) \) (Newton step) are expected to be dense, as the column vectors in these blocks are each a product of a sparse matrix and a dense vector.

It follows from [1, Theorem 5.2.1] that the Euler–Newton predictor-corrector method described above will converge to the solution of the target system if the step sizes are sufficiently small and the approximate start solution is sufficiently close to the actual start solution. Quantifying what it means to have a sufficiently small step size and sufficiently close start solution is still an active area of research, but we will see in Sections 8–10 ample numerical evidence of stable convergence to true solutions (even for singular MEPs). Indeed, the results in Sections 8–10 are the outcomes of thousands of MEPs — a single value in the tables and figures often represents an aggregate over tens or hundreds of runs — and we did not encounter any instance where our implementation of Euler–Newton had failed to converge.
7. Implementation

We implemented our method in both MATLAB and BERTINI/MACULA,Y2, catering respectively to the numerical computing and symbolic computing communities. We have made all our codes available online. The parameters below can be readily changed by the user.

**Inputs:** The inputs are the coefficients $A_{ij} \in \mathbb{C}^{n_i \times n_j}$, $j = 0, 1, \ldots, k$, $i = 1, \ldots, k$, of the polynomial matrices $H_1, \ldots, H_k$, as in Definition 1.1.

**Start solutions:** The solutions to the start system (5.4) are obtained as follows. For each $i$, we set the constant terms of $L_i(\lambda_i)$ to be $-1$, and we generate the other coefficients from the standard complex Gaussian distribution using **randn** (MATLAB) or **random CC** (MACULA,Y2). In our MATLAB implementation, we determine a null vector $q_i \in \ker(\nabla L_i)$ using **null** and a particular solution $p_i$ to $L_i(\lambda_i) = 0$ using **mldivide** (i.e., the backslash operator). We find the associated eigenpairs using **eig**. In our BERTINI/MACULA,Y2 implementation, we use **bertiniZeroDimSolve** (BERTINI) with default settings to determine the solutions to $H_i(\lambda_i)x_i = 0$ and $L_i(\lambda_i) = 0$.

**Continuation:** The main bulk of our computations occur in running the Euler–Newton predictor-corrector method described in Section 6. In our MATLAB implementation, the linear systems in the Euler and Newton steps are solved using **mldivide**. The predictor $p$ relies on a step size $h$, which is updated as we perform the continuation. The Newton iteration continues until the corrector $c$ in (6.1) satisfies $\|c\|_{\infty} < 10^{-9}$ or the number of iterations exceeds eight. The step size $h$ is updated according to the number of Newton iterations $i_{NT}$: if $i_{NT} \leq 2$, we double the step size; if $i_{NT} > 8$, we halve the step size; else we...
keep the same step size. These choices are largely heuristical but may be easily fine-tuned. Our default sets the maximum and minimum step size as $10^{-2}$ and $10^{-6}$ respectively.

On the other hand, BERTINI is specifically designed with the homotopy method in mind and these tasks are built-in and automated; users need only specify the configurations they wish to change in the input file. In our BERTINI/MACaulay2 implementation, we simply issue the command runBertini to use the default configurations.

**Stopping conditions and outputs:** In our MATLAB implementation, we estimate the endpoint of the homotopy as follows: when $t + h > 1$, we perform the Euler step with step size $h = 1 - t$ and we refine our endpoint with Newton’s method until the change in the update is sufficiently small (the default is $\|c\|_\infty < 10^{-9}$) or when the maximum number of iterations is reached (based on numerical experiments the default is set to $\max\{20, k \cdot \max(n_1, \ldots, n_k) + 5\}$). Again, since BERTINI is designed for homotopy method, it already has a variety of built-in options for estimating the endpoint. The so-called “fractional power series endgame” is the default, and is what we used.

Over the next three sections, we present the results of our numerical experiments on MEPs that are (i) randomly generated (Section 8), (ii) from a real-world application (Section 9), and (iii) singular (Section 10). In each section, we compare speed and accuracy of our method with those of Delta and diagonal coefficient homotopy methods whenever possible — any omission is the result of one of these methods failing to work, e.g., Delta method does not apply to singular MEPs.

8. **Numerical results I: Randomly generated MEPs**

In these experiments we randomly generate our inputs $A_{ij} \in \mathbb{C}^{n_i \times n_j}$, $j = 0, 1, \ldots, k$, $i = 1, \ldots, k$, from the standard complex Gaussian distribution. For convenience, we will assume that $n_1 = \cdots = n_k = n$ so that there are just two parameters $k$ and $n$ to consider. We write $N$ for the number of eigenpairs, given by Proposition 3.3.

We will report on the maximum time it takes to track one path, denoted by $t_{\text{PATH}}$, and the average number of Newton iterations during the path tracking, denoted by $\phi(k, n)$. The value of $t_{\text{PATH}}$ is particularly important as path-tracking is a task with high parallelism and $t_{\text{PATH}}$ provides a good estimate of the time it takes to run fiber product homotopy method in parallel. Nevertheless, we are only able to report $t_{\text{PATH}}$ for our MATLAB implementation as BERTINI deals with path-tracking in a more sophisticated and automated manner that offers users no easy way of determining the total track time of one path.

We investigate the stability of our method and the accuracy of our solutions by examining the backward error as defined in [15, Section 3]. The normwise backward error of an approximate eigenpair $(\lambda, x_1, \ldots, x_k)$ of an MEP with coefficients $A_{ij} \in \mathbb{C}^{n_i \times n_j}$, $j = 0, 1, \ldots, k$, $i = 1, \ldots, k$, and polynomial matrices $H_1, \ldots, H_k$, as in Definition 1.1 is given by

$$
\eta(\lambda, x_1, \ldots, x_k) := \min \{\varepsilon \in \mathbb{R} : (H_i(\lambda) + \Delta H_i(\lambda))x_i = 0,
\|\Delta A_{ij}\| \leq \varepsilon\|A_{ij}\|, i = 1, \ldots, k, j = 0, \ldots, k\},
$$

where

$$
\Delta H_i(\lambda) := \Delta A_{i0} - \sum_{j=1}^k \lambda_j \Delta A_{ij}, \quad i = 1, \ldots, k.
$$

To compute $\eta(\lambda, x_1, \ldots, x_k)$ we take advantage of [15 Theorem 2], which says that

$$
\eta(\lambda, x_1, \ldots, x_k) = \max_{i=1, \ldots, k} \left( \frac{\|H_i(\lambda)x_i\|}{\|A_{i0}\| + \sum_{j=1}^k \lambda_j \|A_{ij}\|} \right).
$$

8.1. **Fixed $k$, varying $n$.** Here we fix $k = 3$ and $n_1 = n_2 = n_3 = n$. For each value of $n$ we generate ten three-parameter eigenvalue problems. Note that the expected number of eigenpairs in these problems is $N = n^3$. In Table 1 we see that our method is faster than the timings reported...
for the diagonal coefficient homotopy method in [9]. The Delta method fails for larger values of \( n \) — our MATLAB implementation crashes with out-of-memory error in every instance when \( n \geq 25 \).

| \( n \) | \( N \) | Wall time | \( t_{\text{PATH}} \) | \( \phi(3,n) \) |
|---|---|---|---|---|
| 10 | 1000 | 3.21 | 241.48 | 779.47 | 1.23 | 503 |
| 15 | 3375 | 52.93 | 1093.95 | 2888.73 | 2.15 | 603 |
| 20 | 8000 | 734.07 | 3969.87 | 7857.44 | 2.99 | 614 |
| 25 | 15625 | failed | 10270.85 | 17169.53 | 3.92 | 659 |
| 30 | 27000 | failed | 23937.03 | 32786.64 | 4.46 | 737 |

Table 1. Elapsed timings (in seconds) for Delta method, fiber product homotopy method (averaged over ten runs), and diagonal coefficient homotopy method [9].

Our results for backward errors are presented in Figure 4, where it is clear that fiber product homotopy method (blue plot) has significantly smaller backward error than the Delta method (red plot) in our numerical experiments.

For the experiments in Table 1 and Figure 4, we found all eigenpairs by tracking all start solutions of the start system (5.4). In our next experiment, our goal is to examine the effect of a substantial increase in \( n \) on the speed and accuracy of our fiber product homotopy method. Finding all eigenpairs would have taken too long and serves little purpose. Instead, we randomly generate 100 MEPs for each \( n \) and track one randomly chosen start point among the set of start solutions. The timings in Table 2 are rough estimates of the time it would have taken to find all eigenpairs — these numbers are obtained by multiplying the average wall time taken for one randomly chosen eigenpair by \( n^3 \). The backward errors in Table 2, on the other hand, are for just one randomly chosen eigenpair (and not multiplied by \( n^3 \)). Evidently, increasing \( n \) has negligible effect on the average backward errors, which are all splendidly small — on the order of \( 10^{-15} \) or smaller.

Table 2. Elapsed timings (in seconds) for the fiber product homotopy method (over 100 runs) and the reported backward errors.

| \( n \) | Wall time | Backward error |
|---|---|---|
| | Best | Avg | Worst | Best | Avg | Worst |
| 30 | 0.56 | 1.19 | 5.74 | 6.79 \times 10^{-17} | 1.63 \times 10^{-16} | 2.06 \times 10^{-15} |
| 70 | 1.71 | 3.50 | 9.81 | 9.24 \times 10^{-17} | 1.65 \times 10^{-16} | 6.81 \times 10^{-16} |
| 150 | 11.76 | 24.57 | 96.81 | 1.32 \times 10^{-16} | 1.95 \times 10^{-16} | 2.00 \times 10^{-15} |

8.2. Fixed \( n \), varying \( k \). In this numerical experiment we fix \( n_1 = \cdots = n_k = 3 \) and vary \( k \) from 3, \ldots, 9. The expected number of eigenpairs is then \( N = 3^k \). If the reader is wondering why we do not increase \( k \) in a more drastic manner, note that increasing \( k \) produces a corresponding exponential increase in the number of solutions — for each randomly generated MEP, there are \( 3^k \) eigenpairs. Also, unlike changing \( n \), which just changes the dimension of the problem, changing \( k \) gives a different class of problems — for example, a two-parameter eigenvalue problem is qualitatively different from a one-parameter eigenvalue problem (i.e., a GEP) — and each \( k \) deserves a careful examination. Compared to the numbers for diagonal coefficient homotopy method in [9, Table 3], the numbers in Table 3 show that the fiber product homotopy method is significantly faster and also more stable in the sense that every path converged and did not need to be rerun.

The reader is reminded that fiber product homotopy method tracks \( k \) paths each carrying a copy of the eigenvalue; these \( k \) copies \( \lambda_1, \ldots, \lambda_k \) should all converge to the same eigenvalue \( \lambda_1 = \cdots = \lambda_k \) if the method runs correctly (see Example 5.5). This is indeed the case as the reader can see from the near zero values of \( \max_{2 \leq i \leq k} (\| \lambda_1 - \lambda_i \|_1) \) reported in Table 3.
Figure 4. The log backward error (vertical axis) of our fiber product method and the Delta method are plotted against the eigenvalues (horizontal axis) ordered by increasing norm from left to right. In the bottom right plot, results for Delta method are not shown because it failed with an out-of-memory error.

Table 3. Elapsed timings (in seconds) for Delta method and the two homotopy methods, all in MATLAB. Accuracy for the fiber product homotopy method.

| $k$ | $N$ | Wall time | $\max_{2 \leq i \leq k}(\|\lambda_1 - \lambda_i\|_1)$ | $\phi(k, n)$ | $t_{\text{PATH}}$ |
|-----|-----|-----------|---------------------------------|-------------|-------------|
| 3   | 81  | 0.08      | 17.55                           | 77.63       | 475         | 0.44        |
| 5   | 243 | 0.27      | 75.00                           | 288.24      | 515         | 1.29        |
| 6   | 729 | 1.91      | 398.74                          | 1089.10     | 721         | 2.38        |
| 7   | 2187| 32.07     | 2215.96                         | 3979.94     | 964         | 3.87        |
| 8   | 6561| 1223.25   | 10295.54                        | 13445.96    | 1311        | 10.68       |
| 9   | 19683| 22069.11 | 45630.48                        | 48624.78    | 1520        | 10.91       |

8.3. Comparisons on Bertini. The results in Tables 1 and 3 compare the MATLAB implementation of our new fiber product homotopy method (5.10) to the results for diagonal coefficient
homotopy method (2.2) reported in [9]. Here we will compare them on Bertini. As Bertini is a specialized software designed for homotopy continuation methods, it makes little sense to include a non-homotopy method like Delta method in our comparison on this platform and so we do not.

In this numerical experiment, we look at a two-parameter eigenvalue problem, i.e., $k = 2$, with $n_1 = n_2 = n$. The dimension of the problem is thus $N = n^2$. From Table 4, the fiber product homotopy method is consistently faster than the diagonal coefficient homotopy method in Bertini. We include the timings of our MATLAB implementation of the fiber product homotopy method in the last column for comparison.

Table 4. Elapsed timings (in seconds) for comparing homotopy methods and implementations. Diagonal coefficient homotopy method had failed to terminate after more than 48 hours in the $n = 50$ case.

| $n$ | $n^2$ | Diag. Coeff. | Fiber Prod. | MATLAB |
|-----|-------|---------------|-------------|--------|
| 5   | 25    | 1.14          | 1.12        | 3.16   |
| 15  | 225   | 34.67         | 20.11       | 22.19  |
| 30  | 900   | 1281.76       | 638.24      | 333.31 |
| 50  | 2500  | FAILED        | 17988.0     | 1592.81|

9. Numerical results II: Mathieu’s systems

An example where multiparameter eigenvalue problems surface is Mathieu’s systems, which in turn arises from studies of vibration of a fixed elliptic membrane [19, 23]. We will test our MATLAB implementation of fiber product homotopy method against the Delta method on this problem.

We refer the reader to [19, Section 2] for a discussion of how a coupled system of two-point boundary value problems, representing Mathieu’s angular and radial equations, yields a two-parameter eigenvalue problem (thus $k = 2$) upon Chebyshev collocation discretization. The dimensions of the matrices $n_1$ and $n_2$ correspond to the number of points used in the discretization.

In Figure 5, we present accuracy result for a two-parameter eigenvalue problem with $n_1 = 18$ and $n_2 = 38$ coming from a Mathieu system. The horizontal axis represents the $n_1 n_2 = 684$ eigenvalues, ordered from the smallest to largest by the norm of $\lambda$. The vertical axis represents the size of the backward errors on a log scale. The blue plot shows the backward error of our fiber product homotopy method whereas the red plot is that for the Delta method. With few exceptions, fiber product homotopy produces significantly more accurate results than the Delta method by orders of magnitude. Indeed, every eigenpair computed with our method in this experiment has a backward error that is less than $10^{-15}$.

9.1. Early stopping. The left and right figures in Figure 5 differ by the number of Newton iterations used in the Newton step of the continuation algorithm — the left plot uses the default stopping condition of $81 = \max(20, k \max\{n_1, \ldots, n_k\} + 5)$ Newton iterations whereas the right plot uses an early stopping condition of 50 Newton iterations. As one can see, there is no discernible difference in the backward errors but computing the left plot took 20 minutes whereas computing the right plot took only five. So this indicates that our implementation of the fiber homotopy method may be further fine-tuned to improve speed without sacrificing stability.

5We have also used a Macaulay2 [10] package [4] to produce the relevant input files.
9.2. Shub–Smale α-theory. We employ Shub–Smale α-theory \cite{5} to certify that the Newton steps converge quadratically in a neighborhood of the end point. Briefly, in this theory there is a function α that takes a polynomial system \( f : \mathbb{C}^n \to \mathbb{C}^n \) and an approximate solution \( z \in \mathbb{C}^n \) as its input and returns a positive real number. If \( \alpha(f, z) \) is less than the constant \( (13 - 3\sqrt{17})/4 \), then one has certified quadratic convergence of Newton’s method on the approximate solution \( z \) to the polynomial system \( f \) \cite[Theorem 2, p. 160]{5}. With our default tolerances we are able to certify the smallest 550 of the 684 eigenvalues — see Figure 6.

Figure 6. The computed bound (vertical axis) on log α for the approximate eigenpairs of the Mathieu MEP plotted against the eigenvalues (horizontal axis) ordered from smallest to largest. If the computed bound is below the red dashed line representing the value log\((13 - 3\sqrt{17})/4\), then the eigenpair is certified. Left figure: 81 (default) Newton iterations. Right figure: 50 Newton iterations.

10. Numerical results III: Singular MEPs

One of the thorniest issues in solving MEPs is singularity — a singular MEP \cite[Section 2]{14} is one where the intrinsic dimension \( (d_1, \ldots, d_k) < (n_1, \ldots, n_k) \). Singular MEPs have fewer than
the expected \( n_1 \cdots n_k \) eigenpairs; in particular, they are not regular. That a singular \( k \)-parameter eigenvalue problem presents computational difficulties is already evident in the case \( k = 1 \): a generalized eigenvalue problem with a singular matrix pencil \( A - \lambda B \) is widely known to be a challenging problem computationally. Also, the Delta method fails on all singular MEPs.

An instance where singular MEPs are inevitable is in the linearization of a quadratic multiparameter eigenvalue problem (QMEP) [14]: Given quadratic polynomial matrices

\[
Q_1(\lambda, \mu) := B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02},
\]

\[
Q_2(\lambda, \mu) := C_{00} + \lambda C_{10} + \mu C_{01} + \lambda^2 C_{20} + \lambda \mu C_{11} + \mu^2 C_{02},
\]

where \( B_{ij} \in \mathbb{C}^{n_1 \times n_1} \) and \( C_{ij} \in \mathbb{C}^{n_2 \times n_2}, i, j = 0, 1, 2 \), solve

\[
Q_1(\lambda, \mu)x_1 = 0, \quad Q_2(\lambda, \mu)x_2 = 0
\]

for all possible \( \lambda, \mu \in \mathbb{C} \) and nonzero \( x_1 \in \mathbb{C}^{n_1}, x_2 \in \mathbb{C}^{n_2} \). It is straightforward to generalize this to a quadratic \( k \)-parameter eigenvalue problem but we will only study the case \( k = 2 \) here.

The quadratic two-parameter eigenvalue problem is mathematically equivalent to a two-parameter eigenvalue problem [14]:

\[
H_1(\lambda, \mu) = \begin{bmatrix} B_{00} & B_{10} & B_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & B_{20} & B_{11} \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & B_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix},
\]

\[
H_2(\lambda, \mu) = \begin{bmatrix} C_{00} & C_{10} & C_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & C_{20} & C_{11} \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & C_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}.
\]

Whereas the original coefficient matrices in \( Q_1 \) are in \( \mathbb{C}^{n_1 \times n_1} \), the coefficient matrices of \( H_i \) are in \( \mathbb{C}^{3n_1 \times 3n_1}, i = 1, 2 \). However, the real catch is that the two-parameter eigenvalue problem \((\text{2.2})\) is singular — its intrinsic dimension \((2n_1, 2n_2)\) is strictly smaller than \((3n_1, 3n_2)\).

In our numerical experiment, we set \( n_1 = n_2 = n \) and randomly generate \( B_{ij}, C_{ij} \in \mathbb{C}^{n \times n} \). We will compare the fiber product and diagonal coefficient homotopy methods in Bertini.

**Table 5. Number of divergent paths and elapsed timings (in seconds) for the homotopy methods.** \( t_{\text{PATH}} \) is the time taken to track a single path and is an estimate of the time it would have taken to run fiber product homotopy method in parallel.

| \( n \) | \( N \) | Number of divergent paths | Wall time | \( t_{\text{PATH}} \) |
|---|---|---|---|---|
| \( 4n^2 \) | **Bertini** | **Matlab** | **Bertini** | **Matlab** |
| | Diag. Coeff. | Fiber Prod. | Fiber Prod. | Diag. Coeff. | Fiber Prod. | Fiber Prod. |
| 2 | 16 | 20 | 0 | 0 | 1 | 0 | 2 | 0.23 |
| 5 | 100 | 125 | 0 | 0 | 51 | 10 | 16 | 0.39 |
| 10 | 400 | 500 | 0 | 0 | 2125 | 301 | 134 | 0.92 |
| 20 | 1600 | 2000 | 0 | 0 | FAILED | 11512 | 899 | 1.96 |
| 40 | 6400 | 8000 | 0 | 0 | FAILED | 614720 | 14876 | 5.66 |

The difficulty of singular MEP is conspicuously reflected in our observed results: The diagonal coefficient homotopy method has \( 9n^2 \) start solutions and \( 5n^2 \) of the paths tracked does not converge for all values of \( n \) we tested, which we recorded in Table 5. On the other hand, our fiber product homotopy method did not produce a single divergent path.

We also recorded the results for our Matlab implementation of the fiber product homotopy method. Again we see that none of the paths diverges and had we run the algorithm in parallel on a machine with sufficiently many CPU cores, the value of \( t_{\text{PATH}} \) indicates that we may expect to obtain all eigenpairs in seconds.
11. Condition number of fiber product multiparameter eigenvalue problem

The condition number for an MEP has been defined and studied [15] — we will discuss this in Section 11.3. Nevertheless, recall that condition number [7, Chapter 14] depends on the problem — two different problems that always give the same solution will in general still have different condition numbers. The condition number for MEP as defined in [15] applies to the standard formulation of MEP in (1.2). Since we are solving a different problem (1.4), or more generally (5.8), albeit one that gives the same solutions as (1.2), it will have a different condition number. The condition number for MEP as defined in [15] applies to the standard formulation of MEP in (1.2). Since we are solving a different problem (1.4), or more generally (5.8), albeit one that gives the same solutions as (1.2), it will have a different condition number.

In this section, we will discuss the condition number for the fiber product MEP (1.4) and compare it to the standard condition number for MEP as defined in [15]. To see that they are expected to be quite different, we may think of the common interpretation of condition number as measuring the change in output produced by a change in input. The standard condition number for MEP in [15] measures the change in eigenvalues (output) produced by a change in the coefficient matrices (input). In fiber product homotopy method, we do not deform these coefficient matrices; instead, the relevant notion of condition number is one that measures the change in eigenvalues (output) produced by a change in the linear space defined by $G_1 = \cdots = G_k = 0$ as in (5.8) (input). So in our setting, we require the condition number of a variety intersected with a varying linear subspace.

In the following, by a *projective linear subspace* of a projective space, we mean \( \{ \pi(x) \in \mathbb{P}^n : Mx = 0, x \neq 0 \} \) for some $M \in \mathbb{C}^{n \times (n+1)}$ and where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection. Henceforth, we will write $\mathbb{M} := \{ \pi(\ker(M) \setminus \{0\}) \}$ for the projective linear subspace that $M \in \mathbb{C}^{n \times (n+1)}$ defines in $\mathbb{P}^n$. We write $[x_0 : x_1 : \ldots : x_n]$ for homogeneous coordinates in $\mathbb{P}^n$.

11.1. Intersecting a variety with a varying linear subspace. We briefly review some relevant ideas in [6], on which our condition number in Section 11.2 is based. Let $Z$ be a degree-$p$ variety in $\mathbb{P}^n$ of codimension $n - d$. The *Hurwitz variety* of $Z$ is a subvariety of the Grassmannian $G(n-d,n)$ of $(n-d)$-dimensional projective linear subspaces in $\mathbb{P}^n$ defined by

\[
\mathcal{H}_Z := \{ M \in G(n-d,n) : Z \cap M \text{ does not consist of } p \text{ reduced points} \}.
\]

The Hurwitz variety is an irreducible hypersurface defined by a polynomial called the *Hurwitz form* in the coordinate ring of the Grassmannian [22]. This variety can be regarded as the set of ill-posed instances of the problem of intersecting a variety by a varying linear space [6]. By [6, Definition 1.1 and Theorem 1.4], we have the following definition.

**Definition 11.1.** Let $Z$ be a $d$-dimensional irreducible projective variety in $\mathbb{P}^n$ and $M \in G(n-d,n)$. Let $z \in Z \cap M$. The *intersection condition number* of $M$ at $z$ with respect to $Z$ is

\[
\kappa_Z(M, z) := \begin{cases} 
\frac{1}{\sin \alpha} & \text{if } z \text{ is a smooth point of } Z \text{ and } M \text{ intersects } Z \text{ transversally at } z, \\
\infty & \text{otherwise,}
\end{cases}
\]

where $\alpha$ is the minimum angle between the tangent spaces $T_z(Z)$ and $T_z(M)$.

The elements of $\mathcal{H}_Z$ are precisely the projective linear subspaces where the intersection condition number is infinite [6, Theorem 1.6].

11.2. Condition number of the fiber product MEP. Consider the multiparameter eigenvalue variety $\mathcal{EV}(H_1, \ldots, H_k) \subseteq \mathbb{C}^{k^2}$ in (3.2). Throughout this section, $Z \subseteq \mathbb{P}^{k^2}$ will always denote the projective variety

(11.1) \[
Z := \{ [1: \lambda_1: \ldots : \lambda_k] \in \mathbb{P}^{k^2} : (\lambda_1, \ldots, \lambda_k) \in \mathcal{EV}(H_1, \ldots, H_k) \},
\]

where we have denoted the homogeneous coordinates of $\mathbb{P}^{k^2}$ by

\[
[\lambda_0 : \lambda_{11} : \cdots : \lambda_{kk} : \lambda_{k1} : \cdots : \lambda_{kk}],
\]

\[
\lambda_i
\]
with $\lambda_0$ as the homogenizing coordinate. The defining equations of $\mathcal{Z}$ are homogeneous polynomials in $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that setting $\lambda_0 = 1$ gives the defining equations for $\mathcal{E}(H_1, \ldots, H_k)$ in (3.2).

We will apply the notion of an intersection condition number in Section 11.1 to define a condition number for fiber product MEP.

**Definition 11.2.** Given a fiber product multiparameter eigenvalue problem (1.4), let $Z$ be the projective variety in (11.1), $M_{fp}$ be the projective linear space

\[
M_{fp} := \left\{ [1: \lambda_1: \ldots: \lambda_k] \in \mathbb{P}^k : \lambda_1 = \cdots = \lambda_k \right\}
\]

(11.2)

and $z$ be the point

\[
z := [1: \lambda_1: \ldots: \lambda_k] \in Z \cap M_{fp}.
\]

The **condition number of the fiber product multiparameter eigenvalue problem (1.4)** is given by

\[
\kappa_{fp}(\lambda_1, \ldots, \lambda_k, H_1, \ldots, H_k) := \kappa_Z(M_{fp}, z).
\]

Note that $Z \cap M_{fp}$ consists precisely of points $[1: \lambda_1: \ldots: \lambda_k] \in \mathbb{P}^k$ where $(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ are the eigenvalues of the MEP. We next show how to compute $\kappa_Z(M, z)$ for any projective linear subspace $M$ — as we will see later, for fiber product homotopy method, we would also be interested in $\kappa_Z(M, z)$ for projective linear subspaces $M$ other than $M_{fp}$.

**Theorem 11.3.** Let $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $i = 1, \ldots, k$, $j = 0, 1, \ldots, k$, be the matrices of an MEP as in Definition 11.1. Let $Z$ be the projective variety in (11.1), $M = \pi(\ker(J) \setminus \{0\})$ for some $M \in \mathbb{C}^{(k^2-k) \times (k^2+1)}$, and $z = [1: \lambda_1: \ldots: \lambda_k] \in Z \cap M$ be a smooth point of $Z$. If $x_i$ and $y_i \in \mathbb{C}^{n_i}$ are nonzero vectors\(^\text{6}\) such that

\[
H_i(\lambda_i)x_i = 0, \quad H_i(\lambda_i)^Ty_i = 0, \quad i = 1, \ldots, k,
\]

then the intersection condition number

\[
\kappa_Z(M, z) = \frac{1}{\sin \alpha},
\]

where $\alpha$ is the minimum angle between $M$ and $J = \pi(\ker(J) \setminus \{0\})$,

\[
J := \begin{bmatrix}
y_1^TA_{10}x_1 & y_1^TB_1(x_1) \\
y_2^TA_{20}x_1 & y_2^TB_2(x_2) \\
\vdots & \vdots \\
y_k^TA_{k0}x_k & y_k^TB_k(x_k)
\end{bmatrix} \in \mathbb{C}^{k \times (1+k^2)}
\]

(11.3)

with $B_i(x_i) := -[A_{i1}x_1, \ldots, A_{ik}x_i] \in \mathbb{C}^{n_i \times k}$, $i = 1, \ldots, k$.

**Proof.** By Definition 11.1, it suffices to show that $T_z(Z) = T_z(J)$. First, recall from Definition 3.1 that the multiparameter eigenvalue variety $\mathcal{E}(H_1, \ldots, H_k)$ is a projection of the multiparameter eigenpair variety $\mathcal{E}(H_1, \ldots, H_k)$. So the projective closure of $\mathcal{E}(H_1, \ldots, H_k)$ in $\mathbb{P}^{k^2} \times \mathbb{P}^{n_1-1} \times \ldots \times \mathbb{P}^{n_k-1}$, which we will denote by $\mathcal{E}P(H_1, \ldots, H_k)$, projects onto the variety $Z$.

For a linear polynomial matrix

\[
H(\lambda) = A_0 - \lambda_1A_1 - \cdots - \lambda_kA_k,
\]

we introduce a homogenizing variable $\lambda_0$ and define a homogenized $H$ as

\[
\widetilde{H}(\lambda_0, \lambda) := \lambda_0A_0 - \lambda_1A_1 - \cdots - \lambda_kA_k.
\]

\(^{6}\)These are respectively right and left eigenvectors of the polynomial matrix.
We homogenize \( H_1, \ldots, H_k \) in this manner, write \( \mathbf{w} := (\lambda_0, \lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k) \) for brevity, and then let

\[
\mathbf{H} := \begin{bmatrix}
\hat{H}_1(\lambda_0, \lambda_1)x_1 \\
\vdots \\
\hat{H}_k(\lambda_0, \lambda_k)x_k
\end{bmatrix}.
\]

Write \( \mathbf{H} = [h_1, \ldots, h_{n_1+\cdots+n_k}]^T \) where \( h_i \in \mathbb{C}[\mathbf{w}], \ i = 1, \ldots, n_1 + \cdots + n_k, \) are multihomogeneous polynomials — homogeneous in the variables \( \lambda_0, \lambda_1, \ldots, \lambda_k \) and homogeneous in each of the variables \( x_1, \ldots, x_k. \) Note that the \( h_i \)'s vanish on \( \overline{\mathcal{F}}(H_1, \ldots, H_k); \) in fact, the set of polynomials that vanish on \( \overline{\mathcal{F}}(H_1, \ldots, H_k) \) is given by

\[
\{ h \in \mathbb{C}[\mathbf{w}] : \lambda_0^m h = g_1 h_1 + \cdots + g_{n_1+\cdots+n_k} h_{n_1+\cdots+n_k}, \ g_i \in \mathbb{C}[\mathbf{w}], \ m \in \mathbb{N} \}.
\]

The \((n_1 + \cdots + n_k) \times (1 + k^2 + n_1 + \cdots + n_k)\) Jacobian matrix of partial derivatives,

\[
\nabla \mathbf{H}(\mathbf{w}) = \begin{bmatrix}
\frac{\partial}{\partial \lambda_0} H, \nabla_{\lambda_1} H, \ldots, \nabla_{\lambda_k} H, \nabla_x H, \ldots, \nabla x_k H
\end{bmatrix},
\]

when evaluated at a point

\[
\mathbf{w}_0 := ([1 : \lambda_1 : \cdots : \lambda_k, x_1, \ldots, x_k] = (z, x_1, \ldots, x_k) \in \overline{\mathcal{F}}(H_1, \ldots, H_k),
\]

has the form

\[
\nabla \mathbf{H}(\mathbf{w}_0) = \begin{bmatrix}
A_{10} x_1 & B_1(x_1) & \hat{H}_1(1, \lambda_1) \\
A_{10} x_1 & B_2(x_2) & \hat{H}_2(1, \lambda_2) \\
\vdots & \vdots & \vdots \\
A_{k0} x_k & B_k(x_k) & \hat{H}_k(1, \lambda_k)
\end{bmatrix}.
\]

For any \( L \in \mathbb{C}^{m \times (k^2+1)}, \) if each row of the matrix \( L, O_{m \times (n_1+\cdots+n_k)} \in \mathbb{C}^{m \times (k^2+1+n_1+\cdots+n_k)} \) is in the row space of \( \nabla \mathbf{H}(\mathbf{w}_0), \) then \( L = \pi(\ker(L) \setminus \{0\}) \) is such that \( T_z(Z) \subseteq T_z(L). \) This is because \( Z \) is the projection of \( \overline{\mathcal{F}}(H_1, \ldots, H_k) \) to \( \mathbb{C}^{k^2+1} \) in the first \( k^2 + 1 \) coordinates. With this in mind, we multiply \( \nabla \mathbf{H}(\mathbf{w}_0) \) on the left by

\[
\begin{bmatrix}
y_1^1 \\
y_2^1 \\
\vdots \\
y_k^1
\end{bmatrix} \in \mathbb{C}^{k \times (n_1+\cdots+n_k)}
\]

to obtain the \( k \times (1+k^2+n_1+\cdots+n_k) \) matrix

\[
\begin{bmatrix}
y_1^1 A_{10} x_1 & y_1^1 B_1(x_1) & O_{n_1+\cdots+n_k}^f \\
y_2^1 A_{20} x_1 & y_2^1 B_2(x_2) & O_{n_1+\cdots+n_k}^f \\
\vdots & \vdots & \vdots \\
y_k^1 A_{k0} x_k & y_k^1 B_k(x_k) & O_{n_1+\cdots+n_k}^f
\end{bmatrix} = [J, \ O_{k \times (n_1+\cdots+n_k)}],
\]

showing that \( T_z(Z) \subseteq T_z(J). \) When \( J \) is full rank, the codimensions

\[
\text{codim} \ T_z(J) = k = \text{codim} \ T_z(Z)
\]

when \( z \) is a smooth point. It follows that \( T_z(Z) = T_z(J). \)

Now we will show how we may obtain the required condition number \( \kappa_{fp}(\lambda_1, \ldots, \lambda_k, H_1, \ldots, H_k). \) Consider the following one-parameter family of projective linear spaces induced by a one-parameter
family of matrices: For a fixed \( t \in [0, 1] \), let

\[
M_t := (1 - t) \begin{pmatrix}
-\mathbb{I}_{k-1} & \nabla L_1 \\
-\mathbb{I}_{k-1} & \nabla L_2 \\
\vdots & \ddots \\
-\mathbb{I}_{k-1} & \nabla L_k
\end{pmatrix} + t \begin{pmatrix}
0_{k-1} & \nabla G_1 \\
0_{k-1} & \nabla G_2 \\
\vdots & \ddots \\
0_{k-1} & \nabla G_k
\end{pmatrix} \in \mathbb{C}^{k(k-1) \times (k^2 + 1)},
\]

where \( L_1, \ldots, L_k \) are as in (5.3) and \( G_1, \ldots, G_k \) are as in (5.6). Let

\[
\mathcal{M}_t := \pi(\ker(M_t) \setminus \{0\}) \subseteq \mathbb{P}^{k^2}
\]

be the corresponding projective linear subspace. Note that \( \mathcal{M}_t \) dehomogenizes (by setting \( \lambda_0 = 1 \)) to the affine linear space in \( \mathbb{C}^{k^2} \) defined by

\[
0 = (1 - t)L_1(\lambda_1) + tG_1(\lambda_1, \ldots, \lambda_k), \\
0 = (1 - t)L_k(\lambda_k) + tG_k(\lambda_1, \ldots, \lambda_k),
\]

where the right-hand sides are the last \( k(k - 1) \) linear polynomials in (5.10). Therefore, for any \( t \in [0, 1] \), the projective closure of the set of solutions to \( H_{fp}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k, t) = 0 \) defined in (5.10) projects onto \( \mathcal{Z} \cap \mathcal{M}_t \).

When \( t = 1 \), by (5.7) and (11.2), we have

\[
\mathcal{M}_1 = \{ \{ \lambda_0 : \lambda_1 : \ldots : \lambda_k \} \in \mathbb{P}^{k^2} : \lambda_1 = \cdots = \lambda_k \} = \mathcal{M}_{fp},
\]

and for \( z = [1 : \lambda_1 : \ldots : \lambda_k] \in \mathcal{Z} \cap \mathcal{M}_1 \), we obtain the condition number in Definition 11.2

\[
\kappa_\mathcal{Z}(\mathcal{M}_1, z) = \kappa_{fp}(\lambda_1, \ldots, \lambda_k, H_1, \ldots, H_k).
\]

The intersection condition number \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) for values of \( t \in [0, 1] \) is also useful as it informs us of the conditioning of the subproblems encountered during path tracking in the fiber product homotopy method. The next theorem shows that \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) is almost always finite and Example 11.5 indicates that it is typically small.

**Theorem 11.4.** For any \( t \in [0, 1] \), \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) is finite with probability one.

**Proof.** Since the Hurwitz variety \( \mathcal{H}_\mathcal{Z} \) comprises the projective linear subspaces \( \mathcal{M} \) with \( \kappa_\mathcal{Z}(\mathcal{M}, z) = \infty \), it suffices to show that for any \( t \in [0, 1] \), \( \mathcal{M}_t \notin \mathcal{H}_\mathcal{Z} \) with probability one.

By Theorem 5.4 the fiber product homotopy \( H_{fp} : \mathbb{C}^{k^2} \times (\mathbb{P}^{n_1 - 1} \times \cdots \times \mathbb{P}^{n_k - 1}) \times \mathbb{C} \to \mathbb{C}^{n_1 + \cdots + n_k \times k^2} \) for an MEP (5.10) has a start system chosen correctly with probability one. Thus the paths of solutions to \( H_{fp}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k, t) = 0 \) for \( t \in [0, 1] \) are smooth with probability one.

If \( \mathcal{M}_t \in \mathcal{H}_\mathcal{Z} \), then these paths of solutions would not be smooth as the projective closure of the set of solutions to \( H_{fp}(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_k, t) = 0 \) projects onto \( \mathcal{Z} \cap \mathcal{M}_t \). Thus, with probability one, \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) is finite for \( t \in [0, 1] \).

**Example 11.5 (Conditioning of Mathieu problem I).** We generate an instance of the Mathieu problem in Section 9 with \( n_1 = 18 \) and \( n_2 = 38 \) and apply fiber product homotopy method to obtain the eigenpairs of this two-parameter fiber product eigenvalue problem. We compute the intersection condition number \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) as \( t \) varies from 0 to 1 in every path we tracked. We ran our implementation five times and found that \( \kappa_\mathcal{Z}(\mathcal{M}_t, z) \) does not exceed 21.85 for every \( t \) we encountered in every path and in every run.

In Figure 7 we plot the fiber product MEP condition number \( \kappa_{fp}(\lambda_1, \lambda_2, H_1, H_2) = \kappa_\mathcal{Z}(\mathcal{M}_1, z) \), ordered by increasing norm of \( \lambda_1 \), from a single run. We emphasize that the vertical axis in Figure 7 is in linear scale, i.e., all condition numbers obtained are less than 2.8 with the vast majority less than 1.8. This means that small perturbations of the projective linear space \( \mathcal{M}_1 \) yield small changes in the points of intersection of \( \mathcal{Z} \) with \( \mathcal{M}_1 \), which correspond to the eigenvalues we seek.
Note that we are solving the Mathieu system as a fiber product MEP (1.4) here. We will solve the same Mathieu system as a standard MEP (1.2) in Example 11.6 and compute the corresponding condition number, where we will see very different results.

**Figure 7.** The condition number \( \kappa_{fp}(\lambda_1, \lambda_2, H_1, H_2) \) (vertical axis) is plotted against the eigenvalue \( \lambda_1 \) (horizontal axis) ordered in increasing norm from left to right. We emphasize that the vertical axis is in linear scale, as opposed to the log scale used in Figure 8.

11.3. Condition number of the standard MEP. For comparison, we review the condition number of a standard MEP (1.2) as defined in [15], which captures how small perturbations of the input coefficients matrices \( A_{ij}, i = 1, \ldots, k, j = 0, \ldots, k \), affect the multiparameter eigenvalue \( \lambda = (\lambda_1, \ldots, \lambda_k) \). More precisely, \( \kappa(\lambda, H_1, \ldots, H_k) \) is given by

\[
\limsup_{\varepsilon \to 0} \left\{ \frac{\| \Delta \lambda \|}{\varepsilon} : (A_{i0} + \Delta A_{i0} - \sum_{j=1}^{k} (\lambda_j + \Delta \lambda_j)(A_{ij} + \Delta A_{ij}))(x_i + \Delta x_i) = 0, \right. \\
\left. \| \Delta A_{ij} \| \leq \varepsilon \| A_{ij} \|, i = 1, \ldots, k, j = 0, \ldots, k \right\}
\]

where \((x_1, \ldots, x_k)\) is the corresponding eigenvector, i.e., \( H_i(\lambda)x_i = 0, i = 1, \ldots, k \). We will let \((y_1, \ldots, y_k)\) denotes the corresponding left eigenvector, i.e., \( H_i(\lambda)^Ty_i = 0, i = 1, \ldots, k \).

Let \( \theta_i := \| A_{i0} \| + \sum_{j=1}^{k} \| \lambda_j \| \| A_{ij} \|, i = 1, \ldots, k \), and define the \( \theta \)-weighted norm of \( M \in \mathbb{C}^{k \times k} \) by

\[
\| M \|_\theta := \max \{ \| Mz \|_2 : z \in \mathbb{C}^k, |z_i| = \theta_i, i = 1, \ldots, k \}.
\]

Then by [15, Theorem 6], \( \kappa(\lambda, H_1, \ldots, H_k) = \| M^{-1} \|_\theta \) where

\[
M := \begin{bmatrix}
y_1^*A_{11}x_1 & y_1^*A_{12}x_1 & \cdots & y_1^*A_{1k}x_1 \\
y_2^*A_{21}x_2 & y_2^*A_{22}x_2 & \cdots & y_2^*A_{2k}x_2 \\
\vdots & \vdots & \ddots & \vdots \\
y_k^*A_{k1}x_k & y_k^*A_{k2}x_k & \cdots & y_k^*A_{kk}x_k
\end{bmatrix}.
\]

**Example 11.6** (Conditioning of Mathieu problem II). We revisit the Mathieu problem in Example 11.5 but this time we formulate it as a standard two-parameter eigenvalue problem (1.2). We record its condition number \( \kappa(\lambda, H_1, H_2) \), plotted against \( \lambda \) in increasing norm, in Figure 8. The difference with Example 11.5 is striking — we emphasize that the vertical axis in Figure 8 is in log...
scale, i.e., the condition numbers are all larger than $10^5$ and those near the right end of the plot corresponding to the largest eigenvalues are as large as $10^{15}$.

Figure 8. The condition number $\kappa(\lambda, H_1, H_2)$ (vertical axis) is plotted against the eigenvalue $\lambda$ (horizontal axis) ordered in increasing norm from left to right. We emphasize that the vertical axis is in log scale, as opposed to the linear scale used in Figure 7.

12. Conclusions

In this article, we propose the fiber product homotopy method for solving an MEP: it solves a mathematically equivalent problem, the fiber product MEP, via a homotopy algorithm specifically designed to take advantage of its structure. Extensive numerical experiments show that:

(i) in terms of speed, the fiber product homotopy method outperforms the diagonal coefficient homotopy method on all instances and outperforms the Delta method on large instances (which in fact tends to fail on larger problems);
(ii) in terms of accuracy, the fiber product homotopy method is extremely accurate, producing relative backward errors on the order of $10^{-16}$, especially in comparison with the Delta method, which produces them on the order of $10^{-11}$;
(iii) a particularly noteworthy feature of the fiber product homotopy method is that it is, as far as we know, the only method that reliably works on singular MEPs.

We proffer two insights that explain the strength of the fiber product homotopy method:
(a) it deforms exponentially fewer equations compared to the diagonal coefficient homotopy method;
(b) the problem that it solves, the fiber product MEP, is much better conditioned than the equivalent standard MEP with the same solutions.

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