Amalgamated Codazzi Raychaudhuri identity for foliation.

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Abstract. It is shown how a pure background tensor formalism provides a concise but explicit and highly flexible machinery for the generalised curvature analysis of individual embedded surfaces and foliations such as arise in the theory of topological defects in cosmological and other physical contexts. The unified treatment provided here shows how the relevant extension of the Raychaudhuri identity is related to the correspondingly extended Codazzi identity.

1 Introduction

I wish to thank the organisors of this Penrose festschrift meeting ‘Geometric issues in the foundations of science’ for the opportunity of again expressing appreciation for the beautifully geometric way of perceiving the physical world that Roger Penrose communicated to so many students of my generation. In particular it was Roger’s emphasis[1] on the importance of features that are conformally invariant that lead me, with his help, to develop the systematic use[2] of 2-dimensional conformal projections (the Lorentz signature analogue of Mercator type projections in ordinary terrestrial mapping) of the kind that have since become widely and appropriately known as ‘Penrose diagrams’. An outstanding example of the kind of conformally invariant structure whose analysis was specially developed and applied under Roger’s leadership is that of null-geodesic foliations: in particular, it was his derivation, with Ted Newman[3], of the null limit of the famous divergence identity obtained originally for a timelike flow by Raychaudhuri[4], that provided the essential tool for deriving the singularity theorems that were subsequently developed, first by Roger himself[5], and later on by Stephen Hawking and others[6, 7].

It was the work on singularity theorems[6, 7] that first drew my attention to the original Raychaudhuri equation, whose extension to higher dimensional foliations, in a manner recently suggested by Capovilla and Guven[21], is described in the present article. The unified treatment provided here shows how the extended Raychaudhuri identity is fractially related to the correspondingly extended Codazzi identity. However it is left for future work to complete the corresponding Penrose program, in the sense of treating the corresponding conformally invariant limit, meaning the case of a foliation not by surfaces with a well behaved induced metric such as will be postulated in the present work (which is physically motivated by contexts such as that of neutron star vortex congruences) but by null surfaces (of the kind whose study has been developed by Barrabès and Israel[8]).

The present article is a sequel to a previous Penrose festschrift contribution[9] in which I showed how a pure background tensor formalism provides a concise but explicit and highly flexible machinery for the generalised curvature analysis of individual embedded timelike or
spacelike p-dimensional surfaces in a flat or curved n-dimensional spacetime background. The relevant spacetime metric will, as usual, be denoted here by $g_{\mu\nu}$, with the understanding that the suffixes are interpretable either in the (mathematically sophisticated) sense of Roger’s abstract index system\[10\] or else in the old fashioned concrete sense (with which most physicists are still more familiar) as the labels of components with respect to some set of local coordinates $x^\mu$, $\mu = 0, 1, n-1$. It will be shown here how this machinery can be extended in a natural way so as to treat a smooth foliation by a congruence of such surfaces, for which the complete orthonormal frame bundle characterised by the relevant group of rotations in n dimensions will have a natural reduction to the bundle of naturally adapted frames characterised by the direct product of the subgroup of tangential frame rotations in p dimensions and the complementary subgroup of orthogonal frame rotations in (n-p) dimensions. This reduced frame bundle will be naturally endowed with a preferred – metric preserving but generically non-symmetric – connection, which will have an associated foliation curvature tensor $\mathcal{F}_{\mu\nu\rho\sigma}$ that is generically distinct from the ordinary Riemannian background curvature which will be denoted here by $B_{\mu\nu\rho\sigma}$. The various ways in which this foliation curvature tensor can be projected, orthogonally or tangentially, with respect to the embedded surfaces give relations of which the generalised Codazzi and Raychaudhuri identities are particular cases.

The present approach has been developed to satisfy needs arising in the context of the recent rise of interest in the theory of topological defect structures such as cosmic strings and higher dimensional cosmic membranes, as well as the related phenomenon of vortex foliations in neutron stars, which has lead to the investigation of a wide range of new problems of equilibrium and dynamical evolution in a special or general relativistic framework. Various aspects of these problems\[11, 12, 13, 14, 15, 16\] and in recent years most particularly the requirements of general purpose perturbation analysis \[17, 18, 19, 20, 21, 22\], have shown, and in some cases helped to satisfy, the need to adapt and develop pre-existing mathematical machinery for describing the relevant geometry and particularly the various kinds of curvature that are involved.

As remarked in my preceding Penrose festchrift contribution\[9\], although much of what is needed has in principle been already available in the mathematical literature\[23, 24, 25, 26, 27\], it has often been in a form that is inaccessible or inconvenient for the purposes of physicists, many of whom have remained excessively dependent on obsolete sources such as Eisenhart’s still very influential textbook\[28\] (written in ignorance of the modern concept of generalised curvature, which was at that time under development by Cartan, and which was made familiar to physicists much later by the theory of Yang and Mills). Whereas some treatments have obtained a neatly concise abstract formulation at the expense of flexibility, others have obtained general purpose adaptability at the price of using complicated and potentially confusing reference systems involving specially adapted coordinates and frames that require the simultaneous use of many different kinds of indices.

For the purpose of obtaining an optimal compromise between these two undesirable extremes, the approach\[8, 16, 17, 22\] used here relies as much as possible just on ordinary tensors, as defined with respect to the relevant background space-time with local coordinates $x^\mu$. The advantage of avoiding explicit dependence on a specialised internal coordinate system becomes particularly clear in cases\[16\] where one is concerned with mutual contractions of tensors constructed on distinct but mutually intersecting embedded surfaces whose internal coordinates could not in general be made to be mutually compatible.

The present treatment will employ the same notation as was used in the most mathematically detailed presentation\[9\] of this approach, in which, rather than describing the induced curvature of an embedded spacelike or timelike p-surface, with internal coordinates
say, in terms of the intrinsic version, with components \( R_{ijkl} \) say, of its Riemann tensor, one prefers to describe it in terms of the corresponding background spacetime tensor \( R_{\lambda\mu\nu\rho} \).

The latter is definable (using the abbreviation \( x^\mu_\sigma \) for \( \partial x^\mu / \partial \sigma \)) as the index lowered version – obtained by contraction with the background spacetime metric \( g_{\mu\nu} \) – of the projection \( R_{\lambda\mu\nu\rho} = R^{ijkl} x^\lambda_\lambda x^\mu_\mu x^\nu_\nu x^\rho_\rho \) of the contravariant version of the intrinsic curvature that is obtained by internal index raising, using the inverse, with contravariant components \( \eta^{ij} \) (whose existence depends on the postulate that the surface is spacelike or timelike, but not null) of the induced metric with components \( \eta_{ij} = g_{\mu\nu} x^\mu_\mu x^\nu_\nu \). This background spacetime representation \( R_{\lambda\mu\nu\rho} \) of the internal curvature of the embedded \( p \)-surface has of course to be distinguished from the ordinary \( n \)-dimensional Riemann curvature of the background spacetime itself, whose components will be denoted simply by \( B_{\lambda\mu\nu\rho} \). In the same way, rather than working with the covariant and contravariant intrinsic components \( \eta_{ij} \) and \( \eta^{ij} \) of the induced metric, one prefers to use the corresponding background coordinate components \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \) of the corresponding projected tensor specified by \( \eta^{\mu\nu} = \eta^{ij} x^\mu_\mu x^\nu_\nu \), which is what is referred to as the (first) fundamental tensor of the embedding.

To set up a systematic analysis of curvature in exclusively background tensorial terms, the natural starting point is obviously the fundamental tensor that has just been defined, whose mixed (contra/covariant) version with components \( \eta^\mu_\nu \) is interpretable as a rank-\( p \) tangential projection operator that sends a vector into the tangent subspace of the embedding. The notation \( \perp^\mu_\nu \) will be used here (instead of the less suggestive symbol \( \gamma^\mu_\nu \)) to denote the complementary rank - \( n - p \) operator of lateral projection orthogonal to the surface, whose components will evidently be given in terms of those of the fundamental (tangential projection) tensor by the defining relation

\[
\perp^\mu_\nu + \eta^\mu_\nu = g^\mu_\nu , \tag{1}
\]

since the mixed version \( g^\mu_\nu \) of the metric tensor is of course interpretable as representing the identity operator. As well as having the separate operator properties

\[
\eta^\mu_\rho \eta^\rho_\nu = \eta^\mu_\nu , \quad \perp^\mu_\rho \perp^\rho_\nu = \perp^\mu_\nu \tag{2}
\]

the tensors thus defined will evidently be related by the conditions

\[
\eta^\mu_\rho \perp^\rho_\nu = 0 = \perp^\mu_\rho \eta^\rho_\nu . \tag{3}
\]

## 2 The deformation tensor.

Although it is less detailed, the present article considerably extends the results of the preceeding development\(^\[\text{4}\]\) of the background tensorial analysis of embedding geometry, by considering not just an individual embedded non-null \( p \)-surface by itself, but the extension of any such surface to a smooth foliation by diffeomorphically similar surfaces. For such a foliation there will be corresponding background (not just single \( p \)-surface supported) fields of tensors \( \eta^\mu_\nu \) and \( \perp^\mu_\nu \), that will not only satisfy the relations \((1)\), \((2)\), and \((3)\), but will also (unlike what was supposed in the preceeding Penrose festscrift article\(^\[\text{4}\]\)) have well defined (Riemannian or pseudo-Riemannian) covariant derivatives. These derivatives will be fully determined by the specification of a certain (first) deformation tensor, \( \mathcal{H}_{\mu\nu\rho} \) say, via an expression of the form

\[
\nabla_\mu \eta^\nu_\rho = - \nabla_\mu \perp^\nu_\rho = \mathcal{H}_{\mu\nu\rho} + \mathcal{H}_{\mu\rho\nu} . \tag{4}
\]

It can easily be seen from \((2)\) that the required deformation tensor will be given simply by

\[
\mathcal{H}_{\mu\nu\rho} = \eta^\rho_\sigma \nabla_\mu \eta^\sigma_\nu = - \perp^\sigma_\nu \nabla_\mu \perp^\rho_\sigma . \tag{5}
\]
The middle and last indices of this tensor will evidently have the respective properties of tangentiality and orthogonality that are expressible as
\[ \perp^\sigma_{\nu} H^\rho_{\sigma} = 0, \quad H^\rho_{\sigma} \eta^\sigma_{\nu} = 0. \] (6)

There is no automatic tangentiality or orthogonality property for the first index of the deformation tensor (5), which is thus reducible with respect to the tangential and orthogonally lateral projections (1) to a sum
\[ H^\rho_{\mu \nu} = K^\rho_{\mu \nu} - L^\rho_{\mu \nu} \] (7)
in which such a property is obtained for each of the parts
\[ K^\rho_{\mu \nu} = \eta^\sigma_{\mu} H^\rho_{\sigma \nu}, \quad L^\rho_{\mu \nu} = - \perp^\sigma_{\mu} H^\rho_{\sigma \nu}, \] (8)

which satisfy the conditions
\[ \perp^\sigma_{\mu} K^\rho_{\sigma \nu} = \perp^\rho_{\sigma} K^\sigma_{\mu \nu} = 0 = K^\rho_{\mu \sigma} \eta^\sigma_{\nu} \] (9)

and
\[ \eta^\rho_{\sigma} L^\rho_{\sigma \nu} = \eta^\rho_{\sigma} L^\sigma_{\mu \nu} = 0 = L^\rho_{\mu \sigma} \perp^\sigma_{\nu}. \] (10)

It is evident that the first of these decomposed parts is appropriately describable as the tangential turning tensor, since by (5) and (8) it is given by the expression
\[ K^\rho_{\mu \nu} = \eta^\sigma_{\mu} \nabla^\tau \eta^\tau_{\sigma \nu}, \] (11)
in which the only differentiation involved is contained in the tangential gradient operator \( \eta^\sigma_{\mu} \nabla^\tau \) – which is well defined even for fields whose support is restricted to a single embedded surface – so that, unlike the full deformation tensor \( H^\rho_{\mu \nu} \) of the foliation, the tangential turning tensor \( K^\rho_{\mu \nu} \) is well defined just for an individual embedded p-surface. As such, this turning tensor \( K^\rho_{\mu \nu} \) is identifiable as what has been defined [as the ordinary second fundamental tensor of the particular p-surface passing through the point under consideration.

Up to this point, none of the relations formulated in this section depends on the condition that the (first) fundamental tensor field \( \eta^\mu_{\nu} \) is actually tangential to well behaved p-surfaces rather than just being an arbitrary field of rank-p projection tensors as characterised by the purely algebraic conditions (2). As pointed out in the previous analysis [1], the Frobenius type integrability condition that is necessary and sufficient for the local existence of well behaved p-surfaces tangential to \( \eta^\mu_{\nu} \) is that the second fundamental tensor defined by (9) should have the generalised Weingarten property
\[ K^\rho_{\mu \nu} = K_{(\mu \nu)}^\rho \iff K_{[\mu \nu]}^\rho = 0, \] (12)

(using round and square brackets to denote index symmetrisation and antisymmetrisation respectively) which means that it is symmetric with respect to its two surface tangential indices.

It is to be noticed that the second part of the decomposition (2.4), namely the lateral turning tensor, \( L^\rho_{\mu \nu} \), is expressible by the formula
\[ L^\rho_{\mu \nu} = \perp^\rho_{\sigma} \perp^\tau_{\mu} \nabla^\tau \perp^\sigma_{\nu}, \] (13)

which differs from that in (11) only by the substitution of \( \perp^\nu_{\mu} \) for \( \eta^\nu_{\mu} \). It evidently follows that the necessary and sufficient integrability condition for the local existence of an (n-p)-dimensional foliation orthogonal to the p-dimensional foliation whose existence is guaranteed
by (12) is that this lateral turning tensor should have the analogous symmetry property, which is expressible as the vanishing of the rotation tensor, \( \omega_{\mu\nu}^\rho \) say, that is defined as its antisymmetric part in a decomposition of the form

\[
L^\rho_{\mu\nu} = \omega_{\mu\nu}^\rho + \theta_{\mu\nu}^\rho , \quad \omega_{(\mu\nu)}^\rho = 0 , \quad \theta_{[\mu\nu]}^\rho = 0 .
\]

The only part that remains if the foliation is (n-p) surface orthogonal is the symmetric part, \( \theta_{\mu\nu}^\rho \), which is the natural generalisation of the two index divergence tensor \( \theta_{\mu\nu} \) whose evolution is the subject of the tensorial Raychaudhuri identity discussed by Hawking and Ellis [7] (the original Raychaudhuri identity [4] being obtained by taking the scalar trace). For a 1-dimensional timelike foliation, which will have a unique future directed unit tangent vector \( u^\mu \), the relevant divergence tensor will be obtainable simply as \( \theta_{\mu\nu} = \theta_{\mu\nu}^\rho u^\rho \). The generalised Raychaudhuri identity to be presented (following Capovilla and Guven [21]) in a later section, provides an evolution equation for the three index generalised divergence tensor \( \theta_{\mu\nu}^\rho \) which (unlike the ordinary divergence tensor \( \theta_{\mu\nu} \)) is always well defined whatever the dimension of the foliation.

3 The adapted foliation connection.

Due to the existence of the decomposition whereby a background spacetime vector, with components \( \xi^\mu \) say, is split up by the projectors \([3]\) as the sum of its surface tangential part \( \eta^\mu_\nu \xi^\nu \) and its surface orthogonal part \( \perp^\mu_\nu \xi^\nu \), there will be a corresponding adaptation of the ordinary concept of parallel propagation with respect to the background connection \( \Gamma^\nu_\mu \rho \).

The principle of the adapted propagation concept is to follow up an ordinary operation of infinitesimal parallel propagation by the projection adjustment that is needed to ensure that purely tangential vectors propagate onto purely tangential vectors while purely orthogonal vectors propagate onto purely orthogonal vectors. Thus for purely tangential vectors, the effect of the adapted propagation is equivalent to that of ordinary internal parallel propagation with respect to the induced metric in the embedded surface, while for purely orthogonal vectors it is interpretable as the natural generalisation of the standard concept of Fermi-Walker propagation. For an infinitesimal displacement \( dx^\mu \) the deviation between the actual component variation \( (dx^\nu) \partial_\nu \xi^\mu \) and the variation that would be obtained by the corresponding adapted propagation law will be expressible in the form \( (dx^\nu) D_\nu \xi^\mu \) where \( D \) denotes the corresponding foliation adapted differentiation operator, whose effect will evidently be given by

\[
D_\nu \xi^\mu = \eta^\mu_\rho \nabla_\nu (\eta^\rho_\sigma \xi^\sigma) + \perp^\mu_\rho \nabla_\nu (\perp^\rho_\sigma \xi^\sigma) .
\]

It can thus be seen that this operation will be expressible in the form

\[
D_\nu \xi^\mu = \nabla_\nu \xi^\mu + \alpha^\mu_\nu_\sigma \xi^\sigma = \partial_\nu \xi^\mu + A^\mu_\nu_\sigma \xi^\sigma ,
\]

where the adapted foliation connection components \( A^\mu_\nu_\rho \) are given by the formula

\[
A^\mu_\nu_\rho = \Gamma^\mu_\nu_\rho + \alpha^\mu_\nu_\rho ,
\]

in which the \( \alpha^\mu_\nu_\rho \) are the components of the relevant adaptation tensor, whose components can be seen from \([3]\) to be given by

\[
\alpha^\mu_\nu_\rho = 2H^\mu_\nu_\rho .
\]
The fact that the expression (18) is manifestly antisymmetric with respect to the last two indices of the adaptation tensor makes it evident that, like the usual Riemannian differentiation operator $\nabla$, the adapted differentiation operator $D$ will commute with index raising or lowering, since the metric itself remains invariant under adapted propagation:

$$D_\mu g_{\nu\rho} = 0 .$$  \hspace{1cm} (19)

However, unlike $\nabla$, the adapted differentiation operator has the very convenient property of also commuting with tangential and orthogonal projection, since it can be seen to follow from (4) and (3) that the corresponding operators also remain invariant under adapted propagation:

$$D_\mu \eta^\nu = 0 , \quad D_\mu \perp = 0 .$$  \hspace{1cm} (20)

There is of course a price to be paid in order to obtain this considerable advantage of $D$ over $\nabla$, but it is not exhorbitant: all that has to be sacrificed is the analogue of the symmetry property

$$\Gamma_{[\mu | \rho |} = 0 ,$$  \hspace{1cm} (21)

expressing the absence of torsion in the Riemannian case. For the adapted foliation connection $A_{\mu \nu \rho}$, the torsion tensor defined by

$$\Theta_{\mu \nu \rho} = 2 A_{[\mu | \nu | \rho]} = 2 \alpha_{[\mu | \nu | \rho]} ,$$  \hspace{1cm} (22)

will not in general be zero.

4 \hspace{1cm} The amalgamated foliation curvature tensor.

The curvature associated with the adapted connection introduced by (4.2) in the preceeding section can be read out from the ensuing commutator formula, which, for an arbitrary vector field with components $\xi^\mu$, will take the standard form

$$2D_{[\mu} D_{\nu]} \xi^\rho = \mathcal{F}_{\mu \nu \sigma} \xi^\sigma - \Theta_{\mu \nu \sigma} D_\sigma \xi^\rho ,$$  \hspace{1cm} (23)

in which the torsion tensor components $\Theta_{\mu \nu \rho}$ are as defined by (22) while the components $\mathcal{F}_{\mu \nu \rho \sigma}$ are defined by a Yang-Mills type curvature formula of the form

$$\mathcal{F}_{\mu \nu \rho \sigma} = 2 \partial_{[\mu} A_{\nu]\rho\sigma} + 2 A_{[\mu | \rho \nu | \sigma] \tau} .$$  \hspace{1cm} (24)

Although the connection components $A_{\mu \nu \rho}$ from which it is constructed are not of tensorial type, the resulting curvature components (24) are of course strictly tensorial. This is made evident by evaluating the components (24) of this amalgamated foliation curvature in terms of the background curvature tensor

$$B_{\mu \nu \rho \sigma} = 2 \partial_{[\mu} \Gamma_{\nu \rho]} \sigma + \Gamma_{\mu \tau} \Gamma_{\nu \sigma} \tau - \Gamma_{\nu \tau} \Gamma_{\mu \sigma} \tau ,$$  \hspace{1cm} (25)

and the adaptation tensor $\alpha_{\mu \nu \rho}$ given by (18), which gives the manifestly tensorial expression

$$\mathcal{F}_{\mu \nu \rho \sigma} = B_{\mu \nu \rho \sigma} + 2 \nabla_{[\mu} \alpha_{\nu] \rho} \sigma + 2 \alpha_{[\mu | \rho \sigma} \alpha_{\nu] \tau} \tau .$$  \hspace{1cm} (26)

Although it does not share the full set of symmetries of the Riemann tensor, the foliation curvature obtained in this way will evidently be antisymmetric in both its first and last pairs of indices:

$$\mathcal{F}_{\mu \nu \rho \sigma} = \mathcal{F}_{[\mu \nu | \rho \sigma]} .$$  \hspace{1cm} (27)
Using the formula (18), it can be seen from (6) that the difference between this adapted curvature and the ordinary background Riemann curvature will be given by

\[ F_{\rho\sigma}^{\mu\nu} - B_{\rho\sigma}^{\mu\nu} = 4\mathcal{X}^{\rho\sigma}_{\mu\nu} + 2\mathcal{H}_{[\mu}^{\tau\sigma}\mathcal{H}^{\rho]_{\tau} + 2\mathcal{H}_{[\mu}^{\tau\rho}\mathcal{H}^{\sigma]_{\tau} , \]

where \( \mathcal{X}^{\rho\sigma}_{\mu\nu} \) is what may be termed the second deformation tensor, which is definable by

\[ \mathcal{X}^{\rho\sigma}_{\mu\nu} = \eta^{\rho\sigma} \nabla_{\lambda} \mathcal{H}^{\mu\nu}_{\lambda} . \]

The formula (28) superficially appears to depend on the higher order derivatives involved in \( \mathcal{X}^{\rho\sigma}_{\mu\nu} \), but this is deceptive: the higher derivatives will in fact cancel, by the “amalgamated Codazzi-Raychaudhuri identity” given below.

Since the adapted derivation operator has been constructed in such a way as to map tangential vector fields into purely tangential vector fields, and lateral (surface orthogonal) vector fields into lateral vector fields, it follows that the same applies to the corresponding curvature (24), which will therefore consist of an additive amalgamation of two separate parts having the form

\[ F_{\rho\sigma}^{\mu\nu} = \mathcal{P}_{\rho\sigma}^{\mu\nu} + \mathcal{Q}_{\rho\sigma}^{\mu\nu} , \]

in which the “inner” curvature acting on purely tangential vectors is given by a doubly tangential (surface parallel) projection as

\[ \mathcal{P}_{\rho\sigma}^{\mu\nu} = F_{\rho\sigma}^{\kappa\lambda} \eta^{\mu\nu}_{\kappa\lambda} , \]

while the “outer” curvature acting on purely orthogonal vectors is given by a doubly lateral projection as

\[ \mathcal{Q}_{\rho\sigma}^{\mu\nu} = F_{\rho\sigma}^{\kappa\lambda} \bot_{\kappa\lambda} . \]

It is implicit in the separation expressed by (30) that the mixed tangential and lateral projection of the adapted curvature must vanish:

\[ F_{\rho\sigma}^{\kappa\lambda} \eta^{\mu\nu}_{\kappa\lambda} \bot_{\mu\nu} = 0 . \]

To get back, from the extended foliation curvature tensors that have just been introduced, to their antecedent analogues for an individual embedded surface, the first step is to construct the amalgamated embedding curvature tensor, \( F_{\rho\sigma}^{\mu\nu} \), say, which will be obtainable from the corresponding amalgamated foliation curvature \( F_{\rho\sigma}^{\mu\nu} \) by a doubly tangential projection having the form

\[ F_{\rho\sigma}^{\mu\nu} = \eta^{\alpha\beta}_{\mu\nu} F_{\alpha\beta}^{\rho\sigma} . \]

As did the extended foliation curvature, so also this amalgated embedding curvature will separate as the sum of “inner” and “outer” parts in the form

\[ F_{\rho\sigma}^{\mu\nu} = R_{\rho\sigma}^{\mu\nu} + \Omega_{\rho\sigma}^{\mu\nu} , \]

in which the “inner” embedding curvature is given by another doubly tangential projection as

\[ R_{\rho\sigma}^{\mu\nu} = F_{\rho\sigma}^{\kappa\lambda} \eta^{\mu\nu}_{\kappa\lambda} \eta^{\alpha\beta}_{\mu\nu} P_{\alpha\beta}^{\rho\sigma} , \]

while the “outer” embedding curvature (whose noteworthy property of conformal invariance was pointed out in my previous Penrose festschrift contribution) is given by the corresponding doubly lateral projection as

\[ \Omega_{\rho\sigma}^{\mu\nu} = F_{\rho\sigma}^{\kappa\lambda} \bot_{\kappa\lambda} \eta^{\alpha\beta}_{\mu\nu} Q_{\alpha\beta}^{\rho\sigma} . \]
The formula (28) can be used to evaluate the “inner” tangential part of the foliation curvature tensor as

\[ P_{\mu\nu\rho} = 2 \mathcal{H}_{[\nu} \mathcal{H}^{\rho]} \mathcal{H}_{\sigma] \tau} + B_{\mu\nu\tau} \eta^\kappa_{\rho} \eta^\lambda_{\sigma} \]  

and to evaluate the “outer” orthogonal part of the foliation curvature tensor as

\[ Q_{\mu\nu\rho} = 2 \mathcal{H}_{[\nu} \mathcal{H}^{\rho]} \mathcal{H}_{\tau\sigma} + B_{\mu\nu\tau} \eta^\kappa_{\rho} \eta^\lambda_{\sigma} . \]  

The formula (38) for the “inner” foliation curvature is evidently classifiable as an further extension of the previously derived generalisation[9] for \( R_{\mu\nu\rho\sigma} \) of the historic Gauss equation. Similarly the formula (39) for the “outer” foliation curvature is an analogous extension of the relation[9] for \( \Omega_{\mu\nu\rho\sigma} \) that corresponds to what has sometimes been referred to as the “Ricci equation” but what would seem more appropriately describable as the Schouten equation, with reference to the earliest relevant source with which I am familiar[23], since long after the time of Ricci it was not yet understood even by such a leading geometer as Eisenhart[28].

In much the same way, the non-trivial separation identity (33) can be considered as a generalisation to the case of foliations of the relation that is itself interpretable as an extended generalisation to higher dimensions of the historic Codazzi equation that was originally formulated in the restricted context of 3-dimensional flat space. It can be seen from (28) that this extended Codazzi identity is expressible as

\[ 2\mathcal{X}_{[\mu\nu]} \eta^\rho_{\rho} + B_{\mu\nu\tau} \eta^\kappa_{\rho} \eta^\lambda_{\sigma} = 0 \]  

which shows that the relevant higher derivatives are all determined entirely by the Riemannian background curvature so that no specific knowledge of the second deformation tensor is needed. By doubly tangential projection of the first two indices, the extended generalisation (40) will give back the already familiar version[9] of the generalised Codazzi identity for the individual embedded p-surfaces of the foliation. The corresponding doubly lateral projection would give the analogous result for the orthogonal foliation by (n-p)-surfaces that would exist in the irrotational case for which the lateral turning tensor given by (13) is symmetric. Finally the corresponding mixed tangential and lateral projection of (41) gives an identity that is expressible in terms of foliation adapted differentiation (16) as

\[ \perp_{\mu} D_{\tau} K_{\nu\rho} + \eta^\tau_{\nu} D_{\tau} L_{\mu\rho\sigma} = K_{\nu\mu} \perp_{\tau} \perp_{\rho} + L_{\mu\nu\lambda} \perp_{\sigma} \perp_{\rho} \eta^\alpha_{\nu} \perp_{\beta} B_{\alpha\beta\lambda \nu} \eta^\kappa_{\rho} \perp_{\sigma} . \]  

This last result is interpretable as the translation into the pure background tensorial formalism used here of the recently derived generalisation[21] to higher dimensional foliations of the well known Raychaudhuri equation (whose original scalar version[4], and its tensorial extension[7], were formulated just for the special case of a foliation by 1-dimensional curves). The complete identity (40) is therefore interpretable as an amalgamated Raychaudhuri-Codazzi identity.

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