The design strain and dead mass of energy absorbing materials and structures: mathematical principles and experimental determination

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Abstract

The most important design parameters of energy absorbers are a) the maximal ratio of energy density $W$ to stress $\sigma$, i.e. $(W/\sigma)_{\text{max}}$, misnamed ‘efficiency’, and b) the maximal admissible strain $\varepsilon_{\text{max}}$ at this ratio. This paper models standard energy absorbing elements and determines the theoretical $(W/\sigma)_{\text{max}}$ and $\varepsilon_{\text{max}}$. The theoretical data are then compared to experimental ones, obtained from compression experiments. A method for correcting $(W/\sigma)_{\text{max}}$ of materials with pronounced negative modulus (over a defined strain window) is introduced. The maximal gradient, of $\log W$ versus $\log \sigma$, i.e. at the optimum point, is unity. $(W/\sigma)_{\text{max}}$ ranges from 0.2 to 0.55, and $\varepsilon_{\text{max}}$ from 0.48 to 0.81. Materials with negative modulus have higher $(W/\sigma)_{\text{max}}$ and $\varepsilon_{\text{max}}$ values. Better shock absorbers, with larger $(W/\sigma)_{\text{max}}$, have less ‘dead mass’.

1. Introduction

The optimum of energy absorption of materials and structures is usually determined either graphically or mathematically from stress $\sigma$ – strain $\varepsilon$ curves obtained from force $F$ – deflection $x$ curves of compression tests. Integrating the area under the loading curve yields the energy density $W$, i.e. the energy $E$ absorbed per unit volume. For the graphical method according to Gibson and Ashby [1], $W$ is plotted against $\sigma$ in a double-logarithmic graph. The optimum points are determined with a tangent line or envelope curve at the shoulder points of $W-\sigma$ data taken at different strain rates. Mathematically, the maximal ratio of absorbed energy to exerted force or stress [2] is

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calculated. The latter ratio is denoted 'efficiency' [2], a misnomer, as efficiency refers to an energy ratio (e.g. system efficiency of energy out over energy input) or performance ratio (measured performance to ideal performance), and not to a ratio of energy density \( W \) to stress \( \sigma \). Therefore, the term '\( W/\sigma \) ratio' will be used throughout this paper. Moreover, the quantities of \( E/F \) and \( W/\sigma \) correspond to deflection \( x \) and strain \( \varepsilon \), respectively; more specifically to \( x \) or \( \varepsilon \) of an \( F-x \) or \( \sigma-\varepsilon \) curve if its shape was rectangular (superelastic) or to half-\( x \) or half-\( \varepsilon \) if the loading curve’s shape was triangular (sawtooth; Hookean elastic). The rationale behind the optimum of energy absorption is to maximise the absorbed energy and minimise the applied stress, i.e. to determine \( (W/\sigma)_{\text{max}} \). The optimal design strain or maximal admissible strain \( \varepsilon_{\text{max}} \) is determined from the \( W/\sigma \) curve at the optimal stress \( \sigma_{\text{opt}} \) at \( (W/\sigma)_{\text{max}} \). The optimal design strain \( \varepsilon_{\text{max}} \) occurs at the transition from collapse to densification, and defines the point where the material or structure bottoms out. As the loading curve’s shape of energy absorbing materials or structures ranges between sawtooth and rectangular, it is expected that \( \varepsilon_{\text{max}}/2 < (W/\sigma)_{\text{max}} < \varepsilon_{\text{max}} \). The knowledge of the optimal design strain is required for the development of cushioning structures such as protection equipment (e.g. mats, helmets, etc.) and running shoes.

The aim of this paper is to analyse the strain performance of theoretical shock absorption models and to provide typical performance data of different shock absorbing materials and structures.

2. Mathematical analysis

As shock absorbers consist of buckling elements and/or pneumatic dampers, the equations for these two models (reciprocal functions) are derived and the design strain is calculated. Subsequently, the gradient of the tangent at the optimum point on the \( W-\sigma \) curve (double-log graph) is determined.

2.1. Buckling Element

In a buckling element (pinned-pinned) with sinusoidal deflection pattern (Figure 1a)

\[
y = y_{\text{max}} \sin \left( \frac{\pi}{L-x_0} \right) \frac{x-x_0}{L-x_0} \tag{1}
\]

where \( y \) is the amplitude at any \( x \) (distance along the deflection direction), the maximal amplitude \( y_{\text{max}} \) of the deflection depends on the length of the buckling element \( L \) and on the instantaneous deflection \( x_0 \) in \( x \)-direction. The \( 2^{\text{nd}} \) derivative of Eqn (1) yields the curvature \( k \) at \( y \), which is the inverse of the bending radius \( R \). According to the bending equation, \( 1/R \) equals the ratio of bending moment \( M \) to flexural rigidity (product of modulus \( E \) and \( 2^{\text{nd}} \) moment of area \( I \)).

\[
\left( \frac{d^2 y}{dx^2} \right)_{y_{\text{max}}} = -y_{\text{max}} \left( \frac{\pi}{L-x_0} \right)^2 \sin \left( \frac{\pi}{L-x_0} \right) \frac{x-x_0}{L-x_0} = k = \frac{1}{R} = -\frac{M}{EI} = -\frac{y_{\text{max}} F}{EI} \tag{2}
\]

where \( F \) is the force applied to the buckled element. At \( y_{\text{max}} \), \( x = 0.5(L+x_0) \). Substituting this term into Eqn (2) and solving for \( F \) yields

![Fig. 1. buckling element (a) and air spring (b); L = uncompressed length of the absorber.](image-url)
\[-\frac{y_{\text{max}}^2}{EI} = -y_{\text{max}} \left( \frac{\pi}{L - x_0} \right)^2 \sin \left( \frac{\pi}{2} \right) \text{ and } F = EI \left( \frac{\pi}{L - x_0} \right)^2 \]

(3)

Integrating the force \( F \) with deflection \( x_0 \) yields the energy \( E_B \) absorbed by buckling

\[ E_B = EI \int_0^x \left( \frac{\pi}{L - x_0} \right)^2 \, dy \quad \text{and} \quad EI = E \left( \frac{\pi^2}{L - x_0} - \frac{\pi^2}{L} \right) \]

(4)

Calculating the ratio of \( E_B \) to \( F \) and determining the ratio’s maximum,

\[ \frac{E_B}{F} = \frac{x(L-x)}{L} \text{ and } \frac{d(E_B/F)}{dx} = \frac{L-2x}{L} = 0 \]

(5)

yields the design deflection \( x_{\text{max}} \) and the optimal (maximal) ratio of \( E_B \) to \( F \)

\[ \frac{x_{\text{B}}}{F_{\text{max}}} = \frac{L}{2} \quad \text{...and} \quad \frac{E_B}{F_{\text{max}}} = \frac{x(L-x)}{L} = \frac{L(L-L/2)}{L} = \frac{L}{4} = 0.25L \]

(6)

Eqn (6) implies that the design strain \( \varepsilon_{\text{max}} = 0.5 \) and that the maximal \( W/\sigma \) ratio = 0.25.

2.2. Pneumatic Damper

The gas pressure \( p_G \) of a pneumatic damper or air spring (Figure 1b) is defined as

\[ p_G = p_0 \frac{\varepsilon}{1 - \varepsilon - R} \]

(7)

[3], where \( p_0 \) is the atmospheric pressure (0.1013 MPa) and \( R \) is the relative density (fraction of solid material within the damper). Integrating the pressure \( p \) with strain \( \varepsilon \) yields the gas energy density \( W_G \)

\[ W_G = p_0 \int_0^{\varepsilon} \frac{\varepsilon}{1 - \varepsilon - R} \, d\varepsilon = p_0 \left[ (1-R) \log(1-\varepsilon - R) - \varepsilon \right]_0 \]

(8)

\[ W_G = -p_0 (1-R) \log(1-\varepsilon - R) - p_0 \varepsilon + p_0 (1-R) \log(1-R) \]

(9)

The \( W/\sigma \) ratio or \( W_G/p \) ratio is therefore

\[ \frac{W_G}{p} = (1-\varepsilon - R)(1-R) \log(1-\varepsilon - R) - \varepsilon (1-\varepsilon - R) + (1-\varepsilon - R)(1-R) \log(1-R) \]

\[ \frac{p}{p_0} \]

(10)

Differentiating the \( W_G/p_G \) ratio with \( \varepsilon \) and equating the result to zero yields

\[ (1-R)^2 \log(1-\varepsilon - R) + \varepsilon (\varepsilon + 1-R) = (1-R)^2 \log(1-R) \]

(11)

Solving Eqn (11) for \( \varepsilon \) delivers the yields the design strain \( \varepsilon_{\text{max}} \). As the porosity \( P = 1 - R \)

\[ P^2 \log(P + \varepsilon) + \varepsilon (P + \varepsilon) = P^2 \log(P) \]

(12)

There is apparently no analytical solution for \( \varepsilon \) in Eqn (12), yet, the fact that \( \varepsilon \) at max \( W_G/p_G \) is the larger the smaller \( R \) or the higher \( P \), suggests that \( \varepsilon P \) could be constant. In order to obtain a proof of this assumption, \( \varepsilon P \) is replaced by the constant \( c \) in Eqn (12):

\[ \log(P - \varepsilon) + \varepsilon (P + \varepsilon) / P^2 = \log(P) \quad \rightarrow \quad \log \left[ P \left( 1 - \frac{\varepsilon}{P} \right) \right] + \frac{\varepsilon P}{P + \varepsilon} \left( 1 + \frac{\varepsilon}{P} \right) = \log(P) \quad \rightarrow \]

\[ \log \left[ P(1-c) \right] + c(1+c) = \log(P) \]

\[ P(1-c)^{\omega(1+c)} = P \]

(13)

As \( P \) is cancelled out in Eqn (14), making it independent of \( P \), the constant nature of \( c \) is proven from:

\[ (1-c)e^{\omega(1+c)} = 1 \]

(15)

Solving Eqn (15) for \( c \) yields two solutions: \( c = 0 \) and \( c = 0.6838 \). As \( \varepsilon = cP \) in Eqn (13), the design strain \( \varepsilon_{\text{max}} = 0.6838 \) if \( P = 1 \) and \( R = 0 \). Substituting \( \varepsilon_{\text{max}} \) into Eqn (10) and considering that \( R = 0 \), yields the optimal \( W_G/p \) ratio or \( W/\sigma \) ratio, which is 0.2162. If \( R = P = 0.5 \), \( \varepsilon_{\text{max}} = 0.3419 \) and the optimal (maximal) \( W/\sigma \) ratio = 0.1081.
Interestingly, Eqn (15) finds itself another application, namely for calculation of the equilibrium position of a buoy (equation 76 of [4]).

2.3. Gradient of tangent at optimal $W/\sigma$ ratio

At the optimal (maximal) $W/\sigma$ ratio
\[
\frac{d(W/\sigma)}{d\varepsilon} = 0 \land \frac{d(W/\sigma)}{d\sigma} = 0 \land \frac{d(W/\sigma)}{dW} = 0
\]  
(16)
Thus,
\[
d(W/\sigma) = 0 \rightarrow d[\log_{10}(W/\sigma)] = 0 \rightarrow d[\log_{10} W - \log_{10} \sigma] = 0
\]  
(17)
Considering the equality
\[
d(\log_{10} W - \log_{10} \sigma) = d(\log_{10} W) - d(\log_{10} \sigma)
\]  
(18)
Eqn (18) yields
\[
d (\log_{10} W) - d (\log_{10} \sigma) = 0 \rightarrow d (\log_{10} W) = d (\log_{10} \sigma) \rightarrow \frac{d(d_{log_{10} W})}{d(d_{log_{10} \sigma})} = 1
\]  
(19)
Therefore, when plotting $W$ against $\sigma$ on a double logarithmic graph, then the gradient of tangent at optimal $W/\sigma$ ratio equals unity.

![Fig. 2. stress $\sigma$, energy density $W$, and $W/\sigma$, against strain $\varepsilon$, $\sigma_{max}$, $W_{max}$ and $d_{max}$ are the design parameters at ($W/\sigma_{max}$): a: material with positive modulus; b: material with negative modulus from 0.4 to 0.7 strain; in b, the design parameters are adjusted, as ($W/\sigma_{max}$) is not calculated from the maximal stress occurring before ($W/\sigma_{max}$).](image)

3. Material and methods

The following materials and structures were tested: foams such as ARTi-lage (www.artilage.com), Berkeley foams (UC-Berkeley), poron (www.poroncushioning.com), Solyte (www.asics.com), Speva (www.asics.com), polyethylene foam (www.dow.com), polyurethane foams (erapol.com.au, www.joyce.com.au, www.dow.com), EPS foam (expanded polystyrene), and D3O foams (Aero and Decell; www.d3o.com); structures such as cardboard (www.cartonpallet.com), Skydex (50D, 55D, Skydex Pad; www.skydex.com), ‘MASS’ shock absorber (shockabsorbingmaterial.co.uk, plastic-castle.co.uk), and poron cushioning structures (www.poroncushioning.com).

In total 88 different energy absorbers were compressed with an Instron material testing machine at crosshead speeds of 500 and 5 mm/s. The force-deflection curves were converted to stress-strain curves, and the energy
density and \((W/\sigma)_{\text{max}}\) were calculated numerically. \((W/\sigma)_{\text{max}}\) served to determine \(\sigma_{\text{max}}, W_{\text{max}}\) and \(\varepsilon_{\text{max}}\) (Figure 2a). From the density \(\rho\) and \(\varepsilon_{\text{max}}\), the active mass \(m_{a}/V\) and dead mass \(m_{d}/V\) (per unit volume \(V\)) were calculated: \(m_{a}/V = \rho \varepsilon_{\text{max}}\) and \(m_{d}/V = \rho (1 - \varepsilon_{\text{max}})\), respectively. The dead mass refers to that fraction of the strain (times the density) that is not used up for energy absorption. If the modulus of the material is negative within a defined strain window, and \(V_{\text{max}}\) at the optimum point is smaller than stresses at smaller \(H_{\text{max}}\), then \((W/\sigma)_{\text{max}}, W_{\text{max}}\) and \(\varepsilon_{\text{max}}\) have to be adjusted according to the method shown in Figure 2b.

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**Fig. 3.** a: average data of \((W/\sigma)_{\text{max}}\) vs design strain \(\varepsilon_{\text{max}}\) (X: PU foam company 1; 297: PU foam company 2; AC, PC, UC, IP: PU foam company 3; M1, M2: MASS shock absorber; A18, A28: arti-lage; 50D, 55D: Skydex; B4, B77: Berkley foam; 45, 64, 144: PE foam; o53, o65: Solyte; p53, p65: Speva; P: Poron; D3D, D3A: D3O foams; + & –: data are taken at crosshead speeds of 500 mm/min and 5 mm/min); b: optimal stress \(\varepsilon_{\text{max}}\) vs density (PU1: PU foam company 1; PU2: PU foam company 2; PE: PE foam; D3O: D3O foam); c: \((W/\sigma)_{\text{max}}\) vs dead mass of different absorber categories; d: \((W/\sigma)_{\text{max}}\) vs dead mass of all data (excluding outlier absorbers such as cardboard, spacer fabric and MASS shock absorber [M], \((W/\sigma)_{\text{max}}\) can be explained from the dead mass in 71.35% (logarithmic regression).
4. Results

Figure 3a shows the average values of \((W/\sigma)_{\text{max}}\) against \(\varepsilon_{\text{max}}\). \((W/\sigma)_{\text{max}}\) ranges from 0.2 to 0.55, and \(\varepsilon_{\text{max}}\) from 0.48 to 0.81. From Figure 3a and c it becomes clear that in materials and structures with positive modulus, the best \((W/\sigma)_{\text{max}}\) is two times greater than the worst \((W/\sigma)_{\text{max}}\). Figure 3b reflects the well-known relationship of higher \(\sigma_{\text{max}}\) of denser materials. Figures 3c and 3d show an interesting relationship between \((W/\sigma)_{\text{max}}\) and the ‘dead mass’ (that fraction of the density not used for compression up to \(\varepsilon_{\text{max}}\)): better shock absorbers, i.e. with larger \((W/\sigma)_{\text{max}}\), have less dead mass \((r^2 = 0.7135\) without outliers).

5. Discussion

In closed-cell energy absorbers with different relative densities \(R\), buckling elements and pneumatic dampers are arranged in parallel. Therefore, \(\varepsilon_{\text{max}}\) can be anywhere between 0.5 and 0.68 as predicted from the theoretical models; even \(\varepsilon_{\text{max}}\) smaller than 0.5 is possible if \(R\) is large. In materials and structures with negative modulus, the adjusted \(\varepsilon_{\text{max}}\) is smaller than 0.7. However even the non-adjusted \(\varepsilon_{\text{max}}\) of honeycomb cardboard is larger than 0.5. The reason for this is that cardboard honeycomb structures buckle plastically, which does not follow a sinusoidal deflection pattern. Adjusted \(\sigma_{\text{max}}, W_{\text{max}}\) and \(\varepsilon_{\text{max}}\) are larger than not adjusted ones; however, the adjusted \((W/\sigma)_{\text{max}}\) is smaller than the non-adjusted one.

In cellular energy absorbers, buckling/pneumatic elements (vertical) and spring elements (horizontal) are arranged in series. Therefore, considering that a Hookean spring has \((W/\sigma)_{\text{max}}\) of 0.5, \((W/\sigma)_{\text{max}}\) of a foam can exceed \((W/\sigma)_{\text{max}}\) of 0.25, with extreme values a little under 0.5. In materials with negative modulus, the adjusted \((W/\sigma)_{\text{max}}\) can be larger than 0.5.

The knowledge of \((W/\sigma)_{\text{max}}, \sigma_{\text{max}}, W_{\text{max}}\) and \(\varepsilon_{\text{max}}\) is important for designing personal protective equipment (PPE), and selecting the optimal absorbers for PPE design [5].

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