Tridiagonal pairs, alternating elements, and distance-regular graphs

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Abstract

The positive part $U^+_q$ of $U_q(\widehat{sl}_2)$ has a presentation with two generators $W_0, W_1$ and two relations called the $q$-Serre relations. The algebra $U^+_q$ contains some elements, said to be alternating. There are four kinds of alternating elements, denoted $\{W_{-k}\}_{k\in\mathbb{N}}$, $\{G_{k+1}\}_{k\in\mathbb{N}}$, $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$. The alternating elements of each kind mutually commute. A tridiagonal pair is an ordered pair of diagonalizable linear maps $A, A^*$ on a nonzero, finite-dimensional vector space $V$, that each act in a (block) tridiagonal fashion on the eigenspaces of the other one. Let $A, A^*$ denote a tridiagonal pair on $V$. Associated with this pair are six well-known direct sum decompositions of $V$; these are the eigenspace decompositions of $A$ and $A^*$, along with four decompositions of $V$ that are often called split. In our main results, we assume that $A, A^*$ has $q$-Serre type. Under this assumption $A, A^*$ satisfy the $q$-Serre relations, and $V$ becomes an irreducible $U^+_q$-module on which $W_0 = A$ and $W_1 = A^*$. We describe how the alternating elements of $U^+_q$ act on the above six decompositions of $V$. We show that for each decomposition, every alternating element acts in either a (block) diagonal, (block) upper bidiagonal, (block) lower bidiagonal, or (block) tridiagonal fashion. We investigate two special cases in detail. In the first case the eigenspaces of $A$ and $A^*$ all have dimension one. In the second case $A$ and $A^*$ are obtained by adjusting the adjacency matrix and a dual adjacency matrix of a distance-regular graph that has classical parameters and is formally self-dual.

Keywords. Tridiagonal pair; alternating elements; $q$-Serre relations; distance-regular graph.

2020 Mathematics Subject Classification. Primary: 17B37. Secondary: 05E30.

1 Introduction

Our point of departure is the following remarkable fact. Let $\mathbb{F}$ denote a field, and fix a nonzero $b \in \mathbb{F}$ that is not a root of unity. Define an algebra over $\mathbb{F}$ by generators $W_0, W_1$ and relations

$$[W_0, [W_0, [W_0, W_1]_b]_{b^{-1}}] = 0, \quad [W_1, [W_1, [W_1, W_0]_b]_{b^{-1}}] = 0,$$

where

$$[X, Y] = XY - YX, \quad [X, Y]_b = bXY - YX.$$
Using $W_0, W_1$ and the equations below, we recursively define some elements

\[ \{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}} \] (2)

in the following order:

\[ W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad W_{-2}, \quad W_3, \quad \ldots \]

For $n \geq 1$,

\[ G_n = \sum_{k=0}^{n-1} \frac{W_{-k}W_{n-k}b^{-k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k}b^{-k}}{1 + b^{-n}} + \frac{[W_n, W_0]}{(1 + b^{-n})(1 - b^{-1})}, \]

\[ \tilde{G}_n = G_n + \frac{[W_0, W_n]}{1 - b^{-1}}, \quad W_{-n} = \frac{[W_0, G_n]_b}{b - 1}, \quad W_{n+1} = \frac{[G_n, W_1]_b}{b - 1}. \]

The remarkable fact is that for $k, \ell \in \mathbb{N}$,

\[ [W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \]

\[ [G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0. \]

The above fact is from \[37\] Proposition 5.10. The relations \([1]\) are called the $q$-Serre relations, where $q^2 = b$. The defined algebra is denoted by $U_q^+$, and called the positive part of $U_q(\mathfrak{sl}_2)$; see Definition \[8.1\] below. The elements \([2]\) are called the alternating elements of $U_q^+$ \[37\] Definition 5.1. These elements satisfy the relations in \[37\] Proposition 5.7, \[37\] Proposition 5.10, \[37\] Proposition 5.11, \[37\] Proposition 6.3, \[37\] Proposition 8.1.

The alternating elements get their name in the following way. Let $x, y$ denote noncommuting indeterminates. Let $F$ denote the free algebra with generators $x, y$. By a letter we mean $x$ or $y$. For $n \in \mathbb{N}$, by a word of length $n$ we mean a product of letters $a_1a_2 \cdots a_n$. The words form a linear basis for $F$. In \[31,32\] M. Rosso introduced an algebra structure on $F$ called the $q$-shuffle algebra. For letters $u, v$ their $q$-shuffle product is given by $u \ast v = uv + q^{(u,v)}vu$, where $(u, v) = 2$ if $u = v$ and $(u, v) = -2$ if $u \neq v$. In \[32\] Theorem 15, Rosso gave an injective algebra homomorphism from $U_q^+$ into the $q$-shuffle algebra $F$, that sends $W_0 \mapsto x$ and $W_1 \mapsto y$. By \[37\] Definition 5.2 the homomorphism sends

\[ W_0 \mapsto x, \quad W_{-1} \mapsto xyx, \quad W_{-2} \mapsto yxyyx, \quad \ldots \]
\[ W_1 \mapsto y, \quad W_2 \mapsto yxy, \quad W_3 \mapsto yxyyx, \quad \ldots \]
\[ G_1 \mapsto yx, \quad G_2 \mapsto yxyx, \quad G_3 \mapsto yxyyx, \quad \ldots \]
\[ \tilde{G}_1 \mapsto xy, \quad \tilde{G}_2 \mapsto xxy, \quad \tilde{G}_3 \mapsto xxyyx, \quad \ldots \]

We used this homomorphism to obtain the relations in \[37\] mentioned above.

Our discovery of the alternating elements \[37\] was motivated by the groundbreaking work of Baseilhac/Koizumi \[9\] and Baseilhac/Shigechi \[10\] concerning the $q$-Onsager algebra $O_q$ \[6,36\]. The algebras $U_q^+, O_q$ have a similar structure; they both belong to a family of algebras called the tridiagonal algebras \[36\] Definition 3.9]. The algebra $U_q^+$ is considerably less complicated than $O_q$, and it is natural to view $U_q^+$ as a toy model for $O_q$. With this
In view in mind, we now discuss [9], [10]. In [9] Baseilhac and Koizumi investigate boundary integrable systems with hidden symmetries. In [9, Section 2.1] they use an RKRK reflection equation to define an algebra [9, Line (4)] that is now denoted by $O_q$ and called the alternating central extension of $O_q$. In [10, Definition 3.1] Baseilhac and Shigechi give a presentation of $O_q$ by generators and relations. This presentation resembles [37, Proposition 5.7], [37, Proposition 5.10]. Moreover [10, Proposition 3.1] resembles [37, Proposition 5.11]. After [9] appeared, we tried to understand these works by searching for analogous results and [10] appeared, we tried to understand these works by searching for analogous results about $U_q^+$. In this search we considered the Rosso embedding of $U_q^+$ into a $q$-shuffle algebra, and this lead us to discover the alternating elements of $U_q^+$. We believe that this discovery would not have occurred without the guides [9], [10].

Next we summarize what is known about the alternating elements of $U_q^+$. We will refer to a Poincaré-Birkhoff-Witt (or PBW) basis for $U_q^+$ due to Damiani [15], and a related PBW basis for $U_q^+$ due to Beck [11, Proposition 6.1]. In [37, Theorem 10.1] we showed that the alternating elements $\{W_{-k}\}_{k \in \mathbb{N}}, \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}, \{W_{k+1}\}_{k \in \mathbb{N}}$ form a PBW basis for $U_q^+$; this PBW basis is said to be alternating. In [37, Section 11] we related the alternating PBW basis to the Damiani PBW basis. In [11, Proposition 9.11] we related the alternating PBW basis to the Beck PBW basis. In [37, Theorem 9.15] we expressed the elements $\{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$ in terms of the alternating PBW basis. This expression involves some elements $\{D_n\}_{n \in \mathbb{N}}$ that we defined recursively; in [33, Theorem 1.11] Chenwei Ruan displayed the elements $\{D_n\}_{n \in \mathbb{N}}$ in closed form. In [8, Theorem 2.10], Baseilhac interprets the alternating elements of $U_q^+$ using an RKRK reflection equation. In [8, Section 3] Baseilhac relates the alternating elements of $U_q^+$ to the Drinfeld generators of $U_q(\mathfrak{sl}_2)$. In [38] we used the alternating elements of $U_q^+$ to construct an algebra $\mathcal{U}_q^+$, called the alternating central extension of $U_q^+$. The algebra $\mathcal{U}_q^+$ is discussed in [8,39,40].

In the present paper, we interpret the alternating elements of $U_q^+$ using the concept of a tridiagonal pair (or TD pair); see Definition 3.1 below. Roughly speaking, a TD pair is an ordered pair $A, A^*$ of diagonalizable linear maps on a nonzero finite-dimensional vector space $V$, that each act on the eigenspaces of the other one in a (block) tridiagonal fashion. Let $A, A^*$ denote a TD pair on $V$. Associated with this TD pair are six well-known direct sum decompositions of $V$; these are the eigenspace decompositions of $A$ and $A^*$, along with four decompositions of $V$ that are often called split [19, Section 4]. On each split decomposition of $V$, one of $A, A^*$ acts in a (block) upper bidiagonal fashion, and the other one acts in a (block) lower bidiagonal fashion [21, Lemma 5.1]. As we will see, it is natural to represent the six decompositions of $V$ by the edges of a tetrahedron, with the eigenspace decompositions of $A$ and $A^*$ represented by a pair of opposite edges. The resulting diagram is called the tetrahedron diagram. In our main results, we assume that the TD pair $A, A^*$ has $q$-Serre type [22, Definition 2.6]. Under this assumption $A, A^*$ satisfy the $q$-Serre relations, so $V$ becomes a $U_q^+$-module on which $W_0 = A$ and $W_1 = A^*$. It turns out that the $U_q^+$-module $V$ is irreducible; see Corollary 9.3 below. We describe how the alternating elements of $U_q^+$ act on the six decompositions of $V$ from the tetrahedron diagram. We show that for each decomposition, every alternating element acts in either a (block) diagonal, (block) upper bidiagonal, (block) lower bidiagonal, or (block) tridiagonal fashion. Our main results on this topic are Theorems 9.8, 9.9. In view of the remarkable fact that we began with, it is natural to seek bases for $V$ of the following types:
(i) a basis of common eigenvectors for \( \{ W_k \}_{k \in \mathbb{N}} \);

(ii) a basis of common eigenvectors for \( \{ W_{k+1} \}_{k \in \mathbb{N}} \);

(iii) a basis of common eigenvectors for \( \{ G_{k+1} \}_{k \in \mathbb{N}} \);

(iv) a basis of common eigenvectors for \( \{ \tilde{G}_{k+1} \}_{k \in \mathbb{N}} \).

The above bases exist sometimes, but not always. We investigate two cases in which the bases exist. In the first case, we assume that \( A, A^* \) is a Leonard pair; this means that the eigenspaces of \( A \) and \( A^* \) all have dimension one. We show that for \( k \in \mathbb{N} \) the element \( W_k \) (resp. \( W_{k+1} \)) acts on \( V \) as a linear combination of \( A, I \) (resp. \( A^*, I \)). We obtain a similar result for \( G_{k+1} \) and \( \tilde{G}_{k+1} \). For the linear combinations that show up, we express the coefficients using a generating function. Our main results on this topic are Theorems 10.14, 10.15. For the second case, start with a distance-regular graph \( \Gamma \) that has diameter \( d \geq 3 \) and classical parameters \( (d, b, \alpha, \beta) \) with \( b \neq 1 \) and \( \alpha = b - 1 \); the condition on \( \alpha \) implies that \( \Gamma \) is formally self-dual in the sense of [12, p. 49]. After an affine adjustment, the adjacency matrix and any dual adjacency matrix become a pair of matrices \( A \) and \( A^* \) that satisfy the \( q \)-Serre relations. The corresponding subconstituent algebra \( T \) is generated by \( A, A^* \). It is known that \( A, A^* \) act on each irreducible \( T \)-module as a TD pair of \( q \)-Serre type [24, Lemma 9.4]. We show that each irreducible \( T \)-module has bases of type (i)–(iv) above. Our main results on this topic are Theorems 11.6, 11.7. We finish the paper with some open problems.

This paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we define a TD pair and TD system, and provide some basic facts about these objects. In Section 4 we discuss a TD system from the point of view of flags and decompositions. In Section 5 we define the tetrahedron diagram and use it to describe a TD system. In Section 6 we give some slightly technical comments about flags and decompositions. In Section 7 we describe the tridiagonal relations. In Section 8 we review the algebra \( U_q^+ \) and its alternating elements. Section 9 contains our main results about TD systems of \( q \)-Serre type. Section 10 contains our main results about Leonard systems of \( q \)-Serre type. Section 11 contains our main results about distance-regular graphs. In Section 12 we give some open problems.

2 Preliminaries

We now begin our formal argument. The following concepts and notation will be used throughout the paper. Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots \} \). We will be discussing finite sequences. For \( d \in \mathbb{N} \) a sequence \( x_0, x_1, \ldots, x_d \) will be denoted by \( \{ x_i \}_{i=0}^d \). By the inversion of \( \{ x_i \}_{i=0}^d \), we mean the sequence \( \{ x_{d-i} \}_{i=0}^d \). Let \( \mathbb{F} \) denote a field. Every vector space discussed in this paper is understood to be over \( \mathbb{F} \). Every algebra discussed in this paper is understood to be associative, over \( \mathbb{F} \), and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Throughout the paper, \( V \) denotes a nonzero vector space with finite dimension. The algebra \( \text{End}(V) \) consists of the \( \mathbb{F} \)-linear maps from \( V \) to \( V \). An element \( A \in \text{End}(V) \) is said to be diagonalizable whenever \( V \) is spanned by the eigenspaces of \( A \). Assume that \( A \) is diagonalizable, and let \( \{ V_i \}_{i=0}^d \) denote an
ordering of the eigenspaces of \( A \). For \( 0 \leq i \leq d \) let \( \theta_i \) denote the eigenvalue of \( A \) for \( V_i \). For \( 0 \leq i \leq d \) define \( E_i \in \text{End}(V) \) such that \( (E_i - I)V_i = 0 \) and \( E_i V_j = 0 \) if \( j \neq i \ (0 \leq j \leq d) \). Thus \( E_i \) is the projection from \( V \) onto \( V_i \). We call \( E_i \) the primitive idempotent of \( A \) associated with \( V_i \) (or \( \theta_i \)). By linear algebra (i) \( A = \sum_{i=0}^{d} \theta_i E_i \); (ii) \( E_i E_j = \delta_{i,j} E_i \) (\( 0 \leq i, j \leq d \)); (iii) \( I = \sum_{i=0}^{d} E_i \); (iv) \( V_i = E_i V \) (\( 0 \leq i \leq d \)); (v) \( AE_i = \theta_i E_i = E_i A \) (\( 0 \leq i \leq d \)). Moreover \( E_i = \prod_{0 \leq j \leq d, j \neq i} A - \theta_j I \) (\( 0 \leq i \leq d \)).

Let \( \langle A \rangle \) denote the subalgebra of \( \text{End}(V) \) generated by \( A \). The elements \( \{ A^i \}_{i=0}^{d} \) form a basis for \( \langle A \rangle \), and \( 0 = \prod_{i=0}^{d} (A - \theta_i I) \). Moreover the elements \( \{ E_i \}_{i=0}^{d} \) form a basis for \( \langle A \rangle \).

3 Tridiagonal pairs and tridiagonal systems

In this section we recall the notions of a tridiagonal pair and a tridiagonal system. We give some basic facts about these objects.

**Definition 3.1.** (See [19, Definition 1.1].) A tridiagonal pair (or TD pair) on \( V \) is an ordered pair \( A, A^* \) of elements in \( \text{End}(V) \) that satisfy the following four conditions.

(i) Each of \( A, A^* \) is diagonalizable.

(ii) There exists an ordering \( \{ V_i \}_{i=0}^{d} \) of the eigenspaces of \( A \) such that

\[
A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),
\]

where \( V_{-1} = 0 \) and \( V_{d+1} = 0 \).

(iii) There exists an ordering \( \{ V_i^* \}_{i=0}^{\delta} \) of the eigenspaces of \( A^* \) such that

\[
A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),
\]

where \( V_{-1}^* = 0 \) and \( V_{\delta+1}^* = 0 \).

(iv) There does not exist a subspace \( W \) of \( V \) such that \( AW \subseteq W, A^* W \subseteq W, W \neq 0, W \neq V \).

**Note 3.2.** According to a common notational convention, \( A^* \) denotes the conjugate transpose of \( A \). We are not using this convention. In a TD pair \( A, A^* \) the maps \( A \) and \( A^* \) are arbitrary subject to (i)–(iv) above.

We refer the reader to [17], [19], [25], [30] for background and historical remarks about TD pairs.

**Lemma 3.3.** For a TD pair \( A, A^* \) on \( V \) and scalars \( r, r^*, s, s^* \in \mathbb{F} \) with \( rr^* \neq 0 \), the pair \( rA + sI, r^* A^* + s^* I \) is a TD pair on \( V \).

**Proof.** Routine.
We have been discussing TD pairs. There is a related notion, called a TD system. To define a TD system, we will use the following concept. Let $A, A^*$ denote a TD pair on $V$. An ordering of the eigenspaces of $A$ (resp. $A^*$) is called standard whenever it satisfies (3) (resp. (4)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of $A$. Then by [19, Lemma 2.4], the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. An ordering of the primitive idempotents of $A$ (resp. $A^*$) is said to be standard whenever the corresponding ordering of the eigenspaces of $A$ (resp. $A^*$) is standard.

**Definition 3.4.** (See [19, Definition 2.1], [28, Definition 2.1].) A tridiagonal system (or TD system) on $V$ is a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta)$$

that satisfies the following three conditions:

(i) $A, A^*$ is a TD pair on $V$;

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A$;

(iii) $\{E_i^*\}_{i=0}^\delta$ is a standard ordering of the primitive idempotents of $A^*$.

We have a comment.

**Lemma 3.5.** (See [19, Section 3].) Consider a TD system on $V$:

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta).$$

For scalars $r, r^*, s, s^* \in \mathbb{F}$ such that $rr^* \neq 0$, the sequence

$$(rA + sI; \{E_i\}_{i=0}^d; r^*A^* + s^*I; \{E_i^*\}_{i=0}^\delta)$$

is a TD system on $V$. Moreover, each of the following is a TD system on $V$:

$$\Phi^* = (A^*; \{E_i^*\}_{i=0}^\delta; A; \{E_i\}_{i=0}^d),$$

$$\Phi^\dagger = (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^\delta),$$

$$\Phi^\ddagger = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta).$$

From now until the end of Section 10, we fix a TD system on $V$:

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta).$$

(5)

**Lemma 3.6.** (See [19, Lemma 2.4], [29, Lemma 2.5].) Referring to the TD system $\Phi$, we have

$$E_iA^*E_j = \begin{cases} 0 & \text{if } |i - j| > 1; \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} (0 \leq i, j \leq d),$$

$$E_i^*AE_j^* = \begin{cases} 0 & \text{if } |i - j| > 1; \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} (0 \leq i, j \leq \delta).$$
Definition 3.7. (See [19, Definition 3.1].) Referring to the TD system $\Phi$, for $0 \leq i \leq d$ let $\theta_i$ denote the eigenvalue of $A$ associated with $E_i$. We call $\{\theta_i\}_{i=0}^d$ the eigenvalue sequence of $\Phi$. For $0 \leq i \leq \delta$ let $\theta_i^*$ denote the eigenvalue of $A^*$ associated with $E_i^*$. We call $\{\theta_i^*\}_{i=0}^\delta$ the dual eigenvalue sequence of $\Phi$.

We emphasize that $\{\theta_i\}_{i=0}^d$ are mutually distinct, and $\{\theta_i^*\}_{i=0}^\delta$ are mutually distinct.

Referring to the TD system $\Phi$, it is shown in [19, Lemma 4.5] that $d = \delta$. This result is nontrivial, and can be obtained using some ideas that will play a role in our main results.

We will give a short proof in order to introduce these ideas.

For $0 \leq i \leq \delta$ and $0 \leq j \leq d$, define

$$ V_{i,j} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V). $$

(6)

For example, $V_{0,d} = E_0^* V$ and $V_{\delta,0} = E_0 V$. Define

$$ N = \min\{i + j | V_{i,j} \neq 0\}. $$

(7)

We have $N \leq d$ because $V_{0,d} \neq 0$. We have $N \leq \delta$ because $V_{\delta,0} \neq 0$. For $0 \leq i \leq N$ abbreviate $U_i = V_{i,N-i}$, so that

$$ U_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{N-i} V). $$

(8)

In a moment we will show that $N = d = \delta$.

Lemma 3.8. (See [19, Lemma 4.4].) We have

(i) $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$ for $1 \leq i \leq N$;

(ii) $(A^* - \theta_0^* I)U_0 = 0$;

(iii) $(A - \theta_{N-i} I)U_i \subseteq U_{i+1}$ for $0 \leq i \leq N - 1$;

(iv) $(A - \theta_0 I)U_N = 0$.

Proof. (i) Use (6) along with the following observations.

$$ (A^* - \theta_i^* I)(E_0^* V + \cdots + E_i^* V) = E_0^* V + \cdots + E_{i-1}^* V. $$

By (3),

$$ (A^* - \theta_i^* I)(E_0 V + \cdots + E_{N-i} V) \subseteq E_0 V + \cdots + E_{N-i+1} V. $$

(ii) Setting $i = 0$ in (6) we have $U_0 \subseteq E_0^* V$.

(iii), (iv) Apply (i), (ii) to $\Phi^*$.

Corollary 3.9. We have

(i) $A^* U_i \subseteq U_{i-1} + U_i$ for $1 \leq i \leq N$;
(ii) $A^*U_0 \subseteq U_0$;
(iii) $AU_i \subseteq U_i + U_{i+1}$ for $0 \leq i \leq N - 1$;
(iv) $AU_N \subseteq U_N$.

**Proof.** By Lemma 3.8. 

**Lemma 3.10.** $V = \sum_{i=0}^{N} U_i$.

**Proof.** Define $U = \sum_{i=0}^{N} U_i$. We have $U \neq 0$, since at least one of $\{U_i\}_{i=0}^{N}$ is nonzero by (7). We have $AU \subseteq U$ and $A^*U \subseteq U$ by Corollary 3.9. Therefore $U = V$ in view of Definition 3.1(iv).

**Lemma 3.11.** (See [19, Lemma 4.5].) $N = d = \delta$.

**Proof.** We mentioned below (7) that $N \leq d$ and $N \leq \delta$. By (8) we have $U_i \subseteq E_0V + \cdots + E_{N-1}V$ for $0 \leq i \leq N$. Consequently $\sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_iV$. Observe that

$$\sum_{i=0}^{d} E_iV = V = \sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_iV.$$ 

Therefore $N = d$. By (8) we have $U_i \subseteq E_0^*V + \cdots + E_i^*V$ for $0 \leq i \leq N$. Consequently $\sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_i^*V$. Observe that

$$\sum_{i=0}^{\delta} E_i^*V = V = \sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_i^*V.$$ 

Therefore $N = \delta$.

**Definition 3.12.** We just showed $d = \delta$; this common value is called the *diameter* of $\Phi$. We say that $\Phi$ is *trivial* whenever it has diameter 0.

Note that $\Phi$ is trivial if and only if $\dim V = 1$. For the rest of this paper, we assume that $\Phi$ is nontrivial.

In the next few results, we describe the subspaces $\{U_i\}_{i=0}^{d}$ in more detail.

**Lemma 3.13.** (See [19, Theorem 4.6].) The sum $V = \sum_{i=0}^{d} U_i$ is *direct*.

**Proof.** It suffices to show that $(U_0 + \cdots + U_{i-1}) \cap U_i = 0$ for $1 \leq i \leq d$. Let $i$ be given. By (8),

$$U_0 + \cdots + U_{i-1} \subseteq E_0^*V + \cdots + E_{i-1}^*V, \quad U_i \subseteq E_0V + \cdots + E_{d-i}V.$$ 

By (6),

$$(E_0^*V + \cdots + E_{i-1}^*V) \cap (E_0V + \cdots + E_{d-i}V) = V_{i-1,d-i}.$$ 

Note that $V_{i-1,d-i} = 0$ by (7) and since $i - 1 + d - i = d - 1 < d = N$. By these comments $(U_0 + \cdots + U_{i-1}) \cap U_i = 0$. 

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Lemma 3.14. (See [19, Theorem 4.6].) For $0 \leq i \leq d$,

(i) $U_0 + \cdots + U_i = E_0^i V + \cdots + E_i^i V$;

(ii) $U_i + \cdots + U_d = E_0^i V + \cdots + E_{d-i}^i V$.

Proof. (i) The inclusion $\subseteq$ follows from (8). Next we obtain the inclusion $\supseteq$. Define $X_i = \prod_{l=i+1}^{d} (A^* - \theta^l I)$, and note that $X_i V = E_0^i V + \cdots + E_i^i V$. Using Lemma 3.8(i),(ii) we have $U_0 + \cdots + U_i \supseteq X_i U_j$ for $0 \leq j \leq d$. Therefore $U_0 + \cdots + U_i \supseteq X_i V$. By these comments $U_0 + \cdots + U_i \supseteq E_0^i V + \cdots + E_i^i V$. We have obtained the inclusion $\supseteq$.

(ii) Apply (i) to $\Phi^*$. □

Next we define some maps that will help us describe how the $\{U_i\}_{i=0}^d$ are related to the $\{E_i V\}_{i=0}^d$ and $\{E_i^* V\}_{i=0}^d$. For $0 \leq i \leq d$ define $E_i^\gamma \in \text{End}(V)$ such that $(E_i^\gamma - I) U_i = 0$ and $E_i^\gamma U_j = 0$ if $i \not= j$ ($0 \leq j \leq d$). By linear algebra, $E_i^\gamma E_j^\gamma = \delta_{ij} E_i^\gamma$ for $0 \leq i, j \leq d$, and $I = \sum_{i=0}^d E_i^\gamma$. By Lemma 3.14 or [19, Lemma 5.4] we find that for $0 \leq i < j \leq d$,

$$E_j^\gamma E_i^\gamma = 0, \quad E_j^\gamma E_i^* = 0, \quad E_{d-i} E_j^\gamma = 0, \quad E_i^\gamma E_{d-j} = 0. \quad (9)$$

Consequently for $0 \leq i \leq d$,

$$E_i^\gamma E_i^* E_i^\gamma = E_i^\gamma, \quad E_i^\gamma E_i^* E_i^* = E_i^*, \quad (9)$$

$$E_i^\gamma E_{d-i} E_i^\gamma = E_i^\gamma, \quad E_i^\gamma E_{d-i} E_i^* = E_i. \quad (10)$$

Lemma 3.15. The maps

$$\Psi = \sum_{i=0}^d E_{d-i}^\gamma E_i, \quad \Psi^* = \sum_{i=0}^d E_i^\gamma E_i^*$$

are invertible, and

$$\Psi^{-1} = \sum_{i=0}^d E_{d-i} E_i^\gamma, \quad (\Psi^*)^{-1} = \sum_{i=0}^d E_i^* E_i^\gamma. \quad (11)$$

Moreover,

$$\Psi(E_i V) = U_{d-i}, \quad \Psi^*(E_i^* V) = U_i \quad (0 \leq i \leq d).$$

Proof. First we verify the assertions about $\Psi$. Define $\Psi' = \sum_{i=0}^d E_{d-i} E_i^\gamma$. Using (9), (11) we routinely obtain $\Psi \Psi' = I$ and $\Psi' \Psi = I$. Therefore, $\Psi$ is invertible and $\Psi^{-1} = \Psi'$. Pick an integer $i$ ($0 \leq i \leq d$). We have

$$\Psi(E_i V) = \left( \sum_{j=0}^d E_j^\gamma E_j \right) E_i V = E_{d-i}^\gamma E_i V \subseteq E_{d-i}^\gamma V = U_{d-i},$$

and also

$$\Psi^{-1}(U_{d-i}) = \left( \sum_{j=0}^d E_{d-j}^\gamma E_j \right) U_{d-i} = E_i E_{d-i}^\gamma U_{d-i} \subseteq E_i V.$$
Lemma 3.16. (See [19 Corollary 5.7].) For $0 \leq i \leq d$ the subspaces $E_iV, E_i^*V, U_i$ have the same dimension. Denoting this common dimension by $\rho_i$, we have $\rho_i = \rho_{d-i}$.

Proof. By Lemma 3.15 the subspaces $E_iV, U_{d-i}$ have the same dimension and the subspaces $E_i^*V, U_i$ have the same dimension. Replacing $i$ by $d-i$, we find that $E_{d-i}V, U_i$ have the same dimension and $E_{d-i}^*V, U_{d-i}$ have the same dimension. By these comments, the subspaces $E_iV, E_i^*V$ have the same dimension. Applying this result to $\Phi^*$, we find that $E_iV, E_i^*V$ have the same dimension. The result follows. \qed

We mention a fact about the dimensions $\{\rho_i\}_{i=0}^d$ from Lemma 3.16. We will not use this fact, so we will not dwell on the proof.

Lemma 3.17. (See [19 Corollary 6.6].) Referring to Lemma 3.16 we have $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$.

Definition 3.18. By the shape of $\Phi$ we mean the sequence $\{\rho_i\}_{i=0}^d$ from Lemma 3.16.

A TD system of shape $(1, 1, \ldots, 1)$ is often called a Leonard system [35 Definition 1.4], [36 Lemma 2.2].

4 Decompositions and flags

We continue to discuss the nontrivial TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on $V$. In the previous section we used $\Phi$ to obtain a sequence $\{U_i\}_{i=0}^d$ of subspaces of $V$. An analogous sequence is obtained from $\Phi^i, \Phi^iU, \Phi^i\Phi$. In this section we interpret the resulting sequences in a comprehensive way.

First we define our terms. By a decomposition of $V$ we mean a sequence $\{\mathcal{V}_i\}_{i=0}^d$ of nonzero subspaces whose direct sum is equal to $V$. Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of $V$. For $0 \leq i \leq d$ we call $\mathcal{V}_i$ the $i$th component of the decomposition. For notational convenience, define $\mathcal{V}_{-1} = 0$ and $\mathcal{V}_{d+1} = 0$.

Example 4.1. Each of the following sequences is a decomposition of $V$:

(i) $\{E_iV\}_{i=0}^d$;
(ii) $\{E_i^*V\}_{i=0}^d$;
(iii) the sequence $\{U_i\}_{i=0}^d$ from (8).

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of $V$. For $0 \leq i \leq d$ define $s_i = \dim \mathcal{V}_i$, and note that $s_i \geq 1$. By construction $\sum_{i=0}^d s_i = \dim V$. We call the sequence $\{s_i\}_{i=0}^d$ the shape of the decomposition $\{\mathcal{V}_i\}_{i=0}^d$.

Next we recall the notion of a flag. Let $\{s_i\}_{i=0}^d$ denote a sequence of positive integers whose sum is equal to the dimension of $V$. By a flag on $V$ of shape $\{s_i\}_{i=0}^d$ we mean a sequence $\{F_i\}_{i=0}^d$ of subspaces of $V$ such that both (i) $F_{i-1} \subseteq F_i$ for $1 \leq i \leq d$; (ii) $F_i$ has dimension $s_0 + s_1 + \cdots + s_i$ for $0 \leq i \leq d$. Let $\{F_i\}_{i=0}^d$ denote a flag on $V$. For $0 \leq i \leq d$ we call $F_i$ the
i\textsuperscript{th} component of the flag. Note that $F_d = V$. For notational convenience, define $F_{-1} = 0$ and $F_{d+1} = V$.

The following construction yields a flag on $V$. Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$. Define $F_i = V_0 + V_1 + \cdots + V_i$ for $0 \leq i \leq d$. Then the sequence $\{F_i\}_{i=0}^d$ is a flag on $V$. The shape of this flag is equal to the shape of the decomposition $\{V_i\}_{i=0}^d$. We say that the flag $\{F_i\}_{i=0}^d$ is induced by the decomposition $\{V_i\}_{i=0}^d$.

Next, we recall what it means for two flags to be opposite. Let $\{F_i\}_{i=0}^d$ and $\{F_i'\}_{i=0}^d$ denote flags on $V$. These flags are called opposite whenever there exists a decomposition $\{V_i\}_{i=0}^d$ of $V$ such that $\{V_i\}_{i=0}^d$ induces $\{F_i\}_{i=0}^d$ and $\{V_{d-i}\}_{i=0}^d$ inducs $\{F_i'\}_{i=0}^d$. In this case $V_i = F_i \cap F_{d-i}'$ for $0 \leq i \leq d$.

We now use $\Phi$ to construct four flags on $V$. To keep track of these flags, we give each one a name. Let $\Omega$ denote the set consisting of the four symbols 0, $D$, 0*, $D*$. Each flag gets a name $[u]$ with $u \in \Omega$.

**Definition 4.2.** In each row of the table below, we display a flag on $V$.

| Flag name | i\textsuperscript{th} component of the flag |
|-----------|------------------------------------------|
| [0]       | $E_0V + E_1V + \cdots + E_iV$           |
| [D]       | $E_dV + E_{d-1}V + \cdots + E_{d-i}V$   |
| [0*]      | $E_0'V + E_1'V + \cdots + E_i'V$        |
| [D*]      | $E_d'V + E_{d-1}'V + \cdots + E_{d-i}'V$|

Referring to Definition 4.2, by construction the flags [0], [D] are opposite, and the flags [0*], [D*] are opposite. In fact we have the following.

**Lemma 4.3.** The four flags in Definition 4.2 are mutually opposite.

**Proof.** The flags [0], [D] are opposite, and the flags [0*], [D*] are opposite. To see that the flags [0], [0*] are opposite, consider the decomposition $\{U_i\}_{i=0}^d$ of $V$ from (8). By Lemma 3.14, $\{U_i\}_{i=0}^d$ induces [0*] and $\{U_{d-i}\}_{i=0}^d$ induces [0]. Therefore the flags [0], [0*] are opposite. To finish the proof, apply this result to $\Phi^\uparrow$, $\Phi^\downarrow$, $\Phi^{\uparrow\downarrow}$.

Next, we use the four flags in Definition 4.2 to construct six decompositions of $V$. Pick distinct $u, v \in \Omega$. By Lemma 4.3, the flags $[u]$ and $[v]$ are opposite. Therefore, there exists a decomposition of $V$ that induces $[u]$ and whose inversion induces $[v]$. This decomposition is unique, and denoted by $[u, v]$. Observe that the decomposition $[v, u]$ is the inversion of the decomposition $[u, v]$. Up to inversion, the above construction yields six decompositions of $V$. These decompositions are described in the following example.


Example 4.4. (See [21, Lemma 4.2].) In each row of the table below, we display a decomposition of $V$.

| decomp. name | $i^{th}$ component of the decomposition |
|--------------|----------------------------------------|
| $[0, D]$     | $E_0V$                                 |
| $[0^*, D^*]$ | $E_1^*V$                                |
| $[0^*, 0]$   | $(E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_{d-i}V)$ |
| $[0^*, D]$   | $(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$ |
| $[D^*, 0]$   | $(E_{d-i}^*V + \cdots + E_d^*V) \cap (E_0V + \cdots + E_{d-i}V)$ |
| $[D^*, D]$   | $(E_{d-i}^*V + \cdots + E_d^*V) \cap (E_iV + \cdots + E_dV)$ |

Note 4.5. The decomposition $[0^*, 0]$ from Example 4.4 is the same as the decomposition $\{U_i\}_{i=0}^d$ from [8].

In the next result, we clarify how the six decompositions from Example 4.4 are related to the four flags from Definition 4.2.

Lemma 4.6. (See [21, Lemma 4.3].) Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$ from Example 4.4. Then for $0 \leq i \leq d$ the sums $V_0 + \cdots + V_i$ and $V_i + \cdots + V_d$ are given in the table below.

| decomp. name | $V_0 + \cdots + V_i$ | $V_i + \cdots + V_d$ |
|--------------|-----------------------|-----------------------|
| $[0, D]$     | $E_0V + \cdots + E_iV$ | $E_iV + \cdots + E_dV$ |
| $[0^*, D^*]$ | $E_0^*V + \cdots + E_i^*V$ | $E_i^*V + \cdots + E_d^*V$ |
| $[0^*, 0]$   | $E_0^*V + \cdots + E_i^*V$ | $E_0V + \cdots + E_{d-i}V$ |
| $[0^*, D]$   | $E_0^*V + \cdots + E_i^*V$ | $E_iV + \cdots + E_dV$ |
| $[D^*, 0]$   | $E_{d-i}^*V + \cdots + E_d^*V$ | $E_0V + \cdots + E_{d-i}V$ |
| $[D^*, D]$   | $E_{d-i}^*V + \cdots + E_d^*V$ | $E_dV + \cdots + E_dV$ |

Proof. By the discussion above Example 4.4. \qed

Recall the shape $\{\rho_i\}_{i=0}^d$ of $\Phi$ from Definition 3.18.

Lemma 4.7. (See [21, Lemma 4.4].) Each flag in Definition 4.2 has shape $\{\rho_i\}_{i=0}^d$. Each decomposition in Example 4.2 has shape $\{\rho_i\}_{i=0}^d$.

Proof. By Lemma 3.10 and the construction. \qed

Next, we describe the actions of $A$ and $A^*$ on each of the six decompositions from Example 4.4.

Lemma 4.8. (See [21, Lemma 5.1].) Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$ from Example 4.4. Then for $0 \leq i \leq d$ the actions of $A$ and $A^*$ on $V_i$ are described in the table below.

| decomp. name | action of $A$ on $V_i$ | action of $A^*$ on $V_i$ |
|--------------|-------------------------|--------------------------|
| $[0, D]$     | $(A - \theta_iI)V_i = 0$ | $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ |
| $[0^*, D^*]$ | $AV_i \subseteq V_{i-1} + V_i + V_{i+1}$ | $(A^* - \theta_i^*I)V_i = 0$ |
| $[0^*, 0]$   | $(A - \theta_{d-i}I)V_i \subseteq V_{i+1}$ | $(A^* - \theta_i^*I)V_i \subseteq V_{i-1}$ |
| $[0^*, D]$   | $(A - \theta_iI)V_i \subseteq V_{i+1}$ | $(A^* - \theta_i^*I)V_i \subseteq V_{i-1}$ |
| $[D^*, 0]$   | $(A - \theta_{d-i}I)V_i \subseteq V_{i+1}$ | $(A^* - \theta_{d-i}^*I)V_i \subseteq V_{i-1}$ |
| $[D^*, D]$   | $(A - \theta_iI)V_i \subseteq V_{i+1}$ | $(A^* - \theta_{d-i}^*I)V_i \subseteq V_{i-1}$ |
Proof. For the decompositions \([0, D]\) and \([0^*, D^*]\) the result holds by Definition 3.1. For the decomposition \([0^*, 0]\) the result holds by Lemma 3.8 and Note 4.5. To get the result for the remaining decompositions, apply Lemma 3.8 to \(\Phi^\downarrow, \Phi^\downarrow, \Phi^\downarrow^\downarrow\).

Here is another version of Lemma 4.8.

**Corollary 4.9.** Let \(\{V_i\}_{i=0}^d\) denote a decomposition of \(V\) from Example 4.4. Then for \(0 \leq i \leq d\) the actions of \(A\) and \(A^*\) on \(V_i\) are described in the table below.

| decomp. name | action of \(A\) on \(V_i\) | action of \(A^*\) on \(V_i\) |
|--------------|----------------------------|-------------------------------|
| \([0, D]\)   | \(AV_i \subseteq V_i\)     | \(A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}\) |
| \([0^*, D^*]\)| \(AV_i \subseteq V_i + V_{i+1}\) | \(A^*V_i \subseteq V_i\) |
| \([0^*, 0]\) | \(AV_i \subseteq V_i + V_{i+1}\) | \(A^*V_i \subseteq V_{i-1} + V_i\) |
| \([0^*, D]\) | \(AV_i \subseteq V_i + V_{i+1}\) | \(A^*V_i \subseteq V_{i-1} + V_i\) |
| \([D^*, 0]\) | \(AV_i \subseteq V_i + V_{i+1}\) | \(A^*V_i \subseteq V_{i-1} + V_i\) |
| \([D^*, D]\) | \(AV_i \subseteq V_i + V_{i+1}\) | \(A^*V_i \subseteq V_{i-1} + V_i\) |

Proof. By Lemma 4.8.

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### 5 The tetrahedron diagram

We continue to discuss the nontrivial TD system \(\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_{i-1}\}_{i=0}^d)\) on \(V\). In the previous section we used \(\Phi\) to construct six decompositions of \(V\). In this section we draw a diagram that illustrates how these six decompositions are related.

Let \(\{V_i\}_{i=0}^d\) denote a decomposition of \(V\). We describe this decomposition by the diagram

\[
\bullet \quad \bullet \quad \bullet \quad \ldots \quad \bullet \\
V_0 \quad V_1 \quad V_2 \quad \ldots \quad V_{d-1} \quad V_d
\]

The labels \(V_i\) might be suppressed, if they are clear from the context.

Let \(\{V_i\}_{i=0}^d\) and \(\{V'_i\}_{i=0}^d\) denote decompositions of \(V\). The condition

\[
V_0 + V_1 + \cdots + V_i = V'_0 + V'_1 + \cdots + V'_i \quad (0 \leq i \leq d)
\]

will be described by the diagram

\[
\begin{align*}
V_0 & \quad V_1 & \quad V_2 & \quad \ldots & \quad V_{d-1} & \quad V_d \\
V'_0 & \quad V'_1 & \quad V'_2 & \quad \ldots & \quad V'_{d-1} & \quad V'_d
\end{align*}
\]
To illustrate the above diagram convention, consider the decomposition \( \{U_i\}_{i=0}^d \) of \( V \) from (8). By Lemma 3.14 we have
\[
U_0 + \cdots + U_i = E_0^i V + \cdots + E_i^i V, \quad U_i + \cdots + U_d = E_0^i V + \cdots + E_{d-i}^i V
\]
for \( 0 \leq i \leq d \). The corresponding diagram is shown below:

In Example 4.4 we gave six decompositions of \( V \). The corresponding diagram is shown below:

This diagram is called the tetrahedron diagram of \( \Phi \).

Next we use the tetrahedron diagram to illustrate Corollary 4.9. The following picture shows how \( A \) acts on the decompositions of \( V \) from the tetrahedron diagram, for \( d = 8 \):
The following picture shows how $A^*$ acts on the decompositions of $V$ from the tetrahedron diagram, for $d = 8$:

![Diagram showing $A^*$ action on decompositions]

We will return to the tetrahedron diagram in Section 9.

### 6 Some comments about flags and decompositions

We continue to discuss the nontrivial TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ on $V$. In Section 4 we used $\Phi$ to construct six decompositions of $V$. In Lemma 4.8 and Corollary 4.9 we described how $A$ and $A^*$ act on these six decompositions. Shortly we will consider some additional maps in $\text{End}(V)$, and describe how these maps act on the six decompositions. To prepare for this description, we have some general comments about how maps in $\text{End}(V)$ act on flags and decompositions.

**Definition 6.1.** Let $\{F_i\}_{i=0}^d$ denote a flag on $V$, and let $\psi \in \text{End}(V)$. We say that $\psi$ stabilizes $\{F_i\}_{i=0}^d$ whenever $\psi F_i \subseteq F_i$ for $0 \leq i \leq d$. We say that $\psi$ raises $\{F_i\}_{i=0}^d$ whenever $\psi F_i \subseteq F_{i+1}$ for $0 \leq i \leq d-1$.

For the rest of this section, the following assumption is in effect.

**Assumption 6.2.** Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$. Let $\{F_i\}_{i=0}^d$ denote the flag on $V$ induced by $\{V_i\}_{i=0}^d$, and let $\{F'_i\}_{i=0}^d$ denote the flag on $V$ induced by $\{V_{d-i}\}_{i=0}^d$. Thus

$$F_i = V_0 + \cdots + V_i, \quad F'_i = V_d + \cdots + V_{d-i}$$

for $0 \leq i \leq d$.

**Lemma 6.3.** With reference to Assumption 6.2, the following are equivalent for $\psi \in \text{End}(V)$:

(i) $\psi$ raises $\{F_i\}_{i=0}^d$ and $\{F'_i\}_{i=0}^d$;

(ii) $\psi V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$.

**Proof.** (i) $\Rightarrow$ (ii) We have

$$\psi V_i \subseteq \psi(V_0 + \cdots + V_i) = \psi F_i \subseteq F_{i+1} = V_0 + \cdots + V_{i+1}$$
and also
\[ \psi \mathcal{V}_i \subseteq \psi (\mathcal{V}_i + \cdots + \mathcal{V}_d) = \psi F'_{d-i} \subseteq F'_{d-i+1} = \mathcal{V}_{i-1} + \cdots + \mathcal{V}_d. \]

Therefore
\[ \psi \mathcal{V}_i \subseteq (\mathcal{V}_0 + \cdots + \mathcal{V}_{i+1}) \cap (\mathcal{V}_{i-1} + \cdots + \mathcal{V}_d) = \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}. \]

(ii) \(\Rightarrow\) (i) For \(0 \leq i \leq d-1\) we have
\[ \psi F_i = \psi (\mathcal{V}_0 + \cdots + \mathcal{V}_i) \subseteq \mathcal{V}_0 + \cdots + \mathcal{V}_{i+1} = F_{i+1} \]

and also
\[ \psi F'_i = \psi (\mathcal{V}_d + \cdots + \mathcal{V}_{d-i}) \subseteq \mathcal{V}_d + \cdots + \mathcal{V}_{d-i+1} = F'_{i+1} \]

\[\square\]

**Lemma 6.4.** With reference to Assumption 6.2, the following are equivalent for \(\psi \in \text{End}(V)\):

(i) \(\psi\) raises \(\{F_i\}_{i=0}^d\) and stabilizes \(\{F'_i\}_{i=0}^d\);

(ii) \(\psi \mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}\) for \(0 \leq i \leq d\).

**Proof.** (i) \(\Rightarrow\) (ii) We have
\[ \psi \mathcal{V}_i \subseteq \psi (\mathcal{V}_0 + \cdots + \mathcal{V}_i) = \psi F_i \subseteq F_{i+1} = \mathcal{V}_0 + \cdots + \mathcal{V}_{i+1} \]

and also
\[ \psi \mathcal{V}_i \subseteq \psi (\mathcal{V}_i + \cdots + \mathcal{V}_d) = \psi F'_{d-i} \subseteq F'_{d-i+1} = \mathcal{V}_i + \cdots + \mathcal{V}_d. \]

Therefore
\[ \psi \mathcal{V}_i \subseteq (\mathcal{V}_0 + \cdots + \mathcal{V}_{i+1}) \cap (\mathcal{V}_i + \cdots + \mathcal{V}_d) = \mathcal{V}_i + \mathcal{V}_{i+1}. \]

(ii) \(\Rightarrow\) (i) For \(0 \leq i \leq d-1\) we have
\[ \psi F_i = \psi (\mathcal{V}_0 + \cdots + \mathcal{V}_i) \subseteq \mathcal{V}_0 + \cdots + \mathcal{V}_{i+1} = F_{i+1} \]

and also
\[ \psi F'_i = \psi (\mathcal{V}_d + \cdots + \mathcal{V}_{d-i}) \subseteq \mathcal{V}_d + \cdots + \mathcal{V}_{d-i+1} = F'_{i+1} \]

\[\square\]

**Lemma 6.5.** With reference to Assumption 6.2, the following are equivalent for \(\psi \in \text{End}(V)\):

(i) \(\psi\) stabilizes \(\{F_i\}_{i=0}^d\) and raises \(\{F'_i\}_{i=0}^d\);

(ii) \(\psi \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i\) for \(0 \leq i \leq d\).
Proof. Apply Lemma 6.4 to the decomposition \( \{ V_d^{-i} \}_{i=0}^d \). \( \square \)

Lemma 6.6. With reference to Assumption 6.2, the following are equivalent for \( \psi \in \text{End}(V) \):

(i) \( \psi \) stabilizes \( \{ F_i \}_{i=0}^d \) and \( \{ F'_i \}_{i=0}^d \);

(ii) \( \psi V_i \subseteq V_i \) for \( 0 \leq i \leq d \).

Proof. (i) \( \Rightarrow \) (ii) We have
\[
\psi V_i \subseteq \psi (V_0 + \cdots + V_i) = \psi F_i \subseteq F_i = V_0 + \cdots + V_i
\]
and also
\[
\psi V_i \subseteq \psi (V_i + \cdots + V_d) = \psi F'_{d-i} \subseteq F'_{d-i} = V_i + \cdots + V_d.
\]
Therefore
\[
\psi V_i \subseteq (V_0 + \cdots + V_i) \cap (V_i + \cdots + V_d) = V_i.
\]
(ii) \( \Rightarrow \) (i) For \( 0 \leq i \leq d - 1 \) we have
\[
\psi F_i = \psi (V_0 + \cdots + V_i) \subseteq V_0 + \cdots + V_i = F_i
\]
and also
\[
\psi F'_i = \psi (V_d + \cdots + V_{d-i}) \subseteq V_d + \cdots + V_{d-i} = F'_i.
\]
\( \square \)

7  The tridiagonal relations

We continue to discuss the nontrivial TD system \( \Phi = (A; \{ E_i \}_{i=0}^d; A^*; \{ E'_i \}_{i=0}^d) \) on \( V \). In this section we recall some relations satisfied by \( A, A^* \).

Lemma 7.1. (See [19, Theorem 10.1].) There exists a sequence of scalars \( \beta, \gamma, \gamma^*, \rho, \rho^* \) in \( \mathbb{F} \) such that both
\[
0 = [A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma (AA^* + A^* A) - \rho A^*],
\]
\[
0 = [A^*, A^* A - \beta A^* AA^* + AA^* A^2 - \gamma^* (A^* A + AA^*) - \rho^* A].
\]
The sequence \( \beta, \gamma, \gamma^*, \rho, \rho^* \) is unique if \( d \geq 3 \).

The relations (11), (12) are called the tridiagonal relations.

Lemma 7.2. (See [36, Theorem 4.3].) For the TD system \( \Phi \) and scalars \( \beta, \gamma, \gamma^*, \rho, \rho^* \) in \( \mathbb{F} \), these scalars satisfy (11), (12) if and only if the following (i)–(v) hold:
the expressions
\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}
\]
are both equal to \( \beta + 1 \) for \( 2 \leq i \leq d - 1 \);

(ii) \( \gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1} \) \( (1 \leq i \leq d - 1) \);

(iii) \( \gamma^* = \theta^*_{i-1} - \beta \theta^*_i + \theta^*_{i+1} \) \( (1 \leq i \leq d - 1) \);

(iv) \( \phi = \theta^2_{i-1} - \beta \theta_{i-1} \theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i) \) \( (1 \leq i \leq d) \);

(v) \( \phi^* = \theta^2_{i-1} - \beta \theta^*_i \theta^*_i + \theta^*_i - \gamma^*(\theta^*_{i-1} + \theta^*_i) \) \( (1 \leq i \leq d) \).

Shortly we will impose a restriction on the eigenvalues and dual eigenvalues of \( \Phi \). In order to motivate this restriction, we consider a certain algebra \( U^+_q \). This algebra is discussed in the next section.

8 The algebra \( U^+_q \) and its alternating elements

From now until the end of Section 10, we fix a nonzero scalar \( q \) in the algebraic closure of \( \mathbb{F} \), such that (i) \( q \) is not a root of unity; (ii) \( q^2 \in \mathbb{F} \). We abbreviate \( b = q^2 \). For elements \( X,Y \) in any algebra, recall the notation

\[
[X, Y] = XY - YX, \quad [X, Y]_b = bXY - YX.
\]

Note that

\[
[X, [X, [X, Y]_b]_{b^{-1}}] = X^3Y - (b + b^{-1} + 1)X^2YX + (b + b^{-1} + 1)XYX^2 - YX^3.
\]

We will refer to the \( q \)-deformed enveloping algebra \( U_q(\hat{\mathfrak{sl}}_2) \); see for example [13].

**Definition 8.1.** (See [27] Corollary 3.2.6.) Define the algebra \( U^+_q \) by generators \( W_0, W_1 \) and relations

\[
[W_0, [W_0, [W_0, W_1]_b]_{b^{-1}}] = 0, \quad [W_1, [W_1, [W_1, W_0]_b]_{b^{-1}}] = 0. \quad (13)
\]

We call \( U^+_q \) the **positive part of** \( U_q(\hat{\mathfrak{sl}}_2) \). The relations \([13]\) are called the **\( q \)-Serre relations**.

We will be discussing automorphisms and antiautomorphisms. For an algebra \( \mathcal{A} \), an **automorphism** of \( \mathcal{A} \) is an algebra isomorphism \( \mathcal{A} \to \mathcal{A} \). The **opposite algebra** \( \mathcal{A}^{\text{opp}} \) consists of the vector space \( \mathcal{A} \) and multiplication map \( \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \ (a,b) \mapsto ba \). An **antiautomorphism** of \( \mathcal{A} \) is an algebra isomorphism \( \mathcal{A} \to \mathcal{A}^{\text{opp}} \).

**Lemma 8.2.** (See [27] Lemma 2.3.) **There exists an automorphism** \( \sigma \) of \( U^+_q \) **that swaps** \( W_0, W_1 \). **There exists an antiautomorphism** \( \dagger \) of \( U^+_q \) **that fixes each of** \( W_0, W_1 \).
Lemma 8.3. For nonzero $\lambda_0, \lambda_1 \in \mathbb{F}$ there exists an automorphism of $U_q^+$ that sends $W_0 \mapsto \lambda_0 W_0$ and $W_1 \mapsto \lambda_1 W_1$.

Proof. The $q$-Serre relations are homogeneous in $W_0$ and $W_1$. $\square$

The alternating elements of $U_q^+$ were introduced in [37]. There are four kinds of alternating elements, denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \{W_{k+1}\}_{k \in \mathbb{N}}, \{G_{k+1}\}_{k \in \mathbb{N}}, \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$ (14)

These elements satisfy many relations; see [37, Proposition 5.7], [37, Proposition 5.10], [37, Proposition 5.11], [37, Proposition 6.3], [37, Proposition 8.1]. Some of these relations are listed below.

Lemma 8.4. (See [37, Proposition 5.7].) For $k \in \mathbb{N}$ the following relations hold in $U_q^+$:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - b^{-1})(\tilde{G}_{k+1} - G_{k+1}),$$

$$[W_0, G_{k+1}]_b = [\tilde{G}_{k+1}, W_0]_b = (b - 1)W_{-k-1},$$

$$[G_{k+1}, W_1]_b = [W_1, \tilde{G}_{k+1}]_b = (b - 1)W_{k+2}.$$ (15)

Lemma 8.5. (See [37, Proposition 5.10].) For $k, \ell \in \mathbb{N}$ the following relations hold in $U_q^+$:

$$[W_{-k}, W_{-\ell}] = 0,$$ $[W_{k+1}, W_{\ell+1}] = 0,$

$$[G_{k+1}, G_{\ell+1}] = 0,$$ $[\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0.$

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Lemma 8.6. (See [37, Proposition 8.1].) For $n \geq 1$ the following relation holds in $U_q^+$:

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} b^{-k} = \sum_{k=0}^{n-1} W_{-k} W_{n-k} b^{-k}.$$ (15)

Lemma 8.7. (See [37, Proposition 8.2].) Using the equations below, the alternating elements in $U_q^+$ are recursively obtained from $W_0, W_1$ in the following order:

$$W_0, W_1, G_1, \tilde{G}_1, W_{-1}, W_2, G_2, \tilde{G}_2, W_{-2}, W_3, \ldots$$

For $n \geq 1$,

$$G_n = \frac{\sum_{k=0}^{n-1} W_{-k} W_{n-k} b^{-k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} b^{-k}}{1 + b^{-n}} + \frac{[W_n, W_0]}{(1 + b^{-n})(1 - b^{-1})},$$ (16)

$$\tilde{G}_n = G_n + \frac{[W_0, G_n]}{1 - b^{-1}},$$ (17)

$$W_{-n} = \frac{[W_0, G_n]}{b - 1},$$ (18)

$$W_{n+1} = \frac{[G_n, W_1]}{b - 1}.$$ (19)
**Lemma 8.8.** (See [37, Lemma 5.3].) Consider the maps \( \sigma, \tilde{\sigma} \) from Lemma 8.2. For \( k \in \mathbb{N} \),

(i) the map \( \sigma \) sends

\[
W_{-k} \mapsto W_{k+1}, \quad W_{k+1} \mapsto W_{-k}, \quad G_k \mapsto \tilde{G}_k, \quad \tilde{G}_k \mapsto G_k;
\]

(ii) the map \( \tilde{\sigma} \) sends

\[
W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad G_k \mapsto \tilde{G}_k, \quad \tilde{G}_k \mapsto G_k.
\]

**Lemma 8.9.** Pick nonzero \( \lambda_0, \lambda_1 \in \mathbb{F} \) and consider the corresponding automorphism of \( U_q^+ \) from Lemma 8.3. For \( k \in \mathbb{N} \) the automorphism sends

\[
W_{-k} \mapsto \lambda_0^{k+1} \lambda_1^k W_{-k}, \quad W_{k+1} \mapsto \lambda_0^k \lambda_1^{k+1} W_{k+1}, \quad G_k \mapsto \lambda_0^k \lambda_1^k G_k, \quad \tilde{G}_k \mapsto \lambda_0^k \lambda_1^k \tilde{G}_k.
\]

**Proof.** Use induction on the ordering of the alternating elements given in Lemma 8.7.

We mention some facts for later use.

**Proposition 8.10.** For \( k \in \mathbb{N} \) the following (i)–(iv) hold in \( U_q^+ \).

(i) \( W_{-k} \) satisfies

\[
[W_0, W_{-k}] = 0, \quad [W_1, [W_1, [W_1, W_{-k}]_{b^{-1}}]] = 0.
\]

(ii) \( W_{k+1} \) satisfies

\[
[W_0, [W_0, [W_0, W_{k+1}]_{b^{-1}}]] = 0, \quad [W_1, W_{k+1}] = 0.
\]

(iii) \( G_{k+1} \) satisfies

\[
[W_0, [W_0, G_{k+1}]] = 0, \quad [[G_{k+1}, W_1], W_1] = 0.
\]

(iv) \( \tilde{G}_{k+1} \) satisfies

\[
[[\tilde{G}_{k+1}, W_0], W_0] = 0, \quad [W_1, [W_1, \tilde{G}_{k+1}]_{b^{-1}}] = 0.
\]

**Proof.** (i) The equation \([W_0, W_{-k}] = 0\) is from Lemma 8.5. Observe that

\[
[W_1, [W_1, [W_1, W_{-k}]_{b^{-1}}]]
= [W_1, [W_1, [W_1, W_{-k}]_{b^{-1}}]]
= (1 - b^{-1})[W_1, [W_1, G_{k+1}]_{b^{-1}}] - (1 - b^{-1})[W_1, [W_1, \tilde{G}_{k+1}]_{b^{-1}}]
= (1 - b^{-1})[W_1, [W_1, G_{k+1}]_{b^{-1}}] - (1 - b^{-1})[W_1, [W_1, G_{k+1}]_{b^{-1}}]
= (1 - b^{-1})[W_1, [W_1, G_{k+1}]_{b^{-1}}] + (b^{-1} - 1)[W_1, [W_1, \tilde{G}_{k+1}]_{b^{-1}}]
= (b^{-1} - 1)[W_1, [G_{k+1}, W_1]_{b^{-1}}] + (b^{-1} - 1)[W_1, [W_1, \tilde{G}_{k+1}]_{b^{-1}}]
= (b^{-1} - 1)(1 - b^{-1})[W_1, W_{k+1}]_{b^{-1}} + (b^{-1} - 1)(b - 1)[W_1, W_{k+2}]_{b^{-1}}
= (b^{-1} - 1)(b - 1)[W_1, W_{k+2}]
= 0.
\]
(ii) Apply the automorphism $\sigma$ to each side of the equations in (i) above. 

(iii) We have

$$[W_0, [W_0, G_{k+1}]_b] = (b-1)[W_0, W_{-k-1}] = 0.$$ 

We also have

$$[[G_{k+1}, W_1]_b, W_1] = (b-1)[W_{k+2}, W_1] = 0.$$ 

(iv) Apply the automorphism $\sigma$ to each side of the equations in (iii) above. \qed

9 TD systems of $q$-Serre type

We return our attention to the nontrivial TD system $\Phi = (A; \{E_i\}^d_{i=0}; A^*; \{E^*_i\}^d_{i=0})$ on $V$. Recall that $\Phi$ has eigenvalue sequence $\{\theta_i\}^d_{i=0}$ and dual eigenvalue sequence $\{\theta^*_i\}^d_{i=0}$.

**Definition 9.1.** (See [22, Definition 2.6].) The TD system $\Phi$ is said to have $q$-Serre type whenever $\theta_i = b\theta_{i-1}$ and $\theta^*_i = b^{-1}\theta^*_{i-1}$ for $1 \leq i \leq d$.

We refer the reader to [1–3, 18–23, 26, 36] for background information on TD systems of $q$-Serre type.

From now until the end of Section 10, we assume that $\Phi$ has $q$-Serre type.

**Lemma 9.2.** The elements $A, A^*$ satisfy the $q$-Serre relations

$$[A, [A, [A, A^*]_b]_{b-1}] = 0, \quad [A^*, [A^*, [A^*, A]_b]_{b-1}] = 0. \quad (20)$$

**Proof.** We invoke Lemma 7.2. The scalars

$$\beta = b + b^{-1}, \quad \gamma = 0, \quad \gamma^* = 0, \quad \varrho = 0, \quad \varrho^* = 0 \quad (21)$$

satisfy the conditions (i)–(v) in Lemma 7.2. Using (21) the tridiagonal relations (11), (12) become the relations (20). \qed

**Corollary 9.3.** The vector space $V$ becomes a $U^+_q$-module on which $W_0 = A$ and $W_1 = A^*$. The $U^+_q$-module $V$ is irreducible.

**Proof.** The first assertion follows from Definition 8.1 and Lemma 9.2. The last assertion follows from condition (iv) in Definition 3.1. \qed

Next we consider how the alternating elements of $U^+_q$ act on the six decompositions of $V$ from Example 4.4. We will use the following result.

**Lemma 9.4.** For $0 \leq i, j \leq d$,

$$b\theta_i = \theta_j \quad \text{iff} \quad i = j - 1 \quad \text{iff} \quad \theta^*_i = b\theta^*_j;$$

$$\theta_i = \theta_j \quad \text{iff} \quad i = j \quad \text{iff} \quad \theta^*_i = \theta^*_j;$$

$$\theta_i = b\theta_j \quad \text{iff} \quad i = j + 1 \quad \text{iff} \quad b\theta^*_i = \theta^*_j.$$
Proof. By Definition \[9.1\] and since \( b \) is not a root of unity.

Lemma 9.5. For \( k \in \mathbb{N} \) and \( 0 \leq i, j \leq d \) the following (i)--(iv) hold on the \( U_q^+ \)-module \( V \).

(i) \( W_{-k} \) satisfies
\[
\begin{align*}
E_iW_{-k}E_j &= 0 \quad \text{if } i \neq j, \\
E_i^*W_{-k}E_j^* &= 0 \quad \text{if } |i - j| > 1.
\end{align*}
\]

(ii) \( W_{k+1} \) satisfies
\[
\begin{align*}
E_iW_{k+1}E_j &= 0 \quad \text{if } |i - j| > 1, \\
E_i^*W_{k+1}E_j^* &= 0 \quad \text{if } i \neq j.
\end{align*}
\]

(iii) \( G_{k+1} \) satisfies
\[
\begin{align*}
E_iG_{k+1}E_j &= 0 \quad \text{if } j - i \notin \{0, 1\}, \\
E_i^*G_{k+1}E_j^* &= 0 \quad \text{if } j - i \notin \{0, 1\}.
\end{align*}
\]

(iv) \( \tilde{G}_{k+1} \) satisfies
\[
\begin{align*}
E_i\tilde{G}_{k+1}E_j &= 0 \quad \text{if } i - j \notin \{0, 1\}, \\
E_i^*\tilde{G}_{k+1}E_j^* &= 0 \quad \text{if } i - j \notin \{0, 1\}.
\end{align*}
\]

Proof. For each equation in Proposition \[8.10\] that involves \( W_0 \), multiply each term on the left by \( E_i \) and on the right by \( E_j \). Simplify the result using \( E_iA = \theta_iE_i \) and \( AE_j = \theta_jE_j \) along with \( A = W_0 \). For each equation in Proposition \[8.10\] that involves \( W_1 \), multiply each term on the left by \( E_i^* \) and on the right by \( E_j^* \). Simplify the result using \( E_i^*A^* = \theta_i^*E_i^* \) and \( A^*E_j^* = \theta_j^*E_j^* \) along with \( A^* = W_1 \). By these comments, the following equations hold on \( V \):
\[
\begin{align*}
E_iW_{-k}E_j & (\theta_i - \theta_j) = 0, \\
E_i^*W_{-k}E_j^* (\theta_i^* - \theta_j^*) (b\theta_i^* - \theta_j^*) (b^{-1}\theta_i^* - \theta_j^*) = 0, \\
E_iW_{k+1}E_j & (\theta_i - \theta_j) (b\theta_i - \theta_j) (b^{-1}\theta_i - \theta_j) = 0, \\
E_i^*W_{k+1}E_j^* & (\theta_i^* - \theta_j^*) = 0, \\
E_iG_{k+1}E_j & (\theta_i - \theta_j) (b\theta_i - \theta_j) = 0, \\
E_i^*G_{k+1}E_j^* & (\theta_i^* - \theta_j^*) (b\theta_i^* - \theta_j^*) = 0, \\
E_i\tilde{G}_{k+1}E_j & (\theta_i - \theta_j) (\theta_i - b\theta_j) = 0, \\
E_i^*\tilde{G}_{k+1}E_j^* & (\theta_i^* - \theta_j^*) (\theta_i^* - b\theta_j^*) = 0.
\end{align*}
\]

Evaluate the above equations using Lemma \[9.4\] to get the result.

The following result is a reformulation of Lemma \[9.5\]

Lemma 9.6. We refer to the \( U_q^+ \)-module \( V \) from Corollary \[9.3\] For \( k \in \mathbb{N} \) and \( 0 \leq j \leq d \) the following (i)--(iv) hold.
(i) $W_{-k}$ satisfies
\[ W_{-k}E_j V \subseteq E_j V, \]
\[ W_{-k}E_j^* V \subseteq E_{j-1}^* V + E_j^* V + E_{j+1}^* V. \]

(ii) $W_{k+1}$ satisfies
\[ W_{k+1}E_j V \subseteq E_{j-1} V + E_j V + E_{j+1} V, \]
\[ W_{k+1}E_j^* V \subseteq E_j^* V. \]

(iii) $G_{k+1}$ satisfies
\[ G_{k+1}E_j V \subseteq E_{j-1} V + E_j V, \]
\[ G_{k+1}E_j^* V \subseteq E_{j-1}^* V + E_j^* V. \]

(iv) $\tilde{G}_{k+1}$ satisfies
\[ \tilde{G}_{k+1}E_j V \subseteq E_j V + E_{j+1} V, \]
\[ \tilde{G}_{k+1}E_j^* V \subseteq E_j^* V. \]

We are using the convention
\[ E_{-1} = 0, \quad E_{d+1} = 0, \quad E_{-1}^* = 0, \quad E_{d+1}^* = 0. \]

Proof. (i) Using Lemma 9.5(i) we obtain
\[ W_{-k}E_j V = IW_{-k}E_j V = \sum_{i=0}^{d} E_i W_{-k}E_j V = E_j W_{-k}E_j V \subseteq E_j V, \]
and also
\[ W_{-k}E_j^* V = IW_{-k}E_j^* V = \sum_{i=0}^{d} E_i^* W_{-k}E_j^* V = \sum_{i=j-1}^{j+1} E_i^* W_{-k}E_j^* V \subseteq E_{j-1}^* V + E_j^* V + E_{j+1}^* V. \]

(ii)–(iv) Similar to the proof of (i) above.

Proposition 9.7. We refer to the $U_q^+$-module $V$ from Corollary 9.3. In the table below, we show how the alternating elements of $U_q^+$ act on the flags from Definition 4.2. For $k \in \mathbb{N}$,

| flag name | $W_{-k}$ action | $W_{k+1}$ action | $G_{k+1}$ action | $\tilde{G}_{k+1}$ action |
|-----------|-----------------|------------------|-----------------|------------------|
| [0]       | stabilize       | raise            | stabilize       | raise            |
| [D]       | stabilize       | raise            | raise           | stabilize        |
| [0*]      | raise           | stabilize        | stabilize       | raise            |
| [D*]      | raise           | stabilize        | raise           | stabilize        |

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Proof. Use Lemmas 6.3–6.6 along with Lemma 9.6.

We now give our first main result.

**Theorem 9.8.** We refer to the $U^+$-module $V$ from Corollary 9.5. Let \( \{ \mathcal{V}_i \}_{i=0}^d \) denote a decomposition of $V$ from Example 4.4. Then for $k \in \mathbb{N}$ and $0 \leq i \leq d$ the actions of $W_{-k}$ and $W_{k+1}$ on $\mathcal{V}_i$ are described in the table below.

| decom. name | action of $W_{-k}$ on $\mathcal{V}_i$ | action of $W_{k+1}$ on $\mathcal{V}_i$ |
|-------------|--------------------------------------|--------------------------------------|
| $[0, D]$    | $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_i$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$ |
| $[0^*, D^*]$| $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_i$ |
| $[0^*, 0]$  | $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$ |
| $[0^*, D]$  | $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$ |
| $[D^*, 0]$  | $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$ |
| $[D^*, D]$  | $W_{-k} \mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$ | $W_{k+1} \mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$ |

Proof. For the decompositions $[0, D]$ and $[0^*, D^*]$ use Lemma 9.6(i),(ii). For the remaining four decompositions, use Lemmas 6.4, 6.5 and Proposition 9.7.

Next we use the tetrahedron diagram to illustrate Theorem 9.8. Pick $k \in \mathbb{N}$. The following picture shows how $W_{-k}$ acts on the decompositions of $V$ from the tetrahedron diagram, for $d = 8$:

![Tetrahedron Diagram for $W_{-k}$ Action](diagram1.png)

The following picture shows how $W_{k+1}$ acts on the decompositions of $V$ from the tetrahedron diagram, for $d = 8$:

![Tetrahedron Diagram for $W_{k+1}$ Action](diagram2.png)
We now give our second main result.

**Theorem 9.9.** We refer to the $U^+_q$-module $V$ from Corollary 9.3. Let $\{V_i\}_{i=0}^d$ denote a decomposition of $V$ from Example 4.4. Then for $k \in \mathbb{N}$ and $0 \leq i \leq d$ the actions of $G_{k+1}$ and $\tilde{G}_{k+1}$ on $V_i$ are described in the table below.

| decomp. name | action of $G_{k+1}$ on $V_i$ | action of $\tilde{G}_{k+1}$ on $V_i$ |
|--------------|-------------------------------|---------------------------------|
| $[0, D]$     | $G_{k+1}V_i \subseteq V_{i-1} + V_i$ | $G_{k+1}V_i \subseteq V_i + V_{i+1}$ |
| $[0^*, D^*]$ | $G_{k+1}V_i \subseteq V_{i-1} + V_i$ | $G_{k+1}V_i \subseteq V_i + V_{i+1}$ |
| $[0^*, 0]$   | $G_{k+1}V_i \subseteq V_i$ | $G_{k+1}V_i \subseteq V_{i-1} + V_i + V_{i+1}$ |
| $[D^*, 0]$   | $G_{k+1}V_i \subseteq V_i + V_{i+1}$ | $G_{k+1}V_i \subseteq V_{i-1} + V_i$ |
| $[D^*, D]$   | $G_{k+1}V_i \subseteq V_{i-1} + V_i + V_{i+1}$ | $G_{k+1}V_i \subseteq V_{i-1} + V_i$ |

*Proof.* For the decompositions $[0, D]$ and $[0^*, D^*]$ use Lemma 9.6(iii),(iv). For the remaining four decompositions, use Lemmas 6.3–6.6 and Proposition 9.7. \[\square\]

Next we use the tetrahedron diagram to illustrate Theorem 9.9.

Pick $k \in \mathbb{N}$. The following picture shows how $G_{k+1}$ acts on the decompositions of $V$ from the tetrahedron diagram, for $d = 8$:

![Diagram showing $G_{k+1}$ action](image1)

The following picture shows how $\tilde{G}_{k+1}$ acts on the decompositions of $V$ from the tetrahedron diagram, for $d = 8$:

![Diagram showing $\tilde{G}_{k+1}$ action](image2)

Next, we seek an attractive basis for $V$. Motivated by Lemma 8.5, we seek:
(i) a basis of common eigenvectors for \( \{W_{-k}\}_{k \in \mathbb{N}} \);

(ii) a basis of common eigenvectors for \( \{W_{k+1}\}_{k \in \mathbb{N}} \);

(iii) a basis of common eigenvectors for \( \{G_{k+1}\}_{k \in \mathbb{N}} \);

(iv) a basis of common eigenvectors for \( \{G_{k+1}\}_{k \in \mathbb{N}} \).

Unfortunately, the above bases might not exist. In Sections 10, 11 we will consider some TD systems of \( q \)-Serre type for which the above bases do exist. The following result is used in Section 11.

**Lemma 9.10.** Assume that \( V \) has a basis of type (i) or (ii). Then \( V \) has a basis of type (iii) and a basis of type (iv).

**Proof.** Replacing \( \Phi \) by \( (\Phi^*)^{-1} \) if necessary, we may assume that \( V \) has a basis of type (i). Let \( 0 \neq v \in V \) denote a common eigenvector for \( \{W_{-k}\}_{k \in \mathbb{N}} \). Note that \( v \) is an eigenvector for \( W_0 = A \), so \( v \in E_jV \) for some \( j \) (\( 0 \leq j \leq d \)). For \( k \in \mathbb{N} \) let \( \omega_k \in F \) denote the \( W_{-k} \) eigenvalue for \( v \). Thus \( \omega_0 = \theta_j \). Recall the decomposition \( \{0^*, 0\} \) of \( V \), from Example 4.4 and Note 4.5. Denote this decomposition by \( \{U_i\}_{i=0}^d \). By Lemma 3.14, \( E_0V + \cdots + E_jV = U_{d-j} + \cdots + U_d \) and \( E_0V + \cdots + E_{j-1}V = U_{d-j+1} + \cdots + U_d \). Define the vector \( u = \Psi v \), where \( \Psi \) is from Lemma 3.15. By construction \( u \in U_{d-j} \) and \( u - v \in U_{d-j+1} + \cdots + U_d \). We claim that for \( k \in \mathbb{N} \), \( u \) is an eigenvector for \( G_k \) with eigenvalue \( \omega_k/\omega_0 \). To prove the claim, let \( k \) be given. We show that \( (G_k - \omega_0^{-1}\omega_{-k})u = 0 \). This is immediate if \( k = 0 \), so assume that \( k \geq 1 \). We have \( 0 = U_{d-j} \cap (U_{d-j+1} + \cdots + U_d) \), so it suffices to show that \( (G_k - \omega_0^{-1}\omega_{-k})u \in U_{d-j} \) and \( (G_k - \omega_0^{-1}\omega_{-k})v \in U_{d-j+1} + \cdots + U_d \). We have \( u \in U_{d-j} \), and \( G_k U_{d-j} \subseteq U_{d-j} \) by Theorem 9.9, so \( (G_k - \omega_0^{-1}\omega_{-k})u \in U_{d-j} \). For notational convenience, define \( L_k = [W_0, G_k]/(b - 1) \). By the relation \( [W_0, G_k]_b = (b - 1)W_{-k} \) from Lemma 8.4

\[
L_k = W_{-k} - W_0 G_k.
\]

By Proposition 8.10(iii) we have \( [W_0, L_k]_b = 0 \). Consequently \( L_k E_jV \subseteq E_{j-1}V \), where \( E_{-1} = 0 \). Note that

\[
(G_k - \omega_0^{-1}\omega_{-k})v = W_0^{-1}(W_0 G_k - W_{-k})v = -W_0^{-1}L_k v \in W_0^{-1}L_k E_jV \subseteq W_0^{-1}E_{j-1}V = E_{j-1}V \subseteq E_0V + \cdots + E_{j-1}V = U_{d-j+1} + \cdots + U_d.
\]

We may now argue

\[
(G_k - \omega_0^{-1}\omega_{-k})u = (G_k - \omega_0^{-1}\omega_{-k})(u - v) + (G_k - \omega_0^{-1}\omega_{-k})v \\
\in (G_k - \omega_0^{-1}\omega_{-k})(U_{d-j+1} + \cdots + U_d) + U_{d-j+1} + \cdots + U_d \\
= U_{d-j+1} + \cdots + U_d.
\]

We have shown that \( (G_k - \omega_0^{-1}\omega_{-k})u = 0 \), and the claim is proved. By assumption, \( V \) has a basis of type (i). We apply \( \Psi \) to this basis, and get a new basis for \( V \). By the claim, the new basis has type (iii). We have shown that \( V \) has a basis of type (iii). By a similar argument, \( V \) has a basis of type (iv). \( \square \)
10 Leonard systems of $q$-Serre type

We continue to discuss the nontrivial TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on $V$ of $q$-Serre type. Recall the $U_q^+$-module $V$ from Corollary 9.3. Throughout this section we assume that $\Phi$ is a Leonard system, so that $E_iV$ and $E_i^*V$ have dimension one for $0 \leq i \leq d$. For aesthetic reasons, we further assume that $q \in \mathbb{F}$ and

$$\theta_i = q^{2i-d}, \quad \theta_i^* = q^{d-2i} \quad (0 \leq i \leq d).$$

Under the above assumptions, we describe how the alternating elements of $U_q^+$ act on $V$.

In [18,23] the TD system $\Phi$ is described from the point of view of the $q$-tetrahedron algebra. We will use some results from these descriptions.

**Definition 10.1.** Define $x_{12} = A$ and $x_{30} = A^*$. Define $x_{01} \in \text{End}(V)$ such that for $0 \leq i \leq d$, the $i^{th}$ component of $[D^*, D]$ is an eigenspace of $x_{01}$ with eigenvalue $q^{d-2i}$. Define $x_{23} \in \text{End}(V)$ such that for $0 \leq i \leq d$, the $i^{th}$ component of $[0^*, 0]$ is an eigenspace of $x_{23}$ with eigenvalue $q^{2i-d}$.

**Lemma 10.2.** (See [23, Theorem 10.4], [18, Definition 4.1].) We have

$$\frac{qx_{01}x_{12} - q^{-1}x_{12}x_{01}}{q - q^{-1}} = I, \quad \frac{q^{-1}x_{12}x_{23} - q^{-1}x_{23}x_{12}}{q - q^{-1}} = I,$$

$$\frac{qx_{23}x_{30} - q^{-1}x_{30}x_{23}}{q - q^{-1}} = I, \quad \frac{qx_{30}x_{01} - q^{-1}x_{01}x_{23}}{q - q^{-1}} = I.$$

**Lemma 10.3.** (See [18, Lemma 9.4, Proposition 9.6].) There exists a unique nonzero $\xi \in \mathbb{F}$ such that

$$\xi(x_{01} - x_{23}) = \frac{[x_{30}, x_{12}]}{q - q^{-1}}, \quad \xi^{-1}(x_{12} - x_{30}) = \frac{[x_{01}, x_{23}]}{q - q^{-1}}.$$

Moreover, $\xi$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

Following [18, Definition 9.10], define

$$\Upsilon = (q^{d+1} + q^{-d-1})I.$$

**Lemma 10.4.** (See [18, Lemma 9.11].) We have

$$\Upsilon = \xi(x_{01}x_{23} - I) + qx_{30} + q^{-1}x_{12}, \quad \Upsilon = \xi^{-1}(x_{12}x_{30} - I) + qx_{01} + q^{-1}x_{23},$$

$$\Upsilon = \xi(x_{23}x_{01} - I) + qx_{12} + q^{-1}x_{30}, \quad \Upsilon = \xi^{-1}(x_{30}x_{12} - I) + qx_{23} + q^{-1}x_{01}.$$

Next we use $x_{01}, x_{12}, x_{23}, x_{30}$ to describe how the alternating elements act on $V$.

**Lemma 10.5.** There exists a sequence $\{r_n\}_{n \in \mathbb{N}}$ of scalars in $\mathbb{F}$ such that $r_0 = 1$ and for $n \in \mathbb{N}$ the following hold on the $U_q^+$-module $V$:

$$W_n = r_n x_{12} - q\xi r_{n-1}I, \quad G_n = r_n I - q\xi r_{n-1}x_{23},$$

$$W_{n+1} = r_n x_{30} - q\xi r_{n-1}I, \quad G_n = r_n I - q\xi r_{n-1}x_{01}.$$

In the above lines $r_{-1} = 0$. 27
Proof. We use induction with respect to the ordering of the alternating elements given in Lemma 10.4. We carry out the induction using (16)–(19) and Lemmas 10.2, 10.3. The details are given below.

\( G_n \) (\( n \geq 1 \)): We evaluate the right-hand side of (16). By induction,

\[
\sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k} = \sum_{k=0}^{n-1} (r_k x_{12} - q \xi r_{k-1} I)(r_{n-k-1} x_{30} - q \xi r_{n-k-2} I) q^{n-1-2k}
\]

\[
= x_{12} x_{30} \sum_{k=0}^{n-1} r_k r_{n-k-1} q^{n-1-2k} - q \xi x_{12} \sum_{k=0}^{n-1} r_k r_{n-k-2} q^{n-1-2k}
\]

\[
- q \xi x_{30} \sum_{k=0}^{n-1} r_{k-1} r_{n-k-1} q^{n-1-2k} + q^2 \xi^2 \sum_{k=0}^{n-1} r_{k-1} r_{n-k-2} q^{n-1-2k}.
\]

By Lemma 10.4

\[
x_{12} x_{30} = (1 + \xi q^{d+1} + \xi q^{-d-1}) I - q \xi x_{01} - q^{-1} \xi x_{23}.
\]

By induction,

\[
\sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} q^{n-2k} = \sum_{k=1}^{n-1} (r_k I - q \xi r_{k-1} x_{23})(r_{n-k} I - q \xi r_{n-k-1} x_{01}) q^{n-2k}
\]

\[
= q^2 \xi^2 x_{23} x_{01} \sum_{k=1}^{n-1} r_{k-1} r_{n-k-1} q^{n-2k} - q \xi x_{23} \sum_{k=1}^{n-1} r_{k-1} r_{n-k} q^{n-2k}
\]

\[
- q \xi x_{01} \sum_{k=1}^{n-1} r_k r_{n-k-1} q^{n-2k} + \sum_{k=1}^{n-1} r_k r_{n-k} q^{n-2k}.
\]

By Lemma 10.4

\[
x_{23} x_{01} = (1 + \xi^{-1} q^{d+1} + \xi^{-1} q^{-d-1}) I - q \xi^{-1} x_{12} - q^{-1} \xi^{-1} x_{30}.
\]

By induction and Lemma 10.3

\[
W_n W_0 - W_0 W_n = [r_{n-1} x_{30} - q \xi r_{n-2} I, x_{12}]
\]

\[
= r_{n-1} [x_{30}, x_{12}]
\]

\[
= (q - q^{-1}) \xi r_{n-1} (x_{01} - x_{23}).
\]

Evaluating the right-hand side of (16) using the above comments, we express \( G_n \) is a linear combination of \( x_{01}, x_{12}, x_{23}, x_{30}, I \). In this linear combination the coefficients of \( x_{01}, x_{12}, x_{23}, x_{30} \) are 0, 0, \(-q \xi r_{n-1}, 0\), respectively. Therefore \( G_n + q \xi r_{n-1} x_{23} \) is a scalar multiple of \( I \). Denoting this scalar by \( r_n \), we have \( G_n = r_n I - q \xi r_{n-1} x_{23} \).
\[ \tilde{G}_n \ (n \geq 1): \text{We evaluate (17). By induction and Lemma 10.3,} \]
\[ \tilde{G}_n = G_n + \frac{[W_0, W_n]}{1 - b^{-1}} \]
\[ = r_n I - q \xi r_{n-1} x_{23} + \frac{[x_{12}, r_{n-1} x_{30} - q \xi r_{n-2} I]}{1 - b^{-1}} \]
\[ = r_n I - q \xi r_{n-1} x_{23} + r_{n-1} \frac{[x_{12}, x_{30}]}{1 - b^{-1}} \]
\[ = r_n I - q \xi r_{n-1} x_{23} + q \xi r_{n-1} (x_{23} - x_{01}) \]
\[ = r_n I - q \xi r_{n-1} x_{01}. \]

\[ W_{-n} \ (n \geq 1): \text{We evaluate (18). By induction and Lemma 10.2,} \]
\[ W_{-n} = \frac{[W_0, G_n]}{b - 1} \]
\[ = \frac{[x_{12}, r_n I - q \xi r_{n-1} x_{23}]}{b - 1} \]
\[ = r_n x_{12} - q \xi r_{n-1} \frac{[x_{12}, x_{23}]}{b - 1} \]
\[ = r_n x_{12} - q \xi r_{n-1} I. \]

\[ W_{n+1} \ (n \geq 1): \text{We evaluate (19). By induction and Lemma 10.2,} \]
\[ W_{n+1} = \frac{[G_n, W_1]}{b - 1} \]
\[ = \frac{[r_n I - q \xi r_{n-1} x_{23}, x_{30}]}{b - 1} \]
\[ = r_n x_{30} - q \xi r_{n-1} \frac{[x_{23}, x_{30}]}{b - 1} \]
\[ = r_n x_{30} - q \xi r_{n-1} I. \]

\[ \square \]

Our next main goal is to compute the scalars \( \{r_n\}_{n \in \mathbb{N}} \) from Lemma 10.5. To compute these scalars, it is convenient to make a change of variables.

**Definition 10.6.** For \( n \in \mathbb{N} \) define
\[ r_n^\vee = \sum_{k=0}^{n} r_k r_{n-k} q^{n-2k}. \]  
(22)

Note that \( r_0^\vee = 1 \).

Next we compute \( \{r_n^\vee\}_{n \in \mathbb{N}} \). This is done in the following recursive manner.
Proposition 10.7. For \( n \geq 1 \) we have
\[
0 = r_n^\vee - q(1 + \xi q^{d+1} + \xi q^{-d})r_{n-1}^\vee + q^2\xi(\xi + q^{d+1} + q^{-d-1})r_{n-2}^\vee - q^3\xi^2 r_{n-3}^\vee,
\]
where \( r_0^\vee = 1 \) and \( r_1^\vee = 0 \) and \( r_2^\vee = 0 \).

Proof. Let \( \zeta_n \) denote the expression on the right in (23). We show that \( \zeta_n = 0 \). Evaluate (15) using Lemma 10.5, and simplify the result using Definition 10.6. This yields
\[
0 = r_n^\vee I - qr_{n-1}^\vee (x_{12}x_{30} + q\xi x_{01} + q^{-1}x_{23}) + q^2\xi r_{n-2}^\vee (\xi x_{23}x_{01} + qx_{12} + q^{-1}x_{30}) - q^3\xi^2 r_{n-3}^\vee I.
\]
By Lemma 10.4,
\[
x_{12}x_{30} + q\xi x_{01} + q^{-1}x_{23} = (1 + \xi q^{d+1} + \xi q^{-d-1})I,
\]
\[
\xi x_{23}x_{01} + qx_{12} + q^{-1}x_{30} = (\xi + q^{d+1} + q^{-d-1})I.
\]
By these comments \( \zeta_n I = 0 \), so \( \zeta_n = 0 \).

Example 10.8. We have
\[
r_0^\vee = 1,
\]
\[
r_1^\vee = q + \xi q^{d+2} + \xi q^{-d},
\]
\[
r_2^\vee = q^2 + \xi^2 q^{2d+4} + \xi^2 q^{-2d} + \xi q^{d+3} + \xi q^{1-d} + \xi^2 q^2.
\]
Next we obtain the scalars \( \{r_n\}_{n \in \mathbb{N}} \) from the scalars \( \{r_n^\vee\}_{n \in \mathbb{N}} \). This is done in the following recursive manner.

Proposition 10.9. For \( n \geq 1 \),
\[
r_n = r_n^\vee - \sum_{k=1}^{n-1} r_k r_{n-k} q^{n-2k}.
\]
Proof. This is a reformulation of (22).

Example 10.10. We have
\[
r_0 = 1,
\]
\[
r_1 = q + \xi q^{d+2} + \xi q^{-d},
\]
\[
r_2 = q^2 + \xi^2 q^{2d+4} + \xi^2 q^{-2d} + \xi q^{d+3} + \xi q^{1-d} + \xi^2 q^2.
\]
Next we describe \( \{r_n\}_{n \in \mathbb{N}} \) and \( \{r_n^\vee\}_{n \in \mathbb{N}} \) using generating functions.

Definition 10.11. Define the generating functions
\[
R(t) = \sum_{n \in \mathbb{N}} r_n t^n, \quad R^\vee(t) = \sum_{n \in \mathbb{N}} r_n^\vee t^n.
\]

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Lemma 10.12. We have

\[ R^\vee(t) = R(qt)R(q^{-1}t). \]

Proof. This is a routine consequence of (22).

The recursion (23) looks as follows in terms of generating functions.

Proposition 10.13. We have

\[ R^\vee(t) = \frac{1}{(1 - qt)(1 - q^{d+2}\xi t)(1 - q^{-d}\xi t)}. \]

Proof. Consider the generating function \( G(t) = \sum_{n \in \mathbb{N}} c_n t^n \) such that \( c_0 = r_0^\vee \) and for \( n \geq 1 \), \( c_n \) is equal to the expression on the right in (23). Using Proposition 10.7 we obtain

\[
1 = G(t) = R^\vee(t) - q(1 + \xi q^{d+1} + \xi q^{-d-1})tR^\vee(t) + q^2\xi(\xi + q^{d+1} + q^{-d-1})t^2R^\vee(t) - q^3\xi^2t^3R^\vee(t)
\]

\[
= (1 - qt)(1 - q^{d+2}\xi t)(1 - q^{-d}\xi t)R^\vee(t).
\]

The result follows.

We now give our third main result.

Theorem 10.14. We have

\[ R(qt)R(q^{-1}t) = \frac{1}{(1 - qt)(1 - q^{d+2}\xi t)(1 - q^{-d}\xi t)}. \]

Proof. By Lemma 10.12 and Proposition 10.13.

Next we impose a mild assumption on \( \mathbb{F} \), and give an explicit formula for \( R(t) \). The following is our fourth main result.

Theorem 10.15. Assume that \( \mathbb{F} \) has characteristic 0. Then

\[ R(t) = \exp\left(\sum_{k=1}^{\infty} \frac{1 + q^{k(d+1)}\xi_k + q^{-k(d+1)}\xi_k}{q^k + q^{-k}} \frac{q^k t^k}{k}\right). \]

Proof. Define the generating function

\[ P(t) = \sum_{k=1}^{\infty} \frac{1 + q^{k(d+1)}\xi_k + q^{-k(d+1)}\xi_k}{q^k + q^{-k}} \frac{q^k t^k}{k}. \]

Note that \( P(t) \) has constant term 0. Define \( E(t) = \exp P(t) \), and note that \( E(t) \) has constant term 1. We will show that \( R(t) = E(t) \). By Lemma 10.12 and the construction, \( R(t) \) is the unique generating function over \( \mathbb{F} \) that has constant term 1 and \( R^\vee(t) = R(qt)R(q^{-1}t) \).
Therefore, $R^\nu(t) = E(qt)E(q^{-1}t)$ implies $R(t) = E(t)$. Recall the natural logarithm $\ln$; see for example [41, Section 2]. We have

$$E(qt)E(q^{-1}t) = \exp \left( P(qt) + P(q^{-1}t) \right)$$

$$= \exp \left( \sum_{k=1}^{\infty} \frac{1 + q^{k(d+1)}\zeta^k + q^{-k(d+1)}\zeta^k}{k} q^k t^k \right)$$

$$= \exp \left( \sum_{k=1}^{\infty} q^{k^2} \right) \exp \left( \sum_{k=1}^{\infty} \frac{q^{k(d+2)}\zeta^k k^2}{k} \right) \exp \left( \sum_{k=1}^{\infty} \frac{q^{-kd\zeta^k k^2}}{k} \right)$$

$$= \exp \left( -\ln(1-qt) \right) \exp \left( -\ln(1-q^{d+2}\xi t) \right) \exp \left( -\ln(1-q^{-d}\xi t) \right)$$

$$= (1-qt)^{-1}(1-q^{d+2}\xi t)^{-1}(1-q^{-d}\xi t)^{-1}$$

and consequently $R(t) = E(t)$.

For the sake of completeness, we mention some matrix representations of the maps in Definition 10.1.

**Remark 10.16.** In [18, Section 10] we defined 24 bases for $V$, and for each basis we gave the matrices that represent $x_{01}$, $x_{12}$, $x_{23}$, $x_{30}$. For one of these bases the representing matrices are shown below.

$$\begin{array}{c|cccc}
\text{map} & x_{01} & x_{12} & x_{23} & x_{30} \\
\text{matrix} & S_{q-1}(\xi^{-1}) & ZE_qZ & K_{q-1} & G_{q^{-1}}(\xi) \\
\end{array}$$

The definitions of $Z$, $K_q$, $E_q$, $G_q(\xi)$, $S_q(\xi)$ can be found in [18, Appendix A].

### 11 Distance-regular graphs

Recall the field $\mathbb{R}$ of real numbers. Throughout this section, assume that $\mathbb{F} = \mathbb{R}$.

In the topic of algebraic graph theory, there is a family of finite undirected graphs, said to be distance-regular [4], [5], [12], [14], [24]. We refer the reader to these works, for background information about the concepts and notation used below.

There is a kind of distance-regular graph, said to have classical parameters $(d, b, \alpha, \beta)$; see [12, Section 6.1]. The parameter $d$ is the diameter of the graph [12, p. 433]. The parameters $b, \alpha, \beta$ are real numbers used to describe the intersection numbers of the graph [12, p. 1]. Throughout this section, we fix a distance-regular graph $\Gamma$ that has diameter $d \geq 3$ and classical parameters $(d, b, \alpha, \beta)$ with $b \neq 1$ and $\alpha = b - 1$. The condition on $\alpha$ implies that $\Gamma$ is formally self-dual in the sense of [12, p. 49]. By [12, Proposition 6.2.1] $b$ is an integer and $b \neq 0, b \neq -1$. Note that $b$ is not a root of unity. Let $X$ denote the vertex set of $\Gamma$. Let $\text{Mat}_X(\mathbb{R})$ denote the algebra of matrices that have rows and columns indexed by $X$ and all entries in $\mathbb{R}$. Let $\mathbb{V} = \mathbb{R}^X$ denote the vector space consisting of the column vectors whose
coordinates are indexed by $X$ and whose entries are in $\mathbb{R}$. Note that $\text{Mat}_X(\mathbb{R})$ acts on $V$ by left multiplication. We endow $V$ with the bilinear form $\langle , \rangle$ that satisfies $\langle u, v \rangle = u^T v$ for $u, v \in V$, where $t$ denotes transpose. Note that $\langle , \rangle$ is symmetric. For $B \in \text{Mat}_X(\mathbb{R})$,

$$\langle Bu, v \rangle = \langle u, B^T v \rangle \quad u, v \in V. \quad (24)$$

Let $\mathbb{A} \in \text{Mat}_X(\mathbb{R})$ denote the adjacency matrix of $\Gamma$ [24, Section 7]. The matrix $\mathbb{A}$ is symmetric, and each entry is 0 or 1. For the rest of this section, fix $x \in X$ and let $\mathbb{A}^* = \mathbb{A}^*(x) \in \text{Mat}_X(\mathbb{R})$ denote the dual adjacency matrix of $\Gamma$ with respect to $x$ [24 Section 7]. The matrix $\mathbb{A}^*$ is diagonal. Let $\mathbb{T} = \mathbb{T}(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by $\mathbb{A}, \mathbb{A}^*$. The algebra $\mathbb{T}$ is called the subconstituent algebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$; see [34 Definition 3.3]. By construction, $\mathbb{T}$ is closed under the transpose map.

We comment on the $\mathbb{T}$-modules. By a $\mathbb{T}$-module, we mean a subspace $V \subseteq V$ such that $\mathbb{T}V \subseteq V$. Let $V$ denote a $\mathbb{T}$-module, and let $V'$ denote a $\mathbb{T}$-module contained in $V$. Then by [16, p. 802], the orthogonal complement of $V'$ in $V$ is a $\mathbb{T}$-module. Consequently, each $\mathbb{T}$-module is an orthogonal direct sum of irreducible $\mathbb{T}$-modules. In particular, the $\mathbb{T}$-module $V$ is an orthogonal direct sum of irreducible $\mathbb{T}$-modules. By [19, Example 1.4] the elements $\mathbb{A}, \mathbb{A}^*$ act on each irreducible $\mathbb{T}$-module as a TD pair.

For convenience, we now adjust $\mathbb{A}$ and $\mathbb{A}^*$. By [12 Corollary 8.4.4], for $\mathbb{A}$ and $\mathbb{A}^*$ the roots of the minimal polynomial have the form

$$rb^{-i} + s \quad (0 \leq i \leq d),$$

where $r, s \in \mathbb{R}$ and $r \neq 0$. Define $\mathbb{A}, \mathbb{A}^* \in \text{Mat}_X(\mathbb{R})$ such that

$$\mathbb{A} = \mathbb{A} + sI, \quad \mathbb{A}^* = \mathbb{A}^* + sI.$$

By construction, for $\mathbb{A}$ and $\mathbb{A}^*$ the roots of the minimal polynomial are $\{rb^{-i}\}_{i=0}^d$. By construction, $\mathbb{A}$ and $\mathbb{A}^*$ are symmetric. By construction, the algebra $\mathbb{T}$ is generated by $\mathbb{A}, \mathbb{A}^*$. By [24 Lemma 9.4], both

$$\mathbb{A}^3 \mathbb{A}^* - (b + b^{-1} + 1)\mathbb{A}^2 \mathbb{A}^* \mathbb{A} + (b + b^{-1} + 1)\mathbb{A} \mathbb{A}^* \mathbb{A}^2 - \mathbb{A}^* \mathbb{A}^3 = 0, \quad A^*^3 A - (b + b^{-1} + 1)A^*^2 AA^* + (b + b^{-1} + 1)A^* AA^2 - AA^*^3 = 0.$$

Thus $\mathbb{A}, \mathbb{A}^*$ satisfy the $q$-Serre relations, where $q$ is a complex number such that $q^2 = b$. The scalar $q$ is nonzero, and not a root of unity. We caution the reader that $q \notin \mathbb{R}$ if $b < -1$.

**Lemma 11.1.** There exists an algebra homomorphism $\sharp : U_q^+ \rightarrow \mathbb{T}$ that sends $W_0 \mapsto \mathbb{A}$ and $W_1 \mapsto \mathbb{A}^*$. The map $\sharp$ is surjective.

**Proof.** The matrices $\mathbb{A}, \mathbb{A}^*$ satisfy the $q$-Serre relations. Moreover $\mathbb{A}, \mathbb{A}^*$ generate $\mathbb{T}$. \qed

In Lemma 8.2 we mentioned an antiautomorphism $\dagger$ of $U_q^+$ that fixes each of $W_0, W_1$.

**Lemma 11.2.** The following diagram commutes:

$$\begin{array}{c}
U_q^+ \xrightarrow{\sharp} \mathbb{T} \\
\dagger \downarrow \quad \quad \quad \quad \quad \downarrow t \\
U_q^+ \xrightarrow{\sharp} \mathbb{T}
\end{array}$$

$t = \text{transpose}$.
Proof. Chase the generators $W_0, W_1$ of $U_q^+$ around the diagram, using the fact that $A^t = A$ and $(A^*)^t = A^*$.

Definition 11.3. Recall the alternating elements of $U_q^+$ from Section 8. For each alternating element, we retain the same notation for its image under $\natural$. These images will be called the alternating elements of $\mathbb{T}$.

We just defined the alternating elements

$$\{W_k\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$$

of $\mathbb{T}$. By construction, these elements have all entries in $\mathbb{R}$. In the next two lemmas we emphasize some additional features of these elements.

Lemma 11.4. For $k, \ell \in \mathbb{N}$ the following relations hold in $\mathbb{T}$:

$$[W_k, W_{k+1}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0,$$

$$[G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

Proof. By Lemma 8.5.

Lemma 11.5. Referring to the alternating elements of $\mathbb{T}$, the following hold for $k \in \mathbb{N}$:

(i) $W_k$ and $W_{k+1}$ are symmetric;

(ii) $G_{k+1}$ and $\tilde{G}_{k+1}$ are the transposes of each other.

Proof. By Lemma 8.8(ii) and Lemma 11.2.

We have seen that the alternating elements $\{W_k\}_{k \in \mathbb{N}}$ of $\mathbb{T}$ are mutually commuting, symmetric, and have all entries in $\mathbb{R}$. Therefore the alternating elements $\{W_k\}_{k \in \mathbb{N}}$ of $\mathbb{T}$ can be simultaneously diagonalized. Similar comments apply to the alternating elements $\{W_{k+1}\}_{k \in \mathbb{N}}$ of $\mathbb{T}$.

We now give our fifth main result.

Theorem 11.6. Each irreducible $\mathbb{T}$-module is an orthogonal direct sum of its common eigenspaces for $\{W_k\}_{k \in \mathbb{N}}$, and an orthogonal direct sum of its common eigenspaces for $\{W_{k+1}\}_{k \in \mathbb{N}}$.

Proof. By the comments above the theorem statement, and since the eigenspaces of a real symmetric matrix are mutually orthogonal.

We now give our sixth main result.

Theorem 11.7. Let $V$ denote an irreducible $\mathbb{T}$-module. Then $V$ is a direct sum of its common eigenspaces for $\{G_{k+1}\}_{k \in \mathbb{N}}$, and a direct sum of its common eigenspaces for $\{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$. For both of these direct sums, each summand is nonorthogonal to a unique summand in the other direct sum. Let $W$ and $\tilde{W}$ denote nonorthogonal summands in the first direct sum and second direct sum, respectively. Then for $k \in \mathbb{N}$ the eigenvalue of $G_{k+1}$ for $W$ is equal to the eigenvalue of $\tilde{G}_{k+1}$ for $\tilde{W}$.
Proof. The two direct sums exist by Lemma 9.10 and Theorem 11.6. The other assertions follow from (24) and Lemma 11.5(ii).

**Note 11.8.** (See [24, Example 8.4].) The following distance-regular graphs have classical parameters \((d, b, \alpha, \beta)\) with \(b \neq 1\) and \(\alpha = b - 1\):

(i) the bilinear forms graph [12, p. 280];
(ii) the alternating forms graph [12, p. 282];
(iii) the Hermitean forms graph [12, p. 285];
(iv) the quadratic forms graph [12, p. 290],
(v) the affine \(E_6\) graph [12, p. 340];
(vi) the extended ternary Golay code graph [12, p. 359].

### 12 Directions for further research

In this section we list some conjectures and open problems.

**Problem 12.1.** Referring to the distance-regular graph \(\Gamma\) and the subconstituent algebra \(T\) from Section 11, let \(V\) denote an irreducible \(T\)-module. Consider four bases for \(V\), obtained by taking a common eigenbasis for \(\{W_k\}_{k \in \mathbb{N}}, \{W_{k+1}\}_{k \in \mathbb{N}}, \{G_{k+1}\}_{k \in \mathbb{N}}, \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}\), respectively. Normalize the four bases in an attractive fashion. For all pairs of bases among the four, find the transition matrices. For each of the four bases, find the matrices that represent the alternating elements of \(T\).

**Conjecture 12.2.** Referring to the distance-regular graph \(\Gamma\) and the subconstituent algebra \(T\) from Section 11, for every irreducible \(T\)-module the common eigenspaces for \(\{W_k\}_{k \in \mathbb{N}}\) or \(\{W_{k+1}\}_{k \in \mathbb{N}}\) or \(\{G_{k+1}\}_{k \in \mathbb{N}}\) or \(\{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}\) all have dimension one.

**Problem 12.3.** We refer to the nontrivial TD system \(\Phi\) on \(V\) that has \(q\)-Serre type, as in Section 9. In [21] we used \(\Phi\) to turn \(V\) into a \(U_q(\hat{sl}_2)\)-module. We did this in two ways; see [21, Theorems 13.1,13.2]. For the \(U_q(\hat{sl}_2)\)-module \(V\) in [21, Theorems 13.1], the weight space decomposition is \([0^*, D]\). For the \(U_q(\hat{sl}_2)\)-module \(V\) in [21, Theorem 13.2], the weight space decomposition is \([D^*, 0]\). Assume that \(F\) is algebraically closed and characteristic zero. By [13], the \(U_q(\hat{sl}_2)\)-module \(V\) is a tensor product of evaluation modules. An evaluation module is a direct sum of its weight spaces, and these weight spaces all have dimension one. Thus the \(U_q(\hat{sl}_2)\)-module \(V\) becomes a direct sum, with each summand a tensor product of evaluation module weight spaces. These summands have dimension one. Investigate how these summands are related to the common eigenspaces for \(\{W_k\}_{k \in \mathbb{N}}\) or \(\{W_{k+1}\}_{k \in \mathbb{N}}\) or \(\{G_{k+1}\}_{k \in \mathbb{N}}\) or \(\{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}\). The article [7, Section 4.2] might be useful in this direction.

**Problem 12.4.** Our main results are about TD pairs of \(q\)-Serre type. Find analogous results for general TD pairs.
13 Acknowledgement

The author thanks Pascal Baseilhac for many discussions about $U_q^+$ and its alternating elements.

14 Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

15 Declarations of competing interest

No conflicts of interest.

References

[1] H. Alnajjar and B. Curtin. A family of tridiagonal pairs. *Linear Algebra Appl.* 390 (2004) 369–384.

[2] H. Alnajjar and B. Curtin. A family of tridiagonal pairs related to the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. *Electron. J. Linear Algebra* 13 (2005) 1–9.

[3] H. Alnajjar and B. Curtin. A bilinear form for tridiagonal pairs of $q$-Serre type. *Linear Algebra Appl.* 428 (2008) 2688–2698.

[4] E. Bannai and T. Ito. *Algebraic Combinatorics, I. Association schemes*. Benjamin/Cummings, Menlo Park, CA, 1984.

[5] E. Bannai, E. Bannai, T. Ito, R. Tanaka. *Algebraic Combinatorics*. De Gruyter Series in Discrete Math and Applications 5. De Gruyter, 2021. https://doi.org/10.1515/9783110630251

[6] P. Baseilhac. Deformed Dolan-Grady relations in quantum integrable models. *Nuclear Phys. B* 709 (2005) 491–521; arXiv:hep-th/0404149.

[7] P. Baseilhac. The alternating presentation of $U_q(\hat{gl}_2)$ from Freidel-Maillet algebras. *Nuclear Phys. B*, 967 (2021) Paper No. 115400, 48; arXiv:2011.01572 DOI = 10.1016/j.nuclphysb.2021.115400.

[8] P. Baseilhac. On the second realization for the positive part of $U_q(\hat{\mathfrak{sl}}_2)$ of equitable type. *Lett. Math. Phys.* 112 (2022) Paper No. 2, 28; arXiv:2106.11706. DOI = 10.1007/s11005-021-01502-1.

[9] P. Baseilhac and K. Koizumi. A new (in)finite dimensional algebra for quantum integrable models. *Nuclear Phys. B* 720 (2005) 325–347; arXiv:math-ph/0503036
[10] P. Baseilhac and K. Shigechi. A new current algebra and the reflection equation. *Lett. Math. Phys.* 92 (2010) 47–65; [arXiv:0906.1482v2](http://arxiv.org/abs/0906.1482v2).

[11] J. Beck. Braid group action and quantum affine algebras. *Commun. Math. Phys.* (1994) 555–568; [arXiv:hep-th/9404165](http://arxiv.org/abs/hep-th/9404165).

[12] A. E. Brouwer, A. Cohen, A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.

[13] V. Chari and A. Pressley. Quantum affine algebras. *Commun. Math. Phys.* 142 (1991) 261–283.

[14] E. R. van Dam, J. H. Koolen, H. Tanaka. Distance-regular graphs. *Electron. J. Combin.* (2016) DS22; [arXiv:1410.6294](http://arxiv.org/abs/1410.6294).

[15] I. Damiani. A basis of type Poincare-Birkoff-Witt for the quantum algebra of $\hat{\mathfrak{sl}}_2$. *J. Algebra* 161 (1993) 291–310.

[16] J. Go and P. Terwilliger. Tight distance-regular graphs and the subconstituent algebra. *European J. Combin.* 23 (2002) 793–816.

[17] T. Ito, K. Nomura, P. Terwilliger. A classification of sharp tridiagonal pairs. *Linear Algebra Appl.* 435 (2011) 1857–1884; [arXiv:1001.1812](http://arxiv.org/abs/1001.1812).

[18] T. Ito, H. Rosengren, P. Terwilliger. Evaluation modules for the $q$-tetrahedron algebra. *Linear Algebra Appl.* 451 (2014) 107–168; [arXiv:1308.3480](http://arxiv.org/abs/1308.3480).

[19] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to $P$- and $Q$-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; [arXiv:math.CO/0406556](http://arxiv.org/abs/math.CO/0406556).

[20] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. *J. Pure Appl. Algebra* 188 (2004) 145–160; [arXiv:math.QA/0304244](http://arxiv.org/abs/math.QA/0304244).

[21] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. *Ramanujan J.* 13 (2007) 39–62; [arXiv:math/0310042](http://arxiv.org/abs/math/0310042).

[22] T. Ito and P. Terwilliger. Two non-nilpotent linear transformations that satisfy the cubic $q$-Serre relations. *J. Algebra Appl.* 6 (2007) 477–503; [arXiv:math/0508398](http://arxiv.org/abs/math/0508398).

[23] T. Ito and P. Terwilliger. The $q$-tetrahedron algebra and its finite-dimensional irreducible modules. *Comm. Algebra* 35 (2007) 3415–3439; [arXiv:math/0602199](http://arxiv.org/abs/math/0602199).

[24] T. Ito and P. Terwilliger. Distance-regular graphs and the $q$-tetrahedron algebra. *European J. Combin.* 30 (2009) 682–697; [arXiv:math.CO/0608694](http://arxiv.org/abs/math.CO/0608694).

[25] T. Ito and P. Terwilliger. The augmented tridiagonal algebra. *Kyushu J. Math.* 64 (2010) 8–144; [arXiv:0904.2889](http://arxiv.org/abs/0904.2889).
[26] A. Karan. Tridiagonal pairs of \(q\)-Serre type and their linear perturbations. *J. Algebra* 606 (2022) 74–763; [arXiv:2107.01430](https://arxiv.org/abs/2107.01430).

[27] G. Lusztig. *Introduction to quantum groups*. Progress in Mathematics, 110. Birkhauser, Boston, 1993.

[28] K. Nomura and P. Terwilliger. Sharp tridiagonal pairs. *Linear Algebra Appl.* 429 (2008) 79–99; [arXiv:0712.3665](https://arxiv.org/abs/0712.3665).

[29] K. Nomura and P. Terwilliger. Towards a classification of the tridiagonal pairs. *Linear Algebra Appl.* 429 (2008) 508–518; [arXiv:0801.0621](https://arxiv.org/abs/0801.0621).

[30] K. Nomura and P. Terwilliger. Totally bipartite tridiagonal pairs. *Electron. J. Linear Algebra* 37 (2021) 434–491; [arXiv:1711.00332](https://arxiv.org/abs/1711.00332).

[31] M. Rosso. Groupes quantiques et algèbres de battage quantiques. *C. R. Acad. Sci. Paris* 320 (1995) 145–148.

[32] M. Rosso. Quantum groups and quantum shuffles. *Invent. Math* 133 (1998) 399–416.

[33] C. Ruan. A generating function associated with the alternating elements in the positive part of \(U_q(\widehat{sl}_2)\). Preprint; [arXiv:2204.10223](https://arxiv.org/abs/2204.10223).

[34] P. Terwilliger. The subconstituent algebra of an association scheme I. *J. Algebraic Combin.* 1 (1992), 363–388.

[35] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* 330 (2001) 149–203; [arXiv:math.RA/0406555](https://arxiv.org/abs/math.RA/0406555).

[36] P. Terwilliger. Two relations that generalize the \(q\)-Serre relations and the Dolan-Grady relations. In *Physics and Combinatorics 1999 (Nagoya)*, 377–398, World Scientific Publishing, River Edge, NJ, 2001; [arXiv:math.QA/0307016](https://arxiv.org/abs/math.QA/0307016).

[37] P. Terwilliger. The alternating PBW basis for the positive part of \(U_q(\widehat{sl}_2)\). *J. Math. Phys.* 60 (2019) 071704; [arXiv:1902.00721](https://arxiv.org/abs/1902.00721).

[38] P. Terwilliger. The alternating central extension for the positive part of \(U_q(\widehat{sl}_2)\). *Nuclear Phys. B* 947 (2019) 114729; [arXiv:1907.09872](https://arxiv.org/abs/1907.09872).

[39] P. Terwilliger. The compact presentation for the alternating central extension of the positive part of \(U_q(\widehat{sl}_2)\). *Ars Math. Contemp.* (2022); [arXiv:2011.02463](https://arxiv.org/abs/2011.02463). https://doi.org/10.26493/1855-3974.2669.58c.

[40] P. Terwilliger. The algebra \(U_q^+\) and its alternating central extension \(\tilde{U}_q^+\). Preprint; [arXiv:2106.14884](https://arxiv.org/abs/2106.14884).

[41] P. Terwilliger. Using Catalan words and a \(q\)-shuffle algebra to describe the Beck PBW basis for the positive part of \(U_q(\widehat{sl}_2)\). *J. Algebra* 604 (2022) 162–184; [arXiv:2112.06085](https://arxiv.org/abs/2112.06085).
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