Flux penetration in slab shaped Type–I superconductors

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We study the problem of flux penetration into type–I superconductors with a high demagnetization factor (slab geometry). Assuming that the interface between the normal and superconducting regions is sharp, that flux diffuses rapidly in the normal regions, and that thermal effects are negligible, we analyze the process by which flux invades the sample as the applied field is increased slowly from zero. We find that flux does not penetrate gradually. Rather there is an instability in the process and the flux penetrates from the boundary in a series of bursts, accompanied by the formation of isolated droplets of the normal phase, leading to a multiply connected flux domain structure similar to that seen in experiments.

When a type–I superconductor is placed in a magnetic field less than the bulk upper critical field it exhibits a phase with interpenetrating domains of the normal and superconducting state called the intermediate state. Despite over fifty years of work, a complete description of the physics of this phenomenon has been elusive. The earliest theoretical work on this problem goes back to Landau [1] who studied the problem of the equilibrium configuration of an infinite superconducting slab of thickness $d$ placed in an applied magnetic field $B_{app}$. When $d$ is much larger than the penetration depth $\lambda$ and the coherence length $\xi$, he showed that the Meissner state is unstable to a configuration of alternating superconducting and normal laminae. Other configurations of flux domains, various sample geometries, etc. have been studied both experimentally and theoretically [2], [3](and references therein).

It has been clear since the work of Landau that these structures arise from a competition between the magnetic field energy, the condensation energy of the superconducting regions, and the surface energy of the interface between the normal and superconducting regions. However, the striking fact about most of the experiments is that the regular structures envisioned by Landau are rarely seen. Instead one sees complicated patterns which are strongly dependent on the temperature, disorder and field history of the sample.

In the last few years considerable progress has been made in understanding a related problem: flux penetration into (or expulsion from) long cylindrical samples oriented parallel to the magnetic field [4]. Although interesting flux patterns are seen in these samples as transients in going from a normal to a superconducting state, a steady–state intermediate state, which arises due to demagnetization effects, is not seen. Very recently, Goldstein, Jackson and Dorsey (hereafter referred to as GJD) [5] attempted to analyze the influence of demagnetization on the formation of flux structures for slab–like type–I superconductors. They assumed that transient currents in the normal domains decay very fast (which amounts to the normal state conductivity being zero), and argued that one can then think of the dynamics as a simple gradient descent of the free–energy. When the coherence length $\xi$ is small, the interface between normal and superconducting regions is approximately sharp. The free–energy for a sample placed in an applied magnetic field $B_{app}$ can then be written as

$$F(B_{app}, \Delta) = F_B + F_c + F_s. \quad (1)$$

Here the first term denotes the magnetic free energy, the second term the condensation energy which is $-H_c^2/8\pi$ per unit volume in the superconducting regions, and the third the interfacial energy which is $H_c^2 A \Delta/8\pi$. $\Delta$ is the surface energy parameter which is of the order of the coherence length $\xi$, $H_c$ is the bulk upper critical field and $A$ is the area of the interfaces between the normal and superconducting regions.

GJD [5] assumed that (in addition to a bulk term) the magnetic part of this energy could be written as a long–range interaction between current loops localized on the interfaces between the normal and superconducting regions. This made the problem very similar to the problem of the dynamics of two–dimensional ferrofluid droplets in a magnetic field [6]. In the ferrofluid case it is known [7] that regular shapes evolve continuously into labyrinthine patterns when the applied field is increased adiabatically. In the case of superconductors, GJD showed that a circular flux droplet in a sea of superconducting material (with an area much larger than the equilibrium area) changes into a many armed structure with three–fold coordinated nodes. Such convoluted structures are indeed seen in some experiments [7]. They also calculated the equilibrium periodicity for a laminar structure and got results that are numerically close to Landau’s. While this would suggest that their free–energy captures the physics of this situation, GJD [5] were unable to derive it from the more basic Ginzburg–Landau description.
As shown below, a careful analysis of the magnetic free energy yields an interaction between superconducting domains that cannot be expressed in terms of current carrying loops (at least for low fields), and results in qualitatively different behavior. (Late stages of flux invasion, when the superconducting domains are tall and thin, are discussed later.) It is not difficult to see why superconductors and ferrofluids are so different. When a thin layer of a ferrofluid of thickness $d$ and a characteristic transverse linear extent of $L$ is placed in an applied magnetic field $B_{app}$, to leading order in $d/L$ the field inside (and outside) the ferrofluid is equal to $B_{app}$. This uniform field induces a uniform magnetization within the sample, which (using the equation \[ \nabla \times B = 4\pi j_{ext} + 4\pi \nabla \times M \]) gives rise to a small fringing field near the sample edges. (The fringing field in turn induces a higher order non-uniform magnetization.) To leading order in $d/L$ the induced currents consist of a ribbon flowing around the boundary of the ferrofluid, causing a long-range interaction between different parts of the boundary. On the other hand, for a thin superconductor in the intermediate phase, $B$ is not even approximately equal to $B_{app}$ near the superconducting regions. Outside the sample, just above or below a superconducting domain, $B$ is parallel to the surface, while inside the domain $B$ is zero.

In addition to ribbons of current along the side walls of the superconducting regions, there are also large current sheets on the top and bottom surfaces, dominating the inter-domain interaction. Thus while the basic GJD idea of long-range interactions destabilizing regular patterns is correct, the actual description of the experimental patterns is more complicated. The numerics we report in this paper show flux invading in bursts, pinching off from the boundaries to form droplets. This is qualitatively different from the continuous evolution of GJD.

![FIG. 1. B-field for a slab shaped (a) superconducting domain (b) ferrofluid in a vertical applied field. The distortion of the field is small only for (b).](image)

We now consider $\mathcal{F}_B$ in detail. For an arbitrary sample placed in an applied magnetic field $B_{app}$, the magnetic free-energy is

\[
\mathcal{F}_B = \frac{1}{8\pi} \int d^3x (B^2 - 2\vec{B} \cdot \vec{H})
\]

Here $B$ is the magnetic induction, $\vec{H}$ is the magnetic field and the integral is over all of three dimensional space. It is convenient to cast this equation in a slightly different form: since $\nabla \times \vec{H} = \nabla \times \vec{B}_{app}$, and $B$ is a transverse field, $\vec{H}$ can be replaced by $\vec{B}_{app}$ above. Adding the $B$-independent term $B^{2}_{app}/8\pi$ to the free energy density yields

\[
\mathcal{F}_B = \frac{1}{8\pi} \int d^3x (B - B_{app})^2
\]

Thus in order to evaluate $\mathcal{F}_B$, one has to find the magnetic induction $B$ for a given applied field $B_{app}$. For the superconducting regions, where $B = 0$, the contribution to $\mathcal{F}_B$ is simple. The non-trivial part of the calculation consists of determining the magnetic field in the normal domains and in the space outside. Outside the superconducting regions, one can define a magnetic scalar potential $\phi$ by $B - B_{app} = \nabla \phi$, satisfying $\nabla^2 \phi = 0$. (We assume that the applied magnetic field is not changed rapidly, so that transient currents in the normal regions can be ignored.) Given any configuration of superconducting and normal regions we then have to solve Laplace’s equation outside the superconducting regions. Since the normal component of $B$ is zero at the boundaries, for a flat superconducting slab in a vertical field $B_{app}$ pointing upwards the boundary conditions are (i) $\partial_n \phi = \pm B_{app}$ on the top and bottom surfaces of the superconducting regions and (ii) $\partial_i \phi = 0$ on the interfaces between the normal and superconducting regions, where $\partial_n$ is the normal derivative. (We have assumed that the interfaces are vertical, ignoring fanning out of flux domains at the top and bottom surfaces.)

At this stage, we use the quasi two-dimensional nature of the problem to reduce its computational complexity. For a thin sample, most of $\mathcal{F}_B$, $\frac{1}{8\pi} \int d^3x (\nabla \phi)^2$, is stored outside the sample (and in the normal regions). Integrating by parts transforms this expression into $\frac{1}{8\pi} \int_\Sigma d\phi \nabla \cdot \phi$, where the integral runs over the top and bottom surfaces of the superconducting regions. (On the side walls, the normal derivative of $\phi$ is zero.) In evaluating the surface integral, for a thin sample the top and bottom surfaces of the superconducting regions can be treated as approximately coincident, lying in the $z = 0$ plane. By symmetry, $\phi(z = 0) = 0$, except in the superconducting regions where $\phi(z = 0^+) = -\phi(z = 0^-) \neq 0$. Using the boundary condition $\partial_z \phi(z = 0^+) = -B_{app}$ for the superconducting regions yields

\[
\mathcal{F}_B = \frac{B_{app}}{4\pi} \int d\vec{r} \phi(z = 0^+, \vec{r})
\]

where $\vec{r}$ is a two-dimensional vector. The integral is performed only over the superconducting regions. To obtain $\phi(z = 0^+)$ we solve the inverse problem: if $\phi(z = 0^+)$ were known, $\partial_z \phi(z = 0^+)$ could be found by solving Laplace’s equation with Dirichlet boundary conditions. Using $\partial_z \phi(z = 0^+) = -B_{app}$ yields the condition
The integral on the right hand side is over the superconducting regions. $K$ is given by
\[
K(\vec{r} - \vec{r'}) = \lim_{z \to 0} \frac{1}{2\pi} \frac{2z^2 - |\vec{r} - \vec{r'}|^2}{|\vec{r} - \vec{r'}|^2 + z^2}^{3/2}
\]
whose (two dimensional) Fourier transform is given by $|k|/2\pi$. Since $K$ is convoluted with $\phi$, which is a smooth function, it can be replaced by
\[
K(\vec{r} - \vec{r'}) = -1/[2\pi|\vec{r} - \vec{r'}|^3] \quad |\vec{r} - \vec{r'}| > \epsilon
\]
with a compensating $\delta$–function of strength $1/\epsilon$ at the origin, in the limit $\epsilon \to 0$. (For the numerics presented in this paper, the cutoff $\epsilon$ is effectively the lattice size.)

For an arbitrary pattern of superconducting and normal regions, one can only evaluate $F_B$ numerically using the prescription above. It is not possible to reduce the expression to integrals over domain boundaries. (Note that it is not possible to solve Eq.(4) trivially by Fourier transforming, since the right hand side is only integrated over the superconducting regions.) The special case of a single circular superconducting region can be solved analytically [1], and can be used as a check of the numerics.

We now discuss the numerics. Since the experimental patterns depend a great deal on the field history of the sample, we concentrate on the following question: if one slowly increases the field from zero, how does flux penetrate the sample? We choose a sample with a thickness $d \simeq 12\Delta$ and linear dimension $L = 10d$ (as stated before $\Delta$ is essentially the coherence length $\xi$). In experiments the flux domains branch near the surfaces for $d > O(800\Delta)$, and type–II behavior is seen for $d < O(\Delta)$. Our choice of $d$ avoids both these regimes. Because $L >> d$, we can use the two dimensional formulation of the magnetic boundary value problem to find $F_B$. A typical value for type–I superconductors, $\Delta \simeq 1500\Delta$, corresponds to $L \simeq 15\mu m$. Although this is much smaller than experimental sample sizes, the qualitative aspects of our results should apply to larger samples as well.

We divide the sample into lattices of various sizes; for a $61 \times 61$ lattice the lattice constant is approximately $2.4\Delta$. The plaquettes in the lattice are either superconducting or normal. Apart from the isolated normal droplets, the superconducting–normal boundary consists of an outer interface close to the sample edges. There are two basic moves in the numerics: changing any one superconducting plaquette on the outer interface to a normal plaquette, or vice versa. (The dynamics of the droplets are different, due to flux conservation, and are discussed later.) Both possibilities need to be allowed for in order allow for droplet formation. A plaquette is flipped if there is a force that favors the move; the force is the gradient of the free energy, which is given by the sum in Eq.(4). The surface tension force (from $F_s$) is calculated by smoothing the lattice interface and computing the local curvature.

Trying single plaquette moves requires a special treatment of surface tension in the dynamics, since flipping a single plaquette corresponds to a small sharp protrusion of the interface, with roughly the same forward and lateral extent. For a small lattice constant such a move will always cost a large surface energy compared to the energy gained from the magnetic and condensation terms. Thus in the lattice dynamics whenever the force on a segment of the interface favors flipping a plaquette, it is likely that at the next time step it will be favorable to revert the plaquette to its original state. This is because in a lattice approximation the interface is forced to make larger excursions than it would like to; it would be better for a segment of the interface to move forward by only a fraction of a plaquette, and then let neighboring segments catch up. We use a simple prescription to cure this lattice effect: when a plaquette is flipped, it is not allowed to flip back at the very next time step, although it can flip back thereafter. This should give rise to errors in the pattern only of the order of a lattice constant.

We start with $B_{app} = 0$, when the whole sample is superconducting. We raise the field till one plaquette on the boundary becomes normal. Flipping this plaquette can make it favorable to flip other plaquettes, in which case we let the system evolve till it reaches a stationary state. At this point the field is raised again. This process simulates the adiabatic increase of the magnetic field that we wish to study. Isolated droplets have to be handled differently, since the flux in them is conserved. In our simulations this constraint is obeyed approximately: once a droplet is formed, we move it rigidly in the direction of the force on it till the force is zero. At this stage we adjust the number of plaquettes in the droplet so that the flux in it is as close as possible to its original value.

We now discuss our results. When the field is raised just above the lower critical field, one plaquette becomes normal. If the penetration of flux were gradual, one would expect to have to raise the field further for more flux to enter. Instead we found that at a field just slightly above the lower critical field, the flux penetrates a distance into the sample of the order of twenty times the coherence length before the first droplet pinches off. Increasing the field further produces similar behaviour: much of the evolution is in the form of bursts of magnetic flux penetrating from the boundary which then pinch off to form droplets. This reflects an instability in the process of flux penetration and is the main result of our work. In Fig. 2 we show the patterns seen for $L = 61$ at a field only moderately above the field of first flux penetration. We saw similar patterns at comparable fields for the other lattice sizes. The droplets form near the boundary of the sample and then move towards the center of the sample, leaving a region near the boundary flux–free,
similar to what is seen in experiments. We also saw that the droplets typically shrink when one puts in flux conservation, though this observation may not have much experimental significance in view of the fact that we deal with the collective motion of the droplet approximately.

The formation of droplets is promoted by the discontinuous nature of the flux invasion, which causes relatively large amounts of flux to enter the system at low magnetic fields. A large amount of flux penetration reduces the magnetic forces which tend to drive further flux into the system. It is then possible for a normal region to find it favorable to revert to being superconducting, which is how droplets are formed. At later stages when the applied field is higher such reversion to a superconducting state becomes less likely. Further flux coming in from the boundary would then probably coalesce with the already existing droplets, leading to the labyrinthine patterns seen in some experiments, reminiscent of the patterns obtained by GJD. However, since the superconducting regions become thinner at higher fields, it is not clear whether quasi two-dimensional descriptions are valid here.

Although the existence of droplets is fairly ubiquitous in the experiments, the actual shape of the droplets varies rather widely, from compact droplets in experiments on mercury in the early stages of flux penetration to long laminar structures in experiments on lead. In order to explore this further, one would need to satisfy the constant flux constraint accurately, as well as construct a continuum description (presumably analytical) of the dynamics. A continuum analysis would also be necessary to treat surface tension exactly.

To conclude, we have developed a description of the problem of flux penetration into slab shaped Type-I superconductors based on the sharp interface approximation. Numerical simulations on a lattice show that as the applied magnetic field is increased, flux penetrates in bursts, forming droplets, leading to isolated normal regions. While the multiply connected nature of the patterns has been emphasized in the literature, the instability that we have noticed does not seem to have been reported so far. This instability should be apparent in real time imaging of the process of flux penetration.

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