Calculation of Coefficients of Simplest Normal Forms of Neimark-Sacker and Generalized Neimark-Sacker Bifurcations

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Abstract. The coefficients of the simplest normal forms of both high-dimensional Neimark-Sacker and generalized Neimark-Sacker bifurcation systems were discussed. On the basis of the simplest normal form theory, using appropriate nonlinear transformations, direct computation and the second complete induction, the central manifold equations were further reduced to the simplest normal forms which only contained two nonlinear terms. Theorems were established for the explicit expression of the simplest normal forms in terms of the coefficients of the conventional normal forms of Neimark-Sacker and generalized Neimark-Sacker bifurcations systems.

1. Introduction
Consider a parameter family of discrete dynamical system
\[ x_{n+1} = F_{\mu}(x_n), \quad x_n, x_{n+1} \in \mathbb{R}^n, \mu \in \mathbb{R} \] (1)
where \( F_{\mu} : \mathbb{R}^n \to \mathbb{R}^n, \ F_{\mu} \in C^{2p+4}, (p \geq 1), F_{\mu}(0) = 0, \ A(\mu) = D_x F_{\mu}(0). \) \( A(0) \) has a pair of complex conjugate eigenvalues, \( \lambda_0, \bar{\lambda}_0, \) with \( |\lambda_0| = |\bar{\lambda}_0| = 1, \) and the remaining \( n-2 \) eigenvalue, with \( |\lambda_i| < 1 (i = 1,2,\cdots,n-2). \) Let \( \lambda(\mu) \) is the eigenvalue of \( A(\mu) \) such that \( \lambda(0) = \lambda_0, \)
\( \lambda^n(0) \neq 1, n = 1,2,\cdots,2p+3, \) and \( \frac{d}{d\mu} |\lambda(\mu)| \bigg|_{\mu=0} \neq 0. \)

As we all know, system (1) is a discrete case of Neimark-Sacker bifurcation system. With the aid of the center manifold theory, system (1) is reduced to a two-dimensional discrete dynamical system
\[ x_{n+1} = f_{\mu}(x_n), \quad x_n, x_{n+1} \in \mathbb{R}^2, \mu \in \mathbb{R} \] (2)
where \( f_{\mu} : \mathbb{R}^2 \to \mathbb{R}^2, f_{\mu} \in C^{2p+4}, f_{\mu}(0) = 0, \ D_x f_{\mu}(0) \) has a pair of complex conjugate eigenvalues \( \lambda(\mu), \bar{\lambda}(\mu). \) Since our attention in this paper is focused on the study of normal forms rather than the bifurcation analysis, we will concentrate on the normal forms in which \( \mu = 0. \)

Accordingly,

\footnotesize\textsuperscript{1} to whom any correspondence should be addressed
satisfies that
\[ x_{n+1} = f(x_n) , \quad x_n , x_{n+1} \in \mathbb{R}^2 , f \in C^{2p+4} , f(0) = 0 , \]
In polar coordinates, \( f(x_n) \sim g(r, \theta) \), \( g \) takes the form:
\[ r \mapsto r + \sum_{m=1}^{p} a_{(2m+1)} r^{2m+1} + O(r^{2p+3}) \]
\[ \theta \mapsto \theta + \phi_0 + \sum_{m=1}^{p} a_{2(2m+1)} r^{2m} + O(r^{2p+2}) \]
where \( a_{(2m+1)} , a_{2(2m+1)} , \phi_0 \) can be obtained explicitly from (4). In the following, the effective form of \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 , g \in C^{2p+4} \) in (5) will be only considered.

Neimark-Sacker bifurcation is a phenomenon in which a fixed point of an evolution equation evolves into an invariant manifold. In some known neural network models\(^{[1-2]}\), biological models\(^{[3]}\), economic models\(^{[4]}\) and nonlinear dynamical system\(^{[5-8]}\), this phenomenon appears usually. Some progresses about the study of Neimark-Sacker bifurcations were made in the recent years. Based on the \(^7\), a new very simple proof of Neimark-Sacker bifurcation theorem was obtained in \(^8\). The existence of asymptotic solutions of bifurcation equations was directly proved, which led to the invariant manifolds for the high dimensional maps in \(^9\). According to \(^10\), the simplest normal form of the system associated with generalized Neimark-Sacker singularity of type \( l(l = 1, 2, 3, \cdots) \) up to any order was written as
\[ r \mapsto r + b_{(2l+1)} r^{2l+1} + b_{(4l+1)} r^{4l+1} \]
\[ \theta \mapsto \theta + \phi_0 + b_{(2l)} r^2 + b_{(2l)} r^4 + \cdots + b_{(2(2l+1))} r^{2l} \]
where \( b_{(2l)}, b_{(2l+2)}, \cdots, b_{(2(2l-1))} = 0 , \) if the corresponding \( a_{2(2l)}, a_{2(2l+2)}, \cdots, a_{2(2l-1)} = 0 \).

From (6), it is not difficult to see that the algebra expression between the coefficients of the simplest normal forms and the classical normal forms (or the origin dynamical system) has not been obtained. But, it is of interest to find it. Here, on the basis of \(^10\), using appropriate nonlinear transformations, direct computations and the second complete induction, it was proved that only two nonlinear terms existed in the simplest normal form of Neimark-Sacker and generalized Neimark-Sacker bifurcation systems respectively. Furthermore, the explicit expression of the simplest normal forms in terms of the coefficients of the classical normal forms was shown.

2. Simplest normal form of Neimark-Sacker bifurcation

**Theorem 1** Assume that \( g(r, \theta) \) in (4) is put into the form (5), then the form of \( g(r, \theta) \) in the simplest normal form up to the \( 2p + 1 \)th order associated with Neimark-Sacker bifurcation is given by
\[ g(r, \theta) : \quad r \mapsto \tilde{r} + a_{(3)} \tilde{r}^3 + a_{(5)} \tilde{r}^5 \]
\[ \theta \mapsto \tilde{\theta} + \phi_0 + a_{2(3)} \tilde{r}^2 \]

**Proof** From (5), we know that the conventional normal form only involves odd and even order terms respectively. In order to obtain the simplest normal form, the appropriate nonlinear transformation is introduced at the beginning of the third order terms in the first formula of (5) and the second order terms in the second formula of (5). The nonlinear transformation is given by
\[ \begin{cases} \tilde{r} = r + c_3 r^3 \\ \tilde{\theta} = \theta + c_5 \tilde{r}^2 \end{cases} \]
Assume that applying the transformation in (8), 
\[ g(r, \theta) \]
the normal form up to 3rd order can be expressed
\[
g(r, \theta) : \begin{cases} \bar{r} \mapsto \bar{r} + b_{l(3)} \bar{r}^3 \\ \bar{\theta} \mapsto \bar{\theta} + \phi_0 + b_{2(3)} \bar{r}^2 \end{cases} \tag{9}
\]
Substituting (8) and (9) into (5), and then balancing the same order terms ( in the amplitude
equation), the resulting equation yields
\[
\begin{cases} a_{l(3)} = b_{l(3)} \\ a_{2(3)} = b_{2(3)} \end{cases}
\]
Here, for arbitrary \( c_2 \) and \( c_3 \), the same results
mentioned above can be obtained. But, as it will be seen, these two coefficients \( c_2, c_3 \) can be
appropriately chosen to simplify higher order terms.

Consider the normal form of 5th order terms in the first formula of (5) and the 4th order terms in the
second formula of (5). The nonlinear transformation is given by
\[
\begin{cases} \bar{r} = r + c_1 r^3 + c_2 r^5 \\ \bar{\theta} = \theta + c_2 \bar{r}^2 + c_4 \bar{r}^4 \end{cases} \tag{10}
\]
Suppose that applying this transformation, 
\[ g(r, \theta) \]
the normal form up to the 5th order can be expressed as
\[
g(r, \theta) : \begin{cases} \bar{r} \mapsto \bar{r} + a_{l(5)} \bar{r}^3 + b_{l(5)} \bar{r}^5 \\ \bar{\theta} \mapsto \bar{\theta} + \phi_0 + a_{2(5)} \bar{r}^2 + b_{2(5)} \bar{r}^4 \end{cases} \tag{11}
\]
Substituting (10) and (11) into (5) and balancing the same order terms in the amplitude equation, we
have
\[
\begin{cases} a_{l(5)} = b_{l(5)} \\ a_{2(5)} + 2a_{l(3)}a_{2(3)} + 2c_2 a_{l(3)} = b_{2(5)} + 2a_{2(3)}c_3 \end{cases} \tag{12}
\]
Hence, when \( c_2 = -\frac{a_{2(5)}}{2a_{l(3)}} \) and \( c_3 = 0 \), \( b_{2(5)} = 0 \) is obtained .

Proceeding in the same manner mentioned above, if we select the nonlinear transformation
\[
\begin{cases} \bar{r} = r + c_1 r^5 + c_2 r^7 \\ \bar{\theta} = \theta + c_2 \bar{r}^2 + c_4 \bar{r}^4 + c_6 \bar{r}^6 \end{cases} \tag{13}
\]
\( b_{l(7)} = 0 \) and \( b_{2(7)} = 0 \) can be obtained. In the following, using the second complete induction, it is
proved that when \( 2k + 1 \geq 7 \), \( b_{l(2k+1)} = 0 \) and \( b_{2(2k+1)} = 0 \) by selecting the appropriate nonlinear
transformations. Clearly, \( b_{l(7)} = 0 \) and \( b_{2(7)} = 0 \). Suppose that, when \( 3 \leq k < n \ ( \forall n \in N, n > 3 ) \),
\( b_{l(2k+1)} = 0 \) and \( b_{2(2k+1)} = 0 \) by selecting \( c_{2k-2}, c_{2k-1} \). Next, we will prove that when \( k = n \),
\( b_{l(2k+1)} = 0 \) and \( b_{2(2k+1)} = 0 \) by selecting \( c_{2k-2} \) and \( c_{2k-1} \). In fact, considering the normal form of
\( 2k + 1 \)th order terms in the first formula of (5) and the \( 2k \)th order terms in the second formula of (5)
by the idea and the procedure mentioned above, the equation yields
\[
\begin{cases} 2(k - 2)c_{2k-1}a_{l(3)} = b_{l(2k+1)} - a_{l(2k+1)} + P_1(\ldots) \\ (2k - 2)c_{2k-2}a_{l(3)} = b_{2(2k+1)} - a_{2(2k+1)} + P_2(\ldots) \end{cases} \tag{14}
\]
where \( P_1(\cdots) \) and \( P_2(\cdots) \) denote the summation of the terms which are functions of the known coefficients \( c_2, \cdots, c_{2k-1}, a_{i(1)}, \cdots, a_{i(2k+i)} \) \((i = 1,2)\). So, we have \( b_{1(2k+i)} = 0 \) and \( b_{2(2k+i)} = 0 \) when
\[
c_{2k-1} = -\frac{1}{2(k-2)a_{i(1)}}[a_{i(2k+i)} - P(\cdots)] \quad \text{and} \quad c_{2k-2} = -\frac{1}{(2k-2)a_{i(1)}}[a_{2(2k+i)} - P_2(\cdots)].
\]

3. Simplest normal form of generalized Neimark-Sacker bifurcation

We will concentrate on the generalized Neimark-Sacker bifurcation singularity of type \( l \), namely,
\[
a_{i(1)} = \cdots = a_{i(2l-1)} = 0, \quad a_{i(2l+i)} \neq 0 \quad (2l \leq p, l \in \mathbb{N}, l \geq 2)
\]
in (5). Then two subcases should be considered:

(i) \( a_{2(3)} = \cdots = a_{2(2j-3)} = 0, \ a_{2(2j-1)} \neq 0 \) for \( 2 \leq j \leq l \);

(ii) \( a_{2(3)} = \cdots = a_{2(2j-1)} = 0 \).

**Theorem 2** Assume that \( g(r, \theta) \) in (4) is put into the form (5) with a generalized Neimark-Sacker bifurcation singularity of type \( l \) and \( a_{2(3)} = a_{2(5)} = \cdots = a_{2(2j-3)} = 0 \), but \( a_{2(2j-1)} \neq 0 \) \((1 \leq j \leq l)\), then, if \( 2l \leq p \), the form of \( g(r, \theta) \) in the simplest normal form up to the \( 2p+1 \)th order associated with generalized Neimark-Sacker bifurcation is described by
\[
\begin{align*}
\vec{r} &\mapsto \vec{r} + a_{i(2l+i)} \vec{r}^{2l+1} + a_{i(4l+i)} \vec{r}^{4l+1} \\
\vec{\theta} &\mapsto \vec{\theta} + \phi_0 + a_{2(2j-3)} \vec{r}^{2l-3} + a_{2(2j+3)} \vec{r}^{2l+3} + \cdots + a_{2(2l+i)} \vec{r}^{2l+i}
\end{align*}
\] (15)

**Proof** Using the idea and procedure mentioned in Theorem 1, Theorem 2 will be proved.

Aiming at the characteristics of generalized Neimark-Sacker, we start from the terms of \( 2l+1 \) order in the first formula of (5) and the terms of \( 2j-2 \) order in the second formula of (5). This is the key to obtain the explicit expression of the simplest normal form in terms of the coefficients of the classical normal forms in this paper.

In order to simplify the terms of \( 2k \) \((j-1 \leq k < l)\) order in the second formula of (5), the nonlinear transformation is given by
\[
\begin{align*}
\vec{r} &= r \\
\vec{\theta} &= \theta + c_{2j-1} \vec{r}^{2j-1} + c_{2j} \vec{r}^{2j} + \cdots + c_{2k} \vec{r}^{2k}
\end{align*}
\] (16)

Clearly, selecting \( c_{2k} = 0 \), \( b_{2(2k+i)} = a_{2(2k+i)} \) is obtained respectively.

When \( k \geq l \), the nonlinear transformation is given by
\[
\begin{align*}
\vec{r} &= r + c_{2l+i} \vec{r}^{2l+i} + c_{2l+3} \vec{r}^{2l+3} + \cdots + c_{2k+1} \vec{r}^{2k+1} \\
\vec{\theta} &= \theta + c_{2l} \vec{r}^{2l} + c_{2l+2} \vec{r}^{2l+2} + \cdots + c_{2k} \vec{r}^{2k}
\end{align*}
\] (17)

Applying (17), obviously, when \( k = l \), \( b_{1(l+1)} = a_{1(l+1)} \) and \( b_{2(l+1)} = a_{2(l+1)} \) can be obtained.

And when \( k > l \), the following equation can be obtained
\[
\begin{align*}
2(k-2l)a_{i(2l+i)} c_{2k-2l+i} &= b_{1(l+1)} - a_{i(2l+i)} + P_3(\cdots) \\
2(k-2) c_{2k-2l+i} a_{i(2l+i)} &= b_{2(l+1)} - a_{2(2l+i)} + P_4(\cdots)
\end{align*}
\] (18)

As in other cases, \( P_3(\cdots) \) and \( P_4(\cdots) \) indicate a summation of known coefficients. Hence, when \( k > l \), \( b_{2(2k+i)} = 0 \) is satisfied by selecting
\[ c_{2k-2l} = -\frac{1}{(2k-2l)a_{l(2l+1)}}[a_{2(2l+1)} - P_l(\cdots)]. \]

And when \( k > l \) and \( k \neq 2l \), \( b_{l(2l+1)} = 0 \) is obtained by selecting
\[ c_{2k-2l+1} = -\frac{1}{2(k-2l)a_{l(2l+1)}}[a_{l(2l+1)} - P_l(\cdots)]. \]

Besides, when \( k > l \) and \( k = 2l \), applying (17),
\[ a_{l(4l+1)} + (2l+1)c_{2l+1}a_{l(2l+1)} = b_{l(4l+1)} + (2l+1)c_{2l+1}a_{l(2l+1)} \]
(19)

can be obtained. So, for arbitrary \( c_{2l+1}, b_{l(4l+1)} = a_{l(4l+1)} \)

**Theorem 3** Assume that \( g(r, \theta) \) in (4) is put into the form (5) with a generalized Neimark-Sacker bifurcation singularity of type \( l \) and \( a_{2(3)} = \cdots = a_{2(l-1)} = 0 \), then, if \( 2l \leq p \), the form of \( g(r, \theta) \) in the simplest normal form up to the \( 2p + 1 \)th order associated with generalized Neimark-Sacker bifurcation is described by
\[ g(r, \theta) : \begin{array}{c} r \mapsto r + a_{l(2l+1)} r^{2l+1} + a_{l(4l+1)} r^{4l+1} \\ \theta \mapsto \theta + \phi_0 + a_{2(2l+1)} r^{2l} \end{array} \]
(20)

**Proof** Using the idea and procedure mentioned in Theorem 2, Theorem 3 can be proved.

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4. **References**

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