QUASICONVEXITY AND RELATIVELY HYPERBOLIC GROUPS THAT SPLIT

HADI BIGDELY AND DANIEL T. WISE

Abstract. We explore the combination theorem for a group $G$ splitting as a graph of relatively hyperbolic groups. Using the fine graph approach to relative hyperbolicity, we find short proofs of the relative hyperbolicity of $G$ under certain conditions. We then provide a criterion for the relative quasiconvexity of a subgroup $H$ depending on the relative quasiconvexity of the intersection of $H$ with the vertex groups of $G$. We give an application towards local relative quasiconvexity.

The goal of this paper is to examine relative hyperbolicity and quasiconvexity in graphs of relatively hyperbolic vertex groups with almost malnormal quasiconvex edge groups. The paper hinges upon the observation that if $G$ splits as a graph of relatively hyperbolic groups with malnormal relatively quasiconvex edge groups, then a fine hyperbolic graph for $G$ can be built from fine hyperbolic graphs for the vertex groups. This leads to short proofs of the relative hyperbolicity of $G$ as well as to a concise criterion for the relative quasiconvexity of a subgroup $H$ of $G$.

Bestvina and Feighn proved a combination theorem that characterized the hyperbolicity of groups splitting as graphs of hyperbolic groups [2]. Their geometric characterization is akin to the flat plane theorem characterization of hyperbolicity for actions on CAT(0) spaces, and leads to explicit positive results, especially in an “acylindrical” scenario where some form of malnormality is imposed on the edge groups. The Bestvina-Feighn combination theorem has been revisited multiple times in a hyperbolic setting, and more recently in a relatively hyperbolic context but through diverse methods.

Dahmani proved a combination theorem for relatively hyperbolic groups using the convergence group approach [4]. Later Alibegović proved similar results in [1] using a method generalizing parts of the Bestvina-Feighn approach. Osin reproved Dahman’s result in the general context of relative Dehn functions [19]. Most recently, Mj and Reeves gave a generalization of the Bestvina-Feighn combination theorem that follows Farb’s approach but uses a generalized “partial electrocution” [17]. Their result appears to be a far-reaching generalization at the expense of complex geometric language.

Our own results revisit these relatively hyperbolic generalizations, and we offer a very concrete approach employing Bowditch’s fine hyperbolic graphs. The most natural formulation of our main combination theorem (proven as Theorem 1.4) is as follows:

**Theorem A** (Combining Relatively Hyperbolic Groups Along Parabolics). Let $G$ split as a finite graph of groups. Suppose each vertex group is relatively hyperbolic and each edge group is
parabolic in its vertex groups. Then \( G \) is hyperbolic relative to \( Q = \{Q_1, \ldots, Q_j\} \) where each \( Q_i \)

is the stabilizer of a “parabolic tree”. (See Definition 1.3.)

A simplistic example illustrating Theorem A is an amalgamated product \( G = G_1 \ast_C G_2 \) where each \( G_i = \pi_1 M_i \) and \( M_i \) is a cusped hyperbolic manifold with a single boundary torus \( T_i \). And \( C \) is an arbitrary common subgroup of \( \pi_1 T_1 \) and \( \pi_1 T_2 \). Then \( G \) is hyperbolic relative to \( \pi_1 T_1 \ast_C \pi_1 T_2 \).

We note that Theorem A is more general than results in the same spirit that were obtained by Dahmani, Alibegović, and Osin. In particular, they require that edge groups be maximal parabolic on at least one side, but we do not. We believe that Theorem A could be deduced from the results of Mj-Reeves.

In Section 4 we employ work of Yang [22] on extended peripheral structures, to obtain the following seemingly more natural corollary of Theorem A which is proven as Corollary 4.6:

Corollary B. Let \( G \) split as a finite graph of groups. Suppose

(a) Each \( G_\nu \) is hyperbolic relative to \( P_\nu \);

(b) Each \( G_e \) is total and relatively quasiconvex in \( G_\nu \);

(c) \( \{G_e : e \text{ is attached to } \nu\} \) is almost malnormal in \( G_\nu \) for each vertex \( \nu \).

Then \( G \) is hyperbolic relative to \( \bigcup_\nu P_\nu - \{\text{repeats}\} \).

The “omitted repeats” in the conclusion of Corollary B refer to (some of) the parabolic subgroups of vertex groups that are identified through an edge group.

It is not clear whether Corollary B could be obtained using the method of Dahmani, Alibegović, or Osin. However, we suspect it could be extracted from the result of Mj-Reeves.

Definition 0.1. (Tamely generated) Let \( G \) split as a graph of groups with relatively hyperbolic vertex groups. A subgroup \( H \) is tamely generated if the induced graph of groups \( \Gamma_H \) has a \( \pi_1 \)-isomorphic subgraph of groups \( \Gamma'_H \) that is a finite graph of groups each of whose vertex groups is relatively quasiconvex in the corresponding vertex group of \( G \).

Note that \( H \) is tamely generated when \( H \) is finitely generated and there are finitely many \( H \)-orbits of vertices \( v \) in \( T \) with \( H_v \) nontrivial, and each such \( H_v \) is relatively quasiconvex in \( G_v \). However the above condition is not necessary. For instance, let \( G = F_2 \times \Z_2 \), and consider a splitting where \( \Gamma \) is a bouquet of two circles, and each vertex and edge group is isomorphic to \( \Z_2 \). Then every f.g. subgroup \( H \) of \( F_2 \times \Z_2 \) is tamely generated, but no subgroup containing \( \Z_2 \) satisfies the condition that there are finitely many \( H \)-orbits of vertices \( \omega \) with \( H_\omega \) nontrivial.

The geometric construction proving Theorem A allows us to give a simple criterion for quasiconvexity of a subgroup \( H \) relative to \( Q \). Again, coupling this with Yang’s work, we obtain (as Theorem 4.13) the following criterion for quasiconvexity relative to \( P \):

Main Theorem C (Quasiconvexity Criterion). Let \( G \) be hyperbolic relative to \( P \) where each \( P \in \mathcal{P} \) is finitely generated. Suppose \( G \) splits as a finite graph of groups. Suppose

(a) Each \( G_e \) is total in \( G \);

(b) Each \( G_e \) is relatively quasiconvex in \( G \);

(c) Each \( G_e \) is almost malnormal in \( G \).

Let \( H \leq G \) be tamely generated. Then \( H \) is relatively quasiconvex in \( G \).

Recall that \( G \) is locally relatively quasiconvex if each finitely generated subgroup \( H \) of \( G \) is quasiconvex relative to the peripheral structure of \( G \). I. Kapovich first recognized that hyperbolic
Figure 1. A fine graph $K_G$ for $G = A \ast_C B$ is built from copies of fine graphs $K_A$ and $K_B$ for $A$ and $B$ by gluing new edges together along vertices stabilized by $C$. The parabolic trees of $T$ are images of trees formed from the new edges in $K_G$. We obtain a fine hyperbolic graph $\bar{K}_G$ with finite edge stabilizers as a quotient $K_G \to \bar{K}_G$.

limit groups are locally relatively quasiconvex \cite{12}, and subsequently Dahmani proved that all limit groups are locally relatively quasiconvex \cite{4}.

A group $P$ is \textit{small} if there is no embedding $F_2 \hookrightarrow P$, and $G$ has a \textit{small hierarchy} if it can be built from small subgroups by a sequence of AFP’s and HNN’s along small subgroups (see Definition \cite[3.4]{4}). When $\mathbb{P}$ is a collection of free-abelian groups, the following inductive consequence of Corollary \cite[3.3]{4} generalizes Dahmani’s result.

**Theorem D.** Let $G$ be hyperbolic relative to a collection of Noetherian subgroups $\mathbb{P}$ and suppose $G$ has a small hierarchy. Then $G$ is locally relatively quasiconvex.

Although Theorem D is implicit in Dahmani’s work, we believe Theorem C is new.

The main construction and its application: Although we work in somewhat greater generality, let us focus on the simple case of an amalgamated product $G = A \ast_C B$ where $A, B$ are relatively hyperbolic and $C$ is parabolic on each side. The central theme of this paper is a construction that builds a fine hyperbolic graph $\bar{K}_G$ for $G$ from fine hyperbolic graphs $K_A$ and $K_B$ for $A, B$. (See Figure 1) This is done in two steps: Guided by the Bass-Serre tree, we first construct a graph $K_G$ which is a tree of spaces whose vertex spaces are copies of $K_A$ and $K_B$, and whose edge spaces are ordinary edges. Though $K_G$ is fine and hyperbolic, its edges have infinite stabilizers. We remedy this by quotienting these edge spaces to form the fine hyperbolic graph $\bar{K}_G$. The vertices of $\bar{K}_G$ are quotients of “parabolic trees” in $K_G$. The fine hyperbolic graph $\bar{K}_G$ quickly proves that $G$ is hyperbolic relative to the collection $\mathbb{Q}$ of subgroups stabilizing parabolic trees. Variations on the construction, hypotheses on the edge groups, and interplay with previous work on peripheral structures, leads to a variety of relatively hyperbolic conclusions. The simplest and most immediate in the case above, is that $G$ is hyperbolic relative to $\mathbb{P}_G = \mathbb{P}_A \cup \mathbb{P}_B - \{C\}$ when $C$ is maximal parabolic on each side and $A, B$ are hyperbolic relative to $\mathbb{P}_A$ and $\mathbb{P}_B$.

Our primary application is to give an easy criterion to recognize quasiconvexity. A subgroup $H$ is relatively quasiconvex in $G$ if there is an $H$-cocompact quasiconvex subgraph $\bar{L} \subset \bar{K}_G$ of the fine hyperbolic $G$-graph. The tree-like nature of our graph $\bar{K}_G$, permits us to naturally build the quasiconvex $H$-graph $\bar{L}$. When $H$ is relatively quasiconvex, there are finitely many $H$-orbits of nontrivially $H$-stabilized vertices in the Bass-Serre tree $T$, and each of these stabilizers is relatively quasiconvex in its vertex group. Choosing finitely many quasiconvex subgraphs in the corresponding copies of $K_A$ and $K_B$, we are able to combine these together to form $L$ in $K_G$ and then to form a quasiconvex $H$-subgraph $\bar{L}$ in $\bar{K}_G$. 
We conclude by mentioning the following consequence of Corollary [15] that is a natural consequence of the viewpoint developed in this paper.

**Corollary E.** Let \( M \) be a compact irreducible 3-manifold. And let \( M_1, \ldots, M_r \) denote the graph manifolds obtained by removing each (open) hyperbolic piece in the geometric decomposition of \( M \). Then \( \pi_1 M \) is hyperbolic relative to \( \{ \pi_1 M_1, \ldots, \pi_1 M_r \} \).

As explained to us by the referee, the relative hyperbolicity of \( \pi_1(M) \) was previously proved by Drutu-Sapir using work of Kapovich-Leeb. This previous proof is deep as it uses the structure of the asymptotic cone due to Kapovich-Leeb together with the technical proof of Drutu-Sapir that asymptotically tree graded groups are relatively hyperbolic [13][5].

1. Combining Relatively Hyperbolic Groups along Parabolics

The class of relatively hyperbolic groups was introduced by Gromov [8] as a generalization of the class of fundamental groups of complete finite-volume manifolds of pinched negative sectional curvature. Various approaches to relative hyperbolicity were developed by Farb [6], Bowditch [3] and Osin [20], and as surveyed by Hruska [10], these notions are equivalent for finitely generated groups. We follow Bowditch’s approach:

**Definition 1.1** (Relatively Hyperbolic). A circuit in a graph is an embedded cycle. A graph \( \Gamma \) is **fine** if each edge of \( \Gamma \) lies in finitely many circuits of length \( n \) for each \( n \).

A group \( G \) is **hyperbolic relative to a finite collection of subgroups** \( \mathcal{P} \) if \( G \) acts cocompactly (without inversions) on a connected, fine, hyperbolic graph \( \Gamma \) with finite edge stabilizers, such that each element of \( \mathcal{P} \) equals the stabilizer of a vertex of \( \Gamma \), and moreover, each infinite vertex stabilizer is conjugate to a unique element of \( \mathcal{P} \). We refer to a connected, fine, hyperbolic graph \( \Gamma \) equipped with such an action as a \( (G; \mathcal{P}) \)-graph. Subgroups of \( G \) that are conjugate into subgroups in \( \mathcal{P} \) are parabolic.

**Technical Remark 1.2.** Given a finite collection of parabolic subgroups \( \{A_1, \ldots, A_r\} \), we choose \( \mathcal{P} \) so that there is a prescribed choice of parabolic subgroup \( P_i \in \mathcal{P} \) so that \( A_i \) is “declared” to be conjugate into \( P_i \). This is automatic for an infinite parabolic subgroup \( A \) but for finite subgroups there could be ambiguity. One way to resolve this is to revise the choice of \( \mathcal{P} \) as follows: For any finite collection of parabolic subgroups \( \{A_1, \ldots, A_r\} \) in \( G \), we moreover assume each \( A_i \) is conjugate to a subgroup of \( \mathcal{P} \) and we assume that no two (finite) subgroups in \( \mathcal{P} \) are conjugate. We note that finite subgroups can be freely added to or omitted from the peripheral structure of \( G \) (see e.g. [16]).

**Definition 1.3** (Parabolic tree). Let \( G \) split as a finite graph of groups where each vertex group \( G_v \) is hyperbolic relative to \( \mathcal{P}_v \), and where each edge group \( G_e \) embeds as a parabolic subgroup of its two vertex groups. Let \( T \) be the Bass-Serre tree. Define the parabolic forest \( F \) by:

1. A vertex in \( F \) is a pair \((u, P)\) where \( u \in T^0 \) and \( P \) is a \( G_u \)-conjugate of an element of \( \mathcal{P}_u \).
2. An edge in \( F \) is a pair \((e, G_e)\) where \( e \) is an edge of \( T \) and \( G_e \) is its stabilizer.
3. The edge \((e, G_e)\) is attached to \((\iota(e), \iota(P_e))\) and \((\tau(e), \tau(P_e))\) where \( \iota(e) \) and \( \tau(e) \) are the initial and terminal vertex of \( e \) and \( \iota(P_e) \) is the \( G_{\iota(e)} \)-conjugate of an element of \( \mathcal{P} \) that is declared to contain \( G_e \). Likewise for \((\tau(e), \tau(P_e))\). We arranged for this unique determination in Technical Remark [1.2].
Each component of $F$ is a parabolic tree and the map $F \to T$ is injective on the set of edges, and in particular each parabolic tree embeds in $T$. Let $S_1, \ldots, S_j$ be representatives of the finitely many orbits of parabolic trees under the $G$ action on $F$. Let $Q_i = \text{stab}(S_i)$, for each $i$.

**Theorem 1.4** (Combining Relatively Hyperbolic Groups Along Parabolics). Let $G$ split as a finite graph $\Gamma$ of groups. Suppose each vertex group is relatively hyperbolic and each edge group is parabolic in its vertex groups. Then $G$ is hyperbolic relative to $Q = \{Q_1, \ldots, Q_j\}$.

**Proof.** For $u \in \Gamma^0$, let $G_u$ be hyperbolic relative to $\mathbb{P}_u$ and let $K_u$ be a $(G_u; \mathbb{P}_u)$-graph. For each $P \in \mathbb{P}_u$, following the Technical Remark 1.2 we choose a specific vertex of $K_u$ whose stabilizer equals $P$. Note that, in general there could be more than one possible choice when $|P| < \infty$, but by Technical Remark 1.2 we have a unique choice. Translating determines a “choice” of vertex conjugates.

We now construct a $(G; \mathbb{Q})$-graph $\tilde{K}$. Let $K$ be the tree of spaces whose underlying tree is the Bass-Serre tree $T$ with the following properties:

1. Vertex spaces of $K$ are copies of appropriate elements in $\{K_u : u \in \Gamma^0\}$. Specifically, $K_v$ is a copy of $K_u$ where $u$ is the image of $v$ under $T \to \Gamma$.
2. Each edge space $K_e$ is an ordinary edge, denoted as an ordered pair $(e, G_e)$ that is attached to the vertices in $K_{(e)}$ and $K_{\tau(e)}$ that were chosen to contain $G_e$.

Note that each $G_e$ acts on $K_v$ and there is a $G$-equivariant map $K \to T$. Let $\tilde{K}$ be the quotient of $K$ obtained by contracting each edge space. Observe that $G$ acts on $\tilde{K}$ and there is a $G$-equivariant map $K \to \tilde{K}$. Moreover the preimage of each open edge of $\tilde{K}$ is a single open edge of $K$.

We now show that $\tilde{K}$ is a $(G; \mathbb{Q})$-graph. Since any embedded cycle lies in some vertex space, the graph $\tilde{K}$ is fine and hyperbolic. There are finitely many orbits of vertices in $K$ and therefore finitely many orbits of vertices in $\tilde{K}$. Likewise, there are finitely many orbits of edges in $\tilde{K}$. The stabilizer of an (open) edge of $\tilde{K}$ equals the stabilizer of the corresponding (open) edge in $K$, and is thus finite. By construction, there is a $G$-equivariant embedding $F \hookrightarrow \tilde{K}$ where $F$ is the parabolic forest associated to $G$ and $T$. Finally, the preimage in $K$ of a vertex of $\tilde{K}$ is precisely a parabolic tree and thus the stabilizer of a vertex of $\tilde{K}$ is a conjugate of some $Q_j$. □

We now examine some conclusions that arise when the parabolic trees are small. An extreme case arises when the edge groups are isolated from each other as follows:

**Corollary 1.5.** Let $G$ split as a finite directed graph of groups where each vertex group $G_v$ is hyperbolic relative to $\mathbb{P}_v$. Suppose that:

1. Each edge group is parabolic in its vertex groups.
2. Each outgoing infinite edge group $G_d$ is maximal parabolic in its initial vertex group $G_v$ and for each other incoming and outgoing infinite edge group $G_e$ or $G_d$ or $G_{\bar{d}}$, none of its conjugates lie in $G_e$.

Then $G$ is hyperbolic relative to $\mathbb{P} = \bigcup_v \mathbb{P}_v - \{\text{outgoing edge groups}\}$.

**Proof.** We can arrange for finitely stabilized edges of $F$ to be attached to distinct chosen vertices when they correspond to distinct edges of $T$. Thus, parabolic trees are singletons and/or $i$-pods consisting of edges that all terminate at the same vertex $((v, \mathbb{P}^g))$ where $P \in \mathbb{P}_v$ and $g \in G_v$. Recall that an $i$-pod is a tree consisting of $i$ edges glued to a central vertex. □
Corollary 1.6. Let \( G \) split as a finite graph of groups. Suppose each vertex group \( G_v \) is hyperbolic relative to \( \mathbb{P}_v \). For each \( G_v \) assume that the collection \( \{ G_v : e \text{ is attached to } v \} \) is a collection of maximal parabolic subgroups of \( G_v \). Then \( G \) is hyperbolic relative to \( \mathbb{P} = \bigcup_v \mathbb{P}_v \setminus \{ \text{repeats} \} \). Specifically, we remove an element of \( \bigcup_v \mathbb{P}_v \) if it is conjugate to another one.

The first two of the following cases were treated by Dahmani, Alibegović, and Osin \([1, 4, 19]\):

Corollary 1.7. (1) Let \( G_1 \) and \( G_2 \) be hyperbolic relative to \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \). Let \( G = G_1 *_{P_1 = P_2} G_2 \) where each \( P_i \in \mathbb{P}_i \) and \( P_1 \) is identified with the subgroup \( P_2' \) of \( P_2 \). Then \( G \) is hyperbolic relative to \( \mathbb{P}_1 \cup \mathbb{P}_2 \setminus \{ P_1 \} \).

(2) Let \( G_1 \) be hyperbolic relative to \( \mathbb{P} \). Let \( P_1 \in \mathbb{P} \) be isomorphic to a subgroup \( P_2' \) of a maximal parabolic subgroup \( P_2 \) not conjugate to \( P_1 \). Let \( G = G_1 *_{P_1 t = P_2 t} \) where \( P_1 t = t^{-1} P_1 t \). Then \( G \) is hyperbolic relative to \( \mathbb{P} \setminus \{ P_1 \} \).

(3) Let \( G_1 \) be hyperbolic relative to \( \mathbb{P} \). Let \( P \in \mathbb{P} \) be isomorphic to \( P' \leq P \). Let \( G = G_1 *_{P = P'} \). Then \( G \) is hyperbolic relative to \( \mathbb{P} \cup \langle P, t \rangle \).

Remark 1.8. Note that in this Corollary and some similar results when we say \( P_i \in \mathbb{P}_i \), we mean if \( P_i^k \in \mathbb{P}_i \) then replace \( P_i^k \) by \( P_i \) in \( \mathbb{P}_i \).

Proof. (1): In this case, the parabolic trees are either singletons stabilized by a conjugate of an element of \( \mathbb{P}_1 \cup \mathbb{P}_2 \setminus \{ P_1 \} \), or parabolic trees are \( i \)-pods stabilized by conjugates of \( P_2 \).

(2): The proof is similar.

(3): All parabolic trees are singletons except for those that are translates of a copy of the Bass-Serre tree for \( P*_{P = P'} \). Following the proof of Theorem 1.4 let \( v \in \bar{K} \), if the preimage of \( v \) in \( K \) is not attached to an edge space, then \( G_v \) is conjugate to an element of \( \mathbb{P} \setminus \{ P \} \), otherwise \( G_v \) is conjugate to \( \langle P, t \rangle \).

Example 1.9. We encourage the reader to consider the case of Theorem 1.4 and Corollaries 1.6 and 1.7 in the scenario where \( G \) splits as a graph of free groups with cyclic edge groups. A very simple case is: Let \( G = \langle a, b, t \mid (W^m)^t = W^{m'} \rangle \) where \( W \in \langle a, b \rangle \) and \( m, n \geq 1 \). Then \( G \) is hyperbolic relative to \( \langle W, t \rangle \).

2. Relative Quasiconvexity

Dahmani introduced the notion of relatively quasiconvex subgroup in \([4]\). This notion was further developed by Osin in \([20]\), and later Hruska investigated several equivalent definitions of relatively quasiconvex subgroups \([10]\). Martinez-Pedroza and the second author introduced a definition of relative quasiconvexity in the context of fine hyperbolic graphs and showed this definition is equivalent to Osin’s definition \([16]\). We will study relatively quasiconvexity using this fine hyperbolic viewpoint. Our aim is to examine the relative quasiconvexity of a certain subgroup which are themselves amalgams, and we note that powerful results in this direction are given in \([15]\).

Definition 2.1 (Relatively Quasiconvex). Let \( G \) be hyperbolic relative to \( \mathbb{P} \). A subgroup \( H \) of \( G \) is quasiconvex relative to \( \mathbb{P} \) if for some (and hence any) \((G; \mathbb{P})\)-graph \( K \), there is a nonempty connected and quasi-isometrically embedded, \( H \)-cocompact subgraph \( L \) of \( K \). In the sequel, we sometimes refer to \( L \) as a quasiconvex \( H \)-cocompact subgraph of \( K \).
Let $G$ be a f.g. group that splits as a finite graph of groups $\Gamma$. If each edge group is f.g. then each vertex group is f.g.

**Proof.** Let $G = \langle g_1, \ldots, g_n \rangle$. We regard $G$ as $\pi_1$ of a 2-complex corresponding to $\Gamma$. We show that each vertex group $G_v$ equals $\langle G_v \rangle$ attached to $v \cup \{g \in G_v : g \text{ in normal form of some } g_i\}$. Let $a \in G_v$ and consider an expression of $a$ as a product of normal forms of the $g_i^{\pm 1}$. Then $a$ equals some product $a_1 t_1 a_1^{\pm 1} b_1 t_1^{\pm 1} a_2^{\pm 1} b_2 t_2^{\pm 1} \cdots a_r t_r^{\pm 1} b_r$. There is a disc diagram $D$ whose boundary path is $a^{-1} a_1 t_1 a_1^{\pm 1} b_1 t_1^{\pm 1} a_2^{\pm 1} b_2 t_2^{\pm 1} \cdots a_r t_r^{\pm 1} b_r$. See Figure 2. The region of $D$ that lies along $a$ shows that $a$ equals the product of elements in edge groups adjacent to $G_v$, together with elements of $G_v$ that lie in the normal forms of $g_1, \ldots, g_n$. \hfill \Box

**Remark 2.2.** It is immediate from the Definition [2.1] that in a relatively hyperbolic group, any parabolic subgroup is relatively quasiconvex, and any relatively quasiconvex subgroup is also relatively hyperbolic. In particular, the relatively quasiconvex subgroup $H$ is hyperbolic relative to the collection $\mathcal{P}_H$ consisting of representatives of $H$-stabilizers of vertices of $L \subseteq K$. Note that a conjugate of a relatively quasiconvex subgroup is also relatively quasiconvex. And the intersection of two relatively quasiconvex subgroups is relatively quasiconvex. Specifically, this last statement was proven when $G$ is f.g. in [15], and when $G$ is countable in [10].

Relative quasiconvexity has the following transitive property proven by Hruska for countable relatively hyperbolic groups in [10]:

**Lemma 2.3.** Let $G$ be hyperbolic relative to $\mathcal{P}_G$. Suppose that $B$ is relatively quasiconvex in $G$, and note that $B$ is then hyperbolic relative to $\mathcal{P}_B$ as in Remark 2.2. Then $A \leq B$ is quasiconvex relative to $\mathcal{P}_B$ if and only if $A$ is quasiconvex relative to $\mathcal{P}_G$.

**Proof.** Let $K$ be a $(G; \mathcal{P}_G)$-graph. As $B$ is quasiconvex relative to $\mathcal{P}_G$, there is a $B$-cocompact and quasiconvex subgraph $L \subset K$. Note that $L$ is a $(B; \mathcal{P}_B)$-graph. Let $A \leq B$.

If $A$ is quasiconvex in $B$ relative to $\mathcal{P}_B$, there is an $A$-cocompact quasiconvex subgraph $M \subset L$. Since the composition $L_A \to L_B \to K$ is a quasi-isometric embedding, $A$ is quasiconvex relative to $\mathcal{P}_G$. Conversely, if $A$ is quasiconvex in $G$ relative to $\mathcal{P}_G$, then there is an $A$-cocompact quasiconvex subgraph $M \subset K$. Let $L' = L \cup BM$ and note that $L'$ is $B$-cocompact and hence also quasiconvex, and thus $L'$ also serves as a fine hyperbolic graph for $B$. Now $M \subset L'$ is quasiconvex since $M \subset L$ is quasiconvex so $A$ is relatively quasiconvex in $B$. \hfill \Box

**Remark 2.4.** One consequence of Theorem 1.4 and its various Corollaries, is that when $G$ splits as a graph of relatively hyperbolic groups with parabolic subgroups, then each of the vertex groups is quasiconvex relative to the peripheral structure of $G$. (For Theorem 1.4 this is $Q$, and for Corollary 1.6 this is $P - \{\text{repeats}\}$.) Indeed, $K_v$ is a $G_v$-cocompact quasiconvex subgraph in the fine graph $K$ constructed in the proof.
Theorem 2.6 (Quasiconvexity of a Subgroup in Parabolic Splitting). Let $G$ split as a finite graph $\Gamma$ of relatively hyperbolic groups such that each edge group is parabolic in its vertex groups. (Note that $G$ is hyperbolic relative to $Q = \{Q_1, \ldots, Q_s\}$ by Theorem 1.4.) Let $H \leq G$ be tamely generated. Then $H$ is quasiconvex relative to $Q$. Moreover if each $H_v$ in the Bass-Serre tree $T$ is finitely generated then $H$ is finitely generated.

Proof. Since there are finitely many orbits of vertices whose stabilizers are finitely generated, $H$ is finitely generated. For each $u \in \Gamma^0$, let $G_u$ be hyperbolic relative to $\mathbb{P}_u$ and let $K_u$ be a $(G_u; \mathbb{P}_u)$-graph. Let $K$ be the $(G; \mathbb{Q})$-graph constructed in the proof of Theorem 1.4 and let $\bar{K}$ be its quotient. We will construct an $H$-cocompact quasiconvex, connected subgraph $\bar{L}$ of $\bar{K}$.

Let $T_H$ be the minimal $H$-invariant subgraph of $T$. Recall that each edge of $T$ (and hence $T_H$) corresponds to an edge of $K$. Let $F_H$ denote the subgraph of $K$ that is the union of all edges correspond to edges of $T_H$. Let $\{v_1, \ldots, v_n\}$ be a representatives of $H$-orbits of vertices of $T_H$. For each $i$, let $L_i \hookrightarrow K_{v_i}$ be a $(H \cap G_{v_i}^0)$-cocompact quasiconvex subgraph such that $L_i$ contains $F_H \cap K_{v_i}$. (There are finitely many $(H \cap G_{v_i}^0)$-orbits of such endpoints of edges in $K_{v_i}$.) Let $L = F_H \cup \bigcup_{i=1}^n H L_i$ and let $\bar{L}$ be the image of $L$ under $K \to \bar{K}$. Observe that $L$ is quasiconvex in $K$ since $\bar{K}$ is a “tree union” and each such $L_i$ of $L$ is quasiconvex in $K_{v_i}$. And likewise, $\bar{L}$ is quasiconvex in $\bar{K}$. \hfill $\square$

Corollary 2.7 (Characterizing Quasiconvexity in Maximal Parabolic Splitting). Let $G$ split as a finite graph of countable groups. For each $v$, let $G_v$ be hyperbolic relative to $\mathbb{P}_v$ and let the collection $\{G_e : e$ is attached to $v\}$ be a collection of maximal parabolic subgroups of $G_v$. (Note that $G$ is hyperbolic relative to $\mathbb{P} = \bigcup_v \mathbb{P}_v$ (repeats) by Corollary 1.6) Let $T$ be the Bass-Serre tree and let $H$ be a subgroup of $G$. The following are equivalent:

1. $H$ is tamely generated and each $H_v$ in the Bass-Serre tree $T$ is f.g.
2. $H$ is f.g. and quasiconvex relative to $\mathbb{P}$.

Proof. (1 $\Rightarrow$ 2): Follows from Theorem 1.4 and Theorem 2.6

(2 $\Rightarrow$ 1): Since $H$ is f.g., the minimal $H$-subtree $T_H$ is $H$-cocompact, and so $H$ splits as a finite graph of groups $\Gamma_H$. Since $H$ is quasiconvex in $\mathbb{P}$, it is hyperbolic relative to intersections with conjugates of $\mathbb{P}$. In particular, the infinite edge groups in the induced splitting of $H$ are maximal parabolic, and are thus f.g. since the maximal parabolic subgroups of a f.g. relatively hyperbolic group are f.g. [20]). Each vertex group of $\Gamma_H$ is f.g. by Lemma 2.5.

By Remark 2.4, each vertex group of $G$ is quasiconvex relative to $\mathbb{P}$, and hence each $G_v$ is relatively quasiconvex by Remark 2.2 since it is a conjugate of a vertex group. Thus $H_v = H \cap G_v$ is quasiconvex relative to $\mathbb{P}$ by Remark 2.2. Finally, $H_v$ is quasiconvex in $G_v$ by Lemma 2.3 \hfill $\square$

3. Local Relative Quasiconvexity

A relatively hyperbolic group $G$ is locally relatively quasiconvex if each f.g. subgroup of $G$ is relatively quasiconvex. The focus of this section is the following criterion for showing that the combination of locally relatively quasiconvex groups is again locally relatively quasiconvex.

Recall that $N$ is Noetherian if each subgroup of $N$ is f.g. We now give a criterion for local quasiconvexity of a group that splits along parabolic subgroups.

Theorem 3.1 (A Criterion for Locally Relatively Quasiconvexity). \hspace{1em} (1) Let $G_1$ and $G_2$ be locally relatively quasiconvex relative to $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $G = G_1 *_{P_1 = p_2} G_2$ where each
Proof. (1): By Corollary 1.7, \( G \) is hyperbolic relative to \( \mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\} \). Let \( H \) be a finitely generated subgroup of \( G \). We show that \( H \) is quasiconvex relative to \( \mathbb{P} \). Let \( T \) be the Bass-Serre tree of \( G \). Since \( H \) is f.g., the minimal \( H \)-subtree \( T_H \) is \( H \)-cocompact, and so \( H \) splits as a finite graph of groups \( \Gamma_H \). Moreover, the edge groups of this splitting are f.g. since the edge groups of \( G \) are Noetherian by hypothesis. Thus each vertex group of \( \Gamma_H \) is f.g. by Lemma 2.6. Since \( G_1 \) and \( G_2 \) are locally quasiconvex, each vertex group of \( T_H \) is quasiconvex in its “image vertex group” under the map \( T_H \to T \). Now by Theorem 2.6 \( H \) is quasiconvex relative to \( \mathbb{P} \). The proof of (2) and (3) are similar. \( \square \)

**Definition 3.2** (Almost Malnormal). A subgroup \( H \) is *malnormal* in \( G \) if \( H \cap H^g = \{1\} \) for \( g \notin H \), and similarly \( H \) is *almost malnormal* if this intersection \( H \cap H^g \) is always finite. Likewise, a collection of subgroups \( \{H_i\} \) is *almost malnormal* if \( H_i^g \cap H_j^h \) is finite unless \( i = j \) and \( gh^{-1} \in H_i \). 

**Corollary 3.3.** Let \( G \) split as a finite graph of groups. Suppose

a) Each \( G_v \) is locally relatively quasiconvex;

b) Each \( G_e \) is Noetherian and maximal parabolic in its vertex groups;

c) \( \{G_e : e \text{ is attached to } v\} \) is almost malnormal in \( G_v \), for any vertex \( v \).

Then \( G \) is locally relatively quasiconvex relative to \( \mathbb{P} \), see Corollary 1.6.

3.1. **Small-hierarchies and local quasiconvexity.** The main result in this subsection is a consequence of Theorem 3.1 that employs results of Yang [22] stated in Theorems 4.7 and 4.2 and also depends on Lemma 4.9 which is independent of Section 4. The reader may choose to read this subsection and refer ahead to those results, or return to this subsection after reading Section 4.

**Definition 3.4** (Small-Hierarchy). A group is small if it has no rank 2 free subgroup. Any small group has a length 0 small-hierarchy. \( G \) has a length \( n \) small-hierarchy if \( G \cong A \ast_C B \) or \( G \cong A \ast_{C=C'} B \), where \( A \) and \( B \) have length \( (n-1) \) small-hierarchies, and \( C \) is small and f.g. We say \( G \) has a small-hierarchy if it has a length \( n \) small-hierarchy for some \( n \).

We can define \( \mathcal{F} \)-hierarchy by replacing “small” by a class of groups \( \mathcal{F} \) closed under subgroups and isomorphisms. For instance, when \( \mathcal{F} \) is the class of finite groups, the class of groups with an \( \mathcal{F} \)-hierarchy is precisely the class of virtually free groups.

**Remark 3.5.** The Tits alternative for relatively hyperbolic groups states that every f.g. subgroup is either: elementary, parabolic, or contains a subgroup isomorphic to \( F_2 \). The Tits alternative is proven for countable relatively hyperbolic groups in [8, Thm 8.2.F]. A proof is given for convergence groups in [21]. It is shown in [20] that every cyclic subgroup \( H \) of a f.g. relatively hyperbolic group \( G \) is relatively quasiconvex.
Theorem 3.6. Let $G$ be f.g. and hyperbolic relative to $\mathbb{P}$ where each element of $\mathbb{P}$ is Noetherian. Suppose $G$ has a small-hierarchy. Then $G$ is locally relatively quasiconvex.

Proof. The proof is by induction on the length of the hierarchy. Since edge groups are f.g., the Tits alternative shows that there are three cases according to whether the edge group is finite, virtually cyclic, or infinite parabolic, and we note that the edge group is relatively quasiconvex in each case. These three cases are each divided into two subcases according to whether $G = A \ast C_1 B$ or $G = A \ast C_1 \ast C_2$. Since $C_1$ and $G$ are f.g. the vertex groups are f.g. by Lemma 2.5. Thus, since $C_1$ is relatively quasiconvex the vertex groups are relatively quasiconvex by Lemma 4.9.

When $C_1$ is finite the conclusion follows in each subcase from Theorem 3.1. When $C_1$ is virtually cyclic but not parabolic, then $C_1$ lies in a unique maximal virtually cyclic subgroup $Z$ that is almost malnormal and relatively quasiconvex by [18]. Thus $G$ is hyperbolic relative to $\mathbb{P} = \mathbb{P} \cup \{Z\}$ by Theorem 4.2.

Observe that $C_1$ is maximal infinite cyclic on at least one side, since otherwise there would be a nontrivial splitting of $Z$ as an amalgamated free product over $C_1$. We equip the (relatively quasiconvex) vertex groups with their induced peripheral structures. Note that $C_1$ is maximal parabolic on at least one side and so $G$ is locally relatively quasiconvex relative to $\mathbb{P}$ by Theorem 3.1. Finally, by Theorem 4.7, any subgroup $H$ is quasiconvex relative to the original peripheral structure $\mathbb{P}$ since intersections between $H$ and conjugates of $Z$ are quasiconvex relative to $\mathbb{P}$.

When $C_1$ is infinite parabolic, we will first produce a new splitting before verifying local relative quasiconvexity.

When $G = A \ast C_1 B$. Let $D_a, D_b$ be the maximal parabolic subgroups of $A, B$ containing $C_1$, and refine the splitting to:

$$A \ast D_a (D_a \ast C_1 D_b) \ast D_b B$$

The two outer splittings are along a parabolic that is maximal on the outside vertex group. The inner vertex group $D_a \ast C_1 D_b$ is a single parabolic subgroup of $G$. Indeed, as $C_1$ is infinite, $D_a \supset C_1 \subset D_b$ must all lie in the same parabolic subgroup of $G$. It is obvious that $D_a \ast C_1 D_b$ is locally relatively quasiconvex with respect to its induced peripheral structure since it is itself parabolic in $G$. Consequently $(D_a \ast C_1 D_b) \ast D_b B$ is locally relatively quasiconvex by Theorem 3.1. Therefore $G = A \ast D_a ((D_a \ast C_1 D_b) \ast D_b B)$ is locally relatively quasiconvex by Theorem 3.1.

When $G \equiv A \ast C_1 = C_2$, let $M_i$ be the maximal parabolic subgroup of $G$ containing $C_i$. There are two subsubcases:

$[i \in M_1]$ Then $C_2 \leq M_1$ and we revise the splitting to $G \equiv A \ast D_1 M_1$ where $D_1 = M_1 \cap A$. And in this splitting the edge group is maximal parabolic at $D_1 \subset A$, and $M_1$ is parabolic.

$[i \notin M_1]$ Let $D_i$ denote the maximal parabolic subgroup of $A$ containing $C_i$. Observe that $\{D_1, D_2\}$ is almost malnormal since $D_i = M_i \cap A$. We revise the HNN extension to the following:

$$\left(D_1' \ast C_1 = C_2 A\right) \ast D_1' = D_i$$

where the conjugated copies of $D_1$ in the HNN extension embed in the first and second factor of the AFP.

In both cases, the local relative quasiconvexity of $G$ now holds by Theorem 3.1 as before. □
4. Relative Quasiconvexity in Graphs of Groups

Gersten [7] and then Bowditch [3] showed that a hyperbolic group $G$ is hyperbolic relative to an almost malnormal quasiconvex subgroup. Generalizing work of Martinez-Pedroza [14], Yang introduced and characterized a class of parabolically extended structures for countable relatively hyperbolic groups [22]. We use his results to generalize our previous results. The following was defined in [22] for countable groups.

**Definition 4.1** (Extended Peripheral Structure). A peripheral structure consists of a finite collection $\mathcal{P}$ of subgroups of a group $G$. Each element $P \in \mathcal{P}$ is a peripheral subgroup of $G$. The peripheral structure $\mathcal{E} = \{E_i\}_{i \in I}$ extends $\mathcal{P} = \{P_i\}_{i \in I}$ if for each $i \in I$, there exists $j \in J$ such that $P_i \subseteq E_j$. For $E \in \mathcal{E}$, we let $\mathcal{P}_E = \{P_i : P_i \subseteq E, P_i \in \mathcal{P}, i \in I\}$.

We will use the following result of Yang [22].

**Theorem 4.2** (Hyperbolicity of Extended Peripheral Structure). Let $G$ be hyperbolic relative to $\mathcal{P}$ and let the peripheral structure $\mathcal{E}$ extend $\mathcal{P}$. Then $G$ is hyperbolic relative to $\mathcal{E}$ if and only if the following hold:

1. $\mathcal{E}$ is almost malnormal;
2. Each $E \in \mathcal{E}$ is quasiconvex in $G$ relative to $\mathcal{P}$.

**Definition 4.3** (Total). Let $G$ be hyperbolic relative to $\mathcal{P}$. The subgroup $H$ of $G$ is total relative to $\mathcal{P}$ if: either $H \cap P^g = P^g$ or $H \cap P^g$ is finite for each $P \in \mathcal{P}$ and $g \in G$.

The following is proven in [5]:

**Lemma 4.4.** If $G$ is f.g. and hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$ and each $P_i$ is hyperbolic relative to $\mathcal{E}_i = \{H_1, \ldots, H_{m_i}\}$, then $G$ is hyperbolic relative to $\bigcup_{1 \leq i \leq n} \mathcal{E}_i$.

As an application of Theorem 4.2 we now generalize Corollary 1.7 to handle the case where edge groups are quasiconvex and not merely parabolic.

**Theorem 4.5** (Combination along Total, Malnormal and Quasiconvex Subgroups).

1. Let $G_i$ be hyperbolic relative to $\mathcal{P}_i$ for $i = 1, 2$. Let $C_i \leq G_i$ be almost malnormal, total and relatively quasiconvex. Let $C_i' \leq C_1$. Then $G = G_1 *_{C_i' = C_2} G_2$ is hyperbolic relative to $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 - \{P_2 \in \mathcal{P}_2 : P_2^g \subseteq C_2, \text{ for some } g \in G_2\}$.
2. Let $G_1$ be hyperbolic relative to $\mathcal{P}$. Let $\{C_1, C_2\}$ be almost malnormal and assume each $C_i$ is total and relatively quasiconvex. Let $C_i' \leq C_1$. Then $G = G_1 *_{C_i' = C_2} G_2$ is hyperbolic relative to $\mathcal{P} = \mathcal{P} - \{P_2 \in \mathcal{P}_2 : P_2^g \subseteq C_2, \text{ for some } g \in G_2\}$.

**Proof.** (1): For each $i$, let

$$\mathcal{E}_i = \mathcal{P}_i - \{P \in \mathcal{P}_i : P^g \leq C_i, \text{ for some } g \in G_i\} \cup \{C_i\}$$

Without loss of generality, we can assume that $\mathcal{E}_i$ extends $\mathcal{P}_i$, since we can replace an element of $\mathcal{P}_i$ by its conjugate. We now show that $G_i$ is hyperbolic relative to $\mathcal{E}_i$ by verifying the two conditions of Theorem 4.2. $\mathcal{E}_i$ is malnormal in $G_i$, since $\mathcal{P}_i$ is almost malnormal and $C_i$ is total and almost malnormal. Each element of $\mathcal{E}_i$ is relatively quasiconvex, since $C_i$ is relatively quasiconvex by hypothesis and each element of $\mathcal{P}_i$ is relatively quasiconvex by Remark 2.2.

We now regard each $G_i$ as hyperbolic relative to $\mathcal{E}_i$. Therefore since the edge group $C_2 = C_1'$ is maximal on one side, by Corollary 1.7, $G$ is hyperbolic relative to $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 - \{C_2\}$.
We now apply Lemma 4.4 to show that $G$ is hyperbolic relative to $P$. We showed that $G$ is hyperbolic relative to $E$. But each element of $E$ is hyperbolic relative to $P$ that it contains. Thus by Lemma 4.4, we obtain the result.

(2): The proof is analogous to the proof of (1).

The following can be obtained by induction using Theorem 4.5 or can be proven directly using the same mode of proof.

Corollary 4.6. Let $G$ split as a finite graph of groups. Suppose

(a) Each $G_v$ is hyperbolic relative to $P_v$;
(b) Each $G_e$ is total and relatively quasiconvex in $G_v$;
(c) $\{G_e : e$ is attached to $v\}$ is almost malnormal in $G_v$ for each vertex $v$.

Then $G$ is hyperbolic relative to $\bigcup_v P_v - \{\text{repeats}\}$.

Yang characterized relative quasiconvexity with respect to extensions in [22] as follows:

**Theorem 4.7** (Quasiconvexity in Extended Peripheral Structure). Let $G$ be hyperbolic relative to $P$ and relative to $E$. Suppose that $E$ extends $P$. Then

1. If $H \leq G$ is quasiconvex relative to $P$, then $H$ is quasiconvex relative to $E$.
2. Conversely, if $H \leq G$ is quasiconvex relative to $E$, then $H$ is quasiconvex relative to $P$ if and only if $H \cap E^g$ is quasiconvex relative to $P$ for all $g \in G$ and $E \in E$.

We recall the following observation of Bowditch (see [16, Lem 2.7 and 2.9]).

**Lemma 4.8** ($G$-attachment). Let $G$ act on a graph $K$. Let $p, q \in K^0$ and $e$ be a new edge whose endpoints are $p$ and $q$. The $G$-attachment of $e$ is the new graph $K' = K \cup Ge$ which consists of the union of $K$ and copies $ge$ of $e$ attached at $gp$ and $gq$ for any $g \in G$. Note that $K'$ is $G$-cocompact/fine/hyperbolic if $K$ is.

In the following lemma, we prove that when a relatively hyperbolic group $G$ splits then relative quasiconvexity of vertex groups is equivalent to relative quasiconvexity of the edge groups.

**Lemma 4.9** (Quasiconvex Edges $\iff$ Quasiconvex Vertices). Let $G$ be hyperbolic relative to $P$. Suppose $G$ splits as a finite graph of groups whose vertex groups and edge groups are finitely generated. Then the edge groups are quasiconvex relative to $P$ if and only if the vertex groups are quasiconvex relative to $P$.

**Proof.** If the vertex groups are quasiconvex relative to $P$ then so are the edge groups, since relative quasiconvexity is preserved by intersection (see [10] [15]) in the f.g. group $G$. Assume the edge groups are quasiconvex relative to $P$. Let $K$ be a $(G; P)$ graph and let $T$ be the Bass-Serre tree for $G$. Let $f : K \to T$ be a $G$-equivariant map that sends vertices to vertices and edges to geodesics. Subdivide $K$ and $T$, so that each edge is the union of two length $\frac{1}{2}$ halfedges. Let $v$ be a vertex in $T$. It suffices to find a $G_v$-cocompact quasiconvex subgraph $L$ of $K$.

Let $\{e_1, \ldots, e_m\}$ be representatives of the $G_v$-orbits of halfedges attached to $v$. Let $\omega_i$ be the other vertex of $e_i$ for $1 \leq i \leq m$. Since each $G_{\omega_i} = G_{e_i}$ is f.g. by hypothesis, we can perform finitely many $G_{\omega_i}$-attachments of arcs so that the preimage of $\omega_i$ is connected for each $i$. This leads to finitely many $G$-attachments to $K$ to obtain a new fine hyperbolic graph $K'$. By mapping the newly attached edges to their associated vertices in $T$, we thus obtain a $G$-equivariant map $f' : K' \to T$ such that $M'_i = f'^{-1}(\omega_i)$ is connected and $G_{\omega_i}$-cocompact for each $i$. 

Lemma 4.10 (Total Edges \iff Total Vertices). Let \( G \) be hyperbolic relative to \( \mathbb{P} \). Let \( G \) act on a tree \( T \). For each \( P \in \mathbb{P} \) let \( T_P \) be a minimal \( P \)-subtree. Assume that no \( T_P \) has a finite edge stabilizer in the \( P \)-action. Then edge groups of \( T \) are total in \( G \) if and only if vertex groups are total in \( G \).

Proof. Since the intersection of two total subgroups is total, if the vertex groups are total then the edge groups are also total. We now assume that the edge groups are total. Let \( G_v \) be a vertex group and \( P \in \mathbb{P} \) such that \( P^G \cap G_v \) is infinite for some \( g \in G \). If \( |P^G \cap G_v| = \infty \) for some edge \( e \) attached to \( v \), then \( P \subseteq G_v \), thus \( P \subseteq G_v \subseteq G_v \). Now suppose that \( |P^G \cap G_v| < \infty \) for each \( e \) attached to \( v \). If \( P^G \nsubseteq G_v \), then the action of \( P^G \) on \( gT \) violates our hypothesis. \( \square \)

Remark 4.11. Suppose \( G \) is f.g. and \( G \) is hyperbolic relative to \( \mathbb{P} \). Let \( P \in \mathbb{P} \) such that \( P = A \ast C B \) \([P = A_\ast C \ast B]\) where \( C \) is a finite group. Since \( P \) is hyperbolic relative to \( [A,B] \) \([\{A\}]\), by Lemma 4.4 \( G \) is hyperbolic relative to \( \mathbb{P}' = \mathbb{P} - \{P\} \cup \{A,B\} \) \([\mathbb{P}' = \mathbb{P} - \{P\} \cup \{A\}]\).

We now describe a more general criterion for relative quasiconvexity which is proven by combining Corollary 2.7 with Theorem 4.7.

Theorem 4.12. Let \( G \) be f.g. and hyperbolic relative to \( \mathbb{P} \). Suppose \( G \) splits as a finite graph of groups. Suppose
(a) Each \( G_v \) is total in \( G \);
(b) Each \( G_v \) is relatively quasiconvex in \( G \);
(c) \( \{G_v : e \) is attached to \( v \}\) is almost malnormal in \( G_v \) for each vertex \( v \).

Let \( H \leq G \) be tamely generated subgroup of \( G \). Then \( H \) is relatively quasiconvex in \( G \).

Proof. Technical Point: By splitting certain elements of \( \mathbb{P} \) to obtain \( \mathbb{P}' \) as in Remark 4.11, we can assume that \( G \) is hyperbolic relative to \( \mathbb{P}' \) and each \( G_v \) is hyperbolic relative to the conjugates of elements of \( \mathbb{P}' \) that it contains.

Indeed for any \( P \in \mathbb{P} \), if the action of \( P \) on a minimal subtree \( T_P \) of the Bass-Serre tree \( T \), yields a finite graph \( \Gamma \) of groups some of whose edge groups are finite, then following Remark 4.11 we can replace \( \mathbb{P} \) by the groups that complement these finite edge groups, (i.e. the fundamental groups of the subgraphs obtained by deleting these edges from \( \Gamma \).) Therefore \( G \) is hyperbolic relative to \( \mathbb{P}' \).

No \( P \in \mathbb{P}' \) has a nontrivial induced splitting as a graph of groups with a finite edge group. The edge groups are total relative to \( \mathbb{P}' \) since they are total relative to \( \mathbb{P} \). Therefore by Lemma 4.10 the vertex groups are total in \( G \) relative to \( \mathbb{P}' \). By Lemma 4.9 each vertex group \( G_v \) is relatively quasiconvex in \( G \) relative to \( \mathbb{P} \), therefore by Theorem 4.7 each \( G_v \) is quasiconvex in \( G \) relative to \( \mathbb{P}' \). Thus \( G_v \) has an induced relatively hyperbolic structure \( \mathbb{P}'_v \) as in Remark 2.2. By totality of \( G_v \), we can assume each element of \( \mathbb{P}'_v \) is a conjugate of an element of \( \mathbb{P}' \). And as usual we may omit the finite subgroups in \( \mathbb{P}'_v \).
Step 1: We now extend the peripheral structure of each $G_v$ from $P'_v$ to $P_v$ where
\[
E_v = \{ G_e : e \text{ is attached to } v \} \cup \{ P \in P'_v : P^g \not\in G_e \text{ for any } g \in G_v \}
\]
Almost malnormality of $E_v$ follows from Condition (c) and the totality of the edge groups in their vertex groups which follows by the totality of the edge groups in $G$, also relative quasiconvexity of the new elements $G_e$ is Condition (b). Thus by $G_v$ is hyperbolic relative to $E_v$ by Theorem 4.7.

Step 2: For each $\tilde{v}$ in the Bass-Serre tree, its $H$-stabilizer $H_{\tilde{v}}$ lies in $G_v$ which we identify (by a conjugacy isomorphism) with the chosen vertex stabilizer $G_v$ in the graph of group decomposition. Then $H_v$ is quasiconvex in $G_v$ relative to $E_v$ for each $v$ by Theorem 4.7 since $E_v$ extends $P'_v$ and each $H_v$ is quasiconvex in $G_v$ relative to $P'_v$. Therefore $H$ is quasiconvex relative to $\bigcup E_v$ by Corollary 2.7.

Step 3: $H$ is quasiconvex relative to $P' = \bigcup P'_v$. Since $\bigcup E_v$ extends $P = \bigcup P'_v$, by Theorem 4.7 it suffices to show that $H \cap K^g$ is quasiconvex relative to $P'$ for all $K \in \bigcup E_v$ and $g \in G$. There are two cases:

Case 1: $K \in P'_v$ for some $v$. Now $H \cap K^g$ is a parabolic subgroup of $G$ relative to $P'$ and is thus quasiconvex relative to $P'$.

Case 2: $K = G_e$ for some $e$ attached to some $v$. The group $K$ is relatively quasiconvex in $G_v$, therefore by Lemma 2.5 $K^g$ is also relatively quasiconvex but in $G_{gv}$. Now since $K^g \cap H = K^g \cap H_{gv}$ and $K^g$ and $H_{gv}$ are both relatively quasiconvex in $G_{gv}$, the group $K^g \cap H$ is relatively quasiconvex in $G_{gv}$. Since by Lemma 4.9 $G_{gv}$ is quasiconvex relative to $P'_v$, Lemma 2.5 implies that $K^g \cap H$ is quasiconvex relative to $P'$.

Now $H$ is quasiconvex relative to $P$ by Theorem 4.7 since $P$ extends $P'$. \qed

The following result strengthens Theorem 4.12 by relaxing Condition (c).

**Theorem 4.13 (Quasiconvexity Criterion for Relatively Hyperbolic Groups that Split).** Let $G$ be f.g. and hyperbolic relative to $P$ such that $G$ splits as a finite graph of groups. Suppose

(a) Each $G_e$ is total in $G$;

(b) Each $G_e$ is relatively quasiconvex in $G$;

(c) Each $G_e$ is almost malnormal in $G$.

Let $H \leq G$ be tamely generated. Then $H$ is relatively quasiconvex in $G$.

**Remark 4.14.** By Lemma 2.5 and Remark 2.7, Condition (b) is equivalent to requiring that each $G_e$ is quasiconvex in $G_v$. Also we can replace Condition (c) by requiring $G_e$ to be total in $G_v$.

**Proof.** We prove the result by induction on the number of edges of the graph of groups $\Gamma$. The base case where $\Gamma$ has no edge is contained in the hypothesis. Suppose that $\Gamma$ has at least one edge $e$ (regarded as an open edge). If $e$ is nonseparating, then $G = A*D$ where $A$ is the graph of groups over $\Gamma - e$, and $C, D$ are the two images of $G_e$. Condition (c) ensures that $[C, D]$ is almost malnormal in $A$, and by induction, the various nontrivial intersections $H \cap A^g$ are relatively quasiconvex in $A^g$, and thus $H$ is relatively quasiconvex in $G$ by Theorem 4.12. A similar argument concludes the separating case. \qed

**Corollary 4.15.** Let $G$ be f.g. and hyperbolic relative to $P$. Suppose $G$ splits as a finite graph of groups. Assume:

(a) Each $G_v$ is locally relatively quasiconvex;

(b) Each $G_e$ is Noetherian, total and relatively quasiconvex in $G$;


(c) Each $G_v$ is almost malnormal in $G$.

Then $G$ is locally relatively quasiconvex relative to $\mathbb{P}$.

**Theorem 4.16.** Let $G$ be hyperbolic relative to $\mathbb{P}$. Suppose $G$ splits as a graph $\Gamma$ of groups with relatively quasiconvex edge groups. Suppose $\Gamma$ is bipartite with $\Gamma^0 = V \sqcup U$ and each edge joins vertices of $V$ and $U$. Suppose each $G_v$ is maximal parabolic for $v \in V$, and for each $P \in \mathbb{P}$ there is at most one $v$ with $P$ conjugate to $G_v$. Let $H \leq G$ be tamely generated. Then $H$ is quasiconvex relative to $\mathbb{P}$.

The scenario of Theorem 4.16 arises when $M$ is a compact aspherical 3-manifold, from its JSJ decomposition. The manifold $M$ decomposes as a bipartite graph $\Gamma$ of spaces with $\Gamma^0 = U \sqcup V$. The submanifold $M_v$ is hyperbolic for each $v \in V$, and $M_u$ is a graph manifold for each $u \in U$. The edges of $\Gamma$ correspond to the “transitional tori” between these hyperbolic and complementary graph manifold parts. Some of the graph manifolds are complex but others are simpler Seifert fibered spaces; in the simplest cases, thickened tori between adjacent hyperbolic parts or $I$-bundles over Klein bottles where a hyperbolic part terminates. Hence $\pi_1 M$ decomposes accordingly as a graph $\Gamma$ of groups, and $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_u : u \in U\}$ by Theorem 1.4 or indeed, Corollary 1.5.

**Proof.** Let $K_\omega$ be a fine hyperbolic graph for $G$. Each vertex group is quasiconvex in $G$ by Lemma 4.9, and so for each $u \in U$ let $K_u$ be a $G_u$-quasiconvex subgraph, and in this way we obtain finite hyperbolic $G_\omega$-graphs, and for $v \in V$, we let $K_v$ be a singleton. We apply the Construction in the proof of Theorem 1.4 to obtain a fine hyperbolic $G$-graph $K$ and quotient $\bar{K}$. Note that the parabolic trees are $i$-pods. We form the $H$-cocompact quasiconvex subgraph $L$ by combining $H_\omega$-cocompact quasiconvex subgraphs $K_\omega$ as in the proof of Theorem 2.6. \qed

**Theorem 4.17.** Let $G$ be f.g. and hyperbolic relative to $\mathbb{P}$. Suppose $G$ splits as graph $\Gamma$ of groups with relatively quasiconvex edge groups. Suppose $\Gamma$ is bipartite with $\Gamma^0 = V \sqcup U$ and each edge joins vertices of $V$ and $U$. Suppose each $G_v$ is almost malnormal and total in $G$ for $v \in V$. Let $H \leq G$ be tamely generated. Then $H$ is quasiconvex relative to $\mathbb{P}$.

Theorem 4.17 covers the case where edge groups are almost malnormal on both sides since we can subdivide to put barycenters of edges in $V$.

Another special case where Theorem 4.17 applies is where $G = G_1 *_{G'_1} G_2$ is hyperbolic relative to $\mathbb{P}$, and $C_2 \leq G_2$ is total and relatively quasiconvex in $G$ and almost malnormal in $G_2$.

**Proof.** Following the Technical Point in the proof of Theorem 4.12 by splitting certain elements of $P$ to obtain $P'$ as in Remark 4.11, we can assume that $G$ is hyperbolic relative to $P'$ where each $P' \in P'$ is elliptic with respect to the action of $G$ on the Bass-Serre tree $T$. Since $P$ extends $P'$ and each $G_v \cap P'$ is conjugate to an element of $P'$, we see that each $G_v$ is quasiconvex in $G$ relative to $P'$ by Theorem 4.7 and moreover, since elements of $P'$ are vertex groups of elements of $P$, each $G_v$ is total relative to $P'$. Therefore each $G_v$ is hyperbolic relative to a collection $P'_v$ of conjugates of elements of $P'$.

We argue by induction on the number of edges of $\Gamma$. If $\Gamma$ has no edge the result is contained in the hypothesis. Suppose $\Gamma$ has at least one edge $e$. If $e$ is separating and $\Gamma = \Gamma_1 \sqcup e \sqcup \Gamma_2$ where $e$ attaches $v \in \Gamma_1^0$ to $u \in \Gamma_2^0$ then $G = G_1 *_{G_v} G_2$ where $G_i = \pi_1 (\Gamma_i)$. Each $G_v$ is the intersection of vertex groups and hence quasiconvex relative to $P'$. By Lemma 4.9 the groups $G_1$ and $G_2$ are quasiconvex in $G$ relative to $P'$. Thus $G_i$ is hyperbolic relative to $P'_i$ by Remark 2.2.
Observe that $T$ contains subtrees $T_1$ and $T_2$ that are the Bass-Serre trees of $\Gamma_1$ and $\Gamma_2$, and $T - \hat{G} = \{ gT_1 \cup gT_2 : g \in G \}$. The Bass-Serre tree $\hat{T}$ of $G_1 *_{G_v} G_2$ is the quotient of $T$ obtained by identifying each $gT_i$ to a vertex.

Since $\hat{H}$ is relatively finitely generated, there is a finite graph of groups $\Gamma_{\hat{H}}$ for $\hat{H}$, and a map $\Gamma_{\hat{H}} \to \Gamma$. Removing the edges mapping to $e$ from $\Gamma_{\hat{H}}$, we obtain a collection of finitely many graphs of groups - some over $\Gamma_1$ and some over $\Gamma_2$. Each component of $\Gamma_{\hat{H}}$ corresponds to the stabilizer of some $gT_i$ and is denoted by $H_{gT_i}$, and since that component is a finite graph with relatively quasiconvex vertex stabilizers, we see that each $H_{gT_i}$ is relatively quasiconvex in $G_i$ relative to $P_i$ by induction on the number of edges of $\Gamma_{\hat{H}}$.

We extend the peripheral structure $P'_i$ of $G_i$ to $B_1 = \{ G_i \}$. Note that now each $H_{gT_i}$ is quasiconvex in $G_1$ relative to $B_1$ by Theorem 4.7. Let $B = B_1 \cup P'_2 - \{ P \in P'_2 : P^g \leq G_v, \text{ for some } g \in G_2 \}$.

Observe that $B$ extends $P'$. Since $G_v$ is total and quasiconvex in $G$ relative to $P'$ and $B$ extends $P'$, the group $G_1$ is total and quasiconvex in $G$ relative to $B$ by Theorem 4.7. Therefore $G$ is hyperbolic relative to $B$ by Theorem 4.2.

Since $G_1$ is maximal parabolic in $G$, by Theorem 4.16 $H$ is quasiconvex in $G$ relative to $B$. The graph $\Gamma_H$ shows that $H$ is generated by finitely many hyperbolic elements and vertex stabilizers $H_{gT_i}$ and each $H_{gT_i} = H_{fT_i}$ which we explained above is relatively quasiconvex in $G_i$.

We now show that $H$ is quasiconvex relative to $P'$ and therefore relative to $P$ by Theorem 4.7. Since $B$ extends $P'$, by Theorem 4.7 it suffices to show that $H \cap E^g$ is quasiconvex relative to $P'$ for all $E \in B$ and $g \in G$. There are two cases:

Case 1: $E \in P'_2$. Now $H \cap E^g$ is a parabolic subgroup of $G$ relative to $P'$ and is thus quasiconvex relative to $P'$.

Case 2: $E = G_1$. Then $H \cap E^g$ is quasiconvex relative to $P'_1$ since $(H \cap E^g) = H_{gT_1}$ is quasiconvex in $G_1^g$ relative to $B_1^g = \{ G_1^g \}$. Since $E^g = G_1^g$ is quasiconvex relative to $P'$, Lemma 2.5 implies that $H \cap E^g$ is quasiconvex relative to $P'$.

Now assume that $e$ is nonseparating. Let $u \in U$ and $v \in V$ be the endpoints of $e$. Then $G = G_1 *_{C \times D}$ where $G_1$ is the graph of groups over $\Gamma - e$, and $C$ and $D$ are the images of $G_v$ in $G_1$ and $G_u$, respectively. We first reduce the peripheral structure of $G$ from $P$ to $P'$, and then extend from $P'$ to $B$ with:

$B = \{ G_v \} \cup P' - \{ P \in P' : P^g \leq G_v, \text{ for some } g \in G \}$.

$G$ is hyperbolic relative to $B$ by Theorem 4.2 as $G_v$ is almost malnormal, total, and quasiconvex relative to $P$. The argument follows by induction and Theorem 4.16 as in the separating case. $\square$

Theorem 4.13 suggests the following criterion for relative quasiconvexity:

**Conjecture 4.18.** Let $G$ be hyperbolic relative to $P$. Suppose $G$ splits as a finite graph of groups with f.g. relatively quasiconvex edge groups. Suppose $H \leq G$ is tamely generated such that each $H_v$ is f.g. for each $v$ in the Bass-Serre tree. Then $H$ is relatively quasiconvex in $G$.

When the edge groups are separable in $G$, there is a finite index subgroup $G'$ whose splitting has relatively malnormal edge groups (see e.g. [11, 9]). Consequently, if moreover, the edge groups of $G$ are total, then the induced splitting of $G'$ satisfies the criterion of Theorem 4.13 and we see that Conjecture 4.18 holds in this case. In particular, Conjecture 4.18 holds when
$G$ is virtually special and hyperbolic relative to virtually abelian subgroups, provided that edge groups are also total. We suspect the totalness assumption can be dropped totally.

As a closing thought, consider a hyperbolic 3-manifold $M$ virtually having a malnormal quasiconvex hierarchy (conjecturally all closed $M$). Theorem 4.13 suggests an alternate approach to the tameness theorem, which could be reproven by verifying:

If the intersection of a f.g. $H$ with a malnormal quasiconvex edge group is infinitely generated then $H$ is a virtual fiber.

Acknowledgement: We are extremely grateful to the anonymous f.g. referee whose very helpful corrections and adjustments improved the results and exposition of this paper.

REFERENCES

[1] E. Alibegović. A combination theorem for relatively hyperbolic groups. Bull. London Math. Soc., 37(3):459–466, 2005.
[2] M. Bestvina and M. Feighn. A combination theorem for negatively curved groups. J. Differential Geom., 35(1):85–101, 1992.
[3] B. Bowditch. Relatively hyperbolic groups. pages 1–63, 1999. Preprint.
[4] F. Dahmani. Combination of convergence groups. Geom. Topol., 7:933–963 (electronic), 2003.
[5] C. Drutu and M. Sapir. Tree-graded spaces and asymptotic cones of groups. Topology, 44(5):959–1058, 2005.

With an appendix by Denis Osin and Sapir.
[6] B. Farb. Relatively hyperbolic groups. Geom. Funct. Anal., 8(5):810–840, 1998.
[7] S. M. Gersten. Subgroups of word hyperbolic groups in dimension 2. J. London Math. Soc. (2), 54(2):261–283, 1996.
[8] M. Gromov. Hyperbolic groups. In Essays in group theory, volume 8 of Math. Sci. Res. Inst. Publ., pages 75–263. Springer, New York, 1987.
[9] F. Haglund and D. T. Wise. A combination theorem for special cube complexes. Ann. of Math. To appear.
[10] G. C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Algebr. Geom. Topol., 10(3):1807–1856, 2010.
[11] G. C. Hruska and D. T. Wise. Packing subgroups in relatively hyperbolic groups. Geom. Topol., 13(4):1945–1988, 2009.
[12] I. Kapovich. Subgroup properties of fully residually free groups. Trans. Amer. Math. Soc., 354(1):335–362 (electronic), 2002.
[13] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds. Geom. Funct. Anal., 5(3):582–603, 1995.
[14] E. Martínez-Pedroza. On Quasiconvexity and Relative Hyperbolic Structures. ArXiv e-prints, Nov. 2008.
[15] E. Martínez-Pedroza. Combination of quasiconvex subgroups of relatively hyperbolic groups. Groups Geom. Dyn., 3(2):317–342, 2009.
[16] E. Martínez-Pedroza and D. T. Wise. Relative quasiconvexity using fine hyperbolic graphs. Algebr. Geom. Topol., 11(1):477–501, 2011.
[17] M. Mj and L. Reeves. A combination theorem for strong relative hyperbolicity. Geom. Topol., 12(3):1777–1798, 2008.
[18] D. V. Osin. Elementary subgroups of relatively hyperbolic groups and bounded generation. Internat. J. Algebra Comput., 16(1):99–118, 2006.
[19] D. V. Osin. Relative Dehn functions of amalgamated products and HNN-extensions. 394:209–220, 2006.
[20] D. V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. Mem. Amer. Math. Soc., 179(843):vi+100, 2006.
[21] P. Tukia. Convergence groups and Gromov’s metric hyperbolic spaces. New Zealand J. Math., 23(2):157–187, 1994.
[22] W. Yang. Peripheral structures of relatively hyperbolic groups. ArXiv e-prints, Jan. 2011.
DEPT. OF MATH & STATS., MCGILL UNIVERSITY, MONTREAL, QUEBEC, CANADA H3A 2K6

E-mail address: bigdely@math.mcgill.ca

E-mail address: wise@math.mcgill.ca