Support of Closed Walks and Second Eigenvalue Multiplicity of Regular Graphs

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Abstract

We show that the multiplicity of the second adjacency matrix eigenvalue of any connected $d$-regular graph is bounded by $O(nd^{7/5}/\log^{1/5-o(1)} n)$ for any $d$, and by $O(n \log^{1/2} d/\log^{1/4-o(1)} n)$ when $d \geq \log^{1/4} n$ and the graph is simple. In fact, the same bounds hold for the number of eigenvalues in any interval of width $\lambda_2/\log_d^{1-o(1)} n$ containing the second eigenvalue $\lambda_2$. The main ingredient in the proof is a polynomial (in $k$) lower bound on the typical support of a random closed walk of length $2k$ in any connected regular graph, which further relies on new lower bounds for the entries of the Perron eigenvector of a connected irregular graph.

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1 Introduction

In their recent beautiful work on the equiangular lines problem, Jiang, Tidor, Yao, Zhang, and Zhao [JTY+19] proved the following remarkable result.

Theorem 1.1. If $G$ is a connected graph of maximum degree $\Delta$ on $n$ vertices, then the multiplicity of the second largest eigenvalue of its adjacency matrix $A_G$ is bounded by $O(n \log \Delta / \log \log(n))$.

For their application to equiangular lines, [JTY+19] only needed to show that the multiplicity of the second eigenvalue is $o(n)$, but they asked whether the $O(n / \log \log(n))$ dependence in Theorem 1.1 could be improved, noting a huge gap between this and the best known lower bound of $\Omega(n^{1/3})$ achieved by certain Cayley graphs of $\text{PSL}(2, p)$ (see [JTY+19, Section 4]). Apart from Theorem 1.1, there are as far as we are aware no known sublinear upper bounds on the second eigenvalue multiplicity for any general class of graphs, even if the question is restricted to Cayley graphs (unless one imposes a restriction on the spectral gap; see Section 1.1 for a discussion). As this eigenvalue plays an important role in many areas of mathematics, any general phenomena concerning it are of potentially broad impact.

In this work, we prove significantly stronger upper bounds under the additional assumption that the graph $G$ is regular. Graphs are undirected and allowed to have multiedges and self-loops unless explicitly stated as being simple. Order the eigenvalues of $A_G$ as $\lambda_1(A_G) \geq \lambda_2(A_G) \geq \ldots \geq \lambda_n(A_G)$, and let $m_G(I)$ denote the number of eigenvalues of $A_G$ in an interval $I$.

Theorem 1.2. If $G$ is a connected $d$–regular graph on $n$ vertices with $\lambda_2(A_G) = \lambda_2$, then

$$m_G\left(\left(1 - \frac{\log \log_d n}{\log_d n}\right)\lambda_2, \lambda_2 \right) = \tilde{O}\left(n \cdot \frac{d^{7/5}}{\log^{1/5} n}\right).$$

(1)

If $G$ is simple, then further

$$m_G\left(\left(1 - \frac{\log \log_d n}{\log_d n}\right)\lambda_2, \lambda_2 \right) = \begin{cases} \tilde{O}\left(\frac{d}{n}\right) & \text{when } d = o(\log^{1/4} n) \\ \tilde{O}\left(\frac{n \log^{1/2} d}{\log^{1/4} n}\right) & \text{when } d = \Omega(\log^{1/4} n). \end{cases}$$

(2)

In addition to the improved $O(n / \text{polylog}(n))$ bounds, one difference between our results and Theorem 1.1 is that we control the number of eigenvalues in a small interval containing $\lambda_2$. Though we do not know whether the exponents in (1), (2) are sharp, we show in Section 5.1 that constant degree bipartite Ramanujan graphs have at least $\Omega(n / \log^{3/2} n)$ eigenvalues in the interval appearing in (1), indicating that $O(n / \text{polylog}(n))$ is the correct regime for the maximum number of eigenvalues in such an interval when $d \geq 3$ is a constant. Note that as $\lambda_2 \geq 2 \sqrt{d - 1} (1 - o_n(1))$ for a connected graph by [Nil91], the width of the interval under consideration increases with the degree.

A second difference is that we obtain nontrivial bounds for all $d \leq \exp(\log^{1/2 - \delta} n)$ assuming the graph is simple, whereas Theorem 1.1 requires $d = O(\text{polylog}(n))$. As remarked in [JTY+19], Paley graphs have degree $\Omega(n)$ and second eigenvalue multiplicity $\Omega(n)$, so some bound on the degree is required to control the multiplicity. It is also clear that the multiplicity can be $\Omega(n)$ if the graph is disconnected, by taking many copies of a small graph.

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1All asymptotics are as $n \to \infty$ and the notation $\tilde{O}(\cdot)$ suppresses polyloglog($n$) terms.
A closed walk of length $\ell$ in a graph $G$ is a sequence $\gamma = ((x_0, x_1), (x_1, x_2), \ldots, (x_{\ell-1}, x_\ell))$ such that $x_0 = x_\ell$ and each $(x_i, x_{i+1})$ is an edge in $G$. We refer to the vertex $x_0$ as the root of the walk, and to the number of distinct vertices in $\{x_0, \ldots, x_\ell\}$ as its support. The main ingredient in the proof of Theorem 1.2 is a new lower bound on the typical support of a random closed walk in a regular graph, rooted at any vertex.

**Theorem 1.3.** Suppose $G$ is connected and $d$-regular, $x$ is any vertex in $G$, and $\gamma$ is a uniformly random closed walk in $G$ of length $2k$ rooted at $x$. Then

$$\Pr(\text{support}(\gamma) \leq s) \leq \exp\left(\frac{-k}{65d^7s^4}\right) \quad \text{for} \quad s \leq \frac{1}{4}\left(\frac{k}{d^7 \log d}\right)^{1/5}. \hspace{1cm} (3)$$

If $G$ is simple, then moreover

$$\Pr(\text{support}(\gamma) \leq s) \leq \exp\left(\frac{-k}{100s^3}\right) \quad \text{for} \quad s \leq \min\left\{\frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4},\frac{d}{2}\right\}. \hspace{1cm} (4)$$

It may be tempting to compare Theorem 1.3 with the familiar fact that a random closed walk of length $2k$ on $\mathbb{Z}$ (or in continuous time, a standard Brownian bridge run for time $2k$) attains a maximum distance of $\Omega(\sqrt{k})$ from its origin. However, as seen in Figure 1, there are regular graphs for which a closed walk of length $2k$ from a particular vertex travels a maximum distance of only $\text{polylog}(k)$ with high probability. Theorem 1.3 reveals that nonetheless the number of distinct vertices traversed is always typically $\text{poly}(k)$. We show that regularity is in fact necessary for this phenomenon by analyzing in Section 5.2 an irregular “lollipop” graph for which the typical support of a closed walk from a specific vertex is only $O(\text{polylog}(k))$. We do not know if the specific exponents of $k^{1/5}$ and $k^{1/4}$ appearing in Theorem 1.3 are sharp, but considering a cycle graph shows that it is not possible to do better than $k^{1/2}$.

\[2\]It remains plausible that an averaged version of Theorem 1.3 in which $x$ is random holds for irregular graphs.
Figure 2: An example of a graph where all vertices $u$ that are not of maximum degree have $\psi(u) = \tilde{O}(n^{-5/2})$. The circled sets $X_0$, $X_1$ and $X_2$ will be used in the analysis of the graph in Section 5.3.

Given Theorem 1.3, our proof of Theorem 1.2 follows the strategy of [JTY+19]: since most closed walks in $G$ have large support, the number of such walks may be drastically reduced by deleting a small number of vertices from $G$. By a moment calculation relating the spectrum to closed walks and a Cauchy interlacing argument, this implies an upper bound on the multiplicity of $\lambda_2(A_G)$. The crucial difference is that we are able to delete only $n/polylog(n)$ vertices whereas they delete $n/poly \log \log(n)$.

The key ingredient in our proof of Theorem 1.3 is a result regarding the Perron eigenvector (i.e., the unique, strictly positive eigenvector with eigenvalue $\lambda_1$) of an irregular connected graph, which may be of independent interest.

**Theorem 1.4.** Let $H = (V, E)$ be an irregular connected graph of maximum degree $\Delta$ with at least two vertices, and let $\psi_H$ be the $\ell_2$-normalized Perron vector of $A_H$. Then there is a vertex $u \in V$ with degree strictly less than $\Delta$ satisfying:

$$\psi_H(u) \geq 1/(\Delta \lambda_1(A_H)|V|^{5/2}).$$

(5)

Theorem 1.4 may be compared with existing results in spectral graph theory on the “principal ratio” between the largest and smallest entries of the Perron vector of a connected graph. The known worst case lower bounds on this ratio are necessarily exponential in the diameter of the graph [CG07]. However, the worst case examples involve showing that two vertices separated by a long chain of degree 2 vertices have very different values in the Perron vector [TT15]. Theorem 1.4 articulates that there is always at least one vertex of non-maximal degree for which the ratio is only polynomial in the number of vertices.

The proof of Theorem 1.4 is based on an analysis of hitting times in the simple random walk on $H$ via electrical flows, and appears in Section 2. Combined with a perturbation-theoretic argument, it enables us to show that any small induced subgraph $H$ of the regular and connected graph $G$, which must be irregular, can be extended to a slightly larger subgraph with significantly larger Perron value. This implies that closed walks cannot concentrate on small sets, yielding Theorem 1.3 in Section 3, which is finally used to deduce Theorem 1.2 in Section 4.

We show in Section 5.3 via an explicit example (Figure 2) that the exponent of 5/2 appearing in Theorem 1.4 is sharp up to polylogarithmic factors.
Remark 1.5 (Higher Eigenvalues). An update of the preprint of [JTY+19] generalizes Theorem 1.1 to the multiplicity of the \( j \)th eigenvalue. Our results can also be generalized in this manner by some nominal changes to the arguments in Section 4, but for simplicity we focus on \( \lambda_2 \) in this paper.

1.1 Related work

Eigenvalue Multiplicity. Despite the straightforward nature of the question, relatively little is known about eigenvalue multiplicity of general graphs. As discussed in [JTY+19], if one assumes that \( G \) is a bounded degree expander graph, then the bound of Theorem 1.1 can be improved to \( O(n / \log n) \). On the other hand, if \( G \) is assumed to be a Cayley graph of bounded doubling constant \( K \) (indicating non-expansion), then [LM08] show that the multiplicity of the second eigenvalue is at most \( \exp(\log^2 K) \). In the context of Cayley graphs, one interesting new implication of Theorem 1.2 is that all Cayley graphs of degree \( O(\exp(\log^{1/2 - \delta} n)) \) have second eigenvalue multiplicity \( O(n / \log \delta/2 n) \).

Distance regular graphs of diameter \( D \) have exactly \( D + 1 \) distinct eigenvalues (see [God93] 11.4.1 for a proof). However, besides the top eigenvalue (which must have multiplicity 1), generic bounds on the multiplicity of the other eigenvalues are not known. As expanding graphs have diameter \( \Theta(\log_d n) \), the average multiplicity of eigenvalues besides \( \lambda_1 \) for expanding distance regular graphs is \( \Theta(n/\log_d n) \). It is tempting to see this as a hint that the multiplicity of the second eigenvalue could be \( \Omega(n/\log_d n) \).

Sublinear multiplicity does not necessarily hold for eigenvalues in the interior of the spectrum even assuming bounded degree. In particular, Rowlinson constructed \( d \)-regular graphs with an eigenvalue of multiplicity at least \( n(d - 2)/(d + 2) \) [Row19].

Higher Order Cheeger Inequalities. The results of [LRTV12, LGT14] imply that if a \( d \)-regular graph \( G \) has a second eigenvalue multiplicity of \( m \), then its vertices can be partitioned into \( \Omega(m) \) disjoint sets each having edge expansion \( O(\sqrt{(d - \lambda_2) \log m}) \). Combining this with the observation that a set cannot have expansion less than the reciprocal of its size shows that \( m = O(n/\text{polylog}(n)) \) whenever \( d - \lambda_2(A_G) \leq d/\log^c n \) for any \( c > 1 \), i.e., the graph is sufficiently non-expanding.

Support of Closed Walks. There are as far as we are aware no known lower bounds for the support of a random closed walk of fixed length in a general graph (or even Cayley graph). It is relatively easy to derive such bounds for bounded degree graphs if the length of the walk is sufficiently larger than the mixing time of the simple random walk on the graph; the key feature of Theorem 1.3, which is needed for our application, is that the length of the walk can be taken much smaller.

Entries of the Perron Vector. There is a large literature concerning the magnitude of the entries of the Perron eigenvector of a graph — see [Ste14, Chapter 2] for a detailed discussion of results up to 2014. Rowlinson showed sufficient conditions on the Perron eigenvector for which changing the neighborhood of a vertex increases the spectral radius [Row90]. Cvetković, Rowlinson, and Simić give a condition which, if satisfied, means a given edge swap increases the spectral radius [CRS93]. Cioabă showed that for a graph of maximum degree \( \Delta \) and diameter \( D \), \( \Delta - \lambda_1 > 1/nD \) [Cio07]. Cioabă, van Dam, Koolen, and Lee then showed that \( \lambda_1 \geq (n - 1)^{1/D} \) [CVDKL10]. The results of [VMSK+11] prove a lemma similar to Lemma 3.2, giving upper and lower bounds on the change in
spectral radius from the deletion of edges. However, their result does not imply Lemma 3.2, and we prove a slightly different statement.

1.2 Notation
All logarithms are base $e$ unless noted otherwise.

**Perron Eigenvector.** We use $\psi_H$ to denote the $\ell_2$-normalized eigenvector corresponding to $\lambda_1(A_H)$, which is a simple eigenvalue if $H$ is connected. Note that for connected $H$, $\psi_H$ is strictly positive by the Perron-Frobenius theorem.

**Electrical Flows.** We use $\text{Reff}_H(\cdot, \cdot)$ to denote the effective resistance between two vertices in $H$, viewing each edge of the graph as a unit resistor. See e.g. [DS84] or [Bol13, Chapter IX] for an introduction to electrical flows and random walks on graphs.

**Graphs.** For a graph $G = (V, E)$, we denote by $G_S$ the subgraph induced by $S \subset V$ with adjacency matrix $A_S$; when $G$ is clear from the context, we will write $\psi_S$ instead of $\psi_G$. A simple graph refers to a graph without multiedges or self-loops. An irregular graph is one which is not regular. We assume $d \geq 2$ for all connected regular graphs, since otherwise the graph is just an edge, so $\log d > 0$.

2 Perron Eigenvector of Irregular Graphs

In this section we prove Theorem 1.4, which is a direct consequence of the following slightly more refined result.

**Theorem 2.1 (Large Perron Entry).** Let $H = (V, E)$ be an irregular connected graph of maximum degree $\Delta$. Then there is a vertex $u \in V$ with degree strictly less than $\Delta$ satisfying:

$$\frac{\psi_H(u)}{\psi_H(t)} \geq \frac{1}{\Delta \lambda_1(A_H)|V|^2},$$

(6)

where $t = \arg \max_{w \in V} \psi_H(w)$.

**Proof.** Write $V = M \cup B$, where $B = \{x \in V : \deg(x) < \Delta\}$ and $M = V \setminus B$. If $t \in B$ then we are done, so assume not. Let $\mathbb{P}_x^H(\cdot)$ denote the law of the simple random walk (SRW) $(X_i)_{i=0}^\infty$ on $H$ started at $X_0 = x$, and for any subset $S \subset V$, let $\tau_S := \min \{i : X_i \in S\}$ denote the hitting time of the SRW to that subset; if $S = \{u\}$ is a singleton we will simply write $\tau_u$.

**Step 1.** We begin by showing that there is a vertex $x \in M$ adjacent to $B$ for which the random walk started at $x$ is reasonably likely to hit $t$ before $B$ (we will eventually choose $u$ to be a neighbor of this $x$). To do so, we use the well-known connection between hitting probabilities in random walks and electrical flows. Define a new graph $H' = (V' = V \setminus B \cup \{s\}, E')$ by contracting all vertices in $B$ to a single vertex $s$. Let $f : V' \to [0, 1]$ be the vector of voltages in the electrical flow in $H'$ with boundary conditions $f(s) = 0$, $f(t) = 1$, regarding every edge as a unit resistor. By Ohm’s law, the current flow from $s$ to $t$ is equal to $1/\text{Reff}_H(s, t)$. We have the crude upper bound $\text{Reff}_H(s, t) \leq |V'| \leq |V|$, so the outflow of current from $s$ is at least $1/|V|$. By Kirchhoff’s current law, there must be a flow of at least
where $\partial_H B$ denotes the edge boundary of $B$ in $H$. Appealing to e.g. [Bol13, Chapter IX, Theorem 8], this translates to the probabilistic bound

$$\mathbb{P}_x^{H}(\tau_t < \tau_B) = \mathbb{P}_x^{H'}(\tau_t < \tau_s) = f(x) \geq \frac{1}{|\Delta|^{2}}.$$  \hspace{1cm} (8)

**Step 2.** We now use (8) to show that $\psi_H(x)$ is large. Let $H_1$ denote $H$ with a self-loop added to each vertex\(^3\), and to ease notation let $\mathbb{P}_x(\cdot) := \mathbb{P}_x^{H_1}(\cdot)$ denote the law of the SRW on $H_1$ started at $x$. Note that $\psi_H = \psi_{H_1}$ and

$$\mathbb{P}_x(\tau_t < \tau_B) = \mathbb{P}_x^{H}(\tau_t < \tau_B)$$  \hspace{1cm} (9)

by (8) since adding self-loops to $H$ preserves electrical flows, which determine the probabilities in (9).

Since $H_1$ is connected and non-bipartite, the Perron-Frobenius theorem implies that:

$$\psi_H(w) = \psi_{H_1}(w) = \lim_{k \to \infty} \frac{1^T A^k_{H_1} e_w}{\|1^T A^k_{H_1}\|},$$

for every $w \in V$. Writing $A_{H_1} = (A_{H_1} D^{-1}) D = PD$ for $P$ the Markov transition matrix of the SRW on $H_1$ and $D$ the diagonal matrix of degrees of vertices in $H_1$, we find that

$$1^T A^k_{H_1} e_w = \mathbb{E}_w \prod_{i=0}^{k-1} D(X_i),$$

where $D(X_i)$ is the degree of $X_i$ in $H_1$.

We are interested in the ratio

$$\frac{\psi_H(x)}{\psi_H(t)} = \lim_{k \to \infty} \frac{\mathbb{E}_x \prod_{i=0}^{k-1} D(X_i)}{\mathbb{E}_t \prod_{i=0}^{k-1} D(X_i)},$$  \hspace{1cm} (10)

Fix an integer $k > 0$. The numerator of (10) is bounded as

$$\mathbb{E}_x \prod_{i=0}^{k-1} D(X_i) \geq \mathbb{E}_x \left( \prod_{i=0}^{k-1} D(X_i) |\tau_t < \tau_B \right) \mathbb{P}_x(\tau_t < \tau_B)$$

$$\geq \frac{1}{|\Delta|^{2}} \mathbb{E}_x \left( \prod_{i=0}^{k-1} D(X_i) |\tau_t < \tau_B \right)$$

$$\geq \frac{1}{|\Delta|^{2}} \sum_{\theta = 0}^{k-1} \mathbb{E}_x \left( \prod_{i=0}^{k-1} D(X_i) |\theta = \tau_t, \tau_t < \tau_B \right) \mathbb{P}_x(\tau_t = \theta |\tau_t < \tau_B).$$  \hspace{1cm} (11)

\(^3\)This modification is only to ensure non-bipartiteness; if $H$ is not bipartite we may take $H_1 = H$ and the proof works with $\Delta$ in place of $\Delta + 1$ in (12).
As the degree in $H_1$ of every vertex $X_i$ encountered before hitting $B$ is equal to $\Delta + 1$, each conditional expectation appearing in (11) may be rewritten as
\[
E_x \left( \prod_{i=0}^{k-1} D(X_i) | \theta = \tau_t, \tau_t < \tau_B \right) = E_x \left( \prod_{i=0}^{\theta-1} (\Delta + 1) \cdot \prod_{i=0}^{k-1} D(X_i) | \theta = \tau_t, \tau_t < \tau_B \right)
\]
\[
= (\Delta + 1)^{\theta} E_x \left( \prod_{i=0}^{k-1} D(X_i) | \theta = \tau_t, \tau_t < \tau_B \right)
\]
\[
= (\Delta + 1)^{\theta} E_x \left( \prod_{i=0}^{k-1} D(X_i) | X_{\theta} = t \right)
\]
\[
= (\Delta + 1)^{\theta} E_{\tau} \left( \prod_{i=0}^{k-1} D(X_i) \right)
\]
\[
\geq E_{\tau} \left( \prod_{i=0}^{k-1} D(X_i) \right),
\]
(12)
since $D(X_i) \leq \Delta + 1$ for all $i \leq k - 1$.

Observe that $E_x \tau_B < \infty$ since $H_1$ is connected. Thus,
\[
\sum_{\theta=0}^{k-1} P_x(\tau_t = 0 | \tau_t < \tau_B) = 1 - P_x(\tau_t \geq k | \tau_t < \tau_B)
\]
\[
\geq 1 - \frac{P_x(\tau_B \geq k)}{P_x(\tau_t < \tau_B)}
\]
\[
\geq 1 - \frac{E_x \tau_B}{k} \cdot \Delta |V|^2 \quad \text{by Markov, (8), and (9)}.
\]
Combining this bound with (11) and (12), we have
\[
E_x \left( \prod_{i=0}^{k-1} D(X_i) \right) \geq \frac{1}{\Delta |V|^2} \left( 1 - \frac{\Delta |V|^2 E_x \tau_B}{k} \right) E_{\tau} \left( \prod_{i=0}^{k-1} D(X_i) \right).
\]
Taking the limit as $k \to \infty$ in (10) yields
\[
\frac{\psi_{H}(x)}{\psi_{H}(t)} \geq \frac{1}{\Delta |V|^2}.
\]

**Step 3.** Since $x$ is adjacent to $B$, we can choose a $u \in B$ adjacent to $x$. The eigenvector equation and nonnegativity of the Perron vector now imply $\lambda_1(A_{H}) \psi_{H}(u) \geq \psi_{H}(x)$, whence
\[
\psi_{H}(u) \geq \frac{1}{\lambda_1(A_{H}) \Delta |V|^2} \psi_{H}(t),
\]
as advertised. □

**Remark 2.2.** As the proof shows, the right-hand side of (6) may be replaced with $1/\lambda_1(A_{H}) \partial_H B|R$ where $B$ is the set of vertices of degree strictly less than $\Delta$ in $H$ and $R$ is the maximum effective resistance between two vertices in $H'$. 8
3 Support of Closed Walks

In this section we prove Theorem 1.3, which is an immediate consequence of the following slightly stronger result. Let $W^k_x$ denote the set of closed walks of length $2k$ in $G$ rooted at $x \in V$ with support at most $s$.

**Theorem 3.1 (Implies Theorem 1.3).** If $G$ is connected and $d$–regular, then for every vertex $x \in G$ and $k \in \mathbb{N}$,

$$|W^k_x| \leq \exp\left(-\frac{k}{65d^7s^4}\right) |W^k_x| \quad \text{for} \quad s \leq \frac{1}{4} \left(\frac{k}{d^7 \log d}\right)^{1/5}.$$

(13)

If further $G$ has exactly $h$ self-loops at every vertex and no multi-edges\footnote{This technical assumption is used to handle the case when $|\lambda_n(G)| > \lambda_2(G)$ in Section 4. For Theorem 1.3 we take $h = 0.$}, then

$$|W^k_x| \leq \exp\left(-\frac{k}{100s^3}\right) |W^k_x| \quad \text{for} \quad s \leq \min\left\{\frac{1}{8} \left(\frac{k}{\log d}\right)^{1/4}, \frac{d-h}{2}\right\}.$$

(14)

The proof requires a simple lemma lower bounding the increase in the Perron value of a subgraph upon adding a vertex in terms of the Perron vector.

**Lemma 3.2 (Perturbation of $\lambda_1$).** For any graph $H = (V, E)$ and vertex $u \in V$, the graph $H' = (V \cup \{v\}, E \cup \{(u,v)\})$, which adds a vertex $v$ and the edge $(u,v)$ to $H$, satisfies

$$\lambda_1(A_{H'}) \geq \frac{1}{2} \left(\lambda_1(A_H) + \sqrt{\lambda_1(A_H)^2 + \psi_H(u)^2}\right).$$

Proof. The largest eigenvalue of $A_{H'}$ is at least the maximum of the quadratic form associated with $A_{H'}$ of the unit vectors

$$g_{\alpha}(x) = \begin{cases} \sqrt{1 - \alpha^2} \psi_H(x) & x \in V \\ \alpha & x = v \end{cases}$$

for $0 \leq \alpha \leq 1$. We have $g_{\alpha}^T A_{H'} g_{\alpha} = (1 - \alpha^2)\lambda_1(A_H) + \alpha \sqrt{1 - \alpha^2} \psi_H(u)$ and this quantity is maximized when

$$\alpha = \sqrt{\frac{1}{2} - \frac{\lambda_1(A_H)}{2 \sqrt{\psi_H(u)^2 + \lambda_1(A_H)^2}}}.$$ 

At which

$$g_{\alpha}^T A_{H'} g_{\alpha} = \frac{1}{2} \left(\lambda_1(A_H) + \sqrt{\lambda_1(A_H)^2 + \psi_H(u)^2}\right).$$

□

Combining Lemma 3.2 and Theorem 2.1 yields a bound on the increase of the top eigenvalue of an induced subgraph that may be achieved by adding vertices to it.

**Lemma 3.3 (Support Extension).** For any connected $d$–regular graph $G = (V, E)$ and any connected subset $S \subseteq V$ such that $2 \leq |S| = s < |V|/2$, there is a connected subset $T \subset V$ containing $S$ such that $|T| = 2s$ and

$$\lambda_1(A_T) \geq \lambda_1(A_S) \left(1 + \frac{5}{128d^7s^4}\right).$$
Proof. Define $\lambda_1 := \lambda_1(A_S)$ and note that $\lambda_1 \geq 1$ since $S$ contains at least one edge. As $G$ is connected, $G_S$ cannot be regular and has maximum degree $\Delta \leq d$. Therefore, by Theorem 2.1, there is some vertex $u \in S$ with $\text{deg}_S(u) < \Delta$ and

$$
\psi_S(u)/\psi_S(t) \geq 1/(\Delta \lambda_1 s^2) \geq 1/(d \lambda_1 s^2),
$$

where $t = \arg \max_{v \in S} \psi_S(v)$. As $\psi_S$ is a normalized vector with $s$ entries, $\psi_S(t) \geq 1/\sqrt{s}$. Therefore $\psi_S(u) \geq 1/(d \lambda_1 s^{5/2})$. Take $v$ to be any vertex in $V \setminus S$ that neighbors $u$ in $G$. By Lemma 3.2,

$$
\lambda_1(A_{S \cup \{v\}}) \geq \frac{1}{2} \left( \lambda_1 + \sqrt{\lambda_1^2 + \psi_S(u)^2} \right)
\geq \lambda_1 + \frac{\psi_S(u)^2}{4 \lambda_1} - \frac{\psi_S(u)^4}{16 \lambda_1^4}
\geq \lambda_1 + \frac{1}{6d^2 \lambda_1^4 s^5} \quad \text{as } \psi_S(u)/\lambda_1 \leq 1
\geq \lambda_1 + \frac{1}{6d^6 s^5} \quad \text{since } \lambda_1 \leq d.
$$

(15)

Assuming that $s < |V|/2$, we can iterate this process $s$ times, adding the vertices $\{v_1, \ldots, v_s\}$. At each step we add the vertex $v_i$ and increase the spectral radius of $A_{S \cup \{v_1, \ldots, v_i\}}$ by at least $1/(6d^6(s + i - 1)^5)$. Therefore, defining $T = S \cup \{v_1, \ldots, v_s\}$, we have

$$
\lambda_1(A_T) \geq \lambda_1 + \frac{1}{6d^6} \sum_{i=1}^s \frac{1}{(s + i - 1)^5} \geq \lambda_1 + \frac{5}{128d^6 s^5},
$$

where the last inequality follows from approximating the sum with the integral. As $\lambda_1 \leq d$, this translates to the desired multiplicative bound. \hfill \Box

Proof of Theorem 3.1. We begin by showing (13). Let $\Gamma_x^s$ be the set of connected subgraphs of $G$ with $s$ vertices containing $x$. Choose $S$ to be the maximizer of $e_x^T A^s_S e_x$ among $S \in \Gamma_x^s$, and let $T \in \Gamma_x^{2s}$ be the extension of $S$ guaranteed by Lemma 3.3 to satisfy

$$
\lambda_1(A_T) \geq \left(1 + \frac{5}{128d^7 s^4}\right) \lambda_1(A_S).
$$

Notice that

$$
|W_{2s}^x| \leq \sum_{S' \in \Gamma_x^s} e_x^T A_{S'}^s e_x,
$$

since each walk $\gamma \in W_{2s}^x$ is contained in at least one $S' \in \Gamma_x^s$. Furthermore, $|\Gamma_x^s| \leq d^{2s}$ since each subgraph of $\Gamma_x^s$ may be encoded by one of its spanning trees, which may in turn be encoded by a closed walk rooted at $x$ traversing the edges of the tree. It follows that

$$
|W_{2s}^x| \leq |\Gamma_x^s| e_x^T A_{S'}^s e_x
\leq d^{2s} \lambda_1(A_S)^{2s}
\leq d^{2s} \left(1 + \frac{5}{128d^7 s^4}\right)^{-s} \lambda_1(A_T)^{2s}.
$$

(16)
We claim that for every \( z \in T \),
\[
e^T z A_T^{2k} e_x \geq e^T z A_T^{2k-4s} e_x.
\] (17)

To see this, let \( \pi \) be a path in \( T \) of length \( \ell \leq 2s \) between \( x \) and \( z \), which must exist since \( T \) is connected and has size \( 2s \). Then every closed walk of length \( 2k-2\ell \) in \( T \) rooted at \( z \) may be extended to a walk of length \( 2k \) in \( T \) rooted at \( x \) by attaching \( \pi \) and its reverse. Since all of the walks produced this way are distinct and \( e^T z A_T^{2k-2\ell} e_x \geq e^T z A_T^{2k-4s} e_x \), inequality (17) follows.

Choose \( z \in T \) to be the maximizer of \( e^T z A_T^{2k-4s} e_x \), for which we have:
\[
e^T z A_T^{2k-4s} e_x \geq \frac{1}{2s} \text{Tr}(A_T^{2k-4s}) \geq \frac{\lambda_1(A_T)^{2k-4s}}{2s}.
\]

Combining this with (17) and substituting in (16), we obtain:
\[
|W_2^{2k,s}| \leq d^{2s} \cdot 2s \left( 1 + \frac{5}{128 d^7 s^4} \right)^{-2k} \lambda_1(A_T)^{4s} e^T z A_T^{2k} e_x
\]
\[
\leq d^{2s} \cdot 2s \left( 1 + \frac{5}{128 d^7 s^4} \right)^{-2k} \lambda_1(A_T)^{4s} |W_x^{2k,2s}|.
\]

Applying the inequality \( e^{x/2} \leq 1 + x \) for \( 0 < x < 1 \) and the bound \( \lambda_1(A_T) < d \), we obtain
\[
|W_2^{2k,s}| \leq \exp \left( 6s \log d + \log(2s) - \frac{5k}{128 d^7 s^4} \right) |W_x^{2k,2s}|
\] (18)
which implies
\[
|W_2^{2k,s}| \leq \exp \left( -\frac{k}{65 d^7 s^4} \right) |W_x^{2k,2s}|
\]
whenever
\[
s \leq \frac{1}{4} \left( \frac{k}{d^7 \log d} \right)^{1/5},
\]
establishing (13).

We now show (14) via a small modification of the above proof. Assume \( s \leq (d - h)/2 \). The key observation is that each vertex has at least \( d - h \) edges in \( G \) to other vertices, so in a subgraph of size at most \( 2s - 1 \) every vertex has at least one edge in \( G \) leaving the subgraph. In this case, we can simply choose \( u \in S \), so \( u := \text{arg}\max_{w \in S} \psi_S(w) \) in Lemma 3.3. Therefore (15) can be improved to
\[
\lambda_1(A_{S\cup \{u\}}) \geq \frac{1}{2} \left( \lambda_1 + \sqrt{\lambda_1^2 + \psi_S(u)^2} \right) \geq \lambda_1 + \frac{\psi_S(u)^2}{6 \lambda_1^2} \geq \lambda_1 + \frac{1}{6 \lambda_1^2 s^2}.
\]

Therefore, after adding \( s \) vertices to \( S \) according to the process of Lemma 3.3, we find a set \( T \in \Gamma_x^{2s} \) satisfying
\[
\lambda_1(A_T) \geq \lambda_1 + \frac{1}{6 \lambda_1^2} \sum_{i=1}^{s} \frac{1}{s + i - 1} \geq \lambda_1 + \frac{\log 2}{6 \lambda_1^2} \geq \lambda_1 + \frac{1}{10 \lambda_1^2}.
\]

Using this improved bound, and keeping in mind that \( \lambda_1(A_T) \leq 2s \), we can replicate the argument above to get to the following improvement over (18):
\[
|W_2^{2k,s}| \leq \exp \left( 2s \log d + 4s \log(2s) + \log(2s) - \frac{k}{80 s^3} \right) |W_x^{2k,2s}|.
\]
This implies

\[ |W_x^{2k,s}| \leq \exp\left(7s \log d - \frac{k}{80s^3}\right) |W_x^{2k,2s}| \leq \exp\left(-\frac{k}{100s^3}\right) |W_x^{2k,2s}| \]

whenever

\[ s \leq \frac{1}{8} \left( \frac{k}{\log d} \right)^{1/4}, \]

establishing (14).

\[ \square \]

4 Bound on Eigenvalue Multiplicity

In this section we prove Theorem 1.2, restated below in slightly more detail.

**Theorem 4.1 (Detailed Theorem 1.2).** Let \( G \) be a \( d \)-regular connected graph on \( n \) vertices. If \( 5d \leq \log \frac{1}{7} n / \log \log n \) then

\[ m_G\left(\left(1 - \frac{\log \log_d n}{\log_d n}\right)\lambda_2, \lambda_2\right) = O\left(n \cdot \frac{d^{7/5} \log^{6/5} d \log \log n}{\log^{1/5} n}\right). \]  

(19)

If \( G \) is simple, then further:

\[ m_G\left(\left(1 - \frac{\log \log_d n}{\log_d n}\right)\lambda_2, \lambda_2\right) = \begin{cases} O\left(n \cdot \frac{\log d \log \log_n}{\log^{1/4} n}\right) & \text{when } d \log^{1/2} d \leq \alpha \log^{1/4} n \\ O\left(n \cdot \frac{d \log \log_n}{\log^{1/4} n}\right) & \text{when } d \log^{1/2} d \geq \alpha \log^{1/4} n \end{cases} \]

(20)

for all \( 6d \leq \exp(\sqrt{\log n}) \), where \( \alpha := \frac{4\sqrt{3}}{4} \).

**Proof.** For now, assume that \( |\lambda_1(A_G)| \leq |\lambda_2(A_G)| \). Let \( W_x^{2k} := \cup_{x \in V} W_x^{2k} \) denote the set of all closed walks in \( G \) of length \( 2k \), \( W_x^{2k,s} := \cup_{x \in V} W_x^{2k,s} \) denote such walks with support at most \( s \), and \( W_x^{2k,\geq s+1} := W_x^{2k} \setminus W_x^{2k,s} \).

Set \( k := \frac{1}{2} \log_d n \) and \( c := 2 \log k \) and let \( s \) be a parameter satisfying

\[ |W_x^{2k,s}| \leq e^{-c} |W_x^{2k}| \]  

(21)

to be chosen later. Delete \( cn/s \) vertices from \( G \) uniformly at random. If \( \gamma \in W_x^{2k,\geq s+1} \), the probability that none of the vertices of \( \gamma \) are deleted is at most

\[ \left(1 - \frac{s}{n}\right)^{\frac{cn}{s}} \leq e^{-c}. \]

It then follows by the probabilistic method that there exists a deletion such that the resulting subgraph \( H \) of \( G \) contains at most \( e^{-c} |W_x^{2k,\geq s+1}| \) walks from \( W_x^{2k,\geq s+1} \).

---

\(^5\) If \( d \geq \log^{1/7} n / \log \log n \) then (1) is vacuously true.

\(^6\) If \( d \geq \exp(\sqrt{\log n}) \) then (2) is vacuously true.
Write $\lambda_2 := \lambda_2(A_G)$ and let $m'$ be the number of eigenvalues of $H$ in the interval $[(1 - \epsilon)\lambda_2, \lambda_2]$ for $\epsilon := c/2 \log d(n)$. Since $2k$ is even,

$$m'(1 - \epsilon)^{2k} \lambda_2^{2k} \leq \text{Tr}(A_G^{2k})$$

$$\leq |W^{2k, \leq s}| + e^{-c}|W^{2k, \geq s+1}| \quad \text{by our choice of H}$$

$$\leq e^{-c}|W^{2k}| + e^{-c}|W^{2k, \leq s+1}| \quad \text{by (21)}$$

$$\leq 2e^{-c} \text{Tr}(A_G^{2k})$$

$$\leq 2e^{-c}(n\lambda_2^{2k} + d^{2k})$$

$$\leq 2e^{-c}(n\lambda_2^{2k} + n^{2/3}).$$

We may assume that the diameter of $G$ is at least 4 since otherwise $d \geq n^{1/4}$, making the theorem statement vacuous. Then the Alon-Boppana bound [Nil91] states that $\lambda_2 \geq 1$ implying $n\lambda_2^{2k} \geq n^{2/3}$. Note that for sufficiently large $n$,

$$e \leq \frac{2 \log \log n}{2 \log_d n} \leq \frac{\log d \log \log n}{\log n} < 1/2,$$

since we have assumed $\log d < \sqrt{\log n}$. Thus, $1 - \epsilon \geq e^{-1.5c}$. Combining these facts,

$$m' \lambda_2^{2k} \leq 4e^{3ke-c}n\lambda_2^{2k},$$

yielding

$$m' \leq 4ne^{3ke-c} \leq 4ne^{-c/2} = O\left(\frac{n}{\log_d n}\right).$$

As we created $H$ by deleting $cn/s$ vertices, it follows by Cauchy interlacing that the number of eigenvalues of $A_G$ in $[(1 - \epsilon)\lambda_2, \lambda_2]$ is at most

$$\frac{cn}{s} + O\left(\frac{n}{\log_d n}\right).$$

We now set $s$ depending on the degree of $G$ and whether or not it is simple.

1. If $d = O(\log^{1/7} n/ \log \log n)$, we take

$$s := 1/4 \left(\frac{k}{d^{7/5} \log d}\right)^{1/5}.$$

Applying Theorem 3.1 equation (13) to each $x \in G$ and summing, we then have

$$\frac{|W^{2k, s}|}{|W^{2k}|} \leq \exp\left(-\frac{k}{65d^{7/5}s^4}\right)$$

$$\leq \exp\left(-\Omega\left(\frac{\log_d^{1/5}(n) \log^{4/5} d}{d^{7/5}}\right)\right)$$

$$\leq \exp(-2 \log_d n) = \exp(-c),$$

satisfying (21) for sufficiently large $n$, and we conclude that

$$m_G\left(\left(1 - \frac{\log \log_d n}{\log_d n}\right)\lambda_2, \lambda_2\right) = O\left(n \cdot \frac{d^{7/5} \log_d^{6/5} d \log \log n}{\log^{15/5} n}\right).$$
2. If $G$ is simple and $d \log^{1/2} d < \alpha \log^{1/4} n$ set

$$s := \min \left\{ \frac{1}{8} \left( \frac{k}{\log d} \right)^{1/4}, \frac{d-h}{2} \right\} = \frac{d}{2}$$

with $h = 0$. Applying Theorem 1.3 equation (14) it is easily checked that (21) is satisfied for large enough $n$, yielding a bound of

$$m_G \left( (1 - \frac{\log \log n}{\log d}) \lambda_2, \lambda_2 \right) = O \left( n \cdot \frac{\log d \log \log n}{d} \right).$$

3. If $G$ is simple and $d \log^{1/2} d \geq \alpha \log^{1/4} n$, set

$$s := \min \left\{ \frac{1}{8} \left( \frac{k}{\log d} \right)^{1/4}, \frac{d-h}{2} \right\} = \frac{1}{8} \left( \frac{\log n}{\log^2 d} \right)^{1/4}$$

with $h = 0$. Then (21) is again satisfied by applying Theorem 3.1 equation (14), and we conclude that

$$m_G \left( (1 - \frac{\log \log n}{\log d}) \lambda_2, \lambda_2 \right) = O \left( n \cdot \frac{\log^{1/2} d \log \log n}{\log^{1/4} n} \right).$$

Finally, if $|\lambda_1| > |\lambda_2|$, add $d$ self-loops to every vertex of $G$, arriving at the graph $G'$. Since $A_{G'} = dI + A_G$, for every $1 \leq i \leq n$, $\lambda_i(A_{G'}) = \lambda_i(A_G) + d$. Thus, every eigenvalue of $A_{G'}$ is non-negative and multiplicities are conserved. The proof now proceeds with $G'$ instead of $G$, applying Theorem 3.1 with degree $2d$ and $h = d$. This does not change the asymptotic conclusions of the theorem. \[\square\]

**Remark 4.2.** The proof above can accommodate setting $c := k^\beta$ for small $\beta > 0$, yielding bounds on $m_G(\{(1 - \frac{\log \log n}{\log d}) \lambda_2, \lambda_2\})$, which are worse by a small polylogarithmic factor.

5 Examples

In this final section, we consider examples demonstrating some of the points raised in the introduction regarding the tightness of our results.

5.1 Ramanujan Graphs

We show that Ramanujan graphs have high multiplicity near $\lambda_2$. This means that the bound of $n/\log^{O(1)} n$ of Theorem 1.2 is tight.

**Theorem 5.1** (Friedman [Fri91] Corollary 3.6). Let $G$ be a $d$-regular graph on $n$ vertices. Then

$$\lambda_2(G) \geq 2 \sqrt{d-1} (1 - O(1/\log^2 n)).$$

**Lemma 5.2** (McKay [McK81] Lemma 3). The number of closed walks on the infinite $d$-regular tree of length $2k$ starting at a fixed vertex is $\Theta\left(\frac{4^k (d-1)^k}{k^{3/2}}\right)$. 

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Proposition 5.3. There exists an $\alpha > 0$ such that for fixed $d$, every bipartite $d$-regular bipartite Ramanujan graph $G$ on $n$ vertices satisfies

$$m_G\left(\lambda_2(1 - \alpha \frac{\log \log(n)}{\log(n)}), \lambda_2 \right) = \Omega(n / \log^{3/2}(n)).$$

Proof. By Theorem 5.1,

$$\lambda_2\left(1 - \alpha \frac{\log \log(n)}{\log(n)} \right) \leq 2 \sqrt{d - 1} \left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right),$$

for sufficiently large $n$. Let $k = \beta \log n$ for some constant $\beta$ to be set later and suppose that there are $m$ eigenvalues of $A_G$ in the interval $[\lambda_2 \left(1 - \alpha \frac{\log \log(n)}{\log(n)} \right), \lambda_2]$. Recall that the spectrum of a bipartite graph is symmetric around 0. From Lemma 5.2 it follows that for some universal constant $C$,

$$Cn\left(\frac{4^k(d - 1)^k}{k^{3/2}}\right) \leq \sum_{i=1}^{n} \lambda_i(A_G)^{2k} \leq 2d^{2k} + (n - 2m) \left(2 \sqrt{d - 1} \left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right)\right)^{2k} + 2m(2 \sqrt{d - 1})^{2k}.$$

If we let $\beta$ be sufficiently small and $\alpha > \frac{3}{2}\beta$, rearranging yields

$$m \geq \frac{C4^k(d - 1)^k}{k^{3/2}} - \frac{2d^{2k}}{n} - \frac{2 \sqrt{d - 1} \left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right)^{2k}}{2(2 \sqrt{d - 1})^{2k} \left(1 - \left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right)^{2k}\right)}$$

$$= \Omega\left(1 - \frac{2n^{2\beta - 1}}{k^{3/2}} - \frac{\left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right)^{2k}}{1 - \left(1 - \frac{1}{2} \alpha \frac{\log \log(n)}{\log(n)} \right)^{2k}}\right)$$

$$= \Omega\left(\frac{1}{k^{3/2}} - \frac{1}{e^{\alpha \beta \log \log(n)}}\right)$$

$$= \Omega\left(\frac{1}{k^{3/2}}\right).$$

\[ \square \]

5.2 Lollipop

Here, we show that if we do not assume that our graph is regular, the average support of a closed walk of length $k$ from a fixed vertex is no longer necessarily $k^{\Theta(1)}$. We take the lollipop graph, which consists of a clique of $(d + 1)$ vertices for fixed $d \geq 3$ and a path of length $n$ $\{u_1, \ldots, u_n\}$ attached to a vertex $v$ of the clique, where $n \gg k$. Here $\psi$ and $\lambda_1$ are the Perron eigenvector and eigenvalue of this graph.

Lemma 5.4. $\psi(v) \geq 1 / \sqrt{d + 2}$.  

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Proof. By symmetry, the value on all entries of the clique besides \( v \) are the same. Call this value \( \psi(b) \). Then by the eigenvalue equation we have \( \lambda_1 \psi(b) = \psi(v) + (d - 1)\psi(b) \), so as \( \lambda_1 \geq d \), it must be that \( \psi(v) \geq \psi(b) \).

Similarly, to satisfy the eigenvalue equation, vertices on the path must satisfy the recursive relation
\[
\lambda_1 \psi(u_i) = \psi(u_{i-1}) + \psi(u_{i+1}) \quad 1 \leq i \leq n - 1
\]
\[
\lambda_1 \psi(u_n) = \psi(u_{n-1})
\]
where we define \( v = u_0 \). To satisfy this equation, we must have \( \psi(u_i) \geq (\lambda_1 - 1)\psi(u_{i+1}) \) for each \( i \), so as \( \lambda_1 \geq d \geq 3 \), \( \psi(v) \geq \sum_{i=1}^n \psi(u_k) \). As the Perron vector is nonnegative, \( \psi(v)^2 \geq \sum_{i=1}^n \psi(u_k)^2 \), and
\[
(d + 2)\psi(v)^2 \geq \psi(v)^2 + d\psi(b)^2 + \sum_{i=1}^n \psi(u_k)^2 = 1,
\]
so \( \psi(v) \geq 1/\sqrt{d + 2} \). \( \square \)

**Proposition 5.5.** For \( \ell \geq 2\log(k)/\log(\lambda_1/2) \),
\[
|W_v^{2k, \geq \ell + d + 1}| = O(k^{-2})|W_v^{2k}|.
\]

**Proof.** For a closed walk to have support \( \ell + d + 1 \), it must contain \( u_{\ell} \). For such walks, once the path is entered, at least \( 2\ell \) steps must be spent in the path, as the walk must reach \( u_{\ell} \) and return. Therefore, closed walks starting at \( v \) that reach \( u_{\ell} \) can be categorized as follows. First, there is a closed walk from \( v \) to \( v \). Then there is a closed walk from \( v \) to \( v \) going down the path containing \( u_{\ell} \). On this excursion, the walk can only go forward or backwards, and it spends at least \( 2\ell \) steps within the path. For each of these steps, there are 2 options. If we remain in the path after \( 2\ell \) steps, upper bound the number of choices until returning to \( v \) by \( \lambda_1 \) at each step. After returning to \( v \), the remaining steps form another closed walk. The number of closed walks from \( v \) of length \( i \) is at most \( \lambda_1^i \). Therefore the number of closed walks with an excursion to \( u_{\ell} \) is at most
\[
\sum_{i=0}^{2k} \lambda_1^i 2^{2\ell} \lambda_1^{2k-2\ell-i} = (2k + 1)\lambda_1^{2k-2\ell} 2^{2\ell}.
\]

The total number of closed walks starting at \( v \) is at least \( \psi(v)^2 \lambda_1^n \). Therefore the fraction of closed walks that have support at least \( \ell \) is at most
\[
\frac{(2k + 1)2^{2\ell} \lambda_1^{2k-2\ell}}{\lambda_1^{2k}/(d + 2)} = \frac{(d + 2)(2k + 1)2^{2\ell}}{\lambda_1^{2\ell}}
\]
so for \( \ell \geq 2(\log k)/\log(\lambda_1/2) \), this probability is \( O(k^{-2}) \). \( \square \)

**Remark 5.6.** Instead of adding a path, we can add a tree (as exhibited in Figure 1). According to the same analysis, the probability a walk reaches depth further than \( \Theta(\log k) \) is small. Therefore, in Theorem 1.3 we can not get a sufficient bound on support from passing to depth, but must deal with support itself.
5.3 Mangrove Tree

This section shows that the dependence on $|V|$ in Theorem 1.4 is tight up to polylogarithmic factors. Our example is a path of multiedges containing $n$ vertices, where each multiedge of the path is composed of $d/2$ edges for some even $d$. At both ends of the path, we attach a tree of depth $\log_{d-1} n$. The roots have degree $d/2$ and all other vertices (besides the leaves) have degree $d$. Therefore the only vertices in the graph that are not degree $d$ are the leaves of the two trees. Call this graph $Q$. An example of this graph is shown in Figure 2.

Proposition 5.7. For every vertex $u$ of degree less than $d$, $\psi_Q(u) = 0(n^{-5/2})$, where $O$ suppresses dependence on logarithmic factors and $d$.

Therefore, we cannot hope to do significantly better than our analysis in Lemma 3.3, in which we find a vertex $u$ of non-maximal degree with $\psi(u) \geq 1/(d\lambda_1 n^{5/2})$.

Proof. For simplicity, call $\lambda_1(A_Q) = \lambda_1$ and $\psi_Q = \psi$. By the symmetry of the graph, the value of $\psi$ at vertices in the tree is determined by the distance from the root. Call the entries of $\psi$ corresponding to the tree $r_0, r_1, \ldots, r_\ell$, where the index indicates the distance from the root.

By the discussion in the proof of Kahale [Kah95] Lemma 3.3, if we define

$$\theta := \log \left( \frac{\lambda_1}{2 \sqrt{d-1}} + \sqrt{\frac{\lambda_1^2}{4(d-1)} - 1} \right),$$

then for $0 \leq i \leq \ell$, entries of the eigenvector must satisfy

$$\frac{r_i}{r_0} = \frac{\sinh((\ell + 1 - i)\theta)(d-1)^{-i/2}}{\sinh((\ell + 1)\theta)}$$

where $\ell$ is the depth of the tree.

Therefore, $r_\ell/r_0 = \frac{\sinh(\theta)(d-1)^{-\ell/2}}{\sinh((\ell+1)\theta)}$. Examining the various terms, $\sinh(\theta) \leq d$ and $(d-1)^{-\ell/2} = \frac{1}{\sqrt{n}}$. To bound the third term, we use the definition $\sinh(x) = (e^x - e^{-x})/2$.

$$\sinh((\ell + 1)\theta) \geq \frac{1 - o_n(1)}{2} \left( \frac{\lambda_1}{2 \sqrt{d-1}} + \sqrt{\frac{\lambda_1^2}{4(d-1)} - 1} \right)^{\log_{d-1} n^{1+}}.$$

$\lambda_1$ is at least the spectral radius of the path of length $n$ with $d/2$ multiedges between vertices. This spectral radius is $d \cos(\pi/(n+1))$. This gives

$$\sinh((\ell + 1)\theta) \geq \frac{1 - o_n(1)}{2(2\sqrt{d-1})^{\log_{d-1} n^{1+}}} \left( d(1 - \frac{\pi^2}{2n^2}) + \sqrt{d^2(1 - \frac{\pi^2}{2n^2})^2 - 4d + 4} \right)^{\log_{d-1} n^{1+}}$$

$$\geq \frac{1 - o_n(1)}{2(2\sqrt{d-1})^{\log_{d-1} n^{1+}}} \left( d + d - 2 \right)^{\log_{d-1} n^{1+}} \left( 1 - O\left( \frac{d}{n^2} \right) \right)^{\log_{d-1} n^{1+}}$$

$$\geq \frac{1 - o_n(1)}{2} \exp^{-O(d\log_{d-1} n^{1/2})} \sqrt{n} \geq \frac{\sqrt{n}}{3}.$$
for large enough \( n \). Therefore

\[
\frac{r_\ell}{r_0} = \frac{\sinh(\theta)(d-1)^{\ell/2}}{\sinh((\ell+1)\theta)} \leq \frac{3d}{n}.
\]  

(22)

At this point, we know the ratio between \( r_\ell \) and \( r_0 \), but still need to bound the overall mass of the eigenvector on the tree. A “regular partition” is a partition of vertices \( V = \bigcup_{j=0}^k X_j \) where the number of neighbors a vertex \( u \in X_i \) has in \( X_j \) does not depend on \( u \). We can create a quotient matrix, where entry \( i, j \) corresponds to the number of neighbors a vertex \( u \in X_i \) has in \( X_j \). For an overview of quotient matrices and their utility, see Godsil, [God93, Chapter 5]. In our partition, every vertex in the path is placed in a set by itself. The vertices of each of the two trees are partitioned into sets according to their distance from the two roots. Call the matrix according to this partition \( \tilde{A}_Q \). We denote by \( \tilde{A}_Q(X_i, X_j) \) the entry in \( \tilde{A}_Q \) corresponding to edges from a vertex in \( X_i \) to \( X_j \).

Define \( X_0, \ldots, X_\ell \) as the sets corresponding to vertices in the first tree of distance \( 0, \ldots, \ell \) from the root. For \( 1 \leq j \leq \ell - 1 \), \( \tilde{A}_Q(X_0, X_1) = d/2 \). \( \tilde{A}_Q(X_j, X_{j+1}) = d − 1 \). Moreover, for \( 0 \leq j \leq \ell - 1 \), \( \tilde{A}_Q(X_{j+1}, X_j) = 1 \). All values between vertices in the path are unchanged at \( d/2 \).

Consider the diagonal matrix \( D \) with \( D_{ii} = |X_i|^{-1/2} \). \( D^{-1}\tilde{A}_QD \) is a symmetric matrix. Define \( C := D^{-1}\tilde{A}_QD \) we now have \( C(X_{j+1}, X_j) = C(X_j, X_{j+1}) = \sqrt{d-1} \) for \( 1 \leq j \leq \ell − 1 \), and \( C(X_0, X_1) = C(X_1, X_0) = \sqrt{d}/2 \).

If a vector \( \phi \) is an eigenvector of \( C \), then \( D\phi \) is an eigenvector of \( \tilde{A} \) with the same eigenvalue. By the definition of \( D \) this means

\[
\psi_C(X_i)^2 = \sum_{u \in X_i} \psi_{A_Q}(u)^2.
\]  

(23)

Define \( C_{X_{0: \ell}} \) as the submatrix of \( C \) corresponding the the sets \( \{X_0, \ldots, X_\ell\} \), then extended with zeros to have the same size as \( C \). Every entry of \( C + \frac{d}{2} \sqrt{d-1}C_{X_{0: \ell}} \) is less than or equal to the corresponding entry of the adjacency matrix of a path of length \( n + 2 \log_{d-1}n \) with \( d/2 \) edges between pairs of vertices. Also, \( \psi_C \) is a nonnegative vector. Therefore the quadratic form

\[
\psi_C^T(C + \frac{d}{2} \sqrt{d-1}C_{X_{0: \ell}})\psi_C
\]

is at most the spectral radius of this path. Namely

\[
\psi_C^T(C + \frac{d}{2} \sqrt{d-1}C_{X_{0: \ell}})\psi_C \leq d \cos(\pi/(n + 2 \log_{d-1}n + 1)).
\]

Because \( C \) contains the path of length \( n \), \( \psi_C^T(C) \geq d \cos(\pi/(n + 1)) \). Putting these together yields

\[
\psi_C^T(C_{X_{0: \ell}})\psi_C \leq \frac{\sqrt{d-1}}{d/2 - \sqrt{d-1}} \cdot d(\cos(\pi/(n + 2 \log_{d-1}n + 1)) - \cos(\pi/(n + 1))) \leq \frac{d \sqrt{d-1}}{d/2 - \sqrt{d-1}} \frac{3\pi^2 \log_{d-1}n}{n^3}.
\]  

(24)

Define \( \psi_C(X_{1: \ell}) \) as the projection of \( \psi_C \) on \( [X_1, \ldots, X_\ell] \). \( \psi_C(X_{1: \ell}) = C_{X_{0: \ell}}\psi_C(X_{1: \ell}) \), so

\[
\psi_C^T(C_{X_{0: \ell}})\psi_C \geq \lambda_1 ||\psi_C(X_{1: \ell})||^2 \geq d(\cos(\pi/(n + 1)))||\psi_C(X_{1: \ell})||^2
\]  

(25)

Combining (24) and (25) yields

\[
||\psi_C(X_{1: \ell})||^2 \leq \frac{d \sqrt{d-1}}{d/2 - \sqrt{d-1}} \left( \frac{3\pi^2 \log_{d-1}n}{n^3} \right) / \cos(\pi/n + 1) \leq \frac{21\pi^2 \log_{d-1}n}{n^3}
\]

assuming \( d \geq 4 \) and \( n \) is sufficiently large.
Using (23) and the eigenvalue equation, we obtain

\[
\psi_Q(r_0) = \psi_C(X_0) \leq \lambda_1(A_C)\|\psi_C(X_1)\| \leq d \cdot \frac{5\pi \log^{1/2} n}{n^{3/2}}.
\]

Therefore, according to (22)

\[
r_\ell \leq \frac{15d^2\pi(\log^{1/2} n)}{n^{5/2}}.
\]

\[\square\]

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