Uniqueness in the Calderon problem with partial data for less smooth conductivities

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Abstract
In this paper, we study the inverse conductivity problem with partial data. Moreover, we show that, in dimension $n \geq 3$, the uniqueness of the Calderon problem holds for the $C^1 \cap H^{3/2}$ conductivities.

1. Introduction
In 1980, Calderon [Cal80] considered whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as electrical impedance tomography (EIT). EIT also arises in medical imaging given that human organs and tissues have quite different conductivities [Jos98]. One exciting potential application is the early diagnosis of breast cancer [ZG03]. The conductivity of a malignant tumor is typically $0.2 \, \Omega \cdot m$, which is significantly higher than normal tissue which has been typically measured at $0.03 \, \Omega \cdot m$. Another application is to monitor the pulmonary function [INGC90]. See [Hol05] and the issue of Physiological Measurement [IMS03] for other medical imaging applications of EIT.

We now describe more precisely the mathematical problem.
Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open, bounded domain with $C^2$ boundary $\partial \Omega$, and let $\gamma$ be a strictly positive real-valued function defined on $\Omega$ which gives the conductivity at a given point. Given a voltage potential $f$ on the boundary, the equation for the potential in the interior, under the assumption of no sinks or sources of current in $\Omega$, is

$$\text{div}(\gamma \nabla u) = 0, \quad \text{in} \, \Omega, \quad u|_{\partial \Omega} = f. \quad (1.1)$$

The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial v} |_{\partial \Omega}, \quad (1.2)$$

where $\frac{\partial}{\partial v}$ is the outward normal derivatives at the boundary. For $\gamma \in \text{Lip}(\overline{\Omega})$, $\Lambda_\gamma$ is a well-defined map from $H^{3/2} (\partial \Omega)$ to $H^{-1/2} (\partial \Omega)$. 

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The Calderón problem concerns the inversion of the map \( \gamma \to \Lambda_\gamma \), i.e. whether \( \Lambda_\gamma \) determines \( \gamma \) uniquely and, in that case, how to reconstruct \( \gamma \) from \( \Lambda_\gamma \).

For the uniqueness, it was first proved for smooth conductivities by Sylvester and Uhlmann in their fundamental paper [SU87], which opened the door to studying the Calderón problem.

Now we briefly recall the basic idea in [SU87]. For the \( C^2 \) conductivity \( \gamma \), letting \( u = \gamma^\frac{1}{2} v \) be the solution of (1.1), we can deduce that \( v \) satisfies the Schrödinger equation

\[
(\Delta + q)v = 0, \tag{1.3}
\]

where \( q \) is defined by \( q = \gamma^{-\frac{1}{2}} \Delta \gamma^\frac{1}{2} \). The corresponding Dirichlet-to-Neumann map is defined by \( \Lambda_q(f) = \frac{\partial v}{\partial \nu} \) with the boundary data \( f \). More important, for the \( C^2 \) conductivities \( \gamma_1, \gamma_2 \), \( \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \) implies \( \Lambda_{\eta_1} = \Lambda_{\eta_2} \) for \( q_i = \gamma_i^{-\frac{1}{2}} \Delta \gamma_i^\frac{1}{2} \), \( i = 1, 2 \) (see [KV84, Ale90]). Therefore, we convert the inverse problem for the conductivity equation (1.1) to an equivalent inverse problem for the Schrödinger equation (1.3).

If \( \Lambda_{\eta_1} = \Lambda_{\eta_2} \) and \( (\Delta + q_1)v_1 = 0 \), then a simple calculation shows that

\[
\int_\Omega (q_1 - q_2) v_1 v_2 \, dx = 0. \tag{1.4}
\]

From this discussion, to obtain \( q_1 = q_2 \), we just need to construct enough solutions to the corresponding Schrödinger equations (1.3) such that their products are dense in some sense.

The authors of [SU87] construct these types of complex geometrical optics solutions \( v_i = e^{i\xi_i} (1 + w_i) \), where the \( \xi_i \in C^n \) are chosen so that \( \xi_i \cdot w_i = 0 \), which implies that e\( ^{-i\xi} \) is harmonic and e\( ^{+i\xi} \)e\( ^{-i\xi} \) = e\( ^{i\xi} \) for some fixed frequency \( k \in \mathbb{R}^n \). For \( w_i \), these are the solutions of the following equations:

\[
\Delta_{\xi_i} w_i := \Delta w_i + 2 \xi_i \cdot \nabla w_i = q_i(1 + w_i). \tag{1.5}
\]

In three or more dimensions, we can find that infinite high-frequency solutions \( v_i = e^{i\xi_i} (1 + w_i) \) satisfying the remainders \( w_i \) decay to zero in some sense as \( |\xi_i| \to \infty \), so that the product \( v_1 v_2 \) converges to \( e^{i\xi} \). Uniqueness then follows from Fourier inversion.

For the less smooth conductivities \( \gamma_i \), following this idea, one may find the solutions \( w_i \) of (1.5) in some suitable Sobolev or Besov space by using the contract mapping theorem. Furthermore, the chosen CGO solution \( v_i \) is such that (1.4) makes sense. In view of the above analysis, one may show the uniqueness of the Calderón problem which holds under different types of conductivities. For example, Brown [Bro96] obtained uniqueness under the assumption of \( \frac{3}{2} + \epsilon \) derivatives. Later, uniqueness for exactly \( \frac{3}{2} \) bounded derivatives was shown in [PPU03] and for \( \frac{3}{2} \) derivatives being in \( L^p, p > 2n \), was shown in [BT03].

In another direction, for some special conductivity, Greenleaf, Lassas and Uhlmann [GLU03] obtained global uniqueness for certain conductivities in \( C^{1+\epsilon} \). Later, Kim [Kim08] established global uniqueness for Lipschitz conductivities that are piecewise smooth across polyhedral boundaries.

Following the above idea, the sharpest uniqueness results so far seem to require essentially \( \frac{3}{2} \) derivatives of the conductivity. Recently, Haberman and Tataru [HT11] used a totally new idea to show uniqueness for almost Lipschitz conductivities. The main idea is as follows. Since the symbol of the operator \( \Delta_{\xi_i} \) is \(-|\xi_i|^2 + 2i\xi_i \cdot \xi_i \), Haberman and Tataru introduce Bourgain’s spaces \( X^{\frac{3}{2},\frac{1}{2}} \) [Bou93], which are defined by the norm \( \|u\|_{X^{\frac{3}{2},\frac{1}{2}}} = \|P_{\xi}(\xi)|^\frac{1}{2} \hat{u}(\xi)\|_{L^2} \), where \( P_{\xi}(\xi) = -|\xi|^2 + 2i\xi \cdot \xi, |\xi| = \epsilon \). Furthermore, by the contract mapping theorem, one may find a family of high frequent CGO solutions \( v_i = e^{i\xi_i} (1 + w_i) \) satisfying \( \|w_i\|_{X^{\frac{3}{2},\frac{1}{2}}} \to 0 \), as \( |\xi_i| \to \infty \). The most important thing is that (1.4) makes sense. Uniqueness then follows from Fourier inversion.
We will use this idea to show the uniqueness of the Calderón problem with partial data for the $C^1(\Omega) \cap H^{2,2}(\Omega)$ conductivities in this paper.

Before stating the theorem, let us recall what is known about the uniqueness of the Calderón problem with partial data.

First we introduce some notation. Let $\eta \in S^{n-1}$ and define the subsets

$$\partial \Omega_+ = \{ x \in \partial \Omega, \, v(x) \cdot \eta \geq 0 \}, \quad \partial \Omega_- = \{ x \in \partial \Omega, \, v(x) \cdot \eta < 0 \},$$

where $v(x)$ is the outward normal direction at the boundary point $x$. For $\epsilon > 0$, define further subsets

$$\partial \Omega_{+,\epsilon} = \{ x \in \partial \Omega, \, v(x) \cdot \eta \geq \epsilon \}, \quad \partial \Omega_{-,\epsilon} = \{ x \in \partial \Omega, \, v(x) \cdot \eta < \epsilon \}.$$

Bukhgeim and Uhlmann [BU02] first established the uniqueness of the Calderón problem under the assumptions $\gamma_1, \gamma_2 \in C^2(\Omega)$, $\gamma_1|_{\partial \Omega} = \gamma_2|_{\partial \Omega}$ and $\Lambda_{\gamma_1}|_{\partial \Omega_-} = \Lambda_{\gamma_2}|_{\partial \Omega_-}$.

Later Knudsen [Knu06] generalized their result and established the uniqueness of the Calderón problem under the assumptions $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega)$, $r > 0$, $\gamma_1|_{\partial \Omega} = \gamma_2|_{\partial \Omega}$, $\partial_r \gamma_1|_{\partial \Omega} = \partial_r \gamma_2|_{\partial \Omega}$, and $\Lambda_{\gamma_1}|_{\partial \Omega_-} = \Lambda_{\gamma_2}|_{\partial \Omega_-}$.

Now we state our result as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open, bounded domain with $C^2$ boundary. For $i = 1, 2$, let $\gamma_i \in C^1(\Omega) \cap H^{2,2}(\Omega)$ be a real-valued function and $\gamma_i > c > 0$. Fix $\eta \in S^{n-1}$ and suppose further that $\gamma_1|_{\partial \Omega} = \gamma_2|_{\partial \Omega}$, $\partial_r \gamma_1|_{\partial \Omega} = \partial_r \gamma_2|_{\partial \Omega}$, and for some $\epsilon > 0$,

$$\Lambda_{\gamma_1}|_{\partial \Omega_-} = \Lambda_{\gamma_2}|_{\partial \Omega_-}.$$

Then we have $\gamma_1 = \gamma_2$.

For the above known results of the Calderón problem with partial data, the main idea is that one can use the linear limiting Carleman weight $x \cdot \eta$ in the proof. Indeed, Kenig, Sjöstrand and Uhlmann use a nonlinear limiting Carleman weight to obtain a new type of result in [KSU07]. By combing our approach and Kenig, Sjöstrand and Uhlmann’s [KSU07] idea, it is expected that a generalization of theorem 1.1 can be found.

Our paper is organized as follows. In section 2, we will state the results of approximation of conductivities. In section 3, we will construct the CGO solutions. In section 4, we will give the Carleman estimate for the CGO solutions. In section 5, we will present the proof of theorem 1.1.

### 2. Preliminary results

From the assumption of the conductivities $\gamma_i, i = 1, 2$, in theorem 1.1, we can extend $\gamma_i$ to be the functions in the whole space $\mathbb{R}^n$ such that (i) $\gamma_i(x) \geq C$, (ii) $\gamma_i - 1 \in H^{1,\infty}(\mathbb{R}^n)$ with compact support, (iii) $\gamma_i \in C^1(\mathbb{R}^n)$ and (iv) $\gamma_1 = \gamma_2$ outside of $\Omega$. For the proof of this extension, we recommend readers read the proof of theorem 5.7 in [Shk11].

Although our assumptions on conductivities are a little bit different, the method of the proof still works.

Let $\Psi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative radial function with $\int_{\mathbb{R}^n} \Psi dx = 1$, $\text{spt} \Psi \subset B(0, 1)$ and $\Psi(t) = t^r \Psi(tx)$. Then for $\phi = \log \gamma$, $A = \nabla \log \gamma$, we define $\phi = \Psi A \ast \phi$, $A = \Psi A \ast A$. Clearly, $A_i = \nabla \phi_i$.

Now we state some approximation results. Some of these results are taken from [Sal04] and [Knu06] and some are new. For the new results, we will give the details of the proof.
Lemma 2.1. Suppose $\gamma \in C^1(\mathbb{R}^n)$ and $\gamma - 1 \in H^{\frac{1}{2}, 2}(\mathbb{R}^n)$ with compact support. Then as $t \to \infty$, we have
\[
\| \phi_t \|_{L^\infty} \leq \| \phi \|_{L^\infty}, \quad \| A_t \|_{L^\infty} \leq \| A \|_{L^\infty}, \quad \| D^2 \phi_t \|_{L^\infty} = o(t),
\]
\[
\| \phi_t - \phi \|_{L^\infty} = o\left(\frac{1}{t^\varepsilon}\right), \quad \| A_t - A \|_{L^\infty} = o(1)
\]
and
\[
\| \phi_t - \phi \|_{L^2(\mathbb{R}^n)} = o\left(\frac{1}{t^\varepsilon}\right). \quad \| \phi_t - \phi \|_{H^{1, 2}(\mathbb{R}^n)} = o\left(\frac{1}{t^\varepsilon}\right).
\]
\[
\| \phi_t - \phi \|_{H^{\frac{1}{2}, 2}(\mathbb{R}^n)} = o(1), \quad \| D^\alpha \phi_t \|_{L^2} = o(t^{\frac{1}{2} - |\alpha|}), |\alpha| = 2.
\]

Proof. Since $\gamma \in C^1(\mathbb{R}^n)$ and $\gamma - 1 \in H^{\frac{1}{2}, 2}(\mathbb{R}^n)$ with compact support, $\phi \in C^2_b(\mathbb{R}^n)$ and $A \in C^1_c(\mathbb{R}^n) \cap H^{\frac{1}{2}, 2}(\mathbb{R}^n)$. From lemma 2.1 in chapter 2 of [Sal04], we can obtain $\| \phi_t \|_{L^\infty} \leq \| \phi \|_{L^\infty}$, $\| A_t \|_{L^\infty} \leq \| A \|_{L^\infty}$, $\| D^2 \phi_t \|_{L^\infty} = o(t)$ and $\| A_t - A \|_{L^\infty} = o(1)$.

To obtain $\| \phi_t - \phi \|_{L^\infty} = o\left(\frac{1}{t^\varepsilon}\right)$, by the fundamental theorem of calculus, we have
\[
\begin{align*}
t(\phi_t(x) - \phi(x)) &= t \left( \int_{\mathbb{R}^n} \Psi_t(x - y) \phi(y) \, dy - \phi(x) \right) \\
&= t \left( \int_{\mathbb{R}^n} \Psi(z) \phi \left( x - \frac{z}{t} \right) - \phi(x) \right) \, dz \\
&= \int_{\mathbb{R}^n} \Psi(z) \int_0^1 \nabla \phi \left( x - s \frac{z}{t} \right) \cdot -z \, ds \, dz.
\end{align*}
\]

On the other hand, since $\Psi(z)$ is a radial function, we know
\[
\int_{\mathbb{R}^n} \Psi(z) \nabla \phi(x) \cdot z \, dz = 0.
\]

Hence,
\[
\begin{align*}
t(\phi_t(x) - \phi(x)) &= \int_{\mathbb{R}^n} \Psi(z) \int_0^1 (\nabla \phi \left( x - s \frac{z}{t} \right) - \nabla \phi(x)) \cdot -z \, ds \, dz.
\end{align*}
\]

Since spt$\Psi(z) \subset B(0, 1)$ and $\nabla \phi \in C_c(\mathbb{R}^n)$, uniform continuity gives
\[
\| \phi_t - \phi \|_{L^\infty} = o\left(\frac{1}{t^\varepsilon}\right).
\]

For any $0 \leq s \leq \frac{1}{2}$, by Fourier transform formula, we have
\[
(\phi_t - \phi)(\xi) = \left( \hat{\Psi} \left( \frac{\xi}{t} \right) - 1 \right) \hat{\phi}(\xi).
\]

Hence,
\[
\begin{align*}
t^{\frac{1}{2} - s} \| \left( \hat{\Psi} \left( \frac{\xi}{t} \right) - 1 \right) \|_{L^2} &= \| g \left( \frac{\xi}{t} \right) \|_{L^2} \| \hat{\phi}(\xi) \|_{L^2},
\end{align*}
\]

where $g(z) = \frac{1}{|z|^{\frac{1}{2} - s}} (\hat{\Psi}(z) - 1)$.

Since $\int_{\mathbb{R}^n} \Psi(z) \, dz = 1$ and $\Psi$ is radial, using the Fourier transform formula, we obtain
\[
\hat{\Psi}(0) = 1, \quad \hat{\nabla \Psi}(0) = 0,
\]
which implies $g(z)$ is continuous and bounded with $g(0) = 0$.

Thus, the Lebesgue-dominated convergence theorem gives
\[
\lim_{t \to \infty} \| g \left( \frac{\xi}{t} \right) \|_{L^2} \| \hat{\phi}(\xi) \|_{L^2} = 0.
\]
Then we obtain
\[ \| \phi_t - \phi \|_{L^2(\mathbb{R}^n)} = o\left( \frac{1}{t^{\frac{3}{2}}} \right), \quad \| \phi_t - \phi \|_{H^{\frac{3}{2}}(\mathbb{R}^n)} = o\left( \frac{1}{t^{\frac{3}{2}}} \right), \quad \| \phi_t - \phi \|_{H^{\frac{1}{2}}(\mathbb{R}^n)} = o(1). \]

To prove \( \| D^\alpha \phi_t \|_{L^2} = o(1) \), \( |\alpha| = 2 \), observe
\[ t^{-\frac{1}{2}} \| D^\alpha \phi_t \|_{L^2} = t^{-\frac{1}{2}} \| \xi^\alpha \hat{\phi} \|_{L^2} \leq \| \hat{g} \|_1 \| \xi^\alpha \hat{\phi} \|_{L^2}, \]
where \( \hat{g}(z) = |z| \frac{\hat{\psi}(z)}{\hat{\psi}(1)} \) is continuous and bounded with \( \hat{g}(0) = 0 \).

Then, the Lebesgue-dominated convergence theorem gives that \( \| D^\alpha \phi_t \|_{L^2} = o(1) \), \( |\alpha| = 2 \). Then lemma 2.1 follows. \( \Box \)

Finally, we need the following type of trace inequality for \( W^{1,2}(\Omega) \). Since we cannot easily find this result in the literature, for completeness, we will give the proof of the following lemma.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a \( C^2 \) smooth bounded domain and \( u \in W^{1,2}(\Omega) \). Then there exists a constant \( C \) such that
\[ \int_{\partial \Omega} u^2 \, ds \leq C \left\{ \left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + \int_{\Omega} u^2 \, dx \right\}, \]
where \( C \) depends only on \( \Omega, n \).

**Proof.** Since \( \partial \Omega \) is a \( C^2 \) smooth boundary, we know that the unit outward normal direction \( \nu(x) \) is a \( C^1 \) vector on the boundary. Then, we can find a \( C^1 \) vector \( \rho \) in \( \Omega \) which is an extension of \( \nu(x) \).

Thus, by the divergence theorem, we have
\[ \int_{\partial \Omega} u^2 \, ds = \int_{\Omega} \text{div}(\rho u^2) \, dx = \int_{\Omega} \text{div}(\rho) u^2 \, dx + \int_{\Omega} \rho \cdot \nabla(u^2) \, dx, \]
in view of the H"older inequality, which implies
\[ \int_{\partial \Omega} u^2 \, ds \leq C \left\{ \left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + \int_{\Omega} u^2 \, dx \right\}. \] \( \Box \)

## 3. CGO solutions

Before stating the main result, let us introduce some preliminarily results.

**Proposition 3.1.** [HT11] Let \( v \) and \( w \) be nonnegative weights defined on \( \mathbb{R}^n \). If \( \Phi \) is a fixed rapidly decreasing function, then
\[ \| \Phi \ast f \|_{L^2_w} \leq \min\{ \sup_{\xi} \sqrt{\int J(\xi, \eta) \, d\eta}, \ \sup_{\eta} \sqrt{\int J(\xi, \eta) \, d\xi} \} \| f \|_{L^2_v}, \]
where
\[ J(\xi, \eta) = |\Phi(\xi - \eta)| \frac{v(\xi)}{w(\eta)}. \]
Now we introduce Bourgain’s spaces [Bou93]. Using Haberman and Tataru’s idea in [HT11], we define the spaces $\dot{X}^s_b$ by the norm

$$\|u\|_{\dot{X}^s_b} = \| |P_{\xi}(\xi)|^s \hat{u}(\xi)\|_{L^2},$$

where $P_{\xi}(\xi) = -|\xi|^2 + 2i\xi \cdot \xi$ is the symbol of $\triangle_{\xi}$. In our paper, we will need the spaces $\dot{X}^s_{1/2}$ and $\dot{X}^{-s}_{1/2}$ as well as also make use of the inhomogeneous spaces $X^s_b$ with the norm

$$\|u\|_{X^s_b} = \|(|\xi| + |P_{\xi}(\xi)|^s \hat{u}(\xi)\|_{L^2}.$$

Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and write $\zeta = s(e_1 - ie_2)$, with $e_1, e_2 \in \mathbb{R}^n$ satisfying $e_1 \cdot e_2 = 0$. Taking the open balls $B_0, B$ s.t. $\Omega \subseteq B_0 \subseteq B$ and choosing a Schwartz cutoff function $\Phi_0$ which is equal to 1 on an open ball $B$, we have the following known results which are taken from [HT11].

**Proposition 3.2.** Let $\Phi_0$ be a fixed Schwartz function defined as above, and write $u_B = \Phi_0 u$. Then the following estimates hold with constants depending on $\Phi_0$:

$$\|u_B\|_{\dot{X}^{-s}_{1/2}} \lesssim \|u\|_{\dot{X}^{-s}_{1/2}},$$

$$\|u_B\|_{\dot{X}^{s}_{1/2}} \lesssim \|u\|_{\dot{X}^{s}_{1/2}},$$

$$\|u_B\|_{L^2} \lesssim s^{-1/2} \|u\|_{\dot{X}^{s}_{1/2}}.$$

**Proof.** To prove (3.1), in view of (3.2), it suffices to show that

$$\|u_B\|_{L^2} \lesssim \|u\|_{\dot{X}^{s}_{1/2}}.$$

(3.4)

Observing that the support of $\Phi_0$ is compact, we can find a ball $\tilde{B} \supset \text{spt} \Phi_0$ and a cutoff function $\varphi_\tilde{B} \in C_0^\infty(\tilde{B})$ satisfying $\varphi_\tilde{B} = 1$ on $\text{spt} \Phi_0$. Then we have $u_B = \varphi_\tilde{B} u_B$. By the Fourier transform formula, we know

$$\hat{u}_B = \hat{\varphi}_\tilde{B} \ast \hat{u}_B,$$

where $\hat{\varphi}_\tilde{B}$ is a Schwartz function.

To obtain (3.6), by proposition 3.1 we just need to show that there exists a constant $C$ s.t., $\forall \xi \in \mathbb{R}^n$,

$$\int |\hat{\varphi}_\tilde{B}(\xi - \eta)| \frac{|\xi|}{|P_\xi(\eta)| + s} \, d\eta \leq C.$$

(3.7)

Noting that $|\xi| \leq |\xi - \eta| + |\eta|$ and $\hat{\varphi}_\tilde{B}$ is a Schwartz function, we need only prove

$$\int |\hat{\varphi}_\tilde{B}(\xi - \eta)| \frac{|\eta|}{|P_\xi(\eta)| + s} \, d\eta \leq C.$$

(3.8)
On the other hand, if $|\eta| \gg s$, we know $|P_\zeta(\eta)| + s \gtrsim |\eta|^2$, which implies
\[
\frac{|\eta|}{|P_\zeta(\eta)|^\gamma} \lesssim C,
\] (3.9)
where the constant $C$ is independent of $s$. Clearly, (3.9) implies (3.8).

To prove (3.5), observe that, $\forall \eta \in \mathbb{R}^n$,
\[
\frac{|\eta|^2}{|P_\zeta(0)|^\gamma} \lesssim s.
\] (3.10)
Similarly to the proof of (3.4), we can show that
\[
\|uB\|_{H^1(\Omega)} \lesssim s^{1/2} \|u\|_{\dot{X}^{1/2}_\gamma}.
\] (3.11)
Lemma 3.3 then follows.

Clearly, if $\gamma \in C^1(\bar{\Omega})$ and $u$ is a solution to (1.1), $u$ satisfies
\[
(\Delta - A \cdot \nabla)u = 0 \text{ in } \Omega,
\] (3.12)
where
\[
A = \nabla \log \gamma.
\]
The following analysis is from [Knu06]. For completeness, we will write it down in detail. Indeed, we find the following type of CGO solution:
\[
u(x, \zeta) = e^{-\frac{t}{2}} e^{i \zeta \cdot (1 + w(x, \zeta))}
\]
of (3.12). Here $\phi_t$ is defined as in section 2 and $\zeta \in C^n \setminus 0$ satisfies $\zeta \cdot \zeta = 0$ which implies that $\exp(x \cdot \zeta)$ is harmonic.

We will decompose
\[
\nabla - A \cdot \nabla = -\Delta + (A_t - A) \cdot \nabla - A_t \cdot \nabla
\]
and use the fact that
\[
(\Delta - A_t \cdot \nabla) e^{-\frac{t}{2}} v = e^{-\frac{t}{2}} \left( -\Delta + \frac{1}{2} \nabla \cdot A_t + \frac{1}{4} (A_t)^2 \right) v.
\]
Since $A_t = \nabla \phi_t$, it follows that
\[
(\Delta - A \cdot \nabla) e^{-\frac{t}{2}} v = (\Delta + (A_t - A) \cdot \nabla - A_t \cdot \nabla) e^{-\frac{t}{2}} v
\]
\[
e^{-\frac{t}{2}} \left( -\Delta + (A_t - A) \cdot \nabla + q_t \right) v,
\]
where $q_t = \frac{1}{2} \nabla \cdot A_t - \frac{1}{4} (A_t)^2 + \frac{1}{2} A_t \cdot A_t$.

Hence, the equation for $w(x, \zeta)$ is
\[
(\Delta + (A_t - A) \cdot \nabla + q_t)w = (A_t - A_t) \cdot \zeta - q_t,
\] (3.13)
where
\[
\Delta + 2\zeta \cdot \nabla, \nabla \zeta = \nabla + \zeta.
\]

For a given $k \in \mathbb{R}^n$, we set $\zeta_1 = s \eta_1 + i \left( \frac{k}{2} + r \eta_2 \right)$, $\zeta_2 = -s \eta_1 + i \left( \frac{k}{2} - r \eta_2 \right)$, where $\eta_1, \eta_2 \in S^{n-1}$ satisfy $k \cdot \eta_1 = k \cdot \eta_2 = \eta_1 \cdot \eta_2 = 0$ and $|\eta_1|^2 + r^2 = s^2$. The vectors $\zeta_i$ are chosen so that $\zeta_i \cdot \zeta_i = 0$ and $\zeta_1 + \zeta_2 = k$.

Our goal is to find sequences $s_n, \zeta(n)^i$ such that $s_n \to \infty$ and $\|w(n)^i\|_{\dot{X}^{1/2}_{\zeta(n)^i}} \to 0$, which are the solutions of (3.13) with $t = s_n$. In fact we have the following lemma.
Lemma 3.4. Let $\gamma$ be a $C^1(\mathbb{R}^n)$ function with $\gamma > C > 0$, $r = 1$ outside a ball. Then for fixed $k$, there exists a sequence $\zeta_i^{(n)}$ with $s_n \to \infty$ such that
\[ \|w^{(n)}\|_{\frac{1}{k}^{(n)}} \to 0, \quad \text{as } s_n \to \infty, \]
where $w^{(n)}$ is a solution of (3.13) with $t = s_n$.

Proof. We want to find the solution $w$ of (3.13). Indeed, we just need to find a fixed point of the following operator:
\[ w = \Delta^{-1}((A_k - A) - \nabla w + q_t w) + \Delta^{-1}((A_k - A) \cdot \xi + q_t) \]
in a suitable function space, where $\Delta^{-1}$ means $\Delta^{-1}f = \frac{1}{\|P_1(\xi)\|} \hat{f}(\xi)$.

First of all, we will show that there exists a sequence $\zeta_i^{(n)}$ with $s_n \to \infty$ such that
\[ \|\Delta^{-1}(A_k - A) \cdot \zeta_i^{(n)} + q_{s_n}\|_{\frac{1}{k}^{(n)}} \to 0. \tag{3.14} \]

In fact, since $q_{s_n} = \frac{1}{s} \nabla \cdot A_{s_n} - \frac{1}{2}(A_{s_n})^2 + \frac{1}{2} A \cdot A_{s_n}$ and $A, A_k$ is bounded with uniformly compact support, by proposition (3.2) we have
\[ \|A_{s_n}\|^2_{\frac{1}{k}^{(n)}} = \|\Phi_B(A_{s_n})\|^2_{\frac{1}{k}^{(n)}} \leq \|A_{s_n}\|^2_{\frac{1}{k}^{(n)}} \leq \frac{1}{s_n^2}, \tag{3.15} \]
\[ \|A_{s_n}\|^2_{\frac{1}{k}^{(n)}} = \|\Phi_B(A \cdot A_{s_n})\|^2_{\frac{1}{k}^{(n)}} \leq \|A_{s_n}\|^2_{\frac{1}{k}^{(n)}} \leq \frac{1}{s_n^2}. \tag{3.16} \]

Now, it is only left to estimate $\|
abla \cdot A_{s_n}\|_{\frac{1}{h}^{(n)}}$. In view of lemma 3.1 in [HT11], we know
\[ \int_{B_{0.1}(1)} \frac{1}{s} \int_{s}^{2s} \|\nabla \hat{A}\|_{\frac{1}{h}^{(n)}} d\lambda \, d\eta \to 0, \quad \text{as } s \to \infty. \]

Then there exists a sequence $\zeta_i^{(n)}$ such that
\[ \|\nabla \cdot A\|_{\frac{1}{k}^{(n)}} \to 0, \quad \text{as } s_n \to \infty. \]

Observing $|\nabla \cdot A_{s_n}| = |\nabla \cdot \hat{A}_{s_n}| = ||\nabla \hat{A}(\xi)\|_{L^1} ||\xi \cdot \hat{A}(\xi)\|_{L^1} \leq ||\hat{A}(\xi)\|_{L^\infty} ||\xi \cdot \hat{A}(\xi)\|_{L^1} \to ||\nabla \cdot A||_{L^1} \to 0.$

Combining estimates (3.15–3.17), we have
\[ \|q_{s_n}\|_{\frac{1}{k}^{(n)}} \to 0, \quad \text{as } s_n \to \infty. \]

Secondly, we need to show $\|A - A_{s_n}\|_{\frac{1}{k}^{(n)}} = o\left(\frac{1}{s_n}\right)$.

Indeed, by lemma 3.1 again in [HT11], we can find the same sequence $\zeta_i^{(n)}$ with $s_n \to \infty$ such that
\[ \|A - A_{s_n}\|_{\frac{1}{k}^{(n)}} = \|\Phi_B \nabla (\phi - \phi_{s_n})\|_{\frac{1}{k}^{(n)}} \leq \frac{1}{s_n} \|\phi - \phi_{s_n}\|_{H^{1,2}}. \]

From the assumptions of $\gamma$, it follows that $\phi \in C^1(\mathbb{R}^n)$, $\phi_{s_n} \in C^1(\mathbb{R}^n)$ with uniformly bounded compact support. Hence, lemma 2.1 implies
\[ \|A - A_{s_n}\|_{L^2} = o\left(\frac{1}{s_n}\right). \]
Therefore, (3.14) holds.

In the following, we will show that \( \Delta^{-1}_{\xi_i} ((A_{\eta_n} - A) \cdot \nabla_{\xi_i} w + q_{\eta_n} w) \) is a contract map in \( \tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n) \) when \( s_n \) is large enough. In fact, by corollary 2.1 in [HT11], we know

\[
\| (A_{\eta_n} - A) \cdot \xi_i^{(n)} w \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n) \to \tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \lesssim \| A - A_{\eta_n} \|_{L^\infty},
\]

\[
\| q_{\eta_n} w \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n) \to \tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \lesssim \frac{1}{s_n} \| q_{\eta_n} \|_{L^\infty} \lesssim \frac{1}{s_n} \left( \| A \|_{L^\infty}^2 + \| D^2 \phi_{\eta_n} \|_{L^\infty} \right).
\]

On the other hand, by lemma 2.3 in [HT11], we know that

\[
\| (A_{\eta_n} - A) \cdot \nabla w \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n) \to \tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \lesssim \| A - A_{\eta_n} \|_{L^\infty}.
\]

Then, by lemma 2.1, it follows that

\[
\| \Delta^{-1}_{\xi_i} ((A_{\eta_n} - A) \cdot \nabla_{\xi_i} w + q_{\eta_n} w) \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n) \to \tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} < 1,
\]

when \( s_n \) is large enough.

Using the contract mapping theorem, we know that there exists a sequence \( \xi_i^{(n)} \) such that

\[
\| w^{(n)} \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \to 0, \quad \text{as} \quad s_n \to \infty,
\]

where \( w^{(n)} \) is a solution of (3.13) with \( t = s_n \).

□

\textbf{Lemma 3.5.} Let \( w^{(n)} \) be chosen as in lemma 3.4; then, the following estimates hold:

\[
\| w^{(n)} \|_{L^2(\Omega)} \lesssim \frac{1}{s_n^\frac{1}{4}},
\]

\[
\| w^{(n)} \|_{H^\frac{1}{2}(\Omega)} \lesssim s_n^\frac{1}{4},
\]

\[
\| w^{(n)} \|_{H^1(\Omega)} \lesssim s_n^\frac{1}{2}.
\]

\textbf{Proof.} From the construction of \( w^{(n)} \), by proposition 3.2 and lemma 3.3, we know

\[
\| w^{(n)} \|_{L^2(\Omega)} \leq \| \Phi_B w^{(n)} \|_{L^2} \lesssim \frac{1}{s_n^\frac{1}{4}} \| w^{(n)} \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \tag{3.18}
\]

and

\[
\| \Phi_B w^{(n)} \|_{H^\frac{1}{2}(\Omega)} \lesssim \frac{1}{s_n^\frac{1}{4}} \| w^{(n)} \|_{\tilde{X}_\xi^{-\frac{1}{2}} (\mathbb{R}^n)} \lesssim \frac{1}{s_n^\frac{1}{4}} \tag{3.19}
\]

In view of the definition of \( \Phi_B \), we know that, for \( \Omega \subset B_0 \subset B \), (3.19) implies

\[
\| \Phi_B w^{(n)} \|_{H^\frac{1}{2}(B)} \lesssim \frac{1}{s_n^\frac{1}{4}} \tag{3.20}
\]

Thus, \( w^{(n)} \) is a weak solution of the following equation:

\[
-\Delta w^{(n)} + (2\xi_i^{(n)} + (A_{\eta_n} - A) \cdot \nabla w^{(n)} + (\xi_i^{(n)} \cdot (A_{\eta_n} - A) + q_{\eta_n}) w^{(n)} = (A - A_{\eta_n}) \cdot \xi_i^{(n)} - q_{\eta_n} \quad \text{in} \quad B.
\]
By the classical interior estimate for the Laplace equation, we deduce
\[
\|u^{(n)}\|_{H^2(\Omega_{t^2})} \lesssim \|u^{(n)}\|_{H^2(B_R)} + \|2\zeta_{\gamma}(\nu - A)\cdot \nabla u^{(n)}\|_{L^2(B_R)} \\
+ \|\zeta_{\gamma}(\nu - A) + q_n\|_{H^2(\Omega_{t^2})} + \|(A - A_n)\cdot \zeta_{\gamma} - q_n\|_{L^2(B_R)} \\
\lesssim \|w^{(n)}\|_{H^2(B_R)} + s_n \|w^{(n)}\|_{H^2(\Omega_{t^2})} + s_n \|w^{(n)}\|_{L^2(B_R)}.
\] (3.21)
where we use the relations \(\|A_n\|_{L^\infty} \leq \|A\|_{L^\infty}\) and \(\|D^2 \Phi_n\|_{L^\infty} = o(s_n)\).

Combining (3.20) and (3.21), we have
\[
\|w^{(n)}\|_{H^2(\Omega_{t^2})} \lesssim s_n^2.
\]
The proof is complete. \(\square\)

4. Carleman estimate for CGO solutions

In this section, we will introduce the Carleman estimate for CGO solutions. The first result is taken from [Knu06].

Proposition 4.1. Let \(\xi \in S^{n-1}\) and suppose \(u \in H^2(\Omega)\). Then there exists a constant \(t_0 > 0\) such that for \(t \geq t_0\), we have the estimate
\[
C(t^2 \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2) - C t^2 \int_{\Omega} |u|^2 \, dx - C n \int_{\Omega} \bar{u} \partial_\nu u \, dx + \int_{\Omega} 4t \text{Re}(\partial_\nu u \partial_\nu \bar{u}) \\
- 2t(\nu \cdot \eta)|\nabla u|^2 + 2t^2(\nu \cdot \eta)|u|^2 \, dx \leq \|e^{-x_\nu}(-\Delta)(e^{x_\nu} u)\|_{L^2(\Omega)}^2.
\] (4.1)

In this paper, we need the following type of Carleman estimate.

Lemma 4.2. Let \(\eta \in S^{n-1}\) and suppose \(u \in H^2(\Omega)\). Then there exists a constant \(\delta\) such that for \(\gamma \in C^1(\overline{\Omega})\), we have
\[
C(t^2 \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2) - C t^2 \int_{\Omega} |u|^2 \, dx - C n \int_{\Omega} \bar{u} \partial_\nu u \, dx \\
+ \int_{\Omega} 4t \text{Re}(\partial_\nu u \partial_\nu \bar{u}) - 2t(\nu \cdot \eta)|\nabla u|^2 + 2t^2(\nu \cdot \eta)|u|^2 \, dx \leq \|e^{-x_\nu}(-\Delta + (A_r - A) \cdot \nabla + q_r)(e^{x_\nu} u)\|_{L^2(\Omega)}^2.
\] (4.2)

for \(t \geq t(\gamma) > 0\), where \(t(\gamma)\) is a constant only depending on \(\gamma\).

Proof. We have
\[
\|e^{-x_\nu}(-\Delta)(e^{x_\nu} u)\|_{L^2(\Omega)}^2 \leq 2\|e^{-x_\nu}(-\Delta + (A_r - A) \cdot \nabla + q_r)(e^{x_\nu} u)\|_{L^2(\Omega)}^2 \\
+ 2\|e^{-x_\nu}((A_r - A) \cdot \nabla + q_r)(e^{x_\nu} u)\|_{L^2(\Omega)}^2 \\
\leq 2\|e^{-x_\nu}(-\Delta + (A_r - A) \cdot \nabla + q_r)(e^{x_\nu} u)\|_{L^2(\Omega)}^2 \\
+ 2\|A - A_\gamma\|_{L^\infty}^2 \|\nabla u\|_{L^2(\Omega)}^2 + 2t^2\|A - A_\gamma\|_{L^\infty}^2 \|u\|_{L^2(\Omega)}^2 \\
+ 2\|q_r\|_{L^\infty} \|u\|_{L^2(\Omega)}^2.
\] (4.3)

Since \(\gamma \in C^1(\overline{\Omega})\), from lemma 2.1, we know that \(\|A - A_\gamma\|_{L^\infty} = o(1)\) and \(\|q_r\|_{L^\infty} = o(t)\). Hence, if \(t\) is large enough, the superfluous terms on the right-hand side in (4.3) can be absorbed by the first term on the left-hand side in (4.1). Then (4.2) holds. \(\square\)
5. The uniqueness proof

First, we introduce a boundary integral identity which is from [Knu06].

**Proposition 5.1.** Suppose \( \gamma_i \in C^1(\Omega), i = 1, 2, \) and \( u_1, u_2 \in H^1(\Omega) \) satisfy \( \nabla \cdot (\gamma_i \nabla u_i) = 0 \) in \( \Omega \). Suppose further that \( \bar{u}_1 \in H^1(\Omega) \) satisfies \( \nabla \cdot (\gamma_1 \nabla \bar{u}_1) = 0 \) with \( \bar{u}_1 = u_2 \) on \( \partial \Omega \). Then

\[
\int_{\Omega} \left( \gamma_1 \nabla y_2 \cdot \gamma_1 \nabla y_1 \right) \cdot \nabla (u_1 u_2) \, dx = \int_{\partial \Omega} \gamma_1 \partial_n (\bar{u}_1 - u_2) u_1 \, ds,
\]

where the integral on the boundary is understood in the sense of the dual pairing between \( H^\gamma(\partial \Omega) \) and \( H^{-\gamma}(\partial \Omega) \).

**Remark 5.2.** Proposition 5.1 also holds for \( \gamma_i \in W^{1,\infty} (\Omega) \).

We will make use of the boundary integral identity to show the uniqueness of the Calderón problem with partial data. For this goal, fix \( k \in \mathbb{R}^n \) with \( k \cdot \eta = 0 \) and choose \( I^{(n)} = I^{(n)} \in \mathbb{R}^n \) with \( I^{(n)} \cdot \eta = I^{(n)} \cdot \eta = 0 \) and \( \frac{|I^{(n)}|}{4} + |I^{(n)}|^2 = s_n^2 \). Further define \( \zeta_i^{(n)} = s_n \eta + i \left( \frac{1}{2} + l^{(n)} \right) \) and

\[
\zeta_2^{(n)} = -s_n \eta + i \left( \frac{1}{2} - l^{(n)} \right),
\]

and let \( \phi_i = \nabla \log \gamma_i \) and \( u_i^{(n)} = e^{\frac{\phi_i}{2}} e^{\epsilon \zeta_i^{(n)}} (1 + u_i^{(n)}) \) be the CGO solutions to \( \nabla \cdot (\gamma_i \nabla u_i) = 0 \), where \( w_i^{(n)} \) is chosen as in lemma 3.4. Finally take \( \bar{u}_i^{(n)} \) as the solution to \( \nabla \cdot (\gamma_i \nabla \bar{u}_i^{(n)}) \) with \( u_i^{(n)} = \bar{u}_i^{(n)} \) on \( \partial \Omega \). Hence, proposition 5.1 implies

\[
\int_{\Omega} \left( \gamma_1 \nabla y_2 \cdot \gamma_1 \nabla y_1 \right) \cdot \nabla \left( u_1^{(n)} \bar{u}_2^{(n)} \right) \, dx = \int_{\partial \Omega} \gamma_1 \partial_n (\bar{u}_1^{(n)} - u_2^{(n)}) \bar{u}_1^{(n)} \, ds. \tag{5.1}
\]

We will first prove

\[
\lim_{n \to \infty} \int_{\partial \Omega} \gamma_1 \partial_n (\bar{u}_1^{(n)} - u_2^{(n)}) \bar{u}_1^{(n)} \, ds = 0. \tag{5.2}
\]

The proof of (5.2) is divided into the following three lemmas. For simplicity, we will not write the superscripts of \( u_i^{(n)}, \bar{u}_i^{(n)}, w_i^{(n)} \) and \( \zeta_i^{(n)} \) and the subscript of \( s_n \) again unless otherwise particularly specified.

Introduce the function

\[
u = e^{\phi_1} u_1 - e^{\phi_2} u_2 = u_0 + \delta u,
\]

where

\[
u_0 = e^{\phi_1} (\bar{u}_1 - u_2),
\]

\[
\delta u = (e^{\phi_2} - e^{\phi_1}) u_2.
\]

**Lemma 5.3.** Suppose the assumptions of theorem 1.1 hold and define \( u_2 \) as above. Then

\[
\int_{\partial \Omega} e^{-4 |\nabla \nu|} |\nabla \nu|^2 \, ds = o(1),
\]

\[
\int_{\partial \Omega} e^{-4 |\delta u|} |\delta u|^2 \, ds = o \left( \frac{1}{s^2} \right),
\]

as \( s \to \infty \).

**Proof.** From the definition of \( \delta u \), we know

\[
\int_{\partial \Omega} e^{-4 |\nabla \nu|} |\nabla \nu|^2 \, ds \leq \int_{\partial \Omega} e^{-4 |\nabla \nu|} \left| \nabla \left( e^{\phi_1} - e^{\phi_2} \right) \right|^2 |\nu|^2 \, ds
\]

\[
\quad + \int_{\partial \Omega} e^{-4 |\nabla \nu|} \left( e^{\phi_1} - e^{\phi_2} \right)^2 |\nabla u_2|^2 \, ds =: I + II.
\]

...
For $I$, from the construction of $u_2$, we have

$$I \lesssim \int_{\partial \Omega} |\nabla (e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}})|^2 |1 + w_2|^2 \, ds. \quad (5.3)$$

Since $\gamma_1|_{\partial \Omega} = \gamma_2|_{\partial \Omega}$, $\partial_\gamma \gamma_1|_{\partial \Omega} = \partial_\gamma \gamma_2|_{\partial \Omega}$, by the mean value theorem, (5.3) implies

$$I \lesssim \int_{\Omega} |\nabla (e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}})|^2 \, ds + \int_{\Omega} |\nabla (e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}})|^2 \, ds$$

$$+ \int_{\Omega} |w_2|^2 \, ds \|\nabla (e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}})|^2_{L^\infty} + \int_{\Omega} |w_2|^2 \, ds \|\nabla (e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}})|^2_{L^\infty}$$

$$\lesssim \int_{\Omega} |\nabla \phi_1 - \nabla \phi_1|^2 \, ds + \int_{\Omega} |\phi_1 - \phi_1|^2 \, ds$$

$$+ \int_{\Omega} |\nabla \phi_2 - \nabla \phi_2|^2 \, ds + \int_{\Omega} |\phi_2 - \phi_2|^2 \, ds$$

$$+ \int_{\Omega} |w_2|^2 \, ds \|\nabla (\phi_1 - \phi_1)|^2_{L^\infty} + |\phi_1 - \phi_1|^2_{L^\infty}$$

$$+ \int_{\Omega} |w_2|^2 \, ds \|\nabla (\phi_2 - \phi_2)|^2_{L^\infty} + |\phi_2 - \phi_2|^2_{L^\infty}.$$ 

Hence, it follows that

$$I \lesssim \|A_{1\gamma} - A_1\|^2_{L^\infty} + \|\phi_1 - \phi_1\|^2_{L^\infty} + \|A_2 - A_2\|^2_{L^\infty} + \|\phi_2 - \phi_2\|^2_{L^\infty}$$

$$+ \int_{\Omega} |w_2|^2 \, ds \|A_{1\gamma} - A_1\|^2_{L^\infty} + \|\phi_1 - \phi_1\|^2_{L^\infty}$$

$$+ \int_{\Omega} |w_2|^2 \, ds \|A_2 - A_2\|^2_{L^\infty} + \|\phi_2 - \phi_2\|^2_{L^\infty}. \quad (5.4)$$

On the other hand, by lemmas 2.2 and 3.5, we have

$$\int_{\partial \Omega} |w_2|^2 \, ds \lesssim 1. \quad (5.5)$$

By lemma 2.1, equations (5.4) and (5.5) imply

$I = o(1)$, as $s \to \infty$.

For $II$, from the definition of $u_2$, we know

$$II \lesssim \int_{\Omega} |e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}}|^2 |\nabla w_2|^2 \, ds + \int_{\Omega} |e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}}|^2 |\nabla w_2|^2 \, ds$$

$$+ s^2 \int_{\Omega} |e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}}|^2 |1 + w_2|^2 \, ds + s^2 \int_{\Omega} |e^{\frac{2\pi}{\epsilon}} - e^{\frac{\pi}{\epsilon}}|^2 |1 + w_2|^2 \, ds.$$ 

Hence, it follows that

$$II \lesssim \|\phi_1 - \phi_1\|^2_{L^\infty} \int_{\partial \Omega} |\nabla w_2|^2 \, ds + \|\phi_2 - \phi_2\|^2_{L^\infty} \int_{\partial \Omega} |\nabla w_2|^2 \, ds$$

$$+ s^2 \|\phi_1 - \phi_1\|^2_{L^\infty} \int_{\partial \Omega} |1 + w_2|^2 \, ds + s^2 \|\phi_2 - \phi_2\|^2_{L^\infty} \int_{\partial \Omega} |1 + w_2|^2 \, ds. \quad (5.6)$$

On the other hand, by lemmas 2.2 and 3.5, we have

$$\int_{\partial \Omega} |\nabla w_2|^2 \, ds \lesssim s^2. \quad (5.7)$$

By lemma 2.1, equations (5.5)–(5.7) imply

$II = o(1)$, as $s \to \infty$. 
Hence, we prove
\[
\lim_{s \to \infty} \int_{\partial \Omega} e^{-s|x|} |\nabla u|^2 \, ds = 0.
\]

Noting
\[
\int d\Omega \left( e^{-s|x|} |\delta u|^2 \right) \lesssim \int d\Omega \left( e^{s \frac{1}{2}} - e^{s \frac{2}{3}} \right) (1 + |u|^2) \, dx
\]
and using the above estimates, we can easily obtain
\[
\int_{\partial \Omega} e^{-s|x|} |\delta u|^2 \, ds = o \left( \frac{1}{s^2} \right), \quad \text{as } s \to \infty.
\]

The proof is complete. \(\square\)

**Lemma 5.4.** Suppose the assumptions of theorem 1.1 hold and define \(u_1, u_2\) and \(\tilde{u}_1\) as above.

Then
\[
\int_{\partial \Omega} e^{-s|x|} |\delta u|^2 \, ds = o(1),
\]
as \(s \to \infty\).

**Proof.** Letting \(v = e^{-s|x|} u\) and recalling the Carleman estimate for \(v\), we have
\[
C(s^2 \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2) - C s^2 \int_{\partial \Omega} |v|^2 \, ds + C \int_{\partial \Omega} \bar{v} \partial_{\nu} v \, ds
\]
\[
+ \int_{\partial \Omega} 4s \text{Re}(\partial_{\nu} v \partial_{\nu} \bar{v}) - 2s(\nu \cdot \eta)|\nabla v|^2 + 2s^2 (\nu \cdot \eta)|v|^2 \, ds
\]
\[
\leq \|e^{-s|x|} (-\Delta + (A_{1s} - A_1) \cdot \nabla + q_{1s}) u\|_{L^2(\Omega)}^2.
\]

Hence,
\[
\int_{\partial \Omega} 4\text{Re}(\partial_{\nu} v \partial_{\nu} \bar{v}) - 2(\nu \cdot \eta)|\nabla v|^2 \, ds \lesssim \int_{\partial \Omega} |v|^2 \, ds + \frac{1}{s} \left| \int_{\partial \Omega} \bar{v} \partial_{\nu} v \, ds \right|
\]
\[
+ s^2 \int_{\partial \Omega} |v \cdot \eta||v|^2 \, ds + \frac{1}{s} \|e^{-s|x|} (-\Delta + (A_{1s} - A_1) \cdot \nabla + q_{1s}) u\|_{L^2(\Omega)}^2.
\]

(5.8)

Now we first estimate the terms on the left-hand side of (5.9):

left-hand side of (5.9) = \int_{\partial \Omega} 4\text{Re}(\partial_{\nu} v \partial_{\nu} \bar{v}) - 2(\nu \cdot \eta)|\nabla v|^2 \, ds
\]
\[
+ \int_{\partial \Omega} 4\text{Re}(\partial_{\nu} v \partial_{\nu} \bar{v}) - 2(\nu \cdot \eta)|\nabla v|^2 \, ds =: V + VI.
\]

For \(V\), since \(\tilde{u}|_{\partial \Omega} = u|_{\partial \Omega}\), recalling the definition of \(V\) and using the Young inequality, we have
\[
\int_{\partial \Omega} \left( e^{-s|x|} u \cdot (\frac{\partial u}{\partial v})^2 \right) \, ds - C \int_{\partial \Omega} e^{-s|x|} |\nabla \delta u|^2 \, ds
\]
\[
- Cs^3 \int_{\partial \Omega} e^{-s|x|} |\delta u|^2 \, ds.
\]

(5.10)
For $VI$, observing $\tilde{u}_1|_{\Omega} = u_2|_{\Omega}$ and $\frac{\partial \tilde{u}_1}{\partial n}|_{\partial \Omega_{-s}} = \frac{\partial u_2}{\partial n}|_{\partial \Omega_{-s}}$, we deduce

$$VI \lesssim \int_{\partial \Omega_{-s}}|\nabla v|^2 \, ds \lesssim \frac{s^2}{2} \int_{\partial \Omega_{-s}} e^{-x_2\eta}|u|^2 \, ds + \int_{\partial \Omega_{-s}} e^{-x_2\eta} |\nabla u|^2 \, ds$$

$$\lesssim \frac{s^2}{2} \int_{\partial \Omega_{-s}} e^{-x_2\eta}|\delta u|^2 \, ds + \int_{\partial \Omega_{-s}} e^{-x_2\eta} |\nabla u|^2 \, ds. \quad (5.11)$$

Combining (5.10) and (5.11), we deduce

left-hand side of (5.9) \( \geq \)

$$-Cs^2 \int_{\partial \Omega_{-s}} e^{-x_2\eta} |\delta u|^2 \, ds - C \int_{\partial \Omega_{-s}} e^{-x_2\eta} |\nabla u|^2 \, ds$$

$$\geq \epsilon \int_{\partial \Omega_{-s}} e^{-x_2\eta} \left( \frac{\partial u}{\partial v} \right)^2 \, ds - o(1), \quad (5.12)$$

as $s \to \infty$, where we use lemma 5.3 in the second inequality.

Now we will deal with the terms on the right-hand side of (5.9). For $I$ and $III$, by lemma 5.3, we have

$$I \lesssim s \int_{\partial \Omega} e^{-x_2\eta}|u|^2 \, ds \lesssim s \int_{\partial \Omega} e^{-x_2\eta} |\delta u|^2 \, ds = o\left(\frac{1}{s}\right)$$

and

$$III \lesssim \frac{s^2}{2} \int_{\partial \Omega} e^{-x_2\eta} |\delta u|^2 \, ds \lesssim \frac{s^2}{2} \int_{\partial \Omega} e^{-x_2\eta} |\nabla u|^2 \, ds = o(1).$$

From the definition of $v$, we know

$$\frac{1}{s} \int_{\partial \Omega} \tilde{v} \partial_v v \, ds = \frac{1}{s} \int_{\partial \Omega} e^{-x_2\eta} \delta u \partial_v \left( e^{-x_2\eta} u \right) \, ds$$

$$= \frac{1}{s} \int_{\partial \Omega} e^{-x_2\eta} \delta u e^{-x_2\eta} - sv \cdot \eta \delta u \, ds + \frac{1}{s} \int_{\partial \Omega} e^{-x_2\eta} \delta u e^{-x_2\eta} \partial_v \delta u \, ds.$$

Observing $\tilde{u}_1|_{\Omega} = u_2|_{\Omega}$ and $\frac{\partial \tilde{u}_1}{\partial n}|_{\partial \Omega_{-s}} = \frac{\partial u_2}{\partial n}|_{\partial \Omega_{-s}}$ and using the Young inequality, we have, for any $\epsilon_0 > 0$,

$$II \leq \epsilon_0 \int_{\partial \Omega_{-s}} e^{-x_2\eta} \left( \frac{\partial u}{\partial v} \right)^2 \, ds + C \int_{\partial \Omega} e^{-x_2\eta} |\delta u|^2 \, ds + C \frac{1}{s^2} \int_{\partial \Omega} e^{-x_2\eta} |\nabla \delta u|^2 \, ds$$

$$\lesssim \epsilon_0 \int_{\partial \Omega_{-s}} e^{-x_2\eta} \left( \frac{\partial u}{\partial v} \right)^2 \, ds + o\left(\frac{1}{s^2}\right),$$

as $s \to \infty$, where we use lemma 5.3 in the second inequality.

Finally, we estimate $IV$:

$$IV \leq \frac{1}{s} \int_{\Omega} e^{-x_2\eta} \left( -\Delta + (A_{12} - A_1) \cdot \nabla + q_{1s} \right) \left( e^{x_2\eta} \partial_1 u \right)^2 \, dx$$

$$\leq \frac{1}{s} \int_{\Omega} e^{-x_2\eta} \left( (A_{12} - A_1) - (A_{2s} - A_2) \right) \cdot \nabla \left( e^{x_2\eta} (1 + w_2) + q_{1s} - q_{2s} \right) e^{x_2\eta} (1 + w_2) \, dx,$$

where we use the equations

$$(-\Delta + (A_{1s} - A_1) \cdot \nabla + q_{1s} \partial_1 u_2) = 0.$$
Equation (5.13) implies

\[ IV \lesssim s \int_{\Omega} |A_2 - A_{21}|^2 + |A_1 - A_{11}|^2 \, dx + \frac{1}{s} \int_{\Omega} (|A_2 - A_{21}|^2 + |A_1 - A_{11}|^2) \, dx\]

\[ + \frac{1}{s} \int_{\Omega} |A_2 - A_{21}|^2 + |A_1 - A_{11}|^2 \, dx + \frac{1}{s} \int_{\Omega} |q_2 - q_{11}|^2 \, dx\]

\[ + \frac{1}{s} \int_{\Omega} |q_2 - q_{11}|^2 \, dx =: a + b + c + d + e.\]

By lemma 2.1, we know

\[ a = o(1), \quad d = o(1), \quad \text{as } s \to \infty.\]

For \( b \), by proposition 3.2 and lemma 3.4, we have

\[ b \leq (\|A_2 - A_{21}\|^2 + \|A_1 - A_{11}\|^2) s \int_{\Omega} |w_2|^2 \, dx\]

\[ \lesssim s \int_{\Omega} \|w_2\|^2 \, dx\]

\[ \lesssim \|w_2\|_{L_2}^2 \to 0 \quad \text{as } s \to \infty.\]

Observing \( \|q_2 - q_{11}\|_{L_\infty} = o(s) \) and using the same method as for \( b \), we can obtain

\[ e = o(1), \quad \text{as } s \to \infty.\]

Finally, for \( c \), using lemmas 2.1 and 3.3, we have

\[ c \leq \frac{1}{s} \int_{\Omega} |\nabla w_2|^2 \, dx \leq \frac{1}{s} \int_{\Omega} |\nabla (\Phi_B w_2)|^2 \, dx \lesssim \|w_2\|_{L_2}^2 \to 0, \quad \text{as } s \to \infty.\]

From the above analysis, we obtain

\[ IV = o(1), \quad \text{as } s \to \infty.\]

Combining (5.9), (5.12) and the estimates of I, II, III and IV, we deduce

\[ \lim_{s \to \infty} \int_{\Omega} e^{-x^2} |\partial_x u|^2 \, dx = 0.\]

\[ \square\]

**Lemma 5.5.** Suppose the assumptions of theorem 1.1 hold and define \( u_1, u_2 \) and \( \tilde{u}_1 \) as above. Then

\[ \lim_{s \to \infty} \int_{\Omega} \gamma_1 \partial_x (\tilde{u}_1 - u_2) u_1 \, dx = 0.\]

**Proof.** From the fact \( \frac{\partial}{\partial x} e^{2x} u = e^{2x} \frac{\partial}{\partial x} u \), and (5.5), the Cauchy–Schwarz inequality gives

\[ \left| \int_{\Omega} \gamma_1 \partial_x (\tilde{u}_1 - u_2) u_1 \, dx \right| \lesssim \int_{\Omega} e^{x^2} |\partial_x (\tilde{u}_1 - u_2) u|^2 \, ds \|1 + w_1\|_{L_2(\Omega)}^2.\]

\[ \lesssim \int_{\Omega} e^{x^2} |\partial_x (\tilde{u}_1 - u_2)|^2 \, dx.\]

To estimate \( \int_{\Omega} e^{x^2} |\partial_x (\tilde{u}_1 - u_2)|^2 \, ds \), from the definition of \( u, u_0 \) and \( \delta u \), we deduce

\[ \int_{\Omega} e^{x^2} |\partial_x (\tilde{u}_1 - u_2)|^2 \, dx = \int_{\Omega} e^{x^2} \partial_x (e^{2x} u_0) |\partial_x (e^{2x} u_0)|^2 \, dx\]

\[ \lesssim \int_{\Omega} e^{x^2} |\partial_x u_0|^2 \, dx\]

\[ \lesssim \int_{\Omega} e^{x^2} |\partial_x u|^2 \, dx + \int_{\Omega} e^{x^2} |\partial_x \delta u|^2 \, dx,\]

where we use the relation \( u_0 |\partial_x \delta u = 0 \) in the first inequality.
By lemmas 3.3 and 3.4, inequality (5.14) implies
\[
\lim_{n \to \infty} \int_{\Omega} \gamma_1 \partial_x (\tilde{u}_1 - u_2) u_1 \, dx = 0.
\]

Then lemma 5.5 follows.

**Proof of theorem 1.1.**

Recalling \( u_i^{(n)} = e^{-\frac{\delta_2}{n} \gamma_i} e^{i \zeta_i} (1 + w_i^{(n)}) \), \( i = 1, 2 \), and using proposition 5.1, we know
\[
\left( \frac{1}{\gamma_1} \nabla \psi_2 + \frac{1}{\gamma_2} \nabla \psi_1 \right) \cdot \nabla \left( e^{\frac{\delta_1}{n} \gamma_1} e^{i \zeta_1} (1 + w_1^{(n)}) e^{\frac{\delta_2}{n} \gamma_2} e^{i \zeta_2} (1 + w_2^{(n)}) \right) = \int_{\Omega} \gamma_1 \partial_x (\tilde{u}_1^{(n)} - u_2^{(n)}) u_1^{(n)} \, dx.
\]

Observing \( e^{i \zeta_1} e^{i \zeta_2} = e^{i \gamma_1} \) and \( \lim e^{-\frac{\delta_2}{n} \gamma_1} = \gamma_2^{-1} \), by lemma 5.5, we have
\[
- \int_{\Omega} \left( \frac{1}{\gamma_1} \nabla \psi_2 + \frac{1}{\gamma_2} \nabla \psi_1 \right) \cdot \nabla \left( e^{\frac{\delta_1}{n} \gamma_1} e^{i \zeta_1} (1 + w_1^{(n)}) \right) dx
\]
\[
= \lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{\gamma_1} \nabla \psi_2 + \frac{1}{\gamma_2} \nabla \psi_1 \right) \nabla \left( e^{\frac{\delta_1}{n} \gamma_1} e^{i \zeta_1} (1 + w_1^{(n)}) \right) (w_1^{(n)} + w_2^{(n)} + u_1^{(n)} u_2^{(n)}) dx
\]
\[
= : S_1 + S_2.
\]

For \( S_1 \), recalling \( \| \phi \|_{L^\infty} \leq \| \phi \|_{L^\infty} \) and the definition of \( \Phi_B \), we have
\[
|S_1| \lesssim \lim sup_{n \to \infty} \int_{\Omega} \left( \| \Phi_B w_1^{(n)} \| + \| \Phi_B w_2^{(n)} \| + \| \Phi_B u_1^{(n)} \| | \Phi_B u_2^{(n)} \| \right) \, dx
\]
\[
\lesssim \lim sup_{n \to \infty} \left( \| \Phi_B w_1^{(n)} \|_{L^2} + \| \Phi_B w_2^{(n)} \|_{L^2} + \| \Phi_B u_1^{(n)} \|_{L^2} \| \Phi_B u_2^{(n)} \|_{L^2} \right). \tag{5.16}
\]

On the other hand, by lemmas 3.2 and 3.4, we have
\[
\| \Phi_B u_1^{(n)} \|_{L^2} \lesssim \frac{1}{s_n} \| \Phi_B u_1^{(n)} \|_{X^{i \gamma_1} \zeta_1^{(n)}}.
\]

It follows that \( S_1 = 0 \).

For \( S_2 \), recalling \( \gamma_1, \gamma_2 \in H^{\frac{1}{2}, 2} (\Omega) \), \( \| \phi \|_{L^\infty} \leq \| \phi \|_{L^\infty} \) and \( \| \nabla \phi \|_{L^\infty} \leq \| \nabla \phi \|_{L^\infty} \), we have
\[
|S_2| \lesssim \lim sup_{n \to \infty} \left( \| \Phi_B w_1^{(n)} \|_{H^{\frac{1}{2}, 2}} + \| \Phi_B w_2^{(n)} \|_{H^{\frac{1}{2}, 2}}
\]
\[
+ \| \Phi_B w_1^{(n)} \|_{L^2} \| \nabla (\Phi_B w_2^{(n)}) \|_{L^2} + \| \nabla (\Phi_B w_1^{(n)}) \|_{L^2} \| \Phi_B w_2^{(n)} \|_{L^2} \right). \tag{5.17}
\]

Using proposition 3.2 and lemma 3.3, we have
\[
\| \Phi_B w_1^{(n)} \|_{H^{\frac{1}{2}, 2}} \lesssim \| w_1^{(n)} \|_{X^{i \gamma_1} \zeta_1^{(n)}}.
\]
\[
\| \Phi_B w_1^{(n)} \|_{L^2} \lesssim \frac{1}{s_n} \| w_1^{(n)} \|_{X^{i \gamma_1} \zeta_1^{(n)}}.
\]
\[
\| \nabla (\Phi_B w_1^{(n)}) \|_{L^2} \lesssim s_n \| w_1^{(n)} \|_{X^{i \gamma_1} \zeta_1^{(n)}}.
\]

by lemma 3.4, which implies \( S_2 = 0 \).
Therefore, we have
\[ \int_{\Omega} (\gamma_1 \nabla \gamma_2 \cdot \nabla (\gamma_1 \gamma_2 e^{ik})) \, dx = 0. \]

It follows that
\[ \int_{\Omega} e^{ik} \left( -\frac{ik}{2} \cdot \nabla (\log \gamma_1 - \log \gamma_2) + \frac{1}{4} ((\nabla \log \gamma_1)^2 - (\nabla \log \gamma_2)^2) \right) \, dx = 0 \] (5.18)
for \( k \perp \eta \). However, since the DN maps agree on \( \partial \Omega_{-\epsilon}(\eta) \) for a fixed constant \( \epsilon > 0 \), they also agree on \( \partial \Omega_{-\epsilon}(\eta') \) for \( \eta' \) sufficiently close to \( \eta \) on the unit sphere and for smaller constant \( \epsilon' \). Thus, in particular, (5.18) holds for \( k \) in an open cone in \( \mathbb{R}^n \). Let the distribution \( q \) be equal to \( \frac{1}{2} \Delta (\log \gamma_1 - \log \gamma_2) + \frac{1}{4} ((\nabla \log \gamma_1)^2 - (\nabla \log \gamma_2)^2) \) in \( \Omega \) and zero outside of \( \Omega \). Hence, equation (5.18) implies that the Fourier transform of \( q \) vanishes in an open set. Since \( q \) is compact supported, the Fourier transform is analytic by the Paley–Wiener theorem, and this implies \( q \equiv 0 \), i.e.,
\[ \frac{1}{2} \Delta (\log \gamma_1 - \log \gamma_2) + \frac{1}{4} ((\nabla \log \gamma_1)^2 - (\nabla \log \gamma_2)^2) = 0, \quad \text{in } \Omega \]
and since \( \log \gamma_1 |_{\partial \Omega} = \log \gamma_2 |_{\partial \Omega} \), the uniqueness of the boundary value problem of uniform elliptic equations implies \( \gamma_1 = \gamma_2 \). Theorem 1.1 then holds. \( \square \)

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