A HOFER-LIKE METRIC ON THE GROUP
OF SYMPLECTIC DIFFEOMORPHISMS

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Abstract. Using a "Hodge decomposition" of symplectic isotopies on a compact
symplectic manifold $\mathcal{M}$, we construct a norm on the identity component in the
group of all symplectic diffeomorphisms of $\mathcal{M}$ whose restriction to the group
$\operatorname{Ham}(\mathcal{M}, \omega)$ of Hamiltonian diffeomorphisms is bounded from above by the Hofer
norm. Moreover, $\operatorname{Ham}(\mathcal{M}, \omega)$ is closed in $\operatorname{Symp}(\mathcal{M}, \omega)$ equipped with the topology
induced by the extended norm. We give an application to the $C^0$ symplectic topology.
We also discuss extensions of Oh’s spectral distance.

1. Introduction and statement of the main results

Let $\operatorname{Symp}(\mathcal{M}, \omega)$ denote the group of all symplectic diffeomorphisms of a compact
symplectic manifold $(\mathcal{M}, \omega)$, endowed with the $C^\infty$ compact-open topology, and
$\operatorname{Symp}(\mathcal{M}, \omega)_0 = G_\omega(\mathcal{M})$ the identity component in $\operatorname{Symp}(\mathcal{M}, \omega)$. $\operatorname{Symp}(\mathcal{M}, \omega)_0$
consists of symplectic diffeomorphisms $h$ such that there is a symplectic isotopy
$h_t$ from the identity to $h$. By definition $h_t$ is a symplectic isotopy if the map
$(x, t) \mapsto h_t(x)$ is smooth and for all $t$, $h_t^* \omega = \omega$. We denote by $\operatorname{Iso}(\mathcal{M})$ the set of
all symplectic isotopies, and by $\operatorname{Iso}(\phi)$ the set of all symplectic isotopies from the
identity to $\phi \in \operatorname{Symp}(\mathcal{M}, \omega)_0$.

Let $\operatorname{Ham}(\mathcal{M}, \omega) \subset \operatorname{Symp}(\mathcal{M}, \omega)_0$ be the subgroup of Hamiltonian diffeomor-
phisms. A diffeomorphism $\psi$ is Hamiltonian iff it is the time 1 map of a smooth
family of diffeomorphisms $\psi_t$, such that if

$$\dot{\psi}_t(x) = \frac{d\psi_t}{dt}(\psi_t^{-1}(x)), \quad \psi_0(x) = x \quad (1)$$

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there exists a smooth family of functions \( u_t \) such that

\[
  i_{(\tilde{\psi}_t)} \omega = du_t. \tag{2}
\]

The family of diffeomorphisms \( \psi_t \) above is called a hamiltonian isotopy.

We denote by \( HIso(\phi) \) the set of all hamiltonian isotopies from \( \phi \in Ham(M, \omega) \) to the identity, and by \( HIso(M) \) the set of all hamiltonian isotopies.

In equation (2), \( i_{(\cdot)} \) denotes the interior product: \( i_X \omega \) is the 1-form such that \( i_X \omega(Y) = \omega(X,Y) \). Recall that a symplectic form is a closed 2-form \( \omega \) such that the map assigning to a vector field \( X \) the 1-form \( i_X \omega \) is an isomorphism \( \tilde{\omega} \). For any 1-form \( \alpha \), we denote by \( \alpha^{\#} \) the vector field \( \tilde{\omega}^{-1}(\alpha) \).

The Hofer length of a hamiltonian isotopy \( \psi_t \) is defined as:

\[
  l_H(\psi_t) = \int_0^1 (\max_x u_t(x) - \min_x u_t(x)) \, dt \tag{3}
\]

One also denotes

\[
  \max_x u_t(x) - \min_x u_t(x) = osc(u_t(x))
\]

and call it the oscillation of \( u_t \).

Hence the Hofer length is the mean oscillation of the hamiltonian \( u_t \) of the hamiltonian isotopy \( \Phi = (\phi_t) \).

For \( \psi \in Ham(M, \omega) \), the Hofer norm is defined as:

\[
  \|\psi\|_H = \inf (l_H(\psi_t)) \tag{4}
\]

where the infimum is taken over all hamiltonian isotopies \( \psi_t \in HIso(\psi) \) and \( u_t \) is the function in equation (2).

The Hofer distance between two hamiltonian diffeomorphisms \( \phi \) and \( \psi \) is:

\[
  d_H(\phi, \psi) = \|\phi\psi^{-1}\|_H
\]
It is easy to see that the formula above defines a bi-invariant pseudo-metric but it is very challenging to show that it is not degenerate and hence it is a genuine distance [5],[7],[12],[13].

In this paper we propose a formula for the length of a symplectic isotopy \( \Phi = (\phi_t) \) (5), which generalizes the length of a hamiltonian isotopy (3).

Fix a riemannian metric on \( M \) and consider the Hodge decomposition of \( i(\dot{\phi}_t)\omega \)

\[
i(\dot{\phi}_t)\omega = H_t^\Phi + du_t^\Phi
\]

where \( H_t^\Phi \) and \( u_t^\Phi \) are smooth family of harmonic 1-forms and functions respectively.

We define the length \( l(\Phi) \) of the isotopy \( \Phi \) by:

\[
l(\Phi) = \int_0^1 (|H_t^\Phi| + (max_x(u_t^\Phi) - min_x(u_t^\Phi)) dt
\]

Here \(|H_t^\Phi|\) is the "Euclidean" norm of the harmonic 1-form \( H_t^\Phi \) (see (13), (14)).

This formula reduces to (3) for hamiltonian isotopies. Unfortunately, unlike (3), we do not have

\[ l(\Phi) = l(\Phi^{-1}) \]

where \( \Phi^{-1} = (\phi_t^{-1}) \). We will also write:

\[ l(\Phi) = ||\dot{\phi}_t|| \]

For any \( \phi \in Symp(M,\omega) \), we define the energy \( e_0(\phi) \) of \( \phi \) as:

\[ e_0(\phi) = \inf_{\Phi \in Iso(\phi)} (l(\Phi)) \]

Our main result is the following
Theorem 1.

Let $(M, \omega)$ be a closed symplectic manifold. Consider the map $e : \text{Symp}(M, \omega)_0 \to \mathbb{R} \cup \{\infty\}$

$$e(\phi) = \frac{1}{2}(e_0(\phi) + e_0(\phi^{-1})).$$

Then $e$ is a norm on $\text{Symp}(M, \omega)_0$ whose restriction to $\text{Ham}(M, \omega)$ is bounded from above by the Hofer metric.

Moreover the subgroup $\text{Ham}(M, \omega)$ is closed in $\text{Symp}(M, \omega)$ endowed with the metric topology defined by $e$.

We define a distance on $\text{Symp}(M, \omega)$ by:

$$d(\phi, \psi) = e(\phi\psi^{-1})$$

This distance is obviously right invariant, but not left invariant.

Remark

The fact that (5) reduces to (3) when $\Phi$ is a hamiltonian isotopy implies that

$$e(\phi) \leq \|\phi\|_H$$

for all $\phi \in \text{Ham}(M, \omega)$.

Conjecture The restriction of the norm $e$ to $\text{Ham}(M, \omega)$ is equivalent to the Hofer norm.

What is the interest of our construction? From the Hofer norm, there are easy ways of constructing bi-invariant norms on $\text{Symp}(M, \omega)$. One is given by Han [4]:

fix a positive number $K$ and define

$$\|\phi\|_K = \begin{cases} \min(\|\phi\|_H, K), & \text{if } \phi \in \text{Ham}(M, \omega) \\ K & \text{otherwise.} \end{cases}$$

Another is given by Lalonde-Polterovich [8]:

fix a real number $\alpha$ and define

$$\|\phi\|_\alpha = \sup\{||\phi f\phi^{-1}f^{-1}||_H | f \in \text{Ham}(M, \omega), ||f||_H \leq \alpha\}.$$
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In both cases the restriction of these metrics back to \( Ham(M, \omega) \) gives different topologies on \( Ham(M, \omega) \). In particular \( Ham(M, \omega) \) in these topology has always a finite diameter which is known to be untrue for the Hofer norm in several cases.

Hence the advantage of our construction is that its restriction to \( Ham(M, \omega) \) gives a "better" topology, which may be the same if the conjecture is true. Moreover the "Hofer-like" formula (5) allows to define a distance \( D \) on the space \( Iso(M) \) of symplectic isotopies of \((M, \omega)\). If \( \Phi = (\phi_t) \) and \( \Psi = (\psi_t) \) are symplectic isotopies define:

\[
D(\Phi, \Psi) = \left\| \dot{\phi}_t - \dot{\psi}_t \right\| =: \int_0^1 (|H^{\Phi_t} - H^{\Psi_t}| + osc(u^{\Phi_t} - u^{\Psi_t})) dt. \tag{5'}
\]

It is clear that \( D \) is a distance. Moreover if \( \Phi, \Psi \) are hamiltonian isotopies, then

\[
D(\Phi, \Psi) = l_H(\Phi \Psi^{-1}).
\]

In general the formula above is not true.

The distance formula will be used to the define the symplectic topology on \( Iso(M) \).

2. Hamiltonian and harmonic diffeomorphisms

For each symplectic isotopy \( \Phi = (\phi_t) \), consider the following 1-form:

\[
\Sigma(\Phi) = \int_0^1 (i_{\dot{\phi}_t} \omega) dt \tag{6}
\]

It is shown in [1], ( see also [2]) that the cohomology class \([\Sigma(\Phi) \in H^1(M, \mathbb{R})]\) of the form \(\Sigma(\Phi)\) depends only on the class \([\Phi]\) of \(\Phi\) in the universal covering \(\tilde{G}(M, \omega)\) of \(Symp(M, \omega)_0 = G(M, \omega)\) and that the map \([\Phi] \mapsto [\Sigma(\Phi)]\) is a surjective homomorphism

\[
\tilde{S}: \tilde{G}(M, \omega) \to H^1(M, \mathbb{R}) \tag{7}
\]

The group \(\Gamma = \tilde{S}(\pi_1(G(M, \omega))) \subset H^1(M, \mathbb{R})\) is called the flux group.
In [1], it was observed that $\Gamma$ was discrete in several examples and the author wrote ”I do not know any flux group which is not discrete”. The statement that ”$\Gamma$ is discrete” became known as the ”Flux conjecture”. This conjecture has been recently proved by Ono [11] using Floer-Novikov homology.

**Theorem (Ono).**

Let $(M, \omega)$ be a compact symplectic manifold, then the flux group is discrete.

Consider the induced homomorphism:

$$ S : G(M, \omega) \to H^1(M, \mathbb{R})/\Gamma $$

In [1], [2], it is shown that the Kernel of $S$ coincides with the group $Ham(M, \omega)$ of Hamiltonian diffeomorphisms, and it is a simple group, which coincides with the commutator subgroup $[G(M, \omega), G(M, \omega)]$ of $G(M, \omega)$. We summarize:

$$ Ham(M, \omega) = \text{Ker}S = [G(M, \omega), G(M, \omega)] $$

for all closed symplectic manifolds $(M, \omega)$.

We will need to represent in a unique way cohomology classes; this is achieved by Hodge theory on compact riemannian manifolds.

The Hodge decomposition theorem (see for instance [14]) asserts that any smooth family of $p$-forms $\theta_t$ on a compact oriented riemannian manifold $(M, g)$ can be decomposed in a unique way as

$$ \theta_t = \mathcal{H}_t + d\alpha_t + \delta\beta_t $$

where $\mathcal{H}_t$ is harmonic, i.e $d\mathcal{H}_t = \delta\mathcal{H}_t = 0$. Here $\delta$ denotes the codifferential.

If $d\theta_t = 0$, then $\delta\beta_t = 0$. The forms $\mathcal{H}_t$, $\alpha_t$ and $\beta_t$ depend smoothly on $t$.

The harmonic form $\mathcal{H}_t$ is a unique representative of the cohomology class $[\theta_t] \in H^1(M, \mathbb{R})$ of $\theta_t$. 
Definition 1.

Let $(M, \omega)$ be a compact symplectic manifold, equipped with some riemannian metric. A vector field $X$ on $M$ is said to be a harmonic vector field if $i_X \omega$ is a harmonic form.

A diffeomorphism $\phi$ of $M$ is said to be a harmonic diffeomorphism if there exists a smooth family $\mathcal{H}_t$ of harmonic 1-forms such that $\phi$ is the time 1 map of the symplectic isotopy $\phi_t$ such that

$$
\phi_t = (\mathcal{H}_t)^\#.
$$

We say that $\phi_t$ is a harmonic isotopy.

Let $\text{symp}(M, \omega)$ be the set of symplectic vector fields, $\text{harm}(M)$ the set of harmonic vector fields and $\text{ham}(M, \omega)$ the space of hamiltonian vector fields. If $X \in \text{symp}(M, \omega)$ then $i_X \omega$ is closed. The decomposition $i_X \omega = H + du$ expresses $X$ as

$$
X = H + X_u
$$

where $H = (\mathcal{H})^\#$ is harmonic and $X_u$ is the hamiltonian vector field with hamiltonian $u$.

Hence $\text{symp}(M, \omega)$ is the Cartesian product of $\text{harm}(M)$ and $\text{ham}(M, \omega)$. We give $\text{symp}(M, \omega)$ the product metric :

$$
|X| = |H| + \max_x u(x) - \min_x u(x)
$$

where $|H|$ is the norm on $\text{harm}(M)$ given below:

the space $\text{harm}(M)$, which is isomorphic to the space of harmonic 1-forms is a finite dimensional vector space whose dimension is the first Betti number of $M$.

In this paper, we fix a basis $h_1, \ldots, h_r$ of harmonic 1-forms and consider $(H_i) = (h_i^\#)$ the corresponding basis of $\text{harm}(M)$. We give these 2 vector spaces the following Euclidean metric: if $h = \sum \lambda_i h_i$, $H = \sum \lambda_i H_i$, then

$$
|h| = |H| = \sum |\lambda_i|
$$

where $|H|$ is the norm on $\text{harm}(M)$ given below:
In view of (13), the length formula (5) gives a Finsler metric on $\text{Symp}(M, \omega)$.

**Remark**

The function $u$ in the Hodge decomposition $i_X \omega = \mathcal{H} + du$ is not necessarily normalized. However if in (13) $|X| = 0$, then $|H| = 0$, i.e $i_X \omega = du$ and $\text{osc}(u) = 0$ implies that $u$ is constant, and hence $du = 0$. Therefore $X = 0$.

**Lemma 1.**

Any symplectic isotopy $\Phi = (\phi_t)$ on a compact symplectic manifold $(M, \omega)$ can be decomposed in a unique way as

$$\phi_t = \rho_t \psi_t$$

where $\rho_t$ is a harmonic isotopy and $\psi_t$ is a hamiltonian isotopy. In particular, if $\phi_t$ is a hamiltonian isotopy, then $\phi_t = \psi_t$ and $\rho_t = id_M$.

**Proof.**

By Hodge decomposition theorem $i_{(\dot{\phi}_t)} \omega$ can be decomposed in a unique way as

$$i_{(\dot{\phi}_t)} \omega = \mathcal{H}_t^\Phi + du_t^\Phi$$

where $\mathcal{H}_t^\Phi$ and $u_t^\Phi$ are smooth family of harmonic 1-forms and functions respectively. Let $\rho_t$ be the harmonic isotopy such that $\dot{\rho}_t = (\mathcal{H}_t)^\#$. Set now $\psi_t = (\rho_t)^{-1} \phi_t$. From $\phi_t = \rho_t \psi_t$, we get:

$$\dot{\phi}_t = \dot{\rho}_t + (\rho_t)_* \psi_t$$

Since $i_{(\dot{\phi}_t - \dot{\rho}_t)} \omega = du_t = i_{(X_{u_t})} \omega$ where $X_{u_t}$ is the hamiltonian vector field of $u_t$, we see that

$$\dot{\phi}_t = \dot{\rho}_t + X_{u_t} = \dot{\rho}_t + (\rho_t)_* ((\rho_t)^{-1})_* (X_{u_t})$$

Hence $\dot{\psi}_t = (\rho_t)^{-1}_* (X_{u_t}) = X_{(u_t \circ \rho_t)}$. This shows that $\psi_t$ is a hamiltonian isotopy. 

$\square$
In formula (5), \( \int_0^1 \text{osc}(u^\Phi_\lambda'))dt \) is nothing else than \( l_H(\psi_t) \) and formula (5) can be written
\[
l(\Phi) = \int_0^1 |i(\dot{\rho}_t)\omega|dt + l_H(\psi_t))dt. 
\]
(5' )

3. Proof of theorem 1

Clearly , \( e(\phi) \geq 0 \) for all \( \phi \) and by definition \( e(\phi) = e(\phi^{-1}) \).

To see that the triangular inequality holds, fix a small positive number \( \epsilon \leq 1/8 \) and a smooth increasing function \( a : [0, 1] \rightarrow [0, 1] \) such that \( a|_{[0, \epsilon]} = 0 \) and \( a|_{(1-\epsilon), 1]} = 1 \) and let \( \lambda(t) = a(2t) \) for \( 0 \leq t \leq 1/2 \) and \( \mu(t) = a(2t-1) \) for \( 1/2 \leq t \leq 1 \).

If \( \Phi \in Iso(\phi) \) and \( \Psi \in Iso(\psi) \), we get an isotopy \( \Phi \ast \Psi = (\sigma_t) \in Iso(\phi \psi) \) defined as:
\[
\sigma_t = \begin{cases} 
\phi_{\lambda(t)}, & \text{for } 0 \leq t \leq 1/2 \\
\phi_{\psi \mu(t)}, & \text{for } 1/2 \leq t \leq 1.
\end{cases}
\]

Let \( c(\Phi, \Psi) \) be the set of all isotopies from \( \phi \psi \) to the identity obtained as above.

Clearly :
\[
e_0(\phi \psi) \leq \inf_{R \in c(\Phi, \Psi)}(l(R))
\]

where \( R \in c(\Phi, \Psi) \).

Since
\[
\dot{\sigma}_t = \begin{cases} 
\lambda' \dot{\phi}_{\lambda(t)}, & \text{for } 0 \leq t \leq 1/2 \\
\mu' \dot{\psi}_{\mu(t)}, & \text{for } 1/2 \leq t \leq 1,
\end{cases}
\]

we have:
\[
i(\dot{\sigma}_t)\omega = \begin{cases} 
\lambda' \mathcal{H}_{\lambda(t)} + d(\lambda' u^\Phi_{\lambda(t)}), & \text{for } 0 \leq t \leq 1/2 \\
\mu' \mathcal{H}_{\mu(t)} + d(\mu' u^\Phi_{\mu(t)}), & \text{for } 1/2 \leq t \leq 1,
\end{cases}
\]

Therefore
\[
l(\Phi \ast \Psi) = \int_0^{1/2} (|\lambda' \mathcal{H}_{\lambda(t)}| + \text{osc}(\lambda' u^\Phi_{\lambda(t)}))dt + \int_{1/2}^1 (|\mu' \mathcal{H}_{\mu(t)}| + \text{osc}(\mu' u^\Phi_{\mu(t)}))dt
\]
By the change of variable formula, we get:

\[ l(\Phi \ast \Psi) = l(\Phi) + l(\Psi) \]

Finally,

\[ e_0(\phi) \leq \inf \mathcal{R}(l(\Phi)) \leq \inf \phi l(\Phi) + \inf \psi l(\Psi) = e_0(\phi) + e_0(\psi). \]

Therefore the triangular inequality holds true for \( e_0 \), and hence for \( e \) as well.

Showing that \( e \) is non-degenerate is more delicate. Suppose that \( e_0(\phi) = 0 \).

**Step 1**

The statement \( e_0(\phi) = \inf l(\Phi) = 0 \) means that for every \( N \), there exists an isotopy \( \Phi^N \) from \( \phi \) to the identity such that \( l(\Phi^N) \leq 1/N \).

Thus:

\[ \int_0^1 |\mathcal{H}_t^{\Phi^N}| dt \leq 1/N \tag{16} \]

and

\[ \int_0^1 \text{osc}(u^{\Phi^N}) dt \leq 1/N \]

Hence

\[ |\mathcal{H}(\Phi^N)| = | \int_0^1 \mathcal{H}_t^{\Phi^N} dt | \leq \int_0^1 |\mathcal{H}_t^{\Phi^N}| dt \leq 1/N. \]

For any symplectic isotopy from \( \phi \) to the identity \( \Phi = (\phi_t) \), the 1-form

\[ \mathcal{H}(\Phi) = \int_0^1 \mathcal{H}_t^{\phi} dt \]

is the harmonic representative of the cohomology class \( \tilde{S}(\{\phi_t\}) \).

For any symplectic isotopy \( \Phi = (\phi_t) \) from \( \phi \) to the identity

\[ \mathcal{H}(\Phi^N) - \mathcal{H}(\Phi) = \gamma(\Phi) \in \Gamma \tag{17} \]

since \( \mathcal{H}(\Phi^N) - \mathcal{H}(\Phi) \) is the harmonic representative of the image by \( \tilde{S} \) of the class \([\phi_t^N \ast \phi_{(1-t)}]\) of the loop \( \phi_t^N \ast \phi_{(1-t)} \).
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By (16) and (17), the distance \( d(\mathcal{H}(\Phi), \Gamma) \) from \( \mathcal{H}(\Phi) \) to \( \Gamma \) satisfies:

\[
d(\mathcal{H}(\Phi), \Gamma) \leq |\mathcal{H}(\Phi) - (-\gamma(\Phi^N))| = |\mathcal{H}(\Phi^N)| \leq 1/N
\]

This says that \( (\mathcal{H}(\Phi)) \) is arbitrarily close to \( \Gamma \). Hence \( (\mathcal{H}(\Phi)) \in \Gamma \). This means that \( \phi \in \text{Ker}S = \text{Ham}(M, \omega) \).

The facts that \( \mathcal{H}(\Phi^N) \in \Gamma \) and \( |\mathcal{H}(\Phi^N)| \leq 1/N \) imply that \( \mathcal{H}(\Phi^N) = 0 \) for \( N \) large enough since \( \Gamma \) is discrete (Ono’s theorem).

Fix now an isotopy \( \Phi^N \) such that \( \mathcal{H}(\Phi^N) = 0 \). To simplify the notations, we denote by \( \Phi = (\phi_t) \) the isotopy \( \Phi^N = (\phi^N_t) \).

The Hodge decomposition of the isotopy \( \phi_t \) gives:

\[
\phi_t = \rho_t \mu_t
\]

where \( \rho_t \) is harmonic and \( \mu_t \) is hamiltonian. We have:

\[
i(\dot{\phi}_t)\omega = \mathcal{H}_t + du_t
\]

\[
\dot{\rho}_t = (\mathcal{H}_t)\# = H_t,
\]

\[
\int_0^1 \mathcal{H}_t dt = 0
\]

and

\[
\int_0^1 (|\mathcal{H}_t| + \text{osc}(u_t)) dt \leq 1/N
\]

Hence

\[
\int_0^1 |\mathcal{H}_t| dt \leq 1/N, \int_0^1 \text{osc}(u_t) dt \leq 1/N.
\]  

(18)

**Step 2**

We are now going to deform the isotopy \( \rho_t \) fixing the extremities to a hamiltonian isotopy following [1], proposition II.3.1.

Let \( Z_{(s,t)} \) be the family of symplectic vector fields:

\[
Z_{(s,t)} = t\dot{\rho}_{(s,t)} - 2s(t\int_0^t (i(\dot{\rho}_u)\omega) du)\#.
\]  

(19)
Clearly, $Z_{(0,t)} = 0$ and we have:

$$\int_0^1 i(Z_{(s,t)})\omega ds = 0. \quad (20)$$

Let $G_{(s,t)}$ be the 2-parameter family of diffeomorphisms obtained by integrating $Z_{(s,t)}$ with $t$ fixed, i.e. $G_{(s,t)}$ is defined by the following equations:

$$\frac{d}{ds}G_{(s,t)}(x) = Z_{(s,t)}(G_{(s,t)}^{-1}(x)), G_{(0,t)}(x) = x. \quad (21)$$

By (20), $G_{(1,t)}$ is a Hamiltonian diffeomorphism for all $t$. Since $Z_{(s,1)} = \dot{\rho}_s - 2s(\int_0^1 (i(\dot{\rho}_u)\omega)du) = \dot{\rho}_s, s \mapsto G_{(s,1)}$ is an isotopy from the identity to $G(1,1) = \rho_1$. Hence the $g_t = G_{(1,t)}$ is an isotopy in $Ham(M,\omega)$ from $\rho_1$ to the identity.

Consider the 2-parameter family of vector fields $V_{(s,t)}$ defined by:

$$V_{(s,t)}(x) = \frac{d}{dt}G_{(s,t)}((G_{(s,t)}^{-1}(x))$$

Clearly $\dot{g}_t = V_{(1,t)}$.

We have (see [1], proposition I.1.1):

$$\frac{\partial}{\partial s} V_{(s,t)} = \frac{\partial}{\partial t} Z_{(s,t)} + [V_{(s,t)}, Z_{(s,t)}] \quad (22)$$

We will need the following Proposition.

$$i(V_{(1,t)})\omega = du_t$$

where $u_t = \int_0^1 \omega(Z_{(s,t)}, V_{(s,t)})ds$.

Proof.

From equation 22

$$0 = \frac{\partial}{\partial t} | \int_0^1 i(Z_{(s,t)})\omega ds | = \int_0^1 i(\frac{\partial}{\partial t}(Z_{(s,t)}))\omega ds$$
We used the facts that $V_i ||| H_{OFER-LIKE METRIC ON THE GROUP OF SYMPLECTIC DIFFEOMORPHISMS 13}$

$$\int_0^1 i\left(\frac{\partial}{\partial s}(V_{(s,t)})\right)\omega ds = \int_0^1 (\frac{\partial}{\partial s}i(V_{(s,t)})\omega) ds = \int_0^1 i([Z_{(s,t)}, V_{(s,t)}])\omega ds$$

$$= i(V_{(1,t)})\omega - i(V_{(0,t)})\omega - \int_0^1 i([Z_{(s,t)}, V_{(s,t)}])\omega ds = i(V_{(1,t)})\omega - d(\int_0^1 \omega(Z_{(s,t)}, V_{(s,t)}) ds)$$

We used the facts that $V_{(0,t)} = 0$, $i([Z, V]\omega = L_Z iV\omega - iV L_Z \omega$ and $L_V \omega = L_V \omega = 0$.

□

Step 3: Norm estimates

The harmonic vector fields $\dot{\rho}_t$ can be written as $\dot{\rho}_t = \sum_i \lambda_i(t)H_i$, where $H_i = h_i^\#$ and $(h_i)$ is a basis of harmonic 1-forms. Formula (19) just says:

$$Z_{(s,t)} = \sum_i (t\lambda_i(st) - 2s \int_0^t \lambda_i(u) du)H_i = \sum_i \mu_i(s, t)H_i$$

(23)

Hence:

$$|Z_{(s,t)}| = \sum_i |\mu_i(s, t)| \leq t|\dot{\rho}_{st}| + 2s \int_0^t |H_i| dt$$

$$\leq t|\dot{\rho}_{st}| + 2s \int_0^1 |H_i| dt \leq t|\dot{\rho}_{st}| + 2s/N.$$  

On the other hand, we have:

$$\omega(Z_{(s,t)}, V_{(s,t)}) = (i(Z_{(s,t)})\omega)(V_{(s,t)}) = \sum_i \mu_i(s, t)h_i(V_{(s,t)})$$

Consequently:

$$|\omega(Z_{(s,t)}, V_{(s,t)})| \leq \sum_i |\mu_i(s, t)h_i(V_{(s,t)})|.$$  

Let $||h_i||$ be the sup norm of the 1-forms $h_i$, i.e $||h_i|| = sup_{x \in M}||h_i(x)||$ and $||h_i(x)||$ is the norm of the linear map $h_i(x)$ on the tangent space $T_x M$.

We have:

$$\sum_i |\mu_i(s, t)h_i(V_{(s,t)})| \leq (\sum_i |\mu_i(s, t)||V_{(s,t)}|) E = |Z_{(s,t)}||V_{(s,t)}| E$$

where $E = max\{||h_i||\}$.

Hence

$$|w_t| = |\int_0^1 \omega(Z_{(s,t)}, V_{(s,t)}) ds| \leq \int_0^1 |\omega(Z_{(s,t)}, V_{(s,t)})| ds$$
$$\leq E(\int_0^1 (t|\dot{\rho}_{st}| + 2s/N)|V_{s,t}|)ds.$$  

Let $A = \sup_{s,t}|V_{s,t}|$, then

$$|w_t| \leq AE \int_0^1 (t|\dot{\rho}_{st}| + 2s/N)ds = AE(\int_0^1 (|\dot{\rho}_u|du) + 1/N)$$

$$\leq AE(\int_0^1 (|\dot{\rho}_u|du + 1/N) \leq 2AE/N.$$  

Therefore $\text{osc}(w_t) \leq 4AE/N$, hence the length of the isotopy $\rho_t$ is less or equal to $4AE/N$, and therefore the Hofer norm of $\rho : \|\rho\|_H \leq 4AE/N$, where $\rho = \rho_1$.

**Step 4**

Let $\mathcal{M}$ denote the space of smooth maps $c : I = [0, 1] \to W$, where $W$ is the space of symplectic vector fields on $(M, \omega)$ such that $c(0) = 0$ with the Hofer norm

$$\|c\| = \int_0^1 |c(t)|dt$$

Here $|c(t)|$ is the norm given by formulas 13 and 14.

On the space $\mathcal{M} \times I$ we define the distance $d((c, s), (c', s')) = ((|c - c'|^2 + (s - s')^2)^{1/2}$

Let $\mathcal{N}$ be the space of smooth functions $u : I \times I \to U$, where $U$ is the space of symplectic vector fields with the metric $\|u\| = \sup_{s,t}|u(s, t)|$.

The family of vector fields $V_{s,t}$ above is the image of $\dot{\rho}_t$ by the following map:

$$\mathcal{R} : \mathcal{M} \times I \to \mathcal{N}$$

where $\mathcal{R} = \partial_t \circ I_s \circ a_s$ with

$$a_s : c(t) \mapsto tc(st) - 2s(\int_0^t i(c(u)\omega du)\#$$

$$I_s : U_{s,t} \mapsto G_{s,t} : M \to M$$

where the family of diffeomorphisms $G_{s,t}$ is obtained by integrating in $s$ like in formula 21.

and finally $\partial t : g_{s,t} \mapsto \partial/\partial t(g_{s,t})$ (formula 22).

The mapping $\mathcal{R}$ is a smooth map since all its components are smooth, consequently it is Lipschitz. Therefore there is a constant $K$ such that $d(\mathcal{R}(\dot{\rho}_t, s), (0, 0)) = sup_{s,t}|V_{s,t}| \leq K(||\dot{\rho}_t||^2 + s^2)^{1/2}$ (Observe that $\mathcal{R}(0, 0) = 0$).
Therefore
\[ A = \sup_{s,t} |V_{s,t}| \leq K((1/N)^2 + s^2)^{1/2} \leq K((1/N)^2 + 1)^{1/2} \leq 2K \]

Finally, we get:
\[ ||\rho||_H \leq \frac{4E(K((1/N)^2 + 1)^{1/2}))}{N} \leq C/N \]

where \( C = 8EK \).

Remember now that \( \phi = \rho \mu \) and \( ||\mu||_H \leq 1/N \). Therefore, \( ||\phi||_H \leq (C + 11)/N \) for all \( N \). Hence \( ||\phi||_H = 0 \) and consequently \( \phi = id \). \( \square \)

**Ham(M, \omega)** is closed in **Symp(M, \omega)**

Let \( (h_n) \in Ham(M, \omega) \) be a sequence converging to \( g \in Symp(M, \omega) \). There exists \( N_0 \) such that for all \( N \geq N_0 \), there exists an isotopy \( \Phi^N \in Iso(g^{-1}h_N) \) with length \( l(\Phi^N) \leq 1/N \). By step 1, \( g^{-1}h_N \) is hamiltonian for \( N \) large. Hence \( g \) is also hamiltonian. \( \square \)

4. Applications to the \( C^0 \) symplectic topology

In [10], Oh and Muller defined the group of symplectic homeomorphisms, \( Sympeo(M, \omega) \) as the closure of the group \( Symp(M, \omega) \) of \( C^\infty \) symplectic diffeomorphisms of \( (M, \omega) \) in the group \( Homeo(M) \) of homeomorphisms of \( M \) with the \( C^0 \) topology, and the group \( Hameo(M, \omega) \) of hamiltonian homeomorphisms. The group \( Sympeo(M, \omega) \) has only \( C^0 \) topology induced from \( Homeo(M) \), but \( Hameo(M, \omega) \) has a more involved topology, called the hamiltonian topology, which combines the \( C^0 \) topology and the Hofer topology.

Using our construction, we define a symplectic topology on the space \( Iso(M) \) of symplectic isotopies of \( (M, \omega) \) as follows:

Fix a distance \( d_0 \) on \( M \) (coming from some riemannian metric) and define the distance \( \bar{d} \) on the space \( Homeo(M) \) of homeomorphismes of \( M \) as
\[ \bar{d}(\phi, \psi) = \max\{d(\phi, \psi), d(\phi^{-1}, \psi^{-1})\} \]
where
\[ d(h, g) = \max_x (d_0(h(x), g(x))) \]
for all \( h, g \in \text{Homeo}(M) \).

Then \((\text{Homeo}(M), \overline{d})\) is a complete metric space and its metric topology is just the \( C^0 \) topology. On the space \( \mathcal{P}\text{Homeo}(M) \) of continuous paths \( \lambda : [0,1] \to \text{Homeo}(M) \), we put the metric topology from the distance
\[ \overline{d}(\lambda, \mu) = \sup_{t \in [0,1]} \overline{d}(\lambda(t), \mu(t)). \]
We define the symplectic distance on \( \text{Iso}(M) \) by:
\[ d_{\text{symp}}(\Phi, \Psi) = \overline{d}(\Phi, \Psi) + D(\Phi, \Psi) \]
where \( D \) is given by formula (5’).

We call the symplectic topology on \( \text{Iso}(M) \) the metric topology defined by the above distance. This topology reduces to the "hamiltonian topology" of [10] on paths in \( \text{Ham}(M, \omega) \).

5. Final Remarks

The metric \( e \) obtained here is not an "extension" of the Hofer metric since we do not know if \( e(\phi) = ||\phi||_H \) when \( \phi \in \text{Ham}(M, \omega) \). We only know that \( e(\phi) \leq ||\phi||_H \).

The problem of extending the Hofer norm was considered in [3]. Here we would like to make some remarks about the results of [3].
Extension of Oh’s spectral norm. It is obvious that formulas of the extensions of the Hofer metric given in [3] give in fact extensions for any bi-invariant metric on $Ham(M,\omega)$. Theorem 2 in [3] uses only the properties of bi-invariance and not the Hofer norm. Then theorem 2 of [3] can be rephrased as

Theorem 2.

Let $(M,\omega)$ be a symplectic manifold such that the homomorphism $S$ admits a continuous homomorphic right inverse, then any bi-invariant metric on $Ham(M,\omega)$ extends to a right invariant metric on $Symp(M,\omega)$.

Under the hypothesis of the theorem above, the spectral norm $||.||_\mathcal{O}$ of Oh extends to all of $Symp(M,\omega)_0$. For the definition of Oh’s spectral norm, we refer to [9]. An example where this hypothesis holds is $T^{2n}$ with its natural symplectic form.

Theorem 3.

If $\Gamma = 0$, Oh’s spectral distance extends to $Symp(M,\omega)_0$.

Proof.

Let $\phi_i, i = 1, 2$ two symplectomorphisms and $\Phi_i = (\phi_i^t) \in Iso(\phi_i)$. The harmonic 1-forms $\mathcal{H}(\Phi_i)$ depend only on $\phi_i$. Let $\rho_i$ be the time one of the 1-parameter group generated by $\mathcal{H}(\Phi_i)^\#$, then $\psi_i = \phi_i \rho_i^{-1} \in Ham(M,\omega)$. We define the Oh distance $d_\mathcal{O}$ of $\phi_1$ and $\phi_2$ by:

$$d_\mathcal{O}(\phi_1, \phi_2) = |\mathcal{H}(\Phi_1) - \mathcal{H}(\Phi_2)| + ||\psi_1 \psi_2^{-1}||_\mathcal{O}.$$ 

The cases where $\Gamma = 0$ include oriented compact surfaces of genus bigger than one. More recently, Kedra, Kotschick and Morita [6] found a longer list of compact symplectic manifolds with vanishing flux group.

References

[1] A. Banyaga *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. 53(1978) pp.174–227.

[2] A. Banyaga *The structure of classical diffeomorphisms groups*, Mathematics
and its applications vol 400. Kluwer Academic Publisher’s Group, Dordrecht, The Netherlands (1997).

[3] A. Banyaga, P. Donato Lengths of Contact Isotopies and Extensions of the Hofer Metric Annals of Global Analysis and Geometry 30(2006) 299-312

[4] Z. Han Bi-invariant metrics on the group of symplectomorphisms, Preprint

[5] H. Hofer On the topological properties of symplectic maps, Proc. Royal Soc. Edinburgh 115A (1990), pp.25-38

[6] J. Kedra, D. Kotschick, S. Morita Crossed flux homomorphism and vanishing theorem for flux groups Geom Funct. Anal 16(2006)no 6 1246-1273.

[7] F. Lalonde, D. McDuff The geometry of symplectic energy, Ann. Math. 141 (1995) 349 - 37  

[8] F. Lalonde, L.Polterovich Symplectic diffeomorphisms as isometries of Hofer’s norm Topology 36(1997) 711-727

[9] Y-G. Oh Spectral invariants, analysis of the Floer moduli space, and the geometry of hamiltonian diffeomorphisms, Duke Math. J. 130(2005) 199-295

[10] Y-G. Oh and S. Muller The group of hamiltonian homeomorphisms and $C^0$-symplectic topology J. Symp.Geom. to appear

[11] K. Ono Floer-Novikiv cohomology and the flux conjecture Geom. Funct. Anal.16(2006) no 5 981-1020

[12] L. Polterovich Symplectic displacement energy for Lagrangian submanifolds, Erg. Th and Dynamical Systems 13 (1993), 357-367.

[13] C. Viterbo Symplectic topology as the geometry of generating functions, Math. Annalen 292(1992), 685-710

[14] F. Warner Foundations of differentiable manifolds and Lie groups Scott, Foresman and Company (1971).

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