Immersion in $\mathbb{R}^n$ by complex spinors

Rafael de Freitas Leão* and Samuel Augusto Wainer†

September 5, 2017

Keywords: Immersion, Spinors, Killing Equation

Abstract

A beautiful solution to the problem of isometric immersions in $\mathbb{R}^n$ using spinors was found by Bayard, Lawn and Roth [4]. However to use spinors one must assume that the manifold carries a Spin-structure and, especially for complex manifolds where is more natural to consider Spin$^C$-structures, this hypothesis is somewhat restrictive. In the present work we show how the above solution can be adapted to Spin$^C$-structures.

1 Introduction

The problem of isometric immersions of Riemannian manifolds is a classical and widely studied problem in differential geometry. Since 1998, mainly because of the work of Thomas Friedrich [5], this problem got a new understanding. In [5], since Riemannian 2-manifolds are naturally Spin-manifolds, Friedrich showed that isometric immersions of these 2-manifolds are related with spinors that satisfies a Dirac type equation. Such relation can be understood as a spinorial approach of the standard Weierstrass representation.

Since then, a lot of work has been done to further understand this relation and to extend it to more general spaces than Riemannian 2-manifolds. Some remarkable examples of these contributions are, for example: in 2004 Bertrand Morel [12] extended Friedrich’s spinorial representation of isometric immersions in $\mathbb{R}^3$ to $S^3$ and $\mathbb{H}^3$; in 2008 Marie-Amélie Lawn [8] showed how a given Lorentzian surface $(M^2, g)$ can be isometrically immersed in the pseudo-Riemannian space $\mathbb{R}^{2,1}$ using spinorial techniques; in 2010 Lawn and Julien Roth [9] exhibit a spinorial characterization of Riemannian surfaces isometrically immersed in the 4-dimensional spaces $M^4$, $M^3 \times \mathbb{R}$ ($M \simeq (\mathbb{R}, \mathbb{S}, \mathbb{H})$); using the same spinorial techniques, Lawn and Roth [10] in 2011, presented the spinorial characterization of isometric immersions of arbitrary dimension surfaces in
3-dimensional space forms, thus generalizing Lawn’s work in $\mathbb{R}^{2,1}$; in 2013 Pierre Bayard [2] proved that an isometric immersion of a Riemannian surface $M^2$ in 4-dimensional Minkowski space $\mathbb{R}^{1,3}$, with a given normal bundle $E$ and a given mean curvature vector $\vec{H} \in \Gamma(E)$, is equivalent to the existence of a normalized spinor field $\varphi \in \Gamma(\Sigma M \otimes \Sigma E)$ which is solution of the Dirac equation $D\varphi = \vec{H} \cdot \varphi$

in the surface.

More recently, Bayard, Lawn and Roth [4], studied spinorial immersions of simply connected Spin-manifolds of arbitrary dimension. The main idea is to use the regular left representation of the Clifford algebra on itself, given by left multiplication, to construct a Spin-Clifford bundle of spinors. In this bundle, using the Clifford algebra structure, is possible to define a vector valued scalar product and, combining this product with a spinor field that satisfies a proper equation, define a vector valued closed 1-form whose integral gives a isometric immersion analogous to the Weierstrass representation of surface.

This work, [4], provides a beautiful generalization of the previous work relating the Weierstrass representation to spinors. However, mainly when we are considering complex manifolds, the hypothesis of existence of a Spin-structure is somewhat restrictive. Complex manifolds always have a canonical Spin$^C$-structure that can be used to construct spinor bundles, but the existence of a Spin-structure is related to square roots of the canonical bundle and they do not always exist.

The aim of the present work is to show how the ideas of [4] can be generalized to spinor bundles associated to Spin$^C$-structure, providing a more natural setting to complex manifolds. Precisely we prove:

**Theorem 1.** Let $M$ a simply connected $n$-dimensional manifold, $E \rightarrow M$ a vector bundle of rank $m$, assume that $TM$ and $E$ are oriented and Spin$^C$. Suppose that $B : TM \times TM \rightarrow E$ is symmetric and bilinear. The following are equivalent:

1. There exist a section $\varphi \in \Gamma(N \sum^{adC})$ such that
   $$\nabla^N_{X} \varphi = -\frac{1}{2} \sum_{i=1}^{m} e_i \cdot B(X, e_i) \cdot \varphi + \frac{1}{2} i A^l(X) \cdot \varphi, \quad \forall X \in TM.$$ (1)

2. There exist an isometric immersion $F : M \rightarrow \mathbb{R}^{(n+m)}$ with normal bundle $E$ and second fundamental form $B$.

Furthermore, $F = \int \xi$ where $\xi$ is the $\mathbb{R}^{(n+m)}$-valued 1-form defined by
   $$\xi(X) := \langle \langle X \cdot \varphi, \varphi \rangle \rangle, \quad \forall X \in TM.$$ (2)

2 **Adapted Structures**

Let $E \rightarrow M$ be a hermitian vector bundle over $M$. A Spin$^C$-structure on $E$ is defined by the following double covering
Spin$^C$ is the group defined by 

\[ Spin^C_n = Spin_n \times S^1 \setminus \{ (-1, -1) \}, \]

and $S^1 = U(1) \subset \mathbb{C}$ is understood as the unitary complex numbers. As usual, a Spin$^C$-structure can be viewed as a lift of the transition functions of $E$, $g_{ij}$, to the group Spin$^C$, $\tilde{g}_{ij}$, but now the transition functions are classes of pairs $\tilde{g}_{ij} = [(h_{ij}, z_{ij})]$, where $h_{ij} : U_i \cap U_j \rightarrow Spin_n$ and $z_{ij} : U_i \cap U_j \rightarrow S^1 = U(1)$.

The identity on Spin$^C$ is the class $\{(1,1),(-1,-1)\}$. Because of this, neither $h_{ij}$ or $z_{ij}$ must satisfy the cocycle condition, only the class of the pair. But, $z_{ij}^2$ satisfies the cocycle condition and defines a complex line bundle $L$, associated with the $PS^1$ principal bundle in the above diagram, called the determinant of the Spin$^C$-structure.

The description using transition functions is useful to make clear that Spin$^C$-structures are more general than Spin-structures. In fact, given a Spin-structure

\[ P_{Spin} \rightarrow P_{SO} \]

we immediately get a Spin$^C$-structure by considering $z_{ij} = 1$, in other words, by considering the trivial bundle as the determinant bundle of the structure. On the other hand \[7\], a Spin$^C$-structure produces a Spin-structure iff the determinant bundle has a square root, that is, the functions $z_{ij}$ satisfies the cocycle condition.

Another way where Spin$^C$-structures are natural is when we consider an almost complex manifold $(M,g,J)$. In this case the tangent bundle can be viewed as an $U(n)$ bundle, and the natural inclusion $U(n) \rightarrow SO(2n)$ produces a canonical Spin$^C$-structure on the tangent bundle \[6,13\]. For this canonical structure the determinant bundle is identified with $\wedge^{0,n} M$ and the spinor bundle constructed using an irreducible complex representation of $\mathcal{O}(2n)$ is isomorphic with $\wedge^{0,*} M = \oplus_{k=0}^n \wedge^{0,k} M$. So, various structures on spinors can be described using known structures of $M$.

Unlike the usual case for Spin-structures, a metric connection on $E$ is not enough to produce a connection on $P_{Spin}^c(E)$, for this, we also need a connection on the determinant bundle of the structure to get a connection on $P_{SO}(E) \times P_{S^1}(E)$ and be able to lift this connection to $P_{Spin^c}^c(E)$.

To understand the problem of immersions using the Dirac equation in the case of Spin$^C$-structures, and spinors associated to this structure, we need to understand adapted Spin$^c$-structures on submanifolds. The difference to the...
standard Spin case is that we need to keep track of the determinant bundle. Using the ideas of [1], we can describe the adapted structure.

Consider a Spin\(n + m\)-dimensional manifold \(Q\) and a isometrically immersed \(n\)-dimensional Spin\(^C\) submanifold \(M \hookrightarrow Q\). Let

\[
P_{\text{Spin}^C_{(n+m)}}(Q) \xrightarrow{\Lambda^Q} P_{SO(n+m)}(Q) \times P_{S^1}(Q)
\]

\[
P_{\text{Spin}^C_{(n+m)}}(Q) \xrightarrow{\Lambda^Q_M} P_{SO(n+m)}(Q) \times P_{S^1}(Q)
\]

\[
P_{\text{Spin}^C_n}(M) \xrightarrow{\Lambda^M} P_{SO_n}(M) \times P_{S^1}(M)
\]

be the corresponding Spin\(^C\)-structures. And let the cocycles associated to this structures be, respectively, \(\tilde{g}_{\alpha\beta}\), \(\tilde{g}_{\alpha\beta}|_M\) and \(\tilde{g}_{\alpha\beta}^1\). If we define the functions \(\tilde{g}_{\alpha\beta}^1\) by

\[
\tilde{g}_{\alpha\beta}^1 \tilde{g}_{\alpha\beta}^2 = \tilde{g}_{\alpha\beta}|_M
\]

it is easy to see, using an adapted frame, that the two sets of functions \(\tilde{g}_{\alpha\beta}^1\) and \(\tilde{g}_{\alpha\beta}^2\) commutes. This implies that \(\tilde{g}_{\alpha\beta}^2\) satisfies the cocycle condition, because both \(\tilde{g}_{\alpha\beta}\) and \(\tilde{g}_{\alpha\beta}^1\) satisfies. The cocycles \(\tilde{g}_{\alpha\beta}^2\) are exactly the Spin\(^C\)-structure for the normal bundle \(\nu(M)\). With this construction, if \(L, L_1\) and \(L_2\) denotes, respectively, the determinant bundle of the Spin\(^C\)-structure of \(Q\), \(M\) and \(\nu(M)\) we have the relation

\[
L = L_1 \otimes L_2
\]

Knowing that \(\nu(M)\) has a natural Spin\(^C\)-structure we can use the left regular representation of \(\mathfrak{Cl}(n)\) on itself to construct the following Spin\(^C\)-Clifford bundle (this bundles will act as spinor bundles)

\[
\Sigma^C Q := P_{\text{Spin}^C_{(n+m)}}(Q) \times_{\rho_{(n+m)}} \mathfrak{Cl}_{(n+m)},
\]

\[
\Sigma^C Q \mid_M := P_{\text{Spin}^C_{(n+m)}}(Q) \mid_M \times_{\rho_{(n+m)}} \mathfrak{Cl}_{(n+m)},
\]

\[
\Sigma^C M := P_{\text{Spin}^C_{(n+m)}}(M) \times_{\rho_m} \mathfrak{Cl}_{(m)},
\]

\[
\Sigma^C \nu(M) := P_{\text{Spin}^C\nu(M)} \times_{\rho_m} \mathfrak{Cl}_{(m)}.
\]

Using the isomorphism \(\mathfrak{Cl}_n \otimes \mathfrak{Cl}_m \simeq \mathfrak{Cl}_{(n+m)}\) and standard arguments, [1], we get the relation

\[
\Sigma^C Q \mid_M \simeq \Sigma^C M \otimes \Sigma^C \nu(M) =: \Sigma^{adC}.
\]

Let \(\nabla_{\Sigma^C Q}, \nabla_{\Sigma^C M}\) and \(\nabla_{\Sigma^C \nu}\) be the connection on \(\Sigma^C Q, \Sigma^C M\) and \(\Sigma^C \nu(M)\) respectively, induced by the Levi-Civita connections of \(P_{SO_{(n+m)}}(Q), P_{SO_n}(M)\), and \(P_{SO(m)}(\nu)\). We denote the connection on \(\Sigma^{adC}\) by

\[
\nabla_{\Sigma^{adC}} = \nabla_{\Sigma^{C} M} \otimes \nabla_{\Sigma^{C} \nu} := \nabla_{\Sigma^{C} M} \otimes Id + Id \otimes \nabla_{\Sigma^{C} \nu}.
\]

The connections on these bundle are linked by the following Gauss formula:
\[ \nabla_{X}^{\Sigma}Q \varphi = \nabla_{X}^{\Sigma} \varphi + \frac{1}{2} \sum_{i=1}^{n} e_{i} \cdot B(e_{i}, X) \cdot \varphi, \quad (6) \]

where \( B : TM \times TM \rightarrow \nu(M) \) is the second fundamental form and \( \{e_{1} \cdots e_{n}\} \) is a local orthonormal frame of \( TM \). Here \( "," \) is the Clifford multiplication on \( \Sigma^{C}Q \).

Note that if we have a parallel spinor \( \varphi \) in \( \Sigma^{C}Q \), for example if \( Q = \mathbb{R}^{n+m} \), then Eq. (6) implies the following generalized Killing equation

\[ \nabla_{X}^{\Sigma} \varphi = -\frac{1}{2} \sum_{i=1}^{n} e_{i} \cdot B(e_{i}, X) \cdot \varphi. \quad (7) \]

## 3 Constructing the Immersion

To construct the immersion we need two steps. First we need to construct a vector valued inner product using the Clifford algebra structure of the Spin\(^{C}\)-Clifford bundle. This first step does not change when we consider Spin\(^{C}\)-structures instead of Spin-structures. Therefore we just remember the construction by Bayard, Lawn and Roth \([4]\) in the first subsection.

Second, we need to understand a Gauss type equation on the manifold. For this step the connection on the determinant bundle of the Spin\(^{C}\)-structures is used and we show how the equations can be reformulated to this case. This is the principal part of the proof and is done on subsection 3.2.

### 3.1 A \( \mathbb{C}l(n+m) \)-valued inner product

To make the converse, obtaining an immersion using spinors that satisfies certain equations, we need the following \( \mathbb{C}l(n+m) \)-valued inner product

\[ \tau : \mathbb{C}l(n+m) \rightarrow \mathbb{C}l(n+m) \]

\[ \tau(a \cdot e_{i_{1}} e_{i_{2}} \cdots e_{i_{k}}) := (-1)^{k} \bar{a} e_{i_{k}} \cdots e_{i_{2}} e_{i_{1}}, \quad (9) \]

\[ \tau(\xi) := \bar{\xi}, \quad (10) \]

\[ \langle\langle \cdot, \cdot \rangle\rangle : \mathbb{C}l(n+m) \times \mathbb{C}l(n+m) \rightarrow \mathbb{C}l(n+m) \]

\[ (\xi_{1}, \xi_{2}) \mapsto \langle\langle \xi_{1}, \xi_{2} \rangle\rangle = \tau(\xi_{2})\xi_{1}. \quad (11) \]

\[ \langle\langle (g \otimes s)\xi_{1}, (g \otimes s)\xi_{2} \rangle\rangle = s\bar{s}\tau(\xi_{2})\tau(g)\xi_{1} = \tau(\xi_{2})\xi_{1} = \langle\langle \xi_{1}, \xi_{2} \rangle\rangle, \quad (12) \]

\[ g \otimes s \in \text{Spin}^{C}(n+m) \subset \mathbb{C}l(n+m), \]
so the product is well defined on the Spin$^C$-Clifford bundles, i.e., 
$$\sum^C Q \times \sum^C Q \to Cl_{n+m}$$

$$\varphi_1, \varphi_2 = ([p, [\varphi_1]], [p, [\varphi_2]]) \to \langle\langle [\varphi_1], [\varphi_2] \rangle\rangle = \tau([\varphi_2])[[\varphi_1],$$

where $[\varphi_1], [\varphi_2]$ are the representative of $\varphi_1, \varphi_2$ in the $Spin^C(n + m)$ frame $p \in P_{Spin^C(n + m)}$.

**Lemma 2.** The connection $\nabla^C Q$ is compatible with the product $\langle\langle \cdot , \cdot \rangle\rangle$.

**Proof.** Fix $s = (e_1, ..., e_{n+m}) : U \subset M \subset Q \to P_{SO(n+m)}$ a local section of the frame bundle, $l : U \subset M \subset Q \to P_{S^1}$ a local section of the associated $S^1$-principal bundle, $w^Q : T(P_{SO(n+m)}) \to so(n + m)$ is the Levi-Civita connection of $P_{SO(n+m)}$ and $iA : TP_{S^1} \to i\mathbb{R}$ is an arbitrary connection on $P_{S^1}$, denote by $w^Q(ds(X)) = (w^Q(X)) \in so(n + m), iA(dl(X)) = iA(X)$.

If $\psi = [p, [\psi]]$ and $\psi' = [p, [\psi']]$ are sections of $\sum C Q$ we have:

$$\nabla^C_X \psi = \left[ p, X([\psi]) + \frac{1}{2} \sum_{i < j} w_{ij}(X) e_i e_j \cdot [\psi] + \frac{1}{2} i A^i(X)[\psi] \right],$$

$$\langle\langle \nabla^C_X \psi, \psi' \rangle\rangle = \left[ \psi' \right] \left( X([\psi]) + \frac{1}{2} \sum_{i < j} w_{ij} e_i e_j \cdot [\psi] + \frac{1}{2} i A^i(X)[\psi] \right),$$

$$\langle\langle \psi, \nabla^C_X \psi' \rangle\rangle = \left[ \psi' \right] \left( X([\psi]) + \frac{1}{2} \sum_{i < j} w_{ij} e_i e_j \cdot [\psi'] + \frac{1}{2} i A^i(X)[\psi] \right),$$

then

$$\langle\langle \nabla^C_X \psi, \psi' \rangle\rangle + \langle\langle \psi, \nabla^C_X \psi' \rangle\rangle = \left[ \psi' \right] X(\xi) + \left[ \psi \right] X(\xi),$$

$$\nabla_X \langle\langle \psi, \psi' \rangle\rangle = X(\xi) = X(\xi) = X(\xi) + \nabla X(\xi).$$

**Lemma 3.** The map $\langle\langle \cdot , \cdot \rangle\rangle : \sum^C Q \times \sum^C Q \to Cl_{n+m}$ satisfies:

1. $\langle\langle X \cdot , \varphi \rangle\rangle = - \langle\langle \psi, X \cdot \varphi \rangle\rangle$, $\psi, \varphi \in \sum^C Q$, $X \in TQ$.

2. $\tau(\langle\langle \varphi, \psi \rangle\rangle) = \langle\langle \varphi, \psi \rangle\rangle$, $\psi, \varphi \in \sum^C Q$.

**Proof.** This is an easy calculation:

1. $\langle\langle X \cdot , \varphi \rangle\rangle = \tau(\varphi)[X][\psi] = \tau(\varphi)[X][\psi] = -\tau(\varphi)\tau[X][\psi] = \langle\langle \psi, X \cdot \cdot \varphi \rangle\rangle$
2. \( \tau(\langle \psi, \varphi \rangle) = \tau([\varphi][\psi]) = \tau[\psi][\varphi] = \langle \varphi, \psi \rangle \).

\[ \square \]

Note the same idea, product and properties are valid for the bundles \( \sum C Q \), \( \sum C M, \sum C \nu(M) \), \( \sum C M \otimes \sum C \nu(M) \).

### 3.2 Spinorial Representation of Submanifolds in \( \mathbb{R}^{n+m} \)

Let \( M \) a \( n \)-dimensional manifold, \( E \rightarrow M \) a real vector bundle of rank \( m \), assume that \( TM \) and \( N \) are oriented and \( \text{Spin}^C \). Denote by \( P_{SO_n}(M) \) the frame bundle of \( TM \) and by \( P_{SO_m}(E) \) the frame bundle of \( E \). The respective \( \text{Spin}^C \) structures are defined as

\[
\begin{align*}
\Lambda^{1C} & : P_{\text{Spin}^C_n}(M) \rightarrow P_{SO_n}(M) \times P_{\text{Spin}^C_1}(M), \\
\Lambda^{2C} & : P_{\text{Spin}^C_m}(E) \rightarrow P_{SO_m}(E) \times P_{\text{Spin}^C_1}(E).
\end{align*}
\]

We can define the bundle \( P_{S^1} \) as the one with transition functions defined by product of transition functions of \( P_{S^1}(M) \) and \( P_{S^1}(E) \). It is not difficult to see that there is a canonical bundle morphism: \( \Phi : P_{S^1}(M) \times_M P_{S^1}(E) \rightarrow P_{S^1} \) such that, in any local trivialization, the following diagram commute:

\[
\begin{array}{ccc}
P_{S^1}(M) \times_M P_{S^1}(E) & \xrightarrow{\Phi} & P_{S^1} \\
\downarrow & & \downarrow \\
U_\alpha \times S^1 \times S^1 & \xrightarrow{\phi_\alpha} & U_\alpha \times S^1
\end{array}
\]

where \( \phi_\alpha(x, r, s) = (x, rs), x \in U_\alpha, r, s \in S^1 \).

Fix the following notation

\[
\begin{align*}
\sum adC & : = \sum C M \otimes \sum C E \cong (P_{\text{Spin}^C(n)} \times_M P_{\text{Spin}^C(m)}) \times \mathbb{C}l_{(n+m)}, \\
N \sum adC & : = (P_{\text{Spin}^C(n)} \times_M P_{\text{Spin}^C(m)}) \times \text{Spin}^C_{(n+m)}.
\end{align*}
\]

Here \( iA^1 : TP_{S^1}(M) \rightarrow i\mathbb{R}, iA^2 : TP_{S^1}(E) \rightarrow i\mathbb{R} \) are arbitrary connections in \( P_{S^1}(M) \) and \( P_{S^1}(E) \). Denote a local section by \( s = (e_1, \cdots, e_n) : U \rightarrow P_{SO_n}(M), l_1 : U \rightarrow P_{S^1}(M), l_2 : U \rightarrow P_{S^1}(E), l = \Phi(l_1, l_2) : U \rightarrow P_{S^1} \).

Now \( iA : TP_{S^1} \rightarrow i\mathbb{R} \) is the connection defined by \( iA(d\Phi(l_1, l_2)) = iA_1(dl_1) + iA_2(dl_2) \).

Established this notation we have the following:

**Theorem 4.** Let \( M \) a simply connected \( n \)-dimensional manifold, \( E \rightarrow M \) a vector bundle of rank \( m \), assume that \( TM \) and \( E \) are oriented and \( \text{Spin}^C \). Suppose that \( B : TM \times TM \rightarrow E \) is symmetric and bilinear. The following are equivalent:

7
1. There exist a section \( \varphi \in \Gamma(N\sum^{ad\mathcal{C}}) \) such that
\[
\nabla^{\sum^{ad\mathcal{C}}}_{X} \varphi = -\frac{1}{2} \sum_{i=1}^{e} e_{i} \cdot B(X, e_{i}) \cdot \varphi + \frac{1}{2} i A^{l}(X) \cdot \varphi, \quad \forall X \in TM. \quad (13)
\]

2. There exist an isometric immersion \( F : M \to \mathbb{R}^{(n+m)} \) with normal bundle \( E \) and second fundamental form \( B \).

Furthermore, \( F = \int \xi \) where \( \xi \) is the \( \mathbb{R}^{(n+m)} \)-valued 1-form defined by
\[
\xi(X) := \langle \langle X \cdot \varphi, \varphi \rangle \rangle, \quad \forall X \in TM. \quad (14)
\]

Proof. \( 2) \Rightarrow 1) \) Since \( \mathbb{R}^{n+m} \) is contractible there exists a global section \( s : \mathbb{R}^{n+m} \to P_{\text{Spin}^c(n+m)}, \) with a corresponding parallel orthonormal basis \( h = (E_{1}, \ldots, E_{n+m}) : \mathbb{R}^{n+m} \to P_{\text{SO}(n+m)}, \) and \( l : \mathbb{R}^{n+m} \to P_{\text{S}^1}, \Lambda^{(n+m)}(s) = (h, l). \) Fix a constant \( \[ \varphi \] \in \text{Spin}^c(n + m) \subset \text{Cl}_{(n+m)} \) and define the spinor field \( \varphi = [s, [\varphi]] \in \sum^{c} \mathbb{R}^{n+m} := P_{\text{Spin}^c(n+m)} \times \text{Cl}_{(n+m)}, \) again denote \( w^{Q}(dh(X)) = (w^{h}_{i}(X)) \in so(n + m), iA(dl(X)) = iA^{l}(X) \in i\mathbb{R}, \)
\[
\nabla^{\sum^{c}Q}_{X} \varphi = \left[ s, X([\varphi]) + \left\{ \frac{1}{2} \sum_{i<j} w^{h}_{ij}(X)E_{i}E_{j} + \frac{1}{2} i A^{l}(X) \right\} \right] \cdot [\varphi]
= \left[ s, \frac{1}{2} i A^{l}(X) \cdot [\varphi] \right].
= \frac{1}{2} i A^{l}(X) \cdot \varphi \quad (15)
\]

Finally, restricting \( \varphi \) to \( \Sigma^{ad\mathcal{C}} \) and applying the gauss formula Eq.(6)
\[
\nabla^{\sum^{ad\mathcal{C}}}_{X} \varphi - \nabla^{\sum^{ad\mathcal{C}}}_{X} \varphi = \frac{1}{2} \sum_{i=1}^{e} e_{i} \cdot B(X, e_{i}) \cdot \varphi
\]
\[
\frac{1}{2} i A^{l}(X) \cdot \varphi - \nabla^{\sum^{ad\mathcal{C}}}_{X} \varphi = \frac{1}{2} \sum_{i=1}^{e} e_{i} \cdot B(X, e_{i}) \cdot \varphi
\]
\[
\nabla^{\sum^{ad\mathcal{C}}}_{X} \varphi = -\frac{1}{2} \sum_{i=1}^{e} e_{i} \cdot B(X, e_{i}) \cdot \varphi + \frac{1}{2} i A^{l}(X) \cdot \varphi. \quad (16)
\]

1) \( \Rightarrow 2) \) The idea here is to prove that the 1-form \( \xi \) Eq.(14) gives us an immersion preserving the metric, the second fundamental form and the normal connection. For this purpose, we will present the following lemmas:

Lemma 5. Suppose that \( \varphi \in \Gamma(N\sum^{ad\mathcal{C}}) \) satisfies Eq.(13) and define \( \xi \) by Eq.(14), then

1. \( \xi \) is \( \mathbb{R}^{(n+m)} \)-valued 1-form.

2. \( \xi \) is a closed 1-form, \( d\xi = 0 \)
Proof. 1. If \( \varphi = [p, [\varphi]], X = [p, [X]] \), where \([\varphi]\) and \([X]\) represent \(\varphi\) and \(X\) in a given frame \(s \in P_{\text{Spin}^c(n)} \times P_{\text{Spin}^c(m)}\),

\[
\xi(X) := \tau[\varphi][X][\varphi] \in \mathbb{R}^n \subset Cl_n \subset Cl_n, \text{ because } [\varphi] \in \text{Spin}^C.
\]

2. Suppose that in the point \(x_0 \in M\) \(\nabla^M X = \nabla^M Y = 0\), to simplify write

\[
\nabla_X^e \varphi = \nabla_X \varphi \text{ and } \nabla^M X = \nabla X,
\]

\[
X(\xi(Y)) = \langle Y \cdot \nabla_X \varphi, \varphi \rangle + \langle Y \cdot \varphi, \nabla_X \varphi \rangle = (id - \tau) \langle (Y \cdot \varphi, \nabla_X \varphi) \rangle
\]

\[
= (id - \tau) \langle \langle \varphi, \frac{1}{2} \sum_{j=1}^m Y \cdot e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} A^l(X) iY \cdot \varphi \rangle \rangle,
\]

\[
Y(\xi(X)) = (id - \tau) \langle \langle \varphi, \frac{1}{2} \sum_{j=1}^m X \cdot e_j \cdot B(Y, e_j) \cdot \varphi - \frac{1}{2} A^l(Y) iX \cdot \varphi \rangle \rangle,
\]

from now on

\[
d\xi(X, Y) = X(\xi(Y)) - Y(\xi(X))
\]

\[
= (id - \tau) \langle \langle \varphi, \frac{1}{2} \sum_{j=1}^m [Y \cdot e_j \cdot B(X, e_j) - X \cdot e_j \cdot B(Y, e_j)] \cdot \varphi \\
+ \frac{1}{2} i (A^l(Y)X - A^l(X)Y) \cdot \varphi \rangle \rangle
\]

\[
= (id - \tau) \langle \langle \varphi, C \cdot \varphi \rangle \rangle,
\]

with \(C = \frac{1}{2} \sum_{j=1}^m [Y \cdot e_j \cdot B(X, e_j) - X \cdot e_j \cdot B(Y, e_j)] + \frac{1}{2} A^l(Y) iX - \frac{1}{2} A^l(X) iY\).

Write \(X = \sum_{k=1}^m x_k e_k\); \(Y = \sum_{k=1}^m y_k e_k\) then

\[
\sum_{k=1}^m X \cdot e_k \cdot B(Y, e_k) = \sum_{j=1}^m \sum_{k=1}^m x_k e_k \cdot e_j \cdot B(Y, e_j)
\]

\[
= -B(Y, X) + \sum_{j=1}^m \sum_{k \neq j}^m x_k e_k \cdot e_j \cdot B(Y, e_j)
\]

\[
\sum_{l=1}^m Y \cdot e_k \cdot B(X, e_k) = \sum_{j=1}^m \sum_{k=1}^m y_k e_k \cdot e_j \cdot B(X, e_j)
\]

\[
= -B(X, Y) + \sum_{j=1}^m \sum_{k \neq j}^m y_k e_k \cdot e_j \cdot B(X, e_j)
\]

from what

\[
C = \frac{1}{2} \left[ \sum_{j=1}^m \sum_{k \neq j}^m e_k \cdot e_j \cdot [y_k B(X, e_j) - x_k B(Y, e_j)] \right]
\]

\[
+ i (A^l(Y)X - A^l(X)Y)
\]

\[
\tau([C]) = \frac{1}{2} \left[ \sum_{j=1}^m \sum_{k \neq j}^m [y_k B(X, e_j) - x_k B(Y, e_j)] \right] \cdot e_j \cdot e_k
\]

\[
+ i (A^l(Y) [X] - A^l(X) [Y])
\]

\[
= \frac{1}{2} \left[ \sum_{j=1}^m \sum_{k \neq j}^m e_k \cdot e_j \cdot [y_k B(X, e_j) - x_k B(Y, e_j)] \right]
\]

\[
+ i (A^l(Y) [X] - A^l(X) [Y]) = [C].
\]
Which implies that
\[ d\xi(X,Y) = (id - \tau)\langle \varphi, C \cdot \varphi \rangle = (id - \tau)(\tau[\varphi]\tau[C]\varphi) = 0. \]

From the fact that \( M \) is simply connected and \( \xi \) is closed, from the Poincaré’s Lemma we know that there exists a
\[ F : M \to \mathbb{R}^{(n+m)} \]
such that \( dF = \xi \). The next lemma allows us to conclude the proof of the theorem.

**Lemma 6.** 1. The map \( F : M \to \mathbb{R}^n \), is an isometry.

2. The map
\[ \Phi_E : E \to M \times \mathbb{R}^n \]
\[ X \in E_m \mapsto (F(m), \xi(X)) \]
is an isometry between \( E \) and the normal bundle of \( F(M) \) into \( \mathbb{R}^{(n+m)} \), preserving connections and second fundamental forms.

**Proof.** 1. Let \( X, Y \in \Gamma(TM \oplus E) \), consequently
\[
\langle \xi(X), \xi(Y) \rangle = -\frac{1}{2} (\xi(X)\xi(Y) - \xi(Y)\xi(X))
\]
\[
= -\frac{1}{2} (\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi] - \tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi])
\]
\[
= -\frac{1}{2} \tau[\varphi]([X][Y] - [Y][X]) [\varphi] = \tau[\varphi] (\langle X, Y \rangle) [\varphi]
\]
\[
= \langle X, Y \rangle \tau[\varphi][\varphi] = \langle X, Y \rangle. \quad (19)
\]
This implies that \( F \) is an isometry, and that \( \Phi_E \) is a bundle map between \( E \) and the normal bundle of \( F(M) \) into \( \mathbb{R}^n \) which preserves the metrics of the fibers.

2. Denote by \( B_F \) and \( \nabla'F \) the second fundamental form and the normal connection of the immersion \( F \). We want to show that:
\[
i) \xi(B(X,Y)) = B_F(\xi(X), \xi(Y)),
\]
\[
ii) \xi(\nabla' X \eta) = (\nabla'_{\xi(X)}\xi(\eta)),
\]
for all \( X, Y \in \Gamma(TM) \) and \( \eta \in \Gamma(E) \).

i) First note that:
\[
\tilde{B}_F(\xi(X), \xi(Y)) := \{\nabla'_{\xi(X)}\xi(Y)\}^\perp = \{X(\xi(Y))\}^\perp.
\]
where the superscript \( \bot \) means that we consider the component of the vector which is normal to the immersion. We know that
\[
X(\xi(Y)) = (id - \tau) \left( \left\langle \varphi, \frac{1}{2} \sum_{j=1}^{m} y^{j} e_{j} \cdot B(X,e_{j}) \cdot \varphi - \frac{1}{2} A^{i}(X)iY \cdot \varphi \right\rangle \right)
\]
\[
= (id - \tau) \left( \left\langle \varphi, \frac{1}{2} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} y^{k} e_{k} \cdot e_{j} \cdot B(X,e_{j}) \right) \varphi \right\rangle - A^{i}(X)iY \cdot \varphi \right)
\]
\[
= (id - \tau) \left( \left\langle \varphi, \frac{1}{2} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} y^{k} e_{k} \cdot e_{j} \cdot B(X,e_{j}) \right) + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} y^{k} e_{k} \cdot e_{j} \cdot B(X,e_{j}) - A^{i}(X)iY \cdot \varphi \right\rangle \right)
\]
\[
= (id - \tau) \left( \left\langle \varphi, \frac{1}{2} (-B(X,Y) + D) \cdot \varphi \right\rangle \right),
\] (20)

where
\[
D = \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} y^{k} e_{k} \cdot e_{j} \cdot B(X,e_{j}) - A^{i}(X)iY
\]
\[
\tau[D] = [D].
\]

Consequently
\[
X(\xi(Y)) = \frac{1}{2} (id - \tau) \left( \left\langle \varphi, (-B(X,Y) + D) \cdot \varphi \right\rangle \right)
\]
\[
= -\tau[\varphi] [B(X,Y)][\varphi] = \left\langle \varphi, B(X,Y) \cdot \varphi \right\rangle
\]
\[
= \xi(B(X,Y)).
\]

Therefore we conclude
\[
B^{F}(\xi(X),\xi(Y)) : = B^{F}(\xi(X),\xi(Y)) := \{\nabla^{F}_{\xi(X)}\xi(Y)\}^{\perp} = \{X(\xi(Y))\}^{\perp}
\]
\[
= \{\xi(B(X,Y))\}^{\perp} = \xi(B(X,Y)),
\]
here we used the fact that \( F = \int \xi \) is an isometry: \( B(X,Y) \in E \Rightarrow \xi(B(X,Y)) \in TF(M)^{\perp}. \) Then \( i) \) follows.

\( ii) \) First note that
\[
\nabla^{F}_{\xi(X)}\xi(\eta) = \{X(\xi(\eta))\}^{\perp} = \{X(\langle \eta \cdot \varphi, \varphi \rangle)\}^{\perp}
\]
\[
= \langle \langle \nabla X \eta \cdot \varphi, \varphi \rangle \rangle^{\perp} + \langle \langle \eta \cdot \nabla X \varphi, \varphi \rangle \rangle^{\perp} + \langle \langle \eta \cdot \varphi, \nabla X \varphi \rangle \rangle^{\perp}.
\]

I will show that:
\[
\langle \langle \eta \cdot \nabla X \varphi, \varphi \rangle \rangle^{\perp} + \langle \langle \eta \cdot \varphi, \nabla X \varphi \rangle \rangle^{\perp} = 0.
\]
In fact
\[
\langle\langle \eta \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle \eta \cdot \varphi, \nabla_X \varphi \rangle\rangle = (id - \tau) \langle\langle \eta \cdot \nabla_X \varphi, \varphi \rangle\rangle
\]
\[
= (id - \tau) \langle\langle \left[ \frac{1}{2} \sum_{j=1}^{m} \eta \cdot e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} A^i(X) i \eta \cdot \varphi \right], \varphi \rangle\rangle
\]
\[
= (id - \tau) \langle\langle \left[ -\frac{1}{2} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{k=1}^{n} n^p b^k_j e_j \cdot f_p \cdot f_k - A^i(X) i \eta \right], \varphi, \varphi \rangle\rangle
\]
\[
= (id - \tau) \langle\langle \left[ -\frac{1}{2} \sum_{j=1}^{m} \sum_{p=1}^{n} n^p b^k_j e_j \cdot f_k - \frac{1}{2} A^i(X) i N \right], \varphi, \varphi \rangle\rangle,
\]
from what
\[
\langle\langle N \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle N \cdot \varphi, \nabla_X \varphi \rangle\rangle
\]
\[
= \tau[\varphi] \left[ \frac{1}{2} \sum_{j=1}^{m} \sum_{l=1}^{n} n^l b^j_l e_j ] [\varphi] + \tau[\varphi] \left[ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{m} n^l b^j_l e_j ] [\varphi] \right.
\]
\[
= \tau[\varphi] \left[ \sum_{j=1}^{n} \sum_{l=1}^{m} n^l b^j_l e_j ] [\varphi] = \tau[\varphi] [V] [\varphi] =: \xi(V) \in TF(M)
\]
\[
\Rightarrow \langle\langle \eta \cdot \nabla_X \varphi, \varphi \rangle\rangle = \langle\langle \eta \cdot \varphi, \nabla_X \varphi \rangle\rangle = 0.
\]

In conclusion
\[
\nabla^F_{\xi(X)} \xi(\eta) = \langle\langle \nabla_X \eta \cdot \varphi, \varphi \rangle\rangle \perp \langle\langle \nabla_X \eta \cdot \varphi, \varphi \rangle\rangle \perp = \langle\langle \nabla_X \eta \cdot \varphi, \varphi \rangle\rangle \perp = \xi(\nabla_X \eta).
\]

At the end \( \text{ii)} \) follows.

With these Lemmas the theorem is proved.

References

[1] Bär, C., Extrinsic Bounds for Eigenvalues of the Dirac Operator, Annals of Global Analysis and Geometry, 16(6), 573-596 (1998).

[2] Bayard, P., Lawn, M. A., Roth, J., Spinorial representation of surfaces into 4-dimensional space forms, Ann. Glob. Anal. Geom. 44 433-453 (2013).

[3] Bayard, P., On the spinorial representation of spacelike surfaces into 4-dimensional Minkowski space, Journal of Geometry and Physics, 74, 289-313 (2013).
[4] Bayard, P., M. A. Lawn, J. Roth, Spinorial Representation of Submanifolds in Riemannian Space Forms, to appear Pacific Journal of Mathematics. [arXiv:1505.02935v4 [math-ph]] 2016.

[5] Friedrich, T., On the Spinor Representation of Surfaces in Euclidean 3-space, Jour. of Geom. and Phys. 28, 143-157 (1998).

[6] Friedrich, T., Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics 25, American Mathematical Soc., Providence, 2000.

[7] Hitchin, N., Harmonic Spinors, Advances in Mathematics, 14(1), 1-55 (1974).

[8] Lawn, M. A., Immersions of Lorentzian surfaces in $\mathbb{R}^{2,1}$, Jour. of Geom. and Phys 58 683–700 (2008).

[9] Lawn, M. A., Roth, J., Isometric immersions of Hypersurfaces into 4-dimensional manifolds via spinors, Diff. Geom. Appl. 28 2 205-219 (2010).

[10] Lawn, M. A., Roth, J., Spinorial Characterizations of Surfaces into 3-dimensional Pseudo-Riemannian Space Forms, Math. Phys. Anal. Geom. 14 185-195 (2011).

[11] Lawson, H. B. Jr. and Michelson, M-L., Spin Geometry, Princeton University Press, Princeton, 1989.

[12] Morel, B., Surfaces in $S^3$ and $H^3$ via spinors, Séminaire de Théorie spectrale et géométrie(Grenoble), 23 131-144 (2004-2005).

[13] Nicolaescu, L. I., Notes on Seiberg-Witten Theory, American Mathematical Society, Princeton, 2000.