On testing homogeneity of two covariance matrices with non-normal data

Jieqiong Shen¹,² and Junyan Li¹,³
¹Department of Mathematics, Jincheng College of Sichuan University, Chengdu 611731, China
²College of Mathematics, Sichuan University, Chengdu 610064, China
³Email: Lijunyan1886@126.com

Abstract. In this paper, we consider testing homogeneity of two high-dimensional covariance matrices without the normality assumption. The test statistic is constructed by the standard U-statistics. Furthermore, the asymptotic distribution of the test statistic is derived under some mild assumptions. Simulations show that the proposed test performs well and better in several cases than some existing tests.

1. Introduction

In analyzing data, it is necessary to ascertain if two populations share the same key distributional characteristics, for instance the covariance. Let $X_i = (x_{ij}, \ldots, x_{ip})'$ be independent $p$-dimensional random vectors with mean $\mu_i$ and covariance $\Sigma_i, \ i = 1, 2, j = 1, \ldots, N_i$, where $N_i$ denotes the sample size from the $i$-th population. For example, we wish to test the following hypotheses:

$$H_0 : \Sigma_1 = \Sigma_2 \text{ vs. } H_1 : \Sigma_1 \neq \Sigma_2.$$ 

In discriminant analysis, the outcome of the test determines if we adopt the linear discrimination function or not. Here we just pay attention to the case $p > N_i$, which is called high-dimensional situation, because data with the dimension larger than the sample sizes are increasingly encountered in recent statistical studies, such as Signal Processing [1, 2]. Schott [3] pioneered a testing method based on the Frobenius norm of the difference of covariance matrices, and proposed a test statistic based on the unbiased estimator of $\text{tr}(\Sigma_1 - \Sigma_2)^2$ for the hypotheses (1). Sometimes, it is also of interest to consider only one-sided alternate [4, 5], namely,

$$H_0 : \Sigma_1 = \Sigma_2 \text{ vs. } H_1 : \Sigma_1 > \Sigma_2 \text{ or } \Sigma_1 > \Sigma_2,$$

where for two matrices $P$ and $Q$, $P > Q$ means $P - Q$ is positive definite. The hypothesis test (2) is of relevant interest in Signal Processing. For example, one focuses on the detection of range distributed targets in the presence of diffuse multipath phenomena. In this situation, the signals which are scattered from the target scattering centers are reflected by a glistening surface, and thus received via many propagation paths. The outcome of the testing problem (2) determines if the diffusion multipath is present or not [6, 7].

Specifically, for testing the hypotheses (2), Srivastava [4] proposed a test based on a lower bound on Frobenius norm, where an unbiased estimator of $\text{tr}(\Sigma_1 - \Sigma_2)^2$ was adopted. Realizing the difference between $\text{tr}\Sigma_1$ and $\text{tr}\Sigma_2$ may also exert influence on the difference between $\Sigma_1$ and $\Sigma_2$, Srivastava and
Yanagihara [5] devised a new test statistic by considering a measure of distance
\[ \text{tr} \Sigma_i / (\text{tr} \Sigma_i)^2 - \text{tr} \Sigma_i^2 / (\text{tr} \Sigma_i)^2. \]

It is worth noting that these tests mentioned above heavily rely on the normality assumption, in which an estimator of \(\text{tr} \Sigma_i^2\) based on [8, 9] is adopted. This estimator is unbiased for the normal case, but not generally unbiased. Thus, for the testing problem (1) without the normality assumption, Srivastava et al. [10] modified the test statistic in [3] by providing an unbiased estimator of \(\text{tr} \Sigma_i^2\) for general models. One can also refer to [11] for more non-normal cases.

Motivated by [10], for the testing problem (2), we have a basic idea that it is necessary to relax the normality assumption in [4, 5], because a large number of data do not follow normality. Technically, we propose a new test statistic based on an unbiased estimator of \(\text{tr} \Sigma_i^2\) introduced in [12]. Instead of using this estimator directly, we reconstruct the estimator in [12] to be a standard \(U\)-statistic. The important advantage of this strategy is that the corresponding theory of \(U\)-statistics can be employed conveniently to derive the basic properties of the proposed test statistic.

The paper is organized as follows. In Section 2, the estimator of \(\text{tr} \Sigma_i^2\) proposed by [12] is reconstructed as a standard \(U\)-statistic so that certain properties of the estimator can be obtained by the theory of \(U\)-statistics, then the test procedure for the hypotheses (2) is provided. Section 3 is devoted to comparing the proposed test to some existing tests by simulations. The conclusion is made in Section 4.

2. Test procedure

2.1. The properties of the estimators

To facilitate our analysis, we consider the following general multivariate model, which is similar to the one in [8]:
\[ X_i = \Lambda_i X_{Y_i} + \mu_i, \quad i = 1, 2, j = 1, \ldots, N_i, \]
where \(X_i = (y_{i1}, \ldots, y_{ip})^T\) has independent and identically distributed elements with 
\(E(Y_i) = 0, \text{Cov}(Y_i) = I_p, \text{I}_p\) being the \(p \times p\) identity matrix and \(\Lambda_i\) being a \(p \times p\) matrix with \(\Lambda_i^T \Lambda_i = \Sigma\) and \(\Lambda_i^T A_i = A\). Obviously, the multivariate normal distribution is a special case of the model (3), which indeed covers a large class of multivariate distributions.

Based on [12], an unbiased estimator of \(\text{tr} \Sigma_i^2\) was proposed as follows:
\[
U_i = \frac{1}{N_i(N_i-1)} \sum_{j \neq i} (X_i^T X_i)^2 - \frac{2}{N_i(N_i-1)(N_i-2)} \sum_{j \neq i} X_i^T X_j X_i^T X_j + \frac{1}{N_i(N_i-1)(N_i-2)(N_i-3)} \sum_{j \neq i} X_i^T X_j X_i^T X_j,
\]
where \(\Sigma\) denotes summation over mutually different indices. Actually, by some straightforward calculation, the following lemma indicates that we can reconstruct \(U_i\) such that it yields a standard \(U\)-statistic.

Lemma 2.1 The estimator \(U_i\) given by (4) is identical to
\[
U_i = \frac{1}{N_i(N_i-1)(N_i-2)(N_i-3)} \sum_{j \neq i} \sum_{k \neq j} \frac{1}{12} (B_{ij}^2 + B_{jk}^2 + B_{kj}^2), \quad i = 1, 2,
\]
where \(B_{abcd} = (X_{ia} - X_{ib})^T (X_{ic} - X_{id}).\)

To proceed further, we consider the following assumptions.

Assumption 1 \(p \rightarrow \infty, N_i \rightarrow \infty.\)

Assumption 2 \(E(y_{ij}^3) = 3 + \gamma_i, E(y_{ij}^3) = \kappa_i,\) where \(\gamma_i\) and \(\kappa_i\) are finite constants.
Assumption 3 \( \frac{1}{p} \text{tr} \Sigma_i = O(1), \frac{1}{p} \text{tr} \Sigma^2_i = O(1), \inf_p \frac{\text{tr} \Sigma^4_i}{(\text{tr} \Sigma^2_i)^2} > 0. \)

Assumption 4 \( \frac{\text{tr}(A_i^* A_i^*)}{\text{tr}(A_i \otimes A_i)} \to 0, \frac{\text{tr}(A_i^2 \otimes A_i^2)}{(\text{tr} \Sigma^2_i)^2} \to 0, \) where \(*\) and \(\otimes\) denote the Hadamard and Kronecker products of two matrices, respectively.

Note that, Assumptions 1-4 are mild and justifiable, which are frequently used in high-dimensional testing problems. One can refer to [9], [12, 13] for details.

The following results summarize the essential properties of \(U_i\), which will further be utilized to determine the asymptotic distribution of the proposed test statistic.

**Lemma 2.2** For the estimators \(U_i\) given by (5), we have
\[
\text{E}(U_i) = \text{tr} \Sigma_i^2,
\]
\[
\text{Var}(U_i) = \frac{4}{N_i(N_i-1)(N_i-2)(N_i-3)} \{[(2N_i^3 - 9N_i^2 + 9N_i + 4)\text{tr} \Sigma_i^4 + (N_i^2 - 3N_i + 2)(\text{tr} \Sigma^2_i)^2
\]
\[
+ (N_i^3 - 4N_i^2 + 3N_i) \gamma_i \text{tr}(A_i^* A_i^*) - (2N_i^7 - 14N_i^6 + 24)\kappa_i^2 \text{tr}((A_i^* A_i)A_i^2) + \frac{1}{2}(N_i^2 - 5N_i + 6)\gamma_i^2 \text{tr}(A_i^* A_i)^2 \}
\]
\[
\text{Cov}(U_i, U_j) = 0.
\]

Note that under Assumption 4, the variance of \(U_i\) in Lemma 2.2 can be reduced to
\[
\text{Var}(U_i) = \eta_i + o(\eta_i),
\]
where
\[
\eta_i = \frac{4}{N_i(N_i-1)(N_i-2)(N_i-3)} \{[(2N_i^3 - 9N_i^2 + 9N_i + 4)\text{tr} \Sigma_i^4 + (N_i^2 - 3N_i + 2)(\text{tr} \Sigma^2_i)^2
\]
\[
+ (N_i^3 - 4N_i^2 + 3N_i) \gamma_i \text{tr}(A_i^* A_i^*) - (2N_i^7 - 14N_i^6 + 24)\kappa_i^2 \text{tr}((A_i^* A_i)A_i^2) + \frac{1}{2}(N_i^2 - 5N_i + 6)\gamma_i^2 \text{tr}(A_i^* A_i)^2 \}.
\]

The following lemma establishes the asymptotic normality of the estimators \(U_i, U_j\).

**Lemma 2.3** For the estimators \(U_i\) given by (5), under Assumptions 1-4, we have
\[
V^\frac{1}{2} (U_i - \text{tr} \Sigma_i^2, U_j - \text{tr} \Sigma_j^2)^T \overset{d}{\rightarrow} N_2(0, I_2),
\]
where \(V = \text{diag}(\text{Var}(U_i), \text{Var}(U_j))\), and \(I_2\) is the 2×2 identity matrix.

2.2. The test statistic

Note that, in our situation, the parameter space \(\Theta\) can be viewed as
\[
\Theta = \{(\Sigma_1, \Sigma_2) : \Sigma_1 \text{ and } \Sigma_2 \text{ are positive definite as well as satisfy } \Sigma_1 \geq \Sigma_2 \text{ or } \Sigma_2 \geq \Sigma_1, \}
\]
Define the difference between \(\Sigma_1\) and \(\Sigma_2\) by the distance function \(\delta = \frac{1}{p} (\text{tr} \Sigma_1^2 - \text{tr} \Sigma_2^2)\), which is able to discriminate two covariance matrices, because \(H_0\) holds on the parameter space \(\Theta\) if and only if \(\delta = 0\).

Hence, we construct the test statistic
\[
T = \frac{1}{p^2} (U_i - U_j),
\]
where \(U_i\) and \(U_j\) are given by (5). It is obvious that \(T\) is an unbiased estimator of the distance measure \(\delta\). The asymptotic normality of \(T\) can be established by the following theorem.

**Theorem 2.1** For \(\delta\) and \(T\) given above, under Assumptions 1-4, we have
\[
\Omega^{\frac{1}{2}}(T - \delta) \overset{d}{\rightarrow} N(0, I),
\]
where \(\Omega = \frac{1}{p^2} [\eta_i + \eta_j + o(\eta_i) + o(\eta_j)]\), and \(\eta_i\) is given by (6).

Moreover, we consider the asymptotic variance of \(T\) as
Under Assumption 4, we have $\Omega_0 \to 1$. In particular, under $H_0 : \Sigma_i = \Sigma = \Sigma$, we have

$$\Omega = \Omega_{H_0},$$

where

$$\Omega_{H_0} = \left[ \frac{4(2N_1^2 - 9N_i^2 + 9N_i + 4)}{N_i(N_i - 1)(N_i - 2)(N_i - 3)} + \frac{4(2N_2^2 - 9N_2^2 + 9N_2 + 4)}{N_2(N_2 - 1)(N_2 - 2)(N_2 - 3)} \right]^{p^2},$$

and $\Omega_{H_0}$ is given in Theorem 1 by [14]. From Lemma 1 in [15] and Lemma 2.2, $\hat{\Omega}_{H_0}$ is indeed the consistent estimator of $\Omega_{H_0}$.

Particularly, under $H_0$,

$$\Omega_{H_0}^{1/2} T \xrightarrow{D} N(0, 1).$$

Hence, denote $T_{new} = \hat{\Omega}_{H_0}^{-1/2} T$ and then we can reject $H_0$ in the testing problem (2) if $|T_{new}| > Z_{(1-\alpha)/2}$, where $Z_{(1-\alpha)/2}$ is upper $\alpha/2$ quantile of the standard normal distribution at significance level $\alpha$.

**Remark:** By employing the theory of $U$ -statistics [16, 17], we can prove Lemmas 2.1-2.3 and Theorem 2.1. However, these proofs are lengthy and tedious. We thus omit here.

### 3. Numerical simulations

In this section, we present some results from simulation experiments which are designed to evaluate the performance of the proposed test statistic $T_{new}$, and compare $T_{new}$ with several existing tests in the literatures. For $i = 1, 2, j = 1, \ldots, N_i$, $y_{ij} = (y_{ijk})$ given in the model (3) are generated from the following two distributions:

**Case 1** $y_{ijk} \sim N(0, 1);$  
**Case 2** $y_{ijk} \sim \frac{V_{ijk} - 8}{4}$, $V_{ijk} \sim \chi^2(8)$. 

For simplicity, we generally restrict attention to the case of \( N_1 = N_2 = N \). Set the nominal significant level \( \alpha = 0.05 \), and the Monte Carlo runs are 1000.

We consider the following hypothesis test for two population covariance matrices:

\[
H_0: \Sigma_1 = \Sigma_2 = \Sigma = \rho_1 I_p + \beta_1 (I_p^T)^{-1} I_p \quad \text{vs.} \quad H_1: \Sigma_1 = \rho_1 I_p + \beta_1 (I_p^T)^{-1} I_p, \quad \Sigma_2 = \Sigma.
\]

Here \( \rho_1 = 2.99, \beta_1 = 1.99, \beta = 0.06, \) and \( I_p \) is a \( p \)-vector with all elements ones.

We compare the proposed test \( T_{\text{new}} \) with four tests given respectively in [3] (referred as \( T_1 \)), [4] (referred as \( T_2 \)), [5] (referred as \( T_3 \)), [10] (referred as \( T_4 \)). It is worth pointing out that the test statistics for the testing problem (1) advocated in [3, 10] are indeed suitable for the testing problem (2) because the alternate hypothesis in (2) is a special case of the one in (1). That is the reason why we also compare \( T_{\text{new}} \) with \( T_1 \) and \( T_2 \) here. Tables 1-2 present the Attained Significance Level (ASL) and Attained Power (AP) of all tests. The ASL assesses how close the empirical distributions of those test statistics are to their limiting ones.

From Table 1, we find that for the normal distribution, when \( N \geq 20 \), the ASLs of all tests are reasonable, which are close to the nominal significant level 0.05 as \( N \) and \( p \) both increase. If the dimension is sufficiently larger than the sample sizes, for example, \( p = 200 \) and \( N = 10 \), then the ASLs of the tests \( T_1, T_2, T_3 \) are much lower than 0.05, while the proposed test \( T_{\text{new}} \) still gives the appropriate ASL around 0.05. From Table 2, we find that for the non-normal case, the ASLs of \( T_1, T_2 \) are similar to the results in Table 1, and \( T_1, T_2 \) encounter serious size distortion, while \( T_{\text{new}} \) still maintains reasonable sizes.

Table 1. The ASLs and APs of \( T_1, T_2, T_3, T_4 \) and \( T_{\text{new}} \) for Case 1.

| \( p \) | \( N \) | \( T_1 \) | \( T_2 \) | \( T_3 \) | \( T_4 \) | \( T_{\text{new}} \) | \( \text{ASL} \) | \( \text{AP} \) |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.032 | 0.038 | 0.027 | 0.040 | 0.058 | 0.103 | 0.397 | 0.054 | 0.101 | 0.301 |
| 20 | 20 | 0.051 | 0.061 | 0.038 | 0.059 | 0.067 | 0.155 | 0.847 | 0.060 | 0.146 | 0.859 |
| 30 | 30 | 0.051 | 0.048 | 0.039 | 0.051 | 0.050 | 0.267 | 0.961 | 0.057 | 0.262 | 0.957 |
| 40 | 40 | 0.058 | 0.059 | 0.042 | 0.055 | 0.050 | 0.416 | 1.000 | 0.067 | 0.420 | 0.999 |
| 60 | 60 | 0.028 | 0.018 | 0.017 | 0.036 | 0.059 | 0.088 | 0.426 | 0.035 | 0.094 | 0.346 |
| 20 | 20 | 0.043 | 0.052 | 0.037 | 0.042 | 0.063 | 0.196 | 0.955 | 0.045 | 0.191 | 0.945 |
| 30 | 30 | 0.042 | 0.049 | 0.043 | 0.044 | 0.054 | 0.285 | 0.998 | 0.059 | 0.273 | 0.999 |
| 40 | 40 | 0.052 | 0.048 | 0.041 | 0.052 | 0.048 | 0.443 | 1.000 | 0.059 | 0.433 | 1.000 |
| 120 | 120 | 0.015 | 0.017 | 0.011 | 0.021 | 0.067 | 0.088 | 0.480 | 0.030 | 0.088 | 0.451 |
| 20 | 20 | 0.023 | 0.039 | 0.028 | 0.029 | 0.045 | 0.170 | 0.938 | 0.070 | 0.141 | 0.934 |
| 30 | 30 | 0.043 | 0.039 | 0.047 | 0.045 | 0.049 | 0.279 | 0.999 | 0.089 | 0.266 | 0.998 |
| 40 | 40 | 0.039 | 0.045 | 0.038 | 0.045 | 0.043 | 0.400 | 1.000 | 0.084 | 0.416 | 0.999 |
| 200 | 200 | 0.016 | 0.004 | 0.002 | 0.024 | 0.059 | 0.100 | 0.306 | 0.031 | 0.080 | 0.457 |
| 20 | 20 | 0.036 | 0.024 | 0.029 | 0.038 | 0.045 | 0.165 | 0.908 | 0.088 | 0.174 | 0.917 |
| 30 | 30 | 0.045 | 0.037 | 0.034 | 0.048 | 0.046 | 0.233 | 0.994 | 0.108 | 0.241 | 0.989 |
| 40 | 40 | 0.046 | 0.045 | 0.050 | 0.042 | 0.045 | 0.462 | 0.996 | 0.126 | 0.444 | 0.995 |
| 600 | 600 | 0.028 | 0.016 | 0.020 | 0.038 | 0.054 | 0.102 | 0.460 | 0.040 | 0.098 | 0.460 |
| 20 | 20 | 0.042 | 0.038 | 0.038 | 0.044 | 0.060 | 0.170 | 0.942 | 0.082 | 0.194 | 0.929 |
| 30 | 30 | 0.048 | 0.046 | 0.046 | 0.042 | 0.052 | 0.246 | 0.998 | 0.118 | 0.262 | 0.992 |
| 40 | 40 | 0.052 | 0.048 | 0.042 | 0.054 | 0.047 | 0.480 | 0.999 | 0.138 | 0.446 | 0.998 |
| 1000 | 1000 | 0.026 | 0.018 | 0.018 | 0.032 | 0.056 | 0.126 | 0.472 | 0.042 | 0.096 | 0.470 |
| 20 | 20 | 0.040 | 0.040 | 0.040 | 0.046 | 0.061 | 0.180 | 0.946 | 0.087 | 0.198 | 0.948 |
| 30 | 30 | 0.046 | 0.048 | 0.042 | 0.048 | 0.054 | 0.261 | 0.999 | 0.120 | 0.278 | 0.998 |
| 40 | 40 | 0.054 | 0.047 | 0.052 | 0.050 | 0.054 | 0.487 | 1.000 | 0.137 | 0.452 | 1.000 |
In this paper, we consider the equality test for two covariance matrices in high-dimensional data (normality). The proposed test may be preferred in more general distributions (containing normality and non-normality) has reasonable size in Cases 1-2. Taking into consideration the ASLs as well as APs, it appears that the new test is robust against departure from normality and has accurate significance level as well as a greater improvement in power than some existing tests.

Simulations demonstrate that the new test is robust against departure from normality and has accurate significance level as well as a greater improvement in power than some existing tests. Moreover, the limit distribution of the proposed test statistic is derived under some mild assumptions.

In the future work, the analysis of the power function of the proposed test and seeking some appropriate real examples may be considered.

| Table 2. The ASLs and APs of $T_1, T_2, T_3, T_4$ and $T_{new}$ for Case 2. |
|---|---|---|---|---|---|---|---|---|
| $p$ | $N$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_{new}$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_{new}$ |
| 20 | 10 | 0.081 | 0.107 | 0.059 | 0.031 | 0.053 | 0.094 | 0.259 | 0.061 | 0.093 | 0.273 |
| 20 | 20 | 0.132 | 0.113 | 0.074 | 0.049 | 0.068 | 0.140 | 0.599 | 0.048 | 0.140 | 0.682 |
| 20 | 30 | 0.165 | 0.118 | 0.069 | 0.052 | 0.064 | 0.250 | 0.877 | 0.060 | 0.218 | 0.898 |
| 20 | 40 | 0.172 | 0.120 | 0.085 | 0.066 | 0.075 | 0.325 | 0.958 | 0.049 | 0.340 | 0.970 |
| 20 | 60 | 0.075 | 0.059 | 0.083 | 0.025 | 0.055 | 0.098 | 0.379 | 0.036 | 0.099 | 0.404 |
| 20 | 60 | 0.19 | 0.082 | 0.058 | 0.042 | 0.066 | 0.165 | 0.878 | 0.068 | 0.179 | 0.913 |
| 20 | 60 | 0.150 | 0.107 | 0.061 | 0.040 | 0.067 | 0.322 | 0.988 | 0.051 | 0.307 | 0.990 |
| 20 | 40 | 0.168 | 0.099 | 0.060 | 0.055 | 0.055 | 0.199 | 0.999 | 0.060 | 0.431 | 0.999 |
| 20 | 60 | 0.067 | 0.024 | 0.011 | 0.030 | 0.059 | 0.099 | 0.299 | 0.024 | 0.109 | 0.386 |
| 120 | 20 | 0.114 | 0.077 | 0.046 | 0.038 | 0.052 | 0.187 | 0.885 | 0.045 | 0.179 | 0.898 |
| 120 | 30 | 0.135 | 0.095 | 0.062 | 0.047 | 0.054 | 0.293 | 0.987 | 0.085 | 0.270 | 0.990 |
| 120 | 40 | 0.149 | 0.097 | 0.062 | 0.050 | 0.060 | 0.368 | 0.999 | 0.113 | 0.399 | 0.999 |
| 200 | 10 | 0.066 | 0.014 | 0.008 | 0.020 | 0.062 | 0.125 | 0.236 | 0.030 | 0.124 | 0.349 |
| 200 | 20 | 0.125 | 0.068 | 0.053 | 0.038 | 0.057 | 0.185 | 0.883 | 0.088 | 0.193 | 0.898 |
| 200 | 30 | 0.139 | 0.084 | 0.058 | 0.044 | 0.054 | 0.266 | 0.977 | 0.090 | 0.282 | 0.979 |
| 200 | 40 | 0.144 | 0.088 | 0.064 | 0.045 | 0.057 | 0.359 | 0.998 | 0.133 | 0.384 | 0.998 |
| 600 | 20 | 0.064 | 0.018 | 0.010 | 0.024 | 0.064 | 0.127 | 0.242 | 0.028 | 0.122 | 0.389 |
| 600 | 30 | 0.124 | 0.070 | 0.056 | 0.040 | 0.058 | 0.182 | 0.890 | 0.090 | 0.196 | 0.916 |
| 600 | 40 | 0.140 | 0.089 | 0.060 | 0.048 | 0.054 | 0.272 | 0.982 | 0.092 | 0.284 | 0.990 |
| 600 | 40 | 0.146 | 0.090 | 0.061 | 0.052 | 0.052 | 0.368 | 0.999 | 0.140 | 0.392 | 1.000 |
| 1000 | 20 | 0.066 | 0.010 | 0.012 | 0.026 | 0.067 | 0.128 | 0.270 | 0.032 | 0.126 | 0.398 |
| 1000 | 30 | 0.132 | 0.072 | 0.046 | 0.032 | 0.052 | 0.188 | 0.889 | 0.098 | 0.199 | 0.920 |
| 1000 | 40 | 0.138 | 0.086 | 0.061 | 0.048 | 0.051 | 0.276 | 0.984 | 0.102 | 0.290 | 0.992 |
| 1000 | 40 | 0.148 | 0.092 | 0.062 | 0.054 | 0.048 | 0.382 | 0.999 | 0.146 | 0.398 | 1.000 |

Concerning the powers, the two tables show that the proposed test $T_{new}$ has quite good power, which are largely comparable to $T_3$ in Case 1 and tends to perform better than $T_3$ in Case 2. For instance, in Table 2, we obtain the APs 0.913, 0.878 of $T_{new}$ and $T_3$ when $p = 60, N = 20$. Both $T_{new}$ and $T_3$ are much more powerful than $T_1, T_2, T_4$. It is worth pointing out that $T_1$ has low power despite it has reasonable size in Cases 1-2. Taking into consideration the ASLs as well as APs, it appears that the proposed test may be preferred in more general distributions (containing normality and non-normality).

4. Conclusions and discussion
In this paper, we consider the equality test for two covariance matrices in high-dimensional data without normality assumption. A new test statistic is constructed by the standard U-statistics. Moreover, the limit distribution of the proposed test statistic is derived under some mild assumptions. Simulations demonstrate that the new test is robust against departure from normality and has accurate significance level as well as a greater improvement in power than some existing tests.

In the future work, the analysis of the power function of the proposed test and seeking some appropriate real examples may be considered.

References
[1] Yazdian E, Gazor S and Bastani H 2010 Source enumeration in large arrays using moments of eigenvalues and relatively few samples IET Signal Process 6(7) 689-696
[2] Huang L, Qian C, So H C and Fang J 2015 Source enumeration for large array using shrinkage-based detectors with small samples IEEE Transaction on Aerospace and Electronic Systems 51(1) 344-357
[3] Schott J R 2007 A test for the equality of covariance matrices when the dimension is large relative to the sample sizes *Computational Statistics and Data Analysis* **51**(12) 6535-6542

[4] Srivastava M S 2007 Testing the equality of two covariance matrices and testing the independence of two subvectors with fewer observations than the dimension In *Proceedings of the International Conference on Advances in Interdisciplinary Statistics and Combinatorics*

[5] Srivastava M S and Yanagihara H 2010 Testing the equality of several covariance matrices with fewer observations than the dimension *Journal of Multivariate Analysis* **101**(6) 1319-1329

[6] Fante R L 1991 Cancellation of specular and diffuse jammer multipath using a hybrid adaptive array *IEEE Transactions on Aerospace and Electronic Systems* **27**(5) 823-837

[7] Aubry A, Foglia G and Orlando D 2015 Diffuse multipath exploitation for adaptive radar detection *IEEE Transactions on Signal Processing* **63**(5) 1268-1281

[8] Bai Z and Saranadasa H 1996 Effect of high dimension By an example of a two sample problem *Statistica Sinica* **6**(2) 311-329

[9] Srivastava M S 2005 Some tests concerning the covariance matrix in high dimensional data *Journal of the Japan Statistical Society* **35**(2) 251-272

[10] Srivastava M S, Yanagihara H and Kubokawa T 2014 Tests for covariance matrices in high dimension with less sample size *Journal of Multivariate Analysis* **130** 289-309

[11] Li J and Chen S X 2012 Two sample tests for high-dimensional covariance matrices *The Annals of Statistics* **40**(2) 908-940

[12] Chen S X, Zhang L X and Zhong P S 2010 Tests for high-dimensional covariance matrices *Journal of the American Statistical Association* **105**(490) 810-819

[13] Ahmad M R 2017 Testing homogeneity of several covariance matrices and multi-sample sphericity for high-dimensional data under non-normality *Communications in Statistic Theory and Methods* **46**(8) 3738-3753

[14] Fisher T J, Sun X and Gallagher C M 2010 A new test for sphericity of the covariance matrix for high dimensional data *Journal of Multivariate Analysis* **101**(10) 2554-2570

[15] Tian X, Lu Y and Li W 2015 A robust test for sphericity of high-dimensional covariance matrices *Journal of Multivariate Analysis* **141** 217-227

[16] Lee J 1990 *U-Statistics Theory and Practice* Marcel Dekker, New York NY USA

[17] Lehmann E L 2004 *Elements of Large-Sample Theory* Springer, New York NY USA