On differences between the B- and E-approaches and the implications for the Solar atmosphere

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Abstract. A simple collisional three-component plasma model consisting of electrons, ions, and neutrals with arbitrary collision frequencies and dynamic time scales is considered. It is shown that the usual MHD-approach dealing with magnetic field perturbations can give other results than the approach in which all perturbations are expressed via the perturbed electric field. For the partially ionized plasma with strong collisional coupling of neutrals with ions, magneto sonic (nondamping) and Alfvén (weakly damping) waves modified by the presence of neutrals are obtained. It is shown that the magnetic diffusivity for Alfvén waves appears only due to the longitudinal current connected with the field $E_{1z}$ at the angular propagation of perturbations relatively to the background magnetic field.

The model can be applied to different parts of a solar atmosphere and prominences.

Key words: Sun: atmosphere - Sun: oscillations

1. Introduction

It is known that analytical investigations of the fluid plasma systems are generally done by two methods. In the first one, the electric field perturbation is excluded from equations of motion and corresponding perturbed velocities are expressed through the magnetic field perturbation (e.g., Chandrasekhar 1961). This method can be called the B-approach. It
is mainly used in astrophysics, for some problems of solar physics and geophysics when studying magnetohydrodynamic (MHD) waves. In the second method or in the E-approach, the perturbed velocities of species are expressed through the components of the electric field perturbation (e.g., Mikhailovskii 1975). Such an approach is mainly applied in the theory of plasma physics, in solar physics, and in geophysics also.

However, the B- and E-approaches can lead to different results. For instance, in a resistive fully-ionized electron-ion medium, dispersion relations for MHD waves obtained via the B-approach are given by (e.g., Nekrasov 2009)

\[ \omega^2 + i\omega\eta k^2 - k^2 c_A^2 = 0 \]  
\[ \omega^2 + i\omega\eta k^2 - k_z^2 c_A^2 = 0 \]

for magnetosonic waves and

for Alfvén waves. Here, \( \omega \) is the frequency, \( k^2 = k^2_\perp + k^2_z \), \( k \) is the wavenumber, \( c_A \) is the Alfvén velocity, \( \eta \) is the magnetic diffusivity, the background magnetic field is assumed to be directed along the axis \( z \), subscripts \( \perp \) and \( z \) denote directions relatively to the magnetic field. We can conclude from Equations (1) and (2) that

1. Magnetosonic and Alfvén waves are damped due to the resistivity.
2. The resistivity is isotropic.

At the same time, the corresponding dispersion relation derived through the E-approach has the form (Nekrasov 2009)

\[ (\omega^2 - k^2 c_A^2) \left( \omega^2 + i\omega\eta k^2_\perp - k^2_z c_A^2 \right) = 0. \]  

We see from Equation (3) that
1. Magnetosonic waves are not damped.

2. Alfvén waves are damped due to the resistivity. The resistivity is anisotropic and proportional to $k_\perp^2$. When $k_\perp = 0$, Alfvén waves are also not damped.

These results differ from the ones obtained from Equations (1) and (2).

The real part of the frequency $\omega$ found from Equations (1) or (2), for example, is equal to zero at the cut-off wavenumber $k_{zc} = \pm 2c_A/\eta$ in the case of the longitudinal propagation ($k_\perp = 0$) (Chandrasekhar 1961; Zaqarashvili et al. 2012 and references therein). Zaqarashvili et al. (2012) have shown that the appearance of the cut-off wavenumber is due to some simplifications of the basic equations. They have considered partially ionized plasmas of the solar atmosphere in the two-fluid description, where one component is the charged fluid (electrons and ions) and the other component is the neutral gas. It has been shown that for the time scales longer then the ion-neutral collision time and neglecting the corresponding Hall term one comes to the usual single-fluid MHD equations giving the magnetosonic and Alfvén waves (1) and (2), respectively. Without introducing one of these two simplifications, the cut-off wavenumber is absent.

In their study, Zaqarashvili et al. (2012) have used the $B$-approach. However, as we have seen above, the $E$-approach can give another results. From our point of view, Equation (3) takes into account the physical mechanism of the collisional damping correctly (see below). Therefore, we consider here the three-component plasma consisting of electrons, ions, and neutrals by making of use the $E$-approach in a general form, where the frequencies of collisions between different species are arbitrary. We derive a dispersion relation for perturbations without any simplifications and consider it in a particular case suitable for solar prominences.

The paper is organized as follows. In Section 2, we give the main equations and find the
perturbed velocities in a general form. The components of perturbed current are obtained in Section 3. The wave equation is considered in Section 4. In section 5, the dispersion relations for the longitudinal and angular propagations in the case of strong collisional coupling of neutrals with ions are derived. An applicability of obtained results to the solar atmosphere is discussed in Section 6. In Section 7, we present conclusive remarks.

2. Basic equations and expressions for perturbed velocities

The equations of motion for species that we consider are the following:

\[
\frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e = \frac{q_e}{m_e} \mathbf{E} + \frac{q_e}{m_e c} \mathbf{v}_e \times \mathbf{B} - \nu_{ei} (\mathbf{v}_e - \mathbf{v}_i) - \nu_{en} (\mathbf{v}_e - \mathbf{v}_n), \tag{4}
\]

\[
\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = \frac{q_i}{m_i} \mathbf{E} + \frac{q_i}{m_i c} \mathbf{v}_i \times \mathbf{B} - \nu_{ie} (\mathbf{v}_i - \mathbf{v}_e) - \nu_{in} (\mathbf{v}_i - \mathbf{v}_n), \tag{5}
\]

\[
\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\nu_{ne} (\mathbf{v}_n - \mathbf{v}_e) - \nu_{ni} (\mathbf{v}_n - \mathbf{v}_i). \tag{6}
\]

Here, \(\mathbf{v}_j\) is the velocity of species \(j\), where \(j = e, i, n\) denotes electrons, ions, and neutrals, respectively, \(q_j\) is the charge, \(\nu_{ab}\) is the collision frequency of particle \(a\) with particles \(b\), \(\mathbf{E}\) and \(\mathbf{B}\) are the electric and magnetic fields, and \(c\) is the speed of light in vacuum. For simplicity, we use here the same momentum equations as given in (Zaqarashvili et al. 2012) to pay attention on collisional interactions in partially ionized plasmas in the \(\mathbf{E}\)-approach. We don’t take into account temperatures, viscosity, gravity etc. The plasma and the background magnetic field are assumed to be homogeneous.

The dynamics of neutrals is only determined by collisions with charged particles. After linearization of Equations (4)-(6) and substitution of the perturbed velocity of neutrals \(\mathbf{v}_{n1}\) (the subject 1 here and below denotes the perturbed values) into the linearized equations (4) and (5), we obtain the equation for \(\mathbf{v}_{j1}\), \(j = e, i,\)

\[
\beta_j \frac{\partial \mathbf{v}_{j1}}{\partial t} = \mathbf{F}_{j1} + \frac{q_j}{m_j c} \mathbf{v}_{j1} \times \mathbf{B}_0. \tag{7}
\]
Here,

\[ F_{e1} = \frac{q_e}{m_e} E_1 - \alpha_e (v_{e1} - v_{i1}), \]

\[ F_{i1} = \frac{q_i}{m_i} E_1 - \alpha_i (v_{i1} - v_{e1}) \]

and

\[ \beta_j = 1 + \frac{\nu_{jn0}}{\alpha_n}, \]

\[ \alpha_e = \nu_{e0} + \frac{\nu_{en0} \nu_{ne0}}{\alpha_n}, \]

\[ \alpha_i = \nu_{i0} + \frac{\nu_{in0} \nu_{ne0}}{\alpha_n}, \]

\[ \alpha_n = \partial \frac{\partial}{\partial t} + (\nu_{ne0} + \nu_{n0}). \]

The subscript 0 denotes unperturbed collision frequencies. Solution of Equation (7) is given by

\[ \Omega^2_{j} v_{j1x} = \omega_{cj} F_{j1y} + \beta_j \frac{\partial F_{j1x}}{\partial t}, \]

\[ \Omega^2_{j} v_{j1y} = -\omega_{cj} F_{j1x} + \beta_j \frac{\partial F_{j1y}}{\partial t}, \]

\[ \beta_j \frac{\partial v_{j1z}}{\partial t} = F_{j1z}, \]

where

\[ \Omega^2 = \beta_j^2 \frac{\partial^2}{\partial t^2} + \omega_{cj}^2 \]

and \( \omega_{cj} = q_j B_0 / m_j c \) is the cyclotron frequency. The background magnetic field \( B_0 \) is directed along the axis \( z \).

3. The perturbed current

Let us now find the perturbed current \( j_1 = \sum_j q_j n_{j0} v_{j1} = q_i n_0 (v_{i1} - v_{e1}) \), where we have assumed the condition of quasineutrality \( n_{0e} = n_{0i} = n_0 \) \( (q_e = -q_i) \). Using Equation
(10) for the transverse velocity and taking into account Equation (8), we obtain two equations

\[ aj_1x + bj_1y = \frac{q_i^2 n_0}{m_i} \left( dE_{1y} + f \frac{\partial E_{1x}}{\partial t} \right), \]

\[ aj_1y - bj_1x = \frac{q_i^2 n_0}{m_i} \left( -dE_{1x} + f \frac{\partial E_{1y}}{\partial t} \right), \]  

where notations are introduced

\[ a = 1 + \left( \frac{\alpha_i \beta_i}{\Omega_i^2} + \frac{\alpha_e \beta_e}{\Omega_e^2} \right) \frac{\partial}{\partial t}, \]

\[ b = \frac{\omega_{ci} \alpha_i}{\Omega_i^2} + \frac{\omega_{ce}}{\Omega_e^2}, \]

\[ d = \frac{\omega_{ci}}{\Omega_i^2} + \frac{\omega_{ce} m_i}{\Omega_e^2 m_e}, \]

\[ f = \frac{\beta_i}{\Omega_i^2} + \frac{\beta_e m_i}{\Omega_e^2 m_e}. \]

It is convenient to find a solution of Equation (12) for the value \( 4 \pi (\partial/\partial t)^{-1} j_{1x}, j_{1y} \). Then we obtain

\[ 4 \pi \left( \frac{\partial}{\partial t} \right)^{-1} j_{1x} = \varepsilon_{xx} E_{1x} + \varepsilon_{xy} E_{1y}, \]

\[ 4 \pi \left( \frac{\partial}{\partial t} \right)^{-1} j_{1y} = -\varepsilon_{xy} E_{1x} + \varepsilon_{xx} E_{1y}, \]

where

\[ \varepsilon_{xx} = \frac{\omega_p^2}{(a^2 + b^2)} \left[ bd \left( \frac{\partial}{\partial t} \right)^{-1} + af \right], \]

\[ \varepsilon_{xy} = \frac{\omega_p^2}{(a^2 + b^2)} \left[ ad \left( \frac{\partial}{\partial t} \right)^{-1} - bf \right] \]

and \( \omega_p = (4 \pi q_i^2 n_i / m_i)^{1/2} \) is the ion plasma frequency.

From Equations (8) and (10), we find further the perturbed longitudinal current

\[ j_{1z} = q_i n_0 \left( v_{i1z} - v_{e1z} \right). \]  

Calculations show that

\[ 4 \pi \left( \frac{\partial}{\partial t} \right)^{-1} j_{1z} = \varepsilon_{zz} E_{1z}, \]  

(16)
where
\[ \varepsilon_{zz} = \omega_{pi}^2 \left( \frac{\partial}{\partial t} + \frac{\alpha_i}{\beta_i} + \frac{\alpha_e}{\beta_e} \right)^{-1} \left( \frac{1}{\beta_i^2} + \frac{1}{\beta_e m_e} \right) \left( \frac{\partial}{\partial t} \right)^{-1}. \] (17)

4. Wave equation

Our model is azimuthally symmetrical. Therefore, we can set \( \partial / \partial x = 0 \). Then, from Faraday’s
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]
and Ampere’s
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \]
laws, we can obtain wave equations for perturbations. Using Equations (14) and (16), we find
\[ (n^2 - \varepsilon_{xx}) E_{1x} - \varepsilon_{xy} E_{1y} = 0, \]
\[ \varepsilon_{xy} E_{1x} + (n_z^2 - \varepsilon_{xx}) E_{1y} - n_y n_z E_{1z} = 0, \]
\[ -n_y n_z E_{1y} + (n_y^2 - \varepsilon_{zz}) E_{1z} = 0, \]
where \( n^2 = n_y^2 + n_z^2, \ n_{y,z} = c (\partial / \partial y, z) (\partial / \partial t)^{-1} \).

Components \( \varepsilon_{xx}, \varepsilon_{xy}, \) and \( \varepsilon_{zz} \) given by Equations (15) and (17) have a general form and can be applied at arbitrary correlations between collision frequencies of species and dynamic time scales. Therefore, it is possible to study wave propagation in partially ionized plasmas in different regions, for example, of the solar atmosphere. Below, we consider one specific case.
5. Dispersion relation

For perturbations of the form $E_1 \propto E_{ik} \exp(ik \cdot r - i\omega t)$ Equation (18) becomes algebraic and the determinant of this system gives a dispersion relation in a general form.

5.1. Longitudinal propagation

We first consider waves propagating almost along the background magnetic field when $k_y \approx 0$. The case $k_y = 0$ was treated by Zaqarashvili et al. (2012). Then the dispersion relation is the following:

\[
(n_z^2 - \varepsilon_{xx}) = \pm i\varepsilon_{xy}. \tag{19}
\]

This equation describes, as it is well-known, two circularly-polarized waves: the magnetosonic and Alfvén waves rotating in opposite directions in the plane perpendicular to the magnetic field $B_0$. The main condition for the value $k_y^2$ can be found from $n_y^2 \ll \varepsilon_{xy}$.

To find the values $\varepsilon_{xx}$ and $\varepsilon_{xy}$, we must specify collision frequencies and time scales. We will consider the case in which

\[
\omega \ll \nu_{ni0} \tag{20}
\]

when there is a strong collisional coupling of neutrals with ions. Then, we obtain (see Equation (9)) $\alpha_n = \nu_{ni0}$, $\beta_e = 1 + \nu_{en0}/\nu_{ni0}$, $\beta_i = 1 + \rho_{i0}/\rho_{i0}$, $\alpha_e = \nu_{ei0} + \nu_{en0}$, $\alpha_i = (m_e/m_i)\alpha_e$, where $\rho_{a0} = m_a n_{a0}$. Further, we assume the condition of magnetization (see Equation (11))

\[
\omega_{ci}^2 \gg \beta_i^2 \omega^2, \tag{21}
\]

which is easily satisfied. We note that according to Equation (21) the electrons are also magnetized ($\omega_{ce}^2 \gg \beta_e^2 \omega^2$) because $\nu_{ni0} \gg \nu_{ne0}$. Under these conditions, the values given by
Equation (13) are the following:

\[ a \simeq 1, b = \frac{\alpha_i \beta_i^2 \omega^2}{\omega_{ci}^3} \ll 1, d = \frac{\beta_i^2 \omega}{\omega_{ci}^3}, f = \frac{\beta_i}{\omega_{ci}}. \] (22)

When calculating the value \( a \), we have assumed the additional condition

\[ \omega_{ci}^2 \gg \alpha_i \beta_i \omega. \] (23)

Substituting Equation (22) into Equation (15), we find

\[ \varepsilon_{xx} = \frac{c^2}{c_A^2}, \varepsilon_{xy} = i \varepsilon_{xx} \beta_i \omega / \omega_{ci}, \]

where \( c_A = [B_0^2/4\pi (\rho_i + \rho_n)]^{1/2} \) is the Alfvén velocity.

The dispersion relation (19) takes the form

\[ \omega^2 \pm \beta_i \frac{\omega^3}{\omega_{ci}} - k_z^2 c_A^2 = 0. \] (24)

The second term on the left-hand side of Equation (24) appearing due to the Hall effect (a sum of the ion and neutral inertia) and describing the dispersion is small (see Eq. (21)).

We have obtained the usual equation for Alfvén (magnetosonic) waves modified by the presence of strong neutral-ion collisions. These waves are not damped and have no the cut-off wavenumber.

5.2. Angular propagation

We now consider the case in which \( k_y \neq 0 \). In the region \( n_y^2 \ll \varepsilon_{zz} \), we can neglect the contribution of \( E_{1z} \) in Equation (18). Assuming condition \( n_y^2 \gg \varepsilon_{xy} \), which is opposite to that in Section 6, the terms \( \varepsilon_{xy} E_{1x,y} \) can also be omitted. Then, we obtain

\[ (n^2 - \varepsilon_{xx}) E_{1x} = 0, \] (25)

\[ (n_z^2 - \varepsilon_{xx}) E_{1y} = 0. \]
Equation (25) describes the independent linearly-polarized magnetosonic, $E_{1x} \neq 0$, and Alfvén, $E_{1y} \neq 0$, waves. Taking into account the field $E_{1z}$ for the Alfvén wave (two last equalities in Equation (18)), we find the dispersion relation

$$n_z^2 - \varepsilon_{xx} + n_y^2 \varepsilon_{xx} \varepsilon_{zz} = 0,$$

where $\varepsilon_{zz}$ given by Equation (17) has the form

$$\varepsilon_{zz} = -\frac{\omega_{pe}^2}{\omega(\omega\beta_e + i\alpha_e)},$$

where $\omega_{pe}$ is the electron plasma frequency. We note that the ions don’t contribute to the longitudinal current in our model without the thermal pressure.

We further consider the low-frequency case in which

$$\omega \beta_e \ll \alpha_e.$$

Substituting Equation (27) into Equation (26), we obtain

$$\omega^2 + i\eta_m k_z^2 \omega - k_z^2 c_A^2 = 0,$$

where $\eta_m = c^2 (\nu_ei0 + \nu_en0)/\omega_{pe}^2$ is the magnetic diffusivity modified by the electron-neutral collisions. This Equation is analogous to Equation (3). Formally, we see that for given $k_z$ the real part of the frequency $\omega_r$ becomes zero at $k_{yg}^2 = \pm 2k_z c_A/\eta_m$. However, it is not the cut-off wavenumber in the sense of paper by Zaqarashvili et al. (2012). We emphasize that the contribution of the magnetic diffusivity to Equation (29) appears only due to the longitudinal current connected with the field $E_{1z}$.

6. Discussion

The solar atmosphere, including prominences, is only partially ionized. We now discuss the applicability conditions used in Section 5 to this medium. The main condition for
Equations (24) and (29) is the strong collision coupling between neutrals and ions given by Equation (20). To find $\nu_{ni}$, we consider parameters corresponding to solar quiescent prominences: $n_i = 10^{10}$ cm$^{-3}$, $n_n = 2 \times 10^{10}$ cm$^{-3}$, and $T = 8000$ K, where subscripts $i = p$ and $n$ denote protons and neutrons, respectively (e.g., Zaqarashvili et al. 2012). Then, using the collisional proton-neutron cross section $\sigma_{in} = 5 \times 10^{-15}$ cm$^2$ (Díaz et al. 2012), we find (Braginskii 1965)

$$\nu_{ni} = n_i \frac{8}{3} \left( \frac{1}{\pi m_i} \right)^{1/2} \sigma_{in} \simeq 61 \text{ s}^{-1}. $$

The observed periods for prominence oscillations are in the range between 30 s (Balthasar et al. 1993) and $10-30$ hr (Foullon et al. 2009). Thus, we have the strong neutron-proton collisional coupling.

The second condition to obtain Equation (29) is Equation (28). Calculating $\nu_{en0}$ and $\nu_{ei0}$ (Braginskii 1965), we obtain $\nu_{en0} = 1.84 \times 10^3$ s$^{-1}$ and $\nu_{ei0} = 5.75 \times 10^5$ s$^{-1}$, or $\alpha_e \simeq \nu_{ei0}$ and $\beta_e \simeq 31.16$. Thus, Equation (28) is also satisfied.

For the magnetic field in solar prominences $B_0 = 10$ G, we have $\omega_{ce} = 1.76 \times 10^8$ s$^{-1}$ and $\omega_{ci} = \omega_{cp} = 0.96 \times 10^8$ s$^{-1}$. We see that the condition of magnetization given by Equation (21) and the additional condition defined by Equation (23) are wittingly satisfied.

For parameters given above, the diffusivity $\eta_m$ is equal to $\eta_m = 1.63 \times 10^7$ cm$^2$ s$^{-1}$. Then, the dissipation term in Equation (29) is much less than $k_z^2 c_A^2$, where $c_A = 1.26 \times 10^7$ cm s$^{-1}$, in the case $k_z \gg 1.29 k_y^2$ cm. If formally set $k_z \sim k_y$, we obtain $\lambda_y \gg 8$ cm that is, of course, satisfied. Thus, this wave is the weakly damping Alfvén one.
7. Conclusion

We have considered a simple collisional three-component plasma model consisting of electrons, ions, and neutrals, in which collision frequencies between different species are arbitrary. This model can be applied to different parts of the solar atmosphere and prominences. One of the main purpose of the paper was to show that the usual MHD-approach dealing with the magnetic field perturbations can give other results than the approach in which all the perturbations are expressed via the perturbed electric field. For the partially ionized plasma of solar prominences with strong collisional coupling of neutrals with ions, we have obtained magnetosonic (nondamping) and Alfvén (weakly damping) waves modified by the presence of neutrals. We have shown that the magnetic diffusivity for Alfvén waves appears only due to the longitudinal current connected with the field $E_{1z}$ in the case of angular propagation of perturbations relatively to the background magnetic field.

The values $\varepsilon_{xx}$, $\varepsilon_{xy}$, and $\varepsilon_{zz}$ given by Equations (15) and (17) have a general form and can be applied to arbitrary correlations between collision frequencies of species and dynamic time scales. Therefore, it is possible to study wave propagation in partially ionized plasmas in different regimes.

The results obtained are useful for an investigation of the solar atmosphere and other collisional astrophysical media.

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