Cherenkov Wakefield Radiation from an Open End of a Three-Layer Dielectric Capillary

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Modern trends in beam-driven radiation sources involve interaction of Cherenkov wakefields with open-ended circular waveguide structures having complicated dielectric lining, with a three-layer dielectric capillary recently offered for reducing the radiation divergency being a representative example [1]. This paper presents rigorous approach allowing analytical description of electromagnetic processes occurring when the described structure is excited by single waveguide mode. In other words, corresponding canonical waveguide diffraction problem is considered in rigorous formulation. This is continuation of our recent paper [2] where a simpler case of a two-layer dielectric filling has been considered. Here we use the same analytical approach based on Wiener-Hopf-Fock technique and deal with more complicated case of a three-layer dielectric lining.

I. INTRODUCTION

Dielectric-lined open-ended waveguide structures are considered nowadays as extremely promising for a variety of applications based on Cherenkov effect. In the context of the present paper, one can mention certain success in both dielectric wakefield acceleration [3–8] and development of high-power narrow-band radiation sources including those for Terahertz (THz) frequencies [9–12]. Typical structure for mentioned application is a dielectric capillary – a circular waveguide with a dielectric layer and axial channel for bunch passage. Recently, a promising three-layer modification of mentioned capillary has been offered which essentially reduces the width of the main radiation lobe and therefore enhances considerably the radiated power [1].

For further development of the discussed topics a rigorous approach allowing analytical investigation of both radiation from such open-ended capillary and its excitation by external source (bunch or electromagnetic pulse) would be extremely useful. In our recent papers [2, 13], we have presented an efficient rigorous method for solving circular open-ended waveguide diffraction problems and illustrated this method using the case of uniform dielectric filling and a two-layer lining of the waveguide. Here we deal with more complicated geometry offered in [1] and internal excitation by single waveguide mode. The presented technique can be directly applied to the radiation of CR wakefield generated behind the moving charge in the form of a slow waveguide mode. Moreover, presented rigorous solution can be potentially extended to a beam-driven case (similar to how it has been done for “embedded” structures [14]).

Despite the aforementioned practical importance, the present paper also contributes to the development of rigorous diffraction theory since it deals with a canonical problem, i.e. relatively simple geometric structure (so called “canonical structure”) excited by simple free or guided wave. A series of related problems connected with an open end discontinuity [15–33] or a cross-section discontinuity [34, 35] in waveguides and resonators can be mentioned. However, the diffraction problem with a canonical structure discussed in this paper has not been investigated rigorously up to now.

II. PROBLEM FORMULATION AND GENERAL SOLUTION

We consider an open-ended semi-infinite cylindrical waveguide with radius \(d\) lined with a dielectric \(\varepsilon > 1\) of thickness \(a - b\) and having a layer of thickness \(d - a\) made of dielectric \(\varepsilon > 1\) near the waveguide wall, see Fig. 1 (cylindrical frame \(\rho, \varphi, z\) is used). Both the region outside the waveguide (\(z > 0\) and \(z < 0\), \(\rho > d\) and the inner channel (\(z < 0\), \(\rho < b\)) are filled with vacuum. Waveguide walls have an ideal electric conductivity (PEC). The method used for solution is the same as in [2, 13].

A \(\varphi\)-symmetric TM problem is considered in the harmonic regime with time dependence in the form

\[
H_\varphi(\rho, z, t) = H_{\omega \varphi}(\rho, z) \exp(-i\omega t).
\]

This problem is formulated for the magnitude \(H_{\omega \varphi}\), the nonzero field components can be derived as follows:

\[
E_{\omega \varphi} = \frac{1}{ik_0 \varepsilon} \frac{\partial H_{\omega \varphi}}{\partial z}, \quad E_{\omega z} = \frac{i}{k_0 \varepsilon} \left( \frac{H_{\omega \varphi}}{\rho} + \frac{\partial H_{\omega \varphi}}{\partial \rho} \right),
\]

where \(\bar{\varepsilon} = 1, \varepsilon\) or \(\varepsilon\) depending on the region. In particular, we have \(E_{\omega z} = 0\) for \(\rho = d, z < 0\).

We suppose that single symmetrical \(TM_{0l}\) waveguide mode is incident on the orthogonal open end:

\[
H_{\omega \varphi}^{(i)} = M^{(i)} e^{ik_{zl}z} \begin{cases}
J_1(\rho \sigma_l) / \sigma_l & \text{for } \rho < b, \\
J_1(\rho \sigma_l) / \psi_l(\rho) & \text{for } b < \rho < a, \\
J_1(\rho \sigma_l) / \sigma_l & \text{for } a < \rho < d,
\end{cases}
(3)

\[
\sigma_l = \psi_l(b) \psi_l(a) \begin{cases}
J_1(\rho \sigma_l) / \psi_l(\rho) & \text{for } b < \rho < a, \\
J_1(\rho \sigma_l) / \psi_l(b) & \text{for } a < \rho < d,
\end{cases}
\]

\[
M^{(i)} = \begin{cases}
J_1(\rho \sigma_l) / \sigma_l & \text{for } \rho < b, \\
J_1(\rho \sigma_l) / \psi_l(\rho) & \text{for } b < \rho < a, \\
J_1(\rho \sigma_l) / \sigma_l & \text{for } a < \rho < d,
\end{cases}
\]
where $M^{(1)}$ is an arbitrary amplitude constant, 
\[
\psi_{0m}(\xi) = J_1(\xi \chi_m) Y_0(d \chi_m) - N_1(\xi \chi_m) J_0(d \chi_m),
\]
\[
\psi_m(\xi) = J_1(\xi s_m) \varphi_{2m} - N_1(\xi s_m) \varphi_{1m},
\]
\[
\varphi_{1m} = \sigma_m J_0(b \sigma_m) J_1(b s_m) - s_m e^{-1} J_1(b \sigma_m) J_0(b s_m),
\]
\[
\varphi_{2m} = \sigma_m J_0(b \sigma_m) Y_1(b s_m) - s_m e^{-1} J_1(b \sigma_m) Y_0(b s_m).
\]

$J_\nu$ and $Y_\nu$ are Bessel and Neumann functions of $\nu$-th order, correspondingly, transverse wave numbers $\sigma_m$, $s_m$ and $\chi_m$ are determined by the following dispersion equation
\[
\det \hat{M} = 0,
\]
\[
\hat{M} = \begin{pmatrix}
J_1(s_m) J_0(s_m) & Y_1(s_m) & 0 & 0 \\
\sigma_m J_0(s_m) & -J_1(s_m) & -Y_1(s_m) & 0 \\
0 & 0 & J_0(d \chi_m) & Y_0(d \chi_m)
\end{pmatrix},
\]
longitudinal wave number $k_z$ is connected with $\sigma_m$, $s_m$ and $\chi_m$ as follows:
\[
k_{zm} = \sqrt{k_0^2 - \sigma_m^2} = \sqrt{k_0^2 c^2 - s_m^2} = \sqrt{k_0^2 c^2 - \chi_m^2}, \quad \text{Im} k_{zm} > 0,
\]

$k_0 = \omega/c + i \delta$ ($\delta \rightarrow 0$, which is equivalent to infinitely small dissipation in all areas), $c$ is the light speed in vacuum. From (7) one can express both $s_m$ and $\chi_m$ through $\sigma_m$ and obtain the dispersion relation with respect to a single variable $\sigma_m$. Note that Eq. (3) transforms to the corresponding mode of a two-layer waveguide [2] for $\epsilon \rightarrow \delta$.

The reflected field in the area inside the waveguide ($z < 0, \rho < a$) is decomposed into a series of TM modes propagating in the opposite direction:
\[
H_{\omega \varphi}^{(r)} = \sum_{m=0}^{\infty} M_m e^{-ik_{zm}z},
\]
where \{$M_m$\} are unknown “reflection coefficients” that should be determined. Note that since both the structure and the excitation (3) are uniform in $\varphi$ then only symmetric TM modes are taken into account in the reflected field.

The area outside the waveguide is divided into three subareas “1”, “2”, “3” and “4” (see Fig. 1), where the field is described by Helmholtz equation:
\[
\left[ \frac{\partial^2}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} + \left( k_0^2 - \frac{1}{\rho^2} \right) \right] H_{\omega \varphi}^{(1,2,3,4)} = 0.
\]

We introduce functions $\Psi_{\pm} (\rho, \alpha)$ (hereafter subscripts mean that function is holomorphic and free of poles and zeros in areas $\text{Im} \alpha > -\delta$ and $\text{Im} \alpha < \delta$, correspondingly):
\[
\Psi_{+}^{(1,2,3,4)} (\rho, \alpha) = (2\pi)^{-1} \int_{-\infty}^{\infty} dz H_{\omega \varphi}^{(1,2,3,4)} (\rho, z) e^{i\alpha z},
\]
\[
\Psi_{-}^{(1,2,3,4)} (\rho, \alpha) = (2\pi)^{-1} \int_{0}^{\infty} dz H_{\omega \varphi}^{(1,2,3,4)} (\rho, z) e^{i\alpha z},
\]
and similar transforms of $E_{\omega \varphi}^{(1,2,3,4)}$, for example,
\[
\phi_{+}^{(1,2,3,4)} (\rho, \alpha) = (2\pi)^{-1} \int_{-\infty}^{\infty} dz \frac{k_0}{i} E_{\omega \varphi}^{(1,2,3,4)} (\rho, z) e^{i\alpha z}.
\]

From (12) and (2) we have the following relation between $\Phi$ and $\Psi$:
\[
\phi_{+}^{(1,2,3,4)} (\rho, \alpha) = \frac{\psi_{+}^{(1,2,3,4)} (\rho, \alpha)}{\rho} + \frac{\partial \psi_{+}^{(1,2,3,4)} (\rho, \alpha)}{\partial \rho},
\]
and the same relation between $\Phi_{-}^{(1,2,3,4)} (\rho, \alpha)$ and $\Psi_{-}^{(1,2,3,4)} (\rho, \alpha)$.

From (9) we obtain
\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\kappa^2}{\rho^2} \right) \Psi_{+}^{(1,2,3)} = \left\{ \begin{array}{c} F^{(1,2,3)} \\ 0 \end{array} \right\},
\]
\[
2\pi F^{(1,2,3)} = \frac{\partial H_{\omega \varphi}^{(1,2,3)}}{\partial z} \bigg|_{z=0^+} - i \alpha \frac{H_{\omega \varphi}^{(1,2,3)}}{z=0^+},
\]
where
\[
\kappa = \sqrt{k_0^2 - \alpha^2}, \quad \text{Im} \kappa > 0.
\]
Equation (14) is obtained in a similar way as Eq. (9) in [13].

Functions $F^{(1,2,3)}$ are determined using continuity of $E_{zz}$ and $H_{zz}$ for $z = 0$, $\rho < d$ (see [13] for details), therefore

\begin{equation}
2\pi F^{(1)} = i \left[ (k_{zl} - \alpha) M^{(i)} J_1(\rho \sigma_1)/\sigma_1 - \sum_{m=1}^{\infty} (k_{zm} + \alpha) M_m J_1(\rho \sigma_m)/\sigma_m \right],
\end{equation}

\begin{equation}
2\pi F^{(2)} = i \left[ M^{(i)} \left( \frac{k_{zl}}{\varepsilon} - \alpha \right) \frac{J_1(\gamma \sigma_1)}{\psi_1(\rho)} + \frac{J_1(\beta \sigma_1)}{\psi_1(\rho)} \sigma_1 \psi_1(b) \right] - \sum_{m=1}^{\infty} M_m \left( \frac{k_{zm}}{\varepsilon} + \alpha \right) \frac{J_1(\gamma \sigma_m)}{\psi_m(\rho)} \sigma_m \psi_m(b),
\end{equation}

\begin{equation}
2\pi F^{(3)} = i \left[ M^{(i)} \left( \frac{k_{zl}}{\varepsilon} - \alpha \right) \frac{J_1(\gamma \sigma_1)}{\psi_1(\rho)} + \frac{J_1(\beta \sigma_1)}{\psi_1(\rho)} \sigma_1 \psi_1(b) \right] - \sum_{m=1}^{\infty} M_m \left( \frac{k_{zm}}{\varepsilon} + \alpha \right) \frac{J_1(\gamma \sigma_m)}{\psi_m(\rho)} \sigma_m \psi_m(b).
\end{equation}

General solution of Eq. (14) has the form (see [13] for details):

\begin{equation}
\Psi_+^{(1)}(\rho, \alpha) = C^{(1)} J_1(\rho \kappa) + \Psi_p^{(1)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Psi_+^{(2)}(\rho, \alpha) = C^{(2)} J_1(\rho \kappa) + C^{(2)} Y_1(\rho \kappa) + \Psi_p^{(2)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Psi_+^{(3)}(\rho, \alpha) = C^{(3)} J_1(\rho \kappa) + C^{(3)} Y_1(\rho \kappa) + \Psi_p^{(3)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Psi_+^{(4)}(\rho, \alpha) + \Psi_0^{(4)}(\rho, \alpha) = C^{(4)} H_1^{(1)}(\rho \kappa),
\end{equation}

where $H_1^{(1)}$ is a Hankel function of the first kind of $\nu$-th order, $C^{(1,4)}$ and $C^{(1,2,3)}$ are unknown coefficients.

Particular solutions of the inhomogeneous equations $\Psi_p^{(1,2,3)}$ have the form (see [13] for details)

\begin{equation}
\Psi_p^{(1)}(\rho, \alpha) = \frac{i M^{(i)} J_1(\rho \sigma_1)}{2\pi \sigma_1} \frac{k_{zl}}{k_{zl} - \alpha} - \sum_{m=1}^{\infty} \frac{i M_m J_1(\rho \sigma_m)}{2\pi \sigma_m} \frac{k_{zm} + \alpha}{k_{zm} - \alpha},
\end{equation}

\begin{equation}
\Psi_p^{(2)}(\rho, \alpha) = \frac{i M^{(i)} k_{zl} \alpha}{2\pi} \frac{k_{zl}}{\chi^2 - \sigma^2} \frac{\psi_1(\rho)}{\psi_1(b)} - \sum_{m=1}^{\infty} \frac{i M_m k_{zm} \alpha}{2\pi} \frac{\psi_m(\rho)}{\psi_m(b)},
\end{equation}

\begin{equation}
\Psi_p^{(3)}(\rho, \alpha) = \frac{i M^{(i)} k_{zl} \alpha}{2\pi} \frac{k_{zl}}{\chi^2 - \sigma^2} \frac{\psi_1(\rho)}{\psi_1(b)} - \sum_{m=1}^{\infty} \frac{i M_m k_{zm} \alpha}{2\pi} \frac{\psi_m(\rho)}{\psi_m(b)}.
\end{equation}

From Eq. (13) one obtains corresponding form of general solution for $\Phi$:

\begin{equation}
\Phi_+^{(1)}(\rho, \alpha) = i k_0^{(-1)} C^{(1)} \kappa J_0(\rho \kappa) + \Phi_p^{(1)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Phi_+^{(2)}(\rho, \alpha) = i k_0^{(-1)} C^{(2)} \kappa J_0(\rho \kappa) + \Phi_p^{(2)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Phi_+^{(3)}(\rho, \alpha) = i k_0^{(-1)} C^{(3)} \kappa J_0(\rho \kappa) + \Phi_p^{(3)}(\rho, \alpha),
\end{equation}

\begin{equation}
\Phi_+^{(4)}(\rho, \alpha) + \Phi_0^{(4)}(\rho, \alpha) = i k_0^{(-1)} C^{(4)} \kappa H_1^{(1)}(\rho \kappa),
\end{equation}

where

\begin{equation}
\Phi_p^{(1)}(\rho, \alpha) = \frac{-M^{(i)} J_0(\rho \sigma_1)}{2\pi k_0} \frac{k_{zl}}{k_{zl} + \alpha} + \sum_{m=1}^{\infty} \frac{M_m J_0(\rho \sigma_m)}{2\pi k_0} \frac{k_{zm} - \alpha}{k_{zm} - \alpha},
\end{equation}

\begin{equation}
\Phi_p^{(2)}(\rho, \alpha) = \frac{-M^{(i)} k_{zl} \alpha}{2\pi k_0} \frac{k_{zl}}{\chi^2 - \sigma^2} \frac{\psi_1(\rho)}{\psi_1(b)} - \sum_{m=1}^{\infty} \frac{M_m k_{zm} \alpha}{2\pi k_0} \frac{\psi_1(\rho)}{\psi_1(b)},
\end{equation}

\begin{equation}
\Phi_p^{(3)}(\rho, \alpha) = \frac{-M^{(i)} k_{zl} \alpha}{2\pi k_0} \frac{k_{zl}}{\chi^2 - \sigma^2} \frac{\psi_1(\rho)}{\psi_1(b)} - \sum_{m=1}^{\infty} \frac{M_m k_{zm} \alpha}{2\pi k_0} \frac{\psi_1(\rho)}{\psi_1(b)}.
\end{equation}

\begin{equation}
\phi_{m\ell}(\rho) = J_0(\rho \chi_m) Y_0(\rho \chi_m) - N_0(\rho \chi_m) J_0(\rho \chi_m),
\end{equation}

and $\Phi_p^{(3)}(\rho, \alpha) = 0$ since $\phi_{m\ell}(d) = 0$.

Boundary condition $E_{zz} = 0$ for $\rho = d$, $z < 0$ results in $\Phi_+^{(4)}(d, \alpha) = 0$, and we obtain from (30):

\begin{equation}
C^{(4)} = \frac{k_0 \Phi_0^{(4)}(d, \alpha)}{i \kappa H_1^{(1)}(\kappa d)},
\end{equation}

therefore from Eq. (23)

\begin{equation}
\Psi_+^{(4)}(d, \alpha) + \psi_0^{(4)}(d, \alpha) = \frac{k_0 \Phi_0^{(4)}(d, \alpha)}{i \kappa H_1^{(1)}(\kappa d)},
\end{equation}

To obtain Wiener-Hopf-Fock equation one should express the term $\Psi_0^{(4)}(d, \alpha)$ in Eq. (36) through the $\Phi_+^{(4)}(d, \alpha)$. This can be done using the continuity con-
.conditions for \( \rho = b, a, d, z > 0 \) (see [2] for details):

\[
\Psi_+^{(4)}(d, \alpha) = \frac{k_0 \Phi_+^{(4)}(d, \alpha) J_1(dk_0)}{i \kappa J_0(dk_0)} + \Psi_+^{(3)}(d, \alpha)
\]

\[
- \frac{k_0}{i \kappa d J_0(dk_0)} \left[ b J_1(b \zeta) \left( \Phi_p^{(1)}(b, \alpha) - \Phi_p^{(2)}(b, \alpha) \right)

+ a J_1(a \zeta) \left( \Phi_p^{(2)}(a, \alpha) - \Phi_p^{(1)}(a, \alpha) \right)

- i b \kappa J_0(b \zeta) \left( \Phi_p^{(1)}(b, \alpha) - \Phi_p^{(2)}(b, \alpha) \right)

- i a \kappa J_0(a \zeta) \left( \Phi_p^{(2)}(a, \alpha) - \Phi_p^{(3)}(a, \alpha) \right) \right].
\]

(37)

Similar to [2, 13], the right-hand side of (37) formally possesses pole singularity for \( \alpha = \alpha_m \):

\[
\alpha_m = \sqrt{k_0^2 - j_{2m}^2 d^2}, \quad \text{Im} \alpha_m > 0,
\]

where \( j_{2m} \) is the \( m \)-th zero of Bessel function \( J_0 \) (\( \alpha_m \) are longitudinal wavenumbers of vacuum waveguide of radius \( d \)). However, the function determined by Eq. (37) should be regular in the area \( \text{Im} \alpha > -\delta \). Therefore, this pole singularity at the right-hand side should be eliminated and we obtain the following requirement:

\[
\Phi_+^{(4)}(d, \alpha) = \frac{1}{d} \left[ b J_1(b \zeta) \frac{d}{dk_0} \right] J_1(a \zeta) \frac{d}{dk_0} \left[ \Phi_p^{(1)}(a, \alpha) - \Phi_p^{(2)}(a, \alpha) \right]

\]

(39)

(40)

(41)

(42)

(43)

Substituting Eq. (37) into Eq. (36) and combining the terms proportional to \( \Phi_p^{(3)}(d, \alpha) \) we obtain the following Wiener-Hopf-Fock equation:

\[
2k_0 \Phi_+^{(4)}(d, \alpha) + \Psi_+^{(4)}(d, \alpha) + \frac{1}{d J_0(dk_0)} \frac{i}{2 \pi} \Pi(\alpha) = 0,
\]

(44)

where

\[
\Psi_+^{(4)}(d, \alpha) = \frac{k_0 \Phi_+^{(4)}(d, \alpha) J_1(dk_0)}{i \kappa J_0(dk_0)} + \Psi_+^{(3)}(d, \alpha)
\]

\[
\Phi_+^{(4)}(d, \alpha) \frac{d}{dk_0} \left[ \Phi_p^{(1)}(a, \alpha) - \Phi_p^{(2)}(a, \alpha) \right] \frac{d}{dk_0} \left[ \Phi_p^{(2)}(b, \alpha) - \Phi_p^{(3)}(b, \alpha) \right]
\]

\[
\Psi_+^{(3)}(d, \alpha) = \Psi_+^{(4)}(d, \alpha) + \frac{1}{d J_0(dk_0)} \frac{i}{2 \pi} \Pi(\alpha)
\]

\[
G(\alpha) = \pi a \kappa J_0(a \zeta) H_0^{(1)}(a \zeta).
\]

(45)

(46)

\[
\Pi(\alpha) = b M^i(\zeta_m(\alpha))\frac{1}{k_{z1} + \alpha} - \sum_{m=0}^{\infty} b M_{m\zeta_m(\alpha)}\frac{1}{k_{zm} - \alpha}
\]

(47)

\[
\eta_{1m}(\alpha) = b \phi_{2m}\mu_{1m}(\alpha) - a \phi_{2m}\mu_{1m}(\alpha)
\]

(48)

\[
\Phi_+^{(4)}(d, \alpha) = \frac{1}{4 \pi i k_0 d} \kappa_+(\alpha) G_+(\alpha)
\]

(49)

\[
W_{pm} M_m = M^{(1)} w_p, \quad p = 1, 2, \ldots,
\]

(50)

\[
H^{(1,2,3)}_{w_p}(\rho, z > 0) = \int_{-\infty}^{+\infty} \Phi_+^{(1,2,3)}(\rho, \alpha) e^{-i \alpha z} d\alpha,
\]

(51)

III. EM FIELD DERIVATION

When the set of coefficients \( \{ M_m \} \) is determined, the electromagnetic field can be calculated via the inverse transform over \( \alpha \), in accordance with Eqs. (10) and (11):

\[
H^{(1,2,3)}_{w_p}(\rho, z > 0) = \int_{-\infty}^{+\infty} \Phi_+^{(1,2,3)}(\rho, \alpha) e^{-i \alpha z} d\alpha,
\]

(52)

\[
H^{(4)}_{w_p}(\rho, z) = \int_{-\infty}^{+\infty} \Phi_+^{(4)}(\rho, \alpha) e^{-i \alpha z} d\alpha.
\]
performing calculations similar to those done in [2], one obtains that the field in the region $z > 0$ is described by the unified formula:

$$H_{ω}^{(1,2,3)}(ρ, z > 0) = \sum_{q=1}^{∞} \Pi(-α_q) G_{+}(α_q) j_{0q} L_{q}^{+}(ρ, z)$$

where $L_{q}^{+}$ is given by Eq. (47) in [13].

For the domain “4” one can also obtain the representation convenient for investigation of the far-field:

$$H_{ω}^{(4)}(ρ, z) = \frac{1}{α} \sum_{q=1}^{∞} \Pi(-α_q) G_{+}(α_q) j_{0q} I_{q}^{(4)}$$

where integral

$$I_{q}^{(4)}(ρ, z) = \int_{−∞}^{∞} \frac{α_0(α)}{α} H_{1}^{(1)}(ρ, α) e^{-iαz} dα$$

has been investigated previously (see Eq. (41) in [13]). In particular, integral (54) can be easily calculated asymptotically (see Eq. (51) in [13]) and substituted into (53) to obtain far-field in the region “4”.

**IV. CONCLUSION**

We have presented convenient rigorous analytical approach for calculation of various diffraction processes at the open end (with orthogonal cut) of a circular waveguide with three-layer dielectric lining. This approach can be effectively used for investigation of radiation of Cherenkov wakefield from the open-ended capillary (see Ref. [1]) which is a promising structure for realization of beam-driven THz source with small divergence and high efficiency. Moreover, this approach can be extended to more complicated problems. For example, excitation by a charged particle bunch (in full formulation including both wakefield radiation and transition radiation) can be incorporated into the solution.

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[1] S. Jiang, W. Li, Z. He, Q. Jia, and L. Wang, Opt. Lett. 45, 5416 (2020).
[2] S. N. Galyamin and V. V. Vorobev, IEEE Transactions on Microwave Theory and Techniques, 1 (2022).
[3] E. A. Nanni, W. R. Huang, K. H. Hong, K. Ravi, A. Falahi, G. Moriena, R. J. Dwayne Miller, and F. X. Kärtner, Nature Communications 6, 8486 (2015).
[4] B. D. O’Shea, G. Andonian, S. Barber, K. Fitzmorris, S. Hakimi, J. Harrison, P. D. Hoang, M. J. Hogan, B. Naranjo, O. B. Williams, V. Yakimenko, and J. Rosenzweig, Nature Communications 7, 12763 (2016).
[5] D. Wang, X. Su, L. Yan, Y. Du, Q. Tian, Y. Liang, L. Niu, W. Huang, W. Gai, C. Tang, and S. Antipov, Applied Physics Letters 111, 174102 (2017), https://doi.org/10.1063/1.4999959.
[6] C. Jing, S. Antipov, M. Conde, W. Gai, G. Ha, W. Liu, N. Neveu, J. Power, J. Qiu, J. Shi, D. Wang, and E. Wiesniewski, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 898, 72 (2018).
[7] M. T. Hibberd, A. L. Healy, D. S. Lake, V. Georgiadi, E. J. H. Smith, O. J. Finlay, T. H. Pacey, J. K. Jones, Y. Saveliev, D. A. Walsh, E. W. Snedden, R. B. Appleby, G. Burt, D. M. Graham, and S. P. Jamison, Nature Photonics (2020), 10.1038/s41566-020-0674-1.
[8] H. Tang, L. Zhao, P. Zhu, X. Zou, J. Qi, Y. Cheng, J. Qiu, X. Hu, W. Song, D. Xiang, and J. Zhang, Phys. Rev. Lett. 127, 074801 (2021).
[9] S. N. Galyamin, A. V. Tyukhtin, S. Antipov, and S. S. Baturin, Opt. Express 22, 8902 (2014).
[10] M. Ivanyan, A. Grigoryan, A. Tsakanian, and V. Tsakanov, Phys. Rev. ST Accel. Beams 17, 074701 (2014).
[11] D. Wang, X. Su, Y. Du, Q. Tian, Y. Liang, L. Niu, W. Huang, W. Gai, L. Yan, C. Tang, and S. Antipov, Review of Scientific Instruments 89, 093301 (2018), https://doi.org/10.1063/1.5042006.
[12] L. Zhao, H. Tang, C. Lu, T. Jiang, P. Zhu, L. Hu, W. Song, H. Wang, J. Qiu, C. Jing, S. Antipov, D. Xiang, and J. Zhang, Phys. Rev. Lett. 124, 054802 (2020).
[13] S. N. Galyamin, V. V. Vorobev, and A. V. Tyukhtin, IEEE Transactions on Microwave Theory and Techniques 69, 2429 (2021).
[14] S. N. Galyamin, A. V. Tyukhtin, V. V. Vorobev, A. A. Grigorev, and A. S. Aryshev, Phys. Rev. Accel. Beams 22, 012801 (2019).
[15] L. Weinstein, *The Theory of Diffraction and the Factorization Method: generalized Wiener-Hopf Technique*, Golem Series in Electromagnetics, V. 3 (Golem Press, 1969).
[16] R. Mittra and S. Lee, *Analytical Techniques in the Theory of Guided Waves* (Macmillian, 1971).
[17] W. E. Williams and M. J. Lighthill, *Mathematical Proceedings of the Cambridge Philosophical Society* 52, 322–335 (1956).
[18] G. Voskresenskii and S. Zhurav, *Radiotechnika i Elektronika* 23, 2505 (1978).
[19] T. Johnson and D. Moffatt, *Electromagnetic Scattering by Open Circular Waveguides*, Technical Report 710816-9 (The Ohio State University, 1980).
[20] K. Kobayashi, in *Antennas and Propagation Society Symposium 1991 Digest* (1991) pp. 1054–1057 vol.2.
[21] K. Kobayashi and A. Sawai, Journal of Electromagnetic Waves and Applications 6, 475 (1992).
[22] S. Koshikawa and K. Kobayashi, IEEE Transactions on Antennas and Propagation 45, 949 (1997).
[23] S. Gupta, A. Bhattacharya, and A. Chakraborty, IEE Proceedings - Microwaves, Antennas and Propagation 144, 126 (1997).
[24] D. Kuryliak, S. Koshikawa, K. Kobayashi, and Z. Nazarchuk, in Conference Proceedings 2000 International Conference on Mathematical Methods in Electromagnetic Theory (Cat. No.00EX413), Vol. 2 (2000) pp. 694–696 vol.2.
[25] D. B. Kuryliak, K. Kobayashi, S. Koshikawa, and Z. T. Nazarchuk, in 10th International Conference on Mathematical Methods in Electromagnetic Theory, 2004. (2004) pp. 251–253.
[26] Y. Hameç and I. H. Tayyar, Progress In Electromagnetics Research 44, 143 (2004).
[27] Y. Hameç and I. H. Tayyar, Electromagnetics 25, 245 (2005), https://doi.org/10.1080/02726340590915638.
[28] R. Cicchetti and A. Faraone, Progress In Electromagnetics Research 78, 285 (2008).
[29] S. N. Galyamin, A. V. Tyukhtin, and V. V. Vorobev, Nuclear Instruments and Methods in Physics Research Section B: Beam Interactions with Materials and Atoms 402, 144 (2017).
[30] A. Buyukaksoy, I. H. Tayyar, F. Hacivelioğlu, and G. Uzgoren, in 2007 International Conference on Electromagnetics in Advanced Applications (2007) pp. 649–652.
[31] A. Buyukaksoy, F. Hacivelioğlu, and G. Uzgoren, in 2008 12th International Conference on Mathematical Methods in Electromagnetic Theory (2008) pp. 85–91.
[32] F. Hacivelioğlu, A. Buyukaksoy, G. Uzgoren, and H. Serbest, in 2009 Mediterranean Microwave Symposium (MMS) (2009) pp. 1–4.
[33] I. H. Tayyar and A. Buyukaksoy, in 2011 International Conference on Electromagnetics in Advanced Applications (2011) pp. 580–583.
[34] K. Zaki and A. Atia, IEEE Transactions on Microwave Theory and Techniques 31, 1039 (1983).
[35] K. Zaki, C. Seng-Woon, and C. Chunming, IEEE Transactions on Microwave Theory and Techniques 36, 1804 (1988).