Small Oscillations of a Vortex Ring: Hamiltonian Formalism and Quantization

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Abstract

This article investigates small oscillations of a vortex ring with zero thickness that evolves under the Local Induction Equation (LIE). We deduce the differential equation that describes the dynamics of these oscillations. We suggest the new approach to the Hamiltonian description of this dynamic system. This approach is based on the extension of the set of dynamical variables by adding the circulation $\Gamma$ as a dynamical variable. The constructed theory is invariant under the transformations of the Galilei group. The appearance of this group allows for a new viewpoint on the energy of a vortex filament with zero thickness. We quantize this dynamical system and calculate the spectrum of the energy and acceptable circulation values. The physical states of the theory are constructed with help of coherent states for the Heisenberg-Weyl group.

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1 Introduction

The study of various vortex structures has a long history. In spite of this fact, dynamics of such objects continue to attract interest[1]. Without trying to make a review of the literature on this topic, we will mention only some works that are directly relevant to our research. So, in the work [2] the vortex filament (in the LIE approximation) was described firstly in terms of solutions of non-linear Schrödinger equation. It should also be mentioned that gauge equivalence between the non-linear Schrödinger equation and the continuous Heisenberg spin chain exists[3]. It is well-known that quantization of similar non-linear systems is a complicated problem (see, for example,
As a consequence, the connection with the initial hydrodynamical system (the vortex ring in our case) maybe seems not obvious. That is why the investigations of the small perturbations of certain initial and stable configurations are interesting, both classical and quantum. Let us note here the work [5], where, in particular, the system of small-perturbed straightforward vortex filaments with certain interactions were investigated. In the work [6], the quantization of such filaments was considered. It is also necessary to mention the direction of the study of quantum vortices in superfluid helium [7, 8, 9]. Since our research lies in a different plane, we will not dwell on this in detail.

In this work, we consider the closed evolving curve \( r(\tau, \xi) \) that is defined by the formula

\[
 r(\tau, \xi) = q + R_0 \int_0^{2\pi} [\xi - \eta] j(\tau, \eta) d\eta. 
\]

Both parameter \( \xi \) that parametrizes the curve \( r(\cdot, \xi) \) and evolution parameter \( \tau \) are dimensionless parameters. The constant \( R_0 \) defines the scale of length. The notation \( [x] \) means the integer part of the number \( x/2\pi \), the variables \( q = q(\tau) \) may be \( \tau \)-depended (conditionally, these are the coordinates of the ”mass center”). \( 2\pi \)-periodical vector function \( j(\xi) \in E_3 \) defines unit tangent vector. We postulate that function \( r(\tau, \xi) \) satisfies the LIE equation

\[
 \partial_\tau r(\tau, \xi) = \frac{1}{R_0} \partial_\xi r(\tau, \xi) \times \partial^2_\xi r(\tau, \xi). 
\]

Consequently, the function \( j(\xi) \) satisfies the equation for the continuous Heisenberg spin chain

\[
 \partial_\tau j(\tau, \xi) = j(\tau, \xi) \times \partial^2_\xi j(\tau, \xi). 
\]

The following equalities are fulfilled too:

\[
 \int_0^{2\pi} j_k(\xi) d\xi = 0 \quad (k = x, y, z), 
\]

The original space-time symmetry group for this system is the group \( E(3) \times E_\tau \), where the group \( E(3) \) is the group of motions of space \( E_3 \) and \( E_\tau \) is the group of ”translations” \( \tau \rightarrow \tau + c \).
The standard simplest configuration here will be the vortex ring with radius $R_0$ that moves parallel to $z$-axis with some constant velocity:

$$r_0(\tau, \xi) = q_0 + R_0 \int_0^{2\pi} [\xi - \eta] j_0(\tau, \eta) d\eta. \quad (5)$$

Tangent vector $j_0$ has the following coordinates:

$$j_0(\xi) = \{-\sin \xi, \cos \xi, 0\}. \quad (6)$$

The quantities $q_{0i} = const$ for the indexes $i = x, y$ here; the quantity $q_{0z}$ is some linear function on the variable $\tau$ that will be specified latter.

We consider the small perturbation ($\varepsilon << 1$) of the tangent vector (6) and coordinates $q_0$:

$$q = q_0 + \varepsilon q_{prt}, \quad j(\tau, \xi) = j_0(\xi) + \varepsilon j_{prt}(\tau, \xi). \quad (7)$$

Therefore, we have representation

$$r(\tau, \xi) = r_0(\tau, \xi) + \varepsilon r_{prt}(\tau, \xi).$$

Let us substitute the representation (7) for the vector $j(\xi)$ into equation (3). Taking into account the equality $\partial^2 j_0(\xi) = -j_0(\xi)$ and neglecting the terms of order $\varepsilon^2$, we deduce the equation

$$\partial_\tau j_{prt}(\tau, \xi) = j_0(\tau, \xi) \times [j_{prt}(\tau, \xi) + \partial_\xi^2 j_{prt}(\tau, \xi)]. \quad (8)$$

Thus, we have the linear equation that describes the dynamics of small perturbations for the vortex ring (5). Next, we will not write the "prt" index explicitly, hoping that this will not lead to misunderstandings. It must be emphasized that equalities (4) must be fulfilled strongly for all configurations (perturbed or not perturbed).

The symmetry of the initial configuration $j_0(\tau, \xi)$ makes natural using the cylindrical coordinates $\{\rho, \phi, z\}$. The $z$-axis of such a system coincides with the axis of the unperturbed vortex ring. Let three vectors $\{e_\rho, e_\phi, e_z\}$ denote the local basis of the cylindrical system. Obviously, in special case when $j \equiv j_0$, the parameter $\xi = \phi$. Therefore,

$$j_0 = e_\phi.$$
Because we consider the small perturbations of the function $\mathbf{j}_0$ only, we assume $\xi = \phi$ for all configurations.

Thus, the equation (3) takes the following form in the cylindrical basis:

$$\partial_\tau \mathbf{j}(\tau, \xi) = \left( j_z(\tau, \xi) + \partial_\xi^2 j_z(\tau, \xi) \right) \mathbf{e}_\rho - \left( \partial_\xi^2 j_\rho(\tau, \xi) - 2 \partial_\xi j_\phi(\tau, \xi) \right) \mathbf{e}_z. \quad (9)$$

This equation demonstrates that $\partial_\tau j_\phi(\tau, \xi) \equiv 0$. For example, the initial data $j_\phi(0, \xi) \equiv j_\phi^0$, where $\partial_\xi j_\phi^0 \equiv 0$, lead to the equality

$$j_\phi(\tau, \xi) \equiv j_\phi^0, \quad j_\phi^0 = \text{const}.$$ 

Obviously, the perturbation $\mathbf{j}_0(\xi) \to \mathbf{j}_0(\xi) + \varepsilon \mathbf{j}_\phi^0 \mathbf{e}_\phi$ conserves the form of the original vortex $\mathbf{r}_0$. In our subsequent considerations we will consider the case $j_\phi^0 = \text{const}$ only. Therefore, the perturbation amplitude $\mathbf{j}(\tau, \xi)$ of the vector $\mathbf{j}_0(\tau, \xi)$ has following form in the cylindrical basis:

$$\mathbf{j}(\tau, \xi) = j_\rho(\tau, \xi) \mathbf{e}_\rho + j_\phi(\tau, \xi) \mathbf{e}_\phi + j_z(\tau, \xi) \mathbf{e}_z. \quad (10)$$

The following identity takes place:

$$\mathbf{j}(\tau, \xi) \mathbf{j}_0(\tau, \xi) \equiv j_\phi^0. \quad (11)$$

The notation $\mathbf{j} \mathbf{j}_0$ means an inner product of two vectors. In the future, it is more convenient for us to consider the non-trivial components of the vector $\mathbf{j}$ in a complex-valued form. We introduce the notation $\mathbf{j}$ for the complex perturbation amplitude to avoid any ambiguity. Thus,

$$\mathbf{j} = j_\rho + ij_z.$$

Equation (9) writing in complex form as follows:

$$\partial_\tau \mathbf{j} = -i \partial_\xi^2 \mathbf{j} - \frac{i}{2} \left( \mathbf{j} - \overline{\mathbf{j}} \right). \quad (12)$$

This simple equation can be solved explicitly:

$$\mathbf{j}(\tau, \xi) = \sum_n j_n e^{i[n\xi + n\sqrt{\mu^2 - 1}\tau]}, \quad (13)$$

where $j_n \equiv \text{const}$ and the coefficients $\overline{\mathbf{j}}_{-n}$ and $j_n$ are related to each other as follows:
\[ J_{-n} = 2 \left[ n\sqrt{n^2 - 1} - n^2 + \frac{1}{2} \right] j_n. \]  

It is clear that this solution is stable for \( \tau \to \infty \). Note that the stability of the real vortex rings is a non-trivial problem in general. For example, this problem was investigated in the work [10] for the Norbury rings.

Restrictions (4) are rewriting in terms of the cylindrical coordinates as follow:

\[ \int_0^{2\pi} j_\rho(\xi)e^{\pm i\xi}d\xi = 0, \quad \int_0^{2\pi} j_z(\xi)d\xi = 0. \]  

As regards to the coefficients \( j_0, j_{\pm 1} \) in the solution (13), constraints (15) lead to the formulas

\[ j_0 = \frac{1}{2\pi} \int_0^{2\pi} (j_\rho + ij_z) d\xi = \frac{1}{2\pi} \int_0^{2\pi} j_\rho d\xi, \]  

\[ j_{\pm 1} = \frac{1}{2\pi} \int_0^{2\pi} (j_\rho + ij_z) e^{\mp i\xi}d\xi = \frac{1}{2\pi} \int_0^{2\pi} j_z(\pm \sin \xi + i \cos \xi) d\xi. \]

So, last equalities lead to the restrictions for the coefficients \( j_{\pm 1} \):

\[ j_1 = -j_{-1}. \]  

As regards the coefficient \( j_0 \), the equalities (16) lead to the restriction \( j_0 = J_0 \). Although these restrictions are deduced from the constraint (4), they are consistent with the formula (14).

2 Dynamical invariants and extended set of the variables

The consideration of vortex structure in terms of the equation (2) only is apparently too formal. As an addition to the LIE equation, we postulate in our theory the canonical formulas for the momentum and angular momenta, that was deduced in the fluid dynamics [11]:

\[ \tilde{p} = \frac{1}{2} \int \mathbf{r} \times \mathbf{\omega}(\mathbf{r})dV, \quad \tilde{s} = \frac{1}{3} \int \mathbf{r} \times (\mathbf{r} \times \mathbf{\omega}(\mathbf{r}))dV. \]  

The vector \( \mathbf{\omega}(\mathbf{r}) \) means the vorticity. The fluid density \( \rho \equiv 1 \) here.
As well-known, the vorticity of the closed vortex filament calculates by means of the formula

$$\omega(r) = \Gamma \int_0^{2\pi} \hat{\delta}(r - r(\xi)) \partial_\xi r(\xi) d\xi,$$  \hspace{1cm} (20)

where the symbol $\Gamma$ denotes the circulation and the symbol $\hat{\delta}(\xi)$ means $2\pi$-periodical $3D$ $\delta$-function.

Taking into account the formulae (1), (4) and (20), the following expression for the canonical momentum is deduced:

$$\tilde{p} = R_0^2 \Gamma f, \hspace{1cm} f = \frac{1}{2} \int_{0}^{2\pi} \left( [\xi - \eta] j(\eta) \times j(\xi) d\xi d\eta \right).$$  \hspace{1cm} (21)

A similar formula can be written for the angular momenta $\tilde{s}$; however, we omit the relevant details here. As opposed to formulae for the values $\tilde{p}$ and $\tilde{s}$, the canonical formula for the energy $E$ gives the unsatisfactory result because of the divergence of the integral. We will return to this question later.

Let us substitute the expansion (7) into expression for the vector $f$ (see (21)). Taking into account identity (11), the following formula for the vector $f$ holds:

$$f = \pi (1 + 2 \varepsilon j_0^0) e_z - \varepsilon f_\perp, \hspace{1cm} f_\perp = \int_{0}^{2\pi} j_z(\xi) e_\phi d\xi.$$  \hspace{1cm} (22)

The quantity $f$ includes both unperturbed ($\pi e_z$) and perturbed ($2\pi \varepsilon j_0^0 e_z - \varepsilon f_\perp$) values. Correspondingly, the full momentum $\tilde{p}$ has following form:

$$\tilde{p} = -\varepsilon \tilde{p}_\perp + \tilde{p}_\parallel e_z,$$

where

$$\tilde{p}_\parallel = \pi (1 + 2 \varepsilon j_0^0) R_0^2 \Gamma.$$  \hspace{1cm} (23)

As for the amplitudes of perturbations for the momentum, we have the following formulas:

$$\tilde{p}_z = 2\pi R_0^2 \Gamma j_0^0 \hspace{1cm} (24)$$

$$\tilde{p}_\perp = R_0^2 \Gamma f_\perp.$$  \hspace{1cm} (25)
Next, we intend to construct a Hamiltonian dynamical system that corresponds to the equation (8). For the general case (2), the corresponding approach was proposed by the author in the work [12]. We will briefly recall here the main points of the proposed theory, making the necessary modifications along the way.

In our opinion, the following steps must be done to construct the physically interpreted dynamical system in our case:

1. As mentioned above, we need to supplement the equation (2) with formulas for the momentum \( \tilde{p} \) and the angular momenta \( \tilde{s} \);

2. The variable "velocity of liquid" is absent in our theory. We propose to take into account the dynamics of the surrounding fluid in a minimal way: to declare the value \( \Gamma \) as a dynamic variable, in addition to variables \( j(\xi) \) and \( q \). We denote as \( \mathcal{A} \) this (extended) set of the dynamical variables \( \{ q, j(\xi), \Gamma \} \) constrained by the conditions (4).

3. The theory must contain a sufficient number of dimensional constants. So far, we use the dimensionless "time" \( \tau \). Consequently, the additional dimensional constant \( t_0 \) that defines the scale of time, must be added to the theory in some way. Subsequently, we will express this constant in terms of other dimensional constants that have a clear physical meaning in our model. In addition to the \( R_0 \) and \( t_0 \) constants, the theory should contain a "mass constant" \( m_0 \). This constant will appear in the theory in a completely natural way later on.

As a subtotal, we have the following

**Proposition 1** The set \( \mathcal{A} \) parametrizes the considered dynamical system - the closed vortex filament \( r(\xi) \) that evolving in accordance with the LIE equation. This dynamical system has a momentum \( \tilde{p} \) and angular momenta \( \tilde{s} \) that calculated as prescribed above.

To perform the hamiltonization of our system, we are going to describe the set \( \mathcal{A} \) in terms of other variables. The reasons are following:

- we intend to expand the symmetry group \( E(3) \times E_\tau \) to Galilei group \( \mathcal{G}_3 \) and use the group-theoretical approach for the definition of the energy of our system;
new variables will be more suitable for subsequent quantization that will be fulfilled later in this article.

In addition, the new variables will give the obvious interpretation of the considered dynamical system as some structured particle. It is probably appropriate to mention here the pioneering work [13] in which the observed particles are modeled by vortex structures.

As a first step, we extend the set \( \mathcal{A} \). Let us denote as \( \mathcal{A}' \) the set of the independent variables \((q; \tilde{p}; j(\xi))\). The formula (21) makes the injection \( F: \mathcal{A} \rightarrow \mathcal{A}', \ \text{Ran} \ F \subset \mathcal{A}' \).

On the set \( \mathcal{A}' \) the action of the central extended Galilei group \( \tilde{\mathcal{G}}_3 \) is defined by natural way. We parametrize the elements \( g \in \tilde{\mathcal{G}}_3 \) as follows:

\[
g : (\mathcal{R}, v, a, c; m_0)
\]

where \( \mathcal{R} \in SO(3), \ v, a \in E_3, \ c \in \mathbb{R} \) and the central charge \( m_0 \in \mathbb{R} \). Traditionally, the last parameter is interpreted as "mass of the particle". Before determining the action of this group on the set \( \mathcal{A}' \), we introduce the factor \( m_0/R_0^3 \) in standard hydrodynamical formulas (19) to provide the dimension for the values \( \tilde{p} \) and \( \tilde{s} \) as in classical mechanics:

\[
\tilde{p} \rightarrow p = (m_0/R_0^3)\tilde{p}, \quad \tilde{s} \rightarrow s = (m_0/R_0^3)\tilde{s}.
\] (26)

Taking into account this redefinition, the group action \( \circ \) on the set \( \mathcal{A}' \) is defined as follows:

\[
g \circ (q; p; j(\xi)) = (\mathcal{R}q + vt_0\tau + a; \mathcal{R}p + m_0v; \mathcal{R}j(\xi)),
\]
and \( g \circ (t_0\tau) = t_0\tau + c \).

Next, we introduce the variables

\[
q_i(0) = q_i - \tau(t_0/m_0)p_i, \quad i = x, y, z,
\]
that will be convenient sometimes for using. The curve \( \mathbf{r}(\tau, \xi) \) is reconstructed through variables \((q(0); p; j(\xi))\) in accordance with the formula

\[
\mathbf{r}(\tau, \xi) = q(0) + \tau(t_0/m_0)p + R_0 \int_0^{2\pi} [\xi - \eta] j(\tau, \eta) d\eta.
\] (27)
As a second step, we must introduce certain constraints on the set $\mathcal{A}'$ that define set $\Omega \subset \mathcal{A}'$. Criteria for introducing these constraints - the one-to-one correspondence

$$\mathcal{A} \leftrightarrow \Omega.$$ 

It is clear that we must define two constraints, because we introduce three variables ($p$) instead one variable $\Gamma$.

First of all, we must require that the vectors $p_\perp$ and $f_\perp$ are proportional. In general, it is not true on set $\mathcal{A}'$, because these vectors are independent at this set. For convenience, let us introduce the complex values:

$$p = (p_\perp)_x + i(p_\perp)_y, \quad f = (f_\perp)_x + i(f_\perp)_y.$$ 

In accordance both the definition of the vector $f_\perp$ (see (22)) and the formulas (17) we have the equalities:

$$f = -\int_0^{2\pi} j_z \sin \xi d\xi + i \int_0^{2\pi} j_z \cos \xi d\xi = 2\pi j_{-1}.$$ 

Taking into account the replacement (26), formula (25) takes following form:

$$p = \frac{m_0 \Gamma}{R_0} f = \frac{2\pi m_0 \Gamma}{R_0} j_{-1}.$$ 

Therefore, by virtue of the formula (25) and the complex-valued notations for the corresponding values, we must demand:

$$\exists \lambda \in \mathbb{R} : \quad p = 2\pi \lambda p_0 j_{-1}.$$ 

The variables $p$ and $j_{-1}$ are the complex numbers here and the value $p_0 = m_0 R_0/t_0$. As we will show further, the constant $p_0$ has a clear physical meaning, so it seems natural to use it as \textit{input} constant, instead of a value $t_0$. However, we will use the constant $t_0$ to simplify some formulas.

The condition (30) can also be written in following form:

$$p_x(f_\perp)_y - p_y(f_\perp)_x = 0.$$ 

Of course, this equality (just like equality (30)) is fulfilled identically in the case when our theory is parametrized by the set $\mathcal{A}$. In complex-valued notations the constraint (31) is written as follows:
\[
\Phi_0 \equiv p \overline{j_{-1}} - p j_{-1} = 0.
\]  
(32)

If the quantities \( p \) and \( j_{-1} \) take non-zero values and the condition (32) is fulfilled, the variable \( \Gamma \) can be determined unambiguously through the formulas:

\[
\Gamma = \frac{\lambda R_0^2}{t_0}, \quad |p|^2 - 4\pi^2 p_0^2 \lambda^2 |j_{-1}|^2 = 0.
\]  
(33)

In corresponding zero points the value \( \Gamma = \Gamma_0 \) is still undetermined. Let us substitute the representation (27) for the original vortex \( r_0(\tau, \xi) \) in LIE equation (2). This procedure leads to the equality for the momentum unperturbed vortex ring:

\[
p_z = p_0 = m_0 R_0/t_0.
\]

To deduce this formula we suppose that \( \partial_\tau p = \partial_\tau q(0) = 0 \). These conditions will be coordinated with the subsequent hamiltonization of our dynamical system. In accordance with formula (23) and (26) we have for this case: \( p_0 = \pi m_0 \Gamma_0/R_0 \). Therefore, \( \Gamma_0 = R_0^2/\pi t_0 \).

Let us return to the perturbed case. In this case the value \( \Gamma \) in the formula (24) must be same value as was determined in formula (33). That is why we must write the second constraint:

\[
\Phi_1 \equiv |j_{-1}|^2 p_z^2 - (j_0^0)^2 |p|^2 = 0.
\]  
(34)

In special case when the value \( j_0^0 = 0 \), we have the constraint

\[
\Phi_1 \equiv p_z = 0
\]  
(35)

instead constraint (34). This means that set \( \Omega \) describes the planar system here. We will consider the case \( j_0^0 = 0 \) only. In a general case when \( j_0^0 \neq 0 \), the set \( \Omega \subset A' \) defines by constraints (32) and (34).

Finally, we have the following

**Proposition 2**  
The variables \( j(\xi), q, p \), that are declared as the new fundamental variables, parametrize uniquely considered dynamical system. These variables are constrained by the equalities (32) and (34).
3 Energy and Hamiltonian structure

The straightforward calculation of the energy of a vortex filament is usually performed using the canonical formula [14]

\[ E = \frac{1}{8\pi} \int \int \frac{\omega(r)\omega(r')}{|r - r'|} dV dV' = \frac{\Gamma^2}{8\pi} \int \int \frac{\partial \xi r(\xi)\partial \xi r(\xi')}{|r(\xi) - r(\xi')|} d\xi d\xi', \]

The result is unsatisfactory if the filament has zero thickness: the integral in this formula diverges. The standard approach to solve this problem is to take into account the finite thickness \(a\) of the filament and the subsequent regularization of the integral (see, for example, [15], where the interaction between pairs of quantized vortex rings was studied).

In the proposed approach, we have chosen a different method: the energy of the arbitrary configuration in our model will be considered from the group-theoretical viewpoint. Indeed, the Lee algebra of group \(\tilde{G}_3\) has three Casimir functions:

\[
\hat{C}_1 = m_0 \hat{I}, \quad \hat{C}_2 = \left( \hat{M}_i - \sum_{k,j=x,y,z} \epsilon_{ijk} \hat{P}_j \hat{B}_k \right)^2, \quad \hat{C}_3 = \hat{H} - \frac{1}{2m_0} \sum_{i=x,y,z} \hat{P}_i^2, \]

where \(\hat{I}\) is the unit operator, \(\hat{M}_i, \hat{H}, \hat{P}_i\) and \(\hat{B}_i (i = x, y, z)\) are the respective generators of rotations, time and space translations and Galilean boosts. As it is well known, the function \(\hat{C}_3\) can be interpreted as an “internal energy of the particle”. Because our dynamical system has an “internal degrees of the freedom”, the function \(\hat{C}_3\) can depend on the internal variables. We define these functions as follows

\[ C_3 = E_0 \sum_{n > 1} |j_{-n}|^2 n \sqrt{n^2 - 1}. \]

Here we have introduced into consideration the value \(E_0 = m_0 R_0^2 / t_0^2\) which defines the energy scale in our theory. The choice of the function \(C_3\) will be quite justified after the definition of the Hamiltonian structure.

As a result, the following function on the set \(\Omega\) is a good candidate for the energy:

\[ H_0(p_1, p_2, p_3; j) = \frac{p^2}{2m_0} + E_0 \sum_{n > 1} |j_{-n}|^2 n \sqrt{n^2 - 1}. \]  

\(\text{\textsuperscript{1}}\text{We are considering here the energy of excitations only.}\)
To complete the consideration of energy, we must define the Poisson brackets that are compatible with the dynamics and constraints.

In this article we consider the simplest case that corresponds the value $j_0^0 = 0$. Consequently, the constraint (35) is fulfilled. It is quite natural to add the additional constraint

$$\Phi_2 \equiv q_z - R_0 \tau = 0.$$  

Pursuant to Dirac’s prescriptions about the primacy of Hamiltonian structure, we define such structure axiomatically here. The correspondent definitions are following.

- **Phase space** $\mathcal{H} = \mathcal{H}_3 \times \mathcal{H}_j$. The space $\mathcal{H}_3$ is the phase space of a 3D free structureless particle. It is parametrized by the variables $\mathbf{q}$ and $\mathbf{p}$. The space $\mathcal{H}_j$ is parametrized by the quantities $j_n$ ($n = 0, 1, \ldots$).

- **Poisson structure**:

$$\{ p_i, q_j \} = \delta_{ij}, \quad i, j = x, y, z,$$

$$\{ j_m, \overline{\tau}_n \} = \left( \frac{i}{\mathcal{E}_0} \right) \delta_{mn}, \quad m, n = -1, -2, \ldots$$  

All other brackets vanish. The variable $j_0$ annullates all brackets. Thus, the Poisson structure of the theory is degenerate in general: the value $j_0$ marks the symplectic sheets where the structure will be non-degenerate.

- **Constraints (32), (35) and (37)**. It is clear that constraints (35) and (37) form the pair of second type constraints in Dirac terminology. Moreover the following equalities hold:

$$\{ \Phi_0, \Phi_k \} = 0 \quad k = 1, 2.$$  

Therefore, we can exclude the coordinates $p_z$ and $q_z$ from the phase space $\mathcal{H}_3$ replacing it to the phase space $\mathcal{H}_2$. The last one - the phase space of a free structureless particle on a plane. There are no additional constraints here because the equality $\{ H, \Phi_0 \} = 0$.

- **Hamiltonian**

$$H = H_0 + \ell \Phi_0,$$  

where the function $H_0$ is defined by the formula (36) with replacing $\mathbf{p} \to \mathbf{p}_\perp$. The quantity $\ell$ is the Lagrange factor.
Let's pay attention to the following point. We introduce the set of the new variables $A'$ which is more extensive than the set of original variables. Constraints on set $A'$ were postulated. These constraints lead to a certain arbitrariness in dynamics. That is why the constructed dynamical system is not equivalent to the original one, but it is more extensive. Indeed, let us define the physical (dimensional) time $t = t_0 \tau$. The following Proposition is true:

**Proposition 3** The following Hamilton equations are valid:

\[
\frac{\partial q}{\partial t} = \{H_0, q\} = \frac{p}{m_0}, \quad \frac{\partial p}{\partial t} = \{H_0, p\} = 0, \tag{40}
\]

\[
\frac{\partial j(\tau, \xi)}{\partial t} = \{H_0, j(\tau, \xi)\} = \frac{i}{t_0} \sum_{|n|>1} j_n n \sqrt{n^2 - 1} e^{i[n\xi + n\sqrt{n^2 - 1}\tau]} \tag{41}
\]

Because the Hamiltonian $H$ differs from $H_0$, constructed system is equivalent to original if the Lagrange factor $\ell = 0$.

## 4 Quantization

Numerous articles devoted to the problem of turbulence show that the understanding of this phenomenon is still not full. This statement also fully applies to the turbulence of quantum liquids. Without setting out to review the literature on this issue, we note that a number of authors (see, for example, [16]) assume that the key to understanding this problem is the topological defects of such liquids - vortices. That is why the quantum description of vortices is an important task. The information about the spectrum of energy of such defects for concrete tasks allows investigating certain statistical characteristics of the system. In this paper, we aim to develop a new approach to the quantization of a single closed vortex with zero thickness. The author believes that in the future, the results obtained may provide new opportunities for explaining the behavior of quantum liquids.

The constructed Hamiltonian structure opens up possibilities for quantization of the small perturbations of the vortex ring under study. Firstly, we must define a Hilbert space $H$ of the quantum states of our dynamical system. The structure of the phase space $\mathcal{H}$ lead to the following structure:

\[
\mathcal{H} = H_2 \otimes H_F, \tag{42}
\]
where the symbol $H_2$ denotes the Hilbert space of a free structureless particle on a plane (the space $L^2(\mathbb{R}_2)$ for example) and symbol $H_F$ denotes the Fock space for the infinite number of the harmonic oscillators. The creation and annihilation operators which are defined in the space $H_F$, have standard commutation relations

$$[\hat{a}_m, \hat{a}_n^+] = \hat{I}_F, \quad \hat{a}_m |0\rangle = 0, \quad m, n = 1, 2, \ldots, \quad |0\rangle \in H_F,$$

where the operator $\hat{I}_F$ is unit operator in the space $H_F$. Let us quantize our theory. We must to construct the function $A \rightarrow \hat{A}$, where $A$ denotes some classical variable and $\hat{A}$ denote some operator in the space $H$. Traditionally, we must to demand

$$[\hat{A}, \hat{B}] = -i\hbar \{A, B\}$$

if the quantities $A, B, \ldots$ denote the fundamental variables in our theory. This equality can possess of some "anomalous terms" if the "observables" $A, B$ are the functions of the fundamental variables. These terms depend on a rule of the ordering of non-commuting operators. We will not discuss these issues here [17].

Let us consider the case when $H_2 = L^2(\mathbb{R}_2)$. This case corresponds to the perturbation of a vortex ring in unbounded space. Our postulate of quantization is following:

$$q_{x,y} \rightarrow q_{x,y} \otimes \hat{I}_F, \quad p_{x,y} \rightarrow -i\hbar \frac{\partial}{\partial q_{x,y}} \otimes \hat{I}_F, \quad j_{-n} \rightarrow \sqrt{\frac{\hbar}{t_0E_0}} (\hat{I}_2 \otimes \hat{a}_n),$$

where $n = 1, 2, \ldots$ and operator $\hat{I}_2$ is unit operator in the space $H_2$. Next, we will not write the index $n = 1$ explicitly: $a_1 = \hat{a}$ and so on. This simplification will be justified later. Moreover we will not write the constructions $(\cdots \otimes \hat{I}_F)$ and $(\hat{I}_2 \otimes \cdots)$ explicitly, hoping that this will not lead to misunderstandings.

As a next step, we should to construct the physical subspace $H_{phys} \subset H$. In accordance with Dirac’s prescription, the presence of constraint (32) leads to the equation for the vectors $|\psi\rangle \in H_{phys}$:

$$\hat{\Phi}_0|\psi\rangle = 0.$$

Here it is extremely important to pay attention to the following fact. In a classical theory, all forms of the first-type constraints lead to the same theory: in our case, we can assume $\Phi_0 = 0$ or $\Phi_0^2 = 0$ and so on. This is not the case at all in the quantum version of the model. The different forms of
first type constraints correspond to the different equations for the "physical vectors" in a quantum theory\textsuperscript{2}. For instance, it is clear that the solutions of the equation $\hat{\Phi}_0 |\psi\rangle = 0$ differ from the solutions of the equation $\hat{\Phi}_2^0 |\psi\rangle = 0$. Consequently, we need to supplement the quantization rules with a specific choice of the form of the constraint in classical theory.

Let us investigate this problem in our model in more detail. First of all, we consider the classical constraint in form (30). Taking into account formula (28), we search the vectors $|\psi_{phys}\rangle$ so that

$$\exists \lambda \in \mathbb{R} : (\hat{p} - 2\pi \lambda p_0 \sqrt{\frac{\hbar}{t_0 E_0}} \hat{a}) |\psi_{phys}\rangle = 0.$$  \hspace{1cm} (43)

Let the complex number $p = p_x + ip_y$ is eigenvalue that corresponds to the (generalized) eigenvector $|p\rangle \in H'_2$ of the operator $\hat{p}$. The notation $H'_2$ means that procedure of a rigging of the space $H_2$ must be fulfilled to consider the generalized eigenvectors rigorously \cite{19}. As ansatz for the solutions of the equation (43), we use the following form for the "physical vectors" $|\psi_{phys}\rangle$:

$$|\psi_{phys}(p)\rangle = |p\rangle |\psi_p\rangle, \quad |p\rangle \in H'_2, \quad |\psi_p\rangle \in H_F.$$

Therefore, the equation for the vector $|\psi_p\rangle$ takes following form:

$$\exists \lambda \in \mathbb{R} : (p - 2\pi \lambda p_0 \sqrt{\frac{\hbar}{t_0 E_0}} \hat{a}) |\psi_p\rangle = 0, \quad |\psi_p\rangle \in H_F, \quad p \in \mathbb{C}.$$ \hspace{1cm} (45)

In other words, the vectors $|\psi_p\rangle$ are the eigenvectors of the spectral problem (45).

\textbf{Proposition 4} Let the the vectors $|p/\lambda\rangle \in H_F$ form the system of coherent states\textsuperscript{3} for Heisenberg - Weyl group with algebra

$$[\hat{a}, \hat{a}^+] = \hat{I}, \quad [\hat{a}^+, \hat{I}] = [\hat{a}, \hat{I}] = 0$$

as follows:

$$|p/\lambda\rangle = \exp \left[ (p \hat{a}^+ - \overline{p} \hat{a})/2\pi \lambda \sqrt{\frac{\hbar p_0}{R_0}} \right] |0\rangle, \quad \lambda \in \mathbb{R}, \quad p \in \mathbb{C}.$$ \hspace{1cm} (46)

\textsuperscript{2}Apparently, Dirac’s words are still relevant: "...methods of quantization are all of the nature of practical rules, whose application depends on consideration of simplicity"\textsuperscript{3}See, for example, \cite{18}
Then the equation (45) has solutions
\[ |\psi^\lambda\rangle = |n_1, \ldots, n_k; p/\lambda\rangle \equiv \hat{a}^+_n \hat{a}^+_n \ldots \hat{a}^+_n |p/\lambda\rangle, \quad n_j > 1. \]

This statement can be verified directly because the coherent states $|p/\lambda\rangle$ are eigenvectors of the operator $\hat{a}$ with eigenvalue $p/(2\pi \lambda \sqrt{\hbar p_0/R_0})$. Recall that the parameter $\lambda$ has the meaning of ”dimensionless circulation” in our model.

Let us consider the vectors $|\psi^\lambda_{[n]}(p)\rangle \in H'$:
\[ |\psi^\lambda_{[n]}(p)\rangle = C_{[n]} |p| |n_1, \ldots, n_k; p/\lambda\rangle, \quad \lambda \in \mathbb{R}, \quad [n] = n_1, \ldots, n_k, \quad (47) \]
where numbers $C_{[n]}$ are normalizing coefficients, so that normalizing conditions are fulfilled:
\[ \langle \psi^\lambda_{[n]}(p) | \psi^\lambda_{[m]}(p') \rangle = \delta_{kl} \delta_{n_1m_1} \ldots \delta_{n_km_k} \delta(p_x - p'_x) \delta(p_y - p'_y). \]

Here we should note that the entire set of vectors (47) can’t form a physical subspace $H_{phys} \subset H$ because any superpositions
\[ c_1 |\psi^\lambda_{[n]}(p_1)\rangle + c_2 |\psi^\lambda_{[n]}(p_2)\rangle, \quad \lambda_1 \neq \lambda_2, \quad p_1 = p_2 \]
is not a solution to the spectral problem (45). Moreover we can’t postulate ”superselection rules” here. Indeed, the coherent states $|p/\lambda\rangle$ are not orthogonal for different values of parameter $p/\lambda$. Therefore, the vectors $|\psi^\lambda_{[n]}(p)\rangle$ are not orthogonal for same values $p$ and different values $\lambda$.

We will proceed as follows. Let the vector $|\psi^\star_{[n]}(p)\rangle \in H'$ will be the vector (47) where the value $\lambda = \lambda_0 = 1/\pi$:
\[ |\psi^\star_{[n]}(p)\rangle = C_{[n]} |p| |n_1, \ldots, n_k; p/\lambda_0\rangle, \quad k = 0, 1, 2, \ldots. \quad (48) \]

The state (48) which corresponds $k = 0$, we call as ”ground state” of our theory. Recall that the value $\lambda = \lambda_0$ corresponds to the circulation $\Gamma_0$ of the unperturbed vortex ring: $\Gamma_0 = R_0^2/\pi t_0$.

We declare that the physical subspace $H_{phys}$ is spanned by the following vectors
\[ |\psi_{phys}\rangle = \sum_{n_1, \ldots, n_k}^{n_j > 1} \int dp_x dp_y \varphi_{n_1, \ldots, n_k} (p_x, p_y) |\psi^\star_{[n]}(p)\rangle, \quad (49) \]
where the wave functions $\varphi_{n_1,\ldots,n_k}(p_x,p_y)$ are normalized.

Does the constructed space $H_{\text{phys}}$ describes the states with circulation $\Gamma_0$ only? In our opinion, the properties of coherent states allow us to assume that this is not the case. Indeed, every system of coherent states $|\alpha\rangle$ is an overdetermined system. Because $\langle \alpha_1 | \alpha_2 \rangle \neq 0$ even for different complex numbers $\alpha_1$ and $\alpha_2$, we can conclude that any specific coherent state $|\alpha_0\rangle$ contains "some part" all other coherent states $|\alpha\rangle$, $\alpha \neq \alpha_0$.

Returning to our notation, we have the following expression for the amplitude for any $\lambda \in \mathbb{R}$:

$$
\langle \psi_{[n]}(p) | \psi_{\text{phys}} \rangle = \varphi_{[n]}(p_x,p_y) \exp \left[ \frac{-|p|^2 R_0}{8 \hbar p_0} \left( \frac{\lambda_0}{\lambda} - 1 \right)^2 \right]. \tag{50}
$$

Although $|\psi_{[n]}(p)\rangle \not\in H_{\text{phys}}$ for the case $\lambda \neq 1/\pi$, we suppose that the amplitude (50) defines the probability density to find our dynamical system with circulation $\Gamma = \lambda R_0^2 / t_0$, transverse impulse $p = p_x + i p_y$ and quantum numbers $[n] = \{n_1,\ldots,n_k\}$.

The numbers $[n]$ define the energy of our system. Indeed, in accordance with our classical formulas for the Hamilton function, the quantum expression for the Hamiltonian $\hat{H}$ takes the form

$$
\hat{H} = \frac{\hat{p}_x^{\dagger} \hat{p}_x + \hat{p}_y^{\dagger} \hat{p}_y}{2m_0} + \frac{\hbar}{t_0} \sum_{n>1} \hat{a}_n^{\dagger} \hat{a}_n \sqrt{n^2 - 1}.
$$

The following statement can be proved by direct verification:

**Proposition 5** The vectors $|\psi_{[n]}(p)\rangle = C_{[n]} |p\rangle |n_1,\ldots,n_k; p/\lambda_0\rangle$ are the eigenvectors of operator $\hat{H}$ with eigenvalues

$$
\mathcal{E} = \frac{|p|^2}{2m_0} + \frac{\hbar}{t_0} \sum_{n>1} \sum_{j=1}^k \delta_{n,n_j} n \sqrt{n^2 - 1}. \tag{51}
$$

As regards the constraint in the form (32), the following equality takes place:

$$
\langle \psi_{\text{phys}} | \hat{\Phi}_0 | \psi_{\text{phys}} \rangle = 0.
$$

As a result, the constructed quantum description of the system allows for a visual interpretation of the vortex as a structured particle. The connection
between external degrees of freedom (space $H_2$) and internal degrees of freedom (space $H_F$) is nontrivial due to the constraint (43). As a consequence, the quantum states of such a system are entangled states. The formula (49) demonstrates that the vectors

$$|\psi^+_{[0]}(p)\rangle = |p\rangle|p/\lambda_0\rangle$$

forms the ”set of the ground states” of our system. These vectors corresponds to the vortex ring with the exact value of the transverse momentum $p$, the most probable value of the circulation $\Gamma_0 = R_0^2/\pi t_0$ and the minimal energy $|p|^2/2m_0$.

5 Concluding remarks

This paper has constructed a model describing the classical and quantum dynamics of small perturbations of the vortex ring, which evolves according to the LIE equation. The theory has three-dimensional constants: $R_0$ (radius of unperturbed vortex ring), $p_0$ (momentum of unperturbed vortex ring), and $m_0$ (the central charge for the central extension of Galilei group). We quantized our model as an abstract dynamical system, without any connection with the quantum properties of the surrounded liquid. Of course, taking into account such property is an important area that is presented in the literature (see [20], for example). In the case under consideration, when $H_2 = L^2(R_2)$, the eigenvalues of the momentum operator belong to a continuous set. The same can be said about the energy $\mathcal{E}$ (see (51)) and the circulation $\Gamma$ (see (50)). However, there is no contradiction with the experiment: for example, all the experiments, where the quantization of the circulation was observed [7], correspond to one of the real cases when certain boundary conditions are present. To take into account any boundary conditions in our approach, we should consider the case when $H_2 = L^2(D)$, where the domain $D$ is a certain compact subset of a plane $R_2$. We suppose that corresponding theory leads to the discrete values of energy $\mathcal{E}$ and circulation $\Gamma$. The author hopes to devote the next article to this issue.

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