Entanglement and phase properties of noisy N00N states

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Quantum metrology and quantum information necessitate a profound study of suitable states. Attenuations induced by free-space communication links or fluctuations in the generation of such states limit the quantum enhancement in possible applications. For this reason we investigate quantum features of mixtures of so-called N00N states propagating in atmospheric channels. Firstly, we show that noisy N00N states can still yield a phase resolution beyond classical limitations. Secondly, we identify entanglement of noisy N00N states after propagation in fluctuating loss channels. For doing so, we will construct a family of entanglement probes based on the method of separability eigenvalue equations. Our theoretical analysis formulates explicit bounds which are indispensable for experimental verification of quantum entanglement and applications in quantum metrology.

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I. INTRODUCTION

Quantum mechanics has opened new fields of applications such as quantum metrology [1–2], quantum information science [3], or quantum communication [4]. All these technologies have in common that they employ non-classical properties such as entanglement in order to beat classical restrictions. In quantum metrology such a classical limit is the so-called standard quantum limit or shot noise limit [1,2]. It relates the error of phase estimation $\Delta \varphi$ of a parameter $\varphi$, e.g., the phase, with the energy, e.g., photon number, used for the estimation. Employing classical strategies only, we get the bound $\Delta \varphi \geq 1/\sqrt{n}$, where $n$ denotes the mean photon number. However, using nonclassical states in detection strategies this behavior can be beaten; cf., e.g., [5–7]. High-precession quantum interferometric measurements allow a resolution up to the Heisenberg limit $\Delta \varphi = 1/n$ [1]. Superior phase estimations are also used to answer open questions in physics, e.g., gravitational wave detection [8,9].

One prominent representative of quantum states which reaches the Heisenberg limit is the two-mode, quantum entangled N00N state [10]. The name originates from its notation in Fock bases in the form

$$|\psi_N\rangle = \frac{1}{\sqrt{2}} (|N\rangle \otimes |0\rangle + |0\rangle \otimes |N\rangle). \quad (1)$$

N00N states are Bell-like entangled states. Besides the application in quantum metrology [2] they are also of interest in quantum lithography [10]. By employing N00N states, a phase super-resolution in a Mach-Zehnder interferometer can be achieved. Several strategies to generate such states have been proposed and experimentally implemented; see, e.g., [11–12] and [13–14], respectively.

Besides quantum metrology, secure communication based on quantum key distribution (QKD) is another vital topic of current research [4]. Though QKD is already used commercially, it is still restricted to relatively small transmission range. In systems using optical fibers the loss has a typical magnitude of $\sim 0.2$ dB/km [15] which renders it possible to preserve entanglement up to a $\sim 100$ km propagation distance [16–18]. An alternative approach is the quantum communication through free-space links. QKD in such atmospheric links was demonstrated for an atmospheric channel of 144 km [19,20] and recent experiments show the feasibility of global communication via orbiting satellites; see, e.g., [21–23]. It is noteworthy that the implemented protocols use post-selection strategies. In order to benefit from free-space QKD it is crucial to study the occurring turbulent losses within such a link. The first consistent quantum description of turbulent loss channels was introduced in terms of fluctuating transmission coefficients [24].

QKD employs entangled states as a resource. One of the most frequently used method for determining this quantum phenomena is formulated in terms of measurable entanglement witnesses [25]. A defined method for the construction of such entanglement probes has been formulated for bipartite and multipartite entangled systems, which is based on the so-called separability eigenvalue equations [26,27]. Recent experimental implementations verify its applicability in demonstrating entanglement in complex structured light fields [28,29].

In this contribution we study mixtures of N00N states propagating in atmospheric channels. We show that superior phase-estimation properties can surpass perturbations within interferometric setups without post-selection. We construct a class of witnesses for characterizing entanglement of noisy N00N states. In particular the entanglement properties are studied in the case of turbulent losses occurring in urban free-space links. Surprisingly simple conditions are deduced to infer sub-standard quantum limit behavior and entanglement correlations. These conditions may be used to estimate rough bounds to experimentally allowed ranges of noise to infer such quantum features.

In Sec. [11] we motivate the generalization from pure
N00N states to mixed ones. We study the phase properties of this class of states within an interferometric setup in Sec. III. In Sec. IV we construct entanglement witnesses by solving the separability eigenvalue equations. The entanglement and phase properties are further studied under turbulent loss conditions in Sec. V. A summary and conclusions are given in section VI.

II. NOISY N00N STATES

In this section we motivate the class of noisy N00N states as a generalization of ordinary N00N states. Such a generalization is natural, considering a possible scheme of generating N00N states. Moreover we consider the effects of atmospheric channels as a source of turbulent losses which effect the properties of the states under study. Another source of possible imperfections is phasing, which will be subsequently investigated. Eventually, we will formulate a general family of quantum states which are based on mixed N00N states including all previously considered perturbations.

A. Generation of mixed N00N states

Here we will focus on one previously implemented generation of N00N states by using coherent and squeezed vacuum states at the input ports of a symmetric beam splitter. Such a generation strategy has been studied in theory and has successfully be applied in experiment. It has been shown that any path entangled N00N state with any arbitrary N can in principle be generated in such form with a fidelity of at least 92% \cite{31}. In the so far conducted experiments, e.g. \cite{33,34}, this generation strategy was used in Mach-Zehnder interferometers with the aim to surpass the standard quantum limit. Via coincidence measurements the N-fold super-resolution was demonstrated for up to \(N = 5\). Analyzing the two mode photon number statistics of the state in the interferometer, one observes that the generated state is a mixture of N00N states rather than a pure single N00N state.

This fact motivates the introduction of the mixed N00N states,

\[
\hat{\rho}_{\text{mixed}} = \sum_{N=0}^{\infty} p_N |\psi_N \rangle \langle \psi_N |
\]

with \(p_N\) being a probability distribution of N00N states \(|\psi_N \rangle\) in Eq. (1). For \(N = 0\) we use the convention \(|\psi_0 \rangle = |0,0\rangle\), being the two-mode vacuum state. These mixtures properly describe the experimentally realized states.

B. Turbulent loss channels

A crucial issue using N00N states is the sensitivity with regards to loss. One imagines a loss of a photon in one of the modes – say second mode. In this case the entangled N00N state \(|\psi_N \rangle\) reduces to a non-entangled state, \(|\psi_N \rangle \rightarrow |0, N-1\rangle\). The outcome lost all the rewarding properties of the N00N state and especially its super-resolution in phase measurement. Thus it is of high interest to study the quantum properties of mixed N00N states undergoing losses. Note that the influence of noise effects on metrology scenarios has been studied generally in Refs. \cite{35,37}.

For a single N00N state we get after a constant loss in both modes

\[
\Lambda_{N,\theta}(|\psi_N \rangle \langle \psi_N |) = \frac{\sqrt{\kappa \theta}}{2} \left[ |N,0 \rangle \langle 0,N | + |0,N \rangle \langle N,0 | \right] + \frac{1}{2} \sum_{k=0}^{N} \left( \kappa^k (1-\kappa)^{N-k} |k,0 \rangle \langle k,0 | \right) + \theta^k (1-\theta)^{N-k} |0,k \rangle \langle 0,k |,
\]

where the transmission coefficients \(\sqrt{\kappa}\) and \(\sqrt{\theta}\) describe the loss in the first and second subsystem, respectively. Starting from the turbulent N00N states will be derived representing an attenuated state after propagation through a fluctuating loss media.

The turbulent atmosphere differs from standard constant loss channels due to the fact that the transmission coefficient is not a constant but a random variable. In this way the description of turbulence is included in the probability distribution of the transmission coefficient (PDTC) \cite{24}, denoted as \(\mathcal{P}\). The density operator is obtained by averaging the lossy state with an appropriate PDTC. In a bipartite system, this process reads as

\[
\hat{\rho}_{\text{out}}(\kappa, \theta) = \frac{1}{\sqrt{\kappa \theta}} \int_0^1 d\kappa \int_0^1 d\theta \mathcal{P}(\kappa, \theta) \Lambda_{N,\theta}(\hat{\rho}_{\text{in}}).
\]

In the following we want to focus on the effect of beam wandering because of its well characterized PDTC \cite{38} and due to its importance for urban communication links. Theory and experiment show excellent agreement with each other \cite{39}. For a single spatial-mode, it can be expressed as a log-negative Weibull distribution,

\[
\mathcal{P}(T) = \frac{2S^2}{\sigma^2 T} \left( \frac{2 \ln \frac{T_0}{T}}{T} \right)^{2-1} \exp \left[ -\frac{1}{2 \sigma^2} S^2 \left( \frac{2 \ln \frac{T_0}{T}}{T} \right)^2 \right],
\]

for \(T \in [0, T_0]\) and \(\mathcal{P}(T) = 0\) else. The meaning of the occurring constants are listed in Appendix A. As an example of an atmospheric channel we will consider a quantum channel in an urban environment as in experiments performed in Erlangen \cite{40,41} and theoretically studied in \cite{42}. Note that there were also other experiments performed in this urban quantum free-space channel regime, e.g., \cite{39,43,44}.
In order to derive the state after propagation through the atmosphere, see Eq. (1), we will now apply the PDTC model to the lossy N00N state (3). This can be done in two configurations. First, the two modes could propagate in different directions yielding a joint PDTC in terms of independent single mode ones,

\[ \mathcal{P}(\sqrt{\kappa}, \sqrt{\theta}) = \mathcal{P}(\sqrt{\kappa}) \mathcal{P}(\sqrt{\theta}). \]  

(6)

Second, a co-propagation of the field components could be considered. According to the experiments in Erlangen we might consider the polarization as the two degrees of freedom. In this case the two modes undergo the same turbulence, i.e. the joint PDTC reads as

\[ \mathcal{P}(\sqrt{\kappa}, \sqrt{\theta}) = \mathcal{P}(\sqrt{\kappa}) \delta \left( \sqrt{\kappa} - \sqrt{\theta} \right), \]  

(7)

with \( \delta \) being the Dirac delta distribution.

### C. Additional noise and general state representation

Among other perturbations, dephasing is a harmful influence to the phase sensitivity of N00N states. A dephasing channel can be described by

\[ \hat{\rho}_{\text{out}} = \Lambda_{\text{deph}}(\hat{\rho}_{\text{in}}) = \int_0^{2\pi} d\varphi \, p(\varphi) e^{i\varphi n} \hat{\rho}_{\text{in}} e^{-i\varphi \hat{n}}, \]  

(8)

for a classical phase distribution \( p(\varphi) \). In case of the N00N state, we get for a dephasing in one mode

\[ \mathbb{I} \otimes \Lambda_{\text{deph}}(|\psi_N\rangle \langle \psi_N|) = \frac{1}{2} \left[ |0,N\rangle \langle 0,N| + |N,0\rangle \langle N,0| \right] + \lambda|N,0\rangle \langle 0,N| + \lambda^*|0,N\rangle \langle N,0|, \]  

(9)

with \( \lambda = \int_0^{2\pi} d\varphi \, p(\varphi) e^{i\varphi N} \). Note that, in general, we have reduction of coherence, \( |\lambda| < 1 \). Moreover a full decoherence, \( p(\varphi) = 1/(2\pi) \), yields \( \lambda = 0 \). That is a complete loss of phase sensitivity and entanglement.

Finally we combine all imperfections considered so far which yields a very general class of noisy NOON states. Hence, the result - the density matrix in a Fock expansion - reads as

\[ \hat{\rho} = \rho_{00,00} |0,0\rangle \langle 0,0| \]

\[ + \sum_{i=1}^{\infty} \left[ \rho_{0i,0i} |i,0\rangle \langle i,0| + \rho_{0i,i0} |0,i\rangle \langle 0,i| + \rho_{0i,i0} |0,i\rangle \langle 0,i| + \rho_{ii,i0} |0,i\rangle \langle 0,i| \right] \]

\[ + \sum_{i=1}^{\infty} \left[ \rho_{0i,0i} |i,0\rangle \langle i,0| + \rho_{0i,i0} |0,i\rangle \langle 0,i| \right] \]  

(10)

In this form the first line represents the vacuum contribution to the state being separable and carrying no phase information. The second line includes the diagonal entries of \( \hat{\rho} \). The last line contains the interferences of the state. Due to \( \hat{\rho} = \hat{\rho}^\dagger \), we have \( \rho_{0i,0i} = \rho_{0i,i0}^* \). From the metrology as well as the entanglement point of view, these terms are those which classify the applicability for quantum enhanced tasks.

This general form includes mixtures of N00N states, the propagation effects in turbulent media, and dephasing. In the following, states of the structure will be referred to as noisy N00N states. In the remainder of this contribution we characterize such states in terms of phase resolution and entanglement on a common bases.

### III. Phase properties of noisy N00N states

As we discussed above N00N states are of great interest due to their sub-standard quantum limit behavior in interferometric setups. We will now show that noisy N00N states can still show this property and give analytical conditions for its appearance. Therefore we consider the typically studied measurement that is given by the operator

\[ \hat{A}_M = |0,M\rangle \langle M,0| + |M,0\rangle \langle 0,M|, \]  

(11)

which can be realized via interference measurements at the output ports of a Mach-Zehnder interferometer. The error in phase estimation can be calculated via error propagation as

\[ \Delta \varphi = \frac{\sqrt{\langle (\Delta \hat{A}_M)^2 \rangle}}{d\varphi d(\Delta \hat{A}_M)}. \]  

(12)

Computing the expectation values we get a phase depended error

\[ \Delta \varphi = \frac{1}{M} \left[ \rho_{MM,MM} + \rho_{0M,0M} - \frac{2 \text{Re}(e^{i\varphi} \rho_{0M,MM})}{2 \text{Im}(e^{i\varphi} \rho_{0M,MM})} \right]^{\frac{1}{2}} \]  

\[ = \frac{f(\varphi)}{M}. \]  

(13)

Here we still observe the \( 1/M \) behavior as obtained for a perfect N00N state. However, we have an additional phase dependent scaling factor \( f(\varphi) \).

The phase-interval in which the standard quantum limit is surpassed, \( \Delta \varphi < 1/\sqrt{M} \), can be shown to be determined by

\[ \frac{1}{M} \arcsin(\omega) < \varphi + \frac{\arg(\rho_{MM,MM})}{M} - \frac{\pi}{M} - \frac{1}{M} \arcsin(\omega), \]  

(14)

with \( \omega = \sqrt{\rho_{MM,MM} + \rho_{0M,0M} - 4 \rho_{0M,MM}^2 (M-1)} \).

Note that this interval has a \( \pi/M \) periodicity. In order to have solutions for the inverse sine function, an additional
condition has to be fulfilled:

\[ |\rho_{M,0}|^2 > \frac{1}{M} \rho_{M0,0} + \rho_{0M,0}. \tag{15} \]

For mixtures of N00N states in the form \([2]\), this phase dependency is illustrated in Fig. 1. Here the error in phase estimation \(\Delta \varphi\) for different mixing coefficients \(p_M\) is displayed. Depending on the value of \(p_M\), the standard quantum limit can be surpassed. In this case Eq. (15) reduces to the rather simple expression \(p_M \geq 1/M\). This is a remarkable result as it directly relates the quantum fidelity \(F\) of mixed N00N states \([2]\) with a N00N state with \(M\) photons. Only if \(F = p_M \geq 1/M\) one can observe a scaling of \(\Delta \varphi\) below the standard quantum limit, compare Fig. 1. Note that in the special case \(p_M \rightarrow 1\) one reaches the well known \(1/M\) behavior of the pure N00N state.

Secondly we will study the influence of dephasing on the error in phase resolution. Therefore we will consider a pure N00N state undergoing dephasing, see Eq. [9], modeled by a wrapped Gaussian phase distribution in the form

\[ p(\varphi) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}\delta^2} \exp \left[ -\frac{(\varphi - \varphi_0 + 2k\pi)^2}{2\delta^2} \right], \tag{16} \]

where \(\delta\) controls the strength of the noise. Such a phase noise distribution has already been studied in the context of entanglement in Ref. [45]. Applied to a N00N state,

we get

\[ \lambda = \int_0^{2\pi} d\varphi \frac{1}{\sqrt{2\pi}\delta^2} \exp \left[ -\frac{(\varphi - \varphi_0 + 2k\pi)^2}{2\delta^2} \right] e^{i\varphi N} \]

\[ = e^{i\varphi_0 N} e^{-\frac{i2\pi k^2}{2}}, \tag{17} \]

cf. Eq. [9]. The dependency of the error in phase estimation on the noise parameter \(\delta\) is shown in Fig. 2. One directly observes that \(\Delta \varphi\) increases form the Heisenberg limit, corresponding to \(\delta \rightarrow 0\), with increasing \(\delta\) and even overshoots the standard quantum limit for \(|\lambda|^2 > 1/M\), corresponding to \(\delta > \ln(M)/M\), cf. Eq. (15). Note that for the limit \(\delta \rightarrow \infty\), the distribution (16) is a uniform distribution, in such a case \(\Delta \varphi \rightarrow \infty\) and it is impossible to determine the phase at all.

\section*{IV. ENTANGLEMENT PROBES FOR NOISY N00N STATES}

In this section we will study the entanglement properties of noisy N00N states [10]. First we will briefly recapitulate the method of the separability eigenvalue problem (SEP) since its solutions are used to construct entanglement conditions based on the entanglement witness method [23]. Secondly we will solve the SEP for test operators which are designed to detect entanglement of noisy N00N states. Based on these solutions surprisingly simple entanglement criteria can be formulated.
A. Witnesses and separability eigenvalue problem

Let us start with the bare definition of the separability eigenvalue equations for a general Hermitian operator \( \hat{L} \) [20] [27]. In the bipartite case the separability eigenvalue equations are defined as:

\[
\begin{align*}
\hat{L}_b|a\rangle &= g(a) \\
\hat{L}_a|b\rangle &= g(b)
\end{align*}
\] (18)

with the local projections,

\[
\hat{L}_a = \text{tr}_{A}[\hat{L}(|a\rangle \langle a| \otimes \hat{I}_B)] \quad \text{and} \quad \hat{L}_b = \text{tr}_{B}[\hat{L}(|A\rangle \langle A| \otimes |b\rangle \langle b|)]
\]

The product vectors \(|a, b\rangle\) which solve the SEP are referred to as separability eigenvectors. The corresponding separability eigenvalues (SEV) are represented by the real values \(g\) in (18).

The determination of bipartite entanglement of a quantum state \(\hat{\rho}\) can be probed with an operator \(\hat{L}\) [25]. In particular, the state under study is entangled if and only if there exists such an operator \(\hat{L}\) with

\[
\langle \hat{L} \rangle = \text{tr}(\hat{\rho} \hat{L}) > \sup\{g : g \text{ SEV to (18)}\}. \quad (19)
\]

Since the maximal SEV corresponds to the maximal expectation value of the operator \(\hat{L}\) for all separable states, condition (19) can be understood as follows. The value \(\langle \hat{L} \rangle\) for the entangled state \(\hat{\rho}\) exceeds the bound for separable states. By this approach it is possible to use almost all Hermitian operator \(\hat{L}\) for an entanglement test. The necessary and sufficient condition to be an appropriate entanglement probe is that the eigenspace to the maximal eigenvalue of \(\hat{L}\) does not contain a product vector [28]. Obtaining the maximal SEV requires the solutions of the corresponding separability eigenvalue equations [18]. Due to its coupled structure, this is a challenging task.

B. Solution of the SEP for a class of test operators

Considering the structure of the left hand side of the entanglement condition (19), it seems to be a good choice if \(\hat{L}\) is in the range of the same subspace as \(\hat{\rho}\), see Eq. (10). In this case the expectation value is large which is due to the Cauchy-Schwartz inequality of the scalar product \(\text{tr}(\hat{\rho} \hat{L})\). Thus it is convenient to consider a test operator of the general form

\[
\hat{L} = L^{(0)}|0, 0\rangle \langle 0, 0| + \sum_{i=1}^{\infty} \left[ L_i^{(A)}|i, 0\rangle \langle i, 0| + L_i^{(B)}|0, i\rangle \langle 0, i| + \frac{1}{2} \left( \gamma_i^{(0, 0)}|0, 0\rangle \langle 0, 0| + \gamma_i^{(0, 1)}|0, 1\rangle \langle 0, 1| + \gamma_i^{(1, 0)}|1, 0\rangle \langle 1, 0| + \gamma_i^{(1, 1)}|1, 1\rangle \langle 1, 1| \right) \right]. \quad (20)
\]

Here let us focus on a particular choice of operator, \(\hat{L} = \hat{A}\), which outlines the general approach to solve the SEP. We may define

\[
\hat{A} = \sum_{M \in \mathcal{M}} \hat{A}_M, \quad (21)
\]

with \(\hat{A}_M\) being given in (11) and \(\mathcal{M}\) being a set of positive integers. Note that this operator addresses the interference terms of the noisy N00N states (10). Since the individual terms \(\hat{A}_M\) are obtained in the Mach-Zehnder setup, the expectation value \(\langle \hat{A} \rangle\) is a directly measurable quantity, see also Sec. III.

For an efficient solution of the SEP, we can decompose the first component of the separability eigenvector as

\[
|a\rangle = r|0\rangle + \sqrt{1-r^2}|a'\rangle \quad (22)
\]

with normalized single-mode state \(|a'\rangle = \sum_{N \in \mathcal{M}} a_N|N\rangle\) and \(r \in [0, 1]\). The reduced operator \(\hat{A}_b\) with respect to \(|a\rangle\) is

\[
\hat{A}_b = r \sqrt{1-r^2} \left[ |0\rangle \langle a'| + |a'\rangle \langle 0| \right]. \quad (23)
\]

If \(r \neq 0, 1\) then we get the eigenvectors to the second equation of the SEP (18) as

\[
|b\rangle = \frac{1}{\sqrt{2}} \left[ |0\rangle \pm |a'\rangle \right]. \quad (24)
\]

The reduced operator \(\hat{A}_b\) with respect to \(|b\rangle\) is

\[
\hat{A}_b = \frac{1}{2} \left[ |0\rangle \langle a'| + |a'\rangle \langle 0| \right]. \quad (25)
\]

The initial vector \(|a\rangle\) is an eigenvector of this operator, if and only if \(r = 1/\sqrt{2}\). Additionally this yields the corresponding SEV

\[
g = \pm \frac{1}{2}. \quad (26)
\]

For the trivial choices \(r = 0, 1\) we get an SEV \(g = 0\) for separability eigenvectors \(|a, b\rangle\) having the form

\[
|0, 0\rangle, |0, b'\rangle, |a', 0\rangle, \text{ or } |a', b'\rangle, \quad (27)
\]

with \(0 = \langle a'|b'\rangle = \langle 0|b'\rangle = \langle a'|0\rangle\).

Hence we get for our operator \(\hat{A}\) the separable bound

\[
\sum_{M \in \mathcal{M}} \left[ \langle M, 0|\hat{\rho}|0, M \rangle + \langle 0, M|\hat{\rho}|M, 0 \rangle \right] \leq \frac{1}{2}. \quad (28)
\]

Using the Hermiticity, \(\langle M, 0|\hat{\rho}|0, M \rangle + \langle 0, M|\hat{\rho}|M, 0 \rangle = 2\text{Re}(\langle M, 0|\hat{\rho}|0, M \rangle)\), and the decomposition of noisy separable N00N states (10), we get

\[
\sum_{M \in \mathcal{M}} 2\text{Re}(\rho_{M0,0M}) \leq \frac{1}{2}. \quad (29)
\]
In Ref. [27], it has been shown that local unitary transformations do not change the SEV of an operator. Therefore, the local map $\hat{U}_{\text{loc}} = \sum_{M \in \mathbb{N}} e^{i \varphi_M} |M\rangle\langle M| \otimes \hat{1}_B$ yields a new observable
\[
\hat{A}' = \hat{U}_{\text{loc}} \hat{A} \hat{U}_{\text{loc}}^\dagger
\]
\[
= \sum_{M \in \mathbb{M}} \left[ e^{i \varphi_M} |M, 0\rangle\langle 0, M| + e^{-i \varphi_M} |0, M\rangle\langle M, 0| \right].
\] (30)
Note that the global phase is omitted, i.e., $\varphi_0 = 0$. Consequently the separability constraint (29) rewrites as
\[
2 \sum_{M \in \mathbb{M}} \text{Re}(\rho_{M,0,0M} e^{i \varphi_M}) \leq \frac{1}{2}.
\] (31)

Due to the fact that this condition is true for any choice of local phases $\varphi_M$, we can distill the combined entanglement condition:
\[
\sum_{M \in \mathbb{M}} |\rho_{M,0,0M}| > \frac{1}{4}.
\] (32)
This means (for $M = \mathbb{N} \setminus \{0\}$) that whenever the sum of all absolute values of interference terms in the density matrix exceeds 1/4, we ensure entanglement of noisy N00N states.

The general solution of the SEP for the class of operators in Eq. (20) is derived in Appendix B. This leads to a maximal SEV
\[
g_{\text{max}} = \max_{\varphi \in \mathbb{N}\setminus\{0\}} \{0, L(0), L(A), L(B), g_i, \pm\},
\] (33)
with
\[
g_i = -v_i \pm \sqrt{v_i^2 - u_i w_i},
\]
\[
u_i = \left[ \left( \frac{L(A)_i + L(B)_i - L(0)_i}{2} \right)^2 - 4 \gamma_i^2 \right],
\]
\[
v_i = 2 \gamma_i^2 L(0)_i - \left( \frac{L(A)_i + L(B)_i - L(0)_i}{2} \right)^2,
\]
\[
w_i = \left( \frac{L(A)_i - L(B)_i}{2} \right)^2.
\]
This determines the right-hand side of the entanglement condition (19). The operator family in Eq. (20) allows the construction of more advanced entanglement probes than the particular example (21).

V. APPLICATION: MIXED N00N STATES AND PROPAGATION IN TURBULENT MEDIA

Here we will study the entanglement and phase properties of mixed N00N states after propagating in a turbulent quantum free-space link. In [113], we described the model of the turbulent atmosphere focusing on the case of beam wandering in an urban environment. The entanglement test derived in section IV is used to detect entanglement of the turbulent noisy N00N state. For a fundamental analysis we will begin our studies with a mixed N00N state.

A. Influence of mixing with vacuum

As a first example we aim to study both, entanglement and phase super-resolution, under the influence of mixing. Therefore we study a mixture of a N00N and a vacuum state in the form
\[
\hat{\rho}_p = (1 - p) |0, 0\rangle\langle 0, 0| + p |\psi_N\rangle\langle \psi_N|,
\] (34)
with $p \in [0, 1]$ and $|\psi_N\rangle$ being a N00N state with $N > 0$. The parameter $p$ controls the purity of this state with respect to an ideal N00N state. The mixture of a pure N00N state with vacuum is of particular interest, as it describes the transition from the N00N state to a phase-independent separable one.

![Figure 3](image-url) (Color online) Properties of the mixture (34) of a N00N state with $N = 4$ with vacuum are shown depending on the noise parameter $p$. The expectation value $\langle \hat{A} \rangle = \text{tr} [\hat{A} \hat{\rho}_p]$ and the minimal phase error $\Delta \varphi_{\text{min}}$, given by $A_4$ in Eq. (11), are depicted on the left and right ordinates, respectively. The state is entangled if the line $\langle \hat{A} \rangle$ is greater than the bound 1/4 (dashed line). Phase super-resolution is achieved if the curve $\Delta \varphi_{\text{min}}$ is below the standard quantum limit $1/\sqrt{4} = 0.5$ (dotted line).

In Fig. 3, the noise dependency of the state (34), represented by the parameter $p$, for entanglement and phase super-resolution is displayed. For the determination of entanglement the condition (32) is applied. One directly identifies a linear dependency in $p$ for witnessing entanglement of the state (34). More precisely as long as the vacuum noise contribution is less than 50%, we can still detect entanglement. Note that this result is independent of the photon number $N$ of the N00N state.

Phase super-resolution is also observed in Fig. 3 for a wide $p$-interval. However in this case the bound, the standard quantum limit, and the minimal phase error, $\Delta \varphi_{\text{min}} = \min \{ f(\varphi) / N \}$, cf. Eq. (13), depend on the photon number $N$ of the N00N state. In general the $p$-interval for which phase super-resolution exists increases with increasing $N$. Note that in this example, cf. Fig. 3, one can observe phase super-resolution in a $p$-interval in which entanglement is not observed by our test. This indicates that a phase super-resolution without entanglement might occur, as reported in Ref. [46].
B. Influence of turbulence

Now we will focus on the influence of turbulent losses on an initially pure N00N state. In particular we want to examine the entanglement properties of a N00N state including imperfections due to atmospheric beam wandering, see Sec. II B. Depending on the propagation distance, we study the properties of N00N states in urban quantum free-space links for the parameters $W_0 = 0.98 \text{ mm}$, $C_n = 10^{-17} \text{ m}^{-2/3}$, and $a = W$, cf. also Appendix A. The choice of parameters is done according to the experimental realization in [30, 41]. This set of parameters corresponds to a mean transmission coefficient of $T \approx 0.843$ at a distance of 200 m within one mode. For the entanglement verification, the test operator (21) is used.

In Fig. 4 this entanglement probe is displayed for $N = 2$ in the co-propagating and counter-propagation case, cf. Eq. (7) and (6), respectively. Its dependency on $d$, the distance between the sender and receiver and the two receivers for co- and counter-propagation, respectively, is shown. Here one sees that entanglement of a N00N states can survive in such an urban link and entanglement can be transfered up to a distance of about 600 m. It is important to stress that typically applied post-selection protocols may improve the range of successful entanglement propagation significantly; see, e.g., the approaches in Ref. [38, 47].

A sub-standard quantum limit behavior cannot be observed for certain distances $d$ in Fig. 4 since phase super-resolution of noisy N00N states is only obtained if the matrix entries $\rho_{0N,N0}$ are large enough, cf. Eq. (15). However, they scale with the 2Nth moment of the corresponding PDTC distribution. More generally, each interference term after an atmospheric channel is rescaled with

$$\int_0^1 \int_0^1 d\sqrt{\kappa} d\sqrt{\theta} P(\sqrt{\kappa}, \sqrt{\theta}) \sqrt{\kappa} \sqrt{\theta}^N,$$

see Eqs. [3] and [4]. Since $\kappa, \theta \leq 1$, we get a decreasing interference term for increasing $N$ and for every PDTC $P(\sqrt{\kappa}, \sqrt{\theta})$ – except the ideal case of no loss, i.e., $P(\sqrt{\kappa}, \sqrt{\theta}) = \delta(\sqrt{\kappa} - 1)\delta(\sqrt{\theta} - 1)$.

Let us stress again that the coherences $\rho_{0N,N0}$ are crucial for determining both the phase resolution and entanglement. On the one hand, a small value of $N$ might be preferable in some quantum communication scenarios in turbulent media. On the other hand, a large $N$ value is favorable in the non-perturbed case. Hence, the present method can be used to predict an optimal choice of $N$ conditioned on the actual turbulence properties of a free space link. This example also demonstrates that entanglement of N00N states tolerates turbulent losses better than its phase super-resolution property.

C. Combination of mixing and turbulence effects

Here we investigate an example of a mixed N00N state in a turbulent loss channel. The entanglement properties of polarization degrees of freedom are studied. A co-propagation in the same spatial mode is considered and both modes undergo the same loss. For the atmospheric channel we choose the parameters $W = 0.9 a$, $\sigma = a$ [42], corresponding to a mean transmission coefficient of $T = \sqrt{\langle T^2 \rangle} \approx 0.702$. The example of a mixed N00N state is given by a geometric series-like form

$$\hat{\rho}_{\text{mixed}} = \frac{1 - q}{q} \sum_{N=1}^{\infty} q^N |\psi_N\rangle\langle\psi_N|,$$

with $0 < q < 1$. The entanglement condition given in Eq. (32), is applied to this state including turbulent loss and plotted depending on the parameter $q$ in Fig. 5. The right and left hand side of the entanglement test (32), $\langle \hat{A} \rangle$ and the threshold $1/4$, respectively, are displayed. One observes that for a wide range of $q$ values entanglement is revealed by this test. In two ways this is a notable result. First it exhibits the strength of the used witness method based on the SEP. Second, it shows that entanglement survives in an atmospheric free-space channel even for non-ideal mixed N00N states without post-selection.

VI. SUMMARY AND CONCLUSIONS

Motivated by the stochastic generation of N00N states and by imperfections such as losses and dephasing, we introduced a generalized class of noisy N00N states. For this class of states conditions for sub-standard quantum
in dependence on the parameter $q$ which determines the mixing of N00N states in a geometric series \cite{36}. Additionally the bound $1/4$ of the entanglement condition \cite{32} is displayed by the horizontal line. For each $q$ with $\langle \hat{A} \rangle > 1/4$, entanglement is certified.

limit behavior in an Mach-Zehnder interferometer have been studied depending on certain perturbations of N00N states. Especially the influence of dephasing on phase super-resolution has been considered.

For the general class of noisy N00N states, a family of entanglement tests has been constructed. This has been achieved by solving the underlying coupled set of separability eigenvalue equations. In particular, the same operators which describe the interferometric measurements for phase super-resolution measurements have been used as entanglement witnesses. This renders it possible to use one setup to determine both: entanglement and substandard quantum limit phase measurements.

The propagation of noisy N00N states in atmospheric links – being described as fluctuating loss channels – has been investigated. Here we could identify, for the effect of beam wandering, certain regions of turbulence which partly preserve the quantum features of the perfect N00N states. Applying the here derived entanglement condition, we could identify regions of imperfections – e.g.: mean turbulent loss, propagation distance, impurity of the states – allowing the identification of entanglement without applying post-selection procedures.

In conclusion, our approach allows the verification of quantum properties of light in highly perturbed systems. The described methods yield lower bounds to the survival of quantum features after propagation in free-space communication links. The bounds might be enhanced by using more general entanglement probes as derived in Appendix \textcolor{blue}{B} or employing post-selection strategies. Our technique yields a useful tool for experimentalists to predict the conservation or loss of quantum correlations in terms of entanglement and phase super-resolution. Hence, the outlined method opens the possibility to employ perturbed quantum states in authentic applications, such as high precision measurements in quantum metrology and quantum key distribution for secure quantum communication.

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### Appendix A: Parameters of the atmosphere

In Eq. \textcolor{blue}{5} the log-negative Weibull distribution was given which describes the PDTC of beam wandering. The needed parameters are \textcolor{blue}{38}

\begin{align*}
W &= W_0 \sqrt{1 + \left( \frac{z\lambda}{\pi W_0^2} \right)^2}, \quad (A1) \\
T_0^2 &= 1 - \exp \left[ -2 \frac{a^2}{W^2} \right], \quad (A2) \\
\zeta &= 8 \frac{a^2}{W^2} \exp \left[ -4 \frac{a^2}{W^2} I_1 \left( 4 \frac{a^2}{W^2} \right) \right] \times \left[ \ln \left( \frac{2T_0^2}{1 - \exp \left[ -4 \frac{a^2}{W^2} I_0 \left( 4 \frac{a^2}{W^2} \right) \right]} \right) \right]^{-1}, \quad (A3) \\
S &= a \left[ \ln \left( \frac{2T_0^2}{1 - \exp \left[ -4 \frac{a^2}{W^2} I_0 \left( 4 \frac{a^2}{W^2} \right) \right]} \right) \right]^{-\frac{1}{\zeta}}, \quad (A4)
\end{align*}

and

\[ \sigma^2 \approx 1.919 C_n^2 z^2 (2W_0)^{-\frac{1}{7}}. \tag{A5} \]

The meaning of the remaining parameters are listed in Table \textcolor{blue}{I}.

| Symbol | Meaning |
|--------|---------|
| $W$    | beam-spot radius at the aperture |
| $W_0$  | beam-spot radius at the radiation source |
| $a$    | aperture radius of the detector |
| $z$    | distance between radiation source and detector |
| $S$    | scale parameter |
| $\zeta$ | shape parameter |
| $\sigma^2$ | variance of the beam-center position |
| $C_n^2$ | index of refraction structure constant |
| $\lambda$ | wavelength |
| $I_n$ | $n$th modified Bessel function |

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|}
\hline
Symbol & Meaning \\
\hline
$W$ & beam-spot radius at the aperture \\
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\hline
\end{tabular}
\end{table}
Appendix B: Solution of separability eigenvalue equation

Please note that the following solution of separability eigenvalue equation is straightforward. However, we formulate a detailed calculation which can be followed step by step. We have to solve the SEP for general operators $\hat{L}$ in 
for employing them as entanglement probe given in 
Hence we have to compute all SEVs. Notice that $\hat{L}$ is structurally symmetric for both subsystems. Each subsystem’s eigenvalue equation [18], can be decomposed in Fock-basis into the block form

\[
\begin{pmatrix}
(\square) & (\square) \\
\vdots & \vdots \\
(\square) & (\square)
\end{pmatrix}
= g
\begin{pmatrix}
(\square) \\
\vdots \\
(\square)
\end{pmatrix},
\]

(B1)

In the matrix all entries are zero except those in the boxes. The boxes $\square$ and the rectangles $\mathcal{I}$ correspond to matrices of the form $M = M_{00} \langle 0 | 0 \rangle + M_{11} \langle i | i \rangle$ and similar for $M_{01} \langle 0 | i \rangle$ and $M_{10} \langle i | 0 \rangle$ and vectors of the form $| c \rangle = c_0 | 0 \rangle + c_i | i \rangle$, respectively. Consequently, without loss of generality, it is sufficient to just deal with a problem in two-qubit sub-spaces $\{ | 0 \rangle, | i \rangle \} \otimes \{ | 0 \rangle, | i \rangle \} (i \neq 0)$ with the normalization $| c_0 |^2 + | c_i |^2 = 1$ and

\[
\hat{L}_i | a, b \rangle = g | a, b \rangle + | \chi \rangle, \\
| \chi \rangle = h | a^\perp, b^+ \rangle,
\]

(B3)

where $| \chi \rangle$ is the perturbation term with the bi-orthogonal vector $| a^\perp, b^+ \rangle$, with $\langle a | a^\perp \rangle = \langle b | b^+ \rangle = 0$ and $h \in \mathbb{C}$.

We can directly identify the obvious solutions of $\hat{L}_i$:

\[
g = \langle 0 | 0 \rangle = | a, b \rangle = | 0, 0 \rangle, \\
g = L_i^{(A)} | a, b \rangle = | i, 0 \rangle, \\
g = L_i^{(B)} | a, b \rangle = | 0, i \rangle, \\
g = 0 | a, b \rangle = | i, i \rangle.
\]

(B4-B7)

An appropriate parametrization of all other separability eigenvalue vectors is

\[
| a, b \rangle = \frac{| 0 \rangle + \alpha | i \rangle}{\sqrt{1 + | \alpha |^2}} \otimes \frac{| 0 \rangle + \beta | i \rangle}{\sqrt{1 + | \beta |^2}},
\]

(B8)

and for the corresponding orthogonal vectors

\[
| a^\perp, b^+ \rangle = \frac{-\alpha | 0 \rangle + | i \rangle}{\sqrt{1 + | \alpha |^2}} \otimes \frac{-\beta | 0 \rangle + | i \rangle}{\sqrt{1 + | \beta |^2}}.
\]

(B9)

We now may decompose $\alpha$ and $\beta$ in the form $\alpha = r_a e^{i\varphi}$ and $\beta = r_b e^{i\theta}$ with $r_a, r_b \in \mathbb{R}$. Note that a negative value $r_{a,b}$ corresponds to a $\pi$-phase shift. The parametrization and decomposition in Eq. (B2) leads then to the following components of the second form of the SEP:

\[
L_i^{(A)} = g + h r_a r_b e^{-i(\varphi + \theta)}, \\
L_i^{(B)} = g r_a e^{i\varphi} - h r_b e^{-i\theta}, \\
L_i^{(B)} = g r_a e^{i\varphi} - h r_b e^{-i\varphi},
\]

(B10-B12)

Introducing $h' = h e^{-i(\varphi + \theta)}$ and $\gamma_i' = \gamma_i e^{i(\theta - \varphi)}$, we end up with

\[
L_i^{(A)} = g + h' r_a r_b, \\
L_i^{(A)} = g r_a e^{i\varphi} - h' r_a, \\
L_i^{(B)} = g r_a e^{i\varphi} - h' r_a + h', \\
0 = g r_a r_b + h'.
\]

(B14-B17)

In a first step we will consider the special cases: $\gamma_i = 0$, $| a^\perp \rangle = 0$, $| b^+ \rangle = 0$, $L_0 = 0$, $h = 0$ and $g = 0$. For the case $\gamma_i = 0$, $\hat{L}_i$ is given in its spectral decomposition

\[
\hat{L}_i = L_i^{(0)} | 0, 0 \rangle \langle 0, 0 | + L_i^{(A)} | i, 0 \rangle \langle i, 0 | + L_i^{(B)} | 0, i \rangle \langle 0, i |.
\]

(B18)

and $g$ is given by

\[
g_{\text{max}} = \max_{i \in \mathbb{N}} \{ L_i^{(0)}, L_i^{(A)}, L_i^{(B)} \}.
\]

(B19)

For the case $| a^\perp \rangle = 0$ and $| b^+ \rangle = 0$ we directly get

\[
g = L_i^{(0)} \quad \text{and} \quad | a, b \rangle = | 0, 0 \rangle.
\]

(B20)

The case $L_i^{(0)} = 0$ leads to the condition

\[
0 = h' (1 - (r_a r_b)^2).
\]

(B21)

If $(r_a r_b)^2 = 1$ then the solution is given by

\[
g = \frac{L_i^{(A)} L_i^{(B)} - \gamma_i'^2}{2 \gamma_i' - (L_i^{(A)} + L_i^{(B)})},
\]

(B22)

and

\[
\varphi = \frac{\xi + \arg(g_i)}{2}, \quad \theta = \frac{\xi - \arg(g_i)}{2},
\]

(B23)

with $\xi = \arg(h)$ being a free parameter which leaves $g$ invariant. The second possibility $h' = 0$ which solves leads to $g = 0$, which is only a solution if $\hat{L}_i$ is of such form that $\gamma_i'^2 = L_i^{(A)} L_i^{(B)}$.

From $g = \langle a, b \rangle \hat{L}_i | a, b \rangle \in \mathbb{R}$ and Eq. (B17) follows that $h' \in \mathbb{R}$. Combining Eq. (B15) and (B16) one obtains

\[
\begin{pmatrix}
L_i^{(A)} - g & \gamma_i' + h' \\
\gamma_i' + h' & L_i^{(B)} - g
\end{pmatrix}
\begin{pmatrix}
r_a \\
r_b
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

(B24)
which has a real-valued solution if and only if \( \gamma' \in \mathbb{R} \). More generally, this equation has a solution if and only if

\[
0 = \det \left( \frac{L_i^{(A)} - g}{\gamma_i' + h'} \right) = g^2 - \left( L_i^{(A)} + L_i^{(B)} \right) g + L_i^{(A)} L_i^{(B)} - (\gamma_i' + h')^2. \tag{B25}
\]

If we now compare Eq. (B14) and (B17) we get

\[
\frac{L^{(0)} - g}{h'} = r_a r_b = -\frac{h'}{g} \tag{B26}
\]

which results in

\[
h'^2 = g^2 - g L^{(0)}. \tag{B27}
\]

Subtracting Eq. (B27) from Eq. (B25) one obtains

\[
h' = -g \frac{L_i^{(A)} + L_i^{(B)} - L^{(0)}}{2 \gamma_i'} + \frac{L_i^{(A)} L_i^{(B)} - \gamma_i'^2}{2 \gamma_i'}. \tag{B28}
\]

Together with Eq. (B27) we end up with a quadratic equation for \( g \):

\[
0 = \frac{\left( L_i^{(A)} L_i^{(B)} - \gamma_i'^2 \right)}{2 \gamma_i'}^2 + 2 \frac{\left[ 2 \gamma_i'^2 L^{(0)} - \left( L_i^{(A)} + L_i^{(B)} - L^{(0)} \right) \left( L_i^{(A)} L_i^{(B)} - \gamma_i'^2 \right) \right]}{2 \gamma_i'} g + \left[ \left( L_i^{(A)} + L_i^{(B)} - L^{(0)} \right)^2 - 4 \gamma_i'^2 \right] g^2. \tag{B29}
\]

Eq. (B29) is a quadratic equation of the form

\[
0 = w + 2vg + ug^2. \tag{B30}
\]

Equating coefficients yields \( u, v, w \) and the solutions:

\[
g_{i, \pm} = \frac{-v \pm \sqrt{v^2 - uw}}{u}. \tag{B31}
\]

With these solutions one can identify \( h' \) via Eq. (B28). Inserting this to Eq. (B24) we can evaluate \( r_a \) and \( r_b \) to be

\[
\left( \begin{array}{c}
    \frac{r_a}{r_b}
\end{array} \right) = \pm \frac{1}{\sqrt{2 \Delta}} \left( \begin{array}{c}
    \sqrt{\Delta} \pm \left( L_i^{(A)} - L_i^{(B)} \right) \\
    \sqrt{\Delta} \pm \left( L_i^{(B)} - L_i^{(A)} \right)
\end{array} \right), \tag{B32}
\]

with \( \Delta = \sqrt{\left( L_i^{(A)} - L_i^{(B)} \right)^2 + 4 \left( \gamma_i' + h' \right)^2}. \tag{B33} \)

The corresponding phases can be obtained form the relations \( \gamma' = \gamma e^{i(\varphi - \theta)} \in \mathbb{R} \) and \( h' = h e^{-i(\varphi + \theta)} \in \mathbb{R} \) leading to

\[
\arg(\gamma) = \varphi - \theta \quad \text{and} \quad \arg(h) = \theta + \varphi. \tag{B34}
\]

Thus, the phases are given by

\[
\vartheta = \frac{\xi - \arg(\gamma)}{2}, \quad \varphi = \frac{\xi + \arg(\gamma)}{2} \tag{B35, B36}
\]

with \( \xi = \arg(h) \) being a free parameter which leaves \( g \) invariant, cf. Eq. (B33).

The condition \( r_a, r_b \in \mathbb{R} \) is assured as \( \Delta \) in Eq. (B33) is real-valued and the term \( \Delta \pm \left( L_i^{(A)} - L_i^{(B)} \right) = \Delta \mp \left( L_i^{(B)} - L_i^{(A)} \right) \) in Eq. (B32) is non-negative. Next we also have to consider the case if \( \Delta \) becomes zero. Then, we get the eigenvalue \( g = L_i^{(A)} = L_i^{(B)} \) where additionally \( L_i^{(A)} \geq L^{(0)} \) has to be fulfilled. We may choose

\[
r_a = r_b = \pm \frac{1}{2} \sqrt{1 - \frac{L^{(0)}}{L_i^{(A)}}}. \tag{B37}
\]

The corresponding phases can be chosen arbitrarily. Finally we conclude that

\[
g_{\max} = \max_{i \in N \setminus \{0\}} \left\{ 0, L^{(0)}, L_i^{(A)}, L_i^{(B)}, g_{i, \pm} \right\}. \tag{B38}
\]

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