Ginzburg Landau theory for $d$-wave pairing and fourfold symmetric vortex core structure

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The Ginzburg Landau theory for $d_{x^2-y^2}$-wave superconductors is constructed, by starting from the Gor’kov equation with including correction terms up to the next order of $\ln(T_c/T)$. Some of the non-local correction terms are found to break the cylindrical symmetry and lead to the fourfold symmetric core structure, reflecting the internal degree of freedom in the pair potential. Using this extended Ginzburg Landau theory, we investigate the fourfold symmetric structure of the pair potential, current and magnetic field around an isolated single vortex, and clarify concretely how the vortex core structure deviates from the cylindrical symmetry in the $d_{x^2-y^2}$-wave superconductors.

KEYWORDS: $d$-wave superconductor, vortex core structure, Ginzburg Landau theory

§1. Introduction

Much attention has been focused on a vortex structure in high-$T_c$ superconductors. It is expected to be clarified how the vortex structure of the high-$T_c$ superconductors is different from that of the conventional superconductors. By a number of experimental and theoretical investigations, it is concluded that the symmetry of these superconductors is most likely to be $d_{x^2-y^2}$-wave. Therefore, one of the points is how the vortex structure of $d_{x^2-y^2}$-wave superconductors is different from that of isotropic $s$-wave superconductors. The $d_{x^2-y^2}$-wave pairing, i.e., $k_x^2 - k_y^2$ in momentum space, has fourfold symmetry for the rotation about the $c$-axis. We expect that, reflecting this symmetry, the core structure of an isolated single vortex may break the cylindrical symmetry and show fourfold symmetry in $d_{x^2-y^2}$-wave superconductors.

In their experiments, Keimer et al.\textsuperscript{\textsection} reported an oblique lattice by a small-angle neutron scattering study of the vortex lattice in YBa$_2$Cu$_3$O$_7$ in a magnetic field region of $0.5 \leq H \leq 5$T applied parallel to the $c$-axis. The vortex lattice has an angle of 73° between the two primitive vectors and is oriented such that the nearest-neighbor direction of vortices makes an angle of 45° with the $a$-axis. The oblique lattice was also observed by scanning tunneling microscopy (STM) by Maggio-Aprile et al.\textsuperscript{\textsection} They also observed the elliptic-shaped STM image of the vortex core, and concluded that this oblique lattice cannot be explained by considering only the effect of the intrinsic in-plane anisotropy, that is, the difference of the coherence lengths between $a$-axis and $b$-axis directions.\textsuperscript{\textsection} It is suggested that this deformation from the triangular lattice in $d_{x^2-y^2}$-wave superconductors is due to the effect of the fourfold symmetric vortex core structure.

The fourfold symmetric vortex core structure in $d_{x^2-y^2}$-wave superconductors was so far derived theoretically when the $s$-wave component is induced around a vortex of $d_{x^2-y^2}$-wave superconductivity.\textsuperscript{\textsection} This mixing scenario was mainly studied by Berlinsky et al.\textsuperscript{\textsection} and Ren et al.\textsuperscript{\textsection} in the two-component Ginzburg Landau (GL) theory for $s$- and $d$-wave superconductivity. According to the consideration based on the two-component GL theory, it is possible that the $s$-wave component is coupled with the $d$-wave component through the gradient terms. Therefore, the $s$-wave component may be induced when the $d$-wave order parameter spatially varies, such as near the vortex or interface under certain restricted conditions.\textsuperscript{\textsection} The induced $s$-wave component around the vortex is fourfold symmetric. The resulting vortex structure in $d$-wave superconductors, therefore, exhibits fourfold symmetry.

For this scenario to be applied to high-$T_c$ superconductors, the amplitude of the induced $s$-wave component should be comparable to the already-existing main $d_{x^2-y^2}$-wave component. The amplitude of the induced $s$-wave component is proportional to the $s$-wave component $V_s$ of the pairing interaction.\textsuperscript{\textsection} Then, in the case where $|V_s|$ is negligibly small compared with the dominant $d_{x^2-y^2}$-wave pairing interaction, the induced $s$-wave order parameter is negligible. In this paper, we consider this pure $d_{x^2-y^2}$-wave case, that is, the case when the induced $s$-wave component can be neglected. In our opinion, as the first step to understanding the fourfold symmetric vortex core in $d_{x^2-y^2}$-wave superconductors, the pure $d_{x^2-y^2}$-wave case should be studied before considering the mixing of the

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induced s-wave component. In this case, we have to consider a new scenario to explain the fourfold symmetric vortex core structure.

When the induced s-wave component can be neglected, the order parameter reduces to be one component, i.e., only the \( d_{x^2-y^2} \)-wave component, and the GL equation reduces to the same form as that of the isotropic s-wave case within the conventional GL theory. Within this framework, the vortex core structure remains cylindrically symmetric, and the vortex lattice forms a triangular lattice, even in the \( d_{x^2-y^2} \)-wave superconductors. On the other hand, Ichiki et al.\(^{13}\) showed that the vortex core structure is fourfold symmetric in the pure \( d_{x^2-y^2} \)-wave superconductors by using the quasi-classical Eilenberger theory. This means that we have to modify the conventional GL theory to explain the fourfold symmetric vortex core structure. For this purpose, we consider the following scenario.

Strictly speaking, the conventional GL equation is valid only near the transition temperature \( T_c \). Far from \( T_c \), we have to include several correction terms of the order \( \ln(T_c/T) \), which consist of both the higher powers of the order parameter and higher order derivatives, i.e., the so-called nonlocal terms. Among them, some nonlocal terms break cylindrical symmetry in the \( d_{x^2-y^2} \)-wave pairing case. Therefore, these correction terms lead to the fourfold symmetry of the vortex structure.

So far, these terms neglected in the conventional GL theory were considered to explain the fourfold symmetric behavior of the upper critical field \( H_{c2} \) in the \( ab \)-plane and the deformation of the vortex lattice from a triangular lattice. As for the fourfold symmetric behavior of \( H_{c2} \), Takanaka and Kuboya\(^{13}\) showed it from the GL theory with nonlocal correction terms, and Won and Maki\(^{15}\) from the Gor’kov equation. A stable vortex lattice was discussed by Won and Maki in the pure \( d_{x^2-y^2} \)-wave superconductors. As for the case of the isotropic s-wave pairing but anisotropic density of states at the Fermi surface, Takanaka and Nagashima\(^{14}\) derived the GL equation with including the higher order correction terms, and studied the anisotropy of \( H_{c2} \) and a stable vortex lattice. However, the anisotropic vortex core structure has not been studied so far. In this paper, we discuss the fourfold symmetric vortex core structure by considering the effect of these correction terms.

The purposes of this paper are, first, to construct the GL theory from the Gor’kov equation for the pure \( d_{x^2-y^2} \)-wave superconductors with including correction terms of the order \( \ln(T_c/T) \) and, second, to investigate the fourfold symmetric core structure (i.e., pair potential, current and magnetic field) of an isolated single vortex by using this extended GL framework. We clarify how the vortex core structure deviates from the cylindrical symmetry. Here we consider the case of an isolated single vortex under a magnetic field applied parallel to the \( c \)-axis (or \( z \)-axis) in the clean limit. The Fermi surface is assumed to be two-dimensional, which is appropriate to high-\( T_c \) superconductors, and isotropic in order to clarify effects of the \( d \)-wave nature of the pair potential on the vortex core structure. The additional anisotropy coming from, e.g., the Fermi surface can be incorporated into our extended GL framework.

The rest of this paper is organized as follows. In §2 we construct the GL theory with including correction terms for the \( d_{x^2-y^2} \)-wave pairing. By using this GL theory, we investigate the fourfold symmetric vortex core structure. The pair potential is studied in §3 and current and magnetic field in §4. The summary and discussions are given in §5. We set \( h = c = \hbar = 1 \) throughout the paper.

§2. GL theory for \( d_{x^2-y^2} \)-wave pairing

We consider the GL theory in pure \( d_{x^2-y^2} \)-wave superconductors. The pair potential is given by

\[
\Delta(r, k) = \eta(r) \sqrt{2} \cos 2\theta. \tag{2.1}
\]

Here, \( r = (x, y) = (r \cos \phi, r \sin \phi) \) is the center of mass coordinate and \( k \) the relative coordinate of the Cooper pairs. Now, \( k \) is denoted by an angle \( \theta \) measured from the \( a \)-axis (or \( x \)-axis) in the \( ab \)-plane. To study the effect of correction terms in the order of small parameter \( \ln(T_c/T) \approx 1 - T/T_c \), we consider the GL theory by including the next higher order terms of \( \eta \) and its derivatives other than those in the conventional GL theory. The GL equation and the current density \( j \) are derived as follows from the Gor’kov equation within the weak coupling approximation (see the detailed derivation in Appendix A),

\[
\ln \left( \frac{T_c}{T} \right) \eta - \beta \left( |\eta|^2 - \frac{\eta^2}{2} D^2 \eta \right) + \alpha \beta^2 \left[ \frac{5\eta^2}{6} |\eta|^4 + \frac{\eta^4}{288} \left( 7D_x^4 + D_y^2 D_x^2 + 7D_y^4 \right) \eta \right] - \frac{\eta^2}{12} \left( 4|\eta|^2(D^2 \eta)^* + \eta^2(D^2 \eta)^* + 2\eta |D\eta|^2 + 3(D\eta)^2 \eta^* \right) + \text{(higher order terms)} = 0, \tag{2.2}
\]

\[
j_x = 2j_0 \text{Im} \left[ (D_x \eta)^* + \frac{\eta^3}{48} \alpha \beta \left( 7(D_x^2 \eta)^* - 7(D^2_x \eta)(D_x \eta)^* + (D_x \eta)(D_y^2 \eta)^* + (D_x D_y^2 \eta)^* \eta^* \right) \right. \\
\left. - 3\alpha \beta |\eta|^2 (D_x \eta)^* \eta^* + \text{(higher order terms)} \right], \tag{2.3a}
\]
\[ j_y = 2j_0 \text{Im} \left[ \left( D_y \eta \right) \eta^* + \frac{\eta^2}{48} \alpha \beta \left\{ 7(D_y^3 \eta) \eta^* - 7(D_y^2 \eta)(D_y \eta)^* + (D_y \eta)(D_y^2 \eta)^* + (D_x \eta)(\overline{D_y D_y \eta})^* \right\} \right. \]
\[ \left. - 3 \alpha \beta |\eta|^2 (D_y \eta) \eta^* \right] + \text{(higher order terms)}, \tag{2.3b} \]

where
\[ \alpha = \frac{62 \zeta(5)}{49 \zeta(3)^2} = 0.908..., \quad \beta = \frac{21 \zeta(3)}{16 \pi^2 T^2}, \quad j_0 = -\frac{1}{3} |e| N_F v_F^2 \beta \tag{2.4} \]

with Riemann’s \( \zeta \)-functions \( \zeta(3) \) and \( \zeta(5) \), the Fermi velocity \( v_F \) and the density of states at the Fermi surface \( N_F \). The differential operator \( D \) (\( D^* \)) is defined by \( D = \nabla + i(2\pi/\phi_0) A \) (\( D^* = \nabla - i(2\pi/\phi_0) A \)) with the vector potential \( A \) and the flux quantum \( \phi_0 \). In Eqs. (2.2) and (2.3), \( D^m D^n \) means to take all the permutations of the product, such as \( D_x^m D_y^n \).

It is convenient to discuss Eqs.(2.2) and (2.3) in the following dimensionless unit,
\[ \eta/\eta_0 \rightarrow \eta, \quad r/\xi \rightarrow r, \quad j/(\eta_0^2/\xi) j_0 \rightarrow j \tag{2.5} \]

with the GL coherent length \( \xi = \sqrt{\alpha v_F / 6 \ln(T_c/T)} \) and the energy gap \( \eta_0 = \sqrt{\ln(T_c/T)} / \beta \).

In the dimensionless unit, the GL equation in Eq.(2.2) and \( j \) in Eq.(2.3), respectively, are written as
\[ \eta - \eta |\eta|^2 + D^2 \eta + \alpha \ln \left( \frac{T_c}{T} \right) \left[ \frac{5}{6} |\eta|^4 + \frac{1}{8} \left( 7D_y^4 + D_x^2 D_y^2 + 7D_y^4 \right) \eta \right. \]
\[ - \frac{1}{2} \left\{ 4|\eta|^2 (D^2 \eta) + \eta^2 (D^2 \eta)^* + 2|\eta| D \eta |\eta|^2 + 3(D \eta)^2 |\eta|^* \right\} + O(\{\ln(T_c/T)^2\}) = 0, \tag{2.6} \]

\[ j_x = 2 \text{Im} \left[ \left\{ 1 - 3 |\eta|^2 \alpha \ln \left( \frac{T_c}{T} \right) \right\} \right. \left( D_x \eta \right) \eta^* + \frac{\alpha}{8} \ln \left( \frac{T_c}{T} \right) \left\{ 7(D_y^3 \eta) \eta^* - 7(D_y^2 \eta)(D_x \eta)^* \right\}
\[ \left. + (D_x \eta)(D_y^2 \eta)^* + (D_x \eta)(\overline{D_y D_y \eta})^* \right\} + O(\{\ln(T_c/T)^2\}) \right], \tag{2.7a} \]

\[ j_y = 2 \text{Im} \left[ \left\{ 1 - 3 |\eta|^2 \alpha \ln \left( \frac{T_c}{T} \right) \right\} \left( D_y \eta \right) \eta^* + \frac{\alpha}{8} \ln \left( \frac{T_c}{T} \right) \left\{ 7(D_y^3 \eta) \eta^* - 7(D_y^2 \eta)(D_y \eta)^* \right\}
\[ \left. + (D_y \eta)(D_x^2 \eta)^* + (D_y \eta)(\overline{D_x D_y \eta})^* \right\} + O(\{\ln(T_c/T)^2\}) \right]. \tag{2.7b} \]

In Eqs. (2.6) and (2.7), the neglected terms, which are higher order of \( \eta \) and derivative, are in the order \( \{\ln(T_c/T)^2\}^n \) (\( n \geq 2 \)). We note that, in the limit \( T \rightarrow T_c \), Eqs. (2.6) and (2.7) reduce to those of the conventional GL theory, and are the same forms as those of the conventional s-wave case. This indicates that there are no difference between s-wave and \( d \)-wave pairing within the conventional GL framework. Therefore, the vortex structure for \( d \)-wave pairing remains cylindrically symmetric within the conventional GL framework. The difference between s-wave and \( d \)-wave pairing first appears in the correction terms of the order \( \ln(T_c/T) \). In particular, the \( D^4 \)-terms in the GL equation (2.6) and the \( D^3 \)-terms in the current density (2.7) are seen to break the cylindrical symmetry, and lead to the fourfold symmetry. These correction terms play an important role as the temperature decreases below the transition temperature.

§3. Pair potential around a single vortex

3.1 Symmetry consideration

The \( d_{x^2-y^2} \)-wave pairing, i.e., \( \hat{k}_x^2 - \hat{k}_y^2 = \cos 2\theta \) in momentum space, has fourfold symmetry for the rotation about the \( c \)-axis. Reflecting this property, the symmetry of a single vortex in \( d_{x^2-y^2} \)-wave superconductors is in the class \( D^{(1)}(D_2) \times R \). In the limit \( T \rightarrow T_c \), the GL equation (2.6) reduces to the conventional GL equation, which is the same as that of the isotropic s-wave pairing case. Therefore, following the well known consideration about a single vortex, we obtain the structure of a single vortex as follows in the limit \( T \rightarrow T_c \),

\[ \Delta(r, \mathbf{k}) = \eta(r) \sqrt{2} \cos 2\theta = f_0(r) e^{i2\theta} \sqrt{2} \cos 2\theta, \tag{3.1} \]
which is the isotropic state with the winding number 1 around the vortex. As noted by Volovik, eight elements of the symmetry operations of the function $\Delta(r, k)$ in Eq. (3.1) form the group $D_4(E)$:

$$D_4(E) = \{ E, C_2e^{i\pi}, U_{2,x}R, U_{2,y}e^{i\pi}R, C_4e^{i\pi/2}, C_4^{-1}e^{-i\pi/2}, U_{2,x+y}e^{i\pi/2}R, U_{2,x-y}e^{-i\pi/2}R \}. \tag{3.2}$$

Here, $R$ is the time-reversal, $C_n$ the $(2\pi/n)$-rotation about a $z$-axis, and $U_{2,x}, U_{2,y}, U_{2,x+y}$ and $U_{2,x-y}$ denote the $\pi$-rotation about the $x$-axis, the $y$-axis, the lines $x + y = 0$ and $x - y = 0$, respectively.

On lowering temperature, other components with different winding numbers may be induced in $\eta(r)$. If the symmetry $D_4(E)$ is conserved, the possible pair potential is generally restricted to the following form (brief derivation is given in Appendix B),

$$\eta(r) = \sum_{n=-\infty}^{\infty} f_n(r)e^{i(4n+1)\phi}, \tag{3.3}$$

where $f_n(r)$ is real. In the following, we determine the amplitude $f_n(r)$ from the GL equation (2.6).

### 3.2 Pair potential around a single vortex

We determine the amplitude $f_n(r)$ in Eq. (3.3) up to the order $\ln(T_c/T)$. Therefore we expand $f_n(r)$ in the powers of $\ln(T_c/T)$ as follows,

$$f_n(r) = f_{n}(0)(r) + \alpha \ln(T_c/T)f_{n}^{(1)}(r) + O((\ln(T_c/T))^2), \tag{3.4}$$

where $f_{n}^{(0)}(r) = 0$ for $n \neq 0$ since $\Delta(r, k)$ reduces to Eq. (3.1) in the limit $T \rightarrow T_c$.

We notice that, as we consider the isolated single vortex in the extreme type II superconductors (GL parameter $\kappa \gg 1$), the vector potential in $D$ can be neglected.

Substituting Eq. (3.4) into Eq. (2.6) and writing the differential operators by the cylindrical coordinate, we obtain the following simultaneous differential equations,

$$F_0^{(0)}(r) \equiv f_0^{(0)}(r) + \nabla^2(1) f_0^{(0)}(r) - f_0^{(0)}(r)^3 = 0, \tag{3.5}$$

$$F_0^{(1)}(r) = f_0^{(1)}(r) + \nabla^2(1) f_0^{(1)}(r) - 3 f_0^{(0)}(r)^2 f_0^{(1)}(r) + 3 \frac{1}{4} D_{1}(1) f_0^{(0)}(r)$$

$$- \frac{1}{2} \left( 5 f_0^{(0)}(r)^2 \nabla^2(1) f_0^{(0)}(r) + 5 f_0^{(0)}(r) f_0^{(0)}(r)^2 - \frac{1}{r^2} f_0^{(0)}(r)^3 \right) + \frac{5}{6} f_0^{(0)}(r)^5 = 0, \tag{3.6}$$

$$F_1^{(1)}(r) \equiv f_1^{(1)}(r) + \nabla^2(5) f_1^{(1)}(r) - \left( 2 f_1^{(1)}(r) + f_{-1}^{(1)}(r) \right) f_0^{(0)}(r)^2 + \frac{1}{16} D_{+}(1) f_0^{(0)}(r) = 0, \tag{3.7}$$

$$F_{-1}^{(1)}(r) \equiv f_{-1}^{(1)}(r) + \nabla^2(-3) f_{-1}^{(1)}(r) - \left( 2 f_{-1}^{(1)}(r) + f_{1}^{(1)}(r) \right) f_0^{(0)}(r)^2 + \frac{1}{16} D_{-}(1) f_0^{(0)}(r) = 0, \tag{3.8}$$

$$F_n^{(1)}(r) \equiv f_n^{(1)}(r) + \nabla^2(4n+1) f_n^{(1)}(r) - \left( 2 f_n^{(1)}(r) + f_{-n}^{(1)}(r) \right) f_0^{(0)}(r)^2 = 0, \quad (n \neq 0, \pm 1) \tag{3.9}$$

where the differential operators $\nabla^2(n), D_{1}(n)$ and $D_{\pm}(n)$ are defined as

$$\nabla^2(n) \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2}. \tag{3.10}$$

$$D_1(n) \equiv \frac{d^4}{dr^4} + \frac{2}{r} \frac{d^3}{dr^3} - \frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{r^2} \frac{d}{dr} - n^2 \left( \frac{2}{r^2} \frac{d^2}{dr^2} - \frac{2}{r^3} \frac{d}{dr} + \frac{4}{r^4} \right) + \frac{n^4}{r^4}, \tag{3.11}$$

$$D_{\pm}(n) \equiv \frac{d^4}{dr^4} - \frac{6}{r^2} \frac{d^3}{dr^3} + \frac{15}{2} \frac{d^2}{dr^2} - \frac{15}{r^2} \frac{d}{dr} \pm 4n \left( \frac{1}{r^3} \frac{d^3}{dr^3} - \frac{6}{r^2} \frac{d^2}{dr^2} + \frac{14}{r^3} \frac{d}{dr} - \frac{12}{r^4} \right)$$

$$+ n^2 \left( \frac{6}{r^2} \frac{d^2}{dr^2} - \frac{30}{r^3} \frac{d}{dr} + \frac{44}{r^4} \right) \pm 4n^3 \left( \frac{1}{r^3} \frac{d^3}{dr^3} - \frac{3}{r^4} \right) + \frac{n^4}{r^4}. \tag{3.12}$$

As for the boundary condition,

$$f_0^{(0)}(0) = f_1^{(1)}(0) = 0 \tag{3.13}$$
at the vortex center so that divergence does not occur at \( r \to 0 \), and

\[
f_0^{(0)}(r \to \infty) = 1, \quad f_0^{(1)}(r \to \infty) = \frac{5}{12}, \quad f_n^{(1)}(r \to \infty) = 0 \quad (n \neq 0)
\]  

(3.14)

far from the vortex center, where the \( r \)-dependence of \( \eta(r) \) can be neglected ensuring \( \eta(r) \) to become the bulk value. Equation (3.14) is obtained by setting the terms with the differential operators to be 0 in Eqs. (3.5) - (3.9).

Equation (3.5) corresponds to the conventional GL equation, leading to the well known vortex core structure of the conventional GL theory for \( f_0^{(0)}(r) \). First, we obtain \( f_0^{(0)}(r) \) from Eq. (3.5), and substitute it to Eqs. (3.6) - (3.9). Then, we obtain the correction terms \( f_0^{(1)}(r) \) from Eq. (3.6), and \( f_{\pm 1}^{(1)}(r) \) from Eqs. (3.7) and (3.8). From Eq. (3.9), we obtain \( f_n^{(1)}(r) = 0 \) for \( n \neq 0, \pm 1 \). In the order \( \ln(T_c/T) \), therefore, the components with \( e^{-3i\phi} \) and \( e^{5i\phi} \) are induced in addition to the isotropic components of \( e^{i\phi} \). Components with other winding numbers are induced in the higher order of \( \ln(T_c/T) \), for example, components with \( e^{-7i\phi} \) and \( e^{9i\phi} \) first appear in the order \( (\ln(T_c/T))^2 \).

The concrete form of the solution \( f_i^{(j)}(r) \) in Eqs. (3.5) - (3.9) cannot obtained by the \( r \)- or \( r^{-1} \)-expansion. When we set

\[
f_i^{(j)}(r) = \sum_{m=1}^{\infty} c_{i,m}^{(j)} r^m
\]

(3.15)
to study the solution for \( r \ll 1 \), we obtain the following results for \( m \leq 3 \),

\[
C_{0,m}^{(0)} = c_{0,m}^{(0)} = 0, \quad (m \neq 1, 3) \quad C_{1,m}^{(1)} = 0, \quad (m \neq 3) \quad C_{i,m}^{(1)} = 0 \quad (i \neq 0, -1),
\]

(3.16)

which means that \( f_0(r) = O(r) \), \( f_{-1}(r) = O(r^3) \) and \( f_1(r) = O(r^4) \). However, the other coefficients \( C_{i,m}^{(j)} \) cannot be determined uniquely in this \( r \)-expansion. On the other hand, when we set

\[
f_i^{(j)}(r) = \sum_{m=0}^{\infty} C_{i,m}^{(j)} r^{-m}
\]

(3.17)
to study the solution for \( r \gg 1 \), inconsistency among Eqs. (3.5) - (3.9) occurs in determining \( C_{1,-4}^{(1)} \) and \( C_{-1,-4}^{(1)} \). It suggests that the singular terms such as \( r^{-1/2}e^{-\tau} \) probably exist in addition to \( r^{-m} \)-terms.

3.3 Numerical Results

We solve numerically the differential equations (3.3) - (3.9) by using the so-called relaxation method, which is often used in determining the vortex structure from the GL theory. The relaxation step

\[
f_i^{(j)}(r)_{\text{[new]}} = f_i^{(j)}(r)_{\text{[old]}} + c F_i^{(j)}(r) \quad (c : \text{constant})
\]

(3.18)
is repeated until the condition \( |F_i^{(j)}(r)| \ll 1 \) is fulfilled for each \( r \), where \( F_i^{(j)}(r) \) is defined in Eqs. (3.7) - (3.9).

Figure 1 (a) shows \( f_0^{(1)}(r) \), which is the correction to the factor of the isotropic component \( e^{i\phi} \). As seen from Eq. (3.4), the correction \( \alpha \ln(T_c/T)f_0^{(1)}(r) \) is added to the solution of the conventional GL theory \( f_0^{(0)}(r) \). In Fig. 1 (b), \( f_{1}^{(1)}(r) \) and \( f_{-1}^{(1)}(r) \) are presented, where \( f_{1}^{(1)}(r) \) behaves as \( r^2 \) near the core while \( f_{-1}^{(1)}(r) \) as \( r^4 \). These components induce the fourfold symmetric structure of the pair potential. They both attain a maximum at several coherence length from the origin, and smoothly and slowly fall off. It is to be noted that they never drop exponentially. It indicates that the effect of the fourfold symmetric structure becomes small but remains far from the vortex core, and may affect a vortex-vortex interaction and the formation of a stable vortex lattice. This point may become important when considering a stable vortex lattice and its orientation relative to the underlying crystal in \( d \)-wave superconductors. It is confirmed that the other components \( f_n^{(1)}(r) \) \((|n| \geq 2)\) reduce to 0 in the relaxation method.

Using these solutions, the pair potential around a single vortex is written as

\[
\eta(r) = \left\{ f_0^{(0)}(r) + \alpha \ln(T_c/T) \left( f_0^{(1)}(r) + f_1^{(1)}(r)e^{4i\phi} + f_{-1}^{(1)}(r)e^{-4i\phi} \right) \right\} e^{i\phi} + O(\{\ln(T_c/T)^2 \}^2).
\]

(3.19)

Then, the amplitude of the pair potential around a single vortex is given by

\[
|\eta(r)| = \left\{ \left| f_0^{(0)}(r) + \alpha \ln(T_c/T) \left( f_0^{(1)}(r) + \cos 4\phi \left( f_1^{(1)}(r) + f_{-1}^{(1)}(r) \right) \right) \right| \right\}^2
\]

\[
+ \left\{ \alpha \ln(T_c/T) \sin 4\phi \left( f_1^{(1)}(r) - f_{-1}^{(1)}(r) \right) \right\}^2 \right)^{1/2}
\]

\[
= f_0^{(0)}(r) + \alpha \ln(T_c/T) \left| f_0^{(1)}(r) + \cos 4\phi \left( f_1^{(1)}(r) + f_{-1}^{(1)}(r) \right) \right| + O(\{\ln(T_c/T)^2 \}^2).
\]

(3.20)

As for the amplitude \(|\eta(r)|\) in Eq. (3.20), we see that the deviation from the cylindrical structure behaves like \( \cos 4\phi \)
Fig. 1. The spatial variation of the components of the order parameter around a vortex in the dimensionless unit: (a) $f_0^{(1)}(r)$, (b) $f_1^{(1)}(r)$ (solid curve) and $f_{-1}^{(1)}(r)$ (long dashed curve), associated with the phase factor $e^{i\phi}$, $e^{5i\phi}$ and $e^{-3i\phi}$, respectively.

Fig. 2. The spatial profile of the order parameter around a vortex. (a) The deviation of the amplitude $|\eta(r, \phi)|$ at $\phi = 0^\circ$ from $|\eta(r, \phi)|$ at $\phi = 45^\circ$ along the radial direction, where the factor of $\alpha \ln(T_c/T)$ in Eq. (3.21) is displayed. (b) The correction of the phase, $\Phi(r, \phi)$, at $\phi = 22.5^\circ$, where the factor of $\alpha \ln(T_c/T)$ in Eq. (3.23) is displayed.

Around a vortex in the order $\ln(T_c/T)$. In Fig. 2 (a), we show the $r$-dependence of the difference of the amplitude along the $0^\circ$ direction (along the $x$-axis and $y$-axis) and along the $45^\circ$ direction (along the line $y = \pm x$), i.e.,

$$|\eta(r, \phi = 0)| - |\eta(r, \phi = \pi/4)| = 2\alpha \ln(T_c/T) \left(f_1^{(1)}(r) + f_{-1}^{(1)}(r)\right).$$

(3.21)

In the figure, the factor of $\alpha \ln(T_c/T)$, i.e., $2(f_1^{(1)}(r) + f_{-1}^{(1)}(r))$ is plotted. It indicates that the anisotropy localizes around the core region, and that the amplitude along the $0^\circ$ direction is suppressed compared with that along the $45^\circ$ direction.

As for the phase of the pair potential, we consider the correction of the phase to the leading term which goes like $e^{i\phi}$. Then, we introduce $\Phi(r)$ defined by

$$\eta(r) = |\eta(r)|e^{i\phi + i\Phi(r)}.$$

(3.22)

From Eq. (3.19), the correction of the phase is given by
\[ \Phi(r) = \tan^{-1} \left( \frac{\alpha \ln(T_c/T) \sin 4\phi \left( f_1^{(1)}(r) - f_1^{(-1)}(r) \right)}{f_0^{(0)}(r) + \alpha \ln(T_c/T) \left\{ f_0^{(1)}(r) + \cos 4\phi \left( f_1^{(1)}(r) + f_1^{(-1)}(r) \right) \right\}} \right) \]

\[ = \alpha \ln(T_c/T) \sin 4\phi \left( f_1^{(1)}(r) - f_1^{(-1)}(r) \right) / f_0^{(0)}(r) + O(\ln(T_c/T)^2). \quad (3.23) \]

It indicates that \( \Phi(r) \) behaves like \( \sin 4\phi \) around a vortex in the order \( \ln(T_c/T) \). In Fig. 2 (b), we show the \( r \)-dependence of \( \Phi(r) \) along the 22.5° direction from the \( a \) axis, i.e., \( \Phi(r, \phi = \pi/8) \). In the figure, the factor of \( \alpha \ln(T_c/T) \), i.e., \( (f_1^{(1)}(r) - f_1^{(-1)}(r))/f_0^{(0)}(r) \) is plotted. It is seen that the anisotropy of the phase rather extends compared to that of the amplitude in Fig. 2 (a), and that \( \Phi(r) > 0 \) for the region \( 0 < \phi < \pi/4 \).

The above-mentioned behavior of the pair potential \( \eta(r) \) qualitatively agrees well with that obtained in the quasi-classical Eilenberger theory (see Figs. 1 and 2 in Ref. 12 for comparison).

§4. Current and magnetic field around a vortex

The current density and the magnetic field around a single vortex is considered with the correction terms up to the order \( \ln(T_c/T) \). Substituting Eq. (3.19) to Eq. (2.7), we obtained the radial component, \( j_r = j_x \cos \phi + j_y \sin \phi \), and the rotational component, \( j_\phi = -j_x \sin \phi + j_y \cos \phi \), of the current density as follows,

\[ j_r(r) = \alpha \ln(T_c/T) \sin 4\phi \left( j_1(r) + j_2(r) \right) + O(\ln(T_c/T)^2), \quad (4.1) \]

\[ j_\phi(r) = \frac{2}{r} f_0^{(0)}(r)^2 + \alpha \ln(T_c/T) \left\{ -j_3(r) + \cos 4\phi \left( j_1(r) + j_4(r) \right) \right\} + O(\ln(T_c/T)^2), \quad (4.2) \]

where

\[ j_1(r) \equiv -\frac{5 \alpha}{2r^2} f_0^{(0)}(r)^2 + \frac{2}{r^2} f_0^{(0)}(r)f_0^{(0)}(r) + \frac{1}{2r} \left( f_0^{(0)}(r)^2 - 2f_0^{(0)}(r)f_0^{(0)}(r) \right), \quad (4.3) \]

\[ j_2(r) \equiv 2 \left( f_0^{(0)}(r)f_0^{(1)}(r) - f_0^{(0)}(r)f_0^{(-1)}(r) + f_0^{(1)}(r)f_0^{(0)}(r) - f_0^{(1)}(r)f_0^{(0)}(r) \right), \quad (4.4) \]

\[ j_3(r) \equiv \frac{3}{r^2} f_0^{(0)}(r)^2 - \frac{4}{r^2} f_0^{(0)}(r)f_0^{(0)}(r) + \frac{1}{r} \left( f_0^{(0)}(r)^2 - 2f_0^{(0)}(r)f_0^{(0)}(r) + 3f_0^{(0)}(r)^4 \right) - \frac{4}{r} f_0^{(0)}(r)f_0^{(0)}(r), \quad (4.5) \]

\[ j_4(r) \equiv \frac{4}{r} f_0^{(0)}(r) \left( 3f_0^{(1)}(r) - f_0^{(-1)}(r) \right). \quad (4.6) \]

From Eqs. (4.1) and (4.2), the amplitude of the current density is given by

\[ |\mathbf{j}(r)| = \frac{2}{r} f_0^{(0)}(r)^2 + \alpha \ln(T_c/T) \left\{ -j_3(r) + \cos 4\phi \left( j_1(r) + j_4(r) \right) \right\} + O(\ln(T_c/T)^2), \quad (4.7) \]

which is the same expression as \( j_\phi \) in Eq. (4.2) within the order \( \ln(T_c/T) \). In the limit \( T \to T_c \), Eqs. (4.1), (4.2) and (4.7) give the cylindrically symmetric structure of the conventional GL theory. This profile of \( |\mathbf{j}(r)| \) is shown in Fig. 3 (a) as a function of \( r \).

From Eq. (4.7), we see that the deviation of \( |\mathbf{j}(r)| \) from the cylindrical structure first appears in the order \( \ln(T_c/T) \) and behaves like \( \cos 4\phi \) around a vortex. In Fig. 3 (b), we show the \( r \)-dependence of the difference of \( |\mathbf{j}(r)| \) along the 0° direction and along the 45° direction, i.e.,

\[ |\mathbf{j}(r, \phi = 0)| - |\mathbf{j}(r, \phi = \pi/4)| = 2\alpha \ln(T_c/T) \left( j_1(r) + j_4(r) \right). \quad (4.8) \]

In the figure, the factor of \( \alpha \ln(T_c/T) \), i.e., \( 2(j_1(r) + j_4(r)) \) is plotted. It is seen that \( |\mathbf{j}(r, \phi = 0)| < |\mathbf{j}(r, \phi = \pi/4)| \) near the vortex core and \( |\mathbf{j}(r, \phi = 0)| > |\mathbf{j}(r, \phi = \pi/4)| \) far from the core. It should be noted that this sign change in Fig. 3 (b) occurs at a point where \( f_0^{(1)}(r) \) in Fig. 1 (b) reaches the minimum.

The resulting magnetic field \( \mathbf{b}(r) \) can be found from the Maxwell equation:

\[ \mathbf{j}(r) = \kappa^2 \nabla \times \mathbf{b}(r) \quad (4.9) \]

in the dimensionless unit: Eq. (2.3) and

\[ \mathbf{b}(r)/\sqrt{2}H_c \kappa \to \mathbf{b}(r), \quad (4.10) \]
where $H_c = 1/\kappa^2 2\sqrt{2}$ is the thermodynamic critical field. From Eqs. (4.3), (4.4) and (4.9), the magnetic field, which has only the $z$-component, is given by

$$b_z(r) = b_z(r = 0) - \frac{1}{\kappa^2} \int_0^r \frac{2}{r} f_0^0(r')^2 dr' + \frac{1}{4\kappa^2} \alpha \ln(T_c/T) \left\{ (1 - \cos 4\phi) r \left( j_1(r) + j_2(r) \right) - 4 \int_0^r \left( j_1(r') - j_3(r') + j_4(r') \right) dr' \right\} + O(\ln(T_c/T)^2).$$

In the limit $T \to T_c$, Eq. (4.11) gives the cylindrically symmetric structure of the conventional GL theory. This profile of $b_z(r) - b_z(r = 0)$ is shown in Fig. 4 (a) as a function of $r$.

From Eq. (4.11), we find that the deviation of $b_z(r)$ from the cylindrical structure first appears in the order $\ln(T_c/T)$ and behaves like $\cos 4\phi$ around a vortex. In Fig. 4 (b), we show the $r$-dependence of the difference of $b_z(r)$ along the $0^\circ$ direction and along the $45^\circ$ direction, i.e.,

$$b_z(r, \phi = 0) - b_z(r, \phi = \pi/4) = -\frac{1}{2\kappa^2} \alpha \ln(T_c/T) r \left( j_1(r) + j_2(r) \right).$$

In the figure, the factor of $\kappa^{-2}\alpha \ln(T_c/T)$, i.e., $-r(j_1(r) + j_2(r))/2$ is plotted. It indicates that $b_z(r)$ is large along the $0^\circ$ direction compared with along the $45^\circ$ direction, that is, the magnetic field extends along the $0^\circ$ direction around a vortex. It is noted that, reflecting the sign change in Fig. 3 (b), the anisotropy of $b_z(r)$ becomes weaker far from the vortex core.

The above-mentioned behaviors of $j(r)$ and $b_z(r)$ qualitatively agree well with those obtained in the quasi-classical Eilenberger theory (see Figs. 4 - 7 in Ref. [12] for comparison).

§5. Summary and discussions

First, the GL theory is constructed from the Gor’kov equation with including the correction terms up to the order $\ln(T_c/T)$ in pure $d_{x^2−y^2}$-wave superconductors. In the limit $T \to T_c$, the GL equation and the current density in the $d_{x^2−y^2}$-wave case reduce to the same form as that of the isotropic $s$-wave superconductors. Thus, within the conventional GL framework, the vortex core structure remains cylindrically symmetric even in the $d_{x^2−y^2}$-wave case. The difference between the $s$-wave and $d_{x^2−y^2}$-wave first appears in the correction terms of the order $\ln(T_c/T)$.

Second, by using this extended GL theory, we investigate the fourfold symmetric vortex core structure in the $d_{x^2−y^2}$-wave superconductors. Among the correction terms, there are some terms which break the cylindrical symmetry and have fourfold symmetry, which is the fact reflecting the fourfold symmetry of the Cooper pair, $\vec{k}_x^2 - \vec{k}_y^2$. Then, the vortex core structure becomes fourfold symmetric. We study the fourfold symmetric structure of the pair potential, the current and the magnetic field around an isolated single vortex with the correction terms up to the order $\ln(T_c/T)$. As a result, the amplitude of the pair potential is suppressed along the $0^\circ$ direction compared with...
Fig. 4. The spatial profile of the magnetic field around a vortex. (a) The magnetic field $b_z(r) - b_z(r = 0)$ as a function of $r$ in the limit $T \to T_c$, where the factor of $\kappa^{-2}$ in Eq. (4.11) is displayed. (b) The deviation of $b_z(\phi = 0)$ from $b_z(\phi)$ at $\phi = 45^\circ$ as a function of $r$, where the factor of $\kappa^{-2} \alpha \ln(T_c/T)$ in Eq. (4.12) is displayed. This plot shows the deviation of $b_z(r)$ from the cylindrical structure. Far from the vortex core, this deviation decreases and the fourfold symmetric structure becomes weaker.

along the $45^\circ$ direction. The phase of the pair potential also shows fourfold symmetry around a vortex. Along the $45^\circ$ direction, the amplitude of the current is enhanced near the vortex core and suppressed far away from the vortex core compared with along the $0^\circ$ direction. Reflecting this behavior of the current, the magnetic field is enhanced along the $0^\circ$ direction near the vortex core, and this fourfold symmetry of the magnetic field fades out slowly far from the vortex core. These fourfold symmetric structures of the pair potential, the current and the magnetic field qualitatively agree well with the results by the quasi-classical Eilenberger theory. On lowering temperature, the correction terms of the order $\ln(T_c/T)$ make important roles, and the fourfold symmetric vortex core structure becomes clear.

It should be noted that, while our obtained vortex core structure has similar fourfold symmetry to that predicted by Xu et al. from the two-component GL equation, the origin of the fourfold symmetry is quite different from theirs. In their theory, the fourfold symmetry is induced by the mixing of the $s$-wave component. Then, in the case induced $s$-wave component is negligibly small, i.e., the pure $d_{x^2-y^2}$-wave case, the vortex core structure reduces to cylindrically symmetric one. On the other hand, we consider the pure $d_{x^2-y^2}$-wave case. Even in this case, the vortex core structure exhibits fourfold symmetry in our theory. It is due to the fourfold symmetric terms which are higher order of $\ln(T_c/T)$ and neglected in the conventional GL theory.

In connection with these correction terms, Takanaka and Kuboya and Won and Maki explained the fourfold symmetric behavior of $H_{c2}$ in the $ab$-plane and the deformation of the vortex lattice from a triangular lattice by considering the effect of these terms.

By the extended GL theory with the correction terms, we succeed in reproducing qualitatively the fourfold symmetric vortex core structure obtained by the quasi-classical Eilenberger theory. This fact indicates that the correction terms of the order $\ln(T/T_c)$ are important in the GL framework, when we consider the vortex structure in pure $d_{x^2-y^2}$-wave superconductors. The GL framework presented here is found to be useful to analyze the vortex core structure and to obtain further physical insight. While enormous numerical computations are needed in the Eilenberger theory, the GL theory gives the information about the vortex structure with concise calculations.

As shown in Fig. 1 (b), the effect of the fourfold symmetric vortex core structure becomes small but remains far from the vortex core. Then it may affect a vortex-vortex interaction and the formation of a stable vortex lattice. This point may become important when considering a stable vortex lattice and its orientation relative to the underlying crystal in $d$-wave superconductors, which belongs to our future study.

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Appendix A: Derivation of the GL equation
The GL equation (2.2) and the current expression (2.3) in the $d_{x^2-y^2}$-wave pairing case are derived from the Gor’kov equation with including both the higher power of the pair potential and higher order derivative. Here,
we consider the singlet anisotropic pairing case, in general. Similar derivation was presented by Takahata and Nagashima in the case of isotropic s-wave pairing but anisotropic density of states at the Fermi surface. In our derivation, higher order terms are considered up to the order $D^n\eta^n (m + n \leq 5)$, and the integral $\int dr$ and $\int dk/(2\pi)^2$ are noted by $\sum r$ and $\sum k$, respectively.

As for the anisotropic pairing, the pair potential has the non-local form $\Delta(r, r')$. Thus, in the Gor’kov equation for the Green function $G_{\omega_n}(r, r')$ and the anomalous Green functions $F_{\omega_n}^0(r, r')$ and $F_{\omega_n}^1(r, r')$, the term $\Delta(r)F_{\omega_n}(r, r')$ in the isotropic s-wave pairing case is replaced by $\sum_{r''} \Delta(r, r'')F_{\omega_n}(r'', r')$, and the self-consistent condition is modified to

$$\Delta^*(r, r') = V(r - r') T \sum_{\omega_n} F_{\omega_n}^1(r, r').$$  \hspace{1cm} (A.1)

From the Gor’kov equation, we obtain the relations

$$G_{\omega_n}(r, r') = G_{\omega_n}^0(r, r') - \sum_{r_1, r_1'} G_{\omega_n}^0(r, r_1) \Delta(r_1, r_1') F_{\omega_n}^1(r_1', r'),$$  \hspace{1cm} (A.2)

$$F_{\omega_n}^1(r, r') = \sum_{r_1, r_1'} G_{\omega_n}^0(r, r_1) \Delta^*(r_1, r_1') G_{\omega_n}(r_1', r'),$$  \hspace{1cm} (A.3)

by using the Green function of the normal state,

$$G_{\omega_n}^0(r, r') = \sum_p \frac{1}{\omega_n - \xi_p} e^{i\xi_p(r - r')}.$$  \hspace{1cm} (A.4)

For convenience, we introduce the center of mass coordinate $\mathbf{r} = (r + r')/2$ and the relative coordinate $\mathbf{r}' = r - r'$ of the Cooper pair. Then, we have $r = \mathbf{r} + \mathbf{r}'/2$, $r' = \mathbf{r} - \mathbf{r}'/2$ and $\nabla_r - \nabla_{r'} = 2\nabla_{\mathbf{r}}$. The pair potential and the pairing interaction are, respectively, assumed to be

$$\Delta(\mathbf{r}, k) \equiv \sum_{\mathbf{r}} e^{-ik\cdot\mathbf{r}} \Delta(r, r') = \eta(\mathbf{r}) \phi(k),$$  \hspace{1cm} (A.5)

$$V(k - k') \equiv \sum_{\mathbf{r}} e^{-i(k\cdot\mathbf{r} - k'\cdot\mathbf{r})} V(\mathbf{r}) = \tilde{V} \phi^*(k) \phi(k'),$$  \hspace{1cm} (A.6)

where $\phi(k)$ is a symmetry function, for example, $\phi(k) = \sqrt{2} \cos 2\theta$ in the $d_{x^2-y^2}$-wave pairing case and $\phi(k) = 1$ in the isotropic s-wave pairing case. Then, the Fourier transformation of $\Delta(r, r')$ is given by

$$\Delta(r, r') = \sum_{q, k} e^{i\mathbf{q}\cdot\mathbf{r} + i\mathbf{k}\cdot\mathbf{r}'} \eta(q) \phi(k).$$  \hspace{1cm} (A.7)

From Eqs. (A.5) and (A.6), the self-consistent condition (A.1) is rewritten as

$$\eta^*(r) = \tilde{V} T \sum_{\omega_n} \sum_k \phi(k) \sum_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}'} F_{\omega_n}^1(r, r').$$  \hspace{1cm} (A.8)

On the other hand, the current density is given by

$$j(r) = \frac{i|e|}{m} (\nabla_r - \nabla_{r'}) T \sum_{\omega_n} G_{\omega_n}(r, r') \bigg|_{r = r'} = \frac{2i|e|}{m} T \sum_{\omega_n} \nabla_{\mathbf{r}} G_{\omega_n}(r, r') \bigg|_{\mathbf{r} = 0}.$$  \hspace{1cm} (A.9)

From Eqs. (A.2) and (A.3), $G_{\omega_n}(r, r')$ in Eq. (A.9) and $F_{\omega_n}^1(r, r')$ in Eq. (A.8) are expanded in the power of $\Delta$ as follows,

$$G_{\omega_n}(r, r') = G_{\omega_n}^0(r, r') + G_{\omega_n}^{(2)}(r, r') + G_{\omega_n}^{(4)}(r, r') + \cdots,$$  \hspace{1cm} (A.10)

$$F_{\omega_n}^1(r, r') = F_{\omega_n}^{(1)}(r, r') + F_{\omega_n}^{(3)}(r, r') + F_{\omega_n}^{(5)}(r, r') + \cdots,$$  \hspace{1cm} (A.11)

where

$$G_{\omega_n}^{(2)}(r, r') = -\sum_{r_1, r_2, r_1', r_2'} G_{\omega_n}^0(r, r_1) G_{\omega_n}^0(r_2, r_1') G_{\omega_n}^0(r_2', r') \Delta(r_1, r_1') \Delta^*(r_2, r_2'),$$  \hspace{1cm} (A.12)

$$G_{\omega_n}^{(4)}(r, r') = \sum_{r_1 \sim r_2, r_1' \sim r_2'} G_{\omega_n}^0(r, r_1) G_{\omega_n}^0(r_2, r_1') G_{\omega_n}^0(r_2', r_3) G_{\omega_n}^0(r_4, r_3') G_{\omega_n}^0(r_4', r').$$
\[ F_{\omega n}^{(1)}(r, r') = \sum_{r_1, r_1'} G_{\omega n}^0(r_1, r) G_{\omega n}^0(r_1', r') \Delta^*(r_1, r_1'), \]  
\[ F_{\omega n}^{(3)}(r, r') = \sum_{r_1 \sim r_2, r_1' \sim r_3} G_{\omega n}^0(r_1, r) G_{\omega n}^0(r_1', r_2) G_{\omega n}^0(r_3, r_2') G_{\omega n}^0(r_3, r_4) \Delta^*(r_1, r_1') \Delta^*(r_2, r_2'), \]  
\[ F_{\omega n}^{(5)}(r, r') = \sum_{r_1 \sim r_5, r_1' \sim r_5} G_{\omega n}^0(r_1, r) G_{\omega n}^0(r_1', r_2) G_{\omega n}^0(r_3, r_2') G_{\omega n}^0(r_3, r_4) G_{\omega n}^0(r_5, r_4') \Delta^*(r_1, r_1') \Delta^*(r_2, r_2') \Delta^*(r_4, r_4') \Delta^*(r_5, r_5'). \]
\[ x \eta^*(r_1) \eta(r_2) \eta^*(r_3) \big|_{r_1=r_2=r_3=r} + 6 N_F \tilde{V} A_4 \left( \frac{\phi(k)}{k} \right)^6 \eta^*(r), \]  
(A-21)

\[ j(r) = 2|e| N_F v_F \sum_{n_1, n_2=0}^{\infty} A_{n_1+n_2+1} \left( \tilde{k} - i D_1 \right)^{n_1} (i D_2)^{n_2} \phi(k) \right| \eta^*(r_1) \eta^*(r_2) \big|_{r_1=r_2=r} + 4i |e| N_F v_F A_4 \left( \tilde{k} - 2 \tilde{D}_1 + \tilde{D}_3 + 2 \tilde{D}_4 \right) \phi(k) \right| \eta^*(r_1) \eta^*(r_2) \eta^*(r_3) \eta^*(r_4) \big|_{r_1=r_2=r_3=r_4=r}, \]  
(A-22)

where \( \tilde{D} = v_F k \cdot D \). In Eq. (A-21), we define \( D_4 = -D_2 - D_3^* \).

We substitute \( \phi(k) = \sqrt{2} \cos 2\theta \) (in the \( d_{2-\omega}^2 \)-wave case) and \( k \cdot D = D_x \cos \theta + D_y \sin \theta \), and perform the \( \theta \)-integral. Then, using the relation \( (N_F \tilde{V})^{-1} = \ln(2\omega_D \gamma/\pi T_c) \), we obtain Eqs. (2.2) and (2.3).

We confirm that the GL equation (2.2) and the current expression (2.3) are also derived by expanding the quantized Eilenberger equation in the powers of \( D \) and \( \eta \) following the method presented by Schopohl and Tewordt.

**Appendix**: Derivation of Eq. (3.3)

Around a vortex, in general, \( \eta(r) \) of the pair potential \( \Delta(r, k) \) in Eq. (2.1) can be expanded by \( e^{i l \phi} \) \( (l: \text{integer}) \) as follows,

\[ \eta(r) = \sum_l f_l(r) e^{i l \phi}. \]  
(B-1)

Then, as for the \( U_{2, x} R \) and \( U_{2, x-y} e^{-i \pi/2} R \) in Eq. (B-2), the transformations of \( \Delta(r, k) \) are given by

\[ U_{2, x} R \Delta(r, k) = \sum_l f_l^*(r) e^{i l \phi} \sqrt{2} \cos 2 \theta, \]  
(B-2)

\[ U_{2, x-y} e^{-i \pi/2} R \Delta(r, k) = -\sum_l f_l^*(r) e^{-i (l+1) \pi/2} e^{i l \phi} \sqrt{2} \cos 2 \theta. \]  
(B-3)

If the symmetry \( D_4(E) \) is conserved, Eqs. (B-2) and (B-3) should be equal to \( \Delta(r, k) \). Then, we obtain the relation,

\[ f_l(r) = f_l^*(r), \quad e^{-i (l+1) \pi/2} = -1. \]  
(B-4)

From Eq. (B-4), \( f_l(r) \) is real, and \( l = 4n + 1 \) \( (n: \text{integer}) \). Writing \( f_{l=4n+1}(r) \) as \( f_n(r) \), we obtain Eq. (3.3).