Optimization-based method of solving 2D thermal cloaking problems

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Abstract. We consider inverse problems for 2D heat conduction model concerning with designing cylindrical thermal cloaking shells. The shells are assumed consisting of a finite number of layers every of which is filled with homogeneous isotropic medium. Using the optimization method these inverse problems are reduced to corresponding control problems. Thermal conductivities of the shell layers play the role of passive controls. A numerical algorithm based on the particle swarm optimization is proposed and the results of numerical experiments are discussed. Rigorous optimization analysis shows that high cloaking efficiency of the shell can be achieved using a multilayer shell composed of only three isotropic nature materials with optimally selected thermal conductivities.

1. Introduction

In recent years much attention has been given to creation of tools of material objects cloaking from detection with the help of electromagnetic or acoustic location. Beginning with pioneering paper [1] the large number of publications was devoted to developing different methods of solving the cloaking problems. Transformation optics (TO) is the most popular method of designing cloaking devices. This cloaking methodology obtained the name of direct design because it is based on solving direct problems of electromagnetic scattering.

The first works in this field focused on the electromagnetic cloaking, i.e. cloaking objects from detection by electromagnetic location. Then the main results of the electromagnetic cloaking theory were expanded to an acoustic cloaking and to cloaking magnetic, electric, thermal and other static fields (see, e.g., [2–7]). It should be noted that the invisibility devices (cloaks) designed on the basis of direct strategies possess serious drawbacks. The main one is the difficulty of their technical realization.

That is why the another cloak design strategy began develop recently. It is based on using the optimization method of solving inverse problems. A growing number of papers is devoted to applying this methodology in various cloaking problems. Among them we mention the first papers [8, 9] where numerical optimization algorithms are applied for finding the unknown material parameters of TO-based cloak and papers [10–13] associated with the use of topology optimization and discrete material optimization for solving problems of designing cloaks, concentrators and other thermal functional devices. Papers [14–17] are devoted to theoretical analysis of cloaking problems using the optimization approach. In these papers some important properties of optimal solutions to the cloaking problems under study are established.
depending on the choice of the cost functional and the set of controls with respect to which the cost functional is minimized. In [18,19] the alternative approach is developed for solving acoustic cloaking problems. The papers [20–27] should also be mentioned in which optimization method is used for solving the related inverse problems arizing in mass and heat transfer, acoustics and electromagnetism.

This paper is devoted to solving control problems for the 2D stationary model of heat conduction arising when developing the technologies of designing thermal cloaking devices having the form of the cylindrical shell. It is assumed that the desired cloaking shell consists of finite number of layers every of which is filled with homogeneous isotropic medium. Thermal conductivities of the medium filling every layer play the role of controls. We propose a numerical algorithm of solving these extremum problems which is based on using the particle swam optimization (PSO) (see [28]) according to the scheme proposed in [29–31] and discuss some results of numerical experiments.

2. Statement of direct conductivity problem

We begin with statement of the general direct conduction problem, considered in a rectangle

\[ D = \{ x \in (x, y) : |x| < x_0, |y| < y_0 \} \]

with specified numbers \( x_0 > 0 \) and \( y > 0 \) (see Figure 1). We will assume that an externally applied field \( T^e \) is created by two vertical plates \( x = \pm x_0 \) which are kept at different values \( T_1 \) and \( T_2 \), while the upper and lower plates \( y = \pm y_0 \) are thermally insulated. We assume further that there is a material shell \( (\Omega, \kappa) \) inside \( D \). Here, \( \Omega \) is a circular layer \( a < |x| < b \) and \( \kappa \) is the thermal conductivity of the medium filling the domain \( \Omega \). It is also assumed that the interior \( \Omega_i: |x| < a \) and exterior \( \Omega_e: |x| > b \) of \( \Omega \) are filled with the same homogeneous medium having constant thermal conductivity \( k_b > 0 \) (see Figure 1).

In this case, the direct conduction problem consists of determination of a triplet of functions, namely \( T_i \) in \( \Omega_i \), \( T \) in \( \Omega \), and \( T_e \) in \( \Omega_e \), which satisfy the equations

\[ k_b \Delta T_i = 0 \quad \text{in} \quad \Omega_i, \quad \text{div} (\kappa \nabla T) = 0 \quad \text{in} \quad \Omega, \quad k_b \Delta T_e = 0 \quad \text{in} \quad \Omega_e, \]

with following boundary conditions

\[ T_e|_{x=-x_0} = T_1, \quad T_e|_{x=x_0} = T_2, \quad \frac{\partial T_e}{\partial y}|_{y=\pm y_0} = 0 \]

and the matching conditions on the boundaries of \( \Omega \)

\[ T_i = T, \quad k_b \frac{\partial T_i}{\partial n} = (\kappa \nabla T) \cdot \mathbf{n} \quad \text{at} \quad |x| = a, \quad T_e = T, \quad k_b \frac{\partial T_e}{\partial n} = (\kappa \nabla T) \cdot \mathbf{n} \quad \text{at} \quad |x| = b. \]

In the particular case when \( \kappa = k_b \) so that the entire medium filling the domain \( D \) is homogeneous and isotropic, the solution to (1)–(3) is defined by formula

\[ T^e(x) = \frac{T_2 - T_1}{2} \frac{x}{x_0} + \frac{T_1 + T_2}{2}, \]

which describes the external applied field \( T^e(x) \). Just this field \( T^e \) is used to detect objects located in domain \( D \).

As already mentioned, our goal is to analyze the cloaking problem for the model (1)–(3). Let us remind that inverse cloaking problem in the exact formulation consists in finding thermal conductivity \( \kappa \) providing the following cloaking conditions [6,17]:

\[ T_e = T^e \quad \text{in} \quad \Omega_e, \quad \nabla T_i = 0 \quad \text{in} \quad \Omega_i. \]

These conditions imply that temperature field must be equal to the external field \( T^e \) in \( \Omega_e \) and the gradient \( \nabla T_i \) must be equal to zero in \( \Omega_i \).
3. Layered shell. Statement of control problem

As mentioned above the technical realization of highly anisotropic cloaking shells is associated with great difficulties. One of the ways of overcoming these difficulties consists of using layered shell consisting of \( M \) concentric circular layers \( \Omega_j = \{ r_{j-1} < r = |x| < r_j \}, j = 1, 2, \ldots, M \), where \( r_0 = a, r_M = b \). Each of these layers is filled with a homogeneous and isotropic medium described by constant conductivity \( k_j \), \( j = 1, 2, \ldots, M \). Conductivity \( \kappa \) in this case is defined by

\[
\kappa(x) = \sum_{j=1}^{M} k_j \chi_j(x) \tag{6}
\]

Here, \( \chi_j(x) \) is a characteristic function of \( \Omega_j \), which is equal to 1 in \( \Omega_j \) and to 0 outside \( \Omega_j \).

We remind that our purpose is the numerical analysis of inverse problems for 2D model (1)–(3) arising when developing technologies of designing thermal cloaking devices. In the case of a layered shell with conductivity (6) these problems consist of finding unknown coefficients \( k_j \), \( j = 1, 2, \ldots, M \) in (6) forming a \( M \)-dimensional vector \( k = (k_1, k_2, \ldots, k_M) \) from the cloaking conditions. We will refer to \( k \) as conductivity vector because it consists of conductivities \( k_j \) of separate sublayers \( \Omega_j, j = 1, 2, \ldots, M \) forming the \( M \)-layered shell \( (\Omega, k) \).

By optimization method our inverse problems are reduced to extremum problems of minimization of certain cost functionals which adequately correspond to inverse problems of designing devices for approximate cloaking [17]. In order to formulate the control problems under study, we denote by \( T[k] \equiv T[k_1, \ldots, k_M] \) the solution to the direct problem (1)–(3) corresponding to parameters \( k_j \) in \( \Omega_j, j = 1, 2, \ldots, M \), and to conductivity \( k_b \) in \( \Omega_i \) and \( \Omega_e \). We will assume below that the vector \( k = (k_1, \ldots, k_M) \) belongs to the bounded set

\[
K = \{ k : k_{\min} \leq k_j \leq k_{\max}, j = 1, 2, \ldots, M \} \tag{7}
\]

to which we will refer to as a control set. Here given positive constants \( k_{\min} \) and \( k_{\max} \) are lower and upper bounds of the control set \( K \). Let us define cost functionals

\[
J_c(k) = \frac{\|T[k] - T^e\|_{L^2(\Omega_e)}}{\|T^e\|_{L^2(\Omega_e)}}, \quad J_i(k) = \frac{\|\nabla T[k]\|_{L^2(\Omega_i)}}{\|\nabla T^e\|_{L^2(\Omega_i)}}, \quad J(k) = \frac{1}{2} [J_i(k) + J_c(k)] \tag{8}
\]
where, in particular, \( \| T^e \|^2_{L^2(\Omega_e)} = \int_{\Omega_e} |T^e|^2 \, dx \), \( \| T[k] - T^e \|^2_{L^2(\Omega_e)} = \int_{\Omega_e} |T[k] - T^e|^2 \, dx \), and formulate the following control problem:

\[
J(k) \to \inf, \quad k \in K.
\]  

(9)

It follows from (8) that the value \( J_e(k) \) (or \( J_i(k) \)) describes for any \( k \in K \) the mean square error of fulfilling the first (or the second) condition in (5) while \( J(k) \) describes the mean square error of fulfilling both conditions in (5). Based on this fact one can easily show that the value \( J(k) \) is connected with the cloaking performance of the cloak \( (\Omega, k) \) (see details in [31]): the smaller value \( J(k) \) corresponds to the higher cloaking performance of the cloak \( (\Omega, k) \) and vice versa. Therefore the problem (9) is aimed to finding a conductivity vector (an optimal solution \( k^{opt} \) of (9)) for which corresponding cloak \( (\Omega, k^{opt}) \) possesses the highest cloaking performance.

4. Simulation results

In this section we discuss numerical results obtained when designing multilayer cloaking shells using PSO algorithm. We used the FreeFEM++ software package (www.freefem.org) to solve direct problem (1)–(3). Numerical simulation was performed for the following problem data: \( a = 1 \, \text{m}, \, b = 2 \, \text{m}, \, x_0 = y_0 = 3 \, \text{m}, \, k_b = k_0 = 1 \, \text{W/(m} \cdot \text{K}) \) as in [4]. The role of the externally applied field was played by the field \( T^e \) in (4) at \( T_1 = 100 \, \text{C}, \, T_2 = 0 \, \text{C} \).

In [4] a multilayer cloaking shell consisting only of two alternating materials was proposed. In our notation this shell is defined by the following parameters: \( k_1 = k_3 = \ldots = k_{M-1} = 0.13k_0 \) (corresponds to rubber), \( k_2 = k_4 = \ldots = k_M = 7.87k_0 \) (corresponds to thermal epoxy) and \( k_b = k_0 \) (corresponds to glass). Since all cloak parameters are specified, it remains to us to determine the cloaking performance of the corresponding cloaking shell \( (\Omega, k^{alt}) \) by calculating the values \( J(k^{alt}), \, J_i(k^{alt}) \) and \( J_e(k^{alt}) \). These values together with \( k_M \) are presented for six different shells \( (M = 2, 4, 6, 8, 10 \text{ and } 12) \) in Table 1. Isolines of corresponding temperature field for a six-layer shell \( (M = 6) \) are shown in Figure 2a. It is seen that \( J(k^{alt}) \) takes the value \( 8.0 \times 10^{-2} \) for \( M = 12 \). Besides, the isolines outside cloak are curved. These results indicate low cloaking performance of the above designed cloak \( (\Omega, k^{alt}) \).

\[
\begin{array}{cccc}
M & k_M/k_0 & J(k^{alt}) & J_i(k^{alt}) & J_e(k^{alt}) \\
2 & 7.87 & 1.6 \times 10^{-1} & 3.2 \times 10^{-1} & 7.3 \times 10^{-2} \\
4 & 7.87 & 1.1 \times 10^{-1} & 1.9 \times 10^{-1} & 3.4 \times 10^{-2} \\
6 & 7.87 & 9.3 \times 10^{-2} & 1.6 \times 10^{-1} & 2.3 \times 10^{-2} \\
8 & 7.87 & 8.7 \times 10^{-2} & 1.6 \times 10^{-1} & 1.8 \times 10^{-2} \\
10 & 7.87 & 8.2 \times 10^{-2} & 1.5 \times 10^{-1} & 1.4 \times 10^{-2} \\
12 & 7.87 & 8.0 \times 10^{-2} & 1.5 \times 10^{-1} & 1.2 \times 10^{-2} \\
\end{array}
\]

(10)

Table 1. Numerical results for unoptimized multilayer shells \((k_{min} = 0.13k_0, \, k_{max} = 7.87k_0)\).

In order to design a cloaking shell with higher cloaking performance we apply the optimization method. To this end we solve the control problem (9) for the case when \( k_{min} = 0.13k_0 \) and \( k_{max} = 7.87k_0 \). Our optimization analysis using PSO algorithm showed that optimal values \( k_i^{opt} \) of all parameters \( k_i \) with odd indices \( i = 1, 3, \ldots, M-1 \) coincide with \( k_{min} \), optimal values of parameters \( k_2^{opt}, \ldots, k_{M-2}^{opt} \) with even indices coincide with \( k_{max} \) while \( k_M^{opt} \) changes from 7.87 \( k_0 \) at \( M = 2 \) to 3.48 \( k_0 \) at \( M = 12 \). Thus we have

\[
k_1^{opt} = k_3^{opt} = \ldots = k_{M-1}^{opt} = k_{min}, \quad k_2^{opt} = k_4^{opt} = \ldots = k_{M-2}^{opt} = k_{max}
\]
while optimal value $k_M^{opt}$ of the last control and corresponding values $J(k^{opt}), J_i(k^{opt})$ and $J_e(k^{opt})$ for different $M = 2, 4, ..., 12$ are presented in Table 2. Isolines of corresponding field $T^{opt}$ for a six-layer shell ($M = 6$) are plotted in Figure 2b.

Table 2. Numerical results for optimized multilayer shells ($k_{min} = 0.13 k_0, k_{max} = 7.87 k_0$).

| $M$ | $k_M^{opt} / k_0$ | $J(k^{opt})$ | $J_i(k^{opt})$ | $J_e(k^{opt})$ |
|-----|-------------------|--------------|----------------|----------------|
| 2   | 7.87              | $1.6 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $7.3 \times 10^{-2}$ |
| 4   | 7.87              | $1.1 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $3.4 \times 10^{-2}$ |
| 6   | 3.51              | $8.9 \times 10^{-2}$ | $1.8 \times 10^{-1}$ | $1.5 \times 10^{-5}$ |
| 8   | 3.15              | $8.4 \times 10^{-2}$ | $1.7 \times 10^{-1}$ | $1.1 \times 10^{-5}$ |
| 10  | 3.52              | $7.9 \times 10^{-2}$ | $1.6 \times 10^{-1}$ | $8.7 \times 10^{-6}$ |
| 12  | 3.48              | $7.8 \times 10^{-2}$ | $1.5 \times 10^{-1}$ | $2.8 \times 10^{-6}$ |

Analysis of Table 2 shows that the optimal values $J(k^{opt})$ and $J_i(k^{opt})$ very slowly decrease with increasing the number of layers $M$. In particular, $J(k^{opt})$ decreases from $1.6 \times 10^{-1}$ at $M = 2$ to $7.8 \times 10^{-2}$ at $M = 12$. Temperature isolines outside the cloak in Figure 2b are straight as if there is no shell in the domain. Analysis of Table 2 shows that the use of the optimization method for the same pair $k_{min} = 0.13 k_0$ and $k_{max} = 7.87 k_0$ did not lead to a significant increase in the cloaking performance of the corresponding cloaking shell. We explain this by the low contrast (ratio) $k_{max}/k_{min}$ for used pair ($k_{min}, k_{max}$).

In order to increase significantly the cloaking performance of the designed shell the optimization method should be applied for another pair ($k_{min}, k_{max}$), which has a higher contrast. Choosing a new pair $k_{min} = 0.03 k_0$ (polystyrene), $k_{max} = 116 k_0$ (zinc) and applying PSO we again obtain relations (10) for all controls $k_i^{opt}$ except the last $k_M^{opt}$. The value $k_M^{opt}$

![Figure 2](image-url)

**Figure 2.** Temperature isolines for a six-layer shell with isotropic layers: a) unoptimized shell; b) optimized shell.
together with the values \(J(k^{\text{opt}}), J_i(k^{\text{opt}})\) and \(J_e(k^{\text{opt}})\) for different \(M\) are presented in Table 3.

**Table 3.** Numerical results for optimized multilayer shells \((k_{\text{min}} = 0.03 k_0, k_{\text{max}} = 116 k_0)\).

| \(M\) | \(k_M^{\text{opt}}/k_0\) | \(J(k^{\text{opt}})\) | \(J_i(k^{\text{opt}})\) | \(J_e(k^{\text{opt}})\) |
|------|-----------------|-----------------|-----------------|-----------------|
| 2    | 116             | \(1.7 \times 10^{-2}\) | \(1.3 \times 10^{-2}\) | \(2.1 \times 10^{-2}\) |
| 4    | 6.1             | \(1.0 \times 10^{-3}\) | \(2.1 \times 10^{-3}\) | \(1.3 \times 10^{-5}\) |
| 6    | 8.0             | \(9.6 \times 10^{-5}\) | \(1.7 \times 10^{-4}\) | \(3.2 \times 10^{-5}\) |
| 8    | 9.2             | \(9.0 \times 10^{-6}\) | \(1.8 \times 10^{-5}\) | \(3.4 \times 10^{-7}\) |
| 10   | 9.5             | \(2.2 \times 10^{-6}\) | \(4.2 \times 10^{-6}\) | \(3.1 \times 10^{-7}\) |
| 12   | 9.3             | \(9.7 \times 10^{-7}\) | \(1.7 \times 10^{-6}\) | \(2.8 \times 10^{-7}\) |

It is clearly seen that in this case the optimal values \(J(k^{\text{opt}})\) of the functional \(J\) for all \(M\) are substantially less than in Tables 2 and 3. Thus for the new choice of bounds \(k_{\text{min}}\) and \(k_{\text{max}}\) we obtained the shell with the highest cloaking performance. This can be explained by the fact that for the second test the value \((\text{contrast})\) \(k_{\text{max}}/k_{\text{min}}\) is much larger than for the first test (3867 versus 60). Comparing the values of functional \(J\) in Tables 2 and 3, we come to the conclusion that the cloaking performance of the shell increases with increasing number of layers \(M\) and the ratio \(k_{\text{max}}/k_{\text{min}}\).

Since the optimal values of all control parameters except the last one \(k_M^{\text{opt}}\) are equal to the lower or upper bounds of the control set \(K\), the designing a multilayer shell can be reduced to solving one-parameter control problem for \(k_M\). According to this reduced design, conductivities of all odd layers are initially prescribed to \(k_{\text{min}}\), conductivities of all even layers except \(k_M\) are fixed to \(k_{\text{max}}\) and conductivity of the last layer is found by solving one-parameter control problem for \(k_M\). As a result, we can use only three isotropic materials to construct a multilayer shell if the optimal conductivity \(k_M^{\text{opt}}\) corresponds to some natural material. For example, the value \(k_2^{\text{opt}} = 9.3 k_0\) from Table 3 at \(M = 12\) is the conductivity of quartz. If the choice of materials is limited then one can use the geometric parameters of the shell as additional controls in order to increase the cloaking performance.

**5. Conclusion**

In this paper we studied control problems for 2D model of heat conduction (1)–(3) associated with designing cylindrical layered thermal cloaking shells. Using the optimization method these inverse problems were reduced to corresponding control problems in which the thermal conductivities of layers play the role of controls. For solving our finite-dimensional extremum problems we proposed numerical algorithm based on the particle swarm optimization and discussed simulation results. Optimization analysis showed that the proposed method enables us (for special choice of the control set (7)) to design cloaking shells which possess the best cloaking performance and simultaneously easy technical realization. We note that the proposed method can be used to solve concentrating or heat flux inversion problems both for temperature field and other physical fields.

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