ENERGY DECAY RATES FOR SOLUTIONS OF THE WAVE EQUATION WITH LINEAR DAMPING IN EXTERIOR DOMAIN

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Abstract. In this paper we study the behavior of the energy and the $L^2$ norm of solutions of the wave equation with localized linear damping in exterior domain. Let $u$ be a solution of the wave system with initial data $(u_0, u_1)$. We assume that the damper is positive at infinity then under the Geometric Control Condition of Bardos et al [5] (1992), we prove that:

1. If $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$, then the total energy $E_u(t) \leq C_0 (1 + t)^{-1} I_0$ and $\|u(t)\|_{L^2}^2 \leq C_0 I_0$, where

   $$I_0 = \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2}.$$

2. If the initial data $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$ and verifies

   $$\|d(\cdot)(u_1 + au_0)\|_{L^2} < +\infty,$$

then the total energy $E_u(t) \leq C_2 (1 + t)^{-2} I_1$ and $\|u(t)\|_{L^2}^2 \leq C_2 (1 + t)^{-2} I_1$, where

   $$I_1 = \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2} + \|d(\cdot)(u_1 + au_0)\|_{L^2}^2$$

and

$$d(x) = \begin{cases} |x| & d \geq 3, \\ |x| \ln (B|x|) & d = 2, \end{cases}$$

with $B \inf_{x \in \Omega} |x| \geq 2$.

1. Introduction and statement of the results. Let $O$ be a compact domain of $\mathbb{R}^d$ ($d \geq 2$) with $C^\infty$ boundary $\Gamma$ and $\Omega = \mathbb{R}^d \setminus O$. Consider the following wave equation with localized linear damping

$$\begin{cases} \partial^2_t u - \Delta u + a(x) \partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(0, x) = u_0 \quad \text{and} \quad \partial_t u(0, x) = u_1. \end{cases}$$

(1)

Here $\Delta$ denotes the Laplace operator in the space variables. $a(x)$ is a nonnegative function in $L^\infty(\Omega)$.

Let

$$A = \begin{pmatrix} 0 & I \\ \Delta & -a \end{pmatrix},$$

and

$$H = H_D(\Omega) \times L^2(\Omega),$$

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the completion of \((C^\infty_c(\Omega))^2\) with respect to the norm
\[\|\varphi_0, \varphi_1\|^2_H = \int_\Omega |\nabla \varphi_0|^2 + |\varphi_1|^2 \, dx,\]

A is a unbounded operator on \(H\) with domain
\[D(A) = \{(u_0, u_1) \in H, A u = u\} \in H\}.

Let \(n \in \mathbb{N}\) and \((u_0, u_1) \in D(A^n)\). Linear semigroup theory applied to (1) (see for example [2]), provides existence of a unique solution \(u\) in the class \(u \in C^0([0,T]\times \mathbb{R}^n, D(A^n-k))\), with \(k \leq n\).

Moreover, if \((u_0, u_1)\) is in \(H_0^1(\Omega) \times L^2(\Omega)\), then the system (1), admits a unique solution \(u\) in the class \(u \in C^0([0,T]\times \mathbb{R}^n, H_0^1(\Omega)) \cap C^1([0,T]\times \mathbb{R}^n, L^2(\Omega))\).

Let us consider the energy at instant \(t\) defined by
\[E_u(t) = \frac{1}{2} \int_\Omega \left( |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx.

The energy functional satisfies the following identity
\[E_u(T) + \int_0^T \int_\Omega a(x) |\partial_t u|^2 \, dxdt = E_u(0), \quad (2)\]
for every \(T \geq 0\).

Zuazua [31], Nakao [26], Dehman et al [12], Aloui et al [1] and Joly et al [19] have considered the problem for the Klein-Gordon type wave equations with localized dissipations. For the Klein-Gordon equations the energy functional itself contains the \(L^2\) norm and boundedness of \(L^2\) norm of the solution is trivial. Thus under some assumptions on the support of the damper \(a\) we can show that the energy decays exponentially while for the system (1) the energy decay rate is weaker and more delicate.

In the case of damping localized in a compact set of \(\Omega\) Nakao in [24] proved that the local energy decays exponentially if \(d\) is odd and polynomially if \(d\) is even under the Lions’s geometric condition. Combining the definition of a non-trapping obstacle and the geometric control condition of Bardos et al [5], Aloui et al [2] introduced the exterior geometric control condition which is a sufficient condition for the stabilization of the local energy, more precisely they proved that the local energy decays exponentially if the space dimension \(d\) is odd. In [21] Khenissi showed that the estimate (3) holds if \(d\) is even. Recently in [9], using a nonlinear internal localized damping, Daoulatli obtained various decay rates, depending on the behavior of the damping term. We also mention the result of Daoulatli et al [10] on the behaviors of the local energy for solutions of the Lamé system in exterior domain. Finally, we quote the result of Bchatnia and Daoulatli [4] on the decay rates of the local energy for solutions of the wave equation with localized time dependent damping in exterior domain. On the other hand, Dan-Shibata [8] studied the local energy decay estimates for the compactly supported weak solutions of (1) with \(a(x) = 1\)
\[\int_{\Omega \cap B_R} \left( |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx \leq C(1 + t)^{-d}, \quad (3)\]
where \(B_R = \{x \in \mathbb{R}^d, |x| < R\} \).
If we assume that \( a(x) \geq \epsilon_0 > 0 \) in all of \( \Omega \), then we know that

\[
E_u(t) \leq C_0 (1 + t)^{-1} \quad \text{and} \quad \| u(t) \|_{L^2}^2 \leq C_0, \quad \text{for all} \quad t \geq 0,
\]

for weak solution \( u \) to the system (1) with initial data in \( H_0^1(\Omega) \times L^2(\Omega) \). Nakao in [25] obtained the same estimates in (4) for a damper \( a \) which is positive near some part of the boundary (Lions’s condition) and near infinity.

Furthermore Ikehata and Matsuyama in [18] obtained a more precise decay estimate for the total energy of solutions of the problem (1) with \( a(x) = 1 \) and for weighted initial data

\[
E_u(t) \leq C_2 (1 + t)^{-2} \quad \text{and} \quad \| u(t) \|_{L^2}^2 \leq C_2 (1 + t)^{-1}, \quad \text{for all} \quad t \geq 0.
\]

Especially this estimate seems sharp for \( d = 2 \) as compared with that of [8].

Ikehata in [17] derived a fast decay rate like (5) for solutions of the system (1) with weighted initial data and assuming that \( a(x) \geq \epsilon_0 > 0 \) at infinity and \( O = \mathbb{R}^d \setminus \Omega \) is star shaped with respect to the origin.

Before introducing our results we shall state several assumptions:

**Hyp A:** There exists \( L > 0 \) such that

\[
a(x) > \epsilon_0 > 0 \quad \text{for} \quad |x| \geq L.
\]

**Definition 1.1.** \((\omega, T_0)\) geometrically controls \( \Omega \), i.e. every generalized geodesic travelling with speed 1 and issued at \( t = 0 \), meets the set \( \omega \) in a time \( t < T_0 \).

This condition is called Geometric Control Condition (see e.g. [5]). We shall relate the open subset \( \omega \) with the damper \( a \) by

\[
\omega = \{ x \in \Omega; a(x) > \epsilon_0 \}.
\]

We note that according to [5] and [7] the Geometric Control Condition of Bardos et al is a sharp sufficient condition for the stabilization of the wave equation in bounded domain.

The goal of this paper is to prove that for a damper \( a \) positive near infinity and under the geometric control condition of Bardos et al [5], the estimates in (4) hold for all solutions of the system (1) with initial data in \( H_0^1(\Omega) \times L^2(\Omega) \) and to show that the estimates in (5) hold for all solutions of the system (1) with weighted initial data. Moreover we show that for every \( p \in \mathbb{N}^* \) there exists a initial data in \( H_0^1(\Omega) \times L^2(\Omega) \) such that the solution \( v \) of (1) verifies

\[
E_v(t) \leq C (1 + t)^{-p} \quad \text{and} \quad \| v(t) \|_{L^2}^2 \leq C (1 + t)^{-p+1}, \quad \text{for all} \quad t \geq 0,
\]

and for some \( C > 0 \) depending on the norm of the initial data.

**Theorem 1.2.** We assume that Hyp A holds and \((\omega, T_0)\) geometrically controls \( \Omega \). Then there exists \( C_0 > 0 \) such that the following estimates

\[
E_u(t) \leq C_0 (1 + t)^{-1} I_0 \quad \text{and} \quad \| u(t) \|_{L^2}^2 \leq C_0 I_0, \quad \text{for all} \quad t \geq 0,
\]

hold for every solution \( u \) of (1) with initial data \((u_0, u_1)\) in \( H_0^1(\Omega) \times L^2(\Omega) \), where

\[
I_0 = \| u_0 \|_{H^1}^2 + \| u_1 \|_{L^2}^2.
\]

As a corollary of theorem 1.2 we have:
Theorem 1.3. We assume that Hyp A holds and \((\omega, T_0)\) geometrically controls \(\Omega\). Let \((u_0, u_1)\) in \(D(A^n)\), such that \(u_0 \in L^2(\Omega)\). Then the solution \(u\) of (1) satisfies
\[
E_{\partial^n}u(t) \leq C_n (1 + t)^{-n} I_{0,n} \quad \text{for all } t \geq 0,
\]
and
\[
\|\partial^n u(t)\|_{L^2}^2 \leq C_{n-1} (1 + t)^{-n} I_{0,n-1} \quad \text{for all } t \geq 0,
\]
where \(C_n\) is a positive constant independent of the initial data and
\[
I_{0,p} = \sum_{i=0}^p \|A^i(u_0, u_1)\|_{H^j}^2 + \|u_0\|_{L^2}^2, \quad \text{for } n \in \mathbb{N}^*.
\]

Next we give the result on the decay rates for weighted initial data. We define
\[
d(x) = \begin{cases} |x| & d \geq 3, \\ |x| \ln (B |x|) & d = 2,
\end{cases}
\]
with \(B \inf_{x \in \Omega} |x| \geq 2\).

**Theorem 1.3.** We assume that Hyp A holds and \((\omega, T_0)\) geometrically controls \(\Omega\). Then there exists \(C_2 > 0\) such that the following estimates hold for every solution \(u\) of (1) with initial data \((u_0, u_1)\) in \(H^j_1(\Omega) \times L^2(\Omega)\) which satisfies
\[
\|d(\cdot)(u_1 + au_0)\|_{L^2} < +\infty,
\]
where
\[
I_1 = \|u_0\|_{H^j}^2 + \|u_1\|_{L^2}^2 + \|d(\cdot)(u_1 + au_0)\|_{L^2}^2.
\]

We note that we obtain the same result of [17] when \(O\) is star shaped which is special case of non trapping obstacle. Our geometric condition allows us to consider also trapped obstacles in which the damper control the trapped rays.

Let \(\psi \in C_c^{\infty}(\mathbb{R}^d)\) such that \(0 \leq \psi \leq 1\) and
\[
\psi(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L
\end{cases}.
\]

The key idea of the proof of theorem 1.2 and theorem 1.3 is to use the following auxiliary functional \(X(t) = \int_{\Omega} v(t) \partial_t v(t) dx + \frac{1}{2} \int_{\Omega} a(x) |v(t)|^2 dx + kE_u(t), u\) is a solution of (1) with initial data in \(H^j_1(\Omega) \times L^2(\Omega)\) and \(v = (1 - \psi)u\) where \(k\) is a positive constant. Then to show some observability estimates for the local energy using observability estimates for the wave equation in bounded domain. In bounded domain to prove such estimates we can use the theorem of propagation of singularities of Melrose-Sjöstrand or the notion of microlocal defect measure (see [14] and [13] for the definition and properties of microlocal defect measure) and exploiting their property. Especially the fact that the measure associated to a bounded sequence of solutions of the conservative wave equation propagates along the generalized bicharacteristic flow of Melrose-Sjöstrand.

In the sequel we suppose that the obstacle \(O\) is contained in \(B_{R_0}\) for some positive \(R_0\).

The rest of the paper is organized as follows. The section 2 is devoted to the proof of theorem 1.2 and proposition 1 and in section 3 we give the proof of theorem 1.3.
2. Proof of Theorem 1.2.

2.1. Preliminary results. We prove an observability estimate for the local energy of solutions of the system 1, the proof is based on an observability estimate for the non homogeneous wave equation in bounded domain.

Proposition 2. We assume that Hyp A holds and \((\omega, T_0)\) geometrically controls \(\Omega\). Let \(\delta > 0\) and \(R > R_0\). There exist \(T > T_0\) and \(C_{T, \delta, R} > 0\), such that the following inequality

\[
\int_{t}^{t+T} \int_{\Omega \cap B_R} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) dx ds \\
\leq C_{T, \delta, R} \left( \int_{t}^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 dx ds \right) + \delta E_u(t),
\]

holds for every \(t \geq 0\) and for all \(u\) solution of (1) with initial data \((u_0, u_1)\) in \(H\).

Proof. To prove this result we argue by contradiction: If (6) was false, there would exist a sequence of positive numbers \((t_n)\) and a sequence of solutions \((u_n)\) such that

\[
\int_{t_n}^{t_n+T} \int_{\Omega \cap B_R} |u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2 dx ds \\
\geq n \left( \int_{t_n}^{t_n+T} \int_{\Omega} a(x) |\partial_t u_n|^2 dx dt \right) + \delta E_{u_n}(t_n).
\]

Setting

\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \int_{\Omega \cap B_R} |u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2 dx ds \quad \text{and} \quad v_n = \frac{u_n (t_n + \cdot)}{\lambda_n}.
\]

It is clear that (7), gives

\[
\int_{0}^{T} \int_{\Omega} a(x) \partial_t v_n |v_n|^2 dx dt \overset{n \to +\infty}{\longrightarrow} 0 \quad \text{and} \quad E_{v_n}(0) \leq \frac{1}{\delta}.
\]

Let \(Z_n\) be the solution of the following system

\[
\begin{\array}{l}
\displaystyle \partial_t^2 Z_n - \Delta Z_n = 0 \quad \Omega \times \Omega, \\
\displaystyle Z_n = 0 \quad \Omega \times \Gamma, \\
\displaystyle (Z_n(0), \partial_t Z_n(0)) = \frac{1}{\lambda_n} (u_n(t_n), \partial_t u_n(t_n)).
\end{\array}
\]

Therefore

\[
\begin{\array}{l}
\left( v_n - Z_n \right)(0) = 0 \quad \Omega \times \Gamma, \\
\displaystyle \left( \left( v_n - Z_n \right)(0), \partial_t (v_n - Z_n)(0) \right) = 0.
\end{\array}
\]

So we obtain the following energy identity

\[
E_{v_n - Z_n}(t) = - \int_{0}^{t} \int_{\Omega} a(x) \partial_t v_n(s) \partial_t (v_n - Z_n)(s) dx ds
\]

for \(t \geq 0\). Using Holder’s inequality, we deduce the following energy inequality

\[
\sup_{[0, T]} E_{v_n - Z_n}^{1/2}(s) \leq 2 \left\| a(x) \partial_t v_n \right\|_{L^1([0, T], L^2(\Omega))},
\]

Now using (8) and the fact that \(a(x)\) is a function in \(L^\infty(\Omega)\), we infer that

\[
\sup_{[0, T]} E_{v_n - Z_n}(s) \overset{n \to +\infty}{\longrightarrow} 0.
\]

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On the other hand,

\[
\frac{1}{2} \int_0^T \int_\Omega a(x) |\partial_t Z_n|^2 \, dx \, dt \\
\leq 2T \|a(x)\|_{L^\infty} \sup_{[0,T]} E_{v_n - Z_n} \left( \int_0^T \int_\Omega a(x) |\partial_t v_n|^2 \, dx \, dt. \right)
\]

(8) and (11) combined with the result above, gives

\[
\int_0^T \int_\Omega a(x) |\partial_t Z_n|^2 \, dx \, dt \underset{n \to +\infty}{\longrightarrow} 0.
\]

The system (9) is conservative, therefore

\[
\sup_{[0,T]} E_{Z_n} (t) = E_{Z_n} (0) = E_{v_n} (0) \leq \frac{1}{\delta}.
\]

(13)

Therefore, along a subsequence, \( (Z_n) \) is convergent to a function

\[
Z \in L^2 ([0,T] ; H_D (\Omega)) \text{ and } \partial_t Z \in L^2 ([0,T] ; L^2 (\Omega)),
\]

with respect to the weak topology. Since \( Z \) satisfies

\[
\begin{cases}
\partial_t^2 Z - \Delta Z = 0 & \text{in } [0,T] \times \Omega, \\
Z = 0 & \text{on } [0,T] \times \Gamma, \\
(Z_0,Z_1) \in H_D (\Omega) \times L^2 (\Omega) \\
\partial_t Z (t,x) = 0 & \text{on } [0,T] \times \omega.
\end{cases}
\]

Therefore we have

\[
Z \in C ([0,T] ; H_D (\Omega)) \text{ and } \partial_t Z \in C ([0,T] ; L^2 (\Omega)),
\]

We remind that \( \{ x \in \mathbb{R}^d, \ |x| \geq L \} \subset \omega \). By a classical result of unique continuation (see [29]), we see that there exists \( T_1 > 0 \) such that if \( T > T_1 \), we obtain that \( \partial_t Z \equiv 0 \) on \( [0,T] \times \Omega \). this mean that \( Z (t,x) = Z(x) \) is independent of \( t \). Therefore, we have

\[
\Delta Z = 0 \text{ and } Z \in H_D (\Omega),
\]

we conclude from this that \( Z \equiv 0 \) on \( [0,T] \times \Omega \) (cf. [22, theorem 2.2 p 145]).

Let \( R_2 > R_0 \). Now we will show that

\[
\int_0^T \int_{\Omega \cap B_{R_2}} |Z_n (s,x)|^2 \, dx \, ds \underset{n \to +\infty}{\longrightarrow} 0
\]

Let \( \theta \in C_c^\infty (\mathbb{R}^d) \) such that \( \theta = 1 \) on \( \{ |x| \leq R_2 \} \) and the support of \( \psi \) is contained in \( \{ |x| \leq R_2 + 1 \} \). Setting \( w_n = \theta Z_n \). Using Poincare’s inequality and (13), we infer that \( w_n \) is bounded in

\[
L^\infty \left( (0,T), H^1_0 (\Omega \cap B_{R_2+1}) \right) \cap W^{1,\infty} \left( (0,T), L^2 (\Omega \cap B_{R_2+1}) \right),
\]

since \( Z_n \) converges weakly to zero in \( L^2 ((0,T) \times (\Omega \cap B_{R_2+1})) \), therefore \( w_n \) converges weakly to zero in \( L^2 ((0,T) \times (\Omega \cap B_{R_2+1})) \). Now using Rellich’s theorem we deduce that

\[
\int_0^T \int_{\Omega \cap B_{R_2}} |Z_n (s,x)|^2 \, dx \, ds \leq \int_0^T \int_{\Omega \cap B_{R_2}} |w_n (s,x)|^2 \, dx \, ds \underset{n \to +\infty}{\longrightarrow} 0.
\]
We take $R_1$ and $R_2$ such that, $R_2 > R_1 \geq \max (R+T, 2L)$ and let $\psi \in C_c^\infty (\mathbb{R}^d)$ such that

$$\psi = 1 \text{ on } \left\{ x \in \mathbb{R}^d, \frac{3L}{2} \leq |x| \leq R_1 \right\}$$

and the support of $\psi$ is contained in $\left\{ x \in \mathbb{R}^d, |x| \geq L \right\} \cap \left\{ x \in \mathbb{R}^d, |x| \leq R_2 \right\}$. Let $0 < \epsilon << 1$ and $\phi$ be a nonnegative function in $C_c^\infty (0,T)$ such that

$$\phi(s) = \begin{cases} 0 & \text{on } [0, \epsilon/2] \cup [0, T - \epsilon/2] \\ 1 & \epsilon \leq s \leq T - \epsilon \end{cases}$$

We multiply the Eq (9) by $\varphi \psi^2 Z_n$ and integrate over $(0,T) \times \Omega$, we obtain

$$\int_0^T \int_{\Omega} \varphi (s) \psi^2 (x) |\nabla Z_n (s)|^2 dxds$$

\[
= \int_0^T \int_{\Omega} \varphi' (s) \psi^2 (x) Z_n (s) \partial_t Z_n (s) + \varphi (s) \psi^2 (x) |\partial_t Z_n (s)|^2 dxds + \int_0^T \int_{\Omega} \frac{1}{2} \varphi (s) \Delta \psi^2 (x) |Z_n (s)|^2 dxds.
\]

Using Young’s inequality and the fact that $\varphi$ is in $C_c^\infty (0,T)$, we infer that there exists a positive constant $c$ such that

$$\int_0^T \int_{\Omega} \varphi (s) \psi^2 (x) |\nabla Z_n (s)|^2 dxds$$

\[
\leq c \int_0^T \int_{\Omega} \psi^2 (x) \left( |\partial_t Z_n (s)|^2 + |Z_n (s)|^2 \right) + \Delta \psi^2 (x) |Z_n (s)|^2 dxds.
\]

Since the support of $\psi$ is contained in $\{|x| \geq L\}$ and $a (x) = 1$ on $\{|x| \geq L\}$, we get

$$\int_0^T \int_{\Omega} \varphi (s) \psi^2 (x) |\nabla Z_n (s)|^2 dxds$$

\[
\leq c \int_0^T \int_{\Omega} a (x) |\partial_t Z_n (s)|^2 + \psi^2 (x) |Z_n (s)|^2 + \Delta \psi^2 (x) |Z_n (s)|^2 dxds.
\]

Combining the estimate above with (12) and (14)

$$\int_0^T \int_{\Omega \cap \left\{ \frac{3L}{2} \leq |x| \leq R_1 \right\}} |\nabla Z_n (s)|^2 dxds$$

\[
\leq \int_0^T \int_{\Omega} \varphi (s) \psi^2 (x) |\nabla Z_n (s)|^2 dxds \xrightarrow{n \to +\infty} 0, \quad (15)
\]

we note that in the inequality above we have used the fact that

$$\psi = 1 \text{ on } \left\{ x \in \mathbb{R}^d, \frac{3L}{2} \leq |x| \leq R_1 \right\}$$

and $\varphi = 1$ on $[\epsilon, T - \epsilon]$.

Let $\chi \in C_c^\infty (\mathbb{R}^d)$ such that $\chi = 1$ on $\{|x| \leq 2L\}$ and the support of $\chi$ is contained in $\{|x| \leq R_1\}$. Setting $W_n = \chi Z_n$, then $W_n$ is a solution of the following system

$$\left\{ \begin{array}{l}
\partial_t^2 W_n - \Delta W_n = -2\nabla \chi \nabla Z_n - Z_n \Delta \chi \\
W_n = 0 \\
(W_n (0), \partial_t W_n (0)) = \chi (Z_n (0), \partial_t Z_n (0)).
\end{array} \right.$$

\[\begin{array}{l}
R_+ \times \Omega \cap B_{R_1}, \\
R_+ \times \Gamma \cup \{|x| = R_1\},
\end{array} \]
In addition we have
\[ W_n \in C \left( (0, T) \setminus H^1_0(\Omega \cap B_{R_1}) \right) \cap C^1 \left( (0, T) \setminus L^2(\Omega \cap B_{R_1}) \right). \]
We take \( T > \max(T_0 + 1, T_1) \), therefore \((\omega \cap B_{R_1}, T - 2\epsilon)\) geometrically controls \(\Omega \cap B_{R_1}\). Then using the observability estimate for the non homogenous wave equation in bounded domain (see for example [11]), we obtain
\[ E_{W_n}(\epsilon) \leq C_{\epsilon,T} \left( \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} a(x) |\partial_t Z_n(s)|^2 + |2\nabla \chi \nabla Z_n + Z_n \Delta \chi|^2 \, dx \, ds \right). \]
We note that to prove the estimate above we can use theorem of propagation of singularities of Melrose-Sjöstrand or the notion of microlocal defect measure (see [14] and [13] for the definition and properties of microlocal defect measure).

Since \( \nabla \chi = 0 \) on \( \{|x| \leq 2L\} \) and \( \text{Supp} \chi \subset \{|x| \leq R_1\} \), (15) gives
\[ \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} |\nabla \chi(x) \nabla Z_n(s,x)|^2 \, dx \, ds \xrightarrow{n \to +\infty} 0. \] (16)
So using (14), we infer that
\[ \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} |2\nabla \chi(x) \nabla Z_n(s) + Z_n(s) \Delta \chi(x)|^2 \, dx \, ds \xrightarrow{n \to +\infty} 0. \] (17)
Combining the estimate above with (12), we obtain
\[ E_{W_n}(\epsilon) \leq C \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} a(x) |\partial_t Z_n(s)|^2 + |2\nabla \chi(x) \nabla Z_n(s) + Z_n(s) \Delta \chi(x)|^2 \, dx \, ds \xrightarrow{n \to +\infty} 0. \]
On the other hand, the energy estimate for the non homogeneous wave equation
\[ E_{W_n}(s) \leq 2e^{T} \left( E_{W_n}(\epsilon) + \int_{\epsilon}^{s} \int_{\Omega \cap B_{R_1}} |2\nabla \chi(x) \nabla Z_n(s) + Z_n(s) \Delta \chi(x)|^2 \, dx \, ds \right), \]
for \( s \geq \epsilon \). Integrating the estimate above between \( \epsilon \) and \( T - \epsilon \), we infer that there exists a positive constant \( C = C(T, \epsilon) \)
\[ \int_{\epsilon}^{T-\epsilon} E_{W_n}(s) \, ds \leq C \left( E_{W_n}(\epsilon) + \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} |2\nabla \chi(x) \nabla Z_n(s) + Z_n(s) \Delta \chi(x)|^2 \, dx \, ds \right) \xrightarrow{n \to +\infty} 0. \]
Combining the result above with (14) and using the fact that \( \chi = 1 \) on \( \{|x| \leq 2L\} \), we infer that
\[ \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{2L}} |\nabla Z_n(s)|^2 + |\partial_t Z_n(s)|^2 \, dx \, ds \xrightarrow{n \to +\infty} 0. \]
This result combined with (15), gives
\[ \int_{\epsilon}^{T-\epsilon} \int_{\Omega \cap B_{R_1}} |\nabla Z_n(s)|^2 + |\partial_t Z_n(s)|^2 \, dx \, ds \xrightarrow{n \to +\infty} 0. \] (18)
with \( R_1 \geq R + T \).
Using the finite speed propagation property, we deduce that
\[
\int_{\Omega \cap B} |\nabla Z_\eta (t)|^2 + |\partial_t Z_\eta (t)|^2 \, dx \\
\leq \int_{\Omega \cap B_{R+1}} |\nabla Z_n (t+s)|^2 + |\partial_t Z_n (t+s)|^2 \, dx \\
\leq \int_{\Omega \cap B_{R+1}} |\nabla Z_n (t+s)|^2 + |\partial_t Z_n (t+s)|^2 \, dx
\]
where \( t \in [0,T] \) and \(-t \leq s \leq T - t\). Let \( \eta \) be a nonnegative function in \( C_c^\infty (0,T) \) such that the support of \( \eta \) is contained in \( [\epsilon, T - \epsilon] \) and
\[
\int_0^T \eta (s) \, ds \geq T.
\]
Multiplying both sides of the estimate (19) by \( \eta (s+t) \) and integrating between \(-t \) and \( T-t \), we obtain
\[
\int_{-t}^{T-t} \eta (s+t) \int_{\Omega \cap B_{R+T}} |\nabla Z_\eta (t+s)|^2 + |\partial_t Z_\eta (t+s)|^2 \, dxds \\
\geq \left( \int_{\Omega \cap B} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx \right) \int_{-t}^{T-t} \eta (s+t) \, ds \\
\geq T \int_{\Omega \cap B} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx,
\]
for all \( t \in [0,T] \). Using the fact that \( R_1 \geq R + T \) and making some arrangement, we obtain
\[
\int_{\Omega \cap B} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx \\
\leq \frac{1}{T} \int_0^T \eta (\tau) \int_{\Omega \cap B_{R+T}} |\nabla Z_\eta (\tau)|^2 + |\partial_t Z_\eta (\tau)|^2 \, dxd\tau \\
\leq \frac{c}{T} \int_{-t}^{T-t} \int_{\Omega \cap B_{R_1}} |\nabla Z_\eta (s)|^2 + |\partial_t Z_\eta (s)|^2 \, dxds.
\]
Now passing to the limit and using (18), we deduce that
\[
\int_{\Omega \cap B} |\nabla Z_n (t)|^2 + |\partial_t Z_n (t)|^2 \, dx \xrightarrow{n \to +\infty} 0, \text{ for all } t \in [0,T].
\]
By taking into account of the result above and (11) we deduce that
\[
\int_{\Omega \cap B} |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \, dx \xrightarrow{n \to +\infty} 0, \text{ for all } t \in [0,T].
\]
The fact that the energy of \( v_n \) is bounded, gives
\[
\int_{\Omega \cap B} |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \, dx \leq C, \text{ for all } t \in [0,T] \text{ and } n \in \mathbb{N}.
\]
By the dominated convergence theorem, we infer that
\[
\int_0^T \int_{\Omega \cap B} |\nabla v_n (t)|^2 + |\partial_t v_n (t)|^2 \, dxdt \xrightarrow{n \to +\infty} 0.
\]
On the other hand, Poincaré's inequality combined with (11) and (14), gives
\[
\int_0^T \int_{\Omega \cap B} |v_n (t)|^2 \, dxdt \xrightarrow{n \to +\infty} 0.
\]
So we conclude that

$$1 = \int_0^T \int_{\Omega \cap B_R} |v_n(t)|^2 + |\nabla v_n(t)|^2 + |\partial_t v_n(t)|^2 \, dx \, dt \xrightarrow{n \to +\infty} 0.$$ 

\[\square\]

In order to prove theorem 1.2 we need the following result on the auxiliary function $X$.

**Lemma 2.1.** Let $\psi \in C_c^\infty (\mathbb{R}^d)$ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L \end{cases}$$

Setting $v = (1 - \psi) u$ where $u$ is a solution of (1) with initial data in $H^1_0 (\Omega) \times L^2 (\Omega)$. Let

$$X(t) = \int_\Omega v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_\Omega a(x) |v(t)|^2 \, dx + kE_u(t),$$

where $k$ is a positive constant. We have

$$X(t + T) - X(t) + \int_t^{t+T} E_u(s) \, ds + \left(k - \frac{2}{\epsilon_0}\right) \int_t^{t+T} \int_\Omega a(x) |\partial_t u|^2 \, dx \, ds \leq C \int_t^{t+T} \int_{\Omega \cap B_{2L}} (|u|^2 + |\nabla u|^2 + |\partial_t u|^2) \, dx \, ds. \quad (20)$$

**Proof.** Noting that for each $(u_0, u_1)$ in $H^1_0 (\Omega) \times L^2 (\Omega)$ the solution $u$ of (1) is given as a limit of smooth solution $u_n$ with initial data $(u_{n,0}, u_{n,1})$ smooth such that $(u_{n,0}, u_{n,1}) \xrightarrow{n \to +\infty} (u_0, u_1)$ in $H^1_0 (\Omega) \times L^2 (\Omega)$. Note that

$$\|u_n(t,.) - u(t,.))\|_{H^1} + \|\partial_t u_n(t,.) - \partial_t u(t,.))\|_{L^2} \xrightarrow{n \to +\infty} 0,$$

uniformly on the each closed interval $[0, T]$ for any $T > 0$. Therefore we may assume that $u$ is smooth.

We have $v = (1 - \psi) u$. Then $v$ is a solution of

$$\begin{cases} \partial^2_t v - \Delta v + a(x) \partial_t v = f(t, x) & \mathbb{R}_+ \times \Omega, \\ v = 0 & \mathbb{R}_+ \times \Gamma, \\ (v(0), \partial_t v(0)) = (1 - \psi) (u_0, u_1), \end{cases} \quad (21)$$

with

$$f(t, x) = 2\nabla \psi \nabla u + u \Delta \psi.$$  

Using the fact that $v$ is a solution of (21) and that

$$\frac{d}{dt} E_u(t) = - \int_\Omega a(x) |\partial_t u(t)|^2 \, dx,$$

we deduce that

$$\frac{d}{dt} X(t) = \int_\Omega |\partial_t v(t)|^2 - |\nabla v(t)|^2 \, dx - k \int_\Omega a(x) |\partial_t u(t)|^2 \, dx + \int_\Omega f(t, x) v \, dx.$$
Since
\[ \int_{\Omega} f(t,x) \, v \, dx = \int_{\Omega} (2 \nabla \psi \nabla u + u \Delta \psi) (1 - \psi) \, u \, dx \]
\[ = \int_{\Omega} \nabla \psi \nabla u^2 + u^2 \Delta \psi \, dx - \int_{\Omega} \frac{1}{2} \nabla \psi^2 \nabla u^2 + u^2 \psi \Delta \psi \, dx \]
\[ = \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx. \]
Thus we obtain
\[ \frac{d}{dt} X(t) = \int_{\Omega} |\partial_t v(t)|^2 - |\nabla v(t)|^2 \, dx - k \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx + \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx \]
\[ = 2 \int_{\Omega} |\partial_t v(t)|^2 \, dx - 2E_v(t) - k \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx + \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx \]
\[ \leq 2 \int_{\Omega} |\partial_t v(t)|^2 \, dx - E_u(t) + 2E_w(t) - k \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx \]
\[ + \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx. \]
where \( w = \psi u \). By taking into account that the support of \( (1 - \psi) \) is contained in the set \( \{ x \in \Omega, a(x) > 0 \} \), we infer that
\[ \int_{\Omega} |\partial_t v(t)|^2 \, dx = \int_{\Omega} |(1 - \psi) \partial_t u|^2 \, dx \leq \frac{1}{\epsilon_0} \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx. \]
Which yields
\[ \frac{d}{dt} X(t) + E_u(t) + \left( k - \frac{2}{\epsilon_0} \right) \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx \leq 2E_w(t) + \int_{\Omega} |\nabla \psi|^2 |u|^2 \, dx. \] (22)
We remind that \( w = \psi u \), therefore there exists positive constant \( C > 0 \) such that
\[ E_w(t) \leq C \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \]
This gives
\[ \frac{d}{dt} X(t) + E_u(t) + \left( k - \frac{2}{\epsilon_0} \right) \int_{\Omega} a(x) |\partial_t u(t)|^2 \, dx \]
\[ \leq C \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx. \] (23)
Integrating the estimate above between \( t \) and \( t + T \), we get (20).

2.2. **Proof of Theorem 1.2.** In the sequel \( C, C_T \) and \( C_{T,\delta} \) denote a generic positive constants and any changes from one derivation to the next will not be explicitly outlined.

Let \( \psi \in C^\infty_c (\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and
\[ \psi(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L \end{cases}. \]
Setting \( v = (1 - \psi) u \) where \( u \) is the solution of (1) with initial data in \( H^1_0(\Omega) \times L^2(\Omega) \). Let
\[ X(t) = \int_{\Omega} v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_{\Omega} a(x) |v(t)|^2 \, dx + kE_u(t). \]
According to lemma 2.1,

\[ X (t + T) - X(t) + \int_t^{t+T} E_u(s) \, ds + \left( k - \frac{2}{\epsilon_0} \right) \int_\Omega \int_t^{t+T} a(x) |\partial_t u|^2 \, dx \, ds \]

\[ \leq C \int_t^{t+T} \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds. \quad (24) \]

The estimate (6), gives

\[ \int_t^{t+T} \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds \]

\[ \leq C_{T, \delta} \left( \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, d\tau \right) + \delta E_u(t), \]

From the following estimate

\[ E_u(t) \leq \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, d\tau + \frac{1}{T} \int_t^{t+T} E_u(s) \, ds, \quad (25) \]

we deduce that

\[ \int_t^{t+T} \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds \]

\[ \leq C_{T, \delta} \left( \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, d\tau \right) + \delta \int_t^{t+T} E_u(s) \, ds, \quad (26) \]

(20) and the estimate above gives

\[ X(t+T) - X(t) + (1 - C\delta) \int_t^{t+T} E_u(s) \, ds + \left( k - \frac{2}{\epsilon_0} - C_{T, \delta} \right) \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, ds \leq 0. \quad (27) \]

It is clear that

\[ X(t) \leq \frac{2}{3} ||a||_\infty \int_\Omega |v(t)|^2 \, dx + \left( k + \frac{2}{\epsilon_0} \right) E_u(t) \quad \text{and} \]

\[ X(t) \geq \frac{\epsilon_0}{4} \int_\Omega |v(t)|^2 \, dx + \left( k - \frac{8}{\epsilon_0} \right) E_u(t). \quad (28) \]

We choose \( \delta \) and \( k \) such that

\[ 1 - C\delta = \frac{1}{2}, \]

\[ k - \frac{2}{\epsilon_0} - C_{T, \delta} \geq \epsilon > 0 \text{ and } k - \frac{8}{\epsilon_0} \geq \epsilon. \]

Therefore using (27) we obtain

\[ X(t + T) - X(t) + \frac{1}{2} \int_t^{t+T} E_u(s) \, ds + \epsilon \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, ds \leq 0, \quad (29) \]

Thus

\[ \sum_{i=0}^{n-1} \left( X((i+1)T) - X(iT) + \frac{1}{2} \int_{iT}^{(i+1)T} E_u(s) \, ds + \epsilon \int_{iT}^{(i+1)T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, ds \right) \leq 0, \]

this gives

\[ X(nT) + \frac{1}{2} \int_0^{nT} E_u(s) \, ds + \epsilon \int_0^{nT} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, ds \leq X(0), \quad \text{for all } n \in \mathbb{N}. \]
Using (20) and (26), we conclude that there exists a positive constant $C$ such that
\[
\sup_{\mathbb{R}^+} X(t) + \int_0^{+\infty} E_u(s) \, ds \leq CI_0
\]  
with
\[
I_0 = \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2}.
\]
Since $1 - \psi \equiv 1$ for $|x| \geq 2L$ and
\[
X(t) \geq \frac{\epsilon_0}{4} \int_{\Omega} |v(t)|^2 \, dx \geq \frac{\epsilon_0}{4} \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx,
\]
using (30) we obtain
\[
\sup_{\mathbb{R}^+} \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx \leq \frac{4C}{\epsilon_0} I_0.
\]
Poincare’s inequality and the fact that the energy of $u$ is decreasing gives
\[
\int_{\Omega \cap B_{2L}} |u(t)|^2 \, dx \leq C_L \int_{\Omega} |\nabla u(t)|^2 \, dx \leq C_L E_u(0).
\]
Combining the last two estimates, we get
\[
\sup_{\mathbb{R}^+} \int_{\Omega} |u(t)|^2 \, dx \leq \frac{4C}{\epsilon_0} I_0 + C_L E_u(0) \leq CI_0.
\]
The energy decay estimate follows from (30) and the fact that
\[
(1 + t) E_u(t) \leq E_u(0) + \int_0^{+\infty} E_u(s) \, ds \leq CI_0, \text{ for all } t \geq 0.
\]
This finishes the proof of theorem 1.2, now we give the proof of proposition 1.

2.3. Proof of Proposition 1. Let $n \in \mathbb{N}$ and $u$ solution of (1) with initial data $(u_0, u_1)$ in $D(A^n)$ such that $u_0 \in L^2(\Omega)$. We set $u_n = \partial^n_t u$. First we prove that for all $n \in \mathbb{N}$ there exists $C_n > 0$ such that
\[
\int_0^{+\infty} (1 + s)^n E_{u_n}(s) \, ds \leq C_n I_{0,n}
\]
where
\[
I_{0,n} = \sum_{i=0}^n \left\| A^i \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \right\|^2_H + \|u_0\|^2_{L^2}.
\]
We will prove the result above using induction. For $n = 0$, let $u$ be a solution of (1) with initial data $(u_0, u_1)$ in $D(A^0)$ such that $u_0 \in L^2(\Omega)$. From (30) we infer that there exists $C_0 > 0$, such that
\[
\int_0^{+\infty} E_u(s) \, ds \leq C_0 I_{0,0}.
\]
We assume that for $p \in \mathbb{N}$ there exists $C_p > 0$ such that the following estimate
\[
\int_0^{+\infty} (1 + s)^p E_{u_p}(s) \, ds \leq C_p I_{0,p},
\]
holds, for all solution $u$ of (1) with initial data $(u_0, u_1)$ in $D(A^p)$ such that $u_0 \in L^2(\Omega)$.
Let $u$ be a solution of (1) with initial data $(u_0, u_1)$ in $D(A^{p+1})$ such that $u_0 \in L^2(\Omega)$.
We have \( u_{p+1} = \partial_t^p (\partial_t u) \). Since \((\partial_t u (0), \partial_t^2 u (0)) = A \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \in D \left( \mathcal{A}^p \right) \) and \( \partial_t u (0) = u_1 \in L^2 (\Omega) \). According to (33), we have

\[
\int_0^{+\infty} (1 + s)^p E_{u_{p+1}} (s) \, ds \leq C_p \left( \sum_{i=0}^p \left\| A^{i+1} \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) \right\|_H^2 + \| u_1 \|_{L^2}^2 \right)
\leq C_p I_{0,p+1}. \tag{34}
\]

Let \( \psi \in C_c^\infty (\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and

\[
\psi (x) = \begin{cases}
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L
\end{cases}
\]

Setting \( w = \psi u_{p+1} \) and \( v = (1 - \psi) u_{p+1} \). Let

\[
X (t) = \int_{t \Omega} v (t) \partial_t v (t) \, dx + \frac{1}{2} \int_{t \Omega} a (x) |v (t)|^2 \, dx + k E_{u_{p+1}} (t),
\]

where \( k \) is a positive constant. \( u_{p+1} \) satisfies

\[
\left\{ \begin{array}{rl}
\partial_t^2 u_{p+1} - \Delta u_{p+1} + a (x) \partial_t u_{p+1} = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u_{p+1} = 0 & \text{on } \mathbb{R}_+ \times \Gamma,
\end{array} \right. \qquad (u_{p+1} (0, x), \partial_t u_{p+1} (0, x)) \in H^1_0 (\Omega) \times L^2 (\Omega).
\]

Then we know from (29) that

\[
X (t + T) - X (t) + \frac{1}{2} \int_t^{t+T} E_{u_{p+1}} (s) \, ds + \epsilon \int_t^{t+T} \int_{t \Omega} a (x) |\partial_t u_{p+1}|^2 \, dx \, ds \leq 0. \tag{35}
\]

Multiplying the estimate above by \((1 + t + T)^{p+1}\), we obtain

\[
(1 + t + T)^{p+1} X (t + T) - (1 + t)^{p+1} X (t) + \frac{1}{2} \epsilon \int_t^{t+T} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq C_T (1 + t)^p X (t).
\]

Therefore using (33), (34) and the fact that

\[
X (t) \leq \frac{3}{2} \| a \|_\infty E_{u_{p_+}} (t) + \left( \frac{k + 2}{\epsilon_0} \right) E_{u_{p+1}} (t),
\]

we deduce that for any \( q \in \mathbb{N}^* \)

\[
\frac{1}{2} \int_0^{qT} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq C_T \sum_{i=0}^{q-1} (1 + iT)^p X (iT)
\]

\[
\leq C_T \sum_{i=0}^{q-1} (1 + iT)^p (E_{u_{p+1}} (iT) + E_{u_{p}} (iT))
\]

\[
\leq C_T \sum_{i=0}^{q-1} \int_{iT}^{(i+1)T} (1 + s)^p (E_{u_{p+1}} (s) + E_{u_{p}} (s)) \, ds
\]

\[
+ C_T (E_{u_{p+1}} (0) + E_{u_{p}} (0))
\]

\[
\leq C_T \int_0^{+\infty} (1 + s)^p (E_{u_{p+1}} (s) + E_{u_{p}} (s)) \, ds
\]

\[
+ C_T (E_{u_{p+1}} (0) + E_{u_{p}} (0))
\]

\[
\leq C_{T,p} I_{0,p+1}.
\]
We deduce that
\[ \int_0^{+\infty} (1 + s)^{p+1} E_{u_{p+1}} (s) \, ds \leq C_{p+1} I_{0,p+1}. \]

We remind that we have proved that
\[ \int_0^{+\infty} (1 + s)^n E_{u_n} (s) \, ds \leq C_n I_{0,n}. \]

Now the energy decay estimate follows from the fact that
\[ (1 + t)^{n+1} E_{u_n} (t) \leq E_{u_n} (0) + (n + 1) \int_0^t (1 + s)^n E_{u_n} (s) \, ds \leq C_n I_{0,n}, \]
for all \( t \geq 0 \). By taking into account of the estimate above, we infer that,
\[ (1 + t)^n \| \partial_t^{n-1} u(t) \|_{L^2}^2 \leq 2 (1 + t)^n E_{u_{n-1}} (t) \leq C_1 I_{0,n-1} \text{ for all } t \geq 0, \]

We have \( \partial_t^{n-1} u \) is a solution of the following system
\[
\begin{cases}
\partial_t^{n-1} u - \Delta \partial_t^{n-1} u + a(x) \partial_t^n u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\
\partial_t^{n-1} u = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\
(\partial_t^{n-1} u (0, x), \partial_t^n u (0, x)) \in D(A)
\end{cases}
\]

therefore
\[ (\partial_t^{n-1} u, \partial_t^n u) \in C^1 (\mathbb{R}^+, H). \]

Using Eq (36), we infer that
\[ (1 + t)^n \| \Delta \partial_t^{n-1} u(t) \|_{L^2}^2 \leq C (1 + t)^n \left( E_{u_n} (t) + E_{u_{n-1}} (t) \right) \leq C I_{0,n}. \]

3. Proof of Theorem 1.3. This section is devoted to the proof of theorem 1.3. We begin by giving some preliminary results.

3.1. Preliminary results. The following result is a observability estimate for the weighted local energy of solutions of the system (1).

**Proposition 3.** We assume that Hyp A holds and \((\omega,T_0)\) geometrically controls \(\Omega\). Let \( \delta > 0 \) and \( R > R_0 \). There exist \( T > T_0 \) and \( C_{T,\delta,R} > 0 \), such that the following inequality
\[
\int_t^{t+T} (1 + s) \iint_{\Omega \times B_R} |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \, dx \, ds \\
\leq C_{T,\delta,R} \left( \int_t^{t+T} \int_{\Omega} a(x) |\partial_t u|^2 \, dx \, ds + \delta \left( (1 + t) E_u (t) + \int_t^{t+T} \int_{\Omega} |u|^2 \, dx \, ds \right) \right).
\]

(37)

holds for every \( t \geq 0 \) and for all \( u \) solution of (1) with initial data \((u_0,u_1)\) in \( H^1_0 \times L^2 \).

**Proof.** To prove this result we argue by contradiction: If (37) was false, there would exist a sequence of positive numbers \((t_n)\) and a sequence of solutions \((u_n)\) such that
\[
\int_{t_n}^{t_n+T} \int_{\Omega \times B_R} (1 + s) \left( |u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2 \right) \, dx \, ds \\
\geq n \left( \int_{t_n}^{t_n+T} \int_{\Omega} a(x) (1 + s) |\partial_t u_n|^2 \, dx \, ds \right) \\
+ \delta \left( (1 + t_n) E_{u_n} (t_n) + \int_{t_n}^{t_n+T} \int_{\Omega} |u_n|^2 \, dx \, ds \right).
\]

(38)
We may assume that $t_n \to +\infty$ (if the sequence $t_n$ is bounded we can argue as in the proof of proposition 2). Setting
\[
\lambda_n^2 = \int_{t_n}^{t_n+T} \int_{\Omega \cap B_R} (1 + s) \left( |u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2 \right) \, dx \, ds
\]
and
\[
v_n = \left( 1 + t_n + \cdot \right)^{1/2} u_n \left( t_n + \cdot \right) \frac{\lambda_n}{\lambda_n}.
\]
Therefore from (38) we infer that
\[
\frac{1 + t_n}{\lambda_n^2} E_{u_n} (t_n) + \frac{1}{\lambda_n^2} \int_0^T \int_{\Omega} \left| u_n (t_n + s) \right|^2 \, dx \, ds \leq \frac{1}{\delta},
\]
and
\[
\int_{t_n}^{t_n+T} \int_{\Omega} a (x) \left( 1 + s \right) \left| \partial_t u_n \right|^2 \, dx \, ds \to 0 \quad \text{as} \quad n \to +\infty.
\]
It is clear that $v_n$ is a solution of the following system
\[
\begin{cases}
\partial_t^2 v_n - \Delta v_n + a (x) \partial_t v_n = f_n (t, x) & \text{in } \mathbb{R}_+ \times \Omega, \\
v_n (t, x) = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\
(v_{n,0}, v_{n,1}) \in H^1_0 (\Omega) \times L^2 (\Omega),
\end{cases}
\]
with
\[
f_n (t, x) = \frac{1}{2 \lambda_n} (1 + t_n + t)^{-\frac{1}{2}} \left( a (x) - \frac{1}{2} (1 + t_n + t)^{-1} \right) u_n (t_n + t)
\]
\[
+ \frac{1}{\lambda_n} (1 + t_n + t)^{-\frac{1}{2}} \partial_t u_n (t_n + t).
\]
It is easy to see that (39), gives
\[
\int_0^T \int_{\Omega} \left| \frac{1}{2 \lambda_n} (1 + t_n + t)^{-\frac{1}{2}} \left( a (x) - \frac{1}{2} (1 + t_n + t)^{-1} \right) u_n (t_n + t) \right|^2 \, dx \, dt 
\leq C \left( 1 + t_n \right)^{-1} \frac{1}{\lambda_n} \int_{t_n}^{t_n+T} \int_{\Omega} \left| u_n (s) \right|^2 \, dx \, ds
\leq C \frac{\left( 1 + t_n \right)^{-1}}{\delta}
\]
and
\[
\int_0^T \int_{\Omega} \left| \frac{1}{\lambda_n} (1 + t_n + t)^{-\frac{1}{2}} \partial_t u_n (t_n + t) \right|^2 \, dx \, dt 
\leq \frac{(1 + t_n)^{-1}}{\lambda_n^2} \int_0^T \int_{\Omega} \left| \partial_t u_n (t_n + t) \right|^2 \, dx \, dt
\leq \frac{(1 + t_n)^{-1}}{\lambda_n^2} 2TE_{u_n} (t_n)
\leq \frac{2T (1 + t_n)^{-2}}{\delta}.
\]
We conclude that
\[
\int_0^T \int_{\Omega} \left| f_n (s, x) \right|^2 \, dx \, ds \leq C \frac{(1 + t_n)^{-1}}{\delta} \quad \text{as} \quad n \to +\infty.
\]
(39) gives
\[
\int_0^T \int_\Omega a(x) |\partial_t v_n|^2 \, dx \, dt \\
\leq \|a\|_\infty (1 + t_n)^{-1} \frac{1}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega |u_n(s)|^2 \, dx \, ds \\
+ \frac{2}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega a(x) (1 + s) |\partial_t u_n|^2 \, dx \, ds
\]
\[
\leq \|a\|_\infty (1 + t_n)^{-1} + \frac{2}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega a(x) (1 + s) |\partial_t u_n|^2 \, dx \, ds.
\]

Now using (40), we deduce that
\[
\int_0^T \int_\Omega a(x) |\partial_t v_n|^2 \, dx \, dt \to_{n \to +\infty} 0.
\]

We multiply the equation satisfied by \( u_n \) by \( (1 + t) \partial_t u_n \) and integrating between \( t_n \) and \( t_n + t \), we obtain
\[
(1 + t_n + t) E_{u_n} (t_n + t) - (1 + t_n) E_{u_n} (t_n) = \int_{t_n}^{t_n+t} E_{u_n} (s) \, ds - \int_{t_n}^{t_n+t} \int_\Omega a(x) (1 + s) |\partial_t u_n|^2 \, dx \, ds,
\]
thus exploiting (39), we infer that
\[
\frac{1}{\lambda_n^2} (1 + t_n + t) E_{u_n} (t_n + t) \leq \frac{1}{\lambda_n^2} (1 + t_n) E_{u_n} (t_n) + \frac{1}{\lambda_n^2} \int_{t_n}^{t_n+t} E_{u_n} (s) \, ds
\]
\[
\leq \frac{1}{\delta} + \frac{T}{\delta}, \text{ for all } t \in [0, T].
\]

On the other hand, we have
\[
\int_0^T E_{v_n} (t) \, dt \\
\leq \frac{1}{\lambda_n^2} \int_0^T \left[ (1 + t_n + t) E_{u_n} (t_n + t) + (1 + t_n + t)^{-1} \int_\Omega |u_n(t_n + t)|^2 \, dx \right] \, dt
\]
\[
\leq \frac{C_T}{\delta}.
\]

Let \( Z_n \) be the solution of the following system
\[
\begin{cases}
\partial_t^2 Z_n - \Delta Z_n = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
Z_n = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\
(Z_n(0), \partial_t Z_n(0)) = (v_n(0), \partial_t v_n(0)).
\end{cases}
\]

The hyperbolic energy inequality gives
\[
\sup_{[0,T]} E_{v_n - Z_n} (s) \leq C_T \|a(x) \partial_t v_n + f_n(t,x)\|_{L^2([0,T],L^2(\Omega))}.
\]

Now using (42) and (41), we deduce that
\[
\sup_{[0,T]} E_{v_n - Z_n} (s) \to_{n \to +\infty} 0.
\]

(43)
Turn into account of the estimate above, we obtain
\[ TE_{v_n}(0) = TE_{Z_n}(0) = \int_0^T E_{Z_n}(t) \, dt \]
\[ \leq 2T \sup_{[0,T]} E_{v_n - Z_n}(s) + 2 \int_0^T E_{v_n}(t) \, dt \]
\[ \leq C_{T,\delta} \]
which means
\[ \sup_{[0,T]} E_{Z_n}(s) = E_{v_n}(0) \leq C_{T,\delta}. \]
To complete the proof we argue as in the proof of the proposition 2, by taking into account of (42) and (43).

In the sequel we need the following result.

**Lemma 3.1.** Let \( \psi \in C^\infty_c(\mathbb{R}^d) \) such that \( 0 \leq \psi \leq 1 \) and
\[
\psi(x) = \begin{cases} 
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L 
\end{cases}
\]
Setting \( w = \psi u \) and \( v = (1 - \psi) u \) where \( u \) is a solution of (1) with initial data in \( H_0^1(\Omega) \times L^2(\Omega) \). Let
\[
X(t) = \int_\Omega v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_\Omega a(x) |v(t)|^2 \, dx + kE_u(t),
\]
where \( k \) is a positive constant. We have
\[
(1 + t + T) X(t + T) - (1 + t) X(t) + \int_t^{t+T} (1 + s) E_u(s) \, ds
\]
\[
+ \left( k - \frac{2}{\epsilon_0} \right) \int_t^{t+T} \int_\Omega a(1 + s) |\partial_t u|^2 \, dx \, ds
\]
\[
\leq C \int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s) \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds + \int_t^{t+T} X(s) \, ds.
\]

**Proof.** We may assume that \( u \) is smooth. According to (22)
\[
\frac{d}{dt} X(t) + E_u(t) + \left( k - \frac{2}{\epsilon_0} \right) \int_\Omega a(x) |\partial_t u(t)|^2 \, dx \leq 2E_w(t) + \int_\Omega |\nabla \psi|^2 |u|^2 \, dx.
\]
We multiply the estimate above by \( 1 + t \), we obtain
\[
\frac{d}{dt} (1 + t) X(t) + (1 + t) E_u(t) + \left( k - \frac{2}{\epsilon_0} \right) \int_\Omega a(x) (1 + t) |\partial_t u(t)|^2 \, dx
\]
\[ \leq 2 (1 + t) E_w(t) + (1 + t) \int_\Omega |\nabla \psi|^2 |u|^2 \, dx + X(t).
\]
We have \( w = \psi u \), therefore there exists positive constant \( C > 0 \) such that
\[
E_w(t) \leq C \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx.
\]
This gives
\[
\frac{d}{dt} (1 + t) X(t) + (1 + t) E_u(t) + \left( k - \frac{2}{\epsilon_0} \right) \int_\Omega a(x) (1 + t) |\partial_t u(t)|^2 \, dx
\]
\[ \leq C (1 + t) \int_{\Omega \cap B_{2L}} |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \, dx + X(t).
\]
We integrate the inequality above between \( t \) and \( t + T \), we obtain (44). \( \square \)
3.2. **Proof of Theorem 1.3.** In the sequel, $C$, $C_T$ and $C_{T,\delta}$ denote a generic positive constants and any changes from one derivation to the next will not be explicitly outlined.

Let $\psi \in C_0^\infty (\mathbb{R}^d)$ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 
1 & \text{for } |x| \leq L \\
0 & \text{for } |x| \geq 2L 
\end{cases}.$$ 

Setting $w = \psi u$ and $v = (1 - \psi) u$ where $u$ is a solution of (1) with initial data in $H_0^1 (\Omega) \times L^2 (\Omega)$ such that

$$||d(\cdot) (u_1 + au_0)||_{L^2} < +\infty.$$ 

Let

$$X(t) = \int_{\Omega} v(t) \partial_t v(t) \, dx + \frac{1}{2} \int_{\Omega} a(x) |v(t)|^2 \, dx + k E_u(t),$$

where $k$ is a positive constant. According to (44) we have

$$(1 + t + T) X(t + T) - (1 + t) X(t) + \int_t^{t+T} (1 + s) E_u(s) \, ds$$

$$+ \left( k - \frac{2}{\varepsilon_0} \right) \int_t^{t+T} \int_{\Omega} a(x) (1 + s) |\partial_t u|^2 \, dx \, ds$$

$$\leq C \int_t^{t+T} (1 + s) \int_{\Omega \cap B_{2L}} \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds + \int_t^{t+T} X(s) \, ds.$$ 

Using (37), we infer that

$$\int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s) \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds$$

$$\leq C_{T,\delta} \left( \int_t^{t+T} \int_{\Omega} a(x) (1 + s) |\partial_t u|^2 \, dx \, ds \right)$$

$$+ \delta \left( \int_t^{t+T} \int_{\Omega} |u|^2 \, dx \, ds + (1 + t) E_u(t) \right).$$

Since

$$(1 + t) E_u(t) \leq \int_t^{t+T} \int_{\Omega} a(x) (1 + s) |\partial_t u|^2 \, dx \, ds + \frac{1}{T} \int_t^{t+T} (1 + s) E_u(s) \, ds,$$

then

$$\int_t^{t+T} \int_{\Omega \cap B_{2L}} (1 + s) \left( |u|^2 + |\nabla u|^2 + |\partial_t u|^2 \right) \, dx \, ds$$

$$\leq C_{T,\delta} \left( \int_t^{t+T} \int_{\Omega} a(x) (1 + s) |\partial_t u|^2 \, dx \, ds \right)$$

$$+ C_T \delta \left( \int_t^{t+T} \int_{\Omega} |u|^2 \, dx \, ds + \int_t^{t+T} (1 + s) E_u(s) \, ds \right).$$

Now (44) and the estimate above gives

$$(1 + t + T) X(t + T) - (1 + t) X(t) + (1 - C_T \delta) \int_t^{t+T} (1 + s) E_u(s) \, ds$$

$$+ \left( k - \frac{2}{\varepsilon_0} - C_{T,\delta} \right) \int_t^{t+T} \int_{\Omega} a(x) (1 + s) |\partial_t u|^2 \, dx \, ds$$

$$\leq \int_t^{t+T} X(s) \, ds + C_T \delta \int_t^{t+T} \int_{\Omega} |u|^2 \, dx \, ds.$$
We have
\[ X(t) \leq \frac{3}{2} \| a \|_\infty \int_\Omega |v(t)|^2 \, dx + \left( k + \frac{2}{\epsilon_0} \right) E_u(t) \quad \text{and} \quad \text{(46)} \]
\[ X(t) \geq \frac{\epsilon_0}{4} \int_\Omega |v(t)|^2 \, dx + \left( k - \frac{8}{\epsilon_0} \right) E_u(t). \quad \text{(47)} \]
We choose \( \delta \) and \( k \) such that
\[ 1 - C_T \delta = \frac{1}{2}, \]
\[ k - \frac{2}{\epsilon_0} - C_{T, \delta} \geq \epsilon > 0 \text{ and } k - \frac{8}{\epsilon_0} \geq \epsilon. \]
Thus we get
\[ (1 + t + T) X(t + T) - (1 + t) X(t) + \frac{1}{2} \int_t^{t+T} (1 + s) E_u(s) \, ds \]
\[ + \epsilon \int_t^{t+T} \int_\Omega a(x) (1 + s) |\partial_x u|^2 \, dx \, ds \]
\[ \leq \int_t^{t+T} X(s) \, ds + \int_t^{t+T} \int_\Omega |u|^2 \, dx \, ds. \quad \text{(48)} \]
As in the proof of theorem 1.2, from the estimate above, we deduce that
\[ \sup_{\mathbb{R}_+} ((1 + t) X(t)) + \frac{1}{2} \int_0^{+\infty} (1 + s) E_u(s) \, ds \]
\[ \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds \right). \quad \text{(49)} \]
By taking into account of \( 1 - \psi \equiv 1 \) for \( |x| \geq 2L \), we obtain
\[ \sup_{\mathbb{R}_+} (1 + t) \int_{\{|x| \geq 2L\}} |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds \right). \]
On the other hand, using (31), we get
\[ (1 + t) \int_{\Omega \cap B_{2L}} |u(t)|^2 \, dx \leq C_L (1 + t) \int_\Omega |\nabla u(t)|^2 \, dx \]
\[ \leq C_L (1 + t) E_u(t) \]
\[ \leq CI_0. \]
Combining the two estimates above we deduce that
\[ \sup_{\mathbb{R}_+} (1 + t) \int_\Omega |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^{+\infty} X(s) \, ds + \int_0^{+\infty} \int_\Omega |u|^2 \, dx \, ds + I_0 \right). \quad \text{(50)} \]
Now we need the following result due to Ikehata [17, lemma 2.5]

**Lemma 3.2.** Let \( u \) be a solution of (1) with initial data \((u_0, u_1)\) in \( H_0^1(\Omega) \times L^2(\Omega) \) which satisfies
\[ \|d(\cdot) (u_1 + au_0)\|_{L^2} < +\infty, \]
where
\[ d(x) = \begin{cases} |x| & d \geq 3 \\ |x| \ln (B |x|) & d = 2, \end{cases} \]
with \( B \inf_{x \in \Omega} |x| \geq 2. \) Then there exists \( C > 0, \) such that
\[ \|u(t)\|_{L^2}^2 + \int_0^t \int_\Omega a(x) |u(s,x)|^2 \, dx \, ds \leq C \left( \|u_0\|_{L^2}^2 + \|d(\cdot) (u_1 + au_0)\|_{L^2}^2 \right), \quad \text{(51)} \]
for all \( t \geq 0. \)
We have
\[ \int_0^t \int_\Omega |u|^2 \, dx \, ds \leq \int_0^t \int_{\Omega \cap B_L} |u|^2 \, dx \, ds + \int_0^t \int_{\{|x| \geq L\}} |u|^2 \, dx \, ds \]
\[ \leq C_L \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds + \frac{1}{\epsilon_0} \int_0^t \int_\Omega a(x) |u(s,x)|^2 \, dx \, ds \]
\[ \leq C_L \int_0^t E_u(s) \, ds + \frac{1}{\epsilon_0} \int_0^t \int_\Omega a(x) |u(s,x)|^2 \, dx \, ds. \]

Now using (51) and (30), we get
\[ \int_0^t \int_\Omega |u|^2 \, dx \, ds \leq CI_1, \]
with
\[ I_1 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 + \|d(\cdot)(u_1 + au_0)\|_{L^2}^2. \]

On the other hand, (46) and the fact that
\[ \int_0^t E_u(s) \, ds \leq CI_0 \]
we obtain
\[ \int_0^t X(s) \, ds \leq C \left( \int_0^t \int_\Omega |u|^2 \, dx \, ds + \int_0^t E_u(s) \, ds \right) \]
\[ \leq CI_1. \]

Finally it is clear that,
\[ X(0) \leq CI_1. \]

Therefore from (50), we deduce
\[ \sup_{R_+} (1 + t) \int_\Omega |u(t)|^2 \, dx \leq C \left( X(0) + \int_0^t X(s) \, ds + \int_0^t \int_\Omega |u|^2 \, dx \, ds + I_0 \right) \]
\[ \leq CI_1, \]
and using (49), we obtain
\[ \int_0^t (1 + s) E_u(s) \, ds \leq CI_1. \]

The energy decay estimate follows from the fact that
\[ (1 + t)^2 E_u(t) \leq E_u(0) + 2 \int_0^t (1 + s) E_u(s) \, ds \]
\[ \leq CI_1, \text{ for all } t \geq 0. \]

This finishes the proof of theorem 1.3.

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