On the algebraic types of the Bel-Robinson tensor

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Abstract The Bel-Robinson tensor is analyzed as a linear map on the space of the traceless symmetric tensors. This study leads to an algebraic classification that refines the usual Petrov-Bel classification of the Weyl tensor. The new classes correspond to degenerate type I space-times which have already been introduced in literature from another point of view. The Petrov-Bel types and the additional ones are intrinsically characterized in terms of the sole Bel-Robinson tensor, and an algorithm is proposed that enables the different classes to be distinguished. Results are presented that solve the problem of obtaining the Weyl tensor from the Bel-Robinson tensor in regular cases.

Keywords Bel-Robinson tensor · Gravitational superenergy · Petrov-Bel classification

1 Introduction

With the aim of defining intrinsic states of gravitational radiation Bel [1] [2] [3] introduced a rank 4 tensor which plays an analogous role for gravitational field to that played by the energy tensor for electromagnetism. This superenergy Bel tensor is quadratic in the Riemann tensor and, in the vacuum...
case, is divergence-free. In this case the Bel tensor coincides with the super-energy Bel-Robinson tensor, built with the same expression by replacing the Riemann tensor with the Weyl tensor.

Both, the Bel and the Bel-Robinson tensors have interesting properties that, relative to every observer, allow the definition of a non-negative super-energy density and a super-energy Poynting vector. In the last decade the interest in the super-energy tensors has been on the increase, and a lot of works are devoted to analyzing their properties and to studying their generalization to any dimension and to any physical field \cite{4} (see references therein for an exhaustive bibliography on this subject). In a recent work \cite{5}, where the dynamical laws of super-energy are accurately analyzed, up-to-date references on the Bel and Bel-Robinson tensors can be found.

The algebraic properties of the Bel-Robinson tensor (BR tensor) and its close relationship with the principal null directions of the Weyl tensor were studied by Debever early on \cite{6}, \cite{7}, and a spinorial approach can be found in \cite{8}. But the intrinsic algebraic characterization of a BR tensor was not obtained until the recent work by Bergqvist and Lankien \cite{9}. They give the conditions on the BR tensor playing a similar role to that played by the algebraic Rainich \cite{10} conditions for the electromagnetic field.

Nevertheless, some gaps remain in the algebraic comprehension of the BR tensor. The analogy with the electromagnetic field can help us to understand them. Let us consider an energy tensor $T_{em}$ satisfying the algebraic Rainich conditions:

\begin{equation}
T_{em}^{2} = \chi^2 g, \quad \chi \equiv \frac{1}{2} \sqrt{\text{tr} T_{em}^{2}}.
\end{equation}

For a $T_{em}$ satisfying \cite{11} we know (see for example \cite{11}):

(i) Its algebraic classification: when $\chi \neq 0$, the Segre type of $T_{em}$ is $[(11)(11)]$ and it is associated with a non null electromagnetic field $F$; when $\chi = 0$, the Segre type of $T_{em}$ is $[(31)]$ and it is associated with a null electromagnetic field $F$.

(ii) Its canonical expression in terms of its intrinsic elements, namely, invariant subspaces and scalars: for a non null field, $T$ takes the expression $T_{em} = -\chi \Pi$, where $\Pi$ is a 2+2 structure tensor that determines the two principal planes and the two null principal directions of the electromagnetic field; for a null field, $T$ takes the expression $T_{em} = l \otimes l$, where $l$ is the fundamental vector of the electromagnetic field.

(iii) It determines the electromagnetic field $F$ up to a duality rotation, and the explicit expression of $F$ in terms of $T_{em}$ is also known \cite{11}.

As commented above, Bergqvist and Lankien \cite{9} have found the necessary and sufficient conditions for a rank 4 tensor $T$ to be the BR tensor associated with a Weyl-like tensor $W$. These conditions play for the BR tensor the same role as the Rainich conditions \cite{11} play for the electromagnetic field. On the other hand, the three points satated above for the electromagnetic field could be similarly established for the BR tensor:

(I) Algebraic classification of a BR tensor $T$: the Petrov-Bel classification can be obtained by studying the Weyl tensor as a linear map \cite{12}, \cite{3}. The symmetries of the BR tensor $T$ allow us to consider and analyze it as a linear map on the nine dimensional space of the traceless symmetric tensors. What
algebraic classification follows on from this study? What relationship exists between these BR classes and the Petrov-Bel types of the Weyl tensor?

(II) Canonical form of the BR tensor $T$ in terms of its invariant spaces and scalars: for every algebraic type of the BR tensor, the eigenvalues and eigenvectors should be analyzed, as well as the canonical expression of $T$ in terms of them. What relationship exists between the spaces and scalar the invariants of both, the BR tensor and the Weyl tensor?

(III) Expression of the Weyl tensor in terms of the BR tensor: it is known that the BR tensor $T$ determines the Weyl tensor $W$ up to a duality rotation. But the explicit expression of $W$ in terms of $T$ has not been established.

The goal of this work is to tackle the algebraic problems of the BR tensor stated in the three points above. Here we solve the first one and give preliminary results on the points (II) and (III) which will be fully solved in another work in progress [13].

Some basic properties of the BR tensor $T$ considered as a linear map are presented in section 2 of this work. In section 3 we define the main BR invariant scalars and study their relationship with the scalar invariants of the Weyl tensor. This result is a contribution to point (II) above. We also study the powers of $T$ and obtain a family of identities on the BR tensor that generalize an already known equality. Finally, we determine the characteristic polynomial of $T$.

In section 4 we show that the main BR scalar invariants allow different classes of Weyl tensors to be distinguished. On one hand, they label the three groups of Petrov-Bel types that can be discriminated by the Weyl eigenvalues, namely, types III, N and O with a triple Weyl eigenvalue, types II and D with a double eigenvalue and type I with three different eigenvalues. On the other hand, the main BR scalars also distinguish between five classes of algebraically general Weyl tensors. These classes refine the Petrov-Bel classification and were introduced by McIntosh and Arianrhod [14]. They correspond to particular configurations of the four null principal directions: either they span a 3-plane or they define a frame with permutability properties. A detailed analysis of these ‘degenerate’ type I classes can be found in [15].

In section 5 we consider the classification of the BR tensor taking into account its eigenvalue multiplicity. We show that seven classes appear from this point of view: one with 9 different eigenvalues, two with 6 different eigenvalues, three with 3 different eigenvalues and one with a eigenvalue of multiplicity 9. We characterize every class in terms of the main BR invariant scalars and show that they precisely correspond to the classes of the Weyl tensor considered in the previous section. An arrow diagram that helps us to visualize the different classes and their degenerations is offered at the end of the section.

The study presented in sections 4 and 5 shows that the main BR scalars do not distinguish between Petrov-Bel types III, N and O, and between types II and D. Nevertheless we prove in section 7 that all these Petrov-Bel types match with different classes of the BR tensor and that they can be discriminated by using their minimal polynomial. At the end of this last section we present an algorithm that summarizes the classification of the BR tensor and the intrinsic characterization of every class.
In studying the minimal polynomial in last section we use an expression that relates the $n$-powers of the BR and Weyl tensors. This expression is proved in section 6 and it is valid except for some degenerate cases. For the case $n = 3$ this expression allows us to obtain, in the same section, the Weyl tensor in terms of the BR tensor when none of the main BR scalar invariants vanish. This result partially solves point (III) above. Elsewhere [13] we obtain the Weyl tensor in terms of the BR tensor in the cases that are not considered here.

With the intention of making the mathematical expressions simpler we use, when possible, a global notation that is accurately explained in four appendixes at the end of this work. This also allows us to simplify the calculations by using some properties that are also stated in these appendixes.

It is worth outlining that, in spite of the fact that the BR tensor loses part of the information of the Weyl tensor (a duality rotation), its study as a linear map leads to ten different classes, a richer classification than the six Petrov-Bel types of the Weyl tensor as a linear map. On the other hand, note that from our study follows a characterization of every Petrov-Bel type in terms of the BR tensor, a result that only was known for types N and O [16] [17].

2 The Bel-Robinson tensor: towards its algebraic classification

We shall note $g$ the space-time metric with signature $\{-, +, +, +\}$ and we shall write $W$ to indicate the Weyl tensor. The conventions of signs are those of the book by Stephani et al. [18]. The Bel-Robinson tensor (BR tensor) $T$ is given in terms of the Weyl tensor as:

$$T_{\alpha\beta\mu\nu} = \frac{1}{4} \left( W_{\alpha\beta}^{\rho\sigma} W_{\mu\rho\nu\sigma} + * W_{\alpha\beta}^{\rho\sigma} * W_{\mu\rho\nu\sigma} \right),$$

where $*$ denotes the Hodge dual operator (see Appendix B.9). Note that expression (2) coincides with that given in [8] [17] and it differs in a factor from the original one by Bel [3].

2.1 The Weyl tensor as an endomorphism

A self–dual 2-form is a complex 2-form $F$ such that $*F = iF$. We can associate biunivocally with every real 2-form $F$ the self-dual 2-form $F = \frac{1}{2} (F - i * F)$. In short, here we refer to a self-dual 2-form as a SD bivector. The endowed metric on the 3-dimensional complex space of the SD bivectors is $G = \frac{1}{2} (G - i \eta)$, $\eta$ being the metric volume element of the space-time, and $G$ being the metric on the space of 2-forms, $G = \frac{1}{2} g \wedge g$. Here $\wedge$ denotes the double-forms exterior product (see Appendix A.5).

The algebraic classification of the Weyl tensor $W$ can be obtained [12], [3] by studying the traceless linear map defined by the self–dual (SD) Weyl
tensor $\mathcal{W} = \frac{1}{2}(W - i \ast W)$ on the SD bivectors space. This SD-endomorphism (see notation in Appendix B) has associated the complex scalar invariants:

$$a = \text{Tr} \mathcal{W}^2, \quad b = \text{Tr} \mathcal{W}^3.$$  \hfill (3)

In terms of them, the characteristic equation reads

$$x^3 - \frac{1}{2}ax - \frac{1}{3}b = 0.$$  \hfill (4)

Then, the Petrov-Bel classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when the characteristic equation (4) admits three different roots. If there is a double root $\rho = -\frac{b}{a}$ and a simple one $-2\rho$, the minimal polynomial distinguishes between types D and II. Finally, if all the roots are equal, and so zero, the Weyl tensor is type O, N or III, depending on the minimal polynomial.

### 2.2 The Bel-Robinson tensor as an endomorphism

The expression (2) of the BR tensor may be written in terms of the SD Weyl tensor as:

$$T_{\alpha\mu\beta\nu} = \mathcal{W}_{\alpha}^\rho \cdot \ast \mathcal{W}_{\mu\rho\nu\sigma},$$  \hfill (5)

where $\ast$ stands for complex conjugate. Taking into account the $\circ$-product defined in Appendix D, we have that (5) becomes

$$T = \mathcal{W} \circ \overline{\mathcal{W}}.$$  \hfill (6)

It is easy to show that, for two arbitrary SD-endomorphisms $\mathcal{X}$ and $\mathcal{Y}$, the tensor $Q = \mathcal{X} \circ \mathcal{Y}$ satisfies

$$Q_{\alpha\mu\beta\nu} = Q_{\mu\alpha\beta\nu} = Q_{\beta\nu\alpha\mu}, \quad Q^\circ_{\alpha\beta\nu} = 0.$$  \hfill (7)

As a consequence of these properties, if $S$ is a trace-less symmetric tensor (TLS tensor), then $Q(S)_{\alpha\beta} = Q_{\alpha\beta}^{\mu\nu} S_{\mu\nu}$ is trace-less and symmetric. Thus, we can see $Q$ as a (complex) symmetric linear map on the space of the TLS tensors, that is to say, $Q$ is a TLS-endomorphism (see notation in Appendix C).

Besides, the BR tensor as given by (6) has two important additional properties. Firstly, it is real and, secondly, as a consequence of the traceless property of the SD Weyl tensor, it is a fully symmetric traceless tensor (see Appendix D.7). Then we have:

**Lemma 1** The Bel-Robinson tensor $T$ is a real traceless TLS-endomorphism, that is, a traceless symmetric endomorphism on the 9-dimensional space of the traceless symmetric tensors.
As explained in subsection above, the Petrov-Bel classification follows on by studying the Weyl tensor as a SD-endomorphism. Similarly, we want to obtain the algebraic classification of the BR tensor by analyzing it as a TLS-endomorphism. We start our study by writing the BR tensor in terms of another TLS-endomorphism. If we define

$$\Omega = \mathcal{W} \circ \mathcal{G},$$  \hfill (8)

and we take into account (6) and that $T$ is real, we obtain:

$$T = \Omega \bullet \Omega = \overline{\Omega} \bullet \Omega,$$  \hfill (9)

where $\bullet$ denotes the composition of TLS-endomorphisms (see Appendix C).

The tensor $\Omega$ is not but the restriction of the Weyl tensor acting (with the indices 2 and 4) on the TLS tensor space. Indeed, for a trace-less symmetric tensor $S$, we have

$$\Omega^{\alpha\beta}_{\mu\nu} S^{\mu\nu} = \mathcal{W}^{\alpha}_{\mu} \epsilon^{\beta}_{\nu} S^{\mu\nu}.$$  \hfill (10)

On the other hand, it is easy to show that the tensorial components of $\Omega$ are

$$\Omega^{\alpha\beta}_{\mu\nu} = -\mathcal{W}^{\alpha}_{\alpha(\mu\nu)\beta},$$  \hfill (11)

where ( ) denotes symmetrization.

The tensor $\Omega$ inherits some properties of the SD Weyl tensor. Evidently, it is a TSL-endomorphism (it satisfies (12)) and, moreover, it is traceless and satisfies a SD-like property:

$$\Omega^{\alpha}_{\mu\alpha\nu} = 0, \quad \Omega^{\alpha\beta\gamma\delta}_{\mu\nu} \mathcal{G}^{\gamma\delta} = 0.$$  \hfill (12)

Conversely, if $\Omega$ is a (complex) TLS-endomorphism that satisfies (12) we can reverse the expression (8) and obtain the traceless SD-endomorphism (see property of Appendix D.5):

$$\mathcal{W}^{\alpha}_{\alpha\beta\mu\nu} = \frac{2}{3} \Omega^{\alpha\beta\gamma\delta}_{\mu\nu} \mathcal{G}^{\gamma\delta} = \frac{4}{3} \Omega^{\alpha\beta\gamma\delta}_{\mu\nu} \mathcal{G}^{\gamma\delta} = \mathcal{W}^{\alpha}_{\alpha(\mu\nu)\beta},$$  \hfill (13)

where [ ] denotes antisymmetrization. Thus we have the following.

**Proposition 1** There is a one-to-one correspondence between traceless SD-endomorphisms $\mathcal{W}$ and the (complex) TLS-endomorphism $\Omega$ which satisfy (12). This bijection is given by (8) and (13).

2.3 Can the Weyl tensor be obtained from the Bel-Robinson tensor?

The BR tensor can be obtained from the Weyl tensor by means of the quadratic expression (2), but it is known that this expression is invariant under duality rotation of the Weyl tensor. For the SD Weyl tensor a duality rotation takes the form $e^{i\theta} \mathcal{W}$. Then, the above quoted invariance follows trivially from the expression (6) because,

$$\mathcal{W} \circ \mathcal{W} = [e^{i\theta} \mathcal{W}] \circ [e^{-i\theta} \mathcal{W}].$$
Thus, the accurate question we should pose is: can the Weyl tensor be determined, up to a duality rotation, from the BR tensor? or, to be more precise, is there an explicit algorithm to obtain it? These questions will be analyzed and solved elsewhere [13] for an arbitrary BR tensor. In section 6 we study this question for regular cases. In this study the TLS endomorphism $\Omega$ plays an important role.

3 Powers, scalar invariants and characteristic polynomial of the Bel-Robinson tensor

In order to tackle the algebraic classification of the BR tensor $T$ as a TLS-endomorphism we need to know its characteristic polynomial and its scalar invariants. In this section we obtain them from the powers of $T$.

The powers of the BR tensor can be obtained as $T^n = T^{n-1} \bullet T$. From (9) and as $T$ is real, the $\Omega$ defined in (8) satisfies $\Omega \bullet \Omega = \Omega \bullet \Omega$. Then, from (9) we obtain

**Lemma 2** For $n \geq 1$ it holds $T^n = \Omega^n \bullet \Omega^n$, where $\Omega$ is the endomorphism defined as (8) from the Weyl tensor $\mathcal{W}$.

On the hand we can obtain the TLS-endomorphism $\Omega^n = \Omega \bullet \Omega^{n-1}$ in terms of the SD-endomorphism $\mathcal{W}^n = \mathcal{W} \circ \mathcal{W}^{n-1}$. Indeed, from the definition (8) and putting, in expression (38) of Appendix D.8, $V = \mathcal{W}$, $X = \mathcal{W}^{n-1}$ and $Z = Y = \mathcal{G}$, we obtain by induction:

**Lemma 3** The TLS-endomorphism $\Omega$ given by (8) satisfies

$$\Omega^n = \mathcal{W}^n \circ \mathcal{G}, \quad n \geq 1.$$  

The expression for the powers of the BR tensor and the traces of these powers in terms of the SD Weyl tensor can also be obtained by induction by taking, in expression (38) of Appendix D.8, $V = \mathcal{W}$, $X = \mathcal{W}$, $Y = \mathcal{G}$, and taking into account the property of Appendix D.2. Thus, we have:

**Proposition 2** The powers of the BR tensor and their traces can be computed in terms of the SD Weyl tensor as

$$T^n = \mathcal{W}^n \circ \mathcal{G}, \quad \text{tr} T^n = |\text{tr} \mathcal{W}^n|^2.$$  

From this proposition, and by applying the property of Appendix D.2, we obtain the following result easily:

**Corollary 1** The powers $T^n$ of the BR tensor satisfy:

$$(T^n)_{\alpha \lambda \beta} = \frac{1}{4} \text{Tr} T^n g_{\alpha \beta}.$$  

For $n = 2$ this expression states that $T_{\alpha \lambda \mu \nu} T_{\beta \lambda \mu \nu} = \frac{1}{4} T_{\rho \lambda \mu \nu} T_{\rho \lambda \mu \nu} g_{\alpha \beta}$, an identity already known [6] [8]. Thus (16) generalizes, for an arbitrary $n$, this identity.

As a consequence of proposition 2 the traces of the powers of $T$ are non negative and we can compute them in terms of the traces of the SD Weyl...
tensor. But we also know that $\mathcal{W}$ satisfies the characteristic equation (11), that is:

$$\mathcal{W}^3 = \frac{1}{2} a \mathcal{W} + \frac{1}{3} b G,$$

(17)

where $a$ and $b$ are the main Weyl scalar invariants given in (8).

From (17), the traces of the powers of $\mathcal{W}^n$ for $n > 3$ can be computed in terms of $a$ and $b$. After that, they can be used to compute the traces of $T^n$ by applying proposition 2. More precisely, if we denote

$$\alpha = \frac{1}{2} |a|, \quad \beta = \frac{1}{3} |b|, \quad \mu = \frac{1}{2 \times 3^2} (a^3 \bar{b}^2 + \bar{a}^3 b^2),$$

(18)

we have that the traces of $T^n$ are given by

- $\text{Tr} \ T = 0$
- $\text{Tr} \ T^2 = 4 \alpha^2$
- $\text{Tr} \ T^3 = 9 \beta^2$
- $\text{Tr} \ T^4 = 4 \alpha^4$
- $\text{Tr} \ T^5 = 25 \alpha^2 \beta^2$
- $\text{Tr} \ T^6 = 4 \alpha^6 + 9 \beta^4 + 6 \mu$
- $\text{Tr} \ T^7 = 49 \alpha^4 \beta^2$
- $\text{Tr} \ T^8 = 4 \alpha^8 + 64 \alpha^2 \beta^4 + 12 \alpha^2 \mu$
- $\text{Tr} \ T^9 = 9 \beta^2 (\beta^4 + 9 \alpha^6 + 3 \mu).$

(19)

Note that the Weyl tensor defines four real scalar invariants that can be grouped in the complex ones $a$, $b$ defined in (8) from the SD Weyl tensor. We have remarked in subsection 2.3 that a duality rotation is lost when constructing the BR tensor. So just three scalars survive in $T$: we can see in (19) that two of them, the modulus of $a$ and $b$ determine the traces of $T^2$ and $T^3$. The third one, $a^3 \bar{b}^2 + \bar{a}^3 b^2$, does not appear until $\text{Tr} \ T^6$ is computed. This fact and expressions (19) justify defining the following main BR scalar invariants:

$$\alpha = \frac{1}{2} \sqrt{\text{Tr} T^2}, \quad \beta = \frac{1}{3} \sqrt{\text{Tr} T^3}, \quad \mu = \frac{1}{6} \left( \text{tr} T^6 - 4 \alpha^6 - 9 \beta^4 \right).$$

(20)

Then, the traces of the powers of the BR tensor depend on the BR main scalar invariants as (19). Moreover, these expressions can be used to obtain the characteristic equation of the BR tensor and we achieve the following.

**Proposition 3** The BR tensor satisfies the characteristic equation

$$x^9 - 2 \alpha^2 x^7 - 3 \beta^2 x^6 + \alpha^4 x^5 + \alpha^2 \beta^2 x^4 + (3 \beta^4 - \mu) x^3 + \alpha^2 \beta^4 x - \beta^6 = 0$$

(21)

where $\alpha$, $\beta$ and $\mu$ are the main BR scalar invariants (20). These invariants determine the traces $\text{Tr} T^n$ as (19).

4 Labeling some Weyl types with the main BR scalar invariants

In the next section we study the algebraic types of the BR tensor as TLS-endomorphism. In order to understand the correspondence between these new BR types and the already known Weyl types, we show in this section that some of these Weyl classes can be characterized in terms of the main BR scalar invariants.
When analyzing the Weyl tensor as a SD-endomorphism, the main Weyl invariant scalars distinguish the multiplicity of the Weyl eigenvalues: the algebraically general type I has 3 simple eigenvalues, types II and D have a simple and a double one, and types III, N and O have a triple vanishing eigenvalue. Then, from the characteristic equation (4) we obtain \[18\] \[19\]:

**Lemma 4**

Let \( a = \text{Tr} W^2 \), \( b = \text{Tr} W^3 \) be the main Weyl scalar invariants. Then, the Weyl tensor is

i. Petrov-Bel type I iff \( 6b^2 - a^3 \neq 0 \).

ii. Petrov-Bel types D or II iff \( 6b^2 = a^3 \neq 0 \).

iii. Petrov-Bel types O, N or III iff \( a = 0 = b \).

Note that in order to distinguish between types D and II as well as between types N or III, the minimal polynomial is necessary \[19\]. But, at the moment, we are only interested in scalar conditions.

On the other hand, some classes of algebraically general space-times have been considered in literature. A refinement of the Petrov-Bel classification is based on the Weyl scalar invariant \[14\]:

\[
M = \frac{a^3}{b^2} - 6.
\]  

The invariant \( M \) is not defined if \( b = 0 \) as occurs in types III and N and in algebraically general space-times with a vanishing eigenvalue. But we can extend its validity to these cases if we put \( M = \infty \) whenever \( b = 0 \neq a \), and \( M = 0 \) if \( a = 0 = b \).

This way, it is known \[14\] \[20\] that the cases where \( M \) is real positive or infinity, which are called \( \text{IM}^+ \) or \( \text{IM}^\infty \) respectively, correspond to the case of the Weyl eigenvalues having a real ratio, \( \frac{\rho_i}{\rho_j} \in \mathbb{R} \) (the case \( M = \infty \) means that the Weyl tensor has a vanishing eigenvalue). An equivalent condition in terms of Debever null directions has also been obtained \[14\] \[20\]: \( M \) is real positive or infinity iff the four Debever null directions span a 3-plane.

On the other hand, the case where \( M \) is real negative \( \text{IM}^- \) has also been considered. It corresponds to the property of two of the SD eigenvalues having the same modulus (\( |\rho_i| = |\rho_j| \) for some \( i \neq j \)). Moreover, the case \( M = -6 \) (or \( a = 0 \)) corresponds to all the eigenvalues having the same modulus. These conditions have also been interpreted in terms of permutability properties of the null Debever directions \[15\].

A detailed analysis of these ‘degenerate’ algebraically general classes can be found in \[15\], where they are also interpreted as the space-times where the electric and magnetic parts of the Weyl tensor are aligned for a (non necessary time-like) direction. These classes also contain the purely electric and purely magnetic space-times which have been accurately studied in \[21\].

The scalar \( M \) is homogeneous in the Weyl tensor and thus invariant by duality rotation. Consequently, it should depend on the main BR scalars. Indeed, if we take into account the relations \[18\] between the main Weyl scalars \( a \), \( b \) and the main BR scalars \( \alpha \), \( \beta \) and \( \mu \), we obtain:

\[
81 \beta^6 (\text{Im}[M])^2 = 16 (4\alpha^6 \beta^4 - \mu^2), \quad 9 \beta^4 \text{Re}[M] = 4 \mu - 54 \beta^4.
\]
The first expression shows that $M$ is real if, and only if, $\mu^2 - 4\alpha^6\beta^4 = 0$. After that, if $M$ is real, the second expression gives its sign in terms of the main BR scalars. More precisely, we have:

**Lemma 5** The invariant scalar $M$ is real, if and only if $\mu^2 - 4\alpha^6\beta^4 = 0$. If this condition holds, the Weyl tensor is:

i. Type $IM^+$ ($M$ real positive) iff $2\mu - 3^3\beta^4 > 0$. 
ii. Type $IM\infty$ ($M = \infty$) iff $\beta = 0 \neq \alpha$. 
iii. Type $IM^-$ ($M$ real negative) iff $2\mu - 3^3\beta^4 < 0$. 
iv. Type $IM[-6]$ ($M = -6$) iff $\alpha = 0 \neq \beta$. 
v. Algebraically special ($M = 0$) iff $2\mu - 3^3\beta^4 = 0$.

From now on we denote $I_r$ the class of non ‘degenerate’ type I Weyl tensors, that is to say, those with non real invariant $M$. Then, from lemmas 4 and 5 and relations (18) we obtain the following characterization of some classes of the Weyl tensor in terms of the main BR scalars:

**Proposition 4** If $\alpha$, $\beta$ and $\mu$ are the main BR invariant scalars, then the Weyl tensor is:

i. Type O, N or III iff $\alpha = \beta = 0$. 
ii. Type D or II iff $\mu^2 - 4\alpha^6\beta^4 = 0, \ 2^2\alpha^3 = 3^3\beta^2 \neq 0$. 
iii. Type $IM^+$ iff $\mu^2 - 4\alpha^6\beta^4 = 0, \ 2\mu - 3^3\beta^4 > 0$. 
iv. Type $IM\infty$ iff $\alpha \neq 0 = \beta$. 
v. Type $IM^-$ iff $\mu^2 - 4\alpha^6\beta^4 = 0, \ 2\mu - 3^3\beta^4 < 0$. 
vi. Type $IM[-6]$ iff $\beta \neq 0 = \alpha$. 
vii. Type $I_r$ iff $\mu^2 - 4\alpha^6\beta^4 \neq 0$. 

5 Classifying the Bel-Robinson tensor

In this section we study the BR tensor as a TLS-endomorphism taking into account the eigenvalue multiplicity.

As a consequence of (18), $\Omega$ and $\mathcal{W}$ satisfy the same minimal polynomial, that is, both have the same eigenvalues $\rho_i, \ i = 1, 2, 3$. On the other hand, from expression (10), $\Omega$ and $\mathcal{D}$ are symmetric endomorphisms that commute and, consequently, they have the same eigenvectors. These eigenvectors and their associated eigenvalues should be either real or pairs of complex conjugates and then $T$ will have the same eigenvectors with associated eigenvalues $\rho_i \bar{\rho}_j$. Thus, we obtain the BR eigenvalues without solving the characteristic equation (21).

**Proposition 5** The BR tensor has 3 real eigenvalues $t_i$, and 3 pairs of complex conjugate eigenvalues $\tau_i, \bar{\tau}_i$. They depend on the Weyl eigenvalues $\rho_i$ as

\[ t_i = |\rho_i|^2; \quad \tau_i = \rho_j \bar{\rho}_k, \quad (i,j,k) \ a \ pair \ permutation \ of \ (1,2,3). \]  

(23)

The BR tensor has three main scalar invariants. In accordance with this fact, the BR complex eigenvalues should depend on the three real ones. Indeed, a straightforward calculation shows
Lemma 6 In terms of the real BR eigenvalues $t_i$ the complex ones $\tau_k$ take the expression:

$$\tau_k = p_k + i q, \quad 2p_k \equiv t_k - t_i - t_j, \quad (i, j, k \neq), \quad q^2 \equiv p_1 p_2 + p_2 p_3 + p_3 p_1. \quad (24)$$

Generically, the BR eigenvalues are nine different scalars. Now we analyze the admitted degenerations. If $t_i = t_j \neq t_k$, then $(24)$ implies $p_i = p_j$, that is, $\tau_i = \tau_j$ and $\bar{\tau}_i = \bar{\tau}_j$ and, consequently, $T$ has 3 double eigenvalues. Moreover, expression $(23)$ implies that the Weyl tensor has two eigenvalues with the same modulus, $|\rho_i| = |\rho_j|$, a case corresponding to a Weyl tensor of type $IM^-$. If we avoid a higher degeneration by considering $t_k \neq t_i = t_j$, then $M \neq -6$.

If one of the complex eigenvalues $\tau_i$ is real, we have $\tau_i = \bar{\tau}_i$. Then $(24)$ implies $q = 0$ and consequently, $\tau_k = \bar{\tau}_k$ for every $k = 1, 2, 3$. Thus $T$ has, again, 3 double eigenvalues. Now, from $(23)$, the ratio of the Weyl eigenvalues $\rho_i$ is real for every pair $(i, j)$ because $\rho_i \bar{\rho}_j$ is real. If we avoid higher degenerations we must remove the case of a vanishing Weyl eigenvalue $(M \neq \infty)$ and then the Weyl tensor is type $IM^+$.

Thus, the case of nine different eigenvalues corresponds to a regular type $I$, Weyl tensor. From all these considerations and taking into account proposition 4 we have the followings.

Proposition 6 A BR tensor $T$ has nine different eigenvalues (3 real and 3 pairs of complex conjugate) iff $\mu^2 - 4\alpha^6\beta^4 \neq 0$, that is to say, the Weyl tensor is type $I_\mu$.

Proposition 7 If $\mu^2 - 4\alpha^6\beta^4 = 0$, the BR tensor $T$ has, at the most, six different eigenvalues. Then we have two cases:

(i) $T$ has a double and a simple real eigenvalues, and two double and two simple complex conjugate eigenvalues iff $2\mu - 3^3 \beta^4 < 0$ and $\alpha \neq 0$, that is to say, the Weyl tensor is type $IM^-$ with $M \neq -6$.

(ii) $T$ has three simple and three double real eigenvalue iff $2\mu - 3^3 \beta^4 > 0$ and $\beta \neq 0$, that is to say, the Weyl tensor is type $IM^+$.

Starting from the two cases above with six different eigenvalues we can consider three kinds of further degeneration. The first one follows on by imposing both conditions, $t_i = t_j \neq t_k$ and $\tau_i = \bar{\tau}_i$, which is a degeneration of both, the $IM^-$ $(M \neq -6)$ and $IM^+$ types. Then, from $(24)$ we have $\tau_j = \bar{\tau}_i = \bar{\tau}_j$ and $\tau_k = t_i$ and, consequently, $T$ has four real eigenvalues with multiplicities 4, 1, and 4. Moreover, $(23)$ implies that the Weyl tensor has two equal eigenvalues, $\rho_i = \rho_j$ and, consequently, it is Petrov-Bel type D or II.

Secondly, we can consider a degeneration of the $IM^-$ type by taking the three real eigenvalues as equal, $t_1 = t_2 = t_3$. Then $(24)$ implies $\tau_1 = \bar{\tau}_2 = \tau_3$ and $T$ has three different eigenvalues with multiplicity three, a real one and a pair of complex conjugates. On the other hand, if we avoid a further degeneration, $(23)$ implies that $M = -6$.

Thirdly, we can consider a degeneration of the $IM^+$ type by imposing $\tau_i = t_j$. Then $(24)$ implies $t_j = \tau_i = \tau_k = 0 (i, j, k \neq)$, and $T$ has three real
eigenvalues with multiplicities 5, 2 and 2. From (23) we have $\rho_j = 0$ and then $M = \infty$.

Note that, as a consequence of (24), imposing $\tau_i = t_i$ to the $IM^+$ case leads to vanishing eigenvalues, a more degenerate case that we will consider below. All these considerations and proposition 4 lead to the following.

**Proposition 8** (i) A BR tensor $T$ has three real eigenvalues with multiplicities 4, 1, and 4 iff $\mu^2 - 4\alpha^6\beta^4 = 0$ and $2^2\alpha^3 = 3^3\beta^2 \neq 0$, that is to say, the Weyl tensor is type $II$ or $D$.

(ii) A BR tensor $T$ has three triple eigenvalues (a real and a pair of complex conjugates) iff $\beta \neq 0 = \alpha$, that is to say, the Weyl tensor is type $IM^{-6}$.

(iii) A BR tensor $T$ has three real eigenvalues (with multiplicities 2, 5 and 2) iff $\beta = 0 \neq \alpha$, that is to say, the Weyl tensor is type $IM^{\infty}$.

Finally we can consider a further degeneration from the three cases in proposition above. By using (24) it is easy to show that in any case we reach the highest degeneration: there is a unique vanishing eigenvalue. Then, (23) implies that the Weyl eigenvalues also vanish and, consequently, the Weyl tensor is Petrov-Bel type III, N or O. Thus, from proposition 4 we have:

**Proposition 9** A BR tensor $T$ has a sole vanishing eigenvalue with multiplicity 9 iff $\alpha = \beta = 0$, that is to say, the Weyl tensor is type $III$, N or O.

It is worth remarking that this proposition implies that the Petrov-Bel types $III$, N and O can not be distinguished by taking into account the eigenvalue multiplicity of the BR tensor $T$ or, equivalently, by analyzing its main scalar invariants. We find a similar situation for the Petrov-Bel types $II$ and $D$ as a consequence of the first point in proposition 8. Nevertheless, we will see in the last section that each of these Petrov-Bel types corresponds to a different algebraic BR type and that they can be distinguished by using the minimal polynomial of $T$ as a TLS-endomorphism.

Propositions 7, 8 and 9 classify and characterize the BR tensor taking into account its eigenvalue multiplicity. In order to better visualize the eigenvalue degeneration that relates the different classes, we present the following arrow
diagram. Note that the four files in the diagram correspond to 9, 6, 3 and 1 different eigenvalues.

- \( L \) \( \{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3\} \)
- \( \text{IM}^+ \) \( \{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3\} \)
- \( \text{IM}^- \) \( \{t, t, t_3, \tau, \tau, \tau, \overline{\tau}, \overline{\tau}, \overline{\tau}\} \)
- \( \text{IM}^{\infty} \) \( \{t, t, 0, 0, 0, \tau_3, 0, 0, \tau_3\} \)
- \( \text{II, D} \) \( \{t, t, t_3, \tau, \tau, \tau, \overline{\tau}, \overline{\tau}, \overline{\tau}\} \)
- \( \text{III, N, O} \) \( \{0, 0, 0, 0, 0, 0, 0, 0\} \)

6 Obtaining the Weyl tensor from the BR tensor: regular case

As commented in subsection [2.3] the general problem of obtaining the Weyl tensor, up to a duality rotation, in terms of the BR tensor will be solved elsewhere [13]. The specific algorithm strongly depends on the algebraic type of the Weyl and BR tensors. Here we present an explicit expression which is valid for the Petrov-Bel types \( D \) and \( I \) and for type \( I \) when none of the main Weyl scalar invariants vanish. This result is based in a general expression that relates the \( n \)-powers of the BR and Weyl tensors. On the other hand, this expression for \( n = 2 \) is used in the last section of this work in showing that the minimal polynomial of the BR tensor distinguishes between types \( I I \) and \( D \).

6.1 Gravitational superenergy tensors of order \( n \)

The tensor \( T^n = W^n \circ \mathcal{G} \) for \( n > 1 \) is not, generically, a completely symmetric tensor, because \( W^n \) does not inherit the symmetries of \( W \); it is a SD-endomorphism but it is not trace-free (in general) and so the Bianchi identity is not satisfied. But its traceless part

\[ W_{(n)} = W^n - \omega_{(n)} \mathcal{G}, \quad \omega_{(n)} \equiv \frac{1}{3} \text{Tr} W^n, \]

(25)
has the symmetries of the SD Weyl tensor. Thus, it has an associated superenergy tensor given by:

\[ T_{(n)} \equiv \mathcal{W}_{(n)} \diamond \overline{\mathcal{W}}_{(n)}, \]

that has all the superenergy properties of the BR tensor \( T \). We will say that \( T_{(n)} \) is the BR superenergy tensor of order \( n \).

All the expression presented in section 2 involving the Weyl tensor \( W \) and the BR tensor \( T \) will also be valid for \( W_{(n)} \) and \( T_{(n)} \). Specifically,

\[ T_{(n)} = \Omega_{(n)} \cdot \overline{T}_{(n)}, \quad \Omega_{(n)} \equiv \mathcal{W}_{(n)} \diamond \mathcal{G}. \]  

(26)

From lemma 4 and (25) the last expression above becomes:

\[ \Omega_{(n)} = \Omega^n - \omega_{(n)} \Gamma. \]

If we use this expression to expand the first expression in (26), we obtain the following.

**Proposition 10** The BR superenergy tensor of order \( n \) associated with the Weyl-like tensor \( W_{(n)} \) defined in (25) takes the expression:

\[ T_{(n)} = T^n - \sigma_{(n)} \Omega_{(n)} - \omega_{(n)} \overline{T}_{(n)} - |\omega_{(n)}|^2 \Gamma. \]  

(27)

6.2 A relation between the \( n \)-powers of the Weyl and BR tensors

From the last expression above, we can obtain its self-dual antisymmetric part by contracting the indexes (23) with the SD-identity \( \mathcal{G} \). Then, the left side vanishes due to the whole symmetry of \( T_{(n)} \). On the other hand, the right side can be computed taking into account: first, the property of Appendix D.4, second, the definition (25), and third, that \( \Omega_{(n)} \) has the same properties as \( \Omega \) and, consequently, it satisfies (12) and (13) by replacing \( W \) with \( \mathcal{W}_{(n)} \). Finally, we obtain the following.

**Lemma 7** The power \( n \) of a SD Weyl tensor is related to the power \( n \) of the associated BR-tensor by

\[ 2 (T^n)_{\alpha \mu \nu \beta} \mathcal{G}^{\mu \nu \rho \sigma} = \left[ (\text{Tr } \mathcal{W}^n) \mathcal{W}^n - \frac{1}{2} |\text{Tr } \mathcal{W}^n|^2 \mathcal{G} \right]_{\alpha \beta \rho \sigma}. \]  

(28)

When \( \text{Tr } \mathcal{W}^n = 0 \), then \( \mathcal{W}^n = \mathcal{W}_{(n)} \) and \( T^n = T_{(n)} \) is a completely symmetric tensor and, consequently, (28) is an identity. Nevertheless, when \( \text{Tr } \mathcal{W}^n \) does not vanish, this proposition provides \( \mathcal{W}^n \) from \( T^n \), up to a duality rotation. Indeed, in this case \( \text{Tr } T^n = |\text{Tr } \mathcal{W}^n|^2 \neq 0 \). Then, from (28) we obtain:

**Proposition 11** If \( T \) is a BR-tensor and \( \text{Tr } T^n \neq 0 \), the power \( n \) of the original SD Weyl tensor is given, up to a duality rotation, by

\[ \mathcal{W}^n = e^{i \theta_n} \left[ \frac{2}{\sqrt{\text{Tr } T^n}} T_{(n)} + \sqrt{\text{Tr } T^n} \overline{\mathcal{G}} \right], \quad e^{i \theta_n} = \frac{\text{Tr } \mathcal{W}^n}{|\text{Tr } \mathcal{W}^n|} \]  

(29)

\[ T_{(n)\alpha \beta \rho \sigma} \equiv (T^n)_{\alpha \mu \nu \beta} \mathcal{G}^{\mu \nu \rho \sigma} \]  

(30)
6.3 The Weyl tensor from the BR tensor

Let us suppose now $a = \text{Tr} W^2 \neq 0$ and $b = \text{Tr} W^3 \neq 0$. Then, for $n = 3$ expression (29) can be written as:

$$\frac{b}{|a||b|} W^3 = \frac{1}{|a||b|} \left[ 2 T_{(3)} + \frac{1}{2} |b|^2 G \right].$$

(31)

Then, if we remove $W^3$ by using the characteristic equation (17), and we take into account that $|a|^2 = \text{Tr} T^2$, $|b|^2 = \text{Tr} T^3$, the SD Weyl tensor can be recovered from the BR tensor as follows.

**Theorem 1** If $T$ is a BR tensor and $\text{Tr} T^2 \neq 0$, $\text{Tr} T^3 \neq 0$, then the Weyl tensor can be obtained, up to duality rotation, as

$$W = e^{i\theta} W_0, \quad W_0 = \frac{1}{\sqrt{\text{Tr} T^2 \text{Tr} T^3}} \left[ 4 T_{(3)} + \frac{1}{3} |\text{Tr} T^3| G \right]$$

(32)

$$T_{(3)\alpha\beta\rho\sigma} \equiv (T^3)_{\alpha\mu\beta} G^{\mu\nu} G^{\rho\sigma}$$

(33)

This result partially solves the point (III) stated in the introduction. It does not apply when $\alpha\beta = 0$, that is, for the classes $N$, $III$, $IM^\infty$ and $IM^{-6}$. In these degenerate cases the determination of the Weyl tensor in terms of the BR tensor requires a different analysis that will be tackled elsewhere [13].

On the other hand, these results provide an alternative approach to know when a traceless completely symmetric rank 4 tensor $T$ is the BR tensor associated with a certain Weyl tensor. An answer to this question has been given by Bergqvist and Lankien [9]. From theorem 1 an alternative answer follows when $\alpha$ and $\beta$ do not vanish. Indeed, if for a given $T$ we calculate the SD double 2-form $W_0(T)$ by using expression (32), then $T$ must satisfy the equation $T = W_0(T) \odot W_0(T)$.

7 Complete classification of the Bel-Robinson tensor

At this point we are ready to complete the classification of the BR tensor by considering the minimal polynomial and not only the eigenvalue multiplicity (see section 4). Although a whole analysis of the canonical forms will be tackled elsewhere [13], here we will obtain the necessary results to distinguish between types $N$ and $III$, and between types $II$ and $D$, and to intrinsically characterize them.

To accomplish this goal, we start from the conditions on the Weyl tensor that distinguish these Petrov-Bel types (see, for example, ref. [19]):

**Lemma 8** Let $a = \text{Tr} W^2$, $b = \text{Tr} W^3$ the main Weyl scalar invariants. Then, the Weyl tensor is

i. Petrov-Bel type $N$ iff $W^2 = 0 \neq W$.

ii. Petrov-Bel type $III$ iff $W^3 = 0 \neq W^2$.

iii. Petrov-Bel type $D$ iff $a \neq 0$, $W^2 = \frac{1}{a} W + \frac{2|b|^2}{a} G$. 

If we apply the property of Appendix D.6 to expression (15) we have
\[ W^n = 0 \text{ if, and only if, } T^n = 0. \] Then, considering this property for \( n = 2, 3 \), and as consequence of lemma 8 we can state:

**Proposition 12** Let \( T \) be a non null BR tensor. Then, the Weyl tensor is:

i. Petrov-Bel type N if, and only if, \( T^2 = 0 \).

ii. Petrov-Bel type III if, and only if, \( T^3 = 0 \neq T^2 \).

This proposition shows that Petrov-Bel types \( N \) and \( III \) correspond to different algebraic classes of the BR tensor because they have different minimal polynomial. Moreover, it affords a characterization of these Petrov-Bel types in terms of the sole BR tensor. The result for type \( N \) was given by Bergqvist [17] together with the equivalent condition \( T_{\alpha\beta\mu\nu}T^{\nu\rho\sigma\lambda} = 0 \). This requisite states, equivalently, that the superenergy flow vector is a null vector for every observer, a condition that was already presented as a characterization of the type \( N \) Weyl tensors [10]. Nevertheless, in this work all the conditions that characterize the different algebraic classes are written as equations on the BR tensor as a TLS-endomorphism.

Now let us deal with a type D Weyl tensor. As the SD Weyl tensor always satisfies the characteristic equation (17), we can compute \( T^3 = W^3 \circ \overline{W}^3 \) to get:

\[ T^3 = \alpha^2 T + \beta^2 \Gamma + \frac{1}{6}(a\vec{b}\Omega + \overline{a\vec{b}\Omega}). \] (34)

On the other hand, from the minimal polynomial of a type D Weyl tensor (see point (iii) of lemma 8), we can compute \( T^2 = W^2 \circ \overline{W}^2 \) and we obtain:

\[ \alpha T^2 = \frac{\alpha^2}{3} T + \frac{\alpha^3}{3^2} \Gamma + \frac{1}{6}(a\vec{b}\Omega + \overline{a\vec{b}\Omega}), \quad 3^3 \beta^2 = 2^2 \alpha^3. \] (35)

Then, we can use (35) to remove \( a\vec{b}\Omega + \overline{a\vec{b}\Omega} \) in (34) and we achieve that \( T \) satisfies a polynomial of degree 3. Moreover, we know from proposition 8 that, for a type D Weyl tensor, the BR tensor has 3 different eigenvalues. Thus, this polynomial is the minimal one. The specific calculation leads to the following.

**Lemma 9** For a type D Weyl tensor, the associated BR tensor \( T \) satisfies the minimal polynomial:

\[ T^3 = \alpha T^2 + \frac{2}{3} \alpha^2 T - \frac{2^3}{3^3} \alpha^3 \Gamma, \quad \alpha \neq 0. \] (36)

We want to prove now that the converse is also true, that is, that condition (36) characterize the type D case. Taking the trace of (36) we obtain \( 2^3 \alpha^3 = 3^3 \beta^2 \). Moreover, we can use expression (34), valid for an arbitrary BR tensor, to remove \( T^3 \) in (36), and thus we obtain that \( T \) satisfies (35).

On the other hand, expression (29) for \( n = 2 \) becomes:

\[ W^2 = \frac{2a}{|a|^2} T_{(2)} + \frac{a}{2} \mathcal{G}. \]
Now, if we compute $T_2$ (defined in (30)) putting the expression (35) for $T^2$, and we take into account expressions (12) and (13) and property of Appendix D.4, we obtain that the SD Weyl tensor satisfies an equation of degree 2. Besides, $\alpha$ and, consequently, $a$ does not vanish. Then, the Weyl tensor is type $D$.

Moreover, if (36) does not hold and the main BR scalar invariants satisfy the conditions (i) in proposition 8, the Weyl tensor is type $II$. These scalar conditions can be changed to other equivalent expressions and, finally, we arrive to the following.

**Proposition 13** Let $T$ be a BR tensor. Then, the Weyl tensor is:

i. Petrov-Bel type D if, and only if, $T$ satisfies (36).

ii. Petrov-Bel type II if, and only if, $T$ satisfies $\mu^2 = 4\alpha^6\beta^4 \neq 0$, $2\mu = 3^3 \beta^4$ and it does not satisfy (36).

Again this proposition shows that Petrov-Bel types $D$ and $II$ correspond to different algebraic classes of the BR tensor because they have different minimal polynomial.

Propositions 6, 7, 8, 9, 12 and 13 afford a characterization of the different algebraic classes of the BR tensor. These classes correspond to different algebraic classes of the Weyl tensor, namely, the Petrov-Bel types and some ‘degenerate’ type $I$ cases. All these result can be summarized by the following.

**Theorem 2** For a non vanishing Bel-Robinson tensor $T$ we can distinguish nine different algebraic classes. The following arrow diagram offers an algorithm that distinguishes them. In it, $\alpha, \beta, \mu$ denote the main BR scalar invariants defined in (20).
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A Products and other formulas involving 2-tensors $A$ and $B$

1. Composition as endomorphisms: $A \cdot B$,
   
   $$(A \cdot B)^{\alpha}_{\beta} = A^{\alpha}_{\mu} B^{\mu}_{\beta}$$

2. In general, for arbitrary tensors, $\cdot$ will be used to indicate the contraction of adjacent indexes on the tensorial product.

3. Square and trace as endomorphism
   
   $$A^2 = A \cdot A, \quad \text{tr} A = A^{\alpha}_{\alpha}$$

4. Action on vectors $x, y$ as an endomorphism $A(x)$ and as a bilinear form $A(x, y)$:
   
   $$A(x)^{\alpha} = A^{\alpha}_{\beta} x^{\beta}, \quad A(x, y) = A^{\alpha}_{\beta} x^{\alpha} y^{\beta}$$

5. Exterior product as double 1-forms: $A \wedge B$,
   
   $$(A \wedge B)^{\alpha\beta\mu\nu} = A^{\alpha\mu} B^{\beta\nu} + B^{\alpha\mu} A^{\beta\nu} - A^{\alpha\nu} B^{\beta\mu} - B^{\alpha\nu} A^{\beta\mu}$$

B Products and other formulas involving SD-endomorphisms $\mathcal{X}$ and $\mathcal{Y}$

1. Every self-dual (SD) symmetric double 2-form $\mathcal{X}$ defines a linear map on the 3-dimensional SD bivector space. For short, we will say that $\mathcal{X}$ is a SD-endomorphism.

2. Composition as endomorphisms $\mathcal{X} \circ \mathcal{Y}$:
   
   $$(\mathcal{X} \circ \mathcal{Y})^{\alpha\beta}_{\rho\sigma} = \frac{1}{2} \mathcal{X}^{\alpha\beta}_{\mu\nu} \mathcal{Y}^{\mu\nu}_{\rho\sigma}$$

3. Square and trace as endomorphism:
   
   $$\mathcal{X}^2 = \mathcal{X} \circ \mathcal{X}, \quad \text{Tr} \mathcal{X} = \frac{1}{2} \mathcal{X}^{\alpha\beta}_{\alpha\beta}$$

4. Action (on SD bivectors $\mathcal{F}, \mathcal{H}$) as an endomorphism $\mathcal{X}(\mathcal{F})$, and as a bilinear form $\mathcal{X}(\mathcal{F}, \mathcal{H})$:
   
   $$\mathcal{X}(\mathcal{F})^{\alpha\beta}_{\mu\nu} = 2 \mathcal{X}^{\alpha\beta}_{\mu\nu} \mathcal{F}^{\mu\nu}, \quad \mathcal{X}(\mathcal{F}, \mathcal{H}) = \frac{1}{4} \mathcal{X}^{\alpha\beta}_{\mu\nu} \mathcal{F}^{\alpha\beta} \mathcal{H}^{\mu\nu}$$

5. Metric on the space of SD bivectors (SD-identity):
   
   $$\mathcal{G} = \frac{1}{2} (G - i \eta), \quad \mathcal{G}(\mathcal{F}) = \mathcal{F}$$

6. The SD-identity $\mathcal{G}$ satisfies:
   
   $$\mathcal{G}^{\alpha\beta}_{\mu\nu} \mathcal{G}^{\mu\nu}_{\rho\sigma} = \frac{1}{2} \mathcal{G}^{\alpha\beta}_{\rho\sigma}, \quad \mathcal{G}^{\alpha\beta}_{\mu\nu} \mathcal{G}^{\mu\nu}_{\rho\sigma} = \frac{3}{2} \mathcal{G}^{\alpha\beta}_{\rho\sigma}$$
7. The \((1,3)\)-trace of a SD-endomorphism \(\mathcal{X}\) is proportional to the metric. More precisely:
\[
\mathcal{X}^\mu_{\alpha\mu\beta} = \frac{1}{2} \text{Tr} \mathcal{X} g_{\alpha\beta}
\]

8. A SD-endomorphism \(\mathcal{X}\) is traceless iff it satisfies the algebraic Bianchi identity. More precisely:
\[
\text{Tr} \mathcal{X} = 0 \iff \mathcal{X}^\mu_{\alpha\mu\beta} = 0 \iff \mathcal{X}_{\alpha(\beta\mu\nu)} = 0
\]

9. The metric volume element \(\eta\) is a linear map on the 2-forms space that defines the Hodge dual operator. For a real 2-from \(F\) and a real symmetric double 2-form \(W\):
\[
*F = \eta(F) , \quad *W = \eta \circ W
\]

C Products and other formulas involving TLS-endomorphisms \(T\) and \(R\).

1. Every 4-tensor \(T\) with the symmetries:
\[
T_{\alpha\beta\mu\nu} = T_{\beta\alpha\mu\nu} = T_{\mu\nu\alpha\beta} , \quad T^\alpha_{\alpha\mu\nu} = 0
\]
defines a symmetric linear map on the 9-dimensional space of the traceless symmetric tensors. We say that \(T\) is a TLS-endomorphism.

2. Composition as endomorphisms: \(T \circ R\),
\[
(T \circ R)^{\alpha\beta}_{\rho\sigma} = T^{\alpha\beta}_{\rho\mu} R^{\mu\nu}_{\rho\sigma}
\]

3. Square and trace as endomorphism:
\[
T^2 = T \circ T , \quad \text{Tr} T = T^{\alpha\beta}_{\alpha\beta}
\]

4. Action (on trace-less symmetric tensors \(A, B\)) as an endomorphism \(T(A)\) and as a bilinear form \(T(A, B)\),
\[
T(A)_{\alpha\beta} = T_{\alpha\beta}^{\mu\nu} A_{\mu\nu} , \quad T(A, B) = T_{\alpha\beta\mu\nu} A^{\alpha\beta} B^{\mu\nu}
\]

5. Metric on the space of traceless symmetric tensors (TLS-identity): \(\Gamma\)
\[
\Gamma_{\alpha\beta\mu\nu} = \frac{1}{2} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} , \quad \Gamma(A) = A
\]

D Properties involving TLS-endomorphisms defined by SD-endomorphisms

1. For two SD-endomorphisms \(\mathcal{X}\) and \(\mathcal{Y}\), we shall define the \(\circ\)-product, \(\mathcal{X} \circ \mathcal{Y}\) as:
\[
(\mathcal{X} \circ \mathcal{Y})_{\alpha\mu\beta\nu} = \mathcal{X}_\alpha^{\mu} \circ \mathcal{Y}_{\beta\nu}^{\rho\sigma}
\]

2. If \(\mathcal{X}\) and \(\mathcal{Y}\) are SD-endomorphisms, then \(Q = \mathcal{X} \circ \mathcal{Y}\) is a TSL-endomorphism such that:
\[
Q^\mu_{\alpha\mu\beta} = \frac{1}{4} \text{Tr} Q g_{\alpha\beta} , \quad \text{Tr} Q = \text{Tr} \mathcal{X} \mathcal{Y} \mathcal{Y}
\]

3. The TLS-identity \(\Gamma\) is given in terms of the SD-identity \(\mathcal{G}\) by:
\[
\Gamma = \mathcal{G} \circ \mathcal{G}
\]
4. The TLS-identity $\Gamma$ satisfies:

$$\Gamma_{\alpha\mu\nu\beta} G^{\mu\nu}_{\rho\sigma} = -\frac{3}{4} \delta_{\alpha\beta\rho\sigma}$$

5. If $Q = X \circ \mathcal{U}$, $X$ being a traceless SD-endomorphism, then:

$$Q_{\alpha\mu\nu\beta} G^{\mu\nu}_{\rho\sigma} = 2 Q_{\alpha[\rho\sigma|\beta} = \frac{3}{2} X_{\alpha\beta\rho\sigma}$$

The last two properties follow from identities of Appendix B.

6. If $X$ is a SD-endomorphism, then

$$Q = X \circ \mathcal{U} = 0 \iff X = 0$$

7. The SD-endomorphism $X$ is traceless iff $Q = X \circ \mathcal{U}$ is a real fully symmetric traceless tensor.

This property can be easily shown by using the properties 2 above and 8 of Appendix B.

8. If $V, \mathcal{Z}, X$ and $Y$ are SD-endomorphisms, then:

$$\left( V \circ \mathcal{Z} \right) \cdot \left( X \circ \mathcal{Y} \right) = \left( V \circ X \right) \circ \left( \mathcal{Z} \circ Y \right)$$

This property can be shown by writing in a normalized basis \{U_i\} of the SD-endomorphisms involved,

$$V = \sum_{ij} V_{ij} U_i \otimes U_j$$

and computing the two parts of the expression taking into account that

$$G(U_i, U_j) = -\delta_{ij}, \quad \sqrt{2} U_i \cdot U_j = -i \epsilon_{ijk} U_k, \quad G(U_i, \mathcal{U}_j) = 0.$$ 

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