Static and dynamic properties of heavily doped quantum vortices

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Abstract
Quantum vortices in superfluids may capture matter and deposit it inside their core. By doping vortices with foreign particles one can effectively visualize them and study them experimentally. To acquire a better understanding of the interaction between quantum vortices and matter, and clarify the details of recent experiments, the properties of doped vortices are investigated here theoretically in the regimes where the doping mass becomes close to the total mass of superfluid particles forming a vortex. Such formations are dynamically stable and, possessing both vorticity and enhanced inertia, demonstrate properties that are different from the pure vortex case. The goal of this paper is to define and investigate the universal aspects of heavily doped vortex behavior, which can be realized in different types of quantum mixtures. The proposed 3D model is based on a system of coupled semiclassical matter wave equations that are solved numerically in a wide range of physical parameters. The size, geometry and binding energy of dopants in different regimes are discussed. The coupled motion of a vortex–dopant complex and decoupling conditions are studied. The reconnection of vortices, taken as an example of a fundamental process responsible for the evolution of a quantum turbulent state, is modeled to illustrate the difference between the light and heavy doping cases.

1. Introduction
The interaction of quantum vortices with impurities in superfluids is a nontrivial process giving insight into fundamental physical questions. As was discussed previously [1], the scattering of particles on vortices is an inelastic process, accompanied by an energy redistribution through the emission of Kelvin waves. If the energy of particles is low enough, they can be captured by a vortex, forming a stable complex, which is referred to as a doped vortex in this paper. Being loaded with an additional mass, which can be high, depending on the type of dopants used, the vortex demonstrates properties clearly different from the undoped case.

In dense quantum fluids, such as liquid helium, where the vortex core size is of the order of 1 Å, doping is a widely used experimental technique that makes vortices visible for detectors [2, 3]. Thus, understanding doped vortex dynamics is important for the interpretation of the experimental data on quantum turbulence. In ultracold atomic gases experiments, where multicomponent Bose–Einstein condensates (BEC) can be created and controlled to a great degree of accuracy via Feshbach resonance [4], the appearance of quantized vortices may lead to a peculiar method of matter organization, when a phase-separated fraction of one component is captured by a vortex formed in the other component [5–7]. The behavior of such composite formations represents a fundamental physical interest. Another example that motivates the investigation of doped quantum vortices is connected to a metallic nanowire production technique based on quantum turbulence [8–11]. It is shown experimentally that the ablation of metals in superfluid helium with laser pulses leads to the formation of centimeter-long wires produced from atoms trapped by quantized vortex filaments. Complex elongated nanostructures with a core and a shell made from different materials can also be produced using the quantum vortices based technique [12]. All the mentioned examples, both fundamental and applied, show the importance of an in-depth understanding of particle–vortex scattering and doped vortex dynamics for subsequent progress in this field.
Illuminating experiments in quantum turbulence were performed recently using superfluid helium nanodroplets [13–16]. Vortex filaments in rotating droplets were doped with Ag and Xe atoms and studied using a femtosecond X-ray coherent diffractive imaging technique as well as electron microscopy preceded by surface-deposition of the samples. Several questions were raised in these works, including the origin of unusual shapes of doped helium droplets and distribution of vortices inside the droplet. Although certain aspects were clarified by theorists [17–19], the connection between the rotational motion of a droplet and a dopant still remains unclear [16]. In nanodroplet experiments doping particles are approximately 33 times heavier than fluid atoms and the diameter of each particle is comparable with a vortex core size in helium. This is quite opposite, for instance, in comparison to large and light electron bubbles often used as dopants and well studied in the past both experimentally and theoretically [20–22]. Being captured, heavy particles may influence the vortex motion significantly and theoretical modeling is necessary to understand the details of their behavior.

According to the results of Gordon and colleagues [8, 9], guest atoms in helium above a critical temperature tend to form spherical clusters. In superfluid helium below critical temperature impurities with a certain probability stick together to produce long cylindrical filaments, which are attributed to the presence of quantum vortices. These filaments are stable enough to exist independently, by decoupling from vortices, which allows them to be studied using a surface-deposition technique. The described behavior also reveals the difference between atomic dopants and electron bubbles, and motivates the necessity of theoretical modeling for the better understanding of heavy vortex dynamics.

Taking the listed experiments as a motivation, the goal of this paper is to study theoretically the general case of a quantum vortex interacting with heavy matter, the case that may be realized in different kinds of multicomponent quantum mixtures. The proposed classical matter field-based 3D model (see section 2) is directly applicable to BEC systems, while certain extensions are necessary to describe liquid helium [23]. The model allows us to treat such parameters of the doping substance as mass and volume. It can be used to describe the dynamics of the interaction including scattering, coupled motion and decoupling of doping matter. The theoretical model is introduced in section 2 of the paper, where the universal dimensionless equations and corresponding Hamiltonians are presented. The space of physically relevant parameters typical for atomic dopants (both heavy and relatively light) is also defined and discussed there. An example of the optimized stationary solution representing a heavy quantum vortex with a dopant trapped and distributed homogeneously along its core is shown. In section 3 the defined space of parameters is investigated to check the existence and stability of phase-separated solutions. The cases where doping matter is trapped by vortices and self-trapped (using the terminology of Gross [24]) inside the superfluid bulk are considered. Special attention is paid to the geometry and total energy of solutions, which is used to determine the most favorable configurations. The sizes of doping matter fractions and binding energies between them, and quantum vortices are calculated and discussed. The interval of masses covering the light to heavy doping transition is scanned to check the smoothness of the transition with respect to the properties of the system. After defining a class of steady solutions of interest, the discussion continues in section 4, where the dynamics of the mixture is probed. Coupled vortex–dopant propagation in a superfluid and decoupling conditions are investigated. The reconnection of two heavy vortices, as an example of a common phenomenon in the turbulent regime [25–28], is modeled. The difference between the light and heavy vortex behavior is emphasized. Our conclusions are given in section 5.

2. Theory

To build a model of a heavy quantum vortex the classical field formalism is applied [29]. The nonlinear field equation, also known as the generalized nonlinear Schrödinger equation (gNLSE) [23], has vortical solutions (along with a family of solitary wave solutions) and presents the most complete mathematical model of quantum vortex behavior. The proposed framework is quite universal. It allows us to take into account an equation of state of a superfluid by adding a realistic internal energy functional into the Hamiltonian [23, 30]. An arbitrary form of nonlocality of particle interactions can also be treated [31]. It has been demonstrated that gNLSE based formalisms have a mathematical parallelism with the Landau two-fluid model and can be used to describe not only the condensed phase but also a cloud of thermal excitations, and thus is applicable for the modeling of finite temperatures [32, 33]. In the context of liquid helium modeling it allows us to describe both the superfluid and the normal fluid components [23, 32].

In the present work the simplest form of the equation with cubic nonlinearity and local interactions is used. The superfluid and the doping substance are modeled using a system of coupled nonlinear matter fields. In the present form the equations, being a good model for BEC mixtures [34], does not allow us to model liquid helium well enough, where only about 10% of the fluid is in a condensed state. However, they allow us to track the universal mechanical aspects of the vortex–matter interaction, taking place in idealized quantum mixtures. Similar models were used in the past to understand qualitatively the behavior of electron bubbles in liquid.
helium [22, 24]. Various extensions can be incorporated into the formalism, if necessary, to investigate the effects caused by the nonlocal interactions of particles and finite temperatures.

The Hamiltonian of a mixture of two incoherently interacting superfluids described by the complex valued classical fields $\psi$ and $\varphi$ reads

$$
H = \int \left\{ \frac{\hbar^2}{2m_1} |\nabla \psi|^2 + \frac{\hbar^2}{2m_2} |\nabla \varphi|^2 + g_{11}|\psi|^2|\varphi|^2 + \frac{g_{12}}{2} |\psi|^4 + \frac{g_{12}}{2} |\varphi|^4 - \mu_1|\psi|^2 - \mu_2|\varphi|^2 \right\} dV,
$$

(1)

where the integral is taken over the volume of the system $V$. Masses, chemical potentials and inter-particle interactions are denoted as $m_i$, $\mu_i$ and $g_{ii}$, respectively ($i = 1, 2$). The interaction between two fluids is expressed through the parameter $g_{12}$. The corresponding equations of motion read

$$
-i\hbar \dot{\psi}_i = \frac{\hbar^2}{2m_i} \nabla^2 \psi - g_{11} |\psi|^2 \psi - g_{12} |\varphi|^2 \psi + \mu_1 \psi,
$$

$$
-i\hbar \dot{\varphi}_i = \frac{\hbar^2}{2m_2} \nabla^2 \varphi - g_{12} |\psi|^2 \varphi - g_{12} |\varphi|^2 \varphi + \mu_2 \varphi.
$$

(2)

The fields $\psi$ and $\varphi$, which are associated with the fluid and the doping substance, are assumed to be normalized

$$
\int |\psi|^2 dV = N_1, \quad \int |\varphi|^2 dV = N_2,
$$

(3)

where $N_1$ and $N_2$ are the numbers of particles in the fluid and doping substance, respectively, and $N_2$ is assumed to be much smaller than $N_1$. Chemical potentials $\mu_1$ and $\mu_2$ are connected with the amounts of particles $N_1$ and $N_2$ and are chosen so that the homogenous solutions $\psi_{\infty} = \sqrt{N_1/V}$ and $\varphi_{\infty} = \sqrt{N_2/V}$ fulfill the decoupled ($g_{12} = 0$) system of equations. It is easy to show that $\mu_1 = g_{11} \psi_{\infty}^2$ and $\mu_2 = g_{22} \varphi_{\infty}^2$. Both components are assumed to be bosonic. In this work the accent is on the modeling of heavy cores and mass effects, while probable particles statistics-based phenomena are not considered. Possible models for fermionic liquids are described in the literature [35–38].

The inter-particle interaction is assumed to be repulsive, which is the case for dense superfluids. In BEC experiments the repulsive interaction corresponds to one of the possible regimes controlled via a Feshbach resonance. Interaction parameters $g_{11}$, $g_{22}$ and $g_{12}$ are connected with scattering lengths of particles as follows

$$
g_{11} = \frac{4\pi \hbar^2}{m_1}, \quad g_{22} = \frac{4\pi \hbar^2}{m_2}, \quad g_{12} = \frac{2\pi \hbar^2}{m_2}.
$$

(4)

where $m_{12} = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. The scattering length between the two species for simplicity is taken in the form $l_{12} = l_1/2 + l_2/2$. The model is thus completely defined by seven primary parameters: $m_i$, $l_i$, $N_i$ and $V$ ($i = 1, 2$).

A characteristic length scale in the system is associated with the first fluid (the amount of the second component is small). The healing length reads $\xi = \hbar / \sqrt{2m_1 \psi_{\infty}^2}$, or, being expressed through the primary parameters, $\xi = \sqrt{V/8\pi l_l N_l}$. The healing length is of the order of an angstrom for helium and it can be of the order of micrometers in BEC systems, which corresponds to an approximate size of vortex cores.

For practical computations the equations are transformed to the dimensionless form [39, 40], using the substitutions: $x \rightarrow \xi x$, $t \rightarrow (\xi^2 m_1 / \hbar) t$, $\psi \rightarrow \psi_{\infty} \psi$, $\varphi \rightarrow \varphi_{\infty} \varphi$. Introducing new notations $\delta = m_1 / m_2$, $\mu = \mu_1 / \mu_2$, $\lambda = g_{12} / g_{11}$ and $\gamma = g_{22} / g_{11}$ and rearranging the terms the following system of equations can be derived

$$
-2i\psi = \nabla^2 \psi + (1 - |\psi|^2) \psi - \lambda |\varphi|^2 \psi,
$$

$$
-2i\varphi = \delta \nabla^2 \varphi - \gamma |\varphi|^2 \varphi - \lambda |\psi|^2 \varphi + \mu \varphi,
$$

(5)

with norms given by

$$
\int |\psi|^2 d\theta = 1, \quad \int |\varphi|^2 d\theta = \frac{N_2}{N_1},
$$

(6)

where $\theta \equiv V / \xi^2$ is a dimensionless volume. It is easy to check that the new equation coefficients can be expressed through the primary parameters of the model in the following way

$$
\delta = \frac{m_1}{m_2}, \quad \gamma = \frac{m_1 l_1}{m_2 l_2}, \quad \lambda = \frac{1}{4} \left( 1 + \frac{m_1}{m_2} \right) \left( 1 + \frac{m_2}{m_1} \right) \mu = \frac{N_2 l_2}{N_1 l_1} \frac{m_2}{m_1}.
$$

(7)

In the dimensionless formulation the model contains only relative parameters, namely the number of particles $N_2/N_1$, mass $m_2/m_1$, and scattering length $l_2/l_1$. Together with the dimensionless volume $\theta$ there are four model defining parameters, and three other parameters are concealed in the units. Such a formulation of the model is more universal, since it does not specify the type of the host superfluid, but only the relative parameters of the dopant. Introducing a dimensionless unit of energy $\tilde{E}_0 = \frac{\hbar^2}{2m_1} \psi_{\infty}^2$, the Hamiltonian of the system can be written
in the form
\[ H = \int \left\{ \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} |\psi|^4 + \frac{1}{2} \gamma |\psi|^4 + \lambda |\psi|^2 |\varphi|^2 - |\psi|^2 - \mu |\varphi|^2 \right\} \, d\theta. \] (8)

In the computations the relative amount of particles \( N_2/N_1 = 0.01 \) is taken to study how a small amount of doping matter influences the behavior of vortices. Since dopants are considered to be larger and heavier than the host fluid particles it is assumed that \( m_2 > m_1 \) and \( l_2 > l_1 \). A range of relative parameters is scanned to investigate different types of phase-separated solutions of equation (5). The relative masses \( m_2/m_1 \) are varied from 5 to 45, which covers a reasonable part of the periodical table of elements. Scattering lengths of atoms are usually of the order of few angstroms, and, assuming \( l_2/l_1 \) to vary from 1 to 10, one covers a large class of physically relevant cases. The dimensionless volume is taken as \( \vartheta = 36.5^3 \), which is large enough to accommodate and comfortably study a doped vortex. All the computations are performed using the dimensionless equations (equation (5)). To go back to the physical domain one has to set explicitly the mass of particles \( m_1 \), the scattering length \( l_1 \) and the amount of particles \( N_1 \), which defines explicitly the units and the type of the superfluid.

In the regimes considered in this paper two components of the quantum mixture remain phase-separated, so the usual criterion \( g_{12} \vartheta < g_{11}^2 \) is always fulfilled \[41\]. This expression can also be applied to evaluate a ‘strength’ of the separation. Using the interaction parameters defined by equation (4) the following criterion is derived
\[ S = \left( 1 + \frac{l_2}{l_1} \right)^3 \left( 1 + \frac{m_2}{m_1} \right)^2 - 16 \frac{m_2 l_2}{m_1 l_1} > 0, \] (9)
which contains only the relative parameters. It is easy to check that the values of \( S \) are always positive for the masses and scattering lengths discussed above. Smaller values of \( S \) correspond to the weaker phase separation and stronger solubility of the doping matter.

The system of equations (5) is solved numerically using the fourth order finite difference space discretization scheme and the fourth order Runge–Kutta method for the time propagation. The initial guess for a vortex is given by the expression
\[ \psi(x, y, z) = \frac{x + iz}{\sqrt{x^2 + z^2 + \xi^2}} \varphi_\infty. \] (10)

Doping matter \( \varphi(x, y, z) \) inside the vortex core is assumed to have the form of a filament, which repeats the cylindrical symmetry of the vortex. It is also assumed that dopants can keep the cylindrical form after decoupling (see section 3). Doping matter inside the vortex or in the superfluid bulk is initially assumed to have a gauss-like radial density distribution. Details on the initial state preparation and boundary conditions can be found in the literature \[1, 22\]. The imaginary time propagation technique \[42\] is used to optimize the initial states and obtain stationary characteristics before the dynamical computations start. To control the convergence, the total energy (equation (8)) is monitored and propagation proceeds until the steady state is reached.

The examples of optimized steady solutions of equation (5) are presented in figure 1. Absolute values of classical fields \( |\psi| \) and \( |\varphi| \) (which are interpreted as square roots from particles density) are plotted along the x-axis, while \( y = z = 0 \). Figures 1(a) and (b) illustrate cylindrically self-trapped filament-like doping matter placed in the bulk of the superfluid. It is located in the center of \((x, z)\)-plane and aligned along the \(y\)-axis (the picture is cylindrically symmetric). The gauss-like distribution of the dopant density and corresponding minimum in the fluid density are shown. The different cases in figures 1(a) and (b) correspond to different relative masses and scattering lengths of doping particles. Values of the parameter \( S \) (equation (9)), showing the degree of phase separation in corresponding regimes, are given in the caption. When the doping atoms mass approaches the mass of superfluid particles, the self-trapping regime decays and the phase separation becomes weak, which corresponds to small values of \( S \). A similar effect appears when the relative scattering length becomes small. This topic is discussed in more detail in the next section.

Figure 1(c) demonstrates the doping substance and the fluid density distribution inside the quantum vortex core. The symmetry and orientation of this solution are the same as in the previous example. Doped and undoped vortex density profiles are plotted for the comparison. One can see that the curvature of the density slightly changes for a doped vortex, but this difference is observable only in the vicinity of a vortex core, at distances smaller than the four healing lengths \( \xi \). Equation (10), used as an initial guess for a vortex is plotted for comparison. It overestimates the density in a vortex core slightly, but provides a good approximation at larger distances. Since no coherent (Rabi) coupling is considered in the present model, the doping substance can ‘see’ only the vortex density profile (through the term \(-\lambda |\psi|^2 \varphi \) in equation (5)), but not the phase of \( \psi \), and the vortex works as a potential well, which accommodates the dopant. It provides additional stabilization, so that even in the regimes where no phase separation exist, vortices may isolate the second component of the mixture inside their cores.
3. Stationary results

In the experiments of Gordon and colleagues [8, 9], the existence of long cylindrical metallic filaments in a superfluid is attributed to the presence of quantum vortices, while spherical metallic clusters are usually found in the bulk of a vortex free liquid. In BEC mixtures spherical phase separation is energetically preferable in the absence of vortices. Being captured, doping matter forms cylindrical filaments in the bulk of a vortex free liquid. In BEC mixtures spherical phase separation is energetically preferable, and full energies of corresponding states are compared. The space of parameters addressed in this section is scanned to determine the characteristics of phase-separated regimes. Such regimes are addressed in this section and full energies of corresponding states are compared. The space of parameters addressed in this section is scanned to determine the characteristics of phase-separated regimes.

A localization radius is an important parameter characterizing self-trapped spherical and cylindrical solutions. In the case of single particle it is determined as a result of interplay between few competing terms in the total energy function [44, 45]. For instance, small electrons, being placed into liquid helium, form large bubbles with radiiues of about 16 Å. Thus, the question of the size of doping particles in superfluids is not always a priori clear. Particles size is also closely connected with the binding energy to vortices, since the latter is partially defined by the amount of rotating superfluid substituted from the vortex core by a captured impurity [1]. On the other hand, in the framework of the electron bubble model (both in the bulk and on vortices) [1, 22], while less is known about cylindrical types. The question of the stability of spherical and cylindrical self-trapped solutions is addressed in this section and full energies of corresponding states are compared. The space of parameters defined in the previous section is scanned to determine the characteristics of phase-separated regimes. Such parameters as the size and binding energy, which are important for describing the matter–vortex interaction, are discussed.

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The regime where \( l_f / \hbar = 2 \) and \( m_2 / m_1 = 15, 45 \) (red, green, blue) are shown. Parameter \( S \) takes values 164, 1824 and 17604, respectively. In (b) \( m_2 / m_1 = 35 \) is fixed and \( l_f / \hbar = 1, 4, 8 \) (red, green and blue, respectively). In these regimes \( S = 4624, 30160 \) and 100496, respectively. The comparison between doped and undoped vortex density profiles is shown in (c) (blue and red, respectively). The approximate equation (10) used as a vortex initial condition in the simulation is shown by the green line.

Figure 1. A visualization of self-trapped (a), (b) and vortex-trapped (c) filament-like steady solutions of equation (5). Absolute values of classical fields \( |\psi(x, 0, 0)| \) (solid lines) and \( |\psi(x, 0, 0)|/3 \) (dashed lines) related to the fluid and dopant density distribution are plotted along the x-axis. (a) The regimes where \( l_f / \hbar = 2 \) and \( m_2 / m_1 = 5, 15, 45 \) (red, green, blue) are shown. Parameter \( S \) takes values 164, 1824 and 17604, respectively. In (b) \( m_2 / m_1 = 35 \) is fixed and \( l_f / \hbar = 1, 4, 8 \) (red, green and blue, respectively). In these regimes \( S = 4624, 30160 \) and 100496, respectively. The comparison between doped and undoped vortex density profiles is shown in (c) (blue and red, respectively). The approximate equation (10) used as a vortex initial condition in the simulation is shown by the green line.
delocalized states is plotted there. This can be easily derived from equation (8) by substituting there solutions $\psi_{\infty}$ and $\varphi_{\infty}$:

$$E_{\text{deloc}} = -\frac{g}{2} \left( 1 + \frac{l_2}{l_1} \frac{m_1}{m_2} \left( \frac{N_2}{N_1^2} - \frac{1}{2} \frac{N_2}{N_1} \right) - \frac{1}{2} \frac{N_2}{N_1} \left( 1 + \frac{l_2}{l_1} + \frac{m_1}{m_2} \right) \right).$$  \hspace{1cm} (11)

In the regimes with weak phase separation the total energy of the considered steady states is close to the value given by equation (11). At $m_2 = 5m_1$ all three branches of solutions become hardly distinguishable (figure 2(e)). The energy difference between the ground state and the fully delocalized state can be used along with equation (9) to evaluate the degree of phase separation in the system.

It is shown in figure 2(b) how the relative scattering length $l_2 / l_1$ influences the radius. Well localized solutions exist in the whole interval of physically relevant parameters. Large values of $l_2 / l_1$ correspond to the increased doping particles mutual repulsion and, thus, the larger radius. Corresponding energies are shown in figure 2(f). At small values of $l_2 / l_1$ both spherical and cylindrical solutions approach the energy of the fully delocalized state and the self-trapping decays. In contrast to the previous case (figure 2(a)), the dopant radius does not grow along with the increasing mutual solubility (for small values of $l_2 / l_1$), but decreases slightly (figure 2(b)). It is also worth mentioning that, in the electron bubble model, both the scattering length and mass are very small, but the solution is well localized. One important difference in our case is the existence of the repulsive term in equation (7) proportional to $\gamma$. Assuming it to be zero and using simultaneously physically small values $m_2$ and $l_2$ for an electron, a bubble model with a large localization radius can be obtained. In this paper the atomic doping is considered and the parameters cannot be small.

The particle–vortex binding energy is one of the most important parameters characterizing the interaction. A vortex oriented along the $\gamma$-axis is added in the center of the computational box to study binding. For the box size used, the vortex energy is $E_v = 765.1 \varepsilon_0$. The computations are made for two different conformations: with the dopant distributed along the vortex core and the dopant placed somewhere far from the vortex, then the full energies are subtracted to get the binding $\Delta E_t = E_{t}^{\text{marg}} - E_{t}^{\text{mf}}$. The energy $E_{t}^{\text{marg}}$ of spherically arranged doping matter placed outside the vortex is given for comparison. This procedure is applied in different regimes, the results are presented in figures 2(c) and (d); corresponding energies are plotted in (g) and (h).

The solid red and green curves in figures 2(g) and (h) are very similar to the curves plotted in (c) and (f). The difference is that they are shifted by a value of the vortex energy $E_v$. This proves the fact that the numerical computational box is large enough to accommodate both the impurity and the vortex far enough to exclude a significant interaction. Dashed curves correspond to the dopant placed inside the vortex core. It is clear from the energy diagrams (g) and (h) that the filaments captured by vortices are the most energetically favorable configurations.

While both spherical and cylindrical solutions exist and are stable in the bulk of a superfluid, the situation is different inside vortex cores. It is found that balls, being captured by vortices, slowly deform themselves into

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**Figure 2.** (a), (b) Radii of spherical (red) and cylindrical (green) phase-separated solutions of equation (5). (c), (d) Binding energies of the doping matter with vortices in different regimes. When the range of relative masses $m_2/m_1$ is considered, the relative scattering length $l_2/l_1 = 2$ is fixed. Accordingly, when the range of scattering lengths is scanned, the mass ratio $m_2/m_1 = 35$ is kept constant. All the dependencies are accompanied by the total energy plots for corresponding parameters (e), (f), (g) and (h). Fully delocalized states energies are plotted with blue dashed lines in (e) and (f). In (g) and (h) the total energy is plotted for two cases: heavy doping matter is captured by the vortex (dashed green lines) and the doping filament is placed far from the vortex (solid green lines). The energy of spherically organized dopant placed outside the vortex is shown for comparison (solid red lines).
filaments. This effect is less obvious for fermionic impurity models, since they do not contain the inter-particle nonlinear repulsion term in the equations [1, 22]. This term is proportional to \( \gamma \) and becomes large for small \( m_2 \). According to Gordon and Okuda [8] the rate of guest particle clustering is higher inside the vortex core than in the bulk. In other words, particles move more freely there and the vortex core works as a ‘potential pipe’ in a superfluid.

As it can be seen in figures 2(c) and (d), according to the expectations, there is correlation between the radius and the binding energy, which grows with \( l_2 / l_1 \) and decreases with \( m_2 / m_1 \). The difference appears at lesser masses, where a small maximum for filaments coupling \( \Delta E_{\ell} \) is observed, despite the fact that both \( E_{\text{eff}} \) and \( E_{\text{m}} \) decrease monotonically while \( m_2 / m_1 \) grows. This could be related to the different terms interplay [1] in the Hamiltonian (equation (8)).

Among all the configurations, the filament-like distribution of the doping matter inside the vortex core possesses the largest binding energy since it substitutes most high speed core volume of a rotating superfluid [22]. This binding obviously depends on the vortex length and the number of doping particles. Nevertheless, decoupling of filaments does not necessarily involve the whole length of the core. It is shown in section 4 how partial filament decoupling takes place during the vortex pair reconnection.

To summarize the section it is necessary to stress that the transition from light to heavy atomic dopants (in the regimes with a good phase separation) appears as quite smooth, with no harsh features in the main parameters of the system (figure 2). This allows us to concentrate the study on the influence of mass on the dynamics and comparison between the light and heavy vortices behavior.

4. Dynamics

It was shown experimentally by Zmeev et al [46] that vortex tangles can carry molecules through the superfluid. If the molecules are heavy enough one can expect a certain back influence on a vortex dynamics. In the regimes considered here, when \( N_2 / N_1 = 0.01 \) and \( m_2 / m_1 = 40 \), quantities \( m_1 N_1 \) and \( m_2 N_2 \) are of the same order of magnitude, ie the mass of the dopant is of the same order as the total mass of particles forming a vortex. The dynamical behavior of such heavy composite formations is analyzed in this section.

The computation starts from the initial state, considered in the previous section, where a doping filament trapped by the vortex is located in the center of the computational box (a 2D cross section is depicted in figure 3). Then equation (3) is solved in the moving reference frame. The transformation is realized by applying the following modification: \( \psi_1 \to \psi - (\mathbf{v}_m \cdot \nabla)\psi \), where \( \mathbf{v}_m \) is the speed of a uniform fluid flow, measured in units of \( v_0 = h / \sqrt{\omega m_1} \). To avoid high speed hydrodynamic phenomena such as a vortex rings generation [47] the speed value is taken well below criticality such that \( v_{x\text{on}} = v_{y\text{on}} = 0 \) and \( v_{z\text{on}} = 0.05 \). Such a uniform superfluid flow, obviously, provides no viscous drag on the phase-separated doping matter, but it moves the vortex, which drags the bonded dopant.

The resulting trajectories of doped vortices with different mass parameters are presented in figure 3(a) (to exclude the influence of boundaries the calculations were repeated in a larger box). The higher the doping mass, the larger the deviation from the straight line propagation. The physical nature of the force, pushing the vortex in the direction perpendicular to the propagation direction, can be understood using the Bernoulli principle.
\[ \frac{\rho v^2}{2} + p = \text{constant} \]  \hspace{1cm} (12)

where \( p, \rho \) and \( v \) are the pressure, density and velocity of a fluid at a certain point in space. As a result of summation of the vortex velocity field and the uniform flow field, the absolute values of fluid velocities at different sides of the dopant become different, which causes the pressure difference and the appearance of the corresponding force. The direction of the force can be inverted by changing the sign of \( \mathbf{v}_{un} \), or the winding number of the vortex. The latter case is demonstrated in figure 3(a) by red dashed line. This force is similar to the classical Magnus force, which acts on a rotating cylinder moving in a liquid or gas. The difference with respect to the classical case is that the doping itself does not rotate at all, but the rotational motion of the surrounding superfluid is caused by the quantum vortex attached to the body.

The Magnus force disappears when the doping filament moves together with the flow and their relative velocity is zero. The pressure in this case is the same everywhere around the filament. If the dopant mass is small it can be quickly accelerated by the flow, but it takes more time to accelerate a heavy filament. In the second case the Magnus force acts longer and causes larger deviations. At the same time one can say that it is easier for a heavy doping filament to drag the vortex aside. In the regimes considered in this paper vortex pinning (when the dopant prevents vortices from moving) was not observed.

To evaluate the force mathematically, one may start from the general expression for the Bernoulli force and take into account the conservation law (equation (12)) to write

\[ F = \int_S \rho(r) \mathbf{n}(r) \, ds = \int_S \frac{\rho v^2}{2} \mathbf{n}(r) \, ds, \]  \hspace{1cm} (13)

where the integral is taken over the surface of a body, embedded into the fluid, and \( \mathbf{n}(r) \) is a unit vector normal to the surface \( S \). The body is assumed to be a cylinder with a radius \( R \) and length \( L \). Using the symmetry and introducing the polar coordinates one may write

\[ F = \frac{L}{2} \int_0^{2\pi} \rho(R, \alpha) v^2(R, \alpha) \left( \frac{\cos \alpha}{\sin \alpha} \right) R \, d\alpha \]  \hspace{1cm} (14)

(there is no force component acting along the \( y \)-axis and we consider only \( x \) and \( z \) components). Assuming the classical field in the form \( \psi = |\psi| e^{i\varphi} \), the fluid density and velocity read:

\[ \rho = |\psi|^2 m, v = \frac{\hbar}{m} \nabla \varphi. \]

In equation (14), \( \varphi \) is an absolute value of a sum of the vortex velocity and the uniform flow velocity fields, i.e. \( \varphi = |\mathbf{v}_f + \mathbf{v}_i| \). To compute \( \mathbf{v}_f \) one may use an approximate expression for a vortex order parameter (equation (10)), and neglect the disturbance caused by the presence of the dopant (figure 1). In this case

\[ |\psi|^2 = \frac{\psi^2}{R^2 + \xi^2}, \]  \hspace{1cm} (15)

\[ \nabla \varphi = \frac{1}{R^2} \left( -\frac{z}{x} + \frac{x}{x} \right). \]  \hspace{1cm} (16)

Assuming \( \mathbf{v}_i \) oriented in the negative direction of \( z \)-axis, one gets

\[ v^2 = v^2_x + v^2_y + 2v_y \cdot v_x = \frac{\hbar^2}{m_i^2 R^2} + v^2_x - \frac{2\hbar v_x}{m_i R^2} \]  \hspace{1cm} (17)

and

\[ \frac{\rho v^2}{2} = \frac{\psi^2}{R^2 + \xi^2} \left[ \frac{\hbar^2}{2m_i} + \frac{m_i v^2}{2} - \frac{\hbar v_x}{R^2} \right]. \]  \hspace{1cm} (18)

In the last equation the first two terms in the brackets do not depend on \( \alpha \) and on substitution into the integral in equation (14) vanish. The expression for the force reads

\[ F = -LR^2 \frac{\psi^2}{R^2 + \xi^2} \hbar v_f \int_0^{2\pi} \cos \alpha \left( \frac{\cos \alpha}{\sin \alpha} \right) d\alpha, \]  \hspace{1cm} (19)

where only the \( x \)-component survives the integration

\[ F_x = -L \hbar v_f \frac{\psi^2}{R^2 + \xi^2}. \]  \hspace{1cm} (20)

Having in mind the classical situation of an ideal liquid flowing around a cylindrical obstacle it is assumed that \( v_f \approx 2v_{un} v_p \).

The dynamics of doped vortices in the presented examples is defined by an interplay of two forces: the vortex drag acting on doping filaments and the Magnus force. Such coupled vortex–dopant dynamics has its limits, defined by the binding energy. The decoupling happens when the kinetic energy of the relative movement exceeds the value \( \Delta E_t \) (see figures 3(c) and (d)). Mathematically it could be formulated as
or, expressing everything through the primary parameters

\[ N_1 m_2 v_{\text{dec}}^2 / 2 = \Delta E_I \epsilon_0 / v_0^2, \]

or

\[ v_{\text{dec}} = \sqrt{\frac{\Delta E_I m_1 N_1}{\vartheta m_2 N_2}}. \]  

Using the value of \( \Delta E_I \) presented in figure 2 for heavy dopants with \( m_2 / m_1 = 40 \) one gets \( v_{\text{dec}}^{40} = 0.1 \). This value is larger for relatively light dopants when, for instance, \( m_2 / m_1 = 10 \) and \( v_{\text{dec}}^{40} = 0.2 \). It is easier to ‘shake off’ heavy dopants, since the decoupling happens at lower velocities.

The decoupling process for \( m_2 / m_1 = 40 \) and \( v_{\text{un}} = 0.15 \) is illustrated in figures 3(b) and (c), where 2D cross sections for \( \psi \) and \( \varphi \) absolute values are shown. The vortex and the doping filament completely separate at \( t = 60 \). The moving directions after the separation are shown by the arrows. According to equation (21), the decoupling appears at higher speeds for smaller masses, which is proved by the simulations (not shown). In figure 3(c) the decoupled doping density is slightly stretched along the direction of movement. For light regimes a small density split effect is observed, when a certain portion of the dopant remains attached to the vortex. This effect is similar to the one mentioned in Pshenichnyuk and Berloff [1].

It is known that in a turbulent regime vortex tangles evolve through multiple reconnection events. This is an example of a fundamental process that can be used to illustrate the role of the mass of doping filaments. The reconnection goes differently for light and heavy vortices (see figures 4(b) and (c)). At the early stages of the evolution two perpendicular vortices, separated by a distance 10\( \xi \), start to bend and behave in a similar way to the well-studied undoped case (see figure 4(a)). The process for the light vortex then goes slightly faster: the core interaction starts at \( t = 70 \), while in the heavy case it happens at \( t = 120 \). Comparing the results of reconnection (figures 4(b) and (c)) one may see that in the first case doping filaments inside the cores (shown by dashed gray lines) reconnect following the motion of host vortices. In the second case the rigid frame, produced by heavy doping filaments, remains almost unchanged, while vortices reconnect and a qualitatively different picture is observed. The disconnected matter then continues to evolve separately (figure 4(d)). Thus, in the heavy doping

\[ \text{Figure 4. The reconnection of two orthogonal quantum vortices initially separated by a distance } 10\xi. \text{ The superfluid density isosurface } |\psi(x, y, z)| = \text{constant is plotted. (a) shows an early stage of the evolution for undoped vortices, which looks similar for doped cases. (b) shows the result of reconnection of light vortices. (c) and (d) demonstrate the heavy vortices reconnection result at two different moments of time. Thin dashed lines mark the layout of doping filaments.} \]
case partial decoupling may take place, when only a certain fraction of the doping filament is released from the vortex core. Similar processes are expected to be common, for example, during the laser ablation of heavy metals in superfluids. The presented modeling of heavy vortices could be potentially stimulating for future BEC experiments where the obtained results can be verified under well controlled conditions.

5. Conclusion

In this paper the general questions of a doped quantum vortex properties and behavior are addressed. A 3D mathematical model based on a system of nonlinear matter field equations is used to describe the interaction of quantum vortices with doping matter. Phase-separated solutions (where the doping substance is separated from the rest of the fluid) of different geometries are investigated. The stability of solutions and possible vortex-induced geometry transformations are discussed. Emphasis is placed on the influence of dopant mass on the vortex behavior. The sizes of the doping matter fractions as well as their binding energy to vortices in a wide range of physically relevant parameters are investigated.

It is shown that the motion of doped vortices is influenced by the Magnus force, which acts on the doping particles and drags the vortex in the direction perpendicular to the direction of the flow. The role of the dopant mass in this effect is discussed and the analytical expression for the force is derived. The computed vortex–dopant binding energies are used to formulate the decoupling criteria and simulate the corresponding process. The reconnection process is simulated to demonstrate the difference between the light and heavy vortex behavior. In the first case the dopant readily follows the vortices during the reconnection. In the second case vortices partially decouple from the heavy dopant frame, which keeps its original topology during the simulation time. To conclude the obtained results it should be stressed that heavy dopants cannot be considered as just a passive visualization tool in superfluid experiments, since they can influence significantly the vortex behavior.

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