On line covers of finite projective and polar spaces

Antonio Cossidente¹ · Francesco Pavese²

Received: 4 July 2018 / Revised: 18 November 2018 / Accepted: 17 December 2018 / Published online: 3 January 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract

An $m$-cover of lines of a finite projective space $\text{PG}(r, q)$ (of a finite polar space $\mathcal{P}$) is a set of lines $\mathcal{L}$ of $\text{PG}(r, q)$ (of $\mathcal{P}$) such that every point of $\text{PG}(r, q)$ (of $\mathcal{P}$) contains $m$ lines of $\mathcal{L}$, for some $m$. Embed $\text{PG}(r, q)$ in $\text{PG}(r, q^2)$. Let $\hat{\mathcal{L}}$ denote the set of points of $\text{PG}(r, q^2)$ lying on the extended lines of $\mathcal{L}$. An $m$-cover $\mathcal{L}$ of $\text{PG}(r, q)$ is an $(r-2)$-dual $m$-cover if there are two possibilities for the number of lines of $\mathcal{L}$ contained in an $(r-2)$-space of $\text{PG}(r, q)$. Basing on this notion, we characterize $m$-covers $\mathcal{L}$ of $\text{PG}(r, q)$ such that $\hat{\mathcal{L}}$ is a two-character set of $\text{PG}(r, q^2)$. In particular, we show that if $\mathcal{L}$ is invariant under a Singer cyclic group of $\text{PG}(r, q)$ then it is an $(r-2)$-dual $m$-cover. Assuming that the lines of $\mathcal{L}$ are lines of a symplectic polar space $\mathcal{W}(r, q)$ (of an orthogonal polar space $\mathcal{Q}(r, q)$ of parabolic type), similarly to the projective case we introduce the notion of an $(r-2)$-dual $m$-cover of symplectic type (of parabolic type). We prove that an $m$-cover $\mathcal{L}$ of $\mathcal{W}(r, q)$ (of $\mathcal{Q}(r, q)$) has this dual property if and only if $\hat{\mathcal{L}}$ is a tight set of an Hermitian variety $\mathcal{H}(r, q^2)$ or of $\mathcal{W}(r, q^2)$ (of $\mathcal{H}(r, q^2)$ or of $\mathcal{Q}(r, q^2)$). We also provide some interesting examples of $(4n-3)$-dual $m$-covers of symplectic type of $\mathcal{W}(4n-1, q)$.

Keywords  Finite projective space · Finite polar space · $m$-cover · Two-character set · Tight set

Mathematics Subject Classification  Primary 51E12; Secondary 51E20 · 51A50

Communicated by G. Lunardon.

Francesco Pavese
francesco.pavese@poliba.it

Antonio Cossidente
antonio.cossidente@unibas.it

¹ Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Contrada Macchia Romana, 85100 Potenza, Italy

² Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy
1 Introduction

In this paper we deal with finite projective and polar spaces with special emphasis on finite Hermitian polar spaces. Polar spaces arise from a finite vector space equipped with a non-degenerate reflexive sesquilinear form. A maximal totally isotropic/totally singular subspace (generator) of a polar space is a subspace of the underlying vector space of maximal Witt index.

We will use the term $n$-space to denote an $n$-dimensional projective subspace of the ambient projective space. Also in the sequel we will use the following notation $\theta_{n,q} := \left[\begin{array}{c} n+1 \\ 1 \end{array}\right]_q = q^n + \cdots + q + 1$ and $\theta_{-1,q} = 0$.

The notion of tight set of a finite generalized quadrangle has been introduced by Payne [24] and later on was generalized to finite polar spaces in [13] and in [1]. If $\mathcal{T}$ is a set of points of a finite polar space, then $\mathcal{T}$ is said to be tight if the average number of points of $\mathcal{T}$ collinear with a given point attains a maximum possible value. The size of a tight set $\mathcal{T}$ of a finite polar space $\mathcal{P}$ in $\text{PG}(r,q)$ is $i\theta_{n,q}$, where a generator of $\mathcal{P}$ is an $n$-space, and $\mathcal{T}$ is said to be $i$-tight. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $i$-tight and $j$-tight sets of points of $\mathcal{P}$, respectively. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2 \setminus \mathcal{T}_1$ is a $(j-i)$-tight set. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint, then $\mathcal{T}_1 \cup \mathcal{T}_2$ is an $(i+j)$-tight set. An $i$-tight set is reducible if it contains a smaller $i'$-tight set for some integer $i' < i$. A tight set is irreducible if it is not reducible.

A subset of points $\mathcal{O}$ of a finite polar space $\mathcal{P}$ is an $m$-ovoid if each generator of $\mathcal{P}$ meets $\mathcal{O}$ in $m$ points.

A two-character set $\mathcal{X}$ of $\text{PG}(n,q)$ is a subset of points with the property that the intersection number with any hyperplane only takes two values.

An $m$-cover of lines of a finite projective space $\text{PG}(r,q)$ (of a finite polar space $\mathcal{P}$) is a set of lines $\mathcal{L}$ of $\text{PG}(r,q)$ (of $\mathcal{P}$) such that every point of $\text{PG}(r,q)$ (of $\mathcal{P}$) contains $m$ lines of $\mathcal{L}$, for some $m$. Embed $\text{PG}(r,q)$ in $\text{PG}(r,q^2)$. Let $\tilde{\mathcal{L}}$ denote the set of points of $\text{PG}(r,q^2)$ lying on the extended lines of $\mathcal{L}$.

In Sect. 2 we introduce the following notion. An $m$-cover $\mathcal{L}$ of $\text{PG}(r,q)$ is an $(r-2)$-dual $m$-cover if there are two possibilities for the number of lines of $\mathcal{L}$ contained in an $(r-2)$-space of $\text{PG}(r,q)$. Basing on this notion, we characterize $m$-covers $\mathcal{L}$ of $\text{PG}(r,q)$ such that $\tilde{\mathcal{L}}$ is a two-character set of $\text{PG}(r,q^2)$. In particular, we show that if $\mathcal{L}$ is invariant under a Singer cyclic group of $\text{PG}(r,q)$, then it is an $(r-2)$-dual $m$-cover.

In Sects. 3 and 4 we assume that the lines of $\mathcal{L}$ are lines of a symplectic polar space $\mathcal{W}(r,q)$ (of an orthogonal polar space $\mathcal{Q}(r,q)$ of parabolic type), and similarly to the projective case we introduce the notion of an $(r-2)$-dual $m$-cover of symplectic type (of parabolic type). We prove that an $m$-cover $\mathcal{L}$ of $\mathcal{W}(r,q)$ (of $\mathcal{Q}(r,q)$) has this dual property if and only if $\tilde{\mathcal{L}}$ is a tight set of an Hermitian variety $\mathcal{H}(r,q^2)$ or of $\mathcal{W}(r,q^2)$ (of $\mathcal{H}(r,q^2)$ or of $\mathcal{Q}(r,q^2)$). We also provide some interesting examples of $(4n-3)$-dual $m$-covers of symplectic type of $\mathcal{W}(4n-1,q)$. At the end of the paper examples of transitive $(q+1)$-tight sets of $\mathcal{H}(n,q^2)$, $(n,q) \in \{(4,2), (4,3),(6,2)\}$ that are neither parabolic quadrics or symplectic spaces nor the disjoint union of generators, are provided.

2 $(r-2)$-dual covers of PG$(r,q)$

An $m$-cover of lines of a projective space $\text{PG}(r,q)$ is a set of lines $\mathcal{L}$ of $\text{PG}(r,q)$ such that every point of $\text{PG}(r,q)$ contains $m$ lines of $\mathcal{L}$, where $0 < m < \theta_{r-1,q}$. We have the following results.

 Springer
Lemma 2.1 Let $\mathcal{L}$ be an $m$-cover of $\text{PG}(r, q)$, then

(i) $\mathcal{L}$ contains $m\theta_{r,q}/(q + 1)$ lines,

(ii) every hyperplane of $\text{PG}(r, q)$ contains $m\theta_{r-2,q}/(q + 1)$ lines of $\mathcal{L}$,

(iii) let $\Sigma$ be an $(r - 2)$-space of $\text{PG}(r, q)$, if $x$ is the number of lines of $\mathcal{L}$ contained in $\Sigma$ and $y$ is the number of lines of $\mathcal{L}$ meeting $\Sigma$ in one point, then $x(q + 1) + y = m\theta_{r-2,q}$.

Proof (i) A standard double counting argument on pairs $(P, \ell)$, where $P$ is a point of $\text{PG}(r, q)$ and $\ell$ is a line of $\mathcal{L}$ such that $P \in \ell$, gives $|\mathcal{L}| = m\theta_{r,q}/(q + 1)$. Hence if $r$ is even, then $q + 1$ divides $m$.

(ii) Let denote with $H_i$, $1 \leq i \leq \theta_{r,q}$, the hyperplanes of $\text{PG}(r, q)$ and let $x_i$ be the number of lines of $\mathcal{L}$ contained in $H_i$. Hence the number of lines of $\mathcal{L}$ meeting $H_i$ in one point equals $m\theta_{r,q}/(q + 1) - x_i$. Double counting the pairs $(P, \ell)$, where $P \in H_i, \ell \in \mathcal{L}$ and $P \in \ell$, we obtain

$$m\theta_{r-1,q} = x_i(q + 1) + \frac{m\theta_{r,q}}{q + 1} - x_i.$$ 

It follows that

$$x_i = \frac{m\theta_{r-2,q}}{q + 1}, \quad 1 \leq i \leq \theta_{r,q}.$$ 

(iii) Let $\Sigma$ be an $(r - 2)$-space of $\text{PG}(r, q)$. Let $x$ be the number of lines of $\mathcal{L}$ contained in $\Sigma$ and let $y$ be the number of lines of $\mathcal{L}$ meeting $\Sigma$ in one point. A line meeting $\Sigma$ in a point lies in some hyperplane containing $\Sigma$. Since there are $q + 1$ hyperplanes of $\text{PG}(r, q)$ through $\Sigma$, it follows that

$$y = \left(\frac{m\theta_{r-2,q}}{q + 1} - x\right)(q + 1).$$

An $n$-spread $\mathcal{F}$ of $\text{PG}(r, \mathbb{F})$ is a set of $n$-spaces of $\text{PG}(r, \mathbb{F})$ such that every point of $\text{PG}(r, \mathbb{F})$ is contained in exactly one element of $\mathcal{F}$. In [3] the authors investigate dual $(r-1)/2$-spreads $\mathcal{F}$ of $\text{PG}(r, \mathbb{F})$, $r$ odd, namely spreads with the property that any hyperplane of $\text{PG}(r, \mathbb{F})$ contains exactly one member of $\mathcal{F}$. In particular they show that, if $\mathbb{F}$ is finite, then $S$ is always a dual spread (Theorem 1). Generalizing [3], we say that an $m$-cover $\mathcal{L}$ of $\text{PG}(r, q)$ is dual if every hyperplane of $\text{PG}(r, q)$ contains a constant number of lines of $\mathcal{L}$. From Lemma 2.1(ii), it follows that any $m$-cover is dual. We introduce the following definition.

Definition 2.2 An $m$-cover of lines $\mathcal{L}$ of $\text{PG}(r, q)$ is an $(r - 2)$-dual $m$-cover of $\text{PG}(r, q)$, if every $(r - 2)$-space of $\text{PG}(r, q)$ contains either $\frac{m\theta_{r-4,q}}{q + 1} + q^{r-3}$ lines of $\mathcal{L}$ or $\frac{m\theta_{r-4,q}}{q + 1}$ lines of $\mathcal{L}$.

Let $\mathcal{L}$ be an $m$-cover of $\text{PG}(r, q)$. Embed $\text{PG}(r, q)$ in $\text{PG}(r, q^2)$. For every line $\ell$ of $\text{PG}(r, q)$ we denote by $\tilde{\ell}$ the extended line of $\text{PG}(r, q^2)$, i.e., the unique line of $\text{PG}(r, q^2)$ meeting $\ell$ in $q + 1$ points. Let $\tilde{\mathcal{L}}$ be the set of points of $\text{PG}(r, q^2)$ lying on the extended lines of $\mathcal{L}$. Note that $\text{PG}(r, q)$ is contained in $\tilde{\mathcal{L}}$. The following result gives a characterization of the $m$-covers of lines $\mathcal{L}$ of $\text{PG}(r, q)$ having the property that $\tilde{\mathcal{L}}$ is a two-character set of $\text{PG}(r, q^2)$.

Theorem 2.3 Let $\mathcal{L}$ be an $m$-cover of lines of $\text{PG}(r, q)$ and let $\tilde{\mathcal{L}}$ be the set of points of $\text{PG}(r, q^2)$ lying on the extended lines of $\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is a two-character set of $\text{PG}(r, q^2)$ if and only if $\mathcal{L}$ is an $(r - 2)$-dual $m$-cover of $\text{PG}(r, q)$.
Comparing (2.5) with (2.2), we obtain
\[ |\Pi \cap \tilde{\mathcal{L}}| = (q^2 - q) \frac{m\theta_{r-2,q}}{q + 1} + \theta_{r-1,q}. \]

If the latter case occurs, then let \( \Sigma = \Pi \cap \text{PG}(r, q) \). Two possibilities for \( \Sigma \) occur. Either \( \Sigma \) contains \( \frac{m\theta_{r-4,q}}{q+1} + q^{r-3} \) lines of \( \mathcal{L} \) or \( \Sigma \) contains \( \frac{m\theta_{r-4,q}}{q+1} \) lines of \( \mathcal{L} \). From Lemma 2.1(iii), it follows that there are either \( (q^2 - q)mq^{r-3} + q^{r-2} \) or \( (q^2 - q)mq^{r-3} \) lines of \( \mathcal{L} \) disjoint from \( \Sigma \). Since a line of \( \text{PG}(r, q) \) disjoint from \( \Sigma \), when extended over \( \text{GF}(q^2) \), meets \( \Pi \) in exactly one point, we have that \( |\Pi \cap \tilde{\mathcal{L}}| \) equals either
\[ (q^2 - q) \frac{m\theta_{r-2,q}}{q + 1} + \theta_{r-1,q}, \]

or
\[ (q^2 - q) \frac{m\theta_{r-2,q}}{q + 1} + \theta_{r-2,q}. \]

Vice versa, suppose that every hyperplane of \( \text{PG}(r, q^2) \) contains either \( \alpha \) or \( \beta \) points of \( \tilde{\mathcal{L}} \). Set \( k = |\tilde{\mathcal{L}}| \). It follows that
\[ k = (q^2 - q)|\mathcal{L}| + \theta_{r,q} = (q^2 - q) \frac{m\theta_{r,q}}{q + 1} + \theta_{r,q}. \]

Note that, if \( \Pi \) is a hyperplane of \( \text{PG}(r, q^2) \) meeting \( \text{PG}(r, q) \) in a \( \text{PG}(r-1, q) \), then
\[ \alpha = |\Pi \cap \tilde{\mathcal{L}}| = (q^2 - q) \frac{m\theta_{r-2,q}}{q + 1} + \theta_{r-1,q}. \]

From [21, pp. 30–31], we have that
\[ k^2\theta_{r-2,q^2} + k(1 - \alpha - \beta)\theta_{r-1,q^2} - k\theta_{r-2,q^2} + \alpha\beta\theta_{r,q^2} = 0. \]

Taking into account (2.1), (2.2) and substituting \( k \) and \( \alpha \) in (2.3) we obtain
\[ \beta = (q^2 - q) \frac{m\theta_{r-2,q}}{q + 1} + \theta_{r-2,q}. \]

On the other hand, if \( \Pi \) is a hyperplane of \( \text{PG}(r, q^2) \) meeting \( \text{PG}(r, q) \) in a \( \text{PG}(r-2, q) \), say \( \Sigma \), and \( x \) is the number of lines of \( \mathcal{L} \) contained in \( \Sigma \) and \( y \) is the number of lines of \( \mathcal{L} \) meeting \( \Sigma \) in one point, then
\[ |\Pi \cap \tilde{\mathcal{L}}| = \frac{m\theta_{r,q}}{q + 1} - x - y + (q^2 - q)x + \theta_{r-2,q} = \frac{m\theta_{r,q}}{q + 1} - (m - 1)\theta_{r-2,q} + xq^2. \]

Comparing (2.5) with (2.2), we obtain
\[ x = \frac{m\theta_{r-4,q}}{q + 1} + q^{r-3}, \]

whereas, comparing (2.5) with (2.4), we get
\[ x = \frac{m\theta_{r-4,q}}{q + 1}. \]
If \( r = 3 \), then every \( m \)-cover of \( \text{PG}(3, q) \) is \((r - 2)\)-dual. However this is no longer true if \( r > 3 \), see Remark (3.7).

### 2.1 \( m \)-covers of lines of Singer type

In this section we exhibit some examples of \((r - 2)\)-dual \( m \)-covers of \( \text{PG}(r, q) \). A Singer cyclic group of \( \text{PG}(r, q) \) is a cyclic group \( \mathcal{K} \) of \( \text{PGL}(r + 1, q) \) acting regularly on points of \( \text{PG}(r, q) \) [19]. In [14, Lemmas 2.1, 2.2] the author studies the action of \( \mathcal{K} \) on subspaces of \( \text{PG}(r, q) \) and counts the number of orbits. In particular, when \( \mathcal{K} \) acts on lines of \( \text{PG}(r, q) \) and \( r \) is odd, apart from one orbit which is a line-spread (or a 1-cover), any other orbit of \( \mathcal{K} \) is a \((q + 1)\)-cover of \( \text{PG}(r, q) \). When \( r \) is even all \( \mathcal{K} \)-orbits on lines are \((q + 1)\)-covers. We use the term Singer cover to denote a \( \mathcal{K} \)-orbit of lines of \( \text{PG}(r, q) \).

**Lemma 2.4** Any Singer cover of \( \text{PG}(r, q) \) is an \((r - 2)\)-dual cover of \( \text{PG}(r, q) \).

**Proof** Let \( \mathcal{L} \) be a Singer cover of \( \text{PG}(r, q) \). Embed \( \text{PG}(r, q) \) in \( \text{PG}(r, q^2) \) and let \( \bar{\mathcal{L}} \) be the set of points of \( \text{PG}(r, q^2) \) lying on the extended lines of \( \mathcal{L} \). By considering the action of \( \mathcal{K} \) on points of \( \text{PG}(r, q^2) \), it follows that, when \( r \) is odd, the projective space \( \text{PG}(r, q^2) \) is partitioned into two \( \frac{q^2 - 1}{2} \)-spaces of \( \text{PG}(r, q^2) \), \( \Sigma_i, i = 1, 2, \) and \((q - 1)\theta_{\frac{q^2 - 1}{2}}\) Baer subgeometries, see [22, Theorem 4.1]. Whereas, when \( r \) is even, \( \text{PG}(r, q^2) \) is partitioned into \( \theta_{r, q^2}/\theta_{r, q} \) Baer subgeometries, see [18, Theorem 4.29]. From [25] a simple counting argument shows that a hyperplane of \( \text{PG}(r, q^2) \) (not containing a subspace \( \Sigma_i \)) meets exactly one Baer subgeometry of the above partition in \( \theta_{r, q} \) points. If \( \mathcal{L} \) is the unique 1-cover of \( \text{PG}(r, q) \), \( r \) odd, then \( \bar{\mathcal{L}} \) is the disjoint union of \( \Sigma_1, \Sigma_2 \) and \( q - 1 \) Baer subgeometries, see [9]. If \( \mathcal{L} \) is a \((q + 1)\)-cover, then \( \bar{\mathcal{L}} \) is the union of \( q^2 - q + 1 \) Baer subgeometries. In both cases, \( \bar{\mathcal{L}} \) is a two-character set. The result follows from Theorem 2.3. \( \Box \)

### 3 \((r - 2)\)-dual covers of symplectic type

Let \( \mathcal{W}(r, q) \) be a symplectic polar space of \( \text{PG}(r, q) \), \( r \geq 3 \) odd, and let \( \bot \) denote the associated symplectic polarity. In this section we consider the case when the lines of an \( m \)-cover \( \mathcal{L} \) of \( \text{PG}(r, q) \) are totally isotropic with respect to \( \bot \), i.e., \( m \)-covers of \( \mathcal{W}(r, q) \). We introduce the following definition.

**Definition 3.1** An \( m \)-cover \( \mathcal{L} \) of \( \mathcal{W}(r, q) \) is said to be \((r - 2)\)-dual of symplectic type if

\[
|\{r \in \mathcal{L} : r \subseteq \ell^\bot\}| = \begin{cases} 
\frac{m\theta_{r,q}}{q+1} + q^{r-3} & \text{if } \ell \in \mathcal{L}, \\
\frac{m\theta_{r,q}}{q+1} & \text{if } \ell \notin \mathcal{L}.
\end{cases}
\]

If \( \mathcal{L} \) is the set of all totally isotropic lines of \( \mathcal{W}(r, q) \), from [6, Sect. 2], the set \( \bar{\mathcal{L}} \) is a non-degenerate Hermitian variety \( \mathcal{H}(r, q^2) \) of \( \text{PG}(r, q^2) \) with associated unitary polarity, say \( \perp_h \), see also [20, Table 4.5.A]. In this geometric setting we also consider the embedding of \( \mathcal{W}(r, q) \) in \( \mathcal{W}(r, q^2) \) and, with a slight abuse of notation, we denote by \( \perp \) the symplectic polarity associated to \( \mathcal{W}(r, q^2) \). Let \( \tau \) denote the semilinear involution of \( \mathcal{W}(r, q^2) \) fixing \( \mathcal{W}(r, q) \) pointwise. It can be seen that \( \perp \perp_h = \perp_h \perp = \tau \), i.e., \( \perp \) and \( \perp_h \) are commuting polarities of the ambient projective space. Let \( P \) be a point of \( \mathcal{W}(r, q^2) \). Then the polar hyperplane of \( P \) with respect to \( \perp \) coincides with the polar hyperplane of \( P' = \tau(P) \) with respect to \( \perp_h \). In particular, if \( P \in \mathcal{W}(r, q) \), then \( P^\perp = P^\perp_h \). If \( P \in \mathcal{H}(r, q^2) \setminus \mathcal{W}(r, q) \) and
\( \ell_P \) is the unique extended line of \( \mathcal{W}(r, q) \) through \( P \), then \( \ell_P^\perp \subset P^\perp \). On the other hand, \( P^\perp \) meets \( \mathcal{W}(r, q) \) in an \((r - 2)\)-space and hence \( P^\perp \cap \mathcal{W}(r, q) = \ell_P^\perp \).

A subset of points \( \mathcal{X} \) of \( \mathcal{H}(r, q^2) \) (or of \( \mathcal{W}(r, q^2) \)) is said to be \( i \)-tight if

\[
|P^\perp \cap \mathcal{X}| = \begin{cases}
  i\theta_{r-3, q^2} + q^{r-1} & \text{if } P \in \mathcal{X}, \\
  i\theta_{r-3, q^2} & \text{if } P \notin \mathcal{X}.
\end{cases}
\]

We have the following result.

**Theorem 3.2** Let \( \mathcal{L} \) be an \( m \)-cover of \( \mathcal{W}(r, q) \) and let \( \tilde{\mathcal{L}} \) be the set of points of \( \mathcal{H}(r, q^2) \) lying on the extended lines of \( \mathcal{L} \). Then \( \tilde{\mathcal{L}} \) is an \((m(q^2 - q) + q + 1)\)-tight set of \( \mathcal{H}(r, q^2) \) if and only if \( \mathcal{L} \) is an \((r - 2)\)-dual cover of symplectic type.

**Proof** Assume that \( \tilde{\mathcal{L}} \) is an \((m(q^2 - q) + q + 1)\)-tight set of \( \mathcal{H}(r, q^2) \). Note that if \( P \in \mathcal{H}(r, q^2) \setminus \mathcal{W}(r, q) \) then \( P^\perp \cap \mathcal{W}(r, q) = \ell_P^\perp \), where \( \ell_P \) is the unique extended line of \( \mathcal{L} \) through \( P \). Assume that \( P \in \tilde{\mathcal{L}} \setminus \mathcal{W}(r, q) \). It follows that

\[
|P^\perp \cap \tilde{\mathcal{L}}| = (m(q^2 - q) + q + 1)\theta_{r-3, q^2} + q^{r-1} = x(q^2 - q) + \theta_{r-2, q} + |\mathcal{L}| - x - y,
\]

where \( x \) is the number of extended lines of \( \mathcal{L} \) contained in \( P^\perp \) (hence contained in \( \ell_P^\perp \)) and \( y \) is the number of extended lines of \( \mathcal{L} \) meeting \( P^\perp \) in a point of \( \mathcal{W}(r, q) \). Then, taking into account Lemma 2.1(iii), we have

\[
xq^2 = m\frac{(q^2 - q)\theta_{r-2, q} - \theta_{r, q}}{q + 1} + q^{r-1} = mq^2\frac{\theta_{r-4, q}}{q + 1} + q^{r-1},
\]

and hence \( x = \frac{m\theta_{r-4, q}}{q+1} + q^{r-3} \). If \( P \in \mathcal{H}(r, q^2) \setminus \tilde{\mathcal{L}} \), then

\[
|P^\perp \cap \tilde{\mathcal{L}}| = (m(q^2 - q) + q + 1)\theta_{r-3, q^2} = x(q^2 - q) + \theta_{r-2, q} + |\mathcal{L}| - x - y,
\]

where again \( x \) is the the number of extended lines of \( \mathcal{L} \) contained in \( P^\perp \) (hence contained in \( \ell_P^\perp \)) and \( y \) is the number of extended lines of \( \mathcal{L} \) meeting \( P^\perp \) in a point of \( \mathcal{W}(r, q) \). In this case \( x = \frac{m\theta_{r-4, q}}{q+1} \).

Viceversa, assume that \( \mathcal{L} \) is an \((r - 2)\)-dual cover of symplectic type. If \( P \in \mathcal{W}(r, q) \), then every line of \( \mathcal{L} \) meets \( P^\perp \) in a point of \( \mathcal{W}(r, q) \). Hence

\[
P^\perp \cap \tilde{\mathcal{L}} = m(q^2 - q)\frac{\theta_{r-2, q}}{q + 1} + \theta_{r-1, q} = (m(q^2 - q) + q + 1)\theta_{r-3, q^2} + q^{r-1}.
\]

On the other hand, if \( P \in \mathcal{H}(r, q^2) \setminus \mathcal{W}(r, q) \), then

\[
P^\perp \cap \tilde{\mathcal{L}} = x(q^2 - q) + \theta_{r-2, q} + |\mathcal{L}| - x - y,
\]

where \( x \) is the the number of lines of \( \mathcal{L} \) contained in \( P^\perp \) (hence contained in \( \ell_P^\perp \)) and \( y \) is the number of lines of \( \mathcal{L} \) meeting \( P^\perp \) in a point of \( \mathcal{W}(r, q) \). Hence, \( \tilde{\mathcal{L}} \) is an \((m(q^2 - q) + q + 1)\)-tight set of \( \mathcal{H}(r, q^2) \). \( \square \)

**Corollary 3.3** In \( \mathcal{PG}(r, q) \), \( r \) odd, if \( \mathcal{L} \) is an \((r - 2)\)-dual cover of symplectic type, then \( \mathcal{L} \) is an \((r - 2)\)-dual cover.

**Proof** If \( \mathcal{L} \) is an \((r - 2)\)-dual cover of symplectic type, than \( \tilde{\mathcal{L}} \) is a tight set of \( \mathcal{H}(r, q^2) \), which in turn is a two-character set of \( \mathcal{PG}(r, q^2) \), see [1, Theorem 12]. From Theorem 2.3, \( \mathcal{L} \) is an \((r - 2)\)-dual cover. \( \square \)
Remark 3.4 The tight sets constructed in Theorem 3.2 are all reducible since they always contain the $\theta_{r,q}$ points of $\mathcal{W}(r, q)$ which constitute a $(q + 1)$-tight set of $\mathcal{H}(r, q^2)$.

Analogously to Theorem 3.2 we have the following result.

**Theorem 3.5** Let $\mathcal{L}$ be an $m$-cover of $\mathcal{W}(r, q)$ and let $\tilde{\mathcal{L}}$ be the set of points of $\mathcal{W}(r, q^2)$ lying on the extended lines of $\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is an $(m(q^2 - q) + q + 1)$-tight set of $\mathcal{W}(r, q^2)$ if and only if $\mathcal{L}$ is an $(r - 2)$-dual cover of symplectic type.

**Proof** Assume that $\tilde{\mathcal{L}}$ is an $(m(q^2 - q) + q + 1)$-tight set of $\mathcal{W}(r, q^2)$. By construction, $\tilde{\mathcal{L}}$ is contained in $\mathcal{H}(r, q^2)$. Also $P_{\perp} = \tau(P)^{\perp h}$ and $P \in \tilde{\mathcal{L}}$ if and only if $\tau(P) \in \tilde{\mathcal{L}}$. Hence $\tilde{\mathcal{L}}$ is an $(m(q^2 - q) + q + 1)$-tight set of $\mathcal{H}(r, q^2)$ and, from Theorem 3.2, $\mathcal{L}$ is an $(r - 2)$-dual cover of symplectic type.

Viceversa, assume that $\mathcal{L}$ is an $(r - 2)$-dual cover of symplectic type. If $P \in \mathcal{W}(r, q)$, then

$$|P_{\perp} \cap \tilde{\mathcal{L}}| = |P_{\perp h} \cap \tilde{\mathcal{L}}| = m(q^2 - q) \frac{\theta_{r-2,q}}{q + 1} + \theta_{r-1,q} = (m(q^2 - q) + q + 1)\theta_{r-2,q^2} + q^{r-1}. $$

If $P \in \mathcal{H}(r, q^2) \setminus \mathcal{W}(r, q)$ then $P_{\perp} = \tau(P)^{\perp h}$, where $(P, \tau(P))$ is the unique extended totally isotropic line of $\mathcal{W}(r, q)$ through $P$. Hence $\tau(P) \in \tilde{\mathcal{L}}$ if and only if $P \in \tilde{\mathcal{L}}$. It follows that $|P_{\perp} \cap \tilde{\mathcal{L}}|$ is either

$$(m(q^2 - q) + q + 1)\theta_{r-2,q^2} + q^{r-1}$$

or

$$(m(q^2 - q) + q + 1)\theta_{r-3,q^2},$$

according as $P$ lies or does not lie on $\tilde{\mathcal{L}}$. Finally, if $P \in \mathcal{W}(r, q^2) \setminus \mathcal{H}(r, q^2)$ then $\tau(P) \notin \mathcal{W}(r, q)$ and the line $(P, \tau(P))$ is not a line of $\mathcal{W}(r, q^2)$. Now $P_{\perp} = \tau(P)^{\perp h}$ and $\tau(P)^{\perp h}$ meets $\mathcal{H}(r, q^2)$ in a Hermitian variety $\mathcal{H}(r - 1, q^2)$ that is a $\theta_{r-3,q^2}$-ovoid of $\mathcal{H}(r, q^2)$ [1, Lemma 7]. From Theorem 3.2, $\tilde{\mathcal{L}}$ is an $(m(q^2 - q) + q + 1)$-tight set of $\mathcal{H}(r, q^2)$. Hence, from [1, Corollary 5], we have that

$$|P_{\perp} \cap \tilde{\mathcal{L}}| = |	au(P)^{\perp h} \cap \tilde{\mathcal{L}}| = |\mathcal{H}(r - 1, q^2) \cap \tilde{\mathcal{L}}| = (m(q^2 - q) + q + 1)\theta_{r-2,q^2}. $$

Hence $\tilde{\mathcal{L}}$ is an $(m(q^2 - q) + q + 1)$-tight set of $\mathcal{W}(r, q^2)$. 

\[ \square \]

**Remark 3.6** If $\mathcal{L}$ consists of all the lines of $\mathcal{W}(r, q)$, then $\mathcal{L}$ is an $(r - 2)$-dual $\theta_{r-2,q}$-cover of symplectic type. A more interesting example arises by considering the set of lines $\mathcal{L}$ of the split Cayley hexagon $H(q)$ embedded in $\mathcal{W}(5, q)$, $q$ even. Indeed, from [10] the set $\mathcal{L}$ is a $(q^3 + 1)$-tight set of both $\mathcal{H}(5, q^2)$ and $\mathcal{W}(5, q^2)$. Hence $\mathcal{L}$ is a $3$-dual $(q + 1)$-cover of symplectic type of $\mathcal{W}(5, q)$, $q$ even. A similar result holds true in the odd characteristic case, see Proposition 4.11.

Note that Theorems 3.2 and 3.5 generalize Theorem 2.1 of [23]. Indeed, if $r = 3$, then every $m$-cover of $\mathcal{W}(3, q)$ is $(r - 2)$-dual of symplectic type. However this is no longer true if $r > 3$, see Remark (3.7).

**Remark 3.7** There exist $m$-covers of $\mathcal{PG}(r, q)$ (of $\mathcal{W}(r, q)$) that are neither $(r - 2)$-dual covers nor $(r - 2)$-dual covers of symplectic type. Let $\mathcal{W}(5, 3)$ be a symplectic polar space of $\mathcal{PG}(5, 3)$. The group $\mathcal{PSp}(6, 3)$ fixing $\mathcal{W}(5, 3)$ has two conjugacy classes of subgroups
isomorphic to $\text{PSL}(2, 13)$, see also [4]. With the aid of MAGMA [2] we checked that one of them has 8 orbits on lines of $\mathcal{W}(5, 3)$ of sizes 91, 91, 364, 364, 546, 546, 546, 1092. All of them are $m$-covers of $\mathcal{W}(5, 3)(\mathcal{W}(5, 3))$, for some $m$. The orbits of size 91 are line spreads of $\mathcal{W}(5, 3)$ (of $\mathcal{W}(5, 3)$) that are neither 3-dual nor 3-dual of symplectic type. One of the two orbits of size 364 is a 3-dual cover of symplectic type and hence 3-dual. All the remaining orbits are neither 3-dual covers nor 3-dual covers of symplectic type. Notice that the union of two orbits of size 546 and the orbit of size 1092 is a 3-dual cover of symplectic type.

### 3.1 $(4n - 3)$-dual covers of symplectic type of $\text{PG}(4n - 1, q)$

In this section we provide some instances of $(4n - 3)$-dual covers of symplectic type of $\text{PG}(4n - 1, q)$, $n \geq 2$. The projective symplectic group $\text{PSp}(4n, q)$ acts naturally on the projective space $\Sigma := \text{PG}(4n - 1, q)$ and fixes a symplectic polar space $\mathcal{W}(4n - 1, q)$ with associated symplectic polarity $\perp$. In [15] Dye constructed a spread $\mathcal{F}$ in $\text{PG}(4n - 1, q)$ consisting of totally isotropic lines of the polarity $\perp$, whose stabilizer in $\text{PSp}(4n, q)$ contains a group $G$ isomorphic to $\text{PSp}(2n, q^2)$. From [15, p. 178], the group $G$ fixes a pencil of linear complexes, say $\mathcal{W}_i$, $1 \leq i \leq q + 1$, one of them being $\mathcal{W}(4n - 1, q)$.

In [16, Theorem 1, p. 499] the author also proved that, if $n \geq 2$, the group $G$ has three orbits on totally isotropic lines of $\text{PG}(4n - 1, q)$, namely the spread $\mathcal{F}$, the set $\mathcal{O}_1$ consisting of the lines that are not conjugate to each member of $\mathcal{F}$ that they intersect non-trivially, and the set $\mathcal{O}_2$ consisting of the lines that are conjugate to each member of $\mathcal{F}$ that they intersect non-trivially. An equivalent description of $\mathcal{O}_1$, $i = 1, 2$, is the following: let $\ell$ be a line of $\mathcal{W}(4n - 1, q^2)$, with $\ell \notin \mathcal{F}$ and let $r_1, \ldots, r_{q+1}$ be the lines of $\mathcal{F}$ meeting $\ell$ in one point. Then $(r_1, \ldots, r_{q+1})$ is a solid $T$ and there are two possibilities: either $T \cap \mathcal{W}(4n - 1, q) = \mathcal{W}(3, q)$ and $\ell \in \mathcal{O}_1$ or $T \subset \mathcal{W}(4n - 1, q)$ and $\ell \in \mathcal{O}_2$. The group $G$ acts transitively on points of $\text{PG}(4n - 1, q)$, see [15, Theorem 5]. It follows that $\mathcal{F}$, $\mathcal{O}_1$ and $\mathcal{O}_2$ are $m$-covers of $\mathcal{W}(4n - 1, q)$, for some $m$.

Embed $\Sigma$ in $\text{PG}(4n - 1, q^2)$ and let $\tau$ denote the semilinear involution of $\text{PG}(4n - 1, q^2)$ fixing $\Sigma$ pointwise. Each linear complex $\mathcal{W}_i$ gives rise to a non-degenerate Hermitian variety $\mathcal{H}_i$ of $\text{PG}(4n - 1, q^2)$ and these Hermitian varieties form a pencil of $\text{PG}(4n - 1, q^2)$. Associated with the spread $\mathcal{F}$ are two disjoint $(2n - 1)$-dimensional subspaces of $\mathcal{W}(4n - 1, q^2)$, say $\Sigma_1$ and $\Sigma_2$, that are generators of each Hermitian variety $\mathcal{H}_i$ and that are disjoint from $\Sigma$. The lines of the spread $\mathcal{F}$, when extended over $\text{GF}(q^2)$, meet both $\Sigma_1$ and $\Sigma_2$. In particular, the lines of $\mathcal{F}$ are exactly those lines of $\mathcal{W}(4n - 1, q)$ meeting both $\Sigma_1$ and $\Sigma_2$. Also, $\tau(\Sigma_1) = \Sigma_2$. The group $G$ stabilizes both $\Sigma_1$, $\Sigma_2$ and acts naturally on $\Sigma_1$ fixing a symplectic polar space $\mathcal{W}(2n - 1, q^2) \subset \Sigma_1$. If $g$ is an $(n - 1)$-space of $\Sigma_1$ that is a generator of $\mathcal{W}(2n - 1, q^2)$, then $(g, \tau(g))$ meets $\Sigma$ in a $(2n - 1)$-space $\gamma$. Note that $\mathcal{F}$ induces a line-spread on $\gamma$, and hence $\gamma$ is totally isotropic with respect to each linear complex $\mathcal{W}_i$, $1 \leq i \leq q + 1$. Varying $g$, we get a set $\mathcal{M}$ of size $(q^2 + 1)(q^4 + 1) \ldots (q^{2n} + 1)$, consisting of $(2n - 1)$-spaces of $\text{PG}(4n - 1, q)$ that are totally isotropic with respect to each symplectic space $\mathcal{W}_i$, $1 \leq i \leq q + 1$. It follows that the lines in common to every symplectic space $\mathcal{W}_i$, $1 \leq i \leq q + 1$ are in the union $\mathcal{F} \cup \mathcal{O}_2$, that are all the lines lying in some member of $\mathcal{M}$.

Using [7, Sect. 2.2] it can be seen that, for a point $P \in \Sigma_1 \cup \Sigma_2$, the polar hyperplane of the point $P$ is the same with respect to each Hermitian variety $\mathcal{H}_i$. We denote such a polar hyperplane $P^{-h}$. Let $P$ be a point of $\Sigma_1$ and let $P^{-h}$ be the polar hyperplane of $P$ with respect to the polarity associated with each $\mathcal{H}_i$. The intersection of $P^{-h}$ and $\Sigma_2$ is a $(2n - 2)$-space of $\Sigma_2$, say $S_P$, and we say that $S_P$ corresponds to $P$. Note that the subspace $(P, S_P)$ is a
generator of \( \mathcal{H}_i \), \( 1 \leq i \leq q + 1 \). Take another point \( P' \) of \( \Sigma_1 \), \( P' \neq P \), and suppose that its corresponding subspace \( S_{P'} \) coincides with \( S_P \). Then, the line joining \( P \) with \( P' \) and \( S_P \) would be orthogonal to each other and hence would generate a \( 2n \)-space contained in \( \mathcal{H}_i \), a contradiction. Analogously, if \( P \neq P' \), then \( S_P \neq S_{P'} \). Thus, allowing \( P \) to vary over the points of \( \Sigma_1 \), the construction described above produces a family, say \( \mathcal{P}_1 \), of \( \theta_{2n-1,q^2} \) distinct generators of \( \mathcal{H}_i \), \( 1 \leq i \leq q + 1 \). In a similar way, as \( Q \) varies over the points of \( \Sigma_2 \), one obtains another collection, say \( \mathcal{P}_2 \), of \( \theta_{2n-1,q^2} \) distinct generators of \( \mathcal{H}_i \), \( 1 \leq i \leq q + 1 \). Note that no point \( R \neq (\Sigma_1 \cup \Sigma_2) \) lies in two distinct elements of \( \mathcal{P}_1 \), otherwise \( R \) would lie in two distinct lines meeting both \( \Sigma_1 \) and \( \Sigma_2 \), a contradiction. Hence, by considering the points lying in some element of \( \mathcal{P}_1 \), we get a subset of points of \( \mathcal{H}_i \), \( 1 \leq i \leq q + 1 \), including \( \Sigma_1 \) and \( \Sigma_2 \).

**Theorem 3.8** The set \( \mathcal{R} \) is a \( (q^{4n-2} + 1) \)-tight set of \( \mathcal{H}_i \).

**Proof** Let \( H \) be a tangent hyperplane to \( \mathcal{H}_i \) at a point \( P \). We distinguish several cases.

Assume first that \( P \in \Sigma_1 \) or \( P \in \Sigma_2 \). Then in this case either \( \Sigma_1 \subset H \) or \( \Sigma_2 \subset H \). Let \( \Sigma_1 \subset H \). A similar argument holds if \( \Sigma_2 \subset H \). In this case the intersection between \( H \) and \( \Sigma_2 \) is the \((2n-2)\)-space \( S_P \). Of course, \( H \) meets any other member of \( \mathcal{P}_1 \) in a \((2n-2)\)-space. It follows that \( H \) meets \( \mathcal{R} \) in \( h_1 \) points where

\[
|\mathcal{R}| = (q^{4n-2} + 1)\theta_{2n-1,q^2}
\]

and containing at least \( 2\theta_{2n-1,q^2} + 2 \) generators of \( \mathcal{H}_i \), \( 1 \leq i \leq q + 1 \), including \( \Sigma_1 \) and \( \Sigma_2 \).

Assume that \( P \neq \Sigma_1 \cup \Sigma_2 \). Let \( \ell \) be the unique line containing \( P \) and meeting both, \( \Sigma_1 \) and \( \Sigma_2 \), in a point. Let \( X_i \) be the \((2n-2)\)-space \( H \cap \Sigma_i \), \( i = 1, 2 \).

If \( P \in \mathcal{R} \), then \( \ell \) is a line of \( \mathcal{H}_i \) and \( \langle \ell, X_2 \rangle \) is a member of \( \mathcal{P}_1 \) contained in \( H \). Also \( H \) meets \((\theta_{2n-2,q^2} - 1)\) members of \( \mathcal{P}_1 \) in a \((2n-2)\)-space meeting \( X_1 \) in a point and \( X_2 \) in a \((2n-3)\)-space. The remaining members of \( \mathcal{P}_1 \) have empty intersection with \( X_1 \) and meet \( X_2 \) in a \((2n-3)\)-space and hence each of them shares \( \theta_{2n-2,q^2} - \theta_{2n-3,q^2} \) points with \( H \setminus (\Sigma_1 \cup \Sigma_2) \). It follows that \( H \) meets \( \mathcal{R} \) in \( h_1 \) points where

\[
h_1 = 2\theta_{2n-2,q^2} + \theta_{2n-1,q^2} - \theta_{2n-2,q^2} - 1 + (\theta_{2n-2,q^2} - 1)(\theta_{2n-2,q^2} - \theta_{2n-3,q^2} - 1)
\]

\[
+ (\theta_{2n-1,q^2} - \theta_{2n-2,q^2})(\theta_{2n-2,q^2} - \theta_{2n-3,q^2}) = (q^{4n-2} + 1)\theta_{2n-2,q^2} + q^{4n-2}.
\]

If \( P \in \mathcal{H}_i \setminus \mathcal{R} \) then \( \ell \) is a line that is secant to \( \mathcal{H}_i \) and does not exist a member of \( \mathcal{P}_1 \) through \( P \) contained in \( H \). It follows that \( H \) meets \( \mathcal{R} \) in \( h_2 \) points where

\[
h_2 = 2\theta_{2n-2,q^2} + \theta_{2n-1,q^2} - \theta_{2n-2,q^2} - 1 + (\theta_{2n-2,q^2} - 1)(\theta_{2n-2,q^2} - \theta_{2n-3,q^2} - 1)
\]

\[
+ (\theta_{2n-1,q^2} - \theta_{2n-2,q^2} - 1)(\theta_{2n-2,q^2} - \theta_{2n-3,q^2}) = (q^{4n-2} + 1)\theta_{2n-2,q^2}.
\]

\( \square \)

Let \( \tilde{F}, \tilde{O}_1, \tilde{O}_2 \) be the sets of points of \( \mathcal{H}_i \) lying on the extended lines of \( F, O_1, O_2 \), respectively.

**Proposition 3.9** Each of the sets \( \tilde{F}, \tilde{O}_1, \tilde{O}_2 \) is an \( i \)-tight set of \( \mathcal{H}_i \) with parameter \( i = q^2 + 1, q^{4n-1} - q^{4n-2} + q + 1, q^{4n-2} - q^2 + q + 1 \), respectively.
Proof The set $\tilde{F}$ is the disjoint union of $q-1$ Baer subgeometries and of the two generators $\Sigma_1$ and $\Sigma_2$, [22, Theorem 4.1]. Since a Baer subgeometry embedded in $H_i$, is a $(q+1)$-tight set of $H_i$, see [1, Sect. 5.2], and $\Sigma_i$, $i=1, 2$, is a 1-tight set of $H_i$, we have that $\tilde{F}$ is a $(q^2+1)$-tight set of $H_i$. Moreover $R = \tilde{F} \cup \tilde{O}_2$ and $\tilde{F} \cap \tilde{O}_2 = \Sigma$, hence $\tilde{O}_2$ is a $(q^{4n-2} - q^2 + q + 1)$-tight set of $H_i$. Trivially the whole of $H_i$ can be considered a $(q^{4n-1} + 1)$-tight set of itself, which gives that $\tilde{O}_1$ is a $(q^{4n-1} - q^{4n-2} + q + 1)$-tight set of $H_i$. 

From Theorem 3.2 and the previous proposition we have the following corollary.

Corollary 3.10 Each of the $m$-covers $F$, $O_1$, $O_2$ of $\mathcal{W}(4n-1, q)$ is $(4n-3)$-dual of symplectic type.

4 ($r - 2$)-dual covers of parabolic type

Let $Q(r, q)$ be a parabolic polar space of $\Sigma := PG(r, q)$, $q$ odd, $r \geq 4$ even, and let $\perp$ denote the associated orthogonal polarity. Let $G$ be the group of projectivities of $\Sigma$ isomorphic to $PG_0(r+1, q)$ stabilizing the parabolic polar space $Q(r, q)$. Embed $\Sigma$ in $PG(r, q^2)$ and let $\tau$ denote the semilinear involution of $PG(r, q^2)$ fixing $\Sigma$ pointwise. From [20, Table 4.5.A] the group $G$, considered as a group of projectivities of $PG(r, q^2)$, fixes a non-degenerate Hermitian variety $H(r, q^2)$, with associated unitary polarity $\perp_h$, and a parabolic quadric $Q(r, q^2)$ of $PG(r, q^2)$, where $Q(r, q) \subset Q(r, q^2) \cap H(r, q^2)$. With a slight abuse of notation, we denote by $\perp$ the orthogonal polarity associated to $Q(r, q^2)$. Then $\perp_h \cap \perp_h = \tau$, i.e., $\perp$ and $\perp_h$ are commuting polarities of the ambient projective space. Let $P$ be a point of $Q(r, q^2)$. Then the polar hyperplane of $P$ with respect to $\perp$ coincides with the polar hyperplane of $P' = \tau(P)$ with respect to $\perp_h$. In particular, if $P \in \Sigma$, then $P^\perp = P^\perp_h$.

From [1, Theorem 8], the group $G$ has three orbits on points of $Q(r, q^2)$, namely $Q(r, q^2)$, $\mathcal{O}$ and $\mathcal{O}'$. The orbit $\mathcal{O}$ has size $q(q^r - 1)(q^{r-2} - 1)/(q^2 - 1)$ and consists of the points on the extended lines of $Q(r, q)$ that are not in $\Sigma$, while $\mathcal{O}'$ has size $q^{r-1}(q^r - 1)/(q + 1)$, and consists of the points lying on the extended lines of $\Sigma$ that are external to $Q(r, q)$.

A subset of points $\mathcal{X}$ of $H(r, q^2)$ (or of $Q(r, q^2)$) is said to be $i$-tight if

$$|P^\perp_h \cap \mathcal{X}| = \begin{cases} i\frac{q^r - 1}{2}q^2 + q^{r-2} & \text{if } P \in \mathcal{X}, \\ i\frac{q^r - 1}{2}q^2 & \text{if } P \notin \mathcal{X}. \end{cases}$$

From [1, Theorem 8] each of the three $G$-orbits $Q(r, q)$, $\mathcal{O}$, $\mathcal{O}'$ is an $i$-tight set of $Q(r, q^2)$ with parameter $i = q + 1$, $q^{r-1} - q$, $q^r - q^{r-1}$, respectively. An analogous result will be proved in the first part of this section.

Proposition 4.1 The group $G$ has four orbits on points of $H(r, q^2)$:

1. $Q(r, q)$,
2. $\mathcal{O}$ consisting of points (not in $\Sigma$) on the extended lines of $Q(r, q)$,
3. $\mathcal{E}$ consisting of points of $H(r, q^2)$ lying on the extended lines of $\Sigma$ that are external to $Q(r, q^2)$,
4. $S$ consisting of points of $H(r, q^2) \setminus Q(r, q)$ lying on the extended lines of $\Sigma$ that are secant to $Q(r, q^2)$.

In particular $|S| = |E| = q^{r-1}(q^r - 1)/2$. 

Springer
Proof The group $G$ is transitive on points of $Q(r, q)$. Let $P \in \mathcal{H}(r, q^2) \setminus Q(r, q)$, then $P$ lies on a unique extended line $\ell_P$ of $\Sigma$, where $\ell_P$ meets $Q(r, q)$ in either $q + 1$ or 0 or 2 points. Note that $\ell_P \cap \Sigma$ cannot be tangent to $Q(r, q)$. Indeed a line of $\Sigma$ that is tangent to $Q(r, q)$ at the point $R$, when extended over $GF(q^2)$, is also a tangent line to $\mathcal{H}(r, q^2)$ at the point $R$. On the other hand the stabilizer of $\ell_P$ in $G$ permutes in a single orbit the $q^2 - q$ or $q + 1$ or $q - 1$ points of $\mathcal{H}(r, q^2) \setminus Q(r, q)$ on $\ell_P$, respectively, see also [8, Proposition 2.2]. The result now follows from the fact that the group $G$ is transitive on lines of $\Sigma$ that are either contained in $Q(r, q)$ or external to $Q(r, q)$ or secant to $Q(r, q)$. \hfill \Box

Corollary 4.2 $\mathcal{H}(r, q^2) \cap \Sigma = Q(r, q)$ and $\mathcal{H}(r, q^2) \cap Q(r, q^2) = Q(r, q) \cup O$.

Lemma 4.3 Let $\ell$ be line of $\mathcal{H}(r, q^2)$, then the following possibilities occur:

(i) $\ell$ has $q + 1$ points in common with $Q(r, q)$ and $q^2 - q$ points in common with $O$, 
(ii) $\ell$ has a point in common with $Q(r, q)$ and $q^2$ points in common with $\mathcal{E}$, 
(iii) $\ell$ has a point in common with $Q(r, q)$ and $q^2$ points in common with $S$, 
(iv) $\ell$ has a point in common with $Q(r, q)$, $r \geq 6$, and $q^2$ points in common with $O$, 
(v) $\ell$ is contained in $O$, here $r \geq 8$, 
(vi) $\ell$ has a point in common with $O$ and $q^2$ points in common with $E$, here $r \geq 6$, 
(vii) $\ell$ has a point in common with $O$ and $q^2$ points in common with $S$, here $r \geq 6$, 
(viii) $\ell$ has $2$ points in common with $O$ and $(q^2 - 1)/2$ points in common with both $E$ and $S$, 
(ix) $\ell$ has $(q^2 + 1)/2$ points in common with both $E$ and $S$.

Proof A line of $PG(r, q^2)$ meets $\Sigma$ in 0, 1 or $q + 1$ points. Let $\ell$ be a line of $\mathcal{H}(r, q^2)$. If $|\ell \cap \Sigma| = q + 1$, then $\ell \cap \Sigma$ is a line of $Q(r, q)$ and hence $|\ell \cap O| = q^2 - q$. If $|\ell \cap \Sigma| \neq q + 1$, then either $|\ell \cap Q(r, q)| = 1$ or $|\ell \cap Q(r, q)| = 0$.

If the former case occurs, let $P = Q(r, q) \cap \ell$. Then $\langle \ell, \tau(\ell) \rangle$ is a plane $\pi$ of $PG(r, q^2)$ meeting $\Sigma$ in a Baer subplane $\pi_0$. Note that $P \in \pi_0$ and $\pi \subset P_{\perp} = P_{\perp r}$. It follows that $\pi_0$ share with $Q(r, q)$ either the point $P$, and $\ell$ is a line of type (ii), or two lines of $\pi_0$ through $P$, and $\ell$ is a line of type (iii), or $\pi_0$ is contained in $Q(r, q)$, $r \geq 6$, and $\ell$ is a line of type (iv). Note that the plane $\pi_0$ cannot intersect $Q(r, q)$ in a line through $P$, otherwise we would find a line of $\Sigma$ that is tangent to $Q(r, q)$ and such that when extended over $GF(q^2)$ should be a secant line to $\mathcal{H}(r, q^2)$, a contradiction.

If the latter case occurs, then $|\tau(\ell) \cap \Sigma| = 0$ and $\langle \ell, \tau(\ell) \rangle$ is a solid $\Pi$ of $PG(r, q^2)$ meeting $\Sigma$ in a $PG(3, q)$, say $\Pi_0$. If $\Pi_0 \subset Q(r, q)$, $r \geq 8$, and hence $\Pi \subset \mathcal{H}(r, q^2)$, then $\ell$ is a line of type (v). Assume that $\Pi_0 \not\subset Q(r, q)$. Then the possibilities for $\Pi_0 \cap Q(r, q)$ are listed in [17, Table 15.4] and, from [17, Table 19.1], we can say that $\Pi \cap \mathcal{H}(r, q^2)$ is either a non-degenerate Hermitian surface $\mathcal{H}(3, q^2)$ or $r \geq 6$ and it consists of $q + 1$ planes through a line. Note that, if $\Pi \cap \mathcal{H}(r, q^2)$ consists of $q + 1$ planes through a line $s$, then the line $s$ has to be an extended line of $Q(r, q)$. Indeed $s$ has to be fixed by $\tau$. Moreover, every point of $\mathcal{H}(r, q^2) \cap (\Pi \setminus \Pi_0)$ lies on a unique extended line of $\Pi_0$ and such a line shares 0, 2 or $q + 1$ points with $\Pi_0 \cap Q(r, q)$. Taking into account the fact that the extended lines of $\Pi_0$ meeting both $\ell$ and $\tau(\ell)$ give rise to a line-spread of $\Pi_0$, we have that necessarily one of the possibilities described below occurs. The set $\Pi_0 \cap Q(r, q)$ consists of the line $s$ and $\Pi \cap \mathcal{H}(r, q^2)$ consists of $q + 1$ planes through the line $s$. In this case $\ell$ is a line of type (vi). The set $\Pi_0 \cap Q(r, q)$, $r \geq 6$, consists of two planes through the line $s$ and $\Pi \cap \mathcal{H}(r, q^2)$ consists of $q + 1$ planes through the line $s$. In this case $\ell$ is a line of type (vii). The set $\Pi_0 \cap Q(r, q)$ is a hyperbolic quadric $Q^+(3, q)$ and $\Pi \cap \mathcal{H}(r, q^2)$ is a non-degenerate Hermitian surface $\mathcal{H}(3, q^2)$. In this case $\ell$ is a line of type (viii), see also [5, Proposition 2.2, Lemma 3.1]. Finally, the set $\Pi_0 \cap Q(r, q)$ is an elliptic quadric $Q^-(3, q)$ and $\Pi \cap \mathcal{H}(r, q^2)$ is a non-degenerate Hermitian surface $\mathcal{H}(3, q^2)$. In this case $\ell$ is a line of type (ix). \hfill \Box
Lemma 4.4 (1) Through a point of $Q(r, q)$ there pass $\theta_{r-3,q}$ lines of type (i), $q(q^{r-2} - 1)(q^{r-4} - 1)/(q^2 - 1)$ lines of type (iv), $q^{-3}(q^{r-2} - 1)/2$ lines of type (ii) and $q^{-3}(q^{r-2} - 1)/2$ lines of type (iii).

(2) Through a point of $O$ there pass one line of type (i), $q^2(q^{r-4} - 1)/(q - 1)$ lines of type (iv), $q^3(q^{r-6} - 1)/(q^{r-4} - 1)$ lines of type (v), $q^{-3}(q^{r-4} - 1)/2$ lines of type (vi) and $q^{-3}(q^{r-4} - 1)/2$ lines of type (vii).

(3) Through a point of $E$ there pass $\theta_{r-3,q}$ lines of type (ii), $q(q^{r-2} - 1)(q^{r-4} - 1)/(q^2 - 1)$ lines of type (vi), $q^{-3}(q^{r-2} - 1)/2$ lines of type (viii) and $q^{-3}(q^{r-2} - 1)/2$ lines of type (ix).

(4) Through a point of $S$ there pass $\theta_{r-3,q}$ lines of type (iii), $q(q^{r-2} - 1)(q^{r-4} - 1)/(q^2 - 1)$ lines of type (vii), $q^{-3}(q^{r-2} - 1)/2$ lines of type (viii) and $q^{-3}(q^{r-2} - 1)/2$ lines of type (ix).

Proof Let $P \in \mathcal{H}(r, q^2)$ and, if $P \notin Q(r, q)$, let $\ell_P$ be the unique extended line of $\Sigma$ through $P$. We distinguish several cases.

$P \in Q(r, q)$

Let $P$ be a point of $Q(r, q)$ and let $\Gamma$ be an $(r-2)$-space of $PG(r, q^2)$ contained in $P_{\perp h} = P_{\perp}$, not containing $P$ and such that $\Gamma \cap \Sigma$ is an $(r-2)$-space of $\Sigma$. Then $\Gamma \cap Q(r, q)$ is a parabolic polar space $Q(r-2, q)$ embedded in $Q(r, q)$ and $P_{\perp} \cap Q(r, q)$ is a cone having as vertex the point $P$ and as basis $Q(r-2, q)$. On the other hand $\Gamma \cap \mathcal{H}(r, q^2)$ is a Hermitian polar space $\mathcal{H}(r-2, q^2)$ embedded in $\mathcal{H}(r, q^2)$ and $P_{\perp h} \cap \mathcal{H}(r, q^2)$ is a cone having as vertex the point $P$ and as basis $\mathcal{H}(r-2, q^2)$, where $Q(r-2, q) \subset \mathcal{H}(r-2, q^2)$. Let $G'$ be the group of projectivities of $\Gamma$ isomorphic to PGO$(r-1, q)$ stabilizing the parabolic polar space $Q(r-2, q)$. From Proposition 4.1, the group $G'$ has four orbits on points of $\mathcal{H}(r-2, q^2)$, say $Q(r-2, q), O', E'$ and $S'$, where $Q(r-2, q) \subset Q(r, q)$, $O' \subset O$, $E' \subset E$, $S' \subset S$. Let $R$ be a point of $\mathcal{H}(r-2, q^2)$ and let $g$ be the line of $\mathcal{H}(r, q^2)$ joining $P$ and $R$. If $R \in Q(r-2, q)$, then $g$ is a line of type (i). If $R \in O'$, then the line $\ell_R$ is an extended line of $Q(r-2, q)$. It follows that $(\ell_R, g)$ is a plane $\sigma$ of $\mathcal{H}(r, q^2)$, $r \geq 6$, such that $\sigma \cap \Sigma$ is a Baer subplane and hence $g$ is a line of type (iv). If $R \in E'$, then $\ell_R$ is an extended line of $\Gamma \cap \Sigma$ such that $\ell_R \cap \Sigma$ is external to $Q(r-2, q)$. Since the plane $(P, \ell_R)$ meets $\Sigma$ in a Baer subplane which in turn shares with $Q(r, q)$ solely the point $P$, it follows that $g$ is a line of type (ii). Analogously, if $R \in S'$, we have that $g$ is a line of type (iii).

$P = O$

If $P \in O$, then let $\Gamma'$ be an $(r-4)$-space of $PG(r, q^2)$ contained in $\ell_{P_{\perp h}} = \ell_{P_{\perp}}$, not containing $\ell_P$ and such that $\Gamma' \cap \Sigma$ is an $(r-4)$-space of $\Sigma$. Then $\Gamma' \cap Q(r, q)$ is a parabolic polar space $Q(r-4, q)$ embedded in $Q(r, q)$ and $\ell_{P_{\perp}} \cap Q(r, q)$ is a cone having as vertex the line $\ell_P$ and as basis $Q(r-4, q)$. On the other hand $\Gamma' \cap \mathcal{H}(r, q^2)$ is a Hermitian polar space $\mathcal{H}(r-4, q^2)$ embedded in $\mathcal{H}(r, q^2)$ and $\ell_{P_{\perp}} \cap \mathcal{H}(r, q^2)$ is a cone having as vertex the line $\ell_P$ and as basis $\mathcal{H}(r-4, q^2)$, where $Q(r-4, q) \subset \mathcal{H}(r-4, q^2)$. Note that if a line of $\mathcal{H}(r, q^2)$ through $P$ intersect $Q(r, q)$ in one point, then such a point lies on $(\ell_{P_{\perp}} \cap \Sigma) \setminus \ell_P$. Let $G''$ be the group of projectivities of $\Gamma'$ isomorphic to PGO$(r-3, q)$ stabilizing the parabolic polar space $Q(r-4, q)$. From Proposition 4.1, the group $G''$ has four orbits on points of $\mathcal{H}(r-4, q^2)$, say $Q(r-4, q), O'', E''$ and $S''$, where $Q(r-4, q) \subset Q(r, q)$, $O'' \subset O$, $E'' \subset E$, $S'' \subset S$.

The hyperplane $P_{\perp h}$ meets $\mathcal{H}(r, q^2)$ in a cone having as vertex the point $P$ and as basis $\mathcal{H}(r-4, q^2)$. In particular, $\mathcal{H}(r-2, q^2)$ can be chosen in such a way that $Q(r-4, q) \subseteq \mathcal{H}(r-4, q^2) \subseteq \mathcal{H}(r-2, q^2)$ and that the point $T := \mathcal{H}(r-2, q^2) \cap \ell_P$ belongs to $\Sigma$. Let $\Lambda$
be the \((r-2)\)-space of \(\text{PG}(r, q^2)\) containing \(\mathcal{H}(r-2, q^2)\), then \(\Lambda \cap \Sigma\) is the \((r-3)\)-space of \(\Sigma\) containing \(T\) and \(Q(r-4, q)\). Hence such an \((r-3)\)-space, when extended over \(\text{GF}(q^2)\) coincides with \(\langle T, \Gamma' \rangle\), which meets \(\mathcal{H}(r-2, q^2)\) in the cone having as vertex the point \(T\) and as basis \(\mathcal{H}(r-4, q^2)\). It follows that \(\mathcal{H}(r-2, q^2) \setminus \langle T, \Gamma' \rangle\) consists of \(q^{2r-5}\) points.

Let \(R\) be a point of \(\mathcal{H}(r-2, q^2)\) and let \(g\) be the line of \(\mathcal{H}(r, q^2)\) joining \(P\) and \(R\). If \(R = T\), then \(g = \ell_P\) and \(\ell_P\) is the unique line of type (i) passing through \(P\). Assume that \(R \neq T\). If \(R\) lies on a line joining \(T\) with a point of \(Q(r-4, q)\), then \(g\) is a line of type \(iv\). If \(R\) lies on line joining \(T\) with a point of \(\mathcal{O}''\), then the line \(\ell_R\) is an extended line of a parabolic polar space \(Q(r-4, q)\), \(r \geq 8\), embedded in \(\ell_P \cap \Sigma\) and disjoint from \(\ell_P\). It follows that \(\langle \ell_P, \ell_R \rangle\) is a solid \(\sigma'\) of \(\mathcal{H}(r, q^2)\), \(r \geq 8\), such that \(\sigma' \cap \Sigma\) is a Baer subgeometry isomorphic to \(\text{PG}(3, q)\) and hence \(g\) is a line of type \(v\). If \(R\) lies on a line joining \(T\) with a point of \(\mathcal{E}''\), then \(\ell_R\) is an extended line of \(\ell_P \cap \Sigma\) such that \(\ell_R \cap \Sigma\) is external to \(Q(r, q)\). Since the plane \(\langle P, \ell_R \rangle\) meets \(\Sigma\) in a Baer subplane which in turn shares with \(Q(r, q)\) solely the point \(P\), it follows that \(g\) is a line of type \(vi\). Analogously, if \(R\) lies on a line joining \(T\) with a point of \(\mathcal{S}''\), we have that \(g\) is a line of type \(vii\). Finally, let \(R\) be a point of \(\mathcal{H}(r-2, q^2) \setminus \langle T, \Gamma' \rangle\). Then the line joining \(R\) and \(\tau(R)\) is disjoint from \(\langle T, \Gamma' \rangle\) and hence the solid generated by \(\ell_P\) and the line \(\ell_R\) meets \(Q(r, q)\) in a hyperbolic quadric \(Q^+(3, q)\). Hence \(g\) is a line of type \(viii\).

\[
P \in \mathcal{E}
\]

If \(P \in \mathcal{E}\), then \(|\ell_P \cap Q(r, q)| = 0, \ell_P^\perp \cap Q(r, q) = Q(r-2, q), \ell_P^\perp \cap \mathcal{H}(r, q^2) = \mathcal{H}(r-2, q^2)\), where \(Q(r-2, q) \subset \mathcal{H}(r-2, q^2)\). Moreover \(P^\perp \cap \mathcal{H}(r, q^2)\) in a cone having as vertex the point \(P\) and as basis \(\mathcal{H}(r-2, q^2)\). Let \(G'\) be the group of projectivities of \(\ell_P^\perp \cap \Sigma\) isomorphic to \(\text{PGO}(r-1, q)\) stabilizing the parabolic polar space \(Q(r-2, q)\). From Proposition 4.1, the group \(G'\) has four orbits on points of \(\mathcal{H}(r-2, q^2)\), say \(Q(r-2, q), \mathcal{O}', \mathcal{E}'\) and \(\mathcal{S}'\), where \(Q(r-2, q) \subset Q(r, q), \mathcal{O}' \subset \mathcal{O}, \mathcal{E}' \subset \mathcal{E}, \mathcal{S}' \subset \mathcal{S}\). Let \(R\) be a point of \(\mathcal{H}(r-2, q^2)\) and let \(g\) be the line of \(\mathcal{H}(r, q^2)\) joining \(P\) and \(R\). If \(R \in Q(r-2, q)\), then \(g\) is a line of type \(ii\).

If \(R \in \mathcal{O}'\), then the line \(\ell_R\) is an extended line of \(Q(r-2, q)\). It follows that \(\langle \ell_R, g \rangle\) is a plane \(\sigma\) of \(\mathcal{H}(r, q^2)\) such that \(\sigma \cap \Sigma = \ell_R\) and hence \(g\) is a line of type \(vi\). If \(R \in \mathcal{E}'\), then \(\ell_R\) is an extended line of \(\Sigma\) such that \(\ell_R \cap \Sigma\) is external to \(Q(r, q)\). Let \(\Lambda\) be the solid generated by \(\ell_P\) and \(\ell_R\). Since \(\ell_P^\perp \cap \Lambda = \ell_R\) and \(\ell_R^\perp \cap \Lambda = \ell_P\), we have that \(\Lambda \cap Q(r, q) = Q^+(3, q)\) and \(g\) is a line of type \(viii\). Analogously, if \(R \in \mathcal{S}'\), we have that \(\Lambda \cap Q(r, q) = Q^-(3, q)\) and \(g\) is a line of type \(ix\). A similar argument holds if \(P \in \mathcal{S}\).

\[\square\]

**Theorem 4.5** Each of the four \(G\)-orbits \(Q(r, q), \mathcal{O}, \mathcal{E}, \mathcal{S}\) is an \(i\)-tight set of \(\mathcal{H}(r, q^2)\) with parameter \(i = q + 1, q^{r-1} - q, (q^{r+1} - q^{r-1})/2, (q^{r+1} - q^{r-1})/2\), respectively.

**Proof** Let \(P\) be a point of \(\mathcal{H}(r, q^2)\). Taking into account Lemmas 4.3 and 4.4, we have that

\[
|P^\perp \cap Q(r, q)| = \begin{cases} 
q |Q(r-2, q)| + 1 = (q + 1)q^{r-4} \frac{q^2 - 4}{q^2 - 1}q^2 + q^2 & \text{if } P \in Q(r, q), \\
q + 1 + q^2 \frac{(q^{r-4} - 1)^2}{q^2 - 1} = \left(\frac{q^{r-2} - 1}{q^2 - 1}\right)^2 = (q + 1)q^{r-4}q^2 & \text{if } P \in \mathcal{O}, \\
|Q(r-2, q)| = (q + 1)q^{r-4}q^2 & \text{if } P \in \mathcal{E} \cup \mathcal{S}.
\end{cases}
\]
An $m$-cover of lines of $Q(r, q)$ is a set of lines $\mathcal{L}$ of $Q(r, q)$ such that every point of $Q(r, q)$ contains $m$ lines of $\mathcal{L}$. In this section we investigate properties of $m$-covers of $Q(r, q)$. Similar arguments to that used in the proof of Lemma 2.1 yield the following.

Lemma 4.6 Let $\mathcal{L}$ be an $m$-cover of $Q(r, q)$, $r \geq 4$ even, then

(i) $\mathcal{L}$ contains $m\theta_{r-1,q}/(q + 1)$ lines,

(ii) a hyperplane $H$ of $PG(r, q)$ contains $m\theta_{r-3,q}/(q + 1)$ lines of $\mathcal{L}$, if $H$ is tangent, $m(q^2 - 1)/(q + 1)$, if $H \cap Q(r, q) = Q^+(r - 1, q)$, $m(q^2 + 1)/(q^2 - 1)$, if $H \cap Q(r, q) = Q^-(r - 1, q)$,

(iii) let $\Sigma$ be an $(r - 2)$-space of $PG(r, q)$, where $\Sigma = \ell^\perp$ and $|\ell \cap Q(r, q)| \in \{0, 2, q + 1\}$. If $x$ is the number of lines of $\mathcal{L}$ contained in $\Sigma$ and $y$ is the number of lines of $\mathcal{L}$ meeting $\Sigma$ in one point, then $x(q + 1) + y = m\theta_{r-3,q}$.

We introduce the following definitions.

Definition 4.7 An $m$-cover $\mathcal{L}$ of $Q(r, q)$ is said to be $(r - 2)$-dual of parabolic type I if

$$[[r \in \mathcal{L} : r \subseteq \ell^\perp]] = \begin{cases} \frac{m \theta_{r-5,q}}{q + 1} + qr^{-4} & \text{if } \ell \in \mathcal{L}, \\ \frac{m \theta_{r-5,q}}{q + 1} & \text{if } \ell \notin \mathcal{L} \text{ and } |\ell \cap Q(r, q)| \in \{0, 2, q + 1\}. \end{cases}$$
Definition 4.8  An $m$-cover $L$ of $Q(r, q)$ is said to be $(r - 2)$-dual of parabolic type II if

$$|\{r \in L : r \subseteq \ell \}^\perp| = \begin{cases} \frac{m \theta_{r-5,q}}{q+1} + q^{r-4} & \text{if } \ell \in L, \\ \frac{m \theta_{r-5,q}}{q+1} & \text{if } \ell \not\in L \text{ and } |\ell \cap Q(r, q)| \in \{0, q+1\}. \end{cases}$$

Basing on the definitions above we have the following results.

**Theorem 4.9** Let $L$ be an $m$-cover of $Q(r, q)$ and let $\tilde{L}$ be the set of points of $H(r, q^2)$ lying on the extended lines of $L$. Then $\tilde{L}$ is an $(m(q^2 - q) + q + 1)$-tight set of $H(r, q^2)$ if and only if $L$ is an $(r - 2)$-dual cover of parabolic type I.

**Proof** Assume that $\tilde{L}$ is an $(m(q^2 - q) + q + 1)$-tight set of $H(r, q^2)$. By construction, $\tilde{L}$ is contained in $Q(r, q) \cup \mathcal{O}$. Hence, if $P \in \tilde{L} \setminus Q(r, q)$ then $P^{\perp h} \cap Q(r, q) = \ell^\perp_P$, where $\ell_P$ is the unique extended line of $L$ through $P$. If $P \in \mathcal{O}$, then

$$|P^{\perp h} \cap \tilde{L}| = (m(q^2 - q) + q + 1)\theta_{r-4,q^2} + q^{r-2} = x(q^2 - q) + \theta_{r-3,q} + |L| - x - y,$$

where $x$ is the the number of extended lines of $L$ contained in $P^{\perp h}$ (hence contained in $\ell^\perp_P$) and $y$ is the number of extended lines of $L$ meeting $P^{\perp h}$ in a point of $Q(r, q)$. Then, taking into account Lemma 4.6(iii), we have

$$xq^2 = m \frac{q^{r-2} - q^2}{q^2 - 1} + q^{r-2},$$

and hence $x = \frac{m \theta_{r-5,q}}{q+1} + q^{r-4}$. If $P \in H(r, q^2) \setminus \tilde{L}$, then

$$|P^{\perp h} \cap \tilde{L}| = (m(q^2 - q) + q + 1)\theta_{r-4,q^2} = x(q^2 - q) + \theta_{r-3,q} + |L| - x - y,$$

where again $x$ is the the number of extended lines of $L$ contained in $P^{\perp h}$ (hence contained in $\ell^\perp_P$) and $y$ is the number of extended lines of $L$ meeting $P^{\perp h}$ in a point of $Q(r, q)$. In this case $x = \frac{m \theta_{r-5,q}}{q+1}$.

Viceversa, assume that $L$ is an $(r - 2)$-dual cover of parabolic type I. If $P \in Q(r, q)$, then

$$|P^\perp \cap \tilde{L}| = |P^{\perp h} \cap \tilde{L}| = m(q^2 - q)\frac{\theta_{r-3,q}}{q+1} + \theta_{r-2,q} = (m(q^2 - q) + q + 1)\theta_{r-4,q^2} + q^{r-2}.$$ 

If $P \in H(r, q^2) \setminus Q(r, q)$ then $P^\perp = \tau(P)^{\perp h}$, where $(P, \tau(P))$ is the unique extended line of $\Sigma$ through $P$. Hence $\tau(P) \in \tilde{L}$ if and only if $P \in \tilde{L}$. Since

$$P^{\perp h} \cap \tilde{L} = x(q^2 - q) + \theta_{r-3,q} + |L| - x - y,$$

where $x$ is the number of lines of $L$ contained in $P^{\perp h}$ (hence contained in $\ell^\perp_P$) and $y$ is the number of lines of $L$ meeting $P^{\perp h}$ in a point of $Q(r, q)$, it follows that $|P^{\perp h} \cap \tilde{L}|$ is either

$$(m(q^2 - q) + q + 1)\theta_{r-4,q^2} + q^{r-2},$$

or

$$(m(q^2 - q) + q + 1)\theta_{r-4,q^2},$$

according as $P$ lies or does not lie on $\tilde{L}$. Therefore $\tilde{L}$ is a $(m(q^2 - q) + q + 1)$-tight set of $H(r, q^2)$.

The proof of the following result is similar to that given for Theorem 4.9 and hence we omit it.
Theorem 4.10 Let $L$ be an $m$-cover of $Q(r, q)$ and let $\tilde{L}$ be the set of points of $Q(r, q^2)$ lying on the extended lines of $L$. Then $\tilde{L}$ is an $(mq^2 - q + q + 1)$-tight set of $Q(r, q^2)$ if and only if $L$ is an $(r - 2)$-dual cover of parabolic type $II$.

It is straightforward to remark that an $(r - 2)$-dual cover of parabolic type $I$ is an $(r - 2)$-dual cover of parabolic type $II$ and if $L$ consists of all the lines of $Q(r, q)$, then $L$ is an $(r - 2)$-dual $\theta_{r-3,q}$-cover of parabolic type $I$. A more interesting example is the following.

Let $PGO(7, q)$ be the projective orthogonal group of $Q(6, q)$. The Cartan Dickson Chevalley exceptional group $G_2(q)$ is a subgroup of $PGO(7, q)$. It occurs as the stabilizer in $PGO(7, q)$ of a configuration of points, lines and planes of $Q(6, q)$. The group $G_2(q)$ is also contained in the automorphism group of a classical generalized hexagon, called the split Cayley hexagon and denoted by $H(q)$, see [12]. The points of $H(q)$ are all the points of $Q(6, q)$ and the lines of $H(q)$ are certain lines of $Q(6, q)$. The number of points of $Q(6, q)$ is $(q^6 - 1)/(q - 1)$, and this is also the number of lines and planes of $Q(6, q)$ involved in $H(q)$. The generalized hexagon $H(q)$ has the following properties [12, p. 33]:

- through any point of $H(q)$ there pass $q + 1$ lines of $H(q)$, and any line of $H(q)$ contains $q + 1$ points of $H(q)$;
- the $q + 1$ lines of $H(q)$ through a point of $H(q)$ are contained in a unique plane of $H(q)$;
- a plane of $Q(6, q)$ either contains $q + 1$ lines of $H(q)$ and in this case it is a plane of $H(q)$ or it contains no line of $H(q)$;
- through a line of $H(q)$, there pass $q + 1$ planes of $H(q)$, whereas through a line of $Q(6, q)$ that is not a line of $H(q)$, there is exactly one plane of $H(q)$;
- an elliptic quadric $Q^-(5, q)$ embedded in $Q(6, q)$ contains exactly $q^3 + 1$ pairwise disjoint lines of $H(q)$.

Proposition 4.11 The set of lines of $H(q)$ is a 3-dual cover of parabolic type $I$.

Proof Let $\ell$ be a line of $Q(6, q)$. Then $\ell^\perp$ contains the $q + 1$ planes of $Q(6, q)$ through $\ell$. If $\ell$ is a line of $H(q)$, then every plane through $\ell$ contains $q$ lines of $H(q)$ distinct from $\ell$. If $\ell$ is a line of $Q(6, q)$ not of $H(q)$, then through $\ell$ there is a unique plane of $H(q)$. Hence in the former case $\ell^\perp$ contains $q^2 + q + 1$ lines of $H(q)$, while in the latter case $\ell^\perp$ contains $q + 1$ lines of $H(q)$. Let $\ell$ be a line of $PG(6, q)$ such that $|\ell \cap Q(6, q)| \not\in \{0, 2\}$. Then $\ell^\perp$ meets $Q(6, q)$ in a parabolic quadric $Q(4, q)$. Let $Q^-(5, q)$ be an elliptic quadric contained in $Q(6, q)$ and containing $Q(4, q)$. Since the $q^3 + 1$ lines of $H(q)$ contained in $Q^-(5, q)$ are pairwise disjoint, they give rise to a line-spread of $Q^-(5, q)$ and hence to an ovoid of the dual of $Q^-(5, q)$. On the other hand, the set of lines of $Q(4, q)$ corresponds to a $(q + 1)$-tight set of the dual of $Q^-(5, q)$. It follows that there are exactly $q + 1$ lines of $H(q)$ contained in $\ell^\perp$.

Remark 4.12 The fact that $Q(r, q)$ is a $(q + 1)$-tight set of $H(r, q^2)$ has been already proved in [11, Theorem 4.1]. In the same paper, the authors conjecture that every $(q + 1)$-tight set of $H(4, q^2)$ that is not the union of $q + 1$ lines is the set of points of an embedded $W(3, q)$, or, when $q$ is odd, the set of points of an embedded $W(3, q)$ or $Q(4, q)$. With the aid of MAGMA [2] we were able to construct counterexamples to this conjecture when $q = 2, 3$. Assume that $H(4, q^2)$ has equation $X_{4}^{q+1} + X_{3}^{q+1} + X_{2}^{q+1} + X_{1}^{q+1} = 0$. The point set $S := \{U_1, \ldots, U_5\}$, where $U_i$ is the point of $PG(4, q^2)$ having 1 at the $i$-th position and 0 elsewhere forms a self-polar simplex with respect to $H(4, q^2)$. Then, the 10 lines joining
two points of $S$ are secant lines of $\mathcal{H}(4, q^2)$. Let $G = (C_{q+1})^4 \rtimes \text{Sym}_5$ be the stabilizer of $S$ in $\text{PGU}(5, q^2)$.

$q = 2$

In this case $|G| = 4860$ and $G$ has a unique subgroup, say $H$ of index 6. The group $H$ acts transitively on $S$ and has two orbits $\mathcal{L}_i$, $i = 1, 2$, of size 5 on the 10 secant lines joining two points of $S$:

$$\mathcal{L}_1 := \{U_1U_4, U_1U_5, U_3U_5, U_2U_4, U_2U_3\}$$

$$\mathcal{L}_2 := \{U_4U_5, U_3U_4, U_1U_3, U_2U_5, U_1U_2\}.$$

Each line of $\mathcal{L}_i$ meets $\mathcal{H}(4, 4)$ in 3 points giving rise to a subset of 15 points that is a transitive 3-tight set that is neither union of generators nor a subquadrangle of order 2.

$q = 3$

In this case each of the 10 lines joining two points of $S$ meets $\mathcal{H}(4, 9)$ in 4 points giving rise to a subset of 40 points forming a transitive 4-tight set that is neither union of generators nor a subquadrangle of order 3.

Similarly, using the stabilizer of a self-polar simplex in $\text{PGU}(7, 4)$ we found a 3-tight set of $\mathcal{H}(6, 4)$ consisting of the 63 points on the 21 secant lines joining two points of the simplex.

References

1. Bamberg J., Kelly S., Law M., Penttila T.: Tight sets and $m$-ovoids of finite polar spaces. J. Comb. Theory Ser. A 114(7), 1293–1314 (2007).
2. Bosma W., Cannon J., Playoust C.: The Magma algebra system. I. The user language. J. Symb. Comput. 24, 235–265 (1997).
3. Bruen A., Fisher J.C.: Spreads which are not dual spreads. Can. Math. Bull. 12(6), 801–803 (1969).
4. Buekenhout F.: More geometry for Hering’s $3^6$ : SL(2, 13). In: Advances in Finite Geometries and Designs (Chelwood Gate, 1990), pp. 57–68. Oxford Science Publications, Oxford University Press, New York (1991).
5. Cossidente A.: Some constructions on the Hermitian surface. Des. Codes Cryptogr. 51(2), 123–129 (2009).
6. Cossidente A., King O.H.: On some maximal subgroups of unitary groups. Commun. Algebra 32(3), 989–995 (2004).
7. Cossidente A., Pavese F.: On intriguing sets of finite symplectic spaces. Des. Codes Cryptogr. 86(5), 1161–1174 (2018).
8. Cossidente A., Penttila T.: Hemisystems on the Hermitian surface. J. Lond. Math. Soc. (2) 72(3), 731–741 (2005).
9. Cossidente A., Durante N., Marino G., Penttila T., Siciliano A.: The geometry of some two-character sets. Des. Codes Cryptogr. 46(2), 231–241 (2008).
10. Cossidente A., Marino G., Penttila T.: The action of the group $G_2(q) < \text{PSU}(6, q^2)$, $q$ even, and related combinatorial structures. J. Comb. Des. 21(2), 81–88 (2013).
11. De Beule J., Metsch K.: On the smallest non-trivial tight sets in Hermitian polar spaces. Electron. J. Comb. 24(1), 1–62 (2017).
12. De Wispelaere A.: Ovoids and spreads of finite classical generalized hexagons and applications. PhD thesis, Gent University (2005).
13. Drudge K.: Extremal sets in projective and polar spaces. PhD thesis, The University of Western Ontario (1998).
14. Drudge K.: On the orbits of Singer groups and their subgroups. Electron. J. Comb. 9(1) (2002).
15. Dye R.H.: Partitions and their stabilizers for line complexes and quadrics. Ann. Mat. Pura Appl. 114(4), 173–194 (1977).
16. Dye R.H.: Maximal subgroups of symplectic groups stabilizing spreads. J. Algebra 87, 493–509 (1984).
17. Hirschfeld J.W.P.: Finite Projective Spaces of Three Dimensions. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1985).
2002 A. Cossidente, F. Pavese

18. Hirschfeld J.W.P.: Projective Geometries over Finite Fields. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1998).
19. Huppert B.: Endliche Gruppen, I, Die Grundlehren der Mathematischen Wissenschaften, vol. 134. Springer, Berlin (1967).
20. Kleidman P., Liebeck M.: The Subgroup Structure of the Finite Classical Groups, vol. 129. London Mathematical Society Lecture Note SeriesCambridge University Press, Cambridge (1990).
21. Lane-Harvard E.: New constructions of strongly regular graphs. Ph.D. thesis, Department of Mathematics, Colorado State University, Fort Collins, Colorado (2014).
22. Mellinger K.E.: Classical mixed partitions. Discret. Math. 283(1–3), 267–271 (2004).
23. Pavese F.: Geometric constructions of two-character sets. Discret. Math. 338(3), 202–208 (2015).
24. Payne S.E.: Tight pointsets in finite generalized quadrangles. In: Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, FL, 1987, Congr. Numer. vol. 60, pp. 243–260 (1987).
25. Sved M.: Baer subspaces in the $n$-dimensional projective space. In: Combinatorial Mathematics, X (Adelaide, 1982). Lecture Notes in Mathematics, vol. 1036, pp. 375–391. Springer, Berlin (1983).

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.