Global structure and thermodynamic property of the 4-dimensional twisted Kerr solution

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Abstract

Rotating stringy black hole solutions with non-vanishing dilaton $\phi$, antisymmetric tensor $B_{\mu\nu}$, and $U(1)$ gauge field $A_\mu$ are investigated. Both Boyer-Lindquist-like and Kerr-Schild-like coordinate are constructed. The latter is utilised to construct the analytically extended spacetime. The global structure of the resulting extended spacetime is almost identical to that of the Kerr. In carrying out the analytic extension, the radial coordinate should be suitably chosen so that we can avoid singularity caused by the twisting. The thermodynamic property of the stringy black hole is examined through the injection of test bodies into the black hole. It is shown that one cannot change a black hole configuration into a naked singularity by way of throwing test bodies into the black hole. The global $O(2,3)$ symmetry and the preservation of the asymptotic flatness are discussed. When we impose stationarity, axisymmetry, and asymptotic flatness, there is no other twisting than the one pointed out by A.Sen[12]. All the other elements of $O(2,3)$ either break the asymptotic flatness, or cause only coordinate transformations and gives no physical change.
1 Introduction

Exact solutions in General Relativity have given us simplified, but fairly qualitative features of spacetimes. Theoretical developments of black holes owe exact solutions for their establishment. Solving the Einstein equation exactly, however, is very difficult and only when we impose some symmetry, can we solve it [1]. One of the largest symmetry we often consider is to impose staticity and spherical symmetry. Then the metric and matter fields become functions of one variable of the radial coordinate and the equations of motion become a suit of ordinary differential equations. Some can be solved analytically (and some numerically) to get physically interesting solutions. When one loosens the spherical symmetry down to axisymmetry, the difficulty increases a great deal and direct quadrature is impossible in general. Several remarkable techniques (which include the introduction of potentials [2], superposition of known solutions [3], or transformations of known solutions to new ones [4]) have been devised to circumvent this difficulty. These techniques are unquestionably important for solving the Einstein equation.

On the other hand, the intensive research of string theories in 1980’s [5] has enriched the models including gravity [6]. Characteristic feature of these models, inspired by string theory or Kaluza-Klein theory, is inclusion of the dilaton. The investigation has revealed that these models allow black hole solutions [7][8][9][10] and that there exist transformations [11][12] that generate new solutions from old ones in these models, just as well as one can obtain charged solutions from uncharged ones through the Harrison transformation in Einstein-Maxwell theory [13]. We know from this that the low energy heterotic string theory possesses a global $O(d, d + p)$ symmetry when the spacetime admits $d$ Killing vectors and the theory contains $p U(1)$ gauge fields [12].

This paper is motivated 1) to clarify the global structure and thermodynamic property of the rotating charged black hole solution [12] inspired by the heterotic string theory, and 2) to examine the possibility of producing other black hole solutions by $O(2, 3)$ $(d = 2$ and $p = 1$ of $(d, d + p))$ transformation. As we will show in section 4, analytic extension of the twisted Kerr solution is obtained by introducing a new radial coordinate $R := \sqrt{r^2 + m(\cosh \alpha - 1)r}$, where $\alpha$ is a twist parameter. The global structure and thermodynamic property are almost identical to those of the Kerr. We will show in section 6 that under fairly general conditions, there is only one possibility of the twisting that preserves the asymptotic flatness. That is, $\Omega_{35} := \exp(\alpha(e_{35} + e_{53})) \in O(2, 3)$ is the only one, which was discussed by A.Sen [12].

The organisation of this article is the following. In section 2, we review the result of A.Sen
et al.\cite{12} that the action (2.1) allows the global $O(2,3)$ symmetry when one assumes that the spacetime is stationary and axisymmetric. (We consider only the case of $d = 2$ and $p = 1$.) The twisted Kerr solution is obtained by the action of $\Omega_{35}$ on the Kerr solution.

In section 3, we construct a Boyer-Lindquist-like coordinate and consider the geodesic equations on the twisted Kerr metric background.

A Kerr-Schild-like coordinate is constructed and the global structure of the spacetime is obtained from the analytically extended solution. This is written in section 4.

Section 5 treats the gedanken experiments of throwing test particles\cite{14} into the twisted Kerr solution. We will show that one cannot decrease the area of the horizon by any means of throwing test bodies. Moreover, the condition in which test bodies fall inside the horizon exactly reproduces the thermal structure of the twisted Kerr solution\cite{15}\cite{16}.

In section 6, we discuss the global $O(2,3)$ transformation in view of the asymptotic flatness of the spacetime. Starting with a known solution, we can obtain several solutions by acting elements of $O(2,3)$. However, some elements of $O(2,3)$ break the asymptotic flatness. We will investigate the infinitesimal action of $o(2,3)$ and show that $\Omega_{35}$ is the only transformation that preserves the asymptotic flatness.

Section 7 is devoted to summary and conclusion.

\section{Global $O(2,3)$ symmetry}

In this section, we review the construction of a rotating charged stringy black hole solution by “twisting” the Kerr solution. We consider the following action, which includes the metric $g_{\mu\nu}$, $U(1)$--gauge field $A_\mu$, the dilaton field $\phi$, and the antisymmetric tensor field $B_{\mu\nu}$:

$$S = \int d^4x \sqrt{-g} \left( R - 2(\partial \phi)^2 - e^{-2\phi} F^2 - \frac{1}{12} e^{-4\phi} H^2 \right),$$

(2.1)

where we put

$$H_{\alpha\beta\gamma} := \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta} - 2(A_\alpha F_{\beta\gamma} + A_\beta F_{\gamma\alpha} + A_\gamma F_{\alpha\beta}).$$

It is pointed out by A.Sen et al. that if there are $d$ independent Killing vector fields, the action (2.1) has global $O(d, d + 1)$ symmetry\cite{12}.

To make this symmetry manifest, we temporarily redefine the metric $g_{\mu\nu}$ of eq.(2.1) by $e^{2\phi} g_{\mu\nu}$, and suitably rescale $\phi$ and $A_\mu$ by some constant factors. Then the action (2.1) is equivalent to

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left( R + (\partial \phi)^2 - \frac{1}{8} F^2 - \frac{1}{12} H^2 \right).$$

(2.2)
The $H_{\alpha\beta\gamma}$ is now defined by

$$H_{\alpha\beta\gamma} := \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta} - \frac{1}{4} (A_\alpha F_{\beta\gamma} + A_\beta F_{\gamma\alpha} + A_\gamma F_{\alpha\beta}).$$

When one treats stationary and axisymmetric solutions, which we will do throughout this paper, they have global $O(2,3)$ symmetry. We will use eq. (2.2) only for the purpose of showing the global $O(2,3)$ symmetry. All physical discussions will be made from the viewpoint of the action (2.1) throughout this paper.

Let us first recall the $O(2,3)$ symmetry that generates new solutions from a known one. We assume that the system be a stationary and axisymmetric rotating body. That is, the system has a timelike (at least outside the horizon) coordinate $t$ and spacelike periodic coordinate $\varphi$ such that $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$ are Killing vectors. The “rotating body” means the system is invariant under the simultaneous inversion of the signs of $(t, \varphi)$. (see, for example, ref. [17].)

The metric is then written as

$$g_{\mu\nu} = \left( \begin{array}{cc} \hat{g}_{mn} & 0 \\ 0 & \tilde{g}_{\alpha\beta} \end{array} \right),$$

where both $\hat{g}_{mn}$ and $\tilde{g}_{\alpha\beta}$ are functions of $(r, \theta)$ and independent of $(t, \varphi)$. They are respectively 2 dimensional metrics of subspaces spanned by $(t, \varphi)$ and $(r, \theta)$. Hereafter we will use the symbols of $\hat{\ }$ and $\tilde{\ }$ as the $(t, \varphi)-$ and $(r, \theta)-$ components of tensor fields. Respecting the symmetry of the system, the configuration of the other fields is in the following form:

$$\phi = \phi(r, \theta), \quad B_{\mu\nu} = \left( \begin{array}{cc} \hat{B}_{mn} & 0 \\ 0 & \tilde{B}_{\alpha\beta} \end{array} \right), \quad A_\mu = \left( \begin{array}{c} \hat{A}_m \\ \tilde{A}_\alpha \end{array} \right).$$

Again, these are functions of $(r, \theta)$ only and independent of $(t, \varphi)$.

Some calculations are needed to see the manifest global $O(2,3)$ symmetry. They are all results of straightforward calculations. We show them here in the form of a proposition:

**Proposition (A.Sen [12])**

The identity 1), 2), 3), and 4) hold.

1) $e^{-\phi} \sqrt{-g} (R + (\partial_\mu \phi)(\partial_\nu \phi) g^{\mu\nu})$

$$= e^{-\phi} \sqrt{-g} (R^{(2)} + \frac{1}{4} \text{tr} (\partial_\alpha \hat{g} \cdot \partial_\beta \hat{g}^{-1}) \hat{g}^{\alpha\beta} + \partial_\alpha (\phi - \log \sqrt{-\hat{g}}) \cdot \partial_\beta (\phi - \log \sqrt{-\hat{g}}) \hat{g}^{\alpha\beta})$$

+ total divergence,

where $R^{(2)}$ is the scalar curvature of the 2 dimensional subspace spanned by $(r, \theta)$, and the trace is taken over the $(t, \varphi)-$ space.

2) $\sqrt{-g} e^{-\phi} = \sqrt{-\hat{g}} \sqrt{g} e^{-\phi} = \sqrt{g} e^{-\chi}$, where we put $\chi := \phi - \log \sqrt{-\hat{g}}$. 

3
Here we denote the transposition of a matrix $M$ by $M'$ for simplicity. We have put the matrix $H_\alpha$ as $(H_\alpha)_{mn} := H_{amn}$. The traces are again taken over the $(t, \varphi)$-space.

where we defined $\mathcal{M}$, $k$, $\eta$, and $L$ by

$$
\mathcal{M} := \begin{pmatrix} k' - \eta & \hat{g}^{-1} (k - \eta) & k + \eta & -\hat{A} \\
\hat{A}' & -1 & n' & c \\
-\hat{A}' & n' & c & e \\
\eta & u' & e' & f \end{pmatrix} \in M_5(\mathbb{R}),
$$

$$
k := -\hat{B} - \hat{g} - \frac{1}{4} \hat{A} \hat{A}', \quad \eta := \begin{pmatrix} -1 & 0 \\
0 & 1 \end{pmatrix}, \quad L := \begin{pmatrix} 1 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1 \end{pmatrix}.
$$

Using this proposition, the action (2.2) is now rewritten as

$$
S = \int dt d\varphi \int dr d\theta \sqrt{\hat{g}} e^{-\chi} \left[ R^{(2)} + \partial_\alpha \chi \partial_\beta \chi \hat{g}^{\alpha \beta} - \frac{1}{8} \hat{F}^2 + \frac{1}{32} \text{tr}(\partial_\alpha \mathcal{M} \partial_\beta \mathcal{M} \mathcal{L}) \hat{g}^{\alpha \beta} \right] + \text{total divergence.}
$$

(2.3)

Note that the matrix-valued field $\mathcal{M}$ is constructed from $(\hat{g}_{mn}, \hat{B}_{mn}, \hat{A}_m)$. The rewritten action (2.3) is considered as a theory with fields $(\hat{g}_{\alpha \beta}, \chi, \hat{A}_\alpha, \mathcal{M})$ defined on the 2 dimensional space $(r, \theta)$. Eq.(2.3) clearly possesses the global $O(2,3)$ symmetry

$$
\mathcal{M} \mapsto \Omega \mathcal{M} \Omega', \quad \chi \mapsto \chi \quad \text{(i.e., } \exp[2\Omega(\phi)] = e^{2\phi} \det(\Omega(\hat{g})) / \det(\hat{g}) \text{)}
$$

with $\Omega$ satisfying the condition $\Omega' \mathcal{L} \Omega = \mathcal{L}$. The interpretation between $(\hat{g}_{ab}, \hat{A}_a, \hat{B}_{ab})$ and the matrix-valued field $\mathcal{M}$ is given by

$$
\hat{g}^{-1} = \frac{1}{4} \eta (l - n - n' + c) \eta, \quad k' = 2(-l + n)(l - n - n' + c)^{-1} \eta + \eta, \quad \hat{A} = 2\eta(-l + n)^{-1} u,
$$

(2.4)

and its inverse is

$$
l = (k' - \eta) \hat{g}^{-1} (k - \eta), \quad n = (k' - \eta) \hat{g}^{-1} (k + \eta),
$$

$$
c = (k' + \eta) \hat{g}^{-1} (k + \eta), \quad u = (k' - \eta) \hat{g}^{-1} (-\hat{A}),
$$

$$
e = (k' + \eta) \hat{g}^{-1} (-\hat{A}), \quad f = \hat{A}' \hat{g}^{-1} \hat{A}.
$$

(2.5)
One immediately recognises that not all the elements of $O(2,3)$ preserve the asymptotic flatness even when one starts with an asymptotically flat seed metric. We will consider this problem in section 6. We act

$$\Omega_{35} := \exp(\alpha(e_{35} + e_{53})) = \begin{pmatrix} 1 & 1 & \cosh \alpha & \sinh \alpha \\ 1 & \cosh \alpha & \sinh \alpha & 1 \\ 1 & \sinh \alpha & \cosh \alpha \end{pmatrix} \in O(2,3)$$

on the (untwisted) Kerr solution to obtain the twisted Kerr solution. The untwisted Kerr solution is

$$ds^2 = -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}) dt^2 - \frac{2mra \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dtd\varphi$$

$$+ \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} [(r^2 + a^2)^2 - (r^2 - 2mr + a^2) a^2 \sin^2 \theta] d\varphi^2$$

$$+ \frac{r^2 + a^2 \cos^2 \theta + \beta r}{r^2 + a^2 - 2mr} dr^2 + (r^2 + a^2 \cos^2 \theta + \beta r) d\theta^2,$$

$$B = A = \phi \equiv 0.$$

The result of the action of $\Omega_{35}$ is given by

$$ds^2 = -\frac{r^2 + a^2 \cos^2 \theta - 2mr}{r^2 + a^2 \cos^2 \theta + \beta r} dt^2 - \frac{2(2m + \beta) r a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta + \beta r} dtd\varphi$$

$$+ \frac{(r^2 + \beta r + a^2)^2 - (r^2 - 2mr + a^2) a^2 \sin^2 \theta}{r^2 + \beta r + a^2 \cos^2 \theta} d\varphi^2$$

$$+ \frac{r^2 + a^2 \cos^2 \theta + \beta r}{r^2 + a^2 - 2mr} dr^2 + (r^2 + a^2 \cos^2 \theta + \beta r) d\theta^2,$$

$$A = \frac{-mr \sinh \alpha}{\sqrt{2} (r^2 + a^2 \cos^2 \theta + \beta r)} (dt - a \sin^2 \theta d\varphi),$$

$$B_{t\varphi} = \frac{\beta ra \sin^2 \theta}{r^2 + a^2 \cos^2 \theta + \beta r}, \quad e^{-2\phi} = \frac{r^2 + a^2 \cos^2 \theta + \beta r}{r^2 + a^2 \cos^2 \theta}.$$ 

(2.6)

One can easily see that when the seed metric is spherically symmetric (that is, the Schwarzschild metric), the twisted solution is

$$ds^2 = -(1 - \frac{2m}{r}) \left(1 + \frac{\beta}{r}\right)^{-1} dt^2 + \left(1 + \frac{\beta}{r}\right) \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + \frac{1}{r^2 + 2m + \beta} d\Omega, $$

$$A = \frac{-mr \sinh \alpha}{\sqrt{2} (r^2 + \beta r)} dt, \quad B_{t\varphi} = 0, \quad e^{-2\phi} = 1 + \frac{\beta}{r},$$

(2.7)

where we have put $\beta := (\cosh \alpha - 1)m \geq 0.$ One introduces a new radial coordinate $\bar{r} := r + \beta$ to get

$$ds^2 = -\left(1 - \frac{2m + \beta}{\bar{r}}\right) dt^2 + \left(1 - \frac{2m + \beta}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 \left(1 - \frac{\beta}{\bar{r}}\right) d\Omega,$$
Global structure and physical properties of this metric are discussed in detail in refs. [7] [8].

What we want to emphasise here is that the twisting by $\Omega_{35}$ produces a new singularity at $r = -\beta$ (or $\bar{r} = 0$, see eq. (2.7)). $r = 0$ ($\bar{r} = \beta$) is a singularity that has been primordially existing before twisting. We can roughly say that the twisting $\Omega_{35}$ adds new singularities in the region $r < 0$ of the Schwarzschild metric. We will see in section 4 that the same phenomenon happens in the case of the $\Omega_{35}$ - twisted Kerr metric. New singularities do not appear in $r > 0$ and do only in $r \leq 0$ in the twisted Kerr solution as well as in the twisted Schwarzschild solution.

3 Boyer-Lindquist coordinate

The twisted Kerr solution is expressed in a concise form by “diagonalising” the $(t, \varphi)$ subspace of the metric [18]:

$$ds^2 = -\frac{\Delta}{\rho^2} \left( dt - a \sin^2 \theta d\varphi \right)^2 + \frac{1}{\rho^2} \sin^2 \theta \left( adt - (R^2 + a^2) d\varphi \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2,$$

where we put

$$\beta := m(cosh \alpha - 1), \quad R^2 := r^2 + \beta r,$$
$$\rho^2 := R^2 + a^2 \cos^2 \theta, \quad \Delta := r^2 - 2mr + a^2.$$

The electromagnetic field is concisely written as

$$A = \frac{Qr}{\rho^2} (dt - a \sin^2 \theta),$$

where we put $Q := -m \sinh \alpha / \sqrt{2}$. The expression is almost the same as that of the untwisted Kerr solution except that some of $r^2$’s in the untwisted solution are replaced by $R^2 := r^2 + \beta r$ in the twisted one. Clearly, the zeros of $g_{tt}$ and $\Delta$ are the same in both the twisted and the untwisted. This shows that the positions of ergosurfaces and horizons remain invariant through the twisting by $\Omega_{35} = \exp(\alpha(e_{35} + e_{53}))$.

The Boyer-Lindquist-like coordinate is convenient for calculating the geodesic orbits of test particles in twisted Kerr metric background. It is well known that the geodesic equations are described by the following Hamiltonian:

$$H(x, \pi) := \frac{1}{2} g_{\mu\nu} p_\mu p_\nu = \frac{1}{2} g_{\mu\nu}(\pi - eA)_\mu(\pi - eA)_\nu,$$
where \((x, \pi)\) are canonical coordinates and \(p^\mu = \frac{dx^\mu}{d\lambda} =: \dot{x}^\mu\). The \(\lambda\) is the proper time coordinate. The corresponding Hamilton-Jacobi equation is written as

\[
\frac{\partial S}{\partial \lambda} + H \left( x, \frac{\partial S}{\partial x} \right) = 0.
\]

The inverse metric is easily calculated and results in

\[
g^{\mu\nu} \left( \frac{\partial}{\partial x^\mu} \right) \otimes \left( \frac{\partial}{\partial x^\nu} \right) = -\frac{1}{\Delta \rho^2} ((R^2 + a^2) \partial_t + a \partial_r)^2 + \frac{1}{\rho^2 \sin^2 \theta} (\partial_r + a \sin^2 \theta \partial_\phi)^2
\]

\[
+ \frac{\Delta}{\rho^2} (\partial_r)^2 + \frac{1}{\rho^2} (\partial_\theta)^2.
\]

We put the principal function \(S(x)\) as \(S = \frac{1}{2} \mu^2 \lambda - Et + L \phi + S(r) + s(\theta)\). This means that the test particle under consideration has its square of the norm of 4-velocity \(\mu^2\), electric charge \(e\), angular momentum \(L\), and energy at infinity \(E\). The Hamilton-Jacobi equation is then

\[
\mu^2 - \frac{1}{\Delta \rho^2}(-(R^2 + a^2)E + aL - cQr)^2 + \frac{1}{\rho^2 \sin^2 \theta} (L - E \rho \sin^2 \theta)^2
\]

\[
+ \frac{\Delta}{\rho^2} (S'(r))^2 + \frac{1}{\rho^2} (s'(\theta))^2 = 0.
\]

Multiplying the factor \(\rho^2\), the variables are separated and the equation of motion is reduced to two ordinary differential equations with a separation constant \(\nu\):

\[
(S'(r))^2 = \frac{\nu}{\Delta} + \frac{1}{\Delta^2} \left[-(R^2 + a^2)E + aL - cQr\right]^2 - \frac{R^2 \mu^2}{\Delta},
\]

\[
(s'(\theta))^2 = -\nu - \frac{1}{\sin^2 \theta} [L - E \rho \sin^2 \theta]^2.
\]

Geodesic motions are given by integrating the equations \(\dot{x}^\mu = p^\mu = \left( \frac{\partial S}{\partial x^\nu} \right) g^{\nu\mu}\). We will use this result when we discuss the thermodynamic property of twisted Kerr black hole in section 5.

### 4 Kerr-Schild coordinate

In this section, we construct the Kerr-Schild-like coordinate of the twisted Kerr solution. It shows us that the twisted Kerr solution can be analytically extended to the region of negative values of the radial coordinate just as well as the Kerr or the Kerr-Newmann solutions. We will do the extension by gluing two copies of the metric (4.2). It will also be shown that singularity in the shape of a ring appears on the boundary of the connection of these two copies.
We observed in the preceding section that the twisting of the Kerr solution by $\Omega$ induces, roughly speaking, the replacement $r^2 \rightarrow R^2 = r^2 + \beta r$. This suggests us to modify the coordinate transformation in the untwisted Kerr coordinate and we get to the following coordinate transformation $(t, \phi, r, \theta) \mapsto (\tau, x, y, z)$:

$$x := (R \cos \tilde{\phi} + a \sin \tilde{\phi}) \sin \theta, \quad y := (R \sin \tilde{\phi} - a \cos \tilde{\phi}) \sin \theta,$$

$$z := R \cos \theta, \quad d\tau := dt + dR - \frac{R^2 + a^2}{\Delta} dR, \quad d\tilde{\phi} := d\phi - \frac{a}{\Delta} dR. \quad (4.1)$$

Then the metric is rewritten as

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + dz^2 - \frac{\beta^2}{4R^2 + \beta^2} dR^2$$

$$+ \frac{(2m + \beta)R^2r}{R^4 + a^2 z^2} \left[ d\tau - \frac{zdR}{R} - \frac{R}{R^2 + a^2} (xdx + ydy) - \frac{a}{R^2 + a^2} (xdy - ydx) \right]^2. \quad (4.2)$$

The existence of the term $-\frac{\beta^2}{4R^2 + \beta^2} dR^2$ indicates the deviation from the standard Kerr-Schild form $ds^2 = (\eta_{\mu\nu} + l_{\mu} l_{\nu}) dx^\mu dx^\nu$.

Now, we investigate where the singularity lies in the twisted Kerr spacetime. We follow the Chandrasekhar’s calculation[17]. The metric of the twisted Kerr solution can be rewritten as

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2p} dr^2 + e^{2q} d\theta^2.$$ 

Here $\nu, \psi, \omega, p,$ and $q$ are functions of $(r, \theta)$. In the present case they are given by

$$e^{2\nu} = \frac{\rho^2 \Delta}{\Sigma^2}, \quad e^{2\psi} = \frac{\Sigma^2}{\rho^2} \sin^2 \theta, \quad \omega = \frac{2Mar}{\Sigma^2}, \quad e^{2p} = \frac{\rho^2}{\Delta}, \quad e^{2q} = \rho^2,$$ 

together with

$$\Sigma^2 := (R^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \quad \rho^2 := R^2 + a^2 \cos^2 \theta,$$

$$M := m + \beta/2, \quad \Delta := r^2 - 2mr + a^2, \quad R^2 := r^2 + \beta r.$$ 

After tedious calculations, we get explicit forms of $R_{abij}$’s. Only information we need here is in which points these components blow up. We show here only the result:

$\Sigma^2$ and $\rho^2$ are the only factors that can appear in the denominators of components of $R_{abij}$. In other words, curvature singularities appear only in points on which $\Sigma^2$ or $\rho^2$ (or both) becomes zero.
Thus we have only to regard zeros of $\Sigma^2$ and $\rho^2$. We provide a lemma to conclude that there is no singularity in the region $r > 0$:

**Lemma** \( \{ m \geq 0, \ \beta \geq 0, \ r > 0 \} \implies \Sigma^2 > 0, \ \rho^2 > 0. \)

Before we prove the lemma, we give a remark here. Clearly, $\cosh \alpha - 1 \geq 0$ should hold, or $\alpha$ will not be real and the reality of the electric charge $Q := -m \sinh \alpha / \sqrt{2}$ will be lost. That is, $\beta := m(\cosh \alpha - 1)$ and $m$ should be in the same signature. Total mass (energy) of the system $M := m + \beta/2$ should be positive if the system describes a black hole at all. Therefore both $m \geq 0$ and $\beta \geq 0$ are necessary. The assumption $\{ \beta \geq 0, \ m \geq 0 \}$ in the lemma is quite reasonable in our case. If we prove the lemma, we can say, under fairly general situations, that there is no singularity in the region $r > 0$.

**Proof of Lemma**

$\rho^2 > 0$ immediately follows from $r > 0$, and therefore $R^2 = r^2 + \beta r > 0$ and $a^2 \cos^2 \theta \geq 0$. We prove $\Sigma^2 > 0$ by dividing the region $r > 0$ into two: $\Delta \leq 0$ and $\Delta > 0$. If $\Delta \leq 0$, $\Sigma^2 > 0$ is trivial. If $\Delta > 0$, then

$$
\Sigma^2 = (R^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \geq (R^2 + a^2)^2 - a^2 \Delta = R^4 + a^2 r^2 + 2a^2 \beta r + 2mra^2
$$

holds. The right hand side of this inequality is strictly positive. This completes the proof of the lemma.

Let us denote two copies of the spacetime by $A$ and $B$. Let $A$ be the spacetime defined by $(R, \theta, t, \varphi)$ with $R = + \left[ \frac{1}{2} (\delta + \sqrt{\delta^2 + 4a^2 z^2}) \right]^{1/2} > 0$. Similarly, let $B$ be the spacetime defined by $(R, \theta, t, \varphi)$ with $R = - \left[ \frac{1}{2} (\delta + \sqrt{\delta^2 + 4a^2 z^2}) \right]^{1/2} < 0$. We identify the two discs $\{x^2 + y^2 \leq a^2, z = +0\}$ in $A$ and $\{x^2 + y^2 \leq a^2, z = -0\}$ in $B$. And we do the same thing with $z = +0$ and $z = -0$ interchanged. With these identifications, we obtain an analytically extended solution, in which the radial coordinate $R$ is now defined in $(-\infty, +\infty)$. Singularity appears on $\{ \rho^2 = 0 \cup \Sigma^2 = 0 \} = \{ R = 0, \ \theta = \pi/2 \}$. Obviously, it is in the shape of a ring and it locates on the boundary of the two discs. The global structure is the same as that of the untwisted Kerr.

To conclude, what we have acquired as knowledge of the twisted Kerr solution is:

1) **The twisting $\Omega_{35}$ gives the Kerr solution new singularities. They appear, however, only in**
the region \( r \leq 0 \).

2) The analytically extended spacetime of the twisted Kerr solution is so constructed that one removes the region \( r < 0 \) out of the spacetime \( S := \{(t, \varphi, r, \theta)\} \) to obtain the spacetime \( \tilde{S} := \{(t, \varphi, R, \theta)\} \), and one glues two copies of \( \tilde{S} \).

As we have mentioned at the end of section 2, \( \Omega_{35} \) – twisting of both the Schwarzschild and the Kerr solutions have new singularities in the region \( r \leq 0 \). Despite of these singularities, we can carry out the analytic extension of the twisted Kerr solution by way of the transition of the radial coordinate \( r \) into the new one \( R = \sqrt{r^2 + \beta r} \).

5 Thermal properties of the twisted Kerr black hole

We calculate the Hawking temperature\(^{20}\) and other macroscopic quantities of the twisted Kerr black hole via geodesic motions of test bodies\(^{14}\). We will show that this calculation reproduces the thermodynamic property of black holes\(^{21} \)\(^{22}\).

Before we set out calculation, we draw a rough sketch. We consider a test body with

\[
(\text{mass}=\text{energy, electric charge, angular momentum}) = (\delta M, \delta Q, \delta J),
\]

thrown from infinity and going into the black hole of mass=\( M \), charge=\( Q \), and angular momentum=\( J \). (Let us assume that \( |\delta M| \ll |M|, |\delta Q| \ll |Q|, |\delta J| \ll |J| \).) When the test body goes into the black hole across the horizon, physical quantities of the black hole will change:

\[
(M, Q, J) \longrightarrow (M + \delta M, Q + \delta Q, J + \delta J).
\]

We already know that if the electric charge or angular momentum of the black hole exceed certain limit, horizon will disappear and it will lead to a naked singularity\(^{12}\). Now, a question arises: “Can we change a singularity surrounded by regular horizons into a naked one through the injection of test bodies?” One has to add less \( \delta M \) and more \( \delta Q \) or \( \delta J \) if one wishes to change a black hole into a naked singularity. But, there is obviously a lower limit of \( \delta M \) for given \( \delta Q \) and \( \delta J \), because certain amount of energy is at least needed in order to overcome the repulsive force made by the electric interaction and the effective potential of angular momenta. Unless \( \delta M \) exceeds the lower limit \( (\delta M)_{\text{min}} \), the test body will not get to the black hole horizon and will move back to infinity.

We estimate the \((\delta M)_{\text{min}}\) for given \( \delta Q \) and \( \delta J \) in the twisted Kerr metric background, and we will see that it is impossible to change a black hole into a naked singularity by way of the injection of test bodies. Furthermore, the inequality of \( \delta M, \delta Q, \) and \( \delta J \) can be considered
as a differential system in the \((M, Q, J)\)-space and it can be integrated to obtain the entropy and temperature of the black hole.

First, \((M, Q, J)\) of the solution (2.6) is expressed by the parameters \((m, a, \beta)\):

\[
M = m + \frac{\beta}{2}, \quad J = a(m + \frac{\beta}{2}) = aM, \quad Q = \frac{m \sinh \alpha}{\sqrt{2}}.
\]

Its inverse is given by

\[
m = M - \frac{Q^2}{2M}, \quad a = \frac{J}{M}, \quad \beta = \frac{Q^2}{M}.
\]

As we saw in section 3, the orbit of the test particle under consideration is expressed by the principal function

\[
S(x) = \frac{1}{2} \mu^2 \lambda - t\delta M + \varphi\delta J + S(r) + s(\theta).
\]

Clearly, the minimum for \(\delta M\) is achieved when \(\mu = \frac{d\theta}{d\lambda} = 0\). Hamilton-Jacobi equation (3.2) is now given by

\[
- \frac{1}{\Delta \rho^2} \left[ -(R^2 + a^2)\delta M + a\delta J + Qr\delta Q \right]^2 + \frac{1}{\rho^2 \sin^2 \theta} (\delta J - \delta Ma \sin^2 \theta)^2 + \frac{\Delta}{\rho^2} (S'(r))^2 = 0
\]

The minimum energy \((\delta M)_{\text{min}}\) is evaluated by the condition in which \(\frac{dr}{d\lambda} = g^{rr} S'(r) = 0\) on the horizon \(r = r_+ := m + \sqrt{m^2 - a^2}\). That is,

\[- (R^2 + a^2)|_{r=r_+}(\delta M)_{\text{min}} + a\delta J + Qr_+\delta Q = 0.\]

This means

\[
\delta M \geq \frac{a\delta J + Qr_+\delta Q}{2Mr_+} = \frac{a\delta J}{2Mr_+} + \frac{Q\delta Q}{2M}.
\]

Let us check that it is really impossible to change a black hole into a naked singularity. It suffices to check this when the black hole is extremal: \(m^2 = a^2\), i.e., \(M^2 = (Q^2/2) + J\). Then \(r_+ = J/M\). Putting this equation into (5.1), we get

\[
\delta M \geq \frac{\delta J}{2M} + \frac{Q\delta Q}{2M}.
\]

This is equivalent to the differential of \(M^2 \geq Q^2/2 + J\), which is exactly the condition of the existence of the regular horizon:

\[
\delta(M^2 \geq Q^2/2 + J) \iff 2M\delta M \geq Q\delta Q + \delta J.
\]
Therefore, even if one starts with the extremal black hole, one cannot change \((M, Q, J)\) in such a way as to break the inequality \(M^2 \geq Q^2/2 + J\).

Now, we show that the inequality (5.1) can be “integrated” and it reproduces the thermal property of the black hole. Let us consider the differential system

\[
dM - \frac{Q}{2M} dQ - \frac{J}{2M^2 r_+} dJ = 0
\]

in the 3-dimensional space \(\mathbb{R}^3\) parametrised by \((M, Q, J)\). That is, there is given a surface element which is normal to \(dM - (Q/2M)dQ - (J/2M^2 r_+)dJ\) at each point in \(\mathbb{R}^3\). One can solve (5.2) by multiplying a suitable factor on the both sides of (5.2) so that it satisfies the integrability condition. We choose as the factor

\[
2M + (2M^2 - Q^2) \left[ \left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2 \right]^{-1/2}.
\]

Then (5.2) is integrated and we obtain

\[
d \left\{ M \left[ M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2} \right] \right\} = 0.
\]

This means that the 2-dimensional surface with the defining equation

\[
M \left[ M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2} \right] = \text{const.}
\]

in \(\mathbb{R}^3\) is a solution of (5.2). Noticing that the factor (5.3) is positive, the inequality (5.1) is rewritten as

\[
\delta \left\{ M \left[ M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2} \right] \right\} \geq 0.
\]

This inequality is precisely equivalent to

\[
\delta \left\{ \frac{1}{8\pi} (\text{Area of the horizon}) \right\} \geq 0.
\]

Actually,

\[
\text{Area} = \int_{r=r_+} \sqrt{g_{\theta\theta} g_{\varphi\varphi}} d\theta \wedge d\varphi = 4\pi (R^2 + a^2) \bigg|_{r=r_+} = 4\pi (2m + \beta) r_+
\]

\[
= 8\pi M \left[ M - \frac{Q^2}{2M} + \sqrt{\left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2} \right].
\]
Thus, the gedanken experiments of throwing test particles into the twisted Kerr black hole really reproduce the 2nd law of black hole mechanics.

Now that the entropy $S$ ($1/4$ of the area of the horizon) of the black hole is given in terms of $(M, Q, J)$, one can calculate its differential

$$dS = \left( \frac{\partial S}{\partial M} \right) dM + \left( \frac{\partial S}{\partial J} \right) dJ + \left( \frac{\partial S}{\partial Q} \right) dQ.$$ 

Comparing this with the formula of the 1st law of the black hole mechanics $dM = TdS + \Phi dQ + \Omega dJ$, one obtains

$$T = \frac{\sqrt{\#}}{4\pi M \left( M - \frac{Q^2}{2M} + \sqrt{\#} \right)}, \quad \Omega = \frac{J}{2M^2 \left( M - \frac{Q^2}{2M} + \sqrt{\#} \right)}, \quad \Phi = \frac{Q}{2M}, \quad (5.4)$$

where we put $\# := \left( M - \frac{Q^2}{2M} \right)^2 - \left( \frac{J}{M} \right)^2$.

On the other hand, surface gravity $\kappa$, angular velocity $\Omega_H$, and electrostatic potential $\Phi_H$ on the horizon are given by

$$\frac{1}{4} \kappa := \left| -\frac{1}{2} \nabla_\mu l^\nu \nabla_\mu l_\nu \right|_H = \lim_{r \to r_+} \sqrt{g^{rr}} \partial_r \sqrt{-g_{tt}} |_{\theta = 0} = \frac{\sqrt{\#}}{2M \left( M - \frac{Q^2}{2M} + \sqrt{\#} \right)},$$

$$\Omega_H := \left[ \frac{g_{t\phi}}{g_{\phi\phi}} \right]_H = \frac{J}{2M^2 r_+},$$

$$\Phi_H := \left[ l^\mu A_\mu \right]_H = \frac{Q}{2M}$$

where $l^\mu := \left( \frac{\partial}{\partial t} \right)^\mu + \Omega_H \left( \frac{\partial}{\partial \phi} \right)^\mu$. These are exactly the same as $(5.4)$. The gedanken experiments of test particles are again consistent with the 1st law of black hole mechanics.

6 \hspace{1cm} O(2, 3) symmetry and asymptotic flatness

We have seen that the action (2.1) admits the global $O(2, 3)$ symmetry and that the vacuum Kerr metric is a solution of the equations of motion. One might hope that one can produce a larger class of solutions by acting elements of $O(2, 3)$. However, we immediately realise that not all of the elements of $O(2, 3)$ preserve the asymptotic flatness. And some elements cause merely coordinate transformations and they do not change physical contents. Here a question arises: “how many physically distinct black hole solutions are generated by the $O(2, 3)$ transformation?” We cannot answer this question completely at present, but part of the information can be cast from the infinitesimal $o(2, 3)$ action, which we are going to discuss.
We will show in this section that \( \{ \exp(\alpha_{35}(e_{35} + e_{53})) | \alpha_{35} \in \mathbb{R} \} \subset O(2,3) \) is the only one-parameter subgroup that preserves the asymptotic flatness. And all the other elements of \( O(2,3) \) break the asymptotic flatness or cause only coordinate transformations.

### 6.1 \( o(2,3) \) and \( gl(2,\mathbb{R}) \) Transformations

Let us explain the basic strategy. As we have seen in section 2, the action of \( O(2,3) \) mixes \( \hat{g}_{mn}, \hat{A}_m, \) and \( \hat{B}_{mn} \) \((t,\varphi)-\) components) each other nonlinearly and reproduces their new configuration. On the other hand, \( \tilde{A}_\alpha \) and \( \tilde{B}_{\alpha\beta} \) are left invariant and \( \tilde{g}_{\alpha\beta} \) \((r,\theta)-\) components) are only multiplied by a function \( \det(\Omega(\hat{g}))/\det(\hat{g}) \). So, if the twisting by an element of \( O(2,3) \) breaks the asymptotic behaviour and yet the breaking is “soft” enough to be recovered by a coordinate transformation of \((t,\varphi)\), then one will obtain a favourable new solution.

\[
\begin{pmatrix}
\text{seed solution} \\
\end{pmatrix}
\xrightarrow{O(2,3)}
\begin{pmatrix}
*** \\
*** \\
\end{pmatrix}
\xrightarrow{GL(2,\mathbb{R})}
\begin{pmatrix}
\text{asymptotically flat solutions}
\end{pmatrix}
\]

Here \( GL(2,\mathbb{R}) \) denotes the affine transformation group of \((t,\varphi)\). Since we are considering stationary and axisymmetric solutions, \( GL(2,\mathbb{R}) \) should be a global symmetry. We linearise these transformations to seek well-behaved twisted solutions.

First, we check the asymptotic behaviour. When the spacetime is asymptotically flat, the fields behave as

\[
\begin{align*}
tt &= -1 + \frac{2M}{r} + O(r^{-2}), \quad t\varphi = \frac{-4J}{r}\sin^2 \theta + O(r^{-2}), \\
\varphi\varphi &= r^2 \sin^2 \theta + O(r), \quad rr = 1 + O(r^{-1}), \quad \theta\theta = r^2 + O(r), \\
A_t &= \frac{Q}{r} + O(r^{-2}), \quad A_\varphi = O(r^{-1}), \quad B = O(r^{-1}), \quad \phi = O(r^{-1}).
\end{align*}
\]

And the corresponding behaviour of the matrix-valued field \( M(r,\theta) \) is given by

\[
M = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & p + 2 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & p - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + O(r^{-1});
\]

where \( p := p_2 r^2 + p_1 r + p_0 \) and

\[
p_2 := \lim_{r \to \infty} r^{-2} g_{\varphi\varphi}, \quad p_1 := \lim_{r \to \infty} r^{-1}(g_{\varphi\varphi} - p_2 r^2), \quad p_0 := \lim_{r \to \infty}(g_{\varphi\varphi} - p_2 r^2 - p_1 r).
\]
That is, $p$ is the divergent and constant parts of $g_{\varphi \varphi}$ as the radial coordinate $r$ goes to infinity. We assume that $r$ should remain to be the radial coordinate before and after the twisting. We will go back and discuss this assumption at the end of this section.

We consider a general element of $\Omega \in O(2, 3)$ and $P \in GL(2, \mathbb{R})$, and act $P \cdot \Omega$ on a seed solution. We examine its linearisation $1 \cdot d\Omega + dP \cdot 1$. Now, let us act elements of $o(2, 3)$ and $gl(2, \mathbb{R})$ and check the asymptotic behaviour of infinitesimally twisted solutions. Clearly, $o(2, 3)$ is spanned by the following canonical basis:

$$
o(2, 3) = \left\{ \begin{array}{ccc}
e_{12} + e_{21}, & e_{13} + e_{31}, & -e_{14} + e_{41}, \\
e_{15} + e_{51}, & -e_{23} + e_{32}, & e_{24} + e_{42}, \\
e_{25} + e_{52}, & e_{34} + e_{43}, & e_{35} + e_{53}, \\
e_{45} + e_{54}, & & \\
\end{array}\right\} =: \{d\Omega_{ij} | 1 \leq i < j \leq 5\}_{\mathbb{R}},
$$

where $e_{ij}$’s are matrix units of $M_{5}(\mathbb{R})$. We recall that the action of $O(2, 3)$ on $\mathcal{M}(\hat{g}_{\mu \nu}, \hat{B}_{\mu \nu}, \hat{A}_{\mu})$ is given by $\Omega(\mathcal{M}) = \Omega \mathcal{M} \Omega'$. Therefore its differential is

$$
d\Omega(\mathcal{M}) = d\Omega \mathcal{M} + \mathcal{M} d\Omega' =: d \left( \begin{array}{ccc}
l & n & u \\
'n & c & e \\
u' & e' & f \end{array}\right).
$$

We choose $d\Omega \in o(2, 3)$ as the most general form:

$$
d\Omega = \sum_{1 \leq i < j \leq 3} \alpha_{ij} d\Omega_{ij} \in o(2, 3) \quad (\forall \alpha_{ij} \in \mathbb{R}).
$$

The result of the calculation of $d\Omega(\mathcal{M})$ is given by

\begin{align*}
(d\Omega(\mathcal{M}))_{11} &= O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{12} = -2\alpha_{12} + (\alpha_{12} - \alpha_{14})p + O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{13} &= -4\alpha_{13} + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{14} = -2\alpha_{14} + (\alpha_{12} - \alpha_{14})p + O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{15} &= -4\alpha_{15} + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{22} = 2\alpha_{24}p + O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{23} &= 2\alpha_{23} + (\alpha_{23} + \alpha_{34})p + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{24} = 2\alpha_{24} + O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{25} &= 2\alpha_{25} + (\alpha_{25} + \alpha_{45})p + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{33} = O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{34} &= -2\alpha_{34} + (\alpha_{23} + \alpha_{34})p + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{35} = O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{44} &= 2\alpha_{24}p + O(r^{-1}), \quad (d\Omega(\mathcal{M}))_{45} = -2\alpha_{45} + (\alpha_{25} + \alpha_{45})p + O(r^{-1}), \\
(d\Omega(\mathcal{M}))_{55} &= O(r^{-1}).
\end{align*}

(6.1)

The $GL(2, \mathbb{R})$— action on the coordinate of $(t, \varphi)$

$$
\begin{pmatrix} t \\ \varphi \end{pmatrix} \mapsto P^{-1} \cdot \begin{pmatrix} t \\ \varphi \end{pmatrix}, \quad P \in GL(2, \mathbb{R})
$$
induces \( P(\hat{g}) = P\hat{g}P' \). Its differential is \( dP(\hat{g}) = dP\hat{g} + \hat{g}dP' \), or equivalently, \( dP(\hat{g}^{-1}) = -dP'\hat{g}^{-1} - \hat{g}^{-1}dP \) for \( dP \in gl(2, \mathbb{R}) \). We choose it in a general form as \( dP = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \).

We immediately notice that \( \alpha_{35} \) component does not appear in \( d\Omega(M) \), whereas all the other components do. We see from this that \( \Omega_{35} \) is the only element of \( O(2,3) \) that preserves the asymptotic flatness. But there is still hope that the breaking of the asymptotic condition is just superficial and the asymptotic flatness can be reinstated by some coordinate transformations. We will see in the next subsection, however, that all of such “soft” twisting only result in coordinate transformations.

### 6.2 detailed calculations

In this subsection, we first calculate \( d\hat{g} = (d\Omega + dP)(\hat{g}) \) and \( d\hat{B} = (d\Omega + dP)(\hat{B}) \) to obtain necessary conditions to keep the asymptotic behaviour. These conditions will tell us that only suitable linear combinations of \( (d\Omega_{ij}) \) are allowed. And then we will explicitly calculate the allowed transformations. It will reveal that all the allowed transformations except \( \Omega_{35} \) are equivalent to coordinate transformations and they give no physically new solutions.

First, we check the asymptotic behaviour of \( d\hat{g} = (d\Omega + dP)(\hat{g}) \) and \( d\hat{B} = (d\Omega + dP)(\hat{B}) \). From eq.(2.4), we have
\[
d\hat{g}^{-1} = \frac{1}{4} \eta(dl - dn - dn' + dc)\eta.
\]
This enables us to calculate \( d\hat{g} \) explicitly:
\[
d\hat{g}_{tt} = -2\alpha_{13} - 2\beta_{11} + O(r^{-1})
\]
\[
d\hat{g}_{u\varphi} = -\frac{1}{2}(\alpha_{12} + \alpha_{14} - \alpha_{23} + \alpha_{34}) - \beta_{21} + \frac{1}{2}p(\alpha_{12} - \alpha_{14} + \alpha_{23} + \alpha_{34}) + p\beta_{12} + O(r^{-1})
\]
\[
d\hat{g}_{\varphi\varphi} = 2\alpha_{24}p + 2\beta_{22}p + O(r^{-1}).
\]
We get
\[
\alpha_{13} = -\beta_{11}, \quad \alpha_{12} + \alpha_{14} - \alpha_{23} + \alpha_{34} + 2\beta_{21} = 0,
\]
\[
\alpha_{12} - \alpha_{14} + \alpha_{23} + \alpha_{34} + 2\beta_{12} = 0, \quad \alpha_{24} = -\beta_{22} \tag{6.2}
\]
as necessary conditions in order to maintain the asymptotic behaviour of \( \hat{g}_{mn} \).

After similar but longer calculations, we get the result of \( d\hat{B} \):
\[
d\hat{B}_{u\varphi} = \frac{1}{2}(-\alpha_{12} - \alpha_{14} - \alpha_{23} + \alpha_{34}) - \frac{1}{2}p(-\alpha_{12} + \alpha_{14} + \alpha_{23} + \alpha_{34}) + O(r^{-1}).
\]
This gives
\[
-\alpha_{12} - \alpha_{14} - \alpha_{23} + \alpha_{34} = 0, \quad -\alpha_{12} + \alpha_{14} + \alpha_{23} + \alpha_{34} = 0 \tag{6.3}
\]
as necessary conditions.

Now, let us pay attention to the components of \((d\Omega + dP)(\mathcal{M})_{i5}\). Noticing that \(\lim_{r \to \infty} \mathcal{M}_{i5} = 0\), \(dP(\mathcal{M})_{i5} = O(r^{-1})\) holds. This means that the constant terms and divergent terms which appear in \(d\Omega(\mathcal{M})_{i5}\) cannot be cured by \(dP(\mathcal{M})\). Thus

\[ \alpha_{15} = \alpha_{25} = \alpha_{45} = 0 \quad (6.4) \]

is necessary in view of the asymptotic behaviour.

Gathering all the data of \((6.2)\), \((6.3)\), and \((6.4)\), we have

\[
\begin{align*}
\alpha_{12} &= \alpha_{34} = -\frac{1}{2}(\beta_{12} + \beta_{21}), \\
\alpha_{14} &= -\alpha_{23} = \frac{1}{2}(\beta_{12} - \beta_{21}), \\
\alpha_{15} &= \alpha_{25} = \alpha_{45} = 0, \\
\alpha_{13} &= -\beta_{11}, \\
\alpha_{24} &= -\beta_{22}, \\
\alpha_{35} &\geq \text{free.}
\end{align*}
\]

This shows that the only 5 elements of

\[
\{d\Omega_{12} + d\Omega_{34}, \ d\Omega_{14} - d\Omega_{23}, \ d\Omega_{13}, \ d\Omega_{24}, \ d\Omega_{35}\} \quad (6.6)
\]

are allowed generators. All the other 5 elements break the asymptotic flatness so severely that the breaking cannot be recovered by coordinate transformations.

Next, we examine the twisting generated by the exponentials of the above elements \((6.6)\).

After tedious and boring calculations, one finds that the twisting by \(\exp(\alpha(d\Omega_{12} + d\Omega_{34}))\) is exactly equivalent to the coordinate transformation

\[
(t, \varphi) \mapsto (t \cosh \alpha - \varphi \sinh \alpha, -t \sinh \alpha + \varphi \cosh \alpha),
\]

which is naturally expected from \((6.3)\). The twisting by \(\exp(\alpha(d\Omega_{14} - d\Omega_{23}))\), \(\exp(\alpha d\Omega_{13})\), and \(\exp(\alpha d\Omega_{24})\) all of them result in coordinate transformations as well. The equivalence is listed as:

\[
\begin{align*}
GL(2, \mathbb{R}) &\leftrightarrow O(2, 3) \\
\exp(-\alpha(e_{12} + e_{21})) &\leftrightarrow \exp(\alpha(d\Omega_{12} + d\Omega_{34})) \\
\exp(-\alpha e_{11}) &\leftrightarrow \exp(\alpha d\Omega_{13}) \\
\exp(-\alpha(-e_{12} + e_{21})) &\leftrightarrow \exp(\alpha(d\Omega_{14} - d\Omega_{23})) \\
\exp(-\alpha e_{22}) &\leftrightarrow \exp(\alpha d\Omega_{24}).
\end{align*}
\]  

(6.7)

Notice that the map \(gl(2, \mathbb{R}) \hookrightarrow o(2, 3)\)

\[
\begin{align*}
e_{11} &\mapsto d\Omega_{13}, \\
&\quad e_{12} \mapsto \frac{1}{2}(d\Omega_{14} - d\Omega_{23}) - \frac{1}{2}(d\Omega_{12} + d\Omega_{34}), \\
&\quad e_{22} \mapsto d\Omega_{24}, \\
&\quad e_{21} \mapsto -\frac{1}{2}(d\Omega_{14} - d\Omega_{23}) - \frac{1}{2}(d\Omega_{12} + d\Omega_{34})
\end{align*}
\]

gives the (Lie-algebraic) embedding of \(gl(2, \mathbb{R})\) in \(o(2, 3)\) and it is in tune with the correspondence \((6.7)\).

Thus we are left with only one twisting of \(\exp(\alpha d\Omega_{35})\), which is already well known to us.
6.3 physical interpretation

We conclude this section by some remarks and discussion. The calculations in the last sub-section gives us the information that “$\Omega_{35}$ is the only one-parameter subgroup of $O(2,3)$ that transforms a seed solution into another and at the same time keeps stationarity, axisymmetry, and asymptotic flatness.”

Actually, suppose that one twists a seed solution by the element $\Omega \in O(2,3)$ in the most general form: $\Omega = \Omega(\alpha) = \exp(\sum_{i<j} \alpha_{ij} d\Omega_{ij})$. Arguments of the infinitesimal transformation has shown that one must impose $\sum_{i<j} \alpha_{ij} d\Omega_{ij}$ be in 5 dimensional subspace of $o(2,3)$ spanned by (6.6). Now, if one restricts the domain of $(\alpha_{ij})$ to a suitably small neighbourhood of $0 \in \mathbb{R}^{10}$, an arbitrary element of the form $\exp(\sum_{i<j} \alpha_{ij} d\Omega_{ij})$ is expressed as

$$\exp(\alpha'_{12} d\Omega_{12}) \cdot \exp(\alpha'_{13} d\Omega_{13}) \cdots \exp(\alpha'_{45} d\Omega_{45}).$$

(This is just the diffeomorphism between the normal coordinates of the 1st and 2nd kinds.) Therefore one can suppose without loss of generality that $\Omega(\alpha)$ is in the form of

$$\exp(\alpha_1 (d\Omega_{12} + d\Omega_{34})) \cdot \exp(\alpha_2 d\Omega_{13}) \cdot \exp(\alpha_3 (d\Omega_{14} - d\Omega_{23})) \cdot \exp(\alpha_4 d\Omega_{24}) \cdot \exp(\alpha_5 d\Omega_{35}).$$

Since one already knows that

$$\exp(\alpha_1 (d\Omega_{12} + d\Omega_{34})) \cdot \exp(\alpha_2 d\Omega_{13}) \cdot \exp(\alpha_3 (d\Omega_{14} - d\Omega_{23})) \cdot \exp(\alpha_4 d\Omega_{24})$$

induces only a coordinate transformation, $\Omega(\alpha)$—twisting gives a solution which is equivalent to the $\Omega_{35}$—twisting.

It should be noted that our arguments given here are developed in the vicinity of the identity of $O(2,3)$. Both the linearisation and the diffeomorphism between the normal coordinates of the 1st and 2nd kinds are “local” arguments. So, one cannot deny the possibility of constructing a new solution by way of multiple twists. That is, even though each of the twists breaks the asymptotic condition, they might form a nice transformation as a whole. For example, one can perhaps construct a new solution in such a way that one breaks the asymptotic flatness by twisting by a certain element of $\Omega_1 \in O(2,3)$, and then transforms it along the flow of $\{\exp(\alpha\omega)|\alpha \in \mathbb{R}, \omega \in o(2,3)\}$, and finally twists it by a suitable element of $\Omega_2 \in O(2,3)$ so that the resulting solution can satisfy the asymptotic flatness:

$$\Omega_2 \cdot \exp(\alpha\omega) \cdot \Omega_1 \text{(seed solution)}.$$
We did not consider such cases. The readers will agree that the investigation of the “global” property of the $O(2, 3)$ transformation is far more difficult than that of the local one. We will put off this problem for future investigation.

Finally, let us discuss the assumption of the radial coordinate $r$. We have assumed that $r$ should remain to be a radial coordinate before and after the twisting. One might wonder that we have put too strict assumption on $r$. That is, one might imagine that even if there is some harmful divergence of the field components at the spatial infinity $r = \infty$, it could be amended by suitable coordinate redefinitions of the radial coordinate. Such a miracle, however, seldom happens. Now, let us consider the question “can one recover the asymptotic flatness by introducing a new radial coordinate $\xi = \xi(r)$?”. As we have seen in the previous subsection, all the infinitesimal transformations of $\alpha_{ij} d\Omega_{ij} \in o(2, 3)$, except for $\alpha_{13} d\Omega_{13}$, $\alpha_{24} d\Omega_{24}$, and $\alpha_{35} d\Omega_{35}$, add terms with positive powers of $r$ to $g_{\phi \phi}$ or $\hat{A}_m$. Then these fields will diverge both at $r = \infty$ and $r = 0$, since they contain terms with both negative and positive power of $r$. One will have to give up identifying the spatial infinity as $r = \infty$ or as $r = 0$. Therefore the new radial coordinate $\xi$, if it exists, should satisfy

$$\{\xi = \infty\} = \{r = (a \text{ certain point } q, \text{ which is not } 0 \text{ nor } \infty)\}.$$

On the other hand, asymptotic behaviour demands that

$$\Omega(g)_{\xi \xi}(q) = e^{-2\Omega(\phi)(q)} g_{rr}(q) \left( \frac{dr}{d\xi} \right)^2 (q) \sim 1,$$

$$\Omega(g)_{\theta \theta}(q) = e^{-2\Omega(\phi)(q)} g_{\theta \theta}(q) \sim \xi^2,$$

$$\Omega(g)_{\varphi \varphi} \sim \xi^2 \sin^2 \theta,$$

$$\Omega(A_t)(q) \sim 0, \quad \Omega(A_\varphi)(q) \sim 0, \quad \Omega B_{t \varphi})(q) \sim 0.$$

It is quite difficult to find a new radial coordinate that satisfies such an asymptotic condition.

7 Summary and conclusion

We analysed the twisted Kerr solution. The operation of the twisting $\Omega_{35} = \exp(\alpha d\Omega_{35})$ does not drastically change the global structure of the spacetime. The Kerr-Schild-like coordinate is introduced to show where the singularities lie in the spacetime. With a suitable radial coordinate transformation $r \mapsto R = \sqrt{r^2 + \beta r}$, we can analytically extend the twisted Kerr solution just as we did in the (untwisted) Kerr solution. The twisting generates new singularities in the region $r < 0$. But the transition $r \mapsto R$ naturally cuts away the negative radial part $r < 0$. Analytic continuation is carried out by gluing the two charts of $R \geq 0$ and $R \leq 0$. 

In section 5, we considered null geodesics of the twisted Kerr metric and discussed its thermal behaviour. The gedanken experiments of throwing test particles into the black hole have led us to the consistent description of the thermodynamic-like property of the black hole. That is,

1) Test bodies can cross the horizon and go into the black hole if and only if the condition (5.1) is satisfied. We cannot decrease the area of the horizon by throwing test bodies into the black hole.

2) We cannot change the black hole into a naked singularity by any means of the injection of test bodies.

3) Physical quantities of the black hole \((\kappa, \Omega_H, Q_H)\) are reproduced by “integrating” the condition (5.1).

We have neglected the interaction of the dilaton and the antisymmetric tensor with particles. We considered only null geodesics, so the interaction of the dilaton can be removed[23]. Interaction of the antisymmetric tensor is more subtle. We put this problem for future investigation.

We discussed in section 6 the global \(O(2, 3)\) symmetry and asymptotic flatness. Some elements of \(O(2, 3)\) break the asymptotic flatness and some elements cause only coordinate transformations. Infinitesimal \(o(2, 3)\) action reveals that 5 generators of \(o(2, 3)\) break the asymptotic condition so severely that the breaking cannot be cured by coordinate transformations. And thus we are left with 5 generators listed in (6.6). All the generators except for \(d\Omega_{35}\) cause coordinate transformations and they do not change physics. Therefore we cannot add any more free parameters than mass, angular momentum, and electric charge, because the asymptotically flat, stationary, and axisymmetric solution we already know is the \(\Omega_{35}\)– twisted Kerr solution only. This gives quite an affirmative example of the no-hair conjecture[24] of stationary and axisymmetric stringy black hole solutions.

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