A SINGULAR LIMIT PROBLEM FOR THE KUDRYASHOV-SINELSHCHIKOV EQUATION

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Abstract. We consider the Kudryashov-Sinelshchikov equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to the entropy ones of the Burgers equation. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction

A mixture of liquid and gas bubbles of the same size may be considered as an example of a classic nonlinear medium. The analysis of propagation of the pressure waves in a liquid with gas bubbles is an important problem. Indeed, there are solitary and periodic waves in such mixtures and they can be described by nonlinear partial differential equations like the Burgers, Korteweg-de Vries, and the Burgers-Korteweg-de Vries ones.

Recently, Kudryashov and Sinelshchikov [7] obtained a more general nonlinear partial differential equation to describe the pressure waves in a liquid and gas bubbles mixture taking into consideration the viscosity of liquid and the heat transfer. They introduced the equation

\begin{equation}
\partial_t u + Au\partial_x u + \beta \partial^3_{xxx} u - B\beta \partial_x \left(u\partial^2_{xx} u\right) - C\beta \partial_x u \partial^2_{xx} u - \varepsilon \partial^2_{xx} u - D\beta \partial_x \left(u\partial_x u\right) = 0,
\end{equation}

where $u$ is a density and models heat transfer and viscosity, while $A$, $\beta$, $B$, $C$, $\varepsilon$, $D$ are real parameters. If $B = C = D = 0$, (1.1) reads

\begin{equation}
\partial_t u + Au\partial_x u + \beta \partial^3_{xxx} u - \varepsilon \partial^2_{xx} u = 0,
\end{equation}

which is known as the Korteweg-de Vries-Burgers equation [15]. If also $\varepsilon = 0$, we obtain the Korteweg-de Vries equation [6].

Several results have been obtained in the case $A = 1$, $\beta = 1$, $B = 1$, $\varepsilon = 0$, $D = 0$, in which (1.1) reads

\begin{equation}
\partial_t u + u\partial_x u + \partial^3_{xxx} u - \partial_x \left(u\partial^2_{xx} u\right) - C\partial_x u \partial^2_{xx} u = 0.
\end{equation}

In [13], the author found four families of solitary wave solutions of (1.3) when $C = -3$, or $C = -4$. In [8], the authors discussed the existence of different kinds of traveling wave solutions by using the approach of dynamical systems, according to different phase orbits of the traveling system of (1.3); twenty-six kinds of exact traveling wave solutions are obtained under the parameter choices $C = -3$, $-4$, $1$, $2$. In [4], the authors discussed the bifurcations of phase portraits and investigated exact traveling wave solutions of (1.3) in
We study the no high frequency limit, namely we send $\beta, \varepsilon \to 0$ in (1.5). In this way we pass from (1.5) to the Burgers equation

$$
\begin{cases}
\partial_t u + Au\partial_x u = 0, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
$$

On the initial datum, we assume that

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}).$$

We study the dispersion-diffusion limit for (1.5). Therefore, we consider the following third order approximation

$$
\begin{cases}
\partial_t u_{\varepsilon, \beta} + A u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} + \beta \partial^3_{xxx} u_{\varepsilon, \beta} - B \beta \partial_x \left( u_{\varepsilon, \beta} \partial^2_{xx} u_{\varepsilon, \beta} \right) \\
- C \beta \partial_x u_{\varepsilon, \beta} \partial^2_{xx} u_{\varepsilon, \beta} - \varepsilon \partial^2_{xx} u_{\varepsilon, \beta} = 0, & t > 0, \quad x \in \mathbb{R}, \\
u_{\varepsilon, \beta}(0, x) = u_{\varepsilon, \beta, 0}(x), & x \in \mathbb{R},
\end{cases}
$$

where $u_{\varepsilon, \beta, 0}$ is a $C^\infty$ approximation of $u_0$ such that

$$u_{\varepsilon, \beta, 0} \to u_0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}), \quad 1 \leq p < 4, \quad \text{as } \varepsilon, \beta \to 0,$$

$$\|u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})}^2 + \|u_{\varepsilon, \beta, 0}\|_{L^4(\mathbb{R})}^4 + (\beta + \varepsilon^2) \|\partial_x u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})}^2 \leq C_0, \quad \varepsilon, \beta > 0,$$

and $C_0$ is a constant independent on $\varepsilon$ and $\beta$.

The main result of this paper is the following theorem.

**Theorem 1.1.** Assume that (1.7) and (1.9) hold. If

$$\beta = O(\varepsilon^4),$$

then, there exist two sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}) \cap L^4(\mathbb{R})),
$$

such that

$$u_{\varepsilon_n, \beta_n} \to u \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \quad \text{for each } 1 \leq p < 4,$$

$$u \quad \text{is the unique entropy solution of } (1.5).$$
The paper is organized in four sections. In Section 2, we prove some a priori estimates, while in Section 3 we prove Theorem 1.1. In Appendix, we prove that Theorem 1.1 holds also in the case \( A = (C + \alpha)^n \), where \( \alpha \) is a suitable real number.

2. A priori Estimates

This section is devoted to some a priori estimates on \( u_{\varepsilon, \beta} \). We denote with \( C_0 \) the constants which depend only on the initial data, and with \( C(T) \) the constants which depend also on \( T \).

**Lemma 2.1.** Assume (1.4). For each \( t > 0 \),

\[
\| u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \| \partial_x^2 u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \\
+ 2\beta \varepsilon \int_0^t \| \partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\]

(2.1)

In particular, we have

\[
\| u_{\varepsilon, \beta}(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq C_0 \beta^{-\frac{1}{4}}.
\]

(2.2)

**Proof.** Multiplying (1.3) by \( u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta} \), we have

\[
(u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} + A(u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \\
+ (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \beta \partial_{xxx}^3 u_{\varepsilon, \beta} - B\beta (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t (u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta}) \\
- C\beta (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} - \varepsilon (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta} = 0.
\]

(2.3)

Since

\[
\int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} \, dx \\
= \frac{1}{2} \frac{d}{dt} \left( \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right),
\]

\[
A \int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \, dx = -A\beta \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \, dx,
\]

\[
\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \beta \partial_{xxx}^3 u_{\varepsilon, \beta} \, dx = -\beta \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \partial_x^3 u_{\varepsilon, \beta} = 0,
\]

\[
-B\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t (u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta}) \, dx \\
= B\beta \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \, dx + B\beta \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} (u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta}) \, dx,
\]

\[
-C\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \, dx \\
= -C\beta \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \, dx + C\beta \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} (\partial_x^2 u_{\varepsilon, \beta})^2 \, dx,
\]

\[-\varepsilon \int_{\mathbb{R}} (u_{\varepsilon, \beta} - \beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta} \, dx = \varepsilon \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \varepsilon \| \partial_x^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]
Therefore, (2.1) follows from (1.4), (1.9) and an integration on (0, t).

Thus, from (2.4),

\[
\begin{align*}
&2B\beta^2 \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \beta} \partial_x \left( u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \right) dx = -2B\beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x \left( \partial_{xx}^2 u_{\varepsilon, \beta} \right)^2 dx \\
&= B\beta^2 \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \left( \partial_{xx}^2 u_{\varepsilon, \beta} \right)^2 dx.
\end{align*}
\]

Thus, from (2.2),

\[
\begin{align*}
&\frac{d}{dt} \left( \|u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} \right) + 2\varepsilon \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + 2\beta\varepsilon \|\partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} - 2\beta (A - B + C) \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} dx \\
&\quad + \beta^2 (B + 2C) \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \left( \partial_{xx}^2 u_{\varepsilon, \beta} \right)^2 dx = 0.
\end{align*}
\]

Thanks to (1.4), we have

\[
\begin{align*}
&A - B + C = 0, \\
&B + 2C = 0.
\end{align*}
\]

Therefore, (2.1) follows from (1.4), (1.9) and an integration on (0, t).

Finally, we prove (2.2). Due to (2.1) and the Hölder inequality,

\[
u_{\varepsilon, \beta}(t, x) = 2 \int_{-\infty}^{x} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} dx \leq 2 \int_{\mathbb{R}} |u_{\varepsilon, \beta}| |\partial_x u_{\varepsilon, \beta}| dx \\
\leq 2 \|u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0 \beta^{-\frac{1}{4}}.
\]

Therefore,

\[
|u_{\varepsilon, \beta}(t, x)| \leq C_0 \beta^{-\frac{1}{4}},
\]

which gives (2.2). \qed

Following [2], Lemma 2.2, or [3], Lemma 4.2, we prove the following result.

**Lemma 2.2.** Assume that (1.4) and (1.10) hold. Then, for each \( t > 0 \),

i) the family \( \{ u_{\varepsilon, \beta} \}_{\varepsilon, \beta} \) is bounded in \( L^\infty(\mathbb{R}^+; L^4(\mathbb{R})) \);

ii) the family \( \{ \varepsilon \partial_x u_{\varepsilon, \beta} \}_{\varepsilon, \beta} \) is bounded in \( L^\infty(\mathbb{R}^+; L^2(\mathbb{R})) \);

iii) the families \( \{ \varepsilon \partial_{xx} u_{\varepsilon, \beta} \}_{\varepsilon, \beta} \), \( \{ \varepsilon \sqrt{\varepsilon} \partial_{xx}^2 u_{\varepsilon, \beta} \}_{\varepsilon, \beta} \) are bounded in \( L^2(\mathbb{R}^+ \times \mathbb{R}) \).

Moreover,

\[
\begin{align*}
\beta \int_{0}^{t} \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot) \|_{L^1(\mathbb{R})} ds \leq C_0 \varepsilon^2, & \quad t > 0, \\
\beta^2 \int_{0}^{t} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq C_0 \varepsilon^5, & \quad t > 0.
\end{align*}
\]
\begin{align}
\beta^2 \int_0^t \left\| u_{e,\beta}(s, \cdot) \partial_{xx}^2 u_{e,\beta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds & \leq C_0 \varepsilon^3, \quad t > 0, \\
\beta \int_0^t \left\| u_{e,\beta}(s, \cdot) \partial_x u_{e,\beta}(s, \cdot) \partial_{xx}^2 u_{e,\beta}(s, \cdot) \right\|_{L^1(\mathbb{R})} \, ds & \leq C_0 \varepsilon, \quad t > 0. 
\end{align}

Proof. Let $K$ be a positive constant which will be specified later. Multiplying (1.8) by $K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}$, we have

\begin{align}
(K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_t u_{e,\beta} & + A (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) u_{e,\beta} \partial_x u_{e,\beta} \\
& \quad + \beta (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_{xxx}^3 u_{e,\beta} \\
& \quad - B \beta (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_x (u \partial_{xx}^2 u) \\
& \quad - C \beta (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \\
& \quad - \varepsilon (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_{xx}^2 u_{e,\beta} = 0. 
\end{align}

Observe that

\begin{align}
\int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_t u_{e,\beta} \, dx & = \frac{d}{dt} \left( \int_\mathbb{R} \frac{K}{4} \|u_{e,\beta}(t, \cdot)\|^4_{L^4(\mathbb{R})} + \frac{\varepsilon^2}{2} \|\partial_x u_{e,\beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} \right), \\
A \int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) u_{e,\beta} \partial_x u_{e,\beta} \, dx & = -A \varepsilon^2 \int_\mathbb{R} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx, \\
\beta \int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_{xxx}^3 u_{e,\beta} \, dx & = -3K \beta \int_\mathbb{R} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx, \\
-B \beta \int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_x (u \partial_{xx}^2 u) \, dx \\
& = 3BK \beta \int_\mathbb{R} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + \varepsilon^2 B \int_\mathbb{R} \partial_x \partial_{xx}^2 u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx \\
& = 3BK \beta \int_\mathbb{R} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + \frac{\varepsilon^2 B}{2} \int_\mathbb{R} \partial_{xx}^2 u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx, \\
-C \beta \int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx \\
& = -C \beta \int_\mathbb{R} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + \varepsilon^2 C \int_\mathbb{R} \partial_{xx}^2 u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx, \\
-\varepsilon \int_\mathbb{R} (K u_{e,\beta}^3 - \varepsilon^2 \partial_{xx}^2 u_{e,\beta}) \partial_{xx}^2 u_{e,\beta} \, dx \\
& = 3K \varepsilon \|u_{e,\beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^3 \|\partial_{xx}^2 u_{e,\beta}(t, \cdot)\|^2_{L^2(\mathbb{R})}. 
\end{align}
Therefore, integrating (2.10) over $\mathbb{R}$, from (1.4), we get
\[
\frac{d}{dt} \left( \frac{K}{4} \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^4_{L^4(\mathbb{R})} + \frac{\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \right) 
+ 3K\varepsilon \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon^3 \left\| \partial^2_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} 
= -A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial^2_x u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial^2_x u_{\varepsilon,\beta} dx 
+ \frac{7A}{3} K\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^3 \partial_x u_{\varepsilon,\beta} \partial^2_x u_{\varepsilon,\beta} dx 
\leq \varepsilon^2 |A| \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx + 3K\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 \partial_x u_{\varepsilon,\beta} \left| \partial^2_x u_{\varepsilon,\beta} \right| \partial_x u_{\varepsilon,\beta} dx 
+ \frac{7}{3} K\beta \int_{\mathbb{R}} \left| Au_{\varepsilon,\beta}^3 \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx.
\] (2.11)

Due to the Young inequality,
\[
\varepsilon^2 |A| \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx = \int_{\mathbb{R}} \left| \frac{\varepsilon}{2} \sqrt{3} A u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \frac{\varepsilon}{\sqrt{3}} \partial^2_x u_{\varepsilon,\beta} \right| 
\leq \frac{3\varepsilon A^2}{2} \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^3}{6} \left\| \partial^2_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Hence, from (2.11),
\[
\frac{d}{dt} \left( \frac{K}{4} \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^4_{L^4(\mathbb{R})} + \frac{\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \right) 
+ \varepsilon \left( 3K - \frac{3A^2}{2} \right) \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{5\varepsilon^3}{6} \left\| \partial^2_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} 
\leq 3K\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 \partial_x u_{\varepsilon,\beta} \left| \partial^2_x u_{\varepsilon,\beta} \right| \partial_x u_{\varepsilon,\beta} dx + \frac{7}{3} K\beta \int_{\mathbb{R}} \left| Au_{\varepsilon,\beta}^3 \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx.
\] (2.12)

Observe that, from (1.10),
\[
\beta \leq D_1 \varepsilon^4,
\]
where $D_1$ is a positive constant which will be specified later. It follows from (2.2), (2.13) and the Young inequality that
\[
3K\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 \partial_x u_{\varepsilon,\beta} \left| \partial^2_x u_{\varepsilon,\beta} \right| \partial_x u_{\varepsilon,\beta} dx \leq 3K\beta \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left| \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx 
\leq KC_0 \beta^2 \int_{\mathbb{R}} \left| \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx \leq \int_{\mathbb{R}} \left| \frac{\varepsilon}{\sqrt{3}} \partial^2_x u_{\varepsilon,\beta} \right| dx 
\leq C_0 D_1^2 K^2 \varepsilon \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^3}{6} \left\| \partial^2_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
\frac{7}{3} K\beta \int_{\mathbb{R}} \left| Au_{\varepsilon,\beta}^3 \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx \leq \frac{7}{3} K\beta \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left| Au_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx 
\leq KC_0 \beta^2 \int_{\mathbb{R}} \left| Au_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^2_x u_{\varepsilon,\beta} \right| dx \leq \int_{\mathbb{R}} \left| \frac{\varepsilon}{\sqrt{3}} \partial^2_x u_{\varepsilon,\beta} \right| dx 
\leq C_0 A^2 D_1^2 K^2 \varepsilon \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^3}{6} \left\| \partial^2_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\]
Therefore, we have
\[
\begin{align*}
\frac{d}{dt} \left( \frac{K}{4} \| u_{x, \beta}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{2} \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
&+ \varepsilon \left( 3K - \frac{3A^2}{2} \right) \| u_{x, \beta}(t, \cdot) \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{5\varepsilon^3}{6} \| \partial^2_{xx} u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
&\leq C_0 D_1^2 K^2 \varepsilon \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C_0 A^2 D_1^2 K^2 \| u_{x, \beta}(t, \cdot) \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\

\end{align*}
\]

that is
\[
\frac{d}{dt} \left( \frac{K}{4} \| u_{x, \beta}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{2} \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
+ \varepsilon \left( 3K - \frac{3A^2}{2} - C_0 A^2 D_1^2 K^2 \right) \| u_{x, \beta}(t, \cdot) \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ \frac{\varepsilon^3}{2} \| \partial^2_{xx} u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 D_1^2 K^2 \varepsilon \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 .
\]

We search a constant \( K \) such that
\[
C_0 A^2 D_1^2 K^2 - 6K + 3A^2 < 0.
\]

\( K \) does exist if and only if
\[
3 - C_0 A^4 D_1^2 > 0.
\]

Choosing
\[
D_1 = \frac{1}{\sqrt{C_0 A^2}},
\]

it follows from (2.15) and (2.17) that, there exist \( 0 < K_1 < K_2 \), such that for every
\[
K_1 < K < K_2
\]

(2.15) holds. Hence, from (2.18), choosing \( K_1 < K_3 < K_2 \), we get
\[
\begin{align*}
\frac{d}{dt} \left( \frac{K_3}{4} \| u_{x, \beta}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{2} \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
+ \varepsilon K_4 \| u_{x, \beta}(t, \cdot) \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^3}{2} \| \partial^2_{xx} u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
\leq K_5 \varepsilon \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 ,
\end{align*}
\]

where \( K_4 \) and \( K_5 \) are two fixed positive constants. Integrating (2.19) on \((0, t)\), from (1.9) and (2.1), we have
\[
\begin{align*}
\frac{K_3}{4} \| u_{x, \beta}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{2} \| \partial_x u_{x, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 & \\
+ \varepsilon K_4 \int_0^t \| u_{x, \beta}(s, \cdot) \partial_x u_{x, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \\
+ \frac{\varepsilon^3}{2} \int_0^t \| \partial^2_{xx} u_{x, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds & \\
\leq C_0 + K_5 \varepsilon \int_0^t \| \partial_x u_{x, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds & \leq C_0 (1 + K_5) \leq C_0 .
\end{align*}
\]
Then, 
\[ \|u_{\varepsilon, \beta}(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq C_0, \]
\[ \varepsilon^2 \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0, \]
(2.21) 
\[ \varepsilon \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \]
\[ \varepsilon^3 \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \]
for every \( t > 0 \). Thanks to (2.6), (2.13) and (2.21) and the Hölder inequality,
\[ \varepsilon \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds = \frac{\beta}{\varepsilon^2} \int_0^t \frac{1}{\varepsilon^2} \|\partial_x u_{\varepsilon, \beta}\|_{L^2(\mathbb{R})}^2 ds dx \]
\[ \leq \frac{\beta}{\varepsilon^2} \left( \varepsilon \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right) \frac{1}{\varepsilon^2} \left( \varepsilon^3 \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right)^{\frac{1}{2}} \]
\[ \leq C_0 \frac{\beta}{\varepsilon^2} \leq C_0 D_1 \varepsilon^2, \]
that is (2.6). Due to (2.13) and (2.21),
\[ \beta^2 \int_0^t \|\partial^2_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds = \frac{\beta^2}{\varepsilon^2} \int_0^t \|\partial^2_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0 D_1 \varepsilon^3, \]
which gives (2.7). It follows from (2.6), (2.13) and (2.21) that
\[ \beta^2 \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq \beta^2 \|u_{\varepsilon, \beta}\|_{L^2((0, \infty) \times \mathbb{R})}^2 \int_0^t \|\partial^2_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \]
\[ \leq \frac{\beta^2}{\varepsilon^3} \int_0^t \|\partial^2_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0 D_1 \frac{\varepsilon^6}{\varepsilon^3} \leq C_0 \varepsilon^3, \]
that is (2.8). From (2.1), (2.2), (2.13), (2.21) and the Hölder inequality,
\[ \beta \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \]
\[ \leq \beta \|u_{\varepsilon, \beta}\|_{L^2((0, \infty) \times \mathbb{R})}^2 \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}\|_{L^2(\mathbb{R})}^2 ds \]
\[ \leq C_0 \frac{\beta^2}{\varepsilon^2} \int_0^t \frac{1}{\varepsilon^2} \|\partial_x u_{\varepsilon, \beta}\|_{L^2(\mathbb{R})}^2 ds dx \]
\[ \leq C_0 \frac{\beta^2}{\varepsilon^2} \left( \varepsilon \int_0^t \|u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right) \frac{1}{\varepsilon^2} \left( \varepsilon^3 \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right)^{\frac{1}{2}} \]
\[ \leq C_0 D_1 \frac{\varepsilon^3}{\varepsilon^2} \leq C_0 \varepsilon, \]
which gives (2.9).

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The following technical lemma is needed [10].

Lemma 3.1. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^2 \). Suppose that the sequence \( \{L_n\}_{n \in \mathbb{N}} \) of distributions is bounded in \( W^{-1, \infty}(\Omega) \). Suppose also that
\[ L_n = L_{1, n} + L_{2, n}, \]
where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{\text{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$.

Moreover, we consider the following definition.

**Definition 3.1.** A pair of functions $(\eta, q)$ is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function and $q : \mathbb{R} \to \mathbb{R}$ is defined by

$$q(u) = \int_{0}^{u} A\xi \eta' (\xi) d\xi.$$  

An entropy-entropy flux pair $(\eta, q)$ is called convex/compactly supported if, in addition, $\eta$ is convex/compactly supported.

Following [9], we prove Theorem [1.1].

**Proof of Theorem [1.1].** Let us consider a compactly supported entropy–entropy flux pair $(\eta, q)$. Multiplying (1.8) by $\eta'(u_{\varepsilon, \beta})$, we have

$$\begin{align*}
\partial_t \eta'(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) &= \varepsilon \eta''(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta} - \beta \eta'(u_{\varepsilon, \beta}) \partial_{xxx} u_{\varepsilon, \beta} \\
&\quad - B \beta \eta'(u_{\varepsilon, \beta}) \partial_x (u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta}) - C \beta \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \\
&= I_{1, \varepsilon, \beta} + I_{2, \varepsilon, \beta} + I_{3, \varepsilon, \beta} + I_{4, \varepsilon, \beta} + I_{5, \varepsilon, \beta} + I_{6, \varepsilon, \beta} + I_{7, \varepsilon, \beta},
\end{align*}$$

where

$$\begin{align*}
I_{1, \varepsilon, \beta} &= \partial_x (\varepsilon \eta''(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta}), \\
I_{2, \varepsilon, \beta} &= -\varepsilon \eta'(u_{\varepsilon, \beta}) (\partial_x u_{\varepsilon, \beta})^2, \\
I_{3, \varepsilon, \beta} &= \partial_x (-\beta \eta'(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta}), \\
I_{4, \varepsilon, \beta} &= \beta \eta''(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta}, \\
I_{5, \varepsilon, \beta} &= \partial_x (-B \beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta}), \\
I_{6, \varepsilon, \beta} &= B \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta}, \\
I_{7, \varepsilon, \beta} &= -C \beta \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta}.
\end{align*}$$

We have

$$I_{1, \varepsilon, \beta} \to 0 \quad \text{in} \quad H^{-1}((0, T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.$$

Indeed, thanks to Lemma [2.1]

$$\begin{align*}
\| \varepsilon \eta''(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta} \|_{L^2((0,T) \times \mathbb{R})}^2 &\leq \| \eta'' \|_{L^\infty(\mathbb{R})}^2 \varepsilon^2 \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \\
&\leq \| \eta'' \|_{L^\infty(\mathbb{R})}^2 \varepsilon C_0 \to 0.
\end{align*}$$

We claim that

$$\{I_{2, \varepsilon, \beta}\}_{\varepsilon, \beta > 0} \quad \text{is bounded in} \quad L^1((0,T) \times \mathbb{R}), \quad T > 0.$$

Again by Lemma [2.1]

$$\begin{align*}
\| \varepsilon \eta''(u_{\varepsilon, \beta}) (\partial_x u_{\varepsilon, \beta})^2 \|_{L^1((0,T) \times \mathbb{R})} &\leq \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \\
&\leq \| \eta'' \|_{L^\infty(\mathbb{R})} C_0.
\end{align*}$$

We have that

$$I_{3, \varepsilon, \beta} \to 0 \quad \text{in} \quad H^{-1}((0, T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.$$
Thanks to Lemma 2.2,
\[ \| \beta^2 \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \|_{L^2((0,T) \times \mathbb{R})}^2 \leq \| \eta' \|_{L^\infty(\mathbb{R})}^2 \beta^2 \int_0^T \| \partial_x \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
\[ \leq \| \eta' \|_{L^\infty(\mathbb{R})}^2 \beta^2 \int_0^T \| \partial_x \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
We show that
\[ I_{1, \varepsilon, \beta} \to 0 \text{ in } L^1((0,T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
Again by Lemma 2.2,
\[ \| \beta^2 \eta'(u_{\varepsilon, \beta}) \partial_x \partial_x u_{\varepsilon, \beta} \|_{L^1((0,T) \times \mathbb{R})} \]
\[ \leq \| \eta' \|_{L^\infty(\mathbb{R})}^2 \beta^2 \int_0^T \| \partial_x \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
We claim that
\[ I_{5, \varepsilon, \beta} \to 0 \text{ in } H^{-1}((0,T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
By Lemma 2.2,
\[ \left\| B \beta \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})} \]
\[ \leq |B| \| \eta' \|_{L^\infty(\mathbb{R})} \beta^2 \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \partial_x^2 u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
We have that
\[ I_{6, \varepsilon, \beta} \to 0 \text{ in } L^1((0,T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
Again by Lemma 2.2,
\[ \left\| B \beta \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \right\|_{L^1((0,T) \times \mathbb{R})} \]
\[ \leq |B| \| \eta' \|_{L^\infty(\mathbb{R})} \beta^2 \int_0^T \| \partial_x \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^1(\mathbb{R})}^2 \, ds \]
We claim that
\[ I_{7, \varepsilon, \beta} \to 0 \text{ in } L^1((0,T) \times \mathbb{R}), \quad T > 0, \text{ as } \varepsilon \to 0. \]
By Lemma 2.2,
\[ \left\| C \beta \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \right\|_{L^1((0,T) \times \mathbb{R})} \]
\[ \leq |C| \| \eta' \|_{L^\infty(\mathbb{R})} \beta^2 \int_0^T \| \partial_x \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
Therefore, (1.11) follows from Lemma 3.1 and the $L^p$ compensated compactness. We have to show that (1.12) holds. We begin by proving that $u$ is a distributional solution of (1.6). Let $\phi \in C^\infty(\mathbb{R}^2)$ be a test function with compact support. We have to prove that
\[ \int_0^\infty \int_\mathbb{R} \left( u_0(x) \phi(0, x) + \frac{A u^2}{2} \partial_x \phi \right) \, dt \, dx + \int_\mathbb{R} u_0(x) \phi(0, x) \, dx = 0. \]
We have that
\[
\int_0^\infty \left( u_{\varepsilon_n, \beta_n} \partial_t \phi + \frac{A n^2}{2} \partial_x \phi \right) \, dt \, dx + \int_R u_{0, \varepsilon_n, \beta_n}(x) \phi(0, x) \, dx \\
+ \varepsilon_n \int_0^\infty u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \phi \, dt \, dx + \varepsilon_n \int_0^\infty u_{0, \varepsilon_n, \beta_n}(x) \partial_{xx}^2 \phi(0, x) \, dx \\
+ \beta_n \int_0^\infty u_{\varepsilon_n, \beta_n} \partial_{xxx}^3 \phi \, dt \, dx + \beta_n \int_0^\infty u_{0, \varepsilon_n, \beta_n}(x) \partial_{xxx}^3 \phi(0, x) \, dx \\
= B \beta_n \int_0^\infty u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx - C \beta_n \int_0^\infty \partial_x u_{\varepsilon_n, \beta_n} \partial_{xx}^2 u_{\varepsilon_n, \beta_n} \phi \, dt \, dx.
\]

Let us show that
\[
(3.4) \quad B \beta_n \int_0^\infty u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \to 0.
\]

Fix $T > 0$. Due to (1.10), (2.2), Lemma 2.2 and the Hölder inequality,
\[
|B| \beta_n \int_0^\infty \int_R u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \\
\leq |B| \beta_n \int_0^\infty \int_R |u_{\varepsilon_n, \beta_n}| \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \\
\leq |B| \beta_n \|u_{\varepsilon_n, \beta_n}\|_{L^\infty((0, T) \times \mathbb{R})} \int_0^\infty \int_R \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \\
\leq |B| C_0 \varepsilon_n \frac{3}{2} \|\partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n}\|_{L^2((0, T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2(\text{supp}(\partial_x \phi))} \\
\leq |B| C_0 \varepsilon_n \|\partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n}\|_{L^2((0, T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0, T) \times \mathbb{R})} \\
\leq |B| C_0 \varepsilon_n \to 0,
\]
that is (3.4).

We prove that
\[
(3.5) \quad - C \beta_n \int_0^\infty \partial_x u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \to 0.
\]

Fix $T > 0$. Thanks to Lemma 2.2,
\[
|C| \beta_n \int_0^\infty \int_R \partial_x u_{\varepsilon_n, \beta_n} \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \\
\leq |C| \beta_n \int_0^\infty \int_R |\partial_x u_{\varepsilon_n, \beta_n}| \partial_{xx}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi \, dt \, dx \\
\leq |C| \|\phi\|_{L^\infty(\text{supp}(\phi))} \beta_n \|\partial_x u_{\varepsilon_n, \beta_n}\|_{L^1(\text{supp}(\phi))} \\
\leq |C| \|\phi\|_{L^\infty((0, T) \times \mathbb{R})} \beta_n \|\partial_x u_{\varepsilon_n, \beta_n}\|_{L^1((0, T) \times \mathbb{R})} \\
\leq |C| \|\phi\|_{L^\infty((0, T) \times \mathbb{R})} C_0 \varepsilon_n^2 \to 0,
\]
which gives (3.5). Therefore, (3.2) follows from (1.9), (1.11), (3.3), (3.4) and (3.5).

We conclude by proving that $u$ is the unique entropy solution of (1.6). Fix $T > 0$. Let us consider a compactly supported entropy–entropy flux pair $(\eta, q)$, and $\phi \in C_c^\infty((0, \infty) \times \mathbb{R})$ a non–negative function. We have to prove that
\[
(3.6) \quad \int_0^\infty (\partial_t \eta(u) + \partial_x q(u)) \phi \, dt \, dx \leq 0.
\]
We have

\[
\int_0^\infty \int_\mathbb{R} \left( \partial_x \eta(u_{\varepsilon}, \beta_n) + \partial_x q(u_{\varepsilon}, \beta_n) \right) \phi \, dt \, dx \\
= \varepsilon_n \int_0^\infty \int_\mathbb{R} \left( \eta'(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \right) \phi \, dt \, dx \\
- \varepsilon_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon}, \beta_n) (\partial_x u_{\varepsilon}, \beta_n)^2 \phi \, dt \, dx \\
- \beta_n \int_0^\infty \int_\mathbb{R} \partial_x (\eta'(u_{\varepsilon}, \beta_n) \partial_{xx} u_{\varepsilon}, \beta_n) \phi \, dt \, dx \\
+ \beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \phi \, dt \, dx \\
- B\beta_n \int_0^\infty \int_\mathbb{R} \partial_x (\eta'(u_{\varepsilon}, \beta_n) \partial_{xx}^2 u_{\varepsilon}, \beta_n) \phi \, dt \, dx \\
+ B\beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \phi \, dt \, dx \\
- C\beta_n \int_0^\infty \int_\mathbb{R} \eta' (u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \phi \, dt \, dx \\
\leq -\varepsilon_n \int_0^\infty \int_\mathbb{R} \eta'(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx} u_{\varepsilon}, \beta_n \phi \, dt \, dx \\
+ \beta_n \int_0^\infty \int_\mathbb{R} \eta'(u_{\varepsilon}, \beta_n) \partial_{xx}^2 u_{\varepsilon}, \beta_n \partial_x \phi \, dt \, dx \\
+ \beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \, dt \, dx \\
+ B\beta_n \int_0^\infty \int_\mathbb{R} \eta'(u_{\varepsilon}, \beta_n) \partial_{xx}^2 u_{\varepsilon}, \beta_n \partial_x \phi \, dt \, dx \\
+ B\beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \, dt \, dx \\
- C\beta_n \int_0^\infty \int_\mathbb{R} \eta'(u_{\varepsilon}, \beta_n) \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n \phi \, dt \, dx \\
\leq \varepsilon_n \int_0^\infty \int_\mathbb{R} |\eta'(u_{\varepsilon}, \beta_n)| |\partial_x u_{\varepsilon}, \beta_n| |\partial_{xx} \phi| \, dt \, dx \\
+ \beta_n \int_0^\infty \int_\mathbb{R} |\eta'(u_{\varepsilon}, \beta_n)| |\partial_{xx}^2 u_{\varepsilon}, \beta_n| |\partial_x \phi| \, dt \, dx \\
+ \beta_n \int_0^\infty \int_\mathbb{R} |\eta''(u_{\varepsilon}, \beta_n)| |\partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n| |\phi| \, dt \, dx \\
+ |B| \beta_n \int_0^\infty \int_\mathbb{R} |\eta'(u_{\varepsilon}, \beta_n)| |u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n| |\partial_x \phi| \, dt \, dx \\
+ |B| \beta_n \int_0^\infty \int_\mathbb{R} |\eta''(u_{\varepsilon}, \beta_n)| |u_{\varepsilon}, \beta_n \partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n| |\phi| \, dt \, dx \\
+ |C| \beta_n \int_0^\infty \int_\mathbb{R} |\eta'(u_{\varepsilon}, \beta_n)| |\partial_x u_{\varepsilon}, \beta_n \partial_{xx}^2 u_{\varepsilon}, \beta_n| |\phi| \, dt \, dx.
\]

Hence, from (2.2),

\[
\int_0^\infty \int_\mathbb{R} \left( \partial_x \eta(u_{\varepsilon}, \beta_n) + \partial_x q(u_{\varepsilon}, \beta_n) \right) \phi \, dt \, dx
\]
Due to (1.10), Lemma (2.2) and the Hölder inequality, we claim that

\[ \beta_n \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\varepsilon n, \beta_n}| |\partial_x \phi| dt dx \to 0. \]  

Due to (1.10), Lemma (2.2) and the Hölder inequality,

\[ \beta_n \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\varepsilon n, \beta_n}| |\partial_x \phi| dt dx \leq C_0 \beta_n^3 \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\varepsilon n, \beta_n}| L^2(\text{supp}(\phi)) + C_0 \beta_n^3 \int_0^\infty \int_{\mathbb{R}} |\partial_x \phi| L^2(\text{supp}(\phi)) \]

that is (3.8).
Thanks to Lemmas 2.1, 2.2, and the Hölder inequality,
\[ \beta_3 \int_0^{\infty} \int_{\mathbb{R}} |\partial_x u_{\varepsilon_n, \beta_n} \partial^2_{xx} u_{\varepsilon_n, \beta_n}| \partial_x \phi |dt dx \]
\[ \leq C_0 \varepsilon_n^3 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_{\varepsilon_n, \beta_n} \partial^2_{xx} u_{\varepsilon_n, \beta_n} \|_{L^1(\text{supp} (\phi))} \]
\[ \leq C_0 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon_n \int_0^T \int_{\mathbb{R}} \varepsilon_n^\frac{1}{2} |\partial_x u_{\varepsilon_n, \beta_n} | \varepsilon_n^\frac{1}{2} |\partial^2_{xx} u_{\varepsilon_n, \beta_n} | |dt dx \]
\[ \leq C_0 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon_n \left( \varepsilon_n \int_0^T \| \partial_x u_{\varepsilon_n, \beta_n} (t, \cdot) \|_{L^2(\mathbb{R})} dt \right)^\frac{1}{2} \]
\[ \cdot \left( \varepsilon_n^3 \int_0^T \| \partial^2_{xx} u_{\varepsilon_n, \beta_n} (t, \cdot) \|_{L^2(\mathbb{R})} dt \right)^\frac{1}{2} \]
\[ \leq C_0 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon \to 0, \]
which gives (3.8).

Finally, (3.6) follows from (1.10), (1.11), (3.7), (3.8), (3.9) and Lemmas 2.1 and 2.2. □

**Appendix A. On the case**

Theorem 1.1 holds also in the cases \( A = C^2 \) and \( A = C^{2n} \), with \( C \neq 0 \) and \( n \in \mathbb{N} \).
Indeed, from (2.5), if \( A = C^2 \), we get \( C = -3 \), while if \( A = C^{2n} \), we obtain \( C = -3^{\frac{1}{2n-1}} \).
If \( A = C^{2n+1} \), from (2.5), we get
\[ C^{2n} + 3 = 0, \]
which does not have solutions in \( \mathbb{R} \).

In this section, we prove that Theorem 1.1 holds also in the case \( A = (C + \alpha)^{2n} \), where \( \alpha \) is a suitable real number. We only need to prove the following result

**Lemma A.1. Assume that**

(A.1) \( A = (C + \alpha)^n \).

If

(A.2) \( \alpha \leq 3^{\frac{1}{2n-1}} \left( \frac{1}{2n} \right)^\frac{2n}{2n-1} + \left( \frac{3}{2n} \right)^\frac{1}{2n-1} \),

then (2.1) holds.

**Proof.** We begin by observing that, by (2.5), we have
\[ (C + \alpha)^{2n} + 3C = 0, \]
that is
(A.3) \( (C + \alpha)^{2n} + 3 (C + \alpha) - 3\alpha = 0. \)
Let us consider the following function
(A.4) \( g(X) = X^{2n} + 3X - 3\alpha. \)
We observe that
(A.5) \( \lim_{X \to -\infty} g(X) = \infty, \quad \lim_{X \to \infty} g(X) = \infty. \)
Since \( g'(X) = 2nX^{2n-1} + 3 \), we have that

\[
(A.6) \quad g \text{ is increasing in } \left( -\left( \frac{3}{2n} \right)^{\frac{1}{2n-1}}, \infty \right).
\]

From \((A.2)\),

\[
(A.7) \quad g(X_0) \leq 0, \quad X_0 = -\left( \frac{3}{2n} \right)^{\frac{1}{2n-1}}.
\]

Then, it follows from \((A.5), \ (A.6) \) and \((A.7)\) that the function \( g \) has only two zeros \( X_1 < 0 < X_2 \). Hence, from \((A.1)\),

\[
A = X_1^{2n}, \text{ or } A = X_2^{2n}.
\]

Therefore, arguing as in Lemma 2.1, we have \((2.1)\). \(\square\)

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