Forecasting extreme events in collective dynamics: 
an analytic signal approach to detecting discrete scale invariance

G. M. Viswanathan
Instituto de Física, Universidade Federal de Alagoas, CEP 57072-970, Maceió–AL, Brazil
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A challenging problem in physics concerns the possibility of forecasting rare but extreme phenomena such as large earthquakes, financial market crashes, and material rupture. A promising line of research involves the early detection of precursory log-periodic oscillations to help forecast extreme events in collective phenomena where discrete scale invariance plays an important role. Here I investigate two distinct approaches towards the general problem of how to detect log-periodic oscillations in arbitrary time series without prior knowledge of the location of the moveable singularity. I first show that the problem has a definite solution in Fourier space, however the technique involved requires an unrealistically large signal to noise ratio. I then show that the quadrature signal obtained via analytic continuation onto the imaginary axis, using the Hilbert transform, necessarily retains the log-periodicities found in the original signal. This finding allows the development of a new method of detecting log-periodic oscillations that relies on calculation of the instantaneous phase of the analytic signal. I illustrate the method by applying it to the well documented stock market crash of 1987. Finally, I discuss the relevance of these findings for parametric rather than nonparametric estimation of critical times.

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I. INTRODUCTION

More than a decade of pioneering research involving catastrophic phenomena as diverse as the rupture of high pressure rocket tanks [2], stock market crashes [3] and earthquakes [4] has lent growing credibility [5, 6] to the hypothesis that such extreme events arise due to coherent large-scale collective behaviors observed in self-organizing systems [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13]. An exciting prospect concerns the possibility of prediction or forecasting of catastrophic events based on the observation of discrete scale invariance. This innovative approach, when applied to problems such as the prediction of earthquakes or financial crashes, questions the common assumption that the absence of characteristic scales seen in self-organizing [6, 14] complex systems precludes the possibility of forecasting [6]. Instead, prediction becomes possible due to the appearance of smaller precursory events that in principle can help determine the critical time \( t_c \) of the catastrophic event, which one can interpret as a finite time moveable singularity. One does not directly observe the singularity due to finite size effects. Instead we observe an ultra-large event comparable in magnitude to the system size. The best known specific signature of discrete scale invariance involves log-periodicity [5, 6, 13, 14, 15, 17]. Even a small improvement in the ability to detect log-periodic oscillations may thus have a relatively large impact, with potentially useful applications.

Attempts to forecast extreme events by exploiting discrete scale invariance and log-periodicity implicitly assume an underlying information-carrying property of some component in the signal studied. Indeed, in trying to forecast an event for some variable \( f(t) \) that will occur at a future time \( t = t_c \), with the information available for \( f(t) \) at the present time \( t < t_c \), one implicitly assumes the existence of correlations in the behavior of \( f(t) \). Such correlations imply that the knowledge of the behavior of \( f(t) \) in a certain period necessarily provides information about the behavior at other (e.g., future) times. For a continuous variable \( f \), the maximum degree of correlation arises for holomorphic or analytic \( f(t) \), since one can then use analytic continuation to know \( f(t) \) for all future times. No new information becomes available when time elapses, because \( f(t) \) evolves deterministically. Consider the well known example of classical Hamiltonian systems. The fine grained Gibbs entropy, equivalent to the Shannon information measure, becomes a constant of the motion for such systems due to Liouville's theorem (see ref. [19] for a discussion of other information measures). Indeed, information can increase only if deterministic evolution becomes interrupted in some way, e.g., perhaps by some stochastic process. Such interruptions of deterministic evolution necessarily lead to breaks in analytic behavior. From such considerations, it follows as a logical consequence that the maximum possible rate of transmission of information measured in bits per unit time for a physical communication channel must equal or exceed the rate of occurrence of non-analytic points in the signal [20].

Detecting log-periodicity presents unique challenges. Direct parametric estimation to obtain log-periodic fits can fail due to the presence of extreme fluctuations as well as due to the problem of large degeneracy of the solutions, i.e., there exist too many good fits that approximate the best fit. Parametric methods also fail because often we do not know which underlying distribution to assume. Moreover, the large scale catastrophic events of interest may represent “outliers” that do not follow
the same distribution as the smaller scale events. Hence, most of the research has tended to apply non-parametric methods.

A widely used spectroscopic method for detecting log-periodic oscillations involves changing the variable \( t \) to a log-time \( \tau \equiv \ln(t_c - t) \), and then studying the power spectrum of the new series thus generated \[2\]. Log-periodic oscillations will appear periodic in the log-time \( \tau \). However, the data will no longer appear evenly sampled. Hence standard FFT-based methods do not work and instead one must obtain the spectrum via the Lomb periodogram \[21\], which can handle unevenly sampled points. In practice this method works remarkably well.

Periodic oscillations will appear periodic in the log-time \( \tau \).\textsuperscript{1} Hence standard FFT-based methods do not work without having prior knowledge of \( t_c \). The methods developed here to the study of cooperative economic phenomena will in principle allow us to “fine tune” our estimates on \( t_c \). The methods developed here apply equally to a variety of time series, so a range of applications become possible.

In this context, one of the most dreaded collective phenomena of our times relates to the financial and economic crises that have punctuated our history since the industrial revolution. Economic events, ranging in size and diversity from the Great Depression to the ongoing bursting of the real estate bubble in the US, affect an at least order of magnitude greater number of people than earthquakes or tidal waves. Moreover, humans actively participate in the dynamics of the economy whereas we only passively watch tectonic plate movements. Indeed, financial crises have the potential to affect almost everybody (unlike, e.g., earthquakes). For such reasons, this article limits the application of the new methods developed here to the study of cooperative economic phenomena. Specifically, I have chosen to focus on the classic financial “correction” of 1987, when stock market indices dropped \( \approx 20\% \) in an amazing display of cooperative behavior (i.e., “herding”) of otherwise rational individual agents.

In Section \[II\] I briefly review discrete scale invariance and log-periodicity. In Section \[III\] I develop techniques for detecting discrete scale invariance. In Sections \[IV-V\] I illustrate the method and then apply it to the stock market crash of 1987. Section \[VI\] concludes with a discussion and a summary.

\section{Complex Dimensions and Discrete Scale Invariance}

The concept of dimensionality has undergone successive generalizations: from integer to fractional to negative to complex \[22\]. The fractional and complex \[11\] dimensions have a relation to fractals \[23, 24\] and scale invariance symmetry, i.e., when a system’s property appears unchanged under a transformation of scale. Power laws, such as \( f(t) \sim t^\alpha \), play a fundamental role in describing scale invariance. A change of scale by a factor \( \lambda \) does not alter the power law behavior: \( f(\lambda t) = f(t)\lambda^\alpha \). Real power law exponents usually involve continuous scale invariance. In contrast, complex exponents bear a relation to discrete scale invariance—a discrete rather than continuous symmetry that holds only for certain discrete values of the magnification \( \lambda_c = \lambda^\alpha \). The powerful formalism that emerged from the study and exploitation of scale invariance in physical systems \[22\] became known as the renormalization group (e.g., see ref. \[20\]). These advances have led to the application of fractal concepts and techniques to diverse systems, ranging from the study of anomalous random walks \[27, 28\] and critical points \[29\] to heart dynamics \[30\] and DNA organization \[31, 32\].

To model a catastrophic event that corresponds to a moveable singularity at time \( t = t_c \), we can consider an arbitrary signal \( x(t) \) in terms of the renormalization group formalism as follows:

\[
F(t_c - t) = x(t_c) - x(t) \]

\[ t_c - t' = \phi(t_c - t) \]

where \( \phi \) denotes the flow map and \( t_c \) the critical time \[32, 34\]. The flow map acts like a “zoom,” mapping the time \( t \) to a new time \( t' \). We can then express \( F(t) \) as a sum of a singular part and a non-singular part as follows:

\[
F(t_c - t) = g(t_c - t) + \frac{1}{\mu} F(\phi(t_c - t)) .
\]

Only the singular part contributes to the ultra-large event, whereas the non-singular part only describes normal events. Close to the critical point, we can apply the linear approximation \( \phi(t) = \lambda t \) to obtain the power law solution, which satisfies

\[
\frac{dF(t_c - t)}{d\ln|t_c - t|} = \alpha F(t_c - t) \]

with \( \alpha = \ln(\mu/\ln \lambda) \), i.e., we essentially ignore the non-singular part. In practice this solution guarantees that continuous scale invariance shows up as straight lines on double log plots, with the slope given by \( \alpha \), which plays the role of a fractal dimension or a scaling exponent. If we allow this dimension or exponent to become complex, \( \alpha = z + i\omega \), then the power law \( (t_c - t)^\alpha \) becomes \( (t_c - t)^z \exp[\omega \ln|t_c - t|] \), i.e., a power law modulated by oscillations with angular frequency \( \omega \) in the logarithm of the time—hence the term log-periodic. Discrete scale invariance leads to complex exponents \( \alpha_n = z + i\omega_n \), with \( \omega_n = 2\pi n/\ln \lambda \).

In a number of applications, the first order representation

\[
x(t) = A + B(t_c - t)^z + C(t_c - t)^z \cos[\omega \ln(t_c - t) + \theta] \] (1)

captures enough of the relevant behavior to become useful in forecasting and prediction applications \[33\]. Further
renormalization group symmetry considerations can lead to higher order representations useful in some cases [33]. A different approach to extending Eq. 1 involves the inclusion of higher harmonics. However, the linear approximation leads us to expect the amplitude of the higher order log-periodic corrections to decay exponentially fast as a function of the order \( l \) of the harmonics [3]. The true behavior (i.e., as opposed to the linear approximation) of the higher order harmonics leads to a slower exponential decay of the higher order harmonics. Nevertheless, the first harmonic still provides a good fit and can account for the experimental data. Having reviewed the basics of discrete scale invariance, I next address the problem of detecting it in arbitrary time series.

### III. ANALYTIC BEHAVIOR AND DISCRETE SCALE INVARIANCE

The relationship between analyticity and information flow discussed in Section I has inspired and allowed the development here of methods for detecting discrete scale invariance in arbitrary time series.

#### A. Detecting discrete scale invariance in Fourier space

How can we exploit the role of correlations to detect log-periodicity without prior knowledge of \( t_c \)? The power spectrum, defined as the modulus squared of the Fourier transform of the time series, allows us to measure correlations. Indeed, one could also equivalently define the power spectrum as the Fourier transform of the standard two-point autocorrelation function. As a starting point, let us consider a useful but not widely known fact about log-periodic time series and their Fourier transforms. Discrete scale invariance in a time series can sometimes also become manifest in the frequency domain.

Mathematically, log-periodicity in the time domain appears in the frequency domain because complex exponents also appear in the Fourier transform of the signal. Consider the indefinite Fourier integral

\[
\int dt \exp[i\omega't] \cos [\omega \ln(t_c - t)]
\]

with complex quantities and represents an antiderivative with respect to the integration variable \( t \). Note the terms of the form \( \omega' - i\omega \). In practice we can evaluate this integral with lower and upper integration limits \( t = -\infty \) and \( t = t_c - \epsilon \) to calculate the Fourier transform.

Figs. 1 and 2 show log-periodic time series and their power spectra \( S(f) \), defined as the modulus squared of the Fourier transform of the time series. We clearly see that the log-periodicity in the time domain manifests itself as log-periodicity in the frequency domain, within a range of frequencies. Specifically, the log-periodic scaling breaks down in the spectra at low and high frequencies and the cutoff frequencies depend both on the value of \( \omega \) as well as the temporal separation from the singularity. Except for these high and low frequency cutoffs, discrete scale invariance in the time domain manifests itself as discrete scale invariance in the frequency domain. Note that a similar relation holds for continuous scale invariance, i.e., the Fourier transform of a power law tailed function \( f(t) \sim t^{-\alpha} \) can also have a power law behavior (e.g., at low frequencies).

An increase in the critical time \( t_c \) leads to a decrease in the upper cutoff frequency in the log-periodicity of the spectra. In principle one could exploit this relationship. Indeed, we can show in a straightforward manner that discrete scale invariance in Fourier space will break down near an upper cutoff frequency \( f_{\text{high}} \) given by

\[
\ln[1 + 1/(f_{\text{high}}(t_c - t_{\text{max}}))] \approx 2\pi/\omega ,
\]

where \( t_{\text{max}} \) denotes the largest time contained in the time series. Except for too small \( \omega \), we can approximate

\[
\omega \approx \frac{2\pi(t_c - t_{\text{max}})}{f_{\text{high}}}.
\]

Hence, one could thus estimate \( t_c \) knowing \( \omega \) and the upper cutoff frequency in the spectra. We find that this relationship agrees fairly well for the data shown in Fig. 1. Yet, in practice I expect that this method may not work very well with real data. For realistic time series, the spectrum contains too many other features that drown out the log-periodic behavior. Perhaps one cannot systematically and reliably apply this method to detect log-periodic oscillations in realistic scenarios. Nevertheless, the concept appears to have validity on a fundamental level.

Moreover, these findings offer insight about the potential of investigating analytic behavior for detecting the crucial log-periodicities. Indeed, the basic premise of forecasting based on exploiting “hidden” analytic properties appears valid. Continuing this line of reasoning leads to a second approach. Next, I show below that the analytic signal obtained using the Hilbert transform of a time series can help to isolate the log-periodic signature in arbitrary time series. I briefly outline the method here before describing it detail below. The method involves taking the Hilbert transform of the time series to obtain the quadrature or analytic signal, consisting of the instantaneous amplitude and the instantaneous phase or argument. If the time series behaves purely log-periodically
FIG. 1: Examples of log-periodic signals $f(t) = \cos[\omega \ln(t_c - t)]$ shown for $t = 1, 2, 3, \ldots, 2^{16}$ with $\omega = 30$ and $t_c = 2^{16} + 1$ (a), $t_c = 65600$ (b) and $t_c = 70000$ (c), along with their corresponding power spectra (d,e,f). An increase in the critical time $t_c$ leads to a decrease in the upper cutoff frequency in the log-periodicity of the spectra. In principle, one could thus estimate $t_c$.

In the more realistic case of a time series with a small log-periodic component, the analytic signal phasor will rotate log-periodically in the complex plane not around the origin, but rather around some other point on the complex plane. In principle this “center” or “focus” of log-periodic rotation can itself fluctuate in time, due to the many other components that contribute to the analytic signal. Hence, in the case of a small log-periodic component in the time series, the phase of the analytic signal will not vary as $\sim \omega \ln(t_c - t)$. Instead the phase will have a component that oscillates log-periodically, due to the angle subtended at the origin by the analytic signal phasor tip. The log-periodicity contained in a time series need not necessarily appear in the amplitude of the analytic signal (e.g., consider how a pure log-periodic oscillation has constant amplitude). In contrast, the log-periodicity necessarily appears in the phase of the analytic signal. By studying the instantaneous phase, we may thus enhance or highlight the log-periodicity apparent in time series.

**B. The Hilbert transform**

The Hilbert transform $Hf(x)$ of a function $f(x)$ represents the convolution of the function $f$ with $1/\pi x$. Mathematically, we define the Hilbert transform as a Cauchy principal value,

$$Hf(x) \equiv \lim_{\epsilon \to 0, R \to \infty} \left[ \int_{-R}^{x-\epsilon} dy \frac{f(y)}{y-x} + \int_{x+\epsilon}^{-R} dy \frac{f(y)}{y-x} \right],$$

to avoid the singularity at $x$. Moreover, cancellation towards $\pm \infty$ allows non-integrable functions to have well defined Hilbert transforms. The Hilbert transform also corresponds to the inverse Fourier transform of the product of the Fourier transform of $f(x)$ with $i \text{sgn}(x)$ (where the latter gives the Fourier transform of $1/\pi x$). Hence the Hilbert transform effectively maintains the Fourier amplitudes but shifts all phases by $-\pi/2$. Hence $H^2 = -I$.

The Hilbert transform of $x(t)$ allows us to define an...
FIG. 2: More examples of log-periodic signals \( f(t) = \cos[\omega \ln(t_c - t)] \) shown for \( t = 1, 2, 3, \ldots 2^{16} \) with \( t_c = 2^{16} + 1 \) and \( \omega = 30 \) (a), \( \omega = 20 \) (b) and \( \omega = 10 \) (c), along with their respective power spectra (d,e,f). As a general rule, for low frequencies at least, discrete scale invariance in the time domain also manifests itself in the frequency domain. Notice that variation of the time-domain angular frequency \( \omega \) leads to a change in the log-periodicity observed in the frequency-domain. In principle, one could thus estimate \( t_c \) knowing \( \omega \) and the upper cutoff frequency in the spectra (see. Fig 1). In practice, however, we cannot rely on this method for the more realistic case of noisy data. This limitation motivates other approaches for estimating \( t_c \) for actual experimentally measured time series.

analytic signal

\[
\chi(t) \equiv x(t) + iHx(t) = A(t) \exp[i\varphi(t)],
\]

where \( A(t) \) and \( \varphi(t) \) represent the instantaneous amplitude and phase of the signal. This and related properties have led Hilbert transforms to have important and diverse applications (e.g., see refs. [35, 36, 37]).

C. Analytic signal of log-periodic data

For a pure log-periodic analytic signal

\[
\chi(t) = \exp[i\omega \ln[t_c - t]]
\]

the “unwrapped” instantaneous phase will follow

\[
\varphi(t) = \omega \ln[t_c - t].
\]

Since in Nature we typically observe log-periodicity “decorating” a power law or else hidden in noisy data, we must consider an analytic signal of the form

\[
\chi(t) = \chi_0(t) + a \exp[i\omega \ln[t_c - t]] = A(t) \exp[i\varphi(t)],
\]

with \( a \ll \chi_0(t) \). Let \( \chi_0(t) = A_0(t) \exp[i\varphi_0(t)] \). Then

\[
A(t) = \sqrt{A_0^2(t) + a^2 + 2A_0(t) a \cos [\varphi_0(t) - \omega \ln[t_c - t]]}
\]

and

\[
\varphi(t) = \arctan \left[ \frac{A_0(t) \sin[\varphi_0(t)] + a \sin [\omega \ln[t_c - t]]}{A_0(t) \cos[\varphi_0(t)] + a \cos [\omega \ln[t_c - t]]} \right].
\]

Notice that log-periodicity need not appear in \( A(t) \), by considering for instance, the important case \( A_0(t) = 0 \), for which we obtain \( A(t) = a \) = constant. In contrast, log-periodicity will always appear in the phase \( \varphi(t) \), which contains the information arising from the complex exponents associated with discrete scale invariance. The phase, calculated as an arctan will belong to a single branch on the complex plane, but we can “unwrap”
FIG. 3: (a) Example of a log-periodicity “decorating” a power law, shown for times $t$ prior to the singularity $t_c$. Formula used: $f(t) = \cos(50 \cdot \log 10(11000 - t)) + 1000 \cdot (11000 - t)^{-0.5}$. (b) A typical noisy time series $g(t)$ containing the log-periodic component shown in (a) above, with a log-periodic signal-to-noise amplitude ratio of 2:5. This noisy signal served as test data to illustrate the method. (c) Numerically calculated discrete Hilbert transform of the series shown in (b). (d) Instantaneous phase $\varphi(t)$ of the analytic signal before and after (e) detrending with polynomial regression (of order 4). I obtained the positions of the minima and maxima (shown as triangles) using quadratic regression (parabolic solid lines) applied in the regions of each peak and valley.
FIG. 4: Logarithm of the inter-extrema intervals $\Delta t \equiv t_n - t_{n+1}$ observed in the log-periodic signal shown in Fig. 3(e) versus the index $n$. The regression coefficient obtained leads to a value of $\omega = 21.6$. Compare this with the known value (see Fig. 3(a)) $\omega = 21.71$, corresponding to an error of less than 0.5%.

FIG. 5: The times (open circles) corresponding to the extrema of the log-periodicity shown in Fig. 3(e), as well as the product $\Delta t_n / (1 - \exp[-\pi/\omega])$ of the inter-extrema intervals and the scaling factor of the geometric series of Eq. 5 (open squares). Their sum should equal the critical time $t_c$, and we indeed observe this (filled diamonds). We thus obtain an estimate $t_c = 11040 \pm 370$. Compare this estimate with the known value $t_c = 11000$. The discrepancy lies below 1% and the standard deviation represents less than 5%.

the phase to make it vary outside the conventional range $-\pi/2 \leq \varphi \leq \pi/2$, to avoid abrupt discontinuities at the branch cut.

IV. AN ILLUSTRATIVE EXAMPLE

The technique developed above finds practical application to arbitrary time series. Consider, as an illustrative example, the log-periodic oscillation decorating a power law shown in Fig. 3(a). In a realistic situation, we would never observe such a clean signal. Rather, it would be embedded in noise or added to other types of signals. Fig. 3(b) shows the same log-periodic signal with noise added. Our goal involves finding the critical time from such data.

Let us thus use the signal shown in Fig. 3(b) as our test data. Applying the method described in the previous section, Fig. 3(c) shows the discrete Hilbert transform of the signal and Fig. 3(d),(e) show the instantaneous phase calculated from the analytic signal before and after detrending.

The last plot, shown in Fig. 3(e), permits an estimation of the times of the minima and maxima. I have used quadratic regression fits in the region of the extrema, due to the validity of the parabolic approximation. In principle, other methods may work equally well. Alternatively,
one could estimate the times corresponding to the zeroes rather than the extrema. Yet another possibility includes direct parametric fitting of a log-periodic cosine function. I have not used this direct parametric estimation due to the reasons mentioned in Section III.

Nevertheless, knowledge of the times of the extrema permits parametric estimation of the log-periodic angular frequency $\omega$ (see Table I).

I have used

$$\Delta t_n = \exp[-\pi n/\omega](\exp[-\pi \omega] - 1) \ ,$$  \hspace{1cm} (4)

where $n$ denotes an arbitrary integer index to identify successive extrema and $\Delta t^{(n)} = t_{n+1} - t_n$ represent the inter-extrema intervals. The limit $n \to \infty$ leads to the singularity. The regression coefficient obtained leads to a value of $\omega = 21.6$, compared with the known value $\omega = 21.71$. Notice the remarkable agreement.

Once we have knowledge of $\omega$ and the positions of the extrema (or zeroes), it becomes straightforward to find the critical time. From Eq. (4) it follows that the inter-extrema intervals follow a geometric series defined by

$$\frac{\Delta t_{n+1}}{\Delta t_n} = \exp[-\pi/\omega] \ .$$  \hspace{1cm} (5)

The critical time $t_c$ thus satisfies

$$t_c = t_n + \frac{\Delta t_n}{1 - \exp[-\pi/\omega]} \ .$$  \hspace{1cm} (6)

Fig. 5 shows the times $t_n$, the inter-extrema intervals $\Delta t_n$, and the estimated values for $t_c$. We obtain an estimate for $t_c$ in excellent agreement with the known value.

Previous applications of the Hilbert transform to study log-periodic precursors (e.g., of financial crashes [37]) have assumed prior knowledge of $t_c$. The log-time $\tau = \ln(t_c - t)$ behaves not log-periodically, but periodically, rendering the use of Hilbert transforms useful. In contrast, here we have assumed no prior knowledge of $t_c$. Rather, such knowledge constitutes the goal.

In summary, the method uses the following steps: (i) generation of the analytic signal from the original time series, (ii) extraction of the instantaneous phase and any necessary “unwrapping” of the phase, (iii) detrending with polynomial regression etc., (iv) testing for evidence of log-periodicity using more conventional methods. If applicable, then the final and most important step consists of estimating the location of the moveable time singularity $t_c$ by regression methods.

The above example illustrates step by step the application of the method to arbitrary time series. Below I apply the method to actual empirical data. Indeed, the crucial question concerns whether the method works for actual experimental data. I have chosen for this purpose the stock market crash of 1987, since it represents a well known event in which the collective social behavior of individual economic agents unleashed financial havoc and in which systematic studies have documented the role played by log-periodicities.

V. THE STOCK MARKET CRASH OF 1987

Fig. 6(a) shows data corresponding to approximately 5000 business days of the S&P500 financial index [38] (which has ticker symbol “SPC”). The area in grey appears only for greater visual clarity.

I have chosen for this purpose the stock market crash of 1987, since it represents a well known event in which the collective social behavior of individual economic agents unleashed financial havoc and in which systematic studies have documented the role played by log-periodicities.
TABLE I: Estimates of the log-periodic angular frequency $\omega$ and $t_c$ found from Fig. 7. The first and last points in Fig. 7 $(n = 1, 8)$ deviate further than the middle points from an exponential behavior. Therefore, I have shown three estimates for $\omega$: (i) taking all points into account, (ii) taking only the middle points into account, excluding the first and last points, and (iii) taking all but the first point into account. (The a priori known value is $t_c \approx 9582$.) In all three cases, I dropped the $n = 1$ point to estimate $t_c$, since it clearly does not belong to the same regime as the other maxima, as seen in Fig. 8. Remarkably, all three estimates for $t_c$ surprisingly bear consistency with the actual stock market crash that followed in the middle of October 1987.

| Data points from Fig. 7 | $\omega$ | $t_c$ | Calendar dates |
|-------------------------|---------|-------|----------------|
| $n = 1, 8$ excluded      | 0.076   | $9681 \pm 107$ | 2/10/1987—8/8/1988 |
| $n = 1$ excluded         | 0.110   | $9419 \pm 179$ | 9/6/1986—5/11/1987 |
| All points               | 0.090   | $9548 \pm 86$  | 24/4/1987—29/12/1987 |

VI. CONCLUDING REMARKS

Several points deserve commenting. One thought-provoking point concerns the analysis of the 1987 crash. The results reported here might bear some relation to previous findings. The very large-scale log-periodic oscillations seen in Fig. 6 are not inconsistent with similar conclusions in previous studies. Such results raise a number of questions and the implications merit further study. The financial crisis of 1987, which few had anticipated even one month prior to the crash, really being slowly built up over large time scales spanning years? Were the individual and institutional agents really starting to behave collectively so long before the crash? What about the implication for the individual agents: is our apparently “free will” essentially irrelevant to collective dynamics?

Moreover, in the past few years, financial institutions (and some individuals) have started to use automated trading software to buy and sell financial assets (in real time). Each algorithmic “robo-trader” follows a given set of rules of arbitrary complexity, however other traders (human and robotic) do not know the specific rules and thus cannot exploit them to obtain financial arbitrage. How will the advent of large numbers of robo-traders affect the collective dynamics and what are the implications relating to the probability of financial crises?

Another noteworthy aspect concerns the the general nature of the methods developed here, whose application goes beyond financial data. One of the major difficulties in parametric estimation of log-periodic properties is due to the lack of foreknowledge of $t_c$. It is not inconceivable that even small improvements in the ability to estimate $t_c$ can be of potential use in forecasting research. It would be interesting to apply and further study the method using other data sets. Given the recent application of physical methodologies to study music time series, it is even conceivable that such methods could be applied to obtain quantitative descriptions of aesthetic phenomena in the arts. In fact, wherever cooperative ef-
fects are involved, there is a real possibility that discrete scale invariance plays some role. For example, an interesting question, in this context, is whether the method is applicable to coherent noise phenomena. Concerning the method itself, there is room for further improvement, e.g., corrections for finite size effects of the conventionally used Hilbert transform algorithm. Similarly, one could take into consideration the role of log-periodic harmonics, which no doubt play an important role in such phenomena. Such issues are important but their relevance is secondary to the more significant question of the theoretical basis for the method. The inclusion here of further discussion about such secondary issues would detract from the central focus.

In summary, I have investigated two distinct approaches towards the general problem of how to detect log-periodic oscillations in arbitrary time series without prior knowledge of the location of critical time. The more promising method involves analytic continuation of the signal onto the imaginary axis, using the Hilbert transform. I have shown that the instantaneous phase necessarily retains the log-periodicities found in the original signal and develop a new method of detecting log-periodic oscillations. Initial results of the application of the method to the stock market crash of 1987 motivate further systematic studies to verify how much promise this approach holds for forecasting extreme events.

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