Flavor Neutrinos States

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Abstract

It is shown that a flavor neutrino state that describes a neutrino produced or detected in a charged-current weak interaction process depends on the process under consideration and is appropriate for the description of neutrino oscillations as well as for the calculation of neutrino production or detection rates. Hence, we have a consistent framework for the description of neutrino oscillations and interactions.

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1 Introduction

The standard theory of neutrino oscillations has been derived in the middle 70’s under the assumption that a neutrino produced or detected in a charged-current weak
interaction process together with a charged lepton with flavor $\alpha = e, \mu$ or $\tau$ is described by the flavor state

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle,$$

(1.1)

where $U$ is the unitary mixing matrix of the neutrino fields and $|\nu_k\rangle$ are the Fock states of massive neutrinos (see the review in Ref. [4]). It is then necessary to ask if the flavor state (1.1) is appropriate also for the description of the neutrino production and detection rates. This is a necessary requirement for the validity of the flavor state (1.1), because neutrino production and detection are essential parts of neutrino oscillation experiments.

In this paper we will show that the flavor state (1.1) is appropriate for the description of neutrino production and detection as well as for the description of neutrino oscillations in the plane wave approximation and for experiments which are not sensitive to the dependence of the interaction probability on the different neutrino masses. In general the appropriate flavor state depends on the process in which the neutrino is produced or detected. This fact was already noted in Refs. [5, 6, 7], where the consequences for neutrino oscillations have been discussed. Here we will show that the appropriate flavor state is suitable not only for the description of neutrino oscillations, but also for the description of neutrino production or detection.

For definiteness, we will consider a neutrino produced in the general decay process

$$P_I \rightarrow P_F + \ell_\alpha^+ + \nu_\alpha,$$

(1.2)

where $P_I$ and $P_F$ are hadronic or leptonic initial and final particles and $\ell_\alpha^+$ is a positively charged lepton of flavor $\alpha$, with $\alpha = e, \mu, \tau$. For example, in the pion decay process $\pi^+ \rightarrow \mu^+ + \nu_\mu$ we have $P_I = \pi^+$, $P_F$ is absent and $\alpha = \mu$. The following considerations can be easily modified to take into account a different production process, as well as a detection process.

In Section 2 we consider the flavor neutrino state in the plane wave approximation. In Subsection 2.1 we show that the flavor neutrino state can be used in the calculation of the decay rate of the process (1.2) and in Subsection 2.2 we discuss the derivation of neutrino oscillations in the plane wave approximation. In Section 3 we present the general derivation in the framework of Quantum Field Theory of the flavor neutrino state that describes the neutrino produced in the process (1.2) as a coherent superposition of massive neutrino wave packets, which is necessary in order to describe the localization of the production process and the related energy-momentum uncertainty which allows the coherent production of a superposition of different massive neutrinos. In Subsection 3.1 we show that this flavor neutrino state leads to the correct decay rate for the process (1.2) and in Subsection 3.2 we discuss the implications for neutrino oscillations. Finally, in Section 4 we present our conclusions.

2 Plane Wave Approximation

In neutrino oscillation experiments the energies and momenta of the particles which participate to the neutrino production process are not measured with a degree of accuracy which would allow to determine, through energy-momentum conservation, which massive neutrino is emitted. In this case, a flavor neutrino is a superposition of massive neutrinos.
In the plane wave approach a neutrino with flavor $\alpha$ created in a charged-current weak interaction process is described by the normalized flavor neutrino state \cite{5,6}

$$|
u_\alpha\rangle = \left(\sum_k |A_{\alpha k}|^2\right)^{-1/2} \sum_k A_{\alpha k} |\nu_k\rangle,$$ (2.1)

which is a coherent superposition of massive neutrino states $|\nu_k\rangle$. The coefficient $A_{\alpha k}$ of the massive neutrino state is given by the amplitude of production of $\nu_k$, which, in general, depends on the production process.

In the case of the general decay process (1.2) the amplitude $A_{\alpha k}$ is given by

$$A_{\alpha k} = \langle \nu_k, \ell^+_\alpha, P_F | \hat{S} | P_I \rangle,$$ (2.2)

where $\hat{S}$ is the $S$-matrix operator.

### 2.1 Production Rate

The amplitude of the general decay process (1.2) is given by

$$A = \langle \nu_\alpha, \ell^+_\alpha, P_F | \hat{S} | P_I \rangle = \left(\sum_k |A_{\alpha k}|^2\right)^{-1/2} \sum_k A_{\alpha k}^* \langle \nu_k, \ell^+_\alpha, P_F | \hat{S} | P_I \rangle = \left(\sum_k |A_{\alpha k}|^2\right)^{1/2}.$$ (2.3)

Therefore, the decay probability is given by an incoherent sum of the probabilities of production of different massive neutrinos,

$$|A|^2 = \sum_k |A_{\alpha k}|^2.$$ (2.4)

In other words, the coherent character of the flavor state (2.1) is irrelevant for the decay rate.

It is useful to express the $S$-matrix operator as

$$\hat{S} = 1 - i \int d^4x \mathcal{H}^{CC}_1(x),$$ (2.5)

where we have considered only the first order perturbative contribution of the effective low-energy charged-current weak interaction hamiltonian

$$\mathcal{H}^{CC}_1(x) = \frac{G_F}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \bar{\nu}_\alpha(x) \gamma^\rho \left(1 - \gamma^5\right) \ell_\alpha(x) J_\rho(x) + \text{h.c.}$$

$$= \frac{G_F}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \sum_k U^*_{\alpha k} \bar{\nu}_k(x) \gamma^\rho \left(1 - \gamma^5\right) \ell_\alpha(x) J_\rho(x) + \text{h.c.}.$$ (2.6)

Here $G_F$ is the Fermi constant and $J_\rho(x)$ is the weak charged current that describes the transition $P_I \rightarrow P_F$.

Taking into account the mixing of the neutrino fields in the charged-current weak interaction hamiltonian \cite{2,6}, the amplitude $A_{\alpha k}$ can be written as

$$A_{\alpha k} = U^*_{\alpha k} M_{\alpha k},$$ (2.7)
where $\mathcal{M}_{\alpha k}$ is the matrix element

$$\mathcal{M}_{\alpha k} = -i \frac{G_F}{\sqrt{2}} \int d^4 x \langle \nu_k, \ell_+^\alpha, P_F | \bar{\nu}_k(x) \gamma^\rho (1 - \gamma^5) \ell_\alpha(x) J_\rho(x) | P_I \rangle .$$

(2.8)

For the decay probability (2.4) we obtain

$$|A|^2 = \sum_k |U_{\alpha k}|^2 |\mathcal{M}_{\alpha k}|^2 ,$$

(2.9)

which is an incoherent sum of the probabilities of production of the different massive neutrinos weighted by $|U_{\alpha k}|^2$, in agreement with Refs. [8, 9, 10, 11, 12].

Therefore, the flavor neutrino state (2.1) leads to the correct decay rate for the general decay process (1.2). It is clear that this proof can be easily generalized to any charged-current weak interaction process in which flavor neutrinos are created or destroyed.

If the experiment is not sensitive to the dependence of $\mathcal{M}_{\alpha k}$ on the different neutrino masses, it is possible to approximate

$$\mathcal{M}_{\alpha k} \approx \mathcal{M}_\alpha .$$

(2.10)

In this case, since $\sum_k |U_{\alpha k}|^2 = 1$, we obtain

$$|A|^2 = |\mathcal{M}_\alpha|^2 ,$$

(2.11)

which coincides with the standard decay probability for massless neutrinos if the common scale of neutrino masses is negligible in comparison with the experimental resolution.

As shown in Ref. [13], the decay probability (2.11) can also be obtained starting from the usual flavor state (1.1) obtained from Eq. (2.1) with the approximation (2.10). Indeed, in this case the decay amplitude is given by

$$\mathcal{A} = \langle \nu_\alpha, \ell_+^\alpha, P_F | \hat{S} | P_I \rangle = \sum_k U_{\alpha k} \mathcal{A}_{\alpha k} = \sum_k |U_{\alpha k}|^2 \mathcal{M}_\alpha = \mathcal{M}_\alpha .$$

(2.12)

Let us remark, however, that a derivation of the decay amplitude starting from the usual flavor state (1.1) when the experiment is sensitive to the dependence of $\mathcal{M}_{\alpha k}$ on the different neutrino masses and the approximation (2.10) is not valid would lead to a wrong result.

### 2.2 Neutrino Oscillations

Let us consider a neutrino oscillation experiments in which $\nu_\alpha \rightarrow \nu_\beta$ transitions are studied. Since in this case there are two interaction processes, one for production (P) and the other for detection (D), we consider the two flavor neutrino states

$$|\nu^P_\alpha\rangle = \left( \sum_k |\mathcal{A}^P_{\alpha k}|^2 \right)^{-1/2} \sum_k \mathcal{A}^P_{\alpha k} |\nu_k\rangle ,$$

(2.13)

$$|\nu^D_\alpha\rangle = \left( \sum_k |\mathcal{A}^D_{\alpha k}|^2 \right)^{-1/2} \sum_k \mathcal{A}^D_{\alpha k} |\nu_k\rangle .$$

(2.14)
The amplitude of $\nu_{\alpha} \to \nu_{\beta}$ transitions is given by

$$A_{\alpha\beta}(L, T) = \langle \nu_{\beta}^D | e^{-i\hat{E}T + i\hat{P}L} | \nu_{\alpha}^P \rangle,$$

(2.15)

where $(L, T)$ is the space-time interval between production and detection and $\hat{E}$ and $\hat{P}$ are, respectively, the energy and momentum operators. Since the massive neutrinos have definite masses and kinematical properties, we obtain

$$A_{\alpha\beta}(L, T) = \left( \sum_k |A_{\alpha k}^P|^2 \right)^{-1/2} \left( \sum_k |A_{\beta k}^D|^2 \right)^{-1/2} \sum_k A_{\alpha k}^P A_{\beta k}^D * e^{-iE_kT + ip_kL},$$

(2.16)

with

$$E_k = \sqrt{p_k^2 + m_k^2}.$$

(2.17)

In oscillation experiments the neutrino propagation time $T$ is not measured. In order to express the propagation time $T$ in terms of the known distance $L$ traveled by the neutrino between production and detection, we take into account the fact that neutrinos in oscillation experiments are ultrarelativistic\(^1\). In this case it is possible to approximate $T \approx L$, because in reality neutrinos are described by wave packets [16, 17, 18, 19, 7, 20], which are localized on the production process at the production time and propagate between the production and detection processes with a group velocity close to the velocity of light.

The physical reason why the approximation $T \approx L$ is correct can be understood by noting that, if the massive neutrinos are ultrarelativistic and contribute coherently to the detection process, their wave packets overlap with the detection process for an interval of time $[t - \Delta t, t + \Delta t]$, with

$$t = \frac{L}{\bar{v}} \approx L \left(1 + \frac{\bar{m}^2}{2E^2}\right), \quad \Delta t \sim \sigma_x,$$

(2.18)

where $\bar{v}$ is the average group velocity, $\bar{m}^2$ is the average of the squared neutrino masses, $\sigma_x$ is given by the spatial uncertainties of the production and detection processes summed in quadrature [18] (the spatial uncertainty of the production process determines the size of the massive neutrino wave packets). The correction $L\bar{m}^2/2E^2$ to $t = L$ in Eq. (2.18) can be neglected, because it gives corrections to the oscillation phases which are of higher order in the very small ratios $m_k^2/E^2$. The corrections due to $\Delta t \sim \sigma_x$ are also negligible, because in all realistic experiments $\sigma_x$ is much smaller than the oscillation length $L_{osc} = 4\pi E/\Delta m_k^2$, otherwise oscillations could not be observed [16, 17, 19, 20]. One can summarize these arguments by saying that the approximation $T \approx L$ is correct because the phase of the oscillations is practically constant over the interval of time in which the massive neutrino wave packets overlap with the detection process.

Using the approximation $T \approx L$ the phase in Eq. (2.16) becomes

$$-E_kT + p_k L \approx -(E_k - p_k)L = -\frac{E_k^2 - p_k^2}{\bar{E}_k + p_k} L = -\frac{m_k^2}{\bar{E}_k + p_k} L \approx -\frac{m_k^2}{2E} L,$$

(2.19)

\(^1\)It is known that neutrino masses are smaller than about one eV (see Refs. [14, 15]). Since only neutrinos with energy larger than about 100 keV can be detected (see the discussion in Ref. [7]), in oscillation experiments neutrinos are always ultrarelativistic.
where $E$ is the neutrino energy neglecting mass contributions. It is important to notice that Eq. (2.19) shows that the phases of massive neutrinos relevant for the oscillations are independent from the particular values of the energies and momenta of different massive neutrinos [21, 17, 22, 23, 20], as long as the relativistic dispersion relation (2.17) is satisfied.

The probability of $\nu_\alpha \rightarrow \nu_\beta$ transitions in space is given by

$$P_{\alpha\beta}(L) \simeq \left( \sum_k |A_{\alpha k}^P|^2 \right) \left( \sum_k |A_{\beta k}^D|^2 \right) \sum_{k,j} \mathcal{A}_{\alpha k}^P \mathcal{A}_{\beta k}^D * \mathcal{A}_{\alpha j}^P \mathcal{A}_{\beta j}^D \exp \left(-i \frac{\Delta m_{kj}^2 L}{2E} \right),$$

(2.20)

with $\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$.

Oscillations can be observed at a macroscopic distance $L$ only if the difference between two neutrino masses is much smaller than the neutrino energy $E$. For example, the oscillation length $L_{osc}^{kj} = \frac{4\pi E}{\Delta m_{kj}^2}$ is larger than about 1 m if $\Delta m_{kj}^2 \lesssim 2.5 eV^2$ for $E \simeq 1 MeV$. In this case the difference of neutrino masses can be neglected in the production and detection amplitudes:

$$\mathcal{A}_{\alpha k}^P \simeq U_{\alpha k}^P M_{\alpha}^P, \quad \mathcal{A}_{\beta k}^D \simeq U_{\beta k}^D M_{\beta}^D,$$

(2.21)

where $M_{\alpha}^P$ is the matrix element (2.8) in which the difference of neutrino masses has been neglected, and $M_{\beta}^D$ is a similar matrix element for the detection process. Taking into account these approximations, the transition probability (2.20) reduces to the standard one [1, 2, 3, 4],

$$P_{\alpha\beta}(L) \simeq \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp \left(-i \frac{\Delta m_{kj}^2 L}{2E} \right),$$

(2.22)

which can be obtained starting from the standard production and detection flavor states

$$|\nu_\alpha^P \rangle = \sum_k U_{\alpha k}^* |\nu_k \rangle, \quad |\nu_\beta^D \rangle = \sum_k U_{\beta k}^* |\nu_k \rangle,$$

(2.23)

obtained from Eqs. (2.13) and (2.14) through the approximations (2.21). Therefore, the standard flavor states in Eq. (1.1) and (2.23) are appropriate to describe neutrino oscillations in the plane wave approximation, taking into account that in all neutrino oscillation experiments the dependence of the production and detection probabilities on the different neutrino masses is negligible.

3 Wave Packet Treatment

In Quantum Field Theory the asymptotic final state resulting from the decay of the initial particle $P_I$ in Eq. (1.2) is given by

$$|f\rangle = \hat{S} |P_I\rangle.$$

(3.1)

This final state is an entangled state in which the final particles do not have individual separate properties. However, in practice the decay always occur in a medium where $P_F$

\footnote{We implicitly assume also that energy-momentum conservation allows the coherent production and detection of massive neutrinos. See the discussion after Eq. (3.3).}
and \( \ell_\alpha^+ \) interact very quickly, reducing the final state to a disentangled state \( |\nu_\alpha, \ell_\alpha^+, P_F \rangle \) in which each particle has individual properties. Hence, the final flavor neutrino state is given by

\[
|\nu_\alpha \rangle \propto \langle \ell_\alpha^+, P_F \rangle = \langle \ell_\alpha^+, P_F \mid \hat{S} \mid P_I \rangle.
\] (3.2)

The proportionality sign in Eq. (3.2) is necessary in order to take into account the normalization of the flavor neutrino state.

Inserting a completeness on the left of the right-hand side of Eq. (3.2), we obtain

\[
|\nu_\alpha \rangle \propto \sum_k \int d^3 p \sum_h |\nu_k(\vec{p}, h)\rangle \langle \nu_k(\vec{p}, h), \ell_\alpha^+, P_F \mid \hat{S} \mid P_I \rangle,
\] (3.3)

where \( \vec{p} \) is the neutrino momentum and \( h \) is the neutrino helicity. The normalized flavor neutrino state can be written as

\[
|\nu_\alpha \rangle = \left( \sum_k \int d^3 p \sum_h |A_{\alpha k}(\vec{p}, h)|^2 \right)^{-1/2} \sum_k \int d^3 p \sum_h A_{\alpha k}(\vec{p}, h) |\nu_k(\vec{p}, h)\rangle,
\] (3.4)

which is a coherent superposition of massive neutrino wave packets. The coefficient \( A_{\alpha k}(\vec{p}, h) \) is given by the amplitude of production of \( \nu_k(\vec{p}, h) \):

\[
A_{\alpha k}(\vec{p}, h) = \langle \nu_k(\vec{p}, h), \ell_\alpha^+, P_F \mid \hat{S} \mid P_I \rangle.
\] (3.5)

It is important to notice that if all the particles \( P_I, P_F, \ell_\alpha^+ \) are described by plane waves, the production process is not localized and energy-momentum conservation forbids the coherent production of different massive neutrinos. Therefore, in order to have neutrino oscillations the particles \( P_I, P_F, \ell_\alpha^+ \) must be described by localized wave packets with sufficient energy-momentum uncertainty. If the energy-momentum uncertainty is so small that different massive neutrinos cannot be produced coherently, the state (3.4) becomes effectively an incoherent mixture of massive neutrino wave packets, because the different energy-momentum conservations contained in the amplitudes (3.5) cannot be satisfied simultaneously.

In the discussion of the plane wave approximation in Section 2 we swept this problem under the carpet. However, there are no implications for the neutrino production and detection rates discussed in Subsection 2.1 which, as we have seen in Eq. (2.4), are given by an incoherent sum of the probabilities of production or detection of the different massive neutrinos. On the other hand, the oscillation probability (2.20) has, strictly speaking, no physical meaning, because the exact energy-momentum conservation delta-functions contained in the amplitudes (2.2) for different massive neutrinos are mutually exclusive. However, these delta-functions have been factorized out of the standard oscillation probability (2.22), which acquires a physical meaning as the approximation of the oscillation probability in the limit of negligible wave packet effects, as we will see in Subsection 3.2.

### 3.1 Production Rate

The amplitude of the general decay process \((1.2)\) is given by

\[
A = \langle \nu_\alpha, \ell_\alpha^+, P_F \mid \hat{S} \mid P_I \rangle
\]
\[
\begin{align*}
&= \left( \sum_k \int d^3p \sum_h |A_{\alpha k}(\vec{p}, h)|^2 \right)^{-1/2} \sum_k \int d^3p \sum_h A^*_{\alpha k}(\vec{p}, h) \langle \nu_k(\vec{p}, h), \ell^+_\alpha, P_F | \hat{S} | P_I \rangle \\
&= \left( \sum_k \int d^3p \sum_h |A_{\alpha k}(\vec{p}, h)|^2 \right)^{1/2}.
\end{align*}
\]

Hence, the decay probability is given by
\[
|A|^2 = \sum_k \int d^3p \sum_h |A_{\alpha k}(\vec{p}, h)|^2,
\]
which is an incoherent sum of the probabilities of production of the different massive neutrinos.

Using the first order perturbative expansion of the \( S \)-matrix operator in Eq. (2.5),
the amplitudes \( A_{\alpha k}(\vec{p}, h) \) can be written as
\[
A_{\alpha k}(\vec{p}, h) = U^\dagger_{\alpha k} \mathcal{M}_{\alpha k}(\vec{p}, h),
\]
where \( \mathcal{M}_{\alpha k}(\vec{p}, h) \) are the matrix elements
\[
\mathcal{M}_{\alpha k}(\vec{p}, h) = -i \frac{G_F}{\sqrt{2}} \int d^4x \langle \nu_k(\vec{p}, h), \ell^+_\alpha, P_F | \bar{\nu}(x) \gamma^\mu (1 - \gamma^5) \ell(x) J_\rho(x) | P_I \rangle.
\]

Using the expression (3.8) in Eq. (3.7) it becomes clear that, in agreement with Refs. [8, 9, 10, 11, 12],
the decay probability is given by an incoherent sum of the probabilities of production of the different massive neutrinos
weighted by \( |U_{\alpha k}|^2 \)
\[
|A|^2 = \sum_k |U_{\alpha k}|^2 \int d^3p \sum_h |\mathcal{M}_{\alpha k}(\vec{p}, h)|^2.
\]

Therefore, also in a quantum field theoretical wave packet treatment, a description of the flavor neutrino created
in the process (1.2) with the process-dependent coherent state (3.4) leads to the correct decay rate, for which the coherent character of the superposition on massive neutrinos is irrelevant. This proof can be easily generalized to any charged-current weak interaction process in which flavor neutrinos are created or destroyed.

If the experiment is not sensitive to the dependence of \( \mathcal{M}_{\alpha k}(\vec{p}, h) \) on the neutrino masses, it is possible to approximate
\[
\mathcal{M}_{\alpha k}(\vec{p}, h) \approx \mathcal{M}_\alpha(\vec{p}, h).
\]
In this case we obtain
\[
|A|^2 = \int d^3p \sum_h |\mathcal{M}_\alpha(\vec{p}, h)|^2,
\]
which coincides with the decay amplitude for massless neutrinos in the wave packet treatment if the common scale of neutrino masses is negligible in comparison with the experimental resolution.
3.2 Neutrino Oscillations

Let us consider the two production (P) and detection (D) neutrino states

$$\left| \nu^P_\alpha \right> = N^P_\alpha \sum_k \int d^3 p \sum_h A^P_{\alpha k}(\vec{p}, h) \left| \nu_k(\vec{p}, h) \right>,$$

$$\left| \nu^D_\alpha \right> = N^D_\alpha \sum_k \int d^3 p \sum_h A^D_{\alpha k}(\vec{p}, h) \left| \nu_k(\vec{p}, h) \right>,$$

with the normalization factors

$$N^I_\alpha = \left( \sum_k \int d^3 p \sum_h |A^I_{\alpha k}(\vec{p}, h)|^2 \right)^{-1/2},$$

for I = P, D. The amplitude of $\nu_\alpha \rightarrow \nu_\beta$ transitions is given by

$$A_{\alpha\beta}(\vec{L}, T) = \langle \nu^P_\beta | e^{-i\hat{E}T} e^{i\hat{P} \vec{L}} | \nu^P_\alpha \rangle,$$

where $(\vec{L}, T)$ is the space-time interval between production and detection and $\hat{E}$ and $\hat{P}$ are, respectively, the energy and momentum operators. Using the flavor states (3.13) and (3.14) we obtain

$$A_{\alpha\beta}(\vec{L}, T) = N^P_\alpha N^D_\beta \sum_k \int d^3 p \sum_h A^P_{\alpha k}(\vec{p}, h) A^D_{\beta k}(\vec{p}, h) e^{-iE_k(\vec{p})T + i\hat{P} \vec{L}},$$

with

$$E_k(\vec{p}) = \sqrt{\vec{p} + m_k^2}.$$

The derivation of the neutrino oscillation probability from the explicit values of the production and detection amplitudes in Eq. (3.18) has been discussed in Ref. [7]. Since it is rather complicated, here we consider the approximation

$$A^P_{\alpha k}(\vec{p}, h) A^{D*}_{\beta k}(\vec{p}, h) \simeq U^*_{\alpha k} U_{\beta k} \hat{M}^P_{\alpha k}(\vec{p}, h) \hat{M}^{D*}_{\beta k}(\vec{p}, h) \exp \left[ -\frac{(\vec{p} - \vec{p}_k)^2}{4\sigma_p^2} \right],$$

where $\vec{p}_k$ is the average momentum of the $k^{th}$ massive neutrino component and the exponential takes into account energy-momentum conservation within the momentum uncertainty $\sigma_p$ determined by the widths of the momentum distributions of the wave packets of the particles participation to the production and detection processes (see Ref. [7]). We choose a gaussian form for the momentum distribution in order to be able to perform the integral over $d^3 p$ analytically. This approximation is almost equivalent to the saddle-point approximation performed in Ref. [7] (it is equivalent if $\omega = 1$, with $\omega$ defined in Ref. [7]).

Furthermore, in order to simplify the derivation of the oscillation probability for realistic experimental setups as much as possible, we consider ultrarelativistic neutrinos and we adopt the following assumptions which in practice are always verified: a) $\sigma_p \ll p_k$, with $p_k = |\vec{p}_k|$; b) $\hat{M}^P_{\alpha k}(\vec{p}, h) \hat{M}^{D*}_{\beta k}(\vec{p}, h)$ is a smooth function of $\vec{p}$; c) the experiment is
not sensitive to the dependence of $\tilde{M}_{\alpha k}^p (\vec{p}, h) \tilde{M}_{\beta k}^{D*} (\vec{p}, h)$ on the different neutrino masses. Under these assumptions we can approximate

$$\tilde{M}_{\alpha k}^p (\vec{p}, h) \tilde{M}_{\beta k}^{D*} (\vec{p}, h) \simeq \tilde{M}_{\alpha}^p (\vec{p}_\nu, h) \tilde{M}_{\beta}^{D*} (\vec{p}_\nu, h),$$

(3.20)

where $\vec{p}_\nu$ is the neutrino momentum neglecting mass effects. The energy $E_k (\vec{p})$ can be approximated by

$$E_k (\vec{p}) \simeq E_k + \vec{v}_k (\vec{p} - \vec{p}_k),$$

(3.21)

with $E_k$ given by Eq. (2.17) and

$$\vec{v}_k = \frac{\partial E_k (\vec{p})}{\partial \vec{p}} \bigg|_{\vec{p} = \vec{p}_k} = \frac{\vec{p}_k}{E_k} \simeq 1 - \frac{m^2_k}{2E^2},$$

(3.22)

where $E = |\vec{p}_\nu|$ is the neutrino energy neglecting mass effects. Since, as shown in Ref. [7], in the case of ultrarelativistic neutrinos all the momenta $\vec{p}_k$ are aligned in the direction of $L$, with these approximations, the transition amplitude (3.17) reduces to

$$A_{\alpha\beta} (L, T) \simeq N_{\alpha}^P N_{\beta}^D \sum_k \tilde{M}_{\alpha}^p (\vec{p}_\nu, h) \tilde{M}_{\beta}^{D*} (\vec{p}_\nu, h) \sum_k U_{\alpha k}^* U_{\beta k} \exp \left\{ -iE_k T + ip_k L \right\}$$

$$\times \int d^3p \exp \left\{ -\frac{(\vec{p} - \vec{p}_k)^2}{4\sigma_p^2} + i (\vec{p} - \vec{p}_k) (\vec{L} - \vec{v}_k T) \right\}$$

$$\propto \sum_k U_{\alpha k}^* U_{\beta k} \exp \left\{ -iE_k T + ip_k L - \frac{\left( \vec{L} - \vec{v}_k T \right)^2}{4\sigma_p^2} \right\},$$

(3.23)

with the spatial uncertainty $\sigma_x$ related to the momentum uncertainty $\sigma_p$ by the minimal Heisenberg relation

$$\sigma_x \sigma_p = \frac{1}{2}.$$  

(3.24)

In order to obtain the oscillation probability as a function of the known distance $L$ traveled by the neutrino between production and detection, the probability $P_{\alpha\beta} (L, T) = |A_{\alpha\beta} (L, T)|^2$ must be integrated over the unknown time $T$ [17]. Since the integral over $T$ is gaussian, we easily obtain

$$P_{\alpha\beta} (L) \simeq \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp \left\{ -i \left[ (E_k - E_j) - (p_k - p_j) \right] L \right\}$$

$$\times \exp \left\{ -\left( \frac{\Delta m_{kj}^2 L}{4\sqrt{2}E^2\sigma_x} \right)^2 - \left( \frac{E_k - E_j}{2\sqrt{2}\sigma_p} \right)^2 \right\}.$$  

(3.25)

Using the same method as in Eq. (2.19), the phase $[(E_k - E_j) - (p_k - p_j)] L$ becomes the standard oscillation phase $\Delta m_{kj}^2 L/2E$ in Eq. (2.22). Since for ultrarelativistic neutrinos the energies $E_k$ can be written as $[17] [22] [20]$

$$E_k \simeq E + \xi \frac{m_k^2}{2E},$$

(3.26)
where $\xi$ is a number, usually of order one (see Ref. 7), which depends on the details of the production and detection processes, the oscillation probability can be written as

$$P_{\alpha\beta}(L) = \sum_{k,j} U_{\alpha k}^* U_{\alpha j} U_{\beta k}^* U_{\beta j} \exp \left[ -2\pi i \frac{L}{L_{\text{osc}}^{kj}} - \left( \frac{L}{L_{\text{coh}}^{kj}} \right)^2 - 2\pi^2 \xi^2 \left( \frac{\sigma_x}{L_{\text{osc}}^{kj}} \right)^2 \right] , \quad (3.27)$$

with the standard oscillation lengths

$$L_{\text{osc}}^{kj} = \frac{4\pi E}{\Delta m_{kj}^2} , \quad (3.28)$$

and the coherence lengths 24 25

$$L_{\text{coh}}^{kj} = \frac{4\sqrt{2} E^2}{|\Delta m_{kj}^2| \sigma_x} . \quad (3.29)$$

As promised at the end of the introduction of Section 3, in the limit of negligible wave packet effects, i.e. for $L \ll L_{\text{coh}}^{kj}$ and $\sigma_x \ll L_{\text{osc}}^{kj}$, the oscillation probability in the wave packet approach reduces to the standard one in Eq. (2.22), obtained in the plane wave approximation.

The physical meaning of the coherence and localization terms which appear in Eq. (3.27) in addition to the standard oscillation phase have been already discussed at length in Refs. 17 18 19 7 20 (see also Refs. 26 27 28 29).

In particular, the localization term $\exp \left[ -2\pi^2 \xi^2 (\sigma_x/L_{\text{osc}}^{kj})^2 \right]$ suppresses the oscillations due to $\Delta m_{kj}^2$ if $\sigma_x \gg L_{\text{osc}}^{kj}$. This means that in order to measure the interference of the massive neutrino components $\nu_k$ and $\nu_j$ the production and detection processes must be localized in space-time regions much smaller than the oscillation length $L_{\text{osc}}^{kj}$. In practice this requirement is satisfied in all neutrino oscillation experiments.

The localization term allows to distinguish neutrino oscillation experiments from experiments on the measurement of neutrino masses. As first shown in Ref. 16, neutrino oscillations are suppressed in experiments which are able to measure, through energy-momentum conservation, the mass of the neutrino. Indeed, from the energy-momentum dispersion relation (2.17) the uncertainty of the mass determination is

$$\delta m_k^2 = \sqrt{(2E_k \delta E_k)^2 + (2p_k \delta p_k)^2} \approx 2\sqrt{2} E \sigma_p , \quad (3.30)$$

where the approximation holds for realistic ultrarelativistic neutrinos. If $\delta m_k^2 < |\Delta m_{kj}^2|$, the mass of $\nu_k$ is measured with an accuracy better than the difference $\Delta m_{kj}^2$. In this case the neutrino $\nu_j$ is not produced or detected and the interference of $\nu_k$ and $\nu_j$ which would generate oscillations does not occur. The localization term in the oscillation probability (3.27) automatically suppresses the interference of $\nu_k$ and $\nu_j$, because it can be written as

$$\exp \left[ -2\pi^2 \xi^2 \left( \frac{\sigma_x}{L_{\text{osc}}^{kj}} \right)^2 \right] = \exp \left[ -\xi^2 \left( \frac{\Delta m_{kj}^2}{4\sqrt{2} E \sigma_p} \right)^2 \right] \approx \exp \left[ -\frac{\xi^2}{4} \left( \frac{\Delta m_{kj}^2}{\delta m_k^2} \right)^2 \right] . \quad (3.31)$$

It is important to notice, however, that the validity of this interpretation of the localization term hinges on the validity of the flavor states (3.13) and (3.14) not only for the
description of neutrino oscillations but also for the description of neutrino production and detection, which we have proved in the previous Section. The suppression of oscillations due to the localization term reflects the effective loss of coherence of the flavor states discussed after Eq. (3.5).

4 Conclusions

We have presented a consistent framework for the description of neutrino oscillations and interactions.

We have shown that the flavor neutrino state that describes a neutrino produced or detected in a charged-current weak interaction process depends on the process under consideration and is appropriate for the description of neutrino oscillations as well as for the calculation of neutrino production and detection rates. We have proved these facts both in the plane wave approximation (Section 2) and in the quantum field theoretical wave packet treatment (Section 3).

In the plane wave approximation the flavor neutrino state can be approximated with the standard expression in Eq. (1.1) only for experiments which are not sensitive to the dependence of the interaction probability on the different neutrino masses. This occurs in all neutrino oscillation experiments.

In the quantum field theoretical wave packet treatment the flavor neutrino state takes into account the localization of the neutrino production or detection process and the related energy-momentum uncertainty. The oscillation probability depends on the standard oscillation phase plus additional coherence and localization terms due to the wave packet treatment. The localization term suppresses the oscillations if the energy-momentum uncertainty is so small that only one massive neutrino is produced. This occurs in experiments on the measurement of masses, whose neutrino production rate follow consistently from the same flavor neutrino states employed for the calculation of the oscillation probability.

Finally, we would like to remark that the validity of the process-dependent flavor neutrino states is consistent with the proof presented in Ref. [30] that flavor neutrino Fock spaces [31, 32, 33] are clever mathematical constructs without physical relevance.

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