Some properties of higher-order Daehee polynomials of the second kind arising from umbral calculus

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Abstract

In this paper, we study the higher-order Daehee polynomials of the second kind from the umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

1 Introduction

Let \( k \in \mathbb{Z}_{\geq 0} \). The Daehee polynomials of the second kind of order \( k \) are defined by the generating function to be

\[
\left( \frac{(1 + t) \log(1 + t)}{t} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} \hat{D}_n^{(k)}(x) \frac{t^n}{n!}
\]

(1)

(see [1]).

When \( x = 0 \), \( \hat{D}_n^{(k)} = \hat{D}_n^{(k)}(0) \) are called the Daehee numbers of the second kind of order \( k \).

The Stirling number of the first kind is defined by the falling factorial to be

\[
(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l.
\]

(2)

Thus, by (2), we get

\[
(\log(1 + t))^m = m! \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!}
\]

(3)

(see [2–4]), where \( m \in \mathbb{Z}_{\geq 0} \).

For \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), the Frobenius-Euler polynomials of order \( s \) \((s \in \mathbb{N})\) are given by

\[
\left( \frac{1 - \lambda}{e^t - \lambda} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\lambda) \frac{t^n}{n!}
\]

(4)

(see [1–18]).

When \( x = 0 \), \( H_n^{(s)}(\lambda) = H_n^{(s)}(\lambda|0) \) are called the Frobenius-Euler numbers of order \( s \).
As is well known, the Bernoulli polynomials of order \( k \) (\( \in \mathbb{N} \)) are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}
\]

(see [1–18]).

When \( x = 0 \), \( B_n^{(k)} = B_n^{(k)}(0) \) are called the Bernoulli numbers of order \( k \).

In this paper, we study the higher-order Daehee polynomials of the second kind with umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

2 Umbral calculus

Let \( \mathbb{C} \) be the complex number field and let \( \mathcal{F} \) be the set of all formal power series

\[
\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \mid a_k \in \mathbb{C} \right\}.
\]

Let \( \mathbb{P} = \mathbb{C}[x] \), and let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L | p(x) \rangle \) indicates the action of the linear functional \( L \) on the polynomial \( p(x) \). Then the vector space operations on \( \mathbb{P}^* \) are given by \( \langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle \), and \( \langle cL | p(x) \rangle = c\langle L | p(x) \rangle \), where \( c \) is a complex constant in \( \mathbb{C} \). For \( f(t) \in \mathcal{F} \), the linear functional on \( \mathbb{P} \) is defined by \( \langle f(t) | x^n \rangle = a_n \). Then, in particular, we have

\[
\langle t^k | x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0)
\]

(see [3, 18]), where \( \delta_{n,k} \) is the Kronecker symbol.

Let \( f_L(t) = \sum_{k=0}^{\infty} \frac{|L|x^k}{k!} t^k \). By (6), we get \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \). That is, \( L = f_L(t) \). The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of the formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of the umbral algebra. The order \( o(f(t)) \) of the power series \( f(t) \) (\( f(t) \neq 0 \)) is the smallest integer for which the coefficient of \( t^k \) does not vanish. If \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series; if \( o(f(t)) = 1 \), then \( f(t) \) is called a delta series.

Let \( f(t), g(t) \in \mathcal{F} \) with \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \). Then there exists a unique sequence \( s_n(x) \) (\( \text{deg} s_n(x) = n \)) such that \( \langle g(t)f(t) | s_n(x) \rangle = n! \delta_{n,k} \), for \( n, k \geq 0 \). The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \). For \( f(t), g(t) \in \mathcal{F} \), we have

\[
\langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle = \langle g(t) | f(t)p(x) \rangle.
\]

(7)

From (6), we note that

\[
f(t) = \sum_{k=0}^{\infty} \frac{f(t) | x^k }{k!}, \quad p(x) = \sum_{k=0}^{\infty} \frac{t^k | p(x)}{k!}
\]

(8)
and, by (8), we get
\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x + y) \] (9)
(see [3, 18]).

For \( s_n(x) \sim (g(t), f(t)) \), we have
\[ \frac{ds_n(x)}{dx} = \sum_{i=0}^{n-1} \left( \binom{n}{i} \overline{\mu}(t) |x^{n-i}|s_i(x) \right), \] (10)
where \( \overline{\mu}(t) \) is the compositional inverse of \( f(t) \) with \( f(f(t)) = t \). We have
\[ \frac{1}{g(\overline{\mu}(t))} e^{y\overline{\mu}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all} \ x \in \mathbb{C}, \] (11)
\[ f(t)s_n(x) = n s_{n-1}(x) \quad (n \geq 1), \quad s_n(x) = \sum_{j=0}^{n} \binom{n}{j} g(\overline{\mu}(t))^{-1} \overline{\mu}(t)^{|x^n|} x^j, \] (12)
\[ s_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x)p_{n-j}(y), \] (13)
where \( p_n(x) = g(t)s_n(x) \).
\[ \langle f(t)|x^p \rangle = \langle \partial_x f(t) |p(x) \rangle, \] (14)
with \( \partial_x f(t) = \frac{df(t)}{dt} \), and
\[ s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \quad (n \geq 0) \] (15)
(see [3, 18]).

Let us assume that \( s_n(x) \sim (g(t), f(t)) \) and \( r_n(x) \sim (h(t), l(t)) \). Then we see that
\[ s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0), \] (16)
where
\[ C_{n,m} = \left. \frac{m! \left( \frac{h(\overline{\mu}(t))}{g(\overline{\mu}(t))} \right)^m |x^n|}{\overline{\mu}(t)^m} \right| \] (17)
(see [3, 18]).

3 Higher-order Dahee polynomials of the second kind

By (1), we see that
\[ D^{(k)}_n(x) \sim \left( \left( \frac{e^t - 1}{te^t} \right)^k \right) e^t - 1 \] (18)
From (18), we have

$$
\left( \frac{e^t - 1}{te^t} \right)^k \hat{D}_n^{(k)}(x) \sim (1, e^t - 1) \quad \text{and} \quad (x)_n \sim (1, e^t - 1).
$$

(19)

By (19), we get

$$
\hat{D}_n^{(k)}(x) = \left( \frac{te^t}{e^t - 1} \right)^k (x)_n
$$

$$
= \sum_{m=0}^{n} S_1(n, m) \left( \frac{te^t}{e^t - 1} \right)^k x^m
$$

$$
= \sum_{m=0}^{n} S_1(n, m) e^{kt} B_m^{(k)}(x)
$$

$$
= \sum_{m=0}^{n} S_1(n, m) B_m^{(k)}(x + k).
$$

(20)

From (12) and (18), we have

$$
\hat{D}_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} \left( \frac{(1 + t) \log(1 + t)}{t} \right)^k (\log(1 + t))^j x^n \theta^j,
$$

(21)

where

$$
\left( \frac{(1 + t) \log(1 + t)}{t} \right)^k \frac{\log(1 + t)^j}{|x^n|} = \left( \frac{\log(1 + t)}{t} \right)^k \theta^j |x^n|
$$

$$
= (n)_j \left( \frac{\log(1 + t)}{t} \right)^k \theta^j \sum_{m=0}^{\min\{k, n-j\}} \binom{k}{m} \frac{t^m n^{-j}}{m}
$$

$$
= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \sum_{l=0}^{\infty} \frac{(k+j)!}{(l+k+j)!} S_1(l+k+j, n-j+1) |x^{n-j-m}|
$$

$$
= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{(k+j)!}{(n+k-m)!} S_1(n+k-m, n-j-m)!
$$

$$
= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, n-j-m)}{m!}.
$$

(22)

Therefore, by (21) and (22), we obtain the following theorem.

**Theorem 1** For \( n \in \mathbb{Z}_{\geq 0} \) and \( k \geq 1 \), we have

$$
\hat{D}_n^{(k)}(x) = \sum_{j=0}^{n} \left\{ \binom{n}{j} \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, n-j-m)}{m!} \right\} \theta^j.
$$
By (1) and (6), we get
\[
\hat{D}_n^{(k)}(y) = \left( \sum_{l=0}^{\infty} \frac{\log(1+t)}{t} \right)^k (1+t)^k x^n
\]
\[
= \sum_{0 \leq r \leq \min\{k,n\}} \binom{k}{r} (n) \left( \frac{\log(1+t)}{t} \right)^k (1+t)^k x^{n-r}
\]
\[
= \sum_{0 \leq r \leq \min\{k,n\}} \binom{k}{r}(n) \sum_{0 \leq m \leq n-r} \binom{y}{m}(n-r)_m
\]
\[
\times \sum_{0 \leq n-r-m} \frac{k!S_1(l+k,k)}{(l+k)!} (1+t)^r x^{n-r-m}
\]
\[
= \sum_{0 \leq r \leq n} \sum_{0 \leq m \leq n-r} \binom{n}{r} \binom{y}{m} \frac{(n-r+m-k)}{S_1(n-r-m+k,k)} S_1(n-r-m+k,k)(y)_m.
\]
(23)

Therefore, by (23), we obtain the following theorem.

**Theorem 2** For \( n \geq 0 \), we have
\[
\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \sum_{0 \leq r \leq \min\{n-m\}} \binom{n}{r} \binom{n-r}{m} S_1(n-r-m+k,k)(x)_m
\]
\[
= \sum_{0 \leq m \leq n} \sum_{0 \leq r \leq \min\{n-m\}} \binom{n}{r} \binom{n-r}{m} S_1(n-r+m+k,k)(x)_{n-m}.
\]

From (12) and (18), we have
\[
(e^{t}-1)\hat{D}_n^{(k)}(x) = n\hat{D}_{n+1}^{(k)}(x)
\]
(24)

and
\[
(e^{t}-1)\hat{D}_n^{(k)}(x) = \hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x).
\]

Thus, by (24), we get
\[
\hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x) = n\hat{D}_{n+1}^{(k)}(x) \quad (n \geq 1).
\]
(25)

From (15) and (18), we derive the following equation:
\[
\hat{D}_{n+1}^{(k)}(x) = \frac{x + k \frac{e^t - 1 - t}{t(e^t - 1)}}{t(e^t - 1)} \hat{D}_n^{(k)}(x)
\]
\[
= x\hat{D}_n^{(k)}(x) + ke^{t}\frac{e^t - 1 - t}{t(e^t - 1)} \hat{D}_n^{(k)}(x),
\]
(26)
where
\[
e^{-t} e^{\eta t} \frac{e^{\eta t} - 1}{t(e^{\eta t} - 1)} \hat{D}_n^{(k)}(x)
\]
\[
= \frac{e^{-t} e^{\eta t} - 1}{t(e^{\eta t} - 1)} \sum_{0 \leq j \leq n} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{n-j}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) x^j
\]
\[
= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!\binom{n-j}{m}}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) e^{-t} e^{\eta t} \frac{e^{\eta t} - 1 - t}{t(e^{\eta t} - 1)} x^j
\]
\[
= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{m}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) \frac{e^{-t} e^{\eta t} (e^{\eta t} - 1 - t)}{j+1} x^{j+1}
\]
\[
= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{m}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) \frac{e^{-t} e^{\eta t} (x^j + 1 - B_{j+1}(x))}{j+1}
\]
\[
= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{m}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) \frac{e^{-t} e^{\eta t} (x^j + 1 - B_{j+1}(x))}{j+1}
\]
\[
\times S_1(n + k - m, k + j) x^{j+1} = \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{m}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) x^{j+1} - B_{j+1}(x^j).
\]

Therefore, from (26) and (27), we obtain the following theorem.

**Theorem 3** For \( n \geq 0, k \geq 1 \), we have

\[
\hat{D}_{n+1}^{(k)}(x)
\]
\[
= x\hat{D}_n^{(k)}(x-1) + \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m!(\binom{m}{m})}{(n+1-k)^k_{k+j}}
\]
\[
\times S_1(n + k - m, k + j) \frac{(x-1)^{j+1} - B_{j+1}(x-1)}{j+1}.
\]

Now, we observe that

\[
e^{-t} e^{\eta t} \frac{e^{\eta t} - 1}{t(e^{\eta t} - 1)} \hat{D}_n^{(k)}(x)
\]
\[
= \sum_{j=0}^{n} \binom{n}{j+k} S_1(n + k, j + k) e^{-t} e^{\eta t} \frac{e^{\eta t} - 1 - t}{t(e^{\eta t} - 1)} (x + k)^j
\]
\[
D_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} S_j(n+k,j+k) e^{x} \left( \frac{1-t}{t} \right)^j x^j.
\]

Thus, by (28), we get
\[
\hat{D}_n^{(k)}(x) = x \hat{D}_n^{(k)}(x-1) + k \sum_{j=0}^{n} \binom{n}{j} S_j(n+k,j+k) \left( (x+k-1)^{j+1} - B_{j+1}(x+k-1) \right).
\]

From (10) and (18), we note that
\[
\frac{d}{dx} \hat{D}_n^{(k)}(x) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \hat{D}_n^{(k)}(x).
\]

By (6) and (18), we see that
\[
\hat{D}_n^{(k)}(y) = \left\{ \sum_{l=0}^{\infty} \hat{D}_l^{(k)}(y) \frac{t^l}{l!} x^n \right\} (n \geq 1)
\]

\[
= \left\{ \left( \frac{(1+t) \log(1+t)}{t} \right)^k \left( 1+t \right)^x \right\} (n \geq 1)
\]

\[
= \left( \partial_t \left( \left( \frac{(1+t) \log(1+t)}{t} \right)^k \left( 1+t \right)^x \right) \right) \left( \log(1+t) - \frac{(1+t) \log(1+t)}{t} \right) \frac{x^n}{n}
\]

\[
= \frac{y \hat{D}_n^{(k)}(y-1)}{n} + \frac{k}{n} \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} \left( 1+t \right)^y \left( \log(1+t) - \frac{(1+t) \log(1+t)}{t} \right) \frac{x^n}{n}
\]

\[
= \frac{y \hat{D}_n^{(k)}(y-1)}{n} + \frac{k}{n} \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} \left( 1+t \right)^y \left( \log(1+t) - \frac{(1+t) \log(1+t)}{t} \right) \frac{x^n}{n}
\]

\[
+ \frac{k}{n} \left( \frac{(1+t) \log(1+t)}{t} \right)^k \left( 1+t \right)^x \frac{x^n}{n}
\]

\[
= \frac{y \hat{D}_n^{(k)}(y-1)}{n} + \frac{k}{n} \hat{D}_n^{(k-1)}(y) - \frac{k}{n} \hat{D}_n^{(k)}(y)
\]

\[
+ \frac{k}{n} \sum_{1 \leq l \leq n} \frac{(-1)^{l-1}(n)_l}{l} \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} \left( 1+t \right)^{y^{l-1}}.
\]
Thus, by (30), we get
\[
\hat{D}_n^{(k)}(x) = \frac{n}{n+k} x \hat{D}_{n-1}^{(k)}(x - 1) + \frac{k}{n+k} \hat{D}_n^{(k-1)}(x) \\
+ \frac{k}{n+k} \sum_{1 \leq l \leq n} (-1)^{l-1} \binom{n}{l} (l-1) \hat{D}_{n-l}^{(k-1)}(x).
\] (31)

Therefore, by (31), we obtain the following theorem.

**Theorem 4** For \( n \geq 0, k \geq 1, \) we have
\[
\hat{D}_n^{(k)}(x) = \frac{n}{n+k} x \hat{D}_{n-1}^{(k)}(x - 1) + \frac{k}{n+k} \hat{D}_n^{(k-1)}(x) \\
+ \frac{k}{n+k} \sum_{1 \leq l \leq n} (-1)^{l-1} \binom{n}{l} (l-1) \hat{D}_{n-l}^{(k-1)}(x).
\]

Now, we compute \( \langle \left( \frac{(1+t) \log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \rangle \) in two different ways:

\[
\langle \left( \frac{(1+t) \log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \rangle \\
= \langle \left( \frac{(1+t) \log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \rangle \\
= \sum_{0 \leq l \leq n-m} m! \binom{n}{l+m} S_l(l+m,m)(n-l+m) \left( \frac{(1+t) \log(1+t)}{t} \right)^k | x^{n-l-m} \rangle \\
= \sum_{0 \leq l \leq n-m} m! \binom{n}{l+m} S_l(l+m,m) \hat{D}_l^{(k)} | x^{n-l-m} \rangle. \] (32)

On the other hand,
\[
\langle \left( \frac{(1+t) \log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \rangle \\
= \langle \left( \frac{(1+t) \log(1+t)}{t} \right)^k (\log(1+t))^m | x^{n-1} \rangle \\
= k \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} \left( \log(1+t) + 1 - \frac{(1+t) \log(1+t)}{t} \right) (\log(1+t))^m | x^{n-1} \rangle \\
+ m \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} (1+t)^{-1} (\log(1+t))^m | x^{n-1} \rangle \\
= \frac{k}{n} \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} (\log(1+t))^{m+1} | x^n \rangle \\
+ \frac{k}{n} \left( \frac{(1+t) \log(1+t)}{t} \right)^{k-1} (\log(1+t))^m | x^n \rangle.
\[
-k \left( \frac{(1 + t) \log(1 + t)}{t} \right)^k \frac{(\log(1 + t))^m}{x^n} + m \left( \frac{(1 + t) \log(1 + t)}{t} \right)^k \frac{(1 + t)^{-1}(\log(1 + t))^{m-1}}{x^{n-1}}. \tag{33}
\]

Thus, by (33), we get
\[
\frac{n + k}{n} \left( \frac{(1 + t) \log(1 + t)}{t} \right)^k \frac{(\log(1 + t))^m}{x^n} = \frac{k}{n} \left( \frac{(1 + t) \log(1 + t)}{t} \right)^{k-1} \frac{(\log(1 + t))^{m+1}}{x^n} + \frac{k}{n} \left( \frac{(1 + t) \log(1 + t)}{t} \right)^{k-1} \frac{(\log(1 + t))^m}{x^n} + m \left( \frac{(1 + t) \log(1 + t)}{t} \right)^k \frac{(1 + t)^{-1}(\log(1 + t))^{m-1}}{x^{n-1}}. \tag{34}
\]

From (34), we derive the following equation:
\[
\frac{n + k}{k} \sum_{0 \leq l \leq n - m} ml \binom{n}{l} S_i(n - l, m) \hat{D}_i^{(k)}
\]
\[
= \frac{k}{n} \sum_{0 \leq l \leq n - m-1} (m + 1)! \binom{n}{l} S_i(n - l + 1, m + 1) \hat{D}_i^{(k-1)}
\]
\[
+ \frac{k}{n} \sum_{0 \leq l \leq n - m} ml \binom{n}{l} S_i(n - l, m) \hat{D}_i^{(k-1)}
\]
\[
+ m \sum_{0 \leq l \leq n - m} (m - 1)! \binom{n - 1}{l} S_i(n - l - 1, m - 1) \hat{D}_i^{(k)}(-1). \tag{35}
\]

Therefore, by (35), we obtain the following theorem.

**Theorem 5** For \( n - 1 \geq m \geq 1 \), we have
\[
\sum_{l=0}^{n-m} \binom{n}{l} S_i(n - l, m) \hat{D}_i^{(k)}
\]
\[
= \frac{k(m + 1)}{n + k} \sum_{0 \leq l \leq n - m - 1} \binom{n}{l} S_i(n - l + 1, m + 1) \hat{D}_i^{(k-1)}
\]
\[
+ \frac{k}{n + k} \sum_{0 \leq l \leq n - m} \binom{n}{l} S_i(n - l, m) \hat{D}_i^{(k-1)}
\]
\[
+ \frac{n}{n + k} \sum_{0 \leq l \leq n - m} \binom{n - 1}{l} S_i(n - l - 1, m - 1) \hat{D}_i^{(k)}(-1).
\]

For \( \hat{D}_i^{(k)}(x) \sim ((\frac{e^x - 1}{e^x})^k, e^x - 1) \), and \( (x)_n \sim (1, e^x - 1) \), let us assume that
\[
\hat{D}_i^{(k)}(x) = \sum_{m=0}^{n} C_{n,m}(x)_m. \tag{36}
\]
Then, by (16) and (17), we get
\[
C_{n,m} = \frac{1}{m!}\left(\left(\frac{(1 + t) \log(1 + t)}{t}\right)^k\right)
\]
\[
= \left(\frac{n}{m}\right)^{n-m}
\]
\[
= \left(\frac{n}{m}\right)\hat{\Delta}_{n-m}^{(k)}.
\]
(37)

Therefore, by (36) and (37), we obtain the following theorem.

**Theorem 6** For \( n \geq 0 \), we have
\[
\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \left(\frac{n}{m}\right)\hat{D}_{n-m}^{(k)}(x)_m
\]
\[
= \sum_{0 \leq m \leq n} m! \left(\frac{n}{m}\right)\hat{\Delta}_{n-m}^{(k)}(x)_m.
\]

Now, we consider the following two Sheffer sequences:
\[
\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t}\right)^k, e^t - 1\right)
\]
(38)

and
\[
H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1-\lambda}\right)^s, t\right), \quad s \in \mathbb{N}, \lambda \in \mathbb{C} \text{ with } \lambda \neq 1.
\]

Let
\[
\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\lambda).
\]
(39)

Here
\[
C_{n,m} = \frac{1}{m!(1-\lambda)^y}\left(\left(\frac{(1 + t) \log(1 + t)}{t}\right)^k \log(1 + t)^m (1 - \lambda + t)^x\right)
\]
\[
= \frac{1}{m!(1-\lambda)^y} \sum_{j=0}^{n} \left(\frac{j}{n}\right)(1 - \lambda)^{s-j}(n)_j
\]
\[
\times \left(\left(\frac{(1 + t) \log(1 + t)}{t}\right)^k \log(1 + t)^m (1 - \lambda + t)^x\right)
\]
\[
= \sum_{j=0}^{n-m} \left(\frac{j}{n}\right)(1 - \lambda)^{s-j}(n)_j \sum_{l=0}^{n-j} \left(\frac{n-j}{l+m}\right)S_1(l+m,m)\hat{D}_{n-j-l-m}^{(k)}
\]
\[
= \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \left(\frac{j}{n}\right)(n)_j(1 - \lambda)^{s-j}S_1(n-j-l,m)\hat{D}_{l}^{(k)}.
\]
(40)

Therefore, by (39) and (40), we obtain the following theorem.
Theorem 7 For \( n \geq 0, k \geq 1 \) and \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), we have

\[
\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{j=0}^{n-m} \sum_{l=0}^{n-j} \binom{s}{j} \binom{n-j}{l} (n)_j \right. \\
\times (1 - \lambda)^j S_1(n - j - l, m) \hat{D}_m^{(l)} \left\} H_m^{(s)}(x|\lambda).
\]

We consider the following two Sheffer sequences:

\[
\hat{D}_n^{(k)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^k, e^t - 1 \right), \quad B_n^{(s)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^s, t \right).
\]

Let

\[
\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x). \tag{41}
\]

Here

\[
C_{n,m} = \frac{1}{m!} \left\{ \frac{t^{n-m}}{(1+t)^m} \left( \log(1+t) \right)^m \right\} x^n
\]

\[
= \frac{1}{m!} \left\{ (1+t)^t \frac{t^{n-m}}{(1+t) \log(1+t)} \left( \log(1+t) \right)^m \right\} x^n. \tag{42}
\]

Case 1. For \( s > k \), we have

\[
C_{n,m} = \frac{1}{m!} \left\{ \left( \frac{t}{1+t \log(1+t)} \right)^{s-k} \left( \log(1+t) \right)^m \right\} (1+t)^s x^n
\]

\[
= \frac{1}{m!} \sum_{0 \leq j \leq m} \binom{s}{j} (n)_j \left\{ \left( \frac{t}{1+t \log(1+t)} \right)^{s-k} \left( \log(1+t) \right)^m \right\} x^{n-j}
\]

\[
= \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j \sum_{m \leq l \leq n-j} S_1(l, m) \hat{C}^{(s-k)}_{n-j-l}
\]

\[
\times \left( \frac{n-j}{l} \right) \left\{ \left( \frac{t}{1+t \log(1+t)} \right)^{s-k} \right\} x^{n-j-l}. \tag{43}
\]

where \( \hat{C}^{(s-k)}_{n-j} \) is the \( n \)-th Cauchy number of the second kind of order \( s - k \) (see [14]).

Case 2. For \( s = k \), we have

\[
C_{n,m} = \frac{1}{m!} \left\{ \left( \log(1+t) \right)^m \right\} (1+t)^s x^n
\]

\[
= \frac{1}{m!} \left\{ \left( \log(1+t) \right)^m \right\} \sum_{j=0}^{s} \binom{s}{j} t^j x^n
\]
\[
\sum_{0 \leq j \leq n-m} \binom{s}{j}(n) \sum_{m \leq l} S_l(l, m) \frac{t^l}{l!} x^n.
\]
\[
= \sum_{0 \leq j \leq n-m} \binom{s}{j}(n) S_l(n-j, m).
\]

(44)

Case 3. For \( s < k \), we have

\[
C_{n,m} = \frac{1}{m!} \left( \frac{(1 + t) \log(1 + t)}{t} \right)^{k-s} \left( \log(1 + t) \right)^m |(1 + t)^x|.
\]
\[
= \sum_{0 \leq j \leq n-m, m \leq l \leq n-j} \binom{s}{j}(n) S_l(l, m) \hat{D}_{n-j,l}^{(k-s)}.
\]

(45)

Therefore, by (41), (42), (43), (44), and (45), we obtain the following theorem.

**Theorem 8** Let \( n \geq 0 \), we have:

(I) For \( s > k \), we have

\[
\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \sum_{0 \leq j \leq n-m, m \leq l \leq n-j} \binom{s}{j}(n) S_l(n-j, m) \hat{D}_{n-j,l}^{(k-s)}.
\]

(II) For \( s = k \), we have

\[
\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \sum_{0 \leq j \leq n-m} \binom{s}{j}(n) S_l(n-j, m) B_m^{(k)}(x).
\]

(III) For \( s < k \), we have

\[
\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \sum_{0 \leq j \leq n-m, m \leq l \leq n-j} \binom{s}{j}(n) S_l(n-j, m) \hat{D}_{n-j,l}^{(k-s)}.
\]

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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