On Quasi-periodic Differential Pencils with Jump Conditions Inside the Interval

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Abstract. Non-self-adjoint second-order differential pencils on a finite interval with non-separated quasi-periodic boundary conditions and jump conditions are studied. We establish properties of spectral characteristics and investigate the inverse spectral problem of recovering the operator from its spectral data. For this inverse problem we prove the corresponding uniqueness theorem and provide an algorithm for constructing its solution.

Key words: differential pencils, non-separated boundary conditions, inverse problems

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1. Introduction

Consider the boundary value problem $B$ of the form

$$
y'' + (\rho^2 + \rho p(x) + q(x))y = 0, \quad x \in [0, T],
$$

$$
y(0) = \alpha y(T), \quad y'(0) - (i\rho h' + h)y(0) = \beta y'(T),
$$

$$
y(b_j + 0) = \gamma_j y(b_j - 0), \quad y'(b_j + 0) = \gamma_j^{-1} y'(b_j - 0) + (i\rho \eta'_j + \eta_j)y(b_j - 0), \quad j = 1, N - 1,
$$

where $\rho$ is the spectral parameter, $p(x), q(x)$ are complex-valued functions, and $h'(h', h, \alpha, \beta, \gamma_j$, $\eta'_j, \eta_j$ are complex numbers, $\alpha \beta \gamma_j \neq 0$, $z^\pm_j := \alpha(1 \mp h') + \beta \neq 0$, $\xi_j^\pm := (\gamma_j + (\gamma_j)^{-1})/2 \mp \eta'_j/2 \neq 0$. Assume that $p(x) \in AC[0, T]$ and $q(x) \in L(0, T)$. We study a nonlinear inverse problem of recovering $B$ from its spectral data. For this inverse problem we prove the uniqueness theorem and provide a procedure for constructing its solution.

Inverse spectral problems often appear in mathematics as well as in applications [1-3]. For Sturm-Liouville operators with separated boundary conditions, inverse spectral problems have been studied fairly completely (see the monographs [1-3] and the references therein). Such problems for Sturm-Liouville operators with non-separated boundary conditions investigated in [4-7] and other works.

Differential equations with nonlinear dependence on the spectral parameter arise in various problems of mathematics as well as in applications. In particular, several examples of such spectral problems arising in mechanical engineering are provided in the book [8] of Collatz; see also [9-11], where further references and links to applications can be found. Detailed studies on direct spectral problems for some classes of ordinary differential operators depending nonlinearly on the spectral parameter can be found in various publications, see e.g. [9-11].

Inverse spectral problems for differential pencils, because of their nonlinearity, are more difficult for investigating, and nowadays there are only isolated fragments, not constituting a general picture, in the inverse problem theory for equation (1). Some aspects of the inverse problem theory for pencils under various restrictions were studied in [12]-[16] and other works, but mostly only particular questions are considered there. Inverse problems for general non-selfadjoint boundary value problem (1)-(3) with jump conditions inside the interval have not been studied yet. We note the inverse problem, considered in this paper, appears in the inverse problem theory for differential operators on spatial networks with cycles (see [17-19]) which have many applications in natural sciences and engineering.

Some words about the structure of the paper. The statement of the inverse problem is provided in section 2. In Section 3 properties of the spectrum are established. In particular, the Weyl-type function and the corresponding Weyl sequence are introduced and investigated. In Section 4 we provide the solution of the inverse spectral problem for the boundary value problem $B$. 

2. Statement of the inverse problem.

Denote by \( S(x, \rho) \) and \( C(x, \rho) \) the solutions of equation (1) satisfying jump conditions (3) and the initial conditions

\[
S(0, \rho) = C'(0, \rho) = 0, \quad S'(0, \rho) = C(0, \rho) = 1.
\]

For each fixed \( x \), the functions \( S^{(\nu)}(x, \rho) \) and \( C^{(\nu)}(x, \rho) \), \( \nu = 0, 1 \), are entire in \( \rho \) of exponential type, and \( \langle C(x, \rho), S(x, \rho) \rangle \equiv 1 \), where \( \langle y, z \rangle := yz' - y'z \) is the Wronskian of \( y \) and \( z \). Put \( \varphi(x, \rho) = C(x, \rho) + (i \rho h + h)S(x, \rho) \), \( d(\rho) = S(T, \rho) \), \( d_0(\rho) = C(T, \rho) \). Eigenvalues \( \mathcal{P} = \{\rho_n\}_{n \in \mathbb{Z}} \) of the boundary value problem (1)-(3) coincide with the zeros (counting with multiplicities) of the characteristic function

\[
a(\rho) = \alpha \varphi(T, \rho) + \beta S'(T, \rho) - (1 + \alpha \beta).
\]

Let \( \Lambda := \{n : \ n = \pm 1, \pm 2, \ldots\} = \mathbb{Z} \setminus \{0\} \), and let \( \mathcal{V} = \{\nu_n\}_{n \in \Lambda} \) be zeros (counting with multiplicities) of \( d(\rho) \). Then \( \{\nu_n\}_{n \in \Lambda} \) are the eigenvalues of the boundary value problem \( B \) for Eq. (1) with jump conditions (3) and with the boundary conditions \( y(0) = y(T) = 0 \). The function \( d(\rho) \) is called the characteristic function for \( B \). Without loss of generality, we agree that the enumeration is chosen such that \( \nu_n \neq \nu_k \) if \( nk < 0 \). Let \( m_n \) be the multiplicity of \( \nu_n \) \( (\nu_n = \nu_{n+1} = \ldots = \nu_{n+m_n-1} \) put \( I := \{n \in \Lambda : \nu_{n-1} \neq \nu_n\} \), \( I' := \{n \in I : m_n > 1\} \).

Denote \( D(\rho) = \alpha \varphi(T, \rho) + \beta S'(T, \rho), \ Q(\rho) = \alpha \varphi(T, \rho) - \beta S'(T, \rho) \). Then

\[
D(\rho) = d(\rho) + (1 + \alpha \beta),
\]

\[
\varphi(T, \rho) = \frac{1}{2\alpha} \left(D(\rho) + Q(\rho)\right), \quad S'(T, \rho) = \frac{1}{2\beta} \left(D(\rho) - Q(\rho)\right).
\]

Since \( \varphi(x, \rho)S'(x, \rho) - \varphi'(x, \rho)S(x, \rho) \equiv 1 \), it follows that

\[
Q^2(\rho) = D^2(\rho) - 4\alpha\beta(1 + \varphi'(T, \rho)S(T, \rho)),
\]

and consequently,

\[
\dot{Q}(\rho)Q(\rho) = \dot{D}(\rho)D(\rho) - 2\alpha\beta(\varphi'(T, \rho)S(T, \rho) + \varphi'(T, \rho)S(T, \rho)),
\]

where "dot" denotes derivatives with respect to \( \rho \). Let \( n \in I \). Denote

\[
\omega_n = \begin{cases} 
0, & Q(\nu_n) = 0, \\
+1, & Q(\nu_n) \neq 0, \ \text{arg} Q(\nu_n) \in [0, \pi), \\
-1, & Q(\nu_n) \neq 0, \ \text{arg} Q(\nu_n) \in [\pi, 2\pi),
\end{cases}
\]

\[
\omega_n := d_1^{(\nu)}(\nu_n), \quad \nu = 0, m_n - 1, \quad I_0 = \{n \in I' : \omega_n = 0\}, \quad I_1 = \{n \in I' : \omega_n \neq 0\}.
\]

The sequence \( \Omega = \{\omega_n\}_{n \in I} \cup \{\omega_n\}_{n \in I_0, \nu = 1, m_n - 1} \) is called the \( \Omega \)-sequence for \( L \). We note that if \( I' = \emptyset \) (i.e. \( m_n = 1 \) for all \( n \)), then \( \Omega = \{\omega_n\}_{n \in \Lambda} \).

Let \( \alpha, \beta \) and \( \gamma \) are known a priori and fixed. The inverse problem is formulated as follows.

**Inverse problem 1.** Given \( a(\rho), d(\rho) \) and \( \Omega \), construct \( B \).

Obviously, in general it is not possible to recover all coefficients from (2)-(3). Note that this inverse problem is a generalization of the classical inverse problems for Sturm-Liouville operators [4-7].

3. Properties of the spectral characteristics.

Let \( \Phi(x, \rho) \) be the solution of equation (1) under the jump conditions (3) and the boundary conditions \( \Phi(0, \rho) = 1, \ \Phi(T, \rho) = 0 \). Denote \( M(\rho) := \Phi'(0, \rho) \). The function \( M(\rho) \) is called the Weyl-type function. Clearly,

\[
\Phi(x, \rho) = C(x, \rho) + M(\rho)S(x, \rho),
\]

(8)
\[ M(\rho) = -\frac{d_1(\rho)}{d(\rho)}. \]  

Since \( \langle C(x, \rho), S(x, \rho) \rangle \equiv 1 \), it follows from (8) that

\[ \langle \Phi(x, \rho), S(x, \rho) \rangle \equiv 1. \]  

Denote \( T_k := b_k - b_{k-1}, \ k = \overline{1,N} \). Then \( b_k = T_1 + \ldots + T_k \), \( T = T_1 + \ldots + T_N \). Put \( x_k = x - b_{k-1} \) for \( x \in [b_{k-1}, b_k] \); hence \( x_k \in [0, T_k] \). Let \( S_k(x_k, \rho) \) and \( C_k(x_k, \rho) \) be the solutions of equation (1) on \([b_{k-1}, b_k]\) under the initial conditions

\[ S_k(0, \rho) = C'_k(0, \rho) = 0, \quad S'_k(0, \rho) = C_k(0, \rho) = 1. \]  

For each fixed \( x_k \), the functions \( S^{(\nu)}_k(x_k, \rho) \) and \( C^{(\nu)}_k(x_k, \rho) \), \( \nu = 0, 1 \), are entire in \( \rho \), and

\[ \langle C_k(x_k, \rho), S_k(x_k, \rho) \rangle \equiv 1. \]

**Lemma 1.** The following relations hold for \( k = \overline{1,N-1}, \ \nu = 0, 1 \):

\[ S^{(\nu)}(b_{k+1} - 0, \rho) = \gamma_k S(b_k - 0, \rho) C^{(\nu)}_k(T_{k+1}, \rho) + \gamma_k^{-1} S'(b_k - 0, \rho) S^{(\nu)}_k(T_{k+1}, \rho) \]

\[ + (i \rho \eta_k + \xi_k) S(b_k - 0, \rho) S^{(\nu)}_k(T_{k+1}, \rho), \]  

\[ C^{(\nu)}(b_{k+1} - 0, \rho) = \gamma_k C(b_k - 0, \rho) C^{(\nu)}_k(T_{k+1}, \rho) + \gamma_k^{-1} C'(b_k - 0, \rho) S^{(\nu)}_k(T_{k+1}, \rho) \]

\[ + (i \rho \eta_k + \xi_k) C(b_k - 0, \rho) S^{(\nu)}_k(T_{k+1}, \rho). \]  

Indeed, fix \( k = \overline{1,N-1} \). Let \( x \in [b_k, b_{k+1}] \), i.e. \( x = x_{k+1} + b_k \), \( x_{k+1} \in [0, T_{k+1}] \). Using the fundamental system of solutions \( C_{k+1}(x_{k+1}, \rho), S_{k+1}(x_{k+1}, \rho) \), one has

\[ S^{(\nu)}(x, \rho) = A(\rho) C^{(\nu)}_{k+1}(x_{k+1}, \rho) + B(\rho) S^{(\nu)}_{k+1}(x_{k+1}, \rho), \quad \nu = 0, 1. \]

Taking initial conditions (11) into account we find the coefficients \( A(\rho) \) and \( B(\rho) \), and arrive at (12). Relation (13) is proved similarly.

Denote

\[ \mathcal{E}(x) = \frac{1}{2} \int_0^x p(t) \, dt, \quad \omega = \frac{1}{2 \tau} \int_0^\tau p(t) \, dt, \quad \tau = \text{Im} \rho, \quad G_\delta = \{ \rho : |\rho - \nu_n| \geq \delta \ \forall n \}, \]

\[ \Pi^+ = \{ \rho : \pm \tau > 0 \}, \quad \Pi_\delta^+ = \{ \rho : \arg \rho \in [\delta, \pi - \delta] \}, \quad \Pi_\delta^- = \{ \rho : \arg \rho \in [\pi + \delta, 2\pi - \delta] \}. \]

It is known (see [9]) that there exists a fundamental system of solutions

\{\( Y^{\pm}_1(x, \rho), Y^{\pm}_2(x, \rho) \}\}, \ x \in [0, T], \ \rho \in \Pi^\pm \), of equation (1) with the properties:

1) The functions \( Y^{\pm}_k(x, \rho) \) are regular in \( \rho \in \Pi^\pm \), \( |\rho| > \rho^* \), and are continuous for \( x \in [0, T] \), \( \rho \in \overline{\Pi^\pm} \), \( |\rho| \geq \rho^* \).

2) For \( |\rho| \to \infty \), \( \rho \in \overline{\Pi^\pm} \), \( k = 1, 2, \ \nu = 0, 1, \)

\[ \frac{d^\nu}{dx^\nu} Y^{\pm}_k(x, \rho) = (\rho R_k)^\nu \exp(\rho x + \mathcal{E}(x)) R_k[1], \quad [1] = 1 + O(\rho^{-1}), \quad R_k = (-1)^{k-1}i. \]  

Using (14) and Lemma 1, one gets for \( x \in (b_j, b_{j+1}) \), \( |\rho| \to \infty \), \( \rho \in \Pi^\pm_\delta \):

\[ C^{(\nu)}(x, \rho) = \frac{\xi_1^{\pm} \ldots \xi_j^{\pm}}{2i \rho} (\mp i \rho)^\nu \exp(\mp i (\rho x + \mathcal{E}(x))[1], \]  

\[ S^{(\nu)}(x, \rho) = \mp \frac{\xi_1^{\pm} \ldots \xi_j^{\pm}}{2i \rho} (\mp i \rho)^\nu \exp(\mp i (\rho x + \mathcal{E}(x))[1], \]  

\[ \Phi^{(\nu)}(x, \rho) = \frac{1}{\xi_1^{\pm} \ldots \xi_j^{\pm}} (\pm i \rho)^\nu \exp(\pm i (\rho x + \mathcal{E}(x))[1], \]
In particular, we have for $|\rho| \to \infty$, $\rho \in \Pi_0^\pm$:

$$a(\rho) = \frac{z_0^+}{2}(\xi_1^+ \ldots \xi_{N-1}^+) \exp(\mp i(\rho+\omega)T)[1], \quad d(\rho) = \mp \frac{1}{2i\rho}(\xi_1^+ \ldots \xi_{N-1}^+) \exp(\mp i(\rho+\omega)T)[1]. \quad (18)$$

Moreover, for $x \in [0,T]$, $\rho \in \Pi_0^\pm$:

$$|S^{(\nu)}(x, \rho)| \leq C|\rho|^{\nu-1} \exp(|\tau|x), \quad |C^{(\nu)}(x, \rho)| \leq C|\rho|^\nu \exp(|\tau|x), \quad (19)$$

$$|\Phi^{(\nu)}(x, \rho)| \leq C|\rho|^\nu \exp(-|\tau|x), \quad |M(\rho)| \leq C|\rho|, \quad \rho \in G_\delta. \quad (20)$$

Let $n \in I$. Using (9) we obtain that in a neighborhood of the point $\rho = \nu_n$, the function $M(\rho)$ has the representation

$$M(\rho) = \sum_{\nu=0}^{m_n-1} \frac{M_{n+\nu}}{(\rho - \nu_n)^{\nu+1}} + M_n^*(\rho), \quad (21)$$

where $M_n^*(\rho)$ is regular in $\rho = \nu_n$, and the coefficients $M_{n+\nu}$, $\nu = 0, m_n - 1$ are calculated from $d_1^{(\nu)}(\nu_n)$ and $d^{(\nu+m_n)}(\nu_n)$ for $\nu = 0, m_n - 1$. More precisely,

$$M_{n+m_n-1-\nu} = -\frac{1}{d_{0n}} \left( d_1^{(\nu)} + \sum_{k=0}^{\nu-1} M_{n+m_n-1-k} d_{\nu-k,n} \right), \quad \nu = 0, m_n - 1,$$

$$d_1^{(\nu)} := \frac{1}{\nu!} d_1^{(\nu)}(\nu_n), \quad d_{\nu} := \frac{1}{(\nu+m_n)!} d^{(\nu+m_n)}(\nu_n), \quad \nu = 0, m_n - 1.$$  

In particular, $M_{n+m_n-1} = -d_{0n}/d_0$. If $m_n = 1$ (i.e. $n \in I \setminus I'$), then

$$M_n = -\frac{d_1^{(\nu_n)}}{d_0}, \quad \dot{d}(\rho) := \frac{d}{d\rho} d(\rho). \quad (22)$$

The sequence $\{M_n\}_{n \in \Lambda}$ is called the Weyl sequence. The data $D = \{\nu_n, M_n\}_{n \in \Lambda}$ are called the spectral data. We note that the specification of the spectral data $D$ uniquely determines the Weyl-type function $M(\rho)$ (see [16]).

We consider the following auxiliary inverse problem which is called IP-0.

**IP-0.** Given the spectral data $D = \{\nu_n, M_n\}_{n \in \Lambda}$ and $\gamma_j$, $j = 1, N - 1$, construct $p(x), q(x), x \in (0,T), \quad \eta_j', \eta_j$, $j = 1, N - 1$.

Let us prove the uniqueness theorem for IP-0. For this purpose together with $B$ we consider a boundary value problem $\tilde{B}$ of the same form but with different potentials $\tilde{p}(x), \tilde{q}(x)$, and different coefficients of the boundary and the jump conditions. We agree that if a certain symbol $\theta$ denotes an object related to $B$, then $\tilde{\theta}$ will denote the analogous object related to $\tilde{B}$.

**Theorem 1.** If $D = \tilde{D}$, $\gamma_j = \tilde{\gamma}_j$, $j = 1, N - 1$, then $p = \tilde{p}, q = \tilde{q}$, $\eta_j' = \tilde{\eta}_j', \eta_j = \tilde{\eta}_j$, $j = 1, N - 1$.

**Proof.** Consider the functions

$$P_{11}(x, \rho) = \Phi(x, \rho) \tilde{S}'(x, \rho) - S(x, \rho) \tilde{\Phi}'(x, \rho), \quad P_{12}(x, \rho) = S(x, \rho) \tilde{\Phi}(x, \rho) - \Phi(x, \rho) \tilde{S}(x, \rho). \quad (23)$$

Using (10), we calculate

$$S(x, \rho) = P_{11}(x, \rho) \tilde{S}(x, \rho) + P_{12}(x, \rho) \tilde{S}'(x, \rho), \quad \Phi(x, \rho) = P_{11}(x, \rho) \tilde{\Phi}(x, \rho) + P_{12}(x, \rho) \tilde{\Phi}'(x, \rho). \quad (24)$$
According to (8) and (9), for each fixed $x$, the functions $P_{11}(x, \rho)$ and $P_{12}(x, \rho)$ are meromorphic in $\rho$ with poles at the points $\nu_n$ and $\tilde{\nu}_n$. Denote $G^0_\delta = G_\delta \cap \tilde{G}_\delta$. By virtue of (19), (20) and (23) we get
\[ |P_{12}(x, \rho)| \leq C|\rho|^{-1}, \quad |P_{11}(x, \rho)| \leq C, \quad \rho \in G^0_\delta. \] (25)

It follows from (8) and (23) that
\[ P_{11}(x, \rho) = C(x, \rho)\tilde{S}'(x, \rho) - S(x, \rho)\tilde{C}'(x, \rho) + (M(\rho) - \tilde{M}(\rho))S(x, \rho)\tilde{S}'(x, \rho), \]
\[ P_{12}(x, \rho) = S(x, \rho)\tilde{C}(x, \rho) - C(x, \rho)\tilde{S}(x, \rho) - (M(\rho) - \tilde{M}(\rho))S(x, \rho)\tilde{S}(x, \rho). \]

Since $D = \tilde{D}$, it follows from (21) that for each fixed $x$, the functions $P_{1k}(x, \rho)$ are entire in $\rho$. Together with (25) this yields $P_{12}(x, \rho) \equiv 0$, $P_{11}(x, \rho) \equiv A(x)$, where the function $A(x)$ does not depend on $\rho$. Using (24) we derive
\[ S(x, \rho) \equiv A(x)\tilde{S}(x, \rho), \quad \Phi(x, \rho) \equiv A(x)\tilde{\Phi}(x, \rho). \] (26)

Taking (15)-(17) and (23) into account, we obtain for each fixed $x \in (b_j, b_{j+1})$:
\[ P_{11}(x, \rho) = \frac{1}{2} \left( \frac{\xi^+_j, \ldots, \xi^+_{j-1}}{\xi^+_1, \ldots, \xi^+_N} + \frac{\tilde{\xi}^+_j, \ldots, \tilde{\xi}^+_{j-1}}{\tilde{\xi}^+_1, \ldots, \tilde{\xi}^+_N} \right) \] [1], \quad \rho \in \Pi^+_\delta, \quad |\rho| \to \infty.

Therefore, the function $A(x)$ is piecewise constant (a step-function). Together with (26) this yields $q(x) = \tilde{q}(x), \quad p(x) = \tilde{p}(x), \quad x \in (0, T)$. It follows from (26) that
\[ \frac{\tilde{S}(x, \rho)}{S(x, \rho)} = \frac{\tilde{\Phi}(x, \rho)}{\Phi(x, \rho)}, \]
and consequently, $(\xi^+_j)^2 = (\tilde{\xi}^+_j)^2$, $j = 1, N - 1$. Then $\eta_j = \tilde{\eta}_j, \quad j = 1, N - 1$, and consequently, $\xi^+_j = \tilde{\xi}^+_j, \quad j = 1, N - 1$, $S(x, \rho) \equiv \tilde{S}(x, \rho)$, $\Phi(x, \rho) \equiv \tilde{\Phi}(x, \rho)$, $\eta_j = \tilde{\eta}_j, \quad j = 1, N - 1$. Theorem 1 is proved.

Using the method of spectral mappings [3], one can obtain a constructive procedure for the solution of IP-0 (see [3] for details).

4. Solution of the Inverse problem 1.

Let $a(\rho), d(\rho)$ and $\Omega$ be given. Note that $\alpha, \beta, \gamma_j, j = 1, N - 1$ are known a priori.

The solution of the Inverse problem 1 is constructed as follows.

First we calculate the zeros $\mathcal{V} = \{\nu_n\}_{n \in \Lambda}$ of $d(\rho)$. Using the asymptotics (18) we find $(\xi^+_1 \ldots \xi^+_N-1)$ and $h'$. Taking (5) into account, we construct $D(\rho)$. According to (6) we calculate $Q^2(\nu_n) = D^2(\nu_n) - 4\alpha\beta$, and
\[ Q(\nu_n) = \omega_n \sqrt{D^2(\nu_n) - 4\alpha\beta}, \quad n \in I. \] (27)

Here and below we agree that if $z = |z|e^{i\xi}, \xi \in [0, 2\pi)$, then $\sqrt{z} = |z|^{1/2}e^{i\xi/2}$. We construct $\omega_{n0}$ by
\[ \omega_{n0} = \frac{1}{2\alpha} \left( D(\nu_n) + Q(\nu_n) \right), \] (28)

since $\omega_{n0} = d_1(\nu_n) = C(T, \nu_n) = \varphi(T, \nu_n)$.

We construct the Weyl sequence $\{M_n\}_{n \in \Lambda}$ as follows: 

Case 1. Let $n \in I \setminus I'$ (i.e. $m_n = 1$). Then, in view of (22),
\[ M_n = -\frac{\omega_{n0}}{d(\nu_n)}. \] (29)
Case 2. Let $n \in I_1$ (i.e. $m_n > 1$, $\omega_n \neq 0$). Then it follows from (7) that

$$(\hat{Q}(\rho)Q(\rho))^{(\nu-1)}_{\rho=\nu_n} = (D(\rho)D(\rho))^{(\nu-1)}_{\rho=\nu_n}, \quad \nu = 1, m_n - 1.$$ (30)

Using (30) we find $Q^{(\nu)}(\nu_n), \quad \nu = 1, m_n - 1$. Since

$$d_1(\rho) = \frac{1}{2\alpha} \left(D(\rho) + Q(\rho)\right) - (i\rho' + h)d(\rho),$$

we construct $\omega_{n\nu}, \quad \nu = 1, m_n - 1$ by the formula

$$\omega_{n\nu} = \frac{1}{2\alpha} \left(D^{(\nu)}(\nu_n) + Q^{(\nu)}(\nu_n)\right), \quad \nu = 1, m_n - 1.$$ (31)

Case 3. Let $n \in I_0$ (i.e. $m_n > 1$, $\omega_n = 0$). Then $\omega_{n\nu}, \quad \nu = 1, m_n - 1$ are given a priori.

Thus, we have constructed the Weyl sequence $\{Q_n\}_{n \in \Lambda}$. By solving the auxiliary inverse problem IP-0 we find $p(x), q(x), x \in (0, T), \quad \eta_j, \eta_j, \quad j = 1, N - 1$, and then we calculate the coefficient $h$, using (4). Thus, the following theorem holds.

**Theorem 2.** The specification of $a(\rho), d(\rho)$ and $\Omega$ uniquely determines the boundary value problem $B$. The solution of Inverse problem 1 can be found by the following algorithm.

**Algorithm.** Given $a(\rho), d(\rho)$ and $\Omega$.

1) Calculate zeros $\nu = \{\nu_n\}_{n \in \Lambda}$ of $d(\rho)$.
2) Find $h'$ using (18).
3) Construct $D(\rho)$ by (5).
4) Calculate $Q^2(\nu_n) = D^2(\nu_n) - 4\alpha \beta$, and $Q(\nu_n)$ by (27).
5) Construct $\omega_{n0}$ by (28).
6) Construct $Q^{(\nu)}(\nu_n), \quad n \in I_1, \quad \nu = 1, m_n - 1$ using (30).
7) Find $\omega_{n\nu}, \quad n \in I_1, \quad \nu = 1, m_n - 1$, via (31).
8) Calculate the Weyl sequence $\{M_n\}_{n \geq 1}$ using (29) and the recurrent formula

$$M_{n+m_n-1-\nu} = -\frac{1}{d_{\nu n}} \left(d_{\nu m}^{(\nu)} + \sum_{k=0}^{\nu-1} M_{n+m_n-1-k} d_{\nu-k,n}\right), \quad n \in I, \quad \nu = 0, m_n - 1, \quad d_{\nu m}^{(\nu)} := \frac{1}{\nu!} \omega_{n\nu}.$$  

9) Find $p(x), q(x), x \in (0, T)$ and $\eta_j, \eta_j, \quad j = 1, N - 1$, by solving the inverse problem IP-0.
10) Calculate the coefficient $h$.

Similarly, one can solve the following inverse problem.

**Inverse problem 2.** Given $P, \nu$ and $\Omega$, construct $B$.

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