Communication Efficient Sparsification for Large Scale Machine Learning

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Abstract
The increasing scale of distributed learning problems necessitates the development of compression techniques for reducing the information exchange between compute nodes. The level of accuracy in existing compression techniques is typically chosen before training, meaning that they are unlikely to adapt well to the problems that they are solving without extensive hyper-parameter tuning. In this paper, we propose dynamic tuning rules that adapt to the communicated gradients at each iteration. In particular, our rules optimize the communication efficiency at each iteration by maximizing the improvement in the objective function that is achieved per communicated bit. Our theoretical results and experiments indicate that the automatic tuning strategies significantly increase communication efficiency on several state-of-the-art compression schemes.

1. Introduction
The vast size of modern machine learning problems is shifting the operating regime of optimization from centralized to distributed algorithms. This makes computations manageable but creates huge communication overheads for large dimensional problems (Dean et al., 2012; Seide et al., 2014; Alistarh et al., 2017). This is because distributed optimization algorithms hinge on frequent transmissions of gradients between compute nodes. These gradients are typically huge, since their size is proportional to the model size and state-of-the-art models often have millions of parameters. To get a sense of the communication costs, transmitting a single gradient or stochastic gradient using single precision (32 bits per entry) requires 40 MB for a models with 10 million parameters (which is not uncommon). This means that if we use 4G, then we can expect to transmit roughly one gradient per second. These huge communication costs easily overburden training on collocated servers and become infeasible for federated learning and learning on IoT or edge devices.

To counter these communication overheads, much recent research has focused on compressed gradient methods. These methods achieve communication efficiency by using only the most informative parts of the gradients at each iteration. We may, for example, sparsify the gradient and use only the most significant entries at each iteration, and set the rest to be zero (Alistarh et al., 2017; 2018; Stich et al., 2018; Wen et al., 2017; Wang et al., 2018; Khirirat et al., 2018b; Wangni et al., 2018). We may also quantize the gradients or do some mix of quantization and sparsification (Alistarh et al., 2017; Khirirat et al., 2018a; Magnússon et al., 2017; Wangni et al., 2018; Zhu et al., 2016; Rabbat & Nowak, 2005).

The above references show that compressed gradient methods can achieve huge communication improvements for specific training problems. However, to reap these communication benefits we usually need to carefully tune the level of accuracy of each compressor before training. For example, to sparsify the gradient we need to decide how many gradient components we will use. We cannot expect there to be a universally good compressor that works well on all problems, as suggested by the worst case communication complexity of optimization in (Tsitsiklis & Luo, 1987). There is generally a delicate problem-specific balance between compressing too much or too little. Striking this balance can be achieved by hyper-parameter tuning. However, hyper-parameter tuning is expensive and the resulting tuning parameters will be problem specific. We take another approach and adaptively tune the level of accuracy by adapting to each communicated gradient.

Contributions: We propose Communication-aware Adaptive Tuning (CAT) for general compression schemes. The main idea is to find the optimal tuning for each communicated gradient by maximizing the objective function improvement achieved per bit. We illustrate these ideas on three state-of-the-art compression schemes: a) sparsifica- tion, b) sparsification with quantization and c) stochastic sparsification. In all cases, we first derive descent lemmas specific to the compression, relating the function improvement to the tuning parameter. Using these results we can find the tuning that optimizes the communication efficiency.
We study compressed gradient methods \( D \) where \( I \) is a set of data points and \( L \) is the index set for the \( T \) components of largest magnitude. Sparsification together with quantization has been shown to give good practical performance (Alistarh et al., 2017). In this case, we communicate only the gradient as:

\[
[Q_T(g)]_j = \begin{cases} 
g_j & \text{if } j \in I_T(g) \\
0 & \text{otherwise.} 
\end{cases}
\]

where \( I_T(g) \) is the index set for the \( T \) components of \( g \) with largest magnitude. Sparsification together with quantization has been shown to give good practical performance (Alistarh et al., 2017). In this case, we communicate only the gradient magnitude and the sparsity structure of the gradient where

\[
[Q_T(g)]_j = \begin{cases} 
\|g\| \text{sign}(g_j) & \text{if } i \in I_T(g) \\
0 & \text{otherwise.} 
\end{cases}
\]

It is sometimes advantageous to use stochastic sparsification. In this case, instead of sending the top \( T \) entries, we send on average \( T \) components. We can achieve this by setting

\[
[Q_{T,p}(g)]_j = \frac{g_j}{p_j} \xi_j, \quad (6)
\]

where \( \xi_j \sim \text{Bernoulli}(p_j) \) and

\[
T = \sum_{j=1}^{d} p_j.
\]

Ideally, we would like \( p_j \) to represent the magnitude of \( g_j \), so that if \( |g_j| \) is large relative to the other entries then \( p_j \) should also be large. There are many heuristic methods to choose \( p_j \). For example, if we set \( p_j = |g_j|/\|g\|_q \) with \( q = 2, \infty \), and \( q \in (0, \infty) \) then we get, respectively, the stochastic sparsifications in (Alistarh et al., 2017) with \( s = 1 \), the TernGrad in (Wen et al., 2017), and \( \ell_q \)-quantization in (Wang et al., 2018). We can also find the optimal choice of \( p \), see (Wang et al., 2018) and Section 5 for details.

Experimental results have shown that compressed gradient methods can save a lot of communication in large-scale machine learning (Shi et al., 2019a;b). Nevertheless, we can easily create pedagogical examples where they are no more communication efficient than full gradient descent. For sparsification, consider the function

\[
F(x) = (1/2)||x||^2.
\]

This function, gradient descent with step-size \( \gamma \) converges in one iteration. That results in communication of \( d/T \) floating points (one gradient \( \nabla F(x) \in \mathbb{R}^d \)) to reach any \( \epsilon \)-accuracy. On the other hand, with \( T \)-sparsified gradient method (where \( T \) divides \( d \)) we need \( d/T \) iterations, which also results in \( d \) communicated floating points. In fact, the sparsified method is even worse, because it requires additional communication of \( d \log(d) \) bits to indicate the sparsification pattern.

This means that the benefits of sparsification are not seen on worst case problems, and that traditional worst case analysis (e.g. Khirirat et al. (2018b)) is unable to guarantee improvements in computation complexity. Rather, sparsification is useful for exploiting potential structure that appears in real-world problems. The key in taking advantage of these structures is to choose the correct \( T \) at each iteration. In this paper we illustrate how to choose \( T \) dynamically to optimize the communication efficiency of sparsification.

2.2. Communication

The compressors discussed above have a tuning parameter \( T \), which controls the level of sparsity of the compressed gradient. Our goal is to tune \( T \) adaptively to optimize the communication efficiency. To explain this we need to first discuss the communication involved. Let \( C(T) \) denote the total number of bits communicated per iteration as a function of the parameter \( T \). The value of \( C(T) \) can be split into
payload (actual data) and communication overhead. The payload is the amount of bits that are needed to communicate the compressed gradient. For the sparsification in Equation (4) the payload consumes
\[ P^s(T) = T \times ([\log_2(d)] + \text{FPP}) \text{ bits} \]
(7)
since we need to communicate \( T \) floating points and indicate \( T \) indices in a \( d \) dimensional gradient vector. Here \text{FPP} is our floating point precision, e.g., \text{FPP} = 32 or \text{FPP} = 64 if we use, respectively, single or double precision floating-points. For the sparsification with quantization in Equation (5) the payload consumes
\[ P^{sq}(T) = \text{FPP} + T \times \lfloor \log_2(d) \rfloor \text{ bits}, \]
(8)
since only one floating point is sent per iteration. Our simplest communication model accounts only for the payload,
\[ C(T) = P(T). \]
We call this the payload model. In real-world networks, however, each communication also includes overhead and set-up costs. A more realistic model is therefore affine
\[ C(T) = c_1 P(T) + c_0, \]
(9)
where \( P(T) \) is the payload. Here \( c_0 \) is the communication overhead while \( c_1 \) is the cost of transmitting a single payload byte. For example, if we just count transmitted bits (\( c_1 = 1 \)), a single UDP packet transmitted over Ethernet requires an overhead of \( c_0 = 54 \times 8 \) bits and can have a payload of up to 1472 bytes. In the wireless standard IEEE 802.15.4, the overhead ranges from 23-82 bytes, leaving 51 – 110 bytes of payload before the maximum packet size of 133 bytes is reached (Kozowski & Sosnowski, 2017). Another possibility is to consider a fixed cost per packet
\[ C(T) = c_1 \times \left\lfloor \frac{P(T)}{P_{\text{max}}} \right\rfloor + c_0, \]
(10)
where \( P_{\text{max}} \) is the number of payload bits per packet. The term \( \left\lfloor \frac{P(T)}{P_{\text{max}}} \right\rfloor \) counts the number of packets required to send the \( P(T) \) payload bits, \( c_1 \) is the cost per packet, and \( c_0 \) is the cost of initiating the communication. These are just two examples; ideally, \( C(T) \) should be tailored to the specific communication standard in use.

2.3. Key Idea: Communication-aware Adaptive Tuning
When communicating the compressed gradients we would like to use each bit as efficiently as possible. In optimization terms, we would like the objective function improvement we get for each communicated bit to be as large as possible. In other words, we want to maximize the ratio
\[ \text{Efficiency}(T) = \frac{\text{Improvement}(T)}{C(T)}, \]
(11)
where \( \text{Improvement}(T) \) is the improvement in the objective function when we use \( T \)-sparsification with the given compressor. We will demonstrate how the value of \( \text{Improvement}(T) \) can be obtained from novel descent lemmas and derive dynamic sparsification policies which, at each iteration, find the \( T \) that optimizes \( \text{Efficiency}(T) \). We work out the details for the three gradient compression families and the two communication models introduced above. However, we believe that this idea is general and can help in improving the communication efficiency for many other optimization algorithms and compression techniques.

3. Dynamic Sparsification
We now describe the Communication-aware Adaptive Tuning (CAT) for the sparsified gradient method. The main idea is to find the best \( T \) at each iteration that gives the biggest improvement in the objective function per communicated bit. The objective function improvement is captured by the following measure (we give a formal proof in § 3.1)
\[ \alpha^i(T) = \frac{|Q_T(\nabla F(x^i))|^2}{||\nabla F(x^i)||^2}, \]
where \( Q_T(\cdot) \) is the \( T \)-sparsification operator defined in (4). Then our CAT sparsified gradient method is given by
\[ T^i = \arg \max_{T \in [1,d]} \frac{\alpha^i(T)}{C(T)} \]
(12)
\[ x^{i+1} = x^i - \frac{1}{L} Q_{T^i}(\nabla F(x^i)). \]
(13)
In the first step, described by Equation (12), the algorithm finds the sparsification parameter \( T \) that optimizes the communication efficiency (cf. Equation 11 above). In the second step (Equation 13), the algorithm just performs a standard sparsification using the \( T \) found in the previous step.

To find \( T^i \) at each iteration we need to solve the optimization problem in Equation (12). This is is a one dimensional optimization problem and can hence be solved efficiently. In particular, as we will show, under the affine communication model, the efficiency is quasi-concave and its optimum is easily found; while under the packet-based model, the optimal \( T \) is a multiple of the number of \( \left\lfloor \frac{P(T)}{P_{\text{max}}} \right\rfloor \).

3.1. A Measure of Function Improvement
The next result shows how \( \alpha^i(T) \) captures the guaranteed objective function improvement for a given \( T \):

Lemma 1. Suppose that \( F : \mathbb{R}^d \rightarrow \mathbb{R} \) is (possibly non-convex) \( L \)-smooth and \( \gamma = 1/L. \) Then for any \( x, x^+ \in \mathbb{R}^d \) with
\[ x^+ = x - \gamma Q_T(\nabla F(x)) \]
we have
\[ F(x^+) \leq F(x) - \frac{\alpha(T)}{2L} ||\nabla F(x)||^2 \]
Table 1. The number of iterations needed to find an \(\epsilon\)-accurate solution for gradient descent (GD) and sparsiﬁed gradient descent with constant \(T\) (\(T\)-SGD). We assume that \(F(\cdot)\) is \(L\)-smooth and consider the cases when \(F(\cdot)\) is \(\mu\)-strongly convex (column 2), convex (column 3), and non-convex (column 4). In the strongly convex case we let \(\kappa = L/\mu\) is the condition number. The initial solution accuracy is denoted by \(\epsilon_0 = F(x^0) - F^*\) and \(R = \|x - x^*\|\) for some optimal solution \(x^*\).

| ALGORITHM | \(\mu\)-CONVEX | CONVEX | NONCONVEX |
|-----------|----------------|--------|-----------|
| \(\epsilon\)-MEASURE | \(F(x^*) - F^*\) | \(F(x^*) - F^*\) | \(\|\nabla F(x^*)\|\) |
| GD | \(\kappa \log((\frac{\kappa}{\epsilon})\)) | \(\frac{2L\epsilon^2}{\kappa} + \frac{2\log \kappa}{\kappa}\) | \(\frac{2L\epsilon}{\kappa}\) |
| \(T\)-SGD | \(\frac{1}{\alpha_T} \kappa \log((\frac{\kappa}{\epsilon})\)) | \(\frac{1}{\alpha_T} \frac{2L\epsilon^2}{\kappa} + \frac{2\log \kappa}{\kappa}\) | \(\frac{1}{\alpha_T} \frac{2L\epsilon}{\kappa}\) |
| \(T\)-SGD WC | \(\frac{d}{T} \kappa \log((\frac{\kappa}{\epsilon})\)) | \(\frac{d}{T} \frac{2L\epsilon^2}{\kappa} + \frac{2\log \kappa}{\kappa}\) | \(\frac{d}{T} \frac{2L\epsilon}{\kappa}\) |

where

\[
\alpha(T) = \frac{\|Q_T(\nabla F(x))\|^2}{\|\nabla F(x)\|^2}.
\]

Moreover, there are \(L\)-smooth functions \(F(\cdot)\) for which the inequality is tight for every \(T = 1, \ldots, d\).

This lemma is in the category of descent lemmas, which are standard tools to study the convergence of convex and non-convex functions. In fact, Lemma 1 is a generalization of the standard descent lemma for \(L\)-smooth functions (see for example Proposition A.24 in \(\text{(Bertsekas, 1999)}\)). In particular, if the gradient \(\nabla F(x)\) is \(T\)-sparse (or \(T = d\)) then Lemma 1 gives the standard descent

\[
F(x^{i+1}) \leq F(x^i) - \frac{1}{2L} \|\nabla F(x^i)\|^2.
\]

We may use Lemma 1 to derive the same convergence rate bounds for sparsiﬁed gradient methods as for gradient descent (we do this in Table 1; see explanations in the next subsection).

Lemma 1 is at the heart of our dynamic sparsiﬁcation. It allows us to choose \(T\) adaptively at each iteration to optimize the ratio between the descent in the objective function and communicated bits. In particular, Lemma 1 implies that the descent in the objective function for a given \(T\) is

\[
F(x^{i+1}) - F(x^i) \leq -\frac{\alpha(T)}{2L} \|\nabla F(x^i)\|^2.
\]

Since \(\|\nabla F(x^i)\|^2/(2L)\) is independent of \(T\), the descent per communicated bit is maximized by the \(T\) that maximizes the ratio \(\alpha(T)/C(T)\) (as in Equation (12)). The descent always increases with \(T\) and is bounded as follows.

**Lemma 2.** For \(g \in \mathbb{R}^d\) the function

\[
\alpha(T) = \frac{\|Q_T(g)\|^2}{\|g\|^2}
\]

is increasing and concave when extended to the the continuous interval \([0, d]\). Moreover, \(\alpha(T) \geq T/d\) for all \(T \in \{0, \ldots, d\}\) and there exists a \(L\)-smooth function such that \(\|Q_T(\nabla f(x))\|^2/\|\nabla f(x)\|^2 = T/d\) for all \(x \in \mathbb{R}^d\).

We will explore several consequences of this lemma in the next subsection, but first we make the following observation:

**Proposition 1.** Let \(\alpha(T)\) be increasing and concave. If \(C(T) = \bar{c}_1 T + c_0\), then \(\alpha(T)/C(T)\) is quasi-concave and has a unique maximum on \([0, d]\). When \(C(T) = \bar{c}_1 [T/\tau_{\max}] + c_0\), on the other hand, \(\alpha(T)/C(T)\) attains its maximum for a \(T\) which is an integer multiple of \(\tau_{\max}\).

The proposition demonstrates that the optimization in Equation (12) is easy to solve. For the affine communication model, one can simply sort the elements in decreasing magnitude, initialize \(T = 1\) and increase \(T\) until \(\alpha(T)/C(T)\) decreases. In the packet model, the search for the optimal \(T\) is even more efficient, as one can increase \(T\) in steps of \(\tau_{\max}\).

### 3.2. The Many Benefits of Dynamic Sparsification

Before illustrating the practical benefits of dynamic sparsiﬁcation, we will consider its theoretical guarantees in terms of iteration and communication complexity. To this end, Table 1 compares the iteration complexity of Gradient Descent (GD) and \(T\)-Sparsiﬁed Gradient Descent (\(T\)-SGD) with constant \(T \in [1, d]\) for strongly-convex, convex, and non-convex problems. The table shows how many iterations are needed to reach an accuracy with \(\epsilon > 0\). The results for gradient descent are well known and found, e.g., in \(\text{(Nesterov, 2018)}\), and the worst-case analysis appeared in \(\text{(Khirarat et al., 2018b)}\). The results for \(T\)-sparsiﬁed gradient descent can be derived similarly except with using our Descent Lemma 1 instead of the standard descent lemma. We provide proof in the supplementary material.

Comparing rows 3 and 5 in the table, we see that the worst-case analysis does not guarantee any improvements in the amount of communicated floating points. Although \(T\)-SGD only communicates \(T^2\) out of \(d\) gradient entries in each round, we need to perform \(d/T\) times more iterations with \(T\)-SGD than with SGD, so the two approaches will need to communicate the same number of floating points. In fact, \(T\)-SGD will be worse in terms of communicated bits, since it requires communication of \(T[\log_2(d)]\) additional bits per iteration to indicate the sparsiﬁcity pattern (see Equation (7)).

Let us now turn our attention to our novel analysis shown in row 4 of Table 1. Here, the parameter \(\bar{\alpha}_T\) is a lower bound on \(\alpha^i(T)\) over every iteration, that is

\[
\alpha^i(T) \geq \bar{\alpha}_T \quad \text{for all } i.
\]

Unfortunately, \(\bar{\alpha}_T\) is not useful for algorithm development: we know from Lemma 2 that it can be as low as \(T/d\), and it...
Figure 1. CAT sparsified gradient descent on the RCV1 data set, with 697, 641 data points and 47, 236 features.

is not easy to compute a tight data-dependent bound off-line, since \( \bar{\alpha}_T \) depends on the iterates produced by the algorithm. However, \( \bar{\alpha}_T \) explains why gradient sparsification is communication efficient. In practice, the majority of the gradient energy tends to be concentrated to a few top entries, so \( \alpha^i(T) \) grows rapidly for small values of \( T \) and is much larger than \( T/d \). To illustrate the benefits of sparsification, let us look at the concrete example of logistic regression on the RCV1 data set (a standard ML benchmark with \( d = 47, 236 \) and 697, 641 data points). Figure 1a depicts \( \bar{\alpha}_T \) computed after running 1000 iterations of Gradient Descent and compares it to the worst case bound \( T/d \). The results show a dramatic difference between these two measures. We quantify this difference in terms of their ratio

\[
\text{SpeedUp}(T) = \frac{d}{T} \cdot \frac{1}{\bar{\alpha}_T} = \frac{\bar{\alpha}_T}{T/d}.
\]

Note that this measure is the ratio between row 4 and 5 in Table 1 and hence tells us the hypothetical speedup of sparsification, i.e., the ratio between the number of communicated floating point numbers needed by GD and \( T \)-SGD to reach \( \epsilon \)-accuracy. The figure shows drastic speedup, for small values of \( T \) it is 3 order of magnitudes (we confirm this in experiments below).

Interestingly, the speedup decrease with \( T \) and the optimal speedup is obtained at \( T = 1 \). There is an intuitive explanation for this. Doubling \( T \) means doubling the amount of communicated bits, while the additional gradient components that are communicated are less significant. Thus, the communication efficiency gets worse as we increase \( T \). This suggests that if we optimize the communication efficiency without considering overhead then we should always take \( T = 1 \). In the context of the dynamic algorithm in Equation (12) and (13), this leads to the following result:

**Proposition 2.** Consider the dynamic sparsified gradient algorithm in Equation (12) and (13) with

\[
C(T) = P^\epsilon(T) = T([\log_2(d)] + \text{FPP}),
\]

where \( \text{FPP} \) is the floating point precision. Then the solution to the optimization in Equation (12) is \( T^i = 1 \) for all \( i \).

Figures 1b and 1c depict, respectively, the hypothetical and true values of the total number of bits needed to reach an \( \epsilon \)-accuracy for different communication models. In particular, Figure 1b depicts the ratio \( C(T)/\bar{\alpha}_T \) (compare with Table 1) and Figure 1c depicts the experimental results of running \( T \)-SGD for different values of \( T \). We consider: a) the payload model in Equation (7) with \( C(T) = P^\epsilon(T) \) (dashed lines) and b) the packet model in Equation (10) with \( c_1 = 128 \) bytes, \( c_0 = 64 \) bytes and \( P_{\text{max}} = 128 \) bytes (solid lines). In both cases, the floating point precision is \( \text{FPP} = 64 \). We compare the results with GD (blue lines). As expected, the results show that if we ignore overheads then \( T = 1 \) is optimal, and the improvement compared to GD are of the order of 3 magnitudes. For the packet model, there is a delicate balance between choosing a small or to big. For general communication models it is difficult to find the right value of \( T \) a priori, and the costs of choosing a bad \( T \) can be of many orders of magnitude. To find a good \( T \) we could do hyper-parameter optimization. Perhaps by first estimating \( \bar{\alpha}_T \) from data and then use it to find optimal \( T \). However, this is going to be expensive and, moreover, \( \bar{\alpha}_T \) might not be a good estimate of the \( \alpha^i(T) \) we get at each iteration.

In contrast, our communication aware adaptive tuning rule finds the optimal \( T \) at each iteration without any hyper-parameter optimization. In Figure 1c we show the number of communicated bits needed to reach \( \epsilon \)-accuracy with our algorithm. The results show that for both communication models, our algorithm achieves the same communication

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1For fair comparison, we let the payload for gradient descent be \( d \times \text{FPP} \) per iteration.
We now describe how our CAT tuning rule can be extended to improve the communication efficiency of compressed gradient methods when we use the sparsification together with quantization, i.e., with $Q_T(\cdot)$ given in Equation (5). Alistarh et al. (2017) provide a heuristic rule for choosing $T^i$ dynamically at each iteration $i$. Specifically, they choose $T^i$ so that $I_{T^i}(\nabla F(x^i))$ is the smallest subset such that $\sum_{j \in I_{T^i}(\nabla F(x^i))} \|g_j\|_2 \geq \|g\|_2$. We show below that this rule works quite well if we only consider the payload, but that it is suboptimal for general communication models.

4. Dynamic Sparsification + Quantization

We now describe how our CAT tuning rule can be extended to improve the communication efficiency of compressed gradient methods when we use the sparsification together with quantization, i.e., with $Q_T(\cdot)$ given in Equation (5). Alistarh et al. (2017) provide a heuristic rule for choosing $T^i$ dynamically at each iteration $i$. Specifically, they choose $T^i$ so that $I_{T^i}(\nabla F(x^i))$ is the smallest subset such that $\sum_{j \in I_{T^i}(\nabla F(x^i))} \|g_j\|_2 \geq \|g\|_2$. We show below that this rule works quite well if we only consider the payload, but that it is suboptimal for general communication models.

4.1. Descent Lemma for Sparsification + Quantization

As before, our goal is to choose $T^i$ dynamically by maximizing the communication efficiency per iteration, i.e., the function improvement per bit. To that end, we first need a similar descent lemma for this compression as Lemma 1 was for the sparsification in the last section. By similar arguments as in Lemma 1, we obtain the following result.

Lemma 3. Suppose that $F: \mathbb{R}^d \to \mathbb{R}$ is (possibly non-convex) $L$-smooth. Then for any $x, x^+ \in \mathbb{R}^d$ with

$$x^+ = x - \gamma Q_T(\nabla F(x))$$

where $Q_T(\cdot)$ is as defined in Equation (5) and $\gamma = \sqrt{\beta(T)}/(\sqrt{TL})$ where

$$\beta(T) = \frac{1}{T} \langle \nabla F(x), Q_T(\nabla F(x)) \rangle^2 \|\nabla F(x)\|_2^2. \tag{14}$$

we have

$$F(x^+) \leq F(x) - \frac{\beta(T)}{2L} \|\nabla F(x)\|^2.$$ 

4.2. Dynamic Algorithm and Illustrations

Using the descent Lemma 3 we can apply CAT for this compression similarly as we did for the sparsification in the previous section. In particular, if we set

$$\beta^i(T) = \frac{1}{T} \langle \nabla F(x^i), Q_T(\nabla F(x^i)) \rangle^2 \|\nabla F(x^i)\|_2^2$$

then our CAT sparsification + quantization is given by

$$T^i = \arg\max_{T \in [1,d]} \beta^i(T) \frac{C(T)}{\sqrt{T}}$$

and

$$x^{i+1} = x^i - \gamma^i Q_T(\nabla F(x^i)),$$

We illustrate the algorithm on the RCV1 data set in Figure 2. We compared CAT to the dynamic tuning introduced in (Alistarh et al., 2017). The black lines illustrate the results when we only communicate the payload, i.e., $C(T) = P_{\text{payload}}(T)$ defined in Equation (8). The blue lines are the results for when $C(T)$ follows the packet model in Equation (10) with $c_1 = 128$ bytes, $c_0 = 64$ bytes and $P_{\text{max}} = 128$ bytes. The results show that if we only count the payload, then the two methods are comparable. Our CAT tuning rule outperforms (Alistarh et al., 2017) by only a small margin. This suggests that the heuristic in (Alistarh et al., 2017) is quite communication efficient in the simplest case when only payload is communicated. However, the heuristic rule is agnostic to the actual communication model $C(T)$ that is used. Therefore, we should not expect it to perform well for general $C(T)$. The blue lines show that the CAT is roughly two times more communication efficient than the the dynamic tuning rule in (Alistarh et al., 2017) for the packet communication model.

5. Dynamic Stochastic Sparsification

We finally illustrate how the CAT tuning rule can improve the communication efficiency of stochastic sparsification methods. One of the advantages of stochastic sparsification is its favorable properties that allow us to generalize our theoretical results to stochastic gradient methods and to multi node settings; we illustrate this in Subsection 5.3.

5.1. Descent Lemma for Stochastic Sparsification

Our goal is to choose $T^i$ and $p^i$ dynamically for the stochastic sparsification in Equation (6), maximizing the communication efficiency per iteration, i.e., the function improvement per bit. To that end, we need the following descent lemma, similar to the ones we proved for non-stochastic sparsifications in the last two sections.

$$\beta^i(T) = \frac{1}{T} \langle \nabla F(x^i), Q_T(\nabla F(x^i)) \rangle^2 \|\nabla F(x^i)\|_2^2$$

then our CAT sparsification + quantization is given by

$$T^i = \arg\max_{T \in [1,d]} \beta^i(T) \frac{C(T)}{\sqrt{T}}$$

and

$$x^{i+1} = x^i - \gamma^i Q_T(\nabla F(x^i)),$$
Lemma 4. Suppose that $F: \mathbb{R}^d \to \mathbb{R}$ is (possibly non-convex) $L$-smooth. Then for any $x, x^+ \in \mathbb{R}^d$ with

$$x^+ = x - \gamma Q_{T,p}(\nabla F(x))$$

where $Q_{T,p}(\cdot)$ is defined in (6) and $\gamma = \omega_p(T)/L$ where

$$\omega_p(T) = \frac{||\nabla F(x)||^2}{E[|Q_{T,p}(\nabla F(x))||]^2}.$$  \hspace{1cm} (15)

we have

$$E(F(x^+)) \leq E(F(x)) - \frac{\omega_p(T)}{2L}E[||\nabla F(x)||^2].$$

Similarly as before, we optimize the descent and the communication efficiency by maximizing, respectively, $\omega_p(T)$ and $\omega_p(T)/C(T)$. To optimize $\omega_p(T)$ we can use that

$$E[|Q_{T,p}(\xi)||^2] = \sum_{j=1}^d p_j g_j^2.$$  \hspace{1cm} (16)

It is illustrated in (Wang et al., 2018) how we can obtain the optimal $p$ for a given $T$ efficiently. That is, for fixed $T$ it provides us with the optimal solution $p^*$ to

$$\max_{p \in [0,1]^d} \omega_p(T)$$

subject to $\sum_{j=1}^d p_j = T$. \hspace{1cm} (17)

In the rest of this section we always use $p^*$ and omit $p$ when in $Q_T(\cdot)$ and $\omega(T)$.

5.2. Dynamic Algorithm and Illustrations

We can now design a dynamic tuning to optimize the communication efficiency, based on the descent lemma. In particular, we get the dynamic algorithm

$$x^{i+1} = x^i - \gamma \frac{1}{n} \sum_{j=1}^n Q_T(x^i; \xi_j^i).$$  \hspace{1cm} (17)

Figure 2. CAT sparsification + quantization on the RCV1 data set.

Figure 3. Expected communicated bits to reach $\epsilon$-accuracy for gradient descent with stochastic sparsification.
where $Q(\cdot)$ is the stochastic sparsifier. Here $g_j(x; \xi_j)$ is the stochastic gradient at $x$. We assume that $g_j(x; \xi_j)$ is unbiased and satisfies a bounded variance assumption, i.e.

$$E_x g_j(x; \xi_j) = \nabla f_j(x), \quad \text{and} \quad E_x \|g_j(x; \xi_j) - \nabla F(x)\|^2 \leq \sigma^2, \quad \forall x \in \mathbb{R}^d. \quad (18)$$

where the expectation is with respect to a distribution of local data stored in node $j$. These conditions are standard for analysis of first-order algorithms in machine learning (Feyzmahdavian et al., 2016; Lian et al., 2015). We have the following convergence result.

**Theorem 1.** Suppose that $f_j(\cdot)$ is $L$-smooth for each $j$. Let $\{x^i\}_{i \in \mathbb{N}}$ be the iterates of Algorithm (17) and suppose that there exists $\bar{\omega}$ such that $\omega(T_j^i) \geq \bar{\omega}$ for all $i$ and $j$. Set $\epsilon_0 = \|x^0 - x^*\|$ and $\epsilon_0 = f(x^0) - f(x^*)$. Then

1. **(Non-convex) If**

   $$\gamma = \frac{\bar{\omega}}{2L} \frac{1}{2\sigma^2/\epsilon + 1}$$

   then we find $\min_{t \in [0, i-1]} E[\|\nabla F(x^t)\|^2] \leq \epsilon$ in

   $$i = \frac{2}{\bar{\omega}} \left(1 + \frac{2\sigma^2}{\epsilon L}\right) \frac{2L\epsilon_0}{\epsilon} \text{ iterations.}$$

2. **(Convex) If $F$ is convex and**

   $$\gamma = \frac{\bar{\omega}}{2} \frac{1}{2\sigma^2/\epsilon + L}$$

   then we find $E[F(\sum_{i=0}^{i-1} x^i/i) - F^*] \leq \epsilon$

   $$i = \frac{2}{\bar{\omega}} \left(1 + \frac{2\sigma^2}{\epsilon L}\right) \frac{2L\epsilon_0}{\epsilon} \text{ iterations.}$$

3. **(Strongly convex) If $F$ is $\mu$-strongly convex and**

   $$\gamma = \frac{\bar{\omega}}{2} \frac{1}{2\sigma^2/(\mu\epsilon) + L}$$

   then we find $E[\|x^i - x^*\|^2] \leq \epsilon$ in

   $$i = \frac{2}{\bar{\omega}} \left(1 + \frac{2\sigma^2}{\mu\epsilon L}\right) \log \left(\frac{2\epsilon_0}{\epsilon}\right) \text{ iterations.}$$

**Lemma 5.** For $g \in \mathbb{R}^d$ we have the lower bound

$$\omega(T) = \frac{\|g\|^2}{E[Q_{T,p}(g)]^2} \geq \frac{T}{d}.$$

With these results, we can translate many of the conclusions for deterministic sparsification to stochastic sparsification.

### 6. Experimental Results for Multiple Nodes

We evaluate the performance of the CAT tuning rule for all three compressors discussed in this paper in the multi-node setting on the RCV1 data-set. We compare the results to gradient descent and the sparsification with quantization from Section 4.2 but using the dynamic tuning rule in (Alistarh et al., 2017) instead of CAT. We implement all algorithms in Julia, and run them on 4 nodes using MPI by splitting the data evenly between the nodes. In all cases we use the packet communication model in Equation (10) with $c_1 = 128$ bytes, $c_0 = 64$ bytes and $P_{\text{max}} = 128$ bytes. The results are shown in Figure 4. Our CAT with sparsification together with quantization outperforms all other compression schemes. In particular, CAT for this compression, is roughly 4 times more communication efficient than the dynamic rule in (Alistarh et al., 2017) for the same compression scheme (compare number of bits needed to reach $\epsilon = 0.4$).

**7. Conclusions**

We have proposed communication-aware adaptive tuning to optimize the communication-efficiency of gradient sparsification. The adaptive tuning relies on a data-dependent
measure of objective function improvement, and adapts the compression level to maximize the descent per communicated bit. Unlike existing heuristics, our tuning rules are guaranteed to save communications in realistic communication models. In particular, our rules are more communication-efficient when communication overhead or packet transmissions are accounted for. In addition to centralized analysis, our tuning strategies are proven to reduce communicated bits also in distributed scenarios.

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**8. Appendix**

**A. Proofs of Lemmas and Propositions**

**A.1. Proof of Lemma 1**

By the $L$-smoothness of $F(\cdot)$ and the iterate $x^+ = x - \gamma Q_T(\nabla F(x))$ where $x^+, x \in \mathbb{R}^d$, from Lemma 1.2.3. of (Nesterov, 2018) we have

$$F(x^+) \leq F(x) - \gamma \langle \nabla F(x), Q_T(\nabla F(x)) \rangle + \frac{L\gamma^2}{2} \|Q_T(\nabla F(x))\|^2.$$  

It can be verified that

$$\langle g, Q_T(g) \rangle = \|Q_T(g)\|^2$$

for all $g \in \mathbb{R}^d$ and, therefore, if $\gamma = 1/L$ then we have

$$F(x^+) \leq F(x) - \frac{1}{2L} \|Q_T(\nabla F(x))\|^2.$$  

By the definition of $\alpha(T)$ we have $\|Q_T(\nabla F(x))\|^2 = \alpha(T)\|\nabla F(x)\|^2$, which yields the result.

Next, we prove that there exist $L$-smooth functions $F(\cdot)$ where the inequality is tight. Consider $F(x) = L\|x\|^2/2$. Then, $F$ is $L$-smooth, and also satisfies

$$F(x - \gamma Q_T(\nabla F(x))) = \frac{L}{2}\|x - \gamma Q_T(Lx)\|^2 = \frac{L}{2}\|x\|^2 - \gamma(Lx, Q_T(Lx)) + \frac{L\gamma^2}{2} \|Q_T(Lx)\|^2.$$  

Since $\langle g, Q_T(g) \rangle = \|Q_T(g)\|^2$ by the definition $Q_T(\cdot)$ and $\gamma = 1/L$, we have

$$F(x - \gamma Q_T(\nabla F(x))) = F(x) - \frac{1}{2L} \|Q_T(Lx)\|^2.$$  

Since $\nabla F(x) = Lx$, by the definition of $\alpha(T)$

$$\alpha(T) = \frac{\|Q_T(Lx)\|^2}{\|Lx\|^2} = \frac{\sum_{i \in I_T} x_i^2}{\sum_{i=1}^d x_i^2},$$

where $I_T$ is the index set of $T$ elements with the highest absolute magnitude. Therefore,

$$F(x - \gamma Q_T(\nabla F(x))) = F(x) - \alpha(T) \|\nabla F(x)\|^2.$$  

**A.2. Proof of Lemma 2**

Take $g \in \mathbb{R}^d$ and, without the loss of generality, let $|g_1| \geq |g_2| \geq \ldots \geq |g_d|$ and $g_i \in \mathbb{R}$ (otherwise we may re-order $g$). To prove that $\alpha(T)$ is increasing we rewrite the definition of $\alpha(T)$ equivalently as

$$\alpha(T) = \sum_{j=1}^d g_j^2/\|g\|^2, \quad \text{for} \ T \in \{0, 1, 2, \ldots, d\}.$$  

Notice that $\alpha(T) = 0$ when $T = 0$. Clearly, $\alpha(T)$ is also increasing with $T \in [1, d]$ since each term of the sum $\sum_{j=1}^T g_j^2$ is increasing.

We prove that $\alpha(T)$ is concave by recalling the slope of $\alpha(T)$

$$\frac{d}{dT} \alpha(T) = \frac{g_M^2}{\|g\|^2},$$

for $T \in (M - 1, M)$ and $M = 1, 2, \ldots, d$. Since $|g_1| \geq |g_2| \geq \ldots \geq |g_d|$, the slope of $\alpha(T)$ has a non-increasing slope when $T$ increases. Therefore, $\alpha(T)$ is concave.

We prove the second statement by writing $\|g\|^2$ on the form of

$$\|g\|^2 = \sum_{j \in I_{T^c}(g)} g_j^2 + \sum_{j \in I_T(g)} g_j^2,$$

where $I_{T^c}$ is the index set of $d - T$ elements with lowest absolute magnitude. Applying the fact that $g_j^2 \leq \min_{t \in I_T(g)} g_t^2$ for $j \in I_{T^c}(g)$ and that $\min_{t \in I_T(g)} g_t^2 \leq (1/T) \sum_{t \in I_T(g)} g_t^2$ into the main inequality, we have

$$\|g\|^2 \leq \left(1 + \frac{d - T}{T}\right) \sum_{j \in I_T(g)} g_j^2.$$  

By the definition of $Q_T(g)$, we get

$$\alpha(T) \geq T/d.$$
Finally, we prove the last statement by setting $F(x) = (1/2)x^TAx$ where $A = (L/d)11^T$. Then $F(\cdot)$ is $L$-smooth and its gradient is
\[ \nabla F(x) = \bar{x}1, \]
where
\[ \bar{x} = \frac{1}{d} \sum_{i=1}^{d} x_i. \]
Therefore, $\|Q_T(\nabla F(x))\|^2 = (T/d)\|\nabla F(x)\|^2$.

### A.3. Proof of Proposition 1

The ratio between a non-negative concave function $\alpha(T)$ and a positive affine function $C(T)$ is quasi-concave and semi-strictly quasi-concave (Schaible, 2013; Avriel et al., 2010), meaning that every local maximal point is globally maximal.

Next, we consider $\alpha(T)/C(T)$ when $C(T) = \tilde{c}_1[T/\tau_{\max}] + c_0$. If $T \in ((c - 1)\tau_{\max}, c\tau_{\max}]$, then $\alpha(T) \leq \alpha(c\tau_{\max})$ due to monotonicity of $\alpha(\cdot)$ and $C(T) = C(c\tau_{\max})$, meaning that $\alpha(T)/C(T) < \alpha(c\tau_{\max})/C(c\tau_{\max})$. This implies that $T = c\tau_{\max}$ maximizes $\alpha(T)/C(T)$ for $T \in ((c - 1)\tau_{\max}, c\tau_{\max}]$ and that we can obtain the maximum of $T = c\tau_{\max}$ for some integers $c$.

### A.4. Proof of Proposition 2

Take $g \in \mathbb{R}^d$ and, without the loss of generality, we let $|g_1| \geq |g_2| \geq \ldots \geq |g_d|$ and $g_1 \in \mathbb{R}$ (otherwise we may re-order $g$). Since $C(T) = C \cdot T$ where $C = [\log_2(d)] + \text{FPF}$, we have
\[ T_i = \arg\max_{T \in [1, d]} \frac{\alpha(T)}{C(T)} = \arg\max_{T \in [1, d]} \frac{\sum_{i=1}^{T} g_i^2}{C \cdot T}. \]
Since $\sum_{i=1}^{T} g_i^2 / T \leq g_1^2$, the solution from Equation (12) is $T_i = 1$ for all $i$.

### A.5. Proof of Lemma 3

By using the $L$-smoothness of $F(\cdot)$ (Lemma 1.2.3. of Nesterov, 2018)) and the iterate $x^+ = x - \gamma Q_T(\nabla F(x))$ where $x^+, x \in \mathbb{R}^d$, we have
\[ F(x^+) \leq F(x) - \gamma \langle \nabla F(x), Q_T(\nabla F(x)) \rangle + \frac{L\gamma^2}{2} \|Q_T(\nabla F(x))\|^2. \]
If $Q_T(\nabla F(x))$ has $T$ non-zero elements, then we can easily prove that
\[ \langle \nabla F(x), Q_T(\nabla F(x)) \rangle = \sqrt{T\beta(T)} \cdot \|\nabla F(x)\|^2, \quad \text{and} \quad \|Q_T(\nabla F(x))\|^2 = T \cdot \|\nabla F(x)\|^2. \]
where $\beta(T)$ is defined in Equation (14). Plugging these equations into the above inequality yields
\[ F(x^+) \leq F(x) - \left( \gamma \sqrt{T\beta(T)} - \frac{TL\gamma^2}{2} \right) \|\nabla F(x)\|^2. \]
Setting $\gamma = \sqrt{\beta(T)}/(\sqrt{T}L)$ completes the proof.

### A.6. Proof of Lemma 4

By using the $L$-smoothness of $F(\cdot)$ (Lemma 1.2.3. of (Nesterov, 2018)) and the iterate $x^+ = x - \gamma Q_T, p(\nabla F(x))$ where $x^+, x \in \mathbb{R}^d$, we have
\[ F(x^+) \leq F(x) - \gamma \langle \nabla F(x), Q_T, p(\nabla F(x)) \rangle + \frac{L\gamma^2}{2} \|Q_T, p(\nabla F(x))\|^2. \]
Taking the expectation, recalling the definition of $\omega_p(T)$ in Equation (15), and using the unbiased property of $Q_T, p(\cdot)$ we get
\[ \mathbb{E}[F(x^+)] \leq F(x) - \left( \gamma - \frac{L\gamma^2}{2\omega_p(T)} \right) \mathbb{E}[\|\nabla F(x)\|^2]. \]
Now taking $\gamma = \omega_p(T)/L$ concludes the complete the proof.

### A.7. Proof of Lemma 5

If $Q_T, p(\cdot)$ in Equation (6) has $p_j = T/d$ for all $j$, then
\[ \mathbb{E}[\|Q_T, p(g)\|^2] = \frac{1}{d} \sum_{j=1}^{d} \mathbb{E}[g_j^2] = \frac{d}{T} \|g\|^2. \]
In $\omega(T)$ we use the optimal $p$ that minimizes $\omega_p(T) = \|g\|^2/\mathbb{E}[\|g\|^2]$. This implies that $\omega(T) \geq T/d$.

### B. Proof of Theoretical Results for Table 1

In this section, we provide the iteration and communication complexities of compressed gradient descent (3) with gradient sparsification (4).

#### Proof of Non-convex Optimization

By recursively applying the inequality from Lemma 1 with $x^+ = x^{i+1}$ and $x = x^i$, we have
\[ \min_{t \in [0, i-1]} \|\nabla F(x^t)\|^2 \leq \frac{2L}{i} \sum_{t=0}^{i-1} \frac{1}{\alpha(T)} [F(x^t) - F(x^{i+1})] \]
where the inequality follows from the fact that $\min_{t \in [0, i-1]} \|\nabla F(x^t)\|^2 \leq \sum_{t=0}^{i-1} \|\nabla F(x^t)\|^2 / i$. If there exists $\alpha(T)$ such that $\alpha(T) \geq \bar{\alpha_T}$, then
\[ \min_{t \in [0, i-1]} \|\nabla F(x^t)\|^2 \leq \frac{2L}{i\bar{\alpha_T}} [F(x^0) - F(x^i)]. \]
Since $F(x) \geq F(x^*)$ for $x \in \mathbb{R}^d$,
\[
\min_{i \in [0, i-1]} \|\nabla F(x^i)\|^2 \leq \frac{2L}{i\alpha_T} \epsilon_0,
\]
where $\epsilon_0 = F(x^0) - F(x^*)$. To reach
\[
\min_{i \in [0, i-1]} \|\nabla F(x^i)\|^2 \leq \epsilon.
\]
This means that to reach $\epsilon > 0$ accuracy, the sparsified gradient method (3) needs at most
\[
i \geq \frac{1}{\bar{\alpha}_T} \frac{2L \epsilon_0}{\epsilon}
\]
iterations.

In addition, we recover the iteration complexities of the sparsified gradient method and of classical full gradient method when we let $\bar{\alpha}_T = T/d$ and $\bar{\alpha}_T = 1$, respectively.

**Proof of Convex Optimization**

Before deriving the result, we introduce one useful lemma.

**Lemma 6.** The non-negative sequence $\{V^i\}_{k \in \mathbb{N}}$ generated by
\[
V^{i+1} \leq V^i - q(V^i)^2, \text{ for } q > 0
\]
satisfies
\[
\frac{1}{V^i} \geq \frac{1}{V^0} + iq.
\]

**Proof.** By the fact that $x^2 \geq 0$ for $x \in \mathbb{R}$, clearly $V^{i+1} \leq V^i$. By the proper manipulation, we rearrange the terms in Equation (19) as follows:
\[
\frac{1}{V^{i+1}} - \frac{1}{V^i} \geq q \frac{V^i}{V^{i+1}} \geq q,
\]
where the last inequality follows from the fact that $V^{i+1} \leq V^i$. By the recursion, we complete the proof. \(\square\)

By Lemma 1 with $x^+ = x^{i+1}$ and $x = x^i$, we have
\[
F(x^{i+1}) \leq F(x^i) - \frac{\alpha^i(T)}{2L} \|\nabla F(x^i)\|^2
\]
where $V^i = F(x^i) - F(x^*)$. If there exists $\alpha_T$ such that $\alpha^i(T) \geq \alpha_T$, then by Lemma 6 and by using the fact that $V^0 \geq 0$
\[
V^i \leq \frac{1}{\bar{\alpha}_T} \frac{2LR^2}{i}.
\]
To reach $F(x^i) - F(x^*) \leq \epsilon$, the sparsified gradient methods needs the number of iterations $i$ satisfying
\[
i \geq \frac{1}{\bar{\alpha}_T} \frac{2LR^2}{\epsilon}.
\]

We also recover the iteration complexities of the sparsified gradient method and of classical full gradient method when we let $\bar{\alpha}_T = T/d$ and $\bar{\alpha}_T = 1$, respectively.

**Proof of Strongly Convex Optimization**

By Lemma 1 with $x^+ = x^{i+1}$ and $x = x^i$, we have
\[
F(x^i) \leq F(x^i) - \frac{\alpha^i(T)}{2L} \|\nabla F(x^i)\|^2.
\]
Since $F$ is $\mu$-strongly convex, i.e. $\|\nabla F(x)\|^2 \geq 2\mu[F(x) - F(x^*)]$ for $x \in \mathbb{R}^d$, applying this inequality into the main result we have
\[
F(x^{i+1}) - F(x^*) \leq \left(1 - \frac{\mu \alpha^i(T)}{L}\right) [F(x^i) - F(x^*)].
\]
If there exists $\alpha_T$ such that $\alpha^i(T) \geq \alpha_T$, then by the recursion we get
\[
F(x^i) - F(x^*) \leq \left(1 - \frac{\mu \alpha_T}{L}\right)^i \epsilon_0,
\]
where $\epsilon_0 = F(x^0) - F(x^*)$. To reach $F(x^i) - F(x^*) \leq \epsilon$, the sparsified gradient methods requires the number of iterations $i$ satisfying
\[
\left(1 - \frac{\mu \alpha_T}{L}\right)^i \epsilon_0 \leq \epsilon.
\]
Taking the logarithm on both sides of the inequality and using the fact that $-1/\log(1 - x) \leq 1/x$ for $0 < x \leq 1$, we have
\[
i \geq \frac{1}{\bar{\alpha}_T} \frac{\epsilon_0}{\epsilon}.
\]

We also recover the iteration complexities of the sparsified gradient method and of classical full gradient method when we let $\bar{\alpha}_T = T/d$ and $\bar{\alpha}_T = 1$, respectively.

**C. Proof of Theorem 1**

In this section, we prove the iteration and communication complexities of distributed stochastic gradient descent using stochastic sparsification in Equation (17). We begin by introducing three lemmas which are useful in our analysis.
Lemma 7. Let \( \{x^i\}_{i \in \mathbb{N}} \) be the iterates generated by Algorithm (17) and suppose that there exists \( \bar{\omega} \geq 0 \) such that \( \omega(T^i_j) \geq \bar{\omega} \) for all \( i, j \). Then,

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} Q_{T^i_j} \left( g_j(x^i; \xi_j^i) \right) \right]^2 \leq \frac{2}{\omega} \mathbb{E} \left[ \frac{\|g_j(x^i; \xi_j^i)\|^2}{\omega(T^i_j)} \right] + \frac{2}{\omega} \mathbb{E} \mathbb{E} \left[ \|\nabla F(x^i)\|^2 \right] + \sigma^2.
\]

Proof. Since \( \mathbb{E} \left[ \frac{\|g_j(x^i; \xi_j^i)\|^2}{\omega(T^i_j)} \right] = \frac{\|g_j(x^i; \xi_j^i)\|^2}{\omega(T^i_j)} \), by using Cauchy-Schwarz’s inequality and by the fact that \( \omega(T^i_j) \geq \bar{\omega} \) for all \( i, j \) we have

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} Q_{T^i_j} \left( g_j(x^i; \xi_j^i) \right) \right]^2 \leq \frac{1}{\omega} \sum_{j=1}^{n} \mathbb{E} \left[ g_j(x^i; \xi_j^i) \right]^2.
\]

After utilizing the inequality \( \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \) with \( x = g_j(x^i; \xi_j^i) - \nabla F(x^i) \) and \( y = \nabla F(x^i) \),

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} Q_{T^i_j} \left( g_j(x^i; \xi_j^i) \right) \right]^2 \leq \frac{2}{\omega} \mathbb{E} \left[ (T + \mathbb{E} \|\nabla F(x^i)\|^2) \right]
\]

where \( T = \mathbb{E} \left[ g_j(x^i; \xi_j^i) - \nabla F(x^i) \right] \). By the bounded variance assumption (i.e. \( \mathbb{E} \|g_j(x; \xi_j) - \nabla F(x)\|^2 \leq \sigma^2 \)), we complete the proof.

\[ \Box \]

Lemma 8. Suppose that each component function \( f_j(\cdot) \) is \( L \)-smooth. Let \( \{x^i\}_{i \in \mathbb{N}} \) be the iterates generated by Algorithm (17) and assume that there exists \( \bar{\omega} \geq 0 \) such that \( \omega(T^i_j) \geq \bar{\omega} \) for all \( i, j \). Then,

\[
\min_{t \in [0, i-1]} \mathbb{E} \|\nabla F(x^t)\|^2 \leq \frac{1}{\gamma} + \frac{\epsilon_0}{\gamma} + \frac{(L/\bar{\omega})\gamma}{1 - (L/\bar{\omega})\gamma} + \frac{\sigma^2}{\gamma},
\]

where \( \epsilon_0 = F(x^0) - F(x^*) \) and \( \gamma < \bar{\omega}/L \).

Proof. By Cauchy-Schwarz’s inequality, we can easily show that \( F(x) = (1/n) \sum_{j=1}^{n} f_j(x) \) is also \( L \)-smooth. From the smoothness assumption (Lemma 1.2.3. of (Nesterov, 2018)) and Equation (17),

\[
F(x^{i+1}) \leq F(x^i) - \gamma \langle \nabla F(x^i), g^i \rangle + \frac{L\gamma^2}{2} \|g^i\|^2
\]

where \( g^i = (1/n) \sum_{j=1}^{n} Q_{T^i_j} \left( g_j(x^i; \xi_j^i) \right) \). Taking the expectation, and using Lemma 7 and the unbiased properties of stochastic gradient \( g_j(\cdot) \) and stochastic sparsification \( Q_T(\cdot) \), we have

\[
\mathbb{E} \|x^{i+1} - x^*\|^2 \leq \mathbb{E} \|x^i - x^*\|^2 + \frac{(2/\bar{\omega})\gamma^2}{\gamma} \mathbb{E} \|\nabla F(x^i)\|^2
\]

Since \( F(\cdot) \) is \( L \)-smooth, i.e. for \( x \in \mathbb{R}^d \)

\[
\|\nabla F(x) - \nabla F(y)\|^2 \leq L \langle \nabla F(x) - \nabla F(y), x - y \rangle,
\]

applying this inequality with \( x = x^i \) and \( y = x^* \) into the main result and recalling that \( \nabla F(x^*) = 0 \) we complete the proof.

\[ \Box \]

Proof of Theorem 1.1.

If the step-size is

\[
\gamma = \frac{\bar{\omega}}{2L \bar{\omega}^2/\epsilon + 1},
\]

By rearranging the terms and calling that \( \gamma < \bar{\omega}/L \), we get

\[
\mathbb{E} \|\nabla F(x^i)\|^2 \leq \frac{1}{\gamma} \left( \mathbb{E} \|F(x^i) - \mathbb{E} F(x^{i+1})\| \right)
\]

\[
+ (L/\bar{\omega})\gamma \mathbb{E} \|\nabla F(x^i)\|^2 + \sigma^2.
\]

Since \( \min_{t \in [0, i-1]} \mathbb{E} \|\nabla F(x^t)\|^2 \leq \sum_{t=0}^{i-1} \mathbb{E} \|\nabla F(x^t)\|^2 / i \),

we obtain

\[
\min_{t \in [0, i-1]} \mathbb{E} \|\nabla F(x^t)\|^2
\]

\[
\leq \frac{1}{i(\gamma - (L/\bar{\omega})\gamma)^2} \left( \mathbb{E} F(x^0) - \mathbb{E} F(x^i) \right) + T,
\]

where \( T = \sigma^2 \cdot (L/\bar{\omega})\gamma / (1 - (L/\bar{\omega})\gamma) \). By the fact that \( F(x) \geq F(x^*) \) for \( x \in \mathbb{R}^d \), we complete the proof.

\[ \Box \]
then clearly \( \gamma < \bar{\omega}/L \) and
\[
\frac{L/\bar{\omega}}{1-(L/\bar{\omega})} \sigma^2 \leq \epsilon/2.
\]
From Lemma 8, Algorithm (17) reaches \( \min_{i \in [0,1]} \mathbb{E}[\|\nabla F(x_i)\|^2] \leq \epsilon \) for the number of iterations \( i \) which fulfills
\[
\frac{1}{2} \frac{2L\epsilon_0}{\bar{\omega}} (2\sigma^2/\epsilon + 1) \frac{4\sigma^2/\epsilon + 2}{4\sigma^2/\epsilon + 1} \leq \epsilon/2.
\]
Since \((4\sigma^2/\epsilon + 2)/(4\sigma^2/\epsilon + 1) \leq 2\), the main condition can be rewritten equivalently as
\[
i \geq \frac{4}{\bar{\omega}} \left(1 + \frac{2\sigma^2}{\epsilon}\right) \cdot \frac{2L\epsilon_0}{\epsilon}.
\]
**Proof of Theorem 1.2.**

If \( \gamma < \bar{\omega}/L \), by Lemma 9 and by using the convexity of \( F \), i.e. \( \langle \nabla F(x), x - x^* \rangle \geq F(x) - F(x^*) \) for \( x \in \mathbb{R}^d \) we get
\[
\mathbb{E} \|x^{i+1} - x^*\|^2 \leq \mathbb{E} \|x^i - x^*\|^2 + 2\gamma \sigma^2/\bar{\omega} - 2(\gamma - L\gamma/\bar{\omega}) \mathbb{E}[F(x^i) - F(x^*)]
\]
By rearranging the terms and using the fact that \( F \) is convex, i.e. \( F(\sum_{i=0}^{i-1} x_i) \leq \sum_{i=0}^{i-1} F(x_i) \), we then have
\[
\mathbb{E} \left[ F \left( \frac{1}{i} \sum_{i=0}^{i-1} x_i \right) - F(x^*) \right]
\leq \frac{1}{i} \sum_{i=0}^{i-1} \mathbb{E}[F(x^i) - F(x^*)]
\leq \frac{1}{i} \frac{1}{2\gamma} \frac{\epsilon_0}{1-(L/\bar{\omega})\gamma} + \frac{\gamma \sigma^2/\bar{\omega}}{1-(L/\bar{\omega})\gamma},
\]
where \( \epsilon_0 = \|x^0 - x^*\|^2 \). The last inequality follows from the cancellations of the telescopic series the fact that \( \|x\|^2 \geq 0 \) for \( x \in \mathbb{R}^d \). If the step-size is
\[
\gamma = \frac{\bar{\omega}}{2} \frac{1}{2\sigma^2/\epsilon + L},
\]
then clearly \( \gamma < \bar{\omega}/L \) and
\[
\frac{\gamma \sigma^2/\bar{\omega}}{1-(L/\bar{\omega})\gamma} \leq \epsilon/2.
\]
To reach \( \mathbb{E}\left[ F \left( \sum_{i=0}^{i-1} x_i/i \right) - F(x^*) \right] \leq \epsilon \), Algorithm (17) needs the number of iterations \( i \) satisfying
\[
i \geq \frac{1}{\bar{\omega}} \frac{L}{4\sigma^2/\epsilon + L} \frac{1 + 2(2\sigma^2/\epsilon + L)^2}{2\sigma^2/\epsilon + L} \epsilon_0 \leq \epsilon/2.
\]
Since \((4\sigma^2/\epsilon + 2L)/(4\sigma^2/\epsilon + L) \leq 2\), the main condition can be rewritten equivalently as
\[
i \geq \frac{2}{\bar{\omega}} \left(1 + \frac{2\sigma^2}{\epsilon L}\right) \frac{2L\epsilon_0}{\epsilon}.
\]
**Proof of Theorem 1.3.**

If \( \gamma < \bar{\omega}/L \), by Lemma 9 and strong convexity of \( F(\cdot) \), i.e. \( \langle \nabla F(x), x - y \rangle \geq \mu \|x - y\| \) for \( x, y \in \mathbb{R}^d \) we have
\[
\mathbb{E} \|x^{i+1} - x^*\|^2 \leq \rho \mathbb{E} \|x^i - x^*\|^2 + 2\gamma \sigma^2/\bar{\omega},
\]
where \( \rho = 1 - 2\mu(\gamma - L\gamma/\bar{\omega}) \). By recursively applying the inequality, we get
\[
\mathbb{E} \|x^i - x^*\|^2 \leq \rho^i \epsilon_0 + \frac{\gamma \sigma^2/\bar{\omega}}{\mu(1-L\gamma/\bar{\omega})},
\]
where \( \epsilon_0 = \|x^0 - x^*\|^2 \). If the step-size is
\[
\gamma = \frac{\bar{\omega}}{2} \frac{1}{2\sigma^2/(\mu\epsilon) + L},
\]
then clearly \( \gamma < \bar{\omega}/L \) and
\[
\frac{\gamma \sigma^2/\bar{\omega}}{\mu(1-L\gamma/\bar{\omega})} \leq \epsilon/2.
\]
To reach \( \mathbb{E} \|x^i - x^*\|^2 \leq \epsilon \), Algorithm (17) needs the number of iterations \( i \) which satisfies
\[
i \geq \frac{2}{\bar{\omega}} \frac{(2\sigma^2/(\mu\epsilon) + L)^2}{(2\sigma^2/(\mu\epsilon) + L)^2} \epsilon_0 \leq \epsilon/2.
\]
Taking the logarithm on both sides, and utilizing the fact that \(-1/\log(1-x) \leq 1/x \) for \( 0 < x \leq 1 \), we have
\[
i \geq \frac{2}{\bar{\omega}} \frac{(2\sigma^2/(\mu\epsilon) + L)^2}{(2\sigma^2/(\mu\epsilon) + L)^2} \log \left( \frac{2\epsilon_0}{\epsilon} \right).
\]
Since \((4\sigma^2/(\mu\epsilon) + 2L)/(4\sigma^2/(\mu\epsilon) + L) \leq 2\), the main condition can be rewritten equivalently as
\[
i \geq \frac{2}{\bar{\omega}} \frac{2\sigma^2}{\mu\epsilon} \log \left( \frac{2\epsilon_0}{\epsilon} \right).
\]
**D. Experiments on Logistic Regression over URL**

In this section, we include additional simulations that validate the performance of the CAT tuning rule for three main compressors in the single node setting (one master and one worker) on the URL data set. The master node, located 500 km away from the worker node, is responsible for computing the new decision vector based on the gradient information received, whereas the worker node computes the gradient based on the loss function and the data. The URL data set contains 2.4 million data points and 3.2 million features. All implementations are in the C++ library POLO (Aytekin et al., 2018). We ran the simulations on the logistic regression problem, and set \( \text{FP}_3 = 32 \) and the step-size according
to the descent lemma associated with each compressed gradient algorithm. Figure 5 suggests the better performance when our CAT rules are used. In particular, our rules for sparsification in Equation (4) (CAT-SG) and sparsification with quantization in Equation (5) (CAT-S+Q) guarantee better savings in communicated bits while attaining faster convergence rate than the non-adaptive compressors.

Figure 5. Performance of compressors on logistic regression problems over the URL data set.