AN EXAMPLE OF $\infty$-HARMONIC FUNCTION, WHICH IS NOT $C^2$ ON A DENSE SUBSET

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Abstract. We show that for certain boundary values McShane-Whitney’s minimal-extension-like function is $\infty$-harmonic near the boundary and is not $C^2$ on a dense subset.

Let us consider the strip $\{(u, v) \in \mathbb{R}^2 : 0 < v < \delta\}$. The function we are going to construct will be defined in this strip. Take a function $f \in C^{1,1}(\mathbb{R})$ with $L_f := \|f'\|_{\infty}$ and $L'_f := Lip(f')$. Let us consider an analogue of the minimal extension of McShane and Whitney

$$u(x, d) := \sup_{y \in \mathbb{R}}[f(y) - L||f'(x, d) - (y, 0)||],$$

(1)

where $0 < d < \delta$ and $L > L_f$. Note that in order to get the classical minimal extension of McShane and Whitney we have to take $L = L_f$.

From now on let us fix the function $f$, constants $L > L_f$ and $\delta > 0$. We are going to find some conditions on $\delta > 0$, which will make our statements to be true. The real number $x$ will be associated with the point $(x, \delta) \in \Gamma_\delta := \{(u, v) \in \mathbb{R}^2 : v = \delta\}$ and the real number $y$ with the point $(y, 0) \in \Gamma_0$. In the sequel the values of $u$ on the line $\Gamma_\delta$ will be of our interest and we write $u(x)$ for $u(x, \delta)$.

Proposition 1.

$$u(x) = \sup_{y \in \mathbb{R}}[f(y) - L\sqrt{\delta^2 + (x - y)^2}] = \max_{|y-x| \leq D\delta} |f(y) - L\sqrt{\delta^2 + (x - y)^2}|,$$

(2)

where $D := \frac{2LL_f}{L^2 - L'_f}$.

Proof. From the definition of $u$ we have that

$$f(x) - L\delta \leq u(x)$$

so it is enough to show that if $|x - y| > D\delta$ then

$$f(y) - L\sqrt{\delta^2 + (x - y)^2} < f(x) - L\delta.$$

On the other hand from the boundedness of $f'$ we have

$$f(y) - L\sqrt{\delta^2 + (x - y)^2} \leq f(x) + L_f|x - y| - L\sqrt{\delta^2 + (x - y)^2}.$$

Thus we note that all values of $y$ for which

$$f(x) + L_f|x - y| - L\sqrt{\delta^2 + (x - y)^2} < f(x) - L\delta$$

can be ignored in taking supremum in the definition of $u$.

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We write
\[ L_f |x - y| + L\delta < L\sqrt{\delta^2 + (x - y)^2} \]
and arrive at
\[ L_f^2 |x - y|^2 + 2LL_f\delta |x - y| + L^2\delta^2 < L^2\delta^2 + L^2 |x - y|^2 \]
Thus
\[ 2LL_f\delta < (L^2 - L_f^2) |x - y| \iff |x - y| > D\delta. \]

Let \( y(x) \) be one of the points in \( \{|y - x| \leq D\delta\} \), where the maximum in (2) is achieved
\[ u(x) = f(y(x)) - L\sqrt{\delta^2 + (x - y(x))^2}. \] (3)

**Lemma 2.** If \( \delta > 0 \) is small enough then for every \( x \in \Gamma_\delta \) the point \( y(x) \) is unique and \( y(x) : \mathbb{R} \to \mathbb{R} \) is a bijective Lipschitz map.
\textbf{Proof.} For each \( x \in \Gamma_\delta \) consider the function \( g_x(y) := u(x) + L \sqrt{\delta^2 + (x - y)^2} \) defined on \( \Gamma_0 \). The graph of \( g_x \) is a hyperbola and the graph of any other function \( g_x' \) can be obtained by a translation. Obviously \( f(y) \leq g_x(y) \) on \( \Gamma_0 \) and \( g_x(x) = f(y(x)) \). If at every point \( y \in \Gamma_0 \) the graph of \( f \) can be touched from above by some hyperbola \( g_x(y) \) then we will get the surjectivity of \( y(x) \). To obtain this the following will be enough
\[
g_x''(y) > L_f, \text{ for all } |y - x| \leq D\delta. \tag{4}\]
So we arrive at
\[
\delta < \frac{L}{L_f(1 + D^2)\delta}. \tag{5}\]
Note that also uniqueness of \( y(x) \) follows from \( \mathbb{H} \); assume we have \( y(x) \) and \( \tilde{y}(x) \), then
\[
L_f|y(x) - \tilde{y}(x)| < \left| \int \frac{\tilde{y}(x)}{y(x)} g_x''(t)dt \right| = |f'(y(x)) - f'(\tilde{y}(x))| \leq L_f|y(x) - \tilde{y}(x)|. \]
We have used here that
\[
f'(y(x)) = g_x'(y(x)) = \frac{L(y(x) - x)}{\sqrt{\delta^2 + (y(x) - x)^2}} \tag{6}\]
(derivatives in \( y \) at the point \( y(x) \)).

The injectivity of the map \( y(x) \) follows from differentiability of \( f \). Assume \( y_0 = y(x) = y(\tilde{x}) \), so we have
\[
f(y_0) = g_x(y_0) = g_x(y_0). \]
On the other hand \( f(y) \leq \min(g_x(y), g_x(y)) \) this contradicts differentiability of \( f \) at \( y_0 \).

The monotonicity of \( y(x) \) can be obtained using same arguments; if \( x < \tilde{x} \) then the ‘left’ hyperbola \( g_x(y) \) touches the graph of \( f \) ‘left’ than the ‘right’ hyperbola \( g_x(y) \), since both hyperbolas are above the graph of \( f \).

Now we will prove that \( y(x) \) is Lipschitz. From (6) it follows that
\[
y(x) - x = \frac{\delta f'(y(x))}{\sqrt{L^2 - (f'(y(x)))^2}}. \tag{7}\]
Taking \( Y(x) := y(x) - x \) we can rewrite this as
\[
Y(x) = \frac{\delta f'(Y(x) + x)}{\sqrt{L^2 - (f'(Y(x) + x))^2}} = \delta \Phi(f'(Y(x) + x)), \tag{8}\]
where \( \Phi(t) = \frac{t}{\sqrt{L^2 - t^2}} \). For \( \delta < \frac{(L^2 - L_f^2)^2}{L^2 L_f} \) we can use Banach’s fix point theorem and get that this functional equation has unique continuous solution. On the other hand it is not difficult to check that
\[
\left| Y(x_2) - Y(x_1) \right| \leq \delta C \frac{1}{1 - \delta C},
\]
where \( C = \frac{L^2 L_f^2}{(L^2 - L_f^2)^2} \). \[\square\]

\textbf{Corollary 3.} If \( \delta \) is as small as in the previous Lemma, then the function \( u \) is \( \infty \)-harmonic in the strip between \( \Gamma_0 \) and \( \Gamma_\delta \).
Proof. This follows from the fact that if we take the strip with boundary values \( f \) on \( \Gamma_0 \) and \( u \) on \( \Gamma_\delta \) then McShane-Whitney’s minimal and maximal solutions will coincide, obviously with \( u \). □

Remark 4. We can rewrite \( \text{(7)} \) in the form

\[
x(y) = y - \frac{\delta f'(y)}{\sqrt{L^2 - (f'(y))^2}}, \tag{9}
\]

where \( x(y) \) is the inverse of \( y(x) \). This together with \( \text{(3)} \) gives us the following

\[
u(x(y)) = f(y) - \frac{\delta L^2}{\sqrt{L^2 - (f'(y))^2}}
\]

Using the recent result of O.Savin that \( u \) is \( C^1 \), we conclude that function \( x(y) \) is as regular as \( f' \), so we cannot expect to have better regularity than Lipschitz.

Lemma 5. If \( \delta > 0 \) is as small as above and function \( f \) is not twice differentiable at \( y_0 \), then the function \( u \) is not twice differentiable at \( x_0 := x(y_0) \).

Proof. First note that for all \( x \) and \( y \), such that \( x = x(y) \) we have

\[u'(x) = f'(y).
\]

This can be checked analytically but actually is a trivial geometrical fact; the hyperbola ‘slides’ in the direction of the growth of \( f \) at point \( y \), thus the cone which generates this hyperbola and ‘draws’ with its peak the graph of \( u \) moves in same direction which is the direction of the growth of \( u \) at point \( x = x(y) \).

Now assume we have two sequences \( y_k \to y_0 \) and \( \tilde{y}_k \to y_0 \) such that

\[
\frac{f'(y_k) - f'(y_0)}{y_k - y_0} \to f''(y_0) \quad \text{and} \quad \frac{f'(\tilde{y}_k) - f'(y_0)}{\tilde{y}_k - y_0} \to f''(y_0)
\]

and \( f''(y_0) < f''(y_0) \). Let us define appropriate sequences on \( \Gamma_\delta \) denoting by \( x_k := x(y_k) \) and by \( \tilde{x}_k := x(\tilde{y}_k) \) and compute the limits of

\[
\frac{u'(x_k) - u'(x_0)}{x_k - x_0} \quad \text{and} \quad \frac{u'(\tilde{x}_k) - u'(x_0)}{\tilde{x}_k - x_0}.
\]

We have

\[
\frac{u'(x_k) - u'(x_0)}{x_k - x_0} = \frac{f'(y_k) - f'(y_0)}{y_k - y_0} \frac{y_k - y_0}{x_k - x_0} \frac{y_k - y_0}{x_k - x_0}
\]

the first multiplier converges to \( f''(y_0) \), let us compute the limit of the second one. From \( \text{(10)} \) we get that

\[
\frac{x_k - x_0}{y_k - y_0} \to 1 - \delta \Phi'(f'(y_0)) f''(y_0),
\]

where \( \Phi(t) = \frac{t}{\sqrt{L^2 - t^2}} \). Thus

\[
\frac{u'(x_k) - u'(x_0)}{x_k - x_0} \to \frac{f''(y_0)}{1 - \delta \Phi'(f'(y_0)) f''(y_0)},
\]

and analogously

\[
\frac{u'(\tilde{x}_k) - u'(x_0)}{\tilde{x}_k - x_0} \to \frac{f''(y_0)}{1 - \delta \Phi'(f'(y_0)) f''(y_0)}.
\]
To complete the proof we need to use the monotonicity of the function
\[
\frac{t}{1-\delta C't}, \quad -L'_f < t < L'_f,
\]
where \( \frac{1}{L} < C < \frac{L^2}{(L^2-L'_f)^2}. \)

We would like to note that if the function \( f \) is not \( C^2 \) at a point \( y \) then \( u \) constructed here is not \( C^2 \) on the whole line connecting \( y \) and \( x(y) \). So choosing \( f \) to be not twice differentiable on a dense set we can get a function \( u \) which is not \( C^2 \) on the collection of corresponding line-segments. Note that a similar example is the distance function from a convex set, whose boundary is \( C^1 \) and not \( C^2 \) on a dense subset. Then the distance function is \( \infty \)-harmonic and is not \( C^2 \) on appropriate lines.

Our example has the property of having constant \( |\nabla u| \) on gradient flow curves (lines in our case). It would be interesting to find a general answer to the question:

'What geometry do the gradient flow curves have, on which \( |\nabla u| \) is not constant?'

From Aronsson’s results we know that \( u \) is not \( C^2 \) on such a curve. This is our motivation for the investigation of \( C^2 \)-differentiability of \( \infty \)-harmonic functions.

The author has only one item in the list of references. The history and the recent developments of the theory of \( \infty \)-harmonic functions, as well as a complete reference list could be found in that paper.

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**References**

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