STABILITY OF POSITIVE SOLUTIONS TO BIHARMONIC EQUATIONS ON HEISENBERG GROUP

G. DWIVEDI, J. TYAGI

Abstract. In this note, we establish the stability of positive solution to the following problem

\[
\begin{align*}
\Delta_{H^n}^2 u &= a(\xi)u - f(\xi, u) \quad \text{in } \Omega \\
|u|_{\partial\Omega} &= 0 = \Delta_{H^n} u|_{\partial\Omega},
\end{align*}
\]

on Heisenberg group.

1. Introduction

The aim of this note is to establish the stability of positive solution to the following biharmonic problem on Heisenberg group:

\[
\begin{align*}
\Delta_{H^n}^2 u &= a(\xi)u - f(\xi, u) \quad \text{in } \Omega \\
u > 0 & \quad \text{in } \Omega \\
u = 0 &= \Delta u \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{H}^n \) is an open, smooth and bounded subset, \( a \in L^\infty(\Omega) \) and \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \). The nonlinearities of the type

\[ a(x)u - f(x, u) \]

are known as logistic nonlinearity, see for instance [1, 7, 14, 17, 22, 23] and references therein.

This work is motivated by the recent works on polyharmonic equations, see for instance [2, 10, 16, 19, 24], where the authors obtained stability properties of solution to polyharmonic equation with exponential nonlinearity, see [10, 19] and stability results to biharmonic equation, see [2, 10] and Liouville theorems for stable radial solution for the biharmonic operators, see [24].

To the best of our knowledge there are no results on the stability of positive solution for the biharmonic operator in the Heisenberg group. For the existence of positive solution to problems similar to (1.1) in \( \mathbb{R}^n \), we refer to [3, 9, 11, 12, 13, 21, 27, 28] and the references therein. For existence of positive solution for Kohn-Laplace operator on Heisenberg group, we refer to [6, 29] and references cited therein. For the existence of positive solution for biharmonic equation on Heisenberg group, we refer to [20].

We make the following hypotheses on the nonlinearity \( f \) and weight \( a \):

(H1) Let \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \) and \( C^1 \) in the \( y \) variable and satisfies

\[ f_y(\xi, y) \geq \frac{f(\xi, y)}{y}, \quad \forall \ 0 < y \in \mathbb{R}, \ \forall \ \xi \in \Omega. \]

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(H2) \( a(\xi)s - f(\xi, s) \geq 0 \), for a.e. \( \xi \in \Omega \) and for all \( s \in \mathbb{R} \).

The functional associated with (1.1) is

\[ E : D^2(\Omega) \cap D^1_0(\Omega) \to \mathbb{R} \]

defined by

\[ E(u) = \frac{1}{2} \int_{\Omega} |\Delta_H u|^2 d\xi - \frac{1}{2} \int_{\Omega} a(\xi)u^2 d\xi + \int_{\Omega} F(\xi, u) d\xi, \]

where

\[ F(\xi, s) = \int_0^s f(\xi, t) dt. \]

The weak formulation of (1.1) is the following:

\[ \int_{\Omega} \Delta_H u \Delta_H \phi d\xi = \int_{\Omega} a(\xi)u\phi d\xi - \int_{\Omega} f(\xi,u)\phi d\xi, \quad \forall \phi \in C^2_c(\Omega), \]

where \( C^2_c(\Omega) \) is the space of \( C^2 \) functions in \( \Omega \) having compact support in \( \Omega \). The linearized operator \( L_u \) associated with (1.1) at a given solution \( u \) is defined by following duality:

\[ L_u : v \in D^2(\Omega) \cap D^1_0(\Omega) \to L_u(v) \in (D^2(\Omega) \cap D^1_0(\Omega))^{'}, \]

where

\[ L_u(v) : \psi \in D^2(\Omega) \cap D^1_0(\Omega) \to L_u(v, \psi) \]

and

\[ L_u(v, \psi) = \int_{\Omega} \Delta_H v \Delta_H \psi d\xi - \int_{\Omega} a(\xi)v\psi d\xi + \int_{\Omega} f_u(\xi,u)v\psi d\xi. \]

It is easy to see that \( L_u \) is well-defined and the first eigenvalue of \( L_u \) is given by

\[ \lambda_1 = \inf_{v \in D^2(\Omega) \cap D^1_0(\Omega), v \neq 0} \frac{\int_{\Omega} (L_u(v, v) d\xi)}{\int_{\Omega} v^2 d\xi}. \]

We say that the solution \( u \) of (1.1) is stable if

\[ \int_{\Omega} |\Delta_H v|^2 d\xi - \int_{\Omega} a(\xi)v^2 d\xi + \int_{\Omega} f_u(\xi,u)v^2 d\xi \geq 0 \]

for every \( v \in C^2_c(\Omega) \), see [25] for the definition of stability of solutions to biharmonic problems. Actually, (1.4) implies that the principal eigenvalue of the linearized equation associated with (1.1) is positive and hence the solution \( u \) of (1.1) is stable.

Throughout the article, the space \( D^2(\Omega) \cap D^1_0(\Omega) \) is denoted by \( D \).

The main result of this paper is as follows, which we will prove in the last section.

**Theorem 1.1.** Let (H1)-(H3) hold. Let \( u \) be a positive solution of (1.1). Then \( u \) is stable.

Section 2 deals with some preliminaries on the Heisenberg group. In Section 3, we give the proof of Theorem 1.1.
left-invariant vector fields

where \( x, y, x' \in \mathbb{H}^n \) results which are used in order to prove the main results. The Heisenberg group \( \mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot) \) is the space \( \mathbb{R}^{2n+1} \) with the non-commutative law of product

\[
 (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2((y, x') - (x, y'))),
\]

where \( x, y, x', y' \in \mathbb{R}^n \), \( t, t' \in \mathbb{R} \) and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^n \). The homogeneous dimension of \( \mathbb{H}^n \) is \( Q = 2n + 2 \). This operation endows \( \mathbb{H}^n \) with the structure of a Lie group. The Lie algebra of \( \mathbb{H}^n \) is generated by the left-invariant vector fields

\[
 T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, \ldots, n.
\]

These generators satisfy the non-commutative formula

\[
 [X_i, Y_j] = -4\delta_{ij}T, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.
\]

Let \( z = (x, y) \in \mathbb{R}^{2n} \), \( \xi = (z, t) \in \mathbb{H}^n \). The parabolic dilation

\[
 \delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)
\]

satisfies

\[
 \delta_\lambda (\xi_0, \xi) = \delta_\lambda \xi, \delta_\lambda \xi_0
\]

and

\[
 \|\xi\|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{2}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{2}}
\]

is a norm with respect to the parabolic dilation which is known as Korányi gauge norm \( N(z, t) \). In other words, \( \rho(\xi) = (|z|^4 + t^2)^{\frac{1}{2}} \) denotes the Heisenberg distance between \( \xi \) and the origin. Similarly, one can define the distance between \((z, t)\) and \((z', t')\) on \( \mathbb{H}^n \) as follows:

\[
 \rho(z, t; z', t') = \rho((z', t')^{-1} \cdot (z, t)).
\]

It is clear that the vector fields \( X_i, Y_i, i = 1, 2, \ldots, n \) are homogeneous of degree 1 under the norm \( \|\cdot\|_{\mathbb{H}^n} \) and \( T \) is homogeneous of degree 2. The Korányi ball of center \( \xi_0 \) and radius \( r \) is defined by

\[
 B_{\mathbb{H}^n}(\xi_0, r) = \{\xi : \|\xi^{-1} \xi_0\| \leq r\}
\]

and it satisfies

\[
 |B_{\mathbb{H}^n}(\xi_0, r)| = |B_{\mathbb{H}^n}(0, r)| = r^d |B_{\mathbb{H}^n}(0, 1)|,
\]

where \( |\cdot| \) is the \((2n + 1)\)-dimensional Lebesgue measure on \( \mathbb{H}^n \) and \( d = 2n + 2 \) is the so-called homogeneous dimension of Heisenberg group \( \mathbb{H}^n \). The Heisenberg gradient and Heisenberg Laplacian or the Laplacian-Kohn operator on \( \mathbb{H}^n \) are given by

\[
 \nabla_{\mathbb{H}^n} = (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)
\]

and

\[
 \Delta_{\mathbb{H}^n} = \sum_{i=1}^n X_i^2 + Y_i^2 = \sum_{i=1}^n \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial z^2} \right).
\]
Definition 2.1 ($D^{1,p} (\Omega)$ and $D^{1,p}_0 (\Omega)$ Space). Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{1,p} (\Omega) = \{ u : \Omega \to \mathbb{R} \text{ such that } u, |\nabla_{\mathbb{H}^n} u| \in L^p(\Omega) \}.$$  

$D^{1,p} (\Omega)$ is equipped with the norm

$$\|u\|_{D^{1,p} (\Omega)} = \left( \|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$  

$D^{1,p}_0 (\Omega)$ is the closure of $C^\infty_0 (\Omega)$ with respect to the norm

$$\|u\|_{D^{1,p}_0 (\Omega)} = \left( \int_{\Omega} |\nabla_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$  

Definition 2.2 ($D^{2,p} (\Omega)$ and $D^{2,p}_0 (\Omega)$ Space). Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{2,p} (\Omega) = \{ u : \Omega \to \mathbb{R} \text{ such that } u, |\nabla_{\mathbb{H}^n} u|, |\Delta_{\mathbb{H}^n} u| \in L^p(\Omega) \}.$$  

$D^{2,p} (\Omega)$ is equipped with the norm

$$\|u\|_{D^{2,p} (\Omega)} = \left( \|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} + \|\Delta_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$  

$D^{2,p}_0 (\Omega)$ is the closure of $C^\infty_0 (\Omega)$ with respect to the norm

$$\|u\|_{D^{2,p}_0 (\Omega)} = \left( \int_{\Omega} |\Delta_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$  

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, first, we prove the following lemma:

Lemma 3.1. Let $u \in D^{2}(\Omega) \cap D^{1}_0 (\Omega)$ be a nonnegative weak solution (not identically zero) of

$$(3.1) \quad \Delta_{\mathbb{H}^n}^2 u = a(\xi)u - f(\xi,u) \quad \text{in } \Omega, \quad u = \Delta_{\mathbb{H}^n} u = 0 \quad \text{on } \partial \Omega,$$

where $a$ and $f$ satisfy (H2) then $-\Delta_{\mathbb{H}^n} u > 0$ in $\Omega$ and $u > 0$ in $\Omega$.

Proof. Let $-\Delta_{\mathbb{H}^n} u = v$. Then writing (3.1) into system form, we get

$$(3.2) \begin{cases} -\Delta_{\mathbb{H}^n} u = v \quad \text{in } \Omega, \\ -\Delta_{\mathbb{H}^n} v = a(\xi)u - f(\xi,u) \quad \text{in } \Omega, \\ u = 0 = v \quad \text{on } \partial \Omega. \end{cases}$$

Since $a(\xi)u - f(\xi,u) \geq 0$ in $\Omega$, so by maximum principle [5], we get $v \geq 0$. By strong maximum principle, either $v > 0$ or $v \equiv 0$ in $\Omega$. If $v \equiv 0$, then we have

$$-\Delta_{\mathbb{H}^n} u = 0 \quad \text{in } \Omega; \quad v = 0 \quad \text{on } \partial \Omega.$$  

Again by maximum principle, we get $u \equiv 0$, which is a contradiction and therefore $v > 0$ in $\Omega$ and hence

$$-\Delta_{\mathbb{H}^n} u > 0 \quad \text{in } \Omega.$$  

Again, since $-\Delta_{\mathbb{H}^n} u > 0$ in $\Omega$, by strong maximum principle, we get

$$u > 0 \quad \text{in } \Omega.$$  

□
Remark 3.2. Let \( -\Delta_{H^n} u = v \). Then writing (3.1) into system form, we get

\[
\begin{align*}
-\Delta_{H^n} u &= v \text{ in } \Omega, \\
-\Delta_{H^n} v &= a(\xi) u - f(\xi, u) \text{ in } \Omega, \\
u &= 0 = v \text{ on } \partial\Omega.
\end{align*}
\]

Now, by using Theorem 3.35 [8] for second equation in (3.3), we conclude that \( v \in C^\alpha(\Omega) \) for some \( 0 < \alpha < 1 \). Then by using Theorem 3.9 [26], we get that \( u \in C^{2,\alpha}(\Omega) \). Again applying Theorem 3.35 [8] and Theorem 3.9 [26] for \( u \in C^{2,\alpha}(\Omega) \), we conclude that \( u \in C^{4,\alpha}(\Omega) \).

**Proof of Theorem 1.1:** Since \( u \in D \cap L^\infty(\Omega) \) be a positive solution of (1.1) so by Remark 3.2 \( u \in C^{4,\alpha}(\Omega) \). Now for any \( v \in C^2(\Omega) \), we choose

\[
\phi = \frac{v^2}{u}
\]

as a test function in (1.2). Since

\[
\nabla_{H^n} \phi = \frac{2uv\nabla_{H^n} v - v^2\nabla_{H^n} u}{u^2},
\]

and

\[
\Delta_{H^n} \phi = \frac{2u^3|\nabla_{H^n} v|^2 - 4vu^2\nabla_{H^n} u, \nabla_{H^n} v + 2u^2v|\nabla_{H^n} u|^2 + 2v^3\Delta_{H^n} v - v^2u^2\Delta_{H^n} u}{u^4}
\]

so from (1.2), we get

\[
\int_\Omega \Delta_{H^n} u \left[ \frac{2u^3|\nabla_{H^n} v|^2 - 4vu^2\nabla_{H^n} u, \nabla_{H^n} v + 2u^2v|\nabla_{H^n} u|^2 + 2v^3\Delta_{H^n} v - v^2u^2\Delta_{H^n} u}{u^4} \right] d\xi = \int_\Omega a(\xi)v^2 d\xi - \int_\Omega \frac{f(\xi, u)v^2}{u} d\xi.
\]

This yields that

\[
\int_\Omega \frac{-4v}{u^2}\Delta_{H^n} u \nabla_{H^n} u, \nabla_{H^n} v d\xi + \int_\Omega \frac{2v}{u}\Delta_{H^n} u \Delta_{H^n} v d\xi + \int_\Omega \frac{2}{u}\nabla_{H^n} v|^2\Delta_{H^n} u d\xi - \int_\Omega \frac{v^2}{u^2}|\nabla_{H^n} u|^2 d\xi + \int_\Omega \frac{2v^2}{u^3}|\nabla_{H^n} v|^2 \Delta_{H^n} u d\xi - \int_\Omega a(\xi)v^2 d\xi + \int_\Omega f(\xi, u)v^2 d\xi + \int_\Omega |\Delta_{H^n} v|^2 d\xi - \int_\Omega |\Delta_{H^n} v|^2 d\xi = 0.
\]

On rearranging the terms, we get

\[
\int_\Omega |\Delta_{H^n} v|^2 d\xi - \int_\Omega a(\xi)v^2 d\xi + \int_\Omega \frac{f(\xi, u)}{u}v^2 d\xi = \int_\Omega \left[ |\Delta_{H^n} v|^2 + \frac{4v}{u^2}\Delta_{H^n} u \nabla_{H^n} u, \nabla_{H^n} v - \frac{2v}{u}\Delta_{H^n} u \Delta_{H^n} v - \frac{2v}{u}\nabla_{H^n} v|^2 \Delta_{H^n} u \right] d\xi
\]

\[
= \int_\Omega \left[ \Delta_{H^n} v - \frac{v}{u}\Delta_{H^n} u \right]^2 + \frac{4v}{u^2}\Delta_{H^n} u \nabla_{H^n} u, \nabla_{H^n} v - \frac{2v}{u}\nabla_{H^n} v|^2 \Delta_{H^n} u \right] d\xi
\]

\[
- \frac{2v^2}{u^3}|\nabla_{H^n} v|^2 \Delta_{H^n} u \right] d\xi.
\]

This implies that

\[
\int_\Omega |\Delta_{H^n} v|^2 d\xi = \int_\Omega a(\xi)v^2 d\xi + \int_\Omega \frac{f(\xi, u)}{u}v^2 d\xi
\]
≥ \int_{\Omega} \left[ \frac{4v}{u} \Delta_{H^n} u \nabla_{H^n} u \nabla_{H^n} v - \frac{2}{u} |\nabla_{H^n} v|^2 \Delta_{H^n} u - \frac{2v^2}{u^3} |\nabla_{H^n} u|^2 \Delta_{H^n} u \right] d\xi \\
= \int_{\Omega} \left[ - \frac{2}{u} \Delta_{H^n} u \left( |\nabla_{H^n} v|^2 + \frac{v^2}{u^2} |\nabla_{H^n} u|^2 - \frac{2v \nabla_{H^n} u \nabla_{H^n} v}{u} \right) \right] d\xi \\
= \int_{\Omega} \left( - \frac{2}{u} \Delta_{H^n} u \left( \nabla_{H^n} v - \frac{v}{u} \nabla_{H^n} u \right) \right)^2 d\xi \\
≥ 0.

By Lemma 3.1 we have

\[-\Delta_{H^n} u > 0 \text{ in } \Omega\]

and this implies that

\[\int_{\Omega} |\Delta_{H^n} v|^2 d\xi - \int_{\Omega} a(\xi)v^2 d\xi + \int_{\Omega} \frac{f(\xi, u)}{u} v^2 d\xi ≥ 0.\]

Now, using Hypothesis (H1), we obtain

\[\int_{\Omega} |\Delta_{H^n} v|^2 d\xi - \int_{\Omega} a(\xi)v^2 d\xi + \int_{\Omega} f_u(x, u)v^2 d\xi ≥ 0\]

and therefore \(u\) is stable. This completes the proof of this theorem. \(\square\)

The following remark is in order:

**Remark 3.3.** Let us consider the following problem

\[\begin{align*}
\Delta^2 u &= a(\xi)u - f(\xi, u) \quad \text{in } B, \\
u &= 0 = \frac{\partial u}{\partial \nu} \quad \text{on } \partial B,
\end{align*}\]

where \(B\) denotes unit ball in \(\mathbb{H}^n\), \(a \in L^\infty(B)\) and \(f \in C(\overline{B} \times \mathbb{R}, \mathbb{R})\). Let (H1)-(H2) hold. Then, using the similar lines of proof as in Theorem 1.1, one can prove that \(u\) is stable. For the sake of brevity, we skip the details.

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