GLOBAL STABILITY OF A MULTI-GROUP MODEL WITH VACCINATION AGE, DISTRIBUTED DELAY AND RANDOM PERTURBATION

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ABSTRACT. A multi-group epidemic model with distributed delay and vaccination age has been formulated and studied. Mathematical analysis shows that the global dynamics of the model is determined by the basic reproduction number $R_0$: the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. Lyapunov functionals are constructed by the non-negative matrix theory and a novel grouping technique to establish the global stability. The stochastic perturbation of the model is studied and it is proved that the endemic equilibrium of the stochastic model is stochastically asymptotically stable in the large under certain conditions.

1. Introduction. An epidemic model was proposed by Cooke [1] to describe the disease spread via a vector (such as a mosquito). It is assumed that when a susceptible vector is infected by a person, there is a time delay, $\tau > 0$, during which the infectious agents develop in the vector, and the infected vector becomes itself infectious after the delay. It is also assumed that the vector population size is large enough such that at any time $t$, the infectious vector population is simply proportional to the infectious human population at time $t - \tau$. Let $S(t)$ and $I(t)$ denote the numbers of the human susceptible and infective individuals, respectively. The force of infection at time $t$ is assumed to be given by $\beta S(t)I(t - \tau)$. Beretta and Takeuchi [2] studied the model with distributed delay,

$$\begin{align*}
S'(t) &= \mu - \mu S(t) - \beta S(t) \int_{0}^{\infty} f(\tau)I(t - \tau)d\tau, \\
I'(t) &= \beta S(t) \int_{0}^{\infty} f(\tau)I(t - \tau)d\tau - (\mu + \lambda)I(t), \\
R'(t) &= \lambda I - \mu R,
\end{align*}$$

(1)
where \( \beta \) is the contact rate; \( \mu \) is the birth and death rate; \( \lambda \) is the recovery/removal rate. \( f(\tau) \) represents the proportion of the infectious vector population, and function \( f(\tau) \) is assumed to be non-negative, square integrable on \( \mathbb{R}^+ = [0, +\infty) \) with \( \int_0^{+\infty} f(\tau) d\tau = 1 \) and \( \int_0^{+\infty} \tau f(\tau) d\tau < +\infty \).

One essential assumption in classical compartmental epidemic models is that the individuals are homogeneously mixed, and each individual has the same chance to get infected. More realistic models divide the host population into groups to consider the disease transmission in heterogeneous cases. The groups can be classified according to education levels, ethnic backgrounds, gender, age, professions, communities, or geographic distributions for their diversities in disease transmission. The vital epidemic parameters vary among different groups. Based on these factors, Shu et al. [3] investigated the following general multi-group model with distributed delay:

\[
\begin{align*}
S'_k(t) &= n_k(S_k) - \sum_{j=1}^{n} \beta_{kj} h_k(S_k) \int_{\tau=0}^{\infty} f_j(\tau) g_j(I_j(t-\tau)) d\tau, \\
E'_k(t) &= \sum_{j=1}^{n} \beta_{kj} h_k(S_k) \int_{\tau=0}^{\infty} f_j(\tau) g_j(I_j(t-\tau)) d\tau - (dE_k + \delta_k)E_k, \\
I'_k(t) &= \delta_k E_k - (d_k^I + \gamma_k + \epsilon_k)I_k, \quad k = 1, 2, \cdots, n.
\end{align*}
\]

(2)

The global stability of the unique endemic equilibrium of model (2) was proved by using a graph-theoretical approach and Lyapunov functionals. Global stability results were also obtained for other multi-group epidemic models [4, 5, 6, 7, 8, 9, 10, 11, 12].

Vaccination is one of the commonly used control measures to prevent and reduce the transmission of infectious diseases. The eradication of smallpox has been considered as the most spectacular success of vaccination. Some vaccines can offer lifelong immunity with one dose, while others require boosters to maintain immunity since the acquired immunity may wane with time. It is natural to consider the vaccination and the waning immunity in modeling disease dynamics. Li et al. [13] has investigated the global dynamics of an epidemic model with vaccination for newborns and susceptibles. Blower and McLean [14] have argued that a mass vaccination campaign may increase the severity of disease, if the vaccination is applied to only 50% of the population and the vaccine efficacy is 60%. Xiao et al. [15] assumed that the vaccinated individuals can be infected at a reduced rate compared to the susceptibles. Other mathematical models on vaccination have been studied in [5, 16, 17].

Although waning immunity has been included in several models [5, 13, 15], it was assumed that the rate of the immunity loss is a constant. A better assumption on the waning immunity is that the protection immunity depends on the vaccination age of an individual (the time from the vaccination). The epidemiological models with vaccination age structure can be a suitable choice to describe the dynamics of an infectious diseases with waning immunity after vaccination.

The mathematical models with the chronological age, disease age, and vaccination age have been widely used to describe the impact of the age on the disease evolution [18, 19, 20, 21, 22]. Iannelli et al. [18] have studied an epidemic model with vaccination age by assuming the immunity decreases with the time after vaccination. Li et al. [19] have proposed an epidemic model with vaccination age and treatment to show that backward bifurcation occurs due to a piecewise treatment
function. Duan et al. [20] have simplified the model [19] by assuming no treatment and have obtained the global stabilities of the disease-free equilibrium and the endemic equilibrium.

Motivated by [2, 3, 20], we formulate and study the following multi-group epidemic models.

\[
\begin{align*}
S_k'(t) &= \Lambda_k - (d_k^S + \xi_k)S_k - \sum_{j=1}^{n} \beta_{kj}S_k \int_0^\infty f_j(\tau)g_j(I_j(t-\tau))d\tau \\
&\quad + \int_0^\infty \alpha_k(\theta)v_k(\theta,t)d\theta, \\
\left(\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)v_k(\theta,t) &= -(d_k^V + \alpha_k(\theta))v_k(\theta,t), \\
v_k(0,t) &= \xi_kS_k(t), \quad v_k(\theta,0) = v_{0k}(\theta) \in L^1_+(0, +\infty), \\
E_k'(t) &= \sum_{j=1}^{n} \beta_{kj}S_k \int_0^\infty f_j(\tau)g_j(I_j(t-\tau))d\tau - (d_k^E + \delta_k)E_k, \\
I_k'(t) &= \delta_kE_k - (d_k^I + \gamma_k + \epsilon_k)I_k, \\
R_k'(t) &= \gamma_kI_k - d_k^R R_k.
\end{align*}
\]

Here \(S_k, E_k, I_k, \) and \(R_k \) \((k = 1, 2, \cdots, n)\) denote the numbers of susceptible, latent, infectious, and recovered individuals at time \(t\) in the \(k\)-th group, respectively. Function \(v_k(\theta,t)\) is the age density of vaccinated individuals at time \(t\) in the \(k\)-th group. The kernel function \(f_j(\tau)\) satisfies the conditions \(f_j(\tau) \geq 0\) and \(\int_0^\infty f_j(\tau)d\tau = \alpha_j > 0\). The non-negative constant \(\beta_{kj}\) is the transmission rate due to the contact of susceptible individuals in the \(k\)-th group with infectious individuals in the \(j\)-th group. The new infection occurred in the \(k\)-th group with distributed delays and the nonlinear transmission is given by \(\sum_{j=1}^{n} \beta_{kj}S_k \int_0^\infty f_j(\tau)g_j(I_j(t-\tau))d\tau\), where \(g_j(I_j)\) denotes the force of the infection. The vaccinated compartment is structured by the vaccination age \(\theta\), and it is assumed that the newly vaccinated individuals enter the vaccinated class \(v_k(\theta,t)\) with vaccination age zero. Function \(\alpha_k(\theta)\) is the immunity wane rate, and it is a nonnegative, bounded and continuous function of \(\theta\). For two given vaccination ages \(\theta_1, \theta_2, (0 \leq \theta_1 \leq \theta_2 \leq +\infty)\), the number of the vaccinated individuals with the vaccination age \(\theta\) between \(\theta_1\) and \(\theta_2\) at time \(t\) is \(\int_{\theta_1}^{\theta_2} v_k(\theta,t)d\theta\). The immunity lose rate (the number of individuals moving from the vaccinated class into the susceptible class due to the waning immunity) at time \(t\) is \(\int_0^\infty \alpha_k(\theta)v_k(\theta,t)d\theta\).

Parameters \(d_k^S, d_k^E, d_k^I, d_k^R\) are the natural death rates of \(S_k, E_k, I_k, \) and \(R_k\), respectively. \(\Lambda_k, \xi_k, \delta_k, \gamma_k,\) and \(\epsilon_k\) are the recruitment rate of the susceptible class, the rate of vaccination of the susceptible individuals, the rate at which exposed individuals become infectious, the recovery rate of infectious individuals, and the disease induced mortality in the \(k\)-th group, respectively. We also assume that the function \(g_k(I_k)\) is sufficiently smooth and satisfies following properties [3]

\[
g_k(0) = 0, \quad g_k(I_k) > 0, \quad \text{for } I_k > 0, \\
\lim_{I_k \to 0^+} \frac{g_k(I_k)}{I_k} = b_k, \quad \sup_{I_k > 0} \frac{g_k(I_k)}{I_k} = b_k, \quad k = 1, 2, \cdots, n,
\]

where \(g_k(I_k)\) describes the infectivity of the individuals in \(I_k\) compartment, and it is natural to assume that \(g_k(0) = 0, \quad g_k(I_k) > 0\) for \(I_k > 0\) due to the fact that the disease can not spread if there is no infection. Note that \(\frac{g_k(I_k)}{I_k}\) is the per capita infectivity of the infected individuals in compartment \(I_k\). The assumption \(\sup_{I_k > 0} \frac{g_k(I_k)}{I_k} = b_k\) says that the per capita infectivity is bounded. The limit
\[
\lim_{t \to 0^+} g_k(I_k) = b_k \text{ indicates that the per capital infectivity is the largest at the beginning of the disease outbreak.}
\]

There exist functions \( g_j(I_j) \) satisfying assumptions given in (4). Four examples are listed here:

\[
I_j, \quad \frac{I_j}{A_j + I_j}, \quad \frac{I_j}{1 + \alpha_j I_j}, \quad (0 < m_j \leq 1), \quad 1 - e^{\alpha_j I_j}, \quad (a_j > 0). \tag{5}
\]

Preliminary results on the dynamics of model (3) are presented in Section 2. The global stability of the equilibrium of model (3) is proved by Lyapunov functional and graph theory in Section 3. The stochastic version of the model is derived and its asymptotic behavior is studied in Section 4. A brief summary is given in the concluding section.

2. Preliminaries. Let \( N_k(t) = S_k(t) + V_k(t) + E_k(t) + I_k(t) + R_k(t) \) with \( N_k(0) = N_{0k} \) be the total population size at time \( t \) in the \( k \)-th group, where \( V_k(t) = \int_0^\infty v_k(\theta, t)d\theta \). From equations in (3), we know that \( N_k(t) \) satisfies the differential equation

\[
N_k'(t) = \Lambda_k - d_k^S S_k(t) - d_k^V V_k(t) - d_k^E E_k(t) - d_k^R I_k(t) - d_k^R R_k(t) - \delta_k I_k(t).
\]

The comparison principle implies

\[
N_k(t) \leq N_{0k} e^{-\mu_k t} + \frac{\Lambda_k}{\mu_k} (1 - e^{-\mu_k t}), \quad N_k(t) < \frac{\Lambda_k}{\mu_k},
\]

where \( \mu_k = \min\{d_k^S, d_k^V, d_k^E, d_k^R \} \), and \( N_{0k} < \frac{\Lambda_k}{\mu_k} \).

Integrating the second equation in (3) along the characteristic line \( t - \theta = \) constant yields

\[
v_k(\theta, t) = \left\{ \begin{array}{ll}
\xi_k S_k(t - \theta) \Gamma_{0k}(\theta), & t \geq \theta, \\
v_{0k}(\theta - t) \Gamma_{0k}(\theta) / \Gamma_{0k}(\theta - t), & \theta > t,
\end{array} \right. \tag{6}
\]

where \( \Gamma_{0k}(\theta) = e^{- \int_0^\theta (d_k^S + \alpha_k(s))ds} \). Substituting (6) into the first equation of (3) gives

\[
S_k'(t) = \Lambda_k - d_k^S S_k - \sum_{j=1}^n \beta_{kj} S_k \int_0^\infty f_j(\tau) g_j(I_j(t - \tau))d\tau - \xi_k S_k
\]

\[
+ \int_0^t S_k(t - \theta) \Gamma_k(\theta)d\theta + F_k(t), \tag{7}
\]

where

\[
\Gamma_k(\theta) = \xi_k \alpha_k(\theta) \Gamma_{0k}(\theta), \quad F_k(t) = \int_t^\infty \alpha_k(\theta) v_{0k}(\theta - \tau) \Gamma_{0k}(\theta) d\theta, \quad \lim_{t \to \infty} F_k(t) = 0,
\]

\[
\Gamma_k = \int_0^\infty \Gamma_k(\theta)d\theta = \xi_k (1 - d_k^S \Gamma_{0k}), \quad \Gamma_{0k} = \int_0^\infty \Gamma_{0k}(\theta)d\theta, \quad \Gamma_k = \int_0^\infty \Gamma_k(\theta)d\theta.
\]

After replacing the first equation in (3) by (7) and dropping the equation for \( R_k(t) \), we have the limiting model (8). The qualitative behavior of the limiting
model is equivalent to that of model (3) [23].

$$\begin{align*}
S_k(t) &= \Lambda_k - (d_k^S + \xi_k)S_k - \sum_{j=1}^{n} \beta_{kj}S_k \int_0^\infty f_j(\tau)g_j(I_j(t-\tau))d\tau \\
&\quad + \int_0^\infty \Gamma_k(\theta)S_k(t-\theta)d\theta, \\
E_k(t) &= \sum_{j=1}^{n} \beta_{kj} \int_0^\infty f_j(\tau)g_j(I_j(t-\tau))d\tau - (d_k^E + \delta_k)E_k, \\
I_k(t) &= \delta_k E_k - (d_k^I + \gamma_k + \epsilon_k)I_k.
\end{align*}$$

(8)

We assume that \( \int_{-\infty}^\infty f_j(\tau)e^{\lambda_j\tau}d\tau < \infty \), where \( \lambda_j \) is a positive number. Define the following Banach space of fading memory type [24]:

\[
\mathcal{C}_k = \left\{ \phi \in C((-\infty, 0], \mathbb{R}) : \phi(s)e^{\lambda_k s} \text{ is uniformly continuous for } s \in (-\infty, 0], \text{ and } \sup_{s \leq 0} |\phi(s)|e^{\lambda_k s} < \infty \right\},
\]

with norm \( \|\phi\|_k = \sup_{s \leq 0} |\phi(s)|e^{\lambda_k s} \), and let \( \phi_k \in \mathcal{C}_k \) be such that \( \phi_k(s) = \phi(t+s), s \in (-\infty, 0] \). Consider system (8) in the phase space

\[
X = \prod_{k=1}^{n} (\mathcal{C}_k \times \mathcal{C}_k).
\]

By the fundamental theory of functional differential equations [25], model (8) has a unique solution satisfying the initial conditions

\[
\begin{align*}
S_k(s) &= \phi_{1k}(s), \quad E_k(s) = \phi_{2k}(s), \quad I_k(s) = \phi_{3k}(s), \\
\phi_{ik}(s) &\geq 0, s \in (-\infty, 0], \phi_{ik}(0) > 0, \quad i = 1, 2, 3, \quad k = 1, 2, \ldots, n,
\end{align*}
\]

(9)

where \( (\phi_{11}(s), \phi_{21}(s), \phi_{31}(s), \ldots, \phi_{1n}(s), \phi_{2n}(s), \phi_{3n}(s)) \in X \).

From above analysis, we know that

\[
\Omega = \left\{ (S_1, E_1, I_1, \ldots, S_n, E_n, I_n) \in X \mid S_k + E_k + I_k \leq \frac{\Lambda_k}{\mu_k}, \quad S_k, E_k, I_k \geq 0, \quad k = 1, 2, \ldots, n \right\}
\]

is the positive invariant set of model (8). The global stability of model (8) will be discussed for the solutions with the initial values in \( \Omega \). Once the solution of model (8) is determined, we can obtain \( v_k(\theta, t) \) from (6). The stability of the equilibrium of model (3) is the same as that of model (8). We will focus on the dynamical analysis of the reduced model (8).

Model (8) always has a disease-free equilibrium \( P_0 = (S_1^0, 0, 0, \ldots, S_n^0, 0, 0) \) where \( S_k^0 = \frac{\Lambda_k}{d_k^S + \xi_k d_k^I \Gamma_{ik}} \). The endemic equilibrium \( P_* = (S_1^*, E_1^*, I_1^*, \ldots, S_n^*, E_n^*, I_n^*) \) of model (8) is determined by following system of equations

\[
\begin{align*}
\Lambda_k &= (d_k^S + \xi_k)S_k^* + \sum_{j=1}^{n} \beta_{kj}S_k^*a_jg_j(I_j^*) - \int_0^\infty \Gamma_k(\theta)S_k^*d\theta, \\
\sum_{j=1}^{n} \beta_{kj}S_k^*a_jg_j(I_j^*) &= (d_k^E + \delta_k)E_k^*, \\
\delta_k E_k^* &= (d_k^I + \gamma_k + \epsilon_k)I_k^*.
\end{align*}
\]

(10)
The basic reproduction number is defined as the spectrum radius of matrix $F_0$, i.e.,

$$R_0 = \rho(F_0), \quad F_0 = \left( \frac{\delta_b \beta_{kj} S_0^0 a_j b_j}{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)} \right)_{1 \leq k,j \leq n}.$$ 

It is the expected number of infected individuals produced by a typical infected individual during its entire infectious period [26].

3. Stability analysis of equilibria. $R_0$ may serve as the threshold to describe disease spread. Usually, the disease will extinct in the host population if $R_0 \leq 1$, and the disease will become endemic if $R_0 > 1$.

**Theorem 3.1.** Assume that (4) holds, and $B = (\beta_{kj})_{n \times n}$ is irreducible. If $R_0 \leq 1$, then the disease-free equilibrium $P_0$ of model (8) is globally asymptotically stable.

**Proof.** The irreducibility of $B$ implies that matrix $F_0$ is also irreducible. $F_0$ has a positive left eigenvector $\omega = (\omega_1, \ldots, \omega_n)$ corresponding to the spectral radius $\rho(F_0) = R_0$. Let $I = (I_1, \cdots, I_n)$, $S^0 = (S_1^0, \cdots, S_n^0)$, $c_k = \frac{\omega_k \delta_k}{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)} > 0$, and define

$$U_1 = \sum_{k=1}^{n} c_k \left\{ S_k - S_k^0 - S_k^0 \ln \frac{S_k}{S_k^0} + E_k + \frac{(d_k^E + \delta_k)}{\delta_k} I_k ight. \\
+ \int_0^{\infty} \Gamma_k(\theta) \int_0^{\theta} \left( S_k(t-\theta) - S_k^0 - S_k^0 \ln \frac{S_k(t-\theta)}{S_k^0} \right) ds d\theta \\
+ \sum_{j=1}^{n} \beta_{kj} S_j^0 \int_0^{+\infty} f_j(\tau) \int_{t-\tau}^{t} g_j(I_j(s)) ds d\tau \right\}.$$ 

From the equation $\Lambda_k = (d_k^S + \xi_k) S_k^0 - \int_0^{\infty} \Gamma_k(\theta) S_k^0 d\theta$, we get the derivative of $U_1$ along the solution of (8)

$$\frac{dU_1}{dt} = \sum_{k=1}^{n} c_k \left\{ \frac{(d_k^S + \xi_k)(S_k - S_k^0)^2}{S_k} + \Gamma_k(S_k - S_k^0) + \frac{\Gamma_k S_k^0 (S_k^0 - S_k)}{S_k} \\
+ \int_0^{\infty} \Gamma_k(\theta) S_k^0 \left( 1 + \ln \frac{S_k(t-\theta)}{S_k} - \frac{S_k(t-\theta)}{S_k} \right) d\theta \\
+ \sum_{j=1}^{n} \beta_{kj} S_j^0 \int_0^{+\infty} f_j(\tau) g_j(I_j(t)) d\tau - \frac{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)}{\delta_k} I_k \right\}$$

$$= \sum_{k=1}^{n} c_k \left\{ \frac{(d_k^S + \xi_k d_k^S \Gamma_k S_k^0)(S_k - S_k^0)^2}{S_k} \\
+ \int_0^{\infty} \Gamma_k(\theta) S_k^0 \left( 1 + \ln \frac{S_k(t-\theta)}{S_k} - \frac{S_k(t-\theta)}{S_k} \right) d\theta \right\}$$

$$+ \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{\omega_k \delta_k \beta_{kj} S_j^0 a_j b_j (I_j)}{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)} I_j - \omega_k I_k \right)$$

$$\leq \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{\omega_k \delta_k \beta_{kj} S_j^0 a_j b_j}{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)} I_j - \omega_k I_k \right)$$
where \( Bv \) is the co-factor of the \( k \)-th diagonal entry of \( B \). It can be verified that the largest compact invariant set in \( \Omega \) is the singleton \( \{P_0\} \) is the singleton \( \{P_0\} \). By the LaSalle invariance principle for delay systems \([25, 27, 28]\), we obtain that the equilibrium \( P_0 \) of system (8) is globally asymptotically stable. This completes the proof of Theorem 3.2. \( \square \)

Next, we can prove that the endemic equilibrium \( P_* \) is globally asymptotically stable when it exists. The method is based on the graph approach and Lyapunov functionals \([9, 10, 11]\).

**Theorem 3.2.** Assume that \( B = (\beta_{kj})_{n \times n} \) is irreducible. If \( \mathcal{R}_0 > 1 \), and

\[
\left( g_j(I_j) - \frac{I_j}{I_j} \right) \left( 1 - g_j(I_j^*) \right) \leq 0, \text{ for } I_j > 0, \ j = 1, 2, \cdots, n, \quad (11)
\]

then the endemic equilibrium \( P_* \) of (8) is globally asymptotically stable when it exists.

**Proof.** For convenience of notations, define

\[
\overline{\beta}_{kj} = \beta_{kj}S_k^*a_jg_j(I_j^*), \ 1 \leq k, j \leq n,
\]

and

\[
\overline{B} = \begin{pmatrix}
\sum_{l \neq 1} \overline{\beta}_{1l} & \cdots & \overline{\beta}_{1n} \\
-\overline{\beta}_{12} & \cdots & \cdots & \overline{\beta}_{1n} \\
\vdots & \ddots & \ddots & \vdots \\
-\overline{\beta}_{1n} & \cdots & \cdots & \sum_{l \neq 1} \overline{\beta}_{nl}
\end{pmatrix}.
\]

\( \overline{B} \) is also irreducible. By Lemma 2.1 in \([9]\), the solution space of the linear system \( \overline{B}v = 0 \) has dimension 1 with a base

\[
(v_1, \cdots, v_n) = (c_{11}, \cdots, c_{nn}),
\]

where \( c_{kk} > 0 \) is the co-factor of the \( k \)-th diagonal entry of \( \overline{B} \). We construct the Lyapunov functional

\[
U_2 = \sum_{k=1}^{n} v_k \left\{ S_k(t) - S_k^* - S_k^* \ln \frac{S_k(t)}{S_k^*} + E_k(t) - E_k^* - E_k^* \ln \frac{E_k(t)}{E_k^*} + (dE_k^* + \delta_k) \left( I_k - I_k^* - I_k^* \ln \frac{I_k}{I_k^*} \right) \right.
\]

\[
\left. + \int_{0}^{\infty} \Gamma_k(\theta) \int_{0}^{\theta} \left( S_k(t-\tau) - S_k^* - S_k^* \ln \frac{S_k(t-\tau)}{S_k^*} \right) d\tau d\theta 
\]

\[
+ \sum_{j=1}^{n} \beta_{kj} S_k^* \int_{0}^{\infty} f_j(\tau) \int_{0}^{t} \left( g_j(I_j(s)) - g_j(I_j^*) \right) ds d\tau 
\]

\[
- g_j(I_j^*) \ln \frac{g_j(I_j(s))}{g_j(I_j^*)} \right\}.
\]
Computing the derivative of $U_2$ along the solution of model (8), we obtain

$$
\frac{dU_2}{dt} = \sum_{k=1}^{n} v_k \left\{ \left( 1 - \frac{S^*_k}{S_k} \right) \left( \Lambda_k - d^S_k S_k - \xi_k S_k + \int_{0}^{\infty} \Gamma_k(\theta) S_k(t - \theta) d\theta \right) \\
- \sum_{j=1}^{n} \beta_{kj} S_k(t) \left( \int_{0}^{\infty} f_j(\tau) g_j(I_j(t - \tau)) d\tau \right) \\
+ \left( 1 - \frac{E^*_k}{E_k} \right) \left( \sum_{j=1}^{n} \beta_{kj} S_k(t) \int_{0}^{\infty} f_j(\tau) g_j(I_j(t - \tau)) d\tau \right) \\
- (d^E_k + \delta_k) E_k \right\} + \frac{(d^E_k + \delta_k) I_k}{I_k} \sum_{j=1}^{n} \beta_{kj} S_k \left( \int_{0}^{\infty} f_j(\tau) g_j(I_j(t - \tau)) d\tau \right) \\
- \left( 1 - \frac{I^*_k}{I_k} \right) \left( \delta_k E_k - (d^I_k + \gamma_k + \epsilon_k) I_k \right) + \Gamma_k S_k - \int_{0}^{\infty} \Gamma_k(\theta) S_k(t - \theta) d\theta + \int_{0}^{\infty} \Gamma_k(\theta) S_k^* \ln \frac{S_k(t - \theta)}{S_k} d\theta \\
+ \sum_{j=1}^{n} \beta_{kj} S_k^* \left( \int_{0}^{\infty} f_j(\tau) \left[ g_j(I_j(t)) - g_j(I_j(t - \tau)) \right] d\tau \right) + g_j(I_j^*) \ln \left( \frac{g_j(I_j(t - \tau))}{g_j(I_j(t))} \right) \right\}.
$$

Using the equilibrium equations (10), we have

$$
\frac{dU_2}{dt} = \sum_{k=1}^{n} v_k \left\{ \left( 1 - \frac{S^*_k}{S_k} \right) \left( \Lambda_k - d^S_k S_k - \xi_k S_k \right) + \Gamma_k(S_k - S^*_k) \\
+ \int_{0}^{\infty} \Gamma_k(\theta) S_k^* \left( \frac{S_k(t - \theta)}{S_k} \right) d\theta + \left( 1 - \frac{S^*_k}{S_k} \right) \sum_{j=1}^{n} \beta_{kj} S_k^* a_j g_j(I_j^*) \\
+ (d^E_k + \delta_k) E_k \frac{I_k}{T_k} \sum_{j=1}^{n} \beta_{kj} S_k \left( \int_{0}^{\infty} f_j(\tau) g_j(I_j(t - \tau)) d\tau \right) \\
- \left( 1 - \frac{I^*_k}{I_k} \right) \frac{I_k}{T_k} \left( \delta_k E_k - (d^I_k + \gamma_k + \epsilon_k) I_k \right) \\
+ \sum_{j=1}^{n} \beta_{kj} S_k^* \left( \int_{0}^{\infty} f_j(\tau) g_j(I_j^*) \ln \left( \frac{g_j(I_j(t - \tau))}{g_j(I_j(t))} \right) \right) d\tau \right\} \\
= \sum_{k=1}^{n} v_k \left\{ \left( 1 - \frac{S^*_k}{S_k} \right) \left( \Lambda_k - d^S_k S_k - \xi_k S_k \right) + \Gamma_k(S_k - S^*_k) \\
+ \int_{0}^{\infty} \Gamma_k(\theta) S_k^* \left( \frac{S_k(t - \theta)}{S_k} - \frac{S_k(t - \theta)}{S_k} \right) d\theta \right\}
$$
\[
+ \sum_{k=1}^{n} \sum_{j=1}^{n} v_k \beta_{kj} S^*_k g_j(I^*_j) \int_0^\infty f_j(\tau) \left( 3 - \frac{S_k}{S^*_k} \frac{I_k}{I^*_k} - \frac{E_k I^*_k}{E^*_k I^*_k} \right) d\tau \\
+ \frac{g_j(I^*_j)}{g_j(I^*_j)} + \ln \frac{g_j(I_j(t-\tau))}{g_j(I^*_j)} - S_k E^*_k g_j(I_j(t-\tau)) \right) \frac{S^*_k}{E^*_k g_j(I^*_j)} d\tau \\
= \sum_{k=1}^{n} v_k \left\{ - \frac{(d^S_k + \xi_k d^V_k \Gamma_{ok})(S^*_k - S_k)^2}{S_k} \\
+ \int_0^\infty \Gamma_k(\theta) S^*_k \left( 1 + \ln \frac{S_k(t-\theta)}{S_k} - \frac{S_k(t-\theta)}{S_k} \right) d\theta \right\} \\
+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S^*_k g_j(I^*_j) \int_0^\infty f_j(\tau) \left[ \left( 1 + \ln \frac{S_k E^*_k g_j(I_j(t-\tau))}{S^*_k E_k g_j(I^*_j)} - \frac{S_k E^*_k g_j(I_j(t-\tau))}{S^*_k E_k g_j(I^*_j)} \right) \\
+ \left( 1 + \ln \frac{E_k I^*_k}{E_k I^*_k} \right) \frac{E_k I^*_k}{E_k I^*_k} \right] d\theta \\
+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \tilde{\beta}_{kj} \left[ \left( g_j(I^*_j) I_k - I^*_j \right) \left( 1 - \frac{g_j(I^*_j)}{g_j(I^*_j)} \right) \\
+ \frac{I_j}{I^*_j} - \frac{I_k}{I^*_k} + \ln \frac{g_j(I^*_j) I_k}{g_j(I^*_j) I^*_j} + \frac{g_j(I^*_j) I_k}{g_j(I^*_j) I^*_j} \right] \\
= \sum_{k=1}^{n} v_k \left\{ - \frac{(d^S_k + \xi_k d^V_k \Gamma_{ok})(S^*_k - S_k)^2}{S_k} \\
+ \int_0^\infty \Gamma_k(\theta) S^*_k \left( 1 + \ln \frac{S_k(t-\theta)}{S_k} - \frac{S_k(t-\theta)}{S_k} \right) d\theta \right\}
\]
\[ + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S_k^* g_j(I^*_j) \int_{0}^{\infty} f_j(\tau) \left[ 1 + \ln \frac{S_k}{S_k^*} - \frac{S_k}{S_k^*} + \left( 1 + \ln \frac{S_k E_k^* g_j(I_j(t - \tau))}{S_k E_k^* g_j(I^*_j)} \right) \right] d\tau \]

\[ + \left( 1 + \ln \frac{E_k^* I_k^*}{E_k^* I_k} - \frac{E_k^* I_k^*}{E_k^* I_k} \right) \]

\[ + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \left( \frac{g_j(I_j)}{g_j(I^*_j)} - \frac{I_j}{I^*_j} \right) \left( 1 - \frac{g_j(I^*_j)}{g_j(I_j)} \right) \]

\[ + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \left( 1 + \ln \frac{g_j(I^*_j) I_j}{g_j(I_j) I^*_j} \right) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \ln \frac{I_j I^*_j}{I^*_j I_k} \]

\[ \leq \sum_{k=1}^{n} \sum_{j=1}^{n} v_k \beta_{kj} \left( \frac{I_j}{I^*_j} - \frac{I_k}{I^*_k} \right) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \ln \frac{I_j I^*_j}{I^*_j I_k} \]
graph $Q$ can be formed when each arc of $CQ$ is added to a corresponding rooted tree $T$. It is not difficult to see that the double sum in $H_2$ can be interpreted as a sum over all the arcs in the cycles of all the unicyclic subgraphs $Q$ containing of $G$. Therefore, $H_2$ can be rewritten as

$$H_2 = \sum_Q H_{n,Q},$$

where

$$H_{n,Q} = w(Q) \cdot \sum_{(k,j) \in E(CQ)} \ln \frac{I_j I_k^*}{I_j^* I_k} = w(Q) \cdot \ln \left( \prod_{(k,j) \in E(CQ)} \frac{I_j I_k^*}{I_j^* I_k} \right).$$

Since $E(CQ)$ is the set of arcs of a cycle $CQ$, we have

$$\prod_{(k,j) \in E(CQ)} \frac{I_j I_k^*}{I_j^* I_k} = 1,$$

and thus

$$\ln \left( \prod_{(k,j) \in E(CQ)} \frac{I_j I_k^*}{I_j^* I_k} \right) = 0.$$

This implies that $H_{n,Q} = 0$ for each $Q$, and $H_2 \equiv 0$ for all $I_1, I_2, \cdots, I_n > 0$. From assumption (11) we have $U'_2 \leq 0$. It can be verified that the largest compact invariant subset of set \( \{(S_1, E_1, I_1(), \cdots, S_n, E_n, I_n()) \mid \frac{dU_2}{dt} = 0\} \) is the singleton \( \{P_*\} \). By the LaSalle invariance principle and an argument similar to that in the proof of Theorem 3.2 we know that the equilibrium $P_*$ of model (8) is globally asymptotically stable. \( \square \)

**Remark 1.** (i) It is easy to see that the functions listed in (5) satisfy condition (11). But the following three functions may be excluded by condition (11)

$$I_j e^{-\alpha_j I_j}, \quad \frac{I_j}{1 + \alpha_j I_j^2}, \quad \frac{I_j}{1 + \alpha_j I_j + \beta_j I_j^2}.$$  

(ii) Condition (11) holds if $g_j(0) = 0$, $g'_j(I_j) > 0$, and $g''_j(I_j) \leq 0$ ($I_j > 0$).

4. **Stochastic model.** The nature of epidemic growth and spread is inherently random due to the unpredictability of person-to-person contacts [29], and the population is subject to a continuous spectrum of disturbances [30]. The deterministic approach has some limitations in modelling the transmission of an infectious disease due to environmental noises. Stochastic differential equation (SDE) models have been applied to different infectious diseases in many circumstances [31, 32, 33, 34, 35, 36]. Motivated by [3], we take the randomly fluctuating environment into consideration by stochastic perturbations of white noise type. We study the following stochastic model,

$$\begin{cases}
S'_k(t) = \Lambda_k - d_k^S S_k - \sum_{j=1}^n \beta_{kj} S_k \int_0^\infty f_j(\tau) I_j(t - \tau) d\tau \\
\quad + \sigma_{1k}(S_k - S_k^*) \mathcal{B}_{1k}', \\
E'_k(t) = \sum_{j=1}^n \beta_{kj} S_k \int_0^\infty f_j(\tau) I_j(t - \tau) d\tau - (d_k^E + \delta_k) E_k \\
\quad + \sigma_{2k}(E_k - E_k^*) \mathcal{B}_{2k}', \\
I'_k(t) = \delta_k E_k - (d_k^I + \gamma_k + \epsilon_k) I_k + \sigma_{3k}(I_k - I_k^*) \mathcal{B}_{3k}',
\end{cases} \quad (12)$$
where $B_{1k}(t), B_{2k}(t)$ and $B_{3k}(t)$ ($k = 1, 2, \ldots, n$) are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t; \mathbb{P})$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-Null set). And $\sigma_{ik}^2 > 0$ ($i = 1, 2, 3$) represent the intensities of $B_{ik}$ ($i = 1, 2, 3$), respectively. If there are no noises, i.e. $\sigma_{ik} = 0$ ($i = 1, 2, 3$), then model (12) is

\begin{equation}
\begin{aligned}
S_k(t) &= \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_{0}^{\infty} f_j(\tau) I_j(t - \tau) d\tau,
E_k(t) &= \sum_{j=1}^{n} \beta_{kj} S_k \int_{0}^{\infty} f_j(\tau) I_j(t - \tau) d\tau - (d_k^E + \delta_k) E_k,
I_k(t) &= \delta_k E_k - (d_k^I + \gamma_k + \epsilon_k) I_k.
\end{aligned}
\end{equation}

The basic reproduction number of model (13) is

\[ R_1 = \rho(M_0), \quad M_0 = \left( \frac{\delta_{kj} \beta_{kj} S_{k0} a_j}{(d_k^E + \delta_k)(d_k^I + \gamma_k + \epsilon_k)} \right)_{1 \leq k, j \leq n}, \]

and $S_{k0} = \frac{\Lambda_k}{d_k} (k = 1, 2, \ldots, n)$. The similar arguments in [3] can lead to the following result.

**Theorem 4.1.** If $R_1 \leq 1$, then the disease-free equilibrium of model (13) is globally asymptotically stable. If $R_1 > 1$, then model (13) has an endemic equilibrium $\hat{P}_e$, which is globally asymptotically stable.

$\hat{P}_e$ is also an equilibrium of the stochastic model (12). We will focus on the stability of the equilibrium $\hat{P}_e$ by using Lyapunov functionals. We firstly give some definitions and auxiliary statements.

Consider the $n$-dimensional stochastic functional differential equation

\[ dX(t) = F(X_t, t) dt + G(X_t, t) dB(t), \quad X_0 = \phi(s) \in BC((-\infty, 0], \mathbb{R}^n), \]

where $X_t = X(t+s), s \leq 0$, $BC((-\infty, 0], \mathbb{R}^n)$ is the space of bounded and continuous functions from $(-\infty, 0]$ to $\mathbb{R}^n$ with the norm $||\phi|| = \sup_{s \leq 0} |\phi(s)|$. Suppose that the existence and uniqueness theorem holds, and (14) has a zero solution.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^n)$ be the family of all nonnegative functions $V(X, t)$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ such that they are continuously differentiable twice in $X$ and once in $t$. For a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^n)$, define the operator $\mathcal{L}$ by

\[ \mathcal{L}V(X, t) = V_t(X, t) + V_X(X, t) F(X, t) + \frac{1}{2} \text{trace}[G^T(X, t) \cdot V_{XX} \cdot G(X, t)], \]

where $T$ means the transposition.

**Definition 4.2.** [37] (1) The trivial solution of (14) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta > 0$ such that

\[ P\{|X(t; \phi)| < r, \forall t \geq 0\} \geq 1 - \varepsilon \text{ holds for all } ||\phi|| < \delta. \]

(2) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable, and for every $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that

\[ P\{\lim_{t \to \infty} |X(t; \phi)| = 0, \forall t \geq 0\} \geq 1 - \varepsilon \text{ holds for all } ||\phi|| < \delta. \]

If $R_1 > 1$, then the stochastic system (12) can be centered at its endemic equilibrium $\hat{P}_e(S^*_1, E^*_1, I^*_1, \ldots, S^*_n, E^*_n, I^*_n)$. The change of variables

\[ x_k = S_k - S^*_k, \quad y_k = E_k - E^*_k, \quad z_k = I_k - I^*_k, \]
Assume that \( B = (\beta_{kj})_{n \times n} \) is irreducible and \( R_1 > 1 \). If
\[
\begin{align*}
\sigma_{1k}^2 < 2d_k^S, & \quad \sigma_{2k}^2 < \frac{\sum_{j=1}^n \beta_{kj} S_j^* I_j^* a_j}{d_k^E E_k^* + \sum_{j=1}^n \beta_{kj} (S_k^* + E_k^*) a_j I_j^*}, \\
\sigma_{3k}^2 < \frac{\delta_k E_k^2 (\sum_{j=1}^n \beta_{kj} I_j^* a_j)}{I_k^* \left( d_k^S E_k^* + \sum_{j=1}^n \beta_{kj} (S_k^* + E_k^*) a_j I_j^* \right)},
\end{align*}
\]
then the endemic equilibrium \( \tilde{P}_* \) of (12) is stochastically asymptotically stable.

**Theorem 4.3.** Assume that \( B = (\beta_{kj})_{n \times n} \) is irreducible and \( R_1 > 1 \). If
\[
\begin{align*}
\sigma_{1k}^2 < 2d_k^S, & \quad \sigma_{2k}^2 < \frac{\sum_{j=1}^n \beta_{kj} S_j^* I_j^* a_j}{d_k^E E_k^* + \sum_{j=1}^n \beta_{kj} (S_k^* + E_k^*) a_j I_j^*}, \\
\sigma_{3k}^2 < \frac{\delta_k E_k^2 (\sum_{j=1}^n \beta_{kj} I_j^* a_j)}{I_k^* \left( d_k^S E_k^* + \sum_{j=1}^n \beta_{kj} (S_k^* + E_k^*) a_j I_j^* \right)},
\end{align*}
\]
then the endemic equilibrium \( \tilde{P}_* \) of (12) is stochastically asymptotically stable.

**Proof.** From the stability theory of stochastic functional differential equations, it is sufficient to find a Lyapunov functional \( V(X) \) such that \( LV(X) \leq 0 \) for sufficiently small \( \delta > 0 \) and the identity holds if and only if \( X = 0 \) (see [37]). The endemic equilibrium \( \tilde{P}_* \) satisfies following equations
\[
\begin{align*}
\Lambda_k &= d_k^S S_k^* + \sum_{j=1}^n \beta_{kj} S_k^* a_j I_j^*, \\
\sum_{j=1}^n \beta_{kj} S_k^* a_j I_j^* &= (d_k^E + \delta_k) E_k^*, \\
\delta_k E_k^* &= (d_k^E + \gamma_k + \epsilon_k) I_k^*.
\end{align*}
\]

We define
\[
V_1(X) = \frac{1}{2} \sum_{k=1}^n m_k y_k^2,
\]
where \( m_k (k = 1, 2, \ldots, n) \) are constants to be determined. Itô’s formula leads to
\[
LV_1 = \sum_{k=1}^n m_k y_k \left[ \sum_{j=1}^n \beta_{kj} a_j I_j^* x_k + \sum_{j=1}^n \beta_{kj} S_k^* \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau \right. \\
- (d_k^E + \delta_k) y_k + \sum_{j=1}^n \beta_{kj} x_k \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau \left. + \frac{1}{2} \sum_{k=1}^n m_k \sigma_{2k}^2 y_k^2 \right]
\]
\[
= \sum_{k=1}^n m_k \left[ \sum_{j=1}^n \beta_{kj} a_j I_j^* x_k + \sum_{j=1}^n \beta_{kj} S_k^* y_k \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau \\
+ \sum_{j=1}^n \beta_{kj} x_k y_k \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau - (d_k^E + \delta_k) y_k^2 + \frac{1}{2} \sigma_{2k}^2 y_k^2 \right]
\]

It is easy to see that the stability of \( \tilde{P}_* \) of (12) is equivalent to the stability of zero solution of system (18).
Similarly, from Itô’s formula, we obtain

\[
LV_2 = \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - (d_k^E + \delta_k) y_k \right] 
+ \frac{1}{2} \sum_{k=1}^{n} n_k \left( \sigma_{1k}^2 x_k^2 + \sigma_{2k}^2 y_k^2 \right) 
= \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - \left( d_k^E - \frac{1}{2} \sigma_{1k}^2 \right) x_k \right. 
\left. - \left( d_k^E + \delta_k - \frac{1}{2} \sigma_{2k}^2 \right) y_k^2 - \left( d_k^S + d_k^E + \delta_k \right) x_k y_k \right] 
\]

\[
\leq \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - \left( d_k^E - \frac{1}{2} \sigma_{1k}^2 \right) x_k \right. 
\left. - \left( d_k^E + \delta_k - \frac{1}{2} \sigma_{2k}^2 \right) y_k^2 - \left( d_k^S + d_k^E + \delta_k \right) x_k y_k \right] 
\]

Define

\[
V_2(\mathcal{X}) = \frac{1}{2} \sum_{k=1}^{n} n_k (x_k^2 + y_k^2) \quad \text{and} \quad V_3(\mathcal{X}) = \frac{1}{2} \sum_{k=1}^{n} l_k z_k^2.
\]

Similarly, from Itô’s formula, we obtain

\[
LV_2 = \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - (d_k^E + \delta_k) y_k \right] 
+ \frac{1}{2} \sum_{k=1}^{n} n_k \left( \sigma_{1k}^2 x_k^2 + \sigma_{2k}^2 y_k^2 \right) 
= \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - \left( d_k^E - \frac{1}{2} \sigma_{1k}^2 \right) x_k \right. 
\left. - \left( d_k^E + \delta_k - \frac{1}{2} \sigma_{2k}^2 \right) y_k^2 - \left( d_k^S + d_k^E + \delta_k \right) x_k y_k \right] 
\]

\[
\leq \sum_{k=1}^{n} n_k \left[ -d_k^S x_k - \left( d_k^E - \frac{1}{2} \sigma_{1k}^2 \right) x_k \right. 
\left. - \left( d_k^E + \delta_k - \frac{1}{2} \sigma_{2k}^2 \right) y_k^2 - \left( d_k^S + d_k^E + \delta_k \right) x_k y_k \right] 
\]

Define

\[
V_2(\mathcal{X}) = \frac{1}{2} \sum_{k=1}^{n} n_k (x_k^2 + y_k^2) \quad \text{and} \quad V_3(\mathcal{X}) = \frac{1}{2} \sum_{k=1}^{n} l_k z_k^2.
\]
Finally, we define

\[ V_4 = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} m_k E_k \int_{t}^{\infty} f_j(\tau) \left( \frac{z_j(s)}{I_j^*} \right)^2 \, ds \, d\tau, \]  

and

\begin{align*}
LV_3 &= \sum_{k=1}^{n} \left[ \delta_k y_k - (d_k^S + \gamma_k + \epsilon_k) z_k \right] z_k + \frac{1}{2} \sum_{k=1}^{n} l_k \sigma_{2k}^2 z_k^2 \\
&= \sum_{k=1}^{n} \left[ -\left( \frac{d_k^S}{I_k^*} - \frac{1}{2} \sigma_{2k}^2 \right) z_k + \frac{1}{2} \delta_k E_k^* I_k^* \right] z_k^2 + \frac{1}{2} \delta_k E_k^* I_k^* \left( \frac{y_k}{E_k^*} \right)^2 \tag{23}
\end{align*}

From (21), (22) and (23) we have

\begin{align*}
LV_1 + LV_2 + LV_3 &\leq \sum_{k=1}^{n} \left\{ -n_k \left( d_k^S - \frac{1}{2} \sigma_{1k}^2 \right) x_k^2 \\
&- \left[ \frac{1}{2} m_k \left( E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j - E_k^* E_k^* \sigma_{2k}^2 \right) - \frac{1}{2} l_k \delta_k E_k^* I_k^* \\
&+ \frac{1}{2} n_k \left( 2 E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j - E_k^* E_k^* \sigma_{2k}^2 \right) \right] \left( \frac{y_k}{E_k^*} \right)^2 \\
&- \left( \frac{1}{2} l_k \delta_k E_k^* I_k^* - l_k \frac{1}{2} I_k^* \sigma_{3k}^2 \right) \left( \frac{z_k}{I_k^*} \right)^2 \right\} \tag{24}
\end{align*}

Finally, we define

\[ V_4 = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} m_k E_k \int_{t}^{\infty} f_j(\tau) \int_{t}^{\infty} \left( \frac{z_j(s)}{I_j^*} \right)^2 ds \, d\tau, \tag{25} \]
which gives
\[
LV_4 = \frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \left( \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \right) \left( \frac{z_j}{I_j} \right)^2 - \frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \int_0^\infty f_j(\tau) \left( \frac{z_j(t-\tau)}{I_j} \right)^2 d\tau.
\] (26)

Let \( m_k = \frac{c_{kk}}{E_k^*} \), where \( c_{kk} \) has the similar definition as that given in Section 3. Then we have \( c_{kk} = m_k E_k^* \), and
\[
\frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \left( \frac{z_k}{I_k^*} \right)^2 = \frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \left( \frac{z_k}{I_k^*} \right)^2,
\]
which gives
\[
LV_4 = \frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \left( \frac{z_k}{I_k^*} \right)^2 - \frac{1}{2} \sum_{k=1}^{n} m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \int_0^\infty f_j(\tau) \left( \frac{z_j(t-\tau)}{I_j} \right)^2 d\tau.
\] (27)

Let \( V = V_1 + V_2 + V_3 + V_4 \), from above analysis we obtain
\[
LV = LV_1 + LV_2 + LV_3 + LV_4
\]
\[
\leq \sum_{k=1}^{n} \left\{ -n_k \left( d_k^S - \frac{1}{2} \sigma_{1k}^2 \right) x_k^2 - \frac{1}{2} \left[ m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j + 2m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j - l_k \delta_k E_k^* I_k^* - (m_k + n_k) E_k^* \sigma_2^2 \right] \left( \frac{y_k}{E_k^*} \right)^2 - \frac{1}{2} \left[ l_k \delta_k E_k^* I_k^* - m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j - l_k I_k^* \sigma_3^2 \right] \left( \frac{z_k}{I_k^*} \right)^2 \right. \right. \\
\left. \left. + \left[ m_k \sum_{j=1}^{n} \beta_{kj} I_j^* a_j - n_k \left( d_k^S + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \right) \right] x_k y_k \\
+ m_k \sum_{j=1}^{n} \beta_{kj} x_k y_k \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau \right\} \\
=: L_0 V + \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} x_k y_k \int_0^\infty f_j(\tau) z_j(t-\tau) d\tau,
\] (28)
In (28) we choose $n_k$ such that

$$m_k \sum_{j=1}^{n} \beta_{kj} I_j^* a_j - n_k \left( d_k^S + \sum_{j=1}^{n} \beta_{kj} \frac{S_k^*}{E_k} I_j^* a_j \right) = 0.$$  

Then

$$n_k = \frac{m_k \sum_{j=1}^{n} \beta_{kj} I_j^* a_j}{d_k^S + \sum_{j=1}^{n} \beta_{kj} \frac{S_k^*}{E_k} I_j^* a_j}. \quad (30)$$

Let

$$l_k = \frac{(m_k + n_k) \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j}{\delta_k I_k^*} = \frac{m_k \left( d_k^S + \sum_{j=1}^{n} \beta_{kj} \frac{S_k^* + E_k^*}{E_k} I_j^* a_j \right) \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j}{\delta_k I_k^* \left( d_k^S + \sum_{j=1}^{n} \beta_{kj} \frac{S_k^*}{E_k} I_j^* a_j \right)}. \quad (31)$$

Substituting (30) and (31) into (29) yields

$$L_0 V = \sum_{k=1}^{n} \left\{ - n_k \left[ d_k^S - \frac{1}{2} \sigma_{1k}^2 \right] x_k^2 \right. $$

$$- \frac{1}{2} \left[ m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j + 2 n_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j \right] \left( \frac{y_k}{E_k^*} \right)^2 $$

$$- \frac{1}{2} \left[ l_k \delta_k E_k^* I_k^* - (m_k + n_k) E_k^* \sigma_{2k}^2 \right] \left( \frac{z_k}{I_k^*} \right)^2 $$

$$+ \left[ l_k \delta_k E_k^* I_k^* - m_k E_k^* \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* a_j - l_k I_k^* \sigma_{3k}^2 \right] \left( \frac{z_k}{I_k^*} \right) $$

$$+ \left[ m_k \sum_{j=1}^{n} \beta_{kj} I_j^* a_j - n_k \left( d_k^S + \sum_{j=1}^{n} \beta_{kj} \frac{S_k^*}{E_k} I_j^* a_j \right) \right] x_k y_k \}.$$
\[
-m_k E_k^2 \sigma_{2k}^2 \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} (S_k^* + E_k^*) I_j a_j \right) \left( y_k^* E_k^* \right)^2 \\
- \frac{1}{2} \left[ m_k E_k^2 \sum_{j=1}^{n} \beta_{kj} I_j a_j \right] \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) \\
- \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} (S_k^* + E_k^*) I_j a_j \right) \delta_k \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) \\
\times \left( \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) m_k I_k \sigma_{2k}^2 \left( \frac{\delta_k}{I_k^2} \right)^2 \\
= - \sum_{k=1}^{n} \left( A_k x_k^2 + B_k y_k^2 + D_k z_k^2 \right), \quad (32)
\]

where

\[
A_k = n_k \left[ d_k^S - \frac{1}{2} \sigma_{ik}^2 \right], \quad B_k = \frac{1}{2} \left[ m_k E_k^2 \left( \sum_{j=1}^{n} \beta_{kj} I_j a_j \right) \sum_{j=1}^{n} \beta_{kj} I_j a_j \right] \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) \\
- \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} (S_k^* + E_k^*) I_j a_j \right) \delta_k \left( d_k^S E_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) \\
\times \left( \sum_{j=1}^{n} \beta_{kj} S_k^* I_j a_j \right) m_k I_k \sigma_{2k}^2 \left( \frac{\delta_k}{I_k^2} \right)^2 .
\]

From (19), we have \( A_k > 0, B_k > 0 \) and \( D_k > 0 \). Consequently

\[
LV \leq - \sum_{k=1}^{n} \left( A_k x_k^2 + B_k y_k^2 + D_k z_k^2 \right) \\
+ \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} x_k y_k \int_{0}^{\infty} f_j(\tau) z_j(t - \tau) d\tau.
\]

Assume \( \mathcal{P}\{ |z_j(s)| \leq \rho \} = 1 \) (\( \rho > 0, j = 1, 2, \ldots, n \)). Then

\[
\sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} x_k y_k \int_{0}^{\infty} f_j(\tau) z_j(t - \tau) d\tau \\
\leq \rho \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} |x_k y_k| \leq \frac{1}{2} \rho \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} a_j \left( x_k^2 + y_k^2 \right).
\]
Therefore,

\[ LV \leq - \sum_{k=1}^{n} \left\{ \left[ A_k - \frac{1}{2} \rho \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} a_j \right] x_k^2 + \left[ B_k - \frac{1}{2} \rho \sum_{k=1}^{n} m_k \sum_{j=1}^{n} \beta_{kj} a_j \right] y_k^2 + D_k z_k^2 \right\}. \]

Consequently, for sufficiently small \( \rho > 0 \), we have \( LV < 0 \) except the zero point. The conclusion follows immediately. This completes the proof of Theorem 4.3. \( \square \)

**Remark 2.** By comparing Theorem 4.1 with Theorem 4.3, we can see that if the positive equilibrium of the deterministic model (13) is stable, then the stochastic system (12) will keep this nice property provided the noise is sufficiently small.

**Numerical examples.** Numerical examples are given by Matlab to demonstrate the dynamical behavior of stochastic model (12). The main idea in simulations is the Milstein method [38] by discretizing the differential equations. We do simulations for \( k = 1, 2 \) and \( f_j(\tau) = e^{-\tau}, \tau \geq 0 \). The initial condition is \( I_j(\theta) = v_j e^\theta, \theta \leq 0 \), where \( v_j > 0 \). For \( k = 1, 2 \) and \( f_j(\tau) = e^{-\tau}, \tau \geq 0 \), model (12) is

\[
\begin{align*}
S_k &= \left[ \Lambda_k - \frac{1}{2} e^{-t} \sum_{j=1}^{2} \beta_{kj} S_k v_j - e^{-t} \sum_{j=1}^{2} \beta_{kj} S_k \int_{0}^{t} e^\tau I_j(\tau) d\tau \right] dt - d_k^S S_k + \sigma_{1k}(S_k - S_k^*) dB_{1k}, \\
E_k &= \left[ \frac{1}{2} e^{-t} \sum_{j=1}^{2} \beta_{kj} S_k v_j + e^{-t} \sum_{j=1}^{2} \beta_{kj} S_k \int_{0}^{t} e^\tau I_j(\tau) d\tau \right] dt - (d_k^E + \delta_k) E_k + \sigma_{2k}(E_k - E_k^*) dB_{2k}, \\
I_k &= \left[ \delta_k E_k - (d_k^I + \gamma_k + \epsilon_k) I_k \right] dt + \sigma_{3k}(I_k - I_k^*) dB_{3k}.
\end{align*}
\]

The discretization of model (33) is

\[
\begin{align*}
S_k^{(i+1)} - S_k^{(i)} &= \left[ \Lambda_k - d_k^S S_k^{(i)} S_k^{(i)} - e^{-i\Delta t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} \sum_{l=1}^{i} e^{l\Delta t} I_j^{(i)}(\tau) \Delta t \right. \\
&\quad \left. - \frac{1}{2} e^{-i\Delta t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} v_j \right] \Delta t + \sigma_{1k}(S_k^{(i)} - S_k^*) \sqrt{\Delta t} \xi_{1k}^{(i)} \\
&\quad + \frac{1}{2} \sigma_{1k}^2 S_k^{(i)}(S_k^{(i)} - S_k^*) \left[ (\xi_{1k}^{(i)})^2 - 1 \right] \Delta t, \\
E_k^{(i+1)} - E_k^{(i)} &= \left[ e^{-i\Delta t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} \sum_{l=1}^{i} e^{l\Delta t} I_j^{(i)}(\tau) \Delta t + \frac{1}{2} e^{-i\Delta t} \sum_{j=1}^{2} \beta_{kj} S_k^{(i)} v_j \right] \\
&\quad - (d_k^E + \delta_k) E_k^{(i)} \Delta t + \sigma_{2k}(E_k^{(i)} - E_k^*) \sqrt{\Delta t} \xi_{2k}^{(i)} \\
&\quad + \frac{1}{2} \sigma_{2k}^2 E_k^{(i)}(E_k^{(i)} - E_k^*) \left[ (\xi_{2k}^{(i)})^2 - 1 \right] \Delta t, \\
I_k^{(i+1)} - I_k^{(i)} &= \left[ \delta_k E_k^{(i)} - (d_k^I + \gamma_k + \epsilon_k) I_k^{(i)} \right] \Delta t + \sigma_{3k}(I_k^{(i)} - I_k^*) \sqrt{\Delta t} \xi_{3k}^{(i)} \\
&\quad + \frac{1}{2} \sigma_{3k}^2 I_k^{(i)}(I_k^{(i)} - I_k^*) \left[ (\xi_{3k}^{(i)})^2 - 1 \right] \Delta t.
\end{align*}
\]
where $\xi_{1k}, \xi_{2k}, \xi_{3k}$ ($k = 1, 2$) are independent random variables with normal distribution $\mathcal{N}(0, 1)$, which can be generated numerically by pseudo-random number generators. For simplicity, we always assume that $d_k = d_k^s = d_k^r = d_k$ and parameters take following values:

$$
\begin{align*}
\Lambda_1 &= 10, \quad \beta_{11} = 0.0135, \quad \beta_{12} = 0.1, \quad d_1 = 0.1, \quad \delta_1 = 0.15, \quad \epsilon_1 = 0.05, \quad \gamma_1 = 0.8, \\
\Lambda_2 &= 15, \quad \beta_{21} = 0.0899, \quad \beta_{22} = 0.24, \quad d_2 = 0.15, \quad \delta_2 = 0.15, \quad \epsilon_2 = 0.05, \quad \gamma_2 = 0.75, \\
v_1 &= 5, \quad v_2 = 10, \quad S_1(0) = 50, \quad E_1(0) = 30, \quad S_2(0) = 20, \quad E_2(0) = 5.
\end{align*}
$$

The endemic equilibrium $\tilde{P}_e$ of model (33) is given by $S_1^* = 10.9077, E_1^* = 35.6369, I_1^* = 5.6269, S_2^* = 6.1632, E_2^* = 46.9184, I_2^* = 7.4082$. The basic reproduction number is $R_1 = 14.1371 > 1$. For those parameters, condition (19) of Theorem 4.3 become

$$
\begin{align*}
\sigma_{11}^2 < 0.2000, \quad \sigma_{21}^2 < 0.1750, \quad \sigma_{31}^2 < 0.6650, \\
\sigma_{12}^2 < 0.3000, \quad \sigma_{22}^2 < 0.2506, \quad \sigma_{32}^2 < 0.7936.
\end{align*}
$$

In the absence of noise, i.e. $\sigma_{ik} = 0, i = 1, 2, 3; k = 1, 2$, the global stability of the endemic equilibrium corresponding deterministic model (33) is shown in Figure 1.

In Figure 2 $\sigma_{ij}$ ($i = 1, 2, 3, j = 1, 2$) are taken to be $\sigma_{11} = 0.05, \sigma_{21} = 0.1, \sigma_{31} = 0.15, \sigma_{12} = 0.1, \sigma_{22} = 0.12, \sigma_{32} = 0.15$, satisfying the conditions of Theorem 4.3. The numerical simulation shows the similar stability result to Theorem 4.3. We can see that the stochastic model (33) preserves the stability property of the corresponding deterministic model for small noises. Numerical results for $\sigma_{11} = 0.25, \sigma_{21} = 0.35, \sigma_{31} = 0.5, \sigma_{12} = 0.3, \sigma_{22} = 0.45, \sigma_{32} = 0.55$, satisfying conditions of (34), are given in Figure 3. The endemic equilibrium of model (33) is also asymptotically stable (see Figure 3). The comparison of Figures 2 and 3 shows that the solutions of stochastic model (33) fluctuate at the beginning, and converge to the equilibrium position finally. The fluctuations of the stochastic model (33) may enhance with the increasing noises.

5. Conclusions. We have studied a multi-group SVEIR epidemic model with delays and vaccination age. The global stability of model (8) is established by Lyapunov functionals. The disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and the unique endemic equilibrium is globally asymptotically stable if $R_0 > 1$. We can define $\widetilde{R}_0 = \rho \left( \frac{\beta_{1k} S_0^u a_j}{d_k^s (d_k^l + \gamma_k + \epsilon_k)} \right)_{1 \leq k, j \leq n} = \lim_{\delta_k \to +\infty} R_0$ to investigate the influence of the latent period on $R_0$. It is obvious that $\widetilde{R}_0 > R_0$ and $\frac{\partial \widetilde{R}_0}{\partial \delta_k} > 0$, which implies that latent period has a positive role in disease control: a long latent period may lead to the extinction of the disease. Similarly, the fact that

$$
\begin{align*}
\frac{\partial \widetilde{R}_0}{\partial \xi_k} &= \rho \left( - \frac{\Lambda_k \delta_k \beta_{1k} \beta_{2k} \epsilon_j}{d_k^s (d_k^l + \delta_k)(d_k^l + \gamma_k + \epsilon_k)} \right)_{1 \leq k, j \leq n} := \widetilde{R}_0^* > R_0,
\end{align*}
$$

implies that the vaccination is very helpful to eradicate the disease. Though the immunity of a vaccine may not be permanent, a long immunity period of vaccines is still expected for diseases prevention.
We have investigated a multi-group stochastic model (12) with white noise perturbations. We obtained sufficient conditions for stochastic stability of model (12). The results reveal that the stochastic stability of the endemic equilibrium depends on the magnitude of the noise. Numerical simulations show that the stochastic
model preserves the stability property of the corresponding deterministic model for small noise.

Other factors, such as population migration, can be integrated into the model to make it more realistic. The efficacy of some vaccines may not be 100% though it is assumed that the vaccinated individuals can not be infected. It is more reasonable to assume that the vaccinated individuals can be infected at a reduced rate [15]. Furthermore, we just suppose that the stochastic perturbations are proportional to $S_k - S_k^*$, $E_k - E_k^*$ and $I_k - I_k^*$. It is also interesting to study other types of stochastic perturbations [33] and consider a stochastic system with nonlinear incidence.

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**REFERENCES**

[1] K. L. Cooke, Stability analysis for a vector disease model, *Rocky Mount. J. Math.*, 9 (1979), 31–42.

[2] E. Beretta and Y. Takeuchi, Global stability of an SIR epidemic model with time delays, *J. Math. Biol.*, 33 (1995), 250–260.

[3] H. Y. Shu, D. J. Fan and J. J. Wei, Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission, *Nonlinear Anal.: Real World Appl.*, 13 (2012), 1581–1592.

[4] A. Lajmanovich and J. A. York, A deterministic model for gonorrhea in a nonhomogeneous population, *Math. Biosci.*, 28 (1976), 221–236.

[5] D. Q. Ding and X. H. Ding, Global stability of multi-group vaccination epidemic models with delays, *Nonlinear Anal.: Real World Appl.*, 12 (2011), 1991–1997.

[6] R. Y. Sun and J. P. Shi, Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates, *Appl. Math. Comput.*, 218 (2011), 280–286.

[7] H. Chen and J. T. Sun, Global stability of delay multigroup epidemic models with group mixing and nonlinear incidence rates, *Appl. Math. Comput.*, 218 (2011), 4391–4400.
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[8] T. Kuniya, Global stability of a multi-group SVIR epidemic model, *Nonlinear Anal.: Real World Appl.*, 14 (2013), 1139–1153.

[9] H. Guo, M. Y. Li and Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, *Canad. Appl. Math. Quart.*, 14 (2006), 259–284.

[10] H. Guo, M. Y. Li and Z. Shuai, A graph-theoretic approach to the method of global Lyapunov functions, *Proc. Amer. Math. Soc.*, 136 (2008), 2793–2802.

[11] M. Y. Li and Z. Shuai, Global-stability problem for coupled systems of differential equations on networks, *J. Differential Equations*, 248 (2010), 1–20.

[12] Z. Shuai and P. van den Driessche, Impact of heterogeneity on the dynamics of an SEIR epidemic model, *Math. Biosci. Eng.*, 9 (2012), 399–411.

[13] J. Q. Li, Y. L. Yang and Y. C. Zhou, Global stability of an epidemic model with latent stage and vaccination, *Nonlinear Anal.: Real World Appl.*, 12 (2011), 2163–2173.

[14] S. M. Blower and A. R. McLean, Prophylactic vaccines, risk behavior change, and the probability of eradicating HIV in San Francisco, *Science*, 265 (1994), 1415–1454.

[15] Y. Xiao and S. Tang, Dynamics of infection with nonlinear incidence in a simple vaccination model, *Nonlinear Anal.: Real World Appl.*, 11 (2010), 4154–4163.

[16] X. Y. Song, Y. Jiang and H. M. Wei, Analysis of a saturation incidence SVEIRS epidemic model with pulse and two time delays, *Appl. Math. Comput.*, 214 (2009), 381–390.

[17] G. P. Sahu and J. Dhar, Analysis of an SVEIS epidemic model with partial temporary immunity and saturation incidence rate, *Appl. Math. Model.*, 36 (2012), 908–923.

[18] M. Iannelli, M. Martcheva and X. Z. Li, Strain replacement in an epidemic model with super-infection and perfect vaccination, *Math. Biosci.*, 195 (2005), 23–46.

[19] X. Z. Li, J. Wang and M. Ghosh, Stability and bifurcation of an SIVS epidemic model with treatment and age of vaccination, *Appl. Math. Model.*, 34 (2010), 437–450.

[20] X. C. Duan, S. L. Yuan and X. Z. Li, Global stability of an SVIR model with age of vaccination, *Appl. Math. Comput.*, 226 (2014), 528–540.

[21] F. Hoppensteadt, An age-dependent epidemic model, *J. Franklin Inst.*, 297 (1974), 325–333.

[22] F. Hoppensteadt, *Mathematical Theories of Populations: Demographics, Genetics and Epidemics*, Philadelphia: Society for industrial and applied mathematics, 1975.

[23] R. K. Miller, *Nonlinear Volterra Integral Equations*, W. A. Benjamin, New York, 1971.

[24] F. V. Atkinson and J. R. Haddock, On determining phase spaces for functional differential equations, *Funkcial. Ekvac.*, 31 (1988), 331–347.

[25] J. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, in: *Applied Mathematical Sciences*, Springer-Verlag, New York, 1993.

[26] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, On the definition and the computation of the basic reproduction ratio $R_0$ in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, 28 (1990), 365–382.

[27] J. R. Haddock and J. Terjeki, Liapunov-Razumikhin functions and an invariance principle for functional-differential equations, *J. Differential Equations*, 48 (1983), 95–122.

[28] J. R. Haddock, T. Krisztin and J. Terjeki, Invariance principles for autonomous functional-differential equations, *J. Integral Equations, 10* (1985), 125–136.

[29] S. Spencer, *Stochastic Epidemic Models for Emerging Diseases*, Ph.D. thesis, University of Nottingham, 2008.

[30] J. R. Beddington and R. M. May, Harvesting natural populations in a randomly fluctuating environment, *Science*, 197 (1977), 463–465.

[31] X. R. Mao, G. Marion and E. Renshaw, Environmental noise suppresses explosion in population dynamics, *Stoch Process Appl.*, 97 (2002), 95–110.

[32] N. Dalal, D. Greenhalgh and X. R. Mao, A stochastic model for internal HIV dynamics, *J Math Anal Appl.*, 341 (2008), 1084–1101.

[33] C. Ji, D. Jiang and N. Shi, Multigroup SIR epidemic model with stochastic perturbation, *Phys A: Stat Mech Appl.*, 390 (2011), 1747–1762.

[34] P. S. Mandal, S. Abbas and M. Banerjee, A comparative study of deterministic and stochastic dynamics for a non-autonomous allelopathic phytoplankton model, *Appl. Math. Comput.*, 238 (2014), 300–318.

[35] M. Liu, C. Bai and K. Wang, Asymptotic stability of a two-group stochastic SEIR model with infinite delays, *Commun Nonlinear Sci Numer Simulat.*, 19 (2014), 3444–3453.

[36] Q. S. Yang and X. R. Mao, Stochastic dynamic of SIRS epidemic models with random perturbation, *Math. Biosci. Eng.*, 11 (2014), 1003–1025.
[37] X. R. Mao, *Stochastic Differential Equations and Their Applications*, Chichester: Horwood publishing, 1997.

[38] D. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, 43 (2001), 525–546.

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