THEORETICAL PROPERTIES OF FRactal DIMENSIONS FOR FRactal STRUCTURES

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Abstract. Hausdorff dimension, which is the oldest and also the most accurate model for fractal dimension, constitutes the main reference for any fractal dimension definition that could be provided. In fact, its definition is quite general, and is based on a measure, which makes the Hausdorff model pretty desirable from a theoretical point of view. On the other hand, it turns out that fractal structures provide a perfect context where a new definition of fractal dimension could be proposed. Further, it has been already shown that both Hausdorff and box dimensions can be generalized by some definitions of fractal dimension formulated in terms of fractal structures. Given this, and being mirrored in some of the properties satisfied by Hausdorff dimension, in this paper we explore which ones are satisfied by the fractal dimension definitions for a fractal structure, that are explored along this work.

1. Introduction. The analysis of fractal patterns have growth increasingly during the last years, mainly due to the wide range of applications to diverse scientific areas where fractals have been explored, including physics, statistics, and economics (see, e.g., [9, 11]). It is also worth mentioning that there has also been a special interest for applying fractals to social sciences (see for example [7] and references therein).

It turns out that the key tool to study the complexity of a given system is the fractal dimension, since this is its main invariant which throws quite useful information about the complexity that it presents when being examined with enough level of detail.

We would like also to point out that fractal dimension is usually understood as the classical box dimension, mainly in the field of empirical applications. In fact, its popularity is due to the possibility of its effective calculation and empirical estimation. On the other hand, the Hausdorff dimension also constitutes a powerful analytical model which allows to “measure” the complexity of a system, at least from a theoretical point of view. Nevertheless, though they are defined for any metric (resp. metrizable) space, almost all the empirical applications of fractal dimension are tackled in the context of Euclidean spaces. In addition to that, recall that box dimension is more useful for practical applications, whereas Hausdorff dimension presents “better” analytical properties, due to the fact that its standard...
definition is based on a measure. Indeed, though Hausdorff dimension becomes the most accurate model for fractal dimension, since its definition is quite general, it can result difficult or even impossible to calculate in practical applications.

It is worth mentioning that the application of fractal structures allows to provide new models for a fractal dimension definition on any generalized-fractal space, and not only on the Euclidean ones. This extends the classical theory of fractal dimension to the more general context of fractal structures. In this way, some theoretical results have been shown to generalize the classical models in the context of fractal structures (see, e.g., [13, Theorem 4.15] and [16, Theorem 3.12]). Moreover, we would like to point out that some fractal dimensions for a fractal structure have been already successfully applied in non-Euclidean contexts, where the box dimension cannot be applied (see, e.g., [14, 15]).

Accordingly, when providing a new model to calculate the fractal dimension, it would be desirable that the contributed definition allows to calculate the fractal dimension for a given subset as easy as the box dimension models, though one should be also mirrored in the analytical properties satisfied by the Hausdorff dimension.

Thus, the main goal in this paper is to verify which kind of properties are satisfied by some provided models to calculate the fractal dimension for a fractal structure. To deal with, [9, Chapter 3] provides a useful guide, including a list containing some theoretical properties that should be checked by any new definition of fractal dimension to test its behavior. In particular, we will focus on the following properties: monotonicity (√), finite stability (also F-stability, for short) (√), countable stability (C-stability) (√), 0-dimensionality for countable sets (denoted as 0-countably) (√), and the so-called cl-dim property, which consists of satisfying the following equality: \( \dim(F) = \dim(\overline{F}) \), where \( \overline{F} \) denotes the closure of any subset \( F \) of a given space \( X \), in the classical sense, and \( \dim \) denotes a generic fractal dimension model. Observe that the notation (√) refers the fact that the Hausdorff dimension, which becomes our main theoretical reference model for a fractal dimension, satisfy such properties, which will be technically defined in forthcoming Subsection 2.2.

Following the above, the structure of this paper remains as follows. Section 2 contains some preliminary topics that will be useful along this paper. They include a brief description regarding the classical models for fractal dimension, namely, the basics on both the box and the Hausdorff dimensions, as well as a technical description for the theoretical properties that will be verified afterwards for our models of fractal dimension for a fractal structure. Further, we devote a subsection therein for the basics on fractal structures. In addition to that, forthcoming Section 3 contains a detailed study about the properties that are satisfied by the fractal dimensions for a fractal structure that we explore in this paper. Table 7 provides a schematic summary regarding the overall results. As usual, the absence of a √ sign has been properly justified by an appropriate counterexample.

2. Preliminaries.

2.1. Classical models for fractal dimension. Fractal dimension consists of a single quantity which yields valuable information about the complexity that a given space presents, provided that it is explored with enough level of detail. Next, we recall the definition of the standard box dimension, which is mainly used in empirical applications of fractal dimension due to the easiness of its empirical estimation. As [9, Subsection 3.6] points out, its origins become hard to trace, though it seems that it would have been considered firstly by the Hausdorff dimension pioneers,
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who rejected it due to its lack of theoretical properties. Anyway, the standard definition for box dimension that we recall next, was firstly contributed in [19].

**Definition 2.1.** The (lower/upper) box dimension for any subset $F \subseteq \mathbb{R}^d$ is given as the following (lower/upper) limit:

$$
\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},
$$

where $N_\delta(F)$ is the number of $\delta = 1/2^n$-cubes that intersect $F$, and $n \in \mathbb{N}$. Recall that a $\delta$-cube in $\mathbb{R}^d$ is a set of the form $[k_1 \delta, (k_1 + 1) \delta] \times \ldots \times [k_d \delta, (k_d + 1) \delta]$, with $k_1, \ldots, k_d \in \mathbb{Z}$.

Some alternatives for the box dimension that we have introduced in Definition 2.1 could be consulted in [9, Equivalent definitions 3.1], where equivalent ways to calculate $N_\delta(F)$ are provided. It is worth mentioning that in both [12, Theorem 3.5] and [10, Equivalent definitions 2.1], the equivalence among such alternative approaches to box dimension are shown.

On the other hand, in 1919, Hausdorff used a method developed by Carathéodory some years earlier [8] in order to define the measures that now bear his name, and showed that the middle third Cantor set has positive and finite measure of dimension equal to $\log 2/\log 3$ [18]. A detailed study regarding the analytical properties of both Hausdorff measure and dimension was mainly developed by Besicovitch and his pupils during the XXth century (see, e.g., [5, 6]).

Along this paper, we will define the diameter of a given subset $A$ of any metric space $(X, \rho)$, as usual, by $\text{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$. Next, let us recall the analytical construction of the Hausdorff dimension. Thus, let $(X, \rho)$ be a metric space, and let $\delta$ be a positive real number. For any subset $F$ of $X$, recall that a $\delta$-cover of $F$ is just a countable family of subsets $\{U_i\}_{i \in I}$ such that $F \subseteq \bigcup_{i \in I} U_i$, where $\text{diam}(U_i) \leq \delta$, for all $i \in I$. Hence, let us denote by $C_\delta(F)$ the collection of all $\delta$-covers of $F$. Moreover, let us consider the following quantity:

$$
H^s_\delta(F) = \inf \left\{ \sum_{i \in I} \text{diam}(U_i)^s : \{U_i\}_{i \in I} \in C_\delta(F) \right\}.
$$

We would like also to point out that the next limit always exists:

$$
\mathcal{H}^s_\delta(F) = \lim_{\delta \to 0} H^s_\delta(F),
$$

which is called the $s$-dimensional Hausdorff measure of $F$. Hence, the Hausdorff dimension of $F$ is fully determined as the point $s$ where $\mathcal{H}^s_\delta(F)$ “jumps” from $\infty$ to 0, namely,

$$
\dim_H(F) = \inf\{s : \mathcal{H}^s_\delta(F) = 0\} = \sup\{s : \mathcal{H}^s_\delta(F) = \infty\}.
$$

2.2. **Theoretical properties for a fractal dimension definition.** Next theorem, which can be found along [9, Section 2.2], contains some analytical properties that are satisfied by our key reference for a fractal dimension definition, namely, the Hausdorff dimension. These properties will be used in forthcoming Subsections 3.1-3.3 for comparative purposes regarding our models of fractal dimension for a fractal structure.

**Theorem 2.2.** (1) Monotonicity: if $E \subseteq F$, then $\dim_H(E) \leq \dim_H(F)$.
(2) Finite stability: $\dim_H(E \cup F) = \max\{\dim_H(E), \dim_H(F)\}$. 
(3) Countable stability: if \( \{ F_i \}_{i \in I} \) is a countable collection of sets, then
\[
\dim_H \left( \bigcup_{i \in I} F_i \right) = \sup \{ \dim_H(F_i) : i \in I \}.
\]

(4) Countable sets: if \( F \) is a countable set, then \( \dim_H(F) = 0 \).

(5) In general, it is not satisfied that \( \dim_H(F) = \dim_H(\overline{F}) \).

However, the latter property (or its opposite, as well), which we will refer to as cl-dim, herein, is not desired (at least at a first glance) to be satisfied by a fractal dimension definition. The key reason was given in [9, Subsection 3.2]. Indeed, if \( \dim(F) = \dim(\overline{F}) \), for any subset \( F \) of \( X \), then it turns out that a “small” (countable) set of points can wreak havoc with the dimension, since it may be different from 0. This constitutes a technical reason that makes the box-counting dimension be seriously limited from a theoretical point of view. A proof for the properties contained in Theorem 2.2 regarding the box dimension can be found in [9, Subsection 3.2]. It is worth mentioning, as stated in [9, Chapter 3], that all dimension definitions are monotonic, and most of them are finitely stable. However, some common definitions do not satisfy the countable stability property, and even may throw positive dimensions for countable sets. Indeed, this is the case of box dimension.

2.3. About fractal structures. The concept of fractal structure, which naturally appears in several asymmetric topological topics [20], was first contributed in [1] to characterize non-Archimedeanly quasi-metrizable spaces. Moreover, in [4], it was applied to deal with IFS-attractors. On the other hand, fractal structures constitute a powerful tool to develop new fractal dimension models which allow to calculate the fractal dimension for a wide range of (non-Euclidean) spaces and contexts (see, e.g., [14]).

Recall that a family \( \Gamma \) of subsets of a space \( X \) is called a covering if \( X = \bigcup \{ A : A \in \Gamma \} \). A fractal structure is a countable collection of coverings of a given subset which provides better approximations to the whole space as deeper stages are reached, which we will refer to as levels of that fractal structure.

Let \( \Gamma \) be a covering of \( X \). Then we will denote \( \text{St}(x, \Gamma) = \bigcup \{ A \in \Gamma : x \in A \} \), and \( U_{x\Gamma} = X \setminus \bigcup \{ A \in \Gamma : x \notin A \} \). Furthermore, if \( \Gamma = \{ \Gamma_n : n \in \mathbb{N} \} \) is a countable family of coverings of \( X \), then we will denote \( U_{xn} = U_{xn\Gamma} \), \( U_x^\Gamma = \{ U_{xn} : n \in \mathbb{N} \} \), and \( \text{St}(x, \Gamma) = \{ \text{St}(x, \Gamma_n) : n \in \mathbb{N} \} \).

Let \( \Gamma_1 \) and \( \Gamma_2 \) be any two coverings for \( X \). Thus, \( \Gamma_1 \prec \Gamma_2 \) means that \( \Gamma_1 \) is a refinement of \( \Gamma_2 \), namely, for all \( A \in \Gamma_1 \), there exists \( B \in \Gamma_2 \) such that \( A \subseteq B \). In addition to that, \( \Gamma_1 \prec \prec \Gamma_2 \) denotes that \( \Gamma_1 \prec \Gamma_2 \), and also, that for all \( B \in \Gamma_2 \), \( B = \bigcup \{ A \in \Gamma_1 : A \subseteq B \} \). Hence, a fractal structure on a set \( X \), is a countable family of coverings of \( X \), \( \Gamma = \{ \Gamma_n : n \in \mathbb{N} \} \), such that \( \Gamma_{n+1} \prec \prec \Gamma_n \), for all \( n \in \mathbb{N} \). It is worth mentioning that covering \( \Gamma_n \) is level \( n \) of the fractal structure \( \Gamma \).

Next, we present the definition of a fractal structure on a topological space, as it was first introduced in [1, Definition 3.1].

**Definition 2.3.** Let \( X \) be a topological space. Thus,

1. a pre-fractal structure on \( X \) is a countable family of coverings, \( \Gamma = \{ \Gamma_n : n \in \mathbb{N} \} \), such that \( U_x^\Gamma \) is an open neighborhood base of \( x \), for each \( x \in X \).
(2) Moreover, if $\Gamma_{n+1}$ is a refinement of $\Gamma_n$, such that for all $x \in A$ with $A \in \Gamma_n$, there exists $B \in \Gamma_{n+1}$ such that $x \in B \subseteq A$, then we will say that $\Gamma$ is a fractal structure on $X$.

(3) If $\Gamma$ is a (pre-)fractal structure on $X$, then $(X, \Gamma)$ is said to be a generalized (pre-)fractal space, or merely, a (pre-)GF-space. Furthermore, if there is no doubt about the fractal structure $\Gamma$ we are working with, then we will say that $X$ is a (pre-)GF-space.

To simplify the theory, the levels of any fractal structure $\Gamma$ will not be coverings in the usual sense. Instead of this, we are going to allow that a set can appear twice or more in any level of $\Gamma$. We would like also to point out that a fractal structure $\Gamma$ is said to be finite if all levels $\Gamma_n$ are finite coverings.

It is worth mentioning that if $\Gamma$ is a fractal structure on $X$, and $St(x, \Gamma)$ is a neighborhood base of $x$, for all $x \in X$, then we will call $\Gamma$ a starbase fractal structure. Starbase fractal structures are connected to metrizability (see [2, 3]). A fractal structure $\Gamma$ is said to be finite, if all levels $\Gamma_n$ are finite coverings. A fractal structure $\Gamma$ is said to be locally finite, if for each level $\Gamma_n$ of the fractal structure $\Gamma$, we have that any point $x \in X$ belongs to a finite number of elements $A \in \Gamma_n$. In general, if $\Gamma_n$ has the property $P$, for all $n \in \mathbb{N}$, and $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ is a fractal structure on $X$, then we will say that $\Gamma$ is a fractal structure within the property $P$, and that $(X, \Gamma)$ is a GF-space with the property $P$.

It turns out that any Euclidean space $\mathbb{R}^d$ can be always equipped with a natural fractal structure, which is locally finite and starbase. This was first provided in [12, Definition 3.1]. Next, we recall the description of such a fractal structure, which becomes essential for upcoming sections.

**Definition 2.4.** The natural fractal structure on the Euclidean space $\mathbb{R}^d$ is defined as the countable family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$, whose levels are given by

$$\Gamma_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1 + 1}{2^n} \right] \times \ldots \times \left[ \frac{k_d}{2^n}, \frac{k_d + 1}{2^n} \right] : k_1, \ldots, k_d \in \mathbb{Z} \right\},$$

for each natural number $n$.

In particular, it is also possible to consider a natural fractal structure induced on real subsets from Definition 2.4. For instance, the natural fractal structure (on the real line) induced on the closed unit interval $[0, 1]$, could be defined as the family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$, where its levels are given by $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : k \in \{0, 1, \ldots, 2^n - 1\}\}$, for all $n \in \mathbb{N}$.

3. **Theoretical properties of fractal dimensions for fractal structures.** In this section, we study which properties from those described in Subsection 2.2 are satisfied by some fractal dimension definitions that are made in terms of fractal structures. Due to the different nature of each definition we provide next, we will divide this section into three subsections, namely, a first one that contains a pair of fractal dimension models whose definition is similar to the classical box dimension, a second hybrid model, which is proposed following the spirit of the Hausdorff dimension, but generalizes the box dimension (see [13, Theorem 4.15]), and finally, in Subsection 3.3, three Hausdorff dimension type approaches are contributed. It is worth mentioning that, under certain natural conditions, they generalize the classical Hausdorff dimension (let us refer the reader [16, Theorem 3.12] for such
Table 1. Analytical properties satisfied by classical models of fractal dimension, namely, both box and Hausdorff dimensions. They constitute our starting references for upcoming results regarding our definitions of fractal dimension for fractal structures.

| Monotonicity | F-stability | C-stability | 0-countably | cl.-dim |
|--------------|-------------|-------------|-------------|---------|
| dim \( B \)  | ✓           | ✓           | ✓           | ✓       |
| dim \( H \)  | ✓           | ✓           | ✓           | ✓       |

3.1. Theoretical properties for box dimension type models for a fractal structure. First of all, let us recall our two first models of fractal dimension for a fractal structure, namely, both fractal dimensions I & II. Thus, as it happens with classical box dimension, these models of fractal dimension do not have to exist always. This is the reason for which we have to define them through lower/upper limits.

**Definition 3.1** (Box dimension type models for a fractal structure). Let \( \Gamma \) be a fractal structure on a distance space \((X, \rho)\), \( F \) be a subset of \( X \), and \( N_n(F) \) be the number of elements of \( \Gamma_n \) which intersect \( F \). Hence,

1. The (lower/upper) fractal dimension I for \( F \) is defined as the (lower/upper) limit:
   \[
   \dim_1(F) = \lim_{n \to \infty} \frac{\log N_n(F)}{n \log 2}.
   \]
2. The (lower/upper) fractal dimension II for \( F \) is defined as the (lower/upper) limit:
   \[
   \dim_2(F) = \lim_{n \to \infty} \frac{\log N_n(F)}{-\log \delta(F, \Gamma_n)}.
   \]
   where \( \delta(F, \Gamma_n) = \sup\{\text{diam}(A) : A \in \Gamma_n, A \cap F \neq \emptyset\} \), is the diameter of \( F \) in level \( n \) of the fractal structure.

It turns out that both fractal dimensions I & II do generalize the box dimension in the context of any Euclidean space equipped with its natural fractal structure, as given in Definition 2.4 (see [12, Theorem 4.7]).

**Theorem 3.2.** Let \( \Gamma \) be a fractal structure on \( X \). Thus,

1. both the lower fractal dimension I and the upper fractal dimension I are monotonic.
2. The upper fractal dimension I is finitely stable.
3. Neither the lower fractal dimension I nor the upper fractal dimension I are countably stable.
(4) There exist a countable subset $F$ of $X$, as well as a fractal structure $\Gamma$ on $X$, such that $\dim_1^\Gamma(F) \neq 0$.

(5) There exists a locally finite starbase fractal structure $\Gamma$ defined on a given space $X$, such that $\dim_1^\Gamma(F) \neq \dim_1^\Gamma(F')$, for a certain subset $F \subseteq X$.

Proof. (1) First of all, let $E, F$ be two subsets of $X$, and let us assume that $E \subseteq F$. Hence, it becomes clear that $N_n(E) \leq N_n(F)$, since each element $A \in \Gamma_n$ such that $A \cap E \neq \emptyset$, also satisfies that $A \cap F \neq \emptyset$. This leads to $\dim_1^\Gamma(E) \leq \dim_1^\Gamma(F)$, and the same arguments are valid for upper fractal dimension $I$.

(2) Let $E, F$ be two subsets of $X$. Since the fractal dimension $I$ function is monotonic (due to Theorem 3.2 (1)), then it follows that

$$\max\{\dim_1^\Gamma(E), \dim_1^\Gamma(F)\} \leq \dim_1^\Gamma(E \cup F).$$

On the other hand, note that $N_n(E \cup F) \leq N_n(E) + N_n(F)$ for all $n \in \mathbb{N}$. In addition to that, let $\varepsilon$ be a positive real number, and let $d_1 = \dim_1^\Gamma(E)$. Hence, there exists a natural number $n_1$ such that $N_n(E) \leq 2^{n(d_1 + \varepsilon)}$, for all $n \geq n_1$. Similarly, if $d_2 = \dim_1^\Gamma(F)$, then $N_n(F) \leq 2^{n(d_2 + \varepsilon)}$, for all $n \geq n_2$. Let us assume, without loss of generality, that $d_1 \geq d_2$, and let $m = \max\{n_1, n_2\}$. Thus, we have that $N_n(E) + N_n(F) \leq 2^{n(d_1 + \varepsilon) + 1}$, for all $n \geq m$. Accordingly, the following holds:

$$\dim_1^\Gamma(E \cup F) \leq \lim_{n \to \infty} \frac{\log(N_n(E) + N_n(F))}{n \log 2} \leq \lim_{n \to \infty} \frac{n(d_1 + \varepsilon) + 1}{n} = d_1 + \varepsilon,$$

for all $\varepsilon > 0$. This implies that

$$\dim_1^\Gamma(E \cup F) \leq \max\{\dim_1^\Gamma(E), \dim_1^\Gamma(F)\},$$

which gives the result.

(4) Let us consider $X = [0, 1]$, $F = \mathbb{Q} \cap X$, as well as $\Gamma$ as the natural fractal structure on the real line induced on $X$, whose levels are given by $\Gamma_n = \left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]: k \in \{0, 1, \ldots, 2^n - 1\}\right\}$, for all $n \in \mathbb{N}$. Thus, it becomes clear that $N_n(F) = 2^n$, so $\dim_1^\Gamma(F) = 1$. It is also worth mentioning that Theorem 3.2 (4) implies (3).

(5) Let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure defined on $X = (0, 1) \times \{0\} \cup \{\left\{\frac{1}{2^n}\right\} \times [0, 1] : n \in \mathbb{N}\}$, whose levels are defined as $\Gamma_n = \left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \{0\}: k \in \{0, 1, \ldots, 2^n - 1\}\right\} \cup \left\{\left\{\frac{1}{2^n}\right\} \times \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]: k \in \{0, 1, \ldots, 2^n - 1\}\right\}$, for all natural number $n$. Further, let $F = \bigcup_{k \in \mathbb{N}} \left[\frac{1}{2^n}, \frac{1}{2^n}\right] \times \{0\}$ be a subset of $X$. Thus, it becomes clear that $\overline{F} = [0, 1] \times \{0\}$, so $N_n(F) = 2^n$, and $N_n(\overline{F}) = \infty$, which leads to $\dim_1^\Gamma(F) = 1$, and $\dim_1^\Gamma(\overline{F}) = \infty$.

Following the above, some analytical properties for fractal dimension $II$ are also explored next in a similar fashion to Theorem 3.2.

**Theorem 3.3.** Let $\Gamma$ be a fractal structure on a distance space $(X, \rho)$. Hence,

(1) both the lower fractal dimension $II$ and the upper fractal dimension $II$ are monotonic.

(2) Neither the lower fractal dimension $II$ nor the upper fractal dimension $II$ are finitely stable.
Table 2. Summary regarding the analytical properties that are satisfied by the box dimension type models explored in Subsection 3.1. This includes the classical box dimension, as well as both fractal dimension I & II models.

| Monotonicity | F-stability | C-stability | 0-countably | cl-dim |
|--------------|-------------|-------------|-------------|--------|
| dim_B        | ✓           | ✓           |             |        |
| dim_1        | ✓           | ✓           |             |        |
| dim_2        | ✓           |             |             |        |

(3) Neither the lower fractal dimension II nor the upper fractal dimension II are countably stable.

(4) There exist a countable subset $F$ of $X$, and a fractal structure $\Gamma$ on $X$, such that $\dim_1^2(F) \neq 0$.

(5) There exists a locally finite starbase fractal structure $\Gamma$ defined on a given space $X$, such that $\dim_1^2(F) \neq \dim_1^2(\Gamma)$, for a certain subset $F \subseteq X$.

Proof. Firstly, it becomes clear that both $\dim_1^2$ and $\dim_2^2$ are monotonic. Moreover, since fractal dimension II generalizes fractal dimension I (in the sense of [12, Theorem 4.6]), then any counterexample shown for fractal dimension I remains valid for fractal dimension II. In particular, those given to prove statements (3), (4), and (5) in Theorem 3.2, allow us to justify the corresponding properties for fractal dimension II.

Accordingly, let us be focused on the proof for Theorem 3.3 (2), regarding the finite stability property for fractal dimension II. However, unlike fractal dimension I, the fractal dimension II definition does not satisfy the finite stability, as the following counterexample states. In fact, let $\Gamma_1$ be the natural fractal structure on $C_1$ as an IFS-attractor, where $C_1$ is the middle third Cantor set on $[0,1]$. In addition to that, let also $\Gamma_2$ be a fractal structure on $C_2 = [2,3]$, defined as $\Gamma_2 = \{\Gamma_{2,n} : n \in \mathbb{N}\}$, where $\Gamma_{2,n} = \{[k/2^n, (k+1)/2^n] : k \in \{2^{2n+1}, 2^{2n+1} + 1, \ldots, 3 \cdot 2^n - 1\}\}$, for all $n \in \mathbb{N}$. Thus, let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on $C = C_1 \cup C_2$, where its levels are defined by $\Gamma_n = \Gamma_{1,n} \cup \Gamma_{2,n}$, for all $n \in \mathbb{N}$. Finally, simple calculations lead to $\dim_1^2(C_1) = \log 2/\log 3$, as well as to $\dim_1^2(C_2) = 1$, whereas $\dim_1^2(C) = \log 4/\log 3 > 1$.  

Upcoming Table 2 contains all the information regarding the analytical properties satisfied by the box dimension type models for fractal dimension that have been explored in this subsection. Notice that neither fractal dimension I nor fractal dimension II satisfy the cl-dim property, which could be understood as an advantage of these models with respect to classical box dimension. Further, as it happens with box dimension, fractal dimension I becomes F-stable, though fractal dimension II do not.
3.2. Theoretical properties for a hybrid model of fractal dimension for any fractal structure. The following fractal dimension definition for a fractal structure could be understood as a hybrid model, since its definition is made, somehow, as a discrete version of the Hausdorff dimension, but it also generalizes the classical box dimension.

Definition 3.4. Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, $F$ be a subset of $X$, and assume that $\delta(F, \Gamma_n) \to 0$. Given $n \in \mathbb{N}$, let us consider the next expression:

$$H_{n,3}^s(F) = \inf \left\{ \sum_{i \in I} \text{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_{n,3}(F) \right\},$$

where $\mathcal{A}_{n,3}(F) = \{ \{A \in \Gamma_1 : A \cap F \neq \emptyset\} : l \geq n \}$. Define also

$$H_3^s(F) = \lim_{n \to \infty} H_{n,3}^s(F).$$

Thus, the fractal dimension $\text{dim}_3^s$ of $F$ is defined as the non-negative real number that follows:

$$\text{dim}_3^s(F) = \inf \{ s : H_3^s(F) = 0 \} = \sup \{ s : H_3^s(F) = \infty \}. $$

Unlike it happens with fractal dimensions I & II (recall Definition 3.1), fractal dimension III does always exist (see [13, Remark 4.4]). This is mainly due to the fact that $H_{n,3}^s$ is a monotonic sequence in $n \in \mathbb{N}$. Moreover, [13, Theorem 4.15] shows that fractal dimension III generalizes box dimension (as well as fractal dimensions I & II) in the context of Euclidean spaces equipped with their natural fractal structures. Interestingly, it turns out that fractal dimension III can be estimated as easy as the box dimension in empirical applications. For additional details, we refer the reader to [13], and specially to Theorem 4.7, therein.

Theorem 3.5. Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, and let us assume that $\delta(\Gamma_n) \to 0$. Thus,

1. Fractal dimension III is monotonic.
2. Fractal dimension III is finitely stable.
3. There exists a countable subset $\mathcal{E}$ of $X$, as well as a fractal structure $\Gamma$ on $X$, such that $\text{dim}_3^1(F) \neq 0$.
4. Fractal dimension III is not countably stable.
5. There exists a locally finite starbase fractal structure $\Gamma$ defined on a given space $X$, such that $\text{dim}_3^1(F) \neq \text{dim}_3^1(\overline{F})$ for a certain subset $F \subseteq X$.

Proof. (1) Let $E, F$ be any two subsets of $X$, and let us suppose that $E \subseteq F$. Then $\text{dim}_3^1(E) \leq \text{dim}_3^1(F)$, since $H_{n,3}^s(E) \leq H_{n,3}^s(F)$ for all $n \in \mathbb{N}$.

(2) Let $E, F$ be two subsets of $X$. Firstly, note that $H_{n,3}^s(E \cup F) \leq H_{n,3}^s(E) + H_{n,3}^s(F)$, which leads to $H^s(E \cup F) \leq H^s(E) + H^s(F)$. Moreover, let $s$ be a positive real number such that $H^s(E \cup F) = \infty$, and let us assume, without loss of generality, that $\text{dim}_3^1(E) \leq \text{dim}_3^1(F)$. Hence, $H^s(F) = \infty$, which leads to $\text{dim}_3^1(E \cup F) \leq \text{dim}_3^1(F)$. The opposite inequality is given by Theorem 3.5 (1), since fractal dimension III is monotonic. Consequently, $\text{dim}_3^1(E \cup F) = \max\{\text{dim}_3^1(E), \text{dim}_3^1(F)\}$, which gives the finite stability property for fractal dimension III.

(3) Let $X = [0, 1]$, $F = \mathbb{Q} \cap X$, and $\Gamma$ be the natural fractal structure on the real line induced on $X$. Thus, since fractal dimension III generalizes fractal dimension I in the context of Euclidean spaces equipped with their natural fractal structures.
Table 3. The table above summarizes the analytical properties satisfied by fractal dimension III, which becomes an intermediate model between both the box and the Hausdorff dimensions, and the classical models for fractal dimension.

(see [13, Theorem 4.15]), then the following holds: $\dim_3^3(F) = \dim_3^1(F) = 1$ (recall Theorem 3.2 (4)).

(4) Notice that Theorem 3.5 (3) implies (4). In fact, for all rational number $q_i \in F = \mathbb{Q} \cap [0,1]$, let us denote, firstly, $F_i = \{q_i\}$. Thus, it becomes clear that $\dim_3^3(F_i) = 0$, for all $i \in \mathbb{N}$. This leads to $\sup\{\dim_3^3(F_i) : i \in \mathbb{N}\} = 0$. On the other hand,

$$\dim_3^3(F) = \dim_3^3\left( \bigcup_{i \in \mathbb{N}} F_i \right) = \dim_3^3\left( \bigcup_{q_i \in F} \{q_i\} \right) = 1.$$ 

Accordingly, fractal dimension III does not satisfy the finite stability.

(5) To deal with Theorem 3.5 (5), let us consider the same list $(F, X, \Gamma)$ as provided in counterexample (5) from Theorem 3.2. Hence, $\dim_3^3(F) = 1$, whereas $H_s^0(F) = \sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \infty$, which leads to $\dim_3^3(F) = \infty$.

Thus, fractal dimension III becomes an intermediate model between classical Hausdorff and box dimensions. Moreover, Tables 3 & 4 are provided for comparative purposes. Thus, while the former summarizes the analytical properties that are satisfied by fractal dimension III and both classical Hausdorff and box dimensions, the latter compares the analytical properties satisfied by fractal dimension I, II, and III models that were defined with respect to any fractal structure. Regarding the theoretical properties described in Subsection 2.2, fractal dimension III behaves like fractal dimension I.

3.3. Theoretical properties for Hausdorff dimension type models for a fractal structure. The last step is to provide three models for a fractal dimension definition (with respect to any fractal structure) following the spirit of the Hausdorff dimension. Thus, while the first one is specially interesting, since its description is made in terms of finite coverings (which allowed the authors to contribute the first known overall algorithm to calculate the Hausdorff dimension of any compact Euclidean subset, see [17]), the remaining definitions become close approaches to classical Hausdorff dimension. Hence, upcoming fractal dimension V becomes a discrete version of the latter, whereas fractal dimension VI is proposed in terms of $\delta$-covers.
Table 4. Analytical properties satisfied by fractal dimensions I, II, and III, which have been explored with respect to a fractal structure, in both Subsections 3.1 and 3.2, respectively.

| Monotonicity | F-stability | C-stability | Z-countably |
|--------------|-------------|-------------|-------------|
| $\dim_1^F$  | ✓           | ✓           |             |
| $\dim_2^F$  | ✓           |             |             |
| $\dim_3^F$  | ✓           |             |             |

Theorem 3.7. Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, and let us suppose that $\delta(F, \Gamma_n) \to 0$. Hence,

1. fractal dimension IV is monotonic.
(2) There exist a countable subset $F$ of $X$, and a fractal structure $\Gamma$ on $X$, such that $\dim^4_{\Gamma}(F) \neq 0$.
(3) Fractal dimension IV is not countably stable.
(4) Fractal dimension IV is finitely stable.
(5) Fractal dimension IV satisfies that $\dim^4_{\Gamma}(F) = \dim^4_{\Gamma}(\overline{F})$, for all subset $F$ of $X$, and all fractal structure $\Gamma$ on $X$.

Proof. (1) Let $E, F$ be any two subsets of $X$, and let us assume that $E \subseteq F$. Additionally, let $\{A_i\}_{i \in I}$ be any covering from the family $\mathcal{A}_{n,4}(F)$. Hence, $A_i \in \bigcup_{i \geq n} \Gamma_i$, for all $i \in I$, and we also have that $E \subseteq F \subseteq \bigcup_{i \in I} A_i$, with $\text{Card}(I) < \infty$. Thus, $\{A_i\}_{i \in I} \subseteq \mathcal{A}_{n,4}(E)$. Accordingly, $\mathcal{A}_{n,4}(F) \subseteq \mathcal{A}_{n,4}(E)$, so $\mathcal{H}_{n,4}^s(E) \leq \mathcal{H}_{n,4}^s(F)$, for all $n \in \mathbb{N}$ and all $s > 0$. This leads to $\mathcal{H}_{n,4}^s(E) \leq \mathcal{H}_{n,4}^s(F)$, for all $s > 0$, and consequently, $\dim^4_{\Gamma}(E) \leq \dim^4_{\Gamma}(F)$, as expected.

(2) Let $\Gamma$ be the natural fractal structure on the real line induced on $X = [0, 1]$, and let us also consider $F = \mathbb{Q} \cap X$. In this way, $F$ is a countable set such that $F = [0, 1]$. Hence, $[16$, Theorem 3.13$]$ leads to $\dim^4_{\Gamma}(F) = \dim_H(F) = 1$.

(3) Theorem 3.7 (2) implies (3). In fact, if $F = \mathbb{Q} \cap [0, 1]$, and $\Gamma$ is chosen to be the natural fractal structure induced on $[0, 1]$, then $\dim^4_{\Gamma}(F) = 1$, though $\sup \{\dim^4_{\Gamma}(\{q_i\}) : q_i \in F\} = 0$.

(4) First of all, since fractal dimension IV is monotonic (due to Theorem 3.7 (1)), then it becomes clear that

$$\max \{\dim^4_{\Gamma}(F_1), \dim^4_{\Gamma}(F_2)\} \leq \dim^4_{\Gamma}(F_1 \cup F_2),$$

for any two subsets $F_1, F_2$ of a given space $X$. Thus, next we focus on the opposite inequality. To deal with, let $s$ be a positive real number such that $s > \max \{\dim^4_{\Gamma}(F_1), \dim^4_{\Gamma}(F_2)\}$, and let also $\varepsilon$ be a fixed but arbitrarily chosen positive real number. Thus, since $\mathcal{H}_{4}^s(F_1) = 0$, then we can affirm that there exists a covering $\{A^1_k\}_{k=1}^{l_1}$ for $F_1$ such that $\sum_{k=1}^{l_1} \text{diam}(A^1_k)^s < \varepsilon^2 = \varepsilon^2$. Similarly, since $\mathcal{H}_{4}^s(F_2) = 0$, then there exists a covering $\{A^2_k\}_{k=1}^{l_2}$ for $F_2$ such that $\sum_{k=1}^{l_2} \text{diam}(A^2_k)^s < \varepsilon^2$. Hence,

$$F_1 \cup F_2 \subseteq \left( \bigcup_{k=1}^{l_1} A^1_k \right) \bigcup \left( \bigcup_{k=1}^{l_2} A^2_k \right),$$

where $\sum_{k=1}^{l_1} \text{diam}(A^1_k)^s + \sum_{k=1}^{l_2} \text{diam}(A^2_k)^s < \frac{\varepsilon^2}{2} < \varepsilon$. Consequently, $\mathcal{H}_{4}^s(F_1 \cup F_2) = 0$, for all $s > \max \{\dim^4_{\Gamma}(F_1), \dim^4_{\Gamma}(F_2)\}$. Finally, this implies that $\dim^4_{\Gamma}(F_1 \cup F_2) \leq \max \{\dim^4_{\Gamma}(F_1), \dim^4_{\Gamma}(F_2)\}$, which completes the proof.

(5) The monotonicity of fractal dimension IV (shown in Theorem 3.7 (1)) leads to $\dim^4_{\Gamma}(F) \leq \dim^4_{\Gamma}(\overline{F})$. So let us concentrate in the opposite inequality. Indeed, let $\{A_i\}_{i \in I} \in \mathcal{A}_{n,4}(F)$. Thus, $A_i \in \bigcup_{i \geq n} \Gamma_i$, for all $i \in I$, and $F \subseteq \bigcup_{i \in I} A_i$, with $\text{Card}(I) < \infty$. Hence,

$$\overline{F} \subseteq \bigcup_{i \in I} A_i = \bigcup_{i \in I} \overline{A_i} = \bigcup_{i \in I} A_i,$$

since all the elements in the covering are closed. Accordingly, $\{A_i\}_{i \in I} \in \mathcal{A}_{n,4}(\overline{F})$. This implies that $\mathcal{A}_{n,4}(F) \subseteq \mathcal{A}_{n,4}(\overline{F})$, for all $n \in \mathbb{N}$. Thus, $\mathcal{H}_{n,4}^s(\overline{F}) \leq \mathcal{H}_{n,4}^s(F)$, for all $n \in \mathbb{N}$, and all $s > 0$. This leads to $\mathcal{H}_{n,4}^s(\overline{F}) \leq \mathcal{H}_{n,4}^s(F)$, for all $s > 0$, and hence, $\dim^4_{\Gamma}(\overline{F}) \leq \dim^4_{\Gamma}(F)$. 

An analogous result to Theorem 3.7 is shown next for both fractal dimensions $V$ & $VI$.

**Theorem 3.8.** Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, and let us suppose that $\text{diam}(\Gamma_n) \to 0$. Then

(1) both fractal dimensions $V$ and $VI$ are monotonic.
(2) Both fractal dimensions $V$ and $VI$ are countably stable.
(3) It is satisfied that $\dim^5_f(F) = \dim^6_f(F) = 0$, whether $F$ is a countable subset of $X$.
(4) There exists a locally finite starbase fractal structure $\Gamma$ defined on a suitable space $X$, such that $\dim^k_f(F) = \dim^k_f(\overline{F})$, for a given subset $F \subseteq X$, where $k = 5, 6$.

**Proof.** (1) Let $E, F$ be any two subsets of $X$, such that $E \subseteq F$. In addition to that, let also $\{A_i\}_{i \in I}$ be any covering from the family $\mathcal{A}_{n,5}(F)$. Thus, $A_i \in \bigcup_{i \in I} \Gamma_i$, for all $i \in I$, and it is also satisfied that $E \subseteq F \subseteq \bigcup_{i \in I} A_i$. Accordingly, $\{A_i\}_{i \in I} \in \mathcal{A}_{n,5}(E)$. This allows us to affirm that $\mathcal{A}_{n,5}(F) \subseteq \mathcal{A}_{n,5}(E)$, so $\mathcal{H}_{n,5}(E) \leq \mathcal{H}_{n,5}(F)$, for all $n \in \mathbb{N}$, and all $s > 0$. Therefore, $\mathcal{H}_{5}^s(E) \leq \mathcal{H}_{5}^s(F)$, for all $s > 0$ and hence, $\dim^5_f(E) \leq \dim^5_f(F)$.

Similarly, if $\{A_i\}_{i \in I} \in \mathcal{A}_{\delta,6}(E)$, then $A_i \in \bigcup_{i \in I} \Gamma_i$, for all $i \in I$, where $\text{diam}(A_i) \leq \delta$, and $E \subseteq F \subseteq \bigcup_{i \in I} A_i$. Thus, $\{A_i\}_{i \in I} \in \mathcal{A}_{\delta,6}(E)$, which leads to $\mathcal{A}_{\delta,6}(F) \subseteq \mathcal{A}_{\delta,6}(E)$, for all $\delta > 0$. Hence, $\mathcal{H}_{\delta,6}^s(E) \leq \mathcal{H}_{\delta,6}^s(F)$, for all $\delta > 0$. This leads to $\mathcal{H}_{5}^s(E) \leq \mathcal{H}_{5}^s(F)$, for all $s > 0$, so $\dim^6_f(E) \leq \dim^6_f(F)$.

(2) Firstly, since fractal dimension $V$ is monotonic (due to Theorem 3.8 (1)), then we have that

$$
\sup \{\dim^5_f(F_i) : i \in I\} \leq \dim^5_f \left( \bigcup_{i \in I} F_i \right).
$$

Therefore, we will be focused on the proof for the opposite inequality. In fact, let $s$ be a positive real number such that $s > \sup \{\dim^5_f(F_i) : i \in I\}$, and let $\varepsilon$ be a fixed but arbitrarily chosen positive real number. Thus, given that $\mathcal{H}_{5}^s(F_i) = 0$, for all $i \in I$, then there exists a family of coverings for $F_i$, $\{A_k^i\}_{k \in \mathbb{N}}$, such that $\sum_k \text{diam}(A_k^i)^s < \frac{\varepsilon}{2^i}$, where $F_i \subseteq \bigcup_k A_k^i$, for all $i \in I$. Hence, we have that $\bigcup_{i \in I} F_i \subseteq \bigcup_{i \in I} A_k^i$, as well as it is satisfied that $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \text{diam}(A_k^i)^s \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$. Thus, $\mathcal{H}_{5}^s \left( \bigcup_{i \in I} F_i \right) = 0$, for all $s > \sup \{\dim^5_f(F_i) : i \in I\}$.

Accordingly,

$$
\dim^5_f \left( \bigcup_{i \in I} F_i \right) \leq \sup \{\dim^5_f(F_i) : i \in I\}.
$$

To prove that $\mathcal{H}_{6}^s \left( \bigcup_{i \in I} F_i \right) = 0$, for all $s > \sup \{\dim^6_f(F_i) : i \in I\}$, just recall that $\mathcal{H}_{6}^s$ is a (metric) outer measure (due to [16, Theorem 3.8]). Therefore, $\mathcal{H}_{6}^s \left( \bigcup_{i \in I} F_i \right) \leq \sum_{i \in I} \mathcal{H}_{6}^s(F_i) = 0$, since $\mathcal{H}_{6}^s(F_i) = 0$, for all $i \in I$.

(3) Let $F$ be a countable subset of $X$. Then $F = \bigcup_{i \in I} \{q_i\}$, and since both fractal dimensions $V$ & $VI$ are countably stable by Theorem 3.8 (2), then $\dim^5_f(F) = \sup \{\dim^5_f(\{q_i\}) : i \in I\} = 0$, for $k = 5, 6$.

(4) Let $\Gamma$ be the natural fractal structure induced on $X = [0, 1]$, and let us consider $F = \mathbb{Q} \cap X$. Thus, $F$ is a countable set such that $\overline{F} = [0, 1]$. Hence, Theorem
Monotonicity  F-stability  C-stability  I-countably  cl.-dim

\[ \dim^4_F = \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \]
\[ \dim^5_F = \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \]
\[ \dim^6_F = \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \]
\[ \dim_H = \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \]

Table 5. The table above summarizes the analytical properties satisfied by the Hausdorff dimension type models explored with respect to a fractal structure in Subsection 3.3.

3.8 (3) leads to \( \dim^5_F(F) = \dim^6_F(F) = 0 \). However, \( \dim^5_F(F) = \dim_H(F) = \dim^5_F(F) = 1 \), due to [16, Corollary 3.11].

Upcoming Tables 5-7 provide schematic comparisons regarding the analytical properties that are satisfied by Hausdorff dimension type definitions for a fractal structure, all the fractal dimensions for a fractal structure studied along this paper, and all the fractal dimensions, together with the classical ones, respectively. Surprisingly, fractal dimension IV would not seem to be interesting enough in the light of the properties it satisfies, since they are the same as box dimension does. However, though its definition is made in terms of finite coverings, it holds that both fractal dimension IV and Hausdorff dimension are equal for compact Euclidean subspaces (see [16, Theorem 3.13 & Corollary 3.14 (2)]). This fact allowed the authors to provide the first known procedure to calculate the Hausdorff dimension in practical applications (see [17, Algorithm 3.1]). It is worth mentioning that fractal dimension IV becomes also an intermediate model between the classical fractal dimension definitions (for additional details, we refer the reader to [16, Remark 3.15]). On the other hand, fractal dimensions V & VI also generalize the Hausdorff model in the context of Euclidean subspaces equipped with their natural fractal structures (see [16, Corollary 3.11]).

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### Table 6. Schematic comparison among the theoretical properties for a fractal dimension definition that are satisfied by all the fractal dimension models for a fractal structure that have been studied along Subsections 3.1, 3.2, and 3.3.

| Fractal dimensions | Monotonicity | F-stability | C-stability | 0-countably | cl-dim |
|--------------------|--------------|-------------|-------------|-------------|--------|
| \( \dim^1_H \)    | ✓            | ✓           |             |             |        |
| \( \dim^2_H \)    | ✓            |             |             |             |        |
| \( \dim^3_H \)    | ✓            | ✓           |             |             |        |
| \( \dim^4_H \)    | ✓            | ✓           |             |             | ✓      |
| \( \dim^5_H \)    | ✓            | ✓           | ✓           | ✓           |        |
| \( \dim^6_H \)    | ✓            | ✓           | ✓           | ✓           | ✓      |

### Table 7. The table above contains the analytical properties that have been explored for all the fractal dimension models considered along this paper.

| Fractal dimensions | Monotonicity | F-stability | C-stability | 0-countably | cl-dim |
|--------------------|--------------|-------------|-------------|-------------|--------|
| \( \dim^1_H \)    | ✓            | ✓           |             |             |        |
| \( \dim^2_H \)    | ✓            |             |             |             |        |
| \( \dim^3_H \)    | ✓            | ✓           |             |             |        |
| \( \dim^4_H \)    | ✓            | ✓           |             |             | ✓      |
| \( \dim^5_H \)    | ✓            | ✓           | ✓           | ✓           |        |
| \( \dim^6_H \)    | ✓            | ✓           | ✓           | ✓           | ✓      |
| \( \dim^7_H \)    | ✓            | ✓           | ✓           | ✓           | ✓      |

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