Mathematical modeling of deformations of flexible threads under their dynamic loading in the zone of material plasticity

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Abstract. The object of study is a vertical flexible thread rotating at some distance around a vertical axis with a lateral load. The calculation is made for the dynamic load condition in the form of centrifugal forces depending on the displacement of the thread from the initial position. The geometric and physical nonlinearity of the flexible threads is taken into account. The proposed method is based on the theory of mathematical modeling using differential and integral calculus of functions of one and several variables as well as on the theory of resistance of materials. A method of the nonlinear calculation of the stress-strain state of flexible threads, taking into account the real properties of materials is proposed. The text of the calculation in the computer program MatchCAD for calculations of the thread, its deformation and tension was theoretically described and developed. The calculation results corresponded to the results obtained by the finite element method (the discrepancy is about 1%). The implementation of the method makes it possible to study the operation of structures outside the conditions of elastic work of the material. The comparative results of modeling inertia loaded rotating threads with linear properties of changes in their strength properties and in the case of nonlinearity of their properties are presented.

1. Introduction

Modern technology development foresees the widespread use of wire ropes [1,2], cables [3], wires [4] ropes [5,6] and threads [7] with relative flexibility of the carrier element that significantly affects the operation of the main devices using these elements.

To determine the coordinates of flexible elements and to perform calculations various theoretical approaches are used. Calculation methods based on the application of the finite element method [2,3] have been particularly popular in recent years. However, these methods [4-6] are demanding of the availability of computer capabilities and the chosen method of applying loads [7]. Classical calculations of flexible elements [8,9] require significantly less IT capabilities. In this case the error in the values of the performed calculations does not exceed 5% [10] as a rule.
The available methods for calculating flexible structural elements [10] are meant to determine the parameters of stationary flexible elements. At the same time in technology there are a large number of devices in which the flexible elements are part of some rotors.

Unfortunately methods for calculating such elements are not widely distributed. Therefore a method for calculating a flexible element rotating with an eccentricity which is subject to centrifugal forces has been developed.

Many technological processes in which flexible threads rotating around an axis participate can be reduced to contour movement, namely to its particular case - the state of “apparent rest”. For example, in the textile industry in the process of winding on and off on annular spinnings and twisting frames, a flexible thread rotates around an axis, while keeping the shape of a certain constant curve all the time. In this case the contour movement of the flexible thread is stationary. The concept of stationary contour motion is relative and depends on the choice of the coordinate system. So the movement of a flexible thread is stationary in the coordinate system rotating at a constant angular velocity together with the flexible thread. The shape and a distribution of tension of the flexible thread under consideration remain unaltered over time.

A honey separator (a device for rotating frames with honeycombs for centrifugal removal of honey from honeycombs) has been taken as an example of a similar initial structural element. It has a vertical rotating shaft. At some distance from the shaft frames with tensioned metal threads placed parallel to the rotating shaft are located. Under the action of centrifugal forces cells with threads located in them begin to deform forming a surface remotely resembling a hemisphere. The formation of such a surface contributes to the emptying of honeycombs from the product (honey).

The aim of the work is to develop computational expressions for determining the stresses and deformations of flexible threads moving around a circle with nonlinear elastic properties under the action of transverse centrifugal forces the numerical values of which are affected by the amount of displacement and deformation of the thread.

2. Research methods

At present, as a rule the calculation of flexible threads for an arbitrary external impact is carried out under the assumption that the material has ideally elastic characteristics. However, the real work of a flexible thread and its limiting bearing capacity may differ significantly from the value obtained in the linear formulation of the problem when the material obeys Hooke's law. Therefore, along with taking into account the geometric nonlinearity, when determining the stress-strain state of a flexible thread, the actual task is to pay special attention to the physical nonlinearity of the material.

The object of study is a vertical flexible thread rotating at some distance around a vertical axis with a lateral load.

The subject of the research is the method of mathematical modeling of the stress-strain state of a flexible thread for determination of the actual stresses and lines of the equilibrium state resulting from the action of centrifugal inertia forces taking into account the plastic deformations of the material.

To model the flexible threads the following accepted assumptions were made: the material continuously filled the entire volume of the threads (continuity hypothesis); material properties were the same at all points (uniformity of material properties) and did not depend on direction (isotropy); transverse normal sections of the thread remained flat after deformation (Bernoulli’s hypothesis); the cross section of the thread was small compared to the length of the thread; the change in cross section under tension was not taken into account; only tension (axial force) was taken into account from internal forces, and rigidity was absent during bending and torsion [10].

In the compilation of the calculated expressions of the model the category of the core type element formation has been used. A uniformly distributed load acted along the entire length of the rod. Under the action of centrifugal forces the flexible rod took an equilibrium state in the form of a curve.

The calculation has been made on the basis of the well-known method presented by us in [10], but upgraded by replacing the lateral static load by the applied dynamic load in the form of centrifugal forces.
depending on the displacement of the thread from the initial position and non-linear change in the elastic forces of the thread.

The solution of the problem at modeling in the MathCAD program has been made by drawing up a system of equations from the expressions below and a subsequent resolution with respect to the indicator of interest.

Initial data for conducting numerical researches (figure 1). The elastic body with density \( \rho = 500 \text{ kg/m}^3 \), length \( d = 0.415 \text{ m} \), width \( b = 0.055 \text{ m} \) and thickness \( h = 0.025 \text{ m} \) is rigidly fixed along its entire length to a flexible thread passing inside the body through its center of mass. In turn, a flexible thread with supports installed in a vertical plane with a span of \( l = 0.415 \text{ m} \), section \( A = 12.56 \times 10^{-8} \text{ m}^2 \), is located parallel to the axis of rotation at a distance \( R = 0.18 \text{ m} \). The initial sag and the load causing the initial outline are missing. Characteristics of the material of a flexible thread are taken: temporary resistance – \( \sigma_2 = 800 \text{ MPa} \); the conditional yield strength is \( \sigma_1 = 600 \text{ MPa} \), the relative elongation after rupture is \( \varepsilon_2 = 0.25 \), the modulus of elasticity is \( E = 2.1 \times 10^5 \text{ MPa} \). The stiffness of flexible supports – \( u = 36.63 \text{ kN/m} \).

To check the implementation of the goal on the basis of established and identified dependencies a graphical analysis of changes in values is carried out so as to establish the results of loading the thread and test its performance. The calculation results of the proposed method have been compared with the finite element method.

3. Research result
Let us consider a rotating flexible thread, the design scheme of which is shown in figure 1 with a rigidly attached elastic body (it is conventionally shown by characterizing section with dimensions in width – \( b \), \text{ m} \) and in thickness – \( h \), \text{ m} \) initially straight, having an initial length equal to the span between attachment points, located parallel to the axis of rotation at a certain distance.
We add the following hypotheses to the generally accepted assumptions made in the calculation of flexible threads: the kinetic energy of the attached body during rotation completely turns into work on changing the position of the flexible thread, including elastoplastic deformation and kinematic displacement; gravity, air resistance and body rigidity can be neglected.

Under the assumptions made here a flexible thread will have only one centrifugal inertia force during rotation.

Imagine the kinetic energy of an attached rotating elastic body in the form of work on changing the position of a flexible thread with a perfect, transverse, evenly distributed load $q_1$, which is equivalent to the centrifugal inertia force, and since the load $q_1$ is not known until the moment of its calculation, then all subsequent expressions are written as functions of $q_1$.

The spread in the flexible thread $H_1$ caused by the action of the inertial load is also unknown, therefore it should also be included in the parameters of the functions:

$$\frac{J(H_1, q_1, x) \cdot \omega^2}{2} = \frac{1}{2} \cdot q_1 \cdot \int_{\frac{d}{2}}^{\frac{d}{2}} w(H_1, q_1, x) \, dx, \quad (1)$$

where $J(H_1, q_1, x)$ is the moment of inertia of the body relative to the axis of rotation, kg m$^2$; $\omega$ – angular velocity, rad/s; $q_1$ – evenly distributed load, N/m; $x_d$ – the center of the zone of action of the load $q_1$, m; $d$ is the width of the zone of action of the load $q_1$, m; $w(H_1, q_1, x)$ – deflection function, m; $x$ – current abscissa ($0 \leq x \leq 1$), m; $l$ – span of flexible thread, m.

During rotation changes of the moment of inertia of the elastic body, which attached to the flexible thread, connected with its deformation take place, so the moment of inertia takes the form of a function:

$$\frac{J(H_1, q_1, x)}{2} = \rho \cdot b \cdot h \cdot \int_{0}^{\frac{2H_1}{u}} \left( R + w(H_1, q_1, x) \right) \, dx, \quad (2)$$

where $\rho$ is the density of the body, kg/m$^3$; $b$ – width of the body, m; $h$ — thickness of the body, m; $u$ is the stiffness of the elastic supports, N/m; $R$ is the distance from the axis of rotation to the center of mass of the non-deformed body, m.

The deflection of a flexible thread is a function of the abscissa and is numerically equal to the difference of the ordinates of the initial and final lines of equilibrium in the section under consideration:

$$w(H_1, q_1, x) = y_1(H_1, q_1, x) - y_0(x), \quad (3)$$

where $y_1(H_1, q_1, x)$ is the function of the equilibrium line from the action of the load $q_1$, m; $y_0(x)$ is the function of the equilibrium line of the initial outline, m.

To build a line of equilibrium of a flexible thread the rules of construction of the diagram of the bending moments for the beam are used. The line of balance under the action of the transverse load coincides with the curve of the bending moments of the hinged beam with a span $l$, which is under the action of the same load; at the same time, the ordinates of the plot of moments are reduced by $H_1$ times and deposited from the chord connecting the attachment points of the flexible thread. Mathematically this is written as:

$$w(H_1, q_1, x) = \frac{M_1(q_1, x)}{H_1}, \quad (4)$$

where $M_1(q_1, x)$ is the bending moment in a hinged supported beam with a span $l$ from the action of the load $q_1$, Н·m; $H_1$ – a thrust, i.e. the horizontal component of the support reactions at the attachment points, equal in magnitude to the horizontal component of the longitudinal forces $T_1(x)$ in all sections of the flexible thread, N.

The construction of the line of equilibrium is reduced to the calculation of thrust $H_1$, since the calculation of the bending moment in a hinged supported single-span beam does not cause difficulties.
Mathematically the internal forces arising from bending in a single-span beam under the action of an evenly distributed load $q_1$ throughout the span $l$ can be written as follows:

$$M_i(q_1, x) = x \left( \int_0^l q_i(x) \, dx - \frac{x}{l} \int_0^l q_i(x) \, dx \right) - \frac{x}{l} \int_0^l x \cdot q_i(x) \, dx + \int_0^l x \cdot q_i(x) \, dx;$$

$$Q_i(q_1, x) = \int_0^l q_i(x) \, dx - \frac{l}{2} \int_0^l x \cdot q_i(x) \, dx. \quad (5)$$

A flexible thread, initially straight, having an initial length not exceeding the span and operating on perceiving the transverse load, is a string. In this case, the equation of the sagging curve will be:

$$y_0(x) = 0 \quad (7)$$

The thrust $H_0$, referring to the original equilibrium line, will be equal to zero.

To determine the thrust $H_1$ we use the equation of continuity of deformations:

$$L_0 + \Delta L(H_1, q_1) = L_1(H_1, q_1), \quad (8)$$

where $L_0$ is the initial length, m; $\Delta L(H_1)$ – elastic-plastic deformation, m; $L_1(H_1)$ – final length, m.

The initial length of the flexible thread is determined by the expression:

$$L_0 = \sqrt{\int_0^l \left(1 + \left(\frac{d}{dx} y_0(x)\right)^2\right) \, dx}. \quad (9)$$

To determine the elastic-plastic deformation of a flexible thread a formula based on Hooke's law can no longer be used:

$$\Delta L(H_1, q_1) = \frac{H_1 - H_0}{E \cdot A} \int_0^l \left(1 + \left(\frac{Q_i(q_1, x)}{H_1}\right)^2\right) \, dx, \quad (10)$$

where $H_0$ – thrust caused by the initial load, N; $E$ – modulus of elasticity, Pa; $A$ – cross-sectional area, m$^2$.

Knowing the stress intensity it is necessary to find the relative deformation from the deformation diagram, and determine the desired elongation from it.

Any kind of diagram can act as a material state calculation diagram which determines the relationship between stresses and relative deformations. In this case, the main parametric points of the diagram should be indicated. The main parametric points are the maximum stresses and the corresponding relative deformations, boundary values, etc. As a working diagram of the state of the material, which determines the relationship between stresses and relative deformations, let us take the piecewise linear deformation diagram presented in figure 2.

![Figure 2. The deformation diagram.](image)

In this diagram the elongation of the flexible thread is determined by the formula:

$$\Delta L(H_1, q_1) = \begin{cases} \frac{\sigma(H_1, q_1) \cdot L_0 - \sigma(H_1, q_1)}{E}, & \sigma(H_1, q_1) \leq \sigma_1 \\ L_0 \cdot \frac{\sigma(H_1, q_1) - \sigma_1}{E_1} + \varepsilon_1, & \sigma_1 \leq \sigma(H_1, q_1) \end{cases} \quad (11)$$
where $\sigma(H_1, q_1)$ – extensible stresses, Pa; $E_1$ – hardening module on the 2nd segment of the deformation diagram, Pa.

Extensible stresses are determined from the expression:

$$\sigma(H_1, q_1) = \frac{H_1 - H_0}{A \cdot L_0} \int_0^l \left( 1 + \left( \frac{Q_1(q_1, x)}{H_1} \right)^2 \right) dx. \quad (12)$$

The hardening modulus of the material on the 2nd segment of the strain diagram can be determined from equation [10]:

$$E_1 = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1}. \quad (13)$$

We can write as a function of thrust $H_1$ and the load $q_1$ the length of the flexible thread $L_1$ under the action of the centrifugal force of inertia causing the final contour:

$$L_1(H_1, q_1) = \int_0^{l/2} \left( 1 + \left( \frac{Q_1(q_1, x)}{H_1} \right)^2 \right) dx. \quad (14)$$

After calculating the elongation caused by the action of the centrifugal force of inertia and after finding the initial and final length of the flexible thread, it is possible to solve a system of two nonlinear equations (1) and (8) with two unknowns by the help of numerical methods and thus to find the values of thrust $H_1$ and the current evenly distributed load $q_1$.

After finding the thrust $H_1$ and the load $q_1$, it is possible to determine the longitudinal force arising in cross sections along the length of the flexible thread.

The function of the longitudinal force is equal to:

$$T_i = \sqrt{H_i^2 + Q_1(q_1, x)^2}. \quad (15)$$

Different but statically equivalent loads result in the same stress state in a flexible thread. This state is a linear (uniaxial) stress state:

$$\sigma_1(x) = \left( \frac{T_i(x)}{A} \right). \quad (16)$$

Solution of the test problem is in a linear and physically nonlinear formulation.

Let us consider the problem with the following initial data to compare the proposed methodology for calculating a rotating flexible thread under the action of centrifugal inertial force, which takes into account the development of plastic deformations with the calculation in a linear formulation.

Before carrying out numerical studies of the behavior of a flexible thread, we will check the adequacy of the results obtained using the proposed method. To do this we compare the main parameters of the stress-strain state of a flexible thread determined with the help of the well-known finite element method and the developed technique.

The calculation with the help of the finite element method was performed in the Russian software package LIRA-SAPR version 2015 release R4. The flexible thread was modeled by universal spatial rod finite elements taking into account physical and geometric nonlinearity (type 410), the breakdown has been made into 20 finite elements. The elastic coupling allowing to model the operation of the spring (elastic base) in the direction of the vertical axis has been set as a one-node finite element of the elastic coupling (type 51).

The calculation has been carried out by a nonlinear stepping processor designed to solve physically and geometrically nonlinear problems as well as and contact problems. As a result of mathematical modeling of the stress-strain state of a flexible thread with given geometrical and physico-mechanical characteristics, the proposed method and the finite element method at a rotation frequency of $n = 70$ min$^{-1}$, the corresponding values of the maximum longitudinal force $T_i(0) = T_i(l)$ and mid-span deflection $w(H_1, q_1, l/2)$ have been obtained. Values are summarized in Table 1.
Table 1. Comparison of modeling results

| Criterion                      | $T_1(0)$, $T_1(l)$, H | $w(H_1,q_1,l/2)$, mm |
|-------------------------------|------------------------|-----------------------|
| Finite element method         | 66.18                  | 26.733                |
| Proposed method               | 66.02                  | 26.98                 |
| Error,%                       | 0.25                   | 0.92                  |

As an example of calculation the results of modeling the stress-strain state of a flexible thread using the finite element method are presented in figure 3. The greatest values of the longitudinal forces are 66.180 N with the transverse displacement of the flexible thread 26.733 mm.

To obtain general conclusions, the data obtained in the calculations will be reduced to the following dimensionless form: the factor of utilization of the strength of the cross section $k_{max} = \frac{\sigma_1(l)}{\sigma_2}$; relative deflection $f = \frac{w(H_1,q_1,l/2)}{g \cdot m}$; dynamic factor $k_d = \frac{q_1 \cdot d}{g \cdot m}$, where $g$ is the acceleration of gravity, m/s$^2$; $m$ is the mass attached to the body thread, kg.

Figure 4 shows a graph of the change in the factor of utilization of the strength of the maximum loaded cross-section when changing the speed of rotation of a flexible thread with an attached elastic body, both in the linear and physically non-linear formulation of the problem. Figure 5 shows a graph of the change in the relative deflection in the middle span of a flexible thread and figure 6 presents a
The obtained results suggest that it is necessary to take into account the development of plastic deformations when modeling and studying the stress-strain state of a rotating flexible thread exposed to the centrifugal force of inertia. So from the graph presented in figure 4 it can be seen that when taking into account the true behavior of the material, the complete exhaustion of the bearing capacity with given geometrical and physico-mechanical characteristics of the flexible thread occurs at a rotation frequency \( n \) equal to 243 revolutions per minute, which is almost 2.5 times higher than obtained in the linear formulation of the problem. This allows to determine more accurately the carrying capacity with increasing speed of rotation and, accordingly, the increasing centrifugal force of inertia. This indicates that due to plastic deformations there are hidden reserves of strength the identification of which is of great interest in solving specific practical problems.

At the same time, owing to the additional displacement of the thread relative to the supports of the attachment due to the effect of the lateral load, the centrifugal forces on specific elementary areas significantly increase, which affects the amount of deflection of the thread (figure 5) increasing it.
On the other hand, while preventing the development of a conditional plastic hinge the linear calculation does not allow determining the point beyond which irreversible deformations are formed. So from the graph presented in figure 3 it can be seen that residual deformations appear already at rotational frequency \( n \) equal to 79 revolutions per minute, although linear calculation implies elastic work of the material up to the rotational frequency at which the destruction of the flexible thread occurs.

At the same time it can be seen from the graph shown in figure 6 that taking into account the physically nonlinear behavior of the material under load reduces the dynamic factor by 1.5 times, while increasing the relative deflection, which can be observed on the graph shown in figure 5 due to the development of plastic deformations.

4. Conclusion
The application of finite element methods for such calculations is significantly complicated due to the need to apply an unevenly imposed load across the thread which depends on the coordinate of the position of the thread with its indeterminate position. This increases the requirements for both the capabilities of IT, the conditions for selecting the loading function and the magnitude of the load application. The discrepancy between the results of the numerical modeling of flexible thread indicators by the proposed method compared with the finite element method does not exceed 1%. Thus the proposed method for calculating flexible threads and the results of its use on IT allow for computer modeling of the behavior of the thread with a dynamic view of its loading without increased IT computing capabilities.

The studies of the behavior of a flexible thread under dynamic load using the proposed method in a linear and physically nonlinear formulation of the problem indicate the need to take into account the development of plastic deformations to determine the true stress-strain state of a rotating flexible thread subjected to centrifugal inertial force. This will allow to determine more correctly the carrying capacity at increasing the speed (frequency) of rotation and consequently at increasing the centrifugal force of inertia, since due to the plastic deformations there are hidden reserves of strength of specific materials, the identification of which is of great interest in solving specific practical problems.

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