On the Berwald-Landsberg problem

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Ricardo Gallego Torromé
Departamento de Matemática
Universidade de São Paulo

Abstract

Given a Finsler manifold \((M, F)\), one can define natural averaged Riemannian manifolds living on \(M\) by averaging on the indicatrix \(I_x\) the fundamental tensor \(g\). In this paper we determine the Levi-Civita connection for such averaged Riemannian manifolds. We apply the result to the case when \((M, F)\) is a Landsberg space. Given a particular averaging procedure, the invariance of the averaged metric under certain homotopy in the space of Finsler manifolds over \(M\) is shown. Using such result we prove that any Landsberg space which is of class \(C^4\) is a Berwald space.

1 Introduction

A Finsler space is a smooth manifold \(M\) and a smooth family of Minkowski norms, such that each tangent space \(T_xM\) is endowed with one of the norms of the family. There are several relevant types of Finsler spaces. For instance, if the Minkowski norms are flat in some specific way \([7]\), the Finsler manifold is a Minkowski space. In the case that each of the Minkowski norms at each \(T_xM\) is a scalar product, then \((M, F)\) is a Riemannian manifold. If the punctured tangent spaces endowed with the Riemannian structure associated to the Finsler metric are isometric, it is called a Landsberg space; if such isometries are linear, it is a Berwald space. Regular Minkowski, Riemann, Berwald, and Landsberg spaces are among the Finsler manifolds that have been thoroughly investigated in the literature\([3,7]\).

Among the above special Finsler spaces, regular Landsberg spaces are the most elusive. They were first introduced by G. Landsberg \([16,17,18]\) (in a non-Finsler

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2The hierarchy of such categories is Minkowski \& Riemann \(\subset\) Berwald \(\subset\) Landsberg \(\subset\) Finsler. Typical examples of Berwald spaces are in the category of Randers spaces \([2]\).
framework) around 1907 and despite the intense effort in the last decades in the search for regular Landsberg spaces that are not Berwaldian, there is not known example of those unicorns. Indeed, a number of negative results on their existence came out from such investigations (see [4, 5, 19, 12] for a review of the state of the art until 2007). The problem considered in this paper is on the existence or not of regular Landsberg spaces.

Regularity of the Finsler metric is relevant for the problem, since the lack of enough regularity opens the possibility to examples of Finsler spaces with specified curvature properties. An example of this is the weakly Berwald-Rund surface described in [7, 24]. The lack of regularity of such 2-dimensional (weakly) Finsler spaces circumvents Szabó rigidity theorem on Berwald surfaces, that states that any regular Berwald surface is either Minkowski (in the sense that it endows a Minkowski norm) or Riemannian. With such precedent it is not a surprise that the regularity hypothesis is fundamental to the existence of Landsberg spaces that are not Berwald spaces. Indeed, there are examples of Landsberg spaces which are not regular in the whole slit tangent bundle $N = TM \setminus \{0\}$ [1, 2, 3]. The positiveness condition is also important, since it is also known the existence of non-Berwaldian Landsberg surfaces without the restriction of being regular and positive definite in the whole slit tangent bundle [30].

However, regularity and positiveness hypotheses are used in the generalization of Riemannian results to the Finsler category [7]. The paradigm is the Chern-Gauss-Bonnet theorem, which admits a generalization in the category of regular Landsberg spaces in dimension two [7, 11] and dimension four [6], using particular generalizations of the transgression method [8]. There are general results on higher dimensions and for general Finsler structures [25, 26]. However, it is under some further assumptions, like $(M, F)$ being Landsberg, that one obtains closer formulas to the standard Chern-Gauss-Bonnet formula in Finsler geometry [8]. From this perspective, the problem of the existence and examples of regular Landsberg spaces is relevant.

A natural procedure to investigate general Finsler spaces is through the investigation of averaged geometric quantities and its relation with the original Finslerian quantities. Averaged Finslerian quantities appear as soon as in [11] when considering the Chern-Gauss-Bonnet theorem for Landsberg surfaces. Also relevant, an averaged Riemannian metric was used by Z. Szabó in the classification of Berwald spaces [27]. That construction works for Berwald spaces and it is difficult to see how it can be generalized to other kind of Finsler spaces. More recently, a general theory of averaging Finsler structures was formulated in [13, 15]. In that pre-print, the metric, the connection, parallel transport, torsion and curvatures are averaged, obtaining new geometric objects living on $M$. The averaged of Finslerian geometric structures has been found useful in the analysis of Berwald spaces [15], formulation of deterministic quantum models [14] or in the investigation of geodesic equivalence [15, 21, 22], to put some very different kind of applications.

Indeed, an isotropic averaging procedure was used in an attempt to solve the Berwald-Landsberg problem [28]. However, it was soon recognized that the key argument contained a fundamental gap [20, 29]: that the isotropically averaged metric does not contain enough information to link it with the Chern connection of the original Finsler structures. This link only happens under more restricted
conditions that just being Landsberg [29].

We have found that in the investigation of the Berwald-Landsberg problem it is fundamental to consider an homotopy invariant property of the averaged metric. This avoids the problem of Szabó’s paper [28], since in our approach one only deals with geometric quantities constructed from the averaged metric. After fixing the averaging procedure, one is able to prove that a family of interpolating Finsler metrics \((M, F_t)\) has the same averaged Riemannian metric \(h\). Therefore, the Levi-Civita connection is invariant under an homotopy of Finsler metrics from \(g\) to \(h\). Such invariance is fundamental to prove the following result,

**Theorem 1.1** Let \((M, F)\) be a Landsberg structure with \(F\) of class \(C^4\) on \(N = TM \setminus \{0\}\) and \(\dim(M) \geq 2\). Then \((M, F)\) is a Berwald space.

This gives a negative answer to the problem, at least for spaces with such regularity.

**2 The Berwald-Landsberg problem**

**Basic notions of Finsler geometry.** We follow the notation from [7]. Let \((x, U)\) be a local coordinate system on the smooth \(n\)-dimensional manifold \(M\), with \(U \subset M\) being an open set of \(M\). In this coordinate system the point \(x \in U\) has local coordinates \((x^1, ..., x^n)\). \(TM\) is the tangent bundle of the manifold \(M\). We use Einstein’s convention for up and down repeated indices if the contrary is not stated. Each local coordinate system \((x, U)\) on the manifold \(M\) induces a local natural coordinate system \((x, y, TU)\) on \(TM\) such that a tangent vector \(y = y^i \frac{\partial}{\partial x^i} \in T_x M \subset TM\) with \(x \in U\) has coordinates \((x^1, ..., x^n, y^1, ..., y^n)\). Once a local coordinate system is fixed, we can identify the point \(x\) with its coordinates \((x^1, ..., x^n)\) and the tangent vector \(y \in T_x M\) at \(x\) with its components \(y = (y^1, ..., y^n)\) respect to the basis \(\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}\). The slit tangent bundle is \(\pi : N \rightarrow M\), with \(N = TM \setminus \{0\}\).

We will use the following notion of Finsler manifold in our considerations,

**Definition 2.1** A Finsler manifold \((M, F)\) of class \(C^k\) is a smooth manifold \(M\) together with a non-negative, real function \(F : TM \rightarrow [0, \infty]\) such that

1. \(F\) is of class \(C^k\) on the slit tangent bundle \(N = TM \setminus \{0\}\).
2. Positive homogeneity holds: \(F(x, \lambda y) = \lambda F(x, y)\) for every \(\lambda > 0\).
3. Strong convexity holds: the Hessian matrix

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{2.1}\]

is positive definite on \(N\).

Since we will deal with curvatures, we will consider metrics of type at least of type \(C^4\).
Definition 2.2 Let \((M, F)\) be a Finsler manifold of class \(C^3\). The components of the Cartan tensor are defined as
\[
A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i, j, k = 1, \ldots, n. \tag{2.2}
\]
The formal second kind Christoffel-type symbols \(\gamma^i_{jk}(x, y)\) are defined by the expression
\[
\gamma^i_{jk} = \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{sk}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^s} \right), \quad i, j, k, s = 1, \ldots, n.
\]
The non-linear connection coefficients are
\[
\frac{N^i_j}{F} = \gamma^i_{jk} y^k - A^i_{jk} \gamma^k_{rs} y^r y^s y^s, \quad F \neq 0, \quad i, j, k, r, s = 1, \ldots, n,
\]
where \(A^i_{jk} = g^{il} A_{ljk}\).

Let us consider the local coordinate system \((x, y, U)\) on the tangent bundle \(TM\). An adapted frame is determined by the smooth tangent basis for \(T_uN, u \in N\) defined by the locally defined vector fields [7]
\[
\left\{ \frac{\delta}{\delta x^1} | u, \ldots, \frac{\delta}{\delta x^n} | u, F \frac{\partial}{\partial y^1} | u, \ldots, F \frac{\partial}{\partial y^n} | u \right\}, \quad \frac{\delta}{\delta y^i} | u = \frac{\partial}{\partial x^j} | u - N^i_j \frac{\partial}{\partial y^i} | u, \quad i, j = 1, \ldots, n. \tag{2.3}
\]
\[
\left\{ \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right\} \text{ generates locally the horizontal distribution}; \left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\} \text{ generates locally the vertical distribution}.
\]
The pull-back bundle \(\pi^*TM\) is the maximal subset of the cartesian product \(N \times TM\) such that the diagram
\[
\begin{array}{ccc}
\pi^*TM & \xrightarrow{\pi_2} & TM \\
\pi_1 \downarrow & & \downarrow \pi \\
N & \xrightarrow{\pi} & M
\end{array}
\]
is commutative. The projections on the first and second factors are
\[
\pi_1 : \pi^*TM \rightarrow N, \ (u, \xi) \mapsto u, \quad \pi_2 : \pi^*TM \rightarrow TM, \ (u, \xi) \mapsto \xi.
\]
A local frame for the sections of \(\pi^*TM\) is \(\{\pi^* \frac{\partial}{\partial x^1}, \ldots, \pi^* \frac{\partial}{\partial x^n}\}\). In the case the frame on \(M\) \(\{e_1, \ldots, e_n\}\), the corresponding frame for sections of \(\pi^*TM\) is \(\{\pi^* e_1, \ldots, \pi^* e_n\}\).
A similar construction can be done for the pull-back bundle \(\pi^*T^*M\). In this case a local frame for \(\pi^*T^*M\) is given by the collection of fiber basis \(\{\pi^* \xi^i dx^i, i = 1, \ldots, n\}\).
The dual to the frame (2.3) is the co-frame
\[
\left\{ dx^1, \ldots, dx^n, \frac{\delta y^1}{F}, \ldots, \frac{\delta y^n}{F}, \quad \delta y^i := dy^i + N^i_j (x, y) dy^j \right\}. \tag{2.4}
\]
There is a tangent structure \([32]\), which is an homomorphism \(J \in T^{(1,1)}(\mathbb{T}N, \pi^*TM)\) given in local coordinates by
\[
J : T\mathbb{N} \rightarrow \pi^*TM,
\]
\[
\zeta^i \frac{\partial}{\partial x^i} \bigg|_\xi \mapsto 0,
\]
\[
\zeta^i \frac{\partial}{\partial y^i} \bigg|_\xi \mapsto \zeta^i \pi^* \frac{\partial}{\partial x^i} \bigg|_\xi.
\]  

(2.5)

**Definition 2.3** The fundamental tensor and the Cartan tensor are defined in local coordinate by the following expressions:

1. The fundamental tensor is
\[
g(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \pi^*dx^i \otimes \pi^*dx^j.
\]  

(2.6)

2. The Cartan tensor is
\[
A(x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} \pi^*dx^i \otimes \pi^*dx^j \otimes \pi^*dx^k = A_{ijk} \pi^*dx^i \otimes \pi^*dx^j \otimes \pi^*dx^k.
\]  

(2.7)

Both \(g\) and \(A\) are natural geometric objects. They are tensors along the projection \(\pi : \mathbb{N} \rightarrow \mathbb{M}\). However, it is usual to call them tensors in the Finslerian community and we adopt it.

The Chern connection is defined through the following \([7]\)

**Theorem 2.4** Let \((\mathbb{M}, F)\) be a Finsler manifold. The vector bundle \(\pi_1 : \pi^*TM \rightarrow \mathbb{N}\) admits a unique linear connection \(\nabla\) whose connection 1-forms \(\{\omega^i_j, i, j = 1, \ldots, n\}\) are defined by the relations
\[
\nabla(\pi^*e_j) := (\omega^i_j) \pi^*e_i, \ i, j = 1, \ldots, n
\]  

(2.8)

and are such that the following structure equations hold:

1. Torsion free condition or symmetric condition,
\[
dx^j \wedge w^i_j = 0, \ i, j = 1, \ldots, n.
\]  

(2.9)

2. Almost \(g\)-compatibility condition,
\[
dg_{ij} - g_{kj}w^k_i - g_{ik}w^k_j = 2A_{ijk} \delta y^k_F, \ i, j, k = 1, \ldots, n.
\]  

(2.10)

The connection coefficients of the Chern connection in natural coordinates are denoted by \(\Gamma^i_{jk}(x, y)\). It is notable that if \((\mathbb{M}, F)\) corresponds to a Riemannian manifold, the Chern connection coincides with the Levi-Civita connection of the Riemannian metric \(g\).
Let $\sigma : [a, b] \to M$ be a smooth curve on $M$. Associated with the Chern connection there is a canonical diffeomorphism
\[
\psi_s : T_x M \setminus \{0\} \to T_{\sigma(s)} M \setminus \{0\}
\]
with $y_s$ the unique (for a small enough time $s$) solution of the differential equation,
\[
\dot{y}_s^k + y_s^j \Gamma^k_{ij}(\sigma(s), y_s) \dot{\sigma}^j = 0
\]
with initial condition $\sigma(a) = x$, $y_a = y$. It happens that parallel transport preserves the Finsler norm \[7\],
\[
F(\sigma(s), y_s) = F(x, y).
\]
Given a Finsler manifold of class $C^4$, the curvature endomorphisms of the Chern connection are $C^1$ tensors along the projection $\pi : N \to M$, with two non-zero pieces \[7\]. The $hh$-curvature tensor has components
\[
P^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l}.
\]
This curvature is equivalent to the Riemannian curvature. However, the curvature that will be of interest for the problem is the $hv$-curvature, which components are
\[
P^i_{jkl} = -F \frac{\partial \Gamma^i_{jk}}{\partial y^l}.
\]
The indicatrix at the point $x \in M$ is the compact, strictly convex sub-manifold
\[
I_x := \{ y \in T_x M \mid F(x, y) = 1 \} \subset T_x M.
\]
The indicatrix bundle $\pi : I \to M$ is the fiber bundle over $M$, with fiber $I_x$ diffeomorphic to the standard sphere $S^{n-1} \subset \mathbb{R}^n$. Each indicatrix $I_x \subset N_x$ is endowed with a natural Riemannian metric $g_x$, which is obtained by isometric embedding of the Riemannian metric $(T_x M, g_x)$, where the Riemannian metric $g_x$ on $T_x M$ is defined by the smooth family of scalar products
\[
g_x(y) : T_y T_x M \times T_y T_x M \to \mathbb{R}, \quad (v, v) \mapsto g_{ij}(x, y)v^i v^j, \quad v \in T_y (T_x M).
\]
with $y \in T_x M$. The metric of the isometric embedding $e : I_x \to T_x M$ and the ambient structure $(T_x M, g_x)$ are both denoted by $g_x$.

To each of the Riemannian manifolds of the family $\{(I_x, g_x), x \in M\}$, there is associated a volume form on each indicatrix $I_x$
\[
dvol(x, y) := \sqrt{\det(g(x, y))} \ d\Omega(x, y) := \sqrt{\det(g(x, y))} \sum_{j=1}^n (-1)^{j-1} y^j \ dy^1 \wedge \cdots \wedge dy^{j-1} \wedge
\]
\[
\wedge dy^{j+1} \wedge \cdots \wedge dy^n.
\]
This form is the volume form on \(I_x\) of the embedded Riemannian manifold \((M, g_x)\) [7]. One can define the volume function on \(M\) by

\[
vol : M \rightarrow \mathbb{R}^+, \quad x \mapsto vol(I_x) := \int_{I_x} 1 \sqrt{\det g((x, y))} \, d\Omega(x, y).
\]

**Statement of the Berwald-Landsberg problem.**

We consider Finsler manifolds of class \(C^4\).

**Definition 2.5** A Finsler manifold \((M, F)\) is a Berwald space iff for every point \(x \in M\), there is a coordinate system such that the connection coefficients of the Chern connection live on the base manifold \(M\).

In the following proposition the second statement follows from the fact that for Berwald spaces the Chern connection is an affine, symmetric connection on \(M\).

**Proposition 2.6** For Berwald spaces the following properties hold:

1. The Chern connection defines an affine connection on \(M\) [7].
2. For each point \(x \in M\) there is a local coordinate system on \(M\) such that the connection coefficients of the Chern connection are zero at the point \(x\).
3. For a Berwald space the curvature components \(P_{ijkl}\) are zero. Conversely, if the hv-curvature of the Finsler manifold \((M, F)\) is zero, then it is a Berwald space [7].

We take the following definition of Landsberg space [7].

**Definition 2.7** Let us consider on \(\pi^* TM\) a local frame such that the basis on each fiber \(\pi^{-1}(u) \subset \pi^* TM\) is such that one of the elements of the local frame is \(e_n = y^k \frac{\partial}{\partial x^k}\). A Finsler manifold \((M, F)\) is a Landsberg space if for the hv-curvature \(P\) of the Chern connection, the condition \(P^n_{ijk} = 0\) holds.

The covariant derivative of the Cartan tensor along \(\delta_{\delta x}\) is denoted by \(A_{ijk}\). Let us choose a local orthonormal frame of \(\pi^* TM\) such that \(e_n = y^k \frac{\partial}{\partial x^k}\). Then the Landsberg tensor and its trace are

\[
\hat{A}_{ijk} := A_{ijkl}n, \quad \text{tr} \hat{A} := \hat{A}_{ik}^k e_k, \tag{2.19}
\]

where the matrix \(g^{-1}\) has been used to raise the indices. One has the following result [5, 10].

**Proposition 2.8** The Finsler structure \((M, F)\) is Landsberg iff \(\hat{A}_{ijk} = 0\).

It is well known that a Berwald space is a Landsberg space. The Berwald-Landsberg problem consists on the other possibility: are there regular Landsberg
spaces but not Berwaldian? This is a long-standing open problem in Finsler geometry [5]. Regularity conditions are essential; there are known examples of Landsberg spaces in the category of Randers spaces [1, 2, 3, 30].

Clearly from its definition and from proposition 2.6, for a Landsberg space that is not Berwald, the Chern connection is not an affine connection. We will exploit this fact later.

In this paper we provide a negative answer given by theorem 1.1 to the Berwald-Landsberg problem. This result is a consequence of an homotopy invariance of the Levi-Civita connection coefficients for the averaged metrics combined with a result of Z. Shen on the isometry of the indicatrix for Landsberg spaces [9].

**Averaged Finsler metrics.**

In order to define the averaged metrics we introduce convenient measures on each indicatrix \( N_x = \pi^{-1}(x) \subset \mathbf{N} \). A measure \( f(x, y) \) will be a positive, smooth function on \( \mathbf{N} \). For each \( x \in \mathbf{M} \), the restriction \( f_x(y) := f(x, y) \) defines a positive function on \( \pi^{-1}(x) \).

**Definition 2.9** The averaged \( \langle f \rangle \) associated with the Finsler structure \( (\mathbf{M}, F) \) and the measure \( f \) is

\[
\langle f \rangle(x) := \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x, y) g_{ij}(x, y) \sqrt{\det(g)} \, d\Omega(x, y),
\]

where \( g \) is the fundamental tensor of \( F \).

It is possible to see that indeed this definition is independent of the coordinate chart used and indeed defines a global object \( \langle f \rangle \) on \( \mathbf{M} \) [13]. Averaged objects are indicated between angles. For instance \( \langle g_{ij} \rangle \) stand for the coefficients \( \langle f \rangle_{ij} \) defined by (2.21) using the measure \( f \); the averaged metric will is \( \langle f \rangle g \), etc... If we assume that the measure is fixed, we left only the angles \( \langle \cdot \rangle \) to indicate averaged quantities.

The proof for the following properties are direct from the definition [13],

**Proposition 2.10** The following properties hold:

1. Given \((\mathbf{M}, F)\) and the positive, smooth function \( f \in \mathcal{F}(\mathbf{N}) \), \( \langle f \rangle \) is a Riemannian metric on \( \mathbf{M} \).

2. For a Finsler structure \((\mathbf{M}, F)\) which is Riemannian one has that \( \langle f \rangle = \langle f \rangle g \). The function \( \langle f \rangle \) is given by

\[
\langle f \rangle(x) := \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x, y) \sqrt{\det(g)} \, d\Omega(x, y),
\]

With such normalization, one has that in the Riemannian case \( \langle f \rangle \) is conformal to the initial metric \( g \).
3. The metric \( f_h \) is independent of the local coordinates system where it is expressed.

4. Averaging is a linear operation for arbitrary \( f \).

5. One can extend the averaging to tensors along \( \pi \) of arbitrary type \((p,q)\).

6. The averaging is not a tensorial operation for an arbitrary measure \( f \).

7. Given the isometry groups \( \text{Iso}(g) \) and \( \text{Iso}(h) \) and if the measure \( f \) is invariant under \( \text{Iso}(g) \), then \( \text{Iso}(g) \subset \text{Iso}(h) \).

3 The coefficients of the Levi-Civita connection for \( h \)

Let us consider the Finsler manifold \((M,F)\). Let us adopt a local coordinate system \((x,U)\) on \( M \) and the associated local natural coordinate system on \( TM \). In order to avoid cumbersome notation we denote \( f_h \) simply by \( h \). Then given a positive, smooth function \( f(x,y) \), the metric coefficients for the averaged Riemannian metric \( h \) are

\[
h_{ij}(x) = \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x,y) g_{lj} \sqrt{\det(g)} \ d\Omega
\]

\[
= \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x,y) g_{lj} \sqrt{\det(g)} \sum_{k=1}^{n} y^k (-1)^{(k-1)} dy^1 \wedge \ldots \wedge dy^{k-1} \wedge
\]

\[
\wedge dy^{k+1} \wedge \ldots \wedge dy^n.
\]

In this section we obtain the Christoffel symbols of the metric \( h \),

\[
h^{k}_{\ ij}(x) = \frac{1}{2} h^{kl} \left( \frac{\partial h_{lj}}{\partial x^i} + \frac{\partial h_{li}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^l} \right), \ i,j,k,l = 1,...,n \quad (3.1)
\]

for any Finsler manifold \((M,F)\) in terms of the original fundamental tensor \( g \) and measure \( f(x,y) \). Therefore, we need to calculate derivatives of the form

\[
\frac{\partial h_{ij}}{\partial x^i} = \frac{\partial}{\partial x^i} \left( \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x,y) g_{lj} \sqrt{\det(g)} \sum_{k} y^k (-1)^{(k-1)} dy^1 \wedge \ldots \wedge dy^{k-1} \wedge dy^{k+1} \wedge \ldots \wedge dy^n \right)
\]

\[
= \frac{\partial}{\partial x^i} \left( \frac{1}{\text{vol}(I_x)} \right) \int_{I_x} f(x,y) g_{lj} \sqrt{\det(g)} \sum_{k} y^k (-1)^{(k-1)} dy^1 \wedge \ldots \wedge dy^{k-1} \wedge dy^{k+1} \wedge \ldots \wedge dy^n
\]

\[
+ \frac{1}{\text{vol}(I_x)} \frac{\partial}{\partial x^i} \left( \int_{I_x} f(x,y) g_{lj} \sqrt{\det(g)} \sum_{k} y^k (-1)^{(k-1)} dy^1 \wedge \ldots \wedge dy^{k-1} \wedge dy^{k+1} \wedge \ldots \wedge dy^n \right).
\]
The first partial derivatives of the volume function of the indicatrix was obtained in [9].

**Theorem 3.1** Let \((M, F)\) be a \(C^4\) Finsler manifolds and consider the volume function of the unit tangent sphere \(\text{vol}(I_x)\). Then for any tangent vector \(b^r \partial \partial x^r \in T_xM\) the following formula holds:

\[
b^r \frac{\partial \text{vol}(I_x)}{\partial x^r} = - \int_{I_x} g(tr \dot{A}, b^r \frac{\partial}{\partial x^r}) \sqrt{\det(g(x, y))} d\Omega(x, y). \tag{3.2}\]

In particular

\[
tr \dot{A} = 0 \implies \text{vol}(I_x) \text{ is locally constant.} \tag{3.3}\]

As a consequence one obtains the following expression,

\[
\frac{\partial}{\partial z^i} \bigg|_{z=x} \left( \frac{1}{\text{vol}(I_x)} \right) = - \frac{1}{\text{vol}^2(I_x)} \frac{\partial \text{vol}(I_x)}{\partial x^i}
\]

\[
= \frac{1}{\text{vol}^2(I_x)} \int_{I_x} g(tr \dot{A}, \frac{\partial}{\partial x^i}) \sqrt{\det(g(x, y))} d\Omega(x, y)
\]

and by the definition of the averaging operator \((\cdot)_{f=1} : \mathcal{F}(N) \to \mathcal{F}(M)\) of functions,

\[
\frac{\partial}{\partial x^i} \left( \frac{1}{\text{vol}(I_x)} \right) = \frac{1}{\text{vol}(I_x)} (g(tr \dot{A}, \frac{\partial}{\partial x^i}) )_{f=1}. \tag{3.4}\]

Next, we compute the partial derivative of the integral. In order to perform this calculation, let \(I_x\) be the indicatrix at the point \(z\), which is the point translate along the integral curve of the vector field \(\frac{\partial}{\partial x^i}\) with initial condition \(z(0) = x\). One needs to calculate partial derivatives of the type

\[
\left. \frac{\partial}{\partial z^i} \left( \int_{I_x} f(z, y_z) g_{ij}(z, y_z) \sqrt{\det(g(z, y_z))} d\Omega(z, y_z) \right) \right|_{z=x}, \quad y_z \in T_zM.
\]

A difficulty in the computation of this derivative is that the manifold where the integrations are performed varies with the limit \(z \to x\) in the partial derivative. In the case of an arbitrary Finsler manifold \((M, F)\), let us consider the diffeomorphism \(\hat{\phi}_{xx} : I_x \to I_z, x, z \in U\) defined as follows. The locally defined vector field \(\frac{\partial}{\partial z^i}\) defines a local flow in a neighborhood of \(x \in M\). Therefore, there is a local diffeomorphism \(\hat{\phi}\) on \(TU\) such that the following diagram commutes,

\[
\begin{array}{ccc}
I_x & \xrightarrow{\hat{\phi}_{xz}} & I_z \\
\downarrow \pi & & \downarrow \pi \\
x & \xrightarrow{\hat{\phi}_{xx}} & z.
\end{array}
\]
Note that for a general Finsler manifold $(M, F)$, one has that $\hat{\phi}_{zz}(I_x) \neq I_z$, since $\hat{\phi}_{zx}$ is not always $F$-preserving the norm defined by $F$. However, we consider the scaling transformation

$$pr_z: \hat{\phi}_{zx}(I_x) \rightarrow I_z, \quad y \mapsto \frac{1}{F(y)} y. \quad (3.6)$$

**Lemma 3.2** For close enough $z$ to $x$, the scaling $pr_z$ is a diffeomorphism.

**Proof:** First, note that $\dim(\hat{\phi}_{zx}(I_x)) = \dim(I_x)$, since $\hat{\phi}_{zx}$ is a diffeomorphism. Moreover, since $I_x$ is strictly convex, for $z$ close enough to $x$, the hypersurface $\hat{\phi}_{zx}(I_x)$ is also strictly convex (the contrary is in contradiction with the fact that $I_x$ is strictly convex and that is of co-dimension 1 in $T_xM$). Both $I_x$ and $\hat{\phi}_{zx}(I_x)$ enclose the origin $y = 0$. It follows that $pr_z$ is both injective and surjective. The inverse $pr_z^{-1}$ is also a diffeomorphism. $\square$

The Jacobian matrix of the projection $pr_z$ is

$$\left(\frac{\partial pr_z}{\partial y}\right)_k^j = \left(\frac{1}{F(y)} \delta_k^j - \frac{y^j}{F^2(y)} \frac{\partial F}{\partial y^k}\right). \quad (3.7)$$

When we take the limit $z \rightarrow x$, the Jacobian of $pr_z$ takes the limit value

$$\lim_{z \rightarrow x} \left| \left(\frac{\partial pr_z}{\partial y}\right)_k^j \right| = \lim_{z \rightarrow x} \left| \left(\frac{1}{F(y)} \delta_k^j - \frac{y^j}{F^2(y)} \frac{\partial F}{\partial y^k}\right) \right| = \left| \left(\frac{1}{F(y)} \delta_k^j - \frac{y^j}{F^2(y)} \frac{\partial F}{\partial y^k}\right) \right|(x) = 1.$$

Since we will deal with the limit $\lim_{z \rightarrow x} \left| \left(\frac{\partial pr_z}{\partial y}\right)_k^j \right| = 1$ when calculating integrals on the indicatrix, one can consider directly the commutative diagrams and corresponding diffeomorphism

$$\begin{array}{ccc}
I_x & \xrightarrow{\hat{\phi}_{zx}} & I_z \\
\downarrow & & \downarrow \\
x & \xrightarrow{\hat{\phi}_{zx}} & z,
\end{array} \quad (3.8)$$

with $\phi = pr \circ \hat{\phi}$ as equivalent to the diagram (3.5). Then we will apply the invariance under diffeomorphism of integrals on compact manifolds [31] to the diffeomorphism $\hat{\phi}$ instead of the original $\hat{\phi}_{zx}$.

Recall that the total lift $\hat{X}$ as the vector field whose flow in $TB$ is $(x, \xi, s) \mapsto (\phi_s(x), \phi_s^*(\xi))$, $\xi \in T_xB$. Let us denote the local flow of $\hat{X}$ by the diffeomorphism $\phi$. Then we can re-write the invariance of the integral as the following lemma,
Lemma 3.3 Let $\pi_B: \hat{B} \rightarrow B$ be a fibred manifold such that $\dim(\pi_B^{-1}(x)) = p$ for all $x \in B$ and $\omega$ be a volume form when restricted to each fiber $\pi_B^{-1}(x)$, $X \in \Gamma TB$ and its total lift $\hat{X} \in \hat{B}$. Then

$$X \cdot \left( \int_{\phi(\pi^{-1}(x))} \omega \right) \bigg|_x = \int_{\pi^{-1}(x)} \left( \mathcal{L}_{\hat{X}} \phi^* \omega \right) \bigg|_x.$$  \hspace{1cm} (3.9)

Proof: By the formula of the invariance of the integral on manifolds by diffeomorphism transformations [31], one has that

$$\int_{\phi(\pi^{-1}(x))} \omega \bigg|_x = \int_{\pi^{-1}(x)} \phi^* \omega. \hspace{1cm} (3.10)$$

From the definition of Lie derivative [31],

$$\mathcal{L}_{\hat{X}} \phi^* \omega = \frac{d}{ds} \bigg|_{s=0} (\phi_s^* \omega). \hspace{1cm} (3.11)$$

it follows that

$$X \cdot \left( \int_{\phi(\pi^{-1}(x))} \omega \right) \bigg|_x = X \cdot \left( \int_{\pi^{-1}(x)} \phi^* \omega \right) \bigg|_x = \lim_{s \to 0} X \cdot \left( \int_{\pi^{-1}(x)} \phi_s^* \omega(\phi_s(x)) - \omega(x) \right) \bigg|_x = \int_{\pi^{-1}(x)} \left( \mathcal{L}_{\hat{X}} \phi^* \omega \right) \bigg|_x. \hspace{1cm} \square$$

We can apply lemma 3.3 to the case $B = M$, $\hat{B} = I$ (the indicatrix bundle over $M$), $X = \frac{\partial}{\partial z} x$ and $\phi = \phi_{zx}$, with the following consequences:

$$\left. \frac{\partial}{\partial z} \right|_{z=x} \left( \int_{I_x} f(z, y) g_{ij}(z, y) \sqrt{\det(g(z, y))} d\Omega(z, y) \right) = \left. \frac{\partial}{\partial z} \right|_{z=x} \left( \int_{I_x} \phi^* f(z, y) g_{ij}(z, y) \sqrt{\det(g(z, y))} d\Omega(z, y) \right) = \left. \frac{\partial}{\partial z} \right|_{z=x} \left( \int_{I_x} \phi_{zx}^* (f(z, y) g_{ij}(z, y) \sqrt{\det(g(z, y))} d\Omega(x, y)) \right) = \int_{I_x} \mathcal{L}_{\phi_{zx}} \left( \phi_{zx}^* (f(z, y) g_{ij}(z, y) \sqrt{\det(g(z, y))} d\Omega(x, y)) \right) \bigg|_{z=x}. \hspace{1cm} 12$$
Therefore, the derivative of the metric components $h_{ij}$ is

$$\frac{\partial h_{ij}}{\partial x^i} = \frac{1}{\text{vol}(I_x)} (g(tr \, \dot{A}, \frac{\partial}{\partial x^i}))_1 \int_{I_x} f(x, y) g_{ij}(x, y) \sqrt{\det g(x, y)} d\Omega(x, y)$$

$$+ \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \mathcal{L}_{\frac{\partial}{\partial x^i}} \left( f(z, y_z) \right) \right) \bigg|_{z=x} g_{ij}(x, y) \sqrt{\det g(z, y_z)} d\Omega(x, y)$$

$$+ \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x, y) \mathcal{L}_{\frac{\partial}{\partial x^i}} \left( g_{ij}(x, y) \sqrt{\det g(z, y_z)} d\Omega(x, y) \right) \bigg|_{z=x}. $$

We introduce the endomorphisms

$$\partial : T_x M \rightarrow T_x M, \quad y^i \mapsto \partial^i \cdot y^j := (h^{-1})^i_{\ j} g_{ij} y^j. $$

and define the following coefficients:

$$(\nabla \text{vol}(I_x))_{k \ ij}(x) := \frac{1}{2 \text{vol}(I_x)} \int_{I_x} f(x, y) \left( \partial^k \, g(tr \, \dot{A}, \frac{\partial}{\partial x^i}) + \partial^i \, g(tr \, \dot{A}, \frac{\partial}{\partial x^k}) - h_{ij} h^{kl} g(tr \, \dot{A}, \frac{\partial}{\partial x^l}) \right) \sqrt{\det g} d\Omega(x, y).$$

Let us introduce also the following notation,

$$(\log \sqrt{\det g})_{k \ ij}(x) := \frac{1}{2 \text{vol}(I_x)} \left( \int_{I_x} f(x, y) \left( \mathcal{L}_{\frac{\partial}{\partial x^k}} (\log \sqrt{\det g}) d\Omega \right) \right) \sqrt{\det g}$$

$$+ \int_{I_x} \partial^k \left( \mathcal{L}_{\frac{\partial}{\partial x^i}} (\log \sqrt{\det g}) d\Omega \right) \sqrt{\det g}$$

$$- \int_{I_x} h_{ij} h^{kl} \left( \mathcal{L}_{\frac{\partial}{\partial x^l}} (\log \sqrt{\det g}) d\Omega \right) \sqrt{\det g} \bigg|_{z=x}. $$

There is a contribution to the Christoffel symbols coming from the derivatives of $f(x, y),

$$f^k \ ij(x) := \frac{1}{2 \text{vol}(I_x)} \left( \int_{I_x} \partial^k \, g_{ij} \left( \frac{\partial f}{\partial x^i} g_{sj} + \frac{\partial f}{\partial x^s} g_{is} - \frac{\partial f}{\partial x^l} g_{lj} \right) \sqrt{\det g} d\Omega(x, y) \right).$$

(3.13)

**Proposition 3.4** Let $(M, F)$ be a $C^4$ Finsler manifold and $(M, h)$ the corresponding averaged Riemannian manifold performed with a positive, smooth measure $f \in \mathcal{F}(N)$. Then the coefficients of the Levi-Civita connection $h$ are given by the formula

$$h \Gamma^k \ ij(x, f) = (\nabla \text{vol}(I_x))_{k \ ij}(x) + \partial^k \Gamma^l \ ij f(x) + (\log \sqrt{\det g})_{k \ ij}(x) + f^k \ ij(x) \quad (3.14)$$

for all $i, j, k, l = 1, \ldots, n$.

**Proof:** The right hand side is obtained by rearranging the derivatives that appear in the calculation of $h \Gamma^k \ ij(x, f)$. \qed

Again, if the measure $f$ fixed, one can omit $f$ on the notation.
4 Application to Landsberg spaces

We will apply proposition 3.4 when \((M, F)\) is a Landsberg space. Let us consider \(g_x\) the metric on \(I_x\) induced by isometric immersion on the Riemannian manifold \((T_xM \setminus \{0\}, g_x)\). Then one has the following result [5, 10].

**Theorem 4.1** Let \((M, F)\) be a Landsberg space. Then for each pair of points \(x, z \in M\), the Riemannian manifolds \((I_x, g_x), (I_z, g_z)\) are isometric.

**Remark 4.2** For a Landsberg space each pair of punctured Riemannian tangent spaces \((T_xM \setminus \{0\}, g_x)\) and \((T_zM \setminus \{0\}, g_z)\) are isometric [5, 10], although the isometry is not necessarily linear. If such isometry is linear for each pair of points \(x, z \in M\) and the Finsler function \(F\) is regular on \(N\), the space \((M, F)\) is Berwald.

As a consequence of theorem 3.1 and theorem 4.1,

**Corollary 4.3** For a Landsberg space \((M, F)\) the function \(\text{vol}(I_x)\) is locally constant on \(M\).

**Proof.** It is based on the fact that for Landsberg spaces \(\text{tr}(A) = 0\). □

Given a curve \(\sigma : [a, b] \to M\) it will be useful to consider a particular class of local frames along \(\sigma\). For Landsberg spaces, let us consider the parallel transport

\[\psi_x : T_xM \setminus \{0\} \to T_{\sigma(s)}M \setminus \{0\}\]

which are the local isometries that theorem 4.1 assures to exist [5]. It defines the curves on \(N\) given by \(\bar{\sigma}(s) = (\sigma(s), y_s)\), where \(y_s \in T_{\sigma(s)}M \setminus \{0\}\) is the solution of the parallel transport equation for the non-linear equation (2.13). Then by an analogous reasoning as in [5] (but instead than using the Berwald connection as Bao uses, using the Chern connection), one obtains the following

**Lemma 4.4** Let \(\hat{e}_i, \hat{e}_j \in \Gamma TN\) be parallel sections using the parallel transport (2.12) on \(N\),

\[
\nabla_{\hat{e}_i} \hat{e}_i = 0, \quad \nabla_{\hat{e}_i} \hat{e}_j = 0.
\]

Then for Landsberg spaces,

\[
\frac{d}{dt} \left( \psi_t(g_{\bar{\sigma}}) (\hat{e}_i, \hat{e}_j) \right) = -2A_{lmk} \hat{e}_l^i \hat{e}_m^j \hat{e}_k = 0.
\]

Let us consider a basis \(\{\hat{e}_1(x, y), ..., \hat{e}_n(x, y)\}\) for \(T_xM \setminus \{0\}\). For any curve \(\sigma\) there is a frame along the horizontal lift \(\hat{\sigma}\) such that it is parallel by the Chern connection. For this, note that given a tangent vector \(\zeta \in T_\zeta T_xM \setminus \{0\}\) there is a unique lift to the corresponding fiber of \(\pi^*TM\) (this is using the homomorphism (2.5)) such that [5]

\[
\zeta^i \frac{\partial}{\partial y^i} \mapsto \zeta^i \pi^* \frac{\partial}{\partial x^i}.
\]
Then we can consider the parallel frame along $\sigma$,

$$\{\hat{e}_1(\sigma(s), y_s), \ldots, \hat{e}_n(\sigma(s), y_s)\} = \{\psi_{z\sigma}(s(\hat{e}_1(x, y)), \ldots, \psi_{z\sigma}(s(\hat{e}_n(x, y)))\},$$

(4.3)

In such parallel frame one has that for Landsberg spaces, the components of the fundamental tensor are constant,

$$g_{ij}(\sigma(s), y_s) = g(\hat{e}_i(\sigma(s), y_s), \hat{e}_j(\sigma(s), y_s)), \ i, j = 1, \ldots, n.$$  

(4.4)

This is a direct consequence of formula (4.2) in the case of Landsberg spaces and that we are using frames that are parallel.

**Lemma 4.5** Let $(M, F)$ be a Landsberg space. Given the curve $\sigma : [a, b] \rightarrow M$ there exits a parallel frame along $\sigma$ such that

1. The following condition holds:

$$ (\log \sqrt{\det g})^k_{ij} = 0, \ i, j, k = 1, \ldots, n. $$

(4.5)

2. The following condition holds:

$$ \mathcal{L}_{\hat{X}} d\Omega(x, y) = 0, $$

(4.6)

where $\hat{X}$ is the total lift of $X \in \Gamma TM$ to $\Gamma TN$.

**Proof.** Since the components of the fundamental tensor are constant in a parallel frame along $\sigma : [a, b] \rightarrow N$, one has that

$$ \det(g(x)) = \det(g_x) = \det(g_s) = \det(g(z)) $$

for all $z \in \pi(\hat{\sigma})([a, b])$. Then this is true in any other frame. This proves the first assertion, since this is true for any curve.

To prove the second statement, recall that $dvol(x, y) = \sqrt{\det g} d\Omega(x, y)$. Therefore,

$$ \mathcal{L}_{\hat{X}} \left( \sqrt{\det g} d\Omega(x, y) \right) = \mathcal{L}_{\hat{X}} \left( \sqrt{\det g} d\Omega(x, y) \right) + \sqrt{\det g} \mathcal{L}_{\hat{X}} d\Omega(x, y) $$

$$ = \sqrt{\det g} \mathcal{L}_{\hat{X}} d\Omega(x, y). $$

We use corollary 4.3 and lemma 3.3 to show that for any $X \in \Gamma TM$, one has that

$$ 0 = X \cdot \left( \int_{I_s} \sqrt{\det g} d\Omega(z, y_s) \right) = \int_{I_s} \mathcal{L}_{\hat{X}} \left( \sqrt{\det g} d\Omega(z, y_s) \right) $$

$$ = \int_{I_s} \sqrt{\det g} \mathcal{L}_{\hat{X}} d\Omega(z, y_s). $$

This is for every smooth $X \in \Gamma TM$. The fundamental lemma of variational calculus [23] shows that

$$ \mathcal{L}_{\hat{X}} d\Omega(z, y_s) = 0 $$

in the above frame and therefore in any frame.  $$\square$$

A direct consequence of lemma 4.5 is
Proposition 4.6 Let \((M, F)\) be a Finsler structure and let us consider a local coordinate system \((x, U)\) on \(M\) such that the function \(\text{vol}(I_x)\) and the density \(\det g(x, y)\) are constant on \(U\). Then the coefficients of the Levi-Civita connection with measure \(f\) are

\[
h\Gamma^k_{ij}(x, f) = \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x, y) \sqrt{|\det(g)|} \partial^k I^l_{ij}(x, y) d\Omega(x, y) + f^k_{ij}(x). \tag{4.7}
\]

Proof. One applies lemma 4.5 to the formula (3.14).

Remark 4.7 From the two terms simplified from (3.14) to (4.7), the term \((\vec{\nabla} \text{vol}(I_x))^k_{ij}(x)\) can be avoided by a convenient choice of the normalization. However, the vanishing of the second term \((\log\sqrt{|\det(g)|})^k_{ij}(x)\) requires additional conditions on the Finsler structure \((M, F)\). Being Landsberg is one enough condition.

Given a Finsler manifold with a fundamental tensor \(g\), there is associated a family of Finsler manifolds with fundamental tensors,

\[g_t = (1-t) \varpi(x, y, t) g + t \chi(x, y, t) h, \quad t \in [0, 1], \tag{4.8}\]

for each pair of homogeneous, positive functions in \(y \varpi(x, y)\) and \(\chi(x, y)\). Each \(g_t\) induces a Riemannian manifolds \((I_{xt}, g_{xt})\) on each \(T_xM \setminus \{0\}\) associated with the Finsler function

\[F_t(x, y) := \sqrt{t g_{ij}(x, y, t) y^i y^j}, \tag{4.9}\]

where \(^t g_{ij}(x, y, t)\) are the coefficients of the metric \(g_{xt} := g_t(x, \cdot)\) and

\[I_{xt} := \{y \in T_x M, \text{ s.t. } F_t(x, y) = 1\}. \]

Each of the indicatrix \(I_{xt}\) is compact. The components of the averaged metric for the interpolating Finsler functions \(F_t\) are

\[
h^k_{ij}(x) = \langle ^t g_{ij} \rangle f(x) := \frac{1}{\text{vol}(I_{xt})} \int_{I_{xt}} ^t g_{ij}(x, y) f(x, y, t) \sqrt{|\det(g_{xt})|} d\Omega(x, y), \tag{4.10}\]

for \(t \in [0, 1]\).

Remark 4.8 We will consider measures \(f(x, y)\) that are homogeneous of degree zero in \(y\). Therefore, \(f(x, y, t) = f(x, y)\). This is because any pair of indicatrix \(I_{xt_1}\) and \(I_{xt_2}\) with \(t_1, t_2 \in [0, 1]\) are related by a homothetic transformation.

Proposition 4.9 There is a convenient choice of the pair of homogeneous, positive smooth functions \((\varpi(x, y, t), \chi(x, y, t))\) such that \((M, F_t)\) has the same averaged Riemannian metric \(h\) for all \(t \in [0, 1]\).

\(^3\)Note that \(h\) must be understood as a special kind of Finsler structure, in order to make sense of the above linear combination.
The function $\chi(x,y,t)$ is such that $\chi(x,y,t)$ is such that

$$\langle t g_{ij} \rangle(x) = \frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} t g_{ij} f(x,y) f(x,y) \sqrt{\det(t g_{ij}(x,y))} d\Omega(y)$$

$$= \frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} ((1-t) \bar{\varpi}(x,y,t) g_{ij}(x,y) + t \chi(x,y,t) h_{ij}) f(x,y) \sqrt{\det(g_{xt}(x,y))} d\Omega(x,y).$$

The function $\chi(x,y,t)$ is such that

$$\frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} t \chi(x,y,t) h_{ij} f(x,y) \sqrt{\det(g_{xt})} d\Omega(x,y) = t h_{ij}.$$

This is accomplish if

$$\chi(x,y,t) = \frac{\hbox{vol}(I_{xt})}{\hbox{vol}(I_{x})} \left( \frac{\sqrt{\det(g_{xt}(\bar{\gamma}))}}{\sqrt{\det(g_{x}(\bar{\gamma}))}} \right)^{-1}, \quad t \in [0,1], \quad (4.11)$$

Then a homothetic transformation from $I_{xt}$ to $I_x$ by the factor $\chi(x,y,t)$ implies the following relation,

$$\frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} t \chi(x,y,t) h_{ij} f(x,y) \sqrt{\det(g_{xt})} d\Omega(x,y)$$

$$= \frac{1}{\hbox{vol}(I_{x})} \int_{I_{x}} t h_{ij} f(x,\bar{y}) \sqrt{\det(g_{x})} d\Omega(x,\bar{y})$$

To determine the function $\bar{\varpi}(x,y,t)$ one uses the invariance of integrals

$$\frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} f(x,y) (1-t) \bar{\varpi}(x,y,t) g_{ij} \sqrt{\det(g_{xt})} d\Omega(x,y)$$

$$= \frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} (1-t) \chi(x,\bar{y},t) g_{ij}(x,\bar{y}) f(x,\bar{y}) \sqrt{\det(g_{xt}(\bar{\gamma}))} \left( \frac{\sqrt{\det(g_{xt}(\bar{\gamma}))}}{\sqrt{\det(g_{x}(\bar{\gamma}))}} \right)(x,\bar{y}) d\Omega(x,\bar{y}),$$

where the diffeomorphism invoked in the last equality is the radial projection between $I_{xt}$ to $I_x$. Note that we have used that the integrand is homogeneous of degree zero on $y$. One defines the function

$$\bar{\varpi}(x,y,t) = \frac{\hbox{vol}(I_{xt})}{\hbox{vol}(I_{x})} \left( \frac{\sqrt{\det(g_{xt}(\bar{\gamma}))}}{\sqrt{\det(g_{x}(\bar{\gamma}))}} \right)^{-1}. \quad (4.12)$$

For this choice of $\bar{\varpi}(x,y,t)$ the above integral reduces to

$$\frac{1}{\hbox{vol}(I_{xt})} \int_{I_{xt}} (1-t) \bar{\varpi}(x,\bar{y},t) g_{ij}(x,\bar{y}) f(x,\bar{y}) \sqrt{\det(\bar{g})} d\Omega(x,\bar{y})$$

$$= \frac{1}{\hbox{vol}(I_{x})} \int_{I_{x}} (1-t) g_{ij}(x,\bar{y}) f(x,\bar{y}) \sqrt{\det(\bar{g})} d\Omega(x,\bar{y})$$

$$= (1-t) h_{ij}.$$
Lemma 4.10 Let $f_{j,k}^i(x,t)$ be the $f$-term (3.13) of the Levi-Civita connection associated to the manifolds $(M,F_t)$ and measure $f$ by (3.14). Then $f_{j,k}^i(x,t)$ does not depend on $t \in [0,1]$ for the intermediate fundamental tensors $g_t$ defined by (4.8), (4.11) and (4.12).

Proof. Firstly, note that $\varphi^k t^l g^{ls} = h^{ks}$ does not depend on the parameter $t \in [0,1]$. We have that $f_{k}^{ij}(x,t) := \frac{1}{2\text{vol}(I_{xt})} \left( \int_{I_{x,t}} t^l \varphi^k t^l (g^{ls}) (\frac{\partial f}{\partial x^i} t^l g_{sj} + \frac{\partial f}{\partial x^j} t^l g_{is} - \frac{\partial f}{\partial x^s} t^l g_{ij}) \sqrt{\det(g_{xt}(y))} d\Omega_t(x,y) \right)$. With $g_{ij}$ given by (4.8) and with the pair of functions $(\chi(x,y,t), \varpi(x,y,t))$ by (4.11)-(4.12), the result follows from the linearity, the way the integrals transform when passing from $I_{xt}$ to $I_x$ and from the specific form of the functions $(\chi, \varpi)$. \hfill \Box

Proposition 4.11 Let $(I_{xt}, g_{xt})$ be the indicatrix corresponding to $(M,g_{xt})$ at $x$. Let $(M,F)$ be a Landsberg space. Then for each fixed $t$, each pair of Riemannian metrics $(I_{xt}, g_{xt})$ and $(I_{zt}, g_{zt})$ are isometric for any $x,z \in M$.

Proof. Any isometry of the fiber metric $g_x$ is an isometry of the fiber averaged metric $h_x$. Such isometries leave invariant the functions $(\chi, \varpi)$. Therefore, any isometry of $g$ is an isometry of the corresponding Riemannian metric $g_{xt}$. Then for Landsberg spaces one can apply theorem 4.1. \hfill \Box

Corollary 4.12 Let $(I_{xt}, g_{xt})$ be as above. Then for fixed $t$, the metric functions $\text{vol}(I_{xt})$ and $\det(g_{xt})$ are constant on an open set $U \subset M$.

Proof. This is direct from proposition 4.11. \hfill \Box

Proposition 4.13 Let $(M,F)$ be a Landsberg space. Then for the intermediate spaces $(M,F_t)$, the coefficients of the Levi-Civita connection of $\nabla^h$ are

$$\Gamma^k_{ij}(x,t,f) = \frac{1}{\text{vol}(I_{xt})} \left( \int_{I_{xt}} \varphi^k t^l (g^{ls}) (\frac{\partial f}{\partial x^i} t^l g_{sj} + \frac{\partial f}{\partial x^j} t^l g_{is} - \frac{\partial f}{\partial x^s} t^l g_{ij}) \sqrt{\det(g_{xt}(y))} d\Omega_t(x,y) + f^k_{ij}(x) \right)$$

(4.13)

for all $t \in [0,1]$. In particular, there is natural local coordinate system on $N$ such that

$$\frac{1}{\text{vol}(I_{xt})} \int_{I_{xt}} \varphi^k t^l (g^{ls}) (\frac{\partial f}{\partial x^i} t^l g_{sj} + \frac{\partial f}{\partial x^j} t^l g_{is} - \frac{\partial f}{\partial x^s} t^l g_{ij}) \sqrt{\det(g_{xt}(y))} d\Omega_t(x,y) + f^k_{ij}(x) = 0, \ \forall t \in [0,1].$$

(4.14)

Proof. Since the volume functions $\text{vol}(I_{xt})$ and the determinant of the induced metrics $\det(g_{xt})$ are locally constant on $M$, equation (4.7) implies (4.13). Therefore,

$h^k \Gamma(x)^k_{ij} = h^k \Gamma^k_{ij}(x,0,f) = h^k \Gamma^k_{ij}(x,1,f) = h^k \Gamma^k_{ij}(x,t,f), \ t \in [0,1].$
Since $h\nabla$ is an affine, torsion free connection on $M$, given a point $x \in M$ there is a local coordinate system on $M$ such that at the point $x$ the coefficients of the connection are zero and therefore (4.14) follows.

The following proposition is direct from the above considerations,

**Proposition 4.14** Let $(M, F)$ be a Landsberg space. For any $f \in \mathcal{F}(N)$ homogeneous and positive one has that

1. The following relation holds,

$$h^k_{ij}(x,0,f) = h^k_{ij}(x,t,f) = \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x,y) \sqrt{\det(g)} \vartheta^k_{ij}(x,y) d\Omega(x,y) + f^k_{ij}(x),$$

for all $i, j, k = 1, \ldots, n$, and for each $t \in [0, 1]$.

2. For a Berwald space the following relation holds,

$$h^k_{ij}(x,0,f) = h^k_{ij}(x,t,f) = \gamma^k_{ij}(x,0,f) + f^k_{ij}(x), \forall i, j, k = 1, \ldots, n$$

(4.15)

for each $t \in [0, 1]$ and for the homogenous of degree zero, positive smooth function $f \in \mathcal{F}(N)$. Conversely, if $f(x,y) = 1$ and (4.15) holds, then the Landsberg space $(M, F)$ is Berwald.

**Remark 4.15** Note that the first statement of proposition 4.14 can be applied to any Finsler manifold such that $\text{vol}(I_x)$ and $\det(g_x)$ are locally constants on $M$, once the natural coordinate system is fixed.

Direct from proposition 4.14 one has that

**Lemma 4.16** The following holds:

1. For any arbitrary Landsberg space and for any arbitrary, positive, smooth function $f$, the Levi-Civita of the Riemannian metric $f^k$ is

$$h^k_{ij}(x,0,f) = \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x,y) \sqrt{\det(g)} \gamma^l_{ij}(x,y,f) d\Omega(x,y) + f^k_{ij}(x)$$

for all $i, j, k = 1, \ldots, n$.

2. For $(M, F)$ Landsberg but not Berwald, for the positive, smooth function $f(x,y) = 1$, there is a $y_0 \in I_x$ such that

$$\frac{1}{\text{vol}(I_x)} \int_{I_x} \sqrt{\det(g)} \gamma^l_{ij}(x,y,f) d\Omega(x,y) \neq \gamma^k_{ij}(x,y_0,0,f),$$

(4.16)

for some of the indices $i, j, k = 1, \ldots, n$. 19
Remark 4.17 The second half of lemma 4.16 means that fixed a point \( x \in M \), in the equality (4.15) holds for all the indices, the space is Berwald. This is a characterization of non-Berwaldian regular Landsberg spaces.

Proof. Point 1 is direct from proposition 4.14. The second part is proved by contradiction: if it is not true, for all \( y \in I_x \) one has that the equality holds. Then the space is Berwald, since \( \gamma^k_{ij} \) does not depend on \( y \).

5 Proof of the theorem 1.1

For any Landsberg space \((M, F)\) and fixed indices \( i, j, k \) associated with a local coordinate system \((x, U)\), we can manipulate some integrals as we do below. Firstly, it is clear from the homogeneity properties that for any \( \xi \in I_x \), one has the following relations:

1. For all \( \lambda > 0 \) one has that \( \gamma^k_{ij}(x, \xi) = \gamma^k_{ij}(x, \lambda \xi) \).
2. For all \( \lambda > 0 \),
   \[
   f(x, \xi) \sqrt{\det(g)(x, \xi)} \partial^k_{i} l(x, \xi) \gamma^l_{ij}(x, \xi) = f(x, \lambda \xi) \sqrt{\det(g)(x, \lambda \xi)} \partial^k_{i} l(x, \lambda \xi) \gamma^l_{ij}(x, \lambda \xi).
   \]

Indeed, they hold for any Finsler structure. Let us also consider the compact manifolds

\[
I(x, \lambda) := \{ y \in T_x M, F_{\lambda}(x, y) = \lambda, \lambda > 0 \}.
\]

That is, we are considering the spheres of radius \( \lambda \) for each of the structures \((M, F_{\lambda})\), \( \lambda \in (0, 1] \). Given any \( \lambda > 0 \), for Landsberg spaces it holds the following relation,

\[
h^{\lambda} \Gamma^k_{ij}(x, 0, f) = \frac{1}{\text{vol}(I_x)} \int_{I_x} f(x, y) \sqrt{\det(g)(x, y)} \partial^k_{i} l(x, y) \gamma^l_{ij}(x, y) d\Omega(x, y) + f^k_{ij}(x)
\]

\[
= \frac{1}{\text{vol}(I(x))} \int_{I(x, \lambda)} f(x, \bar{y}) \sqrt{\det(g)(x, \bar{y})} \partial^k_{i} l(x, \bar{y}) \gamma^l_{ij}(x, \bar{y}) d\Omega(x, \bar{y}) + f^k_{ij}(x).
\]

Note that in the first equality where the fact to be \((M, F)\) be Landsberg is used; the second equality is by homogeneity properties of the integrand.

Let us consider the measure \( f(x, y) = 1 \). This simplifies the above expressions, since then \( f^i_{jk} = 0 \). By lemma 4.16 if \((M, F)\) is a Landsberg space which is not Berwald space, there is a \( y_0 \in I_x \) such that for all \( i, j, k = 1, \ldots, n \)

\[
\gamma^k_{ij}(x, y_0, 0, f) \neq \frac{1}{\text{vol}(I_x)} \int_{I_x} \sqrt{\det(g)(x, y)} \partial^k_{i} l(x, y) \gamma^l_{ij}(x, y) d\Omega(x, y)
\]

\[
= \frac{1}{\text{vol}(I(x))} \int_{I(x, \lambda)} \sqrt{\det(g)(x, y)} \partial^k_{i} l(x, y) \gamma^l_{ij}(x, y) d\Omega(x, \bar{y})
\]

\[
= *.
\]

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We will obtain a contradiction with this condition.

Each of the manifolds $I(x, \lambda)$ is compact. Therefore the smooth functions $\gamma_k^ij(x, \bar{y})$ reaches its maximum value at some point $\bar{y}_0$ in $I(x, \lambda)$. Therefore

$$
\gamma^k_{ij}(x, \bar{y}_0(i, j, k)) = \frac{1}{vol(I(x, \lambda))} \int_{I(x, \lambda)} \sqrt{det(g)(x, \bar{y}(i, j, k))} \, \vartheta_i^l \gamma^l_{ij}(x, \bar{y}) \, d\Omega(x, \bar{y})
$$

Since $\vartheta_i^l \gamma^l_{ij}(x, \bar{y})$ are smooth and $I(x, \lambda)$ is compact, then given an arbitrary $\epsilon > 0$ there are a $\lambda > 0$ and a $\lambda > 0$ such that for all $\lambda \xi \in I(x, \lambda)$ with

$$
\|\lambda \xi - \bar{y}_0(i, j, k)\|_{g_x} \leq diam(I(x, \lambda)) = c \pi \lambda = \lambda
$$

one has that by homogeneity of degree zero of the integrand,

$$
|\vartheta_i^l \gamma^l_{ij}(x, \frac{1}{\lambda} \bar{y}_0(i, j, k)) - \vartheta_i^l \gamma^l_{ij}(x, \tilde{\lambda} \xi)| = |\vartheta_i^l \gamma^l_{ij}(x, \frac{1}{\lambda} \bar{y}_0(i, j, k)) - \vartheta_i^l \gamma^l_{ij}(x, \xi)| < \epsilon,
$$

where $\| \cdot \|_{g_x}$ is the norm defined on the manifold vector space $T_x M$ induced by $g_x$ and $diam(I(x, \lambda)) = c \pi \lambda$ is the diameter of the compact manifold $I(x, \lambda)$. Let $y_0(i, j, k)$ be such that $\lambda y_0(i, j, k) = \bar{y}_0(i, j, k)$ for a given $\lambda > 0$. Assuming a local coordinate system where the differences are positive, by homogeneity properties of the functions one can write the condition (5.1) as

$$(\vartheta_i^l \gamma^l_{ij}(x, \frac{1}{\lambda} \bar{y}_0(i, j, k)) - \vartheta_i^l \gamma^l_{ij}(x, \xi)) = (\vartheta_i^l \gamma^l_{ij}(x, \frac{1}{\lambda} \bar{y}_0(i, j, k)) - \vartheta_i^l \gamma^l_{ij}(x, \xi)).$$

By (5.1) one obtains the relation

$$
\frac{1}{vol(I_x)} \int_{I_x} \sqrt{det(g)} \, \vartheta_i^l \gamma^l_{ij}(x, y) \, d\Omega(x, y) - \gamma^k_{ij}(x, y_0(i, j, k))
$$

Since this relation holds for every $\epsilon > 0$, it follows that

$$
\frac{1}{vol(I_x)} \int_{I_x} \sqrt{det(g)} \, \vartheta_i^l \gamma^l_{ij}(x, y) \, d\Omega(x, y) - \gamma^k_{ij}(x, y_0(i, j, k)) = 0.
$$
Therefore,

\[ h\Gamma^k_{ij}(x) - \gamma^k_{ij}(x,y_0(i,j,k)) = 0. \]  \hspace{1cm} (5.2)

Note that the above manipulation of the integrals hold for any Finsler manifold \((M,F)\), since they only make use of smoothness conditions and homogeneity. However, iff \((M,F)\) is Landsberg, \((5.2)\) holds for \(y_0\) independent of the particular values of the indices \((i,j,k)\). In order to show this, let us note that:

1. We have associated the radius of \(I(x,\lambda)\) with the homotopy parameter \(\lambda = t\)
2. For a Landsberg space, the coefficients \(h\Gamma^{ij}_{jk}(x)\), \(i,j,k = 1,\ldots,n\) are invariant under the homotopy \((4.8)\) with \(t = \lambda \in (0,1]\).

Therefore, although \((5.2)\) shows \(y_0(i,j,k)\) depending on \((i,j,k)\), one can take a smaller \(\lambda\) and shows that the situation for any triplet of indices \((i,j,k)\) \((5.2)\) holds for any \(y \in I(x,\lambda)\). Therefore,

\[ h\Gamma^k_{ij}(x) - \gamma^k_{ij}(x,y_{\text{max}}) = 0, \]  \hspace{1cm} (5.3)

where \(\gamma^k_{ij}(x,y_{\text{max}})\) is the maximal value of the function \(\gamma^k_{ij}(x,y)\) in \(I_x\). The same argument can be done for the minimal value of \(\gamma^k_{ij}(x,y)\). Therefore, we get the contradiction with the second statement of the characterization of Landsberg spaces which are not Berwald spaces \((\text{lemma} \ 4.16)\). Therefore, \((M,F)\) must be Berwald.\

Remark 5.1 The fact that \(y_0\) in \((5.2)\) does not depend on the particular indices \(i,j,k\) is a global property of the indicatrix \(I_x\). Also note that we never perform calculation at \(\lambda = 0\), the situation at \(\lambda = 0\) is known by continuity.

The following are direct consequences of the theorem \(1.1\).

**Corollary 5.2** Let \((M,F)\) be a Landsberg space. Then all the Finsler manifolds \(\{(M,F_t), t \in [0,1]\}\) are Berwald.

Combined with a rigidity condition on Berwald surfaces \([27]\).

**Corollary 5.3** Let \((M,F)\) be a Landsberg space. If it is not a Berwald space, then \(F\) cannot be \(C^4\) in the whole \(N\).

The identification of the Landsberg category with the Berwald category implies that the Chern-Gauss-Bonnet theorem and formulas for Landsberg spaces \([5,6,7,11]\) are the same than for Berwald spaces.

This fact suggests the interesting question of which is the category of Finsler spaces where the Chern-Gauss-Bonnet theorem holds \(\text{(apart from the general results from Shen} [25,26], since we can be interested on closer formulas to the Riemannian one).\

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