Analysis of the radiative decays among the bottomonium states

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Abstract
In this article, we perform an systematic study of the radiative transitions among the bottomonium states using the heavy quarkonium effective Lagrangians, and make predictions for the ratios among the radiative decay widths of a special multiplet to another multiplet. The predictions can be confronted with the experimental data in the future.

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1 Introduction
In recent years, the Babar, Belle, CLEO, CDF, D0 and FOCUS collaborations have discovered (or confirmed) a large number of charmonium-like states and revitalized the interest in the spectroscopy of the charmonium states [1][2][3][4]. There are also some progresses in the spectroscopy of the bottomonium states. In 2004, the CLEO collaboration observed the $\Upsilon(3S)$ states in the four-photon decay cascade, $\Upsilon(3S) \rightarrow \gamma \chi_b(2P), \chi_b(2P) \rightarrow \gamma \Upsilon(1D), \Upsilon(1D) \rightarrow \gamma \chi_b(1P), \chi_b(1P) \rightarrow \gamma \Upsilon(1S)$, and obtained the mass $M_{\Upsilon(1D_2)} = (10161.1 \pm 0.6 \pm 1.6) \text{ MeV}$ [5]. In 2008, the Babar collaboration observed the $\eta_b(1S)$ in the radiative decay $\Upsilon(3S) \rightarrow \gamma \eta_b(1S)$ [6], and latter confirmed it in the radiative decay $\Upsilon(2S) \rightarrow \gamma \eta_b(1S)$ [7]. In 2010, the Babar collaboration observed the $\Upsilon(13D_j)$ state through the decay chain $\Upsilon(3S) \rightarrow \gamma \gamma \Upsilon(13D_j) \rightarrow \gamma \gamma \pi^+ \pi^- \Upsilon(1S)$ with $j = 1, 2, 3$, and obtained the mass $M_{\Upsilon(13D_2)} = (10164.5 \pm 0.8 \pm 0.5) \text{ MeV}$ [8]. In 2011, the Belle collaboration reported the observation of the spin-singlet bottomonium states $h_b(1P)$ and $h_b(2P)$, which are produced in the reactions $e^+e^- \rightarrow h_b(nP)\pi^+\pi^-$ with significances of 5.5$\sigma$ and 11.2$\sigma$, respectively [9]. The measured masses are $M_{h_b(1P)} = (9898.25 \pm 1.06^{+1.07}_{-1.03}) \text{ MeV}$ and $M_{h_b(2P)} = (10259.76 \pm 0.64^{+1.48}_{-1.03}) \text{ MeV}$, respectively. Recently, the ATLAS collaboration observed the $\chi_{bJ}(3P)$ with $J = 1, 2$ in the radiative transitions $\chi_{bJ}(3P) \rightarrow \gamma \Upsilon(1S), \gamma \Upsilon(2S)$ in the proton-proton collisions at the Large Hadron Collider (LHC) at the energy $\sqrt{s} = 7 \text{ TeV}$ [10]. The measured mass barycenter is $(10530 \pm 5 \pm 9) \text{ MeV}$, and the hyperfine mass splitting is fixed to the theoretically predicted value of 12 MeV. And more bottomonium states would be observed in the future at the Tevatron, KEK-B, RHIC and LHCb.

On the other hand, there have been several theoretical works on the spectroscopy of the bottomonium states, such as the relativized potential model (Godfrey-Isgur model) [11], the Cornell potential model, the logarithmic potential model, the power-law potential model, the QCD-motivated potential model [12], the relativistic quark model based on a quasipotential approach in QCD [13], the Cornell potential model combined with heavy quark mass expansion [14], the screened potential model [15], the potential non-relativistic QCD model [16], the confining potential model with the Bethe-Salpeter equation [17], etc. In Table 1, we list the experimental values of the bottomonium states compared with some theoretical predictions [9][10][11][15][18].

The charmonium and bottomonium states have analogous properties, the hadronic transitions and radiative transitions among the heavy quarkonium states have been studied by the QCD multipole expansion [19], the nonrelativistic potential model [20][21][22][23], the heavy quarkonium effective theory [24], the coupled-channel approach [21][22], the hybrid approach based on the multipole expansion and heavy quark symmetry [25], etc. In the nonrelativistic potential models, the $E_1$ and $M_1$ transitions among the bottomonium states are usually studied by the following

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\[ \Gamma_{E_1}(n^{2s+1}L_j \rightarrow n'^{2s'+1}L'_{j'} + \gamma) = \frac{4}{3} \alpha e^2 \alpha E_f E_i \delta_{ss'} C_{f1} | \langle n'^{2s'+1}L'_{j'} | r | n^{2s+1}L_j \rangle |^2, \]
\[ \Gamma_{M_1}(n^{2s+1}L_j \rightarrow n'^{2s'+1}L'_{j'} + \gamma) = \frac{4}{3} \alpha e^2 \alpha E_f \delta_{JJ'} + 1 \delta_{LL'} \delta_{ss' \pm 1} | \langle n'^{2s'+1}L'_{j'} | n^{2s+1}L_j \rangle |^2, \]

(1)

where the \( E_\gamma \) is the photon energy, the \( E_f \) is the energy of the final bottomonium state, the \( M_i \) is the mass of the initial bottomonium state, and the angular matrix factor \( C_{fi} \) is

\[ C_{fi} = \max(L, L')(2j' + 1) \left\{ \begin{array}{ccc} L' & J' & s \\ J & L & 1 \end{array} \right\}^2. \]

(2)

The values of the matrix elements \( \langle n'^{2s'+1}L'_{j'} | r | n^{2s+1}L_j \rangle \) and \( \langle n'^{2s'+1}L'_{j'} | n^{2s+1}L_j \rangle \) and their \( \alpha^2/c^2 \) corrections depend on the details of the wave-functions, which are evaluated using a special potential model \([11, 12, 13, 15]\). In Ref. \([26]\), we focus on the traditional charmonium scenario of the new charmonium-like states and study the radiative transitions among the charmonium states with the heavy quarkonium (or meson) effective theory based on the heavy quark symmetry \([24, 27, 28]\), which have been applied to identify the excited heavy quarkonium states \([29, 30, 31, 32, 33]\). In this article, we extend our previous works to study the radiative transitions among the bottomonium states using the heavy quarkonium effective theory.

The article is arranged as follows: we study the radiative transitions among the bottomonium states with the heavy quarkonium effective Lagrangians in Sect.2; in Sect.3, we present the numerical results and discussions; and Sect.4 is reserved for our conclusions.

### 2 The radiative transitions with the heavy quarkonium effective Lagrangians

In the infinite heavy quark mass limit, the heavy quarkonium states do not have heavy quark flavor symmetry and spin symmetry, while for the intermediate heavy quark mass, the heavy quark spin symmetry is expected to make sense. \([34, 35]\). In fact, the \( c \) and \( b \) quarks have large but finite masses, we can construct heavy quarkonium effective Lagrangians based on the heavy quark spin symmetry. In the nonrelativistic QCD for the heavy quark systems, we introduce three typical energy scales \( m_Q, m_Q v, m_Q v^2 \), where \( m_Q \) and \( v \) are the heavy quark masses and velocities respectively, and count the operators with the power orders of \( v \), take the heavy quarkonia as bound states and study them with the nonrelativistic Schrodinger field theory, and apparent Lorentz covariance is lost. In the heavy quark effective theories, we introduce two typical energy scales \( m_Q \) and \( \Lambda_{QCD} \), count the operators with the power orders of \( 1/m_Q \), and take the heavy quarkonia as the basic relativistic quantum fields rather than bound states at the hadron level, the calculations are more simple, and apparent Lorentz covariance is maintained. In the two approaches, the heavy quarkonium states are classified in the same scheme. And the two approaches both have advantages and shortcomings.

The bottomonium states can be classified according to the notation \( n^{2s+1}L_j \), where the \( n \) is the radial quantum number, the \( L \) is the orbital angular momentum, the \( s \) is the spin, and the \( j \) is the total angular momentum. They have the parity and charge conjugation \( P = (-1)^{L+s} \) and \( C = (-1)^{L+s} \), respectively. The states have the same radial quantum number \( n \) and orbital
| States | Experimental data | Theoretical values | Theoretical values |
|--------|-------------------|--------------------|--------------------|
| 1S \(\Upsilon(1^3S_1)\) \(\eta_b(1^3S_0)\) | 9460.30 ± 0.26 | 9460 | 9460 |
|        | 9390.9 ± 2.8     | 9389 | 9400 |
| 2S \(\Upsilon(2^3S_1)\) \(\eta_b(2^3S_0)\) | 10023.26 ± 0.31 | 10016 | 10000 |
|        | 9987             | 9987 | 9980 |
| 3S \(\Upsilon(3^3S_1)\) \(\eta_b(3^3S_0)\) | 10355.2 ± 0.5 | 10351 | 10350 |
|        | 10330            | 10340 | 10340 |
| 4S \(\Upsilon(4^3S_1)\) \(\eta_b(4^3S_0)\) | 10579.4 ± 1.2 | 10611 | 10630 |
|        | 10595            | 10630 | 10630 |
| 5S \(\Upsilon(5^3S_1)\) \(\eta_b(5^3S_0)\) | 10865 ± 8 | 10831 | 10880 |
|        | 10817            | 10880 | 10880 |
| 6S \(\Upsilon(6^3S_1)\) \(\eta_b(6^3S_0)\) | 11019 ± 8 | 11023 | 11100 |
|        | 11011            | 11100 | 11100 |
| 7S \(\Upsilon(7^3S_1)\) \(\eta_b(7^3S_0)\) | 11193 | 11183 | 11183 |
| 1P \(\kappa_b(1^3P_2)\) \(\chi_b(1^3P_1)\) \(\lambda_b(1^3P_0)\) \(\eta_b(1^1P_1)\) | 9912.21 ± 0.26 ± 0.31 | 9918 | 9900 |
|        | 9892.78 ± 0.26 ± 0.31 | 9897 | 9880 |
|        | 9859.44 ± 0.42 ± 0.31 | 9865 | 9850 |
|        | 9898.25 ± 1.06 ± 1.03 | 9903 | 9880 |
| 2P \(\kappa_b(2^3P_2)\) \(\chi_b(2^3P_1)\) \(\lambda_b(2^3P_0)\) \(\eta_b(2^1P_1)\) | 10268.65 ± 0.22 ± 0.50 | 10269 | 10260 |
|        | 10255.46 ± 0.22 ± 0.50 | 10251 | 10250 |
|        | 10232.5 ± 0.4 ± 0.5 | 10226 | 10230 |
|        | 10259.76 ± 0.64 ± 1.03 | 10256 | 10250 |
| 3P \(\kappa_b(3^3P_2)\) \(\chi_b(3^3P_1)\) \(\lambda_b(3^3P_0)\) \(\eta_b(3^1P_1)\) | 10536 ± 5 ± 9 | 10540 | 10540 |
|        | 10524 ± 5 ± 9 | 10524 | 10524 |
| 4P \(\kappa_b(4^3P_2)\) \(\chi_b(4^3P_1)\) \(\lambda_b(4^3P_0)\) \(\eta_b(4^1P_1)\) | 10767 | 10767 | 10767 |
|        | 10753             | 10753 | 10753 |
|        | 10732             | 10732 | 10732 |
|        | 10757             | 10757 | 10757 |
| 5P \(\kappa_b(5^3P_2)\) \(\chi_b(5^3P_1)\) \(\lambda_b(5^3P_0)\) \(\eta_b(5^1P_1)\) | 10965 | 10965 | 10965 |
|        | 10951             | 10951 | 10951 |
|        | 10933             | 10933 | 10933 |
|        | 10955             | 10955 | 10955 |
| 1D \(\Upsilon_3(1^3D_3)\) \(\eta_{b2}(1^1D_2)\) | 10161 ± 0.6 ± 1.6 | 10156 | 10160 |
|        | 10151             | 10150 | 10150 |
|        | 10145             | 10140 | 10140 |
|        | 10152             | 10150 | 10150 |
| 2D \(\Upsilon_2(2^3D_3)\) \(\eta_{b2}(2^1D_2)\) | 10442 | 10442 | 10442 |
|        | 10438             | 10438 | 10438 |
|        | 10432             | 10432 | 10432 |
|        | 10439             | 10439 | 10439 |
| 3D \(\Upsilon_3(3^3D_3)\) \(\eta_{b2}(3^1D_2)\) | 10680 | 10680 | 10680 |
|        | 10676             | 10676 | 10676 |
|        | 10670             | 10670 | 10670 |
|        | 10677             | 10677 | 10677 |
| 4D \(\Upsilon_3(4^3D_3)\) \(\eta_{b2}(4^1D_2)\) | 10886 | 10886 | 10886 |
|        | 10882             | 10882 | 10882 |
|        | 10877             | 10877 | 10877 |
|        | 10883             | 10883 | 10883 |
| 5D \(\Upsilon_3(5^3D_3)\) \(\eta_{b2}(5^1D_2)\) | 11069 | 11069 | 11069 |
|        | 11065             | 11065 | 11065 |
|        | 11060             | 11060 | 11060 |
|        | 11066             | 11066 | 11066 |
momentum $L$ can be expressed by the superfields $J$, $J^\mu$, $J^{\mu\nu}$, etc [34, 35].

$$J = \frac{1 + \hat{y}}{2} \left\{ \Upsilon_\mu \gamma^\mu - \eta_5 \gamma^5 \right\},$$

$$J^\mu = \frac{1 + \hat{y}}{2} \left\{ \chi_2^\mu \gamma_\mu + \frac{1}{\sqrt{2}} \epsilon^{\mu\nu\alpha\beta} v_\alpha \gamma^\beta \chi_\lambda + \frac{1}{\sqrt{3}} (\chi_0 + \frac{1}{\Lambda_{QCD}}) \chi_0 + \frac{1}{\Lambda_{QCD}} \chi_0 \right\} \frac{1 - \hat{y}}{2},$$

$$J^{\mu\nu} = \frac{1 + \hat{y}}{2} \left\{ \Upsilon_3^{\mu\nu} \gamma_\alpha + \frac{1}{\sqrt{6}} \left[ \epsilon^{\mu\alpha\beta\lambda} v_\alpha \gamma^\beta g^{\tau\nu} + \epsilon^{\nu\alpha\beta\lambda} v_\alpha \gamma^\beta g^{\tau\mu} \right] \Upsilon_2^{2\lambda} + \left[ \frac{3}{20} \left( (\gamma^\mu - v^\mu) g^{\mu\nu} + (\gamma^\nu - v^\nu) g^{\nu\mu} \right) - \frac{1}{\Lambda_{QCD}} (g^{\mu\nu} - v^\mu v^\nu) \right] \Upsilon_\alpha + \eta_2^{\mu\nu} \right\} \frac{1 - \hat{y}}{2},$$

(3)

where the $v^\mu$ denotes the four velocity associated to the superfields. We multiply the bottomonium fields $\Upsilon_3^{\mu\nu}$, $\Upsilon_2^{\mu\nu}$, $\Upsilon_\mu$, $\eta_2^{\mu\nu}$, $\chi_2^{\mu\nu}$, $\cdots$ with the factors $\sqrt{M_{\Upsilon_3}}$, $\sqrt{M_{\Upsilon_2}}$, $\sqrt{M_\Upsilon}$, $\sqrt{M_\eta}$, $\sqrt{M_\chi}$, $\cdots$, respectively, and they have dimension of mass $\frac{1}{\Lambda_{QCD}}$. The superfields $J$, $J^\mu$, $J^{\mu\nu}$ are functions of the radial quantum numbers $n$, the fields in a definite superfield have the same $n$, and form a multiplet. The superfields $J^{\mu_1 \cdots \mu_L}$ have the following properties under the parity, charge conjugation, heavy quark spin transformations,

$$J^{\mu_1 \cdots \mu_L} \overset{P}{\longrightarrow} \gamma^0 J^{\mu_1 \cdots \mu_L} \gamma^0,$$

$$J^{\mu_1 \cdots \mu_L} \overset{C}{\longrightarrow} (-1)^{L+1} C [J^{\mu_1 \cdots \mu_L}]^T C,$$

$$J^{\mu_1 \cdots \mu_L} \overset{S}{\longrightarrow} S J^{\mu_1 \cdots \mu_L} S^T,$$

$$v^\mu \overset{P}{\longrightarrow} v^\mu,$$

(4)

where $S, S' \in SU(2)$ heavy quark spin symmetry groups, and $[S, \hat{y}] = [S', \hat{y}] = 0$. The $S$-wave multiplet contains the spin-singlet bottomonium states; while the $P$-wave and $D$-wave multiplets contain both the spin-singlet and spin-triplet bottomonium states, there exist mass splittings among (or between) the bottomonium states in the same multiplet. In the nonrelativistic quark models, we resort to the spin-spin, spin-orbit and tensor interactions to take into account the mass splittings.

In the present case, we can use the superfields $J$, $J^\mu$, $J^{\mu\nu}$ and the Dirac matrix $\sigma^{\mu\nu}$ to construct the heavy quarkonium effective Lagrangians at the hadronic level to account for the mass-splittings in a multiplet [35]. The heavy quark effective Lagrangian can be written as

$$\mathcal{L} = \bar{h}_v i v \cdot D h_v + \frac{1}{2m_Q} \left[ \bar{h}_v (i D_L) h_v + \frac{g}{2} \bar{h}_v \sigma^{\alpha\beta} G_{\alpha\beta} h_v \right] + \cdots ,$$

(5)

where $D^\mu_L = D^\mu - v^\mu v \cdot D$, the $D_\mu$ is the covariant derivative, and the $h_v$ is the heavy quark field. The heavy quark spin symmetry breaking terms are of the order $\mathcal{O}(1/m_Q)$, $\mathcal{O}(1/m_Q^2)$, $\cdots$, and the corresponding corrections to the heavy quarkonium states (or mesons) can be counted as $\Lambda_{QCD}/m_Q$, $(\Lambda_{QCD}/m_Q)^2$, $\cdots$, respectively, here we introduce the scale $\Lambda_{QCD}$ to characterize the bound states. The mass-splittings in a multiplet can also be reproduced with the heavy quarkonium effective Lagrangians at the next-to-leading order [35].

The radiative transitions between the $m$ and $n$ bottomonium states can be described by the
following heavy quarkonium effective Lagrangians \[24, 26, 36, 39\]:

\[
\mathcal{L}_{SS} = \sum_{m,n} \delta(m,n) \text{Tr} [\bar{J}(m)\sigma_{\mu\nu}J(n)] F^{\mu\nu},
\]

\[
\mathcal{L}_{SP} = \sum_{m,n} \delta(m,n) \text{Tr} [\bar{J}(m)J_\mu(n) + \bar{J}_\mu(n)J(m)] V^\mu,
\]

\[
\mathcal{L}_{PD} = \sum_{m,n} \delta(m,n) \text{Tr} [\bar{J}_{\mu\nu}(m)V^\nu(n) + \bar{J}^\nu(n)J_{\mu\nu}(m)] V^\mu,
\]

where \(\bar{J}_{\mu_1...\mu_L} = \gamma^0 J_{\mu_1...\mu_L}^1 \gamma^0\), \(V^\mu = F^{\mu\nu}v_\nu\), the \(F^{\mu\nu}\) is the electromagnetic field tensor, and the \(\delta(m,n)\) are the coupling constants and have different values in the \(\mathcal{L}_{SS}\), \(\mathcal{L}_{SP}\) and \(\mathcal{L}_{PD}\).

The Lagrangians \(\mathcal{L}_{SP}\) and \(\mathcal{L}_{PD}\) preserve parity, charge conjugation, gauge invariance and heavy quark spin symmetry, while the Lagrangian \(\mathcal{L}_{SS}\) violates the heavy quark spin symmetry. The effective Lagrangians \(\mathcal{L}_{SP}\) and \(\mathcal{L}_{PD}\) describing the electric dipole \(E_1\) transitions can be realized in the leading order \(O(1)\) in the heavy quark effective theory, while the effective Lagrangian \(\mathcal{L}_{SS}\) describing the magnetic dipole \(M_1\) transitions can be realized in the next-to-leading order \(O(1/m_Q)\). The corrections to the \(\mathcal{L}_{SP}\), \(\mathcal{L}_{PD}\) and \(\mathcal{L}_{SS}\) come from the next-to-leading order \(O(1/m_Q)\) and the next-to-next-to-leading order \(O(1/m_Q^2)\), respectively. We can construct the corresponding Lagrangians with the superfields \(J, J^\mu, J_{\mu\nu}\), the Dirac matrix \(\sigma_{\mu\nu}\), the electromagnetic field tensor \(F_{\mu\nu}\) and the four-vector \(v^\mu\), and introduce additional unknown coupling constants, which have to be fitted to the precise experimental data in the future, and study the spin symmetry violations in the radiative decays to the bottomonium states in a special multiplet.

In the case of the charmonium states, the \(O(v^2/c^2)\) corrections to the \(E_1\) transitions come from the magnetic quadrupole \(M_2\) and electric octupole \(E_3\) terms; the current average values of the relative amplitudes are \(M_2/\sqrt{E_1^2 + M_2^2 + E_3^2} = (-10.0 \pm 1.5) \times 10^{-2}\) and \(E_3/\sqrt{E_1^2 + M_2^2 + E_3^2} = (1.6 \pm 1.3) \times 10^{-2}\) in the \(\chi_c(1P)\) \(\rightarrow \gamma J/\psi\) decay, \(M_2/\sqrt{E_1^2 + M_2^2 + E_3^2} = (1.0 \pm 1.4) \times 10^{-2}\) and \(E_3/\sqrt{E_1^2 + M_2^2 + E_3^2} = (-5.4 \pm 1.2) \times 10^{-2}\) in the \(\psi(2S)\) \(\rightarrow \gamma \chi_{c2}(1P)\) decay. Although the values differ from the theoretical expectations \(v^2/c^2 \approx 0.3\), the corrections are very small, we expect that the corresponding corrections for the bottomonium states are also very small.

In the screened potential model \[15\], the predictions for the decay widths of the transitions \(\Upsilon(2S) \rightarrow \chi_{bJ}(1P)\gamma\), \(\Upsilon(3S) \rightarrow \chi_{bJ}(2P)\gamma\), \(J = 0, 1, 2\), become better compared to the experimental data with the \(O(v^2/c^2)\) corrections taken into account, the \(O(v^2/c^2)\) corrections are not large; on the other hand, the \(O(v^2/c^2)\) corrections in the radiative transitions \(4S \rightarrow 2P\), \(4S \rightarrow 1P\), \(\cdots\), are very large, and it is not a good approximation for taking the spin symmetry breaking effects perturbatively. At the hadron level, large spin symmetry violations mean that the \(O(1/m_Q)\) corrections are large enough to ruin the leading order approximation, while we do not have enough experimental data to fit the unknown coupling constants in the phenomenological Lagrangians, and those parameters cannot be canceled out with each other to result in independent ratios among the radiative decay widths; we expect that neglecting the next-to-leading order and next-to-next-to-leading order corrections for the \(\mathcal{L}_{SP}\), \(\mathcal{L}_{PD}\) and \(\mathcal{L}_{SS}\) respectively leads to uncertainties of the order \(O(\Lambda_{QCD}/m_b)\); in other words, we expect that the flavor and spin symmetry breaking corrections of the order \(O(1/m_Q)\) to the effective Lagrangians \(\mathcal{L}_{SP}\) and \(\mathcal{L}_{PD}\) are smaller than (or not as large as) the leading order contributions.

In the heavy quark limit, the contributions of the order \(O(1/m_Q)\) are greatly suppressed. For example, the branching ratios of the radiative decay widths of the \(\Upsilon(2S)\) have the hierarchy \(\text{Br}(\Upsilon(2S) \rightarrow \chi_{bJ}(1P)\gamma) \sim 10^{-2} \gg \text{Br}(\Upsilon(2S) \rightarrow \eta_c(1S)\gamma) \sim 10^{-4}\), \(J = 0, 1, 2\) \[18\], although

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the radiative decays $\Gamma(2S) \to \chi_{bJ}(1P)\gamma$ are suppressed in the phase-space. In the case of the charmonium states, the decay widths $\Gamma(\psi' \to \chi_{c0}\gamma) = 29.2$ KeV, $\Gamma(\psi' \to \chi_{c1}\gamma) = 28.0$ KeV, $\Gamma(\psi' \to \chi_{c2}\gamma) = 26.6$ KeV, $\Gamma(\chi_{c0} \to J/\psi\gamma) = 119.5$ KeV, $\Gamma(\chi_{c1} \to J/\psi\gamma) = 295.8$ KeV, $\Gamma(\chi_{c2} \to J/\psi\gamma) = 384.2$ KeV in the $E_1$ transitions are very large compared with the decay widths $\Gamma(\psi' \to \gamma\eta_c) = 1.03$ KeV and $\Gamma(J/\psi \to \gamma\eta_c) = 1.58$ KeV in the $M_1$ transitions [18], as the $M_1$ transitions take place at the order $O(1/m_Q)$. However, it is difficult to count for that the decay widths $\Gamma(\psi' \to \gamma\eta_c) = 1.03$ KeV and $\Gamma(J/\psi \to \gamma\eta_c) = 1.58$ KeV are of the same order by the $O(1/m_Q)$ depression. In the nonrelativistic potential models, the overlap matrix elements $\langle n^{2s+1}S_{j'} | n^{2s+1}S_j \rangle$ in the leading order approximation are 1 and 0 for the transitions $J/\psi \to \gamma\eta_c$ and $\psi' \to \gamma\eta_c$, respectively, the perturbative $\alpha_s$ corrections and relativistic $v^2/c^2$ corrections cannot count for the decay widths $\Gamma(\psi' \to \gamma\eta_c) = 1.03$ KeV and $\Gamma(J/\psi \to \gamma\eta_c) = 1.58$ KeV simultaneously [18], we have to suppose large high order corrections or introduce new mechanisms [3, 37].

In the heavy quarkonium effective theory, the heavy quark spin symmetry cannot count for the analogous decay widths as the $\psi'$ and $J/\psi$ have the same quantum numbers except for the radial numbers and masses, we can use the coupling constants $\delta(2,1)$ and $\delta(1,1)$ to parameterize all those corrections in the nonrelativistic potential models, and take them as free parameters fitted to the experimental data, then use those parameters to study other processes.

From the heavy quarkonium effective Lagrangians $L_{SS}$, $L_{SP}$ and $L_{PD}$, we can obtain the radiative decay widths $\Gamma$,

$$\Gamma = \frac{1}{2j+1} \sum \frac{p_{cm}}{8\pi M^2} |T|^2,$$

where $T$ denotes the scattering amplitude, the $p_{cm}$ (or $k_{\gamma}$) is the momentum of the final states in the center of mass coordinate, the $\sum$ denotes the sum of all the polarization vectors, the $j$ is the total angular momentum of the initial state, and the $M$ is the mass of the initial state. For example, in the radiative decays $\Upsilon(3S)(m^3D_3) \to \chi_{b2}(n^3P_2)\gamma$,

$$|T|^2 = 4M_{\Upsilon_3}M_{\chi_{b2}}\delta^2(m,n)\epsilon_{\alpha\beta\mu\nu}^*\epsilon_{\alpha\beta\mu\nu}^* (g^{\alpha1\beta1} - v^{\alpha1}v^{\beta1}) (k^{\mu1}\epsilon_{\sigma1} - k^{\sigma1}\epsilon_{\mu1}) v_{\sigma1},$$

where the $\epsilon_{\alpha\beta\nu}(\lambda,q)$, $\epsilon_{\mu\nu}(\lambda,p)$ and $\epsilon_{\mu}(\lambda,k)$ are the polarization vectors of the bottomonium states $\Upsilon(3S)(m^3D_3)$, $\chi_{b2}(n^3P_2)$ and the photon, respectively. The summation of the polarization vectors of the total angular momentum $j = 1, 2, 3$ states results in the following three formulae,

$$\sum_{\lambda} \epsilon_{\mu\nu}^* \epsilon_{\alpha\beta} = -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{p^2},$$

$$\sum_{\lambda} \epsilon_{\mu\nu}^* \epsilon_{\alpha\beta} = \frac{\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha}}{2} - \frac{\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}}{3},$$

$$\sum_{\lambda} \epsilon_{\mu\nu}^* \epsilon_{\alpha\beta} = \frac{1}{6} (\bar{g}_{\mu2}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu3}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu2}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu3}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu2}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu3}\bar{g}_{\nu3}\bar{g}_{\rho2})$$

$$+ \frac{1}{15} (\bar{g}_{\mu3}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu2}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu2}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu3}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu2}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu3}\bar{g}_{\nu3}\bar{g}_{\rho2}) + \bar{g}_{\mu2}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu3}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu2}\bar{g}_{\nu3}\bar{g}_{\rho2} + \bar{g}_{\mu3}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu2}\bar{g}_{\nu2}\bar{g}_{\rho3} + \bar{g}_{\mu3}\bar{g}_{\nu3}\bar{g}_{\rho2})$$

we use the FeynCalc to carry out the contractions of the Lorentz indexes.

### 3 Numerical Results

In calculations, the masses of the bottomonium states are taken as the experimental values from the Belle collaboration [9], the ATLAS collaboration [10] and the Particle Data Group [18], see
If the coupled channel effects are large enough to distort the \( b \bar{b} \) configurations and induce some meson-meson components in the wave-functions, the mass-shifts are very large. For example, in the case of the charmonium states, the mass-shifts originate from the coupled channel effects are about \(-(400-500)\) MeV \[38\], we have to take the masses from the special potential quark model as the bare masses, and redefine the bare masses to reproduce the physical masses or the experimental values, the net coupled channel effects lead to the mass-shifts of the order 10 MeV. For the observed bottomonium states, the masses from the screened potential model \[15\] are consistent with the experimental data from the Belle collaboration \[9\], the ATLAS collaboration \[10\] and the Particle Data Group \[18\]. Although the net coupled channel effects result in mass-shifts for the observed bottomonium states, the masses from the screened potential model \[15\] are consistent with the experimental data from the Belle collaboration \[9\], the ATLAS collaboration \[10\] and the Particle Data Group \[18\]. Although the net coupled channel effects result in mass-shifts for

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Table 1; for the unobserved bottomonium states, we take the values from the screened potential model as the physical masses \[15\]. For the bottomonium states above the \( B \bar{B} \) threshold, the masses receive contributions from the intermediate meson-loops, such as the \( B \bar{B}, B^* \bar{B}, B^* \bar{B}^*, \ldots \). If the coupled channel effects are large enough to distort the \( b \bar{b} \) configurations and induce some meson-meson components in the wave-functions, the mass-shifts are very large. For example, in the case of the charmonium states, the mass-shifts originate from the coupled channel effects are about \(-(400-500)\) MeV \[38\], we have to take the masses from the special potential quark model as the bare masses, and redefine the bare masses to reproduce the physical masses or the experimental values, the net coupled channel effects lead to the mass-shifts of the order 10 MeV. For the observed bottomonium states, the masses from the screened potential model \[15\] are consistent with the experimental data from the Belle collaboration \[9\], the ATLAS collaboration \[10\] and the Particle Data Group \[18\]. Although the net coupled channel effects result in mass-shifts for

The numerical values of the radiative decay widths are presented in Tables 2-6, where we retain the unknown coupling constants \( \delta(m,n) \) among the multiplets of the radial quantum numbers \( m \) and \( n \). In general, we expect to fit the parameters \( \delta(m,n) \) to the precise experimental data, however, the experimental data are far from enough in the present time. In Tables 2, 7-10, we present the ratios among the radiative transitions among the bottomonium states.

The CLEO collaboration have observed that the doubly radiative decay \( \psi' \to \gamma \gamma J/\psi \) takes place through the decay cascade \( \psi' \to \gamma \chi_{cj}, \chi_{cj} \to \gamma J/\psi, j = 0, 1, 2 \) with additional tiny non-resonance contributions \[39, 40\]. Recently, the BESIII collaboration observed the first evidence for the direct two-photon transition \( \psi' \to \gamma \gamma J/\psi \) with the branching fraction \((3.3 \pm 0.6^{+0.8}_{-1.1}) \times 10^{-4}\) in a sample of 106 million \( \psi' \) decays collected by the BESIII detector \[41\]. In Ref. \[42\], He et al study the discrete contributions to decay \( \psi' \to \gamma \gamma J/\psi \) due to the \( E_1 \) transitions using the heavy quarkonium effective Lagrangian \[24\]. We expect that the corresponding doubly radiative decay \( \Upsilon(2S) \to \gamma \gamma \Upsilon(1S) \) occurs through the analogous decay cascade \( \Upsilon(2S) \to \gamma \chi_{bj}(1P), \chi_{bj}(1P) \to \gamma \Upsilon(1S) \). Experimentally, the doubly radiative decays \( \Upsilon(3S) \to \gamma \Upsilon(2S) \) and \( \Upsilon(2(1D) \to \gamma \Upsilon(1S) \) have been observed \[15\]. Once the coupling constants \( \delta(m,n) \) in the heavy quarkonium effective Lagrangians \( \mathcal{L}_{SS}, \mathcal{L}_{SP} \) and \( \mathcal{L}_{PD} \) are fitted to the precise experimental data, we can use them to study the doubly radiative decays, \( \Upsilon(3S) \to \gamma \Upsilon(2S), \Upsilon(2(1D) \to \gamma \Upsilon(1S) \), and other physical processes have the radiative transitions as their sub-processes, or study the singly radiative decays. For example, the radiative decays \( \Upsilon(2S) \to \gamma \chi_{bj}(1P) \) and \( \Upsilon(3S) \to \gamma \chi_{bj}(2P) \) have been observed experimentally, we can use them to fit the coupling constants \( \delta(2,1) \) and \( \delta(3,2) \), and make predictions for the decay widths \( \Gamma(\eta_b(2S) \to \gamma \eta_b(1P)) \) and \( \Gamma(\eta_b(3S) \to \gamma \eta_b(2P)) \), see Tables 3,7.

The widths of the radiative transitions of the \( S \)-wave to the \( P \)-wave bottomonium states listed in the Review of Particle Physics are presented in Table 3 \[15\]. From those radiative decay widths, we can obtain the ratios among the radiative decay widths of the \( S \)-wave to the \( P \)-wave bottomonium states, which are presented in Table 7. From the table, we can see that the agreements between the experimental data and the theoretical predictions are rather good, and the heavy quarkonium effective theory in the leading order approximation works rather well. The ratios presented in Tables 2, 7-10 can be confronted with the experimental data in the future at the Tevatron, KEK-B, RHIC and LHCb. In Tables 7-10, we also present values come from the screened potential model for comparison \[15\], and no definite conclusion can be made.

In calculations, we observe that the radiative decay widths are sensitive to the masses of the bottomonium states in some channels. The experimental value of the \( \Upsilon(4S) \) from the Particle Data Group is \( M_{\Upsilon(4S)} = (10579.4 \pm 1.2) \) MeV \[15\], while the predictions of the \( 4S \) states from the screened potential model are \( M_{\Upsilon(4S)} = 10661 \) MeV and \( M_{\eta_b(4S)} = 10595 \) MeV \[15\]. In this article, we take the values \( M_{\Upsilon(4S)} = (10579.4 \pm 1.2) \) MeV and \( M_{\eta_b(4S)} = 10595 \) MeV, i.e. \( M_{\Upsilon(4S)} < M_{\eta_b(4S)} \), the prediction of the ratio \( \frac{\Gamma(\eta_b(4S) \to \eta_b(3P)\gamma)}{\Gamma(\Upsilon(4S) \to \chi_{b2}(3P)\gamma)} \approx 6.297 \) seems rather exotic. Naively, we expect that

\[ 7 \]
the spin-1 states \( \Upsilon(nS) \) have larger masses than the corresponding ones of the spin-0 states \( \eta_b(nS) \).

If we take \( M_{\Upsilon(4S)} > M_{\eta_b(4S)} \), then the ratio \( \Gamma(\eta_b(4S) \to h_b(3P)\gamma) / \Gamma(\Upsilon(4S) \to \chi_b(3P)\gamma) \sim 3 \) seems rather natural, more experimental data are still needed to make better predictions.

The radiative decay widths \( \Gamma \propto k_3^3 \), the uncertainties originate from the masses of the bottomonium states can be estimated as

\[
\frac{\Delta \Gamma}{\Gamma} \approx \frac{\Delta k_3^3}{k_3^3} = \frac{3}{1 - \frac{M_f^2}{M_i^2}} \sqrt{\left( \frac{M_i^2}{M_i} \right)^2 + \left( \frac{M_f^2}{M_f} \right)^2 \left( \frac{\Delta M_i}{M_i} \right)^2 + 4 \left( \frac{M_f^2}{M_f} \right)^2 \left( \frac{\Delta M_f}{M_f} \right)^2},
\]

where we have neglected the terms \( \frac{\Delta M_i}{M_i} \) and \( \frac{\Delta M_f}{M_f} \) not enhanced by the factor \( \frac{1}{1 - \frac{M_f^2}{M_i^2}} \), the subscripts \( i \) and \( f \) denote the initial and final bottomonium states, respectively. The uncertainties of the masses of the bottomonium states \( \Upsilon(5^3S_1) \) and \( \Upsilon(6^3S_1) \) are 8 MeV from the Particle Data Group \(^{[18]}\), and the corresponding relative uncertainties \( \frac{\Delta M}{M} \) are 0.074% and 0.073%, respectively, and expected to result in the largest uncertainties for the decay widths. The ATLAS collaboration fixed the hyperfine mass splitting between the \( \chi_b(3P) \) and \( \chi_b(3P) \) to the theoretically predicted value of 12 MeV, which maybe result in larger uncertainty for the mass barycenter, as the theoretical and experimental values of the hyperfine mass splitting always have differences \(^{[10]}\). On the other hand, the uncertainties of the masses of other bottomonium states are very small, about (or less than) 1 MeV, from the recent Belle data \(^{[9]}\) and the Review of Particle Physics \(^{[18]}\), the relative uncertainties \( \frac{\Delta M}{M} \) are tiny. If the factor \( \frac{1}{1 - \frac{M_f^2}{M_i^2}} \) is not large enough (i.e. the difference between the radial quantum numbers of the initial \( |m| \) and final \( |n| \) states is larger than 1, \( |m - n| > 1 \)), the uncertainties originate from the masses of the bottomonium states are of a few percents, and can be neglected. For example, the uncertainties in the radiative decays of the S-wave to the S-wave and the S-wave to the P-wave bottomonium states are

\[
\begin{align*}
\frac{\Delta \Gamma}{\Gamma} &< 0.9\% \text{ for } \Upsilon(4^3S_1) \to \eta_b(1^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 0.6\% \text{ for } \Upsilon(4^3S_1) \to \eta_b(2^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 1.5\% \text{ for } \Upsilon(4^3S_1) \to \eta_b(3^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 1.9\% \text{ for } \Upsilon(5^3S_1) \to \eta_b(1^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 2.9\% \text{ for } \Upsilon(5^3S_1) \to \eta_b(2^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 4.6\% \text{ for } \Upsilon(5^3S_1) \to \eta_b(3^1S_0) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} &< 9.0\% \text{ for } \Upsilon(5^3S_1) \to \eta_b(4^1S_0) \gamma,
\end{align*}
\]
and

\[
\frac{\Delta \Gamma}{\Gamma} < 0.6\% \quad \text{for} \quad \Upsilon(4^3S_1) \rightarrow \chi_{b1}(1^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 1.3\% \quad \text{for} \quad \Upsilon(4^3S_1) \rightarrow \chi_{b1}(2^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 6.5\% \quad \text{for} \quad \Upsilon(4^3S_1) \rightarrow \chi_{b1}(3^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 2.6\% \quad \text{for} \quad \Upsilon(5^3S_1) \rightarrow \chi_{b1}(1^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 3.9\% \quad \text{for} \quad \Upsilon(5^3S_1) \rightarrow \chi_{b1}(2^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 7.2\% \quad \text{for} \quad \Upsilon(5^3S_1) \rightarrow \chi_{b1}(3^3P_1) \gamma, \\
\frac{\Delta \Gamma}{\Gamma} < 22\% \quad \text{for} \quad \Upsilon(5^3S_1) \rightarrow \chi_{b1}(4^3P_1) \gamma, \tag{12}
\]

respectively. From above equations, we can see that the relative uncertainties of the decay widths of the $\Upsilon(5^3S_1) \rightarrow \eta_b(n^1S_0), \chi_{b1}(n^3P_1)$ are larger than the corresponding ones of the $\Upsilon(4^3S_1) \rightarrow \eta_b(n^1S_0), \chi_{b1}(n^3P_1)$, as the mass of the $\Upsilon(5^3S_1)$ has larger uncertainty [18]. In calculations, we observe that the upper bound of the uncertainties originate from the masses of the $\Upsilon(4^3S_1)$ and $\Upsilon(5^3S_1)$ are about ten (or twenty) percent for the values $|m - n| \leq 1$.

In this article, we have neglected the corrections of the order $O(1/m_Q^2)$ and $O(1/m_Q)$ for the heavy quarkonium effective Lagrangians $\mathcal{L}_{SS}$ (as the spin-flipped Lagrangian $\mathcal{L}_{SS}$ is of the order $O(1/m_Q)$) and $\mathcal{L}_{SP}, \mathcal{L}_{PD}$, respectively, which maybe result in rather large uncertainties. If those corrections can be counted as $\Lambda_{QCD}/m_b$, taking the Lagrangians $\mathcal{L}_{SS}, \mathcal{L}_{SP}$ and $\mathcal{L}_{PD}$ to fit the experimental data to determine the unknown coupling constants can result in uncertainties of the order $O(\Lambda_{QCD}/m_b)$. Such crude estimation maybe not work well, we should bear in mind that the magnitudes of the contributions from the higher order terms in the heavy quarkonium effective theory be determined experimentally.

4 Conclusion

In this article, we extend our previous work on the radiative transitions among the charmonium states to study the radiative transitions among the bottomonium states in an systematic way based on the heavy quarkonium effective theory, and make predictions for ratios among the radiative decay widths of a special multiplet to another multiplet, where the unknown couple constants $\delta(m,n)$ are canceled out with each other. The predictions can be confronted with the experimental data in the future at the Tevatron, KEK-B, RHIC and LHCb.

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Table 2: The ratios of the radiative decay widths of the $S$-wave to the $S$-wave bottomonium states, where the unit of the widths is $\delta^2(m,n)$. 

|       | $\Gamma(\Upsilon \rightarrow \eta_b \gamma)$ | $\Gamma(\eta_b \rightarrow \Upsilon \gamma)$ | $\frac{\Gamma(\eta_b \rightarrow \Upsilon \gamma)}{\Gamma(\Upsilon \rightarrow \eta_b \gamma)}$ |
|-------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $2S \rightarrow 1S$ | 0.091                                        | 0.163                                        | 1.781                                        |
| $3S \rightarrow 1S$ | 0.299                                        | 0.674                                        | 2.254                                        |
| $3S \rightarrow 2S$ | 0.019                                        | 0.034                                        | 1.761                                        |
| $4S \rightarrow 1S$ | 0.532                                        | 1.408                                        | 2.648                                        |
| $4S \rightarrow 2S$ | 0.076                                        | 0.207                                        | 2.711                                        |
| $4S \rightarrow 3S$ | 0.006                                        | 0.017                                        | 2.673                                        |
| $5S \rightarrow 1S$ | 0.952                                        | 2.290                                        | 2.406                                        |
| $5S \rightarrow 2S$ | 0.233                                        | 0.527                                        | 2.261                                        |
| $5S \rightarrow 3S$ | 0.057                                        | 0.113                                        | 1.962                                        |
| $5S \rightarrow 4S$ | 0.008                                        | 0.016                                        | 2.059                                        |
| $6S \rightarrow 1S$ | 1.240                                        | 3.277                                        | 2.643                                        |
| $6S \rightarrow 2S$ | 0.366                                        | 0.973                                        | 2.658                                        |
| $6S \rightarrow 3S$ | 0.118                                        | 0.308                                        | 2.607                                        |
| $6S \rightarrow 4S$ | 0.029                                        | 0.093                                        | 3.158                                        |
| $6S \rightarrow 5S$ | 0.003                                        | 0.004                                        | 1.147                                        |
| $7S \rightarrow 1S$ | 1.620                                        | 4.330                                        | 2.673                                        |
| $7S \rightarrow 2S$ | 0.563                                        | 1.517                                        | 2.697                                        |
| $7S \rightarrow 3S$ | 0.224                                        | 0.597                                        | 2.669                                        |
| $7S \rightarrow 4S$ | 0.079                                        | 0.244                                        | 3.081                                        |
| $7S \rightarrow 5S$ | 0.021                                        | 0.038                                        | 1.839                                        |
| $7S \rightarrow 6S$ | 0.002                                        | 0.005                                        | 2.204                                        |
Table 3: The radiative decay widths of the $S$-wave to the $P$-wave bottomonium states, where the unit is $10^{-4}\delta^2(m,n)$. The wide-hat denotes the experimental values, where the unit is KeV.

| $\Gamma$ | $\gamma_{2} \rightarrow \chi_{2}\gamma$ | $\gamma_{1} \rightarrow \chi_{1}\gamma$ | $\gamma_{0} \rightarrow \chi_{0}\gamma$ | $\eta_b \rightarrow h_b\gamma$ |
|----------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------|
| $2S \rightarrow 1P$ | 2.355 | 2.281 | 1.492 | 2.176 |
| $2S \rightarrow 1P$ | 2.287 ± 0.219 | 2.207 ± 0.222 | 1.215 ± 0.162 | |
| $3S \rightarrow 1P$ | 138.0 | 93.76 | 38.16 | 230.4 |
| $3S \rightarrow 1P$ | < 0.386 | < 0.035 | 0.061 | |
| $3S \rightarrow 2P$ | 1.123 | 1.028 | 0.634 | 1.084 |
| $3S \rightarrow 2P$ | 2.662 ± 0.406 | 2.560 ± 0.337 | 1.199 ± 0.164 | |
| $4S \rightarrow 1P$ | 447.6 | 291.6 | 110.9 | 909.9 |
| $4S \rightarrow 2P$ | 49.28 | 33.40 | 13.59 | 110.7 |
| $4S \rightarrow 2P$ | 1.143 | 0.178 | 0.161 | 0.901 |
| $5S \rightarrow 1P$ | 1223 | 777.3 | 283.1 | 1983 |
| $5S \rightarrow 2P$ | 326.4 | 208.6 | 77.14 | 483.1 |
| $5S \rightarrow 3P$ | 58.35 | 38.88 | 15.55 | 71.10 |
| $5S \rightarrow 4P$ | 1.627 | 1.453 | 0.807 | 0.678 |
| $6S \rightarrow 1P$ | 1854 | 1169 | 419.5 | 3375 |
| $6S \rightarrow 2P$ | 628.4 | 396.5 | 143.3 | 1133 |
| $6S \rightarrow 3P$ | 178.4 | 114.9 | 43.38 | 318.9 |
| $6S \rightarrow 4P$ | 26.72 | 18.80 | 7.829 | 49.22 |
| $6S \rightarrow 5P$ | 0.275 | 0.329 | 0.221 | 0.552 |
| $7S \rightarrow 1P$ | 2770 | 1736 | 614.5 | 5003 |
| $7S \rightarrow 2P$ | 1132 | 707.3 | 251.2 | 2025 |
| $7S \rightarrow 3P$ | 432.2 | 273.2 | 99.66 | 766.9 |
| $7S \rightarrow 4P$ | 124.2 | 81.88 | 31.21 | 223.4 |
| $7S \rightarrow 5P$ | 19.91 | 14.24 | 5.863 | 35.84 |

Table 4: The radiative decay widths of the $P$-wave to the $S$-wave bottomonium states, where the unit is $10^{-4}\delta^2(m,n)$.

| $\Gamma$ | $\gamma_{2} \rightarrow \chi_{2}\gamma$ | $\gamma_{1} \rightarrow \chi_{1}\gamma$ | $\gamma_{0} \rightarrow \chi_{0}\gamma$ | $h_b \rightarrow \eta_b\gamma$ |
|----------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------|
| $1P \rightarrow 1S$ | 87.27 | 76.89 | 60.89 | 121.6 |
| $2P \rightarrow 1S$ | 458.6 | 438.5 | 402.5 | 559.5 |
| $2P \rightarrow 2S$ | 14.76 | 12.55 | 9.232 | 20.13 |
| $3P \rightarrow 1S$ | 1017 | 988.1 | 927.5 | 1181 |
| $3P \rightarrow 2S$ | 126.5 | 118.2 | 103.7 | 148.2 |
| $3P \rightarrow 3S$ | 6.006 | 4.902 | 3.241 | 7.973 |
| $4P \rightarrow 1S$ | 1733 | 1688 | 1602 | 1940 |
| $4P \rightarrow 2S$ | 366.3 | 347.4 | 319.0 | 403.1 |
| $4P \rightarrow 3S$ | 67.28 | 60.86 | 51.94 | 74.70 |
| $4P \rightarrow 4S$ | 6.706 | 5.331 | 3.638 | 4.343 |
| $5P \rightarrow 1S$ | 2536 | 2484 | 2379 | 2787 |
| $5P \rightarrow 2S$ | 711.8 | 683.7 | 644.7 | 766.1 |
| $5P \rightarrow 3S$ | 209.0 | 195.7 | 178.9 | 224.0 |
| $5P \rightarrow 4S$ | 55.67 | 50.00 | 43.23 | 45.56 |
| $5P \rightarrow 5S$ | 1.037 | 0.662 | 0.328 | 2.702 |
Table 5: The radiative decay widths of the $P$-wave to the $D$-wave bottomonium states, where the unit is $10^{-4}\delta^2(m,n)$.

| $\Gamma$ | $\chi_2 \to \bar{\chi}_3 \gamma$ | $\chi_2 \to \bar{\chi}_2 \gamma$ | $\chi_1 \to \bar{\chi}_1 \gamma$ | $\chi_1 \to \bar{\chi}_2 \gamma$ | $\chi_0 \to \bar{\chi}_0 \gamma$ | $h_b \to \eta_2 \gamma$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| 2P $\to$ 1D | 2.066 | 0.321 | 0.032 | 1.089 | 0.580 | 1.160 | 2.155 |
| 3P $\to$ 1D | 74.46 | 12.79 | 0.963 | 58.12 | 21.98 | 73.83 | 86.58 |
| 3P $\to$ 2D | 1.207 | 0.244 | 0.019 | 0.827 | 0.337 | 0.597 | 1.262 |
| 4P $\to$ 1D | 293.8 | 51.27 | 3.676 | 239.4 | 86.16 | 311.1 | 339.6 |
| 4P $\to$ 2D | 47.27 | 8.752 | 0.615 | 38.51 | 13.57 | 44.49 | 52.79 |
| 4P $\to$ 3D | 0.959 | 0.196 | 0.016 | 0.595 | 0.248 | 0.415 | 0.889 |
| 5P $\to$ 1D | 653.1 | 114.8 | 8.061 | 544.6 | 192.2 | 719.2 | 759.9 |
| 5P $\to$ 2D | 188.5 | 34.43 | 2.368 | 159.1 | 54.85 | 197.9 | 215.7 |
| 5P $\to$ 3D | 32.22 | 5.995 | 0.424 | 25.90 | 9.198 | 30.28 | 35.64 |
| 5P $\to$ 4D | 0.719 | 0.149 | 0.012 | 0.429 | 0.176 | 0.307 | 0.649 |

Table 6: The radiative decay widths of the $D$-wave to the $P$-wave bottomonium states, where the unit is $10^{-4}\delta^2(m,n)$.

| $\Gamma$ | $\bar{\chi}_3 \to \chi_2 \gamma$ | $\bar{\chi}_2 \to \chi_2 \gamma$ | $\bar{\chi}_2 \to \chi_1 \gamma$ | $\bar{\chi}_1 \to \chi_1 \gamma$ | $\bar{\chi}_0 \to \chi_0 \gamma$ | $\eta_2 \to h_b \gamma$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1D $\to$ 1P | 14.48 | 3.847 | 14.38 | 0.351 | 6.664 | 12.78 |
| 2D $\to$ 1P | 138.9 | 34.00 | 113.0 | 3.638 | 60.85 | 96.19 | 147.2 |
| 2D $\to$ 2P | 5.302 | 1.237 | 4.632 | 0.123 | 2.331 | 4.460 | 5.852 |
| 3D $\to$ 1P | 400.7 | 98.87 | 317.3 | 10.64 | 172.7 | 258.3 | 416.3 |
| 3D $\to$ 2P | 67.05 | 16.30 | 53.59 | 1.730 | 28.56 | 44.49 | 69.83 |
| 3D $\to$ 3P | 3.063 | 0.704 | 2.697 | 0.069 | 1.330 | 2.687 | 3.322 |
| 4D $\to$ 1P | 781.8 | 193.7 | 610.7 | 20.89 | 335.2 | 487.6 | 804.2 |
| 4D $\to$ 2P | 216.5 | 53.17 | 169.2 | 5.737 | 91.91 | 135.6 | 222.2 |
| 4D $\to$ 3P | 41.97 | 10.15 | 33.61 | 1.079 | 17.92 | 28.49 | 43.37 |
| 4D $\to$ 4P | 1.740 | 0.393 | 1.658 | 0.038 | 0.819 | 1.738 | 2.062 |
| 5D $\to$ 1P | 1261 | 313.5 | 977.0 | 33.69 | 537.8 | 769.1 | 1289 |
| 5D $\to$ 2P | 453.4 | 111.9 | 350.5 | 12.10 | 191.6 | 275.6 | 462.0 |
| 5D $\to$ 3P | 142.4 | 34.85 | 111.4 | 3.751 | 60.26 | 90.07 | 145.3 |
| 5D $\to$ 4P | 27.29 | 6.563 | 22.51 | 0.693 | 11.93 | 19.30 | 29.18 |
| 5D $\to$ 5P | 1.166 | 0.259 | 1.149 | 0.025 | 0.559 | 1.173 | 1.415 |
Table 7: The ratios among the radiative decay widths of the $S$-wave to the $P$-wave bottomonium states, where $\tilde{\Gamma} = \frac{\Gamma}{\Gamma(\Upsilon \rightarrow \chi_{2}\gamma)}$, $\tilde{\Gamma}(\Upsilon \rightarrow \chi_{2}\gamma) = \frac{\Gamma(\Upsilon \rightarrow \chi_{2}\gamma)}{\Gamma(\Upsilon \rightarrow \chi_{2}\gamma)} = 1$, and the wide-hat denotes the experimental values, the values in the bracket come from the screened potential model \cite{15}.

| $\Gamma$       | $\Upsilon \rightarrow \chi_1\gamma$ | $\Upsilon \rightarrow \chi_0\gamma$ | $\eta_b \rightarrow h_6\gamma$ |
|---------------|---------------------------------|---------------------------------|--------------------------------|
| $2S \rightarrow 1P$ | 0.969 [0.846] | 0.634 [0.451] | 0.924 [2.264] |
| $2S \rightarrow 1P$ | 0.965 ± 0.134 | 0.531 ± 0.087 | |
| $3S \rightarrow 2P$ | 0.679 [0.111] | 0.277 [0.040] | 1.670 [4.508] |
| $3S \rightarrow 2P$ | 0.915 [0.803] | 0.565 [0.405] | 0.966 [3.322] |
| $3S \rightarrow 2P$ | 0.962 ± 0.194 | 0.450 ± 0.092 | |
| $4S \rightarrow 3P$ | 0.651 [0.233] | 0.248 [0.001] | 2.033 [6.558] |
| $4S \rightarrow 3P$ | 0.678 [0.002] | 0.276 [0.375] | 2.246 [3.857] |
| $4S \rightarrow 3P$ | 1.245 [1.423] | 1.125 [1.038] | 6.297 [24.81] |
| $5S \rightarrow 4P$ | 0.636 | 0.231 | 1.622 |
| $5S \rightarrow 4P$ | 0.639 | 0.236 | 1.480 |
| $5S \rightarrow 4P$ | 0.666 | 0.266 | 1.218 |
| $5S \rightarrow 4P$ | 0.893 | 0.496 | 0.417 |
| $6S \rightarrow 5P$ | 0.631 | 0.224 | 1.622 |
| $6S \rightarrow 5P$ | 0.631 | 0.228 | 1.803 |
| $6S \rightarrow 5P$ | 0.644 | 0.243 | 1.788 |
| $6S \rightarrow 5P$ | 0.703 | 0.293 | 1.842 |
| $6S \rightarrow 5P$ | 1.194 | 0.802 | 2.007 |
| $7S \rightarrow 6P$ | 0.627 | 0.222 | 1.806 |
| $7S \rightarrow 6P$ | 0.625 | 0.222 | 1.790 |
| $7S \rightarrow 6P$ | 0.632 | 0.231 | 1.775 |
| $7S \rightarrow 6P$ | 0.659 | 0.251 | 1.799 |
| $7S \rightarrow 6P$ | 0.715 | 0.294 | 1.800 |
Table 8: The ratios among the radiative decay widths of the $P$-wave to the $S$-wave bottomonium states, where $\tilde{\Gamma} = \Gamma(\chi_{2} \rightarrow \Upsilon \gamma) \Gamma(\chi_{2} \rightarrow \Upsilon \gamma)^{-1}$, $\tilde{\Gamma}(\chi_{2} \rightarrow \Upsilon \gamma) = \Gamma(\chi_{2} \rightarrow \Upsilon \gamma) \Gamma(\chi_{2} \rightarrow \Upsilon \gamma)^{-1} = 1$, and the values in the bracket come from the screened potential model [15].

| $\Gamma$ | $\chi_{1} \rightarrow \Upsilon \gamma$ | $\chi_{0} \rightarrow \Upsilon \gamma$ | $h_{b} \rightarrow \eta_{b} \gamma$ |
|----------|--------------------------------------|--------------------------------------|-------------------------------|
| $1P \rightarrow 1S$ | 0.881 [0.920] | 0.698 [0.745] | 1.394 [1.114] |
| $2P \rightarrow 1S$ | 0.956 [0.685] | 0.878 [0.360] | 1.220 [1.440] |
| $2P \rightarrow 2S$ | 0.850 [0.972] | 0.625 [0.817] | 1.364 [1.077] |
| $3P \rightarrow 1S$ | 0.972 [0.501] | 0.912 [0.127] | 1.161 [1.399] |
| $3P \rightarrow 2S$ | 0.934 [0.782] | 0.820 [0.533] | 1.172 [1.495] |
| $3P \rightarrow 3S$ | 0.816 [0.898] | 0.540 [0.691] | 1.327 [1.045] |
| $4P \rightarrow 1S$ | 0.974 | 0.924 | 1.120 |
| $4P \rightarrow 2S$ | 0.948 | 0.871 | 1.101 |
| $4P \rightarrow 3S$ | 0.905 | 0.772 | 1.110 |
| $4P \rightarrow 4S$ | 0.795 | 0.543 | 0.648 |
| $5P \rightarrow 1S$ | 0.979 | 0.938 | 1.099 |
| $5P \rightarrow 2S$ | 0.960 | 0.906 | 1.076 |
| $5P \rightarrow 3S$ | 0.936 | 0.856 | 1.072 |
| $5P \rightarrow 4S$ | 0.898 | 0.776 | 0.818 |
| $5P \rightarrow 5S$ | 0.638 | 0.317 | 2.605 |

Table 9: The ratios among the radiative decay widths of the $P$-wave to the $D$-wave bottomonium states, where $\tilde{\Gamma} = \Gamma(\chi_{2} \rightarrow \Upsilon_3 \gamma) \Gamma(\chi_{2} \rightarrow \Upsilon_3 \gamma)^{-1}$, $\tilde{\Gamma}(\chi_{2} \rightarrow \Upsilon_3 \gamma) = \Gamma(\chi_{2} \rightarrow \Upsilon_3 \gamma) \Gamma(\chi_{2} \rightarrow \Upsilon_3 \gamma)^{-1} = 1$, and the values in the bracket come from the screened potential model [15].

| $\Gamma$ | $\chi_{2} \rightarrow \Upsilon_2 \gamma$ | $\chi_{2} \rightarrow \Upsilon_1 \gamma$ | $\chi_{1} \rightarrow \Upsilon_2 \gamma$ | $\chi_{1} \rightarrow \Upsilon_1 \gamma$ | $\chi_{0} \rightarrow \Upsilon_1 \gamma$ | $h_{b} \rightarrow \eta_{b} \gamma$ |
|----------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|-------------------------------|
| $2P \rightarrow 1D$ | 0.156 [0.185] | 0.016 [0.013] | 0.527 [0.722] | 0.281 [0.268] | 0.561 [0.591] | 1.043 [2.371] |
| $3P \rightarrow 1D$ | 0.172 | 0.013 | 0.781 | 0.295 | 0.992 | 1.163 |
| $3P \rightarrow 2D$ | 0.202 [0.190] | 0.016 [0.013] | 0.685 [0.635] | 0.279 [0.241] | 0.494 [0.471] | 1.046 [2.388] |
| $4P \rightarrow 1D$ | 0.175 | 0.013 | 0.815 | 0.293 | 1.059 | 1.156 |
| $4P \rightarrow 2D$ | 0.185 | 0.013 | 0.815 | 0.287 | 0.941 | 1.117 |
| $4P \rightarrow 3D$ | 0.204 | 0.016 | 0.620 | 0.259 | 0.433 | 0.927 |
| $5P \rightarrow 1D$ | 0.176 | 0.012 | 0.834 | 0.294 | 1.101 | 1.164 |
| $5P \rightarrow 2D$ | 0.183 | 0.013 | 0.844 | 0.291 | 1.050 | 1.144 |
| $5P \rightarrow 3D$ | 0.186 | 0.013 | 0.804 | 0.285 | 0.940 | 1.106 |
| $5P \rightarrow 4D$ | 0.207 | 0.016 | 0.596 | 0.245 | 0.426 | 0.903 |
| $\Gamma$    | $\bar{\Gamma}_{2 \rightarrow \chi_{2}\gamma}$ | $\bar{\Gamma}_{2 \rightarrow \chi_{1}\gamma}$ | $\bar{\Gamma}_{2 \rightarrow \chi_{0}\gamma}$ | $\eta_{2 \rightarrow \gamma_h\gamma}$ |
|------------|----------------------------------------------|----------------------------------------------|----------------------------------------------|-----------------------------------|
| 1D $\rightarrow$ 1P | 0.266 [0.240]                               | 0.994 [0.808]                               | 0.024 [0.025]                               | 0.460 [0.420]                    |
| 2D $\rightarrow$ 1P | 0.245 [0.182]                               | 0.814 [1.196]                               | 0.026 [0.013]                               | 0.438 [0.501]                    |
| 2D $\rightarrow$ 2P | 0.233 [0.240]                               | 0.874 [0.761]                               | 0.023 [0.025]                               | 0.440 [0.399]                    |
| 3D $\rightarrow$ 1P | 0.247                                         | 0.792                                         | 0.027                                         | 0.654                            |
| 3D $\rightarrow$ 3P | 0.230                                         | 0.880                                         | 0.022                                         | 0.877                            |
| 4D $\rightarrow$ 1P | 0.248                                         | 0.781                                         | 0.027                                         | 0.624                            |
| 4D $\rightarrow$ 2P | 0.246                                         | 0.781                                         | 0.027                                         | 0.627                            |
| 4D $\rightarrow$ 3P | 0.242                                         | 0.801                                         | 0.026                                         | 0.679                            |
| 4D $\rightarrow$ 4P | 0.226                                         | 0.953                                         | 0.022                                         | 0.999                            |
| 5D $\rightarrow$ 1P | 0.249                                         | 0.775                                         | 0.027                                         | 0.610                            |
| 5D $\rightarrow$ 2P | 0.247                                         | 0.773                                         | 0.027                                         | 0.608                            |
| 5D $\rightarrow$ 3P | 0.245                                         | 0.782                                         | 0.026                                         | 0.633                            |
| 5D $\rightarrow$ 4P | 0.240                                         | 0.825                                         | 0.025                                         | 0.707                            |
| 5D $\rightarrow$ 5P | 0.222                                         | 0.986                                         | 0.021                                         | 1.006                            |

Table 10: The ratios among the radiative decay widths of the D-wave to the P-wave bottomonium states, where $\bar{\Gamma} = \frac{\Gamma_{(\Upsilon_3 \rightarrow \chi_{2}\gamma)}}{\Gamma(\Upsilon_3 \rightarrow \gamma_h\gamma)} \approx 1$, and the values in the bracket come from the screened potential model [15].

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