PROJECTIVE DYNAMICS AND AN INTEGRABLE BOLTZMANN BILLIARD MODEL

LEI ZHAO

ABSTRACT. The aim of this note is to explain the integrability of an integrable Boltzmann billiard model, previously established by Gallavotti and Jauslin [10], alternatively via the viewpoint of projective dynamics. The additional first integral is shown to be equivalent to the energy of an associated system on a hemisphere. We show that this can be further generalized to certain similar billiard models in the plane defined through Kepler-Coulomb problems as well. We propose a family of integrable billiard systems on the sphere as uniform extensions of these integrable planar billiard systems.

1. Introduction

In [10], Gallavotti and Jauslin examined a billiard model derived from the Kepler problem in the plane, with a line not containing the attractive center as wall of reflection. The billiard model is defined through the Kepler dynamics which lie in the other side of the wall as the center with the usual law of reflection in the plane. They show that the billiard system is integrable in the sense that it has another first integral additional to the energy, which confirms a previous conjecture of Gallavotti [9, Appendix D] on its integrability. This integrable Boltzmann model is shown to carry periodic and quasi-periodic dynamics by Felder [8] with algebraic geometric method related to the Poncelet theorem.

This model is a limiting case of a model considered by Boltzmann [5], defined such that an additional (sufficiently large) centrifugal force with strength inverse proportional to the cubic of the distance to the center is added. Boltzmann assumed that this is an ergodic system which illustrates his “ergodic hypothesis”. Actually this can only happen when the additional centrifugal force is sufficiently large, as by the analysis of Gallavotti-Jauslin and Felder, KAM and topological stability of orbits on energy hypersurfaces can be established via application of KAM theory.

In this note we aim to explain the integrability of the integrable Boltzmann Model alternatively with the viewpoint of projective dynamics, as developed and illustrated in [12], [4], [2], [3]. We shall show that the first integrals of Gallavotti-Jauslin is equivalent to the energies of the planar problem and a corresponding spherical problem. Moreover, we explain that this viewpoint leads to certain integrable variants of this integrable Boltzmann model in the plane and on the sphere.

We note that the “projective method” has been previously used to obtain the integrability of the geodesic flow on an ellipsoid [15], [17] and the billiard system inside an ellipsoid [16].

We organize this note as follows: We first recall projective dynamical properties of the Kepler-Coulomb problem in Section 2. In Sections 3 and 4, we define certain billiard systems in the plane (which include the integrable Boltzmann model) and on the sphere with Kepler-Coulomb problem and prove their integrability respectively. Some further remarks are collected in Section 5.

2. Projective Dynamics of Kepler-Coulomb Problems

We start with a mechanical system $(M, g, U)$ on a Riemannian manifold $(M, g)$ with the force function (negative of the potential) $U$. Newton’s equations of motion of this system are given by

$$\nabla_q \dot{q} = \text{grad} \, U,$$

in which $\nabla$ denotes the Levi-Civita connection associated to $g$, and grad denotes the gradient.
Definition 2.1. Two mechanical systems $(M_1, g_1, U_1), (M_2, g_2, U_2)$ are called in correspondence, if there is a diffeomorphism $\phi : M_1 \rightarrow M_2$, such that for any unparametrized orbit $\gamma \subset M_1$ of $(M_1, g_1, U_1)$, the orbit $\phi(\gamma) \subset M_2$ is an unparametrized orbit of $(M_2, g_2, U_2)$ and vice versa.

In other words, two mechanical systems are in correspondence if their orbits in configuration spaces agree up to diffeomorphism and up to time-reparametrization. The correspondence is called non-trivial if the energies of the two mechanical systems are functionally independent. When a mechanical system has a non-trivial corresponding system, then the system itself can be realized as a quasi-bi-Hamiltonian system, i.e. it admits two different Hamiltonian formalisms up to time reparametrization and in this case both Hamiltonians, i.e. both energies, are first integrals of the system [6].

Now let $\phi : M_1 \rightarrow M_2$ be a diffeomorphism and suppose that we have a mechanical system $(M_1, g_1, U_1)$. With these we get a vector field (force field) $\phi, \text{grad} U_1$ on $M_2$. To get a corresponding system of $(M_1, g_1, U_1)$ it is enough to have a function $\rho : M_2 \rightarrow \mathbb{R}$ such that the reparametrized force field $\rho \cdot \phi, \text{grad} U_1$ is the gradient of a function $U_2 : M_2 \rightarrow \mathbb{R}$.

A planar central force problem is a mechanical system $(\mathbb{R}^2 \setminus Z, g_{flat}, U)$ such that the force function $U$ is invariant under the $SO(2)$ action by rotations in $\mathbb{R}^2$ fixing the center $Z \in \mathbb{R}^2$. The mass factor of such a system refers to that of the center, which can nevertheless take both signs to allow both attractive and repulsive forces. The mass of the moving particle is taken as the unit of mass. A central force problem on a sphere is defined analogously.

We now consider two types of diffeomorphisms given by central projections and see how they transform central force problems, following [2] and [1]. The center of projection is always $O = (0, 0, 0) \in \mathbb{R}^3$.

The first type of projection projects affine planes to affine planes in $\mathbb{R}^3$ (See Figure 2). Let $V_1, V_2 \subset \mathbb{R}^3$ be two hyperplanes in $\mathbb{R}^3$ given respectively by the equations $\langle h_1, q \rangle = 1$ and $\langle h_2, q \rangle = 1$, in which $h_1, h_2 \in \mathbb{R}^3 \setminus O \equiv \mathbb{R}^3 \setminus O$. For a central force system on $V_1$, its center $Z_1$ is projected to a point $Z_2 \in V_2$. A point $q_2 \in V_2$ is projected from a point $q_1 \in V_1$ such that $q_2 := h(q_1)^{-1} q_1$. With the equations we determine the function as $h(q_1) = \langle h_2, q_1 \rangle$. We consider the projection in the region where $h(q_1) = \langle h_2, q_1 \rangle = \langle h_1, q_2 \rangle^{-1} > 0$, which projects the half plane $\{ q_1 \in V_1, \langle h_2, q_1 \rangle > 0 \}$ to the half plane $\{ q_2 \in V_2, \langle h_1, q_2 \rangle > 0 \}$.

We compute

$$q_2 = h(q_1)^{-1} (q_1, h(q_1) - \langle \text{grad} h(q_1), q_1 \rangle q_1) = h(q_1)^{-1} (h(q_1) - \langle h_2, q_1 \rangle q_1).$$

We now change time in $V_2$ according to the law $\frac{d}{d\tau} = \frac{d}{dt} h(q_1)^{1/2}$. We denote the time derivative with respect to $\tau$ by $'$. We may then write the above expression as

$$q'_2 = q_1, h(q_1) - \langle h_2, q_1 \rangle q_1.$$
Thus Eq. (1) takes the form
\[ q''_2 = \lambda(q_1)^2(\lambda(q_1)\dot{q}_1 - \langle h_2, \dot{q}_1 \rangle q_1). \]
We also have \( q_1 = \langle h_1, q_2 \rangle^{-1} q_2 \). The right hand side can thus be written in a form depending only on \( q_2 \) and therefore defines a force field on \( V_2 \).

We now assume that \( U_1 = m||q_1 - Z_1||^n \), in which the distance \( \| \cdot \|_1 \) is an Euclidean distance on \( V_1 \). It can be taken as the restriction of the Euclidean distance of \( \mathbb{R}^3 \), but it has to be kept in mind that this is purely auxiliary and we shall not always equip \( \mathbb{R}^3 \) with its Euclidean form.

We thus have
\[ \ddot{q}_1 = \text{grad } U_1 = am||q_1 - Z_1||^{n-2}(q_1 - Z_1). \]
Thus Eq. (1) takes the form
\[
q''_2 = \lambda(q_1)^2(\lambda(q_1)\dot{q}_1 - \langle h_2, \dot{q}_1 \rangle q_1) \\
= am\langle h_1, q_2 \rangle^{-3}\langle h_1, Z_2 \rangle^{-1}||q_2\langle h_1, q_2 \rangle^{-1} - Z_2\langle h_1, Z_2 \rangle^{-1}||^{n-2}(q_2 - Z_2).
\]

To proceed we would like to sort out the factor \( \langle h_1, q_2 \rangle^{-1} \) in the distance appeared in the above expression. On the other hand, the vector \( Z_2\langle h_1, q_2 \rangle^{-1} \in \mathbb{R}^3 \) does not necessarily lie in \( V_2 \).

To define a system in \( V_2 \), it is desired to define a distance on \( V_2 \) such that
\[
||q_2 - Z_2||_2 = \langle h_1, q_2 \rangle ||q_2\langle h_1, q_2 \rangle^{-1} - Z_2\langle h_1, Z_2 \rangle^{-1}||_1.
\]
In order to achieve this, we extend the Euclidean distance \( \| \cdot \|_1 \) in a non-standard way: Each vector \( v \in \mathbb{R}^3 \) is decomposed as \( v = v_1 + c \cdot Z_1 \). We set \( ||v||_2 = ||v_1||_1 \). Consequently we set \( \| \cdot \|_2 \) as the restriction of \( || \cdot ||_1 \) to \( V_2 \). Clearly \( \| \cdot \|_2 \), which is non-degenerate whenever \( V_1 \) and \( V_2 \) are not perpendicular, has the desired property (3).

We thus deduce from Eq. (4), by setting \( m_2 = m\langle h_1, Z_2 \rangle^{-1} \), that
\[
q''_2 = am\langle h_1, Z_2 \rangle^{-1}\langle h_1, q_2 \rangle^{-3}\langle h_1, q_2 \rangle^{-3-|n-2|}||q_2 - Z_2||^{n-2}(q_2 - Z_2) = \langle h_1, q_2 \rangle^{-1-\alpha} \text{grad } m_2||q_2 - Z_2||^2,
\]
which is again the force field of a central force problem with potential when \( \alpha = -1 \), i.e. when the system is the Coulomb-Kepler problem.

We summarize

**Proposition 2.1.** A Kepler-Coulomb problem with mass \( m_1 \) and center \( Z_1 \) in a half-plane in \( V_1 \) is centrally projected to a Kepler-Coulomb problem with mass \( m_2 \) and center \( Z_2 \) in a half-plane in \( V_2 \).

We now discuss central projection from a hemisphere to a plane. We take \( S \) to be the unit sphere in \( \mathbb{R}^3(x, y, z) \). We consider the plane \( V := \{ z = -1 \} \) (so \( h = (0, 0, -1) \)) tangent to \( S \) at its south pole \( (0, 0, -1) \) which we take as the center \( Z \). The central projection from the origin \( O \in \mathbb{R}^3 \) defines a diffeomorphism between the south-hemisphere \( S_{\mathbb{S}^2} := S \cap \{ z < 0 \} \) and \( V \). It is well-known that this projection sends unparametrized geodesics to unparametrized geodesics.

We now let \( q_1 \in V \) and \( q_2 \in S_{\mathbb{S}^2} \) be related by the central projection as:
\[
q_2 = (1 + ||q_1 - Z||^2)^{-1/2}q_1 := \lambda(q_1)^{-1}q_1.
\]
Again we have
\[
\ddot{q}_2 = \lambda(q_1)^2\dot{q}_1, \lambda(q_1) - \langle \text{grad } \lambda(q_1), \dot{q}_1 \rangle q_1,
\]
and we may again change time according to \( \frac{d}{d\tau} = \lambda(q_1)^3 \frac{d}{dt} \) which then leads to the equation
\[
q''_2 = \lambda(q_1)^2(\lambda(q_1)\ddot{q}_1 - \langle \text{grad } \lambda(q_1), \dot{q}_1 \rangle q_1).
\]
The last term of the right hand side of this equation is proportional to \( q_2 \), thus is vertical to \( S_{\mathbb{S}^2} \). It can be seen as a force of constraint which keeps the force to be tangent to \( S_{\mathbb{S}^2} \). By replacing \( \ddot{q}_1 \) with a force field, we see that with this computation, any force field in \( V \) is transformed into a force field of \( S_{\mathbb{S}^2} \) and vice versa.

Now start with the Kepler-Coulomb problem in \( V \), i.e.
\[
\ddot{q}_1 = -m||q_1 - Z||^{-3}(q_1 - Z),
\]
we have

\[ \lambda(q_1)\dot{q}_1 = -m \sin^{-3} \theta \cdot (q_1 - Z), \]

in which \( \theta \) denotes the angle \( \angle ZOq_1 \), whose component tangent to \( S_{SH} \) at \( q_2 \) gives the force, which has norm \( |m| \sin^{-2} \theta \), points toward \( Z \) when \( m > 0 \), and points in the reverse direction when \( m < 0 \). This is a central force system on \( S_{SH} \) with force function \( m \cot \theta \). We conclude that the resulting system is the spherical Kepler-Coulomb problem with mass factor \( m \), a system first defined by Serret [14], on the hemisphere \( S_{SH} \).

In the spherical Kepler-Coulomb problem, the force function \( m \cot \theta \) (and indeed the whole system) extends to the whole \( S \) except at the two poles, which are singular centers for the extended systems, one attractive and the other one repulsive. The sign of the mass factor \( m \) now determines which center is attractive and which is repulsive. The orbits of this system are (possibly degenerate) spherical (conics and more precisely) ellipses. Elliptic, parabolic and hyperbolic orbits of the Kepler-Coulomb problem in \( V \) corresponds respectively to intersections with \( S_{SH} \) of spherical ellipses lying entirely in \( S_{SH} \), tangent to the equator and intersecting transversely the equator respectively. In this sense, the spherical Kepler-Coulomb problem completes and extends the corresponding planar Kepler-Coulomb problem. We refer to [1] for more detailed analysis on the spherical Kepler-Coulomb problem.

We summarize the above analysis in the following proposition:

**Proposition 2.2.** A Kepler-Coulomb problem in \( V \) with mass factor \( m \) and center at the south pole \((0, 0, -1)\) is centrally projected to a spherical Kepler-Coulomb problem in \( S_{SH} \) with mass factor \( m \) and with center at \((0, 0, -1)\).

We now construct a projection in which the south pole is not necessarily the center of the spherical Kepler-Coulomb problem. We let \( Z = (0, a, -1) \in V \) be the center in \( V \), which is the projection of the center \( Z_1 := (0, \frac{a}{\sqrt{1 + a^2}}, -\frac{1}{\sqrt{1 + a^2}}) \) in \( S_{SH} \). With the previous procedure, the spherical Kepler-Coulomb force field is projected to an analytic force field \( F \) in \( V \). To deduce \( F \) without additional computation, we proceed with steps. We set \( h_1 = (0, \frac{a}{\sqrt{1 + a^2}}, -\frac{1}{\sqrt{1 + a^2}}) \). The affine plane \( V_1 = \{q : \langle h_1, q \rangle = 1 \} \) is now tangent to \( S_{SH} \) at \( Z_1 \). We first project from \( S_{SH} \) to \( V_1 \), then from \( V_1 \) to \( V \) (See Figure 2). Note that due to the composition of two different projections, a priori only an open subset of \( S_{SH} \) is projected to an open subset of \( V \). Nevertheless, the previous analysis shows that the force field \( F \) coincides with that of the planar Kepler-Coulomb problem with mass factor \( m \) on an open subset of \( V \). Consequently by analyticity, \( F \) is derived from the Kepler-Coulomb potential with mass factor \( m \) in \( V \).

We thus obtain the following proposition by combining Prop 2.1 and Prop 2.2.
Proposition 2.3. The planar Kepler-Coulomb problem in $V$ with mass factor $m$ and center $(0, a, -1)$ is centrally projected from the spherical Kepler-Coulomb problem on $S_{SH}$ with mass factor $m' = m \sqrt{1 + a^2}$ and center $(0, \frac{a}{\sqrt{1 + a^2}}, -\frac{1}{\sqrt{1 + a^2}})$.

As a consequence, the energies $E_{pl}, E_{sph}$ of the planar and the spherical problems are both first integrals of the planar Kepler-Coulomb problem. They are also functionally independent, as can be observed with their explicit expressions in a chart which we shall describe later.

Moreover, we shall see that $\{E_{pl}, E_{sph}\}$ is equivalent to $\{E_{pl}, D\}$ where $D$ is Gallavotti-Jauslin’s first integral for the integrable Boltzmann billiard model, constructed with an analysis of the geometry of ellipses [10]. Our result shows that the same is true for parabolic and hyperbolic orbits of the planar Kepler-Coulomb problem as well.

We now go through necessary computations. Let $S$ be the unit sphere in $\mathbb{R}^3$ and $V_1$ be the affine plane tangent to $S$ at the point $Z_1 = (0, \frac{a}{\sqrt{1 + a^2}}, -\frac{1}{\sqrt{1 + a^2}})$ which is the center of the spherical Kepler-Coulomb problem with mass $m'$. We equip both $S$ and $V_1$ with the induced metrics from the standard Euclidean metric of $\mathbb{R}^3$. We take $V = \{z = -1\}$, which is tangent to $S$ at $(0, 0, -1)$ on which there defines a Kepler-Coulomb problem with center $Z := (0, a, -1)$ and with mass $m$. The metric $\| \cdot \|_2$ in $V$ will not be the one induced from the standard Euclidean metric of $\mathbb{R}^3$, but the one defined by Prop. [2,1] which we shall now compute.

According to the construction we first extend the Euclidean form $\| \cdot \|_1$ of $V_1$ to a non-standard form $\| \cdot \|_2$ on $\mathbb{R}^3$, by setting for $v \in \mathbb{R}^3$, that $\|v\|_2 = \|v - (v, Z_1)Z_1\|_1$. Now for $v = (x, y, -1) \in V$, we have

$$v - (v, Z_1)Z_1 = (x, y - \frac{a}{1 + a^2}, -1 + \frac{a}{1 + a^2}) = (x, \frac{y - a}{1 + a^2}, \frac{a(y - a)}{1 + a^2}),$$

thus

$$\|v\|_2 = \|v\|_1 = \sqrt{x^2 + \frac{(y - a)^2}{1 + a^2}}.$$  

We therefore have that the energy of the planar Kepler-Coulomb problem in $(V, \| \cdot \|_2)$ is

$$E_{pl, 2} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m}{\sqrt{x^2 + \frac{(y - a)^2}{1 + a^2}}}.$$  

We now express the energy of the spherical Kepler-Coulomb problem in $S_{SH}$ with center $Z_1$ and with mass $m'$ in $V$, by viewing $V$ as a gnomonic chart for $S_{SH}$ as given by the central projection.

In the gnomonic chart $V$, the metric of $S_{SH}$ takes the form

$$\frac{1}{(1 + x^2 + y^2)^2}((1 + y^2)dx^2 - 2xy dy^2 + (1 + x^2)dy^2).$$

The kinetic energy of the system thus takes the form

$$(7) \quad \frac{1}{2(1 + x^2 + y^2)^2}((1 + y^2)x^2 - 2xy y' + (1 + x^2)y'^2)$$

in which $'$ denotes the derivative with respect to the time on $S_{SH}$. The factor of time change is obtained via [5]:

$$(x', y') = (1 + x^2 + y^2)(\dot{x}, \dot{y}).$$

Thus the kinetic energy is expressed equivalently as

$$\frac{1}{2}((1 + y^2)\dot{x}^2 - 2xy \dot{y} + (1 + x^2)\dot{y}^2) = \frac{1}{2}((\dot{x}^2 + \dot{y}^2) + (xy - y\dot{x})^2).$$

We see that this is the sum of the kinetic energy in the plane and half of the square of the angular momentum in the plane with respect to the point $(x, y) = (0, 0)$.  


The potential on the sphere is \(-m' \cot \theta\), where \(\theta\) is the angle between the position of the moving particle \(\frac{1}{\sqrt{x^2 + y^2 + 1}} (x, y, -1)\) on \(S_{SH}\) and the center \(Z_1\). This is expressed in terms of \((x, y)\) as
\[
-\frac{m' a y + 1}{\sqrt{(y - a)^2 + (1 + a^2)x^2}}
\]
thus the spherical energy has expression
\[
E_{sph,2} := \frac{1}{2}((1 + y^2)x^2 - 2xy\dot{y} + (1 + x^2)y^2) - m' \frac{a y + 1}{\sqrt{(y - a)^2 + (1 + a^2)x^2}}
\]
We now normalize the metric in \(V\) to a standard Euclidean one via the affine transformation
\[
(x, y) \mapsto (x = \xi, y = \sqrt{1 + a^2}\eta + a),
\]
which implies \((\dot{x}, \dot{y}) = (\dot{\xi}, \sqrt{1 + a^2}\dot{\eta})\). The planar and spherical energies now respectively take the forms (with relation \(m' = \sqrt{1 + a^2}\))
\[
E_{pl} = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) - \frac{m}{\sqrt{\xi^2 + \eta^2}}
\]
and
\[
E_{sph} = (1 + a^2)(\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) - \frac{m}{\sqrt{\xi^2 + \eta^2}}) + \frac{(1 + a^2)}{2}((\dot{\eta} - \eta\dot{\xi})^2 - \frac{a}{\sqrt{1 + a^2}}(\dot{\xi}\dot{\eta} - \dot{\xi}\eta\dot{\xi} - \eta\dot{\xi}))
\]
Set \(L = \dot{\xi}\dot{\eta} - \eta\dot{\xi}\) the angular momentum with respect to \((\xi, \eta) = (0, 0)\) and \(A_{\eta} = L\dot{\xi} + \frac{mn}{\sqrt{\xi^2 + \eta^2}}\) the \(\eta\)-component of the Laplace-Runge-Lenz vector in the direction. Let \(h = \frac{a}{\sqrt{1 + a^2}}\) be the distance of the center \((0, a, -1)\) to the line \(\mathcal{L} := \{(x, 0, -1)\}\) in \(V_2\) with respect to the metric \(\|\cdot\|_2\), and \(D = L^2 - 2hA_{\eta}\). We have
\[
E_{sph} = (1 + a^2)(E_{pl} + \frac{D}{2}).
\]
Consequently for the planar system, we have that

**Proposition 2.4.** The pair of independent first integrals \(\{E_{pl}, D\}\) is equivalent to the pair of independent first integrals \(\{E_{pl}, E_{sph}\}\).

### 3. Integrable Billiard Models with Potentials in the Plane

We now define certain billiard models in the plane with Kepler-Coulomb problem with mass factor \(m\) and center \(Z\). For this, it is enough to specify the wall of reflection and on which side of the wall the dynamics defining the billiard system takes place. At this generality, not all orbits which start from a point on the wall comes back to hit the wall again, so that the billiard mapping is not always defined. Nevertheless we also accept such cases for uniformity.

Such a billiard system is called *integrable* when there exist two functionally independent conserved quantities. Note that the energy is always a conserved quantity. Therefore the system is integrable when an additional conserved quantity functionally independent from the energy can be found.

We consider the following billiard models with potentials (See Figure 3): We take a Kepler-Coulomb problem in the plane, in which the mass factor can take either positive or negative signs. The wall of reflection is taken either as a circle \(C\) or a line \(\mathcal{L}\), which can take any positions and the dynamics on either sides of the wall can be taken to define the billiard system.

In order to deal with possible collisions with the center \(Z\), we have to invoke a regularization. Standard Levi-Civita regularization [13] does well the work, and gives a continuation of an orbit running into a collision with the center by an orbit ejecting from the collision by elastic bouncing. When the wall of reflection contains the center \(Z\), any orbit which passes through a point in the line of reflection different from \(Z\) will never meet \(Z\), therefore it is also harmless to delete \(Z\) from the line of reflection and consider only the rest, non-collisional orbits.
The integrable Boltzmann model corresponds to the case that the mass factor is negative, with a line as the wall of reflection, and with the dynamics on the side of the line not containing the center $Z$.

**Theorem 3.1.** Any of the planar billiard systems thus defined is integrable. The first integrals are its energy together with the energy of a spherical corresponding system.

**Proof.** The integrability of these systems are explained conveniently using projective dynamics. We let $V = \{ z = -1 \} \in \mathbb{R}^3$ be the plane in which the system is defined. When the wall is a circle $C$, we choose it to have its center at the south pole $(0, 0, -1)$ of $S$. When the wall is a line $L$, we choose it to cross the south pole $(0, 0, -1)$ of $S$ such that $(0, 0, -1)$ is the nearest point to the center of the system $Z$ on $L$. By translation and rotation we may set the line of reflection to $L = \{ (x, 0, -1) \}$, and we may put the center at $(0, a, -1)$.

In any of these cases, the law of reflection preserves the planar kinetic energy. The angular momentum with respect to the south pole is also preserved. As a consequence, the law of reflection also preserves the spherical kinetic energy on the south hemisphere of the unit sphere $S_{SH} \subset \mathbb{R}^3$. Alternatively we may see this directly with the expression of the spherical kinetic energy in a chart (7). Consequently both $E_{pl}$ and $E_{sph}$ are first integrals of the system. The system is therefore integrable.

In case that the wall of reflection is a line, then we may equivalently take $\{ E_{pl}, D \}$ as conserved quantities. □

We note that the integrability of the billiard system with a line of reflection which crosses the center $Z$ has been observed in [9, Appendix D]. The integrable Boltzmann model was investigated in [10] and in [8]. Our result can be regarded as an extension of this result to other systems defined in this section.

4. **Integrable Billiard Models on the Sphere**

The construction in the last section above also suggests a family of integrable billiard systems on the sphere.

We define billiard systems on the sphere with the spherical Kepler-Coulomb problem with a fixed choice of a circle as the wall of reflection, and with the natural law of reflection on the sphere, with dynamics on any side of the circle. Any pair of antipodal points may serve as centers. When they are contained in the circle of reflection, then we may either regularize collisions with the attractive center with elastic bouncing, or remove them from the circle. The repulsive center does not lie in any finite energy hypersurface of the spherical Kepler problem.

**Proposition 4.1.** The spherical billiard systems thus defined are integrable.

**Proof.** First we consider the generic case that the circle of reflection is a small circle, and the centers do not lie on the parallel great circle. This is the case that the projective dynamics argument applies directly. By
rotation we may set the circle wall of reflection to lie horizontally in $S_{SH}$, which is then centrally projected to a planar system in $V$. The small circle of reflection on the sphere is projected to a circle in $V$ and the law of reflections on the sphere and in the plane correspond to each other. Therefore the energy of the corresponding planar system $E_{pl}$ and the energy of the spherical problem $E_{sph}$ are functionally independent conserved quantities of the corresponding mechanical system on the sphere. Moreover, the function $E_{pl}$ extends to the whole sphere. For this, it is helpful to observe that an alternative to $E_{pl}$ is the square of the angular momentum with respect to the vertical $z$-axis (3), which is clearly well-defined on the whole sphere. They are therefore distinct first integrals of the billiard system on the sphere.

The systems that are not yet covered carry some inconvenience for applying projective dynamics argument, since either the great circle parallel to the circle of reflection contains the centers, or the great circle wall of reflection is not contained in a hemisphere to which we may apply central projection. On the other hand, on the sphere they are regularly approximated by systems from the generic case. Their integrability thus follows from a straightforward limiting argument on the sphere.

It is clear that the law of reflection on the sphere $S$ at a horizontal small circle in $S_{SH}$ corresponds to the law of reflection in the plane $V$ at the projection of the horizontal small circle in $V$. Also, in view of the expression of the spherical metric (7), if we put the wall of reflection on the line $L := \{y = 0, z = -1\}$ of the plane $V := \{z = -1\}$, then the law of reflection in this plane corresponds to the law of reflection on the sphere with respect to half of the great circle $\{x^2 + z^2 = 1, z < 0\}$, which is projected to $L$ via the central projection.

Therefore it is natural to consider these spherical systems on $S$ as extensions of all the planar billiard systems proposed in Section 3, with both types, circle or line, of walls of reflections, in a similar way as the spherical Kepler-Coulomb problem extends the corresponding planar Kepler-Coulomb problem. In particular, all orbits of these planar billiard systems are lifted to $S_{SH}$ and are completed in the corresponding spherical systems on the sphere.

5. SOME FURTHER REMARKS

We conclude with a few remarks.

First we remark that with similar construction we may get several families of integrable billiard systems with potentials on the pseudosphere as well.

Next we remark that conformal transformations induce correspondences of mechanical systems [11, 7], which can be used to obtain other integrable billiard systems. As an example, with the conformal mapping $C \to C, z \mapsto z^2$, one gets from the integrable Boltzmann model an integrable billiard system in the plane with the harmonic potential of a pair of isotropic harmonic oscillators, with a hyperbola as wall of reflection. It is then possible to obtain the integrability of the billiard system with harmonic potential and with a line not containing the center as wall of reflection, as has been mentioned in [9, Appendix D], via a limiting process, by approximating the line with a sequence of branches of hyperbolas. Moreover, the method of Darboux [7] can be used to obtain integrable billiard systems on a cone or certain other surfaces of revolutions.
Finally, it can be interesting to better understand the integrable dynamics of all these systems. In particular, it can be interesting to see to which extend the algebraic geometric method in [8] can be applied to the systems on the sphere.

**Acknowledgements** Thanks to A. Albouy for helpful discussions and precise suggestion of references. The author is supported by DFG ZH 605/1-1.

**References**

[1] A. Albouy, Lectures on the Two-Body Problem, in Classical and Celestial Mechanics: The Recife Lectures, Princeton University Press, (2002).
[2] A. Albouy, Projective dynamics and classical gravitation, *Regul. Chaot. Dyn.*, 13:525-542, (2008).
[3] A. Albouy, There is a projective dynamics, *EMS Newsletter*, September:37-43 (2013).
[4] P. Appell, Sur les lois de forces centrales faisant décrire à leur point d’application une conique quelles que soient les conditions initiales, *Amer. J. Math.*, 13:153–158, (1891).
[5] L. Boltzmann, Lösung eines mechanischen Problems, *Wiener Berichte*, 58:1035-1044,(1868). *Wissenschaftliche Abhandlungen*, Vol. I, p. 97-105.
[6] R Brouzet, R Caboz, J Rabenivo and V Ravoson, Two degrees of freedom quasi-bi-Hamiltonian systems, *J. Phys. A: Math. Gen.* 29(9): 2069-2076, (1996).
[7] G. Darboux, Remarque sur la Communication précédente. *Comptes Rendus Acad. Sci. Paris* 108:449-450, (1889).
[8] G. Felder, Poncelet Property and Quasi-periodicity of the Integrable Boltzmann System, arXiv: 2008.03480, (2020).
[9] G. Gallavotti, Ergodicity: a historical perspective. Equilibrium and Nonequilibrium, *Eur. Phys. J. H.* 41:181-259, (2016).
[10] G. Gallavotti, I. Jauslin, A Theorem on Ellipses, an Integrable System and a Theorem of Boltzmann, [arXiv:2008.01955] (2020).
[11] E. Goursat, Les transformations isogonales en Mécanique. *Comptes Rendus Acad. Sci. Paris* 108:446-448, (1889).
[12] G.H. Halphen, Sur les lois de Kepler, *Bulletin de la Société Philomatique de Paris*, 7(1): 89-91, (1878).
[13] T. Levi-Civita, Sur la régularisation du problème des trois corps. *Acta Math.*, 42, 99-144, (1920).
[14] P. Serret, Théorie Nouvelle Géométrique et Mécanique des Lignes à Double Courbure Mallet-Bachelier, Paris, (1860).
[15] S. Tabachnikov, Projectively equivalent metrics, exact transverse linefields and the geodesic flow on the ellipsoid, *Comment. Math. Helv.* 74:306-321, (1999).
[16] S. Tabachnikov, Ellipsoids, complete integrability and hyperbolic geometry, *Moscow Math. J.*, 2:185-198, (2002).
[17] V. S. Matveev, P. Topalov, Geodesic Equivalence and Integrability, *MPIM Preprint Series* No. 74, (1998).

Lei Zhao, University of Augsburg, Augsburg, Germany. Email: lei.zhao@math.uni-augsburg.de.