Representation theory of $\mathfrak{sl}(2|1)$

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Abstract

In this note we present a complete analysis of finite dimensional representations of the Lie superalgebra $\mathfrak{sl}(2|1)$. This includes, in particular, the decomposition of all tensor products into their indecomposable building blocks. Our derivation makes use of a close relation with the representation theory of $\mathfrak{gl}(1|1)$ for which analogous results are described and derived.

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1 Introduction

Since their first systematic discussion \cite{1} in the 1970’s, Lie superalgebras have been studied for a variety of reasons, both in physics and in mathematics. They found applications not only in elementary particle physics (see \cite{2} for an early review) but also to condensed matter problems, mostly in the context of disordered fermions \cite{3} and in particular the quantum Hall effect \cite{4,5} (see also e.g. \cite{6} for further applications to models of statistical physics). During the last years, non-linear $\sigma$-models on supergroups and supercosets have also emerged through studies of string theory in certain Ramond-Ramond backgrounds \cite{7,8,9}. Many special properties of these models, such as the possible presence of conformal invariance without a Wess-Zumino term, originate from peculiar features of the underlying Lie superalgebra \cite{10,11}.

Even though Lie superalgebras are so widely used, their representation theory, and in particular their Clebsch-Gordan decomposition, is far from being fully developed. This may partly be explained by the fact that indecomposable (but reducible) representations occur quite naturally \cite{1,12,13}. Furthermore, many Lie superalgebras are known not to admit a complete classification of all finite dimensional representations \cite{14}. One of the rare exceptions for which such a classification exists are the Lie superalgebras of type $\mathfrak{sl}(n|1)$ \cite{15,16,14,17}.

In this note we shall discuss the representation theory of $\mathfrak{sl}(2|1)$, including a complete list of tensor products of finite dimensional representations with diagonalizable Cartan elements. Thereby, we extend previous partial studies \cite{12,18}. Our derivations are based on a particular embedding of the Lie superalgebra $\mathfrak{gl}(1|1)$. For the purpose of being self-contained we shall therefore commence in section \ref{sec:2} with a short exposition of the Lie superalgebra $\mathfrak{gl}(1|1)$, its finite dimensional representations and their tensor products.

All our new results on $\mathfrak{sl}(2|1)$ are contained in section \ref{sec:4}. First, we investigate how $\mathfrak{sl}(2|1)$ representations decompose after restricting the action to the subalgebra $\mathfrak{gl}(1|1)$. These decompositions exhibit a very close correspondence between atypical representations (short multiplets) of $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}(2|1)$. The latter extends to indecomposable composites of atypical representations. Our results for the decomposition of $\mathfrak{sl}(2|1)$ tensor products into their indecomposable building blocks are stated in the propositions 1, 2 and 4. Proposition 3 states that, modulo projectives, the representation ring of $\mathfrak{sl}(2|1)$ may be embedded into the representation ring of $\mathfrak{gl}(1|1)$.
In a forthcoming publication [19] we shall employ the results of this paper and related ideas in order to determine the tensor products of a large class of \( \mathfrak{psl}(2|2) \)-representations. The latter are relevant for the study of strings in \( \text{AdS}_3 \). In addition, our analysis might possess implications for the construction of new conformal fields theories with \( \mathfrak{gl}(1|1) \) or \( \mathfrak{sl}(2|1) \) superalgebra symmetries (see, e.g., [20, 21, 22]). As in the case of bosonic current algebras, the representation theory of affine Lie superalgebras inherits much of its features from the finite dimensional algebra of zero modes. But in the case of current superalgebras, there remain many unresolved issues, e.g. concerning the modular transformation of characters and the relation of the modular \( S \) matrix to the fusion algebra \([23, 24, 25]\). We hope to come back to these important questions in the future. Let us finally also note that \( \mathfrak{gl}(1|1) \) symmetry has been argued to be an imminent feature of every \( c = 0 \) conformal field theory \([26, 27]\).

2 The Lie superalgebra \( \mathfrak{gl}(1|1) \)

This section is devoted to the representation theory of \( \mathfrak{gl}(1|1) \). Not only will this Lie superalgebra play a crucial role when we determine tensor products of \( \mathfrak{sl}(2|1) \) representations, it can also serve as a very instructive example in which we encounter some of the most interesting phenomena and notions in the representation theory of Lie superalgebras.

2.1 The defining relations

The Lie superalgebra \( \mathfrak{h} = \mathfrak{gl}(1|1) \) is generated by two even elements \( E, N \) and two odd elements \( \psi^\pm \) (we shall follow the physicists convention of [20]). The element \( E \) is central and the fermions \( \psi^\pm \) have opposite charge with respect to \( N \). More explicitly the defining relations read,

\[
[E, \psi^\pm] = [E, N] = 0 \quad [N, \psi^\pm] = \pm \psi^\pm \quad \{\psi^+, \psi^-\} = E .
\] (2.1)

The even subalgebra is thus given by \( \mathfrak{h}^{(0)} = \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \). For later convenience let us also introduce the automorphism \( \omega \) which is defined by its action

\[
\omega(E) = E \quad \omega(N) = -N \quad \omega(\psi^\pm) = \psi^\mp \quad (2.2)
\]
on the generators and extended to the whole Lie superalgebra \( \mathfrak{h} \) by linearity.
2.2 The finite dimensional representations

The indecomposable finite dimensional representations of $\mathfrak{h} = \mathfrak{gl}(1|1)$ have been classified in \cite{1, 28, 16, 17}. We shall start with a short discussion of irreducible representations which may all be obtained from the so-called Kac modules. In this context it is crucial to distinguish between typical and atypical representations \cite{29}, or long and short multiplets. The most striking feature of the latter is that they can be part of larger indecomposable representations. A complete list of such “composites” is provided in the second and third subsection.

2.2.1 Kac modules and irreducible representations

Let us agree to work with a Cartan subalgebra that is spanned by $E$ and $N$. In order to introduce Kac modules we define $\psi^+$ to be a positive root and $\psi^-$ to be a negative root. The Kac modules $\langle e, n \rangle$ are induced highest weight modules over a one-dimensional representation $(e, n)$ of the bosonic subalgebra, where $e \in \mathbb{C}$ and $n \in \mathbb{C}$ are the eigenvalues of $E$ and $N$, respectively. A more explicit description through matrices is,

\begin{align*}
\langle e, n \rangle : & \quad E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad N = \begin{pmatrix} n & 0 \\ 0 & n-1 \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.3)
\end{align*}

Similarly, one can introduce anti-Kac modules $\overline{\langle e, n \rangle}$ by switching the role of positive and negative roots. The corresponding matrix representation reads,

\begin{align*}
\overline{\langle e, n \rangle} : & \quad E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad N = \begin{pmatrix} n-1 & 0 \\ 0 & n \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}. \quad (2.4)
\end{align*}

Note that the modules $\langle e, n \rangle$ and $\overline{\langle e, n \rangle}$ are irreducible if and only if $e \neq 0$ in which case they are also isomorphic. The resulting representations are called typical and they provide the “generic” irreducible representations of $\mathfrak{gl}(1|1)$.

For $e = 0$, on the other hand, one obtains two inequivalent indecomposable representations both of which possess an invariant one-dimensional subspace. If we introduce the notation $\langle n \rangle$ for the one-dimensional irreducible representations specified by

\begin{align*}
E = 0 \quad N = n \quad \psi^\pm = 0 \quad (2.5)
\end{align*}

\footnote{To avoid confusion we stress that the term indecomposable refers to both irreducible as well as reducible but indecomposable representations.}
then we may express the structure of the indecomposable Kac and anti-Kac modules through the following diagrams,

\[
\langle 0, n \rangle : \quad \langle n - 1 \rangle \leftarrow \langle n \rangle \quad \langle 0, n \rangle : \quad \langle n - 1 \rangle \rightarrow \langle n \rangle .
\] (2.6)

Pictures of this type (and certainly much more complicated versions) will appear frequently throughout this text. Let us therefore pause for a moment to recall how we decode their information: Atypical representations from which no arrows emanate correspond to invariant subspaces. If we divide by such a subrepresentation, the resulting quotient is encoded by a new diagram which is obtained from the original by deleting the invariant subspace along with all the adjacent arrows. In the case of (anti-) Kac modules there exists only a single irreducible invariant subrepresentation and the corresponding quotients are irreducible. But we will soon see examples of modules with several invariant subspaces or even whole hierarchies thereof. In such cases, our diagrams may have different floors which are connected by arrows.

2.2.2 Projective covers of atypical irreducible representations

We have observed already that the atypical irreducible representations can be part of larger indecomposables, e.g. of the Kac and anti-Kac modules. The latter can themselves appear as proper submodules of indecomposable structures. There exist certain distinguished indecomposables, however, that admit no further extension. These are the so-called projective covers \(P_h(n)\) of atypical representations that we are going to introduce next.

The representations \(P_h(n)\) are four-dimensional and they are parametrized by one complex parameter \(n\) which features explicitly in the following matrices,

\[
N = \begin{pmatrix}
n & 0 & 0 & 0 \\
0 & n + 1 & 0 & 0 \\
0 & 0 & n - 1 & 0 \\
0 & 0 & 0 & n \\
\end{pmatrix}, \quad \psi^+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad \psi^- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}.
\]

The element \(E\) vanishes identically. It is worth mentioning that \(P_h(0)\) is the adjoint representation of \(\text{gl}(1|1)\). As they stand, the matrices are not very illuminating. In fact, the structure of \(P_h(n)\) is much better understood after translation into our diagrammatic language,

\[
P_h(n) : \quad \langle n \rangle \rightarrow \langle n + 1 \rangle \oplus \langle n - 1 \rangle \rightarrow \langle n \rangle .
\] (2.7)
There is a variant of this pictorial presentation that keeps track of the ordering of the weights, i.e. of the eigenvalues for $N$,

$$
\langle n \rangle \quad \langle n-1 \rangle \quad \langle n+1 \rangle \quad \langle n \rangle
$$

In this diagram, $N$-eigenvalues increase from left to right. Both pictures display the essential features of $\mathcal{P}_h(n)$ very clearly. To begin with, these representations contain a unique irreducible one-dimensional subrepresentation $\langle n \rangle$ in the rightmost position (bottom). This is called the “socle” of $\mathcal{P}_h(n)$ and it is the reason for us to think of the four-dimensional indecomposables as a “cover” of atypical irreducible representations. In addition, we can also find three different types of indecomposable subrepresentations in $\mathcal{P}_h(n)$. These include the two-dimensional (anti-)Kac modules $\langle 0, n+1 \rangle$ and $\langle 0, n \rangle$. But there appears also one new class of three-dimensional indecomposables that we did not meet before. Their diagram is obtained from the above by deleting the representation $\langle n \rangle$ on top along with the arrows that emanate from it. One can go through a similar analysis of factor representations obtained from $\mathcal{P}_h(n)$ with very much the same pattern of results. Let us only point out that the quotient of $\mathcal{P}_h(n)$ by its socle $\langle n \rangle$ provides a new three-dimensional indecomposable representation which is not isomorphic to the one we found among the submodules of $\mathcal{P}_h(n)$.

We have seen that atypical irreducibles sit inside (anti-)Kac modules which in turn appear as subrepresentations of three-dimensional indecomposables. But the sequence of embeddings does not end here. In the next subsection we shall construct two infinite series of indecomposables which are nested into each other such that their $m^{th}$ member appears as an extension of the $(m-1)^{th}$ by a one-dimensional atypical representation. The representation $\mathcal{P}_h(n)$ gives rise to another extension of three-dimensional indecomposables, but this one turns out to be maximal, i.e. no further embedding into a larger indecomposable is possible. Their maximality distinguishes $\mathcal{P}_h(n)$ from all other representations with $E = 0$ and it places them in one group with the typical two-dimensional representations. In more mathematical terms, $\langle e, n \rangle$, $e \neq 0$, and $\mathcal{P}_h(n)$ are known as projective representations of $\mathfrak{gl}(1|1)$, a notion that is particularly important for our investigation of tensor products since the projective representations form an ideal in the representation ring.
2.2.3 Zigzag modules

As we have anticipated at the end of the previous subsection, there exist two different families of indecomposable representations $\mathcal{Z}_d^h(n)$ and $\bar{\mathcal{Z}}_d^h(n)$ which we shall name (anti-)zigzag representations. They are parametrized by the eigenvalue $n \in \mathbb{C}$ of $N$ with the largest real part and by the number $d = 1, 2, 3, \ldots$ of their atypical constituents. On a basis of eigenstates $|m\rangle, m = n, \ldots, n - d + 1$, for the element $N$, the generators of zigzag representations $\mathcal{Z}_d^h(n)$ read

$$N|m\rangle = m|m\rangle, \quad \psi^\pm|m\rangle = \frac{1}{2} (1 + (-1)^{n-m})|m \pm 1\rangle$$

(2.9)

and $E$ vanishes identically. Here we agree that $|m\rangle = 0$ when $m$ is outside the allowed range. Similarly, we can introduce anti-zigzag representations $\bar{\mathcal{Z}}_d^h(n)$ through

$$N|m\rangle = m|m\rangle, \quad \psi^\pm|m\rangle = \frac{1}{2} (1 - (-1)^{n-m})|m \pm 1\rangle$$

(2.10)

The only difference between the formulas (2.9) and (2.10) is in the sign between the two terms for the action of fermionic elements. Note that atypical irreducible representations and (anti-)Kac modules are special cases of (anti-)zigzag representations, in particular we have $\langle n \rangle \cong \mathcal{Z}_1^h(n) \cong \bar{\mathcal{Z}}_1^h(n)$.

Once more we can display the structure of the (anti-)zigzag modules through their associated diagram. In doing so we shall separate two cases depending on the parity of $d$. When $d = 2p$ is even we find

$$\mathcal{Z}_{2p}^h(n) : \quad \langle n - 2p + 1 \rangle \leftarrow \langle n - 2p + 2 \rangle \rightarrow \cdots \rightarrow \langle n - 2 \rangle \leftarrow \langle n - 1 \rangle \leftarrow \langle n \rangle$$

$$\bar{\mathcal{Z}}_{2p}^h(n) : \quad \langle n - 2p + 1 \rangle \rightarrow \langle n - 2p + 2 \rangle \leftarrow \cdots \rightarrow \langle n - 2 \rangle \leftarrow \langle n - 1 \rangle \rightarrow \langle n \rangle$$

Observe that the leftmost atypical constituent is invariant for even dimensional zigzag modules, a property that is not shared by the even dimensional anti-zigzag representations which, by construction, always possess an invariant constituent in their rightmost position. When $d = 2p + 1$ is odd, on the other hand, the corresponding diagrams read

$$\mathcal{Z}_{2p+1}^h(n) : \quad \langle n - 2p \rangle \rightarrow \langle n - 2p + 1 \rangle \leftarrow \cdots \leftarrow \langle n - 2 \rangle \rightarrow \langle n - 1 \rangle \leftarrow \langle n \rangle$$

$$\bar{\mathcal{Z}}_{2p+1}^h(n) : \quad \langle n - 2p \rangle \leftarrow \langle n - 2p + 1 \rangle \rightarrow \cdots \rightarrow \langle n - 2 \rangle \leftarrow \langle n - 1 \rangle \rightarrow \langle n \rangle$$

In this case, both ends of the anti-zigzag modules correspond to invariant subspaces. Tensor products of (anti-)zigzag representations will turn out to depend very strongly on
the parity of $d$. In analogy with our second graphical presentation (2.8) for the projective representations $\mathcal{P}_h(n)$, one may be tempted to change our diagrams for $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ a little bit by moving the sources up such that all arrows run at a 45 degree angle. The resulting pictures explain our name “zigzag module”.

2.2.4 Action of the automorphism on modules

When calculating the tensor products of $\mathfrak{gl}(1|1)$ representations we can save some work by using the additional information that is encoded in the existence of the outer automorphism $\omega$. In fact, given any representation $\mu$ with map $\rho_\mu : \mathfrak{g} \rightarrow \text{End}(V)$ and an automorphism $\omega$, we may define the new representation $\omega(\mu)$ on the same space $V$ through the prescription $\rho_{\omega(\mu)} = \rho \circ \omega$. Depending on the choice of $\mu$, the new representation $\omega(\mu)$ will often turn out to be inequivalent to $\mu$.

Let us briefly work out how the various representations of $\mathfrak{gl}(1|1)$ are mapped onto each other. For the projective representations one easily finds

$$\omega(\langle e,n \rangle) = \langle e, 1-n \rangle \quad \omega(\mathcal{P}_h(n)) = \mathcal{P}_h(-n) \ . \quad (2.11)$$

The second assignment is easily found from the structure (2.8) of the projective cover along with the obvious rule $\omega(\langle n \rangle) = \langle -n \rangle$. A similar argument also determines the action of the automorphism $\omega$ on zigzag representations,

$$\omega(\mathcal{Z}_h^d(n)) = \begin{cases} 
\bar{\mathcal{Z}}_h^d(d-n-1) & \text{for } d \text{ even} \\
\mathcal{Z}_h^d(d-n-1) & \text{for } d \text{ odd} 
\end{cases} \ . \quad (2.12)$$

For anti-zigzag representations the same rules apply with the roles of $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ being switched. What makes these simple observations useful for us is the fact that the fusion of representation respects the action of $\omega$. In other words, if $\mu_3$ is a subrepresentation of $\mu_1 \otimes \mu_2$, then $\omega(\mu_3)$ arises in the tensor product of $\omega(\mu_1)$ and $\omega(\mu_2)$ and their multiplicities coincide.

2.3 Decomposition of $\mathfrak{gl}(1|1)$ tensor products

We are now ready to spell out the various tensor products of finite dimensional representations of $\mathfrak{gl}(1|1)$. Obviously, there are quite a few cases to consider. For the tensor
product of two typical representations one finds
\[\langle e_1, n_1 \rangle \otimes \langle e_2, n_2 \rangle = \begin{cases} \mathcal{P}_h(n_1 + n_2 - 1) & \text{for } e_1 + e_2 = 0 \\ \bigoplus_{p=0}^{1} \langle e_1 + e_2, n_1 + n_2 - p \rangle & \text{for } e_1 + e_2 \neq 0 \end{cases} \] (2.13)

This formula should only be used when \( e_1, e_2 \neq 0 \). Tensor products between atypical Kac modules will appear as a special case below when we discuss the multiplication of zigzag representations.

Next we would like to consider the tensor products involving projective covers \( \mathcal{P}_h \) in addition to typical representations. These are given by
\[\langle e, n \rangle \otimes \mathcal{P}_h(m) = \langle e, n + m + 1 \rangle \oplus 2 \cdot \langle e, n + m \rangle \oplus \langle e, n + m - 1 \rangle \]
\[\mathcal{P}_h(n) \otimes \mathcal{P}_h(m) = \mathcal{P}_h(n + m + 1) \oplus 2 \cdot \mathcal{P}_h(n + m) \oplus \mathcal{P}_h(n + m - 1) \] (2.14)
where we assume once more that \( e \neq 0 \) in the first line. We observe that typical representations and projective covers close under tensor products, in perfect agreement with the general behavior of projective representations.

Tensor products between the projective representations and (anti-)zigzag modules are also easy to spell out
\[\langle e, n \rangle \otimes \mathcal{Z}_d(m) = \langle e, n \rangle \otimes \mathcal{Z}_d(m) = \bigoplus_{p=0}^{d-1} \langle e, n + m - p \rangle \] (2.15)
\[\mathcal{P}_h(n) \otimes \mathcal{Z}_d(m) = \mathcal{P}_h(n) \otimes \mathcal{Z}_d(m) = \bigoplus_{p=0}^{d-1} \mathcal{P}_h(n + m - p) \] (2.16)

On the right hand side, only projective representations appear. We conclude that the latter form an ideal in the representation ring, just as predicted by general results in the theory of Lie superalgebras.

The description of tensor products between (anti-)zigzag representations requires the most efforts since we have to treat various cases separately, depending on the parity of the parameter \( d \).
\[\mathcal{Z}^{2p_1}_h(n_1) \otimes \mathcal{Z}^{2p_2}_h(n_2) = \bigoplus_{\nu_1=0}^{p_1-1} \bigoplus_{\nu_2=0}^{p_2-1} \mathcal{P}_h(n_1 + n_2 - 2\nu_1 - 2\nu_2 - 1) \]
\[\mathcal{Z}^{2p_1}_h(n_1) \otimes \mathcal{Z}^{2p_2}_h(n_2) = \mathcal{Z}^{2p_1}_h(n_1 + n_2) \oplus \mathcal{Z}^{2p_1}_h(n_1 + n_2 - 2p_2 + 1) \oplus \bigoplus_{\nu_1=0}^{p_1-1} \bigoplus_{\nu_2=1}^{p_2-1} \mathcal{P}_h(n_1 + n_2 - 2\nu_1 - 2\nu_2) \quad \text{for } p_1 \leq p_2 \]
\[ Z_{h}^{2p+1}(n_1) \otimes Z_{h}^{2p+1}(n_2) = Z_{h}^{2(p_1+p_2)+1}(n_1 + n_2) \oplus \bigoplus_{\nu_1=0}^{p_1-1} \bigoplus_{\nu_2=1}^{p_2} \mathcal{P}_h(n_1 + n_2 - 2\nu_1 - 2\nu_2), \]

\[ Z_{\tilde{h}}^{2p+1}(n_1) \otimes Z_{\tilde{h}}^{2p+1}(n_2) = Z_{\tilde{h}}^{2(p_2-p_1)+1}(n_1 + n_2 - 2p_1) \oplus \bigoplus_{\nu_1=0}^{p_1-1} \bigoplus_{\nu_2=0}^{p_2} \mathcal{P}_h(n_1 + n_2 - 2\nu_1 - 2\nu_2 - 1) \quad \text{for} \quad p_1 \leq p_2, \]

\[ Z_{h}^{2p+1}(n_1) \otimes Z_{\tilde{h}}^{2p+1}(n_2) = Z_{\tilde{h}}^{2p+1}(n_1 + n_2) \oplus \bigoplus_{\nu_1=1}^{p_1-1} \bigoplus_{\nu_2=1}^{p_2} \mathcal{P}_h(n_1 + n_2 - 2\nu_1 - 2\nu_2). \]

The remaining formulas can either be obtained by applying the outer automorphism \( \omega \) to the ones we have displayed or by a formal conjugation of the above expression in which we replace \( Z \) by \( \tilde{Z} \) (and vice versa) while touching neither their arguments nor the projective part at all. Though we have not found these tensor products in the literature, we would not be surprised if they were known before. In any case, they may be derived by an explicit construction of the vectors that span the corresponding invariant subspaces in each tensor product. Let us also point out that the representation ring of \( \mathfrak{gl}(1|1) \) possesses many different subrings, i.e. there exist many different subsets of representations which close under tensor products. We observe, for example, that (anti-)zigzag modules of any given even length (or even a finite set thereof) can be combined with projective representations to form an ideal in the fusion ring.

### 3 The Lie superalgebra \( \mathfrak{sl}(2|1) \)

This section is devoted to our main theme, the theory of finite dimensional representations of \( \mathfrak{sl}(2|1) \). The latter have been entirely classified \[13, 15, 16, 17\]. This distinguishes \( \mathfrak{sl}(2|1) \) from most other members of the A-series of Lie superalgebras for which a classification is even known to be impossible \[14\]. Here we shall provide a complete list of tensor products of finite dimensional representations of \( \mathfrak{sl}(2|1) \), thereby extending previous partial results by Marcu \[18\]. We shall achieve this with the help of a nice correspondence between the indecomposables of \( \mathfrak{sl}(2|1) \) and \( \mathfrak{gl}(1|1) \) which allows us to employ the results of the previous section.

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2Note that the described conjugation and the application of \( \omega \) are two different operations.
3.1 The defining relations

The even part \( g^{(0)} = \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \) of the Lie superalgebra \( g = \mathfrak{sl}(2|1) \) is generated by four bosonic elements \( H, E^\pm \) and \( Z \) which obey the commutation relations

\[
[H, E^\pm] = \pm E^\pm, \quad [E^+, E^-] = 2H, \quad [Z, E^\pm] = [Z, H] = 0. \quad (3.1)
\]

In addition, there exist two fermionic multiplets \((F^+, F^-)\) and \((\bar{F}^+, \bar{F}^-)\) which generate the odd part \( g^{(1)} \). They transform as \((\pm \frac{1}{2}, \frac{1}{2})\) with respect to the even subalgebra, i.e.

\[
[H, F^\pm] = \pm \frac{1}{2} F^\pm \quad [H, \bar{F}^\pm] = \pm \frac{1}{2} \bar{F}^\pm \\
[E^\pm, F^\pm] = [E^\pm, \bar{F}^\pm] = 0 \quad [E^\pm, F^\mp] = -F^\pm \quad [E^\pm, \bar{F}^\mp] = \bar{F}^\pm \quad (3.2)
\]

\[
[Z, F^\pm] = \frac{1}{2} F^\pm \quad [Z, \bar{F}^\pm] = -\frac{1}{2} \bar{F}^\pm.
\]

Finally, the fermionic elements possess the following simple anti-commutation relations

\[
\{F^\pm, F^\mp\} = \{\bar{F}^\pm, \bar{F}^\mp\} = 0 \quad \{F^\pm, \bar{F}^\pm\} = E^\pm \quad \{F^\pm, \bar{F}^\mp\} = Z \mp H \quad (3.3)
\]

among each other. Formulas (3.1) to (3.3) provide a complete list of relations in the Lie superalgebra \( \mathfrak{sl}(2|1) \).

There are two different decompositions of \( \mathfrak{sl}(2|1) \) that shall play some role in our analysis below. One of them is the following triangular decomposition

\[
g = \mathfrak{g}_+ \oplus \mathfrak{p} \oplus \mathfrak{g}_- , \quad (3.4)
\]

in which the Cartan subalgebra is given by \( \mathfrak{p} = \text{span}(H, Z) \), the positive roots span \( \mathfrak{g}_+ = \text{span}(E^+, F^\pm) \) and the negative roots generate the third subspace \( \mathfrak{g}_- = \text{span}(E^-, \bar{F}^\pm) \).

This decomposition corresponds to a particular choice of the root system. Let us recall that for Lie superalgebras, the latter is not unique.

Another natural decomposition is obtained by collecting all bosonic generators in one subspace while keeping the fermionic generators in two separate sets,

\[
g = \mathfrak{g}^{(1)}_1 \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}_{-1} . \quad (3.5)
\]

Here, \( \mathfrak{g}^{(1)}_1 = \text{span}(F^\pm) \) and \( \mathfrak{g}^{(1)}_{-1} = \text{span}(\bar{F}^\pm) \). By declaring elements of these three subspaces to possess grade \((-1, 0, 1)\), respectively, we can introduce an \( \mathbb{Z} \)-grading in the
universal enveloping algebra. Fermionic elements possess odd grades so that the new grading is consistent with the usual distinction between even and odd generators.

As in our discussion of \( \mathfrak{g}l(1|1) \) above, it will be useful for us to exploit the symmetries of \( \mathfrak{sl}(2|1) \). In this case, they are described by an outer automorphism that acts trivially on the generators \( E^\pm \) and \( H \) while exchanging the barred and unbarred fermionic elements and reversing the sign of \( Z \), i.e.

\[
\Omega : (H, E^\pm, Z, F^\pm, \bar{F}^\pm) \mapsto (H, E^\pm, -Z, \bar{F}^\pm, F^\pm) .
\]  

The existence of this \( \mathbb{Z}_2 \)-automorphism will allow us to determine several tensor products rather easily.

### 3.2 Finite dimensional representations

Since there are different notations floating around in the mathematics \([1]\) and in the physics literature \([12, 30]\) we shall give a short account of the basic constructions of modules and how they are related. Our discussion is restricted to finite dimensional representations in which the Cartan subalgebra can be diagonalized. More general representations have been discussed in \([13, 17]\) and \([31]\). An overview over the representations considered in this paper is given in table 1.

#### 3.2.1 Kac modules and irreducible representations

The basic tool in the construction of irreducible representations are again the Kac modules \([1]\). In the case of \( \mathfrak{g} = \mathfrak{sl}(2|1) \), these form a 2-parameter family \( \{b, j\} \) of \( 8j \)-dimensional representations. We may induce them from the \( 2j \)-dimensional representations \( (b - \frac{1}{2}, j - \frac{1}{2}) \) of the bosonic subalgebra \( \mathfrak{g}^{(0)} \) by applying the generators in \( \mathfrak{g}_1^{(1)} \), i.e. the pair \( F^\pm \) of fermionic elements. Our label \( b \in \mathbb{C} \) denotes a \( \mathfrak{gl}(1) \)-charge and spins of \( \mathfrak{sl}(2) \) are labeled by \( j = \frac{1}{2}, 1, \ldots \). To be more precise, we must first promote the representation space of the bosonic subalgebra to a \( \mathfrak{g}^{(0)} \oplus \mathfrak{g}_1^{(1)} \)-module by declaring that its vectors are annihilated when we act with elements \( \bar{F}^\pm \). Then we can set

\[
\{b, j\} = \text{Ind}_{\mathfrak{g}^{(0)} \oplus \mathfrak{g}_1^{(1)}}^\mathfrak{g} V_{(b - \frac{1}{2}, j - \frac{1}{2})} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^{(0)} \oplus \mathfrak{g}_1^{(1)})} V_{(b - \frac{1}{2}, j - \frac{1}{2})} .
\]

In this formula, \( \mathcal{U}(\mathfrak{g}) \) denotes the universal enveloping algebra of \( \mathfrak{g} \) and \( V \) is the \( 2j \)-dimensional representation space of the bosonic subalgebra, or, to be more precise, of the
extented algebra $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}_−$. Let us emphasize that there is a relative shift in the labels between the representation $\{b, j\}$ of the Lie superalgebra and the corresponding bosonic representation $(b - \frac{1}{2}, j - \frac{1}{2})$. The shift guarantees that the highest eigenvalue of $H$ in the whole module is given by $j$ and it corresponds to the conventions of [12]. Even though the latter seem somewhat unnatural from the point of view of Kac modules we will later encounter some simplifications which justify this choice.

The dual construction which promotes the fermions in $\mathfrak{g}^{(1)}_−$, i.e. the generators $\bar{F}^\pm$, to creation operators yields anti-Kac modules $(b$ and $j$ take the same values as above)

$$\{b, j\} = \text{Ind}_{\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}_−} V_{(b + \frac{1}{2}, j - \frac{1}{2})} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}_−)} V_{(b + \frac{1}{2}, j - \frac{1}{2})}.$$  \hspace{1cm} (3.7)

This bosonic content of (anti-)Kac modules may be read off rather easily form their construction,

$$\{b, j\}_{\mathfrak{g}^{(0)}} = (b - \frac{1}{2}, j - \frac{1}{2}) \otimes \mathcal{U}(\mathfrak{g}^{(1)}_+)|_{\mathfrak{g}^{(0)}}$$

$$\overline{\{b, j\}}_{\mathfrak{g}^{(0)}} = (b + \frac{1}{2}, j - \frac{1}{2}) \otimes \mathcal{U}(\mathfrak{g}^{(1)}_-)|_{\mathfrak{g}^{(0)}}$$

where

$$\mathcal{U}(\mathfrak{g}^{(1)}_{\pm})|_{\mathfrak{g}^{(0)}} = [(0, 0) \oplus (\pm \frac{1}{2}, \frac{1}{2}) \oplus (\pm 1, 0)].$$

The product $\otimes$ on the right hand side denotes the tensor product of $\mathfrak{g}^{(0)}$ representations. For generic values of $b$ and $j$, the modules $\{b, j\}$ and $\overline{\{b, j\}}$ are irreducible and isomorphic. At the points $\pm b = j$, however, they degenerate, i.e. the representations are indecomposable and no longer isomorphic. In fact, Kac and anti-Kac modules are then easily seen to possess different invariant subspaces.

By dividing out the maximal submodule from each Kac module $\{\pm j, j\}$ we obtain irreducible highest weight representation $\{j\}_\pm$ of dimension $4j+1$$^3$. In order to understand their structure in more detail, we emphasize that the representations $\{j\}_+$ with $j = 0, \frac{1}{4}, \ldots$ are constructed from the Kac modules $\{j + \frac{1}{2}, j + \frac{1}{2}\}$ by decoupling the states in the representation $(j + \frac{1}{2}, j + \frac{1}{2}) \oplus (j + 1, j)$ of the bosonic subalgebra. For the representations $\{j\}_-$ with $j = \frac{1}{2}, 1, \ldots$, on the other hand, we start from the Kac modules $\{-j, j\}$ and decouple the bosonic multiplets $(-j, j - 1)$ and $(-j + \frac{1}{2}, j - \frac{1}{2})$. This construction implies that the bosonic content of atypical representations is given by

$$\{j\}_\pm = \begin{cases} (j, j) \oplus (j + \frac{1}{2}, j - \frac{1}{2}) & \text{for } + \text{ and } j = \frac{1}{2}, 1, \ldots \\ (-j, j) \oplus (-j + \frac{1}{2}, j - \frac{1}{2}) & \text{for } - \text{ and } j = \frac{1}{2}, 1, \ldots \end{cases} \hspace{1cm} (3.8)$$

$^3$A similar construction using anti-Kac modules instead of Kac modules leads to the same set of representations.
and by (0) in case of the trivial representation \( \{0\} = \{0\}_+ \). Note that the representations \( \{j\}_\pm \) are labeled by a non-negative \( j \). From time to time we shall adopt a notation in which the label \( \pm \) is traded for a sign in the argument, i.e. we set \( \{l\} = \{\vert l\vert \} \text{sign}(l) \). In case of the trivial representation, this convention amounts to omitting the subscript \( + \). The irreducible representations \( \{b, j\} \) with \( \pm b \neq j \) are called typical. All other irreducibles of the type \( \{j\}_\pm \) are atypical. The 8-dimensional adjoint representation is given by \( \{0, 1\} \), i.e. it is typical.

Let us note in passing that our outer automorphism \( \Omega \) acts on the irreducible representations much in the same way as for \( \mathfrak{gl}(1|1) \) (see eq. (3.6)). It is not difficult to see that

\[
\Omega(\{b, j\}) = \{-b, j\} \quad \quad \Omega(\{j\}_\pm) = \{j\}_\mp .
\]

The second formula will be particularly useful to understand the action of \( \Omega \) on indecomposable representation of \( \mathfrak{gl}(1|1) \).

As a byproduct of the construction of irreducible representations we have seen the first examples of indecomposables of \( \mathfrak{sl}(2|1) \), namely the (anti-)Kac modules \( \{\pm j, j\} \) and \( \{\pm j, j\} \). They are built from two atypical representations such that

\[
\{\pm j, j\} : \quad \{j\}_\pm \longrightarrow \{j - \frac{1}{2}\}_\pm
\]

\[
\{\pm j, j\} : \quad \{j - \frac{1}{2}\}_\pm \longrightarrow \{j\}_\pm .
\]

We shall construct many other indecomposables in the following subsections. Let us also note that Kac and anti-Kac modules are mapped onto each other by the action of our automorphism (3.6).

We wish to stress that in the physics literature the construction of representations originally proceeded along a different line [12]. Here the existence of a state \( |b, j\rangle \) with maximal \( H \)-eigenvalue \( j \) (and \( Z \)-eigenvalue \( b \)) was assumed on which \( E^+, F^+ \) and \( \bar{F}^+ \) acted as annihilators while the generators \( E^-, F^- \) and \( F^- \) have been used to construct the remaining states. The shift in the definition of the Kac module above is reminiscent of these different conventions. Note that the summary on tensor products which can be found in [30] uses the physical conventions of the original articles [13, 18].

### 3.2.2 Projective covers of atypical irreducible modules

When we discussed the representations of \( \mathfrak{gl}(1|1) \) we have already talked about the concept of a projective cover of an atypical representation. By definition, the projective cover of a
representation \( \{ j \}_\pm \) is the largest indecomposable representation \( P_{\mathfrak{g}}^\pm (j) \) which has \( \{ j \}_\pm \) as a subrepresentation (its socle). We do not want to construct these representations explicitly here. Instead, we shall display how they are composed from atypicals. The projective cover of the trivial representation is an 8-dimensional module of the form

\[
P_{\mathfrak{g}}(0) : \quad \{ 0 \} \longrightarrow \{ \frac{1}{2} \}_+ \oplus \{ \frac{1}{2} \}_- \longrightarrow \{ 0 \} . \tag{3.11}
\]

For the other atypical representations \( \{ j \}_\pm \) with \( j = \frac{1}{2}, 1, \ldots \) one finds the following diagram,

\[
P_{\mathfrak{g}}^\pm (j) : \quad \{ j \}_\pm \longrightarrow \{ j + \frac{1}{2} \}_\pm \oplus \{ j - \frac{1}{2} \}_\pm \longrightarrow \{ j \}_\pm . \tag{3.12}
\]

These representation spaces are \( 16j + 4 \)-dimensional. A rather explicit constructions of the modules \( P_{\mathfrak{g}}^\pm (j) \) with \( j \neq 0 \) will be sketched in the next section. Let us also agree to absorb the superscript \( \pm \) on \( P \) into the argument, i.e. \( P_{\mathfrak{g}}^\pm (j) = P_{\mathfrak{g}}(\pm j) \), wherever this is convenient.

### 3.2.3 Zigzag modules

There are two additional sets of indecomposables that are close relatives of the (anti-)zigzag representations of \( \mathfrak{gl}(1|1) \). We shall refer to them as (anti-)zigzag modules of \( \mathfrak{sl}(2|1) \), though based on the shape of their (full) weight diagram it might be more appropriate to call them wedge modules. The (anti-)zigzag modules of \( \mathfrak{sl}(2|1) \) are parametrized by the number \( d \) of their irreducible constituents and by the largest parameter \( b \in \frac{1}{2} \mathbb{Z} \) that appears among the atypical representations in their composition series. For our purposes it will suffice to describe how (anti-)zigzag modules are built from atypical representations

\[
\mathcal{Z}_{\mathfrak{g}}^d (b) : \quad \bigoplus_{l=0}^{\lfloor \frac{1}{2} (d-1) \rfloor} \{ b - l \} \longrightarrow \bigoplus_{l= \frac{1}{2}}^{\lfloor \frac{1}{2} d - \frac{1}{2} \rfloor} \{ b - l \} . \tag{3.13}
\]

\[
\mathcal{Z}_{\mathfrak{g}}^d (b) : \quad \bigoplus_{l= \frac{1}{2}}^{\lfloor \frac{1}{2} d - \frac{1}{2} \rfloor} \{ b - l \} \longrightarrow \bigoplus_{l=0}^{\lfloor \frac{1}{2} (d-1) \rfloor} \{ b - l \} .
\]

Here, the symbol \( \lfloor . \rfloor \) instructs us to take the integer part of the argument. Since we have simplified the diagrammatic presentation of the (anti-)zigzag modules in comparison to their counterparts for \( \mathfrak{gl}(1|1) \), we would like to stress that the structures are identical to the ones before. In particular, every invariant subspace \( \{ b' \} \) is a common submodule of
$$\{0\} = \mathcal{Z}_g^1(0) = \mathcal{Z}_g^1(0) \quad 1 \quad \text{atypical, irreducible}$$
$$\{j\}_\pm = \mathcal{Z}_g^1(\pm j) = \mathcal{Z}_g^1(\pm j) \quad 4j + 1 \quad \text{atypical, irreducible}$$
$$\{b, j\} = \{b, j\}; \; b \neq \pm j \quad 8j \quad \text{typical, irreducible, projective}$$
$$\{\pm j, j\} = \mathcal{Z}_g^2(\pm j), \{\pm j, j\} = \mathcal{Z}_g^2(\pm j) \quad 8j \quad \text{indecomposable}$$
$$\mathcal{P}_g(0) \quad 8 \quad \text{indecomposable, projective}$$
$$\mathcal{P}_g^\pm(j) = \mathcal{P}_g(\pm j); \; j > 0 \quad 16j + 4 \quad \text{indecomposable, projective}$$
$$\mathcal{Z}_g^d(b), \mathcal{Z}_g^d(b) \quad \text{indecomposable}$$

Table 1: A complete list of finite dimensional indecomposable representations of $\mathfrak{sl}(2|1)$ (including irreducibles) with diagonalizable Cartan elements.

both of its neighbors $\{b' + \frac{1}{2}\}$ and $\{b' - \frac{1}{2}\}$ (should they be part of the composition series). Consequently, there exists the same dependence on the parity of the parameter $d$. This also reflects itself in the behavior Kac modules under the action of the automorphism $\Omega$,

$$\Omega(\mathcal{Z}_g^d(b)) = \begin{cases} 
\mathcal{Z}_g^d(\frac{d-1}{2} - b) & \text{for even } d \\
\mathcal{Z}_g^d(\frac{d-1}{2} - b) & \text{for odd } d 
\end{cases} \quad (3.14)$$

Similar formulas apply to (anti-)zigzag modules, only that all the $\mathcal{Z}$ must be replaced by $\tilde{\mathcal{Z}}$ and vice versa. Let us finally point out that (anti-)Kac modules and atypical irreducible representations are just special cases of zigzag representations. The former correspond to the values $d = 2$ and $d = 1$ of the length $d$, respectively.

This concludes our presentation of all finite dimensional representations of $\mathfrak{sl}(2|1)$. Throughout most of our discussion, we have not been very explicit, but in section 4.1.2 we shall see that many of the indecomposable representations of $\mathfrak{sl}(2|1)$ may be induced from representations of $\mathfrak{gl}(1|1)$. Along with our good insights into $\mathfrak{gl}(1|1)$ modules, this then provides us with a rather direct construction of $\mathfrak{sl}(2|1)$ representations.

### 4 Tensor products of $\mathfrak{sl}(2|1)$ representations

In this section, we are going to address the main goal of this note, i.e. we shall determine all tensor products of finite dimensional $\mathfrak{sl}(2|1)$ representations. Our results are partly based on the previous analysis of certain special cases. The second important ingredient
comes with our study of the $\mathfrak{gl}(1|1)$ representation theory which enters through a particular embedding of $\mathfrak{gl}(1|1)$ into $\mathfrak{sl}(2|1)$. We shall describe this embedding first before presenting our findings on the fusion of $\mathfrak{sl}(2|1)$ representations.

4.1 Decomposition with respect to $\mathfrak{gl}(1|1)$

Our main technical observation that will ultimately allow us to decompose arbitrary tensor products of finite dimensional $\mathfrak{sl}(2|1)$ representations is a close correspondence with the representation theory of $\mathfrak{gl}(1|1)$. The latter emerges from a particular embedding of $\mathfrak{gl}(1|1)$ into $\mathfrak{sl}(2|1)$. We shall specify this embedding in the first subsection. As an aside, we are then able to provide a much more explicit construction for many of the $\mathfrak{sl}(2|1)$ representations we have introduced above. Finally, in the third subsection, we explain how finite dimensional representations of $\mathfrak{sl}(2|1)$ decompose when restricted to $\mathfrak{gl}(1|1)$.

4.1.1 Embedding $\mathfrak{gl}(1|1)$ into $\mathfrak{sl}(2|1)$

In order to embed the Lie superalgebra $\mathfrak{gl}(1|1)$ into $\mathfrak{sl}(2|1)$ we shall employ the following regular monomorphism $\epsilon$,

$$
\epsilon(E) = Z - H \quad \epsilon(N) = Z + H \quad \epsilon(\psi^+) = F^+ \quad \epsilon(\psi^-) = \bar{F}^- \quad . \quad (4.1)
$$

There exist different embeddings which arise by concatenating $\epsilon$ with $\Omega$ and/or $\omega$ but we will not consider them since apparently they do not give rise to any new information. Let us point out, though, that $\epsilon$ does not intertwine the actions of the outer automorphism $\omega$ and $\Omega$, i.e. $\Omega \circ \epsilon \neq \epsilon \circ \omega$.

4.1.2 Induced representations from $\mathfrak{gl}(1|1)$

As we have anticipated, we can exploit the relation between $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}(2|1)$ to construct representations of the latter from the former. To this end, we note that the embedding of $\mathfrak{gl}(1|1)$ induces the following decomposition of $\mathfrak{sl}(2|1)$ into eigenspaces of the element $\epsilon(E)$,

$$
\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_{-1} \quad , \quad (4.2)
$$

where $\mathfrak{k}_0 = \mathfrak{gl}(1|1)$, $\mathfrak{k}_1 = \text{span}\{E^+, \bar{F}^+\}$ and $\mathfrak{k}_{-1} = \text{span}\{E^-, F^-\}$ such that $[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$. Given any representation $\rho_0$ of $\mathfrak{gl}(1|1)$ we can thus induce a module of $\mathfrak{sl}(2|1)$ using the elements of $\mathfrak{k}_1$ (or $\mathfrak{k}_{-1}$) as generators. The resulting representation is infinite dimensional but
under certain circumstances one may take a quotient and end up with a finite dimensional representation space. A condition in the choice of the \( \mathfrak{gl}(1|1) \) representation \( \rho_\mathfrak{h} \) arises in particular from considering the \( \mathfrak{sl}(2) \) multiplets within the induced representation. In order for the latter to possess a finite dimensional quotient, the spectrum of the Cartan element \( 2H \) must be integer. Since \( 2H \) is the image of \( N - E \) under the monomorphism \( \epsilon \), we conclude that \( \rho_\mathfrak{h} \) is only admissible if \( \rho_\mathfrak{h}(N - E) \) has integer spectrum. In the case of a typical representation \( \rho_\mathfrak{h} = \langle e, n \rangle \), for example, our condition restricts \( e - n \) to be an integer.

Many \( \mathfrak{sl}(2|1) \)-representations can actually be obtained through such an induction. This applies in particular to the projective covers \( P^\pm_\mathfrak{g}(j) \) with \( j \neq 0 \) which are obtained from \( \rho_\mathfrak{h} = P_\mathfrak{h}(\pm 2j) \). In the case of the (anti-)zigzag modules \( Z^d_\mathfrak{g}(b) \) and \( \tilde{Z}^d_\mathfrak{g}(b) \), we only need to avoid the range \( 0 < 2b < d - 1 \). Outside this interval, we can obtain the (anti-)zigzag representations by induction, using the \( \mathfrak{gl}(1|1) \) representations \( Z^d_\mathfrak{h}(2b) \) and \( \tilde{Z}^d_\mathfrak{h}(2b) \) for \( \rho_\mathfrak{h} \).

What makes the induction particularly interesting for us is another aspect: Suppose we start with a \( \mathfrak{gl}(1|1) \)-representation \( \rho_\mathfrak{h} \) in which \( \rho_\mathfrak{h}(E) = 0 \). Since \( [\epsilon(E), \mathfrak{t}_{\pm 1}] = \pm \mathfrak{t}_{\pm 1} \), our creation operators cannot generate any additional eigenstates of \( \rho_\mathfrak{h}(E) \) with vanishing eigenvalue. In other words, if \( \rho_\mathfrak{g} \) is an \( \mathfrak{sl}(2|1) \) representation which can be obtained by our induction from \( \rho_\mathfrak{h} \) and if \( \rho_\mathfrak{h}(E) = 0 \), then the decomposition of \( \rho_\mathfrak{g} \) into representations of \( \mathfrak{h} \) can only contain typical representations in addition to the representation \( \rho_\mathfrak{h} \) we started with. We shall find that this observations extends to a simple correspondence between atypical representations (and their indecomposable composites) of \( \mathfrak{sl}(2|1) \) and \( \mathfrak{gl}(1|1) \).

### 4.1.3 Decomposition of \( \mathfrak{sl}(2|1) \) representations

Before we decompose representations of \( \mathfrak{g} \) into representations of \( \mathfrak{h} \) we introduce a few new notations that will become quite useful. In particular, we will employ a map \( \mathcal{E} \) which takes irreducible representations of \( \mathfrak{g} \) and turns them into a very specific sum of typical \( \mathfrak{h} \) representations. On atypical representations, \( \mathcal{E} \) is defined by

\[
\mathcal{E}(\{j\}_\pm) = \bigoplus_{n=1}^{2j} (\pm n, \frac{1}{2} \pm (2j + \frac{1}{2} - n)) .
\]  

(4.3)
We shall claim below that $\mathcal{E}(\{j\})$ contains all the typical $\mathfrak{gl}(1|1)$-representations that appear in the decomposition of $\{j\}$. Similarly, we may define

$$\mathcal{E}(\{b, j\}) = \bigoplus_{n=-j}^{j} (\langle b + 1 - n, b + n \rangle \oplus \langle b - n, b + n \rangle)$$

(4.4)
on typical representations $\{b, j\}, b \neq \pm j$. The prime $'$ on the summation symbol instructs us to omit all terms that correspond to atypical representations. We can extend $\mathcal{E}$ linearly to all completely reducible representations of $\mathfrak{sl}(2|1)$.

Another map $S_\theta$ converts indecomposable representations of $\mathfrak{sl}(2|1)$ into semi-simple modules, namely into the sum of all irreducible representations that appear in the decomposition series. Explicitly, we have

$$S_\theta(\mathcal{P}_\theta(j)) = 2\{j\} \oplus \{j - \frac{1}{2}\} \oplus \{j + \frac{1}{2}\},$$

$$S_\theta(\mathcal{Z}_d^d(b)) = \bigoplus_{l=0}^{d-1} \{b - \frac{l}{2}\}.$$ 

(4.5)

The expressions should be compared with our diagrams (3.12) and (3.13) for the projective covers and the (anti-) zigzag modules of $\mathfrak{sl}(2|1)$.

Once this notation is introduced, our decomposition formulas take a particularly simple form. For the atypical representations and their composites one obtains

$$\{j\}_{b} = \langle 2j \rangle \oplus \mathcal{E}(\{j\})$$

$$\mathcal{P}_\theta(j)_{b} = \mathcal{P}_\theta(2j) \oplus \mathcal{E} \circ S_\theta(\mathcal{P}_\theta(j))$$

$$\mathcal{Z}_d^d(b)_{b} = \mathcal{Z}_d^d(2b) \oplus \mathcal{E} \circ S_\theta(\mathcal{Z}_d^d(b)).$$ 

(4.6)

The last relation also holds for anti-zigzag modules if we replace all $\mathcal{Z}$ by $\mathcal{Z}$. Note that, up to typical contributions, there is a one-to-one correspondence between the $\mathfrak{sl}(2|1)$ representations on the left and the $\mathfrak{gl}(1|1)$ representations on the right hand side. Things are slightly more complicated for the typical representations of $\mathfrak{sl}(2|1)$ for which the decomposition is given by

$$\{b, j\}_{b} = \begin{cases} 
\mathcal{E}(\{b, j\}) & \text{for } b \neq -j, \ldots, j, \\
\mathcal{P}_\theta(2b) \oplus \mathcal{E}(\{b, j\}) & \text{for } b = -j + 1, \ldots, j - 1.
\end{cases}$$

(4.7)

Note that in the second case, the image of the symbol $\mathcal{E}$ contains only $4j - 2$ typical representations so that the dimensions match.
4.2 Decomposition of $\mathfrak{sl}(2|1)$ tensor products

We are finally prepared to decompose arbitrary tensor products of finite dimensional $\mathfrak{sl}(2|1)$ representations. Our presentation below is split into three different parts. We shall begin by reviewing Marcu’s results [18] on the decomposition of tensor products between two typical representations and between a typical and an atypical representation. The extension to arbitrary tensor products involving one typical representation is then straightforward. The second subsection contains new results on tensor products in which at least one factor is a projective cover. Finally, we shall decompose arbitrary tensor products of two (anti-)zigzag modules.

4.2.1 Tensor products involving a typical representation

Before presenting Marcu’s results, we would like to introduce some notation that will permit us to rephrase the answers in a much more compact form. To this end, let us define a map $\pi$ which sends representations of the bosonic subalgebra $g^{(0)}$ to typical representations of $\mathfrak{g}$. Its action on irreducibles is given by

$$\pi(b - \frac{1}{2}, j - \frac{1}{2}) = \begin{cases} \{b, j\} & \text{for } b \neq \pm j, \\ 0 & \text{for } b = \pm j. \end{cases}$$  \hspace{1cm} (4.8)

The map $\pi$ may be extended to a linear map on the space of all finite dimensional representations of $\mathfrak{g}^{(0)}$.

The first tensor product we would like to display is the one between two typical representations [18]. In our new notations, the decomposition is given by

$$\{b_1, j_1\} \otimes \{b_2, j_2\} = \pi\left((b_1 - \frac{1}{2}, j_1 - \frac{1}{2}) \otimes \{b_2, j_2\}\right) \oplus \sum_{\text{cases}} \mathcal{P}_g(\pm|b_1 + b_2| \mp \frac{j}{2}) \quad \text{for } b_1 + b_2 = \pm(j_1 + j_2)$$

$$\oplus \begin{cases} \mathcal{P}_g^\pm(|b_1 + b_2|) & \text{for } b_1 + b_2 \in \pm\{\{j_1 - j_2\} + 1, \ldots, j_1 + j_2 - 1\} \\ \mathcal{P}_g(\pm|b_1 + b_2|) & \text{for } b_1 + b_2 = \pm|j_1 - j_2|. \end{cases}$$  \hspace{1cm} (4.9)

Note that neither $j_1$ nor $j_2$ can vanish so that the three cases listed above are mutually exclusive. If none of them applies, the tensor product contains only typical representations. These are computed by the first term. All it requires is the decomposition of typical $\mathfrak{g}$ representations into irreducibles of the bosonic subalgebra (see eq. (3.7)) and a computation of tensor products for representations of $\mathfrak{g}^{(0)} = \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$ which presents
no difficulty. The outcome is then converted into a direct sum of typical representations through our map $\pi$.

Tensor products of typical with atypical representations can also be found in Marcu’s paper. The results are

$$\{b_1, j_1\} \otimes \{j_2\}_\pm = \pi((b_1 - \frac{j_1}{2}, j_1 - \frac{j_2}{2}) \otimes \{j_2\}_\pm|_{g(0)}) \oplus (4.10)$$

This formula can also be used to determine the tensor product of typical representations with any composite of atypical representations, i.e. with the projective covers and the (anti-)zigzag modules. In fact, these tensor products are simply given by

$$\{b, j\} \otimes \mathcal{H} = \{b, j\} \otimes \mathcal{S}_g(\mathcal{H}) \quad \text{for} \quad \mathcal{H} = \mathcal{P}_g(l), \mathcal{Z}_d(l) \text{ or } \mathcal{Z}_d(l). \quad (4.11)$$

Such an outcome is natural since the decomposition of a tensor product of a typical with any other representation is known to be decomposable into typicals and projective covers. One may determine the exact content through the $\mathfrak{gl}(1|1)$ embedding and it is rather easy to see that the answers may always be reduced to the computation of tensor products with atypical irreducibles, as it is claimed in equation (4.11).

### 4.2.2 Tensor products involving a projective cover

This subsection collects all our findings on tensor products involving at least one projective cover. General results guarantee that such tensor products decompose into a sum of projective representations. The result for the tensor product of a projective cover with a typical representation has been spelled out already (see eq. (4.11)). Therefore, we can turn directly to the next case, the product of two projective covers.

**Proposition 1:** The tensor product between two projective covers $\mathcal{P}_g^\pm(j_1), j_1 \geq 0$, and $\mathcal{P}_g(j_2) = \mathcal{P}_{\text{sign}(j_2)}(\{j_2\})$ is given by

$$\mathcal{P}_g^\pm(j_1) \otimes \mathcal{P}_g(j_2) = \pi(H_j^\pm \otimes \mathcal{P}_g(j_2)|_{g(0)}) \oplus (4.12)$$

$$\oplus \mathcal{P}(\pm j_1 + j_2 + \frac{1}{2}) \oplus 2 \cdot \mathcal{P}(\pm j_1 + j_2) \oplus \mathcal{P}(\pm j_1 + j_2 - \frac{1}{2})$$

where $H_j^\pm = (\pm j - \frac{1}{2}, j - \frac{1}{2}) \oplus (\pm (j + \frac{1}{2}) - \frac{1}{2}, j)$ for $j > 0 \quad (4.13)$
and \( H_0 = H_0^\pm = (0,0) \oplus (-1,0) \). In the argument of \( \pi \) the product \( \otimes \) refers to the fusion between representations of the bosonic subalgebra \( g^{(0)} = gl(1) \oplus sl(2) \).

**Proof:** Our claim concerning typical representations in the decomposition requires little comment. Let us only stress that the two bosonic multiplets \((\pm (j + \frac{1}{2}) - \frac{1}{2}, j)\) and \((\pm j - \frac{1}{2}, j - \frac{1}{2})\) that appear in the space \( H_j^\pm \) are the ground states of the two Kac modules from which \( P_g^\pm j \) is composed (see eq. (3.12)). The contributions from projective covers, on the other hand, may be deduced from the embedding of \( gl(1|1) \) along with the formula \((2.14)\) for tensor products of the projective covers \( P_h \).

**Proposition 2:** The tensor product between a projective cover \( P_g^\pm j, j \geq 0, \) and a zigzag module \( Z_g^d(b) \) is given by

\[
P_g^\pm j \otimes Z_g^d(b) = \pi \left( H_j^\pm \otimes Z_g^d(b) \bigg|_{g^{(0)}} \right) \oplus \bigoplus_{p=0}^{d-1} P_g(\pm j + b - \frac{1}{2}p)
\]

where \( H_j^\pm \) is the same as in proposition 1. To determine the tensor product with an anti-zigzag module \( \bar{Z}_g^d(b) \), we replace \( Z_g^d(b) \) by \( \bar{Z}_g^d(b) \).

**Proof:** The statement is established in the same way as proposition 1, using formula \((2.16)\) as input from the representation theory of \( gl(1|1) \).

4.2.3 Tensor products between (anti-)zigzag modules

In the following we shall denote the fusion ring of finite dimensional representations of a Lie superalgebra \( g \) by \( \text{Rep}(g) \). As we remarked before, projective representations of \( g \) form an ideal in \( \text{Rep}(g) \). The latter will be denoted by \( \text{Proj}(g) \). Our results on the decomposition of \( sl(2|1) \) representations into representations of \( gl(1|1) \) imply the following nice result.

**Proposition 3:** Modulo projectives, the representation ring of \( g = sl(2|1) \) may be embedded into the representation ring of \( h = gl(1|1) \), i.e. there exists a monomorphism \( \vartheta \),

\[
\vartheta : \text{Rep}(g)/\text{Proj}(g) \longrightarrow \text{Rep}(h)/\text{Proj}(h) \text{ .}
\]

Note that \( \text{Rep}(g)/\text{Proj}(g) \) is generated by (anti-)zigzag modules. On the latter, the monomorphism \( \vartheta \) acts according to

\[
\vartheta \left( Z_g^d(b) \right) = Z_h^d(2b) \text{ , } \vartheta \left( \bar{Z}_g^d(b) \right) = \bar{Z}_h^d(2b) \text{ .}
\]
PROOF: This proposition is an obvious consequence of the formulas (4.6) for the decomposition of $\mathfrak{sl}(2|1)$ representations into indecomposables of $\mathfrak{gl}(1|1)$. \qed

This proposition can be used to compute the non-projective contributions of tensor products between (anti-)zigzag representations explicitly from our $\mathfrak{gl}(1|1)$ formulas. For the tensor product of two atypical representations one finds in particular

$$\{j_1\} \otimes \{j_2\} = \{j_1 + j_2\} \mod \mathfrak{proj}(\mathfrak{sl}(2|1)).$$

The answer is in agreement with the findings of Marcu who has computed the tensor product of atypical representation in [18]. In fact, the full answer for the tensor product of two atypical representations is encoded in the formulas

$$\{j_1\}_{\pm} \otimes \{j_2\}_{\pm} = \{j_1 + j_2\}_{\pm} \bigoplus_{j = |j_1 - j_2|}^{j_1 + j_2 - 1} \{\pm(j_1 + j_2 + \frac{1}{2}), j + \frac{1}{2}\}, \quad (4.14)$$

$$\{j_1\}_+ \otimes \{j_2\}_- = \{|j_1 - j_2|\}_{\text{sign}(j_1 - j_2)} \bigoplus_{j = |j_1 - j_2| + 1}^{j_1 + j_2} \{j_1 - j_2, j\}. \quad (4.15)$$

Let us agree to denote the sums of typical representations that appear on the right hand side by $\mathcal{T}(\{j_1\}_{\pm}, \{j_2\}_{\pm})$ and $\mathcal{T}(\{j_1\}_+, \{j_2\}_-)$, respectively. Furthermore, we would like to extend $\mathcal{T}$ to a bi-linear map on arbitrary sums of atypical irreducibles. The map $\mathcal{T}$ features in the following decomposition of tensor products between two (anti-)zigzag modules.

**Proposition 4:** Tensor product between two zigzag modules of $\mathfrak{sl}(2|1)$ can be decomposed as follows

$$\mathcal{Z}_d^d(b_1) \otimes \mathcal{Z}_d^d(b_2) = \mathcal{T}(\mathcal{S}_g(\mathcal{Z}_d^d(b_1)), \mathcal{S}_g(\mathcal{Z}_d^d(b_2)) \bigoplus \Theta(\mathcal{Z}_h^d(2b_1) \otimes \mathcal{Z}_h^d(2b_2)). \quad (4.16)$$

The map $\mathcal{T}$ was introduced in the text preceding this proposition and $\mathcal{S}_g$ replaces its argument by a direct sum of irreducibles in the decomposition series (see eqs. (4.3)). $\Theta$ is a linear map that replaces certain $\mathfrak{h}$-representations by $\mathfrak{g}$-representations according to

$$\Theta(\mathcal{P}_h^d(n)) = \mathcal{P}_g^d(\frac{1}{4}n), \quad \Theta(\mathcal{Z}_h^d(n)) = \mathcal{Z}_g^d(\frac{1}{4}n), \quad \Theta(\mathcal{Z}_h^d(n)) = \mathcal{Z}_g^d(\frac{1}{4}n).$$

Analogous formulas apply to tensor product of zigzag with anti-zigzag modules and to the fusion of two anti-zigzag representations.
Proof: The rule that determines the contribution from typical representations is fairly obvious and the (anti-)zigzag representations in the tensor product are a consequence of proposition 3. The terms involving projective covers, finally, can be found through the decomposition into $\mathfrak{gl}(1|1)$ representations. This part is the most subtle, since projective $\mathfrak{gl}(1|1)$ representations can in principle arise through the decomposition of both projective covers and typical $\mathfrak{sl}(2|1)$ representations. To see that projective covers for $\mathfrak{sl}(2|1)$ representations contribute to the decomposition only through the second term, we note that all the atypical components that appear in the tensor product of the $\mathfrak{sl}(2|1)$ zigzag representations are needed to build the image of $\Theta$ on the right hand side of eq. [1.16]. Hence, all projective covers of $\mathfrak{gl}(1|1)$ atypicals that are not found in the restriction of $\Theta(Z^{d_1}_h(2b_1) \otimes Z^{d_2}_h(2b_2))$ must arise from a restriction of typical $\mathfrak{sl}(2|1)$ representations. □

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