ARITHMETIC PROGRESSIONS OF CARMICHAEL NUMBERS
IN A REDUCED RESIDUE CLASS

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ABSTRACT. Fix coprime natural numbers $a,q$. Assuming the Prime $k$-tuple Conjecture, we show that there exist arbitrarily long arithmetic progressions of Carmichael numbers, each of which lies in the reduced residue class $a \mod q$ and is a product of three distinct prime numbers.

1. INTRODUCTION

For any prime number $n$, Fermat’s little theorem asserts that
\[ x^n \equiv x \mod n \quad (x \in \mathbb{Z}). \] (1)

Around 1910, Carmichael initiated the study of composite numbers $n$ with the same property; these integers are now called Carmichael numbers. In 1994 the existence of infinitely many Carmichael numbers was established by Alford, Granville and Pomerance [1]; see also [3].

Since both primes and Carmichael numbers share the property (1), it seems natural to ask whether certain known results about primes can also be proved for Carmichael numbers, and indeed this theme has been explored by several authors. For example, the analogue of Dirichlet’s theorem about the infinitude of primes in a reduced residue class has been established for Carmichael numbers by Wright [6], building on ideas of Banks and Pomerance [2] and of Matomäki [5].

In 2008 a stunning and celebrated work of Green and Tao [4] established the existence of arbitrarily long arithmetic progressions in the primes; their landmark paper in additive number theory at once resolved the longstanding open problem about prime numbers and also an important case of a famous conjecture of Erdős on arithmetic progressions. In the present note, we give a conditional proof of a similar result for Carmichael numbers.

**Theorem 1.** Let $a,q$ be fixed coprime natural numbers. Under the Prime $k$-tuple Conjecture, there exist arbitrarily long arithmetic progressions of Carmichael numbers, each of which lies in the reduced residue class $a \mod q$ and is a product of three distinct prime numbers.

In light of this result, we conjecture that every reduced residue class $a \mod q$ contains arbitrarily long arithmetic progressions of Carmichael numbers.

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1.1. Acknowledgements. The original draft of this manuscript established (under the Prime $k$-tuple Conjecture) the existence of arbitrarily long arithmetic progressions of Carmichael numbers. The author thanks Andrew Granville for sharing a simpler proof (elements of which are incorporated here) and for posing the question as to whether the same result holds true in an arbitrary reduced residue class $a \mod q$.

2. The Prime $k$-tuple Conjecture

A $k$-tuple of linear forms in $\mathbb{Z}[X]$, denoted by

$$\mathcal{H}(X) = \{g_j X + h_j\}_{j=1}^k,$$

is said to be admissible if the associated polynomial $f_{\mathcal{H}}(X) := \prod_{j=1}^k (g_j X + h_j)$ has no fixed prime divisor, that is, if

$$\left| \{n \mod p : f_{\mathcal{H}}(n) \equiv 0 \mod p \} \right| \neq p$$

for every prime $p$. In this note we consider only $k$-tuples for which

$$g_1, \ldots, g_k > 0 \quad \text{and} \quad \prod_{1 \leq i < j \leq k} (g_i h_j - g_j h_i) \neq 0. \quad (2)$$

The Prime $k$-tuple Conjecture asserts that if $\mathcal{H}(X)$ is admissible and satisfies (2), then $\mathcal{H}(n) = \{g_j n + h_j\}_{j=1}^k$ is a $k$-tuple of primes for infinitely many $n \in \mathbb{N}$.

3. Proof of Theorem 1

Lemma 2. Suppose $b, c, d, e \in \mathbb{N}$ satisfy the conditions

$$b, c, d \text{ are coprime in pairs,} \quad (3)$$

$$cde + c + d \equiv 0 \mod b, \quad (4)$$

$$bde + b + d \equiv 0 \mod c, \quad (5)$$

$$bce + b + c \equiv 0 \mod d. \quad (6)$$

If $n \equiv e \mod bcd$ and the numbers

$$r := bn + 1, \quad s := cn + 1 \quad \text{and} \quad t := dn + 1$$

are distinct primes, then $rst$ is a Carmichael number.

Proof. Let $\lambda$ denote the Carmichael function. A composite number $N$ is a Carmichael number if and only if $\lambda(N) \mid N - 1$.

Put $N := rst$. Using (3) we have $\lambda(N) = bcdn$, which divides

$$N - 1 = bcdn^3 + (bc + bd + cd)n^2 + (b + c + d)n$$

if and only if

$$bcd \mid (bc + bd + cd)n + b + c + d.$$ 

Since $n \equiv e \mod bcd$, the last condition follows from (4)–(6). $\square$
Proof of Theorem 1. For any prime \( p \), let \( v_p \) be the standard \( p \)-adic valuation.

Let \( a, q \) be fixed coprime natural numbers, and let \( \hat{a} \) be an integer such that \( a \hat{a} \equiv 1 \mod q \). For any prime \( p \mid q \) let

\[
\alpha_p := v_p(q), \quad \beta_p := v_p(a - 1), \quad \hat{\beta}_p := v_p(\hat{a} - 1),
\]

and put

\[
q_\sharp := \prod_{p \mid q} p^{\alpha_p}, \quad q_\flat := \prod_{p \mid q} p^{\beta_p},
\]

Clearly,

\[
q = q_\sharp q_\flat, \quad \gcd(q_\sharp, q_\flat) = 1, \quad a \equiv \hat{a} \equiv 1 \mod q_\flat.
\]  
(7)

From the theory of valuations it is immediate that \( \beta_p = \hat{\beta}_p \) for any prime \( p \mid q_\sharp \).

We define

\[
n_p := \frac{a - 1}{p^{\alpha_p}} \in \mathbb{N} \quad \text{and} \quad \hat{n}_p := \frac{\hat{a} - 1}{p^{\beta_p}} \in \mathbb{N}
\]

for such primes, and so we have \( \gcd(n_p \hat{n}_p, p) = 1 \).

Let \( e \) be a natural number such that

\[
e \equiv p^{\beta_p} \mod p^{\alpha_p} \quad (p \mid q_\sharp);
\]  
(8)

such integers \( e \) exist by the Chinese Remainder Theorem (CRT). We define

\[
e_p := \frac{e}{p^{\beta_p}} \in \mathbb{N} \quad (p \mid q_\sharp).
\]

Let \( b, c, d \) be large and distinct primes for which the congruences

\[
b e_p \equiv n_p \mod p^{\alpha_p - \beta_p},
\]
\[
c e_p \equiv n_p \mod p^{\alpha_p - \beta_p},
\]
\[
d e_p \equiv \hat{n}_p \mod p^{\alpha_p - \beta_p},
\]

hold for all primes \( p \mid q_\sharp \) (such \( b, c, d \) exist because \( \gcd(e_p n_p \hat{n}_p, p) = 1 \)). After multiplying the above congruences by \( p^{\beta_p} \), we get that

\[
b e + 1 \equiv a \mod p^{\alpha_p},
\]
\[
c e + 1 \equiv a \mod p^{\alpha_p},
\]
\[
d e + 1 \equiv \hat{a} \mod p^{\alpha_p},
\]  
(9)

for all \( p \mid q_\sharp \). Moreover, taking into account (7), the same congruences hold for any prime \( p \mid q_\flat \) provided that

\[
e \equiv 0 \mod q_\flat.
\]  
(10)

Assuming (8) and (10), by the CRT and (9) it follows that

\[
(b e + 1)(c e + 1)(d e + 1) \equiv a \mod q.
\]  
(11)

In the above construction, we choose the three primes \( b, c, d \) large enough so that each one exceeds \( qA \), where

\[
A := \prod_{p \leq k} p.
\]
From now on, we suppose that
\[ e \equiv 0 \mod A \]  
(12)
(since \( \gcd(A, q) = 1 \), this is compatible with the conditions (8) and (10) already imposed on \( e \)). Let
\[ b_j := b + Acdqj \quad (j = 1, \ldots, k), \]
and denote
\[ B := b_1 \cdots b_k cd. \]
Our choices of \( b, c, d, A \) ensure that the numbers \( b_1, \ldots, b_k, c, d \) are coprime in pairs, and \( \gcd(B, q) = 1 \). Therefore, in addition to (8), (10) and (12), using the CRT along with (13) we can further arrange for the integer \( e \) to satisfy the congruence conditions
\[ cde + c + d \equiv 0 \mod b_j, \]
\[ b_jde + b_j + d \equiv 0 \mod c, \]
\[ b_jce + b_j + c \equiv 0 \mod d, \]
(14)
for every \( j \).
To finish the proof, let
\[ Z := \{b_1, \ldots, b_k, c, d\}, \]
let \( \mathcal{H}(X) \) be the \((k + 2)\)-tuple comprised of the linear forms
\[ F_z(X) := z(e + ABqX) + 1 \quad (z \in Z), \]
and put
\[ f_\mathcal{H}(X) := \prod_{z \in Z} F_z(X). \]
Under the Prime \( k \)-tuple Conjecture, the numbers
\[ r_j := b_j(e + ABqm) + 1 \quad (j = 1, \ldots, k), \]
\[ s := c(e + ABqm) + 1, \]
\[ t := d(e + ABqm) + 1, \]
are simultaneously prime for infinitely many \( m \in \mathbb{N} \) provided that \( f_\mathcal{H}(X) \) is admissible and (2) holds (with \( g_i, h_i \) suitably defined). Assuming this for the moment, let \( m \) be one such integer (fixed), and let \( n := e + ABqm \). For each \( j \), using (14) and the fact that \( n \equiv e \mod b_j cd \), Lemma 2 shows that \( r_j st \) is a Carmichael number. Since \( r_1 < \cdots < r_k \) is an arithmetic progression (see (13)), \( r_1 st < \cdots < r_k st \) is an arithmetic progression of Carmichael numbers. Using (11) and (13), we also have
\[ r_j st \equiv a \mod q \quad (j = 1, \ldots, k). \]
Since \( k \) is arbitrary, the theorem follows.

It remains to verify the conditions of the Prime \( k \)-tuple Conjecture.
To see that \( f_\mathcal{H}(X) \) is admissible, observe that for any fixed \( z \in Z \) the set
\[ \mathcal{S}_z := \{n \mod p : z(e + ABqn) + 1 \equiv 0 \mod p\} \]
has cardinality one if \( p \nmid ABq \); for such primes we have \( p > k \) (since \( p \nmid Aq \)), and thus
\[
\left| \{ n \mod p : f_H(n) \equiv 0 \mod p \} \right| = \left| \bigcup_z S_z \right| = k < p,
\]
as required. On the other hand, for primes \( p \mid ABq \) we claim that
\[
ze + 1 \not\equiv 0 \pmod{p} \quad (z \in \mathbb{Z}),
\]
which implies that
\[
\left| \{ n \mod p : f_H(n) \equiv 0 \mod p \} \right| = \left| \bigcup_z S_z \right| = 0 < p.
\]
Indeed, if \( p \mid q \), then (15) is a consequence of (9) and (13). When \( p \mid B \) (in other words, \( p \in \mathbb{Z} \)), (15) follows from (14). If \( p \mid A \), then (15) is implied by (12).

Finally, writing \( F_z(X) = g_z X + h_z \) with \( g_z := zABq \) and \( h_z := ze + 1 \) for each \( z \in \mathbb{Z} \), we have
\[
g_{z_1} h_{z_2} - g_{z_2} h_{z_1} = ABq(z_1 - z_2) \neq 0 \quad (z_1, z_2 \in \mathbb{Z}, \ z_1 \neq z_2),
\]
and (2) follows. \( \square \)

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