Radiated Angular Momentum and Dissipative Effects in Classical Scattering

Aneesh V. Manohar,1 Alexander K. Ridgway,1 and Chia-Hsien Shen1

1Department of Physics 0319, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

We present a new formula for the angular momentum $J^{\mu
u}$ carried away by gravitational radiation in classical scattering. This formula, combined with the known expression for the radiated linear momentum $P^\mu$, completes the set of radiated Poincare charges due to scattering. We parametrize $P^\mu$ and $J^{\mu
u}$ by non-perturbative form factors and derive exact relations using the Poincare algebra. There is a contribution to $J^{\mu
u}$ due to static (zero-frequency) modes, which can be derived from Weinberg’s soft theorem. Using tools from scattering amplitudes and effective field theory, we calculate the radiated $J^{\mu
u}$ due to the scattering of two spinless particles to third order in Newton’s constant $G$, but to all orders in velocity. Our form-factor analysis elucidates a novel relation found by Bini, Damour, and Geralico between energy and angular momentum loss at $O(G^3)$. Our new results have several nontrivial implications for binary scattering at $O(G^3)$. We give a procedure to bootstrap an effective radiation reaction force from the loss of Poincare charges due to scattering.

Introduction. It is crucial to have accurate theoretical modeling of binary coalescence, given the rapid improvement in sensitivity of current and future gravitational-wave detectors. Recently, there has been tremendous progress in solving binary dynamics by utilizing tools in quantum field theory (QFT) building on the pioneering work of nonrelativistic general relativity [1].

The power of QFT-based methods originates from the gauge invariance and Lorentz covariance of scattering observables, which can be extracted from QFT amplitudes via effective field theory (EFT) methods [1-3] and the Kosower-Maybee-O’Connell (KMO) framework [4-5]. This enables tools developed in particle physics to be applied to classical gravity. Scattering results can then be translated into binary bound state ones through the effective-one-body mapping [6-8], EFT method [2-3] and analytic continuation [9-11]. Outputs from this program naturally fit within the post-Minkowskian (PM) framework, which expands in $G$ but keeps all orders in velocity. State-of-the-art results for the conservative PM potential [12-15] and scattering tail effect [16, 17] illustrate the power of this new methodology.

Dissipation is a key feature of binary coalescence that is already present at 2.5 post-Newtonian (PN) order, as can be seen by the radiation reaction (RR) [16-24]. The RR force has been extended to up to 4.5PN accuracy [21-22]. Theoretical predictions for dissipative effects on binary scattering are also relatively less developed. For instance, the waveform [25-35], impulses [26-44] and radiated linear [36-44] and angular momentum [36, 37, 46-48], have been computed to only the leading PM order.

The aim of this Letter is to leverage Poincare symmetry to incorporate dissipation due to radiation into the QFT-based framework. Poincare invariance imposes conservation laws that relate the linear and angular momentum carried away by radiation, $P^\mu$ and $J^{\mu
u}$, to the corresponding loss in the binary system. While the formula for $P^\mu$ is well-known (see also its expression in the KMO form), the standard formula for $J^{\mu
u}$ [49] is less well-understood in scattering scenarios. This is due to the presence of the static mode, which is analogous to the Coulomb mode in electrodynamics (EM) [50]. In this paper, we derive a new formula [5] for $J^{\mu
u}$ in terms of the stress-energy pseudotensor that applies to radiation with arbitrary frequency. The formula manifests the gauge independence and Lorentz covariance of $J^{\mu
u}$. This enables us to parametrize $P^\mu$ and $J^{\mu
u}$ with non-perturbative form factors in Eq. (5) that obey exact constraints imposed by the Poincare algebra. Applying this framework perturbatively in $G$, we calculate $J^{\mu
u}$ to $O(G^3)$ in Eqs. (13) and (15), and find agreement with the literature [46-48]. In particular, we directly derive the remarkable relation [16] between energy and angular momentum loss first found by [47]. Weinberg’s soft theorem [51] greatly simplifies the calculation of the zero-frequency contribution to $J^{\mu\nu}$. Our results, however, disagree with those calculated using standard formula in the rest frame [34-35], due to the subtlety in the static mode.

The radiated Poincare charges have important implications for dissipative binary dynamics. By combining our $O(G^3)$ results for $J^{\mu\nu}$ with those for $P^\mu$ [43-44], one can immediately predict the linear-in-RR correction to the scattering angle and transverse impulse at $O(G^3)$ using the Bini-Damour formula [46-52] and the maps in [47]. In addition, by following the framework in [46-47-52], we bootstrap an effective PM RR force via the balance equations [27-29] modulo total time derivatives, i.e. the so-called Schott terms [53-54].

Radiated Linear and Angular Momentum. Consider a scattering process where the initial state consists of massive particles (referred to as matter), and the final state consists of matter and outgoing gravitational radiation. Poincare symmetry implies that the loss of linear and angular momentum of matter is equal to that carried away by radiation. The radiated linear and angular momentum in the final state are given by

$$P^\mu = \int d^3 x \ T^\mu_{00}, \quad J^{\mu\nu} = \int d^3 x x^{[\mu} T^\nu_{0]} \label{eq:1},$$

where $T^{\mu
u}$ is the stress-energy tensor of the radiation, $a_{[\mu} b_{\nu]} \equiv a^\mu b^\nu - a^\nu b^\mu$, and the integrals are over all space at a fixed time. The global conserved charges are invariant under improvement terms in $T^{\mu\nu}$ [55].
Gravitational radiation is defined as the fluctuation around flat space $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{2\pi G} h_{\mu\nu}$. In what follows, we use the mostly-minus convention. Asymptotically, the radiation can be decomposed into on-shell plane waves labeled by $k^\mu = (\omega, k)$, where $\omega$ is the energy, boldface $k$ denotes spatial momentum, and $k^2 = 0$. After gauge fixing, the radiation field of any frequency can be solved in terms of the stress-energy pseudotensor $\tilde{T}^{\mu\nu}(k)$, which is analogous to the current in EM. $\tilde{T}^{\rho\sigma}(k)$ contains both matter and radiation contributions, unlike the usual stress-tensor which does not contain radiation. $\tilde{T}^{\mu\nu}(k)$ is conserved on shell, i.e. $k_\mu \tilde{T}^{\mu\nu}(k) = 0$. (One can always find such a $\tilde{T}^{\mu\nu}(k)$ [39].) There is an invariance under residual gauge transformations $\tilde{T}^{\mu\nu}(k) \rightarrow \tilde{T}^{\mu\nu}(k) + k^\mu \epsilon^\nu(k) + k^\nu \epsilon^\mu(k)$ where $\epsilon(k) \cdot k = 0$. Crucially, the formula for the radiation field and $J^{\mu\nu}$ are written in terms of $\tilde{T}^{\rho\sigma}(k)$, rather than the transverse-traceless components of the radiation field used in the standard formula [39], which means our results are also applicable to the static mode that contributes to the angular momentum. This is similar to the Coulomb field in EM, which is not in the transverse projection of the vector potential. The radiation field in terms of $\tilde{T}^{\rho\sigma}(k)$ reads

$$h_{\mu\nu}(x) = \sqrt{8\pi G} \int \tilde{d}k \left( P_{\mu\nu\rho\sigma} \tilde{T}^{\rho\sigma}(k) e^{-ik \cdot x} + c.c. \right), \quad (2)$$

where $P_{\mu\nu\rho\sigma}$ is the gauge-dependent projection and $\tilde{d}k = \frac{dk}{(2\pi)^3 2\omega}$ is the Lorentz invariant phase space measure.

Using Einstein’s equations, it is straightforward to relate $T^{\mu\nu}$ to the radiation field. Combining Eqs. [1], [2], and the expression for $T^{\mu\nu}$ in terms of $h_{\mu\nu}(x)$, we obtain the main formulae of this paper,

$$P^\mu = 8\pi G \int \tilde{d}k \, k^\mu \left( T^{\rho\sigma}(k) T_{\rho\sigma}(k) - \frac{1}{2} T^\rho_{\rho}(k) T_\sigma^\sigma(k) \right), \quad (3)$$

$$J^{\mu\nu} = 8\pi G \int \tilde{d}k \left( T^{\rho\sigma}(k) L^{\mu\nu}(k) T_{\rho\sigma}(k) - \frac{1}{2} T^\rho_\rho(k) L^{\mu\nu}(k) T^\sigma_\sigma(k) \right. + 2i T^{\rho\nu}(k) T^{\mu\sigma}(k)), \quad (3)$$

where $L^{\mu\nu} \equiv ik^{[\nu} \partial^{\mu]}$. Note the absence of any explicit time dependence. This completes the set of expressions for the radiated Poincare charges. Since the stress-energy pseudotensor can be derived directly from on-shell amplitudes using the KMO framework [4, 5], our formulation for $J^{\mu\nu}$ meshes well with the QFT-based approach. Analogous formulae for $P^\mu$ and $J^{\mu\nu}$ in EM are given in Appendix [A].

The expressions in Eq. [3] are highly constrained by gauge invariance and the Poincare algebra. The relative factor between the first and last terms in the $J^{\mu\nu}$ integrand is fixed by invariance under residual gauge transformations. These terms are sometimes referred to as the orbital and spin contributions. However, only their combination is gauge invariant, implying that individually, they have no physical meaning [57]. The Poincare algebra imposes the following transformations under the translation $x^\mu \rightarrow x^\mu + a^\mu$,

$$P^\mu \rightarrow P^\mu, \quad J^{\mu\nu} \rightarrow J^{\mu\nu} + a^{[\mu} P^{\nu]}. \quad (4)$$

Since $T^{\mu\nu}(k) \rightarrow T^{\mu\nu}(k)e^{ik \cdot a}$ under translations, the expressions in Eq. [3] indeed obey Eq. [4].

**Form Factor Parametrization.** By parametrizing $P^\mu$ and $J^{\mu\nu}$ in terms of the initial data of binary scattering, one can derive additional constraints on them using the Poincare algebra. The particles are labeled by a Roman subscript $i = 1, 2$ and $m_i, p_i^\mu, b_i^\mu$ correspond to the particle’s mass, initial momentum and impact vector, which, as depicted in Fig. [1] obey $p_i \cdot b_i = 0$. In addition, it is useful to define the relative impact vector $\Delta b^\mu \equiv b_1^\mu - b_2^\mu$, its magnitude $b \equiv \sqrt{-\Delta b^2}$, and $b^\nu \equiv (p_1 \cdot (p_1 + p_2)) b_1^\nu + (p_2 \cdot (p_1 + p_2)) b_2^\nu)/(p_1 + p_2)^2$. The Lorentz-invariant variables are then $m_i, b$, and the relative boost $\sigma \equiv p_1 \cdot p_2/(m_1 m_2)$.

We find that the most general forms of $P^\mu$ and $J^{\mu\nu}$ consistent with Lorentz covariance, the Poincare constraints [1], and particle interchange symmetry are

$$P^\mu = F_1 p_1^\mu + F_2 p_2^\mu + F_3 \Delta b^\mu, \quad J^{\mu\nu} = \tilde{b}^{[\mu} \left( F_1 p_1^{\nu]} + F_2 p_2^{\nu]} + F_3 \Delta b^{\nu]} \right) \quad (5)$$

$$+ \Delta b^{[\mu} \left( G_1 p_1^{\nu]} - G_2 p_2^{\nu]} \right) + H_{12} p_2^\mu p_1^\nu, \quad (6)$$

where $F_i, G_i, H_{12}$ are form factors that are functions of the Lorentz invariants $m_1, m_2, \sigma, b$. Particle interchange symmetry implies that the form factors satisfy

$$F_1 m_1 \leftrightarrow m_2, \quad G_1 m_1 \leftrightarrow m_2, \quad H_{12} m_1 \leftrightarrow m_2, \quad (5)$$

so that the only independent ones are $F_2, F_3, G_2, H_{12}$. We consider two frames in this paper, the center-of-mass (CM) and the frame where particle 1 is initially at rest (referred to as the rest frame hereafter). See Appendix [B] for the initial conditions in each frame. In particular, $\tilde{b}^\mu = 0$ in the CM frame and $b^\mu = 0$ in the rest frame. We denote the components of $J^{\mu\nu}$ in the CM and rest frames as $J_{\text{CM}}^{\mu\nu}$ and $J_{\text{rest}}^{\mu\nu}$, and the initial angular momentum along the $z$ direction as $J_{\text{CM}}$ and $J_{\text{rest}}$. Their form factor expressions are summarized in Eqs. [A-1] and [A-2].

![FIG. 1. The initial configuration of the binary system. The spatial momenta of the two particles are along the $x$ direction, and the impact vectors $b_{1,2}^\mu$ are along the $y$ direction.](image-url)
To obtain the Stress-Energy Pseudotensor, the standard formula \[34, 35, 49\] does not indeed the full result at this order. As we discuss further we find \(\mathcal{J}\) constructed \[58, 59\].

Moreover, at \(\mathcal{O}\) the leading soft limit of \(\mathcal{J}\) is governed by \(\mathcal{O}\) scattering deflections. The bottom row depicts \(\mathcal{J}\) in the rest and CM frames: \(\mathcal{J}\) and \(\mathcal{I}\) are given in Table \(\mathcal{I}\). Interestingly, \(\mathcal{J}\) can be written in terms of the scattering angle \(\chi\) and is independent of short-distance details. We find the leading radiated angular momentum is positive when \(\chi > 0\), i.e., the scattering is attractive.

Matching the above to Eq. \(\mathcal{I}\) gives

\[
\mathcal{G}_{1,2} = \mathcal{G}_{2,2} = \frac{\nu M^2 (2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \mathcal{I}(\sigma),
\]

while all other form factors vanish at this order. Plugging Eq. \(\mathcal{I}\) into Eq. \(\mathcal{I}\) gives the remaining components of \(\mathcal{J}\) in the rest and CM frames:

\[
\frac{\mathcal{I}_{\text{CM,2}}}{E_1 - E_2} = \frac{\mathcal{I}_{\text{rest,2}}}{m_1 - m_2} = \frac{2\nu M^2 (2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \mathcal{I}(\sigma),
\]

\[
\frac{\mathcal{J}_{\text{CM,2}}}{\mathcal{J}_{\text{rest}}} = 2 \frac{\mathcal{J}_{\text{rest,2}}}{\mathcal{J}_{\text{rest}}} = \frac{2\nu M^2 (2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \mathcal{I}(\sigma).
\]

These results can be directly verified using Eq. \(\mathcal{I}\).

Our results for \(\mathcal{J}_{\text{CM,2}}\) agree with Eq. \(\mathcal{I}\) of \[10\]. \(\mathcal{J}_{\text{CM,2}}\) agrees with Eq. \(\mathcal{I}\) of \[45\] modulo an extra term. As a nontrivial check, we computed the angular momentum loss using the known 3.5PN RR force exerted on the matter \[26\]. This predicts the first two orders of the velocity expansion of \(\mathcal{J}\) at \(\mathcal{O}\), and we fully agree in the CM and rest frames. However, we disagree with the expression for \(\mathcal{J}_{\text{rest,2}}\) obtained by using the standard form \[83\] which leads to \(\mathcal{J}_{\text{rest,2}}/\mathcal{J}_{\text{rest}} = J_{\text{CM,2}}/J_{\text{CM}}\). This disagrees with Eqs. \(\mathcal{I}\) and \(\mathcal{I}\) by a factor of 2 because of the subtlety of applying standard formulae away from the CM frame \[83\].
even in the large mass ratio limit, i.e.
the ratio
momentum but, at the same time, zero linear momen-
ting terms. Using this method, we computed the non-

\[ J_{\text{rest,3}}^{12} = -b\left(m_1 m_2 (m_1^2 - m_2^2) - \frac{\chi^2}{\sqrt{\sigma^2 - 1}} I(\sigma) \right) \]

where \(\sigma\) and \(D(\sigma)\) correspond to the non-zero frequency and interference contributions (see Table 1). Using the maps in Eq. (14) we fix \(F_{2,3}, G_{1,3}\) and \(H_{12,3}\) by matching to \(P^\mu, T^\mu\) and Eq. (13).

\[ F_{2,3} = \frac{m_1 m_2}{\sigma + 1} \frac{\nu \epsilon}{M^2}, \quad G_{1,3} = 0, \quad H_{12,3} = -b(m_1^2 - m_2^2) \cdot \frac{\chi^2}{\sigma^2 - 1} I(\sigma), \]

where \(h = \sqrt{1 + 2\nu/\sigma - 1}\).

The form factors in Eq. (14) can be used to translate the rest-frame results into CM ones via Eq. (5). For instance,

\[ J_{\text{CM,3}}^{12} = \frac{\nu M^3}{p_\infty^2} \left[ (\sigma + 2D(\sigma) - \frac{\nu p_\infty \epsilon}{M^2} \frac{1}{h^2} ) \right]. \]

Defining \(J_3 = \frac{\nu}{h^2 p_\infty^2} J_{\text{CM,3}}^{12} \), we find that the combination

\[ h^3 J_3 + \frac{\nu p_\infty^2}{h^2} \epsilon(\sigma) = (\sigma^2 - 1) (\sigma + 2D(\sigma)) \]

only depends on \(\sigma\) but not \(m_i\). This is precisely the relation first observed in 47 by considering the matter impulses at \(O(G^4)\). Here, we obtain the same result as a consequence of the mass scaling at \(O(G^3)\) in Eq. (13) and Lorentz covariance in Eq. (5). Expanding \(J_3\) in small velocity yields

\[ \frac{J_3}{\pi} = \frac{28}{5} \frac{p_\infty^2}{p_\infty^4} + \left( \frac{739}{84} - \frac{163}{15} \nu \right) p_\infty^4 \]

(17)

\[ + \left( \frac{5777}{2520} - \frac{539}{420} \nu + \frac{50}{3} \nu^2 \right) p_\infty^6 \]

\[ + \left( \frac{115769}{126720} + \frac{1469}{504} \nu + \frac{9235}{672} \nu^2 - \frac{553}{24} \nu^3 \right) p_\infty^8 + \ldots \]

The first three terms agree with Eq. (7.21) of 47.

Implications for \(O(G^2)\) Scattering. It was pointed out in 49, 50, 52 that RR effects on the scattering angle and momentum impulses can be extracted from \(P^\mu\) and \(J^{\mu\nu}\).
Define the transverse impulse at $\mathcal{O}(G^4)$ to be $\Delta p_{\perp,4} = \frac{\Delta r}{\delta t} \Delta \phi |_{G^4}$. It can be written as

$$\Delta p_{\perp,4} = \nu M^5 (G/b)^4 \left( c_{b,4}^{\text{cons}} + c_{b,4}^{\text{rr,even}} + c_{b,4}^{\text{rr,odd}} \right), \tag{18}$$

where $c_{b,4}^{\text{cons}}$ is the conservative contribution calculated in $[10, 17]$ using the prescription $[47]$, and $c_{b,4}^{\text{rr,even}}$ and $c_{b,4}^{\text{rr,odd}}$ are the dissipative contributions that are even and odd under time reversal. ($c_{b,4}^{\text{cons}}$ and $c_{b,4}^{\text{rr,odd}}$ are $c_{b,4}^{\text{G4}}$ and $c_{b,4}^{\text{rr,4}}$ in $[47]$). Using the explicit map in $[47]$, $c_{b,4}^{\text{rr,odd}}$ is fixed by $P^\mu$ and $J^{\mu \nu}$ to $\mathcal{O}(G^3)$

$$c_{b,4}^{\text{rr,odd}} = \nu \left[ \frac{\sigma(6\sigma^2-5)}{\sigma^2-1} - \frac{m_1}{M} \frac{2\sigma^2-1}{\sigma+1} \right] \mathcal{E}(\sigma) \frac{p_\infty}{p^\infty} \tag{19}$$

where the mass dependence is consistent with $[68, 69]$. The first three orders of the velocity expansion in Eq. (19) agree with the last line of Eq. (8.6) of $[47]$.

In the high energy limit $\sigma \to \infty$, $c_{b,4}^{\text{rr,odd}}$ is dominated by terms coming from $\mathcal{C}(\sigma)$ and $\mathcal{E}(\sigma)$ and scales as $\sigma^3$. This high-energy behavior is comparable to that of $c_{b,4}^{\text{cons}}$. However, the sum does not cancel and $\Delta p_{\perp,4} \sim G^4 \sigma^3$ in the high energy limit. It would be interesting to see if the contribution from $c_{b,4}^{\text{rr,odd}}$ tames this divergence.

**Radiation Reaction Force.** Dissipative effects on a binary system in a generic orbit can be described by the RR force $F_{RR}$. Let the spinless binary motion lie on the $x-y$ plane. In polar coordinates, $F_{RR} = F_\phi \hat{r} + F_\psi \hat{\phi}$, where $e_\varphi$ and $e_\psi$ are the radial and angular unit vectors, $r$ is the relative distance, and $\phi$ is the polar angle. The energy $E$ and angular momentum $J$ of the binary are not conserved in the presence of $F_{RR}$. Using the formulation in $[47, 52]$, the equations of motion for the CM recoil, and the extension to $\mathcal{O}(G^3)$, can be studied similarly.

One can bootstrap the RR force using the loss of energy and angular momentum due to scattering in the CM frame. Assuming $F_{RR}$ is a vector under spatial parity and odd under time reversal,

$$F_{RR} = c_r p_r \hat{e}_r + c_p p = (c_r + c_p) p_r \hat{e}_r + c_p J \hat{e}_\phi \tag{21}$$

where $p$ is the relative momentum, $p_r = p \cdot \hat{e}_r$, and $c_r$ and $c_p$ are unknown coefficients that are even under time reversal. We also assume that $c_r$ and $c_p$ can be expressed in isotropic gauge, i.e. they only depend on $r$ and $p^2$. Classical power counting yields the ansatz

$$c_r = \frac{G^2}{r^3} c_{r,2} (p^2) + \ldots, \quad c_p = \frac{G^2}{r^3} c_{p,2} (p^2) + \ldots, \tag{22}$$

where the dots denote higher orders in $G$.

Plugging Eqs. (21) and (22) into Eq. (20), and integrating over the conservative trajectories in the CM frame, which to leading order are straight lines, yields the change in $E$ and $J$ after scattering

$$\Delta J = \frac{2G^2 E_1 E_2}{b} c_{p,2} (p_0^2) + \ldots, \tag{23}$$

$$\Delta E = \frac{2G^2 J_{CM}}{3b^3} (c_{r,2} (p_0^2) + 3c_{p,2} (p_0^2)) + \ldots. \tag{24}$$

Conservation of energy and angular momentum implies

$$\Delta J = -J_{CM}^2 \quad \Delta E = -P_{CM}^0. \tag{24}$$

Matching this to Eq. (12) and $P_{CM,2}^0 = 0$ at $\mathcal{O}(G^2)$ fixes the ansatz entirely:

$$c_{r,2} (p_0^2) = -3c_{p,2} (p_0^2), \tag{25}$$

$$c_{p,2} (p_0^2) = -\frac{\nu^2 M^4}{E_1 E_2} (2\sigma^2 - 1) \mathcal{I}(\sigma). \tag{26}$$

This extends the RR force at $\mathcal{O}(G^2)$ to all orders in velocity, which was only derived previously to the first three orders in the velocity expansion $[32]$. The equations of motion for the CM recoil, and the extension to $\mathcal{O}(G^3)$, can be studied similarly.

**Conclusions.** In this Letter, we build a new framework to calculate the radiated angular momentum due to scattering that meshes well with QFT-based methods. Our work opens up many avenues for future work. Some obvious generalizations include dissipative effects in scattering with spin $[69, 70]$, and in gauge $[53, 71]$ and supersymmetric theories $[36, 72]$. It would also be interesting to compare our method with other approaches using soft theorems $[11, 22]$. A crucial next step is to calculate $P^\mu$ and $J^{\mu \nu}$ to $\mathcal{O}(G^4)$. For the bounded binaries, it would be interesting to compare with the flux from analytic continuation $[11]$, and study its impact on waveform models $[73]$. Last but not least, it would be interesting to extend our framework beyond gravitational-wave science, perhaps along the lines of jet observables $[74]$.

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Appendix A: Radiated Linear and Angular Momenta in Electromagnetism

In this section, we derive the analog of Eq. (3) of the main paper in electromagnetism (EM). The EM stress-energy tensor associated with the gauge field $A_\mu$ can be written as

$$T^{\mu\nu} = -F^{\mu}_{\rho} F^{\nu\rho} + \frac{\eta^{\mu\nu}}{4} F^{\rho\sigma} F_{\rho\sigma}, \quad (A1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. Once a gauge is chosen, $A^\mu$ can be expressed in terms of the source current using the classical Green’s function. In momentum space, the gauge field is given by

$$A_\mu(x) = \int \tilde{d}k \ (P_{\mu\nu} \mathcal{J}^\nu(k) e^{-ik\cdot x} + \text{c.c.}), \quad (A2)$$

where $\mathcal{J}^\nu(k)$ is the conserved current satisfying $k_{\mu}\mathcal{J}^\nu(k) = 0$, and $P_{\mu\nu}$ is the gauge-dependent transverse projector. In Lorenz gauge, one replaces $P_{\mu\nu} \mathcal{J}^\nu(k)$ with $\mathcal{J}_\mu(k)$ in Eq. (A2). Like the stress-energy pseudotensor in gravity, the EM source current exhibits a residual gauge freedom: $J^\mu(k) \rightarrow J^\mu(k) + \alpha k^\mu$.

Combining Eq. (1) of the main text and Eqs. (A1) and (A2) above, the EM expressions for $P^\mu$ and $J^{\mu\nu}$ are

$$P^\mu = \int \tilde{d}k \ k^\mu (-\mathcal{J}^{*\rho}(k) \mathcal{J}_\rho(k)), \quad (A3)$$

$$J^{\mu\nu} = \int \tilde{d}k \ (-\mathcal{J}^{*\rho}(k) L^{\mu\nu} \mathcal{J}_\rho(k) - i \mathcal{J}^{*\mu}(k) \mathcal{J}^\nu(k)).$$

Like in gravity, the explicit time dependence and gauge choice drop out and we are left with expressions that only depend on $\mathcal{J}^\mu(k)$. This radiated angular momentum formula is valid for radiation with arbitrary frequency. As a consistency check, the formulae Eq. (A3) are also invariant under the residual gauge transformation and obey the Poincare algebra.

Appendix B: Frame Choices and Form Factor Expressions

In this section, we summarize the initial conditions of the matter in the CM and rest frames. The two frames are related by a boost along the $x$ axis and a translation along the $y$ axis. In both frames, $\Delta b^\mu = (0, 0, -b, 0)$ as depicted in Fig. 1 of the main text.

In the CM frame, the initial conditions are

$$p_1^\mu = (E_1, |p_0|, 0, 0), \quad p_2^\mu = (E_2, -|p_0|, 0, 0), \quad (B1)$$

$$b_1^\mu = \frac{E_2}{E_1 + E_2} \Delta b^\mu, \quad b_2^\mu = -\frac{E_1}{E_1 + E_2} \Delta b^\mu,$$

such that $\tilde{b}^\mu = 0$. Here, $E_i = \sqrt{p_0^2 + m_i^2}$ denotes the initial energy of the scalars. The initial angular momentum lies along the $z$ direction with magnitude $J_{\text{CM}} = |p_0| b$.

In the rest frame, particle 1 is initially at rest and sits at the origin,

$$p_1^\mu = (m_1, 0, 0, 0), \quad p_2^\mu = (\sigma m_2, -p_\infty m_2, 0, 0), \quad (B2)$$

$$b_1^\mu = (0, 0, 0, 0), \quad b_2^\mu = -\Delta b^\mu,$$

such that in this frame

$$\tilde{b}^\mu = -\frac{m_2(m_2 + m_1 \sigma)}{M^2 h^2} \Delta b^\mu. \quad (B3)$$

Recall that $h \equiv ((p_1 + p_2)^2/M^2)^{1/2} = \sqrt{1 + 2\nu(\sigma - 1)}$.

Given the initial conditions, the components of $J^{\mu\nu}$ can be expressed in terms of form factors. The nontrivial components in the CM frame are

$$J^{12}_{\text{CM}} = J_{\text{CM}} (\mathcal{G}_1 + \mathcal{G}_2),$$

$$J^{01}_{\text{CM}} = b (\mathcal{G}_1 E_1 - \mathcal{G}_2 E_2), \quad (B4)$$

and in the rest frame

$$J^{12}_{\text{rest}} = J_{\text{rest}} \left(\mathcal{G}_2 + \frac{m_2(m_2 + m_1 \sigma)}{M^2 h^2} \mathcal{F}_2\right),$$

$$J^{01}_{\text{rest}} = b (\mathcal{G}_1 m_1 - \mathcal{G}_2 m_2 \sigma) \frac{b}{M^2 h^2}(\mathcal{F}_1 m_1 + \mathcal{F}_2 m_2), \quad (B5)$$

$$J^{01}_{\text{rest}} = m_1 m_2 p_\infty \mathcal{H}_{12}.$$  

In particular, we can obtain all form factors by calculating $P^\mu$, $J^{12}_{\text{rest}}$, and $J^{01}_{\text{rest}}$ and using the particle exchange symmetry described in the main text. Evidently, $J^{02}_{\text{rest}}$ and the CM frame results become consistency checks.

Appendix C: Perturbative Expansion in $G$

The PM expansion organizes terms in powers of $G/b$. The coefficients are accurate to all orders in the velocity. We can expand the linear and angular momentum as

$$P^\mu_X = \left(\frac{G}{b}\right)^3 P^\mu_{X,3} + \ldots, \quad (C1)$$

$$J^{\mu\nu}_X = \left(\frac{G}{b}\right)^2 J^{\mu\nu}_{X,3} + \left(\frac{G}{b}\right)^3 J^{\mu\nu}_{X,5} + \ldots, \quad (C2)$$

where $X = \text{CM}$ or rest. The form factors can also be expanded as

$$\mathcal{G}_i = \left(\frac{G}{b}\right)^2 \mathcal{G}_{i,3} + \left(\frac{G}{b}\right)^3 \mathcal{G}_{i,5} + \ldots, \quad (C3)$$

$$\mathcal{F}_i = \left(\frac{G}{b}\right)^3 \mathcal{F}_{i,3} + \ldots,$$

$$\mathcal{H}_{12} = \left(\frac{G}{b}\right)^3 \mathcal{H}_{12,3} + \ldots.$$  

The dots denote higher order corrections in the $G/b$ expansion.
Appendix D: Conservative Scattering Angle

The conservative scattering angle in the CM frame is

$$
\chi = \sum_a \left( \frac{G m_1 m_2}{\mathcal{J}_{CM}} \right)^a \frac{2 \chi_a}{\mathcal{J}_{CM}} , \quad (D1)
$$

$$
\chi_1 = \frac{(2 \sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} , \quad \chi_2 = \frac{3 \pi}{8} \frac{5 \sigma^2 - 1}{\sqrt{1 + 2 \nu(\sigma - 1)}}, \quad (D2)
$$

The impulses in the CM frame are

$$
p_{\mu, l} - p_{\mu}^l = -(p_{\mu, f}^l - p_{\mu}^l) = |p_0| \left( 0, \cos \chi, \sin \chi, 0 \right). \quad (D3)
$$

Appendix E: Stress-energy Pseudotensor at $O(G)$

The full stress-energy pseudotensor at $O(G)$ sourced by binary scattering has been calculated using various methods, including the worldline and the KMAC framework. It can be written as

$$
T^{\mu \nu}(k) = i \int d\ell \delta(2p_1 \cdot l) \delta(2p_2 \cdot \ell) e^{il \cdot \ell - i \ell \cdot b_1 - i \ell \cdot b_2} \mathcal{M}_5^{\mu \nu}(\ell, k), \quad (E1)
$$

where $d\ell = \frac{d^2 \ell}{(2 \pi)^2}$ and $l = k + \ell$. The kernel $\mathcal{M}_5^{\mu \nu}(\ell, k)$ is closely related to the tree-level amplitude $\mathcal{M}_5(\ell, k)$ of five-particle scattering depicted by the diagrams in the bottom row of Fig. 2 of the main text. We use the KMAC formalism to extract $\mathcal{M}_5^{\mu \nu}(\ell, k)$. First, we strip off the graviton polarization $e_{\mu \nu}$ from $\mathcal{M}_5(\ell, k)$ such that $\mathcal{M}_5^{\mu \nu}(\ell, k)$ is a symmetric tensor and reproduces the correct amplitude, $e_{\mu \nu} \mathcal{M}_5^{\mu \nu}(\ell, k) = \mathcal{M}_5(\ell, k)$. Next, we impose that the Ward identity is satisfied even when one index is free, i.e., $k_\mu \mathcal{M}_5^{\mu \nu}(\ell, k) = 0$, to ensure the conservation of the stress-energy pseudotensor, $k_\mu T^{\mu \nu}(k) = 0$. This is made possible by using the pure gauge freedom in $\mathcal{M}_5^{\mu \nu}(\ell, k)$. Finally, we isolate the classical contribution to $\mathcal{M}_5^{\mu \nu}(\ell, k)$ by rescaling $\ell, k \rightarrow (\lambda \ell, \lambda k)$ and expanding in small $\lambda$. For this purpose, it is sufficient to truncate to $O(1/\lambda^2)$. In contrast to the conservative case, there is actually an iteration contribution if one keeps the Feynman $i\epsilon$ in the matter propagators. As expected, this iteration is precisely canceled by the cut contribution in the KMAC approach. Therefore, we can effectively ignore the Feynman $i\epsilon$ when taking the classical limit. This procedure yields

$$
\mathcal{M}_5^{\mu \nu}(\ell, k) = 8\pi G \left( D_1^{\mu \nu} + D_2^{\mu \nu} + D_3^{\mu \nu} \right), \quad (E2)
$$

where

$$
D_1^{\mu \nu} = 2m_1^2 m_2^2 (2\sigma^2 - 1) \left( 2(p_1 \cdot k) p_1^{(\mu \ell')}(\ell \cdot k)p_1^{(\nu \ell')} - (\ell \cdot k)p_1^{(\mu \ell')}p_1^{(\nu \ell')} \right), \quad (E3)
$$

$$
D_2^{\mu \nu} = 2m_1^2 m_2^2 (2\sigma^2 - 1) \left( 2(p_2 \cdot k) p_2^{(\mu \ell')}(\ell \cdot k)p_2^{(\nu \ell')} - (\ell \cdot k)p_2^{(\mu \ell')}p_2^{(\nu \ell')} \right), \quad (E4)
$$

$$
D_3^{\mu \nu} = \frac{16m_1 m_2 \sigma}{\ell^2 \ell^2} \left( (p_2 \cdot k)p_1^{(\mu \ell')}(\ell \cdot k)p_1^{(\nu \ell')} - (\ell \cdot k)p_1^{(\mu \ell')}p_1^{(\nu \ell')} \right)
- \frac{8}{\ell^2 \ell^2} \left( (p_1 \cdot k) p_2^{(\mu \ell')}(\ell \cdot k)p_2^{(\nu \ell')} + (p_2 \cdot k)^2 p_1^{(\mu \ell')}p_1^{(\nu \ell')} \right)
+ \frac{8p_2^{(\mu \ell')}}{\ell^2 \ell^2} \left( (p_2 \cdot k)p_2^{(\nu \ell')}(\ell \cdot k)p_2^{(\mu \ell')} \right)
- \frac{4m_1^2 m_2^2 (2\sigma^2 - 1)}{\ell^2 \ell^2} \ell^\mu \ell^\nu. \quad (E5)
$$

We use $a^{(\mu \nu \ell')} \equiv (a^{\mu \nu} b^\ell + a^{\nu \ell} b^\mu)/2$ in the above. Together with Eq. (8) of the main text, this completes the set of expressions for the stress-energy pseudotensor that are needed to compute $J^{\mu \nu}$ to $O(G^3)$.

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