Probabilistic Relational Reasoning via Metrics

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Abstract—The Fuzz programming language by Reed and Pierce uses an elegant linear type system combined with a monad-like type to express and reason about probabilistic properties, most notably $\varepsilon$-differential privacy. We show how to extend Fuzz to capture more general relational properties of probabilistic programs, with approximate, or $(\varepsilon, \delta)$-differential privacy serving as a leading example. Our technical contributions are threefold. First, we introduce the categorical notion of comonadic lifting of a monad to model composition properties of probabilistic divergences. Then, we show how to express relational properties in terms of sensitivity properties via an adjunction we call the path construction. Finally, we instantiate our semantics to model the terminating fragment of Fuzz extended with types carrying information about other divergences between distributions.

I. INTRODUCTION

Over the past decade, differential privacy has emerged as a robust, compositional notion of privacy, imposing rigorous bounds on what database queries reveal about private data. Formally, a probabilistic database query $f$ is $\varepsilon$-differentially private if, given two pairs of adjacent databases $X_1$ and $X_2$—that is, databases differing in at most one record—we have

\[
\mathbb{P}(f(X_1) \in U) \leq e^{\varepsilon} \mathbb{P}(f(X_2) \in U) \tag{1}
\]

and

\[
\mathbb{P}(f(X_2) \in U) \leq e^{\varepsilon} \mathbb{P}(f(X_1) \in U), \tag{2}
\]

where $U$ ranges over arbitrary sets of query results. Intuitively, the parameter $\varepsilon$ measures how different the result distributions $f(X_1)$ and $f(X_2)$ are—the smaller $\varepsilon$ is, the less the output depends on any single record in the input database.

The strengths and applications of differential privacy have prompted the development of a range of verification techniques, in particular approaches based on linear types such as the Fuzz language of Reed and Pierce \cite{28}. These systems exploit the fact that $\varepsilon$-differential privacy is equivalent to a sensitivity property, a guarantee that applies to arbitrary pairs of databases:

\[
\text{MD}(f(X_1), f(X_2)) \leq \varepsilon \cdot d_{DB}(X_1, X_2), \tag{3}
\]

where $d_{DB}$ measures how similar the input databases are (for instance, via the Hamming or symmetric distance on sets), and MD, the max divergence, is the smallest value of $\varepsilon$ for which \( \text{(3)} \) holds. In other words, $f$ is $\varepsilon$-Lipschitz continuous, or $\varepsilon$-sensitive. Sensitivity has pleasant properties for formal verification; for example, the sensitivity of the composition of two functions is the product of their sensitivities. By leveraging these composition principles, Fuzz can track the sensitivity of a function in its type, reducing a proof of differential privacy for an algorithm to simpler sensitivity checks about its components.

However, not all distance bounds between distributions can be converted into sensitivity properties. One example is $(\varepsilon, \delta)$-differential privacy \cite{11}, a relaxation of $\varepsilon$-differential privacy that allows privacy violations with a small probability $\delta$; in return, $(\varepsilon, \delta)$-differential privacy can allow significantly more accurate data analyses. Superficially, its definition resembles \( \text{(2)} \): a query $f$ is $(\varepsilon, \delta)$-differentially private if, for all pairs of adjacent input databases $X_1$ and $X_2$, we have

\[
\mathbb{P}(f(X_1) \in U) \leq e^{\varepsilon} \mathbb{P}(f(X_2) \in U) + \delta \tag{4}
\]

and

\[
\mathbb{P}(f(X_2) \in U) \leq e^{\varepsilon} \mathbb{P}(f(X_1) \in U) + \delta. \tag{5}
\]

Setting $\delta = 0$ recovers the original definition. Introducing the skew divergence $\text{AD}_\varepsilon$ \cite{4}, $(\varepsilon, \delta)$-privacy is equivalent to a bound $\text{AD}_\varepsilon(f(X_1), f(X_2)) \leq \delta$ for adjacent databases.

Despite the similarity between the two definitions, Fuzz could not handle $(\varepsilon, \delta)$-differential privacy, because it cannot be stated directly in terms of function sensitivity. This is possible for $\varepsilon$-differential privacy because \( \text{(2)} \) can be recast as the bound $\text{MD}(f(X_1), f(X_2)) \leq \varepsilon$, which is equivalent to the sensitivity property \( \text{(3)} \) because the max divergence is actually a proper metric satisfying the triangle inequality. In contrast, the skew divergence does not satisfy the triangle inequality and does not scale up smoothly when the inputs $X_1$ and $X_2$ are farther apart—for instance, an $(\varepsilon, \delta)$-private function $f$ usually does not satisfy $\text{MD}(f(X_1), f(X_2)) \leq 2 \cdot \delta$ when $X_1$ and $X_2$ are at distance 2. Similar problems arise for other properties based on distances that violate the triangle inequality, such as the Kullback-Leibler (KL) and $\chi^2$ divergences.

This paper aims to bridge this gap, showing that Fuzz’s core can already accommodate other quantitative properties, with $(\varepsilon, \delta)$-differential privacy being our motivating application. To handle such properties, we first need a semantics for the probabilistic features of Fuzz. Typically in programming language semantics, probabilistic programs are structured using the probability monad \cite{16,25}, the return operation produces a deterministic distribution that always yields the same value, while the bind operation samples from a distribution and runs another probabilistic computation. However, monads cannot describe the composition principles supported by many useful metrics on probabilities—though the typing rules for distributions in Fuzz resemble the usual monadic rules, their treatment of context sensitivities departs from the expected monadic rules \cite{25}. Accordingly, our first contribution is a notion of comonadic lifting of a monad, which lifts the operations of a monad from a symmetric monoidal closed category (SMCC) to a related refined category. We demonstrate our theory by modeling statistical distance and Fuzz’s max divergence. We...
also propose a graded variant of liftings to encompass other examples, including the KL divergence, $\chi^2$ divergence, and the Hellinger distance.

Our second contribution is a path metric construction that reduces relational properties such as $(\varepsilon, \delta)$-differential privacy to equivalent statements about sensitivity. Concretely, given any reflexive, symmetric relation $R$ on a set $X$, we define the path metric $d_R$ on $X$ by setting $d_R(x_1, x_2)$ to be the length of the shortest path connecting $x_1$ and $x_2$ in the graph corresponding to $R$. The path construction provides a full and faithful functor from the category RSRel of reflexive, symmetric relations into the category Met of metric spaces and 1-sensitive functions. We also show a right adjoint to the path construction—as Met is a symmetric monoidal closed category and RSRel is a cartesian closed category, this adjunction recalls mixed linear and non-linear models of linear logic [8].

Putting these two pieces together, our third contribution is a model of the terminating fragment of Fuzz [28]. We extend the language with new types and typing rules to express interpretations of its deterministic fragment in the category of relations. Section V I introduces a framework of comonadic liftings over monads, and Section IV interprets the terminating fragment of Fuzz with probabilistic constructs.

Shifting gears, Section V explores how probabilistic properties can be modeled in the category of relations. Section VI develops a graded version of our comonadic liftings over relations, to model composition of properties like $(\varepsilon, \delta)$-privacy. In Section VII we consider how to transfer liftings between different categories; our leading example is the path construction, which moves liftings over relations to liftings over metric spaces. As an application, Section VIII extends Fuzz with graded types capable of modeling $(\varepsilon, \delta)$-privacy and other sensitivity properties of divergences.

In Section IX we sketch how our results can be partially extended to model general recursion in Fuzz, by combining the metric CPO semantics of Azevedo de Amorim et al. [3] with the probabilistic powerdomain of Jones and Plotkin [17]. While the probabilistic features and liftings pose no problems, the path construction runs into technical difficulties; we leave this extension as a challenging open problem. Finally, we survey related work (Section XI) and conclude (Section X).
elements $x_1, x_2 \in X$. Thus, the smaller $r$ is, the less the output of a function varies when its input varies. Note that this condition is vacuous when $r = \infty$, so any function between metric spaces is $\infty$-sensitive. When $r = 1$, we speak of a non-expansive function instead. We write $f : X \longrightarrow Y$ to mean that $f$ is a non-expansive function from $X$ to $Y$.

To illustrate these concepts, consider the set of real numbers $\mathbb{R}$ equipped with the Euclidean metric: $d(x,y) = |x - y|$. The doubling function that maps the real number $x$ to $2x$ is $2$-sensitive; more generally, a function that scales a real number by another real number $k$ is $|k|$-sensitive. However, the squaring function that maps each $x$ to $x^2$ is not $r$-sensitive for any finite $r$. The identity function on a metric space is always non-expansive. The definition of $\varepsilon$-differential privacy, as stated in [3], says that the private query $f$ is $\varepsilon$-sensitive.

Figure 1 summarizes basic constructions on metric spaces. Scaling allows us to express sensitivity in terms of non-expansiveness: an $r$-sensitive function from $X$ to $Y$ is a non-expansive function from the scaled metric space $r \cdot X$ (or simply $rX$) to $Y$. The metric on $\infty X$ does not depend on the metric of $X$—note that we define scaling by $\infty$ separately from scaling by a finite number to ensure that $d(x,x) = 0$—so we use this notation even when $X$ is a plain set that does not have a metric associated with it. We consider two metrics on products: one combines metrics by taking their maximum, while the other takes their sum. The metric on disjoint unions places their two sides infinitely apart.

Let Met be the category of metric spaces and non-expansive functions. We write $p : Met \rightarrow Set$ to denote the forgetful functor defined by $pX = |X|$ and $pf = f$. Note that $Set(X,pY) = Met(\infty \cdot X,Y)$, so $\infty \cdot (\_)$ and $p$ form an adjoint pair. The $\times$ metric—used to interpret the connective $\&$—and $+$ metric yield products and sums in Met, with the expected projections and injections. Since the two sides of a sum are infinitely apart, we can define non-expansive functions by case analysis without reasoning about sensitivity across two different branches. The other metric on products, given by $\otimes$, is needed to make the operations of currying and function application compatible with the metric on non-expansive functions defined above. Formally, Met forms a symmetric monoidal closed category with monoidal structure given by $\otimes$, $1$ and exponentials given by $\rightarrow$.

**D. The Fuzz Language**

Fuzz [28] is a type system for analyzing program sensitivity. The language is a largely standard, call-by-value lambda calculus; Fig. 2 summarizes the syntax, types, contexts, and context operations. Types in Fuzz are interpreted as metric spaces, and function types carry a numeric annotation that describes their sensitivity. The type system tracks the sensitivity of typed terms with respect to each of its bound variables, akin to bounded linear logic [15]. Fig. 3 presents the typing rules of the terminating, deterministic fragment of the language.

Following Azevedo de Amorim et al. [3], we can interpret each typing derivation $x_1 : r_1, r_1, \ldots, x_n, r_n \vdash e : \sigma$ as a non-expansive function $[e] : r_1[\tau_1] \otimes \cdots \otimes r_n[\tau_n] \rightarrow [\sigma]$, where types are interpreted homomorphically using the constructions on metric spaces described thus far. In particular, context splitting corresponds to the family of functions $\delta : [\Delta + \Gamma] \rightarrow [\Delta] \otimes [\Gamma]$ defined as $\delta(x) \triangleq (x,1)$, where $x$ and $x_2$ are obtained by removing the components of $x$ that do not appear in $\Delta$ and $\Gamma$, respectively.

In addition to the probabilistic features that we will cover next, the original Fuzz language also includes general recursive types. These pose challenges related to non-termination, which we return to in Section IX.

### III. Extending Semantics to Handle Probabilities

Looking beyond the deterministic fragment discussed earlier, Fuzz offers a monad-like interface for probabilistic programming [25], structured around the operations discussed in [2].

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1. Our interpretation of scaling differs slightly from the one of Azevedo de Amorim et al. [3] in that distinct points in scaled spaces $\infty X$ are infinitely apart. This does not affect the validity of the interpretation; in particular, scaling is still associative, and commutes with $\otimes$, $\&$, and $+$.

| Metric Space | Carrier Set | $d(a,b)$ |
|--------------|-------------|-----------|
| $\mathbb{R}$ | $\mathbb{R}$ | $|a-b|$ |
| 1            | $\{\ast\}$ | 0         |
| $r \cdot X$  | $X$         | $\begin{cases} \inf & \text{if } r \neq \infty \\ 0 & \text{if } r = \infty, a \neq b \\ \infty & \text{if } r = \infty, a = b \end{cases}$ |
| $X \times Y$ | $X \times Y$ | $\max(d_X(a_1,b_1), d_Y(a_2,b_2))$ |
| $X \otimes Y$ | $X \otimes Y$ | $d_X(a_1,b_1) + d_Y(a_2,b_2)$ |
| $X + Y$      | $X \sqcup Y$ | $\begin{cases} d_Y(a,b) & \text{if } a,b \in Y \\ \inf & \text{otherwise} \end{cases}$ |
| $X \rightarrow Y$ | $X \rightarrow Y$ | $\sup_{x \in X} d_Y(a(x),b(x))$ |

Fig. 1: Basic constructions on metric spaces.

\[ e \in E ::= x \mid k \in \mathbb{R} \mid e_1 + e_2 \mid (\_ \mid \lambda x.e \mid e_1 e_2 \mid (e_1, e_2) \mid \text{let } (x,y) = e \text{ in } e' \mid \text{let! } x = e \text{ in } e' \mid \text{inl } e \mid \text{inr } e \mid (\text{case } e \text{ of } \text{inl } x, e_l \mid \text{inr } y, e_r) \mid \sigma, \tau ::= 1 \mid \sigma \rightarrow \tau \mid \sigma \otimes \tau \mid \sigma \& \tau \mid \sigma + \tau \mid 1, \sigma \]

\[ r, s \in \mathbb{R}_{\geq 0}, \quad \Gamma, \Delta ::= \emptyset \mid \Gamma, x : r, \sigma \mid \sigma \otimes \delta \mid r \cdot (\Gamma, x : r, \sigma) \triangleq r \cdot \Gamma, x : r, s \sigma \mid (\Gamma, x : r, \sigma) + (\Delta, x : s, \sigma) \triangleq (\Gamma + \Delta), x : r + s \sigma \mid (\Gamma, x : r, \sigma) + \Delta \triangleq (\Gamma + \Delta), x : r + s \sigma \mid (\Delta, x : s, \sigma) \triangleq (\Gamma + \Delta), x : r + s \sigma \]

Fig. 2: Fuzz syntax, types, contexts, and context operations.
These typing rules resemble those of monadic primitives, and we might hope to interpret them using a monad on metric spaces, but the $\infty$ sensitivities pose problems. To give a rough idea of the issue, we could try to combine interpretations for the two sub-derivations in the \textbf{bind} rule as follows:

\[ [\Delta] \otimes [\Gamma] \otimes [e_1] \otimes [e_2] \rightarrow ([\sigma]) \otimes ([\tau]) \rightarrow [\sigma]. \]

In a typical linear monadic calculus, we could just plug in the internal Kleisli lifting in the arrow marked with "\?". Here, however, the types do not match up, because of the $\infty$ factor.

In the remainder of this section, we develop a theory of \textit{parameterized comonadic liftings} to refine the types of the operations of a monad, encompassing the composition principles of differential privacy in Fuzz, as well as new examples discussed later.

\subsection*{A. Monoidal Refinements}

The $\otimes$ monoidal structure of Met, which lies at the core of Fuzz's linear analysis, is derived from the cartesian monoidal structure of Set. The two are related by the forgetful functor $p$, which is strict monoidal. Following Melliès and Zeilberger \cite{24}, we view $p$ as a \textit{refinement} layering metrics on sets.

\begin{equation}
\text{(Met, 1, $\otimes$)} \xrightarrow{p} \text{Set}
\end{equation}

The above monoidal structures are closed, but exponentials only match up for discrete metric spaces $\kappa \cdot Z$—that is, $(p, p)$ is a map of adjunctions of type

\[ (- \otimes (\kappa \cdot Z)) \rightarrow (\kappa \cdot Z \rightarrow -) \rightarrow (\kappa \cdot Z \rightarrow Z \Rightarrow -). \]

The liftings we will introduce are based on a generalization of this situation.

\textbf{Definition 1.} A \textit{weakly closed monoidal refinement of a symmetric monoidal closed category (SMCC)} $(\mathbb{B}, \mathbb{I}, \otimes, \rightarrow)$ \textit{consists of a symmetric monoidal category} $(\mathbb{E}, \mathbb{I}, \otimes)$ \textit{and an adjunction}

\begin{equation}
\text{(\mathbb{E}, \mathbb{I}, \otimes)} \xrightarrow{p} \mathbb{B}
\end{equation}

such that:

1. $p$ is strict symmetric monoidal and faithful;
2. the \textit{unit} of the adjunction is the identity;
3. for each $X \in \mathbb{B}$, $- \otimes LX$ has a right adjoint $X \adj -$;
4. for each $X \in \mathbb{B}$, $(p, p)$ is a map of adjunctions of type

\[ (- \otimes LX \rightarrow X \adj -) \rightarrow (- \otimes X \rightarrow X \rightarrow -). \]
There are many such monoidal refinements. Since $- \times (\infty \cdot X)$ is equal to $- \otimes (\infty \cdot X)$, it also has $\infty \cdot X \to -$ as a right adjoint, which yields another refinement involving $\text{Met}$:

\[
\text{Met}, 1, \times \xrightarrow{p} \text{Set}
\]

We’ll discuss other examples in Section V where we will use monoidal refinements to extend Fuzz with $(\varepsilon, \delta)$-differential privacy.

Given a weakly closed monoidal refinement as in (11), we write ! for the comonad $L \circ p$. For $X, Y \in \mathbb{E}$ and a morphism $f : pX \to pY$ in $\mathbb{B}$, by $f : X \to Y$ we mean that there exists a (necessarily unique) $\hat{f} : X \to Y$ such that $p\hat{f} = f$. Since the unit of the adjunction is the identity, we have 1) $\forall f \in \mathbb{B}(X, pY). f : LX \to Y$ and 2) $\forall f \in \mathbb{E}(LX, Y). p f : LX \to Y$.

**B. Parameterized !-Liftings**

Consider the following simplified instance of the Fuzz bind rule, which samples $x$ from $y \notin \Delta$ and computes $e$:

\[
y : \Delta \otimes \tau \vdash y : \Delta \quad \Delta, x : \infty \otimes \tau \vdash e : \sigma
\]

\[
\Delta, y : \Delta \otimes \tau \vdash \text{bind } x \leftarrow y; e : \sigma
\]

In a set-theoretic semantics that ignores sensitivities, $\circ$ might correspond to the monad $D$ of discrete probability distributions on $\text{Set}$, and we can use the Kleisli lifting of $[e]$ to interpret the entire derivation. We can refine this interpretation with metrics by lifting $\Delta$—that is, finding a functor $D$ such that $p \circ D = D \circ p$—provided that the following implication holds:

\[
f \in \text{Met}([\Delta] \otimes [\tau], D[\sigma]) \implies f^\dagger \in \text{Met}([\Delta] \otimes D[\tau], D[\sigma])
\]

Abstracting this idea leads to $\otimes$-parameterized !-liftings.

**Definition 2.** A $\otimes$-parameterized !-lifting of $T$ along a weakly closed monoidal refinement $p : \mathbb{E} \to \mathbb{B}$ is a mapping $\hat{T} : [\mathbb{E}] \to [\mathbb{E}]$ such that 1) $p\hat{T}X = TpX$, and 2) the parameterized Kleisli lifting of $\hat{T}$ satisfies

\[
f : Z \otimes LX \to \Delta Y \implies f^\dagger : Z \otimes \Delta X \to \Delta Y
\]

Every $\otimes$-parameterized !-lifting $\hat{T}$ satisfies

\[
\eta_{p\mathbb{E}} : !\mathbb{E} \to \hat{T}\mathbb{E},
\]

which allows us to extend $\hat{T}$ to a functor of type $\mathbb{E} \to \mathbb{E}$. We will show how to use parameterized liftings to interpret probabilistic computations in Fuzz, but we take a brief detour to relate liftings to two equivalent notions:

1) $\otimes$-parameterized $L$-relative liftings of $T$
2) $\otimes$-parameterized assignments of $\mathbb{E}$ on $T$

**C. Parameterized $L$-Relative Liftings**

Relative monads [1] generalize ordinary monads by allowing the unit and Kleisli lifting operations to depend on another functor, akin to the role of the comonad ! in (13) and Definition 2. It is natural to wonder if parameterized liftings are related to relative monads, but the two notions are actually distinct. While parameterized liftings were designed to model a bind rule under a context of variables, the Kleisli lifting of a relative monad can only handle bind in empty contexts, which corresponds to setting $Z$ to $I$ in Definition 2.

In the classical case, we can parameterize bind by combining the Kleisli lifting of a monad with a strength. While a notion of strength also exists for relative monads [32], it was introduced with a different purpose in mind, as an analogue of arrows in functional programming languages. We add extra conditions to relative monads so that they are above $T$, and so that the parameterization is taken into account.

**Definition 3.** A $\otimes$-parameterized $L$-relative lifting of $T$ along $p : \mathbb{E} \to \mathbb{B}$ is a mapping $\Delta : [\mathbb{E}] \to [\mathbb{E}]$ such that 1) $p\Delta X = TX$, and 2) the parameterized Kleisli lifting of $\Delta$ satisfies

\[
\eta_X : LX \to \Delta X \text{ and } 2) f : LX \to \Delta Y \implies f^\dagger : \Delta X \to \Delta Y.
\]

From the faithfulness of $p$, these two properties imply that $\Delta$ is a $L$-relative monad.

**D. Parameterized Assignments**

In statistics and probability theory, various notions of metrics between probability distributions have been studied. Well-known examples are max divergence from differential privacy:

\[
\text{MD}_X(\mu, \nu) = \sup_{i \in X} |\ln \mu(i) - \ln \nu(i)|
\]

and statistical distance:

\[
\text{SD}_X(\mu, \nu) = \frac{1}{n} \sum_{i \in X} |\mu(i) - \nu(i)|,
\]

where $X$ is a set and $\mu, \nu \in DX$ are probability distributions. For the max divergence, we take the absolute difference to be $\infty$ if exactly one of $\mu(i)$ or $\nu(i)$ is 0, and 0 if $\mu(i) = \nu(i) = 0$. We regard MD, SD as mappings of type $[\text{Set}] \to [\text{Met}]$.

The metric assigned to $DX$ can interact with the monad structure of $D$ in different ways. One example is the composability condition of Barthe and Olmedo [4]. Let $\Delta : [\text{Set}] \to [\text{Met}]$ be a mapping such that $p\Delta X = DX$. The composability condition on $\Delta$ states that:

\[
d_{\Delta Y}(f^\dagger \mu, g^\dagger \nu) \leq d_{\Delta X}(\mu, \nu) + \sup_{x \in X} d_{\Delta Y}(f(x), g(x)).
\]

This condition is equivalent to the nonexpansivity of the internalized Kleisli lifting introduced in Eq. 7:

\[
k^D_{\Delta Y} : (\infty \cdot X \to \Delta Y) \otimes \Delta X \xrightarrow{\text{inf}} \Delta Y.
\]

We can formulate this property in our setting of monoidal refinements as $\otimes$-parameterized assignments of $\mathbb{E}$ on $T$. 

Definition 4. A $\otimes$-parameterized assignment of $E$ on $T$ is a mapping $\Delta : |B| \to |E|$ such that $p\Delta X = TX$ and the internal Kleisli lifting $k_l T$ of $T$ in Eq. (7) satisfies

$$k_l X, Y : (X \otimes \Delta Y) \otimes \Delta X \to \Delta Y.$$

(14)

In our terms, the results of Barthe and Olmedo [4] show that MD and SD are $\otimes$-parameterized assignments on $D$.

E. Equivalence of Concepts

To relate the three liftings we have introduced, we organize them as preorders. Let $F : A \to |B|$ be a mapping from a class $A$; this $F$ is typically the object part of a functor to $B$. We define the preordered class $\text{Ord}(p, F)$ by

$$\text{Ord}(p, F) = \{ \hat{F} : A \to |E| \mid p\hat{F}X = FX \},$$

$$\hat{F} \leq \tilde{F} \iff \forall a \in A, \text{id}_{\hat{F}a} : \hat{F}a \to \tilde{F}a.$$

Theorem 1. Consider a weakly closed monoidal refinement [11] and a $\otimes$-strong monad $T$ on $B$. Then we have:

$$\text{!Lift}_{\otimes}(T) \equiv \text{RLift}_{\otimes}(T) = \text{Asign}_{\otimes}(T),$$

where $\text{!Lift}_{\otimes}(T)$ is the subpreorder of $\text{Ord}(p, T \circ p)$ consisting of $\otimes$-parameterized $\text{!}$-liftings of $T$, and $\text{RLift}_{\otimes}(T)$ and $\text{Asign}_{\otimes}(T)$ are the subpreorders of $\text{Ord}(p, T)$ consisting of $\otimes$-parameterized $\text{!}$-relative liftings of $T$ and of $\otimes$-parameterized assignments of $E$ on $T$.

IV. Metric Interpretation of Probabilistic Fuzz

With our notion of lifting, we can extend the metric interpretation of Fuzz to handle probabilistic computations and the rules $\otimes$ and $\odot$ by working in the weakly closed monoidal refinement [10] and lifting the distribution monad $D$:

$$\text{Met} \xrightarrow{\otimes} |B| \xrightarrow{\text{Set}} \text{D}.$$

In fact, we may give different interpretations to Fuzz by varying our choice of lifting. Suppose we take a $\otimes$-parameterized $\text{!}$-lifting $D$ of the monad $D$ on $\text{Set}$ to interpret the type $\text{!}\tau$. (As we will soon see, one example of this kind of lifting models the max divergence in the original Fuzz.) We can interpret return by using [13], setting

$$\text{return} e_1 : [\text{!}\Gamma] \to [\text{!}\tau]$$

$$\text{return} e_2 \triangleq \eta_{\text{d}[\tau]} \circ \infty \circ [e].$$

Since $\text{!}$ commutes with $\otimes$, the first factor is well-typed. We interpret bind as a parameterized Kleisli lifting:

$$\text{bind } x \leftarrow e_1 ; e_2 : [\Delta + \Gamma] \to [\text{!}\tau]$$

$$\text{bind } x \leftarrow e_1 ; e_2 \equiv [e_2] \circ ([\Delta] \otimes [e_1]) \circ \delta.$$

Now, suppose we take $\hat{D}$ to be a $\times$-parameterized $\text{!}$-lifting of $D$. From the equality of metric spaces $Z \otimes !X = Z \times !X$ and the nonexpansivity of $\text{id}_{X, pY} : X \otimes Y \to X \times Y$, $\text{!Lift}_{\times}(D)$ is a subpreorder of $\text{!Lift}_{\otimes}(D)$. Thus, we can also interpret return and bind using this kind of lifting. In fact, exploiting the above properties of $\text{Met}$, we can strengthen the typing rules for bind to:

$$\Gamma \vdash e_1 : \text{!}\tau \quad \Gamma, x : \text{!}\tau \vdash e_2 : \text{!}\sigma$$

$$\Gamma \vdash \text{bind } x \leftarrow e_1 ; e_2 : \text{!}\sigma,$$

(15)

where the context $\Gamma$ is shared between $e_1$ and $e_2$, yielding an “additive” variant of the Fuzz bind rule. Observe that the domain of $[e_2]$ is defined as $[\Gamma] \otimes |!\sigma|$, which is equal to $[\Gamma] \times |!\tau|$. Thus we can apply the parameterized Kleisli lifting to $[e_2]$. The interpretation of $\text{bind } x \leftarrow e_1 ; e_2$ is then given as the following composite:

$$\text{bind } x \leftarrow e_1 ; e_2 = [\Gamma] \circ \text{!}[\Gamma] \times \text{!}[\tau] \circ [e_2],$$

A. Concrete Examples of Metric Interpretations

As we have seen, any parameterized lifting gives rise to an interpretation of Fuzz. However, these interpretations are of little interest without compatible primitive distributions, needed to write non-trivial programs. In this section, we analyze two examples of $\otimes$-parameterized $\text{!}$-liftings (we will see examples of $\times$-parameterization in Section VIII after we introduce graded liftings). The first example is the max divergence, originally adopted in Fuzz to model $\varepsilon$-differential privacy. The second example, statistical distance, is a basic metric on distributions that can be seen as measuring $\delta$ in $(0, \frac{1}{\delta})$-differential privacy.

B. Max Divergence

We first consider the max divergence, identified as a $\otimes$-parameterized $\text{!}$-lifting $D = \text{MD}$. For a primitive distribution, we take the real-valued Laplace distribution, a fundamental building block in differential privacy: given a database query, we can make it differentially private by adding Laplace noise to its result—provided that the scale of the noise is calibrated according to the query’s sensitivity, as measured in terms of a suitable metric on databases. Its density function is given by

$$L(\mu, b)(x) \equiv \frac{1}{2b} \exp \left( -\frac{|x - \mu|}{b} \right),$$

where $\mu \in \mathbb{R}$ and $b > 0$ are parameters controlling the mean and the scale of the distribution. The Laplace distribution induces a discrete distribution $L(\mu, b) \in \overline{D}\mathbb{R}$ by truncating the sample up to some fixed precision (breaking ties arbitrarily). This new distribution is compatible with the max divergence: it satisfies the following max divergence bound:

$$\text{MD}_{\mathbb{R}}(L(\mu, b), L(\mu', b)) \leq \frac{|\mu - \mu'|}{b}.$$

In other words, the mapping $\mu \mapsto L(\mu, b)$ is a $b^{-1}$-sensitive function from $\mathbb{R}$ to $\overline{D}\mathbb{R}$ equipped with the max divergence.\footnote{Technically, this follows from $\varepsilon$-privacy of the Laplace mechanism combined with stability of max divergence under post-processing (see, e.g., Dwork and Roth [10]).}

Fuzz exposes the Laplace distribution as a primitive

$$\text{Laplace} : !\mathbb{R} \to \text{!}\mathbb{R},$$
originally called add_noise in Fuzz. We can interpret this as:
\[
[Laplacian(\varepsilon)] \triangleq \lambda x. \tilde{L}(x, 1/\varepsilon).
\]
The full Fuzz language also provides a type \(\text{set}\) of finite sets with elements drawn from \(\tau\), used to model sets of private data ("databases"). This type is equipped with the Hamming distance, which is compatible with the primitives operations on sets, e.g., computing the size, filtering according to a predicate, etc. Extending the interpretation of Fuzz accordingly, a function of type \(\text{Gamma} \vdash \text{set} \tau \to \circ A\) corresponds to an \(\varepsilon\)-sensitive function from databases to distributions over \(R\) equipped with the max divergence. As we discussed in the Introduction, this sensitivity property is equivalent to \(\varepsilon\)-differential privacy with respect to the adjacency relation relating pairs of databases at Hamming distance at most 1, i.e., databases differing in at most one record.

C. Statistical Distance

We next consider the metric interpretation of Fuzz using the statistical distance (identified as a \(\otimes\)-parameterized \(\text{!}-\)lifting): \(\tilde{D} = \text{SD}\). The boolean-valued Bernoulli distribution models a coin-flip with some bias \(p\). The probability mass function is simply \(B(p)(\text{true}) \triangleq p\) and \(B(p)(\text{false}) \triangleq 1 - p\). It is straightforward to check that the Bernoulli distribution satisfies the following statistical distance bound:
\[
\text{SD}_{\text{SD}}(B(p), B(p')) \leq |p - p'|.
\]
Thus, we can introduce a new term Bernoulli, interpreted as
\[
[\text{Bernoulli}] \triangleq \lambda p. B(p),
\]
a non-expansive function from a new unit interval type \([0, 1]\) with the usual Euclidean distance to the statistical distance assignment \(\text{SD}_{\text{SD}}\). By adding a new distribution type \(\circ \text{SD}_{\tau}\) and interpreting it via this assignment \([\circ \text{SD}_{\tau}] \triangleq \text{SD}_{\text{Gamma}}\), the following typing rule is sound:
\[
\Gamma \vdash \text{Bernoulli} : [0, 1] \to \circ \text{SD}_{\text{Gamma}}
\]

V. Relations and \((\varepsilon, \delta)\)-Differential Privacy

The Fuzz language—and our semantics—supports reasoning about function sensitivity properties, such as \((\varepsilon, 0)\)-differential privacy. The main challenge in handling \((\varepsilon, \delta)\)-differential privacy is that it is unclear how to cast the definition as a function sensitivity property. Recall that \((\varepsilon, \delta)\)-differential privacy is a relational property: a query \(f\) satisfies the definition if for all pairs of adjacent input databases \(X_1\) and \(X_2\), we have \(\text{AD}_{\varepsilon}(f(X_1), f(X_2)) \leq \delta\) where \(\text{AD}_{\varepsilon}\) is the skew divergence. Unlike the max divergence MD used for defining \((\varepsilon, 0)\)-privacy, the skew divergence \(\text{AD}_{\varepsilon}\) does not satisfy the triangle inequality, so it is unclear how to express \((\varepsilon, \delta)\)-differential privacy as function sensitivity.

These considerations suggest that \((\varepsilon, \delta)\)-privacy should be expressed in a category of relations instead of metric spaces, but it would be a shame to require a completely different verification technique just to handle a mild generalization of \((\varepsilon, 0)\)-differential privacy. In the following sections, we build up to a graded extension of Fuzz capable of supporting \((\varepsilon, \delta)\)-privacy and other relational properties, by embedding relations into metric spaces.

As a roadmap, in this section we show how to model \((\varepsilon, \delta)\)-differential privacy in a category of relations. Still working with relations, we then show how to capture the composition properties with graded versions of parameterized liftings (Section VI). Then, we consider how to transfer these structures from relations to metric spaces via the path construction (Section VII). Finally, we extend Fuzz with grading to support relational properties (Section VIII).

A. Reflexive Symmetric Relations

We begin by fixing a category of relations to work in. To smooth the eventual transfer to metrics, which are reflexive and symmetric, we work with reflexive and symmetric relations.

**Definition 5.** The category \(\text{RSRel}\) of reflexive, symmetric relations has as objects pairs \(X = ([X], \sim_X)\) of a carrier set \([X]\) and a reflexive, symmetric relation \(\sim_X \subseteq [X] \times [X]\). We will often use the carrier set \([X]\) to refer to \(X\), and write \(\sim\) when the underlying space is clear.

An arrow \(X \to Y\) is a function from \(X\) to \(Y\) that preserves the relation: \(x \sim x' \implies f(x) \sim f(x')\). Composition is simply regular function composition. For \(X, Y \in \text{RSRel}\) and \(f : [X] \to [Y]\), we write \(f : X^{\circ_\text{RSRel}} Y\) to mean \(f \in \text{RSRel}(X, Y)\).

Figure IV summarizes basic constructions on this category, many of which have analogues in metric spaces. Just like in Met, there are two natural products. The first, \(X_1 \times X_2\), requires pairs to be related pointwise; along with the obvious projections, it yields a categorical product on \(\text{RSRel}\). The second, \(X_1 \otimes X_2\), further requires that related pairs be equal in at least one of their components. Coupled with the terminal object 1, these two products yield symmetric monoidal structure on \(\text{RSRel}\), which are also closed: both \(X \otimes -\) and \(X \times -\) have right adjoints.

There is a forgetful functor \(q : \text{RSRel} \to \text{Set}\) defined by \(qX = [X]\) and \(qf = f\). It has a left adjoint \(M : \text{Set} \to \text{RSRel}\) endowing a set \(X\) with the equality relation. Moreover, the forgetful functor \(q : \text{RSRel} \to \text{Set}\) is a weakly closed monoidal
refinement of the CCC Set for these two products:

\[
\begin{align*}
\text{(RSRel, 1, } \otimes \text{)} & \quad \xrightarrow{q} \quad \text{Set} \\
\text{(RSRel, 1, } \times \text{)} & \quad \xrightarrow{q} \quad \text{Set}
\end{align*}
\]

(16) (17)

B. Differential Privacy in RSRel

An \((\varepsilon, \delta)\)-differentially private program can be interpreted as a relation-preserving map. The definition of differential privacy is parameterized by a set \(db\) of databases, along with a binary adjacency relation \(adj \subseteq db \times db\), which we assume to be symmetric and reflexive. Conventional choices for \(db\) include the set of sets of (or multisets, or lists) of records from some universe of possible data, while \(adj\) could relate pairs of databases at symmetric difference at most 1. We recall the original definition here for convenience.

**Definition 6** (Dwork et al. [11]). Let \(\varepsilon, \delta \in [0, \infty)\). A randomized computation \(f : db \to DX\) is \((\varepsilon, \delta)\)-differentially private if for all pairs of adjacent databases \((d, d') \in adj\) and subsets of outputs \(S \subseteq X\), we have:

\[
\begin{align*}
& f(d)(S) \leq \exp(\varepsilon) \cdot f(d')(S) + \delta \\
& f(d')(S) \leq \exp(\varepsilon) \cdot f(d)(S) + \delta.
\end{align*}
\]

We can track the privacy parameters by attaching the following indistinguishability relation to the codomain of a differentially private algorithm. Given \(\varepsilon, \delta \in [0, \infty)\) and a set \(X\), we define \(\text{DPR}(\varepsilon, \delta)(X) \in \text{RSRel}\) by setting

\[
\text{DPR}(\varepsilon, \delta)(X) \triangleq (DX, \{(\mu, \nu) \mid \forall S \subseteq X, (\mu(S) \leq \exp(\varepsilon) \cdot \nu(S) + \delta) \wedge (\nu(S) \leq \exp(\varepsilon) \cdot \mu(S) + \delta)) \}.
\]

**Proposition 1.** A function \(f : db \to DX\) is \((\varepsilon, \delta)\)-differentially private if and only if \(f : (db, \text{adj}) \to \text{DPR}(\varepsilon, \delta)(X)\).

Like \(\varepsilon\)-differential privacy, \((\varepsilon, \delta)\)-differential privacy behaves well under sequential composition.

**Theorem 2** (Dwork et al. [12]). Let \(f : db \to DX\) and \(g : db \times X \to DY\) be such that

1) \(f\) is \((\varepsilon, \delta)\)-differentially private, and
2) \(g(-, x) : db \to DY\) is \((\varepsilon', \delta')\)-differentially private for every \(x \in X\).

Then the composite function \(d \mapsto g^\dagger(d, f(d))\) is \((\varepsilon + \varepsilon', \delta + \delta')\)-differentially private.

VI. Graded !-Liftings

The previous theorem formalizes an important property of differential privacy: the privacy parameters \(\varepsilon\) and \(\delta\) degrade linearly under sequential composition. In Met (and in Fuzz), the \(\varepsilon\) parameter is reflected in the scale of the domain of a non-expansive map, and the composition principle of \((\varepsilon, 0)\)-privacy translates into composition of non-expansive maps. In RSRel, there is no analogous scaling operation. To track the privacy parameters through composition, we will instead grade the codomain of the relation-preserving map.

Technically, we view Theorem 2 as an instance of a more general composition pattern where the indices of a parameterized relational property are combined with an arbitrary monoid operation. This generality will prove useful later on (Section VIII-B), when modeling the composition behavior of other properties. Inspired by graded extension of monads [18, 23, 51], we extend the equivalent concepts introduced in Section III \(\otimes\)-parameterized !-liftings, L-relative liftings and assignments—with monoid grading.

We again assume a general weakly closed monoidal refinement of a SMCC \((\mathbb{B}, I, \otimes, \_\_\_\_)\), as in [11]. We fix a \(\otimes\)-monad over \(\mathbb{B}\), and a preordered monoid \((M, \leq, 1, \cdot)\). The following three concepts are graded extensions of \(\otimes\)-parameterized !-liftings, L-relative liftings and assignments; by letting \(M = 1\) they all reduce to their non-graded counterparts.

**Definition 7.** An \(M\)-graded \(\otimes\)-parameterized !-lifting is a monotone function \(\hat{T} : (M, \leq) \to \text{Ord}(p, T \circ p)\) such that the parameterized Kleisli lifting Eq. (4) of \(T\) satisfies

\[
f : Z \otimes !X \to \hat{T}\alpha Y \implies f^\dagger : Z \otimes \hat{T}\beta X \to \hat{T} \cdot \alpha Y.
\]

Monotonicity of an \(M\)-graded \(\otimes\)-parameterized !-lifting \(\hat{T}\) means that for any monoid elements \(\alpha \leq \beta\) and any object \(X \in \mathbb{E}\), we have \(\hat{T}\alpha X \leq \hat{T}\beta X\). Regarding the unit, \(\eta_\mathbb{X} : !X \to \hat{T}\alpha X\) holds for any \(\alpha \in M\) and \(X \in \mathbb{E}\). From this, each \(\hat{T}\alpha\) extends to an endofunctor over \(\mathbb{E}\).

**Definition 8.** An \(M\)-graded \(\otimes\)-parameterized L-relative lifting is a monotone function \(\Delta : (M, \leq) \to \text{Ord}(p, T)\) such that the parameterized Kleisli lifting Eq. (6) of \(T\) satisfies

\[
f : Z \otimes LX \to \Delta\alpha Y \implies f^\dagger : Z \otimes \Delta\beta X \to \Delta \cdot \alpha Y.
\]

This has the indistinguishability relation as an instance.

**Proposition 2.** Let \(\mathbb{R}_{\geq 0}^+\) be the additive monoid of nonnegative real numbers. Then in the weakly closed monoidal refinement (17), the indistinguishability relation, regarded as a mapping of type \(\text{DPR} : \mathbb{R}_{\geq 0}^+ \times \mathbb{R}_{\geq 0}^+ \to |\text{RSRel}|\), satisfies

\[
\text{DPR} \in \text{RLift}_\times (\mathbb{D}, \mathbb{R}_{\geq 0}^+ \times \mathbb{R}_{\geq 0}^+).
\]

**Definition 9.** An \(M\)-graded \(\otimes\)-parameterized assignment of \(\mathbb{E}\) on \(T\) is a monotone function \(\Delta : (M, \leq) \to \text{Ord}(p, T)\) such that the internalized Kleisli lifting morphism \(\text{kl}^T\) of \(T\) in Eq. (7) satisfies

\[
\text{kl}^T_{X,Y} : (X \cap \Delta\alpha Y) \otimes \Delta\beta X \to \Delta \cdot \alpha Y.
\]

(18)

For instance, in the weakly closed monoidal refinement (10), an \(M\)-graded \(\otimes\)-parameterized assignment \(\Delta\) of Met on \(\mathbb{D}\) consists of a family of metrics \(d_{\Delta\alpha X}^M\) on \(DX\), indexed by \(\alpha \in M\), such that, for any \(X, Y \in \text{Set}, \mu, \nu \in DX, f, g : X \to DY\) and \(\alpha, \beta \in M\), we have

\[
d_{\Delta(\alpha \cdot \beta)Y}(f^\dagger\mu, g^\dagger\nu) \leq d_{\Delta\alpha X}(\mu, \nu) + \sup_x d_{\Delta\beta Y}(f(x), g(x)).
\]
In this case, assignments encode a family of distances on distributions enjoying the sequential composition theorem for statistical divergences proposed by Barthe and Olmedo [20, Theorem 1] (see also Olmedo [26]).

As expected, the graded versions of the three notions are all equivalent.

**Theorem 3.** Consider a weakly closed monoidal refinement $\langle \mathcal{L}, \ominus \rangle$ and $\otimes$-strong monad $\mathcal{T}$ on $\mathcal{B}$. Let $M$ be a preordered monoid. The following equivalences of preorders hold:

\[
\text{!Lift}_{\mathcal{L}}(\mathcal{T}, M) \equiv \text{RLift}_{\mathcal{L}}(\mathcal{T}, M) = \text{Asign}_{\mathcal{L}}(\mathcal{T}, M),
\]

where $\text{!Lift}_{\mathcal{L}}(\mathcal{T}, M)$ is the pointwise preorder of $\mathcal{L}$-parameterized $\text{!}$-liftings of $\mathcal{T}$, and $\text{RLift}_{\mathcal{L}}(\mathcal{T}, M)$ and $\text{Asign}_{\mathcal{L}}(\mathcal{T}, M)$ are the pointwise preorders of $\mathcal{L}$-parameterized $\mathcal{L}$-relative liftings of $\mathcal{T}$ and assignments on $\mathcal{T}$, respectively.

**VII. TRANSFERS OF ASSIGNMENTS**

So far, we are able to model relational properties and their composition behavior in RSRel. In this section, we will show how to carry out this reasoning in a different category, namely $\text{Met}$. We introduce this idea abstractly, then give a concrete example called the path construction for the special case of RSRel and Met. In our framework, this transfer of structure is induced by a morphism between weakly closed monoidal refinements.

**Definition 10.** Consider two weakly closed monoidal refinements of a SMCC $\mathcal{B}$, and a functor $F: \mathcal{E} \to \mathcal{F}$:

\[
\begin{array}{c}
\xymatrix{
(\mathcal{E}, \mathcal{I}, \odot) & (\mathcal{F}, \mathcal{I}, \odot) \\
\mathcal{L} & \mathcal{B} \\
\ar@{->}[r]^F & \\
\ar@{->}[u]^L \ar@{->}[r]^{L'} \ar@{->}[u]^{p} & \ar@{->}[u]^{p'}
}
\end{array}
\]  

(19)

$F$ is a morphism of weakly closed monoidal refinements if

1) $F$ is strict symmetric monoidal,
2) $(\text{Id}_\mathcal{B}, F): (\mathcal{L} \otimes p) \to (\mathcal{L}' \otimes p')$, and
3) $(F, F): (- \odot L X + L \circ \mathcal{H}) \to (- \odot L' X + \mathcal{H})$

for each $X \in \mathcal{B}$.

We write $F: (\mathcal{E}, \mathcal{I}, \odot, L, p) \to (\mathcal{F}, \mathcal{I}, \odot, L', p')$.

**Theorem 4.** If $F: \mathcal{E} \to \mathcal{F}$ is a morphism of weakly closed monoidal refinements, then $F \circ -$ restricts to a monotone function of type $\text{Asign}_{\mathcal{L}}(\mathcal{T}, M) \to \text{Asign}_{\mathcal{L}}(\mathcal{T}, M)$.

**Path Construction**

We represent the differential privacy of computations in metric spaces through the path construction functor $P: \text{RSRel} \to \text{Met}$. Given an object $X \in \text{RSRel}$, we define an $\mathbb{N}$-valued metric on the underlying set by counting the number of times $\sim_X$ must be composed to relate two points. Such metrics are also known as path metrics, and the corresponding metric spaces are known as path-metric spaces.

**Definition 11.** Let $X \in \text{RSRel}$. The path metric is a metric on $X$ defined as follows: $d(x, x')$ is the length $k$ of the shortest path of elements $x_0, \ldots, x_k$ such that $x_0 \sim x, x_k \sim x'$, and $x_i \sim x_{i+1}$ for every $i \in \{0, \ldots, k-1\}$. If no such sequence exists, we set $d(x, x') \equiv \infty$. We write $PX$ for the corresponding metric space. This definition can be extended to a functor $\text{RSRel} \to \text{Met}$ that acts as the identity on morphisms.

Conversely, we can turn any metric space into an object of RSRel by relating elements at distance at most 1.

**Definition 12** (At most one). Given $X \in \text{Met}$, we define $QX \in \text{RSRel}$ by

\[
QX = (X, \{ (x, x') \mid d(x, x') \leq 1 \}), \quad Qf = f.
\]

**Theorem 5.** The functor $P: \text{RSRel} \to \text{Met}$ is fully faithful, and a left adjoint to $Q: \text{Met} \to \text{RSRel}$.

**Theorem 6.** The path metric functor $P: \text{RSRel} \to \text{Met}$ is a morphism of weakly closed monoidal refinements in two ways:

\[
P: (\text{RSRel}, 1, \ominus, M, q) \to (\text{Set}, 1, \otimes, \infty \cdot -, p).
\]

**VIII. METRIC SEMANTICS OF GRADED FUZZ**

Now, we have all the ingredients we need to extend Fuzz. We fix a preordered monoid $(M, \leq, 1, -)$, augment Fuzz with $M$-graded monadic types $\bigcirc_{\alpha} \tau$, adjust the typing rules (9) and (8) to track the grading, and add a grading subsumption rule.

\[
\begin{align*}
\Gamma \vdash e: \bigcirc_{\alpha} \tau & \quad \alpha \leq \beta \\
\Gamma \vdash e: \bigcirc_{\beta} \tau & \\
\infty \cdot \Gamma \vdash \text{return} e: \bigcirc_{\alpha} \tau \\
\Gamma \vdash e_1: \bigcirc_{\alpha} \tau & \quad \Delta, x: \infty \cdot \tau \vdash e_2: \bigcirc_{\beta} \sigma \\
\Delta + \Gamma \vdash \text{bind} & \quad x \leftarrow e_1; e_2: \bigcirc_{\alpha, \beta} \sigma
\end{align*}
\]

Here, $\alpha, \beta$ are elements of $M$. The metric interpretation in Section [V] can be easily extended when we have an $M$-graded $\ominus$-parameterized $\text{!}$-lifting $D$ of the distribution monad $\mathcal{D}$ along $p: \text{Met} \to \text{Set}$, taking $\llbracket \bigcirc_{\alpha} \tau \rrbracket = D\alpha[\tau]$.

If instead we have a $M$-graded $\times$-parameterized $\text{!}$-lifting $D$ of $\mathcal{D}$ along $p: \text{Met} \to \text{Set}$, we can again interpret the monadic type with $\llbracket \bigcirc_{\alpha} \tau \rrbracket = D\alpha[\tau]$. However, we can adopt a alternative, stronger rule for $\text{bind}$ in place of (22):

\[
\begin{align*}
\Gamma \vdash e_1: \bigcirc_{\alpha} \tau & \quad \Gamma, x: \infty \cdot \tau \vdash e_2: \bigcirc_{\beta} \sigma \\
\Gamma \vdash \text{bind} & \quad x \leftarrow e_1; e_2: \bigcirc_{\alpha, \beta} \sigma
\end{align*}
\]

**A. Modeling $(\varepsilon, \delta)$-Differential Privacy in Graded Fuzz**

The technical machinery to model $(\varepsilon, \delta)$-differential privacy is in place: (1) differential privacy can be expressed in RSRel, (2) the sequential composition property gives a graded assignment structure in RSRel, and (3) this structure can be transferred to Met through the path metric. By Proposition 2 and Theorem 3 in the weakly closed monoidal refinement (17) we have $\text{DPR} \in \text{Asign}_\times (\mathcal{D}, \mathbb{R}^+_{\geq 0} \times \mathbb{R}^+_{\geq 0})$. From Theorem 4, the path construction maps $\text{DPR}$ to a graded $\times$-parameterized assignment of Met on $\mathcal{D}$:

\[
P \circ \text{DPR} \in \text{Asign}_\times (\mathcal{D}, \mathbb{R}^+_{\geq 0} \times \mathbb{R}^+_{\geq 0})
\]
in the weakly closed monoidal refinement [12]. Below we identify \( P \circ \text{DPR} \) as a \( \mathbb{R}^+ \times \mathbb{R}^+ \)-graded \( \times \)-parameterized \(!\)-lifting of \( D \). By posing
\[
\left[ (\varepsilon, \delta)^\tau \right] \triangleq P(\text{DPR}(\varepsilon, \delta)([\tau])),
\]
the following specializations of \((20), (21), \) and \((23)\) are sound:
\[
\begin{align*}
\Gamma \vdash e : \left[ (\varepsilon, \delta)^\tau \right] & \quad \varepsilon \leq \varepsilon' \quad \delta \leq \delta' \\
\Gamma \vdash e : \left[ (\varepsilon', \delta)^\tau \right] & \\
\Gamma \vdash e : \tau \\
\infty \cdot \Gamma \vdash \text{return } e : \left[ (\varepsilon, \delta)^\tau \right]
\end{align*}
\]

Much like we did for the standard \(!\)-liftings in Section IV-A, we can add primitive distributions to the system. The basic building block for \((\varepsilon, \delta)\)-differential privacy is the real-valued Gaussian or normal distribution. Given a mean \( \mu \in \mathbb{R} \) and a variance \( \sigma \in \mathbb{R} \), this distribution has density function
\[
N(\mu, \sigma)(x) \triangleq \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( \frac{(x - \mu)^2}{2\sigma^2} \right).
\]

A result from the theory of differential privacy states that if we have a numeric query \( q : db \to \mathbb{R} \) whose results differ by at most 1 on adjacent databases, then adding Gaussian noise to any fixed precision while preserving privacy; call this truncated distribution \( \tilde{N}(\mu, \sigma) \). The function \( \lambda x. \tilde{N}(x, \sigma) : \mathbb{R} \to DR \) can then be interpreted in RSRel:
\[
\lambda x. \tilde{N}(x, \sigma) : Q\mathbb{R} \xrightarrow{\equiv} \text{DPR}(\varepsilon, \delta)(\mathbb{R})
\]

recalling that \( Q\mathbb{R} \) relates pairs of real numbers that are at most 1 apart under the standard Euclidean distance. The path construction gives a non-expansive map between path-metric spaces:
\[
\lambda x. \tilde{N}(x, s(\varepsilon, \delta)) : PQ\mathbb{R} \xrightarrow{\equiv} \text{DPR}(\varepsilon, \delta)(\mathbb{R})
\]

\( PQ\mathbb{R} \) is a metric space over the real numbers, but with the metric rounded up to the nearest integer; we introduce a corresponding Fuzz type \([\mathbb{R}]\). Then, we can introduce a new Fuzz term \( \text{Gaussian}[\varepsilon, \delta] \) for \( \varepsilon, \delta > 0 \) with the interpretation
\[
\left[ \text{Gaussian}[\varepsilon, \delta] \right] \triangleq \lambda x. N(x, s(\varepsilon, \delta))
\]

and the following sound typing rule:
\[
\begin{align*}
\Gamma \vdash \text{Gaussian}[\varepsilon, \delta] : [\mathbb{R}] & \to [\varepsilon, \delta] \mathbb{R} \\
\end{align*}
\]

We can now capture \((\varepsilon, \delta)\)-privacy via Fuzz types. For instance, suppose the following typing judgment is derivable:
\[
\vdash e : db \to O(\varepsilon, \delta) \tau
\]

where we interpret the type \( db \) as the path-metric space \( P(db, \text{adj}) \). Note that if \( db = \text{set} \sigma \) is the space of sets of \( \sigma \) and we take the Hamming distance as the metric \( d_{DB} \) on this space (as we did in previous examples), then \( (db, d_{DB}) \) is automatically a path metric space for the relation relating any two databases at Hamming distance at most 1. We have a non-expansive map
\[
\left[ e \right] : P(db, \text{adj}) \xrightarrow{\equiv} \text{DPR}(\varepsilon, \delta)([\tau]).
\]

Since the path functor \( P \) is full and faithful (Theorem [5]), we have a relation-preserving map
\[
\left[ e \right] : (db, \text{adj}) \xrightarrow{\equiv} \text{DPR}(\varepsilon, \delta)([\tau])
\]
in RSRel. By Proposition [1] this map satisfies \((\varepsilon, \delta)\)-privacy.

**Typing \((\varepsilon, \delta)\)-Differential Privacy:** We give two examples to demonstrate the type system. Consider the type \( db \to \mathbb{R} \), typically used to model 1-sensitive queries. Applying \( PQ \), we find that we can model 1-sensitive queries with the type \( [db] \to [\mathbb{R}] \). Now \( [db] \) rounds up the metric on \( db \) to the nearest integer, but since \( db \) is already a path metric space, \( [db] \) and \( db \) have the same denotations. Thus, 1-sensitive queries can be interpreted as type \( db \to [\mathbb{R}] \).

Let \( q_1 \) and \( q_2 \) be 1-sensitive queries of type \( db \to [\mathbb{R}] \), and consider the program \( \text{two}_{\cdot\cdot}q \):
\[
\begin{align*}
\lambda db. \text{bind } a_1 & \leftarrow \text{Gaussian}[\varepsilon, \delta](q_1(db)) \\
\text{bind } a_2 & \leftarrow \text{Gaussian}[\varepsilon, \delta](q_2(db)) \\
\text{return } & (a_1 + a_2)
\end{align*}
\]

This program evaluates the first query \( q_1 \) and adds Gaussian noise to the answer, evaluates the second query \( q_2 \) and adds more Gaussian noise, and finally returns the sum of the two noisy answers. By applying the typing rules for the Gaussian distribution, along with the graded monadic rules, we can derive the following type:
\[
\vdash \text{two}_{\cdot\cdot}q : db \to O(2\varepsilon, 2\delta) \mathbb{R}
\]

Though the database \( db \) is used twice in the program, it has sensitivity 1 in the final type. This accounting follows from the bind rule, which allows the contexts of its premises to be shared. The fact that the database is used twice is instead tracked through the grading on the codomain, where the privacy parameters \((\varepsilon, \delta)\) sum up. By soundness of the type system, \( \text{two}_{\cdot\cdot}q \) is \((2\varepsilon, 2\delta)\)-differentially private.

Other types can capture variants of differential privacy. For example, suppose added the queries first and noised just once:
\[
\begin{align*}
\lambda db. \text{let } s & \leftarrow q_1(db) + q_2(db); \text{Gaussian}[\varepsilon, \delta](s)
\end{align*}
\]

We use standard syntactic sugar for let bindings; call this program \( \text{two}_{\cdot\cdot}q' \). We can derive the following type:
\[
\vdash \text{two}_{\cdot\cdot}q' : l_2 db \to O(\varepsilon, \delta) \mathbb{R}
\]

This type is not equivalent to the type for \( \text{two}_{\cdot\cdot}q \). However, we can still interpret it in terms of differential privacy. In general, consider the following judgment:
\[
\vdash e : l_2 db \to O(\varepsilon, \delta) \tau
\]
By soundness, the interpretation is non-expansive:
\[ [e] : 2 \cdot P(db, adj) \xrightarrow{n} P(DPR(\varepsilon, \delta)([[\tau]])). \]

Though the scaling of a path metric is not necessarily a path metric, we can still give a meaning to this judgment. For any two input databases \( (x, x') \in adj \), the distance between \([e]x\) and \([e]x'\) in \( P(DPR(\varepsilon, \delta)([[\tau]])) \) is at most 2. Suppose that the distance is exactly 2 (smaller distances yield stronger privacy bounds). Then, there must exist an intermediate distribution \( y \in D[[\tau]] \) such that
\[ [e]x \sim_{DPR(\varepsilon, \delta)([[\tau]])} y \sim_{DPR(\varepsilon, \delta)([[\tau]])} [e]x'. \]

Unfolding definitions, a small calculation shows that the output distributions must be related by
\[ [e]x \sim_{DPR(2\varepsilon, (1 + \exp(\varepsilon))\delta)([[\tau]])} [e]x'. \]

Hence we have a relation-preserving map
\[ [e] : (db, adj) \xrightarrow{n} DPR(2\varepsilon, (1 + \exp(\varepsilon))\delta)([[\tau]]) \]
and by Proposition 1, the map \([e]—and our program two—is satisfy \((2\varepsilon, (1 + \exp(\varepsilon))\delta)\)-differential privacy.

Internalizing Group Privacy: Differential privacy compares the output distributions when a program is run on two input databases at distance 1. These guarantee can sometimes be extended to cover pairs of inputs at distance \( k \), so-called called group privacy guarantees. Roughly speaking, an algorithm is said to be \((\varepsilon(k), \delta(k))\)-differentially private for groups of size \( k \) if for any two inputs at distance \( k \), the output distributions satisfy the divergence bound for \((\varepsilon(k), \delta(k))\)-differential privacy.

For standard \( \varepsilon\)-differential privacy, group privacy is straightforward: an \( \varepsilon\)-private program is automatically \( k \cdot \varepsilon\)-private for groups of size \( k \). This clean, linear scaling of the privacy parameters is the fundamental reason why the original Fuzz language fits \( \varepsilon\)-differential privacy. In fact, group privacy with linear scaling is arguably a more accurate description of the properties captured by Fuzz—it just so happens that this seemingly stronger property coincides with \( \varepsilon\)-privacy.

In general, however, group privacy guarantees are not so clean. For \((\varepsilon, \delta)\)-differential privacy, the parameters also degrade when inputs are farther apart, but this degradation is not linear. In a sense, our perspective generalizes \((\varepsilon, \delta)\)-differential privacy to group privacy, a notion that better matches the linear nature of Fuzz. For instance, the type \( \tau_{db} \xrightarrow{n} \bigcirc (\varepsilon, \delta) \mathbb{R} \) in the last example represents the group privacy guarantee when \((\varepsilon, \delta)\)-private algorithms are applied to groups of size 2. While we can explicitly compute the corresponding privacy parameters, this unfolded form seems unwieldy to accommodate in Fuzz.

Handling Advanced Composition: The typing rules we have seen so far capture two aspects of \((\varepsilon, \delta)\)-differential privacy: primitives such as the Gaussian mechanism and sequential composition via the bind rule. In practice, \((\varepsilon, \delta)\)-privacy is often needed to apply the advanced composition theorem [13]. While standard composition simply adds up the \((\varepsilon, \delta)\) parameters, the advanced version allows trading off the growth of \( \varepsilon \) with the growth of \( \delta \). By picking a \( \delta \) that is slightly larger than the sum of the individual \( \delta \) parameters, advanced composition ensures a significantly slower growth in \( \varepsilon \).

Unfortunately, the indices in advanced composition do not combine cleanly—they are not given by a monoid operation—and so advanced composition is typically applied to blocks of \( n \) composed programs rather than two programs at a time. As a result, it is not clear how to internalize this principle for all \( n \) as a typing rule in Fuzz, even with our extensions. However, we can internalize limited versions through higher-order primitives that perform a constant number of iterations—say, for composing exactly \( n = 3 \) functions.

B. Modeling Other Divergences

The \((\varepsilon, \delta)\)-differential privacy property belongs to a broader class of probabilistic relational properties: two related inputs lead to two output distributions that are a bounded distance apart, as measured by some divergence on distributions. By varying the divergence, these properties can capture different notions of probabilistic sensitivity. In Appendix I, we show how to internalize other composable divergences [4].

IX. HANDLING NON-TERMINATION

Most of our development would readily generalize to the full Fuzz language, which includes general recursive types (and hence also non-terminating expressions). Azevedo de Amorim et al. [3] modeled the deterministic fragment of Fuzz with metric CPOs—ordered metric spaces that support sound definitions of non-expansive functions by general recursion. We can extend their work to encompass probabilistic features by endowing the Jones-Plotkin probabilistic powerdomain [17] with metrics, much like was done in Section III. Briefly, the order on the probabilistic powerdomain \( E(X) \) is given by
\[ \mu \sqsubseteq \nu \iff \forall U : \mu(U) \leq \nu(U) \]
where \( U \) ranges over the Scott-open sets of the CPO \( X \). The statistical distance and max divergence are all defined continuously and pointwise from the probabilities \( \mu(U) \), and they satisfy the compatibility conditions required for metric CPOs. The proofs that these distances form liftings of the probabilistic powerdomain generalize by replacing sums over countable sets with integrals.

While the simple distances pose no major problem, the same cannot be said about the path metric construction. A natural attempt to generalize relations to CPOs is to require admissibility: relations should be closed under limits of chains to support recursive function definitions. Unfortunately, the notion of admissibility is not well-behaved with respect to relation composition: the composite of two admissible relations may not be admissible. This is an obstacle when defining the path metric, since a path of length \( n \) in the graph induced by the relation is simply a pair of points related by its \( n \)-fold composition. Roughly, because admissible relations fail to compose, the path construction does not yield metric CPOs in general, and does not form a morphism of refinements.
The situation can be partially remedied by categorical arguments. Both the category of reflexive, symmetric admissible relations and the category of metric CPOs can be characterized as fibrations over the category of CPOs with suitably complete fibers. This allows us to define an analog of the path construction abstractly as the left adjoint of the $Q$ functor of Section VII which builds the “at most one” relation. However, this construction does not inherit the pleasant properties of the path metric on sets and functions. More precisely, the proof of Lemma shown in the Appendix, which is instrumental for showing soundness of the bind rule for $(\varepsilon, \delta)$-differential privacy, does not carry over.

X. RELATED WORK

Language-Based Techniques for Differential Privacy: Owing to its clean composition properties, differential privacy has been a fruitful target for formal verification. Our results follow a line of research on the Fuzz programming language, a linear type system for differential privacy conceived by Reed and Pierce [28] and subsequently extended with sized types [14] and algorithmic typechecking [2]. Adaptive Fuzz [34] is a recent extension of the language that features an outer layer for constructing and manipulating Fuzz programs—for instance, using program transformations and partial evaluation—before calling the typechecker and running the query. By tracking the overall privacy level externally, and not in the Fuzz type system, Adaptive Fuzz supports many composition principles for $(\varepsilon, \delta)$-differential privacy, such as the advanced composition theorem and adaptive variants called privacy filters. Our work expresses $(\varepsilon, \delta)$-differential privacy and basic composition in the type system, rather than using a two-level design.

The only type system we are aware of that can capture $(\varepsilon, \delta)$-differential privacy is the HOARe\textsuperscript{2} language proposed by Barthe et al. [5] and later extended to handle other $f$-divergence properties [6]. With a relational refinement type system, HOARe\textsuperscript{2} can naturally express properties like $(\varepsilon, \delta)$-differential privacy. However, this style of reasoning has its own drawbacks. For instance, the input type of a differentially private algorithm typically requires related inputs to be at most a fixed distance apart—private functions can’t be applied when the inputs are farther apart. In contrast, our sensitivity-based approach can reason about differentially private functions applied to inputs that are at any distance apart.

Recently, several variations of differential privacy have been proposed for designing private mechanisms with better accuracy. These variations are motivated by properties of continuous distributions and can be characterized by means of a lifting based on spans [30]. We hope to adapt our approach to reason about these notions of privacy over discrete distributions, as we have for the Laplace mechanism. An interesting problem for future work would be to extend our semantics to continuous distributions, perhaps by leveraging recent advances in probabilistic semantics [53].

Verification of Probabilistic Relational Properties: The last decade has seen significant developments in verification for probabilistic relational properties other than differential privacy. Our work is most closely related to techniques for reasoning about $f$-divergences [4, 26]. Recent work by Barthe et al. [7] develops a program logic for reasoning about a probabilistic notion of sensitivity based on couplings and the Kantorovich metric. Barthe et al. [7] identified path metrics as a useful concept for formal verification, in connection with the path coupling proof technique. Our work uses path metrics for a different purpose: interpreting relational properties as function sensitivity.

As an aside, the path adjunction can be defined via a general construction on enriched categories [19]. Given a monoidal category $\mathcal{V}$ with coproducts, the forgetful functor $\mathcal{V} \to \text{Cat}$ has a left adjoint that generalizes the construction of the free category on a graph. When $\mathcal{V} = ([0, \infty], +, 0)$, a $\mathcal{V}$-category is a metric space that does not satisfy the symmetry axiom, and a $\mathcal{V}$-graph is simply a weighted graph. The at-most-one relation of Definition 12 yields a further adjunction between $\mathcal{V}$-graphs and the category of reflexive relations. The composition of these two adjunctions, when restricted to true symmetric metric spaces and symmetric relations, is precisely the path adjunction.

Categorical Semantics for Metrics and Probabilities: Our constructions build on a rich literature in categorical semantics for metric spaces and probability theory. Most directly, Azevedo de Amorim et al. [3] modeled the non-probabilistic fragment of the Fuzz language using the concept of a metric CPO; we have adapted their model of the terminating fragment of the language to handle probabilistic sampling. Sato [29] introduced a graded relational lifting of the Giry monad for the semantics of relational Hoare logic for the verification of $(\varepsilon, \delta)$-differential privacy with continuous distributions. Our graded liftings are similar to his graded liftings, but the precise relationship is not yet clear. Reasoning about metric properties remains an active area of research [22, 27].

XI. CONCLUSION AND FUTURE DIRECTIONS

We have extended the Fuzz programming language to handle probabilistic relational properties beyond $\varepsilon$-differential privacy, most notably $(\varepsilon, \delta)$-differential privacy and other properties based on composable $f$-divergences. We introduced the categorical notion of parameterized lifting to reason about $(\varepsilon, \delta)$-differential privacy in a compositional way. Finally, we showed how to cast relational properties as sensitivity properties through a path metric construction.

There are several natural directions for future work. Most concretely, the interaction between the path metric construction and non-termination remains poorly understood. While the differential privacy literature generally does not consider non-terminating computations, extending our results to CPOs would complete the picture. More speculatively, it could be interesting to understand the path construction through calculi that include adjunctions as type constructors [20]. This perspective could help smooth the interface between relational and metric reasoning.
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A. Proof of Theorem 1 and Theorem 3

We show the non-graded version (Theorem 1); the graded version (Theorem 3) is similarly provable.

Let $\Delta \in \mathbf{RLift}_\otimes(\mathbb{T})$. We define $\hat{T}_\Delta X = \Delta p X$. We show that it is a $\otimes$-parameterized $!$-lifting, that is,

$$f : X \otimes L p Y \rightarrow \Delta p Z \implies f^\dagger : X \otimes \Delta p Y \rightarrow \Delta p Z.$$

This is true by the assumption.

Conversely, let $\hat{T} \in \mathbf{!Lift}_\otimes(\mathbb{T})$. We define $\Delta_p \hat{T} = \hat{T} \mathbf{LI}$. We show that it is a $\otimes$-parameterized $L$-relative lifting, that is,

$$f : X \otimes \mathbf{LI} \rightarrow \hat{T} \mathbf{LI} \implies f^\dagger : X \otimes \hat{T} \mathbf{LI} \rightarrow \hat{T} \mathbf{LI}.$$

This is also true from $\mathbf{LI} = \mathbf{LI}$ and the assumption.

We show that the above processes are equivalence of preorders. We first have

$$\Delta_p \hat{T} = \hat{T} \mathbf{LI} = \Delta p \mathbf{LI} = \Delta I.$$

Next, we have $\hat{T}_\Delta X = \Delta_p \hat{T} p X = \hat{T} \mathbf{LI} p X \leq \hat{T} X$. We show the converse $\hat{T} X \leq \hat{T} \mathbf{LI} p X$. Since $\hat{T}$ is a $!$-lifting, we have $\eta_{\mathbf{LI} p X} : \mathbf{LI} p X \rightarrow \hat{T} \mathbf{LI} p X$. From $p \mathbf{L} = \mathbf{Id}$, we have $\eta_{\mathbf{LI} X} : \mathbf{LI} X \rightarrow \hat{T} \mathbf{LI} X$. As $\hat{T}$ is a $\otimes$-parameterized $!$-lifting, we obtain

$$\eta_{\mathbf{LI} X} \dagger = \mathbf{Id}_{\mathbf{LI} p X} : \mathbf{LI} X \rightarrow \hat{T} \mathbf{LI} p X,$$

that is, $\hat{T} X \leq \hat{T} \mathbf{LI} X$. Therefore $\hat{T} \simeq \hat{T}_\Delta$ holds in the preorder $\mathbf{LI} \otimes (\mathbb{T})$.

This finishes the equivalence of $\mathbf{LI} \otimes (\mathbb{T})$ and $\mathbf{RLift}_\otimes(\mathbb{T})$.

Next, suppose that $\Delta \in \mathbf{Asign}_\otimes(\mathbb{T})$. We infer:

$$\begin{align*}
\lambda(f) : X &\rightarrow I \dagger \Delta J \\
\lambda(f) \otimes \mathbf{Id}_{TJ} : X \otimes \Delta I &\rightarrow (I \dagger \Delta J) \otimes \Delta I \\
ev^\dagger \circ (\lambda(f) \otimes \mathbf{Id}_{TJ}) : X \otimes \Delta I &\rightarrow \Delta J
\end{align*}$$

At the last step, we use the $\otimes$-strong assignment. Now

$$ev^\dagger \circ (\lambda(f) \otimes \mathbf{Id}_{TJ}) = \mu \circ T(ev) \circ \theta \circ (\lambda(f) \otimes T\mathbf{Id}_J) = \mu \circ T(ev \circ \lambda(f) \otimes \mathbf{Id}_J) \circ \theta = \mu \circ T(f) \circ \theta = f^\dagger.$$

Therefore $\Delta \in \mathbf{RLift}_\otimes(\mathbb{T})$.

Conversely, suppose that $\Delta \in \mathbf{RLift}_\otimes(\mathbb{T})$. Since $p$ is the map of adjunction from $- \otimes \mathbf{LI} \dashv I \dagger \dashv - \dagger \otimes \mathbf{LI} \dashv I \dashv -$, we have

$$ev : (I \dagger \Delta J) \otimes \mathbf{LI} \rightarrow \Delta J.$$

Therefore we obtain

$$ev^\dagger : (I \dagger \Delta J) \otimes \Delta I \rightarrow \Delta J,$$

that is, $\Delta \in \mathbf{Asign}_\otimes(\mathbb{T})$. We finished the proof of the equality $\mathbf{RLift}_\otimes(\mathbb{T}) = \mathbf{Asign}_\otimes(\mathbb{T})$.

B. Proof of Proposition 2

Before the proof, we introduce an auxiliary concept abstracting the composition properties of divergences studied in differential privacy as Theorem 2. This concept is valid when the symmetric monoidal structure $(\mathbb{I}, \otimes)$ assumed on $\mathbb{E}$ in (11) is cartesian.

**Definition 13.** Assume that in (11) the symmetric monoidal structure on $\mathbb{E}$ is cartesian (hence we denote it by $(\mathbb{I}, \times)$). An $M$-graded sequentially composable family of $\mathbb{E}$-objects above $\mathbb{T}$ is a monotone function $\Delta : (M, \leq) \rightarrow \mathbb{Ord}(p, T)$ such that for any $f : Z \rightarrow \Delta p X$ and $g : Z \times LX \rightarrow \Delta p Y$, we have

$$g^\dagger \circ (\mathbf{id}_{pZ}, f) : Z \rightarrow \Delta(\alpha \cdot \beta)Y.$$

the subpreorder of $[(M, \leq), \mathbb{Ord}(p, T)]$ consisting of $M$-graded sequentially composable families of $\mathbb{E}$-objects above $\mathbb{T}$ is denoted by $\mathbf{Comp}(\mathbb{T}, M)$. 

```
Let us see how this definition expands in the weakly closed monoidal refinement \(^{(17)}\), and a monad \(\mathcal{T}\) on Set:

\[
(RSRel, 1, \times) \xrightarrow{q} M \xrightarrow{\delta} \text{Set} \xrightarrow{\delta} \mathcal{T}
\]

An \(M\)-graded sequentially composable family of \(RSRel\)-objects assigns to each set \(TX\) and \(\alpha, \beta \in M\), a reflexive, symmetric relation \(\sim^m_X\) satisfying:

\[
(\forall z \sim z' : f(z) \sim^m g(z')) \\
\land (\forall z \sim z' : g(z, x) \sim^m g(z', x)) \\
\implies \forall z \sim z' : g^+(z, f(z)) \sim^m g^+(z', f(z'))
\]

**Theorem 7. Asign\(_X\)(\(T, M\)) = Comp(\(T, M\)).**

**Proof.** Let \(\Delta \in \text{Comp}(\(T, M\)).\) That is, the following implication holds:

\[
f : Z \to \Delta \alpha X \land g : Z \times LX \to \Delta nY
\]

\[
\implies g^+ \circ \langle \text{id}_{PZ}, f \rangle : Z \to \Delta (m \cdot n)Y
\]

We instantiate the premise of the sequential composition condition with the following data \((X, Y, \text{left unchanged})\):

\[
Z = (X \triangleleft \Delta \beta Y) \times \Delta \alpha X
\]

\[
f = \pi_1 : Z \to \Delta \alpha X
\]

\[
g = \lambda(\text{ev} \circ \langle \pi_1, \pi_2 \rangle) : Z \times LX \to \Delta \beta Y.
\]

We then obtain

\[
g^+ \circ \langle \text{id}, f \rangle : (X \triangleleft \Delta \beta Y) \times \Delta \alpha X \to \Delta(\alpha \cdot \beta)Y;
\]

and we have \(kl = g^+ \circ \langle \text{id}, f \rangle\). Therefore \(\Delta \in \text{Asign}_X(\(T, M\)).\)

Conversely, suppose that \(\Delta \in \text{Asign}_X(\(T, M\)).\) We have the following construction of morphisms in \(\mathbb{E}:\)

\[
g : Z \times LX \to \Delta \beta Y
\]

\[
f : Z \to \Delta \alpha X
\]

\[
(\lambda(g), f) : Z \to (X \triangleleft \Delta \beta Y) \times \Delta \alpha X
\]

\[
g^+ \circ \langle \text{id}, f \rangle = kl \circ (\lambda(g), f) : Z \to \Delta(\alpha \cdot \beta)Y
\]

Therefore \(\Delta \in \text{Comp}(\(T, M\)).\)

As a result, the sequential composability (Theorem \(^{(2)}\)) of differential privacy is equivalent to that the indistinguishability relation is a graded sequentially composable family:

\[
\text{DPR} \in \text{Comp}(\mathbb{D}, \mathbb{R}^+_0 \times \mathbb{R}^+_0) = \text{Asign}_X(\mathbb{D}, \mathbb{R}^+_0 \times \mathbb{R}^+_0)
\]

where \(\mathbb{R}^+_0\) is the additive monoid of nonnegative real numbers.

**C. Proof of Proposition \(^{(4)}\)**

Let us show that the first term is contained in the second. Suppose we have a morphism \(f : PX \to Y.\) Take \(x, x'\) in \(X\) such that \(x \sim x'\). We have \(d(x, x') \leq 1\) in \(PX\), thus \(d(f(x), f(x')) \leq 1\) by non-expansiveness—that is, \(f(x) \sim f(x')\) in \(QY\) by definition. This shows that \(f\) is also a morphism in \(X \to QY.\)

To conclude, we just need to show the reverse inclusion. Suppose we have a morphism \(f : X \to QY.\) Showing that \(f\) is a non-expansive function \(PX \to Y\) is equivalent to showing that, given a path \(x_0 \sim \cdots \sim x_k\) of length \(k\) in \(X, d(x_0, x_k) \leq k.\) Since \(f : X \to QY,\) we know that for every \(i < k\) the relation \(f(x_i) \sim_QY f(x_{i+1})\) holds; by definition, this means that \(d(f(x_i), f(x_{i+1})) \leq 1.\) Applying the triangle inequality \(k - 1\) times, we conclude \(d(f(x_0), f(x_k)) \leq k.\)

**D. Proof of Lemma \(^{(2)}\)**

As the underlying sets are equal, it suffices to show that the metrics are equal. Let \(d_P\) be the metric associated with \(P(\prod_{i} X_i),\) and let \(d_P^{\prod}\) be the metric associated with \(\prod_{i} PX_i.\) Since \(P\) is a functor, we know that the identity function is a morphism in \(P(\prod_{i} X_i) \to \prod_{i} PX_i;\) that is, \(d_P^{\prod}(x, x') \leq d_P(x, x')\) for all families \(x, x' \in \prod_{i} X_i.\) To conclude, we just need to show the opposite inequality. If \(d_P(\prod_{i} x_i) = \infty,\) we are done; otherwise, there exists some \(k \in \mathbb{N}\) such that \(d_P(\prod_{i} x_i, x'_i) \leq k\) for every \(i \in I.\) Since all the relations associated with the \(X_i\) are reflexive, we know that for every \(i\) there must exist a path of related elements \(x_i = x_{i,0} \sim \cdots \sim x_{i,k} = x'_i\) of length exactly \(k,\) by padding one of the ends with reflexivity edges. Therefore, there exists a path of related families \(\prod_{i \in I} x_{i,0} \sim \cdots \sim x_{i,k} = \prod_{i \in I} x'_i\) of length \(k\) in \(\prod_{i} X_i.\) By the definition of the path metric, this means that \(d_P(\prod_{i} x_i, x'_i) \leq k,\) and thus \(d_P(x, x') \leq d_P(\prod_{i} x_i, x'_i),\) as we wanted to show.
E. Proof of Lemma 3

Once again, it suffices to show that both metrics are equal.
For all pairs \((x, y)\) and \((x', y')\) in \(X \times Y\), there are paths between \(x\) and \(x'\) and between \(y\) and \(y'\) whose summed length is at most \(k\) if and only if there is a path between \((x, y)\) and \((x', y')\) of length at most \(k\) in \(X \otimes Y\). For the “only if” direction, if there are such paths \((x_i)_{i \leq k_x}\) and \((y_i)_{i \leq k_y}\), then the sequence
\[(x_0, y_0), (x_1, y_0), \ldots, (x_{k_x}, y_0), (x_{k_x}, y_1), \ldots, (x_{k_x}, y_{k_y})\]
is a path from \((x, y)\) and \((x', y')\) in \(X \otimes Y\). Conversely, suppose that we have a path from \((x, y)\) to \((x', y')\) of length at most \(k\) in \(X \otimes Y\). We can show by induction on the length of the path that there are paths from \(x\) to \(x'\) and from \(y\) and \(y'\) with total length at most \(k\). The base case is trivial. If there is a hop, by the definition of the relation for \(X \otimes Y\), this hop only adds one unit to the length of the path on \(X\) or to the path on \(Y\), which allows us to conclude.

F. Proof of Theorem 4

As a basic sanity check, converting the path metric of a relation back into a relation with the at-most-one operation yields the original relation.

Proposition 3. For any \(X \in RSRel\), \(QP X = X\).

Composing in the opposite order \(PQ X\) does not yield \(X\) in general. Consider, for instance, a metric space \(X = (\{x, y\}, d)\) where \(d(x, y) = 2\). However, \(P\) and \(Q\) do form an adjoint pair.

Proposition 4. For any \(X \in RSRel\) and \(Y \in Met\), we have \(\text{Met}(P X, Y) = RSRel(X, Q Y)\). In particular, the functors \(P\) and \(Q\) form an adjoint pair \(P \dashv Q\), whose unit is identity morphism.

Since the unit is identity, we conclude that \(P\) is full and faithful. In the current setting, this fact can be phrased as follows:

Corollary 1. Let \(X\) and \(Y\) be objects of \(RSRel\). Then \(f: X \rightarrow Y\) if and only if \(f : PX \xrightarrow{\text{eq}} PY\).

This result is the cornerstone of our strategy: relational properties—interpreted as relation-preserving maps in \(RSRel\)—can be translated precisely to non-expansive maps in \(Met\) using the path construction. In other words, a function preserves a relation if and only if it is non-expansive with respect to the corresponding path metrics.

G. Properties of the Path Adjunction

The adjunction \(P \dashv Q\) preserves much—but not all—of the structure in \(RSRel\) and \(Met\). We summarize these properties below. First, discrete spaces are preserved.

Lemma 1. For any set \(X\), \(P(\infty \cdot X) = \infty \cdot X \in Met\), and \(Q(\infty \cdot X) = \infty \cdot X \in RSRel\).

The path functor \(P : RSRel \rightarrow Met\) strictly preserves all products (including infinite ones), and symmetric monoidal structure.

Lemma 2. Let \((X_i)_{i \in I}\) be a family of objects of \(RSRel\). Then \(P(\prod_i X_i) = \prod_i PX_i\). The metric on a general product of metric spaces is defined by taking the supremum of all the metrics.

Lemma 3. For every \(X, Y \in RSRel\), \(P(X \otimes Y) = PX \otimes PY\).

H. Proof of Theorem 6

Let \(\Delta : (M, \leq) \rightarrow \text{Ord}(p, T)\) be an \(M\)-graded \(\otimes\)-parameterized assignment of \(\mathbb{E}\) on \(T\). Since \(p \circ F = p'\), \(F \circ \Delta\) is a monotone function of type \((M, \leq) \rightarrow \text{Ord}(p', T)\). By applying \(F\) to the internal Kleisli lifting morphism Eq. (7), we obtain
\[
kl : (X \otimes (\Delta m) Y) \otimes (\Delta n X) \rightarrow (\Delta (n \cdot m) Y)
\]
This concludes that \(F \circ \Delta : (M, \leq) \rightarrow \text{Ord}(p', T)\) is a \(\otimes\)-parameterized assignment.

I. Modeling Other Divergences in Graded Fuzz

To reason about other divergences besides the skew divergence in \((\epsilon, \delta)\)-differential privacy via graded parameterized !-liftings, we introduce the following conditions.

Definition 14. Let \(H = (\mathbb{R}_{\geq 0}, \leq, u, \bullet)\) be a partially ordered monoid over the non-negative extended reals. A family of divergences \(d_X\) on \(DX\) indexed by sets \(X\) is \(H\)-composable if for any \(f, g : X \rightarrow DY\) and \(\mu, \nu \in DX\), we have
\[
d_Y(f^!(\mu), g^!(\nu)) \leq d_X(\mu, \nu) \bullet \sup_{x \in X} d_Y(f(x), g(x)).
\]
Previously, Barthe and Olmedo [4] proposed weak and strong compositability to study sequential composition properties for the class of \( f \)-divergences. (The skew divergence in \((\varepsilon, \delta)\)-differential privacy is an example of an \( f \)-divergence.) These notions coincide when working with full distributions rather than sub-distributions, as in our settings. Given any family of composable divergences, we can build a corresponding graded \(!\)-lifting of \( RSRel \) on \( \mathcal{D} \).

**Theorem 8.** Let \( d_X \) be an \( H \)-composable family of divergences on \( DX \) and \( q : RSRel \rightarrow \text{Set} \) be the forgetful functor. Define a mapping \( R(d) \) by

\[
R(d)(\delta)(X) \triangleq (DX, \{ (\mu, \nu) \mid d_X(\mu, \nu) \leq \delta \}).
\]

Then \( R(d) \) is a monotone mapping of type \((\mathbb{R}_\infty^{\geq 0}, \leq) \rightarrow \text{Ord}(q, D)\), and is an \( H \)-graded \( \times \)-parameterized assignment of \( RSRel \) on \( \mathcal{D} \).

As usual, we identify \( R(d) \) and the corresponding \( H \)-graded \( \times \)-parameterized \(!\)-lifting of \( \mathcal{D} \) along \( q : RSRel \rightarrow \text{Set} \).

Much like we handled \((\varepsilon, \delta)\)-differential privacy, we can extend Fuzz with composable divergences by applying the path construction to transport graded assignments of \( RSRel \) to \( \text{Met} \). We demonstrate this pattern on several examples, briefly sketching the composition rules, graded assignment structure, and Fuzz typing rules.

**KL Divergence:** The Kullback-Leibler (KL) divergence, also known as relative entropy, measures the difference in information between two distributions. For discrete distributions over \( X \), it is defined as:

\[
KL_X(\mu, \nu) \triangleq \sum_{i \in X} \mu(i) \log \left( \frac{\mu(i)}{\nu(i)} \right),
\]

where summation terms with \( \mu(i) = \nu(i) = 0 \) are defined to be 0, and terms with \( \mu(i) > \nu(i) = 0 \) are defined to be \( \infty \). Note that this divergence is reflexive, but it is not symmetric and does not satisfy the triangle inequality. (It is not immediately obvious, but the KL divergence is always non-negative.) We can define a family of relations that models pairs of distributions at bounded KL divergence. For any \( \alpha \in \mathbb{R} \) and set \( X \), we define:

\[
KL(\alpha)(X) \triangleq R(KL(\alpha)(X)) = (DX, \{ (\mu, \nu) \mid KL_X(\mu, \nu) \leq \alpha, KL_X(\nu, \mu) \leq \alpha \}).
\]

Note that \( \alpha \) need not be an integer—it can be any real number. The relation \( KL(\alpha)(X) \) is reflexive and symmetric, hence an object in \( RSRel \). Barthe and Olmedo [4, Proposition 5] show that KL is \( H \)-composable for the monoid \( H = (\mathbb{R}, \leq, 0, +) \), so \( KL(\alpha)(X) \) is a \( H \)-graded \( \times \)-parameterized assignment of \( RSRel \) on \( \mathcal{D} \) by Theorem 8. By applying the path construction, we get a \( H \)-graded \( \times \)-parameterized assignment of \( \text{Met} \) on \( \mathcal{D} \) which we can use to capture KL divergence as a graded distribution type:

\[
[\bigcirc_{\alpha}^{KL}] \triangleq P(KL(\alpha)([\tau]))
\]

For **bind**, for instance, we obtain the following typing rule:

\[
\begin{array}{c}
\Gamma \vdash e_1 : \bigcirc_{\alpha}^{KL} \tau \\
\Gamma \vdash x : \infty \\
\Gamma \vdash e_2 : \bigcirc_{\beta}^{KL} \sigma \\
\hline
\Gamma \vdash \text{bind } x \leftarrow e_1 ; e_2 : \bigcirc_{\alpha + \beta}^{KL} \sigma
\end{array}
\]

Like we did for the max divergence of differential privacy, we can also introduce primitive distributions and typing rules into Fuzz. For instance, a standard fact in probability theory is that the standard Normal distribution satisfies the bound

\[
KL_{\mathbb{R}}(N(\mu_1, 1), N(\mu_2, 1)) \leq (\mu_1 - \mu_2)^2
\]

If we again discretize this continuous distribution to \( \hat{N}(\mu, 1) \) and interpret the primitive term \([\text{Normal}] = \lambda x. \hat{N}(x, 1)\), the following typing rule is sound:

\[
\begin{array}{c}
\Gamma \vdash \text{Normal} : [\mathbb{R}] \rightarrow \bigcirc_{KL}^{\mathbb{R}} \mathbb{R}
\end{array}
\]

**\( \chi^2 \) Divergence:** We turn next to the \( \chi^2 \) divergence from statistics. For discrete distributions over \( X \), the \( \chi^2 \) divergence is defined as

\[
XD_X(\mu, \nu) \triangleq \sum_{i \in X} \frac{(\mu(i) - \nu(i))^2}{\nu(i)},
\]

where summation terms with \( \mu(i) = \nu(i) = 0 \) are defined to be 0, and terms with \( \mu(i) > \nu(i) = 0 \) are defined to be \( \infty \). Note that this divergence is not symmetric and does not satisfy the triangle inequality. We define a family of reflexive symmetric relations that models pairs of distributions at bounded \( \chi^2 \)-divergence. For any \( \alpha \geq 0 \) and set \( X \), we pose

\[
XD(\alpha)(X) \triangleq R(XD(\alpha)(X)) = (DX, \{ (\mu, \nu) \mid XD_X(\mu, \nu) \leq \alpha, XD_X(\nu, \mu) \leq \alpha \})
\]
Olmedo [26, Theorem 5.4] shows that XD is \( H \)-composable for the monoid \( H = (\mathbb{R}, \leq, 0, +) \), where \( \alpha + \chi \beta = \alpha + \beta + \alpha \beta \). Hence XD\( (\alpha) (X) \) is a \( H \)-graded \( \times \)-parameterized assignment of RSRel on \( D \) by Theorem \( X \). By applying the path construction, we get a \( H \)-graded \( \times \)-parameterized assignment of Met on \( D \) which we can use to interpret a graded distribution type capturing \( \chi^2 \)-divergence:

\[
[\bigcirc^\alpha_X \tau] \triangleq P(XD(\alpha)([\tau]))
\]

The corresponding typing rule for bind becomes:

\[
\Gamma \vdash e_1 : \bigcirc^\alpha_X \tau, \Gamma, x : \tau \vdash e_2 : \bigcirc^\beta_X \sigma
\]

\[
\Gamma \vdash \text{bind} \ x \leftarrow e_1; e_2 : \bigcirc^\alpha_{\tau^{\times} \sigma} \sigma
\]

**Hellinger Distance:** Our last case study is the Hellinger distance, a standard measure of similarity between distributions originating from statistics. Unlike the previous examples, we show how to massage a weak composability property to build a graded assignment structure with a different indexing behavior. For discrete distributions over \( X \), the Hellinger distance is defined as:

\[
\text{HD}_X(\mu, \nu) \triangleq \sqrt{\frac{1}{2} \sum_{i \in X} |\mu(i) - \nu(i)|^2}.
\]

This distance is a proper metric: it satisfies reflexivity, symmetry, and triangle inequality. However, its behavior under composition means that we cannot model it with a standard assignment structure. Instead, we will use a graded assignment. For any \( \alpha \geq 0 \) and set \( X \), we define:

\[
\text{HDR}(\alpha)(X) \triangleq R(\text{HD})(\alpha)(X) = (DX, \{(\mu, \nu) \mid \text{HD}_X(\mu, \nu) \leq \alpha, \text{HD}_X(\nu, \mu) \leq \alpha\})
\]

Note that this is a reflexive and symmetric relation, hence an object in RSRel. Barthe and Olmedo [4, Proposition 5] show the following composition principle:

\[
\text{HD}_Y(f^\dagger \mu, g^\dagger \nu)^2 \leq \text{HD}_X(\mu, \nu)^2 + \sup_{x \in X} \text{HD}_Y(f(x), g(x))^2
\]

Note that this is for squared Hellinger distance. Taking square roots, we have the following composition property for standard Hellinger distance:

\[
\text{HD}_Y(f^\dagger \mu, g^\dagger \nu) \leq \sqrt{\text{HD}_X(\mu, \nu)^2 + \sup_{x \in X} \text{HD}_Y(f(x), g(x))^2}
\]

Hence, HD is \( H \)-composable for the monoid \( H = (\mathbb{R}, \leq, 0, +) \), where \( \alpha + \delta \beta = \sqrt{\alpha^2 + \beta^2} \) and \( \text{HDR}(\alpha)(X) \) has the structure of a \( H \)-graded \( \times \)-parameterized assignment of RSRel on \( D \) by Theorem \( X \). By applying the path construction to this assignment, we get a \( H \)-graded \( \times \)-parameterized assignment of Met on \( D \) with which we can then interpret a graded distribution type capturing Hellinger distance:

\[
[\bigcirc^\alpha_X \tau] \triangleq P(\text{HDR}(\alpha)([\tau]))
\]

The typing rule for bind is then:

\[
\Gamma \vdash e_1 : \bigcirc^\alpha_X \tau, \Gamma, x : \tau \vdash e_2 : \bigcirc^\beta_X \sigma
\]

\[
\Gamma \vdash \text{bind} \ x \leftarrow e_1; e_2 : \bigcirc^\alpha_{\tau^{\times} \sigma} \sigma
\]

Like the other divergences, we can introduce primitives and typing rules capturing the Hellinger distance. For example, the natural-valued Poisson distribution models the number of events occurring in some time interval, if the events happen independently and at constant rate. Given a parameter \( \alpha \in \mathbb{N} \), this distribution has the following probability mass function:

\[
P(\alpha)(n) = \frac{\alpha^ne^{-\alpha}}{n!}
\]

It is known that the Poisson distribution satisfies the following Hellinger distance bound:

\[
\text{HD}_N(P(\alpha), P(\alpha')) = 1 - \exp \left( -\frac{|\sqrt{\alpha} - \sqrt{\alpha'}|^2}{2} \right) \leq \frac{1}{2} |\sqrt{\alpha} - \sqrt{\alpha'}|^2
\]

Given \( \alpha \) and \( \alpha' \) at most 1 apart, this Hellinger distance is at most \( 1/2 \). Hence, we may introduce a type Poisson and interpret it as \([\text{Poisson}] = \lambda x. P(x)\), and the following typing rule is sound:

\[
\Gamma \vdash \text{Poisson} : [\mathbb{R}] \rightarrow \bigcirc_{\mathbb{N}}^{X_D/2} X_D
\]