PHASE MIXING FOR SOLUTIONS TO 1D TRANSPORT EQUATION IN A CONFINING POTENTIAL

SANCHIT CHATURVEDI AND JONATHAN LUK

Dedicated to the memory of Robert Glassey

Abstract. Consider the linear transport equation in 1D under an external confining potential $\Phi$:
\[
\partial_t f + v \partial_x f - \partial_x \Phi \partial_v f = 0.
\]
For $\Phi = \frac{x^2}{2} + \frac{\epsilon x^4}{2}$ (with $\epsilon > 0$ small), we prove phase mixing and quantitative decay estimates for $\partial_t \varphi := -\Delta^{-1} \int f \partial_v f \, dv$, with an inverse polynomial decay rate $O((t)^{-2})$. In the proof, we develop a commuting vector field approach, suitably adapted to this setting. We will explain why we hope this is relevant for the nonlinear stability of the zero solution for the Vlasov–Poisson system in 1D under the external potential $\Phi$.

1. Introduction

Consider the linear transport equation in 1D
\[
\partial_t f + v \partial_x f - \partial_x \Phi \partial_v f = 0, \tag{1.1}
\]
for an unknown function $f : [0, \infty) \times \mathbb{R}_x \times \mathbb{R}_v \to \mathbb{R}_{\geq 0}$ with a smooth external confining potential $\Phi : \mathbb{R} \to \mathbb{R}$.

The following is the main result of this note:

Theorem 1.1. Let $\epsilon > 0$ and $\Phi(x) = \frac{x^2}{2} + \frac{\epsilon x^4}{2}$. Consider the unique solution $f$ to (1.1) with initial data $f \rvert_{t=0} = f_0$ such that
- $f_0 : \mathbb{R}_x \times \mathbb{R}_v \to \mathbb{R}_{\geq 0}$ is smooth, and
- there exists $c_\epsilon > 0$ such that $\text{supp}(f_0) \subseteq \{(x, v) : c_\epsilon \leq \frac{v^2}{2} + \Phi(x) \leq c_\epsilon^{-1}\}$.

Then, for $\epsilon$ sufficiently small, there exists $C > 0$ depending on $\epsilon$ and $c_\epsilon$ such that the following estimate holds:
\[
\sup_{x \in \mathbb{R}} |\partial_t \varphi|(t, x) \leq C(t)^{-2} \sup_{(x, v) \in \mathbb{R}_x \times \mathbb{R}_v} \sum_{|\alpha| + |\beta| \leq 2} |\partial_x^\alpha \partial_v^\beta f_0|(x, v),
\]
where $\varphi$ is defined by
\[
\partial_x^2 \varphi(t, x) = \int_{\mathbb{R}} f(t, x, v) \, dv, \quad \varphi(t, 0) = \partial_x \varphi(t, 0) = 0. \tag{1.2}
\]

A few remarks of the theorem are in order.

Remark 1.2 (Nonlinear Vlasov–Poisson system). The reason that we are particularly concerned with $\partial_t \varphi$ is that it appears to be the quantity relevant for the stability of the zero solution for the nonlinear Vlasov–Poisson system in 1D; see Section 2.

It should be noted that $\varphi$ itself is not expected to decay to 0 (since $\int_{\mathbb{R}} f \, dv \geq 0$). Thus the decay for $\partial_t \varphi$ can be viewed as a measure of the rate that $\varphi$ approaches the limit $\lim_{t \to +\infty} \varphi(t, x)$.

Date: September 29, 2021.
Remark 1.3 (Derivatives of $\partial_t \varphi$). For the applications on the Vlasov–Poisson system, one may also wish to obtain estimates for the derivatives of $\partial_t \varphi$. It is easy to extend our methods to obtain

$$|\partial_x \partial_t \varphi| \lesssim (t)^{-1}, \quad |\partial^2_x \partial_t \varphi| \lesssim 1.$$ 

Notice that these decay rates, at least by themselves, do not seem sufficient for a global nonlinear result.

Remark 1.4 (Phase mixing and the choice of $\Phi$). The result in Theorem 1.1 can be interpreted as a quantitative phase-mixing statement. It is well-known that for

$$\Phi(x) = \frac{x^2}{2},$$

the solution to (1.1) does not undergo phase mixing (see chapter 3 in [6]). It is therefore important that we added the $\frac{e^4}{2}$ term in the definition of the potential.

On the other hand, there are other choices of $\Phi$ for which analogues of Theorem 1.1 hold. We expect that as long as $\Phi$ is even and satisfies the non-degeneracy condition of [23], then a similar decay estimate holds. The particular example we used is only chosen for concreteness.

Remark 1.5 (Method of proof). It is well-known that the linear transport equation (1.1) can be written in action-angle variables, say $(Q, K)$, in which case (1.1) takes the form

$$\partial_t f - c(K) \partial_Q f = 0. \quad (1.3)$$

When $c'(K)$ is bounded away from 0, phase mixing in the sense that $f$ converges weakly to a limit can be obtained after solving (1.3) with a Fourier series in $Q$; see [23]. The point here is that $\varphi$ is a (weighted) integral of $f$ over a region of phase space that is most conveniently defined with respect to the $(x, v)$ (as opposed to the action-angle) variables.

We quantify the strong convergence of $\varphi_t \to 0$ by finding an appropriate commuting vector field $Y$ that is adapted to the action-angle variables. The fact that $\varphi$ is naturally defined as an integral over $v$ in $(x, v)$ coordinates makes it tricky to prove decay using this vector field. Furthermore, we are only able to prove $1/(t)^2$ decay; this is for instance in contrast to the decay of the density for the free transport equation on a torus.

1.1. Related result.

Linear phase mixing results. In the particular context of Theorem 1.1, decay of $\partial_t \varphi$, but without a quantitative rate, can be inferred from the work [23].

There are many linear phase mixing results, the simplest setting for this is the linear free transport equation. This is well-known; see for instance notes [25] by Villani.

One of the most influential work on phase mixing is the groundbreaking paper [21] of Landau wherein he proposes a linear mechanism for damping for plasmas that does not involve dispersion or change in entropy. In the case of $T^d$, this is even understood in a nonlinear setting; see the section on nonlinear results below. The situation is more subtle in $\mathbb{R}^d$, see [4], [16], [17] and [20].

See also [5], [14] and [24] for linear results on related models. In particular, we note that [14] also rely on action-angle variables in their analysis.
Relation with other phase-mixing problems with integrable underlying dynamics. As pointed out in [23], phase space mixing is relevant for the dynamics of kinetic models in many physical phenomena from stellar systems and dark matter halos to mixing of relativistic gas surrounding a black hole. See [11] for related discussions on dark matter halos. We also refer the interested reader to [6] for further background and discussions of phase mixing in other models, including the stability of galaxies.

We hope that the present work would also be a model problem and aid in understanding more complicated systems such as those described in [23]. One particularly interesting problem is the stability of the Schwarzschild solution to the Einstein–Vlasov system in spherical symmetry.

Nonlinear phase mixing results. Nonlinear Landau damping for Vlasov–Poisson on $\mathbb{T}^d$ was first proven in analytic regularity by Mouhot–Villani in their landmark paper [22]. Since then their work has been extended and simplified in [2] and [18].

See also other nonlinear results, e.g. in [1], [3], [8], [15], [19], [26].

Collisional problems with confining potentials. Confining potentials for kinetic equations have been well-studied, particularly for collisional models. Linear stability results can be found in [7], [9], [10], [12] and [13].

In this connection, it would also be of interest to understand how phase mixing effects (studied in the present paper) interact with collisional effects (cf. [1], [8], [24].)

2. The Vlasov–Poisson system

The motivation of our result is the Vlasov–Poisson system:

$$\begin{cases}
\partial_t f + v \partial_x f - (\partial_x \Phi + \partial_x \varphi) \partial_v f = 0, \\
- \partial_x^2 \varphi = \int_{\mathbb{R}} f \, dv.
\end{cases} \tag{2.1}$$

Note that (2.1) can be rewritten as

$$\partial_t f + \{H, f\} = 0, \tag{2.2}$$

where $H$ is the Hamiltonian given by

$$H(x, v) = \frac{v^2}{2} + \Phi(x) + \varphi(t, x). \tag{2.3}$$

Notice that $f \equiv 0$ is a solution to (2.1), and the transport equation (1.1) is the linearization of (2.1) near the zero solution.

One cannot hope that the term $\partial_x \varphi$ in the nonlinear term decays as $t \to +\infty$. (This can be seen by noting that $\int_{\mathbb{R}} f \, dv \geq 0$ pointwise.) At best one can hope that $\partial_x \varphi$ converges to some (non-trivial) limiting profile as $t \to +\infty$. For $f$ satisfying the linear equation (1.1), such convergence (without a quantitative rate) has been shown in [23].

In anticipation of the nonlinear problem, it is important to understand the quantitative convergence. Since $\partial_x \varphi$ does not converge to 0, it is natural to understand the decay rate of $\partial_t \partial_x \varphi$.

As a first step to understand (2.1), we look at the linearized problem (1.1) around the zero solution and prove that we get integrable decay for $\varphi_t$ in the linearized dynamics.
Remark 2.1. Note that the Poisson’s equation above reads
\[-\partial_x^2 \varphi = \rho.\]
In particular, \(\varphi\) is only defined up to a harmonic function, i.e. a linear function \(a x\). In Theorem 1.1, we remove this ambiguity by setting \(\varphi(0) = (\partial_x \varphi)(0) = 0\). Notice that other normalization, e.g., \(\varphi(-\infty) = (\partial_x \varphi)(-\infty) = 0\) would not change the function \(\varphi = \partial_v \varphi\).

3. The action-angle variables

3.1. First change of variables. From now on we will consider the Hamiltonian
\[H = \frac{v^2}{2} + \Phi(x)\]
This is the Hamiltonian for the equations (1.1), which is also (2.3) without the \(\varphi\) (the self-interaction term). As an intermediate step to getting the action-angle variables we use the change of coordinates
\[(t, x, v) \mapsto (t, \chi, H) \quad \text{when } x > 0,
(t, x, v) \mapsto (t, \pi - \chi, H) \quad \text{when } x \leq 0,
\]
where \(\chi := \arcsin \left(\frac{v}{\sqrt{2H}}\right)\).

First we check if the change of variables is well defined by calculating the Jacobian for \(x > 0\),
\[J = \begin{pmatrix}
\partial_t & \partial_x t & \partial_v t \\
\partial_t H & \partial_x H & \partial_v H \\
\partial_t \chi & \partial_x \chi & \partial_v \chi
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \Phi_x & v \\
0 & -\frac{v}{2H} \cdot \frac{\Phi_x}{\sqrt{\Phi}} & \frac{\sqrt{\Phi}}{H}
\end{pmatrix}.
\]
Now
\[\det(J) = \frac{\Phi_x}{\sqrt{\Phi}} \frac{\Phi + v^2/2}{H} = \frac{\Phi_x}{\sqrt{\Phi}}.\]
Similarly for \(x \leq 0\),
\[\det(J) = -\frac{\Phi_x}{\sqrt{\Phi}}.\]
Hence,
\[\det(J) = \text{sign } x \frac{\Phi_x}{\sqrt{\Phi}}.\]
Next by chain rule and using that \(H\) is independent of \(t\), we get
\[\partial_x = \text{sign } x \partial_x \chi \partial_\chi + \partial_x H \partial_H = -\frac{v}{2H} \cdot \frac{\Phi_x}{\sqrt{\Phi}} \partial_\chi + \Phi_x \partial_H,
\]
\[\partial_v = \text{sign } x \partial_v \chi \partial_\chi + \partial_v H \partial_H = \frac{\sqrt{\Phi}}{H} \partial_\chi + v \partial_H.\]
Pluggin this in (2.2), we get the equation

$$\partial_t f - \text{sign} x \Phi_x \sqrt{\Phi} \partial_x f = 0. \quad (3.1)$$

3.2. Second change of variables. The coefficient in front of $\partial_x f$ in (3.1) depends on both $\chi$ and $H$. To take care of this, we reparametrize $\chi$ (in a manner depending on $H$). More precisely, for a fixed $H$, we define $Q(\chi, H)$ such that

$$\frac{dQ}{d\chi} = \frac{c(H)}{a(\chi, H)}, \quad Q(0, H) = 0,$$

where $a(\chi, H) = \text{sign} x \frac{\Phi_x(x)}{\sqrt{\Phi(x)}}$ such that $x = x(\chi, H)$. To fix $c(H)$, we require that for every $H$,

$$2\pi = \int_0^{2\pi} dQ = c(H) \int_0^{2\pi} \frac{1}{a(\chi, H)} d\chi. \quad (3.2)$$

Now we define the change of variables, $((\chi, H) \mapsto (Q, K))$ where $K = H$. Then note,

$$a(\chi, H) \partial_\chi = c(H) \partial_Q$$

and

$$\partial_H = \partial_K + \frac{\partial Q}{\partial H} \partial_Q.$$

Thus in these coordinates, we can rewrite (3.1) as

$$\partial_t f - c(K) \partial_Q f = 0. \quad (3.3)$$

Further, the Jacobian is

$$\begin{pmatrix} \partial_H K & \partial_\chi K \\ \partial_H Q & \partial_\chi Q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c(H)}{a(\chi, H)} & \frac{c(H)}{a(\chi, H)} \end{pmatrix}.$$ 

Note that the determinant is $\frac{c(H)}{a(\chi, H)}$. Further, since

$$a(\chi, H) = \text{sign} x \frac{\Phi_x(x)}{\sqrt{\Phi(x)}} = \sqrt{2} \frac{1 + 2\varepsilon x^2}{\sqrt{1 + \varepsilon x^2}},$$

we have that $a(\chi, H) \approx 1$ when $x$ is in a compact subset of $\mathbb{R}$. As a result the determinant is bounded away from zero. For more details see Lemma 4.3.

4. The commuting vector field

We first define the vector field

$$Y = tc'(H) \partial_Q - \partial_K.$$

In this section we prove that this vector field commutes with the transport operator as in (3.3) and that $|c'(H)| > 0$. 

4.1. **Commutation property.** The following commutation formula is an easy computation and thus we leave out the details.

**Lemma 4.1.** Let $Y = tc'(H) \partial_Q - \partial_K$. Then

$$[\partial_t - c(H) \partial_Q, Y] = 0.$$  

The following is an easy consequence of Lemma 4.1:

**Lemma 4.2.** Let $f$ be a solution to (3.3) with initial data satisfying assumptions of Theorem 1.1. Then

$$\sup_{(t,Q,K) \in [0,\infty) \times \mathbb{T} \times [c_s,c_s^{-1}]} \sum_{\ell \leq 2} |Y^\ell f|(t,Q,K) \leq \sup_{(Q,K) \in \mathbb{T} \times [c_s,c_s^{-1}]} \sum_{\ell \leq 2} |\partial^\ell_K \partial^2_\alpha f_0|(Q,K).$$

**Proof.** By Lemma 4.1, we have that $Y^{\ell} f$ satisfies the transport equation (3.3) for any $\ell \in \mathbb{N} \cap \{0\}$. Hence we get the estimate

$$\sup_{(t,Q,K) \in [0,\infty) \times \mathbb{T} \times [c_s,c_s^{-1}]} \sum_{\ell \leq 2} |Y^\ell f|(t,Q,K) \leq \sup_{(Q,K) \in \mathbb{T} \times [c_s,c_s^{-1}]} \sum_{\ell \leq 2} |\partial^\ell_K f_0|(Q,K).$$

Since the change of variables $(Q,K) \to (x,v)$ is well-defined away from the support of $f$. This plays a key role in the next section ensuring phase mixing.

4.2. **Positivity of $|c'(K)|$.** We prove that $|c'(K)|$ is uniformly bounded below on the support of $f$.

**Lemma 4.3.** For every $c_s < +\infty$, there exists $\epsilon_0 > 0$ such that whenever $\epsilon \in (0,\epsilon_0]$, there is a small constant $\delta > 0$ (depending on $c_s$ and $\epsilon$) such that

$$\inf_{K \in [c_s,c_s^{-1}]} |c'(K)| = \inf_{H \in [c_s,c_s^{-1}]} |c'(H)| \geq \delta.$$  

**Proof.** By definition of $c(H)$, we have

$$\frac{2\pi}{c(H)} = \int_0^{2\pi} \left| \frac{\sqrt{\Phi}}{\Phi'} \right| \, d\chi,$$

so that using $\Phi = \frac{\pi^2}{2} + \frac{x^4}{2}$, we obtain

$$\frac{2\pi}{c(H)} = \int_0^{2\pi} \frac{\sqrt{1 + \epsilon x^2}}{\sqrt{2(1 + 2\epsilon x^2)}} \, d\chi.$$  

Notice that for $H \in [c_s,c_s^{-1}]$, $|x|$ is bounded. It follows that $c(H) \approx 1$. Therefore, to prove strict positivity of $|c'(H)|$, it suffices to prove positivity of $\left| \frac{c'(H)}{c^2(H)} \right|$. Note that

$$\frac{-2\pi c'(H)}{c^2(H)} = \frac{1}{\sqrt{2}} \int_0^{2\pi} \partial_H \left( \frac{\sqrt{1 + \epsilon x^2}}{1 + 2\epsilon x^2} \right) \, d\chi$$

$$= \frac{1}{\sqrt{2}} \int_0^{2\pi} \partial_H x \left[ \frac{\epsilon x}{\sqrt{1 + \epsilon x^2(1 + 2\epsilon x^2)}} - \frac{4\epsilon x \sqrt{1 + \epsilon x^2}}{(1 + 2\epsilon x^2)^2} \right] \, d\chi$$

$$= \frac{1}{\sqrt{2}} \int_0^{2\pi} \partial_H x \left[ \frac{-3\epsilon x - 2\epsilon^2 x^3}{\sqrt{1 + \epsilon x^2(1 + 2\epsilon x^2)^2}} \right] \, d\chi.$$  

Now we calculate $\partial_H x$. First we use the equation, $H = \frac{v^2}{2} + \Phi(x)$. Precisely, we have

$$1 = v \partial_H v + \Phi'(x) \partial_H x.$$
Thus
\[ \partial_H x = \frac{1 - v \partial_H v}{\Phi'} \] (4.2)

Next we use that \[ \frac{v}{\sqrt{2H}} = \sin \chi, \]
\[ 0 = \frac{\partial_H v}{\sqrt{2H}} - \frac{v}{(2H)^{3/2}}. \]
Thus \( \partial_H v = \frac{v}{2H} \). Plugging this into (4.2), we get that
\[ \partial_H x = \frac{1 - \frac{v^2}{2H}}{\Phi'} = \frac{\cos^2 \chi}{\Phi'(x)} = \frac{\cos^2 \chi}{x + 2\epsilon x^3}, \] (4.3)

where in the last equality we used \( \Phi = \frac{x^2 + \epsilon x^4}{2} \).

Plugging (4.3) back into (4.1), we get that
\[ -\frac{2\pi c'(H)}{c^2(H)} = \int_0^{2\pi} \cos^2 \chi \left[ \frac{-x(3\epsilon + 2\epsilon^2 x^2)}{\sqrt{1 + \epsilon x^2}(1 + 2\epsilon x^2)^2} \right] d\chi \]
\[ = -\int_0^{2\pi} \cos^2 \chi \left[ \frac{(3\epsilon + 2\epsilon^2 x^2)}{\sqrt{2(1 + \epsilon x^2)(1 + 2\epsilon x^2)^3}} \right] d\chi. \]

Finally note that since \( |x| \) is bounded on the region of interest, after choosing \( \epsilon_0 \) sufficiently small, we have
\[ \frac{(3\epsilon + 2\epsilon^2 x^2)}{\sqrt{2(1 + \epsilon x^2)(1 + 2\epsilon x^2)^3}} \approx \epsilon, \]
and thus \( |c'(H)| > \delta. \) \( \Box \)

5. Decay for \( \varphi_t \)

In this section we finally prove the decay for \( \varphi_t \) (recall Theorem 1.1).

To keep the notation lean, we will often suppress the explicit dependence on \( t \).

**Lemma 5.1.** For \( f \) satisfying the assumptions of Theorem 1.1, and \( \varphi \) defined as in (1.2), we have the following formula
\[ \varphi_t(x') = \int_0^{x'} \int_R v[f(y, v) - f(0, v)] dv dy. \]

**Proof.** By the continuity equation (following directly from (1.1)), we have that
\[ \rho_t = \int_R v \partial_z f \, dv. \]
Thus
\[ -\partial_z^2 \varphi_t = \rho_t = \int_R v \partial_z f \, dv. \]
Solving the Laplace’s equation (with boundary conditions (1.2)), we get
\[ \varphi_t(x') = \int_0^{x'} \int_0^y \int_R v \partial_z f(z, v) \, dz \, dv \, dy. \]
Integrating by parts in \( z \), we get
\[ \varphi_t(x') = \int_0^{x'} \int_R v[f(y, v) - f(0, v)] \, dv \, dy. \] \( \Box \)
In view of Lemma 5.1, it suffices to bound \( \int_0^t \int \mathbb{R} v f(0, v) \, dv \, dy \) and \( \int_0^t \int \mathbb{R} v f(y, v) \, dv \, dy \), which will be achieved in the next two subsections respectively.

5.1. Decay for the term involving \( f(0, v) \). We first prove decay for \( \int \mathbb{R} v f(0, v) \, dv \). Before proving the main estimate in Proposition 5.3, we first prove a lemma.

**Lemma 5.2.** The level set \( \{ x = 0 \} \) corresponds to the level sets \( \{ Q = \frac{\pi}{2} \} \cap \{ Q = -\frac{\pi}{2} \} \cup \{ (x, v) = (0, 0) \} \).

**Proof.** First note that level set \( \{ x = 0 \} \) corresponds to the level sets \( \{ \chi = \frac{\pi}{2} \} \cap \{ \chi = -\frac{\pi}{2} \} \cup \{ (x, v) = (0, 0) \} \). This is because when \( x = 0 \), \( \Phi(x) = 0 \), and thus by definition (when \( v \neq 0 \))

\[
\chi := \arcsin \left( \frac{v}{\sqrt{2t}} \right) = \arcsin(\pm 1) = \pm \frac{\pi}{2}.
\]

It thus remains to show that

\[
\chi = \pm \frac{\pi}{2} \iff Q = \pm \frac{\pi}{2}.
\]

Fix \( H \), then since \( a(\chi, H) = \frac{\phi}{\sqrt{\Phi}} \) is independent of \( v \), we have

\[
c(H) \int_0^\pi \frac{1}{a(\chi, H)} \, d\chi = c(H) \int_\pi^{2\pi} \frac{1}{a(\chi, H)} \, d\chi.
\]

Further, by the evenness of \( \Phi \), we have

\[
c(H) \int_0^{\pi/2} \frac{1}{a(\chi, H)} \, d\chi = c(H) \int_0^\pi \frac{1}{a(\chi, H)} \, d\chi.
\]

Finally, since we have by construction,

\[
c(H) \int_0^{2\pi} \frac{1}{a(\chi, H)} \, d\chi = 2\pi,
\]

we have that

\[
Q(\chi = \pi/2, H) = c(H) \int_0^{\pi/2} \frac{1}{a(\chi, H)} \, d\chi = \pi/2.
\]

Similarly, \( Q(\chi = -\pi/2, H) = -\pi/2 \). Combining these, we obtain (5.1). \( \square \)

**Proposition 5.3.** For \( f \) satisfying the assumptions of Theorem 1.1, we have the following estimate:

\[
\left| \int_\mathbb{R} v f(0, v) \, dv \right| \lesssim \langle t \rangle^{-2} \sup_{(x, v) \in \mathbb{R} \times \mathbb{R}} \sum_{|\alpha| + |\beta| \leq 2} |\partial_x^\alpha \partial_v^\beta f_0|(x, v).
\]

**Proof.** The transport equation preserves \( L^\infty \) bounds so that by the support properties, we obviously have

\[
\left| \int_\mathbb{R} v f(0, v) \, dv \right| \lesssim \sup_{(x, v) \in \mathbb{R} \times \mathbb{R}} |f_0|(x, v).
\]

In other words, it suffices to prove the desired bound with \( t^{-2} \) instead of \( \langle t \rangle^{-2} \).

Now note that

\[
\int_\mathbb{R} v f(0, v) \, dv = \int_0^\infty v[f(0, v) - f(0, -v)] \, dv.
\]

For clarity of notation, we let

\[
\mathcal{F}(t, Q, K) = f(t, x, v).
\]
Now writing in the \((K, Q)\) variables, and using Lemma 5.2 together with the fact that \(K = H = \frac{x^2}{2}\) when \(x = 0\), we have
\[
\int_{\mathbb{R}} v f(0, v) \, dv = \int_{0}^{\infty} v[f(0,v) - f(0,-v)] \, dv = \int_{0}^{\infty} [\overline{f}(\pi/2, K) - \overline{f}(-\pi/2, K)] \, dK.
\]
By the fundamental theorem of calculus, we have
\[
\int_{0}^{\infty} [\overline{f}(\pi/2, K) - \overline{f}(-\pi/2, K)] \, dK = \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \overline{\partial}_Q \overline{f}(Q, K) \, dK \, dQ.
\]
Next, the Cauchy–Schwarz inequality implies
\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \overline{\partial}_Q \overline{f}(Q, K) \, dK \, dQ = \sqrt{\pi} \left( \int_{-\pi/2}^{\pi/2} \left( \int_{0}^{\infty} \overline{\partial}_Q \overline{f}(Q, K) \, dK \right)^2 \, dQ \right)^{\frac{1}{2}}
\leq \left( \int_{0}^{\infty} \left( \int_{-\pi/2}^{\pi/2} \overline{\partial}_Q \overline{f}(Q, K) \, dQ \right)^2 \, dK \right)^{\frac{1}{2}}.
\]
Now using Poincare’s inequality we get that for any \(\ell \geq 2\)
\[
\left( \int_{-\pi/2}^{\pi/2} \left( \int_{0}^{\infty} \overline{\partial}_Q \overline{f}(Q, K) \, dK \right)^2 \, dQ \right)^{\frac{1}{2}} \leq \left( \int_{0}^{\infty} \left( \int_{-\pi/2}^{\pi/2} \overline{\partial}_Q \overline{f}(Q, K) \, dQ \right)^2 \, dK \right)^{\frac{1}{2}}.
\tag{5.2}
\]
Now take \(\ell = 2\). We write \(\partial_Q = \frac{1}{c(K)\ell} (Y + \partial_K)\) so that
\[
\left( \int_{0}^{2\pi} \left( \int_{0}^{\infty} \overline{\partial}_Q^2 \overline{f}(Q, K) \, dK \right)^2 \, dQ \right)^{\frac{1}{2}}
= \left( \int_{0}^{2\pi} \left( \int_{0}^{\infty} \frac{1}{|\overline{c}'(K)|^2 \ell^2} (Y^2 \overline{f} + 2 \partial_K Y \overline{f} + \partial_K^2 \overline{f})(Q, K) \, dK \right)^2 \, dQ \right)^{\frac{1}{2}}
\leq \frac{1}{\ell^2} \left( \int_{0}^{2\pi} \left( \int_{0}^{\infty} \left( \sum_{k=0}^{\infty} |Y^k \overline{f}| \right)(Q, K) \, dK \right)^2 \, dQ \right)^{\frac{1}{2}}.
\]
where in the last step we have integrated by parts in \(K\) and bounded \(\frac{1}{|\overline{c}'(K)|^2 \ell^2} \frac{1}{|c''(K)| \ell^2}\), etc. using Lemma 4.3 and the smoothness of \(c\).

Finally, since \(f(t, Q, K)\) is non-zero for \(c_s \leq K \leq c_s^{-1}\) and \(Q \in [0, 2\pi]\), we can take supremum in \(K\) and \(Q\) followed by Lemma 4.2 to get the required result. \(\square\)

**Remark 5.4.** Notice that since we can take any \(\ell \geq 2\) in (5.2), we can write each \(\partial_Q = \frac{1}{c(H)\ell} (Y + \partial_K)\) and integrate by parts in \(K\) many times to show that the term in Proposition 5.3 in fact decays faster than any inverse polynomial (depending on smoothness of \(f\))!

In other words, the decay rate that we obtain in Theorem 1.1 is instead limited by the term treated in Proposition 5.7 below.

### 5.2. Decay for the term involving \(f(y, v)\)

We now turn to the other term in Lemma 5.1. Before we obtain the main estimate in Proposition 5.7, we first prove two simple lemmas.

**Lemma 5.5.** Under the change of variables \((x, v) \mapsto (Q, K)\) as in Section 3, the volume form transforms as follows:
\[
dv \, dx = c(K) \, dQ \, dK.
\]
Proof. The Jacobian determinant for the change of variables \((x, v) \mapsto (\chi, H)\) is \(a(\chi, H) = \frac{\phi}{\sqrt{\Phi}}\).

Further the Jacobian determinant for the change of variables \((\chi, H) \mapsto (Q, K)\) is \(c(H) = c(K)\).

\[\text{Lemma 5.6.}\] Let \(\overline{f}(Q, K) = f(x, v)\) as above. There exists a function \(\overline{g}(Q, K)\) such that

\[
\partial^2_Q \overline{g} = \partial_Q \overline{f}
\]

and

\[
\max_{\ell \leq 2} \sup_K \|Y^\ell \overline{g}\|_{L^2_Q} \lesssim \max_{\ell \leq 2} \sup_{Q, K} \|Y^\ell \overline{f}\|_{L^2_Q}.
\]

Proof. We use the Fourier series of \(f\) in \(Q\) to get that

\[
\overline{f}(Q, K) = \sum_{k=-\infty}^{k=\infty} \hat{f}_k(K)e^{ikQ}.
\]

Now we define

\[
\overline{g}(Q, K) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{ik} \hat{f}_k(K)e^{ikQ}.
\]

Then we see that \(\partial^2_Q \overline{g} = \partial_Q \overline{f}\).

Using Plancherel’s theorem we can easily see that

\[
\max_{\ell \leq 2} \sup_K \|Y^\ell \overline{g}\|_{L^2_Q} \lesssim \max_{\ell \leq 2} \sup_K \|Y^\ell \overline{f}\|_{L^2_Q}.
\]

Finally, the result follows by taking supremum in \(Q\).

\[\text{Proposition 5.7.}\] For \(f\) satisfying the assumptions of Theorem 1.1, we have the following estimate:

\[
\int_0^x \int_{\mathbb{R}} v f(t, y, v) \, dv \, dy \lesssim (t)^{-2} \sup_{(x, v) \in \mathbb{R}^2} \sum_{|\alpha| + |\beta| \leq 2} |\partial_x^\alpha \partial_v^\beta f_0|(x, v).
\]

Proof. As in the proof of Proposition 5.3, boundedness is obvious and thus it suffices to prove an estimate with \((t)^{-2}\) replaced by \(t^{-2}\).

We first note that

\[
\int_0^x \int_{\mathbb{R}} v f(y, v) \, dv \, dy = \int_0^x \int_0^\infty v [f(y, v) - f(y, -v)] \, dv \, dy.
\]

Again let

\[
\overline{f}(t, Q, K) = f(t, x, v).
\]
Next we use the change of variables $(x,v) \mapsto (Q,K)$, that $v = \sqrt{2H} \sin \chi$ and Lemma 5.5 followed by the fundamental theorem of calculus to obtain

$$
\int_0^{x'} \int_0^\infty v[f(y,v) - f(y,-v)] \, dy \, dv
= \int_0^{\Phi(x')/2} \int_0^{\pi/2} c(K)\sqrt{2K}S(Q,K)[\overline{f}(Q,K) - \overline{f}(-Q,K)] \, dQ \, dK
+ \int_{\Phi(x')}^{\infty} \int_{\Omega_K}^{\pi/2} c(K)\sqrt{2K}S(Q,K)[\overline{f}(Q,K) - \overline{f}(-Q,K)] \, dQ \, dK
= \int_0^{\Phi(x')} \int_0^{\pi/2} \int_Q^{-Q} c(K)\sqrt{2K}S(Q,K)\partial_Q \overline{f}(Q',K) \, dq' \, dQ \, dK
+ \int_{\Phi(x')}^{\infty} \int_{\Omega_K}^{\pi/2} \int_Q^{-Q} c(K)\sqrt{2K}S(Q,K)\partial_Q \overline{f}(Q',K) \, dq' \, dQ \, dK
=: T_1 + T_2,
$$

where we have defined

- $S(Q,K) := \sin \chi$, and
- $\Omega_K$ to be the angle in $(Q,K)$ coordinates corresponding to angle $\arccos \left( \frac{\Phi(x')}{H} \right)$ in $(\chi,H)$ coordinates.

Now using Fubini’s theorem, we have

$$
T_1 = \int_{-\pi/2}^{\pi/2} \int_0^{\Phi(x')} \left( \int_{-Q'}^Q S(Q,K) \, dQ \right) c(K)\sqrt{2K}\partial_Q \overline{f}(Q',K) \, dK \, dQ'
$$

and

$$
T_2 = \int_{-\pi/2}^{\pi/2} \int_{\Phi(x')}^{\pi/2} \left( \int_{-Q'}^Q S(Q,K) \, dQ \right) c(K)\sqrt{2K}\partial_Q \overline{f}(Q',K) \, dK \, dQ'
$$

where $\Omega_{Q'}$ is such that $(\arccos \left( \frac{\Phi(x')}{H} \right), \Omega_{Q'})$ in $(\chi,H)$ coordinates gets mapped to $([Q'],\Omega_{Q'})$ in $(Q,K)$ coordinates (such an $\Omega_{Q'}$ exists because $\chi = \arccos \left( \frac{\Phi(x')}{H} \right)$ increases as $H$ does and $Q$ is monotone\(^1\) in $\chi$.)

Putting the above together we get,

$$
T_1 + T_2 = \int_{-\pi/2}^{\pi/2} \int_0^{\Phi(x')} \left( \int_{-Q'}^Q S(Q,K) \, dQ \right) c(K)\sqrt{2K}\partial_Q \overline{f}(Q',K) \, dK \, dQ'
+ \int_{-\pi/2}^{\pi/2} \int_{\Omega_K}^{\pi/2} \left( \int_{-Q'}^Q S(Q,K) \, dQ \right) c(K)\sqrt{2K}\partial_Q \overline{f}(Q',K) \, dK \, dQ'.
$$

\(^1\)Since $a(\chi,H) > 0$, we have that $Q$ is monotonically increasing as a function of $\chi$ and vice-versa.
Now we use (5.3) from Lemma 5.6 and that $\partial_Q = \frac{1}{c'(K)}(Y + \partial_K)$ to get that

$$T_1 + T_2 \leq t^{-1} \int_{-\pi/2}^{\pi/2} \int_0^{\delta_{Q'}} \frac{1}{\rho'(K)} \left( \int_{|Q'|}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K}(Y + \partial_K)\partial_Q \overline{g}(Q', K) dK dQ'$$

$$+ t^{-1} \int_{-\pi/2}^{\pi/2} \int_{\delta_{Q'}}^{\pi/2} \frac{1}{\rho'(K)} \left( \int_{\Omega_K}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K}(Y + \partial_K)\partial_Q \overline{g}(Q', K) dK dQ'.$$

Next we integrate by parts in $K$. Since $\Omega_{\delta_{Q'}} = |Q'|$, we see that the boundary terms exactly cancel! Hence,

$$t^{-1} \int_{-\pi/2}^{\pi/2} \int_0^{\delta_{Q'}} \frac{1}{\rho'(K)} \left( \int_{|Q'|}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K}\partial_K \partial_Q \overline{g}(Q', K) dK dQ'$$

$$+ t^{-1} \int_{-\pi/2}^{\pi/2} \int_{\delta_{Q'}}^{\pi/2} \frac{1}{\rho'(K)} \left( \int_{\Omega_K}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K}\partial_K \partial_Q \overline{g}(Q', K) dK dQ'$$

$$= -t^{-1} \int_{-\pi/2}^{\pi/2} \int_0^{\delta_{Q'}} \partial_K \left( \frac{1}{\rho'(K)} \left( \int_{|Q'|}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K} \right) \partial_Q \overline{g}(Q', K) dK dQ'$$

$$- t^{-1} \int_{-\pi/2}^{\pi/2} \int_{\delta_{Q'}}^{\pi/2} \partial_K \left( \frac{1}{\rho'(K)} \left( \int_{\Omega_K}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K} \right) \partial_Q \overline{g}(Q', K) dK dQ'.$$

Since there is no boundary term we can integrate by parts after writing $\partial_Q = \frac{1}{c'(K)}(Y + \partial_K)$ once more. Next note that that $\overline{g}(Q, K)$ is nonzero only for $K \in [c_s, c_s^{-1}]$ and that derivatives of $\frac{c'(K)}{c(K)}$ is bounded as $|c'(K)| \geq \delta$ by Lemma 4.3. Further, $S(Q, K) = \sin \chi$ is smooth as a function of $K$. Thus

$$\sum_{\ell \leq 2} \partial_K^\ell \left( \frac{1}{\rho'(K)} \left( \int_{|Q'|}^{\pi/2} S(Q, K) dQ \right) c(K)\sqrt{2K} \right) \leq 1.$$

By Cauchy–Schwarz in $Q'$ and $K$, we get that

$$T_1 + T_2 \leq \sum_{\ell \leq 2} \sup_K \|Y' \overline{g}\|_{L^2_b}.$$

Finally, an application of (5.4) from Lemma 5.6 followed by Lemma 4.2 gives us the required bound. \hfill \Box

**Proof of Theorem 1.1.** The proof follows by using Lemma 5.1 and combining the estimates from Proposition 5.3 and Proposition 5.7. \hfill \Box

**References**

[1] Jacob Bedrossian. Suppression of plasma echoes and Landau damping in Sobolev spaces by weak collisions in a Vlasov-Fokker-Planck equation. *Ann. PDE*, 3(2):Paper No. 19, 66, 2017.

[2] Jacob Bedrossian, Nader Masmoudi, and Clément Mouhot. Landau damping: paraproducts and Gevrey regularity. *Ann. PDE*, 2(1):Art. 4, 71, 2016.

[3] Jacob Bedrossian, Nader Masmoudi, and Clément Mouhot. Landau damping in finite regularity for unconfined systems with screened interactions. *Comm. Pure Appl. Math.*, 71(3):537–576, 2018.

[4] Jacob Bedrossian, Nader Masmoudi, and Clément Mouhot. Linearized wave-damping structure of Vlasov–Poisson in $\mathbb{R}^3$. *arXiv preprint arXiv:2007.08580*, 2020.

[5] Jacob Bedrossian and Fei Wang. The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field. *J. Stat. Phys.*, 178(2):552–594, 2020.

[6] James Binney and Scott Tremaine. *Galactic dynamics*. Princeton university press, 2011.
[7] Kleber Carrapatoso, Jean Dolbeault, Frédéric Hérau, Stéphane Mischler, Clément Mouhot, and Christian Schmeiser. Special modes and hypocoercivity for linear kinetic equations with several conservation laws and a confining potential, 2021.

[8] Sanchit Chaturvedi, Jonathan Luk, and Toan T Nguyen. The Vlasov–Poisson–Landau system in the weakly collisional regime. arXiv preprint arXiv:2104.05692, 2021.

[9] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for kinetic equations with linear relaxation terms. Comptes Rendus Mathematique, 347(9-10):511–516, 2009.

[10] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. Transactions of the American Mathematical Society, 367(6):3807–3828, 2015.

[11] Paola Dominguez-Fernández, Erik Jiménez-Vázquez, Miguel Alcubierre, Edison Montoya, and Dario Núñez. Description of the evolution of inhomogeneities on a dark matter halo with the Vlasov equation. General Relativity and Gravitation, 49(9):1–39, 2017.

[12] Renjun Duan. Hypocoercivity of linear degenerately dissipative kinetic equations. Nonlinearity, 24(8):2165, 2011.

[13] Renjun Duan and Wei-Xi Li. Hypocoercivity for the linear Boltzmann equation with confining forces. Journal of Statistical Physics, 148(2):306–324, 2012.

[14] Erwan Faou, Romain Horsin, and Frédéric Rousset. On linear damping around inhomogeneous stationary states of the Vlasov-HMF model, 2021.

[15] Erwan Faou and Frédéric Rousset. Landau damping in Sobolev spaces for the Vlasov–HMF model. Archive for Rational Mechanics and Analysis, 219(2):887–902, 2016.

[16] Robert Glassey and Jack Schaeffer. Time decay for solutions to the linearized Vlasov equation. Transport Theory Statist. Phys., 23(4):411–453, 1994.

[17] Robert Glassey and Jack Schaeffer. On time decay rates in Landau damping. Comm. Partial Differential Equations, 20(3-4):647–676, 1995.

[18] Emmanuel Grenier, Toan T. Nguyen, and Igor Rodnianski. Landau damping for analytic and Gevrey data. arXiv preprint arXiv:2004.05979, 2020.

[19] Daniel Han-Kwan, Toan T. Nguyen, and Frédéric Rousset. Asymptotic stability of equilibria for screened Vlasov–Poisson systems via pointwise dispersive estimates. arXiv preprint arXiv:2007.07787, 2020.

[20] Daniel Han-Kwan, Toan T. Nguyen, and Frédéric Rousset. On the linearized Vlasov-Poisson system on the whole space around stable homogeneous equilibria. arXiv preprint arXiv:2006.05723, 2019.

[21] Lev Davidovich Landau. On the vibrations of the electronic plasma. Zh. Eksp. Teor. Fiz., 10:25, 1946.

[22] Clément Mouhot and Cédric Villani. On Landau damping. Acta Math., 207(1):29–201, 2011.

[23] Paola Rioseco and Olivier Sarbach. Phase space mixing in an external gravitational central potential. Classical Quantum Gravity, 37(19):195027, 42, 2020.

[24] Isabelle Tristani. Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime. J. Stat. Phys., 169(1):107–125, 2017.

[25] Cédric Villani. Landau damping. Notes de cours, CEMRACS, 2010.

[26] Brent Young. Landau damping in relativistic plasmas. J. Math. Phys., 57(2):021502, 68, 2016.

(Sanchit Chaturvedi) DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, 450 JANE STANFORD WAY, BLDG 380, STANFORD, CA 94305, USA
Email address: sanchat@stanford.edu

(Jonathan Luk) DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, 450 JANE STANFORD WAY, BLDG 380, STANFORD, CA 94305, USA
Email address: jluk@stanford.edu