A Large Distance Expansion for Quantum Field Theory

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Abstract
Using analyticity of the vacuum wave-functional under complex scalings, the vacuum of a quantum field theory may be reconstructed from a derivative expansion valid for slowly varying fields. This enables the eigenvalue problem for the Hamiltonian to be reduced to algebraic equations. Applied to Yang-Mills theory this expansion leads to a confining force between quarks.

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1 Introduction

I will describe an approach to the eigenvalue problem for the Hamiltonian of a quantum field theory, $\hat{H} |E\rangle = E |E\rangle$, in which states are constructed from their simple large distance behaviour.\(^1\) This is in contrast to the usual approach to, say, Yang-Mills theory, which is built up from simple short-distance behaviour. For simplicity, I will concentrate on scalar field theory, although the results also apply to Yang-Mills theory where the leading order in the expansion which I will describe leads to an area law for the Wilson loop\(^2\) via a kind of dimensional reduction.\(^3\)

In the Schrödinger representation the field operator, $\hat{\varphi}$, is diagonal and its conjugate momentum represented by functional differentiation

$$\langle \varphi | \hat{\varphi}(x) = \langle \varphi | \hat{\varphi}(x), \quad \langle \varphi | \hat{\pi}(x) = -i \frac{\delta}{\delta \varphi(x)} \langle \varphi |, \quad \tag{1}$$

so that the ground state is represented by the wave-functional

$$\langle \varphi | E_0 \rangle = \Psi[\varphi] = e^{W[\varphi]}.$$  

In general $W[\varphi]$ is non-local, but when $\varphi(x)$ varies very slowly on length-scales that are large in comparison to the inverse of the mass of the lightest particle it has a derivative expansion in terms of local functions, e.g. $W = \int dx (a_1 \varphi^2 + a_2 \nabla \varphi \cdot \nabla \varphi + a_3 \varphi^4 \ldots)$. This expansion is the basis of our method. At first glance it would appear to be completely useless, because the internal structure of particles is characterised by much shorter scales, although there is one physically interesting phenomenon that takes place at arbitrarily large distances, the confinement of quarks. I will claim, however, that this large distance behaviour is relevant not just to confinement but to understanding physics on all length scales, because I will show how it may be used to reconstruct the wave-functional $\Psi[\varphi]$ for arbitrary $\varphi(x)$.

2 The Local Expansion

Consider $\langle \varphi | e^{-T\hat{H}} | \tilde{\varphi} \rangle$. According to Feynman this is given by an integral over fields $\phi(x,t)$ that live in a Euclidean space-time bounded by the surfaces $t = 0$, and $t = -T$ on which $\phi$ is equal to $\varphi$ and $\tilde{\varphi}$ respectively. As $T \to \infty$ this matrix element is dominated by the contribution from the ground-state, so

$$\langle \varphi | e^{-T\hat{H}} | \tilde{\varphi} \rangle = \int D\phi e^{-S_E} \sim \Psi[\varphi] e^{-TE_0} \Psi^*[\tilde{\varphi}] = e^{W[\varphi] + W[\tilde{\varphi]} - E_0 T}, \quad \tag{2}$$

where $S_E$ is the Euclidean action. From this we can extract $\Psi[\varphi]$. A different formulation makes the dependence on $\varphi$ more explicit.\(^4\) Define the bra $\langle D |$ so as to be annihilated by $\hat{\varphi}$, then the canonical commutation relations imply that $\langle \varphi | = \langle D | \exp(i \int dx \hat{\pi} \varphi)$. So now

$$\langle \varphi | e^{-T\hat{H}} | \tilde{\varphi} \rangle = \langle D | e^{i \int dx \varphi} e^{-T\hat{H}} e^{-i \int dx \tilde{\varphi}} | D \rangle, \quad \tag{3}$$

which can be written as the functional integral

$$\int D\phi e^{-S_E + \int dx \phi(x,0) \varphi(x) - \int dx \phi(x,-T) \tilde{\varphi}(x)}, \quad \tag{4}$$
The boundary condition on the integration variable, $\phi$, implied by $\langle D \rangle$ is that it should vanish on the boundary surfaces $t = 0$ and $t = -T$. (In replacing $\hat{\pi}$ by $\hat{\phi}$, the time derivative of $\phi$, we should also include delta functions in time, coming from the time-ordering.) So $W[\varphi]$ is the sum of connected Euclidean Feynman diagrams in which $\varphi$ is a source for $\hat{\phi}$ on the boundary. The only major difference from the usual Feynman diagrams encountered in field theory is that the propagator vanishes when either of its arguments lies on the boundary. Using this, Symanzik discovered the remarkable result that in 3 + 1 dimensional $\phi^4$ theory $W[\varphi]$ is finite as the cut-off is removed. For a free scalar field with mass $m$ this gives $W = -\frac{1}{2} \int dx \varphi\sqrt{-\nabla^2 + m^2}\varphi$, so that if the Fourier transform of $\varphi$ vanishes for momenta with magnitude greater than the mass, $W$ can be expanded in the convergent series $-\int dx \left(\frac{m^2}{2}\varphi^2 + \frac{1}{4m}(\nabla \varphi)^2 - \frac{1}{16m^3}(\nabla^2 \varphi)^2 \ldots\right)$. The terms of this expansion are local in the sense that they involve the field and a finite number of its derivatives at the same spatial point. The same is true for an interacting theory in which the lightest particle has non-zero mass, because massive propagators are exponentially damped at large distances so that configuration-space Feynman diagrams are negligible except when all their points are within a distance $\approx 1/m$ of each other. Integrating these against slowly varying sources, $\varphi(x)$, leads to local functions.

3 Reconstructing the Vacuum

For 1 + 1-dimensional scalar theory define the scaled field $\varphi^s(x) = \varphi(x/\sqrt{s})$ where $s$ is real and greater than zero. I will now show that $W[\varphi^s]$ extends to an analytic function of $s$ with singularities only on the negative real axis (at least within an expansion in powers of $\varphi$) from which $W[\varphi]$ can be obtained using Cauchy’s theorem. As $T \to \infty$ in (1), $\Psi$ becomes a functional integral on the Euclidean space-time $t \leq 0$. By rotating the coordinates we can view this instead as a functional integral over the Euclidean space-time $x \geq 0$, so

$$e^{W[\varphi^s]} = \int \mathcal{D}\phi e^{-S^s_E + \int dt \dot{\phi}'(0,t)\varphi^s(t)},$$

where $\phi' = \partial\phi/\partial x$, and $S^s_E$ is the action for the rotated space-time. This can be re-interpreted as the time-ordered expectation value of $\exp \int dt (\varphi^s(t)\dot{\varphi}'(0,t) - \hat{H}^r)$ in the ground-state, $|E^r\rangle$, of the rotated Hamiltonian, $\hat{H}^r$. The time integrals can be done if this is expanded in powers of $\varphi^s$, and the sources Fourier analysed using $\varphi^s(k) = \sqrt{s}\tilde{\varphi}(k\sqrt{s})$. This yields

$$\Psi[\varphi^s] = \sum_{n=0}^{\infty} \int dk_n \ldots dk_1 \varphi^s(k_n) \ldots \tilde{\varphi}(k_1) \delta(\sum_{i}^{n} k_i) \times \sqrt{s}^n \langle E^r_0 | \hat{\phi}'(0) \frac{1}{\sqrt{s}H^r + i(\sum_{i}^{n-1} k_i)} \hat{\varphi}'(0) \ldots \hat{\phi}'(0) \frac{1}{\sqrt{s}H^r + ik_1} \hat{\varphi}'(0) | E^r_0 \rangle. \tag{6}$$

This can now be extended to the complex $s$-plane. Since the eigenvalues of $\hat{H}^r$ are real, the singularities occur for $s$ on the negative real axis. This must also hold for $W[\varphi^s]$, which is the connected part of $\Psi[\varphi^s]$, since any additional singularities could not cancel.
between connected and disconnected pieces. Now define

$$I(\lambda) \equiv \frac{1}{2\pi i} \int_C \frac{ds}{s-1} e^{\lambda(s-1)} W[\varphi^s]$$

(7)

Where $C$ is a very large circle centred on the origin, beginning just below the negative real axis and ending just above. On $C$, $\varphi^s(x) = \varphi(x/\sqrt{s}) \approx \varphi(0)$ and so varies only very slowly with $x$, so here we can use our local expansion. Now collapse the contour to a small circle around $s = 1$, which contributes $W[\varphi]$, and a contour, $C'$, surrounding the negative real axis. When $\Re(\lambda) > 0$ the latter is exponentially suppressed (to check this note that for large $|s|$ we can use the local expansion, and elsewhere on $C'$ the integrand is bounded). Hence $W[\varphi] = \lim_{\Re(\lambda) \to \infty} I$, which is expressed in terms of the local expansion only. In practice we can truncate the series to a finite number of terms and work with a large value of $\lambda$ to get a good approximation.

In the Schrödinger representation for $\varphi^4$ theory the term in the Hamiltonian that needs to be regulated is $\int dx \hat{\pi}^2$. If we introduce a momentum cut-off, $1/\epsilon$, and define $H_\epsilon$ as

$$-\frac{1}{2} \int_{k^2 < 1/\epsilon} dk \frac{\delta^2}{\delta \hat{\varphi}(k)\delta \hat{\varphi}(-k)} + \int dx \left( \frac{1}{2}(\varphi'^2 + M^2(\epsilon)\varphi^2) + \frac{g}{4!}\varphi^4 - \mathcal{E}(\epsilon) \right),$$

(8)

where $M^2$ and $\mathcal{E}$ are known functions that diverge as $\epsilon \downarrow 0$, and $g$ and $E$ are finite, then the Schrödinger equation is $\lim_{\epsilon \downarrow 0} (H_\epsilon - E) \Psi = 0$. This cannot be applied directly to the local expansion since the cut-off refers to short distances, whereas the local expansion is only valid at large distances. However, using the same technique as above, it may be shown that $(H_\epsilon \Psi)[\varphi^s]$ is analytic in the $s$-plane with the negative real axis removed. The small-$s$ and large-$s$ behaviour are related by Cauchy’s theorem, so

$$\lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_C \frac{ds}{s} e^{\lambda s} (\{H_\epsilon - E\} \Psi)[\varphi^s] = 0$$

(9)

This leads to a separate equation for the coefficient of each independent local function of $\varphi$. A good approximation results from working to a finite order in $\lambda$, and taking $\lambda$ large, but finite. Expanding in powers of $g$ reproduces standard perturbative results for short-distance phenomena, but these equations may also be solved without resorting to perturbation theory.

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