Cosmic microwave background anisotropies seeded by coherent topological defects: a semi-analytic approach

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Abstract. We consider a perfectly homogeneous, isotropic and spatially flat universe which undergoes a sudden phase transition producing topological defects. We assume that these defects form a coherent network which scales like the background density during the radiation and matter dominated eras, and describe them in terms of a few free functions. We carefully model the loss of scaling invariance during the transition from radiation to matter dominated era. We choose a simple set for the free functions, compute the microwave background temperature anisotropies generated by such a network, and compare our results to previous calculations.

1. Introduction

The origin of the small inhomogeneities in the cosmic fluids which eventually evolved into the large scale structures that we observe today in the universe is still under debate. Two classes of models are at present in competition. The inflationary models on one hand explain those inhomogeneities by the amplification of quantum fluctuations at the end of an accelerated phase of expansion of the universe. On the other hand topological defects appearing during a phase transition in the early universe can also seed inhomogeneities.

A way to discriminate between those two classes of models is to compute the cosmic microwave background (CMB) anisotropies they predict and confront the results to observations. At present the anisotropies on large angular scales are known from the COBE measurements. On smaller scales, the present balloon and ground experiments are still unprecise but the measurements should improve when the MAP and Planck satellite missions are launched.

On the theoretical front, the CMB anisotropy predictions from the inflationary models (at least the simplest ones) are robust, and can now in many cases be obtained by plugging into the CMBFAST code the initial conditions given by the particular model at hand. On the other hand the predictions from defect scenarios are still unclear, the reason being that the defects, which act as a continuous source of inhomogeneities, are difficult to model. Indeed the description of the formation, evolution and decay of defects requires heavy numerical simulations (see e.g.).
and the CMB anisotropies their networks create seem to be very sensitive to their
detailed structure. Recent results \[16\][17] indicate that defect model predictions do
not agree with the data. It is clear however that more work needs to be done before
rejecting topological defects as a possible explanation for the CMB anisotropies as
they may for example have a more complex internal structure (see e.g. \[18\]) than that
assumed in those numerical models.

Considering the complexity of modeling realistic topological defects, the semi-
analytic approach initiated by Durrer and collaborators \[19\] seems to be a fruitful
compromise. The idea is to model the defect network by a stress-energy tensor which
acts as a source to the linearized Einstein equations and induces inhomogeneities in
the cosmic fluids. The ten components of this stress-energy tensor are then drastically
constrained by a number of physical requirements. In this paper, we shall impose, as
in \[14,20\], that the defect network

(i) is statistically homogeneous and isotropic (hence obeys the copernican principle),

(ii) is created at a phase transition in an up to then perfectly homogeneous
and isotropic universe (hence obeys specific causality \[21\] and matching \[22\] conditions),

(iii) evolves deep in the radiation era and deep in the matter era in a way which is
statistically independent of time (scaling requirement \[14\]).

We shall also make the further assumptions that the network

(iv) satisfies the conservation equations,

(v) is statistically coherent (see e.g. \[22\] and below for a precise definition).

Of all those requirements the latter two are stringent and probably not fulfilled by
realistic defect networks (see e.g. \[14,19,24\]).

Now, as for assumption (v), it is a sensible one to make, as incoherent sources can in
principle be described as a sum of coherent ones \[17\] (see \[3.3\]). As for assumption (iv),
all authors who describe semi-analytically the defects as a fluid have up to now made
it. All authors who consider global defects construct conserved stress energy tensors
(see e.g. \[24\]). Those who study local strings may have to add an extra fluid so that
the total stress energy tensor is conserved (see e.g. \[16,24\]). Only Contaldi et al. \[23\],
as far as we are aware, have questioned it. In order to take into account the network
loss of energy through gravitational radiation or particle production, they have added
an extra fluid which compensates for the non conservation of the defect stress-energy
tensor (which comes from excising small loops from the numerical simulation in order
for the network to scale). The problem with such a modelling (as Contaldi et al.
themselves note) is, first, that the results are very sensitive to the equation of state of
that extra fluid (which is left as a free parameter), and, second, that the energy lost
by the network is transferred exclusively to that extra fluid and not to the background
photons, baryons, Cold Dark Matter (CDM) and neutrinos. Therefore the modelling of
the defect network energy loss is still in a primitive stage and needs to be improved by
a careful analysis of the microphysics involved in the evolution of the network (which
includes friction effects, particle and gravitational radiation production by decaying
loops, as well as electromagnetic interactions with the plasma when the network carries
currents). In this paper, in order to keep arbitrariness to a minimum, we choose to
ignore the network energy losses.
Those five conditions being imposed, the stress-energy tensor of the defects will turn out to depend only on four free functions, two describing its scalar part and two describing respectively its vectorial and tensorial parts. Moreover, for a particular ansatz which guarantees that the defects evolve from the two scaling regimes in such a way that the conservation equations are satisfied, those four functions will, as we shall see, depend only on one variable \( u \) and tend to constants when \( u \to 0 \), and to 0 when \( u \to \infty \).

We shall choose simple ansätze for those four functions (their precise shape can in principle be obtained by numerical simulations) and solve the well-known (see e.g. [25]) linearised Einstein equations which couple the defect network to the cosmic fluid inhomogeneities. Finally we shall compute numerically the CMB anisotropies those inhomogeneities produce.

In agreement with [19] (as well as with [16][20]) we shall see that those anisotropies are very sensitive to the choices made for the free functions describing the defects. Our results therefore confirm previous work but the loss of scaling invariance during the transition is treated more carefully. They are also more complete as they include the vector and tensor contributions.

The paper is organised as follows: in §2 we write down the linearised Einstein equations (in longitudinal gauge) describing the evolution of the cosmic fluid inhomogeneities driven by the defect stress-energy tensor (we suppose that the material content of the universe is a mixture of four perfect fluids: CDM, neutrinos, baryons and photons, the latter two fluids being tightly coupled before decoupling). In §3 we see in a detailed way how the five conditions given above constrain the defect stress-energy tensor. In §4 we solve analytically the linearised Einstein equations deep in the radiation era, when all scales of interest today were larger than the Hubble radius, and we set the initial conditions on the cosmic fluid perturbations, following [22]. In §5 we give the correlation function of the CMB anisotropies in function of the cosmic fluid perturbations, assuming instantaneous decoupling (our computation is slightly different from the standard one — see e.g. [20]). Finally in §6 we compute numerically the multipole coefficients \( C_\ell \) of the CMB anisotropy correlation function using a perfect fluid code and compare our results to previous computations. We discuss the approximations made, to wit perfect fluid description, tight coupling and instantaneous decoupling hypotheses, by comparing our perfect fluid code to a Boltzmann one. We also show how to evaluate analytically the vector and tensor contributions to the \( C_\ell \) for small \( \ell \).

2. Einstein equations

2.1. The background

The universe at large appears to be remarkably homogeneous and isotropic and governed by the gravitational force created by its material content. We suppose that that was always so, and describe it in a first approximation by a Robertson-Walker geometry (that we shall for simplicity suppose spatially flat) whose time evolution satisfies Friedmann’s equations. We suppose that its material content is a mixture of perfect fluids made of decoupled black body radiations of photons and massless neutrinos, of baryons with which the photons were coupled until their temperature dropped to \( \approx 3000 \) K, and of uncoupled “Cold Dark Matter” (CDM). We thus take
the line element to be:

\[ ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^idx^j), \]

where \( \eta \) is conformal time (we set the velocity of light equal to 1), where \( x^i, i = 1, 2, 3 \) are three cartesian coordinates and \( \delta_{ij} \) is the Kronecker symbol, and where \( a(\eta) \) is the scale factor.

The conservation equations for the fluids read:

\[ \rho'_n = -3\mathcal{H}(\rho_n + P_n), \]

where \( \rho_n \) and \( P_n \) are the energy density and pressure of the \( n \)-th fluid and where \( \mathcal{H} \equiv a'/a \) is the comoving Hubble radius, with a prime denoting a derivative with respect to conformal time. The radiation fluids of photons and neutrinos both have a pressure equal to a third of their energy density, and the fluids of baryons and CDM have zero pressure (whence they have become non-relativistic). Hence the two former scale as \( a^{-4} \) and the two latter as \( a^{-3} \).

The Friedmann equation is:

\[ \mathcal{H}^2 = \frac{\kappa a^2}{3\rho_t}, \]

where \( \kappa \equiv 8\pi G \) is Einstein’s constant and \( \rho_t \equiv \sum_n \rho_n \).

Combining eqns. (2) and (3) yields:

\[ h^2 = \frac{1}{x^2}(\Omega^0_r + x\Omega^0_m), \]

with:

\[ \Omega^0_r \equiv \rho^0_\gamma + \rho^0_\nu, \]

\[ \Omega^0_m \equiv \rho^0_b + \rho^0_c, \]

\[ x \equiv a/a_0 \]

\[ h \equiv \mathcal{H}/\mathcal{H}_0, \]

where \( \Omega_n \equiv \rho_n/\rho_t \) (with \( \sum_n \Omega_n = 1 \), where the index zero means the present time and where the indices \( \gamma, \nu, b, c \) refer respectively to the photons, neutrinos, baryons and CDM.

The numerical values of \( \Omega^0_n \) are known from:

(i) the value of the Hubble constant \( H_0 \equiv \mathcal{H}_0/a_0 \), which determines \( \rho^0_\gamma \) via eq. (3) (we shall take \( H_0 = 100h_0 \) km sec\(^{-1}\)Mpc\(^{-1}\) with \( h_0 = 0.5 \)),

(ii) the temperature of the microwave background (2.726 K), which determines \( \rho^0_\nu \) via Stefan’s law,

(iii) electroweak theory, which determines the ratio \( \rho^0_\nu/\rho^0_\gamma = 0.68132 \) (assuming three families of massless, non-degenerate neutrinos),

(iv) nucleosynthesis, which determines \( \rho^0_b \) (we shall take \( \Omega^0_0 h^2_0 = 0.0125 \)).

All that yields:

\[ \Omega^0_\gamma \simeq 9.4 \times 10^{-5}, \]

\[ \Omega^0_\nu \simeq 6.3 \times 10^{-5}, \]

\[ \Omega^0_b = 5.0 \times 10^{-2}, \]

\[ \Omega^0_c \simeq 0.95. \]
The value \( x_d \) of \( x \) at decoupling is given by the ratio of the present temperature of the photon background and its temperature at decoupling: \( x_d \approx 0.9 \times 10^{-3} \). Equality of matter and radiation is defined by \( x_{eq} \equiv \Omega^0_r/\Omega^0_m \) and occurs before decoupling.

Equation (4) can be easily solved to give \( x(\eta) \):

\[
\eta_0 - \eta = \frac{2}{\Omega^0_r H_0} \left[ 1 - \sqrt{\Omega^0_r + x\Omega^0_m} \right].
\]

In the radiation era \( (x \ll x_{eq}) \), \( x \propto \eta \), in the matter era \( (x \gg x_{eq}) \), \( x \propto \eta^2 \) and in both eras \( \mathcal{H} \propto 1/\eta \).

Finally, when the scale factor behaves as \( a(\eta) \propto \eta^q \) with \( q > 0 \) when \( \eta \to 0 \) (which is the case in the standard scenario we consider here) the particle horizon of the point \( x' \) is defined by \( r = \eta \) \( (\text{where } r \equiv |x' - x'|) \) is the comoving spatial distance from \( x' \) and is proportional to the comoving Hubble radius \( \mathcal{H}^{-1} \).

2.2. Perturbation equations

Following Bardeen [27] (see also [28]) we split the perturbations of the geometry and the matter variables into (spatial) scalar, vector and tensor components (for reviews of this formalism, see e.g. [25]). We shall work in the longitudinal (also called newtonian) gauge in which the line element of a perturbed Friedmann-Robertson-Walker space time reads:

\[
ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 - 2\Phi_i dx^i d\eta + [(1-2\Psi)\delta_{ij} + 2E_{ij}] dx^i dx^j].
\]

Here and in the following all barred spatial vectors (such as \( \bar{\Phi} \)) are divergenceless \((\partial_i \bar{\Phi}^i = 0)\), all barred spatial tensors (such as \( \bar{E}_{ij} \)) are divergenceless and traceless \((\partial_i \bar{E}^i = 0, \bar{E}^i = 0)\) and all spatial indices are moved with the Kronecker symbol \( \delta_{ij} \). The six functions \( (\Phi, \Psi, \bar{\Phi}, \bar{\Psi}, \bar{E}_{ij}) \) are respectively the scalar, vector and tensor parts of the metric perturbations in longitudinal gauge.

We shall treat the material content of the universe as a mixture of perfect fluids (see [2] below for a discussion of this hypothesis). The perturbations of the stress-energy tensor of the \( n \)-th perfect fluid in this perturbed universe are:

\[
\delta T^n_{00} = a^2 \rho_n (\delta_n + 2\Phi) \quad (15)
\]
\[
\delta T^n_{0i} = a^2 [\rho_n \bar{\Phi}_i - (\bar{v}_{i,n} + \partial_n v_i)(\rho_n + P_n)] \quad (16)
\]
\[
\delta T^n_{ij} = a^2 P_n [\bar{E}_{ij} + \delta_{ij}(\delta_n - 2\Psi)] \quad (17)
\]

where \( \delta_n = \delta \rho_n / \rho_n \) is its density contrast and \( \partial_n v_i + \bar{v}_{i,n} \) its velocity perturbation.

The stress-energy tensor of the topological defects is in itself a small perturbation (this is the so-called “stiff approximation”, see e.g. [29]). We decompose its components as:

\[
\frac{\kappa}{\mathcal{H}_0} \Theta_{00} = \rho^s, \quad (18)
\]
\[
\frac{\kappa}{\mathcal{H}_0} \Theta_{0i} = - (\bar{v}_i^s + \partial_i v^s), \quad (19)
\]
\[
\frac{\kappa}{\mathcal{H}_0} \Theta_{ij} = \Pi^s_{ij} + \partial_i \bar{\Pi}^s_j + \partial_j \bar{\Pi}^s_i + \delta_{ij} \left( P^s - \frac{1}{3} \Delta \Pi^s \right) + \partial_{ij} \Pi^s. \quad (20)
\]

The ten source functions \( (\rho^s, P^s, v^s, \bar{\Pi}^s_i, (\bar{v}_i^s, \bar{\Pi}^s_i) \) and \( \bar{\Pi}^s_{ij} \) will be discussed later.
We shall write the evolution equations in Fourier space, the Fourier transform of any function \( f(x^i, \eta) \) being defined (formally) as:

\[
\hat{f}(k^i, \eta) = \frac{\mathcal{H}_0^3}{(2\pi)^{3/2}} \int d^3x e^{-ik^ix^i} f(x^i, \eta),
\]

(21)

\[
f(x^i, \eta) = \frac{1}{(2\pi)^{3/2}} \mathcal{H}_0^3 \int d^3k e^{ik^ix^i} \hat{f}(k^i, \eta).
\]

(22)

(The factor \( \mathcal{H}_0^3 \) insures that \( \hat{f} \) and \( f \) have the same dimension.)

As announced in the introduction, we assume that the defects interact only gravitationally with the rest of the matter. Therefore their stress-energy tensor is conserved, which yields for all (but the \( k^i = 0 \)) modes:

\[
\frac{d\hat{\rho}^s}{d\eta} = -3 \hat{\rho}^s - 3 \hat{\rho}^s + k \hat{V}^s,
\]

(23)

\[
\frac{d\hat{V}^s}{d\eta} = -2 \hat{\Phi} + \left(-\hat{\rho}^s + \frac{2}{3} \hat{\pi}^s\right),
\]

(24)

\[
\frac{d\hat{\pi}^s}{d\eta} = -2 \hat{\pi}^s + k \hat{\pi}^s,
\]

(25)

where we have introduced \( k \equiv \sqrt{k^ik^i} \), where a barred vector is orthogonal to \( k^i \) and where

\[
\hat{V}^s \equiv k \hat{v}^s,
\]

(26)

\[
\hat{\pi}^s \equiv k^2 \hat{\Pi}^s,
\]

(27)

\[
\hat{\bar{\pi}}^s_i \equiv k \hat{\bar{\Pi}}^s_i.
\]

(28)

After decoupling the fluids of neutrinos, photons, baryons and CDM are decoupled. Their stress-energy tensors are therefore separately conserved, which yields:

\[
\frac{d\hat{\delta}_\nu^s}{d\eta} = \frac{4}{3} k \hat{V}_\nu, \quad \frac{d\hat{V}_\nu}{d\eta} = -k \left( \hat{\Phi} + \frac{1}{4} \hat{\delta}_\nu \right), \quad \frac{d\hat{\pi}^i_\nu}{d\eta} = 0,
\]

(29)

\[
\frac{d\hat{\delta}_\gamma^s}{d\eta} = \frac{4}{3} k \hat{V}_\gamma, \quad \frac{d\hat{V}_\gamma}{d\eta} = -k \left( \hat{\Phi} + \frac{1}{4} \hat{\delta}_\gamma \right), \quad \frac{d\hat{\pi}^i_\gamma}{d\eta} = 0,
\]

(30)

\[
\frac{d\hat{\delta}_b^s}{d\eta} = k \hat{V}_b, \quad \frac{d\hat{V}_b}{d\eta} = -\mathcal{H} \hat{V}_b - k \hat{\Phi}, \quad \frac{d\hat{\pi}^i_b}{dx} = -\mathcal{H} \hat{\pi}^i_b,
\]

(31)

\[
\frac{d\hat{\delta}_c^s}{d\eta} = k \hat{V}_c, \quad \frac{d\hat{V}_c}{d\eta} = -\mathcal{H} \hat{V}_c - k \hat{\Phi}, \quad \frac{d\hat{\bar{\pi}}^i_c}{d\eta} = -\mathcal{H} \hat{\bar{\pi}}^i_c.
\]

(32)

where we have introduced

\[
\hat{\delta}_n^s \equiv \hat{\delta}_n - 3 \left( 1 + \frac{P_{\nu}}{\rho_n} \right) \hat{\Psi},
\]

(33)

\[
\hat{V}_n^s \equiv k \hat{v}_n.
\]

(34)

Before decoupling the density contrasts and velocity perturbations of the neutrino and CDM energy fluids still obey eqns. (29) and (32) but the baryons and photons being
tightly coupled, eqns. (30) and (31) must be replaced by (using also the fact that the number of baryons is conserved)

\[
\frac{d\hat{\delta}_b}{d\eta} = k\hat{V}_b, \quad \frac{d\hat{\delta}_\gamma}{d\eta} = \frac{4}{3}k\hat{V}_\gamma, (35)
\]

\[
\hat{V}_b = \hat{V}_\gamma, \quad \frac{d\hat{V}_\gamma}{d\eta} = -\mathcal{H}R\hat{V}_\gamma - k \left( \frac{1}{4}(1 - R)\hat{\delta}_\gamma + \hat{\Phi} \right), (36)
\]

\[
\hat{v}_{i,b} = \hat{v}_{i,\gamma}, \quad \frac{d\hat{v}_{i,\gamma}}{d\eta} = -\mathcal{H}R\hat{v}_{i,\gamma}, (37)
\]

where we have introduced \( R \equiv (3\alpha x)/(4+3\alpha x) \), \( \alpha \equiv \Omega_0^b/\Omega_0^\gamma \). (It is clear that imposing that the density contrasts and velocity perturbations of the baryon and photon fluids obey eqns. (35-37) until decoupling and (30,31) after decoupling, all variables being continuous at \( x = x_d \), is a first approximation : see \( \S 6 \) below.)

The Einstein equations give, after linearisation, the scalar and vector metric perturbations algebraically in function of the matter perturbations as

\[
\hat{\Psi} = \frac{-1}{k^2 + 3(H^2 - d\mathcal{H}/d\eta)} \times \\
\left[ 3H^2 \left( \sum_n \Omega_n \left( \hat{\delta}_n^s - 3(1 + \omega_n)\frac{H}{k}\hat{V}_n \right) \right) + H_0^2 \left( \rho^s - 3\frac{H}{k}\hat{V}^s \right) \right], (38)
\]

\[
\hat{\Phi} = \hat{\Psi} - \frac{\hat{\pi}^s}{q^2}, (39)
\]

\[
\hat{\Phi}_s = \frac{2}{k^2} \left( 3H^2 \sum_n \Omega_n (1 + \omega_n)\hat{v}_{i,n} + H_0^2 \hat{\delta}_i^s \right), (40)
\]

(The other Einstein equations are redundant because of Bianchi’s identities. The index \( n \) runs on the four background fluids, and we have set \( \omega_b = \omega_c = 0, \omega_\nu = \omega_\gamma = 1/3 \) and \( q \equiv k/H_0 \).) Finally the tensor metric perturbations are not determined algebraically : they solve the differential equation

\[
\frac{d^2\hat{E}_{ij}}{d\eta^2} - 2\mathcal{H}\frac{d\hat{E}_{ij}}{d\eta} + k^2\hat{E}_{ij} = \mathcal{H}_0^2\hat{\Pi}_{ij}. \tag{41}
\]

where a barred tensor is traceless and orthogonal to \( k^i \).

The background quantity \( \mathcal{H}(\eta) \) being known from eqns. (3, 13) and (29-41) give the evolution of the perturbations in function of \( \eta \) and the comoving wavenumber \( k \), once the source variables (constrained by eqns. (23-25)), are known in function of \( \eta \) and \( k \) (\( \S 3 \)) and once the initial conditions are set (\( \S 4 \)).

3. The stress-energy tensor \( \Theta_{\mu\nu} \) of the sources

3.1. Homogeneity and isotropy constraints

The ten components of the stress-energy tensor \( \Theta_{\mu\nu}(\eta, x^i) \) of the topological defects \( (\mu, \nu = 0, 1, 2, 3) \) are ten statistically spatially homogeneous and isotropic random fields. So are their ten Fourier transforms \( \hat{\Theta}_{\mu\nu}(\eta, k^i) \) (which are complex but such that \( \hat{\Theta}_{\mu\nu}^*(\eta, k^i) = \hat{\Theta}_{\mu\nu}(\eta, -k^i) \)). (We ignore the \( k^i = 0 \) mode which can be absorbed in the background.)
The statistical properties of those ten random fields will be described by their unequal time two-point correlators

\[
\langle \Theta_{\mu\nu}(\eta, x^i)\Theta_{\rho\sigma}(\eta', x'^i) \rangle \equiv \frac{\mathcal{H}_0}{\kappa^2} C_{\mu\nu\rho\sigma}(\eta, \eta', r^i),
\]

where \(\langle \ldots \rangle\) means an ensemble average on a large number of realisations, and where the correlator \(C_{\mu\nu\rho\sigma}\) is a tensor which depends only on \(\eta, \eta'\) and \(r^i \equiv x^i - x'^i\) because of the spatial homogeneity of the distribution. The power spectra of the correlators \(C_{\mu\nu\rho\sigma}\) are defined as

\[
P_{\mu\nu\rho\sigma}(\eta, \eta', k^i) \equiv (2\pi)^{3/2} \hat{C}_{\mu\nu\rho\sigma}(\eta, \eta', k^i),
\]

where a hat denotes a Fourier transform. The power spectra are related to the correlators in Fourier space by

\[
\langle \hat{\Theta}^{\mu}_{\nu}(\eta, k^i)\hat{\Theta}_{\rho\sigma}(\eta', k'^i) \rangle = \delta(k^i - k'^i) \frac{\mathcal{H}_0^4}{\kappa^2} P_{\mu\nu\rho\sigma}(\eta, \eta', k^i).
\]

The spatial isotropy of the distribution now forces the power spectra to be of the form

\[
P_{0000} = A_0, \quad P_{000k} = ikkB_1, \quad P_{00ij} = C_0\delta_{ij} + C_2k_i k_j, \quad P_{000j} = D_0\delta_{ij} + D_2k_i k_j, \quad P_{0ijk} = i\left[ E_1k_i\delta_{jk} + \bar{E}_1(k_j\delta_{ik} + k_k\delta_{ij}) + E_3k_i k_j k_k \right],
\]

\[
P_{ijkl} = F_0\delta_{ij}\delta_{kl} + F_0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + F_2(k_i k_j \delta_{kl} + k_k k_l \delta_{ij}) + F_2(k_j k_k \delta_{il} + k_i k_l \delta_{ij}) + F_2(k_k k_l \delta_{ij} + k_j k_k \delta_{il}) + F_3 k_i k_j k_k k_l,
\]

where \(A_0, B_1\) etc are 14 real functions of \(\eta, \eta'\) and the modulus \(k\) of the spatial vector \(k^i\).

When \(\Theta_{\mu\nu}\) is decomposed into its scalar, vector and tensor components according to eq. (32), then eqns. (43) and (44) yield

\[
\langle \hat{\rho}_{\mu}^{\mu} \hat{\rho}_{\mu} \rangle = A_0, \quad \langle \hat{\rho}_{\mu}^{\mu} \hat{V}_{\mu} \rangle = -kB_1, \quad \langle \hat{\rho}_{\mu}^{\mu} \hat{\pi}_{\mu} \rangle = C_0 + \frac{1}{3}k^2 C_2, \quad \langle \hat{\rho}_{\mu}^{\mu} \hat{V}_{\mu} \rangle = -k^2 C_2, \quad \langle \hat{\rho}_{\mu}^{\mu} \hat{\pi}_{\mu} \rangle = D_0 + k^2 D_2,
\]

\[
\langle \hat{V}_{\mu}^{\mu} \hat{P}_{\mu} \rangle = k \left[ (E_1 + \frac{2}{3}E_2) + \frac{k^2}{3} E_3 \right], \quad \langle \hat{V}_{\mu}^{\mu} \hat{\pi}_{\mu} \rangle = -k(2E_1 + k^2 E_3),
\]

\[
\langle \hat{P}_{\mu}^{\mu} \hat{F}_{\mu} \rangle = \left( F_0 + \frac{2}{3}F_0 \right) + \frac{2}{3}k^2 \left( F_2 + \frac{2}{3}F_2 \right) + \frac{1}{9}k^4 F_4, \quad \langle \hat{P}_{\mu}^{\mu} \hat{\pi}_{\mu} \rangle = -k^2 \left[ (F_2 + \frac{4}{3}F_2) + \frac{1}{3}k^2 F_4 \right],
\]

\[
\langle \hat{\pi}_{\mu}^{\mu} \hat{\pi}_{\mu} \rangle = 3F_0 + 4k^2 \bar{F}_2 + k^4 F_4.
\]
\[
\langle \hat{\psi}_{i}^{\mu} \hat{\psi}_{j}^{\nu} \rangle = D_{0} P_{ij}, \quad (61)
\]
\[
\langle \hat{\psi}_{i}^{\mu} \hat{\eta}_{j}^{\nu} \rangle = -k \hat{E}_{i} P_{ij}, \quad (62)
\]
\[
\langle \hat{\pi}_{i}^{\mu\nu} \hat{\pi}_{j}^{\rho\sigma} \rangle = (F_{0} + k^{2} F_{2}) P_{ij}, \quad (63)
\]
\[
\langle \hat{\Pi}_{ij} \hat{\Pi}_{kl} \rangle = \hat{F}_{0}(P_{ik} P_{jl} + P_{il} P_{jk} - P_{ij} P_{kl}), \quad (64)
\]

where \(\langle \hat{\rho}_{i}^{*}(\eta, k^{i}) \hat{\rho}_{i}(\eta', k'^{i}) \rangle = \delta(k^{i} - k'^{i}) \langle \hat{\rho}_{i}^{*} \hat{\rho}_{i} \rangle\) etc., and where \(P_{ij} = \delta_{ij} - k_{i} k_{j}/k^{2}\). All other correlators are zero so that the scalar, vector and tensor parts of \(\Theta_{\mu\nu}\) fall into statistically independent sets.

We also note that if \((k^{2} F_{2})\) and \((k^{4} F_{4})\) are of higher order in \(k\) than \(\hat{F}_{0}\), then, for small \(k\) \(\mathbb{[7, 21]}\)
\[
\frac{1}{3} \langle \hat{\pi}_{i}^{*} \hat{\pi}_{i} \rangle \simeq \frac{1}{2} \langle \hat{\pi}_{i}^{*} \hat{\pi}_{i} \rangle \simeq \frac{1}{4} \langle \hat{\Pi}_{ij} \hat{\Pi}_{ij} \rangle. \quad (65)
\]

3.2. Causality constraints

Since the network of defects appeared at a definite time the distribution must be, for causality reasons, completely uncorrelated on scales larger than the particle horizon. Therefore, as stressed e.g. by Turok \(\mathbb{[21]}\), the unequal time correlators are strictly zero outside the intersection of the past light-cones, that is:
\[
C_{\mu\nu\rho\sigma}(\eta, \eta', r^{i}) = 0 \quad \text{if} \quad r > \eta + \eta'. \quad (66)
\]

Property \(\mathbb{[6]}\) translates in Fourier space into the fact that the equal time power spectra are white noise on super horizon scales (that is for \(k \eta \ll 1\)). Indeed, because the correlators \(\mathbb{[6]}\) have compact supports their Fourier transforms are \(C^{\infty}\) in \(k^{i}\). Therefore causality forces the fourteen functions \(A_{0}, B_{1}\) etc. to be \(C^{\infty}\) in \(k^{2}\).

Moreover, since within one horizon volume there are almost no defects, those fourteen functions must tend to zero on small scales, that is for \(k \eta \gg 1\).

3.3. Coherence hypothesis

Any distribution of active sources such as topological defects must be homogeneous, isotropic and causal. The hypothesis of statistical coherence, that we shall now make, is stringent but motivated by the fact that any incoherent distribution can in principle be decomposed into a sum of coherent ones \(\mathbb{[17]}\).

By definition two statistically homogeneous random fields \(S_{\alpha}(\eta, k^{i})\) and \(S_{\beta}(\eta, k^{i})\) are statistically coherent if their correlators factorize, that is if
\[
\langle S_{\alpha}(\eta, k^{i}) S_{\beta}(\eta', k'^{i}) \rangle = \delta(k^{i} - k'^{i}) p_{\alpha}(\eta, k^{i}) p_{\beta}(\eta', k^{i}) \quad (67)
\]
for \(a, b = 1, 2\). For our purposes this is equivalent to saying that
\[
\hat{S}_{\alpha}(\eta, k^{i}) = p_{\alpha}(\eta, k^{i}) e(k^{i}) \quad (68)
\]
where \(p_{\alpha}(\eta, k^{i})\) is a real function and where \(e(k^{i})\) is a normalized complex random field such that \(\langle e^{*}(k^{i}) e(k'^{i}) \rangle = \delta(k^{i} - k'^{i})\).

Let us first combine coherence and isotropy, that is \(\mathbb{[51, 54]}\). The four scalar components \(\hat{\rho}_{s}, \hat{V}_{s}, \hat{P}_{s}, \) and \(\hat{\pi}_{s}\) of \(\Theta_{\mu\nu}\) form a statistically independent set. The
hypothesis (68) of maximal coherence implies that
\[
\hat{\rho}^*(\eta, k^i) = p_\rho(\eta, k)e(k^i),
\]
\[
\hat{\nu}^*(\eta, k^i) = p_\nu(\eta, k)e(k^i),
\]
\[
\hat{\phi}^*(\eta, k^i) = p_\phi(\eta, k)e(k^i),
\]
\[
\hat{\pi}^*(\eta, k^i) = p_\pi(\eta, k)e(k^i).
\]
Hence, out of the fourteen functions \(A_0, B_1\) etc appearing in (51-60) ten can be expressed in terms of the four real functions \(p_\alpha\) and the four functions (e.g. \(D_0, E_1, F_0\) and \(F_2\)) that remain arbitrary. More precisely:

\[
A_0 = p_\rho^2,
\]
\[
B_1 = -p_\rho \left(\frac{\nu}{k}\right),
\]
\[
C_0 = p_\rho \left(3pp + \frac{1}{3}p_\pi\right),
\]
\[
C_2 = -p_\rho \left(\frac{p_\pi}{k^2}\right),
\]
\[
E_1 = p_\nu \left(2pp + \frac{1}{3}p_\pi\right),
\]
\[
D_2 = \left(\frac{p_\nu}{k}\right)^2 - \left(\frac{D_0}{k^2}\right),
\]
\[
E_2 = -\left(\frac{p_\nu}{k}\right) \left(\frac{p_\pi}{k^2}\right) - 2\left(\frac{E_1}{k^2}\right),
\]
\[
F_2 = -\frac{p_\pi}{k^2} \left(2pp + \frac{1}{3}p_\pi\right) + \left(\frac{F_0}{k^2}\right),
\]
\[
F_0 = \left(2pp + \frac{1}{3}p_\pi\right)^2 - \left(\frac{F_0}{k^2}\right),
\]
\[
F_4 = \left(\frac{p_\pi}{k^2}\right)^2 - 3\left(\frac{F_0}{k^2}\right) - 4\left(\frac{F_2}{k^2}\right).
\]

The four vector components \(\hat{v}_i^*\) and \(\hat{\pi}_i^*\) of \(\hat{\Theta}_{\mu\nu}\) form yet another statistically independent set. In order to obey (73) as well as (61-63) they must fall into two independent subsets corresponding to two different polarisations. Hence we shall write them as

\[
\hat{v}_i^*(\eta, k^i) = p_\epsilon(\eta, k)e_i(k^i),
\]
\[
\hat{\pi}_i^*(\eta, k^i) = p_{\pi}(\eta, k)e_i(k^i),
\]
\[
e_i(k^i) \equiv l_ie_i(k^i) + \bar{m}e_i(k^i),
\]
where \(\langle e_A^*(k^i)e_B(k^j)\rangle = \delta_{AB}\delta(k^i - k^j)\) with \(A, B\) standing for \(l\) or \(m\), and where \(l^i\) and \(\bar{m}^i\) form with \(k^i/k\) an orthonormal basis (so that \(P_{ij} = l_il_j + \bar{m}_i\bar{m}_j\)). Hence two of the three previously free functions \(D_0, E_1\) and \(F_0\) can be expressed in terms of the two real functions \(p_\pi\) and \(p_\bar{\pi}\) as [see (61-63)]

\[
D_0 = p_\epsilon^2,
\]
\[
E_1 = -\frac{p_\epsilon}{k}p_\pi,
\]
\[
F_0 = p_\pi^2 - k^2F_2.
\]
Finally the two tensor components $\hat{\Pi}_{ij}^{\ast}$ of $\hat{\Theta}_{\mu \nu}$ form a third statistically independent set. In order to obey (84) they cannot be coherent but, on the contrary, independent of each other. Hence we shall write them as

$$\hat{\Pi}_{ij}^{\ast}(\eta, k^i) = p_T(\eta, k)e_{ij}(k^i), \quad (89)$$

$$e_{ij}(k^i) = \hat{e}_{ij}(k^i) + \bar{m}_{ij}(k^i), \quad (90)$$

where $\hat{e}_{ij} = \hat{e}_{ij}^2 - \bar{m}_{ij}$ and $\bar{m}_{ij} = \hat{e}_{ij}$ are the two polarisation tensors, where $\langle e_{ij}^2(k^i)e_B(k^i) \rangle = \delta_{AB}\delta(k^3 - k')$ with $A, B$ standing for $+ \, \text{or} \, \times$, and where (see eqns. [84])

$$\bar{F}_0 = p_2^2. \quad (91)$$

In conclusion combining coherence and isotropy reduces the number of free functions from fourteen to seven: $p_\mu, p_V, p_P, p_\pi, p_\sigma$ and $p_T$.

Let us add now the causality constraint which states that the fourteen functions $A_0$, $B_1$ etc must be $C^\infty$ in $k^2$. It implies [see (73)] that $p_\mu, p_V/k, p_P$ and $p_\pi/k^2$ are $C^\infty$. This in turn implies [see eqns. (88-82)] that $D_0/k^2$, $E_1/k^2$, $F_2/k^2$ and $\bar{F}_0/k^4$ are analytic as well, and therefore, from (84), $p_/k$ and $p_\pi/k^2$ are $C^\infty$. Finally, from (81), $p_T$ is also $C^\infty$.

An immediate consequence of this regularity requirement is that

$$\langle \hat{\pi}_n^\ast \hat{\pi}_n \rangle = p_\pi^2 = O(k^4), \quad (92)$$

$$\langle \hat{\pi}_n^\ast \hat{\pi}_n^\dagger \rangle = 2p_\pi^2 = O(k^4), \quad (93)$$

$$\langle \hat{\Pi}_{ij} \hat{\Pi}_{ij} \rangle = 4p_\pi^2 = O(k^4), \quad (94)$$

so that, contrarily to the general case (84), the anisotropic stress correlators of coherent defects (when $p_\pi \neq p_\sigma \neq p_T$) are not in a definite ratio for small $k$.

A question that can be asked is how can a sum of coherent defects (with anisotropic stress correlators of order $k^4$) leads to an incoherent defect (with anisotropic stress correlator of order $k^0$). In a paper to be submitted [30], we show explicitly how the $k^0$-behaviour of incoherent defects can be obtained as a (infinite) sum of $k^4$-behaving coherent defects. We also show how the causality constraints fade away in the process and how the procedure introduces almost no extra arbitrariness.

3.4. Scaling hypothesis

We shall now make the other assumption that, as long as the evolution of the universe is described by a single scale (to wit the Hubble radius $H^{-1}$), that is deep in the radiation era and deep in the matter era but not during the transition, the dimensionless quantities $\Theta_{\mu \nu}/\rho_0$ are scale invariant, that is such that

$$C_{\mu \nu \mu \nu}(\eta, \eta, r^i) = H^4 F_{\mu \nu}(r^i/H) \quad (95)$$

(using the fact that $\kappa \rho_0 < H^2$ in both eras). This assumption is supported by a number of numerical simulations [14, 15, 16, 31] as well as qualitative arguments [32]. In Fourier space this hypothesis translates as

$$P_{\mu \nu \mu \nu}(\eta, \eta, k^i) = (2\pi)^{3/2} H \hat{F}_{\mu \nu}(k^i/H). \quad (96)$$

This implies [see (13, 31)] that the functions $A_0/H$, $D_0/H$, $F_0/H$, $\bar{F}_0/H$ on one hand, $D_2H$, $F_2H$, $\bar{F}_2H$ on another and finally $F_3H^3$ are all functions of $k/H$ only (when
\( \eta = \eta' \). From eqns. [82-83] one then obtains the behaviour of the equal time correlators \( \langle \hat{\rho}_s^2 \hat{\rho}_s \rangle, \langle \hat{V}_s^* \hat{V}_s \rangle, \langle \hat{P}_s^* \hat{P}_s \rangle, \langle \hat{\pi}_s^* \hat{\pi}_s \rangle, \langle \hat{\pi}_s^* \hat{\pi}_s \rangle \) and \( \langle \hat{\Pi}_{ij}^* \hat{\Pi}_{kl} \rangle \).

Combining this scaling hypothesis with the one of coherence implies [see eqns. (73)-[82]] that all six \( p_u/\sqrt{H} \) are functions of \( k/H \) only.

Therefore, all in all, coherent defects, which scale during the radiation and matter dominated eras, which are causal and statistically homogeneous and isotropic, are described by the following random variables:

\[
\begin{align*}
\hat{\rho}^s &= \sqrt{h} f_1(q/h, x) e(q'), \\
\hat{P}^s &= \sqrt{h} f_2(q/h, x) e(q'), \\
\hat{V}^s &= -\frac{q}{\sqrt{h}} f_3(q/h, x) e(q'), \\
\hat{\pi}^s &= \frac{q^2}{h^{3/2}} f_4(q/h, x) e(q'), \\
\hat{\pi}_i^s &= \frac{q}{\sqrt{h}} f_5(q/h, x) \bar{e}_i(q'), \\
\hat{\Pi}_{ij} &= -\frac{q^2}{h^{3/2}} f_6(q/h, x) \bar{e}_{ij}(q').
\end{align*}
\]

where we have reintroduced the dimensionless quantities \( x \equiv a/a_0, h \equiv H/H_0 \), and where the seven dimensionless functions \( f_{ij}(q/h, x) \) depend only on \( q/h \) deep in the radiation or matter eras, tend to constants on super-horizon scale (that is for \( k \eta \propto q/h \ll 1 \)) and to zero on small scales.

### 3.5. Conservation laws and loss of scaling invariance during the radiation to matter dominated transition

When the defects are described by eqns. [97-103] the conservation equations [23-25] become

\[
\begin{align*}
\frac{1}{2}(1 + 3\omega) \frac{\partial f_3}{\partial u} + \frac{\partial f_3}{\partial x} + \frac{3}{4} (3 + \omega) f_3 &= f_2 - 2 f_4, \\
\frac{1}{2}(1 + 3\omega) \frac{\partial f_1}{\partial u} + \frac{\partial f_1}{\partial x} + \frac{3}{4} (1 - \omega) f_1 &= - 3 f_2 - u^2 f_3, \\
\frac{1}{2}(1 + 3\omega) \frac{\partial f_5}{\partial u} + \frac{\partial f_5}{\partial x} + \frac{3}{4} (3 + \omega) f_5 &= u^2 f_6,
\end{align*}
\]

where we have introduced

\[
\begin{align*}
u \equiv q/h, \\
\omega \equiv \frac{1}{2} \frac{\Omega_0}{\Omega_r + x \Omega_m}.
\end{align*}
\]

Deep in the radiation or matter eras \( \omega \rightarrow 1/3 \) or 0, and scaling solutions independent of \( x \) are readily found for \( f_2, f_4 \) and \( f_6 \) in terms of the four free functions \( f_1, f_3 \) and \( f_5 \). During the transition from the radiation to matter dominated eras, scaling is lost. How can this loss of scaling be modelled, and how is the final result dependent on this modelisation?

Two different approaches have been followed:
(i) Durrer et al. [13] used a “sudden transition” approximation in which $\omega = 1/3$ until $x = x_{eq}$ and $\omega = 0$ afterwards. The problem here is that the functions $f_a$ cannot all be continuous at $x = x_{eq}$ and that energy momentum conservation is violated during the transition†.

(ii) Turok [21], Hu et al. [34], Cheung and Magueijo [35] fix the time dependence of two source functions and determine the other two by integrating (104-106).

We follow a route similar to (ii), which also amounts to making fairly artificial ansätze but which guarantees that the conservation equations are exactly satisfied: we first impose

\begin{align}
 f_1 &= \mathcal{F}(u), \\
 f_3 &= \hat{f}_3(u)g(x), \\
 f_5 &= \hat{f}_5(u)p(x), \\
 f_7 &= \mathcal{Q}(u). \\
\end{align}

It then follows from (104-106) that

\begin{align}
 \hat{f}_3(u) &= -\frac{(1 - \omega)\mathcal{F}}{3(3 + \omega)g + 4x dg/dx} + u^2 \mathcal{G}(u, x) \\
 \hat{f}_5(u) &= \frac{u^2 \mathcal{P}(u, x)}{3(3 + \omega)p + 4x dp/dx}
\end{align}

where $\mathcal{G}(u, x)$ and $\mathcal{P}(u, x)$ are some functions, $C^\infty$ in $u^2$, that we impose to depend on $u$ only

\begin{align}
 \mathcal{G}(u, x) &= \mathcal{G}(u), \quad \mathcal{P}(u, x) = \mathcal{P}(u).
\end{align}

This implies that $g(x)$ and $p(x)$ satisfy the following differential equations

\begin{align}
 (3 + \omega)g + \frac{4}{3} x dg/dx &= 1 - \omega \\
 (3 + \omega)p + \frac{4}{3} x dp/dx &= 1
\end{align}

(up to irrelevant overall constants) with $\omega$ given by (108) and $g(x \to 0) = 1/5$, $p(x \to 0) = 3/10$ (and $g(x \to \infty) = p(x \to \infty) = 1/3$). We hence arrive at the following expressions for the seven functions $f_a$ in terms of the four arbitrary $C^\infty$ functions $\mathcal{F}(u)$, $\mathcal{G}(u)$, $\mathcal{P}(u)$ and $\mathcal{Q}(u)$, with $g(x)$ and $p(x)$ being given by (116-117):

\begin{align}
 f_1 &= \mathcal{F}, \\
 f_2 &= -\frac{1}{4}(1 - \omega)\mathcal{F} - \frac{1}{6}(1 + 3\omega)u \frac{d\mathcal{F}}{du} + \frac{u^2}{9} g\mathcal{F} - \frac{u^4}{3} g\mathcal{G}, \\
 f_3 &= \left[ -\frac{1}{3}\mathcal{F} + \frac{u^2}{3} \mathcal{G} \right] g, \\
 f_4 &= \frac{1}{4}(1 + 3\omega) \left[ (g - 1) \frac{d\mathcal{F}}{u du} - 6Gg - 3gu \frac{dG}{du} \right],
\end{align}

† Note added in proof: In Durrer et al. [33], the loss of scaling is taken into account by interpolating the source functions computed respectively in the radiation and matter era. The fit is performed by using either the function $t_{eq}/(t + t_{eq})$ or $\exp(-t/t_{eq})$ and the results do not depend too much on this interpolation. However, energy conservation is still violated during the transition.
\[ + \frac{1}{6} g F - \frac{9}{8} (1 - \omega) g - \frac{u^2}{2} G g, \]

\[ f_5 = \frac{1}{3} u^2 \mathcal{P} p, \] (122)

\[ f_6 = \left[ \frac{1}{4} + \left( 1 + 3 \omega \right) \frac{p}{3} \right] \mathcal{P} + \frac{1}{6} (1 + 3 \omega) p u \frac{d \mathcal{P}}{du}, \] (123)

\[ f_7 = \mathcal{Q}. \] (124)

In conclusion, we note that in our model both components of the defects stress-energy tensor are generically present, in contrary to e.g. the “pressure model” or the “anisotropic stress model" [34][35].

4. Initial conditions

The set of equations (29-41) with the defects described by eqns.(97-103) and (118-124) will be solved numerically. However, deep in the radiation era when all scales of cosmological interest today were larger than the Hubble radius, an analytic solution exists, which is easily found when adding to the set (29-41) the Einstein equations for scalar perturbations which we had not included because they were redundant (by virtue of the Bianchi identities). Those two extra equations can be written as:

\[ \frac{d}{dx} \Psi = - \frac{\dot{\Phi}}{2 q h} \] (125)

\[ (\Omega^0_r + x \Omega^0_m) \frac{d^2 \Psi}{dx^2} = - \frac{1}{x^{4/3}} \frac{d}{dx} \left( 4 \Omega^0_r + 9 \frac{\Omega^0_m}{2} x \right) \] (126)

Deep in the radiation era, when \( x \Omega^0_m \ll \Omega^0_r \); \( h x \simeq \sqrt{\Omega^0_r} \) (and \( x \propto \eta \)). The wavelengths we are interested in are then much bigger than the Hubble radius: \( q / h \ll 1 \), that is \( q^2 x^2 \ll \Omega^0_r \) (or \( u^2 \ll 1 \)). In that regime the conservation equations for the defects [eqns. (118-124)] yield that the functions \( f_a \) are constants such that

\[ f_1 = - 6 f_2 + O(u^2), \] (127)

\[ f_3 = \frac{2}{5} f_2 + O(u^2), \] (128)

\[ f_5 = \frac{2}{9} u^2 f_6 + O(u^4). \] (129)

Let us first solve the equations for the scalar perturbations. The conservation equations for the fluids (29, 32, 35-37) impose that in the long wavelength limit all the
density perturbations $\hat{\delta}_n^\flat$ are constant, that is random variables independent of time.

\[
\begin{align*}
\hat{\delta}_\nu^\flat & \approx \delta_{\nu}^{\text{in}}, \\
\hat{\delta}_c^\flat & \approx \delta_{c}^{\text{in}}, \\
\hat{\delta}_b^\flat & \approx \delta_{b}^{\text{in}}, \\
\hat{\delta}_\gamma^\flat & \approx \delta_{\gamma}^{\text{in}},
\end{align*}
\]

where $\approx$ means deep in the radiation era and up to terms of order $u^2$. Eq. (126) for $\hat{\Psi}$ is therefore a closed equation in that regime. Its solution is

\[
\hat{\Psi} \approx \left[ \Psi_0 + \Psi_1 x^{3/2} (\hat{f}_4 + \hat{f}_2) e(q^\prime) - \frac{\delta_{m}^{\text{in}}}{8 \Omega_0^r} x \right],
\]

(134)

where $\Psi_0$ and $\Psi_1$ are two integration constant random variables and where we have set $\hat{f}_a \equiv \Omega_0^{b-3/4} \lim_{u \to 0} f_a$. The perturbation $\hat{\Phi}$ then follows from eq. (39)

\[
\hat{\Phi} \approx \left[ \Psi_0 + \Psi_1 x^{3/2} (\hat{f}_4 - \hat{f}_2) e(q^\prime) - \frac{\delta_{m}^{\text{in}}}{8 \Omega_0^r} x \right].
\]

(136)

The Euler equations (29, 32, 35-37) then give the velocity perturbations of the fluids

\[
\begin{align*}
\hat{V}_\nu &= \frac{V_{\nu}^{\text{in}}}{q} \approx V_{\nu}^{\text{in}} - \frac{1}{\sqrt{\Omega_0^r}} \left[ \left( \frac{1}{4} \delta_{\nu}^{\text{in}} + 2 \Psi_0 \right) x - \frac{\Psi_1}{x^2} \right. \\
&\left. + \frac{2}{45} x^{5/2} (4 \hat{f}_2 - 7 \hat{f}_4) e(q^\prime) - \frac{\delta_{m}^{\text{in}}}{8 \Omega_0^r} x \right],
\end{align*}
\]

(137)

\[
\begin{align*}
\hat{V}_c &= \frac{V_{c}^{\text{in}}}{q} \approx V_{c}^{\text{in}} - \frac{1}{\sqrt{\Omega_0^r}} \left[ \Psi_0 \right. \\
&\left. - \frac{\Psi_1}{2 x^{3/2}} \right. \left[ \left( \frac{1}{4} \delta_{c}^{\text{in}} + 2 \Psi_0 \right) x - \frac{\Psi_1}{x^2} \right. \\
&\left. + \frac{2}{63} x^{5/2} (2 \hat{f}_2 - 7 \hat{f}_4) e(q^\prime) - \frac{\delta_{m}^{\text{in}}}{24 \Omega_0^r} x^{3/2} \right],
\end{align*}
\]

(138)

\[
\begin{align*}
\hat{V}_b &= \frac{V_{b}^{\text{in}}}{q} \approx V_{b}^{\text{in}} - \frac{1}{\sqrt{\Omega_0^r}} \left[ \frac{1}{4} \delta_{b}^{\text{in}} + 2 \Psi_0 \right. \\
&\left. \left. \right] x - \frac{\Psi_1}{x^2} \right. \\
&\left. + \frac{2}{45} x^{5/2} (4 \hat{f}_2 - 5 \hat{f}_4) e(q^\prime) - \frac{\delta_{m}^{\text{in}}}{8 \Omega_0^r} x^{3/2} \right].
\end{align*}
\]

(139)

where $V_{\nu}^{\text{in}}, V_{c}^{\text{in}}$ and $V_{b}^{\text{in}}$ are constants of integration. The nine integration constants introduced (the four density perturbations $\delta_{n}^{\text{in}}, \Psi_0, \Psi_1$ and the three initial velocity perturbations $V_{\nu}^{\text{in}}$) are constrained by the remaining equations (125) and (38)

\[
\begin{align*}
\Psi_0 &= - \frac{1}{6 \Omega_0^r} \left( \Omega_0^r \delta_{\gamma}^{\text{in}} + \Omega_0^r \delta_{\nu}^{\text{in}} \right), \\
V_{\nu}^{\text{in}} &= - V_{\gamma}^{\text{in}},
\end{align*}
\]

(140)
so that the system depends on seven integration constants as requested.

Solving the equations for the vector perturbations, deep in the radiation era and for long wavelengths, is straightforward. The conservations equations (29, 32, 35-37) yield

\[ \hat{v}^i_v = \hat{v}^i_{v,\text{in}}, \]
\[ \hat{v}^i_c = \hat{v}^i_{c,\text{in}}, \]
\[ \hat{v}^i_b = \hat{v}^i_{b,\text{in}}, \]
\[ \hat{v}^i_\gamma = \hat{v}^i_{\gamma,\text{in}}, \]

where \( \hat{v}^i_{\text{in}} \) are three constant vectors orthogonal to \( k^i \), and eq. (40) gives

\[ \hat{\Phi}^i \approx \left[ \Phi^i_0 + \Phi^i_1 + \frac{4}{9} \frac{q}{\sqrt{\Omega_0}} x^{5/2} \hat{f}_6 \hat{e}^i(q') \right], \]
\[ \hat{E}^{ij} \approx \left[ E^{ij}_0 + \frac{E^{ij}_1}{x} + \frac{4}{63} \frac{q^2}{\Omega_0} x^{7/2} \hat{r}_e \hat{e}^{ij}(q') \right], \]

where \( \hat{E}^{ij}_0 \) and \( \hat{E}^{ij}_1 \) are two constant traceless tensors orthogonal to \( k^i \).

The solution we have obtained is valid deep in the radiation era, for wavelengths larger than the Hubble radius, but late enough so that the baryons and CDM are non-relativistic. At the epoch of the phase transition which gave rise to the defects all the matter was relativistic and the solution (given in [22]) is (130-146) where one sets \( \Omega_b^0 = \Omega_c^0 = \Omega_\nu^0 = 0 \) and \( \Omega_\gamma^0 \equiv \Omega_r^0 = 1 \). As shown in [22], the simple fact that the defects suddenly appear in an up to then perfectly homogeneous and isotropic universe in such a way that the standard general relativistic matching conditions are satisfied determines some of the constants of integration in terms of the \( \hat{f}_a \). More precisely, two of the seven integration constants which appear in the scalar perturbation set, one of the three integration constant vectors for the vector set and the two integration constant tensors for the tensor set are determined by these matching conditions (see [22] for details). The other constants must be fixed by the physics at the epochs when the baryons and CDM become non-relativistic. For example, if one assumes that the transition is sudden and occurs on a hypersurface of constant density then (see [22]) \( \Psi \) must be continuous and therefore \( \delta^\text{min}_m = 0 \).

It turns out however that almost all those initial conditions which fix the integration constants are “forgotten” at late times, that is long enough after the transitions. This is clearly seen for all the metric perturbations and the scalar velocity perturbations. Indeed, asymptotically [see eqns. (134, 138, 144, 146)],

\[ \hat{\Psi} \rightarrow \frac{2}{9} \Omega_0^3 \hat{f}_4 + \frac{2}{9} \hat{f}_2 e(q'), \]
\[ \hat{\Phi} \rightarrow \frac{1}{9} \Omega_0^3 \hat{f}_2 - \frac{7}{9} \hat{f}_4 e(q'), \]
\[ \hat{\Phi}_1 \rightarrow \frac{4}{9} \frac{q}{\sqrt{\Omega_0}} x^{5/2} \hat{f}_6 \hat{e}_i(q'), \]
\[ \begin{align*}
\hat{E}_{ij} & \rightarrow 4 \frac{q^2}{63 \Omega_r} x^{7/2} \hat{f}_7 \epsilon_{ij}(q'), \\
\hat{V}_\nu \rightarrow \hat{V}_\nu = \hat{V}_\gamma & \rightarrow - \frac{2}{45} \frac{q}{\sqrt{\Omega_r}} x^{5/2} (4 \hat{f}_2 - 5 \hat{f}_4) e(q'), \\
\hat{V}_c & \rightarrow - \frac{2}{63} \frac{q}{\sqrt{\Omega_r}} x^{5/2} (2 \hat{f}_2 - 7 \hat{f}_4) e(q').
\end{align*} \]

In order now to obtain the asymptotic behaviours of the scalar density perturbations one must solve the equations at next order in \( q \). This is easily done: injecting the asymptotic expressions (151-152) for the scalar velocity perturbations into the conservation equations (29, 32, 35-37) one gets

\[ \begin{align*}
\delta_{\nu} \rightarrow \delta_{\gamma} & \rightarrow \frac{4}{3} \delta_{\gamma} \rightarrow - \frac{16}{945 \Omega_r} q^2 (4 \hat{f}_2 - 5 \hat{f}_4) x^{7/2} e(q'), \\
\delta_c & \rightarrow - \frac{4}{441 \Omega_r} q^2 (2 \hat{f}_2 - 7 \hat{f}_4) x^{7/2} e(q').
\end{align*} \]

Finally the solutions (143-144) hold for all wavelengths so that the initial conditions for the vector velocity perturbations are not “forgotten” in the evolution (apart from \( \dot{v}_{ci} \)). We shall take (rather arbitrarily)

\[ \hat{v}_{in}^i = 0. \]

In conclusion, we shall start the numerical integration of the set (29) with the initial conditions (147-155), at \( x = x_{in} \) (in practice \( x_{in} = 10^{-3} \)), for various sets of the constants \( f_2, f_4, f_6, f_8 \) (which will depend on our choices for the functions \( F(u), G(u), P(u) \) and \( Q(u) \)) which describe the defects [see eqns. (118-124)], and all relevant values of \( q \) (in practice \( 5 \times 10^{-2} < q < 5 \times 10^2 \), which corresponds to wavelengths from 10 times to one thousandth of the horizon size today).

An important point to note is that all scalar perturbations, collectively called \( \hat{S}(\eta, k^i) \), are proportional to the same random variable \( e(k^i) \) such that \( \langle e^*(k^i) e(k^i) \rangle = \delta(k^i - k^i) \), and can be written as

\[ \hat{S}(k^i) = \tilde{S}(k) e(k^i) \]

where \( \tilde{S} \) is a function of \( k \) only which is entirely determined once the two functions \( f_1(u) \) and \( G(u) \) describing the scalar components of the defects are known. As for the vector perturbations, collectively called \( \hat{V}_i(k^i) \), they can be written as

\[ \hat{V}_i(k^i) = \tilde{V}(k) \tilde{e}_i(k^i) \]

with \( \langle \tilde{e}_i(k^i) \tilde{e}_k(k^i) \rangle = \delta(k^i - k^i) P_{ik} \). Finally the tensor perturbations, collectively called \( \hat{T}_{ij}(k^i) \), can be written as

\[ \hat{T}_{ij}(k^i) = \tilde{T}(k) \tilde{e}_{ij}(k^i) \]

with \( \langle \tilde{e}_{ij}(k^i) \tilde{e}_{kl}(k^i) \rangle = \delta(k^i - k^i) (P_{ik} P_{jl} + P_{il} P_{jk} - P_{ij} P_{kl}) \). The functions \( \tilde{V}(k) \) and \( \tilde{T}(k) \) are entirely determined once the functions \( P(u) \) and \( Q(u) \) describing the vector and tensor components of the defects is known.
5. The correlation function of the microwave background anisotropies

The energy of a microwave background photon in the direction $\gamma^i$, as measured by an observer $O$ at $x_0^\mu$ is: $E_\gamma^0 = -(g_{\mu\nu}K^\mu u^\nu)|_0$ where $g_{\mu\nu}|_0$ is the metric (14) evaluated at $x_0^\mu$, $K^\mu|_0$ (with $g_{\mu\nu}K^\mu K^\nu = 0$) is the wave vector of the photon at $x_0^\mu$ and $u^\nu|_0$ (with $g_{\mu\nu}u^\mu u^\nu = -1$) is the velocity vector of the observer $O$. $E_\gamma^0$ depends on $K^0|_0$, on the direction of observation $\gamma^i \equiv -(K^i/K)|_0$ and the position and velocity of the observer $O$. Assuming that the photon has followed a null geodesic (no-collision hypothesis) since it has decoupled from the baryons on a surface of constant electronic density or, almost equivalently, constant photon density, it is a standard calculation (see e.g. [36] for a detailed derivation) to relate $E_\gamma^0$ to $E^d = -(g_{\mu\nu}K^\mu u^\nu)|_d$, its energy at decoupling, which is independent of $\gamma^i$ when “measured” by an observer comoving with the baryons. The result is

$$\left(\frac{\Delta T}{T}\right)_\gamma \equiv \frac{E_\gamma^0 - E^0}{E^0} = -\left[\Phi + \frac{1}{4}\delta_\gamma - \gamma^i (\partial_i v + \bar{v}_i)\right]_0 + \left[\Phi + \frac{1}{4}\delta_\gamma - \gamma^i (\partial_i v_b + \bar{v}_i,b)\right]_d + \int_{\eta_d}^{\eta_0} (\Phi' + \Psi' - \gamma^i \bar{v}'_i - \gamma^i \gamma^j E'_{ij}) d\eta$$

where $E^0 \equiv E^d x_d$ ($x_d \equiv a(\eta_d)/a(\eta_0)$), where the index 0 means that the quantity is evaluated today at $(\eta_0, x_0^\mu = 0)$, where the index $d$ means that the quantity is evaluated at $(\eta_d, x_d^\mu = \gamma^i(\eta_0 - \eta_d))$ and where the integrands in the integrals are evaluated on the photon trajectory that is at $x^\mu = (\eta, \gamma^i(\eta_0 - \eta))$. Assuming that the microwave background is and always was a perfect black body radiation, eq. (159) defines its temperature anisotropies.

Eq. (159), where the reception event $x_0^\mu$ is defined (rather arbitrarily) as belonging to a surface of constant photon density, defines $\Delta T/T|_\gamma$ as a gauge invariant quantity. The first term is the sum of a monopole ($-\left[\Phi + \frac{1}{4}\delta_\gamma\right]_0$) which does not depend on $\gamma^i$ and a dipole ($\left[\gamma^i (\partial_i v + \bar{v}_i)\right]_0$) which is a Doppler effect due to the peculiar velocity of the observer. In the following we shall ignore these contributions and define the temperature background anisotropies as

$$\Theta_\gamma = \Theta_\gamma^S + \Theta_\gamma^Y + \Theta_\gamma^T$$

with

$$\Theta_\gamma^S = \left[\Phi + \frac{1}{4}\delta_\gamma - \gamma^i \partial_i v_b\right]_d + \int_{\eta_d}^{\eta_0} (\Phi' + \Psi') d\eta,$$  

$$\Theta_\gamma^Y = -\gamma^i \bar{v}_i,b|_d + \int_{\eta_d}^{\eta_0} \bar{v}'_i d\eta,$$  

$$\Theta_\gamma^T = -\gamma^i \gamma^j E'_{ij} \int_{\eta_d}^{\eta_0} d\eta.$$

The next step consists in expressing the perturbations in terms of their Fourier transforms [see eq. (22)] and to expand $e^{ik.x}$ as

$$e^{ik.x} = \sum_{\ell} \int (2\ell + 1) j_\ell(kx) P_\ell(k.x/kx)$$

(164)
where \(j_\ell\) is a Bessel function, \(P_\ell\) a Legendre polynomial, and where \(kx \equiv \sqrt{k^0 \sqrt{x_i x^i}}\) and \(\bar{k} \cdot \bar{x} \equiv k_\ell x^\ell\). Eqs. (161-163) then read (reintroducing the dimensionless wave vector \(q^\prime \equiv k^i / \lambda_0\))

\[
\Theta_{S,V,T} = \frac{1}{(2\pi)^{3/2}} \sum_\ell i^{\ell} (2\ell + 1) \int q^2 dq \, d\Omega q \, P_\ell(\bar{n}_q \cdot \vec{\gamma}) \, G_{S,V,T}^\prime(q^\prime) \tag{165}
\]

where \(S, V, T\) stand for either \(S, V\) or \(T\), where \(\Omega q\) defines the direction of the vector \(q^i\) in Fourier space, where \(\bar{n}_q \equiv \bar{q}/q\) and where [integrating on \(x \equiv a/a_0\) rather than on \(\eta\) and recalling the notations (150,158)]

\[
G^S_S(q^i) \equiv \hat{G}^S(q) e(q^i), \tag{166}
\]

\[
G^V_V(q^i) \equiv \hat{G}^V(q) \gamma^i \hat{e}_i(q^i), \tag{167}
\]

\[
G^T_T(q^i) \equiv \hat{G}^T(q) \gamma^i \hat{e}_i(q^i), \tag{168}
\]

with

\[
\hat{G}^S(q) = j_\ell(y_d) \left[ \hat{\Phi} + \frac{1}{4} \delta_\gamma \right]_d \left[ - \frac{d}{dy} j_\ell(y) \right]_{y_d}, \tag{169}
\]

\[
\hat{G}^V(q) = -j_\ell(y_d) \hat{v}_b|_d - \int_{x_d}^1 dx j_\ell(y) \frac{d}{dx} \hat{\Phi}, \tag{170}
\]

\[
\hat{G}^T(q) = - \int_{x_d}^1 dx j_\ell(y) \frac{d}{dx} \hat{E}, \tag{171}
\]

where \(y \equiv kr = k(\eta_0 - \eta) = 2q(1 - \sqrt{\Omega_{11}^0 + x \Omega_{12}^0}) / \Omega_{12}^0 \) [see eq. (13)] and where the index “\(d\)” means that the quantity is evaluated at \(x = x_d\).

\(\Theta^S_\gamma, \Theta^V_\gamma,\) and \(\Theta^T_\gamma\) are random variables which fall into statistically independent sets so that the temperature anisotropy correlation function factorizes as

\[
C(\vec{\gamma}_1 \cdot \vec{\gamma}_2) \equiv \langle \Theta_{\vec{\gamma}_1} \Theta_{\vec{\gamma}_2} \rangle = \langle \Theta^S_{\vec{\gamma}_1} \Theta^S_{\vec{\gamma}_2} \rangle + \langle \Theta^V_{\vec{\gamma}_1} \Theta^V_{\vec{\gamma}_2} \rangle + \langle \Theta^T_{\vec{\gamma}_1} \Theta^T_{\vec{\gamma}_2} \rangle \tag{172}
\]

where \(\vec{\gamma}_1 \cdot \vec{\gamma}_2\) is the cosine of the angle between the two directions of observation \(\vec{\gamma}_1\) and \(\vec{\gamma}_2\).

Injecting eqns. (165-171) into eq. (172) one therefore gets

\[
\langle \Theta_{S,V,T}^S \Theta_{S,V,T}^S \rangle = \frac{1}{(2\pi)^{3/2}} \sum_{\ell, \ell'} (-1)^\ell' (2\ell' + 1)(2\ell + 1) \times \int q^2 dq \, d\Omega q \, \hat{G}^{SS}_{\ell} \hat{G}^{SS}_{\ell'} P_\ell(\bar{n}_q \cdot \vec{\gamma}_1) P_{\ell'}(\bar{n}_q \cdot \vec{\gamma}_2) A^{SS}_{S,V,T}, \tag{173}
\]

where

\[
A^S = 1, \tag{174}
\]

\[
A^V = \gamma_1^2 \gamma_2^1 P_{ij} = \vec{\gamma}_1 \cdot \vec{\gamma}_2 - (\bar{n}_q \cdot \vec{\gamma}_1)(\bar{n}_q \cdot \vec{\gamma}_2), \tag{175}
\]

\[
A^T = \gamma_1^2 \gamma_2^1 \gamma_3^k \gamma_4^l (P_{kl} P_{ij} + P_{il} P_{jk} - P_{ij} P_{kl}) \tag{176}
\]

\[
= 2[\vec{\gamma}_1 \cdot \vec{\gamma}_2 - (\bar{n}_q \cdot \vec{\gamma}_1)(\bar{n}_q \cdot \vec{\gamma}_2)]^2 - [1 - (\bar{n}_q \cdot \vec{\gamma}_1)^2][1 - (\bar{n}_q \cdot \vec{\gamma}_2)^2].
\]

Using now the identity [37]

\[
\int d\Omega q \, P_\ell(\bar{n}_q \cdot \vec{\gamma}_1) \times P_{\ell'}(\bar{n}_q \cdot \vec{\gamma}_2) = \frac{4\pi}{2\ell + 1} P_\ell(\vec{\gamma}_1 \cdot \vec{\gamma}_2) \delta_{\ell \ell'}, \tag{177}
\]
one first obtains (see e.g. \[38\])

\[
\langle \Theta_{\gamma_1}^S \Theta_{\gamma_2}^S \rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) C_{\ell}^S \mathcal{P}_\ell(\gamma_1, \gamma_2),
\]

\[C_{\ell}^S = \frac{2}{\pi} \int q^2 dq |\tilde{G}_\ell^S|^2.
\]

where \(\tilde{G}_\ell^S\) is the function of \(q\) given by eq. (169). The other identity

\[
\int (\tilde{n}_q \cdot \gamma) \mathcal{P}_\ell(\tilde{n}_q \cdot \gamma_1) \mathcal{P}_\ell(\tilde{n}_q \cdot \gamma_2) d\Omega_q \]

\[= 4\pi \frac{(\ell + 1)}{(2\ell + 3)(2\ell + 1)} \int \frac{(\ell + 1)(\ell + 2)}{2\ell + 3} \delta_{\ell,\ell+2} \mathcal{P}_{\ell+1}(\gamma_1, \gamma_2)
\]

\[+ 4\pi \frac{(\ell)}{(2\ell + 1)(2\ell - 1)} \int \frac{(\ell)}{2\ell + 3} \delta_{\ell,\ell-2} \mathcal{P}_{\ell-1}(\gamma_1, \gamma_2)
\]

together with standard recursion relations for the Legendre polynomials and Bessel functions \[37\] then yield (see e.g. \[26\])

\[
\langle \Theta_{\gamma_1}^V \Theta_{\gamma_2}^V \rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) C_{\ell}^V \mathcal{P}_\ell(\gamma_1, \gamma_2),
\]

\[C_{\ell}^V = \frac{2}{\pi} \int q^2 dq \left[ \int_{x_d}^1 dx \frac{j_\ell(y)}{y} \frac{d}{dx} \hat{\Phi} \right]^2.
\]

The computation of \(\langle \Theta_{\gamma_1}^T \Theta_{\gamma_2}^T \rangle\) can proceed along similar lines but is more involved. The result is (see e.g. \[38\])

\[
\langle \Theta_{\gamma_1}^T \Theta_{\gamma_2}^T \rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) C_{\ell}^T \mathcal{P}_\ell(\gamma_1, \gamma_2)
\]

with

\[
C_{\ell}^T = \frac{2}{\pi} \int q^2 dq \left[ \int_{x_d}^1 dx \frac{j_\ell(y)}{y^2} \frac{d}{dx} \hat{E} \right]^2.
\]

The coefficients \(C_{\ell}^{S,V,T}\), which will be computed numerically in the next section for various choices of the source functions describing the defects, are the quantities to be compared with observations, through the inverse formula

\[
C_{\ell}^S + C_{\ell}^V + C_{\ell}^T = 2\pi \int_{-1}^{+1} dz C(z) P_\ell(z),
\]

\[
z \equiv \gamma_1 \cdot \gamma_2,
\]

\[C(z) \equiv \langle \Theta_{\gamma_1} \Theta_{\gamma_2} \rangle.
\]

(Observationally, the ensemble average \(\langle ... \rangle\) is replaced by an average on different regions of the celestial sphere. For large angles there are few such independent regions so that the “cosmic variance” problem arises; see e.g. \[40\].)

† We thank Luc Blanchet for showing us how to demonstrate it by means of the STF formalism (see e.g. \[39\]).
6. Numerical results

Two of us (JPU and AR) have developed a numerical code to solve the equations for the perturbations (29-41) and compute the coefficients $C_\ell$ in function of the multipoles $\ell$ (eqns. (179,182,183) when the defects are described by eqns. (97-103) and (118-124), and the initial conditions are given by eqns. (147-155) (the computations start at $x_{in} = 10^{-9}$, $q \in [2 \times 10^{-2} - 2 \times 10^{+3}$ and $\ell \in [2 - 1500]$). This code incorporates Silk damping (that is the decay of the $C_\ell$ at large $\ell$ due to the fact that a fluid description no longer holds on small scales) by a simple multiplication of $G^S$ in (169) by the decaying exponential (see [41]-[43])

$$e^{-2k^2\eta_{rec}^2x^2_s},$$

where $x_s = 0.6\Omega_\text{m}^{1/4}(\Omega_bh_0)^{-1/2}\eta_{rec}^{3/4}$.

As for the fact that decoupling is not instantaneous, we take it into account by convolving eqns. (169-171) with the visibility function (see [26] for a detailed derivation; a simple multiplication by $e^{-k^2\eta_{rec}^2\sigma^2}$ with $\sigma = 0.03$ for standard recombination [12] is too rough an approximation).

Instead of eq. (188) we could have used the better fit proposed by Hu and Sugiyama [44] but it is an analytical fit deduced from a Boltzmann code whereas eq. (188) is obtained from physical arguments [41].

In this section we first test our code and discuss the approximations made. We then give a few examples of solutions and compare the results with previous calculations. (Note that for a better visibility, the results are not normalized to the COBE data at $\ell = 10$ as is usual.)

6.1. No sources and adiabatic conditions: comparison with a Boltzmann code

The code which we have implemented can of course also be used to study scenarios where no defects are present and where adiabatic initial conditions (as imposed by e.g. the standard inflationary scenario) are imposed. One has just to set $\mathcal{F} = \mathcal{G} = \mathcal{P} = \mathcal{Q} = 0$ and replace the initial conditions (147-155) by

$$\dot{\delta}_\nu = \dot{\delta}_\gamma = \frac{4}{3} \delta_0 = \frac{4}{3} \delta_c = -6\Psi_0$$
$$\dot{V}_\nu = \dot{V}_\gamma = \dot{V}_b = \dot{V}_c = -\frac{1}{2q\sqrt{\Omega_r}}\Psi_0$$
$$\dot{\Psi} = \dot{\Phi} = \Psi_0$$

where $\Psi_0$, in the standard chaotic inflationary model, is fixed by the quantization of the inflaton so that (see e.g. [16]) $\Psi_0 \propto q^{-3/2}$. (We restrict ourselves to the scalar perturbations as the vector and tensor contributions are negligible in the standard inflationary scenario — see e.g. [17].)

The $C_\ell^S$ coefficients as computed with our fluid code, with the Boltzmann code we have developed [8], and with CMBFAST [10] are compared on fig. 1 (top). One first notes the excellent agreement between our Boltzmann code with CMBFAST. One also notes that if the position of the Doppler peaks, as given by our fluid code and the Boltzmann ones, coincide, their heights differ by about 15 %, which we consider enough for our present purposes considering that the physics in our fluid code (as well as in e.g. Seljak’s in [42]) and a Boltzmann code differs on four points:
• We use a perfect fluid approximation to describe the various matter and radiation components of the universe,
• We make a tight-coupling approximation, that is we assume that the baryons and photons have exactly the same velocities until decoupling,
• We suppose instantaneous decoupling,
• We correct for those assumptions by an adhoc damping of the scalar perturbations on small scales [see eq. (188)] and we take into account the finite width of the last scattering surface.

whereas a description based on Boltzmann’s equation is used in our Boltzmann code and in CMBFAST. We find that the difference in the height of the peaks is due to these approximations (see [30] for a detailed discussion).

In conclusion a perfect fluid code is useful to find the positions of the peaks predicted by various types of topological defects but can be inaccurate when it comes to their height. In the following we shall use a perfect fluid code and therefore comment only on the position of the peaks. In [30] we shall use our Boltzmann code and shall therefore be able to compare not only the positions but also the heights of the peaks predicted by various defects as well as with those predicted by inflationary scenarios, the latter being also computed using a Boltzmann code.

6.2. The scalar contribution of coherent defects to the CMB anisotropies

The scalar component $C^S_\ell$ of the $C_\ell$ coefficients is given by (179) and is entirely determined once the two functions $F(u)$ and $G(u)$ (with $u \equiv q/h = k/H$) describing the scalar component of the stress-energy tensor of the defects are known.

First fig. 2 shows, on the particular case $G(u) = 0$ and $F(u) \propto \exp(-u^2)$ (fig. 2 bottom), that, contrarily to what happens in inflationary scenarios (fig. 2 top), the integrated Sachs-Wolfe (ISW) contribution to $C^S_\ell$ (that is the integral term in eq. (169)) dominates at low $\ell$. This is in agreement with intuition : in inflationary scenarios the large scale anisotropies are built in the initial conditions, whereas in defect scenarios they are due to line-of-sight gravitational effects. We are aware that the inverse Fourier transform of a gaussian does not meet condition (66) but we note that the Fourier transform of any compact supported $C^\infty$ function decreases faster than any power law.

Fig. 3 illustrates the case when $G = 0$ and $F = \exp(-u^2/L^2)$ for various values of $L$. The most striking feature of these curves is that they do not exhibit a plateau at low $\ell$ when $L \neq 1$. As $\ell$ decreases the first “peak” is shifted to lower multipoles and tends to the position predicted by the standard inflationary scenario (see fig. 1 top). This can be explained by the fact that the defects decay on larger scales.

Fig. 4 illustrates the case when both $F$ and $G$ are proportional to $\exp(-u^2)$, with $G/F = r$. First, as expected, when $r$ is small one recovers fig. 3 bottom or fig. 4 with $L = 1$. Second, when $G \neq 0$, the dominant component to the stress-energy tensor of the defects is their anisotropic stress $\Pi^S$ [see eqns. (118-121 and 97)]. The first Doppler peak is then washed out. This result compares to [17] where incoherent sources are studied, for which the anisotropic stress dominates too. An interesting particular case is when $a \simeq 0.3$ which minimizes the anisotropic stress contribution $f_\Pi$ [see eq. (121)]. The first Doppler peak is then higher than the plateau.

Finally, as a caveat not to comment too seriously on the height of the Doppler peaks when a perfect fluid code is used, fig. 4 bottom compares $C^S_\ell$ as given by the perfect fluid code used here and the Boltzmann code used in [30] for $G(u) = 0$ and
Figure 1. Comparison between our fluid code (solid line), our Boltzmann code (dashed line) and CMBFAST (short-dashed line) in the case of inflation (top); Comparison between our fluid code (solid line) and our Boltzmann code (dashed line) in the case of topological defects (bottom) with $\mathcal{F} = \exp(-u^2)$ and $\mathcal{G} = 0.$
Figure 2. Decomposition of the different contributions of the $C^S_{\ell}$ in the case of inflation (top) and topological defects (bottom) with $F = \exp(-u^2)$ and $G = 0$. In each case, we have represented the Sachs-Wolfe, Doppler, Integrated Sachs-Wolfe contribution, as well as the total contribution (dashed, short-dashed, dotted and solid lines respectively).
Figure 3. The dependence of the coefficients $C_\ell$ in defect scenario where we have varied the scale of decay, $L$, of the defects by choosing $F = \exp(-u^2/L^2)$ and $G = 0$, with $L^2 = 0.5, 1, 2$ (dotted, dashed and solid line respectively).

Figure 4. The effect of the function $G = \exp(-u^2)$. For that purpose we have varied the ratio $r = \frac{G}{F}$ with $r = 0, 0.1, 0.3$ (dotted, dashed and solid line respectively).
(u) \propto \exp(-u^2). The much bigger discrepancy in the height of the peaks in that case as compared to the case of inflation is due to the fact that eq. (188) for Silk damping is valid only if the metric perturbations in the evolution equations for $\delta_\gamma$ can be approximated by constant (see e.g. eq. (98) of Hu and White [26]). It can be shown that this is approximatively true in the case of inflationary scenarios in a flat universe without cosmological constant, but here it is not true, as it can be checked by looking at the ISW contribution. Of course, it is also possible to do some a posteriori corrections by comparing fluid and Boltzman codes and by fitting ad hoc damping terms in order to find a good agreement between the two, see Hu and Sugiyama [26], but we did not find them necessary since we simply wanted to present a semi-analytic formalism. Let us note finally that in the case of the pressure model [34][35], the discrepancy between this (simple) fluid code and an accurate Boltzmann code is less serious, probably because the ISW contribution is smaller.

6.3. The vector and tensor contributions

The vector and tensor components of the $C_\ell$ coefficients [given by eqns. (182,183)] are entirely determined once the functions $P(u)$ and $Q(u)$ describing the vector and tensor components of the stress-energy momentum of the defects are known. Since we supposed $\bar{v}_0 = 0$ they both reduce to their integrated Sachs-Wolfe contributions.

![Figure 5. The vector (dashed line) and the tensor (solid line) contributions to the CMB temperature anisotropies in a model where $P = Q = \exp(-u^2)$](image)

6.4. Analytic estimate of the $C_\ell^{V,T}$ at small $\ell$

As we have seen, only the integrated Sachs-Wolfe effect contributes to the vector and tensor parts of the $C_\ell$. In this section, we estimate them at small $\ell$. We first assume that the last scattering surface is deep enough in the matter dominated era to
approximate $H$ and $u$ by: $H = 2/\eta$ and $u = k\eta/2$. The equations (40) and (41) for $\Phi_i$ and $E_{kl}$ can then be rewritten [using eqns (25, 101–103) and (157–158)] as

\[
\ddot{\Phi} + \frac{4}{\eta} \dot{\Phi} = \kappa \frac{k \eta^{3/2}}{2^{1/2}} f_6(u), \tag{192}
\]

\[
\ddot{E} + \frac{4}{\eta} \dot{E} + k^2 \dot{E} = \kappa \frac{k \eta^{3/2}}{2^{1/2}} \sqrt{2} f_7(u), \tag{193}
\]

the solutions of which are given by

\[
\tilde{\Phi} = \alpha \frac{u^4}{u^2} \int_0^u v^{3/2} f_6(v) dv, \tag{194}
\]

\[
\tilde{E} = \alpha \frac{j_1(u)}{u} + \beta \frac{n_1(u) k^{-3/2}}{2u} \times \int_0^u v^{3/2} \frac{j_1(u)n_1(v) - j_1(v)n_1(u)}{j_1(v)n_0(v) - j_0(v)n_1(v)} f_7(v) dv. \tag{195}
\]

We now make the ansatz that the defects disappear as soon as the $y$ enter the horizon i.e. that $f_6(v) \propto f_7(v) \propto Y(v - 1)$ where $Y$ is the Heavyside function.

Computing (194-195), their time derivative and inserting the result in eqns. (182,183), we obtain that

\[
C^\ell_V \propto \ell (\ell + 1) \int_0^A j_2^2(y) \frac{1}{y^2} y + 1 dy, \tag{196}
\]

\[
C^\ell_T \propto (\ell + 2)! (\ell - 2)! \int_0^A j_2^2(y) \frac{1}{y^2} y + 1 dy, \tag{197}
\]

with $A \equiv \eta_0/\eta_d - 1$. The main contribution to the two integrals comes from the points $y \simeq \ell$ (since the Bessel functions are peaked around that point). In a matter dominated universe, we have that $A = \sqrt{a_0/a_d} - 1 \simeq 32$ and for $\ell < 30$ we will assume that $A \sim \infty$. With that approximation, we are led to compute the integrals

\[
B^\ell_p \equiv \int_0^\infty j_2^2(y) \frac{1}{y^{p-1}} y + 1 dy \quad -1 < p < 2\ell + 2, \tag{198}
\]

with $p = 5$ and $p = 3$ for the tensor and vector modes respectively. Using the relation

\[
A^\ell_p \equiv \int_0^\infty j_2^2(y) y^p dy = \frac{\pi}{2^{2+p}} \frac{\Gamma(1+p)}{\Gamma^2(1+p/2)} \frac{\Gamma(\ell + 1/2 - p/2)}{\Gamma(\ell + 3/2 + p/2)}, \quad -1 < p < 2\ell + 1, \tag{199}
\]

it can be shown that

\[
B^\ell_p = A^\ell_p (1 - R^\ell_p) \tag{200}
\]

with

\[
0 \leq R^\ell_5 \leq \frac{15\pi}{3.2^{11} (\ell - 5/2)}, \tag{201}
\]

\[
0 \leq R^\ell_3 \leq \frac{9\pi}{2^5 (\ell - 3/2)}. \tag{202}
\]
Thus, the $\ell$ dependence of $C_{\ell}^{V,T}$ at small $\ell$ is given by

$$\ell(\ell+1)C_{\ell}^{V} \propto \frac{\ell(\ell+1)}{(\ell+2)(\ell-1)},$$  

(203)

$$\ell(\ell+1)C_{\ell}^{T} \propto \frac{\ell(\ell+1)}{(\ell+3)(\ell-2)}, \quad 3 \leq \ell \leq 20.$$  

(204)

This solution is compared to our numerical results on fig. 6 where $C_{\ell}^{T}/C_{\ell}^{V}$ and its analytic estimate are plotted (when $f_6 = f_7$). The agreement is excellent for $3 \leq \ell \leq 20$ as expected.

This estimation only assumes that we are in a matter dominated universe, that $\eta_0/\eta_d - 1 < 20$ and that $f_6 \propto f_7 \propto Y$. The two first approximations are well satisfied for small multipoles. The ansatz concerning the modelisation of the functions $f_6$ and $f_7$ will affect only the numerical pre-factors in eqns. (196-197) and (203-204), as long as the scale of decay of the defects is comparable with the horizon.

6.5. Conclusions

In this article, we have studied the signature of coherent scaling defects on the cosmic microwave background. For that purpose, we have developed a semi-analytic formalism to describe the topological defect network as well as two numerical codes (based respectively on a fluid and a Boltzmann approach) to compute the scalar, vector and tensor components of the coefficients $C_{\ell}$ of the CMB anisotropy correlation function.
Concerning the scalar component, and contrarily to what happens in standard inflationary scenarii, there is an important contribution from the integrated Sachs-Wolfe term, which does not in general build up as a plateau. In the generic case, this term dominates at low $\ell$ so that the power on large scales is of the same order or even higher than on small scales. Such characteristics do not fit the present observations.

However, one can improve the model in various ways. First one can find special combinations of the arbitrary functions which yield results that are more compatible with the data (see fig. 3 and 30). But of course such a fine tuning needs to be justified. Second one can play with the cosmological parameters. Third one can extend our approach to more realistic defects (incoherent defects, loss of scaling ...). Finally one can consider defects that are produced at the end of an inflationary period (and not, as in this paper, in an up to then perfectly homogeneous and isotropic universe). Various extensions of the standard model of particle physics predict such a possibility (see e.g. 43). We plan to explore those various improvements in future work.

Acknowledgments

We thank David Langlois, Neil Turok and Mairi Sakellariadou for helpful discussions.

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