Asymptotics of Eigenvalues for Differential Operators of Fractional Order

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Abstract

In this paper we deal with a second order multidimensional fractional differential operator. We consider a case where the leading term represented by the uniformly elliptic operator and the final term is the Kipriyanov operator of fractional differentiation. We conduct classification of such a type of operators by belonging of their resolvent to the Schatten-von Neumann class and formulate the sufficient condition for completeness of the root functions system. Finally we obtain an asymptotic formula.

Keywords: Operators of fractional differentiation; Weyl asymptotic; Schatten-von Neumann class; sectorial property; accretive property; operators with a compact resolvent; root vectors.

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1 Introduction and preliminary remarks

Everywhere further in abstract definitions we consider linear operators acting in a separable complex Hilbert space $\mathcal{H}$. Denote by $C, C_i$, $i \in \mathbb{N}_0$ arbitrary positive constants. Let $\mathcal{B}(\mathcal{H})$ be a set of linear bounded operators acting in the Hilbert space $\mathcal{H}$. By $\text{D}(A)$, $\text{R}(A)$, $\text{N}(A)$ we denote the domain of definition, range, and inverse image of zero of the operator $A$ respectively. We use the denotation $\text{null} A := \dim \text{N}(A)$. Denote by $R_A(\lambda)$, $R_A(0) := R_A$ the resolvent of the operator $A$ and let $P(A)$ be the resolvent set of the operator $A$. By $\hat{A}$ we denote the closure of the operator $A$. Using the definition [13, p.46] we define $s$-numbers for a compact operator $A$. By definition, put $s_i(A) = \lambda_i(T)$, $i = 1, 2, ..., r(T)$, where $T = (A^*A)^{1/2}$, $r(T) = \dim \text{R}(T)$. Suppose $s_i = 0$, $i = r(T) + 1, ..., r(T) < \infty$. According to the terminology of the monograph [13] a dimension of the root vectors subspace corresponding to a certain eigenvalue $\lambda_k$ is called algebraic multiplicity of the eigenvalue $\lambda_k$. Denote by $\nu(A)$ the sum of all algebraic multiplicities of the operator $A$. Let $\mathcal{S}_p(\mathcal{H})$, $0 < p < \infty$ be the Schatten-von Neumann class and $\mathcal{S}_\infty(\mathcal{H})$ be the set of
Let $A_R := (A + A^*)/2$, $A_A := (A - A^*)/2i$ be a so-called real and imaginary component of the operator $A$ respectively. Denote by $\mu(A)$ order of the densely defined operator $A$ with a compact resolvent if we have the estimate $s_n(R_A) \leq C n^{-\mu}$, $n \in \mathbb{N}$, $0 \leq \mu < \infty$. Let $\{x_n\}_1^\infty, \{y_n\}_1^\infty$ be sequences consist of positive real numbers. If there exist constants $c_1, c_2 > 0$ such that $c_1 x_n \leq y_n \leq c_2 x_n$, $n \in \mathbb{N}$, then we write $x_n \asymp y_n$. In accordance with the terminology of the monograph [16] the set $\Theta(A) := \{z \in \mathbb{C} : z = (Af, f)_\beta, f \in D(A), \|f\|_\beta = 1\}$ is called a numerical range of the operator $A$. Denote by $\hat{\Theta}(A)$ the closure of the set $\Theta(A)$. We use the definition of the sectorial property given in [16, p.280]. An operator $A$ is called a sectorial operator if its numerical range belongs to a closed sector $\Sigma_\gamma(\theta) := \{\zeta : |\arg(\zeta - \gamma)| \leq \theta < \pi/2\}$, where $\gamma$ is a vertex and $\theta$ is a semi-angle of the sector $\Sigma_\gamma(\theta)$. We shall say that the operator $A$ has a positive sector if $\text{Im} \gamma = 0$, $\gamma > 0$. According to the terminology of the monograph [16] an operator $A$ is called strictly accretive if the following relation holds $\text{Re}(Af, f)_\beta \geq C\|f\|_\beta^2$, $f \in D(A)$. In accordance with the definition [16, p.279] an operator $A$ is called $m$-accretive if the next relation holds $(A + \zeta)^{-1} \in \mathcal{B}(\mathcal{H})$, $\|(A + \zeta)^{-1}\| \leq (\text{Re}\zeta)^{-1}$, $\text{Re} \zeta > 0$. An operator $A$ is called $m$-sectorial if $A$ is sectorial and $A + \beta$ is $m$-accretive for some constant $\beta$. An operator $A$ is called symmetric if one is densely defined and the next equality holds $(Af, g)_\beta = (f, Ag)_\beta$, $f, g \in D(A)$. A symmetric operator is called positive if values of its quadratic form are nonnegative. Denote by $\mathcal{F}_A$, $\| \cdot \|_A$ the energetic space generated by the operator $A$ and the norm on this space respectively (see [21], [43]). In accordance with the notation of the paper [16] we consider a sesquilinear form $t[\cdot, \cdot]$ defined on a linear manifold of the Hilbert space $\mathcal{H}$ (further we use the term form). Denote by $t[\cdot]$ the quadratic form corresponding to the sesquilinear form $t[\cdot, \cdot]$. Denote by $\text{Re} t = (t + t^*)/2$, $\text{Im} t = (t - t^*)/2i$ the real and imaginary component of the form $t$ respectively, where $t[u, v] = t[v, u]$, $D(t^*) = D(t)$. According to these definitions, we have $\text{Re} t[\cdot] = \text{Re} t[\cdot]$, $\text{Im} t[\cdot] = \text{Im} t[\cdot]$. Denote by $t$ the closure of the form $t$. The range of the quadratic form $t[f]$, $f \in D(t)$, $\|f\|_\beta = 1$ is called a range of the sesquilinear form $t$ and is denoted by $\Theta(t)$. A form $t$ is called sectorial if its range belongs to a sector having a vertex $\gamma$ situated at the real axis and a semi-angle $0 \leq \theta < \pi/2$. Suppose $t$ is a closed sectorial form; then a linear manifold $D' \subset D(t)$ is called a core of $t$ if the restriction of $t$ to $D'$ has the closure $t$. Due to Theorem 2.7 [16, p.323] there exist unique $m$-sectorial operators $A_t, A_{\text{Re} t}$ associated with the closed sectorial forms $t, \text{Re} t$ respectively. The operator $A_{\text{Re} t}$ is called a real part of the operator $A_t$ and is denoted by $Re A_t$. Suppose $A$ is a sectorial densely defined operator and $t[u, v] := (Au, v)_\beta$, $D(t) = D(A)$; then due to Theorem 1.27 [16, p.318] the form $t$ is closable, due to Theorem 2.7 [16, p.323] there exists the unique $m$-sectorial operator $T_1$ associated with the form $t$. In accordance with the definition [16, p.325] the operator $T_1$ is called a Friedrichs extension of the operator $A$. Everywhere further, unless otherwise stated, we use the notations of [16], [13], [21], [22], [47].

In accordance with the notation of the paper [21] we assume that $\Omega$ is a convex domain of the $n$ - dimensional Euclidean space $\mathbb{E}^n$, $P$ is a fixed point of the boundary $\partial \Omega$, and $Q(r, \vec{e})$ is an arbitrary point of $\Omega$. Denote by $\vec{e}$ an unit vector having the direction from $P$ to $Q$, by $r = |P - Q|$ an Euclidean distance between the points $P$ and $Q$. We consider the Lebesgue classes
where $C$ acts as follows on the segment in one dimensional case. According to Theorem 2 \[22\] the Kipriyanov operator \[24\], \[27\]. It is easy to see that the Kipriyanov operator coincides with the Marchaud operator \[47\].

Consider Kipriyanov’s fractional differential operator defined by the following formal expression (see \[22\])

$$(\mathcal{D}^\alpha f)(Q) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^r \frac{[f(Q) - f(P + t\vec{e})]}{(r - t)^{n+1}} dt + C_n^{(\alpha)} f(Q) r^{-\alpha}, \quad P \in \partial \Omega,$$

where $C_n^{(\alpha)} = (n - 1)! / \Gamma(n - \alpha)$. The properties of this operator were studied in the papers \[21\]-\[24\],\[27\]. It is easy to see that the Kipriyanov operator coincides with the Marchaud operator \[47\] on the segment in one dimensional case. According to Theorem 2 \[22\] the Kipriyanov operator acts as follows

$$\mathcal{D}^\alpha : H_0^1(\Omega) \to L_2(\Omega), \quad 0 < \alpha < 1, \quad (n = 2, 3, \ldots),$$

The case ($n=1$) is also obtained due to the property of the Marchaud operator (see \[47\]). In accordance with the definition given in the paper \[28\] we consider the directional fractional integrals. By definition, put

$$(\mathcal{I}_{0+}^\alpha f)(Q) := \frac{1}{\Gamma(\alpha)} \int_0^r \frac{f(P + t\vec{e})}{(r - t)^{1-\alpha}} \left(\frac{t}{r}\right)^{n-1} dt, \quad (\mathcal{I}_{d-}^\alpha f)(Q) := \frac{1}{\Gamma(\alpha)} \int_r^\infty \frac{f(P + t\vec{e})}{(t - r)^{1-\alpha}} dt,$$

$$f \in L_p(\Omega), \quad 1 \leq p < \infty.$$
Also, we consider auxiliary operators the so-called truncated directional fractional derivatives (see [28]). By definition, put

\[
(D_{0+}^{\alpha, \varepsilon} f)(Q) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^{r - \varepsilon} \frac{f(Q) r^{n-1} - f(P + \varepsilon t) t^{n-1}}{(r-t)^{\alpha+1} r^{n-1}} dt + \frac{f(Q)}{\Gamma(1 - \alpha)} r^{-\alpha}, \quad \varepsilon \leq r \leq d,
\]

\[
(D_{0-}^{\alpha, \varepsilon} f)(Q) = \frac{f(Q)}{\varepsilon^\alpha}, \quad 0 \leq r < \varepsilon;
\]

\[
(D_{d-}^{\alpha, \varepsilon} f)(Q) = \frac{f(Q)}{\alpha} \left( \frac{1}{\varepsilon^\alpha} - \frac{1}{(d-r)^\alpha} \right), \quad d - \varepsilon < r \leq d.
\]

Now we can consider the directional fractional derivatives. By definition, put

\[
(D_{0+}^{\alpha} f = \lim_{\varepsilon \to 0} (D_{0+}^{\alpha, \varepsilon} f), \quad D_{d-}^{\alpha} f = \lim_{\varepsilon \to 0} (D_{d-}^{\alpha, \varepsilon} f), \quad 1 \leq p \leq \infty.
\]

The properties of these operators are described in detail in the paper [28]. Similarly to the monograph [47] we consider left-side and right-side cases. For instance, \( I_{0+}^{\alpha} \) is called a left-side directional fractional integral and \( D_{d-}^{\alpha} \) is called a right-side directional fractional derivative. We consider a second order differential operator with the Kipriyanov fractional derivative in the final term

\[
Lu := -D_j(a^{ij} D_i f) + \rho D^{\alpha} f,
\]

\[
D(L) = H^2(\Omega) \cap H_0^1(\Omega),
\]

we assume that the coefficients are real-valued and satisfy the following conditions

\[
a^{ij}(Q)_{i,j=1,n} \in C^1(\overline{\Omega}), \quad A = \sup_{Q \in \Omega} \left( \sum_{i,j=1}^n |a_{ij}(Q)|^2 \right)^{1/2}, \quad a^{ij} \xi_i \xi_j \geq a|\xi|^2, \quad a > 0,
\]

\[
\rho(\xi) \in \text{Lip } \mu, \quad \mu > \alpha, \quad 0 < \varrho < \rho(\xi) < \mathcal{P}.
\]

We also consider a formal adjoint operator

\[
L^+ f := -D_i(a^{ij} D_j f) + \mathcal{D}_{d-}^{\alpha} \rho f, \quad D(L^+) = D(L).
\]

For the sake of simplicity, we should formulate the following trivial propositions which are not worth studying in the main part of the paper. Using the results of the the paper [28] it can easily be checked that

\[
\text{Re}(f, \rho D^{\alpha} f)_{L_2(\Omega)} \geq \frac{1}{\eta^2} ||f||_{L_2(\Omega)}^2, \quad \eta \neq 0, \quad f \in H_0^1(\Omega),
\]

for \( \varrho \) sufficiently large. Using condition (1), we obtain

\[
- \text{Re}(D_j[a^{ij} D_i f], f)_{L_2} \geq a ||f||_{H_0^1}^2, \quad f \in D(L),
\]
Combining (4) and (5), we get
\[ \Theta(L), \Theta(L^+) \subset \{ z \in \mathbb{C} : \text{Re}z > C \}, \tag{6} \]
where \( \eta \) depends on parameters of expression (1). Under these assumptions, due to the results of the paper [25], we have
\[ R(L) = R(L^+) = L_2(\Omega). \tag{7} \]
Now it can easily be checked that \( L^+ = L^* \). Hence by virtue of the well-known theorem we conclude that \( R_L^* = R_L \). Combining (6) and (7), we will have no difficulty in establishing the fact \( \hat{L} = L \), see Problem 5.15 [16, p.165]. Further, we use the shorthand notation \( H := L_{R_l}, V := (R_L)_{R_l} \).

Arguing as above, we see that \( H \) is closed and \( R(H) = L_2(\Omega) \). Due to the reasoning of Theorem 4.3 [28], we know that \( H \) is selfadjoint. By virtue of (6), we have
\[ \Theta(H) \subset \{ z \in \mathbb{R} : z > C \}. \tag{8} \]

Using (8), we obtain that \( R_H \) is selfadjoint and positive, see Theorem 3 [11, p.136]. Since \( V \) is symmetric and \( \text{D}(V) = L_2(\Omega) \), then \( V \) is selfadjoint, see Theorem 1 [11, p.136]. In the same way, we conclude that \( V \) is positive. Note that due to Theorem 5.3 [28] the operators \( V \) and \( R_H \) are compact. Further, if it is not stated overwise, we suppose that inequality (4) is fulfilled.

Before our main consideration, we would like to make some remarks. Particularly we will discuss possible methods for achieving prospective results. The eigenvalue problem is still relevant for second order fractional differential operators. Many papers were devoted to this question, for instance the papers [45], [4]-[7], [26]. The singular number problem for a resolvent of the second order differential operator with Riemann-Liuvile’s fractional derivative in the final term is considered in the paper [4]. It is proved that the resolvent belongs to the Hilbert-Shmidt class. The problem of completeness of the root functions system is researched in the paper [5], also a similar problem is considered in the paper [7]. We would like to research a multidimensional case which can be reduced to the cases considered in the papers listed above. For this purpose we deal with the extension of the Kiprianov fractional differential operator considered in detail in the papers [21]-[24]. We will apply a method of researching based on properties of the real component of the operator, but first let us verify an opportunity of using results obtained previously. For instance the studied operator (1) can be represented by a sum and can be considered as a perturbation of one operator by another one. In accordance with this let us say a couple of words on the perturbation theory.

It is remarkable that initially the perturbation theory of selfadjoint operators was born in the works of M. Keldysh [18]-[20] and had been motivated by the works of such famous scientists as T. Carleman [11] and Ya. Tamarkin [50]. Over time, many papers were published within the framework of this theory, for instance F. Browder [8], M. Livshits [31], B. Mukminov [44], I. Glazman [12], M. Krein [39], B. Lidsky [32], A. Marcus [35], [36], V. Matsaev [37]-[39], S. Agmon [41], V. Katzenelson [42]. Nowadays’ there exists a huge amount of theoretical results formulated in the paper of A. Shkalikov [49]. However for using these results we must have a representation of an initial operator by the sum of a main part a so-called non-perturbing operator and an operator-perturbation. It is essentially the main part must be an operator of a special type either selfadjoint or normal operator. If we consider a case where in the representation the main part neither selfadjoint nor normal and we can not approach the required representation in an obvious way, then we can use the other technique based on properties of the real component of the initial operator.
We stress that both summands of sum (11) are neither selfadjoint no normal operators. We also notice that search for a convenient representation of $L$ by a sum of a selfadjoint operator and an operator-perturbation does not seem to be a reasonable way. Now to justify this claim we consider one of possible representations of $L$ by a sum. Consider the Hilbert space $H$ and a selfadjoint strictly accretive operator $T : H \to H$.

**Definition 1.** In accordance with the definition of the paper [49], a quadratic form $a := a[f]$ is called a $T$-subordinated form if the following condition holds

$$|a[f]| \leq b t[f] + M||f||^2_2, \quad D(a) \supset D(t), \quad b < 1, \quad M > 0,$$

where $t[f] = ||T^{1/2}||^2_2, \quad f \in D(T^{1/2})$. The form $a$ is called a completely $T$-subordinated form if besides of (9) we have the following additional condition $\forall \theta > 0 \exists b, M > 0 : b < \varepsilon$.

Let us consider a trivial decomposition of the operator $L$ on the sum $L = 2L_\mathbb{R} - L^+$ and let us use the notation $T := 2L_\mathbb{R}, \quad A := -L^+$. Then we have $L = T + A$. Due to the sectorial property proven in Theorem 4.2 [28] we have

$$|(Af, f)_{L_2}| = \sec \theta f |\text{Re}(Af, f)_{L_2}| = \sec \theta f \frac{1}{2}(T f, f)_{L_2}, \quad f \in D(T),$$

where $0 \leq \theta f \leq \theta, \quad \theta f := |\arg(L^+ f, f)_{L_2}|, \quad L_2 := L_2(\Omega)$ and $\theta$ is a semi-angle corresponding to the sector $\Sigma_{\theta}$. Due to Theorem 4.3 [28] the operator $T$ is $m$-accretive. Hence in consequence of Theorem 3.35 [16] p.281 $D(T)$ is a core of the operator $T^{1/2}$. It implies that we can extend relation (10) to

$$\frac{1}{2} t[f] \leq |a[f]| \leq \sec \frac{1}{2} t[f], \quad f \in D(t),$$

where $a$ is a quadratic form generated by $A$ and $t[f] = ||T^{1/2}||^2_2$. If we consider the case $0 < \theta < \pi/3$, then it is obvious that there exist constants $b < 1$ and $M > 0$ such that the following inequality holds

$$|a[f]| \leq b t[f] + M||f||^2_2, \quad f \in D(t).$$

Hence the form $a$ is a $T$-subordinated form. In accordance with the definition given in the paper [49] it means $T$-subordination of the operator $A$ in the sense of form. Assume that $\forall \varepsilon > 0 \exists b, M > 0 : b < \varepsilon$. Using inequality (11), we get

$$\frac{1}{2} t[f] \leq \varepsilon t[f] + M(\varepsilon)||f||^2_2; \quad t[f] \leq \frac{2M(\varepsilon)}{1 - 2\varepsilon}||f||^2_2, \quad f \in D(t), \quad \varepsilon < 1/2.$$

Using the strictly accretive property of the operator $L$ (see inequality (4.9) [28]), we obtain

$$||f||^2_{H_0^1} \leq C t[f], \quad f \in D(t).$$

On the other hand, using results of the paper [28] it is easy to prove that $H^1_0(\Omega) \subset D(t)$. Taking into account the facts considered above, we get

$$||f||^2_{H^1_0} \leq C||f||_{L_2}, \quad f \in H^1_0(\Omega).$$

It cannot be! It is well-known fact. This contradiction shows us that the form $a$ is not the completely $T$-subordinated form. It implies that we can not use Theorem 8.4 [49]. Note that a reasoning corresponding to another trivial representation of $L$ by a sum is analogous. This rather particular example does not aim to show the inability of using remarkable methods considered in the paper [49] but only creates prerequisite for some valubleness of another method based on using spectral properties of the real component of the initial operator $L$. Now we would like to demonstrate effectiveness of this method.
2 Auxiliary lemmas

It was proved in the paper [28] that the operator $L$ is sectorial. Note that we used Theorem 2 [22] to obtain coordinates of the sector vertex $\gamma$ and values of the sector semi-angle $\theta$, but this theorem is useful in the general case where a considered space is the Nicodemus space. Note that we can improve estimate (5) [22] in the case corresponding to the values of indexes $p = 2, l = 1$, thus making the estimate more convenient for finding values of $\gamma$ and $\theta$.

Lemma 1. We have the following estimate

$$\|D_{0+}^\alpha f\|_{L^2} \leq \mathcal{K}\|f\|_{H^1_0}, \; f \in H^1_0(\Omega), \; \mathcal{K} > 0. \quad (12)$$

Proof. First assume that $f \in C^\infty_0(\Omega)$, then in accordance with Lemma 2.5 [28], we have $D_{0+}^\alpha f = \mathfrak{D}^\alpha f$. Applying the triangle inequality, we get

$$\|\mathfrak{D}^\alpha f\|_{L^2} \leq \frac{\alpha}{\Gamma(1 - \alpha)} \left( \int_\Omega \left( \int_0^r \frac{f(Q) - f(P + \vec{e}t)}{(r - t)^{\alpha+1}} dt \right)^2 dQ \right)^{\frac{1}{2}} + C_n^{(\alpha)} \left( \int_\Omega |f(Q) r^{-\alpha}|^2 dQ \right)^{\frac{1}{2}} = \frac{\alpha}{\Gamma(1 - \alpha)} I_1 + C_n^{(\alpha)} I_2.$$

Making the change of variables, applying the generalized Minkowski inequality, we get

$$I_1 = \left( \int_\Omega \left( \int_0^r \frac{|f(Q) - f(Q - \vec{e}t)|}{t^{\alpha+1}} dt \right)^2 dQ \right)^{\frac{1}{2}} \leq \int_0^\delta t^{-\alpha-1} \left( \int_\Omega |f(Q) - f(Q - \vec{e}t)|^2 dQ \right)^{\frac{1}{2}} dt,$$

where $\delta = \text{diam } \Omega$. Let us rewrite the previous inequality as follows

$$I_1 \leq \int_0^\delta t^{-\alpha-1} \left( \int_\Omega \left( \int_0^t |f'(Q - \vec{e}\tau)| d\tau \right)^2 dQ \right)^{\frac{1}{2}} dt, \; f'(Q) := \lim_{t \to 0} \frac{f(Q - \vec{e}t) - f(Q)}{t}.$$

Using the Cauchy-Schwarz inequality with respect to the inner integral, applying the Fubini theorem, taking into account the fact that the function $f$ vanishes outside of $\Omega$, we obtain

$$I_1 \leq \int_0^\delta t^{-\alpha-1} \left( \int_\Omega |f'(Q - \vec{e}\tau)|^2 d\tau \int_0^t d\tau \right)^{\frac{1}{2}} dt = \int_0^{\delta^{1-\alpha}/2} \left( \int_\Omega \left( \int_0^t |f'(Q - \vec{e}\tau)|^2 dQ \right) d\tau \right)^{\frac{1}{2}} dt \leq \frac{\delta^{1-\alpha}}{1 - \alpha} \|f'\|_{L^2}.$$

Arguing as above, we get

$$I_2 = \left( \int_\Omega \left( \int_0^r |f'(Q - \vec{e}t)| dt \right)^2 dQ \right)^{\frac{1}{2}} \leq \left\{ \int_\Omega \left( \int_0^r |f'(Q - \vec{e}t)|^2 dQ \right)^{\frac{1}{2}} dt \right\} \left\{ \int_\Omega \left( \int_0^r |f'(Q - \vec{e}t)|^{-\alpha} dt \right)^{\frac{1}{2}} dQ \right\} \left\{ \int_\Omega \left( \int_0^r |f'(Q - \vec{e}t)|^{\alpha} dt \right)^{\frac{1}{2}} dQ \right\} \leq \mathcal{K} \|f\|_{H^1_0}.$$


Inequality (14), it is not hard to prove that the sequence \( \{ \} \). Since \( | \) then it is easy to prove that \( \| f \| _{L^2} \leq \| f \| _{H^1/2}, f \in C_0^\infty (\Omega) \). Hence \( \| D_{\alpha}^n f \| _{L^2} \leq \| f \| _{H^1/2}, f \in C_0^\infty (\Omega) \). (13)

Finally, we obtain

\[
I_2 \leq \int_0^t \frac{|f'(Q + \tilde{e}t)|^2 dQ}{2} dt \leq \frac{\delta^{1-\alpha}}{1-\alpha} \| f' \| _{L^2}.
\]

Applying the generalized Minkowski inequality, we get

\[
I_2 \leq \int_0^t \left( \int \frac{|f'(Q - \tilde{e}t)| t^{-\alpha} dt \right) dQ \leq \left( \int \left( \int |f'(Q - \tilde{e}t)| t^{-\alpha} dt \right) dQ \right)^{1/2}.
\]

Now assume that \( f \in H^1_0(\Omega) \), then there exists the sequence \( \{ f_n \} \in C_0^\infty (\Omega) : f_n \overset{H^1_0}{\rightarrow} f \). Applying inequality (14), it is not hard to prove that the sequence \( \{ D_{\alpha}^n f_n \} \) is fundamental with respect to the norm of the space \( L^2(\Omega) \). Hence there exists a limit \( D_{\alpha}^n f_n \overset{L^2}{\rightarrow} \varphi \). By virtue of the continuity property of the directional fractional integral (see Theorem 2.1 [28]), we get \( \mathcal{H}_{0+}^\alpha D_{\alpha}^n f_n \overset{L^2}{\rightarrow} \mathcal{H}_{0+}^\alpha \varphi \). On the other hand, the application of Theorem 2.4 [28], Theorem 2.3 [28] yields \( \mathcal{H}_{0+}^\alpha D_{\alpha}^n f_n = f_n, n \in \mathbb{N} \). Therefore \( f_n \overset{L^2}{\rightarrow} \mathcal{H}_{0+}^\alpha \varphi \). Since \( f_n \overset{L^2}{\rightarrow} f \), then by virtue of the uniqueness property of limit, we obtain \( f = \mathcal{H}_{0+}^\alpha \varphi \). Applying Theorem 2.3 [28], we get \( D_{\alpha}^n f = \varphi \). In accordance with said above, we have \( D_{\alpha}^n f_n \overset{L^2}{\rightarrow} D_{\alpha}^n f \). Passing to the limit on the left and right side of inequality (14) we complete the proof.

Lemma 2. The numerical range \( \Theta(L) \) belongs to a sector \( \Sigma_\gamma(\theta) \), where

\[
\gamma = \eta^{-2} - a - \frac{T}{2} \left( \frac{I}{2} \varepsilon^2 + A \varepsilon \right)^{-1}, \theta = \arctan \left( \frac{T}{2a} \varepsilon + \frac{A}{a} \right), \varepsilon \in (0, \infty).
\]

The parameter \( \varepsilon \) can be chosen so that

\[
\gamma := \begin{cases} 
\gamma < 0, \varepsilon \in (0, \xi), \\
\gamma \geq 0, \varepsilon \in [\xi, \infty), \\
\xi = \sqrt{\left( \frac{A}{I} \right)^2 + an^2 - \frac{A}{I}}.
\end{cases}
\]

In the case \( (\gamma = 0) \), we have the following value of the semi-angle

\[
\theta = \arctan \left( \sqrt{\left( \frac{A}{2a} \right)^2 + \frac{T^2n^2}{4a} + \frac{A}{2a}} \right), I = \mathcal{P} \varepsilon.
\]

Proof. We have the following estimate

\[
|\text{Im}(f, Lf)|_{L^2} \leq \int_{\Omega} \left( a^i D_i u D_j v - a^i D_i v D_j u \right) dQ
\]
It follows that $I_1 \leq A\|f\|_{H_0^2}^2$. Using Lemma 1, the Jung inequality, we obtain

$$|(u, \rho D^\alpha v)_{L^2}| \leq I \|u\|_{L^2} \|v\|_{H_0} \leq \frac{T}{2} \left( \frac{1}{\varepsilon} \|u\|_{L^2}^2 + \varepsilon \|v\|_{H_0}^2 \right);$$

$$|(v, \rho D^\alpha u)_{L^2}| \leq \frac{T}{2} \left( \frac{1}{\varepsilon} \|v\|_{L^2}^2 + \varepsilon \|u\|_{H_0}^2 \right).$$

Hence

$$I_2 \leq |(u, \rho D^\alpha v)_{L^2}| + |(v, \rho D^\alpha u)_{L^2}| \leq \frac{T}{2} \left( \frac{1}{\varepsilon} \|f\|_{L^2}^2 + \varepsilon \|f\|_{H_0}^2 \right).$$

Finally we have the following estimate

$$\text{Im}(f, Lf)_{L^2} \leq \frac{T}{2} \varepsilon^{-1} \|f\|_{L^2}^2 + \left( \frac{T}{2} \varepsilon + A \right) \|f\|_{H_0}^2.$$ 

Thus, taking into account (11), (15), we obtain

$$\text{Re}(f, Lf)_{L^2} - b |\text{Im}(f, Lf)_{L^2}| \geq \left[ a - b \left( \frac{T}{2} \varepsilon + A \right) \right] \|f\|_{H_0}^2 + \left( \eta^{-2} - \frac{T}{2} b \varepsilon^{-1} \right) \|f\|_{L^2}^2, \ b > 0.$$ 

Choosing a value of the parameter $b$ accordingly, we get

$$|\text{Im}(f, [L - \gamma]f)_{L^2}| \leq \frac{1}{b(\varepsilon)} \text{Re}(f, [L - \gamma]f)_{L^2}, \ b(\varepsilon) = a \left( \frac{T}{2} \varepsilon + A \right)^{-1},$$

$$\gamma = \eta^{-2} - \frac{T}{2} b(\varepsilon) \varepsilon^{-1}.$$  

Taking into account that $\Theta(L) = \{ \zeta : \zeta = \omega + \gamma, \omega \in \Theta(L - \gamma) \}$, where $\theta = \arctan\{1/b(\varepsilon)\}$. Relation (15) is proved. Solving system of equations (15) relative to $\varepsilon$, we obtain the positive root $\varepsilon$ corresponding to the value $\gamma = 0$. Also, we obtain description (16) for coordinates of a sector vertex $\gamma$. Let us consider in detail the case $(\gamma = 0)$. In this case $\varepsilon = \xi$, hence $b(\xi) = a \left( \xi T/2 + A \right)^{-1}$. Using the expression for $\xi$, by easy calculation, we get (17).

**Remark 1.** Consider the sector $\Sigma_0(\theta), \theta = \arctan\{1/b(\xi)\}$. Combining Theorem 4.3 [28] with Theorem 3.2 [17], p.268, we obtain $0 \cup (C \setminus \Sigma_0) \subset P(L)$. Using Lemma 7 it is not hard to prove that $\Theta(R_L) \subset \Sigma_0(\theta)$. 

9
3 Main results

The following lemma is a central point of the method based on properties of the real component.

**Lemma 3.** The following relation holds

\[ \lambda_i [(R_L)_{\mathbb{R}}] \asymp i^{-2/n}. \]

**Proof.** First consider the next operators with positive coefficients

\[ Y_k f = -\mu_k \Delta f + \sigma_k f, \quad f \in D(L), \]

\[ \mu_k, \sigma_k = \text{const}, \quad k = 0, 1. \]

It is well known fact that \( Y_k^* = Y_k, \Theta(Y_k) \subset \{ z \in \mathbb{R} : z > C \} \). Choose the coefficients so that \( \mu_0 = a, \sigma_0 = \eta^{-2}, \mu_1 = A + I/2, \sigma_1 = I/2 \), then using (4),(5),(18),(2), we obtain

\[ (Y_0 f, f)_{L_2} \leq (H f, f)_{L_2} \leq (Y_1 f, f)_{L_2}, \quad f \in D(L), \]

where \( H := L_2 \). It is easy to prove that the energetic spaces \( H_{Y_0}, H_{Y_1} \) coincide with the space \( H_0^1(\Omega) \) as sets of elements. Hence by virtue of inequality (21) it can easily be checked that the energetic space \( H_{H} \) coincides with the space \( H_0^1(\Omega) \) as a set of elements. Also it is clear that \( (Y_0 f, f)_{L_2} \geq C \| f \|_{H_0^1}, f \in H_0^1(\Omega) \). Hence due to the Rellich-Kondrashov theorem the considered above energetic spaces are compactly embedded in \( L_2(\Omega) \).

Applying Theorem 1 [43, p.225], we get

\[ \lambda_i(Y_0) \leq \lambda_i(H) \leq \lambda_i(Y_1), \quad i \in \mathbb{N}. \]

(22)

Let us use asymptotic formula (3) [15] for the counting function of the Laplace operator

\[ N_{-\Delta}(\lambda) = \frac{\text{mes}\Omega}{2^n \pi^{n/2} \Gamma(n/2 + 1)} \lambda^{n/2} + O(\lambda^{(n-1)/2} \ln \sqrt{\lambda}), \quad \lambda \to \infty. \]

(23)

Applying formula (5.12) [3], we get

\[ \lambda_i(-\Delta) = 4\pi \left[ \frac{\Gamma(n/2 + 1)}{\text{mes}\Omega} \right]^{2/n} i^{2/n} + \beta(i^{2/n}), \quad \beta(i^{2/n}) = o(i^{2/n}), \quad i \to \infty. \]

(24)

It implies that

\[ \lambda_i(Y_k) = E_k i^{2/n} + \phi_k(i^{2/n}), \quad i \in \mathbb{N}, \]

\[ E_k = 4\pi \mu_k \left[ \frac{\Gamma(n/2 + 1)}{\text{mes}\Omega} \right]^{2/n} ; \quad \phi_k(i^{2/n}) = \beta(i^{2/n}) \mu_k + \sigma_k, \quad k = 0, 1. \]

Applying estimate (22), we get

\[ E_0 i^{2/n} + \phi_0(i^{2/n}) \leq \lambda_i(H) \leq E_1 i^{2/n} + \phi_1(i^{2/n}), \quad i \in \mathbb{N}. \]

(25)

Now we shall show that the following relation holds

\[ \lambda_i [(R_L)_{\mathbb{R}}] \asymp \lambda_i(R_H). \]

(26)
It is proved in Theorem 5.4 that \( H = \text{Re}L \). By virtue of Theorem 4.3 \[28\], we know that the operator \( L \) is m-sectorial, moreover due to Lemma [1] we have \( \Theta(L) \subset \mathcal{L}_0(\theta) \). Hence using Theorem 3.2 \[16\] p.337 we have the following representation

\[
L = H^\frac{1}{2}(I + iB_1)H^\frac{1}{2}, \quad L^* = H^\frac{1}{2}(I + iB_2)H^\frac{1}{2},
\]

(27)

where \( B_i := \{ B_i : L_2(\Omega) \to L_2(\Omega), B_i^* = B_i, \| B_i \| \leq \tan \theta \}, i = 1, 2. \)

Since the set of linear operators generates ring (algebraic structure), then we obtain

\[
Hf = \frac{1}{2} \left[ H^\frac{1}{2}(I + iB_1) + H^\frac{1}{2}(I + iB_2) \right] H^\frac{1}{2} = \frac{1}{2} \left\{ H^\frac{1}{2} [(I + iB_1) + (I + iB_2)] \right\} H^\frac{1}{2} = Hf + \frac{i}{2} H^\frac{1}{2} (B_1 + B_2) H^\frac{1}{2} f, \quad f \in D(L).
\]

Consequently

\[
H^\frac{1}{2} (B_1 + B_2) H^\frac{1}{2} f = 0, \quad f \in D(L).
\]

(28)

Let us show that \( B_1 = -B_2 \). Since the operator \( H \) is m-accretive, then we have \((H + \xi)^{-1} \in B(L_2), \text{Re} \xi > 0. \) Using this fact, we get

\[
\text{Re} \left( [H + \xi)^{-1} Hf, f] \right)_{L_2} = \text{Re} \left( [H + \xi)^{-1} [H + \xi]f, f \right)_{L_2} - \text{Re} \left( \xi [H + \xi]^{-1} f, f \right)_{L_2} \\
\geq \|f\|_{L_2}^2 - |\xi| \cdot \| (H + \xi)^{-1} \| \cdot \|f\|_{L_2}^2 = \|f\|_{L_2}^2 (1 - |\xi| \cdot \| (H + \xi)^{-1} \|), \text{Re} \xi > 0, \quad f \in D(L).
\]

(29)

Applying inequality (3), we obtain

\[
\|f\|_{L_2} \| (H + \xi)^{-1} f \|_{L_2} \geq |(f, [H + \xi]^{-1} f) | \geq (\text{Re} \xi + C_0) \| (H + \xi)^{-1} f \|_{L_2}^2, \quad f \in L_2(\Omega).
\]

It implies that

\[
\| (H + \xi)^{-1} \| \leq (\text{Re} \xi + C_0)^{-1}, \quad \text{Re} \xi > 0.
\]

Combining this estimate and (3), we have

\[
\text{Re} \left( [H + \xi)^{-1} Hf, f] \right)_{L_2} \geq \|f\|_{L_2}^2 \left( 1 - \frac{|\xi|}{\text{Re} \xi + C_0} \right), \quad \text{Re} \xi > 0, \quad f \in D(L).
\]

Applying formula (3.45) \[16\] p.282 and taking into account that \( H^\frac{1}{2} \) is selfadjoint, we get

\[
\left( H^\frac{1}{2} f, f \right)_{L_2} = \frac{1}{\pi} \int_0^\infty \zeta^{-1/2} \left( [H + \xi]^{-1} Hf, f \right)_{L_2} d\zeta \\
\geq \|f\|_{L_2}^2 \cdot C_0 \pi \int_0^\infty \zeta^{-1/2} \cdot \frac{d\zeta}{\zeta + C_0} = \sqrt{C_0} \|f\|_{L_2}^2, \quad f \in D(L).
\]

(30)

Since in accordance with Theorem 3.35 \[16\] p.281 the set \( D(H) \) is a core of the operator \( H^\frac{1}{2} \), then we can extend (30) to

\[
\left( H^\frac{1}{2} f, f \right)_{L_2} \geq \sqrt{C_0} \|f\|_{L_2}^2, \quad f \in D(H^\frac{1}{2}).
\]

(31)
Hence $N(H^{\frac{1}{2}}) = 0$. Combining this fact and (28), we obtain
\[ (B_1 + B_2) H^{\frac{1}{2}} f = 0, \ f \in D(L). \] (32)

In accordance with Theorem 3.35 [16, p.281] the operator $H^{\frac{1}{2}}$ is m-accretive. Hence combining Theorem 3.2 [16, p.268] with (31), we obtain $R(H^{\frac{1}{2}}) = L_2(\Omega)$. Taking into account that $D(H) = D(L)$ is a core of the operator $H^{\frac{1}{2}}$, we conclude that $R(H^{\frac{1}{2}})$ is dense in $L_2(\Omega)$, where $H^{\frac{1}{2}}$ is a restriction of the operator $H^{\frac{1}{2}}$ to $L_2(\Omega)$. Finally, by virtue of (32), we have that the sum $B_1 + B_2$ equals zero on the dense subset of $L_2(\Omega)$. Since these operators are defined on $L_2(\Omega)$ and bounded, then $B_1 = -B_2$. Further, we use the denotation $B_1 := B$. Using the properties of the operator $B$, we get $\|(I \pm iB)f\|_{L_2}\|f\|_{L_2} \geq \text{Re} \ ((I \pm iB)f,f)_{L_2} = \|f\|_{L_2}^2$, $f \in L_2(\Omega)$. Hence
\[ \|(I \pm iB)f\|_{L_2} \geq \|f\|_{L_2}, \ f \in L_2(\Omega). \] (33)

It implies that the operators $I \pm iB$ are invertible. Using the facts proven above: $R(H^{\frac{1}{2}}) = L_2(\Omega)$, $N(H^{\frac{1}{2}}) = 0$, we conclude that there exists the operator $H^{-\frac{1}{2}}$ defined on $L_2(\Omega)$. Taking into account the reasoning given above, using representation (27), we get the following
\[ R_L = H^{-\frac{1}{2}}(I + iB)^{-1}H^{-\frac{1}{2}}, \ R_{L^*} = H^{-\frac{1}{2}}(I - iB)^{-1}H^{-\frac{1}{2}}. \] (34)

Since we have $R_L^* = R_{L^*}$, then
\[ V = \frac{1}{2} (R_L + R_{L^*}), \ V := (R_W)_{R^*}. \] (35)

Combining (34), (35), we get
\[ V = \frac{1}{2} H^{-\frac{1}{2}} [(I + iB)^{-1} + (I - iB)^{-1}] H^{-\frac{1}{2}}. \] (36)

Using the obvious identity $(I + B^2) = (I + iB)(I - iB) = (I - iB)(I + iB)$, by direct calculation we get
\[ (I + iB)^{-1} + (I - iB)^{-1} = (I + B^2)^{-1}. \] (37)

Combining (36), (37), we obtain
\[ V = \frac{1}{2} H^{-\frac{1}{2}}(I + B^2)^{-1}H^{-\frac{1}{2}}. \] (38)

Let us evaluate the form $(V f, f)_{L_2}$. Note that by virtue of Theorem [29, p.174] there exists a unique square root of the selfadjoint operator $R_H$ the selfadjoint operator $\hat{R}$ such that $\hat{R}^2 = R_H$. Using the decomposition $H = H^{\frac{1}{2}} H^{\frac{1}{2}}$, we get $H^{-\frac{1}{2}} H^{-\frac{1}{2}} H = I$. Therefore $R_H \subset H^{-\frac{1}{2}} H^{-\frac{1}{2}}$. On the other hand $D(R_H) = L_2(\Omega)$, hence $R_H = H^{-\frac{1}{2}} H^{-\frac{1}{2}}$. Using the uniqueness property of square root we obtain $H^{-\frac{1}{2}} = \hat{R}$. Note that due to the obvious inequality $\|Sf\|_{L_2} \geq \|f\|_{L_2}, f \in L_2(\Omega)$, $S := I + B^2$ the operator $S^{-1}$ is bounded on the set $R(S)$. Taking into account the reasoning given above, we get
\[ (V f, f)_{L_2} = \left( H^{-\frac{1}{2}} S \right) (H^{-\frac{1}{2}} S f, f)_{L_2} = \left( S^{-1} H^{-\frac{1}{2}} S f, H^{-\frac{1}{2}} f \right)_{L_2} \leq \|S^{-1} H^{-\frac{1}{2}} S f\|_{L_2} \leq \|S^{-1}\| \cdot \|H^{-\frac{1}{2}} f\|_{L_2}^2 \leq \|S^{-1}\| \cdot \|(R_{Hf}, f)_{L_2}\|, f \in L_2(\Omega). \]
Theorem 1. We have the following sufficient and necessary conditions

\[ \text{Proof.} \]

Consider the case \( \lambda_i = 1 \), \( n = 1 \).

\[ F \in L_2 \Rightarrow n < 2p, \quad 1 \leq p < \infty. \]

\[ \text{Proof.} \]

Since we already know that \( R_L = R_L^* \), then it can easily be checked that the operator \( F := R_L^* R_L \) is a selfadjoint positive compact operator. Due to the well-known fact [29, p.174] there exists a unique square root of the operator \( F \), the selfadjoint positive operator \( F^{\frac{1}{2}} \) such that \( F = F^{\frac{1}{2}} F^{\frac{1}{2}} \). By virtue of Theorem 9.2 [29, p.178] the operator \( F^{\frac{1}{2}} \) is compact. Since \( N(F) = 0 \), it follows that \( N(F^{\frac{1}{2}}) = 0 \). Hence applying Theorem [13, p.189], we get that the operator \( F^{\frac{1}{2}} \) has an infinite set of the eigenvalues. Using condition (3), we get

\[ \text{Re}(R_L f, f)_{L_2} \geq C_1 \|R_L f\|_{L_2}^2, \quad f \in L_2(\Omega). \]

Hence

\[ (F f, f)_{L_2} = \|R_L f\|_{L_2}^2 \leq C_1^{-1} \text{Re}(R_L f, f)_{L_2} = C_1^{-1} (V f, f)_{L_2}, \quad V := (R_L)_\Omega. \]

Since we already know that the operators \( F, V \) are compact, then using Lemma 1.1 [13, p.45], Lemma [3, we get

\[ \lambda_i(F) \leq C_1^{-1} \lambda_i(V) \leq C i^{-2/n}, \quad i \in \mathbb{N}. \] (39)

Recall that by definition, we have \( s_i(R_L) = \lambda_i(F^{\frac{1}{2}}) \). Note that the operators \( F^{\frac{1}{2}}, F \) have the same eigenvectors. This fact can be easily proved if we note the obvious relation \( F f_i = |\lambda_i(F^{\frac{1}{2}})|^2 f_i, \quad i \in \mathbb{N} \) and representation for a square root of a selfadjoint positive compact operator

\[ F^{\frac{1}{2}} f = \sum_{i=1}^{\infty} \sqrt{\lambda_i(F)} (f, \varphi_i)_{L_2} \varphi_i, \quad f \in L_2(\Omega), \]

where \( f_i, \varphi_i \) are eigenvectors of the operators \( F^{\frac{1}{2}}, F \) respectively (see (10.25) [29, p.201]). Hence \( \lambda_i(F^{\frac{1}{2}}) = \sqrt{\lambda_i(F)}, \quad i \in \mathbb{N} \). Combining this fact with (39), we get

\[ \sum_{i=1}^{\infty} s_i^p(R_W) = \sum_{i=1}^{\infty} \lambda_i^p(F) \leq C \sum_{i=1}^{\infty} i^{-\frac{p}{n}}. \]

This completes the proof for the case \( (n \geq 2) \).
Consider the case \((n = 1)\). Since \(V\) is positive and bounded, then by virtue of Lemma 8.1 [13, p.126] we have that for any orthonormal basis \(\{\psi_i\}_{i=1}^{\infty} \subset L^2(\Omega)\) the next equalities hold

\[
\sum_{i=1}^{\infty} \text{Re}(R_L\psi_i, \psi_i)_{L^2} = \sum_{i=1}^{\infty} (V\psi_i, \psi_i)_{L^2} = \sum_{i=1}^{\infty} (V\varphi_i, \varphi_i)_{L^2},
\]

where \(\{\varphi_i\}_{i=1}^{\infty}\) is an orthonormal basis of the eigenvectors of the operator \(V\). Due to Lemma 9, we get

\[
\sum_{i=1}^{\infty} (V\varphi_i, \varphi_i)_{L^2} = \sum_{i=1}^{\infty} s_i(V) \leq C \sum_{i=1}^{\infty} i^{-\frac{2}{p}}.
\]

By virtue of Lemma 2 we have \(|\text{Im}(R_L\psi_i, \psi_i)_{L^2}| \leq b^{-1}(\xi) \text{Re}(R_L\psi_i, \psi_i)_{L^2}\). Hence the following series converges, i.e.

\[
\sum_{i=1}^{\infty} (R_L\psi_i, \psi_i)_{L^2} < \infty.
\]

Hence, by definition [13, p.125] the operator \(R_L\) has a finite matrix trace. Using Theorem 8.1 [13, p.127], we get \(R_L \in \mathcal{S}_1\). This completes the proof for the case \((n = 1)\).

Now, assume that \(R_L \in \mathcal{S}_p, 1 \leq p < \infty\). Recall that \(V\) is compact and let us show that the operator \(V\) has a complete orthonormal system of the eigenvectors. Using formula (39), we get

\[
V^{-1} = 2H^{\frac{1}{2}}(I + B^2)H^{\frac{1}{2}}, \quad D(V^{-1}) = R(V).
\]

It can easily be checked that \(D(V^{-1}) \subset D(H)\). Using this fact, we get

\[
(V^{-1}f, f)_{L^2} = 2(SH^{\frac{1}{2}} f, H^{\frac{1}{2}} f)_{L^2} \geq 2\|H^{\frac{1}{2}} f\|_{L^2}^2 = 2(H f, f)_{L^2}, f \in D(V^{-1}), \quad (40)
\]

where \(S = I + B^2\). Since \(V\) is selfadjoint, then due to Theorem 3 [11, p.136] \(V^{-1}\) is selfadjoint. Combining (35) and (40), we obtain \(\Theta(V^{-1}) \subset \{z \in \mathbb{R} : z > C\}\). Since the operator \(H\) has a discrete spectrum (see Theorem 5.3 [28]), then any set bounded with respect to the norm \(\mathcal{S}_H\) is a compact set with respect to the norm \(L^2(\Omega)\) (see Theorem 4 [11, p.220]). Combining this fact with (40), Theorem 3 [13, p.216], we get that the operator \(V^{-1}\) has a discrete spectrum, i.e. it has an infinite set of the eigenvalues \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \rightarrow \infty, \; i \rightarrow \infty\) and a complete orthonormal system of the eigenvectors. Now note that the operators \(V, V^{-1}\) have the same eigenvectors. Therefore the operator \(V\) has a complete orthonormal system of the eigenvectors. Recall that any complete orthonormal system is a basis in separable Hilbert space. Hence the system of the eigenvectors of the operator \(V\) is a basis in the space \(L^2(\Omega)\). Let \(\{\varphi_i\}_{i=1}^{\infty}\) be a complete orthonormal set of the eigenvectors of the operator \(V\). Using inequalities (7.9) [13, p.123], Lemma 3 we get

\[
\sum_{i=1}^{\infty} |s_i(R_L)|^p \geq \sum_{i=1}^{\infty} |(R_L\varphi_i, \varphi_i)_{L^2}|^p \geq \sum_{i=1}^{\infty} |\text{Re}(R_L\varphi_i, \varphi_i)_{L^2}|^p =
\]

\[
= \sum_{i=1}^{\infty} |(V\varphi_i, \varphi_i)_{L^2}|^p = \sum_{i=1}^{\infty} |\lambda_i(V)|^p \geq C \sum_{i=1}^{\infty} i^{-\frac{2p}{n}}.
\]

We claim that \(n < 2/p\). Assuming the converse in the last inequality, we come to contradiction with the imposed condition \(R_L \in \mathcal{S}_p\).
The following theorem establishes a sufficient condition for completeness of the root functions system of the operator $R_L$. This result is formulated for the multidimensional case, however it is remarkable that the obtained sufficient condition gives us an opportunity to claim that a root functions system is always complete in the cases of the dimensions $n = 1, 2$.

**Theorem 2.** If $0 < \theta < \pi/n$, then the root functions system of the operator $R_L$ is complete, where $\theta$ is the semi-angle of the sector $\mathfrak{L}_0(\theta)$.

**Proof.** We mentioned above that $\Theta(R_L) \subset \mathfrak{L}_0(\theta)$. Note that the map $z : \mathbb{C} \to \mathbb{C}$, $z = 1/\zeta$, takes each eigenvalue of the operator $R_L$ to the eigenvalue of the operator $L$. It is also clear that $z : \mathfrak{L}_0(\theta) \to \mathfrak{L}_0(\theta)$. Using the definition [13, p.302] let us consider the following set

$$\tilde{W}_{R_L} := \left\{ z : z = t\xi, \xi \in \tilde{\Theta}(R_L), 0 \leq t < \infty \right\}.$$

It is easy to see that $\tilde{W}_{R_L}$ coincides with a closed sector of the complex plane with the vertex situated at the point zero. Let us denote by $\vartheta(R_L)$ the angle of this sector. It is obvious that $\tilde{W}_{R_L} \subset \mathfrak{L}_0(\theta)$. Therefore $0 \leq \vartheta(R_L) \leq 2\theta$. Let us prove that $0 < \vartheta(R_L)$, i.e. strict inequality holds. If we assume that $\vartheta(R_L) = 0$, then we get $e^{-\text{arg}z} = \varsigma, \forall z \in \tilde{W}_{R_L} \setminus 0$, where $\varsigma$ is a constant independent on $z$. In consequence of this fact we have $\text{Im}\Theta(\varsigma R_L) = 0$. Hence the operator $\varsigma R_L$ is symmetric (see Problem 3.9 [16, p.269]) and by virtue of the fact $D(\varsigma R_L) = L_2(\Omega)$ it is selfadjoint. On the other hand, taking into account the equivalence $R_L^* = R_L$, we have $(\varsigma R_L f, g)_{L_2} = (f, \varsigma R_L^* g)_{L_2}$, $f, g \in L_2(\Omega)$. Hence $\varsigma R_L = \varsigma R_L^*$. In the particular case, we have $\forall f \in L_2(\Omega), \text{Im}f = 0 : \text{Re}\varsigma R_L f = \text{Re}\varsigma R_L^* f, \text{Im}\varsigma R_L f = -\text{Im}\varsigma R_L^* f$. It implies that $N(R_L) \neq 0$. This contradiction concludes the proof of the fact $\vartheta(R_L) > 0$. Let us use Theorem 6.2 [13, p.305] according to which we have the following. If the next two conditions (a) and (b) are fulfilled, then the set of root functions of the operator $R_L$ is complete.

a) $\vartheta(R_L) = \pi/d$, where $d > 1$,

b) for some $\beta$, the operator $B := (e^{i\beta R_L})_\gamma : s_i(B) = o(i^{-1/d}), i \to \infty$.

Let us show that conditions (a) and (b) are fulfilled. Note that due to Lemma 2 we have $0 \leq \theta < \pi/2$. Hence $0 < \vartheta(R_L) < \pi$. It implies that there exists $1 < d < \infty$ such that $\vartheta(R_L) = \pi/d$. Thus condition (a) is fulfilled. Let us choose a certain value $\beta = \pi/2$ in condition (b) and notice that $(e^{i\pi/2 R_L})_\gamma = (R_L)_{\Re}$. Since the operator $V := (R_L)_{\Re}$ is selfadjoint, it follows that $s_i(V) = \lambda_i(V), i \in \mathbb{N}$. In consequence of Lemma 3 we have

$$s_i(V) i^{1/d} \leq C \cdot i^{1/d-2/n}, i \in \mathbb{N}.$$ 

Hence, it is sufficient to show that $d > n/2$ to achieve condition (b). By virtue of the conditions $\vartheta(R_L) \leq 2\theta, \theta < \pi/n$, we have $d = \pi/\vartheta(R_L) \geq \pi/2\theta > n/2$. Hence we obtain $s_i(V) = o(i^{-1/d})$. Since both conditions (a),(b) are fulfilled, then using Theorem 6.2 [13, p.305] we complete the proof.

**Remark 2.** Assume that the root functions system is complete; then using the definition, it is easy to prove that the set of the eigenvalues is infinite. Thus in the cases $(n=1,2)$ we have an infinite set of the eigenvalues.
The theorems proven above is devoted to description of $s$-numbers of the operator $R_L$, but questions related with the eigenvalues asymptotic are still relevant in our work. The following theorem establishes an asymptotic formula.

**Theorem 3.** The following inequality holds

$$
\sum_{i=1}^{k} |\lambda_i(R_L)|^p \leq \sec^p \theta \|S^{-1}\| \sum_{i=1}^{k} \gamma_i^p(i) i^{-2p/n},
$$

\[ (k = 1, 2, ..., \nu(R_L)), \ 1 \leq p < \infty, \]  

where $\theta = \arctan\{1/b(\xi)\}$, $\gamma_0(i) = [\mathcal{E}_0 + \phi_0(i^{2/n})/i^{2/n}]^{-1}$.

Moreover if $\nu(R_L) = \infty$, then the following asymptotic formula holds

$$
|\lambda_i(R_L)| = o\left(i^{-2/n+\varepsilon}\right), i \to \infty, \ \forall \varepsilon > 0.
$$

**Proof.** In accordance with Theorem 6.1 [13, p.81], we have the following inequality for any bounded linear operator $A$ with the compact imaginary component

$$
\sum_{m=1}^{k} |\text{Im} \lambda_m(A)|^p \leq \sum_{m=1}^{k} |s_m(A^\gamma)|^p, \ (k = 1, 2, ..., \nu(A)), \ 1 \leq p < \infty,
$$

where $\nu(A) \leq \infty$ is a sum of all algebraic multiplicities corresponding to the not real eigenvalues of the operator $A$ (see [13, p.79]). We can also verify the following relation by direct calculation

$$
(iA)^\gamma = A_\gamma, \ \text{Im} \lambda_m(iA) = \text{Re} \lambda_m(A), \ m \in \mathbb{N}.
$$

By virtue of the fact $\Theta(R_L) \subset \mathcal{L}_0(\theta)$, we have $\text{Re} \lambda_m(R_L) > 0, \ m = 1, 2, ..., \nu(R_L)$.

Combining this fact with (43), we obtain $\nu_\gamma(iR_L) = \nu(R_L)$. Taking into account the last identity and combining (42), (43), we obtain

$$
\sum_{m=1}^{k} |\text{Re} \lambda_m(R_L)|^p \leq \sum_{m=1}^{k} |s_m(V)|^p, \ (k = 1, 2, ..., \nu(R_L)).
$$

Due to Lemma 2, we have

$$
|\text{Im} \lambda_m(R_L)| \leq \tan \theta \text{Re} \lambda_m(R_L), \ m \in \mathbb{N},$

Hence, we get

$$
|\lambda_m(R_L)| = \sqrt{|\text{Im} \lambda_m(R_L)|^2 + |\text{Re} \lambda_m(R_L)|^2} \leq \sqrt{|\text{Re} \lambda_m(R_L)| \cdot \tan^2 \theta + 1} = \sec \theta |\text{Re} \lambda_m(R_L)|, \ m \in \mathbb{N}.
$$

Combining (44), (45), we obtain

$$
\sum_{m=1}^{k} |\lambda_m(R_L)|^p \leq \sec^p \theta \sum_{m=1}^{k} |s_m(V)|^p, \ (k = 1, 2, ..., \nu(R_L)).
$$

Applying Lemma 8 we complete the proof of inequality (3).
Assume that $\nu(R_L) = \infty$. Note that the series on the right side of (3) converges in the case $(p > n/2)$. It implies that

$$|\lambda_i(R_L)|^{1/p} \to 0, \ i \to \infty.$$ \hspace{1cm} (47)

We can choose a value of $p$ so that for arbitrary small $\varepsilon > 0$, we have $1/p < 2/n$, $1/p > 2/n - \varepsilon$. Hence

$$|\lambda_i(R_L)|^{2/n-\varepsilon} < |\lambda_i(R_L)|^{1/p}, \ i \in \mathbb{N}. $$

Applying relation (47), we complete the proof of the asymptotic formula.

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**References**

[1] Ahiezer N.I., Glazman I.M. Theory of linear operators in Hilbert space. *Moscow: Nauka, Fizmatlit*, 1966.

[2] Agranovich M.S. Sobolev spaces, their generalization and elliptic problems in domains with the smooth and Lipschitz boundary. *Moscow: MCNMO*, 2013.

[3] Agranovich M.S. Spectral problems in Lipshitz mapping areas. *Modern mathematics, Fundamental direction*, 39 (2011), 11-35.

[4] Aleroev T.S. Spectral analysis of one class of non-selfadjoint operators. *Differential Equations*, 20, No.1 (1984), 171-172.

[5] Aleroev T.S., Aleroev B.I. On eigenfunctions and eigenvalues of one non-selfadjoint operator. *Differential Equations*, 25, No.11 (1989), 1996-1997.

[6] Aleroev T.S. On eigenvalues of one class of nonselfadjoint operators. *Differential Equations*, 30, No.1 (1994), 169-171.

[7] Aleroev T.S. On completeness of the system of the eigenfunctions of one differential operator of fractional order. *Differential Equations*, 36, No.6 (2000), 829-830.

[8] Browder F.E. On the eigenfunctions and eigenvalues of the general linear elliptic differential operator. *Proc. Nat. Acad. Sci. U.S.A.*, 39 (1953), 433-439.

[9] Browder F.E. On the spectral theory of strongly elliptic differential operators. *Proc. Nat. Acad. Sci. U.S.A.*, 45 (1959), 1423-1431.

[10] Browder F.E. On the spectral theory of elliptic differential operators I. *Math. Ann.*, 142 (1959), 22-130.
[11] Carleman T. Uber die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen. *Ber. Verh. Sachs. Akad. Leipzig*, 88, No.3 (1936), 119-132.

[12] Glazman I.M. On decomposability in a system of eigenvectors of dissipative operators. *Russian Mathematical Surveys*, 13, Issue 3 (81), (2016), 179-181.

[13] Gohberg I.C., Krein M.G. Introduction to the theory of linear non-selfadjoint operators in Hilbert space. *Moscow: Nauka, Fizmatlit*, 1965.

[14] Friedrichs K. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11 (1958), 238-241.

[15] Fedosov B.V. Asymptotic formulas for the eigenvalues of the Laplace operator in the case of a polyhedron. *Reports of the Academy of Sciences of the USSR*, 157, No. 3 (1964), 536-538.

[16] Kato T. Perturbation theory for linear operators. *Springer-Verlag Berlin, Heidelberg, New York*, 1980.

[17] Kato T. Fractional powers of dissipative operators. *J.Math.Soc.Japan*, 13, No.3 (1961), 246-274.

[18] Keldysh M.V. On the eigenvalues and eigenfunctions of some classes of non-selfadjoint equations. *Reports of the Academy of Sciences of the USSR*, 77, No.1 (1951), 11-14.

[19] Keldysh M.V. On a Tauberian theorem. *Amer. Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 102, No.2 (1973), 133-143.

[20] Keldysh M.V. On completeness of eigenfunctions of some classes of non-selfadjoint linear operators. *Russian Math. Surveys*, 26, No.4 (1971), 15-44.

[21] Kipriyanov I.A. On spaces of fractionally differentiable functions. *Proceedings of the Academy Of Sciences. USSR*, 24 (1960), 665-882.

[22] Kipriyanov I.A. The operator of fractional differentiation and powers of elliptic operators. *Proceedings of the Academy of Sciences. USSR*, 131 (1960), 238-241.

[23] Kipriyanov I.A. On some properties of the fractional derivative in the direction. *Proceedings of the universities. Math., USSR*, No.2 (1960), 32-40.

[24] Kipriyanov I.A. On complete continuity of embedding operators in spaces of fractionally differentiable functions. *Russian Mathematical Surveys*, 17 (1962), 183-189.

[25] Kukushkin M.V. Theorem of existence and uniqueness of a solution for a differential equation of fractional order. *Journal of Fractional Calculus and Applications*, 9 (2) July (2018), 220-228.

[26] Kukushkin M.V. Evaluation of eigenvalues of the Sturm-Liouville problem for a differential operator with the fractional derivative in the final term. *Belgorod State University Scientific Bulletin, Math. Physics.*, 46, No.6 (2017), 29-35.

[27] Kukushkin M.V. On some qualitative properties of the Kipriyanov fractional differentiation operator. *Vestnik of Samara University, Natural Science Series, Math.*, 23, No.2 (2017), 32-43.
[28] Kukushkin M.V. Spectral properties of fractional differentiation operators. *Electronic Journal of Differential Equations*, 2018, No. 29 (2018), 1-24.

[29] Krasnoselskii M.A., Zabreiko P.P., Pustylnik E.I., Sobolevskii P.E. Integral operators in spaces of summable functions. *Moscow: Science, FIZMATLIT*, 1966.

[30] Krein M.G. Criteria for completeness of the system of root vectors of a dissipative operator. *Amer. Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 26, No.2 (1963), 221-229.

[31] Livshits M.S. On spectral decomposition of linear non-selfadjoint operators. *Amer. Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 5, No.2 (1957), 67-114.

[32] Lidskii V.B. Conditions for completeness of a system of root subspaces for non-selfadjoint operators with discrete spectra. *Amer. Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 34, No.2 (1963), 241-281.

[33] Lidskii V.B. Summability of series in the principal vectors of non-selfadjoint operators. *Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 40, No.2 (1964), 193-228.

[34] Lidskii V.B. On the Fourier series expansion in the main functions of the non-selfadjoint elliptic operator. *Math. collection*, 57 (99), No.2 (1962), 137-150.

[35] Markus A.S. On the basis of root vectors of a dissipative operator. *Soviet Math. Dokl.*, 1 (1960), 599-602.

[36] Markus A.S. Expansion in root vectors of a slightly perturbed self-adjoint operator. *Soviet Math. Dokl.*, 3 (1962), 104-108.

[37] Matsaev V.I. On a class of completely continuous operators. *Soviet Math. Dokl.*, 2 (1961), 972-975.

[38] Matsaev V.I. Some theorems on completeness of root subspaces of completely continuous operators. *Soviet Math. Dokl.*, 5 (1964), 396-399.

[39] Matsaev V.I. A method for estimation of the resolvents of non-selfadjoint operators. *Soviet Math. Dokl.*, 5 (1964), 236-240.

[40] Agmon S. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.*, 15 (1962), 119-147.

[41] Agmon S. Lectures on elliptic boundary value problems. *Van Nostrand Math. Stud., 2*, D. Van Nostrand Co., Inc., Princeton, NJ-Toronto-London, 1965.

[42] Katsnelson V.E. Conditions under which systems of eigenvectors of some classes of operators form a basis. *Funct. Anal. Appl.*, 1, No.2 (1967), 122-132.

[43] Mikhlin S.G. Variational methods in mathematical physics. *Moscow Science*, 1970.

[44] Mukminov B.R. Expansion in eigenfunctions of dissipative kernels. *Proceedings of the Academy of Sciences. USSR*, 99, No.4 (1954), 499-502.
[45] NAKHUSHEV A.M. The Sturm-Liouville problem for an ordinary differential equation of second order with fractional derivatives in final terms. Proceedings of the Academy of Sciences. USSR, 234, No.2 (1977), 308-311.

[46] ROZENBLYUM G.V., SOLOMYAK M.Z., SHUBIN M.A. Spectral theory of differential operators. Results of science and technology. Series Modern problems of mathematics Fundamental directions, 64 (1989), 5-242.

[47] SAMKO S.G., KILBAS A.A., MARICHEV O.I. Integrals and derivatives of fractional order and some of their applications. Minsk Science and technology, 1987.

[48] SOBOLEV S.L. Some applications of functional analysis in mathematical physics. Moscow: Nauka, Fizmatlit, 1988.

[49] SHKALIKOV A.A. Perturbations of selfadjoint and normal operators with a discrete spectrum. Russian Mathematical Surveys, 71, Issue 5 (431) (2016), 113-174.

[50] TAMARKIN I.D. On some general problems of the theory of ordinary linear differential equations and on expansion of arbitrary functions in series. Type. M. P. Frolova, Petrograd, 1917.

[51] ZEIDLER E. Applied functional analysis, applications to mathematical physics. Applied mathematical sciences 108, Springer-Verlag, New York, 1995.