Partial regularity result of elliptic systems with Dini continuous coefficients and \( q \)-growth

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Abstract. We establish partial regularity result for vector-valued solutions \( u : \Omega \to \mathbb{R}^N \) to second order elliptic systems of the type:

\[- \text{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega,\]

where the coefficients \( A : \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \) satisfies Dini condition respect to \( (x, u) \) with growth order \( q \geq 2 \). We prove \( C^1 \)-regularity of the solutions outside of singular sets.

Keywords. Nonlinear elliptic systems, Partial regularity, Dini condition, \( A \)-harmonic approximation.

Mathematics Subject Classification (2010): 35J60, 35B65.

1 Introduction

In this paper, we consider the second order nonlinear elliptic systems in divergence form of the following type:

\[- \text{div}(A(x, u, Du)) = f(x, u, Du) \quad \text{in } \Omega. \quad (1.1)\]

Here, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( u \) takes values in \( \mathbb{R}^N \) with coefficients \( A : \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \).

The regularity theory with the growth of \( A(x, \xi, p) \) with respect to \( p \) has been proved by Giaquinta and Modica [10]. They proved that weak solutions of (1.1) has Hölder continuous first derivatives outside of a singular set of Lebesgue measure zero if \((1 + |p|)^{-1}A(x, \xi, p)\) is Hölder continuous in variables \((x, \xi)\) uniformly with respect to \( p \). In [6], Duzaar and Grotowski gave a simplified proof of their result without \( L^q \)-\( L^2 \) estimates for \( Du \). The method of proof also gives the optimal result in one step, i.e. if \((1 + |p|)^{-1}A(x, \xi, p)\) is in \( C^{0, \alpha} \) for some \( 0 < \alpha < 1 \) in \((x, \xi)\) then \( u \) is in \( C^{1, \alpha} \) outside of the singular set. The essential feature is the use of the \( A \)-harmonic approximation lemma (cf. [6], Lemma 2.1); see also Lemma 3.2.

Duzaar and Gastel [5] prove under weaker assumptions on \( A(x, \xi, p) \) with respect to continuity in the variables \((x, \xi)\). More precisely, they assume for the continuity of \( A(x, \xi, p) \) with respect to the variables \((x, \xi)\) that

\[ |A(x, \xi, p) - A(\tilde{x}, \tilde{\xi}, p)| \leq \kappa(|\xi|)\mu \left( |x - \tilde{x}| + |\xi - \tilde{\xi}| \right)(1 + |p|), \quad (1.2)\]

for all \( x, \tilde{x} \in \Omega, \xi, \tilde{\xi} \in \mathbb{R}^N, p \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \), where \( \kappa : [0, \infty) \to [1, \infty) \) is nondecreasing, and \( \mu : (0, \infty) \to [0, \infty) \) is nondecreasing and concave with \( \mu(0+) = 0 \). They also have to require that \( r \mapsto r^{-\alpha} \mu(r) \) is nonincreasing for some \( 0 < \alpha < 1 \), and that

\[ F(r) = \int_0^r \frac{\mu(\rho)}{\rho} d\rho < \infty \quad \text{for some } r > 0. \quad (1.3)\]
They conclude that a bounded weak solution of elliptic system (1.1) satisfying (1.2) and (1.3) is in $C^1$ outside a closed singular set with Lebesgue measure zero.

The condition (1.3) is called Dini condition in the literature, although Dini himself [4] used a slightly weaker conditions a century ago. It had some significance for the theory of linear elliptic partial differential equations in the first half of the century, cf. [12].

Qiu [13] extend the result in [5], which is the result under quadratic growth condition, to the subquadratic case. In this case, the assumptions (1.2) and (1.3) are modified as weaker conditions a century ago. It had some significance for the theory of linear elliptic partial differential equations in the first half of the century, cf. [12].

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In the regularity theory in the subquadratic case, we write $\text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^n))$ for the space of bilinear forms on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$. We denote $c$ a positive constant, possibly varying from line by line. Special occurrences will be denoted by capital letters $K$, $C_1$, $C_2$ or the like.

2 Hypothesis and Statement of Results

Definition 2.1. We define $u \in W^{1,q}(\Omega, \mathbb{R}^N), q \geq 2$ is a weak solution of (1.1) if $u$ satisfies

$$\int_{\Omega} \langle A(x, u, Du), D\varphi \rangle dx = \int_{\Omega} \langle f, \varphi \rangle dx$$

(2.1)

for all $\varphi \in C^\infty_0(\Omega, \mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^N$ or $\mathbb{R}^{nN}$. 2
We assume following structure condition.

(H1) \( A(x, \xi, p) \) is differentiable in \( p \) with continuous derivatives. Moreover, there exists \( L \geq 1 \) such that

\[
|D_p A(x, \xi, p)| \leq L(1 + |p|)^{\gamma - 2} \quad \text{for all } (x, \xi, p) \in \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N);
\]

this infers the existence of a modulus of continuity \( \omega : [0, \infty) \times [0, \infty) \to [0, 1] \) with \( \omega(t, 0) = 0 \) for all \( t \) such that \( t \mapsto \omega(s, t) \) is nondecreasing for fixed \( s \), \( s \mapsto \omega(s, t) \) is concave and nondecreasing for fixed \( t \). \( \omega(s, t) \) also satisfies

\[
|D_p A(x, \xi, p) - D_p A(\bar{x}, \bar{\xi}, \bar{p})| \leq L \omega(\|\xi\| + \|\nu\|, |x - \bar{x}|^2 + |\xi - \bar{\xi}|^2 + |p - \bar{p}|^2) (1 + |p| + |\bar{p}|)^{\gamma - 2}.
\]

for all \( (x, \xi, p), (\bar{x}, \bar{\xi}, \bar{p}) \in \Omega \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \) with \( |\xi| + |p| \leq M \).

(H2) \( A(x, \xi, p) \) is uniformly strongly elliptic i.e., for some \( \lambda > 0 \), \( A(x, \xi, p) \) satisfies

\[
(D_p A(x, \xi, p)\nu, \nu) \geq \lambda |\nu|^2 (1 + |p|)^{\gamma - 2} \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^N, p, \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N);
\]

(H3) There exists a modulus of continuity \( \mu : [0, \infty) \to (0, \infty) \), and a nondecreasing function \( \kappa : [0, \infty) \to [1, \infty) \) such that

\[
|A(x, \xi, p) - A(\bar{x}, \bar{\xi}, \bar{p})| \leq \kappa(|\xi|) \mu\left( |x - \bar{x}| + |\xi - \bar{\xi}| \right) (1 + |p|)^{\gamma - 1}
\]

for all \( x, \xi \in \Omega, \xi, \bar{\xi} \in \mathbb{R}^N, p \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \). Without loss of generality we may assume that

\[
(\mu_1) \mu \text{ is nondecreasing function with } \mu(0+) = 0.
\]

\[
(\mu_2) \mu \text{ is concave; in the proof of the regularity theorem we have to require that } r \mapsto r^{-\alpha} \mu(r) \text{ is nonincreasing for some exponent } \alpha \in (0, 1).
\]

We also require modified Dini’s condition:

\[
(\mu_3) F(r) := \int_0^r \frac{\mu^\beta(\rho)}{\rho} d\rho < +\infty \text{ for some } r > 0 \text{ and } \beta \in (0, 1].
\]

(H4) There exists constants \( a \) and \( b \), with a possibly depending on \( M > 0 \), such that

\[
|f(x, \xi, p)| \leq a(M)|p|^\gamma + b
\]

for all \( x \in \Omega, \xi \in \mathbb{R}^N \) with \( |\xi| \leq M \), and \( p \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \).

Using above structure conditions, we state our main theorem.

**Theorem 2.2.** Let \( u \in W^{1,q}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \) be a bounded weak solution to \((1.1)\) under the structure conditions \((H1), (H2), (H3), (H4), (\mu_1), (\mu_2) \text{ and } (\mu_3)\), satisfying \( \|u\|_{\infty} \leq M \) and \( 2^{(10 - 9q)/2} \lambda > a(M)M \). Then there is a relatively closed set \( \text{Sing} u \subset \Omega \), such that the weak solution \( u \) satisfies \( u \in C^1(\Omega \setminus \text{Sing} u, \mathbb{R}^N) \). Further \( \text{Sing} u \subset \Sigma_1 \cup \Sigma_2 \), where

\[
\Sigma_1 := \left\{ x_0 \in \Omega : \liminf_{\rho \to 0} \int_{\mathcal{B}_\rho(x_0)} |Du - (Du)_{x_0, p}|^q dx > 0 \right\}, \quad \text{and}
\]

\[
\Sigma_2 := \left\{ x_0 \in \Omega : \limsup_{\rho \to 0} |(Du)_{x_0, p}| = \infty \right\}
\]

and in particular, \( \mathcal{L}^n(\text{Sing} u) = 0 \). In addition, for \( \sigma \in [\alpha, 1) \) and \( x_0 \in \Omega \setminus \text{Sing} u \) the derivative of \( u \) has modulus of continuity \( r \mapsto r^{\sigma} + F(r) \) in a neighborhood of \( x_0 \).
3 Some preliminaries

In this section we recall the $\mathcal{A}$-harmonic approximation lemma, and some standard estimates for the proof of the regularity theorem.

First we state the definition of $\mathcal{A}$-harmonic function and present the following version of an $\mathcal{A}$-harmonic approximation lemma which can be retrieved from the corresponding parabolic version in [8, Lemma 2.3]. This lemma allowed us to approximate the weak solution $u$ to the solution of constant coefficients elliptic system in $L^2$ as well as in $L^q$. For more detail about $\mathcal{A}$-harmonic approximation technique, we refer to the survey paper [8].

**Definition 3.1** ([8 Section 1]). For a given $\mathcal{A} \in \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N))$, we say that $h \in W^{1,q}(\Omega, \mathbb{R}^N)$ is an $\mathcal{A}$-harmonic function, if $h$ satisfies

$$
\int_{\Omega} \mathcal{A}(Dh, D\varphi)dx = 0
$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

**Lemma 3.2** ([1 Lemma 2.3]). Let $0 < \lambda \leq L$ and $q \geq 2$ be given. For every $\varepsilon > 0$, there exists a constant $\delta = \delta(n, q, \lambda, L, \varepsilon) \in (0, 1]$ such that the following holds: assume that $\gamma \in [0, 1]$ and that $\mathcal{A}$ is a bilinear form on $\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ with the properties

$$
\mathcal{A}(\nu, \nu) \geq \lambda|\nu|^2, \quad \text{and} \quad \mathcal{A}(\nu, \tilde{\nu}) \leq L|\nu||\tilde{\nu}|, \quad (3.1)
$$

for all $\nu, \tilde{\nu} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Furthermore, let $w \in W^{1,q}(B_\rho(x_0), \mathbb{R}^N)$ be an approximately $\mathcal{A}$-harmonic map in the sense that there holds

$$
\left| \int_{B_\rho(x_0)} \mathcal{A}(Dw, D\varphi)dx \right| \leq \delta \sup_{B_\rho(x_0)} |D\varphi|
$$

for all $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ and that

$$
\int_{B_\rho(x_0)} \{|Dw|^2 + \gamma^q|Dw|^q\}dx \leq \tilde{C}(n, q).
$$

Then there exists an $\mathcal{A}$-harmonic function $h \in C^\infty(B_{\rho/2}(x_0), \mathbb{R}^N)$ that satisfies

$$
\int_{B_{\rho/2}(x_0)} \left\{|Dh|^2 + \gamma^q |Dh|^q\right\}dx \leq \tilde{C}(n, q) \quad (3.2)
$$

and, at the same time,

$$
\int_{B_{\rho/2}(x_0)} \left\{|w-h|_{\rho/2}^2 + \gamma^q |w-h|_{\rho/2}^q\right\}dx \leq \varepsilon. \quad (3.3)
$$

Next is a standard estimates for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [2, Teorema 9.2]. For convenience, we state the estimate in a slightly general form than the original one.

**Theorem 3.3** ([8 Theorem 2.3]). Consider $\mathcal{A}$, $\lambda$ and $L$ as in Lemma 3.2. Then there exists $C_0 \geq 1$ depending only on $n$, $N$, $\lambda$ and $L$ such that any $\mathcal{A}$-harmonic function $h$ on $B_{\rho/2}(x_0)$ satisfies

$$
\left(\frac{\rho}{2}\right)^2 \sup_{B_{\rho/4}(x_0)} |Dh|^2 + \left(\frac{\rho}{2}\right)^4 \sup_{B_{\rho/4}(x_0)} |D^2h|^2 \leq C_0 \left(\frac{\rho}{2}\right)^2 \int_{B_{\rho/2}(x_0)} |Dh|^2dx. \quad (3.4)
$$
We state the Poincaré inequality in a convenient form.

**Lemma 3.4** ([11] Proposition 3.10). There exists $C_P \geq 1$ depending only on $n$ such that every $u \in W^{1,q}(B_{\rho}(x_0), \mathbb{R}^N)$ satisfies
\[
\int_{B_{\rho}(x_0)} |u - u_{x_0,\rho}|^q dx \leq C_P \rho^q \int_{B_{\rho}(x_0)} |Du|^q dx. \tag{3.5}
\]

Using Young’s inequality, we obtain the following estimates.

**Lemma 3.5** ([13] Lemma 3.7). Consider fixed $a, b \geq 0$, $q \geq 1$. Then for any $\varepsilon > 0$, there exists $K = K(q, \varepsilon) \geq 0$ satisfying
\[
(a + b)^q \leq (1 + \varepsilon)a^q + Kb^q. \tag{3.6}
\]

**Lemma 3.6** ([11] Lemma 2.1). For $\delta \geq 0$, and for all $a, b \in \mathbb{R}^n$ we have
\[
4^{-1+2\delta} \leq \int_0^1 \frac{(1 + |sa + (1-s)b|^2)^{\delta/2} ds}{(1 + |a|^2 + |b - a|^2)^{\delta/2}} \leq 4^\delta. \tag{3.7}
\]

In the followings, we write the modulus of continuity $\mu$ as
\[
\eta(t) := \mu^2 \left( \sqrt{t} \right)
\]
by technical reason (cf. (H3)). The conditions $(\mu 1)$, $(\mu 2)$ and $(\mu 3)$ are expressed as

$(\mu 1)$ $\eta$ is continuous, nondecreasing, and $\eta(0) = 0$,

$(\mu 2)$ $\eta$ is concave; and $t \mapsto t^{-\alpha}\eta(t)$ is nonincreasing for the same exponent $\alpha$ as in $(\mu 2)$,

$(\mu 3)$ $\tilde{F}(t) := \left[ 2F \left( \sqrt{t} \right) \right]^2 = \left[ \int_0^t \frac{\eta^2(s)}{s} ds \right]^2 < +\infty$ for some $t > 0$.

Changing $\kappa$ by a constant, but keeping $\kappa \geq 1$, we can also assume that

$(\mu 4)$ $\eta(1) = 1$, implying $t \leq \eta(t) \leq 1$ for $t \in (0, 1]$.

From the fact that $\eta$ is nondecreasing, for $t \leq s$ and $\sigma \leq 1/\alpha$, we deduce $s\eta^\sigma(t) \leq s\eta^\sigma(s)$. For $s \leq t$, we use nonincreasing property of $t^{-\alpha}\eta(t)$ and $\eta(s) \leq 1$, and we obtain $s\eta^\sigma(t) \leq t$. Combining both cases we obtain
\[
s\eta^\sigma(t) \leq s\eta^\sigma(s) + t \quad \text{for } s \in [0, 1], \ t > 0, \ \sigma \leq \frac{1}{\alpha}.
\]

In particular, we have

$(\mu 5)$ $s\eta(t) \leq s\eta(s) + t$ for $s \in [0, 1], \ t > 0$,

$(\mu 6)$ $s\sqrt{\eta(t)} \leq s\sqrt{\eta(s)} + t$ for $s \in [0, 1], \ t > 0$.

From $(\mu 2)$ we infer for $i \in \mathbb{N} \cup \{0\}$, $\theta \in (0, 1/8]$, $t > 0$
\[
\int_{\theta^{2i} t}^{\theta^{2i+1} t} \frac{\eta^2(\tau)}{\tau} d\tau \geq \frac{\eta^2(\theta^{2i} t)}{(\theta^{2i} t)^{\alpha\beta}} \int_{\theta^{2i} t}^{\theta^{2i+1} t} \tau^{(\alpha\beta - 2)/2} d\tau = \frac{2}{\alpha\beta} (1 - \theta^{\alpha\beta}) \sqrt{\eta^2(\theta^{2i} t)},
\]

Consider Lemma 4.1. Note that for all \( k \) with \( C \) which implies
\[
\eta(t) \leq \frac{\alpha^2 \beta^2}{4(1-\theta^{\alpha\beta})^2} \tilde{F}(t)
\]
for all \( t \in [0,1] \). Moreover, for \( t \in [0,1], \theta \in (0,1/8) \), we have
\[
t^{-\alpha} \tilde{F}(t) = t^{-\alpha} \left[ \sqrt{F(\theta t)} + \int_{\theta t}^{t} \sqrt{\tau^{-\alpha} \eta(\tau)} \tau^{(\alpha-2)/2} d\tau \right]^2
\]
\[
\leq t^{-\alpha} \left[ \sqrt{F(\theta t)} + \frac{2}{\alpha} \sqrt{\theta t}^{-\alpha} \eta(\theta t) \left( \sqrt{\theta t} - \sqrt{\theta t}^\alpha \right) \right]^2
\]
\[
\leq \left[ \sqrt{t^{-\alpha} \tilde{F}(\theta t)} + \sqrt{(\theta t)^{-\alpha} \tilde{F}(\theta t)} \frac{1-\theta^{\alpha/2}}{1-\theta^{\alpha}} \right]^2
\]
\[
\leq 4(\theta t)^{-\alpha} \tilde{F}(\theta t).
\]

\[\tag{3.10}\]

\section{Caccioppoli-type inequality}

For \( s, t \geq 0 \) let
\[
\rho_1(s,t) := (1 + t)^{-1}\kappa^{-1}(s+t), \quad G(s,t) := (1 + t)^2\kappa^2(s+t).
\]
Note that \( \rho_1 \leq 1 \) and \( G \geq 1 \).

\textbf{Lemma 4.1.} Consider \( \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \) and \( \xi \in \mathbb{R}^N \) with \( |\xi| \leq M \) fixed. Let \( u \in W^{1,q}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \) be a bounded weak solution to (4.1) under the structure conditions (H1), (H2), (H3), (H4), (\( \eta_1 \)), (\( \eta_2 \)), (\( \eta_3 \)) and (\( \eta_4 \)) with satisfying \( \|u\|_\infty \leq M \) and \( 2^{(1-n)/(2+2)} > a(M)M \). Then for any \( x_0 \in \Omega \) and \( \rho \leq \rho_1(|\xi|, |\nu|) \) such that \( B_\rho(x_0) \subset \Omega \), there holds
\[
\left\{ \int_{B_{\rho/2}(x_0)} \frac{|Du - \nu|^2}{(1 + |\nu|)^2} dx \right\} \left\{ \int_{B_{\rho}(x_0)} \frac{|Du - \nu|^q}{(1 + |\nu|)^q} dx \right\}
\leq C_1 \left\{ \int_{B_{\rho}(x_0)} \left( \frac{|u - \xi - \nu(x - x_0)|^2}{\rho(1 + |\nu|)} + \frac{|u - \xi - \nu(x - x_0)|^q}{\rho(1 + |\nu|)} \right)^{\frac{q}{2}} dx + G(|\xi|, |\nu|)\eta(\rho^2) + (a|\nu| + b)^2 \rho^2 \right\}
\]
\[\tag{4.1}\]

with \( C_1 \geq 1 \) depending only on \( \lambda, q, L, a(M) \) and \( M \).

\textbf{Proof.} Assume \( x_0 \in \Omega \) and \( \rho \leq 1 \) satisfy \( B_\rho(x_0) \subset \Omega \) and \( \rho \leq \rho_1(|\xi|, |\nu|) \). We denote \( \xi + \nu(x - x_0) \) by \( \ell(x) \) and take a standard cut-off function \( \psi \in C_0^\infty(B_\rho(x_0)) \) satisfying \( 0 \leq \psi \leq 1, |D\psi| \leq 4/\rho, \psi \equiv 1 \) on \( B_{\rho/2}(x_0) \). Then \( \varphi := \psi^q(u - \ell) \) is admissible as a test function in (4.1), and obtain
\[
\int_{B_{\rho}(x_0)} \psi^q A(x, u, Du) Du - \nu dx
\]
\[= - \int_{B_{\rho}(x_0)} A(x, u, Du), q\psi^{q-1} D\psi \otimes (u - \ell) dx + \int_{B_{\rho}(x_0)} \langle f, \varphi \rangle dx, \tag{4.2}\]
where $\xi \otimes \zeta := \xi \zeta^\alpha$. We further have

$$- \int_{B_\rho(x_0)} \psi^q (A(x, u, \nu), Du - \nu) dx = \int_{B_\rho(x_0)} A(x, u, \nu), q \psi^{q-1} \psi \otimes (u - \ell) dx - \int_{B_\rho(x_0)} A(x, u, \nu), D\varphi dx,
$$

(4.3)

and

$$\int_{B_\rho(x_0)} A(x_0, \xi, \nu), D\varphi dx = 0.
$$

(4.4)

Adding these equations, from (4.2) to (4.4), we obtain

$$\int_{B_\rho(x_0)} \psi^q \langle A(x, Du, u, \nu), Du - \nu \rangle dx = -\int_{B_\rho(x_0)} \langle A(x, u, \nu), q \psi^{q-1} \psi \otimes (u - \ell) \rangle dx + \int_{B_\rho(x_0)} \int_0^1 \langle A(x, Du, u, sDu + (1 - s)\nu), D\varphi \rangle ds ds + \int_{B_\rho(x_0)} \langle f, \varphi \rangle dx =: I + II + III + IV.
$$

(4.5)

The terms I, II, III and IV are defined above. Using the ellipticity condition (H2) to the left hand side of (4.5), we get

$$\langle A(x, u, Du, u, \nu), Du - \nu \rangle \geq \lambda \|Du - \nu\|^2 \int_0^1 (1 + |sDu + (1 - s)\nu|)^{q-2} ds.$$

Then we estimate above by (3.7) in Lemma 3.6 and obtain

$$\langle A(x, u, Du, u, \nu), Du - \nu \rangle \geq 2^{(12-9q)/2} \lambda \{ (1 + |\nu|)^{q-2} |Du - \nu|^2 + |Du - \nu|^q \}.
$$

(4.6)

For $\varepsilon > 0$ to be fixed later, using (H1) and Young’s inequality, we have

$$|I| \leq \varepsilon \int_{B_\rho(x_0)} \psi^q \{ (1 + |\nu|)^{q-2} |Du - \nu|^2 + |Du - \nu|^q \} dx + c(p, L, \varepsilon) \int_{B_\rho(x_0)} \left\{ (1 + |\nu|)^{q-2} \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^9 \right\} dx.
$$

(4.7)
In order to estimate \( II \), we first use \((H3)\) and \( D\varphi = \psi^q (Du - \nu) + q \psi^{q-1} D\psi \otimes (u - \ell) \), we get

\[
|II| \leq \int_{B_{\rho}(x_0)} \kappa(|\xi| + |\nu| \rho) \mu((1 + |\nu|) \rho) \psi^q |Du - \nu| \, dx \\
+ \int_{B_{\rho}(x_0)} \kappa(|\xi| + |\nu| \rho) \mu((1 + |\nu|) \rho) q \psi^{q-1} \psi^q |Du| \, dx \\
= : I_1 + I_2.
\]

The terms \( I_1 \) and \( I_2 \) are defined above. Using Young’s inequality we estimate \( I_1 \) as

\[
|I_1| \leq \varepsilon \int_{B_{\rho}(x_0)} \psi^q (1 + |\nu|)^{q-2} |Du - \nu|^2 \, dx + \frac{1}{\varepsilon} \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \, dx \\
+ \frac{1}{\varepsilon} \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \kappa^2 (|\xi| + |\nu|) \eta \left( \rho^2 (1 + |\nu|)^2 \kappa^2 (|\xi| + |\nu|) \right) \, dx.
\]

Using the definition of \( G(\cdot, \cdot) \) and the fact that \( \eta(ct) \leq c \eta(t) \) for \( c \geq 1 \), we deduce

\[
|I_1| \leq \varepsilon \int_{B_{\rho}(x_0)} \psi^q (1 + |\nu|)^{q-2} |Du - \nu|^2 \, dx + \frac{1}{\varepsilon} \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \, dx \\
+ \frac{1}{\varepsilon} (1 + |\nu|)^q G(|\xi|, |\nu|) \eta \left( \rho^2 \right).
\]

Similarly we see

\[
|I_2| \leq c(q, \varepsilon) \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \left| \frac{u - \ell}{\rho} \right|^2 \, dx + c(q, \varepsilon) (1 + |\nu|)^q G(|\xi|, |\nu|) \eta \left( \rho^2 \right).
\]

Combining these two estimates and get

\[
|II| \leq \varepsilon \int_{B_{\rho}(x_0)} \psi^q (1 + |\nu|)^{q-2} |Du - \nu|^2 \, dx + c(q, \varepsilon) \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \left| \frac{u - \ell}{\rho} \right|^q \, dx \\
+ c(q, \varepsilon)(1 + |\nu|)^q G(|\xi|, |\nu|) \eta \left( \rho^2 \right).
\] (4.8)

In the same way we derive

\[
|III| \leq \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-1} \kappa(|\xi| + |\nu|) \mu((1 + |\nu|) \rho) \psi^q |Du - \nu| \, dx \\
+ \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-1} \kappa(|\xi| + |\nu|) \mu((1 + |\nu|) \rho) 4q \left| \frac{u - \ell}{\rho} \right| \, dx \\
\leq \varepsilon \int_{B_{\rho}(x_0)} \psi^q (1 + |\nu|)^{q-2} |Du - \nu|^2 \, dx + \varepsilon \int_{B_{\rho}(x_0)} (1 + |\nu|)^{q-2} \left| \frac{u - \ell}{\rho} \right|^2 \, dx \\
+ c(q, \varepsilon)(1 + |\nu|)^q G(|\xi|, |\nu|) \eta \left( \rho^2 \right).
\] (4.9)
For $\varepsilon' > 0$ to be fixed later, using (H4), Lemma 3.5 and Young’s inequality, we have

$$\left| IV \right| \leq \int_{B_{\rho}(x_0)} (a|Du|^q + b)\psi^q|u - \ell| dx$$

$$\leq a(1 + \varepsilon') \int_{B_{\rho}(x_0)} \psi^q|Du - \nu|^q|\nu vert u - \ell| dx + \frac{1}{\varepsilon} \int_{B_{\rho}(x_0)} \left| \frac{u - \ell}{\rho} \right|^2 dx$$

$$+ \varepsilon \int_{B_{\rho}(x_0)} \left\{ aK(q, \varepsilon') \rho|\nu|^{(q+2)/2} \left( 1 + |\nu| \right)^{(q-2)/2} \left| \frac{u - \ell}{\rho} \right|^2 \right\} dx$$

$$\leq a(1 + \varepsilon')(2M + |\nu|\rho) \int_{B_{\rho}(x_0)} \psi^q|Du - \nu|^q dx + \frac{2}{\varepsilon} \int_{B_{\rho}(x_0)} \left( 1 + |\nu| \right)^{q-2} \left| \frac{u - \ell}{\rho} \right|^2 dx$$

$$+ \varepsilon(1 + |\nu|)^2 \rho^2 \left\{ aK(q, \varepsilon') |\nu| + b \right\}^2. \quad (4.10)$$

Combining above estimates, from (4.5) to (4.10), and set $\lambda' = 2^{(12-9q)/2}\lambda \Lambda := \lambda' - 3\varepsilon - a(1 + \varepsilon')(2M + |\nu|\rho)$, this gives

$$\Lambda \int_{B_{\rho}(x_0)} \psi^q \left\{ (1 + |\nu|)^{q-2} |Du - \nu|^2 + |Du - \nu|^q \right\} dx$$

$$\leq c(\lambda, L, \varepsilon) \int_{B_{\rho}(x_0)} \left\{ (1 + |\nu|)^{q-2} \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^q \right\} dx + (1 + |\nu|)^q G(|\xi|, |\nu|) \eta(\rho^2)$$

$$+ \varepsilon(1 + |\nu|)^2 \rho^2 \left\{ aK(q, \varepsilon') |\nu| + b \right\}^2. \quad (4.11)$$

Now choose $\varepsilon = \varepsilon(\lambda, p, a(M), M) > 0$ and $\varepsilon' = \varepsilon'(\lambda, p, a(M), M) > 0$ in a right way (for more precise way of choosing $\varepsilon$ and $\varepsilon'$, we refer to \[6, Lemma 4.1\]), we obtain (4.1).

### 5 Approximatively $A$-harmonic functions

**Lemma 5.1.** Under the same assumption in Lemma 4.1, take $\xi = u_{x_0, p}$. Then for any $x_0 \in \Omega$ and $\rho \leq \rho_1(|\xi|, |\nu|)$ satisfy $B_{\rho}(x_0) \subseteq \Omega$, the inequality

$$\int_{B_{\rho}(x_0)} A(Dv, D\varphi) dx \leq C_2(1 + |\nu|) \left[ \omega^{1/2}(1 + |\nu|) \Phi(x_0, \rho, \nu) \Phi^{1/2}(x_0, \rho, \nu) \right.$$

$$\left. + \Phi(x_0, \rho, \nu) + G(|\xi|, |\nu|) \sqrt{\eta(\rho^2)} + \rho(a|\nu| + b) \right] \sup_{B_{\rho}(x_0)} |D\varphi| \quad (5.1)$$

holds for all $\varphi \in C_0^\infty(B_{\rho}(x_0), \mathbb{R}^N)$. Where

$$v := u - \ell = u - \xi - \nu(x - x_0),$$

$$A(Dv, D\varphi) := \frac{1}{(1 + |\nu|)^{q-1}} \langle D_{\rho}A(x_0, \xi, \nu)Dv, D\varphi \rangle,$$

$$\Phi(x_0, \rho, \nu) := \int_{B_{\rho}(x_0)} \left\{ \frac{|Du - \nu|^2}{(1 + |\nu|)^2} + \frac{|Du - \nu|^2}{(1 + |\nu|)^q} \right\} dx$$

and $C_2 \geq 1$ depending only on $n, q, L$ and $a(M)$. 

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Proof. Assume \( x_0 \in \Omega \) and \( \rho \leq 1 \) which satisfies \( B_{\rho}(x_0) \in \Omega \) and \( \rho \leq \rho_1(|\xi|, |\nu|) \). Without loss of generality we may assume \( \sup_{B_{\rho}(x_0)} |D\varphi| \leq 1 \). Note that this implies \( \sup_{B_{\rho}(x_0)} |\varphi| \leq \rho \leq 1 \). Using the fact that
\[
\int_{B_{\rho}(x_0)} A(x_0, \xi, \nu) D\varphi dx = 0
\]
holds for all \( \varphi \in C_0^\infty(B_{\rho}(x_0), \mathbb{R}^N) \) we deduce
\[
(1 + |\nu|)^{q-1} \int_{B_{\rho}(x_0)} A(Dv, D\varphi) dx
= \int_{B_{\rho}(x_0)} A(x_0, \xi, \nu) A(0) (Du - \nu) d\nu + \int_{B_{\rho}(x_0)} A(x_0, \xi, Du) - A(x, \ell, Du) D\varphi dx
+ \int_{B_{\rho}(x_0)} (f, \varphi) dx
=: I + II + III + IV \tag{5.2}
\]
where terms I, II, III and IV are defined above.

We estimate I using the modulus of continuity \( \omega(\cdot, \cdot) \) from (H1), the Jensen’s inequality and Hölder’s inequality, and we get
\[
|I| \leq c(q, L) \int_{B_{\rho}(x_0)} \int_0^1 \omega(\|\xi\| + |\nu|, |Du - \nu|^2) (1 + |\nu|)^{q-2} |Du - \nu| d\nu dx
\]
\[
\leq c (1 + |\nu|)^{q-1} \int_{B_{\rho}(x_0)} \omega(\|\xi\| + |\nu|, |Du - D\ell|^2) \frac{|Du - \nu|}{1 + |\nu|} \frac{|Du - \nu|^{q-1}}{(1 + |\nu|)^{q-1}} dx
\]
\[
\leq c (1 + |\nu|)^{q-1} \left[ \omega^{1/2}(\|\xi\| + |\nu|, (1 + |\nu|)^2\Phi(x_0, \rho, \nu)) \Phi^{1/2}(x_0, \rho, \nu)
+ \omega^{1/2}(\|\xi\| + |\nu|, (1 + |\nu|)^2\Phi(x_0, \rho, \nu)) \Phi^{1/2}(x_0, \rho, \nu) \right]^q
\]
\[
\leq c (1 + |\nu|)^{q} \left[ \omega^{1/2}(\|\xi\| + |\nu|, \Phi(x_0, \rho, \nu)) \Phi^{1/2}(x_0, \rho, \nu) + \Phi(x_0, \rho, \nu) \right], \tag{5.3}
\]
where \( q_* > 0 \) is the dual exponent of \( q \geq 2 \), i.e., \( q_* = q/(q - 1) \). The last inequality following from the fact that \( a^{1/q} b^{1/q} = a^{1/q} b^{1/q} (q-2)/q \leq a^{1/2} b^{1/2} + b \) holds by Young’s inequality and the fact that \( \omega(s, t) \leq c \omega(s, t) \) for \( c \geq 1 \) which deduce from the concavity of \( t \mapsto \omega(s, t) \).

In the same way, using the modulus of continuity \( \eta(\cdot) \) from (H3), Young’s inequality and, we deduce
\[
|II| \leq 2^{q-2} \kappa(\|\xi\| + |\nu|)(1 + |\nu|)^{q} \sqrt{\eta(\rho^2)}
+ 2^{q-2} \int_{B_{\rho}(x_0)} \kappa(\|\xi\| + |\nu|) \sqrt{\eta(\rho^2(1 + |\nu|^2))} |Du - \nu|^{q-1} dx
\]
\[
\leq 2^{q-2} (1 + |\nu|)^{q} G(\|\xi\|, |\nu|) \sqrt{\eta(\rho^2)} + 2^{q-2} (1 + |\nu|)^{q} \Phi(x_0, \rho, \nu). \tag{5.4}
\]
Here we have used \( \eta^{1/2}(\rho^2(1 + |\nu|^2)) \leq \sqrt{\eta(\rho^2(1 + |\nu|^2))} \) which follows from the nondecreasing property of \( t \mapsto \eta(t) \), (η4) and our assumption \( \rho \leq \rho_1 \leq 1 \).
Similarly, we have, using Young’s inequality, \( H3 \),
\[
|III| \leq c(q) \int_{B_{\rho}(x_\theta)} \kappa(|\xi| + |\nu|) \sqrt{\eta(|u - \ell|^2)} (1 + |\nu|)^{q-1} dx + c(q) \int_{B_{\rho}(x_\theta)} \kappa(|\xi| + |\nu|) \sqrt{\eta(|u - \ell|^2)} Du - \nu |\nu|^{q-1} dx
\]
where the terms III_1 and III_2 are defined above. Using Hölder’s inequality, Jensen’s inequality, \( \eta 6 \) and the Poincaré inequality, we have
\[
III_1 \leq c(q)(1 + |\nu|)^{q-1} \kappa(|\xi| + |\nu|) |\nu|^{1/2} \left( \int_{B_{\rho}(x_\theta)} |u - \ell|^2 dx \right)
\]
\[
\leq c \rho^{-2}(1 + |\nu|)^{q-2} \left\{ \rho^2 (1 + |\nu|)^2 \kappa^2(|\xi| + |\nu|) |\nu|^{1/2} \left( \rho^2 (1 + |\nu|)^2 \kappa^2(|\xi| + |\nu|) \right) + \int_{B_{\rho}(x_\theta)} |u - \ell|^2 dx \right\}
\]
\[
\leq c(q)(1 + |\nu|)^q G(|\xi|, |\nu|) \sqrt{\eta(\rho^2)} + c(n, q)(1 + |\nu|)^q \Phi(x_0, \rho, \nu).
\]
Similarly, we have, using Young’s inequality, \( \eta 5 \) and the Poincaré inequality,
\[
III_2 \leq c(q) \int_{B_{\rho}(x_\theta)} \kappa^2(|\xi| + |\nu|) |\nu|^{q/2} (|u - \ell|^2) dx + c(q) \int_{B_{\rho}(x_\theta)} |Du - \nu|^q dx
\]
\[
\leq c \int_{B_{\rho}(x_\theta)} \left\{ \rho^{-2} \left\{ \kappa^2(|\xi| + |\nu|)^2 |\nu|^{q/2} (\kappa^2(|\xi| + |\nu|) \rho^2) + |u - \ell|^2 \right\} \right\}^{q/2} dx + c(1 + |\nu|)^q \Phi(x_0, \rho, \nu)
\]
\[
\leq c(1 + |\nu|)^q G(|\xi|, |\nu|) \sqrt{\eta(\rho^2)} + c(n, q)(1 + |\nu|)^q \Phi(x_0, \rho, \nu).
\]
Thus we obtain
\[
|III| \leq c(q)(1 + |\nu|)^q G(|\xi|, |\nu|) \sqrt{\eta(\rho^2)} + c(n, q)(1 + |\nu|)^q \Phi(x_0, \rho, \nu). \tag{5.5}
\]
Using \( H4 \) and recall that \( \sup_{B_{\rho}(x_\theta)} |\varphi| \leq \rho \) holds, we have
\[
|IV| \leq \int_{B_{\rho}(x_\theta)} \rho a(|Du - \nu| + |\nu|)^q dx + b \rho
\]
\[
\leq 2^{q-1} a(1 + |\nu|)^q \Phi(x_0, \rho, \nu) + 2^{q-1} b(1 + |\nu|)^q (a |\nu| + b). \tag{5.6}
\]
Combining these estimates, from (5.2) to (5.6), we obtain the conclusion. \( \square \)

6 Proof of the Regularity Theorem

Let write \( \Phi(\rho) = \Phi(x_0, \rho, (Du)_{x_0, \rho}) \) from now on. Now we are in the position to establish the excess improvement.

Lemma 6.1. Assume the same assumption with Lemma 5.1. Let \( \theta \in (0, 1/8) \) be arbitrary and impose the following smallness conditions on the excess:

(i) \( \omega^{1/2}(|u_{x_0, \rho}| + |(Du)_{x_0, \rho}|, \Phi(\rho)) + \sqrt{\Phi(\rho)} \leq \delta \) with the constant \( \delta = \delta(n, q, \lambda, L, \theta^{\alpha+\gamma+2}) \) from Lemma 5.2
and (iii) are satisfied and we rescale $u$

We consider

Proof. We consider $B_\rho(x_0) \subseteq \Omega$ and set $\xi = u_{x_0,\rho}, \nu = (Du)_{x_0,\rho}, \ell(x) = \xi + \nu(x-x_0)$. Assume (i), (ii) and (iii) are satisfied and we rescale $u$ as

$$w := \frac{u - \ell}{(1 + |\nu|)^{\gamma}}.$$ 

Applying Lemma 5.1 on $B_\rho(x_0)$ to $w$ and combining (i), we obtain

$$\int_{B_\rho(x_0)} A(Dw, Dw) dx \leq \left[ \omega^{1/2} \left( |\xi| + |\nu|, \sqrt{\Phi(\rho)} + \sqrt{\Phi(\rho)} + \frac{\delta}{2} \right) \right] \sup_{B_\rho(x_0)} |D\varphi|$$

$$\leq \delta \sup_{B_\rho(x_0)} |D\varphi|$$

for all $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$. Moreover, we have, note that $\gamma \geq C_2\sqrt{\Phi(\rho)}$ holds from the definition of $\gamma$,

$$\int_{B_\rho(x_0)} |Du|^2 + \gamma^{-2} |Du|^p \ dx = \int_{B_\rho(x_0)} \left\{ \frac{|Du - \nu|^2}{\gamma^2(1 + |\nu|)^2} + \gamma^{-2} \frac{|Du - \nu|^q}{\gamma^q(1 + |\nu|)^q} \right\} dx$$

$$\leq \frac{\Phi(\rho)}{\gamma^2} \leq \frac{1}{C_2^2} \leq 1.$$ 

Thus, these two inequalities allow us to apply the $A$-harmonic approximation lemma (Lemma 3.2), to conclude the existence of an $A$-harmonic function $h$ satisfying

$$\int_{B_{\rho/2}(x_0)} \left\{ \left| w - \frac{h}{\rho/2} \right|^2 + \gamma^{-2} \left| w - \frac{h}{\rho/2} \right|^q \right\} dx \leq \theta^{n+q+2}, \quad \text{and} \quad (6.2)$$

$$\int_{B_{\rho/2}(x_0)} \{ |Dh|^2 + \gamma^{-2} |Dh|^q \} dx \leq \tilde{C}, \quad (6.3)$$

where we take $\varepsilon = \theta^{n+q+2}$. From Theorem 5.3 and (6.3) we have

$$\sup_{B_{\rho/2}(x_0)} |D^2h|^2 \leq 4C_0\tilde{C}\rho^{-2}.$$
From this we infer the following estimate for \( s = 2 \) as well as for \( s = q \),
\[
\sup_{B_{\rho/4}(x_0)} |D^2 h|^s \leq c(n, N, \lambda, L, q, s)\rho^{-s}.
\]
For \( \theta \in (0, 1/8] \), Taylor’s theorem applied to \( h \) at \( x_0 \) yields
\[
\sup_{x \in B_{2\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^s \leq c(n, N, \lambda, L, q, s)\theta^{2s} \rho^s.
\]
We have then
\[
\theta \leq c(n, N, \lambda, L, q, s)\theta^2.
\]
Recall that the mean-value of \( u - (\nu + \gamma(1 + |\nu|)Dh(x_0))(x - x_0) \) on \( B_{2\rho}(x_0) \) is \( u_{x_0, 2\rho} \), we have
\[
\begin{align*}
(2\theta\rho)^s \int_{B_{2\rho}(x_0)} |u - u_{x_0, 2\rho} - (\nu + \gamma(1 + |\nu|)Dh(x_0))(x - x_0)|^s dx \\
\leq c(s)(2\theta\rho)^s \gamma^s(1 + |\nu|)^s \int_{B_{2\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^s dx \\
\leq c(n, N, \lambda, L, q, s)(1 + |\nu|)^s \theta^2 \gamma^2.
\end{align*}
\]
By assumption (ii), we infer \( \sqrt{\Phi(\rho)} \leq \theta^n/2 \). This yields
\[
|(Du)_{x_0, \theta, \rho} - \nu| \leq \theta^{-n} \int_{B_{\rho}(x_0)} |Du - \nu| dx \leq \theta^{-n}(1 + |\nu|)\sqrt{\Phi(\rho)} \leq \frac{1}{2}(1 + |\nu|).
\]
Thus, combining with the estimate \( 1 + |\nu| \leq 1 + |(Du)_{x_0, \theta, \rho}| + |(Du)_{x_0, \theta, \rho} - \nu| \), we obtain
\[
1 + |\nu| \leq 2(1 + |(Du)_{x_0, \theta, \rho}|).
\]
Set \( P_0 = \nu + \gamma(1 + |\nu|)Dh(x_0) \). Then Theorem 5.3 (5.3) and assumption (ii) imply
\[
|P_0| \leq |\nu| + |\gamma(1 + |\nu|)Dh(x_0)| \leq |\nu| + |\gamma(1 + |\nu|)\sqrt{\Phi(\rho)}| \leq 1 + |\nu|.
\]
Therefore, combining with (6.6), we have
\[
1 + |P_0| \leq 3(1 + |(Du)_{x_0, \theta, \rho}|).
\]
Applying Caccioppoli-type inequality (Lemma B.1) on \( B_{2\rho}(x_0) \) with \( \xi = u_{x_0, 2\rho} \) and \( \nu = P_0 \) yields
\[
\Phi(\theta, \rho) \leq 6^d \Phi(x_0, \theta, \rho, P_0)
\]
\[
\begin{align*}
\leq 6^d C_1 \int_{B_{2\rho}(x_0)} \left\{ \left| \frac{u - u_{x_0, 2\rho} - P_0(x - x_0)}{2\theta \rho(1 + |P_0|)} \right|^2 + \left| \frac{u - u_{x_0, 2\rho} - P_0(x - x_0)}{2\theta \rho(1 + |P_0|)} \right|^2 \right\} dx \\
+ G(|u_{x_0, 2\rho}|, |P_0|)(2\theta \rho)^2 + (a|P_0| + b)^2 (2\theta \rho)^2.
\end{align*}
\]
Using Hölder’s inequality, the Poincaré inequality and assumption (ii) we have
\[
|u_{x_0,2\theta\rho}| \leq |u_{x_0,\rho}| + \left| \int_{B_{2\rho x_0}(x_0)} (u - u_{x_0,\rho} - \nu(x - x_0)) dx \right|^{1/2} \\
\leq |u_{x_0,\rho}| + \left( \int_{B_{2\rho x_0}(x_0)} |u - u_{x_0,\rho} - \nu(x - x_0)|^2 dx \right)^{1/2} \\
\leq |u_{x_0,\rho}| + (2\theta)^{-n/2} \left( \int_{B_{\rho x_0}(x_0)} |u - u_{x_0,\rho} - \nu(x - x_0)|^2 dx \right)^{1/2} \\
\leq |u_{x_0,\rho}| + \theta^{-n/2} \sqrt{C_F} (1 + |\nu|) \sqrt{\Phi(\rho)} \\
\leq |u_{x_0,\rho}| + \theta^{-n/2} \sqrt{C_F} (1 + |\nu|)|\gamma| \\
\leq |u_{x_0,\rho}| + 1. \quad (6.10)
\]

Set \( H_0(s,t) := C^2(1 + s,1+t) + (a(1+t) + b)^2 \) and using (6.7) we obtain
\[
G(|u_{x_0,2\theta\rho}|, |P_0|) \eta((2\theta\rho)^2) + (a |P_0| + b)^2 (2\theta\rho)^2 \leq H_0(|\xi|, |\nu|) \eta(\rho^2). \quad (6.11)
\]

The definition of \( \gamma \) and \( H_0 \) imply
\[
\gamma^2 \leq 2C_2^2 \left[ \Phi(\rho) + 4\delta^{-2} \left\{ G(|\xi|, |\nu|) \sqrt{\eta(\rho^2)} + \rho(a(1 + |\nu|) + b) \right\}^2 \right] \\
\leq 2C_2^2 \left[ \Phi(\rho) + 8\delta^{-2} H_0(|\xi|, |\nu|) \eta(\rho^2) \right]. \quad (6.12)
\]

Plugging (6.4), (6.11), (6.12) into (6.9), we deduce
\[
\Phi(\theta \rho) \leq \delta^2 C_1 \left[ c(n,N,\lambda,L,q) \theta^2 \gamma^2 + G(|u_{x_0,2\theta\rho}|, |P_0|) \eta((2\theta\rho)^2) + (a |P_0| + b)^2 (2\theta\rho)^2 \right] \\
\leq \delta^2 C_1 \left[ c \theta^2 C_2^2 \left\{ \Phi(\rho) + \delta^{-2} H_0(|\xi|, |\nu|) \eta(\rho^2) \right\} + H_0(|\xi|, |\nu|) \eta(\rho^2) \right] \\
\leq C_3 \left[ \theta^2 \Phi(\rho) + 8\delta^{-2} H_0(|\xi|, |\nu|) \eta(\rho^2) \right],
\]
and this complete the proof.  

For \( \sigma \in [\alpha,1) \) we find \( \theta \in (0,1/8) \) such that \( C_3 \theta^2 \leq \theta^{2\sigma}/2 \). For \( T_0 \geq 1 \) there exists \( \Phi_0 > 0 \) such that
\[
\omega^{1/2} \left( 2T_0, \sqrt{2\Phi_0} \right) + \sqrt{2\Phi_0} \leq \frac{\delta}{2}, \quad (6.13)
\]
\[
2C_4 \left( 1 + 2T_0 \right) \sqrt{2\Phi_0} \leq \theta^\sigma, \quad (6.14)
\]
where \( C_4 := C_3 \left( 1 + \sqrt{C_F} \right) \). Note that \( \Phi_0 < 1 \). Then choose \( 0 < \rho_0 \leq 1 \) such that
\[
C_5 \sqrt{\eta(\rho_0)} \leq \Phi_0, \quad (6.15)
\]
\[
\frac{(1 + 2T_0)(1 + \sqrt{C_F})}{\theta^{\sigma/2}} \sqrt{\frac{C_3 \alpha^2 \beta^2 \tilde{F}(\rho_0^2)}{4(1 - \theta^{\sigma})^2}} \leq \frac{1}{2} T_0, \quad (6.16)
\]
where
\[
C_5 = C_5(n,N,\lambda,L,q,a,M,\alpha,\sigma,T_0) := \frac{2H(2T_0,2T_0)}{2\theta^{2\sigma} - \theta^{2\sigma}}.
\]

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Lemma 6.2. Assume that for some \(T_0 \geq 1\) and \(B_r(x_0) \in \Omega\) we have

(a) \(|u_{x_0,\rho} + |(Du)_{x_0,\rho}| \leq T_0,
(b) \Phi(\rho) \leq \Phi_0,
(c) \rho \leq \rho_0.

Then the smallness conditions (i), (ii) and (iii) are satisfied on \(B_{\sqrt{T_0}}(x_0)\) for \(k \in \mathbb{N} \cup \{0\}\) in Lemma 6.1. Moreover, the limit

\[ \Lambda_{x_0} := \lim_{k \to \infty} (Du)_{x_0,\theta^k \rho} \]

exists, and the inequality

\[ \int_{B_r(x_0)} |Du - \Lambda_{x_0}|^2 dx \leq C_6 \left[ \left( \frac{r}{\rho} \right)^{2\alpha} \Phi(\rho) + \tilde{F}(r^2) \right] \tag{6.17} \]

is valid for \(0 < r \leq \rho\) with a constant \(C_\theta = C_6(n, N, \lambda, L, q, a(M), M, \alpha, \beta, \sigma, T_0)\).

Proof. Inductively we shall derive for \(k \in \mathbb{N} \cup \{0\}\) the following three assertions:

(I) \(\Phi(\theta^k \rho) \leq 2\Phi_0,\)

(II) \(|u_{x_0,\theta^k \rho} + |(Du)_{x_0,\theta^k \rho}| \leq 2T_0,\)

(III) \(\theta^k \rho \leq \rho_1(|u_{x_0,\theta^k \rho}|, |(Du)_{x_0,\theta^k \rho}|).\)

We first note that (I), (II) and (6.13) imply the smallness condition (i), i.e. (i) with \(\theta^k \rho\) instead of \(\rho\). Next we observe that (I), (II), (6.14) and (6.15) yield

\[
(1 + |(Du)_{x_0,\theta^k \rho}|) \left( 2\sqrt{C_6} \right) \gamma(\theta^k \rho) \\
\leq (1 + |(Du)_{x_0,\theta^k \rho}|) \left[ C_3 \sqrt{2\Phi_0 + H(|u_{x_0,\theta^k \rho}|, |(Du)_{x_0,\theta^k \rho}|)} \sqrt{\eta(\rho_0^2)} \right] \\
\leq (1 + 2T_0) \left[ C_3 \sqrt{2\Phi_0 + H(2T_0, 2T_0)} \sqrt{\eta(\rho_0^2)} \right] \\
\leq (1 + 2T_0) \left[ C_3 \sqrt{2\Phi_0 + \frac{2\theta^{2\alpha} - \theta^{2\alpha}}{2} \Phi_0} \right] \\
\leq 2C_3 (1 + 2T_0) \sqrt{2\Phi_0} \\
\leq 1,
\]

Thus we have (ii),. Note that \(C_2 \left( 2\sqrt{C_6} \right) \leq C_3\) and \(\Phi_0 > 1\) are hold from there definition. Finally (iii) is just (Iii).

By (a), (b) and (c), there holds (I),(II) and (III). Now suppose that we have (I),(II) and (III) for \(\ell = 0, 1, \cdots, k-1\) with some \(k \in \mathbb{N}\). Then we can use Lemma 6.1 with \(\rho, \theta \rho, \cdots, \theta^{k-1} \rho\), and yields

\[
\Phi(\theta^k \rho) \leq \left( \frac{1}{2} \theta^{2\alpha} \right)^k \Phi(\rho) + \sum_{\ell=0}^{k-1} \left( \frac{1}{2} \theta^{2\alpha} \right)^\ell H(|u_{x_0,\theta^{k-1-\ell} \rho}|, |(Du)_{x_0,\theta^{k-1-\ell} \rho}|) \eta((\theta^{k-1-\ell} \rho)^2) \\
\leq \left( \frac{1}{2} \theta^{2\alpha} \right)^k \Phi(\rho) + H(2T_0, 2T_0) \sum_{\ell=0}^{k-1} \left( \frac{1}{2} \theta^{2\alpha} \right)^\ell \eta((\theta^{k-1-\ell} \rho)^2).
\]
The nondecreasing property of \( t \mapsto t^{-\alpha} \eta(t) \) and the choice of \( \sigma \) implies

\[
\sum_{\ell=0}^{k-1} \left( \frac{1}{2} \theta^{2\sigma} \right)^\ell \eta \left( \left( \theta^{k-1-\ell} \rho \right)^2 \right) \leq \theta^{-2\alpha} \eta \left( \left( \theta^k \rho \right)^2 \right) \sum_{\ell=0}^{k-1} \left( \frac{1}{2} \theta^{2\sigma-2\alpha} \right)^\ell \leq 2 \eta \left( \left( \theta^k \rho \right)^2 \right) \frac{2}{2 \theta^{2\alpha} - \theta^{2\sigma}}.
\]

Therefore we have

\[
\Phi(\theta^k \rho) \leq \left( \frac{1}{2} \theta^{2\sigma} \right)^k \Phi(\rho) + C_5 \eta \left( \left( \theta^k \rho \right)^2 \right).
\] (6.18)

Keeping in mind of (b), (c) and the choice of \( \rho \), we prove (I_k). We next want to show (II_k). Using the fact that \( \int_{B_2(x_0)} \nu(x - x_0) dx = 0 \) holds for all \( \nu \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \), Hölder’s inequality and the Poincaré inequality, we obtain

\[
|u_{x_0, \theta^k \rho}| \leq |u_{x_0, \theta^{k-1} \rho}| + \left\| \int_{B_{h_k}(x_0)} (u - u_{x_0, \theta^{k-1} \rho} - (Du)_{x_0, \theta^{k-1} \rho} (x - x_0)) dx \right\|
\leq |u_{x_0, \theta^{k-1} \rho}| + \theta^{-n/2} \sqrt{C_P} (1 + |(Du)_{x_0, \theta^{k-1} \rho}|) \Phi^{1/2} (\theta^{k-1} \rho)
\leq |u_{x_0, \rho}| + \theta^{-n/2} \sqrt{C_P} \sum_{\ell=0}^{k-1} (1 + |(Du)_{x_0, \theta^\ell \rho}|) \Phi^{1/2} (\theta^\ell \rho).
\]

Similarly we see

\[
|(Du)_{x_0, \theta^k \rho}| \leq |(Du)_{x_0, \theta^{k-1} \rho}| + \left\| \int_{B_{h_k}(x_0)} (Du - (Du)_{x_0, \theta^{k-1} \rho}) dx \right\|
\leq |(Du)_{x_0, \rho}| + \theta^{-n/2} \sum_{\ell=0}^{k-1} (1 + |(Du)_{x_0, \theta^\ell \rho}|) \Phi^{1/2} (\theta^\ell \rho).
\]

Combining two estimates and using (6.18) and (3.5), we infer

\[
|u_{x_0, \theta^k \rho}| + |(Du)_{x_0, \theta^k \rho}|
\leq |u_{x_0, \rho}| + |(Du)_{x_0, \rho}| + \frac{(1 + \sqrt{C_P})(1 + 2T_0)}{\theta^{n/2}} \sum_{\ell=0}^{k-1} \Phi^{1/2} (\theta^\ell \rho)
\leq |u_{x_0, \rho}| + |(Du)_{x_0, \rho}| + \frac{(1 + \sqrt{C_P})(1 + 2T_0)}{\theta^{n/2}} \sum_{\ell=0}^{k-1} \left\{ \frac{1}{\sqrt{2}} \theta^{\sigma} \right\}^{\ell} \Phi^{1/2} (\rho) + \sqrt{C_5 \eta (\theta^{2\sigma} \rho^2)}
\leq |u_{x_0, \rho}| + |(Du)_{x_0, \rho}| + \frac{(1 + \sqrt{C_P})(1 + 2T_0)}{\theta^{n/2}} \left\{ \sqrt{2} \Phi (\rho) + \sqrt{C_5 \eta (\theta^{2\sigma} \rho^2)} \right\}
\leq T_0 + \frac{(1 + \sqrt{C_P})(1 + 2T_0)}{\theta^{n/2}} \sqrt{\frac{2 \Phi (\rho)}{\sqrt{2} - \theta^\sigma}} + \frac{(1 + \sqrt{C_P})(1 + 2T_0)}{\theta^{n/2}} \sqrt{\frac{C_5 \eta (\theta^{2\sigma} \rho^2)}{4(1 - \theta^{\alpha \beta})^2}}
\leq T_0 + \frac{1}{\sqrt{2} - \theta^\sigma} + \frac{1}{2} T_0
\leq 2T_0.
\]
This proves (II_k). By hypothesis (c), (II_k), (η4), the definition of $H$ and (6.15), we easily derive
\[
(1 + |(Du)_{x_0,\theta^k\rho}|) \kappa(|w_{x_0,\theta^k\rho}| + |(Du)_{x_0,\theta^k\rho}|) \theta^k
\leq H(2T_0, 2T_0) \sqrt{\eta(\rho_0)}
\leq 1.
\]

Thus, we prove (III_k).

We next want to prove that $(Du)_{x_0,\theta^k\rho}$ converges to some limit $\Lambda_{x_0}$ in Hom($\mathbb{R}^n, \mathbb{R}^N$). Arguing as in the proof of (II_k) we deduce for $k > j$
\[
|(Du)_{x_0,\theta^k\rho} - (Du)_{x_0,\theta^j\rho}| \leq \sum_{\ell=j+1}^{k} |(Du)_{x_0,\theta^\ell\rho} - (Du)_{x_0,\theta^{\ell-1}\rho}|
\leq \sum_{\ell=j+1}^{k} \theta^{\ell-j} |(Du)_{x_0,\theta^{\ell-1}\rho}|
\leq \frac{(1 + 2\sqrt{2})}{\theta^{j/2} \sqrt{\theta_0^2 - \theta^j}} + \frac{C_5 \alpha^2 \beta^2 \bar{F}(\theta^j \rho^2)}{4(1 - \theta) \theta^2}.
\]
(6.19)

Taking into account our assumption (η3) we see that $(Du)_{x_0,\theta^k\rho}$ is a Cauchy sequence in Hom($\mathbb{R}^n, \mathbb{R}^N$). Therefore the limit
\[
\Lambda_{x_0} := \lim_{k \to \infty} (Du)_{x_0,\theta^k\rho}
\]
exists and from (6.19) we infer for $j \in \mathbb{N} \cup \{0\}$
\[
|(Du)_{x_0,\theta^j\rho} - \Lambda_{x_0} | \leq |(Du)_{x_0,\theta^j\rho} - (Du)_{x_0,\theta^k\rho}| + |(Du)_{x_0,\theta^k\rho} - \Lambda_{x_0}|
\rightarrow C_7 \sqrt{\theta^{2j} \Phi(\rho) + \bar{F}(\theta^j \rho^2)} \quad \text{(as } k \to \infty)\]

where
\[
C_7 := \sqrt{2} \frac{(1 + 2\sqrt{2})}{\theta^{j/2} \sqrt{\theta_0^2 - \theta^j}} + \frac{C_5 \alpha^2 \beta^2}{4(1 - \theta) \theta^2}.
\]

Combining this with (6.18), and recalling the estimate (3.10) we arrive at
\[
\int_{B_{\theta^j\rho}(x_0)} |Du - \Lambda_{x_0}|^2 dx \leq 2(1 + 2T_0) \Phi(\theta^j \rho) + 2|(Du)_{x_0,\theta^j\rho} - \Lambda_{x_0}|^2
\leq C_8 \left\{ \theta^{2j} \phi(\rho) + \bar{F}(\theta^j \rho^2) \right\}
\]
with
\[
C_8 := 2 \left\{ 1 + 2T_0 + C_7^2 + \frac{C_5 \alpha^2 \beta^2 (1 + 2T_0)}{4(1 - \theta) \theta^2} \right\}.
\]
For $0 < r \leq \rho$ we find $j \in \mathbb{N} \cup \{0\}$ such that $\theta^{j+1}\rho \leq r \leq \theta^j\rho$. Then using the above estimate with (3.10) implies
\[
\int_{B_r(x_0)} |Du - \Lambda_{x_0}|^2 dx \leq \theta^{-n} \int_{B_{\theta^j\rho}(x_0)} |Du - \Lambda_{x_0}|^2 dx
\leq C_8 \theta^{-n} \left\{ \theta^{2j} \Phi(\rho) + \bar{F}(\theta^j \rho^2) \right\}
\leq 4C_8 \theta^{-n-2\sigma} \left\{ \left( \frac{r}{\rho} \right)^{2\sigma} \Phi(\rho) + \bar{F}(r^2) \right\}.
\]
This proves (6.17) with $C_6 := 4C_8 \theta^{-n-2\sigma}$.  

The main theorem (Theorem 2.2) is obtained from Lemma 6.2 by using standard arguments.

Acknowledgments
The author thanks Professor Hisashi Naito for helpful discussions.

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