Comparative Study on Two Methods for Calculating the Gravitational Potential of a Prism

HAN Jiancheng1, SHEN Wenbin1,2,3

1. School of Geodesy and Geomatics, Wuhan University, 129 Luoyu Road, Wuhan 430079, China
2. State Key Laboratory of Information Engineering in Surveying, Mapping and Remote Sensing, Wuhan University, 129 Luoyu Road, Wuhan 430079, China
3. Key Laboratory of Geospace Environment and Geodesy, Ministry of Education, Wuhan University, 129 Luoyu Road, Wuhan 430079, China

© Wuhan University and Springer-Verlag Berlin Heidelberg 2010

Abstract The determination of the gravitational potential of a prism plays an important role in physical geodesy and geophysics. However, there are few literatures that provide accurate approaches for determining the gravitational potential of a prism. Discrete element method can be used to determine the gravitational potential of a prism, and can approximate the true gravitational potential values with sufficient accuracy (the smaller each element is, the more accurate the result is). Although Nagy’s approach provided a closed expression, one does not know whether it is valid, due to the fact that this approach has not been confirmed in literatures. In this paper, a study on the comparison of Nagy’s approach with discrete element method is presented. The results show that Nagy’s formulas for determining the gravitational potential of a prism are valid in the domain both inside and outside the prism.

Keywords gravitational potential of a prism; Nagy approach; discrete element method; numerical comparison study

CLC number P223

Introduction

Potential theory as well as the determination of the gravitational potential of a body (or a layer) with an arbitrary shape is fundamental in physical geodesy. The expressions of the gravitational potentials of some kinds of bodies with special shapes, such as uniform plane, sphere, cylinder, etc., can be found in various literatures, see, e.g., References [1-3]. However, one can hardly find literatures that provide the expressions of the gravitational potential generated by a prism. In fact, the determination of the gravitational potential of a prism are frequently used in many cases, especially in gravity field modeling studies (e.g., using a combination of a number of prisms with different sizes and densities to approximate an irregular layer or body). Nagy et al. (2000) formulated the formulas for expressing the gravitational potential of a prism and its derivatives[4,5,6]. Later, Wang (2003)[7] provided another expression independently. However, the expressions given by Wang (2003) can only deal with the derivatives. In order to use Nagy’s approach to calculate the gravitational potential of a prism, one should know whether these expressions given by Nagy et al. (2000) are valid. Hence, it is necessary to make a thorough comparative study between Nagy’s approach and the discrete element method, since the

► Received on July 15, 2009.
► Supported by the National Natural Science Foundation of China (No.40637034, 40974015); the National 863 Program of China (No.2006AA12Z211).
► HAN Jiancheng, Ph.D candidate of Wuhan University. His research interests include solid geophysics and physical geodesy.
► E-mail: wbshen@sgg.whu.edu.cn
latter is sufficiently accurate when each element is sufficiently small.

1 Nagy’s approach for determining the gravitational potential of a prism

Nagy et al. (2000) presented a closed expression of the gravitational potential and its derivatives (up to the third order) for a prism\[^4,5\]. The definition of the prism is shown in Fig.1.

![Fig.1 Sketch map of the definition of the prism (after Nagy et al. 2000)](image)

The prism is bounded by planes parallel to the coordinate planes, defined by the coordinates \(X_1, X_2, Y_1, Y_2, Z_1, Z_2\), respectively in the Cartesian coordinate system, and the field point \(P\) is denoted by \((X, Y, Z)\). Then, one has

\[
\begin{align*}
x_1 &= X_1 - X_p, & x_2 &= X_2 - X_p \\
y_1 &= Y_1 - Y_p, & y_2 &= Y_2 - Y_p \\
z_1 &= Z_1 - Z_p, & z_2 &= Z_2 - Z_p
\end{align*}
\]

By the well-known Newton’s integral, the gravitational potential of the prism can be expressed as

\[
V(P) = G \rho \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \frac{dx dy dz}{r}
\]

where \(r = \sqrt{x^2 + y^2 + z^2}\) is the distance between the origin and the field point, \(G\) the gravitational constant, and \(\rho\) the density of the prism. The integration of Eq.(2) can be expressed as\[^4\]:

\[
V(P) = G \rho \left[ xy \ln(z + r) + yz \ln(x + r) + xz \ln(y + r) \right. \\
\left. - \frac{x^2}{2} \tan^{-1} \frac{yz}{xr} - \frac{y^2}{2} \tan^{-1} \frac{xz}{yr} - \frac{z^2}{2} \tan^{-1} \frac{xy}{zr} \right]_{x_1}^{x_2} \left[ y_1 \right]_{y_1}^{y_2} \left[ z_1 \right]_{z_1}^{z_2} \left[ \right]_{z_1}^{z_2}
\]

Although the gravitational potential \(V(P)\) is continuous in the entire domain \(R^3\), Eq.(3), which is an analytical solution of Eq.(2), is not defined at certain places in \(R^3\): 8 corners, 12 edges and 6 planes of the prism. If the field point \(P\) is located at one of these special places, some terms of Eq.(3) are not defined, and the relevant terms of Eq.(3) should be set to zero because they have zero limit there. For example, if the field point \(P\) is located at corner \(D\) (see Fig.1), then \(x_1 = y_1 = z_1 = 0\). It should be noted that, the first term on the right-hand side of Eq.(3) is not defined at this point, but its limit exists\[^4\], i.e.,

\[
\lim_{(x,y,z) \to (0,0,0)} xy \ln(z + r) = 0
\]

Hence, if the field point \(P\) is located at corner \(D\), \(V(P)\) can be expressed by the following equation

\[
V(P) = \lim_{(x,y,z) \to (0,0,0)} \frac{G \rho}{r} \left[ xy \ln(z + r) + yz \ln(x + r) + xz \ln(y + r) \right. \\
\left. - \frac{x^2}{2} \tan^{-1} \frac{yz}{xr} - \frac{y^2}{2} \tan^{-1} \frac{xz}{yr} - \frac{z^2}{2} \tan^{-1} \frac{xy}{zr} \right]_{x_1}^{x_2} \left[ y_1 \right]_{y_1}^{y_2} \left[ z_1 \right]_{z_1}^{z_2} \left[ \right]_{z_1}^{z_2}
\]

where \(\epsilon_1, \epsilon_2, \epsilon_3\) are small quantities and \(\epsilon_1 + \epsilon_2 + \epsilon_3 \neq 0\).

Eq. (5) results in nine terms by a simple substitution, because other terms are equal to zero at this limit. Other situations in the case that \(P\) is located the mentioned special places can be dealt with in a similar way. Thus, \(V(P)\) given by Eq. (3) can be extended in the entire domain \(R^3\) continuously\[^4\].

2 Discrete element method

In this section, we simply describe the discrete element method for determining the gravitational potential of a prism. The basic principle of the discrete element method is to divide the given prism into many small prisms and then treat each small prism as a particle. Each particle generates its gravitational potential at the field point \(P\), and the sum of them approaches the gravitational potential of the given prism. The more pieces the given prism is partitioned, the more precise the result is. This is the case that \(P\) lies outside the given prism. In the case that \(P\) lies inside the given prism, we can further divide the prism into many small ones, and it is obvious that \(P\) must be loc-
cated in one of these small prisms (including the surfaces). Treat each small prism as a particle except for the special one in which $P$ is located. We treat the special one as a small ball because a rigorous formula can be easily found in case that the field point lies inside a uniform sphere\cite{1,2}.

3 Comparisons of the results based on two methods

The calculations and comparisons between Nagy’s approach and the discrete element method are executed in a Cartesian coordinate system (see Fig.1). Suppose the coordinate unit is in meters, and set

$$X_1 = 100, \quad X_2 = 150$$

$$Y_1 = 100, \quad Y_2 = 150$$

$$Z_1 = 100, \quad Z_2 = 200$$

This means that the dimension of the concerned prism is $50m \times 50m \times 100m$. Suppose the density of the prism is $2.67g \cdot cm^{-3}$.

3.1 Field point outside the prism

First, suppose the field point $P$ is located at $(105, 105, 10200)$, away from the prism (about 10 km). Divide the given prism into 31250 pieces ($2m \times 2m \times 2m$ for each small piece), and then we can compare the gravitational potentials given by the two methods. The results are listed in Table 1, where “$V_d$” denotes the gravitational potential given by the discrete element method, and “$V_n$” denotes that given by Nagy’s approach.

| Field point far from the prism (m$^2 \cdot s^{-2}$) | $V_d$ | $V_n$ | $V_n - V_d$ | ($V_n - V_d) / V_d$ |
|-----------------------------------------------|------|------|-------------|------------------|
| $4.43 \times 10^{-6}$                         | 4.43 $\times 10^{-6}$ | $-1.90 \times 10^{-15}$ | $-4.30 \times 10^{-10}$ |

Second, suppose the field point $P$ is located at $(105, 105, 98)$ and it is very close to one of the surfaces of the prism (2 m external to one of the planes). Then we compare the gravitational potentials given by the two methods. The results are listed in Table 2.

| Field point close to the prism (m$^2 \cdot s^{-2}$) | $V_d$ | $V_n$ | $V_n - V_d$ | ($V_n - V_d) / V_d$ |
|-----------------------------------------------|------|------|-------------|------------------|
| $m = 31250$                                    | 8.599965 $\times 10^{-4}$ | $-3.847835 \times 10^{-10}$ | $-4.74244 \times 10^{-7}$ |
| $m = 250000$                                   | 8.599962 $\times 10^{-4}$ | $-9.316263 \times 10^{-12}$ | $-1.08328 \times 10^{-8}$ |
| $m = 2000000$                                  | 8.599962 $\times 10^{-4}$ | $-5.890528 \times 10^{-13}$ | $-6.849482 \times 10^{-10}$ |
| $m = 16000000$                                 | 8.599962 $\times 10^{-4}$ | $-3.711072 \times 10^{-14}$ | $-4.315220 \times 10^{-11}$ |

From Tables 1 and 2 we can see that the gravitational potential given by the discrete element method is in agreement with that given by Nagy’s approach: the smaller the given prism is separated, the smaller the difference between the potentials calculated by the two methods. Because the field point $P$ lies outside the given prism (referred to as the outside-case), after dividing the given prism into many small ones, only a few of them are close to the field point, while most of them are far from it. Therefore, the discrete element method makes a good approximation to the real case.

3.2 Field point inside the prism

Suppose the field point $P$ is located at $(105, 105, 123)$, lying inside the prism. In section 3, we have discussed the method to deal with the case that $P$ lies inside the given prism. In order to get more accurate results, we make some improvements. After the given prism is divided into many small prisms, there must be a small prism in which $P$ is located (including the surfaces). Then we divide this particular small prism into 10000 smaller pieces, and $P$ must be located in one of the smaller ones. Treat the one containing the field point as a small ball and the others as particles, and then complete the calculations.

The comparison values between the gravitational potentials given by the two methods are shown in Table 3.

From Table 3 we get the following results. First, the relative accuracies (shown in the 5th column) are a little larger than those given by Table 2. The reason is that, in the case that $P$ lies inside the given prism (referred to as the inside-case), after the partition, a lot of small prisms are very close to the field point. Hence, when these small ones are treated as particles, the calculated errors will be magnified. Second, because the given prism is divided into smaller ones,
the difference between the potentials given by the discrete element method and Nagy’s approach becomes smaller. It is similar to that in the outside-case.

## 3.3 Field point located at special places

By Nagy’s approach, there are 26 special places: 8 corners, 12 edges, and 6 planes of the given prism \(^4\). Suppose the field point \(P\) is located at one of the corners, e.g., corner \(D\) (\(x_1 = y_1 = z_1 = 0\), see Fig.1). Comparing the gravitational potentials given by the two methods, the results are listed in Table 4.

The other corner-cases are similar to the above one and the results are in the same range.

In the case where the field point \(P\) is located at one of the edges, e.g., edge \(AB\) (\(x_1 \neq 0\), \(x_2 \neq 0\), \(y_2 = z_1 = 0\), see Fig.1), and without loss of the generality, taking the coordinates of \(P\) as \((124,150,100)\), the comparison values between the gravitational potentials given by the two methods are shown in Table 5.

The other edge-cases are similar to the above and the results are in the same range.

In the case where the field point \(P\) is located at one of the planes, e.g., \(ABCD\) \((z_1=0\), see Fig.1), taking the coordinates of \(P\) as \((105,105,100)\), the comparison values between the gravitational potentials given by the two methods are listed in Table 6.

The other plane-cases are similar to the above and the results are in the same magnitude.

### Table 3  Field point inside the prism (m²⋅s⁻²)

| Dividing given prism into \(m\) pieces | \(V_d\) | \(V_n\) | \(V_n - V_d\) | \((V_n - V_d)/V_d\) |
|---------------------------------------|--------|--------|---------------|-----------------|
| \(m = 31250\)                         | 1.191022×10⁻³ | -3.992014×10⁻⁵ | -3.351754×10⁻² |                    |
| \(m = 250000\)                        | 1.150847×10⁻³ | 2.553743×10⁻⁷ | 2.219012×10⁻⁴ |                    |
| \(m = 2000000\)                       | 1.151038×10⁻³ | 6.384258×10⁻⁸ | 5.546520×10⁻⁵ |                    |
| \(m = 16000000\)                      | 1.151086×10⁻³ | 1.596058×10⁻⁸ | 1.386566×10⁻⁵ |                    |
| \(m = 250000000\)                     | 1.151100×10⁻³ | 2.555255×10⁻⁹ | 2.219838×10⁻⁶ |                    |

### Table 4  Field point located at one of the corners (m²⋅s⁻²)

| Dividing given prism into \(m\) pieces | \(V_d\) | \(V_n\) | \(V_n - V_d\) | \((V_n - V_d)/V_d\) |
|---------------------------------------|--------|--------|---------------|-----------------|
| \(m = 31250\)                         | 7.986773×10⁻⁴ | 2.480053×10⁻⁴ | 3.105180×10⁻⁵ |                    |
| \(m = 250000\)                        | 7.986959×10⁻⁴ | 6.200033×10⁻⁹ | 7.762696×10⁻⁶ |                    |
| \(m = 2000000\)                       | 7.987005×10⁻⁴ | 1.550002×10⁻⁸ | 1.940655×10⁻⁵ |                    |
| \(m = 16000000\)                      | 7.987017×10⁻⁴ | 3.875003×10⁻¹⁰ | 4.851627×10⁻⁷ |                    |

### Table 5  Field point located at one of the edges (m²⋅s⁻²)

| Dividing given prism into \(m\) pieces | \(V_d\) | \(V_n\) | \(V_n - V_d\) | \((V_n - V_d)/V_d\) |
|---------------------------------------|--------|--------|---------------|-----------------|
| \(m = 31250\)                         | 9.170123×10⁻⁴ | 4.960118×10⁻⁴ | 5.408999×10⁻⁵ |                    |
| \(m = 250000\)                        | 9.170495×10⁻⁴ | 1.240008×10⁻⁴ | 1.352171×10⁻⁵ |                    |
| \(m = 2000000\)                       | 9.170588×10⁻⁴ | 3.100005×10⁻⁹ | 3.380378×10⁻⁶ |                    |
| \(m = 16000000\)                      | 9.170611×10⁻⁴ | 7.499993×10⁻¹⁰ | 8.450902×10⁻⁷ |                    |

### Table 6  Field point located at one of the planes (m²⋅s⁻²)

| Dividing given prism into \(m\) pieces | \(V_d\) | \(V_n\) | \(V_n - V_d\) | \((V_n - V_d)/V_d\) |
|---------------------------------------|--------|--------|---------------|-----------------|
| \(m = 31250\)                         | 8.950021×10⁻⁴ | -1.523807×10⁻⁷ | -1.702573×10⁻⁴ |                    |
| \(m = 250000\)                        | 8.948249×10⁻⁴ | 2.480254×10⁻⁴ | 2.771776×10⁻⁵ |                    |
| \(m = 2000000\)                       | 8.948435×10⁻⁴ | 6.200159×10⁻⁴ | 6.928764×10⁻⁶ |                    |
| \(m = 16000000\)                      | 8.948482×10⁻⁴ | 1.550009×10⁻⁴ | 1.732148×10⁻⁶ |                    |
In summary, by various comparisons we can draw the following conclusion: using the discrete element method, the smaller the given prism is partitioned, the better approximation it makes. That means when the given prism is divided into smaller and smaller ones, it is closer and closer to reality. From Table 1 to Table 6, it is demonstrated that in each Table, \( V_d \) is closer and closer to \( V_n \). Since \( V_d \) becomes closer and closer to reality (as mentioned above), one can safely conclude that \( V_n \) is accurate, and Nagy’s approach is valid.

## 4 Conclusion

From subsections 3.1 and 3.2, we can see that wherever the field point is located (outside, inside or on the special place of the given prism), the gravitational potential given by the discrete element method coincide very well with that given by Nagy’s approach, and consequently one can safely use Nagy’s approach to calculate the gravitational potential of a given prism.

The discrete element method is simple in theory and easy to be realized. This is its advantage. However, there are some disadvantages: (1) this method is some kind of discretization, and consequently it will inevitably lose accuracy in applications; (2) its application efficiency is low and it runs slowly on a computer, which limits its applicability.

The formulas given by Nagy’s approach are formulated in Cartesian coordinates, and their usage implies a planar approximation. Despite this disadvantage, Nagy’s approach has the following advantages: (1) it is closed and more integrated, so it is more accurate; (2) it runs faster and can save more CPU time; (3) it is efficient for many applications, such as the description of density distribution, terrain corrections, etc.\[^{4,6,8-13}\]

Nagy’s approach mainly focuses on determining the gravitational potential of a regular prism. For more complex cases, e.g., in spherical coordinates or the prism has a curved or inclined top, the interested readers may consult the work of Smith et al.\[^{14}\] for more details.

## References

[1] Guan Zelin, Ning Jinseng (1980) The Earth’s figure and its external gravity field[M]. Beijing: Chinese Surveying and Mapping Press (in Chinese)

[2] Guo Junyi (1994) Foundation of physical geodesy[M]. Wuhan: Wuhan Technical University of Surveying and Mapping Press (in Chinese)

[3] Heiskanen W A, Moritz H (1967) Physical geodesy[M]. San Francisco: Freeman and Company

[4] Nagy D, Papp G, Benedek J (2000) The gravitational potential and its derivatives for the prism[J]. *Journal of Geodesy*, 74: 552-560

[5] Nagy D, Papp G, Benedek J (2002) Corrections to “The gravitational potential and its derivatives for the prism”[J]. *Journal of Geodesy*, 76: 475

[6] Kuhn M (2003) Geoid determination with density hypotheses from isostatic models and geological information[J]. *Journal of Geodesy*, 77: 50-65

[7] Wang Qianshen (2003) Gravitology[M]. Beijing: Seismology Press (in Chinese)

[8] Luo Zhicai, Chen Yongqi, Ning Jinseng (2003) Effects of terrain on the determination of high precise local gravimetric geoid[J]. *Geomatics and Information Science of Wuhan University*, 28(3): 340-344

[9] Lou Lizhi (2004) A study on simulated geoid in China and its adjacent regions[D]. Wuhan: Institute of Geodesy and Geophysics, Chinese Academy of Science

[10] Lou Lizhi, Fang Jian, Xu Houze (2006) Effects of interface undulations on simulated geoid[J]. *Journal of Tongji University*(Natural Science), 34(6): 848-852

[11] Fellner J (2007) Development of a software package for visualization of three-dimensional mass distributions and forward gravity modelling[J]. *Journal of Geophysics and Engineering*, 4: 31-39

[12] Makhloof A A (2007) The use of topographic-isostatic mass information in geodetic applications[D].Bonn: Institute of Geodesy and Geoinformation

[13] Heck B, Seitz K (2007) A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling[J]. *Journal of Geodesy*, 81: 121-136

[14] Smith D A, Robertson D, Milbert D G (2001) Gravitational attraction of local crustal masses in spheric coordinates[J]. *Journal of Geodesy*, 74: 783-795