A combinatorial proof of the Burdzy–Pitman conjecture

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Abstract

We prove a sharp upper bound for the number of high degree differences in bipartite graphs: let \((U, V, E)\) be a bipartite graph with \(U = \{u_1, u_2, \ldots, u_n\}\) and \(V = \{v_1, v_2, \ldots, v_n\}\); for \(n \geq k > \frac{n}{2}\) we show that

\[
\sum_{1 \leq i,j \leq n} \mathbb{1}\{|\deg(u_i) - \deg(v_j)| \geq k\} \leq 2k(n - k).
\]

As a direct application we show a slightly stronger, probabilistic version of this theorem and thus confirm the Burdzy–Pitman conjecture about the maximal spread of coherent and independent distributions.

1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We say that a random vector \((X, Y)\) defined on this probability space, is coherent, if there exist sub \(\sigma\)-fields \(\mathcal{G}, \mathcal{H} \subset \mathcal{F}\) and an event \(A \in \mathcal{F}\), such that

\[
X = \mathbb{E}(\mathbb{1}_A | \mathcal{G}), \quad Y = \mathbb{E}(\mathbb{1}_A | \mathcal{H}).
\]

We will also say that the joint distribution of such \((X, Y)\) is coherent on \([0, 1]^2\). Hereinafter, we write \((X, Y) \in C\) or \(\mu \in C\) to indicate that the vector \((X, Y)\) or a distribution \(\mu\) is coherent. Although this notation might be seen as a bit ambiguous, it does not lead to any misunderstandings.

Krzysztof Burdzy and Soumik Pal [1] prove that for any \(\delta \in (\frac{1}{2}, 1]\) and \((X, Y) \in C\) the probability \(\mathbb{P}(|X - Y| \geq \delta)\) of the difference between coherent variables exceeding a given threshold \(\delta\) is bounded above by the quantity \(\frac{2(1-\delta)}{2-\delta}\). They go on to show that this bound is sharp and it is attained by a random vector \((X, Y)\) with \(X\) and \(Y\) being dependent random variables.

Let us denote

\[
C_T = \{(X, Y) : X, Y \in C, X \perp Y\},
\]

as a family of those coherent distributions, which are additionally independent. In this paper we positively answer a related and very natural question raised by Krzysztof Burdzy and Jim Pitman in [2], where they have formulated the following conjecture:

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Conjecture 1.1. For $\delta \in (\frac{1}{2}, 1]$, we have

$$\sup_{(X,Y) \in \mathcal{I}} \mathbb{P}(|X - Y| \geq \delta) = 2\delta(1 - \delta).$$

Let us highlight, that this formalism should be regarded as taking supremum over all probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, all events $A \in \mathcal{F}$ and all pairs of independent sub $\sigma$-fields $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$. Although there are known alternative characterizations of coherent distributions [5, 6, 8], let us quote [2]:

For reasons we do not understand well, these general characterizations seem to be of little help in establishing the evaluations of $\epsilon(\delta)$ [i.e. $\mathbb{P}(|X - Y| \geq \delta)$] discussed above, or in settling a number of related problems about coherent distributions [...].

It is our belief that this is indeed so, because of the underlying combinatorial nature of those problems. Notice that discretization and combinatorial techniques appear already in [1, 4].

Let us briefly describe our approach and the organization of the paper. It is a well-known fact that the properties of two-dimensional coherent vectors are very similar to the properties of degree sequences of bipartite graphs. Remarkable example of this phenomenon can be found in [10]. Therefore, in order to take advantage of the combinatorial nature of the problem, we start by discussing its graph-theoretic version. More precisely, in the next section we prove the following theorem.

Theorem 1.1. Let $G = (U, V, E)$ be a bipartite graph with an equal bipartition, i.e.

$$U = \{u_1, u_2, \ldots, u_n\}, \quad V = \{v_1, v_2, \ldots, v_n\},$$

for some $n \in \mathbb{Z}_+$. For $n \geq k > \frac{n}{2}$ we have

$$\sum_{1 \leq i, j \leq n} 1\{ |\deg(u_i) - \deg(v_j)| \geq k \} \leq 2k(n - k).$$

The proof of the Theorem 1.1 is based on an idea similar to the spread bounding theorem of Erdős et al. – see [7]. In Section 2 we then provide an elementary example showing that the bound (2) is sharp. In what follows in Section 3, we show how to reduce the initial problem to the Theorem 1.1. To this end, we make use of an appropriate sampling construction, similar in spirit to [9]. The key idea is to approximate a fixed coherent distribution with a randomly generated sequence of graphs. We then apply Theorem 1.1 to each of the graphs in the sequence and obtain (1) by passing to the limit.

2 Number of high degree differences in bipartite graphs

Let $G = (U, V, E)$ be a bipartite graph with an equal bipartition, that is a triplet

$$U = \{u_1, u_2, \ldots, u_n\}, \quad V = \{v_1, v_2, \ldots, v_n\},$$

and

$$E \subset U \times V,$$
for some fixed $n \in \mathbb{Z}_+$. Let us also choose a natural number $k$ satisfying $n \geq k > \frac{n}{2}$. Hereinafter, we denote the degree sequences of $G$ as $(\alpha_i)_{i=1}^n$ and $(\beta_j)_{j=1}^n$, i.e. $\alpha_i = \deg(u_i)$ and $\beta_j = \deg(v_j)$ for all $1 \leq i, j \leq n$. Without loss of generality we also assume that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n,$$

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n.$$

We start with an observation similar to the spread bounding theorem of Erdős et al. – see [7].

**Lemma 2.1.** There exist $s, t \in \{1, 2, \ldots, n-k+1\}$ such that $\alpha_s \leq \beta_s + k - 1$ and $\beta_t \leq \alpha_t + k - 1$.

**Proof:** We will prove only the existence of $s$, as the case of $t$ is analogous. Assume for the sake of contradiction that such a number $s$ does not exists. Therefore, the total number of edges incident to $u_1, u_2, \ldots, u_{n-k+1}$ is at least $\beta_k + \beta_{k+1} + \cdots + \beta_n + k(n - k + 1)$. Observe that at least $k(n - k + 1)$ of these edges go to vertices $v_1, v_2, \ldots, v_{k-1}$. Let us denote

$$\tilde{E} := E \cap \left( \{u_1, u_2, \ldots, u_{n-k+1}\} \times \{v_1, v_2, \ldots, v_{k-1}\} \right),$$

and notice that we have just shown that $|\tilde{E}| \geq k(n - k + 1)$. On the other hand, we clearly have

$$|\tilde{E}| \leq (k - 1)(n - k + 1),$$

which is a contradiction. \hfill \square

We now prove the Theorem 1.1. With this result, we establish a natural upper bound on the number of possible pairs of vertices with high degree differences in $G$.

**Proof of Theorem 1.1** For $1 \leq i, j \leq n$, let us call $( i, j )$ an $A$-pair if $\alpha_i \geq \beta_j + k$. Correspondingly, let us call $( i, j )$ a $B$-pair if $\beta_j \geq \alpha_i + k$. Since $k > \frac{n}{2}$, we have $\alpha_i > \frac{n}{2}$ for all $A$-pairs $( i, j )$ and $\alpha_i < \frac{n}{2}$ for all $B$-pairs $( i, j )$. As a consequence, there exists an $i_0 \in \{ 1, 2, \ldots, n+1 \}$ such that:

1. $i \leq i_0 - 1$ for any $A$-pair $( i, j )$,
2. $i \geq i_0$ for any $B$-pair $( i, j )$.

Analogously, there exists $j_0 \in \{ 1, 2, \ldots, n+1 \}$ such that:

3. $j \leq j_0 - 1$ for any $B$-pair $( i, j )$,
4. $j \geq j_0$ for any $A$-pair $( i, j )$.

Observe that by the Lemma 2.1 we also have:

5. for any $A$-pair $( i, j )$ either $i < s$ or $j > s + k - 1$,
6. for any \(B\)-pair pair \((i, j)\) either \(j < t\) or \(i > t + k - 1\).

We now show that the restrictions 1–6, regardless of the initial graph, imply that the total number of \(A\)-pairs and \(B\)-pairs together is at most \(2k(n - k)\). Let us fix \(i_0, j_0 \in \{1, 2, \ldots, n + 1\}\). First, we verify that the optimal values of \(s\) and \(t\) satisfy \(s, t \in \{1, n - k + 1\}\).

Notice that the variable \(s\) appears only in the 5-th condition and thus the value of \(s\) is not important for bounding the number of \(B\)-pairs. Moreover, observe that if \(i_0 \leq n - k + 1\), then for \(s = n - k + 1\) the condition 5. is automatically fulfilled and thus \(s = n - k + 1\) is an optimal value. Similarly, if \(j_0 \geq k + 1\), then for \(s = 1\) the condition 5. is also automatically fulfilled and \(s = 1\) is an optimal value. Finally, let us assume that \(i_0 \geq n - k + 2\) and \(j_0 \leq k\). In this case, the restrictions imposed by the condition 5. remove exactly \((i_0 - s)(s + k - j_0)\) additional pairs. Therefore, as the last expression is a concave function of \(s \in [1, n - k + 1]\), it is minimized in one of the endpoints. Hence we may assume that \(s = 1\) or \(s = n - k + 1\), as desired. Analogously, we show that \(t = 1\) or \(t = n - k + 1\) is optimal. There are four possible cases now:

a. \(s = 1, t = n - k + 1\). We have \(j \geq k + 1\) for all \(A\)-pairs and \(j \leq n - k\) for all \(B\)-pairs \((i, j)\). Thus any \(i\) participates in at most \(n - k\) of \(A\)-pairs and in at most \(n - k\) of \(B\)-pairs. Therefore, since a fixed vertex can not participate in both types of pairs, every \(i\) participates overall in at most \(n - k\) pairs. As a consequence, the total number of pairs does not exceed \(n(n - k) < 2k(n - k)\).

b. \(s = n - k + 1, t = 1\). This case is symmetric to the previous one.

c. \(s = 1, t = 1\). We have \(j \geq k + 1\) for all \(A\)-pairs and \(i \geq k + 1\) for all \(B\)-pairs \((i, j)\). Let us denote \(a := \max(k + 1, j_0)\) and \(b := \max(k + 1, i_0)\). Then the total number of \(A\)-pairs is bounded by \((n - a + 1)(b - 1)\), while the total number of \(B\)-pairs is at most \((n - b + 1)(a - 1)\). Notice, that for \(a, b \in [k + 1, n + 1]\) the sum

\[
S := (n - a + 1)(b - 1) + (n - b + 1)(a - 1),
\]

is bilinear and it is maximized at one of four endpoints. For \(a = b = k + 1\), we get \(S = 2k(n - k)\). For, say \(a = n + 1\), we get \(S = n(n - b + 1) \leq n(n - k) < 2k(n - k)\).

d. \(s = n - k + 1, t = n - k + 1\). This case is analogous to c.

Hence we have shown that the Theorem 1.1 holds in all cases. This ends the proof. \(\square\)

We end this section with an example showing that the upper bound \(2k(n - k)\) in (2) cannot be improved. Note that a straightforward modification of this example shows that \(2\delta(1 - \delta)\) in (1) is also sharp.

**Example 2.1.** Consider \(n, k \in \mathbb{Z}_+,\) with \(n \geq k > \frac{n}{2}\). Let \(G_{n,k} = (U, V, E)\), where \(U = V = \{1, 2, \ldots, n\}\) and

\[
E = \{(u, v) \in U \times V : \max(u, v) \leq k\}.
\]
We clearly have
\[ \sum_{1 \leq i, j \leq n} 1 \left\{ |\deg(u_i) - \deg(v_j)| \geq k \right\} = 2k(n - k). \]

Moreover, one can check that inequality \(2\) becomes an equality exactly for those graphs \(G\) that are isomorphic to \(G_{n,k}\) or to its complement \(\overline{G_{n,k}}\). This follows easily from the proof of Theorem 1.1 and we leave the details to interested reader.

3 Proof of the Burdzy–Pitman conjecture

By \(C_{I}(n)\) we denote the set of those \((X, Y) \in C_{I}\), that both \(X\) and \(Y\) take at most \(n\) different values.

**Proposition 3.1.** Let \((X, Y)\) be coherent and independent, and let \(n\) be a positive integer. Then there exists \((X_n, Y_n) \in C_{I}(n)\), such that \(|X - X_n| \leq \frac{1}{n}\) and \(|Y - Y_n| \leq \frac{1}{n}\), almost surely.

The proof of the above Proposition can be found in [3, 1]. In what follows, fix any \(\delta \in \left(\frac{1}{2}, 1\right]\).

**Proposition 3.2.** To prove the Conjecture 1.1 it is enough to verify it for \(\bigcup_{n=1}^{\infty} C_{I}(n)\).

**Proof:** Fix \((X, Y) \in C_{I}\) and choose \((X_n, Y_n)\) as in Proposition 3.1. By the triangle inequality we get
\[ \mathbb{P}(|X - Y| \geq \delta) \leq \mathbb{P}(|X_n - Y_n| \geq \delta - 2/n). \]

Thus, assuming that the Conjecture 1.1 is true for \(\bigcup_{n=1}^{\infty} C_{I}(n)\), for \(n\) large enough we obtain
\[ \mathbb{P}(|X - Y| \geq \delta) \leq 2(\delta - 2/n)(1 - \delta + 2/n). \]

Passing to the limit ends the proof.

We are now able to prove our main result.

**Proof of Conjecture 1.1.** Fix \((X, Y) \in \bigcup_{n=1}^{\infty} C_{I}(n)\). There exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), independent sub \(\sigma\)-fields \(\mathcal{G}, \mathcal{H} \subset \mathcal{F}\) and an event \(A \in \mathcal{F}\), such that \(X = \mathbb{E}(1_A|\mathcal{G})\) and \(Y = \mathbb{E}(1_A|\mathcal{H})\). Furthermore, for some \(N, M \in \mathbb{Z}_{+}\), we may suppose that \(X\) takes values \(x_1, x_2, \ldots, x_N\) on sets \(G_1, G_2, \ldots, G_N\), respectively of the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and \(Y\) takes values \(y_1, y_2, \ldots, y_M\) on sets \(H_1, H_2, \ldots, H_M\). For simplicity, we can also assume that
\[ \mathcal{G} = \sigma\left(G_1, G_2, \ldots, G_N\right), \]
\[ \mathcal{H} = \sigma\left(H_1, H_2, \ldots, H_M\right), \]
meaning that \(\sigma\)-fields \(\mathcal{G}, \mathcal{H}\) are generated by those disjoint partitions of \(\Omega\). For \(1 \leq i \leq N\) and \(1 \leq j \leq M\), denote the probabilities \(p_i = \mathbb{P}(G_i), q_j = \mathbb{P}(H_j)\) and
\[ \rho_{i,j} = \frac{\mathbb{P}(G_i \cap H_j \cap A)}{\mathbb{P}(G_i \cap H_j)}. \]
Then by the independence we have $P(G_i \cap H_j) = p_i q_j$ and

$$x_i = \sum_{j=1}^{M} q_j \rho_{i,j}, \quad 1 \leq i \leq N,$$

$$y_j = \sum_{i=1}^{N} p_i \rho_{i,j}, \quad 1 \leq j \leq M,$$

which follows from a direct computation.

First, we show how to construct a sequence of bipartite graphs $G_n = (U_n, V_n, E_n)$ with $|U_n| = |V_n| = n$, such that:

1. in $U_n$ we have $p_i n + O(n^{3/4})$ vertices of degree $x_i n + O(n^{3/4})$, $i = 1, 2, \ldots, N$,

2. in $V_n$ we have $q_j n + O(n^{3/4})$ vertices of degree $y_j n + O(n^{3/4})$, $j = 1, 2, \ldots, M$,

where by $O(n^{3/4})$ we denote any quantity bounded in magnitude by $C n^{3/4}$ for some absolute constant $C > 0$, which is uniform in $i$ and $j$.

To this end, let us fix $n$ and without loss of generality assume that $n$ is large. We choose $n$ independent points $u_1, u_2, \ldots, u_n$ in our probability space $(\Omega, F, P)$ and for $1 \leq i \leq n$ denote $\alpha_i = s$ if $u_i \in G_s$. In other words, $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an i.i.d. sample from the set $\{1, 2, \ldots, N\}$ with weights $p_1, p_2, \ldots, p_N$, respectively. We can think about this sample as a randomly generated sequence of labels. Let $A_s = \sum_{i=1}^{n} \mathbb{1}_{\{\alpha_i = s\}}$ be the number of labels equal to $s$, $1 \leq s \leq N$. Observe that $A_s$ is clearly a sum of $n$ independent Bernoulli random variables. Hence, by the well known Hoeffding’s inequality, we have

$$P(|A_s - np_s| \geq nr) \leq 2e^{-2nr^2},$$

for all positive $r$. Consequently, setting $r = n^{-1/4}$ we get

$$P(|A_s - np_s| \geq n^{3/4}) \leq 2e^{-2\sqrt{nr}}.$$

Thus, as $n$ is large, with high probability we have $A_s = np_s + O(n^{3/4})$ for all $1 \leq s \leq N$. In fact, if this would not be the case, we can always reject the labels $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and resample them again. Analogously, we choose points $v_1, v_2, \ldots, v_n$ and generate an i.i.d. sample $(\beta_1, \beta_2, \ldots, \beta_n)$ from the set $\{1, 2, \ldots, M\}$ with weights $q_1, q_2, \ldots, q_M$, respectively. Next, let $B_t = \sum_{j=1}^{n} \mathbb{1}_{\{\beta_j = t\}}$ be the number of labels equal to $t$, $1 \leq t \leq M$. As previously, we can further assume that $B_t = nq_t + O(n^{3/4})$ for all $t$.

Next, conditioned on the labels $(\alpha_i)_{i=1}^{n}$ and $(\beta_j)_{j=1}^{n}$, we generate a random bipartite graph $(U_n, V_n, E_n)$:

1. for $1 \leq i, j \leq n$ independently, generate an indicator variable $Z_{i,j}$ with

$$P_{\alpha, \beta}(Z_{i,j} = 1) = 1 - P_{\alpha, \beta}(Z_{i,j} = 0) = \rho_{\alpha_i, \beta_j},$$

where $(\Omega_{\alpha, \beta}, F_{\alpha, \beta}, P_{\alpha, \beta})$ is a new space on which such independent $(Z_{i,j})_{i,j}$ can be constructed.
2. for \( 1 \leq i, j \leq n \), set \((u_i, v_j) \in E_n\) if \( Z_{i,j} = 1\), or equivalently, let \( Z_{i,j} = 1_{\{(u_i, v_j) \in E_n\}}\). For \( 1 \leq i \leq n \), we can now write
\[
\mathbb{E}_{\alpha, \beta} \deg(u_i) = \mathbb{E}_{\alpha, \beta} \left( \sum_{j=1}^{n} Z_{i,j} \right) = \sum_{t=1}^{M} B_t p_{\alpha_i, t} = \sum_{t=1}^{M} \left( nq_t + O(n^{3/4}) \right) p_{\alpha_i, t},
\]
and hence
\[
\mathbb{E}_{\alpha, \beta} \deg(u_i) = nx_{\alpha_i} + O(n^{3/4}),
\]
where the last line follows from (3). Similarly, for \( 1 \leq j \leq n \), by (4) we get
\[
\mathbb{E}_{\alpha, \beta} \deg(v_j) = ny_{\beta_j} + O(n^{3/4}).
\]
Lastly, again by the Hoeffding’s inequality, we have
\[
\mathbb{P}(\deg(u_i) - \mathbb{E}_{\alpha, \beta} \deg(u_i) \geq n^{3/4}) \leq 2e^{-2\sqrt{n}},
\]
and
\[
\mathbb{P}(\deg(v_j) - \mathbb{E}_{\alpha, \beta} \deg(v_j) \geq n^{3/4}) \leq 2e^{-2\sqrt{n}},
\]
for all \( i, j \in \{1, 2, \ldots, n\} \). Note that the concentration rates (7) and (8) are exponential in \( \sqrt{n} \). Thus, since \( n \) is large, with high probability all these concentrations take place. Then, by (5) and (6), we have
\[
\deg(u_i) = nx_{\alpha_i} + O(n^{3/4}) \quad \text{and} \quad \deg(v_j) = ny_{\beta_j} + O(n^{3/4})
\]
for all \( i, j \in \{1, 2, \ldots, n\} \). This, together with bounds on \((A_s)_{s=1}^N\) and \((B_t)_{t=1}^M\), proves that \( G_n \) does indeed satisfy the structural conditions stated.

In what follows, we add additional subscripts and write \( u_i^{(n)} \) and \( v_j^{(n)} \) for generic elements of \( U_n \) and \( V_n \), respectively. We can now write
\[
\mathbb{P}(|X - Y| \geq \delta) = \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq M} 1_{\{|x_i - y_j| \geq \delta\}} \cdot p_{i,j} q_{j}
\]
\[
= \lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq M} 1_{\{|x_i - y_j| \geq n\delta\}} \cdot \left(p_{i}n + O(n^{3/4})\right) \left(q_{j}n + O(n^{3/4})\right).
\]
Observe, that by the triangle inequality and defining properties of \( G_n = (U_n, V_n, E_n) \), we have
\[
|x_i - y_j| \leq |deg(u_i^{(n)}) - deg(v_j^{(n)})| + 2 \cdot O(n^{3/4}),
\]
for all \( i, j \in \{1, 2, \ldots, n\} \). Thus, we can further estimate the bound (9) by
\[
\leq \lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} 1_{\{|deg(u_i^{(n)}) - deg(v_j^{(n)})| \geq n\delta - 2O(n^{3/4})\}}.
\]
Finally, applying Theorem 1.1 to each of the bipartite graphs \( G_n \), we obtain
\[
\leq \lim_{n \to \infty} \frac{1}{n^2} \cdot 2\left(n\delta - 2O(n^{3/4})\right) \left(n - n\delta + 2O(n^{3/4})\right) = 2\delta(1 - \delta),
\]
which ends the proof. \( \square \)
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References

[1] K. Burdzy and S. Pal. Can coherent predictions be contradictory? Advances in Applied Probability, 53, 2021.

[2] K. Burdzy and J. Pitman. Bounds on the probability of radically different opinions. Electron. Commun. Probab., 25, 2020.

[3] S. Cichomski. Maximal spread of coherent distributions: a geometric and combinatorial perspective. Master’s thesis, University of Warsaw, 2020. available at arXiv:2007.08022 [math.PR].

[4] S. Cichomski and A. Osekowski. The maximal difference among expert’s opinions. Electronic Journal of Probability, 26, 2021.

[5] A. P. Dawid, M. H. DeGroot, and J. Mortera. Coherent combination of experts’ opinions. Test, 4, 1995.

[6] L. E. Dubins and J. Pitman. A maximal inequality for skew fields. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 52, 1980.

[7] P. Erdős, G. Chen, C.C. Rousseau, and R.H. Schelp. Ramsey problems involving degrees in edge-colored complete graphs of vertices belonging to monochromatic subgraphs. European Journal of Combinatorics, 14, 1993.

[8] S. Gutmann, J. H. B. Kemperman, J. A. Reeds, and L. A. Shepp. Existence of probability measures with given marginals. The Annals of Probability, 19, 1991.

[9] L. Lovász. Large Networks and Graph Limits. American Mathematical Society, 2012.

[10] T. Tao. Szemerédi’s regularity lemma revisited. Contributions to Discrete Mathematics, 1, 2006.