ADM canonical formulation with spin
and application to post-Newtonian approximations

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Recently, different methods succeeded in calculating the spin dynamics at higher orders in the post-Newtonian (PN) approximation. This is an essential step toward the determination of more accurate templates for gravitational waves, to be used in future gravitational wave astronomy. We focus on the extension of the ADM canonical formalism to spinning binary black holes. Using the global Poincaré invariance of asymptotically flat spacetimes as the most important guiding consistency condition, this extension can be constructed order by order in the PN approximation. We were able to reach a high order both in the spin power and the PN counting.

Keywords: canonical formalism; post-Newtonian approximation; wave generation and sources; classical black holes; binary and multiple stars

1. ADM Hamiltonian and global Poincaré invariance

In this article we consider asymptotically flat spacetimes and make use of standard (3+1)-splitting of spacetime into a stack of 3-dim. spacelike hypersurfaces. The Einstein equations are split into constraints and evolution equations. The speed of light \( c \) and the gravitational constant \( G \) are put equal to one.

The ADM Hamiltonian \( H \) can basically be expressed in terms of variables which possess standard canonical Poisson brackets (called canonical variables in the following). Here the matter variables \( \hat{z}_a \), \( P_{ai} \), and \( \hat{S}_{a(i)(j)} \) are the canonical position, momentum, and spin tensor of the \( a \)-th object, respectively. \( h_{ij}^{\text{TT}} \) is the transverse-traceless part of the induced metric of the spacelike hypersurfaces and \( \pi_{ij}^{\text{can}} \) its canonical conjugate momentum (related to the extrinsic curvature). In order to calculate the ADM Hamiltonian, one only has to solve the field constraints (in the ADM gauge), not the evolution equations.

The global Poincaré algebra is a consequence of the asymptotic flatness and is represented by Poisson brackets of the corresponding conserved quantities. These quantities are the ADM energy \( H \), total linear momentum \( P_i \), total angular momentum tensor \( J_{ij} \), and the boost vector \( K_i \). The boosts have an explicit dependence on the time \( t \) and can be decomposed as \( K_i = G_i - t P_i \).

2. Hamiltonians linear in the single spin variables

We treat black holes not as vacuum solutions of the Einstein equations. Instead we represent them by Tulczyjew’s singular stress-energy tensor \( T_{\mu\nu} \) (in the covariant spin supplementary condition), which contains the matter variables. The matter variables are evolved by the Mathisson-Papapetrou equations. Then it is straightforward to calculate the ADM energy or the other conserved quantities depending
on the matter variables appearing in $T_{\mu\nu}$, at least up to some order in the post-Newtonian (PN) approximation (using certain regularization techniques). However, the variables appearing in $T_{\mu\nu}$ are, in general, not standard canonical and one has to redefine them. One approach is to find such a redefinition by requiring that the conserved quantities fulfill the global Poincaré algebra. Then it must hold

$$P_i = \sum_a p_{ai} - \frac{1}{16\pi} \int d^3 x \pi^{kTT}_{can} h_{kl,i}^{TT},$$  

(1a)

$$J_{ij} = \sum_a (\hat{z}_a p_{aj} - \hat{z}_d p_{ai}) + \sum_a \hat{S}_{a(i)(j)} - \frac{1}{16\pi} \int d^3 x (x^i \pi^{kTT}_{can} h_{kl,j}^{TT}$$

$$- x^j \pi^{kTT}_{can} h_{kl,i}^{TT}) - \frac{1}{8\pi} \int d^3 x (\pi^{ikTT}_{can} h_{kj}^{TT} - \pi^{jkTT}_{can} h_{ki}^{TT}),$$  

(1b)

which implies that a major part of the Poincaré algebra is fulfilled. Requiring that $P_i$ and $J_{ij}$ take this form indeed fixes the transition to canonical variables in a unique way at the next-to-leading order (NLO). This allowed us to compute the NLO spin-orbit Hamiltonian, which has already been derived earlier and agrees with non-canonical results. A new result was the complete NLO $S_1 S_2$ Hamiltonian which was later confirmed by a different method. Further, $G_i$ was calculated and the full Poincaré algebra was checked.

It was shown that at higher PN orders one also has to redefine the canonical momentum of the gravitational field. By considering Eqs. (1), all variable redefinitions necessary for the next-to-next-to-leading order can be found (up to a canonical transformation). Variable redefinitions leading to canonical variables valid at all orders were recently constructed using an action approach. The used action basically results from a minimal coupling of the flat space one, similar as in previous approaches. Next, all constraints, supplementary, gauge, and coordinate conditions are eliminated from the action. The variable redefinitions in question then transform this action into the form known to produce Hamilton’s equations.

### 3. Hamiltonians non-linear in the single spin variables

If one knows the singular source terms of the constraints in terms of canonical variables, a straightforwardly calculated ADM energy will automatically be the sought-for Hamiltonian. This approach was used for higher orders in the spin for binary black holes. Source terms in canonical variables sufficient for $H_{S_2^1 S_1^p_1}, H_{S_2^2 p_1^*}, H_{S_1^2 p_2}, H_{S_2^1 S_2^p}, H_{S_2^1 S_2^2}, H_{S_2^1 S_2^1}, H_{S_2^1 S_2^1}$ were obtained from the Kerr metric in ADM coordinates. All these Hamiltonians are linear in $G$. The relation between Kerr parameter and $J_{ij}$ is crucial to obtain the canonical spin. Further, a leading order boost was applied to the Kerr metric, which allowed for leading corrections in the momenta.

Another approach is to construct $H$ and $G_i$ depending on canonical variables directly from an ansatz and using the global Poincaré algebra to fix the coefficients, with $P_i$ and $J_{ij}$ still given by Eqs. (1). Ansatzes for $H_{S_2^1 p^2}, H_{S_2^2 p^2}, H_{S_1^p_1}, H_{S_2^2 p_2}, H_{S_2^1 p_2}$,
$H_{S^2_S p_1}, H_{S^2_S p_2}, H_{S^4},$ and $H_{S^4}$ (again all linear in $G$) can indeed be fixed up to canonical transformation by $\{G_i, H\} = P_i$ only. Here also a corresponding ansatz for the source terms was used. The test mass case was used as a check.

The static ($P_{ai}$-independent) source part of the Hamilton constraint at NLO $S^2_1$ is needed to fix the remaining degrees of freedom from the canonical transformation, as well as to get the missing $G^2$-part of the NLO $S^2_1$ Hamiltonian (the linear-in-$G$ part is given by $H_{S^4}$). A general ansatz for these source terms, however, contains only four unknown constants. Two of these constants could be fixed by matching to the Kerr metric in ADM coordinates. The remaining two constants do not contribute to the Hamiltonian due to some cancellations. Finally, the complete NLO $S^2_1$ Hamiltonian can be calculated. The spin evolution agrees with an earlier result and the test mass case was checked. If one wants to use the approach of the last section to derive the NLO $S^2_1$ Hamiltonian, one needs quadrupole corrections to Tulczyjew’s stress-energy tensor. This, however, has not been tried yet.

It should be noted that all Hamiltonians for binary black holes at the formal 2PN order are now known (for the formal counting, see, e.g., Ref. [16] and also Appendix A of Ref. [5]).

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