Iterative Implicit Gradients for Nonconvex Optimization with Variational Inequality Constraints

Harshal D. Kaushik and Ming Jin

Abstract—We propose an implicit gradient based scheme for a constrained optimization problem with nonconvex loss function, which can be potentially used to analyze a variety of applications in machine learning, including meta-learning, hyperparameter optimization, and reinforcement learning. The proposed algorithm is based on the iterative differentiation (ITD) strategy. We extend the convergence and rate analysis of the current literature of bilevel optimization to a constrained bilevel structure with the motivation of learning under constraints. For addressing bilevel optimization using any first-order scheme, it involves the gradient of the inner-level optimal solution with respect to the outer variable (implicit gradient). In this paper, taking into account of a possible large-scale order scheme, it involves the gradient of the inner-level optimal constraints. We further provide error bounds with respect to the true gradients. Further, we provide nonasymptotic rate results.

I. INTRODUCTION

In the work, motivated by applications such as meta-learning, hyperparameter optimization, and reinforcement learning, we are interested in a constrained form of a bilevel optimization problem—a class of optimization problems with equilibrium constraints. In the existing literature, classical meta-learning, hyperparameter optimization problems do not consider constraints. There is an imperative need for constrained learning to incorporate safety, fairness, and other high-level specifications [24], [5]. Main motivation arises from the fact that meta-learning can be formed as a bilevel optimization problem where the “inner” optimization deals with the adaptation for a particular task and the “outer” optimization can be modified for meta-training with the safety constraints in order to restrict biased and risky scenarios. Despite being widely applicable to unconstrained settings and consistent convergence [2], [15], [9], [22], [10], in particular, meta-learning requires backpropagation through the solution of the inner-level optimization problem. This requires high memory requirements and non-trivial computations, which makes optimization-based meta learning difficult to scale to medium or higher datasets.

Popular techniques of addressing the bilevel optimization problem is to reformulate the problem in a form of constrained optimization [14], [19], [23] and then address that by differentiating through the KKT conditions. Consider a linearly constrained optimization problem as follows

$$\min_{x} \left\{ \langle \hat{c}, x \rangle \mid Ax \leq b, Sx \leq t \right\}$$ (1)

where $x \in \mathbb{R}^n$, and $A, S \in \mathbb{R}^{m \times n}$ are the known constraint matrices. For this single level, linear optimization problem, the calculation of implicit gradient is not straightforward. For example, consider parameters $\hat{c}$ in the objective as a function $\theta$. Gradient $\nabla_{\theta} \hat{x}$ can be evaluated by differentiating through the KKT conditions as follows [8]

$$\begin{bmatrix}
-2\gamma & A & S \\
D(\lambda) & D(A\hat{x} - b) & 0 \\
D(\mu) & 0 & D(S\hat{x} - t)
\end{bmatrix}
\begin{bmatrix}
\nabla_{\theta} \hat{x} \\
\nabla_{\theta} \lambda \\
\nabla_{\theta} \mu
\end{bmatrix}
= \begin{bmatrix}
\nabla_{\theta}^2 f(x, \theta) \\
0 \\
0
\end{bmatrix},$$ (2)

where $D(\cdot)$ is a diagonal matrix. In the evaluation of the implicit gradient above, there are certain challenges in the implementation, such as the inversion matrix $H$ becomes hard with increasing the number of constraints. There are no approximate techniques that can simplify the implementation (matrix inversion). Further, in every run, there are certain constraint qualifications that need to be satisfied for matrix $H$ to be invertible.

II. PROBLEM FORMULATION

We consider here a constrained bilevel optimization problem with nonconvex loss function; to be specific, we are interested in optimization problems with variational inequality constraints. The outer-level is a nonconvex optimization problem and the inner-level is the variational inequality (VI) problem. VI is a very powerful tool that can model: (1) complementarity problems, (2) noncooperative games, (3) inner-level optimization problem [17], [18]. Consider a set valued map $Y(x) \subseteq \mathbb{R}^n$ and mapping $F(\cdot, x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ then $\text{SOL}(Y(x), F(\cdot, x))$ is the solution of variational inequality $VI(Y(x), F(\cdot, x))$. Let us first define $\text{SOL}(Y(x), F(\cdot, x))$ in the following

$$\text{SOL}(Y(x), F(\cdot, x)) = \{y \in Y(x) : (F(y, x), z - y) \geq 0 \text{ for all } z \in Y\}.$$ We consider the following problem formulation

$$\minimize_{x \in X} f(y^*(x), x)$$ (P)

$$\text{subject to } y^*(x) \in \text{SOL}(Y(x), F(\cdot, x)).$$

For a case when the inner-level of problem (P) is an unbounded optimization problem, there exists a broad collection of approaches, broadly summarized into two categories: (1) Iterative implicit differentiation and (2) Approximate implicit differentiation [11], [16], [13].
Different from the existing literature for the analysis of bilevel optimization problems, in this work, we utilize a merit function that characterizes the solution of the inner-level VI. In Section III, we introduce the concept of a D-gap function. This is useful in characterizing the solution of the inner-level VI \(Y(F)\) in problem (P). Later in Section IV, we extend the analysis provided in [13], [16] to optimization problems with variational inequality constraints.

**Contribution:** Our contribution is summarized as follows:

1. In this work, we avoid the backpropagation through the inner-level problem in the evaluation of the implicit gradient. We obtain implicit gradient by using the ideas of merit function (D-gap) and fixed-point equations (corresponding to the natural map of VI) in Section III. Our approach is much more general and computationally efficient, compared with the existing approaches [8], [6], [1], [3].

2. Here, we address a constrained bilevel optimization in literature [13], [16]. We consider a class of nonsmooth optimization problems with the VI constraints in problem (P). It can be shown that bilevel optimization is a special case of problem (P) [17].

3. We extend the analysis from [13] and [16] to the class of optimization problems with VI constraints. We provide the error bounds with respect to the true gradients for the implicit gradients and the gradients of objective function. Further, we discuss the nonasymptotic convergence rate for the proposed scheme.

**Notation.** For the sake of brevity, some places we write vector \(y(x)\) as \(Y\), set valued map \(Y(X)\) as \(Y\), and mapping \(F(\cdot, x)\) as \(F(\cdot)\) or simply \(F\). For convenience, Jacobian of mapping \(F : \mathbb{R}^n \to \mathbb{R}^n\) with respect to \(x \in \mathbb{R}^m\) at any \(y \in \mathbb{R}^n\) is denoted with a bold \(\nabla_x F(y) \in \mathbb{R}^{n \times m}\). For any \(f : \mathbb{R}^n \to \mathbb{R}\), we use \(\nabla_x f \in \mathbb{R}^n\) to denote a partial derivative and \(\nabla_x^2 f\) denotes the Hessian matrix of \(f\). For convenience, instead of \(\text{SOL}(Y(x), F(\cdot, x))\), some places we alternatively refer the inner-level solution set by \(S(x)\). For denoting the projection of \(x\) onto set \(X\), we use \(P_X\{x\}\). All the norms are Euclidean for vectors and spectral norm for matrices, unless otherwise specified. We represent the inner product between two vectors by \(\langle \cdot, \cdot \rangle\), whereas for matrices it corresponds to a Frobenius inner product.

Next we provide necessary assumptions on the problem structure.

**Assumption 1.** Consider problem (P). We have the following hold on the problem structure:

(a) For \(x \in X \subseteq \mathbb{R}^m\) and \(y \in Y(x) \subseteq \mathbb{R}^n\), the outer objective function \(f(x, y)\) is continuously differentiable with respect to \(x\) and \(y\).

(b) For any \(x \in X\) and \(y \in Y(x)\), the inner-level map \(F(\cdot, x) : \mathbb{R}^n \to \mathbb{R}^n\) is continuously differentiable and \(\mu\)-strongly monotone with respect to \(y \in Y\).

(c) Set \(X\) and for any \(x \in X\), the set \(Y(x)\) are closed, convex, and bounded.

Next, we provide assumption on the set \(Y\) such that necessary constraint qualification conditions hold. The following assumption comes handy in establishing the continuity of the solution map \(S(x)\) of the inner-level VI in Lemma 7.

**Assumption 2.** Consider problem (P). For the inner-level VI \((Y, F)\), with functional map \(Y(x) \equiv \{y \in \mathbb{R}^n : g_i(x, y) \leq 0\}\) such that for a feasible point \((\bar{x}, \bar{y})\), we have:

(a) There exists vector \(v \in \mathbb{R}^m\) such that \(\langle v, \nabla_y g_i(\bar{x}, \bar{y}) \rangle < 0\), for all \(i \in I(\bar{x})\) where \(I(\bar{x}) \triangleq \{i : g_i(\bar{x}, \bar{y}) = 0\}\).

(b) Consider a neighborhood \(W\) of \((\bar{x}, \bar{y})\). The rank of gradient vectors \(\nabla_y g_i(x, y) : i \in I(\bar{x})\) is constant for any \((x, y)\) in \(W\).

(c) The gradient matrix \(\nabla_y g_i(x, y) : i \in I(\bar{x})\) has a full-row rank.

(d) The matrix formed using Lagrangian \(L(\bar{x}, \bar{y}, \bar{\lambda})\), \(\langle U, L(\bar{x}, \bar{y}, \bar{\lambda}) \rangle\) is nonsingular where \(U\) is the orthogonal basis of the null space of \(\nabla_y g_{I(\bar{x})}(x, y)\).

Problem (P) can be addressed by the first-order method. A general outline for iteration \(k\) is

\[ x_{k+1} := x_k - \gamma \nabla_x f(\gamma^*(x_k), x_k), \]

where \(\nabla_x f(x)\) is \(L\)-continuous and \(\gamma < 1/L\). In the above, calculation of the gradient \(\nabla_x f(\gamma^*(x_k), x_k)\) involves the implicit gradient \(\nabla_x y^*(x_k)\), which can be challenging to estimate. Instead, an approximate version can be obtained from (1) ITD, and (2) Approximate iterative differentiation (AID). In this paper, we extend the analysis of ITD [13], [16] for a class of optimization problems (P) with VI constraints.

**III. ESTIMATION OF IMPPLICIT GRADIENT**

In this section, we will provide necessary preliminary results and obtain the implicit gradient in Theorem 5. We begin this section with a metric to characterize the optimality of a solution to the inner-level VI problem in problem (P).

**Definition 3.** Consider problem (P) and let Assumption 1(b, c) hold on mapping \(F(\cdot, x)\) and set \(Y(x)\), respectively. For scalars \(b > a > 0\), \(y \in \mathbb{R}^n\), the merit function of \(\phi_{ab}(y, x)\) is defined as

\[ \phi_{ab}(y, x) \triangleq \phi_a(y, x) - \phi_b(y, x), \]

where for any \(c > 0\) and a positive definite matrix \(G\), \(\phi_c(y, x)\) is as follows

\[ \phi_c(y, x) \triangleq \sup_{z \in Y} \left\{ \langle F(y, x), y - z \rangle - \frac{c}{2} \langle y - z, G, y - z \rangle \right\}. \]

In the next result, we list an important property of \(\phi_{ab}\), that will be used to characterize the root point.

**Lemma 4 ([7]).** Consider problem (P) and let the merit function \(\phi_{ab}(y, x)\) be given by Definition 3 for any \(y \in Y\) and \(x \in X\). Then the root point \(y_s \in Y \) of \(\phi_{ab}(y, x)\) (i.e. solution to \(\phi_{ab}(y_s, x) = 0\) ) also solves VI(\(Y(x), F(\cdot, x)\) ) and \(y_s \in \text{SOL}(Y(x), F(\cdot, x))\).

In the next result, we will show that the inner-level solution of VI can be neatly obtained by solving a fixed-point equation. Further, we will obtain the implicit gradient.
Theorem 5. Consider problem (P). Let Assumption 1 (b, c) hold on map \( F(\cdot, x) \) and set \( Y(x) \), respectively. Let \( y_s \in Y \) be a solution of the inner-level variational inequality problem, i.e. \( y_s \in \text{SOL}(Y(x), F(\cdot, x)) \). Then
(a) For a scalar \( b > 0 \), we have \( y_s = z_b^*(y_s, x) \).
(b) for scalars \( b > a > 0 \), we obtain the implicit gradient \( \nabla_y z_a^*(y, x) \) as follows
\[
\nabla_y y = \left( \nabla_y z_a^*(y, x), \nabla_y z_b^*(y, x) \right) + \nabla_y z_b^*(y, x),
\]
where \( z_b^*(y, x) \) is the optimal solution of \( \sup_{z \in Y} \left\{ (F(y, x)^T(y - z) - \frac{c}{2}\|y - z\|^2) \right\} \) and terms \( I, 2 \) can be obtained from differentiating through optimization problem (4) [1]

Proof. For any point \( y \in Y \), from Definition 3 and taking \( G \) as an identity matrix, we have the following
\[
\phi_{ab}(y, x) = \phi_a(y, x) - \phi_b(y, x) = \sup_{z \in Y} \left\{ (F(y, x), y - z) - \frac{a}{2}\|x - z\|^2 \right\} - \sup_{z \in Y} \left\{ (F(y, x), y - z) - \frac{b}{2}\|y - z\|^2 \right\}.
\]
Let us now consider \( z_b^*(y, x) \) as the unique optimal solution of \( \sup_{z \in Y} \left\{ (F(y, x)^T(y - z) - \frac{c}{2}\|y - z\|^2) \right\} \) for \( c > 0 \). Therefore, we can now bound equation (7) as the following
\[
\phi_{ab}(y, x) = (F(y, x), y - z_b^*(y, x)) - \frac{a}{2}\|y - z_b^*(y, x)\|^2
\]
\[
- (F(y, x), y - z_b^*(y, x)) + \frac{b}{2}\|y - z_b^*(y, x)\|^2
\]
\[
\geq - \frac{a - b}{2}\|y - z_b^*(y, x)\|^2.
\]
Let us now consider \( y_s \in Y \) as the stationary point. Therefore, from Definition 3 and Lemma 4, we have \( \phi_{ab}(y_s, x) = 0 \). Now from equation (7) and taking into account \( b > a > 0 \), we obtain
\[
y_s = z_b^*(y_s, x).
\]
This shows part (a). Now note that the equation above is a fixed-point equation in \( y \), that is also a function of \( x \). We now differentiate equation (8) and try to obtain the value for implicit gradient \( \nabla_y z_a^*(y, x) \) at the point \( y_s \). We have
\[
\nabla_y y = \nabla_y z_b^*(y(x), x) = (\nabla_y z_b^*(y, x), \nabla_y z_b^*(y, x)) + \nabla_y z_b^*(y, x).
\]

Next we provide a result to make sure the gradient obtained in Theorem 5 exists.

Lemma 6. Provided Assumptions 1 and 2 hold on the structure of problem (P), the implicit gradient provided by equation (5) is unique and exists everywhere.

Proof. Consider the D-gap function, defined in Definition 3. The objective function in problem (4) is strongly concave. From the strong concavity and from the MFCQ condition

Algorithm 1 Iterative Differentiation for Implicit Gradient
1: Consider \( K, T \in N \). Initialize \( x_0, y_0(x_0) \), stepsizes \( \gamma, \beta \)
2: for \( k = 0, 1, 2, \ldots, K \)
3: for \( t = 0, 1, 2, \ldots, T \)
4: \( z_b^*(y_t(x_k), x_k) = \text{argmax}_{z \in Y} \left\{ (F(y_t(x_k), y_t - z) - \frac{b}{2}\|y_t - z\|^2) \right\}.
5: \( y_{t+1}(x_k) := z_b^*(y_t(x_k), x_k). \)
6: Update \( x_{k+1} := P \{ x_k - \beta \nabla_x f(y_{t+1}(x_k), x_k) \} \).
7: end for
8: end for
9: (which is equivalent to Slater CQ for differentiable function g Thm 2.3.8, 3.2.8 in [4], we have the continuously differentiability of the solution map [3].)

Next result discusses smoothness of the solution of the inner-level VI.

Lemma 7 (Theorem 4.2.16 [18]). Let the set valued map \( Y(x) \equiv \{ y \in \mathbb{R}^m : g_i(x, y) \leq 0 \} \) such that Assumption 2 is satisfied. Now let \( (x^*, y^*) \) is the solution of SOL(Y(x), F(x, y)). Then provided necessary constraint qualifications (MFCQ, CRCQ, SCOC [18], [12], [20]) hold. Therefore, \( y(x) \) is unique and \( S(x) \) is continuously differentiable.

Next result, we comments on the Lipschitz continuity of solution map \( S(x) \).

Lemma 8. Consider problem (P). The solution map of the inner-level VI, \( S(x) : X \rightarrow Y \) is Lipschitz continuous with respect to parameter \( 0 < L_S < \infty \).

Proof. From the continuity of the solution map of the inner-level of problem (P) (discussed in Lemma 7) and from the boundness of set \( Y \) (Assumption 1(c)), there exists a scalar \( (L_S < \infty) \) such that \( \|\nabla_x S(x)\| \) is bounded by \( L_S \) for any \( y \in S(x) \subseteq Y \).

IV. ERROR BOUNDS AND RATE ANALYSIS

In this section, we discuss the error bounds on the gradients, obtained from Algorithm 1 and provide the rate results
in Theorem 15. For further analysis we provide here another set of assumptions on the smoothness of $f$ in problem (P).

**Assumption 9.** Consider problem (P). The gradient of the objective function $f(x, y)$ has the following properties:

(a) We assume the Lipschitz smoothness property for $f(\bar{x}, y)$ with respect to $y$, i.e. for any $\bar{x} \in X$, and $y_1, y_2 \in Y$, we have
\[
\|\nabla_x f(\bar{x}, y_1) - \nabla_x f(\bar{x}, y_2)\| \leq L_{f_y} \|y_1 - y_2\|
\]
and
\[
\|\nabla_y f(\bar{x}, y_1) - \nabla_y f(\bar{x}, y_2)\| \leq L_{f_y} \|y_1 - y_2\|.
\]

(b) We assume the Lipschitz continuity for $\nabla_y f(\bar{x}, \bar{y})$ with respect to $x$ for any $\bar{y} \in Y$, i.e. for any $\bar{x}, \bar{x}_2 \in X$, and $y \in Y$, we have
\[
\|\nabla_y f(\bar{x}_1, \bar{y}) - \nabla_y f(\bar{x}_2, \bar{y})\| \leq L_{f_y} \|\bar{x}_1 - \bar{x}_2\|.
\]

(c) Function $f$ is $M$-Lipschitz. For $x_1, x_2 \in X$, we have
\[
\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|.
\]

In the next result, we establish the Lipschitz smoothness constant for $f$ with respect to $x$.

**Lemma 10.** Provided Assumption 9 hold on the objective function of problem (P). For $x_1, x_2 \in X$, we have the following
\[
\|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_2))\| \leq L_f \|x_1 - x_2\|
\]
where $L_f \triangleq L_{f_y} (L_{f_x} + L_{f_y})$.

**Proof.** Consider $\|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_2))\|$. Using the triangle inequality, we can write this as
\[
\|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_1))\| + \|\nabla_x f(x_2, y^*(x_1)) - \nabla_x f(x_2, y^*(x_2))\|.
\]

Next, from Assumption 9(a) and using Cauchy-Schwarz, we bound the above as
\[
\|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_2))\| \leq \|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_1))\| \|\nabla_y y\|
\]
\[
+ L_{f_y} \|y^*(x_1) - y^*(x_2)\|.
\]

Next, from Assumption 9(b), we bound the above as
\[
\|\nabla_x f(x_1, y^*(x_1)) - \nabla_x f(x_2, y^*(x_2))\| \leq L_{f_y} \|x_1 - x_2\| \|\nabla_y y\| + L_{f_x} \|y^*(x_1) - y^*(x_2)\|.
\]

Recalling the Lipschitz continuity of solution map $S(x)$ from Lemma 8, we have the required result.

In the following result, we show that the solution obtained from fixed-point equation is a contraction mapping.

**Lemma 11.** Consider the fixed-point equation obtained from Theorem 5(a). Show that it is a contraction map.

**Proof.** Vector $z^*_k(y, x)$ is an optimal solution, obtained by solving a skewed projection problem (4). Projection operator is a nonexpansive map [7]. From Theorem 12.1.2 in [7], as long as for the contraction coefficient ($\mu$), we have $\mu \leq 2\mu$ for $\mu$-strongly monotone $F$, we have the required result.

The above result is revisited in Assumption 13(d) again. Next, we will discuss main results of this work. In Theorem 15 we show that the update from Algorithm 1 converges to local optimum with $O(1/K)$. Before that, we provide next result for the convergence of the inner-level problem. We tackle this as the convergence of a sequence, generated from iteratively solving the fixed-point equation (9).

**Lemma 12.** Consider problem (P). We show that for some $x \in X$, iterative update of $y_k$, obtained from equation (9) in Algorithm 1 converges to the limit point $y^*$ with an $R$-linear rate, after iteration $k$ of the inner-level loop in Algorithm 1
\[
\|y_k - y^*\| \leq \frac{\phi_{ab}(y_0, x)}{C_1} \left(1 - \frac{C_2}{\sqrt{C_1 + C_2}}\right)^k,
\]
where $C_1, C_2, \text{ and } \delta$ are the nonnegative scalars such that for any $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$ we have
\[
\phi_{ab}(y, x) - \phi_{ab}(z_k^*(y, x), x) \geq C_1 \|y - z_k^*(y, x)\|^2,
\]
\[
\min(\phi_{ab}(y, x), \phi_{ab}(z_k^*(y, x), x)) \leq C_2 \|y - z_k^*(y, x)\|^2,
\]
\[
\|y - z_k^*(y, x)\| \leq \delta.
\]

**Proof.** From the definitions of $C_1, C_2, \delta$, we have
\[
\phi_{ab}(y_k, x) - \phi_{ab}(y_{k+1}, x) \geq C_1 \|y_k - y_{k+1}\|^2,
\]
\[
\phi_{ab}(y_{k+1}, x) \leq C_2 \|y_k - y_{k+1}\|^2.
\]

From the above two, we have the nonnegative sequence $\{\phi_{ab}(y_k, x)\}$ converging to zero. Therefore, we can write
\[
\phi_{ab}(y_{k+1}, x) \leq \frac{C_2}{C_1 + C_2} \phi_{ab}(y_k, x).
\]

For sufficiently large $k$, telescoping the above equation and utilizing the bounds above, we have
\[
C_1 \|y_k - y_{k+1}\|^2 \leq \phi_{ab}(y_k, x) \leq \left(\frac{C_2}{C_1 + C_2}\right)^k \phi_{ab}(y_0, x),
\]
this can be written as
\[
\|y_k - y_{k+m}\| \leq \sqrt{\frac{\phi_{ab}(y_0, x)}{C_1}} \sum_{j=k}^{k+m-1} \left(\frac{C_2}{C_1 + C_2}\right)^j.
\]

Therefore, $\{y_k\}$ is a Cauchy sequence that converges to a limit point ($y^*$). Utilizing the continuity of function $\phi_{ab}$, we have
\[
\|y_k - y^*\| \leq \frac{\phi_{ab}(y_0, x)}{C_1} \left(1 - \frac{C_2}{\sqrt{C_1 + C_2}}\right)^k.
\]
Assumption 13. Consider Problem (P), Assumption 1, and the update obtained in equation (9). We have the following hold:
(a) Jacobians $\nabla_x z^*_b(y, x)$ and $\nabla_y y^*_b(y, x)$ are Lipschitz continuous with constants $L_{x_{in}}$ and $L_{y_{in}}$, respectively.
(b) Considering the boundedness of set $Y$ (Assumption 1(c)), there exists a bound on the update from equation (9), $\|y(x)\| \leq C_{y_{in}}$.
(c) There exists $C'_{x_{in}} > 0$, $\sup_{\|y\| \leq 2C_{y_{in}}} \|\nabla_x z^*_b(y, x)\| \leq C'_{x_{in}}$, where $C_{y_{in}} > 0$.
(d) Referring to Lemma 11, we have $q_x \in (0, 1)$ as the contraction coefficient for $z_b^*(x, x)$ such that $q_x^2 \leq 2\mu$.

In the next result, we will derive the error bound on difference between the implicit gradient obtained from Algorithm 1 and true gradient at solution $y^*(x)$ for the inner-level VI.

**Proposition 14.** Consider problem (P). Let Assumptions 1, and 13 hold. Then we have the error bound for the implicit gradient at the iterative update obtained from equation (9) after iteration $T$, and the gradient of the inner-level fixed-point of the VI in Proposition (P) as follows
\[
\|\nabla_x y_T - \nabla_x y^*\| \leq \left( L_{x_{in}} + \frac{L_{y_{in}} C'_{x_{in}}}{1 - q_x} \right) C_{y_{in}} q_x^{T - 1} T + \frac{C'_{x_{in}} q_x T}{1 - q_x}.
\]

**Proof.** Consider equation (9). Differentiating $y_{T - 1} = z_b^*(y_T, x)$, we have the following at $y_T$ and $y^*$.
\[
\nabla_x y_T = \langle \nabla_y z_b^*(y_T, x), \nabla_x y_T \rangle + \nabla_x z_b^*(y_T, x)
\]
\[
\nabla_x y^* = \langle \nabla_y z_b^*(y^*, x), \nabla_x y^* \rangle + \nabla_x z_b^*(y^*, x).
\]

Substituting the above in $\|\nabla_x y_T - \nabla_x y^*\|$, we have
\[
\|\nabla_x y_T - \nabla_x y^*\|
\leq \|\nabla_x z_b^*(y_T, x) + \nabla_x z_b^*(y_T, x)\| \|\nabla_x y^*\|
+ \|\nabla_x z_b^*(y_T, x)\| \|\nabla_x y_T - \nabla_x y^*\|
+ \|\nabla_x z_b^*(y_T, x) - \nabla_x z_b^*(y^*, x)\|.
\]

Next, from Assumption 1 and 13, we bound the above as
\[
\|\nabla_x y_T - \nabla_x y^*\| \leq \left( L_{x_{in}} + \frac{L_{y_{in}} C'_{x_{in}}}{1 - q_x} \right) \|y_T - y^*\|
+ q_x \|\nabla_x y_T - \nabla_x y_T\|.
\]

Next, utilizing a result on the recursive error bound from Lemma 1, Section 2.2 in [21] and using the following bounds, we establish the required result on error bound.
\[
\|y^* - y_0\| = \|y^*\| \leq C_{y_{in}},
\|\nabla_x y^* - \nabla_x y_0\| \leq \|\nabla_y y^*\| \leq C'_{x_{in}} \frac{q_x}{1 - q_x}.
\]

Next, we will discuss one of the main results of this work. We show that the update from Algorithm 1 converges to local optimum with $O(1/K)$.

**Theorem 15.** Consider problem (P). Let Assumption 1, 9, and 13 hold. Consider the update from step 6 of Algorithm 1. We show that sequence $\{x_k\}$ converges to local optimum with a rate $O(1/K)$ for $K$ iterations
\[
\min_{k \in \{0, \ldots, K\}} \|\nabla_x f(y^*(x_k), x_k)\|^2
\leq f(y(x_0), x_0) - f(y^*(x_{K+1}), x_{K+1})
+ \frac{\beta (1 - \beta L) K}{2} + L_f \left( \frac{\phi_{y_0}(y_0, x_0)}{C_1} \right) + M \left( \frac{\beta^2 + \beta^2 L_f}{2} \right) \left( \frac{C_2}{C_1 + C_2} \right)^{T + 1} + \frac{C'_{x_{in}} q_x^{T + 1}}{1 - q_x}.
\]

**Proof.** Consider problem (P). The total gradient of the objective function is
\[
\nabla_x f(y_{T + 1}(x_k), x_k) = \nabla_x f(y_T(x_k), x_k)
+ \langle \nabla_y f(y_T(x_k), x_k), \nabla_x y_T(x_k) \rangle
= \nabla_x f(y^*(x_k), x_k)
+ \langle \nabla_y f(y^*(x_k), x_k), \nabla_x y^*(x_k) \rangle.
\]

Using the Lipschitz smoothness of $f$, we have
\[
\|\nabla_x f(y_{T + 1}(x_k), x_k) - \nabla_x f(y^*(x_k), x_k)\|
\leq L_f \|y_{T + 1}(x_k) - y^*(x_k)\|
+ M \left( \frac{\beta^2 + \beta^2 L_f}{2} \right) \left( \frac{C_2}{C_1 + C_2} \right)^{T + 1} + \frac{C'_{x_{in}} q_x^{T + 1}}{1 - q_x}.
\]

From the boundedness of set $Y$ and a continuity of $y^*(x_k)$ over set $Y$, we have bound on term 1 as $L_S$. Above equation can be written as
\[
\|\nabla_x f(y_{T + 1}(x_k), x_k) - \nabla_x f(y^*(x_k), x_k)\|
\leq L_f (1 + L_S) \|y_{T + 1}(x_k) - y^*(x_k)\|
+ \frac{M (\beta^2 + \beta^2 L_f)}{2} \left( \frac{C_2}{C_1 + C_2} \right)^{T + 1} + \frac{C'_{x_{in}} q_x^{T + 1}}{1 - q_x}.
\]

Next, we bounds terms 2 and 3 in the above from the results in Lemma 12 and Proposition 14, we have
\[
\|\nabla_x f(y_{T + 1}(x_k), x_k) - \nabla_x f(y^*(x_k), x_k)\|
\leq L_f (1 + L_S) \frac{\phi_{y_0}(y_0, x_k)}{C_1} \left( \frac{1}{1 - q_x} \right) \left( \frac{C_2}{C_1 + C_2} \right)^{T + 1} + \frac{M (\beta^2 + \beta^2 L_f)}{2} \left( \frac{C_2}{C_1 + C_2} \right)^{T + 1} + \frac{C'_{x_{in}} q_x^{T + 1}}{1 - q_x}.
\]
following for any two \( x_k, x_{k+1} \in X \)
\[
f(y^*(x_{k+1}), x_{k+1}) \leq f(y^*(x_k), x_k) \\
+ \langle \nabla_x f(y^*(x_k), x_k), x_{k+1} - x_k \rangle + \frac{L_f}{2} \| x_{k+1} - x_k \|^2.
\]
Substituting the update rule from Algorithm 1, utilizing the nonexpansiveness of the projection mapping, Cauchy-Schwarz inequality, and adding subtracting \( \beta \| \nabla_x f(y^*(x_k), x_k) \| \), we obtain
\[
f(y^*(x_{k+1}), x_{k+1}) \leq f(y^*(x_k), x_k) \\
- \left( \frac{\beta}{2} - \beta^2 L_f \right) \| \nabla_x f(y^*(x_k), x_k) \|^2 \\
+ \left( \frac{\beta}{2} + \beta^2 L_f \right) \| \nabla_x f(y^*(x_k), x_k) - \nabla_x f(y_{T+1}(x_k), x_k) \|^2.
\]
Substituting the bound for term 4, we have
\[
f(y^*(x_{k+1}), x_{k+1}) \leq f(y^*(x_k), x_k) \\
- \left( \frac{\beta}{2} - \beta^2 L_f \right) \| \nabla_x f(y^*(x_k), x_k) \|^2 \\
+ L_f (1 + L_S) \sqrt{\frac{\phi_{ab}(y_n, x_k)}{C_1} \left( \frac{\beta}{2} + \beta^2 L_f \right) \left( \sqrt{\frac{C_2}{C_1 + C_2}} \right)^{T+1} \\
+ M \left( \frac{\beta}{2} + \beta^2 L_f \right) \left( L_{x_in} + L_{y_in} C_{x_in} \frac{C_{y_in} q_T^T (T+1)}{1 - q_T} \right) C_{y_in} q_T^T (T+1) \\
+ \frac{C_{y_in} q_T^T (T+1)}{1 - q_T} \right).
\]
Taking summation on both sides over \( k \) from 0 to \( K \), we have
\[
\min_{k \in \{0,\ldots,K\}} \| \nabla_x f(y^*(x_k), x_k) \|^2 \\
\leq \frac{\| f(y^*(x_0), x_0) - f(y^*(x_{K+1}), x_{K+1}) \|}{\beta \left( \frac{\beta}{2} - \beta L \right) K} \\
+ L_f (1 + L_S) \sqrt{\frac{\phi_{ab}(y_n, x_k)}{C_1} \left( \frac{\beta}{2} + \beta^2 L_f \right) \left( \sqrt{\frac{C_2}{C_1 + C_2}} \right)^{T+1} \\
+ M \left( \frac{\beta}{2} + \beta^2 L_f \right) \left( L_{x_in} + L_{y_in} C_{x_in} \frac{C_{y_in} q_T^T (T+1)}{1 - q_T} \right) C_{y_in} q_T^T (T+1) \\
+ \frac{C_{y_in} q_T^T (T+1)}{1 - q_T} \right) .
\]
Note that the last two terms in the above go to zero with increasing number of inner iteration \( T \). We hereby focus on establishing the nonasymptotic convergence rate of the outer-level update \( \{x_k\} \) from Algorithm 1. Therefore, assuming the inner-level converges R-linearly, we bound the last two terms with \( \epsilon \) and we secure the rate of \( O \left( \frac{1}{T} \right) \).

V. CONCLUSION

In this work, we consider a class of optimization problems with VI constraints, motivated by a variety of applications in machine learning. We propose an implicit gradient scheme based on the ITD strategy. The proposed scheme is an efficient way of obtaining the implicit gradient to compute the gradient of the outer-level objective function. We also provide the error bounds with respect to the true gradients. Further, considering nonconvex objective and a strongly monotone map, we discuss the nonasymptotic convergence and obtain the rate results.

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