Symmetry-Protected Quantum Adiabatic Evolution in Spontaneous Symmetry-Breaking Transitions

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Quantum adiabatic evolution describes the dynamical evolution of a slowly driven Hamiltonian. In most systems undergoing spontaneous symmetry-breaking transitions, the symmetry-protected quantum adiabatic evolution can still appear, even when the two lowest eigenstates become degenerate. Here, a general derivation to revisit the symmetry-dependent transition and the symmetry-dependent adiabatic condition (SDAC) is given. Further, based on the SDAC, an adiabatic-parameter-fixed sweeping scheme is used for achieving fast adiabatic evolution, which is more efficient than the linear sweeping scheme. In the limit of small adiabatic parameter, an analytic inequality is obtained for the ground state fidelity only dependent on the adiabatic parameter. The general statements are then demonstrated via two typical systems. Besides, the robustness of the symmetry-dependent adiabatic evolution against weak symmetry-breaking sources is studied. The findings can be tested via the techniques in quantum annealing and may provide promising applications in practical quantum technologies.

1. Introduction

Quantum adiabatic theorem (QAT) states that a slowly driven system from an initial eigenstate will stay close to the correspondingly instantaneous eigenstate of its Hamiltonian $H(t)$.[1–4] The QAT is the theoretical basis for the Landau–Zener tunneling,[5,6] the perturbative quantum field theory,[7,8] the Berry phase,[9,10] the topological Thouless pumping,[11–17] and the quantum annealing.[18–23] Moreover, the QAT has promising applications in quantum technologies such as quantum state engineering[22,23] and quantum computing.[24–26] Usually, given the instantaneous eigenvalues $\{E_i(t)\}$ and eigenstates $\{|E_i(t)\rangle\}$ of $H(t)$, the QAT requires that $H(t)$ slowly varies according to $|i\dot{H}(t)| |E_m| \ll |E_n - E_m|$ for $n \neq m$.[27–30] If the energy degeneracy does not change, that is, the energy gap between neighboring energy eigenstates[27–29] (or neighboring degenerate energy eigenspaces[30]) is always open, this condition can always be satisfied if the driving is sufficiently slow. However, it is still unclear whether there is an adiabatic condition for the slowly driven system involving degeneracy change.

Spontaneous symmetry-breaking (SSB) is a powerful fundamental concept in understanding continuous phase transitions.[31–34] An SSB takes place when the ground state does not display a symmetry of the physical system. In most systems undergoing SSB transitions, such as the transverse-field quantum Ising model,[31,35–38] the Lipkin–Meshkov–Glick model,[35–48] and the quantized Bose–Josephson junction,[49–55] the two lowest eigenstates vary from non-degenerate to degenerate. However, if the system processes a certain symmetry, the quantum adiabatic evolution can still appear. It is interesting to investigate how to perform an efficient ground state adiabatic evolution in spontaneous symmetry-breaking system and find out the lower bound of the fidelity for the ground state.

Here, we study the slow dynamics in a system driven through a SSB transition. In the driving process, although the instantaneous ground states undergo an SSB, the driven Hamiltonian itself keeps the symmetry unchanged. Due to the driven Hamiltonian conserves the symmetry in the whole process, the population can only transfer between the instantaneous eigenstates of the same symmetry. Therefore, the adiabatic evolution can still appear when the driven system varies according to the symmetry-dependent adiabatic condition (SDAC).[35,42,53,56,57] Based on the SDAC, we use an adiabatic-parameter-fixed sweeping scheme for state preparation. We find this scheme is more efficient than the linear sweeping scheme, and further give an analytical lower bound of the ground state fidelity only dependent on the adiabatic parameter.

In Section 2, we give a general derivation to revisit the symmetry-dependent transition and SDAC. Based on the SDAC, we use an adiabatic-parameter-fixed sweeping scheme for state preparation. In the limit of small $\epsilon$, we obtain an analytic inequality for the ground state fidelity dependent only on the adiabatic parameter $\epsilon$. In Section 3 and Section 4, to illustrate our generic
statements, we consider two typical examples: the single-particle system within a symmetric 1D potential varying from single-well to double-well and the Lipkin–Meshkov–Glick model undergoing an SSB transition. In Section 5, we study the robustness of the symmetry-dependent adiabatic evolution. We show that the symmetry-dependent adiabatic evolution is robust against weak symmetry-breaking sources. In Section 6, we briefly summarize our results.

2. Symmetry-Dependent Evolution and Adiabatic-Parameter-Fixed Sweeping Scheme

We consider a driven quantum system $\hat{H}(R(t))$ with a time-independent symmetry $\hat{Y}$ obeying the commutation relation $[\hat{Y}, \hat{H}(R(t))]=0$. Generally, the Hamiltonian can be given as $\hat{H}(R(t)) = \sum_{i} R(t) \hat{H}_{i}$ with the time-varying parameters $R(t) = [R_{1}(t), R_{2}(t), \ldots, R_{n}(t)]$ and the time-independent operators $\hat{H}_{i}$. Thus, an arbitrary state can be expanded by the instantaneous eigenstates of $\hat{Y}$ and $\hat{H}(R(t))$: $|\psi_{\lambda}(R(t))\rangle$. Here, $E_{\lambda}$ and $\lambda$ stand for the $\lambda$th eigenvalue of $\hat{H}(R(t))$ and the $\lambda$th eigenstate of $\hat{Y}$, respectively.

2.1. Symmetry-Protected Transition

As the symmetry $\hat{Y}$ is a time-independent operator, we have $\frac{\partial}{\partial t} |\psi_{\lambda}(R(t))\rangle = \hat{Y} |\psi_{\lambda}(R(t))\rangle$ and their inner products (with $\langle \phi_{\nu}^{\lambda}(R(t)) \rangle$ satisfying

$$\langle \phi_{\nu}^{\lambda}(R(t)) | \frac{\partial}{\partial t} |\psi_{\lambda}(R(t))\rangle \rangle = 0$$

Due to $i\hbar \frac{\partial}{\partial t} = \hat{H}(R(t))$, we have

$$H_{\text{max}}^{\lambda}(t)(\lambda_{\beta} - \lambda_{\alpha}) = 0$$

with $H_{\text{max}}^{\lambda}(t) = \langle \phi_{\nu}^{\lambda}(R(t)) | \hat{H}(R(t)) |\phi_{\nu}^{\lambda}(R(t))\rangle$ (see Appendix). This means the state transition is protected by the symmetry. For the instantaneous eigenstates of the same symmetry (i.e., $\lambda_{\beta} = \lambda_{\alpha}$), the population may transfer between them. For the instantaneous eigenstates with different symmetries (i.e., $\lambda_{\beta} \neq \lambda_{\alpha}$), even if their instantaneous eigenenergies are degenerate, the population transfer between them is exactly forbidden. The symmetry-protected transition have been mentioned in discussing the dynamics crossing through quantum phase transitions. In the following, we make use of this property and explore the symmetry-dependent adiabatic evolution.

2.2. Symmetry-Dependent Adiabatic Evolution

The symmetry-protected transition is the basis for the following SDAC. Without loss of generality, we revisit the SDAC for degenerate systems, which can be relaxed to the non-degenerate systems. Below, $\mathbb{H}_{m}(R(t))$ denotes the $m$th degenerate subspace of $\hat{H}(R(t))$ with the eigenenergy $E_{m}$ and the degeneracy number $d_{m}$.

We assume the system is driven from an instantaneous eigenstate $|\phi_{m}^{\lambda_{\alpha}}(R(t))\rangle$ in a degenerate subspace $\mathbb{H}_{m}(R(t))$, in which each eigenstate has different symmetry (i.e., $\lambda_{\beta} \neq \lambda_{\alpha}$ if $i \neq j$ for $i,j = [1,2,\ldots,d_{m}]$). Thus, the adiabatic condition for remaining in the same instantaneous eigenstate at time $t + dt$ (where $dt$ is an infinitesimal interval) is given as

$$\epsilon(t) = \max_{m,n} \left| \frac{H_{\text{max}}^{\lambda_{m}}(t)}{E_{m} - E_{n}} \right| \ll 1 \quad \text{with} \quad m \neq n,$$

where $H_{\text{max}}^{\lambda_{m}}(t) = i\hbar \langle \phi_{m}^{\lambda_{m}}(R(t)) | \frac{\partial}{\partial R} |\phi_{m}^{\lambda_{m}}(R(t))\rangle$. $\lambda_{\beta}$ denotes the symmetry and $E_{[m,n]}$ stand for the instantaneous eigenenergies (see the detailed derivation in the Appendix). Since $E_{m}$ and $E_{n}$ belong to different subspaces $\mathbb{H}_{m}(R(t))$ and $\mathbb{H}_{n}(R(t))$, the energy gap does not vanish, that is, $|E_{m} - E_{n}| > 0$. This condition implies that, the adiabaticity of the time-evolution is determined by the energy gap between neighboring instantaneous eigenstates of the same symmetry. The gap with the same symmetry is similar with the relevant gap in constrained quantum annealing. Thus, there is no transition between eigenstates with different symmetries even if their energy gap vanishes. When $d_{m} = 1$, the subspace $\mathbb{H}_{m}(R(t))$ becomes non-degenerate, and the above SDAC keeps valid. If there is no symmetry-dependent behavior, that is, all $\lambda_{\beta}$ have the same value, the SDAC becomes the conventional adiabatic condition.

2.3. Adiabatic-Parameter-Fixed Sweeping Scheme

According to the SDAC (3), adiabatic evolutions may still appear even if the energy gap between nearest neighboring eigenstates vanishes. In a driven system through an SSB transition, in which the two lowest eigenstates change from non-degenerate to degenerate, the dynamics may still evolve arbitrarily close to its instantaneous ground state if there is a finite minimum energy gap between instantaneous eigenstates of the same symmetry. Naively, one can drive a system parameter $R$ linearly with fixed sweeping rate $\dot{R} = \nu$ from the non-degenerate regime across to the degenerate regime. If $\nu$ is sufficiently small, the adiabatic evolution of the ground state can still be achieved with high fidelity. However, this linear sweeping scheme is not timesaving since the energy gaps change dramatically when crossing through SSB transitions.

To perform fast and efficient ground state adiabatic evolution, we change the sweeping rate with time according to the instantaneous energy gaps between the eigenstates of the same symmetry under a fixed adiabatic parameter $\epsilon$ (designed by SDAC (3)). Thus, this scheme is called adiabatic-parameter-fixed sweeping scheme. It is one of the nonlinear sweeping schemes which one the sweeping rate is related to instantaneous energy gaps. Thus, substituting $\frac{\partial}{\partial R(t)} = \frac{\partial}{\partial R(0)} \frac{\partial R(t)}{\partial R(0)} = \frac{\partial R(t)}{\partial R(0)} \nu(t)$ into Equation (3), we can obtain

$$\nu(t) = \min_{\lambda} \left\{ \epsilon \left| E_{\lambda}(R(t)) - E_{\lambda}(R(0)) \right| \frac{\hbar}{\epsilon} \right\}$$

where $E_{\lambda}(R(0))$ is the eigenvalue of $\mathbb{H}_{\lambda}(R(0))$. $\epsilon$ is the difference between the instantaneous eigenenergies.

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Next, we will discuss how the population on ground state varies in the adiabatic-parameter-fixed sweeping scheme. At any instant, the instantaneous state $|\Psi(R(t))\rangle$ can be expanded in terms of the complete basis of $|\phi_{\alpha}^n(R(t))\rangle$.

$$|\Psi(R(t))\rangle = \sum_n \sum_{\alpha} a_{\alpha}^n(t) \exp \left[ \frac{i}{\hbar} \int_0^t \Delta E_n(R(t')) dt' \right] |\phi_{\alpha}^n(R(t))\rangle \quad (5)$$

Here, starting from the initial state $|\Psi(R(0))\rangle = |\phi_{\alpha}^n(R(0))\rangle$ (which is the instantaneous ground state in non-degenerate regime), and vary one of the system parameter $R(t)$ with time to across the quantum phase transition. The time-varying parameter $R(t) = R(0) + \int_0^t \dot{R}(t') dt'$, where $\dot{R}(t)$ is the sweeping rate of the parameter. We define the fidelity $F_n^R(t) = |\langle \Psi(t)|\phi_{\alpha}^n(R(0))\rangle|^2$ between the instantaneous evolved state $|\Psi(t)\rangle$ and the $n$-th instantaneous eigenstate $|\phi_{\alpha}^n(R(0))\rangle$. According to Equation (2), the transitions between states in the same degenerate subspace $\mathcal{H}_n(R(t))$ are forbidden if all degenerate energy eigenstate possess different values of $\lambda_n$. Thus, the evolved state $|\Psi(R(t))\rangle$ is given as

$$|\Psi(R(t))\rangle = \sum_n a_{\alpha}^n(t) \exp \left[ \frac{i}{\hbar} \int_0^t \Delta E_n(R(t')) dt' \right] |\phi_{\alpha}^n(R(t))\rangle \quad (6)$$

From the time-independent Schrödinger equation, the differential equation for the coefficients of Equation (6) can be written as

$$\dot{a}_{\alpha}^n(t) = -\left( \frac{\partial}{\partial t} + \frac{\Delta E_n}{\hbar} \right) a_{\alpha}^n(t) |\phi_{\alpha}^n(R(t))\rangle$$

$$\quad - \sum_{m \neq n} a_{\alpha}^m(t) \exp \left[ \frac{i}{\hbar} \int_0^t \Delta E_m(R(t')) dt' \right] \times |\phi_{\alpha}^n(R(t))\rangle$$

$$\quad \times \langle \phi_{\alpha}^m(R(t')) | \phi_{\alpha}^n(R(t)) \rangle$$

$$\quad \times \frac{\langle \phi_{\alpha}^m(R(t')) | \phi_{\alpha}^n(R(t)) \rangle}{E_n(R(t)) - E_m(R(t))} \quad (7)$$

Hereafter, we choose properly such that the condition $\langle \phi_{\alpha}^m(R(t)) | \phi_{\alpha}^n(R(t)) \rangle = 0$.[29] Now, we start from the ground state, $a_{\alpha}^n(0) = 1$, and according to Equation (7), we have

$$\dot{a}_{\alpha}^n(t) = 1 + \frac{i}{\hbar} \sum_{k=1}^n a_{\alpha}^k(t) \exp \left[ \frac{i}{\hbar} \int_0^t \left( \Delta E_1(R(t')) - \Delta E_n(R(t')) \right) dt' \right]$$

$$\quad \times \langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle$$

$$\quad \times \frac{\langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle}{E_1(R(t')) - E_n(R(t'))}$$

$$\quad - \frac{\partial}{\partial t} \left( \langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle \right)$$

$$\quad \times \frac{\langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle}{E_1(R(t')) - E_n(R(t'))} \quad (8)$$

In general, the nearest eigenstate of the same symmetry $|\phi_{\alpha}^{n+1}(R(t))\rangle$ determines the sweeping rate (4). If $\epsilon$ is very small, we can make an assumption that the whole process only involve $|\phi_{\alpha}^{n+1}(R(t))\rangle$ and $|\phi_{\alpha}^{n+1}(R(t))\rangle$. Under this approximation, Equation (8) can be simplified as

$$\dot{a}_{\alpha}^n(t) = 1 + \frac{i}{\hbar} \sum_{k=1}^n a_{\alpha}^k(t) \exp \left[ \frac{i}{\hbar} \int_0^t \left( \Delta E_1(R(t')) - \Delta E_n(R(t')) \right) dt' \right]$$

$$\quad \times \frac{\langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle}{E_1(R(t')) - E_n(R(t'))}$$

$$\quad \times \frac{\langle \phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t)) \rangle}{E_1(R(t')) - E_n(R(t'))} \quad (9)$$

Noting that $|\exp[\frac{i}{\hbar} \int_0^t \left( \Delta E_1(R(t')) - \Delta E_n(R(t')) \right) dt'| = 1$, and the adiabatic parameter $\epsilon(t) = |\hbar |(\phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t')) \rangle |^2 = \epsilon$ is time-independent, so that $\frac{\partial}{\partial t} |(\phi_{\alpha}^{n+1}(R(t')) | \phi_{\alpha}^{n+1}(R(t')) \rangle | = 0$. Thus, we have

$$\left| a_{\alpha}^n(t) \exp \left[ i \frac{\hbar}{\hbar} \int_0^t \left( \Delta E_1(R(t')) - \Delta E_n(R(t')) \right) dt' \right] \right|$$

$$\leq \left| a_{\alpha}^n(t) \right| \epsilon(t) \frac{\int_0^t \left( \Delta E_1(R(t')) - \Delta E_n(R(t')) \right) dt'}{\epsilon} \quad (10)$$
From Equations (10)–(12) and \( | \int_0^t \dot{a}_m^n(t') dt' | = | a_m^n(t) | \), we can obtain

\[ 1 - 2e|a_m^n(t)| \leq | a_m^n(t) | \]  

Due to probability conservation

\[ |a_m^n(t)|^2 + |a_m^n(t)|^2 = 1 \]  

we have

\[ |a_m^n(t)|^2 = 1 - |a_m^n(t)|^2 \]  

Squaring Equation (13), and substituting Equation (15) into it, we get

\[ |a_m^n(t)|^2 \geq 1 - 4|a_m^n(t)|e + 4|a_m^n(t)|^2 e^2 \]  

\[ 1 - |a_m^n(t)|^2 \geq 1 - 4|a_m^n(t)|e + 4|a_m^n(t)|^2 e^2 \]  

\[ |a_m^n(t)|^2 \leq 4|a_m^n(t)|e - 4|a_m^n(t)|^2 e^2 \]  

\[ |a_m^n(t)|^2 \leq 4|a_m^n(t)|e \]  

Thus, we obtain the inequality for the coefficient \( | a_m^n(t) | \), that is,

\[ |a_m^n(t)| \leq \frac{4e}{(1 + 4e^2)} \]  

or

\[ |a_m^n(t)| \leq \frac{16e^2}{(1 + 4e^2)} \]  

Further, substituting Equation (17) into Equation (13), we finally get the inequality for coefficient \( | a_m^n(t) | \), that is,

\[ |a_m^n(t)| \geq 1 - \frac{8e^2}{(1 + 4e^2)} \]  

or

\[ |a_m^n(t)| \geq \left( 1 - \frac{8e^2}{(1 + 4e^2)} \right)^2 \]  

The inequalities (17)–(20) only hold when \( e \) is sufficiently small. Therefore, in the limit of small \( e \) (i.e., \( e \ll 1 \)), we analytically obtain an inequality between the fidelity of staying in the instantaneous ground state and the adiabatic parameter.

\[ F_m^n(t) \geq \left( 1 - \frac{8e^2}{(1 + 4e^2)} \right)^2 \approx 1 - 16e^2 + O(e^4) \]  

While \( e \) becomes larger, the SDAC (3) gradually breaks, higher excited states (with same symmetry) begin to be populated. Thus, Equation (9) will not satisfy and the inequalities no longer hold.

### 3. Single-Particle in a Symmetric Potential

We first consider a single particle confined within a symmetric 1D potential, which slowly varies from single-well to double-well. Its Hamiltonian reads

\[ \hat{H}_S(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \]  

The first term is the kinetic energy and the second term describes the potential. The time-varying potential \( V(x, t) \) is a superposition of a time-independent harmonic trap and a time-dependent Gaussian barrier, \( V(x, t) = \frac{\omega^2}{2} m x^2 + A(t) e^{-x^2/d^2} \). Here, \( \omega \) is the trapping frequency, \( d \) denotes the barrier width and the barrier height \( A(t) \) varies with time. At \( t = 0 \), \( A = 0 \), \( V(x, t) \) is a harmonic potential (a symmetric single-well potential). When \( A(t) \) increases with time, \( V(x, t) \) gradually becomes a symmetric double-well potential, and the two lowest eigenstates change from non-degenerate to degenerate (or quasi-degenerate for a large but finite \( A(t) \)).

In the whole process, the Hamiltonian (22) keeps the mirror-reflection parity symmetry. That is, \( \hat{H}_S(x, t) \) is invariant under the mirror reflection \( \hat{P} : x \rightarrow -x \). \( [\hat{H}_S(x, t), \hat{P}] = 0 \). Thus \( \hat{P} \) has two eigenvalues \( \pm 1 \) respectively representing even and odd parity. Due to the parity symmetry, the instantaneous eigenstates appear with even and odd parity alternately. Initially, the energy levels are non-degenerate, see Figure 1a. When the barrier height is sufficiently high (i.e., \( A \gg \omega^2 d^2 \)), the neighboring pairs of eigenstates of different parity become quasi-degenerate, see Figure 1b. The quasi-degeneracy is also evidenced by the static energy spectrum versus the barrier height \( A \), see Figure 1c.

Now we discuss how adiabatic evolution appears. Due to the symmetry protected transition, from an initial even-parity eigenstate, the odd-parity instantaneous eigenstates will never be populated and vice versa. According to the SDAC (3), the adiaticity is determined by the minimum energy gap between the instantaneous eigenstates of the same symmetry. Since there always exists a finite gap between the instantaneous eigenstates of the same symmetry, adiabatic evolution may always appear if the system is driven sufficiently slowly.

To show how to achieve adiabatic evolution via designing the sweeping process of the barrier height, we now perform a numerical calculation based upon the Schrödinger equation \( i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H}_S(x, t) \Psi(x, t) \). In our calculation, we use the dimensionless units of \( m = \hbar = \omega = 1 \) and set \( d = \sqrt{2} \). We choose the initial state as the gerade state \( \phi_{GS}^+ \) of \( \hat{H}_S(0) \) with even parity. The barrier height is gradually lifted from \( A(0) = 0 \) to \( A(T) \gg \omega^2 d^2 \). Since the nearest eigenstate with even parity is the second-excited state, the SDAC in this system is

\[ \epsilon(t) = \left| \frac{\langle \phi_{GS}^+ (A(T)) | v_A(t) | \phi_{GS}^+ (A(0)) \rangle}{\sqrt{E_2(A(T)) - E_2(A(0))}} \right| \ll 1 \]  

Thus, the sweeping process is described as \( A(t) = \int_0^t v_A(t') dt' \) with the sweeping rate \( v_A(t) \). Given \( \epsilon \), according to the Equation (4), we have \( v_A(t) = \frac{\epsilon}{|E_2(A(T)) - E_2(A(0))|} \).
The time-evolution sensitively depends on the value of $\varepsilon$. In Figure 1d, for $\varepsilon = 0.1$, we show the fidelities versus the instantaneous barrier height $A(t)$. Although the first gap $E_3(t) - E_4(t)$ gradually vanishes, because the transition is protected by the symmetry, the population in the first-excited state gradually vanishes, because the transition is protected by the symmetry, the population in the first-excited state remains zero during the whole process. Particularly, the population in the ground state $F_0^i(t)$ keeps above 0.95 and only small population is transferred to the second-excited state (characterized by $F_1^i$). To show how slow the sweeping is practical, we plot the final fidelity $F_3^i(T)$ [where $A(T) = 20$] versus the adiabatic parameter $\varepsilon$, see Figure 1e. The final fidelity shows the appearance of adiabatic evolution for sufficiently small $\varepsilon$. Clearly, the curve $F_3^i(T)$ is always above the analytical line $(1 - \frac{8\varepsilon^2}{1 + 4\varepsilon^2})^2$, which confirms the validity of (21).

Meanwhile, we compare the linear sweeping scheme and our adiabatic-parameter-fixed sweeping scheme under the same evolution duration $T$. For the linear sweeping, $A(t) = \nu_A t$, as shown in Figure 2a,b (blue solid lines). While for adiabatic-parameter-fixed sweeping, $A(t) = \frac{\nu_A}{(\nu_A^2 - \nu_B^2)^{1/2}}$, as shown in Figure 2a,b (red dashed lines). Under the same total duration $T$, the adiabatic-parameter-fixed sweeping outperforms the linear sweeping with larger final fidelity of staying in the ground state, see in Figure 2c,d. Besides, the amplitude of the oscillation of $F_3^i(t)$ with adiabatic-parameter-fixed sweeping is much smaller than linear sweeping.

In Figure 3a,b, we show the minimal fidelity $\min[F_3^i(t)]$ and the maximum fidelity $\max[F_3^i(t)]$ versus $\varepsilon$. In Figure 3c,d, we give the time-evolution of $F_3^i(t)$ and $F_3^i(t)$ in the case of $\varepsilon = 0.02$. The fidelity $F_3^i(t)$ and the fidelity $F_3^i(t)$ satisfy the two inequations, that is, $F_3^i(t) \geq (1 - \frac{8\varepsilon^2}{1 + 4\varepsilon^2})^2$ and $F_3^i(t) \leq \frac{16\varepsilon^2}{(1 + 4\varepsilon^2)^2}$.

4. Lipkin–Meshkov–Glick Model

In addition to single-particle systems, symmetry-protected quantum adiabatic evolutions may also appear in many-body quantum systems driven through an SSB transition. Below we consider the Lipkin–Meshkov–Glick model. The Hamiltonian is $H_{LMG}$

$$H_{LMG}(t) = \frac{B(t)}{2} \sum_{i} \sigma_i^z + \frac{J}{2N} \sum_{i<j} \sigma_i^x \sigma_j^x$$

(24)

with the Pauli operators $\sigma_i^x$, the homogeneous spin interaction $J/N$, the time-varying transverse magnetic field $B(t)$, and the total spin number $N$. The Lipkin–Meshkov–Glick model is equivalent to a symmetric Bose–Josephson junction $H_{LMG}$ and have been realized in experiments. This model is invariant under the transformation: $\sigma_i^+ \rightarrow -\sigma_i^-$, $\sigma_i^- \rightarrow -\sigma_i^+$, $\sigma_i^z \rightarrow -\sigma_i^z$. By defining the parity operator, $\hat{P} = e^{-i\pi/2} \sum \sigma_i^z$ for even $N$ and $\hat{P} = -ie^{-i\pi/2} \sum \sigma_i^z$ for odd $N$ which has two eigenvalues $\pm 1$, respectively representing even and odd parity, we have $[H_{LMG}, \hat{P}] = 0$. If the Hamiltonian
**Figure 2.** a,b) The changes of barrier height $A(t)$ with time for adiabatic-parameter-fixed sweeping and linear sweeping. c,d) The comparison between our adiabatic-parameter-fixed sweeping and the linear sweeping within the same evolution duration $T$. The evolved population in the instantaneous ground state $F_1$ when c) $T \approx 17$ and d) $T \approx 33$, respectively.

**Figure 3.** The dynamical evolution of the single-particle system within a symmetric 1D potential varying from single-well to double-well. a) The minimal fidelity in the instantaneous ground state $\min[F_1(t)]$ (adiabatic-parameter-fixed sweeping) versus the adiabatic parameter $\epsilon$. The blue solid lines denote the analytic results for $\min[F_1(t)]$. b) The maximum fidelity in the instantaneous second excited state $\max[F_3(t)]$ (adiabatic-parameter-fixed sweeping) versus the adiabatic parameter $\epsilon$. The blue solid lines denote the analytic results for $\max[F_3(t)]$. c) The evolution of $F_1(t)$ for $\epsilon = 0.02$. d) The evolution of $F_3(t)$ for $\epsilon = 0.02$. 

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Figure 4. The dynamics of the Lipkin–Meshkov–Glick model (24) with $N = 100$ driven through an SSB transition. a) The energy spectrum. b) The populations in the lowest three instantaneous eigenstates $F^+_1, F^-_2, F^+_3$ versus the magnetic field $B(t)$ for $\epsilon = 0.05$. c) The final average fidelity $F^+_1$ versus $\epsilon$, in which the star ($\star$) denotes the starting point of the average. The inset d) shows the population dynamics in the instantaneous ground state $F^+_1$ for different $\epsilon$. The blue solid line denotes the analytic lower bound of $F^+_1(t)$.

is dominated by the first term, the ground state is a paramagnetic state of all spins aligned along the magnetic field $B$. If the Hamiltonian is dominated by the second term and $J < 0$, there appear two degenerate ferromagnetic ground states of all spins in either spin-up or spin-down. Thus any superposition of these two ground states is also a ground state, in which the equal-probability superposition of these two states is known as a GHZ state. [67–69] Below we concentrate on discussing the system of $J < 0$, in which an SSB occurs at the critical point $|B_c/J| = 1$ when $N \to \infty$. Accompanying with the SSB, the two lowest eigenstates change from non-degenerate to degenerate. Initially, the system is dominated by the transverse magnetic field (i.e., $|B| \gg |J|$), the energy levels are non-degenerate, and the eigenstates alternate appear with even and odd parity. Fixing the spin interaction, when the transverse magnetic field $B(t) < B_c$, the neighboring pairs of eigenstates $\{|\phi^\pm_{2n}(t)\rangle, |\phi^\mp_{2n}(t)\rangle\}$ ($n = 1, 2, 3, \ldots$) of different parity become degenerate (or quasi-degenerate for finite $N$). It is worth to mention that, the minimum energy gap between the ground state and the second excited state is inversely proportional to the cube root of particle number, that is, $E_2 - E_1 \propto N^{-1/3}$. [42,74] When $N \to \infty$, the gap $E_2 - E_1 \to 0$ also vanishes, the non-adiabatic excitation will inevitably occur, which is consistent with the studies on Kibble–Zurek mechanism in quantum Ising model [53,56] and Lipkin–Meshkov–Glick model [47,48,57].

For a realistic system, as its $N$ is finite, according to the SDAC (3), adiabatic evolutions may still appear due to there always have finite energy gaps between eigenstates of the same parity. In Figure 4a, we show the energy spectrum and the population...
Thus, the transverse magnetic field is gradually varied from \( B(0) \) to \( B(T) = 0 \) across the transition point \( B = 1 \). The sweeping process is described as \( B(t) = \int_0^t v_B(t')dt' \) with the sweeping rate \( v_B(t) = \frac{|\phi_{E_1}(0)\rangle \langle E_3|\phi_{E_1}(t)|\phi_{E_1}(0)\rangle|^2}{\langle E_3|\phi_{E_1}(0)\rangle \langle \sum_{i=1}^N \sigma_i^+ |\phi_{E_1}(0)\rangle} \) determined according to Equation (4).

In Figure 4b, given \( \epsilon = 0.05 \), we show the fidelities versus the instantaneous magnetic field \( B(t) \). Due to the two lowest instantaneous eigenstates have different symmetry and the system is driven from the ground state, as a result of the symmetry-protected transition, the instantaneous first-excited state is never occupied, see \( F_1(t) \) in Figure 4b. Most of population stays in the instantaneous ground state and only small amount jumps to the instantaneous second-excited state, see \( F_3(t) \) and \( F_3(t) \) in Figure 4b. In general, the final fidelities (at \( B=0 \)) oscillate with adiabatic parameter. To eliminate the oscillation and smooth the curve monotonously, we analyze the final average fidelity \( F_\infty = \frac{1}{T} \int_0^T F_1(t)dt \) with \( T \) denoting the instant that the middle point of the first oscillation after the transition point. This can be used as an indicator for finding out how slow the sweeping is practical. In Figure 4c, we show \( F_1^* \) versus \( \epsilon \). Clearly, \( F_1^* \) is always above the analytical lower bound (the solid blue line) given by (21).

Here, we mention some advantages of our adiabatic-parameter-fixed sweeping scheme. First, the adiabaticity of our adiabatic-parameter-fixed sweeping is better than the one of the linear sweeping. For the same time-evolution duration \( T \), our scheme always has a higher final average fidelity \( F_1^* \), see Figure 5a,b. Second, for a given adiabatic parameter \( \epsilon \), the final average fidelities \( F_1^* \) and \( F_1^* \) remain almost the same for different \( N \) (Figure 5c). Although the energy gap decreases with the total spin number \( N \) (Figure 5d), the final average fidelities are almost independent on \( N \) for a given \( \epsilon \) (Figure 5d).

In Figure 6, we show the change of \( B(t) \) for the linear sweeping (blue solid lines) and the adiabatic-parameter-fixed sweeping (red dashed lines). For a given adiabatic parameter \( \epsilon \), \( B(t) \) changes differently with total particle number \( N \) since the energy spectra are different with \( N \). It is shown that, for the same \( \epsilon \), as \( N \) is getting larger, the required total evolution duration \( T \) is longer.

Finally, we demonstrate the time-evolution of the fidelities for \( N = 10 \) and \( N = 100 \). When \( \epsilon \) is small, the population only occupy between the ground state \( |\phi_{E_1}(t)\rangle \) and the second excited state \( |\phi_{E_2}(t)\rangle \), which coincides with our assumption. The fidelity \( F_1^* \) and the fidelity \( F_3^* \) also satisfy the two inequations, that is, \( F_1^* (t) \geq 1 - \frac{8\epsilon^2}{(1+4\epsilon)^2} \) and \( F_3^* (t) \leq \frac{16\epsilon^2}{(1+4\epsilon)^2} \). The numerical
results confirm our analytical derivation of inequality (21) (Figure 7).

5. Robustness

In realistic experiments, a systematic bias due to experimental imperfections or a time-dependent random bias caused by stochastic noise may break the parity symmetry of the system. Here, we investigate the influences of bias on the symmetry-dependent adiabatic evolution.

First, we discuss the systematic bias due to experimental imperfections, which is independent on time. For the first example of single particle, the bias may cause the imbalance between the two wells, and the system Hamiltonian can be described by

$$\hat{H}_S(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + A(t)e^{-\frac{x^2}{2\sigma^2}} + \Lambda x$$

(26)

where $\Lambda$ corresponds to a gradient that imbalance the double-well. We show the influence of bias in single-particle system in Figure 8. When $\Lambda$ is small, even though the probability of transferring to the first excited state with odd-parity $|\phi_i^{+}\rangle$ is no longer zero, the transition from $|\phi_{E_{1}}^{+}\rangle$ to $|\phi_{E_{2}}^{+}\rangle$ still dominates.

For the second example of Lipkin–Meshkov–Glick model, the bias corresponds to the longitude field. The system Hamiltonian is

$$\hat{H}_{LMG}(t) = B(t) \sum_{i=1}^{N} \sigma_{i}^{x} + J \sum_{i<j}^{N-1} \sigma_{i}^{z} \sigma_{j}^{z} + \eta \sum_{j=1}^{N} \sigma_{j}^{z}$$

(27)

with $\eta$ the bias. Similarly, when $\eta$ is small enough, the signature of symmetry-dependent adiabatic evolution still preserve.

Next, we turn to discuss the influence of time-dependent random fluctuating bias in the Lipkin–Meshkov–Glick model. In this case, the bias fluctuates randomly with time, $\eta \rightarrow \eta(t)$ with $\eta(t) = 0$ and $\eta(t) \in [-\tilde{\eta}, \tilde{\eta}]$, as shown in Figure 9. We find that, the symmetry-dependent adiabatic evolution can occur even when the strength of the randomly fluctuating bias is much larger compared with spin interaction strength $J$.

Further, we study the influence of a fixed bias on the population of the first excited state with different adiabatic parameter $\epsilon$. We find that, the smaller the adiabatic parameter $\epsilon$ is, the higher population that jump to the first excited state. That is, if the parameter changes more slowly, the influences of the parity breaking term is more obvious, as shown in Figure 10.

Our numerical simulation clearly indicates that, even though the bias breaks the parity symmetry, the symmetry-dependent adiabatic evolution can still tolerant small experimental imperfections and random experimental noises, which is robust and feasible in realistic experiments.
Figure 8. Influences of bias in single-particle system when $\varepsilon = 0.05$. The evolution of fidelities under a) $\Lambda = 0.0005$ and b) $\Lambda = 0.001$. The enlarged details for $F_2^-$ and $F_3^+$ under c) $\Lambda = 0.0005$ and d) $\Lambda = 0.001$.

Figure 9. Influences of bias in the Lipkin–Meshkov–Glick model when $\varepsilon = 0.05$. a,b) The evolution of fidelities in the presence of a fixed bias a) $\eta = 0.001$ and b) $\eta = 0.002$. c,d) The evolution of fidelities in the presence of a randomly fluctuating bias with maximal amplitude of c) $\tilde{\eta} = 0.1$ and d) $\tilde{\eta} = 1$. Here, $N = 10$ and $J = -0.05$. 
Figure 10. Influences of a fixed bias on the population of the first excited state with different adiabatic parameter $\varepsilon$. a,b) The evolution of population on the first excited state in single particle system with a fixed bias $\Lambda = 0.001$ and $\Lambda = 0.0005$. c,d) The evolution of population on the first excited state in the Lipkin–Meshkov–Glick model with a fixed bias $\eta = 0.01$ and $\eta = 0.02$. [Here, $N = 10$ and $J = -0.05$.]

6. Conclusions

We have studied the time-evolution dynamics in a slowly driven system with degeneracy change and time independent symmetry. Due to the commutativity between the symmetry and the Hamiltonian, the population transition is protected by the symmetry. We give a general derivation to revisit the symmetry-dependent adiabatic condition (SDAC). Based on SDAC, we use an adiabatic-parameter-fixed sweeping scheme for state preparation and analytically obtain a lower bound for the ground state fidelity dependent only on the adiabatic parameter $\varepsilon$. Our numerical results confirm the analytic bound in both single- and many-particle quantum systems. Even if there exist weak symmetry-breaking sources, such as static bias and stochastic noises, the population transfer may be still dominated by the transitions between states with same symmetry. This means that the symmetry-dependent adiabatic evolution is robust against weak symmetry-breaking sources. Our study is general and can be applied to various systems.

Our adiabatic-parameter-fixed sweeping scheme is simple and the time-dependent sweeping rate can be completely determined by the static energy spectrum. Besides, based on the analytical lower bound of the ground state fidelity, one can easily choose a proper adiabatic parameter for the desired fidelity, and design the sweeping process in advance. Our scheme is different from the state preparation via optimal quantum control.\cite{76} We focus on the adiabatic quantum evolution where the whole process is nearly adiabatic, and the fidelity of staying in a certain target eigenstate is always high. However, for the quantum optimal control methods, the aim is to achieve a final target state with high fidelity. The evolution process involve non-adiabatic excitations. Even for the same target, it may be hard to design the whole process with optimal control methods and consumes much more computing resources, especially when the particle number becomes large.

In particular, our findings can be tested via the techniques in quantum annealing. Our first example can be realized by the quantum annealing of a single superconducting flux qubit\cite{19,20} by switching off the energy bias. Our second example can be implemented by the quantum annealing in a programmable D-wave system\cite{21} from transverse field limit to Ising interaction limit in the absence of local fields. Our work is also related to the recent experiment work for probing quantum criticality and symmetry breaking via Dysprosium atoms.\cite{75} It is possible to verify our theory in that platform. Besides, our findings also can be applied for the constrained quantum annealing\cite{65,77,78} and speedup the adiabatic process in quantum computation.\cite{79,80} Our study will not only deepen the understandings of quantum adiabatic evolution and SSB transitions, but also provide promising applications ranging from quantum state engineering to topological Thouless pumping.

Appendix A

A.1. Derivation of the Symmetry-Protected Transition

In this section, we give the proof of Equations (1) and (2), which describe the symmetry-protected transition. Assume the Hamiltonian $\hat{H}(\mathbf{R}(t)) = \sum_{i=1}^{K} \hat{R}_i(t) \hat{H}_i$ with $K$ time-varying parameters $\mathbf{R}(t) = [R_1(t), R_2(t), R_3(t), \ldots, R_K(t)]$, and has at least one time-independent symmetry $\hat{Y}$ that commutes with the Hamiltonian.
Thus, if \( \hat{H} = \hat{Y} \), the operator \( \hat{Y} \) and the Hamiltonian \( \hat{H}(R(t)) \) have a set of simultaneous eigenstates: \( \{ \phi_{\lambda_n}^{R}(R(t)) \} \). We use \( E_n \) and \( \lambda_n \) stand for n-th and \( \alpha \)-th eigenvalues of \( \hat{H}(R(t)) \) and \( \hat{Y} \), respectively.

We write the eigen-equations,

\[
\hat{H}(R(t))|\phi_{\lambda_n}^{R}(R(t))\rangle = E_n(R(t))|\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A1}
\]

and

\[
\hat{Y}|\phi_{\lambda_n}^{R}(R(t))\rangle = \lambda_n|\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A2}
\]

in which

\[
E_1(R(t)) \leq E_2(R(t)) \ldots \leq E_N(R(t))
\]

with \( n = 1, \ldots, N \).

By differentiating Equation (A2) with respect to time, we obtain

\[
\frac{d}{dt}\left[\hat{Y}|\phi_{\lambda_n}^{R}(R(t))\rangle \right] = \hat{Y} \frac{d}{dt}|\phi_{\lambda_n}^{R}(R(t))\rangle = \lambda_n|\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A3}
\]

where \( |\phi_{\lambda_n}^{R}(R(t))\rangle = \frac{1}{\sqrt{2}}|\phi_{\lambda_n}^{R}(R(t))\rangle \). By taking the inner product with \( \langle \phi_{\lambda_n}^{R}(R(t)) \rangle \), we obtain

\[
\left\langle \phi_{\lambda_n}^{R}(R(t)) \right| \frac{d}{dt}|\phi_{\lambda_n}^{R}(R(t))\rangle \right) = (\lambda_n - \lambda_n) = 0 \tag{A4}
\]

Thus, if \( \lambda_n \neq \lambda_n \), the above equation requests

\[
\left\langle \phi_{\lambda_n}^{R}(R(t)) \right| \frac{d}{dt}|\phi_{\lambda_n}^{R}(R(t))\rangle \right) = 0 \tag{A5}
\]

On the other hand, the time-evolution of the eigenstate \( |\phi_{\lambda_n}^{R}(R(t))\rangle \) obeys the Schrödinger equation,

\[
i\hbar \frac{d}{dt}|\phi_{\lambda_n}^{R}(R(t))\rangle = \hat{H}(R(t))|\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A6}
\]

Thus, substituting Equation (A6) into the Equation (A5), we have

\[
H_{\lambda_n}(t) = \langle \phi_{\lambda_n}^{R}(R(t)) | \hat{H}(R(t)) | \phi_{\lambda_n}^{R}(R(t)) \rangle = 0 \tag{A7}
\]

for \( \lambda_n \neq \lambda_n \).

### A.2. Derivation of the Symmetry-Dependent Adiabatic Condition

In this section, we give the detailed derivation to revisit the symmetry-dependent adiabatic condition (SDAC) [Equation (3) in the main text]. We start from the time-dependent Schrödinger equation,

\[
i\hbar \frac{d}{dt}|\psi(R(t))\rangle = \hat{H}(R(t))|\psi(R(t))\rangle \tag{A8}
\]

At any instant, the instantaneous state \(|\psi(R(t))\rangle\) can be expanded in terms of the complete basis of \(|\phi_{\lambda_n}^{R}(R(t))\rangle\),

\[
|\psi(R(t))\rangle = \sum_n \sum_{\alpha} \alpha_n^{\alpha}(t) \exp \left[ \frac{i}{\hbar} \int_0^t E_n(R(t'))dt' \right] |\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A9}
\]

For non-degenerate situation, there is a one-to-one correspondence between certain \( n \) and \( \alpha \). Thus, the second sum about \( \alpha \) can be cancel. Inserting the above expansion (A9) into the Schrödinger equation (A8), we obtain

\[
\sum_n \sum_{\alpha} \exp \left[ \frac{i}{\hbar} \int_0^t E_n(R(t'))dt' \right] \left\{ \alpha_n^{\alpha}(t) \exp \left[ \frac{i}{\hbar} \int_0^t E_n(R(t'))dt' \right] \right\} \times |\phi_{\lambda_n}^{R}(R(t))\rangle = 0 \tag{A10}
\]

Taking the inner product with \( \langle \phi_{\lambda_n}^{R}(R(t))\rangle\) \( \exp \left[ \frac{-i}{\hbar} \int_0^t E_n(R(t'))dt' \right] \), the differential equation for the coefficients are

\[
\alpha_n^{\alpha}(t) = - \sum_{\alpha} \sum_{\beta} \alpha_n^{\alpha}(t) \exp \left[ \frac{i}{\hbar} \int_0^t \left[ E_n(R(t')) - E_n(R(t')) \right]dt' \right] \times \langle \phi_{\lambda_n}^{R}(R(t))\rangle \langle \phi_{\lambda_n}^{R}(R(t))\rangle \tag{A11}
\]

By using the result of Equation(2) in the main text, substituting Equation (A5) into Equation (A11), we have

\[
\alpha_n^{\alpha}(t) = - \sum_{\alpha} \sum_{\beta} \alpha_n^{\alpha}(t) \exp \left[ \frac{i}{\hbar} \int_0^t \left[ E_n(R(t')) - E_n(R(t')) \right]dt' \right] \times \langle \phi_{\lambda_n}^{R}(R(t))\rangle \langle \phi_{\lambda_n}^{R}(R(t))\rangle \delta_{\lambda_n,\lambda_n} \tag{A12}
\]

Therefore, the Hilbert space of the quantum system can be partitioned into different subspaces according to the eigenvalues of the symmetry operator \( \hat{Y} \), and the transition between the states with different eigenvalues of \( \hat{Y} \) is forbidden in the dynamical evolution process.

Without loss of generality, we first give a general derivation to revisit the SDAC for the degenerate quantum system and then relax it to the non-degenerate cases. In the following, \( \tilde{H}_{\lambda_n}(R(t)) \) denotes the n-th degenerate subspace of \( \hat{H}(R(t)) \) with the eigenenergy \( E_n \). The degeneracy of each subspace \( \tilde{H}_{\lambda_n}(R(t)) \) is denoted by \( d_n \) and each energy eigenstate has different symmetry (i.e., \( \lambda_n \neq \lambda_n \) for \( a \neq \beta \) and \( a, \beta = 1, \ldots, d_n \)). That is, the degenerate energy eigenstates in the subspace \( \tilde{H}_{\lambda_n}(R(t)) \) can be distinguished from each other by the symmetry \( \hat{Y} \). We use \( |\phi_{\lambda_n}^{R}(R(t))\rangle \) to denote the instantaneous eigenstates in n-th degenerate subspace \( \tilde{H}_{\lambda_n}(R(t)) \). According to Equation (A12), the transitions between states in the same degenerate subspace \( \tilde{H}_{\lambda_n}(R(t)) \) are forbidden if all degenerate energy eigenstate possess different values of \( \lambda_n \). Thus, starting from the initial state \(|\psi(0)\rangle = |\phi_{\lambda_n}^{R}(0)\rangle\), the evolved state \(|\psi(R(t))\rangle\) is given as

\[
|\psi(R(t))\rangle = \sum_n \sum_{\alpha} \alpha_n^{\alpha}(t) \exp \left[ \frac{i}{\hbar} \int_0^t E_n(R(t'))dt' \right] |\phi_{\lambda_n}^{R}(R(t))\rangle \tag{A13}
\]
From the time-dependent Schrödinger equation, the differential equation for the coefficients in Equation (6) can be written as

$$\frac{d}{dt}\phi_{E_n}(R(t)) = -iA_{E_n}^a(t)\phi_{E_n}(R(t)) + \sum_{m\neq n} A_{E_m}^a(t)\phi_{E_m}(R(t))$$

and therefore, the SDAC can be written as

$$\langle \phi_{E_n}(R(t))\rangle = \int \phi_{E_n}(R(t))P_0(R(t))dR(t)$$

so that the second term in Equation (A14) can be dropped. Here, $E_m$ and $E_n$ stand for the instantaneous eigenenergies of $H(t)$, and they respectively belong to different subspaces $H_m(R(t))$ and $H_n(R(t))$. Since $i\hbar \frac{d}{dt} = \hat{H}(t)$, $H_{nm}(t) = i\hbar \langle \phi_{E_n}(R(t))|\phi_{E_m}(R(t))\rangle$, and therefore, the SDAC can be written as

$$\epsilon(t) = \max_{(m)} \left| \frac{\hbar\langle \phi_{E_n}(R(t))|\phi_{E_m}(R(t))\rangle}{E_n - E_m} \right| \ll 1 \text{ with } m \neq n$$

By taking the time derivative on both sides, we also obtain

$$\frac{\partial}{\partial t}\langle \phi_{E_n}(R(t))\rangle + \hat{H}(t)|\phi_{E_n}(R(t))\rangle = -\frac{\partial}{\partial t}\langle \phi_{E_m}(R(t))\rangle + \hat{H}(t)|\phi_{E_m}(R(t))\rangle$$

and

$$\langle \phi_{E_n}(R(t))\rangle = \frac{\hbar\langle \phi_{E_n}(R(t))|\phi_{E_m}(R(t))\rangle}{E_n(R(t)) - E_m(R(t))}$$

Therefore, the SDAC (A16) can also be rewritten as

$$\epsilon(t) = \max_{(m)} \left| \frac{\hbar\langle \phi_{E_n}(R(t))|\phi_{E_m}(R(t))\rangle}{[E_n(R(t)) - E_m(R(t))]^2} \right| \ll 1$$

with $m \neq n$ (A19)

A.3. Parity Operator

In the final section, we give a brief introduction about the parity operator $\hat{P}$. The parity operator $\hat{P}$ is defined as an operation of space/spin inversion. The parity operator $\hat{P}$ has the following properties

$$\hat{P}^2 = 1, \quad \hat{P} = P^i$$

As it turns out, the parity operator $\hat{P}$ can only ever have two eigenvalues $\xi = \pm 1$. The parity eigenvalue equations are given as

$$\hat{P}|\xi\rangle_{\text{even}} = +1|\xi\rangle_{\text{even}}$$

and

$$\hat{P}|\xi\rangle_{\text{odd}} = -1|\xi\rangle_{\text{odd}}$$

This implies that the parity eigenstates will either be the same or be the opposite with their original ones under the space/spin inversion. If the sign doesn’t change, the state $|\xi\rangle_{\text{even}}$ is symmetric under space inversion (called even). But, if the sign does change, the state $|\xi\rangle_{\text{odd}}$ is antisymmetric under space inversion (called odd). For different quantum systems, the parity operator has different definitions, but they share common properties [Equation (A20)]. If the parity operator $\hat{P}$ commutes with the Hamiltonian $\hat{H}(R(t))$ of the system, we called the system has the parity symmetry.

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Conflict of Interest

The authors declare no conflict of interest.

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