Sandwiched Volterra Volatility model: Markovian approximations and hedging

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Abstract

We consider stochastic volatility dynamics driven by a general Hölder continuous Volterra-type noise and with unbounded drift. For these so-called SVV-models, we consider the explicit computation of quadratic hedging strategies. While the theoretical hedge is well-known in terms of the non-anticipating derivative for all square integrable claims, the fact that these models are typically non-Markovian provides a challenge in the direct computation of conditional expectations at the core of the explicit hedging strategy. To overcome this difficulty, we propose a Markovian approximation of the model which stems from an adequate approximation of the kernel in the Volterra noise. We study the approximation of the volatility, of the prices and of the optimal mean-square hedge. We provide the corresponding error estimates. The work is completed with numerical simulations.

Keywords: stochastic volatility, sandwiched process, Hölder continuous noise, hedging, Monte Carlo methods

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1 Introduction

A significant number of modern financial market models are characterized by the absence of the Markov property. For a detailed discussion of such a choice of modeling framework, we refer the reader to a recent review \cite{29}; here we only note that the non-Markovian structure is substantiated by several empirical considerations coming from the analysis of financial time series and implied volatility (IV) surfaces.

- A number of studies (see e.g. \cite{10, 11, 17, 35} or \cite{24, 25}) report the presence of long memory in financial markets. Additionally, Comte and Renault \cite{23} note that the observed IV smile amplitude decreases much slower than predicted by standard models, when the time to maturity $T \to \infty$, which can also be interpreted as a manifestation of volatility persistence. These observations resulted in multiple works utilizing long memory processes such as fractional Brownian motion (fBm) with Hurst index $H > 1/2$ for volatility modeling (see e.g. \cite{16, 21, 22} or \cite{58}).

- Other studies (such as \cite{11, 45}) estimate the Hölder order of the market volatility to be in the vicinity of 0.1. Additionally, this idea of modeling volatility by a process with low regularity is supported by the behavior of the short-term skew slope of the at-the-money IV \cite{8, 39}. These facts gave birth to a number of so-called rough volatility models and, while roughness in itself does not necessarily yield the absence of Markovianity, many rough volatility models in the literature (see e.g. \cite{8} Section 7.2) as well as \cite{15, 38, 40, 47} exploit the structure of fBm with $H < 1/2$ and hence are non-Markovian.
It should be noted, however, that fBm-based models are far from perfect, despite being typical choices for introducing long memory/roughness.

(i) In fractional models, there seems to be an inherent contradiction in the choice of the Hurst index $H$: long memory and the behavior of the IV surfaces for longer maturities yield $H > 1/2$ whereas the power law of the IV skew slope and roughness demand $H < 1/2$, see e.g. the discussion in [42, 43] or [7, Section 7.7].

(ii) In rough volatility models (such as rough Stein-Stein [2, 17] or rough Heston [30, 33, 37]), there is often no transparent procedure of transition between physical and pricing measures: it is often not clear whether the volatility process $\sigma = \{\sigma(t), \ t \in [0, T]\}$ hits zero. Hence the integral $\int_0^T \frac{1}{\sigma(t)} ds$ that is typically present in densities of martingale measures may be undefined.

(iii) In general, stochastic volatility models are susceptible to moment explosions, i.e. the expectations of the prices $\mathbb{E}[S^2(T)]$ may be infinite for all big enough $T$, see e.g. [9, 46, 49] or the article “Moment Explosions” in [26]. Such a technical property can be detrimental in many ways: it may result in infinite prices [9, Section 8], invalidate error estimates in numerical schemes [6, Section 4.2], and cause infinite expected utility in optimization problems [48]. In addition, infinite second moments of prices rule out quadratic hedging tools which is a big disadvantage for models that normally produce incomplete markets.

Moreover, even if a fractional model somehow overcomes the problems (i)–(iii), it still produces substantial difficulties from the point of view of stochastic methods, especially in optimization problems. As an example, consider a mean-variance hedging problem of the form

$$\inf_{u} J(u) := \inf_{u} \mathbb{E} \left[ \left( F - \int_0^T u(s) dX(s) \right)^2 \right],$$  \hspace{1cm} (1.1)$$

where $X$ is a square-integrable non-Markovian martingale w.r.t. some filtration $F = \{F_t, \ t \in [0, T]\}$ denoting a discounted risk-neutral price, $F = f(X(T))$ is a square integrable financial claim and the infimum in (1.1) is taken over all $F$-adapted $X$-integrable strategies. From the theoretical perspective, the problem (1.1) is well understood: for martingale discounted prices $X$, the existence of the solution to (1.1) is guaranteed by the celebrated Galtchouk-Kunita-Watanabe decomposition theorem (see e.g. [57, 59]). Furthermore, [28] gives the explicit representation of the optimal hedging strategy: according to [28, Theorem 2.1], the optimal hedging portfolio $u$ minimizing (1.1) can be written as the non-anticipating derivative $DF$ defined as the $L^2$-limit of simple processes

$$DF := L^2 \lim_{|\pi| \to 0} u_{\pi}, \quad u_{\pi} := \sum_{k=0}^{n-1} \mathbb{E} \left[ \left( X(t_{k+1}) - X(t_k) \right) F \bigg| F_{t_k} \right] \mathbb{E} \left[ \left( X(t_{k+1}) - X(t_k) \right)^2 \bigg| F_{t_k} \right]^{-1/2},$$  \hspace{1cm} (1.2)$$

where $|\pi| := \max_{k}(t_k - t_{k-1})$ denotes the mesh of the partition $\pi = \{0 = t_0 < t_1 < ... < t_n = T\}$. Note that (1.2) is explicit in the sense that the hedge is written only in terms of the discounted price model, the information flow of reference, and the claim $F$. Nevertheless, the practical use of (1.2) is still limited: analytical expressions for $DF$ are usually impossible to obtain whereas dependence of $X$ on the past significantly complicates numerical computation of the conditional expectations in (1.2) by e.g. Monte Carlo methods.

**Modeling framework: outline and main features.** In this paper, we consider the numerical aspect of the mean-variance hedging problem (1.1) within the Sandwiched Volterra Volatility (SVV) model initially introduced in [30, 33] in the option pricing setting. Namely, we assume that the price $S = \{S(t), \ t \in [0, T]\}$ of a risky asset has the risk-free dynamics of the form

$$S(t) = S(0) + \int_0^t \nu(s) S(s) ds + \int_0^t Y(s) S(s) \left( \rho dB_1(s) + \sqrt{1 - \rho^2} dB_2(s) \right),$$  \hspace{1cm} (1.3)$$

$$Y(t) = Y(0) + \int_0^t b(s, Y(s)) ds + Z(t),$$  \hspace{1cm} (1.4)$$

$$X(t) = \exp \left\{ - \int_0^t \nu(s) ds \right\} S(t),$$  \hspace{1cm} (1.5)$$

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where $S(0)$ and $Y(0)$ are given constants, $\nu \in C([0,T])$ is a deterministic instantaneous interest rate, $(B_1, B_2) = \{(B_1(t), B_2(t)), \ t \in [0,T]\}$ is a standard 2-dimensional Brownian motion, $\rho \in (-1,1)$, $Z(t) = \int_0^t K(t,s) dB_1(s)$ is a H"older continuous Gaussian Volterra process. The drift $b = b(t,y)$ in (1.4) is unbounded and has an explosive growth to $+\infty$ whenever $y \to \varphi(t)+$ and explosive decay to $-\infty$ whenever $y \to \psi(t)-$, where $0 < \varphi < \psi$ are two deterministic H"older continuous functions. This structure ensures that the volatility process $Y = \{Y(t), \ t \in [0,T]\}$ is sandwiched between $\varphi$ and $\psi$, i.e.

$$0 < \varphi(t) < Y(t) < \psi(t), \quad t \in [0,T].$$

The SVV model (1.3)–(1.5) has several important advantages.

- In line with the recent papers [4] [55], the noise driving the volatility is a general Gaussian Volterra process. This family is very broad and allows bypassing problem (i) by e.g. using several fBms with different Hurst indices as suggested in [33] and [7] Section 7.7 or utilizing multifractional Brownian motion with time-varying local regularity as in [13] [27]. For example, according to [33] Theorem 4.8 and Example 4.9), (1.3)–(1.5) can reproduce the IV skew power law and long memory simultaneously if $\mathcal{K}$ is a linear combination of fractional kernels.

- The lower bound in (1.6) guarantees that the volatility $Y$ stays strictly positive. In general, this property allows modeling the market under the physical measure and switching to the pricing measure when required. In this case, [30] Subsection 2.2] gives the full description of all equivalent local martingale measures.

- The upper bound in (1.6) ensures that the price $S$ has moments of all orders, which allows us to consider the quadratic hedging problem (1.1) in the first place.

- In addition to addressing problems (i)–(iii), the SVV model has efficient simulation schemes preserving the property (1.6) [31] and generates Malliavin differentiable volatility and price [30] [33].

We tackle the numerical computation of (1.2) in three stages.

I. First, we construct a finite-dimensional Markovian approximation to the SVV model (1.3)–(1.4) by using the approach in the spirit of [3] [5] [18] [19]. Namely, we approximate $\mathcal{K}$ with a sequence of degenerate kernels $\{\mathcal{K}_m, m \geq 1\}$

$$\mathcal{K}_m(t,s) = \sum_{i=0}^m e_{m,i}(t)f_{m,i}(s)\mathbb{1}_{s<t}$$

and then exploit the Markovianity of the $(m+2)$-dimensional process $(X_m, Y_m, U_{m,0}, ..., U_{m,m})$, where

$$X_m(t) = X(0) + \int_0^t Y_m(s)X_m(s) \left(\rho dB_1(s) + \sqrt{1-\rho^2}dB_2(s)\right),$$

$$Y_m(t) = Y(0) + \int_0^t b(s, Y_m(s))ds + \sum_{i=1}^m e_{m,i}(t)U_{m,i}(t),$$

$$U_{m,i}(t) = \int_0^t f_{m,i}(s)dB_1(s), \quad i = 0,1,\ldots,m.$$  

II. Second, we plug (1.7) in the conditional expectations from (1.2) and prove that the result converges to $\mathcal{D}F$ in $L^2$.

III. Finally, we employ the Markovian structure of (1.7) to numerically estimate conditional expectations $E[(X_m(t_{k+1}) - X_m(t_k))F(X_m(T)) \mid \mathcal{F}_{t_k}]$ and $E[(X_m(t_{k+1}) - X_m(t_k))^2 \mid \mathcal{F}_{t_k}]$. We propose two algorithms: Nested Monte Carlo (NMC) and Least Squares Monte Carlo (LSMC).

Note that the convergence of (1.7) as $m \to \infty$ is not trivial due to the explosive nature of the drift $b$. For this reason, we have to use our own technique based on pathwise estimates from [32].
Structure of this work. Section 2 contains the detailed description of the SVV model (1.3)–(1.5) and gathers some necessary known results. Section 3 is devoted to Stage I of our plan above: we introduce the approximation (1.7) and prove its convergence to the original SVV model. In Section 4, Stage II is realized: we construct an approximation of the optimal hedging strategy \( D_F \) and study its convergence. Stage III is implemented in Section 5 which concentrates on numerical algorithms for computing the optimal hedge and describes two Monte Carlo approaches for the computation of the non-anticipating derivative; the results are illustrated by simulations. Appendices A and B contain the proofs of some technical results.

2 Sandwiched Volterra Volatility Model

Hereafter we give detailed specifications for the model (1.3)–(1.5) and collect some fundamental results for the upcoming discussion. Before proceeding to the main content, let us introduce some notation used in the sequel.

Notation.

1. The model (1.3)–(1.4) is defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a \(\mathbb{P}\)-augmented filtration \(\mathcal{F} = \{\mathcal{F}_t, \ t \in [0, T]\}, \ T < \infty\), generated by the 2-dimensional Brownian motion \((B_1, B_2) = \{(B_1(t), B_2(t)), \ t \in [0, T]\}.

2. In what follows, by \(C\) we will denote any positive deterministic constant the exact value of which is not important in the context. Note that \(C\) may change from line to line (or even within one line).

3. Throughout the paper, \(L^2(\mathbb{P})\) will denote a standard space of square-integrable random variables. For a given square-integrable martingale \(\eta = \{\eta(t), \ t \in [0, T]\}\), the Hilbert space of stochastic processes such that

\[
\|u\|_{L^2(\mathbb{P} \times [\eta])} := \left( \mathbb{E} \left[ \int_0^T u^2(s)d[\eta](s) \right] \right)^{\frac{1}{2}} < \infty
\]

will be denoted by \(L^2(\mathbb{P} \times [\eta])\).

2.1 Volterra noise

Consider a function \(K: [0, T]^2 \rightarrow \mathbb{R}\) that satisfies the following assumption.

Assumption (K). The function \(K: [0, T]^2 \rightarrow \mathbb{R}\) is a Volterra kernel, i.e. \(K(t, s) = 0\) whenever \(t < s\), and the following conditions hold:

(K1) \(K\) is square integrable, i.e.

\[
\int_0^T \int_0^T K^2(t, s)dsdt = \int_0^T \int_0^t K^2(t, s)dsdt < \infty;
\]

(K2) there exists a constant \(H \in (0, 1)\) such that, for any \(\lambda \in (0, H)\),

\[
\int_0^t (K(t, u) - K(s, u))^2du \leq \ell_3|t - s|^{2\lambda}, \quad 0 \leq s \leq t \leq T,
\]

where \(\ell_3 > 0\) is a constant, possibly dependent on \(\lambda\).

For a fixed \(K\) satisfying Assumption (K) we define a Gaussian Volterra process \(Z = \{Z(t), \ t \in [0, T]\}\) as

\[
Z(t) := \int_0^t K(t, s)dB_1(s), \quad t \in [0, T],
\]

and immediately remark that

1) (K1) ensures that the stochastic integral in (2.1) is well-defined for all \(t \in [0, T]\),
2) \textbf{(K2)} guarantees (see e.g. \cite[Theorem 1 and Corollary 4]{14}) that $Z$ has a modification that is Hölder continuous of any order $\lambda \in (0, H)$. Moreover, for any $\lambda \in (0, H)$, the positive random variable $\Lambda = \Lambda_\lambda$ such that
\[ |Z(t) - Z(s)| \leq \Lambda|t - s|^{\lambda}, \quad t, s \in [0, T], \]
\text{(2.2)}
can be chosen to have moments of all orders. In what follows, such Hölder continuous modification as well as $\Lambda$ with all the moments will be used.

Before proceeding further, let us give two examples of kernels satisfying Assumption $\textbf{(K)}$.

\begin{example} \textbf{(Hölder continuous kernels)} \label{ex1}
Let $\mathcal{K}(t, s) = \mathcal{K}(t - s)\mathbb{1}_{0 \leq s \leq t}$ for some $H$-Hölder continuous function $\mathcal{K}$, $H \in (0, 1)$. Then
\[
\int_0^t (\mathcal{K}(t, u) - \mathcal{K}(s, u))^2 du = \int_0^s (\mathcal{K}(t - u) - \mathcal{K}(s - u))^2 du + \int_s^t (\mathcal{K}(t - u) - \mathcal{K}(0) + \mathcal{K}(0))^2 du \\
\leq C \int_0^s |t - s|^{2H} du + C \int_s^t u^{2H} du + C \int_0^t 2^{2H} \mathcal{K}^2(0) du \\
\leq C|t - s|^{2(H + \frac{1}{2})},
\]
i.e. \textbf{(K2)} is satisfied and the process $Z(t) = \int_0^t \mathcal{K}(t - s)dB_1(s), t \in [0, T]$, has a modification that is Hölder continuous up to the order $H \wedge \frac{1}{2}$. Furthermore, if additionally $\mathcal{K}(0) = 0$, then it is easy to see that
\[
\mathbb{E}[(Z(t) - Z(s))^2] = \int_0^t (\mathcal{K}(t, u) - \mathcal{K}(s, u))^2 du \leq C|t - s|^{2H},
\]
i.e. $Z$ has a modification that is Hölder continuous up to the order $H$.
\end{example}

\begin{example} \textbf{(Fractional kernel)} \label{ex2}
Let $\mathcal{K}(t, s) = \frac{1}{\Gamma(H + \frac{1}{2})} (t - s)^{H - \frac{1}{2}} \mathbb{1}_{0 \leq s \leq t}$ with $H \in (0, 1)$. The process
\[
Z(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t - s)^{H - \frac{1}{2}} dB_1(s), \quad t \in [0, T],
\]
is called the Riemann-Liouville fractional Brownian motion (RL-fBm) and it is well-known (see e.g. \cite[Lemma 3.1]{61}) that, for any $s < t$,
\[
\mathbb{E}[(Z(t) - Z(s))^2] = \frac{1}{\Gamma^2(H + \frac{1}{2})} \left( \int_0^s \left((t - u)^{H - \frac{1}{2}} - (s - u)^{H - \frac{1}{2}}\right)^2 du + \int_0^t u^{2H - 1} du \right) \\
\leq C|t - s|^{2H},
\]
for some constant $C$. Thus \textbf{(K2)} holds and the RL-fBm has a modification that is Hölder continuous up to the order $H$.
\end{example}

\subsection{2.2 Sampled volatility}

For a given pair $\varphi, \psi : [0, T] \to \mathbb{R}$ such that $\varphi(t) < \psi(t), t \in [0, T]$, and any $a_1, a_2 \geq 0$, denote
\[
\mathcal{D}_{a_1, a_2} := \{(t, y) \in [0, T] \times \mathbb{R}_+, y \in (\varphi(t) + a_1, \psi(t) - a_2)\}.
\]
Consider a stochastic process $Y = \{Y(t), t \in [0, T]\}$ defined by \cite[1.4]{14}, where the initial value $Y(0)$ and the drift $b$ satisfy the following assumption.

\begin{assumption} \textbf{(Y)} \label{assumption_y}
There exist $H$-Hölder continuous functions $\varphi, \psi : [0, T] \to \mathbb{R}$, $0 < \varphi(t) < \psi(t), t \in [0, T]$, with $H$ being the same as in \textbf{(K2)} such that
\begin{itemize}
\item[(Y1)] $Y(0)$ is deterministic and $0 < \varphi(0) < Y(0) < \psi(0)$,
\item[(Y2)] $b : \mathcal{D}_{0,0} \to \mathbb{R}$ is continuous and, moreover, for any $\varepsilon \in (0, \min \{1, \frac{1}{2}\|\psi - \varphi\|_\infty\}]$
\[
|b(t_1, y_1) - b(t_2, y_2)| \leq \frac{c_1}{\varepsilon^p} \left(|y_1 - y_2| + |t_1 - t_2|^H\right), \quad (t_1, y_1), (t_2, y_2) \in \mathcal{D}_{\varepsilon, \varepsilon},
\]
where $c_1 > 0$ and $p > 1$ are given constants,
\end{itemize}
\end{assumption}
\[ b(t, y) \geq \frac{c_2}{(y - \varphi(t))^{\gamma}}, \quad (t, y) \in D_{0,0} \setminus D_{y,0}, \]
\[ b(t, y) \leq -\frac{c_2}{(\psi(t) - y)^{\gamma}}, \quad (t, y) \in D_{0,0} \setminus D_{0,0}, \]
where \(y_*, c_2 > 0\) are given constants and \(\gamma > \frac{1}{\eta} - 1\),

(Y3) the partial derivative \(\frac{\partial b}{\partial y}\) with respect to the spacial variable exists, is continuous and
\[
-c_3 \left(1 + \frac{1}{(y - \varphi(t))^{\eta}} + \frac{1}{(\psi(t) - y)^{\eta}}\right) < \frac{\partial b}{\partial y}(t, y) < c_3, \quad (t, y) \in D_{0,0},
\]
for some \(c_3 > 0\) and \(q > 0\).

Processes with such drifts were studied in detail in [32]. For reader’s convenience, we collect some relevant results form [32] regarding the volatility (1.4) in the statement below.

**Theorem 2.3.** Let Assumptions [(K)] and [(Y)] hold. Then
1) the SDE (1.4) has a unique strong solution \(Y\) and (2.5) holds with probability 1;
2) for any \(\lambda \in \left(\frac{1}{\gamma - 1}, H\right)\), where \(H\) is from (K2), there exist deterministic constants \(L_1\) and \(L_2 > 0\) (depending only on \(Y(0)\), the shape of \(b\) and \(\lambda\)) such that, with probability 1,
\[
\varphi(t) + \frac{L_1}{(L_2 + \Lambda)^{\gamma + \eta - 1}} \leq Y(t) \leq \psi(t) - \frac{L_1}{(L_2 + \Lambda)^{\gamma + \eta - 1}}, \quad t \in [0, T],
\]
where \(\Lambda\) is from (2.2) and \(\gamma\) is from (Y3); in particular,
\[
\varphi(t) < Y(t) < \psi(t), \quad t \in [0, T];
\]
3) for any \(r \in \mathbb{R}\),
\[
\mathbb{E}\left[\sup_{t \in [0, T]} \frac{1}{(Y(t) - \varphi(t))^{\eta}}\right] < \infty, \quad \mathbb{E}\left[\sup_{t \in [0, T]} \frac{1}{(\psi(t) - Y(t))^{\eta}}\right] < \infty.
\]

**Remark 2.4.** Having in mind property (2.5), we will refer to the process \(Y = \{Y(t), t \in [0, T]\}\) defined by (1.4) as a sandwiched process.

In addition to Theorem 2.3 we will also use the following result which can be found in [31] Lemma 3.6.

**Lemma 2.5.** Let Assumptions [(K)] and [(Y)] hold. Then, for each \(\lambda \in (0, H)\) there exists a random variable \(\Upsilon = \Upsilon_\lambda > 0\) such that \(\mathbb{E}[\Upsilon^r] < \infty\) for any \(r > 0\) and
\[
|Y(t) - Y(s)| \leq \Upsilon |t - s|^\lambda, \quad t, s \in [0, T].
\]

### 2.3 Price process

Let Assumptions [(K)] and [(Y)] hold, \(Z\) be a Gaussian Volterra process (2.1), and \(Y = \{Y(t), t \in [0, T]\}\) be the sandwiched process (1.4) such that (2.5) holds for some functions \(0 < \varphi < \psi\). We now consider the model introduced in Section 1 with a risk-free asset \(e^\int_0^t \varphi(s)ds\) and a risky asset \(S = \{S(t), t \in [0, T]\}\) defined by (1.3), where \(S(0) > 0\) is some deterministic constant and \(\rho \in (-1, 1)\). Clearly, the discounted price process \(X\) defined by (1.5) satisfies the SDE
\[
X(t) = X(0) + \int_0^t Y(s)X(s) \left(\rho dB_1(t) + \sqrt{1 - \rho^2} dB_2(t)\right), \quad t \in [0, T],
\]
where, for simplicity, we denote \(X(0) := S(0)\). Moreover, the following Theorem holds.

**Theorem 2.6.** ([36, Theorem 2.6])
1) Equations (1.3) and (1.6) both have unique strong solutions and

\[
S(t) = S(0) \exp \left\{ \int_0^t \left( \nu(s) - \frac{Y^2(s)}{2} \right) ds + \int_0^t Y(s) \left( \rho dB_1(t) + \sqrt{1 - \rho^2} dB_2(t) \right) \right\};
\]

\[
X(t) = X(0) \exp \left\{ -\int_0^t \frac{Y^2(s)}{2} ds + \int_0^t Y(s) \left( \rho dB_1(t) + \sqrt{1 - \rho^2} dB_2(t) \right) \right\}.
\]

2) For any \( r \in \mathbb{R} \),

\[
E \left[ \sup_{t \in [0,T]} S^r(t) \right] < \infty, \quad E \left[ \sup_{t \in [0,T]} X^r(t) \right] < \infty. \tag{2.7}
\]

Remark 2.7. Theorem 2.6 implies that \( X = \{X(t), t \in [0,T]\} \) is a square integrable martingale w.r.t. the filtration \( \mathbb{F} = \{ \mathcal{F}_t, t \in [0,T]\} \) generated jointly by \( B_1 \) and \( B_2 \).

Remark 2.8. We stress that (2.7) does not hold in general even in classical stochastic volatility models such as the Heston model (see e.g. [9]). This has repercussions in the possibilities of numerical simulations as well as in various applications such as pricing hedging or portfolio optimization. In our case, the existence of moments of all orders is achieved thanks to the boundedness of \( Y \).

We conclude this Section with a result on regularity of the probability laws of \( S \) and \( X \) presented in [30].

Theorem 2.9. [30] Corollary 3.8 For any \( t \in (0,T] \), the law of \( S(t) \) (and consequently of \( X(t) = e^{-\int_0^t \nu(s) ds} S(t) \)) has continuous and bounded density.

3 Markovian approximation of the SVV model

In general, the SVV model (1.3)–(1.4) is non-Markovian because of the Volterra Gaussian noise in the volatility dynamics. As discussed in the Introduction, while it can be viewed as a positive feature from the modeling perspective, non-Markovianity entails numerous numerical problems. In principle, one could construct Markovian approximations for the SVV model based on the following idea taking inspiration from [31,12]. It is evident that the original kernel \( K \) in (1.4) can be approximated in \( L^2 \) by a sequence of degenerate Volterra kernels

\[
K_m(t,s) = \sum_{i=0}^{\infty} e_{m,i}(t) f_{m,i}(s) \mathbb{1}_{s<t},
\]

where \( \int_0^T e_{m,i}(t) dt < \infty, \int_0^T f_{m,i}(t) dt < \infty, i = 0, \ldots, m, m \geq 0 \). Then one can attempt to define the approximated SVV model as

\[
S_m(t) = S(0) + \int_0^t \nu(s) S_m(s) ds + \int_0^t Y_m(s) S_m(s) dW(s), \tag{3.1}
\]

\[
Y_m(t) = Y(0) + \int_0^t b(s, Y_m(s)) ds + Z_m(t), \tag{3.2}
\]

\[
X_m(t) = e^{-\int_0^t \nu(s) ds} S_m(t), \tag{3.3}
\]

where \( Z_m \) is given by

\[
Z_m(t) := \int_0^t K_m(t,s) dB_1(s) = \sum_{i=0}^{m} e_{m,i}(t) \int_0^t f_{m,i}(s) dB_1(s), \quad t \in [0,T]. \tag{3.4}
\]

In this case, the \((m + 2)\)-dimensional process \((S_m, Y_m, U_{m,0}, \ldots, U_{m,m})\), where

\[
U_{m,i}(t) = \int_0^t f_{m,i}(s) dB_1(s), \quad i = 0, 1, \ldots, m, \tag{3.5}
\]

is Markovian w.r.t. the filtration \( \mathbb{F} = \{ \mathcal{F}_t, t \in [0,T]\} \) generated by \((B_1, B_2)\) and, intuitively, should approximate the original non-Markovian SVV model if \( K_m \) is close enough to \( K \). However, such intuition is not simple to back up analytically due to the explosion (2.3) of the drift \( b \).
• Assumption (Y3) demands \( \gamma > \frac{d}{H} - 1 \), i.e. the order of the drift explosion in (2.3) is coupled with the Hölder regularity of the Volterra noise \( Z \) – changing the kernel may violate (Y3) and hence (3.2) may not have a solution;

• it is not clear why \( Y_m \) approximates \( Y \) in \( L^2 \): the standard reasoning via Gronwall’s inequality fails, again because of the explosive drift.

The goal of this section is to overcome the problems mentioned above and study the convergence of approximations (3.1–3.3).

### 3.1 Admissible approximating kernels

As noted above, we want to replace the kernel \( \mathcal{K} \) with its appropriate \( L^2 \)-approximations \( \mathcal{K}_m, \ m \geq 1 \), and then consider stochastic processes \( Z_m = \{ Z_m(t), \ t \in [0, T] \} \) and \( Y_m = \{ Y_m(t), \ t \in [0, T] \} \) defined by (3.4) and (3.2) respectively. However, we need to make sure that the solution to (3.2) exists, i.e. Theorem 2.3 holds, for all \( m \geq 1 \). Moreover, we will need that bounds of the type (2.4) hold for each \( Y_m \) with constants \( L_1, L_2 \) that do not depend on \( m \). In order to ensure that, we introduce the following assumption.

**Assumption (Km).** The sequence of kernels \( \{ \mathcal{K}_m, \ m \geq 1 \} \) is such that each function \( \mathcal{K}_m : [0, T]^2 \to \mathbb{R} \) is a Volterra kernel, i.e. \( \mathcal{K}_m(t, s) = 0 \) whenever \( t < s \), and

(Km1) for each \( m \geq 1 \), \( \mathcal{K}_m \) is square integrable, i.e.

\[
\int_0^T \int_0^T K_m^2(t, s)dsdt = \int_0^T \int_0^T K_m^2(t, s)dsdt < \infty;
\]

(Km2) for each \( m \geq 1 \) and \( \lambda \in (0, H) \) with \( H \) being from (K2),

\[
\int_0^t (K_m(t, u) - K_m(s, u))^2 du \leq \ell_\lambda^2 (t - s)^{2\lambda}, \quad 0 \leq s \leq t \leq T,
\]

where \( \ell_\lambda > 0 \) is a constant that possibly depends on \( \lambda \) but does not depend on \( m \).

In general, it is not clear whether a given \( L^2 \)-approximation \( \{ \mathcal{K}_m, \ m \geq 1 \} \) of the kernel \( \mathcal{K} \) satisfies Assumption (Km), so one has to check it separately. Luckily, for kernels from Examples 2.1 and 2.2 above, such approximations exist and have relatively simple shapes.

**Example 3.1 (Hölder continuous kernels).** Assume that \( \mathcal{K}(t, s) = \mathcal{K}(t - s)1_{s \leq t} \) with \( \mathcal{K}(0) = 0 \) and \( \mathcal{K} \in C^H([0, T]) \) for some \( H \in (0, 1) \), i.e.

\[
|\mathcal{K}(t) - \mathcal{K}(s)| \leq C_H|t - s|^H, \quad s, t \in [0, T]. \tag{3.6}
\]

Let \( \mathcal{K}_m \) be the corresponding Bernstein polynomial of order \( m \) defined as

\[
\mathcal{K}_m(t) = \frac{1}{T^m} \sum_{i=0}^{m} \mathcal{K} \left( \frac{T_i}{m} \right) t^{m-i} = \frac{1}{T^m} \sum_{i=0}^{m} \left( \frac{T}{m} \right)^i \mathcal{K} \left( \frac{T_j}{m} \right) \left( \frac{m-j}{i-j} \right) t^i \tag{3.7}
\]

and

\[
\mathcal{K}_m(t, s) = \mathcal{K}_m(t - s)1_{s \leq t} = \sum_{i=0}^{m} \mathcal{K}_m(t) \left( \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^{i-j} s^{i-j} \right) 1_{s \leq t} \tag{3.8}
\]

By [74, Proposition 2],

\[
|\mathcal{K}_m(t) - \mathcal{K}_m(s)| \leq C_H|t - s|^H, \quad s, t \in [0, T],
\]

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where $C_H$ is the same as in (3.8) for all $m \geq 1$. Since $K(0) = 0$, $K_m(0) = \kappa_m = 0$ and we can write for any $0 \leq s \leq t \leq T$

$$
\int_0^t (K_m(t-u) - K_m(s-u))^2 du + \int_s^t K_m^2(t-u) du \leq C_H^2 T |t-s|^{2H} + \int_0^{t-s} K_m^2(v) dv \\
\leq C_H^2 T |t-s|^{2H} + C_{H}^2 \int_0^{t-s} v^{2H} dv \leq C_H^2 T |t-s|^{2H} + \frac{C_H^2 2H + 1 |t-s|^{2H+1}}{2H + 1},
$$

where $C$ does not depend on $m$, i.e. Assumption $[K_m]$ holds. Moreover, [34 Theorem 1] gives the estimate

$$
\sup_{t \in [0,T]} |K(t) - K_m(t)| \leq C m^{-\frac{H}{2}},
$$

which implies that

$$
\| K - K_m \|_{L^2([0,T])} \leq C m^{-\frac{H}{2}} \rightarrow 0, \quad m \rightarrow \infty. \quad (3.9)
$$

**Remark 3.2.** Note that each process

$$
Z_m(t) = \int_0^t K_m(t-s) dB_1(s) = \sum_{i=0}^m \kappa_{m,i} \int_0^t (t-s)^i dB_1(s), \quad t \in [0,T],
$$

is actually Hölder continuous of any order $\lambda \in (0,1)$, i.e. the solution to the SDE

$$
Y_m(t) = Y(0) + \int_0^t b(s,Y_m(s)) ds + Z_m(t), \quad t \in [0,T],
$$

exists and is unique for any value $\gamma > 0$ in (Y3) provided that $\varphi$ and $\psi$ are both Lipschitz.

**Example 3.3 (Fractional kernel).** Let $H \in (0, \frac{1}{2})$ and

$$
K(t,s) = K(t-s) \mathbb{1}_{s<t} = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \mathbb{1}_{s<t}, \quad (3.10)
$$

i.e. $Z(t) = \int_0^t K(t,s) dB_1(s)$ is a RL-fBm as discussed in Example 2.2. For such a kernel, an appropriate sequence $\{K_m, m \geq 1\}$ of degenerate $L^2$-approximations satisfying Assumption $[K_m]$ is presented in [3, Section 3]. Namely, let $\mu$ be a measure on $\mathbb{R}^+$ defined as

$$
\mu(\alpha) := \frac{\alpha^{-H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2}) \Gamma(\frac{1}{2} - H)} d\alpha, \quad \sigma_{m,i} := \int_{\tau_{m,i-1}}^{\tau_{m,i}} \mu(\alpha) d\alpha, \quad \alpha_{m,i} := \frac{1}{\sigma_{m,i}} \int_{\tau_{m,i-1}}^{\tau_{m,i}} \alpha \mu(\alpha), \quad i = 1, ..., m,
$$

where $0 = \tau_{m,0} < \tau_{m,1} < ... < \tau_{m,m}$ is such that

$$
\tau_{m,m} \rightarrow \infty, \quad \sum_{i=1}^m \int_{\tau_{m,i-1}}^{\tau_{m,i}} (\alpha_{m,i} - \alpha)^2 \mu(\alpha) \rightarrow 0, \quad m \rightarrow \infty. \quad (3.11)
$$

Denote for $m \geq 1$

$$
K_m(t-s) \mathbb{1}_{s<t} = \sum_{i=1}^m \sigma_{m,i} e^{-\alpha_{m,i} (t-s)} \mathbb{1}_{s<t} = \sum_{i=1}^m (\sigma_{m,i} e^{-\alpha_{m,i} t}) e^{\alpha_{m,i} s} \mathbb{1}_{s<t} =: \sum_{i=1}^m e_{m,i}(t) f_{m,i}(s) \mathbb{1}_{s<t}. \quad (3.12)
$$

By [3 Lemma 5.2], the sequence $\{K_m, m \geq 1\}$ satisfies Assumption $[K_m]$ and, moreover, by [3 Proposition 3.3],

$$
\| K - K_m \|_{L^2([0,T])} \rightarrow 0, \quad m \rightarrow \infty.
$$
Note that the values $0 = \tau_{m,0} < \tau_{m,1} < \ldots < \tau_{m,m}$ satisfying (3.11) can be chosen in multiple ways. For example, as suggested in [3, Subsection 3.2], one can put

$$\tau_{m,i} := \frac{1}{T} \left( \frac{\sqrt{10}(1-2H)}{5-2H} \right)^{\frac{2}{\lambda}}, \quad i = 0, 1, 2, \ldots, m.$$  \hfill (3.13)

In this case,

$$\|K - K_m\|_{L^2([0,T])} \leq C_H m^{-\frac{4\alpha}{\lambda}},$$  \hfill (3.14)

where $C_H$ is a positive constant that depends only on the Hurst parameter $H \in (0, \frac{1}{2})$.

**Remark 3.4.** Putting $e_{m,i}(t) := \sigma_{m,i} e^{-\alpha_{m,i}t}$, $f_{m,i}(s) := e^{\alpha_{m,i}s}$ in Example 3.3, we obtain that each process of the form

$$e_{m,i}(t)U_{m,i}(t) = \sigma_{m,i} \int_0^t e^{-\alpha_{m,i}(t-s)} dB_1(s) =: V_{m,i}(t), \quad t \in [0, T],$$

from (3.5) is an Ornstein-Uhlenbeck process satisfying the SDE

$$V_{m,i}(t) = -\alpha_{m,i} \int_0^t V_{m,i}(s) ds + \sigma_{m,i} B_1(t), \quad t \in [0, T].$$

Just like in Subsection 2.1, we see here that Assumption (Km) together with [14, Theorem 1 and Corollary 4] guarantee that each $Z_m$ has a continuous modification that satisfies the Hölder property for all $\lambda \in (0, H)$. In fact, (Km2) allows to deduce a stronger statement: the Hölder seminorms of $Z_m$, $m \geq 1$, are uniformly bounded in $L^r(\mathbb{P})$. The corresponding result is presented in the next lemma.

**Lemma 3.5.** Let $K$ satisfy Assumption [K] and the sequence $\{K_m, m \geq 1\}$ satisfy Assumption (Km). Then for any constant $\lambda \in (0, H)$ there exist positive random variables $A_m, m \geq 1$, such that for each $m \geq 1$, $r > 0$ and $s, t \in [0, T]$

$$|Z_m(t) - Z_m(s)| \leq A_m |t - s|^{\lambda},$$

and

$$\sup_{m \geq 1} E A_m^r < \infty.$$ \hfill (3.15)

**Proof.** Fix an arbitrary $\lambda \in (0, H)$, choose $\alpha > 1$ such that $\alpha \lambda + \frac{2}{\alpha} < H$ and denote

$$A_m := A_{\lambda, \alpha} \left( \int_0^T \int_0^T \frac{|Z_m(t) - Z_m(s)|^\alpha}{|t - s|^{\alpha \lambda + 2}} dsdt \right)^{\frac{1}{\alpha}}, \quad m \geq 1,$$

with $A_{\lambda, \alpha} := 2^{\frac{\alpha}{2}} \left( \frac{\lambda \alpha + 2}{\alpha} \right)$. By the Garsia-Rodemich-Rumsey inequality (see [44] and [32, Lemma 1.1]), for any $m \geq 1$ and $s, t \in [0, T]$,

$$|Z_m(t) - Z_m(s)| \leq A_m |t - s|^{\lambda},$$

and, moreover, the proof of [14, Theorem 1] implies that $E A_m^r < \infty$ for each $r > 0$. It remains to show that $\sup_{m \geq 1} E A_m^r < \infty$. Take any $r > \alpha$, and observe that Minkowski integral inequality, Gaussian distribution of $Z_m(t) - Z_m(s)$ and (Km2) yield

$$\left( E [A_m^r] \right)^{\frac{1}{r}} \leq A_{\lambda, \alpha}^\alpha \int_0^T \int_0^T \left( E \left[ |Z_m(t) - Z_m(s)|^r \right] \right)^{\frac{1}{r}} dsdt \leq 2^{\frac{r}{2}} \left( \frac{\Gamma \left( \frac{r+1}{\alpha} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2}} A_{\lambda, \alpha}^\alpha \int_0^T \int_0^T \left( \int_0^s (K_m(t, u) - K_m(s, u))^2 du \right)^{\frac{1}{2}} \frac{1}{|t - s|^{\lambda \alpha + 2}} dsdt \leq 2^{\frac{r}{2}} \left( \frac{\Gamma \left( \frac{r+1}{\alpha} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2}} A_{\lambda, \alpha}^\alpha e_{\lambda + \frac{2}{\alpha}} \int_0^T \int_0^T \frac{|t - s|^{\lambda \alpha + 2}}{|t - s|^{\lambda \alpha + 2}} dsdt \leq 2^{\frac{r}{2}} T^2 \left( \frac{\Gamma \left( \frac{r+1}{\alpha} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2}} A_{\lambda, \alpha}^\alpha e_{\lambda + \frac{2}{\alpha}}$$

and thus

$$\sup_{m \geq 1} E [A_m^r] \leq 2^{\frac{r}{2}} T^2 \left( \frac{\Gamma \left( \frac{r+1}{\alpha} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2}} A_{\lambda, \alpha}^\alpha e_{\lambda + \frac{2}{\alpha}} < \infty,$$

which ends the proof. $\square$
3.2 Approximation of the volatility

Next, consider a family of sandwiched process $Y_m = \{Y_m(t), \ t \in [0,T]\}, m \geq 1$, defined by equations of the form (3.2). Note that the conditions of Theorem 2.6 are met for all $m \geq 1$, i.e. each $Y_m$ is well-defined. Since constants $L_1, L_2$ in (2.4) depend only on $Y(0)$, the shape of $b$ and $\lambda \in \left(\frac{1}{1+\gamma}, H\right)$ that can be chosen jointly for all $m \geq 1$, we have that

$$\varphi(t) + \frac{L_1}{(L_2 + \Lambda_m)^{\frac{1}{1+\lambda}-1}} < Y_m(t) < \psi(t) - \frac{L_1}{(L_2 + \Lambda_m)^{\frac{1}{1+\lambda}-1}}, \ t \in [0,T], \ m \geq 1. \quad (3.16)$$

Moreover, the following result is true.

**Lemma 3.6.** Let Assumptions \((K), (Y)\) and \((Km)\) hold. Then, for all $\lambda \in (0, H)$, there exists a sequence of positive random variables $\{Y_m, \ m \geq 1\}$ such that $\sup_{m \geq 1} \mathbb{E}[Y_m] < \infty$ for any $r > 0$ and $|Y_m(t) - Y_m(s)| \leq Y_m|t - s|^\lambda, \ t, s \in [0,T].$

**Proof.** It is enough to prove the claim for any fixed $\lambda \in \left(\frac{1}{1+\gamma}, H\right)$. Taking the corresponding $\Lambda_m$ from Lemma 3.5 and proceeding exactly as in the proof of [31, Lemma 3.6], one can see that

$$|Y_m(t) - Y_m(s)| \leq C \left(1 + \Lambda_m + \frac{(L_2 + \Lambda_m)^{\frac{1}{1+\lambda}-p}}{L_1}\right)|t - s|^\lambda,$$

where $C > 0$ is a deterministic constant that does not depend on $m$, $p$ is from Assumption $(Y2)$ and $L_1, L_2$ are from (3.16). Putting $Y_m := C \left(1 + \Lambda_m + (L_2 + \Lambda_m)^{\frac{1}{1+\lambda}-p}\right)$ and taking into account (3.15), it is easy to see that $\sup_{m \geq 1} \mathbb{E}[Y_m] < \infty$ for any $r > 0$ as required.

The next result gives a pathwise estimate of distance between $Y$ and $Y_m$.

**Theorem 3.7.** Let the processes $Y$ and $Y_m, m \geq 1$, satisfy Assumptions $(K), (Y)$ and $(Km)$. Then, for each $m \geq 1$, there exist a deterministic constant $C > 0$ that does not depend on $m$ or $t$ and a random variable $\xi_m$ that does not depend on $t$ such that, for any $r > 0$,

$$\sup_{m \geq 1} \mathbb{E}[\xi_m] < \infty \quad (3.17)$$

and, for any $t \in [0,T],$

$$|Y(t) - Y_m(t)| \leq C \left(|Z(t) - Z_m(t)| + \xi_m \int_0^t |Z(u) - Z_m(u)|\,du\right).$$

The proof of Theorem 3.7 is rather long and technical, so we present it in Appendix A.

3.3 Approximation of the price

We are now ready to study the error estimates for the approximations of the prices. For this, we first present a result on the moments of the approximating processes. Let us consider the price process $S$ given by (1.3) and denote

$$W(t) := \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t), \ t \in [0,T].$$

Note that Theorem 2.6 immediately implies that for each $m \geq 1$ and all $r \in \mathbb{R}$

$$\mathbb{E} \left[\sup_{t \in [0,T]} S_m^r(t)\right] < \infty, \quad \mathbb{E} \left[\sup_{t \in [0,T]} R_m^r(t)\right] < \infty.$$

However, in the sequel we will require a slightly stronger result summarized in the following Lemma.

**Lemma 3.8.** Under Assumptions $(K), (Y)$ and $(Km)$, for any $r \in \mathbb{R}$

$$\sup_{m \geq 1} \mathbb{E} \left[\sup_{t \in [0,T]} S_m^r(t)\right] < \infty, \quad \sup_{m \geq 1} \mathbb{E} \left[\sup_{t \in [0,T]} R_m^r(t)\right] < \infty.$$
Proof. It is sufficient to prove the result for $S_m$. Using the same argument as in the proof of [30, Theorem 2.6], it is straightforward to show that

\[
E \left[ \sup_{t \in [0,T]} S_m^r(t) \right] \leq C \left( 1 + C_1 \sqrt{T} \sup_{s \in [0,T]} \psi(s) \exp \left( \frac{r^2 T}{2} \max_{s \in [0,T]} \psi^2(s) \right) \right), \tag{3.18}
\]

where

\[
C := S^r(0) \exp \left\{ |r| T \max_{s \in [0,T]} |r(s)| + \frac{|r||(|r|+1)T}{2} \max_{s \in [0,T]} \psi^2(s) \right\}
\]

and the constant $C_1$ comes from the Burkholder-Davis-Gundy inequality. The result follows from the fact that the right-hand side of (3.18) does not depend on $m$.

We are now finally ready to proceed to the main result of the Subsection.

**Theorem 3.9.** Let Assumptions $[K], [Y], [K_m]$ hold and $r \geq 2$ be fixed.

1) If $K, K_m \in L^r([0, |T|^2])$, $m \geq 1$, then there exists a constant $C > 0$ that does not depend on $m$ such that

\[
E \left[ \sup_{t \in [0,T]} |S(t) - S_m(t)|^r \right] \leq C \|K - K_m\|_{L^r([0,|T|^2])}^r, \quad E \left[ \sup_{t \in [0,T]} |X(t) - X_m(t)|^r \right] \leq C \|K - K_m\|_{L^r([0,|T|^2])}^r.
\]

2) If $K, K_m \in L^2([0, |T|^2])$, $m \geq 1$, are of the difference type, i.e.

\[
K(t, s) = K(t - s)1_{s < t}, \quad K_m(t, s) = K_m(t - s)1_{s < t}, \quad t, s \in [0, T],
\]

then there exists constant $C > 0$ that does not depend on $m$ such that

\[
E \left[ \sup_{t \in [0,T]} |S(t) - S_m(t)|^r \right] \leq C \|K - K_m\|_{L^2([0,|T|^2])}^r, \quad E \left[ \sup_{t \in [0,T]} |X(t) - X_m(t)|^r \right] \leq C \|K - K_m\|_{L^2([0,|T|^2])}^r.
\]

Proof. It is sufficient to prove both claims for the non-discounted price $S$.

**Item 1.** For any $t \in [0, T]$, (1.3), (3.1) and the continuity of $\nu$ imply

\[
|S(t) - S_m(t)|^r \leq C \left( \int_0^t |S(s) - S_m(s)|^r ds + \int_0^t (Y(s)S(s) - Y_m(s)S_m(s))dW(s) \right)^r
\]

\[
\leq C \int_0^t \sup_{u \in [0,s]} |S(u) - S_m(u)|^r ds + C \left( \sup_{u \in [0,t]} \left[ \int_0^u (Y(s)S(s) - Y_m(s)S_m(s))dW(s) \right] \right)^r,
\]

whence

\[
\sup_{u \in [0,t]} |S(u) - S_m(u)|^r \leq C \int_0^t \sup_{u \in [0,s]} |S(u) - S_m(u)|^r ds
\]

\[
+ C \left( \sup_{u \in [0,t]} \left[ \int_0^u (Y(s)S(s) - Y_m(s)S_m(s))dW(s) \right] \right)^r.
\]

By the Burkholder-Davis-Gundy inequality,

\[
E \left[ \sup_{u \in [0,t]} |S(u) - S_m(u)|^r \right] \leq C \int_0^t E \left[ \sup_{u \in [0,s]} |S(u) - S_m(u)|^r \right] ds + C E \left[ \left( \int_0^t (Y(s)S(s) - Y_m(s)S_m(s))^2 ds \right)^{\frac{r}{2}} \right] \tag{3.19}
\]

\[
\leq C \int_0^t E \left[ \sup_{u \in [0,s]} |S(u) - S_m(u)|^r \right] ds + C \int_0^t E \left[ |Y(s)S(s) - Y_m(s)S_m(s)|^r \right] ds.
\]
Let us focus on the last summand of (3.19). Using the fact that
\[ \max_{m \geq 1} \max_{t \in [0,T]} |Y_m(t)| \leq \max_{t \in [0,T]} |\psi(t)|, \]
we have
\[
\int_0^t \mathbb{E} \left[ |Y(s)S(s) - Y_m(s)S_m(s)|^r \right] ds \\
\leq C \int_0^t \mathbb{E} \left[ |Y_m(s)|S(s) - S_m(s)|^r \right] ds + C \int_0^t \mathbb{E} \left[ |S(s)|Y(s) - Y_m(s)|^r \right] ds \\
\leq C \int_0^t \mathbb{E} \left[ \sup_{u \in [0,s]} |S(u) - S_m(u)|^r \right] ds + C \int_0^t \mathbb{E} \left[ |S(s)|Y(s) - Y_m(s)|^r \right] ds.
\] (3.20)

Studying again the last summand of (3.20), we obtain by Theorem 3.7 that there exist random variables \( \xi_m \), \( m \geq 1 \), such that (3.17) holds and
\[
\int_0^t \mathbb{E} \left[ |S^r(s)|Y(s) - Y_m(s)|^r \right] ds \\
\leq C \int_0^t \mathbb{E} \left[ |S^r(s)|Z(s) - Z_m(s)|^r \right] ds + C \int_0^t \mathbb{E} \left[ |S^r(s)\xi_m \int_0^s |Z(u) - Z_m(u)|^r du \right] ds.
\] (3.21)

Also, by (3.9) Theorem 2.6,
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} S^{2r}(t) \right] < \infty, \]
so the Hölder inequality as well as the Gaussianity of the random variable \( Z(u) - Z_m(u) \) yield
\[
\int_0^t \mathbb{E} \left[ |S^r(s)|Z(s) - Z_m(s)|^r \right] ds \leq \int_0^t \left( \mathbb{E} \left[ |S^{2r}(s)| \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |Z(s) - Z_m(s)|^{2r} \right] \right)^{\frac{1}{2}} ds \\
\leq C \int_0^t \left( \int_0^s (K(s,u) - K_m(s,u))^2 du \right)^{\frac{1}{2}} ds \leq C \|K - K_m\|_{L^r([0,T]^2)}.
\] (3.22)

Additionally, (3.17) together with (3.9) Theorem 2.6 imply that
\[ \sup_{m \geq 1} \mathbb{E}[\xi_m^{2r} \sup_{t \in [0,T]} S^{2r}(t)] < \infty \]

hence
\[
\int_0^t \mathbb{E} \left[ |S^r(s)\xi_m \int_0^s |Z(u) - Z_m(u)|^r du \right] ds \\
\leq \int_0^t \left( \mathbb{E} \left[ |S^{2r}(s)\xi_m^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |Z(u) - Z_m(u)|^{2r} \right] \right)^{\frac{1}{2}} ds \\
\leq C \int_0^t \int_0^s \left( |Z(u) - Z_m(u)|^{2r} \right)^{\frac{1}{2}} du ds \leq C \int_0^t \int_0^s \left( \int_0^u \left( K(s,u) - K_m(s,u) \right)^2 dv \right)^{\frac{1}{2}} du ds \\
\leq C \|K - K_m\|_{L^r([0,T]^2)}.
\] (3.23)

Therefore, taking into account (3.19)–(3.23), we see that there exists a constant \( C > 0 \) that does not depend on \( m \) or the particular choice of \( t \in [0,T] \) such that
\[ \mathbb{E} \left[ \sup_{u \in [0,t]} |S(u) - S_m(u)|^r \right] \leq C \int_0^t \mathbb{E} \left[ \sup_{u \in [0,s]} |S(u) - S_m(u)|^r \right] ds + C \|K - K_m\|_{L^r([0,T]^2)}. \] (3.24)

and item 1) now follows from the Gronwall’s inequality.

**Item 2.** If \( K \) and \( K_m \), both have the form
\[ K(t,s) = K(t-s)\mathbb{1}_{s \leq t}, \quad K_m(t,s) = K_m(t-s)\mathbb{1}_{s \leq t}, \quad t, s \in [0,T], \]

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Let estimate the error between the solution to the original hedging problem (1.1) and its Markovian counterpart. Non-anticipating derivatives and their connection to the optimization problem (1.1). In Subsection 4.2, we require the numerical computation of which heavily benefits from the Markovian approximation derived in Section 3. Optimal hedging portfolio coincides with the posed and has a solution given by the Galtchouk-Kunita-Watanabe decomposition. In this situation, the unique representation of the form

\[ \text{Theorem 4.2. (28, Theorem 2.1)} \] The non-anticipating derivative \( F \) of European type, i.e. the problem (1.1) is well-posed and has a solution given by the Galtchouk-Kunita-Watanabe decomposition. In this situation, the optimal hedging portfolio coincides with the non-anticipating derivative of \( F \) with respect to the martingale \( X \) the numerical computation of which heavily benefits from the Markovian approximation derived in Section 3.

This Section is organized as follows. In Subsection 4.1, we provide all the necessary details on the non-anticipating derivatives and their connection to the optimization problem (1.1). In Subsection 4.2, we estimate the error between the solution to the original hedging problem (1.1) and its Markovian counterpart. The numerical algorithms for computing optimal hedging strategies are described in the subsequent Section 5.

4 Mean-variance hedging in the SVV model

In this Section, we will deal with the mean-variance hedging problem (1.1) with the SVV model (1.3)-(1.4), where the payoff \( F \) is of European type, i.e. \( F = f(X(T)) \). Note that Theorem 2.6 guarantees that \( F = f(X(T)) \in L^2(P) \) as long as the payoff function is of polynomial growth, i.e. the problem (1.1) is well-posed and has a solution given by the Galtchouk-Kunita-Watanabe decomposition. In this situation, the optimal hedging portfolio coincides with the non-anticipating derivative of \( F \) with respect to the martingale \( X \) the numerical computation of which heavily benefits from the Markovian approximation derived in Section 3.

This Section is organized as follows. In Subsection 4.1, we provide all the necessary details on the non-anticipating derivatives and their connection to the optimization problem (1.1). In Subsection 4.2, we estimate the error between the solution to the original hedging problem (1.1) and its Markovian counterpart. The numerical algorithms for computing optimal hedging strategies are described in the subsequent Section 5.

4.1 Non-anticipating derivative and mean-variance hedging

Let \( \xi \in L^2(P) \) and \( \eta = \{ \eta(t), t \in [0, T] \} \) be a square-integrable martingale w.r.t. a filtration \( \mathcal{G} = \{ \mathcal{G}_t, t \in [0, T] \} \). For an arbitrary partition \( \pi = \{ 0 = t_0 < t_1 < ... < t_n = T \} \) of \([0, T]\) with the mesh \( |\pi| := \max_{k}(t_k - t_{k-1}) \), denote

\[ u_{\pi,k} := \sum_{k=0}^{n-1} u_{\pi,k} \mathbbm{1}_{[t_k, t_{k+1})}, \quad u_{\pi,k} := \frac{\mathbb{E}[\eta(t_{k+1}) - \eta(t_k)\xi | \mathcal{G}_{t_k}]}{\mathbb{E}[\eta(t_{k+1}) - \eta(t_k)^2 | \mathcal{G}_{t_k}]} \], \quad k = 0, 1, ..., n - 1. \tag{4.1} \]

**Definition 4.1.** Consider a monotone sequence of partitions \( \{ \pi_M, M \geq 1 \} \), such that \( |\pi_M| \to 0 \) as \( M \to \infty \). The \( L^2(P \times [\eta]) \)-limit

\[ D\xi := \lim_{|\pi_M| \to 0} u_{\pi_M} \tag{4.2} \]

is called the non-anticipating derivative of \( \xi \) w.r.t. \( \eta \).

It turns out that \( D\xi \) is well-defined and \( \int_0^T D\xi(s)d\eta(s) \) gives the orthogonal \( L^2(P \times [\eta]) \)-projection of \( \xi \) on the subspace of stochastic integrals w.r.t. \( \eta \).

**Theorem 4.2.** (28, Theorem 2.1) The non-anticipating derivative \( D\xi \) is well-defined, i.e. the limit (4.2) exists and does not depend on the particular choice of the partitions. Moreover, any \( \xi \in L^2(P) \) admits a unique representation of the form

\[ \xi = \xi_0 + \int_0^T D\xi(s)d\eta(s), \]

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where \( \xi_0 \in L^2(\mathbb{P}) \) is such that \( \mathcal{D}\xi_0 = 0 \) and \( \mathbb{E}\left[ \xi_0^T \mathcal{D}\xi_0 \right] = 0 \). In other words, the infimum over all \( \mathcal{F} \)-adapted \( \eta \)-integrable strategies

\[
\inf_u \mathbb{E}\left[ \left( \xi - \int_0^T u(s) d\eta(s) \right)^2 \right]
\]

is attained at \( u = \mathcal{D}\xi \).

**Remark 4.3.** By Theorem 4.2, it is easy to see that the linear operator \( \mathcal{D} \) is actually the dual of the Itô integral.

**Notation.** Since the limit (4.2) does not depend on the particular choice of partition \( \{ \pi_M, M \geq 1 \} \), we will use the notation

\( \mathcal{D}\xi := \lim_{|\pi| \to 0} u_\pi \),

meaning that the limit in \( L^2(\mathbb{P} \times [\eta]) \) is taken along an arbitrary sequence of monotone partitions with the mesh going to zero.

Next, let Assumptions [K] and [Y] hold, and \( S \) and \( X \) be defined by (1.3) and (1.5) respectively. Consider the mean-square hedging problem (1.1), where the payoff function \( f: \mathbb{R}_+ \to \mathbb{R} \) satisfies the following assumption.

**Assumption (F).** The payoff function \( f: \mathbb{R}_+ \to \mathbb{R} \) can be represented as \( f = f_1 + f_2 \), where

(i) \( f_1 \) is globally Lipschitz, i.e. there exists \( C > 0 \) such that

\[ |f_1(t) - f_1(s)| \leq C|t - s|, \quad s, t \geq 0; \]

(ii) \( f_2: \mathbb{R}_+ \to \mathbb{R} \) is of bounded variation over \( \mathbb{R}_+ \), i.e.

\[ V(f_2) := \lim_{x \to \infty} V_{[0,x]}(f_2) < \infty, \]

where

\[ V_{[0,x]}(f_2) := \sup_{N} \sum_{j=1}^{N} |f_2(x_j) - f_2(x_{j-1})| \]

and the supremum is taken over all \( N \geq 0 \) and all partitions \( 0 = x_0 < x_1 < ... < x_N = x \).

It is evident that Eq. (2.7) and Assumption (F) imply that \( F = f(X(T)) \in L^2(\mathbb{P}) \), i.e. Theorem 4.2 holds and we have the following corollary.

**Corollary 4.4.** The non-anticipating derivative \( u = \mathcal{D}F \) of \( F \) w.r.t. \( X \) is the minimizer of \( J \) from (1.1) and hence represents the optimal hedging portfolio.

**Remark 4.5.** Consider an arbitrary partition \( \pi = \{0 = t_0 < t_1 < ... < t_n = T\} \). The proof of [28, Theorem 2.1] implies that the pre-limit sum

\[ u_\pi := \sum_{k=0}^{n-1} u_{\pi,k} 1_{(t_k, t_{k+1}]} \]

\[ u_{\pi,k} := \mathbb{E}\left[ (X(t_{k+1}) - X(t_k))F | \mathcal{F}_{t_k} \right] / \mathbb{E}\left[ (X(t_{k+1}) - X(t_k))^2 | \mathcal{F}_{t_k} \right], \quad k = 0, 1, ..., n - 1, \]  \quad (4.3)

is the \( L^2(\mathbb{P} \times [\eta]) \)-orthogonal projection of \( F = f(X(T)) \) onto the subspace generated by stochastic integrals of simple processes

\[ \sum_{k=0}^{n-1} \xi_k 1_{(t_k, t_{k+1}]} \]  \quad (4.4)

w.r.t. \( X \). Note that admissible portfolios in real markets are exactly of this type since there is no technical possibility of real “continuous” trading.

**Remark 4.6.** Note that formula (4.3) is explicit in the sense that the hedge is written only in terms of the discounted price model, the information flow of reference, and the claim \( F \). This formula is in the spirit of the Clark-Haussmann-Ocone (CHO) formula (see e.g. [50, 77]) but we stress that the non-anticipating derivative has several important advantages. Namely, the CHO formula exploits the Malliavin derivative which is tailor for specific noises (e.g. Brownian motion) and claims \( F \) falling in the domain of the Malliavin operator. In turn, formula (4.3) is available for all square integrable claims and all square integrable martingales as discounted prices.
4.2 Non-anticipating derivative and Markovian approximations

The very definition of the non-anticipating derivative \( u = \mathcal{DF} \) describing the hedging portfolio provides a natural approximation of it. Indeed, simple processes \( u_\pi \) given by (4.3) converge in \( L^2(\mathbb{P} \times \mathbb{X}) \) to \( \mathcal{DF} \) and each \( u_\pi \) is itself the optimal hedge in the corresponding class of simple processes (4.4). However, the computation of the conditional expectations in (4.3) is a challenging task that becomes even more complicated since the Volterra noise \( Z \) from (1.4) may have memory (and thus, in general, \( X \) is not Markovian).

In what follows, we will utilize the Markovian approximation derived in Section 3 and compare the non-anticipating derivative \( \mathcal{D}f(X(T)) \) w.r.t. \( X \) with \( \mathcal{D}f(X_m(T)) \) w.r.t. \( X_m \). Note that we are interested in European options with rather complicated payoffs covering the ones with discontinuities, so we will need to compare \( F = f(X(T)) \) with \( F_m := f(X_m(T)) \). In order to do that, the following Theorem will be used.

**Theorem 4.7.** Let \( f = f_1 + f_2 \) satisfy Assumption (F) and \( \xi, \hat{\xi} \in L^r(\mathbb{P}), 1 \leq r < \infty \). Then

1) there exists a constant \( C > 0 \) depending only on \( f_1 \) and \( r \) such that

\[
\mathbb{E}[|f_1(\xi) - f_1(\hat{\xi})|^r] \leq C \mathbb{E}[|\xi - \hat{\xi}|^r];
\]

2) if \( \xi \) has bounded density \( \phi_\xi \), then

\[
\mathbb{E}[|f_2(\xi) - f_2(\hat{\xi})|^r] \leq 3^{r+1} V^r(f_2)(\sup \phi_\xi) \left( \mathbb{E} \left[ |\xi - \hat{\xi}|^r \right] \right) \frac{r}{r+1};
\]

3) if \( \xi \) has bounded density \( \phi_\xi \), then there exists a constant \( C > 0 \) depending only on \( f_1, f_2, r \) and the density \( \phi_\xi \) such that

\[
\mathbb{E}[|f(\xi) - f(\hat{\xi})|^r] \leq C \left( \mathbb{E}[|\xi - \hat{\xi}|^r] + \left( \mathbb{E} \left[ |\xi - \hat{\xi}|^r \right] \right)^{\frac{r}{r+1}} \right).
\]

**Proof.** Item 1) is obvious and follows directly from the Lipschitz condition for \( f_1 \). Item 2) is proved in [12] Theorem 2.4. Item 3) is a combination of 1) and 2). \( \square \)

**Assumption 4.8.** Throughout this Subsection, we always assume that Assumptions (K) and (Y) hold, the sequence of kernels \( \{K_m, m \geq 1\} \) satisfies Assumption (Km) and \( K_m \), \( K_m, m \geq 1 \), are all difference kernels, i.e.

\[
K(t, s) = K(t - s) \mathbb{1}_{s < \tau}, \quad K_m(t, s) = K_m(t - s) \mathbb{1}_{s < \tau}, \quad t, s \in [0, T].
\]

Moreover, we also impose Assumption (F) on the payoff function \( f = f_1 + f_2 \).

To allow for compact writing, denote

\[
W(t) := \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t), \quad t \in [0, T],
\]

where \( \rho \in (-1, 1) \) is from (1.3). Consider \( F_m := f(X_m(T)) \) and observe that for any \( r \in \mathbb{R} \)

\[
\mathbb{E} [F^r] + \sup_{m \geq 0} \mathbb{E} [F_m^r] < \infty \quad (4.6)
\]

by Lemma 3.8. For a given partition \( \pi = \{0 = t_0 < t_1 < \ldots < t_n = T\} \), denote also \( \Delta X(t_k) := X(t_{k+1}) - X(t_k), \Delta X_m(t_k) := X_m(t_{k+1}) - X_m(t_k) \) and consider the non-anticipating derivatives

\[
\mathcal{DF} := \lim_{|\pi| \to 0} u_\pi, \quad \mathcal{DF}_m := \lim_{|\pi| \to 0} u^m_\pi,
\]

where

\[
u_\pi := \sum_{k=0}^{n-1} u_{\pi,k} \mathbb{1}_{(t_k, t_{k+1}]}, \quad u_{\pi,k} := \frac{\mathbb{E} [F \Delta X(t_k) | \mathcal{F}_{t_k}]}{\mathbb{E} [\Delta X(t_k)^2 | \mathcal{F}_{t_k}]}, \quad (4.7)
\]

\[
u_m := \sum_{k=0}^{n-1} u^m_{\pi,k} \mathbb{1}_{(t_k, t_{k+1}]}, \quad u^m_{\pi,k} := \frac{\mathbb{E} [F_m \Delta X_m(t_k) | \mathcal{F}_{t_k}]}{\mathbb{E} [\Delta X_m(t_k)^2 | \mathcal{F}_{t_k}]}, \quad (4.8)
\]
Lemma 4.9. Let Assumptions (K) (Y) and (Km) hold, the payoff function $F$ satisfy Assumption (F) and both $\mathcal{K}$ and $\mathcal{K}_m$, $m \geq 1$, have the form

$$
\mathcal{K}(t, s) = \mathcal{K}(t - s)\mathbb{1}_{s < t}, \quad \mathcal{K}_m(t, s) = \mathcal{K}_m(t - s)\mathbb{1}_{s < t}, \quad t, s \in [0, T].
$$

Then there exists a constant $C > 0$ that does not depend on $m$ such that for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$

$$
\mathbb{E}[u_{\pi, k} - u^{m}_{\pi, k}] \leq \frac{C}{\sqrt{t_{k+1} - t_k}} \left( \left( \mathbb{E} \left[ (F - F_m)^4 \right] \right)^{\frac{1}{4}} + \|\mathcal{K} - \mathcal{K}_m\|_{L^2([0, T])} \right), \quad k = 0, \ldots, n - 1,
$$

where $u_{\pi, k}$ and $u^{m}_{\pi, k}$ are defined in (4.7) and (4.8).

Proof. Fix $k = 0, \ldots, n - 1$ and denote $\Delta := t_{k+1} - t_k$. It is easy to see that

$$
\mathbb{E}[u_{\pi, k} - u^{m}_{\pi, k}] \leq \mathbb{E} \left[ \left| F - F_m \right| \left| \Delta X(t_k) \right| \right] + \mathbb{E} \left[ \left| F_m \right| \left| \Delta X(t_k) - \Delta X_m(t_k) \right| \right] + \mathbb{E} \left[ \left| F_m \right| \left| \Delta X_m(t_k) \right| \right],
$$

(4.9)

$$
=: I_1 + I_2 + I_3.
$$

Now we will deal with each of the above summands separately.

**Step 1: estimation of $I_1$.** By Lemma B.1 there exist constants $C_1, C_2$ (not depending on the partition or $m$) such that

$$
\mathbb{E} \left[ \left| \Delta X(t_k) \right|^2 \right] \geq \Delta C_1 X^2(t_k)
$$

and

$$
\mathbb{E} \left[ \left| \Delta X(t_k) \right|^2 \right] \leq \Delta C_2 \mathbb{E} \left[ X^2(t_k) \right].
$$

Hence, using Hölder inequality and (2.7), one can write

$$
I_1 := \mathbb{E} \left[ \left| F - F_m \right| \left| \Delta X(t_k) \right| \right] \leq \frac{C}{\Delta} \mathbb{E} \left[ \left| F - F_m \right| \left| \Delta X(t_k) \right| X^{-2}(t_k) \right]
$$

$$
\leq \frac{C}{\Delta} \left( \mathbb{E} \left[ \left| \Delta X(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ X^{-8}(t_k) \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F - F_m \right|^4 \right] \right)^{\frac{1}{4}}
$$

(4.10)

$$
\leq \frac{C}{\sqrt{\Delta}} \left( \mathbb{E} \left[ X^2(t_k) \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F - F_m \right|^4 \right] \right)^{\frac{1}{4}} \leq \frac{C}{\sqrt{\Delta}} \mathbb{E} \left[ \left| F - F_m \right|^4 \right]^{\frac{1}{2}},
$$

where $C > 0$ can be chosen to not depend on $m$ or the partition.

**Step 2: estimation of $I_2$.** By Hölder inequality, Lemma B.1 Lemma B.2 and (4.6),

$$
I_2 := \mathbb{E} \left[ \left| F_m \right| \left| \Delta X(t_k) - \Delta X_m(t_k) \right| \right] \leq \frac{C}{\Delta} \left( \mathbb{E} \left[ \left| \Delta X(t_k) - \Delta X_m(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F_m \right| X^{-4}(t_k) \right] \right)^{\frac{1}{2}}
$$

(4.11)

$$
\leq \frac{C}{\sqrt{\Delta}} \left( \mathbb{E} \left[ \left| \Delta X(t_k) - \Delta X_m(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F_m \right| X^{-4}(t_k) \right] \right)^{\frac{1}{2}},
$$

where $C$ does not depend on $m$ or the partition.

**Step 3: estimation of $I_3$.** Using Hölder’s inequality and (4.6), one can verify that

$$
I_3 := \mathbb{E} \left[ \left| F_m \right| \left| \Delta X_m(t_k) \right| \right] \left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F_m \right| X^{-4}(t_k) \right] \right)^{\frac{1}{2}}
$$

(4.12)

$$
\leq \frac{C}{\Delta} \left( \mathbb{E} \left[ F_m^4 \right] \right)^{\frac{1}{4}} \left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^4 \right] \right)^{\frac{1}{4}} \left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F_m \right| X^{-4}(t_k) \right] \right)^{\frac{1}{2}}
$$

$$
\leq \frac{C}{\Delta} \left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^4 \right] \right)^{\frac{1}{4}} \left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| F_m \right| X^{-4}(t_k) \right] \right)^{\frac{1}{2}}.
$$

By the Burkholder–Davis–Gundy and Hölder inequalities, uniform boundedness of $Y_m$ and (2.7),

$$
\left( \mathbb{E} \left[ \left| \Delta X_m(t_k) \right|^4 \right] \right)^{\frac{1}{4}} \leq \mathbb{E} \left[ \left( \int_{t_k}^{t_{k+1}} Y_m(s)X_m(s)ds \right)^4 \right]^{\frac{1}{4}} \leq \left( \mathbb{E} \left[ \left( \int_{t_k}^{t_{k+1}} Y_m^2(s)X_m^2(s)ds \right)^2 \right] \right)^{\frac{1}{2}}
$$

(4.13)

$$
\leq \left( \Delta \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ Y_m^2(s)X_m^2(s) \right] ds \right)^{\frac{1}{2}} \leq \Delta \sqrt{\Delta} \left( \mathbb{E} \left[ \sup_{u \in [0, T]} X_m^4(u) \right] \right)^{\frac{1}{4}} \leq C \sqrt{\Delta}.
$$

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Further, one can verify using Itô formula that

$$
E \left[ (\Delta X_m(t_k))^2 - (\Delta X(t_k))^2 \right] \leq C E \left[ \left( \int_{t_k}^{t_{k+1}} \left( Y_m^2(s)X_m^2(s) - Y^2(s)X^2(s) \right) ds \right)^2 \right] + C E \left[ \left( \int_{t_k}^{t_{k+1}} \left( (X_m(s) - X_m(t_k))Y_m(s)X_m(s) - (X(s) - X(t_k))Y(s)X(s) \right) dW(s) \right)^2 \right].
$$

(4.14)

We study the last two summands separately. It is clear that

$$
E \left[ \left( \int_{t_k}^{t_{k+1}} \left( Y_m^2(s)X_m^2(s) - Y^2(s)X^2(s) \right) ds \right)^2 \right] \lesssim \Delta \int_{t_k}^{t_{k+1}} E \left[ \left( Y_m^2(s)X_m^2(s) - Y^2(s)X^2(s) \right)^2 \right] ds
$$

(4.15)

Taking into account that processes \( \{X^2(t), t \in [0,T]\} \) and \( \{X_m^2(t), t \in [0,T]\} \) satisfy the stochastic differential equations

$$
dX^2(t) = Y^2(t)X^2(t)dt + 2Y(t)X^2(t)dW(t),
$$

$$
dX_m^2(t) = Y_m^2(t)X_m^2(t)dt + 2Y_m(t)X_m^2(t)dW(t),
$$

boundedness of \( Y \), uniform boundedness of \( \{Y_m, m \geq 1\} \) as well as the same argument as in Theorem 3.9 imply that there exists a constant \( C > 0 \) such that

$$
E \left[ \sup_{s \in [0,T]} \left( X_m^2(s) - X^2(s) \right)^2 \right] \leq C \|K - K_m\|_{L^2([0,T])}^2,
$$

whereas, again like in the proof of Theorem 3.9, we have

$$
E \left[ X^2(s) \left( Y_m^2(s) - Y^2(s) \right)^2 \right] = E \left[ X^2(s)(Y_m(s) + Y(s))^2(Y_m(s) - Y(s))^2 \right]
$$

$$
\leq C E \left[ X^2(s)(Y_m(s) - Y(s))^2 \right] \leq C \|K - K_m\|_{L^2([0,T])}^2.
$$

Plugging these estimates in (4.15), we obtain that

$$
E \left[ \left( \int_{t_k}^{t_{k+1}} Y_m^2(s)X_m^2(s) - Y^2(s)X^2(s) ds \right)^2 \right] \leq C \Delta^2 \|K - K_m\|_{L^2([0,T])}^2.
$$

Next, for the second summand of (4.14) we have

$$
E \left[ \left( \int_{t_k}^{t_{k+1}} (X_m(s) - X_m(t_k))Y_m(s)X_m(s) - (X(s) - X(t_k))Y(s)X(s) ds \right)^2 \right]
$$

$$
= E \left[ \left( \int_{t_k}^{t_{k+1}} (X_m(s) - X_m(t_k))Y_m(s)X_m(s) - (X(s) - X(t_k))Y(s)X(s) ds \right)^2 \right]
$$

$$
\leq C \int_{t_k}^{t_{k+1}} E \left[ (X(s) - X(t_k))^2(Y_m(s)X_m(s) - Y(s)X(s))^2 \right] ds
$$

(4.16)

$$
+ C \int_{t_k}^{t_{k+1}} E \left[ (X(s) - X(t_k))^2 \right] \left( E \left[ (Y_m(s)X_m(s) - Y(s)X(s))^2 \right] \right)^{1/2} ds
$$

$$
+ C \int_{t_k}^{t_{k+1}} \left( E \left[ (X(s) - X(t_k))^4 \right] \right)^{1/2} \left( E \left[ X_m^4(s) \right] \right)^{1/2} ds.
$$

Using the same argument as in (4.13), one can verify that there exists a constant \( C > 0 \) (not depending on \( m \) or the partition) such that, for all \( s \in [t_k, t_{k+1}] \),

$$
E \left[ (X(s) - X(t_k))^4 \right] \leq C(s - t_k)^2.
$$

(4.17)
Moreover, using (2.7) and Theorem 3.7 it is easy to show that for any \( s \in [0,T] \)
\[
\mathbb{E} \left[ X^4(s)(Y_m(s) - Y(s))^4 \right] \leq \left( \mathbb{E} \left[ X^8(s) \right] \right)^{\frac{5}{8}} \left( \mathbb{E} \left[ (Y_m(s) - Y(s))^8 \right] \right)^{\frac{1}{8}} \leq C\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^4
\]
whence
\[
\mathbb{E} \left[ (Y_m(s)X_m(s) - Y(s)X(s))^4 \right] \\
\leq C \mathbb{E} \left[ Y_m(s)(X_m(s) - X(s))^4 \right] + C \mathbb{E} \left[ X^4(s)(Y_m(s) - Y(s))^4 \right] \\
\leq C \mathbb{E} \left[ (X_m(s) - X(s))^4 \right] + C \mathbb{E} \left[ X^4(s)(Y_m(s) - Y(s))^4 \right] \leq C\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^4
\]
(4.18)

Next, by Lemma B.2, for all \( s \in [t_k,t_{k+1}] \),
\[
\mathbb{E} \left[ \left( (X_m(s) - X_m(t_k)) - (X(s) - X(t_k)) \right)^4 \right] \leq C(s - t_k)^2\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^4
\]
(4.19)

Plugging the estimates (4.17)–(4.19) into (4.16), we finally obtain that there exists a constant \( C > 0 \) (not depending on \( m \) or the partition) such that
\[
\mathbb{E} \left[ \left( \int_{t_k}^{t_{k+1}} (X_m(s) - X_m(t_k))Y_m(s)X_m(s) - (X(s) - X(t_k))Y(s)X(s) \right) dW(s) \right]^2 \\
\leq C \int_{t_k}^{t_{k+1}} (s - t_k)^2\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^4 ds + C \int_{t_k}^{t_{k+1}} (s - t_k)^2\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^2 \left( \sup_{m \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} X_m^4(s) \right] \right)^{\frac{1}{2}} ds \\
\leq C\Delta^2\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^2
\]
Taking into account all of the above, we can finally write
\[
\mathbb{E} \left[ \left( (\Delta X_m(t_k))^2 - (\Delta X(t_k))^2 \right)^2 \right] \leq C\Delta^2\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^2
\]
and whence we have the estimate
\[
I_3 := \mathbb{E} \left[ |F_m| |\Delta X_m(t_k)| \right] \mathbb{E} \left[ \left( (\Delta X_m(t_k))^2 - (\Delta X(t_k))^2 \right) |F_{t_k} \right] \mathbb{E} \left[ (\Delta X_m(t_k))^2 |F_{t_k} \right] \\
\leq \frac{C}{\Delta^2} \left( \mathbb{E} \left[ |\Delta X_m(t_k)|^4 \right] \right)^{\frac{1}{4}} \left( \mathbb{E} \left[ (\Delta X_m(t_k))^2 - (\Delta X(t_k))^2 \right] \right)^{\frac{3}{4}} \\
\leq \frac{C}{\Delta^2} \sqrt{\Delta} \|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])} \leq \frac{C}{\sqrt{\Delta}} \|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}
\]
which finalizes the proof.  \( \square \)

**Theorem 4.10.** Let Assumptions \([K],[Y],[Y_m]\) hold, the payoff function \( f = f_1 + f_2 \) satisfy Assumption \([F]\), the kernels \( \mathcal{K} \) and \( \mathcal{K}_m \), \( m \geq 1 \), have the form
\[
\mathcal{K}(t,s) = \mathcal{K}(t-s)1_{s < t}, \quad \mathcal{K}_m(t,s) = \mathcal{K}_m(t-s)1_{s < t}, \quad t,s \in [0,T],
\]
and \( \|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])} \to 0 \), \( m \to \infty \). Then the following statements hold.

1) There exists a constant \( C > 0 \) that does not depend on \( m \) such that, for any partition \( \pi = \{0 = t_0 < t_1 < \ldots < t_n = T\} \) with the mesh \( |\pi| := \max_k |t_{k+1} - t_k| \),
\[
\mathbb{E} \left[ \int_0^T |u_\pi(s) - u_\pi^m(s)| ds \right] \leq C\sqrt{\pi}\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^\frac{1}{2}
\]
(4.20)

where \( u_\pi \) and \( u_\pi^m \) are defined by (4.7) and (4.3). In particular, if the partition \( \pi \) is uniform, i.e. \( t_k = \frac{kT}{n} \) and \( |\pi| = \frac{T}{n} \),
\[
\mathbb{E} \left[ \int_0^T |u_\pi(s) - u_\pi^m(s)| ds \right] \leq C\sqrt{n}\|\mathcal{K} - \mathcal{K}_m\|_{L^2([0,T])}^\frac{1}{2}
\]
(4.21)
2) If \( \{m_n, n \geq 1\} \) is such that

\[
n \sqrt{n} \| \mathcal{K} - \mathcal{K}_{m_n} \|_{L^2([0,T])} \to 0, \quad n \to \infty,
\]

then

\[
\mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_{m_n}^{m_n}(s)| ds \right] \to 0, \quad n \to \infty.
\]

**Proof.** By Theorem 2.9 the random variable \( X(T) \) has continuous and bounded density, whence conditions of Theorem 4.7 are fulfilled and

\[
\mathbb{E}[|F - F_m|^4] = \mathbb{E}[(f(X(T)) - f(X_m(T))^4] \leq C \left( \mathbb{E}[|X(T) - X_m(T)|^4] + \mathbb{E}[|X(T) - X_m(T)|^4] \right)^{\frac{1}{2}} \\
\leq C \left( \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])}^4 + \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])}^4 \right) \leq C\| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])},
\]

where the constant \( C \) does not depend on \( m \) or the partition. Furthermore, by Lemma 4.9

\[
\mathbb{E} \left[ \int_0^T |u_\pi(s) - u_{m_n}^{m_n}(s)| ds \right] = \sum_{k=0}^{n-1} \mathbb{E} \left[ |u_{\pi,k} - u_{m_n,k}^{m_n}| (t_{k+1} - t_k) \right] \\
\leq C \sum_{k=0}^{n-1} \sqrt{t_{k+1} - t_k} \left( \left( \mathbb{E}[|F - F_m|^4] \right)^{\frac{1}{2}} + \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])} \right) \\
\leq C n \sqrt{|\pi|} \left( \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])} + \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])} \right) \\
\leq C n \sqrt{|\pi|} \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])},
\]

which gives item 1).

To prove item 2), first note that

\[
\mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_\pi(s)| ds \right] = \mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_\pi(s)| X(s) \frac{1}{Y(s)X(s)} ds \right] \\
\leq \left( \mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_\pi(s)|^2 Y^2(s)X^2(s) ds \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \int_0^T \frac{1}{Y^2(s)X^2(s)} ds \right] \right)^{\frac{1}{2}} \\
\leq C \| \mathcal{D} F - u_\pi \|_{L^2(F \times [X])} \to 0
\]

as \( |\pi| \to 0 \). Whence

\[
\mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_{m_n}^{m_n}(s)| ds \right] \leq \mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_\pi(s)| ds \right] + \mathbb{E} \left[ \int_0^T |u_\pi(s) - u_{m_n}^{m_n}(s)| ds \right] \\
\leq \mathbb{E} \left[ \int_0^T |\mathcal{D} F(s) - u_\pi(s)| ds \right] + C n \sqrt{|\pi|} \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])} \to 0, \quad n \to \infty.
\]

By this, the proof is complete. \( \square \)

We remark that that the exponent \( \frac{1}{4} \) appears in (4.20) and (4.21) exclusively due the estimate (4.5) of Theorem 4.7 that corresponds to the (possibly discontinuous) component \( f_2 \) of \( f \). If \( f_2 \equiv 0 \), i.e. when

\[
\mathbb{E}[|F - F_m|^4] = \mathbb{E}[|f(X(T)) - f(X_m(T))^4] \leq C \mathbb{E}[|X(T) - X_m(T)|^4] \\
\leq C \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])},
\]

Theorem 4.10 can be reformulated as follows.

**Theorem 4.11.** Let Assumptions \( [K], [Y] \) and \( [Km] \) hold, the payoff function \( f = f_1 \) be globally Lipschitz, \( \mathcal{K} \) and \( \mathcal{K}_m, m \geq 1 \), have the form

\[
\mathcal{K}(t,s) = \mathcal{K}(t-s)\mathbb{1}_{s<t}, \quad \mathcal{K}_m(t,s) = \mathcal{K}_m(t-s)\mathbb{1}_{s<t}, \quad t, s \in [0,T],
\]

and \( \| \mathcal{K} - \mathcal{K}_m \|_{L^2([0,T])} \to 0, m \to \infty. \) Then the following statements are true.
1) There exists a constant $C > 0$ that does not depend on $m$ such that for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ with the mesh $|\pi| := \max |t_{k+1} - t_k|$, 
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C n \sqrt{|\pi|} \|K - K_m\|_{L^2([0,T])},
\]
where $u_\pi$ and $u^{m_n}_\pi$ are defined by (3.14) and (4.7). In particular, if $\pi$ is uniform, i.e. $t_k = \frac{kT}{n}$ and $|\pi| = \frac{T}{n}$, 
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C \sqrt{n} \|K - K_m\|_{L^2([0,T])}. \tag{4.22}
\]

2) If $\{m_n, n \geq 1\}$ is such that 
\[
n \sqrt{|\pi|} \|K - K_{m_n}\|_{L^2([0,T])} \to 0, \quad n \to \infty,
\]
then 
\[
E \left[ \int_0^T |DF(s) - u^{m_n}_\pi(s)| ds \right] \to 0, \quad n \to \infty.
\]

**Example 4.12.** Let the kernel $K$ be as in Examples 2.3 and 3.3, i.e. $K(t,s) = K(t-s)1_{s_1<s}$, $K(0) = 0$ and $K \in C^H([0,T])$ for some $H \in (0,1)$. Let $X$ be the corresponding SVV discounted price (1.5), $K_m$ be the Bernstein polynomial approximation of $K$ given by (3.9) and $X_m$ be the approximation of $X$ constructed using $K_m$. Then, taking into account (3.9) and Theorem 3.7, for any $r \geq 2$, 
\[
E \left[ \sup_{t \in [0,T]} |X(t) - X_m(t)|^r \right] \leq C m^{-\frac{5r}{4H}} \to 0, \quad m \to \infty.
\]
Moreover, for a uniform partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$, $t_k = \frac{kT}{n}$, Theorem 4.10 and Theorem 4.17 imply that 
1) if $f = f_1 + f_2$ satisfies Assumption $[F]$ and $m_n := n^\alpha$ for $\alpha > \frac{5}{11}$, then
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C n^{-\frac{5r}{4H} + \frac{1}{2}} \to 0, \quad n \to \infty; \tag{4.23}
\]
2) if $f = f_1$ is globally Lipschitz and $m_n := n^\alpha$ for $\alpha > \frac{5}{11}$, then
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C n^{-\frac{5r}{4H} + \frac{1}{2}} \to 0, \quad n \to \infty. \tag{4.24}
\]

**Example 4.13.** Let the kernel $K$ be the fractional kernel from Examples 2.2 and 3.3, $X$ be the corresponding SVV discounted price (1.5), $K_m$ be the approximation of $K$ with exponentials given by (3.12) with $\tau_{m,i}$, $i = 0, \ldots, m$, $m \geq 1$, given by (3.13). Let also $X_m$ be the approximation of $X$ constructed using $K_m$. Then, taking into account (3.14) and Theorem 3.7, for any $r \geq 2$, 
\[
E \left[ \sup_{t \in [0,T]} |X(t) - X_m(t)|^r \right] \leq C m^{-\frac{5r}{8H}}.
\]
Moreover, for a uniform partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$, $t_k = \frac{kT}{n}$, Theorem 4.10 and Theorem 4.17 imply that 
1) if $f = f_1 + f_2$ satisfies Assumption $[F]$ and $m_n := n^\alpha$ for $\alpha > \frac{20}{21}$, then
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C n^{-\frac{20r}{21} + \frac{1}{2}} \to 0, \quad n \to \infty; \tag{4.25}
\]
2) if $f = f_1$ is globally Lipschitz and $m_n := n^\alpha$ for $\alpha > \frac{5}{8H}$, then
\[
E \left[ \int_0^T |u_\pi(s) - u^{m_n}_\pi(s)| ds \right] \leq C n^{-\frac{5r}{8H} + \frac{1}{2}} \to 0, \quad n \to \infty. \tag{4.26}
\]
5 Monte Carlo computation of the optimal hedge and simulations

In Section 4, we approximated the optimal hedging portfolio $\mathcal{D}f(X(T))$ with $\mathcal{D}f(X_m(T))$, where $X_m$ is a coordinate of $(m+2)$-dimensional Markov process. As noted above, Markovianity is incredibly beneficial for the numerical computation of conditional expectations in (4.8): indeed,

$$
\mathbb{E}[f(X_m(T))(X_m(t_{k+1}) - X_m(t_k)) \mid \mathcal{F}_{t_k}] \\
= \mathbb{E}[f(X_m(T))(X_m(t_{k+1}) - X_m(t_k)) \mid X_m(t_k), Y_m(t_k), U_{m,0}(t_k), ..., U_{m,m}(t_k)] \\
=: \Phi_1(t_k, X_m(t_k), Y_m(t_k), U_{m,0}(t_k), ..., U_{m,m}(t_k)),
$$

and

$$
\mathbb{E}[(X_m(t_{k+1}) - X_m(t_k))^2 \mid \mathcal{F}_{t_k}] \\
= \mathbb{E}[(X_m(t_{k+1}) - X_m(t_k))^2 \mid X_m(t_k), Y_m(t_k), U_{m,0}(t_k), ..., U_{m,m}(t_k)] \\
=: \Phi_2(t_k, X_m(t_k), Y_m(t_k), U_{m,0}(t_k), ..., U_{m,m}(t_k)),
$$

i.e. the computational problem boils down to learning the shape of functions $\Phi_1$ and $\Phi_2$ that can be done via Monte Carlo methods.

In this Section, we propose two algorithms for estimating $\Phi_1$ and $\Phi_2$: Nested Monte Carlo (NMC) and Least-Squares Monte Carlo (LSMC). In order to simulate paths of sandwiched volatility processes (1.4), we use the drift-implicit Euler approximation scheme from [24]. The original and the approximated discounted price processes $X$ and $X_m$ are simulated just like in [20]. We also present simulations of the hedging strategies for a standard European call option. Note that algorithms presented below also work for exotic contracts with discontinuous payoffs, but slower convergence rates require longer computations. All simulations were performed in R programming language on the system with Intel Core i9-9900K CPU and 64 Gb RAM.

5.1 Nested Monte Carlo method

The first and more straightforward way to compute $\Phi_1$ and $\Phi_2$ is the Nested Monte Carlo (NMC) approach that can be summarized as follows (see also Fig. [1]):

1) given $X_m(t_k) = x$, $Y_m(t_k) = y$, $U_{m,0}(t_k) = u_0$, ..., $U_{m,m}(t_k) = u_m$, simulate $N$ independent trajectories $\{X_m^{(i)}(t), \ t \in (t_k, T]\}, i = 1, ..., N$;

2) for each trajectory, compute $f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))$ and $(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2$, $i = 1, ..., N$;

3) put

$$
\hat{\Phi}_1(t_k, x, y, u_0, ..., u_m) := \frac{1}{N} \sum_{i=1}^{N} f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k)),
$$

and

$$
\hat{\Phi}_2(t_k, x, y, u_0, ..., u_m) := \frac{1}{N} \sum_{i=1}^{N} (X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2.
$$

Example 5.1. (Hölder continuous kernel) Consider the SVV model

$$
X(t) = X(0) + \int_0^t Y(s) X(s) \left( \rho dB_1(s) + \sqrt{1 - \rho^2} dB_2(s) \right),
$$

$$
Y(t) = Y(0) + \int_0^t b(s, Y(s)) ds + \int_0^t K(t, s) dB_1(s),
$$

with $X(0) = 5$, $Y(0) = 1$, $\rho = 0.5$, $b(t, y) = \frac{1}{(y-0.01)^2} - \frac{1}{(y+9.8)^2}$ and $K(t, s) = K(t-s) = (t-s)^{0.4}$. The approximation $(X_m, Y_m, U_{m,0}, U_{m,1}, ..., U_{m,m})$ is constructed using the Bernstein polynomial approximation $K_m$ from (3.8).

Figure 2 illustrates the approximation of the SVV model; original kernel and generated sample paths of the corresponding processes are depicted in black whereas approximations using the Bernstein polynomials are depicted in red (of order $m = 10$) and blue (of order $m = 30$). Note that the path of $X$ simulated
The approximation (m approximations are depicted in red (original kernel and generated sample paths of the corresponding processes are depicted in black whereas Example 3.3 and Remark 3.4. Just as in Example 5.1, Figure 4 illustrates approximation of the SVV model; Figure 1: Nested Monte Carlo approach, \( t_k = 0.8, T = 1 \). Given the values \( X_m(t_k), Y_m(t_k), U_{m,1}(t_k), \ldots, U_{m,m}(t_k) \), we simulate a number of trajectories \( X_m^{(i)} \) on \( (t_k,T) \) (blue), use each of those to compute \( f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2 \) and \( (X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2 \). The means of the latter values are then used as approximations of the required conditional expectations.

using the original noise (black) is not visible on Figure 2(d) due to the high degree of overlapping with the approximations (red and blue trajectories). Figure 3 contains a path of the process

\[
\hat{u}_\pi(t) := \sum_{k=0}^9 \Phi_1 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), \ldots, U_{m,m} \left( \frac{k}{10} \right) \right) \Phi_2 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), \ldots, U_{m,m} \left( \frac{k}{10} \right) \right) Y(t),
\]

with the European call payoff function \( f(x) := \max\{x - 4, 0\} \) and the values \( X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), \ldots, U_{m,m} \left( \frac{k}{10} \right) \) coming exactly from the trajectory depicted on Figure 2. In order to estimate

\[
\hat{\Phi}_1 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), \ldots, U_{m,m} \left( \frac{k}{10} \right) \right),
\]

\[
\hat{\Phi}_2 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), \ldots, U_{m,m} \left( \frac{k}{10} \right) \right),
\]

100000 simulations were used for each \( k = 0, 1, \ldots, 9 \).

Example 5.2. (Rough kernel) Consider the SVV model

\[
X(t) = X(0) + \int_0^t Y(s)X(s) \left( \rho dB_1(s) + \sqrt{1 - \rho^2} dB_2(s) \right),
\]

\[
Y(t) = Y(0) + \int_0^t b(s,Y(s))ds + \int_0^t K(t,s)dB_1(s),
\]

with \( X(0) = 5, Y(0) = 1, \rho = 0.5, b(t,y) = 1/y - 1/4, \) and \( K(t,s) = 1/\sqrt{t-s} \). The approximation \((X_m,Y_m,U_{m,1},...,U_{m,m})\) is constructed by approximating the Volterra noise \( Z(t) := \int_0^t (1/\sqrt{t-s}) dB_1(s) \) by a linear combination of standard Ornstein-Uhlenbeck processes as described in Example 3.3 and Remark 3.4. Just as in Example 5.1, Figure 4 illustrates approximation of the SVV model; original kernel and generated sample paths of the corresponding processes are depicted in black whereas approximations are depicted in red (m = 10 summands), green (m = 100 summands) and blue (m = 1000 summands).
(a) Kernels $K$ and $K_m$

(b) Volterra noises $Z$ and $Z_m$

(c) Volatility processes $Y$ and $Y_m$

(d) Price processes $X$ and $X_m$

Figure 2: Approximation of the SVV model with Hölder continuous kernel

Figure 3: Hedging strategy estimated for the path from Fig. 2 for $m = 10$ (red) and $m = 30$ (blue). The corresponding partition is $k/10$, $k = 0, 1, \ldots, 10$. Simulating the red line took 63231 seconds whereas the blue line took 73410 seconds.

Note that rough volatility requires much higher values of $m$ in order to ensure a decent level of
approximation in comparison to Example 5.1. Figure 5 contains a path of the process
\[ \hat{u}_n(t) := \sum_{k=0}^{9} \Phi_1 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), ..., U_{m,m} \left( \frac{k}{10} \right) \right) \Phi \left( \frac{k}{10}, \frac{k+1}{10} \right)(t), \] (5.2)
\[ t \in [0,1], \] where the payoff function \( f(x) := \max\{x - 4, 0\} \) and the values \( X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,1} \left( \frac{k}{10} \right), ..., U_{m,m} \left( \frac{k}{10} \right) \) are exactly the ones from the trajectory depicted on Figure 4. In order to estimate
\[ \hat{\Phi}_1 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), ..., U_{m,m} \left( \frac{k}{10} \right) \right), \]
\[ \hat{\Phi}_2 \left( \frac{k}{10}, X_m \left( \frac{k}{10} \right), Y_m \left( \frac{k}{10} \right), U_{m,0} \left( \frac{k}{10} \right), ..., U_{m,m} \left( \frac{k}{10} \right) \right), \]
100000 simulations were used for each \( k = 0, 1, ..., 9. \)

Figure 4: Approximation of the SVV model with rough fractional kernel
Figure 5: Hedging strategy estimated for the path from Fig. 4 for $m = 10$ (red), $m = 100$ (green), $m = 1000$ (blue) and $m = 2000$ (black). The corresponding partition is $k/10$, $k = 0, 1, \ldots, 10$. The figure also illustrates slower rate of convergence in comparison to the Hölder kernel case: the black and blue lines are close to each other but the red and green lines (corresponding to relatively low values of $m$) differ substantially. Computation time: 24162 seconds for $m = 10$, 25348 seconds for $m = 100$, 38530 seconds for $m = 1000$ and 42431 seconds for $m = 2000$.

5.2 Least squares Monte Carlo method

Despite its simplicity and clear theoretical justification, the nested Monte Carlo approach has a substantial disadvantage: it takes long to compute and thus requires powerful computational resources in order to be used in practice. In order to overcome this issue, one can use the Least Squares Monte Carlo (LSMC) method instead:

1) simulate $N$ independent realizations

$$(X_m^{(i)}, Y_m^{(i)}, U_{m,0}^{(i)}, \ldots, U_{m,m}^{(i)}) = \{(X_m^{(i)}(t), Y_m^{(i)}(t), U_{m,0}(t), \ldots, U_{m,m}(t)), \ t \in [0,T]\},$$

$i = 1, \ldots, N$;

2) for each path $i = 1, \ldots, N$, evaluate

$$(X_m^{(i)}(t_k), Y_m^{(i)}(t_k), U_{m,0}(t_k), \ldots, U_{m,m}(t_k)), \ f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k)), (X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2;$$

3) apply an appropriate non-parametric regression (with a mean squared error loss function) to the generated “dataset” treating

$$(X_m^{(i)}(t_k), Y_m^{(i)}(t_k), U_{m,0}(t_k), \ldots, U_{m,m}(t_k)), \ i = 1, \ldots, N,$$

as “input” and

$$f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k)), (X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2, \ i = 1, \ldots, N,$$

as “output” variables. The resulting estimates $\hat{\Phi}_1(t_k, \cdot), \hat{\Phi}_2(t_k, \cdot)$ of the regression functions are then used to calculate $\tilde{u}_\pi$ on the interval $(t_k, t_{k+1}]$.

Remark 5.3. The Least Squares Monte Carlo method described above can be regarded as a simplified version of the approach suggested in [20, 53, 60] for pricing American options or in [26, 51, 52] for modeling of life insurance companies.

Example 5.4. Fig. 6 contains approximations of the optimal hedging strategy constructed for the path of $X_m$ from Example 5.1 that corresponds to $m = 10$. In order to obtain the dark green line, we simulated the “dataset” containing the “input” variables

$$(X_m^{(i)}(t_k), Y_m^{(i)}(t_k), U_{m,0}(t_k), \ldots, U_{m,m}(t_k)), \ i = 1, \ldots, N,$$
together with the corresponding “output” variables
\[ f(X_m^{(i)}(T))(X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k)), \quad (X_m^{(i)}(t_{k+1}) - X_m^{(i)}(t_k))^2, \quad i = 1, \ldots, N, \]
with \( N = 1000000 \) for each \( k = 0, 1, \ldots, 9 \). In order to compute \( \hat{\Phi}_1(t_k, \cdot), \hat{\Phi}_2(t_k, \cdot) \), we used the idea from [51] and utilized a neural network with 3 hidden layers and 20 nodes in each layer. The red line is exactly the path of \( \hat{u}_\pi \) for \( m = 10 \) from Fig. 3 and is treated as a reference.

Figure 6: Approximation of the optimal hedge using the Nested Monte Carlo method for \( m = 10 \) from Fig. 3 (red) and using the Least Squares Monte Carlo method (dark green).

Note that the Nested approach described in Example 5.1 required 63231 seconds to simulate a path of \( \hat{u}_\pi \) corresponding to a single realization of \((X_m, Y_m, U_{m,0}, \ldots U_{m,m})\) whereas the Least Squares modification took roughly 6 days to simulate the initial dataset and about 5 hours to fit the neural network. Once ready, the actual computations were conducted almost instantly.

Remark 5.5. The neural network architecture in Example 5.4 is not optimal. It is fairly clear from Fig. 6 even though the LSMC Monte Carlo estimate preserves in general the shape of the reference optimal hedge simulated using the NMC approach, the difference between the two is still quite substantial. We emphasize that performance of the NMC method heavily depends on the chosen non-parametric regression method, and therefore a separate investigation on performance of different regression approaches is required.

Remark 5.6. Similar method can theoretically be applied for the SVV model with a rough fractional kernel. However, if one chooses the values of \( \tau_{m,i} \) as in (3.13), one may need a very high dimensionality of the “input” variables vector (over 1000) which requires a dataset with possibly unrealistically huge number of observations. Perhaps a different choice of \( \tau_{m,i} \) (e.g. as in [1]) may improve the situation; a separate analysis on that is needed.

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A Proof of Theorem 3.7

Fix \( t \in [0, T] \) and consider a uniform partition \( 0 = t_0 < t_1 < ... < t_N = t, \, t_n = \frac{n}{N} t, \) of the interval \([0, t]\) such that the mesh \( \Delta_N := \frac{t}{N} \) satisfies the condition

\[
e_{3} \Delta_N < 1, \tag{A.1}
\]

for \( c_{3} \) from (Y4). Denote \( e_{n,m} := Y(t_n) - Y_m(t_n) \) and observe that

\[
e_{N,m} = Y(t_{N-1}) + \int_{t_{N-1}}^{t_N} b(s, Y(s))ds + Z(t_N) - Z(t_{N-1})
\]

\[
- Y_m(t_{N-1}) - \int_{t_{N-1}}^{t_N} b(s, Y_m(s))ds + Z_m(t_N) - Z_m(t_{N-1})
\]

\[
= e_{N-1,m} + \int_{t_{N-1}}^{t_N} (b(s, Y(s)) - b(t_N, Y(t_N)))ds - \int_{t_{N-1}}^{t_N} (b(s, Y_m(s)) - b(t_N, Y_m(t_N)))ds
\]

\[
+ (b(t_N, Y(t_N)) - b(t_N, Y_m(t_N))) \Delta_N + (Z(t_N) - Z(t_{N-1})) - (Z_m(t_N) - Z_m(t_{N-1})).
\]
Note that, for each \( n = 1, \ldots, N \), there exists \( \Theta_{n,m} \) between \( Y(t_n) \) and \( Y_m(t_n) \) such that
\[
b(t_n, Y(t_n)) - b(t_n, Y_m(t_n)) = \frac{\partial b}{\partial y}(t_n, \Theta_{n,m})(Y(t_n) - Y_m(t_n)) = \frac{\partial b}{\partial y}(t_n, \Theta_{n,m})e_{n,m},
\]
and thus
\[
\left(1 - \frac{\partial b}{\partial y}(t_n, \Theta_{N,m})\Delta_N\right)e_{N,m} = e_{N-1,m} + \int_{t_{N-1}}^{t_N} (b(s, Y(s)) - b(t_N, Y(t_N))) \, ds
\]
\[
- \int_{t_{N-1}}^{t_N} (b(s, Y_m(s)) - b(t_N, Y_m(t_N))) \, ds + (Z(t_N) - Z(t_{N-1})) - (Z_m(t_N) - Z_m(t_{N-1})).
\]

Observe that each \( 1 - \frac{\partial b}{\partial y}(t_n, \Theta_{N,m})\Delta_N > 0 \) by (A.1) and denote
\[
\zeta_{0,m} := 1, \quad \zeta_{n,m} := \prod_{k=1}^{n} \left(1 - \frac{\partial b}{\partial y}(t_k, \Theta_{k,m})\Delta_N\right),
\]
\[
\tilde{e}_{n,m} := \zeta_{n,m}e_{n,m} \quad \text{By multiplying the left- and right-hand sides of (A.2) by } \zeta_{N-1,m}, \text{ we obtain:}
\]\[
\tilde{e}_{N,m} = \tilde{e}_{N-1,m} + \zeta_{N-1,m} \int_{t_{N-1}}^{t_N} (b(s, Y(s)) - b(t_N, Y(t_N))) \, ds
\]
\[
- \zeta_{N-1,m} \int_{t_{N-1}}^{t_N} (b(s, Y_m(s)) - b(t_N, Y_m(t_N))) \, ds
\]
\[
+ \zeta_{N-1,m}(Z(t_N) - Z(t_{N-1})) - (Z_m(t_N) - Z_m(t_{N-1}))
\]
\[
= \tilde{e}_{N-2,m} + \sum_{n=N-1}^{N} \zeta_{n-1,m} \int_{t_{n-1}}^{t_n} (b(s, Y(s)) - b(t_n, Y(t_n))) \, ds
\]
\[
- \sum_{n=N-1}^{N} \zeta_{n-1,m} \int_{t_{n-1}}^{t_n} (b(s, Y_m(s)) - b(t_n, Y_m(t_n))) \, ds
\]
\[
+ \sum_{n=N-1}^{N} \zeta_{n-1,m}((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))
\]
\[
= \sum_{n=1}^{N} \zeta_{n-1,m} \int_{t_{n-1}}^{t_n} (b(s, Y(s)) - b(t_n, Y(t_n))) \, ds - \sum_{n=1}^{N} \zeta_{n-1,m} \int_{t_{n-1}}^{t_n} (b(s, Y_m(s)) - b(t_n, Y_m(t_n))) \, ds
\]
\[
+ \sum_{n=1}^{N} \zeta_{n-1,m}((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))
\]
\[
i.e.
\]
\[
e_{N,m} = \sum_{n=1}^{N} \frac{\zeta_{N,m}}{\zeta_{N-1,m}} \int_{t_{n-1}}^{t_n} (b(s, Y(s)) - b(t_n, Y(t_n))) \, ds - \sum_{n=1}^{N} \frac{\zeta_{N,m}}{\zeta_{N-1,m}} \int_{t_{n-1}}^{t_n} (b(s, Y_m(s)) - b(t_n, Y_m(t_n))) \, ds
\]
\[
+ \sum_{n=1}^{N} \frac{\zeta_{N,m}}{\zeta_{N-1,m}}((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))).
\]

Next, observe that, by (Y4) and (A.1), for any \( n = 1, \ldots, N \)
\[
\frac{\zeta_{n-1,m}}{\zeta_{N,m}} = \prod_{k=n}^{N} \left(1 - \frac{\partial b}{\partial y}(t_k, \Theta_{k,m})\Delta_N\right)^{-1} \leq \prod_{k=n}^{N} (1 - c_3\Delta_N)^{-1} \leq (1 - c_3\Delta_N)^{-N}.
\]

Note that \((1 - c_3\Delta_N)^{-N}\) converges as \( N \to \infty \) and hence is bounded w.r.t. \( N \), therefore there exists a (non-random) constant \( C > 0 \) that does not depend on \( n, N \) or \( m \) such that \( \frac{\zeta_{n-1,m}}{\zeta_{N,m}} \leq C \). Using this, one
can easily deduce that

\[
|c_{N,m}| \leq \left| \sum_{n=1}^{N} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} \int_{t_{n-1}}^{t_n} (b(s, Y(s)) - b(t_n, Y(t_n))) \, ds \right| + \left| \sum_{n=1}^{N} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} \int_{t_{n-1}}^{t_n} (b(s, Y_m(s)) - b(t_n, Y_m(t_n))) \, ds \right| + \left| \sum_{n=1}^{N} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} ((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))) \right| (A.3)
\]

\[
\leq C \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} |b(s, Y(s)) - b(t_n, Y(t_n))| \, ds + C \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} |b(s, Y_m(s)) - b(t_n, Y_m(t_n))| \, ds
\]

\[
+ \left| \sum_{n=1}^{N} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} ((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))) \right|
\]

Next, fix an arbitrary \( \lambda \in \left( \frac{1}{1+\gamma}, H \right) \) and observe that (2.4) and (Y2) imply

\[
|b(s_1, Y(s_1)) - b(s_2, Y(s_2))| \leq C(L_2 + \Lambda)^{\frac{1}{1+\gamma}} (|s_1 - s_2|^H + |Y(s_1) - Y(s_2)|)
\]

\[
\leq C(L_2 + \Lambda)^{\frac{1}{1+\gamma}} (|s_1 - s_2|^\lambda + |Y(s_1) - Y(s_2)|)
\]

for all \( s_1, s_2 \in [0, T] \). Therefore, by Lemma 3.6

\[
\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} |b(s, Y(s)) - b(t_n, Y(t_n))| \, ds
\]

\[
\leq C(L_2 + \Lambda)^{\frac{1}{1+\gamma}} \left( \sum_{n=1}^{N} t_{n-1} |s - t_n|^\lambda ds + \sum_{n=1}^{N} t_{n-1} |Y(s) - Y(t_n)|ds \right) (A.4)
\]

\[
\leq C(L_2 + \Lambda)^{\frac{1}{1+\gamma}} \left( \sum_{n=1}^{N} t_{n-1} |s - t_n|^\lambda ds + \mathcal{Y} \sum_{n=1}^{N} t_{n-1} |s - t_n|^\lambda ds \right)
\]

\[
\leq C(L_2 + \Lambda)^{\frac{1}{1+\gamma}} (1 + \mathcal{Y}) \Delta_N^\lambda,
\]

where \( \mathcal{Y} \) is from Lemma 3.6 and has all the moments. Moreover, (3.16) and the same argument as above yield that

\[
\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} |b(s, Y_m(s)) - b(t_n, Y_m(t_n))| \, ds \leq C(L_2 + \Lambda_m)^{\frac{1}{1+\gamma}} (1 + \mathcal{Y}_m) \Delta_N^\lambda
\]

(A.5)

with \( \mathcal{Y}_m \) being from Lemma 3.6 Finally, using Abel summation-by-parts formula, one can write:

\[
\left| \sum_{n=1}^{N} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} ((Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1}))) \right|
\]

\[
= \left| \frac{\zeta_{N-1,m}}{\zeta_{N,m}} (Z(t_N) - Z_m(t_N)) - \sum_{n=1}^{N-1} (Z(t_n) - Z_m(t_n)) \left( \frac{\zeta_{n,m}}{\zeta_{N,m}} - \frac{\zeta_{n-1,m}}{\zeta_{N,m}} \right) \right|
\]

\[
= \left| \frac{\zeta_{N-1,m}}{\zeta_{N,m}} (Z(t_N) - Z_m(t_N)) + \sum_{n=1}^{N-1} \frac{\zeta_{n-1,m}}{\zeta_{N,m}} \frac{\partial b}{\partial y}(t_n, \Theta_{n,m}) (Z(t_n) - Z_m(t_n)) \Delta_N \right|
\]

\[
\leq C |Z(t_N) - Z_m(t_N)| + C \sum_{n=1}^{N-1} \left| \frac{\partial b}{\partial y}(t_n, \Theta_{n,m}) \right| |Z(t_n) - Z_m(t_n)| \Delta_N.
\]

Random variables \( \Theta_{n,m} \) lie between \( Y(t_n) \) and \( Y_m(t_n) \), hence

\[
\Theta_{n,m} - \varphi(t) \geq \frac{L_1}{(L_2 + \Lambda)^{\frac{1}{1+\gamma}}} \wedge \frac{L_1}{(L_2 + \Lambda_m)^{\frac{1}{1+\gamma}}},
\]

\[
\psi(t) - \Theta_{n,m} \geq \frac{L_1}{(L_2 + \Lambda)^{\frac{1}{1+\gamma}}} \wedge \frac{L_1}{(L_2 + \Lambda_m)^{\frac{1}{1+\gamma}}},
\]

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Lemma B.1. Let Assumptions (K), (Y) and (Km) hold. For any pair
\[ \left. \frac{\partial h}{\partial y} (t_n, \Theta_{n,m}) \right| \leq C \left( 1 + (L_2 + \Lambda)^{\frac{1}{q+1}} + (L_2 + \Lambda_m)^{\frac{1}{q+1}} \right), \]
where C is, as always, a deterministic constant that does not depend on n, N or m. Therefore,
\[ \sum_{n=1}^{N-1} \left| \xi_{n,m} (Z(t_n) - Z(t_{n-1})) - (Z_m(t_n) - Z_m(t_{n-1})) \right| \]
\[ \leq C \left( |Z(t_N) - Z_m(t_N)| + \xi_m \sum_{n=1}^{N-1} |Z(t_n) - Z_m(t_n)| \Delta_N \right), \]
where \( \xi_m := 1 + (L_2 + \Lambda)^{\frac{1}{q+1}} + (L_2 + \Lambda_m)^{\frac{1}{q+1}}. \)

Summarizing (A.3) and (A.4)–(A.6), we have that
\[ |Y(t) - Y_m(t)| \leq C(L_2 + \Lambda)^{\frac{1}{q+1}} (1 + Y) \Delta_N^3 + C(L_2 + \Lambda_m)^{\frac{1}{q+1}} (1 + Y_m) \Delta_N^3 \]
\[ + C|Z(t) - Z_m(t)| + C \xi_m \sum_{n=1}^{N-1} |Z(t_n) - Z_m(t_n)| \Delta_N \]
and, moving \( \Delta_N \to 0 \), we obtain
\[ |Y(t) - Y_m(t)| \leq C \left( |Z(t) - Z_m(t)| + \xi_m \int_0^t |Z(s) - Z_m(s)| ds \right). \]
It remains to notice that \( \sup_{m \geq 1} \mathbb{E}[\xi_m^r] < \infty \) for any \( r > 0 \) due to (3.15) from Lemma 3.5.

B Estimates for the increments of \( X \) and \( X_m \)

In this Section, we will provide some technical estimates for the increments of \( X \) and \( X_m \), where \( X \) is the SVV discounted price (1.5) and \( X_m \) is its approximation (3.3). To allow for compact writing, we also denote
\[ W(t) := \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t), \quad t \in [0, T], \]
where \( \rho \in (-1, 1) \) is from (1.3).

Lemma B.1. Let Assumptions (K), (Y) and (Km) hold. For any pair \( 0 \leq t_1 \leq t_2 \leq T \), there exist positive constants \( C_1 < C_2 \) that do not depend on m or the particular choice of \( t_1, t_2 \) such that
\[ C_1 (t_2 - t_1) X^2(t_1) \leq \mathbb{E} \left[ (\Delta X)^2 \mid \mathcal{F}_{t_1} \right] \leq C_2 (t_2 - t_1) X^2(t_1) \]
and
\[ C_1 (t_2 - t_1) X_m^2(t_1) \leq \mathbb{E} \left[ (\Delta X_m)^2 \mid \mathcal{F}_{t_1} \right] \leq C_2 (t_2 - t_1) X_m^2(t_1). \]

Here above \( \Delta X := X(t_2) - X(t_1), \Delta X_m := X_m(t_2) - X_m(t_1). \)

Proof. We only give the proof for the process \( X \); the one for \( X_m \) is identical. By the very definition of \( X \) and Itô lemma, we have
\[ \mathbb{E} \left[ (\Delta X)^2 \mid \mathcal{F}_{t_1} \right] = \mathbb{E} \left[ 2 \int_{t_1}^{t_2} (X(s) - X(t_1))Y(s)X(s) dW(s) \mid \mathcal{F}_{t_1} \right] + \mathbb{E} \left[ \int_{t_1}^{t_2} Y^2(s)X^2(s) ds \mid \mathcal{F}_{t_1} \right] \]
\[ = \int_{t_1}^{t_2} \mathbb{E} \left[ Y^2(s)X^2(s) \mid \mathcal{F}_{t_1} \right] ds. \]
We remark that the above relies on the boundedness of \( Y \) and the moments of \( X \).

Let now \( s > t_1 \) be fixed. Note that, with probability 1,
\[ \mathbb{E} \left[ X^2(s) \mid \mathcal{F}_{t_1} \right] \min_{t \in [0, T]} \varphi(t) \leq \mathbb{E} \left[ Y^2(s)X^2(s) \mid \mathcal{F}_{t_1} \right] \leq \mathbb{E} \left[ X^2(s) \mid \mathcal{F}_{t_1} \right] \max_{t \in [0, T]} \psi(t) \]
(B.2)
and
\[ E[X^2(s) | \mathcal{F}_t] = E \left[ \exp \left\{ 2 \int_0^s Y(u) dW(u) - \int_0^s Y^2(u) du \right\} | \mathcal{F}_t \right], \]

whence, by the boundedness of \( Y \),
\[ E[X^2(s) | \mathcal{F}_t] \leq e^{T \max_{t \leq s} \psi(t)} E \left[ \exp \left\{ \int_0^s 2Y(u) dW(u) - \frac{1}{2} \int_0^s (2Y(u))^2 du \right\} | \mathcal{F}_t \right] \]

Next, by the Novikov criterion, the process
\[ \exp \left\{ \int_0^s 2Y(u) dW(u) - \frac{1}{2} \int_0^s (2Y(u))^2 du \right\}, \quad s \in [0, T], \]
is a martingale, and hence, for \( s > t_1 \),
\[ E \left[ \exp \left\{ \int_0^s 2Y(u) dW(u) - \frac{1}{2} \int_0^s (2Y(u))^2 du \right\} | \mathcal{F}_t \right] = \exp \left\{ \int_0^{t_1} 2Y(u) dW(u) - \frac{1}{2} \int_0^{t_1} (2Y(u))^2 du \right\} = X^2(t_1) \exp \left\{ - \int_{t_1}^s Y^2(u) du \right\}. \]

Therefore,
\[ E \left[ \exp \left\{ \int_0^s 2Y(u) dW(u) - \frac{1}{2} \int_0^s (2Y(u))^2 du \right\} | \mathcal{F}_t \right] \leq X^2(t_1), \]
\[ E \left[ \exp \left\{ \int_0^s 2Y(u) dW(u) - \frac{1}{2} \int_0^s (2Y(u))^2 du \right\} | \mathcal{F}_t \right] \geq X^2(t_1) e^{-T \max_{t \leq s} \psi(t)}, \]

and we can now write
\[ e^{-T \max_{t \leq s} \psi(t)} X^2(t_1) \leq E[X^2(s) | \mathcal{F}_t] \leq e^{T \max_{t \leq s} \psi(t)} X^2(t_1). \] (B.3)

Finally, by (B.1) and (B.3),
\[ C_1(t_2 - t_1) X^2(t_1) \leq E[\Delta X^2 | \mathcal{F}_t] \leq C_2(t_2 - t_1) X^2(t_1) \]
where
\[ C_1 := \min_{t \in [0, T]} \varphi(t) e^{-T \max_{t \leq s} \psi(t)}, \quad C_2 := \max_{t \in [0, T]} \varphi(t) e^{T \max_{t \leq s} \psi(t)}. \]

\[ \Box \]

**Lemma B.2.** Let Assumptions \([K] \), \([Y] \) and \([Km] \) hold and both \( \mathcal{K} \) and \( \mathcal{K}_m \), \( m \geq 1 \), have the form
\[ \mathcal{K}(t, s) = \mathcal{K}(t-s) \mathbf{1}_{s \leq t}, \quad \mathcal{K}_m(t, s) = \mathcal{K}_m(t-s) \mathbf{1}_{s \leq t}, \quad t, s \in [0, T]. \]

For any \( 0 \leq t_1 \leq t_2 \leq T \) and \( r \geq 2 \), there exists a constant \( C > 0 \) that does not depend on \( m \) or the particular choice of \( t_1, t_2 \in [0, T] \) such that
\[ E[|\Delta X - \Delta X_m|^r] \leq C(t_2 - t_1)^{r/2} \|\mathcal{K} - \mathcal{K}_m\|_{L^2([0, T])} \]
with \( \Delta X := X(t_2) - X(t_1) \), \( \Delta X_m := X_m(t_2) - X_m(t_1) \).
Proof. Using uniform boundedness of \( Y \) and \( Y_m \), the Burkholder-Davis-Gundy and Jensen inequalities as well as Theorem 3.7 one can write:

\[
\begin{align*}
\mathbb{E} \left[ |\Delta X - \Delta X_m| \right] &= \mathbb{E} \left[ \left| \int_{t_1}^{t_2} (Y(s)X(s) - Y_m(s)X_m(s))dW(s) \right|^2 \right] \\
&\leq C \mathbb{E} \left[ \int_{t_1}^{t_2} (Y(s)X(s) - Y_m(s)X_m(s))^2 ds \right]^2 \\
&\leq C(t_2 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_2} \mathbb{E} \left[ |Y(s)X(s) - Y_m(s)X_m(s)|^2 \right] ds \\
&\leq C(t_2 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_2} \mathbb{E} \left[ |X(s) - X_m(s)| \right] ds + C(t_3 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_3} \mathbb{E} \left[ |X(s)| |Y(s) - Y_m(s)| \right] ds \\
&\leq C(t_2 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_2} \mathbb{E} \left[ |X(s) - X_m(s)| \right] ds + C(t_2 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_2} \mathbb{E} \left[ |X^r(s)| Z(s) - Z_m(s) \right] ds \\
&\quad + C(t_2 - t_1)^{\frac{7}{2} - 1} \int_{t_1}^{t_2} \mathbb{E} \left[ X^r(s) \xi_m \int_0^s |Z(u) - Z_m(u)| \right] du \right] ds,
\end{align*}
\]

where the random variable \( \xi_m \) does not depend on \( t_1 \) or \( t_2 \). Now, since

\[
\mathbb{E} \left[ \sup_{s \in [0,T]} |X(s) - X_m(s)| \right] \leq C \|\mathcal{K} - \mathcal{K}_m\|_{L^2(0,T)},
\]

by Theorem 3.7 it is clear that

\[
\int_{t_1}^{t_2} \mathbb{E} \left[ |X(s) - X_m(s)| \right] ds \leq C(t_2 - t_1) \|\mathcal{K} - \mathcal{K}_m\|_{L^2(0,T)}.
\]

Next, (2.7) in Theorem 2.6 implies that there exists a constant \( C \) that does not depend on \( t_1 \) or \( t_2 \) such that

\[
\int_{t_1}^{t_2} \mathbb{E} \left[ X^r(s) \right] ds < C,
\]

which yields

\[
\int_{t_1}^{t_2} \mathbb{E} \left[ X^r(s)|Z(s) - Z_m(s)| \right] ds \leq \left( \int_{t_1}^{t_2} \mathbb{E} \left[ X^r(s)^2 \right] ds \right)^\frac{1}{2} \left( \int_{t_1}^{t_2} \mathbb{E} \left[ |Z(s) - Z_m(s)|^2 \right] ds \right)^\frac{1}{2} \leq C(t_2 - t_1) \|\mathcal{K} - \mathcal{K}_m\|_{L^2(0,T)}. \]

Similarly, by (2.7) and (3.17) we can write

\[
\int_{t_1}^{t_2} \mathbb{E} \left[ X^r(s) \xi_m \int_0^s |Z(u) - Z_m(u)| \right] du \right] ds \leq C \int_{t_1}^{t_2} \int_0^s \mathbb{E} \left[ |Z(u) - Z_m(u)|^2 \right] \frac{1}{2} duds \leq C(t_2 - t_1) \|\mathcal{K} - \mathcal{K}_m\|_{L^2(0,T)},
\]

which finalizes the proof.