Deriving relativistic momentum and energy

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Abstract

We present a new derivation of the expressions for momentum and energy of a relativistic particle. In contrast to the procedures commonly adopted in textbooks, the one suggested here requires only the knowledge of the composition law for velocities along one spatial dimension, and does not make use of the concept of relativistic mass, or of the formalism of four-vectors. The basic ideas are very general and can be applied also to kinematics different from the Newtonian and Einstein ones, in order to construct the corresponding dynamics.

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1 Introduction

Every physics student knows that, in Newtonian dynamics, a particle with mass $m$ and velocity $u$ has a momentum

$$ p = m u $$

and a kinetic energy

$$ T = \frac{1}{2} m u^2 . $$

(1.2)

Usually, she does not stop to worry why these quantities are defined just by Eqs. (1.1) and (1.2), and not by other functions such as, e.g., $m u / 2$, $m u^2 u$, or $3 m u^4$. This comfortable situation ends when she attends a first course in Einstein dynamics. The expressions for momentum and kinetic energy,

$$ p = \frac{m u}{\sqrt{1 - u^2 / c^2}} $$

(1.3)

and

$$ T = \frac{m c^2}{\sqrt{1 - u^2 / c^2}} - m c^2 , $$

(1.4)

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now look so unnatural that she wonders about the reasons for choosing such complicated functions of velocity. At this point she can find, basically, three kinds of justifications for the expressions (1.3) and (1.4) in textbooks dealing with relativistic dynamics at an introductory level:

1. Requiring momentum conservation in all inertial frames for an elastic collision where particles are deflected from their initial line of flight \(^1\). Within this approach, the discussion is not entirely trivial, because of the two-dimensional character of the process. Also, it seems odd that one need invoke a second space dimension at all. What happens if we restrict ourselves to study motion along a straight line? There ought to be a way to find \(p\) and \(T\) without going “outside” in the second dimension.

2. Requiring momentum conservation for a head-on elastic collision together with conservation of a “relativistic mass” \(^2\). This circumvents the previous difficulty, but the use of a relativistic mass, and the pedagogical value of such a concept, have been strongly criticised \(^3\). (However, see Ref. \(^4\) for different opinions about this issue.) Of course, what is used in this approach is actually conservation of energy \(E\) (equal to \(T + m c^2\)), but why then should one assume that \(p = E u / c^2\), as done implicitly by the authors in Ref. \(^2\)?

3. Working with four-vectors, so one defines four-momentum just in the same way as the three-momentum of Newtonian theory, but with three-velocity replaced by four-velocity \(^5\). The problem here is that there is no guarantee, a priori, that such a quantity will be conserved for an isolated system. Indeed, the conservation law is usually checked for the case of a simple elastic collision, after four-momentum has been defined. Also, this approach requires the introduction of radically new ideas, hence it is unsuitable for a conceptually elementary presentation of the theory.

There is a fourth approach, which to our knowledge has never been adopted in textbooks,\(^1\) that resembles 1 above but is cleaner and can be consistently applied even in one space dimension only. It is based on the remark that, if energy is conserved in all inertial frames, then “something else” is also conserved. In the non-relativistic regime, this “something else” turns out to coincide with linear momentum. We suggest doing the same at the relativistic level.

We believe that this is the best strategy to adopt in an introductory course, because it relies on the same physical concepts that students are already familiar with from Newtonian mechanics, and does not introduce any new notion like relativistic mass or four-vectors. It focusses on similarities, rather than differences, between Newtonian and Einstein dynamics. Hence, a student is not required to replace physical pictures and mathematical tools with other, quite different ones — minor adaptations are enough, the general scheme remaining the same.

Furthermore, the approach we suggest casts light not only on the expressions (1.3) and (1.4) for momentum and energy in special relativity, but on (1.1) and (1.2) in Newtonian mechanics as well. Indeed, the underlying philosophy is that energy and momentum are nothing else than functions of mass and velocity that, under suitable conditions, happen

\(^1\)Actually, the idea has not been totally ignored; see, e.g., Ref. \(^6\). However, its extant implementations are not as simple as they could, making use of two-dimensional scattering and four-vectors.
to be conserved. This is why we treat in a special way those functions, rather than others. This point of view deserves to be emphasised in a pedagogical exposition, because it provides clear insights on the reasons why momentum and energy are defined the way they are, at the same time demystifying their meaning.

Our starting point is the definition of kinetic energy for a particle, as a scalar quantity whose change equals the work done on the particle. Mathematically, one first defines the power (work per unit time)

$$W := F \cdot u,$$

where $F$ is the total force acting on the particle. Then, using Newton’s second law

$$\frac{dp}{dt} = F,$$

which holds both in the non-relativistic and in the relativistic regimes, one gets also

$$W = \frac{dp}{dt} \cdot u.$$  \hspace{1cm} (1.7)

Finally, one defines a function $T(u)$ such that

$$\frac{dT}{dt} := W$$  \hspace{1cm} (1.8)

(this is possible since using Newton’s second law we have equated power to a purely kinetic quantity), thus obtaining

$$dT = u \cdot dp.$$  \hspace{1cm} (1.9)

In the following, we shall adopt Eq. (1.9) as a fundamental relationship between kinetic energy and momentum, that stands up on its own, independently of any justification like the one based on Eqs. (1.5)–(1.8).

For a system of non-interacting particles, kinetic energy is necessarily additive, since work is. We are then ready to introduce the two physical postulates upon which our discussion is based:

**P1.** The principle of relativity;

**P2.** The existence of elastic collisions between asymptotically free particles.

As we shall see, from these ingredients it follows that in an elastic collision, there is another quantity that is conserved — a vector one, that we identify with momentum. This quantity is linked to kinetic energy through a simple equation containing a function $\varphi$, that follows from the composition law for velocities. This, together with Eq. (1.9) above, allows us to find the explicit dependence of both momentum and energy on velocity. We can then easily construct also the expressions for the Lagrangian and the Hamiltonian for a free particle in an inertial frame. Hence, the entire basis of dynamics turns out to be uniquely determined by the function $\varphi$, a purely kinematical quantity.

We stress again that this procedure works for Newtonian and Einstein mechanics as well, so a student who is already familiar with Newtonian concepts will have no difficulty

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2In the one-dimensional case discussed in this paper. In three dimensions, $\varphi$ is replaced by a matrix of functions $\varphi_{ij}$, where the indices $i$ and $j$ run from 1 to 3.
in following the argument, since no new notion is required. The treatment has therefore 
a unifying conceptual power. In addition, it shows clearly and explicitly why Newtonian 
and Einstein dynamics are different. This happens because the expression we derive for 
momentum depends on the composition law for velocities (through the function \( \varphi \)), which 
is not the same in Galilean and Lorentz kinematics. Hence, Newtonian dynamics is not the 
same as Einstein dynamics only because the underlying kinematics are different. In fact, 
all differences between the two theories can be traced back to this one — another point 
well worth emphasising when teaching the subject. Of course, this is just an alternative 
way for saying that one can construct a relativistic theory based on Galilei invariance, 
or on Lorentz invariance. However, a beginner will find it easier to follow this approach, 
rather than more abstract considerations of formal symmetry.

The generality of the method makes it, in fact, applicable to a wider class of theo-
ries. Indeed, as discussed by Mermin [7], the principle of relativity is compatible with a 
generalised kinematics, that includes the Galilean and Lorentz ones as particular cases. 
One can then apply the techniques discussed in the present paper, in order to construct 
the corresponding dynamics. This will be done systematically in Ref. [8]. Here, we focus 
only on the Newtonian and Einstein cases, which are by far the most important from a 
pedagogical point of view.

The paper is structured as follows. In Sec. 2 we present the basic ideas. In Sec. 3 we 
review the main points behind Mermin’s discussion of the composition law for velocities. 
We do this both for the sake of completeness and for notational convenience, because the 
function \( \varphi \) that plays an essential role in our method also emerges naturally in Mermin’s 
approach to kinematics. However, it should be obvious to the reader that, since the 
core of the paper depends only on the formula for the composition law, an instructor 
who wants to follow our suggestions in the particular cases of Newtonian and Einstein 
mechanics does not necessarily have to present kinematics as in Ref. [7], and can adopt a 
more traditional approach. In Sec. 4 we present the general derivation of the expressions 
for momentum, kinetic energy, the Lagrangian, and the Hamiltonian. Then, in Sec. 5 we 
apply these results to construct the basis of Newtonian and Einstein dynamics. Again, 
an instructor can easily shorten significantly the presentation of this material at his/her 
convenience, according to the level of sophistication of the class. Section 6 contains some 
concluding comments about the different status of energy and momentum conservation, 
and the fact that there are no other conservation laws in one spatial dimension.

With the exception of Sec. 2 we restrict ourselves to considering motion along one 
space dimension. This makes the material accessible to a student with an elementary 
knowledge of calculus. In particular, no knowledge of vector algebra is required, contrary 
to what happens in approaches 1 and 3 above. The extension to three space dimensions is 
almost straightforward, but we prefer to postpone it to another publication for pedagogical 
clarity.

2 Main ideas

Consider the following argument in Newtonian mechanics, originally due to Huygens [9]. 
Conservation of energy in an inertial frame \( K \) during an elastic collision between two
particles with masses \(m_1\) and \(m_2\) gives
\[
\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2.
\] (2.1)

With respect to another inertial frame \(K\), in which \(K\) moves with velocity \(v\), the velocities are \(\bar{u}_1 = u_1 + v\), \(\bar{u}_2 = u_2 + v\), \(\bar{u}'_1 = u'_1 + v\), and \(\bar{u}'_2 = u'_2 + v\). Conservation of energy in \(K\) then implies
\[
\frac{1}{2}m_1(u_1 + v)^2 + \frac{1}{2}m_2(u_2 + v)^2 = \frac{1}{2}m_1(u'_1 + v)^2 + \frac{1}{2}m_2(u'_2 + v)^2.
\] (2.2)

Expanding the squares and using Eq. (2.1), one immediately gets
\[
(m_1u_1 + m_2u_2) \cdot v = (m_1u'_1 + m_2u'_2) \cdot v.
\] (2.3)

Since this must hold for an arbitrary vector \(v\), momentum conservation follows immediately.

The idea behind this derivation of the expression for momentum can easily be generalised to any theory satisfying postulates P1 and P2. More precisely, let \(T(u)\) be the kinetic energy of a particle with velocity \(u\) in an inertial frame \(K\). In an elastic collision,
\[
T_1(u_1) + T_2(u_2) = T_1(u'_1) + T_2(u'_2).
\] (2.4)

(Of course, the kinetic energy will also depend on the particle mass; we keep track of this dependence with the indices 1 and 2 on \(T\).) With respect to \(K\),
\[
T_1(\bar{u}_1) + T_2(\bar{u}_2) = T_1(\bar{u}'_1) + T_2(\bar{u}'_2),
\] (2.5)
where now \(\bar{u} = \Phi(u, v)\) is the composition law for velocities. On expanding Eq. (2.5) in the variable \(v\) and using Eq. (2.4), we find a conservation equation to first order in \(v\), analogous to the one expressed by Eq. (2.3) — although with different coefficients, in general. We can then define linear momentum\(^3\) \(p\) as the first-order coefficient in \(v\). If we know the function \(T(u)\), we can find \(p\).

If we do not already know \(T(u)\), we can define it by requiring that it obey the fundamental relation (1.9). Then, putting together Eq. (1.9) and the one that expresses \(p\) in terms of \(T(u)\), one obtains a simple system of differential equations, from which it is possible to find both \(T(u)\) and \(p(u)\). In the following sections, we implement these ideas in detail, restricting ourselves to considering motion along one spatial dimension.

### 3 Velocity composition law

Suppose that a particle moves with velocity \(u\) with respect to a reference frame \(K\). If \(K\) moves with velocity \(v\) with respect to another reference frame \(K\), the particle velocity \(\bar{u}\) with respect to \(K\) is given by some composition law
\[
\bar{u} = \Phi(u, v).
\] (3.1)

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\(^3\)With this definition, linear momentum turns out to be a one-form rather than a vector, which is very satisfactory from a formal point of view.
Of course, when $\mathcal{K}$ is at rest with respect to $\mathcal{K}$, we have that $\bar{u} = u$. Similarly, if the particle is at rest in $\mathcal{K}$, its velocity with respect to $\mathcal{K}$ is the same of $\mathcal{K}$. More synthetically,

$$\Phi(u, 0) = \Phi(0, u) = u, \quad \forall u. \quad (3.2)$$

Also, a particle at rest in $\mathcal{K}$ has $\bar{u} = 0$, hence it moves with respect to $\mathcal{K}$ with a velocity $v_L$ such that

$$\Phi(v_L, v) = 0, \quad \forall v. \quad (3.3)$$

Similarly, if a particle moves with respect to $\mathcal{K}$ with velocity $u$ but is at rest in $\mathcal{K}$, then the speed $u_R$ of $\mathcal{K}$ with respect to $\mathcal{K}$ must satisfy the relation

$$\Phi(u, u_R) = 0, \quad \forall u. \quad (3.4)$$

Finally, from the relativity principle it follows [7] that $\Phi$ satisfies the associative law, i.e., that

$$\Phi(\Phi(u, v), w) = \Phi(u, \Phi(v, w)), \quad \forall u, v, w. \quad (3.5)$$

Therefore, Eq. (3.1) gives the composition law of a group, with neutral element 0 and left- and right-inverses given by Eqs. (3.3) and (3.4). Combining Eqs. (3.3)–(3.5), one finds that there actually is only one inverse of $v$, say $v'$ — that is, $v_L = v_R =: v'$. Note that the inverse $v'$ of $v$ is not necessarily equal to $-v$.

Let us now define the function

$$\varphi(u) := \left. \frac{\partial \Phi(u, v)}{\partial v} \right|_{v=0}. \quad (3.6)$$

The meaning of $\varphi$ can be found by expanding $\bar{u}$ to the first order in $v$:

$$\bar{u} = u + \varphi(u) v + \mathcal{O}(v^2). \quad (3.7)$$

This is the composition law between an arbitrary velocity $u$ and a velocity $v$ with small magnitude. Since Eq. (3.2) implies $\varphi(0) = 1$, at very low speeds one recovers Galilean kinematics.

Another interesting property of $\varphi$ is that if some velocity, say $C$, is invariant, then $\varphi(C) = 0$. This follows immediately by applying Eq. (3.6) to the condition

$$\Phi(C, v) = C, \quad \forall v, \quad (3.8)$$

which expresses the invariance of $C$.

The function $\varphi$ contains all the information needed to specify $\Phi$. Indeed, it is not difficult to show [7] that $\Phi$ can be written as

$$\Phi(u, v) = h^{-1} \left( h(u) + h(v) \right), \quad (3.9)$$

where

$$\left. \frac{dh(u)}{du} \right|_{u=0} = \frac{1}{\varphi(u)} \quad (3.10)$$

and $h(0) = 0$. As a corollary of Eq. (3.9), one finds that the composition law for collinear velocities is also commutative, i.e.,

$$\Phi(u, v) = \Phi(v, u), \quad \forall u, v. \quad (3.11)$$
It is worth stressing, however, that this property does not hold in general for the composition law of velocities along arbitrary directions in more than one spatial dimension [10].

With further requirements, essentially equivalent to homogeneity of space and time, and spatial isotropy (or better, its one-dimensional counterpart — the physical equivalence of the two orientations in the one-dimensional space), one can further restrict \( \varphi \) to the form

\[
\varphi(u) = 1 - Ku^2,
\]

where \( K \) is a constant [7]. Moreover, the possibility \( K < 0 \) can be excluded on physical grounds.

The case \( K = 0 \) gives the simple Galilean addition of velocities. With \( K > 0 \) one finds

\[
h(u) = \frac{1}{\sqrt{K}} \ln \left( \frac{1 + \sqrt{K} u}{1 - \sqrt{K} u} \right)^{1/2}. \tag{3.13}
\]

This leads to Einstein’s composition law, with the speed of light replaced by \( 1/\sqrt{K} \). Hence, the mathematical structure of Einstein’s composition law is a consequence of the principle of relativity alone, combined with the postulates of homogeneity of space and time, and of spatial isotropy. Remarkably, this was known to Kaluza already in 1924 [11].

An analogous result about the structure of the Lorentz transformation was obtained by von Ignatowsky in 1910 [12], and has been rediscovered many times since [13]. (See also Ref. [14] for a rigorous derivation, and Ref. [15] for clear presentations at the textbook level.)

4 General analysis

We now carry on the programme outlined in Sec. 2 deriving the general expressions for momentum and kinetic energy (Sec. 4.1), the Lagrangian (Sec. 4.2), and the Hamiltonian (Sec. 4.3) for a free particle in an inertial frame, that follow from postulates P1 and P2 when a given composition law for velocities is adopted.

4.1 Momentum and kinetic energy

Let \( T(u) \) be the kinetic energy of a particle in a reference frame where the particle velocity is \( u \). (Of course, \( T(u) \) may depend on some invariant parameters characterising the particle, in addition to its velocity. For example, in Newtonian dynamics it depends on the particle mass.) In an inertial frame, \( T(u) \) is conserved for a free particle, because \( u \) is constant, by the principle of inertia. Then, the total kinetic energy is conserved also for a system of non-interacting particles.

According to postulate P2, there are spatially localised interactions between particles which do not change the total kinetic energy. It is then easy to see that there is another additive quantity which is conserved in any theory in which a relativity principle holds (postulate P1). More specifically, if the composition law for velocities is given by Eq. (3.1), such a quantity is, for a single particle,

\[
p(u) = \varphi(u) \frac{dT(u)}{du}, \tag{4.1}
\]
where $\varphi(u)$ is the function defined by Eq. (3.6).

The proof of this theorem relies on the generalisation of Huygens’ argument outlined in Sec. 2. Let us write energy conservation in two inertial frames $K$ and $\bar{K}$ during a head-on elastic collision between two particles, as in Eqs. (2.4) and (2.5):

$$T_1(u_1) + T_2(u_2) = T_1(u_1') + T_2(u_2'); \quad (4.2)$$

$$T_1(\bar{u}_1) + T_2(\bar{u}_2) = T_1(\bar{u}_1') + T_2(\bar{u}_2'). \quad (4.3)$$

We have used the same functions $T_1$ and $T_2$ in both reference frames because of the relativity principle. We can expand the generic function $T(\bar{u})$ around $v = 0$, and use the property (3.2) to get

$$T(\bar{u}) = T(u) + \frac{dT(u)}{du} \varphi(u) v + O(v^2). \quad (4.4)$$

Doing this for each term in Eq. (4.3) and using Eq. (4.2), then dividing by $v$ and taking the limit for $v \to 0$, we obtain

$$\frac{dT_1(u_1)}{du_1} \varphi(u_1) + \frac{dT_2(u_2)}{du_2} \varphi(u_2) = \frac{dT_1(u_1')}{du_1'} \varphi(u_1') + \frac{dT_2(u_2')}{du_2'} \varphi(u_2'). \quad (4.5)$$

This proves the claim above.

Of course, Eq. (4.1) is not sufficient in order to find an expression for momentum, since the function $T(u)$ is also unknown. However, as already discussed in Secs. 1 and 2, we can define kinetic energy so that its variation gives the work done on the particle — that is, impose the validity of Eq. (1.9). Combining Eq. (1.9) in its one-dimensional version ($dT = u dp$) with Eq. (4.1), we obtain a differential equation for the function $p(u)$:

$$\frac{dp}{du} = \frac{p}{u \varphi(u)} . \quad (4.6)$$

Integrating by separation of variables, one finds the expression for $p(u)$. In general, we can write

$$p(u) = m \exp \int^u du' \frac{1}{u' \varphi(u')} , \quad (4.7)$$

where $m$ is a constant parameter that can vary from particle to particle. Mathematically, $m$ represents the arbitrary constant associated with the general solution of the differential equation (4.6). Physically, it is identified with the particle mass by imposing the Newtonian limit for $u \to 0$.

Finally, one can replace the expression for $p(u)$ into Eq. (4.1), and integrate with the condition $T(0) = 0$ to obtain also the expression for $T(u)$:

$$T(u) = \int_0^u du' \frac{p(u')}{\varphi(u')} . \quad (4.8)$$

4Actually, this argument only shows that the quantity on the right-hand side of Eq. (4.1) is conserved. The most general form of momentum is then

$$p = \lambda \varphi \frac{dT}{du} + \mu T + \nu ,$$

where $\lambda$, $\mu$, and $\nu$ are constants. It turns out that the simplest non-trivial choice $\lambda = 1$, $\mu = \nu = 0$ is the one that leads to Newtonian and Einstein dynamics, while other values of the coefficients correspond to alternative relativistic dynamics (in particular, for $\mu \neq 0$ one finds anisotropic theories). We leave the discussion of these possibilities to a forthcoming, more technical paper [8].
4.2 Lagrangian

The Lagrangian should satisfy the relation

\[ p(u) = \frac{dL(u)}{du}. \tag{4.9} \]

Using Eq. (4.1), we obtain

\[ dL(u) = \varphi(u) \frac{dT(u)}{du} du. \tag{4.10} \]

Obviously, it is only for \( \varphi = 1 \) that \( L = T + \text{const} \), so the Lagrangian for a free particle coincides with the kinetic energy only in Newtonian dynamics.

4.3 Hamiltonian

Equation (1.9) allows us to identify the Hamiltonian for a free particle. Indeed,

\[ u = \frac{dT(u)}{du} \left/ \frac{dp(u)}{du} \right. = \frac{dT(u) du(p)}{dp} = \frac{dT(u(p))}{dp}. \tag{4.11} \]

On the other hand, one of Hamilton’s equations of motion is

\[ u = \frac{dH(p)}{dp}, \tag{4.12} \]

so one can write \( H(p) \) as \( T(u(p)) \), up to a \( u \)-independent additive term. Apart from \( u \), the only other parameter \( T \) depends on is the particle mass \( m \), so we have in general

\[ H(p, m) = T(u(p, m), m) + E_0(m), \tag{4.13} \]

where we have made explicit the dependence of the various quantities on \( m \), and \( E_0(m) \) denotes the value of the Hamiltonian when \( p = 0 \). Since, numerically, \( H \) coincides with the particle energy \( E \), it follows from Eq. (4.13) that

\[ E(u, m) = T(u, m) + E_0(m), \tag{4.14} \]

so \( E_0(m) \) can be interpreted as the particle rest energy.

Of course, the same expression for \( H \) can be obtained as the Legendre transform \[16\] of \( L \).

5 Special cases

Let us apply the general results derived in the previous section to the two cases of pedagogical interest, namely Newtonian and Einstein dynamics.
5.1 Newtonian dynamics

The composition law is simply
\[ \Phi(u, v) = u + v, \]  
(5.1)
so \( \varphi(u) = 1 \). From Eq. (4.7) one then finds immediately \( p(u) = m u \) which, replaced into Eq. (4.8), gives \( T(u) = m u^2 / 2 \).

The Lagrangian coincides with the kinetic energy, as already noted. In order to get the Hamiltonian, we first express velocity as a function of momentum, \( u(p) = p/m \), so
\[ H(p) = \frac{p^2}{2m} + E_0(m). \]  
(5.2)
The choice \( E_0(m) = 0 \) is obviously the simplest. Note that particles of zero mass cannot exist in this theory.

5.2 Einstein dynamics

Einstein’s composition law
\[ \Phi(u, v) = \frac{u + v}{1 + uv/c^2} \]  
(5.3)
corresponds to
\[ \varphi(u) = 1 - u^2/c^2. \]  
(5.4)
(Note that \( \varphi(\pm c) = 0 \), so \( \pm c \) are invariant velocities.) From Eq. (4.7) one finds
\[ p(u) = m u \gamma(u), \]  
(5.5)
where we have defined the Lorentz factor
\[ \gamma(u) := \left(1 - u^2/c^2\right)^{-1/2}. \]  
(5.6)
The expression for the kinetic energy follows immediately on replacing Eq. (5.5) into Eq. (4.8):
\[ T(u) = mc^2 \gamma(u) - mc^2. \]  
(5.7)
The Lagrangian is
\[ L(u) = -mc^2/\gamma(u). \]  
(5.8)
For zero-mass particles, the function \( L \) is ill-defined, and a Lagrangian formulation is not viable.

Inverting Eq. (5.5) we get
\[ u(p) = \frac{pc}{\sqrt{p^2 + m^2c^2}}, \]  
(5.9)
so the Hamiltonian is
\[ H(p) = \sqrt{p^2c^2 + m^2c^4} - mc^2 + E_0(m). \]  
(5.10)
Now, the simplest choice is \( E_0(m) = mc^2 \). In this theory we can treat also zero-mass particles, for which \( H(p) = pc \).
6 Comments

The most important result presented in this paper is the theorem in Sec. 4 — that, in theories obeying postulates P1 and P2, the quantity $p(u)$ defined by Eq. (4.1) is conserved and can be identified with momentum. Broadly, this implies that, in such theories, kinematics “determines” dynamics. As applications, we have shown how to recover the expressions for the quantities on which dynamics is based, in the two cases of Newtonian and Einstein mechanics. Of course, one may consider other types of dynamics as well [8], based on alternative kinematics but still obeying postulates P1 and P2.

Energy conservation in one inertial frame, together with the relativity principle, implies energy conservation in all inertial frames. As we have seen, this leads to momentum conservation. In fact, on replacing Eq. (4.1) into Eq. (4.4), one finds

$$T(\bar{u}) = T(u) + v p(u) + O(v^2), \quad (6.1)$$

which holds in general. In particular, Eq. (6.1) is consistent with the law of transformation for energy both in Newtonian and Einstein dynamics, where

$$T(\bar{u}) = T(u) + v p(u) + \frac{1}{2} mv^2 \quad (6.2)$$

and

$$E(\bar{u}) = \gamma(v) (E(u) + v p(u)) \quad (6.3)$$

respectively, with $E(u) = mc^2 + T(u)$ in the second case. Now, it is well known that energy conservation is related to invariance under time translations, while momentum conservation is related to invariance under space translations. Hence, the relativity principle has, apparently, the effect of generating invariance under space translations from the invariance under time translations. (In other words, homogeneity of time in all inertial frames enforces also homogeneity of space.) This is indeed the case, as one can easily understand thinking that what appears purely as a time displacement in an inertial frame, acquires a spatial component in any other frame with $v \neq 0$ (see Fig. 1). This is true in general, not only for a Lorentz transformation. For example, for a Galilean transformation between two frames $\mathcal{K}$ and $\mathcal{K}$ one has $\bar{x} = x + vt$. Then, if two events have time and space separations $\Delta t \neq 0$ and $\Delta x = 0$ in $\mathcal{K}$, their space separation in the reference frame $\mathcal{K}$ is $\Delta \bar{x} = v \Delta t \neq 0$, as shown in the part of Fig. 1 on the right.

The situation is not symmetric regarding momentum conservation, as one can see already by examining the cases of a Lorentz and a Galilean transformation. In the first case,

$$p(\bar{u}) = \gamma(v) \left( p(u) + v E(u)/c^2 \right) \quad (6.4)$$

implies

$$\frac{dp(u)}{du} \varphi(u) = \frac{E(u)}{c^2}. \quad (6.5)$$

Hence, one can enforce energy conservation by requiring momentum conservation in every inertial frame.\(^5\) On the other hand, in Newtonian mechanics Eq. (6.4) is replaced by

$$p(\bar{u}) = p(u) + m v, \quad (6.6)$$

\(^5\)Interestingly, one can combine Eqs. (6.4) and (6.5) to get a single differential equation for $p$,

$$\frac{d^2 p}{du^2} - \frac{1}{c^2} p = 0,$$
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Figure 1: Events $P$ and $Q$ take place at the same spatial point ($A$) in the reference frame $(t, x)$, but at different points ($B$ and $C$) in the reference frame $(\bar{t}, \bar{x})$. This is true both in Lorentz (left) and in Galilean (right) kinematics.

from which one gets

$$\frac{dp(u)}{du} \varphi(u) = m.$$  \hspace{1cm} (6.7)

Therefore, momentum conservation in all inertial frames now enforces conservation of mass, rather than of energy. This asymmetry is related to the fact that, while under a Lorentz transformation a purely spatial displacement acquires also a time component, this is not true for a Galilean transformation (see Fig. 2). Indeed, since for the latter one has $\bar{t} = t$, it will be $\Delta \bar{t} = \Delta t$ regardless of what $\Delta x$ is.

Of course, there is no reason to stop the analysis in Sec. 4.1 to the first order in $v$. In fact, by considering the second order, then the third order, and so on, an infinite set of conserved quantities can be generated. However, these “extra” quantities are not independent and do not give anything new, as one might also expect noticing that conservation of energy and momentum already exploit all the available symmetries, namely, homogeneity of time and space. It is nevertheless instructive to see explicitly what happens for the two dynamics considered in Sec. 5.

In Newtonian dynamics, at the second order in $v$ one recovers conservation of mass (which is already contained, however, in our implicit assumption that masses do not change during a collision), while at still higher orders all coefficients vanish identically — see Eq. (2.2). In Einstein dynamics, at the second order one finds conservation of energy $E$ which, once again, amounts to conservation of mass when one considers that $T$ is also conserved, by postulate P2. At orders higher than two the situation is a bit more involved. It is convenient first to rewrite the second equation in (6.3) as

$$E(\Phi(u, v)) = E(u) \gamma(v) + p(u) \gamma(v) v.$$  \hspace{1cm} (6.8)

Now, the coefficient $\gamma(v)$ which appears on the right-hand side of Eq. (6.8) is a function of $v^2$, so the coefficients of the expansion of $E(\Phi(u, v))$ in powers of $v$ will all be equal to $E(u)$ where $U = h(u)$, with $h$ given by Eq. (3.13) with $K = 1/c^2$. 

\hspace{1cm}
for even powers, and to $p(u)$ for odd powers. Since $E$ and $T$ differ only by a velocity-independent quantity, one simply recovers the conservation of energy and momentum, alternatively.

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