THE BARGMANN TRANSFORM AND POWERS OF HARMONIC OSCILLATOR ON GELFAND-SHILOV SUBSPACES

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Abstract. We consider the counter images $J(R^d)$ and $J_0(R^d)$ of entire functions with exponential and almost exponential bounds, respectively, under the Bargmann transform, and we characterize them by estimates of powers of the harmonic oscillator. We also consider the Pilipović spaces $S_s(R^d)$ and $\Sigma_s(R^d)$ when $0 < s < 1/2$ and deduce their images under the Bargmann transform.

0. Introduction

The aim of the paper is to characterize the images of the Pilipović spaces $S_s(R^d)$ and $\Sigma_s(R^d)$ under the Bargmann transform when $s < 1/2$, as well as the test function spaces $J_0(R^d)$ and $J(R^d)$ considered in [8], in terms of estimates of powers of the harmonic oscillator. The set $J(R^d)$ consists of all $f \in \mathcal{S}(R^d)$ such that their Hermite series expansions are given by

$$f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha(f) h_\alpha,$$

where

$$|c_\alpha(f)| \leq C r^{|\alpha|} \sqrt{\alpha!},$$

for some constants $r > 0$ and $C > 0$. (See Section 2 for notations.) In the same way, $f$ belongs to $J_0(R^d)$, if and only if for every $r > 0$, there is a constant $C > 0$ such that [2] holds.

The sets $J_0(R^d)$ and $J(R^d)$ are small in the sense that they are continuously embedded in the Schwartz space $\mathcal{S}(R^d)$ and its subspaces $S_{s_1}(R^d)$ and $\Sigma_{s_2}(R^d)$ when $s_1 \geq 1/2$ and $s_2 > 1/2$. Here $S_s(R^d)$ and $\Sigma_s(R^d)$ are the sets of Gelfand-Shilov spaces of Roumieu and Beurling types, respectively, of order $s \geq 0$ on $R^d$. The function spaces $S_s(R^d)$ and $\Sigma_s(R^d)$, increase with the parameter $s \geq 0$, and the following holds true (see [8] for the verifications):

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1In the latest version of [8], the spaces $J_0(R^d)$ and $J(R^d)$ are denoted by $\mathcal{H}_{0,\delta}(R^d)$ and $\mathcal{H}_0(R^d)$, respectively.
• $\mathcal{S}_s(\mathbb{R}^d)$ is non-trivial for every $s \geq 0$ and $\mathcal{S}_s(\mathbb{R}^d)$ consists of all finite linear combinations of Hermite functions. Furthermore, $\mathcal{S}_s(\mathbb{R}^d) = \mathcal{S}_s(\mathbb{R}^d)$ when $s \geq 1/2$, and $\mathcal{S}_s(\mathbb{R}^d) \neq \mathcal{S}_s(\mathbb{R}^d) = \{0\}$ when $s < 1/2$;
• $\Sigma_s(\mathbb{R}^d)$ is non-trivial if and only if $s > 0$. Furthermore, $\Sigma_s(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d)$ when $s > 1/2$, and $\Sigma_s(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}$ when $s \leq 1/2$;
• For every $\varepsilon > 0$, $s \geq 0$, $s_1 < \frac{1}{2}$ and $s_2 \geq \frac{1}{2}$, the inclusions
\[
\Sigma_s(\mathbb{R}^d) \subseteq \mathcal{S}_s(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d) \subseteq \mathcal{J}(\mathbb{R}^d)
\]
and
\[
\mathcal{S}_s(\mathbb{R}^d) \subseteq \mathcal{J}_0(\mathbb{R}^d) \subseteq \mathcal{J}(\mathbb{R}^d) \subseteq \Sigma_{s_2}(\mathbb{R}^d)
\]
are continuous and dense. A similar fact holds for corresponding distribution spaces, after the inclusions have been reversed.

The spaces in (3) and their duals have in most of the cases, convenient images under the Bargmann transform. In fact, in view of [8] it is proved that the Bargmann transform $\mathcal{U}_d$ is injective on all these spaces and their duals, and, among others, that

\[
\mathcal{U}_d(\mathcal{J}_0(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|} \text{ for every } R > 0 \},
\]
\[
\mathcal{U}_d(\mathcal{J}(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|} \text{ for some } R > 0 \},
\]
\[
\mathcal{U}_d(\Sigma_{\frac{1}{2}}(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|^2} \text{ for every } R > 0 \},
\]
\[
\mathcal{U}_d(\Sigma'_{\frac{1}{2}}(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|^2} \text{ for some } R > 0 \},
\]
\[
\mathcal{U}_d(\mathcal{J}'(\mathbb{R}^d)) = A(\mathbb{C}^d),
\]
\[
\mathcal{U}_d(\mathcal{J}'_0(\mathbb{R}^d)) = \bigcup_{r>0} A(B_r(0)).
\]
(See [8] for more comprehensive lists of mapping properties of the spaces in (3) and their duals under the Bargmann transform.) Here $A(\Omega)$ is the set of all analytic functions on the open set $\Omega \subseteq \mathbb{C}^d$, and $B_r(z_0)$ is the open ball with center at $z_0 \in \mathbb{C}^d$ and radius $r > 0$. We remark that one of the reasons for considering $\mathcal{J}_0(\mathbb{R}^d)$ and $\mathcal{J}(\mathbb{R}^d)$ is that $\mathcal{J}_0(\mathbb{R}^d)$ and $\mathcal{J}'(\mathbb{R}^d)$ possess the mapping properties under the Bargmann transform given here above.

In Section 3 we make the list here above more complete by proving that
\[
\mathcal{U}_d(\Sigma_s(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R(\log|z|)^{1-2s}} \text{ for every } R > 0 \},
\]
and
\[
\mathcal{U}_d(\mathcal{S}_s(\mathbb{R}^d)) = \{ F \in A(\mathbb{C}^d) : |F(z)| \lesssim e^{R \log(z) \frac{1}{1-2s}}, \text{ for some } R > 0 \},
\]
when \( 0 < s < 1/2 \).

1. Preliminaries

In this section we recall some basic facts. We start by discussing Pilipović and Gelfand-Shilov spaces and some of their properties. Finally we recall the Bargmann transform and some of its mapping properties.

Let \( 0 < h, s, t \in \mathbb{R} \). Then \( \mathcal{S}_{s,h}(\mathbb{R}^d) \) consists of all \( f \in C^\infty(\mathbb{R}^d) \) such that
\[
\|f\|_{\mathcal{S}_{s,h}} := \sup_{|\alpha| + |\beta|} \frac{|x^\beta \partial^\alpha f(x)|}{h^{(\alpha + \beta)!}}<\infty
\]
is finite. The Gelfand-Shilov spaces \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), of Roumieu and Beurling types respectively, are the sets
\[
\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d),
\]
with inductive and projective topologies, respectively. We remark that \( \Sigma_s(\mathbb{R}^d) \neq \{0\} \), if and only if \( s > 1/2 \), and \( \mathcal{S}_s(\mathbb{R}^d) \neq \{0\} \), if and only if \( s \geq 1/2 \). We refer to [2, 3] for general facts about Gelfand-Shilov spaces, and their duals.

Next we consider spaces which are obtained by suitable estimates of Gelfand-Shilov or Gevrey type, after the operator \( x^\beta \partial^\alpha \) in (4) is replaced by powers of the harmonic oscillator \( H = |x|^2 - \Delta \). More precisely, if \( s \geq 1/2 \) (\( s > 1/2 \)), then Pilipović showed in [6] that \( f \in \mathcal{S}_s(\mathbb{R}^d) \) (\( f \in \Sigma_s(\mathbb{R}^d) \)), if and only if
\[
\sup_{N \geq 0} \|H^N f\|_{L^\infty}^{hN(N!)^{2s}} < \infty,
\]
holds for some \( h > 0 \) (for every \( h > 0 \)). (See also [5, 6] for more general approaches.) On the other hand, \( \mathcal{S}_s(\mathbb{R}^d) \) (\( \Sigma_s(\mathbb{R}^d) \)) is empty when \( s < 1/2 \) (\( s \leq 1/2 \)), while any Hermite function \( h_\alpha \) fulfills (3) for some \( h > 0 \) (for every \( h > 0 \)), when \( s \geq 0 \) (\( s > 0 \)).

For this reason, we let \( \mathcal{S}_s(\mathbb{R}^d) \) (\( \Sigma_s(\mathbb{R}^d) \)) be the set of all \( f \in C^\infty(\mathbb{R}^d) \) such that (3) holds for some \( h > 0 \) (for every \( h > 0 \)). We call \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \) the Pilipović spaces of Roumieu and Beurling types respectively, of order \( s \geq 0 \) on \( \mathbb{R}^d \).

In [4, 5, 8] there are different types of characterizations of the Pilipović spaces. For example, it is here proved that \( f \) belongs to \( \mathcal{S}_s(\mathbb{R}^d) \) (\( \Sigma_s(\mathbb{R}^d) \)) with \( s > 0 \), if and only if (1) holds with
\[
|c_\alpha(f)| \lesssim e^{-r|\alpha|^{1/2s}}
\]
for some \( r > 0 \) (for every \( r > 0 \)).
2. Characterizations of $J_0(R^d)$ and $J(R^d)$ in terms of powers of the harmonic oscillator

In the previous section, $\mathcal{S}_s(R^d)$ and $\Sigma_s(R^d)$ were defined by means of (6). In this section we deduce characterizations of the test function spaces $J_0(R^d)$ and $J(R^d)$, in similar ways.

More precisely we have the following.

**Theorem 1.** Let $f$ be given by (1). Then the following conditions are equivalent:

1. There exists $r > 0$ such that
   \[ |c_\alpha(f)| \lesssim r^{\alpha}(\alpha!)^{-\frac{1}{2}}. \]

2. There exists $r > 0$ such that
   \[ \|H^Nf\|_{L^2} \lesssim 2^N r^{\frac{N}{\log N}} \left( \frac{2N}{\log N} \right)^{N\left(1 - \frac{1}{\log N}\right)}. \]

The same arguments also give the following.

**Theorem 2.** Let $f$ be given by (1). Then the following conditions are equivalent:

1. For every $r > 0$,
   \[ |c_\alpha(f)| \lesssim r^{\alpha}(\alpha!)^{-\frac{1}{2}}. \]

2. For every $r > 0$,
   \[ \|H^Nf\|_{L^2} \lesssim 2^N r^{\frac{N}{\log N}} \left( \frac{2N}{\log N} \right)^{N\left(1 - \frac{1}{\log N}\right)}. \]

We need some preparations for the proofs, and start with the following lemma.

**Lemma 3.** Let $r > 0$. Then, for $N$ large enough, the function
\[ f(t) = \frac{t^{2N(2re)^t}}{t^t}, \quad t > 0, \]
attains its maximum in the interval $[t_1, t_2]$, where
\[ t_\alpha = \frac{2N}{\log N} \left( 1 + \alpha \frac{\log(r \log N)}{\log N + 1} \right), \quad \alpha \geq 0. \] (7)

**Proof.** We may assume that $N > e^{1/r}$. Let
\[ m(t) = \log f(t) = 2N \log t + t \log(2re) - t \log t. \]

Then
\[ m'(t) = \frac{2N}{t} + \log(2r) - \log t \quad \text{and} \quad m''(t) = \frac{2N}{t^2} - \frac{1}{t} < 0. \]

It follows that $m$ is strictly concave, and has at most one local maximum, and if so, it is also a global maximum. Furthermore, $m'(t)$ is...
strictly decreasing and has a possible zero only in the possible point where maximum of \( m(t) \) is attained. Hence it suffices to show that \( m'(t_1) > 0 \) and \( m'(t_2) < 0 \).

We observe that \( m'(t_0) > 0 \) and that the tangent line to the graph of \( m' \) at point \( (t_0, m'(t_0)) \) intersects with the abscissa axis at \( (t_1, 0) \). Since \( m' \) is a convex function it follows that \( m'(t_1) > 0 \).

In order to show that \( m'(t_2) < 0 \), we put
\[
h(s) = s + \left( 1 + \frac{2 \log(rs)}{s + 1} \right) (\log(rs) - s)
\]
\[
= \log(rs) \left( 1 + \frac{2 \log(rs)}{s + 1} - \frac{2s}{s + 1} \right).
\]
Then \( h(s) < 0 \) when \( s = \log N \) is large enough, and by straight-forward computations we get
\[
m'(t_2) = \left( 1 + \frac{2 \log(rs)}{s + 1} \right)^{-1} h(s) - \log \left( 1 + \frac{2 \log(rs)}{s + 1} \right)
\]
\[
< \left( 1 + \frac{2 \log(rs)}{s + 1} \right)^{-1} h(s) < 0,
\]
where the inequalities follow from the fact that \( N > e^{1/r} \) is chosen large enough. Hence \( m'(t_2) < 0 \).

\[\Box\]

**Lemma 4.** Let \( r > 0 \) and let
\[
f(t) = \frac{t^{2N}(2re)^t}{t^t}.
\]
Then there exists a positive and increasing function \( \theta \) on \([0, \infty)\) and an integer \( N_0(r) \) such that
\[
\max_{t>0} f(t) \leq \left( \frac{2N}{\log N} \right)^{2N(1 - \frac{1}{\log N})} (\theta(r) \cdot r)^{\frac{2N}{\log N}},
\]
when \( N \geq N_0(r) \).

**Proof.** Let \( N_0(r) > \max(e^{1/r}, e) \) be such that the conclusions in Lemma 3 are fulfilled when \( N \geq N_0(r) \). Then
\[
\max_{t>0} f(t) = \max_{a \in [1,2]} f(t_a).
\]
Let \( N \geq N_0(r) \),
\[
s = \log N, \quad g(s) = \frac{\log(rs)}{s + 1}, \quad \text{and} \quad g_0(s) = \frac{\log((r + 2)s)}{s + 1}.
\]
Then $s > 1$, $0 < g(s) < g_0(s)$, $t_\alpha = t_0 (1 + \alpha g(s))$ and

\[
\begin{align*}
f(t_\alpha) &= t_\alpha^{2N-t_\alpha} (2re)^{t_\alpha} \\
&= t_0^{2N-t_0(1+\alpha g(s))} (1 + \alpha g(s))^{2N-t_0(1+\alpha g(s))} (2re)^{t_0(1+\alpha g(s))} \\
&= f(t_0) \left( e^{\alpha g(s)} \right)^{-t_0 \log t_0} (1 + \alpha g(s))^{2N-t_0(1+\alpha g(s))} (2re)^{t_0 \alpha g(s)}.
\end{align*}
\]

Since $1 + x \leq e^x$ we obtain

\[
\begin{align*}
f(t_\alpha) &\leq f(t_0) \left( e^{\alpha g(s)} \right)^{-t_0 \log t_0 + 2N} (1 + \alpha g(s))^{-t_0(1+\alpha g(s))} (2re)^{t_0 \alpha g(s)} \\
&\leq f(t_0) \left( e^{\alpha g(s)} \right)^{-t_0 \log t_0 + 2N} (2re)^{t_0 \alpha g(s)}.
\end{align*}
\]

Now

\[
2N - t_0 \log t_0 = \frac{2N}{\log N} \left( \log \left( \frac{\log N}{2} \right) \right) \leq \frac{2N}{\log N} \log s,
\]

which gives

\[
f(t_\alpha) \leq f(t_0) \left( e^{\alpha g(s) \log s} \right)^{\frac{2N}{\log N}} (2re)^{\frac{2N}{\log N} \alpha g(s)} \leq f(t_0) \theta_0(r) \frac{2N}{\log N},
\]

when

\[
\theta_0(r) = \sup_{s > 1} e^{\alpha g_0(s) \log s} (2re)^{\alpha g_0(s)}.
\]

Evidently, $\theta_0$ is positive and increasing on $[0, \infty)$, and since

\[
f(t_0) = \left( \frac{2N}{\log N} \right)^{2N(1-\frac{1}{\log N})} (2re)^{\frac{2N}{\log N}},
\]

(8) holds with $\theta(r) = 2e\theta_0(r)$. \hfill \Box

**Lemma 5.** Let $0 < r < r_0$, $a_1 > 0$, $a_2 \geq 0$ and let $\theta(r)$ be as in Lemma 4. Then

\[
\sum_{k=0}^{\infty} \frac{(a_1^k + a_2^{2N}) \cdot r_{2k}}{k!} \leq C a_1^{2N} \left( \frac{2N}{\log N} \right)^{2N(1-\frac{1}{\log N})} (r_0^2 \cdot \theta(r_0^2))^{\frac{2N}{\log N}} \frac{r_0^2}{r_0^2 - r^2}, \quad (9)
\]

for some constant $C \geq 1$, depending on $a_1$, $a_2$ and $r_0$ only, provided $N$ is chosen large enough.
Proof. First we prove the result in the case \( a_1 = 1 \) and \( a_2 = a \in \mathbb{N} \). By Lemma 4 and using Stirling’s formula we have

\[
\sum_{k=0}^{\infty} \frac{(k + a)^{2N} r^{2k}}{k!} = r^{-2a} \sum_{k=0}^{\infty} \frac{(k + a)^{2N} r^{2(k+a)}}{k!} \leq C_1 r^{-2a} \sum_{k=0}^{\infty} \frac{(k + a)^{2N} (2r^2)^k}{(k+a)!}
\]

\[
= C_1 r^{-2a} \sum_{k=0}^{\infty} \frac{k^{2N}(2r^2)^k}{k!} \leq C_2 r^{-2a} \sum_{k=0}^{\infty} \frac{k^{2N}(2r_0^2 e)^k}{k^k} \leq C_2 r^{-2a} \left( \frac{2N}{\log N} \right)^{2N(1 - \frac{1}{\log N})} \left( \frac{r^2_0}{r_0^2} \cdot \theta(r_0^2) \right)^{\frac{2N}{r_0^2}} \sum_{k=0}^{\infty} \left( \frac{r}{r_0^2} \right)^{2k}
\]

for some constants \( C_1 \) and \( C_2 \) which are independent of \( N \) and \( r_0 \).

For general \( a_1 \) and \( a_2 \), let

\[
s_{a_1,a_2}(N, r) := \sum_{k=0}^{\infty} \frac{(a_1 k + a_2)^{2N} r^{2k}}{k!} \quad \text{and} \quad s_a(N, r) := s_{1,a}(N, r).
\]

Then \( s_{a_1,a_2}(N, r) = a_1^{2N} s_{a_2/a_1}(N, r) \), and hence it suffices to prove (9) in the case \( a_1 = 1 \). Moreover, since \( s_{a_1,a_2}(N, r) \) increases with \( a_1 \) and \( a_2 \), and all factors on the right-hand side of (9) except \( C \) are independent of \( a_1 \) and \( a_2 \), it follows that we may assume that \( a = a_2 \) is an integer, and the proof is complete. □

Remark 6. For a given \( N \in \mathbb{N} \), the Bell number \( B_N \) counts the number of all partitions of a set of size \( N \) and it is given by

\[
B_N = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^{2N}}{k!}.
\]

According to [1, Theorem 2.1],

\[
\frac{1}{e} \sum_{k=1}^{\infty} \frac{k^{2N}}{k!} < \left( \frac{0.792 \times 2N}{\log(2N + 1)} \right)^{2N} \quad N = 1, 2, \ldots
\]

For large values of \( N \) we claim that the estimate of Lemma 5 is significantly better.

For arbitrary

\[
r > 0, \quad 1/e < \lambda << 1 \quad \text{and} \quad 0 < a < 1 - \log \left( \frac{1}{\lambda} \right) = 1 + \log \lambda
\]
we have
\[
\left( \frac{2N}{\log N} \right)^{2N} \left( \frac{r \log N}{2N} \right) \frac{2N}{2N} = \left( \frac{2\lambda N}{\log(2N + 1)} \right)^{2N} \cdot C_N^{2N},
\]
where
\[
\lim_{N \to \infty} C_N \cdot N^a = 0.
\]
In fact,
\[
C_N = \frac{r}{2} \left( \frac{\log(2N + 1)}{\log N} \right)^{\log N} \frac{\log N}{e^{\log N(1+\log(\lambda))}}.
\]
Since
\[
\lim_{N \to \infty} \left( \frac{\log(2N + 1)}{\log N} \right)^{\log N} = 2
\]
and
\[
\lim_{N \to \infty} \frac{N^a \log N}{e^{\log N(1+\log(\lambda))}} = \lim_{N \to \infty} \frac{\log N}{e^{\log N(1+\log(\lambda)-a)}} = 0
\]
we are done.

The lower estimate
\[
\left( \frac{2N}{e \log(2N)} \right)^{2N} \leq \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^{2N}}{k!}
\]
appears in [1, (2.4)]. Since the function \( f(t) = t^{2N-t} \) is increasing in \([1, 1 + \frac{2N}{\log N}]\), we have for large values of \( N \)
\[
f\left( \frac{2N}{\log N} \right) \leq \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^{2N}}{k!}.
\]
Again it is straight-forward to check that
\[
\left( \frac{2N}{e \log(2N)} \right)^{2N} = o\left( f\left( \frac{2N}{\log N} \right) \right) \quad \text{as} \quad N \to \infty.
\]

In the following we apply the previous result to functions \( f \) with Hermite series expansions, given by (I).

Proposition 7. Let \( f \in L^2(\mathbb{R}^d) \) be given by (I) such that
\[
|c_\alpha(f)| \lesssim r^{(|\alpha|/|\alpha|)}^{-\frac{1}{2}}
\]
for some \( r > 0 \). Then, for \( 0 < d \cdot r < r_0 \),
\[
\|H^N f\|_{L^2} \lesssim 2^{2N} \left( \frac{2N}{\log N} \right)^{2N} \left( \frac{r_0^2 \cdot \theta(r_0^2)}{\log N} \right)^{2N} \frac{r_0^2}{r_0^2 - (d \cdot r)^2}.
\]
Proof. From
\[ |c_\alpha(H^N f)| = (2|\alpha| + d)^N |c_\alpha(f)| \]
we obtain
\[
\|H^N f\|_{L^2}^2 = \sum_{\alpha \in \mathbb{N}^d} |c_\alpha(H^N f)|^2 \lesssim \sum_{\alpha \in \mathbb{N}^d} \frac{(2|\alpha| + d)^{2N} \cdot r^{2|\alpha|}}{\alpha!}
\]
\[
\lesssim \sum_{\alpha \in \mathbb{N}^d} \frac{(2|\alpha| + d)^{2N} d^{|\alpha|} r^{2|\alpha|}}{|\alpha|!} \lesssim \sum_{k=0}^{\infty} \frac{(2k + d)^{2N} (d \cdot r)^{2k}}{k!},
\]
and it suffices to apply Lemma 5.

Next we deduce some kind of converse of Proposition 7. For this reason we need the following lemma.

Lemma 8. Let \( r > 1 \),
\[ \psi(t) = e^{t \left( \frac{t}{2} - \log r \right)} + r^2 \log r \quad \text{and} \quad t_0 = 2 \log r. \]
Then there exists a convex and increasing function \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi(0) = 0 \) and \( \varphi(t) = \psi(t) \) for every \( t \geq t_0 \).

Proof. We have
\[ \psi'(t) = e^{t \left( \frac{1 + t}{2} - \log r \right)} \quad \text{and} \quad \psi''(t) = e^{t \left( \frac{1 + t}{2} - \log r \right)}. \]
Hence
\[ \psi''(t) > \psi'(t) > 0 \]
for every \( t \geq t_0 \). On the other hand, the tangent line to the graph of \( \psi \) at point \( (t_0, \psi(t_0)) \) passes through \( (0, 0) \), since \( \psi(t_0) = t_0 \psi'(t_0) \).
Consequently, if \( \varphi : [0, \infty) \to [0, \infty) \) is defined by \( \varphi(0) = 0 \), \( \varphi \) linear in \([0, t_0]\) and \( \varphi(t) = \psi(t) \) for \( t \geq t_0 \), then \( \varphi \) satisfies the required conditions.

Proposition 9. Let
\[ \|H^N f\|_{L^2} \lesssim 2^N \left( \frac{2N}{\log N} \right)^{N(1 - \frac{1}{\log N})} \frac{2N}{r^{2N}}. \]
Then
\[ |c_\alpha(f)| \lesssim r^{\alpha |\alpha| - \frac{|\alpha|}{2} + 1}. \]

Before the proof we recall that the Young conjugate of a convex and increasing function \( \varphi : [0, \infty) \to [0, \infty) \) is the increasing and convex function \( \varphi^* : [0, \infty) \to [0, \infty) \), given by
\[ \varphi^*(s) = \sup_{t \geq 0} (st - \varphi(t)). \]
It turns out that \( (\varphi^*)^* = \varphi \).
Now assume that \( \varphi \) is the same as in Lemma 8. We claim that
\[
k^{\frac{k}{2} - k^{-1}} r^{-k} \lesssim \exp \left( \sup_N (N \log k - \varphi^*(N)) \right),
\]
when \( k \) is large enough.

In fact, for \( k \) large enough we have
\[
\exp (\varphi(\log k)) = k^{\frac{k}{2} - k^{-1}} r^2 \lesssim k^{\frac{k}{2}} r^{-k}.
\]
Moreover, for \( s \in [N, N + 1] \) we have
\[
st - \varphi^*(s) \leq t + (Nt - \varphi^*(N)),
\]
from where it follows
\[
\varphi(t) = \sup_{s \geq 0} (st - \varphi^*(s)) \leq t + \sup_N (Nt - \varphi^*(N)).
\]
In particular,
\[
k^{\frac{k}{2}} r^{-k} \lesssim \exp \left( \log k + \sup_N (N \log k - \varphi^*(N)) \right),
\]
which is the same as (10).

**Proof of Proposition 9.** Let
\[
\varepsilon_N := \left( \frac{\log N}{2N} \right)^{\frac{N}{\log N}},
\]
and let \( \varphi \) be the same as in Lemma 8. Since
\[
|c_\alpha(f)| \leq \inf_N \left( \frac{1}{(2|\alpha|)^\gamma} \|H^N f\|_{L^2} \right),
\]
we have
\[
\frac{1}{|c_\alpha(f)|} \gtrsim \exp \left( \sup_N \left( N \log |\alpha| - \frac{2N}{\log N} \log r - N \log \left( \frac{2N}{\log N} \right) - \log \varepsilon_N \right) \right).
\]
By Lemma 8 and (10) it suffices to prove
\[
\frac{2N}{\log N} \log r + N \log \left( \frac{2N}{\log N} \right) + \log \varepsilon_N \leq \varphi^*(N) + C,
\]
for large values of \( N \), where \( C > 0 \) is a constant which is independent of \( N \) and \( k \). From the definitions it follows that, for \( N \) large enough,
\[
\varphi^*(N) \geq N \log \left( \frac{2N}{\log N} \right) - \varphi \left( \log \left( \frac{2N}{\log N} \right) \right)
\]
\[
= N \log \left( \frac{2N}{\log N} \right) + \log \varepsilon_N + \frac{2N}{\log N} \log r - r^2 \log r
\]
and the lemma is proved. \( \square \)

**Proofs of Theorems 1 and 2.** The results follow immediately from Propositions 7 and 9. \( \square \)
3. Mapping properties of $\Sigma_s(\mathbb{R}^d)$ and $\mathcal{S}_s(\mathbb{R}^d)$ under the Bargmann transform

In [8], complete mapping properties of $\Sigma_s(\mathbb{R}^d)$ and $\mathcal{S}_s(\mathbb{R}^d)$ under the Bargmann transform are deduced when $s \geq 1/2$ and when $s = 0$. Here we show analogous properties in the case $0 < s < 1/2$.

In what follows we let

$$A_s(\mathbb{C}^d) := \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R(\log(|z|))^{1-2s}} \text{ for some } R > 0 \}$$

for some $R > 0$, and

$$A_{0,s}(\mathbb{C}^d) := \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R(\log(|z|))^{1-2s}} \text{ for every } R > 0 \}$$

endowed with the inductive limit topology, and

$$A_{0,s}(\mathbb{C}^d) := \{ e^{R(\log(|z|))^{1-2s}} ; R > 0 \}$$

endowed with the projective limit topology, when $0 < s < 1/2$. Here, $\langle z \rangle := (1 + |z|^2)^{1/2}$, as usual.

We may replace the inequalities in (12) and (13) by suitable (weighted) $L^p$ estimates on the involved entire functions. This is for example a consequence of [7, Theorem 3.2] and the fact that

$$\{ e^{R(\log(|z|))^{1-2s}} ; R > 0 \}$$

is an admissible family of weight functions on $\mathbb{C}^d$ in the sense of [7, Definition 1.4].

Therefore, for any $p \in [1, \infty]$, $A_s(\mathbb{C}^d)$ is the set of all $F \in A(\mathbb{C}^d)$ such that

$$\left( \int_{\mathbb{C}^d} |F(z)|^p \, d\lambda(z) \right)^{1/p} < \infty$$

is true for some $R > 0$, and $A_{0,s}(\mathbb{C}^d)$ is the set of all $F \in A(\mathbb{C}^d)$ such that (14) is true for every $R > 0$ (also in topological sense, and with obvious modifications when $p = \infty$). We also note that $A_{0,s}(\mathbb{C}^d)$ is a Fréchet space.

**Theorem 10.** Let $0 < s < 1/2$. Then the following is true:

1. The Bargmann transform $\mathfrak{H}_d$ is a topological isomorphism from $\mathcal{S}_s(\mathbb{R}^d)$ onto $A_{0,s}(\mathbb{C}^d)$;
2. The Bargmann transform $\mathfrak{H}_d$ is a topological isomorphism from $\Sigma_s(\mathbb{R}^d)$ onto $A_s(\mathbb{C}^d)$.

**Proof.** Let

$$\vartheta_R(\alpha) := \left( \frac{\pi^d}{2^d(|\alpha| + d - 1)!} \int_0^{\infty} e^{-R(\log(r))^{1-2s}} r^{(|\alpha|+d-1)} \, dr \right)^{1/2}$$
and denote by $\mathcal{H}_R^2(\mathbb{R}^d)$ the set of all $f \in \mathcal{S}(\mathbb{R}^d)$ such that $c_\alpha(f)$ in (11) satisfies
\[
\|f\|_{\mathcal{H}_R^2} := \left( \sum_{\alpha \in \mathbb{N}^d} |c_\alpha(f)\vartheta_R(\alpha)|^2 \right)^{1/2} < \infty.
\]
Then it follows from Proposition 3.4 in [8] that $\vartheta_d$ is a topological isomorphism from
\[
\bigcup_{R>0} \mathcal{H}_R^2(\mathbb{R}^d)
\]
on to $A_s(\mathbb{C}^d)$ and from
\[
\bigcap_{R>0} \mathcal{H}_R^2(\mathbb{R}^d)
\]
on onto $A_{0,s}(\mathbb{C}^d)$. Hence, to conclude, it suffices to show the existence of positive constants $c_j$ and $a_j$, $j = 1, 2$ such that
\[
e^{c_1|\alpha|^{\frac{s}{2}}/R^{a_1}} \lesssim \vartheta_R(\alpha) \lesssim e^{c_2|\alpha|^{\frac{s}{2}}/R^{a_2}}, \tag{15}
\]
for every $R > 0$.

In order to prove (15), let
\[
m_\alpha = |\alpha| + d - 1 \quad \text{and} \quad \theta = \frac{1}{1-2s} > 1.
\]
and fix $1 < \mu < \theta$. Then
\[
\vartheta_R(\alpha)^2 \gtrsim \frac{1}{m_\alpha!} \int_{R_{1,\alpha}}^{R_{2,\alpha}} e^{-R(\log r)^\theta} r^{m_\alpha} dr,
\]
where
\[
\log R_{2,\alpha} = \left( \frac{m_\alpha}{\theta R} \right)^{\frac{1-2s}{2s}} \quad \text{and} \quad \log R_{1,\alpha} = \frac{1}{\mu} \log R_{2,\alpha}.
\]
For $R_{1,\alpha} \leq r \leq R_{2,\alpha}$ we have
\[
-R(\log r)^\theta + m_\alpha \log r \geq m_\alpha \log R_{1,\alpha} - R(\log R_{2,\alpha})^\theta
\]
\[
= m_\alpha \left( \frac{m_\alpha}{\theta R} \right)^{\frac{1-2s}{2s}} - R \left( \frac{m_\alpha}{\theta R} \right)^{\frac{1}{2s}} \left( \theta^{\frac{1}{2s}} m_\alpha^{\frac{1}{2s}} R^{1-\frac{1}{2s}} \left( \frac{\theta}{\mu} - 1 \right) \right) \geq C_1 |\alpha|^{\frac{s}{2}}/R^{a_1},
\]
where $a_1 = \frac{1}{2s} - 1$ and $C_1 = \theta^{-\frac{1}{2s}}(\frac{\theta}{\mu} - 1)$.

As $R_{2,\alpha} - R_{1,\alpha} = R_{2,\alpha}(1 - R_{2,\alpha}^{\frac{1}{2s}-1})$ increases with $|\alpha|$ and $\log(m_\alpha!) = o(|\alpha|^{\frac{s}{2}})$ as $|\alpha| \to \infty$, we get (with $c_1 < C_1$ fixed) that
\[
\vartheta_R(\alpha) \geq C_1 e^{c_1|\alpha|^{\frac{s}{2}}/R^{a_1}}.
\]
To prove the other inequality we observe that for each $R > 0$, there is a constant $C > 0$ such that
\[ \vartheta_R^2(\alpha) \leq C \int_{\epsilon}^{\infty} e^{-\frac{R}{2} (\log r)^\theta} g_\alpha(r) \, dr \]
\[ \leq C \sup_{r \geq \epsilon} \left( g_\alpha(r) \right) \left( \int_{\epsilon}^{\infty} e^{-\frac{R}{2} (\log r)^\theta} \, dr \right) , \]
where $g_\alpha(r) = e^{-\frac{R}{2} (\log r)^\theta} r^{m_\alpha}$.

By straight-forward computations it follows that $g_\alpha(r)$ when $r \geq \epsilon$, attains its global maximum for
\[ r_\alpha = \exp \left( \frac{2m_\alpha}{\theta R} \right)^{\frac{1-\frac{1}{2s}}{2}} \]
and that
\[ g_\alpha(r_\alpha) = e^{c_2 m_\alpha^2 / R e^2} , \]
where
\[ c_2 = 2^{\frac{1}{2s} - 1} \theta^{-\frac{1}{2}} (\theta - 1) \quad \text{and} \quad a_2 = \frac{1}{2s} - 1 . \]
Hence, by replacing $c_2$ by a larger constant, if necessary, we get the desired inequality.

\[ \square \]

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