THE VARIATION OF THE MAXIMAL FUNCTION OF
A RADIAL FUNCTION

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Abstract. It is shown for the non-centered Hardy-Littlewood maximal operator \( M \) that \( \| DMf \|_1 \leq C_n \| Df \|_1 \) for all radial functions in \( W^{1,1}(\mathbb{R}^n) \).

1. Introduction

The non-centered Hardy-Littlewood maximal operator \( M \) is defined by setting for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) that

\[
Mf(x) = \sup_{B(z,r) \ni x} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(y)| \, dy =: \sup_{B(z,r) \ni x} \int_{B(z,r)} |f(y)| \, dy
\]

for every \( x \in \mathbb{R}^n \). The centered version of \( M \), denoted by \( M_c \), is defined by taking the supremum over all balls centered at \( x \). The classical theorem of Hardy, Littlewood and Wiener asserts that \( M \) (and \( M_c \)) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p \leq \infty \). This result is one of the cornerstones of the harmonic analysis. While the absolute size of a maximal function is usually the principal interest, the applications in Sobolev-spaces and in the potential theory have motivated the active research of the regularity properties of maximal functions. The first observation was made by Kinnunen who verified \([K]\) that \( M_c \) is bounded in Sobolev-space \( W^{1,p}(\mathbb{R}^n) \) if \( 1 < p \leq \infty \), and inequality

\[
|DM_c f(x)| \leq M_c(\|Df\|)(x) \quad (1.2)
\]

holds for all \( x \in \mathbb{R}^n \). The proof is relatively simple and inequality \((1.2)\) (and the boundedness) holds also for \( M \) and many other variants.

The most challenging open problem in this field is so called ‘\( W^{1,1} \)-problem’: Does it hold for all \( f \in W^{1,1}(\mathbb{R}^n) \), that \( Mf \in W^{1,1}(\mathbb{R}^n) \) and

\[
\| DMf \|_1 \leq C_n \| Df \|_1 ?
\]

This problem has been discussed (and studied) for example in \([AlPe], [CaHu], [CaMa], [HO], [HM], [Ku] \) and \([Ta]\). The fundamental obstacle is that \( M \) is not bounded in \( L^1 \) and therefore inequality \((1.2)\) is not enough to solve the problem. In the case \( n = 1 \) the answer is known to be positive, as was

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proved by Tanaka [Ta]. For \( M_c \) the problem turns out to be very complicated also when \( n = 1 \). However, Kurka [Ku] managed to show that the answer is positive also in this case.

The goal of this paper is to develop technology for \( W^{1,1} \)-problem in higher dimensions, where the problem is still completely open. The known proofs in the one-dimensional case are strongly based on the simplicity of the topology: the crucial trick (in the non-centered case) is that \( Mf \) does not have a strict local maximum outside the set \( \{ Mf(x) = f(x) \} \). This fact is a strong tool when \( n = 1 \) but is far from sufficient for higher dimensions.

The formula for the derivative of the maximal function (see Lemma 2.2 or [L]) has an important role in the paper. It says that if \( Mf(x) = \int_{B} |f|, \quad |f(x)| < Mf(x) < \infty, \) and \( Mf \) is differentiable at \( x \), then

\[
DMf(x) = \int_{B} Df(y) \, dy. \tag{1.3}
\]

From this formula one can see immediately the validity of the estimate (1.2) for \( M \). However, since \( B \) is exactly the ball which gives the maximal average (for \( |f| \)), it is expected that one can derive from (1.3) much more sophisticated estimates than (1.2). In Section 2 (Lemma 2.2), we perform basic analysis related to this issue. The key observation we make is that if \( B \) is as above, then

\[
\int_{B} Df(y) \cdot (y - x) \, dy = 0. \tag{1.4}
\]

In the background of this equality stands a more general principle, concerning other maximal operators as well: if the value of the maximal function is attained to ball (or other permissible object) \( B \), then the weighted integral of \( |Df| \) over \( B \) is zero for a set of weights depending on the maximal operator. We believe that the utilization of this principle is a key for a possible solution of \( W^{1,1} \)-problem.

As the main result of this paper, we employ equality (1.4) to show that in the case of radial functions the answer to \( W^{1,1} \)-problem is positive (Theorem 3.11). Even in this case the problem is evidently non-trivial and truly differs from the one-dimensional case. To become convinced about this, consider the important special case where \( f \) is radially decreasing \((f(x) = g(|x|))\), where \( g : [0, \infty) \to \mathbb{R} \) is decreasing). In this case \( Mf \) is radially decreasing as well and \( Mf(0) = f(0) \). If \( n = 1 \), these facts immediately imply that \( ||DMf||_1 = ||Df||_1 \), but if \( n \geq 2 \) this is definitely not the case: the additional estimates are necessary. This type of estimate for radially decreasing functions can be derived from (1.3) and (1.4), saying that

\[
|DMf(x)| \leq \frac{C_n}{|x|} \int_{B(0,|x|)} |Df(y)||y| \, dy. \tag{1.5}
\]
By using this inequality, the positive answer to \( W^{1,1} \)-problem for radially decreasing functions follows straightforwardly by Fubini Theorem (Corollary 3.1).

For general radial functions, inequality (1.3) turns out to hold only if the maximal average is achieved in a ball with radius comparable to \(|x|\). To overcome this problem, we study the auxiliary maximal function \( M^I \), defined for \( f \in L^1_{loc}(\mathbb{R}^n) \) by

\[
M^I f(x) = \sup_{x \in B(z,r), r \leq |x|/4} \int_{B(z,r)} |f(y)| \, dy,
\]

and prove (Lemma 3.2) that for all radial \( f \in W^{1,1}(\mathbb{R}^n) \) it holds that

\[
||D M^I f||_1 \leq C_n ||D f||_1.
\]  

The proof of this auxiliary result resembles the proof of \( W^{1,1} \)-problem (for \( M \)) in the case \( n = 1 \). As the first step, we prove by straightforward calculation that for the 'endpoint operator' of \( M^I \), defined by

\[
F_{1/4}(x) := \sup_{x \in B(z,|x|/4)} \int_{B(z,|x|/4)} |f(y)| \, dy,
\]

it holds that \( ||D F_{1/4}||_1 \leq C ||D f||_1 \) for all \( f \in W^{1,1}(\mathbb{R}^n) \). Recall again the fact that \( M f \) does not have a local maximum in \( \{ f(x) > |f(x)| \} \), leading to the estimate \( ||D M f||_1 \leq ||D f||_1 \) in the case \( n = 1 \). As a multidimensional counterpart for radial functions, we show that \( M^I f \) does not have a local maximum in \( \{ M^I f(x) > \max\{|f(x)|, f_{1/4}(x)\} \} \) and for every \( k \in \mathbb{Z} \) it holds that

\[
\int_{\{2^k \leq |y| \leq 2^{k+1}\}} D M^I f(y) \, dy \leq C_n \int_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}} |D f(y)| \, dy.
\]

Estimate (1.7) can be easily derived from this fact. The main result follows by combining (1.6) and exploiting the estimate (1.5) in \( \{ M f(x) > M^I f(x) \} \).

**Question.** The analysis presented in this paper raises the interest towards the study of the integrability properties of some conditional maximal operators. As an example, (1.3) and (1.4) yield that \( |D M f(x)| \leq \tilde{M}(D f)(x) \), where \( \tilde{M} \) is defined for all locally integrable gradient fields \( F : \mathbb{R}^n \to \mathbb{R}^n \) by

\[
\tilde{M}F(x) = \sup \left\{ \left| \int_{B(z,r)} F \right| : x \in B(z,r), \int_{B(z,r)} F(y) \cdot (y-x) \, dy = 0 \right\}.
\]

It is clear that \( \tilde{M} \) is bounded by \( M(|F|) \), but does it hold that \( \tilde{M} \) has even better integrability properties than \( M \)? What about the boundedness in the Hardy-space \( H^1 \) or even in \( L^1 \)? Notice that the boundedness of \( \tilde{M} \) in \( L^1 \) would imply the solution to \( W^{1,1} \)-problem. This problem is almost completely open, even in the case \( n = 1 \). Counterexamples would be highly interesting as well.
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2. Preliminaries and general results

Let us introduce some notation. The boundary of the $n$-dimensional unit ball is denoted by $S^{n-1}$. The $s$-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. The volume of the $n$-dimensional unit ball is denoted by $\omega_n$ and the $\mathcal{H}^{n-1}$-measure of $S^{n-1}$ by $\sigma_n$. The weak derivative of $f$ (if exists) is denoted by $Df$. If $v \in S^{n-1}$, then

$$D_v f(x) := \lim_{h \to 0} \frac{1}{h} (f(x + hv) - f(x)),$$

in the case the limit exists.

Definition 2.1. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ let

$$B_x := \{ B(z, r) : x \in \bar{B}(z, r), r > 0, \int_B |f| = Mf(x) \}.$$

It is easy to see that if $f \in L^1(\mathbb{R}^n)$ and $|f(x)| < Mf(x) < \infty$, then $B_x \neq \emptyset$.

The following lemma is the main result of this section. We point out that below (6) is especially useful in the case of radial functions.

Lemma 2.2. Suppose that $f \in W^{1,1}(\mathbb{R}^n)$, $Mf(x) > f(x)$ and $Mf$ is differentiable at $x$. Then

1. For all $v \in S^{n-1}$ and $B \in B_x$, it holds that

$$DMf(x) = \int_B D|f|(y) dy$$

and

$$D_v Mf(x) = \int_B D_v |f|(y) dy.$$

2. If $x \in B$ for some $B \in B_x$, then $DMf(x) = 0$.

3. If $x \in \partial B$, $B = B(z, r) \in B_x$ and $DMf(x) \neq 0$, then

$$\frac{DMf(x)}{|DMf(x)|} = \frac{z - x}{|z - x|}.$$

4. If $B \in B_x$, then

$$\int_B D|f|(y) \cdot (y - x) dy = 0. \quad (2.8)$$

5. If $x \in \partial B$, $B = B(z, r) \in B_x$, then

$$|DMf(x)| = \frac{1}{r} \int_B D|f|(y) \cdot (z - y) dy.$$

6. If $B \in B_x$, then

$$DMf(x) \cdot \frac{x}{|x|} = \frac{1}{|x|} \int_B D|f|(y) \cdot y dy. \quad (2.9)$$
The proof of Lemma 2.2 is essentially based on the following auxiliary propositions.

**Proposition 2.3.** Suppose that \( f \in W^{1,1}(\mathbb{R}^n) \), \( B \) is a ball, \( h_i \in \mathbb{R} \) such that \( h_i \to 0 \) as \( i \to \infty \), and \( B_i = L_i(B) \), where \( L_i \) are affine mappings and
\[
\lim_{i \to \infty} \frac{L_i(y) - y}{h_i} = g(y).
\]
Then
\[
\lim_{i \to \infty} \frac{1}{h_i} \left( \int_{B_i} f(y) \, dy - \int_B f(y) \, dy \right) = \int_B Df(y) \cdot g(y) \, dy. \tag{2.10}
\]

**Proof.** The proof is a simple calculation:
\[
\frac{1}{h_i} \left( \int_{B_i} f(y) \, dy - \int_B f(y) \, dy \right) = \frac{1}{h_i} \left( \int_{L_i(B)} f(y) \, dy - \int_B f(y) \, dy \right) = \frac{1}{h_i} \left( \int_B f(L_i(y)) - f(y) \, dy \right) \approx \int_B \frac{Df(y) \cdot (L_i(y) - y)}{h_i} \, dy \to \int_B Df(y) \cdot g(y) \, dy,
\]
if \( i \to \infty \). \( \square \)

**Lemma 2.4.** Let \( f \in W^{1,1}(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \), \( B \in B_x \), \( \delta > 0 \), and let \( L_h, h \in [-\delta, \delta] \), be affine mappings such that \( x \in L_h(\overline{B}) \) and
\[
\lim_{h \to 0} \frac{L_h(y) - y}{h} = g(y). \tag{2.11}
\]
Then
\[
\int_B D|f|(y) \cdot g(y) \, dy = 0. \tag{2.12}
\]

**Proof.** Let us denote \( B_h := L_h(B) \). By Proposition 2.3 it holds that
\[
\int_B D|f|(y) \cdot g(y) \, dy = \lim_{h \to 0} \frac{1}{h} \left( \int_{B_h} |f|(y) - \int_B |f|(y) \right).
\]
Since \( B \in B_x \) and \( x \in B_h \), the sign of the quantity inside the large parentheses is non-positive for all \( h \in [-\delta, \delta] \). However, the sign of \( 1/h \) depends on the sign of \( h \). The conclusion is that the above equality is possible only if (2.12) is valid. \( \square \)

**Proof of Lemma 2.2**

(1) The claim is counterpart for the formula for \( DM_c f \), which was first time proved in [L]. Suppose that \( B = B(z,r) \in B_x \) and let \( B_h :=...\)
Let $B = B(z + hv, r)$. Then it holds that

$$D_v Mf(x) = \lim_{h \to 0} \frac{1}{h} (Mf(x + hv) - Mf(x))$$

$$\geq \lim_{h \to 0} \frac{1}{h} \left( \int_{B_h} |f(y)| dy - \int_{B} |f(y)| dy \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left( \int_{B_h} |f(y + hv)| - |f(y)| dy \right) = \int_{B_h} D_v f(y) dy.$$  

On the other hand, if $B_h := B(z - hv, r)$, then

$$D_v Mf(x) = \lim_{h \to 0} \frac{1}{h} (Mf(x) - Mf(x - hv))$$

$$\leq \lim_{h \to 0} \frac{1}{h} \left( \int_{B} |f(y)| dy - \int_{B_h} |f(y)| dy \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left( \int_{B} |f(y)| - |f(y + hv)| dy \right) = \int_{B_h} D_v f(y) dy.$$  

These inequalities imply the claim.

(2) If $B \in \mathcal{B}_z$ and $x \in B$, then $y \in B$ if $|y - x|$ is small enough, and thus $Mf(y) \geq Mf(x)$.

(3) Let $B = B(z, r) \in \mathcal{B}_z$, $v \in S^{n-1}$ such that $v \cdot (z - x) = 0$, and $h_i \in (0, \infty)$, $h_i \to 0$ as $i \to \infty$. Moreover, let us denote $B_i := B(z, |z - (x + h_i v)|)$. Then it clearly holds that $x + h_i v \in B_i$ and it is also easy to see that $B_i = L_i(B)$ for an affine mapping $L_i$ given by

$$L_i(y) = y + \left( \frac{|z - (x + h_i v)| - |z - x|}{|z - x|} \right) (y - z).$$

By the assumption $v \cdot (z - x) = 0$ it follows that

$$\lim_{i \to \infty} \frac{L_i(y) - y}{h_i} = (y - z) \lim_{i \to \infty} \left( \frac{|z - (x + h_i v)| - |z - x|}{|z - x|} \right) = 0.$$  

Therefore, Proposition 2.3 implies that

$$\lim_{i \to \infty} \frac{1}{h_i} \left( \int_{B_i} |f(y)| dy - \int_{B} |f(y)| dy \right) = 0.$$  

This shows that $D_v Mf(x) = 0$ for all $v$ orthogonal to $(z - x)$. In particular, it follows that $DMf(x)$ is parallel to $z - x$ or $x - z$. The final claim follows easily by the fact that $Mf(x + h(z - x)) \geq Mf(x)$ if $0 < h \leq 2$.

(4) Let $B \in \mathcal{B}_z$ and $L_h(y) := y + h(y - x)$, $h \in \mathbb{R}$. Then it holds that $L_h$ is affine mapping, $L_h(x) = x$, and so $x \in L_h(B) =: B_h$, and $(L_h(y) - y)/h = y - x$ for all $h \in \mathbb{R}$. Therefore, Lemma 2.4 implies that

$$\int_{B} D|f|(y) \cdot (y - x) dy = 0.$$
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(5) By combining (1), (3) and (4) the claim follows by

\[ |DMf(x)| = DMf(x) \cdot \frac{z - x}{|z - x|} = \int_B D|f|(y) \cdot \frac{z - y}{|z - x|} \, dy \]

\[ = \int_B D|f|(y) \cdot \frac{z - y}{|z - x|} \, dy. \]

(6) The claim follows from (1) and (4).

\[ \blacksquare \]

3. W^{1,1}-problem for radial functions

Radial functions and notation. In what follows, we will interpret a radial function on \( \mathbb{R}^n \) as a function on \((0, \infty)\) in a natural way. To be more precise, if \( f \in W^{1,1}_{loc}(\mathbb{R}^n) \) is radial, it is well-known fact that there exists a continuous function \( \tilde{f} : (0, \infty) \to \mathbb{R} \) such that \( \tilde{f} \) is weakly differentiable,

\[ \int_0^\infty |\tilde{f}'(t)| t^{n-1} \, dt < \infty, \]

and (by a possible redefinition of \( f \) in a set of measure zero) for all \( t \in (0, \infty) \) it holds that \( f(x) = \tilde{f}(t) \) and \( D_{x/|x|} f(x) = \tilde{f}'(t) \) if \( |x| = t \). In what follows, we will simplify the notation and use \( f \) to denote \( \tilde{f} \) as well. To avoid the possibility of misunderstanding, we usually use variable \( t \) and notation \( f' \) (instead of \( Df \)) when we are actually working with \( \tilde{f} \). We also say that \( f \) is radially decreasing if \( f \) is radial and \( f(t_1) \leq f(t_2) \) if \( t_1 > t_2 \). Notice also that if \( f \) is radial then \( Mf \) is also radial.

The following result is an easy consequence of Lemma 2.2.

Corollary 3.1. If \( f \in W^{1,1}(\mathbb{R}^n) \) is radially decreasing, then \( DMf \in W^{1,1}(\mathbb{R}^n) \) and \( ||DMf||_1 \leq C_n ||f||_1 \).

Proof. Since \( f \) is radially decreasing, it is easy to show (the rigorous proof is left to the reader) that if \( Mf(x) \neq 0 \) and \( B \in B_x \), then \( 0 \in \bar{B} \) and \( \bar{B} \subset \bar{B}(0, |x|) \). Especially, we get by Lemma 2.2 (6), that

\[ |DMf(x)| \leq \frac{C_n}{|x|} \int_{B(0,|x|)} |Df(y)||y| \, dy. \]  \[ (3.13) \]
Then the claim follows by Fubini theorem:

$$\int_{\mathbb{R}^n} \left( \frac{1}{|x|} \int_{B(0,|x|)} |Df(y)||y| \, dy \right) \, dx$$

$$= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{\mathbb{R}^n} \frac{\chi_{B(0,|x|)}(y)}{\omega_n |x|^{n+1}} \, dx \right) \, dy$$

$$= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{\{x: |x| \geq |y|\}} \frac{1}{\omega_n |x|^{n+1}} \, dx \right) \, dy$$

$$= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{S^{n-1}} \int_{|y|}^{\infty} \frac{1}{\omega_n t^{n+1}} \, dt \, d\mathcal{H}^{n-1}(t) \right) \, dy$$

$$= \frac{\sigma_n}{\omega_n} \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{|y|}^{\infty} \frac{1}{t^2} \, dt \right) \, dy$$

$$= \frac{\sigma_n}{\omega_n} \int_{\mathbb{R}^n} |Df(y)| \, dy .$$

□

In the case of general radial functions, (1.5) is in general valid (and useful) only for those \( x \) for which the radius of \( B \in B_x \) is comparable to \( |x| \). As it was explained in the introduction, the main auxiliary tool in the case of general radial functions is the following result (recall the definition of \( M^I \) in the introduction):

**Lemma 3.2.** If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial, then \( M^I f \in W^{1,1}(\mathbb{R}^n) \) and \( ||Df||_1 \leq C_n ||Df||_1 \).

Before the actual proof of this result, we prove several auxiliary results. The first of them is well known.

**Proposition 3.3.** Suppose that \( E \subset \mathbb{R} \) is open. Then there exist disjoint intervals \( (a_i, b_i) \) such that \( E = \bigcup_{i=1}^{\infty} (a_i, b_i) \) and \( a_i, b_i \in \partial E \cup \{-\infty, \infty\} \) for all \( i \in \mathbb{N} \).

The following auxiliary result is repeatedly utilized in the proof. The result is well known but we express the proof for readers convenience.

**Lemma 3.4.** Suppose that \( \Omega \subset \mathbb{R}^n \), \( f \in W^{1,1}(\Omega) \) is continuous, \( g : \Omega \to \mathbb{R} \) is continuous and weakly differentiable in \( E := \{x \in \Omega : g(x) > f(x)\} \), and \( \int_{E} |Dg| < \infty \). Then max\{f, g\} is weakly differentiable in \( \Omega \) and

$$D(\max\{f, g\}) = \chi_E Dg + \chi_{\Omega \cap E^c} Df .$$

**Proof.** Suppose that \( \phi \) is a smooth test function, compactly supported in \( \Omega \), \( 1 \leq i \leq n \), \( L(t) = p + te_i \), \( p \in \mathbb{R}^n \), and let \( L \) denote the line \( L(\mathbb{R}) \). By Proposition 3.3, \( E \cap L \) can be written as a union of disjoint and open (in \( \Omega \cap L \)) line segments \( E_j = L((a_j, b_j)) \), \( j \in \mathbb{N} \), such that \( L(a_j), L(b_j) \in \partial E \) (with
respect to $\Omega \cap L$ or $a_j = -\infty$ or $b_j = \infty$. In particular, $f(L(a_j)) = g(L(a_j))$ if $a_j \neq -\infty$ and $f(L(b_j)) = g(L(b_j))$ if $b_j \neq \infty$. Since $\phi$ is compactly supported, it follows that

$$f(L(a_j))\phi(L(a_j)) = g(L(a_j))\phi(L(a_j)) \quad \text{and} \quad f(L(b_j))\phi(L(b_j)) = g(L(b_j))\phi(L(b_j)) \quad \text{for all} \quad j \in \mathbb{N}. $$

Therefore, by using the assumptions for $g$, it holds that

$$\int_{E_j} g(D_i\phi) d\mathcal{H}^1 = \int_{E_j} D_i(g\phi) d\mathcal{H}^1 - \int_{E_j} (D_i g) \phi d\mathcal{H}^1$$

$$= g(L(b_j))\phi(L(b_j)) - g(L(a_j))\phi(L(a_j)) - \int_{E_j} (D_i g) \phi d\mathcal{H}^1$$

$$= f(L(b_j))\phi(L(b_j)) - f(L(a_j))\phi(L(a_j)) - \int_{E_j} (D_i g) \phi d\mathcal{H}^1$$

$$= \int_{E_j} D_i(f \phi) d\mathcal{H}^1 - \int_{E_j} (D_i g) \phi d\mathcal{H}^1$$

$$= \int_{E_j} (D_i f) \phi + f(D_i \phi) - (D_i g) \phi d\mathcal{H}^1$$

for all $j \in \mathbb{N}$. Then

$$\int_{\Omega \cap L} \max\{f, g\}(D_i\phi) d\mathcal{H}^1 = \int_{E \cap L} g(D_i\phi) d\mathcal{H}^1 + \int_{\Omega \cap E \cap L} f(D_i\phi) d\mathcal{H}^1$$

$$= \sum_{j=1}^{\infty} \int_{E_j} g(D_i\phi) d\mathcal{H}^1 + \int_{\Omega \cap E \cap L} f(D_i\phi) d\mathcal{H}^1$$

$$= \int_{E \cap L} (D_i f) \phi + f(D_i \phi) - (D_i g) \phi d\mathcal{H}^1 + \int_{\Omega \cap E \cap L} f(D_i\phi) d\mathcal{H}^1$$

$$= \int_{E \cap L} (D_i f) \phi d\mathcal{H}^1 + \int_{E \cap L} (D_i f) \phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g) \phi d\mathcal{H}^1$$

$$= - \int_{E \cap L} (D_i f) \phi d\mathcal{H}^1 + \int_{E \cap L} (D_i f) \phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g) \phi d\mathcal{H}^1$$

$$= - \int_{E \cap L} (D_i f) \phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g) \phi d\mathcal{H}^1$$

This implies the claim. \qed

**Definition 3.5.** Let $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}$ is open. We say that $x$ is a local strict maximum of $f$ in $(a, b) \subset \Omega$, $-\infty \leq a < b \leq \infty$, if there exist $a', b' \in (a, b)$ such that $a' < x < b'$, $f(t) \leq f(x)$ if $t \in (a', b')$, and $\max\{f(a'), f(b')\} < f(x)$.
Proposition 3.6. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous and \( c \in (a, b) \) such that \( f(c) > \max\{f(a), f(b)\} \). Then \( f \) has a local strict maximum on \( (a, c) \).

Proof. It is easy to see that now any maximum point \( c \) \((f(c) = \max f)\), which is known to exist, is also a local strict maximum of \( f \). \( \square \)

Proposition 3.7. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous and does not have a local strict maximum on \( (a, b) \). Then there exists \( c \in [a, b] \) such that \( f \) is non-increasing on \( [a, c] \) and non-decreasing on \( [c, b] \).

Proof. Since \( f \) is continuous, we can choose \( c \in [a, b] \) such that \( f(c) = \min f \). To show that \( f \) is non-decreasing on \( [c, b] \), let \( c < y_1 < y_2 < b \) and assume, on the contrary, that \( f(y_2) < f(y_1) \). This implies that \( f(y_1) > \max\{f(c), f(y_2)\} \), and thus \( f \) has a strict local maximum on \( (c, y_2) \) by Proposition 3.6. This is the desired contradiction. To show that \( f \) is non-increasing on \( [a, c] \), let \( a < y_1 < y_2 < c \) and assume, on the contrary, that \( f(y_1) < f(y_2) \). This implies that \( f(y_2) > \max\{f(y_1), f(c)\} \), and thus \( f \) has a strict local maximum on \( (y_1, c) \) by Proposition 3.6. This is the desired contradiction. \( \square \)

Let us define for \( 0 < a < b < \infty \) the annular domains
\[
A_n(a, b) := A(a, b) := \{ x \in \mathbb{R}^n : a < |x| < b \} \quad \text{and} \quad A_n[a, b] := A[a, b] := \{ x \in \mathbb{R}^n : a \leq |x| \leq b \}.
\]

Lemma 3.8. If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial, then \( Mf \) does not have a local strict maximum in \( \{ t \in (0, \infty) : Mf(t) > f(t) \} \).

Proof. Suppose, on the contrary, that \( t_0 \in (0, \infty) \) is a local strict maximum of \( Mf \) and \( Mf(t_0) > f(t_0) \). Let us choose
\[
t^- := \sup\{ t < t_0 : Mf(t) < Mf(t_0) \} \quad \text{and} \quad t^+ := \inf\{ t > t_0 : Mf(t) < Mf(t_0) \}.
\]

By the definition of the local strict maximum, it follows that \( t_0 \in [t^-, t^+] \) and
\[
Mf(t) = Mf(t_0) \quad \text{for all} \ t \in [t^-, t^+]. \quad (3.14)
\]
Suppose that \( |x| = t_0 \). Since \( Mf(t_0) > f(t_0) \), it follows that there exist a ball \( B \) such that \( Mf(t_0) = \mathcal{F}_B |f|, \ x \in B \). Suppose first that \( B \not\subset A[t^-, t^+] \). In this case there exists \( \varepsilon > 0 \) such that \( [t^- - \varepsilon, t^-] \subset \{ |y| : y \in B \} \) or \( [t^+, t^+ + \varepsilon] \subset \{ |y| : y \in B \} \). Especially, it follows by the definition of \( M \) that \( Mf(t) \geq \mathcal{F}_B |f| = Mf(t_0) \) if \( t \in [t^- - \varepsilon, t^-] \) or \( t \in [t^+, t^+ + \varepsilon] \), respectively. Obviously this contradicts with the choice of \( t^- \) and \( t^+ \). This verifies that \( B \subset A[t^-, t^+] \). Therefore, it holds by (3.14) that
\[
Mf(y) = Mf(t_0) \quad \text{for all} \ y \in B. \quad (3.15)
\]
However, \( f(t_0) < Mf(t_0) \) also implies that there exists a ball \( B' \) with positive radius such that \( B' \subset B \) and \( f < Mf(t_0) \) in \( B' \). Combining this with \([3.15]\) yields the desired contradiction by

\[
Mf(t_0) = \int_B |f| \leq \frac{1}{|B|} \left( \int_{B \setminus B'} |f| + \int_{B'} |f| \right) < \frac{1}{|B|} \left( \int_{B \setminus B'} Mf + \int_{B'} Mf(t_0) \right) = Mf(t_0).
\]

\( \square \)

Recall the definition of \( f_{1/4} \) (the endpoint operator of \( M^I \), \([1.7]\)) from the introduction. Before showing the boundedness for \( M^I \), we have to prove the boundedness for \( f_{1/4} \).

**Proposition 3.9.** If \( f \in W^{1,1}(\mathbb{R}^n) \), then \( f_{1/4} \in W^{1,1}(\mathbb{R}^n) \) and \( ||Df_{1/4}||_1 \leq C_n ||Df||_1 \).

**Proof.** It is easy to check that \( f_{1/4} \) is Lipschitz outside the origin. Therefore, it suffices to verify the desired norm estimates for \( Df_{1/4} \). We will exploit Proposition \([2.3]\). If \( x \neq 0 \), we are going to show that if \( h > 0 \) is small enough and \( v \in S^{n-1} \), then

\[
\frac{1}{h} |f_{1/4}(x) - f_{1/4}(x + hv)| \leq C_n \int_{B(x,\frac{|x|}{4})} |Df| |y| \, dy. \tag{3.16}
\]

To show this, we may assume that \( f_{1/4}(x) > f_{1/4}(x + hv) \). Suppose that

\[ f_{1/4}(x) = \int_{B(z,|x|/4)} |f(y)| \, dy, \quad x \in \bar{B}(z,|x|/4) =: B, \]

\[ g_h(y) := x + hv + \frac{|x + hv|}{|x|}(y - x) \quad \text{and} \]

\[ B_h := g_h(B) = B(x + hv + \frac{|x + hv|}{|x|}(z - x), |x + hv|/4). \]

Especially, \( x + hv \in B_h \). Moreover, it is easy to compute that

\[
\lim_{h \to 0} \frac{g_h(y) - y}{h} = \lim_{h \to 0} \frac{hv + \left( \frac{|x + hv|}{|x|} - 1 \right)(y - x)}{h} = v + \frac{v \cdot x}{|x|^2} (y - x).
\]

Then it follows by Proposition \([2.3]\) that

\[
\lim_{h \to 0} \frac{f_{1/4}(x) - f_{1/4}(x + hv)}{h} \leq \lim_{h \to 0} \frac{1}{h} \left( \int_B |f(y)| \, dy - \int_{B_h} |f(y)| \, dy \right) = \int_B |Df| (y) \cdot (v + \frac{v \cdot x}{|x|^2} (y - x)) \, dy \leq \int_B |Df| (1 + \frac{|y - x|}{|x|}) \, dy \leq \int_B (1 + \frac{1}{4}) |Df| (y) \, dy \leq C_n \int_{B(x,\frac{|x|}{4})} |Df| |y| \, dy.
\]
This proves (3.16). Then the claim follows (e.g.) by using Fubini Theorem: Let us denote below \( B_x = B(x, \frac{|x|}{2}) \). By the above estimate,

\[
\int_{\mathbb{R}^n} |Df(y)| dy \leq C_n \left( \int_{\{x : \frac{|x|}{4} \leq |x| \leq 2|y|\}} |B_x|^{-1} dx \right) dy \leq C_n \|Df\|_1.
\]

\( \Box \)

The following estimate is well known.

**Proposition 3.10.** If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial and \( 0 < a < b < \infty \), then

\[
\sigma_n a^{n-1} \int_a^b |f'(t)| \, dt \leq \int_{A(a,b)} |Df(y)| \, dy \leq \sigma_n b^{n-1} \int_a^b |f'(t)| \, dt.
\]

**The proof of Lemma 3.2.** Let

\[ g(x) = \max \{f/4(x), |f(x)|\}. \]

By Lemma 3.4 and Proposition 3.9 it follows that \( g \in W^{1,1}(\mathbb{R}^n) \) and \( \|Dg\|_1 \leq C_n \|Df\|_1 \). Let

\[ E := \{x \in \mathbb{R}^n : Mf(x) > g(x)\} \quad \text{and} \quad E_k := E \cap A[2^{-k}, 2^{-k+1}], \quad k \in \mathbb{N}. \]

It is well known that mapping \( Mf \) is locally Lipschitz in \( E \) and, especially, \( D(Mf) \) exists in \( E \). By Lemma 3.4 it suffices to show that \( \int_E |DMf| \leq C_n \|Df\|_1 \).

First observe that since \( |f| \) is radial, it follows that \( Mf \) and \( g \) are radial as well, and continuous in \( \mathbb{R}^n \setminus \{0\} \). In particular, if

\[ E_k^R := \{|x| : x \in E_k\}, \]

then \( x \in E_k \) if and only if \( |x| \in E_k^R \). Since \( E_k^R \) is open, we can write

\[ E_k^R = \bigcup_{i=1}^\infty (a_i, b_i), \]

such that \( a_i < b_i \), \((a_i, b_i)\) are pairwise disjoint and \( a_i, b_i \in \partial E_k^R \). In the other words,

\[ E_k = \bigcup_{i=1}^\infty A(a_i, b_i), \]

and (by the definition of \( E_k \)) for all \( i \in \mathbb{N} \) it holds that

\[ Mf(x) = g(x) \quad \text{if} \quad |x| = a_i > 2^{-k} \quad \text{and} \quad Mf(x) = g(x) \quad \text{if} \quad |x| = b_i < 2^{-k+1}. \]

(3.17)
Moreover, since $M^I f > f$ in $E_k$, Lemma 3.8 says that $M^I f$ does not have a strict local maximum in $E_k^R$. In particular, by Proposition 3.7 there exist $c_i \in (a_i, b_i)$ such that

$$\int_{A(a_i, b_i)} D^I M f(y) \, dy \leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |(M^I f)'(t)| \, dt$$

$$= \sigma_n b_i^{n-1} (M^I f(a_i) - M^I f(c_i) + M^I f(b_i) - M^I f(c_i))$$

$$\leq \sigma_n b_i^{n-1} (M^I f(a_i) - g(c_i) + M^I f(b_i) - g(c_i)) .$$

Combining this with (3.17) implies that if $2^{-k} < a_i < b_i < 2^{-k+1}$, then

$$\int_{A(a_i, b_i)} D^I M f(y) \, dy \leq \sigma_n b_i^{n-1} (g(a_i) - g(c_i) + g(b_i) - g(c_i))$$

$$\leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |g'(t)| \, dt \leq \left(\frac{b_i}{a_i}\right)^{n-1} \int_{A(a_i, b_i)} |Dg(y)| \, dy$$

$$\leq 2^{n-1} \int_{A(a_i, b_i)} |Dg(y)| \, dy .$$

For the case $a_i = 2^{-k}$ or $b_i = 2^{-k+1}$, we employ the fact

$$M^I f(2^{-k}), M^I f(2^{-k+1}) \leq \sup_{y \in A(2^{-k-1}, 2^{-k+2})} g(y)$$

to obtain the estimates ($a_i = 2^{-k}$ or $b_i = 2^{-k+1}$)

$$\int_{A(a_i, b_i)} D^I M f(y) \, dy \leq \sigma_n b_i^{n-1} (M^I f(a_i) - g(c_i) + M^I f(b_i) - g(c_i))$$

$$\leq \sigma_n b_i^{n-1} \int_{2^{-k-1}}^{2^{-k+2}} |g'(t)| \, dt$$

$$\leq 2^{3(n-1)} \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| \, dy .$$

Combining these estimates implies that

$$\int_{E_k} |D^I M f(y)| \, dy = \sum_{i=1}^{\infty} \int_{A(a_i, b_i)} |D^I M f(y)| \, dy$$

$$\leq 2^{n-1} \sum_{i=1}^{\infty} \left[ \int_{A(a_i, b_i)} |Dg(y)| \, dy \right] + 2(2^{3(n-1)}) \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| \, dy$$

$$\leq 2^{3n} \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| \, dy .$$
Theorem 3.11. If \( DMf \) is radial, then \( Mf \in W^{1,1}(\mathbb{R}^n) \) and \( ||DMf||_1 \leq C_n ||Df||_1 \).

Proof. Let 

\[
E := \{ x \in \mathbb{R}^n : Mf(x) > M^I f(x), \ DMf(x) \neq 0 \}.
\]

It is well known that \( Mf \) is locally Lipschitz in \( \{ Mf(x) > f(x) \} \), implying the existence of \( DMf \) in \( \{ Mf(x) > f(x) \} \). Since \( Mf \geq M^I f(x) \), it holds that \( Mf(x) = \max\{ Mf(x), M^I f(x) \} \). Therefore, the theorem follows by Lemmas 3.4 and 3.2, if we can show that

\[
\int_E |DMf(y)| dy \leq C_n ||Df||_1.
\]

To show this, observe first that for all \( x \in E \) there exist \( r_x > \frac{|x|}{4} \) and \( z_x \in \mathbb{R}^n \) such that \( x \in B(z_x, r_x) \subseteq B_x \). Moreover, since \( DMf(x) \neq 0 \), Lemma 2.2 (2) and (3) says that \( x \in \partial B(z_x, r_x) \) and \( DMf(x)/|DMf(x)| = (z_x - x)/|z_x - x| \). On the other hand, \( Mf \) is radial and so \( DMf(x)/|DMf(x)| = \pm x/|x| \). We conclude that

\[
B_x = B(c_x x, |c_x x - x|) \text{ for some } c_x \in \mathbb{R}.
\]

Observe that \( r_x = |c_x x - x| = |c_x - 1||x| > |x|/4 \) by the assumption, and thus \( |c_x - 1| > 1/4 \). Moreover, it holds that \( c_x \geq -1 \). To see this, observe that if \( c_x < -1 \), then \( -x \in B_x \) and, since \( Mf \) is radial, \( B_x \in \mathcal{B}_x \), implying by Lemma 2.2 that \( 0 = DMf(-x) = DMf(x) \), which contradicts with the assumption \( x \in E \). Summing up, we can write \( E = E_+ \cup E_- \), where

\[
E_+ = \{ x \in E : c_x > 1 + 1/4 \} \text{ and } E_- = \{ x \in E : -1 \leq c_x < 3/4 \}.
\]

We are going to use different estimates for \( DMf(x) \) in \( E_+ \) and \( E_- \). Since \( |DMf(x)| = |DMf(x) \cdot \frac{x}{|x|}| \), it follows from Lemma 2.2, 2.3 that

\[
|DMf(x)| \leq \frac{1}{|x|} \int_{B_x} |Df|(y)|y| dy.
\]
This estimate will be used in $E_-$, while in $E_+$ we will use (easier) estimate $|DMf(x)| \leq \int_{B_x} |Df| \, dx$ (Lemma 2.2 (1)). We get that

$$\int_E |DMf(x)| \, dx \leq \int_E \chi_{E_+(x)} |DMf(x)| + \chi_{E_-(x)} |DMf(x)| \, dx$$

$$= \int_E \int_{B_x} \chi_{E_+(x)} \left( \int_{B_x} |Df|(y) \, dy \right) + \chi_{E_-(x)} \left( \int_{B_x} |Df|(y) \frac{|y|}{|x|} \, dy \right) \, dx$$

$$= \int_{\mathbb{R}^n} \chi_{E_+(x)} \left( \int_{B_x} \frac{\chi_{B_y}(y)|Df|(y)}{|B_x|} \, dx \right) + \chi_{E_-(x)} \left( \int_{B_x} \frac{\chi_{B_y}(y)|Df|(y)|y|}{|B_x||x|} \, dx \right) \, dy,$$

If $y \in B_x$ and $x \in E_+$, it follows from the definition of $E_+$ that $|x| \leq |y|$. Moreover, $y \in B_x$ and $x \in E$ imply also that $r_x \geq \max\{|y-x|, \frac{|x|}{4}\} \geq \frac{|y|}{6}$. This implies the estimate

$$\int_{E_+} \frac{\chi_{B_y}(y)}{|B_x|} \, dx \leq \int_{B(0,|y|)} \frac{dx}{\omega_n(|y|/6)^n} \leq C_n, \text{ for all } y \in \mathbb{R}^n.$$

On the other hand, if $x \in E_-$, then $-1 \leq c_x < 3/4$ especially implies that $B_x \subset B(0,|x|)$. Therefore, if $x \in E_-$ and $y \in B_x$, then $y \in B(0,|x|)$, and thus $|x| \geq |y|$. Recall also that $r_x \geq \frac{|x|}{4}$. Combining these yields that

$$\int_{E_-} \frac{\chi_{B_y}(y)}{|B_x|} \, dx \leq |y| \int_{\mathbb{R}^n \setminus B(0,|y|)} \frac{dx}{\omega_n(|x|/4)^{n+1}} = C'_n |y| \int_{|y|}^{\infty} \frac{dt}{t^2} = C'_n,$$

for all $y \in \mathbb{R}^n$. This completes the proof. \qed

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