Stochastic Acceleration in Random Environment: Averaging and Large Deviations Principles*

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Abstract

Consider a system of nonlinear second-order stochastic differential equations with fast-slow processes. Our main effort is to obtain averaging and large deviations principles. If the fast process is a diffusion, neither Lipschitz continuity nor linear growth is needed. Our approach is based on the combinations of the intuition from Smoluchowski-Kramers approximation, the concepts of relatively large deviations compactness, and identification of rate functions. The case that the fast process has no specific structure is also considered. We establish the large deviations principle of the underlying system under the assumption on the local large deviations principles of the corresponding first-order system. Some applications and numerical examples are also provided to illustrate our results.

Keywords. Stochastic acceleration, random environment, second-order stochastic differential equation, large deviations, averaging principle.

Subject Classification. 60F05, 60F10, 60J60,

Running Title. Averaging and Large Deviations of Second-Order Stochastic Differential Equations

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1 Introduction

In this paper, we study the second-order stochastic differential equations of the form

\[
\begin{align*}
\varepsilon^2 \ddot{X}^\varepsilon_t &= F^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) - \lambda^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) \dot{X}^\varepsilon_t, \quad X^\varepsilon_0 = x^\varepsilon_0 \in \mathbb{R}^d, \quad X^\varepsilon_0 = x^\varepsilon_0 \in \mathbb{R}^d, \\
\dot{Y}^\varepsilon_t &= \frac{1}{\varepsilon} b^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) + \frac{1}{\sqrt{\varepsilon}} \sigma^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) \dot{B}^\varepsilon_t, \quad Y^\varepsilon_0 = y^\varepsilon_0 \in \mathbb{R}^l,
\end{align*}
\]

(1.1)

where \( \varepsilon > 0 \) is a small parameter. Note that (1.1) is a two-time-scale and fully nonlinear system.

In the above, for each \( \varepsilon > 0 \), \( F^\varepsilon_t(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \), \( \lambda^\varepsilon_t(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \), \( b^\varepsilon_t(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma^\varepsilon_t(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \) are measurable functions of their arguments \( (t, x, y) \); \( \dot{B}^\varepsilon_t \) is an \( m \)-dimensional vector-valued standard Brownian motions with \( \dot{B}^\varepsilon_t \) being its formal derivative. For each \( \varepsilon > 0 \), the weak solution of (1.1) is defined in the usual way, i.e., there exist a suitable probability space and an adapted Brownian motion \( \dot{B}^\varepsilon_t \) such that there are adapted processes \( (X^\varepsilon, Y^\varepsilon) \) satisfying the system of stochastic integral equations corresponding to (1.1) almost surely.

In this paper, our main effort is devoted to obtaining asymptotic properties of the underlying systems. Under mild conditions, we establish the large deviations principle (LDP for short) for the family of the coupled processes \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) containing the slow process \( X^\varepsilon \) and the occupation measures \( \mu^\varepsilon \) of the fast process \( Y^\varepsilon \). Neither Lipschitz continuity nor growth condition of \( F^\varepsilon, b^\varepsilon, \sigma^\varepsilon \) is assumed. From the LDP of such couples, we can obtain averaging and large deviations principles for \( \{X^\varepsilon\}_{\varepsilon > 0} \). We also consider the case that the structure of the fast process is not explicitly given.

The formulation is generalized from (1.1) to

\[
\varepsilon^2 \ddot{X}^\varepsilon_t = F^\varepsilon_t(X^\varepsilon_t, \xi_t^\varepsilon) - \lambda^\varepsilon_t(X^\varepsilon_t, \xi_t^\varepsilon) \dot{X}^\varepsilon_t,
\]

(1.2)

with \( \xi_t^\varepsilon \) being a general random process without a particular structure.

Our main motivations stem from mathematical physics and statistical mechanics. Consider the motions of a net of particles in a net of random force fields, which is described by the Newton’s law as

\[
\ddot{x}_\varepsilon(t) = \ddot{F}_\varepsilon(t, \omega, x_\varepsilon(t), \dot{x}_\varepsilon(t), \xi_\varepsilon(t)),
\]

where \( x_\varepsilon(t) \) denotes the location of the particles at time \( t \), \( \ddot{F}_\varepsilon \) denotes the random force fields depending on time \( t \), random states \( \omega \), the particle’s locations \( x_\varepsilon \), the particle’s velocities \( \dot{x}_\varepsilon \), and \( \xi_\varepsilon(t) \) the random environments interacting with the system. Such models were considered by Kesten and Papanicolaou in [14, 15] under suitable conditions. We focus on the motions of particles, in which the Reynolds number (see e.g., [23] for a definition) is very small so that inertial effects are negligible compared to the damping force by assuming that

\[
\dddot{x}_\varepsilon(t, \omega, x_\varepsilon(t), \dot{x}_\varepsilon(t), \xi(t)) = \ddot{F}_\varepsilon(t, x_\varepsilon(t), \xi(t)) - \frac{\lambda_\varepsilon(t, x_\varepsilon(t), \xi(t))}{\varepsilon} \dot{x}_\varepsilon(t).
\]

Now, by scaling \( X^\varepsilon_t = x_\varepsilon(t/\varepsilon) \) and \( \xi_t^\varepsilon = \xi_\varepsilon(t/\varepsilon) \), the system can be rewritten as (1.2). As one of the most natural settings for \( \xi_\varepsilon(t) \), one can assume the random process \( \xi_\varepsilon(t) \) is described by a diffusion process and thus, \( \xi_t^\varepsilon \) is a fast diffusion process that is fully coupled with the system, which leads to the system of equations in (1.1). Another important motivation is from the averaging and large deviations principles for systems of stochastic differential equations. System (1.1) can be viewed as the second-order version of the problem considered in [18].

The averaging principle plays an important role in studying heterogeneity, which often occurs in physics as well as in biology, economics, queuing theory, game theory, among others; see, e.g.,
Typically, analyzing and simulating heterogeneous models are much more challenging than the corresponding homogeneous models, in which the heterogeneous property is replaced by its average value. The averaging principle for a system guarantees the validity of this replacement. On the other hand, the LDPs (see [6, 7]), characterizing quantitatively the rare events, play an important role in many areas with a wide range of applications. To mention just a few, they include equilibrium and nonequilibrium statistical mechanics, multifractals, thermodynamics of chaotic systems, among others [23]. By establishing the LDPs for system (1.1) and (1.2), we provide an insight about the motions of (small) particles in random force fields, which is heterogeneous and the heterogeneity is allowed to interact with the system. Not only will it illustrate averaging of the heterogeneity works in this case, but also provide the picture of the dynamics around the averaged system.

From the development of a large deviations theory point of view, much effort has been devoted to the study of the LDPs around the averaged systems of the first-order differential equations under random environment (given by diffusion process or wideband noise) in the setting of fast-slow systems. Such problems have been addressed in [12, 16, 25, 26] under certain settings, in which, the fast process is often not fully coupled with the slow system. Very recently, the question for the fully-coupled system was completely solved in [22]. Some other studies related to these results in this development can be found in [2, 13, 18], while such a system under wideband noise is studied in [16]. In contrast to the systems considered in the aforementioned works, we consider systems (1.1) and (1.2), which are systems of second-order differential equations. To the best of our knowledge, this paper is one of the first works addresses the problem of the LDPs for the second-order equations in random environment that are fully coupled. We establish the LDPs under mild and natural conditions. From a statistical physics point of view, there are some works regarding the stochastic acceleration and the Langevin equations such as [4, 10] for the study of Smoluchowski-Kramers approximation, the work [3] for the LDPs, and [5] for the MDPs (moderate deviations principle) in the absence of the random environment, and [20, 21] for the LDPs of Langevin systems with random environment under certain specific settings.

To establish the desired LDPs for system (1.1), we first establish the LDP for \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \), where \( \mu^\varepsilon \) is defined as a random occupation measure of \( Y^\varepsilon \). The LDPs for the family of either \( \{X^\varepsilon\}_{\varepsilon > 0} \) or \( \{\mu^\varepsilon\}_{\varepsilon > 0} \) can be handled directly by some standard projection techniques in the large deviations theory. The proof relies on the concept of exponential tightness and the property that if a family of random processes is exponentially tight then it is relatively compact in the sense of large deviations theory, i.e., for any subsequence there exists another subsequence that enjoys the LDP with a suitable rate function. It is worth noting that we could not establish a “good” connection between the solution of the second-order equation and its corresponding first-order equation because we do not assume any regularity of \( F^\varepsilon \). Our approach is based on a combination of the approach of Puhaskii in [22] for the first-order coupled system (namely, obtaining the relatively large deviations compactness and then carefully identifying the rate functions), and the intuition of Smoluchowski-Kramers approximation. In particular, after proving the exponential tightness of the family of \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \), we try to estimate \( X^\varepsilon \) along the trajectory of the corresponding first-order coupled system when we identify the rate function.

To establish the desired LDPs for the system under general fast random process (1.2), we use a different approach. We assume that the corresponding first-order equation satisfies the local LDP and stronger assumptions (Lipschitz and linear growth) for \( F^\varepsilon \). To prove the LDP, we show that the family of \( \{X^\varepsilon\}_{\varepsilon > 0} \) is exponentially tight and satisfies the local LDP. The growth-rate conditions allow us to prove the exponential tightness of \( \{X^\varepsilon\}_{\varepsilon > 0} \) and the Lipschitz continuity helps us to establish the local LDP of \( \{X^\varepsilon\}_{\varepsilon > 0} \) from that of the corresponding first-order system.

The rest of the paper is arranged as follows. Section 2 formulates the problem and states our main results. Some specification and numerical examples are given in Section 2.1.1. The proofs of
the main results are given in Section 3 and Section 4.

2 Formulations and Main Results

**Notation.** Throughout the paper, $|·|$ denotes an Euclid norm while $∥·∥$ indicates the operator sup-norm, $C(\mathcal{X}, \mathcal{Y})$ is the space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ and if $\mathcal{Y}$ is an Euclid space, we write $C(\mathcal{X}, \mathcal{Y})$ as $C(\mathcal{X})$ for simplicity. Let $\mathcal{M}(\mathbb{R}^l)$ be the set of finite measures on $\mathbb{R}^l$ endowed with the weak topology, and $\mathcal{P}(\mathbb{R}^l)$ be the set of probability densities $m(y)$ on $\mathbb{R}^l$ such that $m \in W_{loc}^{1,1}(\mathbb{R}^d)$ and $\sqrt{m} \in W_{loc}^{1,2}(\mathbb{R}^l)$, where $W_{loc}^{1,2}(\mathbb{R}^l)$ (resp., $W_{loc}^{1,1}(\mathbb{R}^d)$) is the Sobolev space (resp., local Sobolev space) with suitable exponents, and $C^1_0(\mathbb{R}^l)$ be the space of continuously differentiable functions with compact supports in $\mathbb{R}^l$. Let $C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$ be the subset of $C(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$ of functions $\mu = (\mu_t, t \in \mathbb{R}^+)$ such that $\mu_t - \mu_s$ is an element of $\mathcal{M}(\mathbb{R}^l)$ for $t \geq s$ and $\mu_t(\mathbb{R}^l) = t$. It is endowed with the subspace topology and is a complete separable metric space, being closed in $C(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$.

We define the random process $\mu^\varepsilon = (\mu^\varepsilon_t, t \in \mathbb{R}^+)$ of the fast process $Y^\varepsilon$ by

$$
\mu^\varepsilon_t(A) := \int_0^t 1_A(Y^\varepsilon_s) \, ds, \quad \forall A \in \mathcal{B}(\mathbb{R}^l).
$$

Then, $\mu^\varepsilon$ is a random element of $C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$ and we can regard $(X^\varepsilon, \mu^\varepsilon)$ as a random element of $C(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l)) \times C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$. Note that the elements of $C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$ can be regarded as a $\sigma$-finite measures on $\mathbb{R}^+ \times \mathbb{R}^l$. As a result, we use the notation $\mu(dt, dy)$ for $\mu \in C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$. For a symmetric positive definite matrix $A$ and matrix $z$ of suitable dimensions, we define $∥z∥_A := z^\top A z$. Following Puhaski’s notation, $∥z∥_A$ can be either matrices or numbers, depending on the dimension $z$. We also use $\nabla_x, \nabla_{xx}, \text{div}_x$ to denote the gradient, the Hessian, and the divergence, respectively, with respect to indicated variables. It should be clear from the context.

We will establish the LDP and describe explicitly the rate function for the family $\{X^\varepsilon, \mu^\varepsilon\}_{\varepsilon>0}$ in $C(\mathbb{R}^+, \mathbb{R}^d) \times C^1(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^l))$. The LDP and the rate function of $\{X^\varepsilon\}_{\varepsilon>0}$ are obtained directly by standard projections in the large deviations theory. To proceed, we recall briefly the basic definitions of the LDP. For further references, see [6, 7, 17].

**Definition 2.1.** The family of $\{P^\varepsilon\}_{\varepsilon>0}$ in $\mathcal{S}$ enjoys the LDP with a rate function $\Pi$ if the following conditions are satisfied: 1) $\Pi : \mathcal{S} \to [0, \infty]$ is inf-compact, that is, the level sets $\{z \in \mathcal{S} : \Pi(z) \leq L\}$ are compact in $\mathcal{S}$ for any $L > 0$; and 2) for any open subset $G$ of $\mathcal{S}$,

$$
\liminf_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(G) \geq -\Pi(G) := -\inf_{z \in G} \Pi(f);
$$

and 3) for any closed subset $F$ of $\mathcal{S}$,

$$
\limsup_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(F) \leq -\Pi(F) := -\inf_{z \in F} \Pi(f).
$$

We say that a family of random elements of $\mathcal{S}$ obeys the LDP if the family of their laws obeys the LDP.

2.1 LDP: Fast Diffusion

Our main effort in this section is to consider system (1.1) and to establish LDP for the family of the processes $\{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon>0}$ with $\mu^\varepsilon$ being the empirical process associated with $Y^\varepsilon$ as in (2.1), where $(X^\varepsilon, Y^\varepsilon)$ is a solution of the second-order differential equation with random environment given in (1.1). Such a solution is defined as follows.
One can rewrite (1.1) as

\[
\begin{cases}
X_t^\varepsilon = p_t^\varepsilon, & X_0^\varepsilon = x_0^\varepsilon \in \mathbb{R}^d, \\
\varepsilon^2 p_t^\varepsilon = F_t^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon) - \lambda_t^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)p_t^\varepsilon, & p_0^\varepsilon = x_1^\varepsilon \in \mathbb{R}^d, \\
Y_t^\varepsilon = \frac{1}{\varepsilon}b_t^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon) + \frac{1}{\varepsilon}\sigma_t^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)\tilde{B}_t^\varepsilon, & Y_0^\varepsilon = y_0^\varepsilon \in \mathbb{R}^d.
\end{cases}
\]  

(2.2)

Recall that for each \( \varepsilon > 0 \), the coefficients \( F_t^\varepsilon(x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^d, \lambda_t^\varepsilon(x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}, b_t^\varepsilon(x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^l, \sigma_t^\varepsilon(x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^{l \times m} \) are functions of \((t, x, y); x_0^\varepsilon, x_1^\varepsilon \in \mathbb{R}^d, y_0^\varepsilon \in \mathbb{R}^l \) are initial values that can be random. Throughout the paper, we assume that these functions are measurable and locally bounded in \((t, x, y)\) such that the system of equations (2.2) admits a weak solution \((X^\varepsilon, p^\varepsilon, Y^\varepsilon)\) with trajectories in \( C(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^l) \) for every initial condition \((x_0^\varepsilon, x_1^\varepsilon, y_0^\varepsilon)\); and then the system of equations (1.1) admits a weak solution \((X^\varepsilon, Y^\varepsilon)\) with trajectories in \( C(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^l) \). More precisely, we assume that there exist a complete probability space \((\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon)\) with filtration \( \mathcal{F}_t^\varepsilon = (\mathcal{F}_t^\varepsilon, t \in \mathbb{R}_+) \), a Brownian motion \( (B_t^\varepsilon, t \in \mathbb{R}_+) \) with respect to \( \mathcal{F}^\varepsilon \), processes \( X^\varepsilon = (X_t^\varepsilon, t \in \mathbb{R}_+) \), \( p^\varepsilon = (p_t^\varepsilon, t \in \mathbb{R}_+) \), and \( Y^\varepsilon = (Y_t^\varepsilon, t \in \mathbb{R}_+) \) that are \( \mathcal{F}^\varepsilon \)-adapted and have continuous trajectories satisfying the following equations

\[
\begin{cases}
X_t^\varepsilon = x_0^\varepsilon + \int_0^t p_s^\varepsilon ds, \\
p_t^\varepsilon = p_0^\varepsilon + \frac{1}{\varepsilon^2} \int_0^t \left( F_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) - \lambda_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)p_s^\varepsilon \right) ds, \\
Y_t^\varepsilon = y_0^\varepsilon + \frac{1}{\varepsilon} \int_0^t b_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\varepsilon^2} \int_0^t \sigma_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) dB_s^\varepsilon,
\end{cases}
\]

for all \( t \in \mathbb{R}_+, \mathbb{P}^\varepsilon\)-a.s. It is noted that it may not guarantee the uniqueness of the solution. [To ensure the uniqueness, one may need to require further that the coefficients are Lipschitz continuous, which we do not assume here.] Next, we need some conditions, which are mild and natural, to establish the LDP for the family of coupled processes \( \{X^\varepsilon, \mu^\varepsilon\}_{\varepsilon > 0} \).

**Assumption 2.1.** Assume that for all \( L > 0 \) and \( t > 0 \),

\[
\limsup_{\varepsilon \to 0} \sup_{s \in [0, t]} \sup_{y \in \mathbb{R}^l} \sup_{x \in \mathbb{R}^d, |x| \leq L} \left| F_s^\varepsilon(x, y) + |\lambda_s^\varepsilon(x, y)| + |b_s^\varepsilon(x, y)| + \left| \Sigma_s^\varepsilon(x, y) \right| \right| < \infty,
\]

(2.3)

where \( \Sigma_s^\varepsilon(x, y) := \sigma_s^\varepsilon(x, y)\sigma_s^\varepsilon(x, y)^\top \),

\[
\limsup_{\varepsilon \to 0} \sup_{s \in [0, t]} \sup_{y \in \mathbb{R}^l} \sup_{x \in \mathbb{R}^d, (1 + |x|^2)\lambda_s^\varepsilon(x, y)} \frac{x^\top F_s^\varepsilon(x, y)}{\lambda_s^\varepsilon(x, y)} < \infty,
\]

(2.4)

\[
\limsup_{M \to \infty} \limsup_{\varepsilon \to 0} \sup_{s \in [0, t]} \sup_{y \in \mathbb{R}^l, |y| \geq M} \sup_{x \in \mathbb{R}^d, |x| \leq L} \frac{[b_s^\varepsilon(x, y)]^\top y}{|y|} < 0,
\]

(2.5)

\[
\limsup_{\varepsilon \to 0} \sup_{s \in [0, t]} \sup_{y \in \mathbb{R}^l, |y| \geq M} \sup_{x \in \mathbb{R}^d, |x| \leq L} \left| \nabla_x \lambda_s^\varepsilon(x, y) \right| + \left| \nabla_y \lambda_s^\varepsilon(x, y) \right| + \left| \nabla_{yy} \lambda_s^\varepsilon(x, y) \right| < \infty,
\]

(2.6)

and

\[
\liminf_{\varepsilon \to 0} \inf_{s \in [0, t]} \lambda_s^\varepsilon(x, y) > \kappa_0 > 0.
\]

(2.7)
Remark 1. The condition (2.3) is (locally in \((t, x)\) and globally in \(y\)) boundedness conditions of \(F^\varepsilon, b^\varepsilon\) and \(\Sigma^\varepsilon\). Note that (2.4) is a growth-rate condition, which is milder than linear growth of \(\frac{F^\varepsilon(x, y)}{\lambda^\varepsilon(x, y)}\). [For example, \(\frac{F^\varepsilon(x, y)}{\lambda^\varepsilon(x, y)} = \frac{1}{\varepsilon}\) satisfies this condition but is not linear growth.] Moreover, it does not imply any growth-rate condition for \(F^\varepsilon_t(x, y)\). The condition (2.5) is a stability condition, which in fact is needed for the ergodicity of the fast process. It is noted that we do not require any Lipschitz continuity and growth-rate conditions for these coefficients. Lower boundedness and regularity conditions (2.6) and (2.7) of \(\lambda^\varepsilon_t(x, y)\) are natural and often used in the literature of mathematical physics; see, e.g., [31, 5].

Assume that there are “limit” measurable functions \(F_t(x, y), \lambda_t(x, y), b_t(x, y), \) and \(\sigma_t(x, y)\) of the families of functions \(F^\varepsilon_t(x, y), \lambda^\varepsilon_t(x, y), b^\varepsilon_t(x, y), \sigma^\varepsilon_t(x, y)\) as \(\varepsilon \to 0\), respectively, in the sense that for all \(t > 0\) and \(L > 0\),

\[
\lim \sup_{\varepsilon \to 0} \sup_{x \in [0, t], y \in \mathbb{R}^l, |y| \leq L} \sup_{x \in \mathbb{R}^d, |x| \leq L} \left[ |F^\varepsilon_t(x, y) - F_t(x, y)| + |\lambda^\varepsilon_t(x, y) - \lambda_t(x, y)| \right. \\
+ \left. |b^\varepsilon_t(x, y) - b_t(x, y)| + ||\sigma^\varepsilon_t(x, y) - \sigma_t(x, y)|| \right] = 0. \tag{2.8}
\]

Assumption 2.2. Assume that the function \(b_t(x, y)\) is Lipschitz continuous in \(y\) locally uniformly in \((t, x)\); the functions \(b_t(x, y)\) and \(\Sigma_t(x, y) := \sigma_t(x, y)[\sigma_t(x, y)]^\top\) are continuous in \(x\) locally uniformly in \(t\) and uniformly in \(y\); \(\Sigma_t(x, y)\) is of class \(C^1\) in \(y\), with the first partial derivatives being bounded and Lipschitz continuous in \(y\) locally uniformly in \((t, x)\), and \(\text{div}_y \Sigma_t(x, y)\) is continuous in \((x, y)\). The matrix \(\Sigma_t(x, y)\) is positive definite uniformly in \(y\) and locally uniformly in \((t, x)\). In addition, \(F_t(x, y)\) is locally Lipschitz continuous in \(x\) locally uniformly in \(t\) and uniformly in \(y\). The conditions (2.6) and (2.7) hold for \(\lambda_t\). Moreover, for all \(t > 0\),

\[
\lim_{|y| \to \infty} \sup_{x \in [0, t], y \in \mathbb{R}^d} \frac{|b_t(x, y)|^2 y}{|y|^2} < 0. \tag{2.9}
\]

Rate function. Denote by \(\mathcal{G}\) the collection of \((\varphi, \mu)\) such that the function \(\varphi = (\varphi_t, t \in \mathbb{R}^+) \in C(\mathbb{R}^+, \mathbb{R}^d)\) is absolutely continuous (with respect to the Lebesgue measure on \(\mathbb{R}^+\)) and the function \(\mu = (\mu_t, t \in \mathbb{R}^+) \in C_t(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^d))\), when considered as a measure on \(\mathbb{R}^+ \times \mathbb{R}^l\), is absolutely continuous (with respect to Lebesgue measure on \(\mathbb{R}^+ \times \mathbb{R}^l\)), i.e., \(\mu(ds, dy) = m_s(y)dyds\), and for almost all \(s, m_s(y)\) (as a function of \(y\)) belongs to \(\mathcal{P}(\mathbb{R}^l)\).

For \((\varphi, \mu) \in \mathcal{G}\), \(\mu(ds, dy) = m_s(y)dyds\), define

\[
\mathbb{I}_1(\varphi, \mu) = \int_0^\infty \left[ \sup_{\beta \in \mathbb{R}^d} \beta^\top \left( \varphi_s - \int_{\mathbb{R}^l} F_s(\varphi_s, y) m_s(y)dy \right) \right. \\
+ \sup_{h \in C_0^1(\mathbb{R}^l)} \int_{\mathbb{R}^l} \left( |\nabla h(y)|^2 \left( \frac{1}{2} \text{div}_y (\Sigma_s(\varphi_s, y)m_s(y)) - b_s(\varphi_s, y)m_s(y) \right) \right. \\
- \left. \left. \frac{1}{2} \|\nabla h(y)\|_{\Sigma_s(\varphi_s, y)m_s(y)}^2 \right) m_s(y) \right] ds,
\]

and define \(\mathbb{I}_1(\varphi, \mu) = \infty\) if \((\varphi, \mu) \notin \mathcal{G}\).

Theorem 2.1. Assume that Assumptions 2.1 and 2.2 hold, that the family of initial values \(\{x^\varepsilon_0\}_{\varepsilon > 0}\) obeys the LDP in \(\mathbb{R}^d\) with a rate function \(\mathbb{I}_0\), that \(\lim \sup_{\varepsilon \to 0} |x^\varepsilon_1| < \infty\) a.s., and that the family of initial values \(\{y^\varepsilon_0\}_{\varepsilon > 0}\) is exponentially tight in \(\mathbb{R}^l\). Then the family \(\{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0}\) obtained from (1.1) obeys the LDP in \(C(\mathbb{R}^+, \mathbb{R}^d) \times C_t(\mathbb{R}^+, \mathcal{M}(\mathbb{R}^d))\) with rate function \(\mathbb{I}\) defined as

\[
\mathbb{I}(\varphi, \mu) = \begin{cases} 
\mathbb{I}_0(\varphi_0) + \mathbb{I}_1(\varphi, \mu), & \text{if } (\varphi, \mu) \in \mathcal{G}, \\
\infty, & \text{otherwise.}
\end{cases}
\]
Corollary 2.1. Under the hypotheses of Theorem 2.1, the family \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfies the LDP in \( C(\mathbb{R}_+, \mathbb{R}^d) \) with the rate function \( \mathbb{I}_X \) defined by \( \mathbb{I}_X(\varphi) = \inf_{\mu \in \mathcal{M}(\mathbb{R}_+, \mathbb{M}(\mathbb{R}^d))} \mathbb{I}(\varphi, \mu) \). As an alternative representation, if function \( \varphi = (\varphi_t, t \in \mathbb{R}_+) \in C(\mathbb{R}_+, \mathbb{R}^d) \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}_+ \), then

\[
\mathbb{I}_X(\varphi) = \mathbb{I}_0(\varphi_0) + \int_0^\infty \sup_{\beta \in \mathbb{R}^d} \left[ \beta^T \varphi_s - \sup_{m \in P(\mathbb{R}^d)} \left( \beta^T \int_{\mathbb{R}^d} F_s(\varphi_s, y) m(y) dy \right) \right] ds,
\]

otherwise, \( \mathbb{I}_X(\varphi) = \infty. \)

2.1.1 Specification and Numerical Examples

Zero points of \( \mathbb{I}(\varphi, \mu) \), averaging principle of \( (1.1) \), and its large deviations analysis.

We start with an intuitive discussion on the behavior of \( (1.1) \) as \( \varepsilon \to 0 \). Intuitively, there are two phases as \( \varepsilon \to 0 \). First, \( \varepsilon^2 \) converges fast to 0. Therefore \( X^\varepsilon_t \) will be somewhat close to the solution of the following associated first-order equation

\[
0 = F^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t, \dot{X}^\varepsilon_t) - \lambda^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t, \dot{X}^\varepsilon_t), \quad X^\varepsilon_0 = x^\varepsilon_0 \in \mathbb{R}^d,
\]

\[
Y^\varepsilon_t = \frac{1}{\varepsilon} b_t(X^\varepsilon_t, Y^\varepsilon_t, \dot{X}^\varepsilon_t) + \frac{1}{\varepsilon} \sigma^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) \dot{B}_t, \quad Y^\varepsilon_0 = y^\varepsilon_0 \in \mathbb{R}^l.
\]

Next, \( Y^\varepsilon_t \) converges to its invariant distribution as \( \varepsilon \to 0 \). Precisely, if we let \( \bar{Y}^\varepsilon_t := Y^\varepsilon_t \) then

\[
\bar{X}^\varepsilon_t = \frac{F^\varepsilon_t(\bar{X}^\varepsilon_t, \dot{\bar{Y}}^\varepsilon_t)}{\lambda^\varepsilon_t(\bar{X}^\varepsilon_t, \dot{\bar{Y}}^\varepsilon_t)}, \quad \bar{X}^\varepsilon_0 = x^\varepsilon_0 \in \mathbb{R}^d,
\]

\[
\dot{\bar{Y}}^\varepsilon_t = b^\varepsilon_t(\bar{X}^\varepsilon_t, \dot{\bar{Y}}^\varepsilon_t) + \sqrt{\varepsilon} \sigma^\varepsilon_t(\bar{X}^\varepsilon_t, \dot{\bar{Y}}^\varepsilon_t) \dot{B}_t, \quad \bar{Y}^\varepsilon_0 = y^\varepsilon_0 \in \mathbb{R}^l.
\]

As a consequence, because \( \bar{Y}^\varepsilon_t \) will come to and stay close to its invariant measure as \( \varepsilon \to 0 \), \( X^\varepsilon_t \) will tend to \( \bar{X}_t \), the solution of the following differential equation

\[
\dot{\bar{X}}_t = F/\lambda_t(\bar{X}_t), \quad \bar{X}_0 = \bar{x}_0,
\]

where \( \bar{x}_0 = \lim_{\varepsilon \to 0} x^\varepsilon_0 \) and

\[
\frac{F}{\lambda_t}(x) := \int_{\mathbb{R}^l} \frac{F_t(x, y)}{\lambda_t(x, y)} \nu^{1,x}(dy),
\]

and for each fixed \( (t_1, x) \), \( \nu^{1,x}(dy) \) is the invariant measure of following stochastic differential equation

\[
\dot{\hat{Y}}_t = b_t(x, \hat{Y}_t) + \sigma_t(x, \hat{Y}_t) \hat{B}_t.
\]

The convergence of \( X^\varepsilon_t \) to \( \bar{X}_t \) as \( \varepsilon \to 0 \) forms an averaging principle of \( (1.1) \). However, not only are we interested in the convergence of \( X^\varepsilon \) to \( \bar{X}_t \), but also the tail probability of this convergence, i.e., the rate of the convergence of the probability of the event \( \{|X^\varepsilon - \bar{X}| > \eta\} \) to 0, for any \( \eta > 0. \)
We show that the convergence is exponentially fast. The answer to these questions can be obtained from the LDP for \( \{X^\varepsilon_t\}_{\varepsilon>0} \) and explicit representations of the rate function.

To proceed, we apply our results to make the above intuition rigorous. It is shown (see, e.g., [22, Lemma 6.7]) that \( \mathbb{I}_0(\varphi, \mu) = 0 \) provided that a.e.

\[
\hat{\varphi}_s = \int_{\mathbb{R}^l} \frac{F_s(\varphi_s, y)}{\lambda_s(\varphi_s, y)} m_s(y) dy,
\]

and \( m_s(y) \) satisfies the following equation

\[
\int_{\mathbb{R}^l} \left( \frac{1}{2} \text{tr}(\Sigma_s(\varphi_s, y)[\nabla_y h(y)]) + [\nabla_y h(y)]^T b_s(\varphi_s, y) \right) m_s(y) dy = 0, \quad \text{for all } h \in C_0^\infty(\mathbb{R}^l), \tag{2.10}
\]

and \( \mathbb{I}_0(\varphi_0) = 0 \). Alternatively, \( m_s(\cdot) \) is the invariant density of the diffusion process with the drift \( b_s(\varphi_s, \cdot) \) and the diffusion matrix \( \Sigma_s(\varphi_s, \cdot) \). Therefore, as \( \varepsilon \to 0 \), the trajectories of \( \{X^\varepsilon_t\}_{\varepsilon>0} \) hover around \( X \) with exponential tail probability, where \( X \) is defined as the solution of the following ODE

\[
\dot{X}_t = F'I_t(X_t), \quad X_0 = x_0, \tag{2.11}
\]

with

\[
\frac{F}{\lambda_I(x)} := \int_{\mathbb{R}^l} \frac{F_I(x, y)}{\lambda_I(x, y)} m_I(y) dy,
\]

and \( m_I(\cdot) \) satisfies equation (2.10) and \( x_0 \) satisfying \( \mathbb{I}_0(\varphi_0) = 0 \). More specifically, let

\[
B^c_{\eta}(X) := \left\{ \varphi \in C(\mathbb{R}_+, \mathbb{R}^d) : ||\varphi_t - X_t||_{C(\mathbb{R}_+, \mathbb{R}^d)} := \sum_{n=1}^{\infty} \frac{1}{2^n} \left( 1 \wedge \sup_{t \leq n} |\varphi_t - X_t| \right) \geq \eta \right\},
\]

the LDP in Section 2.1 implies that

\[
\mathbb{P}^\varepsilon(X^\varepsilon \in B^c_{\eta}(X)) \sim e^{-\frac{1}{\varepsilon} \mathbb{I}_X(B^c_{\eta}(X))},
\]

where \( \mathbb{I}_X(B^c_{\eta}(X)) = \inf_{\varphi \in B^c_{\eta}(X)} \mathbb{I}_X(\varphi) \). If we assume that \( X \) is the unique solution of (2.11), it is the unique solution of \( \mathbb{I}_X(\varphi) = 0 \). As a result, \( \mathbb{I}_X(B^c_{\eta}(X)) > 0 \). Indeed, if \( \mathbb{I}_X(B^c_{\eta}(X)) = 0 \), there exists \( \{\varphi_k\}_{k=1}^{\infty} \subset B^c_{\eta}(X) \) such that \( \lim_{k \to \infty} \mathbb{I}_X(\varphi_k) = 0 \). Because of that \( \mathbb{I}_X \) is a rate function, there exists a convergent subsequence (still denoted by \( \varphi_k \)) of \( \{\varphi_k\} \) with limit denoted by \( \varphi \in B^c_{\eta}(X) \). Since \( \mathbb{I}_X \) is lower semi-continuous, \( 0 \leq \mathbb{I}_X(\varphi) = \mathbb{I}_X(\lim_{k \to \infty} \varphi_k) \leq \lim_{k \to \infty} \mathbb{I}_X(\varphi_k) = 0 \). It leads to \( \mathbb{I}_X(\varphi) = 0 \), which is a contradiction. Because \( \mathbb{I}_X(B^c_{\eta}(X)) > 0 \), \( \mathbb{P}(\|X^\varepsilon - X\| > \eta) \to 0 \) exponentially fast for any \( \eta > 0 \).

Some variant representations of the rate function of (1.1). One can write the rate function \( \mathbb{I}(\varphi, \mu) \) as

\[
\mathbb{I}(\varphi, \mu) = \mathbb{I}_0(\varphi_0) + \int_0^\infty \left[ \left( \sup_{\beta \in \mathbb{R}^d} \beta^T \left( \hat{\varphi}_s - \int_{\mathbb{R}^l} \frac{F_s(\varphi_s, y)}{\lambda_s(\varphi_s, y)} m_s(dy) \right) + \mathbb{J}_{s, \varphi_s}(\nu_s) \right) \right] ds,
\]

where \( \nu_s(dy) = m_s(y)dy \) and

\[
\mathbb{J}_{s, \varphi_s}(\nu_s) := \sup_{h \in C_0^\infty(\mathbb{R}^l)} \int_{\mathbb{R}^l} \left[ [\nabla h(y)]^T \left( \frac{1}{2} \text{div}_x (\Sigma_s(\varphi_s, y)m_s(y)) - b_s(\varphi_s, y)m_s(y) \right) - \frac{1}{2} ||\nabla h(y)||_{\Sigma_s(\varphi_s, y)m_s(y)}^2 \right] dy.
\]
Next, we provide a couple of numerical examples. We start with a simple

**Remark**

and in this case, we have

\[ Y_t^{s,x} = b_s(x, Y_t^{s,x}) + \sigma_s(x, Y_t^{s,x})d\tilde{B}_t; \]

see [22] Section 2, Corollary 2.2 and 2.3.

Moreover, if \( I(\varphi, \mu) \) is finite, it is necessary that

\[ \dot{\varphi}_s = \int_{\mathbb{R}^l} \frac{F_s(\varphi_s, y)m_s(y)}{\lambda_s(\varphi_s, y)} \, dy \text{ a.e.,} \]

and in this case, we have

\[ I(\varphi, \mu) = I_0(\varphi_0) + \int_0^\infty J_{s,\varphi_s}(\nu_s)ds. \]

On the other hand, one can also write the rate function (see [22] Section 2, Proposition 2.1)

\[ I(\varphi, \mu) = I_0(\varphi_0) + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^l} \left\| \nabla y m_s(y) - J_{s,m_s(\cdot),\varphi_s(\cdot)}(y) \right\|_{L_\varphi(y)} m_s(y)dyds, \]

with \( \mu(dy, ds) = m_s(y)dyds \), where for each \( s \in \mathbb{R}_+ \), each function \( m_s(\cdot) \) belongs to \( \mathcal{P}(\mathbb{R}^l) \), and \( J_{t,m(\cdot),u} \) is a function defined as follows. With \( L^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_s(y)dy) \) denoting the Hilbert space of all \( \mathbb{R}^d \)-valued functions (of \( y \)) in \( \mathbb{R}^l \) with norm \( \| f \|_{L_{\Sigma,m}}^2 = \int_{\mathbb{R}^l} \| f(y) \|_{\Sigma_{t}(x,y), m_s(y)dy} dy \) and \( L^2_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_s(y)dy) \) the space consisting of functions whose products with arbitrary \( C_0^\infty \)-functions belong to \( L^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_s(y)dy) \), then \( J_{t,m(\cdot),u} \) is defined as a function of \( y \) by

\[ J_{t,m(\cdot),u}(y) = \Pi_{\Sigma_{t}(x,y),m_s(\cdot)}(\Sigma_{t}(x,y))^{-1} (b_t(x,y) - \text{div}_x \Sigma_{t}(x,y)/2)), \]

where \( \Pi_{\Sigma_{t}(x,y),m_{s}(\cdot)} \) maps a function \( \phi(y) \in L^2_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_{s}(y)dy) \) to a function \( \Pi_{\Sigma_{t}(x,y),m_{s}(\cdot)}(\phi(y)) \), which belongs to \( L^1_{0,2}(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_{s}(y)dy) \) and satisfies that for all \( h \in C_0^\infty(\mathbb{R}^l) \),

\[ \int_{\mathbb{R}^l} [\nabla h(y)]^\top \Sigma_{t}(x,y) \Pi_{\Sigma_{t}(x,y),m_s(\cdot)}(\phi(y)) m(y)dy = \int_{\mathbb{R}^l} [\nabla h(y)]^\top \Sigma_{t}(x,y) \phi(y) m(y)dy. \]

If \( \phi(y) \in L^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_{s}(y)dy) \), then \( \Pi_{\Sigma_{t}(x,y),m_s(\cdot)}(\phi(y)) \) is nothing than the orthogonal projection of \( \phi \) onto \( L^1_{0,2}(\mathbb{R}^l, \mathbb{R}^l, \Sigma_{t}(x,y), m_{s}(y)dy) \).

**Remark 2.** In fact, \( I(\varphi, \mu) \) is defined similarly to the rate function of the family of processes \( \{(\tilde{X}^\varepsilon, \mu_{t,\tilde{X}^\varepsilon})\}_{\varepsilon > 0} \), where

\[ \mu_{t,\tilde{X}^\varepsilon}(\mathcal{A}) = \int_0^t 1_\mathcal{A}(Y_s^{\varepsilon,\tilde{X}})ds, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^l), \]

and \((\tilde{X}^\varepsilon, Y_t^{\varepsilon,\tilde{X}})\) is the solution of the following equation

\[
\begin{cases}
\tilde{X}_t^\varepsilon = F_t^\varepsilon(\tilde{X}_t^\varepsilon, Y_t^{\varepsilon,\tilde{X}}), & \tilde{X}_0^\varepsilon = x_0^\varepsilon \in \mathbb{R}^d, \\
Y_t^{\varepsilon,\tilde{X}} = \frac{1}{\varepsilon} b_t^\varepsilon(\tilde{X}_t, Y_t^{\varepsilon,\tilde{X}}) + \frac{1}{\varepsilon^2} \sigma_t^\varepsilon(\tilde{X}_t, Y_t^{\varepsilon,\tilde{X}}) \tilde{B}_t, & Y_0^\varepsilon = y_0^\varepsilon \in \mathbb{R}^l.
\end{cases}
\]

**Numerical examples.** Next, we provide a couple of numerical examples. We start with a simple example, in which the environment \( Y_t^\varepsilon \) is decoupled from the main process \( X_t^\varepsilon \).
**Example 2.1.** Consider the following equation

\[
\begin{aligned}
\varepsilon^2 \dddot{X}^\varepsilon_t &= -(1 + Y^\varepsilon_t)X^\varepsilon_t - \dddot{X}^\varepsilon_t, \\
\dot{Y}^\varepsilon_t &= -\frac{1}{\varepsilon}Y^\varepsilon_t + \frac{1}{\sqrt{\varepsilon}}\dot{B}^\varepsilon_t.
\end{aligned}
\]  

(2.12)

It is readily seen that the invariant measure of $Y^\varepsilon_t$ is $m(y) = \frac{1}{\sqrt{\pi}}e^{-y^2}, y \in \mathbb{R}$. Consider the averaged process $\overline{X}_t$ given by the following system

\[
\overline{X}_t = \int_{\mathbb{R}} X_t(1 + y)m(y)dy.
\]  

(2.13)

Using our results, $X^\varepsilon$ converges to $\overline{X}$ as $\varepsilon \to 0$ in probability and the tail probability of this convergence is exponentially small. For $\varepsilon = 0.005$, the simulated results are shown in picture 1, and as $\varepsilon$ decreases to $\varepsilon = 0.001$, the tail probability is rather small as demonstrated in the picture. [We used state-of-the-art algorithm in non-parametric kernel density estimation introduced by $[1]$ to simulate the numerical examples in this paper.]

![Figure 1](image1.png)

Figure 1: The distributions of the sample paths of the solution $X^\varepsilon$ of (2.12) with $\varepsilon = 0.005$ (left) and $\varepsilon = 0.001$ (right), respectively. The red line on the plane is the averaged system (2.13).

**Example 2.2.** Consider another example

\[
\begin{aligned}
\varepsilon^2 \dddot{X}^\varepsilon_t &= -(1 + (Y^\varepsilon_t)^2)X^\varepsilon_t - \dddot{X}^\varepsilon_t, \\
\dot{Y}^\varepsilon_t &= -\frac{1}{\varepsilon}(1 + (X^\varepsilon_t)^2)Y^\varepsilon_t + \frac{1}{\sqrt{\varepsilon}}(2 + \sin(X^\varepsilon_t))\dot{B}^\varepsilon_t.
\end{aligned}
\]  

(2.14)

For each fixed $x$, the invariant measure of the diffusion

\[
\dot{Y}^\varepsilon,x_t = -\frac{1}{\varepsilon}(1 + x^2)Y^\varepsilon,x_t + \frac{1}{\sqrt{\varepsilon}}(2 + \sin x)\dot{B}^\varepsilon_t,
\]

has the density

\[
m(y) = \frac{\sqrt{1 + x^2}}{\sqrt{\pi}(2 + \sin x)}e^{-\frac{1 + x^2}{(2 + \sin x)^2}y^2}.
\]

Numerical examples of the distribution of $(X^\varepsilon, Y^\varepsilon)$ and its averaged system are given in Figure 2.
Figure 2: The numerical examples of \(2.14\) with \(\varepsilon = 0.001\). The graph on the left is the distribution of sample paths of \(X^\varepsilon\) and the red line is the averaged system \(\overline{X}_t\). The graph on the right is the distribution of sample paths of \((X^\varepsilon, Y^\varepsilon)\) and the red line on the plane is the system \((\overline{X}_t, \overline{Y}_t)\) with \(\overline{Y}_t\) having distribution with the density function given by \(Ce^{\frac{1}{(2 + \sin X^t)^2}}y^2\).

2.2 LDP: General Fast Random Processes

In this section, we treat \((1.2)\). We need the following assumptions.

**Assumption 2.3.** The functions \(F^\varepsilon_t(x, y)\) and \(\lambda^\varepsilon_t(x, y)\) are Lipschitz continuous in \(x\) locally uniformly in \(t\) and globally uniformly in \(y\), and \(\lambda^\varepsilon_t(x, y)\) is bounded below (uniformly) by a positive constant \(\kappa_0\). Either \(F^\varepsilon_t(x, y)\) and \(\lambda^\varepsilon_t(x, y)\) have linear growth in \((t, x)\) globally in \(y\), i.e., there is a universal constant \(\tilde{C}\) such that

\[
|F^\varepsilon_t(x, y)| + |\lambda^\varepsilon_t(x, y)| \leq \tilde{C}(1 + |t| + |x|),
\]  

(2.15)

or \(F^\varepsilon(x, y)\) and \(\lambda^\varepsilon_t(x, y)\) have linear growth in \(x\) locally in \(y\), i.e., the constant \(\tilde{C}\) in (2.15) is uniformly in bounded sets of \(y\) and \(\xi^\varepsilon_t/\varepsilon\) is such that for any \(T > 0\)

\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left( \sup_{0 \leq t \leq T} |\xi^\varepsilon_t/\varepsilon| > L \right) = -\infty.
\]  

(2.16)

**Definition 2.2.** A family of stochastic processes \(\{X^\varepsilon\}_{\varepsilon > 0}\) is said to satisfy the local LDP in \(C([0, 1], \mathbb{R}^d)\) with rate function \(\mathbb{J}\), if for any \(\varphi \in C([0, 1], \mathbb{R}^d)\),

\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in B(\varphi, \delta)) = -\mathbb{J}(\varphi),
\]

where \(B(\varphi, \delta)\) is the ball centered at \(\varphi\) with radius \(\delta\) in \(C([0, 1], \mathbb{R}^d)\). \(\mathbb{J}\) is called local rate function.

**Assumption 2.4.** The family of processes \(\{Z^\varepsilon\}_{\varepsilon > 0}\) given by

\[
\dot{Z}^\varepsilon_t = \frac{F^\varepsilon_t(Z^\varepsilon_t, \xi^\varepsilon_t)}{\lambda^\varepsilon_t(Z^\varepsilon_t, \xi^\varepsilon_t)}, \quad Z^\varepsilon_0 = x^\varepsilon_0,
\]  

(2.17)

satisfies the local LDP with a rate function \(\mathbb{J}\).
Theorem 2.2. Assume that Assumptions 2.3 and 2.4 hold, that the family \( \{x_0^\varepsilon\}_{\varepsilon > 0} \) is exponentially tight, and that \( \limsup_{\varepsilon \to 0} \varepsilon |x_1^\varepsilon| < \infty \) a.s. Then, the family \( \{X^\varepsilon\}_{\varepsilon > 0} \) of (1.2) obeys the LDP in \( C(\mathbb{R}_+, \mathbb{R}^d) \) with rate function \( J \).

Remark 3. We did not assume any regularity and growth-rate conditions of the coefficients of the slow component when dealing with (1.1). However, for general fast random process, it seems to be impossible to use the same assumptions because we do not require any structure for the fast process. As a result, the assumptions in this section are stronger than that of Section 2.1. In particular, we need the Lipschitz continuity and growth-rate conditions of \( F^\varepsilon_t(x, y) \). It is worth noting that we used two totally different approaches for the two cases. If the fast process is a diffusion, thanks to the nice structure of martingales, we can identify the rate function after estimating the exponential moment. Therefore, in the first case, after obtaining the exponential tightness and then relatively LD compactness (see Definition 3.3), our remaining work is to identify the rate functions. In the general case, we use a different approach that relies on the property that exponential tightness and the local LDP imply the full LDP. In this situation, we need to connect directly the solutions of the second-order and the first-order equations.

3 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. In what follows, we use \( C \) to represent a generic positive constant that is independent of \( \varepsilon \). The value \( C \) may change at different appearances; we will specify which parameters it depends on if it is necessary.

The proof of the LDP of \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \), whose methodology is similar to that of the first-order case in [22], relies on the properties that if a family of random elements is exponentially tight then it is sequentially LD relatively compact, i.e., any subsequence contains a further subsequence enjoying the LDP with some rate function. The remaining work is done by carefully identifying the rate functions. To be self-contained, we recall these preliminaries below; see [6, 7, 17] for more detail.

Definition 3.1. The family \( \{\mathbb{P}^\varepsilon\}_{\varepsilon > 0} \) is said to be exponentially tight in the space \( S \) if there exists an increasing sequence of compact sets \( \{K_L\}_{L \geq 1} \) of \( S \) such that

\[
\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon(K_L) = -\infty.
\]

Definition 3.2. The family \( \{\mathbb{P}^\varepsilon\}_{\varepsilon > 0} \) is said to be sequentially LD relatively compact if any subsequence \( \{\mathbb{P}^{\varepsilon_k}\}_{k \geq 1} \) of \( \{\mathbb{P}^\varepsilon\}_{\varepsilon > 0} \) contains a further subsequence \( \{\mathbb{P}^{\varepsilon_{kj}}\}_{j \geq 1} \) which satisfies the LDP with some large deviations rate function as \( j \to \infty \).

Definition 3.3. We say that a family of random elements of \( S \) is sequentially LD relatively compact (resp. exponentially tight) if their laws have the indicated property.

Proposition 3.1. ([22, Theorem 4.1]) If a family \( \{\mathbb{P}^\varepsilon\}_{\varepsilon > 0} \) is exponentially tight then it is sequentially LD relatively compact.

This section is summarized as follows. The exponential tightness of \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) is proved in Section 3.1 by applying the (extended) Puhalskii’s criteria. The large deviations limit points then are identified in Section 3.2. The proof is completed at the end of this section. To proceed, we introduce the following technical lemma, which will be used often in some calculations in the remaining of this section.
Lemma 3.1. For real-valued continuous function \( g(s) \), and real-valued continuously differentiable function \( u(s) \), and real-valued Itô process \( w(s) \), \( w(s) > 0 \) \( \forall s \) with the quadratic variation denoted by \( \langle dw, dw \rangle_s \), we have the following identity

\[
\frac{1}{\varepsilon^2} \int_0^t u(s) \int_0^s e^{-\frac{\varepsilon}{2} \int_s^r w(r') dr'} g(r) dr ds = \int_0^t \frac{u(t) g(s)}{w(s)} ds + \int_0^t \frac{u(t)}{w(s)} \left( \int_0^s e^{-\frac{\varepsilon}{2} \int_s^r w(r') dr'} g(r) dr \right) ds
\]

\[
- \frac{u(t) e^{-\frac{\varepsilon}{2} \int_0^t w(r) dr}}{w(t)} g(s) ds - \int_0^t \frac{u(s) g(s)}{w(s)^2} \left( \int_0^s e^{-\frac{\varepsilon}{2} \int_s^r w(r') dr'} g(r) dr \right) dw(s)
\]

Moreover, the identity \( (3.1) \) still holds if \( g \) is \( \mathbb{R}^d \)-valued function and \( u \) can be either \( \mathbb{R} \)-valued or \( \mathbb{R}^d \)-valued (with the operations corresponding to \( u \) and \( g \) being understood as the inner product in \( \mathbb{R}^d \)).

Proof. Using integration by parts for

\[
u(s) \int_0^s e^{-\frac{\varepsilon}{2} \int_s^r w(r') dr'} g(r) dr \quad \text{and} \quad e^{-\frac{\varepsilon}{2} \int_0^s w(r) dr} \frac{w(s)}{w(s)}
\]

(3.1) follows from standard calculations. \( \square \)

3.1 Exponential Tightness of \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \)

In this section, we establish the exponential tightness of \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) in \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}^l) \times \mathcal{C}(\mathbb{R}_+^l, \mathcal{M}(\mathbb{R}^l)) \).

Theorem 3.1. Suppose that Assumption 2.1 holds, that the family \( \{(x_0^\varepsilon, y_0^\varepsilon)\}_{\varepsilon > 0} \) is exponentially tight, and that \( \limsup_{\varepsilon \to 0} \varepsilon x_1^\varepsilon < \infty \) a.s. Then the family \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) obtained from (1.1) is exponentially tight and sequentially LD relatively compact in \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}^l) \times \mathcal{C}(\mathbb{R}_+^l, \mathcal{M}(\mathbb{R}^l)) \).

Since \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}^l) \times \mathcal{C}(\mathbb{R}_+^l, \mathcal{M}(\mathbb{R}^l)) \) is a closed subset of \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}(\mathbb{R}_+^d, \mathcal{M}(\mathbb{R}^d)) \) and \( \mathcal{P}^\varepsilon((X^\varepsilon, \mu^\varepsilon) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}(\mathbb{R}_+^d, \mathcal{M}(\mathbb{R}^d))) = 1 \), it is sufficient to prove that the family \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) is exponentially tight in \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}(\mathbb{R}_+^d, \mathcal{M}(\mathbb{R}^d)) \). To prove that, it suffices to verify \( \{(X^\varepsilon, \mu^\varepsilon)\}_{\varepsilon > 0} \) satisfying the (extended) Puhalskii’s criteria (see [17] Theorem 3.1 and [8] Remark 4.2, or see also [22] Lemma 4.1]), namely, \( \forall \ell, t > 0 \)

\[
limit_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon \left( \sup_{s \in [0,t]} |X^\varepsilon_s| > L \right) = -\infty, \quad (3.2)
\]

\[
limit_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon \left( \sup_{s \leq s_1 \leq s < s + \delta} |X^\varepsilon_{s_1} - X^\varepsilon_s| > \ell \right) = -\infty, \quad (3.3)
\]

\[
limit_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon \left( \mu^\varepsilon([0, t], \{y \in \mathbb{R}^l : |y| > L\}) > \ell \right) = -\infty, \quad (3.4)
\]

and

\[
limit_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon \left( \sup_{s_1 \in [s, s + \delta]} d(\mu^\varepsilon_{s_1}, \mu^\varepsilon_s) > \ell \right) = -\infty. \quad (3.5)
\]
Remark 4. In general, (3.2)-(3.5) only imply the sequentially exponential tightness (i.e., any subsequence is exponentially tight). However, because \((X^\varepsilon, \mu^\varepsilon)\) is continuous in \(\varepsilon\) in distribution, it is true that \((X^\varepsilon, \mu^\varepsilon)\) is exponentially tight although in our proof, only the sequentially exponential tightness is needed.

From equation (2.2), by the variation of parameter formula, we obtain

\[
 p_t^\varepsilon = x_1^\varepsilon e^{-A_\varepsilon(t)} + \frac{1}{\varepsilon^2} \int_0^t e^{-A_\varepsilon(t,s)} F_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) ds, \tag{3.6}
\]

where for any \(0 \leq s \leq t, \varepsilon > 0,\)

\[
 A_\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda_\varepsilon(X_r^\varepsilon, Y_r^\varepsilon) dr, \quad A_\varepsilon(t) = A_\varepsilon(t, 0).
\]

Proof of (3.2). It is readily seen that

\[
 \ln(1 + |X_t^\varepsilon|^2) - \ln(1 + |x_0^\varepsilon|^2) = \int_0^t \frac{2(X_s^\varepsilon)^\top p_s^\varepsilon}{1 + |X_s^\varepsilon|^2} ds - \frac{1}{\varepsilon^2} \int_0^t \frac{2}{1 + |X_s^\varepsilon|^2} (X_s^\varepsilon)^\top \left( \int_0^s e^{-A_\varepsilon(s,r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon) dr \right) ds. \tag{3.7}
\]

Denote

\[
 v_t^\varepsilon = \frac{2}{1 + |X_t^\varepsilon|^2} (X_t^\varepsilon)^\top \left( \int_0^t e^{-A_\varepsilon(t,r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon) dr \right).
\]

We have from (3.7) and Lemma 3.1 that

\[
 \ln(1 + |X_t^\varepsilon|^2) - \ln(1 + |x_0^\varepsilon|^2) = \int_0^t \frac{2(X_s^\varepsilon)^\top x_1^\varepsilon}{1 + |X_s^\varepsilon|^2} e^{-A_\varepsilon(s)} ds + \int_0^t \frac{2(X_s^\varepsilon)^\top F_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)}{(1 + |X_s^\varepsilon|^2) \lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)} ds

+ \frac{2}{\lambda_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)} \left( \frac{p_s^\varepsilon}{1 + |X_s^\varepsilon|^2} - \frac{2(X_s^\varepsilon)^\top p_s^\varepsilon}{1 + |X_s^\varepsilon|^2} \right)^\top \left( \int_0^s e^{-A_\varepsilon(s,r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon) dr \right) ds

- \int_0^t \frac{v_s^\varepsilon}{\lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)} d\lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) + \int_0^t \frac{v_s^\varepsilon}{[\lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^3} (d\lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon), d\lambda_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)) ds. \tag{3.8}
\]
Combining (3.8) and the Itô Lemma, one has

\[ \ln(1 + |X_t|^2) - \ln(1 + |x_0|^2) \]

\[ = \int_0^t \frac{2\langle X_s \rangle^\top x_s}{1 + |X_s|^2} e^{-A_\varepsilon(s)} ds + \int_0^t \frac{2\langle X_s \rangle^\top F_s(X_s, Y_s)}{(1 + |X_s|^2)\lambda_s(X_s, Y_s)} ds 
+ \int_0^t \frac{2}{\lambda_s(X_s, Y_s)} \left( \frac{p_s}{1 + |X_s|^2} - \frac{2\langle X_s \rangle^\top p_s}{(1 + |X_s|^2)^2} \right)^\top \left( \int_0^s e^{-A_\varepsilon(r, s)} F_r(X_r, Y_r) dr \right) ds 
- \frac{1}{\varepsilon} \int_0^t \frac{v_s^e}{\lambda_s(X_s, Y_s)} \frac{v_s^e}{\lambda_s(X_s, Y_s)} \nabla_x \lambda_s(X_s, Y_s) ds 
- \frac{1}{\varepsilon} \int_0^t \frac{v_s^e}{\lambda_s(X_s, Y_s)} \nabla_y \lambda_s(X_s, Y_s) ds 
- \frac{1}{\varepsilon} \int_0^t \frac{v_s^e}{\lambda_s(X_s, Y_s)} \frac{v_s^e}{\lambda_s(X_s, Y_s)} \nabla_x \lambda_s(X_s, Y_s) \sigma_s(X_s, Y_s) ds 
- \frac{1}{\varepsilon} \int_0^t \frac{v_s^e}{\lambda_s(X_s, Y_s)} \nabla_y \lambda_s(X_s, Y_s) \sigma_s(X_s, Y_s) ds \]

\[ =: K_t^\varepsilon + \frac{1}{\varepsilon} \int_0^t \frac{\nabla_y \lambda_s(X_s, Y_s) \sigma_s(X_s, Y_s)}{\lambda_s(X_s, Y_s)} ds \]

where \( K_t^\varepsilon \) is the remaining in the right-hand side. Therefore, we get

\[ \frac{1}{\varepsilon} \left[ \ln(1 + |X_t|^2) - \ln(1 + |x_0|^2) \right] = \frac{1}{\varepsilon} \tilde{K}_t^\varepsilon + D_t^\varepsilon, \]

where

\[ \tilde{K}_t^\varepsilon := \left[ K_t^\varepsilon + \frac{1}{\varepsilon^2} \int_0^t \frac{|v_s^e|^2}{\lambda_s(X_s, Y_s)} \nabla_y \lambda_s(X_s, Y_s) \sigma_s(X_s, Y_s)^2 ds \right], \]

and

\[ D_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t \frac{\nabla_y \lambda_s(X_s, Y_s) \sigma_s(X_s, Y_s)}{\lambda_s(X_s, Y_s)} ds \]

Let \( \zeta_t^\varepsilon := \inf \{ t \geq 0 : |X_t^\varepsilon| > L \} \). It is obvious that \( \zeta_t^\varepsilon \) is an \( \mathcal{F}_t \)-stopping time. Since \( D_t^\varepsilon \) is a local martingale, we have from (3.9) that

\[ \mathbb{E}^\varepsilon \left\{ \frac{1}{\varepsilon} \left[ \ln(1 + |X_t^\varepsilon(\zeta_t^\varepsilon)|^2) - \ln(1 + |x_0|^2) - \tilde{K}_t^\varepsilon \right] \right\} \leq 1. \]

On the other hand, from (3.6) and Assumption 2.1 and noting

\[ \int_0^t e^{-A_\varepsilon(t, s)} ds \leq \int_0^t e^{-\kappa_0(t-s)} ds \leq \frac{\varepsilon^2}{\kappa_0}, \]

one obtains that there is a finite constant \( C_{t, L} \) depending only on \( t, L \) satisfies that for all sufficiently small \( \varepsilon \),

\[ |p_{t \wedge \zeta_t^\varepsilon}^\varepsilon| \leq C_{t, L} + x_1^2 e^{-A_\varepsilon(t)}. \]
Similarly, we have for $\varepsilon$ small
\[ |\varepsilon| \leq C_{t,L} \int_0^t e^{-\frac{\alpha_0(t-s)}{2\varepsilon^2}} ds \leq \varepsilon^2 C_{t,L}. \] (3.12)

Thus, by combining (3.11), (3.12), the definition of $\hat{K}^{\varepsilon}_{t,L}$, Assumption 2.1 and $\limsup_{\varepsilon \to 0} \varepsilon x_1^\varepsilon < \infty$ yields that as $\varepsilon$ being small
\[ |\hat{K}^{\varepsilon}_{t,L}| \leq C + \varepsilon C_{t,L}, \] (3.13)
where $C$ is some finite constant depending neither $\varepsilon$ nor $t, L$. Therefore, from (3.10) and (3.13), we have that for all $\varepsilon$
\[ \mathbb{E}^\varepsilon \exp \left\{ \frac{1}{\varepsilon} \left[ \ln(1 + |X^\varepsilon_{t,L}|^2) - \ln(1 + |x_0^\varepsilon|^2) - C - \varepsilon C_{t,L} \right] \right\} \leq 1. \]

Thus, one has for any $t, L, N > 0$,
\[ \mathbb{P}^\varepsilon \left( \sup_{s \in [0,t]} |X^\varepsilon_s| > L \right) = \mathbb{P}^\varepsilon (|X^\varepsilon_{t,L}| > L) \]
\[ \leq \mathbb{P}^\varepsilon (|x_0^\varepsilon| > N) + \mathbb{E}^\varepsilon \exp \left\{ \frac{1}{\varepsilon} \left[ \ln(1 + |X^\varepsilon_{t,L}|^2) - \ln(1 + L^2) \right] \right\} 1_{\{|x_0^\varepsilon| \leq N\}} \]
\[ \leq \mathbb{P}^\varepsilon (|x_0^\varepsilon| > N) + \exp \left\{ \frac{1}{\varepsilon} \left[ \ln(1 + N^2) + C + \varepsilon C_{t,L} - \ln(1 + L^2) \right] \right\}. \] (3.14)

From (3.14) and the logarithm equivalence principle [7, Lemma 1.2.15], we obtain that for all $t, N > 0$,
\[ \limsup_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \sup_{s \in [0,t]} |X^\varepsilon_s| > L \right) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^\varepsilon (|x_0^\varepsilon| > N). \] (3.15)

Because $\{x_0^\varepsilon\}_{\varepsilon > 0}$ is exponentially tight and (3.15), we obtain (3.2) for any $t > 0$. □

**Proof of (3.3).** By applying Lemma 3.1 to (3.6), one has
\begin{align*}
X^\varepsilon_t &= x_0^\varepsilon + \int_0^t p^\varepsilon_s ds = x_0^\varepsilon + \int_0^t x_1^\varepsilon e^{-A_\varepsilon(s)} ds + \int_0^t \int_0^s e^{-A_\varepsilon(r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr ds \\
&= x_0^\varepsilon + x_1^\varepsilon \int_0^t e^{-A_\varepsilon(r)} dr + \int_0^t \int_0^s \frac{F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r)}{\lambda^\varepsilon_t(X^\varepsilon_r, Y^\varepsilon_r)} dr ds - \int_0^t \lambda^\varepsilon_t(X^\varepsilon_t, Y^\varepsilon_t) \int_0^s e^{-A_\varepsilon(r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr ds \\
&- \int_0^t \left[ \frac{1}{\lambda^\varepsilon_t(X^\varepsilon_s, Y^\varepsilon_s)^2} \right] \left( \int_0^s e^{-A_\varepsilon(s,r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) d\lambda^\varepsilon_s(X^\varepsilon_s, Y^\varepsilon_s) \\
&+ \int_0^t \left[ \frac{1}{\lambda^\varepsilon_t(X^\varepsilon_s, Y^\varepsilon_s)^3} \right] \left( \int_0^s e^{-A_\varepsilon(s,r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) (d\lambda^\varepsilon_s(X^\varepsilon_s, Y^\varepsilon_s), d\lambda^\varepsilon_s(X^\varepsilon_s, Y^\varepsilon_s))_s.
\end{align*}

\[ (3.16) \]
We obtain from (3.16) and Itô's formula that

\[
X_t^\varepsilon = x_0^\varepsilon + x_1^\varepsilon \int_0^t e^{-A_\varepsilon(r)}dr + \int_0^t \frac{F_t^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)}{\lambda_f^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)}dr - \frac{1}{\lambda_f^\varepsilon} \int_0^t e^{-A_\varepsilon(t, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \\
- \int_0^t \nabla_\varepsilon \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon) \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) ds \\
- \int_0^t \frac{\nabla Y \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)^\top b_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)}{[\lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^2} \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) ds \\
- \frac{1}{\varepsilon} \int_0^t \frac{\nabla Y \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)^\top b_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)}{[\lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^2} \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) ds \\
+ \frac{1}{\varepsilon^2} \int_0^t \frac{[\nabla Y \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)^\top b_s^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^2}{[\lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^3} \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) ds \\
- \frac{1}{\varepsilon^3} \int_0^t \frac{1}{[\lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^3} \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) \left( \nabla Y \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)^\top \sigma_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)dB_s^\varepsilon \right)
=: \overline{K}_t^\varepsilon - \overline{D}_t^\varepsilon,
\]

where

\[
\overline{D}_t^\varepsilon := \frac{1}{\varepsilon} \int_0^t \frac{1}{[\lambda_f^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)]^2} \left( \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr \right) \left( [\nabla Y \lambda_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)]^\top \sigma_f^\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)dB_s^\varepsilon \right),
\]

and \( \overline{K}_t^\varepsilon \) is the remaining in the right-hand side of (3.17). By the regularity of \( \lambda_f^\varepsilon \), it is not difficult to see that

\[
\left| \frac{1}{\lambda_f^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)} \int_0^t e^{-A_\varepsilon(t, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)dr - \frac{1}{\lambda_f^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)} \int_0^s e^{-A_\varepsilon(s, r)} F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)ds \right|
\leq \left| e^{-A_\varepsilon(t)} - e^{-A_\varepsilon(s)} \right| \int_0^s e^{A_\varepsilon(r)} |F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)|dr \\
+ \frac{e^{-A_\varepsilon(t)}}{\lambda_f^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)} \int_s^t e^{A_\varepsilon(r)} |F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)|dr \\
\leq C \frac{t-s}{\varepsilon^2} \sup_{r \in [s, t]} |\lambda_f^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)| \int_0^s e^{-a_\varepsilon(r-s)} |F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)|dr + C \int_s^t e^{-a_\varepsilon(t-r)} |F_r^\varepsilon(X_r^\varepsilon, Y_r^\varepsilon)|dr.
\]

We obtain from definition of \( \overline{K}_t^\varepsilon \), an application of (3.18), and recalling definition of \( \zeta_f^\varepsilon \) that there is a finite constant \( C_L \) depending only on \( L \) such that for all small \( \varepsilon \),

\[
\sup_{s \in [0, T]} \sup_{t \in [s, t]} |\overline{K}_{t \wedge \zeta_f^\varepsilon} - \overline{K}_{s \wedge \zeta_f^\varepsilon}| \leq C_{T, L} \delta, \text{ for all } T > 0, 0 < \delta < 1.
\]

Now, let \( T \), \( \ell \), \( 0, \delta > 0 \) be fixed, and \( L > 0 \) be fixed but otherwise arbitrary. We have that for
any small $\delta$ satisfying $\delta < 1$, $C_{T,L}\delta < \ell/2$ and small $\varepsilon$

\[
P^\varepsilon\left( \sup_{t \in [s,s+\delta]} |X^\varepsilon_t - X^\varepsilon_s| > \ell \right) \leq P^\varepsilon(\zeta^L_t \leq T+1) + P^\varepsilon\left( \sup_{t \in [s,s+\delta]} |X^\varepsilon_t \wedge \zeta^L_t - X^\varepsilon_s \wedge \zeta^L_s| > \ell \right)
\]

\[
\leq P^\varepsilon(\zeta^L_t \leq T+1) + P^\varepsilon\left( \sup_{t \in [s,s+\delta]} |\overline{D}^\varepsilon_{t \wedge \zeta^L_t} - \overline{D}^\varepsilon_{s \wedge \zeta^L_s}| > \frac{\ell}{2} \right)
\]

\[
\leq P^\varepsilon\left( \sup_{t \in [0,T+1]} |X^\varepsilon_t| > L \right) + \sum_{k=1}^{d} P^\varepsilon\left( \sup_{t \in [s,s+\delta]} |\overline{D}^\varepsilon_{t \wedge \zeta^L_t} - \overline{D}^\varepsilon_{s \wedge \zeta^L_s}| > \frac{\ell}{2} \right),
\]

where $\overline{D}^\varepsilon_{t \wedge \zeta^L_t}$ is the $k$-th component of $\overline{D}^\varepsilon_t$, $k = 1, \ldots, d$. It is readily seen that $\{\overline{D}^\varepsilon_{t \wedge \zeta^L_t} - \overline{D}^\varepsilon_{s \wedge \zeta^L_s}\}_{t \geq s}$ is a martingale with the quadratic variations bounded by

\[
\frac{C_L}{\varepsilon} \int_{s \wedge \zeta^L_t}^{T \wedge \zeta^L_t} \int_{0}^{s} e^{-\frac{2\kappa_{l}(s-r)}{\varepsilon}} dr ds \leq \varepsilon C_L \delta.
\]

By the exponential martingale inequality \cite[Theorem 7.4, p. 44]{19}, we have

\[
P^\varepsilon\left( \sup_{t \in [s,s+\delta]} \left| \overline{D}^\varepsilon_{t \wedge \zeta^L_t} - \overline{D}^\varepsilon_{s \wedge \zeta^L_s} \right| > \frac{\ell}{2} \right) \leq P^\varepsilon\left( \sup_{t \in [s,s+\delta]} \left| \overline{D}^\varepsilon_{t \wedge \zeta^L_t} - \overline{D}^\varepsilon_{s \wedge \zeta^L_s} \right| > \frac{\ell}{4} + \frac{\ell}{4\varepsilon C_L \delta} \varepsilon C_L \delta \right)
\]

\[
\leq \exp\left\{ -\frac{\ell^2}{8\varepsilon C_L \delta} \right\}.
\]

Combining (3.20) and (3.21), the logarithm equivalence principle \cite[Lemma 1.2.15]{7} yields that

\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,T]} \varepsilon \log P^\varepsilon\left( \sup_{t \in [s,s+\delta]} |X^\varepsilon_t - X^\varepsilon_s| > \ell \right) \leq \limsup_{\varepsilon \to 0} \varepsilon \log P^\varepsilon\left( \sup_{t \in [0,T+1]} |X^\varepsilon_t| > L \right), \ \forall L > 0.
\]

Letting $L \to \infty$ and using (3.2), we obtain (3.3). \qed

Proof of (3.4) and (3.5). Once we established the exponential tightness of $\{X^\varepsilon\}_{\varepsilon > 0}$, the proof of (3.4) and (3.5) for $\{\mu^\varepsilon\}_{\varepsilon > 0}$, which is in fact the occupation measure of a diffusion, is similar to that of the first-order coupled systems. As a consequence, such proofs can be found in \cite[p. 3134]{22}. \qed

### 3.2 Characterization of Rate Function

Let $\beta(s) \in C(\mathbb{R}_+, \mathbb{R}^d)$ be a step function satisfying that there are $0 = t_0 < t_1 < \cdots < t_m < \infty$ and $\beta_i \in \mathbb{R}, i = 1, \ldots, m$ such that

\[
\beta(s) = \sum_{i=1}^{m} \beta_i \mathbb{1}_{[t_{i-1}, t_i)}(s).
\]

For $\varphi_s \in C(\mathbb{R}_+, \mathbb{R}^d)$ and $\beta(s)$ of the form (3.23), we define

\[
\int_{0}^{t} \beta(s) d\varphi_s := \sum_{i=1}^{m} \beta_i^T (\varphi_{t \wedge t_i} - \varphi_{t \wedge t_{i-1}}).
\]

Now, for each step function $\beta(s)$, each $f(t, x, y)$ representing a (real-valued) $C^{1,2,2}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l)$-function with compact support in $y$ locally uniformly in $(t, x)$, and each $(\varphi, \mu) \in C(\mathbb{R}_+, \mathbb{R}^d) \times \ldots$
Moreover, let $\tau(\varphi, \mu)$ represent a continuous function of $(\varphi, \mu) \in C(\mathbb{R}_+, \mathbb{R}^d) \times C_t(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^l))$ that is also a stopping time relative to the flow $G = (G_t, t \in \mathbb{R}_+)$ on $C(\mathbb{R}_+, \mathbb{R}^d) \times C_t(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^l))$ of the $\sigma$-algebra $G_t$ generated by the mappings $\varphi \mapsto \varphi_s$ and $\mu \mapsto \mu_s$ for $s \leq t$. Let us also assume that $\varphi_{t \wedge \tau(\varphi, \mu)}$ is a bounded function of $(\varphi, \mu)$. It is seen that under Assumption 2.2, $\Phi_t^\beta,f(\varphi, \mu)$ is continuous in $(\varphi, \mu)$. Next, define

$$I^*(\varphi, \mu) = \sup_{\beta, f, \tau} \Phi_{t \wedge \tau(\varphi, \mu)}^\beta,f(\varphi, \mu),$$

where the supremum is taken over $\beta(s), f(s, x, y)$, and $\tau(\varphi, \mu)$ satisfying the requirements as the above and over $t \geq 0$. It is seen that $I^*$ is lower semi-continuous in $(\varphi, \mu)$.

Now, let $\tilde{I}$ be a large deviations limit rate functions or (large deviations) LD limit points of $\{(X^\varepsilon, \mu^\varepsilon)_t\}_{\varepsilon > 0}$ (i.e., a rate function of some subsequence of $\{(X^\varepsilon, \mu^\varepsilon)_t\}_{\varepsilon > 0}$ that obeys the LDP) such that $\tilde{I}(\varphi, \mu) = \infty$ unless $\varphi_0 = \tilde{\varphi}$, where $\tilde{\varphi}$ is a preselected element of $\mathbb{R}^d$. This restriction will be removed in Section 3.3. We will identify the rate functions. For any such a large deviation limit point $\tilde{I}$, we aim to prove $\tilde{I} = I^*$ by showing the upper bound $\tilde{I} \geq I^*$ and the lower bound $\tilde{I} \leq I^*$; see detail in Section 3.2.1 and Section 3.2.2. Moreover, it will be seen that $I^*(\varphi, \mu) = \tilde{I}(\varphi, \mu)$ provided $I^*(\varphi, \mu) < \infty$, $\varphi_0 = \tilde{\varphi}$, and $\tilde{I}_0(\tilde{\varphi}) = 0$. Throughout this section, the assumptions in Theorem 2.1 are always assumed to be satisfied.

**Remark 5.** Note that $I^*$ is defined similarly but not identical as that in the case of first-order coupled systems in [22] although the solution of (1.1) shares the same rate function with the corresponding first-order system. Compared with [22], $I^*$ is defined by taking the supremum over smaller space when we did not allow $\beta$ to be a function of $X$. This modification has an important role in the proof of the lower bound of the LD limits, i.e., the inequality $I^* \leq \tilde{I}$. Otherwise, it would be impossible to control terms containing the derivative $p^\varepsilon$ of $X^\varepsilon$, specially the integral involving $p^\varepsilon$ and the diffusion part of the fast process (see (3.31)). Meanwhile, it would have led to a difficulty in proving the upper bound of the LD limits, i.e., the inequality $I^* \geq \tilde{I}$. However, the effort in this part can be done similarly to that in [22] for the first-order system; see the details in Section 3.2.2.

### 3.2.1 Lower Bound of Large Deviations Limits

This section is devote to proving $I^* \leq \tilde{I}$. We have the following theorem.

**Theorem 3.2.** Let $\tilde{I}$ be a LD limit point of $\{(X^\varepsilon, \mu^\varepsilon)_t\}_{\varepsilon > 0}$. For any $t > 0$, $\beta, f, \tau$ are as given above,

$$\sup_{(\varphi, \mu) \in C(\mathbb{R}_+, \mathbb{R}^d) \times C_t(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^l))} \left( \Phi_{t \wedge \tau(\varphi, \mu)}^\beta,f(\varphi, \mu) - \tilde{I}(\varphi, \mu) \right) = 0. \tag{3.26}$$

Then $I^*(\varphi, \mu) \leq \tilde{I}(\varphi, \mu)$ for all $(\varphi, \mu) \in C(\mathbb{R}_+, \mathbb{R}^d) \times C_t(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^l))$. 

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Proof. For \( \beta(s) \) being of the form (3.23) and \( f(s, x, y) \) being a function with compact support in \( y \) locally uniformly in \( (t, x) \), denote

\[
\Phi_t^{\varepsilon, \beta, f}(\varphi, \mu) := \int_0^t \beta(s) d\varphi_s - \int_0^t \int_{\mathbb{R}^l} \frac{\beta(s)^T F^\varepsilon(s, y)}{\lambda^\varepsilon_s(\varphi_s, y)} \mu(ds, dy)
+ \int_0^t \int_{\mathbb{R}^l} [\nabla_y f(s, \varphi_s, y)]^T b^\varepsilon_s(\varphi_s, y) \mu(ds, dy)
\]

\[
- \frac{1}{2} \int_0^t \int_{\mathbb{R}^l} \text{tr} \left( \Sigma^\varepsilon_s(\varphi_s, y) \nabla_{yy} f(s, \varphi_s, y) \right) \mu(ds, dy)
\]

and

\[
\Gamma_t^{\varepsilon, \beta}(\varphi, \mu) = - \int_0^t \int_{\mathbb{R}^l} [\beta(s)]^T x_t^\varepsilon e^{-A^\varepsilon Y(s)} \mu(ds, dy) + \int_0^t \int_{\mathbb{R}^l} \frac{\beta(t)}{\lambda^\varepsilon_t(\varphi_t, y)} e^{-A^\varepsilon Y(t, s)} F^\varepsilon(s, y) \mu(ds, dy)
\]

\[
+ \int_0^t \int_{\mathbb{R}^l} [\nabla_x \lambda^\varepsilon_s(\varphi_s, y)] \left( \int_0^s e^{-A^\varepsilon Y(s, r)} F^\varepsilon(r, y) dr \right) \mu(ds, dy)
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^l} [\beta(s)]^T \left( \int_0^s e^{-A^\varepsilon Y(s, r)} F^\varepsilon(r, y) dr \right) \left( \frac{\left[ \nabla_Y \lambda^\varepsilon_s(\varphi_s, y) \right]^T b^\varepsilon_s(\varphi_s, y)}{[\lambda^\varepsilon_s(\varphi_s, y)]^2} \right) \mu(ds, dy)
\]

\[
+ \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{R}^l} \left[ [\beta(s)]^T \left( \int_0^s e^{-A^\varepsilon Y(s, r)} F^\varepsilon(r, y) dr \right) \right]^2 \mu(ds, dy)
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^l} \frac{[\beta(s)]^T \left( \int_0^s e^{-A^\varepsilon Y(s, r)} F^\varepsilon(r, y) dr \right)}{[\lambda^\varepsilon_s(\varphi_s, y)]^4} \mu(ds, dy)
\]

\[
- \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^l} \frac{[\beta(s)]^T \left( \int_0^s e^{-A^\varepsilon Y(s, r)} F^\varepsilon(r, y) dr \right)}{[\lambda^\varepsilon_s(\varphi_s, y)]^2} \mu(ds, dy)
\]

where \( A^\varepsilon Y(t, s) := \frac{1}{\varepsilon} \int_s^t \int_{\mathbb{R}^l} \lambda^\varepsilon_r(\varphi_r, y) \mu(dr, dy) \), and

\[
\Psi_t^{\varepsilon, \beta, f}(\varphi, \mu) := f(t, \varphi_t, Y^\varepsilon_t) - f(0, \varphi_0, \mu_0) - \int_0^t \int_{\mathbb{R}^l} \nabla_s f(s, \varphi_s, y) \mu(ds, dy)
\]

\[
- \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^l} \left[ \nabla f(s, \varphi_s, y) \right]^T \varphi_s \mu(ds, dy).
\]
Itô's formula yields that

\[
\begin{align*}
\int_t^s f(s, X_s, Y_s) ds &= f(t, X_t, Y_t) - f(0, x_0, y_0) + \int_0^t \nabla f(s, X_s, Y_s) T p_s ds \\
&\quad + \frac{1}{\varepsilon} \int_0^t \left[ \nabla f(s, X_s, Y_s) \right]^T b_s(X_s, Y_s) ds \\
&\quad + \frac{1}{\varepsilon^2} \int_0^t \left[ \nabla f(s, X_s, Y_s) \right]^T \sigma_s(X_s, Y_s) dB_s \\
&\quad + \frac{1}{2\varepsilon} \int_0^t \text{tr} \left( \Sigma_s(X_s, Y_s) \nabla f(s, X_s, Y_s) \right) ds.
\end{align*}
\]

Moreover, we have from (3.6) and Lemma 3.1 that

\[
\begin{align*}
\int_0^t [\beta(s)]^T X_s ds &= \int_0^t [\beta(s)]^T p_s ds = \int_0^t [\beta(s)]^T x_0 e^{-A_s(s)} ds + \frac{1}{\varepsilon^2} \int_0^t [\beta(s)]^T e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon) dr \\
&\quad - \frac{\beta(t)}{\lambda(X_t, Y_t)} \int_0^t e^{-A_s(s, t)} F^\varepsilon_s(X^\varepsilon_s, Y^\varepsilon_s) ds \\
&\quad - \frac{1}{\varepsilon} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \frac{\nabla \lambda(X_s, Y_s)}{[\lambda(X_s, Y_s)]^2} ds \\
&\quad - \frac{1}{\varepsilon^2} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \frac{[\nabla \lambda(X_s, Y_s)]^T b_s(X_s, Y_s)}{[\lambda(X_s, Y_s)]^2} ds \\
&\quad - \frac{1}{\lambda(X_t, Y_t)} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \left\{ [\nabla \lambda(X_s, Y_s)]^T \sigma_s(X_s, Y_s) \right\} dB_s \\
&\quad - \frac{1}{\varepsilon} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \left\{ [\nabla \lambda(X_s, Y_s)]^T \sigma_s(X_s, Y_s) \right\} dB_s
\end{align*}
\]

Combining (3.30), (3.31), the definition of $\Phi^\varepsilon_{t, \beta} f$, $\Gamma^\varepsilon_{t, \beta}$, $\Psi^\varepsilon_{t, \beta} f$ in (3.27), (3.28), and (3.29), we obtain that

\[
\begin{align*}
\frac{1}{\varepsilon} \left[ \Phi^\varepsilon_{t, \beta}(X^\varepsilon, \mu^\varepsilon) + \Gamma^\varepsilon_{t, \beta}(X^\varepsilon, \mu^\varepsilon) \right] + \Psi^\varepsilon_{t, \beta}(X^\varepsilon, \mu^\varepsilon) \\
&= \frac{1}{\varepsilon} \int_0^t \left[ \nabla f(s, X_s, Y_s) \right]^T \sigma_s(X_s, Y_s) dB_s^\varepsilon - \frac{1}{2\varepsilon} \int_0^t \left\| \nabla f(s, X_s, Y_s) \right\|_{[\Sigma_s(X_s, Y_s)]^2} ds \\
&\quad - \frac{1}{\varepsilon \sqrt{\varepsilon}} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \frac{1}{[\lambda(X_s, Y_s)]^2} ds \\
&\quad - \frac{1}{2\varepsilon^3} \int_0^t \left[ [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \right]^2 ds \\
&\quad + \frac{1}{\varepsilon^2} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \left\{ [\nabla \lambda(X_s, Y_s)]^T \sigma_s(X_s, Y_s) \right\} dB_s \\
&\quad - \frac{1}{\varepsilon} \int_0^t [\beta(s)]^T \left( \int_0^s e^{-A_s(s, r)} F^\varepsilon_r(X^\varepsilon_r, Y^\varepsilon_r) dr \right) \left\{ [\nabla \lambda(X_s, Y_s)]^T \sigma_s(X_s, Y_s) \right\} dB_s
\end{align*}
\]
Since the right-hand side of (3.32) is a local martingale and \( \tau(X^\varepsilon,\mu^\varepsilon) \) is a stopping time with respect to \( \mathcal{F}^\varepsilon \) due to the measurability of \( X_t^\varepsilon,\mu_t^\varepsilon \) with respect to \( \mathcal{F}_t^\varepsilon \), we have that

\[
\mathbb{E}^\varepsilon \exp \left\{ \frac{1}{\varepsilon} \left[ \Phi_{t\wedge \tau}^{\varepsilon,\beta,f}(X^\varepsilon,\mu^\varepsilon) + \Gamma_{t\wedge \tau}^{\varepsilon,\beta}(X^\varepsilon,\mu^\varepsilon) + \varepsilon \Psi_{t\wedge \tau}^{\varepsilon,\beta,f}(X^\varepsilon,\mu^\varepsilon) \right] \right\} = 1. \tag{3.33}
\]

To proceed, we restate the following lemma, which can be found in [22, Theorem 3.3] or [6, Theorem 2.1.10].

**Lemma 3.1.** Assume that the net \( \{ \nu_z \}_{z > 0} \) is exponentially tight and let \( \mathbb{I} \) represent an LD limit point of \( \{ \nu_z \}_{z > 0} \). Let \( \Phi_\varepsilon \) be a net of uniformly bounded real-valued functions on \( \mathcal{S} \) such that \( \int_{\mathcal{S}} \exp\left( \frac{1}{\varepsilon} \Phi_\varepsilon(z) \right) \nu_z(dz) = 1 \). If \( \Phi_\varepsilon \) converges to \( \Phi \) uniformly on compact sets (as \( \varepsilon \to 0 \)) with the function \( \Phi \) being continuous, then \( \sup_{z \in \mathcal{S}}(|\Phi(z) - \mathbb{I}(z)|) = 0 \).

As in the proof of (3.2) and (3.3) in Section 3.1, it is not difficult to obtain from Assumption 2.1 and the fact \( \varphi_{t\wedge \tau(\varphi,\mu)} \) is bounded function of \( (\varphi,\mu) \) that there is a finite constant \( C \), which is independent of \( \varepsilon \) such that for all small enough \( \varepsilon \)

\[
|\Gamma_{t\wedge \tau(\varphi,\mu)}^{\varepsilon,\beta}(\varphi,\mu)| \leq C \varepsilon \text{ uniformly over } (\varphi,\mu).
\]

Similarly, there is a constant \( C \) such that for \( \varepsilon \) sufficiently small,

\[
|\Psi_{t\wedge \tau(\varphi,\mu)}^{\varepsilon,\beta,f}(\varphi,\mu)| < C \text{ uniformly over } (\varphi,\mu).
\]

As a result, one has

\[
\Gamma_{t\wedge \tau(\varphi,\mu)}^{\varepsilon,\beta}(\varphi,\mu) + \varepsilon \Psi_{t\wedge \tau(\varphi,\mu)}^{\varepsilon,\beta,f}(\varphi,\mu) \to 0 \tag{3.34}
\]

as \( \varepsilon \to 0 \) uniformly in compact sets. Finally, by assumption (2.8), we have

\[
\Phi_{t\wedge \tau(\varphi,\mu)}^{\varepsilon,\beta,f}(\varphi,\mu) \to \Phi_{t\wedge \tau(\varphi,\mu)}^{\beta,f}(\varphi,\mu) \tag{3.35}
\]

as \( \varepsilon \to 0 \) uniformly in compact sets. Combining (3.33) and (3.34) and then applying Lemma 3.1 yields (3.26). Then, it follows immediately that \( \mathbb{I}^*(\varphi,\mu) \leq \mathbb{I}(\varphi,\mu) \) for all \( (\varphi,\mu) \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^d) \times \mathcal{C}_1(\mathbb{R}^+,\mathcal{M}(\mathbb{R}^d)) \). The proof is complete.

### 3.2.2 Upper Bound of Large Deviations Limits

Let \( \mathbb{I} \) be a large deviations limit point of \( \{(X^\varepsilon,\mu^\varepsilon)\}_{\varepsilon > 0} \) such that \( \mathbb{I}(\varphi,\mu) = \infty \) unless \( \varphi_0 = \hat{x} \), a preselected element of \( \mathbb{R}^d \). In this section, we aim to prove that \( \mathbb{I}(\varphi,\mu) \leq \mathbb{I}^*(\varphi,\mu) \), for any \( (\varphi,\mu) \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^d) \times \mathcal{C}_1(\mathbb{R}^+,\mathcal{M}(\mathbb{R}^d)) \) such that \( \varphi_0 = \hat{x} \). The completion of the proof will be given later in Section 3.3. With the results established in Sections 3.1 and 3.2.1, this part can be done similarly to that of [22, Section 6, 7, 8] because the rate function has a similar variational representation. Although our \( \mathbb{I}^* \) is defined as the supremum in a smaller space than in [22], we can still prove \( \mathbb{I} \leq \mathbb{I}^* \) by a similar argument as in [22]. We will only sketch the main ideas here, and indicate the difference points only, whereas similar arguments will be referred to [22, Section 6, 7, 8].

It is obvious that it suffices to consider the case \( \mathbb{I}^*(\varphi,\mu) < \infty \). Therefore, we should investigate the regularity of \( (\varphi,\mu) \) provided \( \mathbb{I}^*(\varphi,\mu) < \infty \) first. It is shown in [22, Section 6] that if \( (\varphi,\mu) \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^d) \times \mathcal{C}_1(\mathbb{R}^+,\mathbb{R}^d), \mathbb{I}^*(\varphi,\mu) < \infty \) then \( \mu(ds,dy) = m_s(y)dyds \) and \( \varphi_s \) is absolutely continuous.
so that

Now, we proceed to the proof, which contains two main steps.

**Step 1:** Identify the LD limits at sufficiently regular (dense) points.

(w.r.t Lebesgue measure on \( \mathbb{R}_+ \)), \( m_s(y) \) is a probability density function in \( \mathbb{R}^l \). In this case, \( I^* \) has the following representation

\[
I^*(\varphi, \mu) = \int_0^\infty \left( \sup_{\beta \in \mathbb{R}^l} \left( \beta^T \hat{\varphi}_s - \beta^T \int_{\mathbb{R}^l} \frac{F_s(\varphi_s, y)m_s(y)}{\lambda_s(\varphi_s, y)} \, dy \right) + \sup_{h \in C_0^1(\mathbb{R}^l)} \int_{\mathbb{R}^l} [\nabla y h(y)]^T \left( \frac{1}{2} \text{div}(\Sigma_s(\varphi_s, y)m_s(y)) - b_s(\varphi_s, y)m_s(y) \right) \, dy \right) - \frac{1}{2} \int_{\mathbb{R}^l} \left\| \nabla y h(y) \right\|_{\Sigma_s(\varphi_s, y)}^2 m_s(y) \, dy \right) ds.
\]

(3.36)

In the above, \( L_{1,2}^0(\mathbb{R}^l, \mathbb{R}^l, \Sigma_s(\varphi_s, y), m_s(y)dy) \) represents the closure of the set of the gradients of functions from \( C_0^\infty(\mathbb{R}^l) \) in \( L_2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_s(\varphi_s, y), m_s(y)dy) \), here \( L_2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_s(\varphi_s, y), m_s(y)dy) \) is the Hilbert space of all \( \mathbb{R}^l \)-valued functions (of \( y \)) in \( \mathbb{R}^l \) endowed with the norm \( \| f \|_{\Sigma, m}^2 = \int_{\mathbb{R}^l} \| f(y) \|_{\Sigma_s(\varphi_s, y)}^2 m_s(y)dy \); and for each \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \), each function \( m(\cdot) \) being a probability density function in \( \mathbb{R}^l \), \( \mathcal{J}_{t,m(\cdot),u} \) is a function of \( y \) defined by

\[
\mathcal{J}_{t,m(\cdot),u}(y) = \Pi_{\Sigma_t(x, \cdot),m(\cdot)}(\Sigma_t(x, y)^{-1}(b_t(x, y) - \text{div}_x \Sigma_t(x, y)/2)),
\]

where \( \Pi_{\Sigma_t(x, \cdot),m(\cdot)} \) maps a function \( \phi(y) \in L_2^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_t(x, y), m(y)dy) \) (the space consists of functions whose products with arbitrary \( C_0^\infty \)-functions belong to \( L_2^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_t(x, y), m(y)dy) \)) to a function \( \Pi_{\Sigma_t(x, \cdot),m(\cdot)} \phi(y) \), which belongs to \( L_{1,2}^0(\mathbb{R}^l, \mathbb{R}^l, \Sigma_t(x, y), m(y)dy) \) and satisfies that, for all \( h \in C_0^\infty(\mathbb{R}^l) \),

\[
\int_{\mathbb{R}^l} [\nabla y h(y)]^T \Sigma_t(x, y) \Pi_{\Sigma_t(x, \cdot),m(\cdot)} \phi(y)m(y)dy = \int_{\mathbb{R}^l} [\nabla y h(y)]^T \Sigma_t(x, y) \phi(y)m(y)dy.
\]

If \( \phi(y) \in L_2^2(\mathbb{R}^l, \mathbb{R}^l, \Sigma_t(x, y), m(y)dy) \), then \( \Pi_{\Sigma_t(x, \cdot),m(\cdot)} \phi(y) \) is nothing than the orthogonal projection of \( \phi \) onto \( L_{1,2}^0(\mathbb{R}^l, \mathbb{R}^l, \Sigma_t(x, y), m(y)dy) \). Moreover, it is readily seen from (3.36) that

\[
\text{if } I^*(\varphi, \mu) < \infty \text{ then } \hat{\varphi}_s = \int_{\mathbb{R}^l} F_s(\varphi_s, y) m_s(y)dy.
\]

(3.37)

In addition, the supremum in the last term in (3.36) is attained at

\[
\hat{g}(y) = \frac{\nabla y m_s(y)}{2m_s(y)} - \mathcal{J}_{s,m_s(\cdot),\varphi_s}(y)
\]

(3.38)

so that

\[
I^*(\varphi, \mu) = \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^l} \left\| \frac{\nabla y m_s(y)}{2m_s(y)} - \mathcal{J}_{s,m_s(\cdot),\varphi_s}(y) \right\|_{\Sigma_s(\varphi_s, y)}^2 m_s(y)dyds.
\]

(3.39)

Now, we proceed to the proof, which contains two main steps.

**Step 1:** Identify the LD limits at sufficiently regular (dense) points.
Theorem 3.3. Assume that the assumptions of Theorem \[\text{2.1}\] hold. Let \(\hat{I}\) be a LD limit point of \(\{X^\epsilon, \mu^\epsilon\}\) such that \(\int (\varphi, \mu) = \infty\) unless \(\varphi = \hat{x}\). Let \((\hat{\varphi}, \hat{\mu}) \in \mathcal{G}\) be such that \(\hat{\varphi}_0 = \hat{x}\) and \(\hat{\mu}(ds, dy) = \hat{m}_s(dy)ds\), where \(\hat{m}_s(y)\) has the form

\[
\hat{m}_s(y) = M_s\left(\hat{m}_s(y)\tilde{\gamma}(\frac{|y|}{r}) + e^{-a|y|}\left(1 - \tilde{\gamma}(\frac{|y|}{r})\right)\right),
\]

where \(\hat{m}_s(y)\) is a probability density in \(y\) locally bounded away from zero and belonging to \(\mathcal{C}^1(\mathbb{R}^l)\) as a function of \(y\) with \(|\nabla \hat{m}_s(y)|\) being locally bounded in \((s, y)\), and \(\tilde{\gamma}(y)\) is a nonincreasing \([0, 1]-\)-valued \(\mathcal{C}^1(\mathbb{R}_+)\)-function, with \(y \in \mathbb{R}_+\), that equals 1 for \(y \in [0, 1]\) and equals 0 for \(y \geq 2\); \(r > 0\) and \(a > 0\), and \(M_s\) is the normalizing constant. For given \(\hat{m}_s(y), \tilde{\gamma}(y),\) and \(r\), there exists \(a_0 > 0\) such that for all \(a > a_0\), \(\hat{I}(\varphi, \mu) = \hat{I}(\hat{\varphi}, \hat{\mu})\).

Technical lemmas. Before proving Theorem \[\text{3.3}\] we first need some technical lemmas. For \(\beta, h\) as in Section \[\text{3.2}\] we denote

\[
\tau_N^{\beta, h}(\varphi, \mu) = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \int_{\mathbb{R}^l} \left| \nabla y h(s, \varphi, y) \right| \Sigma(s, \varphi, y) ds, dy + \sup_{s \in [0, t]} |\varphi(s)| + t \geq N \right\}.
\]

Performing integrating by parts in \[\text{3.25}\] yields that

\[
\Phi_t^{\beta, h}(\varphi, \mu) := \int_0^t \left(\beta(s) \dot{\varphi}_s - [\beta(s)]^T \int_{\mathbb{R}^l} F_s(\varphi, y) m_s(y) \lambda_s(\varphi, y) \right) ds \\
+ \int_0^t \left| \nabla y h(s, \varphi, y) \right|^T \left( \frac{1}{2} \text{div}(\Sigma_s(\varphi, y) m_s(y)) - b_s(\varphi, y) m_s(y) \right) dy \\
- \frac{1}{2} \int_0^t \left| \nabla y h(s, \varphi, y) \right|^2 \Sigma_s(\varphi, y) m_s(y) dy ds,
\]

Let

\[
\theta_N^{\beta, h}(\varphi, \mu) := \Phi_N^{\beta, h}(\varphi, \mu),
\]

and for each \(\delta > 0\), \(K_\delta := \{(\varphi, \mu) : \hat{I}(\varphi, \mu) \leq \delta\}\). The following are some technical lemmas needed for the proof of Theorem \[\text{3.3}\].

Lemma 3.2. (\[\text{22}\] Lemma 7.1) (Approximation of \(\tau, \theta\)) Under the following conditions for the boundedness, and the convergence (uniformly in \(K_\delta\)) of \(\{\beta_s^i\}_{i=1}^\infty, \{h_s^i(x, y)\}_{i=1}^\infty\) to \(\beta_s, h_s(x, y)\):

\[
\int_0^N |\beta_s| ds + \int_0^N \sup_{(\varphi, \mu \in K_\delta)} \int_{\mathbb{R}^l} |\nabla h_s(\varphi, y)| m_s(y) dy ds < \infty,
\]

and

\[
\lim_{i \to \infty} \int_0^N |\beta_s - \beta_s^i| ds + \lim_{i \to \infty} \sup_{(\varphi, \mu \in K_\delta)} \int_0^N \int_{\mathbb{R}^l} |\nabla h_s(\varphi, y) - \nabla h_s^i(\varphi, y)|^2 m_s(y) dy ds = 0,
\]

we have the convergence

\[
\tau_N^{\beta, h,i}(\varphi, \mu) \to \tau_N^{\beta, h}(\varphi, \mu) \quad \text{and} \quad \theta_N^{\beta, h,i}(\varphi, \mu) \to \theta_N^{\beta, h}(\varphi, \mu) \quad \text{uniformly in} \ K_\delta.
\]
Proof of Theorem 3.3 Let $a_0$ and then $\tilde{w}_s(x,y)$ be as in Lemma 3.4 for $\tilde{m}_s(y)$. Let $\tilde{h} = 0$ and

$$\tilde{h}_s(x,y) = \frac{1}{2} \ln \tilde{m}_s(y) - \tilde{w}_s(x,y).$$

Then

$$\nabla \tilde{h}_s(x,y) = \frac{\nabla \tilde{m}_s(y)}{2\tilde{m}_s(y)} - \nabla \tilde{w}_s(x,y).$$

We want to apply Lemma 3.3 for $\tilde{h}_s(x,y)$. However, $\tilde{h}_s(x,y)$ might not have a compact support in $y$. Hence, in order to use Lemma 3.3, we need to restrict it to a compact set. Therefore, we shall truncate $\tilde{h}_s(x,y)$. Let $\eta(t)$ represent an $\mathbb{R}_+$-valued nonincreasing $C^0_0(\mathbb{R}_+)$-function such that $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. Let $\tilde{w}_s(x,y) = \tilde{w}_s(x,y)\eta(|y|)$ and

$$\tilde{h}_s(x,y) = \frac{1}{2} \eta \left( \frac{|y|}{\epsilon} \right) \ln \tilde{m}_s(y) - \tilde{w}_s(x,y).$$

As in [22] Lemma 7.4, we can prove that $\tilde{h}_s(x,y)$ satisfies the conditions in Lemma 3.3.

Next, given $N \in \mathbb{N}$, let $t_N^{\tilde{h}_s}$ and $\theta_N^{\tilde{h}_s}$ be defined by the respective equations (3.41) and (3.43) with $\tilde{h} = \tilde{h}_s(x,y)$. Since the functions $\tilde{h} = 0$ and $\tilde{h}_s(x,y)$ satisfy the hypothesis in Lemma 3.3, there exists $(\varphi^{N,i}, \mu^{N,i}) \in \mathcal{G}$ such that $\theta_N^{\tilde{h}_s}(\varphi^{N,i}, \mu^{N,i}) = \tilde{h}(\varphi^{N,i}, \mu^{N,i})$ and $(\varphi^{N,i}, \mu^{N,i}) \in K_{2N+2}$ for all $i$. In particular, $\mu^{N,i}(ds,dy) = m_s^{N,i}(y)dyds$, where $m_s^{N,i}(-)$ belongs to $\mathcal{P}(\mathbb{R}_+)$, and the set $\{(\varphi^{N,i}, \mu^{N,i}), i = 1, 2, \ldots \}$ is relatively compact. Since $\tilde{h}(\varphi^{N,i}, \mu^{N,i}) \geq \tilde{h}(\varphi^{N,i}, \mu^{N,i})$ and $\theta_N^{\tilde{h}_s}(\varphi^{N,i}, \mu^{N,i}) \leq \tilde{h}(\varphi^{N,i}, \mu^{N,i})$, one has

$$\theta_N^{\tilde{h}_s}(\varphi^{N,i}, \mu^{N,i}) = \tilde{h}(\varphi^{N,i}, \mu^{N,i}).$$

Extract a convergent subsequence (still denoting the index by $i$) $\mu^{N,i} \rightarrow \mu^N$ in $C_1(\mathbb{R}_+, \mathcal{M}(\mathbb{R}_+))$ and $\varphi^{N,i} \rightarrow \varphi^N$ in $C(\mathbb{R}_+, \mathbb{R}^d)$. 

Lemma 3.4. ([22] Lemma 7.3) (Regularities and growth-rate properties of a certain (dense) class) Assume $m_s(y)$ is an $\mathbb{R}_+$-valued measurable function and is a probability density in $y$ for almost every $s$ and is bounded away from zero on bounded sets of $(s,y)$ and is in $C^1(\mathbb{R}_+)$, with $|\nabla m_s(y)|$ being locally bounded in $(s,y)$, and $m_s(y) = M_se^{-a|y|}$ ($a > 0$) for all $|y|$ large enough locally uniformly in $s$. There exists an $\alpha_0$ such that if $\alpha > \alpha_0$, there is a $w_s(x,\cdot)$ such that $J_{s,w_s(-)} = \nabla w_s(x,\cdot)$ and satisfies certain regularity and growth-rate properties [22] (7.13)-(7.15)].
Because of (3.46) and (3.42), \( \Pi^*(\varphi^{N,i}, \mu^{N,i}) \) obtains supremum at \( \hat{h}^0_{\lambda}(x, y) \) when \( s \leq \tau_{N}^{0,\hat{h}}(\varphi^{N,i}, \mu^{N,i}) \).
Therefore, by using (3.38), we can characterize \( m^{N,i}_{s} \) (noted that \( \mu^{N,i}(ds, dy) = m^{N,i}_{s}(y)dyds \)) and then show that the convergence of (3.44) and (3.45) in the hypothesis of Lemma 3.2 are satisfied (see [22] (7.46)-(7.48)). Thus, by Lemma 3.2 we have that \( \tau_{N}^{0,\hat{h}}(\varphi^{N,i}, \mu^{N,i}) \rightarrow \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \) as \( i \rightarrow \infty \), and that \( m^{N,i}_{s}(y) \rightarrow \hat{m}_{s}(y) \) in \( \mathbb{L}^1([0, \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N})] \times \mathbb{R}) \). Therefore, \( \mu^{N}(ds, dy) = \hat{m}_{s}(y)dyds \) for almost all \( s \leq \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \).

Using \( \dot{\varphi}^{N,i}_{s} = \int_{\mathbb{R}} \frac{F_{s}(\varphi^{N,i}_{s}, y)\mu^{N,i}(y)}{\lambda_{s}(\varphi^{N,i}_{s}, y)} dy \) due to (3.37) and applying [22] Lemma 6.7, we obtain from the convergence of \( \varphi^{N,i} \rightarrow \varphi^{N} \) in \( \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{d}) \) and \( m^{N,i}_{s}(y) \rightarrow \hat{m}_{s}(y) \) in \( \mathbb{L}^1([0, \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N})] \times \mathbb{R}) \) as \( i \rightarrow \infty \) that \( \dot{\varphi}^{N} = \int_{\mathbb{R}} \frac{F_{s}(\varphi^{N}_{s}, y)\mu^{N}(y)}{\lambda_{s}(\varphi^{N}_{s}, y)} dy \) a.e. for \( s \leq \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \). By the uniqueness, \( \varphi^{N} = \dot{\varphi}^{N} \) for \( s \leq \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \). As a byproduct, \( \dot{\varphi}^{N,i}_{s} \rightarrow \dot{\varphi}^{N}_{s} \) as \( i \rightarrow \infty \) a.e. on \( [0, \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N})] \).

We have proved that \( \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) = \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \) and \( \dot{\varphi}^{N} = \dot{\varphi}^{N} \) so that \( \theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) = \theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \). We can show that
\[
\theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) = \lim_{i \rightarrow \infty} \theta_{N}^{0,\hat{h}}(\varphi^{N,i}, \mu^{N,i}).
\]

Therefore, taking the limit in (3.46), we have \( \Pi^*(\varphi^{N}, \mu^{N}) \geq \theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \geq \hat{H}(\varphi^{N}, \mu^{N}) \), which together with the fact \( \Pi^* \leq \hat{H} \) obtained in previous section implies that
\[
\Pi^*(\varphi^{N}, \mu^{N}) = \theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) = \hat{H}(\varphi^{N}, \mu^{N}).
\]

Therefore, we have for all \( N > 0 \),
\[
\Pi^*(\varphi^{N}, \mu^{N}) = \theta_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) = \hat{H}(\varphi^{N}, \mu^{N}).
\]
(3.47)

From (3.47), the fact \( \varphi^{N} = \varphi^{N} \) until \( \tau_{N}^{0,\hat{h}}(\varphi^{N}, \mu^{N}) \) and the fact \( \hat{H} \) is lower semi-continuous and inf-compact, we obtain
\[
\Pi^*(\varphi^{N}, \mu^{N}) \geq \hat{H}(\varphi^{N}, \mu^{N}).
\]

As a result, we can conclude that \( \Pi^*(\varphi^{N}, \mu^{N}) = \hat{H}(\varphi^{N}, \mu^{N}) \) for all \( (\varphi^{N}, \mu^{N}) \) satisfying the requirements in Theorem 3.3.

**Step 2: Approximating the LD limits in arbitrary points by regular points.** Let \( \hat{H} \) be a LD limit point of \( \{(X^{\varepsilon}, \mu^{\varepsilon})\}_{\varepsilon > 0} \) and be such that \( \hat{H}(\varphi, \mu) = \infty \) unless \( \varphi_{0} = \hat{x} \). In this step, it is proven (see [22] Theorem 8.1]) if \( \Pi^*(\varphi, \mu) < \infty \), then there exists a sequence \((\varphi^{k}, \mu^{k})\) whose elements have the properties as in Theorem 3.3 such that \( (\varphi^{k}, \mu^{k}) \rightarrow (\varphi, \mu) \) and \( \Pi^*(\varphi^{k}, \mu^{k}) \rightarrow \Pi^*(\varphi, \mu) \) as \( k \rightarrow \infty \). Therefore, one has
\[
\hat{H}(\varphi, \mu) \leq \Pi^*(\varphi, \mu) = \lim_{k \rightarrow \infty} \Pi^*(\varphi^{k}, \mu^{k}) = \lim_{k \rightarrow \infty} \hat{H}(\varphi^{k}, \mu^{k}) \geq \hat{H}(\varphi, \mu).
\]

Hence, we have obtained desired properties in this Section.

### 3.3 Completion of the Proofs of Main Results

**Proof of Theorem 2.1** We will complete the proof of Theorem 2.1 by removing the restriction that \( \hat{H}(\varphi, \mu) = \infty \) unless \( \varphi_{0} = \hat{x} \) in Section 3.2 where \( \hat{x} \) is a preselected element such that \( \mathbb{L}_{0}(\hat{x}) = 0 \). This can be done similarly as in [22] Section 9 which will be omitted here. □
4 Proof of Theorem 2.2

The proof of this theorem is based on the fact that the exponential tightness and local LDP implies the full LDP. The following result is well-known in large deviations theory; see, e.g., [6, 7, 17].

**Proposition 4.1.** The exponential tightness and the local LDP for a family \( \{X^\varepsilon\}_{\varepsilon > 0} \) in \( C([0, 1], \mathbb{R}^d) \) with local rate function \( J \) imply the full LDP in \( C([0, 1], \mathbb{R}^d) \) for this family with rate function \( J \).

In what follows, we prove the LDP of \( \{X^\varepsilon\}_{\varepsilon > 0} \) in \( C([0, 1], \mathbb{R}^d) \). It will be seen that it can be extended to the space \( C([0, T], \mathbb{R}^d) \) endowed with the sup-norm topology for any \( T > 0 \). As a consequence, the LDP still holds in \( C([0, \infty), \mathbb{R}^d) \), the space of continuous function on \( [0, \infty) \) endowed with the local supremum topology. (This fact follows from the Dawson-Gärtner theorem; see [7, Theorem 4.6.1], which states that it suffices to check the LDPs in \( C([0, T], \mathbb{R}^d) \) for any \( T \) in the uniform metric.) We will still use \( C \) to represent a generic positive constant that is independent of \( \varepsilon \). The value \( C \) may change at different appearances; we will specify which parameters it depends on if it is necessary.

**Exponential tightness.** We aim to prove (3.2) and (3.3). We have

\[
X^\varepsilon_t = x_0^\varepsilon + \int_0^t x_0^\varepsilon e^{-A_t^\varepsilon(s)}ds + \frac{1}{\varepsilon^2} \int_0^t \int_0^s e^{-A_t^\varepsilon(s, r)} F_s^\varepsilon(X^\varepsilon_r, \xi^\varepsilon_{r/\varepsilon})dr,
\]

(4.1)

where for any \( 0 \leq s \leq t \leq 1, \varepsilon > 0, \)

\[
A_t^\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda^\varepsilon_r(X^\varepsilon_r, \xi^\varepsilon_{r/\varepsilon})dr, \quad A_t^\varepsilon(t) = A_t^\varepsilon(t, 0).
\]

So, we can obtain from some direct calculations and Assumption 2.3 that

\[
|X^\varepsilon_t| \leq |x_0^\varepsilon| + C\varepsilon^2|x_1^\varepsilon| + C \int_0^t \sup_{r \in [0, s]} |F_r^\varepsilon(X^\varepsilon_r, \xi^\varepsilon_{r/\varepsilon})|ds,
\]

(4.2)

and by noting further that \( \int_s^t e^{-\frac{r^2}{2\sigma^2}}dr \leq C\varepsilon^2(1 - e^{-\frac{t^2}{2\sigma^2}}) \leq C\varepsilon\sqrt{|t-s|} \), we get

\[
|X^\varepsilon_t - X^\varepsilon_s| \leq C\varepsilon|x_1^\varepsilon|\sqrt{|t-s|} + C|t-s| \sup_{r \in [s, t]} |F_r^\varepsilon(X^\varepsilon_r, \xi^\varepsilon_{r/\varepsilon})|.
\]

(4.3)

If (2.15) in Assumption 2.3 holds, (3.2) follows immediately from (4.2) and Gronwall’s inequality on noting that \( \lim\sup_{\varepsilon \to 0} \varepsilon|x_1^\varepsilon| < \infty \) a.s., \( \{x_0^\varepsilon\}_{\varepsilon > 0} \) is exponentially tight; and then (3.3) follows from (3.2) and (4.3).

Otherwise, assume that (2.16) holds. Let \( \tilde{C}_N \) be constant in (2.15) uniformly in \( |y| < N \). We get from (1.2) that \( \sup_{t \in [0, 1]} |X^\varepsilon_t| \leq C(\tilde{C}_N + N)e^{\tilde{C}_N} \) provided \( \sup_{t \in [0, 1]} |\xi^\varepsilon_{t/\varepsilon}| < N, |x_0^\varepsilon| < N \). Therefore, for any \( N > 0 \), for \( L > C(\tilde{C}_N + N)e^{\tilde{C}_N} \) one has

\[
\mathbb{P}^\varepsilon(\sup_{t \in [0, 1]} |X_t| > L) \leq \mathbb{P}^\varepsilon(|x_0^\varepsilon| > N) + \mathbb{P}^\varepsilon(\sup_{t \in [0, 1]} |\xi^\varepsilon_{t/\varepsilon}| > N).
\]

Letting \( L \to \infty \) and \( N \to \infty \) and using the logarithm equivalence principle [7, Lemma 1.2.15] and (2.16) in Assumption 2.3 we get (3.2). Thus, we also obtain (3.3).

**Local LDP.** It is noted that we do not assume any structure of \( \xi^\varepsilon_{t/\varepsilon} \). As a result, we could not use the integration by parts (Lemma 3.1) to connect the first-order and the second-order systems. Therefore, we will establish a relationship in a local sense.
For each continuous function $\varphi$, we introduce the auxiliary processes $X_{t}^{\varepsilon,\varphi}$, the solution of the following equation

$$
\varepsilon^{2}\ddot{X}_{t}^{\varepsilon,\varphi} = F_{t}^{\varepsilon}(\varphi_t, \xi_{t/\varepsilon}^{\varepsilon}) - \lambda_{t}^{\varepsilon}(\varphi_t, \xi_{t/\varepsilon}^{\varepsilon})\dot{X}_{t}^{\varepsilon,\varphi}, \quad X_{0}^{\varepsilon} = x_{0}, \quad \dot{X}_{0}^{\varepsilon} = x_{1},
$$

(4.4)

and $Z_{t}^{\varepsilon,\varphi}$, the solution of

$$
Z_{t}^{\varepsilon,\varphi} = \frac{F_{t}^{\varepsilon}(\varphi_t, \xi_{t/\varepsilon}^{\varepsilon})}{\lambda_{t}^{\varepsilon}(\varphi_t, \xi_{t/\varepsilon}^{\varepsilon})}, \quad Z_{0}^{\varepsilon,\varphi} = x_{0}^{0}.
$$

(4.5)

We have from (4.4) and the variation of parameters formula that

$$
X_{t}^{\varepsilon,\varphi} = x_{0} + \int_{0}^{t} x_{1}^{\varepsilon}e^{-A_{t}^{\varepsilon,\varphi}(s,r)}ds + \frac{1}{\varepsilon^{2}}\int_{0}^{t} \int_{0}^{s} e^{-A_{t}^{\varepsilon,\varphi}(s,r)}F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})dr,
$$

(4.6)

where for any $0 \leq s \leq t \leq 1, \varepsilon > 0,$

$$
A_{t}^{\varepsilon,\varphi}(t, s) := \frac{1}{\varepsilon^{2}}\int_{s}^{t} \lambda_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})dr, \quad A_{t}^{\varepsilon,\varphi}(t) = A_{t}^{\varepsilon,\varphi}(t, 0).
$$

From the fact that $A_{t}^{\varepsilon}(s, r), A_{t}^{\varepsilon,\varphi}(s, r) \geq \frac{-\kappa_{0}(s-r)}{\varepsilon^{2}},$ and the property of $\lambda,$ we obtain that

$$
\left|e^{-A_{t}^{\varepsilon}(s, r)} - e^{-A_{t}^{\varepsilon,\varphi}(s, r)}\right| \leq C e^{-\frac{-\kappa_{0}(s-r)}{\varepsilon^{2}}} \cdot \frac{1}{\varepsilon^{2}}\int_{s}^{t} |X_{r}^{\varepsilon} - \varphi_{r}| dr = C e^{-\frac{-\kappa_{0}(s-r)}{\varepsilon^{2}}} \cdot \frac{s - r}{\varepsilon^{2}} \cdot \sup_{r \in [0, s]} |X_{r}^{\varepsilon} - \varphi_{r}|.
$$

(4.7)

A change of variable leads to

$$
\int_{0}^{s} e^{-\frac{-\kappa_{0}(s-r)}{\varepsilon^{2}}} \cdot \frac{s - r}{\varepsilon^{2}} dr = e^{2} \int_{0}^{s} e^{-\kappa_{0}r} r dr \leq C e^{2}.
$$

(4.8)

Therefore, we obtain from the Lipschitz property of the coefficients and (4.7) that

$$
\int_{0}^{s} e^{-A_{t}^{\varepsilon,\varphi}(s,r)}F_{r}^{\varepsilon}(X_{r}^{\varepsilon,\varphi}, \xi_{r/\varepsilon}^{\varepsilon}) - e^{-A_{t}^{\varepsilon}(s,r)}F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon}) \right| dr
\leq \int_{0}^{s} e^{-A_{t}^{\varepsilon}(s,r)} \left|F_{r}^{\varepsilon}(X_{r}^{\varepsilon,\varphi}, \xi_{r/\varepsilon}^{\varepsilon}) - F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon}) \right| dr + \int_{0}^{s} \left|F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})\right| \left|e^{-A_{t}^{\varepsilon}(s,r)} - e^{-A_{t}^{\varepsilon,\varphi}(s,r)}\right| dr
\leq C e^{2} \sup_{0 \leq r \leq s} |X_{r}^{\varepsilon,\varphi} - \varphi_{r}| + C e^{2} \sup_{r \in [0, s]} |F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})| \sup_{0 \leq r \leq s} |X_{r}^{\varepsilon} - \varphi_{r}|.
$$

(4.9)

Combining (4.1), (4.6), and applying (4.9) and noting that $\lim_{\varepsilon \to 0} \varepsilon |x_{1}^{\varepsilon}| < \infty$ leads to

$$
\sup_{s \in [0, t]} |X_{s}^{\varepsilon,\varphi} - X_{s}^{\varepsilon}| \leq C e \sup_{s \in [0, t]} |X_{s}^{\varepsilon} - \varphi_{s}| + C \sup_{r \in [0, t]} |F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})| \int_{0}^{t} \sup_{r \in [0, s]} |X_{r}^{\varepsilon} - \varphi_{r}| ds.
$$

(4.10)

From (4.4) and (4.5), we obtain that

$$
|X_{t}^{\varepsilon,\varphi} - Z_{t}^{\varepsilon,\varphi}| = \left|\int_{0}^{t} \frac{\varepsilon^{2}}{\lambda_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})} \dot{X}_{r}^{\varepsilon,\varphi} dr\right| \leq C e^{2} \sup_{s \in [0, t]} |\dot{X}_{s}^{\varepsilon,\varphi}|
\leq C e^{2} (|x_{1}^{\varepsilon}| + \sup_{r \in [0, t]} |F_{r}^{\varepsilon}(\varphi_r, \xi_{r/\varepsilon}^{\varepsilon})|).
$$

(4.11)
Thus, for small $\varepsilon$ and $\delta > C$ for some constants $C_1$ and $C_2$, by letting $F$ and $N$ be defined as in Assumption 2.3, one also gets from (2.17), (4.5), and the Lipschitz continuity of $\lambda^\varepsilon$, the local LDP of $\{X^\varepsilon\}_{\varepsilon>0}$ follows directly from the local LDP of $\{Z^\varepsilon\}_{\varepsilon>0}$.

If (2.15) in Assumption 2.3 holds, combining (4.10), (4.11), and (4.12), we get

$$\sup_{s \in [0,t]} |X^\varepsilon_s - \varphi_s| \leq C \varepsilon \sup_{s \in [0,t]} |X^\varepsilon_s - \varphi_s| + C \varepsilon (1 + \sup_{r \in [0,1]} |\varphi_r|) + C (1 + \sup_{r \in [0,1]} |\varphi_r|) \sup_{r \in [0,1]} |Z^\varepsilon_r - \varphi_r|$$

and

$$\sup_{s \in [0,t]} |Z^\varepsilon_s - \varphi_s| \leq C \varepsilon (1 + \sup_{r \in [0,1]} |\varphi_r|) + C (1 + \sup_{r \in [0,1]} |\varphi_r|) \sup_{r \in [0,1]} |Z^\varepsilon_r - \varphi_r|$$

Thus, for small $\varepsilon$, one has

$$\sup_{t \in [0,1]} |X^\varepsilon_t - \varphi_t| \leq C_1 \varepsilon + C \varepsilon \sup_{t \in [0,1]} |Z^\varepsilon_t - \varphi_t|, \quad (4.13)$$

and

$$\sup_{t \in [0,1]} |Z^\varepsilon_t - \varphi_t| \leq C_2 \varepsilon + C \varepsilon \sup_{t \in [0,1]} |X^\varepsilon_t - \varphi_t|, \quad (4.14)$$

for some constants $C_{1,\varphi^2}, C_{2,\varphi}$ depending only on $\sup_{r \in [0,1]} |\varphi_r|$ and independent of $\varepsilon$. So, for any $\delta > 0$ we have from (4.13) and (4.14) that

$$\mathbb{P}^\varepsilon \left( \sup_{t \in [0,1]} |X^\varepsilon_t - \varphi_t| < \delta \right) \leq \mathbb{P}^\varepsilon \left( \sup_{t \in [0,1]} |Z^\varepsilon_t - \varphi_t| < 2\delta C_2,\varphi \right), \quad \forall \varepsilon < \delta/C_{2,\varphi},$$

and

$$\mathbb{P}^\varepsilon \left( \sup_{t \in [0,1]} |X^\varepsilon_t - \varphi_t| < \delta \right) \geq \mathbb{P}^\varepsilon \left( \sup_{t \in [0,1]} |Z^\varepsilon_t - \varphi_t| < \frac{\delta}{2C_{1,\varphi}} \right), \quad \forall \varepsilon < \delta/C_{1,\varphi}.$$
References

[1] Z. I. Botev, J. F. Grotowski, D. P. Kroese, Kernel density estimation via diffusion, *Ann. Statist.* 38 (2010), 2916–2957.
[2] A. Budhiraja, P. Dupuis, A. Ganguly, Large deviations for small noise diffusions in a fast Markovian environment, *Electron. J. Probab.* 23 (2018), no. 112, 1–33.
[3] S. Cerrai and M. Freidlin, Large deviations for the Langevin equation with strong damping, *J. Stat. Phys.* 161 (2015), 859–875.
[4] Z. Chen, M. I. Freidlin, Smoluchowski–Kramers approximation and exit problems, *Stoch. Dyn.* 5 (2005), 569–585.
[5] L. Cheng, R. Li, and W. Liu, Moderate deviations for the Langevin equation with strong damping, *J. Stat. Phys.* 170 (2018), 845–861.
[6] J.D. Deuschel and D.W. Stroock, *Large Deviations*, Academic Press, San Diego, 1989.
[7] A. Dembo, O. Zeitouni, *Large Deviations Techniques and Their Applications*, 2nd ed. Jones and Bartlett, Boston, 1998.
[8] J. Feng, T. Kurtz, *Large deviations for stochastic processes*, 2000.
[9] G. Fibich, A. Gavious, and E. Solan, Averaging principle for second-order approximation of heterogeneous models with homogeneous models, *PNAS*, 109 (48) (2012), 19545–19550; https://doi.org/10.1073/pnas.1206867109
[10] M. I. Freidlin, Some remarks on the Smoluchowski-Kramers approximation, *J. Stat. Phys.* 117 (2004), 617–634.
[11] M.I. Freidlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.
[12] A. Guillin, Averaging principle of SDE with small diffusion: Moderate deviations, *Ann. Probab.* 31 (2003), 413–443.
[13] Q. He, G. Yin, Large deviations for multi-scale Markovian switching systems with a small diffusion. *Asymptot. Anal.* 87 (2014), 123–145.
[14] H. Kesten, G. C. Papanicolaou, A limit theorem for turbulent diffusion, *Commun. math. Phys.* 65 (1979), 97-128.
[15] H. Kesten, G. C. Papanicolaou, A limit theorem for stochastic acceleration, *Commun. math. Phys.* 78 (1980), 19-63.
[16] H. J. Kushner, P. Dupuis, *Large deviations estimates for systems with small (wideband) noise effects and applications to stochastic systems and communication theory*, Probability Theory and Applications, 659–662, De Gruyter, 2020.
[17] R. S. Liptser, A. A. Puhalskii, Limit theorems on large deviations for semimartingales, *Stochastics Stochastics Rep.* 38 (1992), 201–249.
[18] R. S. Liptser, Large deviations for two scaled diffusions. *Probab. Theory Related Fields* 106 (1996), 71–104.
[19] X. Mao, *Stochastic differential equations and their applications*, Horwood Publishing chichester, 1997
[20] N. Nguyen, G. Yin, A Class of Langevin Equations with Markov Switching Involving Strong Damping and Fast Switching, *J. Math. Phys.* 61 (2020), 063301.
[21] N. Nguyen, G. Yin, Large deviations principles for Langevin equations in random environment and applications, *J. Math. Phys.*. 62 (2021), 083301.
[22] Anatoli A. Puhalskii, On large deviations of coupled diffusions with time scale separation, *Ann. Probab.* 44 (2016), 3111–3186.
[23] E. M. Purcell, Life at low Reynolds number, *Am. J. Phys.* 45, 3 (1977).
[24] H. Touchette, The large deviation approach to statistical mechanics, *Phys. Rep.* 478 (2009), 1–69.
[25] A. Yu. Veretennikov, On large deviations in the averaging principle for SDEs with a “full dependence”, *Ann. Probab.* 27 (1999), 284–296.
[26] A. Yu. Veretennikov, On large deviations for SDEs with small diffusion and averaging, *Stochastic Process. Appl.* 89 (2000), 69–79.