Noise and disorder effects in a series of birhythmic Josephson junctions coupled to a resonator

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This paper deals the effects of uncorrelated white noise, in a series of Josephson Junctions coupled to a linear RLC resonator. The junction are hysteretic, and hence can be considered birhythmic, that is capable to oscillate at different frequencies for the same set of parameters. Both Josephson Junctions with identical and disordered parameters are considered. With the uniform parameters, the array behaves similarly to single Josephson junctions, also in the presence of noise. The magnitude of the effective energy that characterizes the response to noise becomes smaller as the number of elements of the array increases, making the resonator less stable. Disorder in the parameters drastically changes the physics of the array. The disordered array of Josephson junctions misses the birhythmic properties for large values of the variance of the disorder parameter. Nevertheless, the system remains birhythmic for low values of the disorder parameter. Finally, disorder makes it difficult to locate the separatrix, hinting to a more complex structure of the effective energy landscape.

Keywords: Josephson junctions; Mean First Passage time, Synchronization

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I. INTRODUCTION

Birhythmic (large capacitance) Josephson Junction (JJ) can be considered a birhythmic system, they can produce oscillations at two distinct periods [1, 2]. Birhythmicity is encountered in some biochemical [3–7] and non linear electronic systems [8–13]. In JJ physics, it is encountered in arrays coupled through an external circuit that possesses resonances [14]. In this condition, the array can either oscillate at two frequencies, the one induced by the external resonance that locks together the JJ, or at the spontaneous frequency of JJ where the elements oscillate incoherently and are unable to load the external circuit. Synchronization of JJ oscillators is also of practical importance for applications in which the power of a single JJ does not suffice. In fact, the power of the coherently working junctions can increase as the number of elements of the array increases, making the resonator less stable. Disorder in the parameters drastically changes the physics of the array. The disordered array of Josephson junctions misses the birhythmicity properties for large values of the variance of the disorder parameter. Thus, disorder makes it difficult to locate the separatrix, hinting to a more complex structure of the effective energy landscape.

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II. MODEL OF A SERIE OF JOSEPHSON JUNCTIONS COUPLED TO A RESONATOR

This Section describes the basic model in Sect. II A and the refinements to include noise and disorder in Sect. II B. The numerical method is described in the last part, Sect. II C.

The electrical model consists of the capacitor $C_J$, the resistor $R_J$, and the ideal Josephson element, connected in series. The nonlinear relation between the current and the gauge invariant phase difference $\phi = \phi_1 - \phi_2$ across two superconductors:

$$I_j = I_0 \sin \phi^i,$$  \hspace{1cm} (1)

$$V_j = \frac{h}{2e} \frac{d\phi^i}{dt},$$  \hspace{1cm} (2)

determines that a JJ is an active oscillator that converts a dc current into an ac drive for the RLC resonator. As mentioned in the introduction, RLC coupling in parallel to the array is but a model for the external circuit that embeds the JJs. In fact, at the microwave frequency the lumped elements is an approximation of the distributed elements. In this context, the choice of series and parallel depends upon the relative magnitude of the impedance of the JJ; for low damping the external circuit “looks parallel”, while for high damping the external circuit “looks series” [14]. In this work we concentrate on

A further problem is to find such border that separates the two attractors, even in the limit of negligible noise, for a system of many oscillators, and hence of many degrees of freedom, the separatrix consists of a complicated hyper-surface in $N - 1$ dimensions, if $N$ is the number of degrees of freedom. To make the problem manageable, it has been proposed to approximate the position of the separatrix with the region where the slope of the mean first passage time changes [33]. However, it remains an open problem, the extension of the method to the case of many disordered JJs, that is the subject of the present work.

A. Model of Josephson junctions dynamics

Figure 1 schematically describes the model: a series of underdamped JJ coupled to an RLC resonator. Both the JJs and the resonant circuit are supposed in the temperature controlled vessel, while the bias current is supplied by a device typically at room temperature. In this configuration, the noise from the bias supply dominates respect to the Johnson noise from the resistors $R_J$ and $R$. Alternatively, one could add a random term for each resistor, as done for instance in Ref. [34]. However, the noise is but a tool. Our goal is to determine the pseudo-energy; the principle of minimum energy [29, 31] assures that the contributions from the minimal trajectory determines the height of the trapping potential, and therefore one does not expect substantial changes with a different noise source.

The paper is organized as follows. The next Section describes a serie of underdamped disordered Josephson Junctions; both connected to an RLC resonator and subject to external bias and noise. Section III focuses on the properties of the attractors in a birhythmic Josephson Junction array, and especially in the method to locate the effective separatrix, which is, as mentioned, the essential information to construct the activation energy barrier. The effective potential that determines the global stability of the system is studied. In Sect. IV the effects of disordered parameters are analysed on the birhythmic properties. Section V closes the paper with conclusions.
The current balance:
\[ I_n = I_a + I_n - I_C \Rightarrow I'_j + I_{R_j} + I_{C_j} = I_a + I_n - I_C. \] (3)

The Kirchhoff law for the loop voltage
\[ \sum_{i=1}^{N} V^j_i = V_C + V_R + V_L \] (4)

completes the model, that is thus described by two second order coupled differential equations:
\[
\begin{align*}
\frac{d^2\phi^j}{dt^2} + N \frac{da^j}{dt} + \sin \phi^j + \frac{da^j}{dt} &= I_a + I_n \\
\frac{d^2\phi^j}{dt^2} + \frac{N}{\Omega} \frac{da^j}{dt} + \sqrt{\frac{\beta_L}{\Omega}} q - \frac{1}{\Omega} \sum_{i=1}^{N} \frac{da^i}{dt} &= 0.
\end{align*}
\] (5)

Introducing the Josephson frequency \( \omega_j = \sqrt{2eI_0/C_j}\hbar \), Eqs. (5) can be cast, with normalized time \( \tau = \omega_j t \) and charge \( q = \omega_j \dot{q}/I_0 \), as follows:
\[
\begin{align*}
\frac{d^2\phi^j}{d\tau^2} + N \frac{da^j}{d\tau} + \sin \phi^j + \frac{da^j}{d\tau} &= \gamma_G + \zeta^i(\tau), \\
\frac{d^2\phi^j}{d\tau^2} + \frac{N}{\beta_L} \frac{da^j}{d\tau} + \Omega^2 q - \frac{1}{\beta_L} \sum_{i=1}^{N} \frac{da^i}{d\tau} &= 0,
\end{align*}
\] (6)

where the parameters are defined as
\[ Q = \frac{L\omega_j}{R}, \quad \alpha = \frac{1}{C_j\omega_j} \sqrt{\frac{1}{R_j}}, \quad \beta_L = \frac{\omega_j^2 C_j}{L}, \quad \gamma_G = \frac{I_a}{I_D}, \quad \Omega = \frac{1}{\omega_j}. \] (7)

The described coupling circuit is thus formed by two branches, a JJ and an RLC resonator in a loop, strongly interacting each other, as the coupling is large, \( 1/\beta_L = 100 \). The two branches are thus not completely independent, the resonance of the circuit, for instance, is away from the RLC resonance \( \Omega \), as shall be discussed in connection with Figs. 2 in the next Sect. IIIA.

The statistical features of the noisy term \( \zeta^i \) are determined by:
\[
\begin{align*}
< \zeta^i(\tau) > &= 0, \\
< \zeta^i(\tau); \zeta^j(\tau') > &= 4D\delta(\tau - \tau').
\end{align*}
\] (8)

where \( D = k_BT\omega_j/\left( R_{j}\rho_j^2 \right) \) is the normalized noise intensity \( (k_B) \) is the Boltzman constant and \( T \) the absolute temperature). If the Johnson noise associated to the resistances of each JJ are sizable, one has separated contributions \( \zeta^1, \zeta^2, ..., \zeta^N \) for the junctions. Also, Eq. (5) is changed, for the normalized noise intensity \( D \) is given by the resistance \( R_j \) of the junctions. Instead, if the noise associated to the superconducting JJs are negligible respect to the resonator, then the noise comes from the resonator resistance \( R \) and it is common to all the junctions. Finally, if the noise is due to an external source, it does not obey to the fluctuation-dissipation theorem and it is therefore independent of the resistance.

**B. Model of disorder**

In the fabrication process of several JJ, as in Fig. 1 fabrication tolerances prevent the junctions to be identical. More
time evolution for the free noise coupled JJ model, \( D = 0 \). Here \( q_s \) indicates the hypothesized position of the separatrix. The parameters of simulations are: \( \beta_L = 0.01 \), \( Q = 200 \), \( \Omega = 2.0 \), and \( \alpha = 0.1 \). For (i) and (ii): \( N = 2 \), \( \gamma_G = 1.50 \); for (iii) and (iv): \( N = 5 \), \( \gamma_G = 2.30 \).

The distribution of the critical currents of the JJ is here assumed uniform in an interval \( \pm \varepsilon \) around the average value, that we keep indicating with \( I_0 \). The choice of the uniform distribution arises from two considerations. The actual distribution of the parameters of junctions arrays, is close to Gaussian \([20,21]\). To avoid negative unphysical values of the critical current, the random variable \( \varepsilon \) is assumed uniform in an interval of amplitude \( 2\sigma \):

\[
\varepsilon \in [-\sigma;+\sigma],
\]

and consequently the distribution of the critical currents of the \( i^{th} \) junction reads

\[
I_0^i = I_0 (1 + \varepsilon^i).
\]

As a consequence of the relation \((9)\), the resistance reads

\[
R_j^i = \frac{\Delta}{e I_0 (1 + \varepsilon^i)}.
\]

The normalized parameters \((7)\) are unchanged, but for the dissipation \( \alpha \) that reads:

\[
\alpha^i = \frac{1}{C_j \omega_j} \sqrt{\frac{1}{R_j^i}} = \frac{1}{C_j \omega_j} \sqrt{\frac{\varepsilon I_0 (1 + \varepsilon^i)}{\Delta}} = \frac{1}{C_j \omega_j} \sqrt{\frac{(1 + \varepsilon^i)}{R_j^i}} = \alpha \sqrt{\frac{1}{\frac{\Delta}{e I_0}}}
\]

Finally, the equations for the disordered model Eqs. \((6)\) read

\[
\begin{align*}
\frac{d^2 \phi^i}{d\tau^2} + \alpha (1 + \varepsilon^i) \frac{d\phi^i}{d\tau} + (1 + \varepsilon^i) \sin \phi^i + \frac{dq^i}{d\tau} &= \gamma_G + \zeta^i(\tau), \\
\frac{d^2 q^i}{d\tau^2} + \frac{1}{Q} \frac{dq^i}{d\tau} + \Omega^2 q - \frac{\beta_L}{\beta_s} \sum_{i=1}^{N} \frac{d\phi^i}{d\tau} &= 0.
\end{align*}
\]

A stochastic model for the random parameters \( I_0^i \) and \( R_j^i \) is given by the assumptions \((11)\) and \((12)\), thus, the random parameters depend on a single parameter \( \varepsilon^i \) characterized by the uniform distribution \((10)\), whose variance is \( 2\sigma^2/3 \).

As a further simplification, we have just assume that the parameters are uniformly distributed in the interval \( [-\sigma;+\sigma] \), that is:

\[
\varepsilon^i = -\sigma + 2 (i - 1) \frac{\sigma}{N - 1}.
\]

As the Josephson fabrication tolerances are typically of the order of few percents \([20,21]\), one can safely assume \( \sigma = 0.1\% \div 10\% \).
C. Algorithm for the numerical solution

Equations (8,14) are simulated with the Euler algorithm [35]. Deterministic results have been obtained using the fourth order Runge Kutta algorithm [36]. The stochastic results are averaged over as many realizations as necessary to guarantee convergence in the statistical sense) within 5%. The Gaussian white noise is generated using the Box-Muller algorithm [37] from two random numbers, a and b, which are uniformly distributed on the unit interval [0, 1]. Thus, for each step $\Delta \tau$, $\zeta^n_i$ is obtained as follows:

$$ a = \text{random number}, \quad b = \text{random number}, \quad \zeta^n_i = \sqrt{-4D\Delta \tau \log(a)} \cos(2\pi b) \quad (16) $$

For some values of $N$ the IV curves have been obtained by slowly increasing the current with a step $\Delta \gamma_G = 0.01$, and using the final state at the previous current step as the initial step for the increased (or decreased) current biased (see Fig.2). At each current step a transient of about 1000 normalized time is discarded. The averages are also calculated over the same time. The time step $\Delta \tau$ is, through all simulations, $\Delta \tau = 0.0001$ for the Euler algorithm and $\Delta \tau = 0.01$ for the fourth order Runge Kutta algorithm.

III. DETERMINISTIC ATTRACTORS AND STABILITY PROPERTIES

The starting point to retrieve the effects of noise and disorder in the considered array of JJs coupled to a resonator,

FIG. 4: Example of the switch from the attractor at frequency $\Omega_1$ (locked to the RLC resonator) to the attractor at frequency $\Omega_3$ (unlocked state) under the influence of noise. After the time $\kappa = 12000$ normalized units the system crosses the estimated separatrix $q_s$. The parameters are: $\beta_L = 0.01$, $Q = 200$, $\Omega = 2.0$, $\alpha = 0.1$, $\gamma_G = 1.55$, $D = 0.15$, $N = 2$.

FIG. 5: Average MFPT as a function of a threshold $q_s$ at different values of applied current $\gamma_G$ and noise intensity $D$. It is evident that the method of the knee to identify the separatrix works in a variety of parameters. The parameters are: $\beta_L = 0.01$, $Q = 200$, $\Omega = 2.0$, $\alpha = 0.1$. For (i): $N = 2$, $\gamma_G = 1.50$; (ii): $N = 2$, $\gamma_G = 1.53$; (iii): $N = 5$, $\gamma_G = 2.30$; (iv): $N = 5$, $\gamma_G = 2.36$. 
are the properties of the uniform array. The dynamics of the ordered array is useful to build a first approximation of the system, e.g., to find the hysteretic IV and the two frequencies (locked and unlocked), as well as the approximate limit cycle and so on. This knowledge is essential to reconstruct the response of the system in the actual disordered and noisy case. It is important to notice that the information we are aiming to, the effective energy barrier or quasipotential, is defined in the limit of vanishingly noise, and it is therefore conceivable that the properties retrieved in the noiseless case are valid in the limit in which one calculates the quasipotential.

### A. Attractors properties

In Fig. 2 are shown the IV curves for arrays of 2 and 5 JJs, obtained increasing and decreasing the bias current $\gamma_G$. The vertical dashed line denotes a particular bias point the system exhibits two frequencies, $\Omega_1$ and $\Omega_3$ obtained increasing and decreasing the bias current, respectively. Thus, $\Omega_1$ denotes the frequency on the so-called unperturbed IV curve (some times referred to as McCumber branch) and $\Omega_3$ is related to the resonant frequency of the RLC circuit [19]. However, as the main purpose of this work is to illustrate the method of the quasipotential for JJs as birhythmic circuits, we have chosen a set of parameter that best fits this illustration. In particular, the high value of the coupling strength $1/\beta_L = 100$ tightly couples the JJs parameters to the RLC tanks resonator, that is therefore strongly influenced by the (nonlinear) inductance and other JJ parameters (capacitance and resistance). Thus, the resonance clearly appears to depend upon the number of JJs – see the shift to higher voltage in Fig. 2(ii). The model is anyway birhythmic, either for the case of a single Josephson junction or an array of JJ: The system exhibits oscillations at two distinct periods depending on the initial conditions. Table I summarizes the range of bias current corresponding to JJs locked to the RLC resonant state.

| $N$ | minimum value | maximum value |
|-----|---------------|---------------|
| 2   | 1.47          | 1.72          |
| 5   | 2.28          | 2.42          |

**TABLE I: Range of the bias current corresponding to JJ locked to the RLC resonant state.**
FIG. 8: Normalized IV curves of coupled disordered JJ for both increasing and decreasing bias current $\gamma_G$. The parameters are: $\beta_L = 0.01$, $Q = 200$, $\Omega = 2$, $\alpha = 0.1$, $N = 5$. (i): $\sigma = 1\%$; (ii): $\sigma = 2\%$; (iii): $\sigma = 3\%$; (iv): $\sigma = 4\%$.

FIG. 9: Normalized voltage as a function of the noise variance $\sigma$. The parameter are: $\beta_L = 0.01$, $Q = 200$, $\Omega = 2.0$, $\alpha = 0.1$, $N = 5$. (i): $\gamma_G = 2.30$; (ii): $\gamma_G = 2.36$.

The evolution of the charge $q$ reveals that a sudden switch occurs when the charge suddenly passes from oscillations around a higher value. During this jump, the charge crosses a threshold $q_s$, and for $q > q_s$ the charge increases and then oscillates around a new value (about $q \approx 780$ in the Figure). Thus one can roughly estimate the position of the separatrix; however there is clearly an arbitrary in identify $q_s$. To refine this guess, it is possible to exploit the properties of the MFPT. In fact the best estimate for the position of the separatrix is signaled by the change of the slope of the MFPT $\kappa$ as a function of the threshold $q_s$, as shown in Fig. 5. The rationale is the following: as the threshold $q_s$ approaches the actual position of the separatrix, the MFPT increases exponentially, for the quasi-potential in correspondence of $q_s$ increases. As the separatrix is passed, the MFPT increases much more weakly: beyond the maximum of the quasi-potential the time spent to reach the threshold $q_s$ is but the time to run downhill. Thus, if the threshold point $q_s$ is set before the separatrix the MFPT increases sharply, while it weakly increases when the threshold is beyond the separatrix; therefore from the change of slope one can estimate the separatrix position. The knee of the MFPT, denoted by the vertical dashed line in Fig. 5 is used as an effective separatrix to estimate the energy activation barrier.

The estimate of an energy activation barrier is practically implemented in Fig. 6 for different value of bias current (and
FIG. 10: Normalized IV curves for the disordered coupled JJ model for both increasing and decreasing bias current $\gamma G$. The parameters are: $n_L = 0.01$, $Q = 200$, $\Omega = 2.0$, $\alpha = 0.1$, $N = 5$, $\sigma = 0.1\%$.

FIG. 11: Example of a switch from the locked attractor at the RLC frequency $\Omega_1$ to the unlocked at the McCumber frequency $\Omega_3$, under the influence of noise for the disordered JJ. Between the initial time and the normalized time 5000 system crosses the separatrix $q_s$. The parameters are: $n_L = 0.01$, $Q = 200$, $\Omega = 2$, $\alpha = 0.1$, $\gamma G = 2.30$, $\sigma = 0.1\%$, $D = 0.2$, $N = 5$, $\Delta = 0.01$, $\sigma = 0.01$, $D = 0.1$.

The linear relationship between the logarithm of the escape time $\kappa$ and the inverse of the noise offers the estimate of an effective energy barrier $\Delta U [29, 31]$: 

$$\Delta U = \lim_{D \to 0} D \ln(\kappa).$$

In practical terms, for low enough noise one uses the approximated expression 

$$\Delta U \approx \frac{\Delta \ln(\kappa)}{\Delta(1/D)}.$$  

Equation (18) is very important to characterize with an activation energy the metastable state in the birhythmic region, that is the subject of next Section.

B. Energy barriers

In this Section, we analyzed the behavior of the activation energy as a function of the bias point, see Fig. 2. In the two panels of Fig 7 the common feature is that the activation energy is low ($0.02 \leq \Delta U \leq 0.14$ for (i) and $0.02 \leq \Delta U \leq 0.07$ for (ii)). At the bottom of the step the energy barrier is at a maximum; while the current is increased along the step the energy barrier decreases and almost disappears at the top of the step, for the gap energy decreases when the number of lumped elements increases. Thus the system is less stable as the number of elements increases. According to the lifetime of the RLC-induced step, it decreases along the step when the current bias $\gamma G$ increases independently of the number of the lumped elements, see Fig 6.

IV. EFFECTS OF DISORDER IN THE MODEL

JJs are fabricated with photo-lithographic processes, and are therefore each JJ is different from the other [20, 21]. It is relevant to investigate how the differences in the parameters reflect on the synchronization properties. The effect of disorder, respect to ideal arrays of identical JJs is the subject of this Section.

A. IV Characteristics

To retrieve the IV characteristics of noiseless arrays, Eqs. (14) and (15) are simulated without noise. Figs 8 show the resulting IV for increasing values of the disorder parameter $\sigma$. From the data it is evident that as the disorder parameter increases, birhythmicity disappears, and the system remains birhythmic only for the low values of the disorder parameter, $\sigma < 4\%$. The behavior is confirmed by the diagram of the voltage as a function of disorder in Fig 7. The curves are obtained starting from the RLC locked state for uniform JJ, and then slowly increasing the disorder up to $\sigma = 10\%$. The procedure is then reversed, and the disorder is slowly decreased. In Fig. 3(i) it is evident that disorder induces de-synchronization. This is complementary to the observation that varying the number of junctions the arrays lock to the cavity [19].

B. Attractors properties

As the birhythmicity of the system remains for low values of the disorder $\sigma$, simulations have been performed to investigate the attractors for low disorder values (e.g., $\sigma = 0.1\%, 0.3\%, 0.55\%$...). The properties are very similar, therefore one can focus only on $\sigma = 0.1\%$. The resulting IV curve is displayed in Fig. 10 that is very similar to the uniform case of Fig. 2(ii). Following the behavior display on Figure 5, one finds that for the low disorder case the amplitude of the oscillations at the frequency $\Omega_1$ (the RLC frequency) is much larger than the oscillations at the frequency $\Omega_3$ (the unlocked or McCumber solution [1]). At variance with the uniform case of Fig. 3 the oscillations are much smaller. This is the first effect of the disorder: the unlocked oscillations of the charge almost disappear. With $D \neq 0$, it is evident that the attractors
are still clearly separated. However, the transition from an attractor to the other is much smoother, as shown in Fig. 11. Under the combined effect of noise and disorder the passage from an attractor to another is much less sharp, and it is therefore much more difficult to identify the separatrix \( q_s \). This is the second remarkable effect of disorder: the switch from the energy barrier and the study of the global stability.

V. CONCLUSIONS

This work analyzes the behavior of coherent cooperation of Josephson junctions, a topic of interest for practical reasons (e.g., increase the emitted power \([24, 25]\)) and as prototype for synchronization. Series arrays of identical Josephson Junction coupled through a linear RLC resonator behave qualitatively as a single junction, for instance the system exhibits two clearly distinct frequencies in the locked and unlocked (to the resonator) cases. The approximation of identical Josephson Junction is unrealistic, for the fabrication process produces changes from junction to junction. When disorder and noise are included, some special features of the arrays emerge, in particular the possibility of large excursions that drive the system from the locked to the unlocked state. Such large excursions cannot be treated in the framework of the Kuramoto model, that deals with the local stability properties \([28]\). An alternative approach based on the quasipotential method \([29, 30]\) has proved fruitful for the single junction case, for both the voltage standard application in which the JJ is driven by an external rf source \([32]\) and of a single JJ coupled to a resonator \([33]\). We have extended the application of the method to a series array coupled to an RLC resonator. To make the extension possible it has been necessary to identify the effective separatrix – the passage from the locked to the unlocked phase space region. This effective barrier is difficult to determine, even in the noiseless and ordered case, for the system is high dimensional \((2N + 2\text{ dimension for } N\text{ junctions})\). The employed method is an approximate one: supposing that the separation region is just a plane identified by the single coordinate (the charge on the resonator), it is possible to compute the MFPT to cross this border as a function of the charge threshold, in analogy with the single JJ case, for the MFPT behavior suddenly changes when the threshold is passed. Numerical findings are encouraging: the change in the slope is neat, and can be clearly identified also in the case of multiple JJ, up to \( N = 5 \). Thus, it is possible to compute the effective confining energy for ordered multiple JJ coupled to a resonator. The numerical investigation has revealed two important features. First, the behavior of this effective energy is similar to the case of a single JJ: it is higher at the top of the resonant step, and gets smaller towards the bottom. Second, the effective energy barrier decreases with the number of JJ, making it easier a passage from the locked to the unlocked state when more JJs are present. In the presence of disorder the situation looks more complicated: as the locked and unlocked states get closer to each other, and therefore it is more difficult to identify, with the above method, the effective separatrix. However, for very low disorder variance the method is applicable and it appears that the system retains the qualitative features of the passages from one state to the other. Some limits of the present study are evident. First, this is a proof of principle for relatively few JJ and secondly to a specific configuration of the external load. It would be interesting to extend the investigation to other configurations \([26, 27]\), to investigate the role of the resonator and JJ parameters, and to consider many more JJs. The latter case calls for much more demanding numerical simulations, that presumably are only possible with massive parallel computations, possibly on cheap CUDA hardware \([39]\).

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N=5, $\gamma_G=2.30$
N=2, $\gamma_G=1.55$