Extensions, expansions, Lie algebra cohomology and enlarged superspaces

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Abstract

After briefly reviewing the methods that allow us to derive consistently new Lie (super)algebras from given ones, we consider enlarged superspaces and superalgebras, their relevance and some possible applications.

1 Introduction

Superstring theories, and their low-energy supergravity limits, have made apparent that the original supersymmetry algebra has to be enlarged beyond the restrictions imposed by the Haag-Lopuszański-Sohnius theorem \cite{1}. This was so, in particular, for the following reasons:

- $D = 11$ supergravity \cite{2} may be formulated in a way \cite{3, 4} which suggests that its possible underlying (gauge) group is related to $\text{OSp}(1|32)$ (see also \cite{5, 6} and references therein).

- In situations where the topology is non-trivial, the quasi-invariance under supersymmetry of the Wess-Zumino (WZ) terms of the brane actions results in algebras realized by the conserved supercharges that include additional (topological) charges and that are extensions of the original supersymmetry algebra \cite{7}.

- The existence of solitonic brane solutions of the different supergravities that preserve a fraction of the supersymmetry may be explained from an algebraic point of view by considering more general forms of the algebra (see \cite{8, 9} and references therein), and described by the preon hypothesis \cite{10} (see also \cite{11}).

The lesson to be learnt from these facts is that wherever there is a consistent modification of a given symmetry algebra, it will probably show up in an application. This spirit, in fact, inspired the old search for mixed unitary and kinematical symmetries that was halted by the well known no-go theorems (see \cite{12} for a history of the subject), theorems that were finally bypassed by the realization that fermionic symmetries should be included, and hence by supersymmetry. In fact, if one grants that fermionic spinors exist as the only primary entities, already ordinary supersymmetry is seen to be a natural outcome: it is the result of a central extension of the odd abelian spinor translation group by the group of spacetime translations \cite{13}. Thus, it makes sense to search for supersymmetry algebras beyond the standard superPoincaré algebra (see \cite{14, 3, 7, 15, 16, 17} and references therein).

With this point of view, we shall first review the known methods for obtaining new algebras from given ones, \textit{i.e.} contractions, deformations and extensions of Lie and super Lie algebras, plus a new one (that \textit{includes} contractions) which we have called in \cite{15} Lie (super)algebra expansions. Next we shall concentrate on extensions and expansions, and look for physical applications in both cases.

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2 Four ways to relate and derive Lie (super)algebras

(a) Contractions of Lie (super)algebras
In their simplest Inönü-Wigner (IW) form [19], the contraction of $G$ with respect to a subalgebra $L_0 \subset G$ is performed by rescaling the generators of the coset $G/L_0$, and then by taking a singular limit for the rescaling parameter.

This procedure can be extended to generalized IW contractions in the sense of Weimar-Woods (W-W) [20]. These are defined when $G$ can be split in a sum of vector subspaces $G = V_0 \oplus V_1 \oplus \cdots \oplus V_n = \bigoplus_{s=0}^{n} V_s$, (1)

(V0 being the vector space of the subalgebra $L_0$), such that the following conditions are satisfied:

$$c^i_{ip} = 0 \text{ if } s > p + q \quad \text{i.e.} \quad [V_p, V_q] \subset \bigoplus_{s} V_s, \ s \leq p + q ,$$ (2)

where $i_p$ labels the generators of $G$ in $V_p$, and $c^i_{ip}$ are structure constants of $G$. Then the W-W [20] contracted algebra is obtained by rescaling the group parameters as $g_{ip} \mapsto \lambda g_{ip}$, $p = 0,\ldots,n$, and then by taking a singular limit for $\lambda$. The contracted Lie algebra obtained this way, $G_c$, has the same dimension as $G$. The case $n=1$ corresponds to the simple IW contraction.

Well known examples of contractions that appear in physics include the Galilei algebra as an IW contraction of the Poincaré algebra, the Poincaré algebra as a contraction of the de Sitter algebras, or the characterization of the M-theory superalgebra [8] (ignoring the Lorentz part) as a contraction of $osp(1|32)$.

(b) Deformations
Lie algebra deformations [21] can be regarded, from the physical point of view, as a process inverse to contractions. Mathematically, a deformation $G_d$ of a Lie algebra $G$ is a Lie algebra close, but not isomorphic, to $G$. As in the case of contractions, $G_d$ has the same dimension as $G$. In this case, $G_d$ is rigid or stable under infinitesimal deformations; any attempt to deform it yields an isomorphic algebra. The problem of finite deformations depends on the integrability condition of the infinitesimal deformation; the obstruction is governed by the third cohomology group $H^3(G,G)$ that needs being trivial.

As is known, the Poincaré algebra may be seen as a deformation of the Galilei algebra, a fact that may be viewed as a group theoretical prediction of relativity; $so(4,1)$ and $so(3,2)$ are stabilizations of the Poincaré algebra; $osp(1|4)$ is a deformation of the $N=1$, $D=4$ superPoincaré algebra (for deformations of Lie superalgebras see [22]). Nontrivial central extensions (see (c) below) of Lie algebras may also be considered as deformations or partial stabilizations of trivial ones.

(c) Extensions
In contrast with the procedures (a), (b), the initial data of the extension problem include two algebras $G$ and $A$. A Lie algebra $\tilde{G}$ is an extension of the Lie algebra $G$ by the Lie algebra $A$ if $A$ is an ideal of $\tilde{G}$ and $\tilde{G}/A = G$. As a result, $\dim \tilde{G} = \dim G + \dim A$, so that this process is also ‘dimension preserving’.
Given $\mathcal{G}$ and $\mathcal{A}$, in order to obtain an extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by $\mathcal{A}$ it is necessary to specify first an action $\rho$ of $\mathcal{G}$ on $\mathcal{A}$ i.e., a Lie algebra homomorphism $\rho : \mathcal{G} \rightarrow \text{End} \mathcal{A}$. The different possible extensions $\tilde{\mathcal{G}}$ for $(\mathcal{G}, \mathcal{A}, \rho)$ and the possible obstructions to the extension process are, once again, governed by cohomology [23]. To be more explicit, let $\mathcal{A}$ be abelian. The extensions are governed by $H^2_G(\mathcal{G}, \mathcal{A})$. Some special cases are: 1) trivial action $\rho = 0$, $H^2_G(\mathcal{G}, \mathcal{A}) \neq 0$. These are central extensions, in which $\mathcal{A}$ belongs to the centre of $\tilde{\mathcal{G}}$; they are determined by non-trivial $\mathcal{A}$-valued two-cocycles on $\mathcal{G}$, and non-equivalent extensions correspond to non-equivalent cocycles; 2) non-trivial action $\rho \neq 0$, $H^2_G(\mathcal{G}, \mathcal{A}) = 0$ (semidirect extension of $\mathcal{G}$ by $\mathcal{A}$); and 3) $\rho = 0$, $H^2_G(\mathcal{G}, \mathcal{A}) = 0$ (direct sum of $\mathcal{G}$ and $\mathcal{A}$, $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathcal{A}$, or trivial extension).

Well-known examples of extensions in physics are the centrally extended Galilei algebra, which is relevant in quantum mechanics; the two-dimensional extended Poincaré algebra that allows for a gauge theoretical derivation of the Callan-Giddings-Harvey-Strominger model, or the M-theory superalgebra that, without the Lorentz automorphisms part, is the maximal central extension of the abelian $D = 11$ supertranslations algebra (see Sec. 5.4 and [14][8][10]).

(d) Expansions

Under a different name, Lie algebra expansions were first used in [25], and then the method was studied in general in [18]. The idea is to consider the Maurer-Cartan (MC) equations of the starting Lie algebra $\mathcal{G}$ in terms of the invariant forms on the group manifold, and then perform a rescaling of some of the group parameters $g^i$, $i = 1, \ldots, \dim \mathcal{G}$, by a parameter $\lambda$. Then, one expands the invariant one-forms $\omega^i$ in $\lambda$. Inserting these expansions (polynomials in $\lambda$) in the original MC equations for $\mathcal{G}$,

$$d\omega^i = -\frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k,$$

one obtains a set of equations that have to be satisfied, one for each power of $\lambda$. The problem to be addressed then is how to cut the series expansions of the $\omega^i$’s in such a way that the resulting set of equations remains consistent i.e., closed under $d$, so that it defines the MC equations of a new algebra, the expanded Lie algebra. We do not enumerate all the possibilities here [18]. We shall just mention that, under the W-W conditions [20] for generalized contractions, Eq. (2), and with the corresponding rescaling, the $\{\omega^i\}$ MC forms are divided into $n+1$ sets $\{\omega^{i_p}\}$, $p = 0, 1, \ldots, n$, and the forms $\omega^{i_p}$ corresponding to each subspace in (1) have the expansion

$$\omega^{i_p} = \sum_{s=p}^{\infty} \omega^{i_p,s} \lambda^s,$$

i.e.,

$$\omega^{i_p}(\lambda) = \lambda^p \omega^{i_p,0} + \lambda^{p+1} \omega^{i_p,1} + \ldots$$

(see [18]). If one demands that the maximum power in the expansion of the forms $\{\omega^{i_p}\}$ in the $p$-th subspace is $N_p \geq p$, then consistency requires that

$$N_{q+1} = N_q \quad \text{or} \quad N_{q+1} = N_q + 1 \quad (q = 0, 1, \ldots, n - 1).$$

The new Lie algebras, generated by the MC forms

$$\{\omega^{i_0,0}, \omega^{i_0,1}, \omega^{i_0,N_0}, \omega^{i_1,0}, \omega^{i_1,N_1}, \ldots; \omega^{i_n,n}, \omega^{i_n,N_n}\},$$

are labelled $\mathcal{G}(N_0, N_1, \ldots, N_n)$ and define expansions of the original Lie algebra $\mathcal{G}$. The case $N_p = p$, $\mathcal{G}(0, 1, \ldots, n)$, coincides with the generalized W-W contraction and has the same dimension as the original $\mathcal{G}$; thus, the W-W contraction is a particular expansion. In all other cases the expanded algebra $\mathcal{G}(N_0, N_1, \ldots, N_n)$ is larger than $\mathcal{G}$ [dim $\mathcal{G}(N_0, \ldots, N_n) = \sum_{p=0}^{n} (N_p - p + 1) \dim V_p$], so that the expansion process is not ‘dimension preserving’ (hence its name).

Other interesting cases are those of Lie superalgebras with splittings satisfying the W-W conditions e.g., of the form $\mathcal{G} = V_0 \oplus V_1$ or $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ and such that $V_0$ or $V_0 \oplus V_2$ contain all the bosonic generators and $V_1$ contains the fermionic ones. Then, the expansions of the one-forms in the (dual) subspaces $V_1^*$ ($V_0^*$ and $V_2^*$) of $\mathcal{G}^*$ only contain odd (even) powers of $\lambda$. The consistency conditions for the existence of the $\mathcal{G}(N_0, N_1)$ and $\mathcal{G}(N_0, N_1, N_2)$ expanded superalgebras require that

$$N_0 = N_1 - 1, \quad N_0 = N_1 + 1.$$
and
\[ N_0 = N_1 + 1 = N_2, \quad N_0 = N_1 - 1 = N_2, \quad N_0 = N_1 - 1 = N_2 - 2, \]
respectively.

3 Super–\( p \)–branes and extended superspaces with additional fermionic generators

As mentioned, the standard supersymmetry algebra \( \{Q_\alpha, Q_\beta\} = (CT^\mu)_{\alpha\beta} P_\mu, \ [Q_\alpha, P_\mu] = 0, \) may be viewed \(^1\) as a central extension of the odd abelian algebra \( \{Q_\alpha, Q_\beta\} = 0 \) by the spacetime translations. Other ‘central’ (ignoring the Lorentz part) extensions, with additional bosonic generators, are realized in brane theory and have a topological origin, as shown in \(^7\). Thus, one may ask whether modifying the \( [Q, P] \) commutator by adding new fermionic generators also gives physically relevant supersymmetry algebras.

The first example was the Green algebra \(^2\), which contains an additional fermionic generator, \( Z_\alpha \), that extends centrally the graded translations algebra (superPoincaré without the Lorentz part) provided that the gamma matrices obey an identity that is satisfied only for the number of spacetime dimensions for which superstrings exist. Further examples were given in \(^27, 28, 16\), which gave the form of the spacetime superalgebras underlying the Lie algebra cohomology characterization \(^29\) of the WZ terms of the scalar \( p \)-branes. This led naturally to the consideration of enlarged superspaces that may be seen to have a supergroup extension structure \(^10\). Using them, it is possible to construct the super–\( p \)-brane actions in such a way that the WZ terms become strictly invariant: then, the Chevalley-Eilenberg (CE) Lie algebra cohomology \((2p + 2)\)-cocycles that define the WZ terms of the scalar \( p \)-branes \(^29\) are trivialized \(^27, 10\) (for further work along this line see \(^31, 32\)). These algebras are not central extensions of the starting centrally extended algebra,
\[ \{Q_\alpha, Q_\beta\} = (CT^\mu)_{\alpha\beta} P_\mu + (CT^{\mu_1\ldots\mu_p})_{\alpha\beta} Z_{\mu_1\ldots\mu_p}, \ [Q_\alpha, P_\mu] = 0 = [Q_\alpha, Z_{\mu_1\ldots\mu_p}], \]
although they can be obtained by a step by step process by extending centrally the previous one. In the first step, one extends centrally \(^10\) by adding the new fermionic generators \( Z_{\mu_1\ldots\mu_p-\alpha_1} \), (the case for the Green algebra corresponds to \( p = 1 \)). The resulting algebra can be extended again centrally by bosonic generators of the form \( Z_{\mu_1\ldots\mu_p-2\alpha_1\alpha_2} \); this yields an algebra that is not a central extension of the original one, Eq. \(^10\). The procedure continues \(^27, 28, 10\) by adding centrally more generators of the type \( Z_{\mu_1\ldots\mu_{p-k}\alpha_1\ldots\alpha_k} \), and it ends when one reaches a set of generators where all spacetime indices have been replaced by spinorial ones. Interestingly enough, the existence of these extensions depends on the same gamma matrix identities valid for the \((D, p)\) values that allow for the existence of the given super–\( p \)-brane.

Although the new, extended superspaces (generically denoted \( \tilde{\Sigma} \)) trivialize the WZ terms of the \( p \)-brane actions, their relevance, beyond the topologically non-trivial case, is marginal here since the new superspace group variables corresponding to the new superalgebra generators appear in the action (in the WZ term) through a total derivative, and therefore they do not modify the Euler-Lagrange equations (it will be different for the D\( p \)-branes case below). Let us see this more explicitly. The Lagrangian density of a scalar \( p \)-brane is of the form \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{WZ} \), where \( \mathcal{L}_0 \) is the kinetic part and \( \mathcal{L}_{WZ} \) is the WZ term, given by \( \mathcal{L}_{WZ} db^{p+1} \xi = \phi^* b \), where \( \phi \) is the mapping that locates the \( p \)-brane in rigid superspace, and \( b \) is a \((p + 1)\)-form such that
\[ h = db \propto \Pi^\alpha \wedge (CT_{\mu_1\ldots\mu_p})_{\alpha\beta} \Pi^\beta \wedge \Pi^\mu_1 \wedge \cdots \wedge \Pi^\mu_p, \]
\( \Pi^\alpha \) and \( \Pi^\mu \) being, in the standard flat superspace \( \Sigma \) parametrized by \((x^\mu, \theta^\alpha)\), the invariant one-forms dual to the \( Q_\alpha \) and \( P_\mu \) superalgebra generators respectively. The form \( h = db \) is invariant under supersymmetry transformations, but \( b \) is only quasi-invariant: it cannot be written in terms of \( \Pi^\alpha, \Pi^\mu \) since \( h \) is a non-trivial \((2p + 2)\)-CE cocycle \(^29\). However, there is a form \( \bar{b} \) on the specific extended superspace \( \tilde{\Sigma} \) that differs from \( b \) by a total exterior differential and can be written in terms of the forms \( \Pi^\alpha, \Pi^\mu \) and \( \Pi_{\mu_1\ldots\mu_{p-k}\alpha_1\ldots\alpha_k} \) on \( \tilde{\Sigma} \). Since the new coordinates \( \phi_{\mu_1\ldots\mu_{p-k}\alpha_1\ldots\alpha_k} \) of \( \tilde{\Sigma} \) are not present in \( \bar{b} = h \), they appear trivially in the action.
4 Another example of the use of extensions: D-branes, the M5-brane and worldvolume fields/extended superspace coordinates democracy

The action of the 10-dimensional D-branes \[33,34,35,36\] contains a one-form \(A(\xi)\), the Born-Infeld field, that is directly defined on the worldvolume parametrized by \(\xi = (\tau, \sigma^1, \ldots, \sigma^9)\). Similarly, that of the 11-dimensional M5-brane \[37,38,39\] contains a worldvolume two-form, which we shall also denote \(A(\xi)\). One can use the extended superspaces \(\tilde{\Sigma}\) of Sec. \[6\] to write these forms on the worldvolume also as pull-backs (by \(\phi^*\)) of forms defined on \(\Sigma\). Since the forms \(A(\xi)\) appear non-trivially in the actions, the same happens to the new superspace variables if one writes \(A(\xi) = \phi^*(A)\), for some form \(A\) constructed from forms on a suitable \(\tilde{\Sigma}\) \[16\]. This is an example where the additional coordinates of \(\tilde{\Sigma}\) appear non-trivially.

Let us consider the case of the type IIA Dp–branes, with \(p\) even (the case of the type IIB Dp–branes could be treated similarly \[40,16\]). In the flat case, with vanishing dilaton field, their action can be constructed entirely in terms of the forms of the free differential algebra given by

\[
d\Pi^\alpha = 0 ,
d\Pi^\mu = \frac{1}{2} (CT^\mu)_{\alpha\beta} \Pi^\alpha \wedge \Pi^\beta ,
\]

\[
d\xi = \Pi^\mu \wedge (CT^\mu \Gamma_{11})_{\alpha\beta} \Pi^\alpha \wedge \Pi^\beta ,
\]

where the first two equations are the MC equations for the \(D = 10, N = 2\) superPoincaré algebra, for which the spinors are of Dirac type as corresponds to the IIA case, and \(\mathcal{F}\) is an invariant two-form given by

\[
\mathcal{F} = dA - B , \quad dB = -\Pi^\mu \wedge (CT^\mu \Gamma_{11})_{\alpha\beta} \Pi^\alpha \wedge \Pi^\beta .
\]

Both the form \(\mathcal{F}(\xi)\) that appears in the kinetic and in the (quasi-invariant) WZ term as \(\mathcal{F}_{ij}(\xi)\) \(\mathcal{F}(\xi) = \frac{1}{2} \mathcal{F}_{ij}(\xi) d\xi^i \wedge dx^j\) and \(A(\xi)\) are forms directly defined on the worldvolume. The M5-brane case can be treated similarly by replacing \(\mathcal{F}(\xi)\) by the three-form \(H(\xi) = dA(\xi) - C(\xi)\).

If one can find forms \(\mathcal{F}\) and \(H\) on a suitably extended superspace \(\tilde{\Sigma}\) such that their differentials coincide with those of Eq. \[12\] and with the corresponding ones for the M5-brane respectively, it follows that in both cases \(A(\xi)\) is \(A(\xi) = \phi^*(A)\), where \(A\) is obtained \[16\] by identifying \(\phi^*\mathcal{F}\), \(\phi^*H\) with \(\mathcal{F}(\xi)\), \(H(\xi)\) respectively. The form \(A\) on \(\tilde{\Sigma}\) contains additional coordinates of \(\Sigma\), which are included in \(\mathcal{F}\) (or \(H\)) inside a total derivative. This is achieved using an extended superspace \(\tilde{\Sigma}\), which for the fivebrane is a \(D = 11, p = 2\) extended supergroup (obtained from Eq. \[10\] for \(p = 2\)), and for the case of the D–branes is its dimensional reduction to \(D = 10\) \[16\].

The extended superspace \(\tilde{\Sigma}\) that allows us to describe the Born-Infeld fields also in terms of one-forms on \(\Sigma\) may not be always sufficient (as it is for the D2–branes) to make the D–brane WZ terms strictly invariant. It may be seen (see \[16\] for details) that a larger extended superspace will trivialize the CE \((2p + 2)\)-cocycles although it may correspond to a rather large superalgebra (see \[28\]).

The replacement of \(A(\xi)\) by \(\phi^*(A)\) in the D–brane and fivebrane actions gives models that are classically equivalent to the original ones. This may be seen by noticing that the field equations obtained by varying the original superspace \(\Sigma\) variables and \(A(\xi)\) coincide with those obtained by varying the extended superspace \(\tilde{\Sigma}\) variables in the new action, provided that the induced worldvolume metric is non-degenerate, as it is the case in brane theory. Furthermore, it may be seen \[17\] that there exist the necessary gauge invariances to reduce the number of degrees of freedom of \(\phi^*(A)\) (resp. \(\phi^*(A)\)) to those of \(A_i(\xi)\) (resp. \(A_{ij}(\xi)\)).

The above facts support the worldvolume fields/superspace variables democracy hypothesis \[10,17\], according to which the action of the flat superspace version of superbranes may be written entirely in terms of invariant one-forms defined on a suitably extended superspace \(\tilde{\Sigma}\) group. The fact that D–branes include the dilaton field in their action does not contradict this conjecture.
because the dilaton field in 10 dimensions comes from the K-K reduction of the 11-dimensional metric, and so it may be viewed as an effect of moving to a curved $D = 11$ spacetime. There is also an auxiliary (PST) scalar field $\mathcal{B}$ in the M5-brane action of $\mathcal{A}$, its only role in the covariant action being to account for the required worldvolume self-duality of $\mathcal{A}$.

5 Applications of Lie algebra expansions

5.1 The complete M-theory superalgebra

The statement that the M-theory superalgebra is a contraction of $\text{osp}(1|32)$ actually refers to what may be called the `maximal graded translation algebra' $\Sigma^{(528|32)}$ (in general, $\Sigma^{(\infty|\infty)|n)}$). This has a central extension (of $\{Q_\alpha, Q_\beta\} = 0$ by $[P_{\alpha\beta}, P_{\gamma\delta}] = 0$) structure and is given by

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta} , \quad [Q_\alpha, P_{\beta\gamma}] = 0 , \quad P_{\alpha\beta} = P_{\beta\alpha} , \quad \alpha, \beta = 1, \ldots, 32 ,$$ (14)

the generators $P_{\alpha\beta}$ being central. These may be written as $P_{\alpha\beta} = P_\mu(CT^n)_{\alpha\beta} + Z_{\mu_1, \mu_2} (CT^{\mu_1 \mu_2})_{\alpha\beta} + Z_{\mu_1, \ldots, \mu_5} (CT^{\mu_1 \ldots \mu_5})_{\alpha\beta}$ which is the most general splitting for the symmetric $P_{\alpha\beta}$ in terms of $\text{Spin}(1,10)$ gamma matrices; this expression breaks the general $\text{GL}(32, \mathbb{R})$ invariance of Eq. (14) down to $\text{Spin}(1,10)$. This M-algebra, however, does not include the Lorentz automorphisms part.

The MC equations of $\text{osp}(1|32)$ may be written as follows:

$$d\rho^{\alpha\beta} = -\rho^\alpha \wedge \rho^\beta - \nu^\alpha \wedge \nu^\beta ,$$
$$d\nu^\alpha = -\rho^\alpha \wedge \nu^\beta , \quad \alpha, \beta = 1, \ldots, 32 ,$$ (15)

where the forms $\rho^{\alpha\beta} = \rho^{\beta\alpha}$ dual to $Z_{\alpha\beta}$ are bosonic, and those $\nu^\alpha$ dual to $Q_\alpha$ are fermionic; the indices $\alpha, \beta$ are raised and lowered by means of the $32 \times 32$ charge conjugation matrix $C_{\alpha\beta}$. Let us perform a generalized W-W contraction relative to the splitting $\text{osp}(1|32) = V_0 \oplus V_1 \oplus V_2$, where $V_0 = 0$, $V_1^*$ is generated by the $\nu^s$, and $V_2^*$ is generated by the $\rho^s$. Then if $\nu$ and $\rho$ are rescaled as $\nu \mapsto \lambda \nu$, $\rho \mapsto \lambda^2 \rho$ and the limit $\lambda \to 0$ is taken, one arrives at

$$d\rho^{\alpha\beta} = -\nu^\alpha \wedge \nu^\beta ,$$
$$d\nu^\alpha = 0 ,$$ (16)

which is precisely the dual or MC forms version of the superalgebra $\mathcal{A}$. The dimensions of both $\text{osp}(1|32)$ and the maximal graded translations algebra in $D = 11$, Eq. (14), are the same: $32 \times 33/2 + 32 = 560$. However, the full M-theory superalgebra has the additional $(\frac{11}{2}) = 55$ Lorentz generators, so it is not possible to obtain it by contracting the smaller $\text{osp}(1|32)$ algebra.

Nevertheless, the M-theory superalgebra including the Lorentz part can be obtained as the expansion $\text{osp}(1|32)(2,1,2)$ of $\text{osp}(1|32)$. Let us start by splitting $\text{osp}(1|32) = V_0 \oplus V_1 \oplus V_2$, where now the (dual) space $V_0^*$ is generated by the $\rho^{\mu\nu}$ and $V_2^*$ is generated by $\rho^\mu$ and $\rho^{\mu_1 \ldots \mu_5}$. This is made explicit by writing

$$\rho_{\alpha\beta} = -\frac{1}{32} \left( \rho_\mu CT^\mu - \frac{1}{2} \rho_{\mu\nu} CT^{\mu\nu} + \frac{1}{6} \rho_{\mu_1 \ldots \mu_5} CT^{\mu_1 \ldots \mu_5} \right)_{\alpha\beta} , \quad \mu, \nu = 0, 1, \ldots, 10 .$$ (17)

If, fulfilling condition $\mathcal{B}$, we set $N_0 = 2$, $N_1 = 1$, $N_2 = 2$, this means that we expand $\rho$ and $\nu$ as follows:

$$\nu = \lambda^1 \nu^{(1)} , \quad \rho_{ab} = \rho_{\mu\nu}^{(0)} + \lambda^2 \rho_{\mu\nu}^{(2)} , \quad \rho_\mu = \lambda^3 \rho_\mu^{(2)} , \quad \rho_{\mu_1 \ldots \mu_5} = \lambda^2 \rho_{\mu_1 \ldots \mu_5}^{(2)} .$$ (18)

It is then seen that the new MC equations are precisely the dual of the complete M-theory superalgebra, the Lorentz generators being $\rho_\mu^{(0)}$, and the ‘generalized translations’ being $\rho_\mu^{(2)}$, $\rho_{\mu\nu}^{(2)}$, $\rho_{\mu_1 \ldots \mu_5}^{(2)}$, which can be collected as $\rho_{\alpha\beta}^{(2)}$. Therefore, using the notation of the introduction, it follows that the full M-theory superalgebra is $\text{osp}(1|32)(2,1,2)$. 

6
5.2 Expansions of gauge differential algebras and Chern-Simons Poincaré supergravity in 2 + 1 dimensions

It is known that Poincaré supergravity in 2 + 1 dimensions is a Chern-Simons (CS) gauge theory based on the superPoincaré algebra. It can also be shown that it may be obtained from a (contraction) limit (that setting the cosmological constant equal to zero) of the (2 + 1)-dimensional type (0, 1) anti-de Sitter supergravity [11], which is also a CS theory based on \( sp(2) \oplus osp(1|2) \) (the IW contraction limit involves simultaneously the two algebras). We are going to show here that Poincaré supergravity in \( D = 3 \) may also be obtained from an expansion of a CS model based on \( osp(1|2) \), with an appropriate splitting. This is based on the fact that the expansion method may also be used to expand the gauge theories associated with the original algebra [13].

Let us start from the MC equations of \( osp(1|2) \). These are given also by [15], but now with \( \alpha, \beta = 1, 2 \). The corresponding gauge free differential algebra (FDA) is given in terms of the gauge potentials \( f^{\alpha, \beta} \) and \( \xi^\alpha \) and their curvatures \( \Omega^{\alpha, \beta} = df^{\alpha, \beta} + f^{\alpha, \gamma} \wedge f^{\gamma, \beta} + \xi^\alpha \wedge \xi^\beta \) and \( \Psi^\alpha = d\xi^\alpha + f^{\alpha, \beta} \wedge \xi^\beta \) by the equations defining the curvatures and the Bianchi identities. Using this FDA, one sees that the gauge invariant 4-form

\[
\mathcal{H} = \Omega_{\alpha, \beta} \wedge \Omega_{\beta, \alpha} - 2\Psi_{\alpha} \wedge \Psi^\alpha \tag{19}
\]

is closed. So, if \( B \) is its CS form, \( dB = \mathcal{H} \), it is possible to define a CS model through the action \( \int_{\mathcal{M}^3} B \).

Let us split \( osp(1|2) \) in the form \( osp(1|2) = V_0 \oplus V_1 \), where the dual space \( V_0^* \) contains the one-forms \( \rho^{\alpha, \beta} \), and \( V_1^* \) contains the \( \nu^\alpha \). It may be shown [13] that the expansion of the gauge potentials follows the same pattern as that of the MC forms,

\[
f^{\alpha, \beta} = \sum_{n=0}^{\infty} f^{\alpha, 2n, \lambda^{2n}} , \quad \xi^\alpha = \sum_{n=0}^{\infty} \xi^{\alpha, 2n+1, \lambda^{2n+1}} \tag{20}
\]

and similarly for \( \Omega^{\alpha, \beta} \) and \( \Psi^\alpha \). We now assign physical dimensions to the parameter \( \lambda \). Since we want to make contact with gravity, we would like \( f^{\alpha, \beta, 0} \) to correspond to the Lorentz generators and \( f^{\alpha, \beta, 2} \) to the dreibein forms. This means that \( [\lambda] = L^{-1/2} \). On the other hand, the action for \( D = 3 \) gravity in geometrized units has dimensions of \( L \), so if we expand the CS action in \( \lambda \) we need the term in \( \lambda^2 \) in it in order to obtain a new CS action with the right physical dimensions. The resulting action and its corresponding superalgebra \( osp(1|2)(2, 1) \) (the consistent one that contains all the gauge fields that appear in the action integrand), coincides with the \( D = 3 \) supergravity action and the \( D = 3 \) superPoincaré algebra respectively. Indeed, the \( osp(1|2) \) action is

\[
\int_{\mathcal{M}^3} B = \int_{\mathcal{M}^3} \left( f^{\alpha, \beta} \wedge \Omega_{\alpha, \beta} - 2\xi_{\alpha} \wedge \Psi^\alpha - \frac{1}{3} f^{\alpha, \beta} \wedge f^{\beta, \gamma} \wedge f_{\gamma, \alpha} - f^{\alpha, \beta} \wedge \xi^\beta \wedge \xi_{\alpha} \right) . \tag{21}
\]

Inserting the expansions (20), selecting the \( \lambda^2 \) terms and using that, in three dimensions, one may write

\[
f^{\alpha, \beta, 0} = \frac{1}{4} (CT^{\alpha \beta}) \omega_{ab} , \quad \Omega^{\alpha, \beta, 0} = \frac{1}{4} (CT^{\alpha \beta}) R_{ab} , \quad f^{\alpha, \beta, 2} = - \frac{1}{4} (CT^{\alpha \beta}) \epsilon_{ab} , \quad \xi^{\alpha, 1} = \psi^\alpha , \tag{22}
\]

we obtain the \( D = 3 \) superPoincaré gravity action,

\[
I = \int_{\mathcal{M}^3} \left( \epsilon^{abc} R_{ab} \wedge e_c + 4 \psi_{\alpha} \wedge D \psi^\alpha \right) . \tag{23}
\]

The method may be applied to Chern-Simons supergravities in higher dimensions (see e.g. [5, 6] for an outlook of CS supergravities and further references) to compare e.g., with the standard supergravity [2] and the approach of [3].

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