Efficient Inverse-Free Algorithms for Extreme Learning Machine Based on the Recursive Matrix Inverse and the Inverse LDL^T Factorization

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Abstract—The inverse-free extreme learning machine (ELM) algorithm proposed in [4] was based on an inverse-free algorithm to compute the regularized pseudo-inverse, which was deduced from an inverse-free recursive algorithm to update the inverse of a Hermitian matrix. Before that recursive algorithm was applied in [4], its improved version had been utilized in previous literatures [9], [10]. Accordingly from the improved recursive algorithm [9], [10], we deduce a more efficient inverse-free algorithm to update the regularized pseudo-inverse, from which we develop the proposed inverse-free ELM algorithm 1. Moreover, the proposed ELM algorithm 2 further reduces the computational complexity, which computes the output weights directly from the updated inverse, and avoids computing the regularized pseudo-inverse. Lastly, instead of updating the inverse, the proposed ELM algorithm 3 updates the LDL^T factor of the inverse by the inverse LDL^T factorization [11], to avoid numerical instabilities after a very large number of iterations [12]. With respect to the existing ELM algorithm, the proposed ELM algorithms 1, 2 and 3 are expected to require only $\frac{1}{N}$, $\frac{1}{M}$ and $\frac{1}{M}$ of complexities, respectively, where $M$ is the output node number. In the numerical experiments, the standard ELM, the existing inverse-free ELM algorithm and the proposed ELM algorithms 1, 2 and 3 achieve the same performance in regression and classification, while all the 3 proposed algorithms significantly accelerate the existing inverse-free ELM algorithm.

Index Terms—Extreme learning machine (ELM), inverse-free, fast recursive algorithms, inverse LDL^T factorization, neural networks.

I. INTRODUCTION

The extreme learning machine (ELM) [1] is an effective solution for Single-hidden-layer feedforward networks (SLFNs) due to its unique characteristics, i.e., extremely fast learning speed, good generalization performance, and universal approximation capability [2]. Thus ELM has been widely applied in classification and regression [3].

The incremental ELM proposed in [2] achieves the universal approximation capability by adding hidden nodes one by one. However, it only updates the output weight for the newly added hidden node, and freezes the output weights of the existing hidden nodes. Accordingly those output weights are no longer the optimal least-squares solution of the standard ELM algorithm. Then the inverse-free algorithm was proposed in [4] to update the output weights of the added node and the existing nodes simultaneously, and the updated weights are identical to the optimal solution of the standard ELM algorithm. The ELM algorithm in [4] was based on an inverse-free algorithm to compute the regularized pseudo-inverse, which was deduced from an inverse-free recursive algorithm to update the inverse of a Hermitian matrix.

Before the recursive algorithm to update the inverse was utilized in [4], it had been mentioned in previous literatures [5]–[9], while its improved version had been utilized in [9], [10]. Accordingly from the improved recursive algorithm [9], [10], we deduce a more efficient inverse-free algorithm to update the regularized pseudo-inverse, from which we develop the proposed ELM algorithm 1. Moreover, the proposed ELM algorithm 2 computes the output weights directly from the updated inverse, to further reduce the computational complexity by avoiding the calculation of the regularized pseudo-inverse. Lastly, instead of updating the inverse, the proposed ELM algorithm 3 updates the LDL^T factors of the inverse by the inverse LDL^T factorization proposed in [11], since the recursive algorithm to update the inverse may introduce numerical instabilities in the processor units with the finite precision, which occurs only after a very large number of iterations [12].

This correspondence is organized as follows. Section II describes the ELM model. Section III introduces the existing inverse-free ELM algorithm [4]. In Section IV, we deduce the proposed 3 inverse-free ELM algorithms, and compare the expected computational complexities of the existing and proposed algorithms. Section V evaluates the existing and proposed algorithms by numerical experiments. Finally, we make conclusion in Section VI.

II. ARCHITECTURE OF THE ELM

In the ELM model, the $n$-th input node, the $i$-th hidden node, and the $m$-th output node can be denoted as $x_n$, $h_i$, and $z_m$, respectively, while all the $N$ input nodes, $l$ hidden nodes, and $M$ output nodes can be denoted as $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_N]^T \in \mathbb{R}^N$, $\mathbf{h} = [h_1 \ h_2 \ \cdots \ h_l]^T \in \mathbb{R}^l$, and $\mathbf{z} = [z_1 \ z_2 \ \cdots \ z_M]^T \in \mathbb{R}^M$, respectively. Accordingly the ELM model can be represented in a compact form as

$$\mathbf{h} = f(\mathbf{A}\mathbf{x} + \mathbf{d}) \quad (1)$$

and

$$\mathbf{z} = \mathbf{W}\mathbf{h} \quad (2)$$

where $\mathbf{A} = [a_{in}] \in \mathbb{R}^{l \times N}$, $\mathbf{d} = [d_1 \ d_2 \ \cdots \ d_l]^T \in \mathbb{R}^l$, $\mathbf{W} = [w_{ml}] \in \mathbb{R}^{M \times l}$, and the activation function $f(\bullet)$ is
to avoid over-fitting, the popular Tikhonov regularization \cite{13}, the activation function $f(\bullet)$ can be chosen as linear, sigmoid, Gaussian models, etc.

Assume there are totally $K$ distinct training samples, and let $x_k \in \mathbb{R}^N$ and $z_k \in \mathbb{R}^M$ denote the $k$-th training input and the corresponding $k$-th training output, respectively, where $k = 1, 2, \ldots, K$. Then the input sequence and the output sequence in the training set can be represented as
\[
X = \begin{bmatrix} x_1 & x_2 & \cdots & x_K \end{bmatrix} \in \mathbb{R}^{N \times K},
\]
and
\[
Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_K \end{bmatrix} \in \mathbb{R}^{M \times K},
\]
respectively. We can substitute (3) into (1) to obtain
\[
H = f \left( AX + 1^T \otimes d \right),
\]
where $H = \begin{bmatrix} h_1 \ h_2 \ \cdots \ h_K \end{bmatrix} \in \mathbb{R}^{l \times K}$ is the value sequence of all $l$ hidden nodes, and $\otimes$ is the Kronecker product \cite{4}. Then we can substitute (5) and (4) into (2) to obtain the actual training output sequence
\[
Z = WH.
\]
In an ELM, only the output weight $W$ is adjustable, while $A$ (i.e., the input weights) and $d$ (i.e., the biases of the hidden nodes) are randomly fixed. Denote the desired output as $Y$. Then an ELM simply minimizes the estimation error
\[
E = Y - Z = Y - WH
\]
by finding a least-squares solution $W$ for the problem
\[
\min_W \| E \|_F^2 = \min_W \| Y - WH \|_F^2,
\]
where $\| \bullet \|_F$ denotes the Frobenius norm.

For the problem (11), the unique minimum norm least-squares solution is
\[
W = YH^T (HH^T)^{-1}.
\]
To avoid over-fitting, the popular Tikhonov regularization \cite{13,14} can be utilized to modify (11) into
\[
W = YH^T (HH^T + k_0^2I)^{-1},
\]
where $k_0^2 > 0$ denotes the regularization factor. Obviously (12) is just the special case of (11) with $k_0^2 = 0$. Thus in what follows, we only consider (11) for the ELM with Tikhonov regularization.

III. THE EXISTING INVERSE-FREE ELM ALGORITHM

In machine learning, it is a common strategy to increase the hidden node number gradually until the desired accuracy is achieved. However, when this strategy is applied in ELM directly, the matrix inverse operation in (10) for the conventional ELM will be required when a few or only one extra hidden node is introduced, and accordingly the algorithm will be computational prohibitive. Accordingly an inverse-free strategy was proposed in [4], to update the output weights incrementally with the increase of the hidden nodes. In each step, the output weights obtained by the inverse-free algorithm are identical to the solution of the standard ELM algorithm using the inverse operation.

Assume that in the ELM with $l$ hidden nodes, we add one extra hidden node, i.e., the hidden node $l + 1$, which has the input weight row vector $\bar{a}_{l+1}^T = \begin{bmatrix} a_{l+1,1} \ a_{l+1,2} \ \cdots \ a_{l+1,N} \end{bmatrix} \in \mathbb{R}^N$ and the bias $d_{l+1}$. Then from (5) it can be seen that the extra row $\bar{h}_{l+1}^T = f(\bar{a}_{l+1}^T X + d_{l+1}1^T)$ needs to be added to $H$, i.e.,
\[
H^{l+1} = \begin{bmatrix} H^l & \bar{h}_{l+1}^T \end{bmatrix},
\]
where $H^i$ $(i = l, l + 1)$ denotes $H$ for the ELM with $i$ hidden nodes. In $\bar{a}_{l+1}, \bar{h}_{l+1}, d_{l+1}$ and what follows, we add the overline to emphasize the extra vector or scalar, which is added to the matrix or vector for the ELM with $l$ hidden nodes.

After $H$ is updated by (11), the conventional ELM updates the output weights by (10) that involves an inverse operation. To avoid that inverse operation, the algorithm in [4] utilizes an inverse-free algorithm to update
\[
B = H^T (HH^T + k_0^2I)^{-1}
\]
that is the regularized pseudo-inverse of $H$, and then substitutes (12) into (10) to compute the output weights by
\[
W = YB.
\]
In [4], $B^{l+1}$ (i.e., $B$ for the ELM with $l + 1$ hidden nodes) is computed from $B^l$ iteratively by
\[
B^{l+1} = \begin{bmatrix} \tilde{B}^l & \tilde{b}_{l+1} \end{bmatrix},
\]
where
\[
\tilde{B}^l = \frac{((\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2)I - \bar{h}_{l+1} \bar{h}_{l+1}^T) B^l H^l \bar{h}_{l+1} \bar{h}_{l+1}^T B^l}{(\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2)(\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2 - \bar{h}_{l+1}^T B^l H^l \bar{h}_{l+1})}
\]
\[
+ \frac{(\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2)I - \bar{h}_{l+1} \bar{h}_{l+1}^T)}{\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2},
\]
and $\tilde{b}_{l+1}$, the $(l + 1)^{th}$ column of $B^{l+1}$, is computed by
\[
\tilde{b}_{l+1} = - \frac{\tilde{B}^l H^l \bar{h}_{l+1}}{\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2} + \frac{\bar{h}_{l+1}}{\bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2}.
\]
Let
\[
R = HH^T + k_0^2 I
\]
and
\[
Q = R^{-1} = (HH^T + k_0^2 I)^{-1}.
\]
Then we can write (12) as
\[
B = H^T Q.
\]
From (17) we have
\[
Q^{l+1} = H^T (Q^l)^T + k_0^2 I_{l+1},
\]
into which we substitute (11) to obtain
\[
R^{l+1} = \begin{bmatrix} R^l & p_l^T \end{bmatrix} \begin{bmatrix} \bar{h}_{l+1}^T \bar{h}_{l+1} + k_0^2 \end{bmatrix}^{-1},
\]
where $p_l$, a column vector with $l$ entries, satisfies
\[
p_l = H^l \bar{h}_{l+1}.
\]
TABLE I
COMPARISON OF FLOPS AMONG THE EXISTING AND PROPOSED ELM ALGORITHMS

|                | Updating the Intermediate Results | Updating the Output Weights |
|----------------|-----------------------------------|-----------------------------|
| Existing Alg.  | 16K                               | 2MlK                        |
| Proposed Alg. 1| 6lK                               | 2MK                         |
| Proposed Alg. 2| 2lK                               | 2MK                         |
| Proposed Alg. 3| 2lK                               | 2MK                         |

The inverse-free recursive algorithm computes \( Q^{l+1} = (R^{l+1})^{-1} \) by equations (11), (16), (13) and (14) in \([4]\), which can be written as

\[
Q^{l+1} = \begin{bmatrix} \tilde{Q}^l \\ \tau_l^T \end{bmatrix}, \tag{22}
\]

and

\[
\begin{align}
\tilde{Q}_l &= Q_l + \frac{Q_l p_l p_l^T Q_l}{h_{l+1}^T h_{l+1} + k_0^2 - p_l^T Q_l p_l}, \tag{23a} \\
t_l &= -\frac{h_{l+1}^T h_{l+1} + k_0^2}{p_l^T Q_l}, \tag{23b} \\
\tau_l &= \frac{1}{(\tilde{h}_{l+1}^T \tilde{h}_{l+1} + k_0^2) + h_{l+1}^T h_{l+1} + k_0^2}, \tag{23c}
\end{align}
\]

respectively. Notice that in \((22)\) and \((23)\), \(t_l\) is a column vector with \(l\) entries, and \(\tau_l\) is a scalar.

IV. PROPOSED INVERSE-FREE ELM ALGORITHMS

Actually the inverse-free recursive algorithm by \((22)\) and \((23)\) had been mentioned in previous literatures \([5]–[9]\), before it was deduced in \([4]\) by utilizing the Sherman-Morrison formula and the Schur complement. That inverse-free recursive algorithm can be regarded as the application of the block matrix inverse lemma \([5]\) p.30, and was called the lemma for inversion of block-partitioned matrix \([6] \text{ Ch. 14.12}, [7] \text{ equation (16)}\). To develop multiple-input multiple-output (MIMO) detectors, the inverse-free recursive algorithm was applied in \([7], [8]\), and its improved version was utilized in \([9], [10]\).

A. Derivation of Proposed ELM Algorithms

In the improved version \([9], [10]\), equation \((23)\) has been simplified into \((22)\) equation \((20)\)

\[
\begin{align}
\tau_l &= l/\left( (\sqrt{h_{l+1}^T h_{l+1} + k_0^2}) - p_l^T Q_l p_l \right), \tag{24a} \\
t_l &= -\tau_l Q_l p_l, \tag{24b} \\
\tilde{Q}_l &= Q_l^l + \left( 1/\tau_l \right) t_l t_l^T. \tag{24c}
\end{align}
\]

Accordingly we can utilize \((24)\) to simplify \((16)\) and \((15)\) into

\[
\bar{h}_{l+1} = \tau_l \left( \tilde{h}_{l+1} - B^l p_l \right) \tag{25}
\]

and

\[
\tilde{B}^l = B^l - \bar{h}_{l+1} \tilde{h}_{l+1}^T B^l, \tag{26}
\]

respectively, where \(\tau_l\) can be computed by

\[
\tau_l = l/\left( (\sqrt{h_{l+1}^T h_{l+1} + k_0^2}) - \tilde{h}_{l+1}^T B^l p_l \right). \tag{27}
\]

Moreover, from \((25)\) and \((26)\) we can deduce an efficient algorithm to update the output weight \(W\), i.e.,

\[
W^{l+1} = \begin{bmatrix} \tilde{W}^l & \tilde{w}_{l+1} \end{bmatrix}, \tag{28}
\]

where

\[
\begin{align}
\tilde{W}^l &= W^l - \bar{w}_{l+1} \tilde{h}_{l+1}^T B^l, \tag{29a} \\
\bar{w}_{l+1} &= \gamma \tilde{h}_{l+1} - W^l p_l, \tag{29b}
\end{align}
\]

The derivation of \((25)–(29)\) is in Appendix A.

To further reduce the computational complexity, we can update the unique inverse \(Q\) by \((21), (24)\) and \((22)\), and update the output weight \(W\) by \((28)\) where

\[
\begin{align}
\tilde{W}^l &= W^l + (\bar{w}_{l+1}/\gamma) t_l^T, \tag{30a} \\
\bar{w}_{l+1} &= \gamma \tilde{h}_{l+1} - W^l p_l, \tag{30b}
\end{align}
\]

are also computed from \(t_l\) and \(\tau_l\) in \(Q^{l+1}\). The derivation of \((30)\)

is also in Appendix A.

Since the processor units are limited in precision, the recursive algorithm utilized to update \(Q\) may introduce numerical instabilities, which occurs only after a very large number of iterations \([12]\). Thus instead of the inverse of \(R\) (i.e., \(Q\)), we can also update the inverse \(L D L^T\) factors \([11]\) of \(R\), since usually the \(L D L^T\) factorization is numerically stable \([15]\). The inverse \(L D L^T\) factors include the upper-triangular \(L\) and the diagonal \(D\), which satisfy

\[
L D L^T = Q = R^{-1}. \tag{31}
\]

From \((31)\) we can deduce

\[
L^{-T} D^{-1} L^{-1} = R, \tag{32}
\]

where the lower-triangular \(L^{-T}\) is the conventional \(L D L^T\) factor \([13]\) of \(R\).

The inverse \(L D L^T\) factors can be computed from \(R\) directly by the inverse \(L D L^T\) factorization in \([11]\), i.e.,

\[
\begin{align}
L^{l+1} &= \begin{bmatrix} L^l & \tilde{t}_l \\ 0^l & 1 \end{bmatrix}, \tag{33a} \\
D^{l+1} &= \begin{bmatrix} D^l & 0_l \\ 0_l^T & \tau_l \end{bmatrix}, \tag{33b}
\end{align}
\]

where

\[
\begin{align}
\tilde{t}_l &= -L^l D^l (L^l)^T p_l, \tag{34a} \\
\tau_l &= 1/\left( (\sqrt{h_{l+1}^T h_{l+1} + k_0^2}) - p_l^T L^T D^l (L^l)^T p_l \right). \tag{34b}
\end{align}
\]

We can show that \(\tilde{t}_l\) in \((34a)\) and \(t_l\) in \((24b)\) satisfy

\[
\tilde{t}_l = t_l / \tau_l, \tag{35}
\]

and \(\tau_l\) in \((34b)\) is equal to \(\tau_l\) in \((24c)\), by substituting \((31)\) into \((34a)\) and \((34b)\), respectively. After updating \(L\) and \(D\), we compute the output weight \(W\) by \((30b)\).

\[
W^{l+1} = W^l + \bar{w}_{l+1} \tilde{T}_l, \tag{36}
\]

and \((28)\), where \((36)\) is deduced by substituting \((35)\) into \((30a)\).
### Table II

**Experimental Results of the Existing and Proposed Algorithms for Regression Problems**

| Dataset+ Kernel | Node Number | Weight Error | Output Error (training) | Output Error (testing) | Testing MSE |
|-----------------|-------------|--------------|-------------------------|------------------------|-------------|
| Airfoil+        | 3           | 6e-16, 8e-16 | 8e-15, 1e-14, 1e-14, 1e-14 | 2e-12, 8e-12, 5e-12, 1e-11 | 4.8e-2      |
| Gaussian        | 100         | 2e-11, 3e-11 | 5e-12, 2e-12, 2e-12, 2e-12 | 3e-12, 5e-12, 5e-15, 5e-15 | 1.1e-2      |
| Sigmoid         | 500         | 2e-9, 6e-10  | 1e-10, 5e-11, 3e-7, 2e-11 | 4e-11, 5e-11, 5e-12, 2e-12 | 7.7e-3      |
| Energy+        | 3           | 2e-14, 1e-14 | 2e-14, 4e-14, 4e-14, 4e-14 | 1e-15, 5e-15, 5e-15, 5e-15 | 3.0e-2      |
| Sine           | 100         | 3e-11, 5e-11 | 3e-12, 5e-12, 5e-12, 3e-12 | 5e-12, 6e-12, 5e-12, 5e-12 | 5.0e-3      |
| Protein+       | 500         | 2e-9, 3e-10  | 1e-10, 2e-10, 1e-10, 6e-10 | 3e-12, 4e-12, 4e-12, 6e-12 | 3.7e-3      |
| Triangular     | 3           | 2e-15, 1e-15 | 3e-14, 1e-14, 1e-14, 1e-14 | 2e-14, 4e-14, 4e-14, 4e-14 | 6.5e-2      |
|                | 100         | 2e-11, 2e-11 | 3e-11, 4e-11, 5e-11, 4e-11 | 2e-11, 2e-11, 2e-11, 3e-11 | 5.6e-2      |
|                | 500         | 2e-9, 1e-9   | 3e-6, 1e-9, 3e-6, 1e-9   | 9e-10, 1e-10, 1e-6, 1e-10 | 4.9e-2      |

### Table III

**Speedups in Training Time of the Proposed Algorithms over the Existing Algorithm**

| Dataset+ Kernel | Nodes Number | Speedups Alg. 1 | Alg. 2 | Alg. 3 |
|-----------------|-------------|-----------------|--------|--------|
| Airfoil+        | 100         | 2.43            | 7.99   | 5.66   |
| Gaussian        | 500         | 2.61            | 3.96   | 2.54   |
| Energy+        | 100         | 2.30            | 4.47   | 3.47   |
| Sine           | 500         | 2.51            | 2.32   | 1.55   |
| Housing+       | 100         | 2.73            | 4.64   | 3.32   |
| Protein+       | 500         | 2.77            | 1.92   | 1.41   |
| Triangular     | 100         | 2.54            | 19.04  | 16.28  |
|                | 500         | 2.66            | 22.09  | 19.29  |

### B. Summary and Complexity Analysis of ELM Algorithms

Firstly let us summarize the existing and proposed inverse-free ELM algorithms, which all compute the output \( Z \) by (6), and compute the estimation error \( E \) by (7). In (6) and (7), the output weight \( W \) is required.

The existing inverse-free ELM Algorithm 1 uses (13) and (16) and update the regularized pseudo-inverse \( B \), from which the output weight \( W \) is computed by (15). The proposed Algorithm 1 uses (21), (27), (25), (26) and (14) to update the regularized pseudo-inverse \( B \), from which the output weight \( W \) is computed by (29) and (28). The proposed Algorithm 2 uses (21), (24) and (22) to update the unique inverse \( Q \), from which the output weight \( W \) is computed by (30) and (28). The proposed Algorithm 3 uses (21), (33) and (32) to update the \( LDL^T \) factors of \( Q \), from which the output weight \( W \) is computed by (30b), (36) and (28).

In the remainder of this subsection, we compare the expected flops (floating-point operations) of the existing ELM algorithm in (4) and the proposed ELM algorithms. Obviously \( l_1 l_3 (2l_2 - 1) \approx 2l_1 l_2 l_3 \) flops are required to multiply a \( l_1 \times l_2 \) matrix by a \( l_2 \times l_3 \) matrix, and \( l_1 l_2 l_3 \) flops are required to sum two matrices in size \( l_1 \times l_2 \).

In Table I, we compare the flops of the existing ELM algorithm (4) and the proposed ELM algorithms 1, 2 and 3. As in (4), the flops of the existing ELM algorithm do not include the \( 0(lK) \) entries for simplicity, since usually the ELM has large \( K \) (the number of training examples) and \( l \) (the number of hidden nodes). The flops of the proposed ELM algorithms do not include the entries that are \( 0(lK) \) or \( 0(MK) \). Since usually \( M/l \approx 0 \), it can easily be seen from Table I that with respect to the existing ELM algorithm, the proposed ELM algorithms 1, 2 and 3 only require about \( \frac{3}{8+lM}, \frac{1}{8+lM} \) and \( \frac{1}{8+lM} \) of flops, respectively.

Notice that in the proposed ELM algorithm 1, \( \tilde{h}_i^T, B_i \) computed in (27) and (29a). The dominant computational load of the proposed ELM algorithm 1 comes from (21), (27), (25) and (29b), of which the flops are \( 2Kl, 2Kl, 2KM \) and \( 2KM \), respectively. Moreover, in the proposed ELM algorithms 2 and 3, the dominant computational load comes from (21) and (30b), of which the flops are \( 2KL, 2KM \), respectively.

### V. Numerical Experiments

We follow the simulations in (4), to compare the existing inverse-free ELM algorithm and the proposed inverse-free ELM algorithms on MATLAB software platform under a Microsoft-Windows Server with 128 GB of RAM. We utilize a fivefold cross validation to partition the datasets into training and testing sets. To measure the performance, we employ the mean squared error (MSE) for regression problems, and employ four commonly used indices for classification problems, i.e., the prediction accuracy (ACC), the sensitivity (SN), the precision (PE) and the Matthews correlation coefficient (MCC). Moreover, the regularization factor is set to \( k_0^2 = 0.1 \) to avoid over-fitting.

For the regression problem, we consider energy efficiency dataset (16), housing dataset (17), airfoil self-noise dataset (18), and physicochemical properties of protein dataset (19). For those datasets, different activation functions are chosen, which include Gaussian, sigmoid, sine and triangular. As Table IV in (4), Table II shows the regression performance. In Table II, the weight error and the output error are defined as \( \| W_1 - W_2 \|_F \) and \( \| Z_1 - Z_2 \|_F \), respectively,
where $W_1$ and $Z_1$ are computed by an inverse-free ELM algorithm, and $W_2$ and $Z_2$ are computed by the standard ELM algorithm. We set the initial hidden node number to 2, and utilize the existing and proposed inverse-free ELM algorithms to add the hidden nodes one by one till the hidden node number reaches 500. Table II includes the simulation results for the hidden node numbers 3, 100 and 500.

As observed from Table II, after 1 iteration (i.e., the node number 3), the weight error and the output error are less than $10^{-13}$. For the existing inverse-free ELM algorithm and the proposed algorithms 1 and 3, the weight error and the output error are less than $10^{-10}$ after 98 iterations (i.e., the node number 100), and are not greater than $2 \times 10^{-9}$ after 498 iterations (i.e., the node number 500). However, for the proposed algorithms 2, the weight error and the output error are not greater than $4 \times 10^{-8}$ after 98 iterations, and are not greater than $3 \times 10^{-6}$ after 498 iterations, since the recursive algorithm to update $Q$ introduces numerical instabilities after a very large number of iterations [12]. Overall, the standard ELM, the existing inverse-free ELM algorithm and the proposed ELM algorithms 1, 2 and 3 achieve the same testing MSEs, which have been listed in the last column of Table II.

The speedups in training time of the proposed ELM algorithms 1, 2 and 3 over the existing inverse-free ELM
algorithms are shown in Table III, where we add just one node to reach 100 and 500 nodes, respectively, and we do 1000 simulations to compute the average training time. The speedups are computed by $T_{\text{existing}} / T_{\text{proposed}}$, i.e., the ratio between the training time of the existing ELM algorithm and that of the proposed ELM algorithm. As observed from Table III, all the 3 proposed algorithms significantly accelerate the existing inverse-free ELM algorithm.

For the classification problem, we consider MAGIC Gamma telescope dataset [20], musk dataset [21], adult dataset [22] and diabetes dataset [19]. For each dataset, five activation functions are simulated, i.e., Gaussian, sigmoid, Hardlim, triangular and sine. In the simulations, the standard ELM, the existing inverse-free ELM algorithm and the proposed ELM algorithms 1, 2 and 3 achieve the same performance, which have been listed in Table IV.

Lastly, in Table V we simulate the existing and proposed algorithms on the Modified National Institute of Standards and Technology (MNIST) dataset [23] with 60000 training images and 10000 testing images, to show the performance on big data. To give the testing accuracy, we set the initial hidden node number to 2000, and utilize the existing and proposed ELM algorithms to add hidden nodes one by one till the hidden node number reaches 2200. To give the speedups of the proposed algorithms over the existing algorithm, we compare the training time to reach 2200 nodes by adding one node, and do 500 simulations to compute the average training time.

As observed from Table V, the existing and proposed inverse-free ELM algorithms bear the same testing accuracy, while all the 3 proposed algorithms significantly accelerate the existing inverse-free ELM algorithm. Moreover, from Table V and Table III, it can be seen that usually the proposed algorithm 2 is faster than the proposed algorithm 3, and the proposed algorithm 3 is faster than the proposed algorithm 1.

### VI. Conclusion

To reduce the computational complexity of the existing inverse-free ELM algorithm [4], in this correspondence we utilize the improved recursive algorithm [9, 10] to deduce the proposed ELM algorithms 1, 2 and 3. The proposed algorithm 1 includes a more efficient inverse-free algorithm to update the regularized pseudo-inverse B. To further reduce the computational complexity, the proposed algorithm 2 computes the output weights directly from the updated inverse $Q$, and avoids computing the regularized pseudo-inverse $B$. Lastly, instead of updating the inverse $Q$, the proposed ELM algorithm 3 updates the $LDL^T$ factors of the inverse $Q$ by the inverse $LDL^T$ factorization [11], since the inverse-free recursive algorithm to update the inverse $Q$ introduces numerical instabilities after a very large number of iterations [12]. With respect to the existing ELM algorithm, the proposed ELM algorithms 1, 2 and 3 are expected to require only $\frac{1}{8+5\alpha}$, $\frac{1}{8+5\alpha}$ and $\frac{1}{8+5\alpha}$ of flops, respectively. In the numerical experiments, the standard ELM, the existing inverse-free ELM algorithm and the proposed ELM algorithms 1, 2 and 3 achieve the same performance in regression and classification, while all the 3 proposed algorithms significantly accelerate the existing inverse-free ELM algorithm. Moreover, in the simulations, usually the proposed algorithm 2 is faster than the proposed algorithm 3, and the proposed algorithm 3 is faster than the proposed algorithm 1.

### APPENDIX A

**DERIVATION OF (23), (26), (27), (28), (29) AND (30)**

Substitute (11) and (18) into (12) to obtain

$$B^{l+1} = \left[ (H^l)^T \bar{h}_{i+1} \right] Q^{l+1}.$$  \hspace{1cm} (37)

Substitute (24c) into (22), which is then substituted into (37) to obtain

$$B^{l+1} = \left[ (H^l)^T \bar{h}_{i+1} \right] \left[ Q^l + \frac{1}{\tau} t_t t_t^T \right].$$ \hspace{1cm} (38)

To deduce (25), denote the second entry in the right side of (38) as $\bar{b}_{i+1} = (H^l)^T t_t + \gamma \bar{h}_{i+1}$, (39)

into which substitute (24b) to obtain

$$\bar{b}_{i+1} = -\tau (H^l)^T Q^l p_t + \gamma \bar{h}_{i+1},$$ \hspace{1cm} (40)

and then substitute (19) into (40).

To deduce (26), substitute (24b) into the first entry in the right side of (38), and denote it as $\bar{B}^{l} = (H^l)^T Q^l - (H^l)^T t_t p_t^T Q^l - \gamma \bar{h}_{i+1} p_t^T Q^l$, i.e.,

$$\bar{B}^{l} = (H^l)^T Q^l - (H^l)^T t_t + \gamma \bar{h}_{i+1} p_t^T Q^l.$$ \hspace{1cm} (41)

Then substitute (39) into (41) to obtain

$$\bar{b}^{l} = (H^l)^T Q^l - \bar{b}_{i+1} p_t^T Q^l,$$ \hspace{1cm} (42)

into which substitute (21) to obtain

$$\bar{b}^{l} = (H^l)^T Q^l - \bar{b}_{i+1} \bar{h}_{i+1}^T (H^l)^T Q^l.$$ \hspace{1cm} (43)
Finally we need to substitute (19) into (43).

To deduce (27), substitute (21) into (24a) to obtain $\tau_l = 1 / \left( \tilde{h}_{t+1} \tilde{h}_{t+1}^T + \xi_l^2 - \tilde{h}_{t+1}^T (H^T)^T Q T \phi_l \right)$, into which substitute (19).

By substituting (26) into (14) and substituting (14) into (13), we can deduce (28) where $\tilde{W}_{t+1}$ satisfies (29b) and

$$\tilde{W}^l = Y B^l - Y \tilde{h}_{t+1} \tilde{h}_{t+1}^T B^l,$$

(44)

into which substitute (29b) and (13) to deduce (29a).

To deduce (30a), substitute (19) into (29a) to obtain $\tilde{W}^l = W^l - \tilde{w}_{t+1} \tilde{h}_{t+1}^T Q^l', \tilde{h}_{t+1}^T Q^l', \tilde{h}_{t+1}^T Q^l)$, into which substitute (21) to obtain

$$\tilde{W}^l = W^l - \tilde{w}_{t+1} \tilde{h}_{t+1}^T Q^l,$$

(45)

and then substitute (24b) into (45). Moreover, to deduce (30b), substitute (25) into (29b) to obtain

$$\tilde{w}_{t+1} = \tau_l \left( Y \tilde{h}_{t+1} - Y B^l \phi_l \right),$$

(46)

into which substitute (13).

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