BLOW-UP OF POSITIVE-INITIAL-ENERGY SOLUTIONS FOR NONLINEARLY DAMPED SEMILINEAR WAVE EQUATIONS

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ABSTRACT. We consider a class of semilinear wave equations with both strongly and nonlinear weakly damped terms, 

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u,$$

associated with initial and Dirichlet boundary conditions. Under certain conditions, we show that any solution with arbitrarily high positive initial energy blows up in finite time if \( m < p \). Furthermore, we obtain a lower bound for the blow-up time.

1. INTRODUCTION

In this contribution, we study the blow-up of solutions of the following initial boundary value problem of a semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases}$$

(1.1)

Here, \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). Additionally, we assume that

$$u_0 \in H_0^1(\Omega), \ u_1 \in L^2(\Omega),$$

(1.2)

and \( \omega, \mu, m \) and \( p \) are positive constants, with

$$\begin{cases} 2 < p \leq \frac{2n}{n-2}, & \text{for } n \geq 3, \\ 2 < p < \infty, & \text{for } n = 2. \end{cases}$$

(1.3)

The linear strong damping term \( -\omega \Delta u_t \) appears in models describing Kelvin–Voigt materials that exhibit both elastic and viscous properties, while the nonlinear frictional damping term \( \mu |u_t|^{m-2} u_t \) usually models external friction forces such as air resistance acting on the vibrating structures.

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293
In the absence of strong damping \((\omega = 0)\), the equation in (1.1) reduces to the nonlinearly damped wave equation
\[
 u_{tt} - \Delta u + \mu |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{1.4}
\]
Eq. (1.4) was first studied by Levine [8] in the case of linear weak damping \((m = 2)\). By using the concavity method, he showed that solutions with negative initial energy blow up in finite time. Later, for the case \(m > 2\), by using a different method, Georgiev and Todorova [4] established a global existence result for Eq. (1.4) if \(m \geq p\) and finite time blow-up if \(p > m\) and the initial energy is sufficiently negative.

In the presence of the strong damping term, i.e. \(\omega > 0\), Gazzola and Squassina [3] studied (1.4) for \(m = 2\). They gave a necessary and sufficient condition for blow-up if \(E(0) < d\), where \(d\) is the depth of the potential well. Recently, Yang and Xu [16] gave a sufficient condition for blow-up if \(E(0) > d\). In the case of \(\omega > 0\) and \(m > 2\), Yu [17] gave a necessary and sufficient condition for blow-up when \(E(0) < d\). Boukhatem and Benabderrahmane [2] extended the previous work to a semilinear hyperbolic equation for a uniformly elliptic operator with nonlinear damping and source terms. For results of the same nature, we refer the reader to [1, 6, 5, 9, 12, 14, 15, 18] and the references therein.

In related work, Messaoudi [11] considered
\[
 u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau) \, d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{1.5}
\]
He proved, for \(m < p\), that solutions with \(E(0) < d\) blow up in finite time. Later, by using a modified concavity method, Kafini and Messaoudi [7] established a blow-up result for (1.5) when the damping term is linear. When \(m > 2\), by introducing a new technique, Song [13] obtained a finite time blow-up result for solutions of (1.5) with arbitrarily high initial energy.

In this paper, motivated by the above-cited works, we give sufficient conditions for the finite time blow-up of solutions of (1.1) in both cases: \(E(0) < 0\) and \(E(0) > 0\). Furthermore, we give a lower bound for the blow-up time.

2. Preliminaries

We denote by \(\|\cdot\|_p\) the \(L^p(\Omega)\) norm \((2 \leq p < \infty)\), and by \(\langle \cdot, \cdot \rangle\) the \(L^2\) inner product. The notation \(\langle \cdot, \cdot \rangle\) is used in this paper to denote the duality paring between \(H^{-1}(\Omega)\) and \(H^1_0(\Omega)\). We introduce the energy functional
\[
 E(t) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 - \frac{1}{p} \|u\|_p^p.
\]
By simple calculation we have
\[
 E'(t) = -\mu \|u_t\|_m^m - \omega \|\nabla u_t\|_2^2 \leq 0,
\]
which implies that
\[
 E(t) \leq E(0) \quad \forall t \geq 0,
\]
and
\[
 -E'(t) \geq \mu \|u_t\|_m^m, \quad -E'(t) \geq \omega \|\nabla u_t\|_2^2. \tag{2.1}
\]
Definition 2.1. By solution of problem (1.1) over \([0, T]\) we mean a function 
\[u \in C([0, T], \mathcal{H}_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-1}(\Omega)),\]
with \(u_t \in L^m([0, T], H_0^1(\Omega)),\) such that \(u(0) = u_0, u_t(0) = u_1\) and
\[
\langle u_{tt}(t), \eta \rangle + \int_\Omega \nabla u(t) \nabla \eta \, dx + \omega \int_\Omega \nabla u_t(t) \nabla \eta \, dx + \mu \int_\Omega |u(t)|^{m-2}u_t(t)\eta \, dx = \int_\Omega |u|^p-2u\eta \, dx
\]
for all \(\eta \in H_0^1(\Omega)\) and a.e. \(t \in [0, T].\)

Theorem 2.2. Assume that conditions (1.2) and (1.3) hold. Then the problem (1.1) admits a unique local solution defined on \([0, T]\). Moreover, if
\[T_{\text{max}} := \sup\{T > 0 : u = u(t) \text{ exists on } [0, T]\} < \infty,\]
then
\[\lim_{t \to T_{\text{max}}} \|u(t)\|_q = \infty \quad \text{for all } q \geq 1 \text{ such that } q > \frac{n(p-2)}{2}.\]

3. Blow-up with negative initial energy

In this section we show that the solution of (1.1) blows up in finite time if \(m < p\) and \(E(0) < 0.\) To prove the main result in this section, we define \(H(t) := -E(t)\) and we use the following lemma. For the proof, see [10].

Lemma 3.1. Suppose (1.3) holds. Then we have
\[\|u\|_s^s \leq C [\|H(t)\| + \|u_t\|_2^2 + \|u\|_p^p]\]
for any \(u \in H_0^1(\Omega)\) and \(2 \leq s \leq p.\)

Theorem 3.2. Suppose (1.2) and (1.3) hold. Assume further that \(p > m \geq 2\) and \(E(0) < 0.\) Then the solution of the problem (1.1) blows up in finite time.

Proof. To obtain a contradiction, we suppose that the solution of (1.1) is global; then, for every fixed \(T > 0,\) there exists a constant \(K > 0\) such that
\[\max\{\|\nabla u\|_2^2, \|u_t\|_2^2, \|u\|_p^p\} \leq K \quad \forall t \in [0, T].\]
We have \(H'(t) = -E'(t) \geq 0,\) which together with \(E(0) < 0\) shows that
\[0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p;\]
furthermore,
\[\mu\|u_t\|_m^m \leq H'(t).\]
We now define an auxiliary function
\[L(t) := H^{1-\alpha}(t) + \varepsilon (u_t, u) + \varepsilon \frac{\omega}{2} \|\nabla u\|_2^2,\]
for \(\varepsilon\) small (to be chosen later) and
\[0 < \alpha \leq \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\} < \frac{1}{2}.\]
By taking the derivative of $L(t)$ we obtain

$$L'(t) = (1 - \alpha)H'(t)H^{-\alpha}(t) + \varepsilon(u_{tt}, u) + \varepsilon\|u_t\|_2^2 + \varepsilon\omega(\nabla u_t, \nabla u).$$ \hspace{1cm} (3.4)

Using (3.1), the equation (3.4) takes the form

$$L'(t) = (1 - \alpha)H'(t)H^{-\alpha}(t) - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u_t\|_2^2$$

$$+ \varepsilon\|u\|_p^p - \varepsilon\mu(|u|^{m-2}u_t, u).$$ \hspace{1cm} (3.5)

To estimate the last term in the right-hand side of (3.5), we use Young’s inequality

$$(|u|^{m-2}u_t, u) \leq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{m/(1-m)}\|u_t\|_m^m \quad \forall \delta > 0. \hspace{1cm} (3.6)$$

By taking

$$\delta = \left[kH^{-\alpha}(t)\right]^{1-m},$$

for a large constant $k$ to be chosen later, (3.6) becomes

$$(|u|^{m-2}u_t, u) \leq \frac{1}{m}\left[kH^{-\alpha}(t)\right]^{1-m}\|u\|_m^m + \frac{m-1}{m}kH^{-\alpha}\|u_t\|_m^m. \hspace{1cm} (3.7)$$

Combining (3.5) and (3.7), and using (3.3), yields

$$L'(t) \geq \left[1 - \alpha - \varepsilon k\frac{m-1}{m}\right]H'(t)H^{-\alpha}(t) - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u\|_p^p$$

$$+ \varepsilon\|u_t\|_2^2 - \varepsilon\frac{\mu}{k}[kH^{-\alpha}(t)]^{1-m}\|u\|_m^m. \hspace{1cm} (3.8)$$

By using (3.2), we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq p^{-\alpha(m-1)}\|u\|_p^{p\alpha(m-1)}\|u\|_m^m,$$

and hence by the inequality

$$\|u\|_m \leq C\|u\|_p,$$

we get

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq C p^{-\alpha(m-1)}\|u\|_p^{p\alpha(m-1)+m}. \hspace{1cm} (3.9)$$

Thus, by (3.9) and Lemma 3.1 for $s = p\alpha(m-1) + m \leq p$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq C p^{-\alpha(m-1)}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right].$$

Therefore, in view of the last inequality, (3.8) becomes

$$L'(t) \geq \left[1 - \alpha - \varepsilon k\frac{m-1}{m}\right]H'(t)H^{-\alpha}(t) + \frac{\varepsilon}{2}(p-2)\|\nabla u\|_2^2$$

$$+ \frac{\varepsilon}{2}(p+2)\|u_t\|_2^2 + \varepsilon\left\{pH(t) - \lambda k^{1-m}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]\right\}, \hspace{1cm} (3.10)$$

where

$$\lambda = C p^{-\alpha(m-1)}\frac{\mu}{m}.$$ 

Writing $p = (p+2)/2 + (p-2)/2$ in (3.10) yields

$$L'(t) \geq \gamma_1 H'(t)H^{-\alpha}(t) + \gamma_2 H(t) + \gamma_3\|u_t\|_2^2 + \gamma_4\|u\|_p^p + \gamma_5\|\nabla u\|_2^2, \hspace{1cm} (3.11)$$
where

\[ \gamma_1 = 1 - \alpha - \varepsilon k \frac{m - 1}{m}, \quad \gamma_2 = \varepsilon \left( \frac{p + 2}{2} - \lambda k^{1-m} \right), \]

\[ \gamma_3 = \varepsilon \left( \frac{p + 6}{4} - \lambda k^{1-m} \right), \quad \gamma_4 = \varepsilon \left( \frac{p - 2}{2p} - \lambda k^{1-m} \right), \]

\[ \gamma_5 = \frac{\varepsilon}{4} (p - 2) > 0. \]

We choose now \( k \) large enough such that the coefficients \( \gamma_i \), for \( 2 \leq i \leq 4 \), are positive. Once \( k \) is fixed, we choose \( \varepsilon \) small enough such that \( \gamma_1 > 0 \) and \( L(0) > 0 \).

Hence, the inequality (3.11) becomes

\[ L'(t) \geq A \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right] \quad (3.12) \]

for some constant \( A > 0 \). Consequently, we have \( L(t) \geq L(0) > 0 \) for all \( t \geq 0 \).

Next, by using Hölder’s and Young’s inequalities, we obtain

\[ |(u_t, u)|^{1/(1-\alpha)} \leq \|u_t\|_2^{1/(1-\alpha)} \|u\|_2^{1/(1-\alpha)} \leq C \left[ \|u_t\|_2^{s/(1-\alpha)} + \|u\|_2^{r/(1-\alpha)} \right] \]

for \( 1/s + 1/r = 1 \). We take \( s = 2(1-\alpha) \), which gives

\[ \frac{s}{1-\alpha} = 2 \quad \text{and} \quad \frac{r}{1-\alpha} = \frac{2}{(1-2\alpha)} \leq p. \]

Therefore, by using Lemma 3.1, we obtain

\[ |(u_t, u)|^{1/(1-\alpha)} \leq C \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \quad (3.13) \]

From (3.1) and (3.2), we have

\[ \|\nabla u\|^{2/(1-\alpha)} \leq K^{1/(1-\alpha)} \leq K^{1/(1-\alpha)} \frac{H(t)}{H(0)}. \quad (3.14) \]

So, by using Jensen’s inequality, we get

\[ L(t)^{1/(1-\alpha)} \leq C \left[ H(t) + |(u_t, u)|^{1/(1-\alpha)} + \|\nabla u\|^{2/(1-\alpha)} \right], \]

and by combining it with (3.13) and (3.14), we deduce

\[ L(t)^{1/(1-\alpha)} \leq B \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \quad (3.15) \]

From the inequalities (3.12) and (3.15), we finally obtain the differential inequality

\[ L'(t) \geq DL(t)^{1/(1-\alpha)} \quad (3.16) \]

for some \( D > 0 \). A simple integration of (3.16) over \((0, t)\) immediately yields

\[ L(t) \geq \left[ L^{-\alpha/(1-\alpha)}(0) - \frac{\alpha D}{1-\alpha} t \right]^{1-1/\alpha}, \quad (3.17) \]

which shows that the functional \( L(t) \) blows up in finite time. \( \square \)
Remark 3.3. From (3.17) we obtain the following upper bound of the blow-up time:

\[ T_{\text{max}} \leq \frac{1 - \alpha}{\alpha D} [L(0)]^{1 - 1/\alpha}. \]

4. Blow-up with positive initial energy

In this section, we consider the blow-up of solutions of the problem (1.1) when \( E(0) > 0 \). To prove the main theorem of this paper, we employ the following lemma.

Lemma 4.1. If \( 2 < m < p \) then

\[ \frac{1}{m} ||u||_m^m \leq \frac{s}{2} ||u||_2^2 + \frac{1 - s}{p} ||u||_p^p, \quad \text{where} \quad s = \frac{p - m}{p - 2}. \]

Proof. By the convexity of the function \( u^x/x \) for \( u \geq 0 \) and \( x > 0 \). \( \square \)

Theorem 4.2. Suppose (1.2) and (1.3) hold. Assume further that \( p > m \geq 2 \). If the solution of (1.1) satisfies

\[ (u_t(0), u(0)) > ME(0) > 0 \] (4.1)

for some \( M > 0 \) to be specified later in the proof, then \( u(t) \) blows up in finite time.

Proof. Assume, towards a contradiction, that \( u(t) \) is a global solution of (1.1). Setting \( F(t) := \frac{1}{2} ||u(t)||_2^2 \), it follows from (1.1) that

\[ F''(t) = ||u_t(t)||_2^2 + ||u(t)||_p^p - ||\nabla u(t)||_2^2 - \omega(\nabla u_t, \nabla u) - \mu(||u||^{m-2}u_t, u). \] (4.2)

By using Hölder’s and Young’s inequalities, we estimate the last two terms in the right-hand side of the previous equation as follows:

\[ (\nabla u_t, \nabla u) \leq \eta ||\nabla u||_2^2 + \frac{1}{4\eta} ||\nabla u_t||_2^2, \quad \eta > 0, \]

\[ (||u||^{m-2}u_t, u) \leq \frac{1}{m} \delta^m ||u||_m^m + \frac{m - 1}{m} \delta^m/(1-m) ||u_t||_m^m, \quad \delta > 0. \]

So, by Lemma 4.1 we get

\[ \frac{1}{m} ||u||_m^m \leq \frac{s}{2} \delta^m ||u||_2^2 + \frac{1 - s}{p} \delta^m ||u||_p^p. \]

Hence, (4.2) becomes

\[ F''(t) \geq ||u_t(t)||_2^2 - (1 + \omega \eta) ||\nabla u(t)||_2^2 + \left[ 1 - \frac{\mu(1 - s)}{p} \delta^m \right] ||u(t)||_p^p \]

\[ - \frac{\mu s}{2} \delta^m ||u(t)||_2^2 - \frac{\omega}{4\eta} ||\nabla u_t(t)||_2^2 - \mu \frac{m - 1}{m} \delta^{- m/\tau} ||u_t||_m^m. \]
Adding and subtracting $p(1-\varepsilon)E(t)$ for $\varepsilon \in (0, 1)$ in the right-hand side of the last inequality, and using (2.1) and the Poincaré inequality, we obtain

\[
\frac{d}{dt} \left\{ \frac{dF(t)}{dt} - \left[ \frac{1}{4\eta} + \frac{m-1}{m} \delta^{\frac{m}{m-1}} \right] E(t) \right\} \geq F''(t) + \frac{\omega}{4\eta} \| \nabla u_t(t) \|_2^2 + \mu \frac{m-1}{m} \delta^{\frac{m}{m-1}} \| u_t(t) \|_m^m \\
\geq \| u_t(t) \|_2^2 - (1 + \omega\eta) \| \nabla u(t) \|_2^2 \\
+ \left[ 1 - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|_p^p - \frac{\mu}{2} \delta^m \| u(t) \|_2^2 \\
\geq \left[ 1 + \frac{p}{2}(1-\varepsilon) \right] \| u_t(t) \|_2^2 + \left[ \frac{p}{2}(1-\varepsilon) - (1 + \omega\eta) \right] \| \nabla u(t) \|_2^2 \\
+ \left[ \varepsilon - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|_p^p - \frac{\mu s}{2} \delta^m \| u(t) \|_2^2 - p(1-\varepsilon)E(t) \\
\geq \left[ 1 + \frac{p}{2}(1-\varepsilon) \right] \| u_t(t) \|_2^2 + \left\{ \alpha(\varepsilon)B - \frac{\mu s}{2} \delta^m \right\} \| u(t) \|_2^2 \\
+ \left[ \varepsilon - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|_p^p - p(1-\varepsilon)E(t),
\]

(4.3)

where

\[
\alpha(\varepsilon) = \frac{p}{2}(1-\varepsilon) - (1 + \omega\eta)
\]

and $B$ is the best constant of the Poincaré inequality

\[
\| \nabla u \|_2^2 \geq B \| u \|_2^2.
\]

Therefore, taking $\eta = \varepsilon$ and

\[
\delta = \left[ \frac{pe}{\mu(1-s)} \right]^{1/m},
\]

setting

\[
\gamma_1(\varepsilon) = \frac{1}{4\varepsilon} + \frac{m-1}{m} \left( \frac{1-s}{p\varepsilon} \right)^{-\frac{1}{m-1}}
\]

and substituting in (4.3), we arrive at

\[
\frac{d}{dt} \left[ F'(t) - \gamma_1(\varepsilon) E(t) \right] \geq \left[ 1 + \frac{p}{2}(1-\varepsilon) \right] \| u_t(t) \|_2^2 \\
+ \left\{ \alpha(\varepsilon)B - \frac{ps}{2(1-s)} \varepsilon \right\} \| u(t) \|_2^2 - p(1-\varepsilon)E(t).
\]

Hence, we choose $\varepsilon$ small enough such that

\[
\alpha(\varepsilon)B - \frac{ps}{2(1-s)} \varepsilon > 0.
\]
By using the Schwarz inequality, we have
\[
2 \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right]^{1/2} \left[ \frac{\alpha(\varepsilon)B - \frac{ps}{2(1-s)}}{1 - \varepsilon} \right]^{1/2} (u_t, u) \leq \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right] \|u_t(t)\|_2^2 + \left[ \frac{\alpha(\varepsilon)B - \frac{ps}{2(1-s)}}{1 - \varepsilon} \right] \|u(t)\|_2^2.
\]
Consequently, we obtain
\[
\frac{d}{dt} \left[ F'(t) - \gamma_1(\varepsilon)E(t) \right] \geq \beta(\varepsilon)(u_t, u) - p(1 - \varepsilon)E(t) = \beta(\varepsilon) \left[ F'(t) - \gamma_2(\varepsilon)E(t) \right],
\]
where
\[
\beta(\varepsilon) = 2 \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right]^{1/2} \left[ \frac{\alpha(\varepsilon)B - \frac{ps}{2(1-s)}}{1 - \varepsilon} \right]^{1/2},
\]
\[
\gamma_2(\varepsilon) = \frac{p(1 - \varepsilon)}{\beta(\varepsilon)}.
\]
Since
\[
\alpha(\varepsilon)B - \frac{ps}{2(1-s)} \varepsilon \rightarrow \begin{cases} \frac{2pB}{p-2} > 0 & \text{as } \varepsilon \to 0^+, \\ \frac{B}{1+\omega} - \frac{ps}{2(1-s)} < 0 & \text{as } \varepsilon \to 1^-, \end{cases}
\]
there exists \( \varepsilon_* \in (0, 1) \) such that
\[
\beta(\varepsilon_*) = 0 \quad \text{and} \quad \beta(\varepsilon) > 0 \quad \forall \varepsilon \in (0, \varepsilon_*).
\]
Hence, we have
\[
\gamma_1(\varepsilon) - \gamma_2(\varepsilon) \rightarrow \begin{cases} +\infty & \text{as } \varepsilon \to 0^+, \\ -\infty & \text{as } \varepsilon \to \varepsilon_*^-.
\end{cases}
\]
Therefore, there exists \( \varepsilon_0 \in (0, \varepsilon_*) \) such that \( \gamma_1(\varepsilon_0) = \gamma_2(\varepsilon_0) > 0 \). So, by setting
\[
L(t) = F'(t) - \gamma_1(\varepsilon_0)E(t),
\]
\[
M = \gamma_1(\varepsilon_0),
\]
and by using (4.1), we obtain
\[
L(0) = (u_t(0), u(0)) - \gamma_1(\varepsilon_0)E(0) > (u_t(0), u(0)) - ME(0) > 0.
\]
Moreover, with this choice of \( \varepsilon_0 \), (4.4) becomes
\[
\frac{d}{dt} L(t) \geq \beta(\varepsilon_0) L(t),
\]
which gives
\[
L(t) \geq L(0) e^{\beta(\varepsilon_0) t} \quad \forall t \geq 0.
\]
Since \( u(t) \) is global and \( E(0) > 0 \), by Theorem 3.2, we have that \( E(t) > 0 \) for all \( t \geq 0 \). Hence, we arrive at the inequality
\[
F'(t) \geq L(0) e^{\beta(\varepsilon_0) t} \quad \forall t \geq 0.
\]
By integrating this inequality over \((0, t)\), we get
\[
\|u(t)\|_2^2 = 2F(t) \geq 2F(0) + 2 \frac{L(0)}{\beta(\varepsilon_0)} \left[ e^{\beta(\varepsilon_0)t} - 1 \right] \quad \forall t \geq 0. \tag{4.5}
\]
On the other hand, by using Hölder’s inequality and \((2.1)\), we have
\[
\|u(t)\|_2 \leq \|u(0)\|_2 + \int_0^t \|u_\tau(\tau)\|_m d\tau
\]
\[
\leq \|u(0)\|_2 + C t^{m-1} \int_0^t \|u_\tau(\tau)\|_m^m d\tau
\]
\[
\leq \|u(0)\|_2 + C t^{m-1} \left[ \frac{E(0) - E(t)}{\mu} \right]^{1/m}
\]
\[
\leq \|u(0)\|_2 + C \left[ \frac{E(0)}{\mu} \right]^{1/m} t^{\frac{m-1}{m}},
\]
which clearly contradicts \((4.5)\).

5. LOWER BOUND FOR THE BLOW-UP TIME

In this section, we give a lower bound for the blow-up time \(T_{\text{max}}\). To this end, we define
\[
G(t) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2.
\]

**Theorem 5.1.** Assume that \((1.2)\) and \((1.3)\) hold, and let \(u\) be the solution of \((1.1)\), which blows up at a finite \(T_{\text{max}}\). Then
\[
T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ A_1 + A_2 \sigma^{\beta/2} \right\}^{-1} d\sigma,
\]
where \(\beta, A_1\) and \(A_2\) are positive constants to be determined later in the proof.

**Proof.** By differentiating \(G(t)\) and using \((1.1)\), we obtain
\[
G'(t) = (\nabla u_t, \nabla u) + (u_{tt}, u_t)
\]
\[
= (\nabla u_t, \nabla u) + (\Delta u, u_t) + \omega(\Delta u_t, u_t) - \mu \|u_t\|_m^m + \left( |u|^{p-2} u, u_t \right)
\]
\[
= -\omega \|\nabla u_t\|_2^2 - \mu \|u_t\|_m^m + \left( |u|^{p-2} u, u_t \right).
\]
Thus,
\[
G'(t) \leq -\omega \|\nabla u_t\|_2^2 + \left( |u|^{p-1}, |u_t| \right).
\]
Using Hölder’s inequality, Young’s inequality and the Sobolev inequality
\[
\|v\|_q \leq B_q \|\nabla v\|_q \quad \forall q \in [1, 2^*], \forall v \in H_0^1(\Omega),
\]
we get
\[
(|u|^{p-1}, |u_t|) \leq \|u_t\|^\alpha_p \|u\|^{p-1}_p
\]
\[
\leq \|u_t\|^\alpha_p + C_1 \|u\|^\beta_p
\]
\[
\leq B_p^\alpha \|\nabla u_t\|^\alpha_p + C_2 \|\nabla u\|^\beta_2
\]
\[
\leq A_1 + \|\nabla u_t\|^{\beta/2}_2 + A_2 (G(t))^{\beta/2},
\]
where \(1 < \alpha < 2\) is some positive constant, \(\beta = \alpha(p - 1)/(\alpha - 1)\) and
\[
C_1 = (\alpha - 1)\alpha^{-\alpha/(\alpha - 1)}, \quad C_2 = C_1 B_p^\beta,
\]
\[
A_1 = (2 - \alpha)2^{-2/(2-\alpha)}B_p^{2\alpha/(2-\alpha)}\alpha^{\alpha/(2-\alpha)} \quad A_2 = C_2^{\beta/2}.
\]
Combining (5.2) and (5.1) gives
\[
G'(t) \leq A_1 + A_2 (G(t))^{\beta/2}.
\]
Finally, integrating inequality (5.3) over \((0, T_{\text{max}})\) we get
\[
T_{\text{max}} \geq \int_0^{T_{\text{max}}} \left\{ A_1 + A_2 (G(\tau))^{\beta/2} \right\}^{-1} G'(\tau) d\tau,
\]
and so
\[
T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ A_1 + A_2 \sigma^{\beta/2} \right\}^{-1} d\sigma.
\]

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