Eigen Equation of the Nonlinear Spinor

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Abstract

How to effectively solve the eigen solutions of the nonlinear spinor field equation coupling with some other interaction fields is important to understand the behavior of the elementary particles. In this paper, we derive a simplified form of the eigen equation of the nonlinear spinor, and then propose a scheme to solve their numerical solutions. This simplified equation has elegant and neat structure, which is more convenient for both theoretical analysis and numerical computation.

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Key Words: nonlinear Dirac equation, nonlinear spinor field, eigen equation, quaternion

1 Introduction

Almost all the elementary fermions have spin-$\frac{1}{2}$, which can be naturally described by spinors, so today spinors and spinor representations play a more and more important role in mathematical and theoretical physics. Noticing the limitations of the linear field equation, many physicists such as H. Weyl, W. Heisenberg, once proposed the nonlinear spinor equations[1, 2, 3, 4, 5, 6] to construct a unified field theory for elementary particles. However they have not gotten many definite results due to the mathematical difficulties. The rigorous solutions for some simple dark nonlinear spinor models were

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obtained in [7, 8, 9, 10], and we found it can provide negative pressure to guarantee a singularity-free and accelerating expanding universe [11, 12].

The theoretical proof about the existence of solitons was investigated in [13, 14, 15, 16, 17, 18]. The symmetries and many beautiful conditional exact solutions of the nonlinear spinor, vector and scalar differential equations are collected in [19]. However lots of these exact solutions seem to be non-physical.

The spinor with its own electromagnetic potential was studied in [20, 21, 22, 23, 24], it was disclosed that the nonlinear spinor equations have particle like solution with anomalous magneton, and imply the exact classical mechanics and quantum mechanics for many-body [25, 26, 27].

In this paper we derive a simplified form of the eigen equation with general meaning for nonlinear Dirac equation, and then give a scheme to solve the solution. Denote the Minkowski metric by

\[ \eta_{\mu \nu} = \text{diag}[1, -1, -1, -1], \]

Pauli matrices by

\[ \vec{\sigma} = (\sigma_j) = \{ (0 1) \quad (0 -i) \quad (1 0) \quad (0 -1) \}. \]

(1.1)

Define 4 \times 4 Hermitian matrices as follows

\[ \alpha^\mu = \left\{ \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right), \quad \left( \begin{array}{cc} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{array} \right) \right\}, \gamma = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right), \beta = \left( \begin{array}{cc} 0 & -iI \\ iI & 0 \end{array} \right). \]

(1.2)

In this paper, we adopt the Hermitian matrices (1.2) instead of Dirac matrices \( \gamma^\mu \), because this form is more convenient for calculation.

For the system of a nonlinear spinor field \( \phi \) in the 4\(-d\) potential \( A^\mu \), the Lagrangian describing the motion is generally given by

\[ \mathcal{L} = \phi^+ [\alpha^\mu (\hbar \partial_\mu - eA_\mu) - \mu c \gamma] \phi + F(\gamma, \bar{\beta}), \]

(1.3)

where \( \mu > 0 \) is a constant mass, \( F \) are the nonlinear coupling potential, which is usually the even polynomial of \( \gamma, \bar{\beta} \), and \( \bar{\gamma}, \bar{\beta} \) are the quadratic scalars of \( \phi \) defined by

\[ \bar{\gamma} = \phi^+ \gamma \phi, \quad \bar{\beta} = \phi^+ \beta \phi. \]

(1.4)

We can check \( \bar{\gamma} \) is a true-scalar, but \( \bar{\beta} \) a pseudo-scalar.

The variation of (1.3) with respect to \( \phi^+ \) gives the dynamic equation

\[ \alpha^\mu (\hbar \partial_\mu - eA_\mu) \phi = (\mu c \gamma - \gamma F_\gamma - \beta F_\beta) \phi, \]

(1.5)

where \( F_\gamma = \frac{\partial F}{\partial \gamma}, F_\beta = \frac{\partial F}{\partial \beta} \). In the Hamiltonian form we have

\[ \hbar \partial_\tau \phi = \hat{H} \phi, \quad \hat{H} \equiv \vec{\alpha} \cdot (-\hbar \nabla - e\vec{A}) + eA_0 + (\mu c - F_\gamma) \gamma - F_\beta \beta. \]

(1.6)
In this paper we denote $\vec{A} = (A^1, A^2, A^3)$ to be the spatial part of a contravariant vector $A^\mu$.

It is easy to check that the current conservation law holds $\partial_\mu q^\mu = 0$, so we can take the normalizing condition as follows

$$\int_{R^3} \left| \phi \right|^2 d^3x = 1.$$  \hfill (1.7)

The 4 − d potential produced by spinor $\phi$ itself takes the following form

$$\partial^\alpha \partial_\alpha A^\mu = e\tilde{\alpha}^\mu = e\phi^+ \alpha^\mu \phi.$$  \hfill (1.8)

## 2 Simplification of the Equation

Consider a spinor keeping motionless in an external magnetic field $\vec{B} = (0, 0, B)$. Since the scale of the elementary particle is very small, we take external field $B$ as a constant. By $\nabla \times \vec{A}_{ext} = \vec{B}$, we have the general solution for external vector potential

$$\vec{A}_{ext} = \frac{1}{2} B(-y, x, 0) + \vec{\nabla} \Phi,$$  \hfill (2.1)

where $\Phi(\vec{x})$ is any given smooth function.

In the spherical coordinate system $(r, \theta, \varphi)$, we have

$$\vec{\sigma} \cdot \nabla = \sigma_r \partial_r + (\sigma_\theta \partial_\theta + \sigma_\varphi \partial_\varphi),$$  \hfill (2.2)

where $(\sigma_r, \sigma_\theta, \sigma_\varphi)$ is given by

$$\begin{pmatrix}
\cos \theta & \sin \theta e^{-\varphi i} \\
\sin \theta e^{\varphi i} & -\cos \theta
\end{pmatrix}, \quad \frac{1}{r} \begin{pmatrix}
-\sin \theta & \cos \theta e^{-\varphi i} \\
\cos \theta e^{\varphi i} & \sin \theta
\end{pmatrix}, \quad \frac{1}{r \sin \theta} \begin{pmatrix}
0 & -i e^{-\varphi i} \\
-ie^{\varphi i} & 0
\end{pmatrix}.$$  \hfill (2.3)

Let $\hat{J}$ be the angular momentum operator for the spinor field

$$\hat{J} = \vec{r} \times (-\hbar \nabla) + \frac{1}{2} \hbar \vec{S}, \quad \vec{S} = \text{diag}(\vec{\sigma}, \vec{\sigma}),$$  \hfill (2.4)

then any eigenfunction of $\hat{J}_3 = -\hbar \partial_\varphi + \frac{1}{2} \hbar S_3$ takes the following form

$$\phi = (u_1, u_2 e^{\varphi i}, -iv_1, -iv_2 e^{\varphi i})^T \exp \left( \kappa \varphi i - \frac{mc^2}{\hbar} t i \right)$$  \hfill (2.5)

with $(\kappa = 0, \pm 1, \pm 2, \cdots)$, where $u_k, v_k (k = 1, 2)$ are functions of $r, \theta$ but independent on $\varphi$ and $t$. In this paper the index $T$ stands for transposed matrix.

For any spin-$\frac{1}{2}$ particle, it has a pole axis. If we set the pole axis as coordinate $x^3 = z$, then $\hat{J}_3$ is commutative with the nonlinear Hamilton operator (1.6) by a $U(1)$
gauge transformation for spinor as $e^{\Phi_\gamma}\phi$, which removes the uncertain function from external vector potential (2.1), thereby we have

$$\vec{A}_{\text{ext}} = \frac{1}{2}B(-y, x, 0) = \frac{1}{2}B r \sin \theta (-\sin \varphi, \cos \varphi, 0).$$  \hspace{1cm} (2.6)

For the above symmetric form of vector potential (2.6), substituting (2.5) into (1.6) we can check that all functions $u_k, v_k$ can take real number. For the vector potential produced by $\phi$ itself, we will find below it also takes the form of (2.6). This simplification of $\phi$ may be the essence of the gauge symmetry.

In what follows, we set $\hbar = c = 1$ as units for convenience. Making variable transformation

$$u = u_1(r, \theta) + u_2(r, \theta)i, \quad v = v_1(r, \theta) + v_2(r, \theta)i,$$  \hspace{1cm} (2.7)

then we have

$$\begin{aligned}
\dot{\alpha}_0 &= |u|^2 + |v|^2, \\
\dot{\alpha} &= (\bar{u}v - u\bar{v})i, \\
\dot{\gamma} &= |u|^2 - |v|^2, \\
\dot{\beta} &= \bar{u}v + uv,
\end{aligned}$$  \hspace{1cm} (2.8)

with $\dot{\alpha} = \dot{\alpha}_0(\sin \varphi, \cos \varphi, 0)$. Substituting (2.7) and (2.8) into (1.3) we get the Lagrangian of the eigen states as follows

$$\mathcal{L} = \text{Re} \left[ e^{\theta i} \left( \bar{u}(\partial_r + \frac{i}{r}\partial_\theta)\bar{v} - \bar{v}(\partial_r + \frac{i}{r}\partial_\theta)\bar{u} \right) - \frac{i}{r \sin \theta} \left( \kappa + \frac{1}{2} \right) (\bar{u}v - uv) \right] + (m - eA_0)(|u|^2 + |v|^2) - i e(\bar{u}v - uv)A - \mu(|u|^2 - |v|^2) + F, \hspace{1cm} (2.9)$$

where

$$A = \frac{1}{r}(\vec{r} \times \vec{A}) \cdot \vec{e}_z = (\cos \varphi A_y - \sin \varphi A_x) = A(r, \sin \theta)$$  \hspace{1cm} (2.10)

including both external and inner vector potential. By variation with respect to $\bar{u}, \bar{v}$, we get an elegant equation with double-helix structure

$$\begin{aligned}
e^{\theta i}(\partial_r + \frac{i}{r}\partial_\theta)\bar{u} &= \frac{i}{r \sin \theta}[(\kappa + \frac{1}{2})u - \frac{1}{2}\bar{u}] + (\mu + m - eA_0 - F_\gamma)v + F_\beta u + ieAu, \\
e^{\theta i}(\partial_r + \frac{i}{r}\partial_\theta)\bar{v} &= \frac{i}{r \sin \theta}[(\kappa + \frac{1}{2})v - \frac{1}{2}\bar{v}] + (\mu - m + eA_0 - F_\gamma)u - F_\beta v + ieAv.
\end{aligned}$$  \hspace{1cm} (2.11)

By (2.11) we find that, $\kappa = 0$ corresponds to spin $\frac{1}{2}$ and $\kappa = -1$ corresponds to spin $-\frac{1}{2}$. Generally the coordinates $r$ and $\theta$ can not be separable for nonlinear equation. The energy functional for (2.11) is given by

$$E = 2\pi \int r^2 \sin \theta dr d\theta \left\{ -\text{Re} \left[ e^{\theta i} \left( \bar{u}(\partial_r + \frac{i}{r}\partial_\theta)\bar{v} - \bar{v}(\partial_r + \frac{i}{r}\partial_\theta)\bar{u} \right) \right] - F \right. \left. + \frac{1}{r \sin \theta} \left( \kappa + \frac{1}{2} \right) + eA \right\} i(\bar{u}v - uv) + eA_0(|u|^2 + |v|^2) + \mu(|u|^2 - |v|^2) \right).$$  \hspace{1cm} (2.12)

The eigen solution of (2.11) is just the extreme point of $E$ under the constraint of normalizing condition

$$2\pi \int (|u|^2 + |v|^2) r^2 \sin \theta dr d\theta = 1.$$  \hspace{1cm} (2.13)

(2.13) is also the quantizing condition of the energy spectrum[9, 10].
3 A Scheme for Solving Solution

For general potential $A^\mu$ and $F$, the analytic solution of (2.11) $u$ and $v$ can not be solved. However they can be conveniently expressed by Fourier series of $\theta$, and the equations of the radial functions can be derived by variation principle. For any given integer $N \geq 0$, define $2N+1$ vectors

$$\Gamma(\theta) = (e^{-2N\theta_0}, e^{-2(N-1)\theta_0}, \ldots, e^{2(N-1)\theta_0}, e^{2N\theta_0}), \quad (3.1)$$

$$U(r) = (U_{-N}(r), U_{-(N-1)}(r), \ldots, U_{(N-1)}(r), U_N(r))^T, \quad (3.2)$$

$$V(r) = (V_{-N}(r), V_{-(N-1)}(r), \ldots, V_{(N-1)}(r), V_N(r))^T. \quad (3.3)$$

The eigen solution of (2.11) with even parity must take the form

$$u = \Gamma \cdot U = \sum_{n=-N}^{N} U_n e^{2n\theta_0}, \quad v = \bar{\Gamma} \cdot V e^{\theta_0} = \sum_{n=-N}^{N} V_n e^{-(2n+1)\theta_0}, \quad (3.4)$$

and the eigen solution with odd parity takes

$$u = \Gamma \cdot U e^{\theta_0} = \sum_{n=-N}^{N} U_n e^{(2n+1)\theta_0}, \quad v = \bar{\Gamma} \cdot V = \sum_{n=-N}^{N} V_n e^{-2n\theta_0}. \quad (3.5)$$

In what follows we only consider (3.4). For (3.5) we have similar results.

For the cases $\kappa \neq 0$ and $\kappa \neq -1$, i.e. for the cases with nonzero magnetic quantum number, the solution must have consistent conditions at $\theta = 0, \pi$ as

$$u(r, 0) = u(r, \pi) = \sum_{n=-N}^{N} U_n(r) \equiv 0, \quad v(r, 0) = v(r, \pi) = \sum_{n=-N}^{N} V_n(r) \equiv 0. \quad (3.6)$$

For this case, (3.4) minus (3.6) we get

$$u = \sum_{n=-N}^{N} U_n(e^{2n\theta_0} - 1), \quad v = \sum_{n=-N}^{N} V_n(e^{-2n\theta_0} - 1)e^{\theta_0}. \quad (3.7)$$

By the form (3.7) we have

$$\frac{e^{2n\theta_0} - 1}{2i \sin \theta} = (1 + e^{2\theta_0} + e^{4\theta_0} + \cdots + e^{2(n-1)\theta_0})e^{\theta_0}, \quad (3.8)$$

which removes the singularity of (2.11) at $\theta = 0, \pi$.

For the spin $\frac{1}{2}$ state, i.e. for the case $\kappa = 0$, we can check from (2.11) that the solution $U, V$ are all real functions. But for the spin $-\frac{1}{2}$ state, i.e. for the case $\kappa = -1$, the solution $U, V$ are all pure imaginary functions. In what follows we only consider the real case. For the covariant quadratic forms (2.8), by (3.3) we have

$$\begin{cases} \dot{\alpha}_0 = U^T P U + V^T \bar{P} V, & \dot{\alpha} = U^T (Q^+ - Q)V i, \\ \gamma = U^T P U - V^T \bar{P} V, & \beta = U^T (Q^+ + Q)V, \end{cases} \quad (3.9)$$
where \( P = \Gamma^+ \Gamma \) and \( Q = \Gamma^T \Gamma e^{\theta i} \) are \((2N+1) \times (2N+1)\) matrices with components as
\[
\begin{align*}
P_{m,n} &= \exp[2(n-m)\theta_i], \quad (-N \leq n, m \leq N), \\
Q_{m,n} &= \exp[(2(n+m)+1)\theta_i].
\end{align*}
\]
(3.10)

By (3.9) and (3.10) we have
\[
\begin{align*}
\hat{\alpha}_0 &= \sum_{n,m=-N}^{N} \frac{1}{2} (U_n U_m + V_n V_m) (e^{-2(n-m)\theta_i} + e^{2(n-m)\theta_i}), \\
\hat{\gamma} &= \sum_{n,m=-N}^{N} \frac{1}{2} (U_n U_m - V_n V_m) (e^{-2(n-m)\theta_i} + e^{2(n-m)\theta_i}), \\
\hat{\alpha} &= \sum_{n,m=-N}^{N} U_n V_m (e^{-2(n-m)\theta_i} - e^{2(n-m-1)\theta_i}) e^{\theta i}, \\
\hat{\beta} &= \sum_{n,m=-N}^{N} U_n V_m (e^{-2(n-m)\theta_i} + e^{2(n-m-1)\theta_i}) e^{\theta i}.
\end{align*}
\]
(3.11) \hat{\gamma} \quad (3.12) \hat{\alpha} \quad (3.13) \hat{\beta} \quad (3.14)

The dynamic equation of the potential \( A \) is given by
\[
-\Delta A = e\hat{\alpha} = e[a_0(r) \sin \theta + a_1(r) \sin 3\theta + \cdots],
\]
\[
= e[a_0(r)(e^{-2\theta i} - 1) + a_1(r)(e^{-4\theta i} - e^{2\theta i}) + \cdots] e^{\theta i}.
\]
(3.15)

Substituting (3.4), (3.7), (3.8) and (3.11)\~(3.15) into (2.11) and directly comparing the coefficients of all \( e^{2n\theta i} \), we can easily get a truncated ordinary differential equation of \( U(r), V(r) \). However the convergence this cut-off equation to the original solution needs proof. A more credible method to get the efficient equation of \( U(r), V(r) \) is via variational principle. The variational equation can be obtained by the following procedure. Define operators
\[
\begin{align*}
\hat{T}_u &= \frac{1}{2} \int_0^\pi \Gamma^T(\theta) e^{-\theta i} \sin \theta d\theta, \\
\hat{T}_v &= \frac{1}{2} \int_0^\pi \Gamma^+(\theta) \sin \theta d\theta,
\end{align*}
\]
(3.16)
then \( \hat{T}_u \) left multiplying the first equation of (2.11) and \( \hat{T}_v \) left multiplying the second give the variational equation of \( U(r), V(r) \). The coefficient matrix of \( U'(r) \) and \( V'(r) \) is the same positive definite symmetric matrix with components
\[
M_{n,m} = \frac{1}{2} \int_0^\pi P_{n,m} \sin \theta d\theta = \frac{1}{1 - 4(n-m)^2}.
\]
(3.17)

The other coefficient matrices can also be similarly obtained.

The convergence of expansion (3.4) and the consistent condition (3.6) seem to have closely relation with the structure of nonlinear potential \( F(\hat{\gamma}, \hat{\beta}) \), which reflects
the important properties of the elementary particles such as the exclusion principle. The above procedure is valid for extensive models. The neat and elegant results are profoundly rooted in the quaternionic structure of the physical variables and spacetime[28, 29, 30], so the 3 + 1 dimensional Universe is a miraculous masterpiece with unique feature.

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