Local behavior of positive solutions of higher order conformally invariant equations with a singular set

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Abstract

We study some properties of positive solutions to the higher order conformally invariant equation with a singular set

\[ (-\Delta)^m u = u^{n+2m-2} \quad \text{in } \Omega \setminus \Lambda, \]

where \( \Omega \subset \mathbb{R}^n \) is an open domain, \( \Lambda \) is a closed subset of \( \mathbb{R}^n \), \( 1 \leq m < n/2 \) and \( m \) is an integer. We first establish an asymptotic blow up rate estimate for positive solutions near the singular set \( \Lambda \) when \( \Lambda \subset \Omega \) is a compact set with the upper Minkowski dimension \( \dim^M(\Lambda) < \frac{n-2m}{2} \), or is a smooth \( k \)-dimensional closed manifold with \( k \leq \frac{n-2m}{2} \). We also show the asymptotic symmetry of singular positive solutions suppose \( \Lambda \subset \Omega \) is a smooth \( k \)-dimensional closed manifold with \( k \leq \frac{n-2m}{2} \). Finally, a global symmetry result for solutions is obtained when \( \Omega \) is the whole space and \( \Lambda \) is a \( k \)-dimensional hyperplane with \( k \leq \frac{n-2m}{2} \).

Key words: higher order conformally invariant equations, singular set, local behavior, symmetry, local integral equations.

1 Introduction and main results

In the seminal paper [4], Caffarelli, Gidas and Spruck studied the local behavior of positive solutions to the conformally invariant scalar curvature equation

\[ -\Delta u = u^{n+2} \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^n, \]

with an isolated singularity at the origin. More precisely, they proved that every local positive solution \( u \) is asymptotically radially symmetric near 0, that is, \( u(x) = \bar{u}(|x|)(1 + O(|x|)) \) as \( x \to 0 \) where \( \bar{u}(|x|) \) is the spherical average of \( u \) on the sphere \( \partial B_{|x|}(0) \). Furthermore, they showed that \( u \) has a precise asymptotic behavior near the isolated singularity 0. Subsequent to [4], equation (1.1) and related second-order Yamabe type equations with isolated singularities have attracted a lot of attention; see, for example, [15, 16]
and references therein. The importance of studying the distributional solutions of (1.1) and characterizing the singular set of \( u \) was indicated in the classical work of Schoen and Yau [37, 38] on complete locally conformally flat manifolds. Solutions of (1.1) with an isolated singularity are the simplest examples of those singular distributional solutions. In [6], Chen and Lin studied a more general case when the singular set is not isolated, which is the equation (1.1) in \( B_1 \setminus \Lambda \) with \( \Lambda \) being a higher-dimensional singular set other than a single point. We also refer the reader to [7, 41] for the local estimates of positive solutions near the singular set of second order conformal scalar curvature equation.

In this paper, we are interested in the local behavior of positive solutions of the higher order conformally invariant equation with a singular set \( \Lambda \):

\[
(-\Delta)^m u = u^{n+2m-2m^2} \quad \text{in} \quad \Omega \setminus \Lambda,
\]

where \( \Omega \subset \mathbb{R}^n \) is an open domain, \( \Lambda \) is a closed subset of \( \mathbb{R}^n \), \( 1 \leq m < n/2 \) and \( m \) is an integer. This equation with the critical Sobolev exponent arises as the Euler-Lagrangian equations of Sobolev inequalities [8, 27, 30] and also arises in conformal geometry. More precisely, let \(|dx|^2\) be the Euclidean metric and consider a conformal change \( g := u^{4/n} |dx|^2 \) for some positive smooth function \( u \). Then the fourth order Paneitz operator with respect to the metric \( g \) satisfies

\[
P^g_2 u = u^{n+4/n-4} \Delta^2 (u),
\]

and the \( Q \)-curvature of \( g \) is given by

\[
Q_g = \frac{2}{n-4} P^g_2(1) = 2 \frac{2}{n-4} u^{n+4/n-4} \Delta^2 u.
\]

Hence, each positive solution \( u \) of (1.2) with \( m = 2 \) induces a conformal metric \( g = u^{4/n} |dx|^2 \) which has positive constant \( Q \)-curvature in \( \Omega \setminus \Lambda \). For an introduction to the \( Q \)-curvature problem see, for instance, Hang-Yang [17]. See also Gursky-Malchiodi [14] and Hang-Yang [18] for the recent progresses on the \( Q \)-curvature problem on Riemannian manifolds.

When \( \Omega = \mathbb{R}^n \) and \( \Lambda = \{0\} \) is an isolated singularity, Lin [31] proved that all the singular positive solutions of (1.2) are radially symmetric about \( 0 \) for \( m = 2 \). Frank-König [12] classified all these singular radial solutions, called the Fowler solutions or Delaunay type solutions, using ODE analysis. Recently, the higher order equation (1.2) in the punctured unit ball \( B_1 \setminus \{0\} \) was studied by Jin and Xiong in [20]. In that paper, the authors showed the asymptotic radial symmetry of singular positive solutions and their sharp blow up rate near \( 0 \) under the sign assumptions

\[
(-\Delta)^s u \geq 0 \quad \text{in} \quad B_1 \setminus \{0\}, \quad s = 1, \ldots, m-1.
\]

In [36], for when \( m = 2 \), Ratzkin proved that every local solution \( u \) of (1.2) which satisfies (1.3) has a refined asymptotic behavior near the isolated singularity \( 0 \) based on the classification result of Frank-König [12] and a priori upper estimates of Jin-Xiong [20].

In this paper, we would like to continue the previous study of Jin-Xiong [20] on singular positive solutions of the higher order equation (1.2) in a more general case, that is, \( \Lambda \) is a singular set rather than an isolated point. For the case with a higher-dimensional singular set, the behavior of solutions is expected to be more complicated. To state our first result, we recall the definition...
of the Minkowski dimension (see, e.g., [22][33]). Suppose $E \subset \mathbb{R}^n$ is a compact set, the $\lambda$-dimensional Minkowski $r$-content of $E$ is defined by
\[
\mathcal{M}_r^\lambda(E) = \inf \left\{ nr^\lambda \mid E \subset \bigcup_{k=1}^{n} B(x_k, r), \ x_k \in E \right\},
\]
and the upper and lower Minkowski dimensions are defined, respectively, as
\[
\overline{\dim}_M(E) = \inf \left\{ \lambda \geq 0 \mid \limsup_{r \to 0} \mathcal{M}_r^\lambda(E) = 0 \right\},
\]
\[
\underline{\dim}_M(E) = \inf \left\{ \lambda \geq 0 \mid \liminf_{r \to 0} \mathcal{M}_r^\lambda(E) = 0 \right\}.
\]
If $\overline{\dim}_M(E) = \dim_M(E)$, then the common value, denoted by $\dim_M(E)$, is the Minkowski dimension of $E$. Recall also that for a compact set $E \subset \mathbb{R}^n$, we have the relation $\dim_H(E) \leq \dim_M(E) \leq \dim_H(E)$, where $\dim_M(E)$ is the Hausdorff dimension of $E$.

We will use $B_r(x)$ to denote the open ball of radius $r$ in $\mathbb{R}^n$ with center $x$ and write $B_r$ for short. From now on, without loss of generality, we take the domain $\Omega = B_2$. Firstly, we derive a local estimate of a singular positive solution $u$ near its singular set $\Lambda$ of (1.2).

**Theorem 1.1.** Suppose that $1 \leq m < n/2$ and $m$ is an integer. Let $\Lambda \subset B_{1/2}$ be a compact set with the upper Minkowski dimension $\dim_M(\Lambda)$ (not necessarily an integer), $\overline{\dim}_M(\Lambda) < \frac{n-2m}{2}$, or be a smooth $k$-dimensional closed manifold with $k \leq \frac{n-2m}{2}$. Let $u \in C^{2m}(B_2 \setminus \Lambda)$ be a positive solution of
\[
(-\Delta)^m u = \frac{u^{n+2m}}{n+2m} \quad \text{in } B_2 \setminus \Lambda.
\]
Suppose
\[
(-\Delta)^s u \geq 0 \quad \text{in } B_2 \setminus \Lambda, \ s = 1, \ldots, m-1.
\]
Then there exists a constant $C > 0$ such that
\[
u(x) \leq C [d(x, \Lambda)]^{-\frac{n-2m}{2}}
\]
for all $x \in B_1 \setminus \Lambda$, where $d(x, \Lambda)$ is the distance between $x$ and $\Lambda$.

Assume that $\Lambda \subset B_{1/2}$ is a smooth $k$-dimensional closed manifold with $k \leq \frac{n-2m}{2}$. Let $N$ be a tubular neighborhood of $\Lambda$ such that any point of $N$ can be uniquely expressed as the sum $x + v$ where $x \in \Lambda$ and $v \in (T_x \Lambda)^\perp$, the orthogonal complement of the tangent space of $\Lambda$ at $x$. Denote $\Pi$ the orthogonal projection of $N$ onto $\Lambda$. For small $r > 0$ and $z \in \Lambda$,
\[
\Pi_{r}^{-1}(z) = \left\{ y \in N \mid \Pi(y) = z, \ |y - z| = r \right\}.
\]
We have the following asymptotic symmetry of solutions near the singular set $\Lambda$.

**Theorem 1.2.** Suppose $1 \leq m < n/2$ and $m$ is an integer. Let $u \in C^{2m}(B_2 \setminus \Lambda)$ be a positive solution of (1.4). Suppose that (1.5) holds, and $N$, $\Lambda$ and $\Pi$ are described as above. Then, for $x, x' \in \Pi_{r}^{-1}(z)$, we have
\[
u(x) = \nu(x')(1 + O(r)) \quad \text{as } r \to 0^+,
\]
where $O(r)$ is uniform for all $z \in \Lambda$. 

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Remark that when \( m = 2 \), the positivity of the scalar curvature of the metric \( u^{-4/3}|dx|^2 \) implies that \(-\Delta u > 0\). We also mention that Gursky-Malchiodi [14] studied the positivity of the Paneitz operator and its Green's function under the assumption that the scalar curvature is positive. Since we do not use any special structure of the open ball, \( B_2 \) can be replaced by general domains containing \( B_{1/2} \). Also, both of the above theorems apply to \( \Lambda \subset B_2 \) being compact. When \( \Lambda \) is a single point, Theorems 1.1 and 1.2 have been proved in Jin-Xiong [20].

Now, we give a global symmetry result when \( \Omega \) is the whole Euclidean space and \( \Lambda \) is a lower dimensional hyperplane. Let \( \mathbb{R}^k \) be a \( k \)-dimensional subspace of \( \mathbb{R}^n \) with \( 0 \leq k \leq n - 1 \) being an integer, where \( \mathbb{R}^0 \) denotes the origin \( \{0\} \).

**Theorem 1.3.** Suppose that \( 1 \leq m < n/2 \) and \( m \) is an integer. Let \( 0 \leq k \leq \frac{n-2m}{2} \) and \( u \in C^{2m}(\mathbb{R}^n \setminus \mathbb{R}^k) \) be a nonnegative solution of
\[
(-\Delta)^m u = u^{\frac{n+2m}{n-2m}} \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{R}^k.
\]
Suppose there exists \( x_0 \in \mathbb{R}^k \) such that \( \limsup_{x \to x_0} u(x) = \infty \). Then
\[
u(x', x'') = u(x', \tilde{x}''),
\]
where \( x' \in \mathbb{R}^k \) and \( x'', \tilde{x}'' \in \mathbb{R}^{n-k} \) that \( |x''| = |	ilde{x}''| \). In particular, If \( k = 0 \), i.e., \( \mathbb{R}^k = \{0\} \), and the origin is a non-removable singularity, then \( u \) is radially symmetric and monotonically decreasing about the origin.

When the singular set \( \mathbb{R}^k \) is removable, i.e., \( u \) can be extended as a positive smooth solution in the whole \( \mathbb{R}^n \), the classification of positive solutions of (1.9) was obtained by Caffarelli-Gidas-Spruck [4] for \( m = 1 \), by Lin [31] for \( m = 2 \) and by Wei-Xu [39] for \( m \geq 3 \). We may also see Chen-Li-Ou [8] and Y.Y. Li [27] for the classification of positive smooth solutions of conformally invariant integral equations. When \( \mathbb{R}^k = \{0\} \) and \( \{0\} \) is a non-removable singularity, the radial symmetry of positive solutions of (1.9) was proved by Caffarelli-Gidas-Spruck [4] for \( m = 1 \) and by Lin [31] for \( m = 2 \) via the method of moving planes. Our proof is different from the ones in [4][31] and in fact, it is easy to derive the following global estimate from our proof.

**Corollary 1.4.** Suppose that \( 1 \leq m < n/2 \) and \( m \) is an integer. Let \( u \in C^{2m}(\mathbb{R}^n \setminus \{0\}) \) be a nonnegative solution of
\[
(-\Delta)^m u = u^{\frac{n+2m}{n-2m}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.
\]
If the origin is a non-removable singularity, then there exist two positive constants \( C_1 = C_1(n, m) \) and \( C_2 = C_2(n, m, u) \) such that for all \( x \in \mathbb{R}^n \setminus \{0\} \),
\[
C_2|x|^{-\frac{n-2m}{2}} \leq u(x) \leq C_1|x|^{-\frac{n-2m}{2}}.
\]

For when \( m = 2 \), Corollary 1.4 was proved by Frank-König [12] by ODE analysis, which plays an important role in their classification of all the singular positive solutions in \( \mathbb{R}^n \setminus \{0\} \).

It is well-known that the equation (1.4) is conformally invariant in the following sense. If \( u \) is a solution of (1.4), then its Kelvin transform
\[
u_{x,\lambda}(\xi) = \left(\frac{\lambda}{|\xi - x|}\right)^{n-2m} u \left(x + \lambda^2(\xi - x) / |\xi - x|^2\right)
\]
(1.12)
is also a solution of \((1.4)\) in the corresponding region. Recall also that in the classification of smooth positive solutions of \((1.4)\) in \(\mathbb{R}^n\) by Lin [31] and Wei-Xu [39], one crucial ingredient is that every entire smooth solution of \((1.4)\) satisfies the sign conditions \((1.5)\) in \(\mathbb{R}^n\). This indicates that the sign conditions \((1.5)\) are kept under the Kelvin transform \((1.12)\) for entire solutions. However, for our local equation \((1.4)\), the sign conditions \((1.5)\) may change under the Kelvin transform \((1.12)\). This makes it seem very difficult to prove Theorems 1.1 and 1.2 directly using the Kelvin transform and the moving plane method to the higher order equation \((1.4)\), which is the approach of Chen-Lin [6] to obtain these results when \(m = 1\).

We overcome this difficulty along the similar idea developed in Jin-Xiong [20] when \(\Lambda = \{0\}\), which is inspired by Jin-Li-Xiong [19]. More specifically, we rewrite the differential equation \((1.4)\) into the local integral equation \((1.13)\) below and study the singular solutions of this integral equation. The aim of this paper is to further develop the idea of Jin-Xiong [20] to study the local behavior of positive solutions to the differential equation \((1.4)\) with a general singular set \(\Lambda\). Moreover, we also apply this idea to study the symmetry of global singular solutions of \((1.9)\).

Suppose the dimension \(n \geq 1\), \(0 < \sigma < \frac{n}{2}\) is a real number, and \(\Sigma\) is a closed set in \(\mathbb{R}^n\). We consider the local integral equation

\[
\int_{B_2} \frac{u(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}} + h(x), \quad u > 0, \quad x \in B_2 \setminus \Sigma, \tag{1.13}
\]

where \(u \in L^{\frac{n+2\sigma}{n-2\sigma}}(B_2) \cap C(B_2 \setminus \Sigma)\) and \(h \in C^1(B_2)\) is a positive function. Under the assumptions in Theorem 1.1 one can show \(u \in L_{\text{loc}}^{\frac{n+2\sigma}{n-2\sigma}}(B_2)\) and can rewrite the equation \((1.4)\) locally into the integral equation \((1.13)\) after some scaling (see Theorem 2.4 in Section 2).

Next we state the corresponding results for singular solutions of the integral equation \((1.13)\). Denote \(\mathcal{L}^n\) the \(n\)-dimensional Lebesgue measure on \(\mathbb{R}^n\).

**Theorem 1.5.** Suppose \(n \geq 1\), \(0 < \sigma < n/2\), and \(\Sigma\) is a closed set in \(\mathbb{R}^n\) with \(\mathcal{L}^n(\Sigma) = 0\). Let \(h \in C^1(B_2)\) be a positive function and \(u \in L^{\frac{n+2\sigma}{n-2\sigma}}(B_2) \cap C(B_2 \setminus \Sigma)\) be a positive solution of \((1.13)\). Then there exists a constant \(C > 0\) such that

\[
u(x) \leq C[d(x, \Sigma)]^{-\frac{n-2\sigma}{2}}, \tag{1.14}\]

for all \(x \in B_1 \setminus \Sigma\), where \(d(x, \Sigma)\) is the distance between \(x\) and \(\Sigma\).

When \(0 < \sigma < 1\), equation \((1.13)\) is closely related to the fractional Yamabe equation. Fractional Yamabe equations with isolated singularities were considered in [2, 5, 10], while solutions with a higher dimensional singular set have been studied by Jin-de Queiroz-Sire-Xiong [21], and by Ao-Chan-DelaTorre-Fontelos-Gonzalez-Wei [11] which develops a Mazzeo-Pacard gluing program (see [34]) in the fractional setting. Note that for the singular set \(\Sigma\) of the integral equation \((1.13)\), we only assume that \(\Sigma\) has \(n\)-dimensional Lebesgue measure 0, which is a weaker condition than the singular set of Newton capacity 0 studied by Chen-Lin [6] to the second order equation \((1.1)\) and the singular set of fractional capacity 0 studied by Jin-de Queiroz-Sire-Xiong [21] to the fractional Yamabe equation.

Further, if \(\Sigma\) is a smooth submanifold of \(\mathbb{R}^n\), then we also have the asymptotic symmetry of singular solutions of \((1.13)\).
Theorem 1.6. Suppose \( n \geq 1, 0 < \sigma < n/2, \Sigma \subset \mathbb{R}^n \) is a smooth \( k \)-dimensional closed manifold with \( k \leq n - 1 \), \( N \) is a tubular neighborhood of \( \Sigma \) and \( \Pi \) is the orthogonal projection of \( N \) onto \( \Sigma \) described as before. Let \( h \in C^1(B_2) \) be a positive function and \( u \in L^{\frac{n+2\sigma}{n-2\sigma}}(B_2) \cap C(B_2 \setminus \Sigma) \) be a positive solution of (1.13). Let \( A \subset B_2 \) be a compact subset of \( \Sigma \). Then we have, for \( x, x' \in \Pi_1^{-1}(z) \),

\[
u(x) = u(x')(1 + O(r)) \quad \text{as} \quad r \to 0^+,
\]

(1.15)

where \( O(r) \) is uniform for all \( z \in A \).

In Theorems 1.5 and 1.6, we allow the singular set \( \Sigma \) to intersect the boundary \( \partial B_2 \), which is essential for applying these results to the differential equation (1.4). When \( \Sigma = \{0\} \) is an isolated singularity, Theorems 1.5 and 1.6 were proved by Jin-Xiong [20].

Finally, for the global singular solutions of the differential equation (1.9), under the assumptions of Theorem 1.3 one can also show that \( u \in L^{\frac{n+2\sigma}{n-2\sigma}}_{\text{loc}}(\mathbb{R}^n) \) and all the singular positive solutions of (1.9) satisfy the following integral equation (1.16) (see Theorem 2.8 in Section 2). This has been proved by Chen-Li-Ou [8] to be true for the entire smooth positive solutions of (1.9) in \( \mathbb{R}^n \). Their proof makes use of the sign conditions (1.5) in (1.16) for smooth solutions proved by Wei-Xu [39], which is different from ours. Of course, our proof also works for smooth solutions.

Let \( \mathbb{R}^k \) be a \( k \)-dimensional subspace of \( \mathbb{R}^n \), and let \( \sigma \) be a real number satisfying \( 0 < \sigma < n/2 \). Consider the integral equation

\[
u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x - y|^{n-2\sigma}} dy, \quad x \in \mathbb{R}^n \setminus \mathbb{R}^k.
\]

(1.16)

Then we have the following

Theorem 1.7. Let \( 0 \leq k < n - 1 \) and \( u \in L^{\frac{n+2\sigma}{n-2\sigma}}_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \mathbb{R}^k) \) be a positive solution of (1.16). Suppose there exists \( x_0 \in \mathbb{R}^k \) such that \( \limsup_{x \to x_0} u(x) = \infty \). Then

\[
u(x', x'') = u(x', \tilde{x}''),
\]

where \( x' \in \mathbb{R}^k \) and \( x'', \tilde{x}'' \in \mathbb{R}^{n-k} \) that \( |x''| = |\tilde{x}''| \). In particular, if \( k = 0 \), i.e., \( \mathbb{R}^k = \{0\} \), and the origin is a non-removable singularity, then \( u \) is radially symmetric and monotonically decreasing about the origin.

When \( k = 0 \), Theorem 1.7 was proved by Chen-Li-Ou [9]. In this case (when \( k = 0 \)), it is well-known that \( c|x|^{-\frac{n-2\sigma}{2}} \) is a singular solution of (1.16) for a positive constant \( c \) depending only on \( n \) and \( \sigma \), the other singular solutions, e.g., the Fowler solutions, of (1.16) were obtained by Jin-Xiong [20]. See also [10,12,13] for the existence of Fowler solutions of related differential equations.

Since the integral equations (1.13) and (1.16) are conformally invariant (see Section 3), we shall prove Theorems 1.5, 1.6 and 1.7 using the method of moving spheres introduced by Li-Zhu [29] for differential equations and by Li [27] for integral equations. A difference from the integral equations studied in [8,27] is that our integral equation (1.13) is locally defined, and we need some more delicate estimates. Another difference is that we are concerned with the singular solutions of (1.13) and (1.16). More applications of the method of moving spheres can be found in [5,20,24,26,28,41].
This paper is organized as follows. In Section 2 we show the integral representations for singular positive solutions to the differential equations (1.4) and (1.9), respectively. In Section 3 we prove the upper bounds in Theorems 1.5 and 1.1. In Section 4, we show the asymptotic symmetry of the solutions near the singular set in Theorems 1.6 and 1.2. In Section 5, we show the symmetry results of global singular solutions in Theorems 1.7 and 1.3, where we also give the proof of Corollary 1.4.

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2 Integral representations for singular solutions

2.1 Local singular solutions

In this subsection, we show that every singular positive solution of the differential equation (1.4) satisfies the integral equation (1.13) in some local sense under suitable assumptions. Firstly, we prove that under the assumptions of Theorem 1.1 (1.5 is not needed here), then \( u \in L^{\frac{n+2m}{n-2m}}(B_2) \) and \( u \) is a distributional solution in the entire ball \( B_2 \).

**Proposition 2.1.** Suppose that \( 1 \leq m < \frac{n}{2} \) and \( m \) is an integer. Let \( \Lambda \subset B_{1/2} \) be a compact set with the upper Minkowski dimension \( \overline{\dim}_M(\Lambda) \) (not necessarily an integer), \( \overline{\dim}_M(\Lambda) < \frac{n-2m}{2} \), or be a smooth \( k \)-dimensional closed manifold with \( k \leq \frac{n-2m}{2} \). Let \( u \in C^{2m}(\overline{B_2} \setminus \Lambda) \) be a positive solution of (1.4). Then \( u \in L^{\frac{n+2m}{n-2m}}(B_2) \) and \( u \) is a distributional solution in the entire ball \( B_2 \), i.e., we have

\[
\int_{B_2} u(-\Delta)^m \varphi dx = \int_{B_2} \frac{n+2m}{n-2m} u^m \varphi dx \tag{2.1}
\]

for every \( \varphi \in C_c^\infty(B_2) \).

**Remark 2.2.** In a recent paper [3], Ao, González, Hyder and Wei studied removable singularities and superharmonicity of non-negative solutions to the fractional equation \((-\Delta)^\gamma u = u^{\frac{n+2\gamma}{n-2\gamma}}\) in \( \mathbb{R}^n \setminus \Sigma \), where \( 0 < \gamma < \frac{n}{2} \). Among other things, they proved if \( \Sigma \) is a compact set in \( \mathbb{R}^n \) with the upper Assouad dimension \( \overline{\dim}_A(\Sigma) \) \( < \frac{n-2\gamma}{2} \), and \( u \in L_\gamma(\mathbb{R}^n) \cap L^{\frac{n+2\gamma}{n-2\gamma}}(\mathbb{R}^n \setminus \Sigma) \) is a non-negative solution, then \( u \in L^{\frac{n+2\gamma}{n-2\gamma}}(\mathbb{R}^n) \) and \( u \) is a distributional solution in \( \mathbb{R}^n \). We also remark that for any compact set \( E \subset \mathbb{R}^n \), the relation \( \overline{\dim}_M(E) \leq \overline{\dim}_A(E) \) holds, see, e.g., [22].

**Proof.** We first assume that \( \Lambda \subset B_{1/2} \) be a compact set with the upper Minkowski dimension \( \overline{\dim}_M(\Lambda) < \frac{n-2m}{2} \). Let \( N_r := \{ x \in \mathbb{R}^n | \text{dist}(x, \Lambda) < r \} \) be a tubular \( r \)-neighborhood of \( \Lambda \). We fix a non-negative function \( \rho \in C_c^\infty(B_1) \) satisfying \( \int_{B_1} \rho dx = 1 \). Setting \( \rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right) \) for small \( \varepsilon > 0 \). As in Ao-González-Hyder-Wei [3], we define

\[
\eta_\varepsilon(x) := 1 - \int_{N_{2\varepsilon}} \rho_\varepsilon(x-y) dy. \tag{2.2}
\]

Then \( \eta_\varepsilon \in C^\infty(\mathbb{R}^n) \) is a non-negative function with values in \([0, 1]\), and it satisfies

\[
\eta_\varepsilon(x) = 1 \quad \text{on} \quad N_{3\varepsilon}^c \quad \text{and} \quad \eta_\varepsilon(x) = 0 \quad \text{on} \quad N_\varepsilon. \tag{2.3}
\]
Moreover, we have, for every \( j = 1, 2, \ldots \),
\[
|\nabla^j \eta| \leq C \varepsilon^{-j}.
\] (2.4)

Next we shall use some arguments in Yang [40]. Let \( \varphi_\varepsilon(x) = [\eta_\varepsilon(x)]^q \) with \( q = \frac{n+2m}{2} \). Multiplying both sides of (1.4) by \( \varphi_\varepsilon \) and using integration by parts, we obtain
\[
\int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx = -\int_{\partial B_2} \frac{\partial}{\partial \nu} (\Delta)^{m-1} u dx + \int_{B_2} u (\Delta)^m \varphi_\varepsilon dx
\leq C + C \varepsilon^{-2m} \int_{N_{3\varepsilon} \setminus N_\varepsilon} u(\eta_\varepsilon)^{q-2m} dx
\leq C + C \varepsilon^{-2m} \int_{N_{3\varepsilon} \setminus N_\varepsilon} u(\varphi_\varepsilon)^{\frac{n-2m}{n+2m}} dx
\leq C + C \varepsilon^{-2m} \left( \int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx \right)^{\frac{n-2m}{n+2m}} \left( \int_{N_{3\varepsilon} \setminus N_\varepsilon} 1 dx \right)^{\frac{4m}{n+2m}}.
\]

Now we estimate the integral \( \int_{N_{3\varepsilon} \setminus N_\varepsilon} 1 dx \). Choosing \( \lambda > \dim_M(\Lambda) \) but sufficiently close to \( \dim_M(\Lambda) \) such that \( \frac{4m(n-\lambda)}{n+2m} - 2m \geq 0 \) (equivalently, \( \lambda \leq \frac{n-2m}{2} \)). By Proposition 5.8 of [22], there exist two constants \( C, r_0 > 0 \) such that
\[
H^{n-1}(\partial N_\varepsilon) \leq C \varepsilon^{n-\lambda-1} \quad \text{for all } 0 < \varepsilon < r_0,
\] (2.5)
where \( H^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure on \( \mathbb{R}^n \). Note that the distance function \( d(x) := \text{dist}(x, \Lambda) \) is a 1-Lipschitz function. It follows from Rademacher’s theorem that \( d(x) \) is differentiable \( a.e. \) with \( |\nabla d| = 1 \). Then by the co-area formula (see, e.g., Evans-Gariepy [11]) and (2.5), we have
\[
\int_{N_{3\varepsilon} \setminus N_\varepsilon} 1 dx \leq C \int_0^{3\varepsilon} \left( \int_{\partial N_r} 1 dH^{n-1} \right) dr
\leq C \int_0^{3\varepsilon} r^{n-\lambda-1} dr \leq C \varepsilon^{n-\lambda}
\]
if \( \varepsilon < r_0/3 \). This together with the above estimate yields
\[
\int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx \leq C + C \varepsilon^{\frac{4m(n-\lambda)}{n+2m} - 2m} \left( \int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx \right)^{\frac{n-2m}{n+2m}}
\leq C + C \left( \int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx \right)^{\frac{n-2m}{n+2m}},
\] (2.6)
where we have used the fact \( \frac{4m(n-\lambda)}{n+2m} - 2m \geq 0 \) due to the choice of \( \lambda \). Using Young inequality in the last term in (2.6), we have
\[
\int_{B_2 \setminus N_{3\varepsilon}} u^{\frac{n+2m}{n-2m}} dx \leq \int_{B_2} u^{\frac{n+2m}{n-2m}} \varphi_\varepsilon dx \leq C.
\]
Sending $\varepsilon \to 0$, we obtain
\[ \int_{B_2} u^{n+2m} \frac{n+2m}{n-2m} dx \leq C. \]

When $\Lambda$ is a smooth $k$-dimensional closed manifold with $k \leq \frac{n-2m}{2}$, the proof is very similar to the above, and we only need to notice that in this case (when $\Lambda$ is smooth), $\mathcal{L}^n(\mathcal{N}_\Lambda) \approx \varepsilon^{n-k}$ for $\varepsilon > 0$ small.

Next, we show that $u$ is a distributional solution in $B_2$. For any $\varphi \in C_c^\infty(B_2)$, using $\psi_\varepsilon := \varphi \eta_\varepsilon$ as a test function in (1.4) with $\eta_\varepsilon$ as before gives
\[ \int_{B_2} u^{n+2m} \frac{n+2m}{n-2m} \psi_\varepsilon dx = \int_{B_2} u\eta_\varepsilon (-\Delta)^m \varphi dx + \int_{B_2} uF_\varepsilon(x) dx, \quad (2.7) \]
where each term of $F_\varepsilon(x)$ involves the derivatives of $\eta_\varepsilon$ up to order $2m$, so it satisfies
\[ |F_\varepsilon(x)| \leq C\varepsilon^{-2m} \cdot \chi_{\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon}(x). \quad (2.8) \]

Since $u^{n+2m} \in L^1(B_2)$, by the dominated convergence theorem
\[ \int_{B_2} u^{n+2m} \frac{n+2m}{n-2m} \psi_\varepsilon dx \to \int_{B_2} u^{n+2m} \frac{n+2m}{n-2m} \varphi dx \quad \text{and} \quad \int_{B_2} u\eta_\varepsilon (-\Delta)^m \varphi dx \to \int_{B_2} u(-\Delta)^m \varphi dx \]
as $\varepsilon \to 0$. On the other hand, by Hölder inequality we have
\[ \left| \int_{B_2} uF_\varepsilon(x) dx \right| \leq C \left( \int_{\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon} u^{n+2m} \frac{n+2m}{n-2m} dx \right)^{\frac{n-2m}{n+2m}} \left( \int_{\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon} |F_\varepsilon|^{\frac{n+2m}{n-2m}} dx \right)^{\frac{n-2m}{n+2m}} \]
\[ \leq C\varepsilon^{-2m} \cdot \mathcal{L}^n(\mathcal{N}_\Lambda) \frac{4m}{n+2m} \cdot \left( \int_{\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon} u^{n+2m} \frac{n+2m}{n-2m} dx \right)^{\frac{n-2m}{n+2m}} \]
\[ \leq C\varepsilon^{4m(n-\lambda)/n+2m} \cdot \left( \int_{\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon} u^{n+2m} \frac{n+2m}{n-2m} dx \right)^{\frac{n-2m}{n+2m}} \to 0 \]
as $\varepsilon \to 0$ because of $\frac{4m(n-\lambda)}{n+2m} - 2m \geq 0$ and $\lim_{\varepsilon \to 0} \mathcal{L}^n(\mathcal{N}_\Lambda \setminus \mathcal{N}_\varepsilon) = 0$. Thus, $u$ is a distributional solution in the entire ball $B_2$. \hfill \Box

Suppose $n > 2m$. Let $G_m(x, y)$ be the Green function of $(-\Delta)^m$ on $B_2$ under the Navier boundary condition:
\[ \begin{cases} (-\Delta)^m G_m(x, \cdot) = \delta_x & \text{in } B_2, \\ G_m(x, \cdot) = -\Delta G_m(x, \cdot) = \cdots = (-\Delta)^{m-1} G_m(x, \cdot) = 0 & \text{on } \partial B_2, \end{cases} \quad (2.9) \]
where $\delta_x$ is the Dirac measure to the point $x \in B_2$. Then we have, for any $u \in C^{2m}(B_1) \cap C^{2m-2}(\overline{B_1})$,
\[ u(x) = \int_{B_2} G_m(x, y)(-\Delta)^m u(y) dy + \sum_{i=1}^m \int_{\partial B_2} H_i(x, y)(-\Delta)^{i-1} u(y) dS_y, \]

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where
\[ H_i(x, y) = -\frac{\partial}{\partial y}(-\Delta y)^{m-i}G_m(x, y) \] for \( x \in B_2, y \in \partial B_2 \).

By direct computations, we have
\[ G_m(x, y) = c_{n,m}|x - y|^{2m-n} + A_m(x, y), \tag{2.10} \]
\[ c_{n,m} = \frac{\Gamma(\frac{n}{2} - m)}{2^{2m-2n}\pi^m}, \quad A_m(x, y) \text{ is smooth in } B_2 \times B_2, \quad H_i(x, y) \geq 0, \quad i = 1, \ldots, m. \tag{2.11} \]

**Proposition 2.3.** Assume as in Proposition 2.1. Let \( u \in C^{2m}(B_2 \setminus \Lambda) \) be a positive solution of (1.4). Then
\[ u(x) = \int_{B_2} G_m(x, y)u(y)^{n+2m-2m}dy + \sum_{i=1}^{m} \int_{\partial B_2} H_i(x, y)(-\Delta)^{i-1}u(y)dSy \tag{2.12} \]
for all \( x \in B_2 \setminus \Lambda \).

**Proof.** For any \( x \in B_2 \setminus \Lambda \), define
\[ v(x) = \int_{B_2} G_m(x, y)u(y)^{n+2m-2m}dy + \sum_{i=1}^{m} \int_{\partial B_2} H_i(x, y)(-\Delta)^{i-1}u(y)dSy. \]

Since \( u(y)^{n+2m} \in L^1(B_2) \) and the Riesz potential \( |x|^{2m-n} \) is weak type \( 1, \frac{n}{n-2m} \), \( v \in L^\frac{n}{n-2m}(B_2) \cap L^1(B_2) \). Let \( w = u - v \). Then, by Proposition 2.1, \( w \) satisfies
\[ (-\Delta)^m w = 0 \quad \text{in } B_2 \]
in the distributional sense, i.e., for any \( \varphi \in C_c^\infty(B_2) \),
\[ \int_{B_2} w(-\Delta)^m \varphi dx = 0. \]

By the regularity for polyharmonic functions (see, e.g., Mitrea [35]), we know that \( w \in C^\infty(B_2) \) is smooth and satisfies \( (-\Delta)^m w = 0 \) pointwise in \( B_2 \). Since \( w = -\Delta w = \cdots = (-\Delta)^{m-1}w = 0 \) on \( \partial B_2, w \equiv 0 \) and thus \( u = v \) in \( B_2 \setminus \Lambda \).

Further, if assume additionally that (1.5) holds, then one can show that \( u \) satisfies the integral equation (1.13) in some local sense. Namely,

**Theorem 2.4.** Assume as in Theorem 1.1. Then there exists \( \tau > 0 \) (independent of \( x \in \Lambda \)) such that for any \( x_0 \in \Lambda \) we have
\[ u(x) = c_{n,m} \int_{B_\tau(x_0)} \frac{u(y)^{n+2m}}{|x - y|^{n-2m}}dy + h_1(x) \quad \text{for } x \in B_\tau(x_0) \setminus \Lambda, \tag{2.13} \]
where \( h_1(x) \) is a positive smooth function in \( B_\tau(x_0) \).
Proof. Without loss of generality, we can assume that $u \in C^{2m}(\overline{B}_2 \setminus \Lambda)$ and $u > 0$ in $\overline{B}_2 \setminus \Lambda$. Otherwise, we just consider the equation in a smaller ball.

It follows from the assumptions on the singular set $\Lambda$ that $\mathcal{H}^{n-2}(\Lambda) = 0$ and hence $\text{Cap}(\Lambda) = 0$, where $\mathcal{H}^{n-2}$ is the $(n - 2)$-dimensional Hausdorff measure on $\mathbb{R}^n$ and $\text{Cap}(\Lambda)$ is the Newton capacity of $\Lambda$ (see, e.g., [11]). Since $u > 0$ and $-\Delta u \geq 0$ in $B_1 \setminus \Lambda$, by the maximum principle (see, e.g., Lemma 2.1 of [6])

$$u(x) \geq c_1 := \inf_{\partial B_2} u > 0 \quad \text{for all } x \in \overline{B}_2 \setminus \Lambda.$$ 

By Proposition 2.1, $u^{n+2m/n-2m} \in L^1(B_2)$. Hence, there exists $0 < \tau < \frac{1}{4}$ independent of $z \in B_1$ such that

$$\int_{B_\tau(z)} |A_m(x, y) u(y)\frac{n+2m}{n-2m} dy < \frac{c_1}{2} \quad \text{for all } x \in B_\tau(z) \subset B_{3/2},$$

where $A_m(x, y)$ is as in (2.10). By Proposition 2.3 for every $x_0 \in \Lambda$ we can write

$$u(x) = c_{n,m} \int_{B_{\tau}(x_0)} \frac{u(y)\frac{n+2m}{n-2m}}{|x-y|^{n-2m}} dy + h_1(x), \quad \text{for } x \in B_\tau(x_0) \setminus \Lambda,$$

where

$$h_1(x) = \int_{B_{\tau}(x_0)} A_m(x, y) u(y)\frac{n+2m}{n-2m} dy + c_{n,m} \int_{B_2 \setminus B_{\tau}(x_0)} G_m(x, y) u(y)\frac{n+2m}{n-2m} dy

+ \sum_{i=1}^m \int_{\partial B_2} H_i(x, y)(-\Delta)^{i-1} u(y) dS_y

\geq -\frac{c_1}{2} + \int_{\partial B_2} H_1(x, y) u(y) dS_y

\geq -\frac{c_1}{2} + \inf_{\partial B_2} u = \frac{c_1}{2} > 0 \quad \text{for } x \in B_\tau(x_0).$$

It is easy to check that $h_1$ is a smooth function in $B_\tau(x_0)$ and satisfies $(-\Delta)^n h_1 = 0$ in $B_\tau(x_0)$. This completes the proof. \hfill \square

### 2.2 Global singular solutions

In this subsection, we show that if $0 \leq k \leq \frac{n-2m}{2}$, then every global singular positive solution of (1.9) satisfies the integral equation (1.16).

**Proposition 2.5.** Suppose that $1 \leq m < n/2$ and $m$ is an integer. Let $0 \leq k \leq \frac{n-2m}{2}$ and $u \in C^{2m}(\mathbb{R}^n \setminus \mathbb{R}^k)$ be a nonnegative solution of (1.9). Then $u \in L^{n+2m}_{\text{loc}}(\mathbb{R}^n)$ and $u$ is a distributional solution in $\mathbb{R}^n$.

**Proof.** The proof is similar to that of Proposition 2.1, the only difference is that we have to choose an additional truncation function. Let $R > 0$ and take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on $B_{R/2}$ and $\varphi = 0$ on $B_{R/2}^c$. Denote $\mathcal{N}_r := \{x \in \mathbb{R}^n \mid \text{dist}(x, \mathbb{R}^k) < r\}$. For small $\varepsilon > 0$, as in the proof of Proposition 2.1 consider

$$\eta_\varepsilon(x) := 1 - \int_{\mathcal{N}_{2\varepsilon}} \rho_\varepsilon(x-y) dy,$$

(2.14)
where \( \rho \in C_c^\infty(B_1) \) with \( \int_{B_1} \rho \, dx = 1 \) and \( \rho_{\varepsilon}(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right) \). Then \( \eta_{\varepsilon} \in C_c^\infty(\mathbb{R}^n) \) is a non-negative function and satisfies (2.3) and (2.4). Let \( \varphi_{\varepsilon}(x) = \left(\left[\varphi \eta_{\varepsilon}\right](x)\right)^q \) with \( q = \frac{n+2m}{2} \). Multiplying both sides of (1.9) by \( \varphi_{\varepsilon} \) and using integration by parts, we obtain

\[
\int_{\mathbb{R}^n} u^{\frac{n+2m}{n-2m}} \varphi_{\varepsilon} \, dx = \int_{\mathbb{R}^n} u(-\Delta)^m \varphi_{\varepsilon} \, dx 
\leq C \int_{B_R} u(\varphi_{\varepsilon})^{\frac{n-2m}{n+2m}} \, dx + C \varepsilon^{-2m} \int_{B_R \cap (N_{3\varepsilon} \setminus N_{\varepsilon})} u(\varphi_{\varepsilon})^{\frac{n-2m}{n+2m}} \, dx 
\leq C \left(1 + \varepsilon^{\frac{4m(n-k)}{n+2m}}\right) \left(\int_{\mathbb{R}^n} u^{\frac{n+2m}{n-2m}} \varphi_{\varepsilon} \, dx\right)^{\frac{n-2m}{n+2m}},
\]

from which it follows that

\[
\int_{B_{R/2} \cap N_{3\varepsilon}^c} u^{\frac{n+2m}{n-2m}} \, dx \leq \int_{\mathbb{R}^n} u^{\frac{n+2m}{n-2m}} \varphi_{\varepsilon} \, dx \leq C.
\]

By sending \( \varepsilon \to 0 \), we obtain

\[
\int_{B_{R/2}} u^{\frac{n+2m}{n-2m}} \, dx \leq C.
\]

Thus, \( u \in L_{n-2m}^{n+2m}(\mathbb{R}^n) \) since \( R > 0 \) is arbitrary. The proof of which \( u \) is a distributional solution in \( \mathbb{R}^n \) is very similar to that of Proposition 2.1, so we omit the details.

Now we give some growth estimates for solutions of (1.9) at infinity.

**Lemma 2.6.** Assume as in Proposition 2.5. Let \( u \in C^{2m}(\mathbb{R}^n \setminus \mathbb{R}^k) \) be a non-negative solution of (1.9). Then

\[
\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^\gamma} \, dx < +\infty \quad \text{for every } \gamma > \frac{n+2m}{2} \tag{2.15}
\]

and

\[
\int_{\mathbb{R}^n} \frac{u(x)^{\frac{n+2m}{n-2m}}}{1 + |x|^\gamma} \, dx < +\infty \quad \text{for every } \gamma > \frac{n-2m}{2}. \tag{2.16}
\]

**Proof.** The estimate (2.15) follows from Lemma 5.5 in [3]. Next we show (2.16). By the proof of Lemma 5.5 in [3], we have

\[
\int_{B_R} u^{\frac{n+2m}{n-2m}} \, dy \leq CR^{\frac{n+2m}{n-2m}} \quad \text{for every } R > 0. \tag{2.17}
\]
Therefore, for every \( \gamma > \frac{n-2m}{2} \), using (2.17) we obtain
\[
\int_{\mathbb{R}^n} u(x) \frac{n+2m}{1 + |x|^\gamma} \, dx = \int_{B_1} \frac{u(x) n+2m}{1 + |x|^\gamma} \, dx + \sum_{i=1}^{\infty} \int_{B_{2^i} \setminus B_{2^{i-1}}} \frac{u(x) n+2m}{1 + |x|^\gamma} \, dx
\leq C \int_{B_1} u(x) \frac{n+2m}{n-2m} \, dx + \sum_{i=1}^{\infty} \int_{B_{2^i} \setminus B_{2^{i-1}}} u(x) \frac{n+2m}{n-2m} \, dx \cdot 2^{-\gamma(i-1)}
\leq C + C \sum_{i=1}^{\infty} (2^i)^{\frac{n-2m}{2} - \gamma} < +\infty.
\]

This completes the proof. \( \square \)

Let \( 0 \leq k \leq \frac{n-2m}{2} \) and \( u \in C^2_m(\mathbb{R}^n \setminus \mathbb{R}^k) \) be a nonnegative solution of (1.9). Then by Lemma 2.6 the following function
\[
v(x) := c_{n,m} \int_{\mathbb{R}^n} \frac{u(y) n+2m}{|x-y|^{n-2m}} \, dy \quad (2.18)
\]
is well-defined for every \( x \in \mathbb{R}^n \setminus \mathbb{R}^k \), and it is continuous on \( \mathbb{R}^n \setminus \mathbb{R}^k \). In addition, for any \( R > 0 \), we write \( v = v_{1,R} + v_{2,R} \), where
\[
v_{1,R}(x) = \int_{B_{2R}^c} \frac{u(y) n+2m}{|x-y|^{n-2m}} \, dy \quad \text{and} \quad v_{2,R}(x) = \int_{B_{2R}^c} \frac{u(y) n+2m}{|x-y|^{n-2m}} \, dy.
\]
Since \( u^{n-2m} \in L^1(B_{2R}) \), we have \( v_{1,R} \in L^1(B_R) \). From Lemma 2.6 we easily know \( v_{2,R} \in L^\infty(B_R) \). Hence we obtain \( v \in L^1_{loc}(\mathbb{R}^n) \). Define, for any \( \gamma \in \mathbb{R} \),
\[
L_\gamma(\mathbb{R}^n) := \left\{ u \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\gamma}} \, dx < \infty \right\}.
\]
Moreover, we have the following property for \( v \).

**Lemma 2.7.** Assume as in Proposition 2.3 and \( v \) is defined by (2.18). Then we have \( v \in L_0(\mathbb{R}^n) \).

**Proof.** By Fubini’s theorem, we have
\[
\int_{\mathbb{R}^n} \frac{v(x)}{1 + |x|^\gamma} \, dx = c_{n,m} \int_{\mathbb{R}^n} u(y) \frac{n+2m}{n-2m} \left( \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2m}} \frac{1}{1 + |x|^\gamma} \, dx \right) \, dy. \quad (2.19)
\]
If \( |y| \leq 1 \), then
\[
\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2m}} \frac{1}{1 + |x|^\gamma} \, dx \leq \int_{B_3} \frac{1}{|x-y|^{n-2m}} \, dx + C \int_{B_3^c} \frac{1}{|x-y|^{n-2m}} \frac{1}{1 + |x|^\gamma} \, dx
\leq C < \infty.
\]
If \( |y| > 1 \), then
\[
\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2m}} \frac{1}{1 + |x|^\gamma} \, dx = \sum_{i=1}^{3} I_i,
\]

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where

\[
I_1 = \int_{\{|x|<|y|\}} \frac{1}{|x-y|^{n-2m}} \frac{1}{1+|x|^n} dx \leq \frac{C \ln(1+|y|^n)}{|y|^{n-2m}},
\]

\[
I_2 = \int_{\{|y/2|<|x|<|y|\}} \frac{1}{|x-y|^{n-2m}} \frac{1}{1+|x|^n} dx \leq \frac{C}{|y|^n} \int_{\{|y/2|<|x|<|y|\}} \frac{1}{|x-y|^{n-2m}} dx \leq \frac{C}{|y|^{n-2m}},
\]

and

\[
I_3 = \int_{\{|x|\geq|y|\}} \frac{1}{|x-y|^{n-2m}} \frac{1}{1+|x|^n} dx \leq C \int_{\{|x|\geq|y|\}} \frac{1}{|x|^{2n-2m}} dx \leq \frac{C}{|y|^{n-2m}}.
\]

All of these estimates together with (2.19) and (2.16) give

\[
\int_{\mathbb{R}^n} \frac{v(x)}{1+|x|^n} dx \leq C \int_{\mathbb{R}^n} u(y) \frac{n+2m}{n-2m} dy + C \int_{\mathbb{R}^n} \ln(1+|y|^n) u(y) \frac{n+2m}{n-2m} dy < \infty.
\]

Thus, we obtain \(v \in L_0(\mathbb{R}^n)\). \(\square\)

Estimate (2.16) and Lemma 2.7 are an improvement of Lemma 5.4 and Lemma 5.6 in Aogonzalez-Hyder-Wei, respectively, when the order of the equation in (1.9) is an integer. Now we can show the integral representation for global singular solutions of (1.9).

**Theorem 2.8.** Suppose that \(1 \leq m < n/2\) and \(m\) is an integer. Let \(0 \leq k \leq \frac{n-2m}{2}\) and \(u \in C^{2m}(\mathbb{R}^n \setminus \mathbb{R}^k)\) be a nonnegative solution of (1.9). Then \(u \in L_{loc}^{\frac{n+2m}{n-2m}}(\mathbb{R}^n)\) and \(u\) satisfies

\[
u(x) = c_{nm} \int_{\mathbb{R}^n} u(y) \frac{n+2m}{n-2m} dy \quad \text{for } x \in \mathbb{R}^n \setminus \mathbb{R}^k.\]  

(2.20)

**Proof.** Our proof is inspired by that of Theorem 1.8 in [3], where the superharmonicity property for the fractional Laplacian equations in \(\mathbb{R}^n\) was showed. Define \(v\) as in (2.18). Then \(v \in L_{loc}^1(\mathbb{R}^n)\) and it satisfies

\[
(-\Delta)^m v = u \frac{n+2m}{n-2m} \quad \text{in } \mathbb{R}^n
\]

(2.21)

in the distributional sense, i.e., for any \(\varphi \in C_c^\infty(\mathbb{R}^n)\),

\[
\int_{\mathbb{R}^n} v(-\Delta)^m \varphi dx = \int_{\mathbb{R}^n} u \frac{n+2m}{n-2m} \varphi dx.
\]

Let \(w = u - v\). Then, using Proposition 2.5 we have that \((-\Delta)^m w = 0\) in \(\mathbb{R}^n\) in the distributional sense. By the regularity of polyharmonic functions (see, e.g., [3]), \(w \in C^\infty(\mathbb{R}^n)\) and \((-\Delta)^m w \equiv 0\) in the classical sense. On the other hand, from Lemmas 2.6 and 2.7 we know that both \(u\) and \(v\) belong to \(L_0(\mathbb{R}^n)\) and hence \(w \in L_0(\mathbb{R}^n)\). It follows from a Liouville type theorem (see, e.g., [3,5]) that \(w \equiv 0\) in \(\mathbb{R}^n\). Thus, we have proved that

\[
u(x) = c_{nm} \int_{\mathbb{R}^n} u(y) \frac{n+2m}{n-2m} dy \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in \mathbb{R}^n.
\]

Recall that \(u, v \in C(\mathbb{R}^n \setminus \mathbb{R}^k)\). Therefore (2.20) holds for any \(x \in \mathbb{R}^n \setminus \mathbb{R}^k\). \(\square\)
3 Local estimate near a singular set

In this section, we first prove Theorem 1.5 and then use it and the local integral representation in Subsection 2.1 to show Theorem 1.1.

For \( x \in \mathbb{R}^n \), \( \lambda > 0 \) and a function \( u \), we denote

\[
\xi^{x,\lambda} = x + \frac{\lambda^2 (\xi - x)}{|\xi - x|^2} \quad \text{for} \quad \xi \neq x, \quad \Omega^{x,\lambda} = \{ \xi^{x,\lambda}, \xi \in \Omega \},
\]

and

\[
u_{x,\lambda}(\xi) = \left( \frac{\lambda}{|\xi - x|} \right)^{n-2\sigma} u(\xi^{x,\lambda}).
\]

Note that \( (\xi^{x,\lambda})^{x,\lambda} = \xi \) and \( u_{x,\lambda} x,\lambda = u \). If \( x = 0 \), we use the notation \( u_{\lambda} = u_{0,\lambda} \).

Suppose \( u \in L^{n+2\sigma}_{\nu,\lambda}(B_2) \cap C(B_2 \setminus \Sigma) \) is a positive solution of (1.13), and suppose \( h \in C^1(B_2) \) is a positive function satisfying

\[
|\nabla \ln h| \leq \dot{C} \quad \text{in} \quad B_{3/2}
\]

for some constant \( \dot{C} > 0 \). If we extend \( u \) to be identically 0 outside \( B_2 \), then we have

\[
u(x) = \int_{\mathbb{R}^n} \frac{u(y) n+2\sigma}{|x - y|^{n-2\sigma}} dy + h(x) \quad \text{for} \quad x \in B_2 \setminus \Sigma. \tag{3.1}
\]

Recalling the following two identities (see, e.g., [27]),

\[
\left( \frac{\lambda}{|\xi - x|} \right)^{n-2\sigma} \int_{|z - x| \geq \lambda} u(z) \frac{n+2\sigma}{n-2\sigma} |z|^{n-2\sigma} dz = \int_{|z - x| \leq \lambda} u_{x,\lambda}(z) \frac{n+2\sigma}{n-2\sigma} |z|^{n-2\sigma} dz \tag{3.2}
\]

and

\[
\left( \frac{\lambda}{|\xi - x|} \right)^{n-2\sigma} \int_{|z - x| \leq \lambda} u(z) \frac{n+2\sigma}{n-2\sigma} |z|^{n-2\sigma} dz = \int_{|z - x| \geq \lambda} u_{x,\lambda}(z) \frac{n+2\sigma}{n-2\sigma} |z|^{n-2\sigma} dz, \tag{3.3}
\]

one has

\[
u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} \frac{u_{x,\lambda}(z) n+2\sigma}{|\xi - z|^{n-2\sigma}} dz + h_{x,\lambda}(\xi) \quad \text{for} \quad \xi \in (B_2 \setminus \Sigma)^{x,\lambda}. \tag{3.4}
\]

Thus, for any \( x \in B_1 \) and \( \lambda < 1 \), we have for any \( \xi \in B_2 \setminus (\Sigma \cup \Sigma^{x,\lambda} \cup B_\lambda(x)) \) that

\[
u(\xi) - u_{x,\lambda}(\xi) = \int_{|z - x| \geq \lambda} K(x, \lambda; \xi, z) [u(z) \frac{n+2\sigma}{n-2\sigma} - u_{x,\lambda}(z) \frac{n+2\sigma}{n-2\sigma}] dz + h_{x,\lambda}(\xi) - h(\xi),
\]

where

\[
K(x, \lambda; \xi, z) = \frac{1}{|\xi - z|^{n-2\sigma}} - \left( \frac{\lambda}{|\xi - x|} \right)^{n-2\sigma} \frac{1}{|\xi^{x,\lambda} - z|^{n-2\sigma}}.
\]

It is elementary to check that

\[
K(x, \lambda; \xi, z) > 0 \quad \text{for all} \quad |\xi - x|, |z - x| > \lambda > 0.
\]
Next, we shall prove Theorem 1.5 using the method of moving spheres introduced by Li and Zhu [27, 29] and some blow up arguments developed in Jin-Li-Xiong [19].

**Proof of Theorem 1.5.** Suppose by contradiction that there exists a sequence \( \{x_j\}_{j=1}^\infty \subset B_1 \setminus \Sigma \) such that

\[
d_j := \text{dist}(x_j, \Sigma) \to 0 \quad \text{as } j \to \infty,
\]

but

\[
d_j^{\frac{n-2\sigma}{2}} u(x_j) \to \infty \quad \text{as } j \to \infty.
\]

We may assume, without loss of generality, that \( 0 \in \Sigma \) and \( x_j \to 0 \) as \( j \to \infty \).

Consider

\[
v_j(x) := \left( \frac{d_j}{2} - |x - x_j| \right) \frac{n-2\sigma}{2} u(x), \quad |x - x_j| \leq \frac{d_j}{2}.
\]

Since \( u \) is positive and continuous in \( B_{d_j/2}(x_j) \), we can find a point \( \bar{x}_j \in B_{d_j/2}(x_j) \) satisfying

\[
v_j(\bar{x}_j) = \max_{|x - x_j| \leq \frac{d_j}{2}} v_j(x) > 0.
\]

Let \( 2\mu_j := \frac{d_j}{2} - |\bar{x}_j - x_j| \). Then

\[
0 < 2\mu_j \leq \frac{d_j}{2} \quad \text{and} \quad \frac{d_j}{2} - |x - x_j| \geq \mu_j \quad \forall \ |x - \bar{x}_j| \leq \mu_j.
\]

By the definition of \( v_j \), we have

\[
(2\mu_j)^{\frac{n-2\sigma}{2}} u(\bar{x}_j) = v_j(\bar{x}_j) \geq v_j(x) \geq \mu_j^{\frac{n-2\sigma}{2}} u(x) \quad \forall \ |x - \bar{x}_j| \leq \mu_j. \tag{3.5}
\]

Hence, we have

\[
2^{\frac{n-2\sigma}{2}} u(\bar{x}_j) \geq u(x) \quad \forall \ |x - \bar{x}_j| \leq \mu_j. \tag{3.6}
\]

We also have

\[
(2\mu_j)^{\frac{n-2\sigma}{2}} u(\bar{x}_j) = v_j(\bar{x}_j) \geq v_j(x_j) = \left( \frac{d_j}{2} \right)^{\frac{n-2\sigma}{2}} u(x_j) \to \infty \quad \text{as } j \to \infty. \tag{3.7}
\]

Now, define

\[
w_j(y) = \frac{1}{u(\bar{x}_j)} u\left( \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{1}{n-2\sigma}}} \right), \quad h_j(y) = \frac{1}{u(\bar{x}_j)} h\left( \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{1}{n-2\sigma}}} \right)
\]

in \( \Omega_j \),

where

\[
\Omega_j = \left\{ y \in \mathbb{R}^n : \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{1}{n-2\sigma}}} \in B_2 \setminus \Sigma \right\}.
\]

We extend \( w_j \) to be 0 outside of \( \Omega_j \). Then \( w_j \) satisfies \( w_j(0) = 1 \) and

\[
w_j(y) = \int_{\mathbb{R}^n} \frac{w_j(z) u(\bar{x}_j)^{\frac{n-2\sigma}{n}}}{|y - z|^{n-2\sigma}} \, dz + h_j(y) \quad \text{for } y \in \Omega_j. \tag{3.8}
\]
Moreover, it follows from (3.6) and (3.7) that
\[ \| h_j \|_{C^1(B_{R_j})} \to 0, \quad w_j(y) \leq 2^{\frac{n - 2\sigma}{2}} \text{ in } B_{R_j}, \]
where
\[ R_j := \mu_j u(x_j) \frac{2}{n - 2\sigma} \to \infty \text{ as } j \to \infty. \]

**Claim 1:** There exists a function \( w > 0 \) such that, after passing to a subsequence, \( w_j \to w \) in \( C^\alpha_{\text{loc}}(\mathbb{R}^n) \) for some \( \alpha > 0 \) and \( w \) satisfies
\[ w(y) = \int_{\mathbb{R}^n} \frac{w(z)}{|y - z|^{\frac{n-2\sigma}{2}}} dz \text{ for } y \in \mathbb{R}^n. \] (3.9)

Since for any \( R > 0 \) we have \( w_j(y) \leq 2^{\frac{n - 2\sigma}{2}} \) in \( B_R \) for all large \( j \), by the regularity results in Section 2.1 of [19] there exists \( w \geq 0 \) such that, after passing to a subsequence if necessary,
\[ w_j \to w \text{ in } C^\alpha_{\text{loc}}(\mathbb{R}^n) \]
for some \( \alpha > 0 \). Clearly \( w(0) = 1 \). To prove that \( w \) satisfies the integral equation (3.9), we follow the argument of Proposition 2.9 in [19]. Write (3.8) as
\[ w_j(y) = \int_{B_R} w_j(z) \frac{2^{\frac{n + 2\sigma}{n - 2\sigma}}}{|y - z|^{n-2\sigma}} dz + h_j(R, y) \text{ for } y \in \Omega_j, \]
where
\[ h_j(R, y) = \int_{B_R} w_j(z) \frac{2^{\frac{n + 2\sigma}{n - 2\sigma}}}{|y - z|^{n-2\sigma}} dz + h_j(y). \]

Then, for \( y \in B_{R/2} \) we have
\[ h_j(R, y) = \int_{B_R} \frac{|z|^{n - 2\sigma}}{|y - z|^{n-2\sigma}} w_j(z) \frac{2^{\frac{n + 2\sigma}{n - 2\sigma}}}{|z|^{n-2\sigma}} dz + h_j(y) \]
\[ \leq C \int_{B_R} w_j(z) \frac{2^{\frac{n + 2\sigma}{n - 2\sigma}}}{|z|^{n-2\sigma}} dz + \| h_j \|_{L^\infty(B_{R/2})} \]
\[ \leq Cw_j(0) + \| h_j \|_{L^\infty(B_{R/2})} \]
for all large \( j \). Similarly, for \( y \in B_{R/2} \),
\[ |\nabla h_j(R, y)| \leq C(R) w_j(0) + \| \nabla h_j \|_{L^\infty(B_{R/2})}. \]

From these we get \( \| h_j(R, \cdot) \|_{C^1(B_{R/2})} \leq C(R) \) for all \( j \) large. Thus, after passing to a subsequence, \( h_j(R, \cdot) \to h(R, \cdot) \) in \( C^{1/2}(B_{R/2}) \). Therefore,
\[ h(R, y) = w(y) - \int_{B_R} \frac{w(z) \frac{2^{\frac{n + 2\sigma}{n - 2\sigma}}}{|y - z|^{n-2\sigma}} dz}{|y - z|^{\frac{n-2\sigma}{2}}} \] (3.10)
for \( y \in B_{R/2} \). Moreover, \( h(R, y) \) is nonnegative and non-increasing in \( R \). Notice that when \( R >> |y| \),

\[
\frac{R^{n-2\sigma}}{(R + |y|)^{n-2\sigma}}(h_j(R, 0) - h_j(0)) \leq h_j(R, y) - h_j(y) \leq \frac{R^{n-2\sigma}}{(R - |y|)^{n-2\sigma}}(h_j(R, 0) - h_j(0)).
\]

Let \( j \) tend to \( \infty \), we get

\[
\frac{R^{n-2\sigma}}{(R + |y|)^{n-2\sigma}} h(R, 0) \leq h(R, y) \leq \frac{R^{n-2\sigma}}{(R - |y|)^{n-2\sigma}} h(R, 0),
\]

which implies that \( \lim_{R \to \infty} h(R, y) = \lim_{R \to \infty} h(R, 0) =: c_0 \geq 0 \). Let \( R \) tend to \( \infty \) in (3.10), from Lebesgue’s monotone convergence theorem,

\[
w(y) = \int_{\mathbb{R}^n} w(z) \frac{n+2\sigma}{n-2\sigma} |z|^{n-2\sigma} dz + c_0 \quad \text{for } y \in \mathbb{R}^n.
\]

We claim that \( c_0 = 0 \). If not, then \( w(y) \geq c_0 \) for all \( y \in \mathbb{R}^n \) and thus

\[
1 = w(0) \geq \int_{\mathbb{R}^n} \frac{n+2\sigma}{c_0} |z|^{n-2\sigma} dz = \infty.
\]

This is impossible. Claim 1 is proved.

Since \( w(0) = 1 \), by the classification results in [27] or [8], we have

\[
w(y) = \left( \frac{1 + \mu^2|y_0|^2}{1 + \mu^2|y - y_0|^2} \right)^{\frac{n-2\sigma}{2}} \quad \text{(3.11)}
\]

for some \( \mu > 0 \) and some \( y_0 \in \mathbb{R}^n \).

On the other hand, we will show that, for every \( \lambda > 0 \),

\[
w(\lambda) \leq w(y) \quad \forall |y| \geq \lambda. \quad \text{(3.12)}
\]

By the proof of Theorem 1.1 in [27], (3.12) implies that \( w \equiv constant \). This contradicts to (3.11).

Let us fix \( \lambda_0 > 0 \) arbitrarily. Then for all \( j \) large, we have \( 0 < \lambda_0 < \frac{R_j}{10} \). Denote

\[
\Xi_j := \left\{ y \in \mathbb{R}^n : \bar{x} + \frac{y}{u(\bar{x}, j)^{\frac{2}{n-2\sigma}}} \in B_1 \setminus \Sigma \right\} \subset \Omega_j.
\]

We are going to show that for all sufficiently large \( j \),

\[
(w_j)\lambda_0(y) \leq w_j(y) \quad \forall |y| \geq \lambda_0, \ y \in \Xi_j. \quad \text{(3.13)}
\]

Then (3.12) follows from (3.13) by sending \( j \to \infty \).
It follows from the same arguments as in Lemma 3.1 of [20] that there exists a constant $\bar{r} > 0$ depending only on $n, \sigma, \tilde{C}$ and $\lambda_0$ such that for all large $j$ with $u(\bar{y})^{-n+2\sigma}/r^2 < \bar{r}$ there holds
\[
(h_j)_\lambda(y) \leq h_j(y) \quad \forall \, y \in \Xi_j \setminus B_\lambda, \quad 0 < \lambda \leq \lambda_0 + \frac{1}{2},
\] (3.14)

Claim 2: There exists a real number $\lambda_1 > 0$ independent of (large) $j$ such that for every $0 < \lambda < \lambda_1$, we have
\[
(w_j)_\lambda(y) \leq w_j(y) \quad \text{in} \ \Xi_j \setminus B_\lambda.
\]

Since $w_j \to w$ locally uniformly and $w$ is given in (3.11), we have that $w_j \geq c_0 > 0$ on $B_1$ for all $j$ sufficiently large. On the other hand, from the equation (3.8) and the regularity results in [19] we know that $|\nabla w_j| \leq C_0 < \infty$ on $B_1$ for all $j$ sufficiently large. By the proof of Lemma 3.1 in [20] (see (20) there), there exists a $r_0 > 0$ independent of (large) $j$ such that for all $0 < \lambda \leq r_0$
\[
(w_j)_\lambda(y) \leq w_j(y), \quad 0 < \lambda < |y| \leq r_0.
\] (3.15)

Since $w_j \geq c_0 > 0$ on $B_1$ for all $j$ sufficiently large, we also have
\[
w_j(y) \geq c_0^{n+2\sigma/2} \int_{B_1} |y-z|^{2\sigma-n}dz \geq \frac{1}{C} (1 + |y|)^{2\sigma-n} \quad \text{in} \ \Omega_j
\]
for some constant $C > 0$. Therefore, we can find a small $0 < \lambda_1 \leq r_0$ independent of (large) $j$ such that for every $0 < \lambda < \lambda_1$
\[
(w_j)_\lambda(y) \leq \left(\frac{\lambda_1}{|y|}\right)^{n-2\sigma} \max_{B_{r_0}} w \leq C \left(\frac{\lambda_1}{|y|}\right)^{n-2\sigma} \leq w_j(y) \quad \text{for all} \quad y \in \Omega_j \setminus B_{r_0}.
\]

Together with (3.15), Claim 2 is proved.

We define
\[
\bar{\lambda}_j := \sup \{0 < \mu \leq \lambda_0 \mid (w_j)_\lambda(y) \leq w_j(y), \quad \forall \ |y| \geq \lambda, \ y \in \Xi_j, \ \forall \ 0 < \lambda < \mu\},
\]
where $\lambda_0$ is fixed at the beginning. By Claim 2, $\bar{\lambda}_j$ is well defined and $\bar{\lambda}_j \geq 1 > 0$ for all sufficiently large $j$.

Claim 3: $\bar{\lambda}_j = \lambda_0$ for all sufficiently large $j$.

By (3.2) and (3.3), we have for any $\bar{\lambda}_j \leq \lambda \leq \bar{\lambda}_j + \frac{1}{2}$ and $y \in \Xi_j \setminus B_\lambda$ that
\[
w_j(y) - (w_j)_\lambda(y)
= \int_{B_\lambda^c} K(0, \lambda; y, z) \left( w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} - (w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz + h_j(y) - (h_j)_\lambda(y)
\geq \int_{\Xi_j \setminus B_\lambda} K(0, \lambda; y, z) \left( w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} - (w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz + J(\lambda, w_j, y),
\] (3.16)

where we have used (3.14) and
\[
J(\lambda, w_j, y) = \int_{B_\lambda^c \setminus \Xi_j} K(0, \lambda; y, z) \left( w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} - (w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz
\geq \int_{\Omega_j \setminus \Xi_j} K(0, \lambda; y, z) \left( w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} - (w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz - \int_{\Omega_j^c} K(0, \lambda; y, z)(w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} dz.
\] (3.17)
Let 
\[ \Sigma_j := \left\{ y \in \mathbb{R}^n : \bar{x}_j + \frac{y}{u(\bar{x}_j)^{n-2\sigma}} \in \Sigma \right\}. \]

Then \( \mathcal{L}^n(\Sigma_j) = 0 \). For any \( z \in \mathbb{R}^n \setminus (\Xi_j \cup \Sigma_j) \) and \( \lambda_j \leq \lambda \leq \bar{\lambda}_j + 1 \), we have \( |z| \geq \frac{1}{2} u(\bar{x}_j)^{\frac{2}{n-2\sigma}} \) and thus
\[ (w_j)_\lambda(z) \leq \left( \frac{\lambda}{|z|} \right)^{n-2\sigma} \max_{B_{\lambda_0+1}} w_j \leq C u(\bar{x}_j)^{-2}. \]

By the equation (1.13), we have
\[ u(x) \geq 4^{2\sigma-n} \int_{B_2} u(y)^{\frac{n+2\sigma}{n-2\sigma}} \, dy =: c_1 > 0 \quad \text{for all } x \in B_2 \setminus \Sigma, \tag{3.18} \]
and using the definition of \( w_j \), we obtain
\[ w_j(y) \geq \frac{c_1}{u(\bar{x}_j)} \quad \text{in } \Omega_j \setminus \Xi_j. \tag{3.19} \]

Therefore, for large \( j \),
\[ w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} - (w_j)_\lambda(z)^{\frac{n+2\sigma}{n-2\sigma}} \geq \frac{1}{2} w_j(z)^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } \Omega_j \setminus \Xi_j. \]

Now, we claim that
\begin{align*}
J(\lambda, w_j, y) &\geq \frac{1}{2} \left( \frac{c_1}{u(\bar{x}_j)} \right)^{\frac{n+2\sigma}{n-2\sigma}} \int_{\Omega_j \setminus \Xi_j} K(0, \lambda; y, z) \, dz - C \int_{\Omega_j} K(0, \lambda; y, z) \left( \frac{\lambda}{|z|} \right)^{n+2\sigma} \, dz \\
&\geq \begin{cases} 
C(|y| - \lambda)u(\bar{x}_j)^{-1}, & \text{if } \lambda \leq |y| \leq \bar{\lambda}_j + 1, \\
Cu(\bar{x}_j)^{-1}, & \text{if } |y| > \bar{\lambda}_j + 1, y \in \Xi_j,
\end{cases}
\end{align*}
where \( C \) is a positive constant.

Indeed, since \( K(0, \lambda; y, z) = 0 \) for \( |y| = \lambda \) and
\[ y \cdot \nabla_y K(0, \lambda; y, z) \bigg|_{|y| = \lambda} = (n - 2\sigma)|y - z|^{2\sigma-n-2}(|z|^2 - |y|^2) > 0 \]
for \( |z| \geq \bar{\lambda}_j + 2 \), and using the positivity and smoothness of \( K \) we obtain
\[ \frac{\delta_1}{|y - z|^{n-2\sigma}}(|y| - \lambda) \leq K(0, \lambda; y, z) \leq \frac{\delta_2}{|y - z|^{n-2\sigma}}(|y| - \lambda) \tag{3.21} \]
for \( \bar{\lambda}_j \leq \lambda \leq |y| \leq \bar{\lambda}_j + 1, \bar{\lambda}_j + 2 \leq |z| \leq M < \infty \), where the positive constants \( \delta_1 \) and \( \delta_2 \) are independent of (large) \( j \). Moreover, if \( M \) is large, then
\[ 0 < c_2 \leq y \cdot \nabla_y(|y - z|^{n-2\sigma} K(0, \lambda; y, z)) \leq C_2 < \infty \]
for all \( |z| \geq M, \bar{\lambda}_j \leq \lambda \leq |y| \leq \bar{\lambda}_j + 1 \). Hence, (3.21) also holds for \( \bar{\lambda}_j \leq \lambda \leq |y| \leq \bar{\lambda}_j + 1, |z| \geq M \). On the other hand, by the definition of \( K(0, \lambda; y, z) \), we can verify that for \( |y| \geq \bar{\lambda}_j + 1 \) and \( |z| \geq \bar{\lambda}_j + 2, \)
\[ \frac{\delta_3}{|y - z|^{n-2\sigma}} \leq K(0, \lambda; y, z) \leq \frac{1}{|y - z|^{n-2\sigma}} \tag{3.22} \]
for some $\delta_3 \in (0,1)$ independent of (large) $j$.

Denote $\tau_j := u(\vec{x}_j)^{\frac{2}{n-2\sigma}}$. Then for all sufficiently large $j$, $\lambda \leq |y| \leq \bar{\lambda}_j + 1$ (recall that $\lambda \leq \bar{\lambda}_j + \frac{1}{2}$) we have

$$J(\lambda, w_j, y) \geq \frac{1}{2} \left( \frac{c_1}{u(\vec{x}_j)} \right)^{\frac{n+2\sigma}{n-2\sigma}} \int_{\Omega_j \setminus \Xi_j} \frac{\delta_1}{|y - z|^{n-2\sigma}} (|y| - \lambda) dz$$

$$- C \int_{\Omega_j} \frac{\delta_2}{|y - z|^{n-2\sigma}} (|y| - \lambda) \left( \frac{\lambda}{|z|} \right)^{n+2\sigma} dz$$

$$\geq C(|y| - \lambda)u(\vec{x}_j)^{-\frac{n+2\sigma}{n-2\sigma}} \int_{\{\frac{3}{4} r_j \leq |z| \leq \frac{7}{4} r_j \}} \frac{1}{|y - z|^{n-2\sigma}} dz$$

$$- C(|y| - \lambda) \int_{\{|z| \geq \frac{7}{4} r_j \}} \frac{1}{|y - z|^{n-2\sigma}} \left( \frac{1}{|z|} \right)^{n+2\sigma} dz$$

$$\geq C(|y| - z)u(\vec{x}_j)^{-1} - C(|y| - z)u(\vec{x}_j)^{-\frac{2n}{n-2\sigma}}$$

$$\geq C(|y| - z)u(\vec{x}_j)^{-1},$$

where we have used $\mathcal{L}^n(\Sigma_j) = 0$ for all $j$ and $u(\vec{x}_j) \to \infty$ as $j \to \infty$. Similarly, for $|y| \geq \bar{\lambda}_j + 1$ and $y \in \Xi_j$, we have

$$J(\lambda, w_j, y) \geq Cu(\vec{x}_j)^{-1} - Cu(\vec{x}_j)^{-\frac{2n}{n-2\sigma}} \geq Cu(\vec{x}_j)^{-1}.$$

Thus, (3.20) is verified.

By (3.16) and (3.20), there exists $\varepsilon_1(j) \in (0, \frac{1}{2})$ such that

$$w_j(y) - (w_j)_{\bar{\lambda}_j}(y) \geq J(\bar{\lambda}_j, w_j, y) \geq Cu(\vec{x}_j)^{-1} \geq \frac{\varepsilon_1(j)}{|y|^{n-2\sigma}}$$

for all $|y| \geq \bar{\lambda}_j + 1, y \in \Xi_j$. This, together with the explicit formula of $(w_j)_{\lambda}(y)$, yields that there exists $0 < \varepsilon_2(j) < \varepsilon_1(j)$ such that for any $\lambda_j \leq \lambda \leq \bar{\lambda}_j + \varepsilon_2(j)$,

$$w_j(y) - (w_j)_{\lambda}(y) \geq \frac{\varepsilon_1(j)}{|y|^{n-2\sigma}} + \left( (w_j)_{\bar{\lambda}_j}(y) - (w_j)_{\lambda}(y) \right)$$

$$\geq \frac{\varepsilon_1(j)}{2|y|^{n-2\sigma}} \quad \forall |y| \geq \bar{\lambda}_j + 1, y \in \Xi_j.$$

(3.23)

For $\varepsilon_j \in (0, \varepsilon_2(j))$ which we choose below, by (3.16) and (3.20) we have, for $\bar{\lambda}_j \leq \lambda \leq \bar{\lambda}_j + \varepsilon_j$
and for \( \lambda \leq |y| \leq \bar{\lambda}_j + 1 \),

\[
 w_j(y) - (w_j)_{\lambda}(y) \geq \int_{\lambda \leq |z| \leq |\lambda| + 1} K(0, \lambda; y, z) \left( w_j(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} \right) dz \\
+ \int_{|\lambda| + 2 \leq |z| \leq |\lambda| + 3} K(0, \lambda; y, z) \left( w_j(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} \right) dz \\
\geq -C \int_{\lambda \leq |z| \leq \lambda + \varepsilon_j} K(0, \lambda; y, z)(|z| - \lambda) dz \\
+ \int_{\lambda + \varepsilon_j \leq |z| \leq \bar{\lambda}_j + 1} K(0, \lambda; y, z) \left( (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} \right) dz \\
+ \int_{|\lambda| + 2 \leq |z| \leq |\lambda| + 3} K(0, \lambda; y, z) \left( w_j(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} \right) dz,
\]

where we have used

\[
|w_j(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma}| \leq C(|z| - \lambda)
\]

in the second inequality. By \([3.23]\) there exists \( \delta_j > 0 \) such that

\[
w_j(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} \geq \delta_j \quad \text{for} \quad |\lambda| + 2 \leq |z| \leq |\lambda| + 3.
\]

Since \( \|w_j\|_{C^1(B_{\lambda_j + 2})} \leq C \) (independent of \( j \)), there exists some constant \( C > 0 \) independent of both \( \varepsilon \) and \( j \) such that for \( \bar{\lambda}_j \leq \lambda \leq \bar{\lambda}_j + \varepsilon_j \),

\[
|(w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma} - (w_j)_{\lambda}(z) \frac{n+2\sigma}{n-2\sigma}| \leq C(\lambda - \bar{\lambda}_j) \leq C \varepsilon_j \quad \forall \lambda \leq |z| \leq \bar{\lambda}_j + 1.
\]

For any \( \lambda \leq |y| \leq \bar{\lambda}_j + 1 \), one can estimate the integrals of the kernel \( K \) (or, see \([20]\)):

\[
\int_{\lambda + \varepsilon_j \leq |z| \leq |\lambda| + 1} K(0, \lambda; y, z) dz \leq \int_{\lambda + \varepsilon_j \leq |z| \leq |\lambda| + 1} \left( \frac{1}{|y - z|^{n-2\sigma}} - \frac{1}{|y^{0, \lambda} - z|^{n-2\sigma}} \right) dz \\
+ \int_{\lambda + \varepsilon_j \leq |z| \leq |\lambda| + 1} \left( \frac{\lambda}{|y|^{n-2\sigma}} - 1 \right) \frac{1}{|y^{0, \lambda} - z|^{n-2\sigma}} dz \\
\leq C(\varepsilon_j^{2\sigma-1} + |\ln \varepsilon_j| + 1)(|y| - \lambda)
\]

and

\[
\int_{\lambda \leq |z| \leq \lambda + \varepsilon_j} K(0, \lambda; y, z)(|z| - \lambda) dz \leq \int_{\lambda \leq |z| \leq \lambda + \varepsilon_j} \left( \frac{|z| - \lambda}{|y - z|^{n-2\sigma}} - \frac{|z| - \lambda}{|y^{0, \lambda} - z|^{n-2\sigma}} \right) dz \\
+ \varepsilon_j \int_{\lambda \leq |z| \leq \lambda + \varepsilon_j} \left( \frac{\lambda}{|y|^{n-2\sigma}} - 1 \right) \frac{1}{|y^{0, \lambda} - z|^{n-2\sigma}} dz \\
\leq C(|y| - \lambda) \varepsilon_j^{2\sigma/n} + C \varepsilon_j (|y| - \lambda) \\
\leq C(|y| - \lambda) \varepsilon_j^{2\sigma/n}.
\]
Therefore, using (3.21) we have for \( \lambda < |y| \leq \bar{\lambda}_j + 1 \) that
\[
\begin{align*}
    w_j(y) - (w_j)_\lambda(y) & \geq -C\varepsilon_j^{2\sigma/n} (|y| - \lambda) + \delta_1 \delta_j (|y| - \lambda) \int_{\lambda_j + 2 \leq |z| \leq \lambda_j + 3} \frac{1}{|y - z|} \, dz \\
    & \geq \left( \delta_1 \delta_j c - C\varepsilon_j^{2\sigma/n} \right) (|y| - \lambda) \geq 0
\end{align*}
\]
if \( \varepsilon_j \) is sufficiently small. This and (3.22) contradict to the definition of \( \bar{\lambda}_j \) if \( \bar{\lambda}_j < \lambda_0 \) for sufficiently large \( j \). Thus, we proved the Claim 3.

It follows that (3.12) holds and the proof of Theorem 1.5 is completed. \( \Box \)

Now we combine Theorem 1.5 with the local integral representation in Theorem 2.4 to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( u \in C^2(B_2 \setminus \Lambda) \) be a positive solution of (1.4) satisfying (1.5). By Theorem 2.4 there exists \( 0 < \tau < 1/4 \) independent of \( \Lambda \) such that for any \( x_0 \in \Lambda \), we have (up to the constant \( c_{n,m} \))
\[
    u(x) = \int_{B_{\tau}(x_0)} u(y) \frac{n+2m}{n-2m} \frac{dy}{|x - y|^{n-2m}} + h_1(x) \quad \text{for} \quad x \in B_\tau(x_0) \setminus \Lambda, \quad (3.24)
\]
where \( h_1(x) \) is a positive smooth function in \( B_\tau(x_0) \). For any \( x_0 \in \Lambda \), define
\[
    v(x) = \left( \frac{\tau}{2} \right)^{\frac{n-2m}{2}} u \left( \frac{\tau}{2} x + x_0 \right), \quad h(x) = \left( \frac{\tau}{2} \right)^{\frac{n-2m}{2}} h_1 \left( \frac{\tau}{2} x + x_0 \right) \quad \text{in} \quad B_2 \setminus \Sigma,
\]
where
\[
    \Sigma := \left\{ x \in \mathbb{R}^n : \frac{\tau}{2} x + x_0 \in \Lambda \right\}.
\]
Note that, in general, \( \Sigma \) intersects the boundary \( \partial B_2 \). Then \( v \in L^{\frac{n+2m}{n-2m}}(B_2) \cap C(B_2 \setminus \Sigma) \) and \( v \) satisfies (1.13) with \( \sigma = m \) in \( B_2 \setminus \Sigma \), and \( h \in C^1(B_2) \) is a positive smooth function. Using Theorem 1.5 for \( v \), we know that there exists a constant \( C > 0 \) such that
\[
    v(x) \leq C [\text{dist}(x, \Sigma)]^{-\frac{n-2m}{2}} \quad \text{for all} \quad x \in B_1 \setminus \Sigma.
\]
Rescaling back to \( u \), we have
\[
    u(x) \leq C [\text{dist}(x, \Lambda)]^{-\frac{n-2m}{2}} \quad \text{for all} \quad x \in B_\tau(x_0) \setminus \Lambda.
\]
Since \( \Lambda \) is a compact set, the desired estimate (1.6) follows by a finite covering argument. \( \Box \)

## 4 Asymptotic symmetry for local singular solutions

In this section, we first show the asymptotic symmetry of local singular solutions for the integral equation (1.13) in Theorem 1.6 and then show the same thing for the differential equation (1.4) stated in Theorem 1.2 by combining Theorem 1.6 with the integral representation in Theorem 2.4. We still use the notations introduced at the beginning of Section 3.
Proof of Theorem 1.6. Without loss of generality, we may assume $0 \in \Sigma$. We will show that there exists a small $\varepsilon > 0$ such that for every $x \in \overline{B}_{1/4} \setminus \Sigma$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{for all } y \in B_{3/2} \setminus (B_{\lambda}(x) \cup \Sigma), \quad 0 < \lambda < \text{dist}(x, \Sigma) \leq \varepsilon. \quad (4.1)$$

First of all, by Lemma 3.1 in [20], there exists a positive constant $0 < r_0 < \frac{1}{2}$ depending only on $n$, $\sigma$ and $\| \nabla \ln h \|_{L^{\infty}(B_{3/2})}$ such that for every $x \in B_1$ and $0 < \lambda \leq r_0$ there holds

$$h_{x,\lambda}(y) \leq h(y) \quad \forall |y - x| \geq \lambda, \quad y \in B_{3/2}. \quad (4.2)$$

Moreover, from the proof of Lemma 3.1 in [20] (see (20) there) we know that, for every $x \in \overline{B}_{1/4} \setminus \Sigma$ there exists $0 < r_x < \text{dist}(x, \Sigma)$ such that for all $0 < \lambda \leq r_x$,

$$u_{x,\lambda}(y) \leq u(y), \quad 0 < \lambda < |y - x| \leq r_x. \quad (4.3)$$

By the equation (1.13), we have

$$u(x) \geq 4^{2\sigma-n} \int_{B_2} u(y)^{\frac{n+2\sigma}{n-2\sigma}} dy =: c_1 > 0 \quad \text{for all } x \in B_2 \setminus \Sigma, \quad (4.4)$$

and thus, we can find $0 < \lambda_1 \ll r_x$ such that for all $0 < \lambda \leq \lambda_1$,

$$u_{x,\lambda}(y) \leq u(y), \quad y \in B_{3/2} \setminus (B_{r_1}(x) \cup \Sigma). \quad (4.5)$$

Combining (4.3) with (4.5), we obtain that for every $0 < \lambda \leq \lambda_1$,

$$u_{x,\lambda}(y) \leq u(y), \quad y \in B_{3/2} \setminus (B_{\lambda}(x) \cup \Sigma). \quad (4.6)$$

Therefore,

$$\tilde{\lambda}(x) := \sup \{0 < \mu < \text{dist}(x, \Sigma) \mid u_{x,\lambda}(y) \leq u(y), \forall y \in B_{3/2} \setminus (B_{\lambda}(x) \cup \Sigma),$$

$$\forall 0 < \lambda < \mu \}$$

is well defined for any $x \in \overline{B}_{1/4} \setminus \Sigma$ and is positive.

Next we show that there exists a small $\varepsilon > 0$ such that $\tilde{\lambda}(x) = \text{dist}(x, \Sigma)$ for all $x \in \overline{B}_{1/4}$ and $0 < \text{dist}(x, \Sigma) \leq \varepsilon$. For brevity, we denote $\lambda = \tilde{\lambda}(x)$ in the below.

For any $x \in \overline{B}_{1/4}$ and $\lambda \leq \text{dist}(x, \Sigma) \leq r_0$, by (4.2), (4.2) and (3.3) we have for $y \in B_{3/2}$ that

$$u(y) - u_{x,\lambda}(y) \geq \int_{B_1 \setminus B_{\lambda}(x)} K(x, \lambda; y, z) \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\lambda}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz + J(\lambda, u, y),$$

where

$$J(\lambda, u, y) = \int_{B_2 \setminus B_1} K(x, \lambda; y, z) \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\lambda}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz$$

$$- \int_{B_2} K(x, \lambda; y, z) u_{x,\lambda}(z)^{\frac{n+2\sigma}{n-2\sigma}} dz.$$
For $y \in B_1^c$, $x \in \overline{B}_{1/4}$ and $\bar{\lambda} \leq \lambda < \text{dist}(x, \Sigma) < \frac{1}{10}$, we have
\[
|x + \frac{\lambda^2(y - x)}{|y - x|^2}| - x \leq \frac{4}{3}\lambda^2 < \frac{1}{2}\text{dist}(x, \Sigma)
\]
and
\[
|x + \frac{\lambda^2(y - x)}{|y - x|^2}| \leq \frac{1}{2}\text{dist}(x, \Sigma) + |x| < 1.
\]
Hence
\[
\text{dist} \left( x + \frac{\lambda^2(y - x)}{|y - x|^2}, \Sigma \right) \leq \left| x + \frac{\lambda^2(y - x)}{|y - x|^2} - x \right| + \text{dist}(x, \Sigma) \leq \frac{3}{2}\text{dist}(x, \Sigma)
\]
and
\[
\text{dist} \left( x + \frac{\lambda^2(y - x)}{|y - x|^2}, \Sigma \right) \geq \text{dist}(x, \Sigma) - \left| x + \frac{\lambda^2(y - x)}{|y - x|^2} - x \right| \geq \frac{1}{2}\text{dist}(x, \Sigma).
\]
It follows from Theorem 1.5 that
\[
u \left( x + \frac{\lambda^2(y - x)}{|y - x|^2} \right) \leq C\text{dist}(x, \Sigma)^{-\frac{n-2\sigma}{2}}.
\]
Thus, for all $y \in B_1^c$,
\[
u_{x,\lambda}(y) = \left( \frac{\lambda}{|y - x|} \right)^{n-2\sigma} \nu \left( x + \frac{\lambda^2(y - x)}{|y - x|^2} \right) \leq C\lambda^{n-2\sigma}\text{dist}(x, \Sigma)^{-\frac{n-2\sigma}{2}} \leq C\text{dist}(x, \Sigma)^{-\frac{n-2\sigma}{2}} \leq C\varepsilon^{-\frac{n-2\sigma}{2}}
\]
for any $x \in \overline{B}_{1/4}$ and $\bar{\lambda} \leq \lambda < \text{dist}(x, \Sigma) < \varepsilon < \frac{1}{10}$. By (4.4) we obtain that for any $x \in \overline{B}_{1/4}$,
\[
u_{x,\lambda}(y) \leq C\varepsilon^{-\frac{n-2\sigma}{2}} \leq \frac{c_1}{2} < u(y) \quad \forall y \in B_2 \setminus (B_1 \cup \Sigma), \quad \forall \bar{\lambda} \leq \lambda < \text{dist}(x, \Sigma) \leq \varepsilon
\]
if $\varepsilon$ is sufficiently small.

For $y \in B_1 \setminus (B_\lambda(x) \cup \Sigma), \ x \in \overline{B}_{1/4}$ and $\bar{\lambda} \leq \lambda < \text{dist}(x, \Sigma) < \varepsilon$, by (4.4) and (4.7), using the similar arguments as in proving (3.20) and noticing that $\mathcal{L}^n(\Sigma) = 0$, we have
\[
J(\lambda, u, y) \geq \int_{B_2 \setminus B_1} K(x, \lambda; y, z) \left( \frac{n+2\sigma}{z - x} - C\varepsilon^{-\frac{n+2\sigma}{2}} \right) dz
\]
\[
- C \int_{B_2^c} K(x, \lambda; y, z) \left( \frac{1}{|z - x|} \right)^{n+2\sigma} |\text{dist}(x, \Sigma)|^{-\frac{n+2\sigma}{2}} dz
\]
\[
\geq \frac{1}{2} c_1 \int_{B_2 \setminus B_1} K(x, \lambda; y, z) dz - C\varepsilon^{-\frac{n+2\sigma}{2}} \int_{B_2^c} K(x, \lambda; y, z) \frac{1}{|z - x|^{n+2\sigma}} dz
\]
\[
\geq \frac{1}{2} c_1 \int_{B_2 \setminus B_1} K(x, \lambda; y, z) dz - C\varepsilon^{-\frac{n+2\sigma}{2}} \int_{B_2^c} K(x, \lambda; y - x, z) \frac{1}{|z - x|^{n+2\sigma}} dz
\]
\[
- C\varepsilon^{-\frac{n+2\sigma}{2}} \int_{B_2 \setminus B_{1/4}} K(0, \lambda; y - x, z) dz
\]
\[
\geq C_2(|y - x| - \lambda),
\]
if we let $\varepsilon$ be sufficiently small, where $C_2$ is a positive constant independent of $x$. If $\bar{\lambda} < \text{dist}(x, \Sigma) \leq \varepsilon$ for some $x \in \overline{B}_{1/4}$, using (4.8) and (4.9) with the integral estimates techniques as in the proof of Theorem 1.5, the moving sphere procedure may continue beyond $\bar{\lambda}$ where we get a contradiction. Thus, we obtain $\lambda(x) = \text{dist}(x, \Sigma)$ for $x \in \overline{B}_{1/4}$ and $0 < \text{dist}(x, \Sigma) \leq \varepsilon$, where $\varepsilon$ is sufficiently small. Therefore, (4.1) is proved.

Let $r > 0$ small (less that $\varepsilon^2$), $x_1, x_2 \in \Pi_r^{-1}(z)$ with $z \in \overline{B}_{1/8} \cap \Sigma$ be such that

$$u(x_1) = \max_{\Pi_r^{-1}(z)} u(x), \quad u(x_2) = \min_{\Pi_r^{-1}(z)} u(x).$$

Let $e_1 = x_1 - z$, $e_2 = x_2 - z$, $x_3 = x_1 + \frac{\varepsilon(e_1 - e_2)}{4|e_1 - e_2|}$. Then $e_1, e_2 \in (T_z \Sigma)^\perp$ and thus, $e_2 - e_1 \in (T_z \Sigma)^\perp$. Let $\lambda = \sqrt{\frac{\varepsilon}{4}}(|e_1 - e_2| + \frac{\varepsilon}{4})$. It is easy to check that $0 < \lambda < |x_3 - z| = \text{dist}(x_3, \Sigma) < \varepsilon$ and $|x_3| < 1/4$. From (4.1) we obtain

$$u_{x_3, \lambda}(x_2) \leq u(x_2).$$

On the other hand, the definition of $u_{x_3, \lambda}$ gives

$$u_{x_3, \lambda}(x_2) = \left(\frac{\lambda}{|e_1 - e_2| + \varepsilon/4}\right)^{n-2\sigma} u(x_1) = \left(\frac{1}{4|e_1 - e_2|/\varepsilon + 1}\right)^{n-2\sigma} u(x_1) \geq \left(\frac{1}{8r/\varepsilon + 1}\right)^{n-2\sigma} u(x_1).$$

Thus,

$$\max_{\Pi_r^{-1}(z)} u(x) \leq (8r/\varepsilon + 1)^{-\frac{n-2\sigma}{2}} \min_{\Pi_r^{-1}(z)} u(x).$$

This implies that

$$u(x) = u(x')(1 + O(r)) \quad \text{for all } x, x' \in \Pi_r^{-1}(z) \quad \text{as } r \to 0,$$

where $O(r)$ is uniform for $z \in \overline{B}_{1/8} \cap \Sigma$. The proof of Theorem 1.6 is completed. \hfill \Box

**Proof of Theorem 1.2** The proof is very similar to that of Theorem 1.1. Using Theorems 1.6 and 2.4 by a rescaling argument and a covering argument, there exists a small real number $\varepsilon > 0$ such that

$$\max_{\Pi_r^{-1}(z)} u(x) \leq (8r/\varepsilon + 1)^{-\frac{n-2\sigma}{2}} \min_{\Pi_r^{-1}(z)} u(x)$$

for all $z \in \Lambda$ and small $r > 0$. Thus, we have

$$u(x) = u(x')(1 + O(r)) \quad \text{for all } x, x' \in \Pi_r^{-1}(z) \quad \text{as } r \to 0,$$

where $O(r)$ is uniform for $z \in \Lambda$. Theorem 1.2 is proved. \hfill \Box

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5 Symmetry for global singular solutions

In this section, we prove Theorem 1.7 and then Theorem 1.3 follows immediately by using Theorem 2.8. Finally, we also give the proof of Corollary 1.4.

Proof of Theorem 1.7 Without loss of generality, we assume that

$$\limsup_{x \to 0} u(x) = \infty, \quad (5.1)$$

where $\mathbb{R}^k$ is a $k$-dimensional subspace of $\mathbb{R}^n$ with $0 \leq k \leq n - 1$. Denote $\mathbb{R}^{n-k} = (\mathbb{R}^k)^\perp$ as the orthogonal complement of $\mathbb{R}^k$.

Claim 1. For every $x \in \mathbb{R}^{n-k}\{0\}$, there exists a real number $\lambda_2 \in (0, |x|)$ such that for any $0 < \lambda < \lambda_2$, we have

$$u_{x,\lambda}(y) \leq u(y) \quad \forall \ |y - x| \geq \lambda, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k. \quad (5.2)$$

The proof of Claim 1 consists of two steps.

Step 1. We show that there exists $0 < \lambda_1 < |x|$ such that for any $0 < \lambda < \lambda_1$,

$$u_{x,\lambda}(y) \leq u(y) \quad \forall \ 0 < \lambda \leq |y - x| \leq \lambda_1. \quad (5.3)$$

By Theorem 2.5 in [19], we know that $u \in C^1(\mathbb{R}^n \setminus \mathbb{R}^k)$. Suppose

$$|\nabla \ln u| \leq C_1 \quad \text{in } B_{|x|/2}(x)$$

for some constant $C_1 > 0$. Then we have

$$\frac{d}{dr}(r^{n-2\sigma} u(x + r\theta)) = r^{n-2\sigma} u(x + r\theta) \left( \frac{n - 2\sigma}{2} - r \frac{\nabla u \cdot \theta}{u} \right) \geq r^{n-2\sigma} u(x + r\theta) \left( \frac{n - 2\sigma}{2} - C_1 r \right) > 0 \quad (5.4)$$

for all $0 < r < \lambda_1 := \min\{\frac{n-2\sigma}{2} \cdot \frac{|x|}{2} \}$ and $\theta \in S^{n-1}$. For any $y \in B_{\lambda_1}(x)$, $0 < \lambda < |y - x| \leq \lambda_1$, let $\theta = \frac{y-x}{|y-x|}$, $r_1 = |y - x|$ and $r_2 = \frac{\lambda_1 r_1}{|y-x|}$. Using (5.4) we have

$$r_2^{n-2\sigma} u(x + r_2\theta) < r_1^{n-2\sigma} u(x + r_1\theta).$$

That is,

$$u_{x,\lambda}(y) \leq u(y), \quad 0 < \lambda \leq |y - x| \leq \lambda_1.$$

Step 2. We show that there exists $0 < \lambda_2 < \lambda_1 < |x|$ such that (5.2) holds for all $0 < \lambda < \lambda_2$.

By Fatou lemma,

$$\liminf_{x \to \infty} |x|^{n-2\sigma} u(x) = \liminf_{x \to \infty} \int_{\mathbb{R}^n} \frac{|x|^{n-2\sigma} u(y)}{|x-y|^{n-2\sigma}} \, dy \geq \int_{\mathbb{R}^n} u(y) \frac{n-2\sigma}{|y|^{n-2\sigma}} \, dy > 0.$$
Consequently, there exist two constants $c_1, R_1 > 0$ such that

$$u(y) \geq \frac{c_1}{|y|^{n-2\sigma}} \quad \text{for all } |y| \geq R_1 \text{ and } y \in \mathbb{R}^n \setminus \mathbb{R}^k. \quad (5.5)$$

On the other hand, if $y \in B_{R_1} \setminus \mathbb{R}^k$, then by the equation (1.16) and the positivity of $u$, we obtain

$$u(y) \geq \int_{B_{R_1}} \frac{u(z)^{n-2\sigma}}{|y-z|^{n-2\sigma}} dz \geq (2R_1)^{2\sigma-n} \int_{B_{R_1}} u(z)^{n-2\sigma} dz > 0.$$ 

This, together with (5.5), implies that there exists $C > 0$ such that

$$u(y) \geq \frac{C}{|y-x|^{n-2\sigma}} \quad \forall |y-x| \geq \lambda_1, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k.$$ 

Thus, for sufficiently small $\lambda_2 \in (0, \lambda_1)$ and for any $0 < \lambda < \lambda_2$,

$$u_{x,\lambda}(y) = \left( \frac{\lambda}{|y-x|} \right)^{n-2\sigma} u \left( x + \frac{\lambda^2(y-x)}{|y-x|^2} \right) \leq \left( \frac{\lambda_2}{|y-x|} \right)^{n-2\sigma} \sup_{B_{\lambda_1}(x)} \leq u(y), \ \forall |y-x| \geq \lambda_1, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k.$$ 

Estimate (5.2) follows from (5.3) and the above. Claim 1 is proved.

Now, we can define

$$\lambda(x) := \sup \{ 0 < \mu \leq |x| \mid u_{x,\lambda}(y) \leq u(y), \ \forall |y-x| \geq \lambda, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k, \ \forall 0 < \lambda < \mu \}.$$ 

By Claim 1, $\lambda(x)$ is well defined and $\lambda(x) > 0$.

**Claim 2.** $\lambda(x) = |x|$ for all $x \in \mathbb{R}^{n-k} \setminus \{0\}$.

Suppose $\lambda(x) < |x|$ for some $x \in \mathbb{R}^{n-k} \setminus \{0\}$. For brevity, we will denote $\lambda = \lambda(x)$ in the below. By the definition of $\lambda$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{for all } |y-x| \geq \bar{\lambda}, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k. \quad (5.6)$$

Because of (5.1), we know that $u_{x,\bar{\lambda}}(y) \neq u(y)$. For any $\bar{\lambda} \leq \lambda < |x|, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k$ with $|y-x| \geq \bar{\lambda}$, by (3.2) and (3.3), we have

$$u(y) - u_{x,\lambda}(y) = \int_{|z-x| \geq \lambda} K(x, \lambda; y, z) \left( u(z)^{n+2\sigma} - u_{x,\lambda}(z)^{n+2\sigma} \right) dz.$$ 

It follows from the positivity of the kernel $K$ and $L^n(\mathbb{R}^k) = 0$ that

$$u_{x,\lambda}(y) < u(y) \quad \text{for all } |y-x| \geq \bar{\lambda}, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k.$$
Using Fatou lemma again,

\[
\liminf_{y \to \infty} |y - x|^{n-2\sigma} (u - u_{x,\bar{\lambda}})(y) = \liminf_{y \to \infty} \int_{|z-x| \geq \bar{\lambda}} |y - x|^{n-2\sigma} K(x, \bar{\lambda}; y, z) \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz \\
\geq \int_{|z-x| \geq \bar{\lambda}} \left[ 1 - \left( \frac{\bar{\lambda}}{|z-x|} \right)^{n-2\sigma} \right] \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz > 0.
\]

Consequently, there exist two constants $c_2, R_2 > 0$ such that

\[
(u - u_{x,\bar{\lambda}})(y) \geq \frac{c_2}{|y - x|^{n-2\sigma}} \quad \text{for all } |y - x| \geq R_2, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k, \quad (5.7)
\]

and for $\bar{\lambda} + 1 \leq |y - x| \leq R_2, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k$, by (3.22) we have

\[
u(y) - u_{x,\bar{\lambda}}(y) = \int_{|z-x| \geq \lambda} K(x, \bar{\lambda}; y, z) \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz \\
\geq \int_{\lambda + 2 \leq |z-x| \leq \bar{\lambda} + 8} \frac{\delta_3}{|y - z|^{n-2\sigma}} \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz \\
\geq C_2 \int_{\lambda + 2 \leq |z-x| \leq \bar{\lambda} + 8} \left( u(z)^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}}(z)^{\frac{n+2\sigma}{n-2\sigma}} \right) dz > 0
\]

for some $C_2 > 0$. Combining this with (5.7), we obtain that there exists $\varepsilon_1 \in (0, 1)$ such that

\[
(u - u_{x,\bar{\lambda}})(y) \geq \frac{\varepsilon_1}{|y - x|^{n-2\sigma}} \quad \text{for all } |y - x| \geq \bar{\lambda} + 1, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k. \quad (5.8)
\]

By (5.8) and the explicit formula of $u_{x,\bar{\lambda}}$, there exists $0 < \varepsilon_2 < \varepsilon_1$ such that for all $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_2 < |x|,

\[
(u - u_{x,\bar{\lambda}})(y) \geq \frac{\varepsilon_1}{|y - x|^{n-2\sigma}} + (u_{x,\bar{\lambda}} - u_{x,\bar{\lambda}})(y) \\
\geq \frac{\varepsilon_1}{2|y - x|^{n-2\sigma}} \quad \forall \ |y - x| \geq \bar{\lambda} + 1, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k. \quad (5.9)
\]

For $\varepsilon \in (0, \varepsilon_2)$ which we choose below, we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon$ and for $y \in \mathbb{R}^n \setminus \mathbb{R}^k$ with
\[ \lambda \leq |y - x| \leq \tilde{\lambda} + 1, \]

\[ u(y) - u_{x, \lambda}(y) = \int_{|z - x| \geq \lambda} K(x, \lambda; y, z) \left( u(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma} \right) dz \]

\[ \geq \int_{\lambda \leq |z - x| \leq \lambda + 1} K(x, \lambda; y, z) \left( u(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma} \right) dz \]

\[ + \int_{\lambda + 2 \leq |z - x| \leq \lambda + 3} K(x, \lambda; y, z) \left( u(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma} \right) dz \]

where in the second inequality we have used

\[ |u(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma}| \leq C(|z - x| - \lambda). \]

Because of (5.9), there exists \( \delta > 0 \) such that for all \( \lambda \leq \lambda \leq \tilde{\lambda} + \varepsilon \), we have

\[ u(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma} > \delta \]

\[ \forall \lambda + 2 \leq |z - x| \leq \lambda + 3, \ z \in \mathbb{R}^n \setminus \mathbb{R}^k. \]

It is also easy to see that there exists \( C > 0 \) independent of \( \varepsilon \) such that for all \( \lambda \leq \lambda \leq \tilde{\lambda} + \varepsilon \),

\[ |u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma} - u_{x, \lambda}(z) \frac{n + 2\sigma}{n - 2\sigma}| \leq C(\lambda - \tilde{\lambda}) \leq C\varepsilon \quad \forall \lambda \leq \lambda \leq |z - x| \leq \tilde{\lambda} + 1. \]

Now, by a very similar estimate for the kernel \( K \) as in the proof of Theorem 1.5, we can obtain that, for \( \lambda \leq \lambda \leq \tilde{\lambda} + \varepsilon \) and for \( y \in \mathbb{R}^n \setminus \mathbb{R}^k \) with \( \lambda \leq |y - x| \leq \tilde{\lambda} + 1 \),

\[ u(y) - u_{x, \lambda}(y) \geq (\delta_1 \delta_2 c - C\varepsilon) \frac{2\sigma}{n} )(|y - x| - \lambda) \geq 0 \]

if \( \varepsilon > 0 \) is sufficiently small. This and (5.9) contradict the definition of \( \tilde{\lambda} \). The proof of Claim 2 is completed. Thus, we have shown that for every \( x \in \mathbb{R}^{n-k} \setminus \{0\} \),

\[ u_{x, \lambda}(y) \leq u(y) \quad \forall |y - x| \geq \lambda, \ y \in \mathbb{R}^n \setminus \mathbb{R}^k, \ \forall 0 < \lambda < |x|. \]  

(5.10)

For any unit vector \( e \in \mathbb{R}^{n-k} \), for any \( a > 0 \), for any \( \xi = (y, z) \in \mathbb{R}^n \) with \( y \in \mathbb{R}^k \) and \( z \in \mathbb{R}^{n-k} \) satisfying \((z - ae) \cdot e < 0\), and for any \( R > a \), we have, by (5.10) with \( x = Re \) and \( \lambda = R - a \),

\[ u(y, z) \geq u_{x, \lambda}(y, z) = \left( \frac{\lambda}{|\xi - x|} \right)^{n - 2\sigma} u \left( \frac{x + \lambda^2 (\xi - x)}{|\xi - x|^2} \right). \]

Sending \( R \) to infinity in the above, we obtain

\[ u(y, z) \geq u(y, z - 2(z \cdot e - a)e). \]  

(5.11)
Since \( z \in \mathbb{R}^{n-k} \) and \( \alpha > 0 \) are arbitrary, (5.11) gives the radial symmetry of \( u \) in the \( \mathbb{R}^{n-k} \) variables.

In particular, if \( k = 0 \), then \( u \) is radially symmetric about the origin. Moreover, (5.11) also gives

\[
u(z) = u(z_1, z_2, \ldots, z_n) \geq u_a(z) := u(2a - z_1, z_2, \ldots, z_n) \quad \forall \ z_1 \leq a, \ a > 0.
\]

This implies that \( u \) is also monotonically decreasing about the origin. Theorem 1.7 is established.

\[\square\]

**Proof of Theorem 1.3** It follows from Theorem 2.8 and Theorem 1.7.

\[\square\]

Finally, we give the proof of Corollary 1.4.

**Proof of Corollary 1.4** The proof is just a combination of Theorem 2.8, Theorem 1.7, Theorem 5 of [9] and Theorem 1.1 of [20]. For the reader’s convenience, we include the details. By Theorems 2.8 and 1.7, we know that \( u \) is radially symmetric and monotonically decreasing about the origin and satisfies

\[
u(x) = c_{n,m} \int_{\mathbb{R}^n} \frac{u(y) n+2m}{|x - y|^{n-2m}} \, dy \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Hence, for any \( r > 0 \) and \( \theta \in S^{n-1} \), we have

\[
u(r\theta) \geq c_{n,m} \int_{B_r(0)} \frac{u(y) n+2m}{|r\theta - y|^{n-2m}} \, dy \\
\geq c_{n,m} r^{2m} u(r) \int_0^r \left( \int_{\partial B_1(0)} \frac{1}{|r\theta - s\omega|^{n-2m}} \, d\omega \right) s^{n-1} \, ds \\
= c_{n,m} r^{2m} u(r) \int_0^1 \left( \int_{\partial B_1(0)} \frac{1}{|\theta - t\omega|^{n-2m}} \, d\omega \right) t^{n-1} \, dt \\
= C_1 r^{2m} u(r) \frac{n+2m}{n-2m}
\]

for some uniform constant \( C_1 > 0 \). It follows that

\[
u(r) \leq C_1 r^{-\frac{n-2m}{s}} \quad \text{for all} \ r > 0.
\]

This proves the upper bound in (1.11). On the other hand, by (5.12) we have for any \( s = 1, \ldots, m - 1 \) that

\[
(-\Delta)^s u(x) = c(n, m, s) \int_{\mathbb{R}^n} \frac{u(y) n+2m}{|x - y|^{n-2m+2s}} \, dy \geq 0 \quad \text{in} \ \mathbb{R}^n \setminus \{0\}.
\]

Since \( 0 \) is a non-removable singularity, using Theorem 1.1 of [20] we know there exists \( C_2 = C_2(n, m, u) > 0 \) such that

\[
u(x) \geq C_2 |x|^{-\frac{n-2m}{s}} \quad \text{for} \ x \in \mathbb{R}^n \setminus \{0\}.
\]

This completes the proof of Corollary 1.4.

\[\square\]
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