CERTAIN INVARIANT ALGEBRAIC SETS IN $S^p \times S^q$

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Abstract. Let $S_{p,q}$ be the hypersurface in $\mathbb{R}^{p+q+1}$ defined by the following:

$$S_{p,q} := \left\{(x_1, \ldots, x_{p+1}, y_1, \ldots, y_q) \in \mathbb{R}^{p+q+1} \mid \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right)^2 + \sum_{j=1}^q y_j^2 = 1 \right\},$$

where $a > 1$. We show that $S_{p,q}$ is homeomorphic to the product $S^p \times S^q$.

We consider the polynomial vector field $\mathcal{X} = (P_1, \ldots, P_{p+1}, Q_1, \ldots, Q_q)$ in $\mathbb{R}^{p+q+1}$ which keeps $S_{p,q}$ invariant. Then we study the number of certain invariant algebraic subvarieties of $S_{p,q}$ for the vector field $\mathcal{X}$ if either $p > 1$ or $q > 1$.

1. Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $R_1, \ldots, R_n$ be polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. The following system of differential equations

$$\frac{dx_i}{dt} = R_i(x_1, \ldots, x_n)$$

for $i = 1, \ldots, n$ is called a polynomial differential system in $\mathbb{R}^n$. The differential operator

$$\mathcal{X} = \sum_{i=1}^n R_i \frac{\partial}{\partial x_i}$$

is called the vector field associated to the system (1.1). The degree of the polynomial vector field in (1.2) is defined to be $\max\{\deg(R_i) \mid i = 1, \ldots, n\}$.

When $n = 2$ in (1.1), this differential system has been studied since 1900 possibly because of the second part of the Hilbert 16th problem (see [4] and the references therein).

An invariant algebraic set for (1.2) is a subset $A \subset \mathbb{R}^n$, such that $A$ is the zero set of some $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, and $\mathcal{X} f = K f$ for some $K \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, which is called the cofactor of $f$.

In this paper, we mainly study the number of invariant algebraic sets of a vector field in $S^p \times S^q$. We are primarily interested in the algebraic sets

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obtained by the intersection of $S^p \times S^q$ with hyperplanes and certain hypersurfaces. We obtain an upper bound of the number of possible invariant algebraic sets that are intersections of $S^p \times S^q$ with hypersurfaces of degree one and two. Therefore, many other types of invariant hypersurfaces in $S^p \times S^q$ remain to be explored.

In the case of the torus $S^1 \times S^1$, the maximum number of invariant meridians and parallels are studied in [8] and [12]. Bounds on the number of invariant hyperplanes and co-dimension one spheres for polynomial vector fields in $\mathbb{R}^n$ are obtained in [7] and [2] respectively. In relation to the second part of Hilbert’s sixteenth problem, the maximum number of algebraic limit cycles of a polynomial vector field in $\mathbb{R}^2$ as a function of its degree has been studied in [10], [11] and [16]. The maximum number of straight lines that are invariant for a vector field in the real plane as a function of its degree has been studied in [1] and [14].

The paper is organized as follows. In Section 2, we recall the definition of the extactic algebraic polynomial associated to a vector subspace of the ring of polynomials and the given vector field. We also state some basic properties of the polynomial vector fields.

In Section 3, we show that $S_{p,q}$ is homeomorphic to $S^p \times S^q$. We define meridians and parallels on $S_{p,q}$ analogously to its definition given in [8] for $S^1 \times S^1$. We prove that meridians and parallels are connected if $p, q > 1$ and give an upper bound for the number of invariant meridians and parallels in Theorem 3.3. We give an upper bound for the number of meridians of linear and quadratic vector fields on $S_{p,q}$. The bound for linear vector fields on $S_{p,q}$ can be obtained. We demonstrate that these bounds are close to being tight for the cases $p = 2, 3$ and $\deg(X) \geq 4$. Next we consider the hypersurfaces of the form $\sum_{i=1}^{p+1} a_i x_i^2 = 1$. We show that its intersection with $S_{p,q}$ has one component if $a > 2$, see Proposition 3.10. Then we give an upper bound for the number of invariant hypersurfaces of this type in $S_{p,q}$. We show that the maximum number of invariant parabolic hypersurfaces $a_1 x_1^2 + a_2 x_2 = 0$ in $S_{p,q}$ is $\left\lfloor \frac{m}{2} \right\rfloor$ where $m$ is the degree of the vector field $\mathcal{X}$ on $S_{p,q}$. Moreover, this bound is attained if $p \geq 4$. We discuss a few examples at the end.

In Section 4, we show that the maximum number of invariant meridians on $S_{1,q} (\cong S^1 \times S^2)$ is $2(m - 1)$ where $m$ is the degree of $\mathcal{X}$ on $S_{1,q}$ and that this bound can be reached. We also discuss parallels on $S_{1,q}$, derive a bound on the number of parallels and show that there exists a vector field with at least $(m - 2)$ many invariant parallels.

In Section 5, we show that the maximum number of invariant parallels for $\mathcal{X}$ on $S_{p,1} := S^p \times S^1$ is $2(m - 2)$ where $m = \deg \mathcal{X}$ and also demonstrate that this bound is attained. We also remark that on an upper bound for the number of invariant meridians for $\mathcal{X}$ on $S_{p,1}$ and exhibit a vector field with $(m - 2)$ invariant meridians on $S_{p,1}$. 
2. Invariant Algebraic Sets and Extactic polynomials

In this section, we recall the concept of invariant algebraic sets and extactic polynomials for polynomial vector fields on \( \mathbb{R}^n \) following [8]. Then we discuss some basic properties of extactic polynomials.

Let \( S \) be a hypersurface in \( \mathbb{R}^n \), defined by the zeroes of a non constant polynomial \( h \in \mathbb{R}[x_1, x_2, \ldots, x_n] \) in \( \mathbb{R}^n \). We say that a vector field \( \mathcal{X} \) of the form (1.2) is defined on \( S \) if \( (R_1, R_2, \ldots, R_n) \cdot \nabla h = 0 \), for points on the hypersurface \( S \), which is equivalent to saying that \( \mathcal{X} h = Kh \), for some polynomial \( K \in \mathbb{R}[x_1, x_2, \ldots, x_n] \). This is because at a point \( (a_1, a_2, \ldots, a_n) \) on the hypersurface, \( h(a_1, a_2, \ldots, a_n) = 0 \) by definition. The hypersurface \( S \) is called a regular hypersurface if \( \nabla h \neq 0 \) for all points on \( S \). This hypersurface is called irreducible if \( h \) is irreducible. The degree of the hypersurface is defined to be the degree of \( h \).

In order to study invariant algebraic sets on an algebraic hypersurface \( S \subset \mathbb{R}^n \), one may use the idea of extactic algebraic polynomial. We briefly recall this concept following [7]. Let \( W \) be a \( k \)-dimensional vector subspace of \( \mathbb{R}[x_1, x_2, x_3, \ldots, x_n] \) with basis \( \{v_1, \ldots, v_k\} \). Then the extactic algebraic set of the vector field \( \mathcal{X} \) associated to \( W \) is given by

\[
\mathcal{E} (\mathcal{X}) = \det \begin{pmatrix} \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_k) \\ \vdots & \vdots & & \vdots \\ \mathcal{X}^{k-1}(v_1) & \mathcal{X}^{k-1}(v_2) & \cdots & \mathcal{X}^{k-1}(v_k) \end{pmatrix} = 0,
\]

where \( \mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i)) \). We note that the definition of the extactic algebraic set is independent of the choice of basis of \( W \), see Section 2 of [8].

We recall the definition of the algebraic multiplicity of an irreducible algebraic set given by some polynomial \( f = 0 \) from [13].

**Definition 2.1.** The hypersurface given by \( f = 0 \) has algebraic multiplicity, or simply, multiplicity \( m \) for \( \mathcal{X} \) if \( \mathcal{E}(\mathcal{X}) \neq 0 \) and \( (f)^m \) divides \( \mathcal{E}(\mathcal{X}) \) and for no integer \( m' > m \), \( (f)^{m'} \) divides \( \mathcal{E}(\mathcal{X}) \).

We shall use the following proposition whose proof is similar to the proof of [7, Proposition 1].

**Proposition 2.2.** Let \( \mathcal{X} \) be a polynomial vector field in \( \mathbb{R}^n \) and let \( W \) be a finite dimensional vector sub-space of \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) with \( \dim(W) > 1 \). Then every algebraic invariant set given by \( f = 0 \) for the vector field \( \mathcal{X} \), with \( f \in W \), is a factor of \( E_W(\mathcal{X}) \).

We recall that a function \( g \) is called a first integral of the system in equation (1.2) if \( \mathcal{X} g = 0 \). If \( g \) is a rational function then \( g \) is called a rational first integral. If the system has a first integral, then the system possesses infinitely many invariant algebraic sets as solutions. A proof of this can be found on page 102 of [5]. We quote the following result from [6].
Proposition 2.3. Let $S$ be a regular algebraic hypersurface of degree $d$ in the Euclidean space $\mathbb{R}^{n+1}$. The polynomial vector field $\mathcal{X}$ on $S$ of degree $m > 0$ admits $\binom{n+m}{n+1} - \binom{n+m-d}{n+1} + n$ invariant algebraic hypersurfaces irreducible in $\mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$ if and only if $\mathcal{X}$ has a rational first integral.

3. IN Variant Algebraic sets in $S^p \times S^q$

In this section, we consider the following hypersurface in $\mathbb{R}^{p+q+1}$ defined by the polynomial identity

\[(x_1^2 + x_2^2 + \cdots + x_{p+1}^2 - a^2)^2 + y_1^2 + y_2^2 + \cdots + y_q^2 = 1\]

where $a > 1$ and $p \geq 2$, $q \geq 2$. We denote this hypersurface by $S_{p,q}$. We show that $S_{p,q}$ is homeomorphic to the product $S^p \times S^q$ of two spheres. Fix $n := p + q + 1$. Suppose $0 \leq c \leq 1$ and $y_1^2 + y_2^2 + \cdots + y_q^2 = c$. Then

\[(x_1^2 + x_2^2 + \cdots + x_{p+1}^2 - a^2)^2 = 1 - c\]

This implies

\[x_1^2 + x_2^2 + \cdots + x_{p+1}^2 = \pm \sqrt{1 - c + a^2}\]

Let

\[U_1 := \{(x_1, \ldots, x_{p+1}, y_1, \ldots, y_q) \in \mathbb{R}^n \mid \sum_{i=1}^{p+1} x_i^2 = a^2 + \sqrt{1 - c}, \quad \sum_{j=1}^q y_j^2 = c, \quad 0 \leq c \leq 1\}\]

and

\[U_2 := \{(x_1, \ldots, x_{p+1}, y_1, \ldots, y_q) \in \mathbb{R}^n \mid \sum_{i=1}^{p+1} x_i^2 = a^2 - \sqrt{1 - c}, \quad \sum_{j=1}^q y_j^2 = c, \quad 0 \leq c \leq 1\}\].

Observe that $U_1$ and $U_2$ are homeomorphic to $S^p \times D^q$. They are identified along their boundary

\[S^p \times S^{q-1} = \left\{(x_1, \ldots, x_{p+1}, y_1, \ldots, y_q) \in \mathbb{R}^n \mid \sum_{i=1}^{p+1} x_i^2 = a^2, \quad \sum_{j=1}^q y_j^2 = 1\right\}.\]

Thus

\[S_{p,q} = U_1 \cup_{S^p \times S^{q-1}} U_2 = S^p \times D^q \cup_{S^p \times S^{q-1}} S^p \times D^q \cong S^p \times S^q.\]

Note that the product $S^p \times S^{q-1}$ in the above equation is a subset of $\mathbb{R}^n$, whereas the representation of $S^p \times S^{q-1}$ as in (3.1) says $S^p \times S^{q-1}$ is subset of $\mathbb{R}^{n-1}$. These two representations of $S^p \times S^{q-1}$ are different in two different ambient spaces. We write the vector field of (1.2) as follows

\[\mathcal{X} := \sum_{i=1}^{p+1} P_i \partial_{x_i} + \sum_{j=1}^q Q_j \partial_{y_j}\]

where, $P_i, Q_j \in \mathbb{R}[x_1, \ldots, x_{p+1}, y_1, \ldots, y_q]$ and $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and $\partial_{y_j} = \frac{\partial}{\partial y_j}$ for $i = 1, \ldots, p+1$ and $j = 1, \ldots, q$. One defines the degree vector of this vector field to be $\varpi := (m_1, m_2, \ldots, m_n)$ where $m_i = \deg(P_i)$ for $i = 1, 2, \ldots, p+1$ and $m_{p+1+j} = \deg(Q_j)$ for $j = 1, \ldots, q$. 


Writing out the condition for $\mathcal{X}$ to be invariant on the hypersurface $S_{p,q}$, we get the following equation,

$$4 \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right) \left( \sum_{i=1}^{p+1} x_i P_i \right) + 2 \left( \sum_{j=1}^{q} y_j Q_j \right) = K \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right)^2 + \sum_{j=1}^{q} y_j^2 - 1 .$$

for some $K \in \mathbb{R}[x_1, \ldots, x_n]$.

3.1. **Computation on hyperplanes.** In this subsection, we are interested in invariant algebraic sets determined by the intersections of $S_{p,q}$ and hypersurfaces determined by polynomials of degree one. Thus the intersection is invariant by $\mathcal{X}$ which satisfies (3.3). First we consider the following:

1. The transverse intersection of $S_{p,q}$ and the hyperplane $\sum_{i=1}^{p+1} a_i x_i = 0$ where $a_i \in \mathbb{R}, \forall i$. We may call this intersection a ‘meridian’ on $S_{p,q}$.
2. The transverse intersection of $S_{p,q}$ and the hyperplane $\sum_{j=1}^{q} b_j y_j = c$, for some $c \in (0,1)$ where $b_j \in \mathbb{R}, \forall j$. We may call this intersection a ‘parallel’.

We say that a meridian on $S_{p,q}$ is invariant by the flow of the polynomial vector field $\mathcal{X}$ on $S_{p,q}$ if $\mathcal{X}(\sum_{i=1}^{p+1} a_i x_i) = K(\sum_{i=1}^{p+1} a_i x_i)$ where $a_i \in \mathbb{R}$ for all $i$ and for some $K \in \mathbb{R}[x_1, \ldots, x_{p+1}, y_1, \ldots, y_q]$.

Similarly, one can define an invariant parallel on $S_{p,q}$. We note that these definitions are similar to the invariant meridians and the invariant parallels on $S^1 \times S^1$ of [8].

Without loss of generality, we may assume that $m_1 \geq m_2 \geq \cdots \geq m_{p+1}$ and that $m_{p+2} \geq m_{p+3} \geq \cdots \geq m_n$.

**Proposition 3.1.** Let $S^{n+1}$ be the standard unit $(n+1)$-sphere in $\mathbb{R}^{n+2}$ and $H$ a hyperplane passing through the origin. Then $S^{n+1} \cap H$ is homeomorphic to $S^n$.

**Proof.** The space $H$ is a normed linear space where the norm is induced from the standard norm in $\mathbb{R}^{n+2}$. Then the set of unit vectors of $H$ is $S^{n+1} \cap H$. Since $H$ is a vector space of dim$(n+1)$, $S^{n+1} \cap H$ is an $n$-dimensional sphere. □

**Proposition 3.2.** On $S_{p,q}$ with $p, q \geq 2$, the invariant meridians and parallels have one connected component each.

**Proof.** Let $\sum_{i=1}^{p+1} a_i x_i = 0$ be the hyperplane $H$, which determines a meridian and

$$\sum_{j=1}^{q} y_j^2 = \alpha \in [0, 1] .$$

Then (3.1) gives the following pair of spheres

$$\sum_{i=1}^{p+1} x_i^2 = a^2 \pm \sqrt{1 - \alpha}$$
unless $\alpha = 1$. By Proposition 3.1, the intersection of one of the spheres with the hyperplane $H$, is a $(p - 1) \text{ dimensional sphere}$ for each $\alpha \in [0, 1)$. Thus, by similar argument as in the proof that $S_{p,q}$ is homeomorphic to $S^p \times S^q$, one can show that $S_{p,q} \cap H$ is homeomorphic to $S^{p-1} \times S^q$. Therefore it is connected since $p, q > 1$.

For the case of parallels, let $\sum_{j=1}^{q} b_j y_j = c$ be the hyperplane $H_2$ and

$$\sum_{i=1}^{p+1} x_i^2 - a^2 = \beta \in [-1, 1].$$

Then (3.1) can be written as

$$\sum_{j=1}^{q} y_j^2 = 1 - \beta^2 \in [0, 1].$$

By Proposition 3.1, the intersection of this sphere and $H_2 - c$ is a $q - 1$ dimensional sphere unless $\beta = \pm 1$. If $\beta = \pm 1$, then this intersection is a point. Therefore, the intersection $S_{p,q} \cap H_2$ is homeomorphic to $S^p \times S^{q-1}$ if $\beta \neq \pm 1$ since smooth homotopies preserve transversality. This is connected since $p, q > 1$. □

**Theorem 3.3.** Assume that the vector field $X$ of (3.2) has finitely many invariant algebraic hypersurfaces. If $p, q \geq 2$, then

1. the number of invariant meridians of $X$ in $S_{p,q}$ is at most

$$\binom{p}{2} (m_1 - 1) + \sum_{i=2}^{p+1} m_i + 1.$$

2. the number of invariant parallels of $X$ in $S_{p,q}$ is at most

$$\binom{q}{2} (m_{p+2} - 1) + \sum_{j=1}^{q-1} m_{p+j+1}.$$

**Proof.** For (1). An invariant meridian of $X$ is given by the intersection of a hyperplane of the form $g := \sum_{i=1}^{p+1} a_i x_i = 0$ with $S_{p,q}$. By Proposition 2.2, this hyperplane is invariant for the vector field $X$ if and only if $\sum_{i=1}^{p+1} a_i x_i$ is a factor of the exactic polynomial

(3.4)

$$E_{\{x_1, x_2, \ldots, x_{p+1}\}}(X) = \det \begin{pmatrix} x_1 & x_2 & \cdots & x_{p+1} \\ P_1 & P_2 & \cdots & P_{p+1} \\ X(P_1) & X(P_2) & \cdots & X(P_{p+1}) \\ \vdots & \vdots & \ddots & \vdots \\ X^{p-1}(P_1) & X^{p-1}(P_2) & \cdots & X^{p-1}(P_{p+1}) \end{pmatrix}. $$
Since we have chosen degrees of $P_1, \ldots, P_{p+1}$, in decreasing order, we see that the term
\begin{equation}
\mathcal{X}^{p-1}(P_{p+1}) \cdot \mathcal{X}^{p-2}(P) \cdots \mathcal{X}(P_3) \cdot P_2 \cdot x_1
\end{equation}
has the least degree in the polynomial $\mathcal{E}_{\{x_1, x_2, \ldots, x_{p+1}\}}(\mathcal{X})$. Now the degree of $\mathcal{X}^{p-1}(P_1)$ is $(p-1)(m_1 - 1) + m_1$, the degree of $\mathcal{X}^{p-2}(P_2)$ is $(p-2)(m_1 - 1) + m_2$, and similarly for the other factors in (3.5). Therefore,
\begin{equation}
\deg(\mathcal{X}^{p-1}(P_{p+1}) \cdot \mathcal{X}^{p-2}(P) \cdots \mathcal{X}(P_3) \cdot P_2 \cdot x_1) = (p-1)(m_1 - 1) + m_{p+1} + (p-2)(m_1 - 1) + m_p + \cdots + (m_1 - 1) + m_3 + m_2 + 1
\end{equation}
\begin{equation}
= (m_1 - 1) \sum_{i=1}^{p-1} i + \sum_{j=2}^{p+1} m_j + 1
\end{equation}
\begin{equation}
= \left(\frac{p}{2}\right)(m_1 - 1) + \sum_{i=2}^{p+1} m_i + 1.
\end{equation}

Since the meridians are determined by linear homogeneous polynomials, the number of meridians cannot exceed $\deg(\mathcal{X}^{p-1}(P_{p+1}) \cdot \mathcal{X}^{p-2}(P) \cdots \mathcal{X}(P_3) \cdot P_2 \cdot x_1)$. This proves (1).

For (2), in this case, the extactic polynomial is given by
\begin{equation}
\mathcal{E}_{\{y_1, y_2, \ldots, y_q\}}(\mathcal{X}) = \det\begin{pmatrix}
Q_1 & Q_2 & \cdots & Q_q & 0 \\
\mathcal{X}(Q_1) & \mathcal{X}(Q_2) & \cdots & \mathcal{X}(Q_q) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathcal{X}^{q-2}(Q_1) & \mathcal{X}^{q-2}(Q_2) & \cdots & \mathcal{X}^{q-2}(Q_q) & 0 \\
\mathcal{X}^{q-1}(Q_1) & \mathcal{X}^{q-1}(Q_2) & \cdots & \mathcal{X}^{q-1}(Q_q) & 0
\end{pmatrix}
\end{equation}

Now the result follows using the same argument as in Theorem 3 of [9].
becomes

\[ 4 \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right) \left( \sum_{i=1}^{p+1} x_i P_i \right) + 2 \left( \sum_{j=1}^{q} y_j Q_j \right) = 0. \]  \hspace{1cm} (3.9)

We see that every first degree vector field defined on \( S_{p,q} \) has a first integral. Thus it may possess infinitely many invariant algebraic sets. However, in the following, we show that it has finitely many invariant meridians on \( S_{p,q} \).

For a degree one vector field we see that \( K'(\in \mathbb{R}) \) of (3.8) is a constant since the left hand side has degree one and the term in the brackets on the right has degree one.

Now, a general degree one vector field may be given by

\[ P_i = \sum_{j=1}^{p+1} c_{ij} x_j + c_{i0}, \quad Q_j = \sum_{s=1}^{q} d_{js} y_s + d_{j0}. \]  \hspace{1cm} (3.10)

Observe that in the expression for \( P_i \), there can be no \( y_j \)s. Since if there was, equation (3.9) will have terms of the form \( 4 x_i^2 x_j y_j \) which cannot cancel out and hence will violate (3.9). Similarly, \( Q_j \) cannot have any terms with \( x_i \)s. By similar reasoning we get that \( c_{i0} = d_{j0} = 0 \) for all \( i \) and \( j \).

Substituting \( P_i \) from (3.10) in (3.9) and collecting coefficients of \( x_{i_0} \) for a fixed \( i_0 \) we have

\[ 4 \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right) \left\{ P_{i_0} + \sum_{s=1}^{p+1} c_{si_0} x_s \right\} = 0. \]  \hspace{1cm} (3.11)

Since \( P_{i_0} = \sum_{j=1}^{p+1} c_{i_0 j} x_j \), then (3.11) becomes

\[ 4 \left( \sum_{i=1}^{p+1} x_i^2 - a^2 \right) \left\{ \sum_{j=1}^{p+1} c_{i_0 j} x_j + \sum_{s=1}^{p+1} c_{si_0} x_s \right\} = 0. \]  \hspace{1cm} (3.12)

Reindexing and gathering coefficients of \( x_j \), we get (here we can assume \( \sum_{i=1}^{p+1} x_i^2 \neq a^2 \) since there is a dense open subset of \( S_{p,q} \) with \( \sum_{i=1}^{p+1} x_i^2 \neq a^2 \))

\[ (c_{i_0 j} + c_{j i_0}) = 0. \]

From this, we see that the real matrix determined by \( \{c_{ij}\} \) is skew-symmetric.

Further substituting for \( P_i \) from (3.10) in (3.8), we get

\[ \sum_{i=1}^{p+1} a_{i} \sum_{j=1}^{p+1} c_{ij} x_j = K' \left( \sum_{s=1}^{p+1} a_{s} x_s \right). \]  \hspace{1cm} (3.13)

This can be written as

\[ \sum_{i=1}^{p+1} \sum_{j=1}^{p+1} a_{ij} c_{ij} x_j = K' \left( \sum_{s=1}^{p+1} a_{s} x_s \right). \]  \hspace{1cm} (3.14)
Reindexing and equating coefficients of $x_j$, we have

\[(3.15) \quad \sum_{i=1}^{p+1} a_i c_{ij} = K' a_j.\]

This implies that $(a_1, \ldots, a_{p+1})$ is an eigenvector for the real matrix $(c_{ij})_{1 \leq i,j \leq p+1}$ with eigenvalue $K'$, but since the matrix $(c_{ij})$ is skew symmetric, 0 is the only possible real eigenvalue. This is true because by [15], a real skew symmetric matrix is similar to a matrix of the form

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus 0 \oplus \cdots \oplus 0$$

where the matrices $A_i$ are of the form

$$A_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

This matrix has eigenvalues $\pm \lambda_i t$, where $t = \sqrt{-1}$. $A_i$ is similar over the complex numbers to the matrix

$$\begin{pmatrix} \lambda_i t & 0 \\ 0 & -\lambda_i t \end{pmatrix}.$$

From this we see that a real skew-symmetric matrix is similar to a matrix whose non zero eigenvalues are all imaginary. Hence a real skew-symmetric matrix has 0 as the only real eigenvalue.

Then (3.15) becomes

$$\sum_{i=1}^{p+1} a_i c_{ij} = 0.$$

A real skew symmetric matrix, $A$, is normal, that is, it commutes with its adjoint (which in this case is the transpose). Hence by the Spectral Theorem (Theorem 2.5.4 in page 101 of [3]) $A$ is diagonalisable. So if all eigenvalues of $A$ are zero, $A$ is the zero matrix since it will be similar to the zero matrix. Hence $A$ cannot have all eigenvalues zero, since $X$ is a non zero vector field.

**Theorem 3.4.** A degree one vector field defined on $S_p \times S_q$ can have at most as many meridians as there are real eigenvectors of the matrix $A = (c_{ij})_{1 \leq i,j \leq p+1}$ formed by the coefficients of the vector field $X$.

**Remark 3.5.** Since $A$ cannot have all eigenvalues zero and since the non zero eigenvalues come in pairs, we see that a degree one vector field on $S_p \times S_q$ can have at most $(p - 1)$ meridians. In fact, by starting with a $(p + 1) \times (p + 1)$ skew-symmetric matrix with $(p - 1)$ eigenvalues zero, we can readily construct a vector field with $(p - 1)$ meridians.

**Example 3.6.** For $p = 2$, take the linear vector field $X$ determined by the following matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$
as explained above, from the skew symmetric matrix $A$ we get a vector field $X$, whose $P_i$s are given by

$$P_1 = -x_2, \quad P_2 = x_1 - x_3, \quad P_3 = x_2.$$ 

The eigenvalues of $A$ are $\pm \sqrt{2}, 0$. We know that any meridian for $X$ has to be an eigenvector of $A$ (when the coefficients of the meridian is written in vector form) which means that meridians of $X$ are eigenvectors corresponding to eigenvalue 0 (since we are working over reals). In this case, the unique eigenvector is $(1, 0, 1)$ and hence the unique meridian is given by $\{x_1 + x_3 = 0\}$. Further we see that this matches our bound in Remark 3.5 which is 1.

Now, we look at quadratic vector fields on $S_{p,q}$. Let $(z_1, \ldots, z_{p+q+1}) \in \mathbb{R}^{p+q+1}$ be such that $z_i = x_i$ for $1 \leq i \leq p + 1$ and $z_i = y_i$ for $p + 2 \leq i \leq p + q + 1$.

Put $n = p + q + 1$, then a general quadratic vector field can be given by

$$P_i = \sum_{j=1}^{n} \sum_{k=1}^{n} e_{ijk} z_j z_k + \sum_{j=1}^{n} c_{ij} z_j + c_0 \quad (3.16)$$

$$Q_j = \sum_{s=1}^{n} \sum_{t=1}^{n} f_{jst} z_s z_t + \sum_{s=1}^{n} d_{js} z_s + d_j 0 \quad (3.17)$$

Here, we are interested in meridians and hence the $P_i$ have to satisfy equation (3.8), that is

$$\sum_{i=1}^{p+1} a_i P_i = K'(\sum_{i=1}^{p+1} a_i x_i).$$

We see that for in our case $K'$ is linear:

$$K' = \sum_{l=1}^{n} k_l z_l + k_0$$

Substituting for $P_i$ in (3.8), from (3.16) and equating terms of matching degrees in the left and the right, we get the following for quadratic terms

$$\sum_{i=1}^{p+1} \sum_{j=1}^{p+1} \sum_{k=1}^{p+1} a_i e_{ijk} z_j z_k = \sum_{l=1}^{m} \sum_{s=1}^{n} k_l a_s x_s z_l \quad (3.18)$$

here we may assume that $z_j$ on the left side is equal to $x_j$ since in the expression for $P_i$, both $z_j$ and $z_k$ cannot be in $\{y_1, \ldots, y_q\}$. Then matching indices $j = l$ and $k = s$, we see that $(a_1, \ldots, a_{p+1})$ is an eigenvector for $(e_{ijk}) = A_k$ for each fixed $k$ with eigenvalue $k_l \in \mathbb{R}$. Thus we have proved the following.

**Proposition 3.7.** A quadratic vector field can have at most $e(A)$ meridians where $e(A)$ is the number of real eigenvalues of $A$ counted with multiplicities.
This implies, in particular, that a quadratic vector field on \( S_{p,q} \) can have at most \((p + 1)\) meridians.

Now we look at the case of vector fields of degree greater than or equal to four.

**Theorem 3.8.** Suppose that the vector field \( \mathcal{X} \) of (3.2) has finitely many invariant algebraic hypersurfaces. Then,

1. There exists a vector field of degree \( m \) with \( 3m - 10 \) meridians counted with multiplicity when \( p = 2 \).
2. There exists a vector field of degree \( m \) with \( 6m - 21 \) meridians when \( p = 3 \).

**Proof.** First we construct \( Q_j \)s for the invariant vector field \( \mathcal{X} \). Let \( Q_s \) be given by

\[
Q_1 = x_1(y_1^2 - 1)H, \quad Q_2 = x_1y_1y_2H, \quad Q_3 = x_1y_1y_3H,
\]

\[
Q_4 = x_1y_1y_4H, \quad Q_5 = x_1y_5H, \quad \ldots, \text{and} \quad Q_q = x_1y_qH
\]

where \( H \in \mathbb{R}[x_1, \ldots, x_p, y_1, \ldots, y_q] \).

For (1): Let

\[
D := (\sum_{i=1}^{3} a_i x_i)^{m-3},
\]

and \( P_s \) be given by

\[
P_1 = \frac{1}{2} y_1(x_1^2 - a^2)D \quad P_2 = \frac{1}{2} y_1 x_1 x_2D \quad P_3 = \frac{1}{2} y_1 x_1 x_3D.
\]

Observe that the vector field \( \mathcal{X} \) determined by the above choices for \( P_i \) where \( i = 1, 2, 3 \) and \( Q_j \) for \( j = 1, \ldots, q \) is invariant on \( S_{p,q} \), if we let \( H = D \). To be precise, for these \( P_s \) and \( Q_s \), \( \mathcal{X} \) satisfies equation (3.3) with \( K = 2x_1y_1 \). We shall prove that the vector field \( \mathcal{X} \), defined by these \( P_i \)s and \( Q_j \)s have \( \sum_{i=1}^{3} a_i x_i = 0 \) as a meridian with multiplicity \( 3m - 10 \) for \( \mathcal{X} \). We note that the bound given by Theorem (3.3) is \( 3m \). The polynomial \( \sum_{i=1}^{3} a_i x_i \) belongs to the vector space \( W := \text{span}\{x_1, x_2, x_3\} \). We note that \( \sum_{i=1}^{3} a_i x_i = 0 \) gives a meridian for this vector field \( \mathcal{X} \) if and only \( \sum_{i=1}^{3} a_i x_i \) is a factor of the corresponding extactic polynomial

\[
\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix}
\mathcal{X}(x_1) & \mathcal{X}(x_2) & \mathcal{X}(x_3) \\
\mathcal{X}^2(x_1) & \mathcal{X}(x_2) & \mathcal{X}^2(x_3)
\end{pmatrix}.
\]

In our case, this is the following

\[
\det \begin{pmatrix}
x_1 \\
\frac{1}{2} y_1(x_1^2 - a^2)D \\
g_1D^2 + h_1E
\end{pmatrix}
\]

\[
x_2 \\
\frac{1}{2} y_1 x_1 x_2D \\
g_2D^2 + h_2E
\]

\[
x_3 \\
\frac{1}{2} y_1 x_1 x_3D \\
\frac{1}{2} y_1 x_1 x_3D
\]

where

\[
E := \mathcal{X}(D) = (m - 3)(\sum_{i=1}^{3} a_i x_i)^{m-4} (\sum_{i=1}^{3} a_i x_{\sigma(i)}) \cdot D
\]
which we have written as $g\ g$ the whole term but only to get the multiple of $(\sum_{i=1}^{3} a_i x_i)$ for $i \in \{1, 2, 3\}$. The third row consists of the terms like
\[
X(\frac{1}{2} y_1 (x_1^2-a^2) D) = \frac{1}{2} (y_1^2-1)(x_1^2-a^2) D^2 + \frac{1}{2} x_1 y_1 (x_1^2-a^2) D^2 + \frac{1}{2} y_1 (x_1^2-a^2) E
\]
which we have written as $g_1 D^2 + h_1 E$. We note that we may need to calculate the whole term but only to get the multiple of $(\sum_{i=1}^{3} a_i x_i)$ which is a common factor of the terms in that row. Since $D^2 = (\sum_{i=1}^{3} a_i x_i)^{2m-6}$, in the third row, one sees that $(\sum_{i=1}^{p+1} a_i x_i)^{2m-7}$ is a common factor of each term in the third row of the matrix in (3.19). Also in the second row, $D$ is a common factor, hence
\[
E_W(X) = D \cdot E \cdot h'(x_1, x_2, x_3, y_1)
\]
\[
= (\sum_{i=1}^{3} a_i x_i)^{3m-10} h(x_1, x_2, x_3, y_1)
\]
for some polynomials $h'$ and $h$ in $R[x_1, x_2, x_3, y_1, \ldots, y_q]$. Thus (1) is proved since $(\sum_{i=1}^{3} a_i x_i)$ divides $E_W(X)$ with multiplicity $3m - 10$.

For (2): Let $R := (\sum_{i=1}^{4} a_i x_i)^{m-3}$ and
\[
P_1 = \frac{1}{2} y_1 (x_1^2-a^2) R, \quad P_2 = \frac{1}{2} y_1 x_1 x_2 R, \quad P_3 = \frac{1}{2} y_1 x_1 x_3 R, \quad \text{and} \quad P_4 = \frac{1}{2} y_1 x_1 x_4 R.
\]
Note that the maximum number of meridians possible in this case is $6m - 2$ by Theorem 3.3. Observe that $(\sum_{i=1}^{4} a_i x_i)$ is a polynomial in $W = \text{span}\{x_1, x_2, x_3, x_4\}$ and the corresponding exactic polynomial is
\[
E_W(X) = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ X(x_1) & X(x_2) & X(x_3) & X(x_4) \\ X^2(x_1) & X^2(x_2) & X^2(x_3) & X^2(x_4) \\ X^3(x_1) & X^3(x_2) & X^3(x_3) & X^3(x_4) \end{pmatrix}
\]
In order to compute $E_W(X)$ in our case, let us first determine
\[
T := X(R) = (m - 3)(\sum_{i=1}^{4} a_i x_i)^{2m-7}(\sum_{i=1}^{4} a_i x_{\sigma(i)})
\]
\[
U := X(T) = (m - 3)(2m - 7)(\sum_{i=1}^{4} a_i x_i)^{3m-11}(\sum_{i=1}^{4} a_i x_{\sigma(i)})^2
\]
\[
+ 2(m - 3)(\sum_{i=1}^{4} a_i x_i)^{3m-10}(\sum_{i=1}^{4} a_i x_{\sigma(i)})
\]
for some permutations \( \sigma \) and \( \sigma' \) on the set \( \{1, 2, 3, 4\} \). Now \( \mathcal{E}_W(\mathcal{X}) \) becomes

\[
\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix}
\frac{1}{2}y_1(x_1^2 - a^2) & \frac{1}{2}y_1x_1x_2 & \frac{1}{2}y_1x_2x_3 & \frac{1}{2}y_1x_3x_4 & \frac{1}{2}y_1x_4x_1 \\
 f_{11}R + g_{11}T & f_{12}R + g_{12}T & f_{13}R + g_{13}T & f_{14}R + g_{14}T & f_{15}R + g_{15}T \\
f_{21}R + g_{21}T & f_{22}R + g_{22}T & f_{23}R + g_{23}T & f_{24}R + g_{24}T & f_{25}R + g_{25}T \\
f_{31}R + g_{31}T & f_{32}R + g_{32}T & f_{33}R + g_{33}T & f_{34}R + g_{34}T & f_{35}R + g_{35}T \\
f_{41}R + g_{41}T & f_{42}R + g_{42}T & f_{43}R + g_{43}T & f_{44}R + g_{44}T & f_{45}R + g_{45}T
\end{pmatrix}
\]

where the \( f_{ij}, g_{kl}, h_{st} \in \mathbb{R}[x_1, \ldots, x_{p+1}, y_1, \ldots, y_q] \) for \( i, l, s \in \{3, 4\} \) and \( j, k, t \in \{1, 2, 3, 4\} \). We shall only need the fact that \( U \) has \( \sum_{i=1}^{4} a_i x_i \) raised to the smallest power. Notice that \( R \) is common in each term of the second row, \( (\sum_{i=1}^{4} a_i x_i)^{2m-7} \) is common in each term of the third row, and \( (\sum_{i=1}^{4} a_i x_i)^{3m-11} \) is common in each term of the fourth row of the matrix in (3.20). Therefore the extactic polynomial can be written as

\[
\mathcal{E}_W(\mathcal{X}) = (\sum_{i=1}^{4} a_i x_i)^{6m-21} \cdot h(x_1, x_2, x_3, x_4, y_1)
\]

This proves the claim (2), since \( (\sum_{i=1}^{4} a_i x_i) \) divides \( \mathcal{E}_W(\mathcal{X}) \) with multiplicity \( 6m - 21 \).

**Example 3.9.** Let the vector field \( \mathcal{X} \) be given by

\[
P_1 = \frac{1}{2}y_1(x_1^2 - a^2)G, \quad P_2 = \frac{1}{2}y_1x_1x_2G, \quad \cdots, \quad P_{p+1} = \frac{1}{2}y_1x_1x_{p+1}G
\]

\[
Q_1 = x_1(y_1^2 - 1)G, \quad Q_2 = x_1y_1y_2G, \quad Q_3 = x_1y_1y_3G, \quad Q_4 = x_1y_1y_4G, \quad Q_5 = x_1y_1y_5G, \quad \cdots, \quad Q_q = x_1y_1y_qG.
\]

Where \( G \in \mathbb{R}[x_1, \ldots, x_{p+1}, y_1, \ldots, y_q] \) is some polynomial we can choose of degree \( \deg \mathcal{X} - 3 \). Consider the hyperplane given by \( x_i = c \) where \( c \) is a constant and \( -1 < c < 1 \). We want to look at the invariant algebraic sets formed by the intersection of these hyperplanes with \( S_{p,q} \). The number of connected components of \( \{x_i = c\} \cap S_{p,q} \) is one since this is homeomorphic to \( S^{p-1} \times S^q \) for \( p \geq 2 \), \( q \geq 2 \). In this case, the extactic polynomial is

\[
\mathcal{E}_{\{1, x_1\}}(\mathcal{X}) = \det \begin{pmatrix} 1 & x_1 \\ 0 & P_i \end{pmatrix} = P_i.
\]

We see that the maximum possible number of these invariant hyperplanes is \( m_1(= \deg P_1) \) since that is the maximum possible degree of any \( P_i \) for all \( i \). Letting \( G = \prod_{j=1}^{m_1-1} (x - c_j) \) in the vector field given by (3.21), and if we regard \( x_1 = 0 = x_i \) also as one of the invariant hyperplanes \( (x_1 - a = 0 \text{ and } x_1 + a = 0 \text{ if } i = 1) \), we see that we have \( m_1 - 1 \) invariant algebraic sets of the form under consideration for this choice of the vector field. Similarly, one can do the computation for \( y_j = c \) and an almost tight bound can be obtained.
3.2. Computation on elliptic cylinders. In this subsection, we consider the hypersurfaces, which we shall call elliptic cylinders, of the form

\begin{equation}
\sum_{i=1}^{p+1} a_i x_i^2 = 1
\end{equation}

where \(0 \leq a_i \in \mathbb{R}\) for \(i = 1, \ldots, p + 1\). We shall obtain a bound on the maximum number of invariant algebraic sets in \(S_{p,q}\) which are intersections of \(S_{p,q}\) with the hypersurfaces of the form \((3.23)\).

**Proposition 3.10.** The number of connected components of the intersection \(\{\sum_{i=1}^{p+1} a_i x_i^2 - 1\} \cap S_{p,q}\) is one if \(a^2 > 2\).

**Proof.** We may assume that \(a_1 \neq 0\). Put \(x_1^2 = 1 - (\sum_{i=2}^{p+1} x_i^2) / a_1\), then (3.1) becomes

\[
(1 - (\sum_{i=2}^{p+1} x_i^2) / a_1 + \sum_{i=2}^{p+1} x_i^2 - a^2)^2 + y_1^2 + y_2^2 + \cdots y_q^2 = 1.
\]

So,

\[
(\sum_{i=2}^{p+1} (1 - 1 / a_1) x_i^2 + a^2 - 1)^2 + y_1^2 + y_2^2 + \cdots y_q^2 = 1.
\]

Therefore this is homeomorphic to \(S^{p-1} \times S^q\) if \(a^2 > 2\) and hence has a single connected component. \(\square\)

**Theorem 3.11.** Assume that the vector field of \((3.2)\) has finitely many invariant algebraic hypersurfaces. If \(p \geq 2\), and \(a^2 > 2\), then the maximum number of invariant algebraic sets obtained by the intersection of \(S_{p,q}\) with \((3.23)\) is

\[
\frac{1}{2} \left\{ \left( \frac{p}{2} \right) (m_1 - 1) + \sum_{i=1}^{p+1} m_i \right\}.
\]

**Proof.** By Proposition 2.2 we know that \(\mathcal{g} := \sum_{i=1}^{p+1} a_i x_i^2 - 1\) is invariant for the vector field \(\mathcal{X}\) in \((3.2)\) if and only if \(\mathcal{g}\) is a factor of the exacted polynomial

\begin{equation}
\mathcal{E}_{\{x_1, x_2, \ldots, x_{p+1}\}}(\mathcal{X}) = \det \begin{pmatrix}
1 & x_1^2 & x_2^2 & \cdots & x_{p+1}^2 \\
0 & \mathcal{X}(x_1^2) & \mathcal{X}(x_2^2) & \cdots & \mathcal{X}(x_{p+1}^2) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \mathcal{X}^{p-1}(P_1) & \mathcal{X}^{p-1}(P_2) & \cdots & \mathcal{X}^{p-1}(P_{p+1})
\end{pmatrix}
\end{equation}

\[
= \det \begin{pmatrix}
1 & x_1^2 & x_2^2 & \cdots & x_{p+1}^2 \\
0 & 2x_1 P_1 & 2x_2 P_2 & \cdots & 2x_{p+1} P_{p+1} \\
0 & 2P_1^2 + 2x_1 P_1 & 2P_2^2 + 2x_2 P_2 & \cdots & 2P_{p+1}^2 + 2x_{p+1} P_{p+1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \{\cdots + 2x_1 \mathcal{X}^p(P_1)\} & \{\cdots + 2x_2 \mathcal{X}^p(P_2)\} & \cdots & \{\cdots + 2x_{p+1} \mathcal{X}^p(P_{p+1})\}
\end{pmatrix}.
\]
From the third row onward in the matrix in (3.24), the element of the form \(2x_i\mathcal{X}(P_i)\), has the highest degree for each entry in that row. Consequently, the product of these terms decides the degree of the extactic polynomial. Therefore, in the expression of the determinant in (3.24), the highest degree term is

\[
2^{p+1}(x_1x_2x_3\cdots x_{p+1})x_{p+1}\mathcal{X}(P_1)\mathcal{X}^{p-1}(P_2)\mathcal{X}^{p-2}(P_3)\cdots \mathcal{X}(P_p).
\]

Note that the degree of the polynomial \(\mathcal{X}^{p-i}(P_{i+1})\) is \((p-i)(m_1-1)+m_{i+1}\) for \(i = 0, \ldots, p\). Hence the degree of the expression (3.25) is

\[
\left(\frac{p}{2}\right)(m_1 - 1) + \sum_{i=1}^{p+1} m_i + p + 2.
\]

So the maximum number of quadratic hypersurfaces (3.20) that could be a factor of the extactic polynomial (3.24) is half of \((\frac{p}{2})(m_1 - 1) + \sum_{i=1}^{p+1} m_i\), since \(g\) is not a factor of \((x_1x_2x_3\cdots x_{p+1})x_{p+1}\). Thus we get the result. □

We remark that one can get similar result as in Theorem 3.11 if the considered hypersurfaces are of the form \(\sum_{j=1}^q b_jy_j^2 = c\), where \(c \in (-1, 1)\).

**Example 3.12.** Let the vector field \(\mathcal{X}\) be given by

\[
P_1 = \frac{1}{2}y_1(x_1^2 - a^2)G, \quad P_2 = \frac{1}{2}y_1x_1x_2G, \quad \ldots, \quad P_{p+1} = \frac{1}{2}y_1x_1x_{p+1}G
\]

\[
Q_1 = x_1(y_1^2 - 1)G, \quad Q_2 = x_1y_1y_2G, \quad Q_3 = x_1y_1y_3G, \quad Q_4 = x_1y_1y_4G,
\]

\[
Q_5 = x_1y_1y_5G, \quad \ldots, \quad Q_q = x_1y_1y_qG.
\]

Where \(G \in \mathbb{R}[x_1, \ldots, x_{p+1}, y_1, \ldots, y_q]\) is some polynomial we can choose of degree \(\deg \mathcal{X} = 3\).

Consider the cylindrical hypersurfaces given by \(ax_1^2 + by_1^2 = 1\) for \(0 \neq a, b \in \mathbb{R}\). Once again, we are interested in the invariant algebraic sets formed by the intersection of these hypersurfaces with \(S_{p,q}\). The extactic polynomial is

\[
\mathcal{E}_{\{x_1^2, y_1^2\}}(\mathcal{X}) = \det \begin{pmatrix}
1 & x_1^2 & y_1^2 \\
0 & 2x_1P_1 & 2y_1Q_1 \\
0 & 2P_1^2 + 2x_1\mathcal{X}(P_1) & 2Q_1^2 + 2y_1\mathcal{X}(Q_1)
\end{pmatrix}.
\]

If we let \(G = \prod_{j=1}^r(a_jx_1^2 + b_jy_1^2 - 1)\) in the vector field given by (3.26), we see that the "elliptic cylinders" are invariant. This is because \(G\) is a common factor in the second row, i.e. \(\mathcal{E}_{\{x_1^2, y_1^2\}}(\mathcal{X}) = G \cdot K\) for some polynomial \(K\). We also see that this vector field can have \(\frac{r}{2}\) many invariant algebraic sets formed by the intersection of \(S_{p,q}\) with the hypersurfaces of the form \(ax_1^2 + by_1^2 = 1\). This number is \(\left\lfloor \frac{m_1-3}{2} \right\rfloor\) in our case.
3.3. **Computation on hyperbolic cylinders.** In this subsection, we study the invariant algebraic sets formed by the intersection of $S_{p,q}$ with hyper-surfaces of the form $x_1x_2 = c$ for some constant $c \in \mathbb{R}$. We shall call these invariant sets "hyperbolic Cylinders".

**Proposition 3.13.** The number of connected components of the intersection $\{(x_1x_2 - c)\} \cap S_{p,q}$ is at most four and at least 2.

**Proof.** Note that $\{x_1x_2 - c = 0\}$ has two connected components. One is given by $\{x_1x_2 - c = 0\}$ with $x_1 > 0$ and $x_2 > 0$, and the other is given by $\{x_1x_2 - c = 0\}$ with $x_1 < 0$ and $x_2 < 0$. For its first connected component, we have in the intersection

\[
(c^2/x_2^2 + x_3^2 + \cdots + x_{p+1}^2 - a^2)^2 + y_1^2 + \cdots + y_q^2 = 1.
\]

So,

\[
(3.28) \quad ((c/x_2 + x_2)^2 + x_3^2 + \cdots + x_{p+1}^2 - (a^2 + 2c))^2 + y_1^2 + \cdots + y_q^2 = 1.
\]

For each $x_3, \ldots, x_{p+1}, y_1, \ldots, y_q$ satisfying this equation, we get two distinct $x_2$. Thus the connected component represented by $\text{(3.28)}$ is at most two. For the other connected component of $\{x_1x_2 - c = 0\}$, we get the similar result. Therefore, we get the proof. \(\Box\)

The polynomial $x_1x_2 - c$ belongs to the vector space

\[
W := \text{span}\{1, x_1x_2\}.
\]

We know that a vector field $\mathcal{X}$ has an invariant hyperbolic cylinder if and only if $x_1x_2 - c$ divides the extactic polynomial

\[
(3.29) \quad \mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix}
1 & x_1x_2 \\
\mathcal{X}(1) & \mathcal{X}(x_1x_2)
\end{pmatrix} = \det \begin{pmatrix}
1 & x_1x_2 \\
0 & x_1P_1 + x_1P_2
\end{pmatrix} = x_2P_1 + x_1P_2.
\]

If the degree of $\mathcal{X}$ is $m$, then, we see that the degree of $\mathcal{E}_W(\mathcal{X})$ is $m + 1$. Since each hyperbolic cylinder has degree two, we get the following.

**Proposition 3.14.** The maximum possible number of invariant hyperbolic cylinders is $2(m_1 + 1)$.

Consider the vector field $\mathcal{X}$ given by

\[
(3.30) \quad P_1 = \frac{1}{2}y_1(x_1^2 - a^2)G, \quad P_2 = \frac{1}{2}y_1x_1x_2G, \quad \ldots, \quad P_{p+1} = \frac{1}{2}y_1x_1x_{p+1}G
\]

\[
Q_1 = x_1(y_1^2 - 1)G, \quad Q_2 = x_1y_1y_2G, \quad Q_3 = x_1y_1y_3G, \quad Q_4 = x_1y_1y_4G, \quad Q_5 = x_1y_1y_5G, \quad \ldots \quad Q_q = x_1y_1y_qG.
\]

If we let $G = \prod_{j=1}^{\lfloor (m_1 - 1)/2 \rfloor} (x_s x_t - c_j)$ where $s, t \in \{2, \ldots, p + 1\}$, $s \neq t$, then $\mathcal{E}_W(\mathcal{X}) = G \cdot (y_1x_1x_s x_t)$. Hence making use of Proposition 3.13, there exists $2(m_1 - 3)$ many invariant hyperbolic cylinders for the vector field $\mathcal{X}$ given by $\text{(3.30)}.~\Box$
In this section, we give tight upper bound for the number of invariant hyperplanes of certain types for the vector fields on $S^1 \times S^q$.

We consider the hypersurface $S_{1,q}$ defined by the polynomial identity

$$ (x_1^2 + x_2^2 - a^2)^2 + \sum_{j=1}^q y_j^2 = 1 $$

where $a > 1$. In section 3, we showed that $S_{1,q} \cong S^1 \times S^2$. This defines an embedding of $S^1 \times S^q$ in $\mathbb{R}^{q+2}$. Let us consider the polynomial vector fields of the form (4.2) defined on $S_{1,q} \cong S^1 \times S^q$ which can be written as

$$ P_1 \frac{\partial}{\partial x_1} + P_2 \frac{\partial}{\partial x_2} + \sum_{i=1}^q Q_i \frac{\partial}{\partial y_i} $$

where $P_1, P_2, Q_i \in \mathbb{R}[x_1, x_2, y_1, \ldots, y_q]$. So, we have the following by the definition of a vector field on $S_{1,q}$.

$$ 4(x_1^2 + x_2^2 - a^2)x_1 P_1 + 4(x_1^2 + x_2^2 - a^2)x_2 P_2 + 2 \sum_{j=1}^q y_j Q_j = K \left( (x_1^2 + x_2^2 - 1)^2 + \sum_{j=1}^q y_j^2 - 1 \right) $$

for some $K \in \mathbb{R}[x_1, x_2, y_1, \ldots, y_q]$. We note that a hyperplane in $\mathbb{R}^{q+2}$ is given by $a_1 x_1 + a_2 x_2 + \sum_{j=1}^q b_j y_j = 0$, so one may consider the algebraic sets obtained by its intersection with $S_{1,q}$. First we shall look into invariant algebraic sets obtained by the intersection of $S_{1,q}$ with the following.

(a) Hyperplanes given by $a_1 x_1 + a_2 x_2 = 0$, also called meridians.

(b) Hyperplanes given by $\sum_{j=1}^q b_j y_j = c$ for $-1 < c < 1$, also called parallels.

For the subsets of type (a) to be invariant algebraic sets, we need that $a_1 x_1 + a_2 x_2$ divides the extactic polynomial

$$ \mathcal{E}_{x_1, x_2}(\mathcal{X}) = \det \begin{pmatrix} x_1 & x_2 \\ \mathcal{X}(x_1) & \mathcal{X}(x_2) \end{pmatrix} = \det \begin{pmatrix} x_1 & x_2 \\ P_1 & P_2 \end{pmatrix} = x_1 P_2 - x_2 P_1. $$

For the subsets of type (b) to be invariant algebraic sets, we need that $\sum_{j=1}^q b_j y_j$ divides the extactic polynomial

$$ \mathcal{E}_{y_1, \ldots, y_q} = \det \begin{pmatrix} 1 & y_1 & \cdots & y_q \\ 0 & \mathcal{X}(y_1) & \cdots & \mathcal{X}(y_q) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{X}^{q-1}(y_1) & \cdots & \mathcal{X}^{q-1}(y_q) \end{pmatrix}. $$

**Proposition 4.1.** The numbers of connected components of the intersections $\{a_1 x_1 + a_2 x_2 = 0\} \cap S_{1,q}$ and $\{\sum_{j=1}^q b_j y_j = c\} \cap S_{1,q}$ are two and one respectively.
Proof. For the case \( \{ a_1 x_1 + a_2 x_2 = 0 \} \cap S_{1,q} \), put \( x_2 = -a_1 x_1 / a_2 \), if \( a_2 \neq 0 \), otherwise \( x_1 = -a_2 x_2 / a_1 \). Then (4.1) may have the following form.

\[
(x_1^2(1 + \frac{a_1^2}{a_2^2}) - r^2)^2 + \sum_{j=1}^{q} y_j^2 = 1.
\]

That is

\[
x_1^2(1 + \frac{a_1^2}{a_2^2}) = a^2 \pm \sqrt{1 - \sum_{j=1}^{q} y_j^2}.
\]

This gives

\[
x_1 \sqrt{1 + \frac{a_1^2}{a_2^2}} = \pm \sqrt{a^2 \pm \sqrt{1 - \sum_{j=1}^{q} y_j^2}}.
\]

Since \( a > 1 \), \( x_1 \) takes positive and negative values without ever becoming zero. Thus there are two connected components of \( \{ a_1 x_1 + a_2 x_2 \} \cap S_{1,q} \).

The argument for the case of \( \{ \sum_{j=1}^{q} b_j y_j = c \} \cap S_{1,q} \) is similar to the proof of the second part of Proposition 3.2. \( \square \)

**Theorem 4.2.** Let \( \mathcal{X} \) be a polynomial vector field of degree \( m \) on \( S_{1,q} \) such that there are only finitely many invariant algebraic sets. Then we have the following.

1. There can be at most \( 2(m - 1) \) invariant algebraic sets of type (a).
2. There exists a polynomial vector field, on \( S_{1,q} \) with exactly \( 2(m - 1) \) invariant algebraic sets of type (a).

**Proof.** For (1): Let \( x_1 = r \cos \theta \), and \( x_2 = r \sin \theta \) where \( r \geq 0 \) and \( \theta \in [0, 2\pi] \). With these choice of coordinates the polynomial vector field becomes

\[
\mathcal{X} = \frac{1}{r} (P_1(r \cos \theta, r \sin \theta, y_1, \ldots, y_q)r \cos \theta \\
+ P_2(r \cos \theta, r \sin \theta, y_1, \ldots, y_q)r \sin \theta) \frac{\partial}{\partial r} \\
- \frac{1}{r^2} (P_1(r \cos \theta, r \sin \theta, y_1, \ldots, y_q)r \sin \theta \\
- P_2(r \cos \theta, r \sin \theta, y_1, \ldots, y_q)r \cos \theta) \frac{\partial}{\partial \theta} \\
+ \sum_{i=1}^{q} Q_i \frac{\partial}{\partial y_i}.
\]

This implies that

\[
x_1 P_2 - x_2 P_1 = r^2 \dot{\theta} = (x_1^2 + x_2^2) \dot{\theta}.
\]

Note that the maximum degree of the left hand side of (4.6) is \( m + 1 \) and \( x_1^2 + x_2^2 \) is irreducible over \( \mathbb{R} \). Thus we get that the maximum number of invariant algebraic sets of type (a) is \( m - 1 \). By Proposition 4.1, we know
that the intersection \( \{a_1x_1 + a_2x_2\} \cap S_{1,q} \) has two connected components. Hence we get \( 2(m - 1) \) invariant algebraic sets of type (a). This proves (1).

For (2): We consider the vector field \( \mathcal{X} \) on \( S_{1,q} \) given by
\[
P_1 = x_1y_1 \cdots y_q - x_2G, \quad P_2 = x_2y_1 \cdots y_q + x_1G
\]
and
\[
Q_j = \frac{2}{q} \left( -a^2(x_1^2 + x_2^2) + \sum_{j=1}^{q} y_j^2 - 1 \right) y_1 \cdots y_j \cdots y_q
\]
where the index with a hat is skipped. Taking
\[
G = \prod_{i=1}^{m-1} (a_i x_1 + b_i x_2)
\]
one can see \( G \) is a factor of \( \mathcal{E}_{x_1,x_2}(\mathcal{X}) \) of (4.3). Now \( (a_i x_1 + b_i x_2), 1 \leq i \leq m - 1 \) are \( m - 1 \) invariant algebraic sets of type (a). But we know that the intersection \( \{a_1x_1 + a_2x_2\} \cap S_{1,q} \) has two connected components. This proves (2).

We remark that the proof of the first part of Theorem 4.2 uses the technique of the proof of Theorem 1 in [3].

**Remark 4.3.** An upper bound for the number of invariant parallels of \( \mathcal{X} \) can be given by similar calculation as in Theorem 3.3 (2) which is \( \binom{q}{2} (m_{p+2} - 1) + \sum_{j=1}^{q-1} m_{p+j+1} \).

**Theorem 4.4.** There exists a vector field \( \mathcal{X} \) on \( S_{1,q} \) with \( (m - 1) \) parallels if \( q \) is even and with \( (m - 2) \) parallels if \( q \) is odd.

**Proof.** Let \( q \) be even. Consider the polynomial vector field \( \mathcal{X} \) of degree \( m \) defined by
\[
P_1 = \frac{1}{2} y_1(x_1^2 - a^2)G, \quad P_2 = \frac{1}{2} y_1 x_1 x_2 G, \quad Q_1 = x_1(y_1^2 - 1)G, \quad Q_2 = x_1 y_1 y_2 G, \quad Q_3 = x_1 y_1 y_3 G, \quad \ldots, \quad Q_q = x_1 y_1 y_q G.
\]

Now we choose
\[
G = \prod_{j=1}^{m-3} \left( \sum_{k=1}^{q} (b_{kj} y_k) - c_j \right)
\]
where \( b_{kj}, c_j \in \mathbb{R} \). Note that this vector field is invariant on \( S_{1,q} \) (i.e. \( P_i \) and \( Q_j \) satisfy (4.2)). Observe that \( \left( \sum_{k=1}^{q} (b_{kj} y_k) - c_j \right), 1 \leq j \leq m - 3 \) are \( m - 3 \) invariant algebraic sets of type (b) when \( q \) is even, and \( m - 2 \) when \( q \) is odd. \( \square \)
5. Invariant Algebraic sets on $S^p \times S^1$

In this section, we study certain invariant algebraic subsets of $S^p \times S^1$ for $p \geq 2$. We consider the following hypersurface, denoted by $S_{p,1}$,

\begin{equation}
(\sum_{i=1}^{p+1} x_i^2 - a^2)^2 + z^2 = 1 \tag{5.1}
\end{equation}

where $a > 1$. We shall consider the polynomial vector fields of the form (1.2) defined on $S_{p,1}$ which can be written as

$$\sum_{i=1}^{p+1} P_i \frac{\partial}{\partial x_i} + Q \frac{\partial}{\partial z}.$$ 

Using arguments similar to that in Proposition 3.1, we get that $S_{p,1}$ is homeomorphic to $S^p \times S^1$. We note that when $p = q$, then $S_{p,1}$ and $S_{1,q}$ are homeomorphic. However, their equations say that they are different algebraic subsets of $\mathbb{R}^{p+2} (= \mathbb{R}^{q+2})$.

Next, we shall study hyperplanes in $\mathbb{R}^{p+2}$ given by $\sum_{i=1}^{p+1} a_i x_i + bz = c$. More specifically, we want to look at the invariant algebraic sets obtained by the intersection of $S_{p,1}$ with the following.

(a). Hyperplanes given by $\sum_{i=1}^{p+1} a_i x_i = 0$, also called meridians.
(b). Hyperplanes given by $z = c$, also called parallels.

For the subsets of type (a) to be invariant algebraic sets, we need that $\sum_{i=1}^{p+1} a_i x_i$ divides the extactic polynomial

\begin{equation}
E_{\{x_1, \ldots, x_{p+1}\}}(X) = \det \left( \begin{array}{ccc}
X(x_1) & \cdots & X(x_{p+1}) \\
\vdots & \ddots & \vdots \\
X^p(x_1) & \cdots & X^p(x_{p+1})
\end{array} \right). \tag{5.2}
\end{equation}

For the subsets of type (b) to be invariant algebraic sets, we need that $z - c$ divides the extactic polynomial

\begin{equation}
E_{\{1, z\}}(X) = \det \left( \begin{array}{cc}
1 & z \\
X(1) & X(z)
\end{array} \right) = \det \left( \begin{array}{cc}
1 & z \\
0 & Q
\end{array} \right) = Q. \tag{5.3}
\end{equation}

**Proposition 5.1.** The numbers of connected components of the intersections $\{\sum_{i=1}^{p+1} a_i x_i = 0\} \cap S_{p,1}$ and $\{z = c\} \cap S_{p,1}$ are one and two respectively for $|c| < 1$.

**Proof.** The argument for the case of $\{\sum_{i=1}^{p+1} a_i x_i = 0\} \cap S_{p,1}$ is similar to the proof of the first part of Proposition 3.2.

For the case of $\{z = c\} \cap S_{p,1}$, set $z = c$, then (5.1) becomes

$$\left(\sum_{i=1}^{p+1} x_i^2 - a^2\right)^2 + c^2 = 1$$

and

$$\sum_{i=1}^{p+1} P_i \frac{\partial}{\partial x_i} + Q \frac{\partial}{\partial z}.$$
that is
\[ \sum_{i=1}^{p+1} x_i^2 = a^2 \pm \sqrt{1-c^2} \]
since \( c \) is a fixed constant, this gives two different concentric spheres. Therefore \( \{z = c\} \cap S_{p,1} \) has two connected components, unless \( |c| \geq 1 \). \( \square \)

We note that by the arguments in the proof of Theorem 3.3 (1) we know that the maximum number of invariant algebraic sets of type (a) is
\[ \left( \begin{array}{c} p+1 \end{array} \right) (m_1 - 1) + \sum_{i=2}^{p+1} m_i + 1. \]

**Proposition 5.2.** Let \( \mathcal{X} \) be a vector field on \( S_{p,1} \). Then the maximum number of parallels for \( \mathcal{X} \) is \( 2(m-2) \) where \( m = \deg \mathcal{X} \). Moreover, there is a vector field on \( S_{p,1} \) with exactly \( 2(m-2) \) invariant parallels.

**Proof.** We proceed in a manner similar to the proof of Theorem 1 in [8]. Since \( \mathcal{X} \) is a vector field on \( S_{p,1} \), we have by definition
\[ 4(x_1^2 + \cdots + x_{p+1}^2)(x_1P_1 + \cdots + x_{p+1}P_{p+1}) + 2zQ = K(x_1, \ldots, x_{p+1}, z) \left( \sum_{i=1}^{p+1} x_i^2 - r^2 \right)^2 + z^2 - 1. \]
In particular, this holds for \( x_i = 0, \forall i \), that is
\[ 2zQ = K(0, \ldots, 0, z)(a^4 + z^2 - 1) \]
where \( a > 1 \) and hence \( a^4 + z^2 - 1 \) cannot be factored over \( \mathbb{R} \). Since \( z \) is a factor of the left hand side of (5.4), it has to be a factor of the right hand side too, so
\[ K(0, \ldots, 0, z) = zh(z) \]
for some \( h \in \mathbb{R}[z] \). We know that \( \deg Q \leq m \), hence we see from (5.4) that \( Q \) can have only \( m-2 \) factors of the form \( z-c \) and hence \( E_{1,z}(\mathcal{X}) \) can have only \( m-2 \) factors of the form \( z-c \). Now from Proposition 5.1 we know that parallels come in pairs and hence the maximum number of parallels a vector field of degree \( m \) can have is \( 2(m-2) \).

Consider the vector field \( \mathcal{X} \) given by
\[ P_1 = x_1zH, \quad P_2 = x_2zH, \quad P_3 = x_3zH, \quad \cdots, \quad P_{p+1} = x_{p+1}zH \]
\[ Q = 2(-r^2(\sum_{i=1}^{p+1} x_i^2) - r^2) + z^2 - 1)H \]
where \( P_i, Q \in \mathbb{R}[x_1, x_2, \cdots, x_{p+1}, z] \). Observe that \( \mathcal{X} \) is a vector field on \( S_{p,1} \). Let
\[ H = \prod_{i=1}^{m-2} (z - c_i) \]
for \( c_i \in \mathbb{R} \). Then we see that \( Q = (\text{quadratic terms}) \cdot H \) and from [5.3] we see that \( H \) divides the extactic polynomial and therefore \( \mathcal{X} \) is a vector field with \( m - 2 \) invariant algebraic sets which are intersections of \( S_{p,1} \) with hyperplanes of type (b). But since each intersection of \( S_{p,1} \) with hyperplanes of type (b) have two components by Proposition 5.1 we get the result. \( \square \)

**Proposition 5.3.** There exists a vector field invariant on \( S_{p,1} \) with \( m - 2 \) invariant meridians which are intersections of \( S_{p,1} \) with hyperplanes of type (a).

**Proof.** If we let

\[
H = \prod_{i=1}^{m-2} \left( \sum_{k=1}^{p+1} a_{ik} x_k \right)
\]

in the vector field [5.5], for \( a_{ik} \in \mathbb{R} \), then we claim that \( \mathcal{X} \) is a vector field with \( m - 2 \) invariant algebraic sets which are intersections of \( S_{p,1} \) with hyperplanes of type (a). This is because, in the extactic polynomial given by [5.2], all terms in second row will have \( H \) as a common factor, and hence \( H \) divides the extactic polynomial which implies our claim. \( \square \)

**Example 5.4.** In the special case when \( p = 2 \), we have the following, \( S_{2,1} = S^2 \times S^1 \) which is given by

\[
(w^2 + x^2 + y^2 - a^2)^2 + z^2 = 1.
\]

Consider the invariant vector field of degree \( m \) given by

\[
P_1 = wzH + xyG, \quad P_2 = xzH - 2wyG, \quad P_3 = yzH + wxG, \quad P_4 = 2(-a^2(w^2 + x^2 + y^2 - a^2) + z^2 - 1).
\]

A hyperplane of the form \( aw + bx + cy = 0 \) is an invariant set for the vector field on \( S_{2,1} \) if it divides the extactic polynomial

\[
E_{w,x,y}(\mathcal{X}) = \det \begin{pmatrix}
w & x & y \\
\mathcal{X}(w) & \mathcal{X}(x) & \mathcal{X}(y) \\
\mathcal{X}^2(w) & \mathcal{X}^2(x) & \mathcal{X}^2(y)
\end{pmatrix}.
\]

Computing \( E(\mathcal{X}) \) in our case, we get,

\[
E_{w,x,y}(\mathcal{X}) = 2G^3w^4x^2 + G^3w^2x^4 - 2y^4(2G^3w^2 + (GG_xw + G^3)x^2) \\
+ (2GG_xw^3x^2 + 4G^3w^4 - G^3x^4)y^2 + z^2(Gw^2x^2 - Gx^2y^2) \\
+ ((2GG_xw + G)x^2y^2 - (2GG_xw^3 + Gw^2)x^2y)z.
\]

we see that \( E(\mathcal{X}) \) is of the form \( G \cdot f(w, x, y, G_x, G) \). Now, letting \( G = \prod_{i=1}^{m-2} (a_iw + b_ix + c_iy) \), the \( (m - 2) \) hyperplane sections \( a_iw + b_ix + c_iy \), \( 1 \leq i \leq m \) divide \( E(\mathcal{X}) \) and hence are invariant.
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REFERENCES

[1] Joan C. Artés, Branko Grünbaum, and Jaume Llibre. On the number of invariant straight lines for polynomial differential systems. Pacific J. Math., 184(2):207–230, 1998.
[2] Yudy Bolaños and Jaume Llibre. On the number of N-dimensional invariant spheres in polynomial vector fields of $\mathbb{C}^{N+1}$. J. Appl. Anal. Comput., 1(2):173–182, 2011.
[3] Roger A. Horn and Charles R. Johnson. Matrix analysis. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.
[4] Yu. S. Il’yashenko. Centennial history of Hilbert’s 16th problem [mr1898209]. In Fundamental mathematics today (Russian), pages 135–213. Nezavis. Mosk. Univ., Moscow, 2003.
[5] J. P. Jouanolou. Équations de Pfaff algébriques, volume 708 of Lecture Notes in Mathematics. Springer, Berlin, 1979.
[6] Jaume Llibre and Yudy Bolaños. Rational first integrals for polynomial vector fields on algebraic hypersurfaces of $\mathbb{R}^{n+1}$. Internat. J. Bifur. Chaos Appl. Sci. Engrg., 22(11):1250270, 11, 2012.
[7] Jaume Llibre and João C. Medrado. On the invariant hyperplanes for $d$-dimensional polynomial vector fields. J. Phys. A, 40(29):8385–8391, 2007.
[8] Jaume Llibre and João C. Medrado. Limit cycles, invariant meridians and parallels for polynomial vector fields on the torus. Bull. Sci. Math., 135(1):1–9, 2011.
[9] Jaume Llibre and Adrian C. Murza. Darboux theory of integrability for polynomial vector fields on $S^n$. Dyn. Syst., 33(4):646–659, 2018.
[10] Jaume Llibre, Rafael Ramírez, and Natalia Sadovskaia. On the 16th Hilbert problem for algebraic limit cycles. J. Differential Equations, 248(6):1401–1409, 2010.
[11] Jaume Llibre, Rafael Ramírez, and Natalia Sadovskaia. On the 16th Hilbert problem for limit cycles on nonsingular algebraic curves. J. Differential Equations, 250(2):983–999, 2011.
[12] Jaume Llibre and Salomón Rebollo-Perdomo. Invariant parallels, invariant meridians and limit cycles of polynomial vector fields on some 2-dimensional algebraic tori in $\mathbb{R}^3$. J. Dynam. Differential Equations, 25(3):777–793, 2013.
[13] Jaume Llibre and Xiang Zhang. Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicity. J. Differential Equations, 246(2):541–551, 2009.
[14] Jacek Sokulski. On the number of invariant lines for polynomial vector fields. Nonlinearity, 9(2):479–485, 1996.
[15] D. C. Youla. A normal form for a matrix under the unitary congruence group. Canadian J. Math., 13:694–704, 1961.
[16] Xiang Zhang. The 16th Hilbert problem on algebraic limit cycles. J. Differential Equations, 251(7):1778–1789, 2011.