Resilience: A Criterion for Learning in the Presence of Arbitrary Outliers

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Abstract

We introduce a criterion, resilience, which allows properties of a dataset (such as its mean or best low rank approximation) to be robustly computed, even in the presence of a large fraction of arbitrary additional data. Resilience is a weaker condition than most other properties considered so far in the literature, and yet enables robust estimation in a broader variety of settings, including the previously unstudied problem of robust mean estimation in $\ell_p$-norms.

Keywords: robust learning, outliers, $p$-norm estimation, low rank approximation

1. Introduction

What are the fundamental properties that allow one to robustly learn from a dataset, even if some fraction of that dataset consists of arbitrarily corrupted data? While much work has been done in the setting of noisy data, or for restricted families of outliers, it is only recently that provable algorithms for learning in the presence of a large fraction of arbitrary (and potentially adversarial) data have been formulated in high-dimensional settings (Diakonikolas et al., 2016; Lai et al., 2016; Charikar et al., 2017; Steinhardt et al., 2016; Bhatia et al., 2015). In this work, we formulate a conceptually simple criterion that a dataset can satisfy – resilience – which guarantees that properties such as the mean of that dataset can be robustly estimated even if a large fraction of additional arbitrary data is inserted. For mean estimation under a norm $\| \cdot \|$, we have the following criterion:

**Definition 1 (Resilience)** A set of points $\{x_i\}_{i \in S}$ lying in $\mathbb{R}^d$ is $(\sigma, \epsilon)$-resilient in a norm $\| \cdot \|$ around a point $\mu$ if, for all subsets $T \subseteq S$ of size at least $(1 - \epsilon)|S|$, 

$$\left\| \frac{1}{|T|} \sum_{i \in T} (x_i - \mu) \right\| \leq \sigma.$$  

(1)

In other words, a set of points is resilient if every large subset of the points has a similar mean. As an example, points sampled from a Gaussian $\mathcal{N}(0, I)$ will have $\sigma = \tilde{O}(\epsilon)$ with high probability.

Resilience posits robustness to arbitrary deletions – no adversary can perturb the mean of the dataset by more than $\sigma$, even if they can selectively delete up to $\epsilon|S|$ of the points.

More interestingly for our purposes, resilience turns out to imply robustness to arbitrary insertions as well: if a set of points $S$ is $(\sigma, \epsilon)$-resilient, then even if an adversary is allowed to insert $\mathcal{O}(\epsilon|S|)$ additional points into the set, the mean of the original set can be recovered with error $\mathcal{O}(\sigma)$. For instance, as a special case of our results we can show the following:

**Theorem 2** Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be a collection of points containing a subset $S$ of size at least $0.87n$ that is $(\sigma, \frac{1}{4})$-resilient around $\mu$ in $\ell_p$-norm, for $p \in (1, 2]$. Then, there is an efficient algorithm outputting a $\hat{\mu}$ such that $\|\mu - \hat{\mu}\|_p = \mathcal{O}\left(\frac{\sigma}{\sqrt{p-1}}\right)$. 


Note that $S$ is not known, and no assumptions are made on the points in $[n] \setminus S$. When $\alpha \overset{\text{def}}{=} \frac{|S|}{n}$ lies in $[0.87, 0.99]$, this is consistent with known results on robust estimation in the $\ell_2$-norm (Diakonikolas et al., 2016; Lai et al., 2016; Charikar et al., 2017) while making weaker assumptions and holding for general $\ell_p$-norms. For instance, Lai et al. (2016) assume the $x_i$ are drawn from a distribution with bounded 4th moments, while resilience in $\ell_2$-norm basically corresponds to bounded 1st moments, and is moreover a deterministic condition. More generally, all existing approaches require (at least) that the data have bounded covariance: $\|\Sigma\|_2 \leq \sigma^2$, where $\Sigma$ is the covariance of the $x_i$. For the $\ell_2$-norm, it is straightforward to show that this implies $(\sigma \sqrt{2\epsilon}, \epsilon)$-resilience for all $\epsilon \leq \frac{1}{2}$.

When the fraction of good data $\alpha \approx 1$, we can strengthen our bound to $O\left(\sigma \sqrt{\frac{1-\alpha}{p-1}}\right)$ under slightly stronger conditions. Perhaps surprisingly, we obtain meaningful guarantees even when $\alpha < \frac{1}{2}$ — we show there is a randomized algorithm that outputs a $\hat{\mu}$ satisfying $\|\mu - \hat{\mu}\|_p = O\left(\frac{\sigma}{\alpha^{(p-1)}}\right)$ with probability $\Omega(\alpha)$; this success probability is optimal when $\alpha < \frac{1}{2}$. Finally, beyond $\ell_p$ estimation, our techniques (with an appropriately modified resilience criterion) allow for robustly recovering the best rank-$k$ approximation to a dataset.

**Notation**

Given vectors $a_i$, we will let $A$ denote a matrix with $i$th column $a_i$, and let $A_T$ or $[a_i]_{i \in T}$ denote the submatrix with columns from $T$. We let $X^\dagger$ be the pseudoinverse of $X$, and $\sigma_k(X)$ the $k$th singular value. Let $\mathbb{1}$ denote the all-1s vector. Given $p \in [1, 2]$, we let $q \in [2, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

For a $d \times n$ matrix $X$, we let $\|X\|_{p_1 \rightarrow p_2}$ denote $\max_{v \in \mathbb{R}^n} \|Xv\|_{p_2}/\|v\|_{p_1}$. We let $\|X\|_2 = \|X\|_{2 \rightarrow 2}$ denote operator norm, and $\|X\|_F$ denote Frobenius norm. Observe that $\|X\|_{2 \rightarrow p}^2 = \max_{\|v\|_q \leq 1} \sum_i \langle x_i, v \rangle^2$. We will say that vectors $[x_i]_{i \in S}$ have $k$th moments bounded by $\sigma^k$ in $\ell_p$-norm if $\max_{\|v\|_q \leq 1} \frac{1}{|S|} \sum_{i \in S} |\langle x_i, v \rangle|^k \leq \sigma^k$.

**Results**

We consider the well-studied problem of robust mean estimation in the $\ell_2$-norm, then extend to the previously unstudied setting of estimation in $\ell_p$-norms (for $1 < p \leq 2$). We also give the first result on rank-$k$ estimation with error depending on $\sigma_{k+1}$ (the $k + 1$st singular value) rather than $\sigma_2$.

In each case, our argument takes the following form:

1. First, we formulate a basic resilience criterion such as Definition 1, such that it is clear that estimation/recovery is possible if we ignore computational constraints.

2. Then, we develop an efficient algorithm based on the resilience criterion. This typically involves minimizing a convex reconstruction error, then removing outliers that cause the error to be too large.

We found this separation of statistical and algorithmic concerns useful, and think it will likely prove fruitful for other robust learning problems as well (e.g. robust classification or regression). In our case, it seems to have led to substantially simpler algorithms and analyses relative to previous work.

In addition, we found that resilient sets have a rich geometric structure, at least in the case of mean estimation. Up to constants, Definition 1 is equivalent to having small first moments around $\mu$. Strikingly, we show that any resilient set also has a large core with bounded second moments. This $L^1/L^2$ “powering up” result is crucial in enabling efficient estimation, because it allows us to work with convenient linear algebraic objects such as covariance matrices.
**Results: mean estimation.** To start, we consider robust mean estimation in the $\ell_p$-norm. For our resilience criterion we use Definition 1 above. As we show in Section 2, if we ignore computational efficiency then Definition 1 directly enables robust mean estimation (the rough argument is that any two large resilient sets must have similar means).

Our main theorem on mean estimation in $\ell_p$ norms is the following: Let $x_1, \ldots, x_n \in \mathbb{R}^d$, and let $S \subseteq [n]$ be an unknown set whose mean we would like to estimate, with $|S| = \alpha n$. We have:

**Theorem 3** Suppose that $S$ is $(\sigma, \frac{1}{2})$-resilient around $\mu$ in $\ell_p$-norm, with $p \in (1, 2)$. Then there is an efficient algorithm whose output $\hat{\mu}$ satisfies $\|\mu - \hat{\mu}\|_p = O\left(\frac{\sigma}{\alpha(p-1)}\right)$ with probability $\Omega(\alpha)$.

Moreover, if $\alpha \geq 0.87$ and $S$ is $(\sigma \sqrt{\epsilon}, \epsilon)$-resilient for $\epsilon \in \left[\frac{1-\alpha}{2}, \frac{1}{2}\right]$, the same algorithm has $\|\mu - \hat{\mu}\|_p = O\left(\sigma \sqrt{\frac{1-\alpha}{p-1}}\right)$ with probability 1.

The first part says that we can estimate the mean for any value of $\alpha$, even $\alpha < \frac{1}{2}$. The $\Omega(\alpha)$ success probability is necessary because the data could consist of $\frac{1}{\alpha}$ identical translates of $S$, in which case it is impossible to determine which one is correct. The second part says that as $\alpha \to 1$, the estimation error goes to zero. Note that we can always replace the $\frac{1}{p-1}$ factor with $\log(d)$, since the $\ell_p$ and $\ell_{1+\frac{1}{\log(d)}}$ norms are equivalent to within constant factors when $p$ is close to 1.

En route to proving Theorem 3, we establish the following geometric proposition, which allows us to replace the resilience criterion with a more convenient linear algebraic condition.

**Proposition 4** If $S$ is $(\sigma, \frac{1}{2})$-resilient in $\ell_p$-norm and $1 < p \leq 2$, then $S$ contains a set $S_0$ of size at least $\frac{1}{2}|S|$ with bounded 2nd moments: $\|\mathbf{x}_i - \mu\|_{2-p} \leq 10\sigma \sqrt{\frac{|S_0|}{p-1}}$.

We found this result quite striking — while $(\sigma, \frac{1}{2})$-resilience is equivalent to having bounded 1st moments (see Lemma 10), Proposition 4 shows that we get a stronger 2nd moment bound essentially for free. The fact that this can hold with no dimension-dependent factors is far from obvious. In fact, if we replace 2nd moments with 3rd moments or take $p > 2$ then the analog of Proposition 4 is false: we incur polynomial factors in the dimension even if $S$ is the standard basis of $\mathbb{R}^d$ (see Section A for details). The proof of Proposition 4 involves minimax duality, Khintchine’s inequality, and the strong convexity of $\ell_p$ norms. We can also strengthen Proposition 4 to yield $S_0$ of size $(1 - \epsilon)|S|$.

**Results: rank-$k$ recovery.** Our final set of results relate to recovering a rank-$k$ approximation to the data in the presence of arbitrary outliers. We first define the goal: given a set of points $[x_i]_{i \in S}$, let $X_S$ be the matrix whose columns are the $x_i$. Our goal is to obtain a low-rank matrix $P$ such that $\|(I - P)X_S\|_2$ is not much larger than $\sigma_{k+1}(X_S)$, where $\sigma_{k+1}$ denotes the $k + 1$st singular value.

As before, we start by formulating an appropriate resilience criterion:

**Definition 5 (Rank-resilience)** A set of points $[x_i]_{i \in S}$ in $\mathbb{R}^d$ is $\delta$-rank-resilient if for all subsets $T$ of size at least $(1 - \delta)|S|$, we have $\text{col}(X_T) = \text{col}(X_S)$ and $\|X_T^\dagger X_S\|_2 \leq 2$, where $\dagger$ is the pseudoinverse and col denotes column space.

Rank-resilience says that the variation in $X$ should be sufficiently spread out: there should not be a direction of variation that is concentrated in only a $\delta$ fraction of the points.

We can perform efficient rank-$k$ recovery even in the presence of a $\delta$ fraction of arbitrary data:

**Theorem 6** Let $\delta \leq \frac{1}{2}$. If a set of $n$ points contains a set $S$ of size $(1 - \delta)n$ that is $\delta$-rank-resilient, then there is an efficient algorithm for recovering a matrix $P$ of rank at most $15k$ such that $\|(I - P)X_S\|_2 = O(\sigma_{k+1}(X_S))$. 
The power of Theorem 6 comes from the fact that the error depends on $\sigma_{k+1}$ rather than e.g. $\sigma_2$, which is what previous results yielded. This distinction is crucial in practice, since most data have a few (but more than one) large singular values followed by many small singular values. Note that in contrast to Theorem 3, Theorem 6 only holds when $S$ is relatively large: at least $(1 - \delta)n$ in size.

Related Work

A number of authors have recently studied robust estimation and learning in high-dimensional settings: Lai et al. (2016) study the problem of mean and covariance estimation, while Diakonikolas et al. (2016) focus on estimating Gaussian and binary product distributions, as well as mixtures thereof, though their results also apply to mean and covariance estimation of sub-Gaussian distributions. Charikar et al. (2017) recently showed that robust estimation is possible even when the fraction $\alpha$ of “good” data is less than $\frac{1}{p}$. We refer to these papers for an overview of the broader robust estimation literature, but highlight in particular Bhatia et al. (2015), who study linear regression, as well as Steinhardt et al. (2016), who study a crowdsourcing setting. In addition, Diakonikolas et al. (2017) provide a case study of various robust estimation methods in a genomic setting.

We can compare our results to what is known for estimation in the $\ell_2$-norm. Here, when $\alpha \approx 1$, Lai et al. (2016) obtain error $\tilde{O}(\sigma \sqrt{T - \alpha})$ assuming bounded 4th moments, while Diakonikolas et al. (2016) obtain error $\tilde{O}(\sigma(1 - \alpha))$ under sub-Gaussianity. Our criterion for obtaining error $\tilde{O}(\sigma \sqrt{T - \alpha})$ is weaker than bounded 2nd moments, and so improves upon Lai et al. (2016).

When $\alpha < \frac{1}{2}$, the only point of comparison is Charikar et al. (2017), who obtain error $\tilde{O}\left(\frac{\sigma}{\sqrt{\alpha}}\right)$ under bounded 1st moments. We obtain a bound of $O\left(\frac{\sigma}{\sqrt{\alpha}}\right)$ under $(\sigma, \frac{1}{\alpha})$-resilience, which essentially corresponds to bounded 1st moments as well. Our results are thus weaker in the $\ell_2$-norm but hold more generally. We think Charikar et al.’s argument could be extended to $\ell_p$ norms, but only with considerable effort. Our approach is simple enough that the generalization to $\ell_p$ norms is obvious.

Finally, low rank estimation was studied by Lai et al. (2016), but their bounds depend on the maximum eigenvalue $\|\Sigma\|_2$ of the covariance matrix. Our bound appears to be the first to provide robust recovery guarantees in terms of lower singular values of $\Sigma$.

2. Resilience and Robustness: Information-Theoretic Sufficiency

Recall the definition of resilience: $S$ is $(\sigma, \epsilon)$-resilient if $\|\frac{1}{|T|} \sum_{i \in T} (x_i - \mu)\| \leq \sigma$ whenever $T \subseteq S$ and $|T| \geq (1 - \epsilon)|S|$. Here we show that, if we ignore computational efficiency, resilience leads directly to an algorithm for robust mean estimation. In what follows, we use $\sigma_*(\epsilon)$ to denote the smallest $\sigma$ such that $S$ is $(\sigma, \epsilon)$-resilient.

**Proposition 7** Suppose that $x_1, \ldots, x_n \in \mathbb{R}^d$ contains a set $S$ of size $\alpha n$ that is resilient around $\mu$ (where $S$ and $\mu$ are both unknown). Then it is possible to recover a $\hat{\mu}$ such that:

- If $\alpha > \frac{1}{2}$, then $\|\mu - \hat{\mu}\| \leq 2\sigma_*(\frac{1-\alpha}{\alpha}).$
- In general, $\|\mu - \hat{\mu}\| \leq 2\sigma_*(1 - \frac{\alpha}{2})$ with probability at least $\frac{\alpha}{2}$.

The first part says that robustness to an $\epsilon$ fraction of outliers (i.e., if $\alpha = 1 - \epsilon$) depends on resilience to a $\frac{1-\alpha}{\alpha}$ fraction of deletions. The second part says that if the number of outliers is much larger, such that the number of good points is only $\alpha \ll 1$, then we want $S$ to be resilient even if we remove all but an $\frac{\alpha}{2}$ fraction of the points.
Proof (Proposition 7) We prove Proposition 7 via a constructive (albeit exponential-time) algorithm. To prove the first part, let $S'$ be any $(\sigma, \frac{1 - \alpha}{\alpha})$-resilient set of size $\alpha n$, and let $\mu'$ be the corresponding mean vector. We claim that $\mu'$ is sufficiently close to $\mu$.

Indeed, let $T = S \cap S'$, which by the pigeonhole principle has size at least $(2\alpha - 1)n = \frac{2\alpha - 1}{\alpha} |S| = (1 - \frac{1 - \alpha}{\alpha}) |S|$. Therefore,

$$\left\| \frac{1}{|T|} \sum_{i \in T} (x_i - \mu) \right\| \leq \sigma_* \left( \frac{1 - \alpha}{\alpha} \right).$$

But by the same argument, $\left\| \frac{1}{|S'|} \sum_{i \in S'} (x_i - \mu') \right\| \leq \sigma_* \left( \frac{1 - \alpha}{\alpha} \right)$ as well. By the triangle inequality, $\| \mu - \mu' \| \leq 2 \sigma_* \left( \frac{1 - \alpha}{\alpha} \right)$, which completes the first part of the proposition.

The second part of the proposition is similar, but requires us to consider multiple resilient sets $S_i$ rather than a single $S'$. Let $S_1, \ldots, S_m$ be a maximal collection of subsets of $[n]$ such that:

1. $|S_j| = \alpha n$ for all $j$.
2. $S_j$ is $(\sigma, 1 - \frac{\alpha}{2})$-resilient around $\mu_j$.
3. $|S_j \cap S_{j'}| \leq \frac{\alpha^2 n}{2}$ for all $j \neq j'$.

We claim that at least one of the $\mu_j$ is close to $\mu$. Indeed, by maximality of the collection $\{S_j\}_{j=1}^m$, we must have $|S \cap S_j| \geq \frac{\alpha^2 n}{2}$ for some $j$. But then letting $T = S \cap S_j$ as before, we find that $|T| \geq \frac{\alpha}{2} |S|$ and hence $\| \mu - \mu_j \| \leq 2 \sigma_* \left( 1 - \frac{\alpha}{2} \right)$. Therefore, outputting one of the $\mu_j$ at random, with probability $\frac{1}{m}$ we are within the desired distance of $\mu$.

It remains to bound $m$. By the principle of inclusion-exclusion, we have $n \geq |S_1 \cup \cdots \cup S_m| \geq \sum_{j=1}^m |S_j| - \sum_{1 \leq j < j' \leq m} |S_j \cap S_{j'}| \geq \alpha mn - \frac{\alpha^2 n}{2} \binom{m}{2}$. Simple algebra shows that $m \leq \frac{2}{\alpha}$, which completes the proof. ■

3. Efficient Recovery: $\ell_2$ Case

In this section, we prove a warm-up to Theorem 2 which focuses on the $\ell_2$-norm, and assumes that the data has bounded second moments. This assumption is stronger than resilience, e.g. second moments bounded by $\sigma^2$ implies $\sigma_*(\epsilon) \leq \sqrt{\frac{1}{1 - \epsilon}} \sigma$; see Section B. Our warm-up result is:

Proposition 8 Let $x_1, \ldots, x_n \in \mathbb{R}^d$, and let $S$ be a subset of size $\alpha n$ with bounded 2nd moments: $\| [x_i - \mu]_{i \in S} \|_2 \leq \sigma \sqrt{|S|}$, where $\mu$ is the mean of $S$. Then there is an efficient randomized algorithm (Algorithm 1) which with probability $\Omega(\alpha)$ outputs a parameter $\hat{\mu}$ such that $\| \mu - \hat{\mu} \|_2 = O \left( \frac{\sigma}{\alpha} \right)$. Moreover, if $\alpha \geq \frac{3}{2}$ then $\| \mu - \hat{\mu} \|_2 = O \left( \sigma \sqrt{1 - \alpha} \right)$ with probability 1.

At the heart of Algorithm 1 is the following optimization problem:

\begin{align*}
\text{minimize} \quad & \| X - XW \|_2^2 \\
\text{subject to} \quad & 0 \leq W_{ji} \leq \frac{1}{\alpha n} \quad \forall i, j, \quad \sum_j W_{ji} = 1 \quad \forall i.
\end{align*}

(3)

Here $X \in \mathbb{R}^{d \times n}$ is the data matrix $[x_1 \cdots x_n]$. The idea is to re-construct each $x_i$ as an average of $\alpha n$ other $x_j$. Note that by assumption we can always re-construct each element of $S$ using the mean of $S$, and have small error. Intuitively, any element that cannot be re-constructed well must not lie in $S$, and can be
safely removed. We do a soft form of removal by maintaining weights \( c_i \) on the points \( x_i \) (initially all 1), and downweighting points with high reconstruction error. We also maintain an active set \( A \) of points with \( c_i \geq \frac{1}{2} \).

Informally, Algorithm 1 for estimating \( \mu \) takes the following form:

1. Solve the optimization problem (3).
2. If the optimum is \( \gg \sigma^2 n \), then find the columns of \( X \) that are responsible for the optimum being large, and downweight them.
3. Otherwise, if the optimum is \( \mathcal{O}(\sigma^2 n) \), then take a low rank approximation \( W_0 \) to \( W \), and return a randomly chosen column of \( XW_0 \).

The hope in step 3 is that the low rank projection \( XW_0 \) will be close to \( \mu \) for the columns belonging to \( S \).

The choice of operator norm is crucial: it means we can actually expect \( XW \) to be close to \( X \) (on the order of \( \sigma \sqrt{n} \)). In contrast, the Frobenius norm would scale as \( \sigma \sqrt{nd} \).

Finally, we note that

\[
\|X - XW\|_2^2 = \max_{Y \succeq 0, \text{tr}(Y) \leq 1} \sum_{i=1}^{n} (x_i - Xw_i)^\top Y(x_i - Xw_i),
\]

which is the form we use in Algorithm 1.

**Proof (Proposition 8)** We need to show two things: (1) that the outlier removal step removes many more outliers than good points, and (2) that many columns of \( XW_0 \) are close to \( \mu \).

**Outlier removal.** To analyze the outlier removal step (step 2 above, or lines 5-6 of Algorithm 1), we make use of the following general lemma:

**Algorithm 1** Algorithm for recovering the mean of a set with bounded 2nd moments in \( \ell_2 \)-norm.

1. Initialize \( c_i = 1 \) for all \( i = 1, \ldots, n \) and \( A = \{1, \ldots, n\} \).
2. Let \( Y \in \mathbb{R}^{d \times d} \) and \( W \in \mathbb{R}^{A \times A} \) be the maximizer/minimizer of the saddle point problem

\[
\max_{Y \succeq 0, \text{tr}(Y) \leq 1} \min_{0 \leq W_{ji} \leq \frac{1}{n(2+\alpha)n} \sum_j W_{ji} = 1} \sum_{i\in A} c_i (x_i - X_A w_i)^\top Y(x_i - X_A w_i). \quad (4)
\]

3. if \( \sum_{i\in A} c_i (x_i - X_A w_i)^\top Y(x_i - X_A w_i) > 4n\sigma^2 \) then
4. Let \( \tau_i = (x_i - X_A w_i)^\top Y(x_i - X_A w_i) \).
5. For \( i \in A \), replace \( c_i \) with \( \left( 1 - \frac{1}{\tau_{\max}} \right) c_i \), where \( \tau_{\max} = \max_{i \in A} \tau_i \).
6. For all \( i \) with \( c_i < \frac{1}{2} \), remove \( i \) from \( A \).
7. Go back to line 2.
8. **end if**
9. Let \( W_1 \) be the result of zeroing out all singular values of \( W \) that are greater than 0.9.
10. Let \( Z = X_A W_0 \), where \( W_0 = (W - W_1)(I - W_1)^{-1} \).
11. if rank(\( Z \)) = 1 then
12. Output the average of the columns of \( X_A \).
13. else
14. Output a column of \( Z \) at random.
15. **end if**
Lemma 9. For any scalars $\tau_i$ and $a$, suppose that $\sum_{i \in A} c_i \tau_i \geq 4a$ while $\sum_{i \in S \cap A} c_i \tau_i \leq \alpha a$. Then the following invariants are preserved by lines 5-6 of Algorithm 1: (i) $\sum_{i \in S} (1 - c_i) \leq \frac{n}{4} \sum_{i=1}^{n} (1 - c_i)$, and (ii) $|S \cap A| \geq \frac{a(2 + \alpha)}{4 - \alpha} n$.

Lemma 9 says that we downweight points within $S$ at least 4 times slower than we do overall (property i), and in particular we never remove too many points from $S$ (property ii). This lemma is not new (cf. Lemma 4.5 of Charikar et al. (2017)) but for completeness we prove it in Section C.

We want to show that we can take $a = n \sigma^2$ in Lemma 9, or in other words that $\sum_{i \in S \cap A} c_i \tau_i \leq \alpha n \sigma^2$. Note that for a fixed $Y$, each of the $w_i$ are optimized independently, so we can bound $\tau_i$ by substituting any feasible $w_i$. We will choose $W_{ji} = \frac{\|j \in S \cap A\|}{|S \cap A|}$, in which case $X_A w_i = \hat{\mu}$, where $\hat{\mu}$ is the average of $x_j$ over $S \cap A$. Then we have

$$\sum_{i \in S \cap A} c_i \tau_i \leq \sum_{i \in S \cap A} c_i (x_i - \hat{\mu})^\top Y (x_i - \hat{\mu}) \quad (6)$$

$$\leq \sum_{i \in S \cap A} c_i (x_i - \mu)^\top Y (x_i - \mu) \quad (7)$$

$$\leq \sum_{i \in S} (x_i - \mu)^\top Y (x_i - \mu) \leq \alpha n \sigma^2 \text{tr}(Y) \leq \alpha n \sigma^2 \quad (8)$$

as desired. Here (i) is because the covariance around the mean ($\hat{\mu}$) is smaller than around any other point ($\mu$).

Analyzing $XW_0$. It remains to analyze $XW_0$. We will show that on the columns in $S \cap A$, the Frobenius norm $\|XW_0 - \mu I\|_F$ is small. At a high level, it suffices to show that $W_0$ has low rank (so that Frobenius norm is close to spectral norm) and that $XW_0$ and $X$ are close in spectral norm (note that $X$ and $\mu I = \text{diag}(\mu)$ are close by assumption, at least within $S$).

To bound $\text{rank}(W_0)$, note that the constraints in (4) imply that $\|W\|_F^2 \leq \frac{4 - \alpha}{\alpha(2 + \alpha)}$, and so at most $\frac{4 - \alpha}{0.81 \alpha(2 + \alpha)}$ singular values of $W$ can be greater than 0.9. Importantly, at most 1 singular value can be greater than 0.9 if $\alpha \geq \frac{3}{2}$, and at most $O(\frac{1}{\alpha})$ can be in general. Therefore, $\text{rank}(W_0) \leq O(\frac{1}{\alpha})$.

Next, we show that $X_A$ and $Z = X_A W_0$ are close in operator norm. Indeed, $X_A - Z = X_A (I - W_0) = X_A (I - W) (I - W_1)^{-1}$, hence:

$$\|X_A - Z\|_2 = \|X_A (I - W) (I - W_1)^{-1}\|_2 \quad (9)$$

$$\leq \|X_A (I - W)\|_2 \| (I - W_1)^{-1}\|_2 \quad (10)$$

$$\leq 10 \|X_A (I - W)\|_2 \quad (11)$$

$$\leq 10 \sqrt{2} \|X_A (I - W) \text{diag}(c_A)^{1/2}\|_2 \quad (11)$$

Here (i) is because all singular values of $W_1$ are less than 0.9, (ii) is because $\text{diag}(c_A)^{1/2} \geq \frac{1}{\sqrt{2}} I$, and (iii) is by the condition in the if statement (line 3 of Algorithm 1), since the sum on line 3 is equal to $\|X_A (I - W) \text{diag}(c_A)^{1/2}\|_2^2$. 

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Combining the previous two observations, we have
\[ \sum_{i \in S \cap A} \| z_i - \mu \|^2 \leq (\text{rank}(Z) + 1) \| [z_i - \mu]_{i \in S \cap A} \|^2 \leq (\text{rank}(Z) + 1) (\| [z_i - x_i]_{i \in S \cap A} \|^2 + \| [x_i - \mu]_{i \in S \cap A} \|^2)^2 \]
\[ \leq (\text{rank}(Z) + 1) (\| [z_i - x_i]_{i \in S \cap A} \|^2 + \| [x_i - \mu]_{i \in S \cap A} \|^2)^2 \]
\[ \leq (\text{rank}(Z) + 1) \left( 20 \sqrt{2 \sigma} + \sqrt{2 \sigma} \right)^2 = O \left( \frac{\sigma^2 n}{\alpha^3} \right). \]
Here (i) uses the preceding bound on \( \| X_A - Z \|_2 \), together with the 2nd moment bound \( \| [x_i - \mu]_{i \in S} \|^2 \leq \sqrt{\alpha \sigma} \). Note that \( \text{rank}(Z) \leq \text{rank}(W_0) = O \left( \frac{1}{\alpha} \right) \).

Since \( |S \cap A| = \Omega(\alpha n) \) by Lemma 9, the average value of \( \| z_i - \mu \|^2 \) over \( S \cap A \) is \( O \left( \frac{\sigma^2}{\alpha^3} \right) \), and hence with probability at least \( \frac{|S \cap A|}{|A|} = \Omega(\alpha) \), a randomly chosen \( z_i \) will be within distance \( O \left( \frac{\sigma}{\alpha} \right) \) of \( \mu \), which completes the first part of Proposition 8.

For the second part, when \( \alpha \geq \frac{3}{4} \), recall that we have \( \text{rank}(W_0) = 1 \), and that \( W_0 = (W - W_1)(I - W_1)^{-1} \). One can then verify that \( \mathbb{I}^\top W_0 = \mathbb{I}^\top \). Therefore, \( W_0 = u \mathbb{I}^\top \) for some \( u \). Letting \( \tilde{\mu} = X_A u \), we have \( \| X_A - \tilde{\mu} \mathbb{I}^\top \|^2 \leq 20 \ell \sigma \) by (12). In particular, \( A \) is resilient (around its mean; see Section B) with \( \sigma(e) \leq 20 \ell \sigma \leq 40 \ell \sigma \) for \( e \leq \frac{1}{2} \). Thus by the proof of Proposition 7 and the fact that \( |A| \geq |S \cap A| \geq \frac{\alpha(2+\alpha)}{4-\alpha} n \geq (1 - \frac{5}{3}(1 - \alpha)) n \), the mean of \( A \) will be within distance \( O(\sigma \sqrt{1 - \alpha}) \) of \( \mu \), as desired.

4. Powering up Resilience: Finding a Resilient Core

In this section, we prove the key “powering up” result (Proposition 11) which says that every resilient set contains a core with bounded second moments. While the previous section focused on the \( \ell_2 \) case, this section and the next will focus on general \( \ell_p \) norms (\( 1 < p \leq 2 \)).

First recall the definition of resilience (Definition 1) in \( \ell_p \)-norms: a set \( S \) is \( (\sigma, \epsilon) \)-resilient if for every set \( T \subseteq S \) of size \( (1 - \epsilon)|S| \), we have \( \| \frac{1}{|T|} \sum_{i \in T} (x_i - \mu) \|_p \leq \sigma \). For \( \epsilon = \frac{1}{2} \), resilience in a norm is equivalent to having bounded first moments in the dual norm:

**Lemma 10** Suppose that \( S \) is \( (\sigma, \frac{1}{2}) \)-resilient in \( \ell_p \)-norm, and let \( q = \frac{p}{1 - p} \). Then \( S \) has 1st moments bounded by \( 3\sigma \) in \( \ell_q \)-norm: \( \frac{1}{|S|} \sum_{i \in S} |\langle x_i - \mu, v \rangle| \leq 3\sigma \| v \|_q \) for all \( v \in \mathbb{R}^d \).

Conversely, if \( S \) has 1st moments bounded by \( \sigma \) in \( \ell_q \)-norm, it is \( (2\sigma, \frac{1}{2}) \)-resilient in \( \ell_p \)-norm.

The proof is routine and can be found in Section E. Supposing a set has bounded 1st moments, we will show that it has a large core with bounded second moments. This next result is not routine:

**Proposition 11** Let \( S \) be any set with 1st moments bounded by \( \sigma \) in \( \ell_q \)-norm. Then for \( q \geq 2 \), there exists a core \( S_0 \) of size at least \( \frac{1}{2}|S| \) with 2nd moments bounded by \( 32\sigma^2(q - 1) \). That is, \( \frac{1}{|S_0|} \sum_{i \in S_0} |\langle x_i - \mu, v \rangle|^2 \leq 32\sigma^2(q - 1) \| v \|_q^2 \) for all \( v \in \mathbb{R}^d \).

The assumptions seem necessary: such a core does not exist for \( \ell_q \)-norms with \( q \leq 2 \), or with bounded 3rd moments for \( q = 2 \) (see Section A). The proof of Proposition 11 uses minimax duality, Khintchine’s inequality, and the strong smoothness of \( \ell_q \)-norms.

**Proof (Proposition 11)** Without loss of generality take \( \mu = 0 \) and suppose that \( S = [n] \). We can pose the problem of finding a resilient core as an integer program:
\[ \min_{e \in \{0,1\}^n, \|e\|_1 \geq \frac{n}{2}, \|v\|_q \leq 1} \max \frac{1}{n} \sum_{i=1}^n c_i |\langle x_i, v \rangle|^2. \]
Here the variable $c_i$ indicates whether the point $i$ lies in the core $S_0$. By taking a continuous relaxation and applying a standard duality argument, we obtain the following:

**Lemma 12** Suppose that for all $m$ and all vectors $v_1, \ldots, v_m$ satisfying $\sum_{j=1}^m \|v_j\|_q^2 \leq 1$, we have

$$\frac{1}{n} \sum_{i=1}^n \left( \sum_{j=1}^m |\langle x_i, v_j \rangle| \right)^2 \leq B. \tag{17}$$

Then the value of (16) is bounded above by $8B^2$.

The proof is straightforward and deferred to Section D. Now, to bound (17), let $s_1, \ldots, s_m \in \{-1, +1\}$ be i.i.d. random sign variables. We have

$$\frac{1}{n} \sum_{i=1}^n \sqrt{\sum_{j=1}^m |\langle x_i, v_j \rangle|} \leq \mathbb{E}_{s_1:m} \left[ \frac{\sqrt{2}}{n} \sum_{i=1}^n \left( \sum_{j=1}^m s_j \langle x_i, v_j \rangle \right) \right] \tag{18}$$

$$= \mathbb{E}_{s_1:m} \left[ \frac{\sqrt{2}}{n} \sum_{i=1}^n \left( \langle x_i, \sum_{j=1}^m s_j v_j \rangle \right) \right] \tag{19}$$

$$\leq \mathbb{E}_{s_1:m} \left[ \frac{\sqrt{2}\sigma}{n} \sum_{j=1}^m \left( \sum_{j=1}^m s_j v_j \right) \right] \tag{20}$$

$$\leq \sqrt{2}\sigma \mathbb{E}_{s_1:m} \left[ \left( \sum_{j=1}^m s_j v_j \right)^2 \right]^{\frac{1}{2}}. \tag{21}$$

Here (i) is Khintchine’s inequality (Haagerup, 1981) and (ii) is the assumed first moment bound. It remains to bound (21). The key is the following inequality, which amounts to asserting smoothness of the $\ell_q$-norm, and is well-known (c.f. Lemma 17 of Shalev-Shwartz (2007)):

**Lemma 13** For any $x$, $y$, and $q \geq 2$, we have $\frac{1}{2}(\|x + y\|_q^2 + \|x - y\|_q^2) \leq \|x\|_q^2 + (q - 1)\|y\|_q^2$.

Applying Lemma 13 inductively to $\mathbb{E}_{s_1:m} \left[ \left( \sum_{j=1}^m s_j v_j \right)^2 \right]$, we obtain

$$\mathbb{E}_{s_1:m} \left[ \left( \sum_{j=1}^m s_j v_j \right)^2 \right] \leq (q - 1) \sum_{j=1}^m \|v_j\|_q^2 \leq q - 1. \tag{22}$$

Combining with (21), we have the bound $B \leq \sigma \sqrt{2(q - 1)}$, which yields the desired result.

### 4.1. Finding Resilient Cores when $\alpha \approx 1$

Lemma 10 together with Proposition 11 show that a $(\sigma, \frac{1}{\sqrt{2}})$-resilient set has a core with bounded 2nd moments, which combined with Proposition 8 from the previous section (as well as its analog for $\ell_p$ norms in the next section) almost yields Theorem 3. The main looseness is that Proposition 11 only exploits resilience for $\epsilon = \frac{1}{\sqrt{2}}$, and hence is weak when $\epsilon \approx 0$. In particular, it only yields a core $S_0$ of size $\frac{1}{4}|S|$, while we might hope to find a much larger core of size $(1 - \epsilon)|S|$.
Here we tighten Proposition 8 to make use of finer-grained resilience information. Recall that we let \( \sigma_*(\epsilon) \) denote the resilience over sets of size \((1 - \epsilon)|S|\). For a given \( \epsilon \), our goal is to construct a core \( S_0 \) of size \((1 - \epsilon)|S|\) with small second moments. The following key quantity will tell us how small the second moments can be:

\[
\bar{\sigma}_*(\epsilon) \triangleq \sqrt{\int_{\epsilon/2}^{1/2} u^{-2}\sigma_*(u)^2 \, du}.
\] (23)

The following proposition, proved in Section F, says that \( \bar{\sigma}_* \) controls the 2nd moments of \( S_0 \):

**Proposition 14** Let \( S \) be any resilient set in \( \ell_p \)-norm, with \( 1 < p \leq 2 \). Then for any \( \epsilon \leq \frac{1}{2} \), there exists a core \( S_0 \) of size \((1 - \epsilon)|S|\) with 2nd moments in \( \ell_q \) bounded by \( O(\sigma_*(\epsilon)^2) \) for \( q = \frac{p}{p-1} \).

The proof is similar to Proposition 11, but requires more careful bookkeeping.

To interpret \( \bar{\sigma}_* \), suppose that \( \sigma_*(\epsilon) = \sigma \epsilon^{1-1/r} \) for some \( r \in [1, 2) \), which roughly corresponds to having bounded \( r \)th moments. Then \( \bar{\sigma}_*^2(\epsilon) = \sigma^2 \int_{\epsilon/2}^{1/2} u^{-2/r} \, du \leq \frac{2}{2/r-1}(\frac{1}{2})^{2/r-1} \). If \( r = 1 \) then a core of size \((1 - \epsilon)|S|\) might require second moments as large as \( \sigma_\epsilon^2 \); on the other hand, as \( r \to 2 \) the second moments can be almost as small as \( \sigma^2 \). In general, \( \bar{\sigma}_*(\epsilon) \) is \( O(\sigma \epsilon^{1/2-1/r}) \) if \( r \in [1, 2) \), is \( O(\sigma \sqrt{\log(1/\epsilon)}) \) if \( r = 2 \), and is \( O(\sigma) \) if \( r > 2 \).

5. Efficient Recovery: \( \ell_p \) case

We are now ready to prove our main theorem on robust mean estimation in \( \ell_p \)-norms. The proof consists of combining the powering up results in Section 4 (Propositions 11, 14) with a generalization of Proposition 8 to \( \ell_p \)-norms. We first give the generalization of Proposition 8:

**Proposition 15** Let \( x_1, \ldots, x_n \in \mathbb{R}^d \), and let \( S \) be a set of size \( \alpha n \) with mean \( \mu \) and 2nd moments bounded by \( \sigma^2 \) in \( \ell_q \)-norm: \( \|[x_i - \mu]|_{i \in S} \|_{2 \rightarrow p} \leq \sigma \sqrt{|S|} \). Then there is an efficient randomized algorithm which with probability \( \Omega(\alpha) \) outputs a parameter \( \hat{\mu} \) such that \( \|\mu - \hat{\mu}\|_p = O\left(\frac{\sigma}{\alpha \sqrt{p-1}}\right) \). Moreover, if \( \alpha \geq \frac{4}{3} \) then \( \|\mu - \hat{\mu}\|_p = O\left(\sigma \sqrt{1 - \alpha}\right) \) with probability 1.

The algorithm is almost identical to Algorithm 1, with two changes. The first (minor) change is that on line 3, the quantity \( 4n\sigma^2 \) is replaced with \( 2\pi n\sigma^2 \). The second and more important change is that in the optimization (4), the constraint \( \text{tr}(Y) \leq 1 \) is replaced with \( \|\text{diag}(Y)\|_{q/2} \leq 1 \) (note these are equal when \( q = 2 \)). We therefore end up solving the saddle point problem

\[
\max_{Y \succeq 0, \|\text{diag}(Y)\|_{q/2} \leq 1} \min_{0 \leq W_{ji} \leq \frac{4-\alpha}{\alpha \pi(4-\alpha)^{\alpha}}} \sum_{i \in A} c_i (x_i - X_A w_i) \top Y (x_i - X_A w_i). \tag{24}
\]

While Algorithm 1 essentially minimized the quantity \( \|X - XW\|_2^2 \), this new algorithm can be thought of as minimizing \( \|X - XW\|_{2 \rightarrow p}^2 \). However, the \( 2 \to p \) norm is NP-hard to compute exactly for \( p \in [1, 2) \) (Steinberg, 2005). The optimization (24) therefore employs a relaxation of the \( 2 \to p \) norm. The result below, which follows from Theorem 3 of Nesterov (1998), asserts this:

**Theorem 16 (Nesterov)** For a matrix \( A \in \mathbb{R}^{d \times n} \), let \( f(A) = \|A\|_2^2 \) and let \( g(A) = \max_{Y \succeq 0, \|\text{diag}(Y)\|_{q/2} \leq 1} \sum_{i=1}^n a_i \top Y a_i \). Then \( f(A) \leq g(A) \leq \frac{\sqrt{p}}{\sqrt{2}} f(A) \).
Note that the first inequality is trivial since $vv^T$ is a feasible value of $Y$; the second inequality is established using a generalization of Grothendieck’s inequality. We now prove Proposition 15.

**Proof (Proposition 15)** The proof is similar to Proposition 8, so we only provide a sketch of the differences. First, the condition of Lemma 9 still holds, now with $a$ equal to $\frac{2}{3}n\sigma^2$ rather than $n\sigma^2$ due to the approximation in Theorem 16. (This is why we needed to change line 3.)

Next, we need to modify equations (9-12) to hold for the $2 \to p$ norm rather than operator norm:

$$\|X_A - Z\|_{2\to p} = \|X_A(I - W)(I - W_1)^{-1}\|_{2\to p} \leq \|X_A(I - W)\|_{2\to p} \|I - W_1\|_{2\to p} \leq 10\|X_A(I - W)\|_{2\to p} \leq 10\sqrt{2}\|X_A(I - W)\| \text{diag}(c_A)_{1/2}\|_{2\to p} \leq 20\sqrt{\pi}n\sigma. \quad (28)$$

Here (i) is from the general fact $\|AB\|_{p\to q} \leq \|A\|_{r\to q}\|B\|_{p\to r}$, and the rest of the inequalities follow for the same reasons as in (9-12).

We next need to modify equations (13-15). This can be done with the following inequality:

**Lemma 17** For any matrix $A$ of rank $r$ and any $p \in (1, 2]$, we have $\sum_{i=1}^{n} \|a_i\|_p^p \leq \frac{\sqrt{\pi}}{p-1} \|A\|_{2\to p}^2$.

This generalizes the inequality $\|A\|_F^2 \leq \text{rank}(A) \cdot \|A\|_2^2$. Using Lemma 17 (which we prove below), we have

$$\sum_{i \in S \cap A} \|z_i - \mu\|_p^2 \leq \frac{\text{rank}(Z) + 1}{p-1} \|\|Z_i - X_i\|_{2\to p}\|^2 \leq \frac{\text{rank}(Z) + 1}{p-1} \left(\|z_i - x_i\|_{2\to p}^2 + \|x_i - \mu\|_{2\to p}^2\right)^2$$

$$= O\left(\frac{\sigma^2 n}{\alpha(p-1)}\right). \quad (30)$$

Thus as before, if we choose $z_i$ at random, with probability $\Omega(\alpha)$ we will output $z_i$ with $\|z_i - \mu\|_p = O \left(\frac{\sigma}{\alpha\sqrt{p-1}}\right)$. This completes the first part of the proposition.

For the second part, by the same reasoning as before we obtain $\tilde{\mu}$ with $\|X_A - \tilde{\mu}I^\top\|_{2\to p} = O\left(\sqrt{n}\sigma\right)$, which implies that $A$ is resilient in the $\ell_p$-norm with $\sigma(\varepsilon) = O\left(\sigma\sqrt{\varepsilon}\right)$ for $\varepsilon \leq \frac{1}{2}$. The mean of $A$ will therefore be within distance $O\left(\sigma\sqrt{1 - \alpha}\right)$ of $\mu$ as before, which completes the proof.

We finish by proving Lemma 17.

**Proof (Lemma 17)** Let $s \in \{-1, +1\}^n$ be a uniformly random sign vector. We will compare $E_s[\|As\|_p^2]$ in two directions. Let $P$ be the projection onto the span of $A$. On the one hand, we have $\|As\|_p^2 = \|APs\|_p^2 \leq \|A\|_{2\to p}^2 \|Ps\|_2^2$, and hence $E_s[\|As\|_p^2] \leq E_s[\|Ps\|_2^2] \|A\|_{2\to p}^2 = \text{rank}(A) \|A\|_{2\to p}^2$. On the other hand, similarly to (22) we have $E_s[\|As\|_p^2] \geq (p - 1) \sum_{i=1}^{n} \|a_i\|_p^2$ by the strong convexity of the $\ell_p$-norm. Combing these yields the desired result.

By putting together the preceding results, we obtain Theorem 3. See Section G for details.

### 6. Robust Low-Rank Recovery

In this section we present our results on rank-$k$ recovery. We first justify the definition of rank-resilience (Definition 5) by showing that it is information-theoretically sufficient for (approximately) recovering the best rank-$k$ subspace. Then, we show that this subspace can be efficiently recovered and provide an algorithm for doing so.
Algorithm 2 Algorithm for recovering a rank-\( k \) subspace.

1: Initialize \( c_i = 1 \) for all \( i = 1, \ldots, n \) and \( \mathcal{A} = \{1, \ldots, n\} \). Set \( \lambda = \frac{(1-\delta)n\sigma^2}{k} \).
2: Let \( Y \in \mathbb{R}^{d \times d} \) and \( Q \in \mathbb{R}^{A \times A} \) be the maximizer/minimizer of the saddle point problem
\[
\max_{Y \geq 0, \text{tr}(Y) \leq 1} \min_{Q \in \mathbb{R}^{n \times n}} \sum_{i \in \mathcal{A}} c_i [(x_i - X_A q_i)^\top Y (x_i - X_A q_i) + \lambda \|q_i\|_2^2].
\]
(32)
3: if \( \sum_{i \in \mathcal{A}} c_i [(x_i - X_A q_i)^\top Y (x_i - X_A q_i) + \lambda \|q_i\|_2^2] > 8n\sigma^2 \) then
4: Let \( \tau_i = (x_i - X_A q_i)^\top Y (x_i - X_A q_i) + \lambda \|q_i\|_2^2 \).
5: For \( i \in \mathcal{A} \), replace \( c_i \) with \( (1 - \frac{\tau}{\tau_{\max}})c_i \), where \( \tau_{\max} = \max_{i \in \mathcal{A}} \tau_i \).
6: For all \( i \) with \( c_i < \frac{1}{2} \), remove \( i \) from \( \mathcal{A} \).
7: Go back to line 2.
8: end if
9: Let \( Q_1 \) be the result of zeroing out all singular values of \( Q \) greater than 0.9.
10: Output \( P = X_A Q_0 X_A^\top \), where \( Q_0 = (Q - Q_1)(I - Q_1)^{-1} \).

6.1. Information-Theoretic Sufficiency

Let \( X_S = [x_i]_{i \in S} \). Recall that \( \delta \)-rank-resilience asks that \( \text{col}(X_T) = \text{col}(X_S) \) and \( \|X_S^\top X_S\|_2 \leq 2 \) for \( |T| \geq (1 - \delta)|S| \). This is justified by the following:

Proposition 18 Let \( S \subseteq [n] \) be a set of points of size \((1 - \delta)n\) that is \( \frac{\delta}{1-\delta} \)-rank-resilient. Then it is possible to output a rank-\( k \) projection matrix \( P \) such that \( \|(I - P)X_S\|_2 \leq 2\sigma_{k+1}(X_S) \).

Proof Find the \( \frac{\delta}{1-\delta} \)-rank-resilient set \( S' \) of size \((1 - \delta)n\) such that \( \sigma_{k+1}(X_{S'}) \) is smallest, and let \( P \) be the projection onto the top \( k \) singular vectors of \( X_{S'} \). Then we have \( \|(I - P)X_{S'}\|_2 = \sigma_{k+1}(X_{S'}) \leq \sigma_{k+1}(X_S) \). Moreover, if we let \( T = S \cap S' \), we have \( \|(I - P)X_T\|_2 \leq \|(I - P)X_{S'}\|_2 \leq \sigma_{k+1}(X_S) \) as well. By pigeonhole, \( |T| \geq (1 - 2\delta)n = (1 - \frac{\delta}{1-\delta})|S| \). Therefore \( \text{col}(X_T) = \text{col}(X_S) \), and \( \|(I - P)X_S\|_2 = \|(I - P)X_TX_T^\top X_S\|_2 \leq \|(I - P)X_T\|_2\|X_T^\top X_S\|_2 \leq 2\sigma_{k+1}(X_S) \) as claimed.

6.2. Efficient Recovery

We now address the question of efficient recovery. For convenience let \( \sigma = \sigma_{k+1}(X_S)/\sqrt{|S|} \).

Theorem 19 Let \( \delta \leq \frac{1}{3} \). If a set of \( n \) points contains a subset \( S \) of size \((1 - \delta)n\) that is \( \delta \)-rank-resilient, then there is an efficient algorithm (Algorithm 2) for recovering a matrix \( P \) of rank at most \( 15k \) such that \( \|(I - P)X_S\|_2 = O(\sigma\sqrt{|S|}) = O(\sigma_{k+1}(X_S)) \).

Algorithm 2 is quite similar to Algorithm 1, but we include it for completeness. The proof is also similar to the proof of Proposition 8, but there are enough differences that we provide details below.

Proof (Theorem 19) First, we show that Lemma 9 holds with \( a = 2n\sigma^2 \). As before, the \( q_i \) can be optimized independently for each \( i \), so \( \sum_{i \in S \cap \mathcal{A}} c_i \tau_i \) is upper-bounded by its value at any \( Q \). Let \( Q^* \) be the projection onto the top \( k \) (right) singular vectors of \( X_{S \cap \mathcal{A}} \). Then we have \( \sum_{i \in S \cap \mathcal{A}} \lambda \|q_i^*\|_2^2 \leq \lambda \|Q^*\|_2^2 = \lambda k \). We also have \( \sum_{i \in S \cap \mathcal{A}} c_i (x_i - X q_i^*)^\top Y (x_i - X q_i^*) \leq \sigma_{k+1}(X_{S \cap \mathcal{A}})^2 \leq \sigma^2|S| \leq (1 - \delta)n\sigma^2 \). Together these imply that \( \sum_{i \in S \cap \mathcal{A}} c_i \tau_i \leq (1 - \delta)n\sigma^2 + \lambda k = 2(1 - \delta)n\sigma^2 \), so we can indeed take \( a = 2n\sigma^2 \). Consequently, \( |S \cap \mathcal{A}| \geq \frac{(1-\delta)(3-\delta)}{3+\delta} n \geq (1 - \frac{5\delta}{3})n \).
Now, when we get to line 9, we have \( \|X_A(I - Q)\text{diag}(c_A)^{1/2}\|_2^2 + \lambda\|Q\|_F^2 \leq 8n\sigma^2 \), and in particular each of the two terms individually is bounded by \( 8n\sigma^2 \). Therefore, \( \|Q\|_F^2 \leq \frac{8n\sigma^2}{\lambda} = \frac{8k}{1 - \delta} \), and so \( Q_0 \) will have rank at most \( \frac{8k}{0.81(1 - \delta)} \leq 15k \). Moreover, \( \|X_A(I - Q)\|_2 \leq \sqrt{2}\|X_A(I - Q)\text{diag}(c_A)^{1/2}\|_2 \leq 4\sqrt{n}\sigma \). We then have

\[
\|(I - P)X_A\|_2 = \|(I - X_AQ_0X_A^\dagger)X_A\|_2 = \|X_A(I - Q_0)X_A^\dagger X_A\|_2 \leq \|X_A(I - Q_0)\|_2 \leq \|X_A(I - Q)(I - Q_1)^{-1}\|_2 \leq 10\|X_A(I - Q)\|_2 \leq 40\sqrt{n}\sigma = O\left(\sigma\sqrt{|S|}\right),
\]

By the same argument as Proposition 18, since \(|S \cap A| \geq \frac{1 - 5\delta/3}{1 - \delta} |S| \geq (1 - \delta)|S|\), it follows that \( \|(I - P)X_S\|_2 \leq 2\|(I - P)X_{S \cap A}\|_2 = O\left(\sigma\sqrt{|S|}\right) \) as well, which completes the proof.

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Appendix A. Counterexample to Analogues of Proposition 11

In this section we show:

**Proposition 20** Let $S = \{e_1, \ldots, e_n\}$, where the $e_i$ are the standard basis in $\mathbb{R}^n$. Then:

1. $S$ is $(2n^{1/p-1}, \frac{1}{2})$-resilient around 0 in $\ell_p$-norm, for all $p \in [1, \infty]$.

2. Any subset $T$ of $\frac{n}{2}$ elements of $S$ has $\max_{\|v\|_q \leq 1} \frac{1}{|T|} \sum_{i \in T} |\langle e_i, v \rangle|^k = (n/2)^{\max(-1, k(\frac{1}{p}-1))}$.

This implies in particular that for large $n$, an analog of Proposition 11 can only hold if $k(1/p - 1) \geq -1$, which implies that $k \leq \frac{1}{p-1} = q$. For $k = 2$, this implies $q \geq 2$, and for $q = 2$ this implies $k \leq 2$. We do not know whether an analog of Proposition 11 holds for e.g. $k = q = 3$.

**Proof (Proposition 20)** For the first part, note that the mean of any subset of $\frac{n}{2}$ elements of $S$ will have all coordinates lying in $[0, \frac{2}{n}]$. Therefore, the $\ell_p$-norm of the mean is at most $\frac{2}{n} \cdot n^{1/p} = 2n^{1/p-1}$, as claimed.

For the second part, assume without loss of generality that $T = \{e_1, \ldots, e_{n/2}\}$. The optimization then reduces to $\max_{\|v\|_q \leq 1} \frac{2}{n} \|v\|_k^k$, where $v \in \mathbb{R}^{n/2}$. For the optimal $v$, it is clear that $\|v\|_k = 1$ if $k \geq q$, and $\|v\|_k = (n/2)^{\frac{1}{k}}$ if $k < q$, whence $\frac{2}{n} \|v\|_k^k = \frac{2}{n} \cdot (n/2)^{\max(0, 1-\frac{k}{q})} = (n/2)^{\max(-1, 1-\frac{1}{p})}$, as claimed.

Appendix B. Basic Properties of Resilience

**Lemma 21** Suppose that $\frac{1}{|S|} \sum_{i \in S} |x_i - \mu, v|^2 \leq \sigma^2 \|v\|_q^2$ for all $v \in \mathbb{R}^d$, where $\mu$ is the empirical mean of the $x_i$. Then $S$ is $(\sigma \sqrt{\frac{4}{1-\epsilon}}, \epsilon)$-resilient around $\mu$ in $\ell_p$-norm for all $\epsilon$.

**Proof** Without loss of generality take $\mu = 0$. The condition is equivalent to $\|X\|_2 \leq \sigma \sqrt{|S|}$. For any set $T$ define the vector $w \in \mathbb{R}^n$ by $w_i = \frac{\|e_i\|_T}{|T|} - \frac{1}{|S|}$. Then $\|Xw\|_p = \|\mu_T - \mu\|_p$, where $\mu_T$ is the mean over $T$. On the other hand, if $|T| = (1-\epsilon)n$ then $\|Xw\|_p \leq \|X\|_2 \|w\|_2 \leq \sigma \sqrt{|S|} \cdot \sqrt{|T| (\frac{1}{|T|} - \frac{1}{|S|})^2 + (|S| - |T|) \frac{1}{|S|^2}} = \sigma \sqrt{\frac{4}{1-\epsilon}}$, which yields the desired result.

Appendix C. Proof of Lemma 9

Assuming that (i) and (ii) hold prior to line 5 of the algorithm, we need to show that they continue to hold.

By assumption, we have $\sum_{i \in S \cap A} c_i \tau_i \leq \alpha a$, while $\sum_{i \in A} c_i \tau_i \geq 4a$. But note that the amount that $\sum_{i \in S} c_i$ decreases in step 3 is proportional to $\sum_{i \in S \cap A} c_i \tau_i$, while the amount that $\sum_{i=1}^n c_i$ decreases is proportional to $\sum_{i \in A} c_i \tau_i$. Therefore, the former decreases at most $\frac{\alpha}{4}$ times as fast as the latter, meaning that (i) is preserved.
Since (i) is preserved, we have $\sum_{i \in S}(1 - c_i) \leq \frac{\alpha}{4}(\sum_{i \in S}(1 - c_i) + \sum_{i \notin S}(1 - c_i))$. Re-arranging yields $\sum_{i \in S}(1 - c_i) \leq \frac{\alpha(1 - \alpha)}{4 - \alpha}n$. In particular, $\# \{ i \mid c_i \leq \frac{1}{2} \} \leq \frac{2\alpha(1 - \alpha)}{4 - \alpha}n$, so at most $\frac{2\alpha(1 - \alpha)}{4 - \alpha}n$ elements of $S$ have been removed from $\mathcal{A}$ in total. This implies $|S \cap \mathcal{A}| \geq \frac{\alpha(2 + \alpha)}{4 - \alpha}n$, and so (ii) is preserved as well. 

Appendix D. Proof of Lemma 12

We start by taking a continuous relaxation of (16), asking for weights $c_i \in [0, 1]$ rather than $\{0, 1\}$:

$$\min_{c \in [0, 1]^n, \|c\|_1 \geq \frac{3n}{4}} \max_{\|v\|_q \leq 1} \frac{1}{n} \sum_{i=1}^{n} c_i |\langle x_i, v \rangle|^2. \tag{38}$$

Note that we strengthened the inequality to $\|c\|_1 \geq \frac{3n}{4}$, whereas in (16) it was $\|c\|_1 \geq \frac{n}{2}$. Given any solution $c_{1:n}$ to (38), we can obtain a solution $c'$ to (16) by letting $c'_i = \mathbb{I}[c_i \geq \frac{1}{2}]$. Then $c'_i \in \{0, 1\}$ and $\|c'||_1 \geq \frac{n}{2}$. Moreover, $c'_i \leq 2c_i$, so $\frac{1}{n} \sum_{i=1}^{n} c'_i |\langle x_i, v \rangle|^2 \leq \frac{2}{n} \sum_{i=1}^{n} c_i |\langle x_i, v \rangle|^2$ for all $v$. Therefore, the value of (16) is at most twice the value of (38).

Now, by the minimax theorem, we can swap the min and max in (38) in exchange for replacing the single vector $v$ with a distribution over vectors $v_j$, thus obtaining that (38) is equal to

$$\lim_{m \to \infty} \max_{\alpha \geq 0, \|v_j\|_q \leq 1} \min_{\|c\|_1 \geq \frac{3n}{4}} \frac{1}{n} \sum_{i=1}^{n} c_i \sum_{j=1}^{m} \alpha_j |\langle x_i, v_j \rangle|^2. \tag{39}$$

By letting $v'_j = \alpha_j v_j$, the above is equivalent to optimizing over $v_j$ satisfying $\sum_j \|v_j\|_q^2 \leq 1$:

$$\lim_{m \to \infty} \max_{\|v_1\|_q^2 + \cdots + \|v_m\|_q^2 \leq 1} \min_{c \in [0, 1]^n, \|c\|_1 \geq \frac{3n}{4}} \frac{1}{n} \sum_{i=1}^{n} c_i \sum_{j=1}^{m} |\langle x_i, v_j \rangle|^2. \tag{40}$$

For any $v_1, \ldots, v_m$, we will find $c$ such that the above sum is bounded. Indeed, define $B(v_{1:m})$ to be $\frac{1}{n} \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{m} |\langle x_i, v_j \rangle|^2}$. Then take $c_i = \mathbb{I}[\sum_{j=1}^{m} |\langle x_i, v_j \rangle|^2 < 16B^2]$, which has $\|c\|_1 \geq \frac{3n}{4}$ by Markov’s inequality, and for which $\sum_i c_i \sum_j |\langle x_i, v_j \rangle|^2 \leq 4B(v_{1:m})^2$.

Therefore, the value of (38) is bounded by $\max_{m, v_{1:m}} 4B(v_{1:m})^2$, and so the value of (16) is bounded by $\max_{m, v_{1:m}} 8B(v_{1:m})^2$, which yields the desired result. 

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Appendix E. Proof of Lemma 10

For a fixed $v$, let $S_+ = \{i \mid \langle x_i - \mu, v \rangle > 0 \}$, and define $S_-$ similarly. Either $S_+$ or $S_-$ must have size at least $\frac{1}{2}|S|$, without loss of generality assume it is $S_+$. Then we have

$$\frac{1}{|S|} \sum_{i \in S} |\langle x_i - \mu, v \rangle| = \frac{1}{|S|} \left( 2 \sum_{i \in S_+} \langle x_i - \mu, v \rangle - \sum_{i \in S} \langle x_i - \mu, v \rangle \right)$$

$$\leq \frac{1}{|S|} \left( 2 \| \sum_{i \in S_+} (x_i - \mu) \|_{p,q} + \| \sum_{i \in S} (x_i - \mu) \|_{p,q} \right)$$

$$\leq \left( 2 \frac{|S_+|}{|S|} + 1 \right) \sigma \| v \|_q \leq 3 \sigma \| v \|_q,$$

which completes the first part. In the other direction, for any set $T$, we have

$$\left\| \frac{1}{|T|} \sum_{i \in T} (x_i - \mu) \right\|_p = \max_{\|v\|_q = 1} \frac{1}{|T|} \sum_{i \in T} \langle x_i - \mu, v \rangle$$

$$\leq \max_{\|v\|_q = 1} \frac{1}{|T|} \sum_{i \in S} |\langle x_i - \mu, v \rangle| \leq \frac{|S|}{|T|} \sigma \leq 2 \sigma,$$

as was to be shown.

Appendix F. Proof of Proposition 14

The proof mirrors that of Proposition 11, but with more careful bookkeeping. Instead of (38), we consider the minimax problem

$$\min_{c \in [0,1]^n, \|c\|_1 \leq (1-\epsilon/2)n} \max_{\|v\| \leq 1} \frac{1}{n} \sum_{i=1}^n c_i |\langle x_i, v \rangle|^2.$$  

The only difference is that $\frac{3}{4}n$ has been replaced with $(1 - \epsilon/2)n$ in the constraint on $\|c\|_1$. We then end up needing to bound

$$\sum_j \max_{\|v_j\| \leq 1} \frac{1}{n} \sum_{k=1}^{(1-\epsilon/2)n} \sum_{j=1}^m |\langle x_{ik}, v_j \rangle|^2.$$  

Here we suppose that the indices $i$ are arranged in order $i_1, i_2, \ldots, i_n$ such that the output of the inner sum is monotonically increasing in $i_k$ (in other words, the outer sum over $k$ is over the $(1 - \frac{\epsilon}{2})n$ smallest values
of the inner sum, which corresponds to the optimal choice of $c$). Now we have

\[
\frac{1}{n} \left(\frac{1}{n-k+1} \sum_{i=k}^{n} \left| \langle x_i, v_j \rangle \right|^2 \right)^{\frac{(i)}{2}} \leq \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{n-k+1} \sum_{i=k}^{n} \left| \langle x_i, v_j \rangle \right|^2 \right)^{\frac{(i)}{2}} 
\]

(48)

\[
\leq \frac{2}{n} \sum_{k=1}^{n} \left( \frac{1}{n-k+1} \sum_{l=k}^{n} \left| \sum_{j=1}^{m} \langle x_i, \sum_{j=1}^{m} s_j v_j \rangle \right| \right)^{\frac{(ii)}{2}} 
\]

(49)

\[
\leq \frac{2}{n} \sum_{k=1}^{n} \left( \frac{3n \min(\sigma_s(n^{-k+1}), \sigma_s(\frac{1}{n}))}{n-k+1} \sum_{j=1}^{m} s_j v_j \right)^{\frac{(iii)}{2}} 
\]

(50)

\[
\leq \frac{18q}{n} \sum_{k=1}^{n} \left( \frac{\min(\sigma_s(n^{-k+1}), \sigma_s(\frac{1}{n}))}{n-k+1} \right)^{\frac{2}{2}}. 
\]

(51)

Here (i) bounds the $k$th smallest term by the average of the $n-k+1$ largest terms, (ii) is Khinchine’s inequality, and (iii) is the following lemma which we prove later:

**Lemma 22** If $S$ is resilient around $\mu$ and $T \subseteq S$ has size at most $\epsilon |S|$, then $\sum_{i \in T} |\langle x_i - \mu, v \rangle| \leq 3|S||v||q \cdot \min(\sigma_s(\epsilon), \sigma_s(\frac{1}{n}))$.

Continuing, we can bound (51) by

\[
18q \int_{0}^{1-\frac{1}{n}} \left( \frac{\min(\sigma_s(1-u), \sigma_s(\frac{1}{n}))}{1-u-1/n} \right)^{2} \, du \leq 18q \left( \frac{1}{\frac{1}{2} - \frac{1}{n}} \right)^{2} \int_{\epsilon/2}^{1} \left( \frac{\min(\sigma_s(u), \sigma_s(\frac{1}{n}))}{u} \right)^{2} \, du 
\]

(52)

\[
\leq O(1) \cdot q \cdot \left( \int_{\epsilon/2}^{1/2} u^{-2\sigma_s(u)^2} \, du + 2\sigma_s(\frac{1}{n})^2 \right) 
\]

(53)

\[
= O \left( q \cdot (\sigma_s(e)^2 + \sigma_s(\frac{1}{n})^2) \right). 
\]

(54)

Since the $\sigma_s(\epsilon)$ term dominates, this completes the bound on (46), from which the remainder of the proof follows identically to Proposition 11.

We end this section by proving Lemma 22.

**Proof (Lemma 22)** Note that the $\sigma_s(\frac{1}{n})$ part of the bound was already established in Lemma 10, so it suffices to show the $\sigma_s(\epsilon)$ part. Let $\bar{\mu}$ be the mean of $S$. Note that we must have $\|\mu - \bar{\mu}\|_p \leq \sigma_s(\epsilon)$ by definition. We can thus replace $\mu$ with $\bar{\mu}$ in the statement of Lemma 22 at the cost of changing the left-hand-side by at most $|T|\sigma_s(\epsilon)\|v\|_q$. Now let $T_+$ be the elements of $T$ with $\langle x_i - \bar{\mu}, v \rangle > 0$ and define $T_-$ similarly. We have

\[
\sum_{i \in T} |\langle x_i - \mu, v \rangle| = \sum_{i \in T_+} \langle x_i - \mu, v \rangle - \sum_{i \in T_-} \langle x_i - \mu, v \rangle 
\]

(55)

\[
= - \sum_{i \in S \setminus T_+} \langle x_i - \mu, v \rangle + \sum_{i \in S \setminus T_-} \langle x_i - \mu, v \rangle 
\]

(56)

\[
\leq \left\| \sum_{i \in S \setminus T_+} (x_i - \mu) \right\|_p \|v\|_q + \left\| \sum_{i \in S \setminus T_-} (x_i - \mu) \right\|_p \|v\|_q 
\]

(57)

\[
\leq 2|S|\sigma_s(\epsilon)\|v\|_q. 
\]

(58)
which yields the desired result.

Appendix G. Proof of Theorem 3

We first analyze the case when \( \alpha \neq 1 \). Lemma 10 and Proposition 11 imply that if \(|S| = \alpha n\) and \(S\) is \((\sigma, \frac{1}{2})\)-resilient in \(\ell_p\)-norm around \(\mu\), then there is a set \(S_0\) of size \(\Omega(\alpha n)\) with second moments around \(\mu\) bounded by \(O(\sigma^2 q)\) in \(\ell_q\)-norm. Letting \(\mu'\) be the mean of \(S_0\), Proposition 15 implies that with probability \(\Omega(\alpha)\) we output a \(\hat{\mu}\) with \(\|\hat{\mu} - \mu'\|_p = O\left(\frac{\sigma}{\alpha} \sqrt{\frac{q}{p-1}}\right)\). Moreover, resilience of \(S\) implies that \(\|\mu - \mu'\|_p \leq \sigma\), hence \(\|\hat{\mu} - \mu\|_p = O\left(\frac{\sigma}{\alpha(p-1)}\right)\) as well, as claimed.

For \(\alpha \geq 0.87\), note that if \(\sigma_*(\epsilon) \leq \sigma \sqrt{\epsilon}\) for \(\epsilon \in \left[\frac{1-\alpha^2}{2}, \frac{1}{2}\right]\), then Proposition 14 and the following discussion imply that we can find a set \(S_0\) of size \(\alpha|S| = \alpha^2 n\) with 2nd moments around \(\mu\) bounded by \(O\left(\sigma^2 q \log\left(\frac{1}{1-\alpha}\right)\right)\). Therefore, Proposition 15 implies that we can output a \(\hat{\mu}\) with \(\|\hat{\mu} - \mu'\|_p = O\left(\sigma \sqrt{(1-\alpha^2)q \log\left(\frac{1}{1-\alpha}\right)}\right)\) provided that \(\alpha^2 \geq \frac{3}{4}\), which is true if \(\alpha \geq 0.87\). In this case resilience of \(S\) implies that \(\|\mu - \mu'\|_p \leq \sigma_*(\frac{1-\alpha}{\alpha}) = O(\sigma \sqrt{1-\alpha})\), so \(\|\hat{\mu} - \mu\|_p = O\left(\sigma \sqrt{q(1-\alpha)}\right) = O\left(\sigma \sqrt{\frac{1-\alpha}{p-1}}\right)\) as well, which completes the proof.