Yang-Mills instantons in Kähler spaces
with one holomorphic isometry

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Abstract

We consider self-dual Yang-Mills instantons in 4-dimensional Kähler spaces with one holomorphic isometry and show that they satisfy a generalization of the Bogomol’nyi equation for magnetic monopoles on certain 3-dimensional metrics. We then search for solutions of this equation in 3-dimensional metrics foliated by 2-dimensional spheres, hyperboloids or planes in the case in which the gauge group coincides with the isometry group of the metric (SO(3), SO(1,2) and ISO(2), respectively). Using a generalized hedgehog ansatz the Bogomol’nyi equations reduce to a simple differential equation in the radial variable which admits a universal solution and, in some cases, a particular one, from which one finally recovers instanton solutions in the original Kähler space. We work out completely a few explicit examples for some Kähler spaces of interest.

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1 Introduction

There is a well-known relation between Yang-Mills instantons in four flat Euclidean dimensions and static magnetic monopoles of the Yang-Mills-Higgs theory in four flat Lorentzian dimensions. A particular case that has focused most of the research is the relation between (anti-) selfdual instantons and monopoles satisfying the Bogomol’nyi equation \[1\] in \(E^3\). These are usually known as BPS magnetic monopoles because the SU(2) ’t Hooft-Polyakov monopole \[2, 3\] satisfies it in the Prasad-Sommerfield limit \[4\]. Both the (anti-) selfduality condition and the Bogomol’nyi equation are first-order equations that imply that the action is extremized locally and the second-order Euler-Lagrange equations are automatically satisfied. On the one hand, first-order equations are easier to solve than second-order ones, and, for instance, this allowed Protogenov to construct all the spherically-symmetric SU(2) BPS magnetic monopoles using the so-called hedgehog ansatz, that exploits the relation between the isometry group of the solution and the gauge group \[5\]. On the other, these, as many other interesting first-order equations, naturally arise in the context of supersymmetric theories, when one searches for field configurations preserving some unbroken supersymmetries. Supersymmetric solutions have many interesting properties, which makes them worth studying for their own sake.² The possibility of constructing solutions, such as the one in Ref. \[7\], describing an arbitrary number of magnetic monopoles in static equilibrium is one of the most remarkable ones, and has been exploited to construct dyonic, non-Abelian multi-black-hole solutions in 4-dimensional Super-Einstein-Yang-Mills theories \[8\].

In Ref. \[9\] Kronheimer showed that the above relation between (anti-) selfdual instantons in \(E^4\) and monopoles satisfying the Bogomol’nyi equation in \(E^3\) could be extended to a relation between (anti-) selfdual instantons in 4-dimensional hyperKähler spaces admitting a triholomorphic isometry, usually known as Gibbons-Hawking spaces \[10, 11\], and, again, monopoles satisfying the Bogomol’nyi equation in \(E^3\). This map between instantons and monopoles is surjective and from a given BPS monopole solution one can construct an instanton solution in every Gibbons-Hawking space, which is characterized by an additional function \(H\), harmonic in \(E^3\), which is not part of the monopole fields. Thus, one can use all the spherically-symmetric SU(2) BPS magnetic monopoles found by Protogenov to construct instantons with SO(3) symmetry in any Gibbons-Hawking space, for instance.³

This mechanism has been used in the context of the construction of timelike supersymmetric solutions of 5-dimensional Super-Einstein-Yang-Mills theories (non-Abelian

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¹The time coordinate is irrelevant in this problem because of the restriction to static monopoles.

²For a review on field configurations and solutions with global or local unbroken supersymmetries see, for instance, Ref. \[6\].

³The relation does not preserve the regularity of the solutions in either sense. In particular, the well-known BPST SU(2) instanton \[12\] on \(E_3^4\), characterized by the choice \(H = 1/r\), gives rise to the Protogenov solution known as coloured monopole, which is singular \[13\]. On the other hand, the rest of the spherically-symmetric BPS monopoles (including the globally regular ’t Hooft-Polyakov one) give rise to badly-behaved instantons solutions in the same GH space and one must consider other GH spaces \[14\].
black holes and rings, microstate geometries and global instantons [15, 16, 7, 17, 18]) because the spacetime metrics of these 5-dimensional supersymmetric solutions are constructed using a 4-dimensional hyperKähler metric (often called “base space metric”) in terms of which a piece of the 2-form field strengths of the theory is forced by supersymmetry to be selfdual [19].

In 5-dimensional supergravities with Abelian gaugings via Fayet-Iliopoulos terms the base-space metric is forced to be Kähler in supersymmetric solutions [20] and, if there are additional non-Abelian gaugings of the isometries of the real Special scalar manifold (i.e. we are dealing with a Super-Einstein-Yang-Mills theory with an additional Abelian gauging that introduces a non-trivial scalar potential, among other things), one faces the problem of finding selfdual instantons in Kähler spaces.4 Just as in the hyperKähler case, it is convenient to have a “parametrization” of the class of metrics under consideration in order to find a set of differential equations for the problem. The space of hyperKähler metrics is very large and finding a generic one for a hyperKähler metric, in terms of a small number of functions satisfying some relations is too complicated or impossible. The restriction to GH metrics, which depend on just one independent function, transforms the selfduality condition into a set of differential equations which, in the end, can be identified with the Bogomol’nyi equations. Alternatively, one can just view the requirement of the existence of a triholomorphic isometry as a condition necessary to dimensionally reduce the equations along the isometric direction preserving the hyperKähler structure.

In the Kähler case it is natural to assume the existence of a holomorphic isometry along which a dimensional reduction can be performed preserving the Kähler structure (and supersymmetry as well, in the supersymmetric context). In Ref. [22] and references therein, it was shown that these metrics can be written in terms of essentially two real functions related by a differential equation (see Eq. (2.1)) and in this paper we are going to make use of this result to transform the selfduality equations of a Yang-Mills field on these metrics into a set of differential equations which ultimately can be seen as a generalization of the Bogomol’nyi equations in some 3-dimensional space (Section 2).

While the physical interpretation of these 3-dimensional equations is not as transparent as those obtained in the hyperKähler space (in particular, it is not clear that they correspond to BPS magnetic monopole solutions in general), they provide an excellent starting point to construct instanton solutions. Thus, in Section 3 we are going to use the hedgehog ansatz and generalizations thereof adequate for other gauge groups in order to simplify the equations and obtain explicit instanton solutions in some simple Kähler spaces of interest in gauged 5-dimensional supergravity, such as \(\mathbb{C}P^2\). Section 4 contains our conclusions.

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4This is not just a very twisted academic problem: this kind of theories arise naturally in Type II Superstring compactifications to 5 dimensions [21].
2 Generalized Bogomol’nyi equations

Any 4-dimensional Kähler metric admitting a holomorphic isometry can be written as [22]

\[ ds_4^2 = H^{-1} (dz + \chi)^2 + H \left\{ (dx^2)^2 + W^2 (\bar{x}) [(dx^1)^2 + (dx^3)^2] \right\}, \] (2.1)

with the functions \( H \) and \( W \), and the 1-form \( \chi \), independent of \( z \) and satisfying the constraint\(^5\)

\[ \check{\iota}_3 d\chi = dH + H \partial_2 \log W dx^2, \] (2.2)

where \( \check{\iota}_3 \) is the Hodge dual in the 3-dimensional manifold

\[ ds_3^2 = (dx^2)^2 + W^2 (\bar{x}) [(dx^1)^2 + (dx^3)^2]. \] (2.3)

The integrability condition of this constraint is a \( W \)-dependent deformation of the Laplace equation for \( H \) on \( E^3 \)

\[ \partial_1 \partial_2 H + \partial_2 \partial_3 (W^2 H) + \partial_3 \partial_1 H = 0. \] (2.4)

Thus, we can construct Kähler metrics with a holomorphic isometry by choosing some function \( W \), solving the above integrability condition for \( H \) and then solving the constraint Eq. (2.2) for the 1-form \( \chi \).\(^6\) Observe that the choice \( W = 1 \) yields hyper-Kähler metrics with a triholomorphic isometry, also known as Gibbons-Hawking metrics [10, 11].

We are interested in Yang-Mills fields \( A^I \) in the above space which are \( z \)-independent (at least in some gauge) and whose 2-form field strengths

\[ F^I = dA^I + \frac{1}{2} \varepsilon_{JK} A^J \wedge A^K, \] (2.6)

are self-dual

\[ F^I = *_4 F^I. \] (2.7)

Here \( *_4 \) is the Hodge operator in the full 4-dimensional metric Eq. (2.1), with the orientation \( \varepsilon^{1234} = +1. \)

\(^5\)Underlined indices refer to the coordinate basis.

\(^6\)We can also construct metrics of this kind starting with an arbitrary real function \( \mathcal{K}(x^1, x^2, x^3) \) and computing directly

\[ H = \partial_2^2 \mathcal{K}, \quad W^2 = -H^{-1} (\partial_1^2 + \partial_3^2) \mathcal{K}, \] (2.5)

\[ \chi_1 = -\partial_3 \partial_2 \mathcal{K}, \quad \chi_3 = \partial_2 \partial_1 \mathcal{K}, \]

which solve all the above equations in coordinates in which \( \chi_2 = 0. \)
Following Kronheimer, who considered the hyper-Kähler case \((W = 1)\) in Ref. [9], we decompose \(A^I\) as
\[
A^I = -H^{-1}\Phi^I(dz + \chi) + \bar{A}^I, \tag{2.8}
\]
and substituting into the self-duality equation (2.7) one finds that it is equivalent to the following generalization of the Bogomol’nyi equation [1]7
\[
\dot{x}_3 \Phi^I = \Phi^I \partial_2 \log W^2 dx^2, \tag{2.9}
\]
where the \(\dot{}\) sign in the field strength and the covariant derivative refers to the 3-dimensional Yang-Mills connection \(\bar{A}^I\).

In general, the right-hand side of this equation does not have a clear geometric or field-theoretic meaning and, therefore, there is no obvious relation between the equation and the Yang-Mills-Higgs action in some 4-dimensional spacetime: unlike what happens with the usual Bogomol’nyi equation, it does not seem to be related to the extremization of an action of this kind and it does not guarantee that the corresponding second order Yang-Mills-Higgs equations of motion are satisfied. As explained in the introduction, this does not make them completely useless or meaningless, because they can arise in more complex theories such as 5- and 4-dimensional gauged supergravities.

Many of the most interesting Kähler metrics in this class are characterized by a function \(H\) that only depends on \(x^2\), which we will denote by \(\varrho\) from now on. Equation (2.4) implies that \(W\) is of the form [23]
\[
W^2(\varrho) = \frac{6}{\varrho} \Phi_1(x^1, x^3) + \frac{1}{\varrho} \Phi_2(x^1, x^3). \tag{2.10}
\]
We will consider, for the sake of simplicity, the case in which either \(\Phi_1 = 0\) or \(\Phi_2 = 0\) so that, calling the surviving function \(\Phi\), \(W^2\) can be written as
\[
W^2 = \Psi(\varrho) \Phi(x^1, x^3), \quad \text{where} \quad \Psi(\varrho) \equiv \frac{\varrho^e}{H(\varrho)}. \tag{2.11}
\]
Thus, the metrics in this class are completely determined by an arbitrary function of \(\varrho\) (either \(\Psi\) or \(H\)) and an arbitrary function \(\Phi\) of \(x^1, x^3\).

For these metrics, the Bogomol’nyi equation (2.9) takes the more geometric expression
\[
\Psi \dot{x}_3 \Phi^I = \bar{\mathcal{D}}(\Psi \Phi^I) = 0, \tag{2.12}
\]
which can be derived from the Yang-Mills-Higgs action with a Higgs field \(\Phi^I = \Psi \Phi^I\) in a 1 + 3 (spacetime) metric of the form
\[
ds_{1+3}^2 = g_{tt} dt^2 - \Psi^2(\varrho) ds_3^2, \tag{2.13}
\]
7The standard Bogomol’nyi equation is defined in Euclidean 3-dimensional space.
for any time-independent $g_{tt}$ by the usual squaring of the action arguments.

We are not going to follow this line of reasoning any further here. Instead, we will just try to find some explicit solutions to the above Bogomol’nyi equation and the corresponding instantons for some interesting Kähler metrics.

### 3 Generalized hedgehog ansatz and solutions

As a further simplification, we are going to restrict ourselves to the case in which the 2-dimensional metric $\Phi(x^1, x^3)[(dx^1)^2 + (dx^3)^3]$ is maximally symmetric. The three distinct possibilities, namely the round sphere $S^2$, the hyperbolic plane $H^2$ and the Euclidean plane $E^2$, are encompassed by the function

$$\Phi(k)(x^1, x^3) = \frac{4}{\{1 + k[(x^1)^2 + (x^3)^2]\}^2},$$

(3.1)

for the values of the parameter $k$ being $+1$, $0$ and $-1$, respectively. For this particular kind of metrics, the 1-form $\chi$ that occurs in the metric Eq. (2.1) is given by

$$\chi = \epsilon \chi(k), \quad \text{with} \quad \chi(k) = \frac{2(x^3dx^1 - x^1dx^3)}{1 + k[(x^1)^2 + (x^3)^2]},$$

(3.2)

The generic coordinate change

$$x^1 = k^{-1/2} \tan(k^{1/2}\theta/2) \cos \varphi, \quad x^2 = \varphi, \quad x^3 = k^{-1/2} \tan(k^{1/2}\theta/2) \sin \varphi,$$

(3.3)

brings the 3-dimensional metric (2.3) into the form

$$d\hat{s}_3^2 = dq^2 + \Psi(q)d\Omega^2(k), \quad \text{where} \quad d\Omega^2(k) = d\theta^2 + k^{-1} \sin^2(k^{1/2}\theta)d\varphi^2,$$

(3.4)

and $\chi(k)$ to the form

$$\chi(k) = k^{-1}[\cos(k^{1/2}\theta) - 1]d\varphi.$$

(3.5)

It is natural to search for monopole solutions in gauge groups which coincide with the isometry group of the maximally symmetric 2-dimensional spaces that foliate the 3-dimensional metrics that we are considering here: $\text{SO}(3)$, $\text{ISO}(2)$ and $\text{SO}(1,2)$, respectively. Observe that, while the non-semisimple group $\text{ISO}(2)$ is just a mere curiosity, the non-compact group $\text{SO}(1,2)$ actually occurs in supergravity theories without any of the pathologies that arise in Yang-Mills(-Higgs) theories because these theories have scalar-dependent kinetic matrices which make compatible $\text{SO}(1,2)$ symmetry with positive-defined kinetic energies.

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8 The $k = 0$ case should be seen as the limit $k \to 0$. 

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For instance, in $\mathcal{N} = 1, d = 5$ supergravities, the vector fields kinetic terms are of the form

$$\sim a_{IJ}(\phi)F^I \wedge \star F^J.$$  \hfill (3.6)

In theories with $\text{SO}(1,2)$ symmetry the scalar fields and, hence, the kinetic matrix $a_{IJ}(\phi)$ transform under that group so that this kinetic term is positive definite and invariant. For the kind of field configurations that we are considering (selfdual instantons living in some 4-dimensional Euclidean submanifold) the contribution to the action of this term would be a positive-definite generalization of the usual instanton number $\sim a_{IJ}(\phi)F^I \wedge F^J$.

In these theories, there is another term relevant for this discussion: the r.h.s. of the equations of motion of the vector fields contains a term of the form

$$\sim C_{IJK}F^I \wedge F^J,$$  \hfill (3.7)

where $C_{IJK}$ is a constant, symmetric tensor, invariant under the gauge group. This term will contain the $\text{SO}(1,2)$ Killing metric and will be proportional to the non-definite positive “instanton number”, but the sign of these terms is irrelevant for the consistency of the theory and the $\text{SO}(1,2)$ selfdual instanton solutions are of potential interest in consistent theories.

For the gauge group $\text{SO}(3)$, the so-called hedgehog ansatz leads to the construction of all the magnetic monopole solutions with this geometry [5]. Here we propose a generalization of this ansatz that encompasses the three cases we are considering. In terms of the structure constants $f_{IJ}^K$ of these groups\(^9\) and the coordinates $y^I = y^I(\varrho, \theta, \varphi)$

$$\begin{align*}
y^1 &= \varrho k^{-1/2} \sin (k^{1/2} \theta) \cos \varphi, \\
y^2 &= \varrho \cos (k^{1/2} \theta), \\
y^3 &= \varrho k^{-1/2} \sin (k^{1/2} \theta) \sin \varphi,
\end{align*}$$  \hfill (3.8)

the generalized hedgehog ansatz for the Higgs and Yang-Mills fields can be written in the form

$$\begin{align*}
\Phi^I &= F(\varrho) \frac{y^I}{\varrho}, \\
A^I &= J(\varrho) f_{JK}^I \frac{y^K}{\varrho} d \left( \frac{y^K}{\varrho} \right),
\end{align*}$$  \hfill (3.9)

where $F(\varrho)$ and $J(\varrho)$ are two functions to be determined.

Plugging the ansatz into the Bogomol’nyi equation (2.12), we get the following system of first-order differential equations involving $F, J$ and the metric function $\Psi$:

\(^9\)For $k = \pm 1, 0$ the structure constants are given by $f_{IJ}^K = \epsilon_{IJK} \eta^{LK}$, where $\langle \eta^{IJ} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$. 

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\[(\Psi F)' = k J (2 + g J), \]
\[J' = F (1 + g J).\]  

where primes stand for derivatives with respect to \(\varrho\).

In Section 3.2 we are going to search for explicit solutions of this system, but, before, we are going to show the form of the 4-dimensional instanton fields in terms of the functions that appear in these equations and in the Kähler metric.

### 3.1 Instanton fields

Given a solution of Eqs. (3.10), \(F(\varrho), J(\varrho)\) for a Kähler metric characterized by the function \(\Psi(\varrho)\) and the parameters \(\epsilon = 0, 1\) and \(k = \pm 1, 0\)

\[ds_4^2 = \frac{\Psi}{\varrho^\epsilon}(dz + \epsilon \chi(k))^2 + \frac{\varrho^\epsilon d\varrho^2}{\Psi} + \varrho^\epsilon d\Omega^2,\]  

with \(d\Omega^2\) and \(\chi(k)\) given by Eq. (3.4) and Eq. (3.5), respectively, the instanton field is given by

\[
A^1 = -\frac{\Psi F}{\varrho^\epsilon} k^{-1/2} \sin (k^{1/2}\theta) \cos \varphi [dz + \epsilon \chi(k)]
\]

\[+ J \sin \varphi d\theta + k^{-1/2} \sin (k^{1/2}\theta) \cos (k^{1/2}\theta) \cos \varphi d\varphi,\]

\[
A^2 = -\frac{\Psi F}{\varrho^\epsilon} \cos (k^{1/2}\theta) [dz + \epsilon \chi(k)] - k J \left[ k^{-1/2} \sin (k^{1/2}\theta) \right]^2 d\varphi,
\]

\[
A^3 = -\frac{\Psi F}{\varrho^\epsilon} k^{-1/2} \sin (k^{1/2}\theta) \sin \varphi [dz + \epsilon \chi(k)]
\]

\[+ \frac{I}{\Psi} \left[ \cos \varphi d\theta - k^{-1/2} \sin (k^{1/2}\theta) \cos (k^{1/2}\theta) \sin \varphi d\varphi \right].\]

This is our main result, but we can elaborate it a bit more.

For \(\epsilon = 1\) and \(k \neq 0\), one can also write the instanton fields in a more compact form by using a generalization of the Maurer-Cartan forms\(^\text{10}\)

\[v^1 = -\sin \varphi d\theta + k^{1/2} \sin (k^{1/2}\theta) \cos \varphi d\bar{z},\]

\[v^2 = d\varphi + k \cos (k^{1/2}\theta) d\bar{z},\]

\[v^3 = \cos \varphi d\theta + k^{1/2} \sin (k^{1/2}\theta) \sin \varphi d\bar{z},\]  

\(^{10}\)Here we are using a shifted coordinate \(\bar{z} = z - k^{-1} \varphi\).
which satisfy $dv^I = -\frac{1}{2} f^{IJK} v^J \wedge v^K$ and in terms of which the instanton field reads

$$A^I = -\frac{k\Psi F}{\rho} v^I + \left( J - k \frac{\Psi F}{\rho} \right) u^I ,$$  \hspace{1cm} (3.14)

where the $u^I$'s are given by

$$u^1 = \sin \varphi \, d\theta + k^{-1/2} \sin(k^{1/2}\theta) \cos(k^{1/2}\theta) \cos \varphi \, d\varphi ,$$

$$u^2 = -k(k^{-1/2} \sin(k^{1/2}\theta))^2 \, d\varphi ,$$

$$u^3 = -\cos \varphi \, d\theta + k^{-1/2} \sin(k^{1/2}\theta) \cos(k^{1/2}\theta) \sin \varphi \, d\varphi .$$ \hspace{1cm} (3.15)

and satisfy $du^I = f^{IJK} u^J \wedge u^K$.

Before we give an expression for the associated field strengths, we find convenient to introduce a basis of three self-dual 2-forms

$$B^i = e^i \wedge e^i + \frac{1}{2} \epsilon_{jk} e^j \wedge e^k ,$$ \hspace{1cm} (3.16)

where $e^a$ is the Vierbein basis of the four-dimensional metric

$$e^a = \frac{\Psi^{1/2}}{\rho^{\epsilon/2}} [dz + \epsilon \chi(k)] , \quad e^1 = \rho^{\epsilon/2} d\theta ,$$

$$e^2 = \frac{\rho^{\epsilon/2}}{\Psi^{1/2}} d\rho , \quad e^3 = \rho^{\epsilon/2} k^{-1/2} \sin(k^{1/2}\theta) \, d\varphi .$$ \hspace{1cm} (3.17)

Then, in terms of these three self-dual three forms the field strengths are
\[
F^1 = \frac{\Psi^{1/2}J'}{q^e} \cos(k^{1/2}\theta) \cos \varphi B^1 + \left(\frac{\Psi F}{q^e}\right)' k^{-1/2} \sin(k^{1/2}\theta) \cos \varphi B^2
\]
\[
\quad - \frac{\Psi^{1/2}J'}{q^e} \sin \varphi B^3 ,
\]
\[
F^2 = -k \frac{\Psi^{1/2}J'}{q^e} k^{-1/2} \sin(k^{1/2}\theta) B^1 + \left(\frac{\Psi F}{q^e}\right)' \cos(k^{1/2}\theta) B^2 ,
\]
\[
F^3 = \frac{\Psi^{1/2}J'}{q^e} \cos(k^{1/2}\theta) \sin \varphi B^1 + \left(\frac{\Psi F}{q^e}\right)' k^{-1/2} \sin(k^{1/2}\theta) \sin \varphi B^2
\]
\[
\quad + \frac{\Psi^{1/2}J'}{q^e} \cos \varphi B^3 .
\]

For later use, it is interesting to have an explicit expression for \(\text{Tr} F \wedge F \sim \eta_{IJ} F^I \wedge F^J\), even if in most cases there is no well-defined notion of instanton number (density). One has

\[
\eta_{IJ} F^I \wedge F^J = d^4x \sqrt{|g|} \left\{ \frac{1}{2} \left[ \left(\frac{\Psi F}{q^e}\right)'\right]^2 + 4k \frac{\Psi (J')^2}{q^e} \right\}
\]
\[
= d^4x \sqrt{|g|} \frac{2}{g^2 q^{2e}} \left[ \left( K' - \epsilon \frac{K}{q} \right)^2 + 2 \frac{K^2}{\Psi} (K' + k) \right]
\]
\[
= d^4x \sqrt{|g|} \frac{2}{g^2 q^{2e}} \left[ \left( G - 1 - \epsilon \frac{k q}{K} \right)^2 + 2k \frac{K^2 G}{\Psi} \right]
\]
\[
= \frac{2}{g^2} d \left[ k \frac{(G - 1) K}{q^e} - \epsilon \frac{K^2}{2 q^{e+1}} \right] \wedge k^{-1/2} \sin(k^{1/2}\theta) dz \wedge d\theta \wedge d\varphi .
\]

where we have defined the functions

\[
K \equiv g^* F , \quad G \equiv (1 + gJ)^2 ,
\]
for reasons that will become clear in the next section, and where, in the second line, it has been assumed that \(k \neq 0\).
For $k = 1$

$$\int F^I \wedge F^I = \frac{8\pi T}{g^2} \left[ \frac{(G - 1)K}{\varrho^e} - \frac{\epsilon}{2 \varrho^{e+1}} \right]_{\varrho_0}^{\varrho},$$

(3.21)

where $T$ is the period of $z$ and $\varrho_0, \varrho_F$ the limits of integration of $\varrho$, which depend on the chosen Kähler space.

### 3.2 Solutions

Let us now go back to the solutions of the system Eqs. (3.10). We consider the $k = 0$ and $k \neq 0$ separately.

#### 3.2.1 The $k = 0$ case

In this case, the first equation of (3.10) can be integrated directly for arbitrary $\Psi(\varrho)$, giving

$$F(\varrho) = \frac{K_0}{g\Psi(\varrho)},$$

(3.22)

where $K_0$ is an integration constant. Plugging this result into the second equation we get

$$J(\varrho) = C e^{\mathcal{I}(\varrho)} - \frac{1}{g},$$

(3.23)

where $C$ is another integration constant and

$$\mathcal{I}(\varrho) \equiv K_0 \int_{\varrho_0}^{\varrho} \frac{du}{\Psi(u)}.$$  

(3.24)

For instance, the metric of $\mathbb{CP}^2$ can be written in the $k = 0$ form with $\Psi = 4\varrho^3/\ell^2$ and $H = \varrho/\Psi$, (i.e. $\epsilon = 1$) [23] and, therefore, we have

$$F(\varrho) = \frac{\lambda}{8\varrho^3}, \quad J = Ce^{-\frac{\lambda}{2\varrho^2}} - \frac{1}{g}.$$  

(3.25)

#### 3.2.2 The $k \neq 0$ case

The system Eqs. (3.10) can be simplified with the change of variables (3.20), after which it takes the form

$$K' = k(G - 1),$$

$$\Psi G' = 2KG.$$  

(3.26)
The first equation in (3.26) can be used to eliminate $G$ in the second one, which leads to a second order equation that only involves the variable $K$:

$$\Psi K'' - 2KK' - 2kK = 0.$$  \hfill (3.27)

Given the function $\Psi(\varrho)$ corresponding to a Kähler metric in the class we are considering, this equation determines $K$. Observe that we can turn around the problem and choose some arbitrary $K(\varrho)$ and then find the Kähler manifold in which it defines a selfdual instanton by computing directly

$$\Psi = \frac{2K(K' + k)}{K''}. \hfill (3.28)$$

There is a simple solution to Eq. (3.27)\textsuperscript{11} which is valid for any $\Psi$, with $K'' = 0$: $K = K_0 - k\varrho$, $G = 0$. The functions $F$ and $J$ that appear in the hedgehog ansatz Eq. (3.9) are given by\textsuperscript{12}

$$F(\varrho) = \frac{K_0 - k\varrho}{8\Psi(\varrho)}, \quad J = -\frac{1}{8}. \hfill (3.29)$$

This solution corresponds to a fixed point of the system Eqs. (3.26).

In order to find more solutions we need to know $\Psi(\varrho)$. In many interesting cases $\Psi(\varrho)$ is a polynomial of order $N$, and, in particular, with $N = 3$. If $\Psi(\varrho)$ is a polynomial of order $N$ we can assume that $K$ is also a polynomial whose order must be $N - 1$ for Eq. (3.27) to have solutions, in general. The differential equation becomes a set of algebraic equations relating the coefficients of the polynomial $K$ to those of the polynomial $\Psi$. For $N = 3$, if

$$\Psi(\varrho) = \Psi_0 + \Psi_1\varrho + \Psi_2\varrho^2 + \Psi_3\varrho^3, \hfill (3.30)$$

$$K(\varrho) = K_0 + K_1\varrho + K_2\varrho^2,$$

one readily finds the following relations ($\Psi_3 \neq 0$, by assumption)

$$K_2 = \frac{\Psi_3}{2},$$

$$K_1 = \frac{\Psi_2 - k}{3}, \hfill (3.31)$$

$$K_0 = \frac{9\Psi_1\Psi_3 - 2(\Psi_2 + 2k)(\Psi_2 - k)}{18\Psi_3},$$

and one constraint for the coefficients of $\Psi(\varrho)$

$$9\Psi_1\Psi_3(\Psi_2 + 2k) - 2(\Psi_2 + 2k)^2(\Psi_2 - k) = 27\Psi_0\Psi_3^2, \hfill (3.32)$$

\textsuperscript{11}This solution is also available in the $k = 0$ case and corresponds to the choice of the integration constant $C = 0$.

\textsuperscript{12}In this case $k = 1/k$ and, in the form in which we are giving this general solution, it is automatically valid for the $k = 0$ case.
which has to be understood just as the condition that $\Psi$ has to satisfy in order for Eq. (3.27) to admit a solution in which $K$ is a second order polynomial.

Since $J$ is a real function, the second of Eqs. (3.20) $G$ must be a positive definite function. The first of Eqs. (3.26) and Eqs. (3.31) tell us that it is given by

$$G = 1 + kK' = 1 + kK + 2kK_2\varrho = \frac{2 + k\Psi_2}{3} + k\Psi_3\varrho,$$

so that $\varrho$ is restricted to the interval

$$\varrho > -\frac{k(2 + k\Psi_2)}{3\Psi_3} \quad \text{for} \quad k\Psi_3 > 0,$$

$$\varrho < -\frac{k(2 + k\Psi_2)}{3\Psi_3} \quad \text{for} \quad k\Psi_3 < 0.$$

The metric of $\mathbb{CP}^2$ can be written in the $k = \pm 1$ form with a $\Psi$ which is the cubic polynomial

$$\Psi = \varrho^2(k + 4\varrho/\ell^2)$$

and, since $\Psi$ must also be positive in the metric, the variable $\varrho$ is restricted to

$$\varrho > -k\ell^2/4,$$

which is compatible with the restriction found above only for $k = 1$.

## 4 Conclusions

We have completed our program of finding simple equations that selfdual instanton solutions in Kähler spaces with one holomorphic isometry have to satisfy, generalizing Kronheimer’s work in the hyperKähler case. We have also constructed some explicit solutions in some Kähler spaces of particular interest from the point of view of 5-dimensional Abelian-gauged supergravity. In passing, we have generalized the hedgehog ansatz to some non-spherical symmetries and gauge groups different from $SU(2)$. There is little work in the literature on non-compact gaugings and we think these results will allow us to find interesting solutions in those cases.

We have not analyzed the regularity of the solutions we have obtained because, ultimately, they are going to be part of a complicated 5-dimensional gauge field defined in a 5-dimensional spacetime whose regularity does not depend on the regularity of each of its building blocks. There are perfectly regular 5-dimensional solutions (microstate geometries) built over singular base spaces like ambipolar GH spaces [7, 24]. And most regular, charged, extremal black holes use Coulomb-like 1-form fields which
are singular at a point which, in the end, turns out to be not a point but a regular horizon. The same mechanism saves the regularity of the 4-dimensional non-Abelian black holes which bear BPS magnetic monopole fields different from the ’t Hooft-Polyakov one. Thus, one should not be worried about the possible singularities of the instantons until the full 5-dimensional supergravity solution is constructed. For the same reason (and also because the Kähler spaces that we are considering are not compact) we have not computed the instanton number. In a forthcoming publication [21] we will put to use the results obtained in this work, which we hope will also be useful in other contexts.

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