Finite Magnetic Flux Tube
as a
Black&White Dihole

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Abstract

A finite-length magnetic vortex line solution is derived within the context of (4-dim) dilaton gravity. We approach the Bonnor metric at the Einstein-Maxwell limit, and encounter the "flux tube as (Euclidean) Kerr horizon" at the Kaluza-Klein level. Exclusively for string theory, the magnetic flux tube world-sheet exhibits a 2-dim black&white dihole structure.

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Dilaton gravity is gaining momentum once again. Several years ago it was the revival of the Kaluza-Klein (KK) idea which served to focus attention on the scalar dilaton (≡ the local scale of the extra-dimensional manifold) couplings induced at the effective 4-dim low energy theory. At present, the resumed interest in dilaton gravity is triggered by string theory, where dilaton couplings (with a slightly reduced strength in comparison with KK) appear rather mandatory when facing the gravitational consequences of the theory.

In its simplest version, that is without a scalar potential, the dilaton Einstein-Maxwell action takes the form

\[ \int d^4x \sqrt{-\det g} \left( R + \frac{1}{4} e^{-2k\eta} g^{\mu\nu} g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} + 2g^{\mu\nu} \partial_\mu \eta \partial_\nu \eta \right) \]  

Depending on the value of the dilaton coupling constant \( k \), the three special cases of interest are the following: \( k = 0 \) takes us back to the Einstein-Maxwell limit, \( k = 1 \) is suggested by string theory treated to the lowest order in the world sheet and string loop expansion, and \( k = \sqrt{1 + \frac{2}{N}} \) is associated with the \( M_4 \otimes S_N \) KK theory. Of particular interest in the latter category is the original 5-dim (that is \( N = 1 \)) KK scheme, for which \( k = \sqrt{3} \).

Traditionally, priority has always been devoted to black hole solutions. In this paper, however, we are a priori after a different type of solutions (although we recapture black/white holes as a string theory bonus), namely axially symmetric solutions which represent finite-length magnetic flux tubes. Being finite, such a magnetic flux tube must connect a monopole anti-monopole pair. The work reported here generalizes in some respect a preliminary \( k = \sqrt{3} \) KK analysis (for the KK monopole solution, see ref. 6). We find it therefore pedagogical to review first this special case, for which there happens to exist a simple (5-dim) geometric interpretation. So, let our starting point be the 5-metric

\[ ds_5^2 = -dt^2 + ds_{Kerr}^2, \]  

where \(-dt^2\) is supplemented by the well known 4-dim (Euclidean) Kerr metric \( ds_{Kerr}^2 \) expressed in the terms of \((x^5, r, \theta, \phi)\). To squeeze out the effective 4-dim theory, one first integrates out the \( x^5 \) circle, and then invokes the Weyl-scaling dimensional reduction procedure

\[ ds_5^2 = \phi^{-\frac{1}{3}} ds_4^2 + \phi^{\frac{2}{3}} (dx^5 + A_\mu dx^\mu)^2. \]

Following such a standard way of identifying the physical fields \( g_{\mu\nu}(x), A_\mu(x), \phi(x) = \exp(-2\sqrt{3}\eta) \), and invoking the explicit form of the (Euclidean) Kerr metric \( ds_{Kerr}^2 \), one recovers the \( k = \sqrt{3} \) limit of our forthcoming eqs.(5-7).
The time is ripe now to present our exact analytic solution which extremizes (for arbitrary $k$) the dilaton Einstein-Maxwell action. Unfortunately, due to length limitations, we only sketch here some technical highlights; the detailed derivation will be presented elsewhere. In deriving this solution we have closely followed the Chandrasekhar prescription of dealing with stationary, axially symmetric Einstein equations. The reason is practical. Although we have a priori content ourselves to a diagonal metric, the corresponding equations of motion happen to be the same as the original Kerr equations, only with replacements $\Omega \equiv \text{Kerr angular velocity} \rightarrow \sqrt{1 + k^2} A_\varphi$, and $\psi \equiv (1/2) \log(-g_{\varphi\varphi}/g_{tt}) \rightarrow \psi + 2k\eta$.

In turn, taking into account the latter modifications, one is still led to the complex Ernst equation. The so-called variable phase Ernst solution then fixes $A_\varphi$ and $\psi + 2k\eta$ combination. The only new equation in the game

$$\partial_r (\Delta \partial_r \chi) + \left(\frac{1}{\sin \theta}\right) \partial_\theta (\delta \frac{1}{\sin \theta} \partial_\theta \chi) = 0$$

is to be solved for $\chi \equiv (\frac{1}{2}) \log(-g_{\varphi\varphi}/g_{tt}) - (\frac{2}{k})\eta$. The solution $e^\chi = (\Delta \delta)^\lambda$ depends on some constant of integration $\lambda$ which, at the later stage, must be chosen as $\lambda = \frac{1}{2}$ to assure asymptotic flatness. Here, we use of the familiar Boyer-Lindquist notations (in their Euclidean version)

$$\Delta(r) = r^2 - 2Mr - a^2, \quad \rho^2(r, \theta) = r^2 - a^2 \cos^2 \theta, \quad \delta = \sin^2 \theta.$$  

Collecting the various pieces together, we finally obtain

$$ds^2 = \left(\frac{\Delta + a^2 \delta}{\rho^2}\right)^{\frac{1}{1+k^2}} [ -dt^2 + \Delta \delta \left(\frac{\rho^2}{\Delta + a^2 \delta}\right)^{\frac{4}{1+k^2}} d\varphi^2 +$$

$$+ \left(\frac{\rho^2}{\Delta + (M^2 + a^2) \delta}\right)^{\frac{4}{1+k^2}} \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2\right)],$$

$$A_\varphi = \frac{1}{4aMr\delta},$$

$$\eta = \frac{k}{1 + k^2 \log \frac{\rho^2}{\Delta + a^2 \delta}}.$$  

Our general solution is clearly asymptotically-flat. Moreover, for $M = 0$, one approaches a flat space-time conveniently described by prolate coordinate system (where
$a$ is recognized as the focal length). The long distance behavior of our solution is governed by its mass $m$, by a magnetic dipole moment $\mu$, and (for $k \neq 0$) by some scalar hypercharge as well. In particular,

$$m = \frac{2M}{1 + k^2}, \quad \mu = \frac{4aM}{\sqrt{1 + k^2}}.$$  \hspace{1cm} (8)

For $k = 0$, to be referred to as the Einstein-Maxwell limit, our solution passes its second consistency check (the first being the $k = \sqrt{3}$ limit) by converging to the well known Bonnor metric $^{10}$. Examining the Riemann tensor, curvature singularities are detected at $\rho^2 = 0$ and also along $\Delta + a^2 \delta = 0$. As far as $\Delta = 0$ is concerned, the situation is a bit tricky, and by far more important for the sake of our discussion. If $\delta \neq 0$ (that is $\theta \neq 0, \pi$) the apparent singularity at $\Delta = 0$ turns out to be an artifact (for arbitrary $k$), reflecting nothing but an ill-defined coordinate system. If $\delta = 0$, on the other hand, and for $k \neq 1$ (to be discussed separately), the so-called end-points $\theta = 0$ and $\theta = \pi$ of the $\Delta = 0$ surface forcefully catch the $\Delta + a^2 \delta = 0$ source singularity!

Approaching the short distance regime, the first surface of interest is defined by the outer zero of $\Delta(r) = 0$, namely

$$r_H = M + \sqrt{M^2 + a^2}.$$  \hspace{1cm} (9)

The geometrical meaning of $r_H$ can be revealed, to a certain extend, within the KK embedding $^5$, see eq. (2), of the $k = \sqrt{3}$ solution. In such a 5-dim framework, the $r = r_H$ surface is identified as the Euclidean extension of the Kerr rotating event horizon, suffering at most conic singularities at the $\theta = 0, \pi$ poles. The fact that this surface (modulo its North and South poles) stays non-singular even at the 4-dim level, and this is true for arbitrary $k$, may indicate a deeper yet quite obscure flux-tube ↔ horizon connection.

The crucial observation now is the overall $\Delta$ factor that $d\varphi^2$ has picked up. The geometrical significance of this is clear: as $r \to r_H$, the invariant circumferencial length vanishes

$$\int_0^{2\pi} \sqrt{g_{\varphi\varphi}} d\varphi \to 0,$$  \hspace{1cm} (10)

thereby marking the axis of axial symmetry. This is to be regarded as a physically non-trivial continuation of a crucial feature borrowed from the underlying prolate coordinate
system (recall the flat $M = 0$ case, where prolate $r = a$ is just a line connecting the two focuses, whereas $r > a$ are ellipsoids of revolution). Altogether, $r = r_H$ defines an 1+1-dim (rather than 2+1-dim) sub-manifold characterized by the following induced world-sheet metric

$$ds^2_H = \left(\frac{a^2 \delta}{\rho_H^2}\right)^{\frac{1}{1+k^2}} [-dt^2 + \left(\frac{\rho_H^2}{(M^2 + a^2)\delta}\right)^{\frac{3-k^2}{1+k^2}} \rho_H^2 d\theta^2],$$

where $\rho_H^2 = r_H^2 - a^2 \cos^2 \theta$.

The electromagnetic gauge field $A_\varphi(r, \theta)$, given by eq.(6), nicely fits into the game. First of all, the $k$ - effect on $A_\varphi$ (and similarly on $\eta$ ) is remarkably simple, just an overall $(1 + k^2)^{-1/2}$ factor. And more important, as $r \to r_H$

$$\int_0^{2\pi} A_\varphi d\varphi \to 4\pi g \equiv \frac{8\pi}{\sqrt{1+k^2}} \frac{Mr_H}{a},$$

indicating magnetic flux confinement. Our $\Delta(r) = 0$ string has become a magnetic vortex line, stretched between a monopole anti-monopole pair of magnetic charges $\pm g$ at its endpoints $\theta = 0, \pi$. The medium surrounding the magnetic flux tube is paramagnetically polarized. The associated magnetic permeability

$$\exp(2k\eta) = (1 + \frac{2Mr}{\Delta(r) + a^2 \sin^2 \theta})^{\frac{2k^2}{1+k^2}},$$

ranges from unity at spatial infinity up to a blow-up at the monopole locations.

Given the world - sheet metric eq.(11), the invariant length $l_k$ of our vortex string can be calculated by integrating $ds_H$ ( keeping $t$ frozen). $l_k$ gets infinite as $k \to 0$, diverging like $k^{-2}$, thereby presenting a major drawback for the Bonnor dipole . The invariant length is finite, however for any non-vanishing $k$. Of particular interest is the $k = 1$ case, representing string theory, for which $l_1 = (2ar_H/\sqrt{M^2 + a^2})E(a/r_H)$, where $E(a/r_H)$ denotes a complete elliptic integral.

Let us now have a closer look at the world-sheet metric of our magnetic flux tube, and reveal the (2-dim) nature of its monopole ($\theta = 0, \pi$) source singularities. For small $\theta$, that is in the neighborhood of the North pole, we reparametrize $\theta \approx \epsilon^{\frac{1}{2k^2}}$, to conveniently arrive at

$$ds^2_H \approx -\epsilon^{\frac{2k^2}{1+k^2}} (\omega_k dt)^2 + d\epsilon^2,$$
\( \omega_k \) being a world-sheet constant easily extracted from eq. (11). Such an approximate metric suffices to make our point. Curvature singularity is encountered as \( \epsilon \to 0 \), unless \( k = 1 \) (recall that the Ricci scalar behaves like \( R \approx 2(1-k^2)/k^2\epsilon^2 \)). Moreover, the exclusive feature of the \( k = 1 \) two metric (in its Hawking \( t \to i\tau \) extension \(^{11}\)) is that the singularity is only conical, and can be regarded an artifact provided \( \tau \) is identified with a period of

\[
\Delta \tau = \frac{2\pi}{\omega} = 4\pi M \left( 1 + \frac{M}{\sqrt{M^2 + a^2}} \right).
\]

(15)

Altogether, we are led to the remarkable conclusion that \( \theta = 0 \) acts as a world-sheet event horizon. A 2-dim hole has emerged. In fact, there are two of them in the game, as the analysis for \( \theta = \pi \) proceeds on entirely equal footing. Taking into account local light-cone considerations, this may suggest (to be verified soon) the (2-dim) interpretation of a black & white dihole. Hypothetical thermodynamic effects of the \( \tau \)-periodicity lie beyond the scope of the present paper.

Owing to its topological origin, the \( \Delta \tau \) periodicity must globally characterize the flux tube world-sheet. In other words, the exact \( k = 1 \) two-metric

\[
a^{-2}ds_H^2 = -\frac{\sin^2 \theta}{r_H^2 - a^2 \cos^2 \theta} dt^2 + \frac{r_H^2 - a^2 \cos^2 \theta}{M^2 + a^2} d\theta^2
\]

(16)
calls for an analytic Kruskal-type extension \(^{12}\). We thus listen to the \( \tau \)-periodicity message, and set accordingly \( u = f(\theta) \cosh \omega t, \: v = f(\theta) \sinh \omega t \), requiring \( f(\theta) \) to be such that

\[
da_N^2 = 4 \cos^4 \frac{\theta}{2} \frac{2\omega^2}{\sqrt{M^2 + a^2}} \cos \theta (du_N^2 - dv_N^2),
\]

and alternatively \( f_S(\theta) \equiv f_N(\pi - \theta) \) for which

\[
da_S^2 = 4 \sin^4 \frac{\theta}{2} \frac{2\omega^2}{\sqrt{M^2 + a^2}} \cos \theta (du_S^2 - dv_S^2).
\]

(17)

(18)
f\(_{N,S}(\theta)\) is regular at the North (South) pole, but apparently troublesome at the opposite pole. A Wu-Yang-like construction \(^{13}\) of matching patches along \( \theta = \pi/2 \) is quite welcome.
Figure 1: (Caption) World–sheet null geodesics in the (retarded) Lemaitre representation. Notice the different fates of Northerly versus Southerly directed ‘light’–rays.

To gain more insight into the structure of local light cone, a synchronous form of metric is in order. This can be achieved using the language of retarded (or advanced) Lemaitre coordinates $T = t + \int \frac{r}{1 - \Gamma^2} d\xi$ and $R = t + \int \frac{dr}{\Gamma(1 - \Gamma^2)}$, where $\Gamma^2(\xi) \equiv \frac{c^2 - 1}{c^2 - \xi^2}$, $\xi(T, R)$ is nothing but the extended $\cos \theta$, and $c = r_H/a \geq 1$. The 2-dim line element gets transformed into

$$ds^2_H = -dT^2 + \Gamma^2(\xi)dR^2.$$  (19)

which is furthermore free of apparent singularities and (by being extensible to include the real singular points) geodesically complete. The light cone generatrix slope $dT/dR$ varies from infinity at the $\xi = \pm c$ singularities, to unity at $\xi = \pm 1$ horizons, down to minimal $\pm c^{-1}\sqrt{c^2 - 1}$ at the central point $\xi = 0$.

The various categories of null geodesics, illustrated in Fig.1, confirm our black&white dihole interpretation. In particular, notice the different fates of oppositely directed ”light”-rays through (say) $\zeta = 0$. The Northerly directed ray will unavoidably cross the black horizon on its way to singularity, whereas the Southerly directed one just accumulates on the white horizon $^{15}$.

Given the arrow of time, one can thus tell, on (2-dim) geodesic grounds, a magnetic monopole from an anti-monopole. Had we invoked the advanced (rather the retarded) Lemaitre coordinates, we would have faced a rather similar picture, save for a black $\leftrightarrow$ white role exchange.

One should wonder by now which of the above 2-dim features persists at the full 4-dim theory. We would like to know whether our black&white dihole is merely a 2-dim
reality, or is it in fact a legitimate 4-dim creature? To answer the latter question in the affirmative, at least locally, we expand our \( k = 1 \) four-metric around the apparent (say) North singularity using \( r \approx r_H + \frac{1}{2} \epsilon^2 \sqrt{M^2 + a^2} \) and \( \sin \theta \approx \theta \). As advertised, the result is manifestly flat, free of curvature singularities, but with a nontrivial double conic interplay. Namely, for \( \epsilon \ll \theta \ll 1 \) we have

\[
ds^2 \approx ds_H^2 + (r_H^2 - a^2)[\frac{a^2}{M^2 + a^2}d\epsilon^2 + \frac{M^2 + a^2}{a^2} \epsilon^2 d\varphi^2], \tag{20}
\]

whereas \( \theta \ll \epsilon \ll 1 \) implies

\[
ds^2 \approx (r_H^2 - a^2)d\Omega^2 + [-\frac{M^2 + a^2}{r_H^2 - a^2} \epsilon^2 dt^2 + (r_H^2 - a^2)de^2]. \tag{21}
\]

The good news is that, independently on how one approaches the \((\epsilon, \theta)\)-origin, the imaginary-time conic singularity is again removable by means of the same old periodicity condition eq. (15). This in turn allows to maintain the geometrical essence of event horizon. The novel effect, however, is the Vilenkin-like cosmic string defect \(^{16}\). Once recognizing the \( \Delta \varphi = 2\pi \) periodicity, on asymptotic and/or eq. (21) grounds, a typical conical structure accompanies eq.(20).

This completes the presentation of a field theoretical configuration which can be loosely referred to as a cosmic magnet. A non-trivial dilaton coupling is the crucial ingredient needed for driving the invariant length finite. On the technical side, we have recover the Bonnor metric at the Einstein-Maxwell limit, and the "flux tube as a (Euclidean) Kerr horizon" at the Kaluza-Klein embedding. Effective string theory gets singled out when analyzing the flux tube world-sheet geometry. It is exclusively for string theory that our flux tube serendipitously resembles a (2-dim) black&white dihole. The latter physical interpretation, analytically accepted by the full 4-dim parent theory, is supported by a Kruskal (imaginary time periodic) extension, as well as by the Lemaitre (synchronous light cone and geodesically complete) representation.
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