Mesoscopic Systems With Fixed Number of Electrons

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Abstract

In this paper, we study the physics of mesoscopic systems with noninteracting, but fixed number of electrons. From a technical point of view, this means a discussion of the differences between the canonical and the grand canonical ensemble (fixed versus fluctuating number of particles). Such a discussion is not trivial since the grand canonical ensemble is the most convenient basis for the statistics of identical particles and one has to spend labour in order to retrieve the canonical ensemble. Specifically, we are considering ensembles of mesoscopic systems with disorder, either by atomic defects or by fluctuations in their geometric definitions and we discuss various forms of disorder averages.

05.30, 73.35, 72.10
I. INTRODUCTION

In the following, we will investigate some properties of noninteracting electrons in mesoscopic systems. We take into account that irregularities in the preparation introduce disorder on an atomic scale and also, in the geometric definition of the samples. We will explore the consequences of strict conservation of particle number and we will find that some details in the disorder averaging procedure may become important. Such a peculiar behavior has been emphasized by various authors; we wish to mention here Shklovskii [1] as well as Imry [2]. From a technical point of view, our paper is meant to discuss the differences between the canonical and the grand canonical statistical ensemble.

We will be concerned first with (A) thermodynamic properties on the average [2], secondly with (B) kinetic behavior [1], and thirdly with (C) stochastic properties of thermodynamic quantities [3]. Clearly, in the first and third case, it is suffice to know the positions of the single particle levels $\epsilon_\lambda$. We will argue that the position of the levels is also most decisive in the second case.

(A) According to standard arguments, thermodynamic properties may be derived from the grand canonical potential $\Omega(T, \mu)$ which is given by

$$\Omega = \sum_\lambda (-T) \ell n \left[ 1 + e^{-\left(\epsilon_\lambda - \mu\right)/T} \right]$$  \hspace{1cm} (1)

We introduce the single particle density of states per spin

$$\mathcal{D}(E) = \frac{1}{s} \sum_\lambda \delta(E - \epsilon_\lambda)$$  \hspace{1cm} (2)

where $s = (2S + 1)$ is the spin degeneracy. (In previous publications, we used to start with the density of states per spin and unit volume $\mathcal{N}(E) = \mathcal{D}(E)/V$, following a convention of previous decades, where one has been interested in bulk properties.) Then, the expression for the grand canonical potential can be written as

$$\Omega = s \int dE \mathcal{D}(E)(-T) \ell n \left[ 1 + e^{-\left(E - \mu\right)/T} \right]$$  \hspace{1cm} (3)
Following Ref. [1], we will also study the photoabsorption of mesoscopic particles. The cross-section for this process can be written [1] in the form

\[ \sigma = \frac{4\pi^2e^2}{3c} \sum_{\lambda\lambda'} (\epsilon_{\lambda'} - \epsilon_{\lambda}) | R_{\lambda\lambda'} |^2 (n_{\lambda} - n_{\lambda'}) \delta(h\omega - \epsilon_{\lambda'} + \epsilon_{\lambda}) \] (4)

where \( n_{\lambda} \) is the population of the single particle state \( | \lambda > \); and \( eR_{\lambda\lambda'} = e < \lambda | \hat{R} | \lambda' > \) the matrix element of the electronic dipole moment. It seems (see Refs. [4] - [6]), that there is not much structure in \( | R_{\lambda\lambda'} |^2 \) so that one may replace it by an average value

\[ \sigma_0 = \frac{4\pi^2e^2}{3ch} < | R_{\lambda\lambda'} |^2 > \] (5)

(which also includes an average with respect to the disorder), without distorting the general structure of the result to be obtained. Assuming that the population is equal to the Fermi function, we obtain

\[ \sigma = \sigma_0 h^2 \omega \int dEdE' \mathcal{D}(E)\mathcal{D}(E') \left[ f(E - \mu) - f(E' - \mu) \right] \delta(h\omega - E' + E) \] (6)

Due to the discreteness of the levels in mesoscopic samples, the \( \omega \)-dependence of the cross section \( \sigma(\omega) \) consists of \( \delta \)-spikes for each individual sample and it is only the average with respect to the ensemble of samples, that is, the disorder averaged cross section \( < \sigma(\omega) > \), which may be expected to be a continuous function of the frequency.

As a rule, the density of states is a fluctuating quantity which consists of a leading part \( \mathcal{D}^0(E) \) and a remainder

\[ \mathcal{D}(E) = \mathcal{D}^0(E) + \mathcal{D}^1(E) \] (7a)

where

\[ \mathcal{D}^0(E) = < \mathcal{D}(E) > \] (7b)

is equal to the average with respect to the disorder. Note that in the literature, the following abbreviation occurs quite frequently

\[ \mathcal{R}_1(E) = \mathcal{D}^0(E) \] (7c)
(one level correlation functions); and if systematics is required, we will resort to that terminology.

We mention in passing that in some problems where no disorder appears explicitly, the average can be taken with respect to an energy window or it may represent a kind of integration with respect to quantum numbers.

As a rule, $\mathcal{D}^0$ is insensitive to genuine quantum phenomena which may result e.g. from an applied magnetic field or from a magnetic flux threading a mesoscopic ring.

If this is the case, the disorder averaged grand canonical potential

$$\Omega^0 := \langle \Omega \rangle = s \int dE \mathcal{D}^0(E)(-T) \ell n \left[ 1 + e^{-(E-\mu)/T} \right]$$

has lost any sensitive dependence on external parameters. Indeed, this turns out to be true as far as the response (persistent currents) to magnetic flux of mesoscopic rings is concerned. On the other hand, it is found that there is a measurable effect in a theory where the number of electrons is kept fixed [2,7].

(C) The fact that the sensitive part of $\Omega$ is, on the average, zero does not mean that $\Omega$ is zero for each realization. Rather, it means that its contributions fluctuate in magnitude and in sign; loosely speaking, one may say that plus and minus signs occur in about equal numbers. The simplest quantity which can be used as a measure of the stochastic (sample to sample) fluctuations, is the mean square value. In previous publications (see references in [3]), it has been considered to be sufficient for non-interacting electrons, to calculate the mean square $\langle \Omega^2 \rangle$, which means stochastic fluctuations at fluctuating numbers of electrons. We will show how stochastic fluctuations can be calculated for fixed number of electrons. A preliminary estimate of the resulting expression indicates that significantly larger values are then obtained as compared with the standard procedure.

For an initiation to a discussion of the photoabsorption problem, we calculate the disorder averaged cross section in a crude approximation $\langle \sigma \rangle^{uc}$ where in Eq. (6) the averaged product of the density of states is replaced by the product of the averages ($uc$: uncorrelated; that is, disregarding correlations). If $\hbar \omega << \mu$, we may even put $\mathcal{D}^0(E) \approx \mathcal{D}^0(\mu^0)$ where
\[ \mathcal{D}^0(\mu^0) =: \mathcal{D}_F \]

is the average density of states at a (typical) Fermi level \( \mu^0 \). Then, we obtain

\[ <\sigma>^{uc} = \hbar^2 \omega^2 s \mathcal{D}_F^2 \sigma_0 \]

The paper is organized as follows. Thermodynamic properties (A) of mesoscopic systems with fixed number of electrons are investigated in Section II, III, and IV on the basis of (i) Legendre Transformation; (ii) Coulomb blockade; and (iii) pinning of the Fermi level to a single particle state. Section V is devoted to the problem of photoabsorption, whereas, stochastic fluctuations are studied in Section VI. The paper concludes with a discussion in Section VII. Appendix A presents a calculation of correlation functions in the cooperon-diffusion approximation for persistent currents and Appendix B contains a collection of results of the random matrix theory.

**II. CANONICAL ENSEMBLE BY LEGENDRE TRANSFORMATION**

The mathematical conveniences provided by the grand canonical ensemble (fluctuating number of particles) are evident and there is no need to consult text books. Certainly, there are differences in the results obtained for the above mentioned ensemble and the canonical ensemble (fixed number of particles) but they are said to become relatively small in the thermodynamic limit.

However, there are phenomena in mesoscopic systems which depend only on a relatively small number of electrons. This suggests that one should work entirely within the canonical ensemble; in fact, this restriction should be taken very serious since in all processes the particle number is conserved.

At low temperatures (at \( T = 0 \), to be precise) where thermal fluctuations are negligible, the grand canonical potential \( \Omega(T = 0, \mu) \) and the free energy \( F(T = 0, N) \) are directly related by a Legendre transformation. However, this is true only before the averages with respect to the disorder have been taken; therefore, the transformation has to be done for
each realization of the disorder separately. This program can be carried through comparably easy, on the basis of the decomposition (7), provided that the fluctuating part $D^1(E)$ in the density of states is small. We will find that the first nontrivial correction is of second order in $D^1$. In what follows, we will relax the condition $T = 0$ and we will perform the Legendre transformation from $\Omega(T, \mu)$ to $F(T, N)$ on the basis of the above mentioned expansion [7].

In this context, it is convenient to formulate the Legendre transformation as follows

$$F(T, N) = \max_\mu \left\{ \Omega(T, \mu) + \mu N \right\}$$

Clearly, a necessary condition for a maximum is

$$\frac{\partial \Omega}{\partial \mu} + N = 0 \Rightarrow \mu = \mu(T, N)$$

Corresponding to the decomposition (7) of the density of states we have $\Omega = \Omega^0 + \Omega^1$, where $\Omega^0 = <\Omega>$ is given in Eq. (8) and where

$$\Omega^1 = s \int dE D^1(E)(-T) \ln \left[ 1 + e^{-(E - \mu)/T} \right]$$

Let us put

$$\mu = \mu^0 + \mu^1$$

where $\mu^0$ is the zero order approximation such that

$$- \left. \frac{\partial \Omega^0}{\partial \mu} \right|_{\mu^0} = N \quad (15a)$$

and let us assume that we have found the leading order Legendre transform

$$F^0(T, N) = \Omega^0(\mu^0) + \mu^0 N \quad (15b)$$

According to what we have said in the Introduction, $F^0$ does not depend on the quantum effects we are interested in; therefore, its detailed form is not important here. In the next order, we have

$$- \left. \frac{\partial \Omega^0}{\partial \mu} \right|_{\mu^0 + \mu^1} - \left. \frac{\partial \Omega^1}{\partial \mu} \right|_{\mu^0} = N \quad (16a)$$
Making use of the relation (cf. Eq. (9))
\[-\frac{\partial^2 \Omega^0}{\partial \mu^2} \bigg|_{\mu^0} = \frac{\partial N}{\partial \mu} \bigg|_{\mu^0} = s D_F\] (16b)
we obtain
\[s D_F \mu^1 + s \int dE D^1(E) f(E - \mu^0) = 0\]
\[\mu^1 = \frac{1}{D_F} \int dE D^1(E) f(E - \mu^0)\] (17)

We conclude that through second order
\[F = F^0 + \Omega^1(\mu^0) + \Delta F\] (18)

where
\[\Delta F = \max_{\mu^1} \left\{ \frac{1}{2} \frac{\partial^2 \Omega^2}{\partial \mu^2}(\mu^1)^2 + \frac{\partial \Omega^1}{\partial \mu} \mu^1 \right\}\] (19)

and where the derivatives are taken at \(\mu = \mu^0\). Consequently,
\[\Delta F = -\frac{1}{2} \frac{\partial^2 \Omega^0}{\partial \mu^2}(\mu^1)^2\]
\[= \frac{1}{2} \frac{1}{s D_F} \left[ s \int dE D^1(E) f(E - \mu^0) \right]^2\] (20)

For convenience, we introduce here
\[\Delta = 1/s D_F\] (21)

which is the mean level separation and
\[\delta N = s \int dE D^1(E) f(E - \mu^0)\] (22)

which is the disorder induced fluctuation in the particle number calculated for the grand canonical ensemble. Thus, we may write
\[\Delta F = \frac{1}{2} \Delta (\delta N)^2\] (23)
We recognize that the calculation of the disorder average $\langle \Delta F \rangle$ requires the knowledge of the two-level correlation function

$$\tilde{R}_2(E_1, E_2) = \langle D(E_1) D(E_2) \rangle$$

which is related to the cumulant as follows

$$\tilde{Y}_2(E_1 E_2) = -\tilde{R}_2(E_1, E_2) + R_1(E_1) R_1(E_2)$$

$$= - \langle D^1(E_1) D^1(E_2) \rangle$$

(24b)

For orientation, we mention some properties of this correlator. We expect that for large separation of the energies the correlations vanish. Hence,

$$\int dE \tilde{Y}_2(E, E') = 0$$

(25)

In general, we expect only a weak dependence on the absolute values of the energies; hence, for energies near the Fermi level, the correlator may be assumed to depend only on the energy difference

$$\tilde{Y}_2(E, E') = \tilde{Y}_2(E - E')$$

$$= \tilde{Y}_2(|E - E'|)$$

(26)

where the second line follows from the symmetry $E \leftrightarrow E'$.

Making use of Eq. (25) and then of Eq. (26), we arrive at

$$\langle (\delta N)^2 \rangle = s^2 \int dE dE' f(E - \mu^0) f(-E' + \mu^0) \tilde{Y}_2(E - E')$$

$$= s^2 2T \int du \frac{1}{e^{u/T} + 1} \left( \ell_n - \frac{1 + e^{u/2T}}{1 + e^{-u/2T}} \right) \tilde{Y}_2(u)$$

(27)

In the limit $T \to 0$, the expression simplifies to

$$\langle (\delta N)^2 \rangle = -s^2 \int_0^0 du u \tilde{Y}_2(u)$$

$$= s^2 \int_0^0 du u \tilde{Y}_2(u)$$

(28)
In retrospect, one comment is in order. We should recognize that the two energy integrations in the expression for $\langle (\delta N)^2 \rangle$ collect contributions from a large range of order Fermi energy. Therefore, it is not obvious that the translational invariant form (26), for the correlator, is admissible. For justification, note that firstly, the application of the sum rule (25) allows to introduce substantial restrictions in one energy integration. Still, the one energy integration in Eq. (28) may diverge. There, we should keep in mind that, we are interested here in quantum effect which occur at low energies. Therefore, it is possible to subtract irrelevant parts and to arrive at a convergent expression one is interested in.

For a demonstration, see the discussion on persistent currents, where a metallic ring is threaded by a magnetic flux $\phi$ (see appendix A). There, it is found that the phase sensitive part of $\langle \Delta F \rangle$ is periodic in $\phi$ with period $\frac{1}{2}\phi_0$ where $\phi_0 = 2\pi\hbar c/e$ is the flux quantum. It is also found that the Fourier components of $\langle \Delta F \rangle$, that is, the phase sensitive parts of $\langle \Delta F \rangle$, are of the order of the mean level spacing $\Delta$.

III. CANONICAL ENSEMBLE BY COULOMB BLOCKADE

Here, we discuss the consequences of the fact that the electrons carry the charge ($-e$) and that the capacitance $C$ of a mesoscopic sample is small. (In case of a spherical geometry, $C$ is proportional to the diameter.)

Suppose that there is an excess charge ($-e$)($N - N_i$) where $N$ is the number of the electrons and $N_i$ the effective number of the positive ions. Then, the electric potential $\phi$ is non-zero and equal to

$$\phi = C(-e)(N - N_i)$$

Accordingly, the single particle levels undergo a change

$$\epsilon_\lambda \rightarrow \epsilon_\lambda - e\phi$$

which can alternatively be expressed as a change in the chemical potential
\[ \mu \rightarrow \mu + e\phi \] (30b)

For a discussion of the thermodynamics of the system, it is most convenient to use \( \phi \) instead of \( N \) as an independent variable. In this case, the charging energy \( C\phi^2/2 \) appears with a minus sign in the expression for the thermodynamic potential; thus

\[ \Omega(T, \mu, \phi) = \Omega(T, \mu + e\phi) - \frac{1}{2}C\phi^2 \] (31)

where the “free” thermodynamic potential \( \Omega(T, \mu) \) is given by Eq. (3).

In thermal equilibrium, \( \Omega(T, \mu, \phi) \) is maximal with respect to \( \phi \); hence

\[ \Omega_C(T, \mu) = \min_\phi \Omega(T, \mu, \phi) \] (32)

and the ensuing argumentation is similar to the one below Eq. (11). Clearly, a necessary condition for a minimum is \( \partial \Omega(T, \mu, \phi)/\partial \phi = 0 \); that is

\[ -e\left(N(T, \mu, \phi) - N_i\right) - C\phi = 0 \] (33)

where the contribution of the positive ions has been subtracted “by hand”.

Following the decomposition (7) of the density of states, we have \( \Omega^0 + \Omega^1 \) as well as \( N = N^0 + N^1 \). Let us now assume that

\[ N^0(T, \mu, \phi = 0) = N_i \] (34)

Note that this assumption corresponds to what has been expressed by Eq. (15a) where the “typical” Fermi level \( \mu^0 \) is defined; for convenience, we have omitted the superscript zero and we have put \( \mu = \mu^0 \) in (34).

As previously, we consider the fluctuating contribution \( N^1 \) (which is proportional to \( D^1 \)) as well as \( \phi \) to be relatively small. As a consequence, Eq. (31) can be expanded as follows

\[ \Omega(T, \mu, \phi) = \Omega^0 - N^1 \cdot (e\phi) - \frac{1}{2} \frac{\partial N^0}{\partial \mu} (e\phi)^2 - \frac{1}{2}C\phi^2 \] (35)

where the arguments of the thermodynamic quantities on the right side are \((T, \mu)\). Calculating the minimum of (35), we find
\[ \Omega_C = \Omega^0 + \Delta \Omega_C \]  

where [8]

\[ \Delta \Omega_C(T, \mu) = \frac{1}{2} \frac{1}{C/e^2 + sD_F} \left( N^1(T, \mu) \right)^2 \]  

In the context of the Thomas-Fermi theory, one may consider the replacement \( C/e^2 \rightarrow C/e^2 + sD_F \) as a manifestation of screening.

In the limit where the charging energy \( E_C \) is much larger than the mean level distance, that is

\[ E_C = \frac{e^2}{2C} \gg \frac{1}{sD_F} = \Delta \]  

we arrive at

\[ \Delta \Omega_C(T, \mu) = \frac{1}{2} \Delta \left( N^1(T, \mu) \right)^2 \]  

which agrees exactly with \( \Delta F \) of Eq. (23) if the identity \( N^1 = \delta N \) – see Eq. (22) – is taken into account.

The physics which has lead us to Eq. (38) has been called in Ref. [7] global charge neutrality. We wish to add that it is quite legitimate – if not obvious – to consider the mesoscopic sample connected to an electron reservoir. Of course, differences in the work function between reservoir and sample – which occur in a real situation – may require a proper redefinition of the typical Fermi level \( \mu^0 \).

**IV. PINNING OF THE FERMI ENERGY TO A SINGLE PARTICLE LEVEL**

The theory which we will put down below is based on a statistical concept of the filling factors of a Fermi system in the presence of disorder. At \( T = 0 \), that is, in the ground state, all single particle levels are fully occupied which satisfy \( \epsilon_{\lambda} \leq \epsilon_F \); the remaining levels \( (\epsilon_{\lambda} > \epsilon_F) \) are unoccupied. Of course, the Fermi energy \( \epsilon_F \) has to be chosen such that the particle number \( N \) which follows from this choice, satisfies the specifications.
At that point one may ask: what are the specifications? In a mesoscopic sample of size, say $10^3$ atomic distances, is there the number of electrons exactly $10^9$, or perhaps $10^9 + 10^2$ ($10^0, 10^1, 10^3, ...$)? Physical considerations suggest that the result should not depend on such details.

Therefore, we propose to average with respect to the particle number in a reasonable window. A convenient way to realize such an ensemble will be to select Fermi-energies at random such that

$$\epsilon_F = \epsilon_{\tilde{\lambda}} + \delta$$

(39)

where $\epsilon_{\tilde{\lambda}}$ may be any level in a reasonable range (which will be specified below); and where $\delta$ is a positive infinitesimal [9]. The above choice implies that the levels $\epsilon_{\lambda} \leq \epsilon_{\tilde{\lambda}}$ only are occupied; and that the remaining ones are empty. We will call this construction the Fermi-level pinning ensemble (FLPE).

At that point, one may ask: what is the difference then of this ensemble and the grand canonical ensemble? The answer is implicitly given by arguments found in Ref. [1]. Accordingly, the grand canonical ensemble (GCE at $T = 0$) may be characterized by a random selection of the chemical potential. This implies that the transition from the last occupied level to the first unoccupied one will preferably take place when the energetic distance is large. In contrast to it, the ensemble (FLPE) introduced above guarantees that such a transition occurs between any pair of levels with equal probability.

Again, we start from the expression (1) for the grand canonical ensemble; however, we wish to add that only the limit $T \to 0$ (ground state) is, strictly speaking, consistent with the argumentation above. Inserting relation (39), we obtain the following relation for the “pinned” grand canonical potential

$$\Omega_P = \sum_{\lambda} (-T)\ell n \left[ 1 + e^{-(\epsilon_{\lambda} - \epsilon_{\tilde{\lambda}} - \delta)/T} \right]$$

(40)

As a rule, the average properties of a Fermi-system do not depend on the absolute values of the single particle energies. Therefore, we remove the arbitrariness in the selection of the
pinning level $\epsilon_\lambda$ by sampling with a weight function $P(\epsilon_\lambda)$ which is centered at the typical Fermi energy, that is, $\epsilon_\lambda = \mu^0$ (see Eq. (15)) and which has a support much smaller than $\mu^0$ but much larger than the mean level spacing $\Delta$. Thus, we obtain for the average potential

$$< \Omega_P > = \frac{\sum_\lambda P(\epsilon_\lambda)}{\sum_\lambda \langle \epsilon_\lambda \rangle} < \sum_\lambda P(\epsilon_\lambda)(-T)\ln \left[ 1 + e^{-(\epsilon_\lambda - \epsilon_\lambda - \delta)/T} \right]$$

(41)

Note that we have taken the average separately for numerator and denominator. One may justify this procedure in view of the assumed properties of the weight function $P(\epsilon_\lambda)$. Nevertheless, this procedure may give rise to delicate questions since the properties we are interested in may depend on one electron only.

Inserting the definition (2) of the density of states, and observing the relations (7) and (24), we obtain

$$< \Omega_P > = \frac{s}{\int dE R_1(\bar{E})P(\bar{E})} \int dE d\bar{E} P(\bar{E}) \tilde{R}_2(E, \bar{E})(-T)\ln \left[ 1 + e^{-(E-\bar{E}-\delta)/T} \right]$$

$$= \Omega^0 + < \Delta \Omega_P >$$

(42)

where $\Omega^0$ is defined by Eq. (8) and where

$$< \Delta \Omega_P > = \frac{-s}{\int dE R_1(\bar{E})P(\bar{E})} \int dE d\bar{E} P(\bar{E}) \tilde{Y}_2(E, \bar{E})(-T)\ln \left[ 1 + e^{-(E-\bar{E}-\delta)/T} \right]$$

(43a)

By and large, the $E$ integration above collects contributions from a large range of order Fermi energy. Therefore, we may not insert at once the translational invariant form (26) for the correlator. On the other hand, the $\bar{E}$ integration covers only the small range $| \bar{E} - \mu^0 | \sim \Delta$ and there, we may assume an appropriate form of translational invariance. Thus, we may write

$$< \Delta \Omega_P > = \frac{-s}{D_F} \int dE d\bar{E} P(\bar{E}) \tilde{Y}_2(E, \bar{E}, \mu^0 - \delta)(-T)\ln \left[ 1 + e^{-(E-\mu^0)/T} \right]$$

$$= \frac{-s}{D_F} \int dE \tilde{Y}_2(E, \mu^0 + \delta)(-T)\ln \left[ 1 + e^{-(E-\mu^0)/T} \right]$$

(43b)

Note that for $T \to 0$, we have $(-T)\ln [1 + e^{-(E-\mu^0)/T}] \to \theta(\mu^0 - E)(\mu^0 - E)$. Therefore, the small quantity $\delta$ is irrelevant. Furthermore, since we are interested only in quantum effects, which occur at low energies (“phase sensitive contributions”), we may now make use of the
translation invariant form (26). Introducing the integration variable \( u = E - \mu^0 \) we arrive at

\[
< \Delta \Omega_P > = -\Delta s^2 \int du \tilde{Y}_2(u)(-T)\ell n \left[ 1 + e^{-u/T} \right] \quad (44)
\]

Surprisingly, we recognize that

\[
< \Delta \Omega_P(T = 0) > = -\Delta s^2 \int^0_0 du u \tilde{Y}_2(u) \quad (45)
\]

is just twice as large as \( < \Delta F(T = 0) > \) as given by Eqs. (23) and (28). This apparent discrepancy can be understood as follows. We recall that essentially one is interested in the dependence of \( \Omega = \Omega(y) \) on an external parameter \( y \). Such a dependence is given when the single particle energies \( \epsilon_\lambda = \epsilon_\lambda(y) \) also depend on this parameter. Thus

\[
\frac{\partial \Omega}{\partial y} = \sum_\lambda \frac{\partial \epsilon_\lambda}{\partial y} f(\epsilon_\lambda - \mu) =: X(y) \quad (46)
\]

Clearly, the \( y \)-dependence will also show up in the correlation functions. Therefore, we define

\[
\tilde{\gamma}_2(E, \bar{y}; E, y) = -< \mathcal{D}^1(E, \bar{y})\mathcal{D}^1(E, y) > \quad (47)
\]

where

\[
\mathcal{D}(E, y) = \frac{1}{s} \sum_\lambda \delta(E - \epsilon_\lambda(y)) \quad (48)
\]

and \( \mathcal{D}^1 = \mathcal{D} - \mathcal{D}^0 \). (We repeat that \( \mathcal{D}^0 = < \mathcal{D} > \) is not expected to depend on \( y \) in an essential way.) For energies not far from the Fermi energy, the correlator will be stationary with respect to energy

\[
\tilde{\gamma}_2(\bar{E}, \bar{y}; E, y) = \tilde{\gamma}_2(E - \bar{E}; \bar{y}, y) = \tilde{\gamma}_2(E - \bar{E}; y, \bar{y}) \quad (49)
\]

and thus, we may generalize Eq. (44) as follows

\[
< \Delta \Omega_P(y, \bar{y}) > = -\Delta s^2 \int du \tilde{Y}_2(u; y, \bar{y})(-T)\ell n \left[ 1 + e^{-u/T} \right] \quad (50)
\]
According to what has been said above, the quantity we are interested in is

\[ < X(y) > = \left. \frac{\partial}{\partial y} < \Omega_P(y, \bar{y}) > \right|_{\bar{y} = y} \]  

(51)

Now, if \( \Omega_P \) were symmetric in \( y, y' \), we would be able to write

\[ < X(y) > = \frac{1}{2} \frac{\partial}{\partial y} < \Omega_P(y, y) > \]  

(52)

and to remove thus the discrepancy, at least as far as physically measurable quantities were concerned.

Comparing the structure of \( < \Delta \Omega_P > \) and of \( < \delta N(\bar{y}) \delta N(y) > \), one recognizes that such a symmetry exists at \( T = 0 \). For \( T \neq 0 \), one finds such a symmetry for \( \tilde{Y}_2(E - \bar{E}; y, \bar{y}) \) in the case of a metal ring threaded by a flux (persistent currents; see appendix A). Of course, one should always keep in mind that both procedures (Legendre transformation and pinning of the Fermi-level) have a sound basis only for \( T = 0 \).

In conclusion, we have to point out one peculiarity of the Fermi-level pinning ensemble. We should not expect any change in the final result if the selection \( \epsilon_F = \epsilon_{\lambda}(y) + \delta \) of Eq. (39) is replaced by \( \epsilon_F = \epsilon_{\lambda}(\bar{y}) + \delta \). This expectation is supported by the fact that the energy levels as a function of \( y \) do not cross in disordered (“chaotic”) systems. Therefore, the set of levels defined by \( P(\epsilon_{\lambda}(y)) \gtrsim 0 \) should be the same as \( P(\epsilon_{\lambda}(\bar{y})) \) provided that the weight function \( P \) is sufficiently broad on a range \( | \epsilon_{\lambda}(y) - \epsilon_{\lambda}(\bar{y}) | \sim \Delta \). However, one finds in the case of persistent currents such chances (see appendix A). Presently, we cannot understand this dependence on such a detail in the preparation of the Fermi-level pinning ensemble.

For sake of completeness, we wish to mention in this context that Kamenev and Gefen [10] have discussed a gedanken experiment, where the system is prepared in a grand canonical state at \( \bar{y} \) whence it is transferred adiabatically to the parameter value \( y \) (see the discussion in appendix A).
V. PHOTOABSORPTION WITH FERMI LEVEL PINNED

In what follows, we will discuss the consequences of the ansatz (39) when inserted into relation (6) for the photoabsorption. In a first step, we replace $µ = \epsilon_\chi + \delta$ in the argument of the Fermi functions. Next, we take an average with respect to the pinning levels by sampling with the weight function $P(\epsilon_\chi)$. Thus, we obtain the following expression for the photoabsorption

$$\sigma = \sigma_0 \hbar^2 \omega_s \int dE dE' d\bar{E} D(E) D(E') D(\bar{E}) P(\bar{E})$$

$$\times \left[ f(E - \bar{E} - \delta) - f(E' - \bar{E} - \delta) \right] \delta(\hbar \omega - E' + E)$$

(53)

Considering the disorder average $<\sigma>$, we argue as previously, that it is possible to average numerator and denominator separately. In this process of averaging, the three-level correlator appears

$$\tilde{R}_3(E_1, E_2, E_3) = <D(E_1) D(E_2) D(E_3)>$$

(54)

appears. Since we are interested only in energies close to the typical Fermi energy $\mu^0$, the correlator $R_3$ may be considered to be invariant with respect to a translation along the energy axis. Thus, we obtain

$$<\sigma> = \sigma_0 \hbar^2 \omega D_F \int dEdE' \tilde{R}_3(E, E', \mu - \delta)$$

$$\times \left[ f(E - \mu^0) - f(E' - \mu^0) \right] \delta(\hbar \omega - E' + E)$$

(55)

For sake of simplicity, we take the limit $T \to 0$ where the Fermi function becomes a step function. In this case

$$<\sigma> = \sigma_0 \hbar^2 \omega D_F \int_{-\hbar \omega}^{0} du \tilde{R}_3(u + \mu^0, u + \hbar \omega + \mu^0, \mu^0 - \delta)$$

(56)

If we were to disregard correlations, $R_3 \to D_F^3$, we would recover Eq. (10).

It will now be necessary to study some general properties of this correlation function. Firstly, we define the three level cumulant as follows
\[ \tilde{Y}_3(E_1, E_2, E_3) = \tilde{R}_3(E_1, E_2, E_3) \]
\[ + R_1(E_1) \tilde{Y}_2(E_2, E_3) \]
\[ + R_1(E_2) \tilde{Y}_2(E_1, E_3) \]
\[ + R_1(E_3) \tilde{Y}_2(E_1, E_2) \]
\[ - R_1(E_1) R_1(E_2) R_1(E_3) \]  
(57)

Again, we expect that for large separation of the energies the correlations vanish; hence
\[ \int dE_3 \tilde{Y}_3(E_1, E_2, E_3) = 0 \]  
(58)

It is possible to find definite forms for the correlation function within the random matrix theory [11,12]. In our notation, we follow most closely the article of Bohigas [12].

However, there is one feature, the autocorrelation namely, which does not seem to have received sufficient attention in the past. For instance, consider Eq. (24a) and insert definition (4). One obtains
\[ \tilde{R}_2(E, E') = R_2(E, E') + \delta(E - E') R_1(E) \]
\[ R_2(E, E') = \frac{1}{\delta^2} \sum_{\lambda \neq \lambda'} < \delta(E - E_\lambda) \delta(E - E_{\lambda'}) > \]  
(59)

Correspondingly, Eq. (24b) is of the form
\[ \tilde{Y}_2(E, E') = -\tilde{R}_2(E, E') + R_1(E) R_1(E') \]
\[ = Y_2(E, E') - \delta(E - E') R_1(E) \]  
(60)

such that
\[ \int dE' Y_2(E, E') = R_1(E) \]  
(61)

Autocorrelations play an important role in photoabsorption. In thermodynamics, however, they are irrelevant as one may convince oneself by inspection of Eq. (28) and Eq. (44).

For energies close to the center of the energy band, that is, for energies close to the Fermi level
\[E = E - \mu^0\]
\[E' = E' - \mu^0\]
\[\{\} \simeq 0\]

we have
\[\mathcal{R}_1(E \simeq \mu^0) = \mathcal{R}_1(E \simeq 0) = \mathcal{D}_F\]

Furthermore
\[\tilde{Y}_2(E, E') = \tilde{Y}_2(E - E')\]
\[\tilde{Y}_2(E - E') = Y_2(E - E') - \delta(E - E')R_1(0)\]

where the following rules of notation have been introduced: Correlators with script letters: arbitrary energies \(E_i\); correlators with roman letters: small energies \(\mathcal{E}_i\). In this limit, the correlators are invariant with respect to translations. Correlators with (without) tilde: with (without) autocorrelations.

For reasons of simplicity, the following relations pertain to the Gaussian unitary ensemble (GUE) of the random matrix theory. There, one has
\[Y_2(E - E') = R^2_1(0)s^2(x - x')\]

where
\[s(x) = \frac{\sin \pi x}{\pi x}\]
\[x_i = \frac{\mathcal{E}_i}{s\Delta}\]

We convince ourselves by explicit calculation that
\[\int dE' \ Y_2(E - E') = R_1(0)\]

as it should be.

A detailed inspection shows that the three level cumulant of Eq. (57)
\[\tilde{Y}_3(E_1, E_2, E_3) = \tilde{Y}_3(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)\]
comprises the following $\delta$-function contributions

$$\tilde{Y}_3(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = Y_3(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$$

$$-\delta(\mathcal{E}_1 - \mathcal{E}_2)\tilde{Y}_2(\mathcal{E}_2 - \mathcal{E}_3)$$

$$-\delta(\mathcal{E}_2 - \mathcal{E}_3)\tilde{Y}_2(\mathcal{E}_3 - \mathcal{E}_1)$$

$$-\delta(\mathcal{E}_3 - \mathcal{E}_1)\tilde{Y}_2(\mathcal{E}_1 - \mathcal{E}_2)$$

$$-2\delta(\mathcal{E}_1 - \mathcal{E}_2)\delta(\mathcal{E}_2 - \mathcal{E}_3)R_1(0) \tag{67b}$$

Using the above decomposition, we conclude that Eq. (58) assumes the form

$$\int d\mathcal{E}_3 \ Y_3(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = 2Y_2(\mathcal{E}_1 - \mathcal{E}_2) \tag{68}$$

For the GUE specifically, we have

$$Y_3(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = R_3^3(0)2s(x_1 - x_2)s(x_2 - x_3)s(x_3 - x_1) \tag{69}$$

For a control, we calculate

$$\int dx_3 s(x_2 - x_3)s(x_3 - x_1) = s(x_1 - x_2) \tag{70}$$

and thus, we find Eq. (68) confirmed.

In accordance with the notation introduced above, we put

$$\tilde{R}_3(E_1, E_2, E_3) = \tilde{R}_3(E_1, E_2, E_3) \tag{71a}$$

and find the explicit form

$$\tilde{R}_3(E_1, E_2, E_3) = Y_3(E_1, E_2, E_3)$$

$$-R_1(0)\left\{Y_2(E_2 - E_3) + Y_2(E_1 - E_3) + Y_2(E_1 - E_2)\right\}$$

$$+\delta(E_1 - E_2)\left[R_1^2(0) - Y_2(E_2 - E_3)\right]$$

$$+\delta(E_2 - E_3)\left[R_1^2(0) - Y_2(E_3 - E_1)\right]$$

$$+\delta(E_3 - E_1)\left[R_1^2(0) - Y_2(E_1 - E_2)\right]$$

$$+R_1(0)\delta(E_1 - E_2)\delta(E_2 - E_3) + R_3^3(0) \tag{71b}$$
We insert this form in the relation (56) for photoabsorption. Since \( < \sigma > \) is an even function of \( \omega \), we take \( \omega > 0 \). Firstly, we recognize that the contribution with a product of two \( \delta \)-function in Eq. (71b) drops out. Next, we also recognize that the infinitesimal \( \delta \) is important for an exact definition of the contributions, which the remaining \( \delta \)-functions collect from the end point of the integration interval.

Using the specific forms of Eq. (65) and Eq. (69) and normalizing the cross-section \( < \sigma > \) to the uncorrelated one of Eq. (10), we may put the result in the following form

\[
< \sigma >_{uc} = \frac{2}{x} s(x) \int_{-x/2}^{x/2} dy \left[ s(y + x/2) s(y - x/2) \right] - s^2(x) + \frac{2}{x^2} \int_{-x}^{x} dy \ s^2(y) + 1 \tag{72a}
\]

where

\[
x = \frac{\hbar \omega}{s\Delta} \tag{72b}
\]

For small and large frequencies, the above relation reduces to

\[
< \sigma >_{uc} = \begin{cases} 
\frac{(2\pi^2/3)\hbar \omega/s\Delta}{1 + s\Delta/\hbar \omega} & \hbar \omega << \Delta \\
\frac{(\pi^2/3) - (\pi^4/30)(\hbar \omega/s\Delta)^2}{1 + s\Delta/\hbar \omega} & \hbar \omega >> \Delta 
\end{cases} \tag{73}
\]

By numerical integration, we have calculated the cross-section for intermediate frequencies. The result is shown in Fig. 1.

The same type of analysis can also be done for the Gaussian Orthogonal Ensemble (GOE) of the random matrix theory. This is outlined in appendix B. In this case

\[
< \sigma >_{uc} = \begin{cases} 
\frac{(\pi^2/3) - (\pi^4/30)(\hbar \omega/s\Delta)^2}{1 + s\Delta/\hbar \omega} & \hbar \omega << \Delta \\
1 + s\Delta/\hbar \omega & \hbar \omega >> \Delta 
\end{cases} \tag{74}
\]

For intermediate frequencies, the cross-section is found by numerical integration; for a graphical representation see Fig. 1.

The limiting forms (73) and (74) are in agreement with Shklovskii’s [1] conclusions.
VI. STOCHASTIC FLUCTUATIONS

Up till now, it seems to be an accepted procedure in mesoscopic physics to calculate stochastic fluctuations within the grand canonical ensemble. Thus, one calculates the connected correlator of the grand canonical potentials (see also Eq. (47))

\[ \langle \Omega(T, \mu, y)\Omega(T, \mu, y') \rangle_c = \langle \Omega \Omega' \rangle - \langle \Omega \rangle \langle \Omega' \rangle \]

\[ = -s^2 \int dEdE' \hat{Y}_2(E, y; E', y')T^2 \left( \ln \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ln \left[ 1 + e^{-(E'-\mu)/T} \right] \right) \quad (75) \]

According to Eq. (46), the correlator \( \langle X(y)X(y') \rangle_c \) for the physical quantity \( X(y) \) can be obtained by operating with \( \partial^2/\partial y \partial y' \) on Eq. (75).

As an illustration, such a calculation is presented in appendix A for the case of persistent currents. One obtains for the phase sensitive part of \( \langle \Omega \Omega' \rangle_c \) by order of magnitude \( \sim E_c \), where \( E_c = D/L^2 \) is the Thouless energy in the diffusive limit. Note that \( E_c \gg \Delta \) in experiments of the usual type.

However, with the results and with the insight obtained in the discussions of the preceding sections, we may quite well ask what the stochastic fluctuations are when calculated for the Fermi level pinning ensemble. Considering Eq. (40), we write down in a first step

\[ \Omega_P(y, \bar{y})\Omega_P(y', \bar{y}) = \sum_{\lambda \lambda'} T^2 \left( \ln \left[ 1 + e^{-(\epsilon_{\lambda}(y)+\delta)/T} \right] \right) \left( \ln \left[ 1 + e^{-(\epsilon_{\lambda'}(y')-\epsilon_{\lambda}(y)+\delta)/T} \right] \right) \quad (76) \]

Next, we perform an average of \( \epsilon_\lambda \) in a reasonable energy range on the basis of the sampling function \( P(\epsilon_\lambda) \). As previously, we claim that it is possible to take the disorder average for numerator and denominator separately. Thus, we obtain

\[ \langle \Omega_P(y, \bar{y})\Omega_P(y', \bar{y}) \rangle = \int \frac{dE dE' d\bar{E} d\bar{E}' \hat{P}(\bar{E}) \hat{D}(E, y) \hat{D}(E', y') \hat{D}(\bar{E}, \bar{y})}{\hat{D}_F} \left( \ln \left[ 1 + e^{-(E-\bar{E})/T} \right] \right) \left( \ln \left[ 1 + e^{-(E'-\bar{E})/T} \right] \right) \quad (77) \]

As far as the energy integration is concerned, we wish to recall the argumentation in connection with Eq. (43). Accordingly, we write (typical Fermi-energy \( \mu_0 \to \mu \)
\[ < \Omega_P(y, \bar{y}) \Omega_P(y', \bar{y'}) > = \frac{s^2}{D_F} \int dEdE' < \mathcal{D}(E, y) \mathcal{D}(E', y') \mathcal{D}(\mu + \delta, \bar{y}) > \]
\[ \times T^2 \left( \ell n \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ell n \left[ 1 + e^{-(E'-\mu)/T} \right] \right) \]  

(78)

Since \[ < \Omega(\mu, y) \Omega(\mu, y') > \] can be obtained rather easily, it is advantageous to calculate the difference
\[ < \Omega_P(y, \bar{y}) \Omega_P(y', \bar{y'}) > - < \Omega(\mu, y) \Omega(\mu, y') > \]
\[ = s^2 \int dEdE' \left\{ \frac{1}{D_F} \tilde{R}_3(E, y; E', y'; \mu + \delta, \bar{y}) - \tilde{R}_2(E, y; E', y') \right\} \]
\[ \times T^2 \left( \ell n \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ell n \left[ 1 + e^{-(E'-\mu)/T} \right] \right) \]  

(79)

For a definition of \( \tilde{R}_2 \) and \( \tilde{R}_3 \), see Eqs. (24) and (54), respectively. As far as the difference of the correlators in the curly brackets is concerned, we make use of Eq. (57) and obtain
\[ \frac{1}{D_F} \tilde{R}_3(E, y; E', y'; \mu + \delta, \bar{y}) - \tilde{R}_2(E, y; E', y') \]
\[ = \frac{1}{D_F} \tilde{Y}_3(E, y; E', y'; \mu + \delta, \bar{y}) \]
\[ - \frac{1}{D_F} \tilde{Y}_2(E', y'; \mu + \delta, \bar{y}) - \frac{1}{D_F} (E') \tilde{Y}_2(E, y; \mu + \delta, \bar{y}) \]  

(80)

As emphasized repeatedly, we are considering systems where \( R_1(E) = \langle \mathcal{D}(E, y) \rangle \) does not depend on the external parameter \( y \).

We insert the last term of Eq. (80) in Eq. (79) and find
\[ -\frac{s^2}{D_F} \int dEdE' \mathcal{R}_1(E') \tilde{Y}_2(E, y; \mu - \delta, \bar{y}) T^2 \left( \ell n \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ell n \left[ 1 + e^{-(E'-\mu)/T} \right] \right) \]
\[ = \Omega^0(\mu) < \Delta \Omega_P(y, \bar{y}) > \]  

(81)

The last line follows from Eqs. (8), (43) and (50). Similarly, the second last term of Eq. (8) contributes with \( \Omega^0(\mu) < \Delta \Omega_P(y', \bar{y'}) \).

At this, we recall that ultimately, we are interested in the correlator of the physical quantity \( X(y) \). This means that we should calculate
\[ < X(y)X(y') >_{P, \bar{y}} = \frac{\partial^2}{\partial y \partial y'} < \Omega_P(y, \bar{y}) \Omega_P(y', \bar{y}) > \]  

(82)
(The subscript \(P\) indicates how the average is taken and \(\bar{y}\) is the parameter value at which the Fermi level pinning ensemble has been prepared.) It is now important to realize that terms of the type (81) do not contribute to the correlator (82).

For sake of transparent formulae, we will omit terms of the type (81); in the following equalities are meant to be relations of equivalence in the sense of Eq. (82). Therefore, we may write

\[
\begin{align*}
\langle \Omega_P(y, \bar{y})\Omega_P(y', \bar{y}) \rangle - \langle \Omega(\mu, y)\Omega(\mu, y') \rangle &= \frac{s^2}{D_F} \int dE dE' \tilde{Y}_3 \left( E, y; E', y'; \mu + \delta, \bar{y} \right) T^2 \left( \ell n \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ell n \left[ 1 + e^{-(E'-\mu)/T} \right] \right) \\
&= \left( \frac{s^2}{D_F} \right) \int dE dE' \tilde{Y}_3 \left( E, y; E', y'; \mu + \delta, \bar{y} \right) T^2 \left( \ell n \left[ 1 + e^{-(E-\mu)/T} \right] \right) \left( \ell n \left[ 1 + e^{-(E'-\mu)/T} \right] \right)
\end{align*}
\]  

(83)

Note that for \(T \to 0\), the last factors in the integrand are \(\epsilon E \theta(-\epsilon')\theta(-\epsilon')\) where \(\epsilon, \epsilon'\) are energies measured from the Fermi level as defined in Eq. (62). Clearly, large values of \(\epsilon, \epsilon'\) are important in the integral (83), and a simple RMT theory fails if not a prescription how to extract phase sensitive contribution is supplied. As an alternative, one may resort to the diagrammatic theory and to the cooperon-diffuson expansion. A preliminary analysis suggests that the expression (83) is of the order \(E_c^2 \left( L/\ell \right)^2 (\Delta^2/\gamma E_c)\) where \(L\) is the circumference and \(\ell\) the mean free path of the metallic ring. Furthermore, \(\gamma\) the rate at which phase coherence is destroyed. Since \(\langle \Omega(\mu, y)\Omega(\mu, y') \rangle \sim E_c^2\), the estimate above indicates that in metals of short mean free path the stochastic fluctuations may be considerably larger within the Fermi level pinning ensemble, that is, when calculated for a fixed number of electrons as compared with calculations for fluctuating number of electrons.

**VII. DISCUSSION**

We recall the discussion on the existence and on the size of persistent currents in mesoscopic metallic rings threaded by a magnetic flux \(\phi\) which took place just a few years ago [2]. It has been found that for non-interacting electrons, the persistent currents are exponentially small [13] when calculated for the grand canonical ensemble (GCE) in contrast to calculations for the canonical ensemble (CE). There one finds persistent currents, (when
expressed in terms of energies) of order \( \Delta \), that is, the mean spacing of the electronic level. Nevertheless, even this result is small and it is presumably correct to say that the size of the persistent currents depends on just one electron.

In such a case, it is necessary to have reliable information on the distribution of the electronic levels \( \epsilon_\lambda \). A basic assumption is that atomic disorder or fluctuations in the geometric definitions exist in all samples which are prepared identically on a mesoscopic scale. Correlation functions (with respect to this disorder), of these levels can be calculated in an approximation by a diagrammatic method [14] or in the random matrix theory [11,12]. A comprehensive approach is provided by the supersymmetry technique [15], but technically, it is not so easy to handle.

Two main subjects are discussed in this paper: (A) Thermodynamics; (B) Dynamics; (C) Stochastic Fluctuations.

Concerning (A), there is no need to emphasize, that for systems of identical particles, a theory can be formulated and carried through most elegantly in the grand canonical ensemble. Thus, there is a possibility (i) to extract from this information thus obtained the canonical ensemble by a Legendre transformation. In a second procedure (ii), one starts from the idea of a Coulomb blockade where the charging energy \( e^2/2C \) (\( C \) is the capacitance of the mesoscopic sample) fixes the number of electrons to be equal to the (effective) number of ions provided that \( e^2/2C \gg \Delta \). A third possibility (iii) is based on the concept of pinning the Fermi level to a single particle level. In this context, one argues that effectively the grand canonical ensemble can be understood as a random selection of Fermi energies which prefers configurations where the energy separation between last occupied and first empty state is large. In contrast to it, the Fermi level pinning ensemble provides an unbiased choice.

By and large we have found that (at \( T = 0 \)) the three methods do lead to the same result as far as the thermodynamics is concerned. One open question remains. In terms of persistent current: Is there a difference in the persistent currents of a flux \( \phi \) when the system has been prepared at a flux \( \bar{\phi} \neq \phi \)? Some comments can be found at the end of appendix A. In this context, the following problem (which at first sight seems to be none at all),
should also be considered. We recall Eq. (15) where \( \mu^0 \) is defined by 
\[- \partial \Omega^0 / \partial \mu \bigg|_{\mu = \mu^0} = N.\]
A relation of similar structure is Eq. (34). The question is: how is it possible to find \( \mu^0 \)
such that \( N \) is an integer? Clearly, within the present formalism, we cannot guarantee \( N \)
to be an integer. On the other hand, such a detail may be very important since we have
convinced ourselves that it may be only one electron which contributes to the phenomenon
we are interested in.

As an example of dynamics (B), we have studied the disorder average cross section
\( < \sigma(\omega) > \) for the photoabsorption of a mesoscopic sample as proposed in Ref. [1]. According
to a qualitative analysis of Shklovskii [1], the cross section \( < \sigma(\omega) > \) is reduced by a factor
\( (\hbar \omega / \Delta) \) (\( \ll 1 \) for small frequencies) as compared with a naive calculation in the grand
canonical ensemble. We have calculated \( < \sigma(\omega) > \) using the concept (iii) of Fermi level
pinning. There, we have obtained agreement with Shklovskii’s ideas within the random
matrix theory. Furthermore, the present theory allows us to cover all frequencies.

It seems to be tempting to calculate photoabsorption within the Coulomb blockade con-
cept (ii). However, in a naive treatment, correlation functions of very high order (\( \rightarrow \infty \))
seems to be required; a situation which calls for a more detailed discussion. In this context,
we wish to mention also an approach to this dynamic problem which has been put forward
by Kamenev et al. [16], which agrees with the limiting form proposed by Shklovskii [1].

Stochastic fluctuations (C) seem to be very important in mesoscopic physics. Though the
average response may be small, there are in general large sample to sample fluctuations of this
response. As a simple measure of these fluctuations, the root-mean-square of the response is
most useful. Usually, the r.m.s is calculated within the grand canonical ensemble. We have
studied this quantity for Fermi level pinning ensemble (iii). In the case of persistent currents,
we have outlined an estimate which indicates a considerable enhancement is possible in the
fluctuations of persistent currents, when calculated for fixed number of electrons.

At the end, we wish to draw attention to measurements where the magnetic moment
of (singly connected) mesoscopic samples has been measured by Lévy et al. [17]. Again,
the question of differences between canonical and grand canonical ensemble arises. In the
presence of strong disorder (diffusive limit), the paper by Altshuler et al. [18], seems to provide a complete answer. On the other hand, if there is only weak atomic disorder, the problem is more complicated [19]. A new physical situation arises when samples are free from intrinsic defects (Aharonov-Bohm ballistic billiards), and where disorder appears only in the form of differences in the geometric definitions of the samples. This situation has been discussed by Ullmo et al. [20]. We, together with Yu. N. Ovchinnikov, are also working on this problem, analytically as well as numerically. It seems to be a very difficult problem; one could say, that it means to catch the one active electron among the million inert ones.

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APPENDIX A:

According to Altshuler and Shklovskii [14], the two level correlation functions can be expanded in two types of impurity ladder diagrams that are called diffusons and cooperons. For this expansion to be valid, a condition has to be satisfied which is that the energetic separation of the two levels to be considered has to be much larger than the mean level separation

$$|E - E'| > > \Delta$$

(A1)

In the framework of this diagramatic theory, one obtains a two-diffuson contribution which is as follows

$$\langle \mathcal{D}^1(E)\mathcal{D}^1(E') \rangle^{2D} = \frac{1}{2\pi^2} \sum_q \text{Re} \frac{1}{[-i(E - E') + \gamma + Dq^2]^2}$$

(A2)
In the relation above, $D\vec{q}^2$ represents the eigenvalues of the diffusion operator $-D\nabla^2$ for the sample in consideration and $\gamma$ means the rate at which phase coherence is destroyed. In case of time reversal symmetry, the two-cooperon contributions $<\mathcal{D}^1(E)\mathcal{D}^1(E')>^{2C}$ is the same as expression (A2).

We call a sample to be effectively of zero dimension if the zero eigenvalue of $-D\nabla^2$ is the only important one. In this case

$$<\mathcal{D}^1(E)\mathcal{D}^1(E')>^{2D} = \frac{1}{2\pi^2} \text{Re} \frac{1}{(E - E' + i\gamma)^2}$$

(A3)

According to the definition (24), the expression (A3) has to be considered as the two-diffuson approximation of the two-level correlator $-\tilde{Y}_2(E, E')$. We also observe that this approximation satisfies the sum rule (25). Furthermore, it agrees with the GUE result (65) of the random matrix theory for large energy separation $|E - E'| >> \Delta; \gamma$, if we replace there the fast oscillatory term $\sin^2 \pi x$ by its average value $\frac{1}{2}$. For small energies, however, the agreement is bad.

As mentioned above, the two-cooperon contribution is the same for a system with perfect time reversal symmetry, $\tilde{Y}_2^{2C} = \tilde{Y}_2^{2D}$. On the other hand, if time reversal symmetry is completely lost, the two-cooperon contribution is zero, $\tilde{Y}_2^{2C} = 0$. In the random matrix theory, the two cases correspond to GOE and GUE, respectively.

Despite its deficiency, the cooperon-diffuson expansion is useful since it can easily be generalized to one and higher dimensional samples. In the discussion of persistent currents in metallic rings threaded by a magnetic flux $\phi$, one considers frequently an effectively one-dimensional closed loop of length $L$; as a consequence, the eigenvalues $D\vec{q}^a$ of the diffusion operator can be found by replacing

$$\vec{q} \rightarrow \frac{2\pi}{L}(n + \varphi)$$

(A4a)

where $\varphi = 0$ for diffusons whereas for cooperons

$$\varphi = \frac{\phi}{\phi_0} ; \quad \phi_0 = \frac{2\pi hc}{e}$$

(A4b)
We remark that the ansatz (A4) neglects any penetration of the magnetic field into the area of the metallic ring. Therefore, the cooperon contribution remains oscillating even for very large flux and no transition to the GUE ensemble of random matrix theory will take place.

Presently, we are interested in the thermodynamics only in so far as the flux dependence is concerned. Hence, only the two-cooperon contribution $\tilde{Y}_2^{2C}$ matters. Considering Eqs. (23) and (28), we recognize that the $\delta$-function contribution to $\tilde{Y}_2$ is irrelevant. We will also see later, that neither the $\eta$-regularisation of the low energy dependence nor the phase breaking rate $\gamma$ is not essential in the present problem. Thus, we may write

$$Y_2^{2C}(u; \varphi) = \frac{1}{2\pi^2} \sum_n \text{Re} \frac{1}{[u + iD(2\pi/L)^2(n + \varphi)]^2}$$

Note that $Y_2^{2C}$ is an even function of $\varphi$; moreover, it is as periodic in $\varphi$ with period 1. Therefore, a Fourier expansion is appropriate and we put

$$Y_2^{2C}(u; \varphi) = \sum_m C_m(u) e^{2\pi im \varphi}$$

$$C_m(u) = \int_{-\infty}^{+\infty} d\varphi e^{-2\pi im \varphi} \frac{1}{2\pi^2 \text{Re} [u + iD(2\pi/L)^2 \varphi^2]^2}$$

Explicitly, one finds

$$C_m(u) = \frac{1}{8\pi^2 E_c^2} \text{Re} \left[ \frac{e^{-i\pi/4}}{v^{3/2}} - i \left| \frac{m}{v} \right| e^{i|m|\alpha(v)} \right]$$

where

$$E_c = D/L^2$$

is the Thouless energy and where

$$v = u/E_c$$

$$\alpha(v) = \pi/2 - (\pi/4)\text{sgn}v$$

Next, we consider Eq. (28) and calculate the cooperon ($\varphi = 2\phi/\phi_0$) contribution
\[ < (\delta N(\phi))^2 > = \sum_m B_m e^{2\pi im2\phi/\phi_0} \]

\[ B_m = s^2 \int_0^\infty du \ u C_m(u) = \frac{s^2}{2\pi^2} \left| \frac{1}{m} \right| \]

We observe that the divergence of \( C_m(u) \propto u^{-3/2} \) is irrelevant in the integration and also, that the integral converges for large values of \( u \) if \( m \neq 0 \). This last point may be considered as a demonstration of the rule that only low energies (here: \( |u| \leq E_c \)) contribute to phase sensitive quantities.

For sake of completeness, we give the expression for the phase sensitive part of the free energy. Inserting expression (A8) in Eq. (28), we have

\[ < \Delta F > = \sum_{m=1}^\infty \frac{s^2}{2\pi^2} \Delta \cos \frac{2\pi m 2\phi}{\phi_0} \]

Since the persistent current \( I(\phi) \) can be calculated according to

\[ I(\phi) = -c \frac{\partial}{\partial \phi} < \Delta F > \]

we obtain a result that can be expressed as follows

\[ I(\phi) =: J(\phi) \]

\[ J(\phi) = \sum_{m=1}^\infty \frac{s^2}{2\pi^2} \cdot \frac{2e}{\hbar} \cdot \Delta \frac{m}{2\pi m} \frac{2\phi}{\phi_0} \]

Following the argumentation which lead to Eq. (47), we also define a two-level correlator which depends on the two fluxes \( \phi \) and \( \bar{\phi} \). In this case, cooperons as well as diffusons contribute to the phase sensitive quantities; and we obtain the appropriate correlators if we substitute \( \varphi = (\phi + \bar{\phi})/\phi_0 \) for cooperons and \( \varphi = (\phi - \bar{\phi})/\phi_0 \) for diffusons in expression (A6). (Note the symmetry \( \phi \leftrightarrow \bar{\phi} \).) From these relations, it follows that

\[ < \Delta \Omega P(T = 0; \phi, \bar{\phi}) > = \Delta < \delta N(\phi)\delta N(\bar{\phi}) > \]

\[ = \sum_{m=1}^\infty \frac{s^2}{\pi^2 m} \left[ \cos \frac{2\pi m \phi + \bar{\phi}}{\phi_0} + \cos \frac{2\pi m \phi - \bar{\phi}}{\phi} \right] \]

Evidently,
\[ I_P(\phi, \bar{\phi}) = -c \frac{\partial}{\partial \phi} \Delta \Omega_P(\phi, \bar{\phi}) > = \mathcal{J} \left( \frac{1}{2} (\phi + \bar{\phi}) \right) + \mathcal{J} \left( \frac{1}{2} (\phi - \bar{\phi}) \right) \] (A12)

As expected, the above result agrees with Eq. (A10) only if \( \bar{\phi} = \phi \).

Next, we comment the paper of Kamenev and Gefen [10]. They consider an experiment where the system is prepared in a grand canonical state at the flux \( \bar{\phi} \) and then transferred adiabatically to a state with flux \( \phi \). Their result for the persistent current can be expressed as follows

\[ I(\phi, \bar{\phi}) = -c \frac{1}{2} \Delta < (\delta N(\phi) - \delta N(\bar{\phi}))^2 > = \mathcal{J}(\phi) - \mathcal{J} \left( \frac{1}{2} (\phi + \bar{\phi}) \right) - \mathcal{J} \left( \frac{1}{2} (\phi - \bar{\phi}) \right) \] (A13)

In conclusion, we note the interesting relation

\[ I_P(\phi, \bar{\phi}) + I(\phi, \bar{\phi}) = \mathcal{J}(\phi) \] (A14)

which we do not understand presently in physical terms.

As far as the problem of stochastic fluctuations is concerned, it is not difficult to calculate the connected part \( < \Omega(\phi)\Omega(\phi') >_c \) of the correlator for the grand canonical potential in the diffuson-cooperon approximation outlined above (see also the standard results for noninteracting electrons quoted in Ref. [3]). Accordingly, one finds for the grand canonical ensemble at \( T = 0 \), the following result

\[ < \Omega(\phi)\Omega(\phi') >_c = \sum_{m=1}^{\infty} \frac{96}{\pi^2 m^5} E_c^2 \cos \frac{2\pi m \phi}{\phi_0} \cos \frac{2\pi m \phi'}{\phi_0} \] (A15)

Note that the fluctuations \( \sim E_c \) are much larger than the mean values \( \sim \Delta \).

**APPENDIX B:**

We outline our calculation of the disorder averaged the normalized cross-section \( \Sigma \) for the GOE. Making use of Eqs. (71b) and (72b), we obtain the relation

30
\[
\frac{<\sigma>}{<\sigma>^{\infty}} = \frac{1}{x} \int_{-x}^{0} dy \ Y_3(y, y + x, 0) \\
-\frac{1}{x} \int_{-x}^{x} dy \ Y_2(y) + \left(1 + \frac{2}{x}\right) (1 - Y_2(x))
\] (B1)

In the RMT, the \( n \) point cumulant function is given by the expression

\[
Y_n(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n) = \frac{1}{2} \text{Tr} \sum_p (\sigma(x_{12})\sigma(x_{23})\sigma(x_{n1})),
\] (B2)

where \( x_{ij} = x_i - x_j \) and \( \sum_p \) denotes the sum over the subsets of the symmetric group with respect to the group of cycle permutation. We note that in the present problems, these permutations produce analytic forms which are identical.

The matrices \( \sigma(x) \) are given by

\[
\sigma(x) = \begin{pmatrix}
    s(x) & D(x) \\
    J(x) & s(x)
\end{pmatrix}
= \begin{pmatrix}
    s(x) & \frac{\partial}{\partial x} s(x) \\
    \int_0^x s(t) dt - \frac{1}{2} \text{sgn}(x) & s(x)
\end{pmatrix},
\] (B3)

where \( s(x) \) is found in Eq. (65b). For the two point cumulant we get

\[
Y_x(x) = s^2(x) - D(x) J(x)
\] (B4)

The three point cumulant can be calculated according to

\[
Y_3(y, y + x, 0) = 2s(x)s(y + x)s(y) \\
+s(x)D(y + x)J(y) + s(x)J(y + x)D(y) \\
+D(x)s(y + x)J(y) + D(x)J(y + x)s(y) \\
+J(x)D(y + x)s(y) + J(x)s(y + x)D(y).
\] (B5)

We insert these quantities in equation (B1) and after some manipulations, we obtain the limiting cases, Eq. (74). Numerical calculation have been done accordingly.
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FIGURES

FIG. 1. Average cross section $< \sigma >$ of photoabsorption for an ensemble of mesoscopic samples, as a function of the photoenergy $\hbar \omega$, normalized to the mean level distance $\Delta$ (spin degeneracy $s=2$). The cross section is given in the form of the ratio $< \sigma > / < \sigma >^{uc}$, where $< \sigma >^{uc} \propto \omega^2$ is the uncorrelated quantity. Dotted line: Gaussian orthogonal ensemble; dashed line: Gaussian unitary ensemble (GOE and GUE, respectively, of random matrix theory).
Figure 1.