AN EXISTENCE THEORY FOR NONLINEAR EQUATIONS ON METRIC GRAPHS VIA ENERGY METHODS

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Abstract. The purpose of this paper is to develop a general existence theory for constrained minimization problems for functionals defined on function spaces on metric measure spaces \((M, d, \mu)\). We apply this theory to functionals defined on metric graphs \(G\), in particular \(L^2\)-constrained minimization problems of the form

\[
E(u) = \frac{1}{2} a(u, u) - \frac{1}{q} \int_K |u|^q \, dx,
\]

where \(q > 2\) and \(a(\cdot, \cdot)\) is a suitable symmetric, sesquilinear form on some function space on \(G\) and \(K \subseteq G\) is given. We show how the existence of solutions can be obtained via decomposition methods using spectral properties of the operator \(A\) associated with the form \(a(\cdot, \cdot)\) and discuss the spectral quantities involved. An example that we consider is the higher-order variant of the stationary NLS (nonlinear Schrödinger) energy functional with potential \(V \in L^2 + L^\infty(G)\)

\[
E^{(k)}(u) = \frac{1}{2} \int_G |u^{(k)}|^2 + V(x)|u|^2 \, dx - \frac{1}{p} \int_K |u|^q \, dx
\]

defined on a class of higher-order Sobolev spaces \(H^k(G)\) that we introduce. When \(K\) is a bounded subgraph, one has localized nonlinearities, which we treat as a special case. When \(k = 1\) we also consider metric graphs with infinite edge set as well as magnetic potentials. Then the operator \(A\) associated to the linear form is a Schrödinger operator, and in the \(L^2\)-subcritical case \(2 < q < 6\), we obtain generalizations of existence results for the NLS functional as for instance obtained by Adami, Serra and Tilli [JFA 271 (2016), 201-223], and Cacciapuoti, Finco and Noja [Nonlinearity 30 (2017), 3271–3303], among others.

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1. Introduction

In recent years, there has been a growth of interest in functionals on metric graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the stationary NLS (Nonlinear Schrödinger) energy functional

$$E_{\text{NLS}}(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 \, dx - \frac{\mu}{q} \int_{\mathcal{G}} |u|^q \, dx, \quad u \in H^1(\mathcal{G}), \quad \|u\|^2_{L^2} = 1, \quad q > 2, \quad \mu > 0$$

and associated ground states of the stationary NLS energy functional, i.e. minimizers for the constrained minimization problem

$$E_{\text{NLS}}(\mathcal{G}) := \inf_{\|u\|^2_{L^2} = 1} E_{\text{NLS}}(u, \mathcal{G}), \quad 2 < q < 6.$$ 

Such minimizers are solutions to the stationary nonlinear Schrödinger equation on $\mathcal{G}$ given by

\[
\begin{cases}
-u'' + \lambda u = \mu |u|^{q-2} u & \text{edgewise,} \\
\text{u is continuous on } \mathcal{G} \text{ and satisfies the Kirchhoff condition} \\
\sum_{e \in \mathcal{E} : e \succ v} \frac{\partial u}{\partial \nu}(v) = 0, & \forall v \in \mathcal{V},
\end{cases}
\]

where we recall that $e \succ v$ denotes the relation that the edge $e$ is adjacent to the vertex $v \in \mathcal{V}$ and $\frac{\partial u}{\partial \nu}|_e(v)$ denotes the inward pointing derivative at $v$ towards the interior of the edge $e$. While in the simplest case of the real line, existence of minimizers in (1.2) can be deduced by standard techniques, on general noncompact graphs existence results are not as easy to obtain due to the lack of a concept of translation invariance. In [AST15] it was shown on the one hand that under certain topological configurations the problem does not admit a minimizer; on the other, in a later paper the same authors derive an existence principle based on a comparison inequality:

**Theorem 1.1** ([AST16]). Let $\mathcal{G}$ be a noncompact metric graph with finitely many edges and $2 < q < 6$. Assume

$$E_{\text{NLS}}(\mathcal{G}) < E_{\text{NLS}}(\mathbb{R}),$$

then there exists a minimizer for $E_{\text{NLS}}(\mathcal{G})$.

This result can be used to obtain existence results on concrete graphs $\mathcal{G}$ via construction of so called competitors, i.e. test functions $u \in H^1(\mathcal{G})$ for which $E_{\text{NLS}}(u, \mathcal{G}) < E_{\text{NLS}}(\mathbb{R})$. This allows to deduce existence of minimizers in certain situations as shown in [Ten16] and [AST17].

A variant of this problem with potential was considered in [CFN17] and [Cac18], where the energy functional was given by

$$E_{\text{V,NLS}}(u) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 + V|u|^2 \, dx - \frac{\mu}{q} \int_{\mathcal{G}} |u|^q \, dx, \quad \|u\|^2_{L^2} = 1.$$ 

In [CFN17] the existence of minimizers of (1.4) was related to the existence of eigenvalues of the Schrödinger operator $-\Delta + V$ below the essential spectrum:

**Theorem 1.2** ([CFN17]). Let $\mathcal{G}$ be a noncompact metric graph with finitely many edges and $V \in L^1 + L^\infty(\mathcal{G})$ with $V_- \in L^r(\mathcal{G})$ for $r \in [1, 1 + \frac{2}{q-2}]$ and $2 < q \leq 6$. Assume

$$\inf \sigma(-\Delta + V) < \inf \sigma_{\text{ess}}(-\Delta + V).$$
Then there exists $\mu^* > 0$ such that for $\mu \in (0, \mu^*)$ the functional (1.4) is bounded below and the associated constrained minimization problem

$$E_{\text{NLS}}^V := \inf_{u \in H^1(G)} E_{\text{NLS}}^V(u)$$

admits a minimizer.

**Theorem 1.3** ([Cac18]). Let $G$ be a noncompact metric graph with finitely many edges and $V \in L^1 + L^\infty(G)$ satisfying the assumptions in Theorem 1.2. Let

$$\Sigma_0 := \inf \sigma(-\Delta + V) < 0, \quad \gamma_q := \inf_{u \in H^1(G)} \frac{1}{2} \int_\mathbb{R} |u'|^2 + V|u|^2 \, dx - \frac{1}{q} \int_G |u|^q \, dx < 0.$$

Then we have existence of minimizers of

$$E_{\text{NLS}}^V = \inf_{u \in H^1(G)} E_{\text{NLS}}^V(u)$$

for $0 < \mu \leq \left( \frac{\Sigma_0}{\gamma_q} \right)^{\frac{q}{2} - \frac{4}{q}}$.

Our goal in this paper is threefold. Firstly, we develop a general existence theory in a far more abstract setting which can be applied to a variety of problems as for example $E_{\text{NLS}}$ and $E_{\text{NLS}}^V$, but which is not limited to metric graphs. For example, the existence theory may be also applied to functionals defined on function spaces on combinatorial graphs or general domains in $\mathbb{R}^n$. We then use this existence theory to obtain generalizations of the results in [AST16] and [CFN17] by considering more general graphs, higher-order derivatives in the functionals, as well as different variants of the problems. We will also discuss the spectral quantities appearing in (1.5), which replaces the inequality (1.3) in Theorem 1.1 for the existence results. Variants we will consider include the case of decaying potentials and localized nonlinearities, i.e. we replace the set of integration in the term corresponding to the nonlinearity by a bounded subgraph $K \subset G$. Unlike [AST16] our proof of Theorem 1.1 does not involve rearrangement inequalities, which allows us to consider potentials and deduce a similar existence result in the case of decaying potentials.

Let us be now more precise about the abstract setting we will consider. Let $(\mathcal{M}, d, \mu)$ be a nonempty metric measure space. Assume $p \in [1, \infty]$ and let $X(\mathcal{M}) \subset L^p(\mathcal{M})$ be a Banach space continuously and locally compactly imbedded in $L^p(\mathcal{M})$, i.e. for any relatively compact, connected subset $K$, the restriction $X(K)$ is compactly imbedded in $L^p(K)$. In the case of metric graphs a prototype would be $H^1(G)$, but we will also apply this to higher-order Sobolev spaces $H^k(G)$ with $k \in \mathbb{N}$. We will consider functionals $E \in C(X(\mathcal{M}), \mathbb{R})$ with $E(0) = 0$ for which the mapping

$$t \mapsto E_t := \inf_{u \in X(\mathcal{M})} E(u)$$

is continuous for $t \geq 0$. It is easy to verify that these properties are satisfied by (1.1) and (1.4). Our first result is a dichotomy result for a class of such functionals:

**Theorem 1.4** (Dichotomy). Let $c > 0$ and $(\mathcal{M}, d, \mu)$ be a metric measure space. Let $X(\mathcal{M})$ and $E$ be as above. Assume additionally

(1) $E$ is strictly subadditive, i.e.

$$E_{t_1 + t_2} < E_{t_1} + E_{t_2}, \quad \forall t_1, t_2 > 0,$$
(2) $E$ is weak limit superadditive, i.e. for any weakly convergent sequence $u_n \rightharpoonup u$ in $X(M)$ there exists a subsequence such that

$$\limsup_{n \to \infty} E(u_n) \geq E(u) + \limsup_{n \to \infty} E(u_n - u)$$

If there exists a weakly convergent minimizing sequence $u_n \rightharpoonup u$ in $X(M)$ for

(1.6) $$E_c = \inf_{u \in X(M)} E(u),$$

then either

(i) $u = 0$ or
(ii) $u_n \to u$ in $L^p(M)$ and $u \neq 0$ is a minimizer of $E_c$.

Observe that under our assumptions, if $u_n \rightharpoonup 0$ in $X(M)$, then

$$u_n \to 0 \quad \text{on } L^p_{\text{loc}}(G).$$

For functionals as considered in Theorem 1.4 this means that either minimizing sequences of (1.6) vanish to infinity, or converge towards minimizers in an appropriate sense. In the first case, we will say that minimizing sequences are vanishing. In this case one cannot deduce existence of minimizers without any additional information. This gives rise to our second result, which gives a sufficient condition for the existence of minimizers. Since the mass of vanishing sequences moves outside of any precompact set of $M$ it is natural to separate the supports of the functions $u_n$ into an expanding part $O_n^{(1)}$ around some fixed precompact set and a part $O_n^{(2)}$ retreating to infinity via sequences of partitions of unity

$$\{\Psi_{O_n^{(1)}}, \Psi_{O_n^{(2)}}\}.$$

If we then require that such a sequence of partitions of unity does not increase the energy functional $E$ under decomposition for vanishing sequences $u_n \rightharpoonup 0$ in Theorem 1.4, i.e.

$$\limsup_{n \to \infty} E(v_n) \geq \limsup_{n \to \infty} E(\Psi_{O_n^{(1)}v_n}) + \limsup_{n \to \infty} E(\Psi_{O_n^{(2)}v_n}),$$

then we say $E$ is superadditive with respect to this vanishing-compatible sequence of partitions of unity and obtain the existence principle:

**Theorem 1.5.** Let $c > 0$ and $K$ be a precompact subset of $M$. Assume $E \in C(X(M), \mathbb{R})$ is

- strictly subadditive,
- weak limit superadditive,
- superadditive with respect to a vanishing-compatible sequence of partitions of unity, and

$$E_c < \tilde{E}_c := \lim_{n \to \infty} \inf_{u \in X(M)} E(u),$$

where $K_n := \{x \in G|d(x, K) < n\}$ is the expanding ball around $K$. Assume there exists a weakly convergent minimizing sequence $u_n \rightharpoonup u$, then $u \neq 0$ is a minimizer of $E_c$. 

It will turn out that the functionals (1.1) and (1.4) as considered in [AST16] and [CFN17] satisfy the prerequisites of this theory. In fact, we will apply Theorem 1.5 to a natural generalization of (1.4), namely the higher-order stationary NLS energy functional

\[
E^{(k)}(u) = \frac{1}{2} \int_G |u^{(k)}|^2 + V|u|^2 \, dx - \frac{\mu}{q} \int_G |u|^q \, dx, \quad \mu > 0, \quad 2 < q < 4k + 2, \quad V \in L^2 + L^\infty(G)
\]

and consider the ground state problem

\[
E^{(k)} = \inf_{u \in H^k(G)} \frac{1}{2} \int_G |u^{(k)}|^2 + V|u|^2 \, dx - \frac{\mu}{q} \int_G |u|^q \, dx.
\]

Here, we write \(V \in L^2 + L^\infty\) to mean that \(V\) admits a decomposition \(V = V_2 + V_\infty\) such that \(V_2 \in L^2(G)\) and \(V_\infty \in L^\infty(G)\). When \(k = 1\) the energy functional (1.7) reduces to the stationary NLS energy functional and we derive conditions for which the theory is applicable. Minimizers of (1.8) satisfy the stationary higher-order nonlinear Schrödinger equation

\[
\begin{align*}
(-1)^k u^{(2k)} + (V + \lambda) u &= \mu |u|^{q-2} u, \quad \forall e \in E \\
u^{(i)} &\in C(G) \quad \text{for all } i \leq 2k - 1 \text{ even} \quad \text{(Continuity)} \\
\sum_{e : e \succ v} u^{(k)}(v) &= 0 \quad \forall i \leq 2k - 1 \text{ odd} \quad \forall v \in V \\
&(\text{Kirchhoff condition}).
\end{align*}
\]

While to the best of our knowledge this functional has not yet been considered on metric graphs, the stationary higher-order nonlinear Schrödinger equation on the real line of 4th order is for instance related to traveling wave solutions of the nonlinear higher-order Schrödinger equation for the pulse envelope with higher-order dispersion as shown in [Kru19, §II]. For combinatorial locally finite graphs a discussion on the existence of solutions of the nonlinear higher-order Schrödinger equation of 4th order was for instance considered very recently in [HSZ19].

A minor difficulty in defining (1.8) is that one needs to define higher-order Sobolev spaces \(H^k(G)\), as to date no standard way to define these spaces has emerged. We will define them in such a way that the formal Poly-Laplacean

\[
A = (-\Delta)^k + V
\]

\(D(A) = H^{2k}(G)\) is a self-adjoint operator on \(L^2(G)\). We remark that the choice is not necessarily unique. A discussion of self-adjoint realizations for the Bilaplacian on metric graphs can be for instance found in [GM17]. One major result on the existence of minimizers in (1.8) is as follows:

**Theorem 1.6.** Let \(G\) be a noncompact metric graph with finitely many edges. Assume that either

(i) there exists \(V = V_2 + V_\infty\) such that \(V_2 \in L^2(G)\) and \(V_\infty \in L^\infty(G)\) and \(V_\infty(x) \to 0 \quad (x \to \infty)\)

on all edges of infinite length, or
(ii) $A = (-\Delta)^k + V$ admits a ground state, i.e. $\inf \sigma(A)$ is an eigenvalue. Then $E^{(k)}$ is strictly subadditive, and if additionally

\begin{equation}
E^{(k)} < \tilde{E}^{(k)} := \lim_{n \to \infty} \inf_{u \in H^k(G)} E^{(k)}(u),
\end{equation}

then $E^{(k)}$ admits a minimizer.

Analogously to (1.2) and (1.4), we will refer to the minimizers as ground states. Theorem 1.6 generalizes Theorem 1.1 since (1.1) satisfies the prerequisites of Theorem 1.6. Indeed, one can show with a test function argument (see Example 3.16) that if $G$ is a metric graph with finitely many edges then

\begin{equation}
E_{\text{NLS}}(u, G) = E_{\text{NLS}}(u) = \inf_{u \in H^1(G)} \frac{1}{2} \| u \|^2_{L^2} + V|u|^2 dx - \mu \int_K |u|^q dx, \quad \| u \|^2_{L^2} = 1,
\end{equation}

and we recover Theorem 1.1.

Under the assumption that eigenvalues exist below the essential spectrum, i.e.

\begin{equation}
\inf \sigma((-\Delta)^k + V) < \inf \sigma_{\text{ess}}((-\Delta)^k + V),
\end{equation}

by a perturbation argument one can ensure that (1.10) is satisfied for small nonlinearities and deduce a generalization of Theorem 1.2:

**Theorem 1.7.** Let $G$ be a noncompact metric graph with finite edge set. If

\begin{equation}
\inf \sigma((-\Delta)^k + V) < \inf \sigma_{\text{ess}}((-\Delta)^k + V)
\end{equation}

then (1.7) admits a ground state for sufficiently small $\mu > 0$.

For $G = \mathbb{R}$ and $V \equiv 0$, due to translation invariance the inequalities in Theorem 1.6 and Theorem 1.7 cannot be satisfied. However, here one can exploit the translation invariance to obtain a similar existence result, for which we are unaware of any reference in the literature:

**Theorem 1.8.** Let $V \equiv 0$ and $G = \mathbb{R}$, then (1.7) admits a ground state for all $\mu > 0$. In particular, there exists a solution $u \in C^\infty(\mathbb{R})$ to the stationary higher-order NLS equation

\begin{equation}
(-1)^k u^{(2k)} + \lambda u = \mu |u_e|^{q-1} u,
\end{equation}

with $\| u \|^2_{L^2(\mathbb{R})} = 1$ for some $\lambda \in \mathbb{R}$.

The results in Theorem 1.1 and Theorem 1.2 were shown for metric graphs with finitely many edges, which we refer to as finite graphs throughout the paper. Such graphs consist of a finite number (possibly zero) of edges of infinite length, i.e. half-lines, which we call rays, and a complement, which is compact, and which we will call the core of the graph. [CFN17], [Cac18] call such graphs starlike graphs (see also Figure 1). Our theory also allows us to handle more general graphs. For the next result we will consider a class of graphs with countable edge set, which is finite when restricted to any precompact subset. We will to refer to such graphs as locally finite graphs in the following. On locally finite graphs we consider the following variant of the NLS energy functional

\begin{equation}
E_{\text{NLS}}^{(k)}(u) = \frac{1}{2} \int_G \left\{ \frac{d}{dx} + M \right\} u^2 + V|u|^2 dx - \frac{\mu}{p} \int_K |u|^q dx, \quad \| u \|^2_{L^2} = 1,
\end{equation}

for $k$ a non-negative integer.
where $K \subseteq G$ is a subgraph of $G$. In this context, we consider the magnetic Schrödinger operator with external potential

$$A^M = \left( i \frac{d}{dx} + M \right)^2 + V$$

with its natural domain of definition, which we describe in detail in §5.1.

The following theorem is an analog of Theorem 1.7. Interestingly, if one considers localized nonlinearities, i.e. $K$ is a bounded subgraph of $G$, then the existence result can be shown independent of the parameter $\mu > 0$ in the nonlinearity:

**Theorem 1.9.** Let $G$ be a noncompact locally finite graph and $K \subseteq G$ a connected subgraph. Suppose $A^M = \left( i \frac{d}{dx} + M \right)^2 + V$ admits a ground state that does not vanish identically on $K$.

(i) If $\inf \sigma(A^M) < \inf \sigma_{ess}(A^M)$, then

$$E_{NLS}^{(K)} := \inf_{u \in H^1(G)} \left\{ \frac{1}{2} \int_G \left( i \frac{d}{dx} + M \right)^2 u^2 + V|u|^2 \, dx - \frac{\mu}{q} \int_K |u|^q \, dx \right\}$$

admits a minimizer for sufficiently small $\mu > 0$.

(ii) If $K$ is a bounded subgraph of $G$, then minimizers exist for all $\mu > 0$.

In §7 we are going to show that for a tree graph $G$ the ground states of Schrödinger operators with magnetic potential do not vanish anywhere on $G$. Then, given a decaying potential $V \in L^2 + L^\infty(G)$ satisfying $V = V_2 + V_\infty$, such that $V_2 \in L^2(G)$ and $V_\infty \in L^\infty(G)$, we show:

**Theorem 1.10.** Let $G$ be a noncompact locally finite tree graph with finitely many vertices of degree 1 and suppose that $V \in L^2 + L^\infty$ satisfies (1.14). Then (1.12)
admits a minimizer if

\[ E_{\text{NLS}}^{(K)} = \inf_{u \in H^1(G), \|u\|_{L^2}^2 = 1} \frac{1}{2} \int_G \left( \frac{i}{dx} + M \right) u^2 + V|u|^2 dx - \frac{\mu}{q} \int_K |u|^q dx < E_{\text{NLS}}(\mathbb{R}). \]

In particular, if

\[ \inf \sigma \left( \left( \frac{i}{dx} + M \right)^2 + V \right) < 0, \]

then we have existence of minimizers of \( E_{\text{NLS}}^{(K)} \) for \( 0 < \mu \leq (\Sigma_0/\gamma q)^{-\frac{q}{2}} \) as in Theorem 1.3.

This paper is organized as follows. In §2 we introduce higher-order Sobolev spaces and obtain inequalities on Sobolev spaces on metric graphs including variants of Sobolev and Gagliardo–Nirenberg inequalities. We also discuss basic properties of these spaces such as density results and a characterization of \( W^{1,\infty} \) via uniformly bounded Lipschitz functions. In §3 we build the existence theory that is the foundation for all of our results. Theorem 1.4 is shown in §3.1 and Theorem 1.5 is shown in §3.2. In §3.3 we develop an existence result for translation-invariant functionals. In §4 we discuss the application of this existence theory to \( E^{(k)} \) on finite graphs and prove Theorem 1.6 and Theorem 1.7. In §3.3 we develop an existence result for translation-invariant functionals. In §4.1 we formalize the problem and show basic properties of the functional. In §4.2 we construct suitable partitions of unity and prove a decomposition formula in §4.3, which we use to show that the existence theory is applicable. In §4.4 we prove Theorem 1.8 and show in §4.5 that the existence theory is applicable for decaying potentials. In §5 we discuss existence results for ground states of \( E_{\text{NLS}}^{(K)} \), where \( K \subseteq \mathcal{G} \), on locally finite graphs. In §6 we discuss the energy inequality that is essential for the existence theory and relate it to spectral estimates by developing a Persson type theory for the operators in (1.9) and (1.13). This will also conclude the proof of Theorem 1.9. In particular, we discuss sufficient conditions for the potential \( V \) such that (1.11) is satisfied. In §7 we finish the paper with an application of the existence results to infinite metric trees via reduction of the problem to one without magnetic potential and prove Theorem 1.10.

Let us finish the introduction by mentioning a few other recent results on related topics. For a general reference on metric graphs we refer to [BK13]. For a broad overview of spectral theory of operators we refer to [RS80]. We refer to [EKMN18] for a recent article on spectral theory for metric graphs with infinitely many edges. The stationary energy functional

\[ E_{\text{NLS}}^{(K)}(u) = \frac{1}{2} \int_G |u'|^2 dx - \frac{\mu}{q} \int_K |u|^q dx, \quad \|u\|_{L^2}^2 = 1. \]

with \( K = \mathcal{G} \) was considered in [ACFN12], [AST15], [AST16], [AST17] among others. A variant of the problem with localized nonlinearities in the \( L^2 \)-subcritical case was considered in [Ten16] and for the \( L^2 \)-critical case extended in [DT18b] and [DT18a], where the domain of integration in the nonlinearity is taken to be a bounded subgraph \( K \). A very recent survey on results on the stationary NLS energy functional with localized nonlinearity can be found in [BCT19]. The main tool for the existence theorems regarding existence of ground states of the stationary NLS energy functional is the constructions of so called competitors, i.e. test functions or a sequence of test functions that establish the energy inequality of Theorem 1.1. Recently, classes of graphs that do not necessarily consist of finitely many edges
have also been considered. For instance, [DST19] deals with a certain class of infinite tree graphs, which fall into the category of the locally finite graphs that we consider here. We would also like to mention the results obtained by [AP19] for the NLS energy functional with growing potentials for a class of general metric graphs satisfying certain volume growth assumptions using a generalized Nehari approach.

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2. Sobolev spaces on graphs

In this section we define metric graphs and define Sobolev spaces on metric graphs and prove Sobolev inequalities on these spaces.

2.1. Metric graphs. Let $G = (V, E)$ be a metric graph, where each edge $e \in E$ is associated with an interval $I_e$ of length $l_e \in (0, \infty]$, where $I_e = [0, \infty)$ if $l_e = \infty$ and $I_e = [0, l_e]$ otherwise. We assume every vertex to be at least of degree one. For every $e \in E$ joining two vertices we associate each of the vertices to $0$ and $l_e < \infty$ respectively on the interval $I_e$. However, we always assume that the half-line $I_e = [0, \infty)$, which we also call a ray, is attached to the remaining part of the graph at $x_e = 0$, and the vertex of the graph corresponding to $x_e = \infty$ is called a vertex at infinity. In particular, there are no edges between vertices at infinity. We denote by $E_\infty \subset E$ the set of all rays.

A connected metric graph $G$ admits a natural structure of a metric space. The metric is the shortest distance measured along the edges of the graph. A more detailed introduction to metric graphs can be found for instance in [BK13].

We consider two classes of noncompact graphs:

**Definition 2.1.** Let $G$ be a connected metric graph. Then we say

1. $G$ is a **finite graph** if there are at most finitely many edges.
2. $G$ is a **locally finite graph** if $E$ is a countable set such that any bounded subset of the graph intersects at most finitely many edges

We will always take our graphs to be connected, locally finite noncompact metric graphs. Note that a finite graph $G$ is compact if and only if the graph $G$ does not admit any rays, that is, there are no edges of $G$ that are half-lines. Note that a locally finite graph is compact, if and only if the graph is bounded. We see immediately that all compact locally finite metric graphs are finite, and all finite graphs are locally finite. In particular for compact graphs the notions in Definition 2.1 coincide.

2.2. First-order Sobolev spaces. Let $G$ be any locally finite graph. Here and in the rest of the article denote

$$u_e := u|_{I_e}.$$
We denote by $C(G)$ the set of continuous complex-valued functions in $G$ and define for $1 \leq p < \infty$

\[
L^p(G) = \left\{ u \in \bigoplus_{e \in E} L^p(I_e) \ \bigg\| u \bigg\|_p := \sum_{e \in E} \| u_e \|_p < \infty \right\},
\]

\[
W^{1,p}(G) = \left\{ u \in C(G) \ \bigg| \ u_e \in W^{1,p}(I_e) \wedge \| u \|_{W^{1,p}} := \sum_{e \in E} \| u_e \|_{W^{1,p}(I_e)} < \infty \right\}.
\]

Then we set $H^1(G) = W^{1,2}(G)$ as usual.

For $q = \infty$ we need to adapt the definition above slightly:

\[
L^\infty(G) = \left\{ u \in \bigoplus_{e \in E} L^\infty(I_e) \ \bigg\| u \bigg\|_\infty := \max_{e \in E} \| u_e \|_\infty < \infty \right\},
\]

\[
W^{1,\infty}(G) = \left\{ u \in C(G) \ \bigg| \ u_e \in W^{1,\infty}(I_e) \wedge \| u \|_{W^{1,\infty}} := \max_{e \in E} \| u_e \|_{W^{1,\infty}(I_e)} < \infty \right\}.
\]

2.3. Gagliardo–Nirenberg inequality with magnetic potential. The goal of this section is to show a Gagliardo–Nirenberg inequality for locally finite, connected metric graphs. With future applications in mind we actually consider a modified Gagliardo–Nirenberg inequality:

**Proposition 2.2.** Let $G$ be a locally finite, connected metric graph and $M \in C(G)$. For $p \in [2, \infty)$ there exists a constant $C > 0$ independent of $M$ such that

\[
(2.1) \quad \| u \|_p^p \leq C \left\| \left( i \frac{d}{dx} + M \right) u \right\|_2^{\frac{p-2}{2}} \| u \|_2^{\frac{p+2}{2}},
\]

for all $u \in H^1(G)$.

**Remark 2.3.** The inequality reduces to the usual Gagliardo–Nirenberg inequality when $M \equiv 0$.

**Proof of Proposition 2.2.** Suppose $G$ is a tree graph at first. Then using the unitary gauge transform $G : H^1(G) \to H^1(G)$ (see also §7 for details) we deduce that (2.1) is equivalent to

\[
\| u \|_p^p \leq C \left\| u' \right\|_2^{\frac{p-2}{2}} \| u \|_2^{\frac{p+2}{2}},
\]

which can be shown via symmetrization methods as shown in [ASTL15]. In particular, the constant $C > 0$ can be chosen independent of $M$. Cutting the graph at a discrete set of points on the metric graph, i.e. we can find a tree graph $\tilde{G}$ such that identifying a discrete set of points on the graph results in a graph isometric isomorph to $G$. Moreover, since $H^1(G)$ imbeds isometrically into $H^1(\tilde{G})$, (2.1) also holds for $H^1(\tilde{G})$ and the constant $C > 0$ can be chosen independent of $M \in C(G)$. □

Similarly, one can also show the following Sobolev inequality.

\[1\text{This was shown for finite graphs, but the proof can be simply adapted since we consider locally finite graphs.}\]
Proposition 2.4. Let $\mathcal{G}$ be a locally finite, connected metric graph and $M \in C(\mathcal{G})$. Let $p \in [2, \infty]$ then there exists a constant $C > 0$ independent of $M$ such that

$$\|u\|_p \leq C \left( \int_{\mathcal{G}} \left( i \frac{d}{dx} + M \right) u \right)^2 dx + \int_{\mathcal{G}} |u|^2 dx \right)^{1/2},$$

for all $u \in H^1(\mathcal{G})$.

Proof. The aproach is similar as before, indeed we can use the known result in absence of $M$ and use a gauge transform to show that (2.2) holds with a constant $C > 0$ independent of the potential $M \in C(\mathcal{G})$. □

2.4. Higher-order Sobolev spaces. In this section we introduce the notion of higher-order Sobolev spaces on graphs for $p \in [1, \infty)$. Let $\mathcal{G}$ be a locally finite metric graph. One naive way of doing so is simply defining it analogously as in $W^{1,p}(\mathcal{G})$

$$\widetilde{W}^{k,p}(\mathcal{G}) := \left\{ u \in C(\mathcal{G}) \big| u_e \in W^{k-1}(I_e) \quad \forall e \in E \quad \text{and} \quad \|u\|_{W^{k,p}} := \sum_{e \in E} \|u_e\|_{W^{k,p}(I_e)}^p < \infty \right\}.$$

Then for $u \in \widetilde{W}^{k,p}(\mathcal{G})$ we always have $u_e \in C^{k-1}(I_e)$ for all $e \in E$. However, we also want to specify a condition on the higher-order derivatives. We define

$$W^{k,p}(\mathcal{G}) = \{ u \in \widetilde{W}^{k,p}(\mathcal{G}) \big| u^{(j)} \in C(\mathcal{G}) \quad \forall j \leq k - 1 \text{ even} \quad \text{and} \quad \sum_{e \ni v} \frac{\partial^j}{\partial \nu^j} u_e(v) = 0 \quad \forall j \leq k - 1 \text{ odd} \quad \forall v \in V \},$$

where $e : e \succ v$ denotes the set of edges $e$ adjacent to a vertex $v$. This definition is natural in the sense that if we consider a dummy vertex $\hat{v}$ of degree 2, i.e. subdividing an edge $e \in \mathcal{E}$ connecting two vertices $v_1, v_2$ into two edges $e_1, e_2$ connecting $v_1, \hat{v}$ and $\hat{v}, v_2$ respectively such that the total length of the graph is preserved, then the Kirchhoff condition simply reduces to a continuity statement of the derivatives. As usual we define $\widetilde{H}^k(\mathcal{G}) = \widetilde{W}^{k,2}(\mathcal{G})$ and $H^k(\mathcal{G}) = W^{k,2}(\mathcal{G})$.

Remark 2.5. While the Sobolev spaces as defined here are domains of self-adjoint realizations of differential operators on $L^2(\mathcal{G})$, the definitions are not necessarily canonical. We refer to [GM17] for a discussion on self-adjoint extension of the Bilaplacian, and a discussion for $W^{2,p}$ spaces on graphs.

In this context, we are going to define some useful related spaces:

$$\widetilde{W}_0^{k,p}(\mathcal{G}) := \{ u \in \widetilde{W}^{k,p}(\mathcal{G}) \big| u^{(l)}(v) = 0, \quad \forall 1 \leq l \leq k - 1, \quad \forall v \in V \},$$

$$\widetilde{W}^{k,p}_c(\mathcal{G}) := \left\{ u \in \widetilde{W}^{k,p}(\mathcal{G}) \big| \text{supp}(u) \text{ compact} \right\}.$$
Of special importance will be the following test function spaces:
\[
\tilde{C}^\infty(G) := \bigcap_{k \in \mathbb{N}} \tilde{W}^{k,\infty}(G)
\]
\[
\tilde{C}^\infty_b(G) := \bigcap_{k \in \mathbb{N}} \tilde{W}^{k,\infty}_0(G)
\]
\[
\tilde{C}^\infty_c(G) := \bigcap_{k \in \mathbb{N}} \tilde{W}^{k,\infty}_c(G).
\]

2.5. **Higher-order Gagliardo–Nirenberg inequality for finite metric graphs.**
Let \( \mathcal{G} \) be a finite metric graph. Consider the norm on \( H^k(\mathcal{G}) \) defined as
\[
|u|_{H^k}^2 := \int_\mathcal{G} |u^{(k)}|^2 + |u|^2 \, dx
\]
Then due to the Gagliardo–Nirenberg interpolation inequality on intervals (see e.g. [Leo17, Theorem 7.41]) applied edgewise (2.3)
\[
|u|_{H^k}^2 \leq \|u\|_{H^k}^2 \leq C|u|_{H^k}^2
\]
and we conclude that \( \| \cdot \|_{H^k} \) and \( | \cdot |_{H^k} \) are equivalent norms in \( H^k(\mathcal{G}) \).

**Proposition 2.6.** Let \( \mathcal{G} \) be a finite metric graph. Then
\[
\|u\|_p^p \leq C\|u\|_2^{(2k-1)p+2} 2k |u|_{H^k}^{p-2}
\]

**Proof.** From the Gagliardo–Nirenberg inequality on metric graphs and Gagliardo–Nirenberg interpolation inequality on intervals we compute
\[
\|u\|_p^p \leq C_1 \|u\|_2^{p+1} \|u'\|_2^{p-1} \leq C_k \|u\|_2^{(2k-1)p+2} 2k |u|_{H^k}^{p-2}.
\]

\[\square\]

2.6. **On the density of Sobolev spaces.**

**Proposition 2.7.** Let \( \mathcal{G} \) be a finite, connected metric graph and \( p \in [1, \infty) \), then \( W^{m,p}(\mathcal{G}) \) is dense in \( W^{m,n,p}(\mathcal{G}) \) for \( m \geq n \geq 1 \).

**Remark 2.8.** For \( n = 0 \) this corresponds to the fact that \( W^{m,p} \) is dense in \( L^p \) for \( m \geq 1 \).

**Proof of Proposition 2.7.** It suffices to prove that \( W^{k+1,p}(\mathcal{G}) \) is dense in \( W^{k,p}(\mathcal{G}) \). To this end, let \( u \in W^{k}(\mathcal{G}) \) arbitrary and \( u_n \) be an edgewise approximating sequence in \( \oplus_{e \in \mathcal{E}} C^\infty(I_e) \cap W^{k+1,p}(I_e) \) such that
\[
\sum_{e \in \mathcal{E}} \|u_n - u|_e\|_{W^{k,p}} \leq \frac{1}{2^n}
\]
for all \( n \in \mathbb{N} \). The general idea is to construct sequences \( v_n \in \oplus_{e \in \mathcal{E}} W^{k+1,p}(I_e) \) such that \( u_n + v_n \in W^{k+1,p} \) and
\[
u_n + v_n \xrightarrow{W^{k,p}} u \quad (n \to \infty).
\]
For fixed $v \in V$ and for $n$ satisfying
\[
\frac{2}{n} \leq \min_{e \in E} |I_e|
\]
for all $e \succ v$ and $\hat{k} \in \{0, \ldots, k\}$ we define
\[
v_{n,\hat{k},v}(x)|_e = \begin{cases} 
\frac{c_{n,\hat{k},v}}{k!} x_v^{\hat{k}} (1 - nx_v)^{k+1}, & \text{for } x \in e \text{ with } x_v := \text{dist}(x, v) \leq \frac{1}{n} \\
0, & \text{otherwise.}
\end{cases}
\]
where $c_{n,\hat{k},v}$ is given by
\[
\begin{align*}
\text{for } \hat{k} = 0 : & \quad c_{n,0,v} = u - u_n(0) \\
\text{for } 1 \leq \hat{k} \leq k - 1 : & \quad c_{n,\hat{k},v} = \frac{1}{k} \sum_{\ell=0}^{k-1} (k+1)(-n)^\ell c_{n,\ell,v}.
\end{align*}
\]
We can extend the functions $v_{n,\hat{k},v}$ by zero on the rest of the graph. With the Leibniz rule for $1 \leq \ell \leq k + 1$ we compute
\[
(2.5) 
\]
\[
\sum_{\ell=0}^{k-1} \sum_{\mu=0}^{k} v_{n,\ell,v} \in W^{k+1,p}(I_e)
\]
satisfies $\bar{u}_n + \sum_{\ell=0}^{k-1} \sum_{\mu=0}^{k} v_{n,\ell,v} \in W^{k+1,p}(G)$ since $u_n + \bar{u}_n$ coincides in all $k - 1$ derivatives with $u$ by construction. Indeed, the restrictions of the $k$th derivatives at the vertices are of rank $\leq 2|E|$. Then we can find $c_{n,k,v}$ for all $v \in V$
\[
v_{n,k,v}|_e(x) = \begin{cases} 
\frac{c_{n,k,v}}{k!} x_v^{k+1}, & \text{for } x \in e \text{ with } x_v \leq \min\{n^{-1}, c_{n,k,v}^2\} \\
0, & \text{otherwise.}
\end{cases}
\]
such that
\[
u_n + \bar{u}_n + \sum_{v \in V} v_{n,k,v} \in W^{k+1,p}(G).
\]
By assumption (2.4) we deduce by applying the Sobolev imbedding edgewise
\[
\sum_{e \in E} \|u_n - u\|_e^{C,k-1} \leq \frac{C}{2^n}
\]
for all $n \in \mathbb{N}$ and some $C > 0$ and satisfies by construction
\[
(2.6) \quad (-n)^\ell c_{n,k,v} \to 0 \quad (n \to \infty)
\]
for all $1 \leq \ell \leq k$ and $v \in V$. By a change of variables we then compute for $0 \leq m \leq \ell \leq k$

\[
\frac{1}{n^{\ell-m}} \int_{I_k} x^{k-m}(1-nx_v)^{k+m+1-\ell} \, dx_v = n^{\ell-1-k} \int_{0}^{1} x^{k-m}(1-t)^{k+m+1-\ell} \, dt
\]

\[
c_{n,k,v} \max\{n, c_{n,k,v}^2\} \int_{I_k} x^{k-m}(1 - \max\{n, c_{n,k,v}^2\} x_v)^{k+m+1-\ell} \, dx_v = c_{n,k,v} \min\{n^{-1}, c_{n,k,v}^{-2}\} \int_{0}^{1} x^{k-m}(1-t)^{k+m+1-\ell} \, dt \to 0 \quad (n \to \infty)
\]

and with (2.5) and (2.6) we conclude

\[
\left\| u - \left[ u_n + \tilde{v}_n + \sum_{v \in V} v_{n,k,v} \right] \right\|_{W^{k,p}} \leq \left\| u - u_n \right\|_{W^{k,p}} + \left\| \tilde{v}_n + \sum_{v \in V} v_{n,k,v} \right\|_{W^{k,p}} \to 0 \quad (n \to \infty).
\]

\[
\square
\]

**Lemma 2.9.** Let $\mathcal{G}$ be a locally finite, connected metric graph and $p \in [1, \infty)$. Then $W^{1,p}_c(\mathcal{G}) = \{ u \in W^{1,p}(\mathcal{G}) | \text{supp } u \text{ is bounded} \}$ is dense in $W^{1,p}(\mathcal{G})$.

**Proof.** Let $K$ be a bounded, connected subgraph of $\mathcal{G}$. For $R > 0$ set

\[
K_R := \{ x \in \mathcal{G} | \text{dist}(x, K) < R \}.
\]

We define the cut-off functions $\psi_n$ via

\[
\tilde{\psi}_n := \frac{1}{n} \max\{n, \text{dist}(x, K_n)\}, \quad \psi_n := 1 - \tilde{\psi}_n
\]

For all $u \in W^{1,p}(\mathcal{G})$ one then computes

\[
\limsup_{n \to \infty} \left\| u - \psi_n u \right\|_{W^{1,p}} \leq \limsup_{n \to \infty} \left[ \int_{\mathcal{G} \setminus K_n} \left| \frac{d}{dx} \tilde{\psi}_n u \right|^p \, dx + \int_{\mathcal{G} \setminus K_n} \left| \tilde{\psi}_n u \right|^p \, dx \right] \]

\[
\leq \limsup_{n \to \infty} \left[ \frac{2^p}{n^p} \int_{\mathcal{G} \setminus K_n} |u|^p \, dx + 2^p \int_{\mathcal{G} \setminus K_n} \left| \tilde{\psi}_n u \right|^p \, dx + \int_{\mathcal{G} \setminus K_n} \left| \tilde{\psi}_n u \right|^p \, dx \right] = 0,
\]

where in the equation we used

\[
\int_{\mathcal{G} \setminus K_n} \left| \tilde{\psi}_n u \right|^p \, dx \leq \int_{\mathcal{G} \setminus K_n} |u|^p \, dx \to 0 \quad (n \to \infty).
\]

As such $\psi_n u \to u$ in $W^{1,p}(\mathcal{G})$ as $n \to \infty$. \quad \square

**Proposition 2.10.** Let $\mathcal{G}$ be a locally finite, connected metric graph and $p \in [1, \infty)$. Then $W^{2,p}_c(\mathcal{G}) = \{ u \in W^{2,p}(\mathcal{G}) | \text{supp } u \text{ is bounded} \}$ is dense in $W^{1,p}$. 
We conclude with Proof. Let \( u \in W^{1,p}(\mathcal{G}) \). By Lemma \ref{lem:compact_embedding} we can find a sequence \( u_n \in W^{1,p}_c(\mathcal{G}) \) with \( u_n \to u \) in \( W^{1,p} \). Then by Proposition \ref{prop:approximation} for each \( n \) we find a sequence \( u_{n,m} \in W^{2,p}(\mathcal{G}) \), after extending by zero on the whole graph, converging towards \( u_n \) in \( W^{1,p}(\mathcal{G}) \) as \( m \to \infty \). Then one can construct a sequence in \( W^{2,p}_c(\mathcal{G}) \) converging to \( u \) in \( W^{1,p} \) by a diagonal argument. \( \square \)

2.7. Characterization of \( W^{1,\infty} \). We give a characterization of \( W^{1,\infty} \) on locally finite, connected metric graphs in the following:

**Proposition 2.11.** Let \( \mathcal{G} \) be a locally finite, connected metric graph. Then \( W^{1,\infty}(\mathcal{G}) = C^{0,1}_b(\mathcal{G}) \) is the set of uniformly bounded, Lipschitz continuous functions.

**Proof.** Assume \( u \in W^{1,\infty}(\mathcal{G}) \). Let \( x, y \in \mathcal{G} \) and \( \gamma \) be a path of length \( L(\gamma) \) connecting \( x, y \), parametrized by arc length. In the first step let us assume \( u \in C^1 \) edgewise, then using the continuity of \( u \) we have

\[
|u(x) - u(y)| \leq \int_0^{L(\gamma)} |u'(\gamma)| \, d|\gamma| \leq \max_t |u'(\gamma(t))| L(\gamma).
\]

Due to density this holds also for \( W^{1,\infty}(\mathcal{G}) \). Taking the infimum over all paths connecting \( x, y \) we conclude

\[
|u(x) - u(y)| \leq \|u'\|_\infty \text{dist}(x, y)
\]

and thus \( u \in C^{0,1}_b(\mathcal{G}) \). On the other hand, let \( f \in C^{0,1}_b(\mathcal{G}) \), then

\[
\frac{|u(x) - u(y)|}{\text{dist}(x, y)} \leq L
\]

for some constant \( L > 0 \). On each edge \( e \in \mathcal{E} \) then \( u \in W^{1,\infty}(I_e) \) and \( u' \) exists a.e. and

\[
\|u'\|_\infty \leq L.
\]

We conclude \( u \in W^{1,\infty}(\mathcal{G}) \) since \( u \) is also uniformly bounded by assumption. \( \square \)

3. A General Existence Principle

In this section we derive an existence theory for ground states of functionals as in \([1.7]\) and \([1.12]\). To do so, we derive a more general existence principle for functionals on function spaces on metric measure spaces, which we will apply later to the functionals introduced before to discuss the existence of minimizers.

3.1. A Dichotomy Result. In the following we work with a slightly more abstract space \( X(\mathcal{M}) \), namely a function space on a metric measure space \( \mathcal{M} \).

**Assumption 3.1.** Let \( p \in [1, \infty) \). Let \( (\mathcal{M}, d) \) be a metric space with a locally finite Borel measure \( \mu \) on \( \mathcal{M} \). Assume \( X = X(\mathcal{M}) \subset L^p(\mathcal{M}) \) is a nontrivial Banach function space continuously and locally compactly imbedded in \( L^p(\mathcal{M}) \), i.e. \( \mathcal{M} \) restricted to

\[
K_R(y) := \{ x \in \mathcal{M} : \text{dist}(x, y) \leq R \}
\]

is compactly imbedded in \( L^p(K_R(y)) \) for all \( R > 0 \) and \( y \in \mathcal{M} \).

**Remark 3.2.** Our prototype to satisfy Assumption \ref{ass:general_existence} is \( X(\mathcal{G}) = H^1(\mathcal{G}) \) where \( \mathcal{G} \) is a connected, locally finite metric graph. However, it is for instance also satisfied by \( X(\Omega) = H^1(\Omega) \) for a domain \( \Omega \subset \mathbb{R}^n \) with \( n \in \mathbb{N} \).
Definition 3.3. Let \( p \geq 2 \) and let \( \mathcal{M} \) and \( X = X(\mathcal{M}) \) be as in Assumption 3.1. Let \( E \in C(X(\mathcal{M}), \mathbb{R}) \) and
\[
E_t := \inf_{u \in X(\mathcal{M})} \frac{E(u)}{\|u\|_p^p} = t
\]
be bounded from below for each \( t \geq 0 \) and \( E(0) = 0 \). We say:
(1) \( t \mapsto E_t \) is strictly subadditive if
\[
E_{t_1 + t_2} < E_{t_1} + E_{t_2}, \quad \forall t_1, t_2 > 0.
\]
(2) \( E \) is weak limit superadditive in \( X \) if for all \( c > 0 \) any weakly convergent minimizing sequence \( u_n \rightharpoonup u \) in \( X(\mathcal{M}) \) of \( E_c \) satisfies up to a subsequence
\[
\limsup_{n \to \infty} E(u_n) \geq E(u) + \limsup_{n \to \infty} E(u_n - u).
\]

Theorem 3.4. Let \( p \in [2, \infty) \), \( c > 0 \) and let \( \mathcal{M}, X = X(\mathcal{M}) \) be as in Assumption 3.1. Let \( E \in C(X(\mathcal{M}), \mathbb{R}) \) be a weak limit superadditive functional in \( X \). Let
\[
t \mapsto E_t = \inf_{u \in X(\mathcal{M})} \frac{E(u)}{\|u\|_p^p} = t
\]
be a strictly subadditive, continuous function of \( t \in [0, c] \). Let \( u_n \) be a minimizing sequence of \( E_c \), and assume there exists \( u \in X \) such that up to a subsequence \( u_n \rightharpoonup u \). Then either \( u \equiv 0 \) or \( u_n \to u \) in \( L^p(\mathcal{M}) \) and \( u \neq 0 \) is a minimizer.

Remark 3.5. Theorem 3.4 gives rise to a dichotomy. If the requirements of Theorem 3.4 are satisfied, then a minimizing sequence satisfies either \( u_n \to 0 \) in \( X \) or there exists an \( L^p \) convergent subsequence converging to a minimizer of \( E_c \). Up to a subsequence, this implies pointwise convergence almost everywhere. On the other hand, \( u_n \to 0 \) in \( X(\mathcal{M}) \) implies \( \|u_n\|_{L^p(K)} \to 0 \) on any bounded subset \( K \) of \( \mathcal{G} \), but since \( \|u_n\|_p^p = c \) for all \( n \in \mathbb{N} \) the mass needs to move outside any compact set.

Definition 3.6. In virtue of Theorem 3.4 we say a minimizing sequence of \( E_c \) is vanishing if \( u_n \to 0 \) in \( X \) and non-vanishing otherwise.

Proof of Theorem 3.4. Let \( u_n \rightharpoonup u \) in \( X \) with \( u \neq 0 \). Then since \( u_n \to u \) in \( L^p_{\text{loc}} \) we deduce \( u \neq 0 \) and
\[
c \geq \|u\|_p^p > 0.
\]
Up to a subsequence \( u_n \to u \) pointwise almost everywhere, and from the Brezis-Lieb Lemma (see [BL83]) we conclude
\[
\|u\|_p^p + \limsup_{n \to \infty} \|u_n - u\|_p^p = c.
\]
By weak limit superadditivity and strict subadditivity of \( t \mapsto E_t \) we deduce that up to a subsequence
\[
E_c \geq E(u) + \limsup_{n \to \infty} E(u - u_n)
\geq E\|u\|_p^p + \limsup_{n \to \infty} E\|u - u_n\|_p^p
\geq E\|u\|_p^p + E\limsup_{n \to \infty} \|u_n - u\|_p^p \geq E_c.
\]
where equality is only attained when \( \|u\|_p^p = c \) and \( \limsup_{n \to \infty} \|u_n - u\|_p^p = 0 \). Thus \( \|u\|_p^p = c \) and we conclude
\[
E_c = E(u)
\]
and $u$ is a minimizer of $E_c$. 

3.2. Vanishing sequences and Ionization Energies. As in the previous section we consider $\mathcal{M}$ to be a metric measure space and $X(\mathcal{M}) \subset L^p(\mathcal{M})$ to be a Banach space which is locally compactly imbedded in $L^p(\mathcal{M})$. In the following we want to introduce partitions of unity and therefore assume the following:

**Assumption 3.7.** Let $(\mathcal{M}, d)$ be a metric space with locally finite Borel measure $\mu$ on $\mathcal{M}$. We assume $X(\mathcal{M})$ to be a subspace of the space of $\mu$-measurable functions on $\mathcal{M}$ and $Y(\mathcal{M})$ to be a set of real-valued functions on $\mathcal{M}$ such that $X(\mathcal{M})$ is invariant with respect to multiplication of elements in $Y(\mathcal{M})$.

**Example 3.8.** Let $\mathcal{G}$ be a locally finite metric graph. If $X(\mathcal{G}) = H^1(\mathcal{G})$ and $Y = W^{1,\infty}(\mathcal{G})$, then $X(\mathcal{G}), Y(\mathcal{G})$ are imbedded in $C(\mathcal{G})$ and $X(\mathcal{G})$ is invariant by multiplications of $Y(\mathcal{G})$, i.e. elements in $X(\mathcal{G})$ are invariant by pointwise multiplication of elements in $Y(\mathcal{G})$, and for $f \in W^{1,\infty}(\mathcal{G})$ and $g \in H^1(\mathcal{G})$

$$(fg)' = f'g + g'f.$$ 

In this section we will characterize a property of functionals with regards to partitions of unity on metric spaces.

**Definition 3.9.** Let $Y$ be as in Assumption 3.7. Assume $\cup_{O \in \mathcal{O}} O = \mathcal{G}$ is a locally finite open covering $\mathcal{O}$ of $\mathcal{M}$. Then we say a family of nonnegative functions $\psi_O \in Y(\mathcal{G})$ is a partition of unity on $\mathcal{M}$ if

$$\sup \psi_O \subset O, \ \forall O \in \mathcal{O} \land \bigcup_{O \in \mathcal{O}} \sup \psi_O = \mathcal{G} \land 0 \leq \psi_O \leq 1$$

and

$$\sum_{O \in \mathcal{O}} \psi_O(x) \neq 0 \text{ for all } x \in \mathcal{M}$$

and

$$\Psi_O(x) = 1, \ \forall x \in \sup \psi_O \setminus \bigcup_{\partial \in \mathcal{O} \setminus \{O\}} \sup \psi_{\partial}.$$ 

In the following we define for $R > 0$ the open and closed $R$-neighborhoods of a subset $K \subset \mathcal{M}$ by

$$(3.1) \quad K_R := \{x \in \mathcal{M} | \text{dist}(x, K) < R\}$$

$$K_R := \{x \in \mathcal{M} | \text{dist}(x, K) \leq R\}.$$ 

**Example 3.10.** Let $\mathcal{G}$ be a locally finite graph. Let $K$ be some compact, connected subgraph. A simple example for a partition of unity in $W^{1,\infty}(\mathcal{G}) = C_0^1(\mathcal{G})$ on a locally finite metric graph subordinate to $K_2, \mathcal{G} \setminus K_1$ is given by

$$\psi(x) = \max\{\text{dist}(\mathcal{G} \setminus K_2, x), 1\}, \quad \tilde{\psi}(x) = 1 - \psi.$$ 

In the following we will characterize a property of functionals with regards to partitions of unity.

**Definition 3.11.** Let $k \in \mathbb{N}$ and let $\mathcal{M}$ be a metric space. Let $K$ be a bounded subset of $\mathcal{M}$ and $K_n$ be defined by (3.1) for $n \in \mathbb{N}$. We say a sequence of open coverings $\mathcal{O}_n = \{O_n^{(1)}, \ldots, O_n^{(k)}\}$ consisting of $k$ open subsets (not necessarily connected) is vanishing-compatible, if

$$K_n \cap O_n^{(i)} = \emptyset, \ \forall i \in \{2, \ldots, k\}$$

and $O_n^{(1)}$ is bounded for all $n$. 

□
In particular, $K_n \subset O_n^{(1)}$. That is, for a sequence of open coverings $O_n = \{O_n^{(1)}, \ldots, O_n^{(k)}\}$ all its members except $O_n^{(1)}$ move away from $K$. Furthermore, this notion does not depend on the choice of $K$, i.e. up to a subsequence any sequence of open coverings is vanishing-compatible for any other $K$.

**Definition 3.12.** Let $k \in \mathbb{N}$ and $O_n = \{O_n^{(1)}, \ldots, O_n^{(k)}\}$ be a vanishing-compatible sequence of open coverings. Then we say $E \in C(X(M), \mathbb{R})$ is $k$-superadditive with respect to a sequence of partitions of unity $\{\psi_O\}_{O \in O_n}$ if for any vanishing sequence $(v_n)$ up to a subsequence

$$
\limsup_{n \to \infty} E(v_n) \geq \sum_{i=1}^k \limsup_{n \to \infty} E(\psi_{O_n^{(i)}} v_n).
$$

**Example 3.13.** Let $G$ be a locally finite metric graph and let $K$ be a compact subgraph of $G$. Recall on $G$ the open covering $O$ as in Example 3.10 with partition of unity $\psi, \tilde{\psi}$. Similarly, we define the sequence of partitions of unity $\psi_n = 1/n \max\{\text{dist}(G \setminus K, x), n\}$, $\tilde{\psi}_n = 1 - \psi_n$ and gives a sequence of partitions of unity with regards to a vanishing-compatible sequence of open coverings.

This gives rise to our second main result:

**Theorem 3.14.** Let $p \in [2, \infty)$, $c > 0$ and let $(M, \mu)$, $X(M)$ and $Y(M)$ satisfy Assumption 3.7 and Assumption 3.7. Let $K$ be a bounded, connected, nonempty set in $G$. Let $E \in C(X(M), \mathbb{R})$, such that

$$
t \mapsto E_t = \inf_{\|u\|_p = t} E(u)
$$

is continuous and assume $E$ to be $2$-superadditive with respect to a sequence of partitions of unity $\{\psi_O\}_{O \in O_n}$ in $Y(M)$ subordinate to a vanishing-compatible sequence of open coverings $O_n = (O_n^{(1)}, O_n^{(2)})$. If there exists a minimizing sequence which is vanishing, then

$$
E_c = \lim_{R \to \infty} \inf_{u \in X(M), \|u\|_p = c \text{ supp } u \subset M \setminus K_R} E(u) =: \tilde{E}_c.
$$

**Proof.** Let $u_n$ be a vanishing sequence. Assume $(O_n^{(1)}, O_n^{(2)})$ to be such that $K \subset O_n^{(1)}$ and $O_n^{(1)}$ is bounded.

For each fixed $m \in \mathbb{N}$ we have

$$
\int_{O_n^{(1)}} |u_n|^p \, d\mu \to 0 \quad (n \to \infty).
$$

Then for any $m \in \mathbb{N}$ we find an $n_m$, such that for $n > n_m$

$$
\int_{O_n^{(1)}} |u_n|^p \, d\mu \leq \frac{1}{m}.
$$
Using a diagonal argument we deduce the existence of a subsequence of \( u_n \), still denoted by \( u_n \), such that
\[
\int_{O_n^{(1)}} |u_n|^p \, d\mu \to 0 \quad (n \to \infty).
\]
In particular,
\[
0 \leq \int_{O_n^{(1)}} |\psi_{O_n^{(1)}u_n}|^p \, d\mu \leq \int_{O_n^{(1)}} |u_n|^p \, d\mu
\]
\[
c - \int_{O_n^{(1)}} |u_n|^p \, d\mu \leq \int_{O_n^{(2)}} |\psi_{O_n^{(2)}u_n}|^p \, d\mu \leq \int_{\mathcal{M}} |u_n|^p \, d\mu = c
\]
and we obtain
\[
\int_{O_n^{(1)}} |\psi_{O_n^{(1)}u_n}|^p \, d\mu \to 0 \quad (n \to \infty)
\]
\[
\int_{O_n^{(2)}} |\psi_{O_n^{(2)}u_n}|^p \, d\mu \to c \quad (n \to \infty).
\]
Then by superadditivity we have
\[
E_c = \lim_{n \to \infty} E(u_n)
\]
\[
\geq \limsup_{n \to \infty} E\left( \psi_{O_n^{(2)}u_n} \right) \geq \tilde{E}_c.
\]
This concludes the inequality \( E_c \geq \tilde{E}_c \). The reverse inequality is trivial since
\[
E_c \leq \inf_{\substack{u \in X(G), \|u\|_p = c \\ \text{supp } u \subset G \setminus K_R}} E(u)
\]
for all \( R > 0 \).

\[\square\]

**Corollary 3.15.** Suppose the assumptions in Theorem 3.4 and Theorem 3.14 are satisfied and
\[
E_c < \tilde{E}_c,
\]
then a minimizer of \( E_c \) exists, and any minimizing sequence for \( E_c \) admits a subsequence converging in \( L^p \) towards a minimizer of \( E_c \).

Throughout the rest of the paper given a functional \( E \in C(X(G), \mathbb{R}) \) we define the corresponding threshold energy
\[
(3.2) \quad \tilde{E}_c := \lim_{R \to \infty} \inf_{\substack{u \in X(G), \|u\|_p = c \\ \text{supp } u \subset \mathcal{G} \setminus K_R}} E(u).
\]
In the case of many-body quantum particle systems, this quantity refers to the ionization energy (see [Gri04] and [Sim83]). For this reason, throughout of the rest of the paper we are going to refer to the quantity in (3.2) also as the ionization threshold or ionization energy.

**Example 3.16.** Let \( G \) be a locally finite, connected noncompact metric graph with at least one ray. Consider the NLS energy functional as considered in [AST16]
\[
E_{\text{NLS}}(u, G) = \int_G |u'|^2 \, dx - \frac{\mu}{p} \int_G |u|^p \, dx.
\]
Consider the minimization problem

\[ \tilde{E}_{\text{NLS}}(G) := \lim_{n \to \infty} \inf_{u \in H^1(G), \|u\|_2^2 = 1} E_{\text{NLS}}(u,G). \]

Then in the context of decaying potentials, we will see in §4.5 that the functional \(E_{\text{NLS}}\) satisfies the prerequisites of Theorem 3.4 and Theorem 3.14. Moreover, we will characterize the ionization threshold of \(\tilde{E}_{\text{NLS}}(G)\). As discussed in Remark 4.14 we will show

\[ \tilde{E}_{\text{NLS}}(G) := \lim_{n \to \infty} \inf_{u \in H^1(G), \|u\|_2^2 = 1, \text{supp } u \subset G \setminus K_n} E_{\text{NLS}}(u,G) \leq E_{\text{NLS}}(\mathbb{R}). \]

If \(G\) is a finite graph we have equality in (3.4) due to Lemma 4.13. In particular Corollary 3.15 gives a generalization of the result in [AST16] where it was shown that if \(G\) is finite then minimizers of (3.3) exist if

\[ E_{\text{NLS}}(G) < E_{\text{NLS}}(\mathbb{R}). \]

Since (3.4) does not guarantee existence by Corollary 3.15 under the assumption (3.5) one cannot necessarily extend this result to general locally finite graphs. But as shown in Example 7.4 for a class of infinite tree graphs, one can show the reverse inequality

\[ \tilde{E}_{\text{NLS}}(G) \geq E_{\text{NLS}}(\mathbb{R}). \]

In particular one can derive for such graphs satisfying (3.6) existence of minimizers of \(E_{\text{NLS}}(G)\) under assumption (3.5).

3.3. An existence result on the real line for translation-invariant functionals. Assume \(X(\mathbb{R}) \subset L^p(\mathbb{R})\) is a Banach space on \(\mathbb{R}\). Let \(E \in C(X,\mathbb{R})\) be a translation invariant functional, i.e. if \(T_\lambda u(x) = u(x - \lambda)\) then

\[ E(u) = E(T_\lambda u) \]

for all \(\lambda \in \mathbb{R}\). The space \(\mathbb{R}\) can be understood as a graph consisting of two half-lines. In this context we consider \(K = \{0\}\) to be the compact core of this particular graph.

**Theorem 3.17.** Let \(p \in [2, \infty), c > 0, \) let \(E \in C(X(\mathbb{R}),\mathbb{R})\) be translation invariant and satisfy Assumption 3.1 and Assumption 3.7. Let

\[ t \mapsto E_t = \inf_{u \in X(\mathbb{R}), \|u\|_p^p = t} E(u) \]

be a strictly subadditive functional in \(X(\mathbb{R})\). Assume \(E\) to be superadditive with respect to a sequence of 3-partitions of unity \(\{\psi_O\}_{O \in \mathcal{O}_n}\) subordinate to the vanishing-compatible sequence of open coverings

\[ \mathcal{O}_n = \{(-2n, 2n), (n, \infty), (-\infty, -n)\}, \]

then it admits a minimizer.

**Proof.** By Theorem 3.14 we only need to construct non-vanishing minimizing sequences. Assume \(u_n\) to be a minimizing sequence of the functional \(E_c\). Then we
may construct such a sequence by using the translation invariance of the functional. Indeed, we may assume up to translation invariance
\[
\int_0^\infty |u_n|^p \, dx \to \frac{c}{2} \quad (n \to \infty)
\]
\[
\int_{-\infty}^0 |u_n|^p \, dx \to \frac{c}{2} \quad (n \to \infty).
\]
Assume \(u_n\) is vanishing. Then since \(u_n \to 0\) in \(L_{\text{loc}}^\infty\) (up to a subsequence) due to a diagonal argument, we have that
\[
\int_\mathbb{R} |\psi(-2n,2n)u_n|^p \, dx \to 0 \quad (n \to \infty)
\]
\[
\int_\mathbb{R} |\psi(n,\infty)u_n|^p \, dx \to \frac{c}{2} \quad (n \to \infty)
\]
\[
\int_\mathbb{R} |\psi(-n,-\infty)u_n|^p \, dx \to \frac{c}{2} \quad (n \to \infty)
\]
Then using the subadditivity of the functional and the strict subadditivity of \(t \mapsto E_t\) we conclude
\[
E_c = \lim_{n \to \infty} E(u_n)
\]
\[
\geq \limsup_{n \to \infty} E(\psi(-\infty,-n)u_n) + \limsup_{n \to \infty} E(\psi(n,\infty)u_n)
\]
\[
\geq E_{c/2} + E_{c/2} > E_c.
\]
By contradiction after translating the \(u_n\) if necessary we can find a non-vanishing subsequence. Passing to a further subsequence there exists a weakly convergent subsequence in \(H^1\) that converges up to a further subsequence to a minimizer by Theorem 3.4.

Example 3.18. Recall the NLS energy functional from Example 3.16 on the real line
\[
E_{\text{NLS}}(\mathbb{R}) = \inf_{u \in H^1(\mathbb{R}), \|u\|_2^2 = 1} E_{\text{NLS}}(u, \mathbb{R}).
\]
Then as discussed in Example 3.16 we have
\[
\tilde{E}_{\text{NLS}}(\mathbb{R}) = E_{\text{NLS}}(\mathbb{R}).
\]
In particular inequality (3.5) cannot be satisfied and we cannot apply Corollary 3.15. On the other hand, the prerequisites of Theorem 3.17 are satisfied. In particular this ensures existence of ground states for all \(\mu > 0\). More explicitly, one can show that ground states of the NLS energy functional on the real line are unique up to translation invariance of the functional and are the well-known soliton solutions.

4. Existence of Ground states of a class of Nonlinear Equations with Polylaplacian on Finite Graphs

In this section, we give a first application of the results derived in §3 on finite graphs. In this context we show a decomposition formula for the Polylaplacian.
4.1. Formulation of the problem. Let $\mathcal{G}$ be a connected, finite metric graph and let $K$ be a connected subgraph of $\mathcal{G}$. For $k \in \mathbb{N}$ consider the energy functional

$$E(k) = \frac{1}{2} \int_{\mathcal{G}} |u(k)|^2 + V |u|^2 \, dx - \frac{\mu}{p} \int_{\mathcal{G}} |u|^p \, dx,$$

with $2 < p < 4k + 2$ and $V \in L^2 + L^\infty$ and for $c > 0$ consider the minimization problem

$$(4.1) \quad E_c^{(k)} := \inf_{\substack{u \in H^k \mid \|u\|^2_{L^2} = c}} E^{(k)}(u).$$

Lemma 4.1. Let $\mathcal{G}$ be a finite connected metric graph. The functional $E^{(k)}$ under the $L^2$-constraint $\|\cdot\|^2_{L^2} = c$ is bounded below for $2 < p < 4k + 2$ and $c > 0$. Moreover, for each $0 < \varepsilon < 1$ there exists a $C_\varepsilon > 0$, such that

$$E^{(k)}(u) \geq 1 - \frac{\varepsilon}{2} \int_{\mathcal{G}} |u(k)|^2 + V |u|^2 \, dx - C_\varepsilon.$$ 

Proof. Let $\varepsilon > 0$. Consider a decomposition of $V \in L^2 + L^\infty$ such that

$$V = V_2 + V_\infty, \quad \|V_2\|_2 \leq \varepsilon.$$

Then

$$\int_{\mathcal{G}} \left| u(k) \right|^2 - \|V_\infty\|_\infty \|u\|^2 \, dx - \varepsilon \|u\|_4^2 \leq \int_{\mathcal{G}} \left| u(k) \right|^2 + V |u|^2 \, dx \leq \int_{\mathcal{G}} \left| u(k) \right|^2 + \|V_\infty\|_\infty \|u\|^2 \, dx + \varepsilon \|u\|_4^2.$$

By the Sobolev inequality we infer

$$\|u\|_4^2 \leq C_1 \|u\|_{H^k}^2 \leq C_2 \left( \left| u(k) \right|^2 + |u|_2^2 \right).$$

Adding a constant to the potential if necessary we have that

$$\|u\| := \left( \int_{\mathcal{G}} \left| u(k) \right|^2 + V |u|^2 \, dx \right)^{1/2}$$

defines an equivalent norm on $H^k$.

From Proposition 2.6 we have

$$\|u\|_{L^p}^p \leq C \|u\|_{L^2(\mathcal{G})}^{(2k-1)p+2} \|u\|^{p-2} \frac{2k+2}{2k},$$

for some $C > 0$. Let $0 < \varepsilon < 1$, then with Young’s inequality we infer for all $u \in H^k(\mathcal{G})$ with $\|u\|_2^2 = c$

$$\frac{\mu}{p} \|u\|_{L^p}^p \leq \frac{\varepsilon}{2} \|u\|^2 + C_{\varepsilon,c}$$

for some $C_{\varepsilon,c} > 0$ and we obtain

$$E^{(k)} \geq 1 - \frac{\varepsilon}{2} \int_{\mathcal{G}} |u(k)|^2 + V |u|^2 \, dx - C_{\varepsilon,c}$$

for $2 < p < 4k + 2$. \qed
Proposition 4.2. Let $G$ be a finite, connected metric graph. Assume $u \in H^k$ is a minimizer of $E^{(k)}$, then $u \in H^{2k}$ and there exists $\lambda \in \mathbb{R}$ such that
\[(4.2) \quad (-1)^k u^{(2k)}_e + (V + \lambda) u_e = \mu |u_e|^{p-1} u_e \]
for all $e \in \mathcal{E}$.

Proof. Since $E^{(k)} \in C^1(H^k, \mathbb{R})$ and the $L^2$ constraint is also $C^1$, and $u$ is a constrained critical point we can compute the Gâteaux derivative
\[
\int_G (u^{(k)}(\eta) - u|u|^{p-2} \eta) \, dx + \int_G (V + \lambda) u \eta \, dx = 0, \quad \forall \eta \in H^k(G)
\]
where $\lambda$ is a Lagrange multiplier. Fixing an edge $e$, then with $\eta \in C^\infty(I_e)$ and integration by parts we deduce (4.2) for each $e \in \mathcal{E}$ and by elliptic regularity $u \in \tilde{H}^{2k}$.

Fixing now $v \in V$ and taking $\eta \in H^k$ to be locally supported near $v$ and not supported at any other vertex, then by integration by parts we deduce
\[
\sum_{j=1}^k (-1)^j \sum_{e \ni v} \partial^{(k+j-1)} u^{(k-j)}_{e^{(k-j)}} \partial^{(k-j)} u^{(k-j)}_{e^{(k-j)}} v_e \partial^{(k-j)} u^{(k-j)}_{e^{(k-j)}} v_e = 0.
\]
Since the choice $\eta \in H^k$ is arbitrary we deduce
\[
\sum_{O \in \mathcal{O}} \Psi^2_O = 1.
\]

4.2. Partitions of unity in $\hat{C}^\infty_b$. Let $G$ be any locally finite, connected graph and $\mathcal{O}$ be a finite covering of $\mathcal{O}$. We construct a partition of unity in $\hat{C}^\infty_b(G)$ by choosing appropriate partitions of unities subordinate to the covering. One rather different “normalization” of the usual partition of unity will be especially useful in applications:

Lemma 4.3. Let $G$ be a metric graph. Consider any finite open covering $\mathcal{O}$ of $G$. There exists a partition of unity subordinate to $\mathcal{O}$ in $\hat{C}^\infty_b$ satisfying
\[(4.3) \quad \sum_{O \in \mathcal{O}} \Psi^2_O = 1.\]

Proof. Consider any smooth partition of unity $\{\psi_O\}_{O \in \mathcal{O}}$ on the graph subordinate to the open covering $\mathcal{O}$ satisfying
\[
\sum_{O \in \mathcal{O}} \Psi_O = 1.
\]

Then we may define
\[
\Psi_O := \frac{\psi_O}{\sqrt{\sum_{O \in \mathcal{O}} \psi_O^2}}
\]
for all $O \in \mathcal{O}$, which is smooth restricted as functions on all edges since $\sum_{O \in \mathcal{O}} \Psi^2_O (y) \neq 0$ for all $y \in \mathcal{G}$. Furthermore, it is constant in a neighborhood of any vertex and we infer $\Psi_O \in \hat{C}^\infty_b$. By construction we conclude
\[
\sum_{O \in \mathcal{O}} \Psi^2_O = 1.
\]
Remark 4.4. The normalization in (4.3) replaces in this context the typical normalization, where one assumes
\[ \sum_{O \in \mathcal{O}} \psi_O \equiv 1. \]
Throughout the rest of the paper we will only work with partitions of unity that satisfy the normalization (4.3).

Example 4.5. Let \( \mathcal{G} \) be a finite, connected metric graph with core \( K = \mathcal{G} \setminus \mathcal{E}_\infty \). Consider on \( \mathcal{G} \) the open covering \( \mathcal{O} \) given by \( K_2 \) and \( \mathcal{G} \setminus K_1 \), where \( K_1 \) and \( K_2 \) are the neighborhoods of \( K \) given as in (3.1), such that \( \mathcal{G} \setminus K_1 \) only consists of disjoint rays. Consider the partition of unity subordinate to \( \mathcal{O} \) from Lemma 4.3 given by \( \psi_K, \{ \psi_e \}_{e \in \mathcal{E}_\infty} \) respective to \( K_2 \) and \( \mathcal{G} \setminus K_1 \), then we define slight modifications
\[ \psi_{K,R}(x) = \begin{cases} 1, & \text{on } K \\ \psi_K(x/R), & \text{on all rays } e \in \mathcal{E}_\infty; \end{cases} \]
\[ \psi_{e,R}(x) = \begin{cases} 0, & \text{on } \mathcal{G} \setminus \{ e \} \\ \psi_e(x/R), & \text{on } e \in \mathcal{E}_\infty. \end{cases} \]
Then by definition, \( \{ \psi_{O,n} \}_{O \in \mathcal{O}} \) is a vanishing-compatible sequence of partitions of unity subordinate to the open coverings given by \( K_2 \) and \( G \setminus K_1 \). By Lemma 4.3 there exists a sequence of partitions of unity \( \Psi_n := \Psi_{K,n}, \tilde{\Psi}_n := \sum_{e \in \mathcal{E}_\infty} \Psi_{e,n} \) in \( \widehat{C}_b^\infty \) satisfying
\[ \Psi_n^2 + \tilde{\Psi}_n^2 \equiv 1. \]

4.3. A decomposition formula. In the following we identify a given function \( f \in \widehat{C}_b^\infty(\mathcal{G}) \) with its corresponding multiplication operator \( M_f \phi := f \phi \). Let \( A \) be an operator such that \( fD(A) \subset D(A) \), then we can define the commutator \( [A,f] = Af - fA \) and
\[ fAf = f^2A + f[A,f] \]
\[ fAf = Af^2 + [A,f]f. \]
Averaging the two preceding equations we conclude
\[ fAf = \frac{1}{2}(f^2A + Af^2) + \frac{1}{2}(f[A,f] - [A,f]f). \]

Lemma 4.6. Let \( \mathcal{G} \) be a locally finite, connected metric graph. Assume \( \{ \psi_k \}_{k=1}^N \) is a family of function in \( \widehat{C}_b^\infty(\mathcal{G}) \) with \( 0 \leq \psi_k \leq 1 \) for all \( k \in \{1, \ldots, k \} \) and
\[ \sum_{k=1}^N \psi_k^2 \equiv 1. \]
Assume \( D(A) \) is invariant under multiplication by elements in \( \widehat{C}_b^\infty(\mathcal{G}) \), then
\[ A = \sum_{k=1}^N \psi_k A \psi_k - \frac{1}{2}(\psi_k[A,\psi_k] - [A,\psi_k]\psi_k). \]

Proof. Follows immediately with (4.4). \( \square \)
We refer to (4.5) as a decomposition formula for $A$. In the following, we develop a decomposition formula for the Polylaplacian $A = (-\Delta)^k$.

Let $G$ be a finite metric graph and let $k \in \mathbb{N}$. In the following, we define the Polylaplacian $A = (-\Delta)^k$ on $G$ as an operator $A : D(A) \subset L^2(G) \to L^2(G)$ given by
\[
((-\Delta)^k u)_e := (-\Delta)^k u_e := (-1)^k u_e^{(2k)}
\]
and we conclude
\[
D(A) = H^{2k}(G).
\]

**Lemma 4.7.** Let $G$ be a locally finite connected graph. Let $A = (-\Delta)^k$ with $D(A) = H^{2k}$, then

(i) $fD(A) \subset D(A)$ for all $f \in \widetilde{C}_b^\infty(G)$.

(ii) Let $f \in \widetilde{C}_b^\infty(G)$, then the operator $fA$ is given by
\[
(fAf) \phi = \frac{1}{2}(f^2A + Af^2)\phi
\]
\[
+ \frac{(-1)^{k+1} 2^{k-1} 2^{k-m} (2k)_{m+n} f(m) f(n) \phi^{(2k-m-n)}}{m!n!}
\]
for all $\phi \in D(A)$.

**Proof.** We apply Leibniz’ formula and compute
\[
[A, f] \phi = (-\Delta)^k f \phi - f(-\Delta)^k \phi
\]
\[
= (-1)^k \sum_{m=1}^{2k} \binom{2k}{m} f^{(m)} \phi^{(2k-m)}.
\]
Then we apply Leibniz’ formula again and compute
\[
([A, f] f) \phi = (-1)^k \sum_{m=1}^{2k} \sum_{n=0}^{2k-m} \binom{2k}{m} \binom{2k-m}{n} f^{(m)} f^{(n)} \phi^{2k-m-n}
\]
\[
= (-1)^k \sum_{m=1}^{2k} \sum_{n=0}^{2k-m} \frac{(2k)_{m+n} f^{(m)} f^{(n)} \phi^{(2k-m-n)}}{m!n!}
\]
and we conclude
\[
\frac{1}{2}(f[A, f] - [A, f] f) \phi = \frac{(-1)^{k+1} 2^{k-1} 2^{k-m} (2k)_{m+n} f^{(m)} f^{(n)} \phi^{(2k-m-n)}}{m!n!}.
\]
The statement follows upon combining this with (4.5). \qed

Given the core $K = G \setminus E_\infty$ of $G$ and $R > 0$ we define
\[
D_R := \{ \phi \in D(A) | \text{supp}(\phi) \subset G \setminus K_R \}
\]
\[
\Sigma_R := \inf \{ \langle \phi, A\phi \rangle | \phi \in D_R, \| \phi \|_2^2 = 1 \},
\]
where $K_R$ was defined in (3.1).

For $R = 0$ we set
\[
D_0 := D(A)
\]
\[
\Sigma_0 := \inf \{ \langle \phi, A\phi \rangle | \phi \in D(A), \| \phi \|_2^2 = 1 \}.
\]
The last relevant quantity, which we will discuss later in (4.3) is
\[
\Sigma := \lim_{R \to \infty} \Sigma_R = \sup_{R > 0} \Sigma_R.
\]
Lemma 4.8. Let \( G \) be a finite, connected metric graph and let \( V \in L^2 + L^\infty(G) \). \( E^{(k)} \) is weak limit superadditive, superadditive with respect to the partition of unity in Example 4.5. Assume \( A = (-\Delta)^k + V \) admits a ground state, then \( t \mapsto E_t \) as defined in (4.1) is strictly subadditive.

Proof. Weak limit superadditivity. We showed in the proof of Lemma 4.1 that

\[
\|u\| = \left( \int_G \left| u^{(k)} \right|^2 + V|u|^2 \, dx \right)^{1/2}
\]

defines an equivalent norm on \( H^k \) upon adding a constant to \( V \).

Assume \( u_n \rightharpoonup u \) in \( H^k \) weakly, then up to a subsequence, which we still denote by \( u_n \), by the Brezis–Lieb Lemma and weak convergence

\[
\limsup_{n \to \infty} \|u_n\|^2 = \|u\|^2 + \limsup_{n \to \infty} \|u - u_n\|,
\]

\[
\limsup_{n \to \infty} \int_G |u_n|^p \, dx = \int_G |u|^p \, dx + \limsup_{n \to \infty} \int_G |u - u_n|^p \, dx.
\]

Then

\[
\limsup_{n \to \infty} E^{(k)}(u_n) = E^{(k)}(u) + \limsup_{n \to \infty} E^{(k)}(u - u_n)
\]

and \( E^{(k)} \) is weak limit superadditive.

Superadditivity with respect to a sequence of partitions of unity. Finally we need to show superadditivity with respect to the partition of unity \( \{\Psi_n, \overline{\Psi}_n\} \) in Example 4.5. Let \( u_n \to 0 \) be a vanishing sequence with \( \|u_n\|_{L^2}^2 = c \), then

\[
\|\Psi_n u_n\|_{L^2}^2 + \|\overline{\Psi}_n u_n\|_{L^2}^2 = c
\]

and up to a subsequence, still denoted by \( u_n \),

\[
\lim_{n \to \infty} \int_{K_{2n}} |u_n|^p \, dx = 0, \quad \lim_{n \to \infty} \int_{K_{2n}} |u_n|^2 \, dx = 0
\]

\[
\lim_{n \to \infty} \int_G |u_n|^p \, dx = \liminf_{n \to \infty} \int_{K_{2n}} |u_n|^p \, dx = \limsup_{n \to \infty} \int_G |u_n|^p \, dx.
\]

Then from

\[
0 \leq \liminf_{n \to \infty} \int_{K_{2n}} |\Psi_n u_n|^p \, dx \leq \limsup_{n \to \infty} \int_{K_{2n}} |\Psi_n u_n|^p \, dx \leq \lim_{n \to \infty} \int_{K_{2n}} |u_n|^p \, dx
\]

\[
0 \leq \liminf_{n \to \infty} \int_{K_{2n}} |\overline{\Psi}_n u_n|^p \, dx \leq \limsup_{n \to \infty} \int_{K_{2n}} |\overline{\Psi}_n u_n|^p \, dx \leq \lim_{n \to \infty} \int_{K_{2n}} |u_n|^p \, dx
\]

we deduce

\[
\lim_{n \to \infty} \int_G |\Psi_n u_n|^p \, dx = \lim_{n \to \infty} \int_{K_{2n}} |\Psi_n u_n|^p = 0
\]

\[
\lim_{n \to \infty} \int_{K_{2n}} |\overline{\Psi}_n u_n|^p \, dx = 0.
\]

and in particular

\[
\lim_{n \to \infty} \int_G |u_n|^p \, dx = \lim_{n \to \infty} \int_{G\setminus K_{2n}} |\overline{\Psi}_n u_n|^p \, dx = \lim_{n \to \infty} \int_G |\overline{\Psi}_n u_n|^p.
\]

\( H^{2k} \) is dense in \( H^k \) by Proposition 2.7 and we may assume that there exists a minimizing sequence in \( H^{2k} \). Let \( u_n \) be a minimizing sequence in \( H^{2k} \), then
by Lemma 4.7 we conclude passing to a subsequence, still denoted by $u_n$, using integration by parts and Young’s inequality

$$\sum_{i=1}^{2k-1} \sum_{j=1}^{2k-m} \frac{(2k)_{m+n}}{i!j!} \left| \left\langle \Psi_n^{(i)} \Psi_n^{(j)} u_n^{(2k-i-j)}, \phi \right\rangle \right|_{L^2} \leq \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-m} \frac{(2k)_{m+n}}{i!j!} \left| \left\langle \frac{d^{k-j}}{dx^{k-j}} \Psi_n^{(i)} \Psi_n^{(j)} u_n^{(k-i)}, \phi \right\rangle \right|_{L^2} \leq \frac{C}{n^2} \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-m} \frac{(2k)_{m+n}}{i!j!} \|u_n\|_{H^k} \to 0 \quad (n \to \infty),$$

and we infer

$$E^{(k)} = \lim_{n \to \infty} \frac{1}{2} \langle u_n, A u_n \rangle + \frac{\mu}{p} \|u_n\|_p^p = \limsup_{n \to \infty} \frac{1}{2} \langle \Psi_n u_n, A \Psi_n u_n \rangle + \frac{\mu}{p} \|\Psi_n u_n\|_p^p + \limsup_{n \to \infty} \frac{1}{2} \langle \widetilde{\Psi}_n u_n, A \tilde{\Psi}_n u_n \rangle + \frac{\mu}{p} \|\tilde{\Psi}_n u_n\|_p^p = \limsup_{n \to \infty} E^{(k)}(\Psi_n u_n) + \limsup_{n \to \infty} E^{(k)}(\tilde{\Psi}_n u_n)$$

and $E^{(k)}$ is (super-)additive with respect to the partition of unity $\{\Psi_n, \tilde{\Psi}_n\}$ in Example 4.5.

Subadditivity. To show the subadditivity, note that

$$(4.10) \quad E_t^{(k)} = \inf_{u \in H^1} \left\{ \frac{1}{2} \int_G |u|_2^2 + V|u|^2 \, dx - t \frac{\mu}{p} \int_G |u|^p \, dx \right\}. $$

We deduce the property by showing that $t \mapsto E_t^{(k)}$ is a concave function. Indeed, the scaling defines a concave function and passing to the limit we deduce concavity of the functional. Hence,

$$(4.11) \quad E_t^{(k)} \geq t E^{(k)}_1, \quad t \in [0, 1],$$

so that

$$E_t^{(k)} + E_{1-t}^{(k)} \geq E^{(k)}_1, \quad t \in [0, 1].$$

For the strictness in the inequality it suffices to show strictness in the inequality (4.11). Assume

$$E_t^{(k)} = t E^{(k)}_1$$

for some $t \in (0, 1)$ and let $u_n$ be a minimizing sequence for $E_t^{(k)}$, then in particular due to (4.10)

$$\int_G |u_n|^p \, dx \to 0 \quad (n \to \infty).$$
By density we may assume \( u_n \in D(A) \) and we deduce
\[
E_t^{(k)} = \lim_{n \to \infty} \frac{1}{2} \langle Au_n, u_n \rangle - \frac{\mu}{p} \int_G |u|^p \, dx
\]
\[
= \lim_{n \to \infty} \frac{1}{2} \langle Au_n, u_n \rangle \geq \frac{\Sigma_0 t}{2}.
\]
Now suppose that a ground state to \( E^{(k)} \) exists, i.e. there exists \( u \in D(A) \) with \( \|u\|^2 = t \) and \( \langle Au, u \rangle = \Sigma_0 t \), then
\[
E_t^{(k)} \leq \frac{1}{2} \langle Au, u \rangle - \frac{\mu}{p} \int_G |u|^p \, dx < \frac{\Sigma_0 t}{2}.
\]
This is a contradiction, and we conclude strict subadditivity in this case.

\[ \square \]

**Proposition 4.9.** Assume \( G \) is a finite, connected metric graph and \( \Sigma_0 < \Sigma \) as defined in (4.8) and (4.9). Then there exists \( \hat{\mu} > 0 \), such that for all \( \mu \in (0, \hat{\mu}) \)
\[ \tilde{\Sigma}_0^{(\mu,k)} := \inf_{u \in D(A)} E_t^{(k)}(u) < \inf_{u \in D_R(A)} E_t^{(k)}(u) =: \tilde{\Sigma}(\mu,k) \]
for some \( R > 0 \).

**Proof.** Without loss of generality we may assume \( \Sigma_0 > 0 \) as we otherwise can simply add a constant to the functional. In particular,
\[
\|u\| = \left( \int_G |u(k)|^2 + V|u|^2 \, dx \right)^{1/2}
\]
defines an equivalent norm on \( H^k \) as shown in Lemma 4.1. Let \( \varepsilon > 0 \). By Proposition 2.6
\[
\|u\|_L^p \leq C \|u\|_{L^2(G)}^{(2k-1)p+2} \|u\|_L^{\frac{p+2}{2p}}
\]
for some \( C > 0 \). In particular, for sufficiently small \( \mu \) with Young’s inequality we have
\[
\frac{\mu}{p} \|u\|_L^p \leq \frac{\varepsilon}{2} \int_G |u(k)|^2 + V|u|^2 \, dx + \tilde{C} \varepsilon
\]
for sufficiently small \( \mu > 0 \) we deduce
\[
E_t^{(k)}(u) \geq \frac{1 - \varepsilon}{2} \left( \int_G |u(k)|^2 + V|u|^2 \, dx \right) - \tilde{C} \varepsilon.
\]
Hence,
\[
\tilde{\Sigma}^{(\mu,k)} - \tilde{\Sigma}_0^{(\mu,k)} \geq \frac{1 - \varepsilon}{2} \Sigma - \frac{\varepsilon}{2} - \frac{1}{2} \Sigma_0 = \frac{1}{2} (\Sigma - \Sigma_0) - \frac{\varepsilon}{2} \left( \tilde{C} + \Sigma \right).
\]
Since \( \varepsilon \) can be chosen arbitrarily small and \( \Sigma > \Sigma_0 > 0 \), we have for sufficiently small \( \mu \)
\[
\tilde{\Sigma}^{(\mu,k)} > \tilde{\Sigma}_0^{(\mu,k)}.
\]
\[ \square \]
Theorem 4.10. Let $\mathcal{G}$ be a finite, connected graph and let $c > 0$. Assume $\Sigma_0 < \Sigma$ as defined in (4.8) and (4.9), then $E_c^{(k)}(\mathcal{G})$ admits a minimizer for $\mu \in (0, \hat{\mu}]$ as in Proposition 4.9 and let $\hat{\mu}$ be as in Proposition 4.9. Then $E_c^{(k)}$ admits a minimizer for any $\mu \in (0, \hat{\mu})$.

Proof. By Lemma 4.1 any minimizing sequence admits a weakly convergent subsequence. $E^{(k)}(\mathcal{G})$ satisfies the prerequisites of Theorem 3.4 and Theorem 3.14 with $X(\mathcal{G}) = H^k(\mathcal{G})$ and $Y(\mathcal{G}) = C^\infty_b(\tilde{\mathcal{G}})$ by Lemma 4.8. Then due to Proposition 4.9 the energy inequality in Corollary 3.15 is satisfied. In particular we deduce existence of a minimizer of $E^{(k)}(\mathcal{G})$ under the assumptions of the statement.

4.4. The case $V \equiv 0$ on the real line.

Consider the infimization problem on the real line

$$E^{(k)}(\mathbb{R}) = \inf_{u \in H^1(\mathbb{R})} \frac{1}{2} \int_{\mathbb{R}} |u^{(k)}|^2 \, dx - \frac{\mu}{p} \int_{\mathbb{R}} |u|^p \, dx.$$ 

This is the special case when $V \equiv 0$ and $\mathcal{G} = \mathbb{R}$ in (4.1). In this case $E_c$ admits a minimizer due to Theorem 3.17:

Theorem 4.11. Let $V \equiv 0$ and $\mathcal{G} = \mathbb{R}$. The minimization problem

$$E^{(k)}(\mathbb{R}) = \inf_{u \in H^1(\mathbb{R})} \frac{1}{2} \int_{\mathbb{R}} |u^{(k)}|^2 \, dx - \frac{\mu}{p} \int_{\mathbb{R}} |u|^p \, dx$$

admits a minimizer for all $\mu > 0$. Furthermore, any minimizer of $E_c(\mathbb{R})$ is $C^\infty_b$ and satisfies the Euler–Lagrange equation

$$(-1)^k u^{(2k)} + \lambda u = \mu |u|^{p-2} u,$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

Proof. Due to Lemma 4.8 it suffices to show that

$$t \mapsto E_t^{(k)} = \inf_{u \in H^1(\mathbb{R})} \frac{1}{2} \int_{\mathbb{R}} |u^{(k)}|^2 \, dx - \frac{\mu}{p} \int_{\mathbb{R}} |u|^p \, dx.$$

is strictly subadditive, then the prerequisites of Theorem 3.17 are satisfied. The Euler–Lagrange equation is satisfied because of Proposition 4.2. The regularity is due to elliptic regularity and a bootstrap argument.

For a contradiction, assume as in the proof of Lemma 4.8

$$E_t = tE_1$$

for some $t \in [0, c]$. Assume $u_n$ is a minimizing sequence for $E_t$, then as in the proof of Lemma 4.8 we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} |u_n|^p \, dx = 0.$$ (4.12)

Given a test function $\phi \in C^\infty_c(\mathbb{R})$ with $\|\phi\|^2_2 = 1$, we define the rescaling for $\lambda > 0$

$$\phi_\lambda := \lambda^{1/2} \phi(\lambda x).$$

Then $\|\phi_\lambda\|^2_2 = 1$ for all $\lambda > 0$. Then

$$E^{(k)}(\phi_\lambda) = \lambda^{2k} \int_{\mathbb{R}} |\phi^{(k)}|^2 \, dx - \lambda^{\frac{p}{2}-1} \int_{\mathbb{R}} |\phi|^p \, dx.$$
In particular for sufficiently small $\lambda > 0$
\[ E^{(k)}(\phi_\lambda) < 0. \]
On the other hand with (4.12)
\[ E^{(k)}(u_n) = \lim_{n \to \infty} E^{(k)}(u_n) \leq \lim_{n \to \infty} \int_\mathcal{G} |u_n^{(k)}|^2 \, dx \geq 0 \]
and we conclude subadditivity due to the contradiction. This concludes the proof. \( \square \)

**Corollary 4.12.** Let $V \equiv 0$. Then
\[ E^{(k)}(\mathbb{R}) < 0. \]

**4.5. Decaying potentials.** In the following we study the minimization problem on finite metric graphs $\mathcal{G}$ under the assumption that $V = V_2 + V_\infty$ with $V_2 \in L^2(\mathcal{G}), V_\infty \in L^\infty(\mathcal{G})$ such that
\[ V_\infty(x) \to 0 \quad (x \to \infty) \]
on all of the rays. Consider the quantities
\[
\tilde{\Sigma}(\mu, k) = \inf_{\phi \in D(A)} E^{(k)}(u)
\]
\[
\tilde{\Sigma}(\mu, k) = \lim_{R \to \infty} \inf_{\phi \in D(R(A)), \| \phi \|^2_{L^2} = 1} E^{(k)}(u)
\]
\[
E^{(k)}(\mathbb{R}) = \inf_{u \in H^1(\mathbb{R}), \| u \|^2_{H^1(\mathbb{R})} = 1} \left( \frac{1}{2} \int_\mathbb{R} |u^{(k)}|^2 \, dx - \frac{\mu}{p} \int_\mathbb{R} |u|^p \, dx \right).
\]

**Lemma 4.13.** Let $\mathcal{G}$ be a finite metric graph and assume that $V \in L^2 + L^\infty(\mathcal{G})$ satisfies (4.13). Then
\[ \tilde{\Sigma}(\mu, k) = E^{(k)}(\mathbb{R}). \]

**Proof.** Assume $\phi$ is a minimizer of $E^{(k)}(\mathbb{R})$, which exists due to Theorem 4.11. Due to density, we can consider a minimizing sequence $u_n$ for $E^{(k)}(\mathbb{R})$ in $C^\infty_c(\mathbb{R})$ satisfying $\| u_n \|^2 = 1$, such that $u_n \to \phi$ strongly in $H^1$ as $n \to \infty$. Now by translation invariance we may assume that $u_n$ is supported in $[n, \infty)$ for $n \in \mathbb{N}$. Identifying the half-line with one of the rays of $\mathcal{G}$, we may consider $u_n$ as a function in $H^1(\mathcal{G})$. Then
\[
\int_\mathcal{G} V|u_n|^2 \, dx = \int_{\mathcal{G} \setminus K_n} V|u_n|^2 \, dx
\]
\[
\leq C \left( \sup_{x \in \mathcal{G} \setminus K_n} |V_\infty(x)|^2 + \int_{\mathcal{G} \setminus K_n} |V_2|^2 \, dx \right) \to 0 \quad (n \to \infty)
\]
and we compute
\[
E^{(k)}(\mathbb{R}) = \lim_{n \to \infty} E^{(k)}(u_n)
\]
\[
= \lim_{n \to \infty} \frac{1}{2} \int_\mathcal{G} |u_n^{(k)}|^2 \, dx - \frac{\mu}{p} \int_\mathcal{G} |u|^p \, dx
\]
\[
= \lim_{n \to \infty} E^{(k)}(u_n) \geq \tilde{\Sigma}(\mu, k).
\]
On the other hand given a minimizing sequence \( u_n \) for \( \tilde{\Sigma}(\mu, k) \), such that \( \text{supp} \ u_n \subset G \setminus K_n \) then the functions in the sequence are supported on each of the rays and

\[
\left| \int_G V |u_n|^2 \, dx \right| = \left| \int_{G \setminus K_n} V |u_n|^2 \, dx \right| \\
\leq C \left( \sup_{x \in G \setminus K_n} |V_\infty(x)| + \left( \int_{G \setminus K_n} |V_2|^2 \, dx \right)^{1/2} \right) \to 0 \quad (n \to \infty).
\]

By density we can consider a collection of sequences \( u_n^{(1)}, \ldots, u_n^{(|E_\infty|)} \) in \( C^\infty_c(\mathbb{R}) \) and choose them to have disjoint supports. Then if we define

\[
\tilde{u}_n := \sum_{i=1}^{|E_\infty|} u_n^{(i)}.
\]

Then with (4.14) we compute

\[
\tilde{\Sigma}(\mu, k) = \lim_{n \to \infty} \sum_{i=1}^{E_\infty} E^{(k)}_i(u_n^{(i)}) \\
= \lim_{n \to \infty} E^{(k)}(\tilde{u}_n) \geq E^{(k)}(\mathbb{R}).
\]

**Remark 4.14.** Suppose \( G \) is a locally finite metric graph with at least one ray. Then the inequality

\[
\tilde{\Sigma}(\mu, k) \leq E^{(k)}(\mathbb{R})
\]

can still be shown as in the proof of Lemma 4.13 using the test function argument on the half-line.

**Theorem 4.15.** Let \( G \) be a finite metric graph. Assume \( V \in L^2 + L^\infty(G) \) satisfies

(4.13), then \( E^{(k)} \) is strictly subadditive and if

\[
\tilde{\Sigma}_{-t}^{(\mu, k)} < E^{(k)}(\mathbb{R})
\]

then there exists a minimizer to the minimization problem.

**Proof.** By Lemma 4.1 any minimizing sequence admits a weakly convergent subsequence. By Lemma 4.8 and Lemma 4.13 it suffices to prove the strict subadditivity of \( E^{(k)} \). As in Lemma 4.8 we can argue by contradiction. Assume namely that

\[
E^{(k)}_t = tE^{(k)}_1
\]

for some \( t \in (0, 1) \) and let \( u_n \) be a minimizing sequence for \( E^{(k)}_t \), then in particular

\[
\int_G |u_n|^p \, dx \to 0 \quad (n \to \infty).
\]

But then \( u_n \) is a vanishing sequence and passing to a subsequence still denoted by \( u_n \), we deduce with superadditivity with respect to a sequence of partitions of
unity as defined in Example 4.5
\[ E_t^{(k)} = \limsup_{n \to \infty} E^{(k)}(\Psi_n u_n) + \limsup_{n \to \infty} E^{(k)}(\widetilde{\Psi}_n u_n) \]
\[ \geq \frac{1}{2} \lim_{n \to \infty} \inf_{\phi \in H^1} \langle Au, u \rangle_{L^2} \]
\[ \geq -C \lim_{n \to \infty} \left( \sup_{x \in \Gamma \setminus K_n} |V(x)| \right) + \left( \int_{\Gamma \setminus K_n} |V|^2 \, dx \right)^{1/2} = 0, \]

since \( \|u_n\|_{H^1} \leq C \) for some \( C > 0 \) by Lemma 4.1. On the other hand, by Lemma 4.13 and Corollary 4.12
\[ \tilde{\Sigma}^{(\mu, k)} = E^{(k)}(\mathbb{R}) < 0 \]
and by contradiction we deduce strict subadditivity.

Hence, the prerequisites of Theorem 3.4 and Theorem 3.14 are satisfied with \( X(\mathcal{G}) = H^k(\mathcal{G}) \) and \( Y(\mathcal{G}) = C_0^\infty(\mathcal{G}) \). Then due to Proposition 4.9 the energy inequality in Corollary 3.15 is satisfied. In particular we deduce existence of a minimizer of \( E^{(k)}(\mathcal{G}) \) under the stated assumptions. \( \square \)

Example 4.16. Let \( \mathcal{G} \) be a finite metric graph and let \( V \in L^2 + L^\infty \) satisfy (4.13). Similarly as in Lemma 4.13 we can show
\[ \Sigma = \lim_{R \to \infty} \inf_{\phi \in D_R(A)} \langle \phi, A\phi \rangle_{L^2} = 0. \]

In particular if \( \Sigma_0 < 0 \), then by Theorem 4.10 there exists \( \tilde{\mu} > 0 \), such that for \( \mu \in (0, \tilde{\mu}) \) there exists a minimizer to \( E^{(1)} \). As in [Cac18] one can show due to scaling properties that
\[ \Sigma_0^{(\mu, 1)} < \Sigma_0 \leq \gamma_\mu \mu^{\frac{d}{4-\gamma}} = E^{(1)}(\mathbb{R}) \]
for some \( \gamma_\mu < 0 \) and \( 0 < \mu \leq \left( \Sigma_0 / \gamma_\mu \right)^{\frac{4-\xi}{2}} \). In particular, we can deduce existence of minimizers for \( E^{(1)} \) and \( 0 < \mu \leq \left( \Sigma_0 / \gamma_\mu \right)^{\frac{6+\xi}{2}} \) by Theorem 4.15.

5. Locally Finite Graphs

In this section, we study the NLS energy functional with potentials on more general graphs. We show a decomposition formula for the form associated with the magnetic Schrödinger operator and adapt previous arguments by introducing a suitable sequence of partitions of unity in the case of locally finite graphs.

5.1. Formulation of the problem. Consider the Schrödinger operator with potentials \( M \in H^1 + W^{1,\infty}(\mathcal{G}) \) and \( V \in L^2 + L^\infty(\mathcal{G}) \) satisfying natural vertex conditions on \( \mathcal{G} \):
\[ A = \left( i \frac{d}{dx} + M \right)^2 + V \]
\[ D(A) = \left\{ u \in C(\mathcal{G}) \left| u_e \in H^2(e), \quad \forall e \in \mathcal{E} \right. \right\} \]
\[ \wedge \sum_{e \ni v} \left( i \frac{\partial}{\partial v} + M \right) u_e(v) = 0 \].
Consider the NLS functional
\[ E_{\text{NLS}}^{(K)}(u) := \frac{1}{2} \int_G \left( i \frac{d}{dx} + M \right) u^2 + V |u|^2 \, dx - \frac{\mu}{p} \int_K |u|^p \, dx \]
where \( V \in L^2 + L^\infty \) and \( K \) is a not necessarily bounded subgraph of \( G \). Define the corresponding minimization problem
\[ E_{\text{NLS}} := \inf_{u \in H^1(G) \atop \|u\|_2^2 = 1} E_{\text{NLS}}^{(K)}(u) \]
similarly as in Section 4.1:
- The localized case, when \( K \) is a bounded subgraph of \( G \);
- The global case, when \( K = G \) is the whole graph. In this case, we drop the argument and simply define
\[ E_{\text{NLS}}(u) := \frac{1}{2} \int_G \left( i \frac{d}{dx} + M \right) u^2 + V |u|^2 \, dx - \frac{\mu}{p} \int_G |u|^p \, dx \]
and
\[ E_{\text{NLS}} := \inf_{u \in H^1(G) \atop \|u\|_2^2 = 1} E_{\text{NLS}}(u). \]
We define quantities analogous to (4.7), (4.8) and (4.9). Given a bounded subgraph of \( G \) and \( R > 0 \) we define
\[ D_R := \{ \phi \in D(A) \atop \text{supp}(\phi) \subseteq G \setminus K_R \} \]
\[ \Sigma_R := \inf \{ \langle \phi, A\phi \rangle \atop \phi \in D_R, \|\phi\|_2^2 = 1 \}. \]
where \( K_R \) was defined in (3.1),
\begin{align*}
D_0 &:= D(A) \\
\Sigma_0 &:= \inf \{ \langle \phi, A\phi \rangle \atop \phi \in D(A), \|\phi\|_2^2 = 1 \}
\end{align*}
and
\begin{align*}
\Sigma &:= \lim_{R \to \infty} \Sigma_R = \sup_{R > 0} \Sigma_R.
\end{align*}
Most results can be extended simply to this case following the previous proofs. The principal difficulty lies in establishing superadditivity with respect to a suitable sequence of partitions of unity. We give a construction of such a sequence of partitions of unity in the following.

5.2. Partitions of unity in \( W^{1,\infty}(G) \). Here we give an important example for a partition of unity in \( W^{1,\infty}(G) = C^{0,1}(G) \). Given any partition of unity in \( W^{1,\infty}(G) \) one can always find a renormalization as in Lemma 4.3.

**Lemma 5.1.** Let \( G \) be a connected, locally finite metric graph. Consider any finite open covering \( \mathcal{O} \) of \( G \). Then there exists a partition of unity in \( W^{1,\infty}(G) \) subordinate to \( \mathcal{O} \) satisfying
\[ \sum_{\phi \in \mathcal{O}} \Psi_\phi^2 \equiv 1. \]
Proof. Consider a partition of unity \( \{ \psi_O \}_{O \in \mathcal{O}} \) on the graph subordinate to the open covering \( \mathcal{O} \). Then we define

\[
\Psi_O := \frac{\psi_O}{\sqrt{\sum_{O \in \mathcal{O}} \psi_O^2}}
\]

for all \( O \in \mathcal{O} \). As a product of uniformly bounded Lipschitz continuous functions, \( \Psi_O \) is also one; and by Proposition 2.11 we conclude \( \psi_O \in W^{1,\infty}(\mathcal{G}) \). Moreover,

\[
\sum_{O \in \mathcal{O}} \psi_O^2 \equiv 1 \text{ by construction.}
\]

□

Example 5.2. Let \( \mathcal{G} \) be a locally finite graph and let \( K \) be some bounded, connected subgraph. Let \( X(\mathcal{G}) = H^1(\mathcal{G}) \) and \( Y(\mathcal{G}) = W^{1,\infty}(\mathcal{G}) \). Then \( X(\mathcal{G}), Y(\mathcal{G}) \) satisfy Assumption 3.1 and Assumption 3.7. Recall the partition of unity in Example 3.10

\[
\psi(x) = \max\{ \text{dist}(\mathcal{G} \setminus K_2, x), 1 \}, \quad \tilde{\psi}(x) = 1 - \psi.
\]

We construct a sequence of partitions of unity via

\[
\psi_n(x) = \frac{1}{n} \max\{ \text{dist}(\mathcal{G} \setminus K_{2n}, x), n \}, \quad \tilde{\psi}_n(x) = 1 - \psi_n.
\]

By Lemma 5.1 we can rescale them in such a way that

\[
\Psi_n^2 + \tilde{\Psi}_n^2 \equiv 1.
\]

By definition the partitions \( K_{2n}, \mathcal{K} \setminus K_n \) are vanishing-compatible and \( \Psi_n, \tilde{\Psi}_n \) is a vanishing-compatible sequence of partitions of unity.

Definition 5.3. Let \( f \in C^{0,1}(\mathcal{G}) \). We call a point \( x \in \mathcal{G} \) a Kirchhoff point of \( f \) if one of the following holds:

1. \( x \in \mathcal{V} \) is a vertex of degree \( d_x \neq 2 \), the derivatives \( f'_e(x) \) exist for all \( e \succ x \), and \( f \) satisfies the Kirchhoff condition

\[
\sum_{e \succ x} \frac{\partial}{\partial \nu} f_e(x) = 0,
\]

2. \( x \in \mathcal{G} \) is an interior point of an edge (equivalently, a dummy vertex of degree 2), and \( f \) is differentiable at \( x \).

We call the set

\[
\mathcal{N}_f = \mathcal{G} \setminus \{ x \in \mathcal{G} : x \text{ is a Kirchhoff point of } f \}
\]

the non-Kirchoff set of \( f \).

Remark 5.4. The sequence constructed in Example 4.5 do not work here, since the functions are not in \( W^{1,\infty}(\mathcal{G}) \). We are going to consider the sequence of partitions of unity in Example 5.2 instead. This concrete sequence has some interesting properties, such that for all \( n \in \mathbb{N} \)

\[
\| \psi'_n \|_{L^\infty} = \frac{1}{n} \quad \| \tilde{\psi}'_n \|_{L^\infty} = \frac{1}{n}
\]

and in particular

\[
\| \Psi'_n \|_{L^\infty} \leq \frac{C}{n} \quad \| \tilde{\Psi}'_n \|_{L^\infty} \leq \frac{C}{n}
\]

for a \( C = C(\mathcal{G}) \) only dependent on the graph.
5.3. A decomposition formula. For the Schrödinger operator with magnetic potential

\[ \tilde{A} = \left( i \frac{d}{dx} + M \right)^2, \]

\[ D(\tilde{A}) = \tilde{H}^2, \]

one can show as in Section 4.3, see Lemma 4.7:

**Lemma 5.5.** Let \( G \) be a locally finite connected metric graph. Let

\[ \tilde{A} := \left( i \frac{d}{dx} + M \right)^2, \]

\[ D(\tilde{A}) := \tilde{H}^2(G), \]

edgewise defined, i.e.

\[ (\tilde{A}\phi)_e = \tilde{A}\phi_e. \]

Then \( \tilde{A} \) defines an unbounded operator on \( L^2(G) \) and satisfies

(i) \( fD(\tilde{A}) \subset D(\tilde{A}) \) for all \( f \in \tilde{C}^\infty(G) \).
(ii) Let \( f \in \tilde{C}^\infty(G) \), then the operator \( f\tilde{A}f \) is given by

\[ f\tilde{A}f = \frac{1}{2} \left( f^2\tilde{A} + \tilde{A}f^2 \right) + |f'|^2 \] (5.4)

**Proof.** The proof is analogous to the one in Lemma 4.7. \( \square \)

**Remark 5.6.** [5.4] does not uniquely determine an operator. Indeed [5.4] is the special case of (4.6) when \( k = 1 \). In particular, formula (4.6) in the case \( k = 1 \) holds for all self-adjoint realizations of the magnetic Schrödinger operators (e.g. (6.2)) and independent of the choice of \( M \in H^1 + W^{1,\infty}(G) \).

We will be interested in a decomposition lemma on the form associated to \( A \) as given in (5.1).

**Lemma 5.7.** Let \( G \) be a locally finite, connected metric graph and \( a(\cdot,\cdot) \) be the symmetric bilinearform given by

\[ a(u,v) := \int_G \left( i \frac{d}{dx} + M \right) u \left( i \frac{d}{dx} + M \right) v \ dx \]

for \( u,v \in H^1(G) \) Then for \( f \in W^{1,\infty}(G) \cap \tilde{C}^\infty(G) \) we have

\[ a(fu,fv) = \frac{1}{2} \left( a_G(u,f^2v) + a(f^2u,v) \right) + \langle |f'|^2u,v \rangle_{L^2(G)} \] (5.5)

**Proof.** By Proposition 2.10 we may assume \( u,v \in H^2_x(G) \) and \( fu,fv \in \tilde{H}^2(G) \cap H^1_x(G) \). Integrating by parts on an arbitrary bounded subgraph \( K \) containing \( \text{supp } u \)
and supp $v$ we compute

$$a(fu,fv) = -\int_K \left( f\hat{A} + \hat{A}f \right) uv + f''^2 u v \, dx$$

$$= -\int_K \left( \frac{1}{2} \left( f^2 \hat{A} + \hat{A} f^2 \right) u + |f''|^2 u \right) v \, dx$$

$$+ \sum_{v \in N_f \cap K} \sum_{e \ni v} \left( i \frac{d}{dx} + M \right) fu \, f(v)$$

$$= \frac{1}{2} \left( a(u,f^2 v) + a(f^2 u,v) \right) + \int_G |f''|^2 \pi v \, dx$$

$$- \sum_{v \in N_f \cap K} \sum_{e \ni v} \frac{1}{2} \left( i \frac{d}{dx} + M \right) f^2 u \, (v(v))$$

$$- \sum_{v \in N_f \cap K} \sum_{e \ni v} \frac{1}{2} \left( i \frac{d}{dx} + M \right) f^2 v \, (v(v))$$

$$+ \sum_{v \in N_f \cap K} \sum_{e \ni v} \left( i \frac{d}{dx} + M \right) fu \, f(v)$$

$$= \frac{1}{2} \left( a(u,f^2 v) + a(f^2 u,v) \right) + \int_G |f''|^2 v \, dx$$

and the statement follows by density. \qed

5.4. Existence of NLS ground state for a class of Schrödinger operators.

5.4.1. The localized setting. In the following we study the localized case. We also remark that some of the lemmas will also apply to the global case. For $t > 0$ we define

$$E^{(K)}_{t} := \inf_{u \in H^1(G), \|u\|_{L^2}^2 = t} E^{(K)}_{NLS}(u)$$

Lemma 5.8. Let $G$ be a connected locally finite metric graph. Let $K$ be a not necessarily bounded subset of $G$. The functional $E^{(K)}_{NLS}$ under $L^2$-constraint $\|u\|_{L^2}^2 = 1$ is bounded below for $2 < p < 6$.

Proof. From the Gagliardo–Nirenberg inequality (2.1) we have

$$\int_K |u|^p \, dx \leq \int_G |u|^p \, dx$$

$$\leq \varepsilon \int_G \left| \left( i \frac{d}{dx} + M \right) u \right|^2 + V|u|^2 \, dx + C_\varepsilon \int_G |u|^2 \, dx$$

and therefore

$$E^{(K)}_{NLS}(u) \geq (1 - \varepsilon) \int_G \left| \left( i \frac{d}{dx} + M \right) u \right|^2 + V|u|^2 \, dx - C_\varepsilon \geq -C_\varepsilon.$$

for all $u \in H^1(G)$ satisfying $\|u\|_2^2 = 1$. \qed
Lemma 5.9. Let $G$ be a locally finite, connected metric graph. Assume $A = (i\frac{d}{dx} + M)^2 + V$ admits a ground state, then

$$E_t^{(K)} = \inf_{u \in H^1(G)} E_{\text{NLS}}^{(K)}(u) \leq \frac{\Sigma_0 t}{2}.$$ 

The inequality is strict if the ground state does not vanish identically on $K$.

Proof. Assume $u$ is a ground state of $A = (i\frac{d}{dx} + M)^2 + V$ with $\|u\|^2_{L^2} = t$, then

$$E_{\text{NLS}}^{(K)}(u) = \frac{\Sigma_0 t}{2} - \frac{\mu}{p} \int_K |u|^p \, dx \leq \frac{\Sigma_0 t}{2}$$

and the inequality is strict if $u$ is not identically vanishing on $K$. In particular

$$\inf_{u \in H^1} E_{\text{NLS}}^{(K)} \leq \frac{\Sigma_0 t}{2}$$

with strictness in the inequality if there exists a ground state, which is not identically vanishing on $K$. □

Lemma 5.10. Let $G$ be a locally finite, connected metric graph and let $K$ be any subgraph. Assume $A = (i\frac{d}{dx} + M)^2 + V$ admits a ground state that is not identically vanishing on $K$, then the functional $E_{\text{NLS}}^{(K)}$ is weak limit superadditive, superadditive with respect to the partition of unity in Example 5.2 and $t \mapsto E_t$ as defined in (5.6) is strictly subadditive.

Proof. With the Minkowski inequality we have

$$\left( \int_G |u'|^2 \, dx \right)^{1/2} - \left( \int_G |M|^2 |u|^2 \, dx \right)^{1/2} \leq \left( \int_G \left| (i\frac{d}{dx} + M) u \right|^2 \, dx \right)^{1/2} \leq \left( \int_G |u'|^2 \, dx \right)^{1/2} + \left( \int_G |M|^2 |u|^2 \, dx \right)^{1/2}.$$ 

Adding a constant to the potential similarly as in the proof of Lemma 4.8 we may assume

$$\|u\|_{2,M,V} = \left( \int_G \left| (i\frac{d}{dx} + M) u \right|^2 + V |u|^2 \, dx \right)^{1/2}$$

to define an equivalent norm on $H^1$.

Weak limit superadditivity. Assume $u_n \to u$ weakly in $H^1$, then up to a subsequence by the Brezis–Lieb Lemma and weak limit superadditivity (similarly as in the proof of Lemma 4.8)

$$\limsup_{n \to \infty} E_{\text{NLS}}^{(K)}(u_n) = E_{\text{NLS}}^{(K)}(u) + \limsup_{n \to \infty} E_{\text{NLS}}^{(K)}(u - u_n)$$

and $E_{\text{NLS}}$ is weak limit superadditive.

Superadditivity with respect to a sequence of partitions of unity. For the superadditivity, since $u_n$ is vanishing, up to a subsequence

$$\|\Psi_n u_n\|^p_p - \|u_n\|^p_p \to 0 \quad (n \to \infty).$$
Then using the decomposition formula \((5.5)\) we compute, similarly as in the proof of Lemma 4.8:

\[
\limsup_{n \to \infty} E^{(K)}_{\text{NLS}}(u_n) = \limsup_{n \to \infty} \frac{1}{2} a(u_n, u_n) - \frac{\mu}{p} \|u_n\|^p_p \\
\geq \limsup_{n \to \infty} a(\tilde{\Psi}_n u_n, \tilde{\Psi}_n u_n) - \frac{\mu}{p} \|\tilde{\Psi}_n u_n\|^p_p \\
+ a(\Psi_n u_n, u_n) - \frac{\mu}{p} \|\Psi_n u_n\|^p_p \\
= \limsup_{n \to \infty} E^{(K)}_{\text{NLS}}(\tilde{\Psi}_n u_n) + E^{(K)}_{\text{NLS}}(\Psi_n u_n).
\]

**Subadditivity.** To show the subadditivity, note that

\[
E^{(K)}_{t} = t \inf_{u \in H^1} \frac{1}{2} \int_G \left( (\frac{d}{dx} + M)^2 u + V|u|^2 \right) dx - t^\frac{p-2}{2} \frac{\mu}{p} \int_K |u|^p dx 
\]

We deduce the property by showing that \(t \mapsto E^{(K)}_{t}\) is a concave function. Indeed, the scaling defines a concave function and in the limit we deduce concavity of the functional. In particular

\[
E^{(K)}_{t} \geq tE^{(K)}_{1}, \quad t \in [0, 1].
\]

Then

\[
E^{(K)}_{t} + E^{(K)}_{1-t} \geq E^{(K)}_{1}, \quad t \in [0, 1].
\]

For the strictness in the inequality it suffices to show strictness in the inequality \((5.8)\). Assume

\[
E^{(K)}_{t} = tE^{(K)}_{1}
\]

for some \(t \in (0, 1)\) and let \(u_n\) be a minimizing sequence for \(E_{t}\), then in particular due to \((5.7)\)

\[
\int_K |u_n|^p dx \to 0 \quad (n \to \infty).
\]

Then by density we may assume \(u_n \in D(A)\) and we infer

\[
E^{(K)}_{t} = \lim_{n \to \infty} E^{(K)}_{\text{NLS}}(u_n) \\
\geq \frac{1}{2} \limsup_{n \to \infty} \langle Au_n, u_n \rangle \geq \frac{\Sigma_0 t^2}{2},
\]

which is a contradiction to the inequality in Lemma 5.9. 

**Theorem 5.11.** Let \(G\) be a connected, locally finite metric graph. Assume \(A = (i\frac{d}{dx} + M)^2 + V\) admits a ground state, which is not identically vanishing on \(K\), then \(E^{(K)}_{\text{NLS}}\) admits a minimizer for all \(\mu > 0\).

**Proof.** For \(R > 0\) sufficiently large since \(K\) is considered to be bounded

\[
\frac{1}{2} \inf_{u \in D(A)} \frac{1}{2} \langle Au, u \rangle = \inf_{u \in D(A)} \frac{1}{2} \langle Au, u \rangle \\
\geq \inf_{u \in D(A)} \frac{1}{2} \langle Au, u \rangle = \frac{\Sigma_0}{2}
\]

\[
\inf_{u \in D(A)} \frac{1}{2} \langle Au, u \rangle = \frac{\Sigma_0}{2}
\]
In particular with Lemma 5.9 we have
\[ E_{NLS}^{(K)} < \lim_{R \to \infty} \inf_{\|u\|^2_{L^2} = 1} E_{NLS}(u) =: \widetilde{E}_{NLS}. \]

Due to Lemma 5.10 the requirements of Theorem 3.4 and 3.14 are satisfied and up to a subsequence any minimizing sequence admits a strong limit in \( L^p \) such that the limit achieves the minimum in \( E_{NLS}^{(K)} \).

□

Remark 5.12. If \( M \equiv 0 \) then we can assume that a ground state of \( A \) is nonnegative and in fact by Hopf’s maximum principle positive everywhere. In particular, any ground state of \( A \) is not identically vanishing on any subset of \( \mathcal{G} \). We will see in §6 that \( \Sigma_0 < \Sigma \) implies the existence of ground states of \( A \). In particular, this condition becomes obsolete in this case.

5.4.2. The global setting \( K = \mathcal{G} \). Consider now the global case, where we consider the functional
\[ E_{NLS}(\phi) = \frac{1}{2} \int_{\mathcal{G}} \left( \left( i \frac{d}{dx} + M \right) \phi \right)^2 + V |\phi|^2 \, dx - \frac{\mu}{p} \int_{\mathcal{G}} |\phi|^p \, dx, \quad \|\phi\|_{L^2} = 1. \]

In the global case Lemma 5.10 applies since any ground state of the magnetic Schrödinger operator \( A = (i \frac{d}{dx} + M)^2 + V \) is not identically zero. In the following we give a criterion for existence of ground states with regards to these quantities.

Proposition 5.13. Assume \( \mathcal{G} \) is a locally finite, connected metric graph and \( \Sigma_0 < \Sigma \). Then there exists \( \hat{\mu} > 0 \), such that for all \( \mu \in (0, \hat{\mu}) \),
\[ \widetilde{\Sigma}^{(\mu)} := \inf_{\phi \in D(A)} E_{NLS}(\phi) < \lim_{R \to \infty} \inf_{\phi \in D_{R}(A)} E_{NLS}(\phi) =: \widetilde{\Sigma}^{(\mu)}. \]

Proof. W.l.o.g. \( \Sigma_0 > 0 \); otherwise we simply add a constant to the potential \( V \). Let \( 0 < \varepsilon < 1 \) arbitrary, which we will only fix later. With Proposition 2.2 we deduce (similarly as in Proposition 4.9) that for sufficiently small \( \mu > 0 \)
\[ E_{NLS}(\phi) \geq \frac{1 - \varepsilon}{2} \left( \int_{\mathcal{G}} \left( \left( i \frac{d}{dx} + M \right) \phi \right)^2 + V |\phi|^2 \, dx \right) - \frac{C \varepsilon}{2}. \]

Then
\[ \widetilde{\Sigma}^{(\mu)} - \widetilde{\Sigma}^{(\mu)}_0 \geq \frac{1 - \varepsilon}{2} \Sigma - \frac{\varepsilon}{2} \Sigma_0 = \frac{1}{2} (\Sigma - \Sigma_0) - \frac{\varepsilon}{2} \left( \tilde{C} + \Sigma \right). \]

Since \( \varepsilon \) can be chosen arbitrarily small, we have for sufficiently small \( \mu \)
\[ \widetilde{\Sigma}^{(\mu)} > \widetilde{\Sigma}^{(\mu)}_0. \]

Lemma 5.14. Let \( \mathcal{G} \) be a locally finite, connected metric graph. Assume \( A = \left( i \frac{d}{dx} + M \right)^2 + V \) admits a ground state, then
\[ E_t = \inf_{u \in H^1(\mathcal{G})} E_{NLS}(u) < \frac{\Sigma_0 t}{2}. \]
Proof. In the nonlocalized case, we can proceed analogously to before. Given a ground state \( u \in H^2 \) we simply compute analogously as in Lemma 5.9
\[
E_t < \frac{\Sigma_0 t}{2}. 
\]
\( \square \)

Lemma 5.15. Assume \( G \) is a locally finite, connected metric graph and \( \Sigma_0 < \Sigma \). Then \( E_{NLS} \) is weak limit superadditive, superadditive with respect to the sequence of partitions of unity in Example 4.5 and \( t \mapsto E_t \) defines a strictly subadditive functional.

Proof. The proof is analogous to the one in Lemma 5.10 by simply replacing \( K \) with the whole graph. \( \Sigma_0 < \Sigma \), as we will see later, implies by Theorem 6.4
\[
\inf \sigma \left( \left( i \frac{d}{dx} + M \right)^2 + V \right) < \inf \sigma_{ess} \left( \left( i \frac{d}{dx} + M \right)^2 + V \right). 
\]
In particular, there exist discrete eigenvalues below the essential spectrum and \( A \) admits a ground state. \( \square \)

Theorem 5.16. Let \( G \) be a locally finite, connected metric graph. Assume \( \Sigma_0 < \Sigma \), then for \( \mu \in (0, \hat{\mu}) \) as in Proposition 4.9
\[
E_{NLS}(\phi) = \frac{1}{2} \int_G \left| \left( i \frac{d}{dx} + M \right) \phi \right|^2 + V|\phi|^2 \, dx - \frac{\mu}{p} \int_G |\phi|^p \, dx, \quad \|\phi\|_2 = 1. 
\]
admits a minimizer.

Proof. By Lemma 5.15 the requirements of Theorem 3.14 are satisfied. Furthermore the energy inequality in Corollary 3.15 is satisfied by Proposition 5.13 and we infer the statement. \( \square \)

6. On the Threshold condition

In this section we study the quantities \( \Sigma_0 \) and \( \Sigma \) that appeared in the applications of the previous sections.

6.1. Relation to essential spectrum. For details on the definitions and characterizations of the essential spectrum we refer to [RS80, §VII].

6.1.1. Polylaplacians. Let \( G = (V, E) \) be a connected metric finite graph in this first part of the section. In particular \( G \) consists of a compact core \( K \) with rays \( E_\infty \subset E \) attached to \( K \). Consider the Polylaplacian on \( G \) defined edgewise on \( H^2_k(G) \):
\[
A = (-\Delta)^k, \quad D(A) = H^{2k}(G) 
\]
Combining Lemma 4.6 and the abstract decomposition formula in Lemma 4.7 we have the decomposition formula for the Polylaplacian:

Lemma 6.1. Let \( G = (V, E) \) be a connected finite graph and assume \( \{\Psi_1, \ldots, \Psi_N\} \) to be a partition of unity subordinate to an open covering \( \mathcal{O} = \{O_1, \ldots, O_n\} \) satisfying
\[
\sum_{k=1}^N \Psi_k^2 = 1. 
\]
Then
\[ A\phi = \sum_{j=1}^{k} \Psi_j A\Psi_j \phi + \frac{(-1)^k}{2} \sum_{m=1}^{2k-2} \sum_{n=1}^{2k-m-n} \frac{(2k+m+n)!}{m!n!} \Psi_j^{(m)} \Psi_j^{(n)} \phi^{(2k-m-n)} \]
for all \( \phi \in D(A) \).

We recall that \( K = G \setminus E_\infty \) is the core of the graph and that for \( R > 0 \)
\[ D_R = \{ \phi \in D(A) | \text{supp}(\phi) \subset G \setminus K_R \} \]
\[ \Sigma_R = \inf \{ \langle \phi, A\phi \rangle | \phi \in D_R, \|\phi\|_2 = 1 \}. \]

Since \( D(A) \) is nontrivial and invariant under multiplication by test functions in \( C_\infty \), the set \( D_R \) is nonempty.

For \( R = 0 \) we set
\[ D_0 = D(A) \]
\[ \Sigma_0 = \inf \{ \langle \phi, A\phi \rangle | \phi \in D(A), \|\phi\|_2 = 1 \} \]
and recall that
\[ \Sigma = \lim_{R \to \infty} \Sigma_R = \sup_{R > 0} \Sigma_R. \]

In the following we characterize the quantities that were central to the existence theorems in the existence results before. Since \( A \) is self-adjoint one can show
\[ \Sigma_0 = \inf \sigma(A). \]

**Theorem 6.2.** Assume \( G \) is a finite metric graph. Let \( A \) be a self-adjoint, nonnegative operator on \( L^2(G) \) that satisfies the decomposition formula \((6.1)\). Additionally let \( f(A + i)^{-1} \) be compact for all \( f \in C_\infty(G) \). Then
\[ \Sigma = \inf \sigma_{\text{ess}}(A). \]

**Proof.** \( \inf \sigma_{\text{ess}}(A) \geq \Sigma \). Let \( \lambda \in \sigma_{\text{ess}}(A) \) and let \( (\phi_n) \) be an associated Weyl sequence satisfying \( \|\phi_n\|_2 = 1 \). Consider the vanishing-compatible sequence of partitions of unity \( \Psi_n, \overline{\Psi}_n \) from Example 4.5.

Since \( \Psi_R(A + i)^{-1} \) is compact for all \( R > 0 \), and since \( (A + i)\phi_n \to 0 \) as \( n \to \infty \) we deduce that
\[ \|\Psi_R\phi_n\|_2 = \|\Psi_R(A + i)^{-1}(A + i)\phi_n\|_2 \to 0 \quad (n \to \infty) \]
and passing to a subsequence, still denoted by \( \phi_n \), we may assume
\[ \|\Psi_n\phi_n\|_2 = \|\Psi_n(A + i)^{-1}(A + i)\phi_n\|_2 \to 0. \]
Furthermore, with \((2.3)\) we deduce that
\[ \|\phi_n\|_{H^2} \leq C|\phi_n|_{H^2} = C \left( \|A\phi_n\|_2^2 + \|\phi_n\|_2^2 \right)^{1/2} \]
is uniformly bounded. Since \( \phi_n \) is a Weyl sequence for \( \lambda \in \sigma_{\text{ess}}(A) \) with the decomposition formula in Lemma \((6.1)\) we then compute
\[ \lambda = \lim_{n \to \infty} \langle \phi_n, A\phi_n \rangle_{L^2} \]
\[ = \lim_{n \to \infty} \langle \Psi_n\phi_n, A\Psi_n\phi_n \rangle_{L^2} + \langle \overline{\Psi}_n\phi_n, A\overline{\Psi}_n\phi_n \rangle_{L^2} + O \left( \frac{1}{n^2} \right) \]
\[ \geq \lim_{n \to \infty} \sum_{e \in E_\infty} \langle \overline{\Psi}_n\phi_n, A\overline{\Psi}_n\phi_n \rangle_{L^2} \geq \lim_{n \to \infty} \Sigma_n = \Sigma. \]
Since $\lambda \in \sigma_{\text{ess}}(A)$ was arbitrary, we conclude $\inf \sigma_{\text{ess}}(A) \geq \Sigma$.

Assume for a contradiction that $\inf \sigma_{\text{ess}}(A) \geq \Sigma + 3\varepsilon$ with $\varepsilon > 0$. Then $\sigma(A) \cap (-\infty, \Sigma + 2\varepsilon]$ is discrete and since $A$ is bounded from below, the spectral projector $P_\Sigma := P_{(-\infty, \Sigma + 2\varepsilon]}$ is of finite rank. Assume $\phi_n \in D_n(A)$ is a sequence such that

$$\langle \phi_n, A\phi_n \rangle \leq \Sigma + \varepsilon$$

and $\phi_n \to 0$ in $L^2$. Then since $(A + \Sigma + 2\varepsilon)P_\Sigma$ is a compact operator and

$$(A + \Sigma + 2\varepsilon)P_\Sigma \phi_n \to 0 \quad (n \to \infty).$$

Hence

$$\langle \phi_n, A\phi_n \rangle_{L^2} = \langle \phi_n, A(1 - P_\Sigma)\phi_n \rangle + \langle \phi_n, AP_\Sigma \phi_n \rangle_{L^2}$$

$$\geq (\Sigma + 2\varepsilon)\langle \phi_n, (1 - P_\Sigma)\phi_n \rangle_{L^2} + \langle \phi_n, AP_\Sigma \phi_n \rangle_{L^2}$$

$$\geq \Sigma + 2\varepsilon + \langle \phi_n, (A + \Sigma + 2\varepsilon)P_\Sigma \phi_n \rangle_{L^2}.$$

Passing to the limit we conclude

$$\lim_{n \to \infty} \inf \langle \phi_n, A\phi_n \rangle_{L^2} \geq \Sigma + 2\varepsilon$$

and the statement follows by contradiction. \hfill \Box

6.1.2. Schrödinger operators and IMS formula on locally finite graphs. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a locally finite graph throughout the rest of the section. Consider the Schrödinger operators with potentials $M$ and $V$ satisfying natural vertex conditions on $\mathcal{G}$:

$$A = \left( i \frac{d}{dx} + M \right)^2 + V$$

(6.2) $D(A) = \left\{ u \in C(\mathcal{G}) \middle| u_e \in H^2(e), \quad \forall e \in \mathcal{E} \wedge \sum_{e \succ v} \left( i \frac{d}{dx} + M \right) u_e(v) = 0, \quad \forall v \in \mathcal{V} \right\}.$$

In the case of magnetic Schrödinger operators

$$A = \left( i \frac{d}{dx} + M \right)^2 + V$$

on domains $\Omega \subset \mathbb{R}^N$ the decomposition formula can be obtained via the IMS formula\(^2\)

$$A = \sum_{j=1}^{k} \Psi_k A \Psi_k + |\Psi_k'|^2 \quad \text{where} \quad \sum_{j=1}^{k} \Psi_k^2 \equiv 1.$$\(^2\)

When considering locally finite graphs, we may not use the approach as before because general locally finite graphs do not necessarily contain a core and we need to adapt the theory using sequences of partitions of unity as in Example 5.2.

\(^2\)According to [Sim83] due to Israel Michael Sigal.
Definition 6.3 (IMS formula on locally finite graphs). Let $\mathcal{G}$ be a locally finite, connected metric graph. Let $A : D(A) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G})$ be a densely defined, self-adjoint operator and assume $a(\cdot, \cdot)$ is the associated symmetric, sesquilinear form, defined on $H^1(\mathcal{G})$. We say $A$ satisfies the IMS formula if for all $f \in W^{1,\infty}(\mathcal{G}) \cap C^\infty(\mathcal{G})$

\begin{equation}
(6.3) \quad a(fu, fv) = \frac{1}{2} \left( a(u, f^2 v) + a(f^2 u, v) \right) + \langle |f'|^2 u, v \rangle_{L^2}, \quad \forall u, v \in D(A).
\end{equation}

We showed in Lemma 5.7 that the magnetic Schrödinger operator in (5.1) satisfies the IMS formula (6.3); and in particular the following result applies:

Theorem 6.4. Assume $\mathcal{G}$ is a locally finite, connected metric graph. Let $A$ be a self-adjoint, nonnegative operator on $L^2(\mathcal{G})$ that satisfies the IMS formula (6.3). Additionally let $f(A + i)^{-1}$ be compact for all $f \in C^{0,1}_c \cap C^\infty(\mathcal{G})$; then

$$
\Sigma = \inf_{\sigma_{\text{ess}}(A)} \sigma_{\text{ess}}(A).
$$

Proof. $\inf_{\sigma_{\text{ess}}(A)} \sigma_{\text{ess}}(A) \leq \Sigma$ follows by an abstract argument analogous to the argument in Theorem 6.2. To establish $\inf_{\sigma_{\text{ess}}(A)} \sigma_{\text{ess}}(A) \leq \Sigma$ consider $\lambda \in \sigma_{\text{ess}}(A)$ and let $(\phi_n)$ be an associated Weyl sequence satisfying $\|\phi_n\|_2^2 = 1$.

Consider the vanishing-compatible sequence of partitions of unity $\Psi_n, \overline{\Psi}_n$ from Example 5.2.

Since $\overline{\Psi}_n^R(A + i)^{-1}$ is compact for all $R > 0$, and since $(A + i)\phi_n \to 0$ as $n \to \infty$ we deduce

$$
\|\overline{\Psi}_n^R\phi_n\|_2 = \|\overline{\Psi}_n^R(A + i)^{-1}(A + i)\phi_n\|_2 \to 0 \quad (n \to \infty)
$$

and passing to a subsequence, still denoted by $\phi_n$, we may assume

$$
\|\Psi_n\phi_n\|_2 = \|\Psi_n(A + i)^{-1}(A + i)\phi_n\|_2 \to 0.
$$

Since $\phi_n$ is a Weyl sequence for $\lambda \in \sigma_{\text{ess}}(A)$, with the decomposition formula in Lemma 6.1 we then compute

$$
\lambda = \lim_{n \to \infty} \langle \phi_n, A\phi_n \rangle_{L^2} = \lim_{n \to \infty} a(\Psi_n\phi_n, \Psi_n\phi_n) + a(\overline{\Psi}_n\phi_n, \overline{\Psi}_n\phi_n) + O \left( \frac{1}{n^2} \right)
$$

$$
\geq \lim_{n \to \infty} \sum_{e \in E} a(\Psi_n\phi_n, \Psi_n\phi_n) \geq \lim_{n \to \infty} \Sigma_n = \Sigma.
$$

Since $\lambda \in \sigma_{\text{ess}}(A)$ was arbitrary, we conclude $\inf_{\sigma_{\text{ess}}(A)} \sigma_{\text{ess}}(A) \geq \Sigma$. \hfill \Box

6.2. Sufficient conditions for the threshold condition for the Polylaplacian. In this section we obtain criteria for the threshold condition for the operator

$$
A = (-\Delta)^k + V
$$

$$
D(A) = H^{2k}
$$

We start with the case $k = 1$ for general locally finite graphs satisfying a volume growth assumption. For $k \geq 2$ we restrict ourselves to finite graphs.
Proposition 6.5. Let $\mathcal{G}$ be a locally finite, connected metric graph and let $K$ be a connected, precompact subgraph. We suppose additionally the volume assumption
\begin{equation}
|K_{2n} \setminus K_n| = o(n^2) \quad (n \to \infty).
\end{equation}
Assume $\sigma_{\text{ess}}(-\Delta + V) \subset [0, \infty)$ and assume additionally either

(i) $V \in L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ and

\begin{equation*}
\int_{\mathcal{G}} V \, dx < 0
\end{equation*}

(ii) or $V < 0$ on $\mathcal{G}$.

Then $\Sigma_0 < \Sigma$ (as defined in (5.2) and (5.3)) and there exists $\hat{\mu} > 0$ such that the minimization problem
\begin{equation}
E^{(1)} = \inf_{u \in H^1_{K_n} \cap L^2(\mathcal{G})} \int_{\mathcal{G}} \left( |\nabla u|^2 + V |u|^2 \right) dx = 0
\end{equation}

admits a minimizer for $\mu \in (0, \hat{\mu})$.

Proof. Consider as a test function $\Psi_n$ as defined in Example 5.2, then we only need to show that for $n$ sufficiently high, the Rayleigh quotient
\begin{equation*}
R[\Psi_n] := \frac{\int_{\mathcal{G}} |\nabla \Psi_n|^2 + V |\Psi_n|^2 \, dx}{\int_{\mathcal{G}} |\Psi_n|^2 \, dx} < 0.
\end{equation*}

Indeed, since $\|\Psi_n\|^2_{L^\infty} \leq O(\frac{1}{n^2})$ as $n \to \infty$ we deduce
\begin{equation*}
\|\Psi_n\|_2^2 \leq \|\Psi_n\|^2_{L^\infty} |K_{2n} \setminus K_n| \to 0 \quad (n \to \infty).
\end{equation*}

If $V < 0$ then for sufficiently large $n$ and $\varepsilon > 0$ sufficiently small
\begin{equation*}
\int_{\mathcal{G}} V |\Psi_n|^2 \, dx \leq -|\{x \in \mathcal{G} : V(x) \leq -\varepsilon\}| \varepsilon < 0.
\end{equation*}

If $\int_{\mathcal{G}} V \, dx < 0$, then
\begin{equation*}
\liminf_{n \to \infty} \int_{\mathcal{G}} V |\Psi_n|^2 \, dx = \int_{\mathcal{G}} V \, dx < 0
\end{equation*}
by dominated convergence. In particular for $n$ large enough
\begin{equation*}
\int_{\mathcal{G}} V |\Psi_n|^2 \leq \frac{1}{2} \int_{\mathcal{G}} V \, dx < 0.
\end{equation*}

We deduce $R[\Psi_n] < 0$ and thus $\inf \sigma((-\Delta)^k + V) < 0$. Then $\Sigma_0 < \Sigma$ and we conclude the existence of minimizers of (6.5) by Theorem 5.16. \hfill \Box

Similarly, for $k \geq 2$ we have:

Proposition 6.6. Let $\mathcal{G}$ be a finite, connected metric graph and $k \geq 1$. Assume $
\sigma_{\text{ess}}((-\Delta)^k + V) \subset [0, \infty)$ and assume additionally either

(i) $V \in L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ and

\begin{equation*}
\int_{\mathcal{G}} V \, dx < 0
\end{equation*}

(ii) or $V < 0$ on $\mathcal{G}$.
Then $\Sigma_0 < \Sigma$ (as defined in (5.2) and (5.3)) and there exists $\hat{\mu} > 0$, such that the minimization problem

$$E^{(k)} = \inf_{u \in H^1} \frac{E^{(k)}(u)}{\|u\|_2^2 = 1}$$

admits a minimizer for $\mu \in (0, \hat{\mu})$.

**Proof.** The proof is analogous to the proof in Proposition 6.5. We only need to replace the test functions $\Psi_n$ with the ones in Example 4.5. Then

$$\|\Psi^{(k)}_n\|_{\infty} \leq \frac{1}{n^{2k}} C$$

with $C$ independent of $n$. We infer the result as in the proof of Proposition 6.5. □

**Remark 6.7.** If $V$ is a relativly compact perturbation of $(-\Delta)^k$, i.e.

$$V \left((-\Delta)^k + i\right)^{-1}$$

is compact, then $\inf \sigma_{\text{ess}}((-\Delta)^k + V) = 0$ and we deduce

$$\inf \sigma_{\text{ess}}((-\Delta)^k + V) \subset [0, \infty).$$

We finish the section by giving a criterion for the potential $V$ such that

$$\Sigma = \lim_{n \to \infty} \inf_{u \in D(A)} \|u\|_2^2 = 1, \text{ supp } u \subset \mathcal{G} \setminus K_n$$

with (6.7) in particular implies

$$\sigma_{\text{ess}}((-\Delta)^k + V) \subset [0, \infty).$$

Consider decaying potentials $V = V_2 + V_\infty$ with $V_2 \in L^2(\mathcal{G})$ and $V_\infty \in L^\infty(\mathcal{G})$ such that

$$\sup_{x \in \mathcal{G} \setminus K_n} |V_\infty(x)| \to 0 \quad (n \to \infty).$$

**Proposition 6.8.** Let $\mathcal{G}$ be a locally finite metric graph. Assume $V \in L^2 + L^\infty$ satisfying (6.6). Let $A = (-\Delta)^k + V$, then

$$\Sigma = \lim_{n \to \infty} \inf_{u \in H^k(\mathcal{G})} \|u\|_2^2 = 1, \text{ supp } u \subset \mathcal{G} \setminus K_n$$

and $\langle u_n, Au_n \rangle_{L^2} \to \Sigma$. 

**Proof.** Assume $u_n$ is a minimizing sequence, such that $\|u_n\|^2_{L^2}, \text{ supp } u \subset \mathcal{G} \setminus K_n$ and

$$\langle u_n, Au_n \rangle_{L^2} \to \Sigma.$$

With (2.3) we deduce that

$$\|u_n\|_{H^k} \leq C \left(\langle u_n, Au_n \rangle_{L^2}^2 + \|u_n\|_2^2\right)$$

is uniformly bounded. Integrating by parts and using (6.7) we infer

$$\int_\mathcal{G} |u_n^{(k)}|^2 + V|u_n|^2 \text{ d}x \geq \int_\mathcal{G} |u_n^{(k)}|^2 \text{ d}x - \tilde{C} \left(\left(\int_{\mathcal{G} \setminus K_n} |V|^2 \text{ d}x\right)^{1/2} + \sup_{x \in \mathcal{G} \setminus K_n} |V_\infty(x)|\right).$$
We have
\[
\left( \int_{G \setminus K_n} |V|^2 \, dx \right)^{1/2} + \sup_{x \in G \setminus K_n} |V_\infty(x)| \to 0 \quad (n \to \infty).
\]
Thus,
\[
\Sigma = \lim_{n \to \infty} \langle u_n, Au_n \rangle_{L^2} \geq 0.
\]

7. Application: Schrödinger operators with magnetic potentials on infinite tree graphs

In certain cases as discussed in [BK13, §2.6] the gauge transform \(G\), as defined below, unitarily transforms the Schrödinger operator with magnetic potential into a Schrödinger operator without magnetic potential, and the NLS functional under gauge transform reduces to a problem without magnetic potential. We may use the results from Section 6.2 to show existence of minimizers of the NLS functional with magnetic potential.

For infinite tree graphs in the context of locally finite, connected metric graphs it is particularly easy to see this. In this context, let \(G\) be an infinite tree graph. Given a vertex \(v\) we can define the gauge transform \(G\) radially. For any \(x \in G\), let \(\gamma\) be a simple path from \(v\) to \(x\) parametrized by arc length, then
\[
G: u(x) \mapsto e^{i \int_{\gamma} M \, dy} u(x).
\]
Assume \(A^M = (i \frac{d}{dx} + M)^2 + V\) admits a ground state. In this particular case since
\[
G^{-1}A^M G = -\Delta + V = A^0,
\]
this is equivalent to the assertion that \(A^0\) admits a ground state. Indeed, let \(u_M\) be a ground state to \(A^M\), then
\[
A^0 G^{-1} u_0 = G^{-1} A^M u_M = \Sigma G^{-1} u_M
\]
and \(G^{-1} u_M\) is a ground state of \(A^0\). Then we may assume \(u_0 > 0\) by phase invariance and the maximum principle. Then \(u_M\) does not vanish anywhere. In particular independent of \(M \in H^1 + W^{1,\infty}(G)\)
\[
\Sigma_0^M = \inf_{u \in D(A^M)} \langle A^M u, u \rangle = \inf_{u \in D(A^0)} \langle A^0 u, u \rangle = \Sigma_0
\]
\[
\Sigma_R^M = \inf_{u \in D_R(A^M)} \langle A^M u, u \rangle = \inf_{u \in D_R(A^0)} \langle A^0 u, u \rangle = \Sigma_R
\]
and in Section 6.2 we gave sufficient conditions for \(\Sigma_0 < \Sigma\).

**Proposition 7.1.** Assume \(G\) is an infinite tree graph, connected and locally finite. Assume \(K\) is a bounded subgraph of \(G\) and \(-\Delta + V\) admits a ground state, then the minimization problem
\[
E_{\text{NLS}}^{(K)} = \inf_{u \in H^1(G)} \frac{1}{2} \int_G \left( i \frac{d}{dx} + M \right) u^2 + V |u|^2 \, dx - \frac{\mu}{p} \int_K |u|^p \, dx
\]
admits a minimizer for all $\mu \in \mathbb{R}$.

**Proof.** This follows immediately from Theorem 5.11 and the unitary equivalence of the problem in absence of a magnetic potential under the gauge transform. \hfill \box

**Proposition 7.2.** Assume $G$ is a infinite tree graph, locally finite and connected. Assume $K$ is any unbounded subgraph and $\Sigma_0 < \Sigma$ then there exists $\hat{\mu} > 0$, such that the infimization problem

$$E_{NLS} = \inf_{\phi \in H^1(G)} \frac{1}{2} \int_G \left( i \frac{d\phi}{dx} + M \right) \phi^2 + V|\phi|^2 \, dx - \int_K \mu |\phi|^p \, dx,$$

admits a minimizer for all $\mu \in (0, \hat{\mu})$.

**Proof.** This follows immediately from Theorem 5.16 and the unitary equivalence of the problem in absence of a magnetic potential under the gauge transform \hfill \box

For decaying potentials in \S 6.2 we discussed criteria such that $\Sigma_0 < \Sigma$ is satisfied. Indeed, for any given locally finite graph, one can construct decaying potentials in the following way:

**Example 7.3.** Let $G$ be a locally finite, connected graph and $K$ a bounded, connected subgraph. Consider the higher-order Schrödinger operator $A = (-\Delta)^k + V$ with potential $V$. We define a potential $V$ a.e. via

$$V \bigg|_{K_{2n} \setminus K_n} = -\frac{1}{2^{n-1} |K_{2n} \setminus K_n|}, \quad n \geq 2$$

on each “annulus” $K_{2n} \setminus K_n$. Then $V \in L^2 \cap L^1(G)$,

$$\int_G V \, d\mu = -\sum_{n=0}^{\infty} \frac{1}{2^n} < 0$$

and by Proposition 6.8 we infer $\inf \sigma_{ess}(A) \geq 0$. In particular, if $G$ is an infinite tree graph satisfying the volume growth assumption (6.4), then the prerequisites in Proposition 6.5 are satisfied as well and we have

$$\Sigma_0 < \Sigma.$$

In particular Proposition 7.1 and Proposition 7.2 are applicable and there exists $\hat{\mu} > 0$ such that

$$E_{NLS}^{(K)} = \inf_{u \in H^1(G)} \frac{1}{2} \int_G \left( i \frac{du}{dx} + M \right) u^2 + V|u|^2 \, dx - \frac{\mu}{p} \int_K |u|^p \, dx$$

admits a minimizer for $\mu \in (0, \hat{\mu})$. If $K \subset G$ is precompact, then minimizers exist for all $\mu > 0$.

For a certain class of infinite tree graphs we can in a similar way as in Example 4.16 give an explicit $\hat{\mu}$ such that for $\mu \in (0, \hat{\mu}]$ the minimization problem $E_{NLS}$ admits a minimizer.
Example 7.4. Consider an unrooted tree graph \( G \) as considered for instance in [DST19], i.e. there are no vertices of degree 1 apart of vertices at infinity. Such trees in particular satisfy the \((H)\)-condition formulated in [AST15] in the special case of finite graphs:

\((H)\) For every point \( x \in G \), there exist two injective curves \( \gamma_1, \gamma_2 : [0, +\infty) \to G \) parametrized by arc length, with disjoint images except on a discrete set of points, and such that \( \gamma_1(0) = \gamma_2(0) = x \).

and by rearrangement methods as in [DST19] one can show for decaying potentials \( V = V_2 + V_\infty \) with \( V_2 \in L^2(G) \) and \( V_\infty \in L^\infty(G) \) satisfying

\[
\sup_{x \in G \setminus K_n} |V_\infty(x)| \to 0 \quad (n \to \infty)
\]

that

\[
\tilde{\Sigma}(\mu) = \lim_{n \to \infty} \inf_{u \in H^1(G), \|u\|^2 = 1, \text{supp } u \subset G \setminus K_n} E_{\text{NLS}}^V(u) \\
\geq \lim_{n \to \infty} \inf_{u \in H^1(G), \|u\|^2 = 1, \text{supp } u \subset G \setminus K_n} E_{\text{NLS}}^0(u) \geq E_{\text{NLS}}(\mathbb{R}),
\]

where by Remark 4.14 one has equality if \( G \) contains a half line.

When \( V \equiv 0 \), by strictness in the rearrangement inequality one can prove nonexistence results similarly as in [AST15]. On the other hand, under the assumption \( \Sigma_0 = \inf \sigma(-\Delta + V) < 0 \), as discussed in Example 4.16 we have thus the existence of minimizers of \( E_{\text{NLS}} \) for

\[
\mu \in \left[ 0, \left( \frac{\Sigma_0}{\gamma_p} \right)^{\frac{6-\gamma}{\gamma}} \right].
\]

Remark 7.5. The arguments in Example 7.4 can be applied to all graphs that satisfy the \((H)\)-condition. One can even consider more general graphs as long they satisfy the following weaker version of the \((H)\)-condition:

\((\bar{H})\) There exists a precompact set \( K \subset G \), such that for every point \( x \in G \setminus K \), there exist two injective curves \( \gamma_1, \gamma_2 : [0, +\infty) \to G \) parametrized by arc length, with disjoint images except on a discrete set of points, and such that \( \gamma_1(0) = \gamma_2(0) = x \).

Example 7.6. Consider the graph consisting of two half-lines and a pendant edge joined at a single vertex (see also Figure 2), then the graph satisfies the \((\bar{H})\)-condition but not the \((H)\)-condition and the existence result from Example 7.4 as discussed in Remark 7.5 is still applicable.

\[
\begin{array}{c}
\infty \\
\ell \\
0 \\
\infty
\end{array}
\]

Figure 2. The graph consisting of two half-lines and a pendant edge as an example of a graph that satisfies the the \((\bar{H})\)-condition but not the \((H)\)-condition.
We finish this section by proving Theorem 1.10.

Proof of Theorem 1.10. Let $G$ be a locally finite metric tree graph that contains at most finitely many vertices of degree 1. Then there exists a connected, precompact set $K \subset G$ that contains all vertices of degree 1 by assumption. Consider the set $\mathcal{G}$ of points $x \in G$, such that there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ parametrized by arc length, with disjoint images except on a discrete set of points, and such that $\gamma_1(0) = \gamma_2(0) = x$. In particular, if $x \in \mathcal{G}$, then

$$\text{im} \gamma_1, \text{im} \gamma_2 \subset \mathcal{G}.$$

1st Case: $\mathcal{G} \neq \emptyset$. Then by assumption $\mathcal{G} \setminus \mathcal{G}$ contains at most finitely many connected components. Moreover of the connected components is precompact. Otherwise one could construct an injective curve $\gamma_1 : [0, +\infty) \to \mathcal{G} \setminus \mathcal{G}$ for all $x \in \mathcal{G} \setminus \mathcal{G}$ and since we assumed $\mathcal{G} \neq \emptyset$, we can construct $\gamma_2 : [0, +\infty) \to \mathcal{G} \setminus \mathcal{G}$. This would then imply that $\mathcal{G} \setminus \mathcal{G}$ is necessarily precompact. Since $\mathcal{G}$ is a tree graph, this also implies that each connected component of $\mathcal{G} \setminus \mathcal{G}$ contains necessarily a vertex of degree 1. In particular, $\mathcal{G} \setminus \mathcal{G}$ admits at most finitely many connected components and is precompact. By construction, $\mathcal{G}$ satisfies the (H)-condition and hence $G$ satisfies the $(\bar{H})$-condition. Then as in Example 7.4 we have

$$\bar{\Sigma}(\mu) = \lim_{n \to \infty} \inf_{\|u\|^2=1, \text{supp} u \subset G \setminus K_n} E_{NLS}^G(u) \geq \lim_{n \to \infty} \inf_{\|u\|^2=1, \text{supp} u \subset G \setminus K_n} E_{NLS}^0(u) \geq E_{NLS}(\mathbb{R})$$

and we obtain existence of minimizers of $E_{NLS}$ for

$$\mu \in \left[0, \left(\frac{\Sigma_0}{\gamma_p}\right)^{\frac{6-p}{4}}\right].$$

2nd Case: $\mathcal{G} = \emptyset$. In particular for each $x \in G$ there exists only one connected component of $\mathcal{G}$ that contains a vertex at infinity. Assume $K$ is a precompact set that contains all vertices of degree 1, then by assumption for any $x \in G \setminus K$ the connected components of $G \setminus \{x\}$ consist of a compact core graph containing all vertices of degree 1 and a half-line. In particular, $G$ is a finite graph and Example 4.16 yields the existence of minimizers of $E_{NLS}$ for

$$\mu \in \left[0, \left(\frac{\Sigma_0}{\gamma_p}\right)^{\frac{6-p}{4}}\right].$$

□

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