Functorial aggregation

David I. Spivak

Abstract

Aggregating data in a database could also be called “integrating along fibers”: given functions \( \pi: E \to D \) and \( s: E \to R \), where \((R, \otimes)\) is a commutative monoid, we want a new function \((\otimes s)_\pi: D \to R\) that sends each \(d \in D\) to the “sum” of all \(s(e)\) for which \(\pi(e) = d\). The operation lives alongside querying—or more generally data migration—in typical database usage: one wants to know how much Canadians spent on cell phones in 2021, for example, and such requests typically require both aggregation and querying. But whereas querying has an elegant category-theoretic treatment in terms of parametric right adjoints between copresheaf categories, a categorical formulation of aggregation—especially one that lives alongside that for querying—appears to be completely absent from the literature.

In this paper we show how both querying and aggregation fit into the “polynomial ecosystem”. Starting with the category \textbf{Poly} of polynomial functors in one variable, we review the relatively recent results of Ahman-Uustalu and Garner, which showed that the framed bicategory \textbf{Cat}^\sharp of comonads in \textbf{Poly} is precisely the right setting for data migration: its objects are categories and its bicomodules are parametric right adjoints between their copresheaf categories. We then develop a great deal of theory, compressed for space reasons, including local monoidal closed structures, a coclosure to bicomodule composition, and an understanding of adjoints in \textbf{Cat}^\sharp. Doing so allows us to derive interesting mathematical results, e.g. that the ordinary operation of transposing a span can be decomposed into the composite of two more primitive operations, and then finally to explain how aggregation arises, alongside querying, in \textbf{Cat}^\sharp.

Keywords: Aggregation, category, copresheaf, polynomial functor, parametric right adjoint, polynomial comonad, cofunctor, comodule, coclosure, profunctor, framed bicategory, database, formal category theory.

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1 Introduction

Alongside querying, aggregating data is one of the most important operations in real-world databases, but it has not yet been given an adequate category-theoretic treatment. The basic operation is akin to “integrating along compact fibers” in differential geometry, and for databases it’s quite simple to explain.

1.1 Aggregating and querying

Suppose given a function \( \pi: E \to D \), say \( E \) is the set of employees in a company, \( D \) is the set of departments, and every employee \( e \in E \) works in a department \( \pi(e) \in D \).\(^1\)

Now if there is a function \( s: E \to \mathbb{R} \), assigning a salary \( s(e) \) to each employee, we can aggregate the salaries of the employees in each department to get a total department salary. That is, there should be an induced function \( (\text{sum } s)_{\pi}: D \to \mathbb{R} \) as in the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{s} & \mathbb{R} \\
\pi \downarrow & & \uparrow \\
D & \xrightarrow{(\text{sum } s)_{\pi}} & \mathbb{R}
\end{array}
\]  

where (1) is not intended to commute. The aggregate function is given by

\[
(\text{sum } s)_{\pi}(d) := \sum_{\pi(e)=d} s(e). \tag{2}
\]

In general, we can replace \( \mathbb{R} \) with any commutative monoid, and call the induced map aggregation.

\(^1\)We should also assume that the fibers are compact in the sense that for every \( d \in D \), the preimage \( \pi^{-1}(d) \subseteq E \) is a finite set.
Aggregation is arguably the most valuable practical operation one performs on databases; it is what allows the CEO to ask “how much did Canadians spend on cell phones in 2021”. It is what allows us to count the rows in a table, e.g. how many rooms are vacant in a hotel, etc. In fact, since the free commutative monoid on a set $E$ is the monoid $M(E)$ of multisets in $E$, perhaps the most important aggregation function is the “group-by” operation

$$
\begin{array}{ccc}
E & \rightarrow & M(E) \\
\pi & \downarrow & \lessgtr \\
D & \rightarrow & \text{groupBy}_\pi
\end{array}
$$

sending each $d \in D$ to the set $\text{groupBy}_\pi(d) := \{ e \in E \mid \pi(e) = d \}$. With $\text{groupBy}_\pi : D \rightarrow M(E)$ in hand, one can use maps out of $M(E)$ to do things like plot graphs of data, etc.

In terms of usefulness, the only database operation that contends with aggregation is that of querying. Roughly speaking, querying a database is asking for rows that agree in a certain way, e.g. asking for “all the pairs of people with the same favorite book”. In other words, queries—or more precisely conjunctive queries—are limit operations, such as pullbacks

$$
\begin{array}{ccc}
Q & \rightarrow & P \\
\downarrow & \searrow & \downarrow \\
P & \rightarrow & B
\end{array}
$$

Whereas aggregation uses a commutative monoid to combine all rows within a single table, querying is a row-by-row matching operation across tables. More generally, one can take disjoint unions of conjunctive queries; we call these duc-queries. For example, in the US, every city is in a state and every county is also in a state, so we have a cospan of sets

$$
\text{city} \rightarrow \text{state} \leftarrow \text{county}
$$

and one can perform a duc-query on such data, e.g. asking for the set

$$(\text{city} \times_{\text{state}} \text{city}) + (\text{city} \times_{\text{state}} \text{county}) + (\text{county} \times_{\text{state}} \text{county})$$

As hinted at in the above example, databases can be modeled by small categories and copresheaves on them. The category $C$ lays out the organizational form (called the schema) of the database, and the functor $X : C \rightarrow \text{Set}$ provides the data itself. We refer objects in the category (topos)

$$C\text{-Set} := \text{Set}^C = \text{Fun}(C, \text{Set})$$

as $C$-instances. This formulation can be found in [Spi12] and when categories are replaced by limit or even coproduct-limit sketches, it can be found much earlier in [JR02]. This story of categorical databases, together with a notion of attributes that we’ll discuss later, has been implemented in open source code [Sch+17; PLF21].
Unlike the sketch-formulation, databases as categories and copresheaves opens up the possibility of using the left and right Kan extensions,

$$\Sigma_F \quad \Delta_F$$

induced by a functor $F: C \to D$, to migrate data. It turns out that the general $\Sigma$ operation is not as useful as one might think, because it can quotient data, and doing so can result in unwanted behavior. For example, the resulting quotients often tend to equate different strings in such a way that across the entire database we have an equality of strings “$A” = “B”! As one can imagine, freely interchanging arbitrary instances of “a” with “b” causes problems. Colimits are useful in mathematics, e.g. for quotienting groups, but somehow they seem not to be as useful in database practice. However, a special case is useful, namely when $F$ is a discrete opfibration (etale): in this case $\Sigma_F$ simply performs coproducts rather than general colimits. The mathematics works out more nicely too: given a setup

$$H \quad G \quad F$$

of categories and functors, with $F: B \to D$ assumed to be etale, the composite

$$\Sigma_F \circ \Pi_G \circ \Delta_H : C\textbf{-Set} \to D\textbf{-Set}$$

is a parametric right adjoint functor, which we call a pra-functor for short, and every prafunctor can be written in the above form. Prafunctors are closed under composition; in the case that $B$, $C$, $D$, and $E$ are discrete categories, prafunctors are exactly the polynomial functors as in [GK12].

We sometimes refer to prafunctors as data migration functors, because a prafunctor as in (5) migrates data from $C$ to $D$ by assigning to each object $d \in D$ a duc-query on $C\textbf{-Set}$. For example when $D = 1$, the constraint that $F$ be etale forces $B = B$ to be discrete, and the resulting prafunctor $C\textbf{-Set} \to \textbf{Set}$ is nothing but a disjoint union (indexed by the set $B$) of conjunctive queries, i.e. limits, as induced by the right adjoint $\Pi_G \circ \Delta_F : C\textbf{-Set} \to B\textbf{-Set}$.

Thus the world of duc-queries—and more generally data-migration functors—has beautiful category theory attached to it; in fact this is an understatement. In the last half-decade (2016 – 2021) we’ve seen a remarkable breakthrough in this theory, due to two results:

- Ahman-Uustalu: polynomial comonads are precisely categories, and
- Garner: bicomodules between polynomial comonads are prafunctors.

All this theory takes place in the framed bicategory, which we denote by $\textbf{Cat}^\#$, of comonoids and comodules in $\textbf{Poly}$. It means that the category of polynomials in one

$^{2}$Note that the $\Sigma$ used here is intended as a colimit, and not intended to be the same symbol as the $\Sigma$ used in (2).
variable is a sufficiently powerful setting that its comonoids and comodules are exactly what are needed for data migration, i.e. that these native notions capture both the categories $C$ and $D$ and the profunctors $\Sigma_F \circ \Pi_C \circ \Delta_H$.

For me personally, the Ahman-Uustalu-Garner results opened up a huge playground that combined and generalized much of my previous work on interacting dynamical systems—which will not appear in this paper—and databases. But in particular, it has allowed me to answer a question that has been plaguing me for over a decade, namely how to understand aggregation categorically; this is the subject of the present paper. Articulating the way aggregation and data migration live together in one mathematical world relies on a great deal of structure available in the polynomial ecosystem. In the next sections I will introduce these structures, but before leaving the introduction I want to say why aggregation was so difficult to get right.

There are several issues that make aggregation difficult. First, whereas querying is functorial, aggregation is not. A map of instances (copresheaves) on $C$ is a natural transformation; for example inserting a new row into a table is natural. If we query a database before and after someone inserts a new employee $e$ into $E$, we'll get a map between the results. But if we aggregate before and after an insertion, there will be no relation between the results: the new sum of salaries may be more or (if the new employee is paid negatively) less than the original. Thus there will be no natural transformation between the aggregate before the insertion and the aggregate afterwards: a map $I \to J$ in $C\text{-Set}$ does not induce any sort of maps between their aggregates, just as there is no sensible map between $3$ and $3 + 4 - 1$. How might we rectify this? Are we to imagine attributes as living in codiscrete categories? This does not work because then one could replace one value with another at will (like “$a$”$=$“$b$” again), accordingly changing all the data in the database, and call the results isomorphic; this is not a good solution. And since aggregation is not functorial in the same way that querying is, in what way could the querying and aggregation stories possibly interoperate?

The second thing that makes aggregation difficult is that it’s so easy. Someone might say “Why don’t you just do it? If you want to add all the salaries up, just use the fact that $M$ is a commutative monoid and add them up!” In some sense this is correct. Without knowing what constraints to put on ourselves, especially once we know that aggregation is not functorial in the way querying is, we lack a good sense of boundaries by which to know what sort of structures are off-limits, and what sort of structures are fair game.

Luckily, the polynomial ecosystem is so rich with structure that it invites the possibility that one needs nothing else. It invites one to constrain themselves to only those mathematical moves that can be expressed cleanly within that language. For example, we will see that even taking the transpose of a span or the opposite of a category—often considered to be atomic operations—each arise as the composite of two more basic moves, a monoidal closure and an adjunction. The Poly ecosystem provides exactly the right sort of articulate constraints to formulate aggregation category-theoretically alongside querying.
1.2 Main theorem

The main theorem is not very difficult to prove, if one is allowed to proceed ad hoc. However we proceed according to a “purity of methods” constraint [DA11], i.e. a principled approach in which we allow ourselves to use only universal and monoidal constructions in $\mathsf{Cat}$. To do so we must develop many new and unexpected results within the long-beloved theory of polynomial functors, including the above “splitting of the atom”, whereby the assumed-to-be-atomic operation of taking an opposite category is split into two more primitive operations. This and other results—such as recovering the full internal subcategory spanned by a bundle [Jac99], again using only a universal operation—demonstrate the abundance of structure in the polynomial ecosystem and hence its capacity as a setting for category theory and its applications.

To state the theorem, we need one piece of notation and one construction. Given a copresheaf $X : C \to \mathbf{Set}$, let $|X| : \text{Ob}(C) \to \mathbf{Set}$ denote its restriction to objects. Given an assignment $M : \text{Ob}(C) \to \mathbf{Set}$, where we write $M_i := M(i)$, we construct a new assignment $\Pi_C M : \text{Ob}(C) \to \mathbf{Set}$, given by

$$(\Pi_C M)_j := \prod_{f : i \to j} M_i$$

This operation is a comonad; it is easy to check that there is an induced map $\varepsilon : \Pi_C M \to M$, given by projection along the identity $\text{id}_i : i \to i$ component, and a map $\delta : \Pi_C M \to \Pi_C \Pi_C M$ given by composition, that together satisfy the usual comonad laws.

**Theorem** (Proved as Corollary 3.3.3). Let $C$ be a category, $M : \text{Ob}(C) \to \mathbf{Set}$ an objectwise assignment as above, and suppose that each $M_i$ comes equipped with a commutative monoid structure $(M_i, \oplus_i)$. For any finitary copresheaf $X : C \to \mathbf{Fin}$ and map $\alpha : |X| \to M$, there is an induced map $\oplus\alpha : |X| \to \Pi_C M$, such that

$$M \xleftarrow{\varepsilon} \Pi_C M \xrightarrow{\delta} \Pi_C \Pi_C M$$

commutes; i.e. $(\oplus\alpha) \circ \varepsilon = \alpha$ and $\oplus\alpha \circ \delta = \oplus(\oplus\alpha)$.

**Example 1.2.1.** Suppose we say every employee works in a department, and every department is part of a college, so we have a category

$$C = \begin{array}{ccc}
\bullet_E & \overset{w}{\longrightarrow} & \bullet_D \\
\downarrow & \searrow & \downarrow p \\
\bullet_C & \downarrow & \bullet_C 
\end{array}$$

and a finitary copresheaf $X : C \to \mathbf{Fin}$. Suppose further that $M_D = M_C = 1$ are trivial monoids, that $M_E = (\mathbb{R}, \text{sum})$, where sum takes a finite set of reals to their sum, and that every employee gets a salary $s : X_E \to \mathbb{R}$. Then $\text{sum } s$ is standing as the $\oplus\alpha$ from (7), and we have maps

$$(\text{sum } s)_{\text{id}} : X_E \to \mathbb{R} \quad (\text{sum } s)_w : X_D \to \mathbb{R} \quad (\text{sum } s)_{wp} : X_C \to \mathbb{R}.$$
What are these?

The map \((\text{sum } s)_w : X_D \to \mathbb{R}\) assigns each department \(d \in X_D\) its “total salary”: the sum of the salaries of the employees who work in \(d\). Similarly, \((\text{sum } s)_{p \circ w} : X_C \to \mathbb{R}\) assigns each college \(c \in X_C\) the sum of the salaries of the employees who work in \(c\), and (7) says that this number will be equal to the sum of the total salaries for the departments that are part of it,

\[
(\text{sum } s)_{w \circ p}(c) = (\text{sum } s)_w(p)(c).
\]

Similarly, (7) says that \((\text{sum } s)_{id_c}(e) = s(e)\). ♦

The proof of Corollary 3.3.3 will take place entirely within \(\mathbb{C}\), using just monoidal and closure structures, universal properties, and the list polynomial:

\[
u := \sum_{N \in \mathbb{N}} y^N.
\]

We make the following into a remark so that the reader can return to it more easily.

**Remark 1.2.2 (Proof ingredients).** Below we list the main stepping stones in proving Theorem 3.3.1 and Corollary 3.3.3. We do not expect the reader to understand these ingredients very well at this point, but instead to get the general gist of the development.

1. In Theorem 2.3.1 we’ll explain that the framed bicategory \(\mathbb{C}\) has small categories as objects and profunctors as horizontal arrows.
2. In Proposition 2.4.6 we’ll show that composition in \(\mathbb{C}\) has a co-closure \([\_\_\_]\)

\[
\text{hom}(p, r \circ q) \cong \text{hom}\left(\begin{bmatrix} q \\ p \end{bmatrix}, r\right)
\]

and that \(\begin{bmatrix} p \\ p \end{bmatrix}\) has a natural category structure for each \(p \in \text{Poly}\).
3. In Section 2.7 we’ll provide formulas for a local monoidal closed structure on \(\mathbb{C}\): each horizontal hom-category \(\mathbb{C}(c, d)\) is naturally monoidal closed, with a “tensor-hom adjunction” between \((-c \otimes d -)\) and \(c[-, -]_d\).
4. In Section 2.8 we’ll show that \(\text{Span}\) is a subcategory of \(\mathbb{C}\) and that every horizontal map \(p\) in it has a right adjoint \(p^\dagger\) in \(\mathbb{C}\). Since categories are monads in \(\text{Span}\), this gives a second way (see 1 above) to view any category as being inside \(\mathbb{C}\); we’ll see various relations between these two viewpoints.
5. In Definition 2.8.3 we’ll assign a horizontal dualizing object \(\perp\) to each pair of sets \(C, D \in \text{Span}\), referring to \(C[p, \perp]_D\) as the dual of \(p\), denoted \(p^\vee\).
6. In Corollary 2.8.7 we’ll show that the transpose of any span \(s\) is its right adjoint’s dual \(s^\top = s^\dagger\). Hence, as previously mentioned, the usual transpose of a span can be derived as the composite of two more primitive operations. From here, we see also that the opposite of a category is similarly derived as a composite of two more primitive notions.

\[\text{We reserve the letter } u \text{ for it, because we will see that it corresponds to the universe of finite sets.}\]
7. In Corollary 3.2.1, with \( u \) as in Eq. (8), we will show that \[ u^\vee \] is equivalent to the category of finite sets, so finite sets are essentially a derived notion, as well, given \( u \). We’ll see that \( u^\vee(1) \) has the structure of a copresheaf on \( \text{Fin} \), corresponding to the inclusion \( \text{Fin} \to \text{Set} \). This in turn allows us to internally define what it means for a copresheaf \( X : C \to \text{Set} \) to be finitary (i.e. \( X_i \in \text{Fin} \) for all \( i \in C \)): namely that \( X \) is a pullback of \( u^\vee(1) \) along a functor \( C \to \text{Fin} \).

8. In Lemma 3.2.4 we will show that a commutative monoid \((M, \odot)\) can be identified with a \( \text{Fin} \)-copresheaf of the form \( u(M) \). This says that for any span \( B \leftarrow A \to M \), where \( A, B \in \text{Fin} \), we get an induced map \((\odot)_f : B \to M\), and that this construction respects identities and composition in place of \( f \).

9. In Section 3.3, we will put these pieces together to prove our main theorem, Theorem 3.3.1, and in particular the key diagram, (63).

1.3 Plan of the paper

We will move quickly, because there is a lot to cover. Two-thirds of the paper is contained in Section 2, where we discuss all the structures in \( \text{Cat}^\sharp \) that will be relevant to us.

We begin by introducing the category \( \text{Poly} \) of polynomial functors in one variable and the framed bicategory \( \text{Cat}^\sharp \) of comonoids in \( \text{Poly} \); this is what we have been referring to as the polynomial ecosystem.\(^4\) By Ahman-Uustalu’s result, the objects in \( \text{Cat}^\sharp \) can be identified with categories \( C, D \), and by Garner’s result, the horizontal maps in \( \text{Cat}^\sharp \) can be identified with parametric right adjoint functors \( C\text{-Set} \to D\text{-Set} \) between their copresheaf categories.

We then discuss two very important subcategories of \( \text{Cat}^\sharp \). The first is the full sub framed bicategory spanned by the discrete categories. We show that this is equivalent to Gambino-Kock’s framed bicategory \( \text{PolyFun}_{\text{Set}} \). Inside of this we find \( \text{Span} \subseteq \text{Cat}^\sharp \), the usual framed bicategory of spans in \( \text{Set} \). Since categories are monads in \( \text{Span} \), we see the ordinary framed bicategory of categories, functors, and profunctors as living in the polynomial ecosystem. In this chapter we also discuss a local monoidal closed structure on \( \text{Cat}^\sharp \). We show how this structure lets us perform the operation of transposing a span as a composite of two other, more primitive operations.

In Section 3, we put the pieces together to explain aggregation. The explanation will use everything discussed up to this point.

1.4 Acknowledgments

I’d like to thank Nick Smith for insisting on something I already knew but had lost courage to pursue: that aggregation is really worth understanding. I thank Simon

\(^4\)Perhaps we should instead refer to the virtual equipment \( \text{Mod}(\text{Cat}^\sharp) \) of monoids and modules in \( \text{Cat}^\sharp \) as the polynomial ecosystem, but aside from the special case of monoids and modules in \( \text{Span} \subseteq \text{Cat}^\sharp \), we will not use the generality of \( \text{Mod}(\text{Cat}^\sharp) \) in the present paper.
Henry, Josh Meyers, David Jaz Myers, and Nelson Niu for contributions that directly helped this research effort (see Propositions 2.1.14, 2.1.17, and 2.2.12 and Lemma 2.3.13). Thanks to Kevin Arlin for catching an error. I also benefited from conversations with Nathanael Arkor, Steve Awodey, Spencer Breiner, Brendan Fong, Richard Garner, Joachim Kock, Sophie Libkind, Joe Moeller, Juan Orendain, Evan Patterson, Todd Trimble, and Ryan Wisnesky.

This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-20-1-0348.

2 The framed bicategory $\mathbf{Cat}^\#$

Throughout this paper we will work in $\mathbf{Set}$ as our ambient category. Everything we say should generalize straightforwardly to any topos with a natural numbers object, using the standard semantics of dependent type theory (see e.g. [Mai05]), but we leave that to the interested reader.

**Notation 2.0.1.** We will use upper-case letters, e.g. $A, B, \ldots, Z \in \mathbf{Set}$ to denote sets. We are less careful with lower-case letters, which we use both for polynomials, e.g. $p, q \in \mathbf{Poly}$ and for elements of sets, e.g. $i, j \in I$. We denote generic categories using cursive font e.g. $\mathcal{C}, \mathcal{D}$, and named categories using bold, e.g. $\mathbf{Set}$ and $\mathbf{Poly}$. We denote double categories using blackboard bold font on their first letter, e.g. $\mathbf{Cat}^\#$, $\mathbf{Span}$, $\mathbf{PolyFun}$, etc.

Given $n \in \mathbb{N}$ we abuse notation and write $n \in \mathbf{Set}$ to also denote the set

$$n := \{ 1', \ldots, n' \}.$$

If we ever need to be careful, we will denote that set by $\bar{n}$, as is more standard. In particular, 0 denotes the empty set and 1 denotes a singleton set; more carefully, these would be $\emptyset = \varnothing$ and $\bar{1} = \{ '1' \}$.

Given a function $f : I \to J$ and an element $i \in I$ we may write $f i$ to denote $f(i) \in J$. For sets $S, T$, we denote the set of functions between them by $T^S := \mathbf{Set}(S, T)$ as usual; e.g. $T^2$ is the set functions $2 \to T$, etc. We sometimes write $\simeq$ to denote the presence of a canonical isomorphism, so we can write $3^0 = 1$ and $3^1 = 3$, but we should *not* write $3^2 = 9$.

We denote the coproduct (we generally say *sum*) of two objects using $+$, e.g. $S + T$, and given a set $I$ and sets $S_i$ for each $i \in I$, we denote their coproduct (*sum*) by $\sum_{i \in I} S_i$.\footnote{In (2), we used $\sum$ to indicate a sum of real numbers. We will never use the symbol in that way again; from now on, $\sum$ indicates a coproduct. Of course, its cardinality will be the sum of the individual cardinalities, but that’s the only connection.}

We denote the product of two sets either using $\times$ or simply by juxtaposing them

$$ST := S \times T$$

when confusion should not arise. We denote the $I$-indexed product of sets by $\prod_{i \in I} S_i$.
\[ \sum_{i \in I} p_i, \text{ and } \prod_{i \in I} p_i \text{ for } p, q, p_i \text{ in any category } C \text{ in which } p, q \text{ have a coproduct, product, etc.} \]

We often use the name of an object to denote the identity morphism on it, \( C = \text{id}_C \). We use \( \mathcal{F} \) to denote diagrammatic composition and \( \circ \) to denote Leibniz-ordered composition, e.g. given maps \( c \to d \to e \), we have \( f \circ g = g \circ f \) and may write either.

Given a category \( C \), we denote the category of \( C \)-copresheaves (functors \( C \to \text{Set} \)) by \( C-\text{Set} \). Given a copresheaf \( P : C \to \text{Set} \), we denote its category of elements by \( \text{El}_C(P) \in \text{Cat} \). We refer objects \( (i, x) \in \text{Ob}(\text{El}_C(P)) \) as elements of \( P \), i.e. an element of \( P \) consists of an object \( i \in \text{Ob}(C) \) and an element \( x \in P(i) \).

When a set \( P \) is endowed with additional structure (e.g. it is the set of elements of a copresheaf \( C \to \text{Set} \)), we will often denote this structured object by its carrier \( P \). We call this carrier notation. So, for example if \( Q \) is the set of elements of a another \( C \)-copresheaf, then the product copresheaf has elements \( P \times_{\text{Ob}(C)} Q \). Because we allow ourselves to denote the copresheaf by its carrying set, we must denote this product copresheaf by \( P \times_{\text{Ob}(C)} Q \).

It is standard mathematical practice to denote a group \((G, e, \ast)\) by its carrier \( G \). We call this carrier notation and follow that convention strictly throughout the article. Thus when a polynomial \( p \) is endowed with extra structure, we will generally denote it simply by \( p \). Thus one can trust that all displayed polynomials really are the carrier of whatever structured object to which they refer.

We denote an adjunction \( F \dashv G \) by

\[
\begin{array}{c}
C \xrightarrow{F} D \\
\leftarrow \ \ G
\end{array}
\]

The 2-arrow always points along the left adjoint, because it denotes the direction of both the unit and the counit of the adjunction

\[ \eta: C \to \mathcal{F} \; \text{ and } \; \epsilon: G \; \mathcal{F} \to D. \]

We will often denote \( F \)'s left adjoint (if it has one) by \( F^\perp \) and denote \( F \)'s right adjoint (if it has one) by \( F^\perp \).

### 2.1 The category Poly of polynomials in one variable

**Definition 2.1.1** (Polynomial functor). Given a set \( S \), we denote the corresponding representable functor by

\[ y^S := \text{Set}(S, -): \text{Set} \to \text{Set}, \]

e.g. \( y^S(X) := X^S \). In particular \( y = y^1 \) is the identity and \( y^0 = 1 \) is constant singleton.

A polynomial functor is a functor \( p: \text{Set} \to \text{Set} \) that is isomorphic to a sum of representables, i.e. for which there exists a set \( I \), a set \( p[i] \in \text{Set} \) for each \( i \in I \), and an

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\(^6\)One remembers this notation in two ways: \( ^\perp \) looks more like an L and \( ^\perp \) looks more like an r, and in prafunctors, \( ^\perp \) will tend to push things down from the exponent to the base, whereas \( ^\perp \) will tend to push things up from the base to the exponent. But we’ll also usually remind the reader which is which.
We call \( \mathcal{C} \) the set of \( p \)-positions, and for each position \( i \in \mathcal{C} \) we call \( p[i] \) the set of \( p \)-directions at \( i \).

A morphism \( p \to q \) of polynomial functors is simply a natural transformation between them. We denote the category of polynomial functors by \( \text{Poly} \).

For any polynomial \( p = \sum_{i \in \mathcal{C}} y^p[i] \), we have a canonical isomorphism \( p(1) \cong \mathcal{C} \); hence we generally denote \( p \) by
\[
p = \sum_{i \in p(1)} y^p[i].
\]

The following remark will be crucial; many of the proofs rely on the reader’s ability to verify calculations on morphisms of polynomials, which are made much easier by understanding morphisms combinatorially.

**Remark 2.1.2 (Combinatorial description of polynomial maps).** As verified below, we have the following combinatorial description of morphisms in \( \text{Poly} \):
\[
\text{Poly}(p, q) = \prod_{i \in p(1)} \sum_{j \in q(1)} \text{Set}(q[j], p[i]).
\]

In other words, to specify a morphism \( \varphi \colon p \to q \) is to specify a way to take a \( p \)-position \( i \in p(1) \) and provide both a \( q \)-position \( j \in q(1) \) and a function \( q[j] \to p[i] \) from \( q \)-directions at \( j \) backwards to \( p \)-directions at \( i \). The map on positions is just the component of \( \varphi \) at the set \( 1 \), so we denote it \( \varphi_1 \) as below-left; we denote the map backwards on directions using a \( \# \)-symbol,\(^7\) as below-right:
\[
\varphi_1 \colon p(1) \to q(1) \quad \text{and} \quad \varphi^\#(i, -) \colon q[j] \to p[i]
\]
where \( \varphi_1(i) = j \). This could also be written
\[
\text{Poly}(p, q) = \sum_{\varphi_1 \in \text{Set}(p(1), q(1))} \prod_{i \in p(1)} \text{Set}(q[\varphi_1 i], p[i]).
\]

The formula in (10) just comes from the definition of coproduct, the Yoneda lemma,

\(^7\)For readers familiar with basic algebraic geometry, the \((\varphi_1, \varphi^\#)\) notation is meant to evoke the standard notation for maps of ringed spaces \((X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\): a map \( f \colon X \to Y \) and a map of sheaves \( f^\#: \mathcal{O}_Y \to \mathcal{O}_X \) [Har77, Section II.2]. This is quite analogous; indeed one can view \( p \in \text{Poly} \) as a sheaf of sets on the discrete space \( p(1) \), assigning to each \( l \subseteq p(1) \) the set of sections \( \prod_{i \in l} p[i] \). Then a map of polynomials is just as for ringed spaces: forward on points, backwards on sections.
and the fact that $\textbf{Poly}$ is a full subcategory of all functors $\textbf{Set} \rightarrow \textbf{Set}$:

\[
\text{Poly}(p, q) = \text{Poly}\left( \sum_{i \in p(1)} y^p[i], \sum_{j \in q(1)} y^q[j] \right)
\]
\[
= \prod_{i \in p(1)} \text{Poly}\left( y^p[i], \sum_{j \in q(1)} y^q[j] \right)
\]
\[
= \prod_{i \in p(1)} \left( \sum_{j \in q(1)} y^q[j] \right)(p[i])
\]
\[
= \prod_{i \in p(1)} \sum_{j \in q(1)} \text{Set}(q[j], p[i]).
\]

\[\diamondsuit\]

**Example 2.1.3.** In functional programming, a *lens* is a sort of morphism between pairs of sets, often denoted $\begin{bmatrix} T \\ S \end{bmatrix} \rightarrow \begin{bmatrix} B \\ A \end{bmatrix}$. It consists of two functions:

\[S \rightarrow A \quad \text{and} \quad S \times B \rightarrow T.\] (13)

Rather than saying how they compose, we can give the whole story at once: The category of lenses is isomorphic to the full subcategory of $\textbf{Poly}$ spanned by the monomials. Indeed, one can read off from (10) that a map $S y^T \rightarrow A y^R$ consists of the two functions displayed in (13).

We will generalize this in Example 2.1.15, seeing that these objects arise as a special case of a coclosure operation in $\textbf{Poly}$.

\[\diamondsuit\]

Here is yet another way to understand maps between polynomials. If $p \in \textbf{Poly}$ is a polynomial, let $\dot{p}$ denote its derivative in the usual calculus sense. Then the positions $\dot{p}(1)$ of the derivative are the directions of the original $p$:

\[\dot{p}(1) \cong \sum_{i \in p(1)} p[i].\]

Thus we can think of a polynomial as a *bundle* (function between sets):

\[\begin{array}{c}
\text{\dot{p}(1)} \\
\downarrow \pi_p \\
p(1)
\end{array}\] (14)

where the fiber over $i \in p(1)$ is $p[i]$. We usually dispense with the $\dot{p}(1) \rightarrow p(1)$ notation, instead denoting the bundle simply as $E \overset{\pi}{\twoheadrightarrow} B$.

**Proposition 2.1.4.** Let $p, p'$ be polynomials and $E \rightarrow B$ and $E' \rightarrow B'$ the corresponding bundles. A morphism $\varphi: p \rightarrow p'$ corresponds to a diagram of the form

\[
\begin{array}{ccc}
E & \xleftarrow{\varphi^E} & \bullet \\
\downarrow \pi & & \downarrow \psi \\
B & & B'
\end{array}
\]

\[
\begin{array}{ccc}
E' & \xrightarrow{\varphi'^E} & E' \\
\downarrow \pi' & & \downarrow \pi' \\
B & \xrightarrow{\varphi'_B} & B'
\end{array}
\]
Proof. After verifying that the pullback object (denoted $\bullet := B \times_{B'} E'$) can be identified with $\sum_{i \in B} p'[q_1(i)]$, the result is just a rephrasing of Eq. (11). \qed

We say that a polynomial $p$ is constant if it is of the form $p \cong A = Ay^0$ and linear if it is of the form $p \cong Ay$ for some set $A \in \text{Set}$. These correspond to bundles of the form $0 \to A$ and $A \to A$, respectively.

**Definition 2.1.5** (Outfacing polynomial). Let $C$ be a category. For any object $a \in \text{Ob}(C)$, define the set $C[a]$ of $a$-outfacing maps by

$$C[a] := \sum_{a' \in \text{Ob}(C)} C(a, a'),$$

i.e. the set of morphisms in $C$ whose domain is $a$.

We define the **outfacing polynomial** $c \in \text{Poly}$ of $C$ to be\(^8\)

$$c := \sum_{a \in \text{Ob}(C)} y^{C[a]}.$$

The outfacing polynomial $c$ of a category $C$ will be a major player in our story: we will see in Section 2.2 that a category can be identified with a comonoid structure on its outfacing polynomial. But we leave the outfacing polynomial aside for the moment and return to the theory of $\text{Poly}$.

**Proposition 2.1.6.** A functor $p : \text{Set} \to \text{Set}$ is a polynomial iff it preserves connected limits.

Proof. Coproducts preserve connected limits and products preserve all limits, so polynomials preserve connected limits. In the other direction, suppose $F$ preserves connected limits. Let $F' : \text{Set} \cong \text{Set}/1 \to \text{Set}/F(1)$ be the induced map to the slice over $F(1)$, so that $F$ is the composite of $F'$ and a functor $\text{Set}/F(1) \to \text{Set}$, the latter of which is given by a coproduct of $F(1)$-many sets. Note that $F'$ preserves all limits, not just connected ones. Hence to show that $F$ is a polynomial (a coproduct of representables), it suffices to show that any limit-preserving functor $\text{Set} \to \text{Set}$ is representable. This is easy to check: it has a left adjoint $L$, and the representing set is $L(1)$. \qed

**Proposition 2.1.7** (Composition monoidal structure $(\text{Poly}, y, \circ)$). The composite of two polynomial functors $p, q$ is again polynomial; we denote it $p \circ q := p \circ q : \text{Set} \to \text{Set}$.\(^9\) It is given by the formula

$$p \circ q \cong \sum_{i \in p(1)} \sum_{j : \text{Ob}(p[i]) \to q(1)} y^{\sum_{d \in \text{Ob}(i)} q[j(d)]}.$$ 

\(^8\)It is slightly bad notational practice to denote a construction using a change of font or capitalization, as we do here by denoting $C$’s outfacing polynomial by $c$. Fortunately, the problem will not haunt us because we will not have much need for the $C$ notation below. Indeed, we will encode all of the category structure of $C$ into a comonoid structure on $c$.

\(^9\)We are thinking of polynomials as objects rather than morphisms in a category, namely $\text{Poly}$. We use $\circ$ for the composition (also known as substitution) monoidal structure on $\text{Poly}$, to reserve the symbol $\bullet$ for composition of morphisms.
Proof. The composite of two connected-limit-preserving functors is also connected-limit-preserving, so the result follows by Proposition 2.1.6. The formula is simply a use of the distributivity law:

\[ p \circ q = \sum_{i \in p(1)} \prod_{d \in p[i]} \sum_{j \in q(1)} y^{q(j)} \approx \sum_{i \in p(1)} \prod_{d \in p[i]} \sum_{j \in q(1)} y^{q[j(d)]} = \sum_{i \in p(1)} \sum_{j : p[i] \to q(1)} \sum_{d \in p[i]} q[j(d)] \]

\( \square \)

In particular, the evaluation of a polynomial \( p \) at a set \( X \) can be denoted \( p \triangleleft X \), e.g. \( p(1) = p \triangleleft 1 \). Note that if \( X \) is constant then \( X \triangleleft p \cong X \) for any \( p \). Very rarely, we will write \( p \triangleright q \) to denote \( q \triangleleft p \). As functors \( p \triangleleft q = p \circ q \) and \( p \triangleright q = p \triangleright q \).

We can refer to (co)monoids for the composition monoidal structure \((y, \triangleleft)\) as polynomial (co)monads, since they are technically (co)monoid objects in a category of endofunctors \( \text{Set} \to \text{Set} \).

Corollary 2.1.8. The category \( \text{Poly} \) of polynomial functors is closed under limits.

Proof. Given a diagram \( p : J \to \text{Poly} \) of polynomial functors, one can compose with the fully faithful functor \(|-| : \text{Poly} \to \text{Fun}(\text{Set}, \text{Set})\), take the limit of \(|p|\) (which is “pointwise”) and ask if the result \( \lim |p| \) is a polynomial. If it is, then \( \lim |p| \) also serves as a limit, \( \lim p \) in \( \text{Poly} \).

The fact that \( \lim |p| \) is a polynomial follows from Proposition 2.1.6, since the limit of connected-limit-preserving functors preserves connected limits. \( \square \)

The category \( \text{Poly} \) also has all colimits, not just coproducts. However coequalizers are not generally preserved by the forgetful functor \( \text{Poly} \to \text{Fun}(\text{Set}, \text{Set}) \), and we will not need them in this paper.

Lemma 2.1.9 (Composition preserves equalizers). Assume that \( e \to p \rightrightarrows q \) is an equalizer and that \( r \) is a polynomial; then both of the following are also equalizers:

\[ (e \triangleleft r) \to (p \triangleleft r) \rightrightarrows (q \triangleleft r) \quad \text{and} \quad (r \triangleleft e) \to (r \triangleleft p) \rightrightarrows (r \triangleleft q). \]

Proof. Both demonstrations use the fact that limits are taken pointwise. For the first, we have that \((e \triangleleft r \triangleleft X) \to (p \triangleleft r \triangleleft X) \rightrightarrows (q \triangleleft r \triangleleft X)\) is an equalizer for any set \( X \), by assumption. For the second we additionally use Proposition 2.1.6: since an equalizer is a connected limit, \( r \) preserves it, i.e. the following is an equalizer for any set \( X \):

\[ (r \triangleleft e \triangleleft X) \to (r \triangleleft p \triangleleft X) \rightrightarrows (r \triangleleft q \triangleleft X). \]

\( \square \)

The following two results are quite useful; we invite the reader to check them using Eq. (10).

Proposition 2.1.10. The categorical sum and product of polynomials \( p + q \) and \( pq \) correspond to the usual algebraic sum and product of polynomials.

The category \( \text{Poly} \) is cartesian closed, but we have found surprisingly little use for that fact. Much more useful to us is the following, which will be generalized in Section 2.7.
Proposition 2.1.11 (Monoidal-closed structure \((\text{Poly}, y, \otimes, [\_, -])\)). The category \(\text{Poly}\) has a symmetric monoidal closed structure with unit \(y\), with multiplication denoted \(\otimes\), called Dirichlet product, and given by the formula
\[
p \otimes q := \sum_{i \in p(1)} \sum_{j \in q(1)} y^{p[i] \times q[j]},
\]
and with closure given by the formula
\[
[p, q] := \sum_{\varphi: p \to q} y^{\varphi_1(i)} \sum_{i' \in p(1)} q[\varphi_1(i')]
\]

Proof. It is easy to see directly from the formula that \(\otimes\) is associative, unital, and symmetric. The proof that \(- \otimes p\) is left adjoint to \([p, -]\) is an elementary calculation using Remark 2.1.2. We refer the interested reader to [SN22] (or our generalization in Proposition 2.7.3) for the details, but we give the evaluation map \(p \otimes [p, q] \to q\) here for the reader’s convenience. Given positions \(i \in p(1)\) and \(\varphi: p \to q\), evaluation returns \(j := \varphi_1(i) \in q(1)\) together with the function
\[
eval[i] := \sum_{i' \in p(1)} q[\varphi_1(i')]
\]
given by sending \(e \in q[j]\) to \((q^\#(i, e), i, e)\). \Box

Example 2.1.12 (Duality in \(\text{Poly}\)). For any set \(A\), we have the following isomorphisms
\[
[A y, y] \cong y^A \quad \text{and} \quad [y^A, y] \cong A y.
\]
(15)

For general \(p\), there is a natural map \(p \to [(p, y), y]\), but it is not an isomorphism unless \(p\) is linear or representable, i.e. unless it’s already as shown in (15).

We think of \(y\) as a dualizing object, and say that \(A y\) is the dual of \(y^A\) and vice versa. In other words, there are embeddings \(\text{Set} \to \text{Poly}\) and \(\text{Set}^p \to \text{Poly}\) given by sending \(A\) to \(A y\) and to \(y^A\) respectively, and the functor \([- , y]\) is a contravariant isomorphism between these two subcategories.

We will see that this is part of a much larger story in Theorem 2.8.4. \diamond

We continue to investigate the Dirichlet and composition products \(\otimes, \ast\).

Proposition 2.1.13 (Duoidality \((\ast \otimes \ast) \to (\otimes \ast \otimes))\). Dirichlet product is naturally colax monoidal with respect to composition product. That is, for any polynomials \(p, q, p', q' \in \text{Poly}\) there is a natural map
\[
(p \ast q) \otimes (p' \ast q') \longrightarrow (p \otimes p') \ast (q \otimes q')
\]
(16)

which, along with the unique map \(y \to y\) between their units, forms a duoidal structure on \(\text{Poly}\).
Proof sketch. We supply the displayed map (16) and leave the rest to the reader. We need to give an element of the following type:

\[
\sum_{i \in p(1)} \sum_{j: p[i] \to q(1)} \sum_{i' \in p'(1)} j': p'[i'] \to q'(1) \sum_{y^{d \in p}[i]} \sum_{d' \in p'[i']} q[j][xq'[j' i']]
\]

\[
\downarrow \phi
\]

\[
\sum_{(i, i') \in (p \otimes p')(1)} (j, j'): (p \otimes p')((i, i')) \to (q \otimes q')(1)
\]

where recall that \( ji := j(i) \) denotes function application. Define \( \phi \) on positions by

\[
(i, j, i', j') \mapsto (i, i', (d, d') \mapsto (ji, j'i'))
\]

and on directions by the obvious bijection

\[
\sum_{(d, d') \in (p \otimes p')((i, i'))} (q \otimes q')[(ji, j'i')] \equiv \sum_{d \in p[1]} \sum_{d' \in p'[1']} q[ji] \times q'[j'i']
\]

We learned the following from Josh Meyers; it will be generalized in Proposition 2.4.6.

Proposition 2.1.14 (Coclosure for \( \otimes \)). The composition operation has a co-closure. That is, for every \( p, q \in \text{Poly} \) we can define a polynomial \( \begin{bmatrix} q \\ p \end{bmatrix} \in \text{Poly} \) and an adjunction

\[
\text{Poly}(p, p' \times q) \cong \text{Poly} \left( \begin{bmatrix} q \\ p \end{bmatrix}, p' \right).
\]

This polynomial is given by the formula

\[
\begin{bmatrix} q \\ p \end{bmatrix} := \sum_{i \in p(1)} y^{q(p[i])}.
\]

Proof. The result follows from the combinatorial description (10) in the sense that on both sides of the adjunction isomorphism (17), the required set-theoretic maps are exactly the same: start with \( i \in p(1) \), assign some \( i' \in p'(1) \) and then, for any \( d' \in p'[i'] \) assign some \( j \in q(1) \) and then, for any \( e \in q[j] \) assign some \( d \in p[i] \).

Example 2.1.15 (Lens notation as coclosure). Suppose \( P, Q \) are sets. Then one reads off from (18) that

\[
\begin{bmatrix} Q \\ P \end{bmatrix} = Py^Q,
\]

recovering our notation from the example on lenses, Example 2.1.3.

In the following example, we use the more careful notation \( N := \{1', \ldots, N'\} \) from Notation 2.0.1, so that the universe polynomial for finite sets (8) will be denoted\(^{10}\)

\[
u := \sum_{N \in \mathbb{N}} y^N.
\]

\(^{10}\) We call \( \nu \) the universe polynomial, because it is a universe in the sense of Awodey’s paper “Natural models of Homotopy Type Theory” [Awo14], from which the inspiration for our heavy use of \( \nu \) in this paper, e.g. in Example 2.1.16, arose. See also Awodey’s very clear video explanation.
Example 2.1.16 (Toward self co-closure of \( u \) as skeleton of \( \text{Fin}^{\text{op}} \)). Let \( u \) be the universe polynomial for finite sets as in (19). We claim that \( \begin{bmatrix} u \\ u \end{bmatrix} \) is the out-facing polynomial for a skeleton of \( \text{Fin}^{\text{op}} \). Let’s first define that skeleton, call it \( \mathcal{F} \).

The objects of \( \mathcal{F} \) are natural numbers \( \text{Ob}(\mathcal{F}) := \mathbb{N} \), and the morphisms \( \mathcal{F}(M, N) \) are the functions \( N \rightarrow M \).\(^{11}\) Thus, according to Definition 2.1.5, the out-facing polynomial \( f \in \text{Poly} \) of \( \mathcal{F} \) is

\[
f := \sum_{M \in \mathbb{N}} y^{\sum_{N \in \mathbb{N}} \prod_{n \in \mathbb{N}} M}.
\]

But \( u(M) \cong \sum_{N \in \mathbb{N}} \prod_{n \in \mathbb{N}} M \), so we read off from (18) that \( \begin{bmatrix} u \\ u \end{bmatrix} \) is indeed isomorphic to \( f \).

In Section 2.2 we will see that polynomial comonoids can be identified with categories. Then we will be able to complete what we started in Example 2.1.16 above: seeing that \( \begin{bmatrix} u \\ u \end{bmatrix} \) is a skeleton of \( \text{Fin}^{\text{op}} \). Indeed, this will follow from Proposition 2.1.17 below, which we learned the following from David Jaz Myers.

Proposition 2.1.17. For any \( p \in \text{Poly} \), there is a natural comonoid structure on \( \begin{bmatrix} p \\ p \end{bmatrix} \).

Proof. This is purely formal; when a monoidal operation, e.g. \( \ast \), has a (co-)closure operation, then applying the (co-)closure to \( p \) with itself will be a (co-)monoid.

For concreteness, we define a map \( p \rightarrow \begin{bmatrix} p \\ p \end{bmatrix}^{\text{eq}} \ast p \) (and hence a map \( \begin{bmatrix} p \\ p \end{bmatrix} \rightarrow \begin{bmatrix} p \\ p \end{bmatrix}^{\text{eq}} \)) for \( n \in \mathbb{N} \): when \( n = 0 \) it is \( \text{id}_p \), when \( n = 1 \) it is the adjunction isomorphism (17) applied to the identity, and for higher \( n \) one uses induction and the \((n = 1)\)-case. The counitality and coassociativity follow from the naturality of the adjunction isomorphism. \( \square \)

Example 2.1.18. For any set \( S \in \text{Set} \), Proposition 2.1.17 says that \( \text{Set}^{\text{op}} \) has a comonoid structure. It is the so-called costate comonad from functional programming, i.e. it is the comonad one gets from product-hom adjunction \( \text{Set}(S \times -, -) \cong \text{Set}(-, -^S) \).

The following will be generalized in Proposition 2.5.4.

Proposition 2.1.19 (Adjunctions in \( \text{Poly} \)). A polynomial functor \( p \) is a left adjoint iff it is linear; \( q \) is a right adjoint iff it is representable.

Proof. First, for any set \( P \in \text{Set} \), we have

\[
\text{Set}(P X, Y) \cong \text{Set}(X, Y^P)
\]

so \( P_y \) is left adjoint to \( y^P \); this proves one direction.

Now assume that \( p \) is a left adjoint. Since it preserves coproducts, the natural map \( p(1) + p(1) \rightarrow p(2) \) is an isomorphism, i.e. the pointwise diagonal map

\[
\sum_{i \in p(1)} 2 = \sum_{i \in p(1)} 1p[i] + 1p[i] \rightarrow \sum_{i \in p(1)} 2p[i]
\]

\(^{11}\) Again, we may denote functions \( N \rightarrow M \) simply as \( N \rightarrow M \) in the future.
is a bijection, but then for each \( i \in p(1) \) the diagonal map \( 2 \to 2^{p[i]} \) is a bijection, which means \( p[i] = 1 \). Hence \( p \) is linear, i.e. of the form \( p = Py^1 \) for a set \( P \), as desired.

Now we know that if \( q \) is a right adjoint, its left adjoint must be linear, so by the first paragraph it must be representable. \( \square \)

### 2.2 \( \text{Cat}^\# = \text{Comonoids in Poly} \)

In 2016, Ahman and Uustalu proved an amazing result [AU16]:

*Polynomial comonads are exactly small categories.*

There is a caveat in that the morphisms are different—morphisms of comonoids are not functors, and this is strange at first—but up to isomorphism, the objects are the same. Moreover, this fact has an elementary proof: one can just check that categories and polynomial comonads consist of the same set-theoretic data and the same equations.

**Definition 2.2.1 (Polynomial comonad).** A polynomial comonad, equivalently a comonoid in \((\text{Poly}, y, \rhd)\), is a triple \((c, e, \delta)\), where \( c \in \text{Poly} \) is a polynomial and \( e : p \to y \) and \( \delta : p \to p \rhd p \) are maps, making these diagrams commute:

\[
\begin{array}{ccc}
p & \xrightarrow{\delta} & p \\
p \xleftarrow{p \rhd e} & & \xrightarrow{e \rhd p} & \xrightarrow{p \rhd} & p \\
\end{array}
\]

A morphism of polynomial comonads is a morphism of polynomials \( \varphi : c \to d \) such that these diagrams commute:

\[
\begin{array}{ccc}
c & \xrightarrow{\varphi} & d \\
\downarrow e & & \downarrow e \\
y & = & y
\end{array}
\quad
\begin{array}{ccc}
c & \xrightarrow{\varphi} & d \\
\downarrow \delta & & \downarrow \delta \\
c \rhd c & \xrightarrow{\varphi \rhd c} & d \rhd d
\end{array}
\]

\[\diamond\]

For the time being we denote the category of comonoids in \( \text{Poly} \) by \( \text{Comon}(\text{Poly}) \), but once we get to Theorem 2.2.5, we will replace that name by \( \text{Cat}^\# \).

Recall from Definition 2.1.5 that we denote by \( C[a] \) the set of all morphisms in \( C \) that are out-facing from \( a \in \text{Ob}(C) \).

**Definition 2.2.2 (Cofunctor).** Let \( C \) and \( D \) be categories. A cofunctor \( F : C \to D \) consists of:

\[\text{This result amazed me, the author; some others I’ve spoken to have found it less surprising than I did.}\]

\[\text{The slash in } F : C \to D \text{ is not intended to make any connection to profunctors, though they will enter the story later. Instead, it is just to indicate that the flow of information—from } C \text{ to } D \text{ and then back—is unusual, and to leave room for the usual notation } C \to D \text{ to refer to honest functors. We will denote profunctors by } C \to D.\]
1. a function $F: \text{Ob}(C) \to \text{Ob}(D)$ on objects and
2. a function $F^\sharp_a: D[Fa] \to C[a]$ backwards on morphisms, for each $a \in \text{Ob}(C)$, satisfying the following conditions:
   i. $F^\sharp_a(\text{id}_{Fa}) = \text{id}_a$ for any $a \in \text{Ob}(C)$;
   ii. $F(\text{cod} F^\sharp_a(g)) = \text{cod} g$ for any $a \in \text{Ob}(C)$ and $g \in D[F(a)]$; and
   iii. $F^\sharp_{\text{cod} F^\sharp(a,g)}(g_2) \circ F^\sharp_a(g_1) = F^\sharp_a(g_2 \circ g_1)$ for composable arrows $g_1, g_2$ out of $Fa$.

In other words, $F^\sharp$ preserves identities, codomains, and compositions.

We denote by $\text{Cat}^\sharp$ the category of categories and cofunctors.

It is often convenient to write $F^\sharp$ as a dependent function of two variables, writing $F^\sharp(a, g)$ to denote $F^\sharp_a(g)$. It is dependent in the sense that the type of $g$ depends on $a$, namely $g \in D[Fa]$.

**Example 2.2.3** (Etale maps (discrete op-fibrations)). For categories $C$ and $D$, a discrete opfibration between them can be characterized as a functor $F: C \to D$ for which the induced map $C[a] \to D[Fa]$, sending outfacing morphisms from $a \in C$ to out-facing morphisms from $Fa \in D$, is a bijection for each object $a \in C$. But then we can define $F^\sharp_a: D[Fa] \to C[a]$ to be the inverse of this bijection. One checks easily that $(\text{Ob}(F), F^\sharp)$ satisfies the conditions of being a cofunctor $C \to D$. Thus every discrete opfibration is a cofunctor with the special property that $F^\sharp_a$ is a bijection for each $a \in C$.

We refer to discrete opfibrations as etale, because the word is shorter, prettier, and evokes the correct mental picture: there is a “local homeomorphism” between the out-facing maps from $a$ and those from its image $Fa$.

**Example 2.2.4** (Very well-behaved lenses). Let $S, T \in \text{Set}$ be sets, and consider the associated codiscrete categories $\delta, \tau$, i.e. those which have a unique morphism between any two objects. We can canonically identify any morphism in $\delta[s]$ with its codomain, so have $\delta[s] = S$ for any $s \in S$.

We now read off that a cofunctor $S \to \tau$ consists of a function $F: S \to T$ and a function $F^\sharp: S \times T \to T$ satisfying three laws:
1. $F^\sharp(s, F(s)) = s$
2. $F(F^\sharp(s, t)) = t$
3. $F^\sharp(F^\sharp(s, t_1), t_2) = F^\sharp(t_2)$.

Functional programmers will recognize these as the three lens laws for what are sometimes called very well-behaved lenses.

**Theorem 2.2.5** (Categories are polynomial comonads). For any category $C$, the out-facing polynomial (see Definition 2.1.5)

$$c = \sum_{i \in \text{Ob}(C)} y^{C[i]}$$

has a natural comonoid structure; $\epsilon: c \to y$ corresponds to the assignment of an identity morphism to each object, and $\delta: c \to c \circ c$ corresponds to the assignment of a codomain and composition formula for morphisms. The comonoid laws say that the codomain of identities and composites are as they should be, and that composition is unital and associative.

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Finally, morphisms of comonoids are precisely cofunctors, so there is an isomorphism of categories

\[ \text{Cat}^\# \cong \text{Comon}(\text{Poly}). \]

We will not prove Theorem 2.2.5 here; we refer the reader to [AU16] or [SN22], or to this video.

**Example 2.2.6** (Monoids are polynomial comonads with representable carriers). If \((M, e, \ast)\) is a monoid—i.e. a category with one object—its out-facing polynomial is \(y^M\).
The counit \(y^M \to y\) corresponds to the element \(e \in M\), the comultiplication map \(y^M \to y^M \otimes y^M\) corresponds to a function \(\ast: M \times M \to M\), and the diagrams in (20) correspond to the unital and associative conditions.

In particular, the full subcategory of \(\text{Cat}^\#\) spanned by the representables is isomorphic to the opposite of the category of monoids. Indeed, morphisms \(y^M \to y^N\) satisfying (21) are functions \(N \to M\) respecting identity and multiplication.

**Example 2.2.7** (Discrete categories are polynomial comonads with linear carriers). For any set \(S\), there is a unique comonoid structure on \(S^y\). Indeed, there is a unique map of polynomials \(S^y \to y\) to serve as \(\epsilon\), and the counitality and coassociativity laws force the map \(S^y \to (S^y \circ S^y) \cong S^{2y}\) to correspond to the diagonal \(S \to S^2\).

It is not hard to see that \(S^y^1\) corresponds to the discrete category on \(S\)-many objects, as each object \(s \in S\) has exactly one out-facing morphism which must then be the identity on \(s\). In particular the polynomials \(0, y \in \text{Poly}\) each have a unique comonoid structure, corresponding to the empty and the one-morphism category, respectively. A constant \(X \in \text{Set} \subseteq \text{Poly}\) is the carrier of a comonoid iff \(X = 0\).

A cofunctor \(F: C_y \to D_y\) between discrete categories sends objects in \(C\) to objects in \(D\), and that’s all of the data since the map backwards on morphisms must preserve identities. Hence we can identify the cofunctor \(F\) with a function \(C \to D\).

**Example 2.2.8** (Full internal subcategory). For any \(p \in \text{Poly}\), we know from Proposition 2.1.7 that

\[ F_p := \left[ \begin{array}{c} p \\ p \end{array} \right] \cong \sum_{i \in p(1)} y^{\sum_{i' \in p(1)} \text{Set}(p[i'], p[i])} \]

(22)

has a comonoid structure and hence a category structure \(F_p \in \text{Cat}\) by Theorem 2.2.5. Its object set is \(\text{Ob}(F_p) = p(1)\); from an object \(i \in p(1)\) the out-facing morphisms are pairs \((i', f) \in \sum_{i' \in p(1)} \text{Set}(p[i'], p[i])\). One checks from the definitions of \(\epsilon, \delta\) for this comonoid that the identity on \(i\) is \((i, \text{id}_{p[i]}\), the codomain of \((i', f)\) is \(i'\), and the composite of \((i', f)\) and \((i'', f')\) is \((i'', f \circ f')\). Hence, the set of maps \(i \to i'\) is given by

\[ F_p(i, i') \cong \text{Set}(p[i'], p[i]). \]

So as a category, \(F_p\) comes with a fully faithful functor to \(\text{Set}^{op}\), given by sending \(i \mapsto p[i]\). It may not be best to say that \(F_p\) is a “subcategory” of \(\text{Set}^{op}\) because there may be \(i \neq i'\) with \(p[i] \cong p[i']\). However, the opposite of \(F_p\) is called the full internal subcategory spanned by \(p\) in [Jac99].
We will be able to get the full internal category itself—not just its opposite—later in Corollary 3.2.1. But for the moment, we can conclude what we started in Example 2.1.16: by (22), we find that \( \mathcal{F}_u := \begin{bmatrix} u & u \\ u & u \end{bmatrix} \) is indeed the skeleton of \( \text{Fin}^{\text{op}} \) as predicted there.

The costate comonoid \( \mathcal{F}_S := \begin{bmatrix} S & S \\ S & S \end{bmatrix} \) from Example 2.1.18 corresponds to the category with object-set \( S \) and a unique morphism between any two objects. Indeed, the fully faithful functor from it to \( \text{Set}^{\text{op}} \) sends every object to \( \emptyset \), since \( S \cong S y^0 \).

We recall a theorem of Shulman [Shu08, Theorem 11.5], by which we will see that polynomial comonads form the objects of a framed bicategory (also known as a proarrow equipment), i.e. a double category with extra structure.

**Theorem 2.2.9** (Shulman). Suppose \( C \) is a framed bicategory such that for every \( c_1, c_2 \in \text{Ob}(C) \), the horizontal hom-category \( C(c_1, c_2) \) has equalizers, and horizontal composition in \( C \) preserves equalizers. Then there is a framed bicategory \( \text{Comod}(C) \) whose objects are comonoids in \( C \), whose vertical morphisms are comonoid morphisms, and whose horizontal morphisms are bicomodules.

**Corollary 2.2.10** (The framed bicategory of interest: \( \text{Cat}^\# \)). There is a framed bicategory \( \text{Cat}^\# := \text{Comod}(\text{Poly}) \) whose vertical category is \( \text{Cat}^\# \) as in Definition 2.2.1.

**Proof.** There is a double category \( P \) with one object, one vertical arrow (the identity), and whose horizontal bicategory is \( \text{Poly} \), where horizontal units are given by \( y \) and horizontal composition is given by \( \triangleleft \). Since all vertical morphisms are identity, \( P \) is also a framed bicategory. By Corollary 2.1.8, \( P \) has local equalizers and by Lemma 2.1.9 they are preserved by \( \triangleleft \). Thus we are done by Theorem 2.2.9. \( \square \)

We saw in Theorem 2.2.5 that comonoids in \( \text{Poly} \) are categories and maps between them are cofunctors. This justifies denoting \( \text{Comod}(\text{Poly}) \) as \( \text{Cat}^\# \). Thus for categories \( c, d \), we continue to denote the set of vertical arrows in \( \text{Cat}^\# \) from \( c \) to \( d \) by \( \text{Cat}^\#(c, d) \). For now we denote the category of horizontal arrows between them by \( \text{Comod}_d \).\(^{14}\)

We should spell out what a bicomodule is, since they are in some sense the main character in our story.

**Definition 2.2.11** (Bicomodule in \( \text{Cat}^\# \)). Let \( c, d \) be polynomial comonads. A \((c, d)\)-bicomodule, denoted \( \begin{array}{c} c \leftarrow m \leftarrow d \\ \lambda \rho \end{array} \) \( \text{Comod}_d \).

\(^{14}\)Once we have more theory under our belt we will denote it by \( c\cdot\text{Set}[d] := \text{Comod}_d \).
making the following diagrams commute in $\text{Poly}$:

\[
\begin{array}{c}
c \lessdot m \xleftarrow{\lambda} m \\
\downarrow \epsilon \lessdot m \\
m \xRightarrow{\rho} m \ast d \\
\downarrow m \epsilon \\
m \\
\end{array} \quad \quad \quad
\begin{array}{c}
c \lessdot m \xleftarrow{\lambda} m \\
\downarrow \epsilon \lessdot m \\
\downarrow \delta \lessdot m \\
\downarrow c \lessdot m \xleftarrow{c \lessdot \lambda} c \lessdot m \\
\end{array}
\]

(23)

\[
\begin{array}{c}
m \xrightarrow{\rho} m \ast d \\
\downarrow \rho \\
m \ast d \xrightarrow{\rho \ast d} m \ast d \\
\end{array} \quad \quad \quad
\begin{array}{c}
m \xrightarrow{\rho} m \ast d \\
\downarrow \lambda \ast d \\
c \lessdot m \xrightarrow{c \rho} c \lessdot m \ast d \\
\end{array}
\]

(24)

\[
\begin{array}{c}
m \xrightarrow{\rho} m \ast d \\
\downarrow \lambda \ast d \\
c \lessdot m \xrightarrow{c \rho} c \lessdot m \ast d \\
\end{array}
\]

(25)

Just $c \lessdot m \xleftarrow{\lambda} m$ satisfying (23) is called a left $c$-comodule; just $m \xrightarrow{\rho} m \ast d$ satisfying (24) is called a right $d$-comodule; and (25) is a coherence between these structures.

Given cofunctors $\alpha : c \to c'$ and $\beta : d \to d'$, and bicomodules $(m, \lambda, \rho) \in \text{Comod}_d$ and $(m', \lambda', \rho') \in \text{Comod}_{d'}$, a 2-cell in $\text{Cat}^d$ that fills in the square (left) is defined to be a polynomial map $\varphi : m \to m'$ making both of the squares (right) commute:

\[
\begin{array}{c}
c \lessdot m \xrightarrow{\varphi} c' \lessdot m' \\
\downarrow \alpha \lessdot m \\
c' \lessdot m' \\
\end{array} \quad \quad \quad
\begin{array}{c}
c \lessdot m \xrightarrow{\varphi} c' \lessdot m' \\
\downarrow \alpha \lessdot m \\
c' \lessdot m' \xleftarrow{\varphi} m' \lessdot m' \\
\end{array}
\]

(26)

Composition of bicomodules is as follows. Suppose given $c \lessdot m \xleftarrow{n} e$. Their composite, which we denote $m \lessdot e$ is carried by the following equalizer in $\text{Poly}$:

$$m \lessdot e \to m \lessdot n \Rightarrow m \lessdot d \lessdot n.$$  

Here the top and bottom map $m \lessdot n \to m \lessdot n$ are $m \lessdot \lambda_n$ and $\rho_m \lessdot n$ respectively. We give both $(c, e)$-bicomodule structure maps on $m \lessdot d \lessdot n$ at once by composing the above diagram with $c$ and $e$ to form the bottom row of the following diagram:

\[
\begin{array}{c}
m \lessdot d \lessdot n \\
\downarrow \lambda \varphi \\
c \lessdot m \lessdot d \lessdot n \\
\end{array} \quad \quad \quad
\begin{array}{c}
m \lessdot d \lessdot n \\
\downarrow \lambda \varphi \\
c \lessdot m \lessdot d \lessdot n \xrightarrow{\rho n} c \lessdot m \lessdot d \lessdot n \\
\end{array}
\]

(27)

The bottom row is an equalizer by Lemma 2.1.9, and hence we obtain the dashed map. We leave to the reader that it satisfies the bicomodule axioms. ♦

Bicomodules seem to include a lot of data; what does it all mean? Explaining this is the subject of the next section. But for now, we can give a special case: that bicomodules $c \lessdot 0$ are essentially $c$-copresheaves, a result I learned from Simon Henry.
**Proposition 2.2.12** (Copresheaves as bicomodules to 0). For any category \( c \in \mathsf{Cat}^{\#} \), there is an equivalence of categories

\[
c\text{-}\mathsf{Set} \cong _c\mathsf{Comod}_0.
\]

Moreover, the carrier of any bicomodule \( c \xleftarrow{m} \xrightarrow{a} 0 \) is constant: \( m = M \) for some \( M \in \mathsf{Set} \); namely \( M \cong \text{Ob}(\mathsf{El}_c M') \) is isomorphic to the set of elements of the copresheaf \( M' \): \( c \to \mathsf{Set} \) corresponding to \( c \xleftarrow{M} \xrightarrow{a} 0 \).

**Proof.** We begin with the fact that \( M \) must be constant. By (24) with \( d = 0 \), the composite \( m \to m \xleftarrow{a} 0 \xrightarrow{m \circ} m \) is the identity on \( m \), and it follows that \( m \to m \xleftarrow{a} 0 \) is a monomorphism of functors, and hence that \( m = M \) is constant.

We proceed to the first statement, first on objects. In fact, we provide an isomorphism of categories \( _c\mathsf{Comod}_0 \cong \mathsf{Et}_c \), where \( \mathsf{Et}_c \) is the category of etale maps into \( c \), and rely on the well-known result that the category of elements construction provides an equivalence \( \mathsf{El}_c : c\text{-}\mathsf{Set} \to \mathsf{Et}_c \).

By definition, \( \lambda : M \to c \xleftarrow{a} M \) is a function

\[
\lambda : M \to \sum_{a \in c(1)} \prod_{f \in [a]} M,
\]

meaning that for each \( j \in M \) we have a pair \( \lambda(j) = (a_j, m_j) \), where \( a_j \in c(1) \) is an object in \( c \) and \( m_j : c[a_j] \to M \) takes every \( a_j \)-outfacing map \( f : a_j \to a_j' \) to an element that we more conveniently denote \( m(j, f) := m_j(f) \in M \). We know by (23) that \( m(j, \text{id}) = j \) and that \( m(j, f \circ g) = m(m(j, f), g) \), and the coherence condition (25) is vacuous since \( M \xleftarrow{a} 0 \cong M \). This is precisely the data needed to define an etale map into \( c \) for which the domain category has object-set \( M \) and the map \( M \to c(1) \) on objects sends \( j \mapsto a_j \).

Finally, by (26), a morphism of \( (c, 0) \)-bicomodules consists of a function \( \varphi : M \to M' \) such that for any \( j \in M \) with \( j' := \varphi(j) \), we have \( a_j = a_{j'} \) and \( \varphi(m(j, f)) = m'(j', f) \). In other words, it corresponds to a functor commuting with the etale maps into \( c \). \( \Box \)

**Example 2.2.13** (Terminal and representable \( c \)-sets). Let \( c \in \mathsf{Cat}^{\#} \) be a category. The terminal \( c \)-set assigns the set \( 1 \in \mathsf{Set} \) to each object \( a \in c(1) \). We thus see that its set of elements is isomorphic to \( c(1) \in \mathsf{Set} \), and hence it corresponds to a bicomodule of the form \( c \xleftarrow{c(1)} \xrightarrow{0} 0 \).

For any \( a \in c(1) \), the functor \( c \to \mathsf{Set} \) represented by \( a \) has elements \( c[a] \). Hence it corresponds to a bicomodule of the form \( c \xleftarrow{c[a]} \xrightarrow{0} 0 \) by Proposition 2.2.12. \( \diamond \)

### 2.3 Bicomodules are prafunctors

We have seen in Definition 2.2.2 that vertical morphisms in the double category \( \mathsf{Cat}^{\#} \) are somewhat exotic. Thus one might be tempted to consider the whole double category as exotic, or mostly irrelevant to the canon of category theory, if it weren’t for the second amazing result in this story, that of Garner, discussed in his HoTTTEST video of 2019.
Theorem 2.3.1 (Horizontal morphisms in $\mathbb{Cat}^+$ are prafunctors). Let $c$ and $d$ be categories, and let $c$-$\textbf{Set}$ and $d$-$\textbf{Set}$ denote the corresponding copresheaf categories. There is an equivalence

$$c\text{Comod}_d \cong \text{Praf}(d$-$\textbf{Set}, c$-$\textbf{Set})$$

where $\text{Praf}$ denotes the category of parametric right adjoint functors. Moreover, this equivalence commutes with composition.

Thus the framed bicategory $\mathbb{Cat}^+$ has categories as objects, cofunctors as vertical morphisms, and parametric right adjoints as horizontal morphisms.

Recall that a functor $F : d$-$\textbf{Set} \to c$-$\textbf{Set}$ between copresheaf categories is a parametric right adjoint iff the induced map $d$-$\textbf{Set} \to c$-$\textbf{Set}/F(1)$ is a right adjoint, where $1 \in d$-$\textbf{Set}$ is the terminal object. Before we explain the above theorem, it will help to define the notion of duc-query (“disjoint union of conjunctive queries”) and give a characterization of prafunctors in terms of them.

Definition 2.3.2 (Duc-query). Let $d \in \mathbb{Cat}^+$ be a category. A duc-query on $d$ is a functor $p : d$-$\textbf{Set} \to \textbf{Set}$ of the form

$$p(X) \cong \sum_{i \in I} d$-$\textbf{Set}(P_i, X)$$

(28)

where $I \in \textbf{Set}$ is a set and $P_i : d \to \textbf{Set}$ is a copresheaf for each $i \in I$. A morphism of duc-queries is a natural transformation of functors. We denote the category of duc-queries on $d$ by $\text{Set}[d]$.

We say that $F$ is a conjunctive query if $I = 1$. ◊

The notation $\text{Set}[d]$ for the category of duc-queries on $d$ is supposed to invoke (a categorification of) the polynomial ring $\mathbb{Z}[y_1, \ldots, y_D]$ on $D$-many variables, and when $d = D y$ is discrete, we will see that this intuition is correct, as long as the coefficient ring $(\mathbb{Z}, 0, +, 1, \times)$ is replaced by the rig category $(\text{Set}, 0, +, 1, \times)$. More generally for nondiscrete $d$, one should imagine objects in $\text{Set}[d]$ as polynomials whose exponents are not mere sets, but $d$-sets.

Example 2.3.3 (Further exemplifying the notation $\text{Set}[d]$). When $d = 0$, there is a canonical isomorphism $0$-$\textbf{Set} = 1$ (there is one functor from the empty category to $\textbf{Set}$), so a duc-query on $d$ is a functor $0$-$\textbf{Set} \to \textbf{Set}$, which is just a set, and there are no further restrictions. So polynomials in 0 variables are sets: $\text{Set}[0] \cong \text{Set}$.

When $d = y$, a duc-query on $d$ is a functor $\text{Set} \to \textbf{Set}$ that can be written as a coproduct of representables $\text{Set} \to \textbf{Set}$. In other words $\text{Set}[y] \cong \text{Poly}$. Suppose $d := 2y$ is the discrete category on a two-element set. Then $X \in d$-$\textbf{Set}$ can be identified with a pair of sets $(X_1, X_2)$, so a duc-query on $d$ can be rewritten

$$\sum_{i \in I} d$-$\textbf{Set}(P_i, X) \cong \sum_{i \in I} X_1^{P_1} \times X_2^{P_2}$$

for arbitrary sets $P_1, P_2 \in \text{Set}$, and this is just the form of an arbitrary polynomial in two variables. So $\text{Set}[2y]$ is the category of polynomial functors in two variables. ◊
Example 2.3.4. Recall the cospan category from (3)
\[ d := \text{city} \to \text{state} \leftarrow \text{county}. \]
Rather than the duc-query written there, let’s use a slightly less symmetric one (so that
the ideas stand out better):\(^{15}\)
\[ (\text{city} \times_{\text{state}} \text{city}) + \text{city} + \text{state} \]
Given sets of cities, states, and counties, and functions between them as indicated, this
query can be written as a bicomodule \( y \xleftarrow{p} d \) with carrier
\[ p = y^{\{\text{city1, state, city2}\}} + y^{\{\text{city, state}\}} + y^{\{\text{state}\}} \cong y^3 + y^2 + y. \]
It is the right-module structure \( \rho: m \to m \triangleleft d \) that makes the text meaningful, i.e. gives
a \( d \)-coaction (a \( d \)-copresheaf structure) on the three-element set \( \{\text{city1, state, city2}\} \), on
the two-element set \( \{\text{city, state}\} \), and on the one-element set \( \{\text{state}\} \).

Proposition 2.3.5. For any category \( d \), the following categories are equivalent:
1. the category \( \text{Set}[d] \) of duc-queries on \( d \), as in Definition 2.3.2;
2. the free coproduct completion \( \text{Coco}(d\text{-Set}^{\text{op}}) \) of \( d\text{-Set}^{\text{op}} \);
3. the category \( \text{Praf}(d\text{-Set, Set}) \) of prafunctors \( d\text{-Set} \to \text{Set} \); and
4. the category \( \text{CLP}(d\text{-Set, Set}) \) of connected-limit-preserving functors \( d\text{-Set} \to \text{Set} \).

Later we will add a fifth element, the category \( \text{Comod}_d \) of right \( d \)-comodules, to the list; the equivalence with it follows from Theorem 2.3.1, which we prove on page 26.

Proof. By Definition 2.3.2, one can understand \( \text{Set}[d] \) as the free coproduct completion
of the full subcategory of functors \( d\text{-Set} \to \text{Set} \) spanned by those of the form
\( d\text{-Set}(P, -) \), which by Yoneda is exactly \( d\text{-Set}^{\text{op}} \). Thus we have shown \( 1 \Leftrightarrow 2 \). Since
each of \( \text{Set}[d] \), \( \text{CLP}(d\text{-Set, Set}) \), and \( \text{Praf}(d\text{-Set, Set}) \) is a full subcategory of the functors
\( d\text{-Set} \to \text{Set} \), it suffices to show that all three have the same essential image.

By definition, a functor \( F: d\text{-Set} \to \text{Set} \) is a prafunctor iff the induced map \( F': d\text{-Set} \to \text{Set}/F(1) \) is a right adjoint. The functor \( \text{Set}/F(1) \to \text{Set} \) is given by taking coproducts. Since coproducts and right adjoints preserves connected limits, we have \( 3 \Rightarrow 4 \).
We also have \( 4 \Rightarrow 3 \) because if \( F: d\text{-Set} \to \text{Set} \) preserves connected limits then
\( F': d\text{-Set} \to \text{Set}/F(1) \) preserves all limits and hence is a right adjoint.

We also have \( 1 \Rightarrow 4 \), that duc-queries preserve connected limits, because they
are coproducts of representables: representables preserve all limits and coproducts
preserve connected limits.

Thus it suffices to show \( 3 \Rightarrow 2 \), that if \( F \) is a prafunctor then it is isomorphic
to a duc-query. Given a prafunctor \( F \), let \( I := F(1) \) so that \( F \) is the composite of
\( d\text{-Set} \xrightarrow{F'} \text{Set}/I \to \text{Set} \), where the second map is given by coproduct. It suffices to show
that each projection \( F'_i: d\text{-Set} \to \text{Set} \) is represented by some \( F'_i(X) \cong d\text{-Set}(P_i, X) \) for
some \( P_i \in d\text{-Set} \). Since \( F' \) and hence \( F'_i \) preserves limits, let \( L_i: \text{Set} \to d\text{-Set} \) be its left
adjoint. Then indeed one checks that \( P_i := L_i(1) \) serves as a representing object. \( \square \)

\(^{15}\)For the curious reader, the duc-query written in (4) has carrier \( 3y^3 \) because each of the three cases is
represented by a \( d \)-set with three elements.
We will use the janus notation \( c\text{-}\mathbf{Set}[d] \) to denote the category of functors \( c \to \mathbf{Set}[d] \); it generalizes both \( 1\text{-}\mathbf{Set}[d] \cong \mathbf{Set}[d] \) and \( c\text{-}\mathbf{Set}[0] \cong c\text{-}\mathbf{Set} \) from Notation 2.0.1 and Definition 2.3.2. Now because limits in \( c\text{-}\mathbf{Set} \) are computed pointwise, Proposition 2.3.5 easily generalizes to the following.

**Corollary 2.3.6.** For categories \( c \) and \( d \), the following categories are equivalent:

1. the category \( c\text{-}\mathbf{Set}[d] \) of functors \( c \to \mathbf{Set}[d] \);
2. the category of functors \( c \to \mathbf{Coco}((d\text{-}\mathbf{Set})^{op}) \);
3. the category \( \mathbf{Praf}(d\text{-}\mathbf{Set}, c\text{-}\mathbf{Set}) \) of prafunctors \( d\text{-}\mathbf{Set} \to c\text{-}\mathbf{Set} \); and
4. the category \( \mathbf{CLP}(d\text{-}\mathbf{Set}, c\text{-}\mathbf{Set}) \) of connected-limit-preserving functors \( d\text{-}\mathbf{Set} \to c\text{-}\mathbf{Set} \).

And once we sketch the proof of Theorem 2.3.1, we will have also added a fifth element, the category \( c\text{-}\mathbf{Comod}_d \) of \( (c, d) \)-bicomodules, to this list.

**Notation 2.3.7** (Indexed component \( m_a \) of a bicomodule). Suppose given a bicomodule \( c \xleftarrow{m} d \), and consider the composite

\[
m \xrightarrow{\lambda} c \otimes m \xrightarrow{c \otimes 1} c \otimes 1.
\]

We denote by \( m_a \) its pullback along \( a : 1 \to c(1) \)

\[
m_a \xrightarrow{a} m \xrightarrow{a} n \xrightarrow{a} c(1)
\]

for any element \( a \in c(1) \). Then we have an isomorphism of polynomials

\[
m \cong \sum_{a \in c(1)} m_a.
\]  
(29)

Moreover, each \( m_a \in \mathbf{Poly} \) naturally has a \( d \)-comodule structure \( y \xleftarrow{m_a} d \) inherited from \( m \). Indeed \( m_a \) can be written as a composite

\[
y \xrightarrow{y[a]} c \xleftarrow{m_a} d
\]

We will use this notation \( m_a \) often below and refer to it as the \( a \)-indexed component of \( m \).

**Example 2.3.8** (Hom-sets \( C(a, a') \) as components \( c[a]_{a'} \)). Suppose given a category \( C \) and the corresponding object \( c \in \mathbf{Cat}^\sharp \). For any object \( a \in c(1) \), we have a representable copresheaf \( c \xleftarrow{[a]} c \) inherited from \( m \). For any \( a' \in c(1) \), its \( a' \)-indexed component is the hom-set

\[
c[a]_{a'} \cong C(a, a').
\]

\[\Diamond\]

**Proof sketch for Theorem 2.3.1.** We have an equivalence \( d\text{-}\mathbf{Set} \cong d\text{-}\mathbf{Comod}_0 \) by Proposition 2.2.12, i.e. a bicomodule \( d \xleftarrow{X} 0 \) is essentially a copresheaf \( d \to \mathbf{Set} \) with elements \( X \). Hence horizontal composition with \( m \in d\text{-}\mathbf{Comod}_d \) induces a functor \( d\text{-}\mathbf{Comod}_0 \to d\text{-}\mathbf{Comod}_0 \). Let’s denote this functor \( F_m := (m \otimes -) \colon d\text{-}\mathbf{Set} \to c\text{-}\mathbf{Set} \).
The claims to prove are that \( F_m \) is a prafunctor, that all prafunctors between copresheaf categories arise from some such \( m \), and that morphisms \( m \to m' \) of bicomodules coincide with natural transformations between prafunctors. A complete proof of this will not appear in this document but Garner has sketched the argument in his HoTTTEST video, and a full proof will appear in [GS22]; for now we simply explain how the correspondence works.

Let \( m \cong \sum_{a \in \mathcal{C}(1)} m_a \) as in (29). We claim that \( m_a \) has the structure of a duc-query on \( d \) and it varies functorially over \( a \in \mathcal{C}(1) \); once established, this gives us our functor \( (a \mapsto m_a) : c \to \mathcal{Set}[d] \), which is enough by Corollary 2.3.6. Again, we do not establish it here, but simply explain the basic idea. Noting that \( m_a(1) \subseteq m(1) \) and that \( m_a[j] = m[j] \) for all \( j \in m_a(1) \), we write

\[
m = \sum_{a \in \mathcal{C}(1)} \sum_{j \in m_a(1)} y^{m[j]}.
\]

Then one calculates using (24) that the right-comodule structure on \( m \) endows \( m[j] \) has the structure of a \( d \)-Set

\[
m[j] \in d\text{-Set}
\]

for any \( j \in m(1) \), and thus \( m_a \) is a duc-query on \( d \). To be clear, the associated prafunctor \( F_m : d\text{-Set} \to c\text{-Set} \) assigns to each \( X \in d\text{-Set} \) and \( a \in \mathcal{C}(1) \) the set

\[
F_m(X)(a) \cong \sum_{j \in m_a(1)} d\text{-Set}(m[j], X).
\]

The functoriality of \( (a \mapsto m_a) : c \to \mathcal{Set}[d] \) is established by Eqs. (23) and (25).

The final statement of the theorem—that \( \mathbf{Cat}^d \) has categories as objects, cofunctors as vertical morphisms, and parametric right adjoints as horizontal morphisms—is just a restatement of the above, together with Corollary 2.2.10. \( \square \)

Let \( c \leftarrow \xymatrix{m \ar[r] & d \ar[r] & e} \) be bicomodules, considered as elements \( m \in c\text{-Set}[d] \) and \( n \in d\text{-Set}[e] \). Under the correspondence from Theorem 2.3.1, their composite \( m \circ d n \in c\text{-Set}[e] \) has carrier polynomial

\[
(m \circ_d n) := \sum_{i \in \mathcal{C}(1)} \sum_{\varphi \in \mathcal{Set}(m[i], n(1))} y^{\left( \text{colim}_{x \in m[i]} n[\varphi(x)] \right)}
\]

where \( \text{colim}_{x \in m[i]} n[\varphi(x)] \) denotes the colimit of the \( e \)-sets \( n[\varphi(x)] \), taken over elements \( x \in \text{El}_d m[i] \).

**Notation 2.3.9.** Now that we have established the equivalence

\[
c\text{Comod}_d \cong c\text{-Set}[d],
\]

adding a fifth equivalent category to Corollary 2.3.6, we will generally default to the notation \( c\text{-Set}[d] \) for all five. For example given \( m \in c\text{-Set}[d] \) we can think of it as a functor \( m : c \to \mathcal{Set}[d] \), as a functor \( m : c \to \text{Coco}((d\text{-Set})^{\text{op}}) \), as a prafunctor \( m : d\text{-Set} \to c\text{-Set} \), as a connected limit preserving functor \( m : d\text{-Set} \to c\text{-Set} \), or as a bicomodule \( c \xymatrix{m \ar[r] & d} \). In particular, we have

\[
c\text{-Set} := c\text{-Set}[0] \cong \text{Fun}(c, \mathcal{Set}) \quad \text{and} \quad \mathcal{Set}[d] := y\text{-Set}[d] \cong \text{Coco}((d\text{-Set})^{\text{op}}).
\]
Example 2.3.10. One can see \textup{Poly} inside of \mathbb{Cat} as the full sub framed bicategory spanned by the object \( y \). We saw in Example 2.2.7 that \( y \) corresponds to the discrete category on one object, so by Theorem 2.3.1, we see that \( \textup{Poly} \cong \textup{Set}[y] \).

\[ m = \sum_{i \in m(1)} y^{m[i]} \quad (33) \]

Remark 2.3.11 (Unreasonably effective notation). Let \( c \leftarrow \leftarrow d \) be a bicomodule. The notation for the underlying polynomial

is surprisingly well-suited to the viewpoint of \( m \) as a functor \( d\text{-Set} \to c\text{-Set} \).

First, the notation \( m(1) \) in (33) could refer to either

- the set one obtains by applying the polynomial functor \( m: \text{Set} \to \text{Set} \) to the terminal object \( 1 \in \text{Set} \), or
- the \( c\text{-Set} \) one obtains by applying the prafunctor \( m: d\text{-Set} \to c\text{-Set} \) to the terminal object \( 1 \in d\text{-Set} \).

It turns out that the first is always the set of elements of the second, fitting our notation all along; see Notation 2.0.1 and Proposition 2.2.12. Thus the ambiguity disappears.

Second, for each element \( i \in m(1) \), we saw in (31) that the set \( m[i] \) is endowed by \( \rho \) with the additional structure of (the elements of) a \( d\text{-Set} \). In other words the notation \( y^{m[i]} \) in (33) could refer to either

- the \( i \)th component \( \text{Set}(m[i], -) \) of \( m \in \text{Poly} \), represented by \( m[i] \) as a set, or
- the \( i \)th component \( d\text{-Set}(m[i], -) \) of \( m: d\text{-Set} \to c\text{-Set} \) represented by \( m[i] \) as a \( d\text{-Set} \),

and again the first is always the set of elements of the second, eliminating the ambiguity.

In other words, the notation for the polynomial \( m \) as in (33) very much suits as notation for the bicomodule \( c \leftarrow \leftarrow d \), in that it is articulated appropriately as a sum of “higher level” representables \( m[i] \in d\text{-Set} \).

Just as in Remark 2.1.2, it is useful to have a more “combinatorial” description of morphisms in \( c\text{-Set}[d] \); indeed, the fact that everything in this paper is combinatorial or “elementary”—i.e. that it can be calculated by considering set-theoretic elements—makes it much more tractable.

Remark 2.3.12 (Combinatorial description of bicomodule maps). For categories \( c, d \), we have the following combinatorial description of morphisms between \( p, q \in c\text{-Set}[d] \):

\[
c\text{-Set}[d](p, q) \cong \sum_{q_1 \in c\text{-Set}(p(1), q(1))} \int_{(a, i) \in \text{El}_{c \to p(1)}} d\text{-Set}(q[q_1 i], p[i]). \quad (34)\]

Here \( \int \) denotes an end. One can see that this formula generalizes our simpler formula (12) in the case that \( c = d = y \).

The following lemma says that extension in the framed bicategory \( \mathbb{Cat} \) is easy: the carrier polynomial does not change. We learned this from Nelson Niu.

\[\footnote{\text{We say “higher-level” because of the potential confusion here. A representable \( d \)-set is usually \( d(b, -) \) for some object \( b \in d(1) \). Here we mean that an arbitrary \( d \)-set, say \( X: d \to \text{Set} \), itself represents a functor \( d\text{-Set}(X, -): d\text{-Set} \to \text{Set} \).} }\]
Lemma 2.3.13 (Cocartesian squares). For any categories $c, c', d, d' \in \mathbf{Cat}^\#$, bicomodule $c \leftarrow m \rightarrow d$ with structure maps $\lambda$ and $\rho$, and diagram (left):

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) {$c$};
  \node (b) [right of=a] {$d$};
  \node (c) [below of=a] {$c'$};
  \node (d) [below of=b] {$d'$};

  \draw[->] (a) -- (b) node[midway,above] {$\beta$};
  \draw[->] (a) -- (c) node[midway,above] {$\alpha$};
  \draw[->] (b) -- (d) node[midway,above] {};
  \draw[->] (c) -- (d) node[midway,above] {};\end{tikzpicture}
\end{array}
\]

the extension bicomodule $c' \leftarrow m \rightarrow d'$ again has carrier $m$, as above-right, and the structure maps $\lambda', \rho'$ are the composites:

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) {$c$};
  \node (b) [right of=a] {$m$};
  \node (c) [below of=a] {$c'$};
  \node (d) [below of=b] {$m \leftarrow d'$};

  \draw[->] (a) -- (b) node[midway,above] {$\lambda$};
  \draw[->] (a) -- (c) node[midway,above] {$\alpha m$};
  \draw[->] (b) -- (d) node[midway,above] {$\lambda'$};
  \draw[->] (c) -- (d) node[midway,above] {$\rho'$};\end{tikzpicture}
\end{array}
\]

Proof. The bicomodules corresponding to $\alpha$ and $\beta$ are of the form $c' \leftarrow c \rightarrow c$ and $d \leftarrow d \rightarrow d'$ respectively, where the left and right structure maps are shown here:

$\alpha m \rightarrow c \leftarrow c \rightarrow c$ and $d \leftarrow d \rightarrow d \rightarrow d' \leftarrow d'$. We need to show that the extension of $m$ along $\alpha$ and $\beta$, i.e. the composite

$c' \leftarrow c \rightarrow m \leftarrow d \rightarrow d'$

again has carrier $m$ and that the bicomodule structure maps are as in (35).

To establish that $m$ carries the composite, we show that the following are equalizers

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) {$m$};
  \node (b) [right of=a] {$c \leftarrow m$};
  \node (c) [below of=b] {$m \leftarrow d$};

  \draw[->] (a) -- (b) node[midway,above] {$\lambda$};
  \draw[->] (b) -- (c) node[midway,above] {$\delta \alpha m$};
  \draw[->] (a) -- (c) node[midway,above] {$\delta \alpha m$};\end{tikzpicture}
\end{array}
\]

By (23) and (24), we know that each of $\lambda$ and $\rho$ is the section of a retraction (we’ll use that later) and that each equalizes the displayed fork. Thus it remains to show that they have the universal property. Suppose $\ell \rightarrow c \leftarrow m$ and $r \rightarrow m \leftarrow d$ also equalize the forks. Then since the following diagrams commute

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) {$m$};
  \node (b) [right of=a] {$c \leftarrow m$};
  \node (c) [below of=b] {$m \leftarrow d$};

  \draw[->] (a) -- (b) node[midway,above] {$\lambda$};
  \draw[->] (b) -- (c) node[midway,above] {$\delta \alpha m$};
  \draw[->] (a) -- (c) node[midway,above] {$\delta \alpha m$};\end{tikzpicture}
\end{array}
\]

and the righthand composites are identity, we find that the left-hand composites $\ell \rightarrow m$ and $r \rightarrow m$ provide the universal map. These maps are unique because $\lambda$ and $\rho$ are sections, hence monomorphisms. In particular, we have now shown that the carrier of the cocartesian bicomodule is indeed $m$. 

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We now show that the structure maps $\lambda': m \to c' \triangleleft m$ and $\rho': m \to m \triangleleft d'$ for the extension bicomodule are computed as in (35). The two cases are similar so we consider only the latter. To do so, we follow Eq. (27) and consider the following diagram

\[
\begin{array}{c}
m \xrightarrow{\rho} m \triangleleft d \\
\downarrow \\
m \triangleleft d' \xrightarrow{\rho \triangleleft \beta} m \triangleleft d \triangleleft d'
\end{array}
\]

The bottom row is also an equalizer, and the dashed map—which is unique among maps making the square commute—is the definition of $\rho'$ from (27). The dotted map $m \triangleleft \beta$ also commutes with the right-hand triangle as shown, so we indeed have $\rho' = \rho \triangleleft (m \triangleleft \beta)$ as in (35), completing the proof. □

2.4 Adjunctions, profunctors, functors, and coclosures in \( \mathbf{Cat}^\# \)

We have now have a good enough working knowledge of \( \mathbf{Cat}^\# \) that we can begin both to locate familiar friends from category theory—profunctors, spans, etc.—and to discover novel universal structures, such as coclosures and local monoidal closed structures, within it.

Recall that an adjunction in a double category is a pair of horizontal arrows in opposite directions with the usual unit and counit maps satisfying triangle-equations, all within the double category.

**Proposition 2.4.1** (Adjoints in \( \mathbf{Cat}^\# \)). Let $c \triangleleft d$ be a bicomodule and consider the corresponding functor $F: \mathbf{d-Set} \to \mathbf{c-Set}$.

- If $F$ has a right adjoint $\mathbf{c-Set} \to \mathbf{d-Set}$ within the category of large categories, then $F$ has a right adjoint in \( \mathbf{Cat}^\# \).
- If $F$ has a left adjoint $L: \mathbf{c-Set} \to \mathbf{d-Set}$ within the category of large categories, then in general one cannot conclude that $F$ has a left adjoint in \( \mathbf{Cat}^\# \). However, if $L$ is a profunctor, then $L$ is left adjoint to $F$ in \( \mathbf{Cat}^\# \).

**Proof.** This follows from the fact the definition of \( \mathbf{Praf}(\mathbf{d-Set}, \mathbf{c-Set}) \subseteq \mathbf{Fun}(\mathbf{d-Set}, \mathbf{c-Set}) \), namely that this subcategory is full for each $c, d \in \mathbf{Cat}^\#$. □

Recall the notation $m_a$ from Notation 2.3.7.

**Definition 2.4.2** (Conjunctive bicomodule). We say that a bicomodule $c \triangleleft d$ is conjunctive if for each $a \in c(1)$ the duc-query $m_a$ is conjunctive in the sense of Definition 2.3.2, i.e. if $m_a(1) = 1$ for each object $a \in c(1)$. ◊

**Proposition 2.4.3.** The following are equivalent:

1. the bicategory of conjunctive bicomodules $c \triangleleft d$ in \( \mathbf{Cat}^\# \),
2. the usual bicategory of profunctors $P: c^{op} \times d \to \mathbf{Set}$, and
3. the category of right adjoints $\mathbf{d-Set} \to \mathbf{c-Set}$. 

Proof. A profunctor $c \to d$ is a functor $F : c^{op} \times d \to \text{Set}$, which by currying can be considered as a functor $c \to (d\text{-Set})^{op}$, and hence a functor $c \to (d\text{-Set})^{op} \to \text{Coco}((d\text{-Set})^{op})$. By Corollary 2.3.6 profunctors correspond to those bicomodules $c \xrightarrow{f} d$ having the form

$$f \cong \sum_{a \in c(1)} y^F(a,-)$$

for $F(a,-) : d \to \text{Set}$ arbitrary. Since $f_a := y^F(a,-)$ is representable for each $a$, i.e. $f_a(1) = 1$, these are exactly the conjunctive queries. It is easy to check that all of the relevant structures of bicategories (2-morphisms, horizontal composites) agree, establishing the equivalence $1 \iff 2$.

To see that $1 \iff 3$, it suffices to show that the functor $d\text{-Set} \to c\text{-Set}$ given on $X \in d\text{-Set}$ and $a \in c(1)$ by

$$(f \ll_d X)_a := d\text{-Set}(F(a,-), X)$$

preserves limits in the $X$ variable. Since limits in $c\text{-Set}$ are taken pointwise, it suffices to notice that $(f \ll_d X)_a$ preserves limits in $X$. \hfill \Box

**Example 2.4.4** (Profunctors and right adjoints). Every profunctor $c^{op} \times d \to \text{Set}$ induces a right adjoint $d\text{-Set} \to c\text{-Set}$, but only some of them are right adjoints in $\text{Cat}^\#$.

Indeed, let $d = \begin{tikzpicture}[baseline=-.5ex]
\node (1) at (0,0) {$\bullet^1$};
\node (2) at (1,0) {$\bullet^2$};
\draw (1) edge (2);
\end{tikzpicture}$ be the parallel-arrows category. The diagonal functor $\text{Set} \to d\text{-Set}$ corresponds to a profunctor $d \to 1$ and hence to a bicomodule of the form $d \xleftarrow{2y} y$.\footnote{The carrier is $2y$ because each of the two objects in $d$ is assigned the conjunctive query $\text{Set}(1, \_)$.) As a profunctor $d^{op} \times 1 \to \text{Set}$, it is constant 1.} It has a left adjoint functor $d\text{-Set} \to \text{Set}$, but this left adjoint (given by taking the coequalizer of sets $X_1 \rightrightarrows X_2$) is itself not a profunctor. One can see that by the fact that coequalizers do not preserve connected limits. \hfill \Diamond

**Proposition 2.4.5** (Functors give adjoint bicomodules). Let $c, d \in \text{Cat}^\#$ and $C, D \in \text{Cat}$ their associated categories. For any functor $F : C \to D$, there is a corresponding adjunction

$$
\begin{array}{ccc}
c & \cong & d \\
\Delta_F & \leftarrow & \Pi_F \\
\downarrow & & \downarrow \\
C & \vDash & D
\end{array}
$$

If $F$ is etale, then $\Delta_F$ has a further left adjoint in $\text{Cat}^\#$,

$$
\begin{array}{ccc}
c & \cong \cong & d \\
\Sigma_F & \leftarrow & \Delta_F \\
\downarrow & & \downarrow \\
C & \dashv & D
\end{array}
$$

Proof. Given $F : C \to D$, the functor $\Delta_F : D\text{-Set} \to C\text{-Set}$ is left adjoint to $\Pi_F : C\text{-Set} \to D\text{-Set}$. Moreover both are right adjoints, hence profunctors $c \xleftarrow{\Delta_F} d$ and $d \xrightarrow{\Pi_F} c$, and the first result follows from Proposition 2.4.1.

In general $\Delta_F : D\text{-Set} \to C\text{-Set}$ has a left adjoint $\Sigma_F : C\text{-Set} \to D\text{-Set}$ which assigns the colimit of a certain comma category to each $a \in \text{Ob} C$. When $F$ is etale this diagram has a discrete final subcategory, and hence can be computed as a coproduct. As such $\Sigma_F$ is a profunctor and we again apply Proposition 2.4.1. \hfill \Box
The following generalizes Proposition 2.1.14.

**Proposition 2.4.6 (Coclosure).** Let \( c, d, e \in \text{Cat}^\# \) be categories. Then for any diagram as shown left

\[ \begin{array}{ccc}
  c & \xrightarrow{p} & d \\
  \downarrow{q} & & \downarrow{p}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
  c & \xrightarrow{p'} & d \\
  \downarrow{q} & & \downarrow{p}
\end{array} \]

the category consisting of bicomodules \( c \leftarrow p' \rightarrow d \) and 2-cells \( \varphi : p \to p' \triangleleft d \), as shown right, has an initial object, which we denote \( c \leftarrow q \rightarrow d \).

**Proof.** The formula for the carrier of \( \begin{bmatrix} q \\ p \end{bmatrix} \) is:

\[ \left[ \begin{array}{c} q \\ p \end{array} \right] := \sum_{a \in c(1)} \sum_{i \in p_a(1)} y^{q \circ p[i]} \]  \( \tag{36} \)

We begin by explaining how to read it, and then show that it indeed has the structure of a \((c, d)\)-bicomodule.

First note that the right \( e \)-comodule structure on \( p \) endows each \( p[i] \) with the structure of an \( e\text{-Set} \). Thus we can consider it as a bicomodule \( e \leftarrow p[i] \rightarrow 0 \) at which point the composite \( q \triangleleft p[i] \) in (36) makes sense. Similarly, the left \( d \)-comodule structure on \( q \) endows \( q(1) \) with a \( d \)-set structure, and \( q[j] \) varies contravariantly over \( j \in \text{El}_d q(1) \). Hence we find a canonical \( d\text{-Set} \)-structure on the composite

\[ q \triangleleft p[i] \cong \sum_{j \in q(1)} e\text{-Set}(q[j], p[i]) \]

for each \( i \in p(1) \). This composite varies contravariantly over \( i \in \text{El}_c p(1) \). Indeed, given a map \( f : a \to a' \) in \( c \), we get a map we’ll temporarily call \( f_\ast : p_a(1) \to p_{a'}(1) \). This gives a map of \( e \)-sets \( p[f_i] \to p[i] \), and hence a map \( q \triangleleft p[f_i] \to q \triangleleft p[i] \). Thus we have established a \((c, d)\)-bicomodule structure on \( \begin{bmatrix} q \\ p \end{bmatrix} \).

We next give the map \( p \to \begin{bmatrix} q \\ p \end{bmatrix} \triangleleft q \) in \( c\text{-Set}[e] \). Its type is

\[ \int_{a \in c(1) \atop \text{El}_c p(1) \atop \text{El}_d q(1)} \prod_{i \in p_a(1)} \sum_{i' \in p_{a'}(1)} \sum_q \lim_{(b,j,w) \in \text{El}_d(q \circ p[i'])} e\text{-Set}(q[j], p[i]). \]

As complicated as it looks, giving an element of this type is straightforward. Indeed, we simply use

\[ a \mapsto i \mapsto ((i, w) \mapsto j), (b, j, w) \mapsto x \mapsto w(x)) \]

where \( a \in c(1), i' := i \in p_a(1), j \in q(1), w \in e\text{-Set}(q[j], p[i']), \varphi(j, w) := j, b \in d(1), x \in q[j], \) and \( w(x) \in p[i'] = p[i] \).
It remains to show that the above is universal, so suppose given $c \xleftarrow{\varphi'} d$ and a map $\psi: p \to p' \xrightarrow{q} q$ in $c\text{-}\mathbf{Set}[e]$. It has type

$$\psi \in \int_{a \in c(1)} \prod_{i \in p_a(1)} \sum_{i' \in p'_a(1)} \sum_{p'_{[i'],q(1)}} \lim_{(h,w') \in \text{El}_{e'} p'[i']} c\text{-}\mathbf{Set}(q[w'], p[i]).$$

In order to give a map $\left[\begin{array}{c} q \\ p \end{array}\right] \to p'$ in $c\text{-}\mathbf{Set}[d]$ we need something of type

$$\int_{a \in c(1)} \prod_{i \in p_a(1)} \sum_{i' \in p'_a(1)} d\text{-}\mathbf{Set}(p'[i'], q \searrow p[i]),$$

and one can check that this type is isomorphic to that defining $\psi$. We leave the remaining details to the reader. \hfill \Box

### 2.5 Gambino-Kock’s framed bicategory $\mathbf{PolyFun} \cong \mathbf{Cat}^\#_{\text{disc}}$

In [GK12], Gambino and Kock define a framed bicategory $\mathbf{PolyFun} := \mathbf{PolyFun}_{\mathbf{Set}}$ whose objects are sets, whose vertical morphisms are functions, whose horizontal morphisms $D \to C$ are bridge diagrams

$$\begin{array}{ccc}
D & \xleftarrow{f} & E \\
& \searrow & \downarrow g \\
& & B \\
& \swarrow & \downarrow h \\
& & C
\end{array}$$

(37)

and whose 2-cells are isomorphism classes of diagrams having the following form

$$\begin{array}{ccc}
D & \xleftarrow{f} & E & \xrightarrow{g} & B & \xrightarrow{h} & C \\
& & \downarrow \varphi & \Rightarrow & \downarrow \varphi \\
& \downarrow q & \Rightarrow & \downarrow r \\
D' & \xleftarrow{f'} & E' & \xrightarrow{g'} & B' & \xrightarrow{h'} & C'
\end{array}$$

(38)

That is, if the bottom and top row are given, and the left and right side functions $q, r$ are given, then the 2-cell consists of $\varphi$ and $\varphi^\#$.

Before showing that this is isomorphic to the full sub framed bicategory $\mathbf{Cat}^\#_{\text{disc}} \subseteq \mathbf{Cat}^\#$ spanned by the discrete objects, the following remark will be useful.

**Remark 2.5.1** (Combinatorial descriptions in $\mathbf{Cat}^\#_{\text{disc}}$). Let $C, D \in \mathbf{Set}$ be sets. The combinatorial description of maps from Remark 2.3.12 simplifies greatly when the categories involved, namely $c = C y$ and $d = D y$, are discrete. By abuse of notation, we will denote the category of bicomodules between them by

$$C\text{-}\mathbf{Set}[D] := (C y)\text{-}\mathbf{Set}[D y].$$

\[\text{\footnote{In fact they define $\mathbf{PolyFun}_E$ for any locally cartesian closed category $E$ and the proof below presumably goes through, but we have not checked it carefully.}}\]
An object in $C\text{-Set}[D]$ consists of $C$-many polynomials in $D$-many (discrete) variables. Then we have the following combinatorial description of morphisms between $p, q \in C\text{-Set}[D]$:  

$$C\text{-Set}[D](p, q) \cong \sum_{\varphi_1 \in C\text{-Set}(p(1), q(1))} \prod_{i \in p(1)} D\text{-Set}(q[\varphi_1 i], p[i]). \tag{39}$$

The combinatorial description (32) of composition $C_y \leftarrow m \rightarrow D_y \leftarrow n \rightarrow E_y$ also simplifies from a colimit to a coproduct in the discrete categories case, becoming:

$$(m \triangleleft_{D_y} n) := \sum_{i \in m(1)} \sum_{q \in d\text{-Set}(m[i], n(1))} \sum_{x \in m[i]} n[\varphi(x)]. \tag{40}$$

\begin{theorem}[Translation between bridge diagrams and bicomodules] We can identify a bridge diagram as in (37) with a bicomodule $C_y \leftarrow m \rightarrow D_y \leftarrow n \rightarrow E_y$ between the corresponding discrete categories. Writing $E[b] := g^{-1}(b)$, the carrier of the bicomodule is

$$m \cong \sum_{b \in B} y^{E[b]}.$$

This mapping extends to an isomorphism of framed bicategories:\footnote{It may be more accurate to say $\mathbb{P}\text{olyFun}^{op} \cong \mathbb{C}\text{at}_{\text{disc}}^{op}$ i.e. use the horizontal opposite of $\mathbb{P}\text{olyFun}$ in (41), depending on how one thinks about the arrows $\leftarrow \leftrightarrow \rightarrow$}.  

$$\mathbb{P}\text{olyFun} \cong \mathbb{C}\text{at}_{\text{disc}}^{\#} \tag{41}$$

\end{theorem}

\textbf{Proof.} We first construct the mapping $\mathbb{P}\text{olyFun} \rightarrow \mathbb{C}\text{at}_{\text{disc}}^{\#}$. The two framed bicategories have the same objects and the same vertical morphisms (see Example 2.2.7). For a horizontal morphism $D \leftarrow f \rightarrow E \rightarrow g \rightarrow B \rightarrow h \rightarrow C$, we must define the bicomodule structure on $m := \sum_{b \in B} y^{E[b]}$. We define the left comodule structure map $\lambda: m \rightarrow C_y \triangleleft m = C \times m$ to be the graph of $m \rightarrow B \rightarrow C$, i.e. identity on the second factor. This is forced by the left-module conditions (23). We define the right comodule structure map

$$\rho: \sum_{b \in B} y^{E[b]} \rightarrow \sum_{b \in B} \prod_{E[b]} D_y$$

to be identity on $B$, so that by Yoneda it is given by an element of $\prod_{b \in E[b]} D \times E[b]$, i.e. a function $E[b] \rightarrow D \times E[b]$. We define it to be the graph of the composite $E[b] \rightarrow E \rightarrow B$, which we denote $f_b: E[b] \rightarrow B$. Again this is forced by the right comodule condition.\footnote{The left-right coherence (25) is vacuous in this case. In general, it is there to force the assignment $c \rightarrow \text{Set}[d]$ to be functorial in $c$, which is vacuous when $c = C_y$ is discrete.}
We now check that the functor $D\text{-Set} \to C\text{-Set}$ described by the bridge diagram, namely $X \mapsto \Sigma_h \Pi_g \Delta_f(X)$, is the same as that described by $m$ as in (30). Letting $B_i := h^{-1}(i)$, the profunctor described by $m$ sends $X \in D\text{-Set}$ and $i \in C$ to

$$\sum_{b \in B_i} D\text{-Set}(E[b], X).$$

Since $\Sigma_h(Y)(i) = \sum_{b \in B_i} Y(b)$ for any $Y : B \to \text{Set}$, it suffices to show that $D\text{-Set}(E[b], X) \simeq \Pi_g \Delta_f(X)(b)$ for each $b \in B$, and indeed this holds:

$$D\text{-Set}(E[b], X) \cong \prod_{j \in D} \text{Set}(f^{-1}_b(j), X(j)) \cong \prod_{e \in E[b]} X(f_be) \cong \Pi_g \Delta_f(X)(b).$$

This implies that horizontal composition works the same in both bicategories.

To complete the proof, we need to show that the 2-cells in $\mathbb{P}\text{olyFun} \cong \mathbb{C}at^{\#}_{\text{disc}}$ agree. Consider a $\mathbb{P}\text{olyFun}$ 2-cell as in (38). The top and bottom maps correspond to bicomodules

$$m = \sum_{b \in B} y^E[b] \quad \text{and} \quad m' = \sum_{b' \in B'} y'^E[b']$$

respectively. Since $\mathbb{C}at^{\#}_{\text{disc}} \subseteq \mathbb{C}at^{\#}$ is defined to be 2-full, a 2-cell in it is just a map of bicomodules, which is a map of carriers $\phi : m \to m'$ satisfying certain conditions. The middle two columns of (38) are an instance of Proposition 2.1.4 (turned sideways) and hence provide this map of carriers. We leave the rest to the reader.  

\begin{lemma}
For sets $C, D \in \text{Set}$, every right (resp. left) adjoint $D\text{-Set} \to C\text{-Set}$ is a right (resp. left) adjoint in $\mathbb{C}at^{\#}$.
\end{lemma}

\begin{proof}
Every right adjoint $D\text{-Set} \to C\text{-Set}$ is in particular a parametric right adjoint, so by Theorem 2.5.2 it can be represented as $\Pi_g \Delta_f$ for some $D \xleftarrow{f} E \xrightarrow{g} C$. This has a left adjoint, namely $\Sigma_f \Delta_g$. Since functors, such as $f : E \to D$, between discrete categories are etale, we see by Proposition 2.4.1 that $\Sigma_f \Delta_g$ is a left adjoint in $\mathbb{C}at^{\#}$.  
\end{proof}

\begin{proposition}[Adjoint bicomodules in $\mathbb{C}at^{\#}_{\text{disc}}$]
Let $C, D \in \text{Set}$, and let $C_y \xleftarrow{m} D_y$ be a bicomodule. Then

1. $m$ is a left adjoint in $\mathbb{C}at^{\#}$ iff $m$ is linear, i.e. iff $m \cong M_y$ for some $M \in \text{Set}$;
2. $m$ is a right adjoint in $\mathbb{C}at^{\#}$ iff $m$ is conjunctive, i.e. iff $m_a(1) = 1$ for each $a \in C$.

The right adjoint of $C_y \xleftarrow{M_y} D_y$, is

$$D_y \xleftarrow{\sum_{b \in D} y^X[b]} C_y$$

for some set $X_b \in \text{Set}$. The left adjoint of $C_y \xleftarrow{\sum_{a \in C} y^m[a]} D_y$ is

$$D_y \xleftarrow{\sum_{b \in D} \sum_{a \in C} y^{m[a]}[b]} C_y.$$

\end{proposition}
Proof. For 1, if \( m = M y \) then in particular \( m[i] = 1 \) is a one-element set for each \( i \in m(1) \). This corresponds to bridge diagrams (37) in which \( E = B \) and \( g = \text{id} \), in which case the associated polynomial is \( \Sigma_i \Delta_f \), a left adjoint. All left adjoint functors \( D\text{-Set} \to C\text{-Set} \) are left adjoints in \( \text{Cat}^d \) by Lemma 2.5.3.

For 2, recall that by Proposition 2.4.3, \( m \) is a profunctor iff \( m_a(1) = 1 \) for each \( a \in C \), i.e. for which \( m(1) \xrightarrow{\sim} c(1) \) is a bijection. This corresponds to a bridge diagram (37) for which \( B = C \) and \( h = \text{id} \). This corresponds to a functor of the form \( \Pi_g \circ \Delta_f \), which has left adjoint \( \Sigma_f \Delta_g \). Thus we are done by Proposition 2.4.1.

The last two claims follow from the above and Theorem 2.5.2. In particular, linear and conjunctive bicomodules (respectively) correspond to bridge diagrams of the form

\[
D \leftarrow M \implies M \xrightarrow{h} C. \quad \text{and} \quad D \xleftarrow{f'} M \xrightarrow{g} C \implies C
\]

The unspecified set \( X_b \) in the proposition statement is given by \( f^{-1}(b) \).

Following Notation 2.0.1, we will denote the left adjoint of \( F \) by \( F^\dagger \) and the right adjoint of \( F \) by \( F^\ddagger \).

**Example 2.5.5.** Given an polynomial \( p \in \text{Poly} \), its bundle form \( \hat{p}(1) \to p(1) \) as in (14) can be captured inside of \( \text{Cat}^d \) in two ways:

\[
p(1)y \xleftarrow{p} y \quad \text{and} \quad y \xleftarrow{p^\dagger} p(1)y.
\]

Indeed, since \( p(1)y \) and \( y \) correspond to discrete categories, Theorem 2.5.2 shows us how to identify these with bridge diagrams; respectively they are

\[
1 \leftarrow \hat{p}(1) \xrightarrow{\pi} p(1) = p(1) \quad \text{and} \quad p(1) \xleftarrow{\pi} \hat{p}(1) = \hat{p}(1) \to 1
\]

Here, if \( p = \sum_{i \in p(1)} y^{p[i]} \) then \( p^\dagger = \sum_{i \in p(1)} p[i]y \). As a functor \( \text{Set} \to p(1)\text{-Set} \), the former sends \( X \) to the \( p[i] \)-fold products \( X^{p[i]} \) for \( i \in p(1) \). As a functor \( p(1)\text{-Set} \to \text{Set} \), the latter sends \( X \to p(1) \) to its pullback along \( \hat{p}(1) \to p(1) \).

These two representations correspond to two sorts of maps one can imagine between bundles. Indeed, consider 2-cells of the form

\[
p(1)y \xleftarrow{p} y \\
\downarrow \quad \downarrow f^\ddagger \\
q(1)y \xleftarrow{q} y
\]

\[
y \xleftarrow{p^\dagger} p(1)y \\
\downarrow \quad \downarrow f^\ddagger \\
y \xleftarrow{q^\dagger} q(1)y
\]

The data \( f : p(1) \to q(1) \) is the same in both cases. One checks that \( f^\ddagger \) is the data of a map \( q[f[i]] \to p[i] \) for each \( i \in p(1) \); in other words the left-hand side is just another representation of a map of polynomials \( p \to q \). On the other hand, \( f^\dagger \) is the data of a map \( p[i] \to q[f[i]] \) for each \( i \in p(1) \); in other words it is just a commutative square.
Here are set-theoretic representations of the 2-cells above:

\[
p(1) \xleftarrow[p]{\Downarrow} \hat{p}(1) \xrightarrow{f^t} \hat{p}(1) \xrightarrow{\Downarrow\hat{f}} p(1)
\]

\[
q(1) \xleftarrow[q]{\Downarrow} \hat{q}(1) \xrightarrow{\Downarrow\hat{f}} \hat{q}(1) \xrightarrow{\Downarrow\hat{f}} q(1)
\]

The above recovers the operation referred to in [SM20, Section 4] as the Dirichlet transform between polynomials and Dirichlet series. Whereas in that reference the operation was regarded as purely syntactic, we are now able to see that it has a universal property: it is an adjunction \( * \xrightarrow{\phi} \phi^t \).

It is well-known that categories are monads in \( \mathbb{S} \text{pan} \). We have already alluded to the fact that spans between sets correspond to bicomodules with linear carrier (between comonoids with linear carriers), so for example \( c(1)_y \xleftarrow{\hat{c}(1)y} c(1)_y \) in the proposition below represents a span, and the monoid structure is giving us a category. In Section 2.8 we will discuss all this in detail, but for now we leave it in the background.

**Proposition 2.5.6** (Right-adjoint comonads, left-adjoint monads). Let \( c \in \mathcal{C} \text{at} \) be a category. Then \( c \) can be considered as a right adjoint bicomodule \( c(1)_y \xleftarrow{\hat{c}(1)y} c(1)_y \) equipped with a comonoid structure. Its left adjoint has carrier \( c^+ \cong \hat{c}(1)_y \) and as such is naturally endowed with a monoid structure.

**Proof.** Let \( C := c(1) \). To give \( c = \sum_{a \in C} y^c[a] \) the structure of a \( (C_Y, C_Y) \)-bicomodule, it suffices to provide a function \( C \to C \) defining \( \lambda \), and a function \( c[a] \to C \) for each \( a \), defining \( p \); the rest is forced by the coherence conditions. For the former we use the identity, and for the latter we use the codomain map cod, sending each \((f : a \to a') \in c[a] \) to \( a' \). Thus we see that \( C_Y \xleftarrow{c} C_Y \) is conjunctive, and hence a right adjoint by Proposition 2.5.4.

Next we want to give the comonoid maps

\[
\begin{array}{ccc}
C_Y & \xrightarrow{c} & C_Y \\
\downarrow & & \downarrow \\
C_Y & \xrightarrow{\phi} & C_Y
\end{array}
\quad
\begin{array}{ccc}
C_Y & \xrightarrow{c} & C_Y \\
\downarrow & & \downarrow \\
C_Y & \xrightarrow{\phi} & C_Y
\end{array}
\]

The counit \( \epsilon \) is more straightforward; by Eq. (39) we just need an element \( f \in c[a] \) for each \( a \in C \) whose codomain is again \( a \); we use the identities. In other words, on carriers \( \epsilon \) here agrees with the counit on \( c \). Similarly, using the formula for the composite in (40), we see that to give a map \( c \to (c \circ_c c) \) is to assign to every pair \((x, y) \in \sum_{x \in [a]} c[\text{cod}(x)] \) an element of \( c[a] \); we use the composite \( x \circ y \). These choices indeed make \((c, \epsilon, \delta)\) a comonoid on \( C_Y \), and to arrive there all of our choices were in fact forced upon us.
Again by Proposition 2.5.4, the left adjoint of \( c \) has carrier \( \sum_{a' \in C} \sum_{a \in C} \sum_{f \in c[a], y(a)} y(1), \) which as a polynomial is isomorphic to \( \hat{c}(1) y \). Recall from Example 2.3.8 that \( c[a]_{a'} \) consists of the morphisms \( a \to a' \), so this bicomodule corresponds to the span

\[
C \xleftarrow{\text{cod}} \hat{C} \xrightarrow{\text{dom}} C
\]

where \( \hat{C} := \hat{c}(1) \). In general for bicategories, if the carrier of a comonoid has a left adjoint, then this left adjoint carries the structure of a monoid; this completes the proof.

\[\square\]

**Corollary 2.5.7.** Let \( c \in \text{Cat}^f \) be a category with objects \( C := c(1) \), and recall the comonoid structure on \( C_y \leftarrow c \xleftarrow{c} C_y \) and monoid structure on \( C_y \leftarrow c^\dagger \xleftarrow{c^\dagger} C_y \) from Proposition 2.5.6. The following structures are equivalent on a set \( X \in \text{Set} \):

1. \( X \) is the element-set of a functor \( X: c \to \text{Set} \),
2. \( X \) has the structure of a bicomodule of the form \( c \leftarrow X \xleftarrow{X} 0 \),
3. \( X \) has the structure of a bicomodule of the form \( C_y \leftarrow c \xleftarrow{c} C_y \) comodule structure,
4. \( X \) has the structure of a bicomodule of the form \( C_y \leftarrow X \xleftarrow{X} 0 \), equipped with a further \( C_y \leftarrow c^\dagger \xleftarrow{c^\dagger} C_y \) module structure,
5. \( X_y \) has the structure of a bicomodule of the form \( C_y \leftarrow c \xleftarrow{c} C_y \) comodule structure,
6. \( X_y \) has the structure of a bicomodule of the form \( C_y \leftarrow X_y \xleftarrow{X_y} y \), equipped with a further \( C_y \leftarrow c^\dagger \xleftarrow{c^\dagger} C_y \)-module structure.

We explain our terminology for \#3 above (\#4, 5, 6 are similar), in case it’s useful. When we say that \( C_y \leftarrow X \xleftarrow{X} 0 \) is equipped with a further \( C_y \leftarrow c \xleftarrow{c} C_y \) comodule structure, we mean that there is a map \( \zeta: X \to c \leftarrow c_{C_y} X \) making the diagrams below commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\zeta} & c \leftarrow c_{C_y} X \\
\downarrow & & \downarrow \epsilon_{C_y} \zeta \\
X & & X \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\zeta} & c \leftarrow c_{C_y} X \\
\downarrow \delta_{C_y} \zeta & & \downarrow \delta_{C_y} X \\
c \leftarrow c_{C_y} X & \xrightarrow{c \epsilon_{C_y} \zeta} & c \leftarrow c_{C_y} c \leftarrow c_{C_y} X
\end{array}
\]

**Proof.** The equivalence of 1 and 2 was given in Proposition 2.2.12. The equivalences of 2, 3, and 5 are straightforward calculations. The equivalence of 3 and 4 is just a property of adjunctions in a bicategory (in this case, \( \text{Cat}^f \)): a map \( \varphi: X \to c \leftarrow c_{C_y} X \) can be identified with a map \( \psi: c^\dagger \leftarrow c_{C_y} X \to X \), under which the comodule conditions on \( \varphi \) correspond to the module conditions on \( \psi \). The equivalence of 5 and 6 is similar. \[\square\]

### 2.6 Dirichlet product on comonoids and bicomodules

Our next goal is to define a local monoidal structures on the horizontal hom-categories \( c_{-}\text{Set}[d] \) for each \( c, d \in \text{Cat}^f \); we will do this in Section 2.7. It appears that the fastest
route to doing so is to define an external structure on \( \mathbf{Cat}^\# \), one that is a weakening of the notion of monoidal structure. Let’s get reacquainted with the situation for \( \mathbf{Poly} \) first.

Recall that in Corollary 2.2.10 we considered \((\mathbf{Poly}, y, \triangleleft)\) to be a bicategory \( \mathcal{P} \) with one object, and hence as a double category. But we note that the Dirichlet monoidal product \( \otimes \) on \( \mathbf{Poly} \) does not endow \( \mathcal{P} \) with a monoidal double category structure in the sense of [Shu08, Section 9]. Indeed, horizontal composition \( \triangleleft \) interacts duoidally with \( \otimes \) (Proposition 2.1.13), meaning that the functor

\[ \otimes : \mathbf{Poly} \times \mathbf{Poly} \to \mathbf{Poly} \]

is lax with respect to horizontal composition. Said another way, the interchange map

\[ \begin{array}{ccc}
\bullet & \otimes & \bullet \\
\downarrow & & \downarrow \\
q_1 \triangleleft q_2 & & q_1 \triangleleft q_2
\end{array} \]

is directed as shown, and not generally an isomorphism. The same holds for \( \mathbf{Cat}^\# \).

**Proposition 2.6.1.** Dirichlet product provides a normal colax double functor

\[ \otimes : \mathbf{Cat}^\# \times \mathbf{Cat}^\# \to \mathbf{Cat}^\#. \]

**Proof.** Given two polynomial comonads \( c, d \in \mathbf{Cat}^\# \), the Dirichlet product of their carriers is again a comonad, essentially by duoidality (Proposition 2.1.13):

\[ c \otimes d \to (c \triangleleft c) \otimes (d \triangleleft d) \to (c \otimes d) \triangleleft (c \otimes d). \]

and it is functorial with respect to comonoid morphisms (cofunctors).

Given two horizontal maps \( c_1 \leftarrow p \leftarrow c_2 \) and \( d_1 \leftarrow q \leftarrow d_2 \), we define their Dirichlet product to be

\[ c_1 \otimes d_1 \leftarrow p \otimes q \leftarrow c_2 \otimes d_2 \]

where the left and right coherence maps arise from duoidality as follows

\[ (c_1 \triangleleft p) \otimes (d_1 \triangleleft q) \xrightarrow{\lambda_1 \otimes \lambda_2} p \otimes q \xrightarrow{p_1 \otimes p_2} (p \triangleleft c_2) \otimes (q \triangleleft d_2). \]

The Dirichlet product of 2-cells is straightforward.

The \( \otimes \) unit bicomodule on \( c \) is \( c \), i.e. the composition unit; thus \( \otimes \) is normal. But just as in (43), the interchange law is only lax: i.e. there is a natural map

\[ (p_1 \otimes p_2) \otimes (q_1 \otimes q_2) \]

Indeed, this follows from duoidality and the fact that the bottom polynomial \( (p_1 \otimes q_1)_{(c_2 \otimes d_2)} (p_2 \otimes q_2) \) is an equalizer. We leave the remaining details to the reader. \( \square \)
**Proposition 2.6.2** (c ⊗ d is the usual categorical product). For any categories $c, d \in \mathbf{Cat}^a$, their Dirichlet product $c \otimes d$ agrees with the usual product $C \times D$ of the associated categories $C, D \in \mathbf{Cat}$. Similarly the $\otimes$-unit $y$ corresponds to the terminal object in $\mathbf{Cat}$.

**Proof.** The outfacing polynomial of the Dirichlet product is given by

$$c \otimes d = \sum_{(a,b) \in C \times D} y^{c[a] \otimes d[b]}.$$  

In other words an object $(a,b) \in (C \otimes D)(1)$ can be identified with an object $(a,b) \in \text{Ob}(C \times D)$, and an outfacing morphism $(f,g) \in (C \otimes D)[(a,b)]$ can be identified with an outfacing morphism $(f,g) \in (C \times D)[(a,b)]$. The fact that identities, codomains, and compositions work as expected follows from (44) and duoidality.  

We record the following for future use.

**Proposition 2.6.3.** For any category $c \in \mathbf{Cat}^a$ there are adjunctions

$$c \otimes c \cong \begin{array}{ccc} & y & c \\ \Delta & \otimes & \Pi \end{array}$$

Both the counit $\Delta_c \llcorner_{\otimes c} \Pi \cong c$ and the unit $y \cong \Pi_e \llcorner_{\otimes c} \Delta$ are isomorphisms. Moreover there are coherent associativity and unitality isomorphisms for $\Pi$ and $\Delta$:

$$(\Pi \otimes c) \llcorner_{\otimes c} \Pi \cong (c \otimes \Pi) \llcorner_{\otimes c} \Pi \text{ and } (c \otimes \Pi_c) \llcorner_{\otimes c} \Pi \cong c \cong (\Pi_c \otimes c) \llcorner_{\otimes c} \Pi$$

$$\Delta \llcorner_{\otimes c} \Delta_c \cong \Delta \llcorner_{\otimes c} \Delta \text{ and } (c \otimes \Delta) \llcorner_{\otimes c} \Delta_c \cong c \cong (\Delta \otimes c) \llcorner_{\otimes c} \Delta$$

**Proof.** For any category $C$ there is a fully faithful diagonal functor $\delta: C \rightarrow C \times C$ and an essentially surjective functor $e: C \rightarrow 1$. These satisfy associativity and unitality equations making $C$ a monoid in $(\mathbf{Cat}, 1, \times)$. The result now follows from Propositions 2.4.5 and 2.6.2.  

**2.7 Local monoidal closed structure**

Now that we understand how $\otimes$ works on comonoids and bimodules, and given the adjunctions from Proposition 2.6.3, we can define what we actually need for our story, namely the local monoidal closed structures. We begin with the local monoidal structure.

**Proposition 2.7.1** (Local symmetric monoidal structure). Given categories $c, d \in \mathbf{Cat}^a$ and bimodules $c \llcorner_{\otimes d} \llcorner_{\otimes d} d$, define their internal monoidal product $p_c \otimes d q$ as the composite

$$c \llcorner_{\otimes d} c \otimes c \llcorner_{\otimes d} d \otimes d \llcorner_{\otimes d} d.$$  

Similarly, define the internal monoidal unit $I_{c,d}$ by the composite $c \llcorner_{\otimes d} y \llcorner_{\otimes d} d$. The monoidal product and monoidal unit have carrier polynomials

$$p_{c \otimes d} q := \sum_{a \in C(1)} \sum_{(i,j) \in p_a(1) \times q_d(1)} y^{p[i] \otimes q[j] (1)} \quad \text{ and } \quad I_{c,d} := c(1)y^{(1)} \quad (45)$$  

respectively. Together these form a symmetric monoidal structure on $c\text{-}\mathbf{Set}[d]$.  

40
\textbf{Proof.} The fact that these definitions form a symmetric monoidal structure is immediate from the results of Section 2.6. One computes using (32) and Proposition 2.6.3 that the carriers of the monoidal product and unit are as in (45). \qed

Using (45), the bicomodule structure on \( p \circ d_q \) can be understood piece by piece as a duc-query, \( p \circ d_q : \text{d-set} \rightarrow \text{c-set} \). It assigns to each object \( a \in c(1) \) a sum of conjunctive queries \((p \circ d_q)[(i, j)]\), indexed by \((i, j) \in p_a(1) \times q_a(1)\). The \( \text{d-set} \) associated to the conjunctive query at \((i, j)\) is

\[(p \circ d_q)[(i, j)] \cong p[i] \times d(j) q[j],\]

i.e. the product of \( p[i] \) and \( q[j] \) taken as functors \( d \rightarrow \text{set} \). Indeed, the category of elements for that product has objects \( p[i] \times d(j) q[j] \).

\textbf{Proposition 2.7.2 (Duoidality).} The local \( \otimes \)-monoidal structures interact “duoidally” under composition. That is, given \( c \xrightarrow{p, p'} d \xrightarrow{q, q'} e \) there is a natural map

\[(p \otimes_d q) \circ_c (p' \otimes_d q') \rightarrow (p \circ_d p') \otimes_d (q \circ_c q'), \tag{46}\]

as well as natural maps of the form:

\[I_{c,e} \twoheadrightarrow I_{c,d} \otimes_d I_{d,e} \quad \text{and} \quad I_{c} \circ_c I_{c} \twoheadrightarrow I_{c} \quad \text{and} \quad I_{c,e} \rightarrow I_{c} \tag{47}\]

where \( I_{c,d} := c(1) y_{d}(1) \) is the \( c \otimes d \) unit and \( I_{c} := \sum_{a \in c(1)} y_{c[a]} \) is the \( c \) unit. These satisfy the usual axioms.\textsuperscript{21} \[\Box\]

\textbf{Proof.} This follows from Proposition 2.6.3. Indeed we define (46) using only the unit of the \( \Pi_{\otimes} \natural \Delta_{\otimes} \) adjunction:

Similarly we define the first two maps in (47) using the unit and counit of the same adjunction. The third map \( I_{c,e} \rightarrow I_{c} \) can be defined by the rest of the structure; see [nLa21]. We leave the verification of the usual axioms to the interested reader. \[\Box\]

\textsuperscript{21}By the usual axioms, we mean those expressing that \( \text{Cat}^{\#} \) is a pseudo-category object in the 2-category \( \text{MonCat} \) of monoidal categories and lax monoidal functors, just as a duoidal category is a pseudo-monoid object in \( \text{MonCat} \); see [nLa21].
To obtain the local closures \( c[-,-] \) for \( c \otimes d \), we do not rely on general theory, but instead simply compute what they must be.

**Proposition 2.7.3 (Local closure).** Given a set \( C \in \text{Set} \), a category \( d \in \text{Cat}^2 \), and bicomodules \( C_y \xrightarrow{q,r} d \), define the polynomial

\[
C_y[q,r]_d := \sum_{a \in C} \sum_{q \in \text{Set}[d]_{(q_a,r_a)}} y^{\sum_{j \in q_a(1)} r[j]}.
\]

(48)

It has a natural \((C_y,d)\)-bicomodule structure, and this operation is functorial. Moreover, for any \( C_y \xrightarrow{p} d \) there is a natural isomorphism

\[
C_y \text{-Set}[d](p, C_y[q,r]_d) \cong C_y \text{-Set}[d](p, C_y[q,r]_d)
\]

making \( C_y[-,-] \) a closure for \( - \otimes d \).

**Proof.** By Theorem 2.3.1 and Corollary 2.3.6 we can establish a \((C_y,d)\)-bicomodule structure on \( C_y[q,r]_d \) by establishing a \( d \)-set structure on the coproduct \( \sum_{a \in C} [r_a(1)] r_a(q) \) for each \( a \in C \) and map \( q : q_a \to r_a \). But doing so is easy: each \( r_a[k] \) is endowed with a \( d \)-set structure by the assumption that \( r \) is a \((C_y,d)\)-bicomodule, and thus the coproduct has an induced \( d \)-set structure as well. The functoriality (contravariant in \( q \), covariant in \( r \)) follows from the fact that everything in Eq. (48) is functorial in that way.

To prove the adjunction isomorphism (49), we recall that \( p[i] \times_{d(1)} q[j] \in d\text{-Set} \) denotes the cartesian product and calculate:

\[
C\text{-Set}[d](p, C_y \otimes_d q, r) = \prod_{a \in C} \text{Set}[d]\left( \sum_{(i,j) \in p_a(1) \times q_a(1)} y^{p[i] \times_{d(1)} q[j]}, \sum_{k \in r_a(1)} y^r[k] \right)
\]

\[
= \prod_{a \in C} \prod_{i \in p_a(1)} \prod_{j \in q_a(1)} \sum_{k \in r_a(1)} d\text{-Set}(r[k], p[i] \times_{d(1)} q[j])
\]

\[
= \prod_{a \in C} \prod_{i \in p_a(1)} \prod_{j \in q_a(1)} \prod_{k \in r_a(1)} d\text{-Set}(r[k], q[j]) \times d\text{-Set}(r[k], p[i])
\]

\[
= \prod_{a \in C} \prod_{i \in p_a(1)} \prod_{q \in \text{Set}[d]_{(q_a,r_a)}} \prod_{j \in q_a(1)} d\text{-Set}(r[q], p[i])
\]

\[
= \prod_{a \in C} \prod_{i \in p_a(1)} \prod_{q \in \text{Set}[d]_{(q_a,r_a)}} \sum_{j \in q_a(1)} d\text{-Set} \left( \sum_{j \in q_a(1)} r[q], p[i] \right)
\]

\[
= C\text{-Set}[d](p, C_y[q,r]_d).
\]

\( \square \)

Some readers may have wondered if the local closure in Proposition 2.7.3 extends from discrete categories \( C_y \) to arbitrary categories \( c \). It does, but the result will not be necessary for, nor show up again in, this paper.
2.7.4. The local closure (48) from Proposition 2.7.3 extends to an arbitrary (not necessarily discrete) category \( c \) in place of \( C \), by the formula

\[
c[q, r]_d := \sum_{a \in \mathcal{C}(1)} \sum_{\varphi \in \mathcal{C}(d)[c[a] \times \mathcal{C}(1), q]} \sum_{y \in \mathcal{C}(1)} r_y[y(id_a, j)]
\]

for \( c \rightsquigarrow q \leftarrow d \). Here, the representable \( c-\text{Set} \) denoted \( c[a] \) is serving as a constant prafunctor \( d-\text{Set} \to c-\text{Set} \). That is, \( c[a] \times \mathcal{C}(1) q \) evaluated at \( a' \in \mathcal{C}(1) \) is the coproduct of \( q_y \) indexed by the set \( c[a]_d' \) of maps \( a \to a' \) in \( c \).

Proving that the above formula gives is a closure for \( c \otimes_d \) as in Proposition 2.7.1 requires a better understanding of the interaction between ends and dependent types than we have found references for and hence feel comfortable to assume. But a key step is to show that for any \( P \in c-\text{Set} \), we have an isomorphism

\[
c-\text{Set}\left(P, \sum_{a \in \mathcal{C}(1)} c-\text{Set}(d)[c[a] \times \mathcal{C}(1), q, r]\right) \cong \sum_{\varphi_1 \in c-\text{Set}(P \times \mathcal{C}(1), q(1), r(1))} \int_{(a, i, j) \in \mathcal{E}(P \times \mathcal{C}(1), q(1))} d-\text{Set}(r[\varphi_1(i, j)], q[j])
\]

2.8 The sub framed bicategory \( \mathcal{S} \text{pan} = \mathcal{C} \text{at}_{\text{lin}}^\# \)

Recall that the framed bicategory \( \mathcal{S} \text{pan} \) of spans has sets \( C, D \) as objects, functions \( f: C \to C' \) as vertical morphisms, spans \( C \leftarrow S \to D \) as horizontal morphisms, and diagrams (left) as 2-cells (right)

\[
\begin{array}{ccc}
C & \leftarrow & S & \rightarrow & D \\
\downarrow & & \downarrow & & \downarrow \\
C' & \leftarrow & S' & \rightarrow & D'
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & D \\
\downarrow & & \downarrow \\
C' & \overset{\gamma'}{\leftarrow} & D'
\end{array}
\]

Recall that we refer to a polynomial \( p \) as linear if it is of the form \( p = Py \) for some set \( P \). We refer to a comonoid or bicomodule as linear if its carrier polynomial is linear. Let \( \mathcal{C} \text{at}_{\text{lin}}^\# \subseteq \mathcal{C} \text{at}^\# \) denote the sub framed bicategory consisting of linear polynomial comonads, all vertical maps between them, and linear bicomodules as horizontal maps.

Proposition 2.8.1 (\( \mathcal{S} \text{pan} \) as linears). There is an isomorphism of framed bicategories

\[
\mathcal{S} \text{pan} \cong \mathcal{C} \text{at}_{\text{lin}}^\#.
\]

Proof. The isomorphism identifies squares in \( \mathcal{S} \text{pan} \) as to the left with squares in \( \mathcal{C} \text{at}_{\text{lin}}^\# \) as to the right:

\[
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & D \\
\downarrow & & \downarrow \\
C' & \overset{\gamma'}{\leftarrow} & D'
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
C_y & \overset{\gamma_y}{\rightarrow} & D_y \\
\downarrow & & \downarrow \\
C'_y & \overset{\gamma'_y}{\leftarrow} & D'_y
\end{array}
\]

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Indeed, we can view spans as bridge diagrams (37) for which \( g = \text{id} \) and \( S := E = B \), and by Theorem 2.5.2 these correspond to linear polynomials since \( E[b] = 1 \) for each \( b \in B \). As for morphisms between them, the middle column of (38) reduces in this case to a map \( S = B \rightarrow B' = S' \) making the outer rectangles commute, which is in agreement with maps of spans. \( \square \)

**Corollary 2.8.2.** The framed bicategory \( \mathcal{S} \text{pan} \) can be regarded as the left adjoints in \( \mathbf{Cat}^\#_{\text{disc}} \).

**Proof.** This follows from Propositions 2.5.4 and 2.8.1. \( \square \)

Recall from Proposition 2.5.4 that the corresponding right adjoints are the conjunctive bicomodules \( C_y \triangleleft m \triangleleft D_y \), i.e. those for which \( m_a = 1 \) for all \( a \in C \). We temporarily denote the category of these by \( \mathbf{Cat}^\#_{\text{disc,con}}(C_y, D_y) \). Thus taking adjoints gives an equivalence of categories

\[
\mathbf{Cat}^\#_{\text{disc,con}}(C_y, D_y) \cong \mathbf{Cat}^\#_{\text{lin}}(D_y, C_y)^{\text{op}}. \tag{50}
\]

We will use the notation \( \mathbf{Cat}^\#_{\text{disc,con}} \) once more in Theorem 2.8.4 where we will obtain an equation like (50) but with \( \mathbf{Cat}^\#_{\text{lin}}(D_y, C_y)^{\text{op}} \) replaced by \( \mathbf{Cat}^\#_{\text{lin}}(C_y, D_y)^{\text{op}} \).

**Definition 2.8.3** (Dualizing object). Let \( C, D \in \mathbf{Set} \) be sets, and consider the terminal span \( C \leftarrow (C \times D) \rightarrow D \) as a linear bicomodule

\[
C_y \triangleleft C_{Dy} \triangleleft D_y.
\]

We denote it \( c \perp_D \in \mathbf{C-Set}[D] \), or simply by \( \perp \) if \( C, D \) are clear from context, and refer to it as the dualizing object. \( \diamond \)

In \( \mathbf{Poly} = \mathbf{y-Set}[y] \), the dualizing object is \( y \). Note that for any \( C, D, E \), we have \( (C \perp_D) \triangleleft_D (D \perp_E) \cong (C \perp_E) \). The following theorem generalizes Example 2.1.12.

**Theorem 2.8.4** (Linear-conjunctive duality). Let \( C, D \in \mathbf{Set} \) be sets, \( c := C_y \) and \( d := D_y \) the associated discrete categories, and \( \perp \) the dualizing object. The operation \( c[-, \perp]_d \) can be considered as a functor in two directions, as follows:

\[
\begin{align*}
\mathbf{Cat}^\#_{\text{disc,con}}(c, d) & \xleftarrow{\phantom{\circ}} \mathbf{Cat}^\#_{\text{lin}}(c, d)^{\text{op}} \\
& \xrightarrow{\phantom{\circ}} \mathbf{Cat}^\#_{\text{disc,con}}(c, d)
\end{align*}
\]

and these are mutually inverse equivalences of categories. We denote this operation \( m \mapsto m^\vee \).

**Proof.** First note that the operation \( c[-, \perp]_d \) is a functor \( \mathbf{Cat}^\#(c, d) \rightarrow \mathbf{Cat}^\#(c, d)^{\text{op}} \) by definition, so we need only show that it is an isomorphism when restricted to conjunctive and linear bicomodules.

Let \( m, n \in \mathbf{C-Set}[D] \), and suppose that \( m \) is linear and that \( n \) is conjunctive. Then by definition we have isomorphisms

\[
m \cong \sum_{a \in C} \sum_{i \in m_a(1)} y^{b(i)} \quad \text{and} \quad n \cong \sum_{a \in C} y^{n[a]}
\]
where \( b : m(1) \rightarrow D \) is a function and \( n[a] \in \text{D-Set} \) (i.e. there is a function \( n[a] \rightarrow D \)) for each \( a \in C \). Starting with \( m \), one checks using Eq. (48) that
\[
c_y[m, \text{CD}y]_{Dy} \cong \sum_{a \in C} \sum_{\varphi \in \text{Set}[D][m, Dy]} y^{\sum_i m_a(1)(\varphi(j))}. \tag{51}
\]
To see that this is conjunctive when \( m \) is linear, it suffices to show that \( \text{Set}[D](m, Dy) = 1 \). This is straightforward, either using Eq. (39) since \( Dy \in \text{D-Set} \) is terminal and each \( m[i] = 1 \), or using bridge diagrams Eq. (38) with \( C = C' = 1, E' = B' = D = D', E = B \) if one prefers.

Moving on to \( n \), one checks again using Eq. (48) that
\[
c_y[n, \text{CD}y]_{Dy} \cong \sum_{a \in C} \sum_{\varphi \in \text{Set}[D][\psi_n[a], Dy]} y^{\varphi_1(1)} \tag{52}
\]
which is clearly linear, as all the exponents have one element (namely \( \varphi_1 \) applied to the unique position \( ! \) in the polynomial \( y^{\psi_n[a]} \)).

Finally, to see that the two constructions are mutually inverse, notice that \( \{\varphi(j)\} \) is a one-element set for any \( j \in m_a(1), \) so the exponent of (51) is \( m_a(1) \). The second sum-index in (52) is \( \text{Set}[D](\psi_n[a], Dy) \) and it remains only to show that it is isomorphic to \( n[a] \). Again by (39) we calculate the following for any \( N \in \text{D-Set} \):
\[
\text{Set}[D](\psi_N, Dy) \cong \sum_{b \in D} \text{D-Set}({\{b\}}, N) \cong \sum_{b \in D} N_b \cong N. \quad \Box
\]

In other words, for linear and conjunctive bicomodules \( C_y \leftarrow_{m} \to D_y \), dualizing just moves the coefficients to exponents and vice versa,
\[
\left( \sum_{a \in C} M_a y \right)^{\vee} \cong \sum_{a \in C} y^{M_a} \quad \text{and} \quad \left( \sum_{a \in C} y^{M_a} \right)^{\vee} \cong \sum_{a \in C} M_a y
\]
for any span of sets \( C \leftarrow M \to D \).

**Remark 2.8.5** (Duals in terms of \( \Delta, \Sigma, \Pi \)). One can understand the duality from Theorem 2.8.4 in terms of the perhaps more familiar \( \Delta, \Sigma, \Pi \) operations. Namely, dualizing switches \( \Pi \circ \Delta \) to \( \Sigma \circ \Delta \) and vice versa. That is, given a span \( C \leftarrow M \to D \), the prafunctor corresponding to the bicomodule \( C_y \leftarrow_{M_y} \to D_y \) is \( \Sigma_f \circ \Delta_g : \text{d-Set} \to \text{c-Set} \). Its dual \( C_y \leftarrow_{(M_y)^{\vee}} \to D_y \) corresponds to \( \Pi_f \circ \Delta_g \). \( \Diamond \)

**Lemma 2.8.6.** For any span of the form \( A = A \to B \), the corresponding bicomodule \( A_y \leftarrow_{A_y} \to B_y \) is self-dual: \( (A_y)^{\vee} \cong A_y \).

**Proof.** The bicomodule \( A_y \leftarrow_{A_y} \to B_y \) is both conjunctive and linear, and so we can use either (51) or (52) to prove the result. \( \Box \)

As mentioned in the introduction, transposing a span—switching \( C \leftarrow M \to D \) to \( D \leftarrow M \to D \), which is usually considered as purely syntactic—in fact splits up into the composite of two more primitive operations, each with a universal property.
Corollary 2.8.7 (Transpose spans). Given a span of sets $C \leftarrow M \rightarrow D$ considered as a bicomodule $C^y \leftarrow M_y \rightarrow D_y$, the following are equivalent:

- its transpose $D^y \leftarrow (M^y)^\top \rightarrow C^y$,
- its right-adjoint’s dual $\left((M^y)^\top\right)^\vee$, and
- its dual’s left-adjoint $\left((M^y)^\vee\right)^\dagger$.

Proof. This follows from Proposition 2.5.4 and Theorem 2.8.4. □

Remark 2.8.8 (Transpose as adjoint dual, in terms of $\Delta, \Sigma, \Pi$). As in Remark 2.8.5, consider the span $C \leftarrow M \rightarrow D$ and corresponding bicomodule $C^y \leftarrow M_y \rightarrow D_y$, which corresponds to $\Sigma_f \circ \Delta_g : D\text{-Set} \rightarrow C\text{-Set}$. Its right adjoint $D^y \leftarrow (M^y)^\dagger \rightarrow C^y$ corresponds to $\Pi_g \circ \Delta_f$, and its right adjoint’s dual corresponds to $\Sigma_g \circ \Delta_f$. Alternatively, its dual $C^y \leftarrow (M^y)^\vee \rightarrow D^y$ corresponds to $\Pi_f \circ \Delta_g$, and hence its dual’s left adjoint is again $\Sigma_g \circ \Delta_f$. In both cases the result, $\Sigma_g \circ \Delta_f$ corresponds to the transpose of the original span. ∆

Corollary 2.8.9 (Opposite categories). For a category $c \in \text{Cat}^\#$, the span $c(1)_y \leftarrow c^\vee \rightarrow c(1)_y$

has a natural monoid structure, which corresponds to $c^{\text{op}}$ in the sense that a $c^{\vee}$-module $c(1)_y \leftarrow X \rightarrow 0$ can be identified with a functor $c^{\text{op}} \rightarrow \text{Set}$.

Moreover, the adjoint-dual and the dual-adjoint of $c$

$c(1)_y \leftarrow (c^\vee)^\dagger \rightarrow c(1)_y$ and $c(1)_y \leftarrow X \rightarrow c(1)_y$

are isomorphic, and each has a natural comonoid structure that also denotes the category $c^{\text{op}}$.

Proof. We know by Proposition 2.5.6 that the left adjoint $c(1)_y \leftarrow c^\dagger \rightarrow c(1)_y$ has a monoid structure. For any $a \in c(1)$, we can identify the set $(c^\dagger)_x(1)$ with the set of incoming morphisms $a \leftarrow a'$; see (42). We also have by Corollary 2.5.7 that a $c^\dagger$-module $c(1)_y \leftarrow X \rightarrow 0$ can be identified with a functor $c \rightarrow \text{Set}$.

Now by Corollary 2.8.7, the transpose of $c^\dagger$ is $(c^\dagger)^\top \cong (c^\dagger)^{\vee \dagger} \cong c^{\vee}$, because the dual of the right adjoint of the left adjoint of $c$ is just the dual of $c$. Since $c^\dagger$ has a monoid structure, it is well-known and easy to check directly that its transpose $c^{\vee}$ also has a monoid structure, and that it corresponds to the opposite of $c^{\dagger}$. This completes the first claim.

We have now shown that the dual $c^{\vee}$ of a comonoid is a span representing the opposite of $c$, so since $c^\dagger = c^{\vee \dagger} = c^{\dagger \vee}$ is a span representing $c$ itself (again by Proposition 2.5.6), we see that the comonoid $c^{\dagger \vee}$ represents the opposite of $c$. Similarly $c^{\vee} = (c^{\dagger \vee})^\dagger$ is a span representing the opposite of $c$, so $(c^{\dagger \vee})^\dagger$ is a comonoid representing the opposite of $c$. Thus they are isomorphic, completing the proof. □
**Proposition 2.8.10.** For any pair of spans $A_y \xleftarrow{M_y} B_y \xrightarrow{N_y} C_y$, there is an isomorphism of bicomodules

$$(M_y)^\vee \triangleleft_{B_y} (N_y)^\vee \cong (M_y \triangleleft_{B_y} N_y)^\vee.$$ 

**Proof.** By explicit calculation from Proposition 2.7.3 and Eq. (40), we have

$$(M_y)^\vee \triangleleft_{B_y} (N_y)^\vee \cong A_y[M_y, \perp]_{B_y} A_y[N_y, \perp]_{C_y} \cong A_y[M_y \triangleleft_{B_y} N_y, \perp]_{C_y} \cong (M_y \triangleleft_{B_y} N_y)^\vee.$$

**Remark 2.8.11.** Suppose that $A_y \xleftarrow{c} A_y$ is a comonoid and $A_y \xrightarrow{m} A_y$ is a monoid. Then it is easy and completely formal to check that there is an induced monoid structure on $A_y[c, m]_{A_y}$. Putting $C := c(1)$, this gives another way to understand the monoid structure on $c^\vee \cong c[y]_C$ from Corollary 2.8.9, since $\perp \cong C^2 y$ has a natural monoid structure (corresponding to the codiscrete category on $C$).

Similarly, there is a universal way of obtaining the comonad structure on $c^{\text{op}}$ in Corollary 2.8.9. Indeed, starting with the monoid structure on $M_y \cong c^+$, use Proposition 2.8.10 to produce a map $(M_y)^\vee \triangleleft_{C_y} (M_y)^\vee \cong (M_y \triangleleft_{C_y} M_y)^\vee$. Since the dual operation $\bullet \mapsto \bullet^\vee$ is contravariant on 2-cells, this implies that if $M_y$ is a monad then $(M_y)^\vee$ is a comonad. \hfill \Box

We end this whirlwind tour of $\text{Cat}^\text{bi}$ by recording two more results, which we will need for our main theorem. Recall the coclosure operation from Proposition 2.4.6.

**Lemma 2.8.12.** For any sets $A, B, C$ and bicomodules as shown left, there is a canonical map as shown right:

![Diagram](image_url)

**Proof.** Begin with the map $p \rightarrow [q] \triangleleft_{B_y} q$ coming from the universal property. Since $\bullet^\vee = [\bullet, \perp]$ is a contravariant functor, we obtain the second map below

$$[q, p] \triangleleft_{B_y} q^\vee \rightarrow [q, p] \triangleleft_{B_y} q^\vee \rightarrow p^\vee.$$ 

The composite is the desired map, so it remains to explain the first map. We obtain it from Propositions 2.7.2 and 2.7.3, duoidality and local closure:

$$\left(\left([q, p], \perp\right) \triangleleft_{B_y} [q, \perp]\right)_{A_y \otimes C_y} \left([q, p] \triangleleft_{B_y} q\right) \rightarrow \left(\left([q, p], \perp\right)_{A_y \otimes B_y} \left[q\right]\right)_{B_y} \triangleleft_{B_y} \left([q, \perp] \triangleleft_{B_y} C_y q\right) \rightarrow \perp \triangleleft_{B_y} \perp \cong \perp.$$

\hfill \Box
**Proposition 2.8.13** (Left closure of \( \text{Span} \)). Let \( A, B, C, X, Y \in \text{Set} \) and suppose given spans as shown left:

\[
\begin{array}{c}
A_y & \xleftarrow{W} & B_y & \xrightarrow{X} & C_y \\
\downarrow & & \downarrow & & \downarrow \\
A_y & \xleftarrow{Y} & C_y
\end{array}
\]

The category of spans \( A_y \xleftarrow{W} B_y \) and 2-cells \( W_y \xunderleftarrow{B_y} X_y \rightarrow Y_y \), as shown right, has a terminal object of the form

\[
A_y \xleftarrow{\left( X_y \right)^\vee} B_y.
\]

**Proof.** To see that \( w_0 := \left[ \left( X_y \right)^\vee \right]^\vee \) is linear (i.e. that it is a span), one checks (36): since \( Y_y \) is linear, its dual \( (Y_y)^\vee \) is conjunctive, so \( \left[ \left( X_y \right)^\vee \right]^\vee \) is too, and hence its dual is linear by Theorem 2.8.4. We obtain the map \( w_0 \xunderleftarrow{B_y} X_y \rightarrow Y_y \) from Lemma 2.8.12. It remains to show that \( w_0 \) is universal.

Suppose given a map \( W_y \xunderleftarrow{B_y} X_y \rightarrow Y_y \). Again by Theorem 2.8.4, it can be identified with a map \( (Y_y)^\vee \rightarrow (W_y \xunderleftarrow{B_y} X_y)^\vee \), and by Proposition 2.8.10 there is an isomorphism \( (W_y \xunderleftarrow{B_y} X_y)^\vee \cong (W_y)^\vee \xunderleftarrow{B_y} (X_y)^\vee \). Thus we have \( (Y_y)^\vee \rightarrow (W_y)^\vee \xunderleftarrow{B_y} (X_y)^\vee \) and, by the universal property of the coclosure, a unique map

\[
\left[ \left( X_y \right)^\vee \right]^\vee \rightarrow (W_y)^\vee
\]

making the requisite diagram commute. The desired map \( W_y \rightarrow w_0 \) is its dual. \( \square \)

### 3  Database aggregation

In this section we will use the formal structures we’ve developed in Section 2 to consider database aggregation. Before getting started, it may be helpful for the reader to return to Remark 1.2.2, where we listed nine ingredients in our proof of the main theorem. We have now completed the first six, and luckily the remaining three will be much easier.

#### 3.1  Schemas, instances, and aggregation

Our first step is to define database schemas and instances on them. The story we will tell is almost certainly just the beginning, in terms of the theory of aggregation. In particular, our notion of database schema in Definition 3.1.1 is not the most general nor the most common one, as we will explain in Remark 3.1.2. It was chosen for our purpose in this paper, which is simply to show that the most basic aspect of aggregation can be defined in terms of the universal structures (monoidal structures, adjoints, etc.) in \( \text{Cat}^\#$, and to leave the development of the theory for later work; indeed, the paper is already long enough.
**Definition 3.1.1** (Database schema, database instance). A database schema (or simply schema) consists of a pair \((C, M)\), where \(C\) is a small category and \(M: \text{Ob}(C) \to \text{CommMon}\) is a function assigning to each object \(a \in C\) a commutative monoid \((M_a, \otimes_a)\).

An instance on \((C, M)\) consists of a pair \((X, \alpha)\), where \(X: C \to \text{Set}\) is a copresheaf and \(\alpha_a: X_a \to M_a\) is a function for each object \(a \in \text{Ob}(C)\). We say that \(X\) is finitary if \(X_a \in \text{Fin}\) is a finite set for each \(a\).

A instance morphism \((X, \alpha) \to (X', \alpha')\) is a morphism \(X \to X'\) of copresheaves that commutes with the maps to \(M\).

In keeping with the rest of the paper, we prefer to work entirely within \(\text{Cat}^\#$\). Thus a database schema consists of \(c \in \text{Cat}^\#\) and \(c(1)_y \xleftarrow{M} 0\), and an instance on it consists of \(X, \zeta\), and \(\alpha\) in the diagram below:

\[
\begin{array}{c}
\xymatrix{
X \\
\downarrow^\zeta \\
\downarrow^c \\
\downarrow^a \\
X \\
\downarrow^M \\
0
}
\end{array}
\]

where \((X, \zeta)\) is a \(c\)-comodule as in Corollary 2.5.7. The above does not yet say that \(M_a\) has a commutative monoid structure for each \(a \in c(1)\). This will occur in Lemma 3.2.4.

**Remark 3.1.2.** Our notion of attributes here is a bit different than one finds in other category-theoretic work on databases. Again, our notion is intended just to get a formalization of aggregation off the ground. But it is worth saying a bit more.

In [JRW02]\(^{22}\), the authors take an attribute on some \(a \in C\) to just be a map \(a \to \Pi V_a 1\) to an \(V_a\)-indexed coproduct of \(1\)'s. This puts the attributes “directly into” the schema, whereas for implementations it often seems preferable to have each attribute point to an external data type in the programming language. Indeed, this is what’s done in [Sch+17]—which has been implemented by a coauthor of that paper, Ryan Wisnesky—where the attributes on each object \(a \in C\) are given by an algebra \(P_a\) of a fixed multi-sorted theory Type (the programming language), and the \(P_a\) vary functorially in \(a\).

The implementation in [PLF21] does not use this much structure for attributes; instead it defines them to be given by a profunctor \(P: C \to T\) where \(T\) is a discrete category equipped with a map \(V: T \to \text{Set}\). In other words, for each \(a \in C\) and \(t \in T\) we have a set \(P(a, t)\), and these vary functorially in \(a\).

Our definition is most closely aligned with the latter. But there are several differences. First, we only have one attribute for each object \(a\), so it’s as though we multiplied all the attributes together:\(^{23}\)

\[
M_a := \prod_{t \in T} \prod_{p \in P(a, t)} V(t).
\]

\(^{22}\)This article was prescient in the sense that EA (entity-attribute) sketches are coproduct-limit sketches, which seem quite related to prafunctors (indexed disjoint unions of conjunctive queries). One could investigate relationships between EA sketches and algebras for monads \(c \xleftarrow{m} c\).

\(^{23}\)This can actually be generalized a bit, as we will mention in Remark 3.3.2.
Second, we don’t ask our attributes to vary in \(a\), or to say it another way, they will vary freely: we just form the composite \(c(1)_y \leftrightarrow \cdot \cdot \cdot \leftrightarrow M \leftrightarrow 0\) and a copresheaf \(X\) with a map \(|X| \to M\) can be identified with a map of copresheaves \(X \to c \times c(1)_y M\).

Third, we ask something additional, not seen in any of the above, namely that each attribute set \(M_a\) be equipped with a commutative monoid structure; this is so that we can aggregate.\(^{24}\)

Our goal is to prove Theorem 3.3.1: that if \(c \leftrightarrow X \leftrightarrow 0\) is finitary, then for any morphism \(f: a \to a'\) in \(c[a]\), there is an induced aggregation map \(X_a \to M_a:\)

\[
\begin{array}{c}
X_a \xrightarrow{a} M_a \\
\downarrow X_f \\
X_{a'} \xleftarrow{(a)_{f}}
\end{array}
\]

and that these satisfy reasonable conditions under composition of \(f\)'s.

To do this, we will begin by framing both the finiteness of the \(X_a\) and the commutative monoid structures on the \(M_a\) in terms of the category \(\text{Fin}\) of finite sets.

### 3.2 Two roles for finite sets

The polynomial of lists, which in Eq. (19) we denoted

\[
u := \sum_{N \in \mathbb{N}} y^N
\]

with \(N := \{1', \ldots, N'\}\), will play an important role in what is to follow because every finite set is in bijection with some such \(N\). In particular, we will consider the dual bicomodules

\[
u(1)_y \leftrightarrow \cdot \cdot \cdot \leftrightarrow y \quad \text{and} \quad \nu(1)_y \leftrightarrow \nu(1')_y.
\]

Note that by Example 2.5.5, there is an isomorphism

\[
u(1)'_y \cong \sum_{N \in \mathbb{N}} N y \equiv \hat{u}(1)_y.
\]

This object will play an important role in the story below.

**Corollary 3.2.1.** There is a monad structure on the linear bicomodule (span)

\[
u(1)_y \xleftarrow{\begin{array}{c}u \end{array}} \nu(1)_y
\]

which, interpreted as a category, is a skeleton of \(\text{Fin}\).

\(^{24}\)To go from a set-attribute with values in \(V\) to a commutative monoid attribute, the free thing to do is replace \(V\) by \(M(V)\), the monoid of multisets in \(V\). Thus any of the other approaches can be replaced by one in our formulation that enables the “group-by” operation in database theory.
Proof. As shown in Example 2.2.8, we have an equivalence \( \left[ \begin{array}{c} u \\ \end{array} \right] \approx \text{Fin}^\text{op} \). The result now follows from Corollary 2.8.9. \( \square \)

There is a canonical functor \( \text{Fin} \to \text{Set} \), namely the inclusion. This copresheaf is represented by an algebra structure on \( u^\vee \):

\[
\begin{align*}
\left[ \begin{array}{c} u \\ \end{array} \right] & \leftarrow u(1)y \\
\uparrow & \\
\downarrow \varphi \\
\left[ \begin{array}{c} u^\vee \\ \end{array} \right] & \leftarrow y
\end{align*}
\]

(56)

The above map \( \varphi \), which has type

\[
\varphi: \prod_{N \in \text{c}(1)} \prod_{M \in \text{c}(1)} f: M \to N \prod_{m \in M} \text{N}
\]

is induced by Lemma 2.8.12; one can check that it is given by \( \varphi(N, M, f, m):= f(m) \).

Theorem 3.2.2 (Classifying object for finitary copresheaves). Let \( c \xleftarrow{Xy} y \) be a copresheaf (see Corollary 2.5.7). Then the following are equivalent:

1. \( Xy \) is finitary, i.e. for each \( i \in \text{c}(1) \) the set \( X_i \) is finite
2. there exists a function \( r^X \gamma^1: \text{c}(1) \to u(1) \) such that the following square is cartesian:

\[
\begin{align*}
c(1)y & \xleftarrow{Xy} y \\
r^X \gamma & \downarrow \text{cart} \\
u(1)y & \xleftarrow{u^\vee} y
\end{align*}
\]

(57)

3. there exists a monad map (2-cell) \( r^X \gamma^1: \text{c}^+ \to \left[ \begin{array}{c} u \\ \end{array} \right]^\vee \) as shown here:

\[
\begin{align*}
c(1)y & \xleftarrow{c^+} c(1)y \\
r^X \gamma & \downarrow \text{cart} \\
u(1)y & \xleftarrow{u^\vee} u(1)y
\end{align*}
\]

(58)

for which the \( c^+ \)-module induced by the \( \left[ \begin{array}{c} u \\ \end{array} \right]^\vee \)-module \( u(1)y \xleftarrow{u^\vee} y \) (see 54) is isomorphic to \( Xy \).

25The category \( \text{Fin} \) plays an important role in topos theory—its copresheaf category \( \text{Fun}(\text{Fin}, \text{Set}) \) is the object classifier—and the inclusion functor \( \text{Fin} \to \text{Set} \) is the “generic object”. We will not use these ideas here.

26The notation \( r^X \gamma^1 \) as shown does not make the vertical maps explicit. If we wanted to do so, Shulman suggests in [Shu08] that we annotate the arrow on top and bottom, namely as \( r^X \gamma^1: \text{c}^+ \xleftarrow{r^X \gamma} \left[ \begin{array}{c} u \\ \end{array} \right]^\vee \).
Before beginning the proof, note that the type of the 2-cell in (58) is

$$\xymatrix{\Gamma X_1 : \prod_{i \in (1)} \prod_{j \in (1)} \prod_{f : j \to i} \text{Set}(u[\Gamma X^\gamma(j)], u[\Gamma X^\gamma(i)])}$$

(59)

In other words, it somehow contains all the data of the copresheaf $X : c \to \text{Set}$.

**Proof.** We have that (3) $\Rightarrow$ (2) by definition: the $c^+$-module induced by $u^\vee$ in the sense of (3) has as its carrier the cartesian horizontal arrow from (57). We also have that (1) $\Leftrightarrow$ (2) because for each $N \in u(1)$, the set $(u^\vee)_{N} \equiv \{1', \ldots , N'\}$ is finite and every finite set is isomorphic to some $(u^\vee)_N$; see (55). As an aside, we can now reduce (59) to something more readable:

$$\xymatrix{\Gamma X_1 : \prod_{i \in (1)} \prod_{j \in (1)} \prod_{f : j \to i} \text{Set}(X_j, X_i).}$$

To show that (2) $\Rightarrow$ (3) we need to find such an element, compatibly with the monad structure on $c^+$. The idea is that to do so is exactly what it means for $c \xleftarrow{\gamma} y$ to be a (finite) copresheaf. But our goal is to prove this using universal properties in $\text{Cat}^\text{op}$.

By Corollary 2.5.7, we can regard the copresheaf $X$ as a module

$$\xymatrix{c^+ \ar@/^/[r]^c \ar@/_/[r]_y & c(1)y \ar@/^/[d] \ar@/_/[d] \ar[r] & y \ar@/^/[d] \ar@/_/[d] \ar[r] & Xy \ar@/^/[d] \ar@/_/[d] \ar[r] & y \ar@/^/[d] \ar@/_/[d] \ar[r] & y.}$$

and by (2), this module structure is equivalent to (or “induced by”, in the terminology of the theorem statement) a 2-cell as shown left:

$$\xymatrix{c(1)y & \ar[r]^-{\gamma} & c(1)y & \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y & \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y.}$$

But by Proposition 2.8.13 the 2-cell in the left-hand diagram factors uniquely as in the right-hand diagram, where the dashed map is $\begin{bmatrix} u & u \end{bmatrix}$ and the 2-cell $\varphi$ is from (56). We leave the verification of the coherence laws for this $c^+$-algebra structure to the reader. \hfill \Box

**Remark 3.2.3** (Classifying copresheaf maps). The following will be useful in future work, but we do not go further into it here; thus we leave it without proof.

Given a category $c$ and a morphism $\varphi \in c\text{-Set}(W, X)$ of finite copresheaves, there is an induced 2-cell

$$\xymatrix{c(1)y & \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y \ar[r]^-{\gamma} & c(1)y.}$$

(60)

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satisfying the following equation, where the unlabeled horizontals are $\left[ \begin{array}{c} u' \\ u \end{array} \right]$:

\[
\begin{array}{c}
c(1)y \\
r_X y \\
u(1)y \\
u(1)y
\end{array}
\quad \rightarrow
\begin{array}{c}
c(1)y \\
r_X y \\
u(1)y \\
u(1)y
\end{array}
\quad =
\begin{array}{c}
c(1)y \\
r_X y \\
u(1)y \\
u(1)y
\end{array}
\quad \rightarrow
\begin{array}{c}
c(1)y \\
r_X y \\
u(1)y \\
u(1)y
\end{array}
\]

and the 2-cells are as in (58), (60), and Corollary 3.2.1.

The other use of finiteness is its role in the behavior of commutative monoids. We denote the operation of the monoid in an unbiased way using $\oplus$ which can be applied to lists of arbitrary finite length. Note that it does not depend on the order of the list. We begin with a technical lemma that coerces commutative monoids into the Poly ecosystem.

**Lemma 3.2.4 (Commutative monoids in terms of Fin).** Let $A$ be a set. Given an assignment of a commutative monoid $(M_a, \otimes_a)$ to each $a \in A$, one obtains

1. a profunctor $\otimes : \text{Fin}^{\text{op}} \rightarrow A$ of the form $(n, a) \mapsto (M_a)^n$ and

2. a $u(1)y \leftarrow u(1)y$ module of the form $u(1)y \leftarrow u \mapsto y \rightarrow M_y \rightarrow A_y$.

In fact conditions (1) and (2) are equivalent. Moreover, a 2-cell of the form

\[
\begin{array}{c}
A_y \\
x \\
u(1)y \\
u(1)y
\end{array}
\quad \rightarrow
\begin{array}{c}
A_y \\
x \\
u(1)y \\
u(1)y
\end{array}
\]

(61)

can be identified with a natural number $X_a \in \mathbb{N}$ and a function $[X_a] \rightarrow M_{a}$, for each $a \in A$.

**Proof.** The set $M$ and and bicomodule $y \leftarrow M_y \rightarrow A_y$ in (2) are given by the coproduct $M := \sum_{a \in A} M_a$ and the projection $M \rightarrow A$. We may assume $A = 1$, since everything is just $A$-many copies of the $A = 1$-case.

Suppose given a monoid $(M, \otimes)$. We need a functor $\text{Fin} \rightarrow \text{Set}$ of the form $n \mapsto M^n$, that is, we need to give a function $M^I \rightarrow M^J$ for each function $f : I \rightarrow J$ between finite sets. This is the same as a function $M^J \rightarrow M$ for each $j \in J$, and we use $(m_1, \ldots, m_I) \mapsto \otimes(m_1, \ldots, m_I)$; this justifies the notation $\otimes : \text{Fin}^{\text{op}} \rightarrow \text{Set}$.

To prove the equivalence of (1) and (2), we need to understand the data of a module of the following form

\[
\begin{array}{c}
[\begin{array}{c} u' \\ u \end{array}]], u(1)y \leftarrow u \mapsto y \\
\otimes \\
u(1)y \\
u(1)y
\end{array}
\quad \rightarrow
\begin{array}{c}
[\begin{array}{c} u' \\ u \end{array}]], u(1)y \leftarrow u \mapsto y \\
\otimes \\
u(1)y \\
u(1)y
\end{array}
\]

(62)
we include the $A$ in this diagram simply to remind the reader where it lives; we will continue to assume $A = 1$. The map $\otimes'$ has type

$$\otimes' : \prod_{f \in u(1)} \prod_{l \in u(1)} \prod_{i \in I} \prod_{j \in J} \prod_{m \in M^I} M^J.$$ 

Thus, dropping [brackets], $\otimes'$ is the data of a map $M^I \to M^J$ for every function $I \to J$ between finite sets. Additionally asking $\otimes'$ to be a $[u_{-}]_\vee$-module is asking this data to be functorial in $\text{Fin}$. Thus we see that $\otimes'$ can be identified with $\otimes$ and that (1) $\iff$ (2).

To complete the proof, we just need to understand the diagram (61). The function $X$ picks out an element $X(a) \in u(1) = \mathbb{N}$ for each $a \in A$, and one reads off that the type of $\alpha$ is then

$$\alpha : \prod_{a \in A} \prod_{x \in u[X(a) \vee]} M_a$$

i.e. a function $X_a \to M_a$ for each $a \in A$, as desired. $\square$

### 3.3 Aggregation

In the theorem below we continue to denote the multi-ary commutative monoid operation by $\otimes$. In keeping with carrier notation 2.0.1, if $c \leftarrow X \to 0$ is a bicomodule, we use the same symbol to denote the restriction $c(1)_y \leftarrow X \to 0$, since they have the same underlying set of elements.

Recall from Theorem 3.2.2 that a finitary instance on $c$ can be identified with a map of monads $c^+ \to [u_{-}]_\vee^y$: indeed, the latter is just a functor from $c$ to (a skeleton) of $\text{Fin}$.

**Theorem 3.3.1 (Aggregation).** Let $(c, M)$ be a database schema and $(X, \alpha)$ a finitary instance on it. Then there is an induced map of modules (shown right) relative to the monad map $\gamma_X : c^+ \to [u_{-}]_\vee^y$ (shown left):

$$
\begin{array}{ccc}
c(1)_y \leftarrow c^+ \to c(1)_y & & c(1)_y \leftarrow c^+ \to c(1)_y \\
\gamma_X \downarrow & & \gamma_X \downarrow \\
u(1)_y \leftarrow [u_{-}]_{\gamma_X} \to u(1)_y & & \gamma_X \downarrow \quad \gamma_X \downarrow \\
\end{array}
$$

Proof. The desired map $\otimes \alpha$ is defined to be the composite,

$$
\begin{array}{ccc}
c(1)_y \leftarrow c^+ \to c(1)_y & & c(1)_y \leftarrow c^+ \to c(1)_y \\
\gamma_X \downarrow & & \gamma_X \downarrow \\
u(1)_y \leftarrow [u_{-}]_{\gamma_X} \to u(1)_y & & \gamma_X \downarrow \quad \gamma_X \downarrow \\
\end{array}
$$

$$
\begin{array}{ccc}
c(1)_y \leftarrow c^+ \to c(1)_y & & c(1)_y \leftarrow c^+ \to c(1)_y \\
\gamma_X \downarrow & & \gamma_X \downarrow \\
u(1)_y \leftarrow [u_{-}]_{\gamma_X} \to u(1)_y & & \gamma_X \downarrow \quad \gamma_X \downarrow \\
\end{array}
$$

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Here, the unlabeled horizontal is $[\begin{array}{c} u \\ u \end{array}]^\vee$, the 2-cell labeled $\r X \uparrow 1$ is the monad map from Theorem 3.2.2, the 2-cell labeled $\alpha$ can be identified with the instance $(X, \alpha)$ by Definition 3.1.1 and Lemma 3.2.4, and the 2-cell $\odot$ defines $M$ as a commutative monoid, as in (62).

It may be clear that this composite $\odot \alpha: c^+ \to u \prec M y$ is a module map relative to the map of monads $c^+ \to [\begin{array}{c} u \\ u \end{array}]^\vee$ since its domain $c^+$ is the free $c^+$-module on $c(1)y$. However we will spell it out for convenience in proving the final corollary. The argument is clearest when written in string diagram form (see [Mye16]).

Here is a picture of the key diagram (63), defining $\odot \alpha$:

\[
\begin{array}{c}
\r X \uparrow 1 \\
\end{array}
\]

where the identity $c(1)y$ is—as usual for bicategory string diagrams—not shown. To prove it is a map of modules, we consider its interaction with the left hand square in the following diagram, where the unlabeled vertical maps are $\r X \uparrow 1$, as before:

\[
\begin{array}{c}
c(1)y \leftarrow c^+ \leftarrow c(1)y \leftarrow c^+ \leftarrow c(1)y \leftarrow c(1)y \\
\downarrow \quad \r X \uparrow 1 \quad \downarrow \quad \r X \uparrow 1 \quad \downarrow \quad \r X \uparrow 1 \\
u(1)y \leftarrow [\begin{array}{c} u \\ u \end{array}]^\vee \leftarrow u(1)y \leftarrow [\begin{array}{c} u \\ u \end{array}]^\vee \leftarrow u(1)y \leftarrow u(1)y \leftarrow M y \leftarrow c(1)y
\end{array}
\]

Here is a string diagram representation of it:

\[
\begin{array}{c}
\r X \uparrow 1 \quad \r X \uparrow 1 \\
\end{array}
\]

Now we show the computation that the proper pieces commute. Indeed, leaving off

27The diagrams in [Mye16] arise from cells in the double category by first taking “Poincare duals” (objects become 2-cells, 1-cells get transposed, and 2-cells become 0-cells or “beads”), and then taking the transpose of the whole picture. Here we do almost the same thing: we do the first part but not the “transpose of the whole picture” part. Note that identity 1-cells can be left out, just like in string diagrams for monoidal categories. This is why $c(1)y$ is missing from the righthand side of (64).
intermediate labels, Theorem 3.2.2 implies the following equations:

\[
\begin{align*}
\tau X^a & = \tau X^a \quad \text{implies the following equations:} \\
\ne & (66)
\end{align*}
\]

The argument that \( \otimes \alpha \) preserve identities is similar, though easier. Thus \( \otimes \alpha \) from (63) is a map of modules, completing the proof. \( \square \)

**Remark 3.3.2.** There is an easy way to generalize Theorem 3.3.1 at almost no conceptual cost. It starts by generalizing the assignment of a function \( \text{Ob}(C) \rightarrow \text{CommMon} \) in the definition of schema (Definition 3.1.1) and then saying what an instance on such a schema would be.

For the generalized schema, one can first choose a set of commutative monoids, say a set \( T \) and a function \( M : T \rightarrow \text{CommMon} \), and a choice of span \( c(1) \leftarrow P \rightarrow T \). This equips each object \( a \in c(1) \) with some set of the monoids from \( T \) rather than just one. Then to specify an instance on the schema, one specifies a copresheaf \( \otimes \alpha : X \rightarrow \text{Set} \), for which each component \( X_a \) is assigned a function \( \alpha_a : X_a \rightarrow \prod_{p \in P,} M_{|p|} \) to all the associated monoids. This data is contained in a square

\[
\begin{array}{c}
\begin{array}{c}
\tau X^a \quad \text{c(1)y} \\
\tau X^a \quad \text{u(1)y}
\end{array} \\
\begin{array}{c}
P_y \\
\theta \alpha
\end{array}
\end{array}
\]

The generalized aggregation theorem says that for any morphism \( b \leftarrow a \) we get a map \( \otimes \alpha : X_b \rightarrow \prod_{p \in P,} M_{|p|} \). This is proved in exactly the same way as Theorem 3.3.1 is, simply replacing the \( c(1)y \) with \( T y \) in the right-hand column of every diagram. \( \diamond \)

**Corollary 3.3.3.** Let \( (c, M) \) be a database schema and \( (X, \alpha) \) a finitary instance on it. Then there is a map \( \otimes \alpha : X \rightarrow \Pi_{C} M \) such that the following commutes:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\tau X^a \\
\tau X^a
\end{array} \\
\begin{array}{c}
\text{c(1)y} \\
\text{u(1)y}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
P_y \\
\theta \alpha
\end{array}
\end{array}
\end{array}
\]

The notation \( \Pi_{C} M \) was defined in (6).

Proof. This follows directly from Theorem 3.3.1, in particular the axioms of the module structure shown in (66).

The first diagram in Eq. (66) says that for any morphisms \( k \leftarrow j \leftarrow i \) in \( c \), one first obtains functions \( X_k \leftarrow X_j \leftarrow X_i \rightarrow M_j \) Then one aggregates to obtain \( X_k \leftarrow X_j \rightarrow M_i \)

\[ \]

\[ \text{The notation } \Pi_{C} M \text{ was defined in (6).} \]

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and aggregates again to obtain $X_k \to M_i$. The second diagram in Eq. (66) says that it is the same to first compose the functions to obtain $X_k \leftarrow X_i \to M_i$ and then aggregate to obtain $X_k \to M_i$. And the third diagram in Eq. (66) says that it is again the same to first compose the morphisms $k \leftarrow i$, then obtain the functions $X_k \leftarrow X_i \to M_i$, and then aggregate to obtain $X_k \to M_i$.

The same holds for identities, and together these constitute what was to be proven. □

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