Log-Infinitely Divisible Multifractal Processes

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Received: 8 July 2002 / Accepted: 17 December 2002
Published online: 14 April 2003 – © Springer-Verlag 2003

Abstract: We define a large class of multifractal random measures and processes with arbitrary log-infinitely divisible exact or asymptotic scaling law. These processes generalize within a unified framework both the recently defined log-normal Multifractal Random Walk processes (MRW) [33, 3] and the log-Poisson “product of cylindrical pulses” [7]. Their construction involves some “continuous stochastic multiplication” [36] from coarse to fine scales. They are obtained as limit processes when the finest scale goes to zero. We prove the existence of these limits and we study their main statistical properties including non-degeneracy, convergence of the moments and multifractal scaling.

1. Introduction

Fractal objects and the related concept of scale-invariance, are now generally used in many fields of natural, information or social sciences. They have been involved in a large amount of empirical, as well as theoretical studies concerning a wide variety of problems. The scale-invariance property of a stochastic process is usually quantified by the scaling exponents \(\zeta_q\) associated with the power-law behavior of the order \(q\) moments of the “fluctuations” at different scales. More precisely, for a 1D random process\(^1\) \(X(t)\), let us consider the order \(q\) absolute moment of the “fluctuation” \(\delta_l X(t)\) at scale \(l\):

\[
m(q, l) = \mathbb{E} \left( |\delta_l X(t)|^q \right),
\]

where the “fluctuation” process \(\delta_l X(t)\) is assumed to be stationary and \(\mathbb{E} (\cdot)\) stands for the mathematical expectation. Usually, the fluctuation \(\delta_l X(t)\) is chosen to be the increment of \(X(t)\) at time \(t\) and scale \(l\):

\[
\delta_l X(t) = X(t + l) - X(t),
\]

\(^1\) We will exclusively consider, in this paper, real valued random functions of a 1D continuous “time” variable \(t\). Though the extension to higher dimensions is rather natural, this problem will be addressed in a forthcoming study.
but it can be also defined as a wavelet coefficient [31, 2, 32]. The $\zeta_q$ exponents are defined from the power-law scaling

$$m(q, l) = K_q l^{\zeta_q}, \forall l \leq T. \quad (2)$$

When the $\zeta_q$ function is linear, i.e., $\zeta_q = qH$, the process is referred to as a monofractal process with Hurst exponent $H$. In that case the scaling can extend over an unbounded range of scales (one can have $T = +\infty$). Examples of monofractal processes are the so-called self-similar processes like (fractional) Brownian motion or $\alpha$-stable motion [39].

When the function $\zeta_q$ is non-linear, it is necessarily a concave function and $T < +\infty$. In that case the process is called a multifractal process. Let us remark that this definition of multifractality relies upon the scaling properties of increment absolute moments. An alternative definition refers to the point-wise fluctuations of the regularity properties of sample paths (see e.g. [7, 20]). Sometimes, one can establish an exact equivalence between these two definitions within the so-called multifractal formalism.

Let us note that the scaling equation (2) refers to an exact continuous scale invariance. Weaker forms of scale invariance are often used, notably asymptotic scale invariance that assumes that the scaling holds only in the limit $l \to 0^+$:

$$m(q, l) \sim K_q l^{\zeta_q}, \text{ when } l \to 0^+. \quad (3)$$

The discrete scale invariance only assumes that the scaling holds for a discrete subset of scales $l_n$ (with $l_n \to 0$ when $n \to +\infty$):

$$m(q, l_n) = K_q l_n^{\zeta_q}. \quad (4)$$

The paradigm of multifractal processes that satisfy discrete scale invariance are the Mandelbrot multiplicative cascades [25] or the recently introduced wavelet cascades [1]. In Mandelbrot construction (the principle is the same for wavelet cascades), $l_n = 2^{-n}$ and a sequence of probability measures $M_{ln}(dt)$ is built recursively. $M_{ln}(dt)$ is uniform on dyadic intervals $\mathcal{I}_{n,k} = [k2^{-n}, (k + 1)2^{-n}]$ and is obtained from $M_{ln-1}(dt)$ using the cascading rule:

$$M_{ln}(dt) = W_{n,k} M_{ln-1}(dt), \text{ for } t \in \mathcal{I}_{n,k}, \quad (5)$$

where the weights $W_{n,k}$ are i.i.d. positive random variables such that $\mathbb{E} (W_{n,k}) = 1$. The convergence and regularity properties of such construction have been studied extensively [21, 11, 8, 16, 17, 30, 5, 6] (from a general point of view, convergence of multiplicative constructions to singular measures have been studied in the Gaussian case in [22] and in the Lévy stable case in [13]). Despite the fact that multiplicative discrete cascades have been widely used as reference models in many applications, they possess many drawbacks related to their discrete scale invariance, mainly they involve a particular scale ratio (e.g. $\lambda = 2$) and they do not possess stationary fluctuations (this comes from the fact that they are constructed on a dyadic tree structure).

The purpose of this paper is to define a new class of continuous time stochastic processes with stationary fluctuations and that are multifractal in the sense that they verify exact or asymptotic continuous scaling (Eqs. (2) or (3)) with a non-linear $\zeta_q$ spectrum. Though, as pointed out in [27, 26], continuous time multifractal processes with continuous scale invariance are obviously appealing from both fundamental and modeling aspects, until very recently such processes were lacking. From our knowledge, only the recent works by Bacry et al. [33, 3] and Barral and Mandelbrot [7] refer to a precise