The Quantisation of Charges

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The question as to whether the integrality of the spectrum of observed electric charges is due to a quantum effect has fascinated theoretical physicists throughout the last century. It leads to unanswered questions at the heart of quantum field and its role as a framework for particle physics.

As you have heard, it is one hundred years ago since Planck introduced his famous constant when he proposed that the energy of components of black body radiation of angular frequency \( \omega \) occurred in discrete quanta:

\[
E = \hbar \omega,
\]

(1)

in modern notation. This revolutionary proposal resolved problems with the intensity spectrum of the radiation, and led eventually to the development of quantum mechanics, a theory in which the constant \( \hbar \) plays a pivotal role. According to this new theory, because the electromagnetic field responsible for the black body radiation is dynamical, it ought to be subjected to the principles of quantisation. These provide a well-defined technical procedure and yield the result that this field had quantum excitations that should be identified as particles, in this case the photon [1]. Then it follows that the energy of a photon associated with an electromagnetic wave of frequency \( \omega \) is indeed given by the above expression (1).

This correspondence between particles and fields has been very much extended, for example, to the electron and its wave function, regarded as a field. Indeed similar results apply to all the known particles discovered in high energy particle accelerators and in cosmic rays. Basic theory in modern day particle physics theory exploits this formalism.

There is another more obvious pattern observed first in atomic physics and applicable to all particle physics. Atoms (unless they are ionised) are electrically neutral despite the fact that a number of electrons orbit around their nuclear core. It follows that the atomic nucleus must carry an electric charge that exactly cancels that of the electrons and hence equals a negative integer times the electron charge. In fact the nucleus is composed of neutrons which are chargeless and protons carrying an electric charge equal and opposite to that of the electron. Indeed all observed particles, such as the muon, the pions, and so on, carry an electric charge that is an integer multiple of the electron charge

\[
q = nq_0, \quad n = 0, \pm 1, \pm 2, \ldots
\]

(2)

This integrality property of electric charges in appropriate units is one of the most striking features of particle physics. Planck’s constant is missing from the relation (2) yet it looks so much like a quantum phenomenon that it is natural to wonder whether there
is some way to deduce it from the principles of quantum mechanics, just as (1) has been. This question leads to an exciting story that I want to tell you about. It is a story still unfinished, one that probes to the very heart of quantum field theory and its concepts, leading to the conclusion that there is still much to learn about the quantum theory, particularly when it is applied to the most fundamental questions, namely of finding a relativistic description of unified particle interactions.

With the electron charge \( q_0 \) it is useful to construct a dimensionless number that is important in quantum electrodynamics. It is called the “fine structure constant”:

\[
\alpha_{\text{structure}} = \frac{q_0^2}{2\pi\hbar c} \sim \frac{1}{137},
\]

Its experimental value is small as indicated. In the early days of quantum theory it was hoped that this number could be determined. Many crazy but unsuccessful ideas were advanced to this end, for example by Eddington, and a more recent group theoretic idea was described at this meeting. But one of these seemingly crazy attempts has survived, remaining tantalising, influential and pervasive ever since. This is the proposal that Dirac made in 1931 [2].

He considered the possibility that somewhere in nature there existed a magnetic monopole, a new sort of particle carrying a magnetic charge, \( g \), and not yet seen. It was known previously that there was no problem in modifying Maxwell’s equations to accommodate this new sort of charge. However there could be a problem with the newly discovered quantum theory and indeed Dirac found that, in general, there was. This difficulty could be circumvented only if the following relation held for any pair of particles with electric charge \( q \) and magnetic charge \( g \), respectively:

\[
qg = 2\pi\hbar cn, \quad n = 0, \pm 1, \pm 2, \ldots
\]

The argument leading to this “Dirac quantisation condition” has been steadily refined and it is clear that it depends upon very little, just the idea that the electrically charged particles possess quantum wave functions that can be pieced together in a consistent way, making careful use of the gauge principle. No particular equations of motion are needed nor other details.

Let \( q_0 \) be the observed electric charge of smallest magnitude (and hence the electron charge) and \( g_0 \) the possible magnetic charge of smallest non-vanishing magnitude. Then it is reasonable that these satisfy (4) with \( n = 1 \) (though \( n \) could have a higher value):

\[
q_0g_0 = 2\pi\hbar c
\]

and that

\[
q = nq_0, \quad n = 0, \pm 1, \pm 2, \ldots \quad (6a)
\]

\[
g = mg_0, \quad m = 0, \pm 1, \pm 2, \ldots \quad (6b)
\]

Thus the integrality pattern of electric charge does follow from a quantum principle so it is now seen to be perfectly legitimate to talk of charge quantisation. But this is at a
price, namely that there has to be a magnetic charge somewhere in the universe, not yet observed. Nevertheless Dirac found this result gratifying and it is still true today, after seventy years development in quantum field theory, that this simple argument remains the best explanation of the integrality relations (2) for electric charge.

Dirac’s original intention was to determine \( \alpha \), (3), numerically. Instead he found the relation following from (5):

\[
\left( \frac{q_0^2}{2\pi \hbar c} \right) \left( \frac{g_0^2}{2\pi \hbar c} \right) = \left( \frac{g_0 g_0}{2\pi \hbar c} \right)^2 = 1.
\]

(7)

So, according to (3), the magnetic “fine structure constant”, \( g_0^2/2\pi \hbar c \), is approximately 137, significantly larger than unity and hence what is called “strong”, in contrast with (3), the electric fine structure constant, which, being much less than unity, is said to be weak.

Until recently the main computational tool in quantum field theory was what is known as perturbation theory and it relied on the dimensionless coupling, such as the electric fine structure constant, (3), being smaller than unity. The situation just described, with electric and magnetic charges, is remarkably symmetrical between the two, despite the disparity in their dimensionless magnitudes. Maybe a quantum field theory could be constructed in which this symmetry between the two roles is maintained. If so, perturbative calculations could be performed exploiting the smallness of the electric coupling, (3) and might yield information about the magnetic features despite their strong coupling. This is a very tantalising prospect and, surprisingly, it seems to be true, at least in suitable cases.

The onus is of course to find such a quantum field theory and it must incorporate much additional structural information about the charged particles such as the ones we have mentioned. In particular their masses and internal structure will have to come into play.

In the last forty years or so the subject of quantum field theory has burgeoned under the stimulation of new ideas (confirmed in part by experimental data) for describing all the elementary particle interactions in a unified way. The simplest concept of unification is to extend Maxwell’s equations to a more complicated, non-linear system associated with more charge operators as well as the electromagnetic one, denoted \( Q \). These charges form a closed algebra under commutation, called a Lie algebra and can be exponentiated to yield a Lie group, called the gauge or Yang-Mills group [3]. The nonlinearity of the modified Maxwell equations is reflected in the fact that this group is non-abelian, namely that the order of multiplication within the group matters.

The simplest new possibility is to have just three charges in all, \( T_1, T_2 \) and \( T_3 \) say. Then the only possible non-abelian group consists of rotations in a three-dimensional space. This has to be an internal space rather than the real space in which we live. But, as in the latter case, the three charges must satisfy the angular momentum commutation relations

\[
[T_1, T_2] = iT_3 \quad \text{etc.}
\]

(8)

The charge operator \( Q \) must be one of these, or, more precisely, must be a linear combination of them. Thus it has to be an angular momentum operator about some direction in the internal three-space. It could, for example be proportional to \( T_3 \), except that it would
not be satisfactory to select a particular direction by such a fiat as the choice would not be symmetrical in the sense of the gauge symmetry which is the new guiding principle.

To accommodate this principle it is necessary to incorporate three new scalar fields \((\phi_1, \phi_2, \phi_3)\) that form a vector in the three dimensional space thereby creating a signpost that designates the electromagnetic direction (at each of the points of space-time for which they are defined).

Then the candidate electric charge operator is proportional to the charge selected thus:

\[
Q \sim \phi T.
\]

But since the charges are to be non-zero, this is not allowed to vanish, even in the vacuum. This means that the fields \(\phi\) cannot vanish there, unlike normal fields. Consequently they are what is known as Higgs fields [4], more realistic examples of which are being sought with such urgency at CERN.

So the quantum field theory under consideration has now become what is called a spontaneously broken gauge theory comprising both gauge and Higgs fields. The Higgs fields will perform their original dedicated role and contribute mass for the gauge particles. The photon will remain massless, as is desirable, while the other two, called \(W^\pm\), antiparticles of each other, will have a specific mass (as is also desirable) and function as intermediate vector bosons for an (oversimplified) toy model of weak or radioactive interactions. This model is known as the Georgi-Glashow model [5].

The three charge operators \(T\) can be thought of as matrices and this therefore gives a matrix for \(Q\), by (9). Its eigenvalues are proportional to the electric charges carried by the various corresponding particles of the theory and are automatically quantised, by the quantum theory of angular momentum applied to the commutation relations (8). Thus charge quantisation is apparently achieved painlessly without a magnetic monopole in sight.

Now comes a surprise due to two features of the theory that have already been mentioned, first, that the equations of motion are highly non-linear and, secondly, that the Higgs field, \(\phi\), is able to “swivel around” from point to point of space-time whilst remaining in its ground state or vacuum, and maintaining eigenvalues or physical electric charges that do not vary from point to point. The result is that the equations possess classical solutions that can be regarded as “solitons”, so that the energy density remains localised in regions of space. Indeed these are analogues of the solitons of sine-Gordon theory familiar in space-times of two dimensions. Those simple solitons can be demonstrated with a toy made of pins and rubber as one of the speakers showed, whereas the present ones can only be demonstrated by explicit analytic solutions that make clear that there are no sorts of singularity. What happens is that in order for a classical solution to have finite energy, the scalar Higgs field evaluated at large distances in space, that is on the two-sphere at infinity, must take values in its ground state or vacuum which itself forms a two-sphere in the three dimensional internal space. Thus the asymptotic Higgs field provides a map between a pair of two-spheres and these maps are characterised by an integer, called a degree, that furnishes a higher dimensional generalisation of the winding number that classifies maps between two one-spheres, or circles. This degree cannot be changed by the supply of any finite amount of energy and so provides a sort of topological quantum number for
the soliton configurations. Something very similar happens for the sine-Gordon solitons. The single soliton (or anti-soliton) corresponds to the smallest value of this degree, \( \pm 1 \), when one sphere is transported bodily onto the other with or without reflection, with the asymptotic Higgs’ field swivelling appropriately. The soliton stability is guaranteed by the conservation of the topological quantum number.

This topological number ought to have a more direct physical interpretation and indeed it does, for it is proportional to the magnetic charge defined as a flux out of the asymptotic sphere of the magnetic components of the Maxwell field in the unified theory. These are the famous results that ’t Hooft and Polyakov found independently in 1974 [6].

This magnetic charge automatically satisfies the Dirac quantisation condition (4), even though quantum theory has not been explicitly invoked. Furthermore the associated monopole particle, and its antiparticle, \( M^{\pm} \), are endowed with a mass whose expression tantalisingly resembles that possessed by the \( W^{\pm} \) gauge particles, by virtue of the Higgs mechanism mentioned earlier.

The first conclusion is that Dirac’s explanation of charge quantisation is triumphantly vindicated. At first sight it seemed as if the idea of unification provided an alternative explanation, avoiding monopoles, but this was illusory as magnetic monopoles were indeed lurking hidden in the theory, disguised as solitons.

This raises an important conceptual point. The magnetic monopole here has been treated as bona fide particle even though it arose as a soliton, namely as a solution to the classical equations of motion. It therefore appears to have a different status from the “Planckian particles” considered hitherto and discussed at the beginning of the lecture. These arose as quantum excitations of the original fields of the initial formulation of the theory, products of the quantisation procedures applied to these dynamical variables (fields).

The point is that once quantum theory is fully applied, as it must be, the solitons too will appear as particles in the only real sense that concept has, namely that enunciated by Wigner in 1939, technically as irreducible representations of the Poincaré group of space-time transformations [7]. The particles appearing by virtue of the two mechanisms, quantum excitation and soliton, must therefore have an entirely equivalent status. The apparent difference is just an artefact of the way the quantum field theory has been formulated. This point was first made clear by Skyrme in his discussion forty years ago of the solitons of the sine-Gordon model in space-times of two dimensions [8]. He found that there do exist field operators of which the solitons are quantum excitations. They are related to the original sine-Gordon field by what is now recognised as a vertex operator construction. Furthermore, as Mandelstam showed [9], they satisfy the equations of motion of what is called the massive Thirring theory. As a result this theory is quantum equivalent to the sine-Gordon theory.

Two questions now arise naturally, (1) the construction of the field operators “creating” the magnetic monopole solitons \( M^{\pm} \), and (2) the identification of the equations of motion that these fields satisfy. In space-times of the interesting dimension, 4, that is under consideration, unlike the easier dimension, 2, the first question is still far too difficult to answer. But it is possible to hazard a guess for the answer to the second question and then check some of the predicted consequences, thereby accumulating circumstantial
The toy unified quantum field theory model so far developed displays a surprising degree of symmetry between the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \), the electric and magnetic charges \( q_0 \) and \( g_0 \) and the corresponding charged particles, \( W^\pm \), the heavy gauge particles, and \( M^\pm \), the heavy monopoles. The way the masses of these particles depend on their charges is surprisingly similar as are other common properties not described here.

This suggests that \( M^\pm \) are very similar to \( W^\pm \) and hence likewise a pair of heavy gauge particles in a spontaneously broken \( SO(3) \) gauge theory, exactly like the original one but with a strong magnetic coupling replacing the weak electric coupling and related by (7). In this alternative description \( W^\pm \) would now occur as soliton solutions. This was the conjecture made by Claus Montonen and myself in 1977 [10] and subsequent developments seem to have vindicated the idea. All the evidence is confirmatory, at least once a suitable degree of supersymmetry is included [11]. Supersymmetry, a symmetry mixing bosons and fermions, is easily achieved by adding a few extra fields and particles without changing anything so far described. The benefit is that there are minimal quantum corrections to the formulae mentioned and that there is a natural mechanism for the monopole solitons to acquire the spin quanta carried by gauge particles.

Actually the evidence points to an even richer structure than this. There are extra soliton states called dyons, with the same magnetic charges as \( M^\pm \) but also carrying electric charges. There are even dyon bound states of higher magnetic charge arising by a totally new mechanism identified by Ashoke Sen [12].

Before explaining the ramifications of this it is important to mention that the toy theory being developed possesses a second dimensionless parameter, called \( \theta \) because it is angular in nature [13], that was overlooked in early work and appeared innocuous anyway. This can be combined with the fine structure constant (3) to form a natural complex, dimensionless variable:

\[
\tau = \frac{\theta}{2\pi} + \frac{2\pi i \hbar}{q_0^2},
\]

so that the imaginary part is simply the inverse of the fine structure constant (3) and, hence, intrinsically positive.

The consequence of the extra richness of the particle spectrum of the quantum field theory just mentioned is that it appears, as Sen suggested, that there are really an infinite number of quantum equivalent reformulations of the theory, not just the two already mentioned. Each of these formulations is distinguished by the assignment of field operators to one particular dyon/antidyon pair (or \( W^\pm \) pair). The equations of motion satisfied are always those of the supersymmetric spontaneously broken \( SO(3) \) gauge theory but the couplings \( \tau \) differ. The relation between two choices always takes the form

\[
\tau \rightarrow \tau' = \frac{A\tau + B}{C\tau + D}, \quad AD - BC = 1,
\]

where the coefficients \( A, B, C \) and \( D \) are integers.

The transformations (11) form an infinite discrete group, called by mathematicians \( PSL(2, \mathbb{Z}) \), or simply, the modular group. It is an easy and instructive exercise to check
that these transformations always preserve the positivity of the fine structure constant, as they should.

That these transformations are quantum is made plain by the appearance of Planck’s constant in (10). A particularly dramatic way of stating the result is to say that the one quantum field theory has an infinite number of classical limits that are classically inequivalent. Although in each of these classical limits $\hbar$ always tends to zero, there is, of course, no paradox since what is held constant during the limiting process varies from case to case.

It is helpful to illustrate by a particular choice of the transformation (11), called $S$, given by

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
$$

By (11), this transformation, $S$, sends $\tau$ to $-1/\tau$. If $\theta$ vanishes this means that we have recovered equation (7) (with $g_0$ replaced by $q_0'$). This was the original transformation between weak and strong coupling.

An interesting historical aside is that a transformation like this was discovered as early as 1941, by Kramers and Wannier [14], in what appeared to be a quite different physical context. They showed that there was a symmetry of the partition function of the Ising model in space of two dimensions with respect to interchange of high and low temperatures. They argued that the fixed point of this transformation had to be the critical temperature at which a phase transition occurred. This was one of the first significant results in the theory of phase transitions.

We have come to the end of the story for now, and I want to sum up and add some broad conclusions. The main point is that the question of charge quantisation has proven unexpectedly deep and has led quantum theory into unexplored territory. A certain class of quantum field theories, of which one example was developed above, display a surprisingly rich spectrum of particle states characterised by quantum numbers that were not inserted initially. As a consequence there are many quantum equivalent reformulations of the one basic theory appearing quite different classically. The reformulations considered were self-similar differing only in coupling strengths, a fact that can be exploited computationally. Obviously the situation requires a deeper understanding. In particular it would be desirable to see explicit field transformations relating the different reformulations. These could be intrinsically interesting as they promise to generalise the vertex operator constructions that have proved so important in string theory and the representation theory of infinite dimensional algebras.

It is intriguing that the relevant class of quantum field theory is so close to physically realistic unified gauge theories. Maybe nature is that way. But then the question as to the whereabouts of the magnetic monopole becomes acute. Perhaps they really are too heavy to pair produce or maybe there is yet another theoretical twist. One promising scenario is that they simply vanish from sight by condensing into the vacuum [15]. The beauty of this is that the resulting condensate can have a very desirable consequence, namely the confinement of quarks, by means of a dual Meissner effect [16].

Despite the above questions and uncertainties, the framework of ideas has found fertile ground in superstring theory, mainly because of the dominant role of supersymmetry there.
The structure is even richer because quantum consistency requires superstrings to live in space-times of ten dimensions and the charge carriers have to be extended objects of varying dimensions, called branes.

The final conclusion is happy. There is clearly much to learn and it promises to involve new physics and new mathematics.

I would like to thank the organisers for the opportunity to give this talk. I have tried to emphasise the conceptual points and I realise this has been at the expense of the technical details. More of these, together with additional references to the original literature, can be found in accounts I have given elsewhere [17]. I do apologise to the experts for some oversimplifications, and in particular for a wayward factor of two.

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