Absolute continuity of the spectrum of coupled identical systems on 1D lattices

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June 7, 2018

Abstract

We prove that the spectrum of the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^2)$

$$
(\psi_{n,m}) \mapsto -(\psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1}) + V_n \psi_{n,m},
$$

(1)

$(n, m) \in \mathbb{Z}^2$, $\{V_n\} \in \ell^\infty(\mathbb{Z})$

is absolutely continuous.

1 Introduction

In this paper we study the spectrum of the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^2)$

$$
H := -\Delta + V,
$$

(2)

where $-\Delta$ is the discrete Laplacian acting on $\ell^2(\mathbb{Z}^2)$ by

$$
(-\Delta \psi)_{n,m} = -(\psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1}) \quad \forall \ (n, m) \in \mathbb{Z}^2
$$

and $V = \{V_n\}_{n \in \mathbb{Z}}$ is a sequence of real numbers defining a multiplication operator by

$$
\psi_{n,m} \mapsto V_n \psi_{n,m} \quad \forall \ (n, m) \in \mathbb{Z}^2.
$$

We emphasize that $V$ is independent of $m$, so that the operator (2) can be interpreted as describing infinitely many identical chains, each one labeled by the index $m$ and coupled to the others by a discrete laplacian along the $m$ direction.

Our main result is the following one:

**Theorem 1.1.** Let $V = \{V_n\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$; then (2) has absolutely continuous spectrum.

We will also prove the following dispersive estimate:

**Theorem 1.2.** Under the hypotheses of Theorem 1.1 let $\psi_0 \in \ell^1(\mathbb{Z}^2)$; then $\exists \ C > 0$ such that

$$
\|e^{i(\Delta + V)t}\psi_0\|_{\ell^\infty(\mathbb{Z}^2)} \leq \frac{C}{\langle t \rangle^{1/2}} \|\psi_0\|_{\ell^1(\mathbb{Z}^2)} \quad \forall \ t \in \mathbb{R}.
$$

(3)

Of course from Theorem 1.2 one can deduce standard Strichartz estimates (see [KT98]).

The main examples we have in mind are the case of a quasi-periodic potential, where $V_n := aV(\omega n + \theta)$,

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with $V \in C^\infty(T^1)$ and $a \in \mathbb{R}$, and the case where $\{V_n\}_{n \in \mathbb{Z}}$ is a random sequence. For both cases it is known that, under suitable assumptions, the spectrum of the unidimensional operator

$$\psi_n \mapsto -(\psi_{n+1} + \psi_{n-1}) + V_n \psi_n, \quad \psi = \{\psi_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$$

(4)
is pure point. See [FSWS86] for the latter and for instance [Jit99] in the case of the almost-Mathieu operator, namely $V(\theta) = a \cos(\theta)$, and [Eli97] for the proof of pure pointness under more generic hypotheses on the quasi-periodic potential $V$.

Theorem 1.1 shows that coupling infinitely many chains of type (4), the spectrum becomes absolutely continuous, independently of the spectral nature of the operator in (4). Furthermore, by Theorem 1.2 one gets dispersion. On the contrary, as shown in [EKMY02] for quasi-periodic potentials, pure pointness persists if the number of chains we couple is finite.

The proof of our results is very easy; nevertheless, we think that they could help to clarify the behavior of two dimensional chains. We recall that if, instead of considering in finitely many cores $D_i \subseteq \mathcal{H}_i$ such that the linear span of $D_1 \otimes D_2$ is also a core for $\mathcal{H}$ and, for all pure tensors $\psi = \psi_1 \otimes \psi_2$ with $\psi_i \in D_i$, one has

$$H \psi = A_1 \psi_1 \otimes \psi_2 + \psi_1 \otimes A_2 \psi_2.$$ 

By Fox76 the following result holds.

**Theorem 2.2.** Let $H$ be a separable selfadjoint operator on $\mathcal{H}$, then the projection valued measure $P$ associated to $H$ is the tensor convolution measure, defined by the relation

$$\langle P(\cdot) \psi_1 \otimes \psi_2, \phi_1 \otimes \phi_2 \rangle_{\mathcal{H}} = \langle P_1(\cdot) \psi_1, \phi_1 \rangle_{\mathcal{H}_1} \ast \langle P_2(\cdot) \psi_2, \phi_2 \rangle_{\mathcal{H}_2},$$

(5)

where $P_i$ are the projection valued measures associated to $A_i$, $i = 1, 2$.

We will apply this theorem to our case exploiting the structure $\ell^2(\mathbb{Z}^2) = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ of the space. Indeed one immediately sees that the operator (2) is separable with parts

$$(A_1 \chi)_n = -(\chi_{n+1} + \chi_{n-1}) + V_n \chi_n, \quad \chi = \{\chi_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$$

and $A_2 \equiv \Delta_m$ the discrete laplacian in the $m$ direction, namely

$$(-\Delta_m \phi)_m = - (\phi_{m+1} + \phi_{m-1}) \quad \phi = \{\phi_m\}_{m \in \mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

In order to exploit Theorem 2.2 we first give a result on the convolution of two bounded measures, one of which is absolutely continuous.

**Proposition 2.3.** Let $m$, $n$ be complex finite Borel measures. If $m$ is absolutely continuous with respect to Lebesgue measure, then their convolution $m * n$ is absolutely continuous with respect to Lebesgue measure.

2 Absolute Continuity of the Spectrum

We exploit the main result of Fox76 dealing with selfadjoint operators in Hilbert spaces $\mathcal{H}$ with the structure of tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

First we recall the definition of separable operators.

**Definition 2.1.** A selfadjoint operator $H$ on $\mathcal{H}$ is called separable with parts $A_1$, $A_2$, if there exist cores $D_i \subseteq \mathcal{H}_i$ such that the linear span of $D_1 \otimes D_2$ is also a core for $\mathcal{H}$ and, for all pure tensors $\psi = \psi_1 \otimes \psi_2$ with $\psi_i \in D_i$, one has

$$H \psi = A_1 \psi_1 \otimes \psi_2 + \psi_1 \otimes A_2 \psi_2.$$
Proof. Let $f \in L^1(d(m \ast n))$, with $mn$ the product measure. One has

$$\int_{\mathbb{R}} f dm \ast n = \int_{\mathbb{R}^2} f(x + y) \, dm(x) \, dn(y)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \, dm(z - y) \, dn(y)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \, g(z - y) \, dz \, dn(y)$$

$$= \int_{\mathbb{R}} f(z) \left( \int_{\mathbb{R}} g(z - y) \, dn(y) \right) \, dz,$$

where $G(z) = \int_{\mathbb{R}} g(z - y) \, dn(y) \in L^1(dz)$, since

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(z - y) \, dn(y) \right| \, dz \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g(z - y)| \, d|n|(y) \right) \, dz$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g(z - y)| \, dz \right) \, d|n|(y)$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g(z)| \, dz \right) \, d|n|(y)$$

$$= \left( \int_{\mathbb{R}} |g(z)| \, dz \right) \, |n(\mathbb{R})| < +\infty.$$ 

\[ \Box \]

Proof of Theorem 1.1. By density argument, it is sufficient to prove absolute continuity of the real Borel measure $\langle P(\cdot) \psi, \psi \rangle_{\mathcal{H}}$ for the set of all finite linear combinations of pure tensors,

$$\psi = \sum_{j=1}^{N} \alpha_j \chi_j \otimes \phi_j, \quad N > 0, \quad \alpha \in \mathbb{C}, \quad \chi_j, \phi_j \in L^2(\mathbb{Z}).$$

We apply Theorem 2.2 to get

$$\langle P(\cdot) \psi, \psi \rangle_{L^2(\mathbb{Z})} = \sum_{j,k=1}^{N} \pi_j \alpha_k \langle P(\cdot) \chi_j \otimes \phi_j, \chi_k \otimes \phi_k \rangle_{L^2(\mathbb{Z})}$$

$$= \sum_{j,k=1}^{N} \pi_j \alpha_k \langle P_1(\cdot) \chi_j, \chi_k \rangle_{L^2(\mathbb{Z})} \ast \langle P_2(\cdot) \chi_j, \chi_k \rangle_{L^2(\mathbb{Z})},$$

where $P_1$ is the projection valued measure associated to $A_1$ and $P_2$ is associated to $-\Delta_m$.

Thus we deal with a linear combination of the complex measures $\langle P_1(\cdot) \chi_j, \chi_k \rangle_{L^2(\mathbb{Z})} \ast \langle P_2(\cdot) \chi_j, \chi_k \rangle_{L^2(\mathbb{Z})}$, whose absolute continuity follows from Proposition 2.3 and from the absolute continuity of the measures $\langle P_2(\cdot) \chi_j, \chi_k \rangle_{L^2(\mathbb{Z})}$. 

\[ \Box \]

3 Dispersive Estimates

In the following, if $p \in [1, \infty]$ we will denote with $\ell^p_n$ (respectively, $\ell^p_m$) the $\ell^p$ space of a complex valued sequence with respect to its integer index $n$ (respectively, its $\ell^p$ space with respect to its integer index $m$). Furthermore, $\ell^p_{n,m}$ will denote the norm of complex sequence with respect to both the indexes $n, m$.

Proof of Theorem 3.2 Denote, by abuse of notation, $A_1 = A_1 \otimes 1$ and $-\Delta_m = 1 \otimes (-\Delta_m)$, then such operators strongly commute and therefore one has

$$e^{i(\Delta + V)t} = e^{i(-\Delta_m + A_1)t} = e^{iA_1}e^{-\Delta_m t} \quad \forall \, t \in \mathbb{R}.$$ 

3
The dispersive estimate for $-\Delta + V$ then follows from the analogous dispersive estimate for the one-dimensional discrete laplacian $-\Delta_m$ (see [SK05]): let $\phi \in \ell^1(\mathbb{Z})$; then $\exists C > 0$ such that

$$\|e^{-i\Delta_m t}\phi\|_{\ell^\infty(\mathbb{Z})} \leq C(t)^{-\frac{1}{3}}\|\phi\|_{\ell^1(\mathbb{Z})}.$$

Indeed, $\forall t \in \mathbb{R}$, if $\psi_0 \in \ell^1_{n,m}$ one has

$$\|e^{(-\Delta + V)t}\psi_0\|_{\ell^\infty_{n,m}} = \|e^{A_1 t} (e^{-i\Delta_m t}\psi_0)\|_{\ell^\infty_{n,m}} \leq \|\|e^{iA_1 t} (e^{-i\Delta_m t}\psi_0)\|_{\ell^2_{n,m}} \leq C(t)^{-\frac{1}{3}} \|\psi_0\|_{\ell^1_{n,m}}.$$