CONDITIONAL SYMMETRIES OF NONLINEAR THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

AEEMAN FATIMA\textsuperscript{a}, F. M. MAHOMED\textsuperscript{b} AND CHAUDRY MASOOD KHALIQUE\textsuperscript{a}

\textsuperscript{a}International Institute for Symmetry Analysis and Mathematical Modeling
North-West University, Mafikeng Campus
P Bag X2046, Mafikeng, South Africa

\textsuperscript{b}School of Computer Science and Applied Mathematics
DST-NRF Centre of Excellence in Mathematical and Statistical Sciences
University of the Witwatersrand, Johannesburg
Wits 2050, South Africa

Abstract. In this work, we take as our base scalar second-order ordinary differential equations (ODEs) which have seven equivalence classes with each class possessing three Lie point symmetries. We show how one can calculate the conditional symmetries of third-order non-linear ODEs subject to root second-order nonlinear ODEs which admit three point symmetries. Moreover, we show when scalar second-order ODEs taken as first integrals or conditional first integrals are inherited as Lie point symmetries and as conditional symmetries of the derived third-order ODE. Furthermore, the derived scalar nonlinear third-order ODEs without substitution are considered for their conditional symmetries subject to root second-order ODEs having three symmetries.

1. Introduction. Lie classical theory of symmetries for differential equations is an inspiring source for various generalizations aimed at finding ways for obtaining reductions and solutions of differential equations [21]. The historical origin of group theory dates back to the work of Evaristé Galois in 1831. He was the first who really linked the algebraic solution of equations to their group properties. Galois proved that there exists polynomial equations whose groups are not solvable and the associated equations are not solvable by means of radicals. Inspired by Galois theory, Lie tried to do for differential equations what Galois had done for polynomial equations. Lie developed a highly algorithmic method for the solutions of differential equations. He studied groups of continuous transformations. Lie demonstrated that many techniques for finding solutions of differential equations can be unified and extended by considering the symmetry group structure of such equations. Today, the Lie symmetry approach to differential equations is widely applied in various fields.

With the passage of time as more and more researchers tried to go forward with Lie theory, certain limitations came up in the Lie approach. They discovered that the theory of Lie groups is not complete for the integrability of differential equations. In the case of ODEs, on the one hand a differential equation may admit a sufficient number of symmetry transformations but may not yet be solvable by quadratures via...
the Lie classical approach. A particular example of this is a variable coefficient linear second-order ODE which has eight Lie point symmetries but is not integrable by quadratures. On the other hand, an ODE or a system of ODEs has general solution in terms of quadratures without possessing any non-trivial symmetry generator [1]. These insights have given rise to extensions to different types of symmetries.

One of the significant extension of the classical Lie approach for partial differential equations (PDEs) is that of the non-classical or what is frequently called conditional symmetries. The conditional symmetry or non-classical symmetry approach has its roots in the work of Bluman and Cole [3]. Nevertheless, this approach is based on the classical Lie scheme generally speaking, but the resulting symmetries may not be any Lie symmetry of the equation in question – these are called conditional symmetries as one requires that not all solutions be mapped into themselves by the symmetries. There are equations occurring in applications that do not possess classical symmetries but have proper conditional symmetries. Therefore, in recent years, the interest in the conditional symmetry approach has intensified. Information on the nonclassical method and related topics can be found in [31]-[30]. In the past two decades, the theoretical background of non-classical symmetry has been widely investigated and the non-classical symmetry technique has been effectively applied to find the exact solutions of many partial differential equations of mathematical physics [32]-[7]. A literature survey witnesses that a considerable amount of papers have been devoted to the study of conditional symmetry properties of PDEs. Our interest here though is on the conditional symmetry of ODEs which has not enjoyed much exposure. However, some useful investigations in this direction are made in the studies [13]-[34].

Linearization plays a significant role in the theory of differential equations. Scalar first-order ODEs can always be linearized by point transformation. Lie proved that the necessary and sufficient condition for a scalar nonlinear second-order ODE to be linearizable is that it must have eight Lie point symmetries. He exploited the fact that all scalar linear second-order ODEs are equivalent under point transformations; that is every linearizable scalar ODE is reducible to the free particle equation. Lie developed invariant criteria and took this approach no further but considerably later Chern [4]-[5] extended the analysis to a class of scalar third-order ODEs by using contact transformations. It was not much later that results were obtained using point transformations by Grebot [15]. Mahomed and Leach [27] proved that for $n$th-order ODEs, $m \geq 3$, there was no unique class of linearizable ODEs. Instead there were three equivalence classes with $m + 1, m + 2$, or $m + 4$ infinitesimal symmetry generators. Neut and Petitot [29] dealt with linearizable third-order ODEs by use of the Cartan method. Ibragimov and Meleshko [16] further focused on the linearizability criteria for third-order ODEs using the Lie approach. Though the procedure followed by [16] was the same as that used by Lie, the calculations become much more complicated and algebraic computation is needed. Afterwards, in [17]-[36] the authors used the point and contact transformations to find the criteria for the linearizability of fourth-order scalar ODEs. Meleshko [28] provided a simple approach to reduce a class of third-order ODEs to second order ODEs. If the reduced equations satisfy the Lie linearizability criteria, the original equation can then be solved by linearization.

The classification of ODEs via conditional symmetries was proposed in [24] in which the definitions of conditional symmetry and linearizability of scalar ODEs
subject to lower order root ODEs were provided. Later the authors in [25] discussed the invariant criteria for conditional linearizability of third-order equations subject to root second-order Lie linearizable ODEs. They have shown that certain omissions in the ODE literature can to good effect be filled by the recent development of conditional linearizability. This certainly applies in the deficiency of certain cases when the ODEs are not Lie linearizable but conditionally linearizable. The conditional linearizability of fourth-order semi-linear ODEs has been investigated in [26]. Further investigations on conditional linearization of systems of second and third-order ODEs were performed in [37]-[22]. All of these studies have been carried out subject to lower second-order Lie linearizable root ODEs.

In this work, we further study conditional symmetries of ODEs from the algorithmic viewpoint. Recently in [11], we have provided the rigorous definition of conditional symmetries of ordinary differential equations and provided an algorithm to compute such symmetries. A proposition has been proved in [11] which provides criteria as to when the symmetries of the root system of ODEs are inherited by the derived higher-order system. The criteria when the symmetries of the root linear ODEs are inherited by the derived scalar linear higher-order ODEs and even order linear system of ODEs have been discussed in detail in [11]. Furthermore in [11] it has been shown that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves.

The present study continues the research which was carried out in [11] by using as base scalar second-order ODEs which have seven equivalence classes and possess three-dimensional Lie algebras. We investigate that all the Lie point symmetries of the second-order ODEs considered as first integrals or just derived without substitution of the integral are not necessarily inherited but can be viewed as conditional symmetries of the derived ODE. Further we discuss when the derived non-linear third-order ODEs do not have all symmetries inherited from the root ODEs.

We commence by giving a precise definition of conditional symmetries (see [11]) adapted to third-order ODEs.

**Definition 1.** A nonlinear third-order ODE is conditionally classifiable by a symmetry algebra \( A \) with respect to a nonlinear second-order ODE called the root ODE if and only if the third-order ODE jointly with the second-order ODE forms an overdetermined compatible system (so the solutions of the third-order ODE reduces to the solutions of the second-order ODE) and the second-order ODE has symmetry algebra \( A \) which is the conditional symmetry algebra of the nonlinear third-order ODE.

We now present the algorithm for computing the conditional symmetries of a scalar third-order ODE of the form

\[
y''' = f(x, y, y', y''),\tag{1}
\]

where \( x \) is the independent variable, \( y \) the dependent variable and \( y', y'' \) denote the derivatives of \( y \) with respect to \( x \).

Let \( X \) be the vector field of dependent and independent variables given by

\[
X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y},\tag{2}
\]

where \( \xi \) and \( \eta \) are the coefficient functions of the vector field \( X \).
Suppose that the vector field $X$ is a conditional symmetry generator of equation (1) subject to a nonlinear second-order ODE

$$y'' = g(x, y, y').$$

(3)

Then the conditional symmetry condition

$$X[^3][y''' - f(x, y, y', y'')]|_{y''' - f = 0, y'' - g = 0} = 0,$$

(4)

holds. Here $X[^3]$ denotes the 3rd prolongation of the generator $X$ given by

$$X[^3] = X + \sum_{j=1}^{3} \zeta_j \partial / \partial y^{(j)},$$

(5)

in which

$$\zeta_j = D_x(\zeta_{j-1}) - y^{(j)}D_x(\xi), \ j = 1, 2, 3, \zeta_0 = \eta$$

(6)

and $D_x$ is the usual total differentiation operator with respect to $x$.

2. Examples of conditional symmetries. In this section, we present two examples of third-order ODEs. One is conditionally linearizable with respect to a Lie linearizable second-order ODE. The other has conditional $sl(2, \mathbb{R})$ symmetry subject to an Ermakov equation which admits the $sl(2, \mathbb{R})$ algebra.

1. First we consider the example of differentiating the linearizable equation [25]

$$y'' + xy'^3 + \frac{2}{x}y' = 0,$$

(7)

which results in

$$y''' - 3x^2y'^5 - 7y'^3 - \frac{6}{x^2}y' = 0.$$

(8)

We take $c = 0, d = 0, g = f - 2b = 0$ and $h = a - 2e = \frac{2}{x}$ in the class considered in [25] and select $b = 0$ and $e = -\frac{1}{2}$. Then $f = 0$ and $a = 0$. It is easily verified that the conditions are met and that it is not in any of the classes of [29, 16, 28] and cannot be linearized according to the methods discussed in these. Equation (7) reduces to the simplest linear form $\frac{dv}{du} = 0$ by use of the transformation $u = x \cos y$ and $v = x \sin y$. The solution is given in terms of two arbitrary constants. Hence the third-order ODE (8) is conditionally linearizable by a point transformation in $u$ and $v$ subject to the second-order root ODE (7).

2. Consider the second-order Ermakov equation [10]

$$y'' + \alpha y^{-3} = 0, \ \alpha = \text{const.}$$

(9)

The above equation has the following symmetry generators (which forms the $sl(2, \mathbb{R})$ algebra)

$$X_1 = \partial_x, \ X_2 = x\partial_x + \frac{1}{2}y\partial_y, \ X_3 = x^2\partial_x + xy\partial_y.$$  

(10)

Now differentiating (9) we obtain

$$y^4y''' - 3\alpha y' = 0.$$  

(11)

It is obvious that $X_1$ is the inherited symmetry of (11). Now consider the generator $X_2$ with its second prolongation

$$X_2[^2] = x\partial_x + y\partial_y - \frac{1}{2}y'\partial_{y'} - \frac{3}{2}y''\partial_{y''}.$$  

(12)
Operating (12) on (9), we deduce
\[ X_2^{[2]}[y'' + \alpha y^{-3}] = -\frac{3}{2}(y'' + \alpha y^{-3}), \] (13)
with \( \lambda = -(3/2) \) which is constant (see Proposition 1 of [11]). Hence \( X_2 \) is an inherited symmetry generator of (11) by Proposition 1. Now consider the generator \( X_3 \) with its prolongation
\[ X_3^{[2]} = x^2 \partial_x + xy \partial_y + (y - xy')\partial_y - 3xy''\partial_{y''}. \] (14)
By making use of (14) on ODE (9), we easily obtain
\[ X_3^{[2]}[y'' + \alpha y^{-3}] = -3x[y'' + \alpha y^{-3}], \] (15)
and this results in \( \lambda = -3x \). This shows that by using \( X_3^{[2]} \) on (9) we are able to determine \( X_3 \) as the symmetry generator of (11); hence \( X_3 \) is a proper conditional symmetry generator of (11) with \( \lambda = -3x \), a non-constant, by Proposition 1.

Now by eliminating \( \alpha \) from (9) we arrive at
\[ yy''' + 3y'y'' = 0. \] (16)
Operating (12) on (16), we find
\[ X_2^{[3]}[yy''' + 3y'y'']_{(16)} = 0 \Rightarrow -2[yy''' + 3y'y'']_{(16)} = 0. \] (17)
This is an inherited symmetry generator of (11). Note that Proposition 1 does not apply as we have substituted \( \alpha \). Making use of (14) on ODE (16), we obtain
\[ X_3^{[3]}[yy''' + 3y'y'']_{(16)} = 0. \] (18)
which gives
\[ [-3x(yy''' + 3y'y'')]_{(16)} = 0, \] (19)
and thus \( X_3 \) is a proper symmetry generator of ODE (11). Here all symmetries are inherited by the derived ODE.

3. **Conditional symmetries of third-order ODEs subject to second-order ODEs as first integrals.** In this section, we consider scalar second-order ODEs in their canonical forms, taken as first integrals of the respective third-order ODEs, which admit three symmetry generators. We must have that all the symmetries of the second-order ODEs considered as first integrals are inherited as point symmetries of the third-order ODEs [18].

Consider the following table of second-order ODEs in canonical form that possess Lie point symmetries [23].

In Table I, \( g \) is an arbitrary function of its arguments and \( A \) constant.

We briefly review the following definitions and results of [18] which shows that the symmetry of a first integral is the symmetry of the equation itself.

**Definition 2** (see [18]). The differential \((n-1)\)-form
\[ w = T^i(x, u, u_1, ..., u_r - 1) \frac{\partial}{\partial x^i}(dx^1 \wedge ... \wedge dx^n) \] (20)
is called a conserved form of
\[ E^\beta(x, u, u_1, ..., u_r) = 0, \beta = 1, ..., p \leq m, \] (21)
if
\[ Dw = 0. \] (22)
Theorem 1 (see [18]). Suppose that

\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^a \frac{\partial}{\partial u^a} + \zeta_{i_1i_2} \frac{\partial}{\partial u_{i_1i_2}} + \ldots \]  

(23)

where \( \xi^i (i = 1, 2, \ldots, n) \) and \( \eta^a (a = 1, 2, \ldots, m) \) are differential functions and the additional coefficients are \( \zeta_{i_1i_2} = D_i(\eta^a) - u^a D(\xi^j) \), \( \zeta_{i_1\ldots i_s} = D_i(\zeta_{i_1\ldots i_{s-1}}) - u^a D_{i_1\ldots i_s} \), \( D_{i_1\ldots i_s} \), \( s > 1 \).

The operator \( X \) is a Lie-Bäcklund symmetry generator of the system (21) such that the conserved form \( w \) of (21), given by (20) is invariant under \( X \). Then

\[ X(T^i) + T^i D_j(\xi^j) - T^i D_j(\xi^j), i = 1, \ldots, n \]  

(24)

where \( D_i = \xi^i \frac{\partial}{\partial x^i} + u^a \frac{\partial}{\partial u^a} + \ldots \) is the total derivative with respect to \( x^i \).

Note. For any differential function \( f \), \( Df = D_i f dx^i \) and for any \( k \)-form \( w = f_{i_1i_2\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \), \( Dw = D_i f_{i_1i_2\ldots i_k} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \).

The above Theorem 1, in the case of ODEs, shows that if we consider a second-order ODE considered as a first integral \( A = I(x, y, y') \), with three symmetries, of a third-order ODE and take \( D_{i}A = 0 \) on \( A = I(x, y, y', y'') \), we obtain the same symmetries of the third-order ODE.

We focus on Table I.

Consider the representative ODE for \( L_{3:3}^1 \) from Table I. The canonical equation written as a first integral is

\[ A = y'' e^{y'}, A \neq 0 \]  

(25)

and has the generators

\[ X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x \partial_x + (x + y) \partial_y. \]  

(26)


Table II

Inherited symmetries of derived scalar third-order equations

| Representative 2nd-order ODE, $A \neq 0$ | Derived 3rd-order ODE | Inherited algebra |
|------------------------------------------|-----------------------|-------------------|
| $y'' = A e^{-y}$ | $y'' + y'^2 = 0$ | $L_{3,5}$ |
| $y'' = A y^{(a-2)/(a-1)}$ | $y'y'' - \frac{a-2}{a} y'^2 = 0$ | $L_{3,6}$ |
| $y'' = A (1 + y^2)^{1/2} \epsilon b \arctan y'$ | $y'' - 3y^{1/2} y'^2 = 0$ | $L_{3,7}$ |
| $xy'' = Ay^3 - \frac{1}{2} y'$ | $y'y'' - 3y'^2 = 0$ | $L_{3,8}$ |
| $xy'' = y' + y'^2 + A (1 + y^2)^{1/2}$ | $y'' + y'' y'^2 - 3y y'^2 = 0$ | $L_{3,9}$ |
| $xy'' = y' - y'^2 + A (1 - y^2)^{1/2}$ | $y'' - y'' y'^2 - 3y y'^2 = 0$ | $L_{3,10}$ |
| $y'' = AK^{3/2}$ | $y'' = \frac{3y}{2} y K^{-1} D_y K$ | $L_{3,11}$ |
| where $K = \frac{1 + y^2 + (y - xy')^2}{1 + x^2 + y^2}$ | | |

The derived ODE ((25) is taken as a first integral) corresponding to (25) is [20]

$$y'' + y'^2 = 0.$$ (27)

It follows that $X_1$, $X_2$ and $X_3$ are the symmetries of (27) by Theorem 1. Hence the derived third-order ODE corresponding to (25) has the same symmetries as the second-order ODE. We have the following table of results.

In Table II, second-order ODEs, for $A \neq 0$, are taken as first integrals of the derived third-order ODEs. However, if we take the second-order ODEs as conditional first integrals, i.e. $A = 0$, then we have the following consequences. For $L_{3,5}$, $y'' = 0$, $y''' = 0$ and all $L_{3,7}^I$ symmetries are inherited by the derived equation $y''' = 0$. The same applies to $L_{3,6}^I$. For the case $L_{3,7}^I$, $X_3$ is the only proper conditional symmetry of the third-order equation. The other two are inherited. This is seen as we have $y'' = 0$ and the derived ODE is $y''' = 0$. Here $X_{3}^{[3]}$ $y''' = 0$ results in $3y'^2 + 2(b + y')y''' = 0$ which is the case provided that both the second and third order ODEs are satisfied. In a similar manner for all three realizations $L_{3,8}$, viz. $L_{3,8}^I$, $L_{3,8}^{II}$ and $L_{3,8}^{III}$ which, for $A = 0$, have second-order ODEs $xy'' + y'/2 = 0$, $xy'' = y' + y'^2$ and $xy'' = y' - y'^2$, we have that only $X_3$ is a proper conditional symmetry of the derived ODEs $xy''' + 3y''/2 = 0$, $xy''' = 3y'^2 y''$ and $xy''' = -3y'^2 y''$. The other two are inherited. For $L_{3,9}$, $y'' = 0$, $y''' = 0$ and all $L_{3,9}$ symmetries are proper conditional symmetries of the derived equation.

Hence we can state a proposition as follows.

**Proposition 1.** The scalar second-order ordinary differential equations in the real plane with three symmetries as in Table II taken as first integrals have their point symmetries inherited for the derived third-order equations. However, if the second-order ODEs are conditional first integrals for $A = 0$, then the derived third-order ODEs have symmetries inherited only for $L_{3,3}$ and $L_{3,7}^I$. For the other realizations except $L_{3,9}$, the third symmetry $X_3$ is a proper conditional symmetry of the third-order ODE. For $L_{3,9}$, all symmetries of the derived third-order equation are conditional.

We have seen that in all cases $X_1$, $X_2$ and $X_3$ remain as inherited symmetries of the third-order ODEs with corresponding first integrals as nonlinear second-order root ODEs having the same symmetries which are first integrals. The situation is
4. **Conditionals symmetries subject to nonlinear second-order ODEs with 3 symmetries.** Conditional linearizability of scalar third-order and fourth-order ODEs subject to Lie linearizable second-order ODEs were completely and invariantly characterized in terms of the coefficients of the root equation in the papers [25, 26] wherein the derived third-order ODE before and after substitution of the second derivative were investigated. Here we focus on the conditional symmetries of nonlinear third-order ODEs subject to second-order nonlinear root ODEs which are not taken as usual first integrals.

Again by considering $L_{I3;3}$, on differentiating the root equation (25) with respect to $x$ with $A \neq 0$, yields

$$y''' + Ay'' e^{-y} = 0. \tag{28}$$

The conditional symmetries are given in Table I. Here clearly $X_1$ and $X_2$ are inherited symmetries of (28). Finally we consider the generator $X_3$ with its second prolongation and apply it to the base equation. This gives rise to

$$X_3^{[2]} [y'' - Ae^{-y}] = -(y'' - Ae^{-y}). \tag{29}$$

Here we have $\lambda = -1$ and hence $X_3$ is also an inherited symmetry by Proposition 1 (see [11]).

For $L_{I3;6}$ we again consider its root ODE from Table I. The differentiated third-order ODE is

$$y''' - \left(\frac{a - 2}{a - 1}\right)Ay'' y'^{\frac{1}{2}} = 0, \tag{30}$$

with conditional symmetry generators given in Table I. Here again $X_1$ and $X_2$ are inherited symmetries of the nonlinear ODE (30). Finally consider the symmetry $X_3$ with prolongation. We end up with

$$X_3^{[2]} [y'' - Ay'^{\frac{1}{2}}] = (a - 2)[y'' - Ay'^{\frac{1}{2}}] = 0. \tag{31}$$

We have that $\lambda = a - 2$ and thus $X_3$ is also an inherited symmetry of (30) by Proposition 1, [11].

For $L_{I3;7}$, the root ODE of Table I upon differentiation with $A \neq 0$ yields

$$y''' - Ay''(3y' + b)(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'} = 0. \tag{32}$$

The conditional symmetry generators are given in Table I. Obviously $X_1$ and $X_2$ are the inherited symmetry generators of (32). Finally we consider $X_3$ and its prolongation and applying it on the root equation we have

$$X_3^{[2]} [y'' - A(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'}] = -(b + 3y') y'' - A(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'}, \tag{33}$$

with $\lambda = -(3y' + b)$ which is non-constant, so $X_3$ is a proper conditional symmetry of (32).

For $L_{I3;8}$, we consider its root ODE and differentiating it with respect to $x$ with $A \neq 0$ gives

$$xy''' + \frac{3}{2} y'' - 3Ay'' y'^{2} = 0, \tag{34}$$

with conditional symmetry generators presented in Table I. Now operating the second prolongation of $X_2$ on the base equation results in

$$X_2^{[3]} [xy'' - Ay'^{3} + \frac{1}{2} y'] = -1[xy'' - Ay'^{3} + \frac{1}{2} y'], \tag{35}$$
which shows that \( X_2 \) is an inherited symmetry generator of ODE (34) with \( \lambda = -1 \).

Finally, we consider \( X_3 \) and with its prolongation and applying on the base ODE results in
\[
X_3^{[2]} [xy'' - Ay'^3 + \frac{1}{2} y'] = -2(y + 3xy') [xy'' - Ay'^3 + \frac{1}{2} y'],
\]
with \( \lambda = -2(y + 3xy') \) which shows that \( X_3 \) is a proper conditional symmetry generator of (34).

For \( L_{3,8}^{III} \), the third-order ODE with \( A \neq 0 \) is given by
\[
xy'' + 3y'' y^2 - 3Ay'y''(1 + y'^2)^{\frac{1}{2}} = 0.
\]
Again we consider \( X_2 \) and its prolongation and operating it yields
\[
X_2^{[2]} [xy'' - y' - y^3 - A'(1 + y'^2)^{\frac{1}{2}}] = -[xy'' - y' - y^3 - A'(1 + y'^2)^{\frac{1}{2}}],
\]
with \( \lambda = -1 \) which shows that \( X_2 \) is an inherited symmetry generator of (37).

Finally, we consider \( X_3 \) and its prolongation and now apply this to find
\[
X_3^{[2]} [xy'' - y' - y^3 - A'(1 + y'^2)^{\frac{1}{2}}] = -2(y + 3xy') [xy'' - y' - y^3 - A'(1 + y'^2)^{\frac{1}{2}}],
\]
with \( \lambda = -2(y + 3xy') \). This shows that \( X_3 \) is a proper conditional symmetry generator of (37).

For \( L_{3,8}^{III} \), the third-order ODE with \( A \neq 0 \) is
\[
xy'' + 3y'' y^2 + 3Ay'y''(1 - y'^2)^{\frac{1}{2}} = 0.
\]
For the operator \( X_2 \) and its prolongation acting on the base ODE we have
\[
X_2^{[2]} [xy'' - y' + y^3 - A(1 - y'^2)^{\frac{1}{2}}] = -[xy'' - y' + y^3 - A(1 - y'^2)^{\frac{1}{2}}],
\]
with \( \lambda = -1 \), thereby showing that \( X_2 \) is an inherited symmetry generator of (40).

Now for \( X_3 \) and its prolongation we have
\[
X_3^{[2]} [xy'' - y' + y^3 - A'(1 - y'^2)^{\frac{1}{2}}] = -2(y + 3xy') [xy'' - y' + y^3 - A'(1 - y'^2)^{\frac{1}{2}}],
\]
with \( \lambda = -2(y + 3xy') \). Therefore \( X_3 \) is a proper conditional symmetry of (40) when we totally differentiate the base ODE with \( A \neq 0 \).

For \( L_{3,9}^{III} \), the derived third-order ODE inherits all the symmetries as conditional symmetries. We see this as follows. The derived ODE of the root equation for the \( L_{3,9}^{III} \) realization is
\[
y''' = \frac{3}{2} A \frac{(1 + y' + (y - xy')^2)^{1/2} D_x [1 + y' + (y - xy')^2]}{1 + x^2 + y^2}.
\]
The representative second-order ODE has the three symmetries as given in Table I which are
\[
X_1 = (1 + x^2) \partial_x + xy \partial_y, \quad X_2 = xy \partial_x + (1 + y^2) \partial_y, \quad X_3 = y \partial_x - x \partial_y.
\]
For each symmetry in (44) we check if it is inherited by the derived ODE (43). In the case \( X_1 \) we have that its second prolongation is
\[
X_1^{[2]} = X_1 + (y - xy') \partial_{y''} - 3xyy'' \partial_{y''}.
\]
Acting with this on the root ODE we find that \( \lambda = -3x \) and thus \( X_1 \) is a proper conditional symmetry of (43). Likewise for \( X_2 \) we have its second prolongation given by
\[
X_2^{[2]} = X_2 + (yy' - xy'^2) \partial_{y''} - 3xyy'' \partial_{y''}.
\]
and this results in $\lambda = -3y'y'$ implying that $X_2$ is a proper conditional symmetry of our derived third-order ODE (43). Lastly the second prolongation of $X_3$ is

$$X_3^{[2]} = X_3 - (1 + y'^2)\partial_{y'} - 3y'y''\partial_{y''}.$$  

Here we see that $\lambda = -3y'y'$. Thus $X_3$ is a proper conditional symmetry of (43). In this final case we see that all symmetries are proper conditional symmetries of (43).

Here we have seen that in all cases except for $L_{3;9}$, $X_1$ and $X_2$ remain as inherited symmetries and the derived nonlinear ODEs have less inherited symmetries than the root ODEs whereas $X_3$ is the inherited symmetry for $L_{3;3}^I$ and $L_{3;6}^I$; it is a proper conditional symmetry for all the remaining cases besides $L_{3;9}$. For $L_{3;9}$ we have that all symmetries are proper conditional symmetries.

We have the following result.

**Proposition 2.** The scalar second-order ODEs in the real plane in Table I with three symmetries which do not have $L_{3;9}$ symmetry taken as representative equations when derived to third-order ODEs have their symmetries inherited for $L_{3;3}^I$ and $L_{3;6}^I$ only. For the other realizations except $L_{3;9}$, the third symmetry $X_3$ is a proper conditional symmetry of the third-order equation and hence the Lie point symmetries reduce in number from three to two in these cases. For the representative equation of $L_{3;9}$, all symmetries become conditional symmetries of the derived third-order ODE.

5. **Concluding remarks.** In this work, we have shown by means of an algorithm as to how one can calculates the conditional symmetries of nonlinear third-order ODEs subject to nonlinear second-order ODEs (this is adapted from [11]). Few simple examples of scalar nonlinear third-order ODEs subject to root second-order ODEs were first considered for their conditional symmetries.

We have then also investigated that all the symmetries of the second-order ODEs when taken as first integrals or conditional first integrals are preserved as inherited symmetries or more generally become conditional symmetries (see Proposition 1). Furthermore, we have shown that the derived nonlinear ODEs without substitution have in general less inherited symmetries as compared to that of the root nonlinear second-order ODEs as in Proposition 2.

**Acknowledgments.** FMM thanks the NRF of South Africa for a research grant. AF is grateful to the DST-NRF CoE-MaSS of South Africa and North-West University for a postdoctoral fellowship.

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Received December 2016; revised April 2017.

E-mail address: emanfatima81@yahoo.com
E-mail address: Fazal.mahomed@wits.ac.za
E-mail address: Masood.Khalique@nwu.ac.za