Chaotic fluctuations in graphs with amplification

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Abstract

We consider a model for chaotic diffusion with amplification on graphs associated with piecewise-linear maps of the interval. We investigate the possibility of having power-law tails in the invariant measure by approximate solution of the Perron-Frobenius equation and discuss the connection with the generalized Lyapunov exponents $L(q)$. We then consider the case of open maps where trajectories escape and demonstrate that stationary power-law distributions occur when $L(q) = r$, with $r$ being the escape rate. The proposed system is a toy model for coupled active chaotic cavities or lasing networks and allows to elucidate in a simple mathematical framework the conditions for observing Lévy statistical regimes and chaotic intermittency in such systems.

Keywords: Chaotic map, Power-law distributions, Diffusion and amplification on graphs, Generalized Lyapunov exponents

1. Introduction

Dynamical systems defined on graphs are subject of current research, due to the many applications to model complex interacting units with non uniform connectivity \[1\]. Also, one of the features of complex systems is the possibility of display non-Gaussian fluctuations that make large rare events very relevant. The distributions of the observables can have fat-tailed statistics leading to domination of a single event and lack of self-averaging of measurements. In active systems where fluctuating amplification can occur, even rare trajectories can generate large-sized fluctuations. This is well-known for multiplicative stochastic processes \[2\] and chaotic dynamical systems that display intermittency and multifractality \[3\].

Countless example are present in the physical, biological and even social sciences. A case of experimental relevance is provided by optical media with diffusion and amplification of light, as it occurs in random lasers where heavy-tailed distributions of emission intensities, characterized by Lévy-stable statistics \[4\] have been predicted \[5\] (see also \[6, 7\]) and confirmed in experiments \[8, 9, 10\].

An experimental system that encompasses properties of a dynamical systems on graphs and non trivial statistics is the lasing network, recently introduced in \[11\]. It consists of active and passive optical fibers, connected to form a graph structure. The connectivity induces a form of topological disorder and can be viewed as a discrete random laser, with a controllable complexity. The presence of the optical gain and disorder induce wild emission fluctuations whose origin is not fully understood \[11\]. So a related question is how such fluctuations relate to the network structure and connectivity.

The existence of fat tails is intimately related to the the possibility for a spontaneous fluctuation to grow well beyond the average. The indicators to quantify this are the finite-time and generalized Lyapunov exponents \[12\]. For multiplicative noise they been shown that they are useful tools to yield an intuitive derivation of the form of the probability distribution’s tail in the presence of additive noise \[13\].

Another possibility to have stationary fat-tailed statistics for multiplicative growing processes is to consider resetting, namely a random process where the variable is set to a given value with some given protocol \[14\]. The cases of stochastic partial differential equations like the Kardar-Parisi-Zhang equation of fluctuating interfaces has been als considered \[15\].

In the present paper we study a simple chaotic map that couples chaotic diffusion and random amplification. Nonlinear maps are thoroughly investigated as mathematically simple model to analyze the connection between macrolaws and microscopic chaos \[16\]. It can be regarded as a toy model for coupled active chaotic cavities or the lasing networks mentioned above \[11\]. The idea is that light rays can be treated as particles undergoing chaotic diffusion and amplification. Indeed, the classical dynamics of particles on graphs is a chaotic type of diffusive process \[17\]. Trajectories of a particle on a graph, undergoing scattering at its vertices, are in one-to-one correspondence with the ones of one-dimensional piecewise chaotic maps \[17, 18\].

The model is simple enough to allow for a very detailed analysis, demonstrating power-law distributions of the invariant measure. It allows to elucidate in a simple mathematical framework the conditions for observing
Lévy statistical regimes and chaotic intermittency in such systems. It also serves as an example to demonstrate the usefulness of generalized Lyapunov exponents to assess the possibility of power-law fluctuations. Moreover we extend the concepts to the case of open systems (like chaotic repellers), a case that, to our knowledge, has not been studied in this terms.

In Section 2 the map model is presented and its relation with the physical systems is sketched. The stationary invariant measure is computed along with an effective master equation. The connection between power-law and generalized Lyapunov exponents is discussed in Section 3. This relation is extended to the case of open maps in Section 4. The connection between the model and the calculation of the spectrum of the lasing network is given in the Appendix.

2. Map model

We consider the following map

\[
\begin{align*}
  x_{n+1} &= f(x_n) \\
  E_{n+1} &= g(x_n)E_n + s
\end{align*}
\]

where \( f \) is chaotic with a positive Lyapunov exponent \( \lambda_1 \).

For definiteness, let us consider \( x \) to belong to the unit interval and \( E_n, g \) positive and \( s \geq 0 \) and small.

As suggested by the notation, one can imagine \( x_n \) to describe couples of the position of a "ray" undergoing chaotic motion during which it acquires an "energy" that increases or decreases according to whether \( g \) is larger or smaller than one. Thus chaotic diffusion and amplification are coupled since the acquired energy depends on the trajectory. In Fig.1 we sketch a physical reference system inspired from the lasing network experimentally studied in 11-13.

The term \( s \) represents some form of energy injection and is needed to avoid that \( E_n = 0 \) is not and "absorbing" point. The case \( s = 0 \) leads to a non-stationary distribution that for large times is log-normal. This is readily understood as the variable \( \log E_n = z_n \) performs a discrete-time biased random walk 20. The average velocity \( \langle \log g(x_n) \rangle \equiv \lambda_2 \) is the (second) Lyapunov exponent of 1. The situation is drastically different in the case of \( s > 0 \), that act as a source term. If \( \lambda_2 < 0 \) the variable \( z_n \) is attracted towards the source and this yield a stationary measure. In the stochastic case this mechanism of repulsion has been shown to yield power-law decaying stationary distributions 21-23. Similar considerations apply for extended stochastic systems like the non-linear diffusion equation with multiplicative noise 24. On the other hand, for \( \lambda_2 > 0 \) the variable grows and some form of saturating mechanism is needed to ensure unbounded motion (more on this below).

To keep the analysis as simple as possible we consider the case of the piecewise-linear map

\[
f(x) = \begin{cases} 
  \frac{1}{p}x & 0 \leq x \leq p/2 \\
  \frac{1}{1-p}x + \frac{1-2p}{2(1-p)} & p/2 < x \leq 1/2 \\
  \frac{1}{1-p}x - \frac{1}{2(1-p)} & 1/2 < x \leq 1 - p/2 \\
  \frac{1}{2}x + 1 - 1/p & 1 - p/2 < x \leq 1 
\end{cases}
\]

see Fig.1. If we consider the motion of a particles on the graph there drawn, \( f \) can be derived exactly as a suitable Poincaré section as described in 17. The map is everywhere expanding and is invariant for \( x \to 1 - x \), but it is straightforward to generalize to an asymmetric case and/or more complex graphs.

Since the invariant measure of the map is constant its the Lyapunov exponent \( \lambda_1 = -p \log p - (1-p) \log(1-p) \). So \( \lambda_1 > 0 \) but it is vanishingly small for \( p \) approaching 0 and 1 where the map has weakly unstable orbits. Also, let us consider a piece-wise constant gain function \( g(x) \)

\[
g(x_n) = \begin{cases} 
  g & 0 \leq x_n \leq 1/2 \\
  l & 1/2 < x_n < 1 
\end{cases}
\]

where \( g \geq 1 \) and \( 0 < l \leq 1 \). This is merely a choice of simplicity and it entails that the sequence of multipliers \( g(x_n) \) is in one-to-one correspondence with symbolic dynamics of the map \( f \). Even in this simple example, for \( p \neq 1/2 \) the

![Figure 1: Left: the chaotic map with \( p = 0.25 \). Right: a sketch of two coupled chaotic cavities as in a double-ring lasing network connected by a coupler that transmits with a given probability (no reflections). One cavity contains an active amplifying medium, the other is dissipative. In the graph interpretation, the map \( f \) can be derived exactly as a suitable Poincaré section as done in 17.](image)

Let us focus on the case where the solution does not diverge namely \( \langle \log g(x) \rangle < 0 \). Before entering the mathematical analysis, in Fig.2 we report some representative

1Thus the "physical" time \( t_n \) corresponding to the \( n \)th iteration of the map and depends on the length on each bond. For instance, for the graph in Fig.1 with bond lengths \( L_1 \) and \( L_2 \), \( t_{n+1} = t_n + T(x_n) \cdot T(x_n) = T_1 \) or \( T_2 \) for \( x_n < 1/2 \) and \( x_n > 1/2 \) respectively (\( T \) being the particle velocity, \( T_{1,2} = L_{1,2}/v \)). This description can be generalized to arbitrary graphs associated with Markov dynamics, see 17.
time-series of the the variable $E_n$. The dynamics is highly intermittent with large-amplitude spikes lasting tenths of iterates. A finite value of the source term insures that the intermittent with large-amplitude spikes lasting tenths of time.

Figure 2: Time series of $E_n$, $g = 1.2$, $l = 0.8$, $s = 10^{-3}$ for different values of $p$. The trajectory is highly intermittent with large excursion of short duration.

We expect that this equation is valid when chaotic diffusion is sufficiently rapid to ensure homogeneization of the measure on the time scale faster than the typical growth. It has a form of a master equation for probabilities of the variable $E$ on each side of the interval. Also, a standard Kramers-Moyal expansion may be used to show that it corresponds to a set of coupled Langevin equations with multiplicative noise for the energies on the two sides of the interval. The connection between this equation and the one used in the calculation of the spectrum of the las-

3. Generalized Lyapunov exponents

The generalized Lyapunov exponents $L(q)$ define the growth of the $q$th moment of the perturbation $\delta u$ of a dynamical system, which evolves according to the linearized equation of motion, let $R(\tau) = ||\delta u(t+\tau)||/||\delta u(t)||$ be the response function after a time $\tau$ to a disturbance at time $t$. Then, for large times $R(q) \sim \exp(L(q)\tau)$ where the overline denote a time average. If $L(q) > 0$ for large enough $q$ then there is a finite probability that a small perturbation grow very large. Moreover, the deviation of $L(q)$ from a linear behavior in $q$ signal an intermittent dynamics.

The condition for power-law stationary tails can be obtained from generalized Lyapunov exponents. In the present case we are interested in the generalized exponents associated with the $E_n$ variable when no source term is...
present and unbounded growth or decay at large possible. Actually, using equations [13] with \( s = 0 \) write the evolution map for the moments \( \langle E_{q}^{n} \rangle \) and \( L \) as the logarithm of its largest eigenvalue,

\[
L(q) = \log \frac{\langle g^{q} + l^{q} \rangle}{\mu} + \sqrt{\mu^{2}(\langle g^{q} + l^{q} \rangle)^{2} - 4(2p - 1)}
\]

In the simplest case \( p = 1/2 \) multipliers are uncorrelated and one indeed gets the value \( L(q) = \log \left( \frac{g^{q} + l^{q}}{2} \right) \). A moment multiplier is just the arithmetic average of \( g^{q} \). Note that, by construction, the standard Lyapnon vector \( \lambda_{2} = L'(q = 0) = \log(g)/2 \) does not depend on \( p \) while the \( L(q) \) do.

The stability condition implies that the Lyapunov exponent is negative. On the other hand, \( L(q) \approx q \log \mu \) large and positive. So a positive solution for \( L(q) \) exist and coincides with the condition for a power-law given for to [13] for \( q_{c} = \alpha \). In the Gaussian approximation this is seen immediately since in this case

\[
L(q) \approx -|\lambda_{2}|q + \mu q^{2}
\]

where \( \mu \) is the variance of \( \lambda_{2} \). In this approximation \( |\lambda_{2}|/\mu \) that makes transparent the fact that fluctuations in the gain have to be of the same order as \( \lambda_{2} \) to observe large fluctuations. This is in agreement with the general scenario described in [13].

In Fig[13] we report the generalized Lyapunov exponents and the distributions of the \( z_{n} \) variables. The estimated exponents are in very good agreement with the simulations. For instance in the case \( p = 0.6 \) the numerical histogram yield an exponent 0.66 to be compared with the value \( q_{c} = 0.71 \).

When the condition of stability is violated, some further mechanism of saturation is needed to have a steady distribution. In this case the term \( s \) can be ignored. For instance one may consider a nonlinear term in the form of a chaotically-driven logistic map

\[
\begin{align*}
x_{n+1} &= f(x_{n}) \\
E_{n+1} &= g(x_{n})E_{n} - E_{n}^{2}
\end{align*}
\]

This a particular case of the systems thoroughly studied in [20, 28] and will not thus be considered any further here. In presence of the nonlinear term, \( \lambda_{2} \) remains negative and the invariant measure remains broad but has an exponential cutoff at large \( E_{n} \). Detailed predictions on the nature of the chaotic intermittency close to the transition can be given [20, 28, 29], including universality of power spectra and Lyapunov exponents. In the next section, we consider an alternative possibility for obtaining steady-state power laws even in the unstable case.

Figure 4: The generalized Lyapunov exponents \( L(q) \) for different values of the map parameter \( p \) and the distributions of \( \log E_{n} \). An exponential tail of \( P(z) \sim \exp(-az) \) correspond to a power-law decay \( E^{-1-\alpha} \).

4. Open maps

We now discuss another possibility to have steady fluctuations with power-law tails namely an open setup where the trajectory are allowed to escape (and be re-injected). The idea is that the distribution of the values of \( E_{n} \) is in
this case determined by the combined effect of the fluctuations of growth rates (as measured by finite-time Lyapunov exponents) and the statistical distribution of the escape events.

Let us consider the growth of a perturbation over a finite time $\tau$, $E \propto \exp(\lambda(\tau)\tau)$ where $\lambda(\tau)$ is the finite-time Lyapunov exponent \cite{7}. Defining $z = \log E$ its distribution $\mathcal{Q}(z)$ is given by

$$\mathcal{Q}(z) = \int \int d\lambda d\tau \delta(z - \lambda(\tau)\tau) \mathcal{P}(\tau) \mathcal{P}(\lambda, \tau)$$

basically an average of the growth rates on the distribution of escape times $\mathcal{P}(\tau)$. As usual, for large $\tau$ we introduce the large-deviation function of the form

$$\mathcal{P}(\lambda, \tau) \sim \exp(-U(\lambda)\tau)$$

In most cases, the distribution of escape times is Poissonian $\mathcal{P}(\tau) = r \exp(-r\tau)$ where $r$ is the escape rate. Substituting this expression in equation (9), the resulting integral can be evaluated using the saddle-point approximation: if we denote by $\lambda_n$ the saddle point, one obtains the condition $\lambda_n U'(\lambda_n) - U(\lambda_n) = r$. Then, recalling that the generalized exponents are the Legendre transform of the large-deviation function $L(q) = q\lambda - U(\lambda)$ with $q = U'(\lambda)$ one obtains that the asymptotic decay of the distribution

$$\mathcal{Q}(z) \sim \exp(-q_z z); \quad L(q_z) = r$$

Changing back to the original variable $E$ one obtains again a power-law tail, $E^{-1-q_z}$ for large $E$. This last expression generalizes the one given above and confirms that also in the open setup the generalized exponents can be used to estimate the power-law decay.

A consequence of the above is that, in the open case we can also consider the unstable case $\lambda_2 > 0$ and expect stationary fat-tailed distributions. Indeed, the equation \cite{12} has a solution $q_2$ relative close to zero. In the Gaussian approximation $L(q) = \lambda_2 q + \mu q^2$ and for small $r$ one has $q_2 \approx r/\lambda_2$. This nicely fits with the estimate given in \cite{6} for observing Lévy fluctuations in amplifying diffusive media with absorbing boundaries, upon identifying $1/r$ with the average residence time in the medium and $\lambda_2$ with the typical amplification time.

To verify the above argument we consider first the simpler case of the map \cite{19} (with $s = 0$) undergoing a stochastic resetting dynamics. With some preassigned small probability $r$ (which represents the escape rate) the variable $E_n$ is reset to some arbitrary value (with no modification on the $x_n$ dynamics).

The second case is a deterministic type of resetting, where the $x_n$ dynamics is given by the map, see Fig.5:

$$f(x) = \begin{cases} 
2x & 0 \leq x \leq 1/4 \\
(a(x - 1/4) + \frac{1}{3}) & 1/4 < x \leq 1/2 \\
(a(x - 3/4) + \frac{1}{3}) & 3/4 < x \leq 1 \\
2x - 1 & 3/4 < x \leq 1 
\end{cases}$$

The graph is in Fig.5 along with a sketch of the physical situation where particles are allowed to escape from the graph. For $a > 2$ there is escaping region in the interval $(1/4 - 1/2a, 3/4 - 1/2a)$. Thus for $a \to 2^+$ the escape rate from the associated chaotic repellor is $r \approx (a - 2)/4$. Whenever the particle trajectory escapes from the unit interval, we reset $E_n$ to one and $x_n$ to a random uniformly-

Figure 5: Left: the chaotic map \cite{19} for $a > 2$. Right: a sketch of the physical realization the open system in the case of two chaotic cavities connected by a coupler with leaks that allow escape of rays with some probability.

In Fig.6 we considered both examples in the unstable and unstable regimes. The distributions are clearly different from the one of the closed map given in Fig.4. As expected, the deterministic and stochastic case are similar. The generalized Lyapunov exponents as given by formula \cite{11} are also reported. In both cases, the distributions have double-exponential shape with rates in excellent agreement with the one given by \cite{11}, represented graphically in the leftmost panels of Fig.6. In the deterministic case with $a$ not too close from 2, the escape rate has been estimated numerically.

We conclude with a remark on the finer-scale structure of the distribution. A feature of the model is that the $z_n$ variable occurs almost in discrete values. Fig.7 shows that the distribution has a finer structure with narrow peaks almost equally-spaced (see inset of Fig.7). That should be contrasted with the case of the closed map where $z_n$ has continuous values and a smooth distribution.

5. Conclusions

Motivated by recent experiments of lasing networks \cite{11, 12}, we have introduced a toy map model describing the effect of chaotic diffusion and amplification on a graph structure. Since the motion is purely classical, it should apply when the wavelength is small with respect of the bond lengths. Evidence of large fluctuations and intermittency for the lasing network has been indeed been provided both experimentally and by Monte-Carlo simulation \cite{11}.
Chaotic diffusion and amplification yield multiplicative fluctuations and power-law steady distributions. In some regimes the variance can diverge leading to Lévy-like statistics. We have confirmed that the Generalized Lyapunov exponents can give a precious hint on the statistics, both in the stable and unstable cases. We have extended this concept to open systems through equation (9) that connects the Generalized exponents with the escape rate. This result should apply under quite general conditions, as demonstrated by the case of random resetting dynamics.

Acknowledgements
I acknowledge Stefano Gelli for contributing to the initial stage of this work.

Appendix
In [11] the steady state modes of the lasing networks have been computed using an approach extending the one used for quantum graphs [30]. This is accomplished imposing that a suitable network matrix $N = SP$ has an eigenvalue equal to one. Physically, $S$ is the scattering matrix of the optical couplers (splitters) and $P$ is the so-called propagation matrix along the optical fibers and contains both the metric information on the bond length than the gain coefficients [10].
To clarify the connection with the map model studied here, let us first consider expressing equation (4) in the "physical" time $t$ (neglecting the term $s$)

$$P_1(E, t) = \frac{p}{g} P_1 \left( \frac{E}{g}, t - T_1 \right) + 1 - \frac{p}{l} P_2 \left( \frac{E}{l}, t - T_2 \right)$$

$$P_2(E, t) = \frac{(1 - p)}{g} P_1 \left( \frac{E}{g}, t - T_1 \right) + \frac{p}{l} P_2 \left( \frac{E}{l}, t - T_2 \right)$$

where $T_{1,2} = L_{1,2} / v$ are the travel times (see the footnote in the main text). Taking the Laplace transform in $t$ and introducing the averages

$$h(E, z) = \int_0^\infty h(E, t) e^{-z t} dt, \quad I_{1,2} \equiv \int_0^\infty E P_{1,2}(E, z) dE$$

one obtains the condition

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = W G \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},$$

where

$$W \equiv \begin{pmatrix} p & 1 - p \\ 1 - p & p \end{pmatrix}, \quad G \equiv \begin{pmatrix} g e^{z T_1} & 0 \\ 0 & g e^{-z T_2} \end{pmatrix}$$

and $W$ is recognized to be the stochastic matrix for a random walk on a graph with two states. To have non-trivial solutions we impose det($WG - 1$) = 0 that determines all possible values of $z$. This equation is obtained by the condition that $N$ has a eigenvalue one, by taking $|N|^2$, i.e. the matrix whose elements are the square moduli of it. The stochastic matrix of the graph is thus the square modulus on the scattering matrix $S$ of the coupler $W = |S|^2$ while $G = |P|^2$. Viewed in this way one can recognize the similarity with the "quantization" procedure outlined in [11, 18] where the classical stochastic transition matrix is replaced by an unitary one describing a quantum map. The generalization of the above to arbitrary graphs is straightforward.

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