Mirror Symmetry for Two Parameter Models – II

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ABSTRACT

We describe in detail the space of the two Kähler parameters of the Calabi–Yau manifold \(\mathbb{P}_{4}^{(1,1,1,6,9)}\)\textsuperscript{[18]} by exploiting mirror symmetry. The large complex structure limit of the mirror, which corresponds to the classical large radius limit, is found by studying the monodromy of the periods about the discriminant locus, the boundary of the moduli space corresponding to singular Calabi–Yau manifolds. A symplectic basis of periods is found and the action of the \(\text{Sp}(6,\mathbb{Z})\) generators of the modular group is determined. From the mirror map we compute the instanton expansion of the Yukawa couplings and the generalized \(N = 2\) index, arriving at the numbers of instantons of genus zero and genus one of each degree. We also investigate an \(\text{SL}(2,\mathbb{Z})\) symmetry that acts on a boundary of the moduli space.

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1. Introduction

The Kähler and complex structure parameters that describe the possible deformations of a Calabi-Yau manifold determine the dynamics of the corresponding string compactification. The kinetic energy terms and Yukawa couplings in the low-energy effective theory, for example, depend on these moduli. Quantities that depend on the Kähler parameters are subject to non-perturbative instanton corrections that, in virtue of mirror symmetry, may be found by a straightforward calculation relating to the complex structure sector of the mirror space. In fact, mirror symmetry leads to a complete geometrical description of the full moduli space of Calabi-Yau manifolds. In this article we continue the study, initiated in a companion paper [1], of the mirror map for the moduli spaces of two-parameter Calabi–Yau manifolds. Our work is a further step towards establishing general results beyond the one-parameter case considered in Refs. [2-7].

Our aim here is to apply the methods developed in [1] to describe the Kähler moduli of the Calabi–Yau manifold $\mathbb{P}^{(1,1,1,6,9)}$ [18]. We first study the complex structure space of its mirror to determine its singularities and characterize the point corresponding to the large complex structure limit. In order to find the explicit mirror map between the flat Kähler coordinates and the complex structure parameters, we obtain and study a basis of periods of the mirror manifold. This basis is constructed from the fundamental period found by direct integration of the holomorphic 3-form. The periods are multivalued functions over the moduli space and this implies the existence of a modular or duality group whose generators correspond to the monodromy of the period basis about the various singularities in moduli space. Using methods similar to [8], we calculate the full duality group for this example. The periods also satisfy certain differential equations that we use to rederive the monodromy. The monodromy properties permit us to identify the large complex structure limit which is the mirror of the large radius limit of the original manifold. With this information we are then able to choose a basis of flat coordinates in which the duality generators belong to $\text{Sp}(6, \mathbb{Z})$. The relation between the flat coordinates and the periods defines the mirror map that we use to derive instanton expansions of the Yukawa couplings and the $\mathcal{N} = 2$ index of Bershadsky et al. [9]. It is interesting that some of the expansion coefficients, which correspond to numbers of rational and elliptic curves, are negative and we explain this fact as a consequence of the existence of continuous families of instantons. A number of two and three parameter examples, including the present one, have been discussed recently from a somewhat different perspective in Ref. [10].

This article closely parallels Ref. [1]. The layout is the following. We study in §2 the geometry of Calabi–Yau hypersurfaces in $\mathbb{P}^{(1,1,1,6,9)}$. The embedding space has an orbifold singularity along a surface which intersects the Calabi–Yau hypersurface along a certain curve $C$. The resolution of the singular curve introduces a surface; every point of $C$ having
been blown up into an instanton. We obtain in this way a continuous family of instantons. We find a homology basis and compute certain invariants that are needed later in relation to the instanton expansions. In §3 we describe the moduli space of the mirror manifold and find the loci where the Calabi–Yau manifold is singular. These loci constitute the boundary of the moduli space. We also use the methods of toric geometry to discuss the compactification of moduli space and locate the point corresponding to the large complex structure limit. In §4 we obtain a basis of periods and derive the analytic properties that are needed in §5 to determine the monodromies under transport around the singularities. In this chapter we also compute monodromy from the Picard-Fuchs equations. In §6 we find the flat coordinates and the explicit mirror map that is used in §7 to compute the instanton expansions for the Yukawa couplings and the $N = 2$ index. We also investigate an $\text{SL}(2,\mathbb{Z})$ symmetry of a boundary component of the moduli space. Finally, in §8 we verify and explain the significance of some of the instanton numbers that we have found.
2. Geometry of CY hypersurfaces in $\mathbb{P}^{(1,1,1,6,9)}$

2.1. Linear systems

We consider Calabi-Yau threefolds $\mathcal{M}$ which are obtained by resolving singularities of degree 18 hypersurfaces $\hat{\mathcal{M}} \subset \mathbb{P}^{(1,1,1,6,9)}$. A typical defining polynomial for such a hypersurface is

$$p = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2.$$ 

The singularities occur along $x_1 = x_2 = x_3 = 0$, which is a curve in $\mathbb{P}^{(1,1,1,6,9)}$ that meets the $p = 0$ hypersurface in the single point $[0, 0, 0, -1, 1]$. Notice that $\mathbb{P}^{(1,1,1,6,9)}$ has quotient singularities by a group of order 3 along this curve. For example, the chart with $x_4 = 1$ is described as points $[x_1, x_2, x_3, 1, x_5]$ modulo the equivalences

$$[x_1, x_2, x_3, 1, x_5] \sim [\lambda x_1, \lambda x_2, \lambda x_3, 1, \lambda^9 x_5],$$

where $\lambda^6 = 1$. The subgroup of the group of equivalences which fixes the curve $x_1 = x_2 = x_3 = 0$ is the subgroup generated by $\lambda^2$, and this gives rise to quotient singularities by a group of order 3. Notice that the polynomial $x_1 x_2 x_3$ is invariant under this subgroup, and describes a divisor which vanishes simply along the curve once the quotient has been taken. If this curve is blown up on $\mathbb{P}^{(1,1,1,6,9)}$ (which will blow up the corresponding point on the threefold), the singularities of the threefold are resolved. A single exceptional divisor $E$ is created during this process.

We now study linear systems on $\mathcal{M}$. We first consider the linear system $|L|$ generated by the polynomials $x_1, x_2,$ and $x_3$ of degree 1. This linear system maps $\mathcal{M}$ to $\mathbb{P}^2$, with the inverse image of a point in $\mathbb{P}^2$ being an elliptic curve

$$x_4^3 + x_5^2 = \text{constant}.$$ 

Consequently, $L^3 = 0$.

The second linear system we study, denoted by $|H|$, is generated by all polynomials of degree 3. One of those polynomials is $x_1 x_2 x_3$, and as we have already noticed, that polynomial vanishes simply at the singular point of $\hat{\mathcal{M}}$. It follows that on $\mathcal{M}$, the divisor $3L + E$ belongs to our linear system $|H|$. (The coefficient “1” of $E$ in that expression corresponds to the “simple” vanishing of the polynomial.) Notice that polynomials of degree 6, 9, ... will give divisors in the linear systems $|2H|$, $|3H|$, ....

Intersection products are now computed as follows. We work in the affine chart $x_1 = 1$ (which is a smooth chart on $\mathbb{P}^{(1,1,1,6,9)}$). If we set $x_2$, $x_3$, and $x_4$ to constant values, we describe an intersection of three divisors in the linear systems $|L|$, $|L|$, and
Figure 2.1: The space $\mathbb{P}_4^{(1,1,1,6,9)[18]}$ contains a curve $C$ which is fixed by a $\mathbb{Z}_2$ action. In the resolved space the singular curve is replaced by a surface $E = H - 3L$ that is ruled by instantons.

$|2H|$, respectively. The number of points in the intersection is the number of solutions to $x_5^2 = \text{constant}$, that is, 2. We conclude that

$$L \cdot L \cdot (2H) = 2.$$  

A similar argument with $x_2$, $x_4$ and $x_5$ produces

$$L \cdot (2H) \cdot (3H) = 18.$$  

Finally, if we set $x_4$ and $x_5$ to constant values, and also impose a general cubic equation on $x_2$ and $x_3$, we will calculate the intersection of $H$, $2H$, and $3H$. The intersection number is the number of common solutions to $x_2^{18} + x_3^{18} = \text{constant}$ and a general (inhomogeneous) cubic in $x_2$, $x_3$; by Bezout’s theorem, there are $3 \cdot 18$ solutions. We conclude

$$H \cdot (2H) \cdot (3H) = 54.$$  

To summarize, the intersection numbers are:

$$H^3 = 9, \quad H^2 \cdot L = 3, \quad H \cdot L^2 = 1, \quad L^3 = 0.$$  

4
Let \( l \) be the intersection of \( L \) and \( E \), and let \( h \) be the intersection of 2 divisors from \(|L|\) (i.e., one of the elliptic curves mentioned above). Using the equivalence \( l = L \cdot E = L \cdot H - 3L^2 \), we easily calculate the following intersection numbers,

\[
\begin{align*}
L \cdot l &= 1 \\
L \cdot h &= 0 \\
H \cdot l &= 0 \\
H \cdot h &= 1 .
\end{align*}
\]

This leads to an identification of the Kähler cone as being the cone generated by \( L \) and \( H \).

\[\text{2.2. Chern classes}\]

We presently compute some intersections that will be of use later in relation to the instantons of genus one. For a smooth divisor \( D \subset \mathcal{M} \), we use the notation \( c_2(D) \) to note the second Chern class of the surface \( D \). The notation \( c_2 \) is reserved for the second Chern class of \( \mathcal{M} \). For the surface \( L \) we have \( c_2(L) = 36 \) whereas for the exceptional divisor \( E \), we have \( c_2(E) = 3 \).

The desired results can be obtained by repeated application of the formula

\[
c_2 \cdot D = c_2(D) - D^3 .
\]

In particular,

\[
c_2 \cdot L = c_2(L) - L^3 = 36 .
\]

Furthermore, we have

\[
c_2 \cdot E = c_2(E) - E^3 = 3 - (H - 3L)^3 = -6 ,
\]

and hence

\[
c_2 \cdot H = c_2 \cdot (3L + E) = 3(36) - 6 = 102 .
\]
3. The Moduli Space of the Mirror

3.1. Basic facts

The most concrete approach to the moduli space of the mirror of $\mathbb{P}_4^{(1,1,1,6,9)}[18]$ begins with the orbifolding construction of [11], according to which the mirror of $\mathbb{P}_4^{(1,1,1,6,9)}[18]$ may be identified with the family of Calabi-Yau threefolds of the form $\{p = 0\}/G$, where $G \cong \mathbb{Z}_6 \times \mathbb{Z}_{18}$ is the group with generators

$$
(\mathbb{Z}_6; 0, 1, 3, 2, 0),
$$
$$
(\mathbb{Z}_{18}; 1, -1, 0, 0, 0),
$$

and $p$ is a $G$-invariant quasi-homogeneous polynomial of weighted degree 18. The most general possible form of $p$ is

$$
p = a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^2 x_2^2 x_3^2 + a_2 x_1^3 x_2^3 x_3 + a_3 x_1^4 x_2^4 x_3^2 + a_4 x_1^6 x_2^6 x_3^6 + a_5 x_1^{18} + a_6 x_2^{18} + a_7 x_3^{18} + a_8 x_4^3 + a_9 x_5^2. \tag{3.1}
$$

Multiplying $p$ by a nonzero scalar does not affect the hypersurface $\{p = 0\}$, so we should regard the parameter space of such hypersurfaces as forming a $\mathbb{P}_9$ with homogeneous coordinates $[a_0, \ldots, a_9]$.

The action of the automorphism group of $\mathbb{P}_4^{(1,1,1,6,9)}[18]/G$ establishes isomorphisms among hypersurfaces defined by different equations. This automorphism group includes the scaling symmetries $x_j \mapsto \lambda_j x_j$, which induce an action on the coefficients of $p$ of the form $a_k \mapsto \lambda^{m(k)} a_k$ for appropriate multi-indices $m(k)$. By using these scaling symmetries, five of the coefficients of $p$ may be set to 1.

In the examples studied in [1], these scaling symmetries accounted for the full automorphism group of the ambient space. This is not the case in the present example, however. There are additional automorphisms of $\mathbb{P}_4^{(1,1,1,6,9)}[18]/G$ which arise from the possibility of modifying the homogeneous coordinates of high weight by addition of nonlinear terms involving the homogeneous coordinate of low weight, in a $G$-invariant manner. The most general automorphism of this type takes the form

$$
x_1 \rightarrow x_1
$$
$$
x_2 \rightarrow x_2
$$
$$
x_3 \rightarrow x_3
$$
$$
x_4 \rightarrow x_4 + a (x_1 x_2 x_3)^2
$$
$$
x_5 \rightarrow x_5 + b (x_1 x_2 x_3) x_4 + c (x_1 x_2 x_3)^3. \tag{3.2}
$$
Such automorphisms should be used to set certain coefficients of \( p \) to zero. The problem arises of how to select the monomials in \( p \) which are to be retained with nonzero coefficients. A very general procedure for doing this was proposed in [12]; in the example at hand, that procedure instructs us to attempt to make a transformation of the form (3.2) which sets \( a_1, a_2, \) and \( a_3 \) to zero. When \( a_8a_9 \neq 0 \), this can be done using (3.2) with coefficients

\[
\begin{align*}
  a &= -\frac{\xi^2 - 4a_1a_9 + a_0^2}{12a_8a_9} \\
  b &= \frac{\xi - a_0}{2a_9} \\
  c &= \frac{a_0\xi^2 - 12a_2a_8a_9 + 4a_0a_1a_9 - a_0^3}{24a_8a_9^2},
\end{align*}
\]

where

\[
\xi = \sqrt[4]{-48a_3a_8a_9^2 + 16a_7^2a_9^2 + a_0^4 + 24a_0a_2a_8a_9 - 8a_0^2a_1a_9}.
\]

The fact that these transformations can be found verifies, for this particular example, the “dominance property” discussed in [12].

Applying (3.2) with coefficients (3.3), and then using scaling symmetries, we may reduce (3.1) to the form

\[
p = x_1^{18} + x_2^{18} + x_3^{18} + x_4^{2} + x_5^{2} - 18\psi x_1 x_2 x_3 x_4 x_5 - 3\phi x_1^6 x_2^6 x_3^6
\]

(3.4)

provided that \( a_5a_6a_7a_8a_9 \neq 0 \) after applying (3.2)). We have introduced factors of \(-18\) and \(-3\) into (3.4) for later convenience; in addition, to simplify some later formulas, we sometimes replace \( \psi \) by

\[
\rho \overset{\text{def}}{=} (342)^{1/3} \psi.
\]

The natural parameter space for polynomials of the form (3.4) would appear at first sight to be the \( \mathbb{C}^2 \) with coordinates \((\rho, \psi)\); however, the transformations used to bring (3.1) to the form (3.4) were not unique, and this non-uniqueness must now be accounted for. First, there are scaling symmetries which preserve the form of (3.4). These will define an enlargement \( \hat{G} \) of the group \( G \) consisting of elements \( g = (\alpha^{a_1}, \alpha^{a_2}, \alpha^{a_3}, \alpha^{6a_4}, \alpha^{9a_5}) \) acting as:

\[
(x_1, x_2, x_3, x_4, x_5; \psi, \phi) \mapsto (\alpha^{a_1}x_1, \alpha^{a_2}x_2, \alpha^{a_3}x_3, \alpha^{6a_4}x_4, \alpha^{9a_5}x_5; \alpha^{-a_1}\psi, \alpha^{-6a_4}\phi),
\]

where \( a = a_1 + a_2 + a_3 + 6a_4 + 9a_5 \), where \( \alpha^{a_i}, i = 1, 2, 3 \), are 18th roots of unity, \( \alpha^{6a_4} \) is a 3rd root of unity, and \( \alpha^{9a_5} \) is a 2nd root of unity\(^1\). This group acts on the family of

\(^1\) We do not require that the product of these roots of unity be 1, since we have ‘corrected’ the equation by an appropriate action on the coefficients.
weighted projective hypersurfaces \( \{ p = 0 \} \), and induces an action on the parameter space \( \{(\rho, \phi)\} \) by a \( \mathbb{Z}_{18} \) whose generator \( A \) acts by

\[
A : (\rho, \phi) \mapsto (\alpha \rho, \alpha^6 \phi)
\]

where \( \alpha = e^{2\pi i/18} \).

Second, there are transformations (3.2) which preserve the form of (3.4). These transformations are generated by a transformation \( \mathcal{I} \) whose coefficients are \( a = 54\psi^2, b = 9(1 - i)\psi \) and \( c = 486\psi \). The induced action on the parameter space is

\[
\mathcal{I} : (\rho, \phi) \mapsto (i\rho, \phi + \rho^6).
\]  

(3.5)

It is often convenient to use \( \phi + \frac{1}{2}\rho^6 \) as a coordinate in place of \( \phi \). For the action of \( \mathcal{I} \) in such coordinates is simply

\[
\mathcal{I} : (\rho, \phi + \frac{1}{2}\rho^6) \mapsto (i\rho, \phi + \frac{1}{2}\rho^6).
\]

It is important to observe that the automorphism \( A^9\mathcal{I}^2 \) acts trivially on the parameter space. The corresponding transformation of the \( x_j \)'s is

\[
(x_1, x_2, x_3, x_4, x_5) \mapsto (-x_1, x_2, x_3, x_4, x_5 - 18\psi x_1 x_2 x_3 x_4).
\]

This is an \( R \)-symmetry, which acts as \( -1 \) on the holomorphic 3-form of each Calabi–Yau in the family. Now it is easy to see that the subgroup of the automorphism group generated by \( A^2 \) and \( \mathcal{I} \) contains all of the actual symmetries of the parameter space, and does not contain the \( R \)-symmetry \( A^9\mathcal{I}^2 \). So to construct the moduli space, it would suffice to consider the quotient by only the automorphisms from that subgroup.

The moduli space can be described explicitly in terms of invariant functions for the actions of \( A \) and \( \mathcal{I} \) on the original parameter space \( \mathcal{C}^2 = \{(\rho, \phi)\} \). First consider the action of

\[
A^3 : (\rho, \phi) \mapsto (\alpha^3 \rho, \phi).
\]

The invariant functions under this transformation are generated by \( \rho^6 \) and \( \phi \), and the quotient of the original \( \mathcal{C}^2 \) by the \( \mathbb{Z}_6 \) generated by \( A^3 \) is again a smooth \( \mathcal{C}^2 \).

The action of the transformation \( A \) on the invariant functions \( \rho^6 \) and \( \phi \) is via

\[
A : (\rho^6, \phi) \mapsto (\alpha^6 \rho^6, \alpha^6 \phi),
\]

and this generates a \( \mathbb{Z}_3 \). The quotient space, which we call the “simplified moduli space” using the terminology of [12], has a singularity at the origin. To obtain the actual moduli space, we would need to quotient this “simplified” space by the automorphism \( \mathcal{I} \).

We wish to compactify the moduli space in order to study the monodromy around boundary divisors. We will work primarily with compactifications of the simplified moduli
space, since these can be analyzed using the methods of toric geometry. Compactifications of the actual moduli space can be obtained by taking the quotient by \( \mathcal{I} \) of our “simplified” compactifications.

An initial compactification of the simplified moduli space can be made once we recognize that the \( \mathbb{Z}_3 \)-quotient singularity in this space is precisely the same as the one in the weighted projective plane \( \mathbb{P}^2_{(3,1,1)} \). A compactification can then be made by associating to \((\rho^6, \phi)\) the point in \( \mathbb{P}^2_{(3,1,1)} \) with homogeneous coordinates \([1, \rho^6, \phi]\). Other representatives of the same point (when \( \rho \) or \( \phi \) is not zero) are \([\rho^{-18}, 1, \rho^{-6} \phi]\) and \([\phi^{-3}, \rho^6 \phi^{-1}, 1]\). We let \([x, y, z]\) denote the general point, in homogeneous coordinates.

The curves in \( \mathbb{P}^2_{(3,1,1)} \) which represent singular Calabi-Yau spaces (and so constitute the boundary of the simplified moduli space) are as follows:

1. \( C_{\text{con}} \)—a locus on which the Calabi-Yau acquires a conifold point—described in affine coordinates as
   \[
   C_{\text{con}} = \{(\rho, \phi) \mid (\rho^6 + \phi)^3 = 1\}
   \]
   or in projective coordinates by its homogeneous equation \((y + z)^3 = x\);

2. \( B_{\text{con}} \)—another locus on which the Calabi-Yau acquires a conifold point—described in affine coordinates as
   \[
   B_{\text{con}} = \{(\rho, \phi) \mid \phi^3 = 1\}
   \]
   or in projective coordinates by its homogeneous equation \(z^3 = x\). The loci \( B_{\text{con}} \) and \( C_{\text{con}} \) are interchanged under the action of \( \mathcal{I} \).

3. \( D_\infty \)—the boundary, where \((\rho, \phi) \to \infty\), of the original \((\rho, \phi)\) space—defined by the homogeneous equation \(x = 0\); and

4. \( D_0 \) (the fixed point set of \( \mathcal{A}^3 \))—the Calabi-Yau spaces corresponding to which will acquire additional singularities during the quotient by the enlarged group \( \hat{G} \)—defined by the affine equation \( \rho = 0 \) or the homogeneous equation \( y = 0 \).

These meet in the following points:

- \([1, 0, 1]\), the common point of intersection of \( D_0, B_{\text{con}}, \) and \( C_{\text{con}} \),
- \( P_+ = [1, \alpha^6 - 1, 1] \) and \( P_- = [1, \alpha^{-6} - 1, 1] \), the two points of intersection of \( C_{\text{con}} \) and \( B_{\text{con}} \) through which \( D_0 \) does not pass,
- \([0, -1, 1]\), the point of triple tangency between \( C_{\text{con}} \) and \( D_\infty \),
- \([0, 1, 0]\), the point of triple tangency between \( B_{\text{con}} \) and \( D_\infty \), and
- \([0, 0, 1]\), the point of intersection of \( D_\infty \) and \( D_0 \).

Also of interest is the point \( P_0 = [1, 0, 0] \), which is the singular point of \( \mathbb{P}^2_{(3,1,1)} \).
We sketch the curves showing their intersections in Fig. 3.1. Note that the points of triple tangency are depicted as simple tangencies in the diagram.

3.2. Considerations of toric geometry

To describe $\mathbb{P}_2^{(3,1,1)}$ as a toric variety, we consider first the smooth affine chart whose points are represented by homogeneous coordinates $[\phi^{-3}, \rho^6\phi^{-1}]$. The functions $\phi^{-3}$ and $\rho^6\phi^{-1}$ furnish coordinates in this chart, and among all rational monomials $(\phi^{-3})^a(\rho^6\phi^{-1})^b$, the ones which are holomorphic in this first chart satisfy $a \geq 0, b \geq 0$. In the other two charts, the corresponding conditions are $a \geq 0, -3a - b \geq 0$ (for the other smooth chart, with coordinates $\rho^{-18}$ and $\rho^{-6}\phi$), and $b \geq 0, -3a - b \geq 0$ (for the singular chart described in terms of $\rho^6$ and $\phi$). The resulting toric diagram is shown in Fig. 3.2. Note that the vectors $(1, 0)$ and $(0, 1)$ in the toric diagram correspond to the divisors $D_\infty$ and $D_0$, respectively. (The divisor corresponding to the vector $(-3, -1)$ does not lie on the boundary of the moduli space.)

An alternative compactification—the compactification of the simplified moduli space described by the “secondary fan” [13,14]—is well-adapted for rapidly locating the large
complex structure limit point(s) (using the “monomial-divisor mirror map”—cf. [15,12].) First, the large radius limit points of the mirror moduli space are located, and then mirror symmetry is used to identify the corresponding large complex structure limit points. The computation proceeds as follows.

We give a toric description of the desingularization of $\mathbb{P}^{(1,1,1,6,9)}$ (using the embedding

$$(\tau_1, \tau_2, \tau_3, \tau_4) \mapsto [1, \tau_1, \tau_2, \tau_3, \tau_4]$$

of the torus $T = (\mathbb{C}^*)^4$ into $\mathbb{P}^{(1,1,1,6,9)}$) by means of a fan in $N_\mathbb{R} = \text{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$. The one-dimensional cones in this fan are spanned by the vectors

\[
\begin{align*}
v_1 &= (-1, -1, -6, -9) \\
v_2 &= (1, 0, 0, 0) \\
v_3 &= (0, 1, 0, 0) \\
v_4 &= (0, 0, 1, 0) \\
v_5 &= (0, 0, 0, 1) \\
v_6 &= (0, 0, -2, -3)
\end{align*}
\]

which are ordered so that, under the identification of edges in the fan with the “toric” divisors in the toric variety, the first five vectors $v_i$, $i = 1, \ldots, 5$ correspond to the proper transforms of $x_i = 0$ and the last vector $v_6$ corresponds to the exceptional divisor. (Note that $v_6$ is the average of $v_1$, $v_2$, $v_3$; this corresponds to the fact that $x_1 = x_2 = x_3 = 0$ has been blown up.) The ‘big’, i.e. top dimensional, cones which describe the blown-up

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.2.png}
\caption{Toric diagram for $\mathbb{P}^{(3,1,1)}$.}
\end{figure}
\( \mathbb{P}^{(1,1,6,9)} \) are:

\[
\begin{align*}
\text{span}\{v_1, \ldots, \tilde{v}_i, \ldots, v_5\} & \quad \text{for } i = 1, 2, 3 \\
\text{span}\{v_1, \ldots, \tilde{v}_i, \ldots, \tilde{v}_j, \ldots, v_5\} & \quad \text{for } i = 1, 2, 3 \text{ and } j = 4, 5 .
\end{align*}
\]

To compute the secondary fan, we need to find a basis for the set of all relations 
\( a_1 v_1 + \cdots + a_6 v_6 = 0 \). A convenient basis is furnished by the rows of

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & -3 \\
0 & 0 & 0 & -2 & -3 & -1
\end{pmatrix}
\]

The one-dimensional cones in the secondary fan are then spanned by the columns of the matrix (3.6), together with an additional column \( \begin{pmatrix} 0 \\ -6 \end{pmatrix} \) whose entries take the form \( -\sum a_j \) for each relation \( \sum a_j v_j \). (We may regard this additional column as corresponding to the zero-vector \( v_0 = (0, 0, 0, 0) \).) We may as well take the secondary fan to be spanned by the vectors

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};
\]

this is illustrated in Fig. 3.3.

**Figure 3.3:** The secondary fan.

We see that the divisor \( L \) corresponds to \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), the divisor \( E \) to \( \begin{pmatrix} -3 \\ -1 \end{pmatrix} \), and so the divisor \( H \) to \( \begin{pmatrix} 0 \\ -1 \end{pmatrix} \). It follows that the Kähler cone corresponds to the fourth quadrant.

This same secondary fan now determines a compactification of the (simplified) complex structure moduli space of the mirror. Under the monomial-divisor mirror map \([15, 12]\), the
divisor determined by the vector \( v_j \) corresponds to the monomial with coefficient \( c_j \) in the general polynomial

\[
c_1 x_1^{18} + c_2 x_2^{18} + c_3 x_3^{18} + c_4 x_4^3 + c_5 x_5^2 + c_6 x_1^6 x_2^6 x_3 + c_0 x_1 x_2 x_3 x_4 x_5.
\]

On the one hand, using the torus action to get this polynomial into the form (3.4):

\[
x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 + c_0 c_1^{-1/18} c_2^{-1/18} c_3^{-1/18} c_4^{-1/3} c_5^{-1/2} x_1 x_2 x_3 x_4 x_5
\]

we see that

\[
-18 \psi = c_0 c_1^{-1/18} c_2^{-1/18} c_3^{-1/18} c_4^{-1/3} c_5^{-1/2},
\]

\[
-3 \phi = c_1^{-1/3} c_2^{-1/3} c_3^{-1/3} c_6.
\]

On the other hand, the vectors \((1,0)\) and \((0,-1)\) are the edges of the fourth quadrant in the diagram, which is the mirror of the Kähler cone and hence should correspond to a large complex structure limit point. (We will verify later that this point satisfies the appropriate monodromy conditions.) To find the coordinates near that point, we note that \((1,0)\) and \((0,-2)\) correspond to the monomials \(x_1^{18}\) and \(x_4^3\), respectively, and we use the torus action to put the polynomial into the form:

\[
c_1 c_2 c_3 c_6^{-3} x_1^{18} + x_2^{18} + x_3^{18} + c_0^{-3} c_4 c_5^{3/2} c_6^{1/2} x_4^{3} + x_5^2 + x_1^6 x_2^6 x_3 + x_1 x_2 x_3 x_4 x_5.
\]

The coordinates near the large complex structure limit point are then given by

\[
(c_1 c_2 c_3 c_6^{-3}, [c_0^{-3} c_4 c_5^{3/2} c_6^{1/2}]) = ((-3 \phi)^{-3}, (-18 \psi)^{-6} (-3 \phi))
\]

\[
= (-3^{-3} \phi^{-3}, -2^{-4} 3^{-3} \rho^{-6} \phi).
\]

To relate these coordinates to the toric description of \( \mathbb{P}^{(3,1,1)}_2 \) given above, we use the torus action one final time to put the polynomial into the form

\[
c_0^{-18} c_1 c_2 c_3 c_4 c_5^6 x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 + c_0^{-6} c_4 c_5^3 c_6 x_1^6 x_2^6 x_3 + x_1 x_2 x_3 x_4 x_5.
\]

This time, the remaining monomials \(x_1^{18}\) and \(x_1^6 x_2^6 x_3\) correspond to the vectors \((1,0)\) and \((-3,-1)\) in the secondary fan—precisely the same vectors as in the toric diagram for \( \mathbb{P}^{(3,1,1)}_2 \). Moreover, the corresponding coordinates

\[
(c_0^{-18} c_1 c_2 c_3 c_4 c_5^6, c_0^{-6} c_4 c_5^3 c_6) = (2^{-12} 3^{-12} \rho^{-18}, -2^{-4} 3^{-3} \rho^{-6} \phi)
\]

agree with those of \( \mathbb{P}^{(3,1,1)}_2 \) up to some irrelevant constants.
3.3. The resolved moduli space

In order to search for possible additional large complex structure limit points, we need a compactification of the simplified moduli space in which the boundary is a divisor with normal crossings. This can be constructed by blowing up the original $\mathbb{P}^2(3,1,1)$ compactification. We first do toric blowups. The singular point $P_0$ of $\mathbb{P}^2(3,1,1)$ can be resolved by simply blowing it up; this introduces the vector $(-1,0)$ into the toric diagram, Fig. 3.2. We denote the corresponding exceptional divisor by $E_0$.

The point of intersection of $B_{con}$ and $D_\infty$ can then be made into a normal crossings point by additional toric blowups. We need three such blowups to reach normal crossings: the first has exceptional divisor $E_1$ corresponding to the toric vector $(-2,-1)$, the second has exceptional divisor $F_1$ corresponding to the toric vector $(-1,-1)$, and the third has exceptional divisor $G_1$ corresponding to the toric vector $(0,-1)$. Since this last vector also occurs in the secondary fan, we see that the large complex structure limit point found in the previous subsection lies at the intersection of $G_1$ and $D_\infty$. 

![Diagram](image)

**Figure 3.4:** The simplified moduli space resolved so that all the components of the discriminant locus have normal crossings.
The remaining blowups are non-toric. There are three blowups to be made at the intersection of $C_{\text{con}}$ and $D_{\infty}$, leading to exceptional divisors $E_2$, $F_2$, and $G_2$. The final blowup needed is of the point $[1,0,1]$ lying at the intersection of $D_0$, $B_{\text{con}}$ and $C_{\text{con}}$; the exceptional divisor for this blowup is denoted by $E_3$.

The resolved simplified moduli space is sketched in Fig. 3.4. This space is invariant under the action of $\mathcal{I}$, which exchanges $B_{\text{con}}$, $E_1$, $F_1$, and $G_1$ with $C_{\text{con}}$, $E_2$, $F_2$, and $G_2$, respectively. The intersection of $G_2$ and $D_{\infty}$ must therefore also be a large complex structure limit point.
4. The Periods

4.1. The fundamental period

We take for the holomorphic three–form the quantity

$$\Omega = \psi \frac{-2^3 3^5}{(2\pi i)^3} \frac{x_5 dx_1 dx_2 dx_3}{(3x_1^2 - 18 \psi x_1 x_2 x_3 x_5)}.$$  

(The expression $3x_1^2 - 18 \psi x_1 x_2 x_3 x_5$ in the denominator arises from $\frac{\partial p}{\partial x_4}$, with $p$ as in (3.4).) The numerical factor has been introduced to simplify later expressions; the factor of $\psi$ ensures that $\Omega$ is invariant under the extended group $\hat{G}$. This holomorphic three–form $\Omega$ is not invariant under $I$, and indeed it is impossible to find a holomorphic three–form which is invariant under both $I$ and $\hat{G}$, since the transformation $A^0 I^2$ is an $R$-symmetry. However, we can find a holomorphic three–form which is invariant under both $A^2$ and $I$; we take it to be

$$\hat{\Omega} \overset{\text{def}}{=} \rho^3 (\phi + \frac{1}{2} \rho^6) \Omega.$$  

This one will determine a holomorphic three–form on the actual moduli space (which is the quotient of the $(\rho, \phi)$ parameter space by $A^2$ and $I$).

A fundamental period $\varpi_0(\psi, \phi)$ of the holomorphic 3-form $\Omega$ can be found by direct integration as explained in [16]. (Properties of the corresponding fundamental period

$$\hat{\varpi}_0(\psi, \phi) = \rho^3 (\phi + \frac{1}{2} \rho^6) \varpi_0(\psi, \phi)$$

of the 3-form $\hat{\Omega}$ can be deduced from a study of $\varpi_0(\psi, \phi)$.) In our case we find the expansion

$$\varpi_0(\psi, \phi) = \sum_{n,m=0}^{\infty} \frac{(18n + 6m)! (-3\phi)^m}{(9n + 3m)! (6n + 2m)! (n!)^3 m! (18\psi)^{18n+6m}},$$

which converges for sufficiently large $\psi$. A useful form may be obtained by setting $k = 3n + m$ in the sum and summing over $k$ and $n$. The fundamental period can then be rewritten in the form

$$\varpi_0(\phi, \psi) = \sum_{k=0}^{\infty} \frac{(6k)!}{k!(2k)!(3k)!} \left( -\frac{3}{18^6 \psi^6} \right)^k U_k(\phi)$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + \frac{1}{6}) \Gamma(k + \frac{5}{6})}{(k!)^2} \rho^{-6k} U_k(\phi),$$

(4.2)
where
\[ U_k(\phi) = \phi^k \sum_{n=0}^{[\frac{k}{3}]} \frac{(-1)^n k!}{(n!)^3 \Gamma(k - 3n + 1)(3\phi)^{3n}} \] (4.3)

The function \( U_k(\phi) \) is a polynomial of degree \( k \) but we shall need to extend its definition to complex values \( \nu \) of \( k \). To this end note that the sum in (4.3) can be taken run to \( \infty \) since the factor \( 1/\Gamma(k - 3n + 1) \) vanishes automatically for \( n > \left[ \frac{k}{3} \right] \) so we take
\[
U_\nu(\phi) = \Gamma(\nu + 1) \phi^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(\nu - 3m + 1)(m!)^3 (3\phi)^{3m}} , \quad |\phi^3| > 1 , \quad 0 < \arg \phi < 2\pi/3
\]
\[
= \phi^\nu \, _3F_2 \left( -\frac{\nu}{3}, \frac{1 - \nu}{3}, \frac{2 - \nu}{3}; 1, 1; \phi^{-3} \right).
\] (4.4)

The second equality follows by use of the multiplication formula for the \( \Gamma \)-function.

A set of linearly independent periods can be chosen among from the functions
\[ \varpi_j(\psi, \phi) \overset{\text{def}}{=} \varpi_0(\alpha^j \psi, \alpha^6 j \phi) ; \quad j = 0, \ldots, 17 \] (4.5)

To find explicit expressions for these periods we must first extend \( \varpi_0(\psi, \phi) \) to small \( \psi \). This can be done by writing either expansion in (4.2) as a contour integral and then deforming the contour appropriately. The result is
\[
\varpi_0(\psi, \phi) = -\frac{1}{6} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right)(-3^{\frac{n}{3}} 2\psi)^n}{\Gamma(n) \Gamma(1 - \frac{n}{3}) \Gamma(1 - \frac{n}{2})} \ U_{-\frac{n}{3}}(\phi)
\] (4.6)

Due to the factors \( \Gamma(1 - \frac{n}{3}) \) and \( \Gamma(1 - \frac{n}{2}) \) in the denominator the summation index actually runs over \( n = 6k + r \), \( r = 1, 5 \). Hence,
\[
\varpi_0(\psi, \phi) = \frac{1}{3\pi} \sum_{r=1,5} \sin \frac{\pi r}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^2(k + \frac{r}{6})}{\Gamma(k+1) \Gamma(k + \frac{r}{3})} \rho^{6k+r} U_{-(k+\frac{r}{3})}(\phi)
\] (4.7)

The basis periods are then given by
\[
\varpi_{3a+\sigma}(\psi, \phi) = \frac{1}{3\pi} \sum_{r=1,5} \alpha^{3ar} \sin \frac{\pi r}{3} \xi_r^\sigma(\psi, \phi) ; \quad a = 0, 1 \quad \sigma = 0, 1 ,
\] (4.8)

where
\[
\xi_r^\sigma = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^2(k + \frac{r}{6})}{\Gamma(k+1) \Gamma(k + \frac{r}{3})} \rho^{6k+r} U_{-(k+\frac{r}{3})}^\sigma(\phi) , \quad \left| \frac{\rho^6}{\phi - \omega^{-\tau}} \right| < 1 ,
\] (4.9)

and we have also defined
\[ U_\nu^\sigma(\phi) = \omega^{-\nu\sigma} U_\nu(\omega^\sigma \phi) , \quad \sigma = 0, 1, 2 ,
\] (4.10)
with $\omega = e^{2\pi i/3}$. These latter functions are initially only defined in an angular sector of the $\phi$-plane, but we extend them by analytic continuation throughout a cut $\phi$-plane with branch cuts chosen to run out radially from the cube roots of unity.

The above expressions imply relations among the $\varpi_j$. It is straightforward to see that

$$
\varpi_j - \varpi_{j+3} + \varpi_{j+6} = 0 \\
\varpi_j + \varpi_{j+9} = 0 .
$$

As expected, only six of the $\varpi_j$ are linearly independent. These may be chosen to be the first five and we introduce the vector

$$
\varpi = \begin{pmatrix}
\varpi_0 \\
\varpi_1 \\
\varpi_2 \\
\varpi_3 \\
\varpi_4 \\
\varpi_5
\end{pmatrix}
$$

It is worth recording also the fact that the six functions $\varpi_j(\psi, 0)$, $j = 0, \ldots, 5$, remain linearly independent even when $\phi = 0$. Near $\psi = \infty$ these are linear combinations of the six functions

$$
1, \log \psi, \log^2 \psi, \log^3 \psi, \psi^{-6}, \psi^{-12} .
$$

Near $\psi = \psi_0 = (3^{4/2})^{-1/3}$, there are five analytic combinations plus one with leading singular behavior

$$
g(\psi) \log(\psi - \psi_0) ,
$$

where $g(\psi) = (\psi - \psi_0) + \cdots$, is itself a period.

4.2. The function $U_\nu$

The differential equation satisfied by $U_\nu$ follows from the general form of the hypergeometric equation of third order. We find

$$
(1 - \phi^3) \frac{d^3 U_\nu}{d\phi^3} + 3(\nu - 1)\phi^2 \frac{d^2 U_\nu}{d\phi^2} - (3\nu^2 - 3\nu + 1)\phi \frac{dU_\nu}{d\phi} + \nu^3 U_\nu = 0 .
$$

In order to find a basis of solutions we first write a series for $U_\nu(\phi)$ that converges near $\phi = 0$. This is accomplished by writing the sum in (4.4) as a Barnes’ integral and continuing to small $\phi$:

$$
U_\nu(\phi) = \frac{3^{1-\nu} \omega^{\frac{\nu}{3}}}{\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{\Gamma(m-\nu) (3\omega \phi)^m}{\Gamma^2(1 - m-\nu) m!} , \ |\phi^3| < 1 .
$$

We note in passing some useful properties of $U_\nu(\phi)$ that follow immediately from this series.
• Values at $\phi = 0$

$$U_\nu(0) = \frac{2\pi}{\sqrt{3}} e^{i\pi\nu/3} \frac{1}{\Gamma^2(1 + \frac{\nu}{3}) \Gamma(1 - \frac{\nu}{3}) \Gamma(\frac{2-\nu}{3})}$$  \hspace{1cm} (4.14)

Hence, for $n = 0, 1, 2, \ldots$,

$$U_{3n}(0) = \frac{(-1)^n(3n)!}{3^{3n}(n!)^3} \quad \text{and} \quad U_{3n+1}(0) = U_{3n+2}(0) = U_{-(3n+3)}(0) = 0 . \hspace{1cm} (4.15)$$

• Recurrence relation

$$\frac{dU_\nu}{d\phi} = \nu U_{\nu-1} \hspace{1cm} (4.16)$$

A set of three solutions to the differential equation is given by the functions $U^\sigma_\nu(\phi)$ defined in (4.10). The Wronskian of these solutions is

$$\text{Wr}[U^0_\nu, U^1_\nu, U^2_\nu] = -\frac{27i}{2\pi^3} e^{-i\pi\nu} \sin^2(\pi\nu) (1 - \phi^3)^{\nu-1}$$

from which we see that these solutions are linearly independent except at the integers where the Wronskian has a double zero. We shall be concerned with finding a basis of solutions that remains linearly independent even at the integers. First however it is useful to note that near $\phi = 1$ there is a multivalued solution of the form

$$y_\nu(\phi) = -\frac{\sqrt{3}}{2\pi(\nu + 1)} (\phi - 1)^{\nu+1} \{1 + O(\phi - 1)\} + \text{analytic} \hspace{1cm} (4.17)$$

the prefactor having been chosen so as to simplify later expressions. There are two other solutions that are single valued and we may complete the specification of $y_\nu$ by requiring that it be single valued in neighborhoods of $\phi = \omega$ and $\phi = \omega^2$. We set also

$$y^\sigma_\nu(\phi) = \omega^{-\nu\sigma} y_\nu(\omega^\sigma \phi) \hspace{1cm} (4.18)$$

which are multivalued about the points $\phi = \omega^{-\sigma}$ but single valued at the other cube roots of unity. Near $\phi = \omega^{-\sigma}$ we have the asymptotic behavior

$$y^\sigma_\nu(\phi) \sim -\frac{\sqrt{3}}{2\pi(\nu + 1)} \omega^{\sigma(\phi - \omega^{-\sigma})^{\nu+1} + \text{analytic}} .$$

The solution $U_\nu(\phi)$ can be expressed in terms of the $y^\sigma_\nu(\phi)$:

$$U_\nu(\phi) = \sum_{\tau=0}^{2} \gamma^{\tau}_\nu y^{\tau}_\nu(\phi)$$

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in terms of certain coefficients $\gamma_\nu^\tau$ which may be computed from the large $m$ behavior of the series \((4.13)\). In this series we observe that

\[
\frac{\Gamma\left(\frac{m-\nu}{3}\right)}{\Gamma^2\left(1-\frac{m-\nu}{3}\right)} = \frac{1}{\pi^2} \Gamma^3\left(\frac{m-\nu}{3}\right) \sin^2\left(\frac{m-\nu}{3}\right)
\sim -\frac{3^{\frac{2}{3}m+\nu}}{2\pi} \Gamma(m - \nu - 1) \left(\omega^{m-\nu} - 2 + \omega^{-m+\nu}\right) \left[1 + \mathcal{O}(m^{-1})\right].
\]

Substituting the leading term into the series \((4.13)\) we obtain linear combinations of the binomial series for the quantities \((\omega^\sigma - \phi)^{\nu+1}\), $\sigma = 0, 1, 2$. Remembering that we have defined the functions in the $\phi$-plane with cuts that run out radially from the cube roots of unity we have

\[
(1 - \omega^\sigma \phi)^{\nu+1} = -e^{-i\pi\nu} \omega^{(\nu+1)\sigma} (\phi - \omega^{-\sigma})^{\nu+1}, \quad \sigma = 0, 1, 2.
\]

In this way we see that near the points $\phi^3 = 1$ we have the asymptotic behavior

\[
U_\nu(\phi) \sim -\frac{\sqrt{3}}{2\pi (\nu + 1)} \left\{ (\phi - 1)^{\nu+1} - 2\omega(\phi - \omega^{-1})^{\nu+1} + \omega^2(\phi - \omega^{-2})^{\nu+1} \right\}.
\]

Thus we find

\[
U_\nu(\phi) = y_0^\nu - 2y_1^\nu + y_2^\nu
\]

from which we may read off the coefficients $\gamma_\nu^\tau$. More generally the functions $U_\nu^\sigma(\phi)$ may be expanded in terms of the $y_\nu^\tau(\phi)$

\[
U_\nu^\sigma(\phi) = \sum_{\tau=0}^{2} \gamma_\nu^{\sigma,\tau} y_\nu^\tau(\phi) = \sum_{\tau=0}^{2} \gamma_\nu^{\tau} y_\nu^{\sigma+\tau}(\phi).
\]

We see from their definitions that the $U_\nu^\sigma(\phi)$ and the $y_\nu^\sigma(\phi)$ are not periodic in $\sigma$ in fact

\[
U_\nu^{\sigma+3}(\phi) = e^{-2\pi i\nu} U_\nu^\sigma(\phi), \quad y_\nu^{\sigma+3}(\phi) = e^{-2\pi i\nu} y_\nu^\sigma(\phi).
\]

With this in mind we are able to read of the coefficients $\gamma_\nu^{\sigma,\tau}$ from \((4.19)\)

\[
\gamma_\nu^{\sigma,\tau} = \begin{pmatrix}
1 & -2 & 1 \\
-e^{-2\pi i\nu} & 1 & -2 \\
-2e^{-2\pi i\nu} & e^{-2\pi i\nu} & 1
\end{pmatrix}.
\]

We are now able to return to the question of finding linearly independent solutions. Note first that the function

\[
\tilde{V}_\nu(\phi) \overset{\text{def}}{=} \sum_{\sigma=0}^{2} U_\nu^\sigma(\phi) = (1 - e^{-2\pi i\nu}) \left( y_0^\nu(\phi) - y_1^\nu(\phi) \right)
\]

\[
(4.21)
\]
vanishes at the integers and that

\[ V_\nu(\nu) \overset{\text{def}}{=} \frac{\hat{V}_\nu(\nu)}{1 - e^{-2\pi i \nu}} \] (4.22)

has a nonvanishing limit. Taking say \( U_\nu^0, U_\nu^1 \) and \( V_\nu \) as a basis improves the situation insofar as the Wronskian \( \text{Wr}[U_\nu^0, U_\nu^1, V_\nu] \) now vanishes only to first order at the integers. Since it still vanishes there must be a linear relation between these solutions at the integers. We define

\[ \hat{W}_\nu(\nu) = 3V_\nu(\nu) - 2U_\nu(\nu) - U_\nu^1(\nu) = (1 - e^{-2\pi i \nu}) y_\nu^0(\nu) \] (4.23)

and we see that \( \hat{W}_\nu(\nu) \) also vanishes at the integers but that

\[ W_\nu(\nu) \overset{\text{def}}{=} \frac{\hat{W}_\nu(\nu)}{1 - e^{-2\pi i \nu}} \] (4.24)

has a nonvanishing limit. We now check that

\[ \text{Wr}[U_\nu, V_\nu, W_\nu] = \frac{27i}{(2\pi)^3} e^{i\pi \nu} (1 - \phi^3)^\nu - 1 \]

so the solutions \( U_\nu(\nu), V_\nu(\nu) \) and \( W_\nu(\nu) \) are always linearly independent. These functions have nonvanishing limits at the negative integers as we have seen. At the positive integers they have poles although this fact will not concern us in the following.

We wish now to discuss the asymptotic behavior of the solutions as \( \nu \to \infty \) for fixed \( \phi \). It is easy to see that the functions \( (\phi - \omega^{-\sigma})^{\nu+1} \) solve the differential equation (4.12) for large \( \nu \). A general solution is then a linear combination of these expressions with coefficients that depend on \( \nu \). The coefficients may be fixed from a knowledge of the monodromy of the solution about the points for which \( \phi^3 = 1 \). Thus, for example, the relation

\[ y_\nu^\sigma(\nu) \sim \omega^\sigma (\phi - \omega^{-\sigma})^{\nu+1} \]

which we have already met as a relation that is valid as \( \phi \to \omega^{-\sigma} \) is valid also for all \( \phi \) in the asymptotic limit \( \nu \to \infty \). The asymptotic behavior of the functions \( U_\nu^\sigma(\phi) \) follows in virtue of (4.19). These considerations will shortly permit us to write integral representations that may be used to continue the periods \( \varpi_j(\psi, \phi) \) to large \( \psi \). Notice that

\[ \zeta_\nu^\sigma = \int_{\Gamma} \frac{d\mu}{2i \sin \pi (\mu + \frac{5}{6})} \frac{\Gamma^2(-\mu)}{\Gamma(-\mu + \frac{1}{6}) \Gamma(-\mu + \frac{5}{6}) \rho^{-6\mu} U_\mu^\sigma(\phi)} \] (4.25)

with the contour \( \Gamma_+ \) enclosing the poles on the negative \( \mu \)-axis.

In order to obtain an integral representation valid for large \( \rho \) we wish to rotate the contour so as to run parallel to the imaginary axis. This requires a consideration of the
convergence of the integrals and the contribution of the arcs at infinity. For $\sigma = 0$ the arcs at infinity give a vanishing contribution so for this case we have
\[
\xi^0_\sigma = \int_\Gamma \frac{d\mu}{2i \sin \pi (\mu + \frac{5}{6})} \frac{\Gamma^2(-\mu)}{\Gamma(-\mu + \frac{1}{6})\Gamma(-\mu + \frac{5}{6})} \rho^{-6\mu} U_\mu(\phi)
\]
For $\sigma \neq 0$ it is not possible to rotate the contour through the second quadrant without modifying the integrand. Note however that the value of the integral is unchanged if we replace $U_\mu^\sigma(\phi)$ in (4.25) by
\[
\tilde{U}^\sigma_{\mu,r}(\phi) \overset{\text{def}}{=} U_\mu^\sigma(\phi) - e^{i\pi r/6} \frac{\sin \pi (\mu + \frac{5}{6})}{\sin \pi \mu} f^\sigma_\mu(\phi)
\]
with $f^\sigma_\mu(\phi)$ a function that has no poles in the left half $\mu$-plane and has zeros at the integers. The factor
\[
e^{i\pi r/6} \frac{\sin \pi (\mu + \frac{5}{6})}{\sin \pi \mu}
\]
ensures that the new term does not contribute to the poles and tends to unity as $\mu \to \infty$ along a ray in the second quadrant. By suitable choice of $f^\sigma_\mu$, we are able to rotate the contour. The convergence of the integral is most easily studied by writing the $U_\mu^\sigma(\phi)$ in terms of the $y^\sigma_\mu(\phi)$-basis. The quantities that have to be cancelled arise from the exponential entries in the matrix $\gamma^\sigma_{\mu,\tau}$ (4.20). It is now easily seen that we should choose
\[
\begin{align*}
f^0_\mu &= 0 \\
f^1_\mu &= -\tilde{W}_\mu \\
f^2_\mu &= \tilde{V}_\mu + \tilde{W}_\mu.
\end{align*}
\]
In fact in terms of the $y^\sigma_\mu(\phi)$-basis we have
\[
\tilde{U}^\sigma_{\mu,r}(\phi) = \sum_\tau \tilde{\gamma}^\sigma_{\mu,\tau} y_\mu^\tau(\phi) , \quad \tilde{\gamma}^\sigma_{\mu,\tau} = \begin{pmatrix}
1 & -2 & 1 \\
\frac{e^{i\pi r/3}}{-2e^{i\pi r/3}} & 1 & -2 \\
\frac{e^{-i\pi r/3}}{e^{-i\pi r/3}} & 1 & 1
\end{pmatrix} .
\]
(4.26)
Gathering these results together and performing the sum over $r$ we find the following integral representations for the periods:
\[
\begin{align*}
\varpi_0 &= \int_\Gamma \frac{d\mu}{4\pi^2 i} \frac{\Gamma(-\mu)\Gamma(\mu + \frac{1}{6})\Gamma(\mu + \frac{5}{6})}{\Gamma(1 + \mu)} \rho^{-6\mu} U^\sigma_\mu \\
\varpi_1 &= \int_\Gamma \frac{d\mu}{8\pi^3} \Gamma^2(-\mu)\Gamma(\mu + \frac{1}{6})\Gamma(\mu + \frac{5}{6}) \rho^{-6\mu} \left[ 2i \sin \pi \mu (U^1_\mu + W_\mu) + e^{-3\pi i\mu} W_\mu \right] \\
\varpi_2 &= \int_\Gamma \frac{d\mu}{8\pi^3} \Gamma^2(-\mu)\Gamma(\mu + \frac{1}{6})\Gamma(\mu + \frac{5}{6}) \rho^{-6\mu} \left[ 2i \sin \pi \mu (U^2_\mu - V_\mu - W_\mu) - e^{-3\pi i\mu} (V_\mu + W_\mu) \right] \\
\varpi_{3+\sigma} &= \int_\Gamma \frac{d\mu}{8\pi^3} \Gamma^2(-\mu)\Gamma(\mu + \frac{1}{6})\Gamma(\mu + \frac{5}{6}) \rho^{-6\mu} e^{\pi i\mu} U^\sigma_\mu , \quad \sigma = 0, 1, 2.
\end{align*}
\]
These representations are valid when
\[-\pi < \arg \left( \frac{\rho^6}{\phi - \omega - \tau} \right) < \pi.\]

4.3. Expansions for large $\psi$

We will see later that the mirror map can be written in terms of the quantities $\varpi_0$, $\varpi_3$, and $\sum_\sigma \varpi_{3+\sigma}$ and we will need the explicit form of their expansions for large $\psi$. If $\left| \frac{\rho^6}{\phi - \omega - \tau} \right| > 1$ the contours may be deformed so as to enclose the poles on the positive $\mu$-axis and the large $\psi$ series are obtained by summing over the residues at $\mu = 0, 1, \ldots$.

For the quantity $\sum_\sigma \varpi_{3+\sigma}$ there is a simple integral representation

$$
\sum_\sigma \varpi_{3+\sigma} = \int_\Gamma \frac{d\mu}{4\pi^2 i} \frac{\Gamma(-\mu)\Gamma(\mu+\frac{1}{6})\Gamma(\mu+\frac{5}{6})}{\Gamma(1+\mu)} \rho^{-6\mu} V_\mu.
$$

On evaluating the residues we find the following series (for the case of $\varpi_0$ this merely reproduces the original definition (4.2))

$$
\varpi_0 = \sum_{k=0}^{\infty} \frac{(6k)!}{k!(2k)!(3k)!(18\psi)^{6k}} U_k(\phi)
$$

$$
\sum_\sigma \varpi_{3+\sigma} = \varpi_0 - \frac{3}{2\pi i} \varpi_0 \log(18\psi)^6 + \frac{3}{2\pi i} \sum_{k=0}^{\infty} \frac{(6k)!}{k!(2k)!(3k)!(18\psi)^{6k}} \left[ A_k U_k(\phi) + Y_k(\phi) \right]
$$

$$
\varpi_3 - \frac{1}{3} \sum_\sigma \varpi_{3+\sigma} = \frac{1}{6} \varpi_0 + \frac{1}{2\pi i} \varpi_0 \log(3\phi) + \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{(6k)!}{k!(2k)!(3k)!(18\psi)^{6k}} \left[ N_k(\phi) \right],
$$

where in these series

$$
A_k = 6\Psi(6k+1) - 3\Psi(3k+1) - 2\Psi(2k+1) - \Psi(k+1)
$$

$$
Y_k(\phi) = \phi^k k! \sum_{n=0}^{\left[ \frac{k}{3} \right]} (-1)^n \frac{(n)!}{(n)!^3 (k-3n)! (3\phi)^{3n}} [\Psi(1+k) - \Psi(1+n)]
$$

$$
N_k(\phi) = \frac{\partial U_k}{\partial \mu} - Y_k - U_k \log \phi
$$

$$
= \phi^k k! \left\{ \sum_{n=0}^{\left[ \frac{k}{3} \right]} \frac{(-1)^n [\Psi(n+1) - \Psi(k-3n+1)]}{(n)!^3 (k-3n)! (3\phi)^{3n}} + \sum_{\left[ \frac{k}{3} \right]+1}^{\infty} \frac{(-1)^{k+1} (3n-k-1)!}{(n)!^3 (3\phi)^{3n}} \right\}
$$

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Notice that $A_0 = Y_0 = 0$.

The following table records the various functions that are associated to the fundamental period that we have introduced in this section

| Functions          | Where Defined          |
|--------------------|------------------------|
| $U_\nu$, $U_\nu^\sigma$ | (4.4), (4.10)         |
| $y_\nu$, $y_\nu^\sigma$  | (4.17), (4.18)         |
| $\hat{V}_\nu$, $V_\nu$     | (4.21), (4.22)         |
| $\hat{W}_\nu$, $W_\nu$     | (4.23), (4.24)         |
| $Y_k$, $N_k$             | (4.29)                 |
| $\varpi_0$, $\varpi_j$, $\xi_\nu^\sigma$ | (4.1), (4.5), (4.8), (4.9) |

**Table 4.1:** Functions associated to the periods and their definitions.
5. Monodromy Calculations

5.1. Generalities

We wish now to study the effect of the various monodromy operations on the period vectors \( \varpi \) and \( \hat{\varpi} \) associated to the 3-forms \( \Omega \) and \( \hat{\Omega} \). To do this consistently we must choose a basepoint. That is, we make a common choice of basepoint for the curves that encircle the various components of the discriminant locus. A change of basepoint induces a conjugation of the matrices. Thus if, with respect to some basepoint, the monodromy matrices are denoted by \( M_a \), \( a = 1, 2, \ldots \) then under a change of basepoint the matrices become \( g^{-1}M_ag \) for some \( g \in \text{Sp}(6, \mathbb{Z}) \). We choose our basepoint to be a point for which \( \psi \) is large, \( \phi \) is small and \( 0 < \arg \psi < \frac{2\pi}{18} \). We shall refer to such curves as having a basepoint at \( \infty \). A useful technique for computing some of the matrices, which we illustrate in the following, involves use of the integral representations (4.27).

5.2. Monodromy about \( \psi = 0 \) and \( \psi = \infty \)

Fix \((\psi, \phi)\) with \( \phi \) small and consider the curve \((e^{i\theta} \psi, e^{6i\theta} \phi)\) as \( \theta \) varies in the range \( [0, \frac{2\pi}{18}] \). This is a closed curve on the simplified moduli space in virtue of the identifications on the parameter space. In virtue of the above discussion we are most interested in the case that \( \psi \) is large since this curve has a basepoint at \( \infty \). However before examining this case let us take \( \psi \) small, we can say that this curve has a basepoint at the origin. From (4.5) and (4.6) we see that the monodromy of \( \varpi \) along this same curve is given by \( A \) def \( A_{10}T^2 \) (which has the same effect on the parameter space as does \( A \), but preserves the 3-form \( \hat{\Omega} \)). Notice that the action of \( A' \) on \( \hat{\varpi} \) has precisely the same matrix \( A \) as the action of \( A \) on \( \varpi \).

Consider now a similar curve with basepoint at \( \infty \). This corresponds to an operation which we denote by \( T^{-1}_\infty \) (the inverse accounts for the fact that a curve in the \( \psi \)-plane that winds about \( \psi = 0 \) in the positive sense winds about \( \psi = \infty \) in the negative sense). The matrix corresponding to this operation may be computed by computing the effect on the
integral representations (4.27). We see that after traversing the curve

\[ \rho^{-6\mu} U_\mu(\phi) \mapsto \rho^{-6\mu} U_\mu^{\sigma+1}(\phi) \]
\[ \rho^{-6\mu} V_\mu(\phi) \mapsto \rho^{-6\mu} (V_\mu(\phi) - U_\mu(\phi)) \]
\[ \rho^{-6\mu} W_\mu(\phi) \mapsto \rho^{-6\mu} (W_\mu(\phi) - V_\mu(\phi)) . \]

In these relations the second and third follow from the first. If we now make these replacements in the integral representations (4.27), compute the new residues (it suffices to compute the residues at \( \mu = 0, 1, 2 \)) and compare with the original residues we find the matrix corresponding to \( T_\infty \)

\[
T_\infty : \omega \to T_\infty \omega ; \quad T_\infty = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 & 0 & 1 \\
-2 & 1 & 2 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} . \tag{5.2}
\]

An identical analysis applies to the periods \( \widehat{\omega} \), and the matrix for the action of \( T_\infty \) on \( \widehat{\omega} \) is again \( T_\infty \).

5.3. The operation \( \mathcal{I} \)

To see the effect of \( \mathcal{I} : (\rho, \phi) \mapsto (i\rho, \phi + \rho^6) \) recall from Section 4 that the periods are linear combinations of the quantities

\[ \xi_\sigma^r = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^2(k + \frac{r}{6})}{\Gamma(k+1)\Gamma(k+\frac{r}{3})} \rho^{6k+r} U_\sigma^{-(k+\frac{r}{3})}(\phi) \]

for \( r = 1 \) and \( r = 5 \). The simplest way to proceed is to take \( \rho \) small and to consider only the \( k = 0 \) terms in the sums. In this way we see that \( \mathcal{I} : \xi_\sigma^r \mapsto i\xi_\sigma^r \) and hence that

\[ \mathcal{I} : \omega_j \mapsto i\omega_j . \]

Thus the effect of \( \mathcal{I} \) on \( \omega \) is not to effect a symplectic transformation but to multiply the periods by a gauge factor. However, when we apply \( \mathcal{I} \) to the other period vector \( \widehat{\omega} \), we find that \( \mathcal{I} \) leaves it invariant.

It is instructive to see the effect of \( \mathcal{I} \) in detail when \( \rho \) is not small. To this end we write the \( \xi_\sigma^r \) as integrals

\[ \xi_\sigma^r = \rho^r \int_0^1 d\lambda \lambda^{\frac{r}{6}}(1 - \lambda)^{-1} U_\sigma^{\sigma+1}(\phi + \lambda \rho^6) . \tag{5.3} \]
as is easily verified by expanding the integrand in powers of $\lambda$, integrating term by term by means of the $B$-function formula

$$\int_0^1 d\lambda \lambda^{a-1}(1-\lambda)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and using the recurrence relation (4.16) which is equivalent to

$$(-1)^k \frac{\Gamma(z)}{\Gamma(k+z)} \left(\frac{d}{d\phi}\right)^k U^\sigma_{\frac{\rho}{\phi}}(\phi) = U^\sigma_{k-\frac{\rho}{\phi}}(\phi).$$

The effect of $I$ on the RHS of (5.3) is to multiply the factor of $\rho^r$ by $i$ and to effect the change $\lambda \rightarrow 1-\lambda$ in the integrand but this leaves the value of the integral unchanged.

5.4. Monodromy about $C_{con}$

To compute the monodromy about the conifold, we notice that near $C_{con}$ the periods have the structure

$$\mathfrak{w}_j(\psi, \phi) = c_j g(\psi, \phi) \log(\rho^6 - \phi - 1) + \text{analytic},$$

where $g(\psi, \phi)$ is itself a period analytic in the neighborhood of $C_{con}$. To evaluate the coefficients $c_j$, we set $\phi = 0$, so that $g(\psi, 0) = (\psi - \psi_0) + \cdots$ as follows from the behavior of the periods around $\psi_0$. The logarithmic piece in $\partial \mathfrak{w}_j / \partial \psi$ is then extracted by using Stirling’s formula in the series expansion. Up to a constant that can be absorbed in $g$, we find

$$c_j = (1, 1, -2, 1, 0, 0) ; \quad j = 0, \cdots, 5$$

The next step is to express $g$ in terms of the $\mathfrak{w}_j$. An argument parallel to that made in [2] shows that $\mathfrak{w}_0 \mapsto \mathfrak{w}_1$ and

$$g = \frac{i}{2\pi c_1}(\mathfrak{w}_1 - \mathfrak{w}_0).$$

Thus, under transport about the conifold $\mathfrak{w}$ transforms as

$$\mathcal{T} : \mathfrak{w} \rightarrow T\mathfrak{w} ; \quad T = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (5.4)$$

The other period vector $\hat{\mathfrak{w}}$ transforms in the same way, again with matrix $T$, since the prefactor $\rho^3(\phi + \frac{1}{2}\rho^6)$ is single-valued near the conifold locus.
In this case we get the same result whether we take the basepoint to be at \(\infty\) or at the origin. Bearing in mind that the abstract operations compose in the reverse order to the matrices we may write a relation between the operations \(A\), \(T\) and \(T_{\infty}\):

\[
T_{\infty} = (TA)^{-1}, \quad T_{\infty} = (AT)^{-1}.
\]

(A similar relation holds among \(A', T\), and \(T_{\infty}\) acting on \(\hat{\omega}\).) This relation has a simple interpretation. It is easy to see, by setting \(\phi = 0\), that \(A\) and \(T_{\infty}^{-1}\) differ by a curve that winds about \(C_{\text{con}}\). Note also that \(A\) can be realised also as a combination of monodromy operations that have basepoints at \(\infty\).

### 5.5. Monodromy about \(B_{\text{con}}\)

We shall compute the monodromy about \(B_{\text{con}}\) from the integral representations for the periods. To this end we make some observations regarding the monodromy of the functions \(U_{\sigma}^\mu\).

Let us denote by \(\Delta_\mu\) the solution to (4.12) corresponding to the first term on the RHS of (4.17) so that

\[
y_\mu(\phi) = \Delta_\mu(\phi) + \text{analytic}, \quad \Delta_\mu(\phi) = -\frac{\sqrt{3}}{2\pi(\mu + 1)}(\phi - 1)^{\mu+1}\left\{1 + O(\phi - 1)\right\}.
\]

Continuing \(y_\mu\) about \(\phi = 1\) gives the result

\[
y_\mu(\phi) \mapsto y_\mu(\phi) + (e^{2\pi i\mu} - 1)\Delta_\mu(\phi)
\]

from which it follows that

\[
U_{\mu}^\sigma(\phi) \mapsto U_{\mu}^\sigma(\phi) + \gamma^\sigma_0(e^{2\pi i\mu} - 1)\Delta_\mu(\phi) .
\]

(5.5)

On the other hand we can work through an argument that is by now familiar. We take \(x\) real and \(x > 1\) then

\[
U_{\mu}^1(x + i\epsilon) \mapsto U_{\mu}^1(x - i\epsilon) = \omega^{-\mu}U_{\mu}^0(\omega(x - i\epsilon)) = U_{\mu}^0(x + i\epsilon)
\]

with \(\epsilon\) an infinitesimal and the last equality following from the series (4.13). By comparing this relation with (5.5) we find an expression for \(\Delta_\mu\) and hence that

\[
U_{\mu}^\sigma(\phi) \mapsto U_{\mu}^\sigma(\phi) + \gamma^\sigma_0 e^{2\pi i\mu}(U_{\mu}^0(\phi) - U_{\mu}^1(\phi))
\]

and by making these replacements in (4.27) we obtain the monodromy matrix

\[
\mathcal{B} : \varpi \to \mathcal{B}\varpi ; \quad \mathcal{B} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & -2 & 2 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 2 & 1
\end{pmatrix}
\]

(5.6)
Again, the same matrix describes the action on $\mathbf{w}$.

Notice that $\mathbf{w}_0 \mapsto \mathbf{w}_0$, reflecting the fact that it is single valued near $\phi = 1$. We do not expect $B$ to be an independent generator and we easily see this to be the case. Comparing $B$ with $T$ we see that

$$B: \mathbf{w}_j \to \mathbf{w}_j + b_j(\mathbf{w}_3 - \mathbf{w}_4)$$

$$T: \mathbf{w}_j \to \mathbf{w}_j + c_j(\mathbf{w}_0 - \mathbf{w}_1)$$

and from this we see that

$$B = A^{-3}TA^3.$$  

5.6. Global considerations

We give now a more global interpretation of our monodromy calculations. Calculating monodromies around all loops in the moduli space $\mathcal{M}$ (with a fixed basepoint $P$) determines the monodromy representation of our theory, which is a homomorphism from the fundamental group $\pi_1(\mathcal{M}, P)$ to the group of linear transformations $\text{Gl}(6, \mathbb{C})$. (The image will lie in $\text{Sp}(6, \mathbb{Z})$ if the basis of periods has been chosen appropriately.) The image is the full group of duality transformations of the theory.

We use the technique of the Zariski–van Kampen theorem (see for example Ref. [17]) to calculate the fundamental group of the moduli space. (A similar method was employed in Ref. [8].) For this purpose we return to the $\mathbb{P}^2_{3,1,1}$ model of the compactification of the simplified moduli space. The fundamental group $\pi_1(\mathcal{M}, P)$ is an index 2 subgroup of the fundamental group of the simplified moduli space, the extra generator corresponding to $I$. Since the monodromy transformation corresponding to $I$ is trivial on the period vector $\mathbf{w}$, we need only consider the simplified moduli space in calculating the duality transformations.

We continue to use homogeneous coordinates $[x, y, z]$ on $\mathbb{P}^2_{3,1,1}$, and consider the affine coordinate chart with $y = 1$ which has coordinates $(x, z)$. The projection to the $z$-axis has as fibers the complex curves $F_{z_0} := (z = z_0)$ (with $z_0$ a constant), and $x$ serves as a coordinate on any of these curves. For general values of $z_0$, the discriminant locus meets the fiber $F_{z_0}$ in precisely three points: $F_{z_0} \cap D_\infty$ at $x = 0$, $F_{z_0} \cap B_{\text{con}}$ at $x = z_0^3$, and $F_{z_0} \cap C_{\text{con}}$ at $x = (z_0 + 1)^3$. The values of $z_0$ at which some of these points come together—namely $z_0 = 0, -1, (-3 \pm \sqrt{-3})/6$—will play an important rôle. The latter two correspond to the points $P_-$ and $P_+$, respectively.

Let us choose a base point $P$ for which the corresponding value of $z_0$ is not one of the special values. The proof of the Zariski–van Kampen theorem guarantees that the fundamental group of the complement of the discriminant locus is generated by loops in the fiber $F_{z_0}$, based at $P$, around the 4 points $x = 0, x = z_0^3, x = (z_0 + 1)^3$, and $z_0 = \infty$. But such a loop can be written as a product

\[ A \text{ priori, since our fibration is not generic at } z_0 = \infty, \text{ we should also include a loop in the base of the fibration, around } z_0 = \infty. \]
When we choose $P$ to be the basepoint “at infinity” (with $\psi$ large, $\phi$ small, and $0 < \arg \psi < \frac{2\pi}{18}$), and choose the loops appropriately, the corresponding monodromy transformations are $D_{\infty}$, $B$, $T$, and $A$, represented by matrices $D_{\infty}$, $B$, $T$, and $A$. ($D_{\infty}$ has not appeared previously; we shall calculate it shortly.) Our choice of loops is indicated in Fig. 5.1, valid when $z_0 = \varepsilon e^{i\theta}$ with $\varepsilon$ and $\theta$ small and positive. Also shown in the figure are the branch cuts used in our analysis of the periods.

![Figure 5.1: The branch cuts, and the choice of loops for $z_0 = \varepsilon e^{i\theta}$, with $\varepsilon$ and $\theta$ small and positive](image)

To calculate the monodromy transformation $D_{\infty}$, recall that our convention is that the composite $L_1L_2$ of the monodromy transformations along loops $L_1$ and $L_2$ describes the monodromy along a loop which first traverses $L_1$ and then traverses $L_2$. If the transformation $L_j$ is represented by the matrix $L_j$ (i.e., if it maps the period vector $\Pi$ to $L_j \Pi$), then $L_1L_2$ is represented by the matrix $L_2L_1$—the matrices compose in the opposite order. Applying this to our situation, we see that $D_{\infty}B = T_{\infty}$, and so $D_{\infty}$ is represented by the matrix

$$D_{\infty} = B^{-1}T_{\infty} = B^{-1}T^{-1}A^{-1}.$$  

We have previously observed the relation

$$B = A^{-3}TA^3;$$

if we combine this with our expression for $D_{\infty}$ we find

$$D_{\infty} = A^{-3}T^{-1}A^3T^{-1}A^{-1}. $$

of loops around the finite special $z_0$-values, and such loops in the base are equivalent to loops in the fiber.
This demonstrates very explicitly the interesting result that the duality group for our family is generated by the matrices $A$ and $T$.

It is instructive to verify the relations in the fundamental group as given by the Zariski–van Kampen theorem. First, as we let $z_0$ wind once about 0, $z_0^3$ will wind three times about 0. The corresponding relation is $(\mathcal{D}_\infty B)^3 B (\mathcal{D}_\infty B)^{-3} = B$, or in matrix form

$$T^{-3}_\infty B T^3_\infty = B.$$

This is easily verified.

Next, we move $z_0$ to the value $z_0 = \varepsilon e^{i\theta}$, through values in the upper half plane. The contours will deform as we do so; to describe the deformation it is convenient to change generators and replace $\mathcal{D}_\infty$ by $\hat{\mathcal{D}}_\infty = B^{-1} \mathcal{D}_\infty B = B^{-1} T_\infty$, as shown in Fig. 5.2. The new contours deform as shown in the top half of Fig. 5.3. To describe the braiding relations, it is easiest to change the generators as shown in the bottom half of the figure. Thus, for this $z_0$-value the generator $B$ is replaced by

$$\bar{B} = \hat{\mathcal{D}}^{-1}_\infty B \hat{\mathcal{D}}_\infty = T^{-1}_\infty B T_\infty.$$

We can move $z_0$ from $\varepsilon e^{i(\theta + 2\pi/3)}$ to $(-3 + \sqrt{-3})/6$ without introducing any further braiding; if we then let $z_0$ wind once around $(-3 + \sqrt{-3})/6$, we find that the intersections with $B_{\text{con}}$ and $C_{\text{con}}$ wind once around each other, leading to the relation $\bar{B} T = T \bar{B}$. The matrix form of this relation

$$T (T^{-1}_\infty B T^{-1}_\infty) = (T^{-1}_\infty B T^{-1}_\infty) T$$

is easily verified. The relation at $z_0 = (-3 - \sqrt{-3})/6$ is similar.

Finally, we move $z_0$ to the value $z_0 = \varepsilon e^{i\pi}$, still through values in the upper half plane. This time, the contours from Fig. 5.2 deform as shown in the top half of Fig. 5.4,
and we change generators as shown in the bottom half. The old generators $B$ and $\hat{D}_{\infty}$ are replaced by new generators $\tilde{B}$ and $\hat{D}_{\infty}$, with $\tilde{B}$ as above and

$$\tilde{D}_{\infty} = \mathcal{T}^{-1}_{\infty} \hat{D}_{\infty} \mathcal{T}_{\infty} = \mathcal{T}^{-1}_{\infty} B^{-1} \mathcal{T}^{2}_{\infty}.\$$

If we allow $z_0$ to wind once around $-1$, we find that the intersection with $C_{\text{con}}$ winds three times around 0, leading to the relation $(\tilde{D}_{\infty} \mathcal{T})^3 \mathcal{T} (\hat{D}_{\infty} \mathcal{T})^{-3} = \mathcal{T}$, or in matrix form,

$$(\mathcal{T} \mathcal{T}_{\infty}^2 B^{-1} \mathcal{T}_{\infty}^{-1})^{-3} \mathcal{T} (\mathcal{T} \mathcal{T}_{\infty}^2 B^{-1} \mathcal{T}_{\infty}^{-1})^3 = \mathcal{T}.\$$

This can also be verified by multiplying the corresponding matrices.
Figure 5.4: The loops deformed to $z_0 = \varepsilon e^{i\pi}$ (top half), and another set of generators at the same $z_0$ value (bottom half)

5.7. The large complex structure limit

Further to the discussion of [18,1] we identify a large complex structure limit, for the general case of $n$ parameters, as a point in the parameter space where $n$ codimension 1 hypersurfaces in the (compactification of the) moduli space meet transversely in a point and the monodromies $S_i$, of a basis of independent periods, about these boundary divisors satisfy certain characteristic properties. The matrices $R_i = S_i - 1$ satisfy the properties

\begin{align*}
    i. \quad [R_i, R_j] &= 0 \\
    ii. \quad R_i R_j R_k &= \tilde{g}_{ijk} Y \\
    iii. \quad R_i R_j R_k R_l &= 0
\end{align*}  \quad (5.7)
where $Y$ is a nonzero matrix independent of $i$ and the $\tilde{g}_{ijk}$ are the “topological” limiting values of the Yukawa couplings, predicted to coincide with the intersection numbers in the cohomology of the mirror. These relations give then a characterization of the large complex structure limit and the mirror map that is basis-independent, provided that the cubic form defined by the coefficients $\tilde{g}_{ijk}$ is sufficiently nondegenerate to be a candidate for an intersection form on a Calabi–Yau threefold. Among these nondegeneracy conditions is the following:

**iv.** For each $i$, there exist $j$ and $k$ such that $\tilde{g}_{ijk} \neq 0$.

We apply this criterion to locate the large complex structure limit points for our family, using the compactification depicted in Fig. 3.4. The first boundary point to consider is the point $P_-$, at which the local monodromy matrices are $\tilde{B} = T_\infty B T_\infty^{-1}$, and $T$. Since

$$(\alpha(\tilde{B} - 1) + \beta(T - 1))^2 = 0$$

for all $\alpha$, $\beta$, these matrices violate condition *iv*, so $P_-$ cannot be a large complex structure limit point. It follows that $P_+ = \mathcal{I}(P_-)$ is not a large complex structure limit point either.

We next consider the boundary points which map to $[0, 1, 0]$ in $\mathbb{P}_2^{(3,1,1)}$ (i.e. the origin in the $(x, z)$-plane). Monodromy calculations for these points are displayed in Table 5.1. For each point, loops are described which lie in curves transverse to the divisors meeting at the point in question. Most of these transverse curves map to curves through the origin in the $(x, z)$-plane, and the corresponding monodromy takes the form $T^k_{\infty}$ for some $k$ (depending on how many times the loop winds about the origin). The other transverse curves are of the form $z = \varepsilon$, and loops on these are identified as in the previous subsection.

Condition *iii* implies that each monodromy transformation near a large complex structure limit point must be unipotent. Now $T^2_{\infty}$ includes $e^{2\pi i/3}$ among its eigenvalues, so it cannot be unipotent; therefore, neither $E_1 \cap F_1$ nor $F_1 \cap G_1$ can be a large complex structure limit point. Eliminating $G_1 \cap B_{\text{con}}$ is more tricky, but if we calculate the expressions

$$(B - 1)^2 = 0 \quad \text{and} \quad (B - 1)(T^3_{\infty} - 1) = 0,$$

we see a violation of condition *iv*. Moreover, by applying $\mathcal{I}$ we deduce that none of $E_2 \cap F_2$, $F_2 \cap G_2$ or $G_2 \cap C_{\text{con}}$ is a large complex structure limit point.

However, $G_1 \cap D_{\infty}$ is a large complex structure limit point, which we will study further in the next section. Applying $\mathcal{I}$, we find another such point $G_2 \cap D_{\infty}$ with an identical structure. (In fact, although these two points appear distinct in our “simplified” moduli space, they are simply two different representatives of the same point in the true moduli space.)

To quickly verify that none of the other boundary points in Fig. 3.4 is a large complex structure limit point, we turn to an alternate method.
\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Point} & \text{Divisor} & \text{Transverse Curve} & \text{Loop} & \text{Monodromy} \\
\hline
E_1 \cap F_1 & E_1 & x = \varepsilon z & |z| = \varepsilon^2 & T_\infty \\
F_1 & x = \varepsilon^{-1} z^2 & |z| = \varepsilon^2 & T^2_\infty \\
\hline
F_1 \cap G_1 & F_1 & x = \varepsilon z^2 & |z| = \varepsilon^2 & T^2_\infty \\
G_1 & x = \varepsilon^{-1} z^3 & |z| = \varepsilon^2 & T^3_\infty \\
\hline
G_1 \cap D_\infty & G_1 & x = \varepsilon z^3 & |z| = \varepsilon & T^3_\infty \\
D_\infty & z = \varepsilon & |x| = \varepsilon^4 & D_\infty \\
\hline
G_1 \cap B_{\text{con}} & G_1 & x = (1 + \varepsilon) z^3 & |z| = \varepsilon & T^3_\infty \\
B_{\text{con}} & z = \varepsilon & |x - \varepsilon^3| = \varepsilon^4 & B \\
\hline
\end{array}
\]

**Table 5.1:** Monodromy calculations for points mapping to \([0, 1, 0]\).

### 5.8. The Picard–Fuchs equations

In this subsection, we use the differential equations satisfied by the cohomology classes of \(\mathcal{M}\) to calculate monodromy around the divisors of the compactification of the moduli space described in Section 3, and finish the verification that there are no additional large complex structure limit points.

These differential equations can be obtained as explained in Refs.\([3,4,19–22]\). In the notation of \([4]\), we choose the basis for \(H^3(\mathcal{M})\) corresponding to the choice of monomials

\[
\begin{align*}
x_0 x_1 x_2 x_3 x_4 x_5 \\
x_0^2 x_1^2 x_2 x_3 x_4 x_5 \\
x_0^2 x_1^2 x_2^2 x_3 x_4 x_5 \\
x_0^3 x_1^3 x_2^3 x_3^3 x_4 x_5 \\
x_0^3 x_1^3 x_2^3 x_3 x_4^2 x_5 \\
x_0^4 x_1^4 x_2^4 x_3^3 x_4^2 x_5
\end{align*}
\]

The differential equations take the matrix form

\[
\frac{\partial R}{\partial \psi} = R M_\psi , \quad \frac{\partial R}{\partial \phi} = R M_\phi
\]  

(5.8)
The matrices $M_\psi, M_\phi$ can be determined as described in Refs. [4,19]. We find

$$M_\psi = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{\psi}{108\Delta} \\
0 & 0 & 54\psi & 0 & 0 & -\frac{\psi(122\rho^6 + 57\phi)}{6\Delta} \\
-2^23^7\psi^3 & 0 & 0 & 0 & 0 & -\frac{2^23^513\psi^5}{\Delta} \\
0 & 0 & 0 & 0 & 54\psi & 3\psi(31\rho^{12} + 57\rho^6\phi + 21\phi^2) \\
0 & -2^23^7\psi^3 & -2^33^{10}\psi^5 & 0 & 0 & 2\cdot 3^6\psi^5(242\rho^6 + 237\phi) \\
0 & 0 & 0 & -2^23^7\psi^3 & -2^33^{10}\psi^5 & -\frac{2^43^{10}\psi^5(\rho^6 + \phi)^2}{\Delta} \\
\end{pmatrix}$$

$$M_\phi = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{1944Z} & \frac{\psi^2(\rho^{12} + 3\rho^6\phi + 3\phi^2)}{648Z\Delta} \\
-3 & 0 & 0 & -\frac{(13\rho^6 + 57\phi)}{108Z} & 0 & -\frac{\alpha\psi^2}{36Z\Delta} \\
0 & 0 & 0 & -\frac{279\psi^4}{2Z} & 0 & -\frac{\beta}{1944Z\Delta} \\
0 & -3 & 0 & \frac{(\rho^{12} + 3\rho^6\phi + 21\phi^2)}{6Z} & 0 & \frac{\gamma\psi^2}{2Z\Delta} \\
0 & 0 & -3 & \frac{2\cdot 3^6\psi^4(\rho^6 + 2\phi)}{Z} & 0 & \frac{\delta}{108Z\Delta} \\
0 & 0 & 0 & -\frac{2\cdot 3^6\psi^4(\rho^{12} + 3\rho^6\phi + 3\phi^2)}{Z} & -3 & -\frac{\epsilon}{6Z\Delta} \\
\end{pmatrix}$$
where we have defined

\[ \rho = (162)^{1/3} \psi \]

\[ \Delta = (\rho^6 + \phi)^3 - 1 \]

\[ Z = 1 - \phi^3 \]

\[ \alpha = 13\rho^{18} + 96\rho^{12}\phi + 210\rho^6\phi^2 + 62\phi^3 + 109 \]

\[ \beta = 31\rho^{18} + 93\rho^{12}\phi + 93\rho^6\phi^2 - 125\phi^3 + 125 \]

\[ \gamma = \rho^{24} + 6\rho^{18}\phi + 33\rho^{12}\phi^2 + 42\rho^6\phi^3 + 30\rho^6 + 9\phi^4 + 54\phi \]

\[ \delta = 18\rho^{24} + 90\rho^{18}\phi + 162\rho^{12}\phi^2 - 116\rho^6\phi^3 + 224\rho^6 - 201\phi^4 + 201\phi \]

\[ \epsilon = \rho^{30} + 6\rho^{24}\phi + 15\rho^{18}\phi^2 - 17\rho^{12}\phi^3 + 35\rho^{12} - 60\rho^6\phi^4 + 69\rho^6\phi - 33\phi^5 + 33\phi^2 \]

The asymptotic behavior of solutions can be calculated along each of the boundary divisors in the moduli space. When this is done, we learn that the monodromy around \( E_0 \) includes \( e^{2\pi i/18} \) among its eigenvalues, while the monodromies around \( E_3 \) and \( D_0 \) include \( e^{2\pi i/6} \) among their eigenvalues. It follows that none of these monodromies can be unipotent, so no large complex structure limit points can occur along these divisors. This takes care of all remaining normal crossing boundary points.
6. The Mirror Map

6.1. Flat coordinates and symplectic basis

We wish to find the explicit map between the (extended) Kähler-cone of $\mathcal{M}$ and the space of complex structures of its mirror. To this end we introduce the period vector

$$\Pi = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ z^0 \\ z^1 \\ z^2 \end{pmatrix}, \quad G_a = \frac{\partial G}{\partial z^a} \quad (6.1)$$

such that the new periods correspond to a basis that is integral and symplectic. In other words we need to find a homology basis $(A^a, B^b)$, $a, b = 0, 1, 2$ with

$$A^a \cap A^b = 0 \quad , \quad B_a \cap B_b = 0 \quad , \quad A^a \cap B_b = \delta^a_b .$$

The components of $\Pi$ are then given by

$$z^a = \int_{A^a} \Omega \quad , \quad G_a = \int_{B_a} \Omega .$$

(We are working near the toric large complex structure limit point, where it is appropriate to use $\Omega$ to represent the 3-form, and $\varpi$ to describe its periods.) We may choose $A^0$ to be the torus corresponding to our fundamental period $\varpi_0$ and $B_0$ to be the three–sphere that shrinks to zero at the conifold. Thus

$$z^0 = \varpi_0 \quad \text{and} \quad G_0 = \varpi_1 - \varpi_0 .$$

As explained in [2], $A^0$ and $B_0$ meet in a single point. For a given choice of symplectic basis $(A^a, B^b)$ there will be a constant real matrix $m$ such that

$$\Pi = m \varpi .$$

On the Kähler side the analogue of the decomposition into $A$ and $B$ cycles is the decomposition of a vector

$$\Pi = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ w^0 \\ w^1 \\ w^2 \end{pmatrix}, \quad F_a = \frac{\partial F}{\partial w^a} \quad (6.2)$$
with respect to $H^0 \oplus H^2 \oplus H^4 \oplus H^6$. The generators of $H^0$ and $H^6$ are special and we identify them with $A^0$ and $B_0$. The flat structure here is identified with the natural flat structure on the Kähler-cone

$$B + iJ = t^j e_j$$

(6.3)

where the $e_j$ are a basis for $H^2(\mathcal{M}, \mathbb{Z})$ and $t^j = w^j / w^0$. For the case in hand we may take $e_j = (H, L)$.

Mirror symmetry implies that the vectors $\Pi$ and $\mathcal{I}$ are equal up to an $Sp(6, \mathbb{Z})$ transformation. This observation allows the mirror map to be determined. From our discussion in Section 5 we know that there must exist monodromies $S_i$ corresponding to $t^j \to t^j + \delta_i^j$. Hence, there must exist periods such that their ratios translate by an integer under certain monodromies satisfying (5.7). Since these relations are basis independent, we can work directly with the vector $\varpi$ and use our results in Section (6.4).

We have argued previously that the relevant monodromies are those associated with transport about the curves that meet transversely at the large complex structure point. The monodromies of $\varpi$ about these curves are

$$G_1 = T^3_\infty \quad , \quad D_\infty = (ATB)^{-1} = B^{-1}T_\infty \quad .$$

which we identify as $S_1 = G_1$ and $S_2 = D_\infty$. We then set

$$R_1 = T^3_\infty - 1 \quad , \quad R_2 = B^{-1}T_\infty - 1$$

and we check that

$$[R_1, R_2] = 0$$

$$R_1^3 = 9Y \quad , \quad R_2^3 = 3Y \quad , \quad R_1R_2^2 = Y \quad , \quad R_2^3 = 0$$

$$R_1Y = 0 \quad , \quad R_2Y = 0$$

with $Y$ a certain matrix. We see that $R_1$ and $R_2$ have the same algebra as $H$ and $L$ and conclude that $G_1 \cap D_\infty$ indeed corresponds to the large complex structure limit.

To determine the flat coordinates we look for ratios of periods, $t^1$ and $t^2$, that under $S_1$ and $S_2$ transform as $t^j \to t^j + \delta_i^j$. In this way we obtain

$$t^1 = \frac{\varpi_3 - \varpi_0}{\varpi_0} \quad , \quad t^2 = \frac{\varpi_4 + \varpi_5 - 2\varpi_3 + 2\varpi_0}{\varpi_0}$$

(6.4)

where we have chosen additive constants so as to simplify later constructions. These relations constitute the mirror map, they express the flat coordinates in terms of the $\psi$ and $\phi$.

Our next task is to find the symplectic basis. Following a procedure that is presented in detail in [1], we first identify $\Pi$ with $\Pi$. This leaves twelve undetermined parameters in
the matrix \( m \), corresponding to the unknown periods \( F_1 \) and \( F_2 \). To fix these parameters we make use of the prepotential

\[
\mathcal{F} = -\frac{1}{6w^0} \left( 9(w^1)^3 + 9(w^1)^2 w^2 + 3w^1(w^2)^2 \right) + \frac{1}{2} \left( \alpha (w^1)^2 + 2\beta w^1 w^2 + \gamma (w^2)^2 \right) + \left( \delta - \frac{3}{4} \right) w^0 w^1 + \epsilon w^0 w^2 \right) + \xi (w^0)^2 + \cdots ,
\]

to construct the vector \( \Xi \) and derive the monodromy matrices associated to \( t^j \rightarrow t^j + \delta_i^j \).

These \( S_i \) are related to \( S_i \) in the \( \varpi \) basis by \( S_i = m S_i m^{-1} \). Implementing these conditions gives all of the undetermined parameters in \( m \) as linear combinations of the parameters \((\alpha, \beta, \ldots, \epsilon)\).

The next step is to require that the monodromy matrix \( T = m T m^{-1} \) be symplectic. This determines \( \delta \) and \( \epsilon \)

\[
\delta = 5 \quad , \quad \epsilon = \frac{3}{2} .
\]

It remains to find \( \alpha, \beta, \gamma \). The matrix \( A = m A m^{-1} \) must also be integral and symplectic. It is symplectic for all values of the parameters but is integral only if \( \gamma \) is an integer while \( \alpha \) and \( \beta \) are half–integers. We take

\[
\alpha = \frac{9}{2} \quad , \quad \beta = \frac{3}{2} \quad , \quad \gamma = 0 .
\]

Any other choice is related to this by an \( \text{Sp}(6, \mathbb{Z}) \) transformation. The result is that the matrix \( m \) is given by

\[
m = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & -2 & 1 & 1
\end{pmatrix} .
\]

For completeness we also record the monodromies in the symplectic basis,

\[
A = \begin{pmatrix}
\begin{array}{cccccc}
-2 & 0 & 1 & -3 & -1 & 0 \\
-2 & 1 & 0 & -2 & 3 & 1 \\
-1 & 1 & -2 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & -3 & -1 & 1 & 0 \\
2 & -3 & 9 & 2 & 0 & 1
\end{array}
\end{pmatrix} , \quad T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} .
\]

These are the generators of the the duality group \( \mathcal{D} \subset \text{Sp}(6, \mathbb{Z}) \).
Section 6.2. Inversion of the mirror map

Our aim is to obtain \( \psi(t^1, t^2) \) and \( \phi(t^1, t^2) \). To begin we express the relations (6.4) as expansions valid for large \( \rho \). To this end, notice that the flat coordinates

\[
t^1 = \frac{\varpi_3 - \varpi_0}{\varpi_0}, \quad t^2 = \sum_\sigma \frac{\varpi_3 + \sigma - 3\varpi_3 + 2\varpi_0}{\varpi_0}
\]

can be expressed entirely in terms of the quantities calculated in (4.28) and (4.29). This leads to the expansions

\[
2\pi i t^1 = -\pi i - \log \left( \frac{(18\psi)^6}{3\phi} \right) + \frac{1}{\varpi_0} \sum_{k=0}^{\infty} \frac{(6k)!(3\psi)^6}{k!(2k)!(3k)!} \left[ A_k U_k(\phi) + Y_k(\phi) + N_k(\phi) \right]
\]

\[
2\pi i t^2 = 3\pi i - \log(3\phi^3) - \frac{3}{\varpi_0} \sum_{k=0}^{\infty} \frac{(6k)!(3\psi)^6}{k!(2k)!(3k)!} N_k(\phi)
\]

To proceed, we introduce ‘large complex structure’ coordinates

\[
X_1 = \frac{(18\psi)^6}{3\phi}; \quad X_2 = (3\phi^3)
\]

which, up to signs, are the inverses of the coordinates found by toric methods in Section 3. We also define functions

\[
u_k(\phi) \overset{\text{def}}{=} \phi^{-k} U_k(\phi) ; \quad \tilde{u}_k(\phi) \overset{\text{def}}{=} \phi^{-k} \left( A_k U_k(\phi) + Y_k(\phi) + N_k(\phi) \right), \quad \nu_k(\phi) \overset{\text{def}}{=} \phi^{-k} N_k(\phi)
\]

and set

\[
q_1 = e^{2\pi i t_1}, \quad q_2 = e^{2\pi i t_2}.
\]

In this way we are able to rewrite the mirror map in a form that is amenable to iterative inversion

\[
q_1 = -\frac{1}{X_1} \exp \left\{ \frac{1}{\varpi_0} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{k!(2k)!(3k)!} X_1^k \tilde{u}_k(X_2) \right\}
\]

\[
q_2 = -\frac{1}{X_2} \exp \left\{ -\frac{3}{\varpi_0} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{k!(2k)!(3k)!} X_1^k \nu_k(X_2) \right\}
\]

and in these expressions \( \varpi_0 \) is to be expanded in the form

\[
\varpi_0 = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{k!(2k)!(3k)!} X_1^k u_k(X_2).
\]
The coordinates $X_1$ and $X_2$ may be regarded as automorphic functions of the duality group. The mirror map (6.6) is naturally inverted to give these functions as expansions in the uniformizing variables $q_1$ and $q_2$. Thus to third order:

\[
X_1 = -\frac{1}{q_1} (1 + 312q_1 + 2q_2 + 10260q_1^2 - 540q_1q_2 - q_2^2 \\
- 901120q_1^3 + 120420q_1^2q_2 + 20q_2^3 + \cdots)
\]

\[
X_2 = -\frac{1}{q_2} (1 + 180q_1 - 6q_2 + 11610q_1^2 + 180q_1q_2 + 27q_2^2 \\
+ 514680q_1^3 - 150120q_1^2q_2 - 5040q_1q_2^2 - 164q_2^3 + \cdots)
\]

(6.7)

Notice that the large radius limit $\Im t^j \to \infty$ manifestly corresponds to the large complex structure limit $X_j \to \infty$. 

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7. Instanton Expansions

7.1. Yukawa couplings

The four Yukawa couplings \( y_{\alpha\beta\gamma} \), where \( \alpha, \beta, \gamma \) run over \( \psi \) and \( \phi \), can be computed by means of a calculation in the ring of the defining polynomial [23]. Multiplying three deformations of \( p \) and reducing the result modulo the Jacobian ideal of \( p \) identifies the couplings through the relation

\[
\partial_\alpha p \partial_\beta p \partial_\gamma p \simeq y_{\alpha\beta\gamma} \frac{h}{\langle h \rangle} \tag{7.1}
\]

where \( h \) denotes the determinant of the matrix of second derivatives of \( p \). \( \langle h \rangle \) is a normalization factor, independent of the parameters, that can be fixed from our knowledge of the periods since we also have the relation

\[
y_{\alpha\beta\gamma} = -\Pi^T \Sigma \partial_{\alpha\beta\gamma} \Pi = -\varpi^T \sigma \partial_{\alpha\beta\gamma} \varpi \tag{7.2}
\]

with

\[
\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = m^T \Sigma m .
\]

Using the expansions (4.8) for small \( \psi \) to fix the normalization and the ring result (7.1) we have

\[
y_{\psi\psi\psi} = -\frac{i}{1536 \pi^3} \frac{(18\psi)^{15}}{\tilde{\phi}^3 - 1} \quad y_{\psi\psi\phi} = -\frac{i}{128 \pi^3} \frac{(18\psi)^{10}}{\phi^3 - 1}
\]

\[
y_{\psi\phi\phi} = -\frac{3i}{32 \pi^3} \frac{(18\psi)^5}{\phi^3 - 1} \quad y_{\phi\phi\phi} = -\frac{9i}{8 \pi^3} \left( \frac{1}{\phi^3 - 1} - \frac{1}{\phi^3 - 1} \right)
\]

with

\[
\tilde{\phi} \overset{\text{def}}{=} \phi + \rho^6 .
\]

If we denote the parameters by \( \varphi^\alpha \) and form the tensor \( y = y_{\alpha\beta\gamma} d\varphi^\alpha d\varphi^\beta d\varphi^\gamma \) then we find the surprisingly simple expression

\[
y = \frac{9i}{8 \pi^3} \left\{ \frac{d\phi^3}{\phi^3 - 1} - \frac{d\tilde{\phi}^3}{\tilde{\phi}^3 - 1} \right\} .
\]
We compute now the couplings in the flat basis, that is in the coordinate basis \((t^1, t^2)\) in which \(w^0 = 1\). This introduces a factor of \(1/w_0^2\) in addition to the usual tensor transformation rules. From the inverse mirror map we find expansions in the variables \(q_j\):

\[
y_{111} = 9 + 540q_1 + 4860q_1^2 + 15120q_1^3 - 1080q_1q_2 + 1146960q_1^2q_2 + 2700q_1q_2^2 + \cdots
\]

\[
y_{112} = 3 - 1080q_1q_2 + 573480q_1^2q_2 + 5400q_1q_2^2 + \cdots
\]

\[
y_{122} = 1 - 1080q_1q_2 + 286740q_1^2q_2 + 10800q_1q_2^2 + \cdots
\]

\[
y_{222} = 0 + 3q_2(1 - 360q_1 + 47790q_1^2 - 15q_2 + 7200q_1q_2 + 244q_2^2 + \cdots)
\]

These expansions are compatible with the general form, established in Ref. [24], which is

\[
y_{abc} = \bar{y}_{abc} + \sum_{j,k=0}^{\infty} \frac{c_{abc}(j,k) n_{jk} q_j^j q_k^k}{1 - q_1^1 q_2^k}
\]

(7.3)

with the quantities \(c_{abc}\) given by

\[
\begin{pmatrix}
c_{111} \\
c_{112} \\
c_{122} \\
c_{222}
\end{pmatrix} = \begin{pmatrix}
j^3 \\
j^2 k \\
j k^2 \\
k^3
\end{pmatrix}
\]

Values for the instanton numbers \(n_{jk}\) are displayed in Table 7.1. These numbers have been found independently in [10].

| \(j\) | \(k=0\) | \(k=1\) | \(k=2\) | \(k=3\) | \(k=4\) | \(k=5\) | \(k=6\) | \(k=7\) |
|---|---|---|---|---|---|---|---|---|
| 0 | \(\ast\) | 3 | -6 | 27 | -192 | 1695 | -17064 | 188454 |
| 1 | 540 | -1080 | 2700 | -17280 | 154440 | -1640520 | 19369800 |
| 2 | 540 | 143370 | -574560 | 5051970 | -57879900 | 751684050 |
| 3 | 540 | 204071184 | 74810520 | -913383000 | 13593850920 |
| 4 | 540 | 21772947555 | -49933059660 | 224108858700 |
| 5 | 540 | 1076518252152 | 7772494870800 |
| 6 | 540 | 33381348217290 |
| 7 | 540 |

**Table 7.1:** Numbers of instantons of type \((j,k)\) for \(1 \leq j + k \leq 7\). The numbers \(n_{j0}\) are equal to 540 for all \(j\).
7.2. An SL(2, \mathbb{Z}) action on $D_\infty$

A curious fact is apparent from Table 7.1; we see that $n_{j0} = 540$ for all $j$. Related to this is the fact that when $q_2 = 0$ the couplings take the values:

\[
y_{111} = 9 + 540 \sum_{k=0}^{\infty} \frac{k^3 q^k}{1-q^k} = \frac{27}{4} + \frac{9}{4} E_2
\]
\[
y_{112} = 3
\]
\[
y_{122} = 1
\]
\[
y_{222} = 0
\]

Where

\[
E_2(t) \overset{\text{def}}{=} 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m+nt)^4}
\]
\[
= 1 + 240 \sum_{k=0}^{\infty} \frac{k^3 q^k}{1-q^k}, \quad q = e^{2\pi it}
\]

is the Eisenstein function of weight two[25]. Recall that a function, $f$, is automorphic of weight $m$ if under an SL(2, \mathbb{Z}) transformation

\[
t \mapsto \tilde{t} = \frac{at+b}{ct+d}
\]

it transforms according to the rule

\[
f(t) \mapsto f(\tilde{t}) = (ct+d)^{2m} f(t).
\]

In order to see the SL(2, \mathbb{Z}) action we set

\[
t = t_1
\]
\[
s = \frac{3}{2} t_1 + t^2
\]

and observe that

\[
A^3 t = \frac{\varpi_6 - \varpi_3}{\varpi_3} = -\frac{\varpi_0}{\varpi_3} = -\frac{1}{t+1}
\]
\[
T^3 \infty t = t + 1.
\]

So the operations $A^3$ and $T^3$ generate an SL(2, \mathbb{Z}) when acting on $t$. We set

\[
r = e^{2\pi is} = q_1^\frac{4}{q_2}.
\]

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The limit \( q_2 \to 0 \) with \( q_1 \) finite is the limit \( r \to 0 \) with \( q_1 \) finite. From (6.6) we see that when \( r \) is small we have the asymptotic relation

\[
    r \sim \frac{\text{const.}}{\rho^9 \phi^2}.
\]

(7.4)

So the locus \( r = 0 \) is \( D_\infty \) and we see from (7.4) that

\[
    \mathcal{A}^3 r = r \quad \text{and} \quad \mathcal{T}_\infty^3 r = -r
\]

and hence that the locus \( r = 0 \) is preserved by \( \mathcal{A}^3 \) and \( \mathcal{T}_\infty^3 \).

With respect to the coordinates \( t \) and \( s \) the couplings become

\[
    \begin{align*}
    y_{ttt} &= \frac{9}{4} E_2(t) \\
    y_{tts} &= 0 \\
    y_{tss} &= 1 \\
    y_{sss} &= 0.
    \end{align*}
\]

(7.5)

Now the holomorphic three-form \( \Omega \) is invariant under the \( \text{SL}(2,\mathbb{Z}) \) generators in the gauge \( w^0 = \varpi_0 \). Achieving the gauge \( w^0 = 1 \) requires dividing by \( \varpi_0 \) which is not invariant under \( \mathcal{A}^3 \). In fact

\[
    \mathcal{A}^3 \varpi_0 = \varpi_3 = (t+1)\varpi_0
\]

so in the new gauge \( \Omega \) has weight \(-\frac{1}{2}\). Furthermore

\[
    \frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} = (ct+d)^2 \frac{\partial}{\partial t}
\]

so each \( t \)-derivative counts as weight one. Thus we see that \( y_{ttt} \) has weight two. The relations (7.5) show that the Yukawa coupling, in the gauge \( w^0 = 1 \), is nonsingular on \( D_\infty \). This was perhaps to be expected since the loci \( B_{\text{con}} \) and \( C_{\text{con}} \) where the coupling is singular do not intersect \( D_\infty \) in the resolved moduli space. Turning the argument around; if we assume that the coupling is nonsingular then since the only automorphic function of weight two that is regular on the upper half \( t \)-plane and bounded as \( q \defeq e^{2\pi i t} \to 0 \) is \( E_2 \) it must be the case that \( y_{ttt} \) is proportional to \( E_2 \). The coupling \( y_{tts} \) has weight one and must vanish since there is no automorphic function of weight one. The coupling \( y_{tss} \) has weight zero and must be a constant since the only automorphic function of weight zero that is bounded as \( q \to 0 \) is a constant. Finally \( y_{sss} \) has weight \(-1\) and must vanish since there is no automorphic function of this weight.

We may also write the large complex structure variable

\[
    X_1 = \frac{(18\psi)^6}{3\phi} = \frac{432\rho^6}{\phi}
\]
in terms of automorphic functions. Clearly \( X_1 \) is invariant under the generators \( A^3 \) and \( T^3_\infty \), when \( q_2 = 0 \), so we expect \( X_1 \) to be related to the \( J \)-invariant

\[
12^3 J(t) = \frac{1}{q_1} + 744 + 19688 q_1 + \cdots.
\]

Now \( t_1 \), being a ratio of periods, is invariant under \( I \) so \( J(t) \) is also invariant. We see however that \( X_1 \) is not invariant. In fact we have

\[
IX_1 = -432 \frac{X_1}{X_1 + 432}, \quad T^2 X_1 = X_1.
\]

The basic invariant is

\[
X_1 + IX_1 = \frac{X_1^2}{X_1 + 432}
\]

any other invariant combination being a function of this one. We therefore expect a relation of the form

\[
\frac{X_1^2}{X_1 + 432} = f(J)
\]

with \( f \) a rational function of \( J \). From (6.4) we see that \( X_1 \sim -q_1^{-1} - 312 \) as \( q_1 \to 0 \) and this information is sufficient to determine that \( f(J) = -12^3 J \). Thus the relation is

\[
\frac{X_1^2}{X_1 + 432} = -12^3 J(t)
\]

or equivalently

\[
\frac{\rho^6}{\phi} = -2J(t) \left\{ 1 + \sqrt{1 - \frac{1}{J(t)}} \right\}.
\]

7.3. Instantons of genus one

Following Bershadsky et al. [9] we consider the index \( F_1 \) defined by a certain path integral and whose topological limit for two-parameter Calabi–Yau threefolds is given by

\[
F_{1}^{\text{top}} = \log \left[ \left( \frac{\psi}{\varphi_0} \right)^{5-12} \frac{\partial(\psi,\phi)}{\partial(t^1,t^2)} \right] + \text{const.}, \quad (7.6)
\]

The holomorphic function \( f \) is determined by requiring regularity of \( F_{1}^{\text{top}} \) at smooth points of moduli space and by imposing the large radius limit condition

\[
F_{1}^{\text{top}} \sim -\frac{2\pi i}{12} c_2 \cdot (B + iJ) \quad (7.7)
\]

\[
= -\frac{2\pi i}{12} c_2 \cdot (t^1 H + t^2 L)
\]

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The regularity conditions are most easily implemented by using the mirror moduli. Thus, $F_{1}^{\text{top}}$ can only diverge at the singular loci $\tilde{\phi}^{3} = 1$ and $\phi^{3} = 1$. Furthermore $F_{1}^{\text{top}}$ must be regular at $\psi = 0$ since the corresponding manifold is nonsingular.

Since $\varpi_{0} \sim \psi$ for $\psi$ small, we conclude that $f$ must have the form

\[ f = (\tilde{\phi}^{3} - 1)^{a} (\phi^{3} - 1)^{b} \psi^{c} \]

(7.8)

where the exponent $c$ is fixed by the behavior of the Jacobian $\frac{\partial (\psi, \phi)}{\partial (t_{1}, t_{2})}$ at $\psi = 0$. From the mirror map (6.4) and the period expansions (4.8) we find that the leading term of this Jacobian is $\psi^{-3}$ hence $c = 3$. The remaining exponents are then determined by the large radius limit condition. From (7.6), (7.8) and the inverse mirror map (6.7), we see that

\[ F_{1}^{\text{top}} \sim -\frac{2\pi i}{12} \left\{ [108 + 36a] t^{1} + [40 + 12a + 12b] t^{2} \right\} . \]

For the model $\mathbb{P}_{4}^{(1,1,1,6,9)}[18]$ we have

\[ \chi = -540 , \quad c_{2} \cdot H = 102 , \quad \text{and} \quad c_{2} \cdot L = 36 \]

hence we find

\[ a = -1/6 \quad \text{and} \quad b = -1/6 . \]

In virtue of mirror symmetry $F_{1}^{\text{top}}$ enjoys an expansion

\[ F_{1}^{\text{top}} = -\frac{2\pi i}{12} c_{2} \cdot (B + iJ) + \text{const.} - \sum_{jk} \left[ 2d_{jk} \log \eta(q_{1}^{j}q_{2}^{k}) + \frac{1}{6} n_{jk} \log (1 - q_{1}^{j}q_{2}^{k}) \right] , \]  

(7.9)

where $\eta$ denotes the Dedekind $\eta$-function\(^{3}\) and $d_{jk}$ and $n_{jk}$ are the numbers of instantons of genus one and genus zero.

Comparing this expansion with the expansion that results from substituting the explicit form for $f$ that we have found in (7.6) we find values for the $d_{jk}$ displayed in Table 7.2.

\(^{3}\) Note that, as observed in [9], shifting $\frac{\partial}{\partial \tau} \log \eta$ by a constant does not affect the final outcome. We did this for simplicity in [1].
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{$j$} & \textbf{$k=0$} & \textbf{$k=1$} & \textbf{$k=2$} & \textbf{$k=3$} & \textbf{$k=4$} & \textbf{$k=5$} & \textbf{$k=6$} & \textbf{$k=7$} \\
\hline
0 & * & 0 & 0 & -10 & 231 & -4452 & 80958 & -1438086 \\
1 & 3 & -6 & 15 & 4764 & -154662 & 3762246 & -82308270 & \\
2 & 0 & 2142 & -8568 & -1079298 & 48907800 & -1510850250 & \\
3 & 0 & -280284 & 2126358 & 152278992 & -9759419622 & \\
4 & 0 & -408993990 & 521854854 & -16704086880 & \\
5 & 0 & -44771454090 & 1122213103092 & \\
6 & 0 & -2285308753398 & \\
7 & 0 & \\
\hline
\end{tabular}
\end{center}
\caption{Values $d_{jk}$ of genus one instantons for $1 \leq j + k \leq 7$}
\end{table}
8. Verification of Some Instanton Numbers

In this section, we verify selected instanton numbers which occur in Table 7.1 and Table 7.2. In the process, we observe some new phenomena which did not arise in [1]: how to calculate some instanton contributions from the topology of singular instanton moduli spaces, and the occurrence of negative instanton contributions for rational curves.

The first fact we will use is the following. Suppose that a complete family of Gorenstein curves is parameterized by a nonsingular manifold $B$ of dimension $b$. Then the instanton contribution is the Chern number $c_b(\Omega_B^1)$, where $\Omega_B^1$ is the holomorphic cotangent bundle of $B$. Equivalently, this is $(-1)^b e(B)$, where $e(B)$ is the Euler characteristic of $B$.

There are a few ingredients used in establishing this fact. First we show that the deformation-theoretic obstruction bundle in this situation is just $\Omega_B^1$. Next, we show that a deformation of almost complex structure gives rise to a $C^\infty$ section of the obstruction bundle; the zero locus of such a section gives the parameter values for which the corresponding curve deforms to a pseudo-holomorphic curve on the infinitesimally nearby almost complex manifold. It can be shown by McDuff’s transversality theorem [26] that for rational curves, the generic deformation yields only finitely many curves that deform, and furthermore that there are no higher-order obstructions, i.e. the curves that infinitesimally deform actually deform in a sufficiently small but finite deformation.

For the first assertion, we assume for simplicity of exposition that we have a smooth curve $C \subset X$. Consider the normal bundle $N$ of $C$ in $X$. For the assertion about the form of the obstruction bundle, we first assert that the obstruction bundle is the natural bundle with fiber equal to $H^1(N)$ \footnote{The statement probably remains true even if the restriction to Gorenstein curves is removed.}. The Calabi-Yau condition leads to $\wedge^2 N \simeq \Omega_C^1$. Also, deformation theory describes $H^0(N)$ as the space of first order deformations of $C$ inside $X$, and $H^1(N)$ as the space of obstructions. Serre duality gives an isomorphism

$$H^0(N) \otimes H^1(N) \to H^1(\wedge^2 N) \simeq H^1(\Omega_C^1) \simeq \mathbb{C}.$$  

The last isomorphism is canonically fixed by sending the fundamental class of a point to 1. Since $H^0(N)$ is canonically isomorphic to the tangent space to $B$ at the point of $B$ corresponding to $C$, it follows that $H^1(N)$ is canonically isomorphic to the cotangent space of $B$. A generic deformation of almost complex structure of $X$ induces an obstruction class in $H^1(N)$ whose vanishing is necessary and sufficient for the first-order deformation of $C$ to a pseudo-holomorphic curve on the nearby almost complex manifold. As the curve varies over the parameter space $B$, we get a smooth section of $\Omega_B^1$ (whose value at $C$ \footnote{This was asserted without proof in [27]; we will sketch the proof of this presently.}
is the corresponding element of \( H^1(N) \), with the indicated identifications. We conclude by observing that \( c_0(\Omega^1_B) \) is the number of zeros (counted with multiplicity) of a general section of \( \Omega^1_B \). We will see below that negative multiplicities are possible.

We next turn to the calculation of the obstruction bundle. To do this, it will be helpful to adapt the deformation theory of almost complex structures [28] to our situation.

We allow the almost complex structure of \( X \) to vary by varying the holomorphic cotangent space \( T^*1,0 \). We accomplish this by varying the projection map \( T^*_C \to T^*1,0 \) to a family of projection maps

\[
\pi_t : T^*_C \to T^*1,0 \tag{8.1}
\]

and for each \( t \) defining the deformed holomorphic cotangent space \( T^*_t1,0 = \ker(\pi_t) \).

The adjoint map of tangent spaces

\[
\pi_t^* : T_{0,1} \to T_C \tag{8.2}
\]

has its image annihilated by \( T^*_t1,0 \); hence it is the deformed antiholomorphic tangent space \( T^t_{01} \). After taking complex conjugates to get \( T^t_{1,0} \), it is easy to write down a projection with kernel \( T^t_{1,0} \), giving a convenient description of \( T^t_{1,0} \).

To describe \( T_t \), it suffices to give \( \pi_t|_{T^*1,0} : T^*1,0 \to T^*0,1 \). In local holomorphic coordinates \( z_i \) on \( X \) we describe this data by a tensor

\[
A = A^i_j \frac{\partial}{\partial z_i} \otimes dz^j, \tag{8.3}
\]

and almost complex structures near \( X \) are parametrized by tensors \( A \) as in (8.3) which are near \( 0 \).

Carrying out the computation outlined above, we find that \( T^t_{1,0} \) is the kernel of the operator \( P_t \) with

\[
P_t \left( \frac{\partial}{\partial z^i} \right) = -A^i_j \frac{\partial}{\partial z^j} \quad P_t \left( \frac{\partial}{\partial z^j} \right) = \frac{\partial}{\partial z^j}. \tag{8.4}
\]

For ease of exposition we only illustrate the deformation theory of rational curves, contenting ourselves with a few comments about what changes in the case of elliptic curves.

So we consider a holomorphic map

\[
f : \mathbb{P}^1 \to X \tag{8.5}
\]

Using a local coordinate \( w \) on \( \mathbb{P}^1 \), we express \( f \) as \( z^i = f^i(w) \) locally. Now, vary the almost complex structure as a function of a parameter \( t \). We have \( A^i_j = A^i_j(t) \) and \( z^i = f^i(w, t) \).

The pseudo-holomorphicity condition is \( P_t f_* \left( \frac{\partial}{\partial w} \right) = 0 \). This becomes

\[
\frac{\partial f^\tau}{\partial w} \frac{\partial}{\partial z^\tau} - \frac{\partial f^j}{\partial w} A^\tau_j \frac{\partial}{\partial z^\tau} = 0. \tag{8.6}
\]
Taking complex conjugates, multiplying by $d\bar{w}$, and differentiating at $t = 0$, we get

$$\overline{\partial}(f'^i \frac{\partial}{\partial z^i}) = f^*(A').$$ \hspace{1cm} (8.7)

Here $f'^i$ means $\frac{\partial f^i}{\partial t}(w, 0)$ and

$$A' = \frac{\partial A^i}{\partial t}(z, 0) \frac{\partial}{\partial z^i} \otimes dz^j.$$ \hspace{1cm} (8.8)

Thus $A'$ is a $(0, 1)$ form on $\mathbb{P}^1$ with values in $f^*(T_{1,0})$. Equation (8.7) says that $A'$ represents the zero class in

$$H_{\overline{\partial}}^{0,1}(f^*(T_{1,0})) \simeq H^1(f^*T_{1,0}).$$

However, from the exact sequence

$$0 \to T_{\mathbb{P}^1} \to f^*(T_{1,0}) \to N \to 0$$ \hspace{1cm} (8.9)

and vanishing of $H^1(T_{\mathbb{P}^1})$, we conclude that $H^1(f^*T_{1,0}) \simeq H^1(N)$, and so the obstruction section $A'$ may be thought of as lying in the claimed obstruction bundle.

In this way, we get a $(C^\infty)$ obstruction section of $\Omega_B^1$. By McDuff’s transversality theorem, this section vanishes at finitely many points if the deformation is generically chosen. The Euler class (or top Chern class) of $\Omega_B^1$ computes the number of such zeros, where each zero is counted with multiplicity $\pm 1$ since the section is not holomorphic in general. Note that the space of pseudoholomorphic maps for the generic almost complex structure carries a preferred orientation [29]. There is then an induced orientation on the limiting set of maps. If this orientation differs from the orientation determined by the complex structure, then the associated multiplicity is $-1$. It can be seen that this multiplicity agrees with the multiplicity arising from the description of the limiting curves as the zero locus of the obstruction section.

For elliptic curves, there is a complication: elliptic curves have moduli, so the elliptic curve $E$ used as the source of a map analogous to (8.5) must be allowed to vary with the parameter $t$. This can be accomplished by allowing the local parameter $w$ for $E$ depend on $t$. In doing so, extra data is introduced corresponding to deformations of the complex structure of $E$; these are parametrized by $H^1(T_E)$, where $T_E$ is the holomorphic tangent bundle of $E$. Now even if the obstruction element of $H^1(f^*(T_{1,0}))$ does not vanish, we may be able to deform $E$ and so modify the obstruction section by an element of $H^1(T_E)$. Thus the true obstruction data lives in the quotient of $H^1(f^*(T_{1,0}))$ by the image of $H^1(T_E)$ given by the cohomology of the exact sequence (8.9); this sequence also tells us that the obstruction space is just $H^1(N)$.

We now immediately can verify some of the numbers in Table 7.1. For $n_{01}$, we must enumerate curves $C$ with $C \cdot H = 0$ and $C \cdot L = 1$. The first equality implies that $C$ is
contained in the exceptional divisor $E \simeq \mathbb{P}^2$; and the second equality shows that $C$ is a line in this $\mathbb{P}^2$. Since the lines in $\mathbb{P}^2$ are parametrized by (the dual) $\mathbb{P}^2$, we verify

$$n_{01} = c_2(\Omega^1_{\mathbb{P}^2}) = 3.$$ 

Similarly, $n_{02}$ counts the contribution of conics in $E \simeq \mathbb{P}^2$. Since conics are parametrized by $\mathbb{P}^5$, we have

$$n_{02} = c_5(\Omega^1_{\mathbb{P}^5}) = -6.$$ 

We interpret this as follows: given a general deformation of almost complex structure, at least 6 of the conics will deform to pseudo-holomorphic curves on the nearby almost complex manifold. Each deformed curve has a multiplicity of $\pm 1$ determined by its intrinsic orientation, and the algebraic sum of these multiplicities is $-6$. Unfortunately, the moduli space of rational curves of degree $k$ in $\mathbb{P}^2$ is more complicated for $k > 2$, so this is as much as we can say here without more work.

We can also verify the ratios $n_{1k}/n_{10}$ for $k = 1$ and 2. If $C$ satisfies $C \cdot H = 1$ and $C \cdot L = k$, then $C \cdot E = 1 - 3k < 0$, recalling that $E = H - 3L$. Thus $C$ has a component which is contained in $E$, and it follows immediately that $C$ is a union of a rational curve $C'$ with $C' \cdot H = 1$ and $C' \cdot L = 0$ and a degree $k$ curve in $E \simeq \mathbb{P}^2$. Note that $C' \cdot E = 1$, so that $C'$ meets $E$ in a unique point $p$. So for each curve $C'$ of type $(1, 0)$, we can take a degree $k$ curve $D$ in $E$ passing through $p$ to get a connected curve $C = C' \cup D$ (degenerate instantons must be connected, cf Appendix to [9]). The degree $k$ curves are parametrized by $\mathbb{P}^1$ for $k = 1$ and by $\mathbb{P}^4$ for $k = 2$. Since we get the same parameter space for any curve of type $(1, 0)$, we can verify

$$\frac{n_{11}}{n_{10}} = c_1(\mathbb{P}^1) = 2$$

and

$$\frac{n_{12}}{n_{10}} = c_4(\mathbb{P}^4) = -5.$$ 

Finally, we check that $n_{10} = 540$, and give some supporting geometric evidence for $n_{j0} = 540$ for all $j \geq 1$. A curve $C$ of type $(1, 0)$ satisfies $C \cdot L = 0$, hence is an elliptic curve, a fiber of the fibration discussed in section 2. We want to see when the elliptic curve can acquire a singularity, to allow it to be the image of a holomorphic map from $\mathbb{P}^1$. Recalling that $C$ is obtained by fixing the values of $x_1$, $x_2$, and $x_3$, we get an equation for $C$ of the form

$$ax_5^2 + bx_4^3 + cx_4x_5 + dx_4^2 + ex_5 + fx_4 + g = 0,$$

the constants being determined by the equation for $\hat{M} \subset \mathbb{P}^{(1,1,1,6,9)}$ and by the $x_1, x_2, x_3$ coordinates. It is easy to change coordinates to arrive at the form

$$x_5^2 + x_4^3 + fx_4 + g = 0.$$
Letting \((x_1, x_2, x_3)\) vary in its parameter space \(\mathbb{P}^2\), we realize that \(f = f(x_1, x_2, x_3)\) has degree 12, and \(g = g(x_1, x_2, x_3)\) has degree 18. The discriminant of our family of curves is

\[4f^3 - 27g^2 = 0\]  \hspace{1cm} (8.10),

a plane curve \(B\) of degree 36. \(B\) has cuspidal singularities at the 216 points where \(f = g = 0\). Since \(B\) therefore does not have a cotangent bundle, it is easier to perform the obstruction analysis on \(\tilde{B}\), the normalization of \(B\). By the genus formula, \(\tilde{B}\) has genus \((35)(34)/2 - 216 = 379\). A careful analysis shows that the obstruction section of \(\Omega^1_{\tilde{B}}\) has extraneous zeros at the points of \(\tilde{B}\) which lie over the singularities of \(B\). So the number of curves which deform in a generic deformation is \(c_1(\Omega^1_{\tilde{B}}) - 216 = 540\). An analogous calculation has been done recently for a different Calabi-Yau manifold in [30].

For \(j > 1\) there are no rational curves of type \((j, 0)\), so our instanton calculation is detecting degenerate instantons of type \((j, 0)\). One possible description is as follows. We have a family of rational curves parametrized by \(B\). All of these curves have arithmetic genus 1, but are instantons (i.e. rational) because of the singularity that each curve contains. If we assign to each curve the multiplicity \(j\) (in other words take the local equation of the curve inside the total space of the family, then raise this to the power \(j\)), this gives a family of curves with multiplicity \(j\) parametrized by the same moduli space \(B\). The adjunction formula gives that each of these curves again has arithmetic genus 1. It is natural to speculate that the singularity on each curve gives rise to an interpretation of the multiple curves as degenerate instantons. If this were true, then the geometric calculation of \(n_{j0}\) would be identical to the calculation of \(n_{10}\) above. A complete verification of this approach must wait for future work.

Let us now turn to Table 7.2. For type \((0, k)\), we have already seen that they are parametrized by curves of degree \(k\) in \(E \simeq \mathbb{P}^2\). Such curves are rational if \(k < 3\), but are elliptic for \(k = 3\). The cubic curves are parametrized by \(\mathbb{P}^9\). Hence

\[d_{01} = d_{02} = 0, \quad d_{03} = c_9(\Omega^1_{\mathbb{P}^9}) = -10.\]  \hspace{1cm} (8.11)

We have already seen that curves of type \((1, 0)\) are elliptic and are parametrized by \(\mathbb{P}^2\). So \(d_{10} = c_2(\Omega^1_{\mathbb{P}^2}) = 3\). The same argument that we used for rational curves shows that \(d_{11} = d_{10} \cdot c_1(\Omega^1_{\mathbb{P}^1}) = -6\) and \(d_{12} = d_{10} \cdot c_4(\Omega^1_{\mathbb{P}^4}) = 15\).

We also have a consistency check between \(d_{13}\) and \(n_{13}\). Curves of type \((1, 3)\) are unions of curves of type \((1, 0)\) and of type \((0, 3)\). Since a curve of type \((1, 0)\) meets \(E\) once, we see that the moduli space of curves of type \((1, 3)\) with a fixed \((1, 0)\) component is isomorphic to the space of rational cubic curves containing a fixed point. This moduli space therefore contributes \(n_{13}/n_{10} = -32\). But elliptic curves come in two types: the \((1, 0)\) could be rational and the \((0, 3)\) elliptic, or vice versa. The elliptic curves through a point are parametrized by a \(\mathbb{P}^8\) hence

\[d_{13} = d_{10} \cdot (-32) + n_{10} \cdot c_8(\Omega^1_{\mathbb{P}^8}) = -96 + 4860 = 4764.\]
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