Which traces are spectral?

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Abstract

Among ideals of compact operators on a Hilbert space we identify a subclass of those closed with respect to the logarithmic submajorization. Within this subclass, we answer the questions asked by Pietsch [22] and by Dykema, Figiel, Weiss and Wodzicki [8]. In the first case, we show that Lidskii-type formulae hold for every trace on such ideal. In the second case, we provide the description of the commutator subspace associated with a given ideal. Finally, we prove that a positive trace on an arbitrary ideal is spectral if and only if it is monotone with respect to the logarithmic submajorization.

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1. Introduction

Let $H$ be a separable Hilbert space and let $\mathcal{L}(H)$ be the algebra of all bounded operators on $H$. The set $\mathcal{L}_1$ of all trace class operators is an ideal in $\mathcal{L}(H)$. It carries a special functional — the classical trace $\text{Tr}$. There is also the description of $\text{Tr}$ as the sum of eigenvalues.

$$\text{Tr}(T) = \sum_{n=0}^{\infty} \lambda(n, T), \quad T \in \mathcal{L}_1. \quad (1)$$

Here, $\lambda(T) = \{\lambda(n, T)\}_{n \geq 0}$ is the sequence of eigenvalues of a compact operator $T$. This result was shown by von Neumann in [21] for self-adjoint operators and

\text{Non-zero eigenvalues are repeated according to their algebraic multiplicity and arranged so that $\{|\lambda(n, T)|\}_{n \geq 0}$ is a decreasing sequence. If there are only finitely many (or none) non-zero eigenvalues, then all the other components of $\lambda(T)$ are zeros.}

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then by Lidskii in [19] in general case. Formula (1) is now known as Lidskii formula (see e.g. [25]).

Fix an orthonormal basis in the Hilbert space $H$. The subalgebra of $\mathcal{L}(H)$ consisting of all diagonal operators with respect to this basis is naturally isomorphic to the algebra $l_\infty$ of all bounded complex sequences. Further, we always identify the algebra $l_\infty$ with this diagonal subalgebra. Thus, the notations $x \in \mathcal{L}(H)$ (or $x \in \mathcal{I}$ for some ideal $\mathcal{I}$ in $\mathcal{L}(H)$) make perfect sense for an element $x \in l_\infty$.

Identifying the sequence $\lambda(T)$ with an element of $\mathcal{L}(H)$, we can write Lidskii formula as $\text{Tr}(T) = \text{Tr}(\lambda(T))$ for all $T \in \mathcal{L}_1$. A natural question concerning the extension of this formula to other ideals and traces on these ideals has been treated in a number of publications (see e.g. [3, 4, 10, 11, 14, 16, 20, 22, 24]).

In what follows, $\mathcal{I}$ is an ideal in $\mathcal{L}(H)$ and $\varphi$ is a trace on $\mathcal{I}$, i.e. a linear functional $\varphi: \mathcal{I} \to \mathbb{C}$ satisfying the condition

$$\varphi(AB) = \varphi(BA), \quad A \in \mathcal{I}, B \in \mathcal{L}(H).$$

The following problem was stated by Pietsch (see p.9 in [22]).

**Question 1.** For which traces $\varphi$ on an ideal $\mathcal{I}$ do we have

$$\varphi(T) = \varphi(\lambda(T)), \quad T \in \mathcal{I}? \quad (2)$$

A given trace $\varphi$ on the ideal $\mathcal{I}$ satisfying (2) is called spectral.

Study of traces in general and Question 1 in particular are closely related to the description of the commutator subspace of an ideal $\mathcal{I}$ in $\mathcal{L}(H)$. The latter subspace (denoted by $\text{Com}(\mathcal{I})$) is a linear span of the elements $AB - BA, A \in \mathcal{I}, B \in \mathcal{L}(H)$. The following question was asked in [8] (see also [7]).

**Question 2.** Does the commutator subspace admit a description in spectral terms?

Note that, for an operator $T \in \mathcal{I}$, we have $T \in \text{Com}(\mathcal{I})$ if and only if all traces on $\mathcal{I}$ vanish on $T$. Thus, if Question 1 is answered in positive (in a sense that all traces on $\mathcal{I}$ are spectral) then, for $T \in \mathcal{I}$, we have $T \in \text{Com}(\mathcal{I})$ if and only if $\lambda(T) \in \text{Com}(\mathcal{I})$. Hence, a positive answer to Question 1 implies a positive answer to Question 2 and vice versa.

For normal operators, Question 2 was answered in the affirmative in [15] (see Theorem 3.1 there) and in [8] (see Theorem 5.6 there) for arbitrary ideals.

**Theorem 3.** A normal operator $N \in \mathcal{I}$ belongs to $\text{Com}(\mathcal{I})$ if and only if $C\lambda(N) \in \mathcal{I}$.

Here, $C: l_\infty \to l_\infty$ is Cesaro operator defined by

$$Cx = (x(0), \frac{x(0) + x(1)}{2}, \frac{x(0) + x(1) + x(2)}{3}, \cdots), \quad x = (x(0), x(1), x(2), \cdots) \in l_\infty.$$
**Theorem 4.** Let an ideal \( \mathcal{I} \) be geometrically stable. An operator \( T \in \mathcal{I} \) belongs to \( \text{Com}(\mathcal{I}) \) if and only if \( C\lambda(T) \in \mathcal{I} \).

Recall that an ideal \( \mathcal{I} \) is called geometrically stable \([15]\) if we have

\[
\left\{ \left( \prod_{m=0}^{n} \mu(k, T) \right)^{1/(n+1)} \right\}_{n \geq 0} \in \mathcal{I}, \quad T \in \mathcal{I}.
\]

Here, \( \mu(T) = \{\mu(n, T)\}_{n \geq 0} \) is the sequence of singular values of a compact operator \( T \), that is, the sequence \( \lambda(|T|) \).

Motivated by Theorem 4, the authors of \([7]\) asked whether the same assertion holds in an arbitrary ideal (see Problem 5.1 in \([7]\)). In order to answer this question and to extend Kalton’s result, we need a concept of logarithmic submajorization\(^2\) and the class of ideals closed with respect to the latter.

**Definition 5.** If \( A, B \in \mathcal{L}(H) \), then the operator \( B \) is logarithmically submajorized by the operator \( A \) (written \( B \prec \log A \)) if

\[
\prod_{k=0}^{n} \mu(k, B) \leq \prod_{k=0}^{n} \mu(k, A), \quad n \geq 0.
\]

**Definition 6.** An ideal \( \mathcal{I} \) is said to be closed with respect to the logarithmic submajorization if \( B \prec \log A \in \mathcal{I} \) implies that \( B \in \mathcal{I} \).

Every geometrically stable ideal is closed with respect to the logarithmic submajorization (see Lemma 35 below). The converse assertion is shown to be true for countably generated ideals by Dykema and Kalton (see Proposition 1.1 and Theorem 1.3 in \([9]\)). However, this is not the case for general ideals. Indeed, we show (see Theorem 36 below) that the class of ideals closed with respect to the logarithmic submajorization is strictly wider than that of geometrically stable ideals.

Our first main result extends Theorem 4 to a wider class of ideals and answers Question 2 (or Problem 5.1 in \([7]\)) in the setting of ideals closed with respect to the logarithmic submajorization.

**Theorem 7.** Let an ideal \( \mathcal{I} \) be closed with respect to the logarithmic submajorization and let \( T \in \mathcal{I} \). We have \( T \in \text{Com}(\mathcal{I}) \) if and only if \( C\lambda(T) \in \mathcal{I} \).

Theorem 7 implies the following Lidskii-type result on traces (in particular, the first part of Theorem 8 provides an alternative\(^3\) proof of \([11]\)). Furthermore, the second part of Theorem 8 shows that the class of ideals closed with respect to the logarithmic submajorization is optimal for Lidskii formulae \([2]\).
Theorem 8. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a trace on $\mathcal{I}$.

(a) If $\mathcal{I}$ is closed with respect to the logarithmic submajorization, then $\varphi(T) = \varphi(\lambda(T))$ for all $T \in \mathcal{I}$.

(b) If $\mathcal{I}$ is not closed with respect to the logarithmic submajorization, then there exists an operator $T \in \mathcal{I}$ such that $\lambda(T) \notin \mathcal{I}$. In particular, the equality (2) makes no sense in this case.

In applications, especially in noncommutative geometry, one is mostly interested in positive traces (see e.g. [4, 5, 6, 11, 20, 27]). We refer the reader to the papers [3, 24] and to the book [20] for the treatment of Lidskii formula for positive traces. For such traces, we are able to present a complete solution of Question 1: a positive trace on an ideal $\mathcal{I}$ is spectral if and only if it is monotone with respect to the logarithmic submajorization.

Definition 9. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a positive trace on $\mathcal{I}$. The trace $\varphi$ is said to be monotone with respect to the logarithmic submajorization if $\varphi(B) \leq \varphi(A)$ for all $0 \leq A, B \in \mathcal{I}$ with $B \prec\prec_{\log} A$.

Later we show (see Lemma 33) that for every ideal $\mathcal{I}$, there exists the least ideal $\text{LE}(\mathcal{I})$ which contains $\mathcal{I}$ and which is closed with respect to the logarithmic submajorization.

Our second main result can be read as follows.

Theorem 10. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a positive trace on $\mathcal{I}$.

(a) If $\varphi(T) = \varphi(\lambda(T))$ for all $T \in \mathcal{I}$ with $\lambda(T) \in \mathcal{I}$, then the trace $\varphi$ is monotone with respect to the logarithmic submajorization.

(b) If $\varphi$ is monotone with respect to the logarithmic submajorization, then $\varphi$ extends to a positive trace on $\text{LE}(\mathcal{I})$. In particular, $\varphi(T) = \varphi(\lambda(T))$ for all $T \in \mathcal{I}$.

Theorem 10 leads to the following surprising corollary. While a positive trace may not necessarily be monotone with respect to the Hardy-Littlewood submajorization (see [17, 20, 26]), it is always monotone with respect to the logarithmic submajorization (on the ideals closed with respect to the latter).

Corollary 11. If an ideal $\mathcal{I}$ is closed with respect to the logarithmic submajorization (e.g. in quasi-Banach ones), then every positive trace on $\mathcal{I}$ is monotone with respect to the logarithmic submajorization.

We should note, however, that the converse assertion to Corollary 11 is false. In Theorem 36 we present a principal ideal which is not closed with respect to the logarithmic submajorization and such that every positive trace on it is monotone with respect to the logarithmic submajorization (and, therefore, spectral by Theorem 10).

*LE stands for the “logarithmic envelope”.*
2. Preliminaries

2.1. Eigenvalues and singular values

For all compact operators $A, B \in \mathcal{L}(H)$, we have (see e.g. Corollary 2.3.16 in [20])

$$
\mu(A + B) \leq \sigma_2(\mu(A) + \mu(B)).
$$

(3)

Here, the dilation operator $\sigma_n : l_\infty \to l_\infty$ is defined as follows.

$$
\sigma_n x = (x(0), \cdots, x(0), x(1), \cdots, x(1), \cdots),
$$

$x = (x(0), x(1), \cdots) \in l_\infty$.

The following inequality relating eigenvalues with singular values is due to Weyl (see Lemma II.3.3. in [13] or Weyl’s original paper [28]).

Theorem 12. For every compact operator $T$, we have

$$
\prod_{k=0}^n |\lambda(k, T)| \leq \prod_{k=0}^n \mu(k, T), \quad n \geq 0.
$$

(4)

Equivalently, $\lambda(T) \prec \prec \log \mu(T)$.

The following assertion due to Dykema and Kalton [9] may be viewed as a converse to Theorem 12.

Lemma 13. If $x = \mu(x) \in c_0$ and $|y| = \mu(y) \in c_0$ are such that $y \prec \prec \log x$, then there exists a compact operator $T \in \mathcal{L}(H)$ such that $\lambda(T) = y$ and $\mu(T) \leq x$.

A class of all positive traces on an ideal $\mathcal{I}$ admits an equivalent description in terms of singular value sequences.

Lemma 14. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi : \mathcal{I} \to \mathbb{C}$ be a linear mapping. The functional $\varphi$ is a positive trace if and only if $\varphi(B) \leq \varphi(A)$ for all $0 \leq A, B \in \mathcal{I}$ such that $\mu(B) \leq \mu(A)$.

Proof. Let $\varphi$ be a positive trace. First, we prove that $\varphi(A) = \varphi(\mu(A))$ for every positive $A \in \mathcal{I}$. Indeed, there exists an isometry $U \in \mathcal{L}(H)$ such that $A = U\mu(A)U^*$ and $U^*U = 1$. Therefore,

$$
A - \mu(A) = U\mu(A) \cdot U^* - U^* \cdot U\mu(A) \in \text{Com}(\mathcal{I})..
$$

In particular, we have $\varphi(A) = \varphi(\mu(A))$.

If now $A, B \in \mathcal{I}$ are positive operators such that $\mu(B) \leq \mu(A)$, then

$$
\varphi(B) = \varphi(\mu(B)) \leq \varphi(\mu(A)) = \varphi(A).
$$

5A variant of Lemma 14 for symmetrically normed ideals can be found in Lemma 2.7.4 of [20].
This proves necessity.

We now prove sufficiency. Let \(0 \leq A \in \mathcal{I}\) and let \(U \in \mathcal{L}(H)\) be a unitary operator. We have \(\mu(U^{-1}AU) = \mu(A)\), which yields (see the assumption) \(\varphi(U^{-1}AU) = \varphi(A)\). By linearity, the latter equality holds for every \(A \in \mathcal{I}\) (i.e., without assumption of positivity). Substituting \(UA\) instead of \(A\), we obtain \(\varphi(AU) = \varphi(UA)\). Since every operator \(B \in \mathcal{L}(H)\) is a linear combination of 4 unitaries, it follows that \(\varphi(AB) = \varphi(BA)\) for every \(A \in \mathcal{I}\) and every \(B \in \mathcal{L}(H)\). Thus, \(\varphi\) is a trace. The assertion that \(\varphi \geq 0\) is immediate. \(\square\)

The following lemma (used only in Section 6 below) demonstrates that every positive trace on the “non-summable” ideals is singular.

**Lemma 15.** Let \(I \not\subset \mathcal{L}_1\) and let \(\varphi\) be a positive trace on \(I\).

(a) The trace \(\varphi\) is singular, that is \(\varphi\) vanishes on the finite rank operators.

(b) If \(A, B \in \mathcal{I}\) are such that \(\mu(k, B) = o(\mu(k, A))\) as \(k \to \infty\), then \(\varphi(B) = 0\).

**Proof.** Select a positive operator \(A \in \mathcal{I}\) such that \(A /\in \mathcal{L}_1\). Let \(P_k, k \geq 0\), be rank one projections such that \(A = \sum_{k=0}^{\infty} \mu(k, A)P_k\).

Here, the convergence is taken in the strong operator topology. For every \(n \geq 0\), it follows from Lemma 14 that

\[
\varphi(A) \geq \varphi\left(\sum_{k=0}^{n} \mu(k, A)P_k\right) = \sum_{k=0}^{n} \mu(k, A)\varphi(P_k) = \sum_{k=0}^{n} \mu(k, A)\varphi(P_0).
\]

If \(\varphi(P_0) \neq 0\), then, the sum at the right hand side tends to \(\infty\) as \(n \to \infty\). This contradicts to the assumption \(\varphi(A) < \infty\).

Hence, \(\varphi\) vanishes on every finite rank projection. Since every finite rank operator is a linear combination of finite rank projections, the assertion (a) follows.

In order to prove (b), we may assume without loss of generality that \(B \geq 0\). Fix \(\varepsilon > 0\) and select \(N\) such that, for every \(n \geq N\) we have \(\mu(n, B) \leq \varepsilon \mu(n, A)\). In particular, \(\mu(B) \leq \varepsilon \mu(A) + \mu(B)\chi_{[0,N)}\). Using Lemma 14 and part (a), we infer that \(\varphi(B) \leq \varepsilon \varphi(A)\). Since \(\varepsilon\) is arbitrarily small, the assertion (b) follows. \(\square\)

Since Hilbert spaces \(H^{\oplus n}\) and \(H\) are isometrically isomorphic, it is often convenient to identify them so that the operator \(A_1 \oplus \cdots \oplus A_n, A_k \in \mathcal{L}(H), 1 \leq k \leq n,\) also belongs to \(\mathcal{L}(H)\). We note the equalities

\[
\lambda(A^{\oplus n}) = \sigma_n \lambda(A), \quad \mu(A^{\oplus n}) = \sigma_n \mu(A), \quad n \geq 1,
\]

which are frequently used in the text. For every trace on \(\mathcal{I}\) and for every \(x \in l_\infty\) such that \(x \in \mathcal{I}\) we have \(\varphi(\sigma_n x) = \varphi(x^{\oplus n}) = n \varphi(x), n \geq 1\).
2.2. Ringrose theorem

An operator $N \in \mathcal{L}(H)$ is said to be normal if $NN^* = N^*N$. A compact operator $Q \in \mathcal{L}(H)$ is said to be quasi-nilpotent if $\lambda(Q) = 0$. The following result belongs to Ringrose (see Chapter 4 in [23]).

**Theorem 16.** For every compact operator $T \in \mathcal{L}(H)$, there exists a compact normal operator $N \in \mathcal{L}(H)$ and compact quasi-nilpotent operator $Q \in \mathcal{L}(H)$ such that $T = N + Q$ and $\lambda(T) = \lambda(N)$.

2.3. Hardy-Littlewood submajorization

The following submajorization was introduced by Hardy and Littlewood. We refer the reader to the book [20] for details.

**Definition 17.** Let $0 \leq A, B \in \mathcal{L}(H)$. We say that the operator $B$ is submajorized by the operator $A$ (written $B \ll A$) if

$$\sum_{k=0}^{n-1} \mu(k, B) \leq \sum_{k=0}^{n-1} \mu(k, A), \quad n > 0.$$  

One of the important features of Hardy-Littlewood submajorization is its nice behaviour with respect to the linear structure of $\mathcal{L}(H)$. For every compact positive $A, B \in \mathcal{L}(H)$, we have (see e.g. Theorems 3.3.3 and 3.3.4 in [20])

$$A + B \ll \mu(A) + \mu(B) \ll 2\sigma_{1/2}\mu(A + B).$$  

(5)

Here, $\sigma_{1/2} : l_\infty \to l_\infty$ is an operator defined as follows.

$$\sigma_{1/2}x = \left(\frac{x(0) + x(1)}{2}, \frac{x(2) + x(3)}{2}, \cdots, x = (x(0), x(1), \cdots) \in l_\infty.$$  

2.4. Uniform submajorization

The following definition, introduced originally in [18] (see also [20]), plays a major role in our treatment of traces.

**Definition 18.** Let $0 \leq A, B \in \mathcal{L}(H)$. We say that the operator $B$ is uniformly submajorized by the operator $A$ (written $B \triangleleft A$) if there exists $\lambda \in \mathbb{N}$ such that

$$\sum_{k=\lambda m}^{n-1} \mu(k, B) \leq \sum_{k=m}^{n-1} \mu(k, B), \quad \lambda m < n.$$  

Uniform submajorization also behaves nicely with respect to the linear structure of $\mathcal{L}(H)$. One can strengthen the inequalities (5) as follows. For every compact positive $A, B \in \mathcal{L}(H)$, we have (see e.g. Lemma 3.4.4 in [20])

$$A + B \triangleleft \mu(A) + \mu(B) \triangleleft 2\sigma_{1/2}\mu(A + B).$$  

(6)

Uniform submajorization is a stronger condition than Hardy-Littlewood submajorization introduced in the preceding subsection. Theorem 19 below describes the convex hull of the set $\{0 \leq z \in l_\infty : \mu(z) \leq \mu(x)\}$ in terms of uniform submajorization (see Theorem 3.4.2 in [20] or Theorem 5.4 in the original paper [18]).
Theorem 19. Let $0 \leq x, y \in l_\infty$.

(a) If $y$ belongs to a convex hull of the set $\{0 \leq z \in l_\infty : \mu(z) \leq \mu(x)\}$, then $y < x$.

(b) If $y, x, \varepsilon > 0$, the $(1 - \varepsilon)y$ belongs to a convex hull of the set $\{0 \leq z \in l_\infty : \mu(z) \leq \mu(x)\}$.

An importance of uniform submajorization for studying of traces may be seen from the following strengthening of Lemma 14.

Lemma 20. Let $\mathcal{I}$ be an ideal and let $\varphi : \mathcal{I} \to \mathbb{C}$ be a linear mapping. The functional $\varphi$ is a positive trace on $\mathcal{I}$ if and only if $\varphi(B) \leq \varphi(A)$ for all $0 \leq A, B \in \mathcal{I}$ such that $B \prec A$.

Proof. Let $\varphi$ be a positive trace on $\mathcal{I}$ and let $0 \leq A, B \in \mathcal{I}$ be such that $B \prec A$. Fix $\varepsilon \in (0, 1)$. By Theorem 19, there exist $n \geq 1$, positive sequences $z_k \in c_0$ and positive constants $\lambda_k \in (0, 1)$, $1 \leq k \leq n$ such that $\mu(z_k) \leq \mu(A)$ for every $1 \leq k \leq n$, and such that

$$(1 - \varepsilon)\mu(B) \leq \sum_{k=1}^{n} \lambda_k z_k, \quad \sum_{k=1}^{n} \lambda_k = 1.$$

Observe that

$$(1 - \varepsilon)\varphi(\mu(B)) \leq \sum_{k=1}^{n} \lambda_k \varphi(z_k)$$

by the assumption and Lemma 14. Applying Lemma 14 again, we infer that $\varphi(B) = \varphi(\mu(B))$ and $\varphi(z_k) \leq \varphi(A)$ for $1 \leq k \leq n$. Hence, $(1 - \varepsilon)\varphi(B) \leq \varphi(A)$. Since $\varepsilon$ is arbitrarily small, it follows that $\varphi(B) \leq \varphi(A)$. This proves necessity.

In order to prove sufficiency, choose $0 \leq A, B \in \mathcal{I}$ such that $\mu(B) \leq \mu(A)$. It follows from Definition 15 that $B \prec A$ and, therefore, by the assumption, $\varphi(B) \leq \varphi(A)$. An application of Lemma 14 completes the proof of sufficiency.

3. Main technical inequalities

The main result of this section is Theorem 24 below. It is the core component of our proof of Theorem 8.

Define a non-linear homogeneous operator $S : l_\infty \to l_\infty$ by setting

$$(Sx)(k) = \mu(k, x)(1 + \frac{1}{k + 1} \log(\prod_{m=0}^{k} \frac{\mu(m, x)}{\mu(k, x)})^{k + 1}), \quad n \geq 0. \quad (7)$$

The operator $S$ is a technical device used in the next section to estimate the action of Cesaro operator $C$ on the eigenvalue sequences of real and imaginary parts of a quasi-nilpotent operator. In this section, we obtain main technical estimate for this operator.

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6 A variant of Lemma 20 for symmetrically normed ideals was proved in Lemma 4.2.5 in [20].
Lemma 21. For every $x \in l_\infty$, we have $Sx = \mu(Sx)$.

Proof. Without loss of generality, we may assume that $x = \mu(x)$. We have to prove $(Sx)(k + 1) \leq (Sx)(k)$, $k \geq 0$. First, note that, for every constant $C > 0$, the function

$$x \rightarrow x(1 + \frac{k + 1}{k + 2} \log \left( \frac{C}{x} \right)), \quad x \in [0, C]$$

is increasing. Fixing the values $x(0), \ldots, x(k)$, and setting $C = (\prod_{m=0}^{k} x(m))^{1/(k+1)}$, we infer that the function

$$x(k + 1) \rightarrow (Sx)(k + 1) = x(k + 1)(1 + \frac{k + 1}{k + 2} \log \left( \frac{C}{x(k + 1)} \right))$$

is also increasing. Thus, for given $x(0), \ldots, x(k)$, the function $x(k + 1) \rightarrow (Sx)(k + 1)$ attains its maximal value when $x(k + 1)$ takes its maximal value, which is $x(k)$. Therefore,

$$(Sx)(k + 1) \leq x(k)(1 + \frac{k + 1}{k + 2} \log \left( \frac{C}{x(k)} \right)) = x(k)(1 + \frac{1}{k + 2} \log \left( \prod_{m=0}^{k} x(m) \right)).$$

It is immediate that the right hand side of the latter inequality does not exceed $(Sx)(k)$. This proves the assertion. □

Lemma 22. If $x = \mu(x) \in l_\infty$, then

(a) For every $n \geq 0$ and for every $k \geq n$, we have

$$(Sx)(k) \leq x(n)(1 + \frac{1}{k + 1} \log \left( \frac{\prod_{m=0}^{n} x(m)}{x(n)^{n+1}} \right)).$$

(b) For every $n \geq 0$ and for every $k \leq n$, we have

$$(Sx)(k) \leq x(k)(1 + \frac{1}{k + 1} \log \left( \frac{\prod_{m=0}^{n} x(m)}{x(n)^{n+1}} \right)).$$

Proof. Despite the similarity of both estimates, they require essentially different proofs.

(a) Fixing the values $x(0), \ldots, x(k - 1)$, and arguing as in the proof of Lemma 21 we infer that the function $x(k) \rightarrow (Sx)(k)$ is increasing. Hence,

$$(Sx)(k) \leq x(k - 1)(1 + \frac{1}{k + 1} \log \left( \frac{\prod_{m=0}^{k-1} x(m)}{x(k - 1)^{k}} \right)).$$

Fixing the values $x(0), \ldots, x(k - 2)$, and repeating the argument above, we infer that the function of the variable $x(k - 1) \in [0, x(k - 2)]$ standing at the right hand side of the preceding inequality is increasing. Hence,

$$(Sx)(k) \leq x(k - 2)(1 + \frac{1}{k + 1} \log \left( \frac{\prod_{m=0}^{k-2} x(m)}{x(k - 2)^{k-1}} \right)).$$

Repeating the argument for $k - 2, k - 3, \ldots, n + 1$, we conclude the proof.
(b) Since both sides of the inequality are homogeneous with respect to $x$, we may assume without loss of generality that $x(n) = 1$. It follows that $x(k) \geq 1$, $1 \leq k \leq n$, and, therefore,

$$\prod_{m=0}^{k} \frac{x(m)}{x(k)^{n+1}} \leq \prod_{m=0}^{k} x(m) \leq \prod_{m=0}^{n} x(m) = \prod_{m=0}^{n} \frac{x(m)}{x(n)^{n+1}}.$$ 

Combining the preceding estimate with the Definition of $S$ given in (7) yields the assertion.

**Lemma 23.** For every $u > 0$ and for every positive integer $n$, we have

$$\prod_{k=0}^{2n} \left( 1 + \frac{u}{k+1} \right) \leq 2^{2n+u+2}.$$ 

**Proof.** Let $u \in [m - 1, m]$ with $m \in \mathbb{N}$. We have

$$\prod_{k=0}^{2n} \left( 1 + \frac{u}{k+1} \right) \leq \prod_{k=0}^{2n} \left( 1 + \frac{m}{k+1} \right) \leq \frac{\prod_{k=0}^{2n} (m + k + 1)}{\prod_{k=0}^{2n} (k+1)} = \frac{(2n+m+1)!}{m!(2n+1)!} \leq 2^{2n+m+1} \leq 2^{2n+u+2}.$$ 

**Theorem 24.** If $x = \mu(x) \in l_{\infty}$, then $(Sx) \ll \log 4(x \oplus x)$.

**Proof.** Fix $n \geq 0$ and denote

$$s = \prod_{m=0}^{n} \frac{x(m)}{x(n)^{n+1}}.$$ 

By Lemma 22 and (1), we have

$$\prod_{k=0}^{2n+1} (Sx)(k) \leq \prod_{k=0}^{n} x(k) \left( 1 + \frac{1}{k+1} \log(s) \right) \cdot \prod_{k=n+1}^{2n+1} x(n) \left( 1 + \frac{1}{k+1} \log(s) \right) =$$

$$= \left( \prod_{k=0}^{n} x(k) \right)^{2} \cdot s^{-1} \prod_{k=0}^{2n+1} \left( 1 + \frac{1}{k+1} \log(s) \right)$$

and

$$\prod_{k=0}^{2n} (Sx)(k) \leq \prod_{k=0}^{n} x(k) \left( 1 + \frac{1}{k+1} \log(s) \right) \cdot \prod_{k=n+1}^{2n} x(n) \left( 1 + \frac{1}{k+1} \log(s) \right) =$$
\[
\left( \prod_{k=0}^{n-1} x(k) \right)^2 x(n) \cdot s^{-1} \prod_{k=0}^{2n} \left( 1 + \frac{1}{k+1} \log(s) \right).
\]

Since \( s \geq 1 \), it follows from Lemma 23 that

\[
\prod_{k=0}^{2n+1} \left( 1 + \frac{1}{k+1} \log(s) \right) \leq \prod_{k=0}^{2n+2} \left( 1 + \frac{1}{k+1} \log(s) \right) \leq 2^{2n+4+\log(s)} \leq 4^{2n+2}s
\]
and

\[
\prod_{k=0}^{2n} \left( 1 + \frac{1}{k+1} \log(s) \right) \leq 2^{2n+2+\log(s)} \leq 4^{2n+1}s.
\]

Therefore,

\[
\prod_{k=0}^{2n+1} (Sx)(k) \leq 4^{2n+2} \prod_{k=0}^{n} x(k)^2 = 4^{2n+2} \prod_{k=0}^{2n+1} \mu(k, x \oplus x). \tag{8}
\]
and

\[
\prod_{k=0}^{2n} (Sx)(k) \leq 4^{2n+1} \prod_{k=0}^{n-1} x(k)^2 x(n) = 4^{2n+1} \prod_{k=0}^{2n} \mu(k, x \oplus x). \tag{9}
\]

Since \( n \geq 0 \) is arbitrary, the assertion follows by combining (8) and (9). \( \square \)

4. Lidskii formula for traces on ideal closed with respect to the logarithmic submajorization

In this section, we prove Theorems 7 and 8.

The idea of the important estimate below belongs to Kalton (see Theorem 2.7 in [15]). We refer the reader to Chapter 5 in the book [20] for the detailed proof.

**Theorem 25.** If \( Q \) is a compact quasi-nilpotent operator, then

\[
| \sum_{|\lambda| > 1, \lambda \in \sigma(\Re Q)} \lambda | \leq 400 \sum_{|\lambda| > 1, \lambda \in \sigma(2eQ)} \log(\lambda).
\]

Here, \( \Re Q \) is the real part of \( Q \). A similar assertion holds for the imaginary part \( \Im Q \).

**Lemma 26.** For every quasi-nilpotent compact operator \( Q \in \mathcal{L}(H) \), we have

\[
|C\lambda(\Re Q)| \leq 200S((2eQ)^{\oplus 2}).
\]

A similar assertion holds for \( \Im Q \).
Proof. Since $\Re Q$ is self-adjoint, we infer from (3) that, for every $n \geq 0$,
$$|\lambda(n, \Re Q)| = \mu(n, \Re Q) = \mu(n, \frac{1}{2}(Q + Q^*)) \leq \mu(n, Q^{\oplus 2}) \leq \mu(n, (2eQ)^{\oplus 2}).$$

Set
$$m(n) = \max\{m \geq 0 : |\lambda(m, \Re Q)| > \mu(n, 2e(Q^{\oplus 2}))\}, \quad n \geq 0.$$ 

By the inequality above, we have $m(n) \leq n$. Clearly,
$$\sigma(\Re Q) \cap \{\lambda : |\lambda| > \mu(n, 2e(Q^{\oplus 2}))\} = \{\lambda(k, \Re Q)\}_{k=0}^{m(n)}$$
and
$$|\sum_{k=0}^{n} \lambda(k, \Re Q)| \leq \sum_{k=m(n)+1}^{n} |\lambda(k, \Re Q)| + \sum_{k=0}^{m(n)} |\lambda(k, \Re Q)|.$$

Therefore,
$$|\sum_{k=0}^{n} \lambda(k, \Re Q)| \leq (n+1)\mu(n, (2eQ)^{\oplus 2}) + \sum_{\lambda \in \sigma(\Re Q), |\lambda| > \mu(n, (2eQ)^{\oplus 2})} |\lambda|. \quad (10)$$

In order to estimate the second summand at the right hand side of (10), we define an operator $Q_0$ by setting
$$Q_0 = \frac{1}{\mu(n, (2eQ)^{\oplus 2})} Q^{\oplus 2}.$$

It is clear that $\lambda(Q^{\oplus 2}) = \sigma_2 \lambda(Q) = 0$ and, therefore, $Q_0$ is quasi-nilpotent operator. Applying Theorem [25] we obtain
$$|\sum_{\lambda \in \sigma(\Re Q), |\lambda| > \mu(n, (2eQ)^{\oplus 2})} \lambda| = \frac{1}{2} \mu(n, (2eQ)^{\oplus 2}) \sum_{\lambda \in \sigma(\Re Q), |\lambda| > 1} |\lambda| \leq 200\mu(n, (2eQ)^{\oplus 2}) \sum_{k=0}^{n} \log(2e\mu(k, Q_0)) = 200\mu(n, (2eQ)^{\oplus 2}) \sum_{k=0}^{n} \log(\frac{\mu(k, (2eQ)^{\oplus 2})}{\mu(n, (2eQ)^{\oplus 2})}).$$

Using the latter estimate together with (10), we obtain
$$|\sum_{k=0}^{n} \lambda(k, \Re Q)| \leq (n+1)\mu(n, (2eQ)^{\oplus 2}) + 200\mu(n, (2eQ)^{\oplus 2}) \log(\prod_{k=0}^{n} \frac{\mu(k, (2eQ)^{\oplus 2})}{\mu(n, (2eQ)^{\oplus 2})}).$$

Dividing both sides by $(n+1)$ and appealing to the definition of $S$ given in (7) yields the assertion. \qed

The following proposition is the key to the proofs of Theorems [7] and [8]. Its proof crucially depends on Theorem [24].

**Proposition 27.** For every quasi-nilpotent compact operator $Q \in \mathcal{L}(H)$, we have
$$C\lambda(\Re Q) \prec \log (1600eQ)^{\oplus 4}, \quad C\lambda(\Im Q) \prec \log (1600eQ)^{\oplus 4}.$$
Proof. The assertion follows from consecutive application of Lemma 26 and Theorem 24.

We are now ready to prove Theorems 7 and 8.

of Theorem 7. Fix an operator $T \in \mathcal{I}$. By Theorem 16, $T = N + Q$, where

$N$ is normal, $\lambda(N) = \lambda(T)$ and $Q$ is quasi-nilpotent. It follows from (11) that

$\mu(N) = |\lambda(T)| \ll \log T$. Since $\mathcal{I}$ is closed with respect to the logarithmic submajorization, it follows that $N \in \mathcal{I}$. Thus, $Q = T - N$ also belongs to $\mathcal{I}$, and so does the operator $Q \oplus 4$.

Since $\mathcal{I}$ is closed with respect to the logarithmic submajorization, it follows from Proposition 27 that $C\lambda(\Re Q) \in \mathcal{I}$ and $C\lambda(\Im Q) \in \mathcal{I}$. It follows from Theorem 3 that both $\Re Q \in \Com(\mathcal{I})$ and $\Im Q \in \Com(\mathcal{I})$. Hence, $Q \in \Com(\mathcal{I})$.

The preceding paragraph yields the following conclusion: the operators $T$ and $N$ simultaneously belong or do not belong to $\Com(\mathcal{I})$. By Theorem 3, the normal operator $N$ belongs to $\Com(\mathcal{I})$ if and only if $C\lambda(N) \in \mathcal{I}$. Since $\lambda(N) = \lambda(T)$ (by construction), the assertion follows.

of Theorem 8. Fix an operator $T \in \mathcal{I}$. By Theorem 16, $T = N + Q$, where $N$ is normal, $\lambda(N) = \lambda(T)$ and $Q$ is quasi-nilpotent. Repeating the argument in the proof of Theorem 7, we infer that $N \in \mathcal{I}$ and that $Q \in \Com(\mathcal{I})$. Hence, $Q \in \Com(\mathcal{I})$.

The preceding paragraph yields the following conclusion: the operators $T$ and $N$ simultaneously belong or do not belong to $\Com(\mathcal{I})$. By Theorem 3, the normal operator $N$ belongs to $\Com(\mathcal{I})$ if and only if $C\lambda(N) \in \mathcal{I}$. Since $\lambda(N) = \lambda(T)$ (by construction), the assertion follows.

5. Lidskii formula vs logarithmic submajorization

The main aim of this section is to furnish the proof of Theorem 10. The following definition provides an extension of a positive trace $\varphi$ given originally on the ideal $\mathcal{I}$ to the positive cone of the algebra $\mathcal{L}(H)$. We emphasize that $\varphi(A)$ does not have to be finite for a particular $0 \leq A \in \mathcal{L}(H)$. In Theorem 10 this obstacle is overcome by exploiting monotonicity of our trace with respect to the logarithmic submajorization.

Definition 28. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a positive trace on $\mathcal{I}$. For every positive $A \in \mathcal{L}(H)$, set

$$\varphi(A) = \sup \{ \varphi(B) : 0 \leq B \leq A, \ B \in \mathcal{I} \}. \quad (11)$$

The next lemma shows that in the Definition 28 the standard order can be replaced with the uniform submajorization.
Lemma 29. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a positive trace on $\mathcal{I}$. For every positive $A \in \mathcal{L}(H)$, we have
\[
\varphi(A) = \sup \{ \varphi(B) : 0 \leq B \subset A, \ B \in \mathcal{I} \}.
\]

Proof. Fix $\varepsilon > 0$. It follows from Theorem 19 that there exist positive sequences $z_k \in l_\infty$, $1 \leq k \leq n$, and positive constants $\lambda_k$, $1 \leq k \leq n$, such that $\mu(z_k) \leq \mu(A)$ and
\[
(1 - \varepsilon)\mu(B) = \sum_{k=1}^{n} \lambda_k z_k, \quad \sum_{k=1}^{n} \lambda_k = 1.
\]
Without loss of generality, $\lambda_k > 0$ for all $1 \leq k \leq n$. It follows that $z_k \in \mathcal{I}$ for all $1 \leq k \leq n$. By Definition 28, we have $\varphi(z_k) \leq \varphi(A)$. Hence,
\[
(1 - \varepsilon)\varphi(B) = \sum_{k=1}^{n} \lambda_k \varphi(z_k) \leq \varphi(A).
\]
Since $\varepsilon > 0$ is arbitrarily small, the assertion follows.

The following lemma shows that the extension of $\varphi$ given in Definition 28 is additive with respect to the direct sum operation.

Lemma 30. Let $\mathcal{I}$ be an ideal in $\mathcal{L}(H)$ and let $\varphi$ be a positive trace on $\mathcal{I}$. For every positive operators $A_1, A_2 \in \mathcal{L}(H)$, we have
\[
\varphi(A_1 \oplus A_2) = \varphi(A_1) + \varphi(A_2).
\]

Proof. If $B_1, B_2 \in \mathcal{I}$ are such that $0 \leq B_1 \leq A_1$ and $0 \leq B_2 \leq A_2$, then $0 \leq B_1 \oplus B_2 \leq A_1 \oplus A_2$. Hence,
\[
\sup \{ \varphi(B_1 \oplus B_2) : 0 \leq B_1 \leq A_1, \ 0 \leq B_2 \leq A_2, \ B_1, B_2 \in \mathcal{I} \} \leq \varphi(A_1 \oplus A_2).
\]
It follows from Definition 28 that
\[
\varphi(A_1) + \varphi(A_2) \leq \varphi(A_1 \oplus A_2).
\]

In order to prove the converse inequality, let $B \in \mathcal{I}$ be such that $0 \leq B \leq A_1 \oplus A_2$. Let $p$ be the support projection of $A_1 \oplus 0$ and let $U = p + i(1 - p)$. Since $U$ is unitary, it follows that $\varphi(C) = \varphi(U^{-1}CU)$ for every operator $C \in \mathcal{I}$. Note that
\[
Bp - U^{-1}(Bp)U = Bp - U^{-1}Bp = (1 + i)(1 - p)Bp.
\]
Therefore, $\varphi((1 - p)Bp) = 0$ and, similarly, $\varphi(pB(1 - p)) = 0$. Hence,
\[
\varphi(B) = \varphi(pBp) + \varphi((1 - p)B(1 - p)).
\]
On the other hand, we have

\[ pBp \leq p(A_1 \oplus A_2)p = A_1 \oplus 0, \quad (1-p)B(1-p) \leq (1-p)(A_1 \oplus A_2)(1-p) = 0 \oplus A_2. \]

Setting \( B_1 = pBp \) and \( B_2 = (1-p)B(1-p) \), we have \( \varphi(B) = \varphi(B_1 + B_2) \) and \( 0 \leq B_1 \leq A_1 \oplus 0, \ 0 \leq B_2 \leq 0 \oplus A_2 \). Therefore,

\[ \varphi(A) = \sup\{\varphi(B) : \ 0 \leq B \leq A_1 \oplus A_2, \ B \in \mathcal{I}\} \leq \sup\{\varphi(B_1) + \varphi(B_2) : \ 0 \leq B_1 \leq A_1 \oplus 0, \ 0 \leq B_2 \leq 0 \oplus A_2, \ B_1, B_2 \in \mathcal{I}\} = \sup\{\varphi(B_1) : \ 0 \leq B_1 \leq A_1, \ B_1 \in \mathcal{I}\} + \sup\{\varphi(B_2) : \ 0 \leq B_2 \leq A_2, \ B_2 \in \mathcal{I}\}. \]

Thus, by Definition 28, we have

\[ \varphi(A_1 \oplus A_2) \geq \varphi(A_1) + \varphi(A_2). \quad (13) \]

The assertion follows from (12) and (13).

Finally, we are prepared to show that the extension of a positive trace \( \varphi \) is a trace on every ideal \( \mathcal{J} \) on which this extension takes finite values.

**Proposition 31.** Let \( \mathcal{I} \) be an ideal in \( \mathcal{L}(H) \) and let \( \varphi \) be a positive trace on \( \mathcal{I} \). If an ideal \( \mathcal{J} \supset \mathcal{I} \) is such that \( \varphi(A) < \infty \) for every \( 0 \leq A \in \mathcal{J} \) (where \( \varphi(A) \) is as in Definition 28), then \( \varphi \) is a positive trace on \( \mathcal{J} \).

**Proof.** Let \( 0 \leq A_1, A_2 \in \mathcal{J} \). Applying the inequality (6) to the positive operators \( A_1 \oplus 0 \) and \( 0 \oplus A_2 \) (and, separately, to the positive operators \( A_1, A_2 \)) we obtain

\[ A_1 \oplus A_2 < \mu(A_1) + \mu(A_2) < 2\sigma_{1/2}\mu(A_1 + A_2) \]

and

\[ A_1 + A_2 < \mu(A_1) + \mu(A_2) < 2\sigma_{1/2}\mu(A_1 \oplus A_2). \]

By Lemma 29, we have

\[ \varphi(A_1 \oplus A_2) \leq \varphi(2\sigma_{1/2}\mu(A_1 + A_2)) = \varphi(A_1 + A_2) \]

and

\[ \varphi(A_1 + A_2) \leq \varphi(2\sigma_{1/2}\mu(A_1 \oplus A_2)) = \varphi(A_1 \oplus A_2). \]

It follows now from Lemma 30 that

\[ \varphi(A_1 + A_2) = \varphi(A_1 \oplus A_2) = \varphi(A_1) + \varphi(A_2). \]

Hence, \( \varphi \) extends to a positive linear functional on \( \mathcal{J} \). By Lemma 14, this functional is a trace.

The following inequalities, though important per se, play a crucial role in the proof of the subsequent Proposition 33.

**Lemma 32.** Let \( A_k, B_k \in \mathcal{L}(H) \), \( k = 1, 2 \), be such that \( B_k \ll_{\log} A_k \). We have
(a) \( B_1 \oplus B_2 \prec \log A_1 \oplus A_2 \).

(b) \( B_1 + B_2 \prec \log 2(A_1 \oplus A_2)^{\otimes 2} \).

Proof. In fact, second assertion follows easily from the first one.

(a) We have
\[
\prod_{k=0}^{n} \mu(k, B_1 \oplus B_2) = \sup_{n_1+n_2=n+1} \left( \prod_{k=0}^{n_1-1} \mu(k, B_1) \right) \left( \prod_{k=0}^{n_2-1} \mu(k, B_2) \right) \leq \\
\leq \sup_{n_1+n_2=n+1} \left( \prod_{k=0}^{n_1-1} \mu(k, B_1) \right) \left( \prod_{k=0}^{n_2-1} \mu(k, B_2) \right) = \prod_{k=0}^{n} \mu(k, A_1 \oplus A_2).
\]

(b) It follows from obvious inequalities
\[
\mu(B_1) \leq \mu(B_1 \oplus B_2), \quad \mu(B_2) \leq \mu(B_1 \oplus B_2)
\]
combined with (3) and part (a) that
\[
\mu(B_1 + B_2) \leq 2\sigma_2 \mu(B_1 \oplus B_2) \prec \log 2\sigma_2 \mu(A_1 \oplus A_2) = 2\mu((A_1 \oplus A_2)^{\otimes 2}).
\]

Proposition 33. For every ideal \( \mathcal{I} \), there exists the least ideal \( LE(\mathcal{I}) \) containing \( \mathcal{I} \) and closed with respect to the logarithmic submajorization. This ideal can be defined as follows
\[
LE(\mathcal{I}) = \{ B \in \mathcal{L}(H) : B \prec \log A \text{ for some } A \in \mathcal{I} \}.
\]

Proof. If \( B_1, B_2 \in LE(\mathcal{I}) \), then there exist \( A_1, A_2 \in \mathcal{I} \) such that \( B_1 \prec \log A_1 \) and \( B_2 \prec \log A_2 \). We have \( A_1 \oplus A_2 \in \mathcal{I} \) and, therefore, \( (A_1 \oplus A_2)^{\otimes 2} \in \mathcal{I} \). It follows from Lemma 32 that \( B_1 + B_2 \prec \log 2(A_1 \oplus A_2)^{\otimes 2} \in \mathcal{I} \). In particular, \( B_1 + B_2 \in LE(\mathcal{I}) \). Thus, \( LE(\mathcal{I}) \) is a linear space.

We now show that \( LE(\mathcal{I}) \) is an ideal in \( \mathcal{L}(H) \). Indeed, for every \( B \in LE(\mathcal{I}) \), there exists an operator \( A \in \mathcal{I} \) such that \( B \prec \log A \). If now \( C \in \mathcal{L}(H) \), then
\[
\mu(BC) \leq \|C\|_\infty \mu(B) \prec \log \|C\|_\infty A \in \mathcal{I}.
\]
Hence, \( BC \in LE(\mathcal{I}) \). Similarly, \( CB \in LE(\mathcal{I}) \) and, therefore, \( LE(\mathcal{I}) \) is an ideal in \( \mathcal{L}(H) \).

It is clear that \( LE(\mathcal{I}) \) is closed with respect to the logarithmic submajorization and that every ideal closed with respect to the logarithmic submajorization and containing \( \mathcal{I} \) must also contain \( LE(\mathcal{I}) \). \( \square \)
Lemma 34. Let \( \varphi \) be a positive trace on the ideal \( \mathcal{I} \). For every \( T \in \mathcal{I} \), we have \( |\varphi(\Re T)| \leq \varphi(|T|) \).

Proof. By Lemma 4.3 of [12], there exist partial isometries \( U, V \in \mathcal{L}(H) \) such that

\[
|\Re T| \leq \frac{1}{2}(U|T|U^* + V|T|V^*).
\]

Note that

\[
\mu(U|T|U^*) \leq \mu(|T|), \quad \mu(V|T|V^*) \leq \mu(|T|) = \mu(|T|).
\]

Since \( \varphi \) is a positive trace, it follows from Lemma 14 that

\[
|\varphi(\Re T)| \leq \varphi(|\Re T|) \leq \frac{1}{2}(\varphi(U|T|U^*) + \varphi(V|T|V^*)) \leq \varphi(|T|) - \varphi(|T|).
\]

We are now ready to prove the second main result of the paper.

of Theorem 11 First, we prove (i). Let \( \varphi \) be a positive spectral trace on the ideal \( \mathcal{I} \) and let positive operators \( A, B \in \mathcal{I} \) be such that \( T \prec \prec \log A \). By Lemma 12 there exists \( T \in \mathcal{L}(H) \) such that \( \lambda(T) = \mu(B) \) and \( \mu(T) \leq \mu(A) \). By the assumption, the trace \( \varphi \) is spectral and, therefore, we have \( \varphi(B) = \varphi(\lambda(T)) = \varphi(T) \). In particular, \( \varphi(T) \in \mathbb{R} \) and, therefore, \( \varphi(T) = \varphi(\Re T) \). By Lemma 34 we have \( \varphi(B) = \varphi(\Re T) \leq \varphi(|T|) \). Since \( \mu(T) \leq \mu(A) \), it follows that \( \varphi(B) \leq \varphi(|T|) \leq \varphi(A) \). This proves (i).

Now, we prove (ii). Let \( \varphi \) be a positive trace on \( \mathcal{I} \) which is monotone with respect to the logarithmic submajorization. If \( 0 \leq C \in \text{LE}(\mathcal{I}) \), then there exists \( A \in \mathcal{I} \) such that \( C \prec \prec \log A \). If now \( 0 \leq B \in \mathcal{I} \) is such that \( B \prec \prec \log A \), then, obviously, \( B \prec \prec \log A \). Hence, by the assumption, \( \varphi(B) \leq \varphi(A) \). If we define \( \varphi \) on \( \text{LE}(\mathcal{I}) \) by formula (11), then \( \varphi(C) \leq \varphi(A) < \infty \). By Proposition 31 \( \varphi \) is a positive trace on \( \text{LE}(\mathcal{I}) \). By Theorem 8.63, \( \varphi(T) = \varphi(\lambda(T)) \) for every \( T \in \mathcal{I} \). This proves (ii). \( \square \)

6. Examples

The following lemma shows that every geometrically stable ideal is closed with respect to the logarithmic submajorization. Theorem 36 (c) shows that the converse is false.

Lemma 35. Every geometrically stable ideal is closed with respect to the logarithmic submajorization.

Proof. Let \( A \in \mathcal{I} \) and let \( B \in \mathcal{L}(H) \) be such that \( B \prec \prec \log A \). We have

\[
\mu(n, B) \leq (\prod_{k=0}^{n} \mu(k, B))^{1/(n+1)} \leq (\prod_{k=0}^{n} \mu(k, A))^{1/(n+1)}, \quad n \geq 0.
\]

The assertion follows now from the definition of a geometrically stable ideal. \( \square \)
Theorem 36. Define an operator $A \in \mathcal{L}(H)$ by setting

$$
\mu(A) = \sup_{n \geq 0} 2^{-2^{3n}} \chi_{[0,2^{2\cdot 3n})}.
$$

For the principal ideal $\mathcal{I}_A$ generated by $A$, we have

(a) Every positive trace $\varphi$ on $\mathcal{I}_A$ extends to a positive trace on $\text{LE}(\mathcal{I}_A)$. In particular, $\varphi$ is spectral and monotone with respect to logarithmic submajorization.

(b) Ideal $\mathcal{I}_A$ is not closed with respect to the logarithmic submajorization. Moreover, there exists a trace (non-positive) on $\mathcal{I}$ which is not spectral.

(c) The ideal $\text{LE}(\mathcal{I}_A)$ fails to be geometrically stable.

Lemma 37 and Proposition 38 below are needed for the proof of Theorem 36 (a).

Lemma 37. Let $A$ be as in Theorem 36 and let a compact operator $A_0 \in \mathcal{L}(H)$ be defined by setting

$$
\mu(A_0) = \sup_{n \geq 0} 2^{-2^{3(n+1)}} \chi_{[0,2^{3n+2^{3n}})}.
$$

We have $A_0 \in \mathcal{I}_A$ and $\varphi(A_0) = 0$ for every positive trace on $\mathcal{I}_A$.

Proof. Clearly, $\mu(A_0) \leq \mu(A) \in \mathcal{I}_A$. For every $l \geq 0$, we have

$$
\sigma_{2^l} \mu(A_0) = \sup_{n \geq 0} 2^{-2^{3(n+1)}} \chi_{[0,2^{3n+l+2^{3n}})}.
$$

Note that $3n + l + 2^{3n} \leq 2^{3(n+1)}$ for $n \geq l$. It follows that

$$
\sigma_{2^l} \mu(A_0) \leq \sup_{0 \leq n < l} 2^{-2^{3(n+1)}} \chi_{[0,2^{3n+l+2^{3n}})} + \sup_{n \geq 0} 2^{-2^{3(n+1)}} \chi_{[0,2^{2^{3(n+1)}})}.
$$

Hence,

$$
\sigma_{2^l} \mu(A_0) \leq \sigma_{2^l} \sup_{0 \leq n < l} 2^{-2^{3(n+1)}} \chi_{[0,2^{3n+l+2^{3n}})} + \mu(A).
$$

Since $\varphi$ is a positive trace, it follows from Lemma 14 that

$$
2^l \varphi(A_0) \leq 2^l \varphi(\sup_{0 \leq n < l} 2^{-2^{3(n+1)}} \chi_{[0,2^{3n+l+2^{3n}})}) + \varphi(A).
$$

By Lemma 15, the first term at the right hand side is $0$. Therefore, $\varphi(A_0) \leq 2^{-l} \varphi(A)$. Since $l$ is arbitrarily large, the assertion follows. \qed

---

7That is, $\mu(k, A) = 2^{-2^{3n}}$ for all $k \in [2^{3(n-1)}, 2^{2\cdot 3n})$. 

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Proposition 38. Let $A$ be as in Theorem 36 and let $A_0$ be as in Lemma 37. If $B \in \mathcal{I}_A$ is such that $B \prec \prec \log A$, then there exists $l \geq 0$ such that
\[
\mu(B) \leq 2\mu(A_0) + 256\sigma_2\mu(A) + \sigma_2(\mu^4(A)).
\] (15)

Proof. Since $\mathcal{I}_A$ is a principal ideal and since $B \in \mathcal{I}_A$, it follows from (3) that there exists $l \geq 1$ such that $\mu(B) \leq 2^l\sigma_2\mu(A)$. We verify the inequality (15) on the following intervals.
\[
[2^{l+2^3n}, 2^{3n+2^3n}), [2^{l+2^3n}, 2^{l+2^3n}), [2^{3n+2^3n}, 2^{23(n+1)}), [2^{23n}, 2^{1+2^3n}).
\]

Here, it is presumed that, when $3n \leq l$, we do not consider first interval at all. For every $k \in [2^{l+2^3n}, 2^{3n+2^3n})$, it follows from the definition of $\sigma_2$ that
\[
\mu(k, B) \leq (2^l\sigma_2\mu(A)) \leq 2^l\mu(k, A) = 2^l\mu(A).
\]

For every $k \in [2^{l+2^3n}, 2^{l+2^3n})$, it follows from the assumption $B \prec \prec \log A$ that
\[
\mu(k, B) \leq \left( \prod_{m=0}^{2^{l+2^3n}-1} \mu(m, B) \right)^{2^{-1-2^3n}} \leq \left( \prod_{m=0}^{2^{l+2^3n}-1} \mu(m, A) \right)^{2^{-1-2^3n}} \leq \left( \prod_{m=2^{3n}}^{2^{l+2^3n}} \mu(m, A) \right)^{2^{-1-2^3n}} = 2^{-4\cdot 2^3n} = (\sigma_2\mu^4(A))(k).
\]

For every $k \in [2^{3n+2^3n}, 2^{23(n+1)})$, it follows from the assumption $B \prec \prec \log A$ that
\[
\mu(k, B) \leq \left( \prod_{m=0}^{k} \mu(m, B) \right)^{1/(k+1)} \leq \left( \prod_{m=0}^{k} \mu(m, A) \right)^{1/(k+1)} \leq \left( \prod_{m=2^{3n}}^{k} \mu(m, A) \right)^{1/(k+1)} = 2^{-2^{3(n+1)}(k+1)/k+1)} \leq 2^{8-2^{3(n+1)}} = 256\mu(k, A).
\]

Arguing similarly, we infer that, for every $k \in [2^{23n}, 2^{1+2^3n})$, we have
\[
\mu(k, B) \leq \left( \prod_{m=0}^{2^{3n}-1} \mu(m, B) \right)^{2^{-2^{3n}}} \leq \left( \prod_{m=0}^{2^{3n}-1} \mu(m, A) \right)^{2^{-2^{3n}}} \leq \left( \prod_{m=2^{3(n-1)}}^{2^{3n}-1} \mu(m, A) \right)^{2^{-2^{3n}}} = 2^{-8-2^{3n}} = (256\sigma_2\mu(A))(k).
\]

A combination of all 4 preceding estimates yields the assertion. □
Define the nonlinear homogeneous mapping $T : l_\infty \to l_\infty$ by setting

$$(T x)(k) = \left( \prod_{m=0}^{k} \mu(m, x) \right)^{1/(k+1)}, \quad k \geq 0.$$  \hfill (16)

Lemma 39, Lemma 40, and Lemma 41 below are used in the proof of Theorem 36 (c).

**Lemma 39.** Operator $T$ has the following properties.

(a) for every $x \in l_\infty$, $Tx = \mu(Tx)$.

(b) for every $x, y \in l_\infty$, we have $y \lesssim \log x$ if and only if $Ty \leq Tx$.

(c) for every $x \in l_\infty$ we have $\sigma_N Tx \leq T(\sigma_N x) \leq \sigma_{2N}Tx$.

(d) for every $x \in l_\infty$, $Tx \lesssim \log N \sigma_N x$ implies that $T^2x \leq N\sigma_{2N}Tx$.

(e) for every $x \in l_\infty$, $T^2x \leq N\sigma_N Tx$ implies that $Tx \lesssim \log N \sigma_N x$.

**Proof.** The claims (a) and (b) are obvious.

In order to prove the claim (c), let $m \geq 0$ and let $0 \leq r \leq N - 1$. It follows from the first assertion that

$$\left( \prod_{m=0}^{mN} \mu(k, x) \right)^{1/(mN+1)} = \left( \prod_{k=0}^{(m+1)N-1} \mu(k, \sigma_N x) \right)^{1/(mN+r)} \leq \left( \prod_{k=0}^{mN+r} \mu(k, \sigma_N x) \right)^{1/(mN+r)}.$$

Equivalently, (setting $n = mN + r$), we have

$$(\prod_{k=0}^{\lfloor n/23 \rfloor} \mu(k, x))^{1/(\lfloor n/23 \rfloor + 1)} \leq \left( \prod_{k=0}^{n} \mu(k, \sigma_N x) \right)^{1/(n+1)}.$$

Since $(\sigma_N z)(n) = z(\lfloor n/23 \rfloor)$ for all $z \in l_\infty$, we rewrite the preceding inequality as $\sigma_N Tx \leq T(\sigma_N x)$. The proof of the inequality $T(\sigma_N x) \leq \sigma_{2N} Tx$ is similar.

The claim (d) follows from consecutive application of (b) and (c). The claim (e) assertions follows from consecutive application of (c) and (b). \hfill \square

**Lemma 40.** For every $2^{2^3n} \leq k < 2^{2^3(n+1)}$, we have

$$2^{\frac{2^3\gamma_n}{k + 1}} \leq 2^{2^3(n+1)} (T \mu(A))(k) \leq 2^{1 + \frac{2^3\gamma_n}{k + 1}}.$$

Here, $\gamma_n = 2^{3n+2^{2n}}$.

**Proof.** Fix $k$ and $n$ satisfying the assumption. By the definition of $A$, we have

$$\prod_{m=0}^{k} \mu(m, A) = \frac{1}{4} \prod_{s=1}^{n} \prod_{m=2^{3s}(s-1)}^{2^{3s}-1} \mu(m, A) \cdot \prod_{m=2^{3n}}^{k} \mu(m, A) = \frac{1}{4} \prod_{s=1}^{n} \prod_{m=2^{3s}(s-1)}^{2^{3s}-1} \mu(m, A).$$
that

\[ \text{it follows from Lemma 40 (with } k \text{ for all sufficiently large } n \text{, we obtain} \]

\[ 2^{(k+1)2^{3(n+1)}} \prod_{m=0}^{k} \mu(m, A) = \frac{1}{4} \prod_{s=1}^{n} 2^{8\gamma_{s-1} - \gamma_s} \cdot 2^{8\gamma_n} = 2^{14} \cdot \prod_{s=1}^{n} 2^{7\gamma_s}. \]

The assertion follows now from the inequality

\[ 2^{7\gamma_n} \leq \prod_{s=1}^{n} 2^{7\gamma_s} \leq 2^{7\gamma_n + 7(n-1)\gamma_{n-1}} \leq 2^{7\gamma_n + k - 13}, \]

where we used the estimate \( 7(n-1)\gamma_{n-1} \leq k - 13 \), which holds for every \( n \) and for every \( k \in [2^{3n}, 2^{3(n+1)}] \).

\[ \square \]

**Lemma 41.** For every \( l \geq 1 \), the inequality \( T^2 \mu(A) \leq 2^l \sigma_2 T \mu(A) \) fails.

**Proof.** We will demonstrate that the inequality above fails at the point \( \gamma_n - 1 \) for all sufficiently large \( n \) (we use the abbreviation \( \gamma_n = 2^{3n+2^n} \) from Lemma 40). It is obvious that

\[ (T^2 \mu(A))(\gamma_n - 1) = 2^{-2^{3(n+1)}} \left( \prod_{m=0}^{\gamma_n-1} 2^{2^{3(n+1)}} (T \mu(A))(m) \right)^{1/\gamma_n}. \]

Using the fact (guaranteed by the inequality \( T \mu(A) \geq \mu(A) \) and Lemma 40) that

\[ 2^{2^{3(n+1)}} (T \mu(A))(m) \geq \begin{cases} 1, & m < 2^{2^{3n}} \\ \frac{2^{\gamma_n}}{2^{3n}}, & 2^{2^{3n}} \leq m < 2^{2^{3(n+1)}} \end{cases} \]

we infer that

\[ (T^2 \mu(A))(\gamma_n - 1) \geq 2^{-2^{3(n+1)}} \left( \prod_{m=2^{2^{3n}}}^{\gamma_n-1} 2^{7\gamma_n} \right)^{1/\gamma_n} = 2^{-2^{3(n+1)}} \cdot 2^{\sum_{m=2^{2^{3n}}}^{\gamma_n-1} \frac{\gamma_n}{m+1}}. \]

Since

\[ \sum_{m=2^{2^{3n}}}^{\gamma_n-1} \frac{1}{m+1} \geq \log \left( \frac{\gamma_n}{2^{2^{3n}}} \right) - 2^{-2^{3n}} \geq n, \]

it follows that

\[ (T^2 \mu(A))(\gamma_n - 1) \geq 2^{7n - 2^{3(n+1)}}. \]

On the other hand, it follows from Lemma 40 (with \( k = 2^{3n-l}+2^{3n} - 1 \) that

\[ (2^l \sigma_2 T \mu(A))(\gamma_n - 1) = 2^l (T \mu(A))(2^{3n-l}+2^{3n} - 1) \leq 2^l + 1 \cdot 2^{7\gamma_n - 2^{3(n+1)}}. \]

Thus, for \( n \geq 2^l + 1 \), we have

\[ (T^2 \mu(A))(\gamma_n - 1) \geq (2^l \sigma_2 T \mu(A))(\gamma_n - 1). \]

\[ \square \]
We are now ready to prove main result of this section.

of Theorem 36. First, we prove (a). In what follows, \( A_0 \) is as in Lemma 37. Let \( 0 \leq B \in I_A \) is such that \( B \prec \log A \). If \( \varphi \) is a positive trace on \( I_A \), then we infer from Proposition 38 that

\[
\varphi(B) \leq 512\varphi(A) + 2^l \varphi(\mu^4(A)) + 2^l \varphi(A_0).
\]

By Lemma 37, \( \varphi(A_0) = 0 \). Since \( \mu^4(k, A) = o(\mu(k, A)) \) as \( k \to \infty \), it follows from Lemma 15 that \( \varphi(\mu^4(A)) = 0 \). Hence, \( \varphi(B) \leq 512\varphi(A) \). Repeating the argument in the proof of Theorem 10 (b), we obtain that \( \varphi \) extends to a positive trace on \( LE(I_A) \). We keep denoting this extension by \( \varphi \).

By Theorem 8 (a), the positive trace \( \varphi \) on the ideal \( LE(I_A) \) is spectral. Applying Theorem 10 (a), we infer that \( \varphi \) is monotone with respect to the logarithmic submajorization. Thus, the original trace \( \varphi \) on \( \mathcal{I} \) is also monotone with respect to the logarithmic submajorization.

We now turn to (b). This assertion is, in fact, proved in Example 1.5 of [9]. More precisely, it is shown there that there exists a quasi-nilpotent operator \( Q \in I_A \) such that \( Q \not\in \text{Com}(I_A) \). By Zorn lemma, there exists a linear functional \( \varphi \) on \( I_A \) such that \( \varphi(Q) = 1 \) and such that \( \varphi \) vanishes on \( \text{Com}(I_A) \). Hence, \( \varphi \) is a (non-positive) trace on \( I_A \) such that \( \varphi(Q) = 1 \). Since \( Q \) is quasi-nilpotent, it follows that \( \lambda(Q) = 0 \) and, therefore, \( \varphi(\lambda(Q)) = 0 \). However, \( \varphi(Q) = 1 \) by construction. Hence, \( \varphi \) is not spectral. This proves (b).

Finally, we prove (c). More precisely, we will prove that

\[
\{ (\prod_{m=0}^n \mu(k, A))^{1/(n+1)} \}_{n \geq 0} \notin LE(I_A).
\]

In terms of the operator \( T \) introduced in (16), the assertion above can be written as \( T\mu(A) \notin LE(I_A) \). Assume the contrary. It follows from (14) that there exists \( B \in I_A \) such that \( T\mu(A) \prec \log B \). Since \( I_A \) is a principal ideal, it follows that \( \mu(B) \leq 2^l \sigma_{2l} \mu(A) \) for some \( l \geq 1 \). Hence, \( T\mu(A) \prec 2^l \sigma_{2l} \mu(A) \). By Lemma 39 (d), we have \( T^2\mu(A) \leq 2^l \sigma_{2l+1} T\mu(A) \). However, the latter inequality fails by Lemma 41. This proves (c).

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