ON THE WHITTAKER PLANCHEREL THEOREM FOR REAL REDUCTIVE GROUPS

NOLAN R. WALLACH

Abstract. The purpose of this article is to give the first complete proof of the Whittaker Plancherel Theorem. The proof uses Harish-Chandra’s Plancherel Theorem for a real reductive group and its exposition can be used as an introduction to Harish-Chandra’s ideas. The proof follows the basic ideas in the author’s original attempt in his second volume on real reductive groups. An error in the calculation of the Whittaker Transform of a Harish-Chandra wave packet is fixed using a result of Raphaël Beuzart-Plessis.

1. Introduction

This paper has its roots in the final chapter of my Real Reductive Groups II. The point of that chapter was to show how to use the Harish-Chandra Plancherel theorem for real reductive groups to decompose the unitarily induced representation from a generic one dimensional character of a maximal unipotent subgroup (the Whittaker Plancherel Theorem). This work has been widely cited by researchers in the theory of automorphic forms but, unfortunately, there are several serious gaps in the main arguments. The first was indicated in [vdBK]. Which made me aware that the theory of cusp forms in the Whittaker case is much more subtle than in the Harish-Chandra theory which was basic to his proof of the Plancherel Theorem for Real Reductive Groups. On fixing that error in the book the first referee found a new gap which was promulgated from the original. The referee of the second version of the manuscript found a related gap which was also implicit in the original version in [RRGII] as did the referee of the third version. This version gives a complete proof of the theorem based on the basic ideas in my original proof augmented by a powerful convergence theorem of Beuzart-Plessis [B].

My two volumes on Real Reductive Groups were written over a period of 15 years starting in 1977, with the first volume appearing in 1987 and the second in 1992 the work on Whittaker vectors was done in the early 1980’s at the same time that Harish-Chandra was doing his work. Recently V.S.Varadarajan and R. Gangolli have edited Volume
V [H5] of Harish-Chandra’s collected work and have, in a heroic piece of work, organized a collection of his unpublished notes of his work on the Whittaker Plancherel Theorem as part III of the volume. As it turns out neither of us completed the proof of the theorem although we did independently find the same statements down to the same definition of the Whittaker Schwartz space. The key missing ingredient has to do with the completeness of the space of wave packets and its implication that the discrete spectrum is an appropriate projection of the discrete spectrum of $L^2(G)$.

Here is a description of the flow of this paper. The purpose of Section 2 is to set up the notation used in the rest of the paper. Section 3 gives a description of Harish-Chandra’s work on what he calls cusp forms for the group, in the context of the Harish-Chandra Schwartz space, and their relationship with the discrete series. Section 4 begins the development of the Whittaker version of the Schwartz space. Theorem 17, combined with Corollary 20 in that section replaces the error found by [vdBK]. Section 5 gives a proof of the Whittaker version of Harish-Chandra’s theorem asserting that an eigenfunction for the bi-invariant differential operators that is in the Schwartz space is a cusp form. Section 6 gives an exposition of the Harish-Chandra Plancherel theorem. The reader should be warned that in [RRGII] the Harish-Chandra Plancherel density, $\mu(\omega, i\nu)$, needs several normalizing constants. In this paper we will be “sloppy” about normalizing constants and measures. These aspects were handled in [RRGII]. The density we write as $\mu(\omega, \nu)$ and it contains all of these normalizations. Theorem 31 is the most important part of the Harish-Chandra Plancherel Theorem for the purposes of the proof of the Whittaker version of the theorem. Section 7 involves a major deviation from [RRGII] giving a decomposition of the Whittaker Schwartz space in terms of Theorem 31. Section 8 gives a description of the holomorphic continuation of the Jacquet integrals. It also proves key tempered estimates for this analytic continuation. Section 9 contains a critical result on the Fourier transform of wave packets leading our proof of the relationship between the discrete series for $G$ and the Whittaker discrete series. The paper next approaches the continuous spectrum. This is where the result in [B] plays an important role. Section 12 contains our version of the Whittaker Plancherel Formula. This version is new to the paper and is based on a preliminary result in Section 11 that is also new to this paper. In Section 13 we show that in the case of right $K$–fixed functions the new statement coincides with the implication of the one in [RRGII].
I heartily thank the readers of the first, second and third versions of this paper for their detailed discussion of key flaws. Many aspects of the second volume of [RRGI] were “written in a vacuum”. [RRGI] contained many complicated results, but almost all of them could be given an exposition in a graduate course or a technical seminar and, in fact, they were. [RRGII] was a different story. A year long graduate course could not handle the shear technical difficulty of either the Whittaker Plancherel theorem or the Harish-Chandra Plancherel Theorem. Thus the added filter was not possible for these highly technical, difficult theorems. The readers played the role of an exceptionally qualified, helpful and critical audience. Also, I placed arXiv:2206.14773 on the arXiv with the hope that someone would come forth with a general proof of the convergence theorem that was critical to my way of calculating the calculation of the Jacquet integral of a wave packet. I thank Raphaël Beuzart-Plessis for responding with a reference to his proof and thus closing a long painful story. In addition I would like to thank Hervé Jacquet for his many clarifying suggestions, perceptive criticism and for his kindness.

2. Notation

We will be studying the class of real reductive groups $G$ as in [BW] and [RRGI], [RRGII]. That is, there is an algebraic subgroup $G_1$ of $GL(n, \mathbb{R})$ (for some $n$) invariant under transpose such that $G$ is a finite covering of an open subgroup of $G_1$. Let $p : G \to G_1$ be the covering map.

Let $K$ be a maximal compact subgroup of $G$ (which we can take as $p^{-1}(G_1 \cap O(2n))$) and let $\theta$ be the corresponding Cartan involution. Let $G = KA_oN_o$ be an Iwasawa decomposition of $G$. Let $P_o$ be the normalizer of $N_o$ in $G$. The minimal parabolic subgroups are the $G$ conjugates of $P_o$. Since $G = KP_o$ they are all $K$ conjugate. A parabolic subgroup of $G$ is a subgroup that is its own normalizer containing a minimal parabolic subgroup. Let $P$ be a parabolic subgroup. We set $M_P = P \cap \theta P$ and $N_P$ equal to the unipotent radical of $P$. Then $P = M_PN_P$ is called a ($K$-) standard Levi decomposition. Let $C$ denote the center of $M_P$. If $a_P = \{x \in \text{Lie}(C) | \theta x = -x\}$ then we set $A_P = \exp(a_P)$. Note that

$$\exp : a_P \to A_P$$

is an isomorphism of Lie groups ($a_P$ a Lie group under addition). Define (as usual) $\log : A_P \to a_P$ to be the inverse of $\exp$. Set $^*M_P = \cap \ker \mu,$
the intersection is over all Lie group homomorphisms

\[ \mu : M_P \to \mathbb{R}_{>0}, \]

then \( M_P = \circ M_P A_P \) and \( P = \circ M_P A_P N_P \) with unique decomposition. Note that \( \circ M_P \) is in the class of groups that we are studying for all parabolic subgroups including \( G \). We will also use the notation \( \circ G \) for \( \circ M_G \). The above decomposition is usually called a Langlands decomposition. We will use the notation \( \overline{P} = \theta P \) the standard opposite parabolic to \( P \). We note that \( \overline{P} = \circ M_P A_P \theta (N_P) \) and we write \( \theta (N_P) = \overline{N} \). Applying this standard material to \( P_o \) and using the notation \( X_{P_o} = X_o \) for \( X = M, A, N, a \) we have \( P_o = \circ M_o A_o N_o \). Also the Iwasawa decomposition implies that \( G = N_o A_o K \) with unique decomposition. So if \( g \in G \) then \( g = n(g)a(g)k(g) \) with \( n(g) \in N_o, a(g) \in A_o, k(g) \in K \). We will call a parabolic subgroup standard if it contains \( P_o \).

As usual, \( U(Lie(G)) \) denotes the universal enveloping algebra of \( Lie(G) \). We look upon \( U(Lie(G)) \) as the left invariant differential operators on \( G \). We use the standard notation \( Z_G(Lie(G)) \) for \( Ad(G) \)–invariants in \( U(Lie(G)) \) and \( Z(Lie(G)) \) for the center of \( U(Lie(G)) \). Note that since \( G \) has a finite number of connected components \( Z(Lie(G)) \) is a finitely generated \( Z_G(Lie(G)) \) module under multiplication.

Let \( n = \dim N_o \) and let \( \mu (g) = \wedge^n Ad(g) \) and let

\[ V = \text{Span}\mu (g) \wedge^n Lie(N_o) \subset \wedge^n Lie(G) \]

then \( (\mu, V) \) is a real representation of \( G \). On \( Lie(G) \) we put the inner product on \( Lie(G) \) such that if \( \theta X = -X \) then \( adX \) is self adjoint and such that \( K \) acts orthogonally. It naturally induces an inner product on \( \wedge^n Lie(G) \). Let \( \|...\| \) denote the operator norm on \( \text{End}(\wedge^n Lie(G)) \). Let \( G = A_G \circ G \) be the \( K \)-standard Langlands decomposition of \( G \). Define

for \( g \in G, g = ag_1 \) with \( a \in A_G, g_1 \in \circ G \), \( \|g\| = e^{[\log a]} \|\mu (g_1)\| \|v\|^{\frac{2}{n}} \)

Then \( \|...\| \) satisfies the following four properties

1. \( \|xy\| \leq \|x\| \|y\| \), \( x, y \in G \).
2. \( \|x\| = \|x^{-1}\|, x \in G \).
3. The sets \( B_r = \{ x \in G \|x\| \leq r \} \) for \( 0 < r < \infty \) are compact.
4. \( \|k_1 x k_2\| = \|x\| \) for \( k_1, k_2 \in K, x \in G \).

Any function that satisfies these properties will be called a norm. Using these properties one has

5. If \( (\pi, H) \) is a Banach representation of \( G \) with operator norm \( \|...\|_H \), then there exist \( C > 0 \) and \( r > 0 \) such that \( \|\pi(x)\|_H \leq C \|x\|^r, x \in G \) (see [RRGI] 2. A.2. p. 71).

The norm we just constructed will be called the standard norm.
We note that for 1. implies that \( \|e\| \geq 1 \) the identity element in \( G \) and so

\[
1 \leq \|e\| = \|gg^{-1}\| \leq \|g\|^2
\]

thus \( \|g\| \geq 1 \). Also note

**Lemma 1.** There exists a constant \( C > 0 \) such that if \( a \in A_o \) then

\[
C \|\log a\| \leq \log \|a\| \leq \|\log a\|.
\]

If \( P \) is a parabolic subgroup of \( G \) and if \( aP \) is above then we define (as usual) \( \rho_P(H) = \frac{1}{2} \text{tr}(\text{ad}(H)_{|\text{Lie}(NP)}) \) for \( H \in aP \). We will set \( \rho = \rho_P \).

If \( \lambda \in a^*_o \) then we set \( e^\lambda = e^{\log a} \).

**Lemma 2.** Assume that \( G \) has compact center then if \( \|...\| \) is the standard norm on \( G \) and \( H \in a^+_{P} \) then \( \|\exp H\| = e^{\rho_o(H)} \) with

\[
\rho_o(H) = \frac{1}{2} \text{tr}(H)_{|\text{Lie}(N_o)}
\]

Let \( P \) be a parabolic subgroup of \( G \) and let \((\sigma, H^\sigma)\) be a Hilbert representation of \( ^oM_P \) such that the underlying \((\text{Lie}(^oM_P), ^oM_P \cap K)\) is admissible and finitely generated and \((\sigma, H^\sigma)\) is unitary as a representation of \( ^oM_P \cap K \) and let \( \nu \in (a^*_o)_{\mathbb{C}} \). We form the smooth induced representation \( I^\infty_{\sigma,\nu} \) as follows: The underlying Fréchet space is

\[
I^\infty_{\sigma} = \{ f : K \to H^\infty_{\sigma} | f \text{ is } C^\infty \text{ and } f(mk) = \sigma(m)f(k), m \in ^oM_P \cap K, k \in K \}.
\]

If \( f \in I^\infty_{\sigma} \) then we define

\[
(\pi_{P,\sigma,\nu} f)(k) = pf_{\nu}(kg)
\]

with

\[
pf_{\nu}(namk) = a^{\nu + \rho_P} \sigma(m)f(k)
\]

if \( n \in N_P, a \in A_P, m \in ^oM_P \) and \( k \in K \), \( \rho_P(h) = \frac{1}{2} \text{tr}h_{|\text{Lie}(NP)} \), and

\[
\langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle_{\sigma} \, dk.
\]

The Hilbert space \( I_{\sigma} \) is the completion of \( I^\infty_{\sigma} \) with respect to this inner product and the operators \( \pi_{P,\sigma,\nu} \) extend to yield a Hilbert representation of \( G \).

A fundamental inequality of Harish-Chandra (c.f. [RRGI], Lemma 4.A.2.3 (2) with the misprint \( \|\log n\| \) replaced by \( \|\log \|n\| \), see also Lemma 16 which is part (1) of the Lemma) implies
Lemma 3. There exist $C_1, C_2 > 0$ such that
\[
C_1(1 + \log \|n\|) \leq (1 + \rho_{P_o}(\log a_{P_o}(n))) \leq C_2(1 + \log \|n\|)
\]
for all $n \in N_o$.

3. The Harish-Chandra Schwartz Space

We keep the notation of the previous section. The purpose of this one is to give a tour of Harish-Chandra’s most profound results involving the discrete series. We follow the discussion in [RRGI] Chapter 7.

First we recall the definition of the Harish-Chandra Schwartz space. If $f \in C^\infty(G), k \in \mathbb{R} \geq [0, \infty), x, y \in U(\text{Lie}(G))$ then set
\[
p_{k,x,y}(f) = \sup_{g \in G} \left( (1 + \log \|g\|)^k \Xi(g)^{-1} |R_yL_x f(g)| \right).
\]

Here $R_y$ is the right regular action so $R_yf = yf$ as a left invariant differential operator and $L_x$ is the left regular action (so a right invariant differential operator) and $\Xi$ is Harish-Chandra’s basic spherical function which in particular is $K$ bi-invariant. The key facts we need about $\Xi$ are in [RRGI], Section 4.5.

1. If $g \in G, \text{write } g = n(g)a(g)k(g)$ with $n(g) \in N_o, a(g) \in A_o$ and $k(g) \in K$. Then $\Xi(g) = \int_K a(kg)^{\rho_o}dk$ (the integration is over normalized invariant measure on $K$).

2. There exist $C, d > 0$ such that if $h \in \mathfrak{a}^*_+$ then
\[
e^{-\rho(h)} \leq \Xi(\exp(h)) \leq Ce^{-\rho(h)}(1 + \|h\|)^d.
\]

3. $\Xi(g) > 0$ and there exists $r > 0$ such that ($dg$ is a Haar measure on $G$)
\[
\int_G (1 + \log \|g\|)^{-r} \Xi(g)^2dg < \infty.
\]

4. If $P$ is a parabolic subgroup of $G$ and $N = N_P$ then there exists $r$ such that ($dn$ is a Haar measure on $N$)
\[
\int_N (1 + \log \|n\|)^{-r} \Xi(n)dn < \infty.
\]

5. Since it is a $K$ bi-invariant spherical function it satisfies
\[
\int_K \Xi(xky)dk = \Xi(x)\Xi(y)
\]

(indeed this is one of the definitions of zonal spherical function).  

6. $\Xi(xy) = \int_K a(kx^{-1})^{\rho}a(ky)^{\rho}dk$ for $x, y \in G$ (c.f. [RRGI] (2) p. 147).
7. Assume that $G$ has compact center and $||\cdot||$ is the standard norm (see Section 2). Then Lemma 2 implies that there exist $C_1, C_2, d > 0$ such that

$$C_1||g||^{-1} \leq \Xi(g) \leq C_2||g||^{-1}(1 + \log||g||)^d$$

since $G = K \exp(\mathfrak{a}_o^+)K$.

Using 6. one sees easily that if $Y$ is a compact subset of $G$ then there exist positive constants $M_Y, L_Y$ such that if $y \in Y$ then

$$L_Y\Xi(x) \leq \Xi(xy) \leq M_Y\Xi(x).$$

Indeed, we can choose $L_Y, M_Y$ such that $0 < L_Y < a(ky)^\rho \leq M_F$ for $y \in Y$ since $KY$ is compact and $a(x)^\rho > 0$. Thus proves

$$L_Y \int_K a(kx^{-1})^\rho dk \leq \Xi(xy) \leq M_Y \int_K a(kx^{-1})^\rho dk = M_Y\Xi(x).$$

We will say that $f \in C^\infty(G)$ satisfies the weak inequality if there exist $C, d$ such that

$$|f(g)| \leq C\Xi(g)(1 + \log||g||)^d.$$  

The Harish-Chandra Schwartz space, $\mathcal{C}(G)$, is the subspace of $C^\infty(G)$ consisting of those functions $f$ such that

$$p_{k,x,y}(f) < \infty$$

for all $x, y \in U(Lie(G))$ and all $k \geq 0$ endowed with the topology given by the semi-norms $p_{k,x,y}$. With this topology $\mathcal{C}(G)$ is a Fréchet space and an algebra under convolution.

In this context the basis of Harish-Chandra’s “philosophy of cusp forms” is encapsulated in the following sequence of results. Let $P$ be a standard parabolic subgroup of $G$. We define for $f \in \mathcal{C}(G)$

$$f^P(m) = a^{-\rho_P} \int_{N_P} f(nm)dn$$

for $m \in M_P$ such that the integral converges. Until further notice all of the coming results in this section are due to Harish-Chandra. We will give references to [RRGI].

**Theorem 4.** (c.f. [RRGI] Theorem 7.2.1) If $f \in \mathcal{C}(G)$ then the integral defining $f^P$ converges absolutely and uniformly in compacta of $M$ and defines an element of $\mathcal{C}(M)$. Furthermore, the map $f \mapsto f^P$ is continuous from $\mathcal{C}(G)$ to $\mathcal{C}(M)$.

This follows from the following result of Harish-Chandra (c.f. [RRGI] p. 233.). Set $\Xi^{0M_P}$ equal to the analogue of $\Xi$ for $^{0M_P}$ and $K \cap M_P$ then
**Proposition 5.** If $u, v > 0$ are given then there exists $r > 0$ and $C_{u,v}$ such that if $m \in \mathfrak{m}_{0}^{\mathbb{M}_{P}}$, $a \in A_{P}$

$$a^{-p_{P}} \int_{N_{P}} \Xi(nam)(1 + \log \|nam\|)^{-r}dn \leq \Xi_{0}\mathbb{M}_{P}(m)(1 + \log \|m\|)^{-u}(1 + \log \|a\|)^{-v}.$$  

If $f^{P} = 0$ for all proper parabolic subgroups of $G$ then we call $f$ a cusp form. Let $Z(\text{Lie}(G))$ denote the center of $U(\text{Lie}(G))$ then

**Theorem 6.** (c.f. [RRGI], Theorem 7.7.2) If $f \in C(G)$ is such that $\dim Z(\text{Lie}(G))f < \infty$ then $f$ is a cusp form.

Let $\hat{K}$ be the set of equivalence classes of irreducible continuous representations of $K$. Let $\gamma \in \hat{K}$ and let $\chi_{\gamma}$ and $d(\gamma)$ be respectively the character and dimension of any representative of $\gamma$. Defining for $f \in C(G)$

$$E_{\gamma}(f)(g) = f_{\gamma}(g) = d(\gamma) \int_{K} f(gk)\chi_{\gamma}(k^{-1})dk$$

then under the right regular action of $K$ on $C(G)$ the function $f_{\gamma}$ transforms as a representative of $\gamma$. The following result is easily derivable from [RRGI] Corollary 3.4.7 and is used in the proof of the preceding theorem.

**Theorem 7.** If $f \in C(G)$ then $f_{\gamma} \in C(G)$ and the series

$$\sum_{\gamma \in \hat{K}} f_{\gamma}$$

converges to $f$ in $C(G)$. Furthermore, if $G$ has compact center and $\dim Z(\text{Lie}(G))f < \infty$ then the $(\text{Lie}(G), K)$ modules

$$U(\text{Lie}(G)_{C})\text{Span}_{C}R_{K}f_{\gamma}$$

are admissible and are finite direct sum of the underlying $(\text{Lie}(G), K)$ modules of irreducible square integrable representations.

The last part of the above theorem needs an explanation. An irreducible unitary representation of $G$, $(\pi, H)$, is said to be square integrable if the matrix coefficients

$$g \mapsto \langle \pi(g)v, w \rangle$$

are square integrable for all $v, w \in H$. In fact, all one needs is one non-zero square integrable matrix coefficient.
Theorem 8. (c.f. Theorem 5.5.4 [RRGI]) Assume that $G$ has compact center. If $(\pi, H)$ is an irreducible square integrable representation of $G$ and if $v, w$ are $K$-finite elements of $H$ then the corresponding matrix coefficient, $f(g) = \langle \pi(g)v, w \rangle$, is in $\mathcal{C}(G)$ and $\dim Z_G(\text{Lie}(G)) f = 1$.

Using the Casselman-Wallach (CW) theorem in form of (Theorem 11.8.2 [RRGII]) we have the following strengthening of the above theorem.

Corollary 9. If $(\pi, H)$ is an irreducible square integrable representation of $G$ and if $v, w \in H^K$ then the function $g \mapsto -\int_G f(g) \pi(g)v dg = \pi(f)v$ is in $\mathcal{C}(G)$.

Proof. We recall the space $\mathcal{S}(G)$ (see 7.1.2 [RRGI]) the space of all elements of $C^\infty(G)$ such that $\sup_{g \in G} \|g\|^d |R_xf(g)| < \infty$ for all $x \in U(\text{Lie}(G))$ (thought of as a left invariant differential operator) and all $d$. Then $\mathcal{S}(G)$ is a convolution algebra. If $(\pi, H)$ is a Hilbert representation of $G$ then we can define an action of $\mathcal{S}(G)$ on $H^K$ by

$$\int_G f(g)\pi(g)v dg = \pi(f)v.$$

Theorem 11.8.2 in [RRGII] implies that if $(\pi, H)$ is irreducible and admissible then $\mathcal{S}(G)$ acts algebraically irreducibly on $H^K$. Thus in particular, $\pi(\mathcal{S}(G))u = H^K$ if $u \in H_K$ and $u \neq 0$. Now assume that $(\pi, H)$ is as in the previous theorem. If $v, w \in H^K$ and $u \in H_K$ is such that $u \neq 0$ then exist $f_1, f_2 \in \mathcal{S}(G)$ such that $\pi(f_1)u = v, \pi(f_2)u = w$. So if $c_{v,w}(g) = \langle \pi(g)v, w \rangle$ then

$$c_{v,w} = \tilde{f}_2 * c_{u,u} * f_1$$

with $\tilde{f}_1(g) = f_1(g^{-1})$. Since $\mathcal{S}(G) \subset \mathcal{C}(G)$, and $\mathcal{C}(G)$ is closed under convolution $c_{v,w} \in \mathcal{C}(G)$. □

A simple limiting argument implies that if $\mathcal{C}_\pi(G)$ is the closure in $\mathcal{C}(G)$ of the span of the $c_{v,w}$ with $v, w$ $K$-finite elements of $H$ then

Corollary 10. With the notation above, the space $\mathcal{C}_\pi(G)$ is contained in the space of cusp forms.

We are now ready to close the circle and describe one of Harish-Chandra’s deepest results. We set $\mathcal{E}_2(G)$ equal to the set of irreducible square integrable representations of $G$ (obviously, in light of Schur’s Lemma, $\mathcal{E}_2(G) = \emptyset$ if the center of $G$ is not compact). If $\omega \in \mathcal{E}_2(G)$ and if $(\pi, H)$ is a representative of $\omega$ then we set $\mathcal{C}_\omega(G) = \mathcal{C}_\pi(G)$.
Theorem 11. Assume that $G$ has compact center. Then the space of cusp forms on $G$ is the topological direct sum

$$
\bigoplus_{\omega \in \mathcal{E}_2(G)} \mathcal{C}_\omega(G).
$$

This follows from Harish-Chandra’s difficult converse to Theorem 6.

Theorem 12. (c.f. [RRGI] Theorem 7.7.6) Assume that the center of $G$ is compact. If $f$ is a cusp form on $G$ that is $K$–finite then $\dim Z(\text{Lie}(G))f < \infty$.

4. The Schwartz space adapted to Whittaker models

In this section we study a parallel theory to that of the previous section for so called Whittaker functions. We retain the notation of the preceding section. Let $\chi : N_o \to S^1$ be a unitary character. We say that $\chi$ is generic if its differential is non-zero on every $A_o$ weight space in $\text{Lie}(N_o)/[\text{Lie}(N_o), \text{Lie}(N_o)]$. We set $C^\infty(N_o \backslash G; \chi)$ equal to the space of smooth functions on $G$ such that $f(ng) = \chi(n)f(g)$ for $n \in N_o$ and $g \in G$.

We define a unitary representation $L^2(N_o \backslash G; \chi)$ as follows: We fix an invariant measure on $N_o$ and take the corresponding right invariant measure on $N_o \backslash G$, $dg$. The Hilbert space is the space of all measurable (with respect to Haar measure) functions on $G$ such that

1. $f(ng) = \chi(n)f(g)$ for $n \in N_o$ and $g \in G$.
2. $\|f\|^2 = \int_{N_o \backslash G}|f(g)|^2dg = \int_{A_o \times K}|f(ak)|^2a^{-2\rho}da < \infty (\rho = \rho_o$, up to normalization of $da$ on $A$).

If $f \in L^2(N_o \backslash G; \chi)$ then we define $\pi_\chi(g)f(x) = f(xg)$ for $x, g \in G$. This defines a unitary representation of $G$.

The right-most formula in 2. suggests the generalization of the Harish-Chandra Schwartz space in this context. We define

$$p_{d,x}(f) = \sup_{g \in G}(a(g)^{-\rho}(1 + \|\log(a(g))\|)^d|xf(g)|$$

for $d \in \mathbb{R}$ and $x \in U(\text{Lie}(G))$. Then $C(\text{Lie}(N_o \backslash G; \chi))$ is the space of all $f \in C^\infty(N_o \backslash G; \chi)$ such that $p_{d,x}(f) < \infty$ for all choices of $d$ and $x$. We endow $C(\text{Lie}(N_o \backslash G; \chi))$ with the topology defined by the semi-norms $p_{d,x}$. This defines the Fréchet space given in [RRGII] Chapter 15. In light of Lemma 11 the following semi–norms

$$q_{d,x}(f) = \sup_{g \in G}(a(g)^{-\rho}(1 + \log|a(g)|)^d|xf(g)|$$

are equivalent to the $p_{d,x}$ we will use one or the other depending on convenience. We will also need another similar space. Here we use the
(Langlands) decomposition

\[ G = A_G \circ G \]

With \( A_G \) a connected subgroup of \( A_o \) and \( oG \) has compact center. If \( d, r \geq 0, x \in U(\text{Lie}(G)) \) and \( g = ag_o \) with \( a \in A_G \) and \( g_o \in oG \) then we define for \( f \in C^\infty(N_o \setminus G; \chi) \)

\[ q_{d,k,x}(f) = \sup_{g \in oG, a \in A_G} a(g)^{-\rho}(1 + \log \| a(g_o) \|)^k(1 + \log \| a \|)^{-d} |xf(a(g_o))|. \]

We set for each \( d \geq 0, \mathcal{B}_d(N_o \setminus G; \chi) \) equal to the space of all \( f \in C^\infty(N_o \setminus G; \chi) \) such that \( q_{d,k,x}(f) < \infty \) for all \( k \geq 0 \) and \( x \in U(\text{Lie}(G)) \).

Then

\[ \mathcal{B}(N_o \setminus G; \chi) = \lim_{\rightarrow} \mathcal{B}_d(N_o \setminus G; \chi) \]

is a LF space. We note that if \( G \) has compact center then \( \mathcal{B}(N_o \setminus G; \chi) = \mathcal{C}(N_o \setminus G; \chi) \).

We have

**Lemma 13.** \( \mathcal{C}(N_o \setminus G; \chi) \subset L^2(N_o \setminus G; \chi) \).

**Proof.** If \( f \in \mathcal{C}(N_o \setminus G; \chi) \) then then for all \( d > 0 \)

\[ |f(ak)| \leq C_d a^\rho (1 + \| \log a \|)^{-d} \]

Thus

\[ \int_{A_o \times K} |f(ak)|^2 a^{-2\rho} da dk \leq C_d \int_{A_o \times K} a^{2\rho} (1 + \| \log a \|)^{-d} a^{-2\rho} da dk. \]

Take \( d \) so large that

\[ \int_{A_o} (1 + \| \log a \|)^{-d} da < \infty. \]

\[ \square \]

**Proposition 14.** If \( f \in \mathcal{C}(G) \) then the the integral

\[ f_\chi(g) = \int_{N_o} \chi(n)^{-1} f(ng)dn \]

converges absolutely and uniformly on compacta in \( g \) to an element of \( \mathcal{C}(N_o \setminus G; \chi) \). Furthermore, the map defined by \( T_\chi(f) = f_\chi \) is a continuous map from \( \mathcal{C}(G) \) to \( \mathcal{C}(N_o \setminus G; \chi) \).

**Proof.** If \( n \in N_o, a \in A_o \) and \( k \in K \) then

\[ |f(nak)| \leq p_{d,1,1}(f) \Xi(na)(1 + \log \| na \|)^{-d}. \]
We note that there is an orthonormal basis of $\mathbb{R}^{2n}$ such that relative to that basis $\mu(n) = I + X$ with $X$ upper triangular with zeroes on the main diagonal and $\mu(a)$ is diagonal. Thus

$$\mu(na) = \mu(a) + X\mu(a)$$

and so if $a \in A_o \cap ^\circ G$ then $|an| \geq |a|$. If $a_1 \in A_G$ and $a_2 \in A_o \cap ^\circ G$ then $\|a_1 a_2 n\| = \|a_1\| \|a_2 n\| \geq \|a_1\| \|a_2\| = \|a_1 a_2\|$ so $|an| \geq |a|$ for $a \in A_o$. Also $|n| = |a^{-1} an|$ $\leq |a^{-1}| |an| = |a| |an|$ $1 \leq |an|^2$. This implies that

$$(1 + \log |an|) \geq (1 + \log |a|)$$

and

$$(1 + \log |an|) \geq \frac{(1 + \log |n|)}{2}.$$ 

So if $d > 0$

$$(1 + \log |an|)^{-d} \leq 2^\frac{d}{2} (1 + \log |a|)^{-\frac{d}{2}} (1 + \log |n|)^{-\frac{d}{2}}.$$ 

This and Proposition 5 imply that for all $\hat{d} > 0$ and $n_1 \in N_o$, $a \in A_o$ and $k \in K$

$$a^{-\rho}|f_\chi(n_1 ak)| \leq a^{-\rho} \left| \int_{N_o} \chi(n)^{-1} f(\chi(n_1 ak) \, dn \right| \leq a^{-\rho} \int_{N_o} |f(n_1 na)| \, dn$$

$$= a^{-\rho} \int_{N_o} |f(na)\, dn \leq p_{d,1,1}(f) a^{-\rho} \int_{N_o} \Xi(na) (1 + \log |na|)^{-\hat{d}} \, dn \leq p_{d,1,1}(f) 2^\frac{d}{2} (1 + \log |a|)^{-\frac{d}{2}}.$$ 

for $d$ sufficiently large. This implies that for large $\hat{d}$

$$q_{\frac{d}{2},1}(f_\chi) \leq 2^\frac{d}{2} p_{d,1,1}(f).$$

If $x \in U(Lie(G))$ then $(xf)_\chi = x(f_\chi)$. Thus

$$q_{\frac{d}{2},x}(f_\chi) \leq 2^\frac{d}{2} p_{d,1,x}(f).$$

The now come to a result that $an$ is the analogue of Lemma 15.3.2 in [RRGII]. The latter lemma has an error in its proof (pointed out in [vdBK] who also show that it cannot be true as stated).
If \( f \in C(N_o \backslash G; \chi) \) and if \( P \) is a standard parabolic subgroup (i.e. \( P_o \subset P \) ) \( P = oM_P A_P N_P \) (as in Section 2) and recall that we set \( N_P = \theta(N_P) \) then we define for \( m \in oM_P \) and \( a \in A_P \)

\[
f^P(ma) = a^o \int_{N_P} f(\bar{n}ma) d\bar{n}.
\]

We need the following simple lemma and to recall a key equality of Harish-Chandra in the proof of the replacement to the defective lemma.

**Lemma 15.** Let \( V \) be a real inner product space. Let \( V = V_1 \oplus V_2 \) an orthogonal direct sum. Let \( P \) denote the orthogonal projection of \( V \) onto \( V_1 \). Assume that we have \( u \in V_1, v \in V_2 \) and \( w \in V \) such that there exists \( 0 < C < 1 \) such that

\[
1 + \|Pw\| \geq C(1 + \|w\|).
\]

Then

\[
1 + \|u + v + w\| \geq \frac{C(1 + \|w\|)}{1 + \|u\|}
\]

and

\[
(1 + \|u + v + w\|)^2 \geq \frac{C(1 + \|v\|)}{(1 + \|w\|)}.
\]

**Proof.** Since \( P(v) = 0, \)

\[
1 + \|u + v + w\| \geq 1 + \|u + P(w)\|.
\]

Note that if \( x, y \in V \) then

\[
1 + \|x + y\| \geq \frac{1 + \|x\|}{1 + \|y\|}.
\]

Indeed,

\[
1 + \|x\| = 1 + \|x + y - y\| \leq 1 + \|x + y\| + \|y\| \leq (1 + \|x + y\|)(1 + \|y\|).
\]

Applying this to the content of the lemma we have

\[
1 + \|u + v + w\| \geq 1 + \|u + P(w)\| \geq \frac{1 + \|Pw\|}{1 + \|u\|} \geq \frac{C(1 + \|w\|)}{1 + \|u\|}.
\]

This proves the first inequality. Also note that

\[
(1 + \|u + v + w\|)(1 + \|w\|) \geq (1 + \|u + v\|)(1 + \|v\|).
\]

By the above

\[
(1 + \|u + v + w\|)(1 + \|w\|) \leq C^{-1}(1 + \|u + v + w\|)^2 (1 + \|u\|)
\]

proving the second inequality. \( \square \)
The next lemma is due to Harish-Chandra (c.f. [RRGI] Lemma 4.A.2.3 (1)).

**Lemma 16.** If $P$ is a standard parabolic subgroup then there is a constant $0 < C_1 < 1$ such that if $\bar{n} \in \bar{N}_P$ then

$$1 - \rho_P(\log a(\bar{n})) \geq C_1(1 + \|\log a(\bar{n})\|).$$

**Theorem 17.** If $f \in \mathcal{B}(N_o \setminus G; \chi)$ then the integral defining $f^P(ma)$ converges absolutely and uniformly for $m \in ^oM_P$ and $a \in A_P$ in compacta. Furthermore, the map $f \mapsto f^P$ is continuous from $\mathcal{B}(N_o \setminus G; \chi)$ to $\mathcal{B}(N_o \cap M_P \setminus M_P; \chi|_{N_o \cap M}).$

**Proof.** Since the estimates on $A_G$ have no effect on the the integrals, we may assume that $G$ has compact center and we need to prove the results for $f \in \mathcal{C}(N_o \setminus G; \chi).$ If $p \geq 0, \bar{n} \in \bar{N}_P, m \in ^oM_P, a \in A_P$ then

$$|f(\bar{n}ma)| \leq a(\bar{n}ma)^p(1 + \|\log a(\bar{n}ma)\|)^{-p_q_p f}(f).$$

The Iwasawa decomposition for $m$ relative to $K_1 = K \cap M_P, A_1 = A_o \cap ^oM_P, N_1 = N_o \cap M_P$ is $m = n_1a_1k_1.$ Thus since $A_P$ is in the center of $M_P$ we have

$$\bar{n}ma = \bar{n}a_1a_1k_1$$

so if we set $b = an_1a_1$ then

$$a(\bar{n}ma) = a(an_1a_1b^{-1}\bar{n}b) = aa_1a(b^{-1}\bar{n}b).$$

We will now apply the previous lemmas to get some inequalities. Set $\mathfrak{a} = \text{Lie}(A_o \cap ^oM_P).$ We note that

$$\mathfrak{a}_o = \mathfrak{a}_p \oplus ^*\mathfrak{a}$$

orthogonal direct sum. If we set $u = \log a, v = \log a_1$ and $w = \log a(b^{-1}\bar{n}b)$ and use the lemma of Harish-Chandra quoted above the hypotheses of Lemma [15] are satisfied with $P$ (sorry about the two meanings of $P$) the projection of $\mathfrak{a}_o$ onto $\mathfrak{a}_p$ in particular Harish-Chandra’s lemma implies that there exists $C_1 > 0$ such that

$$1 + \|P \log a(\bar{n})\| \geq C_1(1 + \|\log a(\bar{n})\|)$$

if $\bar{n} \in \bar{N}_P.$ So we have

$$1 + \|\log a(\bar{n}ma)\| \geq \frac{C_1(1 + \|\log a(b^{-1}\bar{n}b)\|)}{1 + \|\log a\|}$$

and

$$\frac{(1 + \|\log a(\bar{n}ma)\|)^2 \geq C_1(1 + \|\log a_1\|)}{1 + \|\log a\|}.$$ 

Hence, writing $p = 2k + d$

$$a(\bar{n}ma)^p(1 + \log \|a(\bar{n}ma)\|)^{-p} =$$
\[ (aa_1)^{\rho} a(b^{-1}\bar{n}b)^{\rho} (1 + \log \|aa_1 a(b^{-1}\bar{n}b)\|)^{-p} \leq \]
\[ C_1^{-(k+d)} (aa_1)^{\rho} a(b^{-1}\bar{n}b)^{\rho} (1 + \|\log a\|)^{d+k} \times \]
\[ (1 + \|\log a_1\|)^{-k} (1 + \|\log a(b^{-1}\bar{n}b)\|)^{-d} = \]
\[ C_1^{-(k+d)} (aa(m))^{\rho} a(b^{-1}\bar{n}b)^{\rho} (1 + \|\log a\|)^{d+k} \times \]
\[ (1 + \|\log a(m)\|)^{-k} (1 + \|\log a(b^{-1}\bar{n}b)\|)^{-d} \]

Obviously, if \( \Omega \) is a compact subset of \( M_P \) then there is a constant \( C_{\Omega,d} \) such that
\[ 0 \leq C_1^{-(k+d)} a^{\rho} (1 + \|\log a\|)^d a(m)^{\rho} (1 + \|\log a(m)\|)^{-d} \leq C_{\Omega,d} \]
if \( ma \in \Omega \). Thus
\[ \int_{N_P} |f(\bar{n}ma)| d\bar{n} \leq \]
\[ C_{\Omega,d} q_{d,1}(f) \int_{N_P} a(b^{-1}\bar{n}b)^{\rho} (1 + \log \|a(b^{-1}\bar{n}b)\|)^{-d} d\bar{n} = \]
\[ C_{\Omega,d} q_{d,1}(f) a^{-2\rho} \int_{N_P} a(\bar{n})^{\rho} (1 + \log \|a(\bar{n})\|)^{-d} d\bar{n}. \]

If \( d \) is sufficiently large this integral converges (c.f. [RRGI] Theorem 4.5.4) to \( C_2 \). Now if we put together everything we have proved so far we have
\[ |f^P (ma)| \leq C_2^{-d} q_p,1(f) a(m)^{\rho} (1 + \log \|a(m)\|)^{-k} (1 + \log \|a\|)^{d+k}. \]
So
\[ q_{d,k,1} (f^P) \leq C_1^{-d} q_{2k,d,1}(f). \]
Replacing \( f \) by \( xf \) for \( x \in U(Lie M_P) \) yields \( q_{d,k,x} (f^P) \leq C_1^{-d} q_{d+2k,x}(f) \).
Thus \( f \mapsto f^P \) is continuous from \( \mathcal{C}(N_0 \setminus G; \chi) \) to \( \mathcal{B}(N_0 \cap M \setminus G; \chi|_{N_0 \cap M}) \).

The next results can be proved using the arguments in the proofs of the two preceding theorems.

**Corollary 18.** Let \( P \) be a standard parabolic subgroup of \( G \) and set \( \overline{N} = N_P = \emptyset N_P. \) If \( f \in \mathcal{C}(G) \) then the integral
\[ \int_{N_0 \times \overline{N}} |f(n_o \bar{n})| dn_0 d\bar{n} \]
is convergent and defines a continuous seminorm on \( \mathcal{C}(G) \).
Corollary 19. Let $P$ be a standard parabolic subgroup of $G$ and set $N_P = \theta N_P$. If $f \in \mathcal{C}(G)$, $m \in ^0 M_P$ and $a \in A_P$ then then the integral
\[
\phi(ma) = \int_{N_P \times N_P} f(n\tilde{n}ma)d\tilde{n}
\]
converges absolutely and uniformly in compacta of $M$ and satisfies the inequalities
\[
|L_xR_y\phi(ma)| \leq q_{x,y,d,k}(f)\Xi_{\theta M_P}(m)(1 + \log \|m\|)^{-k}(1 + \|\log a\|)^d
\]
with $d,k > 0, x,y \in U(\mathfrak{g})$ and $q_{x,y,d,k}$ is a continuous semi-norm on $\mathcal{C}(G)$.

Corollary 20. Assume that $G$ has compact center. Let $P$ be a standard parabolic subgroup of $G$. Let $H \in \text{Lie}(A_P)$ be such that $\rho(H) \leq 0$ and let $m \in ^0 M_P$. If $f \in \mathcal{C}(N_\chi \setminus G; \chi)$ then for each $d > 0$ there exists $C_k$ such that
\[
|f^P(\exp(H)m)| \leq C_d(1 - \rho(H))^{-d}a(m)^\rho.
\]

Proof. We begin the argument as we did in the proof Theorem 17 with the same notation. Let $a = \exp(H)$ and $b = an_1a_1$
\[
a(\tilde{n}ma)^\rho(1 + \|\log a(\tilde{n}ma)\|)^{-d} = (aa_1)^\rho a(b^{-1}\tilde{n}b)^\rho(1 + \|\log (aa_1a(b^{-1}\tilde{n}b))\|)^{-d}.
\]
We note that
\[
\rho_P(\log (aa_1a(b^{-1}\tilde{n}b))) = \rho_P(\log(a)) + \rho_P(\log a(b^{-1}\tilde{n}b))
\]
and there exists $0 < C < 1$ such that
\[
1 + \|\log a(\tilde{n}ma)\| \geq C(1 - \rho_P(\log (aa_1a(b^{-1}\tilde{n}b))).
\]
Thus
\[
1 + \|\log a(\tilde{n}ma)\| \geq C(1 - \rho_P(\log(a)) - \rho_P(\log a(b^{-1}\tilde{n}b)))
\]
and $\rho_P(\log a(b^{-1}\tilde{n}b)) \leq 0$ so
\[
1 + \|\log a(\tilde{n}ma)\| \geq C(1 - \rho_P(\log a))
\]
and
\[
1 + \|\log a(\tilde{n}ma)\| \geq C(1 - \rho_P(\log a(b^{-1}\tilde{n}b)).
\]
This implies that
\[
a(\tilde{n}ma)^\rho(1 + \|\log a(\tilde{n}ma)\|)^{-k-d} \leq C_1^{-k-d}(aa_1)^\rho a(b^{-1}\tilde{n}b)^\rho(1 - \rho_P(\log a))^{-k}(1 - \rho_P(a(b^{-1}\tilde{n}b))^{-d}.
\]
Harish-Chandra has shown that if $d > 0$ is sufficiently large then
\[
\int_N a(\tilde{n})^\rho(1 - \rho_P(\log a(\tilde{n}))^{-d} < \infty
\]
(c.f. Theorem 4.5.4 [RRGI]) We have

\[ |f^P(ma)| \leq a^\rho \int_{N} |f(\tilde{n}ma)| d\tilde{n} \leq C (-k)^{-d} \int_{N} (1 + \rho_P(\log a))^{-k} a(b^{-1}\tilde{n}b)^\rho (1 - \rho_P(a(b^{-1}\tilde{n}b)) - d d\tilde{n}.

If \( d \) is sufficiently large then the integral converges in light of the lemma above and the argument in Theorem 17. The estimate follows from the formula. \( \square \)

Let \( \alpha_1, \ldots, \alpha_l \) be the simple roots of the parabolic \( P_o \) relative to \( A_o \).

**Lemma 21.** We assume that \( G \) has compact center and that \( \chi \) is generic. If \( f \in C(N_o \setminus G; \chi) \), \( h \in a_o \), \( m_i \in \mathbb{Z}_{\geq 0} \) is given for \( 1 \leq i \leq l \) and \( d > 0 \) then there is a continuous seminorm \( q = q(m_i, d) \) on \( C(N_o \setminus G; \chi) \) such that

\[ |f(\exp(h)k)| \leq e^{-\sum_{i=1}^l m_i \alpha_i(h)} (1 + \|h\|)^{-d} e^{\rho_o(h)} q(f). \]

**Proof.** Let \( F = \{ i | m_i > 0 \} \) and let \( x_1, \ldots, x_n \) be a basis of \( \text{Lie}(G) \). If \( X \in \text{Lie}(G) \) and if \( k \in K \) then we can write \( Ad(k)X = \sum a_i(k, X)x_i \). Note that

\[ |a_i(k, X)| \leq C \|X\| \]

for all \( k \in K \). Now let \( X_i \) be an element of the \( \alpha_i \) root space in \( n_o \) such that \( d\chi(X_i) = z_i \neq 0 \). Then

\[ f(\exp(h)k) = z_i^{-1} L_{X_i} f(\exp(h)k) = z_i^{-1} \frac{d}{dt|_{t=0}} f(\exp(tX_i) \exp(h)k) = \]

\[ z_i^{-1} \frac{d}{dt|_{t=0}} f(\exp(h) \exp(tAd(\exp(-h))X_i)k) = \]

\[ z_i^{-1} \frac{d}{dt|_{t=0}} f(\exp(h) \exp(te^{-\alpha_i(h)}X_i)k) = \]

\[ z_i^{-1} \frac{d}{dt|_{t=0}} f(\exp(h) \exp(te^{-\alpha_i(h)}Ad(k^{-1})X_i)k) = \]

\[ e^{-\alpha_i(h)} z_i^{-1} \sum a_j(k^{-1}, X_i)x_j f(\exp(h)k). \]

Iterating this argument yields and expression

\[ f(\exp(h)k) = e^{-\sum_{i \in F} m_i \alpha_i(h)} Z(k) f(ak) \]

with \( Z \) a smooth function from \( K \) to \( L = U_{\sum_{i \in F} m_i (\text{Lie}(G))} \) with \( U^j(\text{Lie}(G)) \) the standard filtration. If we choose a basis of \( L, y_1, \ldots, y_r \) then we have

\[ Z(k) = \sum_{17} b_i(k)y_i. \]
Thus we have
\[ |f(\exp(h)k)| \leq e^{-\sum_{i \in I} m_i \alpha_i(h)} \sum_j C_j |y_j f(\exp(h)k)| \leq e^{-\sum_{i \in I} m_i \alpha_i(h)} (1 + \|h\|)^{-d} e^{\rho_o(h)} \sum_j C_j q_{d,y_j}(f). \]

Corollary 22. Assume that \( \chi \) is generic then if \( f \in C(N_o \backslash G; \chi) \) and \( x \in U(g) \) then
\[ |xf(g)| \leq q_{x,d,m}(f) \Xi(a(g)) a(g)^{-\sum_{i \in F} r_i \alpha_i(1 + \log \|a(g)\|)^{-d}. \]

with \( r_i \in \mathbb{Z}_{\geq 0} \).

Proof. If \( a \in A_o \) then \( a = \exp(h) \) with \( h \in -sa_o^T \) for some \( s \in W(A_o) \). Harish-Chandra’s estimates (2. in the previous section) imply that
\[ a^{s \rho_o} \leq \Xi(a) \leq C a^{s \rho_o}(1 + \log \|a\|)^d. \]

Now the desired inequality follows from \( s \rho_o(h) = \rho_o(h) - \sum_i m_i \alpha_i(h) \) with \( m_i \geq 0 \). The previous lemma completes the proof.

We have seen in assertion 7. section 3 that if \( G \) has compact center then the standard norm, \( \|\ldots\| \), on \( G \) such that there exist \( C_1, C_2, r > 0 \) such that
\[ C_1 \|g\|^{-1} \leq \Xi(g) \leq C_2 \|g\|^{-1} (1 + \log \|g\|)^r. \]

Theorem 23. Assume that \( \chi \) is generic. Let \( f \in C(N_o \backslash G; \chi) \) be right \( K \)-finite and let \( \varphi \in C_c^\infty(N_o) \) then \( \psi(nak) = \varphi(n) f(ak) \) for \( n \in N_o, a \in A_o \) and \( k \in K \) defines an element of \( C(G) \).

To prove this result we will use the following lemma.

Lemma 24. Let \( \psi \in C_c^\infty(G) \) be such that
\[ \psi(nak) = \sum_{a \in A_o} \varphi_i(n) f_{ij}(a) \gamma_i(k), n \in N_o, a \in A_o, k \in K. \]

Assume that \( \varphi_i \in C_c^\infty(N_o), \gamma_i \in C_c^\infty(K) \) and \( f_{ij} \in C_c^\infty(A_o) \) satisfies
\[ |uf_{ij}(a)| \leq C_{u,r,d} a^{-\sum r_i \alpha_i} \|a\|^{-1} (1 + \log \|a\|)^{-d} \]
for all \( u \in U(A_o) \) and \( r_i \in \mathbb{Z}_{\geq 0} \). Then
\[ |\psi(g)| \leq C_d \|g\|^{-1} (1 + \log \|g\|)^{-d} \]
and if \( X \in \text{Lie}(G) \) then both \( R_X \psi \) and \( L_X \psi \) have decompositions as above relative to the Iwaswa decomposition.
Proof. Let $\Omega$ be the union of the supports of the $\varphi_i$. There exist constants $c_3$ and $c_4$ such that if $n \in \Omega$ then
\begin{equation*}
\|n\| \leq c_3, |\varphi_i(n)| \leq c_4, 1 \leq i \leq d.
\end{equation*}
Thus if $a \in A_o$ then
\begin{equation*}
\|na\| \leq \|n\| \|a\| \leq c_3 \|a\|
\end{equation*}
and
\begin{equation*}
\|a\| = \|n^{-1}na\| \leq \|n^{-1}\| \|na\| \leq c_3 \|na\|.
\end{equation*}
Thus, since $\|gk\| = \|g\|$ for $k \in K$, for each $d$ there exists a constant, $B_d$, such that
\begin{equation*}
|\psi(nak)| \leq c_4 B_d \|a\|^{-1} (1 + \log \|a\|)^{-d} \leq c_4 c_3^{-1} B_d \|nak\|^{-1} (1 + \log \|nak\|)^{-d}.
\end{equation*}
We now prove that each of $L_X \psi(g)$ and $R_X \psi(g)$ satisfy the hypotheses of the lemma. We first consider the right derivatives. If $X \in \text{Lie}(G)$ then
\begin{equation*}
R_X \psi(nak) = \frac{d}{dt_{t=0}} \psi(na \exp(t \text{Ad}(k) X) k)
\end{equation*}
we choose a basis of $\text{Lie}(G)$, $X_1, ..., X_n$ with $X_i$ in the $\beta_i$ root space for the action of $a_o$ on $\text{Lie}(N_o)$ if $i \leq r$, $X_i \in a_o$ for $r < i \leq r + l$ and $X_i \in \text{Lie}(K)$ for $i > r + l$. Then
\begin{equation*}
\text{Ad}(k)X = \sum_i c_i(k, X) X_i.
\end{equation*}
Hence
\begin{equation*}
R_X \psi(nak) = \sum_i c_i(k, X) \frac{d}{dt_{t=0}} \psi(na \exp(t X_i) k).
\end{equation*}
It is enough to prove that each of the functions
\begin{equation*}
\frac{d}{dt_{t=0}} \psi(na \exp(t X_p) k)
\end{equation*}
satisfies the hypotheses. If $p \leq r$ then
\begin{equation*}
\frac{d}{dt_{t=0}} \psi(na \exp(t X_p) k) = \frac{d}{dt_{t=0}} \psi(n \exp(t \text{Ad}(a) X_p) k)
\end{equation*}
\begin{equation*}
= a^{\beta_p} \sum \frac{d}{dt_{t=0}} \varphi_i(n \exp(X_p)) f_{ij}(a) \gamma_j(k).
\end{equation*}
Which is of the right form since $\beta_i = \sum s_{ij} \alpha_j$ with $s_{ij} \in \mathbb{Z}_{\geq 0}$. If $r < p \leq r + l$ then
\begin{equation*}
\frac{d}{dt_{t=0}} \psi(na \exp(t X_p) k) = \sum_{i,j} \gamma_j(k) \varphi_i(n) \frac{d}{dt_{t=0}} f_{ij}(a \exp(t X_p)).
\end{equation*}
which is of the right form. Note that the decomposition is obvious if \( p > r + l \).

We now take a basis \( Y_i = X_i, i \leq r + l \) and \( Y_i \) in the \(-\beta_i\) root space with \( \beta_i \) as above \( i > r + l \). We now look at the left derivative. Noting that

\[
\text{Ad}(n)^{-1}X = \sum_i d_i(n, X)Y_i
\]

\[
L_X \psi(nak) = \sum d_p(n, X) \frac{d}{dt}_{t=0} \psi(n \exp(tY_p)ak).
\]

If \( p \leq r + l \) then it is obvious that \( \frac{d}{dt}_{t=0} \psi(n \exp(tY_p)ak) \) has the desired decomposition. Suppose that \( p > r + l \) then

\[
\frac{d}{dt}_{t=0} \psi(n \exp(tY_i)ak) = a^{\beta_i} \frac{d}{dt}_{t=0} \psi(na \exp(tZ_i)k).
\]

We leave the rest to the reader.

\[\square\]

**Proof.** (of the theorem)

\[
\psi(nak) = \varphi(n)f(ak)
\]

since \( f \) is \( K\)-finite \( f(gk) = \sum f_i(g)\gamma_i(k) \) with \( \gamma_i \in C^\infty(K) \) and \( f_i \in C(N_o \backslash G; \chi) \). Corollary 22 implies that \( \psi \) satisfies the hypotheses of the previous lemma. An iteration of the lemma implies the theorem.

\[\square\]

**Theorem 25.** Assume that \( \chi \) is generic. Let \( \mathcal{C}(G)_K \) (resp. \( \mathcal{C}(N_o \backslash G; \chi)_K \)) denote the right \( K \) finite elements of \( \mathcal{C}(G) \) (resp. \( \mathcal{C}(N_o \backslash G; \chi) \)). Let \( T_\chi \) be as in Theorem 13 then the map

\[
T_\chi : \mathcal{C}(G)_K \to \mathcal{C}(N_o \backslash G; \chi)_K
\]

is surjective.

**Proof.** Let \( f \in \mathcal{C}(N_o \backslash G; \chi)_K \) and let \( \varphi \in C^\infty_c(N_o) \) be such that

\[
\int_{N_o} \chi(n)^{-1}\varphi(n)dn = 1.
\]

Then define \( \psi(nak) = \varphi(n)f(ak) \) or \( n \in N_o, a \in A_o \) and \( k \in K \). We have seen that \( \psi \in \mathcal{C}(G)_K \). Also

\[
\psi_\chi(nak) = \chi(n)f(ak) = f(nak).
\]
We will use the following result later in this paper.

\textbf{Lemma 26.} Let \( \omega \) be a compact subset of \( A_0 \) and let \( P = {}^o M_P A_P N_P \) be a standard parabolic subgroup with standard Langlands decomposition. If \( \mu \in a^* \) satisfies \( (\mu, \alpha) \geq 0 \) for all \( \alpha \in \Phi(P, A_P) \) then there exists a constant \( C_{\mu, \omega} > 0 \) such that if \( \bar{n} \in N_\omega K \) with \( \bar{n} \in N_P, m \in {}^o M_P \) and \( k \in K \) then \( a^\mu \geq C_{\mu, \omega} \).

\textbf{Proof.} The set of \( \mu \) in the statement is the convex hull of the elements \( \lambda \in a^* \) obtained as follows: Let \( F \) be a finite dimensional representation of \( G \) such that \( {}^o M_P \) acts trivially on \( F \) and \( a \in A_P \) acts by \( a^\lambda \) on \( F \). It is therefore sufficient to prove the result for such \( \lambda \) with \( F \) the corresponding representation. Put a \( K \)-invariant inner product on \( F \) such that the elements of \( a \) act as Hermitian operators. Let \( \|...\| \) be the corresponding norm on \( F \). Let \( v_o \) be a unit vector in \( F \). There exist constants \( C_1, C_2 \) such that if \( u \in \omega \) then
\[
0 < C_1 \leq \|uv_o\| \leq C_2 < \infty.
\]
Thus, if \( \bar{n} \in \bar{N}, a \in A_P, m \in {}^o M_P, n \in N, u \in \omega, k \in K \) and if \( \bar{nm} = nuk \) then
\[
\|(\bar{n}am)^{-1}v_o\| = \|u^{-1}v_o\| \leq C_2.
\]

Also,
\[
C_2 \geq \|(\bar{n}am)^{-1}v_o\| = \|a^{-1}(am)\bar{n}(am)^{-1})v_o\| \geq \|a^{-1}v_o\| = a^{-\lambda}.
\]

Here, we have used the fact that if \( \bar{n} \in \bar{N} \) then \( \bar{n}v_o = v_o + z \) with \( \langle v_o, Az \rangle = 0 \). \( \Box \)

5. The space of Whittaker cusp forms

We retain the notation of the previous sections.

If \( f \in C(N_0 \backslash G; \chi) \) then we say that \( f \) is a cusp form if \( (R_k f)^P = 0 \) for all proper standard parabolic subgroups \( P \) and all \( k \in K \). We leave the following lemma as an exercise

\textbf{Lemma 27.} If \( f \) is a cusp form then \( (R_g f)^P = 0 \) for all proper standard parabolic subgroups \( P \) and all \( g \in G \).

We set \( {}^o C(N_0 \backslash G; \chi) \) equal to the space of cusp forms in \( C(N_0 \backslash G; \chi) \). Here is the analogue of Theorem 6 in this context. Due to the error in [RRGII] this result was also left without a proof.

\textbf{Theorem 28.} If \( f \in C(N_0 \backslash G; \chi) \) and \( \dim Z(\text{Lie}(G)) f < \infty \) then \( f \) is a cusp form.
Proof. It is enough to prove that $f^p_\gamma = 0$ for all standard parabolic subgroups and all $\gamma \in \hat{K}$. So we will assume $f$ is $K$-finite. We may assume that the center of $G$ is compact. If $P$ is a parabolic subgroup of $G$ we have $\bar{P} = {}^o M_P A_P \bar{N}_P$. We note that

$$V = U(Lie(G)) \text{Span}_\mathbb{C} R_K f$$

is a finitely generated, admissible $(Lie(G), K)$–module. We also note that if $\phi \in Lie(\bar{N}_P)V$ then $\phi^P = 0$. Also $V/Lie(\bar{N}_P)V$ is an admissible, finitely generated $(Lie(^o M_P), ^o M_P \cap K)$–module ([RRGII] Corollary 3.7.2) so $\text{dim Span}_\mathbb{C}(A_P u) < \infty$ if $u \in V/Lie(\bar{N}_P)V$. This implies that $f^P(\exp(tH)m)$ is an exponential polynomial in $t$ for each $m \in {}^o M_P$ and $H \in Lie(A_P)$ i.e.

$$f^P(\exp(tH)m) = \sum_j e^{\mu_j t} p_j(t, m)$$

with $p_j(t, m)$ a polynomial in $t$ and $\mu_j \in \mathbb{C}$ distinct. But

$$f^P \in B(N_o \cap M_p \backslash G; \chi|_{N_o \cap M_p})$$

which implies that there exists $r$ and $C$ such that for all $t$

$$|f^P(\exp(tH)m)| \leq C(1 + |t|)^r.$$ 

Hence $\mu_j = i\nu_j$ with $\nu_j \in \mathbb{R}$. Now Corollary [20] implies that if $\rho(H) > 0$ and $t > 0$ and $d > 0$ than there exists $C_{m,d}$ such that

$$\sum_j e^{-i\nu_j t} p_j(-t, m) \leq C_{m,d}(1 + t)^{-d}.$$

Thus $\sum_j e^{-i\nu_j t} p_j(t, m) = 0$ for all $t$ which implies the theorem. \qed

Theorem 29. Assume that $\chi$ is generic and $G$ has compact center. If $H \subset L^2(N_o \backslash G; \chi)$ is a closed, invariant, irreducible subspace then $H^\infty \subset \mathcal{C}(N_o \backslash G; \chi)$.

Proof. In [RRGII] Theorem 15.3.5 a proof is given that

$$H_K^\infty \subset \mathcal{C}(N_o \backslash G; \chi).$$

The completion of $H_K^\infty$ in $L^2(N_o \backslash G; \chi)^\infty$ is a smooth Fréchet representation of moderate growth as is the completion in $\mathcal{C}(N_o \backslash G; \chi)$. The Casselman-Wallach theorem [RRGII],11.6.7 implies that the two completions are the same. Now the previous theorem implies this theorem. \qed

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In order to carry out the rest of the theory of Whittaker cusp forms we need to recall Harish-Chandra’s decomposition of the Schwartz space of $G$ and our results on the analytic continuation of Jacquet integrals.

6. The Harish-Chandra Plancherel Theorem.

Throughout this section we will assume that $G$ has compact center. The purpose of this section is to describe Harish-Chandra’s decomposition of $\mathcal{C}(G)$ relative to conjugacy classes of associate standard parabolic subgroups of $G$. If $P_1$ and $P_2$ are parabolic subgroups of $G$ then they are associate if there exists $g \in G$ with $gM_{P_1}g^{-1} = M_{P_2}$. A parabolic subgroup, $P$, of $G$ is said to be cuspidal if $^oM_P$ contains a compact Cartan subgroup. Let $\mathcal{P} = \mathcal{P}(G)$ denote the set of conjugacy classes of associate cuspidal parabolic subgroups. Then one can show that up to conjugacy all Cartan subgroups of $G$ can be obtained from elements of $\mathcal{P}$ as $[P] \longmapsto T_PA_P$ where $P$ is a representative of $[P]$ and $T_P$ is a compact Cartan subgroup of $M_P$. If we further, divide by $M_P$ conjugacy then the correspondence is bijective. Thus we can either talk about the set of all conjugacy classes of Cartan subgroups of $G$ or the set of associativity classes of cuspidal, standard parabolic subgroups of $G$.

The distributional form Harish-Chandra Plancherel theorem is (we will explain the notation after the statement).

**Theorem 30.** (c.f. [RRGII] Theorem 13.4.1) Let $\delta$ be the Dirac distribution at the identity element $e$ in $G$. If $f \in \mathcal{C}(G)$ then

$$
\delta(f) = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\mathcal{M}_P)} d(\sigma) \int_{\mathfrak{a}_P^*} \Theta_{P,\sigma,iv}(f) \mu(\sigma, \nu) d\nu
$$

with $d(\sigma)$ the formal degree of $\sigma$.

Now for the explanations. The set $\mathcal{E}_2(\mathcal{M}_P)$ is the set of equivalence classes of irreducible square integrable representations of $^oM_P$, $[\sigma]$ is the equivalence class of $\sigma$. $\Theta_{P,\sigma,iv}$ is the character of the induced representation $I_{P,\sigma,iv}$ initially defined for $f \in C_c^\infty(G)$ as $\text{tr}(\pi_{P,\sigma,iv}(f))$ and shown by Harish-Chandra to extend to a continuous functional on $\mathcal{C}(G)$. $\mu(\sigma, \nu)$ is for each $\sigma$ a non-negative analytic function on $\mathfrak{a}_P^*$ that is of polynomial growth in $\nu$. We should note that the function $\mu$ depends on $[P]$ and the normalizations of the Haar measures of all of the groups involved. Also, the parameter $[P]$ is the conjugacy class of $A_P$ in Harish-Chandra [H3] and [RRGII].
This monumental achievement is one of the most important theorems of the twentieth century. Harish-Chandra’s steps leading to its proof led to the deepest results on intertwining operators, square integrable representations, regular singular differential equations, classification problems,... We will need the following implication.

**Theorem 31.** (c.f. [RRGII], Theorem 13.4.6 (1)) Let \( P \in [P] \in P \).

Let for \( \sigma \in \mathcal{E}_2(oM_P), v, w \in (I_\sigma)_K \) and \( \alpha \) in the (usual) Schwartz space of \( a^*_P, S(a^*_P) \),

\[
\varphi_{P,\sigma,\alpha,v,w}(g) = \int_{a^*_P} \langle \pi_{P,\sigma,\alpha}(g)v, w \rangle \alpha(\nu)\mu(\sigma,\nu)d\nu.
\]

Then \( \varphi_{\sigma,v,w} \in \mathcal{C}(G) \). Let \( \mathcal{C}_P(G) \) denote the closure in \( \mathcal{C}(G) \) of the span of

\[
\{ \varphi_{P,\sigma,\alpha,v,w} | \sigma \in \mathcal{E}_2(oM_P), v, w \in (I_\sigma)_K, \alpha \in S(a^*_P) \}.
\]

Then \( \mathcal{C}(G) \) is the orthogonal (relative to the \( L^2 \)-inner product) direct sum

\[
\bigoplus_{[P] \in P} \mathcal{C}_P(G).
\]

**7. A decomposition of \( \mathcal{C}(N_o \setminus G; \chi) \)**

**Lemma 32.** Let \( \varphi, \psi \in \mathcal{C}(G) \) then

\[
\int_{N_o \setminus G} \int_{N_o} \int_{N_o} |\varphi(n_1g)\psi(n_2g)|dgdn_1dn_2 < \infty.
\]

**Proof.** The proof of Proposition 14 used only the estimates on the absolute value of an element of \( \mathcal{C}(G) \) and is true for the trivial character of \( N_o \). Set

\[
\varphi_1(g) = \int_{N_o} |\varphi(ng)|dn.
\]

Then the proof of Proposition 14 shows that

\[
\varphi_1(g) \leq C_d a(g)^\rho(1 + \|\log a(g)\|)^{-d}
\]

for all \( d > 0 \). Also (using the proof of Lemma 13)

\[
\int_G \varphi_1(g)|\psi(g)|dg = \int_{N_o \setminus G} \int_{N_o} \varphi_1(g)|\psi(ng)|dndg =
\]

\[
\int_{N_o \setminus G} \varphi_1(g)\psi_1(g)dg < \infty
\]

with

\[
\psi_1(g) = \int_{N_o} |\psi(ng)|dg.
\]

□
Proposition 33. Let \( \chi \) be a character of \( N_o \). If \( \varphi \in C_{[P],[\omega]}(G) \) and \( \psi \in C_{[Q],[\eta]}(G) \) and \( ([P],[\sigma]) \neq ([Q],[\eta]) \) then
\[
\langle T_\chi \varphi, T_\chi \psi \rangle = 0.
\]

Proof. We are looking at the integral
\[
\int_{N_o \setminus G} \int_{N_o} \chi(n_1)^{-1} \varphi(n_1g)dn_1 \int_{N_o} \chi(n_2) \overline{\psi(n_2g)}dn_2dg.
\]
This integral converges absolutely by the previous lemma. We also note that it can be written
\[
\int_G \varphi(g) \int_{N_o} \chi(n) \overline{\psi(ng)}dg
\]
which also converges absolutely. Hence it can be rewritten as
\[
\int_{N_o} \chi(n) \int_G \varphi(g) \overline{\psi(ng)}dgdn = \int_{N_o} \chi(n)^{-1} \int_G \varphi(g) \overline{L(n)\psi(g)}dgdn.
\]
But \( L(n)\psi \in C_{[Q],[\eta]}(G) \), and \( \langle C_{[P],[\omega]}(G), C_{[Q],[\eta]}(G) \rangle = 0. \)

Theorem 34. Assume that \( \chi \) is generic. The space \( C(N_o \setminus G; \chi) \) is the completion of the orthogonal direct sum
\[
\bigoplus_{[P] \in P} \bigoplus_{[\omega]} T_\chi C_{[P],[\omega]}(G).
\]

Proof. The Proposition combined with Theorem 25 and the continuity of \( T_\chi \) imply the theorem.

Lemma 35. If \( \varphi \in C(G) \) and if \( P \) is a parabolic subgroup of \( G \) containing \( P_o \) then
\[
(T_\chi \varphi)^P(ma) = \int_{N_o} \chi(n_o) (L(n_o)\varphi)^P (ma)dn_o.
\]

Proof. Let \( \varphi \in C(G) \). Then we calculate \( (\varphi_\chi)^P(ma) \) for \( m \in {}^o M_P, a \in A_P \).
\[
(\varphi_\chi)^P(ma) = a^{\rho_P} \int_{N_P} \int_{N_o} \chi(n_o)^{-1} \varphi(n_o\tilde{n}ma)dn_od\tilde{n}.
\]
The integral converges absolutely so we may interchange the order of integration. This implies that
\[
(\varphi_\chi)^P(ma) = a^{\rho_P} \int_{N_o} \chi(n_o)^{-1} \int_{N_P} \varphi(n_o\tilde{n}ma)dn_od\tilde{n} = a^{\rho_P} \int_{N_o} \chi(n_o) \int_{\tilde{N}_P} \varphi(n_o^{-1}\tilde{n}ma)dn_od\tilde{n} = \int_{N_o} \chi(n_o) (L(n_o)\varphi)^P (ma)dn_o.
\]
8. THE HOLOMORPHIC CONTINUATION OF JACQUET INTEGRALS

The main purpose of this section is to describe our work on the analytic continuation of Jacquet integrals. The proofs of the main results are complicated. We will refer to the appropriate places in [RRGII] Section 15.4. In addition the section includes some new estimates on Jacquet integrals outside the range of convergence that are needed in the Whittaker Plancherel theorem.

We will assume that \( \chi \) is generic throughout this section. If \((\pi, H)\) is a Hilbert representation of \( G \) then the space of Whittaker vectors on \( H^\infty \) is

\[
Wh_\chi(H^\infty) = \{ \lambda \in (H^\infty)' | \lambda \circ \pi(n) = \chi(n)\lambda, n \in N_o \}.
\]

Since a finitely generated \((\text{Lie}(G), K)\) module is finitely generated as a \( U(\text{Lie}(N_o)) \) module (c.f. [RRG] 3.7.2 p.96) we have

Lemma 36. (RRGII, Lemma 15.4.3) If \((\pi, H)\) is admissible and finitely generated then \( \dim Wh_\chi(H^\infty) < \infty \).

Theorem 37. Let \((\pi, H)\) be an irreducible square integrable representation of \( G \). If \( w \in H_K \) then

\[
\lambda_w(v) = \int_{N_o} \chi(n)^{-1} \langle \pi(n)v, w \rangle \, dn
\]

is absolutely convergent for \( v \in H^\infty \) and defines an element of \( Wh_\chi(H^\infty) \). Furthermore, \( Wh_\chi(H^\infty) = \{ \lambda_w | w \in H_K \} \).

Proof. Let \( w \in H_K \). If \( v \in H^\infty \) then the function \( T_w(v)(g) = \langle \pi(g)v, w \rangle \) is in \( \mathcal{C}(G) \). Furthermore, \( T_w : H^\infty \rightarrow \mathcal{C}(G) \) is continuous. Thus the first part of our assertion follows from Proposition [13].

We now prove the second assertion. Let, for \( \gamma \in \hat{K} \), \( E_\gamma \) denote the orthogonal projection of \( H \) onto its \( \gamma \)-isotypic component, \( H(\gamma) \). Then

\[
\lambda_w \circ E_\gamma(v) = \langle v, z_\gamma \rangle
\]

with \( z_\gamma \in H(\gamma) \). Let \( \| \cdot \| \) be a standard norm.

First note that

1. If \( \lambda \in Wh_\chi(H^\infty) \) then the function \( g \rightarrow \lambda(\pi(g)v) \) is in \( \mathcal{C}(N_o \setminus G; \chi) \).

To prove this assertion we first note that Lemma 15.2.3 in [RRGII] (the Lemma should have the hypothesis that \( \chi \) is generic) it is proved that \( \lambda \) is tame for every minimal parabolic subgroup \( Q \). Also Theorem 5.5.4 combined with Proposition 5.1.2 in [RRG] implies that there exists \( \varepsilon > 0 \) and \( C \) such if \( a \in A \) and \( \alpha(a) \geq 1 \) for all \( \alpha \in \Phi(Q, A_Q) \) then if \( v, w \in H_K \)

\[
|\langle \pi(a)v, w \rangle| \leq C_{v,w}a^{-(1+\varepsilon)\rho_Q}.
\]
Theorem 15.2.5 in [RRGII] now implies that there exists $\varepsilon > 0$ and $p$, a
continuous seminorm on $H^\infty$ such that if $a$ is as above then

$$\lambda(\pi(a)v) \leq p(v)a\rho_Q^{-\varepsilon\rho_Q}, v \in H^\infty.$$ 

This inequality implies 1.

Consider

$$I_w(\lambda)(v) = \int_{N_0 \backslash G} \lambda(\pi(g)v)\lambda_w(\pi(g)v) dg =$$

$$\int_G \lambda(\pi(g)v) \langle \pi(g)v, w \rangle \, dg.$$ 

Which converges by 1. Note that

$$\varphi \mapsto \int_G \lambda(\pi(g)v)\varphi(g) dg$$

defines a continuous functional on $C(G)$. This implies that if we set

$$\varphi_\gamma(g) = d(\gamma) \int_K \chi_\gamma(k)^{-1}\varphi(kg) dk$$

(here $d(\gamma)$ is the dimension, and $\chi_\gamma$ is the character of $\gamma$) then the series

$$\sum_{\gamma \in \hat{K}} \varphi_\gamma$$

converges to $\varphi$ in $C(G)$. If $\varphi(g) = \langle \pi(g)v, w \rangle$ then the Schur orthogonality relations (for $K$) imply that

$$\int_G \lambda(\pi(g)v) \langle \pi(g)v, w \rangle \, dg = \sum_{\gamma \in \hat{K}} \int_G \lambda(\pi(g)v) \langle E_\gamma \pi(g)v, w \rangle \, dg =$$

$$\sum_{\gamma \in \hat{K}} \int_G \lambda(E_\gamma \pi(g)v) \langle \pi(g)v, E_\gamma w \rangle \, dg =$$

$$\sum_{\gamma \in \hat{K}} \int_G \lambda(E_\gamma \pi(g)v) \langle E_\gamma \pi(g)v, w \rangle \, dg =$$

$$\frac{1}{d(\pi)} \sum_{\gamma \in \hat{K}} \|v\|^2 \langle z_\gamma, w \rangle = \frac{1}{d(\pi)} \|v\|^2 \overline{\lambda(w)}.$$ 

(\ast)

Here $d(\pi)$ is the formal degree of $\pi$ relative to our choice of Haar measure on $G$. The formula (\ast) implies that if $I_w(\lambda)(v) = 0$ for all $w$ for some $v \neq 0$ then $\lambda = 0$. Since $\dim Wh_\chi(H^\infty) < \infty$ the second assertion follows. \[\square\]
**Corollary 38.** (To the proof) If $\omega \in \mathcal{E}_2(G)$ then there exists an inner product $(\ldots, \ldots)_\omega$ on $W_h(H^\omega)$ such that if $\lambda, \mu \in W_h(H^\omega)$ and $v, w \in H^\omega$ then

$$\int_{N_0 \backslash G} \lambda(\pi_\omega(g)v)\overline{\mu(\pi_\omega(g)w)} \, dg = (\lambda, \mu)_\omega \langle v, w \rangle$$

In particular, if $\|v\| = 1$ then

$$(\lambda, \mu)_\omega = \frac{1}{c(\lambda, \mu)} \langle v, w \rangle$$

independent of $v$. Furthermore if $w \in H^\omega$ then

$$(\lambda, \lambda)_\omega = \frac{1}{d(\omega)} \chi(w).$$

**Proof.** First note that if $\lambda, \mu \in W_h(H^\omega)$ then

$$\int_{N_0 \backslash G} \lambda(\pi_\omega(g)v)\overline{\mu(\pi_\omega(g)w)} \, dg$$

defines an $G$–invariant Hermitian form on $H^\omega$. Thus

$$\int_{N_0 \backslash G} \lambda(\pi_\omega(g)v)\overline{\mu(\pi_\omega(g)w)} \, dg = c(\lambda, \mu) \langle v, w \rangle$$

with $c(\lambda, \mu)$ Hermitian. If $v \neq 0$ then Span$_C(\pi_\omega(G)v)$ is dense in $H^\omega$ so if $\lambda \neq 0$ then $\lambda(\pi_\omega(g)v)$ is not identically 0. Hence, if $\lambda \neq 0$, $c(\lambda, \lambda) > 0$. Hence, $(\ldots, \ldots)$ defines an inner product on $W_h(H^\omega)$ which we denote by $(\lambda, \mu)_\omega$. Let $v \in H^\omega$ then

$$\int_{N_0 \backslash G} \lambda(\pi_\omega(g)v)\overline{\lambda(w)(\pi_\omega(g)v)} \, dg = \int_{N_0 \backslash G} \lambda(\pi_\omega(g)v)\chi(n) \langle w, \pi_\omega(n(g)v) \rangle \, dndg =$$

$$\int_{N_0 \backslash G} \int_{N_0} \lambda(\pi_\omega(n(g)v)) \langle w, \pi_\omega(n(g)v) \rangle \, dndg = \int_{G} \lambda(\pi_\omega(g)v)\overline{\langle \pi_\omega(g)v, w \rangle} \, dg =$$

$$\frac{1}{d(\omega)} \langle v, w \rangle \lambda(w)$$

by formula (*) in the proof of the preceding theorem. \qed

Let $P$ be a standard parabolic subgroup of $G$ with Langlands decomposition $P = {}^oM_P A_P N_P$. If $(\sigma, H_\sigma)$ is a Hilbert representation of $^oM_P$ that is finitely generated and admissible, $\lambda \in W_h(x^oM_P \cap N_0)(H^\omega)$ and $f \in I^\omega$ and $\nu \in (a^*_P)_C$ ($a_P = \text{Lie}(A_P)$) then we consider the integral

$$J(P, \sigma, \nu)(\lambda)(f) = \int_{N_0} \chi(n)^{-1} \lambda(\pi_\nu(n)) \, dn.$$
We set for \( r \in \mathbb{R} \)
\[
(a^*_P)^r = \{ \nu \in (a^*_P)^{\mathbb{C}} \mid \text{Re}(\nu, \alpha) < r, \alpha \in \Phi(P, A_P) \}.
\]
We have ([RRGII] Lemma 15.6.5, p.398)

**Lemma 39.** Let \( \sigma, \lambda \) be as above then there exists \( c = c_\sigma \) such that if \( \nu \in (a^*_P)^c \) the the integral \( J(P, \sigma, \nu)(\lambda)(f) \) converges absolutely for all \( f \in I^\infty_\sigma \). Furthermore, the map \( \nu \mapsto J(P, \sigma, \nu)(\lambda)(f) \) is holomorphic on \( (a^*_P)^c \) for all \( \lambda \in Wh_{\chi[a^*_{M_P \cap N_o}]}(H^\infty_{\sigma}) \) and all \( f \in I^\infty_\sigma \).

The next two results are the main results on the holomorphic continuation. The first ([RRGII] Theorem 15.6.5, p. 399) is

**Theorem 40.** Let \( (\sigma, H_\sigma) \) be an irreducible, admissible Hilbert representation of \( a^* M_P \). Let \( \lambda \in Wh_{\chi[a^*_{M_P \cap N_o}]}(H^\infty_{\sigma}) \). If \( f \in I^\infty_\sigma \) then the map
\[
\nu \mapsto J(P, \sigma, \nu)(\lambda)(f)
\]
has a holomorphic continuation to \( (a^*_P)^c \). Furthermore, for all \( \nu \in (a^*_P)^c \), \( J(P, \sigma, \nu) \) defines a linear bijection between \( Wh_{\chi[a^*_{M_P \cap N_o}]}(H^\infty_{\sigma}) \) and \( Wh_\chi(I^\infty_{P, \sigma, \nu}) \).

And the second ([RRGII], Theorem 15.4.1 p. 381) whose proof and that of the previous theorem comprise more than 20 complicated pages (pp. 382-405).

**Theorem 41.** Assume that \( (\sigma, H_\sigma) \) is a square integrable representation of \( a^* M_P \) then the constant \( c_\sigma \) can be taken to be 0 and we have
\[
\nu, f \mapsto J(P, \sigma, \nu)(\lambda)(f)
\]
defines a continuous map on \( (a^*_P)^{\mathbb{C}} \times I^\infty_\sigma \) that is holomorphic on \( \nu \) and linear in \( f \). Furthermore, This map extends to a weakly holomorphic map of \( (a^*_P)^c \) to \( I^\infty_\sigma \). Finally, for each \( \nu \in (a^*_P)^c \), \( J(P, \sigma, \nu) \) defines a linear bijection between \( Wh_{\chi[a^*_{M_P \cap N_o}]}(H^\infty_{\sigma}) \) and \( Wh_\chi(I^\infty_{P, \sigma, \nu}) \).

Our next task is to derive a tempered estimate, in \( \nu \), on \( J(P, \sigma, i \nu) \). Assume that \( G \) has compact center. Let \( P = N_P A_P a^* M_P \) be a \( P \) standard parabolic subgroup of \( G \) with standard Langlands decomposition. As usual, \( \bar{P} = \bar{N}_P A_P a^* M_P \). If \( g \in G \) then \( g = \bar{n} a m k \) with \( \bar{n} \in \bar{N}_P, a \in A_P, m \in a^* M_P, k \in K \) and we use the notation \( n_P(g) = \bar{n}, a_P(g) = a, m_P(g) = m \) and \( k_P(g) = k \). Then as is usually noted \( n_P \) and \( a_P \) define smooth functions but only \( m_P(g)k_P(g) \) defines a function.

Fix \( (\sigma, H_\sigma) \) and irreducible square integrable representation of \( a^* M_P \). Let \( \pi_\nu \) denote \( \pi_{\bar{P}, \sigma, \nu} \) and \( u_\nu \) denote \( \bar{P} u_\nu \) for \( u \in I^\infty_\sigma \). We now estimate
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\( J(P, \sigma, \nu)(v) \) for \( v \in I_\sigma^\infty, \lambda \in Wh_{\chi|_{N_\sigma \cap M_P}}(H_\sigma^\infty) \) and \( \text{Re}(\nu, \alpha) < 0 \) for all \( \alpha \in \Phi(P, A) \). We have

\[
J(P, \sigma, \nu)(\lambda)(v) = \int_{N_P} \chi(n)^{-1} \lambda(v(n))dn = \int_{N_P} \chi(n)^{-1} a_P(n)^{\nu-\rho P} \lambda(m_P(n))v(k_P(n)))dn.
\]

Thus

\[
|J(P, \sigma, \nu)(\lambda)(v)| \leq \int_{N_P} a_P(n)^{\text{Re}\nu-\rho P} |\lambda(m_P(n))v(k_P(n)))|dn \leq \int_{N_P} a_P(n)^{\text{Re}\nu-\rho P} \Xi(a(m_P(n)))dn \parallel \lambda \parallel q(v)
\]

by Corollary 22, here \( q \) is a continuous seminorm on \( I_\sigma^\infty \). We have

Lemma 42.

\[
\|J(P, \sigma, \nu)(\lambda)(v)\| \leq \int_{N_P} a_P(n)^{\text{Re}\nu-\rho P} |\lambda(m_P(n))v(k_P(n)))|dn \leq \|\lambda\| q(v) \int_{N_P} a(n)^{\text{Re}\nu-\rho P} dn.
\]

with \( q \) a continuous seminorm on \( I_\sigma^\infty \).

Let \( F \) be the irreducible finite dimensional representation of \( G \) such that \( ^G M_P \) acts trivially on \( F^{a_P} \) and \( A_P \) acts by \( a^4 \rho \). (See the example in 10.2.1 in [RRGII]). We note that \( F = F^{a_P} \oplus \bar{n}_P F \) as an \( M_P \)-module. Thus

\[
F/\bar{n}_P F \cong F^{a_P}
\]

as an \( M_P \)-module. We have a surjective homomorphism

\[
I_{P,\sigma,\nu-4\rho} \otimes F \to I_{P,\sigma,\nu}^\infty
\]

this induces an injective map

\[
Wh_{\chi}(I_{P,\sigma,\nu}^\infty) \to Wh_{\chi}(I_{P,\sigma,\nu-4\rho}^\infty \otimes F).
\]

We also note that we have a map

\[
\Gamma : Wh_{\chi}(I_{P,\sigma,\nu-4\rho}^\infty) \otimes F^* \to Wh_{\chi}(I_{P,\sigma,\nu-4\rho}^\infty \otimes F)
\]

(see Theorem 15.5.7 p.393 [RRGII]). This map is defined on the space \( \hat{Wh}_{\chi}(I_{P,\sigma,\nu-4\rho}^\infty) \otimes F^* \) with

\[
\hat{Wh}_{\chi}(I_{P,\sigma,\nu-4\rho}^\infty) = \{ T \in (I_{P,\sigma,\nu-4\rho}^\infty)^* | (X - d\chi(X))^kT = 0, X \in n_\sigma \}
\]

which is a \( \mathfrak{g} = \text{Lie}(G) \)-module and is given as follows. Let \( H \in a_P \) be such that \( \alpha(H) = 1 \) for \( \alpha \) the non-zero restriction of a simple root to
Then the eigenvalues of $H$ are of the form $4\rho_P(H) - k$ with $k \geq 0$ and $k \in \mathbb{Z}$. We set $F_k^e$ equal to the $4\rho_P(H) - k$ eigenspace of $H$ acting on $F^e$ and let $p_k$ be the projection of $F^e$ onto $F_k^e$. Then

$$
\Gamma = \sum L_k(I \otimes p_k)
$$

with $L_k \in U(g)$ depending only on $\chi$ and $k$ (see the definition of $\Gamma_k$ in 15.5.7 in [RRGI]). Thus, if $\lambda_1, ..., \lambda_d$ is a basis of $\text{Wh}_\chi|\text{No} \cap \text{MP}((H^)\infty\sigma)$ and if $f_1, ..., f_r$ is a basis of $F^e$ compatible with the grade (i.e. $f_j \in F^e_{k(j)}$) then the elements

$$
L_{k(j)}(j_{P,\sigma,i\nu-4\rho_P}(\lambda_i) \otimes f_j), 1 \leq i \leq d, 1 \leq j \leq r
$$

form a basis of $\text{Wh}_\chi(I_{P,\sigma,i\nu-4\rho_P}^\infty \otimes F)$. This basis can be written

$$
\sum_{p,q} a_{ij,pq}(\nu)J(P,\sigma,\nu-4\rho_P)(\lambda_p) \circ d_{ij} \otimes f_q
$$

with $a_{ij,pq}$ a polynomial in $\nu$ and $d_{ij}$ is a differential operator on $I_\sigma^\infty$ corresponding to the action of . Now applying the map

$$
T : I_{P,\sigma,i\nu-4\rho_P}^\infty \otimes F \rightarrow I_{P,\sigma,\nu}^\infty
$$

which is a composition of two maps $T_1$ and $T_2$ with

$$
T_2(\varphi \otimes f)(g) = \varphi(g) \otimes g\nu
$$

which maps

$$
I_{P,\sigma,i\nu-4\rho_P}^\infty \otimes F \rightarrow I_{P,\sigma,i\nu-4\rho_P}^\infty \otimes F_{1,\rho}
$$

(here $\sigma_\nu$ is the $\bar{P}$ representation on $H_\sigma^\infty$ with $\bar{N}_P$ acting trivially, $A_P$ acting by $a \mapsto a^{-\rho_P+\nu-4\rho_P}$) and

$$
T_1(h)(g) = (I \otimes Q)(h(g))
$$

with $Q$ the natural surjection $F \rightarrow F/\bar{n}_P F$. Note that neither of these maps depend on $\nu$.

With this in mind we have our tempered estimate:

**Theorem 43.** There exists $m > 0$ and a continuous seminorm, $\gamma_1$ on $I_\sigma^\infty$ such that if $u \in I_\sigma^\infty$ then

$$
|J(P,\sigma,i\nu)(\lambda)(u)| \leq \gamma_1(u)(1 + \|\nu\|)^m \|\lambda\|
$$

for $\nu \in \mathfrak{a}^*$.  

**Proof.** We note that we can chose a $K$–Fréchet, summand $Z$ of the space $I_\sigma^\infty \otimes F$ such then $T$ is a $K$–isomorphism of $Z$ to $I_\sigma^\infty$. Let $u_1, ..., u_r$ be the dual basis to the basis $f_1, ..., f_r$ of $F$. If $z \in Z$ then

$$
z = \sum z_i \otimes u_i.$$
Thus
\[ J(P, \sigma, i\nu)(\lambda)T(z) = \sum_{ij} b_{ij}(\nu, \lambda)J(P, \sigma, i\nu - 4\rho)\circ d_{ij}z_j \]

Applying the above Lemma we have
\[ |J(P, \sigma, i\nu)(\lambda)(T(z))| \leq \|\lambda\| \sum_{ij} \gamma(d_{ij}z_j) (1 + \|\nu\|)^m \|\lambda_i\| \int_{N_P} a(n)^{-5\rho} dn. \]

This implies that there exists a continuous seminorm, \( \gamma_1 \), on \( I^\infty_{\sigma} \) such that
\[ |J(P, \sigma, i\nu)(\lambda)(T(z))| \leq \|\lambda\| \gamma_1(Tz) (1 + \|\nu\|)^m. \]

\[ □ \]

**Corollary 44.** Let \( \alpha \in S(\mathfrak{a}^*) \) then if \( \lambda \in Wh_{\chi,|M_P \cap N_o}(H^\infty_{\sigma}), v, w \in I^\infty_{\sigma} \) then
\[ \int_{\mathfrak{a}^*} J(P, \sigma, i\nu)(\lambda)(\pi_P,\sigma,i\nu(g)v)\alpha(\nu)d\nu \]
converges absolutely and defines an element of \( C^\infty(N_o\backslash G; \chi) \).

We will see that if \( \alpha \in S(\mathfrak{a}^*)\mu(\sigma, i\cdot) \) then the integral in the Corollary defines an element of \( C(\mathfrak{n}_o\backslash G; \chi) \).

**Theorem 45.** Let \( \omega \subset \mathfrak{a}_P^* \) be a compact set then there exists \( r > 0 \) and a continuous seminorm, \( \gamma_1 \) on \( I^\infty_{\sigma} \), and \( d \) such that if \( u \in I^\infty_{\sigma} \) then
\[ |J(P, \sigma, i\nu)(\lambda)(\pi_P,\sigma,i\nu(ak)u)| \leq \gamma_1(u) \|\lambda\| a^\rho(1 + \|\log a\|)^d (1 + \|\nu\|)^r \]
for \( \nu \in \omega \) and \( a \in A_o, k \in K \).

**Proof.** One proves this by establishing the dependence on parameters of the asymptotic parameters using the method of 12.4.8,12.4.9 and 12.4.10 on the argument in 15.2.4 used to prove Theorem 15.2.5 all in [RRGII]. Here the key aspect of the argument is to prove the estimates in the end of 12.4.9 and 12.4.10 using the same methodology involving systems ordinary differential equations depending on parameters as in 12.4.8. \[ □ \]

9. **Whittaker cusp forms and the Harish-Chandra discrete series**

**Theorem 46.** Let \( P \) be a standard parabolic subgroup of \( G \). Let \( (\sigma, H_{\sigma}) \in [\sigma] \in \mathcal{E}_2(M_P) \). For \( \lambda \in Wh_{\chi,|M_P \cap N_o}(H^\infty_{\sigma}) \), and \( \alpha \in C^\infty_c(\mathfrak{a}_P^*) \) set
\[ \Psi(P, \alpha, \sigma, \lambda, \nu)(g) = \int_{\mathfrak{a}_P^*} J(P, \sigma, i\nu)(\lambda)(\pi_P,\sigma,i\nu(g)v)\alpha(\nu)d\nu. \]
If \( f \in C(N_0 \setminus G, \chi) \) then
\[
\int_{N_0 \setminus G} \Psi(P, \alpha, \sigma, \lambda, v)(g) f(g) dg = \int_{N_0 \cap M_P \setminus \sigma M_P \times K \times A} a^{-i\nu} \alpha(\nu) \lambda(\sigma(m)v(k))(R(k)f)^P(ma) d\nu dmdk da.
\]

Proof. Since both sides of this equation are continuous in \( f \) it is enough to prove the equation for \(|f| \in C_c(N_0 \setminus G)\). Set \( \rho = \rho_P \). If \( z \in \mathbb{C} \)
\[
\varphi(z, g) = \int_{\sigma_P} J(P, \sigma, i\nu - z\rho)(\lambda)(\pi_{P, \sigma, i
u-z\rho}(g)v)\alpha(\nu) d\nu
\]
which is entire in \( z \).

Also
\[
\int_{N_0 \setminus G} \varphi(z, g) f(g) dg
\]
is entire in \( \bar{z} \). Thus
\[
\int_{N_0 \setminus G} \Psi(P, \alpha, \sigma, \lambda, v)(g) f(g) dg = \lim_{\varepsilon \to +0} \int_{N_0 \setminus G} \varphi(\varepsilon, g) f(g) dg.
\]

If \( \varepsilon > 0 \) then
\[
\int_{N_0 \setminus G} J(P, \sigma, i\nu - \varepsilon\rho)(\lambda)(\pi_{P, \sigma, i\nu-\varepsilon\rho}(g)v) f(g) dg = \int_{N_0 \setminus G} \lambda \left( \int_{N_P} \chi(n)^{-1} v_{i\nu-\varepsilon\rho}(ng) dn \right) f(g) dg.
\]

We note \( N_0 = N_P N_0 \cap M_P \) and the expression \( n_0 = n^* \) with \( n_P \in N_P \)
and \( n^* \in N_0 \cap M_P \) is unique. Now,
\[
\int_{N_0 \setminus G} \lambda \left( \int_{N_P} \chi(n)^{-1} v_{i\nu-\varepsilon\rho}(ng) dn \right) f(g) dg = \int_{N_0 \cap M_P \setminus G} \lambda \left( \int_{N_P} v_{i\nu-\varepsilon\rho}(ng) dn f(ng) dn \right) dg = \int_{N_0 \cap M_P \setminus G} \lambda \left( \int_{N_P} v_{i\nu-\varepsilon\rho}(g) f(g) dg \right) dg.
\]

We now include the integral over \( \alpha \) and have
\[
\int_{N_0 \times N_0 \cap M_P \setminus \sigma M_P \times A \times K \times \alpha_P^*} \frac{\alpha(\nu)a^{-2p_P} \lambda(v_{i\nu-\varepsilon\rho}(\bar{\nu} mak))}{(\bar{R}(k)f) (\bar{\nu} ma) d\nu dmdk da} = \int_{N_0 \times N_0 \cap M_P \setminus \sigma M_P \times A \times K \times \alpha_P^*} \frac{\alpha(\nu)a^{-2p_P} a^{\rho P - i\nu - \varepsilon\rho} \lambda(\sigma(m)v(k))}{(R(k)f)^P (ma) d\nu dmdk da}.
\]
Lemma 26 implies that there exists $C > 0$ such that if $a \in A_P$ and if $m \in \mathbb{M}_P$ and if $(R(k)f)^P(ma) \neq 0$ then $a^p \geq C > 0$. Thus $a^{-\varepsilon_p} \leq C^{-\varepsilon}$ so we can apply dominated convergence the limit as $\varepsilon \to +0$ to deduce the theorem. □

**Lemma 47.** Let $\nu_o \in \mathfrak{a}_P^*$ and $\alpha_j \in C_c^\infty(\mathfrak{a}_P^*)$ for $j = 1, 2, \ldots$ be non-negative valued functions satisfying

a) $\text{supp } \alpha_{j+1} \subset \text{supp } \alpha_j$ and $\bigcap_{j \geq 1} \text{supp } \alpha_j = \{\nu_o\}$.

b) $\int_{\mathfrak{a}_P^*} \alpha_j(v) dv = 1$.

In the notation of the previous theorem set

$$T_j(f) = \int_{N_o \setminus G} \overline{\Psi(\alpha_j, \sigma, v, \lambda)(g)} f(g) dg$$

for $f \in \mathcal{C}(N_o \setminus G, \chi)$. Then

$$\lim_{j \to \infty} T_j(f) = \int_{N_o \setminus G} \overline{J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)} f(g) dg.$$

**Proof.** Let $\omega$ be the support of $\alpha_1$ then Theorem 45 implies that there exists a constants $C, d$ depending on $\omega$ and $\nu$ such that such that

$$|J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(ak)v)| \leq Ca_P^{\rho}(1 + \log \|a\|)^d.$$  

We write

$$T_j(f) - \int_{N_o \setminus G} \overline{J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)} f(g) dg = \int_{N_o \setminus G} \int_{\mathfrak{a}_P^*} (J(P, \sigma, iv)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv}(g)v) - J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)) \alpha_j(v) dv f(g) dg$$

To prove the limit formula we note that if $\varepsilon > 0$ there exists a compact subset $U \subset N_o \setminus G$ such that

$$\left| \int_{U} \int_{\mathfrak{a}_P^*} (J(P, \sigma, iv)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv}(g)v) - J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)) \alpha_j(v) dv f(g) dg \right| < \frac{\varepsilon}{2}$$

for all $j$. Indeed for each $r > 0$ there exists $B$ such that if $\nu \in \text{supp } a_1$ then

$$|J(P, \sigma, iv)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv}(g)v) - J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)| \leq Ba_P^{\rho}(1 + \log \|a_P(g)\|)^d$$

Thus for each $r$ there exists $B_r$ such that

$$|J(P, \sigma, iv)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv}(g)v) - J(P, \sigma, iv_o)(\lambda)(\pi_{\mathfrak{p}, \sigma, iv_o}(g)v)| \|f(g)\| \leq B_r a(g)^{2\rho}(1 + \log \|a(g)\|)^{d-r}.$$
Let $r$ be large enough that
\[
\int_{N_0 \setminus G} a(g)^{2r}(1 + \log ||a(g)||)^{d-r} dg < \infty.
\]
If $\delta > 0$ is given then there exists $U_\delta \subset N_0 \setminus G$ compact such that
\[
\int_{N_0 \setminus G - U_\delta} a(g)^{2r}(1 + \log ||a(g)||)^{d-r} dg < \delta.
\]
Thus
\[
\left| \int_{N_0 \setminus G - U_\delta} (J(P, \sigma, iv)(\pi_{\mathcal{T}, \sigma, iv}(g)v) - J(P, \sigma, iv_0)(\pi_{\mathcal{T}, \sigma, iv}(g)v))\alpha_j(v)d\nu f(g)dg \right| \leq B_r \delta \int_{\mathfrak{a}_p^*} \alpha_j(v)d\nu = A_r \delta.
\]
Let $\delta$ be such that $B_r \delta < \frac{\varepsilon}{2}$ and $U = U_\delta$. To complete the proof of the lemma we must prove that there exists $j_o$ such that if $j \geq j_o$ then
\[
\left| \int_{U} \int_{\mathfrak{a}_p^*} (J(P, \sigma, iv)(\pi_{\mathcal{T}, \sigma, iv}(g)v) - J(P, \sigma, iv_0)(\pi_{\mathcal{T}, \sigma, iv}(g)v))\alpha_j(v)d\nu f(g)dg \right| < \frac{\varepsilon}{2}.
\]
Since $U \times \text{supp} \alpha_1$ is compact and
\[
g, v \mapsto J(P, \sigma, iv)(\pi_{\mathcal{T}, \sigma, iv}(g)v) - J(P, \sigma, iv_0)(\pi_{\mathcal{T}, \sigma, iv}(g)v))\alpha_j(v)f(g)
\]
is continuous a standard compactness argument proves that given $\mu > 0$ there exist $V$ a neighborhood of $\nu_0$ in $\mathfrak{a}_p^*$ such that
\[
|J(P, \sigma, iv)(\lambda)(\pi_{\mathcal{T}, \sigma, iv}(g)v) - J(P, \sigma, iv_0)(\lambda)(\pi_{\mathcal{T}, \sigma, iv}(g)v))f(g)| < \mu
\]
if $(g, v) \in U \times V$. Thus if the support of $\alpha_{j_o}$ is $V$ and if $j \geq j_o$ then
\[
\left| \int_{U} \int_{\mathfrak{a}_p^*} (J(P, \sigma, iv)(\pi_{\mathcal{T}, \sigma, iv}(g)v) - J(P, \sigma, iv_0)(\lambda)(\pi_{\mathcal{T}, \sigma, iv}(g)v))\alpha_j(v)d\nu f(g)dg \right| \leq \mu \int_{\mathfrak{a}_p^*} \alpha_j(v)d\nu.
\]
So, (say) take $\mu = \frac{\varepsilon}{4}$. 

**Theorem 48.** If $(\pi, H)$ is an irreducible unitary subrepresentation of $L^2(N_0 \setminus G, \chi)$ then there exists an irreducible square integrable representation $(\sigma, V)$ and $\lambda \in \text{Wh}_\chi(V^\infty)$ such that
\[
H^\infty = \{g \mapsto \lambda(\sigma(g)v)|v \in V^\infty\}.
\]
Proof. We have seen in [RRGII] Lemma 15.1.1 p. 365 that the support of \( L^2(N_o \setminus G, \chi) \) as an abstract representation is contained in the set of tempered representations. Thus it is enough to show that if \( (\sigma, V) \) is an irreducible tempered representation such that there exists a unitary intertwining operator \( L : V \to L^2(N_o \setminus G, \chi) \) then \( \sigma \) must be square integrable on \( G \). We note that if we set for \( v \in V_\infty \), \( \lambda(v) = L(v)(e) \) (e the identity element of \( G \)) then \( \lambda \in Wh_{\chi}(V_\infty) \) and \( L(v)(g) = \lambda(\pi(g)v) \) defines an element of \( ^oC(N_o \setminus G; \chi) \) (Theorem 29). Since \( (\sigma, V) \) is tempered there exists \( (P, \mu, v) \) a standard parabolic pair with \( \mu \neq G \), \( (\mu, H_\mu) \) a square integrable irreducible representation of \( ^oM \) and \( \nu_o \in a^*_\mu \) such that \( (\sigma, V) \) is equivalent with a direct summand of \( H^\infty\bar{P}, \sigma, i\nu_o \) (c.f, [RRGII] Proposition 5.2.5) thus Theorem 41 implies that there exists \( u \in H^\infty\bar{P}, \mu, i\nu_o \) and \( \tau \in Wh_{\chi}|_{MP \cap N_o}(H^\infty_{\mu}) \) such that
\[
L(v)(g) = J(P, \sigma, i\nu)(\tau)(\pi_{\bar{P}, \mu, i\nu_o}(g)u), \ g \in G.
\]
We have in the notation of the previous lemma
\[
\lim_{j \to \infty} \int_{N_o \setminus G} \Psi(P, \alpha_j, \mu, u, \tau)(g) L(u)(g) dg = \\
\int_{N_o \setminus G} J(P, \sigma, i\nu_o)(\tau)(\pi_{P, \mu, i\nu_o}(g)u) L(u)(g) dg = \|L(u)\|^2.
\]
On the other hand by the preceding theorem
\[
\int_{N_o \setminus G} \Psi(P, \alpha_j, \mu, v, \tau)(g) L(u)(g) dg = \\
\int_{N_o \cap MP \setminus ^oMP \times K \times A} \int_{a^*_\mu} a^{-i\nu_o} \alpha_j(v) \lambda(\sigma(m)v(k))(R(k)L(u))^P (ma) d\nu dmdkda.
\]
Which is 0 because \( L(u) \in ^oC(N_o \setminus G; \chi) \). This implies the theorem. \( \Box \)

10. First steps on the continuous spectrum

Beuzart-Plessis’ Proposition B.3.1 in [B] has as an immediate consequence

**Proposition 49.** Let \( P_o \) be a minimal parabolic subgroup of \( G \) and \( N_o \) (as usual) its unipotent radical
\[
\int_{[N_o, N_o]} a_{P_o}^{-r}(1 + \log \|n\|)^rdn < \infty
\]
for all \( r \).
Note that if the domain of integration is $N_o$ rather than $[N_o, N_o]$ then
\[ \int_{N_o} a_{\rho_o}(n)^{-\rho_o} dn = \infty \]

The result in [3] is actually much stronger. We have stated the result in this form because it is all that we need and we have an elementary proof of this result for $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ and real reductive groups of split rank 1 and several groups of rank 2 is given in [W3]. Since this result will be the basis of our determination of the continuous spectrum the use of this form of the proposition leads to an elementary proof for the important special case of $GL(n)$.

We will be using notation from the material preceding Lemma 1 in section 2 except that $v$ will denote a unit vector in $\wedge^{\dim N_o} \text{Lie}(N_o)$ also $\|...\|$ the standard norm defined therein.

**Corollary 50.** Let $\text{Lie}(N_o) = \text{Lie}([N_o, N_o]) \oplus Z$ with $\text{Ad}(A_o)Z = Z$. There exists $s \in \mathbb{R}$ and for each $\omega \subset G$ a compact subset a constant $C_\omega$ such that if $X \in Z$ and if $g \in \omega$ such that if $d$ is given there exists $C_d$ such that
\[ \int_{[N_o, N_o]} a_P(n \exp Xg)^{-\rho_o}(1 + \log \|n\|)^d dn \leq C_\omega C_d (1 + \|X\|)^s. \]

**Proof.** If $v$ is as as above and if $g \in G$ then
\[ \|g^{-1}v\| = a_{\rho_o}(g)^{2\rho_o}. \]
Thus if $n \in [N_o, N_o], g \in \omega, X \in Z$ then
\[ a_{\rho_o}(n \exp Xg)^{2\rho_o} = \|g^{-1} \exp(-X) n^{-1}v\|. \]
The inequality
\[ \|n^{-1}v\| = \|\exp Xg^{-1} \exp(-X) n^{-1}v\| \leq \|\exp Xg\| \|g^{-1} \exp(-X) n^{-1}v\| \]
implies that
\[ a_{\rho_o}(n \exp Xg)^{-\rho_o} \leq \|\exp Xg\|^2 a_{\rho_o}(n)^{-\rho_o} \leq \|\exp X\|^2 \|g\|^2 a_{\rho_o}(n)^{-\rho_o}. \]
Noting that $\|\exp X\|^2$ is a polynomial in $X$ the Corollary follows. $\square$

We note that if $P$ is a standard parabolic subgroup then

**Lemma 51.** There exist $k$ and $C > 0$ such that $\|n\|^k \|^\omega m_P(n)\| \leq \|n\|^k$ for $n \in N_P$.

**Proof.** $n = \bar{n}_P(n) a_P(n)^\omega m_P(n) k_P(n)$ so we have
\[ \|n\| = \|\bar{n}_P(n) a_P(n)^\omega m_P(n)\| \geq \|a_P(n)^\omega m_P(n)\| \geq \|a_P(n)\|^{-1} \|^\omega m_P(n)\|. \]
thus
\[ \|a_P(n)\| n \| \geq \|^o m_P(n)\|. \]

Note that
\[ \|a_P(n)\| \leq C_1 \|n\| \]
for some \( C_1 > 0 \) and \( r \) take \( k = r + 1 \) and \( C = C_1^{-1} \).

**Lemma 52.** Let \((\pi, H_\pi)\) be an admissible Hilbert representation of \(G\) and let \(\chi\) be a generic character of \(N_o\). Then there exists a finite subset \(S_{\chi,\pi} = \{x_1, x_2, ..., x_r\} \subset U(\text{Lie}(G))\) such that if \(H_{\chi,\pi}\) is the Hilbert space completion of \((H_\pi)_K\) with respect to the inner product
\[ \langle v, w \rangle_{\chi,\pi} = \sum_{i=1}^{r} \langle x_i v, x_i w \rangle_{\pi} + \langle v, w \rangle_{\pi} \]
then if \(\lambda \in Wh_\chi(H_\pi^\infty)\) then \(\lambda\) extends to a continuous functional on \(H_{\chi,\pi}\). Furthermore, \(H_{\chi,\pi}\) is \(\pi(G)\) invariant and \((\pi, H_1)\) is a Hilbert representation of \(G\).

**Proof.** Let \(\lambda_1, ..., \lambda_m\) be a basis of \(Wh_\chi(H_\pi^\infty)\). Then for each \(i\) there exist \(S_i \subset U(\text{Lie}(G))\) such that \(\lambda_i\) extends to a continuous functional on the Hilbert space completion of \((H_\pi)_K\) with respect to
\[ (v, w)_i = \sum_{x \in S_i} \langle xv, xw \rangle. \]
Take \(\{x_1, x_2, ..., x_r\}\) to be a basis of the (finite dimensional) span of \(\cup_{i=1}^{m} \text{Ad}(G)S_i\). \(\square\)

Let \(\chi\) be a generic character of \(N_o\) and let \(P\) be a standard parabolic subgroup of \(G\) and let \((\sigma, H_\sigma)\) be an irreducible square integrable representation of \(\circ M_P\). We consider the integral if \(u \in I_\sigma^\infty\)
\[ j_{\sigma,\mu}(u) = \int_{N_P} \chi(n)^{-1} u_\mu(n)dn. \]

Here and in the rest of this section if \(P\) and \(\sigma\) are understood then we write \(u_\mu\) for \(P u_{\sigma,\mu}\).

We will now apply the above lemmas with \(G\) replaced by \(\circ M_P\) and \(\pi\) replaced by \(\sigma\) a square integrable representation of \(\circ M_P\). To simplify notation we will use the notation \(\chi^*\) for \(\chi|_{N_o} \cap M_P\).

**Lemma 53.** Let \(\langle...,\rangle_{\chi^* ,\sigma}\) be as in the previous lemma for \(\sigma\). There exists \(c_\sigma\) such that if \(\mu \in (a_P)^*_C\) and
\[ \text{Re}(\mu, \alpha) < -c_\sigma(\rho_P, \alpha) \]
for \( \alpha \in \Phi(P, A_P) \) then
\[
\int_{N_P} \|u_\mu(n)\|_{\chi, \sigma}^p \, dn < \infty
\]
if \( u \in I_\sigma^\infty \).

**Proof.** We have for \( n \in N_P \)
\[
u_\mu(n) = a_P(n)^{-\rho + \Re \mu} \sigma(o m_P(n))u(k_P(n)),
\]
thus
\[
\|u_\mu(n)\|_{\chi, \sigma}^* \leq a_P(n)^{-\rho + \Re \mu} \|\sigma(o m_P(n))\|_{\chi, \sigma} \|u(k_P(n))\|_{\chi, \sigma}^*. 
\]
Thus
\[
\|u_\mu(n)\|_{\chi^*, \sigma} \leq C_2 a_P(n)^{-\rho + \Re \mu} \|\sigma(o m_P(n))\|_{\chi^*, \sigma} \leq C_2 a_P(n)^{-\rho + \Re \mu} \|o m_P(n)\|^s 
\]
\[
\leq C_2 C_3 a_P(n)^{-\rho + \Re \mu} \|n\|^{sr}.
\]
by Lemma 51 and the fact that there exist \( C \) and \( c \) such that \( \|\sigma(m)\|_{\chi, \sigma} \leq C \|m\|^c \). Harish-Chandra’s Lemma (see Lemma 3) implies that
\[
1 + \rho(\log a_P(n)) \geq C_4 (1 + \log \|n\|).
\]
This implies that there exists \( C_5, r \) such that
\[
a_P(n)^p \geq C_5 \|n\|^p.
\]
thus if \( \Re(\mu, \alpha) < -t(\rho_P, \alpha) \) then
\[
a_P(n)^{-\rho + \Re \mu} \leq a_P(n)^{-\rho + \Re \mu} \leq C_5^{-(1+t)} \|n\|^{-p(1+t)}.
\]
Hence
\[
\|u_\mu(n)\| \leq C_6 \|n\|^{-p(1+t)} \|n\|^{sr}.
\]
Since there exists \( q \) such that
\[
\int_{N_P} \|n\|^{-q} \, dn < \infty.
\]
The lemma follows. \( \Box \)

In light of this lemma we see that \( j_{\sigma, \mu}(u) \) is holomorphic in \( \mu \) with values in \( H_{\chi^*, \sigma} \) on the set
\[
\{\mu \in \mathfrak{a}_C^* \ | \ \Re(\mu, \alpha) < -c_\sigma(\rho_P, \alpha) \text{ for } \alpha \in \Phi(P, A_P)\}.
\]
Define \( (H_\sigma^\infty)_{\chi^*} \) to be the closure of the space
\[
\{\sigma(n) - \chi(n) \mid u \in H_\sigma^\infty, n \in N_o \cap M_P\}.
\]
in \( H_\sigma^\infty \). Also set \( (H_{\chi^*, \sigma})_{\chi^*} \) equal to the closure of
\[
\{\sigma(n) - \chi(n) \mid u \in H_{\chi^*, \sigma}, n \in N_o \cap M_P\}.
\]
in $H_1$. Then $(H^\infty_\sigma)_{\chi'} \subset (H_{\chi^\sigma})_{\chi'}$, so we have a natural continuous map $\iota : H^\infty_\sigma / (H^\infty_\sigma)_{\chi'} \rightarrow H_1 / (H_1)_{\chi'}$.

**Lemma 54.** The space $H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$ is finite dimensional and $\iota$ is bijective.

**Proof.** We note that $H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$ is a Fréchet space and $H_1 / (H_1)_{\chi'}$ is a Hilbert space so in both cases the continuous duals separate the points. By the definition of $H_1$ the inclusion of $H^\infty_\sigma$ into $H_1$ is continuous and its image is dense. Also, if $\lambda \in \left( (H_{\chi^\sigma})_{\chi'} \right)'$ then $\lambda$ pulls back to $\tilde{\lambda}$ on $H_1$ as an element of $\text{Wh} \chi^\sigma(H^\infty_\sigma)$. By the definition of $H_1$, the element $\tilde{\lambda}|_{H^\infty_\sigma}$ is in $\text{Wh} \chi^\sigma(H^\infty_\sigma)$. The result now follows form the definition of $H^\infty_{\chi^\sigma}$.

We will identify $(H^\infty_\sigma / (H^\infty_\sigma)_{\chi'})'$ with $\text{Wh} \chi^\sigma(H^\infty_\sigma)$ and thus write $\lambda$ for $\tilde{\lambda}$.

Let $p$ be the canonical projection of $H_{\chi^\sigma}$ onto $H_{\chi^\sigma} / (H_{\chi^\sigma})_{\chi'}$, and let $\tau$ be the inverse map to $\iota$. We define for $(\mu, \alpha) < -c_\sigma(\rho, \alpha), \alpha \in \Phi(P, A_P)$,

$$\tilde{j}_{\sigma, \mu}(u) = \tau(p(j_{\sigma, \mu}(u)).$$

Then for each $u \in H^\infty_\sigma$ this defines a holomorphic map in $\mu$ of

$$\left\{ \mu \in (a_P)_C^* \mid (\mu, \alpha) < -c_\sigma(\rho, \alpha), \alpha \in \Phi(P, A_P) \right\}$$

to $H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$.

**Proposition 55.** If $u \in H^\infty_\sigma$ then the map $\mu \mapsto \tilde{j}_{\sigma, \mu}(u)$ initially defined on

$$\left\{ \mu \in (a_P)_C^* \mid (\mu, \alpha) < -c_\sigma(\rho, \alpha), \alpha \in \Phi(P, A_P) \right\}$$

extends to a holomorphic map of $(a_P)_C^*$ to $H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$. Furthermore, the map

$$(a_P)_C^* \times H^\infty_\sigma \rightarrow H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$$
given by

$$\mu, u \mapsto \tilde{j}_{\sigma, \mu}(u)$$

is continuous and holomorphic in $\mu$.

**Proof.** If $\mu \in (a_P)_C^*$, $(\mu, \alpha) < -c_\sigma(\rho, \alpha), \alpha \in \Phi(P, A_P)$, and if $\lambda \in \text{Wh} \chi^\sigma(H_1)$ then $\lambda(j_{\sigma, \mu}(u)) = J(P, \sigma, \mu)(\lambda)(u)$. The observation that $\lambda|_{(H_1)_{\chi'}} = 0$ combined with the above lemmas now imply that $J(\tilde{j}_{\sigma, \mu}(u)) = J(P, \sigma, \mu)(\lambda)(u)$. Let $w_1, ..., w_r$ be a basis of $H^\infty_\sigma / (H^\infty_\sigma)_{\chi'}$ and let $\lambda_1, ..., \lambda_r$ be the dual basis in $\text{Wh} \chi^\sigma(H^\infty_\sigma)$ then

$$\tilde{j}_{\sigma, \mu}(u) = \sum_{i=1}^r J(P, \sigma, \mu)(\lambda_i)(u)w_i.$$
This formula implements the holomorphic continuation and proves the indicted properties of it. □

**Theorem 56.** Let $\alpha \in \mathcal{S}(\mathfrak{a}_p^\ast)$, $\beta(\nu) = \mu(\sigma, \nu)\alpha(\nu)$ (recall that $\mu(\sigma, \nu)$ is the Harish-Chandra Plancherel density) and $v, w \in (I_\infty)_K$ then

$$
\int_{N_0} \chi(n_o)^{-1} \int_{\mathfrak{a}_p^\ast} \langle \pi_{iv}(n_o g) v, w \rangle \beta(\nu) d\nu d n_o
$$

$$
= \int_{\mathfrak{a}_p^\ast} J(P, \sigma, iv)(\lambda_{\sigma, iv}(w))(\pi_{iv}(g) v) \beta(\nu) d\nu.
$$

**Proof.** Fix $g \in G$. We are computing

$$
\int_{N_0} \chi(n_o)^{-1} \int_{\mathfrak{a}_p^\ast} \int_{N_P} \langle v_{iv}(n_o g), w_{iv}(n) \rangle d n \beta(\nu) d\nu d n_o.
$$

We will do the calculation indirectly. Corollary 50 and Corollary 68 in the Appendix imply that there exist $d$ and $C_{g,v,w}$ such that for all $z \in \mathbb{C}$ with $\text{Re} \, z \geq 0$ if $X \in Z$ (as in Corollary 50) then

$$
\left| \int_{[N_0, N_o]} \int_{\mathfrak{a}_p^\ast} \int_{N_P} \langle v_{iv-\nu \rho_P}(n_o \exp X g), w_{iv-\nu \rho_P}(n) \rangle d n \beta(\nu) d\nu d n_o \right|
$$

$$
\leq \left| \int_{\mathfrak{a}_p^\ast} \beta(\nu) d\nu \right| C_{g,v,w}(1 + \|X\|)^d.
$$

We fix $v, w, g$ and $\alpha$ and write $C_{g,v,w} \left| \int_{\mathfrak{a}_p^\ast} \beta(\nu) d\nu \right|$ as $C$. Set for $\text{Re} \, z \geq 0$ and $X \in Z$

$$
\phi_z(X) = \int_{[N_o, N]} \int_{\mathfrak{a}_p^\ast} \int_{N_P} \langle v_{iv-\nu \rho_P}(n_o \exp X g), w_{iv-\nu \rho_P}(n) \rangle d n \beta(\nu) d\nu d n_o
$$

Here $\phi_z(X)$ is holomorphic in $z$ for $\text{Re} \, z > 0$ and continuous for $\text{Re} \geq 0$. If $\psi \in \mathcal{S}(Z)$ then

$$
|\phi_z(X)\psi(X)| \leq C(1 + \|X\|)^d |\psi(X)|.
$$

with $C$ (given above) independent of $\text{Re} \, z \geq 0$ and $\psi$. This allows us to define the tempered distributions

$$
T_z(\psi) = \int_Z \phi_z(X)\psi(X) dX.
$$

for $\text{Re} \, z \geq 0$. The estimate above allows us to use dominated convergence to see that

$$
\lim_{\varepsilon \to 0^+} T_\varepsilon = T_0
$$

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weakly in $S(Z)$. Recalling that if $\mathcal{F}$ denotes the Fourier transform on $S(Z)$, which we will write as
\[
\mathcal{F}(\psi)(Y) = \int_Z e^{iB(\theta Y, X)} \psi(X) dX,
\]
(here $dX$ is normalized so that $\mathcal{F}^{-1}(\psi)(X) = \mathcal{F}(\psi)(-X)$) and using the usual formula for the Fourier transform of a tempered distribution as $\mathcal{F}(T) = T \circ \mathcal{F}^{-1}$. We have
\[
\mathcal{F}(T)(\mathcal{F}(\psi)) = T(\psi).
\]
The continuity of the Fourier transform implies that
\[
\lim_{\varepsilon \to 0^+} \mathcal{F}(T_\varepsilon) = \mathcal{F}(T_0)
\]
We also note that Fubini's theorem implies that $\phi_z \in L^1(Z)$ for $\text{Re} \ z \geq 0$ thus
\[
\mathcal{F}(T_z)(\psi) = \int_Z \mathcal{F}(\phi_z)(X) \psi(X) dX.
\]
and $\mathcal{F}(\phi_z)$ is a continuous bounded function on $Z$.
Note that if $\text{Re} \ z \geq 0$, $Y \in Z$ and
\[
\chi_Y(\exp(X)) = e^{-iB(\theta Y, X)}
\]
then
\[
h_z(Y) = \int_{N_o} \chi_Y(n_o)^{-1} \int_{\mathfrak{a}_p^*} \int_{N_P} \langle v_{\nu - z\rho P}(nn_o g), w_{\nu - z\rho P}(n) \rangle d\beta(\nu) dn_o
\]
then $h_z$ is continuous for $\text{Re} \ z \geq 0$ and holomorphic for $\text{Re} \ z > 0$ and
\[
\mathcal{F}(T_z)(\psi) = \int_Y h_z(Y) \psi(Y) dY.
\]
Let $U$ be the set of all $Y \in Z$ such that $\chi_Y$ is a generic character, the $U$ is open and dense in $Z$. We fix $Y \in U$ and write $\chi = \chi_Y$ and will now compute
\[
h_z(Y) = \int_{N_o} \chi(n_o)^{-1} \int_{\mathfrak{a}_p^*} \int_{N_P} \langle v_{\nu - z\rho P}(nn_o g), w_{\nu - z\rho P}(n) \rangle d\beta(\nu) dn_o
\]
under the assumption that $\text{Re} \ z > c_\sigma$ where this total integral is absolutely convergent. So we may integrate in any order. Thus if $\text{Re} \ z > c_\sigma$, continue with fixed $Y \in U$ and set $\chi = \chi_Y$ then $(N_M = N_o \cap M_P$ and note that $N_P N_M = N_P$) consider
\[
h_z(Y) = \int_{N_P \times N_M} \chi(n_P n_M)^{-1} \times
\int_{\mathfrak{a}_p^*} \int_{N_P} \langle \sigma(n_M) v_{\nu - z\rho P}(n_M^{-1}n_M n_M g), w_{\nu - z\rho P}(n) \rangle d\beta(\nu) dn_p dn_M.
\]
We first integrate over $N_P$ and use the substitution $n_P \to n^{-1}n_P$ to find that
\[ h_z(Y) = \int_{N_P \times N_M} \chi(n_Pn_M)^{-1} \times \]
\[ \int_{a_P} \int_{N_P} \chi(n) \left\langle \sigma(n_M)v_{iv-zp}(n^{-1}n_Pn_Mg), w_{iv-zp}(n) \right\rangle dn \beta(\nu)dvdn_Pdn_M, \]
\[ = \int_{N_P \times N_M} \chi(n_Pn_M)^{-1} \times \]
\[ \int_{a_P} \int_{N_P} \chi(n) \left\langle \sigma(n_M)v_{iv-zp}(n_Pg), w_{iv-zp}(n) \right\rangle dn \beta(\nu)dvdn_Pdn_M \]
\[ = \int_{a_P} \int_{a_P} \chi_{i \sigma, iv-zp}(w)(j_{i \sigma, iv-zp}(\pi_{iv}(g)v))\beta(\nu)d\nu \]
\[ = \int_{a_P} \int_{a_P} \chi_{i \sigma, iv-zp}(w)(j_{i \sigma, iv-zp}(\pi_{iv}(g)v))\beta(\nu)d\nu \]
This implies that the value only depends on the image of $j_{i \sigma, iv-zp}(v)$ (resp. $j_{i \sigma, iv-zp}(w)$) in the finite dimensional space
\[ H_{\sigma}^\infty / (H_{\sigma}^\infty)_{\chi|N_0 \cap M_P} = H_{\chi|N_0 \cap M_P, \sigma} / \left( H_{\chi|N_0 \cap M_P, \sigma} \right)_{\chi|N_0 \cap M_P, \sigma} \]
We therefore have: if $\text{Re } z > c_\sigma$ then using the notation $J_{\chi_Y}(P, \sigma, \nu)$ and $\tilde{j}^{Y}_{\sigma, \nu}$ to take into account the dependence on $Y$ and noting that both are holomorphic in $\nu$ and continuous in $Y$ we have for $\text{Re } z > c_\sigma$
\[ h_z(Y) = \int_{a_P} J_{\chi_Y}(P, \sigma, iv-zp)(\lambda_{j_{i \sigma, iv-zp}(w)})(\pi_{iv}(g)v))\beta(\nu)d\nu \]
if $Y \in U$. The holomorphy for $\text{Re } z > 0$ and continuity for $\text{Re } z \geq 0$ imply that the equation is true for $\text{Re } z \geq 0$. If $\psi \in C_c^\infty(U)$ (which is contained in $\mathcal{F}(\mathcal{S}(Z))$) then by the above
\[ \lim_{\varepsilon \to 0^+} \int_Z h_{\varepsilon}(Y)\psi(Y)dY = \int_Z h_0(Y)\psi(Y)dY. \]
The holomorphy of the Jacquet integral for regular characters now implies that if $\chi = \chi_Y$, $Y \in U$ then
\[ h_0(Y) = \int_{a_P} J_{\chi_Y}(P, \sigma, ivz)(\lambda_{\tilde{j}^{Y}_{i \sigma, iv}(w)})(\pi_{iv}(g)v))\beta(\nu)d\nu. \]
Which is the content of the theorem. □
In the proof of Proposition 55 let \( \lambda_1, ..., \lambda_r \) be a basis of \( Wh_{\chi|\mathcal{M}_p \cap N_0}(H_\sigma^\infty) \) and let \( w_1, ..., w_r \in H_\sigma^\infty \) project to \( H_\sigma^\infty / (H_\sigma^\infty)_{\chi|N_0 \cap M_p} \) and satisfy \( \lambda_i(w_j) = \delta_{ij}, i, j = 1, ..., r \). If \( w \in I_\sigma^\infty \) then
\[
\tilde{j}_{\sigma, i\nu}(w) \equiv J(P, \sigma, i\nu)(\lambda_i)(w)w_i \mod (H_\sigma^\infty)_{\chi|N_0 \cap M_p}.
\]
Hence
\[
\lambda_{\tilde{j}_{\sigma, i\nu}}(w) = \sum J(\chi, \sigma, i\nu)(\lambda_i)(w)\lambda_{wi}.
\]
We also note that if \( \lambda_1, ..., \lambda_r \) is an orthonormal basis relative to the inner product in Corollary 38 then \( \lambda_{wi} = \lambda_i \). We have proved

**Corollary 57.** If \( \alpha \in S(a_p^\ast) \) and \( v, w \in I_\sigma \) set
\[
\psi(g) = \psi(\alpha, v, w)(g) = \int_{a_p} \langle \pi_{i\nu}(g)v, w \rangle \sigma(v) \mu(\sigma, iv)dv
\]
then if \( \lambda_1^\epsilon, ..., \lambda_r^\epsilon \) is an orthonormal basis of \( Wh_{\chi|\mathcal{M}_p \cap N_0}(H_\sigma^\infty) \) then (recalling that
\[
\int_{N_0} \chi(n)^{-1} \psi(ng) dn
\]
\[
\psi_\chi(g) = \sum_{i=1}^r \int_{a_p} J(\chi, P, \sigma, i\nu)(\lambda_i^\epsilon)(w)J(\chi, P, \sigma, i\nu)(\lambda_i^\epsilon)(\pi_{P, \sigma, i\nu}(g)v) \sigma(v) \mu(\sigma, iv) dv.
\]

**11. The Whittaker Plancherel Theorem first form**

We assume that \( G \) has compact center and that \( \chi \) is generic.

If \( F \subset \hat{K} \) is a finite set then define \( \mathcal{C}(G)_F \) to be the set of elements of \( \mathcal{C}(G) \) such that
\[
\sum_{\gamma \in F} d(\gamma) \int_K f(gk) \chi_\gamma(k^{-1}) dk = f(g), g \in G
\]
and let \( \mathcal{C}(G) \) be those elements of \( \mathcal{C}(G) \) such that
\[
\sum_{\gamma \in F} d(\gamma) \int_K f(gk) \chi_\gamma(k) dk = f(g), g \in G
\]
Let \( P \) be a standard cuspidal parabolic subgroup and let \( \sigma \in \mathcal{E}_2(\mathcal{O} M_P) \). Let \( E_\gamma \) be the orthogonal projection onto \( I_\sigma(\gamma) \) (the \( \gamma \)-isotypic component). We set \( \pi_\nu = \pi_{P, \sigma, \nu} \). If \( F \subset \hat{K} \) then we write \( E_F = \sum_{\gamma \in F} E_\gamma \).

If \( f \in \mathcal{C}(G)_F \) then if \( f \in H \mathcal{C}(G) \cap \mathcal{C}(G)_F \) then
\[
\pi_\nu(f) = E_H \pi_\nu(f) E_F
\]
Thus if \( \{v_\gamma^\sigma\} \) is an orthonormal basis of \( I_\sigma \) such that \( v_\gamma^\sigma \in I_\sigma(\gamma) \) and if \( S_\sigma(F) = \{i | i \in F \} \) then \( \{v_\gamma^\sigma\}_{i \in S_\sigma} \) is an orthonormal basis of \( I_\sigma(F) = \sum_{\gamma \in F} I_\sigma(\gamma) \). Before we state our key result we need a bit more notation.
If we need to indicate the dependence of \( J(P, \sigma, \nu) \) on \( \chi \) we will write \( J_\chi(P, \sigma, \nu) \). If \( f \in \mathcal{C}(G) \) then set
\[
T_{P,\sigma}(f)(g) = \int_{a_P} \text{tr}(\pi_{P,\sigma,\nu}(L_g^{-1}f))\mu(\sigma, \nu)d\nu.
\]

**Theorem 58.** Let \( P \) be a standard cuspidal parabolic subgroup and let \( \sigma \in \mathcal{E}_2(\sigma) \). Let \( \{\lambda_1^\sigma, \ldots, \lambda_r^\sigma\} \) be an orthonormal basis of \( Wh(H_\sigma^\infty) \) relative to the inner product given in Corollary 37. Let \( F \) be a finite subset of \( \hat{K} \), let \( f \in \mathcal{C}(G)_F \) and let \( \{v_\sigma^\nu\} \) be as above an orthonormal basis of \( (I_\sigma)_K \) compatible with the isotypic decomposition then
\[
T_{P,\sigma}(f)\chi(g) = \sum_{i=1}^{r_\sigma} \sum_{l \in S_\sigma(F)} \int_{a_P^*} J_{\chi^{-1}}(P, \sigma, \nu)(\lambda_i^\sigma)(\pi_{P,\sigma,\nu}(f)v_l) \times
\]
\[
J_{\chi^{-1}}(P, \sigma, \nu)(\lambda_i^\sigma)(\pi_{P,\sigma,\nu}(g)v_l)\mu(\sigma, \nu)d\nu.
\]

**Proof.** Both sides of the equation are continuous in \( f \in \mathcal{C}(G)_F \). Since the left \( K \)-finite elements of \( \mathcal{C}(G)_F \) are dense in \( \mathcal{C}(G)_F \) it is enough to prove the equation for \( f \in H\mathcal{C}(G) \cap \mathcal{C}(G)_F \) any finite \( H \subset \hat{K} \). By definition
\[
T_{P,\sigma}(f)(g) = \sum_{l \in S_\sigma(F)} \int_{a_P^*} \langle \pi_{P,\sigma,\nu}(g^{-1})\pi_{P,\sigma,\nu}(f)v_l, v_l \rangle \mu(\sigma, \nu)d\nu
\]
note that this is a finite sum and is equal to (\( \psi \) is as in Corollary 47)
\[
\sum_{l \in S_\sigma(F), j \in S_\sigma(H)} \int_{a_P^*} \langle \pi_{P,\sigma,\nu}(g^{-1})v_j, v_l \rangle \langle \pi_{P,\sigma,\nu}(f)v_l, v_j \rangle \mu(\sigma, \nu)d\nu
\]
\[
= \sum_{l \in S_\sigma(F), j \in S_\sigma(H)} \int_{a_P^*} \langle v_j, \pi_{P,\sigma,\nu}(g)v_l \rangle \langle \pi_{P,\sigma,\nu}(f)v_l, v_j \rangle \mu(\sigma, \nu)d\nu
\]
\[
= \sum_{l \in S_\sigma(F), j \in S_\sigma(H)} \psi(\alpha_{ij}, v_l, v_j)(g).
\]
With \( \psi \) as in Corollary 57 and \( \alpha_{ij}(\nu) = \langle v_j, \pi_{P,\sigma,\nu}(f)v_l \rangle \) which implies that
\[
T_{P,\sigma}(f)_\chi(g) = \sum_{l \in S_\sigma(F), j \in S_\sigma(H)} \psi(\alpha_{ij}, v_l, v_j)\chi^{-1}(g)
\]
\[
= \sum_{i=1}^{d} \sum_{j \in S_\sigma(F)} \sum_{l \in S_\sigma(F)} \int_{a_P^*} J_{\chi^{-1}}(P, \sigma, \nu)(v_j) \times
\]
\[
J_{\chi^{-1}}(P, \sigma, \nu)(\pi_{\nu}(g)v_l)\langle \pi_{P,\sigma,\nu}(f)v_l, v_j \rangle \mu(\sigma, \nu)d\nu
\]
Let $P$ be a $P_o$ standard parabolic subgroup of $G$ with standard Langlands decomposition $M_P A_P N_P$, $\sigma$ an irreducible square integrable representation of $M_p$ and and $\{ \lambda_\sigma^1, ..., \lambda_\sigma^r \}$ an orthonormal basis of $Wh_{\chi | M_P \cap N_o}$. If $f \in \mathcal{C}(G)$ is right $K$–finite we define

$$W_{P,\sigma}(f)(g) = \sum_{i=1}^{r_{\sigma}} \sum_{l \in S_\sigma(F)} \int_{a_P^*} J_{\chi^{-1}}(P, \sigma, i\nu)(\pi_{P,\sigma,i\nu}(f)v_l) \overline{J_{\chi^{-1}}(P, \sigma, i\nu)(\pi_{\sigma,i\nu}(g)v_l)} \mu(\sigma, i\nu) dv.$$

The following is the distributional version of the Whittaker Plancherel Theorem

**Corollary 59.** If $f$ is a right $K$–finite element of $\mathcal{C}(G)$ then

$$f_{\chi} = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\sigma M_P)} d(\sigma) W_{P,\sigma}(f).$$

Note that this is a finite sum since an $\sigma M_P \cap K$–type can appear in only a finite number of elements of $\mathcal{E}_2(\sigma M_P)$.

**Proof.** Harish-Chandra’s Plancherel Theorem (Theorem 30) implies that in the notation of the previous theorem

$$f = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\sigma M_P)} d(\sigma) T_{P,\sigma}(f).$$

Thus

$$f_{\chi} = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\sigma M_P)} d(\sigma) T_{P,\sigma}(f)_{\chi} = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\sigma M_P)} d(\sigma) W_{P,\sigma}(f).$$

by the previous theorem.

**12. Whittaker Plancherel Theorem**

In this section we will use Theorem 59 to derive the spectral decomposition of a right $K$–finite element of $\mathcal{C}(N_o \setminus G; \chi)$ in the case when $G$ has compact center and $\chi$ is generic. The key lemma in the proof is
Lemma 60. Assume $\chi$ is generic. Let $P$ be a standard cuspidal parabolic subgroup and let $\sigma \in \omega \in \mathcal{E}_2(A_P)$. Let $\psi \in C^\infty(N_o \setminus G; \chi)$ be such that $|\psi| \in C_c(N_o \setminus G)$ and let $\varphi \in C^\infty_c(N_o)$ is such that

$$\int_{N_o} \chi(n)^{-1}\varphi(n)dn = 1.$$ 

Set $f(nak) = \varphi(n)\psi(ak)$ for $n \in N_o, a \in A_o, k \in K$. If $\lambda \in Wh_{\chi^{-1}|_{N_o \cap M_P}}(H^\infty)$ and $\nu \in I_{\sigma}$ then

$$J_{\chi^{-1}}(P,\sigma,iv)(\lambda)(\pi_{P,\sigma,i\nu}(f)v) = \int_{N_o \setminus G} J_{\chi^{-1}}(P,\sigma,iv)(\lambda)(\pi_{P,\sigma,i\nu}(g)v)\psi(g)dg$$

Proof. Note that $\nu \mapsto J_{\chi^{-1}}(P,\sigma,\nu)(\lambda)(\pi_{P,\sigma,\nu}(f)v)$ is holomorphic on $(a_P^\sigma)_G$. Thus if we prove that

$$J_{\chi^{-1}}(P,\sigma,\nu)(\pi_{P,\sigma,\nu}(f)v) = \int_{N_o \setminus G} J_{\chi^{-1}}(P,\sigma,\nu)(\pi_{P,\sigma,\nu}(g)v)\psi(g)dg$$

if $\text{Re}(\nu,\alpha) < 0$ for $\alpha \in \Phi(P,A_P)$ then the formula will be true for all $\nu$ since both sides of the equation are holomorphic in $\nu$. We calculate (setting $w_\nu = \bar{\nu}w_\nu$ for $w \in I_{\sigma}$)

$$J_{\chi^{-1}}(P,\sigma,\nu)(\pi_{P,\sigma,\nu}(f)v) = \int_{N_P} \chi(n)\lambda(\pi_{P,\sigma,\nu}(f)v)_n(n)dn$$

$$= \int_{N_P} \chi(n)\lambda(f(g)v_\nu(ng))dg)dn.$$ 

Note that $N_o = N_M \cap N_P$ with $N_M = N_o \cap M_P$ and the map $N_M \times N_P \rightarrow N_o$ given by multiplication is a diffeomorphism. There exists $z \in (H^\infty)_K$ such that if $u \in H^\infty$ then

$$\lambda(u) = \lambda_\ast(u) = \int_{N_M} \langle \sigma(n_M)u, z \rangle \chi(n_M)dn_M.$$ 

Thus

$$\int_{N_P} \chi(n)\lambda(\int_G f(g)v_\nu(ng))dg)dn =$$

$$\int_{N_P} \chi(n) \int_{N_M} \chi(n_M) \langle \sigma(n_M)(\int_G f(g)v_\nu(ng))dg), z \rangle dn$$

$$= \int_{N_P} \chi(n) \int_{N_M} \chi(n_M) \times$$

$$\langle \sigma(n_M)(\int_{N_o \times A_o \times K} f(n_oa_o\nu)(nn_oa_o\nu)a_o^{-2\rho}dn_oa_o\nu), z \rangle dn$$

$$= \int_{N_P} \int_{N_M} \chi(nn_M) \times$$
Note that under the condition on \( \nu \) and \( \psi \) this integral converges absolutely so it can be calculated in any order. So we are calculating

\[
\int_{N_o} \chi(n_1) \left( \int_{N_o \times A_o \times K} \varphi(n_o) \psi(a_o k) v_r(n_1 n_o a_o k), z \right) a_o^{-2\rho} d\alpha_a d\rho d\kappa.
\]

Let \( P \) be a standard cuspidal parabolic subgroup  and let \( \sigma \in \omega \in \mathcal{E}_2(\rho M_P) \). If \( \psi \in C(N_o \setminus G; \chi) \) and \( \lambda \in Wh_{\chi^{-1}|_{N_o \cap M_P}}(H^\infty_\sigma), v \in I^\infty_\sigma \) then set

\[
\mathcal{W}_{P,\sigma}^\chi(\psi)(\lambda, v, \nu) = \int_{N_o \setminus G} J_{\chi^{-1}}(P, \sigma, iv)(\lambda)(\pi_{P,\sigma,iv}(g)v)\psi(g)dg.
\]

Before we can state the Whittaker Plancherel Theorem we need more notation. If \( P \) is a standard Parabolic subgroup of \( G \) and \( \sigma \in \mathcal{E}_2(\rho M_P) \) choose \( \lambda_1^\sigma, ..., \lambda_r^\sigma \) an orthonormal basis of \( Wh_{\chi^{-1}|_{N_o \cap M_P}}(H^\infty_\sigma) \) and \( \{ v_i^\sigma \} \) an orthonormal basis of \( (I^\infty_\sigma)_K \) that respects the \( K \)-isotypic decomposition.

**Theorem 61.** Assume that \( \chi \) is generic. If \( \psi \) is a right \( K \)-finite element of \( C(N_o \setminus G; \chi) \) then

\[
\psi = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{E}_2(\rho M_P)} \frac{d(\sigma)}{48} \sum_{i=1}^{r_\sigma} \int_{A^\sigma_P} \mathcal{W}_{P,\sigma}^\chi(\psi)(\lambda_i^\sigma, v_i^\sigma, \nu) J_{\chi^{-1}}(P, \sigma, iv)(\lambda_i^\sigma)(\pi_{P,\sigma,iv}(g)v_i) \mu(\sigma, iv) d\nu.
\]
Proof. First we assume that $|\psi| \in C_c(N_o \backslash G)$ and let $f$ be as in the previous lemma. Then, since, $f_\chi = \psi$ Theorem 59 implies that $$\psi = \sum_{[P] \in \mathcal{P}} \sum_{[\sigma] \in \mathcal{H}(\alpha M_P)} d(\sigma) W_{P,\sigma}(f).$$ Now $$W_{P,\sigma}(f)(g) = \sum_{i=1}^{d_P} \sum_{l} d(\sigma) \int_{a_{\mathfrak{p}_P}} J_{\chi^{-1}}(P, \sigma, i\nu) (\chi_P^\sigma)(\pi_{P,\sigma,i\nu}(f) v_{\chi}^\sigma) \times \overline{J_{\chi^{-1}}(P, \sigma, i\nu)(\chi_P^\sigma)(\pi_{P,\sigma,i\nu}(g) v_{\chi}^\sigma)} \mu(\sigma, i\nu) d\nu.$$ But the formula is a finite sum depending only on the $K$-types that occur in $\text{Span}_{R_K} \psi$ and the terms in the sum are continuous on $C_c(N_o \backslash G; \chi)$ so the formula is true on all of $C_c(N_o \backslash G; \chi)$. The theorem now follows from the definition of $W_{P,\sigma}(\psi)(\lambda, v, \nu)$. $\square$

13. The Case of $C(N_o \backslash G/K; \chi)$

The purpose of this section is to derive from the main theorem the corresponding result for $$C(N_o \backslash G/K; \chi) = \{f \in C(N_o \backslash G/K; \chi)| f(gk) = f(g), g \in G, k \in K\}.$$ If $\psi \in C(N_o \backslash G/K; \chi)$ then $W_{P,\sigma}(\psi) = 0$ if $P \neq P_0$ and $\sigma$ not 1 the trivial representation of $\mathfrak{m}$. A Set $$K^\chi_{\nu}(g) = J_{\chi}(P, \sigma, i\nu)(\pi_{P_0,1,\nu}(g)1).$$ If $\text{Re}(\nu, \alpha) < 0$ for all $\alpha \in \Phi^+(A_o)$ then $$K^\chi_{\nu}(g) = \int_{N_o} \chi(n_o)^{-1} a_{\mathfrak{p}_o}(n_og)^{-\rho} dn_o.$$ So $$\overline{K^\chi_{\nu}(g)} = K_{T\nu}^{\chi^{-1}}(g).$$ Hence, if we set $$T(\psi(\nu)) = \int_{N_o \backslash G} \psi(g) \overline{K_{i\nu}^\chi(g)} dg$$ for $\nu \in \mathfrak{a}_o^*$ then the Whittaker Plancherel theorem implies that $$\psi(g) = |W(A_o)^{-1} c_A^{-1} \gamma_A \int_{\mathfrak{a}_o^*} T(\psi(\nu)) \overline{K_{i\nu}^\chi(g)} \frac{d\nu}{c(i\nu)c(-i\nu)}.\] Here we have included all of the normalizing constants (see p. 427 [RRGII]) assumed in $\mu(1, i\nu)$ and used the relation between $\mu(1, i\nu)$ (1 denoting the trivial one dimensional representation of $\mathfrak{m}$) and the
Harish-Chandra $c$-function. This formula is the same as the one in 15.10.2 in [RRGI].

**Appendix A. Tempered estimates**

In this section we collect some estimates that will be needed in the proofs of the main results. The first result is a recalls a technique of [CHH]. This result was also used by Sun in [S] which contains versions of many of the results in this appendix, We include full details of the argument in [CHH] and our version of the results in [S] since we need the independence of parameters in Corollary 64. Let $G$ be a real reductive group and let $(R, L^2(G))$ denote the right regular representation of $G$ on $L^2(G)$.

**Theorem 62.** Let $f, h \subset L^2(G)$ be continuous and be respectively in the $K$--isotypic components for $\gamma, \mu \in \hat{K}$ relative to $R$. Then

$$|\langle R(g)f, h \rangle| \leq d(\gamma)d(\mu)\Xi(g)\|f\|\|g\|$$

here the norm is the $L^2$--norm.

**Proof.** If $p \in C(G)$ then set

$$\tilde{p}(g) = \max_{k \in K} |p(gk)| .$$

1. If in addition $\int_K p(gk)\chi_\gamma(k^{-1})dk = d(\gamma)p(g)$ for all $g \in G$ then

$$|\tilde{p}(g)| \leq d(\gamma)\left(\int_K |p(gk)|^2 dk\right)^{\frac{1}{2}} .$$

Indeed,

$$p(gk) = d(\gamma)\int_K p(gku)\overline{\chi_\gamma(u)}du .$$

so

$$|p(gk)| \leq d(\gamma)\left(\int_K |p(gku)|^2 du\right)^{\frac{1}{2}} \left(\int_K |\chi_\gamma(u)|^2 du\right)^{\frac{1}{2}} .$$

This clearly implies the assertion.

2. If $p \in L^2(G)$ is as in 1. then

$$\|\tilde{p}\| \leq d(\gamma)\|p\| .$$

To see this integrate the squares of the two sides of the inequality in 1.

3. If $f, g$ are as above then

$$|\langle R(g)f, h \rangle| \leq \left\langle R(g)\tilde{f}, \tilde{h} \right\rangle .$$

This is obvious.
We are left with showing that if \( \alpha, \beta \in C(G/K) \cap L^2(G) \) are positive functions then
\[
\langle R(g)\alpha, \beta \rangle \leq \Xi(g) \|\alpha\| \|\beta\|.
\]
Let \( P_o \) be a minimal parabolic subgroup of \( G \) then if \( \delta \) is the modular function of \( P_o \) and if \( u \in L^1(G) \) then
\[
\int_G u(x)dx = \int_K \int_{P_o} \delta(p)^{-1}u(pk)dpdk
\]
(here \( dp \) is an appropriate choice of right invariant measure on \( P_o \)). Thus we have
\[
\langle R(g)\alpha, \beta \rangle = \int_K \int_{P_o} \delta(p)^{-1}\alpha(pkg)\beta(p)dpdk
\]
\[
\leq \int_K \left( \int_{P_o} \delta(p)^{-1}\alpha(pkg)^2dp \right)^{\frac{1}{2}} dk \|\beta\|.
\]
If \( x \in G \) then write \( x = p(x)k(x) \) then
\[
\int_K \left( \int_{P_o} \delta(p)^{-1}\alpha(pkg)^2dp \right)^{\frac{1}{2}} dk = \int_K \left( \int_{P_o} \delta(p)^{-1}\alpha(ppk(kg))kg)^2dp \right)^{\frac{1}{2}} dk
\]
\[
= \int_K \delta(kg)^{\frac{1}{2}}dk \left( \int_{P_o} \delta(p)^{-1}\alpha(p)^2dp \right)^{\frac{1}{2}} = \Xi(g) \|\alpha\|.
\]

\[\square\]

**Theorem 63.** (compare \[S\]) Let \((\pi, H)\) be an irreducible square integrable representation of \( G \) then there exists a continuous semi-norm, \( q \), on \( H^\infty \) such that
\[
|\langle \pi(g)v, w \rangle| \leq \Xi(g)q(v)q(w)
\]
for all \( g \in G \).

**Proof.** Let \( u \in H \) be a unit vector. If \( T(z)(g) = \sqrt{d(\pi)} \langle \pi(g)z, u \rangle \) then the Schur orthogonality relations imply that \( T \) is an isometry hence
\[
|\langle \pi(g)v, w \rangle| = |\langle R(g)T(v), T(w) \rangle|
\]
\[
\leq d(\gamma)d(\tau) \|T(v)\| \|T(w)\| \Xi(g) = d(\gamma)d(\tau) \|v\| \|w\| \Xi(g)
\]
Thus if \( v, w \in H^\infty \) and \( v = \sum_{\gamma \in K} v_\gamma \) and \( w = \sum_{\gamma \in K} w_\gamma \) then
\[
|\langle \pi(g)v, w \rangle| = \left| \sum_{\gamma, \tau} \langle \pi(g)v_\gamma, w_\tau \rangle \right| \leq \sum_{\gamma, \tau} |\langle \pi(g)v_\gamma, w_\tau \rangle|
\]
\[
\leq \Xi(g) \sum_{\gamma, \tau \in K} d(\gamma)d(\tau) \|v_\gamma\| \|w_\tau\|.
\]
We note that if $C_K$ is the Casimir operator of $K$ and if $\lambda_{\gamma}$ is the eigenvalue of $C_K$ on corresponding to $\gamma$ and if $v \in H^\infty$ then
$$\|d\pi((I + C_K)^{r})v\|^2 = \sum (1 + \lambda_{\gamma})^r \|v_{\gamma}\|^2.$$ 
This implies that
$$\|v_{\gamma}\| \leq (\lambda_{\gamma} + 1)^{-\frac{r}{2}} \|(I + C_K)^{r})v\|$$
so
$$\sum_{\gamma \in \hat{K}} d(\gamma) \|v_{\gamma}\| \leq \|d\pi((I + C_K)^{r})v\| \sum_{\gamma \in \hat{K}} d(\gamma)(1 + \lambda_{\gamma})^{-\frac{r}{2}}.$$ 
If $r$ is sufficiently large then the series converges. So define
$$q(v) = \sum_{\gamma \in \hat{K}} d(\gamma) \|v_{\gamma}\|.$$

\[\square\]

**Corollary 64.** Let $P$ be a standard parabolic subgroup, if $\sigma$ is a square integrable representation of $o^M_P$ and if $v, w$ are continuous maps of $K$ to $H^\infty$ that are elements of $\text{Ind}^K_{M_P \cap K}(\sigma|_{M_P \cap K})$ then
$$\left|\langle \pi_{i\nu}(g)v,w \rangle\right| \leq \Xi(g)r(v)r(w)$$
with $r(v) = \max_{k \in K} q_{M_P}(v(k))$. Note that $r$ depends on $\sigma$ and not on $\nu$.

**Proof.**
$$\langle \pi_{i\nu}(g)v,w \rangle = \int_K \langle v_{i\nu}(kg),w(k) \rangle dk$$
$$= \int_K a_P(kg)^{i\nu - \rho} \langle \sigma(m(kg))v(k(kg)),w(k) \rangle dk.$$ 
So
$$\left|\langle \pi_{i\nu}(g)v,w \rangle\right| \leq \int_K a_P(kg)^{-\rho\nu} \left|\langle \sigma(m(kg))v(k(kg)),w(k) \rangle\right| dk$$
$$\leq \int_K a_P(kg)^{-\rho\nu} \Xi_{o^M_P}(m(kg))q(v(k(kg)))q(w(k)) dk$$
$$\leq \int_K a_P(kg)^{-\rho\nu} \Xi_{o^M_P}(m(kg))dkr(v) r(w).$$
We note that
$$\int_K a_P(kg)^{-\rho\nu} \Xi_{o^M_P}(m(kg)) dk = \Xi(g).$$
Indeed
$$\int_K a_P(kg)^{-\rho\nu} \Xi_{o^M_P}(m(kg)) dk$$
\[
= \int_K a_P(kg)^{-\rho_P} \int_{K \cap M} a_{M \cap P_o}(k_1 m(kg))^{-\rho_{M \cap P_o}} dk_1 dk.
\]

Reversing the order of integration and noting that \( k_1 m(kg) = m(k_1 kg) \) and

\[
a_P(g)^{-\rho_P} a_{P_o}(m(g))^{-\rho_{M \cap P_o}} = a_P(g)^{-\rho_o}
\]
yields

\[
\int_K a_P(kg)^{-\rho_P} a_{M \cap P_o}(m(kg))^{-\rho_{M \cap P_o}} dk = \Xi(g).
\]

The corollary follows. \(\square\)

Let \( P \) be a standard parabolic subgroup and let \( \bar{P} \) be the standard opposite parabolic subgroup to \( P \). Let \( \bar{N} \) be the unipotent radical of \( \bar{P} \) and let \( \bar{P} = o M_P A_P \bar{N}_P \) be its \( K \)–standard Langlands decomposition.

Write for \( g \in G \),

\[
g = \bar{n}(g)a_P(g)m_P(g)k_{\bar{P}}(g)
\]

\( \bar{n}(g) \in \bar{N}_P, a_P(g) \in A_P, m_P(g) \in o M_P, k_{\bar{P}}(g) \in K \) with the usual ambiguity in \( m_P \) and \( k_{\bar{P}} \).

**Lemma 65.** Let \( \omega \subseteq G \) be a compact subset then there exists a positive constant \( C_\omega \) such that if \( g \in \omega \) then

\[
C_\omega^{-1} a_P(x)^{\rho_P} \leq a_P(xg)^{\rho_P} \leq C_\omega a_P(x)^{\rho_P}.
\]

**Proof.** Let \( r = \dim \bar{n}_P \) consider the \( G \) module \( V = \Lambda^r \mathfrak{g} \). Let \( \| \cdots \|_V \) denote the norm on \( V \) and the operator norm on \( \text{End}(V) \) corresponding to extension of the inner product \( \langle X, Y \rangle = -B(X, \theta Y) \), \( X, Y \in \mathfrak{g} \) to \( \Lambda^r \mathfrak{g} \). If \( \xi \) is a unit vector in \( \Lambda^r \bar{n}_P \) and \( g \in G \) then

\[
g^{-1} \xi = a_P(g)^{2\rho_P} k_{\bar{P}}(g)^{-1} \xi.
\]

Thus

\[
\| g^{-1} \xi \|_V = a_P(g)^{2\rho_P}.
\]

Thus

\[
a_P(xg)^{2\rho_P} = \| g^{-1} x^{-1} \xi \| \leq \| g \|_V \| x^{-1} \xi \|_V = \| g \|_V a_P(g)^{2\rho_P},
\]

Also

\[
\| x^{-1} \xi \|_V = \| gg^{-1} x^{-1} v \|_V \leq \| g \|_V \| g^{-1} x^{-1} \xi \|_V
\]

so

\[
a_P(xg)^{2\rho_P} \geq \| g \|_V^{-1} a_P(xg)^{2\rho_P}.
\]

Let \( C_\omega = \max_{g \in \omega} \| g \|_V \). \(\square\)
Proposition 66. Notation as in the previous corollary. If \( v, w \in I^\infty_\sigma \), if \( \omega \) is a compact subset of \( G \) and if \( g \in \omega, n_o \in N_o \) and \( 0 \leq \varepsilon \leq 1 \) then there exist constants \( C'_\omega, C''_\omega \) such that

\[
\left| \int_{N_P} \langle v_{iv-\varepsilon \rho p}(nn_o g), w_{iv-\varepsilon \rho p}(n) \rangle \, dn \right| \leq C'_\omega \Xi(n_o g)r(v)r(w) \leq C''_\omega \Xi(n_o g)r(v)r(w).
\]

Here \( v_{\nu} = \bar{v}_{\rho, \sigma, \nu} \).

Proof. We have

\[
\int_{N_P} \langle v_{iv-\varepsilon \rho p}(nn_o g), w_{iv-\varepsilon \rho p}(n) \rangle \, dn
\]

\[
= \int_{N_P} a_P(nn_o g)^{iv-(1+\varepsilon)\rho p} a_P(n)^{-iv-(1+\varepsilon)\rho p} \langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle \, dn
\]

thus taking absolute values we have

\[
\left| \int_{N_P} \langle v_{iv-\varepsilon \rho p}(nn_o g), w_{iv-\varepsilon \rho p}(n) \rangle \, dn \right|
\]

\[
\leq \int_{N_P} a_P(nn_o g)^{-(1+\varepsilon)\rho p} a_P(n)^{-(1+\varepsilon)\rho p} \left| \langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle \right| \, dn.
\]

The previous lemma implies that this expression is

\[
\leq C^{2+\varepsilon}_\omega \int_{N_P} a_P(nn_o)^{-(1+\varepsilon)\rho p} a_P(n)^{-(1+\varepsilon)\rho p} \left| \langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle \right| \, dn
\]

\[
\leq C^{2+\varepsilon}_\omega \int_{N_P} a_P(nn_o)^{-\rho p} a_P(n)^{-\rho p} \left| \langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle \right| \, dn
\]

since \( \alpha_p(n)^{\rho p} \geq 1 \) for \( n \in N_o \). Now applying the lemma again this expression is

\[
\leq C^{3+\varepsilon}_\omega \int_{N_P} a_P(nn_o)^{-\rho p} a_P(n)^{-\rho p} \left| \langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle \right| \, dn.
\]

Also Theorem \( \ref{63} \) implies

\[
|\langle \sigma(m_P(nn_o g))v(k(nn_o g)), \sigma(m_P(n))w(k(n)) \rangle| \leq \Xi_{M_p}(m_P(n)^{-1}m_P(nn_o g))r(v)r(w).
\]

To complete the proof we need to show that

\[
\int_{N_P} a_P(n)^{-\rho p} a_P(n)^{-\rho p} \Xi_{M_p}(m_P(n)^{-1}m_P(nn_o g))dn = \Xi(g).
\]

Indeed, note then

\[
\Xi(g) = \langle \pi_{\rho, \sigma, 1, 0}(g)1, 1 \rangle.
\]

Set \( \sigma = \pi_{\rho, \sigma, 1, 0} \) then induction in stages implies that if \( v \in I^\infty_\sigma \) corresponds to 1 then using the notation

\[
v_0 = \bar{v}_{\rho, \sigma, 0}
\]

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we have
\[ \Xi(g) = \langle \pi_{\bar{P}_0 \cap M_p, \sigma, 0}(g)v, v \rangle = \int_{N_p} \langle v_0(n g), v_0(n) \rangle \, dn \]
\[ = \int_{N_p} a_P(n g)^{-\rho_P} a_P(n)^{-\rho_P} \langle \sigma(n g)1, \sigma(n)1 \rangle \, dn \]
\[ = \int_{N_p} a_P(n g)^{-\rho_P} a_P(n)^{-\rho_P} \langle \sigma(n g)1, \sigma(n)^{-1}1 \rangle \, dn \]
\[ = \int_{N_p} a_P(n g)^{-\rho_P} a_P(n)^{-\rho_P} \Xi_M(n g, 1) \, dn \]
\]
\[ \square \]

Observing that if \( v_o \) is a unit vector in \( \Lambda^n \text{Lie} (\bar{N}_o) \), with \( n = \dim \bar{N}_o \), and if \( g \in G \)
\[ a_{\bar{P}_o}(g) \rho_o \leq \| g^{-1} v_o \| \leq \| g^{-1} \| = \| g \| . \]
Using inequality 7. in section 3 we have

**Lemma 67.** \( \Xi(g) \leq C_2 a_{\bar{P}_o}(g)^{\rho_o} (1 + \log \| g \|)^d. \)

This leads to the following variant of the above proposition.

**Corollary 68.** Notation as in the previous corollary. If \( v, w \in I^\infty_\sigma \), if \( \omega \) is a compact subset of \( G \) and if \( g \in \omega, n_o \in N_o \) and \( \varepsilon \geq 0 \) then
\[ \left| \int_{N_p} \langle v_{\omega-\varepsilon \rho_P}(n n_o g), w_{\omega-\varepsilon \rho_P}(n) \rangle \, dn \right| \leq C_2 C_o a_{\bar{P}_o}(n_o)^{-\rho_o} (1 + \log \| n_o \|)^d, \quad r(v)r(w). \]

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