$A_0$ Condensation, Nielsen’s Identity and Effective Potential of Order Parameter

V. Skalozub*
Oles Honchar Dnipro National University, Dnipro, 49010 Ukraine
*e-mail: Skalozubv@daad-alumni.de
Received June 11, 2021; revised August 27, 2021; accepted August 28, 2021

Abstract—In high temperature SU(2) gluodynamics, the condensation of the zero component gauge field potential $A_0 = \text{const}$ and its gauge-fixing dependence are investigated. The $A_0$ and Polyakov’s loop $\langle L \rangle$ are mutually related. The two-loop effective potential $W(A_0, \xi)$ is recalculated in the background relativistic $R_\xi$ gauge. It depends on the parameter $\xi$, has a nontrivial minimum and satisfies Nielsen’s identity. These signs mean gauge invariance of the condensation phenomenon. We express $W(A_0, \xi)$ in terms of $\langle L \rangle$ and obtain the effective potential of order parameter $W(A_0^\text{cl})$ which is independent of $\xi$ and has a nontrivial minimum position. Hence the $A_0^\text{cl}$ condensate value is detected. We show that the equation relating $A_0$ and observable $A_0^\text{cl}$ coincides with the special characteristic orbit in the $(A_0, \xi)$-plane along which the $W(A_0, \xi)$ is $\xi$-independent. In this way the link between these two gauge invariant descriptions is established. The minimum value of Polyakov loop, the two-loop Debye mass and thermodynamical pressure are calculated. We also show that $A_0^\text{cl}$ condensate stabilizes the charged gluon spectrum in the chromomagnetic field $H(T)$ which is spontaneously generated at high temperature due to Savvidy’s mechanism. These two fields form a self-consistent background of plasma. Comparison with results of other authors (and other approaches) is given.

Keywords: $A_0$ condensate, Nielsen’s identity, effective potential, order parameter, gauge invariance
DOI: 10.1134/S1547477121070116

INTRODUCTION

Investigations of the deconfinement phase transition and quark-gluon plasma are in the center of modern high energy physics. This state of matter is created at high temperature due to asymptotic freedom of strong interactions. In quantum field theory, the order parameter for the deconfinement phase transition is Polyakov’s loop, which is zero at low temperature and nonzero at high temperatures $T > T_d$, where $T_d \sim 160–180$ MeV is the deconfinement phase transition temperature (for critical temperature estimations see, for instance, [1]). Standard information on the deconfinement phase transition is adduced, in particular, in [2–4].

In $SU(2)$, the Polyakov loop is defined as (in general case see [5]):

$$\langle L \rangle = \frac{1}{2} \frac{\beta}{\beta} \text{Tr} P \exp \left( i g \oint_0^\beta dx A_0(x_4, \vec{x}) \right).$$ (1)

Here, $g$ is coupling constant, $A_0(x_4, \vec{x}) = A_0^a(x_4, \vec{x}) \frac{\sigma^a}{2}$ is the zero component of the gauge field potential, $\sigma^a$ is Pauli’s matrix, $\beta$ is inverse temperature and integration is going along the fourth direction in the Euclidean space-time.

The Polyakov loop violates the color center group symmetry $Z_3$ that could result in spontaneous generation of color static potential $A_3^\text{cl} = \text{const}$ and non conservation of color charge $Q^3$. Investigations of these phenomena, which have started forty years ago, are carried out using various approaches including calculation of the effective potentials in quantum field theory in continuum and on the lattice, Monte Carlo lattice simulations. The $A_0$ condensate is an important dynamical parameter regulating infrared region of momenta of gauge fields at finite temperature. First it has been derived within two-loop effective potential $W(A_0, T)$ [6]. Practically simultaneously it was observed that in a relativistic background $R_\xi$ gauge the effective potential and the value of the condensed field detected as its minimum position $(A_0)_{\text{min}}$ are gauge-fixing dependent. That caused numerous discussions about gauge invariance of the $A_0$ condensation phenomenon as whole. In Refs. [7, 8] it was claimed that

738
only zero condensate value is compatible with $\xi$-independence. The other conclusion was found by the present author within Nielsen’s identity approach [9], [10]. It has been demonstrated that $W(A_0, \xi)$ satisfies Nielsen’s identity [11, 12]. Hence gauge invariance follows. The review paper on these and related calculations and obtained results is [13].

Recently in Ref. [14] for $SU(N)$ gluodynamics a nonperturbative procedure has been found for removing $\xi$-dependency within a constrained potential method. It includes special resummations of perturbation series. The obtained effective potential is gauge-fixing independent and has a nontrivial minimum, that proves gauge invariance of the condensate. In Ref. [15] the deconfinement phase transition was investigated in perturbation theory in one-loop approximation in $SU(N)$ gluodynamics in a modified Landau—De Witt gauge and the phase transition type, in particular, was determined.

The above mentioned results, calculated by different methods, and existing in the literature discrepancies stimulate an investigation, which explains the latter and relates the former ones. This is the main goal of the present paper. To realize that we derive a link between Nielsen’s identity for the two-loop effective potential [9] and Belyaev’s effective potential of order parameter [7]. This is key point because, as we show below, a very perspective idea to express the effective potential $W(A_0, \xi)$ in terms of the observable gauge invariant parameter $\langle L \rangle$ was realized with calculation errors. Hence incorrect conclusion about the absence of the condensation at two-loop order followed.

Since necessary for what follows standard calculations have been reported in numerous papers, we present them in brief and concentrate on the most important points. In next section, to have consistent presentation, we give information on the effective potential and Nielsen’s identity calculations. In Sect. 3 we calculate the relation between the $A_0$ and $A^{(0)}_0$—the observable (physical) value of the $A_0$ condensate, which follows from calculation of $\langle L \rangle$. In course of these calculations we correct the results of Ref. [7], and find the gauge-fixing independent effective potential $W_L(A^{(0)}_0)$, its minimum position and the value of the Polyakov loop in this vacuum. As application we calculate thermodynamical pressure and Debye’s gluon mass. Discussions and concluding remarks are given in the last section. In Appendix we carry out the calculation of the one-loop correction to the Polyakov loop omitted in Ref. [7], where only final result has been adduced. Also we present information about Bernoulli’s polynomials.

1. EFFECTIVE POTENTIAL AND NIELSEN’S IDENTITY

Let us consider $SU(2)$ gluodynamics in the Euclidean space time placed in the background field $A_0^\beta = A_0 \delta_{\mu\nu} \delta^{\alpha 3} = \text{const}$ described by the Lagrangian

$$\beta L = \frac{1}{4} (g_\mu^a)^2 + \frac{1}{2} \left[ (\bar{D}_\mu A_\mu)^a \right]^2 - \bar{C} D_\mu D^\mu C. \quad (2)$$

The gauge field potential $A^{a}_\mu = Q^{a}_\mu + \bar{A}^{a}_\mu$ is decomposed in quantum and classical parts. The covariant derivative in Eq. (2) is $(\bar{D}_\mu A_\mu)^a = (\bar{\partial}_\mu \bar{\delta}^{a\beta} - g e^{abc} \bar{A}^b_\mu) A^a_\mu$, $g_{\mu\nu} = (\bar{D}_\mu Q_\nu)^a - (\bar{D}_\nu Q_\mu)^a = g e^{abc} \bar{Q}^b_\mu Q^c_\nu$. $g$ is a coupling constant, internal index $a = 1, 2, 3$. The Lagrangian of ghost fields $\bar{C}, C$ is determined by the background covariant derivative $D_\mu (\bar{A})$ and the total one $D_\mu (\bar{A} + Q)$. As in Refs. [7, 16], we introduce the charged basis of fields:

$$A^0_\mu = A^3_\mu, \quad A^\pm_\mu = \frac{1}{\sqrt{2}} (A^1_\mu \pm i A^2_\mu), \quad C^0 = C^3, \quad C^\pm = \frac{1}{\sqrt{2}} (C^1 \pm i C^2). \quad (3)$$

In this basis a scalar product is $x^a x^b = x^+ x^- + x^- x^+ + x^0 x^0$, and the structure constants are: $\epsilon^{abc} = 1$ for $a = "+", \ b = "-", \ c = "0"$. Feynman’s rules are the usual ones for the theory at finite temperature with modification: in the background field $\sum_{k_0} e^{i k_0 x}$ in all loops of the fields $Q^a_\mu, C^\pm$. Here, $n = 0, \pm 1, \pm 2, \ldots$. This frequency shift must be done not only in propagators but also in three particle vertexes.

Carrying out standard calculations we obtain the two-loop effective potential

$$W(x) = W^{(0)}(x) + W^{(2)}(x),$$

$$\beta^4 W^{(0)}(x) = \frac{2}{3} \pi^2 \left[ B_0 (0) + 2 B_1 \left( \frac{x}{2} \right) \right],$$

$$\beta^4 W^{(2)}(x) = \frac{1}{2} \pi^2 \left[ B_2 \left( \frac{x}{2} \right) + 2 B_1 (0) B_2 \left( \frac{x}{2} \right) \right] + \frac{2}{3} \pi^2 (1 - \xi) B_3 \left( \frac{x}{2} \right) + B_1 \left( \frac{x}{2} \right). \quad (4)$$

where $B_i(x)$ are Bernoulli’s polynomials defined modulo 1 presented in Appendix, $x = \frac{gAJ}{\pi}$. This expression coincides with calculated already in Refs. [7, 9]. In what follows we consider the interval $0 \leq x \leq 2$. 

PHYSICS OF PARTICLES AND NUCLEI LETTERS  Vol. 18  No. 7  2021
Let us investigate the minima of it. We apply an expansion in powers of $g$ and get
\[ \beta^4 W_{\text{min}} = \beta^4 W(0) - \frac{1}{192\pi^2} (3 - \xi)^2 g^4, \]
where the first term is the value at zero field. Actually, an expansion parameter determined from the ratio of two- and one-loop contributions equals to $\frac{g^2}{8\pi}$, and therefore sufficiently large coupling values are permissible. As we see, both the minimum position and the minimum energy value are gauge-fixing dependent. Hence the gauge invariance of the $A_0$ condensation phenomenon is questionable.

This problem was solved within Nielsen’s identity method in [9, 10] for $SU(2)$ and $SU(3)$ gluodynamics and in [17, 18] for $QCD$ with quarks. Since this approach is important for what follows, we describe it in short here.

In Ref. [19] Nielsen’s identity for a general type effective potential has been derived:
\[ \delta' W(\phi) = W'_i \delta\chi^i (\phi), \tag{6} \]
which describes a variation of $W(\phi)$ due to variation of the gauge fixing term $F^a(\phi)$. In Eq. (6) $\phi'$ is gauge field, $\phi_0'$ denotes a vacuum value of $\phi'$, comma after $W$ means variation derivative with respect to corresponding variable. Variation $\delta\chi^i$ describes change of field $\phi$ due to special gauge transformation which compensates variation of a classical action appearing after variation of gauge-fixing function $F^a(\phi) \to F^a(\phi) + \delta F^a(\phi)$.

In field theory $\delta\chi^i$ is calculated from equation [19]:
\[ \delta\chi^i = - \left\{ D'_a(\phi) \Delta^i_\mu(\phi) \delta' F^\beta(\phi) \right\}, \tag{7} \]
where $\langle O(\phi) \rangle$ denotes functional average of $O(\phi)$. In this expression $D'_a(\phi)$ is generator of gauge group, $\Delta^i_\mu(\phi)$ is propagator of ghost fields, $\delta' F^\beta(\phi)$ is variation of gauge-fixing term.

In our case according Eq. (2) $\delta' F^\beta(\phi) = -\frac{1}{2} \left( D'_a(\phi) Q_\mu^a \delta^{\xi \phi} \xi \right)_i$, $D'_a$ is covariant derivative. In Ref. [9], Eq. (26), the expression was derived (for more details on calculations and discussions for $SU(3)$ case see in Refs. [10, 18]):
\[ \delta\chi^0 = \frac{g}{4\pi^2} B_1 \left( \frac{x}{2} \right) \delta\xi, \tag{8} \]
Nielsen’s identity for two-loop EP reads
\[ \frac{dW}{d\xi} = \frac{\partial W^{(2)}}{\partial \xi} + \frac{\partial W^{(1)}}{\partial x} \frac{\partial x}{\partial \xi} = 0, \tag{9} \]
where in the order $-g^2$ the derivative $\frac{\partial x}{\partial \xi}$ equals to $\frac{\delta \chi^0}{\delta \phi_0} \times \left( \frac{\delta \beta}{\delta \xi} \right)$ in Eq.(8). The latter factor comes from definition of $x = \frac{gA_0}{\beta}$. Since $W^{(2)}$ has the order $g^2$, and $W^{(1)} - g^0$, the Eq.(9) states that $W(x, \xi)$ does not change along the characteristic curve
\[ x = x' + \frac{g^2}{4\pi^2} B_1 \left( \frac{x'}{2} \right) (\xi - \xi') \tag{10} \]
in the plain of variables $(x, \xi)$. $\xi$ is an arbitrary integration constant. In the given approximation, this is a straight line. Thus, there is the set of orbits where $W(x')$ is gauge-fixing independent. Along them a variation in $\xi$ is compensated by the special variation of $x'$.

2. EFFECTIVE POTENTIAL

OF ORDER PARAMETER

In this section, following Ref. [7], express the effective potential (4) in terms of $\langle L \rangle$. We call it “effective potential of order parameter” $W(x)$. In $SU(2)$ group, in tree approximation, the Polyakov loop is expressed in terms of $x$ as follows: $\langle L \rangle = \cos \left( \frac{\pi x}{2} \right)$. This formula can be used to relate physical value of Polyakov loop and classical (observable) condensate value with accounting for radiation corrections:
\[ \langle L \rangle = \cos \left( \frac{\pi x_0}{2} \right) = \cos \left( \frac{\pi x}{2} \right) + \Delta \langle L \rangle. \]
The quantum correction was calculated in one-loop order (Eq. (10) in Ref. [7]).
\[ \Delta \langle L \rangle = - \frac{g^2}{4\pi^2} \sin \left( \frac{\pi x}{2} \right) \times \int \frac{dk}{k_0} \left[ 1 - \frac{1}{(k_0)^2} - \frac{1}{(k_0)^2 + k^2} \right], \tag{11} \]
where the notations are introduced:
\[ \int dk = \int d^3k \frac{1}{(2\pi)^3} \left( \frac{1}{\beta} \sum_{m=0}^{\infty} \right), \tag{12} \]
Eq. (11) is crucial for what follows.

In Ref. [7] the calculation of $\Delta \langle L \rangle$ was not presented in detail and only the final expression for the effective potential (Eq. (17)) has been given and analyzed. We fill in this gap below.
The second integral in Eq. (11) is well known, it is expressed in terms of Bernoulli’s polynomials [16, 18],

\[
I_2 = -\frac{(\xi - 1)}{4\pi \beta} B_1 \left(\frac{x}{2}\right).
\]  

(13)

We reduce calculation of the first integral, \(I_1\), to the previous one. Introducing the notation \(k^2_1 = (k^2_0) + k^2\), we write its integrand as follows

\[
\text{Intgd. } I_1 = \frac{1}{k^2_0} \frac{1}{((k^2_0)^2 + k^2)}
\]

\[
= \frac{k^2_0}{((k^2_0)^2 + k^2)^2 (1 - \frac{k^2}{k^2})}
\]

\[
= \frac{k^2_0}{((k^2_0)^2 + k^2)^2 (1 + \sum_{l=1}^{\infty} \frac{(k^2)^l}{(k^2)})}.
\]  

(14)

Hence, the first term in \(I_1\) coincides (up to the factor \((\xi - 1)\)) with \(I_2\). The other terms are also positive. So, the sings of \(I_1\) and \(I_2\) must be the same.

Now, instead summing series over \(l\) in Eq. (14), we return back to the initial expression for \(I_1\) and calculate it by using a standard procedure. This is presented in Appendix. The result is

\[
I_1 = -\frac{1}{2\pi \beta} B_1 \left(\frac{x}{2}\right).
\]  

(15)

Substituting \(I_1\) and \(I_2\) in Eq. (11), we obtain finally

\[
\Delta \langle L \rangle = \frac{g^2}{16\pi^2} B_1 \left(\frac{x}{2}\right) (\xi + 1).
\]  

(16)

Just this formula should be used in order to express “nonphysical field” \(x\) in terms of “classical observable one”, \(x_{cl}\).

Note again that in Ref. [7] the corresponding calculations have not been presented. Only the final effective potential of order parameter, which (as it is not difficult to verify) corresponds to the factor \((\xi - 3)\) (Eq. (17)) is adduced. But according to Eq. (11) the opposite signs of \(I_1\) and \(I_2\) are impossible. Factor \(\xi - 3\) can be obtained only for positive sign in Eqs. (15), (32).

Actually, to get the correct results in Ref. [7], we have to replace the parameter \((\xi - 3)\) by \((\xi + 1)\) in all the expressions. In particular, the relation between \(x\) and \(x_{cl}\) looks as follows (compare with Eq. (13) in Ref. [7]):

\[
x = x_{cl} + \frac{g^2}{4\pi^2} B_1 \left(\frac{x_{cl}}{2}\right) (\xi + 1).
\]  

(17)

Within Nielsen’s identity approach, this formula corresponds to the choice in Eq. (10) \(x = x_{cl}\) and \(\zeta = -1\). Along this orbit the EP is gauge-fixing independent and expressed in terms of \(\langle L \rangle\). In such a way these two methods are related.

Inserting Eq. (17) in Eq. (4) and expanding \(B_1 \left(\frac{x_{cl}}{2}\right)\) in powers of \(g^2\), we obtain \(W_L(x_{cl}) = W_L^{(0)}(x_{cl}) + W_L^{(2)}(x_{cl})\), where the first term is obtained from \(W^{(0)}(x)\) by means of substitution \(x \rightarrow x_{cl}\) and the second is

\[
\beta^4 W_L^{(2)}(x_{cl}) = \frac{g^2}{2} \left[ B_2^0 \left(\frac{x_{cl}}{2}\right) + 2 B_2(0) B_1 \left(\frac{x_{cl}}{2}\right) B_1 \left(\frac{x_{cl}}{2}\right) \right].
\]  

(18)

In the \(W_L(x_{cl})\), the \(\xi\)-dependent terms are mutually cancelled, as it should be and demonstrate gauge-fixing independence.

We also note that the final expression for \(W_L(x_{cl})\) can be obtained from \(W(x)\) Eq. (4) formally (omitting described intermediate steps) by means of the next substitutions: \(x \rightarrow x_{cl}\) and \(\xi \rightarrow \zeta = -1\). As a result, according Eq. (5) we get for the minimum values

\[
\beta^4 W_L(x_{cl}) \big|_{\min} = \beta^4 W_L^{(0)}(0) - \frac{1}{12\pi^2} g^4,
\]  

\[
x_{cl}|_{\min} = \frac{g^2}{2\pi^2}.
\]  

(19)

Thus, the EP \(W_L(x_{cl})\) has a nonzero minimum position and does not depend on \(\xi\). The condensation happens at the two-loop level. The minimum value of PL (corresponding to the physical states) equals to:

\[
\langle L \rangle = \cos \left(\frac{g^2}{4\pi}\right).\]  

In contrast, in Ref. [7] the value \(\langle L \rangle = \pm 1\) was obtained.

The expression \(-W_L(x_{cl})\) gives a thermodynamical pressure in the plasma. The first term is

\[
\beta^4 W_L^{(0)}(0) = -0.657974 + \frac{g^2}{24}.
\]  

The function \(W_L(x_{cl})\) can be used for calculating Debye’s mass of neutral gluons defined as

\[
m^2_D = \frac{d^2 W_L(x_{cl})}{dA^2_0} \bigg|_{A_0 = 0} ;
\]  

(20)

it should be reminded that, \(x_{cl} = \frac{g A^2_0}{\pi T}\). We get

\[
m^2_D = \frac{2}{3} g^2 T^2 + \frac{5}{4} g^4 T^2.
\]  

(21)

Here, first term is well known one-loop contribution and the second is two-loop correction.

To complete, we note that the \(A_0\) condensation is derived within the correlation of the one- and two-loop effective potentials. Whereas asymptotic freedom at high temperature is realized due to the relation of the
tree-level and one-loop contributions to the effective potential. Formally, the latter results in the replacement of coupling constant $g^2 \rightarrow \tilde{g}^2 - \frac{g^2}{\log(T/T_0)}$, $T_0$ is a reference temperature. In both cases, the ratio of the relevant terms is $\sim \frac{g^2}{4\pi^2}$. Hence it follows that at high temperature we can substitute $g^2 \rightarrow \tilde{g}^2$ in all the above formulas, in particular, in Eq. (19).

Thus, the value of the order parameter PL in the minimum of the effective potential is

$$\langle L \rangle = \cos\left(\frac{\tilde{g}^2}{4\pi}\right).$$

(22)

It gives a possibility for determination of the deconfinement phase transition and its type. Accounting for the explicit expression for the one-loop effective coupling $\alpha_s = \frac{\tilde{g}^2}{4\pi}$ in the $SU(2)$ case

$$\alpha_s = \frac{\alpha_s}{1 + \frac{11}{3\pi}\alpha_s \log(T/T_0)}$$

(23)

we see that the Polyakov loop is continuously decreasing with temperature lowering and becomes zero at $\tilde{g}^2 = \frac{\pi}{2}$. This signals a confinement. If we set $T_0 = T_0$ the value of the ratio $W^{(2)}/W^{(0)}$ is $\sim 1/2$, that is in the range of applicability of perturbation theory. The phase transition is second order, as it is well known for this gauge group. We stress once again that due to the smallness of the expansion parameter our perturbation effective potential of order parameter is suitable for investigating the confinement phase transition.

3. $A_0$ CONDENSATE AND STABILIZATION OF CHROMOMAGNETIC FIELD

During recent years it was realized that in plasma strong temperature dependent chromomagnetic fields have to be spontaneously generated [20–22]. These phenomena are related to asymptotic freedom in covariantly constant fields and take place even at zero temperature [23]. They are realized because for such fields an infrared region of strengths has to be unstable. In fact, this is Savvidy’s vacuum at finite temperature. In contrast to the zero temperature case, at finite temperature magnetic field stabilization takes place. Moreover, the $A_0$ classical fields act as stabilizing factors [21, 24]. As a result, the background of plasma is not a perturbation vacuum. The actual one is formed out of gauge field condensates. Moreover, the presence of magnetic fields is an other independent signal of the phase transition. In general, we have to take into consideration both type of condensates.

At high temperature, when perturbation methods are reliable, it is possible to analyze the role of the $A_0$ condensate as infrared stabilizing factor. Note that this idea first was proposed in Ref. [21]. Now, when the value of the condensate is calculated (19), we can verify it using the result of Ref. [25], Eq. (19), where the spontaneously created chromomagnetic field was computed for $SU(2)$ gluodynamics:

$$(gH)_{cl} = \frac{g^4}{4\pi^2} T^2.$$  

(24)

On the other hand, the value of the $A_0$ condensate following from Eq. (19) is

$$(A_0)_{cl_{\text{min}}} = \frac{\tilde{g}^2}{2\pi} T.$$  

(25)

Hence, for the charged gluon spectrum in the classical background fields directed along third axis in both internal and usual spaces, we get

$$\left(p_t^2 + \frac{g^2}{2\pi} T^2\right)^2 + p_3^2 + (2n - 1) \frac{g^4}{4\pi^2} T^2,$$  

(26)

where $p_t$ is momentum along the field $H(T)$ and $n = 0, 1, 2, ..$ is Landau level number. In the ground state $n = 0$ and the $T$-depend term in the first brackets cancels the negative last one. This important non-trivial cancellation is not expected in advance. It takes place only for $(A_0)_{cl}$ derived from the gauge invariant effective potential of order parameter. The spectrum is stable. In contrast, for $A_0 = 0$ the well known unstable mode $p_t^2 + p_3^2 - g H$ is reproduced. $A_0$ condensate stabilizes infrared region of momenta $p_t \rightarrow 0$. It is worth noting that in Ref. [24] the stabilization was detected in simulations on the lattice and the values of the $A_0 \neq 0$ condensate and corresponding magnetic fields have been detected for a number of plasma temperatures. Here we presented it for the values of fields obtained in the consistent analytic gauge invariant calculations. To complete this section, we note that in Ref. [28] a thermal gluo-magnetic vacuum of $SU(N)$ gauge theory has been detected on the lattice. And recently in Ref. [29] the zero magnetic mass of Abelian chromomagnetic field was estimated, also in lattice calculations. Thus this field is the inevitable constituent of the high temperature plasma.

4. DISCUSSION AND CONCLUSIONS

Two main conclusions follow from the above considerations. First, we found a simple correlation between Nielsen’s identity method for the effective potential (4) and the effective potential of order parameter $W_L(x_{cl})$. To link them, we have to substitute in Eq. (4) $x \rightarrow x_{cl}$, $\xi \rightarrow \zeta = -1$. We see that the effec-
tive potential satisfying Nielsen’s identity is $\xi$-independent on numerous orbits in the $(x, \xi)$-plane. The only free parameter is the value $\xi$ in Eq. (10). The found choice, which relates these approaches for describing gauge invariance, is $\xi = -1$. Second, we calculated the observable “classical” values of the condensate $x_{\text{cl min}}$ Eq. (19). In fact, we just realized the Belyaev’s idea of expressing the effective potential in terms of $\langle L \rangle$.

It is worth saying a few words about the orbit with $\xi = 3$ in Eq. (10), that corresponds to the case considered in Ref. [7]. This is a special case where $x_{\text{cl}} = 0$ at two-loop order. Actually, zero value here separately follows either for the one-loop effective potential or the two-loop one plus the terms $-g^2$ coming from the expansion of $B_0 \left( \frac{x}{2} \right)$ in the former effective potential. Looking at Eq. (5), in the gauge $\xi = 3$ we have to expect $A_0$ condensation in three-loop approximation. But in general basically this is two-loop phenomenon. The standard loop expansion method is well applicable for this problem.

As we noted in Introduction, in Ref. [14] the $\xi$-independence of the $A_0$ has been established within the constrained effective potential approach. It is close to the EP of order parameter and results in the same conclusions. The main difference between them consists in the calculation procedures applied. In the former one the Polyakov loop is considered as a dynamical parameter. In the latter the dynamical parameter is $A_0$ external field which is a solution to local field equations. This field is expressed though $\langle L \rangle$ after actual calculation of the EP $W(A_0)$. It is also important that qualitatively these results are in agreement with the ones derived by means of lattice QCD methods [26], [27]. In these papers the effective potential for the Polyakov loop has been studied in a gauge invariant lattice formulation.

In is also interesting to compare our results with the ones in Ref. [15] where the deconfinement phase transition in SU(2) gluodynamics was investigated by a perturbation method. Special modified Landau-DeWitt gauge was applied. It differs from the original one by an addition mass term $L_m$, introduced in a standard way

$$L_m = \frac{1}{2} m^2 Q_\mu^c Q_\mu^c.$$

This term was considered by the authors as an infrared cut-off for gluon low momenta. The quantization has been carried out by means of the background field method. In fact, such a simple introducing of mass term for Yang–Mills fields has to result in nonrenormalizable theory, which is nonunitary. This will be discussed in details in other place. Here in short we mention that the only way to preserve renormalizability is spontaneous symmetry breaking when, in particular, the contribution of the longitudinal components of massive vector field is cancelled by the contributions of scalar fields introduced via the Higgs-Kibble mechanism. Without such cancellations the theory should be cut-off at large momenta. The limit $m \to 0$ is singular. All these details we leave behind the present paper and compare the results relevant to our investigation. Note that in Ref. [15] the applicability of perturbation methods in deconfinement phase transition is also motivated by the smallness of the expansion parameter $\frac{g^2}{4\pi^2}$ in the range of the transition. The presence of the $A_0$ condensate in the high temperature phase was also observed. In contrast to our investigation the low temperature phase has been taken into consideration, that is impossible in our case and our results are reliable till the temperatures when the Polyakov loop becomes zero. Remind that this happens when $g^2 = 2\pi^2$ and the expansion parameter equals $\frac{1}{2}$. Qualitatively they are in agreement with each other. It is also very important that the presence of the $A_0$ condensate stabilizes temperature dependent magnetic fields in the plasma. Hence it follows that the stable plasma background is formed out of two classical condensates $A_0$ and $H(T)$. These fields occupy the whole volume of plasma. The magnetic mass of neutral gluons is zero but electric Debye’s mass is finite and given by Eq. (21).

Summarizing all together we note that gauge independence of the $A_0$ condensation is proved. It introduces the dynamical parameter regulating behavior of gauge fields at infrared momenta. The simplicity of dealing with constant background potentials $A_0$ makes them useful and convenient objects for different applications in QCD after deconfinement phase transition.

The author grateful Michael Bordag and Oleg Borisenko for constructive remarks and suggestions.

**APPENDIX**

Let us calculate $I_1$. For the temperature sum we use the representation

$$S^{(0)} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} F(k_n)$$

$$= \frac{1}{4\pi^2} \int \cot \left( \frac{1}{2} \beta \omega \right) F(\omega) d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega.$$

This term was considered by the authors as an infrared cut-off for gluon low momenta. The quantization has been carried out by means of the background field method. In fact, such a simple introducing of mass term for Yang–Mills fields has to result in nonrenormalizable theory, which is nonunitary. This will be discussed in details in other place. Here in short we mention that the only way to preserve renormalizability is spontaneous symmetry breaking when, in particular, the contribution of the longitudinal components of massive vector field is cancelled by the contributions of scalar fields introduced via the Higgs-Kibble mechanism. Without such cancellations the theory should be cut-off at large momenta. The limit $m \to 0$ is singular. All these details we leave behind the present paper and compare the results relevant to our investigation. Note that in Ref. [15] the applicability of perturbation methods in deconfinement phase transition is also motivated by the smallness of the expansion parameter $\frac{g^2}{4\pi^2}$ in the range of the transition. The presence of the $A_0$ condensate in the high temperature phase was also observed. In contrast to our investigation the low temperature phase has been taken into consideration, that is impossible in our case and our results are reliable till the temperatures when the Polyakov loop becomes zero. Remind that this happens when $g^2 = 2\pi^2$ and the expansion parameter equals $\frac{1}{2}$. Qualitatively they are in agreement with each other. It is also very important that the presence of the $A_0$ condensate stabilizes temperature dependent magnetic fields in the plasma. Hence it follows that the stable plasma background is formed out of two classical condensates $A_0$ and $H(T)$. These fields occupy the whole volume of plasma. The magnetic mass of neutral gluons is zero but electric Debye’s mass is finite and given by Eq. (21).

Summarizing all together we note that gauge independence of the $A_0$ condensation is proved. It introduces the dynamical parameter regulating behavior of gauge fields at infrared momenta. The simplicity of dealing with constant background potentials $A_0$ makes them useful and convenient objects for different applications in QCD after deconfinement phase transition.

The author grateful Michael Bordag and Oleg Borisenko for constructive remarks and suggestions.
where \( k_0 = \frac{2\pi n}{\beta} \) and contour \( C \) goes counterclockwise around the real axis in complex \( \omega \)-plane where the poles at \( k_0 \) are located. The second integral removes a zero temperature contribution. To calculate the contour integral, we have to extend \( C \) to infinity and calculate the sum of residuums,

\[
I^{(0)} = -\frac{1}{2} \sum \text{Res} \left[ \cot \left( \frac{1}{2} \right) F(\omega) \right].
\]

The sign “−” reflects that the poles of \( F(\omega) \) are passed round clockwise in the \( \omega \)-plane.

In our case (see Eq. (11)), \( F(\omega) \) has two simple poles: \( \omega = \pm \frac{gA_0}{i} \), and we get

\[
I^{(0)}_1 = \frac{i}{4k^2} \left[ \coth \left( \frac{1}{2} \beta (x + \frac{gA_0}{i}) \right) - \coth \left( \frac{1}{2} \beta (x - \frac{gA_0}{i}) \right) \right]
\]

(30)

Introducing the notations \( X_1 = e^{i\beta \frac{gA_0}{i}} \), \( X_2 = e^{-i\beta \frac{gA_0}{i}} \), the Eq. (30) can be written in the form

\[
I^{(0)}_1 = \frac{i}{2k^2} \sum_{n=1}^{\infty} \left( X_1^n - X_2^n \right)
\]

(31)

Then, performing momentum integration and summing up the series, we obtain

\[
I_1 = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{\beta n} \sin (gA_0 \beta n) = -\frac{1}{2\pi \beta} B_1\left( \frac{3}{2} \right)
\]

(32)

where \( x = \frac{gA_0}{\beta} \). The Bernoulli’s polynomials defined modulo 1 are

\[
B_1(x) = x - \frac{x}{2|x|}, \quad B_2(x) = x^2 - |x| + \frac{1}{6},
\]

\[
B_3(x) = x^3 - \frac{3x}{2|x|} + \frac{1}{2}, \quad B_4(x) = x^4 - 2|x|^2 + \frac{1}{30}.
\]

At \( x = 0 \) the \( B_1(x) \) is defined to be 0.

REFERENCES

1. Ya. Aoki, Sz. Borsanyi, S. Duerr, Z. Fodor, S. D. Katz, S. Krieg, and K. Szabo, “The QCD transition temperature: Results with physical masses in the continuum limit II,” J. High Energy Phys., No. 6 (2009).

2. D. J. Gross, R. D. Pisarski, and L. G. Yaffe, “QCD and instantons at finite temperature,” Rev. Mod. Phys. 53, 43–80 (1981).

3. J. I. Kapusta, Finite-Temperature Field Theory (Cambridge Univ. Press, Cambridge, 1989).

4. M. Le Bellac, Thermal Field Theory (Cambridge Univ. Press, Cambridge, 1996).

5. M. Srednicki and L. Susskind, “Colored monopoles on the lattice,” Nucl. Phys. B 179, 239–252 (1981).

6. R. Anishetty, “Colour singlet ensemble,” J. Phys. G: Nucl. Phys. 10, 439–445 (1984).

7. V. M. Belyaev, “Order parameter and effective potential,” Phys. Lett. B 254, 153–157 (1991).

8. K. Enqvist and K. Kajantie, “Hot gluon matter in a constant A0-background,” Zeitschr. Phys. C 47, 291–295 (1990).

9. V. V. Skalozub, “Gauge independence of the hot A0-condensate,” Mod. Phys. Lett. A 7, 2895–2903 (1992).

10. V. V. Skalozub, “Nielsen’s identity and gluon condensation at finite temperature,” Phys. Rev. D 50, 1150–1156 (1994).

11. N. K. Nielsen, “On the gauge dependence of spontaneous symmetry breaking in gauge theories,” Nucl. Phys. B 101, 173–188 (1975).

12. R. Fukuda and T. Kugo, “Gauge invariance in the effective action and potential,” Phys. Rev. D: Part. Fields 13, 3469–3484 (1976).

13. O. A. Borisenko, J. Boháčik, and V. V. Skalozub, “A0 condensate in QCD,” Fortschr. Phys. 43, 301–348 (1995).

14. Ch. P. Korthals Altes, H. Nishimura, R. D. Pisarski, and V. V. Skokov, “Conundrum for the free energy of a holonomous gluonic plasma at cubic order,” Phys. Lett. B 803, 135336 (2020).

15. U. Reinosa, J. Serreau, M. Tisser and N. Wschebor, “Deconfinement transition in SU(N) theories from perturbation theory,” Phys. Lett. B 742, 61–66 (2015).

16. V. M. Belyaev and V. L. Eletsky, “Two-loop free energy for finite temperature SU(3) gauge theory in a constant external field,” Zeitschr. Phys. C 45, 355–359 (1990).

17. V. V. Skalozub, “Gauge invariance of the gluon field condensation phenomenon in finite temperature QCD,” Int. J. Mod. Phys. A 9, 4747–4758 (1994).

18. V. V. Skalozub and I. V. Chub, “Two-loop contribution of quarks to the condensate of the gluon field at finite temperatures,” Phys. At. Nucl. 57, 324–328 (1994).

19. R. Kobes, G. Kunstatter, and A. Rebhan, “Gauge dependence identities and their application at finite temperature,” Nucl. Phys. B 355, 1–37 (1991).

20. V. V. Skalozub and A. Yu. Tishchenko, “The effective electromagnetic interaction in a dense fermionic medium in QED (2+1),” Phys. Lett. B 387, 835–840 (1996).

21. A. O. Starinets, A. S. Vshivtsev, and V. C. Zhukovskii, “Color ferromagnetic state in SU(2) gauge-theory at finite temperature,” Phys. Lett. B 322, 403–412 (1994).
22. V. Demchik and V. Skalozub, “Spontaneous magnetization of a vacuum in the hot universe and intergalactic magnetic fields,” Phys. Part. Nucl. 46, 1–23 (2015).

23. G. K. Savvidy, “Infrared instability of the vacuum state of gauge theories and asymptotic freedom,” Phys. Lett. B 71, 133 (1977).

24. V. Demchik and V. Skalozub, “Spontaneous creation of chromomagnetic field and A(0)-condensate at high temperature on a lattice,” J. Phys. A 41, 164051 (2008).

25. V. Skalozub and M. Bordag, “Once more on a color ferromagnetic vacuum state at finite temperature,” Nucl. Phys. B 576, 430–441 (2000).

26. O. A. Borisenko, V. K. Petrov, and G. M. Zinovjev, “A0 condensate in high-temperature phase of lattice QCD,” Phys. Lett. B 264, 166–172 (1991).

27. O. A. Borisenko and J. Boháčik, “Invariant measure in hot gauge theories,” Phys. Rev. D: Part. Fields 56, 5086–5096 (1997).

28. N. O. Agasian, “Thermal gluo-magnetic vacuum of SU(N) gauge theory,” Phys. Lett. B 562, 257–264 (2003).

29. N. Kolomyiets and V. Skalozub, “The color structure of gluon field magnetic mass,” Int. J. Mod. Phys. A 34, 1950053 (2019).