Optimal exponentials of thickness in Korn’s inequalities for parabolic and elliptic shells

Peng-Fei YAO

Key Laboratory of Systems and Control
Institute of Systems Science, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100190, P. R. China
School of Mathematical Sciences
University of Chinese Academy of Sciences, Beijing 100049, China
e-mail: pfyao@iss.ac.cn

Abstract We establish Korn’s interpolation inequalities and the rigidity results of the strain tensor of the middle surface for the parabolic and elliptic shells and show that the best constant in Korn’s inequalities scales like $h^{3/2}$ for the parabolic shell and $h$ for the elliptic shell, removing the main assumption that the middle surface of the shell is given by one single principal coordinate in the literature and, in particular, including the closed elliptic shell.

Keywords Korn’s inequality, shell, nonlinear elasticity, Riemannian geometry

Mathematics Subject Classifications (2010) 74K20(primary), 74B20(secondary).

1 Introduction and Main Results

Korn’s inequalities have arisen in the investigation of the boundary value problem of linear elastostatics, [19, 20] and have been proven by different authors, e.g., [7, 16, 17, 18, 28]. Some generalized versions of the classical second Korn inequality have been recently proven in [1, 5, 26, 27]. The optimal exponential of thickness in Korn’s inequalities for thin shells represents the relationship between the rigidity and the thickness of a shell when the small deformations take place since Korn’s inequalities are linearized from the geometric rigidity inequalities under the small deformations ([6]). Thus it is the best Korn constant...
in the Korn inequality that is of central importance (e.g., [4, 21, 22, 23, 24, 25]). Moreover, it is ingenious that the best Korn constant is subject to the Gaussian curvature. The one for the parabolic shell scales like $h^{3/2}$ ([10, 11]), for the hyperbolic shell, $h^{4/3}$ ([14]) and for the elliptic shell, $h$ ([14]). All those results were derived under the main assumption that the middle surface of the shell is given by a single principal coordinate system in order to carry out some necessary computation. This assumption is

$$S = \{ r(z, \theta) \mid (z, \theta) \in [1, 1 + l] \times [0, \theta_0] \}, \quad (1.1)$$

where the properties

$$\nabla_{\partial z} \vec{n} = \kappa_z \partial z, \quad \nabla_{\partial \theta} \vec{n} = \kappa_\theta \partial \theta \quad \text{for} \quad p \in S$$

hold.

In the case of the parabolic or hyperbolic shell, a principal coordinate only exists locally (Proposition 2.1). There is even no such a local existence for the elliptic shell. However, the assumption (1.1) in [10, 11, 14] can be removed if the Bochner technique is employed to perform some necessary computation. The Bochner technique provides us the great simplification in computation, for example, see [31] or [33]. Here we remove the assumption (1.1) to obtain that the optimal exponentials are $3/2$ and $1$ for the parabolic shell and the elliptic shell, respectively. In particular, the closed elliptic shell is included here. The case of the hyperbolic shell is treated in [35] where we show that the optimal exponential is $4/3$ without the assumption (1.1).

Let $M \subset \mathbb{R}^3$ be a $C^3$ surface with the induce metric $g$ and a normal field $\vec{n}$. Let $S \subset M$ be an open bounded set with a regular boundary $\partial S$. We consider a shell with thickness $h > 0$

$$\Omega = \{ x + t\vec{n}(x) \mid x \in S, \ -h < t < h \}.$$  

Let $\kappa$ be the Gaussian curvature of $M$. We say that $\Omega$ is parabolic if

$$\kappa(x) = 0, \quad |\Pi(x)| > 0 \quad \text{for} \quad x \in \overline{S}, \quad (1.2)$$

where $\Pi = \nabla \vec{n}$ is the second fundamental form of $M$. If

$$\kappa(x) > 0 \quad \text{for} \quad x \in \overline{S}, \quad (1.3)$$

then $\Omega$ is said to be elliptic.

Set

$$H^1_0(\Omega, \mathbb{R}^3) = \{ y \in H^1(\Omega, \mathbb{R}^3) \mid y|_{\Sigma_0} = 0 \},$$

where

$$\Sigma_0 = \{ x + t\vec{n}(x) \mid x \in \partial S, \ |t| \leq h \}.$$
Here it can happen that \( \partial S = \emptyset \), for example, to a closed elliptic shell, for which \( H^1_0(\Omega, \mathbb{R}^3) = H^1(\Omega, \mathbb{R}^3) \).

All the norm \( \| \cdot \| \) in this paper is that of \( L^2(\Omega) \), unless it is specified.

**Theorem 1.1** *(Korn’s interpolation inequalities)* There are \( C > 0, h_0 > 0 \), independent of \( h > 0 \), such that
\[
\| \nabla y \|^2 \leq C \left( \frac{1}{h} \| \langle y, \vec{n} \rangle \| \| \text{sym} \, \nabla y \| + \| y \|^2 + \| \text{sym} \, \nabla y \|^2 \right) \tag{1.4}
\]
for all \( h \in (0, h_0) \) and \( y \in H^1(\Omega, \mathbb{R}^3) \) with \( \langle y, \vec{n} \rangle |_{\Sigma_0} = 0 \) where
\[
\text{sym} \, \nabla y = \frac{1}{2} (\nabla y + \nabla^T y).
\]

We have the following.

**Theorem 1.2** Let \( \Omega \) be parabolic. There are \( C > 0, h_0 > 0 \), independent of \( h > 0 \), such that
\[
\| \nabla y \|^2 \leq \frac{C}{h^{3/2}} \| \text{sym} \, \nabla y \|^2, \tag{1.5}
\]
for all \( h \in (0, h_0) \) and \( y \in H^1_0(\Omega, \mathbb{R}^3) \).

**Theorem 1.3** Let \( \Omega \) be elliptic. There are \( C > 0, h_0 > 0 \), independent of \( h > 0 \), such that
\[
\| \nabla y \|^2 \leq \frac{C}{h} \| \text{sym} \, \nabla y \|^2 \tag{1.6}
\]
for all \( h \in (0, h_0) \) and \( y \in H^1_0(\Omega, \mathbb{R}^3) \).

In particular, we have

**Corollary 1.1** If \( \Omega \) is a closed elliptic shell, then there is \( C > 0 \) such that
\[
\min_{A \in \text{so(3)}} \| \nabla y - A \|^2 \leq \frac{C}{h} \| \text{sym} \, \nabla y \|^2 \tag{1.7}
\]
for any \( y \in H^1(\Omega, \mathbb{R}^3) \), where \( \text{so(3)} \) is the set of all \( 3 \times 3 \) skew matrices.

**Theorem 1.4** The exponentials of the thickness in (1.5) and (1.6)-(1.7) are optimal, respectively, for the parabolic shell and the elliptic shell, respectively.

**Remark 1.1** The interpolation inequality (1.4) is given in [10, 11, 14] under the assumption (1.1) and extended in [12] to the case that there is a local principal coordinate for each \( p \in S \). The inequalities (1.5) and (1.6) are given in [11, 12] and [14], respectively, under the assumption (1.1).
2 Proofs of Main Results

2.1 Proof Theorem 1.1

Let \((M, g)\) be a Riemannian manifold. Let \(T\) be a 2-order tensor field on \((M, g)\) and let \(X\) be a vector field on \((M, g)\). We define the inner multiplication of \(T\) with \(X\) to be another vector field, denoted by \(i(X)T\), given by

\[
\langle i(X)T, Y \rangle = T(X, Y) \quad \text{for} \quad Y \in M_p, \quad p \in M, \quad g = \langle \cdot, \cdot \rangle.
\]

For any \(y \in H^1(\Omega, \mathbb{R}^3)\), we decompose \(y\) into

\[
y(z) = W(x, t) + w(x, t)\tilde{n}(x) \quad \text{for} \quad z = x + t\tilde{n}(x) \in \Omega, \quad x \in S, \quad |t| < h,
\]

where \(u = \langle y, \tilde{n} \rangle\) and \(U(\cdot, t)\) is a vector field on \(S\) for \(|t| < h\). It follows from (2.1) that

\[
\nabla_{\alpha + t\nabla\tilde{n}_\alpha} y = D_{\alpha} W + w\nabla_{\alpha} \tilde{n} + [\alpha(w) - \Pi(W, \alpha)]\tilde{n} \quad \text{for} \quad \alpha \in S_x,
\]

\[
\nabla_{\tilde{n}} y = W_t(x, t) + w_t(x, t)\tilde{n}(x) \quad \text{for} \quad x \in S, \quad |t| < h,
\]

where \(\nabla\) and \(D\) are the covariant differentials of the dot metric in \(\mathbb{R}^3\) and of the induced metric in \(S\), respectively, and \(W_t = \partial_t W\) and \(w_t = \partial_t w\). We need to deal with the relations between \(\nabla\) and \(D\) carefully.

By defining \(\nabla\tilde{n}\tilde{n} = 0\), we introduce an 2-order tensor \(p(y)\) on \(\mathbb{R}^3_x\) by

\[
p(y)(\tilde{\alpha}, \tilde{\beta}) = \langle \nabla\nabla\tilde{n}_\tilde{\alpha} y, \tilde{\beta} \rangle \quad \text{for} \quad \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^3.
\]

We have

**Lemma 2.1** Let \(y \in H^2(\Omega, \mathbb{R}^3)\) be given in (2.1). Then

\[
|\nabla y + tp(y)|^2 = |DW + w\Pi|^2 + |Dw - i (W)\Pi|^2 + |W_t|^2 + w_t^2,
\]

\[
|\text{sym} \nabla y + t \text{sym} p(y)|^2 = |\Upsilon(y)|^2 + \frac{1}{2}|X(y)|^2 + w_t^2,
\]

where

\[
\Upsilon(y) = \text{sym} DW + w\Pi, \quad X(y) = Dw - i (W)\Pi + W_t.
\]

**Proof** Let \(x \in S\) be given. Let \(\tau_1, \tau_2\) be an orthonormal basis of \(S_x\). Then \(\tau_1, \tau_2,\) and \(\tilde{n}(x)\) forms an orthonormal basis of \(\mathbb{R}^3_x\). From (2.3) and (2.3), we have

\[
|\nabla y + tp(y)|^2 = \sum_{i,j=1}^{2} (\nabla_{\tau_i + t\nabla\tilde{n}_\tau_i} y, \tau_j)^2 + \sum_{i=1}^{2} (\langle \nabla_{\tau_i + t\nabla\tilde{n}_\tau_i} y, \tilde{n} \rangle)^2 + |\nabla_{\tilde{n}} y, \tilde{n} \rangle|^2
\]

\[
= |DW + w\Pi|^2 + |Dw - i (W)\Pi|^2 + |W_t|^2 + w_t^2,
\]

\[
|\text{sym} \nabla y + t \text{sym} p(y)|^2 = |\Upsilon(y)|^2 + \frac{1}{2}|X(y)|^2 + w_t^2,
\]

\[
where
\]

\[
\Upsilon(y) = \text{sym} DW + w\Pi, \quad X(y) = Dw - i (W)\Pi + W_t.
\]
| \text{sym } \nabla y + t \text{sym } p(y) |^2 = \sum_{i,j=1}^{2} \left[ \frac{1}{2} \left( \langle \nabla_{\tau_i + t \tau_j} y, \tau_j \rangle + \langle \nabla_{\tau_j + t \tau_i} y, \tau_i \rangle \right) \right]^2 \\
+ \sum_{i=1}^{2} \left[ \frac{1}{2} (\langle \nabla_{\tau_i} y, \hat{n} \rangle + \langle \nabla_{\hat{n}} y, \tau_i \rangle) \right]^2 + \langle \nabla_{\hat{n}} y, \hat{n} \rangle^2 \\
= |\text{sym } DW + w\Pi|^2 + \frac{1}{2} | Dw - i(W)\Pi + W_t|^2 + w_t^2. \\
\square

\textbf{Remark 2.1} \ Y(y) \text{ and } DX \text{ are called the strain tensor and the curvature tensor of the middle surface, respectively, see [15].}

\textbf{Lemma 2.2} \text{ Let } w \in H^2(\Omega) \text{ be a function. Then the following formulas hold true.}

(i) \ \Delta w = \Delta_g w + w_t \text{tr}_g \Pi + w_{tt};

(ii) \ \langle \nabla_{\hat{n}} \nabla w, \nabla w \rangle = \langle \nabla w_t, \nabla w \rangle - \Pi(Dw, Dw);

(iii) \ \Delta w_t = \hat{n}(\Delta w) + (\Delta \hat{n})(w) + 2\Pi(D^2 w) + 2w_t|\Pi|^2;

(iv) \ \text{div } [w_t \iota(Dw)\Pi] = w_t(\Pi, D^2 w) + \Pi(Dw, Dw) + w_tD(\text{tr}_g \Pi)(w), \text{ where } \Delta \text{ and } \text{div are the Laplacian and the divergence of the dot metric in } \mathbb{R}^3, \text{ respectively, and } \text{tr}_g \text{ is the trace of the induced metric } g \text{ in } S. \text{ Moreover, } \Delta \hat{n} \text{ is a vector field on } S.

\textbf{Proof} \text{ Let } x \in S \text{ be given. Let } E_1, E_2 \text{ be a frame field normal at } x \in S, \text{ i.e.,}

\langle E_i, E_j \rangle = \delta_{ij} \quad \text{in some neighbourhood of } x \text{ on } S, \quad (2.8)

\nabla_{E_i} \hat{n} = \lambda_i E_i, \quad D_{E_i} E_j = 0 \quad \text{at } x \text{ for } 1 \leq i, j \leq 2. \quad (2.9)

Then

\nabla_{E_i} E_j = D_{E_i} E_j - \Pi(E_i, E_j) \hat{n} \quad \text{in some neighbourhood of } x \text{ on } S, \quad (2.10)

\nabla_{E_i} E_j = -\lambda_i \delta_{ij} \hat{n} \quad \text{at } x, \quad \lambda_i = \Pi(E_i, E_i)(x). \quad (2.11)

We have

\Delta w = \nabla^2 w(E_1, E_1) + \nabla^2 w(E_2, E_2) + \nabla^2 w(\hat{n}, \hat{n}) = E_1 E_1(w) - \nabla_{E_1} E_1(w) \\
+ E_2 E_2(w) - \nabla_{E_2} E_2(w) + \hat{n}\hat{n}(w) - \nabla_{\hat{n}} \hat{n}(w) \\
= \Delta_g w + w_t \text{tr}_g \Pi + w_{tt}.

In addition, we obtain

\nabla_{\hat{n}} \nabla w = \sum_{i=1}^{2} \hat{n}(\langle \nabla w, E_i \rangle) E_i + \hat{n}(\langle \nabla w, \hat{n} \rangle) \hat{n} = \sum_{i=1}^{2} \nabla^2 w(E_i, \hat{n}) E_i + \nabla^2 w(\hat{n}, \hat{n}) \hat{n} \\
= \sum_{i=1}^{2} [E_i \hat{n}(w) - \nabla_{E_i} \hat{n}(w)] E_i + w_t \hat{n} = \nabla w_t - \sum_{i=1}^{2} \Pi(Dw, E_i) E_i,
which yields the formula in (ii).

Using the symmetry of $\nabla^3 w$ and the formulas (2.8)-(2.11), we have

$$\nabla^3 w(E_i, E_i, \bar{n}) = \nabla^3 w(\bar{n}, E_i, E_i) = E_i\nabla^2 w(\bar{n}, E_i) - \nabla^2 w(\nabla E_i \bar{n}, E_i) - \nabla^2 w(\bar{n}, \nabla E_i E_i)
= E_i\{E_i(w_t) - \langle \nabla w, \nabla E_i \bar{n} \rangle \} - \nabla^2 w(\nabla E_i \bar{n}, E_i) - \nabla^2 w(\bar{n}, \nabla E_i E_i)
= E_i E_i(w_t) - \nabla E_i E_i(w_t) + \nabla E_i E_i(\langle \nabla w, \bar{n} \rangle) - \langle \nabla w, \nabla E_i \nabla E_i \bar{n} \rangle
- 2\lambda_i \nabla^2 w(E_i, E_i) - \nabla^2 w(\bar{n}, \nabla E_i E_i)
= \nabla^2 w_t(E_i, E_i) - 2\Pi(E_i, E_i)[E_i E_i(w) + \lambda_i \bar{n}(w)] - \langle \nabla w, \nabla E_i \nabla E_i \bar{n} \rangle,$$

from which it follows that

$$\bar{n}(\Delta w) = \nabla_{\bar{n}} \left[ \sum_{i=1}^{2} \nabla^2 w(E_i, E_i) + \nabla^2 w(\bar{n}, \bar{n}) \right] = \sum_{i=1}^{2} \nabla^3 w(\bar{n}, E_i, E_i) + \nabla^2 w_t(\bar{n}, \bar{n})$$

$$= \Delta w_t - 2\langle \Pi, D^2 w \rangle - 2w_t \| \Pi \|^2 - \langle \nabla w, \Delta \bar{n} \rangle,$$

where the following formula has been used

$$\Delta \bar{n} = \sum_{i=1}^{2} (\nabla_{E_i} \nabla_{E_i} \bar{n} - \nabla_{\nabla_{E_i} E_i} \bar{n}) + \nabla_{\bar{n}} \nabla_{\bar{n}} \bar{n} - \nabla_{\nabla_{\bar{n}} \bar{n}} \bar{n} = \sum_{i=1}^{2} \nabla_{E_i} \nabla_{E_i} \bar{n} \quad \text{at} \quad z = x + t\bar{n}.$$

Finally, we have

$$\langle \Delta \bar{n}, \bar{n} \rangle = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i} \bar{n}, \bar{n} \rangle = 0 \quad \text{for} \quad z = x + t\bar{n}(x),$$

that is, $\Delta \bar{n} \in S_x$. \hfill \Box

**Lemma 2.3** Let $y \in H^2(\Omega, \mathbb{R}^3)$ be given in (2.1) and let $\mathcal{Y}(y)$ and $X(y)$ be given in (2.7). Then

$$\Delta w = \text{div} \ X(y) + \text{tr}_g i(W)D\Pi - [\text{tr}_g \mathcal{Y}(y)]_t + 2w_t \text{tr}_g \Pi + w_{tt}, \quad (2.12)$$

for $z = x + t\bar{n} \in \Omega$, where $\text{div}$ is the divergence of the dot metric in $\mathbb{R}^3$.

**Proof** Let $x \in S$ be given. Let $E_1$, $E_2$ be a frame field normal at $x$ in $S$ such that (2.8)-(2.11) hold. Then $E_1$, $E_2$, and $\bar{n}(x)$ forms an orthonormal frame at $z = x + t\bar{n}(x)$. Using (i) in Lemma 2.2 and

$$E_i \langle X, E_i \rangle = \langle \nabla_{E_i} X, E_i \rangle + \langle X, \nabla_{E_i} E_i \rangle = 0, \quad \langle \nabla_{\bar{n}} X, \bar{n} \rangle = 0,$$

we have

$$\Delta w(z) = E_1 \langle Dw, E_1 \rangle + E_2 \langle Dw, E_2 \rangle + w_t \text{tr}_g \Pi + w_{tt}
= E_1 \langle Dw - i(W)\Pi + W_t, E_1 \rangle + E_2 \langle Dw - i(W)\Pi + W_t, E_2 \rangle + \langle \nabla_{\bar{n}} X, \bar{n} \rangle
+ E_1 [\Pi(W, E_1) - \langle W_t, E_1 \rangle] + E_2 [\Pi(W, E_2) - \langle W_t, E_2 \rangle] + w_t \text{tr}_g \Pi + w_{tt}
= \text{div} \ X(y) + D\Pi(W, E_1) + D\Pi(W, E_2) + \Pi(D_{E_1} W, E_1) + \Pi(D_{E_2} W, E_2)
- DW_t(E_1, E_1) - DW_t(E_2, E_2) + w_t \text{tr}_g \Pi + w_{tt}
= \text{div} \ X(y) + \text{tr}_g i(W)D\Pi - [\text{tr} \mathcal{Y}(y)]_t + 2w_t \text{tr}_g \Pi + w_{tt},$$

6
where the following formulas have been used
\[ DW_t(E_i, E_i) = \bar{n}(DW(E_i, E_i)) + \Pi(D_{E_i}W, E_i) \quad \text{at} \quad z = x + t\bar{n}(x). \]

We need the following lemma from [13].

**Lemma 2.4** ([13]) Assume \( \lambda \in (0, 1] \), \( 0 \leq a < b \) and \( f : [a, b] \to \mathbb{R} \) is absolutely continuous. Then the inequality holds:
\[
\int_{a+\lambda(b-a)}^{b} f^2(t)dt \leq \frac{2}{\lambda} \int_{a}^{a+\lambda(b-a)} f^2(t)dt + 4 \int_{a}^{b} (b-t)^2 f'^2(t)dt. \tag{2.13}
\]

For \( f \in L^2(\Omega) \), we have
\[
\int_{\Omega} f^2(z)dz = \int_{-h}^{h} \int_{S} f^2(x + t\bar{n}(x))(1 + t \text{tr} \Pi + t^2 \kappa)dgdt,
\]
where \( \kappa \) is the Gaussian curvature. It follows that
\[
(1 - Ch) \int_{-h}^{h} \int_{S} f^2(x + t\bar{n})dgdt \leq \int_{\Omega} f^2(z)dz \leq (1 + Ch) \int_{-h}^{h} \int_{S} f^2(x + t\bar{n})dgdt. \tag{2.14}
\]

In the sequel, we sometimes use the norm
\[
\|f\|^2 = \int_{-h}^{h} \int_{S} f^2dgdt \quad \text{for} \quad h \quad \text{small} \tag{2.15}
\]

instead of the norm
\[
\|f\|^2 = \int_{\Omega} f^2dz.
\]

The next lemma is the key to our analysis that is the 3-dimensional version of [12, Lemma 4.5]. In the 2-dimensional case [12, Lemma 4.5] establishes the inequality (2.17) without the assumption (2.16) below.

**Lemma 2.5** There is a constant \( C > 0 \), independent of \( h > 0 \), such that any harmonic function \( w \in C^1(\Omega) \) with
\[
w|_{\Sigma_0} = 0 \tag{2.16}
\]
fulfills the inequality
\[
\|Dw\|^2 \leq C(\frac{1}{h}\|w\|w_t\| + \|w_t\|^2). \tag{2.17}
\]

**Proof** Using (2.16) and (2.14), we have
\[
\int_{\Omega(t)} |\nabla w|^2dgdt \leq C \int_{\Omega(t)} |\nabla w|^2dz = C \int_{\Omega(t)} \text{div} w\nabla wdz - C \int_{\Omega(t)} w\Delta wdz
\]
\[
= C(\int_{\Sigma_+(t)} w\nabla w_t d\Sigma_+ - \int_{\Sigma_-(t)} w\nabla w_t d\Sigma_-)
\]
\[
\leq C \int_{S} (|w(x + t\bar{n})w_t(x + t\bar{n})| + |w(x - t\bar{n})w_t(x - t\bar{n})|)dg, \tag{2.18}
\]
for $t \in (0, h)$, where
\[
\Omega(t) = S \times (-t, t), \quad \hat{\Omega}(t) = \{ x + s\bar{n} \mid x \in S, |s| < t \}, \quad \Sigma_{\pm}(t) = S \times \{ \pm t \}.
\]

We integrate (2.18) in $t$ over $(h/2, h)$ to obtain, by (2.14),
\[
\int_{-h/2}^{h/2} \int_S |\nabla w|^2 dgdt = \int_{\Omega(h/2)} |\nabla w|^2 dgdt \leq \frac{C}{h} \int_{-h}^{h} \int_S |w_t|dgdt. \tag{2.19}
\]

Let
\[
f(t) = |\nabla w| \quad \text{for} \quad t \in (0, h).
\]

Using (ii) in Lemma 2.2, we have
\[
|f'(t)| \leq \frac{1}{f(t)} |(\langle \nabla w_t, \nabla w \rangle - \Pi(Dw, Dw))| \leq |\nabla w_t| + C|\nabla w|.
\]

Applying $f$ to Lemma 2.4 with $\lambda = 1/2, a = 0, \text{ and } b = h$, we obtain
\[
\int_{h/2}^{h} |\nabla w|^2 dt \leq 4 \int_{0}^{h/2} |\nabla w|^2 dt + 4 \int_{0}^{h} (h - t)^2 |\nabla w_t|^2 dt + Ch^2 \int_{0}^{h} |\nabla w|^2 dt.
\]

Integrating the above inequality in $x$ over $S$ yields, by (2.19),
\[
\int_{h/2}^{h} \int_S |\nabla w|^2 dgdt \leq C(\frac{1}{h} \int_{\Omega} |w_t|dgdt + \int_{S \times (-h, h)} \rho^2(t)|\nabla w_t|^2 dgdt + h^2 \int_{-h}^{h} |\nabla w|^2 dt), \tag{2.20}
\]

where
\[
\rho(t) = h - t \quad \text{for} \quad t \in (0, h); \quad \rho(t) = h + t \quad \text{for} \quad t \in (-h, 0).
\]

It follows from (iii) in Lemma 2.2 that
\[
\rho^2 w_t|2(\Pi, D^2 w) + 2w_t|\Pi|^2| = \rho^2 w_t\Delta w_t = \text{div} (\rho^2 w_t \nabla w_t) - \rho^2 |\nabla w_t|^2 - 2\rho \rho' w_t w_{tt},
\]

from which we obtain, by (vi) in Lemma 2.2,
\[
\rho^2 |\nabla w_t|^2 = \text{div} (\rho^2 w_t \nabla w_t) - 2\rho \rho' w_t w_{tt} - 2\rho^2 w_t^2 |\Pi|^2
\]
\[
- 2\rho \{ \text{div} [w_t i(Dw)\Pi] - \Pi(Dw_t, Dw_t) - w_t D(\text{tr}_g \Pi)(w) \}
\]
\[
= \text{div} \{ \rho^2 w_t |\nabla w_t - 2i(Dw)\Pi| \} - 2\rho \rho' w_t w_{tt}
\]
\[
+ 2\rho^2 [\Pi(Dw, Dw_t) + w_t D(\text{tr}_g \Pi)(w) - w_t^2 |\Pi|^2].
\]

Thus we have
\[
\int_{S \times (-h, h)} \rho^2(t)|\nabla w_t|^2 dgdt \leq C \int_{\Omega} \rho^2(t)|\nabla w_t|^2 dz
\]
\[
= C \int_{\Omega} \{ -2\rho \rho' w_t w_{tt} + 2\rho^2 [\Pi(Dw, Dw_t) + w_t D(\text{tr}_g \Pi)(w) - w_t^2 |\Pi|^2] \} dz
\]
\[
\leq C (\|\rho \nabla w_t\|\|w_t\| + \rho \|\nabla w_t\| |\nabla w| + h^2 |\nabla w||w_t| + h^2 |w_t|^2),
\]

8
which yield
\[ \| \rho \nabla w_t \|^2 \leq C(\| w_t \|^2 + h^2 \| \nabla w \|^2). \tag{2.21} \]

Combining (2.20) and (2.21), we have
\[ \int_{h/2}^{h} \int_S |\nabla w|^2 dgdt \leq C(\frac{1}{h} \| w_t \| + \| w_t \|^2 + h^2 \| \nabla w \|^2). \]

A similar argument yields
\[ \int_{-h/2}^{-h} \int_S |\nabla w|^2 dgdt \leq C(\frac{1}{h} \| w_t \| + \| w_t \|^2 + h^2 \| \nabla w \|^2). \]

Thus (2.17) follows. \qed

**Proof of Theorem 1.1** Let
\[ I(y) = \nabla y + tp(y), \tag{2.22} \]

where \( p(y) \) is given in (2.4).

**Step 1** Let \( \hat{w} \) be the solution to problem
\[
\begin{aligned}
\Delta \hat{w} &= 0 \quad \text{in} \quad \Omega, \\
\hat{w} &= w \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

Then
\[ \| w - \hat{w} \| \leq C h \| \nabla (w - \hat{w}) \|. \tag{2.23} \]

\[ \Delta w = \text{div} \ X(y) + \text{tr}_g i (W) \text{D} \Pi - \text{tr}_g \text{Y}(y) \text{tr}_g \Pi + 2w_t \text{tr}_g \Pi + w_{tt} \]

It follows from (2.12) that
\[
\begin{aligned}
|\nabla (w - \hat{w})|^2 &= \text{div} [(w - \hat{w}) \nabla (w - \hat{w})] - (w - \hat{w}) \Delta w \\
&= \text{div} [(w - \hat{w}) \nabla (w - \hat{w})] - \text{div} [(w - \hat{w}) X] + \langle D(w - \hat{w}), X \rangle \\
&\quad + \{ (w - \hat{w}) \text{tr}_g \text{Y}(y) - 2w \text{tr}_g \Pi \}_t - (w - \hat{w}) \text{tr}_g \Pi - (w - \hat{w}) \text{tr}_g \text{Y}(y) - 2w \text{tr}_g \Pi \\
&\quad - [(w - \hat{w}) w_t]_t + (w - \hat{w})_t w_t - (w - \hat{w}) \text{tr}_g \text{i} (W) \text{D} \Pi.
\end{aligned}
\]

We integrate the above identity over \( \Omega \) in \( z = x + t \hat{n} \) to have, by (2.6),
\[
\begin{aligned}
|\nabla (w - \hat{w})|^2 &\leq |\nabla (w - \hat{w})| \| \| X(y) \| \| + \| \text{tr}_g \text{Y}(y) - 2w \text{tr}_g \Pi \| + \| w_t \| \| + \| w_t \| \| y \| \| + C \| w - \hat{w} \| \| y \| \| \\
&\leq C \| \nabla (w - \hat{w}) \| (\| \text{sym} I(y) \| + h \| y \|),
\end{aligned}
\]

that is,
\[ \| \nabla (w - \hat{w}) \| \leq C(\| \text{sym} I(y) \| + h \| y \|). \tag{2.24} \]
Using (2.24), (2.17), (2.23) and (2.6), we obtain
\[
\|Dw\|_2^2 \leq C\|\nabla (w - \hat{w})\|_2^2 + C\|D\hat{w}\|_2^2
\]
\[
\leq C\|\nabla (w - \hat{w})\|_2^2 + C\frac{1}{h}\|\hat{w} - w\|\|w\| + C\|\hat{w}_t - w_t\| + C\|w_t\|_2
\]
\[
\leq C\|\nabla (w - \hat{w})\|_2^2 + C\frac{1}{h}\|w\|\|\text{sym} I(y)\| + h\|y\| + C\|w_t\|_2
\]
\[
\leq C(\frac{1}{h}\|\langle y, \vec{n} \rangle\|\|\text{sym} I(y)\| + \|y\|^2 + \|\text{sym} I(y)\|^2).
\] (2.25)

Thus we have
\[
\|W_t\|_2^2 \leq C\|Dw - i(W)\Pi + W_t\|_2^2 + C\|Dw\|^2 + C\|W\|^2
\]
\[
\leq C(\frac{1}{h}\|\langle y, \vec{n} \rangle\|\|\text{sym} I(y)\| + \|y\|^2 + \|\text{sym} I(y)\|^2).\] (2.26)

From [2, Theorem 1.1], there is a constant \(C > 0\) such that
\[
\int_S |DW|^2 dg \leq C \int_S (|\text{sym} DW|^2 + |W|^2) dg.
\] (2.27)

It follows from (2.25)–(2.27) and (2.5)–(2.6) that
\[
\|I(y)\|^2 \leq C(\frac{1}{h}\|\langle y, \vec{n} \rangle\|\|\text{sym} I(y)\| + \|y\|^2 + \|\text{sym} I(y)\|^2).
\] (2.28)

**Step 2** From (2.4), we have
\[
|p(y)| \leq C|\nabla y| \quad \text{for} \quad z = x + t\vec{n} \in \Omega.
\]
Then
\[
(1 - Ch)\|\nabla y\| \leq \|I(y)\| \leq (1 + Ch)\|\nabla y\|,
\]
\[
\|\text{sym} \nabla y\| - Ch\|\nabla y\| \leq \|\text{sym} I(y)\| \leq \|\text{sym} \nabla y\| + Ch\|\nabla y\|.
\]

Thus the inequality (1.4) follows from (2.28). \(\square\)

### 2.2 Proofs Theorems 1.2 and 1.3

Let \(\mathcal{X}(S)\) be the set of all vector fields on \(S\). For any \(X, Y \in \mathcal{X}(S)\), the curvature operator \(R_{XY}\) is defined by
\[
R_{XY} = -D_X D_Y + D_Y D_X + D_{[X,Y]},
\]
where \([\cdot, \cdot]\) is the Lie product. The Ricci identity reads
\[
D^2 T(\cdots, X, Y) = D^2 T(\cdots, Y, X) + R_{XY}(T)(\cdots),
\] (2.29)
where \(T\) is a k-order tensor field. This formula can help us to exchange the order of the second-order covariant differential of a k-order tensor field.
Let \( x \in S \) be given and let \( e_1, e_2 \) be an orthonormal basis of \( M_x \) with the positive orientation in the induced metric \( g \). For any \( W \in H^1(S, \mathcal{X}(S)) \), we denote a 2-form \( \sigma(W) \) on \( S \) by
\[
\sigma(W) = D_{e_1}W \wedge_g D_{e_2}W \text{ at } x,
\]
where \( \wedge_g \) is the exterior product of the induced metric \( g \) on \( S \). Then \( \sigma(W) \) is well defined. In fact, let \( \hat{e}_1, \hat{e}_2 \) be another orthonormal basis with the positive orientation. Suppose that
\[
e_1 = \alpha_{11} \hat{e}_1 + \alpha_{12} \hat{e}_2, \quad e_2 = \alpha_{21} \hat{e}_1 + \alpha_{22} \hat{e}_2.
\]
Then
\[
(\alpha_{ij})(\alpha_{ij})^T = I, \quad \det(\alpha_{ij}) = 1
\]
where \( I \) is the identity matrix in \( \mathbb{R}^2 \). It follows that
\[
D_{e_1}W \wedge D_{e_2}W = (\alpha_{11}D_{\hat{e}_1}W + \alpha_{12}D_{\hat{e}_2}W) \wedge (\alpha_{21}D_{\hat{e}_1}W + \alpha_{22}D_{\hat{e}_2}W) \\
= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})D_{\hat{e}_1}W \wedge D_{\hat{e}_2}W = D_{\hat{e}_1}W \wedge D_{\hat{e}_2}W.
\]
Then there is a function \( \varphi \) on \( S \), independent of the choice of orthonormal base, such that
\[
\sigma(W) = \varphi(x)\mathcal{E} \quad \text{for } x \in S, \tag{2.30}
\]
where \( \mathcal{E} \) is the volume element of the induced metric \( g \).

**Lemma 2.6** For any \( W \in H^1(S, \mathcal{X}(S)) \), we have
\[
2 \int_S \varphi dg = \int_S \kappa |W|^2 dg + \int_{\partial S} [2\langle W, \mu \rangle \tau \langle W, \tau \rangle + \langle D_\tau \mu, \tau \rangle |W|^2] d\partial S, \tag{2.31}
\]
where \( \varphi \) is given in (2.30) and \( \mu \) and \( \tau \) are the outside normal and the tangential along the boundary \( \partial S \) in the induced metric \( g \), respectively.

**Proof** For \( W \) given, we denote a vector field \( B(W) \) on \( S \) by
\[
B(W) = (W \wedge \imath (e_2)D^TW)(e_1, e_2)e_1 - (W \wedge \imath (e_1)D^TW)(e_1, e_2)e_2 \quad \text{for } x \in S, \tag{2.32}
\]
where \( e_1, e_2 \) is an orthonormal basis of \( M_x \) and \( D^TW \) is the transpose of \( DW \). It is easy to check that the definition of \( B(W) \) is independent of the choice of \( e_1, e_2 \).

Since \( D_\tau \mu = \langle D_\tau \mu, \tau \rangle \tau \) and \( D_\tau \tau = -\langle D_\tau \mu, \tau \rangle \mu \) on the boundary \( \partial S \), we have
\[
\langle B(W), \mu \rangle = (W \wedge \imath (\tau)D^TW)(\mu, \tau) = \langle W, \mu \rangle \langle D_\tau W, \tau \rangle - \langle W, \tau \rangle \langle D_\tau W, \mu \rangle \\
= \langle W, \mu \rangle \tau \langle W, \tau \rangle - \langle W, \tau \rangle \tau \langle W, \mu \rangle - \langle W, \mu \rangle \langle W, D_\tau \tau \rangle + \langle W, \tau \rangle \langle W, D_\tau \mu \rangle \\
= 2\langle W, \mu \rangle \tau \langle W, \tau \rangle - \tau (\langle W, \mu \rangle \langle W, \tau \rangle) + |W|^2 \langle D_\tau \mu, \tau \rangle. \tag{2.33}
\]
Let \( x \in S \) be given. Let \( E_1, E_2 \) be a frame field normal at \( x \) with the positive orientation. Then

\[
D_{E_i}E_j = 0 \quad \text{at} \quad x \quad \text{for} \quad 1 \leq i, j \leq 2.
\]

It follows (2.30), (2.29), and (2.32) that

\[
\varphi(x) = \sigma(W)(E_1, E_2) = \langle D_{E_1}W, E_1 \rangle \langle D_{E_2}W, E_2 \rangle - \langle D_{E_2}W, E_1 \rangle \langle D_{E_1}W, E_2 \rangle
\]

\[
= E_1(\langle W, E_1 \rangle \langle D_{E_2}W, E_2 \rangle) - \langle U, E_1 \rangle D^2W(E_2, E_2, E_1)
- E_2(\langle W, E_1 \rangle \langle D_{E_1}W, E_2 \rangle) + \langle W, E_1 \rangle D^2W(E_2, E_1, E_2)
\]

\[
= E_1(\langle W, E_1 \rangle \langle D_{E_2}W, E_2 \rangle - \langle W, E_2 \rangle \langle D_{E_2}W, E_1 \rangle) + E_1(\langle W, E_2 \rangle \langle D_{E_2}W, E_1 \rangle)
+ E_2(-\langle W, E_1 \rangle \langle D_{E_1}W, E_2 \rangle + \langle W, E_2 \rangle \langle D_{E_1}W, E_1 \rangle) - E_2(\langle W, E_2 \rangle \langle D_{E_1}W, E_1 \rangle) + \kappa \langle W, E_1 \rangle^2
\]

\[
= \operatorname{div} g B(W) + \langle D_{E_1}W, E_2 \rangle \langle D_{E_2}W, E_1 \rangle - \langle D_{E_2}W, E_2 \rangle \langle D_{E_1}W, E_1 \rangle
+ \langle W, E_2 \rangle \langle D^2W(E_1, E_2, E_1) - D^2W(E_1, E_1, E_2) \rangle + \kappa \langle W, E_1 \rangle^2
\]

\[
= \operatorname{div} g B(W) - \varphi(x) + \kappa |W|^2. \tag{2.35}
\]

Thus (2.31) follows from (2.35) and (2.33).

In the sequel, for a vector field \( W \in \mathcal{X}(S) \), we denote

\[
W_i = \langle W, E_i \rangle, \quad W_{ij} = DW(E_i, E_j) = \langle D_{E_j}W, E_i \rangle \quad \text{for} \quad 1 \leq i, j \leq 2,
\]

where \( E_1, E_2 \) is an orthonormal frame on \( S \). From (2.34), we have

\[
\varphi(x) = W_{11}W_{22} - W_{12}W_{21}, \tag{2.36}
\]

where \( \varphi \) is given in (2.30). Moreover, if \( f \) is a function, we denote

\[
W(f) = \langle W, Df \rangle.
\]

We need the following.

**Lemma 2.7** Let \( M \) be of \( C^3 \). Let \( \lambda(q) \) be a principal curvature for each \( q \in M \). Let \( p \in M \) be given. Suppose that there is a neighbourhood \( \mathcal{N} \) of \( p \) such that the following assumptions hold.

(i) \( \lambda \in C^1(\mathcal{N}) \);

(ii) the algebraic multiplicity of \( \lambda(q) \) = the geometric multiplicity = 1 for all \( q \in \mathcal{N} \).

Then there exists locally a \( C^1 \) vector field \( X \) such that

\[
\nabla_X \tilde{n} = \lambda X \quad \text{in a neighbourhood of} \ p.
\]
Proof Let \( \psi : \mathcal{N} \to \mathbb{R}^2 \) be a local coordinate at \( p \) with \( \psi(q) = (x_1, x_2) \) and \( \psi(p) = 0 \). Consider the matrices

\[
A(x) = \left( a_{ij}(x) \right), \quad \nabla_{\partial x_i} \vec{n} = a_{1i}(x) \partial x_1 + a_{2i}(x) \partial x_2 \quad \text{for} \quad 1 \leq i, j \leq 2.
\]

From (ii)

\[
\text{rank} \left( \lambda(x) \delta_{ij} - a_{ij}(x) \right) = 1 \quad \text{for} \quad x \quad \text{in a neighbourhood of} \quad 0.
\]

We may assume that

\[
\left( \lambda(0) - a_{11}(0), -a_{12}(0) \right) \neq 0.
\]

Thus

\[
\left( \lambda(x) - a_{11}(x), -a_{12}(x) \right) \neq 0 \quad \text{for} \quad x \quad \text{in a neighbourhood of} \quad 0.
\]

Let

\[
X = a_{12}(x) \partial x_1 + [\lambda(x) - a_{11}(x)] \partial x_2.
\]

Obviously, the above \( X \) meets our need. \( \square \)

For each \( p \in M \), we denote by \( Q : M_p \to M_p \) the rotation by \( \pi/2 \) along the clockwise direction, which is very useful in the case of the negative curvature, see [34]. For any \( \alpha \in M_p, \alpha, Q\alpha \) forms an orthonormal basis on \( M_p \).

Proposition 2.1 Let \( p \in M \) be given. Suppose that there are two different principal curvatures, \( \lambda_1 \neq \lambda_2 \), at \( p \). Then there exists a local principal coordinate \( \psi = x \) around \( p \), i.e.,

\[
\nabla_{\partial x_i} \vec{n} = \lambda_i \partial x_i \quad \text{in a neighbourhood of} \quad p \quad \text{for} \quad i = 1, 2.
\]

Proof From Lemma 2.7 there is a vector field \( X \) with \( |X| = 1 \) such that

\[
\nabla_X \vec{n} = \lambda_1 X \quad \text{in a neighbourhood of} \quad p. \quad (2.37)
\]

Let \( Y = QX \). Then \( X, QX \) forms an orthonormal basis. Thus

\[
\nabla_Y \vec{n} = \lambda_2 Y \quad \text{in a neighbourhood of} \quad p. \quad (2.38)
\]

We claim there exist functions \( f_1 \) and \( f_2 \) such that

\[
[f_1 X, f_2 Y] = 0. \quad (2.39)
\]

We define a curve by

\[
\alpha'(t) = X(\alpha(t)) \quad \text{for} \quad t \in (-\varepsilon, \varepsilon), \quad \alpha(0) = p.
\]

13
Then for \( t \in (-\varepsilon, \varepsilon) \) given, we solve problem
\[
\beta_s(t, s) = Y(\beta(t, s)) \quad \text{for} \quad s \in (-\varepsilon_1, \varepsilon_1), \quad \beta(t, 0) = \alpha(t). \tag{2.40}
\]

Since
\[
\det \left( \beta_t(0, 0), \beta_s(0, 0) \right) = \det \left( X(p), Y(p) \right) = \pm 1 \neq 0,
\]
the map \( \psi(\beta(t, s)) = (t, s) \) forms a local coordinate at \( p \) with (2.40) true. We let
\[
f_1(\beta(t, s)) = e^{\int_{-\varepsilon}^s \langle D_X Y, X(\beta(t, s)) \rangle \, ds} \quad \text{for} \quad (t, s) \in (-\varepsilon, \varepsilon) \times (-\varepsilon_1, \varepsilon_1).
\]

Then \( f_1 \) satisfies
\[
Y(f_1) = f_1 \langle D_X Y, X \rangle. \tag{2.41}
\]

Similarly, there is a function \( f_2 \) such that
\[
X(f_2) = f_2 \langle D_Y X, Y \rangle. \tag{2.42}
\]

(2.39) follows from (2.41) and (2.42).

Next, we define a curve by
\[
\varsigma'(t) = f_1(\varsigma(t))X(\varsigma(t)), \quad \varsigma(0) = p.
\]

Then define
\[
\eta_s(t, s) = f_2(\eta(t, s))Y(\eta(t, s)), \quad \eta(t, 0) = \varsigma(t).
\]

Then (2.39) implies that \( \hat{\psi}(\eta(t, s)) = (t, s) \) is a local coordinate such that
\[
\partial t = f_1 X, \quad \partial s = f_2 Y.
\]

Next, we consider a rigidity lemma on the strain tensor of the middle surface. In the case of the parabolic or the hyperbolic, it has established in [10]-[14] when the middle surface is given by a single principal coordinate. In the case of the elliptic shell, it has been given in [3] if the middle surface consists of a single coordinate. Here we treat it coordinates free, which particularly includes the case of the closed elliptic shells.

**Proposition 2.2** Suppose \( \Omega \) is a parabolic shell. Then there is \( C > 0 \) such that
\[
\|W\|_{L^2(S)}^2 \leq C \|Y(y)\|_{L^2(S)}(\|Y(y)\|_{L^2(S)} + \|w\|_{L^2(S)}) \tag{2.43}
\]
for any \( y = W + w\mathbf{n} \in H^1_0(S, \mathbb{R}^3) \).
Proof Let \( \hat{S} \) be a bounded open region on \( M \) such that
\[
\overline{S} \subset \hat{S}; \quad \kappa(x) = 0, \quad \nabla \vec{n} \neq 0 \quad \text{for} \quad x \in \hat{S}. \tag{2.44}
\]
For \( y \in H^1_0(S, \mathbb{R}^3) \), we extend \( y \in H^1_0(\hat{S}, \mathbb{R}^3) \) by
\[
y = 0 \quad \text{for} \quad x \in \hat{S}/S.
\]
In the above sense, we have
\[
H^1_0(S, \mathbb{R}^3) \subset H^1_0(\hat{S}, \mathbb{R}^3). \tag{2.43}
\]
Thus (2.43) follows from Lemma 2.8 below. \( \square \)

Lemma 2.8 Let \( \hat{S} \subset M \) be such that (2.44) hold. Let \( p \in \hat{S} \) be given and \( \gamma > 0 \) be given small. Then exist a neighbourhood \( \mathcal{N} \) of \( p \) and constants \( C > 0 \), independent of \( \gamma \), and \( C \gamma > 0 \), such that
\[
\|W\|_{L^2(\mathcal{N})}^2 \leq C \gamma \|W\|_{L^2(\hat{S})}^2 + C \gamma \|\Upsilon(y)\|_{L^2(\hat{S})}(\|\Upsilon(y)\|_{L^2(\hat{S})} + \|w\|_{L^2(\hat{S})}) \tag{2.45}
\]
for any \( y = W + w\vec{n} \in H^1_0(\hat{S}, \mathbb{R}^3) \).

Proof From Lemma 2.7 there is a vector field \( X \) with \( |X| = 1 \) such that (2.37) and (2.38) hold for \( x \) in a neighbourhood of \( \gamma \), where \( \lambda_1 = \text{tr}_g \Pi, \lambda_2 = 0 \), and \( Y = QX \). It follows from (2.38) that
\[
\nabla_Y X = D_Y X = a Y, \quad \nabla_Y Y = D_Y Y = \langle D_Y Y, X \rangle X = -a X, \tag{2.46}
\]
where \( a = \langle D_Y X, Y \rangle \).

Let \( \alpha(\cdot) : (-\varepsilon, \varepsilon) \to \hat{S} \) be the curve with
\[
\alpha(0) = x, \quad \alpha'(t) = X(\alpha(t)) \quad \text{for} \quad t \in (-\varepsilon, \varepsilon).
\]
Then we define \( \beta : (-\varepsilon, \varepsilon) \times (-\varepsilon_1, \varepsilon_1) \) by
\[
\beta(t, s) = Y(\beta(t, s)) \quad \text{for} \quad (t, s) \in (-\varepsilon, \varepsilon) \times (-\varepsilon_1, \varepsilon_1); \quad \beta(t, 0) = \alpha(t) \quad \text{for} \quad t \in (-\varepsilon, \varepsilon).
\]
Since
\[
\det \left( \beta_t(0, 0), \beta_s(0, 0) \right) = \det \left( X(p), QX(p) \right) = \pm 1 \neq 0,
\]
the map \( \psi(\beta(t, s)) = (t, s) \) forms a coordinate at \( p \). We set
\[
\mathcal{N} = \{ b(t, s) \mid (t, s) \in (-\varepsilon, \varepsilon) \times (-\varepsilon_1, \varepsilon_1) \},
\]
where \( \varepsilon > 0 \) and \( \varepsilon_1 > 0 \) are small enough.

Step 1 We claim that, for each \( t \in (-\varepsilon, \varepsilon) \) fixed,
(1) the curve $\beta(t, \cdot)$ has no self-intersection point for $s \in (-\varepsilon, \varepsilon_1)$;
(2) the vector fields $X$ and $Y$ and the curve $\beta(t, \cdot)$ can be simultaneously extended to outside of $\hat{S}$ from both directions, i.e., there are $s_-(t) < 0$ and $s_+(t) > 0$ satisfying

$$\beta(t, s_\pm(t)) \in \partial\hat{S};$$

For convenience, we denote $\beta(s) = \beta(t, s)$. Let

$$\beta(s) = \beta_1(s)X + \beta_2(s)Y + b_3(s)\bar{n} \quad \text{for} \quad s \in (-\varepsilon_1, \varepsilon_1).$$

Using (2.38) and (2.46), we have

$$\beta'(s) = \beta'_1(s)X + \beta'_2(s)Y + \beta'_3(s)\bar{n} + \beta_1(s)\nabla_Y X + \beta_2(s)\nabla_Y Y + \beta_3(s)\nabla_Y \bar{n},$$

which yields, since $\beta'(s) = Y$,

$$\beta'_1(s) - a\beta_2(s) = 0, \quad \beta'_2(s) + a\beta_1(s) = 1, \quad \beta'_3(s) = 0. \quad (2.47)$$

On the other hand, using the formula

$$\nabla_X \nabla_Y \bar{n} = \nabla_Y \nabla_X \bar{n} + \nabla_{[X,Y]} \bar{n},$$

and from (2.37) and (2.38), we obtain

$$[Y(\lambda_1) + \lambda_1([X, Y], X)]X + \lambda_1 aY = 0,$$

that is, $a = 0$, since $\lambda_1 \neq 0$. It follows from (2.47) that

$$\beta(s) = \beta_1(0)X + [\beta_2(0) + s]Y + \beta_3(0)\bar{n} \quad \text{for} \quad s \in (-\varepsilon_1, \varepsilon_1),$$

which proves (1) and (2) by Lemma 2.7.

**Step 2** Let $\varphi$ be given in (2.30). From (2.37), (2.38), and (2.36), we have

$$|\Upsilon(y)|^2 = [D_W(X, X) + \lambda w]^2 + \frac{1}{2}[D_W(X, Y) + D_Y(X, Y)]^2 + [D_W(Y, Y)]^2$$

$$\geq \frac{1}{2}\{|D_W(X, Y)|^2 + [D_W(Y, X)]^2\} - \varphi + D_W(X, X)D_W(Y, Y)$$

$$= \frac{1}{2}\{|D_W(X, Y)|^2 + [D_W(Y, X)]^2\} - \varphi + [\Upsilon(y)(X, X) - \lambda w]\Upsilon(y)(Y, Y),$$

that is,

$$[D_W(X, Y)]^2 + [D_W(Y, X)]^2 \leq C|\Upsilon(y)|(|\Upsilon(y)| + |w|) + 2\varphi. \quad (2.48)$$

**Step 3** For $t \in (-\varepsilon, \varepsilon)$ given, from Step 1, we have

$$|W|^2 = 2\int_{s_-(t)}^{s_+(t)} |D_YW, W| ds = 2\int_{s_-(t)}^{s_+(t)} [\langle W, X \rangle D_W(X, Y) + \langle W, Y \rangle D_W(Y, Y)] ds$$

$$\leq \gamma \int_{s_-(t)}^{s_+(t)} |W|^2 ds + C\gamma \int_{s_-(t)}^{s_+(t)} \{|D_W(X, Y)|^2 + |\Upsilon(y)|^2\} ds,$$
for $\gamma > 0$ small. We integrate the above inequality in $(t, s)$ over $(-\epsilon, \epsilon) \times (-\epsilon_1, \epsilon_1)$ to have, by (2.48) and (2.31),

\[
\int_N |W|^2 dg \leq \gamma \|W\|^2_{L^2(\hat{S})} + C \gamma \int_{-\epsilon}^{\epsilon} \int_{s_{-1}(t)}^{s_1(t)} \{|D\phi|^2 + |\phi|^2\} ds dt
\]

\[
\leq \gamma C \|W\|^2_{L^2(\hat{S})} + C \gamma \int_{-\epsilon}^{\epsilon} \int_{s_{-1}(t)}^{s_1(t)} \{|\phi|^2 + |\phi|^2\} ds dt
\]

\[
\leq \gamma C \|W\|^2_{L^2(\hat{S})} + C \gamma \int_{S} \{|\phi|^2 + |\phi|^2\} ds
dg
\]

The proof is complete. 

\[\square\]

**Proposition 2.3** Let $S$ be elliptic. Then there is $C > 0$ such that

\[
\|D\phi\|^2_{L^2(S)} + \|w\|^2_{L^2(S)} \leq C \|\phi\|^2_{L^2(S)} + C \int_S \phi dg
\]

for any $y = W + w\hat{n} \in H^1(S, \mathbb{R}^3)$. 

It follows from Proposition 2.3 immediately that

**Corollary 2.1** Let $S$ be elliptic.

(i) If $|\partial S| > 0$, then there is $C > 0$ such that

\[
\|y\|^2_{L^2(S)} \leq C \|\phi\|^2_{L^2(S)}
\]

for any $y = W + w\hat{n} \in H^1(S, \mathbb{R}^3)$.

(ii) If $S$ is a closed surface, then there is $C > 0$ such that, for any $y = W + w\hat{n} \in H^1(S, \mathbb{R}^3)$, there exists an infinitesimal identity $y_0 \in H^1(S, \mathbb{R}^3)$, satisfying

\[
\|y - y_0\|^2_{L^2(S)} \leq C \|\phi\|^2_{L^2(S)}.
\]

**Proof of Proposition 2.3** Let $p \in S$ be given. Let $e_1, e_2$ be an orthonormal basis of $M_p$ with the positive orientation such that

\[
\nabla_{e_i} \hat{n} = \lambda_i e_i \quad \text{at} \quad p \quad \text{for} \quad 1 \leq i, j \leq 2.
\]

Let $E_1, E_2$ be a frame field normal at $p$ such that

\[
E_1(p) = e_1, \quad E_2(p) = e_2.
\]

Then

\[
\langle E_i, E_j \rangle = \delta_{ij} \quad \text{a neighbourhood of} \quad p.
\]
Using the above formulas, we compute at $p$, for $\varepsilon > 0$ and $\varsigma > 0$ small,

$$|\Upsilon(y)|^2 = |W_{11} + \lambda_1 w|^2 + \frac{1}{2} |W_{12} + W_{21}|^2 + |W_{22} + \lambda_2 w|^2$$

$$= W_{11}^2 + W_{22}^2 + \frac{1}{2} |W_{12} + W_{21}|^2 + 2(\lambda_1 W_{11} + \lambda_2 W_{22}) w + |\Pi|^2 w^2$$

$$\geq W_{11}^2 + W_{22}^2 + \frac{1}{2} |W_{12} + W_{21}|^2 - 2 \frac{1}{|\Pi|^2 - \varepsilon} (\lambda_1 W_{11} + \lambda_2 W_{22})^2 + \varsigma w^2$$

$$= \frac{1}{|\Pi|^2 - \varepsilon} [(\lambda_2^2 - \varepsilon) W_{11}^2 - 2\kappa W_{11} W_{22} + (\lambda_1^2 - \varepsilon) W_{22}^2] + \frac{1}{2} |W_{12} + W_{21}|^2 + \varsigma w^2$$

$$\geq \frac{1}{|\Pi|^2 - \varepsilon} \{ (\lambda_2^2 - \varepsilon) W_{11}^2 - 2[\kappa - (|\Pi|^2 - \varepsilon)\varsigma] W_{11} W_{22} + (\lambda_1^2 - \varepsilon) W_{22}^2 \}$$

$$+ \varsigma (W_{12}^2 + W_{21}^2) - 2\varsigma \varphi(p) + \varsigma w^2$$

$$= \frac{1}{|\Pi|^2 - \varepsilon} \{ \sigma (|W_{11}|^2 + |W_{22}|^2) + (\sqrt{\lambda_2^2 - \varepsilon - \sigma} W_{11} - \sqrt{\lambda_1^2 - \varepsilon - \sigma} W_{22})^2 \}$$

$$+ \varsigma (W_{12}^2 + W_{21}^2) - 2\varsigma \varphi(p) + \varsigma w^2,$$

(2.52)

where $W_{ij} = DW(E_i, E_j)$, $\varphi$ is given in (2.36), and $\sigma > 0$ is given through the formula

$$(\lambda_2^2 - \varepsilon - \sigma)(\lambda_1^2 - \varepsilon - \sigma) = [\kappa - (|\Pi|^2 - \varepsilon)\varsigma]^2,$$

when $\varepsilon > 0$ and $\varsigma > 0$ are small enough.

We integrate (2.52) over $S$ to obtain (2.49) from Lemma 2.6.

Proofs of Theorems 1.2 and 1.3

By a similar argument as in [10, 14], we combine Theorem 1.1 with (2.43) and (2.51), respectively, to complete the proofs. □

2.3 Proof Theorem 1.4; Ansatz

Here we use the norm (2.15).

(i) Let $\Omega$ be parabolic. From Proposition 2.1, a local principal coordinate exists on $S$. In such a principal coordinate an ansatz has been constructed in [10, Theorem 3.3].

(ii) Let $\Omega$ be elliptic. Set

$$\kappa_0 = \sup_{p \in S} \kappa(p).$$

Let $p_0 \in S$ be given and let $\sigma_0 > 0$ be such that

$$\mathcal{B}(p_0, \sigma_0) \subset S, \quad \frac{\sin \sqrt{\kappa_0 t}}{\sqrt{\kappa_0 t}} \geq \frac{1}{2} \quad \text{for} \quad t \in [0, \sigma_0],$$

where $\mathcal{B}(p_0, \sigma_0)$ is the geodesic plate in the induced metric $g$ centered at $p_0$ with radius $\sigma_0$. Let $\varphi \in C^2_0(S)$ be such that

$$\varphi(p) = 1 \quad \text{for} \quad p \in \mathcal{B}(p_0, \sigma_0).$$
Let $\rho(p) = d_g(p, p_0)$ be the distance from $p \in S$ to $p_0$ in the induced metric $g$ on $M$. We set

$$y = W + w\hat{n}, \quad w = \phi \cos(\rho), \quad W = -tDw, \quad \phi = \frac{1}{h^{1/2}}.$$ 

Denote $B(\sigma_0)$ by the plate in $M_{p_0}$ centered at the origin with radius $\sigma_0$. Let $dx$ be the volume element in $M_{p_0}$. From the volume comparison theorem, we have

$$\int_S w^2 dg \geq \int_{B(p_0, \sigma_0)} \cos^2(\phi) dg \geq \frac{1}{2} \int_{|x|<\sigma_0} \cos^2(\phi|x|) dx \geq \frac{1}{2} \int_{|x|<\sigma_0} \cos^2(\phi|x|) dx \geq \frac{(m+1)h^{1/2} \pi^2}{4} \geq \frac{\pi^2}{4} \left(\frac{\sigma_0}{\pi} - 3\frac{h^{1/2}}{4}\right),$$ (2.53)

where

$$m = \lfloor \sigma_0 \frac{h^{1/2}}{\pi} - \frac{3}{4} \rfloor.$$

Moreover, we have

$$Dw = -\phi \sin(\phi) D\rho, \quad D^2 w = -\phi^2 \cos(\phi) D\rho \otimes D\rho - \phi \sin(\phi) D^2 \rho,$$

that yield

$$|Dw|^2 \leq \frac{C}{h}, \quad |D^2 w|^2 \leq \frac{C}{h^2} \quad \text{for} \quad p \in S.$$ (2.54)

Noting that $|D\rho| = 1$, by a similar computation as in (2.53), we obtain

$$\frac{\sigma_1}{h} \leq \int_S |Dw|^2 dg \leq \frac{C}{h}. \quad (2.55)$$

In addition, a simple computation shows that

$$\|\nabla y + tp(y)\|^2 = h \int_S (w^2 |\Pi|^2 + 2|Dw|^2) dg + \frac{h^3}{12} \int_S (|D^2 w|^2 + |i(Dw)\Pi|^2) dg, \quad (2.56)$$

$$\|\text{sym} \nabla y + t\text{sym} p(y)\|^2 = h \int_S w^2 |\Pi|^2 dg + \frac{h^3}{12} \int_S \left(\frac{1}{4} |i(Dw)\Pi|^2 + |D^2 w|^2\right) dg. \quad (2.57)$$

Finally, it follows from (2.53)-(2.57) that

$$\frac{\|\nabla y\|^2}{\|\text{sym} \nabla y\|^2} \sim \frac{C}{h}.$$  \hspace{1cm} \Box$$

**Conflict of interest statement**

There is no conflict of interests.

**Ethical approval:** This article does not contain any studies with human participants or animals performed by the author.
References

[1] S. Bauer, P. Neff, D. Pauly, G. Starke, Some Poincar type inequalities for quadratic matrix fields. Proc. Appl. Math. Mech. 13, 359-360 (2013).

[2] W. Chen and J. Jost, A Riemannian version of Korn’s inequality. Calc. Var. Partial Differential Equations 14 (2002), no. 4, 517-530.

[3] P. G. Ciarlet and V. Lods, On the ellipticity of linear membrane shell equations. J. Math. Pures Appl. (9) 75 (1996), no. 2, 107-124.

[4] D. Cioranescu, O. Oleinik, and G. Tronel. On Korns inequalities for frame type structures and junctions. C. R. Acad. Sci. Paris Ser. I Math., 309(9):591-596, 1989.

[5] S. Conti, G. Dolzmann, S. Mller, Korns second inequality and geometric rigidity with mixed growth conditions. Calc. Var. Partial Differ. Equ. 50, 437-454 (2014).

[6] G. Friesecke, R. James, S. Muller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. Commun. Pure Appl. Math. 55, 1461-1506 (2002).

[7] K. O. Friedrichs, On the boundary-value problems of the theory of elasticity and Korns inequality. Ann. Math. 48(2), 441-471 (1947).

[8] G. Friesecke, R. James, S. Muller, A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence. Arch. Ration. Mech. Anal. 180(2), 183-236 (2006).

[9] G. Geymonat, E. Sanchez-Palencia, On the rigidity of certain surfaces with folds and applications to shell theory. Arch. Ration. Mech. Anal. 129(1), 11-45 (1995)

[10] Y. Grabovsky and D. Harutyunyan, Korn inequalities for shells with zero Gaussian curvature. Ann. Inst. H. Poincar Anal. Non Linaire 35 (2018), no. 1, 267-282.

[11] —, Exact scaling exponents in Korn and Korn-type inequalities for cylindrical shells. SIAM J. Math. Anal. 46 (2014), no. 5, 3277-3295.

[12] D. Harutyunyan, On the Korn interpolation and second inequalities for shells with non-constant thickness, arXiv:1709.04572 [math.AP].

[13] —, New asymptotically sharp Korn and Korn-like inequalities in thin domains. Journal of Elasticity, 117(1), pp. 95-109, 2014.

[14] —, Gaussian curvature as an identifier of shell rigidity. Arch. Ration. Mech. Anal. 226 (2017), no. 2, 743-766.
[15] W. T. Koiter, On the nonlinear theory of thin elastic shells I, Proc. Kon. Ned. Akad. Wetensch. B69 (1966), 1-17.

[16] V. A. Kondratiev, O. A. Oleinik, Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities. Usp. Mat. Nauk 43(5), 55-98 (1988).

[17] —, On Korns inequalities. C. R. Math. Acad. Sci. Paris, Sr. I 308, 483-487 (1989).

[18] R. V. Kohn, New integral estimates for deformations in terms of their nonlinear strains. Arch. Ration. Mech. Anal. 78(2), 131-172 (1982).

[19] A. Korn, Solution gnrale du problme dquilibre dans la thorie de llasticit dans le cas o les erts sont donn la surface. Ann. Fac. Sci. Toulouse 10, 165-269 (1908).

[20] —, Uber einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen. Bull. Int. Cracovie Akademie Umiejet, Classe des Sci. Math. Nat., 705-724 (1909).

[21] M. Lewicka and S. Muller. On the optimal constants in korns and geometric rigidity estimates, in bounded and unbounded domains, under neumann boundary conditions. Indiana Univ. Math. J. 65 (2016), no. 2, 377-397.

[22] S. A. Nazarov, Weighted anisotropic Korns inequality for a junction of a plate and a rod. Sbornik: Mathematics, 195(4):553-583, 2004.

[23] —, Korn inequalities for elastic junctions of massive bodies, thin plates, and rods. Russian Mathematical Surveys, 63(1):35, 2008.

[24] R. Paroni and G. Tomassetti, Asymptotically exact Korns constant for thin cylindrical domains. Comptes Rendus Mathematique, 350(15):749-752, 2012.

[25] —, On Korns constant for thin cylindrical domains. Mathematics and Mechanics of Solids, 19(3):318-333, 2014.

[26] P. Neff, D. Pauly, K.-J. Witsch, A canonical extension of Korns first inequality to \( H(\text{Curl}) \) motivated by gradient plasticity with plastic spin. C. R. Math. Acad. Sci. Paris, Sr. I 349, 1251-1254 (2011).

[27] —, A Korns inequality for incompatible tensor fields. In: Proceedings in Applied Mathematics and Mechanics, 6 June 2011.

[28] L. E. Payne, H. F. Weinberger, On Korns inequality. Arch. Ration. Mech. Anal. 8, 89-98 (1961).
[29] E. Sanchez-Palencia, Statique et dynamique des coques minces. II. Cas de flexion pure inhibée. Approximation membranaire. C. R. Acad. Sci. Paris Sr. I Math. 309(7), 531-537 (1989)

[30] M. Spivak, A comprehensive introduction to differential geometry. Vol. III. Second edition. Publish or Perish, Inc., Wilmington, Del., 1979. xii+466 pp. ISBN: 0-914098-83-7.

[31] H. Wu, The Bochner Technique in Differential Geometry, Mathematical Reports, Vol. 3, part 2, Harwood Academic Publishers, London-Paris, 1988.

[32] P. F. Yao, Space of Infinitesimal Isometries and Bending of Shells, 2012, arXiv:1310.5384.

[33] —, Modeling and control in vibrational and structural dynamics. A differential geometric approach. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton, FL, 2011.

[34] —, Linear Strain Tensors on Hyperbolic Surfaces and Asymptotic Theories for Thin Shells, arXiv:1708.07202 [math-ph].

[35] —, Linear Strain Tensors and Optimal Exponential of thickness in Korn’s Inequalities for Hyperbolic Shells, arXiv:1807.11115 [math-ph].