Local Stability of the Free Additive Convolution

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We prove that the system of subordination equations, defining the free additive convolution of two probability measures, is stable away from the edges of the support and blow-up singularities by showing that the recent smoothness condition of Kargin is always satisfied. As an application, we consider the local spectral statistics of the random matrix ensemble $A + U B U^*$, where $U$ is a Haar distributed random unitary or orthogonal matrix, and $A$ and $B$ are deterministic matrices. In the bulk regime, we prove that the empirical spectral distribution of $A + U B U^*$ concentrates around the free additive convolution of the spectral distributions of $A$ and $B$ on scales down to $N^{-2/3}$.

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1. Introduction

One of the basic concepts of free probability theory is the free additive convolution of two probability laws in a non-commutative probability space; it describes the law of the sum of two free random variables. In the case of a bounded self-adjoint random variable, its law can be identified with a probability measure of compact support on the real line. Hence the free additive convolution of two probability measures is a well-defined concept and it is characteristically different from the classical convolution.

In this paper, we prove a local stability result of the free additive convolution. A direct consequence is the continuity of the free additive convolution in a much stronger topology than established earlier by Bercovici and Voiculescu [10]. A second application of our stability result is to establish a local law on a very small scale for the eigenvalue density of a random matrix ensemble $A + U B U^*$ where $U$ is a Haar distributed unitary or orthogonal matrix and $A$, $B$ are deterministic $N$ by $N$ Hermitian matrices.

The free additive convolution was originally introduced by Voiculescu [36] for the sum of free bounded noncommutative random variables in an algebraic setup (see Maassen [32] and by Bercovici and Voiculescu [10] for extensions to the unbounded case). The Stieltjes transform of the free additive convolution is related to the Cauchy-Stieltjes transforms of the original measures by an elegant analytic change of variables. This subordination phenomenon was first observed by Voiculescu [38] in a generic situation and extended to full generality by Biane [14]. In fact, the subordination equations, see (2.5)-(2.6) below, may directly be used.

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to define the free additive convolution. This analytic definition was given independently by Belinschi and Bercovici [4] and by Chistyakov and Götze [18]; for further details we refer to, e.g., [39, 27, 2].

Kargin [30] pointed out that the analytic approach to the subordination equations, in contrast to the algebraic one, allows one to effectively study how free additive convolution is affected by small perturbations; this is especially useful to treat various error terms in the random matrix problem [31]. The basic tool is a local stability analysis of the subordination equations. In [30], Kargin assumed a lower bound on the imaginary part of the subordination functions and a certain non-degeneracy condition on the Jacobian that holds for generic values of the spectral parameter. While these so-called smoothness conditions hold in many examples, a general characterization was lacking. Our first result, Theorem 2.5, shows that the smoothness conditions hold wherever the absolutely continuous part of the free convolution measure is finite and nonzero. In particular, local stability holds unconditionally (Corollary 2.6) and, following Kargin’s argument [30], we immediately obtain the continuity of the free additive convolution in a stronger sense; see Theorem 2.7.

The random matrix application of this stability result, however, goes well beyond Kargin’s analysis [31] since our proof is valid on a much smaller scale. To explain the new elements, we recall how free probability connects to random matrices.

The following fundamental observation was made by Voiculescu [37] (later extended by Dykema [29] and Speicher [35]): if $A = A^{(N)}$ and $B = B^{(N)}$ are two sequences of Hermitian matrices that are asymptotically free with eigenvalue distributions converging to probability measures $\mu_\alpha$ and $\mu_\beta$, then the eigenvalue density of $A + B$ is asymptotically given by the free additive convolution $\mu_\alpha \boxplus \mu_\beta$. One of the most natural ways to ensure asymptotic freeness is to consider conjugation by independent unitary matrices. Indeed, if $A$ and $B$ are deterministic (may even be chosen diagonal) with limit laws $\mu_\alpha$ and $\mu_\beta$, then $A$ and $UBU^*$ are asymptotically free if $U = U^{(N)}$ is a Haar distributed matrix; see [37] and many subsequent works, e.g., [34, 40, 15, 33, 19]. In particular, the limiting spectral density of the eigenvalues of $H = A + UBU^*$ is given by $\mu_\alpha \boxplus \mu_\beta$.

The conventional setup of free probability operates with moment calculations. An alternative approach [33] proves the convergence of the resolvent at any fixed spectral parameter $z \in \mathbb{C}^+$. Both approaches give rise to weak convergence of measures, in particular they identify the limiting spectral density on macroscopic scale.

Armed with these macroscopic results, it is natural to ask for a local law, i.e., for the smallest possible $(N$-dependent) scale so that the local eigenvalue density on that scale still converges as $N$ tends to infinity. Local laws have been somewhat outside of the focus of free probability before Kargin’s recent works. After having improved a concentration result for the Haar measure by Chatterjee [17] by using the Gromov-Milman concentration inequality, Kargin obtained a local law for the ensemble $H = A + UBU^*$ on scale $\eta \gg (\log N)^{-1/2}$ [29], i.e., slightly below the macroscopic scale. Recently in [31], he improved this result down to scale $\eta \gg N^{-1/2}$ under the above mentioned smoothness condition. In Theorem 2.8 we prove the local law down to scale $\eta = \text{Im } z \gg N^{-1/3}$ without any additional assumption.

To achieve this short scale, we effectively use the positivity of the imaginary parts of the subordination functions by localizing the Gromov–Milman concentration inequality within the spectrum. Since the subordination functions are obtained as the solution of a system of self-consistent equations whose derivation itself requires bounds on the subordination functions, the reasoning seems circular. We break this circularity by a continuity argument (similarly as in [23]) in which we reduce the imaginary part of the spectral parameter in very small steps, use the previous step as an a priori bound and show that the bound does not deteriorate by using the local stability result, Theorem 2.6.
Finally, we remark that the local stability result is also a key ingredient in [3], where we were able to prove a local law down to the smallest possible scale $\eta \gg N^{-1}$, but with a weaker error bound than in Theorem 2.8; see Remark 2.4 for details.

1.1. Notation. We use the symbols $O(\cdot)$ and $o(\cdot)$ for the standard big-O and little-o notation. We use $c$ and $C$ to denote positive numerical constants. Their values may change from line to line. For $a, b > 0$, we write $a \lesssim b$, $a \gtrsim b$ if there is $C \geq 1$ such that $a \leq Cb$, $a \geq C^{-1}b$ respectively. We write $a \sim b$, if $a \lesssim b$ and $a \gtrsim b$ both hold. We denote by $\|v\|$ the Euclidean norm of $v \in \mathbb{C}^N$. For an $N \times N$ matrix $A \in M_N(\mathbb{C})$, we denote by $\|A\|$ its operator norm and by $\|A\|_2 := \sqrt{\langle A, A \rangle}$ its Hilbert-Schmidt norm, where $\langle A, B \rangle := \text{Trace}(AB^*)$, for $A, B \in M_N(\mathbb{C})$. Finally, we denote by $\text{tr}A$ the normalized trace of $A$, i.e., $\text{tr}A = \frac{1}{N} \text{Trace}A$.

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2. Main results

2.1. Free additive convolution. In this subsection, we recall the definition of the free additive convolution. Given a probability measure $\mu$ on $\mathbb{R}$, its \textit{Stieltjes transform}, $m_\mu$, on the complex upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im} z > 0\}$ is defined by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C}^+. \quad (2.1)$$

We denote by $F_\mu$ the negative reciprocal Stieltjes transform of $\mu$, i.e.,

$$F_\mu(z) := -\frac{1}{m_\mu(z)}, \quad z \in \mathbb{C}^+. \quad (2.2)$$

Observe that

$$\lim_{\eta \nearrow \infty} \frac{F_\mu(\eta i)}{i \eta} = 1, \quad (2.3)$$

as follows easily from (2.1). Note, moreover, that $F_\mu$ is an analytic function on $\mathbb{C}^+$ with non-negative imaginary part. Conversely, if $F : \mathbb{C}^+ \to \mathbb{C}^+$ is an analytic function such that $\lim_{\eta \nearrow \infty} F(i\eta)/i\eta = 1$, then $F$ is the negative reciprocal Stieltjes transform of a probability measure $\mu$, i.e., $F(z) = F_\mu(z)$, for all $z \in \mathbb{C}^+$; see, e.g., [1].

The free additive convolution is the binary operation on probability measures on $\mathbb{R}$ characterized by the following result.

\begin{prop}[Theorem 4.1 in [4], Theorem 2.1 in [18]] Given two probability measures $\mu_1$ and $\mu_2$ on $\mathbb{R}$, there exist unique analytic functions, $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$, such that,

(i) for all $z \in \mathbb{C}^+$, $\text{Im} \omega_1(z)$, $\text{Im} \omega_2(z) \geq \text{Im} z$, and

$$\lim_{\eta \nearrow \infty} \frac{\omega_1(\eta i)}{i \eta} = \lim_{\eta \nearrow \infty} \frac{\omega_2(\eta i)}{i \eta} = 1; \quad (2.4)$$

(ii) for all $z \in \mathbb{C}^+$,

$$F_{\mu_1}(\omega_2(z)) - \omega_1(z) - \omega_2(z) + z = 0, \quad (2.5)$$

$$F_{\mu_2}(\omega_1(z)) - \omega_1(z) - \omega_2(z) + z = 0.$$  

\end{prop}

*All probability measures considered will be assumed to be Borel.
It follows from (2.4) that the analytic function $F : \mathbb{C}^+ \to \mathbb{C}^+$ defined by
\[
F(z) := F_{\mu_1}(\omega_2(z)) = F_{\mu_2}(\omega_1(z)),
\] satisfies (2.3). Thus $F$ is the negative reciprocal Stieltjes transform of a probability measure $\mu$, called the free additive convolution of $\mu_1$ and $\mu_2$, usually denoted by $\mu = \mu_1 \boxplus \mu_2$. Note that (2.6) shows that the roles of $\mu_1$ and $\mu_2$ are symmetric and thus $\mu_1 \boxplus \mu_2 = \mu_2 \boxplus \mu_1$. The functions $\omega_1$ and $\omega_2$ of Proposition 2.1 are called subordination functions and $F$ is said to be subordinated to $F_{\mu_1}$, respectively to $F_{\mu_2}$.

We mention that Voiculescu [30] originally introduced the free additive convolution in a different, algebraic manner. The equivalent analytic definition based on the existence of subordination functions (taken up in Proposition 2.1 above) was introduced in [4, 18].

We next recall some basic examples. Choosing $\mu_1$ arbitrary and $\mu_2$ as a single point mass at $b \in \mathbb{R}$, it is easy to check that $\mu_1 \boxplus \mu_2$ simply is $\mu_1$ shifted by $b$. We exclude this uninteresting case by henceforth assuming that $\mu_1$ and $\mu_2$ are both supported at more than one point. Choosing $\mu_1 = \mu_2 = \mu$ as the Bernoulli distribution
\[
\mu = (1 - \xi)\delta_0 + \xi \delta_1,
\] the free additive convolution is explicitly given by (see e.g., (5.5) of [33])
\[
(\mu \boxplus \mu)(x) = \frac{\sqrt{(\ell_+ - x)(x - \ell_-)} + (1 - 2\xi)\delta_0(x) + (2\xi - 1)\delta_2(x)}{\pi x(2 - x)},
\] $x \in \mathbb{R}$, where $\ell_{\pm} := 1 \mp 2\sqrt{\xi(1 - \xi)}$ and where $(\cdot)_+$ denotes the positive part. Observe that $\mu \boxplus \mu$ has a nonzero absolutely continuous part and, depending on the choice of $\xi$, a point mass. Another important choice for $\mu_2$ is Wigner’s semicircle law $\mu_{sc}$. For arbitrary $\mu_1$, $\mu_1 \boxplus \mu_{sc}$ is then purely absolutely continuous with a bounded density\(^1\) that is real analytic wherever positive [13].

Returning to the generic setting, the atoms of $\mu_1 \boxplus \mu_2$ are identified as follows. A point $c \in \mathbb{R}$ is an atom of $\mu_1 \boxplus \mu_2$, if and only if there exist $a, b \in \mathbb{R}$ such that $c = a + b$ and $\mu_1(\{a\}) + \mu_2(\{b\}) > 1$; see [Theorem 7.4, [11]]. For another interesting properties of the atoms of $\mu_1 \boxplus \mu_2$ we refer the reader to [12]. The boundary behavior of the functions $F_{\mu_1 \boxplus \mu_2}, \omega_1$ and $\omega_2$ has been studied by Belinschi [5, 6, 7] who proved the next two results. For simplicity, we restrict the discussion to compactly supported probability measures.

**Proposition 2.2** (Theorem 2.3 in [5], Theorem 3.3 in [6]). Let $\mu_1$ and $\mu_2$ be compactly supported probability measures on $\mathbb{R}$, none of them being a single point mass. Then the functions $F_{\mu_1 \boxplus \mu_2}, \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ extend continuously to $\mathbb{R}$.

Belinschi further showed in Theorem 4.1 in [6] that the singular continuous part of $\mu_1 \boxplus \mu_2$ is always zero and that the absolutely continuous part, $(\mu_1 \boxplus \mu_2)^{ac}$, of $\mu_1 \boxplus \mu_2$ is always nonzero. We denote the density function of $(\mu_1 \boxplus \mu_2)^{ac}$ by $f_{\mu_1 \boxplus \mu_2}$.

We are now ready to introduce our notion of regular bulk, $S_{\mu_1 \boxplus \mu_2}$, of $\mu_1 \boxplus \mu_2$. Informally, we let $S_{\mu_1 \boxplus \mu_2}$ be the open set on which $\mu_1 \boxplus \mu_2$ admits a continuous density that is strictly positive and bounded from above. For a formal definition we first introduce the set
\[
U_{\mu_1 \boxplus \mu_2} := \text{int} \left\{ \supp (\mu_1 \boxplus \mu_2)^{ac} \setminus \{ x \in \mathbb{R} : \lim_{\eta \searrow 0} F_{\mu_1 \boxplus \mu_2}(x + i\eta) = 0 \} \right\}.
\] (2.8)
Note that $U_{\mu_1 \boxplus \mu_2}$ does not contain any atoms of $\mu_1 \boxplus \mu_2$. By Privalov’s theorem the set \( \{ x \in \mathbb{R} : \lim_{\eta \searrow 0} F_{\mu_1 \boxplus \mu_2}(x + i\eta) = 0 \} \) has Lebesgue measure zero. In fact, an even stronger statement applies for the case at hand. Belinschi [7] showed that if $x \in \mathbb{R}$ is such that

\(^1\)All densities are with respect to Lebesgue measure on $\mathbb{R}$. 

Proposition 2.3 (Theorem 3.3 in [6]). Let $\mu_1$ and $\mu_2$ be as above and fix any $x \in \mathcal{U}_{\mu_1} \oplus \mu_2$. Then $F_{\mu_1, \mu_2}, \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ extend analytically around $x$. In particular, the density function $f_{\mu_1, \mu_2}$ is real analytic in $\mathcal{U}_{\mu_1} \oplus \mu_2$ wherever positive.

The regular bulk is obtained from $\mathcal{U}_{\mu_1} \oplus \mu_2$ by removing the zeros of $f_{\mu_1, \mu_2}$ inside $\mathcal{U}_{\mu_1} \oplus \mu_2$.

Definition 2.4. The regular bulk of the measure $\mu_1 \oplus \mu_2$ is defined as the set

$$
\mathcal{B}_{\mu_1 \oplus \mu_2} := \mathcal{U}_{\mu_1 \oplus \mu_2} \setminus \{x \in \mathcal{U}_{\mu_1 \oplus \mu_2} : f_{\mu_1, \mu_2}(x) = 0\}.
$$

2.2. Stability Result. To present our results it is convenient to recast (2.5) in a compact form: For generic probability measures $\mu_1, \mu_2$ as above, let the function $\Phi_{\mu_1, \mu_2} : (\mathbb{C}^+)^3 \to \mathbb{C}^2$ be given by

$$
\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) := \begin{pmatrix} F_{\mu_1}(\omega_1) - \omega_1 - \omega_2 + z \\ F_{\mu_2}(\omega_1) - \omega_1 - \omega_2 + z \end{pmatrix}.
$$

(2.10)

Considering $\mu_1, \mu_2$ as fixed, the equation

$$
\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0,
$$

(2.11)

is equivalent to (2.5) and, by Proposition 2.1, there are unique analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+, z \mapsto \omega_1(z), \omega_2(z)$ satisfying (2.4) that solve (2.11) in terms of $z$. We use the following conventions: We denote by $\omega_1$ and $\omega_2$ generic variables on $\mathbb{C}^+$ and we denote, with a slight abuse of notation, by $\omega_1(z)$ and $\omega_2(z)$ the subordination functions solving (2.11) in terms of $z$. When no confusion can arise, we simply write $\Phi$ for $\Phi_{\mu_1, \mu_2}$.

We call the system (2.11) linearly S-stable at $(\omega_1, \omega_2)$ if

$$
\Gamma_{\mu_1, \mu_2}(\omega_1, \omega_2) := \left\| \begin{pmatrix} -1 & \frac{1}{F_{\mu_1}'(\omega_1)} - 1 \\ \frac{1}{F_{\mu_2}'(\omega_1)} - 1 & -1 \end{pmatrix} \right\| \leq S,
$$

(2.12)

for some constant $S$. Especially, the partial Jacobian matrix, $D\Phi(\omega_1, \omega_2)$, of (2.10) given by

$$
D\Phi(\omega_1, \omega_2) := \left( \begin{array}{cc} \frac{\partial \Phi}{\partial \omega_1}(\omega_1, \omega_2, z) & \frac{\partial \Phi}{\partial \omega_2}(\omega_1, \omega_2, z) \\ \frac{\partial \Phi}{\partial \omega_1}(\omega_1, \omega_2, z) & \frac{\partial \Phi}{\partial \omega_2}(\omega_1, \omega_2, z) \end{array} \right) = \left( \begin{array}{cc} -1 & \frac{1}{F_{\mu_1}'(\omega_1)} - 1 \\ \frac{1}{F_{\mu_2}'(\omega_1)} - 1 & -1 \end{array} \right),
$$

admits a bounded inverse at $(\omega_1, \omega_2)$. Note that $D\Phi(\omega_1, \omega_2)$ is independent of $z$.

Our first main result shows that the system (2.11) is linearly stable and that the imaginary parts of the subordination functions are bounded below in the regular bulk. We require some more notation: For $a, b \geq 0$, $b \geq a$, and an interval $\mathcal{I} \subset \mathbb{R}$, we introduce the domain

$$
\mathcal{S}_2(a, b) := \{z = E + i\eta \in \mathbb{C}^+ : E \in \mathcal{I}, a \leq \eta \leq b\}.
$$

(2.13)

Theorem 2.5. Let $\mu_1$ and $\mu_2$ be compactly supported probability measures on $\mathbb{R}$, and assume that neither is supported at a single point and that at least one of them is supported at more than two points. Let $\mathcal{I} \subset \mathcal{B}_{\mu_1} \oplus \mu_2$ be a compact non-empty interval and fix some $0 < \eta_M < \infty$.

Then there are two constants $k > 0$ and $S < \infty$, both depending on the measures $\mu_1$ and $\mu_2$, on the interval $\mathcal{I}$ as well as on the constant $\eta_M$, such that following statements hold.
(i) The imaginary parts $\Im \omega_1$ and $\Im \omega_2$ of the subordination functions associated with $\mu_1$ and $\mu_2$ satisfy
\[
\min_{z \in \mathcal{S}_2(0, \eta_M)} \Im \omega_1(z) \geq 2k, \quad \min_{z \in \mathcal{S}_2(0, \eta_M)} \Im \omega_2(z) \geq 2k.
\]
(ii) The system $\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0$ is linearly $S$-stable at $(\omega_1(z), \omega_2(z))$ uniformly in $\mathcal{S}_2(0, \eta_M)$, i.e.,
\[
\max_{z \in \mathcal{S}_2(0, \eta_M)} \Gamma_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z)) \leq S.
\]

Remark 2.1. The assumption that neither of $\mu_1$, $\mu_2$ is a point mass guarantees that the free additive convolution is not a simple translate. The case when both, $\mu_1$ and $\mu_2$ are combinations of two point masses is special and its discussion is postponed to Section 7.

Theorem 2.5 has the following local stability result as corollary.

Corollary 2.6. Let $\mu_1$, $\mu_2$ and $\mathcal{S}_2(0, \eta_M)$ be as in Theorem 2.5. Fix $z_0 \in \mathbb{C}^+$. Assume that the functions $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{r}_1$, $\tilde{r}_2 : \mathbb{C}^+ \to \mathbb{C}$ satisfy $\Im \tilde{\omega}_1(z_0) > 0$, $\Im \tilde{\omega}_2(z_0) > 0$ and
\[
\Phi_{\mu_1, \mu_2}(\tilde{\omega}_1(z_0), \tilde{\omega}_2(z_0), z_0) = \tilde{r}(z_0),
\]
with $\tilde{r}(z) := (\tilde{r}_1(z), \tilde{r}_2(z))^T$. Let $\omega_1$, $\omega_2$ be the subordination functions solving the system $\Phi_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z), z) = 0$, $z \in \mathbb{C}^+$.

Then there exists a (small) constant $\delta_0 > 0$ such that whenever we have
\[
|\tilde{\omega}_1(z_0) - \omega_1(z_0)| \leq \delta_0, \quad |\tilde{\omega}_1(z_0) - \omega_1(z_0)| \leq \delta_0,
\]
we also have
\[
|\tilde{\omega}_1(z_0) - \omega_1(z_0)| \leq 2S|\tilde{r}(z_0)|, \quad |\tilde{\omega}_2(z_0) - \omega_2(z_0)| \leq 2S|\tilde{r}(z_0)|.
\]
The constant $\delta_0 > 0$ depends on $\mu_1$ and $\mu_2$, on the interval $\mathcal{I}$ as well as on $\eta_M$.

We omit the proof of Corollary 2.6 from Theorem 2.5, since it follows directly from Proposition 4.1 in Section 4 below.

2.3. Applications. We next explain two main applications of the stability estimates obtained in Theorem 2.5.

2.3.1. Continuity of the free additive convolution. Our first application shows that the free additive convolution is a continuous operation when the image is equipped with the topology of local uniform convergence of the density in the regular bulk; see (2.23). Bercovici and Voiculescu (Proposition 4.13 of [10]) showed that the free additive convolution is continuous with respect to weak convergence of measures. More precisely, given two pairs of probability measures $\mu_A$, $\mu_B$ and $\mu_\alpha$, $\mu_\beta$ on $\mathbb{R}$, the measures $\mu_A \boxplus \mu_B$ and $\mu_\alpha \boxplus \mu_\beta$ satisfy
\[
d_L(\mu_A \boxplus \mu_B, \mu_\alpha \boxplus \mu_\beta) \leq d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta),
\]
where $d_L$ denotes the Lévy distance. In particular, weak convergence of $\mu_A$ to $\mu_\alpha$ and weak convergence of $\mu_B$ to $\mu_\beta$ imply weak convergence of $\mu_A \boxplus \mu_B$ to $\mu_\alpha \boxplus \mu_\beta$.

Using the Stieltjes transform, we can easily link (2.19) to the systems of equations in (2.5), respectively in (2.10). Using integration by parts and the definition of the Stieltjes transform, a direct computation reveals that there is a numerical constant $C$ such that
\[
|m_{\mu_A \boxplus \mu_B}(z) - m_{\mu_\alpha \boxplus \mu_\beta}(z)| \leq \frac{C}{\eta} \left(1 + \frac{1}{\eta}\right) d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta),
\]
\[
\leq \frac{C}{\eta} \left(1 + \frac{1}{\eta}\right) \left(d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta)\right), \quad \eta = \Im z,
\]

(2.20)
for all $z \in \mathbb{C}^+$, where we used (2.19) to get the second line. Note that the estimate in (2.20) deteriorates as $\eta$ approaches the real line. Our next result strengthens (2.20) as follows. We consider the measure $\mu_\alpha \boxplus \mu_\beta$ as “reference” measure (in the sense that it locates the regular bulk) while $\mu_A$, $\mu_B$ are arbitrary probability measures and show that the Lévy distances bound $|m_{\mu_\alpha \boxplus \mu_\beta}(E + i\eta) - m_{\mu_\alpha \boxplus \mu_\beta}(E + i\eta_0)|$ uniformly in $\eta$, for all $E$ inside the regular bulk of $\mu_\alpha \boxplus \mu_\beta$.

**Theorem 2.7.** Let $\mu_\alpha$ and $\mu_\beta$ be compactly supported probability measures on $\mathbb{R}$, and assume that neither is supported at a single point and that at least one of them is supported at more than two points. Let $I \subset B_{\mu_\alpha \boxplus \mu_\beta}$ be a compact non-empty interval and fix some $0 < \eta_M < \infty$. Let $\mu_A$ and $\mu_B$ be two arbitrary probability measures on $\mathbb{R}$.

Then there are constants $b > 0$ and $Z < \infty$, both depending on the measures $\mu_\alpha$ and $\mu_\beta$, on the interval $I$ as well as on the constant $\eta_M$, such that whenever

$$d_L(\mu_\alpha, \mu_\beta) + d_L(\mu_B, \mu_\beta) \leq b$$

(2.21)

holds, then

$$\max_{z \in S_T(0, \eta_0)} |m_{\mu_\alpha \boxplus \mu_\beta}(z) - m_{\mu_\alpha \boxplus \mu_\beta}(z)| \leq Z (d_L(\mu_\alpha, \mu_\beta) + d_L(\mu_B, \mu_\beta)) ,$$

(2.22)

holds, too.

Note that $\max_{z \in S_T(0, \eta_0)} |m_{\mu_\alpha \boxplus \mu_\beta}(z)| < \infty$ by compactness of $I$ and analyticity of $m_{\mu_\alpha \boxplus \mu_\beta}$ in $I$. Thus the Stieltjes-Perron inversion formula directly implies that $(\mu_\alpha \boxplus \mu_\beta)_{ac}$ has a density, $f_{\mu_\alpha \boxplus \mu_\beta}$, inside $I$ and that

$$\max_{x \in I} |f_{\mu_\alpha \boxplus \mu_\beta}(x) - f_{\mu_\alpha \boxplus \mu_\beta}(x)| \leq Z (d_L(\mu_\alpha, \mu_\beta) + d_L(\mu_B, \mu_\beta)) ,$$

(2.23)

provided that (2.21) holds, where $f_{\mu_\alpha \boxplus \mu_\beta}$ is the density of $(\mu_\alpha \boxplus \mu_\beta)_{ac}$.

**Remark 2.2.** The estimate (2.22) was recently given by Kargin [30] under the assumption that (2.14) and (2.15) hold for all $z \in S_T(0, \eta_M)$, i.e., under the assumption that the conclusions of our Theorem 2.5 hold. It is quite surprising that one can directly set $\text{Im } z = 0$ in (2.22). As first noted by Kargin, this is due to the regularizing effect of $\omega_\alpha$, $\omega_\beta$ and to the global uniqueness of solutions to (2.5) for arbitrary probability measures.

### 2.3.2. Application to random matrix theory

We now turn to an application of Theorem 2.5 in random matrix theory. Let $A \equiv A^{(N)}$ and $B \equiv B^{(N)}$ be two sequences of $N \times N$ deterministic real diagonal matrices, whose empirical spectral distributions are denoted by $\mu_A$ and $\mu_B$ respectively, i.e.,

$$\mu_A := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i}, \quad \mu_B := \frac{1}{N} \sum_{i=1}^{N} \delta_{b_i},$$

(2.24)

where $A = \text{diag}(a_i)$, $B = \text{diag}(b_i)$. The matrices $A$ and $B$ depend on $N$, but we omit this fact from the notation. Let $\omega_A$ and $\omega_B$ denote the subordination functions associated with $\mu_A$ and $\mu_B$ by Proposition 2.1.

We assume that there are deterministic probability measures $\mu_\alpha$ and $\mu_\beta$ on $\mathbb{R}$, neither of them being a single point mass, such that the empirical spectral distributions $\mu_A, \mu_B$ converge weakly to $\mu_\alpha, \mu_\beta$, as $N \to \infty$. More precisely, we assume that

$$d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \to 0 ,$$

(2.25)

as $N \to \infty$. Let $\omega_\alpha$, $\omega_\beta$ denote the subordination functions associated with $\mu_\alpha$ and $\mu_\beta$. 
Let $U$ be an independent $N \times N$ Haar distributed unitary matrix (in short Haar unitary) and consider the random matrix

$$H \equiv H^{(N)} := A + U B U^*.$$  

(2.26)

We introduce the Green function, $G_H$, of $H$ and its normalized trace, $m_H$, by setting

$$G_H(z) := \frac{1}{H - z}, \quad m_H(z) := \text{tr} G_H(z),$$  

(2.27)

$z \in \mathbb{C}^+$. We refer to $z$ as the spectral parameter and we often write $z = E + i\eta$, $E \in \mathbb{R}$, $\eta > 0$. Recall the definition of $S_\ell(a, b)$ in (2.13). We have the following local law for $m_H$.

**Theorem 2.8.** Let $\mu_\alpha$ and $\mu_\beta$ be two compactly supported probability measures on $\mathbb{R}$, and assume that neither is only supported at one point and that at least one of them is supported at more than two points. Let $I \subset B_{\mu_\alpha \boxplus \mu_\beta}$ be a compact non-empty interval and fix some $0 < \eta_M < \infty$. Assume that the sequences of matrices $A$ and $B$ in (2.26) are such that their empirical eigenvalue distributions $\mu_A$ and $\mu_B$ satisfy (2.25). Fix any small $\gamma > 0$ and set $\eta_m := N^{-2/3 + \gamma}$.

Then we have the following uniform estimate: For any (small) $\epsilon > 0$ and any (large) $D$, 

$$\mathbb{P} \left( \bigcup_{z \in S_\ell(\eta_m, \eta_M)} \left\{ |m_H(z) - m_{\mu_A \boxplus \mu_B}(z)| > \frac{N^\epsilon}{N^{2/3}} \right\} \right) \leq \frac{1}{N^D},$$  

(2.28)

holds for $N \geq N_0$, with some $N_0$ sufficiently large, where we write $z = E + i\eta$.

Using standard techniques of random matrix theory, we can translate the estimate (2.28) on the Green function into an estimate on the empirical spectral distribution of the matrix $H$. Let $\lambda_1, \ldots, \lambda_N$ denote the ordered eigenvalues of $H$ and denote by

$$\mu_H := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$$  

(2.29)

its empirical spectral distribution. Our result on the rate of convergence of $\mu_H$ is as follows.

**Corollary 2.9.** Let $I \subset B_{\mu_\alpha \boxplus \mu_\beta}$ be a compact non-empty interval. Then, for any $E_1 < E_2$ in $I$, we have the following estimate. For any (small) $\epsilon > 0$ and any (large) $D$ we have

$$\mathbb{P} \left( \left| \mu_H([E_1, E_2]) - \mu_{A \boxplus B}([E_1, E_2]) \right| > \frac{N^\epsilon}{N^{2/3}} \right) \leq N^{-D},$$  

(2.30)

for $N \geq N_0$, with some $N_0$ sufficiently large.

We omit the proof of Corollary 2.9 from Theorem 2.8, but mention that the normalized trace $m_H$ of the Green function and the empirical spectral distribution $\mu_H$ of $H$ are linked by

$$m_H(z) = \text{tr} G_H(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z} = \int_{\mathbb{R}} \frac{d\mu_H(x)}{x - z}, \quad z \in \mathbb{C}^+.$$  

Corollary 2.9 then follows from a standard application of the Helffer-Sjöstrand functional calculus; see e.g., Section 7.1 of [22] for a similar argument.

Note that assumption (2.25) does not exclude that the matrix $H$ has outliers in the large $N$ limit. In fact, the model $H = A + U B U^*$ shows a rich phenomenology when, say, $A$ has a finite number of large spikes; we refer to the recent works in [8, 9, 16, 31].

**Remark 2.3.** Our results in Theorem 2.8 and Corollary 2.9 are stated for $U$ Haar distributed on the unitary group $U(N)$. However, they also hold true (with the same proofs) when $U$ is Haar distributed on the orthogonal group $O(N)$.
Remark 2.4. In [3], we derive, with a different approach, the estimate (with the notation of (2.28))
\[
P \left( \bigcup_{z \in \mathcal{S}(\eta_m, \eta_H)} \left\{ |m_H(z) - m_{\mu, \eta}(z)| > \frac{N^c}{\sqrt{N\eta}} \right\} \right) \leq \frac{1}{N^D}, \tag{2.31}
\]
for \( N \geq N_0 \), with some \( N_0 \) sufficiently large, and with \( \eta_m = N^{-1+\gamma} \). In fact, we obtain estimates for individual matrix elements of the resolvent \( G_H \) as well. Comparing with (2.28), we see that we can choose \( \eta \) in (2.31) almost as small as \( N^{-1} \) at the price of losing a factor \( \sqrt{N} \). The stability and perturbation analysis in [3] rely on the optimal results in Theorem 2.5 and Theorem 2.7 as well as in Sections 3-5 of the present paper.

2.4. Organization of the paper. In Section 3, we consider the stability of the system (2.5) when at least one of the measures \( \mu_1 \) and \( \mu_2 \) is supported at more than two points and we give the proof of Theorem 2.5. In Section 4, we consider perturbations of the system (2.5) and derive results that will be used in the proof of Theorem 2.8 and also in [3]. In Section 5, we prove Theorem 2.7. In Section 6 we consider the random matrix setup and prove Theorem 2.8. In the final Section 7, we separately settle the special case when both \( \mu_1 \) and \( \mu_2 \) are combinations of two point masses and give the results analogous to Theorem 2.5, Theorem 2.7 and Theorem 2.8 for that case.

3. Stability of the system (2.11) and proof of Theorem 2.5

In this section, we discuss stability properties of the system (2.11), with \( \mu_1 \), \( \mu_2 \) two compactly supported probability measures satisfying the assumptions of Theorem 2.5.

Lemma 3.1. Let \( \mu_1 \), \( \mu_2 \) be two probability measures on \( \mathbb{R} \) neither of them being supported at a single point. Then there is, for any compact set \( \mathcal{K} \subset \mathbb{C}^+ \), a strictly positive constant \( 0 < \sigma(\mu_1, \mathcal{K}) < 1 \) such that the reciprocal Stieltjes transform \( F_{\mu_1} \) (see (2.2)) satisfies
\[
\text{Im} \ z \leq (1 - \sigma(\mu_1, \mathcal{K})) \text{Im} \ F_{\mu_1}(z), \quad \forall z \in \mathcal{K}. \tag{3.1}
\]
Similarly, there is \( 0 < \sigma(\mu_2, \mathcal{K}) < 1 \) such that (3.1) holds with \( \mu_2 \) and \( F_{\mu_2} \), respectively.

Assume in addition that \( \mu_1 \) is supported at more than two points. Then there is, for any compact set \( \mathcal{K} \subset \mathbb{C}^+ \), a strictly positive constant \( 0 < \tilde{\sigma}(\mu_1, \mathcal{K}) < 1 \) such that
\[
|F'_{\mu_1}(z) - 1| \leq (1 - \tilde{\sigma}(\mu_1, \mathcal{K})) \frac{\text{Im} \ F_{\mu_1}(z) - \text{Im} \ z}{\text{Im} \ z}, \quad \forall z \in \mathcal{K}, \tag{3.2}
\]
where \( F'_{\mu_1}(z) \equiv \partial_z F_{\mu_1}(z) \).

Proof of Lemma 3.1. Assuming by contradiction that inequality (3.1) saturates (with vanishing constant \( \sigma(\mu_1, \mathcal{K}) = 0 \), for some \( z \in \mathcal{K} \subset \mathbb{C}^+ \)), we have \( \text{Im} \ F_{\mu_1}(z) = \text{Im} \ z \) for some \( z \), thus \( F_{\mu_1}(z) = z - a, \ a \in \mathbb{R}, \ i.e., \mu_1 = \delta_a \). This shows (3.1).

To establish (3.2), we first note that the analytic functions \( F_{\mu_j} : \mathbb{C}^+ \to \mathbb{C}^+ \), \( j = 1, 2 \), have the Nevanlinna representations
\[
F_{\mu_j}(z) = a_{F_{\mu_j}} + z + \int_{\mathbb{R}} \frac{1 + zx}{x - z} \, d\rho_{F_{\mu_j}}(x), \quad j = 1, 2, \quad z \in \mathbb{C}^+, \tag{3.3}
\]
where \( a_{F_{\mu_j}} \in \mathbb{R} \) and \( \rho_{F_{\mu_j}} \) are finite Borel measures on \( \mathbb{R} \). Note that the coefficients of \( z \) on the right-hand side are determined by (2.3). From (3.3) we see that
\[
|F'_{\mu_1}(z) - 1| = \left| \int_{\mathbb{R}} \frac{1 + x^2}{(x - z)^2} \, d\rho_{F_{\mu_1}}(x) \right|, \quad z \in \mathbb{C}^+, \tag{3.4}
\]
as well as
\[
\frac{\text{Im } F_{\mu_1}(z) - \text{Im } z}{\text{Im } z} = \int_{\mathbb{R}} \frac{1 + x^2}{|x - z|^2} \, d\rho_{F_{\mu_1}}(x), \quad z \in \mathbb{C}^+.
\] (3.5)

Hence, assuming by contradiction that inequality (3.2) saturates (with \( \tilde{\sigma}(\mu_1, \mathcal{K}) = 0 \), for some \( z \in \mathcal{K} \)), we must have
\[
\int_{\mathbb{R}} \frac{1 + x^2}{|x - z|^2} \, d\rho_{F_{\mu_1}}(x) = \int_{\mathbb{R}} \frac{1 + x^2}{|x - z|^2} \, d\rho_{F_{\mu_1}}(x),
\] for some \( z \in \mathcal{K} \), implying that \( \rho_{F_{\mu_1}} \) is either a single point mass or \( \rho_{F_{\mu_1}} = 0 \). In the latter case, we have \( F_{\mu_1}(z) = a_{\mu_1} + z \) and we conclude that \( \mu_1 \) must be single point measure, but this is excluded by assumption. Thus \( \rho_{F_{\mu_1}} \) is a single point mass, i.e., there is a constant \( d_{\mu_1} \in \mathbb{R} \) such that \( F_{\mu_1}(z) = a_{F_{\mu_1}} + z + (1 + zd_{\mu_1}^2)/(d_{\mu_1} - z) \), \( z \in \mathcal{K} \). It follows that \( \mu_1 \) is a convex combination of two point measures yielding a contradiction. This shows (3.2). \[\square\]

3.1. Bounds on the subordination functions. Let \( \mu_1, \mu_2 \) be as above and let \( \omega_1(z), \omega_2(z) \) be the associated subordination functions. Recall that we rewrite the defining equations (2.5) for \( \omega_1 \) and \( \omega_2 \) in the compact form \( \Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0 \) introduced in (2.11).

We first provide upper bounds on the subordination functions \( \omega_1(z), \omega_2(z) \). Our proof relies on the assumption that \( \mu_1, \mu_2 \) are compactly supported, i.e., that there is a constant \( L < \infty \) such that
\[
\text{supp } \mu_1 \subset [-L, L], \quad \text{supp } \mu_2 \subset [-L, L].
\] (3.7)

Recall from Theorem 2.5 that we fixed a compact non-empty interval \( \mathcal{I} \subset B_{\mu_1, \mu_2} \). Since the density \( f_{\mu_1, \mu_2} \) is real analytic inside the regular bulk by Proposition 2.3 and since \( \mathcal{I} \) is compact, there exists a constant \( \kappa_0 > 0 \) such that
\[
0 < \kappa_0 \leq \min_{x \in \mathcal{I}} f_{\mu_1, \mu_2}(x).
\] (3.8)

Fixing a constant \( 0 < \eta_M < \infty \), it further follows that there is a constant \( M < \infty \) such that
\[
\max_{z \in S_{\mathcal{I}}(0, \eta_M)} |m_{\mu_1, \mu_2}(z)| \leq M.
\] (3.9)

**Lemma 3.2.** Let \( \mu_1, \mu_2 \) be two compactly supported probability measures on \( \mathbb{R} \) satisfying (3.7), for some \( L < \infty \), and assume that both are supported at more than one point. Let \( \mathcal{I} \subset B_{\mu_1, \mu_2} \) be a compact non-empty interval. Then there is a constant \( K < \infty \) such that
\[
\max_{z \in S_{\mathcal{I}}(0, \eta_M)} |\omega_1(z)| \leq \frac{K}{2}, \quad \max_{z \in S_{\mathcal{I}}(0, \eta_M)} |\omega_2(z)| \leq \frac{K}{2}.
\] (3.10)

The constant \( K \) depends on the constant \( \kappa_0 \) and on the interval \( \mathcal{I} \) as well as on the measures \( \mu_1, \mu_2 \) through the constants \( \kappa_0 \) in (3.8) and the constant \( L \) in (3.7).

**Proof.** We start by noticing that there is a constant \( \kappa_1 > 0 \) such that
\[
\text{Im } m_{\mu_1} \in \mu_2(z) = \int_{\mathbb{R}} \frac{\eta d(\mu_1 \boxplus \mu_2)(x)}{(x - E)^2 + \eta^2} \geq \int_{\mathcal{I}} \frac{\eta f_{\mu_1, \mu_2}(x)dx}{(x - E)^2 + \eta^2} \geq \kappa_1,
\] uniformly in \( z = E + i\eta \in S_{\mathcal{I}}(0, \eta_M) \), where we used (3.8). Thus by subordination we have
\[
\min_{z \in S_{\mathcal{I}}(0, \eta_M)} |m_{\mu_1} \boxplus \mu_2(z)| = \min_{z \in S_{\mathcal{I}}(0, \eta_M)} \left| \int_{\mathbb{R}} \frac{d\mu_1(x)}{a - \omega_2(z)} \right| \geq \kappa_1,
\] (3.12)
since \( m_{\mu_1} \boxplus \mu_2(z) = 1/F_{\mu_1, \mu_2}(z) \) by (2.6).

On the other hand, \( \mu_1 \) is supported on the interval \([-L, L] \); see (3.7). Hence, using (3.12), \( |\omega_2(z)| \) must be bounded from above on \( S_{\mathcal{I}}(0, \eta_M) \). Interchanging the roles of the indices 1 and 2, we also get that \( |\omega_1(z)| \) is bounded from above on \( S_{\mathcal{I}}(0, \eta_M) \). \[\square\]
Having established upper bounds on the subordination functions, we show that their imaginary parts are uniformly bounded from below on the domain $S_T(0, \eta_M)$. The proof relies on inequality (3.1).

**Lemma 3.3.** Let $\mu_1, \mu_2$ be two probability measures on $\mathbb{R}$ satisfying (3.7), for some $L < \infty$, and assume that neither of them is only supported at a single point. Let $I \subset B_{\mu_1, \mu_2}$ be a compact non-empty interval. Then there is a strictly positive constant $k > 0$ such that

$$\min_{z \in S_T(0, \eta_M)} \text{Im} \omega_1(z) \geq 2k, \quad \min_{z \in S_T(0, \eta_M)} \text{Im} \omega_2(z) \geq 2k. \quad (3.13)$$

**Remark 3.1.** The constant $k$ in (3.13) depends on the interval $I$ through the constants $\kappa_0$ in (3.8) and $M$ in (3.9). It further depends on $\eta_M$, as well as on $\sigma(\mu_1, K_2)$ and $\sigma(\mu_2, K_1)$ in (3.1), with

$$K_i = \{ u \in \mathbb{C}^+ : u = \omega_i(z), z \in S_T(0, \eta_M) \}, \quad i = 1, 2. \quad (3.14)$$

**Proof of Lemma 3.3.** First note that there is $\kappa_1 > 0$ such that $\text{Im} m_{\mu_1, \mu_2}(z) \geq \kappa_1$ for all $z \in S_T(0, \eta_M)$; c.f., (3.11). Moreover, there is $M < \infty$ such that $|m_{\mu_1, \mu_2}(z)| \leq M$ for all $z \in S_T(0, \eta_M)$; c.f., (3.9). Recall from (2.5) and (2.6) that

$$\omega_1(z) + \omega_2(z) = z - \frac{1}{m_{\mu_1, \mu_2}(z)}, \quad z \in \mathbb{C}^+. \quad (3.15)$$

Hence, considering the imaginary part, we notice from (3.9) that there is $\kappa_2 > 0$ such that

$$\min_{z \in S_T(0, \eta_M)} (\text{Im} \omega_1(z) + \text{Im} \omega_2(z)) \geq \kappa_2. \quad (3.16)$$

It remains to show that $\text{Im} \omega_1$ and $\text{Im} \omega_2$ are separately bounded from below. To do so we invoke (3.1) and assume by contradiction that $\text{Im} \omega_1(z) \leq \epsilon$, for some small $0 \leq \epsilon < \kappa_2/2$. We must thus have $\text{Im} \omega_2(z) \geq \kappa_2/2$. Since $\mu_1$ is assumed not to be a single point mass, Lemma 3.1 assures that

$$\text{Im} F_{\mu_1} (\omega_2(z)) \geq \frac{\text{Im} \omega_2(z)}{1 - \sigma(\mu_1, K_2)}, \quad z \in S_T(0, \eta_M), \quad (3.17)$$

with $0 < \sigma(\mu_1, K_2) < 1$, where $K_2$ denotes the image of $S_T(0, \eta_M)$ under the map $\omega_2$ (which is necessarily compact by Lemma 3.2). On the other hand, (2.11) implies

$$\text{Im} F_{\mu_1} (\omega_2(z)) = \text{Im} \omega_2(z) + \text{Im} \omega_1(z) - \text{Im} z, \quad z \in \mathbb{C}^+. \quad (3.18)$$

Since $\text{Im} \omega_1(z) \geq \text{Im} z$, by Proposition 2.1, we get, by comparing (3.18) and (3.17), a contradiction with the assumption that $\text{Im} \omega_1(z) \leq \epsilon$, for sufficiently small $\epsilon$. Repeating the argument with the roles of the indices 1 and 2 interchanged, we get (3.13). \hfill \Box

### 3.2. Linear stability of (2.11)

Having established lower and upper bounds on the subordination functions $\omega_1, \omega_2$, we now turn to the stability of the system $\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0$. Remember that we call the system linearly $S$-stable at $(\omega_1, \omega_2)$ if $\Gamma_{\mu_1, \mu_2}(\omega_1, \omega_2) \leq S$, where $\Gamma_{\mu_1, \mu_2}$ is defined in (2.12).

**Lemma 3.4.** Let $\mu_1, \mu_2$ be two probability measures on $\mathbb{R}$ satisfying (3.7) for some $L < \infty$. Assume that neither of them is a single point mass and that at least one of them is supported at more than two points. Let $I \subset B_{\mu_1, \mu_2}$ be a compact non-empty interval.

Then, there is a finite constant $S$ such that

$$\max_{z \in S_T(0, \eta_M)} \Gamma_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z)) \leq S, \quad (3.19)$$
and

\[ \max_{z \in S_2(0, \eta_\lambda)} |\omega'_1(z)| \leq 2S, \quad \max_{z \in S_2(0, \eta_\lambda)} |\omega'_2(z)| \leq 2S, \]  \hspace{1cm} (3.20)

where \( \omega_1(z), \omega_2(z) \) are the solutions to \( \Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0 \).

**Remark 3.2.** Lemma 3.4 is the first instance where we use that at least one of \( \mu_1 \) and \( \mu_2 \) is supported at more than two points. For definiteness, we assume that \( \mu_1 \) is supported at more than two points. The constant \( S \) in (3.19) depends on the interval \( I \) through the constant \( \kappa_0 \) in (3.8), on the constants \( \eta_\lambda, L \) in (3.7), \( \sigma(\mu_1, K_2) \) and \( \sigma(\mu_2, K_2) \), as well as on \( \tilde{\sigma}(\mu_1, K_2) \) of (3.2) with \( K_2 \) defined in (3.14).

**Proof of Lemma 3.4.** Using (2.12) and Cramer’s rule, \( \Gamma = \Gamma_{\mu_1, \mu_2}(\omega_1, \omega_2, z) \) equals

\[ \Gamma = \frac{1}{1-(F'_{\mu_1}(\omega_2)-1)(F'_{\mu_2}(\omega_1)-1)} \left| \begin{array}{ccc} -1 & -1 & 0 \\ \frac{-F'_{\mu_1}(\omega_1(z)) + 1}{\omega_1(z)} & \frac{-F'_{\mu_2}(\omega_1(z)) + 1}{\omega_1(z)} \\ (3.21) \end{array} \right|. \]

As above, we assume for definiteness that \( \mu_1 \) is supported at more than two points. We first focus on \( F'_{\mu_1}(\omega_2) \). Recalling the definition of \( K_2 \) from Remark 3.1 and invoking (3.2), we obtain

\[ |F'_{\mu_1}(\omega_2(z)) - 1| \leq (1 - \tilde{\sigma}(\mu_1, K_2)) \frac{\text{Im} F'_{\mu_1}(\omega_2(z)) - \text{Im} \omega_2(z)}{\text{Im} \omega_2(z)}, \]  \hspace{1cm} (3.22)

for all \( z \in S_2(0, \eta_\lambda) \), where \( 0 < \tilde{\sigma}(\mu_1, K_2) < 1 \). Abbreviating \( \tilde{\sigma} \equiv \tilde{\sigma}(\mu_1, K_2) \) and using \( \Phi_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z), z) = 0 \), we thus have

\[ |F'_{\mu_1}(\omega_2(z)) - 1| \leq (1 - \tilde{\sigma}) \frac{\text{Im} \omega_1(z)}{\text{Im} \omega_2(z)}, \quad z \in S_2(0, \eta_\lambda). \]  \hspace{1cm} (3.23)

Reasoning in the similar way (c.f., (4.9)), we also obtain

\[ |F'_{\mu_2}(\omega_1(z)) - 1| \leq \frac{\text{Im} \omega_2(z)}{\text{Im} \omega_1(z)}, \quad z \in \mathbb{C}^+, \]  \hspace{1cm} (3.24)

where the inequality may saturate here since we do not exclude \( \mu_2 \) being supported at two points only. Multiplying (3.23) and (3.24), we get

\[ \max_{z \in S_2(0, \eta_\lambda)} |F'_{\mu_1}(\omega_2(z)) - 1||F'_{\mu_2}(\omega_1(z)) - 1| \leq 1 - \tilde{\sigma}. \]  \hspace{1cm} (3.25)

Using Lemma 3.2 and Lemma 3.3, we also have from (3.23) and (3.24) that

\[ \max_{z \in S_2(0, \eta_\lambda)} |F'_{\mu_i}(\omega_j(z)) - 1| \leq \frac{K}{4k}, \quad \{i, j\} = \{1, 2\}. \]  \hspace{1cm} (3.26)

Hence, bounding the operator norm by the Hilbert-Schmidt norm in (3.21), we obtain by (3.26) and (3.25) that

\[ \max_{z \in S_2(0, \eta_\lambda)} \Gamma_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z)) \leq \frac{\sqrt{\tilde{\sigma}}}{\tilde{\sigma}} \left( 1 + \left( \frac{K}{4k} \right)^2 \right)^{1/2} = S, \]  \hspace{1cm} (3.27)

with finite constant \( S \). This proves (3.19).

The estimates in (3.20) follow by differentiating the equation \( \Phi_{\mu_1, \mu_2}(\omega_1(z), \omega_2(z), z) = 0 \) with respect to \( z \). We get

\[ \left( \begin{array}{cc} -1 & F'_{\mu_1}(\omega_2(z)) - 1 \\ F'_{\mu_2}(\omega_1(z)) - 1 & -1 \end{array} \right) \left( \begin{array}{c} \omega'_1(z) \\ \omega'_2(z) \end{array} \right) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right). \]  \hspace{1cm} (3.28)

From (3.19) we know that \( \Phi \) is uniformly \( S \)-stable and we get (3.20) by inverting (3.28). \( \square \)
Remark 3.3. The crucial estimate in the proof above is (3.25). An alternative proof of (3.25) under the assumptions that both $\mu_1$ and $\mu_2$ are supported at more than two points was pointed out by an anonymous referee. From (2.5) we observe that the subordination function $\omega_1(z)$ appears, for fixed $z \in \mathbb{C}^+$, as the fixed point of the map $F_z: \mathbb{C}^+ \to \mathbb{C}^+$,

$$u \mapsto F_z(u) := F_{\mu_1}(F_{\mu_2}(u) - u + z) - F_{\mu_2}(u) - u + z.$$  

Indeed, assuming that $\omega_1, \mu_1, \mu_2$ are supported at least at three points (so that $F_{\mu_1}(z) - z$ and $F_{\mu_2}(z) - z$ are not Möbius transformations), the fixed point $\omega_1(z)$ is attracting as was shown in [4]. Thus, for any fixed $k > 0$, the Schwarz–Pick Theorem and (2.5) imply that for any compact subset $\tilde{K}$ of $\{z \in \mathbb{C}^+ : \text{Re}\omega_1(z), \text{Re}\omega_2(z) \geq 2k\}$ there is a constant $\tilde{\sigma}(\tilde{K}) < 1$ such that $|F_{\mu_1}\omega_1(z) - 1||F_{\mu_2}\omega_2(z) - 1| \leq \tilde{\sigma}(\tilde{K}) < 1$, for any $z \in \tilde{K}$. Thus, under the assumption that $\mu_1$ and $\mu_2$ are both supported at least at three points, (3.25) follows from Lemma 3.2 and Lemma 3.3.

Collecting the results of this section, we obtain the proof of Theorem 2.5.

**Proof of Theorem 2.5.** Lemma 3.3 proves (2.14). Lemma 3.4 proves (2.15). \qed 

4. Perturbations of the system (2.11)

In this section, we study perturbations of the system $F_{\mu_1,\mu_2}(\omega_1, \omega_2, z) = 0$, where $\mu_1, \mu_2$ denote general compactly supported probability measures on $\mathbb{R}$. The main results of this section, Proposition 4.1 below, is used repeatedly in the continuity argument to prove Theorem 2.8. Yet, as noted in Corollary 2.6, it is of interest itself and it is also used in [3].

**Proposition 4.1.** Fix $z_0 \in \mathbb{C}^+$. Assume that the functions $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{r}_1, \tilde{r}_2 : \mathbb{C}^+ \to \mathbb{C}$ satisfy

$$\text{Im}\tilde{\omega}_1(z_0) > 0, \text{Im}\tilde{\omega}_2(z_0) > 0$$

and

$$\Phi_{\mu_1,\mu_2}(\tilde{\omega}_1(z_0), \tilde{\omega}_2(z_0), z_0) = \tilde{r}(z_0),$$

where $\tilde{r}(z) := (\tilde{r}_1(z), \tilde{r}_2(z))^\top$. Assume moreover that there is $\delta \in [0, 1]$ such that

$$|\tilde{\omega}_1(z_0) - \omega_1(z_0)| \leq \delta, \quad |\tilde{\omega}_2(z_0) - \omega_2(z_0)| \leq \delta,$$

where $\omega_1(z), \omega_2(z)$ solve the unperturbed system $F_{\mu_1,\mu_2}(\omega_1, \omega_2, z) = 0$. Assume that there is a constant $S$ such that $\Phi$ is linearly $S$-stable at $(\omega_1(z_0), \omega_2(z_0))$, and assume in addition that there are strictly positive constants $K$ and $k$ with $k > \delta$ and with $k^2 > \delta KS$ such that

$$0 < 2k \leq \text{Im}\omega_1(z_0) \leq K, \quad 0 < 2k \leq \text{Im}\omega_2(z_0) \leq K.$$

Then we have the bounds

$$|\tilde{\omega}_1(z_0) - \omega_1(z_0)| \leq 2S||\tilde{r}(z_0)||, \quad |\tilde{\omega}_2(z_0) - \omega_2(z_0)| \leq 2S||\tilde{r}(z_0)||.$$

**Proof.** Combining (4.3) and (4.2) with $\delta < k$, we get

$$\text{Im}\tilde{\omega}_1(z_0) \geq k, \quad \text{Im}\tilde{\omega}_2(z_0) \geq k.$$

Next, we bound higher derivatives of $F_i \equiv F_{\mu_i}, i = 1, 2$. We first note that by the Nevanlinna representation (3.3) we have

$$\frac{\text{Im}F_i(\omega)}{\text{Im}\omega} = 1 + \int_{\mathbb{R}} \frac{1 + x^2}{|x - \omega|^2} \, d\rho_{F_i}(x), \quad \omega \in \mathbb{C}^+, \quad i = 1, 2.$$  

On the other hand, we also have from (3.3) that

$$|F'_i(\omega) - 1| \leq \int_{\mathbb{R}} \frac{1 + x^2}{|x - \omega|^2} \, d\rho_{F_i}(x), \quad \omega \in \mathbb{C}^+, \quad i = 1, 2.$$
and analogously for higher derivatives, $n \geq 1$,

$$
|F_i^{(n)}(\omega)| \leq \int_{\mathbb{R}} \frac{1 + x^2}{|x - \omega|^{n+1}} \, d\rho_i(x) \leq \frac{1}{(\text{Im} \, \omega)^{n-1}} \int_{\mathbb{R}} \frac{1 + x^2}{|x - \omega|^2} \, d\rho_i(x),
$$

(4.8)

$\omega \in \mathbb{C}^+$, $i = 1, 2$. Thus, combining (4.7), (4.6) and (2.11) we get

$$
|F_i'(\omega_j(z)) - 1| \leq \frac{\text{Im} \, F_i(\omega_j(z)) - \text{Im} \, \omega_j(z)}{\text{Im} \, \omega_j(z)} = \frac{\text{Im} \, \omega_j(z) - \text{Im} \, z}{\text{Im} \, \omega_j(z)}, \quad \{i, j\} = \{1, 2\},
$$

(4.9)

$z \in \mathbb{C}^+$, and similarly, starting from (4.8),

$$
|F_i^{(n)}(\omega_j(z))| \leq \frac{\text{Im} \, \omega_j(z) - \text{Im} \, z}{(\text{Im} \, \omega_j(z))^n}, \quad z \in \mathbb{C}^+, \quad \{i, j\} = \{1, 2\}.
$$

(4.10)

Let $\Omega_i(z) := \tilde{\omega}_i(z) - \omega_i(z)$, $i = 1, 2$, and $\Omega := (\Omega_1, \Omega_2)^T$. Fixing $z = z_0$ and Taylor expanding $F_i(\tilde{\omega}_2(z_0))$ around $\omega_2(z_0)$ we get

$$
F_i'(\omega_2(z_0))\Omega_2(z_0) - \Omega_1(z_0) - \Omega_2(z_0) = \tilde{r}_1(z) - \sum_{n \geq 2} \frac{1}{n!} F_i^{(n)}(\omega_2(z_0))\Omega_2(z_0)^n.
$$

(4.11)

Recalling that $\|\Omega(z_0)\|/2k \leq \delta/k < 1$ and using (4.10) together with (4.3), we obtain from (4.11) the estimate

$$
|F_i'(\omega_2(z_0))\Omega_2(z_0) - \Omega_1(z_0) - \Omega_2(z_0)| \leq \|\tilde{r}_1(z)\| + \frac{K}{4k^2}\|\Omega(z_0)\|^2,
$$

(4.12)

and the analogous expansion with the roles of the indices 1 and 2 interchanged. We therefore obtain from (2.12) and from solving the linearized equation that

$$
\|\Omega(z_0)\| \leq S\|\tilde{r}(z_0)\| + \frac{KS}{4k^2}\|\Omega(z_0)\|^2.
$$

(4.13)

Thus, we have the dichotomy that either $\|\Omega(z_0)\| \leq 2S\|\tilde{r}\|$ or $2(KS)^{-1}k^2 \leq \|\Omega(z_0)\|$. Since $k^2 > \delta KS$ by assumption, the second alternative contradicts $\|\Omega(z_0)\| \leq 2\delta$. This proves the estimates in (4.4). \qed

In Proposition 4.1 we assumed the a priori bound $|\tilde{\omega}_i - \omega_i| \leq \delta$; see (4.2). The next lemma shows that we may drop this assumption, for spectral parameters $z$ with sufficiently large imaginary part, at the price of assuming effective lower bounds on $\text{Im} \, \tilde{\omega}_i$. This statement will be used as an initial input to start the continuity argument in Section 6.

**Lemma 4.2.** Assume there is a (large) $\tilde{\gamma}_0 > 0$ such that for any $z \in \mathbb{C}^+$ with $\text{Im} \, z \geq \tilde{\gamma}_0$ the analytic functions $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{r}_1$, $\tilde{r}_2 : \mathbb{C}^+ \rightarrow \mathbb{C}$ satisfy

$$
\text{Im} \, \tilde{\omega}_1(z) - \text{Im} \, z \geq 2\|\tilde{r}(z)\|, \quad \text{Im} \, \tilde{\omega}_2(z) - \text{Im} \, z \geq 2\|\tilde{r}(z)\|.
$$

(4.14)

and

$$
\Phi_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z), z) = \tilde{r}(z),
$$

(4.15)

where $\tilde{r}(z) := (\tilde{r}_1(z), \tilde{r}_2(z))^T$.

Then there is a constant $\eta_0 > 0$, with $\eta_0 \geq \tilde{\gamma}_0$, such that

$$
|\tilde{\omega}_1(z) - \omega_1(z)| \leq 2\|\tilde{r}(z)\|, \quad |\tilde{\omega}_2(z) - \omega_2(z)| \leq 2\|\tilde{r}(z)\|,
$$

(4.16)

on the domain $\{z \in \mathbb{C}^+ : \text{Im} \, z \geq \eta_0\}$, where $\omega_1$ and $\omega_2$ are the subordination functions associated with $\mu_1$ and $\mu_2$. The constant $\eta_0$ depends on the measures $\mu_1$ and $\mu_2$, and on the function $\tilde{r}$ through the constant $\tilde{\gamma}_0 > 0$. 


Proof. Recall the Nevanlinna representation (3.3) for $F_{\mu_1}$ and $F_{\mu_2}$. Since $\mu_1$ and $\mu_2$ are compactly supported, we have, as $\Im \omega \nearrow \infty$,

$$F_{\mu_1}(\omega) - \omega = a_1 + O(|\omega|^{-1}), \quad F_{\mu_2}(\omega) - \omega = a_2 + O(|\omega|^{-1}),$$

(4.17)

with $a_1 \equiv a_{F_{\mu_1}}$ and $a_2 \equiv a_{F_{\mu_2}}$. There are thus $\tilde{s}_1, \tilde{s}_2 : \mathbb{C}^+ \to \mathbb{C}$ such that

$$\Phi_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z), z) = \begin{pmatrix} a_1 + \tilde{s}_1(z) - \tilde{\omega}_1(z) + z \\ a_2 + \tilde{s}_2(z) - \tilde{\omega}_2(z) + z \end{pmatrix} = \begin{pmatrix} \tilde{r}_1(z) \\ \tilde{r}_2(z) \end{pmatrix},$$

(4.18)

with

$$\tilde{s}_1(z) = O(|\tilde{\omega}_2(z)|^{-1}), \quad \tilde{s}_2(z) = O(|\tilde{\omega}_1(z)|^{-1}),$$

(4.19)

as $\Im z \nearrow \infty$. It follows immediately that $\tilde{\omega}_1(z) = O(\Im z)$ and $\tilde{\omega}_2(z) = O(\Im z)$, as $\Im z \nearrow \infty$. Thus, recalling the definition of $\Gamma_{\mu_1, \mu_2}$ in (2.12), we get

$$\Gamma_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z)) = 1 + O(\eta^{-2}),$$

(4.20)

as $\eta = \Im z \nearrow \infty$. In particular, we obtain

$$\|((D\Phi)^{-1}\Phi)(\tilde{\omega}_1(z), \tilde{\omega}_2(z), z)\| \leq \|\Gamma_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z))\|\|\Phi(\tilde{\omega}_1(z), \tilde{\omega}_2(z), z)\| \leq 2\|\tilde{r}(z)\|,$$

(4.21)

for $\Im z$ sufficiently large. From (4.8) and (4.17), we also get

$$|F^{(2)}_{\mu_i}(\omega)| \leq \frac{\Im F_{\mu_i}(\omega) - \Im \omega}{(\Im \omega)^2} = O((\Im \omega)^{-3}), \quad \omega \in \mathbb{C}^+, \quad i = 1, 2,$$

(4.22)

as $\Im \omega \nearrow \infty$. Thus the matrix of second derivatives of $\Phi$ given by

$$D^2\Phi(\omega_1, \omega_2) := \begin{pmatrix} \frac{\partial^2 \Phi}{\partial \omega_1^2}(\omega_1, \omega_2, z) & \frac{\partial^2 \Phi}{\partial \omega_2^2}(\omega_1, \omega_2, z) \\ \frac{\partial^2 \Phi}{\partial \omega_1 \partial \omega_2}(\omega_1, \omega_2, z) & 0 \end{pmatrix} = \begin{pmatrix} 0 & F^{(2)}_{\mu_i}(\omega_2) \\ F^{(2)}_{\mu_2}(\omega_1) & 0 \end{pmatrix},$$

satisfies $\|D^2\Phi(\tilde{\omega}_1(z), \tilde{\omega}_2(z))\| = O(\Im z)^{-3}$, as $\Im z \nearrow \infty$. Hence, choosing $\eta_0 > 0$ sufficiently large, we achieve that

$$s_0 := 2\|\tilde{r}(z)\| \|D^2\Phi(\tilde{\omega}_1(z), \tilde{\omega}_2(z))\| < \frac{1}{2},$$

on the domain \( \{ z \in \mathbb{C}^+ : \Im z \geq \eta_0 \} \). Thus, by the Newton-Kantorovich theorem (see, e.g., Theorem 1 in [25]), there are for every such $z$ unique $\tilde{\omega}_1(z), \tilde{\omega}_2(z)$ such that $\Phi_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z), z) = 0$, with

$$|\tilde{\omega}_i(z) - \tilde{\omega}_i(z)| \leq \frac{1 - \sqrt{1 - 2s_0}}{s_0} \|\tilde{r}(z)\| \leq 2\|\tilde{r}(z)\|, \quad i = 1, 2.$$

(4.23)

Finally, we note that $\Im \tilde{\omega}_1(z) = \Im \tilde{\omega}_1(z) - \Im \tilde{\omega}_1(z) + \Im \tilde{\omega}_1(z) \geq \Im z$, by (4.14), for $\Im z \geq \eta_0$. Similarly, $\Im \tilde{\omega}_2(z) \geq \Im z$, for $\Im z \geq \eta_0$. It further follows that $\Gamma_{\mu_1, \mu_2}(\tilde{\omega}_1(z), \tilde{\omega}_2(z)) \neq 0$, for all $z \in \mathbb{C}^+$ with $\Im z \geq \eta_0$, thus $\tilde{\omega}_1(z)$ and $\tilde{\omega}_2(z)$ are analytic on \( \{ z \in \mathbb{C} : \Im z > \eta_0 \} \) since $F_{\mu_1}$ and $F_{\mu_2}$ are. Finally, using (4.17) with $\omega = \tilde{\omega}_1, \omega = \tilde{\omega}_2$ respectively, we see that

$$\lim_{\eta \nearrow \infty} \frac{\Im \tilde{\omega}_1(\eta)}{\eta} = \lim_{\eta \nearrow \infty} \frac{\Im \tilde{\omega}_1(\eta)}{\eta} = 1.$$

Thus, by the uniqueness claim in Proposition 2.1, $\tilde{\omega}_1(z), \tilde{\omega}_2(z)$ agree with $\omega_1(z), \omega_2(z)$ on the domain \( \{ z \in \mathbb{C}^+ : \Im z \geq \eta_0 \} \). This proves (4.16) from (4.23). \( \square \)
5. Proof of Theorem 2.7

In the setup of Theorem 2.7 we have two pairs of probability measures on \( \mathbb{R} \), \( \mu_\alpha, \mu_\beta \) and \( \mu_A, \mu_B \), where we consider \( \mu_\alpha, \mu_\beta \) as “reference” measures (in the sense that they satisfy the assumptions of Theorem 2.7), while \( \mu_A, \mu_B \) are arbitrary. Under the assumptions of Theorem 2.7 we can apply Theorem 2.5 with the choices \( \mu_\alpha = \mu_1 \) and \( \mu_\beta = \mu_2 \).

Recall from (2.13) the definition of the domain \( S_\tau(a, b), a \leq b \).

**Lemma 5.1.** Let \( \mu_A, \mu_B \) and \( \mu_\alpha, \mu_\beta \) be the probability measures from (2.24) and (2.25) satisfying the assumptions of Theorem 2.7. Let \( \omega_A, \omega_B \) and \( \omega_\alpha, \omega_\beta \) denote the associated subordination functions by Proposition 2.1. Let \( I \subset B_{\mu_\alpha, \mu_\beta} \) be a compact non-empty interval. Fix \( 0 < \eta_M < \infty \).

Then there are a (small) constant \( b_0 > 0 \) and a (large) constant \( K_1 < \infty \), both depending on the measures \( \mu_\alpha \) and \( \mu_\beta \), on the interval \( I \) and on the constant \( \eta_M \), such that whenever

\[
d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b_0,
\]

holds, then

\[
|\omega_A(z) - \omega_\alpha(z)| \leq K_1 \frac{d_L(\mu_A, \mu_\alpha)}{(\text{Im } \omega_\alpha(z))^2} + K_1 \frac{d_L(\mu_B, \mu_\beta)}{(\text{Im } \omega_\beta(z))^2},
\]

\[
|\omega_B(z) - \omega_\beta(z)| \leq K_1 \frac{d_L(\mu_A, \mu_\alpha)}{(\text{Im } \omega_\alpha(z))^2} + K_1 \frac{d_L(\mu_B, \mu_\beta)}{(\text{Im } \omega_\beta(z))^2},
\]

hold uniformly on \( S_\tau(0, \eta_M) \). In particular, choosing \( b \leq b_0 \) sufficiently small and assuming that \( d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b \), we have,

\[
\max_{z \in S_\tau(0, \eta_M)} |\omega_A(z)| \leq K, \quad \max_{z \in S_\tau(0, \eta_M)} |\omega_B(z)| \leq K,
\]

\[
\min_{z \in S_\tau(0, \eta_M)} \text{Im } \omega_A(z) \geq k, \quad \min_{z \in S_\tau(0, \eta_M)} \text{Im } \omega_B(z) \geq k,
\]

where \( K \) and \( k \) are the constant from Lemma 3.2 and Lemma 3.3, respectively.

**Remark 5.1.** Armed with the conclusions of Theorem 2.5, our proof follows closely the arguments of [30]. We further remark that the main argument in the proof of Lemma 5.1 is different from the ones given in Section 4: it crucially relies on the global uniqueness of solutions on the upper half plane for both systems, \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha, \omega_\beta, z) = 0 \) and \( \Phi_{\mu_A, \mu_B}(\omega_A, \omega_B, z) = 0 \), asserted by Proposition 2.1.

**Proof of Lemma 5.1.** We first write the system \( \Phi_{\mu_A, \mu_B}(\omega_\alpha(z), \omega_\beta(z), z) = 0 \) as

\[
\Phi_{\mu_A, \mu_B}(\omega_\alpha(z), \omega_\beta(z), z) = r(z), \quad z \in \mathbb{C}^+,
\]

with

\[
r(z) \equiv \begin{pmatrix} r_A(z) \\ r_B(z) \end{pmatrix} := \begin{pmatrix} F_{\mu_A}(\omega_\beta(z)) - F_{\mu_\alpha}(\omega_\beta(z)) \\ F_{\mu_B}(\omega_\alpha(z)) - F_{\mu_\beta}(\omega_\alpha(z)) \end{pmatrix}.
\]

From Lemma 3.3, we know that the imaginary parts of the subordination functions \( \omega_A, \omega_B \) are uniformly bounded from below on \( S_\tau(0, \eta_M) \). Next, integration by parts reveals that for any probability measures \( \mu_1 \) and \( \mu_2 \),

\[
|m_1(z) - m_2(z)| \leq c \frac{d_L(\mu_1, \mu_2)}{\text{Im } z} \left( 1 + \frac{1}{\text{Im } z} \right), \quad z \in \mathbb{C}^+,
\]

with some numerical constant \( c \); see, e.g., [31]. Thus,

\[
|F_{\mu_A}(\omega_\beta(z)) - F_{\mu_\alpha}(\omega_\beta(z))| \leq c \frac{|m_A(\omega_\beta(z)) - m_\alpha(\omega_\beta(z))|}{|m_A(\omega_\beta(z))m_\alpha(\omega_\beta(z))|} \leq c \frac{d_L(\mu_A, \mu_\alpha)}{(\text{Im } \omega_\beta(z))^2},
\]

(5.7)
with a new constant $C$ that depends on the lower bound of $\text{Im} m_{\mu_a}(\omega_\beta(z)) = \text{Im} m_{\mu_a \equiv \mu_\beta}(z)$ which is strictly positive on $\mathcal{S}_\tau(0, \eta_M)$; c.f., (3.12). Here we used

$$\text{Im} m_{\mu_A}(\omega_\beta(z)) \geq \text{Im} m_{\mu_A}(\omega_\beta(z)) - |\text{Im} m_{\mu_A}(\omega_\beta(z)) - \text{Im} m_{\mu_A}(\omega_\beta(z))| \geq \frac{1}{2} \text{Im} m_{\mu_A}(\omega_\beta(z)),$$

as follows from (5.6) for small enough $d_L(\mu_A, \mu_\alpha) \leq b_0$. Repeating the argument with the roles of $A$ and $B$ interchanged, we arrive at

$$|r_A(z)| \leq C \frac{d_L(\mu_A, \mu_\alpha)}{(\text{Im} \omega_\beta(z))^2}, \quad |r_B(z)| \leq C \frac{d_L(\mu_B, \mu_\beta)}{(\text{Im} \omega_\alpha(z))^2}, \quad z \in \mathcal{S}_\tau(0, \eta_M), \quad (5.8)$$

for some constant $C$. Recalling the definition of $\Gamma$ in (2.12), we get for sufficiently small $b_0$,

$$\Gamma_{\mu_A, \mu_B}(\omega_\alpha, \omega_\beta) \leq 2\Gamma_{\mu_A, \mu_\beta}(\omega_\alpha, \omega_\beta) \leq 2S, \quad (5.9)$$

where $S$ is from Lemma 3.4, and where we also use Lemma 3.3 and the assumption $d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b_0$. The Newton–Kantorovich theorem then implies (c.f., the proof of Lemma 4.2 for a similar application) that there are $\hat{\omega}_A(z), \hat{\omega}_B(z)$ satisfying

$$\Phi_{\mu_A, \mu_B}(\hat{\omega}_A(z), \hat{\omega}_B(z), z) = 0, \quad z \in \mathcal{S}_\tau(0, \eta_M), \quad (5.10)$$

and

$$|\omega_\alpha(z) - \hat{\omega}_A(z)| \leq 2\|r(z)\|, \quad |\omega_\beta(z) - \hat{\omega}_B(z)| \leq 2\|r(z)\|, \quad (5.11)$$

$z \in \mathcal{S}_\tau(0, \eta_M)$. Invoking (5.8), (5.11) and Lemma 3.3, we see that $\hat{\omega}_A(z) \in \mathbb{C}^+$ and $\hat{\omega}_B(z) \in \mathbb{C}^+$, for any $z \in \mathcal{S}_\tau(0, \eta_M)$ if $b_0$ is sufficiently small. Yet, by the global uniqueness of solutions asserted in Proposition 2.1, we must have $\hat{\omega}_A(z) = \omega_A(z), \hat{\omega}_B(z) = \omega_B(z), z \in \mathbb{C}^+$. Together with (5.11) and (5.8) this implies (5.2) and concludes the proof. Then, choosing $b_0$ sufficiently small, (5.3) and (5.4) are direct consequences of (5.2), Lemma 3.2 and Lemma 3.3. \hfill \square

With the aid of Lemma 3.4, we prove the stability of the system $\Phi_{\mu_A, \mu_B}(\omega_A, \omega_B, z) = 0$.

**Corollary 5.2.** Under the assumptions of Lemma 5.1, there is a (small) constant $b_1 > 0$, depending on the measures $\mu_\alpha$ and $\mu_\beta$, on the interval $I$ and on the constant $\eta_M$, such that

$$d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b_1$$

implies

$$\max_{z \in \mathcal{S}_\tau(0, \eta_M)} \Gamma_{\mu_A, \mu_B}(\omega_A(z), \omega_B(z)) \leq 2S \quad (5.13)$$

and

$$\max_{z \in \mathcal{S}_\tau(0, \eta_M)} |\omega_A'(z)| \leq 4S, \quad \max_{z \in \mathcal{S}_\tau(0, \eta_M)} |\omega_B'(z)| \leq 4S, \quad (5.14)$$

where $\omega_A(z), \omega_B(z)$ satisfy $\Phi_{\mu_A, \mu_B}(\omega_A(z), \omega_B(z), z) = 0$ and $S$ is the constant in Lemma 3.4.

**Proof.** Let $\Gamma \equiv \Gamma_{\mu_A, \mu_B}(\omega_A(z), \omega_B(z))$. Analogously to (3.21), we have

$$\Gamma = \frac{1}{1 - (F_{\mu_A}'(\omega_B) - 1)(F_{\mu_B}'(\omega_A) - 1)} \left\| \begin{pmatrix} -1 & -F_{\mu_A}'(\omega_B(z)) + 1 \\ 0 & -1 \end{pmatrix} \right\|. \quad (5.15)$$

Using the bounds (5.3) and (5.4) for sufficiently small $b_1$, we follow, mutatis mutandis, the proof of Lemma 3.4 to get (5.13). The estimates in (5.14) then follow as in Lemma 3.4. \hfill \square

We are now ready to complete the proof of Theorem 2.7.
Proof of Theorem 2.7. Recall that \( m_{\mu_A} \mathbb{E}_{\mu_B}(z) = m_{\mu_A}(\omega_B(z)), z \in \mathbb{C}^+ \). We first note that
\[
|m_{\mu_A}(\omega_B(z)) - m_{\mu_A}(\omega_B(z))| \leq C \frac{d_L(\mu_A, \mu_A)}{\Im \omega_B(z)} \left( 1 + \frac{1}{\Im \omega_B(z)} \right),
\]
for some numerical constant \( C \); c.f., (5.6). Thus using (5.4) we get
\[
|m_{\mu_A}(\omega_B(z)) - m_{\mu_A}(\omega_B(z))| \leq K_2 k^{-2} d_L(\mu_A, \mu_A), \quad z \in S_k(0, \eta_M),
\]
for some numerical constant \( K_2 \). Choosing \( b \) as in Lemma 5.1 and assuming that \( d_L(\mu_A, \mu_A) + d_L(\mu_A, \mu_B) \leq b \), we get from (5.2) that
\[
|m_{\mu_A}(\omega_B(z)) - m_{\mu_A}(\omega_B(z))| \leq K_1 k^{-4} ((d_L(\mu_A, \mu_A) + d_L(\mu_A, \mu_B)), \quad z \in S_k(0, \eta_M).
\]
Setting \( Z := K_1 k^{-4} + K_2 k^{-2} \) we thus obtain (2.22).

Remark 5.2. Note that under the assumptions of Theorem 2.7, we have, for \( d_L(\mu_A, \mu_A) + d_L(\mu_A, \mu_B) \leq b \), the bounds
\[
\kappa_1/2 \leq |m_{\mu_A} \mathbb{E}_{\mu_B}(z)| \leq 1/k,
\]
uniformly on \( S_k(0, \eta_M) \) with \( \kappa_1 > 0 \) from (3.12) and \( k > 0 \) from (5.4).

6. Proof of Theorem 2.8

Before we immerse into the details of the proof of Theorem 2.8, we outline how Theorem 2.5 and the local stability results of Section 4 in combination with concentration estimates for the unitary groups lead to the local law in (2.28).

6.1. Outline of proof. We briefly outline of our proof when \( U \) is Haar distributed on \( U(N) \). Since we are interested in the tracial quantity \( m_H \) of \( \tilde{H} = A + U B U^* \), we may replace \( \tilde{H} \) by the matrix
\[
\tilde{H} := V A V^* + U B U^*,
\]
where \( V \) is another Haar unitary independent from \( U \). By cyclicity of the trace we have \( m_H = m_{\tilde{H}} \) and we study \( m_{\tilde{H}} \) below. We emphasize that this replacement is a convenient technique which is not essential to our proof.

Using the shorthand
\[
\tilde{A} := V A V^*, \quad \tilde{B} := U B U^*,
\]
we introduce the Green functions
\[
G_{\tilde{A}}(z) := (\tilde{A} - z)^{-1}, \quad G_{\tilde{B}}(z) := (\tilde{B} - z)^{-1}, \quad z \in \mathbb{C}^+.
\]
For a given \( N \times N \) matrix \( Q \), we introduce the function
\[
f_Q(z) := \text{tr} Q G_{\tilde{H}}(z), \quad z \in \mathbb{C}^+,
\]
where \( G_{\tilde{H}} = (\tilde{H} - z)^{-1} \) is the Green function of \( \tilde{H} \). We define the approximate subordination functions, \( \omega_{\tilde{A}} \) and \( \omega_{\tilde{B}} \), by setting
\[
\omega_{\tilde{A}}(z) := z - \frac{E f_{\tilde{A}}(z)}{E m_{\tilde{H}}(z)}, \quad \omega_{\tilde{B}}(z) := z - \frac{E f_{\tilde{B}}(z)}{E m_{\tilde{H}}(z)}, \quad z \in \mathbb{C}^+,
\]
where the expectation \( E \) is with respect to both Haar unitaries \( U \) and \( V \). From the identity \((\tilde{H} - z) G_{\tilde{H}}(z) = 1, z \in \mathbb{C}^+ \), we then obtain the relation
\[
\omega_{\tilde{A}}(z) + \omega_{\tilde{B}}(z) - z = -\frac{1}{E m_{\tilde{H}}(z)}, \quad z \in \mathbb{C}^+,
\]
reminiscent to (c.f., (2.5)–(2.6))

\[ \omega_A(z) + \omega_B(z) - z = -\frac{1}{m_{ABB}(z)}, \quad z \in \mathbb{C}^+. \]

For the proof of Theorem 2.8, we decompose

\[ m_{\tilde{H}}(z) - m_{ABB}(z) = (m_{\tilde{H}}(z) - \mathbb{E}m_{\tilde{H}}(z)) + (\mathbb{E}m_{\tilde{H}}(z) - m_{ABB}(z)), \quad (6.7) \]

where we abbreviate \( m_{ABB} \equiv m_{\mu_A \mu_B} \). To control the fluctuation part, \( m_{\tilde{H}}(z) - \mathbb{E}m_{\tilde{H}}(z) \), we rely on the Gromov–Milman concentration inequality [26] for the unitary group; see (6.22) below. To control the deterministic part, we first note that, by (6.6) and \( m_{ABB}(z) = m_A(\omega_B((z))) \), bounding \( |\mathbb{E}m_{\tilde{H}}(z) - m_{ABB}(z)| \) amounts to bounding \( |\omega_A^c(z) - \omega_A(z)| \) and \( |\omega_B^c(z) - \omega_B(z)| \). We then show that \( \omega_A^c(z) \) and \( \omega_B^c(z) \) are both in the upper-half plane and satisfy

\[ \Phi_{\mu_A, \mu_B}(\omega_A^c(z), \omega_B^c(z), z) = r(z), \quad z \in S_T(\eta_m, \eta_M), \quad (6.8) \]

for some small error \( r(z) \in \mathbb{C}^+ \), i.e., we consider (6.8) as a perturbation of the system \( \Phi_{\mu_A, \mu_B}(\omega_A(z), \omega_B(z), z) = 0 \); c.f., (2.10). The formal derivation of (6.8) goes back to Pastur and Vasilchuk [33]. Using Proposition 4.1 (with rough a priori estimates on \( |\omega_A^c(z) - \omega_A(z)| \) and \( |\omega_B^c(z) - \omega_B(z)| \) obtained from the continuity argument below) and stability results of Theorem 2.5 and of Section 5, we then bound \( |\omega_A^c(z) - \omega_A(z)| \) and \( |\omega_B^c(z) - \omega_B(z)| \) in terms of \( r(z) \).

In sum, for fixed \( z \in \mathbb{C}^+ \), our proof includes two parts: (i) estimation of the error \( r(z) \) in (6.8) and (ii) concentration for \( m_{\tilde{H}}(z) \) around \( \mathbb{E}m_{\tilde{H}}(z) \). Both parts rely on the estimates

\[ \mathbb{E}m_{\tilde{H}}(z), \omega_A^c(z), \omega_B^c(z) \sim 1, \quad \text{Im} \omega_A^c(z), \text{Im} \omega_B^c(z) \gtrsim 1, \quad z \in S_T(\eta_m, \eta_M). \quad (6.9) \]

Note that the quantities in (6.9) are obtained from the Green function of \( \tilde{H} \) by averaging with respect to the Haar measure. Similar bounds for \( m_{ABB}, \omega_A \) and \( \omega_B \) were obtained in Section 5. These latter quantities are defined directly from \( \mu_A \) and \( \mu_B \) via Proposition 2.1.

To establish (6.9), we use a similar continuity argument as was used for Wigner matrices in [24]: For \( \text{Im} z = \eta_M \) sufficiently large, the estimates in (6.9) directly follow from definitions. For \( z = E + i\eta \), with \( E \in \mathbb{T} \) fixed, we decrease \( \eta = \eta_M \) down to \( \eta = \eta_m \) in steps of size \( O(N^{-5}) \), where, at each step, we invoke parts (i) and (ii). However, a direct application of the Gromov–Milman concentration inequality for part (i) does not allow to push \( \eta \) below the mesoscopic scale \( \eta = N^{-1/2} \). Indeed, the Gromov–Milman inequality is effective if \( L^2/N = o(1) \), where \( L \) is the Lipschitz constant of \( m_{\tilde{H}}(z) \) with respect to the Haar unitary \( V \). It is roughly bounded by \( \sqrt{\text{tr}[G_{\tilde{H}}(z)]^4/N} \), which in turn is trivially bounded by \( 1/\sqrt{N\eta^{\gamma}} \), giving the \( \eta \geq N^{-1/2+\gamma}, \gamma > 0 \), threshold. However, in reality, the random quantity \( \sqrt{\text{tr}[G_{\tilde{H}}(z)]^4/N} \) is typically of order \( 1/\sqrt{N\eta^{\gamma}} \) as follows by combining the deterministic estimate \( \text{tr}[G_{\tilde{H}}(z)]^4 \leq \eta^{-3}\text{Im} m_{\tilde{H}}(z) \) with a probabilistic order one bound for \( \text{Im} m_{\tilde{H}}(z) \).

Our key novelty here is to capitalize on this latter information. We introduce a smooth cutoff that regularizes \( m_{\tilde{H}}(z) \) and then apply the Gromov–Milman inequality for this regularized quantity. With the bound \( 1/\sqrt{N\eta^{\gamma}} \) for the Lipschitz constant, we get concentration estimates down to scales \( \eta \geq N^{-2/3+\gamma}, \gamma > 0 \).

6.1.1. Notation. The following notation for high-probability estimates is suited for our purposes. A slightly different form was first used in [21].

**Definition 6.1.** Let

\[ X = (X^{(N)}(v) : N \in \mathbb{N}, v \in \mathcal{V}^{(N)}), \quad Y = (Y^{(N)}(v) : N \in \mathbb{N}, v \in \mathcal{V}^{(N)}) \quad (6.10) \]
be two families of nonnegative random variables where $\mathcal{V}^{(N)}$ is a possibly $N$-dependent parameter set. We say that $Y$ stochastically dominates $X$, uniformly in $v$, if for all (small) $\epsilon > 0$ and (large) $D > 0$, 
\[
\mathbb{P} \left( \bigcup_{v \in \mathcal{V}^{(N)}} \left\{ X^{(N)}(v) > N^D Y^{(N)}(v) \right\} \right) \leq N^{-D}, \tag{6.11}
\]
for sufficiently large $N \geq N_0(\epsilon, D)$. If $Y$ stochastically dominates $X$, uniformly in $v$, we write $X \prec Y$. If we wish to indicate the set $\mathcal{V}^{(N)}$ explicitly, we write that $X(v) \prec Y(v)$ for all $v \in \mathcal{V}^{(N)}$.

6.2. Localized Gromov–Milman concentration estimate. In this subsection, we derive concentration bounds for some key tracial quantities. They are tailored for the continuity argument of Subsection 6.3 used to complete the proof of Theorem 2.8. The argument works with $U, V$ independent and both Haar distributed on $U(N)$ or on $O(N)$. Below, $\mathbb{E}$ denotes the expectation with respect Haar measure.

In the rest of this section, we let $\mathcal{I} \subset B_{\|\cdot\|_{\mu,\gamma}}$ denote the compact non-empty subset fixed in Theorem 2.8. Also recall from Theorem 2.8 that we set $\eta_m = N^{-2/3+\gamma}, \gamma > 0$. Below we choose the constant $\eta_M \sim 1$ to be sufficiently large at first, but from the proof it will be clear that we can eventually choose $0 < \eta_M \ll \infty$ arbitrary. Recall from (6.4) the notation $f_Q$, where $Q$ is an arbitrary $N \times N$ matrix.

**Proposition 6.2.** Let $Q$ be a given $N \times N$ deterministic matrix with $\|Q\| \lesssim 1$. Fix $E \in \mathcal{I}$ and $\tilde{\eta} \in [\eta_m, \eta_M]$. Then
\[
\Im m_{\tilde{\eta}}(E + i\eta) \prec 1, \quad \forall \eta \in [\tilde{\eta}, \eta_M], \tag{6.12}
\]
implies the concentration bound
\[
|f_{VQV^*}(E + i\tilde{\eta}) - \mathbb{E} f_{VQV^*}(E + i\tilde{\eta})| \ll \frac{1}{\sqrt{N^2\eta}}. \tag{6.13}
\]
The same concentration holds with $VQV^*$ replaced by $QUQ^*$.

**Proof.** For fixed $E \in \mathcal{I}$, we consider $z = E + i\eta \in \mathbb{C}^+$ as a varying spectral parameter and use $\tilde{z} = E + i\tilde{\eta}$ for the specific choice in the lemma. By the definition of $f_{(\cdot)}$ and cyclicity of the trace, we have
\[
f_{VQV^*}(z) = \text{tr} VQV^*(VAV^* + UBU^* - z)^{-1} = \text{tr} Q(A + V^*UBU^*V - z)^{-1}, \tag{6.14}
\]
where $\text{tr} (\cdot)$ stands for the normalized trace. For simplicity, we denote
\[
W := V^*U, \quad H := A + WBW^*, \quad G_H(z) := (H - z)^{-1}. \tag{6.15}
\]
Observe that $W$ is Haar distributed on $U(N)$, respectively $O(N)$, too. By cyclicity of the trace we have $\text{tr} G_H(z) = \text{tr} G_H(z) = m_H(z)$. According to (6.14) and (6.15), we may regard in the sequel $f_{VQV^*}$ as a function of the Haar unitary matrix $W$ by writing
\[
h(z) = h_W(z) := f_{VQV^*}(z).
\]
For any fixed (small) $\epsilon > 0$, let $\tilde{\chi}$ be a smooth cutoff supported on $[0, 2N^\epsilon]$, with $\tilde{\chi}(x) = 1, x \in [0, N^\epsilon]$, and with bounded derivatives. Since $m_H(z) = \text{tr} G_H(z)$, we can regard $m_H(z)$ as a function of $W$ and write
\[
\chi(z) = \chi_W(z) := \tilde{\chi}(\text{Im} m_H(z)). \tag{6.16}
\]
We then introduce a regularization, \( \tilde{h}_W \), of \( h_W \) by setting

\[
\tilde{h}(z) = \tilde{h}_W(z) := h_W(z) \prod_{n=0}^{\lceil -\log_2 \eta \rceil} \chi_W(E + i2^n \eta).
\]  

(6.17)

We will often drop the \( W \) subscript from the notations \( h_W(z) \), \( \tilde{h}_W(z) \) and \( \chi_W(z) \) but remember that these are random variables depending on the Haar unitary \( W \).

We will use assumption (6.12) at dyadic points, \( i.e., \) that

\[
\text{Im} m_H(E + i2^l \eta) < 1, \quad 0 \leq l \leq \lceil -\log_2 \eta \rceil,
\]  

(6.18)

(recall that \( m_H(z) = m_H^*(z) \) so we may drop the tilde in the subscript of \( m \)). Hence, by (6.16) and (6.18) we see that, for arbitrary large \( D > 0 \),

\[
\prod_{l=0}^{\lceil -\log_2 \eta \rceil} \chi(E + i2^l \eta) = 1, \quad \text{i.e.,} \quad \tilde{h}(\eta) = h(\eta),
\]  

(6.19)

with probability larger than \( 1 - N^{-D} \), for \( N \) sufficiently large (depending on \( \epsilon \) and \( D \)).

Taking the trivial bound \(|Q|/\hat{h}^2\) for \( h(\eta) \) and for \( \tilde{h}(\eta) \) into account, we also have

\[
\mathbb{E} \tilde{h}(\eta) - \mathbb{E} h(\eta) = O(N^{-D+1}).
\]  

(6.20)

To prove (6.13), it therefore suffices to establish the concentration estimate

\[
\left| \tilde{h}(\eta) - \mathbb{E} \tilde{h}(\eta) \right| < \frac{1}{\sqrt{N^{2}\eta^{1}}},
\]  

(6.21)

for the regularized quantity \( \tilde{h}(\eta) \).

To verify (6.21), we use the Gromov–Milman concentration inequality [26] (see Theorem 4.4.27 in [2] for similar applications) which states the following. Let \( M(N) = \text{SO}(N) \) or \( SU(N) \) endowed with the Riemann metric \(|ds|_2\) inherited from \( M_N(\mathbb{C}) \) (equipped with the Hilbert–Schmidt norm). If \( g : (M(N),|ds|_2) \to \mathbb{R} \) is an \( L \)-Lipschitz function satisfying \( \mathbb{E} g = 0 \), then

\[
\mathbb{P}(|g| > \delta) \leq e^{-\frac{\delta^2}{c}}, \quad \forall \delta > 0,
\]  

(6.22)

with some numerical constant \( c > 0 \) (depending only on the symmetry type and not on \( N \)). Here \( \mathbb{P} \) and \( \mathbb{E} \) are with respect Haar measure on \( M(N) \).

In order to apply (6.22) to the function \( W \mapsto \tilde{h}_W(\eta) = \tilde{h}(\eta) \), we need to control its Lipschitz constant. To that end, we define the event

\[
\Omega(\eta) \equiv \Omega_E(\eta) := \{ \text{Im} m_H(E + i2^n \eta) \leq 2N^\epsilon : \forall n \in \mathbb{N}_0 \}.
\]  

(6.23)

To bound the Lipschitz constant, we need to bound quantities of the form \( \text{tr} |G_H(\eta)|^k \) restricted to the event \( \Omega(\eta) \). Let \( (\lambda_i(\mathcal{H})) \) denote the eigenvalues of \( \mathcal{H} \) and introduce

\[
\mathcal{I}_n := [E - 2^n \eta, E + 2^n \eta] \cap \mathcal{I}, \quad N_n := |\{ i : \lambda_i(\mathcal{H}) \in \mathcal{I}_n \}|, \quad n \in \mathbb{N}_0.
\]

Since \( H \) and \( \mathcal{H} \) are unitarily equivalent, their empirical eigenvalue distributions are the same, \( \mu_{H} ; \text{c.f., } (2.29) \). Using the definition of the Stieltjes transform we have, for all \( n \in \mathbb{N}_0 \), the estimate

\[
N_n = N \int_{\mathcal{I}_n} d\mu_H \leq \frac{3}{2} N \cdot 2^{n+1} \eta \int_{E-2^n \eta}^{E+2^n \eta} \frac{\eta d\mu_H(x)}{(x-E)^2 + \eta^2} \leq 3N \cdot 2^n \eta \text{Im} m_H(\eta).
\]

Thus we have on the event \( \Omega(\eta) \) that

\[
N_n \leq 2^n N^{1+\epsilon} \eta, \quad \forall n \in \mathbb{N}_0.
\]  

(6.24)
By the spectral theorem, we can bound
\[ \text{tr } |G_H(\tilde{z})| \lesssim \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\lambda_i(H) - E| + \eta}. \]  
(6.25)

Then we observe (with the convention $\mathcal{I}_{-1} = \emptyset$) that
\[ \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\lambda_i(H) - E| + \eta} = \frac{1}{N} \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} 1(\lambda_i \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}) \frac{1}{|\lambda_i(H) - E| + \eta}, \]
where we used $\|H\| \leq C$ to truncate the sum over $n$ at $[c \log N]$. We then bound
\[ 1(\Omega(\tilde{\eta})) \frac{1}{N} \sum_{n=0}^{\infty} \sum_{\lambda_i \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}} 1(\Omega(\tilde{\eta})) \frac{1}{n} \frac{|\lambda_i(H) - E| + \eta}{2^\eta} \lesssim N^c \log N, \]
where we used (6.24), i.e., with (6.25) we arrive at
\[ 1(\Omega(\tilde{\eta})) \text{tr } |G_H(\tilde{z})| \lesssim N^c \log N. \]  
(6.26)

Using the spectral decomposition of $H$ we see that
\[ \text{tr } |G_H(\tilde{z})|^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\lambda_i(H) - E|^2 + \tilde{\eta}^2} = \frac{1}{N} \sum_{i=1}^{N} \frac{\tilde{\eta}}{|\lambda_i(H) - E| + \tilde{\eta}} = \frac{\text{Im } m_H(\tilde{z})}{\tilde{\eta}}, \]  
(6.27)
where we also used that $\text{tr } G_H(\tilde{z}) = \text{tr } G_H(\tilde{z}) = m_H(\tilde{z})$. Thus, we bound
\[ 1(\Omega(\tilde{\eta})) \text{tr } |G_H(\tilde{z})|^k \leq 1(\Omega(\tilde{\eta})) \tilde{\eta}^{-k+1} \text{Im } m_H(\tilde{z}) \lesssim N^c \tilde{\eta}^{-k+1}, \quad \forall k \geq 2. \]  
(6.28)

Having established (6.26) and (6.28), we proceed to estimate the Lipschitz constant of $\tilde{h}(\tilde{z})$ as a function of $W$. Let $\mathfrak{su}(N)$ and $\mathfrak{so}(N)$ denote the (fundamental representations in $M_N(\mathbb{C})$ of the) Lie algebras of $SU(N)$ and $SO(N)$ respectively. Let $\mathfrak{m}$ stand for either $\mathfrak{su}(N)$ or $\mathfrak{so}(N)$. Note that $X \in \mathfrak{m}$ satisfies $X^* = -X$. Since $SU(N)$ and $SO(N)$ are matrix groups the Lie bracket of $\mathfrak{su}(N)$ and $\mathfrak{so}(N)$ respectively is given by the commutator in the matrix algebras. For fixed $X \in M_N(\mathbb{C})$, we let $\text{ad}_X : M_N(\mathbb{C}) \to M_N(\mathbb{C})$, $Y \mapsto \text{ad}_X(Y) := XY - YX$. For $X \in \mathfrak{m}$ and $t \in \mathbb{R}$, we may write $e^{t\text{ad}_X(WBW^*)} = (e^{tX}W)B(e^{tX}W)^*,$ where we used that $X^* = -X$. Further note that
\[ \frac{d}{dt} e^{t\text{ad}_X(WBW^*)} = e^{t\text{ad}_X(WBW^*)} \text{ad}_X(WBW^*). \]  
(6.29)

For $X \in \mathfrak{m}$ with $\|X\|_2 = 1$, we let
\[ G_H(z, tX) := \left( A + e^{t\text{ad}_X(WBW^*)} - z \right)^{-1}, \quad t \in \mathbb{R}, \]
and denote accordingly
\[ m_H(z, tX) := \text{tr } G_H(z, tX), \quad \chi(z, tX) := \hat{\chi}(\text{Im } m_H(z, tX)), \]
\[ \tilde{h}(z, tX) := \text{tr } QG_H(z, tX) \prod_{l=0}^{\lfloor \log_2 \tilde{\eta} \rfloor} \chi(E + i2^l \eta, tX), \]
with $\chi(z, 0) \equiv \chi(z), \tilde{h}(z, 0) \equiv \tilde{h}(0)$, etc.
Evaluating the derivative of \( \tilde{h}(z, tX) \) with respect to \( t \) at \( t = 0 \) we get

\[
\frac{\partial}{\partial t} \tilde{h}(z, tX) \bigg|_{t=0} = - \text{tr} \left( QG_H(\tilde{z}) \text{ad}_X(WB^*)G_H(\tilde{z}) \right) \prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta) \\
- \text{tr} \left( QG_H(\tilde{z}) \right) \left( \sum_{j=0}^{[-\log_2 \eta]} \prod_{l \neq j} \chi(E + i2^l \eta) \cdot \phi(E + i2^l \eta) \right) \\
\times \text{Im} \text{tr} \left( G_H(E + i2^l \eta) \text{ad}_X(WB^*)G_H(E + i2^l \eta) \right),
\]

(6.30)

where we used (6.29) and where we introduced \( \phi(z) := \tilde{\chi}'(\text{Im} m_H(z)) \), with \( \tilde{\chi}' \) the derivative of \( \tilde{\chi} \). Recalling (6.16) and the definition of the cutoff \( \tilde{\chi} \), we note the bounds

\[
\prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta) \leq 1,
\]

(6.31)

\[
\sum_{j=0}^{[-\log_2 \eta]} \prod_{l \neq j} \chi(E + i2^l \eta) \cdot \phi(E + i2^l \eta) = O(\log N).
\]

On the event \( \Omega^c(\tilde{\eta}) \), the complementary event to \( \Omega(\tilde{\eta}) \), we further have the identities

\[
\prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta) = 0,
\]

(6.32)

\[
\sum_{j=0}^{[-\log_2 \eta]} \prod_{l \neq j} \chi(E + i2^l \eta) \cdot \phi(E + i2^l \eta) = 0.
\]

It thus suffices to bound (6.30) on the event \( \Omega(\tilde{\eta}) \). We bound the first term on the right side of (6.30) as

\[
1(\Omega(\tilde{\eta})) \left| \text{tr} \left( QG_H(\tilde{z}) \text{ad}_X(WB^*)G_H(\tilde{z}) \right) \prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta) \right|
\]

\[
\leq 1(\Omega(\tilde{\eta})){\frac{1}{N}} \| \text{ad}_X(WB^*) \|_2 \| G_H(\tilde{z})QG_H(\tilde{z}) \|_2 \prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta),
\]

(6.33)

where we used cyclicity of the trace and Cauchy–Schwarz inequality. Next, note that \( \| \text{ad}_X(WB^*) \|_2 \leq 2 \| B \| \| X \|_2 \leq 2 \| B \| , \) where we used the definition of \( \text{ad}_X \), \( \| W \| \leq 1 \) and \( \| X \|_2 = 1 \). Similarly, we have \( \| G_H(\tilde{z})QG_H(\tilde{z}) \|_2 \leq \| Q \| \| G_H(\tilde{z})G_H(\tilde{z}) \|_2 \). Thus from (6.33),

\[
1(\Omega(\tilde{\eta})) \left| \text{tr} \left( QG_H(\tilde{z}) \text{ad}_X(WB^*)G_H(\tilde{z}) \right) \prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta) \right|
\]

\[
\leq 2 \| B \| \| Q \| 1(\Omega(\tilde{\eta})){\left( \frac{\text{tr} |G_H(\tilde{z})|^4}{N} \right)}^{\frac{1}{2}} \prod_{l=0}^{[-\log_2 \eta]} \chi(E + i2^l \eta)
\]

\[
\lesssim \left( \frac{N^e}{N^\eta} \right)^{\frac{1}{2}},
\]

(6.34)

where we used (6.28) with \( k = 4 \) in the last step.
To handle the second term on the right side of (6.30), we use (6.31) and (6.26) to get

$$1(\Omega(\tilde{\eta})) |\text{tr}(QG_H(\tilde{z}))| \sum_{j=0}^{[-\log_3 \tilde{\eta}]} \prod_{l=0}^{[-\log_2 \tilde{\eta}]} \chi(E + i2^l \tilde{\eta}) \cdot \phi(E + i2^l \tilde{\eta})$$

$$\times \text{Im} \text{tr} \left( G_H(E + i2^{l} \tilde{\eta}) \text{ad}_X(\mathcal{W} \mathcal{B} \mathcal{W}^*) G_H(E + i2^l \tilde{\eta}) \right)$$

$$\leq 1(\Omega(\tilde{\eta})) |Q||\text{tr}|G_H(\tilde{z})| \sum_{j=0}^{[-\log_3 \tilde{\eta}]} \prod_{l=0}^{[-\log_2 \tilde{\eta}]} \chi(E + i2^l \tilde{\eta}) \cdot |\phi(E + i2^l \tilde{\eta})|$$

$$\times 2 \|B\| \|Q\| \left( \frac{|\text{tr}|G_H(E + i2^{l} \tilde{\eta})|^4}{N} \right)^{\frac{1}{2}}$$

$$\lesssim \left( \frac{N^{4e}}{N^{\eta}} \right)^{\frac{1}{2}}.$$  (6.35)

Combining (6.35) and (6.34) we obtain, for any $X \in \mathfrak{su}(N)$ or $\mathfrak{so}(N)$ with $\|X\|_2 = 1$, that

$$\left| \frac{\partial}{\partial t} \tilde{h}(\tilde{z}, tX) \right|_{t=0} \lesssim \left( \frac{N^{4e}}{N^{\eta}} \right)^{\frac{1}{2}},$$  (6.36)

i.e., the Lipschitz constant of $\tilde{h}(\tilde{z})$ as a function of $W$ is bounded by $C \left( \frac{N^{4e}}{N^{\eta}} \right)^{1/2}$, for some constant $C$ depending only on $\|B\|$ and $\|Q\|$. Thus, taking

$$g = \tilde{h}(z) - \mathbb{E} \tilde{h}(z), \quad \mathcal{L} = C \left( \frac{N^{4e}}{N^{\eta}} \right)^{\frac{1}{2}}, \quad \delta = N^{3e}/\sqrt{N^{2}\tilde{\eta}^3}$$

in (6.22), and choosing $\varepsilon > 0$ sufficiently small, we get (6.21). Together with (6.19) and (6.20) this implies (6.13).

6.3. Continuity argument. In this subsection, we often omit $z \in \mathbb{C}^+$ from the notation. Let $U$ and $V$ be independent and both Haar distributed on either $U(N)$ or $O(N)$. Recalling the notation in Section 6.1, we set

$$\Delta_A(z) := - (\mathbb{E}[m_H(z)])G_H(z) - (\mathbb{E}[f_H(z)])G_A(z)G_H(z),$$

$$\Delta_B(z) := - (\mathbb{E}[m_H(z)])G_H(z) - (\mathbb{E}[f_H(z)])G_B(z)G_H(z), \quad z \in \mathbb{C}^+, \quad (6.37)$$

where we introduced $\mathbb{E} X := X - \mathbb{E} X$, for any random variables $X$. Using the left-invariance of Haar measure, one derives the identities

$$\mathbb{E}[G_H \otimes \tilde{A}G_H] = \mathbb{E}[\tilde{A}G_H \otimes \tilde{G}_H], \quad \mathbb{E}[G_H \otimes \tilde{B}G_H] = \mathbb{E}[\tilde{B}G_H \otimes \tilde{G}_H];$$

see Theorem 7 in [33] or Appendix A of [31] for proofs. Taking the partial trace for the first component of the tensor products, we get

$$\mathbb{E} G_H(z) = \mathbb{E} \tilde{A}(\omega(z)) + \delta_A(z), \quad \delta_A(z) := \frac{1}{\mathbb{E} m_H(z)} \mathbb{E} G_A(\omega(z))(\tilde{A} - z) \Delta_A(z),$$

(6.38)

where $\omega(z)$ is defined in (6.5), we used (6.6), and where we implicitly assumed that $\text{Im} \omega(z) > 0$. This last assumption will be verified along the continuity argument. Then, we set

$$r_A(z) := - \frac{\text{tr} \delta_A(z)}{\text{tr} G_A(\omega(z))(\text{tr} G_A(\omega(z)) + \text{tr} \delta_A(z))},$$

(6.39)
and define $\delta^c_B(z)$ and $r^c_B(z)$ in the same way by swapping the roles of $A$ and $B$. Using (6.38), (6.6), we eventually obtain, under the assumption that $\text{Im} \omega_A^c(z) > 0$ and $\text{Im} \omega_B^c(z) > 0$,

$$\Phi_{\mu_A, \mu_B}(\omega_A^c(z), \omega_B^c(z), z) = r^c(z), \quad z \in \mathbb{C}^+,$$  

(6.40)

with $r^c(z) = (r^c_A(z), r^c_B(z))^\top$.

**Lemma 6.3.** Fix $E \in \mathcal{I}$ and any $\tilde{\eta} \in [\eta_m, \eta_M]$. Set the notation $z = E + i\eta$ and $\tilde{z} = E + i\tilde{\eta}$. Suppose that

$$|\omega_A^c(z) - \omega_A(z)| + |\omega_B^c(z) - \omega_B(z)| \leq N^{-\gamma}, \quad \forall \eta = \text{Im} \ z \in [\tilde{\eta}, \eta_M].$$  

Moreover, assume that for the event

$$\Xi(\tilde{\eta}) \equiv \Xi_E(\tilde{\eta}) := \left\{ |m_H(z) - m_{A\#B}(z)| \leq N^{-\gamma} : z = E + i\eta, \ \forall \eta \in [\tilde{\eta}, \eta_M] \right\}$$

we have

$$\mathbb{P}(\Xi(\tilde{\eta})) \geq 1 - N^{-D} (1 + N^5 (\eta_M - \tilde{\eta})),$$  

(6.42)

for any $D > 0$ if $N \geq N_1(D)$. Then, for any $\epsilon > 0$, the estimates

$$|r^c_A(\tilde{z})| + |r^c_B(\tilde{z})| \leq \frac{N^\epsilon}{N^2 \tilde{\eta}^4},$$  

(6.43)

$$|\omega_A^c(\tilde{z}) - \omega_A(\tilde{z})| + |\omega_B^c(\tilde{z}) - \omega_B(\tilde{z})| \leq \frac{N^\epsilon}{N^2 \tilde{\eta}^4},$$  

(6.44)

$$|\text{Im} m_H(\tilde{z}) - m_{A\#B}(\tilde{z})| \leq \frac{N^\epsilon}{N^2 \tilde{\eta}^4},$$  

(6.45)

hold for any $N \geq N_2(\epsilon)$. Moreover, for any $\epsilon, D > 0$, the event

$$\Theta(\tilde{\eta}) \equiv \Theta_E(\tilde{\eta}) := \Xi_E(\tilde{\eta}) \cap \left\{ |m_H(z) - m_{A\#B}(z)| \geq \frac{N^\epsilon}{\sqrt{N^2 \tilde{\eta}^4}} \right\}$$  

(6.46)

satisfies

$$\mathbb{P}(\Theta(\tilde{\eta})) \leq N^{-D},$$  

(6.47)

if $N \geq N_3(\epsilon, D)$. The threshold functions $N_1, N_2, N_3$ depend only on $\mu_A, \mu_B$, the speed of convergence in (2.25) and they are uniform in $\tilde{\eta} \in [\eta_m, \eta_M]$ and $E \in \mathcal{I}$.

We postpone the proof of Lemma 6.3 and prove Theorem 2.8 first.

**Proof of Theorem 2.8.** We start with observing that it is sufficient to prove a version of (2.28) where the real part of the spectral parameter $E$ is fixed. This version asserts that there is a large ($N$-independent) $\eta_M$, to be fixed below, such that for any (small) $\epsilon > 0$ and (large) $D$, and any fixed $E \in \mathcal{I}$,

$$\mathbb{P} \left( \bigcup_{z \in S_E(\eta_m, \eta_M)} \left\{ |m_H(z) - m_{\mu_A \# \mu_B}(z)| \geq \frac{N^\epsilon}{N(\text{Im} z)^{3/2}} \right\} \right) \leq \frac{1}{N^D},$$  

(6.48)

holds for $N \geq N_0$, i.e., the set $S_E(\eta_m, \eta_M)$ in (2.28) is replaced with $S_E(\eta_m, \eta_M) := \{ E + i\eta : \eta \in [\eta_m, \eta_M] \}$. The threshold $N_0$ depends on $\epsilon, D, \mu_A, \mu_B, \mathcal{I}$ and on the speed of convergence in (2.25).

Indeed, by introducing the discretized lattice version

$$\tilde{S}_E(a, b) := S_E(a, b) \cap N^{-5} \{ \mathbb{Z} \times i\mathbb{Z} \}$$
of the spectral domain $S_2(a, b)$ (c.f., (2.13)) and by taking a union bound, we see that (6.48) implies
\[
P \left( \bigcup_{z \in S_2(\eta_m, \eta_m)} \left\{ \left| m_H(z) - m_{\mu_A \mu_B}(z) \right| > \frac{N^\epsilon}{N |\Im z|^{3/2}} \right\} \right) \leq \frac{C}{N^{D-\delta}}. \tag{6.49}
\]

Thanks to the Lipschitz continuity of the Stieltjes transforms $m_H(z)$ and $m_{\mu_A \mu_B}(z)$ with Lipschitz constant $\eta^{-2} = (\Im z)^{-2} \leq N^2$, for any $\Im z \geq \eta_m$, we see that (2.28) follows from (6.49) after a small adjustment of $\epsilon$ and $D$ that were anyway arbitrary.

From now on we fix $E \in I$ and our goal is to prove (6.48). We will use Lemma 6.3. In the first step we verify that the assumptions of this lemma hold for $\tilde{\eta} = \eta_M$, i.e., that (6.41) and (6.42) hold for $z = E + i\eta_M$. In the second step, we successively use Lemma 6.3 to reduce $\tilde{\eta}$ steps by steps of size $N^{-5}$ until we have verified (6.41)–(6.42) down to $\tilde{\eta} = \eta_m$. Then (6.48) will follow from a final application of Lemma 6.3 combined with discretization argument similar to the one above, but this time the $\eta$ variable instead of the $E$ variable.

**Step 1. Initial bound.** First we note that since $\mu_A$ and $\mu_B$ are compactly supported, $\|H\|$ is deterministically bounded, we thus have $\Im m_H(E + i\eta_M) \leq (\eta_M)^{-1} \leq 1$ assuming $\eta_M \geq 1$.

Following the main argument in the proof of Proposition 6.2, we have the concentration inequality
\[
|f_{VQV^*}(E + i\eta_M) - E f_{VQV^*}(E + i\eta_M)| < \frac{1}{\sqrt{N^2 \eta_M}}, \tag{6.50}
\]
uniformly for any deterministic $Q$ with $\|Q\| \lesssim 1$. The analog concentration holds with $V$ replaced by $U$. Using (6.50) with $Q = I$ (i the identity matrix), we have $\| \Im m_H(E + i\eta_M) \| \times N^{-1}$. Hence, it suffices to show that
\[
\| \Im m_H(E + i\eta_M) - m_A \mu_B(E + i\eta_M) \| < \frac{1}{N}. \tag{6.51}
\]

Recalling the definitions of $\omega_A^\eta$ and $\omega_B^\eta$ in (6.5), we have, with $z = E + i\eta_M$, the expansion
\[
\omega_A^\eta(z) = z - \frac{\Im \tr \tilde{A} G^\eta_B(z)}{\tr G^\eta_B(z)} = z - \frac{\tr A z^{-1} + \Im \tr \tilde{A} (\tilde{A} + \tilde{B}) z^{-2} + O(z^{-2})}{z^{-1}},
\]
as $\eta_M \rightarrow \infty$. Thus using the assumption $\tr A = 0$ we get
\[
\Im \omega_A^\eta(E + i\eta_M) - \Im \eta_M = \frac{\tr A^2 \eta_M + \Im \tr \tilde{A} \tilde{B} \eta_M}{|E + i\eta_M|^2} + O \left( \frac{1}{\eta_M^2} \right),
\]
as $\eta_M \rightarrow \infty$. Next, since $V$ and $U$ are independent, we have
\[
\Im \tr VAV^* UBU^* = \Im E[VAV^*] E [UBU^*] = \tr A \tr B = 0,
\]

since $\tr A = \tr B = 0$ by assumption. Thus
\[
\Im \omega_A^\eta(E + i\eta_M) - \Im \eta_M = \frac{\tr A^2 \eta_M}{|E + i\eta_M|^2} + O \left( \frac{1}{\eta_M^2} \right), \tag{6.52}
\]
as $\eta_M \rightarrow \infty$. Since $\tr A^2 > 0$, we achieve by choosing $\eta_M$ sufficiently large (but independent of $N$) that
\[
\Im \omega_A^\eta(E + i\eta_M) - \Im \eta_M \geq \frac{1}{2} \frac{\tr A^2 \eta_M}{|E + i\eta_M|^2}, \tag{6.53}
\]
and the analogue estimate holds with $A$ replaced by $B$. In particular, we have, for such $\eta_M$,\[
\Im \omega_A^\eta(E + i\eta_M) \gtrsim 1, \quad \Im \omega_B^\eta(E + i\eta_M) \gtrsim 1, \tag{6.54}
\]
and $\omega_A^\eta(E + i\eta_M) \sim 1, \omega_B^\eta(E + i\eta_M) \sim 1.$
To show (6.51), we apply Lemma 4.2 to the system (6.40). Having established (6.53), it suffices to show that
\[ |r_A^*(E + i\eta M)| \prec \frac{1}{N}, \quad |r_B^*(E + i\eta M)| \prec \frac{1}{N}, \] (6.55)
since then we have, for $N$ sufficiently large and $\eta M$ as above that, for any fixed $\varepsilon \in [0, 1)$,
\[ N^\varepsilon \frac{1}{N} \leq \frac{1}{2} \text{tr} A^2 \eta M \leq \text{Im} \omega_A^*(E + i\eta M) - \text{Im} \eta M, \] (6.56)
and similarly with $B$ replacing $A$. In particular, combining (6.55) and (6.56), we see that assumption (4.14) of Lemma 4.2 (with the choice $\tilde{r} = r^\varepsilon$) is satisfied for $N$ sufficiently large (with high probability). Consequently, we see that (6.41) (even with $N^{-1+\varepsilon}$ instead of $N^{-\gamma}$ in the latter) hold for $z = E + i\eta M$. Finally, the equations
\[ \text{Em}_H(z) = \frac{1}{z - \omega_A^*(z) - \omega_B^*(z)}, \quad m_{A\boxplus B}(z) = \frac{1}{z - \omega_A(z) - \omega_B(z)}, \] (6.57)
together with the concentration estimate (6.50) yield (6.42).

It remains to justify (6.55). Since $i\eta M \text{m}_H(E + i\eta M) = O(\eta M^{-1})$, we have $\text{Em}_H(E + i\eta M) \sim 1$. In addition, from (6.54) it follows that $m_A(\omega_A^*(E + i\eta M)) \sim 1$ and $m_B(\omega_B^*(E + i\eta M)) \sim 1$. Recalling the definition in (6.39), we see that (6.55) is equivalent to
\[ |\text{tr} \delta_A^*(E + i\eta M)| \prec \frac{1}{N}, \quad |\text{tr} \delta_B^*(E + i\eta M)| \prec \frac{1}{N}. \] (6.58)
By the definitions of $\delta_A^*$, $\delta_B^*$ in (6.38), and $\Delta_A$, $\Delta_B$ in (6.37), it is easy to obtain (6.58) by using (6.50) and Cauchy–Schwarz. This completes Step 1, i.e., the verification of (6.41)–(6.42) for $\tilde{\eta} = \eta M$.

Step 2. Induction. Recall that $\omega_A$, $\omega_B$ and $m_{A\boxplus B}$ (see Lemma 5.1) are uniformly bounded and $\omega_A^*(z)$, $\omega_B^*(z)$, $m_A(z)$, $\omega_A(z)$, $\omega_B(z)$ and $m_{A\boxplus B}(z)$ are Lipschitz continuous with a Lipschitz constant bounded by $|\text{Im} z|^{-2} \leq N^2$, for any $|\text{Im} z| \geq \eta m$. Applying Lemma 6.3 to conclude (6.44) with the choice $\epsilon = \gamma/10$, we see that if (6.41) and (6.42) hold for some $\tilde{\eta}$, then (6.41) also holds for $\tilde{\eta}$ replaced with $\tilde{\eta} - N^{-5}$ as long as $\tilde{\eta} \geq \eta m$. Moreover, by the Lipschitz continuity of $m_H$ and $m_{A\boxplus B}$, notice that
\[ \Xi(\tilde{\eta} - N^{-5}) \supset \Xi(\tilde{\eta}) \setminus \Theta(\tilde{\eta}). \] (6.59)
Thus, if (6.42) holds for some $\tilde{\eta}$, then (6.59) and (6.47) imply that (6.42) also holds for $\tilde{\eta}$ replaced with $\tilde{\eta} - N^{-5}$. Using Step 1 as an initial input with the choice $\tilde{\eta} = \eta M$, and applying the above induction argument $O(N^5)$ times by reducing $\tilde{\eta}$ with stepsize $N^{-5}$, we see that (6.41) and (6.42) hold for all $\tilde{\eta}_k \in [\eta m, \eta M]$ of the form $\tilde{\eta}_k = \eta M - k \cdot N^{-5}$ with some integer $k$. Applying Lemma 6.3 once more for these $\tilde{\eta}_k$, but now with an arbitrary $\epsilon > 0$, we conclude from (6.42) and (6.47) that
\[ |m_H(E + i\tilde{\eta}_k) - m_{A\boxplus B}(E + i\tilde{\eta}_k)| \prec \frac{1}{\sqrt{N^5 \tilde{\eta}_k}}, \quad k = 0, 1, \ldots, k_0, \] (6.60)
where $k_0$ is the largest integer with $\tilde{\eta}_k \geq \eta m$. The uniformity of (6.60) in $k$ follows from the fact that the threshold functions $N_k$ in Lemma 6.3 are independent of $\tilde{\eta}$. Clearly $k_0 = O(N^5)$, so taking a union bound of (6.60), compensating the combinatorial factor $CN^5$ by replacing $D$ by $D - 5$, and slightly adjusting $\epsilon$ to extend the control from the set $\{ z = E + i\tilde{\eta}_k : k \leq k_0 \}$ to all $z \in \mathcal{S}_E(\eta m, \eta M)$, we obtain (6.48). \hfill \Box

It remains to prove Lemma 6.3.
Proof of Lemma 6.3. First we notice that $E \in \mathcal{I}$ and (2.25) imply that for all sufficiently large $N$, the bounds (5.3)-(5.4) hold. Together with (6.41) they imply that

$$\omega_A(\tilde{z}), \omega_B(\tilde{z}) \sim 1, \quad \text{Im} \omega_A(\tilde{z}), \text{Im} \omega_B(\tilde{z}) \gtrsim 1,$$

moreover, using (6.6) we also get

$$\frac{1}{\mathbb{E} m_H(\tilde{z})} \lesssim 1. \quad (6.62)$$

We start with (6.43). Thanks to symmetry, we only need to estimate $|r_A(\tilde{z})|$. By (6.61) we have

$$\|G_A(\omega_B(\tilde{z}))\| = \|G_A(\omega_B^c(\tilde{z}))\| \lesssim 1. \quad (6.63)$$

Furthermore, $\omega_B^c(\tilde{z}) \sim 1$ and $\text{Im} \omega_B^c(\tilde{z}) \gtrsim 1$ imply $m_A(\omega_B^c(\tilde{z})) \sim 1$.

We next claim that

$$\mathbb{E}[\text{tr}(G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})\Delta A(\tilde{z}))] \leq \frac{Nc}{N^2 \eta^2}, \quad (6.64)$$

for any $\epsilon > 0$ if $N \geq N_0(\epsilon)$ is large enough, uniformly for $\tilde{\eta} \in [\eta_m, \eta_M]$. Assuming (6.64) and recalling the definition of $\delta_A$ and $r_A$ in (6.38)-(6.39), from (6.62) we get the first estimate in (6.43).

Next, we prove (6.64). By the definitions in (6.37), we have

$$\mathbb{E} \left[ \text{tr}(G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})\Delta A(\tilde{z})) \right] = -\mathbb{E} \left[ \mathbb{E}[m_H(\tilde{z})] \text{tr}(G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})G_B(\tilde{z})) \right]$$

$$- \mathbb{E} \left[ \mathbb{E}[f_B(\tilde{z})] \text{tr}(G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z})) \right]. \quad (6.65)$$

We rewrite the two terms on the right side separately as covariances,

$$\mathbb{E} \left[ \mathbb{E}[m_H(\tilde{z})] \text{tr}(G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})G_B(\tilde{z})) \right] = \text{Cov} \left( m_H(\tilde{z}), \text{tr} \left( G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})G_B(\tilde{z}) \right) \right),$$

$$\mathbb{E} \left[ \mathbb{E}[f_B(\tilde{z})] \text{tr} \left( G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z}) \right) \right] = \text{Cov} \left( f_B(\tilde{z}), \text{tr} \left( G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z}) \right) \right), \quad (6.65)$$

respectively,

$$\mathbb{E} \left[ \mathbb{E}[f_B(\tilde{z})] \text{tr} \left( G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z}) \right) \right] = \text{Cov} \left( f_B(\tilde{z}), \text{tr} \left( G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z}) \right) \right),$$

where $\text{Cov}(X, Y) := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$, for arbitrary random variables $X$ and $Y$.

Given (6.42) and the uniform boundedness of $m_{AEB}(z)$ from (5.16), we see that (6.12) is satisfied and we can apply Proposition 6.2 using different choices for $Q$. Together with Cauchy–Schwarz inequality $|\text{Cov}(X, Y)|^2 \leq \mathbb{E}[|X|^2] \cdot \mathbb{E}[|Y|^2]$, we get

$$\left| \text{Cov} \left( m_H(\tilde{z}), \text{tr} \left( G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})G_B(\tilde{z}) \right) \right) \right| \lesssim \frac{1}{N^2 \eta^2},$$

$$\left| \text{Cov} \left( f_B(\tilde{z}), \text{tr} \left( G_A(\omega_B^c(\tilde{z}))G_B(\tilde{z}) \right) \right) \right| \lesssim \frac{1}{N^2 \eta^2}. \quad (6.66)$$

More specifically, for the first line of (6.66), we chose $Q = I$ and $Q = G_A(\omega_B^c(\tilde{z}))(\tilde{A} - \tilde{z})$; for the second line we chose $Q = B$ and $Q = G_A(\omega_B^c(\tilde{z}))$, where we also used the facts $\tilde{A} = VAV^*$ and $\tilde{B} = UBU^*$. Here, we also implicitly used (6.63). Then, (6.64) follows from (6.66), which in turn proves (6.43).

Next, using Proposition 4.1, (6.40) and (6.43) we immediately get (6.44). Moreover, since $\text{Im} \omega_A(z), \text{Im} \omega_B(z) \gtrsim \text{Im} z$, we have $|z - \omega_A(z) - \omega_B(z)| \gtrsim 1$. Together with (6.44) and (6.57) this implies (6.45). Notice that (6.42) together with the uniform bound on $m_{AEB}$ imply the condition (6.12) in Proposition 6.2. Thus, finally (6.46) and (6.47) follow from (6.45) and the concentration inequality (6.13). This completes the proof of Lemma 6.3. \qed
7. Two point mass case

In this section, we discuss stability properties of the free additive convolution \( \mu_\alpha \boxplus \mu_\beta \) when both \( \mu_\alpha \) and \( \mu_\beta \) are convex combinations of two point masses. The analogous result to Theorem 2.5 is given in Proposition 7.2 below. Applications of that result in the spirit of Theorems 2.7 and 2.8 are then stated in Proposition 7.3 and Proposition 7.4. When we refer to the results in Sections 2-4, we will henceforth regard \( \mu_1 \) and \( \mu_2 \) as \( \mu_\alpha \) and \( \mu_\beta \), respectively, unless specified otherwise.

7.1. Stability in the two point mass case. Without loss of generality (up to shifting and scaling), we assume that

\[
\mu_\alpha = \xi \delta_1 + (1-\xi) \delta_0, \quad \mu_\beta = \zeta \delta_0 + (1-\zeta) \delta_0, \quad \theta \neq 0,
\]

\[
\xi, \zeta \in [0, \frac{1}{2}], \quad \xi \leq \zeta, \quad (\theta, \xi, \zeta) \neq (-1, \frac{1}{2}, \frac{1}{2}). \tag{7.1}
\]

Here we exclude the case \((\theta, \xi, \zeta) = (-1, \frac{1}{2}, \frac{1}{2})\) since it is equivalent to \((\theta, \xi, \zeta) = (1, \frac{1}{2}, \frac{1}{2})\) under a shift. Note that the latter is a special case of \(\mu_\alpha = \mu_\beta\).

Set

\[
\ell_1 := \min \left\{ \frac{1}{2} \left( 1 + \theta - \sqrt{(1-\theta)^2 + 4\theta r_+} \right), \frac{1}{2} \left( 1 + \theta - \sqrt{(1-\theta)^2 + 4\theta r_-} \right) \right\},
\]

\[
\ell_2 := \max \left\{ \frac{1}{2} \left( 1 + \theta - \sqrt{(1-\theta)^2 + 4\theta r_+} \right), \frac{1}{2} \left( 1 + \theta - \sqrt{(1-\theta)^2 + 4\theta r_-} \right) \right\},
\]

\[
\ell_3 := \min \left\{ \frac{1}{2} \left( 1 + \theta + \sqrt{(1-\theta)^2 + 4\theta r_+} \right), \frac{1}{2} \left( 1 + \theta + \sqrt{(1-\theta)^2 + 4\theta r_-} \right) \right\},
\]

\[
\ell_4 := \max \left\{ \frac{1}{2} \left( 1 + \theta + \sqrt{(1-\theta)^2 + 4\theta r_+} \right), \frac{1}{2} \left( 1 + \theta + \sqrt{(1-\theta)^2 + 4\theta r_-} \right) \right\},
\]

where we introduced

\[
r_{\pm} := \xi + \zeta - 2\xi \zeta \pm \sqrt{4\xi \zeta (1-\xi)(1-\zeta)}. \tag{7.2}
\]

Note that \(\ell_1 < \ell_2 \leq \ell_3 < \ell_4\). The following result, taken from [28], describes the regular bulk of \(\mu_\alpha \boxplus \mu_\beta\) in the setting of (7.1). Recall that \(f_{\mu_\alpha \boxplus \mu_\beta}\) denotes the density of \((\mu_\alpha \boxplus \mu_\beta)^{ac}\).

**Lemma 7.1.** Let \(\mu_\alpha\) and \(\mu_\beta\) be as in (7.1). Then the regular bulk is given by

\[
\mathcal{B}_{\mu_\alpha \boxplus \mu_\beta} = (\ell_1, \ell_2) \cup (\ell_3, \ell_4),
\]

in case \(\mu_\alpha \neq \mu_\beta\), while in case \(\mu_\alpha = \mu_\beta\) it is given by

\[
\mathcal{B}_{\mu_\alpha \boxplus \mu_\beta} = (\ell_1, \ell_4). \tag{7.4}
\]

**Proof.** Choose the diagonal matrices \(A\) and \(B\) with spectral distribution \(\mu_A = \xi_N \delta_1 + (1-\xi_N) \delta_0\) and \(\mu_B = \zeta_N \delta_0 + (1-\zeta_N) \delta_0\) respectively, with \(\xi_N := \lfloor \xi N \rfloor / N\) and \(\zeta_N := \lfloor \zeta N \rfloor / N\), where \(\lfloor \cdot \rfloor\) denotes the integer part. Recall from (7.1) that \(\xi \leq \zeta\) and \(\xi + \zeta \leq 1\). From Theorem 1.1 of [28], we first observe that the \(\theta\) and \(0\) are eigenvalues of the matrix \(H = A + U B U^*\), \(U\) a Haar unitary, with multiplicities \(N(\zeta_N - \xi_N)\) and \(N(1-\zeta_N - \xi_N)\), respectively. The remaining \(2\xi_N N\) eigenvalues of \(H\) may be obtained via a two-fold transformation from the eigenvalues, \(\ell_i\), of a \(\xi_N N\)-dimensional Jacobi ensemble as

\[
r_j^\pm := \frac{1}{2} \left( 1 + \theta \pm \sqrt{(1-\theta)^2 + 4\ell_j} \right), \quad j = 1, \ldots, \xi_N N, \tag{7.5}
\]
and then identifying the eigenvalues of $H$ as the set $\{ \tau_j^+ \} \cup \{ \tau_j^- \} \cup \{ 0, \theta \}$. In addition, the weak limit of $\frac{1}{N} \sum_j \delta_{\tau_j}$, as $N \to \infty$, admits a density given by

$$f(x) = \frac{1}{2\pi\xi} \frac{\sqrt{(r_+ - x)(x - r_-)}}{x(1 - x)} \mathbf{1}_{[r_- , r_+)}(x), \quad x \in \mathbb{R}, \quad (7.6)$$

where $r_+$ and $r_-$ are defined in (7.2). Since the limiting spectral distribution of $H$ is given by $\mu_a \boxplus \mu_\beta$, we see that $(\mu_a \boxplus \mu_\beta)^{ac}$ agrees with the weak limit of the measure $\frac{1}{N} \sum_j (\delta_{\tau_j^+} + \delta_{\tau_j^-})$, as $N \to \infty$. Using this information together with (7.5) and (7.6), one deduces that $\text{supp} \ (\mu_a \boxplus \mu_\beta)^{ac} = [\ell_1, \ell_2] \cup [\ell_3, \ell_4]$. It then follows from the explicit form of the limiting distribution of the Jacobi ensemble that $f_{\mu_a \boxplus \mu_\beta}$ is bounded and strictly positive inside its support. This proves (7.3).

In the special case $\mu_a = \mu_\beta$, we have $\ell_2 = \ell_3 = 1$ and thus $\text{supp} \ (\mu_a \boxplus \mu_\beta)^{ac} = [\ell_1 , \ell_4]$, with $\ell_1 = 1 - 2\sqrt{\xi(1 - \xi)}$ and $\ell_4 = 1 + 2\sqrt{\xi(1 - \xi)}$. In fact, the density of $(\mu_a \boxplus \mu_\beta)^{ac}$ equals

$$f_{\mu_a \boxplus \mu_\beta}(x) = \frac{1}{\pi} \frac{\sqrt{(\ell_4 - x)(x - \ell_1)}}{x(2 - x)}, \quad x \in (\ell_1 , \ell_4); \quad (7.7)$$

see (5.5) of [33] for instance. Then (7.4) follows directly.

Our main task in this section is to show the following result on the stability of the system $\Phi_{\mu_a, \mu_\beta}(\omega, \omega_\beta, z) = 0$ in the setting (7.1). Recall the definition of $\Gamma_{\mu_a, \mu_\beta}$ in (2.12).

**Proposition 7.2.** Let $\mu_a$ and $\mu_\beta$ be as in (7.1). Let $\mathcal{I} \subset B_{\mu_a \boxplus \mu_\beta}$ be a compact non-empty interval. Fix $0 < \eta_M < \infty$. Then, there are constants $k > 0, K < \infty$ and $S < \infty$, depending on the constants $\xi, \zeta, \theta, \eta_M$ and on the interval $\mathcal{I}$, such that the subordination functions possess the following bounds:

$$\min_{z \in S_\mathcal{I}(0, \eta_M)} \mathbf{Im} \omega(z) \geq 2k, \quad \min_{z \in S_\mathcal{I}(0, \eta_M)} \mathbf{Im} \omega_\beta(z) \geq 2k, \quad (7.8)$$

$$\max_{z \in S_\mathcal{I}(0, \eta_M)} |\omega(z)| \leq \frac{K}{2}, \quad \max_{z \in S_\mathcal{I}(0, \eta_M)} |\omega_\beta(z)| \leq \frac{K}{2} \quad (7.9)$$

Moreover, we have the following bounds:

(i) If $\mu_a \neq \mu_\beta$,

$$\max_{z \in S_\mathcal{I}(0, \eta_M)} \Gamma_{\mu_a, \mu_\beta}(\omega(z), \omega_\beta(z)) \leq S. \quad (7.10)$$

(ii) If $\mu_a = \mu_\beta$,

$$\Gamma_{\mu_a, \mu_\beta}(\omega(z), \omega_\beta(z)) \leq \frac{S}{|z - 1|}; \quad (7.11)$$

holds uniformly on $S_\mathcal{I}(0, \eta_M)$.

**Remark 7.1.** As an immediate consequence of Proposition 7.2 and (3.28), we obtain for $\mu_a \neq \mu_\beta$ the bounds $\max_{z \in S_\mathcal{I}(0, \eta_M)} |\omega(z)| \leq 2S, \max_{z \in S_\mathcal{I}(0, \eta_M)} |\omega_\beta(z)| \leq 2S$ with $\mathcal{I}$ as in (7.10). For $\mu_a = \mu_\beta$, we get $|\omega(z)| \leq \frac{2S}{|z - 1|}$, uniformly on $S_\mathcal{I}(0, \eta_M)$ as in (7.11).

**Remark 7.2.** In the case $\mu_a = \mu_\beta$, we note from Lemma 7.1 (c.f., (7.7)) that the point $E = 1$ is in the regular bulk $B_{\mu_a \boxplus \mu_\beta}$. However, $m_{\mu_a \boxplus \mu_\beta}(1 + i0)$ is unstable under small perturbations. For instance, let

$$\mu_A = \xi \delta_1 + (1 - \xi)\delta_0, \quad \mu_B = (\xi - \varepsilon)\delta_1 + (1 - \xi + \varepsilon)\delta_0,$$

for some small $\varepsilon > 0$. Then, according to Theorem 7.4 of [11], $\mu_A \boxplus \mu_B$ has a point mass $\varepsilon \delta_1$. Hence, even though (2.25) (i.e., $d_\mathcal{I}(\mu_B, \mu_\beta) \to 0$, as $\varepsilon \to 0$) is satisfied, $m_{\mu_A \boxplus \mu_B}(z)$ contains
a singular part $\frac{1}{(z - \xi_\ast)}$, which blows up as $|z - 1| = o(\varepsilon)$. This explains, on a heuristic level, the bound in (7.11) and shows why the $\mu_\alpha = \mu_{\beta}$ case at energy $E = 1$ is special even though the density $f_{\mu_\alpha \| \mu_{\beta}}$ is real analytic in a neighborhood of $E = 1$.

**Remark 7.3.** Consider a more general setup with $\mu_\alpha = \xi_\delta + \tilde{\mu}_\alpha$ and $\mu_{\beta} = (1 - \xi_\delta)\delta_0 + \tilde{\mu}_{\beta}$, for some constants $\xi \in (0,1)$, $a, b \in \mathbb{R}$ and for some Borel measures $\tilde{\mu}_\alpha$ and $\tilde{\mu}_{\beta}$ with $\tilde{\mu}_\alpha(\mathbb{R}) = 1 - \xi$ and $\tilde{\mu}_{\beta}(\mathbb{R}) = \xi$. Analogously to the discussion in Remark 7.2, we note that $m_{\mu_\alpha \| \mu_{\beta}}(a + b + 10)$ is unstable under small perturbations. However, from Lemma 3.4, we know that the system $\Phi_{\mu_\alpha, \mu_{\beta}}(\omega_\alpha, \omega_{\beta}, z) = 0$ is linearly $S$-stable in the regular bulk under the assumptions of Theorem 2.5. That means, if neither $\mu_\alpha$ nor $\mu_{\beta}$ is supported at a single point and at least one of them is supported at more than two points, then the point $E = a + b$ cannot lie in the regular bulk $B_{\mu_\alpha \| \mu_{\beta}}$. Thus, only in the special case $\mu_\alpha = \mu_{\beta}$ with $\mu_\alpha$ as in (7.1), there is an unstable point, up to scaling and shifting given by $E = 1$, inside the regular bulk $B_{\mu_\alpha \| \mu_\alpha}$.

**Proof of Proposition 7.2.** Estimates (7.8) and (7.9) follow from Lemma 3.2 and Lemma 3.3. To show statement (i), we recall from the proof of Lemma 3.4 that $\Phi_{\mu_\alpha, \mu_{\beta}}(\omega_\alpha, \omega_{\beta}, z) = 0$ is linearly $S$-stable at $(\omega_\alpha, \omega_{\beta})$ if

\[
1 - (F'_\alpha(\omega_{\beta}) - 1)(F'_\beta(\omega_\alpha) - 1) \geq c,
\]

for some strictly positive constant $c$. We now show that (7.12) holds for the case $\mu_\alpha \neq \mu_{\beta}$ in the setup of (7.1). Using henceforth the shorthand $F_\alpha \equiv F_{\mu_\alpha}$, $F_{\beta} \equiv F_{\mu_{\beta}}$, we compute

\[
F_\alpha(z) = \frac{z(1 - z)}{1 - \xi - z}, \quad F_\beta(z) = \frac{z(\theta - z)}{(\theta - \theta_\xi - z)}, \quad z \in \mathbb{C}^+.
\]

Then it is easy to obtain

\[
F'_\alpha(z) - 1 = \frac{\xi - \xi^2}{(1 - \xi - z)^2}, \quad F'_\beta(z) - 1 = \frac{\theta^2(\zeta - \xi_\xi^2)}{(\theta - \theta_\xi - z)^2},
\]

and

\[
|F'_\alpha(z) - 1| = \frac{\text{Im} F_\alpha(z) - \text{Im} z}{\text{Im} z}, \quad |F'_\beta(z) - 1| = \frac{\text{Im} F_\beta(z) - \text{Im} z}{\text{Im} z}.
\]

Consequently, we have (c.f., (3.18))

\[
\left| (F'_\alpha(\omega_{\beta}(z)) - 1)(F'_\beta(\omega_\alpha(z)) - 1) \right| = \frac{(\text{Im} \, \omega_\alpha(z) - \text{Im} z)(\text{Im} \, \omega_{\beta}(z) - \text{Im} z)}{\text{Im} \omega_\alpha(z) \text{Im} \omega_{\beta}(z)}
\]

for any $z \in \mathbb{C}^+$. Hence, for $z \in \mathcal{S}_T(\eta_0, \eta_M)$ with some small but fixed $\eta_0 > 0$ to be chosen below, we trivially get (7.12) from (7.15). It remains to discuss the regime $z \in \mathcal{S}_T(0, \eta_0)$. Then (7.13) together with (2.5) implies that

\[
\omega_\beta(1 - \omega_\beta) = \frac{\omega_\alpha(\theta - \omega_\alpha)}{1 - \xi - \omega_\beta}, \quad \omega_\beta(1 - \omega_{\beta}) = \frac{\omega_\alpha(\theta - \omega_\alpha)}{1 - \xi - \omega_{\beta}} = \omega_\alpha + \omega_{\beta} - z.
\]

Denote $s := 1 - \xi - \omega_\beta$ and $t := \theta - \theta_\xi - \omega_\alpha$. From (7.14) we then have

\[
(F'_\alpha(\omega_{\beta}) - 1)(F'_\beta(\omega_\alpha) - 1) = \frac{(\xi - \xi^2)(\theta^2\zeta - (\theta_\xi)^2)}{(st)^2}.
\]

Using (7.16), some algebra reveals that

\[
\frac{1}{st} = \frac{1}{\xi - \xi^2} + \frac{\xi + \theta - \theta_\xi - z}{(\xi - \xi^2)t}, \quad \frac{1}{st} = \frac{1}{\theta^2(\zeta - \xi^2)} + \frac{\theta_\xi + 1 - \xi - z}{\theta^2(\zeta - \xi^2)s}.
\]

(7.18)
Owing to (7.15) and \((\xi - \xi^2)(\theta^2 \xi - (\theta \xi)^2) > 0\) (recall that \(\xi, \xi \in (0, 1/2] \text{ and } \theta \neq 0\)), it suffices to show that
\[
|\text{Im}(st)| \geq c, \tag{7.19}
\]
in order to prove (7.12). Note that, from the definitions of \(s\) and \(t\), together with (7.8) and (7.9), we have
\[
|\text{Im} s|, |\text{Im} t| \geq c, \quad |s|, |t| \leq C. \tag{7.20}
\]
Since \(\mu_\alpha \neq \mu_\beta\), there exists a positive constant \(d\) such that \(\max\{|\xi - \xi|, |\theta - 1|\} \geq d\). It is then elementary to work out that
\[
\max\{|\xi - \xi^2 - \theta^2(\xi - \xi^2)|, |(2\xi - 2\theta \xi + \theta - 1)|\} \geq d_1, \tag{7.21}
\]
for some positive constant \(d_1 \equiv d_1(\xi, \xi, \theta) > 0\), since the special case \((\theta, \xi, \xi) = (-1, 1, 1/2)\) is also excluded in the setting (7.1). For brevity, we adopt the notation
\[
\varphi := \frac{\theta \xi + 1 - \xi - z}{\theta^2(\xi - \xi^2)s}, \quad \psi := \frac{\xi + \theta \xi - z}{(\xi - \xi^2)t}.
\]
Then, according to (7.18) we have
\[
\text{Re} \frac{1}{st} = \text{Re} \psi - \frac{1}{\xi - \xi^2} = \text{Re} \varphi - \frac{1}{\theta^2(\xi - \xi^2)}, \quad \text{Im} \frac{1}{st} = \text{Im} \psi = \text{Im} \varphi. \tag{7.22}
\]
If \(|(\xi - \xi^2) - \theta^2(\xi - \xi^2)| \geq d_1\) holds in (7.21), then (7.22) implies that
\[
|\text{Re} \psi - \text{Re} \varphi| \geq d_2, \tag{7.23}
\]
for some positive constant \(d_2 \equiv d_2(\xi, \xi, \theta)\). For small enough \(\eta_0 = \eta_0(\xi, \xi, \theta)\), we then get
\[
\text{Re} \psi - \text{Re} \varphi = \frac{(\xi + \theta - \theta \xi - E)\text{Re} t + O(\eta_0) - (\theta \xi + 1 - \xi - E)\text{Re} s + O(\eta_0)}{(\xi - \xi^2)|t|^2 - \theta^2(\xi - \xi^2)|s|^2},
\]
which, together with (7.20) and (7.23), implies that
\[
\max\{|\theta \xi + 1 - \xi - E|, |\xi + \theta - \theta \xi - E|\} \geq d_3, \tag{7.24}
\]
for some positive constant \(d_3 \equiv d_3(\xi, \xi, \theta)\). If, on the other hand, \(|(2\xi - 2\theta \xi + \theta - 1)| \geq d_1\) holds in (7.21), we get (7.24) by triangle inequality. Either way, (7.24) follows from (7.21), for sufficiently small, but fixed, \(\eta_0 > 0\).

Next, using (7.20) and (7.24), we see that there is a constant \(c > 0\) such that, for sufficiently small \(\eta_0\), for all \(z \in S_2(0, \eta_0)\), we have \(\max\{|\text{Im} \varphi|, |\text{Im} \psi|\} \geq c\). Since \(\text{Im} \varphi = \text{Im} \psi\) by (7.22), (7.19) holds on \(S_2(0, \eta_0)\). Therefore, (7.19) holds on all of \(S_2(0, \eta_1)\). So, if \(\mu_\alpha \neq \mu_\beta\), the system \(\Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha, \omega_\beta, z) = 0\) is linearly \(S\)-stable with some finite \(S\).

We next prove statement (ii) where \(\mu_\alpha = \mu_\beta\) and thus \(\theta = 1, \xi = \xi\). From (7.16), we see that \(\omega_\alpha = \omega_\beta\) satisfies the equation
\[
\omega_\alpha(1 - \omega_\alpha) = 2\omega_\alpha - z. \tag{7.25}
\]
Solving (7.25) for \(\omega_\alpha(z)\) we get
\[
\omega_\alpha(z) = \omega_\beta(z) = \frac{1}{2} \left(z - 1 + 2(1 - \xi) + \sqrt{(z - 1)^2 - 4(1 - \xi)}\right), \tag{7.26}
\]
where the square root is chosen such that \(\omega_\beta(z) \to i\sqrt{1 - \xi}\), as \(z \to 1\). Substituting (7.26) into (7.17), together with the \(\theta = 1, \xi, s = t = 1 - \xi - \omega_\alpha\), yields
\[
(F_\alpha'(\omega_\beta(z)) - 1)(F_\beta'(\omega_\alpha(z)) - 1) = \frac{4(\xi - \xi^2)^2}{(z - 1 + \sqrt{(z - 1)^2 - 4(1 - \xi)(\xi - \xi^2)})^2}.
\]
Then it is elementary to check that
\[ |1 - (F'_\alpha(\omega_\beta(z)) - 1)(F'_\beta(\omega_\alpha(z)) - 1)| \gtrsim |z - 1|, \quad z \in S(0, \eta_M), \]
which further implies \( \Gamma_{\mu_\alpha, \mu_\beta}(\omega_\alpha(z), \omega_\beta(z)) \lesssim 1/|z - 1|. \) Hence (7.11) is proved. \( \square \)

7.2. Applications of Proposition 7.2. Analogously to Theorem 2.5, we have two main applications of Proposition 7.2. The first one is the following modification of Theorem 2.7. Let \( \mu_\alpha, \mu_\beta \) be as in (7.1) and let \( \mu_A, \mu_B \) be arbitrary probability measures on \( \mathbb{R} \). Recall the domain \( S_\mathcal{Z}(a, b) \) introduced in (2.13). For given (small) \( \varsigma > 0 \), we set
\[ S_\mathcal{Z}(a, b) := \left\{ z \in S_\mathcal{Z}(a, b) : \varsigma |z - 1| \geq \max \left\{ \sqrt{d_L(\mu_A, \mu_\alpha)}, \sqrt{d_L(\mu_B, \mu_\beta)} \right\} \right\}. \]

**Proposition 7.3.** Let \( \mu_\alpha, \mu_\beta \) be as in (7.1). Let \( \mathcal{Z} \subset B_{\mu_\alpha, \mu_\beta} \) be a compact non-empty interval. Let \( \mu_A, \mu_B \) be two probability measures on \( \mathbb{R} \). Fix \( 0 < \eta_M < \infty \). Then there are constants \( b \) and \( Z \) and satisfying the condition
\[ d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b \]
implies
\[ \max_{z \in S_\mathcal{Z}(0, \eta_M)} |m_{\mu_\alpha, \mu_\beta}(z) - m_{\mu_\alpha, \mu_\beta}(0)| \leq Z \left( d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \right), \]
in case \( \mu_\alpha \neq \mu_\beta \), respectively
\[ |m_{\mu_\alpha, \mu_\beta}(z) - m_{\mu_\alpha, \mu_\beta}(0)| \leq \frac{Z}{|z - 1|} \left( d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\alpha) \right), \]
uniformly on \( S_\mathcal{Z}(0, \eta_M) \) with \( \varsigma \leq \varsigma_0 \), for some \( \varsigma_0 > 0 \), in case \( \mu_\alpha = \mu_\beta \). The constants \( b \) and \( Z \) depend only on the constants \( \xi, \zeta, \theta \) and on the interval \( \mathcal{Z} \), while \( \varsigma_0 \) also depends on \( b \).

**Proof.** Having established Proposition 7.2, the proof of (7.29) is the same as that of Theorem 2.7. To establish (7.30), we mimic the proof of Theorem 2.7 with \( S \) replaced by \( \overline{S_\mathcal{Z}(0, \eta_M)} \).

We only give a sketch here. Similarly to (5.9), using (7.8) and (7.11), we have with \( b \) in (7.28) sufficiently small that
\[ \Gamma_{\mu_A, \mu_B}(\omega_\alpha(z), \omega_\alpha(z)) \lesssim \frac{1}{|z - 1|}, \quad z \in S_\mathcal{Z}(0, \eta_M). \]
As in the proof of Lemma 5.1, we rewrite the system \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha(z), \omega_\alpha(z), z) = 0 \) as \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha(z), \omega_\alpha(z), z) = r(z) \) with \( ||r(z)|| \) satisfying the bound (5.8). From the uniqueness of the solution to \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha, \omega_{\beta}, z) = 0 \) and (7.31), we get
\[ |\omega_A(z) - \omega_\alpha(z)| \lesssim ||r(z)||/|z - 1|, \]
\[ |\omega_B(z) - \omega_\alpha(z)| \lesssim ||r(z)||/|z - 1|, \quad z \in S_\mathcal{Z}(0, \eta_M), \]
via the Newton-Kantorovich theorem. Note that the inequality \( ||r(z)|| \lesssim \varsigma^2 |z - 1|^2 \) is needed to guarantee that the first order term dominates over the higher order terms in the Taylor expansion of \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_A, \omega_{\beta}, z) \) around \( \Phi_{\mu_\alpha, \mu_\beta}(\omega_\alpha, \omega_\beta, z) \). This is the reason why we restrict our discussion on the set \( S_\mathcal{Z}(0, \eta_M) \). In addition, thanks to (7.32) we see that (5.3) and (5.4) still hold with \( S_\mathcal{Z}(0, \eta_M) \) replaced by \( S_\mathcal{Z}(0, \eta_M) \). Then the remaining parts of the proof of (7.30) are the same as the counterparts in the proof of Theorem 2.7. \( \square \)
The second application of Proposition 7.2 gives the following local law for the Green function in the random matrix setup from Subsection 2.3.2. Fix any \( \gamma > 0 \). We introduce a sub-domain of \( S_2^\gamma(a, b) \) by setting
\[
S_2^\gamma(a, b) := S_2^\gamma(a, b) \cap \left\{ z \in \mathbb{C} : |z - 1| \geq \frac{N^\gamma}{\sqrt{N\eta M}} \right\}.
\] (7.33)

**Proposition 7.4.** Let \( \mu_A, \mu_B \) be as in (7.1). Assume that the empirical eigenvalue distributions \( \mu_A, \mu_B \) of the sequences of matrices \( A, B \) satisfy (2.25). Fix any \( 0 < \eta M < \infty \), any small \( \gamma > 0 \) and set \( \eta M = N^{-\frac{3}{4} + \gamma} \). Let \( \mathcal{I} \subset B_{\mu_A, \mu_B} \) be a compact non-empty interval. Then we have the following conclusions.

(i) If \( \mu_A \neq \mu_B \), then
\[
\max_{z \in S_2^\gamma(\eta M, \eta M)} |m_H(z) - m_A \oplus B(z)| \lesssim \frac{1}{N \eta^{3/2}}.
\]

(ii) If \( \mu_A = \mu_B \), then, for any fixed (small) \( \zeta > 0 \),
\[
|m_H(z) - m_A \oplus B(z)| \lesssim \frac{1}{|z - 1|} \frac{1}{N \eta^{3/2}}
\]
uniformly on \( S_2^\gamma(\eta M, \eta M) \).

**Proof of Proposition 7.4.** Note that, in the proof of Theorem 2.8, the only place where we use the assumption that at least one of \( \mu_A \) and \( \mu_B \) is supported at more than two points is Lemma 3.4; in particular in (3.25). Hence, it suffices to mimic the proof of Theorem 2.8 with Lemma 3.4 replaced by Proposition 7.2. Then the proof of the case \( \mu_A \neq \mu_B \) is exactly the same as that of Theorem 2.8. It suffices to discuss the case \( \mu_A = \mu_B \) below.

Analogously to Corollary 5.2, with the aid of (7.31) and (7.32), we show that
\[
\Gamma_{\mu_A, \mu_B}(\omega_A(z), \omega_B(z)) \lesssim \frac{1}{|z - 1|}, \quad z \in S_2^\gamma(0, \eta M).
\] (7.34)

Then, we use a continuity argument, based on Lemma 4.2 and Proposition 4.1 with \( S \) replaced by \( S_2^\gamma \) therein, to deduce from (6.40) that \( |\omega_i(z) - \omega_i(z)| \lesssim |\tau(z)| / |z - 1| \), \( i = A, B \), on \( S_2^\gamma(\eta M, \eta M) \). The remaining parts of the proof are the same as in Theorem 2.8. This completes the proof of part (ii) of Proposition 7.4. \( \square \)

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