Kauffman Monoids

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Abstract
This paper gives a self-contained and complete proof of the isomorphism
of freely generated monoids extracted from Temperley-Lieb algebras with
monoids made of Kauffman’s diagrams.

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0 Introduction

Kauffman monoids (i.e. semigroups with unit) are multiplicative structures
extracted from Temperley-Lieb algebras, which play a very prominent role in
knot theory, in low-dimensional topology, in topological quantum field theories,
in quantum groups and in statistical mechanics. The label “Temperley-Lieb” is
derived from a paper in this last field [10]. The axioms of Kauffman monoids
(see Section 1 below) are, however, not stated explicitly in that paper. They may be found in Jones’ paper \[4, \S 4.1.4\], and they appear in full light in many works of Kauffman’s \[3, \S 4\], \[7, pp. 431-433\], \[8, IV\], \[10, I.7, I.16, II.8\], \[11, IV\], \[12, II\], \[14, \S 7.2, pp. 8-9\], \[2, Section 3\], \[13, Section 6\]—in particular, in \[9\], which is most detailed. Although it is tempting to call the monoids in question “Temperley-Lieb monoids”, we believe “Kauffman monoids” is a fairer denomination, doing justice to the author who unearthed these structures (cf. \[5, p. 324\]).

Kauffman monoids are seldom, if ever, separated from Temperley-Lieb algebras, which are much richer structures, and not much attention is paid usually to the completeness of the standard axiomatics of Kauffman monoids with respect to their standard geometrical interpretation. Although this completeness, which consists in showing that the freely generated Kauffman monoids are isomorphic to the connection monoids of \[9\], is well known, we have been unable to find in the literature a self-contained and exhaustive proof. Among the papers we know, Kauffman’s \[9\] and \[13, Section 6\], together with an aside in Jones’ paper \[4, \S 4.1.4\], whose content may also be found in \[3, \S 2.8\], are nearest to the mark. There are, however, some uncertainties in \[9\] concerning the appropriate normal form for terms of Kauffman monoids (see the proof of Theorem 4.3 on p. 442, where the normal form mentioned is not Jones’, and is not unique; see also p. 434 of the same paper). All these papers still leave pretty much work to the reader.

We have been readers who have done the work left to us. The completeness proof for Kauffman monoids we are going to present here is, we believe, self-contained and thorough. Moreover, it presents some aspects of Kauffman monoids that may be novel and worth recording. We have not encountered elsewhere the block formulation of freely generated Kauffman monoids of Section 1. We use this formulation to reduce terms syntactically to normal form, which is an essential step in the completeness proof. In this formulation of Kauffman monoids all axioms are tied to reduction to the normal form. We have also not found elsewhere a rigorous proof, such as we present in Section 3, that all elements of connection monoids can be generated from what Kauffman called hooks (and which we shall call diapsides). The best we have found is a proof in \[13, Section 6\], which leaves some details to the reader, and a sketchy proof in \[15, Theorem 26.10, \S 26, Chapter VIII, pp. 251-253\]. In \[1, Proposition 4.1.3\] one may find a proof of something more general, and somewhat more complicated. Our proof exhibits some difficulties, which we think cannot be evaded, and it also sheds some light on the normal form of terms. We have not found elsewhere Lemma 4, the key lemma of Section 4 and of the whole completeness proof.
1 The monoids $K_n$

The Kauffman monoid $K_n$, for $n \geq 2$, has for every $i \in \mathbb{N}$ such that $1 \leq i \leq n-1$ a generator $h^i$, called a diapsis (plural diapsides), and also the generator $c$, called the circle. (Kauffman in [9] called diapsides “hooks”, and that is where the label $h$ comes from; our Greek neologism means “double arc”.) The terms of $K_n$ are defined inductively by stipulating that the generators and $1$ are terms, and that if $t$ and $u$ are terms, then $(tu)$ is a term (as usual, we shall omit the outermost parentheses of terms).

The monoid $K_n$ is freely generated from the generators above so that the following equations hold between terms of $K_n$:

\[(1) \quad 1t = t1 = t, \]

\[(2) \quad t(uv) = (tu)v, \]

\[(h1) \quad h^i h^j = h^j h^i, \quad \text{for } |i - j| \geq 2, \]

\[(h2) \quad h^i h^{i \pm 1} h^i = h^i, \]

\[(hc1) \quad h^i c = ch^i, \]

\[(hc2) \quad h^i h^l = ch^i. \]

For $1 \leq j \leq i \leq n-1$, let the block $h^{[i,j]}$ be defined as $h^i h^{i-1} \ldots h^j$. The block $h^{[i,i]}$, which is defined as $h^i$, will be called singular. Let $c^1$ be $c$, and let $c^{l+1}$ be $c^l c$.

A term is in Jones normal form iff it is either of the form $c^l h^{[b_1,a_1]} \ldots h^{[b_k,a_k]}$ for $l, k \geq 1$, $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_k$, or of the form $h^{[b_1,a_1]} \ldots h^{[b_k,a_k]}$ for $k \geq 1$, $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_k$, or of the form $c^l$ for $l \geq 1$, or it is the term $1$ (see [4, §4.1.4]). For the sake of definiteness, we require that in the Jones normal form all parentheses are associated to the left (but another arrangement of parentheses would do as well). In the reduction to Jones normal form of Lemma 1 below we shall not bother with trivial considerations concerning parentheses. The associativity equation (2) guarantees that we can move them at will (as it dispensed us from writing parentheses in $(h2)$).

That every term of $K_n$ is equal to a term in Jones normal form will be demonstrated with the help of an alternative formulation of $K_n$, called the block formulation, which is obtained as follows. Besides the circle $c$, we take as generators the blocks $h^{[i,j]}$ instead of the diapsides, we generate terms with these generators, $1$ and multiplication, and to the equations (1) and (2) we add the equations

\[(hI) \quad \text{for } j \geq k + 2, \quad h^{[i,j]} h^{[k,l]} = h^{[k,l]} h^{[i,j]}, \]

\[(hII) \quad \text{for } i \geq l \text{ and } |k - j| = 1, \quad h^{[i,j]} h^{[k,l]} = h^{[i,l]}, \]

\[(hcI) \quad h^{[i,j]} c = ch^{[i,j]}, \]

\[(hcII) \quad h^{[i,j]} h^{[j,l]} = ch^{[i,l]}. \]
We verify first that with \( h^i \) defined as the singular block \( h^{[i,i]} \) the equations \((h1), (h2), (hc1)\) and \((hc2)\) are instances of the new equations of the block formulation. The equation \((h1)\) is \((hI)\) for \( i = j \) and \( k = l \), the equation \((h2)\) is \((hII)\) for \( i = j = l \) and \( k = i + 1 \), or \( i = k = l \) and \( j = i - 1 \), the equation \((hc1)\) is \((hcI)\) for \( i = j \), and the equation \((hc2)\) is \((hcII)\) for \( i = j = l \). We also have to verify that in the new axiomatization we can deduce the definition of \( h^{[i,j]} \) with diapsides replaced by singular blocks; namely, we have to verify

\[
h^{[i,j]} = h^{[i,i]} h^{[i-1,i-1]} \ldots h^{[j+1,j+1]} h^{[j,j]},
\]

which readily follows from \((hcII)\) for \( j = k + 1 \). To finish showing that the block formulation of \( \mathcal{K}_n \) is equivalent to the old formulation, we have to verify that with blocks defined via diapsides we can deduce \((hl), (hII), (hcI)\) and \((hcII)\) from the old equations, which is a straightforward exercise.

We can deduce the following equations in \( \mathcal{K}_n \) for \( j + 2 \leq k \):

\[
\begin{align*}
(hIII.1) & \quad h^{[i,j]} h^{[k,l]} = h^{[k-2,l]} h^{[i,j+2]} & \text{if } i \geq k \text{ and } j \geq l, \\
(hIII.2) & \quad h^{[i,j]} h^{[k,l]} = h^{[i,l]} h^{[k,j+2]} & \text{if } i < k \text{ and } j \geq l, \\
(hIII.3) & \quad h^{[i,j]} h^{[k,l]} = h^{[k-2,j]} h^{[i,l]} & \text{if } i \geq k \text{ and } j < l,
\end{align*}
\]

which is also pretty straightforward. Then we prove the following lemma.

**Lemma 1** Every term of \( \mathcal{K}_n \) is equal in \( \mathcal{K}_n \) to a term in Jones normal form.

**Proof.** We shall give a reduction procedure that transforms every term into a term in Jones normal form, every reduction step being justified by an equation of \( \mathcal{K}_n \). (In logical jargon, we establish that this procedure is strongly normalizing—namely, that any sequence of reduction steps terminates in a term in normal form.)

Take a term in the block formulation of \( \mathcal{K}_n \), and let subterms of this term of the forms

\[
h^{[i,j]} h^{[k,l]}, \text{ for } i \geq k \text{ or } j \geq l,
\]

\[
h^{[i,j]} c,
\]

\(1, t, \mathbf{t}1\)

be called redexes. A reduction of the first sort consists in replacing a redex of the first form by the corresponding term on the right-hand side of one of the equations \((hI), (hII), (hcI), (hIII.1), (hIII.2)\) and \((hIII.3)\). (Note that the terms on the left-hand sides of these equations cover all possible redexes of the first form, and the conditions of these equations exclude each other.) A reduction of the second sort consists in replacing a redex of the second form by the right-hand side of \((hcI)\), and, finally, a reduction of the third sort consists in replacing a redex of one the forms in the third line by \( t \), according to equation (1).
Let the weight of a block \( h_{i,j} \) be \( i - j + 2 \). For any subterm \( h_{i,j} \) of a term \( t \) in the block formulation of \( K_n \), let \( \rho(h_{i,j}) \) be the number of subterms \( h_{k,l} \) of \( t \) on the right-hand side of \( h_{i,j} \) such that \( i \geq k \) or \( j \geq l \). The subterms \( h_{k,l} \) are not necessarily immediately on the right-hand side of \( h_{i,j} \) as in redexes of the first form: they may also be separated by other terms. For any subterm \( c \) of a term \( t \) in the block formulation \( K_n \), let \( \tau(c) \) be the number of blocks on the left-hand side of this \( c \).

The complexity measure of a term \( t \) in the block formulation is \( \mu(t) = (n_1, n_2) \) where \( n_1 \geq 0 \) is the sum of the weights of all the blocks in \( t \), and \( n_2 \geq 0 \) is the sum of all the numbers \( \rho(h_{i,j}) \) for all blocks \( h_{i,j} \) in \( t \) plus the sum of all the numbers \( \tau(c) \) for all circles \( c \) in \( t \) plus the number of occurrences of 1 in \( t \). The ordered pairs \((n_1, n_2)\) are well-ordered lexicographically.

Then we check that if \( t' \) is obtained from \( t \) by a reduction, then \( \mu(t') \) is strictly smaller than \( \mu(t) \). With reductions of the first sort we have that if they are based on \((hI)\), then \( n_2 \) diminishes while \( n_1 \) doesn’t change, and if they are based on the remaining equations, then \( n_1 \) diminishes. With reductions of the second and third sort, \( n_2 \) diminishes while \( n_1 \) doesn’t change.

So, by induction on the complexity measure, we obtain that every term is equal to a term without redexes, and it is easy to see that a term is without redexes if it is in Jones normal form. \( q.e.d. \)

Note that for a term \( e^i h_{[b_1,a_1]} \ldots h_{[b_k,a_k]} \) in Jones normal form the number \( a_i \) is strictly smaller than \( \mu(t) \). With reductions of the first sort we have that if they are based on \((hI)\), then \( n_2 \) diminishes while \( n_1 \) doesn’t change, and if they are based on the remaining equations, then \( n_1 \) diminishes. With reductions of the second and third sort, \( n_2 \) diminishes while \( n_1 \) doesn’t change.

So, by induction on the complexity measure, we obtain that every term is equal to a term without redexes, and it is easy to see that a term is without redexes if it is in Jones normal form. \( q.e.d. \)

Remark 1 If in a term in Jones normal form a diapsis \( h^i \) occurs more than once, then in between any two occurrences of \( h^i \) we have an occurrence of \( h^{i+1} \) and an occurrence of \( h^{i-1} \).

A normal form dual to Jones' is obtained with blocks \( h_{i,j} \) where \( i \leq j \), which are defined as \( h^i h^{i+1} \ldots h^j h^j \). Then in \( e^i h_{[a_1,b_1]} \ldots h_{[a_k,b_k]} \) we require that \( a_1 > \ldots > a_k \) and \( b_1 > \ldots > b_k \). The length of this new normal form will be the same as the length of Jones'. As a matter of fact, we could take as a term in normal form many other terms of the same reduced length as terms in the Jones normal form. For all these alternative normal forms we can establish the property of Remark 1.

2 The \( n \)-diagrams

A one-manifold with boundary is a topological space whose points have open neighbourhoods homeomorphic to the real intervals \((-1,1)\) or \([0,1)\), the boundary points having the latter kind of neighbourhoods. For \( n \geq 2 \) a natural number and \( a > 0 \) a real number, let \( R_{n,a} \) be the rectangle \([0,n+1] \times [0,a] \).
Let \([0, n+1] \times \{a\} \subseteq R_{n,a}\) be the top of \(R_{n,a}\) and let \([0, n+1] \times \{0\} \subseteq R_{n,a}\) be the bottom of \(R_{n,a}\).

An \(n\)-diagram \(D\) is a compact one-manifold with boundary with a finite number of connected components embedded in a rectangle \(R_{n,a}\) such that the intersection of \(D\) with the top of \(R_{n,a}\) is \(t(D) = \{(i, a) \mid i \in \mathbb{N} \cap [1, n]\}\), the intersection of \(D\) with the bottom of \(R_{n,a}\) is \(b(D) = \{(i, 0) \mid i \in \mathbb{N} \cap [1, n]\}\), and \(t(D) \cup b(D)\) is the set of boundary points of \(D\).

It follows from this definition that every \(n\)-diagram has \(n\) components homeomorphic to \([0, 1]\), which are called threads, and a finite number of components homeomorphic to \(S^1\), which are called circular components. The threads and the circular components make all the connected components of an \(n\)-diagram. All these components are mutually disjoint.

Every thread has two end points that belong to the boundary \(t(D) \cup b(D)\). When one of these end points is in \(t(D)\) and the other in \(b(D)\), the thread is transversal. A transversal thread is vertical when the first coordinates of its end points are equal. A thread that is not transversal is a cup when both of its end points are in \(t(D)\), and it is a cap when they are both in \(b(D)\). It is clear that the following holds.

**Remark 2** In every \(n\)-diagram the number of cups is equal to the number of caps.

For example, an 11-diagram in \(R_{11,10}\) looks as follows:
We say that two \( n \)-diagrams \( D_1 \) in \( R_{n,a} \) and \( D_2 \) in \( R_{n,b} \) are equivalent, and write \( D_1 \simeq D_2 \), iff there is a homeomorphism \( h : D_1 \rightarrow D_2 \) such that \( h(i,0) = (i,0) \) and \( h(i,a) = (i,b) \). It is clear that this defines indeed an equivalence relation between \( n \)-diagrams.

Equivalence classes of \( n \)-diagrams are the sort of structure that make what Kauffman in [9, pp. 440-441] calls a connection monoid. Kauffman’s diagram monoids of [9, p. 440] are generated from equivalence classes of particular \( n \)-diagrams, such as we consider in Section 3. It simplifies matters that equivalence of \( n \)-diagrams can be defined in terms of homeomorphisms, as it was done here, rather than in terms of ambient isotopies, as with other sorts of tangles, which involve crossings.

If \( i \) stands for \((i,a)\) and \(-i\) stands for \((i,0)\), we may identify the end points of each thread of an \( n \)-diagram in \( R_{n,a} \) by a pair of integers in \([-n,n] - \{0\}\). Then it is easy to see the following.

**Remark 3** The \( n \)-diagrams \( D_1 \) and \( D_2 \) are equivalent iff

(i) the end points of the threads in \( D_1 \) are identified with the same pairs of integers as the end points of the threads in \( D_2 \), and

(ii) \( D_1 \) and \( D_2 \) have the same number of circular components.

Let us say that the closed interval \([x,y] = \{r \in \mathbb{R} \mid x \leq r \leq y\}\) is proper iff \(x < y\). We say that \([a,b]\) encloses \([c,d]\) iff \(a < c < d < b\). We may then identify the equivalence class of an \( n \)-diagram by a set of \( n \) proper intervals in \([-n,n] - \{0\}\) with end points in \(\mathbb{Z}\) such that any two distinct intervals in the set are either disjoint or one of these two intervals encloses the other; we must moreover specify a natural number that stands for the number of circular components in the \( n \)-diagrams of our equivalence class. Each interval with end points in \(\mathbb{Z}\) may, of course, be identified with a pair of integers. So we may identify the equivalence class of an \( n \)-diagram by the pair \((\Theta,l)\) where \(\Theta\) is a set of \( n \) pairs of integers in \([-n,n] - \{0\}\) that satisfies the conditions stated above in terms of intervals and \(l\) is the number of circular components.

Parenthetical words are finite sequences of the symbols ( and ) defined inductively as follows:

- the empty word is a parenthetical word;
- if \(\alpha\) is a parenthetical word, then \((\alpha)\) is a parenthetical word;
- if \(\alpha\) and \(\beta\) are parenthetical words, then \(\alpha\beta\) is a parenthetical word.

It is clear that parenthetical words with \(2n\) symbols are in one-to-one correspondence with equivalence classes of \( n \)-diagrams without circular components.

Let an \( o \)-monoid be a monoid with an arbitrary unary operation \( o \), and consider the free \( o \)-monoid \( \mathcal{M} \) generated by the empty set of generators. The free \( o \)-monoid \( \mathcal{M} \) is isomorphic to parenthetical words where the unit is the empty word, multiplication is concatenation, and the operation \( o \) is putting in parentheses. So equivalence classes of \( n \)-diagrams without circular components may be conceived as elements of \( \mathcal{M} \).
The set of equivalence classes of $n$-diagrams is endowed with the structure of a monoid in the following manner. Let the unit $n$-diagram $I$ be $\{(i, y) \mid i \in \mathbb{N} \cap [1,n] \text{ and } y \in [0,1]\}$ in $R_{n,1}$. So $I$ has no circular component, and all of its threads are vertical transversal threads. We draw $I$ as follows:

For two $n$-diagrams $D_1$ in $R_{n,a}$ and $D_2$ in $R_{n,b}$ let the composition of $D_1$ and $D_2$ be defined as follows:

$$D_2 \circ D_1 = \{(x, y+b) \mid (x, y) \in D_1\} \cup D_2.$$ 

It is easy to see that $D_2 \circ D_1$ is an $n$-diagram in $R_{n,a+b}$.

Let $D_1$ be an $n$-diagram in $R_{n,a}$, and suppose $D_1 \simeq D_3$ with the homeomorphism $h_1 : D_1 \to D_3$, and $D_2 \simeq D_4$ with the homeomorphism $h_2 : D_2 \to D_4$. Then $D_2 \circ D_1 \simeq D_4 \circ D_3$ with the homeomorphism $h : D_2 \circ D_1 \to D_4 \circ D_3$ defined as follows. For $p^1$ the first and $p^2$ the second projection, let

$$h(x, y) = \begin{cases} (p^1(h_1(x, y - a_2)), p^2(h_1(x, y - a_2)) + a_4), & \text{if } y > a_2 \\ h_2(x, y), & \text{if } y \leq a_2. \end{cases}$$

So the composition $\circ$ defines an operation on equivalence classes of $n$-diagrams. We can then establish that

1. $I \circ D \simeq D \circ I \simeq D$,
2. $D_3 \circ (D_2 \circ D_1) \simeq (D_3 \circ D_2) \circ D_1$.

The equivalences of (1) follow from the fact that $I \circ D$, $D \circ I$ and $D$ have the same number of circular components, because $I$ has no circular component, and that their threads may be identified with the same pairs of integers, because all the threads of $I$ are vertical transversal threads. Then we apply Remark 3. For the equivalence (2), it is clear that $D_3 \circ (D_2 \circ D_1)$ is actually identical to $(D_3 \circ D_2) \circ D_1$. 

8
Let $a(n)$ be $\max\{5, (n-1)(n-2)/2\}$. (The precise value of $a(n)$ could be varied, and we have chosen one of infinitely many possibilities.) We say that an $n$-diagram is normal iff it is in $R_{n,a(n)}$, each of its transversal threads is a straight line segment, and each of its cups and caps is a semicircle. It is clear that two normal $n$-diagrams without circular components are equivalent iff they are equal. (To make this equivalence hold even in the presence of circular components we can find a definite place in $R_{n,a(n)}$ for an arbitrary finite number of circles that will not intersect with each other and with other components.)

Every $n$-diagram $D$ is equivalent to a normal $n$-diagram, which is a handy representative for the equivalence class of $D$. A normal 11-diagram equivalent to the 11-diagram from the beginning of this section looks as follows:

3 Generating $n$-diagrams

For $i \in \mathbb{N} \cap [1,n-1]$, the diapsidal $n$-diagram $H^i$ is the normal $n$-diagram without circular components, with a single cup with the end points $(i,a(n))$ and $(i+1,a(n))$, and a single cap with the end points $(i,0)$ and $(i+1,0)$; all the other threads are transversal threads orthogonal to the $x$ axis. A diapsidal $n$-diagram $H^i$ looks as follows:
The *circular* $n$-diagram $C$ is the $n$-diagram that differs from the unit $n$-diagram $I$ by having a single circular component, which for the sake of definiteness we may choose to be a circle of radius, let us say, $1/4$, with centre $(1/2, 1/2)$.

If the end points of a thread of an $n$-diagram are $(i, x)$ and $(j, y)$, where $x, y \in \{0, a\}$, let us say that this thread *covers* a pair of natural numbers $(m, l)$, where $1 \leq m < l \leq n$, if $\min\{i, j\} \leq m$ and $l \leq \max\{i, j\}$. Then we can establish the following.

**Remark 4** In every $n$-diagram, every pair $(m, m+1)$, where $1 \leq m < n$, is covered by an even number of threads.

*Proof.* For an $n$-diagram $D$ the cardinality of the set $P = \{(i, x) \in t(D) \cup b(D) \mid i \leq m\}$ is $2m$. Every thread of $D$ that covers $(m, m+1)$ has a single end point in $P$, and other threads of $D$ have 0 or 2 end points in $P$. Since $P$ is even, the remark follows. *q.e.d.*

If the end points of a thread of an $n$-diagram are $(i, x)$ and $(j, y)$, let the *span* of the thread be $|i - j|$. Let the span $\sigma(D)$ of an $n$-diagram $D$ be the sum of the spans of all the threads of $D$. Remark 4 entails that the span of an $n$-diagram is an even number greater than or equal to 0; the span of $I$ is 0. It is clear that equivalent $n$-diagrams have the same span. It is also easy to see that the following holds.

**Remark 5** If an $n$-diagram has cups, then it must have at least one cup whose span is 1. The same holds for caps.

We can now prove the following lemma.

**Lemma 2** Every $n$-diagram is equivalent to an $n$-diagram generated from $I$, $C$ and the diapsidal $n$-diagrams $H^i$, for $1 \leq i \leq n - 1$, with the operation of composition $\circ$.

*Proof.* Take an arbitrary $n$-diagram $D$ in $R_{n,a}$, and let $D$ have $l \geq 1$ circular components. Let $C^1$ be $C$, and let $C^{l+1}$ be $C^1 \circ C$. It is clear that $D$ is equivalent to an $n$-diagram $C^l \circ D_1$ in $R_{n,a+l}$ with $D_1$ an $n$-diagram in $R_{n,a}$ without circular components. If $l = 0$, then $D_1$ is $D$. It simplifies matters if we assume that $D_1$ is a normal $n$-diagram, but this is not essential.

Then we proceed by induction on $\sigma(D_1)$. If $\sigma(D_1) = 0$, then $D_1 \simeq I$, and the lemma holds. Suppose $\sigma(D_1) > 0$. Then there must be a cup or a cap in $D_1$, and by Remarks 2 and 5, there must be at least one cup of $D_1$ whose span is 1. The end points of such cups are of the form $(i, a)$ and $(i+1, a)$. Then select among all these cups that one where $i$ is the greatest number; let that number be $j$, and let us call the cup we have selected $v_j$. By Remark 4, there must be at least one other thread of $D_1$, different from $v_j$, that also covers $(j, j+1)$. We have to consider four cases, which exclude each other:
(1) \((j, j + 1)\) is covered by a cup \(\xi\) of \(D_1\) different from \(\nu_j\), whose end points are \((p, a)\) and \((q, a)\) with \(p < q\);

(2.1) \((j, j + 1)\) is not covered by a cup of \(D_1\) different from \(\nu_j\), but it is covered by a transversal thread \(\xi\) of \(D_1\) whose end points are \((p, a)\) and \((q, 0)\) with \(p < q\);

(2.2) same as case (2.1) save that the end points of \(\xi\) are \((p, 0)\) and \((q, a)\) with \(p < q\);

(3) \((j, j + 1)\) is covered neither by a cup of \(D_1\) different from \(\nu_j\), nor by a transversal thread of \(D_1\), but it is covered by a cap \(\xi\) of \(D_1\), whose end points are \((p, 0)\) and \((q, 0)\) with \(p < q\).

In cases (1) and (2.1) we select among the threads \(\xi\) mentioned the one where \(p\) is maximal, and in cases (2.2) and (3) we select the \(\xi\) where \(p\) is minimal. (We obtain the same result if in cases (1) and (2.2) we take \(q\) minimal, while in (2.1) and (3) we take \(q\) maximal.)

We build out of the \(n\)-diagram \(D_1\) a new \(n\)-diagram \(D_2\) in \(R_{n,a}\) by replacing the thread \(\nu_j\) and the selected thread \(\xi\), whose end points are \((p, x)\) and \((q, y)\), where \(x, y \in \{0, a\}\), with two new threads: one whose end points are \((p, 0)\) and \((q, 0)\) with \(p < q\). We can easily check that \(D_1 \simeq D_2 \circ H^j\). This is clear from the following picture:

Neither of the new threads of \(D_2\) that have replaced \(\nu_j\) and \(\xi\) covers \((j, j + 1)\), and \(\sigma(D_1) = \sigma(D_2) + 2\). So, by the induction hypothesis, \(D_2\) is equivalent to an \(n\)-diagram \(D_3\) generated from \(I, C\) and \(H^i\) with \(\circ\), and since \(D_1 \simeq D_3 \circ H^j\), this proves the lemma. \(q.e.d.\)
Note that we need not require in this proof that in \( v_j \) the number \( j \) should be the greatest number \( i \) for cups with end points \((i, a)\) and \((i+1, a)\). The proof would go through without making this choice. But with this choice we shall end up with a composition of \( n\)-diagrams that corresponds exactly to a term of \( K_n \) in Jones normal form.

4 \( K_n \) is the monoid of \( n\)-diagrams

Let \( D_n \) be the set of \( n\)-diagrams. We define a map \( \delta : K_n \rightarrow D_n \) as follows:

\[
\begin{align*}
\delta(h^i) &= H^i, \\
\delta(c) &= C, \\
\delta(1) &= I, \\
\delta(tu) &= \delta(t) \circ \delta(u).
\end{align*}
\]

We can then prove the following.

**Lemma 3** If \( t = u \) in \( K_n \), then \( \delta(t) \simeq \delta(u) \).

**Proof.** We have already verified in Section 2 that we have replacement of equivalents, and that the equations (1) and (2) of the axiomatization of \( K_n \) are satisfied for \( I \) and \( \circ \). It just remains to verify \((h1), (h2), (hc1)\) and \((hc2)\), which is quite straightforward. q.e.d.

Let \([D_n]\) be the set of equivalence classes \([D]\) of the \( n\)-diagrams \( D \). This set is a monoid whose unit is \([I]\) and whose multiplication is defined by taking that \([D_1][D_2]\) is \([D_1 \circ D_2]\). If \( \delta' : K_n \rightarrow [D_n] \) is defined as \( \delta \) on the generators of \( K_n \), save that \( H^i \) and \( C \) are replaced by \([H^i]\) and \([C]\), and if \( \delta'(1) = [I]\) and \( \delta'(tu) = \delta'(t) \circ \delta'(u) \), then Lemma 3 guarantees that \( \delta' \) is a homomorphism. Lemma 2 guarantees that \( \delta' \) is onto. To establish that \( \delta' \) is an isomorphism it remains only to show that \( \delta' \) is one-one.

Let a transversal thread in an \( n\)-diagram be called \textit{falling} iff its end points are \((i, a)\) and \((j, 0)\) with \( i < j \). If the end points of a thread of an \( n\)-diagram \( D \) are \((i, a)\) and \((j, x)\) with \( x \in \{0, a\} \) and \( i < j \), then we say that \((i, a)\) is a \textit{top slope point} of \( D \). If the end points of a thread of \( D \) are \((i, x)\) and \((j, 0)\) with \( x \in \{0, a\} \) and \( i < j \), then we say that \((j, 0)\) is a \textit{bottom slope point} of \( D \). Each cup has a single top slope point, each cap has a single bottom slope point, and each falling transversal thread has one top slope point and one bottom slope point. Other transversal threads have no slope points. So we can ascertain the following.

**Remark 6** In every \( n\)-diagram the number of top slope points is equal to the number of bottom slope points.
Remember that by Remark 2 the number of cups is equal to the number of caps.

Let \((a_1, a), \ldots, (a_k, a)\) be the sequence of all top slope points of an \(n\)-diagram \(D\), ordered so that \(a_1 < \ldots < a_k\), and let \((b_1 + 1, 0), \ldots, (b_k + 1, 0)\) be the sequence of all bottom slope points of \(D\), ordered so that \(b_1 < \ldots < b_k\) (as we just saw with Remark 6, these sequences must be of equal length). Then let \(T_D\) be the sequence of natural numbers \(a_1, \ldots, a_k\) and \(B_D\) the sequence of natural numbers \(b_1, \ldots, b_k\).

**Remark 7** The sequence \(T_{\delta(h^{[i,j]})}\) has a single member \(j\) and the sequence \(B_{\delta(h^{[i,j]})}\) has a single member \(i\).

This is clear from the \(n\)-diagram \(\delta(h^{[i,j]})\), which is equivalent to an \(n\)-diagram of the following form:

![Diagram](attachment:diagram.png)

provided \(0 < j < i < n\) (in other cases we simplify this diagram by omitting some transversal threads).

**Remark 8** If \(D_1 \simeq D_2\), then \(T_{D_1} = T_{D_2}\) and \(B_{D_1} = B_{D_2}\).

This follows from Remark 3.

Then we can prove the following lemmata.

**Lemma 4** If \(t\) is the term \(h^{[b_1, a_1]} \ldots h^{[b_k, a_k]}\) with \(a_1 < \ldots < a_k\) and \(b_1 < \ldots < b_k\), then \(T_{\delta(t)}\) is \(a_1, \ldots, a_k\) and \(B_{\delta(t)}\) is \(b_1, \ldots, b_k\).

**Proof.** We proceed by induction on \(k\). If \(k = 1\), we use Remark 7. If \(k > 1\), then, by the induction hypothesis, the lemma has been established for the term \(h^{[b_1, a_1]} \ldots h^{[b_{k-1}, a_{k-1}]}\), which we call \(t'\). So \(T_{\delta(t')}\) is \(a_1, \ldots, a_{k-1}\).

Since in \(\delta(h^{[b_k, a_k]})\) every point in the top with the first coordinate \(i < a_k\) is the end point of a vertical transversal thread, and since \(\delta(t) = \delta(t') \circ \delta(h^{[b_k, a_k]})\), the beginning of the sequence \(T_{\delta(t)}\) must be \(a_1, \ldots, a_{k-1}\). To this sequence we have to add \(a_k\) because \(\delta(t')\) inherits the cup of \(\delta(h^{[b_k, a_k]})\). This shows immediately that \(a_k + 1\) is not in \(T_{\delta(t)}\). It remains to show that for no \(i \geq a_k + 2\) we can have in \(\delta(t)\) a top slope point with the first coordinate \(i\).
If \( i > b_k + 1 \), then every point in the top with the first coordinate \( i \) is the end point of a vertical transversal thread in both \( \delta(h^{[b_k,a_k]}) \) and \( \delta(t') \). So \( i \) is not in \( T_{\delta(t')} \). It remains to consider \( i \) for \( a_k + 2 \leq i \leq b_k + 1 \). Every point in the top with this first coordinate \( i \) is the end point of a transversal thread in \( \delta(h^{[b_k,a_k]}) \) whose other end point is \((i - 2, 0)\). If \( i \) were to be added to \( T_{\delta(t')} \), the number \( i - 2 \) would be in \( T_{\delta(t')} \), but this contradicts the fact that \( T_{\delta(t')} \) ends with \( a_{k-1} \). So \( T_{\delta(t')} \) is \( a_1, \ldots, a_k \).

To show that \( B_{\delta(t')} \) is \( b_1, \ldots, b_k \) we reason analogously by applying the induction hypothesis to \( h^{[b_2,a_2]} \ldots h^{[b_k,a_k]} \). \( q.e.d. \)

**Lemma 5** If \( t \) and \( u \) are terms of \( \mathcal{K}_n \) in Jones normal form and \( \delta(t) \simeq \delta(u) \), then \( t \) and \( u \) are the same term.

**Proof.** Let \( t \) be \( c^l h^{[b_1,a_1]} \ldots h^{[b_k,a_k]} \) and let \( u \) be \( c^j h^{[d_1,c_1]} \ldots h^{[d_m,c_m]} \). If \( l \neq j \), then \( \delta(t) \) is not equivalent to \( \delta(u) \) by Remark 3, because \( \delta(t) \) and \( \delta(u) \) have different numbers of circular components. If \( a_1, \ldots, a_k \) is different from \( c_1, \ldots, c_m \), or \( b_1, \ldots, b_k \) is different from \( d_1, \ldots, d_m \), then \( \delta(t) \) is not equivalent to \( \delta(u) \) by Lemma 4 and Remark 8. \( q.e.d. \)

**Lemma 6** If \( \delta(t) \simeq \delta(u) \), then \( t = u \) in \( \mathcal{K}_n \).

**Proof.** Suppose \( t = u \) doesn’t hold in \( \mathcal{K}_n \). Let \( t' \) and \( u' \) be terms in the Jones normal form such that \( t = t' \) and \( u = u' \) in \( \mathcal{K}_n \). Such terms exist according to Lemma 1. Then \( t' \) and \( u' \) must be different terms; otherwise \( t = u \) would hold in \( \mathcal{K}_n \). By Lemma 5 we have that \( \delta(t') \) is not equivalent to \( \delta(u') \). Since by Lemma 3 we have \( \delta(t) \simeq \delta(t') \) and \( \delta(u) \simeq \delta(u') \), we can conclude that \( \delta(t) \) is not equivalent to \( \delta(u) \). \( q.e.d. \)

Lemma 6 guarantees that \( \delta' : \mathcal{K}_n \to [D_n] \) is one-one, and so \( \delta' \) is an isomorphism.

We can now conclude that for every term \( t \) of \( \mathcal{K}_n \) there is a unique term \( t' \) in Jones normal form such that \( t = t' \) in \( \mathcal{K}_n \). Otherwise, if \( t \) were equal in \( \mathcal{K}_n \) to two different terms \( t' \) and \( t'' \) in Jones normal form, by Lemma 3 we would have \( \delta(t) \simeq \delta(t') \) and \( \delta(t) \simeq \delta(t'') \), and hence also \( \delta(t') \simeq \delta(t'') \). But this contradicts Lemma 5.

This solves the word problem for Kauffman monoids. To check whether \( t = u \) in \( \mathcal{K}_n \) just reduce \( t \) and \( u \) to Jones normal form, according to the procedure of the proof of Lemma 1, and then check whether the normal forms obtained are equal. However, to reduce a term of \( \mathcal{K}_n \) to Jones normal form, now that we have established that \( \delta' \) is an isomorphism, we can proceed more efficiently with \( n \)-diagrams than with the syntactical method of the proof of Lemma 1.

From Remark 1 in Section 1 it follows that in the \( n \)-diagram \( \delta(t) \) of a term \( t \) in Jones normal form, or in any alternative reduced normal form, such as those envisaged after Remark 1, we will have no threads with bulges like the following:
Reduction to normal form involves getting rid of these bulges, and this is done more easily diagrammatically than syntactically. To reduce a term \( t \) to Jones normal form we first draw the \( n \)-diagram \( \delta(t) \). Then we replace \( \delta(t) \) by a normal \( n \)-diagram \( D \) such that \( D \simeq \delta(t) \). This is where bulges get eliminated.

Our proof of Lemma 2 in Section 3 then gives a procedure for building out of \( D \) the term \( t' \) in Jones normal form such that \( \delta(t') \simeq D \), Lemma 6 guaranteeing that \( t' \) is equal in \( K_n \) to the original \( t \). We could apply this procedure to \( \delta(t) \) directly, but it is easier to apply it to the normal \( n \)-diagram \( D \).

There is, however, a procedure handier than that, which also yields a term in Jones normal form out of a normal \( n \)-diagram. This procedure (suggested by Figure 16 of [9, p. 434], and detailed in [13, Section 6], in mirror image) is illustrated in the following picture, based on the example from the end of Section 2:
The term \( t' \) in Jones normal form such that \( \delta(t') \) is equivalent to the 11-diagram whose threads are dotted is

\[ c^6 h^{[3,1]} h^{[4,4]} h^{[7,7]} h^{[9,8]} h^{[10,9]} \]

Each solid staircase in the picture corresponds to a block of the normal form.

However, the most simple procedure to obtain out of \( t \) the term \( t' \) in Jones normal form such that \( t = t' \) in \( \mathcal{K}_n \) is to draw \( \delta(t) \) and recognize in it the top and bottom slope points, from which we immediately obtain \( t' \). And to check whether \( t = u \) in \( \mathcal{K}_n \), it is enough to check whether \( \delta(t) \simeq \delta(u) \), which we can do without mentioning the Jones normal form, though we relied essentially on this normal form in order to demonstrate that \( \delta' \) is an isomorphism.

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