Abstract: In the framework of a renormalizable quantum field theory and taking the example of neutral kaons, $CP$ violation is shown to be a dynamical consequence of the anomalous scaling of the meson fields; it is also shown to be compatible with the normality condition $[M, M^\dagger] = 0$ for the renormalized mass matrix.

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1 Introduction

$CP$ violation for pseudoscalar mesons is a well detected experimental fact which is parametrized, in the quark framework, by a suitable mixing matrix \[1\]. It stays a phenomenon \textit{per se}, which has been lacking a fundamental understanding, or, at least, a connection with other properties of field theory.

In the framework of a renormalizable quantum field theory, indirect $CP$ violation for neutral kaons (or similar systems) is shown in this letter to be directly connected to the anomalous dimension of the neutral kaon fields, and, so, to scaling violations.

The basic remark which straightforwardly leads to this result is that the physical (different) masses $m_S$ and $m_L$ of the neutral mesons are not the roots of the characteristic equation of a single constant effective mass matrix \[2\], but are the poles of the full renormalized propagator of this binary system. The solutions of the self-consistent equations $p^2 = m^2(p^2)$, the $m^2(p^2)$’s being the eigenvalues of the full renormalized self-energy. Their evolution in the (small) interval $p^2 \in [m^2_S, m^2_L]$ is accordingly driven by renormalization group equations, with the consequence that no more than one of them can be a $CP$ eigenstate as soon as they scale with anomalous dimensions.

2 Demonstration

2.1 Physical masses and eigenstates

Let $M^{(2)}(p^2)$, with dimension $[\text{mass}]^2$, be the full renormalized self-energy $2 \times 2$ matrix for the system of neutral kaons in the ($K^0$, $\bar{K}^0$) basis; the corresponding full inverse propagator writes

$$L(p^2) = \mathbb{I} p^2 - M^{(2)}(p^2), \quad \mathbb{I} = \text{diag}(1, 1).$$

Accounting for unstable kaons requires $M^{(2)}(p^2)$ to be non-hermitian, but we insist that, \textit{for all} $p^2$, it is normal \[1\] such that its right- and left-eigenvectors coincide \[3\] \[2\]; we demand furthermore that it satisfies the condition for $CPT$ invariance, \textit{i.e.} its two diagonal elements are identical \[1\]; using the more convenient notation $z$ for the complex $p^2$, $M^{(2)}$ writes accordingly

$$M^{(2)}(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & a(z) \end{pmatrix}, \quad \text{with } b(z) \overline{b(\bar{z})} = c(z) \overline{c(\bar{z})}. \quad (2)$$

The physical masses \[3\] $z_1 \equiv m^2_S$ and $z_2 \equiv m^2_L$ are solutions of the self consistent equations $z - \lambda_\pm(z) = 0$, $\lambda_\pm(z)$ being the two solutions $\lambda_\pm(z) = a(z) \pm \sqrt{b(z)c(z)}$ of $\det(M^{(2)}(z) - \lambda \mathbb{I}) = 0$, such that one has \[4\]

$$\text{for } K_S : z_1 = a(z_1) - \sqrt{b(z_1)c(z_1)}, \quad \text{for } K_L : z_2 = a(z_2) + \sqrt{b(z_2)c(z_2)}.$$ \quad (3)

The existence of other (spurious) solutions is irrelevant for what follows; one only needs $z_1 \neq z_2$.

The components $u_\pm(z)$ and $v_\pm(z)$ of the eigenvectors of $M^{(2)}(z)$ (and $L(z)$) satisfy

$$\frac{v_\pm(z_\pm)}{u_\pm(z_\pm)} = \pm \frac{c(z_\pm)}{b(z_\pm)}, \quad (4)$$

\[1\] \textit{i.e.} it commutes with its hermitian conjugate $[M, M^\dagger] = 0$.

\[2\] No physically satisfying interpretation of a mismatch between the two could ever be given; that our result fits in the case of a normal matrix is thus welcome.

\[3\] They are complex as soon as the kaons are unstable \[1\].

\[4\] Of course, which is attributed to which is irrelevant.
and thus 5

\[
\text{for } K_S : \quad \frac{v_-(z_1)}{u_-(z_1)} = -\sqrt{\frac{c(z_1)}{b(z_1)}} = -\frac{z_1 - a(z_1)}{b(z_1)}; \quad \text{for } K_L : \quad \frac{v_+(z_2)}{u_+(z_2)} = +\sqrt{\frac{c(z_2)}{b(z_2)}} = +\frac{z_2 - a(z_2)}{b(z_2)}.
\]

The condition for the normality of \( M^{(2)} \) at the right of (2) entails

\[
\left| \frac{v_+(z)}{u_+(z)} \right|^2 = 1.
\]

### 2.2 Renormalization group equations

We want to connect \( v_+(z_2)/u_+(z_2) \) to \( v_-(z_1)/u_-(z_1) \).

The elements of the renormalized full self-energy matrix \( M^{(2)} \), that we noted \( a(z), b(z) \) and \( c(z) \) depend in reality not only on the external momentum \( p \), but also of (dimensionless) coupling constants \( g_i \), masses \( m_j \), (dimensionless) gauge parameters \( \xi_k \), and of a renormalization scale \( \mu \); this is why, in the following, we adopt instead the notation \( a(p, g_i, m_j, \alpha_k, \mu) \) etc, with \( z = p^2 \). Defining \( y_j = \frac{m_j}{\mu} \) and \( t = \ln x \), \( a(p, g_i, m_j, \alpha_k, \mu) \) satisfies the renormalization group equation (see appendix for the demonstration)

\[
\left( -\frac{\partial}{\partial t} + \sum_i \beta_i(g_i, y_j, \xi_k, \epsilon) \frac{\partial}{\partial g_i} - \sum_j \left( 1 + \gamma_j(g_i, y_j, \xi_k, \epsilon) \right) y_j \frac{\partial}{\partial y_j} + \sum_k \beta_{ak}(g_i, y_j, \xi_k, \epsilon) \frac{\partial}{\partial \xi_k} \right) a(e^t p, g_i, y_j, \xi_k, \mu) = 0,
\]

where the \( \beta_i \)'s, \( \gamma_j \)'s, \( \beta_{ak} \)'s, and the anomalous dimensions \( \gamma_{K^0} \) and \( \overline{\gamma_{K^0}} \) of the mesons are defined by

\[
\frac{d g_i}{d \mu} = g_i \beta_i(g_i, y_j, \xi_k, \epsilon), \quad \frac{\mu}{m_j} \frac{d m_j}{d \mu} = -\gamma_j(g_i, y_j, \xi_k, \epsilon), \quad \frac{\mu}{\xi_k} \frac{d \xi_k}{d \mu} = \beta_{ak}(g_i, y_j, \xi_k, \epsilon),
\]

\[
\frac{d Z_{K^0}(\mu, \cdots)}{d \mu} = \gamma_{K^0}(g_i, y_j, \xi_k, \epsilon), \quad \frac{d Z_{\overline{K^0}}(\mu, \cdots)}{d \mu} = \overline{\gamma_{K^0}(g_i, y_j, \xi_k, \epsilon)};
\]

\( Z_{K^0}(\mu, \cdots) \) and \( Z_{\overline{K^0}}(\mu, \cdots) \) are the renormalization constants for the wave functions of \( K^0 \) and \( \overline{K^0} \). 6

The solution of (7) writes \((x = e^t)\)

\[
a(e^t p, g_i, y_j, \xi_k, \mu) = \left[ x^2 a(p, \overline{g_i}, \overline{y_j}, \overline{\xi_k}, \mu) e^{-\int_0^t dt' \gamma_{K^0}(\overline{g_i}(t'), \overline{y_j}(t'), \overline{\xi_k}(t'), \mu)} \right],
\]

where the “running” \( \overline{g_i}, \overline{y_j} \) and \( \overline{\xi_k} \) are defined by (at the extreme right are the initial conditions)

\[
\frac{d \overline{g_i}(t, g_j)}{d t} = \overline{g_i}(\overline{g_j}); \quad \overline{g_i}(0, g_j) = g_i;
\]

\[
\frac{d \overline{y_j}(t, g_i)}{d t} = -(1 + \gamma_j(\overline{g_i})) \overline{y_j}; \quad \overline{y_j}(0, g_i) = y_j;
\]

\[
\frac{d \overline{\xi_k}(t, g_i)}{d t} = \beta_{ak}(\overline{g_i}); \quad \overline{\xi_k}(0, g_i) = \xi_k.
\]

The equations for \( b \) and \( c \) are similar to (7) with the only replacement of \( \gamma_{K^0} \) by \( \frac{\gamma_{K^0} + \overline{\gamma_{K^0}}}{2} \).

5The other eigenvectors \( \begin{pmatrix} u_+(z_1) \\ v_+(z_1) \end{pmatrix} \) and \( \begin{pmatrix} u_-(z_2) \\ v_-(z_2) \end{pmatrix} \) respectively of \( M^{(2)}(z_1) \) and \( M^{(2)}(z_2) \) are spurious 2.

6The renormalized and bare elements of \( M^{(2)} \) are connected by \( a(\mu, \cdots) = Z_{K^0}(\mu, \cdots) a_0, b(\mu, \cdots) = \sqrt{Z_{K^0}(\mu, \cdots) Z_{\overline{K^0}}(\mu, \cdots) b_0}, c(\mu, \cdots) = \sqrt{Z_{K^0}(\mu, \cdots) Z_{\overline{K^0}}(\mu, \cdots) c_0}. \)
Let \( x = p/p_1 \). Since \( z = p^2 \) and \( p_1^2 = m_S^2 \), (9) yields (somewhat lightening the notations)

\[
a(xp_1, g_i, y_j, \xi_k, \mu) = \frac{z}{m_S^2} a(p_1, \bar{g}_i, \bar{y}_j, \bar{\xi}_k) e^{-\int_0^{(1/2)\ln(z/m_S^2)} dt' \gamma_{K^0}(\bar{g}_i, g_i, \cdots)},
\]

\[
b(xp_1, g_i, y_j, \xi_k, \mu) = \frac{z}{m_S^2} b(p_1, \bar{g}_i, \bar{y}_j, \bar{\xi}_k) e^{-\int_0^{(1/2)\ln(z/m_S^2)} dt' (1/2)(\gamma_{K^0}(\bar{g}_i, g_i, \cdots) + \gamma_{\bar{K}^0}(\bar{g}_i, g_i, \cdots)).}
\]

(11)

In particular, for \( z = z_2 = m_L^2 \), using (5), one gets (with a still more lightened notation)

\[
\frac{v_+(z_2)}{u_+(z_2)} = \frac{z_1 - a(z_1)}{b(z_1)} e^{-\int_0^{(1/2)\ln(m_L^2/m_S^2)} dt' \gamma_{K^0}(t')}.
\]

(12)

2.3 The case of a small mass splitting. Result

Now, we use \( m_L^2/m_S^2 \approx 1 \) and develop the exponentials at the first non-trivial order to get \(^7\)

\[
\frac{v_+(z_2)}{u_+(z_2)} \approx \frac{v_+(z_1)}{u_+(z_1)} \left( 1 + \frac{a(z_1)}{z_1 - a(z_1)} \delta_a + \delta_b \right) = -\frac{v_-(z_1)}{u_-(z_1)} \left( 1 + \frac{a(z_1)}{z_1 - a(z_1)} \delta_a + \delta_b \right),
\]

with \( \delta_a = \int_0^{(1/2)\ln(m_L^2/m_S^2)} dt' \gamma_{K^0}(t') \) and \( \delta_b = \int_0^{(1/2)\ln(m_L^2/m_S^2)} dt' \frac{\gamma_{\bar{K}^0}(t') + \gamma_{K^0}(t')}{2} \).

In the small interval \([0, (1/2)\ln(m_L^2/m_S^2)]\), \( \gamma_{K^0} \) and \( \gamma_{\bar{K}^0} \) are expected to keep practically constant, such that \( \delta_a \approx (1/2)\gamma_{K^0} \ln(m_L^2/m_S^2) \approx (\Delta m_K^2/2m_K)\gamma_{K^0}, \delta_b \approx (1/4)(\gamma_{K^0} + \gamma_{\bar{K}^0}) \ln(m_L^2/m_S^2) \approx (\Delta m_K^2/4m_K^2)(\gamma_{K^0} + \gamma_{\bar{K}^0}) \), with \( \Delta m_K^2 = m_L^2 - m_S^2 \).

Using again \( \Delta m_K^2/m_K^2 \ll 1 \), we can approximate, using (3), \( a(z_1) \approx a(z_2) \approx (z_1 + z_2)/2 \approx m_K^2 \) and, consequently \( z_1 - a(z_1) \approx -\Delta m_K^2/2 \), such that

\[
\frac{v_+(z_2)}{u_+(z_2)} \approx -\frac{v_-(z_1)}{u_-(z_1)} \left( 1 - \gamma_{K^0} - \frac{\gamma_{\bar{K}^0}}{2} \right)
\]

(14)

becomes independent of the mass spectrum. If the \( K^0 \) and \( \bar{K}^0 \) fields, related in quantum field theory by hermitian conjugation, scale in the same way, their anomalous dimensions match and (14) rewrites

\[
\frac{v_+(z_2)}{u_+(z_2)} \approx \frac{v_+(z_1)}{u_+(z_1)} (1 - 2\gamma_{K^0}).
\]

(14) entails that indirect \( CP \) violation is an unavoidable dynamical phenomenon in renormalizable theories with an anomalous scaling of the fields; suppose indeed that one of the mass eigenstate is a \( CP \) eigenstate, i.e. either \( v_+(z_2)/u_+(z_2) = \pm 1 \) or \( v_-(z_1)/u_-(z_1) = \pm 1 \); the other ratio, in order to satisfy the constraint (6), must acquire an imaginary part: the corresponding state cannot be a \( CP \) eigenstate.

3 Conclusion

A general framework has been provided for the study of \( CP \) violation, which is not limited to neutral kaons and extends to other species of particles. A straightforward consequence is that all and only those of the fundamental interactions which give rise to violations of scaling are expected to contribute. Furthermore, by showing that the renormalized mass matrix can be normal, all problems arising from the difference between right- and left-eigenstates have been wiped out.

\(^7\)We use the property demonstrated in (3) that, for all \( z \), \( v_-(z)/u_-(z) = -v_+(z)/u_+(z) \).
A Demonstration of equation (7)

The argumentation follows classical literature (see for example [4]). Since \( a(p, g_i, m_j, \xi_k, \mu) \) has dimension \([\text{mass}]^2\), it satisfies

\[
a(xp, g_i, m_j, \xi_k, \mu) = \mu^2 \tilde{a}(x^2 \frac{z}{\mu^2}, g_i, m_j \mu, \xi_k, \mu),
\]

where \( \tilde{a} \) is an homogeneous function of degree 0 in \( x, m_j, \mu \); similar equations for \( b \) and \( c \) define the homogeneous functions \( \tilde{b} \) and \( \tilde{c} \). Euler’s theorem applied to \( \tilde{a} \) yields

\[
\left( x \frac{\partial}{\partial x} + \sum_j m_j \frac{\partial}{\partial m_j} + \mu \frac{\partial}{\partial \mu} - 2 \right) a(xp, g_i, m_j, \xi_k, \mu) = 0,
\]

and similar equations for \( b \) and \( c \). On the other side, since the bare quantities \( a_0, b_0 \) and \( c_0 \) do not depend on \( \mu \)

\[
\mu \frac{\partial}{\partial \mu} a_0(p, g_{i0}, m_{j0}, a_{k0}, \epsilon) = 0, \text{ etc}
\]

where we have supposed the regularization done by going to \( 4 - \epsilon \) dimensions. This yields (see notations and footnote in subsection 2.2)

\[
\left( \mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial g_i} + \sum_j \gamma_j m_j \frac{\partial}{\partial m_j} + \sum_k \beta_{ak} \frac{\partial}{\partial \xi_k} \right) a(p, g_i, m_j, \xi_k, \mu) = \gamma_K a(p, g_i, m_j, \xi_k, \mu),
\]

which rewrites

\[
\left( \mu \frac{\partial}{\partial \mu} + \sum_i \beta_i g_i \frac{\partial}{\partial g_i} - \sum_j \gamma_j m_j \frac{\partial}{\partial m_j} + \sum_k \beta_{ak} \frac{\partial}{\partial \xi_k} \right) a(p, g_i, m_j, \xi_k, \mu) = \gamma_K a(p, g_i, m_j, \xi_k, \mu).
\]

(19) also applies to \( a(xp, \cdots, \mu) \); replacing there \( \mu \frac{\partial a(xp, \cdots, \mu)}{\partial \mu} \) by its value extracted from (16) gives equation (7).

References

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