Multi-scale Deep Neural Network (MscaleDNN) Methods for Oscillatory Stokes Flows in Complex Domains

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Summary. In this paper, we study a multi-scale deep neural network (MscaleDNN) as a meshless numerical method for computing oscillatory Stokes flows in complex domains. The MscaleDNN employs a multiscale structure in the design of its DNN using radial scalings to convert the approximation of high frequency components of the highly oscillatory Stokes solution to one of lower frequencies. The MscaleDNN solution to the Stokes problem is obtained by minimizing a loss function in terms of $L^2$ norm of the residual of the Stokes equation. Three forms of loss functions are investigated based on vorticity-velocity-pressure, velocity-stress-pressure, and velocity gradient-velocity-pressure formulations of the Stokes equation. We first conduct a systematic study of the MscaleDNN methods with various loss functions on the Kovasznay flow in comparison with normal fully connected DNNs. Then, Stokes flows with highly oscillatory solutions in a 2-D domain with six randomly placed holes are simulated by the MscaleDNN. The results show that MscaleDNN has faster convergence and consistent error decays than normal fully connected DNNs in the simulation of Kovasznay flow for all four tested loss functions. More importantly, the MscaleDNN is capable of learning highly oscillatory solutions while the normal DNNs fail to converge.

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Key words: deep neural network, Stokes equation, multi-scale, meshless methods.

1 Introduction

Numerical methods for incompressible flow is one of the major topics in computational fluid dynamics, which has been intensively studied over last five decades. Various techniques have been proposed to address the incompressibility condition of the flow, including projection methods \cite{4} \cite{18}, Gauge methods \cite{6}, and time splitting methods \cite{13}, among others. Finite element and spectral element methods \cite{3} are mostly used to

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discretize the Navier-stokes equation where special attentions are needed for the design of solution spaces for the velocity and pressure variables to satisfy the Babuska and Brezzi inf-sup condition due to the saddle point problem with respect to these primitive variables [8]. Besides, for large scale engineering applications, body-fitted mesh generations for 3-D objects and efficient solvers for the resulting linear systems have been a major issue for computational resources.

The emerging deep neural network (DNN) has found many applications beyond its traditional applications such as image classification and speech recognition. Recent work in extending DNN to the field of scientific and engineering computing has shown much promise [7] [9] [17]. DNN based numerical methods are usually formulated as an optimization problem where the loss function could be an energy functional as in a Ritz formulation of a self-adjoint differential operator [7] or simply the least squared mean of the residual of the PDEs [10] [2] [11]. The DNN technique provides a powerful approximation method to represent solutions of high dimensional variables while the traditional finite element and spectral methods encounter the well known curse of dimensionality problem. There are several advantages of using DNN to approximate the solution of the incompressible flows. Firstly, the stochastic optimization algorithm employed by DNN based methods relies on loss calculated on randomly sampled points in the solution domain rather than over any unstructured mesh fitting the geometry of complex objects in the fluid problem. This feature renders the DNN-based methods for the PDEs a truly mesh-less method. Secondly, due to the capability of the DNN in handling high dimensional functions, the approximation of a time dependent solution can be carried out in the temporal-spatial four dimensional space, thus eliminating the need of time marching schemes with strict stability requirement on the time steps by the traditional CFD algorithms. Thirdly, boundary conditions for the fluid problems can be simply enforced by introducing penalty terms in the loss function and no need to find and implement appropriate and non-trivial boundary conditions for pressure [16] or vorticity variables in corresponding formulations of the Stokes or Navier-Stokes equations.

Normal fully connected DNNs commonly used for image classification and data science applications have been shown to be ineffective in learning high frequency contents of the solution as illustrated in recent works on DNNs’ frequency dependent convergence [19]. Unfortunately, fluid flow at high Reynolds number will contain many scales, which is the hallmark of the onset of turbulent flow from a laminar one. Therefore, in order to make the DNN based approaches to be competitive numerical methods, in terms of resolution power, compared to the well known spectral [3] and spectral element methods [12], it is important to develop new classes of DNNs which can represent scales of drastic disparities arising from the study of turbulent flows. For this purpose, we have recently developed strategies to speed up the convergence of DNNs in learning high frequency content of the solutions of PDEs. Two new DNNs have been proposed: a PhaseDNN [2] and a MscaleDNN [11]. The PhaseDNN uses a series of phase shifts to convert high frequency contents to the low frequency range before the learning is carried out. This method has been shown to be very effective in simulating high frequency Helmholtz
equations in acoustic wave scattering. On the other hand, the MscaleDNN uses a radial scaling technique in the frequency domain (or a corresponding scaling in the physical domain) to convert solution content of a range of higher frequency to a lower frequency one, which will be learned quickly with a small size DNN and then scaled back in the physical space to approximate the original solution content. MscaleDNN is more effective to handle higher dimensional PDEs and has already been shown to be superior over traditional fully connected DNNs for solving Poisson-Boltzmann equation in complex and singular domains [11]. In this paper, we will extend the MscaleDNN approach to find the solution of Stokes problem as a first step to develop DNN based numerical methods for time-dependent incompressible Navier-Stokes equations.

The rest of the paper is organized as follows. In section 2, we will present the structure of the MscaleDNN to be used for solving the Stokes problems. Section 3 will propose several loss functions for training, based on three different first order system reformulations of the Stokes equation. A benchmark test on a low frequency Kovasznay flow will be conducted in section 4 to evaluate the performance of normal fully connected DNN and Mscale DNN as well as different loss functions. Section 5 will present the numerical results for highly oscillatory Stokes flows with multiple frequencies. Finally, a conclusion and discussion of future work are given in Section 6.

## 2 Mult-scale DNN (MscaleDNN)

In a recent work [11], a multi-scale DNN was proposed, which is formed by a series of parallel normal sub-neural networks each of them will receive a scaled version of the input and their outputs will then be combined to make the final output of the MscaleDNN (refer to Fig. 1). The individual sub-network in the MscaleDNN with a scaled input is designed to approximate a segment of frequency content of the targeted function and the effect of the scaling is to convert a specific high frequency content to a lower frequency range so the learning can be accomplished much quickly. Recent work [19] on the frequency dependence of the DNN convergence shows much faster convergence occurs for the low frequency function than one with higher frequencies, the MscaleDNN takes advantage of this property. In addition, in order to produce scale separation and identification capability of a MscaleDNN, we borrowed the idea of compact mother scaling and wavelet functions in the wavelet theory [5], and found that the activation functions with a localized frequency profile works better than the normal ReLU activation function.

Fig. 1 shows the schematics of a MscaleDNN consisting of a sum of $n$ subnetworks with each scaled input passing through a subnetwork of the following form

$$f_\theta(x) = W^{[L-1]} \sigma(\cdots(W^1 \sigma( W^0(x) + b^0 ) + b^1) \cdots + b^{[L-1]}, \quad (2.1)$$

with the following activation function

$$\sigma(x) = \sin(x), \quad (2.2)$$
where $W^{[1]}$ to $W^{[L-1]}$ and $b^{[1]}$ to $b^{[L-1]}$ are the weight matrices and bias unknowns, respectively, to be optimized during the training of the networks. For the input scales, we could select the scale for the $i$-th sub-network to be $i$ (as shown in Fig. 1) or $2^{i-1}$. For more details on the design and discussion of the MscaleDNN, please refer to [11].

For comparison studies in this paper, we will define a “normal” network as an one fully connected DNN with the same total number of neurons as the MscaleDNN, but without multi-scale features. We would perform extensive numerical experiments to examine the effectiveness of different settings and use an efficient one to solve complex problems. All DNN models are trained by Adam [14].

3 Loss functions and the MscaleDNN for Stokes Problem

The following Stokes problem in two dimensional (2-D) space will be solved by the MscaleDNN,

$$-\nu \Delta u + \nabla p = f, \quad \text{in} \quad \Omega,$$

$$\nabla \cdot u = 0, \quad \text{in} \quad \Omega,$$  \hspace{1cm} (3.1) \hspace{1cm} (3.2)

$$u = g, \quad \text{on} \quad \partial \Omega,$$ \hspace{1cm} (3.3)

where $\Omega$ is an open bounded domain in $\mathbb{R}^2$, and the boundary condition $g$ satisfies the compatibility condition

$$\int_{\partial \Omega} g \cdot nds = 0.$$ \hspace{1cm} (3.4)

The MscaleDNN solution will be found as in the traditional least square finite element method [1] where the DNN-solution is obtained by minimizing a loss function in terms of the residual of the Stokes problem (3.1). Similar to the least square finite element method
for solving Stokes problem, we first reformulate (3.1)-(3.3) into a first order system. There are various possible ways of recasting (3.1) into a first order system, and we will focus on the following three popular approaches used in the construction of least square finite element methods [1] as follows.

- **Vorticity-velocity-pressure (ωVP) formulation:** The first approach introduces the vorticity variable, a scalar quantity for 2-D flows,

\[
\omega = \nabla \times \mathbf{u} = \partial_x u_y - \partial_y u_x,
\]

arriving at a vorticity-velocity-pressure (ωVP) system:

\[
\nu \nabla \times \omega + \nabla p = f, \quad \text{in} \quad \Omega, \tag{3.6}
\]

\[
\omega = \nabla \times \mathbf{u}, \quad \text{in} \quad \Omega, \tag{3.7}
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega. \tag{3.8}
\]

For this formulation, a total of three MscaleDNNs will be used: one for the scalar vorticity \(\omega\), one for the velocity vector \(\mathbf{u}\) where the output \(y = \mathbf{u}\) in Fig. 1, and one for the scalar pressure \(p\).

- **Velocity-stress-pressure (VSP) formulation:** The second approach introduces a stress tensor

\[
T = \sqrt{2} \nu (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2, \tag{3.9}
\]

while a velocity-stress-pressure (VSP) system

\[
-\sqrt{2} \nu \nabla \cdot T + \nabla p = f, \quad \text{in} \quad \Omega, \tag{3.10}
\]

\[
T = \frac{\sqrt{2} \nu}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top), \quad \text{in} \quad \Omega, \tag{3.11}
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega. \tag{3.12}
\]

is obtained.

For this formulation, a total of three MscaleDNNs will be used: one for the velocity vector \(\mathbf{u}\) where the output \(y = \mathbf{u}\) in Fig. 1, one for the stress tensor \(T\) where the output \(y = T\) in Fig. 1, and one for the scalar pressure \(p\).

- **Velocity-gradient of velocity-pressure (VgVP) formulation:** The third approach introduces a variable \(\mathbf{U} = \nabla \mathbf{u}\) (by taking gradient on each component of the velocity field), which leads to a velocity gradient-velocity-pressure (VgVP) system

\[
-\nu \nabla \cdot \mathbf{U} + \nabla p = f, \quad \text{in} \quad \Omega, \tag{3.13}
\]

\[
\mathbf{U} = \nabla \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{3.14}
\]

For this formulation, a total of three MscaleDNNs will be used: one for the velocity vector \(\mathbf{u}\) where the output \(y = \mathbf{u}\) in Fig. 1, one for the tensor of the gradient of velocity \(\mathbf{U}\) where the output \(y = \mathbf{U}\) in Fig. 1, and one for the scalar pressure \(p\).
Based on the first order systems above, we can define loss functions for MscaleDNN machine learning algorithms, accordingly.

Let $u(x, \theta_u), p(x, \theta_p), \omega(x, \theta_\omega), T(x, \theta_T), U(x, \theta_U)$ represent the DNN solutions of the reformulated first order systems, and corresponding loss functions are defined as

$$L_{\omega VP}(\theta_u, \theta_p, \theta_\omega) := \| \nu \nabla \times \omega + \nabla p - f \|_{\partial \Omega}^2 + \| \nabla \cdot u - \omega \|_{\Omega}^2 + \beta_1 \| u - g \|_{\partial \Omega}^2,$$

$$L_{VSP}(\theta_u, \theta_p, \theta_T) := \| \nu \nabla \cdot T - \nabla p + f \|_{\Omega}^2 + \| \nabla u + \nabla u^\top - T \|_{\Omega}^2 + \| \nabla \cdot u \|_{\Omega}^2 + \beta_2 \| u - g \|_{\partial \Omega}^2,$$

$$L_{VgVP}(\theta_u, \theta_p, \theta_U) := \| \nu \nabla \cdot U - \nabla p + f \|_{\Omega}^2 + \| \nabla u - U \|_{\Omega}^2 + \| \nabla \cdot u \|_{\Omega}^2 + \beta_3 \| u - g \|_{\partial \Omega}^2. \quad (3.15)$$

In this paper, the loss functions in (3.15) are named as $\omega$VP-loss, VSP-loss and VgVP-loss, accordingly. These loss functions will be compared with the simple loss function using the original Stokes equation:

$$L_{VP}(\theta_u, \theta_p) := \| \nu \Delta u - \nabla p + f \|_{\Omega}^2 + \| \nabla \cdot u \|_{\Omega}^2 + \beta \| u - g \|_{\partial \Omega}^2. \quad (3.16)$$

which is named as VP-loss. And for this formulation, a total of two MscaleDNNs will be used: one for the velocity vector $u$ where the output $y = u$ in Fig. 1, and one for the scalar pressure $p$.

### 4 Kovasznay flow in a square domain

As a benchmark test, we first consider the Stokes problem in the domain $\Omega = [0,2] \times [-0.5,1.5]$ with an exact solution coinciding with the analytical solution of the incompressible Navier-Stokes equations obtained by Kovasznay [15], i.e.,

$$u_1 = 1 - e^{\lambda x_1} \cos(2\pi x_2), \quad u_2 = \frac{\lambda}{2\pi} e^{\lambda x_1} \sin(2\pi x_2), \quad p = \frac{1}{2} e^{2\lambda x_1}, \quad (4.1)$$

where

$$\lambda = \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}, \quad Re = \frac{1}{\nu}.$$

The source term $f$ is obtained by substituting the exact solution into the Stokes equation (3.1). We set the viscosity $\nu = 0.1$ and investigate the performance of algorithms using a fully connected DNN and MscaleDNNs. In the simulations of this benchmark problem, all MscaleDNNs are set to have six scales: \{ $x, 2x, 4x, 8x, 16x, 32x$ \} and their sub networks of a fully connected DNN for each scale all have 4 hidden layers and 50 neurons in each hidden layer. On the other hand, the fully connected DNN is set to have 4 hidden layers and 300 neurons in each hidden layer. Therefore, the total number of neurons in the fully connected DNN and MscaleDNNs are the same. Nevertheless, the fully connected DNN does have more connectivity with more parameters.
For monitoring the accuracy, we define $\ell^2$-errors

\[
\text{Err}(u) = \left( \frac{1}{N} \sum_{j=1}^{N} |u^{\text{DNN}}(x_j) - u(x_j)|^2 \right)^{\frac{1}{2}}, \quad \text{Err}(p) = \left( \frac{1}{N} \sum_{j=1}^{N} |p^{\text{DNN}}(x_j) - p(x_j)|^2 \right)^{\frac{1}{2}},
\]

(4.2)

between the DNN solution \( \{u^{\text{DNN}}(x), p^{\text{DNN}}(x)\} \) and a given reference solution \( \{u(x), p(x)\} \). Here \( \{x_j = (x_{j1}, x_{j2})\}_{j=1}^{N} \) are locations of a uniform 200 \( \times \) 200 mesh of the domain \( \Omega \).

Figure 2: Normal DNN with different loss functions.

Figure 3: MscaleDNN with different loss functions.

The DNN solutions obtained by minimizing different loss functions in (3.15)-(3.16) are compared in Fig. 2-3. The results show that both fully connected DNN and MscaleDNNs converge in 300 epochs with any one of the loss function in (3.15). However, the simple VP-loss in (3.16) has a very poor performance no matter if the fully connected DNN or the MscaleDNNs are used. In particular, both fully connected DNN and MscaleDNNs can not produce reasonable results within 300 epochs if the VP-loss function is used.

We also compare the performance of the normal DNN and MscaleDNN for the three loss functions in Fig. 4-6. The results show that the MscaleDNNs have much faster convergence no matter which loss function is used. Actually, MscaleDNNs can achieve
much better accuracy than normal DNN as we can see in Fig. 4(b)-6(b). The MscaleDNN solutions obtained by minimizing the VSP-loss are compared with exact solution along the line $y = 0.7$ in Fig. 7. It is clear that the MscaleDNN solutions match very well with the exact solutions.
5 Oscillatory Kovasznay flows in a domain with multiple cylindrical voids

The MscaleDNN is more powerful than a normal DNN due to the former’s capability on solving complicate problems with oscillatory solutions. Here, we consider the Stokes flow in the domain $\Omega = [0,2] \times [-0.5,1.5]$ with 6 cylindrical holes (refer to Fig. 8) centered at

$$(0.5,0.0), (1.25,-0.2), (1.3,0.4), (0.5,1.1), (1.2,0.9), (1.6,1),$$

inside the domain. The radius of the cylinders are set to be 0.2,0.15,0.8,0.2,0.18,0.15, respectively. We will test two exact solutions with highly oscillatory velocity fields. All examples are set to run 1500 epochs using Adam.

Adaptive learning rates: We have found that reducing learning rate as the training progresses can have a noticeable improvement in the reduction of loss. The learning rate
of the first 500 epochs is set to be 0.001. Then, the learning rate will be reduced by a factor
of 10 after each 500 epochs. The change of learning rate can be seen clearly in the history
of losses later.
In the results below, the $\ell^2$-errors defined in (4.2) are computed again with 34,072
randomly picked points in the computational domain.

5.1 Two frequency solution

The first case has an exact solution given by

$$u_1 = 1 - e^{\lambda x_1} \cos(2n\pi x_1 + 2m\pi x_2),$$
$$u_2 = \frac{\lambda}{2m\pi} e^{\lambda x_1} \sin(2n\pi x_1 + 2m\pi x_2) + \frac{n}{m} e^{\lambda x_1} \cos(2n\pi x_1 + 2m\pi x_2),$$
$$p = \frac{1}{2} e^{2\lambda x_1}, \quad \lambda = \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}, \quad Re = \frac{1}{\nu},$$

(5.1)

with frequencies $n = 50, m = 55$. In the simulations of this example, all MscaleDNNs are set
to have 11 scales: \{x, 2x, \ldots, 2^{10} x\} and the fully connected DNN embedded in the Mscale
DNNs for each scale is set to have 8 hidden layers and 100 neurons in each hidden layer.
The MscaleDNN solutions of $v_x$ is compared with the exact $v_x$ in Fig. 9-11. Error of the
MscaleDNN approximation for $v_x$ using different losses are depicted in Fig. 12. Here, we
plot the solutions along the line $y = 0.7$ which does not cross any of cylinders inside the
domain. We can see that the $\omega VP$-loss or $V_g VP$-loss with MscaleDNN can produce very
accurate solutions in just 1500 epochs while the VSP-loss needs more learning to achieve
similar accuracy.

For comparison, we also test a fully connected DNN with 8 hidden layers and 1100
neurons in each hidden layer so the total number of neurons in the fully connected DNN
and the MscaleDNN are the same. The losses and $\ell^2$-errors obtained by minimizing different loss functions in (3.15) are compared in Fig. 13-15. For this highly oscillatory solution, the normal fully connected DNN can not learn anything in 1500 epochs. However, the MscaleDNN can converge very fast within 1500 epochs.
5.2 Multiple frequency solutions

Our second test problem will be a case where the Stokes solution has multiple frequencies with a more complicated flow fields as follows,

\[
\begin{align*}
  u_1 &= 2 - e^{\lambda x_1} \cos(70\pi x_1 + 60\pi x_2) - e^{\lambda x_1} \cos(80\pi x_1 + 90\pi x_2), \\
  u_2 &= \frac{\lambda}{60\pi} e^{\lambda x_1} \sin(70\pi x_1 + 60\pi x_2) + \frac{7}{6} e^{\lambda x_1} \cos(70\pi x_1 + 60\pi x_2) \\
  &\quad + \frac{\lambda}{90\pi} e^{\lambda x_1} \sin(80\pi x_1 + 90\pi x_2) + \frac{8}{9} e^{\lambda x_1} \cos(80\pi x_1 + 90\pi x_2), \\
  p &= \frac{1}{2} e^{2\lambda x_1}, \quad \lambda = \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}, \quad Re = \frac{1}{\nu}.
\end{align*}
\]

For this test, all MscaleDNNs are set to have 10 scales: \{x, 2x, \ldots, 2^9 x\} and the fully connected DNN embedded in the MscaleDNNs for each scale is set to have 8 hidden layers.
and 100 neurons in each hidden layer. The MscaleDNN solutions of $v_x$ are compared with the exact $v_x$ in Fig. 9-11. Error of the MscaleDNN approximation for $v_x$ using different losses are depicted in Fig. 12. Here, we plot the solutions along the line $y = 0.7$ which does not cross any of cylinders inside the domain. We can see that the $\omega VP$-loss or $VgVP$-loss with the MscaleDNN can obtain very accurate solutions within 1500 epochs. Again, the VSP-loss need more learning to achieve similar accuracy.

For comparison, we also test a normal fully connected DNN with 8 hidden layers and 1000 neurons in each hidden layer so the total number of neurons in the fully connected DNN and the MscaleDNN are the same. The losses and $\ell^2$-errors obtained by minimizing different loss functions in (3.15) are compared in Fig. 20-22.
Figure 16: Exact $v_x$ and its MscaleDNN approximation with loss function $L_{V_{VP}}(\theta_u, \theta_p, \theta_\omega)$.

Figure 17: Exact $v_y$ and its MscaleDNN approximation with loss function $L_{VSP}(\theta_u, \theta_p, \theta_T)$.

6 Conclusion and future work

In this paper, we have studied the MscaleDNN methods for solving highly oscillatory Stokes flow in complex domains and demonstrated the capability of the MscaleDNN as a meshless numerical method for simulating flows in complex domains. Several least square formulations of the Stokes equations using different forms of first order systems are used to construct the loss functions for the MscaleDNN learning. The numerical results have clearly demonstrated the increased resolution power of the MscaleDNN to capture the fine structures in the flow fields when the normal fully connected network with the
same overall sizes fail to converge at all. The MscaleDNN shows the potential of DNN machine learning as a potential alternative numerical method to traditional finite element methods with the obvious advantage of not needing expensive mesh generations and matrix solvers as in the case of traditional mesh-based numerical methods.

There are many unresolved issues for solving Navier-Stokes equation, among them the most important one is to understand the convergence property of the MscaleDNN learning. The structure of MscaleDNN is amendable to adaptive selections of scales by either adding or removing a scale dynamically during learning, future work will be done to explore this feature as well as applying the MscaleDNN to 3-D time-dependent incompressible flows.
Figure 20: Comparison of a normal DNN and the MscaleDNN with loss function $L_{\omega VP}(\theta_u, \theta_p, \theta_\omega)$.

Figure 21: Comparison of a normal DNN and the MscaleDNN with loss function $L_{VSP}(\theta_u, \theta_p, \theta_T)$.

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References

[1] P. B. Bochev and M. D. Gunzburger, Finite element methods of least-squares type, SIAM Rev., 40 (1998), pp. 789-837.
Figure 22: Comparison of a normal DNN and the MscaleDNN with loss function $L_{VgVP}(\theta_u, \theta_p, \theta_U)$.

[2] W. Cai, X.G. Li, and L.Z. Liu. A phase shift deep neural network for high frequency approximation and wave problems. to appear in SIAM J. Scientific Computing, arXiv:1909.11759, 2019.
[3] C. Canuto, M. Hussain, A. Quarteroni, And T. Zang, Spectral Methods in Fluid Dynamics (Springer-Verlag, New York/Berlin, 1987).
[4] A.J. Chorin, On the convergence of discrete approximations to the Navier-Stokes equations, Mnh. Comp. 23, 341-353 (1969).
[5] I. Daubechies, Ten lectures on wavelets. Society for industrial and applied mathematics; 1992 Jan 1.
[6] W. N. E, J. G. Liu, Gauge method for viscous incompressible flows. Communications in Mathematical Sciences. 2003;1(2):317-32.
[7] W. N. E and B. Yu, The deep Ritz method: A deep learning-based numerical algorithm for solving variational problems. Communications in Mathematics and Statistics, 6(1):1-12, 2018.
[8] Girault V, Raviart PA. Finite element methods for Navier-Stokes equations: theory and algorithms. Springer Science & Business Media; 2012 Dec 6.
[9] J.Q. Han, A. Jentzen, and W. N. E, Solving high-dimensional partial differential equations using deep learning. Proceedings of the National Academy of Sciences, 115(34):8505–8510, 2018.
[10] X. Jin, S. Cai, H. Li, G.E. Karniadakis, NSFnets (Navier-Stokes Flow nets): Physics-informed neural networks for the incompressible Navier-Stokes equations. arXiv preprint arXiv:2003.06496. 2020 Mar 13.
[11] Z.Q. Liu, W. Cai, and Z.Q. John Xu, Multi-scale Deep Neural Network (MscaleDNN) for Solving Poisson-Boltzmann Equation in Complex Domains, submitted to CiCP, 2020.
[12] G. E. Karniadakis, Spectral element simulations of laminar and turbulent flows in complex geometries, Appl. Numer. Math. 6, 85 (1989).
[13] G.E. Karniadakis, M. Israeli, SA Orszag, High-order splitting methods for the incompressible Navier-Stokes equations. Journal of computational physics. 1991 Dec 1;97(2):414-43.
[14] D. P. Kingma and J. Ba, Adam: A Method for Stochastic Optimization, preprint, https://arxiv.org/abs/1412.6980, 2014.
[15] L. I. G. Kovasznay, Laminar flow behind a two-dimensional grid, Proc. Camb. Philol. Soc., 44 (58), 1948
[16] S. A. Orszag, M. Israeli, And M. 0. Deville, Boundary conditions for incompressible flows, J. Sci. Comput. 1 No. 1, 75 (1986).
[17] M. Raissi, P. Perdikaris, and G. E. Karniadakis, Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. Journal of Computational Physics, 378:686–707, 2019.
[18] R. Temam, Sur l’approximation de la solution des equations de Navier-Stokes par la methode des fractionnaires II. Arch. Rational Mech. Anal., 33:377–385, 1969.
[19] Z.Q. John Xu, Y. Y. Zhang, T. Luo, Y. Y. Xiao, and Z. Ma. Frequency principle: Fourier analysis sheds light on deep neural networks. Accepted by Communications in Computational Physics, arXiv:1901.06523, 2019.