Research Article

Smoothing Connected Ball Bézier Curves by Energy Minimization

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In this paper, we aim at smoothing two connected ball Bézier curves from $C^{r-1}$ to $C^r$ ($r \geq 1$) by minimizing the energies of the curves. We propose the algorithms based on internal energy minimization and curve attractor minimization. Then, we combine the internal energy and the curve attractor and give the algorithm based on combined energy minimization. All algorithms are established by solving bi-objective minimizations. Some numerical examples show that the proposed algorithms are effective, making them useful for smoothing 3D objects constructed by connected ball Bézier curves.

1. Introduction

It is known that every object in nature has thickness. In the field of computer-aided design (CAD) and computer graphics (CG), a curve or a surface without thickness is just a mathematical abstraction or simplification to represent the shapes of objects (see [1]). Because a ball curve (see [2]) can be viewed as an ordinary curve with variable thickness in geometry, it can be used in modeling tubular shapes with variable thickness widely existing in nature. According to definition of the ball curve, specific models could be designed by concretizing the basis. For example, the ball B-spline curve (BBSC) derived from B-spline basis has attracted much attention in recent years. Owing to the fact that BBSC is defined by replacing control points in the B-spline curve with control balls, most algorithms can be easily obtained by applying the B-spline curve to the BBSC. Without providing an exhaustive survey, we list some studies where the proposed algorithms relate to BBSC (see, e.g., [2–7]). Since the Bézier curve and surface (see [8]) have long been a basic tool for geometric design and shape representation and have been widely used in practical engineering (see [9, 10]), the ball Bézier curve derived from the Bernstein basis is also worthy of attention. In this paper, we focus on the ball Bézier curve (BBC).

Although the BBC can be used to represent tubular shapes with variable thickness, we may need to connect two or more BBCs for designing some complex objects in practical applications. To make the connected BBCs have overall smoothness, they should satisfy certain continuity conditions. Generally, there are two metrics for the continuity between connected parametric curves: parametric continuity (also called $C^n$ continuity) and geometric continuity (also called $G^n$ continuity) (see [11]). The smooth continuity between connected BBCs we are concerned with here is the parametric continuity. Let us consider the issues. If two BBCs satisfy $C^{r-1}$ connection, how can we smooth the connected BBCs from $C^{r-1}$ to $C^r$? Theoretically, we can modify the control balls of the connected BBCs so as to make the corresponding continuity conditions hold, but the modification of control balls without specific objectives may not meet the needs of practical applications. Recently, some objective functionals have been proposed for modifying the shape of curves. These objective functionals can be roughly divided into two categories: the internal energy of curves and the external energy of curves. The internal energy of a curve oftentimes includes three forms: the stretch energy, the strain energy, and the curvature variation energy (see [12–17]). The external energy of a curve usually includes two forms: the point attractor and the curve attractor (see [17]). The curve attractor minimization is applied to smooth two
connected curves defined by control points (see [17]), and the same methodology was extended to smooth two con-
nected ball curves defined by control balls (see [2]). In this
paper, we, respectively, use the internal energy minimization
and the curve attractor minimization to smooth two con-
nected BBCs. We think that the energy of each segment of
two connected BBCs should be as small as possible, rather
than considering that the total energy of two connected
BBCs should be as small as possible. Therefore, we use bi-
objective minimization to achieve this goal.

The main contributions of this paper are as follows:

(a) We give a scheme for smoothing two connected
BBCs from $C^0$ to $C^1$ by minimizing the internal
energy. The scheme makes the internal energy of
each segment of two connected BBCs as small as
possible, rather than the total internal energy of two
connected BBCs as small as possible.

(b) We present a scheme for smoothing two connected
BBCs from $C^0$ to $C^1$ by minimizing the curve
attractor. The scheme synchronously minimizes the
distance between each adjusted control point and its
original position, instead of minimizing the total
distance between all adjusted control points with
their original positions.

c) We combine the internal energy minimization and
the curve attractor minimization to smooth two
connected BBCs from $C^0$ to $C^1$ in a new way, which
extends the application of previous methods.

The remainder of this paper is organized as follows. In
Section 2, we give the preliminaries for the issues to be
discussed. In Section 3, we present the algorithm for
smoothing two connected BBCs from $C^0$ to $C^1$ by
minimizing the internal energy. In Section 4, we propose
the algorithm for smoothing two connected BBCs from $C^0$ to $C^1$
by minimizing the curve attractor. In Section 5, we combine
the internal energy and the curve attractor to smooth two
connected BBCs from $C^0$ to $C^1$ and conclude the corre-
sponding algorithm. In Section 6, we bring some numerical
examples to illustrate the effectiveness of the proposed al-
gorithms. Finally, the conclusion is provided in Section 7.

2. Preliminaries

A ball Bézier curve (BBC) of degree $n$ can be defined as (see
[2])

$$\langle B \rangle (t) = \sum_{i=0}^{n} B_{i,n}(t) \langle b_i; r_i \rangle, \quad (1)$$

where $0 \leq t \leq 1$, $B_{i,n}(t) = \binom{n}{i} (1 - t)^{n-i} t^i$, and $\langle b_i; r_i \rangle$ is the
control ball which represents a ball centered at $b_i(x_i, y_i, z_i)$
with radius $r_i$. Geometrically, a BBC can be viewed as a
Bézier curve with variable thickness.

Since $\sum_{i=0}^{n} B_{i,n}(t) \langle b_i; r_i \rangle = \sum_{i=0}^{n} B_{i,n}(t) b_i \sum_{i=0}^{n} B_{i,n}(t) r_i$, a BBC can be regarded as having two parts: the
skeloton (or center curve) $c(t) = \sum_{i=0}^{n} B_{i,n}(t) b_i$, which is a
Bézier curve and the radius function $r(t) = \sum_{i=0}^{n} B_{i,n}(t) r_i$, which is a Bézier scalar function. Then, equation (1) can be
simplified as

$$\langle B \rangle (t) = \langle c(t); r(t) \rangle. \quad (2)$$

Figure 1 shows an illustration of a quadratic BBC.

From equation (2), a BBC of degree $(m)$ and a BBC of
degree $(n)$ can be expressed as

$$\langle B_1 \rangle (t) = \langle c_1(t); r_1(t) \rangle, \quad (3)$$
$$\langle B_2 \rangle (t) = \langle c_2(t); r_2(t) \rangle,$$

where

$$c_1(t) = \sum_{i=0}^{m} B_{i,m}(t) p_i,$$
$$r_1(t) = \sum_{i=0}^{m} B_{i,m}(t) r_i,$$
$$c_2(t) = \sum_{j=0}^{n} B_{j,n}(t) q_j,$$
$$r_2(t) = \sum_{j=0}^{n} B_{j,n}(t) s_j.$$

Since the expression of BBC is very similar to that of
standard Bézier curve, we can easily obtain the $C^n$ continuity
conditions of connecting two BBCs from that of the standard
Bézier curve. Here, we only give the $C^1$ continuity conditions
of connecting two BBCs which is shown in the following
theorem without proof.

**Theorem 1.** The $C^1$ continuity conditions of two connected
BBCs $\langle B_1 \rangle (t)$ and $\langle B_2 \rangle (t)$ are given by

$$q_0 = p_m = d,$$
$$s_0 = r_m = R,$$
$$q_1 = m + n - d - \frac{m}{n} p_{m-1},$$
$$s_1 = m + n - R - \frac{m}{n} r_{m-1},$$

where $d$ and $R$, respectively, represent the center and radius of
the connecting control ball.

**Remark 1.** The conditions $q_0 = p_m$ and $s_0 = r_m$ in equation
(5) imply that two BBCs should reach $C^0$ connection first, which
means combining the end of $\langle B_2 \rangle (t)$ with the be-

**Remark 2.** Actually, we can consider smoothing two con-
nected BBCs from $C^{r-1}$ to $C^r (r \geq 1)$. Without loss of

Complexity
In this section, we adopt internal energy to smooth two connected BBCs from $C^0$ to $C^1$ in the following sections. The methods of smoothing two connected BBCs from $C^{-1}$ to $C'$ ($r \geq 2$) could refer to the process of that from $C^0$ to $C^1$.

3. Internal Energy Minimization

In this section, we adopt internal energy to smooth two connected BBCs from $C^0$ to $C^1$. In most cases, the internal energy of a parametric curve $b(t)$ can be approximately expressed as $E_k = \int_0^1 \| b'(t) \|^2 dt$, $k = 1, 2, 3$, where $k$ denotes the derivative of order and $\| \cdot \|$ represents the 2-norm.

Since a BBC $\langle B \rangle (t)$ is considered to contain two parts, the internal energy of $\langle B \rangle (t)$ can be expressed as two separate parts:

$$E_k = \int_0^1 \| c^{(k)}(t) \|^2 dt,$$

$$R_k = \int_0^1 \| r^{(k)}(t) \|^2 dt.$$  (6) (7)

From equation (5), we only need to adjust the control balls $\langle p_{m-1}; r_{m-1} \rangle$ and $\langle q_1; s_1 \rangle$ for smoothing the two connected BBCs from $C^0$ to $C^1$. Let $\langle p_{m-1}; r_{m-1} \rangle$ denote the adjusted position of the control ball $\langle p_{m-1}; r_{m-1} \rangle$, and $\langle q_1; s_1 \rangle$ be that of the control ball $\langle q_1; s_1 \rangle$. Let $\Omega_1 = \{0, 1, \ldots, m-2, m\}$ and $\Omega_2 = \{0, 2, \ldots, n-1, n\}$ denote the index set of the control balls which keep unchanged. Then, from equations (3), (6), and (7), the internal energy of each segment of the two connected BBCs after smoothing can be expressed as

$$E_{1,k}(\mathbf{p}_{m-1}) := \int_0^1 \| c_1^{(k)}(t) \|^2 dt = \int_0^1 \| f_1^{(k)}(t) + B_{m-1,m}(t)\mathbf{p}_{m-1} \|^2 dt$$

$$= \int_0^1 \| f_1^{(k)}(t) \|^2 dt + 2\mathbf{p}_{m-1} \cdot \int_0^1 f_1^{(k)}(t)B_{m-1,m}(t)dt + \| \mathbf{p}_{m-1} \|^2 \int_0^1 (B_{m-1,m}(t))^2 dt,$$

$$R_{1,k}(\mathbf{r}_{m-1}) := \int_0^1 \| r_1^{(k)}(t) \|^2 dt = \int_0^1 \| g_1^{(k)}(t) + B_{m-1,m}(t)\mathbf{r}_{m-1} \|^2 dt$$

$$= \int_0^1 \| g_1^{(k)}(t) \|^2 dt + 2\mathbf{r}_{m-1} \cdot \int_0^1 g_1^{(k)}(t)B_{m-1,m}(t)dt + \| \mathbf{r}_{m-1} \|^2 \int_0^1 (B_{m-1,m}(t))^2 dt,$$

$$E_{2,k}(\mathbf{q}_1) := \int_0^1 \| c_2^{(k)}(t) \|^2 dt = \int_0^1 \| f_2^{(k)}(t) + B_{1,n}(t)\mathbf{q}_1 \|^2 dt$$

$$= \int_0^1 \| f_2^{(k)}(t) \|^2 dt + 2\mathbf{q}_1 \cdot \int_0^1 f_2^{(k)}(t)B_{1,n}(t)dt + \| \mathbf{q}_1 \|^2 \int_0^1 (B_{1,n}(t))^2 dt,$$

$$R_{2,k}(\mathbf{s}_1) := \int_0^1 \| r_2^{(k)}(t) \|^2 dt = \int_0^1 \| g_2^{(k)}(t) + B_{1,n}(t)\mathbf{s}_1 \|^2 dt$$

$$= \int_0^1 \| g_2^{(k)}(t) \|^2 dt + 2\mathbf{s}_1 \cdot \int_0^1 g_2^{(k)}(t)B_{1,n}(t)dt + \| \mathbf{s}_1 \|^2 \int_0^1 (B_{1,n}(t))^2 dt,$$  (8)

where “.” means the inner product of two vectors, and
possible, the bi-objective minimizations can be obtained as
\[ f_1(t) = \sum_{i \in \Omega_1} B_{i,m}(t)p_i, \]
\[ g_1(t) = \sum_{n \in \Omega_2} B_{i,m}(t)r_i, \]
\[ f_2(t) = \sum_{j \in \Omega_2} B_{j,n}(t)q_j, \]
\[ g_2(t) = \sum_{j \in \Omega_2} B_{j,n}(t)s_j. \]

Since we believe that the internal energy of each segment of two connected BBCs after smoothing should be as small as possible, the bi-objective minimizations can be obtained as follows:
\[
\min \left( E_{i,k} (\bar{\mathbf{p}}_{m-1}), E_{2,k} (\bar{\mathbf{q}}_i) \right)^T, \tag{10}
\]
\[
\min \left( R_{1,k} (\bar{r}_{m-1}), R_{2,k} (\bar{s}_1) \right)^T. \tag{11}
\]

Because equation (5) should be satisfied when smoothing the two connected BBCs from \( C^0 \) to \( C^1 \), we have
\[
\bar{q}_i = \frac{m+n}{n} d - \frac{m}{n} \bar{\mathbf{p}}_{m-1}, \tag{12}
\]
\[
\bar{s}_1 = \frac{m+n}{n} R - \frac{m}{n} \bar{r}_{m-1}. \tag{13}
\]

It is not difficult to find that \( \bar{q}_i \) can be regarded as a function of \( \bar{\mathbf{p}}_{m-1} \), and \( \bar{s}_1 \) can be regarded as a function of \( \bar{r}_{m-1} \). For convenience, we set the two functions as \( \alpha (\bar{\mathbf{p}}_{m-1}) \) and \( \beta (\bar{r}_{m-1}) \).

By substituting (12) into (10), we have
\[
\min \left( E_{i,k} (\bar{\mathbf{p}}_{m-1}), E_{2,k} (\bar{\mathbf{p}}_{m-1}) \right)^T, \tag{14}
\]
where \( E_{2,k} (\bar{\mathbf{p}}_{m-1}) = E_{2,k} (\alpha (\bar{\mathbf{p}}_{m-1})) \).

Note that the radii of the control balls should be non-negative, that is, \( \bar{r}_{m-1} \geq 0 \) and \((m+n)nR - (m+n)\bar{r}_{m-1} \geq 0 \), namely, \( 0 \leq \bar{r}_{m-1} \leq (m+n/m)R \). Here, the radius equal to 0 means that a control ball degenerates to be a point. By substituting (13) into (11) and adding the constraint with radius, we have
\[
\min_{r_{m-1} \in D} \left( R_{1,k} (\bar{r}_{m-1}), R_{2,k} (\bar{r}_{m-1}) \right)^T, \tag{15}
\]
where \( R_{2,k} (\bar{r}_{m-1}) = R_{2,k} (\beta (\bar{r}_{m-1})) \), \( D = \{ \bar{r}_{m-1} \in R | 0 \leq \bar{r}_{m-1} \leq (m+n/m)R \} \).

Generally, (14) and (15) can be transformed into the single-objective minimizations:
\[
\min E_{i,k} (\bar{\mathbf{p}}_{m-1}) = \lambda_k E_{i,k} (\bar{\mathbf{p}}_{m-1}) + (1 - \lambda_k) E_{2,k} (\bar{\mathbf{p}}_{m-1}), \tag{16}
\]
\[
\min R_{k} (\bar{r}_{m-1}) = \omega_k R_{1,k} (\bar{r}_{m-1}) + (1 - \omega_k) R_{2,k} (\bar{r}_{m-1}), \tag{17}
\]

where \( \lambda_k, 0 \leq \lambda_k \leq 1 \) and \( \omega_k, 0 \leq \omega_k \leq 1 \) are the weights. If the solution of equation (16) is unique, it must be the noninferior solution of equation (14), and so is the relationship between equations (15) and (17) (see, e.g., [18]). We discuss equations (16) and (17) separately.

Before solving equation (18), we need to determine the values of \( \lambda_k \). To this end, we consider the single-objective minimizations
\[
\min E_{i,k} (\bar{\mathbf{p}}_{m-1}), \tag{18}
\]
\[
\min E_{2,k} (\bar{\mathbf{p}}_{m-1}). \tag{19}
\]

**Lemma 1.** The solution of equation (18), denoted by \( \bar{\mathbf{p}}_{m-1}^1 \), is
\[
\bar{\mathbf{p}}_{m-1}^1 = \frac{m+n}{m} d + \frac{n}{m} \int_0^1 \left( B_{1,n}(t) \right) \, dt.
\]

The solution of equation (19), denoted by \( \bar{\mathbf{p}}_{m-1}^2 \), is
\[
\bar{\mathbf{p}}_{m-1}^2 = \frac{m+n}{m} d + \frac{n}{m} \int_0^1 \left( B_{1,n}(t) \right) \, dt.
\]

**Proof.** The gradients of \( E_{i,k} (\bar{\mathbf{p}}_{m-1}) \) expressed in equation (8) can be calculated by
\[
\frac{\partial E_{i,k} (\bar{\mathbf{p}}_{m-1})}{\partial \bar{\mathbf{p}}_{m-1}} = 2 \int_0^1 f_{1}^{(k)}(t) B_{1,m-1,n}(t) \, dt + 2 \bar{p}_{m-1} \int_0^1 \left( B_{1,n}(t) \right) ^2 \, dt.
\]

Then, we have \((\partial^2 E_{i,k} (\bar{\mathbf{p}}_{m-1})/\partial \bar{\mathbf{p}}_{m-1}^2) = 2 e\int_0^1 (B_{1,m-1,n}(t))^2 \, dt\), where \( e \) is the unit vector. Since \(\int_0^1 (B_{1,m-1,n}(t))^2 \, dt \) always holds, equation (18) has a unique solution which can be solved by \((\partial E_{i,k} (\bar{\mathbf{p}}_{m-1})/\partial \bar{\mathbf{p}}_{m-1}) = 0 \). We then gain equation (20) by computing from equation (22).
The gradients of $E_{2,k}(\bar{p}_{m-1})$ expressed by equations (8) can be calculated by
\[
\frac{\partial E_{2,k}(\bar{p}_{m-1})}{\partial \bar{p}_{m-1}} = \frac{-2m}{n} \left( \int_0^1 f_1^{(k)}(t)B_{1,n}^{(k)}(t)dt \right)
+ \left( \frac{m+n}{n} - \frac{m}{n} \bar{p}_{m-1} \right) \int_0^1 \left( B_{1,n}^{(k)}(t) \right)^2 dt.
\]
(23)

Because equation (19) has a unique solution which can be solved by $(\partial E_{2,k}(\bar{p}_{m-1})/\partial \bar{p}_{m-1}) = 0$, we then obtain equation (24) by computing from equation (23).

Next, we determine the value of $\lambda_k$ in equation (16) by using the sorting algorithm (see e.g., [18]) described in Algorithm 1.

We then obtain the following theorem.

**Theorem 2.** Given two BBCs $\langle B_1 \rangle (t)$ and $\langle B_2 \rangle (t)$ that satisfy $C^\infty$ connection. For smoothing the two connected BBCs from $C^0$ to $C^1$, the centers of the adjusted control balls which minimized the internal energy $E_k(\bar{p}_{m-1})$ are given by
\[
\begin{cases}
\bar{p}_{m-1} = -\frac{B}{A} \\
\bar{q}_1 = \frac{m+n}{n} - \frac{m}{n} \bar{p}_{m-1},
\end{cases}
\]
(24)

where
\[
A = \lambda_k \int_0^1 \left( B_{m-1,m}^{(k)}(t) \right)^2 dt + \frac{m^2}{n^2} \left( 1 - \lambda_k \right) \int_0^1 \left( B_{1,n}^{(k)}(t) \right)^2 dt,
\]
\[
B = \lambda_k \left[ \int_0^1 f_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt - \frac{m}{n} \left( 1 - \lambda_k \right) \right]
+ \left( \int_0^1 f_2^{(k)}(t)B_{1,n}^{(k)}(t)dt + \frac{m+n}{n} - \frac{m}{n} \right) \int_0^1 \left( B_{1,n}^{(k)}(t) \right)^2 dt.
\]
(25)

**Proof.** The gradients of $E_k(\bar{p}_{m-1})$ expressed in equation (16) can be calculated by
\[
\frac{\partial E_k(\bar{p}_{m-1})}{\partial \bar{p}_{m-1}} = \lambda_k \frac{\partial E_{1,k}(\bar{p}_{m-1})}{\partial \bar{p}_{m-1}} + \left( 1 - \lambda_k \right) \frac{\partial E_{2,k}(\bar{p}_{m-1})}{\partial \bar{p}_{m-1}} = 2A\bar{p}_{m-1} + 2B.
\]
(26)

Since equation (16) has a unique solution which can be solved by $\partial E_k(\bar{p}_{m-1})/\partial \bar{p}_{m-1} = 0$, we then get equation (24) by computing from equation (26) and combining it with equation (12).

Analogously, we should determine the values of $\omega_k$ before solving equation (17). Hence, the following single-objective minimizations are considered:
\[
\min_{\bar{r}_{m-1} \in D} R_{1,k}(\bar{p}_{m-1}),
\]
(27)

**Lemma 2.** The solution of equation (27), denoted by $\bar{r}_{m-1}$, is given by the cases as follows:

(i) If $\bar{r}_{m-1} \in D$, we have $\bar{r}_{m-1} = \bar{r}_{m-1}^{[1]}$, where $\bar{r}_{m-1}^{[1]} = -\int_0^1 g_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt/\int_0^1 \left( B_{m-1,m}^{(k)}(t) \right)^2 dt$.

(ii) If $\bar{r}_{m-1}^{[1]} \notin D$ and $R_{1,k}(0) \leq R_{1,k}(m + m/nR)$, we have $\bar{r}_{m-1}^{[1]} = 0$.

(iii) If $\bar{r}_{m-1}^{[1]} \notin D$ and $R_{1,k}(0) > R_{1,k}(m + m/nR)$, we have $\bar{r}_{m-1}^{[1]} = m + n/R$.

The solution of equation (28), denoted by $\bar{r}_{m-1}^{[2]}$, is given by the cases as follows:

(i) If $\bar{r}_{m-1}^{[2]} \in D$, we have $\bar{r}_{m-1}^{[2]} = \bar{r}_{m-1}^{[1]}$, where $\bar{r}_{m-1}^{[2]} = m + n/mR + m/n \int_0^1 g_2^{(k)}(t)B_{1,n}^{(k)}(t)dt/\int_0^1 \left( B_{1,n}^{(k)}(t) \right)^2 dt$.

(ii) If $\bar{r}_{m-1}^{[2]} \notin D$ and $R_{2,k}(0) \leq R_{2,k}(m + m/nR)$, we have $\bar{r}_{m-1}^{[2]} = 0$.

(iii) If $\bar{r}_{m-1}^{[2]} \notin D$ and $R_{2,k}(0) > R_{2,k}(m + m/nR)$, we have $\bar{r}_{m-1}^{[2]} = m + n/mR$.

**Proof.** From equation (8), we have
\[
\frac{dR_{1,k}(\bar{r}_{m-1})}{d\bar{r}_{m-1}^{[1]}} = 2\int_0^1 g_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt + 2\bar{r}_{m-1} \int_0^1 \left( B_{m-1,m}^{(k)}(t) \right)^2 dt.
\]
(29)

By computing from equation (29), the unique stable point of $R_{1,k}(\bar{r}_{m-1})$, denoted by $\bar{r}_{m-1}^{[1]}$, can be obtained. There are two cases.

**Case 1.** $\bar{r}_{m-1}^{[1]} \in D$. Because $d^2R_{1,k}(\bar{r}_{m-1})/d(\bar{r}_{m-1})^2$ $= 2\int_0^1 \left( B_{m-1,m}^{(k)}(t) \right)^2 dt > 0$, the solution of equation (27) must be $\bar{r}_{m-1}^{[1]} = \bar{r}_{m-1}^{[1]}$.

**Case 2.** $\bar{r}_{m-1}^{[1]} \notin D$. That means $R_{1,k}(\bar{r}_{m-1})$ is monotone. If $R_{1,k}(0) \leq R_{1,k}(m + m/nR)$, the solution of equation (27) should be $\bar{r}_{m-1}^{[1]} = 0$; else, the solution of equation (27) should be $\bar{r}_{m-1}^{[1]} = m + n/R$. Thus, we obtain the solution of equation (27). In a similar way, we can reach the solution of equation (28).

The value of $\omega_k$ in equation (17) can be determined by using the sorting algorithm (see e.g., [18]) described in Algorithm 2.

We then obtain the following theorem.

**Theorem 3.** Given two BBCs $\langle B_1 \rangle (t)$ and $\langle B_2 \rangle (t)$ that satisfy $C^\infty$ connection. For smoothing the two connected BBCs from $C^0$ to $C^1$, the radii of the adjusted control balls which minimized the internal energy $R_k(\bar{p}_{m-1})$ are given by the cases as follows:

(i) If $-b/a \in D$, the radii of the adjusted control balls are
Complexity

Algorithm 1: Determining the value of $\lambda_k$ in equation (16).

Step 1. Compute $e^j_{jk} = R_{jk}(\mathbf{r}_{m-1}) - R_{jk}(\mathbf{r}_{m-1})$, $i, j = 1, 2$.
Step 2. Let $h_{jk} = \sum_{j=1}^{m} e^j_{jk}$, $i = 1, 2$.
Step 3. Compute $\alpha_{jk} = g_{jk}(g_{jk} + g_{j2} + g_{j3})$, $i = 1, 2$.
Step 4. If $\lambda_{jk} \geq \alpha_{jk}$, the weight is taken as $\lambda_k = \alpha_{jk}$; else, the weight is taken as $\lambda_k = \lambda_{jk}$.

Algorithm 2: Determining the value of $\omega_k$ in equation (17).

\[
\begin{align*}
\mathbf{r}_{m-1} &= -\frac{b}{a}, \\
\bar{s}_1 &= \frac{m + n}{n} R - \frac{m_n}{n} \mathbf{r}_{m-1},
\end{align*}
\]

where

\[
a = \omega_k \int_0^1 \left( B_{m-1}^{(k)}(t) \right)^2 dt + \frac{m_n}{n} (1 - \omega_k) \int_0^1 \left( B_{m}^{(k)}(t) \right)^2 dt,
\]

\[
b = \omega_k \int_0^1 g_{1}^{(k)}(t) R_{m-1}^{(k)}(t) dt
\]

\[
- \frac{m_n}{n} (1 - \omega_k) \left( \int_0^1 g_{2}^{(k)}(t) R_{1}^{(k)}(t) dt + \frac{m + n}{n} R \int_0^1 \left( B_{m}^{(k)}(t) \right)^2 dt \right).
\]

\[
(ii) \text{ If } -b/a \notin D \text{ and } R_k(0) \leq R_k(m + n/mR), \text{ the radii of the adjusted control balls are}
\]

\[
\begin{align*}
\mathbf{r}_{m-1} &= 0, \\
\bar{s}_1 &= \frac{m + n}{n} R.
\end{align*}
\]

\[
(iii) \text{ If } -b/a \notin D \text{ and } R_k(0) > R_k(m + n/mR), \text{ the radii of the adjusted control balls are}
\]

\[
\begin{align*}
\mathbf{r}_{m-1} &= \frac{m + n}{m} R, \\
\bar{s}_1 &= 0.
\end{align*}
\]

Proof. From equation (17), we have

\[
\frac{dR_k(\mathbf{r}_{m-1})}{d\mathbf{r}_{m-1}} = \omega_k \frac{dR_k(\mathbf{r}_{m-1})}{d\mathbf{r}_{m-1}} + (1 - \omega_k) \frac{dR_{2k}(\mathbf{r}_{m-1})}{d\mathbf{r}_{m-1}}
\]

\[
= 2b + 2b.
\]

By computing from equation (34), the unique stable point of $R_k(\mathbf{r}_{m-1})$, denoted by $\mathbf{r}^*_m(\mathbf{r}_{m-1})$, is $\mathbf{r}^*_m = -b/a$. There are two cases.

Case 1. $\mathbf{r}^*_m \in D$. Because $d^2R_k(\mathbf{r}_{m-1})/d(\mathbf{r}_{m-1})^2 = 2a > 0$, we then get equation (30) by combining it with equation (13).

Case 2. $\mathbf{r}^*_m \notin D$. That means $R_k(\mathbf{r}_{m-1})$ is monotonous.

We roughly describe the methods of smoothing two connected BBCs from $C^0$ to $C^1$ by minimizing the internal energy as shown in Algorithm 3.

Remark 3. It is noted that the methods in [2] took the total internal energy of two connected BBCs as the minimization target. Different from [2], we make the internal energy of each segment of two connected BBCs as small as possible, rather than considering the total internal energy of two connected BBCs as small as possible.

4. Curve Attractor Minimization

In this section, we use external energy to smooth two connected BBCs from $C^0$ to $C^1$. The point attractor and the curve attractor are two widely used external energies of a curve. The point attractor pulls a curve towards a given point, and the curve attractor pulls a curve towards a given curve. Since we expect to smooth the connected BBCs to the new reconnected BBCs, we adopt the curve attractor to optimize the control balls.

In fact, the curve attractor used for a curve defined by equation (33) by combining it with equation (13).

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In fact, the curve attractor used for a curve defined by equation (33) by combining it with equation (13).

By computing from equation (34), the unique stable point of $R_k(\mathbf{r}_{m-1})$, denoted by $\mathbf{r}^*_m(\mathbf{r}_{m-1})$, is $\mathbf{r}^*_m = -b/a$. There are two cases.

Case 1. $\mathbf{r}^*_m \in D$. Because $d^2R_k(\mathbf{r}_{m-1})/d(\mathbf{r}_{m-1})^2 = 2a > 0$, we then get equation (30) by combining it with equation (13).

Case 2. $\mathbf{r}^*_m \notin D$. That means $R_k(\mathbf{r}_{m-1})$ is monotonous.

If $R_k(0) \leq R_k(m + n/mR)$, we can get equation (32) by combining it with equation (13); else, we can get equation (33) by combining it with equation (13).

We roughly describe the methods of smoothing two connected BBCs from $C^0$ to $C^1$ by minimizing the internal energy as shown in Algorithm 3.

Remark 3. It is noted that the methods in [2] took the total internal energy of two connected BBCs as the minimization target. Different from [2], we make the internal energy of each segment of two connected BBCs as small as possible, rather than considering the total internal energy of two connected BBCs as small as possible.
Step 1. Input two BBCs that satisfy $C^0$ connection.
Step 2. Select the value of $k$ to determine which kind of internal energy is adopted.
Step 3. Use Algorithm 1 to determine the weight in equation (18) and compute the centers of the adjusted control balls by Theorem 2.
Step 4. Use Algorithm 2 to determine the weight in equation (19) and compute the radii of the adjusted control balls by Theorem 3.
Step 5. Output two BBCs that satisfy $C^1$ connection.

**Algorithm 3:** Smoothing two connected BBCs from $C^0$ to $C^1$ by minimizing the internal energy.

$$F_1(\overline{p}_{m-1}) = \|\overline{p}_{m-1} - p_{m-1}\|^2,$$

$$S_1(\overline{r}_{m-1}) = |\overline{r}_{m-1} - r_{m-1}|^2,$$

$$F_2(\overline{p}_{m-1}) = \|q_1 - q_1\|^2 - \frac{m + n}{n}d - m\overline{p}_{m-1} - q_1,$$

$$S_2(\overline{r}_{m-1}) = |\overline{s}_1 - s_1|^2 = \left|\frac{m + n}{n}R - m\overline{r}_{m-1} - s_1\right|^2.$$

We then get the bi-objective minimizations:

$$\min (F_1(\overline{p}_{m-1}), F_2(\overline{p}_{m-1}))^T,$$

$$\min_{\overline{r}_{m-1} \in D} (S_1(\overline{r}_{m-1}), S_2(\overline{r}_{m-1}))^T.$$

We transform equations (39) and (40) into the single-objective minimizations:

$$\min F(\overline{p}_{m-1}) = \eta F_1(\overline{p}_{m-1}) + (1 - \eta)F_2(\overline{p}_{m-1}),$$

$$\min S(\overline{r}_{m-1}) = \rho S_1(\overline{r}_{m-1}) + (1 - \rho)S_2(\overline{r}_{m-1}),$$

where $\eta$, $(0 \leq \eta \leq 1)$ and $\rho$, $(0 \leq \rho \leq 1)$ are the weights.

To solve equation (41), we consider the single-objective minimizations:

$$\min F_1(\overline{p}_{m-1}),$$

$$\min F_2(\overline{p}_{m-1}).$$

From equations (35) and (37), we can easily get the lemma as follows.

**Lemma 3.** The solution of equation (43), denoted by $\overline{p}_{m-1}^1$, is $\overline{p}_{m-1}^1 = p_{m-1}$. The solution of equation (44), denoted by $\overline{p}_{m-1}^1$, should satisfy $m + n/d - m\overline{p}_{m-1}^2 = q_1$, that is, $\overline{p}_{m-1}^2 = m + n/d - n/mq_1$.

We then can determine the value of $\eta$ in equation (41) referring to Algorithm 1. Here we do not bring the detailed description of this. Furthermore, we get the solution of equation (41) reached by the theorem as follows.

**Theorem 4.** Given two BBCs $(B_1)$ and $(B_2)$ at $(t)$ that satisfy $C^0$ connection. For smoothing the two connected BBCs from $C^0$ to $C^1$, the centers of the adjusted control balls which minimized the curve attractor $F(\overline{p}_{m-1})$ are given by

$$\overline{p}_{m-1} = \frac{\eta r^2_{m-1} + (1 - \eta)(m(m + n)d - mnq_1)}{(1 - \eta)m^2 + m^2},$$

$$\overline{q}_1 = \frac{m + n}{n}d - m\overline{p}_{m-1}.$$

**Proof.** The gradients of $F(\overline{p}_{m-1})$ expressed in equation (41) can be calculated by

$$\frac{\partial F(\overline{p}_{m-1})}{\partial \overline{p}_{m-1}} = \eta \frac{\partial F_1(\overline{p}_{m-1})}{\partial \overline{p}_{m-1}} + (1 - \eta) \frac{\partial F_2(\overline{p}_{m-1})}{\partial \overline{p}_{m-1}},$$

$$= 2\eta(\overline{p}_{m-1} - \overline{p}_{m-1}) - 2(1 - \eta)\left(\frac{m + n}{n}d - m\overline{p}_{m-1} - q_1\right).$$

Since equation (41) has a unique optimal solution which can be solved by $\frac{\partial F(\overline{p}_{m-1})}{\partial \overline{p}_{m-1}} = 0$, we then get equation (45) by computing from equation (46) and combining it with equation (12).

Similarly, we consider the following single-objective minimizations for solving equation (44):

$$\min_{\overline{r}_{m-1} \in D} S_1(\overline{r}_{m-1}),$$

$$\min_{\overline{r}_{m-1} \in D} S_2(\overline{r}_{m-1}).$$

From equations (36) and (38), it is not difficult to obtain the following lemma.

**Lemma 4.** The solution of equation (47), denoted by $\overline{r}_{m-1}^1$, is given by the cases as follows:

(i) If $r_{m-1} \in D$, we have $\overline{r}_{m-1}^1 = r_{m-1}.$

(ii) If $r_{m-1} \notin D$ and $S_1(0) \leq S_1(m + n/R)$, we have $\overline{r}_{m-1}^1 = 0.$

(iii) If $r_{m-1} \notin D$ and $S_1(0) > S_1(m + n/R)$, we have $\overline{r}_{m-1}^1 = m + n/R.$

The solution of equation (48), denoted by $\overline{r}_{m-1}^2$, is given by the cases as follows:

(i) If $(m + n/R - n/m\overline{q}_1) \in D$, we have $\overline{r}_{m-1}^2 = m + n/R - n/m\overline{q}_1.$

(ii) If $(m + n/R - n/m\overline{q}_1) \notin D$ and $S_2(0) \leq S_2(m + n/R)$, we have $\overline{r}_{m-1}^2 = 0.$
Theorem 5. Given two BBCs \( \langle B_1 \rangle (t) \) and \( \langle B_2 \rangle (t) \) that satisfy \( C^0 \) connection. For smoothing the two connected BBCs from \( C^0 \) to \( C^1 \), the radii of the adjusted control balls which minimized the curve attractor \( S(\bar{r}_{m-1}) \) are given by the cases as follows:

(i) If \( n^2 r_{m-1} + m(m+n)R - mns_i/m^2 + n^2 \in D \), the radii of the adjusted control balls are

\[
\begin{align*}
\bar{r}_{m-1} &= \frac{n^2 r_{m-1} + m(m+n)R - mns_i}{m^2 + n^2}, \\
\bar{s}_1 &= \frac{m + n}{n} R.
\end{align*}
\]

(ii) If \( n^2 r_{m-1} + m(m+n)R - mns_i/m^2 + n^2 \notin D \) and \( S(0) \leq S(m + n/mR) \), the radii of the adjusted control balls are

\[
\begin{align*}
\bar{r}_{m-1} &= \frac{m + n}{m} R, \\
\bar{s}_1 &= 0.
\end{align*}
\]

(iii) If \( n^2 r_{m-1} + m(m+n)R - mns_i/m^2 + n^2 \notin D \) and \( S(0) > S(m + n/mR) \), the radii of the adjusted control balls are

\[
\begin{align*}
\bar{r}_{m-1} &= \frac{n^2 r_{m-1} + m(m+n)R - mns_i}{m^2 + n^2}, \\
\bar{s}_1 &= \frac{m + n}{n} R.
\end{align*}
\]

Lemma 5. The solution of equation (56), denoted by \( \hat{p}^1_{m-1} \), is combining it with equation (13); else, we can get equation (51) by combining it with equation (13).

Remark 4. It is noted that the methods in [17] took the total distance (i.e., curve attractor) between all adjusted control points with their original positions as the minimization target. Different from [17], we synchronously minimize the distance between each adjusted control point and its original position, instead of minimizing the total distance between all adjusted control points and their original positions.

5. Combined Energy Minimization

In this section, we consider combining the internal energy and the curve attractor to smooth two connected BBCs from \( C^0 \) to \( C^1 \). We express the combined energy of each segment of the two connected BBCs after smoothing as follows:

\[
\begin{align*}
G_{1,k}(\hat{p}_{m-1}) &= E_{1,k}(\hat{p}_{m-1}) + F_{1}(\hat{p}_{m-1}), \\
T_{1,k}(\bar{r}_{m-1}) &= R_{1,k}(\bar{r}_{m-1}) + S_{1}(\bar{r}_{m-1}), \\
G_{2,k}(\hat{p}_{m-1}) &= E_{2,k}(\hat{p}_{m-1}) + F_{2}(\hat{p}_{m-1}), \\
T_{2,k}(\bar{r}_{m-1}) &= R_{2,k}(\bar{r}_{m-1}) + S_{2}(\bar{r}_{m-1}),
\end{align*}
\]

where \( E_{i,k}(\hat{p}_{m-1}) \) and \( R_{i,k}(\bar{r}_{m-1}) \), \( (i = 1, 2) \) are the internal energies used in Section 3 and \( F_{i}(\hat{p}_{m-1}) \) and \( S_{i}(\bar{r}_{m-1}) \), \( (i = 1, 2) \) represent the curve attractor used in Section 4.

Thereby, we obtain the bi-objective minimizations:

\[
\min_{r_{m-1} \in D} \begin{pmatrix} G_{1,k}(\hat{p}_{m-1}), G_{2,k}(\hat{p}_{m-1}) \end{pmatrix}^T,
\]

Analogously, we transform equations (54) and (55) into the single-objective minimizations:

\[
\begin{align*}
\min_{r_{m-1} \in D} T_{1,k}(\bar{r}_{m-1}) &= T_{2,k}(\bar{r}_{m-1}),
\end{align*}
\]

where \( \xi_k, \ (0 \leq \xi_k \leq 1) \) and \( \xi_k, \ (0 \leq \xi_k \leq 1) \) are the weights.

To solve equation (56), we consider the single-objective minimizations:

\[
\begin{align*}
\min_{r_{m-1} \in D} G_{1,k}(\hat{p}_{m-1}), \\
\min_{r_{m-1} \in D} G_{2,k}(\hat{p}_{m-1}).
\end{align*}
\]
Step 1. Input two BBCs that satisfy $C^0$ connection.
Step 2. Determine the value of $\eta$ in equation (40) referring to Algorithm 1 and compute the centers of the adjusted control balls by Theorem 4.
Step 3. Determine the value of $\rho$ in equation (41) referring to Algorithm 2 and compute the radii of the adjusted control balls by Theorem 5.
Step 4. Output two BBCs that satisfy $C^1$ connection.

**Algorithm 4**: Smoothing two connected BBCs from $C^0$ to $C^1$ by minimizing the curve attractor.

\[
\bar{p}_{m-1}^1 = \frac{p_{m-1} - \int_0^1 f_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt}{1 + \int_0^1 (B_{m-1,m}^{(k)}(t))^2 dt}. \tag{60}
\]

The solution of equation (58), denoted by $\bar{p}_{m-1}^2$, is

\[
\bar{p}_{m-1}^2 = n\int_0^1 f_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt + (m + n)d\left(1 + \int_0^1 (B_{m-1,m}^{(k)}(t))^2 dt\right) - nq_1.
\]

**Proof.** The gradients of $G_{1,k}(\bar{p}_{m-1}^1)$ can be calculated by

\[
\frac{\partial G_{1,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} = \frac{\partial E_{1,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} + \frac{\partial F_1(\bar{p}_{m-1}^1)}{\partial p_{m-1}}
\]

\[
= 2\int_0^1 f_1^{(k)}(t)B_{m-1,m}^{(k)}(t)dt + 2\bar{p}_{m-1}
\]

\[
+ \int_0^1 (B_{m-1,m}^{(k)}(t))^2 dt + 2(p_{m-1} - \bar{p}_{m-1}). \tag{62}
\]

Since equation (58) has a unique solution which can be solved by $\frac{\partial G_{1,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} = 0$, we then get equation (60) by computing from equation (62).

The gradients of $G_{2,k}(\bar{p}_{m-1}^1)$ can be calculated by

\[
\frac{\partial G_{2,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} = \frac{\partial E_{2,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} + \frac{\partial F_2(\bar{p}_{m-1}^1)}{\partial p_{m-1}}.
\]

Because equation (59) has a unique solution which can be solved by $\frac{\partial G_{2,k}(\bar{p}_{m-1}^1)}{\partial p_{m-1}} = 0$, we then get equation (61) by computing from equation (63).

Then, we can determine the values of $\xi_k$ in equation (56) referring to Algorithm 1. Here we do not provide detailed description of this. Furthermore, we reach the solution of equation (56) given by the theorem as follows.

**Theorem 6.** Given two BBCs $\langle B_1 \rangle$, $\langle B_2 \rangle$ that satisfy $C^0$ connection. For smoothing the two connected BBCs from $C^0$ to $C^1$, the centers of the adjusted control balls which minimized the combined energy $G_k(\bar{p}_{m-1}^1)$ are given by

\[
\begin{aligned}
\bar{p}_{m-1}^1 &= -\frac{D}{C}, \\
q_1 &= \frac{m + n}{n}d - \frac{m}{n}p_{m-1} - q_1,
\end{aligned}
\tag{64}
\]

where

\[
C = \xi_k\left(1 + \int_0^1 (B_{m-1,m}^{(k)}(t))^2 dt\right) + \frac{m^2}{n^2}(1 - \xi_k)\left(1 + \int_0^1 (B_{1,n}^{(k)}(t))^2 dt\right),
\]

\[
D = \xi_k\left(\int_0^1 f_2^{(k)}(t)B_{m-1,m}^{(k)}(t)dt - p_{m-1}\right)
\]

\[
- \frac{m}{n}(1 - \xi_k)\left(\int_0^1 f_2^{(k)}(t)B_{m-1,m}^{(k)}(t)dt + \frac{m + n}{n}d\left(1 + \int_0^1 (B_{1,n}^{(k)}(t))^2 dt\right) - q_1\right). \tag{65}
\]

**Proof.** The gradients of $G_k(\bar{p}_{m-1}^1)$ can be calculated by
\[
\frac{\partial G_k(\bar{P}_{m-1})}{\partial \bar{P}_{m-1}} = \xi_k \frac{\partial G_{1,k}(\bar{P}_{m-1})}{\partial \bar{P}_{m-1}} + (1 - \xi_k) \frac{\partial G_{2,k}(\bar{P}_{m-1})}{\partial \bar{P}_{m-1}} = 2C\bar{P}_{m-1} + 2D.
\]

Since equation (56) has a unique solution which can be solved by \(\frac{\partial G_k(\bar{P}_{m-1})}{\partial \bar{P}_{m-1}} = 0\), we then get equation (64) by computing from equation (66) and combining it with equation (12).

For determining the values of \(\xi_k\) in equation (57), we consider the single-objective minimizations:

\[
\min_{r_{m-1} \in D} T_{1,k}(\bar{r}_{m-1}),
\]

\[
\min_{r_{m-1} \in D} T_{2,k}(\bar{r}_{m-1}).
\]

Lemma 6. The solution of equation (67), denoted by \(\bar{r}_{m-1}^1\), is given by the cases as follows:

(i) If \(\bar{r}_{m-1}^1 \in D\), we have \(\bar{r}_{m-1}^1 = \bar{r}_{m-1}^1\), where

\[
\bar{r}_{m-1}^1 = \bar{r}_{m-1} - \int_0^1 g^{(k)}_{1,m}(t)dt / [1 + \int_0^1 B^{(k)}_{m-1,m}(t)^2 dt].
\]

(ii) If \(\bar{r}_{m-1}^1 \notin D\) and \(T_{1,k}(0) \leq T_{1,k}(m + n/mR)\), we have \(\bar{r}_{m-1}^1 = 0\).

(iii) If \(\bar{r}_{m-1}^1 \notin D\) and \(T_{1,k}(0) > T_{1,k}(m + n/mR)\), we have \(\bar{r}_{m-1}^1 = m + n/mR\).

The solution of equation (68), denoted by \(\bar{r}_{m-1}^2\), is given by the cases as follows:

(i) If \(\bar{r}_{m-1}^2 \in D\), we have \(\bar{r}_{m-1}^2 = \bar{r}_{m-1}^2\), where

\[
\bar{r}_{m-1}^2 = \frac{n \int_0^1 g^{(k)}_{2,m}(t)B^{(k)}_{m-1,m}(t)dt + (m + n)R(1 + \int_0^1 B^{(k)}_{1,m}(t)^2 dt)}{n(1 + \int_0^1 (B^{(k)}_{1,m}(t))^2 dt)}.
\]

(ii) If \(\bar{r}_{m-1}^2 \notin D\) and \(T_{2,k}(0) \leq T_{2,k}(m + n/mR)\), we have \(\bar{r}_{m-1}^2 = 0\).

(iii) If \(\bar{r}_{m-1}^2 \notin D\) and \(T_{2,k}(0) > T_{2,k}(m + n/mR)\), we have \(\bar{r}_{m-1}^2 = m + n/mR\).

Proof. By computing from equations (8) and (36), we have

\[
\frac{dT_{1,k}(\bar{r}_{m-1})}{d\bar{r}_{m-1}} = 2 \int_0^1 g^{(k)}_{1,m}(t)B^{(k)}_{m-1,m}(t)dt + 2\bar{r}_{m-1}
\]

\[
\int_0^1 (B^{(k)}_{m-1,m}(t))^2 dt + 2(\bar{r}_{m-1} - r_{m-1}).
\]

By computing from equation (70), the unique stable point of \(T_{1,k}(\bar{r}_{m-1})\), denoted by \(\bar{r}_{m-1}^1\), can be obtained. There are two cases.

Case 1. \(\bar{r}_{m-1}^1 \in D\). Because \(d^2 T_{1,k}(\bar{r}_{m-1})/d(\bar{r}_{m-1})^2 > 0\), the solution of equation (67) must be \(\bar{r}_{m-1}^1 = \bar{r}_{m-1}^1\).

Case 2. \(\bar{r}_{m-1}^1 \notin D\). That means \(T_{1,k}(\bar{r}_{m-1}^1)\) is monotonous. If \(T_{1,k}(0) \leq T_{1,k}(m + n/mR)\), the solution of equation (67) should be \(\bar{r}_{m-1}^1 = 0\); else, the solution of equation (67) should be \(\bar{r}_{m-1}^1 = m + n/mR\). Thus, we obtain the solution of (67). In a similar, the solution of equation (68) can be reached.

We then can determine the values of \(\xi_k\) in equation (57) referring to Algorithm 2. The detailed description is not provided here. Furthermore, the solution of equation (57) can be gained by the following theorem.

Theorem 7. Given two BBCs \(\langle B_1 \rangle (t)\) and \(\langle B_2 \rangle (t)\) that satisfy \(C^0\) connection. For smoothing the two connected BBCs from \(C^0\) to \(C^1\), the radii of the adjusted control balls which minimized the internal energy \(T_k(\bar{r}_{m-1})\) are given by the cases as follows:

(i) If \(-(d/c) \in D\), the radii of the adjusted control balls are

\[
\begin{align*}
\bar{r}_{m-1} & = \frac{d}{c}, \\
\bar{s}_1 & = \frac{m + n}{n} R - \frac{m}{n}\bar{r}_{m-1},
\end{align*}
\]

where

\[
c = \xi_k \left( 1 + \int_0^1 (B^{(k)}_{m-1,m}(t))^2 dt \right) + \frac{m^2}{n^2}(1 - \xi_k) \left( 1 + \int_0^1 (B^{(k)}_{1,m}(t))^2 dt \right),
\]

\[
d = \xi_k \left( \int_0^1 g^{(k)}_{1,m}(t)B^{(k)}_{m-1,m}(t)dt - r_{m-1} \right) - \frac{m}{n}(1 - \xi_k) \left( \int_0^1 g^{(k)}_{1,m}(t)B^{(k)}_{1,m}(t)dt \right) + \frac{m + n}{n} R \left( 1 + \int_0^1 (B^{(k)}_{1,m}(t))^2 dt \right) - \bar{s}_1.
\]

(ii) If \(-d/c \notin D\) and \(T_{k}(0) \leq T_{k}(m + n/mR)\), the radii of the adjusted control balls are

\[
\begin{align*}
\bar{r}_{m-1} & = 0, \\
\bar{s}_1 & = \frac{m + n}{n} R.
\end{align*}
\]
Complexity

(iii) If \(-d/c \notin D \text{ and } T_k(0) > T_k(m + n/mR)\), the radii of the adjusted control balls are
\[
\begin{align*}
\bar{r}_{m-1} &= \frac{m + n}{m} R, \\
\bar{s}_1 &= 0.
\end{align*}
\] (74)

Proof. Because
\[
\frac{dT_k(\bar{r}_{m-1})}{dr_{m-1}} = \xi_k \frac{dT_{1,k}(\bar{r}_{m-1})}{dr_{m-1}} + (1 - \xi_k) \frac{dT_{2,k}(\bar{r}_{m-1})}{dr_{m-1}}
\] (75)

By computing from equation (75), the unique stable point of \(T_k(\bar{r}_{m-1})\), denoted by \(\bar{r}_{m-1}^*\), is \(\bar{r}_{m-1}^* = -d/c\). There are two cases.

Case 1. \(\bar{r}_{m-1}^* \in D\). Because \(d^2T_k(\bar{r}_{m-1})/dr_{m-1}^2 = 2c > 0\), we then get equation (71) by combining it with equation (13).

Case 2. \(\bar{r}_{m-1}^* \notin D\). That means \(T_k(\bar{r}_{m-1})\) is monotonous. If \(T_k(0) \leq T_k(m + n/mR)\), we can get equation (73) by combining it with equation (13); else, we can get equation (74) by combining it with (13).

We describe the methods of smoothing two connected BBCs from \(C^0\) to \(C^1\) by minimizing the combined energy as shown in Algorithm 5.

Remark 5. Although the internal energy minimization and the curve attractor minimization have been used to smooth two connected curves, respectively, there are no studies to combine them to smooth two connected curves. We combine these two kinds of minimizations in a new way, which extends the applications of previous methods.

6. Numerical Examples

In this section, we present some examples to illustrate the effectiveness of the proposed methods. All the numerical examples are implemented by MATLAB.

Example 1. Given a quartic BBC \(\langle \mathbf{B}_1 \rangle(t)\) with control balls
\[
\begin{align*}
\langle \mathbf{p}_0; r_0 \rangle &= \langle -10, 1, 0; 0.05 \rangle, \\
\langle \mathbf{p}_1; r_1 \rangle &= \langle -7, 0, 1; 0.06 \rangle, \\
\langle \mathbf{p}_2; r_2 \rangle &= \langle -5, 1, 0; 0.04 \rangle, \\
\langle \mathbf{p}_3; r_3 \rangle &= \langle -1, 1, 1; 0.02 \rangle, \\
\langle \mathbf{p}_4; r_4 \rangle &= \langle 0, 1, 0; 0.03 \rangle,
\end{align*}
\] (66)

and a quintic BBC \(\langle \mathbf{B}_2 \rangle(t)\) with control points
\[
\begin{align*}
\langle \mathbf{q}_0; s_0 \rangle &= \langle 0, 1, 0; 0.03 \rangle, \\
\langle \mathbf{q}_1; s_1 \rangle &= \langle 2, 1, 1; 0.04 \rangle, \\
\langle \mathbf{q}_2; s_2 \rangle &= \langle 5, 1, 0; 0.06 \rangle, \\
\langle \mathbf{q}_3; s_3 \rangle &= \langle 7, 0, 1; 0.05 \rangle, \\
\langle \mathbf{q}_4; s_4 \rangle &= \langle 8, 1, 0; 0.02 \rangle, \\
\langle \mathbf{q}_5; s_5 \rangle &= \langle 10, 0, 0; 0.01 \rangle.
\end{align*}
\] (77)

It implies that the two BBCs satisfy \(C^0\) connection. We smooth \(\langle \mathbf{B}_1 \rangle(t)\) and \(\langle \mathbf{B}_2 \rangle(t)\) from \(C^0\) to \(C^1\) by internal energy minimization. By calculation, the adjusted control balls can be obtained as follows:

(a) When \(k = 1\), we have
\[
\langle \mathbf{p}_1; \bar{r}_3 \rangle = \left\langle \frac{227}{24}, \frac{253}{96}, \frac{6400}{48} \right\rangle,
\] (78)

(b) When \(k = 2\), we have
\[
\langle \mathbf{q}_1; \bar{s}_1 \rangle = \left\langle \frac{809}{193}, \frac{4562}{4447}, \frac{179}{26/1295} \right\rangle,
\] (79)

(c) When \(k = 3\), we have
\[
\langle \bar{\mathbf{p}}; \bar{r}_3 \rangle = \left\langle \frac{1661}{456}, \frac{713}{887}, \frac{21081}{277400} \right\rangle,
\] (80)

\[
\langle \bar{\mathbf{q}}; \bar{s}_1 \rangle = \left\langle \frac{2678}{919}, \frac{695}{887}, \frac{1}{144115188075855870} \right\rangle.
\]

The connection BBCs before and after smoothing are shown in Figure 3.

Example 2. Given a quadratic BBC \(\langle \mathbf{B}_1 \rangle(t)\) with control balls
\[
\begin{align*}
\langle \mathbf{p}_0; r_0 \rangle &= \langle -4, 0, 0; 0.1 \rangle, \\
\langle \mathbf{p}_1; r_1 \rangle &= \langle -2, 1, 1; 0.05 \rangle, \\
\langle \mathbf{p}_2; r_2 \rangle &= \langle 0, 1, 0; 0.02 \rangle,
\end{align*}
\] (81)

and a cubic BBC \(\langle \mathbf{B}_2 \rangle(t)\) with control points
\[
\begin{align*}
\langle \mathbf{q}_0; s_0 \rangle &= \langle 0, 1, 0; 0.02 \rangle, \\
\langle \mathbf{q}_1; s_1 \rangle &= \langle 2, 1, 1; 0.05 \rangle, \\
\langle \mathbf{q}_2; s_2 \rangle &= \langle 5, 1, 0; 0.04 \rangle, \\
\langle \mathbf{q}_3; s_3 \rangle &= \langle 7, 0, 1; 0.02 \rangle.
\end{align*}
\] (82)
Step 1. Input two BBCs that satisfy \( C^0 \) connection.
Step 2. Select the value of \( k \) to determine which kind of combined energy is adopted.
Step 3. Determine the value of \( \xi_k \) in equation (54) referring to Algorithm 1 and compute the centers of the adjusted control balls by Theorem 6.
Step 4. Determine the value of \( \zeta_k \) in equation (55) referring to Algorithm 2 and compute the radii of the adjusted control balls by Theorem 7.
Step 5. Output two BBCs that satisfy \( C^1 \) connection.

Algorithm 5: Smoothing two connected BBCs from \( C^0 \) to \( C^1 \) by minimizing the combined energy.

Figure 3: Smoothing two connected BBCs from \( C^0 \) to \( C^1 \) by minimizing the internal energy. (a) \( C^0 \) connection. (b) \( C^1 \) connection (\( k = 1 \)). (c) \( C^1 \) connection (\( k = 2 \)). (d) \( C^1 \) connection (\( k = 3 \)).

It implies that the two BBCs satisfy \( C^0 \) connection. We smooth \( \langle B_1 \rangle (t) \) and \( \langle B_2 \rangle (t) \) from \( C^0 \) to \( C^1 \) by the curve attractor minimization. By calculation, the adjusted control balls can be obtained as follows:

\[
\langle \hat{p}_1; \hat{r}_1 \rangle = \left\langle \frac{5}{2}, 1, \frac{1}{4}, \frac{7}{260} \right\rangle,
\]

\[
\langle \hat{q}_1; \hat{s}_1 \rangle = \left\langle \frac{5}{3}, 1, \frac{1}{6}, \frac{65}{63} \right\rangle.
\]
The connection BBCs before and after smoothing are shown in Figure 4.

**Example 3.** Given a quartic BBC \( \{ B_1 \} (t) \) with control balls
\[
\begin{align*}
\langle p_0; r_0 \rangle &= (-8,0,1; 0.04), \\
\langle p_1; r_1 \rangle &= (-6,1,0; 0.05), \\
\langle p_2; r_2 \rangle &= (-3,0,1; 0.02), \\
\langle p_3; r_3 \rangle &= (-2,1,1; 0.02), \\
\langle p_4; r_4 \rangle &= (0,0,1; 0.04),
\end{align*}
\]
and a quintic BBC \( \{ B_2 \} (t) \) with control points
\[
\begin{align*}
\langle q_0; s_0 \rangle &= (0,0,1; 0.04), \\
\langle q_1; s_1 \rangle &= (3,1,1; 0.04), \\
\langle q_2; s_2 \rangle &= (4,0,1; 0.03), \\
\langle q_3; s_3 \rangle &= (6,1,0; 0.05), \\
\langle q_4; s_4 \rangle &= (10,0,1; 0.02), \\
\langle q_5; s_5 \rangle &= (12,0,0; 0.03).
\end{align*}
\]

It implies that the two BBCs satisfy \( C^0 \) connection. We smooth \( \{ B_1 \} (t) \) and \( \{ B_2 \} (t) \) from \( C^0 \) to \( C^1 \) by the combined energy minimization. By calculation, the adjusted control balls can be obtained as follows:

(a) When \( k = 1 \), we have
\[
\begin{align*}
\langle \bar{p}_3; \bar{r}_3 \rangle &= \left( \frac{-813}{293}, \frac{113}{5414}, \frac{758}{737}, \frac{147}{800} \right), \\
\langle \bar{q}_1; \bar{s}_1 \rangle &= \left( \frac{1727}{778}, \frac{253}{15152}, \frac{643}{658}, \frac{24}{415} \right).
\end{align*}
\]

(b) When \( k = 2 \), we have
\[
\begin{align*}
\langle \bar{p}_3; \bar{r}_3 \rangle &= \left( \frac{1279}{309}, \frac{-751}{680}, \frac{253}{1323}, \frac{9}{100} \right), \\
\langle \bar{q}_1; \bar{s}_1 \rangle &= \left( \frac{4063}{1227}, \frac{751}{850}, \frac{808}{2991}, \frac{1}{72057594037927936} \right).
\end{align*}
\]

(c) When \( k = 3 \), we have
\[
\begin{align*}
\langle \bar{p}_3; \bar{r}_3 \rangle &= \left( \frac{7304}{2999}, \frac{439}{13162}, \frac{1056}{1093}, \frac{9}{200} \right), \\
\langle \bar{q}_1; \bar{s}_1 \rangle &= \left( \frac{2831}{1453}, \frac{461}{17277}, \frac{1024}{997}, \frac{9}{250} \right).
\end{align*}
\]

The connection BBCs before and after smoothing are shown in Figure 5.
7. Conclusion

In this paper, we focus on smoothing two connected BBCs from $C^0$ to $C^1$ by optimizing their control balls. We propose the algorithms based on internal energy minimization, curve attractor minimization, and combined energy minimization. All the algorithms are established on the basis of solving bi-objective minimizations. The internal energy minimization can be applied in the applications which require the energy of each segment of the connected BBCs after smoothing to be as small as possible. The curve attractor minimization can be applied in the applications which require the adjusted control balls of each segment of the connected BBCs after smoothing not to be far away from their original positions. The combined energy minimization can be utilized in the applications which require each segment of the connected BBCs after smoothing to be as optimal as possible in both targets. Numerical examples show the effectiveness of the proposed algorithms. The proposals would help to improve the smooth continuity of connected BBCs which is often encountered in 3D modeling.

We give the smoothing schemes of two connected BBCs from $C^0$ to $C^1$, and the schemes of smoothing two connected BBCs from $C^{r-1}$ to $C^r$ ($r \geq 2$) could refer to the process of that from $C^0$ to $C^1$. However, we do not present the schemes of smoothing two connected BBCs from $C^{r-1}$ to $C^{r+1}$ ($r \geq 1$). We have tried to use the proposed ideology to smooth two connected BBCs from $C^0$ to $C^2$, but the effect is not satisfactory. This will be our next study issue.

Data Availability

The data used for the numerical analysis are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] H. Cao, H. Zheng, and G. Hu, “Adjusting the energy of Ball surfaces by modifying unfixed control balls,” Numerical Algorithms, vol. 89, no. 2, pp. 749–768, 2022.

[2] H. Cao, H. Zheng, G. Hu, and M. Abbas, “Adjusting the energy of Ball curves by modifying movable control balls,” Computational and Applied Mathematics, vol. 40, no. 3, p. 76, 2021.

[3] H. Seah and Z. Wu, “Ball b-spline based geometric models in distributed virtual environments,” in Proceedings of workshop towards semantic virtual environments, pp. 1–8, Villars, Switzerland, March 2005.

[4] Z. Wu, H. Seah, and M. Zhou, “Skeleton based parametric solid models: ball B-spline surfaces,” in Proceedings of the IEEE International Conference on Computer-Aided Design and Computer Graphics, pp. 421–424, Beijing, China, October 2007.

[5] Q. Fu, Z. Wu, X. Wang, M. Zhou, and J. Zheng, “An algorithm for finding intersection between ball B-spline curves,” Journal of Computational and Applied Mathematics, vol. 327, pp. 260–273, 2018.

[6] Z. Wu, X. Wang, Y. Fu et al., “Fitting scattered data points with ball B-Spline curves using particle swarm optimization,” Computers & Graphics, vol. 72, pp. 1–11, 2018.

[7] X. Liu, X. Wang, Z. Wu, D. Zhang, and X. Liu, “Extending ball B-spline by B-spline,” Computer Aided Geometric Design, vol. 82, Article ID 101926, 2020.

[8] G. Farin, Curves and Surfaces for CAGD: A Practical Guide, Academic Press, Cambridge, MA, USA, 2002.

[9] G. Hu, C. Bo, G. Wei, and X. Qin, “Shape-adjustable generalized Bézier surfaces: construction and it is geometric continuity conditions,” Applied Mathematics and Computation, vol. 378, Article ID 125215, 2020.

[10] G. Hu, C. Bo, and X. Qin, “Continuity conditions for tensor product Q-Bézier surfaces of degree (m, n),” Computational and Applied Mathematics, vol. 37, no. 4, pp. 4237–4258, 2018.

[11] G. Wang, G. Wang, and J. Zheng, Computer Aided Geometric Design, Springer, Beijing, China, 2001, in Chinese.

[12] R. C. Veltkamp and W. Wesselink, “Modeling 3D curves of minimal energy,” Computer Graphics Forum, vol. 14, no. 3, pp. 97–110, 1995.

[13] G. Xu, G. Wang, and W. Chen, “Geometric construction of energy-minimizing Bézier curves,” Science China Information Sciences, vol. 54, no. 7, pp. 1395–1406, 2011.

[14] G. Xu, Y. Zhu, L. Deng, G. Wang, B. Li and Kin-chuen Hui, and K. Hui, “Efficient construction of B-Spline curves with minimal internal energy,” Computers, Materials & Continua, vol. 58, no. 3, pp. 879–892, 2019.

[15] J. Li, “Combined internal energy minimizing planar cubic Hermite curve,” Journal of Advanced Mechanical Design, Systems, and Manufacturing, vol. 14, no. 7, Article ID JAMDSM0103, 2020.

[16] W. Veltkamp and R. C. Veltkamp, “Interactive design of constrained variational curves,” Computer Aided Geometric Design, vol. 12, no. 5, pp. 533–546, 1995.

[17] I. Röth, A. Röth, “Adjusting the energies of curves defined by control points,” Computer-Aided Design, vol. 107, pp. 77–88, 2019.

[18] C. Lin and J. Dong, The Methods and Theories of Multi-Objective Optimizations, Jilin Education Press, Changchun, Jilin, 1992, in Chinese.