Actively Learning Concepts and Conjunctive Queries under $\mathcal{EL}'$-Ontologies

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Abstract

We consider the problem to learn a concept or a query in the presence of an ontology formulated in the description logic $\mathcal{EL}'$, in Angluin’s framework of active learning that allows the learning algorithm to interactively query an oracle (such as a domain expert). We show that the following can be learned in polynomial time: (1) $\mathcal{EL}$-concepts, (2) symmetry-free $\mathcal{ELI}$-concepts, and (3) conjunctive queries (CQs) that are chordal, symmetry-free, and of bounded arity. In all cases, the learner can pose to the oracle membership queries based on ABoxes and equivalence queries that ask whether a given concept/query from the considered class is equivalent to the target. The restriction to bounded arity in (3) can be removed when we admit unrestricted CQs in equivalence queries. We also show that $\mathcal{EL}$-concepts are not polynomial query learnable in the presence of $\mathcal{ELI}$-ontologies.

1 Introduction

In logic based knowledge representation, a significant bottleneck is the construction of logical formulas such as description logic (DL) concepts, queries, and ontologies, as it is laborious and expensive. This is particularly true if the construction involves multiple parties because logic expertise and domain knowledge are not in the same hands. Angluin’s model of exact learning, a form of active learning, is able to support the construction of logical formulas in terms of a game-like collaboration between a learner and an oracle [Angluin, 1987b; Angluin, 1987a]. Applied in knowledge representation, the learner can be a logic expert and the oracle a domain expert that is interactively queried by the learner. Alternatively, the oracle can take other forms such as a set of labeled data examples that in some way represents the formula to be learned. The aim is to find an algorithm that, when executed by the learner, constructs the desired formula in polynomial time even when the oracle is not able to provide most informative answers. Landmark results from active learning state that such algorithms exist for learning propositional Horn formulas and finite automata [Angluin et al., 1992; Angluin, 1987a].

The aim of this paper is to study active learning of DL concepts and of conjunctive queries (CQs) in the presence of an ontology. Concepts are the main building block of ontologies [Baader et al., 2017] and learning them is important for ontology engineering. CQs are very prominent in ontology-mediated querying where data stored in an ABox is enriched with an ontology [Bienvenu et al., 2014]. We concentrate on the $\mathcal{EL}$ family of DLs which underlies the OWL EL profile of the OWL 2 ontology language [Krotzsch, 2012] and is frequently used in biomedical ontologies such as SNOMED CT. We consider ontologies formulated in the DLs $\mathcal{EL}'$ and $\mathcal{ELI}$ where $\mathcal{EL}'$ extends $\mathcal{EL}$ with range restrictions and $\mathcal{ELI}$ extends $\mathcal{EL}'$ with inverse roles. In both DLs, concepts can be viewed as a tree-shaped conjunctive query, and from now on we shall treat them as such. In fact, it is not uncommon to use concepts as queries in ontology-mediated querying, which provides an additional motivation for learning them.

We now describe the learning protocol in detail. It is an instance of Angluin’s model, which we do not repeat here in full generality. The aim is to learn a target CQ $q_T(\bar{x})$ in the presence of an ontology $\mathcal{O}$. The learner and the oracle both know and agree on the ontology $\mathcal{O}$, the arity of $q_T$, and the concept and role names that are available for constructing $q_T$; we assume that all concept and role names in $\mathcal{O}$ can be used also in $q_T$. The learner can ask two types of queries to the oracle. In a membership query, the learner provides an ABox $\mathcal{A}$ and a candidate answer $\bar{a}$ and asks whether $\mathcal{A}, \mathcal{O} \models q_T(\bar{a})$; the oracle faithfully answers “yes” or “no”. In an equivalence query, the learner provides a hypothesis CQ $q_H$ and asks whether $q_H$ is equivalent to $q_T$ under $\mathcal{O}$; the oracle answers “yes” or provides a counterexample, that is, an ABox $\mathcal{A}$ and tuple $\bar{a}$ such that $\mathcal{A}, \mathcal{O} \models q_T(\bar{a})$ and $\mathcal{A}, \mathcal{O} \not\models q_H(\bar{a})$ (positive counterexample) or vice versa (negative counterexample). When we learn a restricted class of CQs such as $\mathcal{EL}$-concepts, we assume that only CQs from that class are admitted in equivalence queries. We are then interested in whether there is a learning algorithm that constructs $q_T(\bar{x})$, up to equivalence under $\mathcal{O}$, such that at any given time, the running time of the algorithm is bounded by a polynomial in the sizes of $q_T$, of $\mathcal{O}$, and of the largest counterexample given by the oracle so far. This is called polynomial time learnability. A weaker requirement is polynomial query learnability where only the sum of the sizes of the queries posed to the oracle up to the current time point has to be bounded by such a polynomial.
Our main results are that the following can be learned in polynomial time under $\mathcal{EL}^r$-ontologies: (1) $\mathcal{EL}$-concepts, (2) $\mathcal{ELI}$-concepts that are symmetry-free, and (3) CQs that are chordal, symmetry-free, and of bounded arity. In Point (2), symmetry-freeness means that there is no subconcept of the form $\exists r,(C \cap \exists r^{-} D)$ with $r$ a role name, a condition that has recently been introduced in [Jung et al., 2020], in a slightly less general form where $r$ can also be an inverse role. In Point (3), chordal means that every cycle of length at least four that contains at least one quantified variable has a chord and symmetry-free means that the CQ contains no atoms $r(x_1, y), r(x_2, y)$ such that $x_1 \neq x_2, y$ is a quantified variable, neither $r(x_1, y)$ nor $r(x_2, y)$ occur on a cycle, and there is no atom $s(z, z)$ for any $z \in \{x_1, x_2, y\}$. An analysis of well-known benchmarks for ontology-mediated querying suggests that the resulting class $\mathcal{CQ}^{\text{cf}}$ of CQs is sufficiently general to include many relevant CQs that occur in practical applications. Our proofs crucially rely on the use of a finite version of the universal model that is specifically tailored to the class $\mathcal{CQ}^{\text{cf}}$. We also show that the restriction to bounded arity can be removed from Point (3) when we admit unrestricted CQs as the argument to equivalence queries. Proving this requires very substantial changes to the learning algorithm.

In addition, we prove several negative results. First, we show that none of the classes of CQs in Points (1) to (3) can be learned under $\mathcal{EL}$-ontologies using only membership queries or only equivalence queries (unless $P = NP$ in the latter case). Note that polynomial time learning with only membership queries is important because it is related to whether CQs can be characterized up to equivalence using only polynomially many data examples [ten Cate and Dalmau, 2020]. We also show the much more involved result that none of the classes of CQs in Points (1) to (3) is polynomial query learnable under $\mathcal{ELI}$-ontologies. Note that while polynomial time learnability cannot be expected because subsumption in $\mathcal{ELI}$ is ExpTime-complete, there could well have been a polynomial time learning algorithm with access to an oracle (in the classical sense) for subsumption/query containment under $\mathcal{ELI}$-ontologies that attains polynomial query learnability. Our result rules out this possibility.

Proof details are in the appendix.

Related work. Learning $\mathcal{EL}$-ontologies, rather than concepts or queries, was studied in [Konev et al., 2018; Konev et al., 2016]. It turns out that $\mathcal{EL}$-ontologies are not polynomial time learnable while certain fragments thereof are. In contrast, we attain polynomial time learnability also under unrestricted $\mathcal{ELI}$-ontologies. See also the surveys [Lehmann and Völker, 2014; Ozaki, 2020] and [Ozaki et al., 2020] for a variation less related to the current work. It has been shown in [ten Cate et al., 2013; ten Cate et al., 2018] that unions of CQs (UCQs) are polynomial time learnable, and the presented algorithm can be adapted to CQs. Active learning of CQs with only membership queries is considered in [ten Cate and Dalmau, 2020] where among other results it is shown that $\mathcal{ELI}$-concepts can be learned in polynomial time with only membership queries when the ontology is empty. PAC learnability of concepts formulated in the DL CLASSIC, without ontologies, was studied in [Cohen and Hirsh, 1994b; Cohen and Hirsh, 1994a; Frazier and Pitt, 1996].

2 Preliminaries

Concepts and Ontologies. Let $N_C$, $N_R$, and $N_I$ be countably infinite sets of concept names, role names, and individual names, respectively. A role $R$ takes the form $r$ or $r^-$ where $r$ is a role name and $r^-$ is called an inverse role. If $R = s^-$ is an inverse role, then $R^-$ denotes the role name $s$.

An $\mathcal{EL}$-concept is formed according to the syntax rule

$$C, D ::= \top \mid A \mid C \land D \mid \exists R.C$$

where $A$ ranges over $N_C$ and $R$ over roles. An $\mathcal{EL}$-concept is an $\mathcal{EL}$-concept that does not use inverse roles.

An $\mathcal{ELI}$-ontology $O$ is a finite set of concept inclusions (CIs) $C \sqsubseteq D$ where $C$ and $D$ range over $\mathcal{ELI}$-concepts. An $\mathcal{EL}$-ontology is an $\mathcal{ELI}$-ontology where inverse roles occur only in the form of range restrictions $\exists r^-, \top \sqsubseteq C$ with $C$ an $\mathcal{EL}$-concept. Note that domain restrictions $\exists r, \top \sqsubseteq C$ can be expressed already in $\mathcal{EL}$. An $\mathcal{EL}$-ontology is an $\mathcal{ELI}$-ontology that does not use inverse roles. An $\mathcal{ELI}$-ontology is in normal form if all CIs in it are of one of the forms

$$A_1 \sqcap A_2 \sqsubseteq A, A_1 \sqsubseteq \exists r_1 A_2, \exists r_1 A_1 \sqsubseteq A_2, \exists r^-, \top \sqsubseteq A$$

where $A_1, A_2$ are concept names or $\top$. An ABox $A$ is a finite set of concept assertions $A(a)$ and role assertions $r(a, b)$ where $A \in N_C \cup \{\top\}$, $r \in N_R$, and $a, b \in N_I$. We use $\text{ind}(A)$ to denote the set of individual names that are used in $A$ and may write $r^-(a, b)$ in place of $r(b, a)$. An ABox is a ditree if the directed graph $(\text{ind}(A), \{(a, b) \mid r(a, b) \in A\})$ is a tree and there are no multi-edges, that is, $(r(a, b), s(a, b)) \in A$ implies $r = s$.

The semantics is defined as usual in terms of interpretations $I$, which we define to be a (possibly infinite and non-empty) set of concept and role assertions. We use $\Delta^I$ to denote the set of individual names in $I$, define $A^I = \{a \mid A(a) \in I\}$ for all $A \in N_C$, and $r^I = \{(a, b) \mid r(a, b) \in I\}$ for all $r \in N_R$. The extension $C^I$ of $\mathcal{ELI}$-concepts $C$ is then defined as usual [Baader et al., 2017]. This definition of interpretation is slightly different from the usual one, but equivalent; its virtue is uniformity as every ABox is a (finite) interpretation. An interpretation $I$ satisfies a CI $C \sqsubseteq D$ if $C^I \subseteq D^I$, and a (concept or role) assertion $a \in I$ if $a \in \Delta^I$ or $a$ has the form $\top(a)$. We say that $I$ is a model of an ontology/ABox if it satisfies all concept inclusions/assertions in it and write $O \models C \sqsubseteq D$ if every model of the ontology $O$ satisfies the CI $C \sqsubseteq D$.

A signature is a set of concept and role names, uniformly referred to as symbols. For any syntactic object $O$ such as an ontology or an ABox, we use $\text{sig}(O)$ to denote the symbols used in $O$ and $|O|$ to denote the size of $O$, that is, the length of a word representation of $O$ in a suitable alphabet.

CQs and Homomorphisms. A conjunctive query (CQ) takes the form $q(\bar{x}) \leftarrow \varphi(\bar{x}, \bar{y})$ where $\varphi$ is a conjunction of concept atoms $A(x)$ and role atoms $r(x, y)$ with $A \in N_C$ and $r \in N_R$. We may write $r^-(x, y)$ in place of $r(y, x)$.

Note that the tuple $\bar{x}$ used in the head $q(\bar{x})$ of the CQ may contain repeated occurrences of variables. When we do
not want to make the body $\varphi(x, y)$ explicit, we may denote $q(\bar{x}) \leftarrow \varphi(x, y)$ simply with $q(\bar{x})$. We refer to the variables in $\bar{x}$ as the answer variables of $q$, and to the variables in $\bar{y}$ as the quantified variables. When we are not interested in order and multiplicity, we treat $\bar{x}$ and $\bar{y}$ as sets of variables. We use $\text{var}(q)$ to denote the set of all variables in $\bar{x}$ and $\bar{y}$. The arity of $q$ is the length of tuple $\bar{x}$ and $q$ is Boolean if it has arity zero. Every $\text{CQ} \ q(\bar{x}) \leftarrow \varphi(x, y)$ gives rise to a $\text{ABox}$ (and thus interpretation) $\mathcal{A}_q$ obtained from $\varphi(x, y)$ by viewing variables as individual names and atoms as assertions. A $\text{CQ}$ is a ditree if $\mathcal{A}_q$ is.

A homomorphism $h$ from interpretation $\mathcal{I}_1$ to interpretation $\mathcal{I}_2$ is a mapping from $\Delta^{\mathcal{I}_1}$ to $\Delta^{\mathcal{I}_2}$ such that $d \in \Delta^{\mathcal{I}_1}$ implies $h(d) \in \Delta^{\mathcal{I}_2}$ and $(d, e) \in r^{\mathcal{I}_1}$ implies $(h(d), h(e)) \in r^{\mathcal{I}_2}$. For $d_1$, a tuple over $\Delta^{\mathcal{I}_1}$, $i \in \{1, 2\}$, we write $\mathcal{I}_1, d_1 \rightarrow \mathcal{I}_2, d_2$ if there is a homomorphism $h$ from $\mathcal{I}_1$ to $\mathcal{I}_2$ with $h(d_1) = d_2$. With a homomorphism from a $\text{CQ}$ to an interpretation $\mathcal{I}$, we mean a homomorphism from $\mathcal{A}_q$ to $\mathcal{I}$.

Let $q(\bar{x}) \leftarrow \varphi(x, y)$ be a $\text{CQ}$ and $\mathcal{I}$ an interpretation. A tuple $\bar{d} \in (\Delta^{\mathcal{I}})^{|\bar{x}|}$ is an answer to $q$ on $\mathcal{I}$, written $\mathcal{I} \models q(\bar{d})$, if there is a homomorphism $h$ from $q$ to $\mathcal{I}$ with $h(\bar{x}) = \bar{d}$. Now let $\mathcal{O}$ be an $\mathcal{EL}C^\Omega$-ontology and $\mathcal{A}$ an $\text{ABox}$. A tuple $\bar{a} \in \text{ind}(\mathcal{A})^{\bar{a}}$ is an answer to $q$ on $\mathcal{A}$ under $\mathcal{O}$, written $\mathcal{A}, \mathcal{O} \models q(\bar{a})$, if $\bar{a}$ is an answer to $q$ on every model of $\mathcal{O}$.

For $q_1$ and $q_2$ $\text{CQs}$ of the same arity $n$ and $\mathcal{O}$ an $\mathcal{EL}C^\Omega$-ontology, we say that $q_1$ is contained in $q_2$ under $\mathcal{O}$, written $q_1 \subseteq_\mathcal{O} q_2$, if for all $\text{ABoxes}$ $\mathcal{A}$ and $\bar{a} \in \text{ind}(\mathcal{A})^{\bar{a}}$, $\mathcal{A}, \mathcal{O} \models q_1(\bar{a})$ implies $\mathcal{A}, \mathcal{O} \models q_2(\bar{a})$. We call $q_1$ and $q_2$ equivalent under $\mathcal{O}$, written $q_1 \equiv_\mathcal{O} q_2$, if $q_1 \subseteq_\mathcal{O} q_2$ and $q_2 \subseteq_\mathcal{O} q_1$.

Every $\mathcal{EL}C^\Omega$-concept can be viewed as a unary tree-shaped $\text{CQ}$ in an obvious way. For example, the $\mathcal{EL}C^\Omega$-concept $\exists s \top \sqcap \exists r. B$ yields the $\text{CQ}$ $q(x) \leftarrow A(x) \wedge s(x, y) \wedge r(x, z) \wedge B(z)$. We use ELQ to denote the class of all $\mathcal{EL}C^\Omega$-concepts viewed as a $\text{CQ}$, and likewise for ELIQ and $\mathcal{EL}C^\Omega$-concepts.

Important Classes of $\text{CQs}$. We next define a class of $\text{CQs}$ that we show later to admit polynomial time learnability under $\mathcal{EL}C^\Omega$-ontologies, one of the main results of this paper. Let $\mathcal{A}$ be an $\text{ABox}$. A path in $\mathcal{A}$ from $a$ to $b$ is a sequence $p = R_0(a_0, a_1), \ldots, R_{n-1}(a_{n-1}, a_n) \in \mathcal{A}$, $n \geq 0$, such that $a_0 = a$ and $a_n = b$. We say that $p$ is a cycle of length $n$ if $a_0 = a_n$, all assertions in $p$ are distinct, and all of $a_0, \ldots, a_{n-1}$ are distinct. A chord of cycle $p$ is an assertion $R(a_i, a_j)$ with $0 \leq i, j < n-1$ and $i \not\equiv j \pmod{n}$.

In Point 2, $r$ is a role name and thus there are no restrictions on ‘inverse symmetries’; $\varphi$ may contain atoms $r(y_1, x), r(y_2, x)$ with $y_1 \neq y_2$, then $x$ is an answer variable or one of the atoms occurs on a cycle or $\varphi$ contains an atom $s(z, z)$ for some $z \in \{x, y_1, y_2\}$.

We believe that $\text{CQ}^\text{sf}$ includes many relevant $\text{CQs}$ that occur in practical applications. To substantiate this, we have analyzed the 65 queries that are part of three widely used benchmarks for ontology-mediated querying, namely Fishmark, LUBM\textsuperscript{3}, and NPD [Bail et al., 2012; Lutz et al., 2013; Lanti et al., 2015]. We found that more than 85% of the queries fall into $\text{CQ}^\text{sf}$ while less than 5% fall into $\text{ELIQ}^\text{sf}$.

Universal Models. Let $\mathcal{A}$ be an $\text{ABox}$ and $\mathcal{O}$ an $\mathcal{EL}C^\Omega$-ontology. The universal model of $\mathcal{A}$ and $\mathcal{O}$, denoted $\mathcal{U}_{\mathcal{A}, \mathcal{O}}$, is the interpretation obtained by starting with $\mathcal{A}$ and then ‘chasing’ with the CI’s in the ontology which adds (potentially infinite) ditrees below every $a \in \text{ind}(\mathcal{A})$. The formal definition is in the appendix. The model is universal in that $\mathcal{U}_{\mathcal{A}, \mathcal{O}} \models q(\bar{a})$ iff $\mathcal{A}, \mathcal{O} \models q(\bar{a})$ for all $\text{CQs}$ $q(\bar{x})$ and tuples $\bar{a} \in \text{ind}(\mathcal{A})^{\bar{a}}$. It can be useful to represent universal models in a finite way, as for example in the combined approach to ontology-mediated querying [Lutz et al., 2009]. Here, we introduce a finite representation that is tailored towards our class $\text{CQ}^\text{sf}$.

The 3-compact model $\mathcal{C}_{3, \mathcal{O}}$ of $\mathcal{A}$ and $\mathcal{O}$ is defined as follows. Let $\text{sub}(\mathcal{O})$ be the set of all concepts in $\mathcal{O}$, closed under subconcepts. $\mathcal{C}_{3, \mathcal{O}}$ uses the individual names from $\mathcal{A}$ as well as individual names of the form $c_{a,i,r,C}$ where $a \in \text{ind}(\mathcal{A})$, $0 \leq i \leq 4$, $r$ is a role name from $\mathcal{O}$, and $C \in \text{sub}(\mathcal{O})$. For every role name $r$ we use $C^r$ to denote the conjunction over all $C$ such that $\exists r, \top \subseteq C \subseteq \mathcal{O}$, and $\top$ if the conjunction is empty. Let $i \equiv 0 \bmod{4} + 1$. Define $\mathcal{C}_{3, \mathcal{O}} := \mathcal{A} \cup \{A(a) \mid \mathcal{A}, \mathcal{O} \models A(a)\} \cup$$\{A(c_{a,i,r,C}) \mid C \sqcap C^r \subseteq \mathcal{A}\} \cup$$\{r(a, c_{a,0,0,r,C}) \mid \mathcal{A}, \mathcal{O} \models \exists r.C(a)\} \cup$$\{r(c_{a,i,s,C}, c_{a,i+1,r,C'}) \mid C' \sqcap C^r \subseteq \exists r.C'\}$.

There is a homomorphism from $\mathcal{U}_{\mathcal{A}, \mathcal{O}}$ to $\mathcal{C}_{3, \mathcal{O}}$ that is the identity on $\text{ind}(\mathcal{A})$, but in general not vice versa. Nevertheless, $\mathcal{C}_{3, \mathcal{O}}$ is universal for $\text{CQ}^\text{sf}$.

Lemma 1. Let $\mathcal{A}$ be an $\text{ABox}$ and $\mathcal{O}$ an $\mathcal{EL}C^\Omega$-ontology. Then $\mathcal{C}_{3, \mathcal{O}}$ is a model of $\mathcal{A}$ and $\mathcal{O}$ such that for every $\text{CQ}$ $q(\bar{x}) \in \text{CQ}^\text{sf}$ and $\bar{a} \in \text{ind}(\mathcal{A})^{\bar{a}}$, $\mathcal{C}_{3, \mathcal{O}} \models q(\bar{a})$ iff $\mathcal{A}, \mathcal{O} \models q(\bar{a})$. 


$C^3_{A,O}$ is defined so as to avoid spurious cycles of length at most 3 while larger spurious cycles are irrelevant for CQs that are chordal. This explains the superscript $^3$ and enables the lemma below. $C^3_{A,O}$ also avoids spurious predecessors connected via different role names. Spurious predecessors connected via the same role name cannot be avoided, but are irrelevant for CQs that are symmetry-free.

**Lemma 2.** Every cycle in $C^3_{A,O}$ of length at most three consists only of individuals from $\text{ind}(A)$. We also use the direct product $T_1 \times T_2$ of interpretations $T_1$ and $T_2$, defined in the standard way (see appendix). For tuples of individuals $\bar{a} = (a_{i,1}, \ldots, a_{i,n})$, $i \in \{1, 2\}$, we set $\bar{a}_1 \otimes \bar{a}_2 = ((a_{1,1}, a_{2,1}), \ldots, (a_{1,n}, a_{2,n}))$.

### 3 Learning under $\mathcal{EL}^r$-Ontologies

We establish polynomial time learnability results under $\mathcal{EL}^r$-ontologies for the query classes $\text{CQ}_{\text{csf}, \mathcal{O}}$, $\text{ELQ}$, and $\text{ELIQ}_{\text{csf}}$. For $\text{CQ}_{\text{csf}}$, we additionally have to assume that the arity of CQs to be learned is bounded by a constant or that unrestricted CQs are not polynomial time learnable with only equivalence queries. When speaking of equivalence queries, we generally imply that the CQs used in such queries must be from the class of CQs to be learned. If this is not the case and unrestricted CQs are admitted in equivalence queries, then we speak of CQ-equivalence queries. When using CQ-equivalence queries, the learned representation of the target query is a CQ, but need not necessarily belong to $\mathcal{C}$ (though it is equivalent to a query from $\mathcal{C}$). For $w \geq 0$, let $\text{CQ}_{\text{csf}}^w$ be the restriction of $\text{CQ}_{\text{csf}}$ to CQs of arity at most $w$. The following are the main results obtained in this section.

**Theorem 1.**

1. $\text{ELQ}$- and $\text{ELIQ}_{\text{csf}}$-queries are polynomial time learnable under $\mathcal{EL}^r$-ontologies using membership and equivalence queries;

2. for every $w \geq 0$, $\text{CQ}_{\text{csf}}^w$-queries are polynomial time learnable under $\mathcal{EL}^r$-ontologies using membership and equivalence queries;

3. $\text{CQ}_{\text{csf}}^w$-queries are polynomial time learnable under $\mathcal{EL}^r$-ontologies using membership and CQ-equivalence queries.

Before providing a proof of Theorem 1, we show that both membership and equivalence queries are needed for polynomial learnability. Let $A^\forall$ denote the class of unary CQs of the form $q(x) \leftarrow A_1(x) \land \cdots \land A_n(x)$, and let a conjunctive ontology be an $\mathcal{EL}$-ontology without role names.

**Theorem 2.**

1. $A^\forall$-queries are not polynomial query learnable under conjunctive ontologies using only membership queries;

2. $\text{ELQ}$-queries are not polynomial time learnable (without ontologies) using only CQ-equivalence queries unless $P = \text{NP}$.

Note that Points 1 and 2 of Theorem 2 imply the same statements for all relevant query classes, that is, $\text{ELQ}$, $\text{ELIQ}_{\text{csf}}$, $\text{CQ}_{\text{csf}}^w$, and $\text{CQ}_{\text{csf}}^w$ for all $w \geq 1$, and CQ, in place of the classes mentioned in the theorem. In particular, Point 2 implies that unrestricted CQs are not polynomially learnable with only equivalence queries in the classical setting (without ontologies) unless $P = \text{NP}$, even when only unary and binary relations are admitted, see [Cohen, 1995; Haussler, 1989; Hirata, 2000] for related results. The proof of Point 1 follows basic lower bound proofs for abstract learning problems [Angel, 1987b] and enables connections between active learning and inseparability questions studied in [Funk et al., 2019; Jung et al., 2020; Funk, 2019].

#### 3.1 Reduction to Normal Form

We show that the ontology under which we learn can w.l.o.g. be assumed to be in normal form. It is well-known that every $\mathcal{EL}^r$-ontology $\mathcal{O}$ can be converted into normal form by introducing fresh concept names [Baader et al., 2017]. We use such a conversion to show that, for the relevant classes of CQs, a polynomial time learning algorithm under $\mathcal{EL}^r$-ontologies in normal form can be converted into a polynomial time learning algorithm under unrestricted $\mathcal{EL}^r$-ontologies. Care has to be exercised as the fresh concept names can occur in membership and equivalence queries. From now on, we thus assume that ontologies are in normal form.

**Proposition 1.** Let $Q \in \{\text{ELQ}, \text{ELIQ}_{\text{csf}}, \text{CQ}_{\text{csf}}^w \mid w \geq 0\}$. If queries in $Q$ are polynomial time learnable under $\mathcal{EL}^r$-ontologies in normal form using membership and equivalence queries, then the same is true for unrestricted $\mathcal{EL}^r$-ontologies.

#### 3.2 Algorithm Overview

We start with proving Points 1 and 2 of Theorem 1. Thus let $Q \in \{\text{ELQ}, \text{ELIQ}_{\text{csf}}, \text{CQ}_{\text{csf}}^w \mid w \geq 0\}$. The algorithm that establishes polynomial time learnability of queries from $Q$ under $\mathcal{EL}^r$-ontologies is displayed as Algorithm 1. We next explain some of its details.

**Algorithm 1 Learning queries $q_T$ from $\text{ELQ} / \text{ELIQ}_{\text{csf}} / \text{CQ}_{\text{csf}}^w$ under an $\mathcal{EL}^r$-ontology $\mathcal{O}$.**

```
procedure LEARNQ
   q_H(\bar{x}) := refine(q_H(\bar{x}_0))
   while q_H \not\equiv_{\mathcal{O}} q_T (equivalence query) do
      Let A, a be the positive counterexample returned
      and let q_H(\bar{x}') be C_{A,w},C_3,O viewed as a CQ
      with answer variables \bar{x}' = \bar{x} \otimes \bar{a}
      q_H(\bar{x}) := refine(q_H(\bar{x}'))
   return q_H(\bar{x})
```

symbols from $\Sigma$. Note that $q^\perp \in CQ^w$ for all $w$, but $q^\perp$ is neither in ELQ nor in ELIQ$^d$.

If $q_i(x_1), q_2(x_2), \ldots$ are the hypotheses constructed during a run of the algorithm, then for all $i \geq 1$:

1. $q_i \in Q$ and $q_i \subseteq \sigma q_T$;
2. $q_i \subseteq q_{i+1}$ and $q_i \not\equiv \sigma q_{i+1}$;
3. $|\var(q_i)| \leq |\var(q_T)|$.

Taken together, Points 1 and 2 mean that the hypotheses approximate the target query from below in an increasingly better way and Point 3 is crucial for proving that we must reach $q_T$ after polynomially many steps. The fact that $\mathcal{O}$ is in normal form is used to attain Point 3.

Point 1 also guarantees that the oracle always returns a positive counterexample $A, \bar{a}$ to the equivalence query used to check whether $q_H \not\equiv Q q_T$ in the while loop. The algorithm extracts the commonalities of $q_H(x)$ and $A, \bar{a}$ by means of a direct product with the aim of obtaining a better approximation of the target. The same is done in the case without ontologies [ten Cate et al., 2013] where $A_{q_H} \times A$ (viewed as a CQ) is the new hypothesis, but this is not sufficient here as it misses the impact of the ontology. The product $U_{A_{q_H, \sigma}} \times U_{A, \sigma}$ would work, but need not be finite. So we resort to $C^3_{A_{q_H, \sigma}} \times C^3_{A, \sigma}$ instead, viewed as a CQ $q_H'(x')$. This new hypothesis need not belong to $Q$, so we call the subroutine refine detailed in the subsequent section to convert it into a new hypothesis $q_H(x) \in Q$ such that $q_H'(x) \subseteq \sigma q_H \subseteq \sigma q_T$. The initial call to refine serves the same purpose as $q^\perp(x_0)$ need not be in $Q$, depending on the choice of $Q$.

It is not immediately clear that the described approach achieves the containment in Point 2 since $C^3_{A_{q_H, \sigma}} \times C^3_{A, \sigma}$ is potentially too strong as a replacement of $U_{A_{q_H, \sigma}} \times U_{A, \sigma}$; in particular, there might be cycles in the former product that do not exist in the latter. What saves us, however, is that the CQ $q_H$ constructed by refine belongs to $Q$ while the models $C^3_{A_{q_H, \sigma}} \times C^3_{A, \sigma}$ are universal for $Q$ as per Lemma 1.

### 3.3 The refine Subroutine

The refine subroutine gets as input a CQ $q_H'(x')$ that does not need to be in $Q$, but that satisfies $q_H' \subseteq \sigma q_T$. It produces a query $q_H(x)$ from $Q$ such that $q_H \subseteq \sigma q_H \subseteq \sigma q_T$ and $|\var(q_H)| \leq |\var(q_T)|$. For notational convenience, we prefer to view $q_H'(x')$ as a pair $(A, \bar{a})$ where $A = A_{q_H'}$ and $\bar{a} = x'$. Let $n_{\text{max}}$ denote the maximum length of a chordless cycle in any query in $Q$, that is $n_{\text{max}} = 0$ for $Q \in \{\text{ELQ}, \text{ELIQ}^d\}$ and $n_{\text{max}} = 3$ for $Q = CQ^w$, $w \geq 0$. We shall use the following.

**Minimize.** Let $B$ be an ABox and $\bar{b}$ a tuple such that $B, \mathcal{O} \models q_T(\bar{b})$. Then minimize $(B, \bar{b})$ is the ABox $B'$ obtained from $B$ by exhaustively applying the following operations:

1. Choose $c \in \text{ind}(B) \setminus \bar{b}$ and remove all assertions that involve $c$. Use a membership query to check whether, for the resulting ABox $B^-, \mathcal{O} \models q_T(\bar{b})$. If so, proceed with $B^-$ in place of $B$.
2. Choose $r(a, b) \in B$ and use a membership query to check whether $B \setminus \{r(a, b)\}, \mathcal{O} \models q_T(\bar{b})$. If so, proceed with $B \setminus \{r(a, b)\}$ in place of $B$.

The refine subroutine builds a sequence $(B_1, \bar{b}_1), (B_2, \bar{b}_2), \ldots$ starting with $(B_1, \bar{b}_1) = (\text{minimize}(A, \bar{a}), \bar{a})$ and exhaustively applying the following step:

**Expand.** Choose a chordless cycle $R_o(a_0, a_1), \ldots, R_{n-1}(a_{n-1}, a_n)$ in $B_i$ with $n > n_{\text{max}}$ and, in case that $Q = CQ^w$,

$$\{a_0, \ldots, a_{n-1}\} \not\subseteq \bar{b}_i.$$  

Let $B'_i$ be the ABox obtained by doubling the length of the cycle: start with $B_i$, introduce copies $a'_0, \ldots, a'_{n-1}$ of $a_0, \ldots, a_{n-1}$, and then

- remove all assertions $R(a_{n-1}, a_0)$;
- add $B(a'_0)$ if $B(a_i) \in B_i$;
- add $R(a'_i, c)$ if $R(a_i, c) \in B_i$ with $0 \leq i < n$ and $c \in \text{ind}(B_i) \setminus \{a_0, \ldots, a_{n-1}\}$;
- add $R(a'_i, a'_j)$ if $R(a_i, a_j) \in B_i$ with $0 \leq i, j < n$ and $\{i, j\} \not\subseteq \{0, n-1\}$;
- add $R(a_{n-1}, a'_0)$ and $R(a'_{n-1}, a_0)$ if $R(a_{n-1}, a_0) \in B_i$.

A similar construction is used in [Konev et al., 2016]. Let $\tau_i$ be the set of tuples $\bar{b}$ obtained from $\bar{b}_i = (b_1, \ldots, b_k)$ by replacing any number of components $b_j$ by $b'_j$. Use membership queries to identify $\bar{b}_{i+1} \in \tau_i$, $B'_i, \mathcal{O} \models q_T(\bar{b}_{i+1})$ and set $B_{i+1} = \text{minimize}(B'_i, \bar{b}_{i+1})$. We prove in the appendix that such a $\bar{b}_{i+1}$ always exists and that the Expand step can only be applied polynomially many times. The resulting $(B_n, \bar{b}_n)$ viewed as a CQ with answer variables $\bar{b}_n$ is chordal, but not necessarily symmetry-free. To establish also the latter, we compute a sequence of ABoxes $B_n, B_{n+1}, \ldots$, by exhaustively applying the following step:

**Split.** Choose $r(a, b), r(c, b) \in B_i$ such that $b \not\equiv \bar{b}_n$ and neither $r(a, b)$ nor $r(c, b)$ occurs on a cycle. Construct $B'_i$ by removing $r(a, b)$ from $B_i$, taking a fresh individual $b'$, and adding $B(b')$ for all $B(b) \in B_i$ and $S(d, b')$ for all $S(d, b) \in B_i$ with $S(d, b) \not\equiv r(c, b')$. If $B'_i, \mathcal{O} \models q_T(\bar{b}_n)$, then $B_{i+1} = \text{minimize}(B'_i, \bar{b}_n)$.

We prove in the appendix that only polynomially many applications are possible and that, for $B_n$ the resulting ABox, $(B_n, \bar{b}_n)$ viewed as a CQ is chordal and symmetry-free. Moreover, it is in ELQ if $q_T$ is, and likewise for ELIQ$^d$. Refine returns this CQ as its result. Note that the running time of refine depends exponentially on $\tau$ due to the brute force search for a tuple $\bar{b}_{i+1} \in \tau_i$ in the Expand step.

### 3.4 Unbounded Arity

To prove the remaining Point 3 of Theorem 1, we have to deal with CQs of unbounded arity and cannot use the refine subroutine presented in Section 3.3. We thus introduce a second version of refine that works rather differently from the previous one. We give an informal description, full details are in the appendix.

Recall that refinement starts with the product $P = C^\perp_{A_{q_H, \sigma}} \times C^3_{A, \sigma}$. In Section 3.3, we blow up cycles in $P$, not distinguishing the ABox part and the existentially generated part of the 3-compact models involved. The second version of refine instead unravels the existentially generated

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1 This is because $CQ^w$ admits cycles that consist only of answer variables while ELQ and ELIQ$^d$ do not.
part of the two 3-compact models inside the product $P$. A full such unraveling would eventually result in $U_{A_{mp}} \otimes \times U_{A_{op}}$, but we interleave with a Minimize step as in Section 3.3 and thus obtain a finite initial piece thereof. Unlike in the previous version of refine, we do not have to redefine the answer variables at all (but note that they may still change outside of refine when we take the product).

The above suffixes for target CQs from $\text{CQ}^{\text{ad}}$ in which every variable is reachable from an answer variable. In the general case, disconnected Boolean components might be present (or emerge during unraveling and minimization) that are never unraveled. To address this, we subsequently apply the original version of refine to such components, avoiding the Splitting step and leaving the already unraveled parts untouched. Note that the exponential blowup in the arity is avoided because the original refine is only applied to Boolean subqueries. However, the resulting queries are not guaranteed to be in $\text{CQ}^{\text{ad}}$. We can thus not rely on Lemma 1 as before which is why we need $\text{CQ}$-equivalence queries.

4 Learning under $\text{ELI}$-Ontologies

When we replace $\text{EL'}$-ontologies with $\text{ELI}$-ontologies, polynomial time learnability can no longer be expected since containment between ELQs under $\text{ELI}$-ontologies is ExpTime-complete [Baader et al., 2008]. In contrast, polynomial query learnability is not ruled out and it is natural to ask whether there is a polynomial time learning algorithm with access to an oracle (in the classical sense) for query containment under $\text{ELI}$-ontologies. Note that such an algorithm would show polynomial query learnability. We answer this question to the negative and show that polynomial query learnability cannot be attained under $\text{ELI}$-ontologies for any of the query classes considered in this paper. This is a consequence of the following result, which also captures learning of unrestricted CQs.

Theorem 3. $\text{EL}$-concepts are not polynomial query learnable under $\text{ELI}$-ontologies with membership queries and $\text{CQ}$-equivalence queries.

For the proof, we use the $\text{ELI}$-ontologies $\mathcal{O}_n, n \geq 1$, given in Figure 1. There, $\tau = s$ and $\tau = r$. Every $\mathcal{O}_n$ is associated with a set $\mathcal{H}_n$ of $2^n$ potential target concepts of the form

$$\exists \sigma_1 \ldots \exists \sigma_n, \exists \sigma^n.A$$

where $\exists \sigma^n$ denotes the $n$-fold nesting of $\exists \sigma$. The idea of the proof is to show that if there was an algorithm for learning $\text{EL}$-concepts under $\text{ELI}$-ontologies such that, at any given time, the sum of the sizes of all (membership and CQ-equivalence) queries asked to the oracle is bounded by a polynomial $p(n_1, n_2, n_3)$ with $n_1$ is the size of the target query, $n_2$ is the size of the ontology, and $n_3$ is the size of the largest counterexample seen so far, then we can choose $n$ large enough so that the learner needs more than $p(n_1, n_2, n_3)$ queries to distinguish the targets in $\mathcal{H}_n$ under $\mathcal{O}_n$ if the oracle uses a ‘sufficiently destructive’ strategy to answer the queries. Such a strategy is presented in the appendix, we only give one example that highlights a crucial aspect.

Assume that the learner poses as an equivalence query the $\text{EL}$-concept $C_H = \exists \sigma_1 \ldots \exists \sigma_n, \exists \sigma^n.A$. Then the oracle returns “no” and positive counterexample $\mathcal{A} = \{K_0(a_0), W_1^0(a_0), \ldots, W_n^0(a_0)\}$. It is instructive to verify that $\mathcal{A}, \mathcal{O} \models C_H(a_0)$ for all $C_H \in \mathcal{H}_n \setminus \{C_H\}$ while $\mathcal{A}, \mathcal{O} \not\models C_H(a_0)$ as this illustrates the use of inverse roles in $\mathcal{O}_n$.

5 Conclusion

We conjecture that our results can be extended from $\text{EL}'$-ontologies to $\text{ELI}'$-ontologies, thus adding role inclusions. In contrast, we do not know how to learn in polynomial time unrestricted $\text{ELI}$-concepts under $\text{EL}$-ontologies, or symmetry-free CQs under $\text{EL}$-ontologies. We would not be surprised if these indeed turn out not to be learnable in polynomial time. It is an interesting question whether our results can be generalized to symmetry-free CQs that admit chordless cycles of length bounded by a constant larger than three. This would require the use of a different kind of compact universal model.

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A Appendix Preliminaries

We introduce some additional preliminaries that are needed for the lemmas and proofs in the appendix.

Let $O$ be an $\mathcal{EL}'$-ontology in normal form. We say that an ABox $A$ is $O$-saturated if $A, O \models A(a)$ implies $A(a) \in A$ for all concept names $A$ and $a \in \text{ind}(A)$.

Universal Models. The universal model of $A$ and $O$, denoted $U_{A,O}$, is the interpretation defined as follows. For every role name $r$, we use $C_r$ to denote the conjunction over all $C$ such that $\exists r^- \cdot T \subseteq C \subseteq O$; note that $C_r = T$ if there is no such range restriction in $A$. A trace for $A$ and $O$ is a sequence $t = a_1r_1C_1a_2r_2C_2 \ldots a_nC_n$, $n \geq 0$, such that $a \in \text{ind}(A)$, $\{\exists r_1C_1, \ldots, \exists r_nC_n\} \subseteq \text{sub}(O)$, $A, O \models \exists r_1C_1(a)$, $O \models C_1 \cap C_n \subseteq \exists r_{i+1}C_{i+1}$ for $1 \leq i < n$. Let $T$ denote the set of all traces for $A$ and $O$. Then

$$U_{A,O} := A \cup \{A(a) \mid A, O \models A(a)\} \cup \{A(rC) \mid rC \in \mathcal{T} \text{ and } O \models C \cap C_r \subseteq A\} \cup \{r(t, trC) \mid trC \in \mathcal{T}\}$$

The following two lemmas are the main properties of $U_{A,O}$. They connect the notions of the universal model, homomorphisms, and queries and are well known.

Lemma 3. Let $A$ be an ABox and $O$ an $\mathcal{EL}'$-ontology. Then

1. $U_{A,O}$ is a model of $A$ and $O$;
2. for every $CQ q(\bar{x})$ and $\bar{a} \in \text{ind}(A)[\bar{a}]$, $U_{A,O} \models q(\bar{a})$ iff $A, O \models q(\bar{a})$.

Lemma 4. For an $\mathcal{EL}'$-ontology $O$ and $CQ$s $q_1(\bar{x}_1)$ and $q_2(\bar{x}_2)$ of the same arity, the following are equivalent:

1. $q_1 \subseteq \alpha q_2$.
2. there is a homomorphism $h$ from $q_2$ to $U_{A_{\alpha}O}$ with $h(\bar{x}_1) = \bar{x}_2$.

Let $I_1, I_2$ be interpretations and let $h$ be a mapping from $\Delta^{I_1}$ to $\Delta^{I_2}$. The image of $h$, denoted $\text{img}(h)$, is the set $\{b \in \Delta^{I_2} \mid \exists a : h(a) = b\}$. Let $g$ be a homomorphism from an ABox $A$ to a universal model $U_{A,O}$. Then let $g^*$ be defined to be a mapping from $\text{ind}(A)$ to $\text{ind}(O)$ by setting $g^*(a) = b$ when $g(a)$ is a trace of shape $bw$ for $w$ a potentially empty sequence $r_1C_1 \ldots r_nC_n$.

Direct Products. The direct product of interpretations $I_1$ and $I_2$ is the interpretation $I_1 \times I_2$ defined as

$$\{\top(a_1, a_2) \mid a_i \in \Delta^{I_i} \text{ for } i \in \{1, 2\}\} \cup \{A(a_1, a_2) \mid A(a_1) \in I_1, a_2 \in I_2\} \cup \{r((a_1, a_2), (b_1, b_2)) \mid r(a_1, b_1) \in I_1, a_2 \in I_2\} \text{ for } i \in \{1, 2\}.$$ 

If $\tilde{A} = (d_1, \ldots, d_n) \in \Delta^{I_i} \text{ for } i \in \{1, 2\}$, then we use $\tilde{A} \otimes \tilde{B}$ to denote the tuple $((d_1, 1, d_2), \ldots, (d_1, n, d_2))$.

The following are some basic facts about products that are straightforward to show.

Lemma 5. Let $T, I_1$ and $I_2$ be interpretations. Then

1. for $i \in \{1, 2\}$ there is a homomorphism $h$ from $I_1 \times I_2$ to $I_i$ such that $h(d_1, d_2) = d_i$ for all $(d_1, d_2) \in \Delta^{I_1} \times \Delta^{I_2}$;
2. if for all $i \in \{1, 2\}$ there is a homomorphism $h_i$ from $I_i$ to $I_1$ with $h_i(d) = d_i$, then there is a homomorphism $h$ from $I_1 \times I_2$ with $h(d) = (d_1, d_2)$;
3. if $I_1$ and $I_2$ are models of $O$, then so is $I_1 \times I_2$.

Simulations. A simulation from interpretation $I_1$ to interpretation $I_2$ is a relation $S \subseteq \Delta^{I_1} \times \Delta^{I_2}$ that satisfies the following conditions:

1. if $A(d) \in I_1$ and $(d, e) \in S$, then $A(e) \in I_2$;
2. if $r(d, d') \in I_1$ and $(d, e) \in S$, then there is an $r(e, e') \in I_2$ with $(d', e') \in S'$.

We further say that $S$ is a simulation from $I_1$ to $I_2$ if $S$ is a simulation from $I_1$ to $I_2$ with $(d_1, d_2) \in S$; we write $I_1, I_2 \models S$ if such a simulation $S$ exists. The following are some basic facts about simulations and homomorphisms that are standard to proof.

Lemma 6. Let $I_1$ be an interpretation and $d_i \in \Delta^{I_i}$, $i \in \{1, 2\}$. Then $I_1 \models d_1 \models \Delta^{I_2}$, and $d_1 \models \Delta^{I_2}$. Then $I_1, I_2 \models S$, $I_1, I_2 \models S$ implies $d_1 \models \Delta^{I_2}$, and $d_1 \models \Delta^{I_2}$.

Lemma 7. Let $O$ be an $\mathcal{EL}'$-ontology and $A_i, A_i$ ABoxes. Then

1. every homomorphism $h$ from $A_1$ to $A_2$ can be extended to a homomorphism $h'$ from $U_{A_1,O}$ to $U_{A_2,O}$ such that $\alpha \notin \text{ind}(A_1)$ is a trace $a = b_w$, then $h'(a)$ is a trace of shape $h(a)w'$, and in particular, $\alpha \notin \text{ind}(A_1)$ implies $h'(a) \notin \text{ind}(A_2)$;
2. if $a_1, a_2 \subseteq A_2, a_2$, then $U_{A_1, O} \models A_2, a_2 \subseteq U_{A_2, O}, a_2$.

Lemma 8. Let $A_1, A_2$ be ABoxes, $a_i \in \text{ind}(A_i)$ for $i \in \{1, 2\}$, and $O$ an $\mathcal{EL}'$-ontology. If there is a simulation $S$ from $A_1$ to $A_2$ with $(a_1, a_2) \in S$, then $A_1, O \models C(a_1)$ implies $A_2, O \models C(a_2)$ for all $\mathcal{EL}'$-concepts $C$.

Now we conclude with a proof of a central property of the 3-compact model.

Lemma 2. Every cycle in $C^3_{A,O}$ of length at most three consists only of individuals from $\text{ind}(A)$.

Proof. The statement is clear by construction of $C^3_{A,O}$ for cycles of length 1. It is also clear for cycles of length 2 since for any pair of individuals of which at least one is of the form $c_{a,i,r,C}$, the ABox $C^3_{A,O}$ contains at most one assertion that involves both of them.

Now for cycles of length 3. Assume to the contrary of what is to be shown that $C^3_{A,O}$ contains a cycle of length 3 that contains an individual not from $\text{ind}(A)$. First assume that there is an individual $a \in \text{ind}(A)$ on the cycle. Since all individuals of the form $c_{a,i,r,C}$ that are on the cycle are adjacent to $a$ on the cycle, $b = a$ and $i = 0$ for all such $c_{b,i,r,C}$. This implies that $a$ is the only individual from $a \in \text{ind}(A)$ on the cycle. But then the cycle contains two distinct individuals of the form $c_{a,b,r,C}$ that are connected by an edge, which is never the case in $C^3_{A,O}$.

Now assume that the cycle contains only individuals of the form $c_{b,i,r,C}$. Then all these individuals are connected in $C^3_{A,O}$ by an edge. This is impossible due to the use of the index $i$ in the construction of $C^3_{A,O}$. □
To simplify some of our proofs, it is useful to consider a strengthening of the condition of symmetry-freeeness from the definition of CQ_{sf}. We say that a CQ is strongly symmetry-free if \( \varphi \) contains atoms \( r(y_1,x), r(y_2,x) \), then \( x \) is an answer variable or one of the atoms occurs on a cycle. Thus, the possibility that \( \varphi \) contains an atom \( s(z,z) \) for some \( z \in \{ x, y_1, y_2 \} \) from the original definition of symmetry-freeenness is excluded. In the following, we show that every CQ from CQ_{sf} is equivalent to one that is strongly symmetry-free.

Two CQs \( q_1 \) and \( q_2 \) of the same arity are equivalent, written \( q_1 \equiv q_2 \), if \( q_1 \equiv q_2 \) for the empty ontology.

**Lemma 9.** For every \( q \in \text{CQ}_{sf} \), there is a \( q' \in \text{CQ}_{sf} \) such that \( q \equiv q' \) and \( q' \) is strongly symmetry-free.

**Proof.** To construct \( q' \), start from \( q \). Then introduce, for every quantified variable \( x \) that occurs in an atom of the form \( r(x,y) \) in \( q \), a fresh quantified variable \( x' \) and add the atom \( B(x',x) \) for every atom \( B(x,y) \) in \( q \) and \( S(y,x') \) for every atom \( S(y,x) \) in \( q \). We say that \( x' \) is a copy of \( x \).

It is clear that \( q \equiv q' \) as there is a homomorphism from \( q' \) to \( q \) and by Lemma 4. Moreover, \( q' \) is strongly symmetry-free. To see this, assume that \( q' \) contains atoms \( r(x_1,y), r(x_2,y) \) with \( y \) a quantified variable. We distinguish three cases.

First assume \( r(x_1,y), r(x_2,y) \) are already in \( q \). Since \( q \) is symmetry-free, \( q \) also contains an atom of the form \( s(x_1,x_1), s(x_2,x_2) \), or \( s(y,y) \). In the first case, \( q' \) contains the cycle \( r(x_1,y), r(y,x_1)' \), \( s(x_1,x_1) \) and thus atom \( r(x_1,y) \) occurs on a cycle in \( q' \). In the second case, atom \( r(x_2,y) \) occurs on a cycle in \( q' \) and in the third case both atoms do.

Now assume that \( r(x_1,y) \) is not in \( q \). Then \( x_1 \) is a copy of a variable \( x_1' \) in \( q \) or \( y \) is a copy of a variable \( y_0 \) in \( q \) (or both). In the first case, \( q' \) contains a cycle of the form \( r(x_1,y)' \), \( r(y,x_1)' \), \( s(x_1,x_1') \) and thus atom \( r(x_1,y) \) occurs on a cycle in \( q' \). In the second case, \( q' \) contains a cycle of the form \( r(x_1,y)' \), \( s(y,y_0)' \), \( r(y_0,x_1) \) and thus again atom \( r(x_1,y) \) occurs on a cycle in \( q' \).

The case that \( r(x_2,y) \) is not in \( q \) is symmetric.

It might seem that we should change the definition of the class CQ_{sf} to be based on strong symmetry-freeeness. This, however, is not possible because the CQ produced by the first version of refine is only symmetry-free, but not strongly symmetry-free. We can also not use the construction from the proof of Lemma 9 as part of refine to attain strong symmetry-freeeness as this interferes with minimization, that is, it would no longer be guaranteed that \( |\text{var}(q_H)| \leq |\text{var}(q_T)| \).

**B Proof of Lemma 1**

The following is easy to show, details are omitted.

**Lemma 10.** Let \( A \) be an ABox and \( O \) an \( \mathcal{EL}' \)-ontology. Further let \( c_{a,i,r,C} \in \Delta^3_{A,O} \) and \( trC \in \Delta^4_{A,O} \). Then \( c_{a,i,r,C} \in \Delta^3_{A,O} \) and \( trC \in \Delta^4_{A,O} \). Then \( c_{a,i,r,C} \in \Delta^3_{A,O} \) and \( trC \in \Delta^4_{A,O} \).

**Lemma 1.** Let \( A \) be an ABox and \( O \) an \( \mathcal{EL}' \)-ontology. Then \( C_{A,O}^A \) is a model of \( A \) and \( O \) such that for every CQ \( q \in \text{CQ}_{sf} \) and \( \bar{a} \in \text{ind}(\mathcal{A})[\bar{x}] \), \( C_{A,O}^A \models q(\bar{a}) \iff A,O \models q(\bar{a}) \).

**Proof.** It is not difficult to prove that \( C_{A,O}^A \) is indeed a model of \( A \) and \( O \), details are omitted. Let \( q(\bar{x}) \in \text{CQ}_{sf} \) and \( \bar{a} \in \text{ind}(\mathcal{A})[\bar{x}] \). We have to show that \( C_{A,O}^A \models q(\bar{a}) \iff A,O \models q(\bar{a}) \). By Lemma 9, we can assume w.l.o.g. that \( q \) is strongly symmetry-free. The “if” direction is trivial by Lemma 3 and because there is an obvious homomorphism from \( U_{A,O} \) to \( C_{A,O}^A \) that is the identity on \( \text{ind}(\mathcal{A}) \). We thus concentrate on “only if”. Let \( q(\bar{x}) \in \text{CQ}_{sf} \) and assume that \( C_{A,O}^A \models q(\bar{a}) \). Then there is a homomorphism \( h \) from \( q \) to \( C_{A,O}^A \) with \( h(\bar{x}) = \bar{a} \). In what follows, we construct a homomorphism \( g \) from \( q \) to \( U_{A,O} \) with \( g(\bar{x}) = \bar{a} \). Thus, \( A,O \models q(\bar{a}) \) as required.

To start the definition of \( g \), set \( g(x) = h(x) \) whenever \( h(x) \in \text{ind}(\mathcal{A}) \). It follows from the construction of \( C_{A,O}^A \) and \( U_{A,O} \) that \( g \) is a homomorphism from the restriction of \( q \) to the domain of \( g \) to \( U_{A,O} \).

As a consequence of Lemma 2, if a variable \( x \) occurs on a cycle of length 1 or 2 in \( q \), then \( g(x) \) is defined at this point. We next define \( g(x) \) for all variables \( x_0 \) that are on a cycle \( R_{0}(x_0,x_1), R_{1}(x_1,x_2), R_{0}(x_2,x_0) \) of length 3 in \( q \). Assume that \( g(x_0) \) was not yet defined. It follows from Lemma 2 that \( h(x_1) = h(x_2) \in \text{ind}(\mathcal{A}) \), and thus \( A \) contains a reflexive \( R_{1} \)-cycle on \( h(x_1) \), \( R_{0} = R_{0}^{-1} \), and \( h(x_0) \notin \text{ind}(\mathcal{A}) \). Let \( h(x_1) = a \). By construction of \( C_{A,O}^A \), \( h(x_0) = c_{a,o,c,r} \) for some \( C \) and where \( r = R_{0} \) if \( R_{0} \) is a role name and \( r = R_{2} \) otherwise. Set \( g(x_0) = arC \). Also after the extension, \( g \) is a homomorphism from the restriction of \( q \) to the (now extended) domain of \( g \) to \( U_{A,O} \). This is easily seen to be a consequence of the definition of the extension and of the construction of \( C_{A,O}^A \) and \( U_{A,O} \).

At this point, \( q(x) \) is defined for all variables \( x \) that occur on a cycle in \( q \). Assume that \( x \) is such a variable. If \( x \) is an answer variable, then \( g(x) \) is clearly already defined. Otherwise chordality of \( q \) implies that \( x \) also occurs on a cycle of length at most 3 and thus \( g(x) \) has been defined above. It remains to define \( g(x) \) for variables \( x \) that do not occur on a cycle.

Let \( q' \) be the subquery of \( q \) consisting of all atoms that contain at least one variable \( x \) with \( g(x) \) undefined at this point. We argue that

1. \( q' \) is a disjoint union of ditrees such that
2. if \( g(x) \) is defined for a variable \( x \) in \( q' \), then \( x \) is the root of a ditree.

First note that

(*) none of the atoms \( r(x_1,x_2) \) in \( q' \) occur on a cycle in \( q \).

In fact, if an atom \( r(x_1,x_2) \) in \( q' \) occurs on a cycle in \( q \), then \( g(x_1) \) and \( g(x_2) \) are already defined and thus \( r(x_1,x_2) \) is not part of \( q' \).

For Point 1, first observe that \( q' \) does not contain a cycle. In fact, any cycle \( C \) in \( q' \) is also a cycle in \( q \), so by (*) \( q' \) does not contain any of the atoms in \( C \). To establish Point 1, it remains to show that \( q' \) contains no atoms \( r_1(x_1,y), r_2(x_2,y) \) with \( x_1 \neq x_2 \). By definition of \( q' \), one of \( g(x_1), g(y) \) and one of \( g(x_2), g(y) \) must be undefined. There are two cases:

- \( g(y) \) is undefined.
- Then \( y \) is a quantified variable and \( r_1 = r_2 \), the latter because \( r_1(h(x_1),h(y)), r_2(h(x_2),h(y)) \subseteq C_{A,O}^A \).
$h(y) \notin \text{ind}(A)$ as $g(y)$ is undefined, and by definition of $C^3_{A,O}$. Moreover, by $(\ast)$ none of $r_1(x_1,y), r_2(x_2,y)$ occurs on a cycle in $q$. Thus, $q$ is not strongly symmetry-free, a contradiction.

- $g(x_1), g(x_2)$ are undefined.

Then $h(x_1)$ and $h(x_2)$ are not in $\text{ind}(A)$. From $r_1(h(x_1), h(y)), r_2(h(x_2), h(y)) \subseteq C^3_{A,O}$ and the definition of $C^3_{A,O}$, we obtain $h(y) \notin \text{ind}(A)$. We can now argue as in the previous case that $y$ is a quantified variable and $r_1 = r_2$, and again obtain a contradiction to $q$ being strongly symmetry-free.

Now for Point 2. It suffices to observe that if $r(x, y)$ is an atom in $q'$, then $g(y)$ is undefined. Assume to the contrary that $g(y)$ is already defined. By choice of $q'$, it follows that $g(x)$ is undefined. As $g(y)$ is defined, one of the following applies:

- $h(y) \in \text{ind}(A)$.
  Then $r(h(x), h(y)) \in C^3_{A,O}$ and the definition of $C^3_{A,O}$ imply that $h(x) \in \text{ind}(A)$, in contradiction to $g(x)$ being undefined.

- there is an atom $s(x', y)$ in $q$ with $h(x') \in \text{ind}(A)$, and $h(y) \notin \text{ind}(A)$.
  From $h(x') \in \text{ind}(A)$, $h(y) \notin \text{ind}(A)$, $s(h(x'), h(y)), r(h(x), h(y)) \in C^3_{A,O}$ and the definition of $C^3_{A,O}$, we obtain $h(x') = h(x)$. But then $h(x) \in \text{ind}(A)$, in contradiction to $g(x)$ not yet being defined.

We next traverse the dites in $q'$ in a top-down fashion to extend $g$. The initial piece of $g$ constructed so far is such that for all variables $x$, $h(x) = c_{a,1,r,C}$ implies that $g(x)$ is of the form $trC$. We shall maintain this invariant during the extension of $g$.

To extend $g$, repeatedly and exhaustively choose atoms $r(x, y) \in q'$ with $g(x)$ defined and $g(y)$ undefined. Then $h(y) \notin \text{ind}(A)$ and thus $h(y)$ has the form $c_{a,1,r,C}$. Define $g(y)$ to be $g(y')$. If $h(x) \in \text{ind}(A)$, then it is immediate by definition of $C^3_{A,O}$ and $U_{A,O}$ that $r(h(x), h(y)) \in C^3_{A,O}$ implies $r(g(x), g(y)) \in U_{A,O}$. If $h(x) \notin \text{ind}(A)$, we need to additionally invoke Lemma 10, applied to $h(x) = c_{a',x',r',C'}$, and to $g(x) = tr'C'$. By construction, $g$ satisfies all binary atoms in $q'$ and thus in $g$. All unary atoms are satisfied, too, because of the invariant mentioned above and by definition of $C^3_{A,O}$ and $U_{A,O}$. \qed

C Proof of Theorem 2

Theorem 2.

1. $AQ^\vee\text{-queries are not polynomial query learnable under conjunctive ontologies using only membership queries;}$

2. $ELQ\text{-queries are not polynomial time learnable (without ontologies) using only CQ-equivalence queries unless } P = NP.$

Proof. For Point 1, we use a proof strategy that is inspired by basic lower bound proofs for abstract learning problems due to Angluin [Angluin, 1987b]. Here, it is convenient to view the oracle as an adversary who maintains a set $S$ of candidate target concepts that the learner cannot distinguish based on the queries made so far. In our case, $S$ of $AQ^\vee$-queries. We have to choose $S$ and the ontology $O$ carefully so that each membership query removes only few candidate targets and after a polynomial number of queries there is still more than one candidate that the learner cannot distinguish.

For each $n \geq 1$, let

$$O_n = \{A_i \cap A'_i \subseteq A_1 \cap A'_1 \cap \cdots \cap A_n \cap A'_n \mid 1 \leq i \leq n\}$$

and

$$S_n = \{q(x) \leftarrow \alpha_1(x) \land \cdots \land \alpha_n(x) \mid \alpha_i \in \{A_i, A'_i\} \text{ for all } i \text{ with } 1 \leq i \leq n\}.$$ 

The set $S_n$ contains $2^n$ queries.

Assume to the contrary of what is to be shown that $AQ^\vee$-queries are polynomial query learnable under conjunctive ontologies. Then there exists a learning algorithm and polynomial $p$ such that the number of membership queries is bounded by $p(n_1, n_2)$, where $n_1$ is the size of the target query $q_T$ and $n_2$ is the size of the conjunctive ontology. We choose $n$ such that $2^n > p(r_1(n), r_2(n))$, where $r_1$ is a polynomial such that every query $q(x) \in S_n$ satisfies $|\{q(x)\}| = r_1(n)$ and $r_2$ is a polynomial such that $r_2(m) > |O_n|$ for every $m \geq 1$.

Now, consider a membership query posed by the learning algorithm with ABox and answer variable $(A, a)$. The oracle responds as follows:

1. if $A, O_n \models q(a)$ for no $q(x) \in S_n$, then answer no
2. if $A, O_n \models q(a)$ for a single $q(x) \in S_n$, then answer no and remove $q(x)$ from $S_n$
3. if $A, O_n \models q(a)$ for more than one $q(x) \in S_n$, then answer yes.

Note that the third response is consistent since $A$ must then contain $A_i(a)$ and $A'_i(a)$ for some $i$ and thus $O_n$ implies that $a$ is an answer to every query in $S_n$. Moreover the answers are always correct with respect to the updated set $S_n$. Thus the learner cannot distinguish the remaining candidate queries by answers to queries posed to far.

It follows that the learning algorithm removes at most $p(r_1(n), r_2(n))$ many queries from $S_n$. By the choice of $n$, at least two candidate concepts remain in $S_n$ after the algorithm is finished. Thus the learner cannot distinguish between them and we have derived a contradiction.

For Point 2, we exploit a classic connection between active learning with equivalence queries and certain separability problems. We start by recalling the latter. A labeled KB takes the form $K = (\mathcal{O}, A, P, N)$ with $\mathcal{O}$ an ontology, $A$ an ABox, and $P, N \subseteq \text{ind}(A)^n$ sets of positive and negative examples, respectively, all of them tuples of the same length $n$. A query $q(x)$ of arity $n$ separates $K$ if

1. $A, \mathcal{O} \models q(\bar{a})$ for all $\bar{a} \in P$, and
2. $A, \mathcal{O} \not\models q(\bar{a})$ for all $\bar{a} \in N$.
Every choice of ontology language $\mathcal{L}$ and query language $\mathcal{Q}$ gives rise to an $\mathcal{L}$, $\mathcal{Q}$-separability problem which is to decide, given a labeled KB $\mathcal{K} = (\mathcal{O}, \mathcal{A}, P, N)$ with $\mathcal{O}$ formulated in $\mathcal{L}$, whether there is a query $q(x) \in \mathcal{Q}$ which separates $P$ and $N$. We are going to concentrate on the case where the ontology is empty, which we simply refer to as $\mathcal{Q}$-separability. For simplicity, we then drop the ontology from labeled KBs. It was shown in [Jung et al., 2020; Funk, 2019] that ELQ-separability is NP-hard. An analysis of the proof reveals a class $\mathcal{C}$ of labeled KBs $(\mathcal{A}, P, N)$ for which ELQ-separability is NP-hard and a polynomial $t$ such that the following conditions are satisfied:

1. if $\mathcal{K}$ is ELQ-separable, then there is a separating ELQ-query of size $t(n)$, where $n = ||\mathcal{K}||$;
2. $\mathcal{A}$ is a disjoint union of ditrees of depth 1 and the elements of $P$ and $N$ are the roots of these ditrees;
3. only a single role name $r$ is used.

Condition 2 implies the following.

Claim 1. Given an ABox $\mathcal{A}$ that satisfies the properties given in Condition 2, an $a \in \text{ind}(\mathcal{A})$, and a unary CQ $q(x)$, it can be decided in polynomial time whether $\mathcal{A} \models q(a)$.

We only sketch the proof. To check whether there is a homomorphism $h$ from $q(x)$ to $\mathcal{A}$ with $h(x) = a$, we treat each maximal connected component of $q$ separately. For the component that contains $x$, we start with setting $h(x) = a$. We then repeatedly extend $h$ to variables $y$ such that $q$ contains some atom $r(y, z)$ or $r(z, y)$ with $h(z)$ already defined. If there are atoms of both forms, then $\mathcal{A} \not\models q(a)$. If there is an atom $r(y, z)$ and $h(z)$ is a non-root in $\mathcal{A}$, then $h(y)$ is the unique predecessor of $h(z)$ in $\mathcal{A}$, and if $h(z)$ is a root, then $\mathcal{A} \not\models q(a)$. If there is an atom $r(z, y)$, then $\mathcal{A} \not\models q(a)$ if $h(z)$ is a non-root. Otherwise, we consider all atoms $A(z) \in q$ and set $h(y)$ to some successor $c$ of $h(z)$ in $\mathcal{A}$ such that $A(c) \in \mathcal{A}$ for all these atoms; if there is no such successor, then again $\mathcal{A} \not\models q(a)$. For components that do not contain $x$ we start with an arbitrarily chosen variable, iterate over all individuals in $\mathcal{A}$ as targets, and for each target proceed as described above.

Now assume that ELQ-queries are polynomial time learnable using only CQ-equivalence queries. Then there exists a learning algorithm $L$ for ELQ-queries and a polynomial $p$ such that at any time, the running time of $L$ so far is bounded by $p(t(n), n)$, where $t$ is the size of the target query $q_T$ and $n$ is the size of the largest counterexample seen so far. We show how to use $L$ to construct an algorithm $L'$ that decides ELQ-separability for the class of labeled KBs $\mathcal{C}$ in polynomial time.

The new algorithm $L'$ takes as input a labeled KB $\mathcal{K} = (\mathcal{A}, P, N) \in \mathcal{C}$. Let $n = ||\mathcal{K}||$. $L'$ then runs $L$ for at most $p(t(n), n)$ steps.

Whenever $L$ asks a CQ-equivalence query with $q_H(x)$ as the hypothesis, $L'$ answers it by testing whether $q_H$ separates $\mathcal{K}$. More precisely, $L'$ checks whether $\mathcal{A} \models q_H(a)$ for all $a \in P$ and $\mathcal{A} \not\models q_H(a)$ for all $a \in N$. Note that, by the above claim, this is possible in time polynomial in $||\mathcal{K}||$. If a check fails for some $a \in P$, then $L'$ answers the equivalence query by giving $(\mathcal{A}, a)$ as a positive counterexample to $L$. If a check fails for some $a \in N$, then $L'$ answers the equivalence query by giving $(\mathcal{A}, a)$ as a negative counterexample to $L$. If all checks succeed, then $L'$ terminates and returns “separable”.

If $L$ terminates returning a learned ELQ-query $q_H$, then $L'$ tests whether $q_H$ separates $\mathcal{K}$. If this is the case, then $L'$ returns “separable”. If $L'$ does not terminate within $p(t(n), n)$ steps or returns a query that does not separate $\mathcal{K}$, then $L'$ returns “not separable”. The following claim shows correctness of $L'$.

Claim 2. $L'$ returns “separable” iff $\mathcal{K}$ is ELQ-separable.

Proof of Claim 2. The “only if” direction follows directly from the fact that $L'$ only returns “separable” if there is a separating CQ. For the “if” direction, assume that $\mathcal{K}$ is not ELQ-separable. By Condition 1 above, there is an ELQ-query $q_T$ of size $t(n)$ that separates $\mathcal{K}$. Note that $q_T$ is consistent with the counterexamples that $L'$ provides to $L$. Since by assumption $L'$ is able to learn any $\mathcal{Q}$-query of size $t(n)$ with counterexamples of size $n \leq p(t(n), n)$ steps, it must within this number of steps either ask an equivalence query with a hypothesis that separates $\mathcal{K}$ (but may not be equivalent to $q_T$), or return a $\mathcal{Q}$-query that is equivalent to $q_T$. In both cases $L'$ returns “separable”.

D Proofs for Section 3.1

It is well-known that every $\mathcal{EL'}$-ontology $\mathcal{O}$ can be converted into an $\mathcal{EL'}$-ontology $\mathcal{O}'$ in normal form by introducing additional concept names [Baader et al., 2017]. For the reduction, it is convenient to use a suitable form of conversion. An $\mathcal{EL'}$-ontology $\mathcal{O}_2$ is a conservative extension of an $\mathcal{EL'}$-ontology $\mathcal{O}_1$ if $\text{sig}((\mathcal{O}_1) \subseteq \text{sig}(\mathcal{O}_2)$, every model of $\mathcal{O}_2$ is a model of $\mathcal{O}_1$, and for every model $\mathcal{I}_1$ of $\mathcal{O}_1$, there exists a model $\mathcal{I}_2$ of $\mathcal{O}_2$ such that $\mathcal{S}^{\mathcal{E}_1} = \mathcal{S}^{\mathcal{E}_2}$ for all symbols $S \notin \text{sig}(\mathcal{O}_2) \setminus \text{sig}(\mathcal{O}_1)$.

Lemma 11. Given an $\mathcal{EL'}$-ontology $\mathcal{O}$, one can compute in polynomial time an $\mathcal{EL'}$-ontology $\mathcal{O}'$ in normal form such that:

1. $\mathcal{O}'$ is a conservative extension of $\mathcal{O}$;
2. $\text{sig}(\mathcal{O}') = \text{sig}(\mathcal{O}) \cup \{X_C \mid C \in \text{sub}(\mathcal{O})\}$,
3. $\mathcal{O}' \models X_C \equiv C$ for each $C \in \text{sub}(\mathcal{O})$.

Proof. Introduce a fresh concept name $X_C$ for every $C \in \text{sub}(\mathcal{O})$ and define $\mathcal{O}'$ to contain, for every $C \in \text{sub}(\mathcal{O})$, the following concept inclusions and range restrictions:

- $X_C \sqsubseteq C, C \sqsubseteq X_C$ if $C$ is a concept name or $\top$;
- $X_C \sqsubseteq X_{D_1}, X_C \sqsubseteq X_{D_2},$ and $X_{D_1} \sqcap X_{D_2} \sqsubseteq X_C$ if $C = D_1 \sqcap D_2$;
- $X_C \sqsubseteq \exists r.X_D$ and $\exists r.X_D \sqsubseteq X_C$ if $C = \exists r.D$;
- $X_C \subseteq X_D$ for each concept inclusion $C \sqsubseteq D \in \mathcal{O}$;
- $\exists r.\top \sqsubseteq X_C$ for each range restriction $\exists r.\top \sqsubseteq C \in \mathcal{O}$.

$\mathcal{O}'$ is in normal form and can be computed in polynomial time. Moreover, it can be verified that Points 1–3 hold. □
A CQ $q'$ can be obtained from a CQ $q$ by attaching ditrees if $q'$ can be constructed by choosing variables $x_1, \ldots, x_n$ from $q$ and Boolean ditree CQs $q_1, \ldots, q_n$ whose sets of variables are pairwise disjoint and disjoint from the set of variables in $q$, and then taking the union of $q$ and $q_1, \ldots, q_n$, identifying the root of $q_i$ with $x_i$ for $1 \leq i \leq n$. A class of CQs $Q$ is closed under attaching ditrees if every CQ $q'$ that can be obtained from a $q \in Q$ by attaching ditrees is also in $Q$. Note that all of $Q^{ed}$, ELQ, and ELIQ$^{ed}$ are closed under attaching ditrees. We prove the following generalization of Proposition 1.

**Proposition 2.** Let $Q$ be a class of CQs closed under attaching ditrees. If queries in $Q$ are polynomial time learnable under E$L^+$-ontologies in normal form using membership and equivalence queries, then the same is true for unrestricted E$L^+$-ontologies.

**Proof.** Let $L'$ be a polynomial time learning algorithm for $Q$ under ontologies in normal form. We show how $L'$ can be modified into an algorithm $L$ that is able to learn $Q$ under unrestricted ontologies in polynomial time.

Given an E$L^+$-ontology $O$ and a finite $\Sigma \subseteq N_C \cup N_R$ such that $\text{sig}(O) \subseteq \Sigma$ and $\text{sig}(q_t) \subseteq \Sigma$, algorithm $L$ first computes the ontology $O'$ in normal form as per Lemma 11, choosing the fresh concept names $X_C$ so that they are not from $\Sigma$. It then runs $L'$ on $O'$ and $\Sigma' = \Sigma \cup \{X_C \mid C \in \text{sub}(O)\}$; note that $\text{sig}(O') \subseteq \Sigma'$ as required. In contrast to the learning algorithm, the oracle still works with the original ontology $O$. To bridge this gap, algorithm $L$ adopts modifications during the run of $L'$, as follows.

First, whenever $L'$ asks a membership query $A', O' \models q_I(\bar{a})$, $L$ instead asks the membership query $A, O \models q_T(\bar{a})$, where $A$ is obtained from $A'$ by replacing each assertion $X_C(x)$ with the $C$ viewed as an ABox, identifying the root with $x$.

By the following claim, the answer to the modified membership query coincides with that to the original query.

**Claim 1.** $A', O' \models q_I(\bar{a})$ iff $A, O \models q_T(\bar{a})$ for all CQs $q$ that use only symbols from $\Sigma$.

**Proof of Claim 1.** For “if”, suppose that $A, O \models q_T(\bar{a})$ and let $I$ be a model of $A$ and $O'$. Then $I$ is a model of $O'$ since $O'$ is a conservative extension of $O$. By Property 3 of Lemma 11, $I$ is a model of $A$. Hence $I \models q_I(\bar{a})$ as required. For “only if”, suppose that $A', O' \models q_I(\bar{a})$ and let $I$ be a model of $A$ and $O$. Since $O'$ is a conservative extension of $O$, there is a model $I'$ of $O'$ that coincides with $I$ on all symbols from $\Sigma$. By Property 3 of Lemma 11, $I'$ is a model of $A'$. Since $\text{sig}(q) \subseteq \Sigma$ and $I'$ and $I$ coincide on $\Sigma$, it follows that $I \models q_I(\bar{a})$ as required.

Second, whenever $L'$ asks an equivalence query $q^t_{IT} \equiv_{O'} q_T$, $L$ instead asks the equivalence query $q^t_{IT} \equiv_{O} q_T$, where $q^t_{IT}$ is obtained from $q^t_{IT}$ by replacing each assertion $X_C(x)$ with the Boolean ditree CQ obtained from ELQ $C$ by quantifying the root, identifying the root with $x$. Since $Q$ is closed under attaching ditrees, $q^t_{IT}$ can be used in an equivalence query. Furthermore, when the counterexample returned is $A$, the algorithm replaces it with the restriction $A|_{\Sigma}$ to signature $\Sigma$ before passing it on to $L'$.

Applying the following claim to both $q^t_{IT}$ and $q^t_{IT} = q_T$, the answer to the modified equivalence query coincides with that to the original query.

**Claim 2.** Let $q'$ be a CQ that uses only symbols from $\Sigma'$ and let $q$ be obtained from $q'$ by replacing each assertion $X_C(x)$ with the Boolean ditree CQ obtained from ELQ $C$ by quantifying the root, the root identified with $x$. Then $A|_{\Sigma} : O' \models q^t_{IT}(\bar{a})$ iff $A, O \models q_T(\bar{a})$ for all ABoxes $A$.

**Proof of Claim 2.** For “if”, suppose $A, O \models q_T(\bar{a})$ and let $I$ be a model of $A|_{\Sigma}$ and $O'$. Since $q$ and $O$ contain only symbols from $\Sigma$, $A|_{\Sigma}, O \models q_T(\bar{a})$. Since $O'$ is a conservative extension of $O$, $I$ is also a model of $O$. Thus $I \models q_T(\bar{a})$ by Property 3 of Lemma 11, $I \models q_T(\bar{a})$ follows as required.

For “only if”, suppose $A|_{\Sigma}, O' \models q_T(\bar{a})$ and let $I$ be a model of $A$ and $O$. Since $O$ contains only symbols from $\Sigma$, $I|_{\Sigma}$ is a model of $A|_{\Sigma}$ and since $O'$ is a conservative extension of $O$, there is a model $I'$ of $O'$ that coincides on all symbols from $\Sigma$ with $I|_{\Sigma}$. Thus $I' \models q_T(\bar{a})$ and by Property 3 of Lemma 11, $I' \models q_T(\bar{a})$ since $q$ uses only symbols from $\Sigma$, as required. $\square$

## E Proofs for Section 3.3

We analyze central properties of the refine subroutine. Recall that we first construct a sequence $(B_1, b_1), (B_2, b_2), \ldots$ using the Expand and Minimize steps. With $B_i$, $i \geq 1$, we denote the result of only applying the Expand step to $B_i$, but not the Minimize step. Also recall that the fresh individuals introduced in $B_i$ are denoted with $a^i$ in case that the original individual was $a$. The following can easily be shown.

**Lemma 12.** Let $i \geq 1$. Then $B_i, a \preceq B_i, a$ and $B_i, a \preceq B_i, a$, for all $a \in \text{ind}(B_i)$, and $B_i, a^i \preceq B_i, a$ and $B_i, a \preceq B_i, a^i$, for all fresh individuals $a^i$.

We start with proving properties of the Expand/Minimize phase.

**Lemma 13.** For all $i \geq 1$, the following properties hold:

1. $B_i, O \models q_T(b_i)$.
2. $B_i, O \models q_T(b)$ for some $b \in \tau_i$.

**Proof.** We prove both points simultaneously by induction on $i$. For Point 1, the case $i = 1$ is immediate since $(B_1, b_1) = \text{minimize}(A, a)$ and $A, O \models q_T(\bar{a})$, and the case $i > 1$ is an immediate consequence of the inductive hypothesis (Point 2), the choice of $b_i$, and the definition of the Minimize step.

For Point 2, the induction start and step are identical. Thus let $i \geq 1$. Assume that $B_i$ was obtained from $B_i$ by expanding cycle $R_0(a_0, a_1), \ldots, R_{n-1}(a_{n-1}, a_n)$. By Point 1, there is a homomorphism $h$ from $q_T$ to $U_{B_i, O}$ with $h(x) = b_i$. We construct a homomorphism $\varphi$ from $q_T$ to $U_{B_i, O}$ with $\varphi(x) = b$ for some $b \in \tau_i$, which yields $B_i, O \models q_T(\bar{a})$ as desired. Let us partition $\varphi(q_T)$ into sets $M_0, M_1, M_2$ such that:

- $x \in M_0$ if $h(x) \in \{a_0, \ldots, a_{n-1}\}$, that is, $h(x)$ lies on the expanded cycle;
- $x \in M_1$ if $h(x) \notin \text{ind}(B_i)$, that is, $h(x)$ is in the part of $U_{B_i, O}$ generated by existential quantification;
• all other variables are in \( M_2 \).

We start with setting
\[
g(x) = h(x) \quad \text{for all } x \in M_2.
\]

To define \( g(x) \) for the variables in \( x \in M_0 \), we first construct an auxiliary query \( q_T \) of treewidth 1. If \( Q \in \{ \text{ELQ}, \text{ELIQ}^{sf} \} \), then \( q_T^0 \) is simply \( q_T \). Now assume that \( Q = \text{CQ}^{sf} \). Then \( q_T^0 \) is obtained by setting the variables in \( M_0 \) and then exhaustively choosing and identifying variables \( x_1, x_2 \) such that

1. there is a cycle \( R_0(z_0, y_1), R_1(y_1, y_2), R_2(y_2, y_0) \) with \( \{ x_1, x_2 \} \subseteq \{ y_0, y_1, y_2 \} \subseteq M_0 \) and
2. \( h(x_1) = h(x_2) \)

The result of identifying an answer variable and a quantified variable in \( C \) contains a chord in \( C \). Then \( H(x_1) \) exists for each \( x_1 \in \text{var}(q_T^0) \) during the construction of \( q_T^0 \) (note that this implies \( h(y) = h(x) \)).

By definition, \( g(x) \in \{ h(x), h(x)' \} \) for all \( x \in M_0 \). Thus, \( g(x) \in \tau \), as announced.

It remains to define \( g(x) \) for the variables \( x \in M_1 \). By definition of \( M_1 \), \( h(x) \) is a trace \( cw \) with \( c \in \text{ind}(B_i) \) and \( \nu \neq \emptyset \), that is, \( x \) is mapped into the subtree below \( c \) in \( U_{B_i, \mathcal{O}} \).

Now do the following:

• if there is a path in \( q_T \) from some variable \( z \in M_0 \) to \( x \), then choose such \( z \), such that the path is shortest (thus, \( h(z) = c \) and \( g(z) \) has already been defined) and set \( g(x) = g(z) w \);

• otherwise, set \( g(x) = h(x) \).

This is well-defined since, due to Lemma 12, the following holds:

1. for each \( c \in \text{ind}(B_i) \), the subtrees below \( c \) in \( U_{B_i, \mathcal{O}} \) and in \( U_{B_i, \mathcal{O}}^f \) are identical.
2. for \( 0 \leq j < n \), the subtree below \( a_j \) in \( U_{B_j, \mathcal{O}} \) and the subtree below \( a_j' \) in \( U_{B_j, \mathcal{O}}^f \) are identical.

Set \( \overline{b} = g(\overline{x}) \). To prove that \( B_i, \mathcal{O} \models q_T(\overline{b}) \), it remains to show the following.

Claim. \( g \) is a homomorphism from \( q_T \) to \( U_{B_i, \mathcal{O}}^f \).

Proof of the claim. Let \( A(x) \) be a concept atom in \( q_T \). Then \( A(h(x)) \in U_{B_i, \mathcal{O}} \). If \( h(x) \in \text{ind}(B_i) \), then \( g(x) \) was defined such that, by Lemma 12, \( B_i, h(x) \leq B_i, g(x) \). By Lemma 7, \( U_{B_i, \mathcal{O}}^f, h(x) \leq U_{B_i, \mathcal{O}}^f, g(x) \) and thus by Lemma 6 \( A(g(x)) \in U_{B_i, \mathcal{O}}^f \). If \( h(x) \notin \text{ind}(B_j) \) then the remark before the claim and the definition of \( g(x) \) ensures that \( A(g(x)) \in U_{B_i, \mathcal{O}}^f \).

Now let \( R(x_1, x_2) \) be a role atom in \( q_T \). We distinguish cases according to \( x_1, x_2 \) belonging to \( M_0, M_1, M_2 \).

- If \( x_1, x_2 \in M_0 \), then \( q_T^0 \) contains an atom \( R(x_1', x_2') \) such that each \( x_i \) was identified with \( x_i' \) during the construction of \( q_T^0 \). If \( x_1' \neq x_2' \), then \( R(g(x_1'), g(x_2')) \in B_i^f \), as argued in the definition of \( g \) for variables from \( q_T^0 \).

- By definition and the construction of \( B_i^f \), the same is true when \( x_1 = x_2 \). We have \( g(x_1) = g(x_2) \) for \( i \in \{ 1, 2 \} \). Thus \( R(g(x_1'), g(x_2')) \in U_{B_i, \mathcal{O}}^f \) as required.

- If \( x_1, x_2 \in M_1 \), then \( h(x_1) = bv \) and \( h(x_2) = bw \) for some \( b \in \text{ind}(B_i) \) and some non-empty \( v, w \), and \( R(h(x_1), h(x_2)) \in U_{B_i, \mathcal{O}} \). By definition of \( g \), we have \( g(x_1) = bv \) and \( g(x_2) = bw \) for some \( b \in \{ b, b' \} \).
Lemma 14. For all $i \geq 1$,

1. $B_i$ is $O$-saturated;
2. if $h$ is a homomorphism from $q_T$ to $U_{B_i,O}$ with $h(\bar{x}) = \bar{b}_i$, then $\text{ind}(B_i) \subseteq \text{img}(h^*)$;
3. $B_{i+1}, \tilde{b}_{i+1} \rightarrow B_i, \tilde{b}_i$;
4. $|\text{ind}(B_{i+1})| > |\text{ind}(B_i)|$.

Proof. We prove Point 1 by induction on $i$. For $i = 1$, recall that the initial ABox $A$ is of the form $C_A^{a_{\text{ABox}}_i} \times C_A^{A_{\text{ABox}}}$ or $A_{\text{query}}$ for all uses of the subrouting refine and that $B_1, \tilde{b}_1$ minimize $(A, a)$. In the first case, both $C_A^{a_{\text{ABox}}_i}$ and $C_A^{A_{\text{ABox}}}$ are $O$-saturated and thus their product is also $O$-saturated by Lemma 5 Point 3. In the second case $A_{\text{query}}$ is $O$-saturated since it contains $A(x_0)$ for all concept names $A \in \Sigma$. Moreover, the Minimize step does not remove any concept assertions. For the induction step, suppose $B_{i+1}, \tilde{b}_{i+1} \rightarrow B_i, \tilde{b}_i$. By monotonicity, $B_i, O \models A(\tilde{a})$ where $B_i$ is the result of applying the Expand step to $B_i$ before the Minimize step. By Lemma 12, $B_i, \tilde{a} \preceq B_i, a$ and thus $B_i, O \models A(a)$ by Lemma 8. By induction, we know that $A(a) \in B_i$, and the application of the rules ensures that $A(\tilde{a}) \in B_{i+1}$.

For Point 2, let $h$ be a homomorphism from $q_T$ to $U_{B_i,O}$ with $h(\bar{x}) = \bar{b}_i$, and suppose that there is an $a \in \text{ind}(B_i)$ that is not in $\text{img}(h^*)$. Let $B'$ be the result of removing from $B_i$ all assertions that involve $a$. We show that

$(*)$ $h$ is a homomorphism from $q_T$ to $U_{B_i,O}$ which witnesses that $B', O \models q_T(\tilde{b}_i)$. Hence, $a$ is dropped during the Minimize step, in contradiction to $a \in \text{ind}(B_i)$. To see that $(*)$ holds, first note that for all $b, b' \in \text{ind}(B_i) \setminus \{a\}$, the following holds by Point 1 and construction of universal models:

(a) $A(b) \in U_{B_i,O}$ iff $A(b) \in U_{B_i,O}$;
(b) $r(b, b') \in U_{B_i,O}$ iff $r(b, b') \in U_{B_i,O}$.

From (a), in turn, it follows that the subtree in $U_{B_i,O}$ below each $b \in \text{ind}(B_i) \setminus \{a\}$ is identical to the subtree in $U_{B_i,O}$ below $b$. Now $(*)$ is an easy consequence.

For Points 3 and 4, define a mapping $g$ from $\text{ind}(B_{i+1})$ to $\text{ind}(B_i)$ by taking $g(a) = a$ for all $a \in \text{ind}(B_i) \cap \text{ind}(B_{i+1})$ and $g(a') = a$ for all $a' \in \text{ind}(B_{i+1}) \setminus \text{ind}(B_i)$. For Point 3, we verify the following Claim.

Claim 1. $g$ is a homomorphism from $B_{i+1}$ to $B_i$ with $g(\tilde{b}_{i+1}) = \tilde{b}_i$.

Proof of Claim 1. If $A(a) \in B_{i+1}$, then $A(a) \in B_i$ by definition of the Minimize step, and Lemma 12 implies that $A(g(a)) \in B_i$, as required. If $r(a, b) \in B_{i+1}$, then $r(a, b) \in B_i$. The definition of the Expand step then yields $r(g(a), g(b)) \in B_i$, as required.

For Point 4, it suffices to show that $g$ is surjective, but not injective.

Claim 2. $g$ is surjective.

Proof of Claim 2. Suppose that $g$ is not surjective. Then $\text{ind}(B_i) \not\subseteq \text{img}(g)$. By Lemma 13 Point 1, there is a homomorphism $h_1$ from $q_T$ to $U_{B_i,O}$ with $h_1(\bar{x}) = \tilde{b}_i$. Let $h_2$ be the extension of $g$ to a homomorphism from $U_{B_i,O}$ to $U_{B_i,O}$ as in Lemma 7 Point 1. Then $\text{img}(h_2) = \text{img}(g)$. Composing $h_1$ and $h_2$ yields a homomorphism $h_3$ from $q_T$ to $U_{B_i,O}$ with $h_3(\bar{x}) = \tilde{b}_i$, but with $\text{ind}(B_i) \not\subseteq \text{img}(h_3)$, in contradiction to Point 2.

For an injective and surjective function, we use $g^-$ to denote its inverse.

Claim 3. If $g$ is injective, then $r(a, b) \in B_i$ implies $r(g^-(a), g^-(b)) \in B_{i+1}$.

Proof of Claim 3. Suppose to the contrary that there is an $r(a, b) \in B_i$ with $r(g^-(a), g^-(b)) \notin B_{i+1}$. Since $g$ is injective, it is then also a homomorphism from $B_{i+1}$ to $B_i \setminus \{r(a, b)\}$ and using composition-of-homomorphisms argument as in the proof of Claim 2, we find a homomorphism $h$ from $q_T$ to $U_{B_i,O \setminus \{r(a, b)\}}$. Hence $r(a, b)$ is dropped during the Minimize step, in contradiction to $r(a, b) \in B_i$.

Claim 4. $g$ is not an injective homomorphism.

Proof of Claim 4. Let $R_0(a_0, a_1), \ldots, R_{n-1}(a_{n-1}, a_n) \in B_i$ be the chordless cycle that is expanded during the construction of $B_{i+1}$ from $B_i$. Recall that $a_0 = a_n$. Without loss of generality, assume that $R_{n-1} = r_{n-1}$ is a role name, but not an inverse role. Suppose for contradiction that $g$ is injective. The construction of $g$, together with $g$ being surjective and injective, implies that exactly one of $a_j, a'_j$ is in $\text{ind}(B_{i+1})$ for all $j$ with $0 \leq j \leq n$.

Assume that $a_{n-1} \in \text{ind}(B_{i+1})$ (the case $a'_n \in \text{ind}(B_{i+1})$ is analogous) and thus $g(a_{n-1}) = a_{n-1}$. We prove
by induction on $i$ that $a_i \notin \text{ind}(B_{i+1})$ for $0 \leq i < n$, thus obtaining a contradiction to $a_{n-1} \in \text{ind}(B_{i+1})$.

For the induction start, assume to the contrary of what is to be shown that $a_0 \in \text{ind}(B_{i+1})$. Then $g(a_0) = a_0$ and $r_{n-1}(a_{n-1}, a_0) \in B_i$. Let $B_i$ implies $r_{n-1}(a_{n-1}, a_0) \in B_{i+1}$ by Claim 3, in contradiction to the definition of the Expand step.

For the induction step, let $i \geq 0$. We know that $a_{i-1} \notin \text{ind}(B_{i+1})$ and thus $a_{i-1} \in \text{ind}(B_{i+1})$. Then $g(a_{i-1}) = a_i$ and $R_{i-1}(a_{i-1}, a_i) \in B_i$ and Claim 3 yield $R_{i-1}(a_{i-1}, a_i) \in B_i$, in contradiction to the definition of the Expand step. \hfill \Box

It is proved as part of Lemma 17 below that the Expand/Minimize phase terminates after polynomially many steps, let $(B_n, \bar{b}_n)$ be the result.

We next construct a sequence $B_n, B_{n+1}, \ldots$ using the Split and Minimize steps. With $B'_i$, $i \geq n$, we denote the result of only applying the Split step to $B_i$, but not the Minimize step.

**Lemma 15.** $B_i, \mathcal{O} \models q_T(\bar{b}_n)$ for all $i \geq n$.

**Proof.** We show the lemma by induction on $i$. For $i = n$, this is a consequence of Point 1 of Lemma 13. For $i > n$, it is immediate from the induction hypothesis and the facts that a split is only taking place if $B'_i, \mathcal{O} \models q_T(\bar{b}_n)$, and that the Minimize step preserves this. \hfill \Box

**Lemma 16.** For all $i \geq n$,

1. $B_i$ is $O$-saturated;
2. if $h$ is a homomorphism from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h(\bar{x}) = \bar{b}_n$, then $\text{ind}(B_i) \subseteq \text{img}(h^*)$;
3. $B_{i+1}, \bar{b}_n \rightarrow B_i, \bar{b}_n$;
4. $|\text{ind}(B_{i+1})| > |\text{ind}(B_i)|$.

**Proof.** We show Point 1 by induction over $i$. For $i = n$, this follows from Lemma 14 Point 1. For the induction step, suppose $B_{i+1}, \mathcal{O} \models A(\bar{a})$ for some $\bar{a} \in \text{ind}(B_{i+1})$, with $\bar{a}$ either in $\text{ind}(B_i)$ or $A'$ for some $a \in \text{ind}(B_i)$. Let $B'_i$ be the result of applying the Split step, but not yet the Minimize step. Then $B'_i, \mathcal{O} \models A(\bar{a})$ by monotonicity. By Lemma 12 we have $B'_i, \bar{a} \leq B_i, a$ and thus $B_i, \mathcal{O} \models A(a)$ by Lemma 8. By the induction hypothesis, we have $A(a) \in B_i$ and the definition of the Split step ensures $A(a) \in B_{i+1}$.

For Point 2, let $h$ be a homomorphism from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h(\bar{x}) = \bar{b}_n$, and suppose that there is an $a \in \text{ind}(B_i)$ that is not in $\text{img}(h^*)$. Let $B'$ be the result of removing from $B_i$ all assertions that involve $a$. We show that

\[
(*) \ h \text{ is a homomorphism from } q_T \text{ to } U_{B', \mathcal{O}}
\]

which witnesses that $B', \mathcal{O} \models q_T(\bar{b}_n)$. Hence $a$ is dropped during the Minimize step, in contradiction to $a \in \text{ind}(B_i)$. To see that $(*)$ holds, first note that for all $b, b' \in \text{ind}(B_i) \setminus \{a\}$, the following holds by Point 1 and construction of universal models:

1. $A(b) \in U_{B_i, \mathcal{O}}$ iff $A(b) \in U_{B_i, \mathcal{O}}$;
2. $r(b, b') \in U_{B_i, \mathcal{O}}$ iff $r(b, b') \in U_{B_i, \mathcal{O}}$.

From Point 1, in turn it follows that the subtree in $U_{B_i, \mathcal{O}}$ below each $b \in \text{ind}(B_i) \setminus \{a\}$ is identical to the subtree in $U_{B_i, \mathcal{O}}$ below $b$. In summary, $(*)$ follows.

For Points 3 and 4, recall that $B_{i+1}$ is the result of applying the Split and Minimize step to $B_i$. Let $b \in \text{ind}(B_i)$ be the individual that is duplicated by the Split step and let $b'$ be the fresh individual. We define a mapping $h$ from $\text{ind}(B_{i+1})$ to $\text{ind}(B_i)$ by taking $h(a) = a$ for all $a \in \text{ind}(B_i) \cap \text{ind}(B_{i+1})$ and $h(b') = b'$ if $b' \in \text{ind}(B_{i+1})$, that is, $b'$ was not removed during minimization. Clearly, we have $h(\bar{b}_n) = \bar{b}_n$. To establish Point 3, we argue that $h$ is a homomorphism. First, let $A(a) \in B_{i+1}$. By construction of $B_{i+1}$, we also have $A(h(a)) \in B_i$. Now, let $r(a, c) \in B_{i+1}$. By definition of the Split step, also $r(h(a), h(c)) \in B_i$.

For Point 4, it suffices to verify that $h$ is surjective but not injective.

**Claim 1.** $h$ is surjective.

**Proof of Claim 1.** Assume to the contrary that there is a $a \in \text{ind}(B_i)$ such that $a \notin \text{img}(h)$. By Lemma 7 Point 1, $h$ can be extended to a homomorphism $h_1$ from $U_{B_i, \mathcal{O}}$ to $U_{B_{i+1}, \mathcal{O}}$ with $h_1(\bar{b}_n) = \bar{b}_n$, such that $\text{img}(h_1^*) = \text{img}(h^*)$. Composing $h_1$ and a homomorphism $h_2$ from $q_T$ to $U_{B_{i+1}, \mathcal{O}}$ with $h_2(\bar{x}) = \bar{b}_n$ (which exists by Lemma 15) yields a homomorphism $h_3$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h_3(\bar{x}) = \bar{b}_n$ such that $a \notin \text{img}(h_3^*)$, in contradiction to Point 2.

**Claim 2.** $h$ is not injective.

**Proof of Claim 2.** Assume to the contrary that $h$ is injective. Then at most one of $b \in \text{ind}(B_{i+1})$ or $b' \in \text{ind}(B_{i+1})$. Again, $h$ can be extended to a homomorphism $h_1$ from $U_{B_{i+1}, \mathcal{O}}$ to $U_{B_{i+1}, \mathcal{O}}$ by Lemma 7 Point 1. Composing $h_1$ and a homomorphism $h_2$ from $q_T$ to $U_{B_{i+1}, \mathcal{O}}$ with $h_2(\bar{x}) = \bar{b}_n$ (exists by Lemma 15) yields a homomorphism $g$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $g(\bar{x}) = \bar{b}_n$. Recall that there is a symmetry $r(a, b), r(c, b) \in B_i$. If neither $b$ or $b'$ are in $\text{ind}(B_{i+1})$, then $b \notin \text{img}(g^*)$, in contradiction to Point 2. If $b \in \text{ind}(B_{i+1})$, then the Minimize step removed $b'$ in the construction of $B_{i+1}$. It follows that there is a homomorphism $g'$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $g'(\bar{x}) = \bar{b}_n$, such that there is no $r(x, y) \in q_T$ with $r(g'(x), g'(y)) = r(c, b)$. This contradicts $B_i$ being the result of the Minimize step. The case for $b' \in \text{ind}(B_{i+1})$ is symmetric. Thus both $b$ and $b'$ are in $\text{ind}(B_{i+1})$. \hfill \Box

We analyze the time requirement of the refine subroutine.

**Lemma 17.** $\text{refine}(q(\bar{x}))$ can be computed in time polynomial in $|q_T| + |q|$ (but exponential in $ar$) using membership queries.

**Proof.** Let $(B_1, \bar{b}_1), (B_2, \bar{b}_2), \ldots$ be the sequence constructed by the Expand/Minimize phase. By Lemma 14 Point 2, $|\text{ind}(B_i)| \leq |\text{var}(q_T)|$ for all $i \geq 1$. By Lemma 14 Point 4, the number of individuals in the ABoxes $B_i$ increases in every step. Thus the number $n$ of steps is at most $|\text{var}(q_T)|$. Now let $B_n, B_{n+1}, \ldots$ be the sequence constructed by the Split/Minimize phase. We can argue in the same way, using Lemma 16 Point 2 and Lemma 16 Point 4 that the number $m - n$ of steps is at most $|\text{var}(q_T)|$.

It remains to show that every step runs in polynomial time. For this, let $\Omega = \text{sig}(q)$ be the set of concept and role names
that occur in the input query \( q \). Clearly, \( |\Omega| \leq |q| \). Note that none of the applied operations introduces new concept or role names, that is, \( \text{sig}(B_i) \subseteq \Omega \), for all \( i \).

For Minimize this is the case, because at most \( |\text{ind}(B_i)| \) membership queries are posed in operation (1) and at most \( |\Omega| \cdot |\text{ind}(B_i)| \) membership queries are posed in operation (2).

For Expand, note that chordless cycles of length \( n > n_{\text{max}} \) can be identified in time polynomial in \( |\text{ind}(B_i)| \leq |\text{var}(q_T)| \).

We then need at most \( 2^n \) membership queries to identify the right tuple \( b_{i+1} \in T_i \).

Finally, for Split, observe that there are at most \( |\Omega| \cdot |\text{ind}(B_i)|^3 \) possible triples \( r(a, b), r(c, b) \in B_i \). Thus, at most as many membership queries are posed.

Before we show that the result of \( \text{refine} \) is always in the desired class, we give an example that demonstrates the necessity of the Split step. Let

\[
q_T(x_1, x_2) \leftarrow A(x_1) \land B(x_2) \land r(x_1, x'_2) \land r(x_2, x'_2)
\]

be the target query and let

\[
q(y_1, y_2) \leftarrow A(y_1) \land B(y_2) \land r(y_1, y) \land r(y_2, y)
\]

be the input to \( \text{refine} \). Then the result of the Expand and Minimize phase is \( q(y_1, y_2) \) which is not symmetry-free. Thus, the Split step is needed.

**Lemma 18.** If \( q_T(y) \in Q \) for \( Q \in \{\text{ELQ}, \text{ELIQ}^p, \text{CQ}^p \mid w \geq 0\} \), then \( \text{refine}(q_T(x)) \in Q \) for every \( CQ \) \( q(x) \).

**Proof.** Let \( p(y) = \text{refine}(q_T(x)) \). Assume that there is a symmetry \( r(x_1, x), r(x_2, x), x_1 \neq x_2 \) in \( p(y) \) such that \( x \notin y \), none of the atoms occurs on a cycle, and there is no atom \( s(z, z) \) for any \( z \in \{x, x_1, x_2\} \). Note that \( x_1 \neq x_2 \) due to the last condition, for \( i \in \{1, 2\} \).

Recall that the query \( p(y) \) is the result of exhaustively applying the steps Split and Minimize. Thus, for every homomorphism \( h \) from \( q_T \) to \( U_{A_\sigma, C} \) with \( h(x) = y \), there must be atoms \( r(y_1, y), r(y_2, y) \) in \( q_T \) such that \( h(y) = x, h(y_1) = x_1, h(y_2) = x_2 \).

It follows that \( y \) is not an answer variable of \( q_T \). Furthermore there is no atom \( s(y', y') \) for \( y' \in \{y_1, y_2, y\} \) in \( q_T \) since otherwise there must be an atom \( s(h(y'), h(y')) \) in \( p \).

Thus, \( q_T \) is in \( \text{CQ}^p \) for all choices of \( Q \), at least one of the atoms \( r(y_1, y), r(y_2, y) \) must occur on a cycle. Assume that \( r(y_1, y) \) occurs on a cycle in \( q_T \), the case for \( r(y_2, y) \) is similar. Since \( q_T \) is chordal, \( r(y_1, y) \) must also be part of a cycle \( r(y_1, y), S_1(y_1, y_2), S_2(y_3, y_1) \) of length three. Consider the atoms \( r(h(y_1), h(y_2)) = r(x_1, x), S_1(h(y_1), h(y_3)) = S_1(x, h(y_3)), \) and \( S_2(h(y_3), h(y_1)) = S_2(h(y_3), x_1) \) which occur in \( p \). We distinguish cases.

- If \( h(y_3) \notin \{x, x_1\} \), then \( r(x_1, x), S_1(x, h(y_3)), S_2(h(y_3), x_1) \) is a cycle of length three in \( p \) which contains \( r(x_1, x) \), contradicting our initial assumption.

- If \( h(y_3) = x \), then \( S_1(x, x) \) is an atom in \( p \), contradicting our initial assumption.

- If \( h(y_3) = x_1 \), then \( S_2(x_1, x_1) \) is an atom in \( p \), contradicting our initial assumption.

Thus, \( p \) is symmetry-free.

It remains to show that \( p(y) \) is chordal if \( q_T \) is, and an ELIQ if \( q_T \) is. Let \( p'(y) \) be the intermediate query obtained after the first phase of Expand and Minimize. By non-applicability of Expand, there is no chordless cycle \( R_0(x_0, x_1), \ldots, R_{n-1}(x_{n-1}, x_n) \) in \( p'(y) \) of length \( n > n_{\text{max}} \) and in case of \( Q = \text{CQ}^p \), \( \{x_0, \ldots, x_{n-1}\} \notin \mathcal{Z} \). For \( Q = \text{CQ}^p \) this means that every cycle in \( p'(y) \) of length at least four that contains at least one quantified variable has a chord. For \( Q \in \{\text{ELQ}, \text{ELIQ}^p\} \), with \( n_{\text{max}} = 0 \), this means that \( p'(y) \) does not contain any cycle at all. By Lemma 14 Point 2, we have \( \text{var}(p') \subseteq \text{img}(h^*) \), for all homomorphisms \( h \) from \( q_T \) to \( U_{A_\sigma, C} \) with \( h(x) = \mathcal{Z} \). Since \( q_T \) is connected and tree-shaped, \( p' \) must be connected and tree-shaped. Moreover in the case of \( q_T \in \text{ELIQ}^p \), all variables \( a \in \text{img}(h^*) \) are reachable by a path \( r_0(a_0, a_1), \ldots, r_{n-1}(a_{n-1}, a) \) from the root \( a_0 \), thus \( p' \) is ditee-shaped in this case. Thus \( p'(y) \in Q \) in all cases.

It remains to observe that the Split operation preserves these properties, thus \( p(y) \) is as required.

**Lemma 19.** Let \( C \) be \( \mathcal{E} \mathcal{L}^\mathcal{C} \)-ontology. Let \( A_1 \) and \( A_2 \) be ABoxes and \( a_i, i \in \{1, 2\} \), be tuples of individuals from \( A_1 \) of the same length. Moreover, let \( q(z) = C_{A_1, C} \times C_{A_2, C} \) viewed as \( CQ \) with answer variables \( z = a_1 \otimes a_2 \) and let \( p(z) \) be the result of \( \text{refine}(q(z)) \) with respect to some target query \( q_T(y) \). Then there is a homomorphism \( h_{\sigma} \) from \( p(x) \) to \( U_{A_\sigma, C} \) with \( h_{\sigma}(x) = (a_i) \), for \( i \in \{1, 2\} \).

**Proof.** By Lemma 14 Point 3, there is a homomorphism \( h \) from \( p \) to \( C_{A_1, C} \times C_{A_2, C} \) with \( h(x) = z \) and it is a property of products that there are homomorphisms \( g_{\sigma} \) from \( C_{A_1, C} \times C_{A_2, C} \) to \( C_{A_\sigma, C} \) with \( g_{\sigma}(z) = a_i \). Composing \( h \) and \( g_{\sigma} \) yields homomorphisms \( q_{\sigma} \) from \( p \) to \( C_{A_\sigma, C} \) with \( q_{\sigma}(x) = a_i \). It follows then, since \( p(x) \in \text{CQ}^p \) by Lemma 1 that there are also homomorphisms \( h_{\sigma} \) from \( p(x) \) to \( U_{A_\sigma, C} \) with \( h_{\sigma}(x) = a_i \), as required.

**F Proofs for Section 3.4.**

We start with describing the second version of the refine subroutine in full detail. It gets as input a \( CQ \) \( q_H(x') \) such that \( q_H \subseteq q_T \) and produces a \( CQ \) \( q_H(x') \) such that \( q_H \subseteq q_T \) and \( \text{var}(q_H) \subseteq \text{var}(q_T) \). The initial call to \( \text{refine} \) in Algorithm 1 is dropped when this version of refine is used. Thus, the argument \( q_H(x') \) is always the product of two 3-compact models. For notational convenience, we prefer to view \( q_H(x') \) as a pair \((A, a)\) where \( A = C_{A_1, C} \times C_{A_2, C} \) and \( a = a_1 \otimes a_2 \). We know that \( A_\sigma, C \models q_T(a) \) for \( i \in \{1, 2\} \).

As in the first version of refine, minimization is a crucial ingredient. However, we minimize in a slightly different way here.

**Minimize.** Let \( B \) be an ABox that contains all individuals from \( a_1 \otimes a_2 \). Then minimize \( B \) is the ABox \( B' \) obtained from \( B \) by exhaustively applying the following operation: choose \( c \in \text{ind}(B) \setminus (a_1 \otimes a_2) \) and remove all assertions that involve \( c \). Use a membership query to check whether, for
the resulting ABox $B'$, $B'\models q_T(\bar{a}_1 \circ \bar{a}_2)$. If this is the case, proceed with $B'$ in place of $B$.

The modified refine subroutine constructs a sequence of ABoxes $B_1, B_2, \ldots$ starting with 

$$B_1 = \minimize(C_{A_1,\circ}^3 \times C_{A_2,\circ}^3)$$

and such that $B_i, \mathcal{O} \models q_T(\bar{a}_1 \circ \bar{a}_2)$ for all $i \geq 1$. Note that in contrast to the first version of refine, the individuals in the answer tuple (which correspond to the answer variables) are never modified.

Each ABox $B_{i+1}$ is obtained from $B_i$ by a local unraveling. All individuals in $B_1$ are pairs $(c_1, c_2)$ and the same shall be true for the individuals in the ABoxes $B_2, B_3, \ldots$. Informally, unraveling replaces components $c_i$ that are individuals from $\text{ind}(C_{A_i,\circ}^3) \setminus \text{ind}(A_i)$ with corresponding individuals from $\text{ind}(\mathcal{U}_{A_i,\circ}) \setminus \text{ind}(A_i)$ in a step-by-step fashion. To make this formal, we call $(c_1, c_2) \in \text{ind}(B_i)$ unraveled if $c_i \in \text{ind}(\mathcal{U}_{A_i,\circ})$ for each $i \in \{1, 2\}$. Note that $(c_1, c_2) \in B_1$ and $c_i \notin \text{ind}(\mathcal{U}_{A_i,\circ})$ implies that $c_i$ is of the form $c_{n,i,s,c}$. The same will be true for all ABoxes $B_2, B_3, \ldots$. We now describe the unraveling step.

**Unravel.** Remove every assertion $r((c_1,c_2), (d_1,d_2)) \in B_i$ with $(c_1, c_2)$ unraveled and $(d_1, d_2)$ not unraveled. Let $d'_1 = d_1$ if $d_1$ occurs in $\mathcal{U}_{A_i,\circ}$ and $d'_j = c_{r,C}$ if $d_j = c_{n,i,s,c}$, for $j \in \{1, 2\}$. Compensate by adding the following assertions:

- $r((c_1, c_2), (d'_1, d'_2))$;
- $A(d'_1, d'_2)$ for all $A(d_1, d_2) \in B_i$;
- $r((d'_1, d'_2), (e_1, e_2))$ for all $r((d_1, d_2), (e_1, e_2)) \in B_i$.

We call $(d'_1, d'_2)$ a copy of $(d_1, d_2)$. Note that unraveling might introduce several copies of the same original element $(d_1, d_2)$ and that $(d_1, d_2)$ might or might not be present after unraveling, the latter being the case when $r((c_1, c_2), (d_1, d_2))$ is the only assertion that mentions $(d_1, d_2)$.

After unraveling, we apply the Minimize step and the resulting ABox is $B_{i+1}$.

We prove later that the Unravel step can only be applied polynomially many times, let the resulting ABox be $B_n$. Let $I$ denote the set of all individuals in $B_n$ that are reachable from some individual in $\text{ind}(A_1) \times \text{ind}(A_2)$ in the directed graph $G_{B_n} = (\text{ind}(B_n), \{(a, b) \mid r(a, b) \in B_n\})$. It is easy to see that all individuals in $I$ are unraveled. However, the restriction of $B_n$ to $\text{ind}(B_n) \setminus I$ might contain individuals that are not unravelled.

For example, consider the $\mathcal{EL}'$-ontology $\mathcal{O} = \{A \subseteq \exists r.A, B \subseteq \exists r.B\}$ and the boolean target query $q_T \iff r(x_1, x_2) \land r(x_2, x_3) \land r(x_3, x_4)$. At some point during the learning algorithm, the refine subroutine might be called with $A = C_{A_1,\circ}^3 \times C_{A_2,\circ}^3$ with $A_1 = \{A(a)\}$ and $A_2 = \{B(b)\}$. By construction of the 3-compact model, $A$ contains a cycle of length 4, consisting of individuals $(c_{a,i,r,A}, c_{b,i,r,B})$ for $i \in \{1, \ldots, 4\}$, that is reachable from from $\text{ind}(A_1) \times \text{ind}(A_2) = \{(a, b)\}$. The first Minimize step might then remove all individuals from $A$ that are not on the cycle, since there is a homomorphism $h$ from $q_T$ to $\mathcal{U}_{A,\circ}$ with $h(x_i) = (c_{a,i,r,A}, c_{b,i,r,B})$ for $i \in \{1, \ldots, 4\}$. Since only the cycle consisting of not unraveled individuals remains, the Unravel step cannot be applied.

To deal with this issue, we apply the original refine subroutine from Section 3.3 to $B_n$ (with $n_{\max} = 3$), resulting in a sequence $(B_n, b_n), (B_{n+1}, b_{n+1}), \ldots$ where $b_n = a_1 \circ a_2$, in a slightly adapted way:

1. the individuals in $I$ are not touched, that is, no cycle that involves an individual from $I$ is considered in the Expansion step nor is any assertion removed during the Minimize step that contains an individual from $I$;

2. as a consequence, the Expansion step cannot involve individuals in $a_1 \circ a_2$, and thus the exponential blowup in the arity is avoided. In fact, $b_n = b_{n+1} = \ldots$.

3. the Splitting step is not applied.

We now analyze the second version of refine, starting with the following lemma.

**Lemma 20.** Let $i \geq 1$. Every cycle in $B_i$ of length at most three consists only of individuals from $\text{ind}(A_1) \times \text{ind}(A_2)$.

**Proof.** We prove the lemma by induction on $i$. In the induction start, $B_1 = \minimize(C_{A_1,\circ}^3 \times C_{A_2,\circ}^3)$. If $B_1$ contains a cycle $r((a_1, a_2), (a_1, a_2))$ of length 1 with $a_i \notin A_i$ for some $i \in \{1, 2\}$, then $r(a_1, a_1)$ is a cycle of length 1 in $C_{A_1,\circ}^3$, which is not the case by Lemma 2. Next assume that $B_1$ contains a cycle $r_0((a_1, a_2), (b_1, b_2)), r_1((b_1, b_2), (a_1, a_2))$ of length 2. Assume w.l.o.g. that $a_1 \notin \text{ind}(A_1)$. If $a_1 = b_1$, then $r_0(a_1, a_1)$ is a cycle of length 1 in $C_{A_1,\circ}^3$, but this is not the case by Lemma 2. If $a_1 \neq b_1$, then $r_0(a_1, b_1), r_1(b_1, a_1)$ is a cycle of length 2 in $C_{A_1,\circ}^3$, which again contradicts Lemma 2. If $B_1$ contains a cycle $r_0((a_1, a_2), (b_1, b_2)), r_1((b_1, b_2), (c_1, c_2)), r_2((c_1, c_2), (a_1, a_2))$ of length 3, we can argue similarly that $C_{A_1,\circ}^3$ contains a cycle of length 1 or 3 that involves an individual not in $\text{ind}(A_1)$, again obtaining a contradiction.

For the induction step, we show that both the Minimize step and the Unravel step do not create cycles of length 1, 2, or 3 that involve individuals not from $\text{ind}(A_1) \times \text{ind}(A_2)$. Since the Minimize step only removes assertions, it cannot create any new cycles. For the Unravel step, let the lemma hold for $B_i$ and let $B$ be $B_i$ after the Unravel step. Let $r_0((a_1, a_2), (b_1, b_2)), r_1((b_1, b_2), (c_1, c_2)), r_2((c_1, c_2), (a_1, a_2))$ be a new cycle of length 3 in $B$. Since the cycle is new, one of $(a_1, a_2), (b_1, b_2)$ or $(c_1, c_2)$ must be $(d'_1, d'_2)$, a new individual created by the Unravel. But by the definition of the Unravel step, replacing $(d'_1, d'_2)$ with $(d_1, d_2)$ in the cycle must yield a cycle in $B_i$, which contradicts the induction hypothesis. The same argument can be applied to cycles of length 1 and 2. □

Recall that we first construct a sequence $B_1, B_2, \ldots$ using the Unravel and Minimize steps. With $B_i, i \geq 1$, we denote the result of only applying the Unravel step to $B_i$, but not the Minimize step.

**Lemma 21.** For all $i \geq 1$, $B_i, a \leq B_i, a$ and $B_i, a \leq B_i, a$ for all $a \in \text{ind}(B_i) \cap \text{ind}(B_i)$ and $B_i, a' \leq B_i, a$ and $B_i, a' \leq B_i, a'$ for all copies $a' \in \text{ind}(B_i) \setminus \text{ind}(B_i)$ of some $a \in \text{ind}(B_i)$.
Proof. Define a relation $S \subseteq \text{ind}(B_i) \times \text{ind}(B'_i)$ by taking:

- $(a, a') \in S$, for all $a \in \text{ind}(B_i) \cap \text{ind}(B'_i)$, and
- $(a, a') \in S$, for all copies $a' \in \text{ind}(B'_i)$ of some element $a \in \text{ind}(B_i)$.

It is routine to verify that $S$ serves as witness for the claimed simulations from $B_i$ to $B'_i$, and its inverse $S^{-1}$ serves as witness for the claimed simulations from $B'_i$ to $B_i$.

The next lemma is the most intricate to prove in the analysis of the second version of refine.

**Lemma 22.** For all $i \geq 1$, $B_i, O \models qr(\bar{a}_1 \otimes \bar{a}_2)$.

**Proof.** We prove the lemma by induction on $i$. The induction start is immediate since $U_{A_i, O} \times U_{A_d, O} \models qr(\bar{a}_1 \otimes \bar{a}_2)$ and there is a homomorphism from $U_{A_i, O} \times U_{A_d, O}$ to $C_{A_i, O}^3 \times C_{A_d, O}^3$ that is the identity on $\bar{a}_1 \otimes \bar{a}_2$. Thus, $C_{A_i, O}^3 \times C_{A_d, O}^3 \models qr(\bar{a}_1 \otimes \bar{a}_2)$. It remains to note that $B_1 = \text{minimize}(C_{A_1, O}^3 \times C_{A_2, O}^3)$ and that the Minimize step preserves $B_1, O \models qr(\bar{a}_1 \otimes \bar{a}_2)$.

For the induction step, consider $B_{i+1}$ with $i \geq 1$. By induction hypothesis, there is a homomorphism $h$ from $qr$ to $U_{B_i, O}$ with $h(\bar{x}) = \bar{a}_1 \otimes \bar{a}_2$. By Lemma 9, we can assume that $h$ is strongly symmetry-free. Let $B$ be the result of applying the unrolling step to $B_i$, and let $U$ be the set of all individuals $(d_1, d_2) \in \text{Ind}(B_i)$ such that some assertion $r(\bar{(c_1, c_2)}, (d_1, d_2))$ was removed in $r$-step. Note that if $(d_1, d_2) \in U$, then $(d_1, d_2) \notin \text{Ind}(A_i) \times \text{Ind}(A_j)$. In what follows, we construct a homomorphism $g$ from $qr$ to $U_{B_i, O}$ with $(g(\bar{x}) = \bar{a}_1 \otimes \bar{a}_2)$. Thus, $B, O \models qr(\bar{a}_1 \otimes \bar{a}_2)$. By definition of the Minimize step, this implies $B_{i+1}, O \models qr(\bar{a}_1 \otimes \bar{a}_2)$ as desired.

We first observe the following, which can be proved by a straightforward induction on $j$.

**Claim 1.** For all $j \geq 0$, if $R((c_1, c_2), (d_1, d_2)) \in B_j$ with $(c_1, c_2)$ unrolled and $(d_1, d_2)$ unrolled, then $R$ is a role name, but not an inverse role.

For a variable $x$ in $qr$, let us denote with $V_x$ the set of all atoms $R(x, y) \in qr$ such that $h(y) \in \text{Ind}(B_i)$ is unrolled.

We observe the following.

**Claim 2.** Let $x \in \text{var}(qr)$ such that $h(x) = (d_1, d_2) \in U$. Then there is a role name $r$ such that all atoms in $V_x$ are of shape $r(y, x)$ and one of the following is the case:

(i) $V_x$ is a singleton;

(ii) $d_1$ has the form $c_{0,0,r,C}$ and for every $r(y, x) \in V_x$, $A_2$ contains an assertion $r(b', d_2)$ with $h(y) = (b, b')$;

(iii) $d_2$ has the form $c_{0,0,r,C}$ and for every $r(y, x) \in V_x$, $A_1$ contains an assertion $r(b', d_1)$ with $h(y) = (b', b)$;

(iv) $d_1$ has the form $c_{0,1,r,C_1}$, $d_2$ has the form $c_{0,2,r,C_2}$, and $h(y) = (b_1, b_2)$ for every $r(y, x) \in V_x$.

**Proof of Claim 2.** To show the first part, let $R((y_1, x), S((y_2, x)) \in V_x$. Since $h(x) = (d_1, d_2)$ is not unrolled, but $h(y_1)$ and $h(y_2)$ are unrolled, $R$ and $S$ are role names by Claim 1. Moreover, $(d_1, d_2)$ not being unrolled means that at least one one of the $d_i$ takes the shape $c_{a,k,r,C}$ for some role name $r$. By definition of $C_{A_i, O}^3$, for every $s(d, c_{a,k,r,C}) \in C_{A_i, O}^3$, we have $s = r$. Hence, for every $s(d, (d_1, d_2)) \in B_i$ we have $s = r$ as well. Thus, $R = S = r$ and all assertions in $V_x$ are based on the same role name $r$.

Now for the second part. Assume that Case (i) does not apply. Then we find $r((y_1, x), r((y_2, x)) \in V_x$ with $y_1 \neq y_2$. Since $qr$ is strongly symmetry-free and $x$ is not an answer variable (which follows from $h(x) \in U$) one of the atoms, say $r((y_1, x)$, occurs on a cycle $p$ in $qr$. Since $qr$ is chordal, we can assume that $p$ has length at most three. Since $h$ is a homomorphism from $qr$ to $U_{B_i, O}$, the ‘$h$-image of $p$’ contains a cycle $p'$ of length at most three in $B_i$. By Lemma 20, $p'$ consists only of elements from $\text{ind}(A_i) \times \text{ind}(A_j)$. Thus $h(x)$ cannot be involved in $p'$, since $h(x) = (d_1, d_2)$ is not unrolled. Consequently, the cycle $p$ has to be of the shape $r((y_1, x), r((x, z), s((y_1, z))$ and $h(y_1) = h(z) = (b_1, b_2) \in \text{ind}(A_i) \times \text{ind}(A_j)$. It follows that we must have $i = 1$ since for $i > 1$, all successors of elements of $\text{ind}(A_i) \times \text{ind}(A_j)$ are unrolled. We distinguish the following cases:

- $d_1 \in \text{ind}(A_1)$ and $d_2 \in \text{ind}(A_2)$.

Impossible because $(d_1, d_2)$ is not unrolled.

- $d_1$ has shape $c_{0,1,0,r,C_1}$ and $d_2$ has shape $c_{0,2,0,r,C_2}$. By definition of the models $C_{A_1, O}$ and since in $B_1 \subseteq C_{A_1, O}^3 \times C_{A_2, O}^3 \cdot (b_1, b_2)$ is the unique unrolled $r$-predecessor of $(d_1, d_2)$ in $B_i = B_1$. Then we are in Case (iv).

- $d_1$ has shape $c_{0,1,0,r,C}$ and $d_2 \in \text{ind}(A_2)$.

Then $b_1$ is the unique $r$-predecessor of $d_1$ in $C_{A_1, O}$ that can appear in the first component of an unrolled element. Let $r((y, x), \in V_x$. Because $h(y)$ is unrolled and $i = 1$, $h(y) \in \text{ind}(A_1) \times \text{ind}(A_2)$. Since $B_1 \subseteq C_{A_1, O}^3 \times C_{A_2, O}^3$, $r((y, x)) \in V_x$ thus implies that there is an assertion $r(b', d_2) \in A_2$ such that $h(y) = (b_1, b')$. Thus, we are in Case (ii).

- $d_2$ has shape $c_{0,2,0,r,C}$ and $d_1 \in \text{ind}(A_1)$.

We argue as in the previous case, but end up in Case (iii).

This finishes the proof of Claim 2. For the next claim, we associate with every variable $x \in \text{var}(qr)$ with $h(x) \in \text{Ind}(B_i)$ the set $Z_x$ that consists of all variables $y \in \text{var}(qr)$ such that $qr$ contains a path $R_0(z_0, z_1), \ldots, R_{m-1}(z_{m-1}, z_m)$ from $x$ to $y$ where $h(z_1), \ldots, h(z_m)$ are all located in the subtree of $U_{B_i, O}$ rooted at $h(x)$, but are different from $h(x)$.

**Claim 3.** For all $y \in \text{var}(qr)$ with $h(y) \notin \text{Ind}(B_i)$, there is at most one $z \in \text{var}(qr)$ with $y \in Z_x$ and $h(x) \in U$.

**Proof of Claim 3.** Suppose that $y \in \text{var}(qr)$ with $h(y) \notin \text{Ind}(B_i)$ and that there are distinct variables $x_1, x_2 \in \text{var}(qr)$ with $y \in Z_{x_1}$ and $h(x_1) \in U$ for $j \in \{1, 2\}$. Let

$$p_1 = R_0(z_0, z_1), \ldots, R_{m-1}(z_{m-1}, z_m)$$

and

$$p_2 = S_0(z_0', z_1'), \ldots, S_{m-1}(z_{m-1}', z_m')$$

be paths in $qr$ from $x_1$ to $y$ and from $x_2$ to $y$, respectively, such that $h(z_j) \neq h(x_1)$ for all $j \in \{1, \ldots, n\}$ and
We analyze the structure of the paths $p_1$ and $p_2$. Let us first verify that all $R_1$ and all $S_j$ can be assumed to be role names. We do this explicitly only for the $R_1$. Let $S$ denote the subtree of $U_{B_1}$ rooted at $h(x_j)$, that is, the restriction of $U_{B_1}$ to all traces that start with $h(x_j)$, including $h(x_j)$ itself. By construction of $U_{B_1}$, $S$ is a ditere. After $R_0$ is a role name since $R_0(h(x_1), h(z_1)) \in S$, $h(x_1)$ is the root of $S$, and $h(z_1)$ in $S$. Now, let $\ell$ be minimal such that $R_\ell$ is an inverse role $r^-$ and consider the atoms $R_{\ell-1}(z_{\ell-1}, z_\ell), r^-(z_{\ell}, z_{\ell+1})$ in $\tau_T$. Since $h$ is a homomorphism and $S$ is a ditere, we know that $R_{\ell-1} = r$, and thus there are atoms $r(z_{\ell-1}, z_\ell), r(z_{\ell+1}, z_\ell)$ in $\tau_T$.

Now, if $z_{\ell-1} = z_{\ell+1}$, we can drop these two atoms from the path. Otherwise, since $\tau_T$ is strongly symmetry-free and $z_\ell$ is not an answer variable (as $h(z_\ell)$ is in $S$ but different from its root), one of these atoms occurs on a cycle $p$ in $\tau_T$. Let us assume that this is atom $r(z_{\ell-1}, z_\ell)$, the case of atom $r(z_{\ell+1}, z_\ell)$ is analogous. Since $\tau_T$ is chordal, we can assume that $p$ has length at most three. Since $h$ is a homomorphism from $\tau_T$ to $U_{B_1}$, the image of $p$ contains a cycle $p'$ of length at most three in $U_{B_1}$. Even if $h$ is not injective, the cycle $p'$ must contain $h(z_\ell)$ or $h(z_{\ell-1})$. However, both possibilities lead to a contradiction. If $p'$ contains $h(z_\ell)$, then $h(z_\ell) \in \text{ind}(A_1) \times \text{ind}(A_2)$ by Lemma 20 but this is not the case since $h(z_\ell)$ is in $S$ and different from $h(x_1)$. If $p'$ contains $h(z_{\ell-1})$, then $h(z_{\ell-1})$ must be $h(x_1)$, and $p'$ witnesses that $h(x_1) \in \text{ind}(A_1) \times \text{ind}(A_2)$, in contradiction to $h(x_1) \in U$.

At this point, we have established that all $R_1$ and $S_j$ are role names $r_j, s_j$. Since $S$ is a ditere, it follows that $m = n$ and $r_j = s_j$ for all $j$. Since $z_0 \neq z'_0$ and $z_n = z'_m$ there is some $\ell > 0$ such that $z_{\ell} = z'_\ell, z_{\ell-1} \neq z_{\ell-1}'$. But then $\tau_T$ contains atoms $r(z_{\ell-1}, z_\ell), r(z_{\ell-1}' , z_\ell)$. This leads to a contradiction in the same way as above. This finishes the proof of Claim 3.

We now define the required homomorphism $g$ in four stages, as follows.

1. Define $g(x) = h(x)$ for all $x \in \text{var}(\tau_T)$ such that $h(x) \in \text{ind}(B_1) \setminus U$ or $h(x)$ is in the subtree below some element $d \not\in U$.

2. For every $x \in \text{var}(\tau_T)$ with $h(x) = (d_1, d_2) \in U$, we distinguish cases according to Claim 2:
   (a) If $V_x = \emptyset$, then define $g(x) = h(x)$. We argue that this is well-defined, that is, $h(x) \in \text{ind}(B)$. Suppose to the contrary that $h(x) \not\in \text{ind}(B)$. By definition of the unraveling operation, this can only be the case if $B_1$ contains only a single assertion that mentions $h(x)$ and this assertion is of shape $r((c_1, c_2), h(x))$ with $(c_1, c_2)$ unraveled. Since $x$ has to occur in some atom in $\tau_T$ and $h$ is a homomorphism, $x$ occurs in an atom $r(z, x) \in \tau_T$ such that $h(z) = (c_1, c_2)$. Hence, $r(z, x) \in V_x = \emptyset$, contradiction.

   (b) If Case (i) applies and $V_x = \{r(y, x)\}$, define $g(x)$ to be the copy $(d'_1, d'_2)$ of $(d_1, d_2)$ introduced when unraveling $r(h(y), h(x)) \in B_1$.

   (c) If $V_x \neq \emptyset$ and Case (ii) applies (but Case (i) does not), then define $g(x)$ to be the copy $(b r C_2, d_2)$ of $(d_1, d_2)$, where $b, C$ are as in Case (ii) of Claim 2.

   (d) If $V_x \neq \emptyset$ and Case (iii) applies (but Case (i) does not), analogously define $g(x)$ to be the copy $(d_1, b r C)$.

   (e) If $V_x \neq \emptyset$ and Case (iv) applies (but Case (i) does not), define $g(x)$ to be the copy $(b_1 r C_1, b_2 r C_2)$ where $b_1, b_2, C_1, C_2$ are as in Case (iv).

3. For every $x$ with $h(x) \in U$ and every $y \in Z_{x}$, $h(y)$ is a trace that starts with $h(x)$, e.g., the definition of $U_{B_1}$. Define $g(y)$ to be the same trace, but with the first element $h(x)$ replaced by $g(x)$. It can be verified that $g(y)$ is indeed an element in $U_{B_1}$ using the fact that, by Lemma 21, the subtrees below $g(y)$ and $h(x)$ in $U_{B_1}$ and $U_{B_2}$, respectively, are identical.

4. For every $y$ with $h(y)$ in the subtree below some $(d_1, d_2) \in U$ but different from $(d_1, d_2)$, and such that $y \notin Z_{x}$ for all $x$ with $h(x) \in U$, choose some copy $(d'_1, d'_2)$ of $(d_1, d_2)$ and define $g(x)$ to be the trace $h(x)$ with the first element $(d_1, d_2)$ replaced by $(d'_1, d'_2)$.

It is easy to see that the four stages above define $g(x)$ for all $x \in \text{var}(\tau_T)$.

**Claim 4.** $g$ is a homomorphism from $\tau_T$ to $U_{B_1}$ with $g(\tilde{a}) = \tilde{a}_1 \otimes \tilde{a}_2$.

**Proof of Claim 4.** For $g(\tilde{a}) = \tilde{a}_1 \otimes \tilde{a}_2$, observe that $h(x) \in \text{ind}(A_1) \times \text{ind}(A_2)$ for every $x \in \tilde{a}$ while $U \cap \text{ind}(A_1) \times \text{ind}(A_2) = \emptyset$. Thus, Stage 1 of the definition of $g$ implies $g(\tilde{a}) = (h(x))$.

Now, let $A(x) \in \tau_T$ and thus $A(h(x)) \in U_{B_1}$. We distinguish the following cases:

- If $g(x)$ was defined in Stage 1, then $g(x) = h(x)$. First assume that $g(x) \in \text{ind}(B_1)$. By Lemma 21, we have $B_1, h(x) \leq B_1, g(x)$ and thus, by Lemma 7 Point 2 $U_{B_1}, h(x) \leq U_{B_1}, g(x)$. Hence, also $A(g(x)) \in U_{B_1}$ by Lemma 6. Now assume that $g(x) \not\in \text{ind}(B_1)$. Then $h(x) = g(x)$ is a trace and traces in $U_{B_1}$ and $U_{B_2}$ that end with the same concept $C$ must satisfy the same concept names.

- If $g(x)$ was defined in Stage 2, then $g(x) = (d'_1, d'_2)$ is a copy of $h(x) = (d_1, d_2)$ or $g(x) = h(x)$. By Lemma 21, we have $B_1, h(x) \leq B_1, g(x)$, thus $A(g(x)) \in U_{B_1}$ by Lemmas 6 and 7.

- If $g(x)$ was defined in Stage 3 or 4, then $h(x)$ and $g(x)$ are both traces that end with the same concept $C$ and, by construction of universal models, thus make true the same concept names. Consequently, $A(h(x)) \in U_{B_1}$ implies $A(g(x)) \in U_{B_1}$.

Finally, let $r(x, y) \in \tau_T$ and thus $r(h(x), h(y)) \in U_{B_1}$. We distinguish the following cases:

- It cannot be that both $h(x)$ and $h(y)$ are elements of $U$, by definition of the unraveling step.
• If both \( h(x) \) and \( h(y) \) are not elements of \( U \), then both \( g(x) \) and \( g(y) \) were defined in the same stage, one of Stage 1, 2, and 3. We can then argue very similar to the case of concept atoms that \( r(g(x), g(y)) \in U_{\delta \cdot} \).
• If \( h(x) = (d_1, d_2) \in U \) and \( h(y) \notin U \), then we distinguish cases:
  - If \( h(y) \notin \text{ind}(B_i) \), then it is an \( r \)-successor of \( (d_1, d_2) \) in the tree below \( (d_1, d_2) \) in \( U_{\delta \cdot} \). Thus \( g(y) \) was defined in Stage 3. If \( h(y) \) is trace \( (d_1, d_2) rC \), then \( g(y) \) is trace \( (d'_1, d'_2) rC \).
  - If \( h(y) \in \text{ind}(B_i) \) is not unruled, then by definition of the unraveling step, we have \( r((d'_1, d'_2), h(y)) \in B \) for all copies \( (d'_1, d'_2) \) of \( (d_1, d_2) \), and \( r((d_1, d_2), h(y)) \in B \). We know that \( g(x) \) was defined in Stage 2 and is either \( h(x) \) or some copy thereof, and \( h(y) \) was defined in Stage 1, thus \( g(y) = h(y) \). Consequently, \( r(g(x), g(y)) \in B \subseteq U_{\delta \cdot} \).
  - It cannot be the case that \( h(y) \in \text{ind}(B_i) \) is unravelled: By Claim 2, \( S \) is a role name for every atom \( S(z, x) \in qr \) such that \( h(z) \) is unruled. However, this is not the case for the atom \( r^{-1}(y, x) \in qr \) we started with.
• If \( h(x) \notin U \) and \( h(y) = (d_1, d_2) \in U \), then \( h(x) \in \text{ind}(B_i) \) since \( r(h(x), h(y)) \in U_{\delta \cdot} \) and by definition of universal models. We distinguish cases according to Claim 2:
  - If \( V_y = \emptyset \), then \( g(y) = h(y) \), by Stage 2(a).
  - If \( h(x) \in \text{ind}(B_i) \setminus U \), we have \( g(x) = h(x) \), by Stage 1. Hence, \( r(g(x), g(y)) \in B \subseteq U_{\delta \cdot} \).
  - If Case (i) applies and \( V_y = \{r(x, y)\} \) with \( h(x) \) unruled, then \( g(y) \) was defined in Stage 2(b) and \( r(g(x), g(y)) \in B \subseteq U_{\delta \cdot} \).
  - If Case (ii) applies to \( V_y \), then \( d_1 \) has the form \( c_{0, 0} r_{C, 2} \) and for every \( r(z, y) \in V_y \), \( A_2 \) contains an assertion \( r(b', d_2) \) with \( h(z) = (b', b') \). Moreover, \( g(x) \) was defined in Stage 2(c) and \( g(y) = (brC, d_2) \).
  - If \( h(x) \) is unruled, then \( h(x) = g(x) = (b', b') \). By definition of the unraveling, \( r(b, b'), (brC, d_2) \in B \). Hence, \( r(g(x), g(y)) \in B \subseteq U_{\delta \cdot} \).
  - If \( h(x) \) is not unruled, then it was defined in Stage 1 and \( h(x) = g(x) \). By definition of the unraveling, \( r(h(x), (brC, d_2)) \in B \).
  - If Case (iii) applies to \( V_y \), the argument is symmetric.
  - If Case (iv) applies to \( V_y \), then \( d_1 \) has the form \( c_{b_1, 0, r_{C, 2}} d_2 \) has the form \( c_{b, 0, r_{C, 2}} \) and \( g(y) \) was defined in Stage 2(e) and \( g(y) = (b_1 rC_1, b_2 rC_2) \). Since \( h(x) \) is unruled, we have \( g(x) = h(x) \) by Stage 1, and the definition of the unraveling yields \( r(g(x), g(y)) \in B \).

This finishes the proof of Claim 4 and thus of the lemma.
refine subroutine, that is, using the Expand and Minimize steps. With $B'_i$, $i \geq n$, we denote the result of only applying the Expand step to $B_i$, but not the Minimize step.

**Lemma 24.** For all $i \geq n$,

1. $B_i, \mathcal{O} \models q_T(\bar{a}_i \otimes \bar{a}_2)$;
2. $B_i$ is $O$-saturated;
3. if $h$ is a homomorphism from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h(\bar{y}) = \bar{a}_1 \otimes \bar{a}_2$, then $\text{ind}(B_i) \subseteq \text{img}(h^*)$;
4. $B_{i+1}, \bar{a}_1 \otimes \bar{a}_2 \rightarrow B_i, \bar{a}_1 \otimes \bar{a}_2$.
5. $|\text{ind}(B_{i+1})| > |\text{ind}(B_i)|$.

**Proof sketch.** Point 1 is a direct consequence of Lemma 13. Points 2 to 4 can be proved in the same way as Points 1 to 4 of Lemma 14. While the proofs of Points 2 and 5 go through without modification, a slight extension is required for the proof of Point 3 in the case that $i > 1$. There, we start with a homomorphism $h$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h(\bar{x}) = \bar{b}_i$, and suppose that there is an $a \in \text{ind}(B_i)$ that is not in $\text{img}(h^*)$. If $a$ is not reachable from some individual in $\text{ind}(A_1) \times \text{ind}(A_2)$ in $B_i$ viewed as a directed graph, then we can argue as in the proof of Point 2 of Lemma 14, that is, obtain a contradiction against exhaustive application of Minimize to $B_{i-1}$. If $a$ is reachable, however, this does not work as we do not apply the Minimize step to such individuals in the modified version of the refine subroutine.

However, by Point 4, there is a homomorphism $h$ from $B_i$ to $B_1$ with $h(\bar{a}_1 \otimes \bar{a}_2) = \bar{a}_1 \otimes \bar{a}_2$. By Point 1 there is a homomorphism $h_1$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h_1(\bar{x}) = \bar{a}_1 \otimes \bar{a}_2$. Let $h_2$ be the extension of $h$ to a homomorphism from $U_{B_i, \mathcal{O}}$ to $U_{B_1, \mathcal{O}}$ as in Lemma 7 Point 1. Then $\text{img}(h_2^* \otimes C) = \text{img}(h^*)$.

Composing $h_1$ and $h_2$ yields a homomorphism $h_3$ from $q_T$ to $U_{B_i, \mathcal{O}}$ with $h_3(\bar{x}) = \bar{a}_1 \otimes \bar{a}_2$, but with $\text{ind}(B_i) \not\subseteq \text{img}(h_3^*)$, in contradiction to Point 2 of Lemma 23.

**Lemma 25.** refine$(q(\bar{x}))$ can be computed in time polynomial in $|\text{var}(q)| + ||q||$ using membership queries.

**Proof.** We first note that the length of the sequence $B_1, B_2, \ldots$ computed in the Unravel/Minimize phase is bounded by $|\text{var}(q_T)| + 1$. Indeed, the following is easy to prove by induction on $i$.

**Claim.** Let $i \geq 1$. Then every individual in $B_i$ that is reachable in the directed graph $G_{B_i} = (\text{ind}(B_i), \{(a, b) \mid r(a, b) \in B_i\})$ from some individual in $\text{ind}(A_1) \times \text{ind}(A_2)$ on a path of length at most $i - 1$ is unraveled.

Since every individual in $B_i$ that is reachable from some individual in $\text{ind}(A_1) \times \text{ind}(A_2)$ is reachable on a path of length at most $|\text{ind}(B_i)|$ and $|\text{ind}(B_i)| \leq |\text{var}(q_T)|$ by Point 2 of Lemma 23, it follows that the Unravel step is thus no longer applicable to $B_{n+2}$ for $n = |\text{var}(q_T)|$.

Next observe that the length of the sequence $B_n, B_{n+1}$ computed in the Expand/Minimize phase is also bounded by $|\text{var}(q_T)|$ as we have for all $i \geq n, |\text{ind}(B_i)| \leq |\text{var}(q_T)|$, by Lemma 24 Point 3, and $|\text{ind}(B_i)| < |\text{ind}(B_{i+1})|$, by Lemma 24 Point 5.

It remains to show that every step runs in polynomial time. First note that, for all $i \geq 1$, we have $|\text{ind}(B_i)| \leq |\text{var}(q_T)|$ by Lemma 23 Point 2 and Lemma 24 Point 3. Moreover, by definition of the Unravel/Minimize steps, we have $\text{sig}(B_i) \subseteq \text{sig}$, for all $i$, where $\Omega = \text{sig}(q)$ and thus $|\text{var}(q_T)| \leq |||q|||$. Applying the Unravel step to $B_i$ thus takes time polynomial in $|\text{ind}(B_i)|$ and $|\text{var}(q_T)|$.

The resulting ABox $B'$ is such that $|\text{ind}(B')| \leq |\text{ind}(B_i)|^2 \cdot |\text{var}(q_T)|$. The number of membership queries needed in the minimization step is thus bounded by $|\text{var}(q_T)|^2 \cdot |||q|||$.

For expand, note that chordless cycles of length $n > n_{\text{max}}$ can be identified in time polynomial in $|\text{ind}(B_i)|$, and constructing the ABox $B_{i+1}$ from $B_i$ is clearly also possible in polynomial time.

**Lemma 26.** Let $\mathcal{O}$ be an $\mathcal{EL}'$-ontology. Let $A_1$ and $A_2$ be ABoxes and $a_i, x_i \in \{1, 2\}$, be tuples of individuals from $A_i$ of the same length. Moreover, let $q(\bar{z})$ be $C_{A_i, \mathcal{O}} \times C_{A_j, \mathcal{O}}$ viewed as CQ with answer variables $\bar{z} = \bar{a}_1 \otimes \bar{a}_2$ and let $p(\bar{x})$ be the result of refine$(q(\bar{z}))$ with respect to some target query $q_T(\bar{y})$. Then there is a homomorphism $h_i$ from $p(\bar{x})$ to $U_{A_i, \mathcal{O}}$ with $h_i(\bar{x}) = (a_i)$, for $i \in \{1, 2\}$.

**Proof.** Let $B_m$ be the result of refine$(q(\bar{z}))$ before it is turned into the CQ $p(\bar{x})$. Further, let $B'$ denote the restriction of $B_m$ to all individuals that are reachable from an individual in $\text{ind}(A_1) \times \text{ind}(A_2)$ in $B_m$, and let $B''$ be the restriction of $B_m$ to all individuals that are not reachable. Thus $B'' = B' \cup B''$.

It suffices to show that, for $i \in \{1, 2\}$, there is a homomorphism $h_i'$ from $B''$ to $U_{A_i, \mathcal{O}}$.

For the former, note that all individuals in $B''$ are unraveled. Thus $B'' \subseteq U_{A_1, \mathcal{O}} \times U_{A_2, \mathcal{O}}$ and the identity is a homomorphism $h_i'$ from $B$ to $U_{A_1, \mathcal{O}} \times U_{A_2, \mathcal{O}}$ with $h_i'(\bar{y}) = a_i \otimes \bar{a}_2$.

Projection to the left and right components yields the homomorphisms $h_i''$ as required.

For the latter, note that none of the individuals in $B''$ is unraveled. In fact, this follows from two obvious properties of the Unravel step and 3-compact canonical models:

- if the Unravel step is not applicable to an ABox $B_i$ and $r(a, b) \in B_i$ with $a$ unraveled, then $b$ is unraveled too;
- if Unraveling (and Minimization) is repeatedly applied to an ABox $C_{A_1, \mathcal{O}} \times C_{A_2, \mathcal{O}}$, then it never produces any fact $r(a, b)$ with $b$ unraveled, but $a$ not unraveled (because there are no $r$-edges from individuals of the form $a_{i, i, s, c}$ to individuals from $\text{ind}(A_i)$ in $C_{A_i, \mathcal{O}}$).

Thus, the identity is a homomorphism from $B''$ viewed as a Boolean CQ to $C_{A_1, \mathcal{O}} \times C_{A_2, \mathcal{O}}$. Projection to the left and right components yields a homomorphism $g_i''$ from $B''$ to $C_{A_i, \mathcal{O}}$ for $i \in \{1, 2\}$. By definition of the Expansion and Minimize step and its use in refine, it is clear that $B''$ is chordal. From Lemma 2, it thus follows that $B''$ viewed as a CQ is an ELIQ. Using the construction of $C_{A_i, \mathcal{O}}$ and $U_{A_i, \mathcal{O}}$, it is now straightforward to convert the homomorphism $g_i''$ from ELIQ $B''$ to $C_{A_i, \mathcal{O}}$ into the desired homomorphism $h_i''$ from $B''$ to $U_{A_i, \mathcal{O}}$.

**G Proof of Theorem 1**

Let $q_0(\bar{x}_0), q_1(\bar{x}_1), \ldots$ be the sequence of hypotheses generated by the algorithm.
Lemma 27. For all $i \geq 0$:

1. $q_i \subseteq \odot q_{r'}$;
2. $q_i \subseteq \odot q_{i+1}$;
3. $q_{i+1} \nsubseteq \odot q_i$.

Proof. Point 1 is a consequence of Lemma 13 Point 1, for the first refine-operation, and Lemma 24 Point 1, for the second refine-operation.

For Point 2, recall that $q_{i+1} = \text{refine}(q'_H(x))$ where $q'_H(x)$ is $C^3_{A,q_i} \times C^3_{A',q_i}$, for some positive counterexample $A, \bar{a}$, viewed as CQ with answer variables $\bar{x} = \bar{x}_i \otimes \bar{a}$. In case of the first refine-operation there is a homomorphism $h$ from $q_{i+1}$ to $U_{A,q_i} \otimes$ with $h(x_{i+1}) = x_i$, by Lemma 19. Lemmas 22 and 24 give us this homomorphism for the second refine-operation. By Lemma 4, we obtain $q_i \subseteq \odot q_{i+1}$.

For Point 3, assume to the contrary that $q_{i+1} \nsubseteq \odot q_i$. Then there is a homomorphism $h_1$ from $q_i$ to $U_{A_{q_{i+1}}^1, \odot}$ with $h_1(\bar{x}_i) = \bar{x}_{i+1}$. Recall once more that $q_{i+1} = \text{refine}(q'_H(x))$ where $q'_H(x)$ is $C^3_{A,q_i} \times C^3_{A',q_i}$, for some positive counterexample $A, \bar{a}$, viewed as CQ with answer variables $\bar{x} = \bar{x}_i \otimes \bar{a}$. In case of the first refine-operation there is a homomorphism $h_2$ from $q_{i+1}$ to $U_{A,q_i}$ with $h_2(\bar{x}_{i+1}) = \bar{a}$, by Lemma 19. Again, Lemmas 22 and 24 show that $h_2$ also exists for the second-refine-operation. By Lemma 7 Point 1 $h_2$ can be extended to a homomorphism $h_2'$ from $U_{A_{q_{i+1}}^1, \odot}$ to $U_{A,q_i}$ with $h_2'(\bar{x}_{i+1}) = \bar{a}$. Composing $h_1$ and $h_2$ yields a homomorphism $h$ from $q_i$ to $U_{A,q_i}$ with $h(x) = \bar{a}$. Thus $A, \odot \models q_i(a)$, in contradiction to $A, \odot \models q_i$, a being a positive counterexample.

We next observe that the sizes of $q_0, q_1, \ldots$ are non-decreasing.

Lemma 28. For all $i \geq 0$:

1. $\var(q_i) \subseteq \im(h^*)$ for every homomorphism $h$ from $q_{i+1}$ to $U_{A_{q_{i+1}}^1, \odot}$ with $h(\bar{x}_{i+1}) = \bar{x}_i$;
2. $|\var(q_i)| \leq |\var(q_{i+1})|$

Proof. For Point 1, let $h$ be a homomorphism from $q_{i+1}$ to $U_{A_{q_{i+1}}^1, \odot}$ with $h(\bar{x}_{i+1}) = \bar{x}_i$. By Lemma 7 Point 1, we can extend $h$ to a homomorphism from $U_{A_{q_{i+1}}^1, \odot}$ to $U_{A,q_i}$ without adding individuals from $\var(q_i)$ to $\im(h^*)$. By Point 1 of Lemma 27, there is a homomorphism $h'$ from $q_{r'}$ to $U_{A_{q_{i+1}}^1, \odot}$ with $h(\bar{x}) = \bar{x}_{i+1}$. We can compose $h$ and $h'$ into a homomorphism $g$ from $q_{r'}$ to $U_{A_{q_{i+1}}^1, \odot}$ with $g(\bar{x}) = \bar{x}_i$. We then obtain $\var(q_i) \subseteq \im(g^*)$ by Lemma 14 Point 2 for the first refine-operation or Lemma 24 for the second version. Since $\im(g^*) \subseteq \im(h^*)$, it follows that $\var(q_i) \subseteq \im(h^*)$.

Point 2 is a consequence of Point 1. In fact, $q_i \subseteq \odot q_{i+1}$ implies via Lemma 4 that there is a homomorphism $h$ from $q_{i+1}$ to $U_{A_{q_{i+1}}^1, \odot}$ with $h(\bar{x}_{i+1}) = \bar{x}_i$. Point 1 yields $\var(q_i) \subseteq \im(h^*)$ and thus $|\var(q_i)| \leq |\var(q_{i+1})|$. \qed

Lemma 29. $q_i \equiv \odot q_{r'}$ for some $i \leq p(|\var(q_{r'})| + |\Sigma|)$ for some polynomial $p$.

Proof. By Lemma 14 Point 2 in case of the first refine-operation or by Lemma 24 Point 3 in case of the second, we have $|\var(q_i)| \leq |\var(q_{r'})|$ for all $i \geq 0$.

Let $q_i, q_{\ell}, \ell \leq u$, be a subsequence of $q_1, q_2, \ldots$ such that $|\var(q_i)| = \cdots = |\var(q_{u})|$. By Point 2 of Lemma 28, it suffices to show that the length $u - \ell$ of any such sequence is bounded by a polynomial in $|\var(q_{r'})|$ and $|\Sigma|$. Let $h_{r'}$ for $i \in \{\ell + 1, \ldots, u\}$, be the homomorphisms from $q_i$ to $U_{A_{q_{i+1}}^1, \odot}$ that exist due to Lemma 27 Point 2.

Note that $h_{r'}$ is a bijection between $\var(q_i)$ and $\var(q_{i-1})$. Denote with $V_i$ the set of all quantified variables $x \in \var(q_i)$ which do not occur in a role atom, and define $U_i = \var(q_i) \setminus V_i$. Let us further denote with $q^2$ the restriction $q_i(\bar{x})$ of a query $q$ to a single variable $x \in \var(q)$. Clearly, $q_i$ can be written as

$$q_i(\bar{x}_i) \leftarrow q_i(\bar{u}_i) \land \bigwedge_{x \in V_i} q_i^2 \quad q_i(\bar{x}_i) \leftarrow q_i(\bar{u}_i) \land \bigwedge_{x \in V_i} q_i^2$$

Notice that, by definition, each $q_i^2$, $x \in V_i$ is a query without answer variables.

Claim 1. $x \in U_i$ implies $h_{r'}^i(x) \in U_{i-1}$.

Proof of Claim 1. Let $x \in U_i$. If $x$ is an answer variable then $h_{r'}(x) = h_{r'}^i(x)$ is an answer variable and thus in $U_{i-1}$. Suppose now that there is a role atom $R(z, x)$ in $q_i$ and consider the assertion $R(h_{r'}(z), h_{r'}(x)) \in U_{A_{q_{i+1}}^1, \odot}$ which exists since $h_{r'}$ is a homomorphism. Since $h_{r'}$ is a bijection, $z = x$ or $h_{r'}^i(z) \neq h_{r'}^i(x)$.

- In the first case, we obtain $R(y, y) \in U_{A_{q_{i+1}}^1, \odot}$ for $y = h_{r'}(z) = h_{r'}^i(x)$. The definition of the universal models yields $y \in \var(q_{i-1})$. $R(y, y)$ occurs in $q_{i-1}$, and $h_{r'}^i(z) = h_{r'}(x) \in U_{i-1}$.
- In the second case, we obtain $h_{r'}^i(z) = h_{r'}(z) \in \var(q_{i-1})$, $h_{r'}^i(x) = h_{r'}(x) \in \var(q_{i})$, and $R(h_{r'}(z), h_{r'}(x)) \in U_{i-1}$. Hence, $x \in U_{i-1}$.

This finishes the proof of Claim 1.

Claim 1 implies $|U_i| \leq |U_{i-1}|$ for every $i \in \{\ell + 1, \ldots, u\}$. We now consider the set $\{q_i, q_{i+1}, \ldots, q_u\}$ with $|U_i| = \cdots = |U_{u'}| = \cdots = |V_{u'}|$. Since $|U_i| \leq |\var(q_{r'})|$, for all $i$, it suffices to show that the length of such a sequence is bounded by a polynomial in $|\var(q_{r'})|$ and $|\Sigma|$.

Claim 2. For every $i \in \{\ell', \ldots, u'\}$,

1. $h_{r'}$ is a bijection between $V_i$ and $V_{i-1}$, and
2. $h_{r'}$ is a bijection between $U_i$ and $U_{i-1}$.

Proof of Claim 2. The first point is a consequence of Claim 1 and the facts that $|V_{i-1}| = |V_i|$. By $h_{r'}$ is a bijection between $\var(q_i)$ and $\var(q_{i-1})$. For the second point, suppose that some $x \in U_{i-1}$ is not $h_{r'}(y)$ for some $y \in U_i$. Thus, there is some $y$ such that $h_{r'}^i(y) = x$ and $h_{r'}(y)$ is strictly in the sub-tree rooted at $x$ in $U_{A_{q_{i+1}}^1, \odot}$. Since $h_{r'}$ is a bijection, $y$ is the unique variable such that $h_{r'}(y)$ is in the sub-tree rooted at $x$. We claim that $y \in V_i$. This implies $x \in V_{i-1}$, by the first point, and contradicts the assumption $x \in U_{i-1}$.
contradiction. Otherwise, there is some $R(y,z)$ in $q_i$. Since $h_i(y)$ is strictly below $x$, this leads to a contradiction as follows. If $z \neq y$, then $h_i(z) = h_i(y)$ contradicts the fact that $h_i$ is a bijection. If, on the other hand, $z = y$, then $h_i$ is not a homomorphism since there is no self-loop $R(h_i(y), h_i(y))$ in $U_{A_{n-1}, O}$, by definition of the universal model. This finishes the proof of Claim 2.

Sanctioned by the first point in Claim 2, in what follows we assume for the sake of readability that $h_i(x) = x$ for all $x \in V_i$ and $i \in \{\ell' + 1, \ldots, u'\}$. Hence, $V_{\ell'} = \cdots = V_{u'}$. Now, observe that, for all $i \in \{\ell' + 1, \ldots, u'\}$, one of the following is the case:

1. the inverse of $h_i$ is not a homomorphism from $q_{i-1}\{u_{i-1}\}$ to $q_i\{u_i\}$;  
2. there is some $x \in V_i$ such that $q_i^x \not\subseteq q_i x^i$.

Indeed, if neither Point 1 nor Point 2 is satisfied then $q_i \equiv_O q_{i-1}$ in contradiction to Point 3 of Lemma 27. It thus remains to bound the number of times each of these points can be satisfied along $q_\ell, \ldots, q_{u'}$. We start with Point 1.

Claim 3. The number of $i \in \{\ell' + 1, \ldots, u'\}$ such that the inverse of $h_i$ is not a homomorphism from $q_{i-1}\{u_{i-1}\}$ to $q_i\{u_i\}$ is at most $(|\var(q_T)|^2 + |\var(q_T)|) \cdot |\Sigma|$.

Proof of Claim 3. Let $i$ be as in the claim. By Point 2 of Claim 2, $h_i$ is a bijective homomorphism from $q_i\{u_i\}$ to $q_{i-1}\{u_{i-1}\}$. Hence, the number $n_i$ of atoms in $q_{i-1}\{u_{i-1}\}$ is at most the number $n_{i-1}$ of atoms in $q_{i-1}\{u_{i-1}\}$. As the inverse of $h_i$ is not a homomorphism, we have $n_i < n_{i-1}$. Since the maximal number of atoms in $q_i$ is bounded by $|\Sigma| \cdot |\var(q_T)|^2 + |\Sigma| \cdot |\var(q_T)|$, the claim follows.

Now for Point 2.

Claim 4. Let $x \in V_{\ell'}$. The number of $i \in \{\ell' + 1, \ldots, u'\}$ such that $q_i^x \not\subseteq q_i x^i$ is at most $|\Sigma|^2$.

Proof of Claim 4. Let $I$ be the set of all $i$ as in the claim. Let $i \in I$. We distinguish two cases.

(A) $h_i(x) = x$.

Since $h_i$ is a homomorphism, the number $n_i$ of atoms in $q_i^x$ is at most the number $n_{i-1}$ of atoms in $q_{i-1}x^i$. From $q_{i-1}^x \not\subseteq q_{i-1} x^{i-1}$, it follows that $n_i < n_{i-1}$.

(B) $h_i(x) \neq x$, that is, $h_i(x)$ is strictly in the subtree below $x$.

By definition of the universal model and since $O$ is in normal form, there is an atom $A(x)$ in $q_i^x$ such that there is a homomorphism from $q_i^x$ to $U_{A(x), O}$. We claim that $A(x)$ is not an atom in any query $q_j^x$ with $j > i$ and $j \in I$. Indeed, if $A(x)$ occurs in $q_j^x$ for such $j$, then $q_j^x \not\subseteq q_i^x$. The homomorphisms $h_{i+1}, \ldots, h_{i-j}$ witness that $q_i^x \not\subseteq q_{i+j} x^j \subseteq \cdots \subseteq q_j^x$, and thus all these queries are actually equivalent, in contradiction to the definition of $I$.

Next observe that, by what was said in Point (A), Point (A) can happen only $|\Sigma|$ times without Point (B) happening in between. Moreover, Point (B) can happen only $|\Sigma|^2$ times overall. We thus have that the size of $I$ is bounded by $|\Sigma|^2$. This finishes the proof of Claim 4.

Since $|V_{\ell'}| \leq |\var(q_T)|$, we obtain that the length of the sequence $q_\ell, \ldots, q_{u'}$ is bounded by 

$$(|\var(q_T)|^2 + |\var(q_T)|) \cdot |\Sigma| + |\var(q_T)| \cdot |\Sigma|^2,$$

where the first summand accommodates the number of $i$ where Point 1 is satisfied and the second summand accommodates the number of $i$ where Point 2 is satisfied.

H Proofs for Section 4

Theorem 3. $\mathcal{EL}$-concepts are not polynomial query learnable under $\mathcal{EL}$-ontologies with membership queries and $CQ$-equivalence queries.

Proof. Assume to the contrary of what is to be shown that $\mathcal{EL}$-concepts are polynomial query learnable under $\mathcal{EL}$-ontologies when unrestricted CQs can be used in equivalence queries. Then there exists a learning algorithm and a polynomial $p$ such that at any time, the sum of the sizes of the inputs to membership and equivalence queries made so far is bounded by $p(n_1, n_2, n_3)$, where $n_1$ is the size of $C_T$, $n_2$ is the size of $O$, and $n_3$ is the size of the largest counterexample seen so far.

We choose $n$ such that $2^n > p(q_1(n), q_2(n), q_3(n)) + 1$ where $q_1, q_2, q_3$ are polynomials such that for every $n \geq 1$, $q_1(n) \geq |H|$ for all $H \in H_n, q_2(n) \geq |O_n|$, and $q_3(n)$ bounds from above the size of all counterexamples returned by the oracle that we craft below.

Consider now $O_n$ and $H_n$ as defined in Section 4. We let the oracle maintain a set of hypotheses $H$, starting with $H = H_n$, and then proceeding to subsets thereof, in such a way that at any point in time the learner cannot distinguish between any of the candidate targets in $H$.

More precisely, consider a membership query with ABox $A$ and individual $a_0$. The oracle responds as follows:

1. if $A, O_n \models L_0(a_0)$, then answer yes;
2. if $A, O_n \not\models K_0(a_0)$ and there are $\sigma_1, \ldots, \sigma_n \in \{r, s\}$ with $A, O_n \models W_i^{\sigma_i}(a_0)$ for $1 \leq i \leq n$, then answer yes and remove $\exists \sigma_1 \cdots \exists \sigma_n. \exists r^n.A$ from $H$;
3. otherwise, answer no and remove all $H$ with $A, O_n \models H(a_0)$ from $H$.

Higher up rules have higher priority, e.g., Case 2 is applied only if Case 1 does not apply. It is not hard to verify that the answers are correct regarding the hypothesis set $H$ that remains after the answer is given.

Now consider an equivalence query with $CQ q_H(x_0)$. The oracle responds as follows:

1. if $L_0(a_0), O_n \not\models q_H(a_0)$, then return $\{L_0(a_0)\}$ as positive counterexample;
2. if $\{\top(a_0), L_0(a_1)\} \models q_H(a_0)$, then return $\{\Top(a_0), L_0(a_1)\}$ as negative counterexample;
3. if there are $\sigma_1, \ldots, \sigma_n \in \{r, s\}$ such that $K_0(a_0), W_1^{\sigma_1}(a_0), \ldots, W_n^{\sigma_n}(a_0), O_n \not\models q_H(a_0)$, then choose such $\sigma_1, \ldots, \sigma_n$, return $\{K_0(a_0), W_1^{\sigma_1}(a_0), \ldots, W_n^{\sigma_n}(a_0)\}$ as positive counterexample, and remove $\exists \sigma_1 \cdots \exists \sigma_n. \exists r^n.A$ from $H$ (if present).
Again, higher up rules have higher priority and the answers are always correct with respect to the updated set \( \mathcal{H} \).

For Case 3, we remark that the counterexample \( A = \{ K_0(a_0), W_1^{\sigma_1}(a_0), \ldots, W_n^{\sigma_n}(a_0) \} \) is such that \( A, O_n \models \exists \sigma_1^1 \cdot \exists \sigma_2^2 \cdot \exists \sigma_n^n . A(a_0) \) for all \( \sigma_1^1 \cdot \sigma_2^2 \cdot \cdots \cdot \sigma_n^n \in \{ r, s \}^n \) except \( \sigma_1^1 \cdot \sigma_2^2 \cdot \cdots \cdot \sigma_n^n = \sigma_1^1 \cdot \sigma_2^2 \cdot \cdots \cdot \sigma_n^n \).

We argue that the cases are exhaustive. Assume that \( A \) contains no \( r/s \)-paths \( p_1, p_2 \) of length \( i \) that end at the same individual and such that \( p_1 \) starts with an \( r \)-edge while \( p_2 \) starts with an \( s \)-edge.

In fact, the existence of such \( r/s \)-paths and the connectedness of \( A \) implies \( A, O_n \models D(a_0) \), thus \( A, O_n \models L_0(a_0) \).

We now sketch the construction of a model \( I \) of \( A \) and \( O_n \). Let \( W \subseteq \{ 0, \ldots, n \} \) contain \( i \) if \( A \) contains an assertion \( W_j^\sigma(a) \) for some \( \sigma \in \{ r, s \} \) and \( a \in \text{ind}(A) \). The following interpretations are used as building blocks for \( I \):

- an \( L_i \)-path, \( n \leq i \leq 2n \), is an \( r/s \)-path of length \( 2n - i \) that makes \( L_{i+j} \) true at the node at distance \( j \in \{ 0, \ldots, n - i \} \) from the start of the path and that makes \( A \) true at the end of the path;
- a \( K_i \)-path, \( n \leq i \leq 2n \), is defined as follows; let \( \ell \) be maximal such that \( \{ i - n, \ldots, \ell \} \subseteq W \); then a \( K_i \)-path is an \( r/s \)-path of length \( \ell - (i - n) \) that makes \( K_{i+j} \) true at the node at distance \( j \in \{ 0, \ldots, \ell \} \) from the start of the path and that makes \( A \) true at the node at distance \( 2n - i \) (if it exists); in addition, the start of the path might make true any of the concept names \( V_j^\sigma \), \( \sigma \in \{ r, s \} \) and \( 1 \leq j \leq n \), which are then all also made true by all other nodes on the path;
- \( L_i \)-tree, \( 1 \leq i \leq n \), is a binary \( r/s \)-tree of depth \( n - i \) that makes \( L_{i+j} \) true at every node on level \( j \in \{ 0, \ldots, n - i \} \); in addition, every node on level \( n - i \) is the start of an \( L_i \)-path;
- \( K_i \)-tree, \( 1 \leq i \leq n \), is a binary \( r/s \)-tree of depth \( n - i \) that makes \( K_{i+j} \) true at every node on level \( j \in \{ 0, \ldots, n - i \} \) and \( V_j^\sigma \) at every node on level at least \( j \) that is a \( \sigma \)-successor of its parent; in addition, every node on level \( n - i \) is the start of a \( K_0 \)-path; moreover, the root might make true any of the concept names \( V_j^\sigma \), \( \sigma \in \{ r, s \} \) and \( 1 \leq j \leq i \), which are then all also made true by all other nodes in the tree.

In all of the above, any node that has an incoming \( r/s \)-path of length \( i \in \{ 1, \ldots, 2n \} \) that starts with \( \sigma \in \{ r, s \} \) is additionally labeled with concept name \( U_j^\sigma \). Moreover, the beginning of the path/root of the tree may be labeled with concept names of the form \( U_j^\sigma \). Then any node on depth \( i + j \), with \( i + j \leq 2n \), is labeled with \( U_{i+j}^\sigma \).

Now, the announced model \( I \) is constructed by starting with \( A \) and doing the following:

1. exhaustively apply all concept inclusions in \( O_n \) that have a concept name on the right-hand side;
2. at every \( a \in \text{ind}(A) \), attach an infinite tree in which every node has two successors, one for each role name \( r, s \), and in which no concept names are made true;
3. if \( a \in L_i^r, 0 \leq i < n \), then attach at \( a \) an \( L_i \)-tree;
4. if \( a \in K_i^r, 0 \leq i < n \), then attach at \( a \) a \( K_i \)-tree;
5. if \( a \in L_i^s, n \leq i \leq 2n \), then attach at \( a \) an \( L_i \)-path;
6. if \( a \in K_i^s, n \leq i \leq 2n \), then attach at \( a \) a \( K_i \)-path;
7. if \( W_j^\sigma(a) \in \mathcal{A} \) for some \( a \), then make \( W_j^\sigma \) true everywhere in \( I \).
By going over the concept inclusions in $O_n$ and using Properties (a) and (b), it can be verified that $\mathcal{I}$ is indeed a model of $O_n$. In particular, the inclusions $W^*_T \cap W^*_s \subseteq L_0$ are satisfied since there is no $d \in \Delta^2$ with $d (W^*_T \cap W^*_s)$; if there was such a $d$, then by construction of $\mathcal{I}$ there are assertions $W^*_T (a)$ and $W^*_s (b)$ in $A$, in contradiction to the connectedness of $A$ and $\mathcal{A}$, $\mathcal{O}_n \not\models L_0 (a_0)$. For the Cls $U^*_T \cap U^*_s \subseteq L_0$, we argue that there is no $d \in \Delta^2$, with $d (U^*_T \cap U^*_s)$. To see this, note that there are no $U^*_T (a), U^*_s (a) \in A$ for any $a$ as otherwise $A, \mathcal{O}_n \models L_0 (a_0)$. Now consider Step 1 of the construction of $\mathcal{I}$ and assume that it adds some $a \in \text{ind}(A)$ to $(U^*_T \cap U^*_s)^2$. But this means that $A, \mathcal{O}_n \models U^*_T (a)$ and $A, \mathcal{O}_n \models U^*_s (a)$, in contradiction to $A, \mathcal{O}_n \not\models L_0 (a_0)$, due to connectedness of $A$. Given that there is no $d \in (U^*_T \cap U^*_s)^2$ for any $a$ after Step 1, it is readily checked that the elements $d$ added in Steps 2-6 also satisfy $d \not\in (U^*_T \cap U^*_s)^2$.

We now use $\mathcal{I}$ to prove the claim. Let $H'$ be the set of all $H \in H_n$ with $A, \mathcal{O}_n \models H (a_0)$. With each $H \in H'$, we associate an $\sigma_H \in \text{ind}(A)$ as follows. Let $H = \exists \sigma_1 \ldots \sigma_{2n} A$. Then $\mathcal{I}$ contains a path from $a_0$ to some element $d_H \in A^2$ that is labeled $\sigma_1 \ldots \sigma_{2n}$. If $d_H \in \text{ind}(A)$, then $a_H = d_H$. Otherwise, $d_H$ is in a path or tree attached to some $a \in \text{ind}(A)$. Set $\sigma_H = a$. To show that $|A| \geq |H \in H_n | A, \mathcal{O}_n \models H (a_0)|$ as required, it suffices to prove that $a_H \neq a_H$ whenever $H \neq H'$. Thus let $H, H' \in H'$ with $H \neq H'$, $H = \exists \sigma_1 \ldots \sigma_{2n} A$, and $H' = \exists \sigma'_1 \ldots \sigma'_{2n} A$. Assume to the contrary of what is to be shown that $a_H = a_H'$. We distinguish four cases:

• $d_H = a_H, d_H = a_H'$. Then there is a path from $a_0$ to $a_H$ in $\mathcal{I}$ labeled $\sigma_1 \ldots \sigma_{2n}$ and a path from $a_0$ to $a_H'$ labeled $\sigma'_1 \ldots \sigma'_{2n}$. By construction of $\mathcal{I}$, these paths must already exist in $A$. From $H \neq H'$, we thus obtain a contradiction to (e).

• $d_H = a_H, d_H = a_H'$. By construction of $\mathcal{I}$, $d_H \neq a_H$ implies that $A, \mathcal{O}_n \models L_i (a_{H})$ or $A, \mathcal{O}_n \models K_i (a_{H})$ for some $i$ with $0 \leq i < 2n$. Moreover, $d_H = a_H$ implies $A, \mathcal{O}_n \models A (a_H)$. Thus $A, \mathcal{O}_n \models D (a_H)$. By the connectedness of $A$, we obtain $A, \mathcal{O}_n \models D (a_H)$, thus $A, \mathcal{O}_n \models L_0 (a_0)$ in contradiction to (c).

• $d_H = a_H, d_H = a_H'$. Symmetric to previous case.

• $d_H = a_H, d_H \neq a_H'$. We first show that $d_H$ and $d_H'$ are in an $L_i$-tree, $0 \leq i < n$. Assume to the contrary that $d_H$ is the case of $d_H'$ is symmetric). Then it occurs on level $2n - i$ in the tree, since $d_H \in A^2$. Since an $L_i$-tree was attached to $a_H$, we must have $A, \mathcal{O}_n \models L_i (a_H)$. Moreover, there is an $r/s$-path in $\mathcal{I}$ from $a_0$ to $a_H$ of length $i$, the prefix of $\sigma_1 \ldots \sigma_{2n}$ of this length. By construction of $\mathcal{I}$, this path must already be in $A$. This is in contradiction to (c).

We next show that $d_H$ and $d_H'$ are in a $K_i$-tree, $0 \leq i < n$. Assume to the contrary that $d_H$ is (the case of $d_H'$ is symmetric). Then it occurs on level $2n - i$ in the tree, since $d_H \in A^2$. By definition of such trees (and the attached paths), this implies that $\forall \mathcal{I} = \{1, \ldots, n\}$ and thus $A$ contains assertions $W^*_T (a_1), \ldots, W^*_s (a_n)$. We must further have $A, \mathcal{O}_n \models K_i (a_H)$ and there is an $r/s$-path in $\mathcal{I}$, thus in $A$ from $a_0$ to $a_H$ of length $i$. This is in contradiction to (d).

Thus, $d_H$ and $d_H'$ are both in an $L_i$-path or in a $K_i$-path, $n \leq i < 2n$. If they are in different such paths, then $A, \mathcal{O}_n \models D (a_H)$, which is in contradiction to (c) as $A$ is connected. Thus, they must be in the same $L_i$-path or the same $K_i$-path. Since each such path contains a single element $d$ with $d \in A^2$, we obtain $d_H = d_H'$. From $H \neq H'$, it thus follows that there are two different paths of length $i$ in $\mathcal{I}$ from $a_0$ to $a_{H'}$, the prefixes of this length of $\sigma_1 \ldots \sigma_{2n}$ and of $\sigma'_1 \ldots \sigma'_{2n}$. This is in contradiction to (e).

This finishes the proof of the claim.

We can use the claim to show the following invariant:

(*) at any point in time, the sum $m$ of the sizes of the inputs to membership and equivalence queries made so far is not smaller than the number of candidates that were removed from $\mathcal{H}$.

Note in fact that only Cases 2 and 3 of membership queries and Cases 3 and 6 of equivalence queries may remove candidates from $\mathcal{H}$ and that they all remove only one candidate for each query posed with the exception of Case 3 of membership queries which can remove multiple candidate. However, the claim implies that the number of removed candidates in Case 3 of membership queries is bounded from above by the size of the query posed.

It is clear that there is a polynomial $q_3$ such that the size of all counterexamples returned by the oracle is at most $q_3 (n)$. The overall sum of the sizes of posed membership and equivalence queries is bounded by $p (q_1 (n), q_2 (n), q_3 (n))$. It thus follows from (*) that at most $p (q_1 (n), q_2 (n), q_3 (n))$ candidate concepts have been removed from $\mathcal{H}$. By the choice of $n$, at least two candidate concepts remain in $\mathcal{H}$ after the algorithm finished. Thus, the learner cannot distinguish between them, and we have derived a contradiction. □