Further analysis of the budgets of the dissipation tensor $\varepsilon_{ij}$ in turbulent plane channel flow

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Abstract

Recent DNS results (Gerolymos and Vallet 2016 J. Fluid Mech. 807 386–418) have provided data for the terms in the transport equations for the components of the dissipation tensor $\varepsilon_{ij}$ in low-Reynolds turbulent plane channel flow. The present paper extends the previous results by a detailed analysis of the behaviour of various mechanisms in the $\varepsilon_{ij}$-transport equations (production, diffusion, redistribution, destruction), with particular emphasis on the component-by-component comparison with the corresponding mechanisms in the transport equations for the Reynolds-stresses $r_{ij}$. The splitting of the pressure terms for the wall-normal components into redistribution and pressure-diffusion reveals substantially different behaviour near the wall. The wall-asymptotics of different terms in the transport equations are studied in detail, and examined using the DNS data. Both DNS data and wall-asympotic analysis show that the anisotropy of the destruction-of-dissipation tensor $\varepsilon_{\alpha\alpha}$ is fundamentally different from that of $r_{ij}$ or $\varepsilon_{ij}$, never approaching the two-component state at the solid wall.

Keywords: turbulence, wall and boundary-layer flows, DNS, dissipation tensor

(Some figures may appear in colour only in the online journal)

1. Introduction

Transport equations (Chou 1945) of 1-point and 2-point statistics are essential both in understanding turbulence dynamics (Tennekes and Lumley 1972) and in providing the theoretical
foundations for turbulence modelling (Schiestel 2008). The fluctuating-velocity-covariance (2-moment) tensor \( r_{ij} := u_i u'_j \), which defines the Reynolds-stresses \(-\rho \eta_{ij}\), is governed by well known transport equations (Mansour et al 1988, (1), p 17) where the dissipation tensor \( \varepsilon_{ij} \) represents the destruction of \( r_{ij} \) by molecular friction (viscosity). The dissipation tensor \( \varepsilon_{ij} \) also follows transport equations (Gerolymos and Vallet 2016a, (3.3), p 403) where the destruction-of-dissipation tensor \( \varepsilon_{ij} \) represents the destruction of \( \varepsilon_{ij} \) by molecular viscosity. Of course \( \varepsilon_{ij} \) is governed in turn by its own transport equation where appears its own destruction-rate, and so on to correlations of higher derivatives of the fluctuating velocity.

The budgets of the \( r_{ij} \)-transport equations (1a) have been studied extensively using DNS (Mansour et al 1988, Moser et al 1999, Sillero et al 2013). Closure of noncomputable terms in (1a), along with a transport equation for some scalar scale-determining variable (Jones and Launder 1972, Launder and Spalding 1974, Wilcox 1988, Menter 1994, Jakirlić and Hanjalić 2002) has led (Launder et al 1975) to the development of second-moment closures (SMCs) or Reynolds-stress models (RSMs). Several models of this family have been assessed for the computation of complex 3D flows (Gerolymos and Vallet 2001, Jakirlić et al 2007, Cécora et al 2015) and are increasingly used to predict practical 3D configurations (Eisfeld 2015). Comparisons with measurements (Rumsey 2010) demonstrate the predictive improvement of 7-equation RSMs against standard 2-equation approaches, especially in presence of separation and/or secondary flows (Gerolymos et al 2010, Gerolymos and Vallet 2016b) but also highlight remaining challenges. In general RSMs cannot return the correct wall-asymptotic behaviour for all of the components of the Reynolds-stress tensor (Yakovenko and Chang 2007), and privileging the wall-normal components improves log-law prediction (Gerolymos et al 2012). An even more difficult challenge is to correctly mimic the \( Re \)-dependence of the near-wall maxima of the diagonal Reynolds-stresses which is revealed by DNS results (Lee and Moser 2015). Finally, the hysteretic behaviour of the separation-and-reattachment process (Gerolymos et al 1989) may require additional specific lag-treatments (Olsen and Coakley 2001).

The correct prediction of near-wall anisotropy (Durbin 1993) and of lengthscale anisotropy in general (Lumley et al 1999) is necessary to meet these challenges. The replacement of the scalar scale-determining equation used in classical RSMs (Wilcox 2006, Schiestel 2008) by transport equations for the individual components of \( \varepsilon_{ij} \) has been suggested to overcome the unsatisfactory \( \text{a posteriori} \) performance of algebraic \( \varepsilon_{ij} \)-closures (Gerolymos et al 2012). Detailed DNS data of the \( \varepsilon_{ij} \)-transport equations (1b) are necessary to achieve this goal.

Scrutiny of the budgets of the scalar \( \varepsilon \)-equation (\( \varepsilon := \frac{1}{2} \varepsilon_{mm} \)) provided by DNS (Mansour et al 1988) has proved particularly useful in improving the closure of this equation (Lai and So 1990, Rodi and Mansour 1993, Jakirlić and Hanjalić 2002). On the other hand, very little work has been done concerning the budgets of the tensorial \( \varepsilon_{ij} \)-equations (1b). In a recent work (Gerolymos and Vallet 2016a) we have generated DNS data of \( \varepsilon_{ij} \)-budgets for low-\( Re \) turbulent plane channel flow and discussed the behaviour of various terms in (1b), with particular emphasis on the 4 production mechanisms.

The purpose of the present work is to further analyse \( \varepsilon_{ij} \)-budgets in turbulent plane channel flow, and in particular the similarities and differences with respect to \( r_{ij} \)-budgets. In section 2 we define the terms in the transport equations for \( r_{ij} \) and \( \varepsilon_{ij} \), and calculate the wall-asymptotic behaviour of different terms in the \( \varepsilon_{ij} \)-transport equations (1b) for the particular case of turbulent plane channel flow. These analytical results are used (section 3) to assess very-near-wall DNS data. In section 3 we use DNS data (section 3.1) to compare \( r_{ij} \)-budgets with \( \varepsilon_{ij} \)-budgets (section 3.2) and to analyse the splitting of the pressure term \( \Pi_{ij} \) in (1b) into a redistributive and a conservative term (section 3.3). In section 4 we compare the anisotropy and associated anisotropy invariant mapping (AIM) of the Reynolds-stresses \( r_{ij} \), their
dissipation $\varepsilon_i$ and the destruction-of-dissipation $\varepsilon_{ii}$, which exhibits a notably different componentality near the wall. Finally, in section 5, we summarise the main results of the present work.

2. Transport equations and wall asymptotics

Consistent with the DNS data, we study incompressible flow with a Newtonian constitutive relation in an inertial frame (Gerolymos and Vallet 2016a). We use a Cartesian reference-frame $x_i \in \{x, y, z\}$, note $u_i \in \{u, v, w\}$ the corresponding components of the velocity vector, and use Reynolds decomposition into averaged $(\bar{\cdot})$ and fluctuating $(\cdot)'$ quantities, we note $t$ the time, $\rho \equiv \text{const}$ the density, $p$ the pressure, $\nu \equiv \text{const}$ the kinematic viscosity, and $\mu = \rho \nu \equiv \text{const}$ the dynamic viscosity.

2.1. Transport equations

Straightforward manipulations of the fluctuating momentum (B.7) and of the fluctuating continuity (B.3) equations and of their gradients lead to the transport equations for $r_{ij} := u_i' u_j'$ (Mansour et al. 1988, (1), p 17)

\[
\frac{\rho_j}{c_j} \frac{\partial u_i' u_j'}{\partial t} + \rho_j \frac{\partial u_i' u_j'}{\partial x_{\ell}} - \rho_j \frac{\partial \bar{u}_i}{\partial x_{\ell}} \bar{u}_j = \rho_j \frac{\partial}{\partial x_{\ell}} \left( \mu_j \frac{\partial \bar{u}_i}{\partial x_{\ell}} + (-\rho_j \nu u_i' u_j' - u_i' \frac{\partial \bar{p}}{\partial x_{\ell}}) - \frac{\partial \bar{u}_i}{\partial x_{\ell}} \frac{\partial \bar{u}_j}{\partial x_{\ell}} \right) \quad (1a)
\]

and $\varepsilon_{ij}$ (Gerolymos and Vallet 2016a, (3.3), p 403)

\[
\frac{\rho_j}{c_j} \frac{\partial \varepsilon_{ij}}{\partial t} + \rho_j \frac{\partial \varepsilon_{ij}}{\partial x_{\ell}} = \rho_j \frac{\partial}{\partial x_{\ell}} \left( \mu_j \frac{\partial \varepsilon_{ij}}{\partial x_{\ell}} - \rho_j \left( \frac{2}{\mu_j} \frac{\partial u_i' u_j'}{\partial x_{\ell}} \frac{\partial \bar{u}_j}{\partial x_{\ell}} + \frac{\partial \bar{u}_j}{\partial x_{\ell}} \frac{\partial \bar{u}_i}{\partial x_{\ell}} \right) \right) \quad (1b)
\]

which were reproduced here for completeness.
The common origin of (1a), (1b) leads to analogous mechanisms in both transport equations, where convection by the mean flow ($C_{ij}$, $C_{y}$) is balanced by 5 mechanisms: diffusion by molecular viscosity ($d^{(a)}_{ij}$, $d^{(a)}_{y}$), turbulent diffusion (mixing) by the fluctuating velocity field $u'_{ij}$ ($d^{(a)}_{ij}$, $d^{(a)}_{y}$), production by various mechanisms ($P_{ij}$, $P_{y} = P^{(1)}_{ij} + P^{(2)}_{ij} + P^{(3)}_{ij} + P^{(4)}_{ij}$), the fluctuating-pressure mechanisms ($\Pi_{ij}$, $\Pi_{y}$), and destruction by molecular viscosity ($\varepsilon_{ij}$, $\varepsilon_{y}$). Of course the tensorial componentality (Lumley 1978, Kassinos et al 2001, Simonsen and Krogstad 2005) and the scaling (Tennekes and Lumley 1972, pp 88–92) of various terms in (1b) differs from that of the corresponding terms in (1a).

2.2. Wall asymptotics

Before studying the present DNS data for the $\varepsilon_{y}$-transport budgets (3.2), it is useful to summarise the theoretically expected (appendix B) asymptotic behaviour of various terms in the viscous sublayer, or, formally, as $y^+ \to 0$. Inner scaling (Buschmann and Gad-el-Hak 2007, ...) is consistently used in these calculations (appendix B). Wall-asymptotics of the terms in (1b) which only involve fluctuating velocities and their derivatives ($d^{(a)}_{ij}$, $d^{(a)}_{y}$, $P^{(3)}_{ij}$, $\varepsilon_{ij}$) can be readily obtained from the Taylor-series expansions (Riley et al 2006, section 4.6, pp 136–141) of $u_{ij}^{+}$ in the wall-normal direction $y^+$

\[
(\cdot)^{+} \sim \left( \lambda^{a}_{ij}(x^{+}, z^{+}, t^{+}) + A^{(a)}_{ij}(x^{+}, z^{+}, t^{+}) y^{+} + B^{(a)}_{ij}(x^{+}, z^{+}, t^{+}) y^{+2} \\
+ C^{(a)}_{ij}(x^{+}, z^{+}, t^{+}) y^{+3} + D^{(a)}_{ij}(x^{+}, z^{+}, t^{+}) y^{+4} + \ldots \right)
\]

under the constraints of the no-slip condition at the wall (A.1a) and of the fluctuating continuity equation (section B.1). On the contrary, determination of the wall-asymptotics of terms in (1b) which contain the fluctuating pressure and its derivatives ($\Pi_{ij}$) or the mean-flow velocities and their derivatives ($C_{ij}$, $P^{(1)}_{ij}$, $P^{(2)}_{ij}$, $P^{(3)}_{ij}$), requires specific simplifications implied by the fully developed plane channel flow conditions (A.1)–(A.3), in line with the analysis of the budgets of $r_{ij}$ and $\varepsilon$ in Mansour et al (1988). Using (2), (B.4), (B.5), along with specific results (B.6)–(B.12) applicable to plane channel flow satisfying conditions (A.1)–(A.3), readily yields the wall-asymptotic expansions (tables 1, 2) of various terms in the $\varepsilon_{y}$-transport equation (1b). The homogeneity relations (A.3) were used, when applicable to simplify these expressions. The plane channel flow identity $B^{(a)}_{ij}/C_{ij}$ (B.11b) was used in $\varepsilon_{ij}^{+}$, $d^{(a)+}_{ij}$ and $\Pi_{ij}$ (table 1), while the plane channel flow identity (B.12) was used to replace $B^{(a)}_{ij}/C_{ij} / \Pi_{ij}$ in $\Pi_{ij}$ (table 1). These results (tables 1, 2) are used in the analysis of the DNS data (section 3).

3. Turbulent plane channel flow budgets

DNS data generated for plane channel flow (section 3.1) illustrate how corresponding mechanisms in the transport equations of $r_{ij}$ (1a) or $\varepsilon_{ij}$ (1b) contribute to the budgets of different components (section 3.2). In direct analogy to $r_{ij}$-transport (Mansour et al 1988), the fluctuating-pressure mechanisms in $\varepsilon_{y}$-transport (1b), $\Pi_{y}$, can be analysed (section 3.3) as the sum of a traceless redistributive term $\phi_{ij}$ and a conservative pressure-diffusion part $d^{(a)}_{ij}$.

3.1. DNS computations

The DNS computations from which the present data were extracted are described in Gerolymos and Vallet (2016a). They were obtained for low $Re_{tu} \approx 180$ plane channel flow
Table 1. Asymptotic (as \( y^+ \to 0 \)) expansion (2) of various terms \( (d_{ij}^{(p)}_w, d_{ij}^{(p)}_e, d_{ij}^{(p)}_f, \phi_{ij}, \Pi_{ij}) \) in the \( \varepsilon_{ij} \)-transport equation (1b), in wall-units (Gerolymos and Vallet 2016a, (A3), p 414), for the particular case of plane channel flow (A.1)–(A.3).

\[
\begin{align*}
\frac{d_{ij}^{(p)}_w}{d_{ij}^{(p)}_e} & \sim 4(6A_{ij}^{(p)}C_{ij}^{(p)} + 4B_{ij}^{(p)} + (\nabla A_{ij}^{(p)})^2) + 24(4A_{ij}^{(p)}D_{ij}^{(p)} + 6B_{ij}^{(p)} + (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\frac{d_{ij}^{(p)}_e}{d_{ij}^{(p)}_f} & \sim 4(3A_{ij}^{(p)}C_{ij}^{(p)} + 4B_{ij}^{(p)} + (\nabla A_{ij}^{(p)})^2) + 12(4A_{ij}^{(p)}D_{ij}^{(p)} + 6B_{ij}^{(p)} + (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\frac{d_{ij}^{(p)}_f}{\Pi_{ij}} & \sim 16B_{ij}^{(p)} + 144B_{ij}^{(p)} + (\nabla A_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\Pi_{ij} & \sim 8B_{ij}^{(p)} + 8 \left( 3C_{ij}^{(p)} + 2B_{ij}^{(p)} \right) - (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\Pi_{ij} & \sim 8B_{ij}^{(p)} + 8 \left( 3C_{ij}^{(p)} + 2B_{ij}^{(p)} \right) - (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\Pi_{ij} & \sim 8B_{ij}^{(p)} + 8 \left( 3C_{ij}^{(p)} + 2B_{ij}^{(p)} \right) - (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\phi_{ij} & \sim \Pi_{ij}^+, \quad \forall y^+ \\
\phi_{ij} & \sim 8B_{ij}^{(p)} + 4(6A_{ij}^{(p)} + (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\phi_{ij} & \sim 16B_{ij}^{(p)} + 144B_{ij}^{(p)} + (\nabla A_{ij}^{(p)}) + (\nabla B_{ij}^{(p)})^2)y^+ + O(y^{+2}) \\
\phi_{ij} & \sim \Pi_{ij}^+, \quad \forall y^+ 
\end{align*}
\]

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Table 2. Asymptotic (as \( y^+ \to 0 \)) expansions (2) of the various mechanisms of production \( P \equiv P^{(1)} + P^{(2)} + P^{(3)} + P^{(4)} \) and of the destruction-of-dissipation \( \varepsilon \), appearing in the \( i_{ij} \)-transport equations (1b), in wall-units (Gerolymos and Vallet 2016a, (A3), p 414), for the particular case of plane channel flow (A.1)–(A.3).

\[
\begin{align*}
\varepsilon_{xx}^+ & \sim 8(2B^{(1)}_{xx} + (\nabla A^{(1)}_{xx} y^+)^2) + 32(3 B^{(2)}_{xx} C^{(2)}_{xx} + (\nabla A^{(2)}_{xx})^2 - (\nabla B^{(2)}_{xx})^2) y^+ + O(y^{+2}) \\
\varepsilon_{yy}^+ & \sim 16B^{(1)}_{yy} B^{(1)}_{yy} + 16(3B^{(2)}_{yy} C^{(2)}_{yy} + (\nabla A^{(2)}_{yy})^2 - (\nabla B^{(2)}_{yy})^2) y^+ + O(y^{+2}) \\
\varepsilon_{zz}^+ & \sim 16 B^{(1)}_{zz} y^+ + 96 B^{(1)}_{zz} C^{(1)}_{zz} y^+ + O(y^{+2}) \\
\varepsilon_{zz}^+ & \sim 8(2B^{(1)}_{zz} + (\nabla A^{(1)}_{zz})^2) + 32(3B^{(2)}_{zz} C^{(2)}_{zz} + (\nabla A^{(2)}_{zz})^2 - (\nabla B^{(2)}_{zz})^2) y^+ + O(y^{+2})
\end{align*}
\]

\[
\begin{align*}
P_{xx}^{(1)} & \sim -8 A^{(1)}_{xx} B^{(1)}_{xx} y^+ + \frac{4 (A^{(0)}_{xx} B^{(0)}_{xx} - 12 A^{(1)}_{xx} C^{(1)}_{xx} - 16 B^{(1)}_{xx} C^{(1)}_{xx})}{Re_{\infty}} y^{+2} + O(y^{+3}) \\
P_{yy}^{(1)} & \sim -8 B^{(1)}_{yy} B^{(1)}_{yy} y^{+2} + 8 \left( \frac{4 A^{(2)}_{yy} B^{(2)}_{yy} - 3 B^{(1)}_{yy} C^{(1)}_{yy}}{Re_{\infty}} \right) y^{+3} + O(y^{+4}) \\
P_{yy}^{(2)} & = P_{zz}^{(2)} = 0 \quad \forall y^+ \\
P_{yy}^{(3)} & \sim -4 \frac{\partial B^{(3)}_{yy}}{\partial x} y^{+2} - 4 \left( \frac{2 C^{(2)}_{yy} + \partial A^{(2)}_{yy}}{Re_{\infty}} - \frac{1}{Re_{\infty}} \right) B^{(1)}_{yy} C^{(1)}_{xx} y^{+3} + O(y^{+4}) \\
P_{yy}^{(4)} & \sim -4 \frac{\partial C^{(3)}_{yy}}{\partial x} y^{+3} - 4 \left( \frac{2 \partial C^{(2)}_{yy}}{\partial x} \right) B^{(1)}_{yy} C^{(1)}_{xx} y^{+4} + O(y^{+5}) \\
P_{zz}^{(3)} & = P_{zz}^{(3)} = 0 \quad \forall y^+ \\
P_{zz}^{(4)} & \sim -12 A^{(2)}_{zz} B^{(3)}_{zz} y^+ - 4 \left( 6 A^{(2)}_{zz} C^{(2)}_{zz} + 8 A^{(2)}_{zz} B^{(2)}_{zz} B^{(1)}_{zz} - A^{(3)}_{zz} B^{(1)}_{zz} + \frac{\partial A^{(3)}_{zz}}{\partial x} + \frac{\partial A^{(3)}_{zz}}{\partial x} \right) y^{+2} + O(y^{+3}) \\
P_{yy}^{(5)} & \sim -2 \left( 8 A^{(3)}_{yy} B^{(3)}_{yy} + A^{(3)}_{yy} - A^{(3)}_{yy} B^{(1)}_{yy} + \frac{\partial A^{(3)}_{yy}}{\partial x} + \frac{\partial A^{(3)}_{yy}}{\partial x} \right) y^{+3} + O(y^{+4}) \\
& -4 \left( 16 A^{(2)}_{yy} B^{(2)} C^{(2)}_{yy} + 8 B^{(1)}_{yy} B^{(3)}_{yy} - B^{(1)}_{yy} B^{(1)}_{yy} + \frac{\partial A^{(3)}_{yy}}{\partial x} + \frac{\partial A^{(3)}_{yy}}{\partial x} \right) \left( 6 A^{(2)}_{yy} C^{(2)}_{yy} + 8 A^{(2)}_{yy} B^{(2)}_{yy} B^{(1)}_{yy} - A^{(3)}_{yy} B^{(1)}_{yy} + \frac{\partial A^{(3)}_{yy}}{\partial x} + \frac{\partial A^{(3)}_{yy}}{\partial x} \right) y^{+4} + O(y^{+5}) \\
P_{zz}^{(4)} & \sim -12 A^{(2)}_{zz} B^{(3)}_{zz} y^+ - 4 \left( 6 A^{(2)}_{zz} C^{(2)}_{zz} + 8 A^{(2)}_{zz} B^{(2)}_{zz} B^{(1)}_{zz} - A^{(3)}_{zz} B^{(1)}_{zz} + \frac{\partial A^{(3)}_{zz}}{\partial x} + \frac{\partial A^{(3)}_{zz}}{\partial x} \right) y^{+2} + O(y^{+3})
\end{align*}
\]
using a very-high-order (Gerolymos et al 2009) finite-volume solver (Gerolymos et al 2010) which has been thoroughly validated by comparison with available (Moser et al 1999, Hoyas and Jiménez 2008, Vreman and Kuerten 2014a, 2014b, 2016, Lee and Moser 2015) 1-point and 2-point DNS data (Gerolymos et al 2010, 2013, Gerolymos and Vallet 2014, 2016a).

The terms in $\varepsilon_{ij}$-transport (1b) contain correlations of 1-order-higher derivatives of fluctuating quantities compared to the corresponding terms in $r_{ij}$-transport (1a). Therefore, terms in the $\varepsilon_{ij}$-transport equations (1b) are more sensitive to computational truncation errors (Gerolymos 2011), requiring finer grids to achieve the same accuracy as the corresponding terms in the $r_{ij}$-transport equations (1a). Furthermore, scaling analysis (Tennekes and Lumley 1972, pp 88–92) substantiates that terms in $\varepsilon_{ij}$-transport (1b) are generally related with Taylor-microscale and/or Kolmogorov-scale structures, again suggesting that finer grids are required to obtain these terms than $\varepsilon_{ij}$ itself. Accordingly, the computational grid resolution (figures 1, 4) was high both streamwise ($\Delta x^+ \approx 5.6$) and spanwise ($\Delta z^+ \approx 1.9$) to correctly predict the details of the elongated near-wall structures (Gerolymos et al 2010, figures 12–15, pp 802–805). Finally, several of the terms in $\varepsilon_{ij}$-transport (1b) present important variations in the viscous sublayer ($0 < y^+ \lesssim 3$; figure 1), requiring a fine wall-normal grid, not only at the wall ($\Delta y^+_w \approx 0.22$ was found sufficient), but with weak cell-size stretching to ensure good resolution in the entire near-wall region ($N_{yi,10} = 26$ points in the region $0 \lesssim y^+ < 10$) and actually throughout the entire channel up to the centerline ($\Delta y^+_c \approx 3.1$). The streamwise resolution is similar to the finest grid used in Vreman and Kuerten (2016) while the present spanwise resolution is roughly twice finer. On the other hand, the present wall-normal resolution is roughly twice coarser compared to Vreman and Kuerten (2016). Although Vreman and Kuerten (2016) did not study the dissipation tensor, their data include the terms in the transport-equations for the variances of the velocity-derivatives (Vreman and Kuerten 2014b), which can be combined (Gerolymos and Vallet 2016a) to obtain the transport equations for the diagonal terms [$\varepsilon_{xx}$, $\varepsilon_{yy}$, $\varepsilon_{zz}$] (but not for the shear term $\varepsilon_{xy}$). The 2 sets of data are in very good agreement (Gerolymos and Vallet 2016a, figures 8, 9, pp 410, 411).

Correlations in (1b) were computed using order-4 inhomogeneous-grid interpolating polynomials (Gerolymos 2012) and sampled at every iteration ($\Delta \tau^+ = \Delta t^+ \approx 0.0059$) for an observation interval $t_{obs} \approx 1113$. Because of the relatively short observation interval, the pressure term $\Pi_{eq}$ (1b) which contains the highly intermittent pressure-Hessian (Vreman and Kuerten 2014b, figure 12, p 21), was calculated from the identity $\Pi_{eq} = d_{eq}^{(h)} + \phi_{eq}$ (4). The rhs terms in (4) only involve fluctuating pressure-derivatives and converge much faster.

### 3.2. $\varepsilon_{ij}$ versus $r_{ij}$ budgets

Comparison (figure 1) of the budgets of the Reynolds-stresses $r_{ij}$ (1a) with those of the dissipation tensor $\varepsilon_{ij}$ (1b), for plane channel flow (appendix A.2), reveals fundamental differences, both in the relative importance of various mechanisms in the budgets of each component and in the componentality of corresponding mechanisms.

Regarding the importance of different mechanisms in the budgets, it is noticeable that the pressure term $\Pi_{eq}^{xx}$ is negligibly small both for the streamwise $\varepsilon_{xx}$ and the spanwise $\varepsilon_{zz}$ components (figure 1). This difference is especially important in the budgets of the spanwise components, $r_{zz}^{xx}$ and $r_{zz}^{zz}$. For the spanwise stress $r_{zz}^{zz}$, in plane channel flow (A.1)–(A.3) there is no production mechanism ($P_{zz}^{zz} = 0 \ \forall \ y^+$) and gain comes mainly from the redistributive action of $\Pi_{zz}^{zz}$ (figure 1). On the contrary, for the spanwise dissipation $\varepsilon_{zz}^{zz}$ gain comes mainly from the production terms $P_{zz}^{zz} + P_{zz}^{zz}$ (A.5d), the pressure term $\Pi_{zz}^{zz}$ being very weak.
Figure 1. Budgets, in wall-units (Gerolymos and Vallet 2016a, (A3), p 414), of the transport equations for the dissipation tensor $\varepsilon_{ij}$ (1b) and for the Reynolds-stresses $r_{ij}$ (1a), from the present DNS computations of turbulent plane channel flow ($Re_\tau \approx 180$), plotted against the inner-scaled wall-distance $y^+$ (logscale and linear wall-zoom).
in the transport equations for the dissipation tensor \( \varepsilon_{ij} \) and for the Reynolds-stresses \( r_{ij} \) (1a), from the present DNS computations of turbulent plane channel flow (\( Re_n \approx 180 \)), plotted against the inner-scaled wall-distance \( y^+ \) (logscale and linear wall-zoom).

(figure 1). Comparison of the componentality of \( \Pi_{ij}^+ \) with that of \( \Pi_{ij}^+ \) (figure 2) reveals that, although all the components of each tensor are of the same order-of-magnitude, \( \Pi_{ij}^+ \) is consistently weaker than the other components of \( \Pi_{ij}^+ \) contrary to \( \Pi_{ij}^+ \) which is the largest component of \( \Pi_{ij}^+ \) near the wall (\( y^+ \leq 10 \); figure 2). Another important difference is observed in the limiting behaviour of \( \Pi_{ij}^+ \) and \( \Pi_{ij}^+ \) both of which are \( \approx 0 \) at the wall (table 1) whereas \( \Pi_{ij}^+ \) \( \approx 0 \) because of the no-slip condition (A.1a).

The \( y^+ \)-distribution (figure 3) of the destruction-of-dissipation tensor \( \varepsilon_{ij}^+ \) (1b) differs substantially from that of the dissipation tensor \( \varepsilon_{ij}^+ \) (1a). Away from the wall, the streamwise components \( \varepsilon_{xx}^+ \) and \( \varepsilon_{yy}^+ \) are in both cases much larger than the other components. Near the wall \( \varepsilon_{xx}^+ \) forms a small plateau (\( y^+ \in [8, 12] \); figure 3) and then increases as \( y^+ \to 0 \), reaching its global maximum at the wall, remaining by far the largest component of \( \varepsilon_{ij}^+ \) \( y^+ \) (figure 3). On the contrary, \( \varepsilon_{zz}^+ \) reaches its global maximum at \( y^+ \approx 7 \) and then decreases as \( y^+ \to 0 \). At the same time \( \varepsilon_{zz}^+ \) sharply increases near the wall, the 2 components crossing each other at \( y^+ \approx 0.7 \) (figure 3) to reach \( [\varepsilon_{zz}^+]_{\text{wall}}^+ \approx [\varepsilon_{zz}^+]_{\text{wall}}^+ \). The wall-asymptotic expansion of \( \varepsilon_{zz}^+ \), as \( y^+ \to 0 \), shows (table 2) that all of the \( \varepsilon_{ij}^+ \)-components are \( \approx 0 \) at the wall in contrast to \( \varepsilon_{ij}^+ \), for which \( [\varepsilon_{yy}^+]_{\text{wall}}^+ = [\varepsilon_{yy}^+]_{\text{wall}}^+ = 0 \) (Mansour et al. 1988, 16, 21, pp 21–22). Another difference in the componentality of the 2 tensors (figure 3) is that while \( \varepsilon_{xy}^+ < 0 \ \forall \ y^+ \in [0, \delta^+] \), \( \varepsilon_{xx}^+ \leq 0 \ \forall \ y^+ \leq 3 \) changes sign further away from the wall (\( \varepsilon_{xx}^+ > 0 \ \forall \ y^+ \geq 3 \); figure 3). Therefore, while \( -\varepsilon_{xy}^+ > 0 \ \forall \ y^+ \in [0, \delta^+] \) is a loss mechanism in the budgets of \( r_{xy}^+ < 0 \ \forall \ y^+ \in [0, \delta^+] \) (figure 1), this is not the case for \( -\varepsilon_{xy}^+ \) which is, in the major part of the channel (\( y^+ \geq 3 \); figure 1), a gain mechanism in the \( \varepsilon_{xy}^+ \)-budgets. The componentality differences between \( r_{ij}^+ \), its dissipation \( \varepsilon_{ij}^+ \) and the destruction-of-dissipation \( \varepsilon_{ij}^+ \) are further studied in section 4.

The most striking componentality difference concerns the production mechanisms, \( P_{ij}^+ \) (1a) and \( P_{ij}^+ \) (1b). In plane channel flow, all of the components of \( P_{ij}^+ \) are generally \( \approx 0 \) and contribute as gain to the corresponding \( \varepsilon_{ij}^+ \) component (figure 1), contrary to \( P_{ij}^+ \) in plane channel flow.
Mansour et al. (1988). The production mechanisms (1b) \( P^{(1)+}_{ij} \) (by the direct action of the components of \( \varepsilon_{ij} \) on the mean velocity-gradient) and \( P^{(3)+}_{ij} \) (related to the mean velocity-Hessian) have a similar componentality in plane channel flow (A.1–A.3), but this is not the case for the second production by mean velocity-gradient mechanism \( P^{(2)+}_{ij} \) nor for the production by the triple correlations of fluctuating velocity-gradients \( P^{(4)+}_{ij} \), both of which are generally \( \neq 0 \) for all of the components (Gerolymos and Vallet 2016a, figure 6, p 407).

At the wall (\( y^+ = 0 \)), production \( P^{+}_{ij} \) and turbulent diffusion by the fluctuating velocities \( d^{(w)}_{ij} \) are 0

\[
(\text{Tables 1, 2}) \implies [P^{+}_{ij}]_w = [d^{(w)}_{ij}]_w = 0
\]

so that the wall-budgets of the \( \varepsilon_{ij} \)-transport equations (A.4), (A.5) reduce to

\[
(\text{Tables 1, 2}) \implies [d^{(p)}_{ij}]_w + [\Pi^{+}_{ij}]_w = [\varepsilon_{ij}]_w.
\]

In the particular case of the wall-normal diagonal component \([\Pi^{+}_{yy}]_w = 0 \) (table 1), implying \([d^{(p)}_{yy}]_w = [\varepsilon_{yy}]_w = 16 B_v^{T+2} \) (tables 1, 2). Notice also that, by (B.5), the halftrace \( \frac{1}{2}[\Pi^{+}_{yy}]_w \) (table 1) = \( 8 B_v^{T+2} \) in agreement with Mansour et al. (1988, (24), p 24).

### 3.3. Redistribution and pressure-diffusion

In exact analogy with \( r_{ij} \)-transport (1a), where by application of the product-rule of differentiation (Riley et al. 2006, section 2.12, pp 44–46), the velocity/pressure-gradient correlation \( \Pi_{ij} \) (1a) can be split into pressure diffusion \( d^{(p)}_{ij} \) and a redistributive term \( \phi_{ij} \).
In SMCs, the pressure term \( \Pi_{ij} \) can be split into pressure diffusion \( d^{(p)}_{ij} \) and a redistributive term \( \phi_{ij} \), viz

\[
\Pi_{ij} = \frac{\partial}{\partial x_k} \left( -\beta_{ij} u_j' u_i' - \delta_{ij} u_i' u_i' \right) + p' \left( \frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j} \right)
\]

with

\[
\phi_{mm} = 0 \implies \Pi_{mm} = d^{(p)}_{mm} = \frac{\partial}{\partial x_k} (-2u_t' u_i')
\]

the pressure term \( \Pi_{e_y} \) in (1b) can be split into pressure diffusion \( d^{(p)}_{e_y} \) and a redistributive term \( \phi_{e_y} \), viz

\[
\Pi_{e_y} = \frac{\partial}{\partial x_k} \left( -2\beta_{e_y} \frac{\partial u_j'}{\partial x_k} - 2\delta_{e_y} u_j' u_i' \right) + 2\nu \frac{\partial p'}{\partial x_k} \left( \frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j} \right)
\]

with

\[
\phi_{e_{mn}} = 0 \implies \Pi_{e_{mn}} = d^{(p)}_{e_{mn}} = \frac{\partial}{\partial x_k} \left( -4\epsilon u_t' \frac{\partial u_i'}{\partial x_k} \right)
\]

Because of the incompressible fluctuating continuity (B.3), \( \phi_{ij} \) (5a) is traceless, exactly like \( \phi_{ij} \) (4). Therefore it does not appear in the transport equation for the dissipation-rate \( \varepsilon \) of the turbulence kinetic energy (Mansour et al. 1988, (23), p 23) and has a redistributory role among components of \( \varepsilon_{ij} \). In SMCs, \( \phi_{ij} \) (4) occupies a central place (Launder et al. 1975, Speziale et al. 1991, Gerolymos et al. 2012, Jakirlić and Hanjalić 2013) in modelling work, because pressure diffusion \( d^{(p)}_{ij} \) is absent in homogeneous flows. It is therefore interesting to investigate (figure 4) the splitting (5) of \( \Pi_{e_y} \) in comparison with the splitting of \( \Pi_{ij} \) (4). Since only y-components of second-moments of fluctuating quantities are \( \neq 0 \) in plane channel flow (A.3) the splittings (4), (5) are only relevant for the wall-normal and the shear components (in plane channel flow \( d^{(p)}_{e_y} = d^{(p)}_{xx} = d^{(p)}_{zz} = 0 \))

As already observed in the analysis of \( r_{ij} \)-transport (Mansour et al. 1988), pressure diffusion is generally weak away from the wall, so that (figure 4) both \( \Pi_{e_y} \approx \phi_{e_y} \approx y^+ \lesssim 10 \) and \( \Pi_{e_y} \approx \phi_{e_x} \approx y^+ \lesssim 10 \). These approximate equalities also apply for the shear components, but for higher \( y^+ \gtrsim 30 \) (figure 4). This implies that modelling \( \phi_{e_y} \) in lieu of \( \Pi_{e_y} \) in the log-region of the velocity profile (Coles 1956) could be a reasonable working choice, exactly like in \( r_{ij} \)-transport models (Launder et al. 1975). On the other hand, nearer to the wall \( (1 \lesssim y^+ \lesssim 10; \text{figure 4}) \) the splittings of \( \Pi_{e_x} \) (5) and \( \Pi_{ij} \) (4) are quite different. Regarding \( \Pi_{ij} \), both \( \Pi_{ix} \) and \( \Pi_{iy} \) are very small for \( y^+ \lesssim 5 \), so that \( \phi_{ij} \approx d^{(p)}_{ij} \approx y^+ \lesssim 8 \) and \( \phi_{ij} \approx -d^{(p)}_{ij} \approx y^+ \lesssim 4 \) (figure 4), but this does not apply to \( \Pi_{e_y} \). Notice also that while \( \Pi_{e_y} \approx \phi_{e_y} \approx y^+ \lesssim 4 \) (figure 4). These differences in near-wall behaviour between \( \Pi_{e_y} \) and \( \Pi_{ij} \) should be kept in mind in modelling efforts of the pressure terms in differential \( \varepsilon_{ij} \)-transport closures.
4. Destruction-of-dissipation tensor \( \varepsilon_{xy} \)

The diagonal components and traces of the 3 tensors

\[
\begin{align*}
\varepsilon_{xx} &= 2\nu \frac{\partial u_x'}{\partial x} \frac{\partial u_x'}{\partial x} \quad \varepsilon_{yy} = \frac{1}{2} \varepsilon_{xx}, \\
\varepsilon_{zz} &= 4\nu \frac{\partial^2 u_z'}{\partial x^2} \frac{\partial^2 u_z'}{\partial x^2} \quad \varepsilon_{zz} = \frac{1}{2} \varepsilon_{xx}.
\end{align*}
\]

are positive in every frame-of-reference. Therefore these tensors are positive-definite (Gerolymos and Vallet 2016a) implying that the invariants (Rivlin 1955) of the corresponding
traceless anisotropy tensors (Gerolymos et al 2012)

\[
b_{ij} = \frac{u_i u_j}{2k} - \frac{1}{3} \delta_{ij}; \quad \Pi_b = -\frac{1}{2} b_{mk} b_{km}; \quad \Pi_{b} = \frac{1}{3} b_{mk} b_{km} b_{lm}, \quad (7a)
\]

\[
b_{ij} = \frac{e_i}{2\varepsilon} - \frac{1}{3} \delta_{ij}; \quad \Pi_{b} = \frac{1}{2} b_{mk} b_{mk}; \quad \Pi_{b} = \frac{1}{3} b_{mk} b_{mk} b_{mk}, \quad (7b)
\]

\[
b_{xx} = \frac{e_{xx}}{2\varepsilon} - \frac{1}{3} \delta_{xx}; \quad \Pi_{b_{xx}} = \frac{1}{2} b_{xx} b_{xx}; \quad \Pi_{b_{xx}} = \frac{1}{3} b_{xx} b_{xx} b_{xx}, \quad (7c)
\]

lie within Lumley (1978) realisability triangle in the (III, –II)-plane (Gerolymos and Vallet 2016a). Lumley (1978) flatness parameters

\[
A = 1 + 27\Pi_{b} + 9\Pi_{b_{xx}}; \quad A_r = 1 + 27\Pi_{b_{xx}} + 9\Pi_{b_{xx}},
\]

are bounded in the interval [0, 1] (Lumley 1978), between the two-component (2-C) limit corresponding to the value 0 and the isotropic componentality corresponding to the value 1 (Simonsen and Krosgstad 2005). It is well known (Mansour et al 1988) that at the wall both \(r\) and \(\varepsilon\) reach the 2-C limit at the wall. It was recently shown (Gerolymos and Vallet 2016a) that the 2-C limit at the wall is approached quadratically (\(A_r \approx \gamma^{-1} \approx 0\)). This result was obtained by calculating the wall-asymptotic expansions of \(b\) (Gerolymos and Vallet 2016a, table 1, p 392) and of \(b_{xx}\) (Gerolymos and Vallet 2016a, table 2, p 393) and of their invariants. However, as shown previously (figure 3) \(\varepsilon_{xx}\) is not 2-C at the wall, where all of its components are generally \(\neq 0\) (tables 2, 3).

These differences in behaviour are better understood by considering (figure 5) the anisotropy tensors \(\{b, b_{xx}, b_{xx}\}\) and their invariants (7). Although the shear components \(\{r_{xy}^0, e_{xy}^0, e_{xy}^0\}\) are invariably much smaller than the traces \(\{k^+ , e^0 , e^0\}\) (6), their anisotropy (figure 5) highlights some fundamental differences between the 3 tensors. The shear Reynolds-stress \(r_{xy}^0 < 0 \forall y^+ \in [0, \delta^+]\) (sign \(r_{xy} = \text{sign} \ b_{xy}\); figure 5), whereas \(e_{xy}^0 < 0 \forall y^+ \in [0, \delta^+]\) is close to 0 at \(y^+ \approx 25\) (figure 5), \(e_{xy}^0\) exhibiting a radically different behaviour (figures 3, 5). The wall-asymptotic expansion of \(b_{xx}\) (table 4) confirms that \(\varepsilon_{xx}\) is not 2-C at the wall, contrary to \(b\) (Gerolymos and Vallet 2016a, table 1, p 392) and \(b_{xx}\) (Gerolymos and Vallet 2016a, table 2, p 393). This is clearly shown by the \(y^+\)-distribution of the corresponding flatness parameter (7d) \(A_{r_{xx}} > 0 \forall y^+\) (figure 5), which reaches its minimum value \(\approx 0.03\) at \(y^+ \approx 5\), then increasing to \([A_{r_{xx}}]_w \approx 0.185\). These differences in near-wall behaviour are also particularly visible in the \(y^+\)-distribution of the anisotropy invariants (figure 5) and in the AIM of \(\varepsilon_{xx}\) (figure 6). The locus of \(\Pi_{b_{xx}} - \Pi_{b_{xx}}\) does not reach the 2-C boundary (figure 6). Instead, near the wall, \([\Pi_{b_{xx}} - \Pi_{b_{xx}}]\) reaches the axisymmetric disk-like boundary of Lumley (1978) realisability triangle (figure 6), roughly corresponding to \(y^+ \approx 0.7\) where \(e_{xy}^+ = e_{xy}^+\) (figure 3) and \(b_{xy} = b_{xx}\) (figure 5). For \(y^+ \leq 0.7\), the locus of \(b_{xx}\) in the (III, –II)-plane returns toward the interior of Lumley (1978) realisability triangle (figure 6). The contrasting behaviour of \(b_{xx}\) compared to \(b\) and \(b_{xx}\) (figures 5, 6) further highlights the complexity of near-wall turbulence, where 2-C componentality at the wall applies to both \(r\) and \(\varepsilon\) but not to \(\varepsilon_{xx}\). Examination of the wall-asymptotic behaviour of various terms in the \(e_{xy}\)-budgets (tables 1, 2) reveals that neither \(d_{yy}^{ou}\) nor \(\Pi_{r_{xx}}\) are 2-C at the wall, in line with (3b), whereas \(P_{xy}\) and \(d_{yy}^{ou}\) are 2-C at the wall (tables 1, 2). Notice in particular the wall-behaviour of \(\Pi_{r_{xy}}\), for which \([\Pi_{r_{xy}}]_w^+ = 0\) while \([\Pi_{r_{xy}}]_w^+ = 0\) (table 1). Notice also that, at the wall, \(e_{xy}^{-1} e_{xy}^{-1}\) defines, by
\[ \epsilon_{yy}^{\perp} \sim 8(2 \mathbf{R}_{ww}^{\perp} \mathbf{B}^{\perp}) + (\nabla A_{y}^{r}) + 32(3 \mathbf{R}_{ww}^{\perp} C_{y}^{\perp} + (\nabla A_{y}^{r}) \cdot (\nabla B_{y}^{r})^{y} y) + O(y^{2}) \]
\[ \epsilon_{xx}^{\perp} \sim 16 \mathbf{B}_{ww}^{\perp} + 16(3 \mathbf{B}_{ww}^{\perp} C_{y}^{\perp} + (\nabla A_{y}^{r}) \cdot (\nabla B_{y}^{r})^{y} y) + O(y^{2}) \]
\[ \epsilon_{xy}^{\perp} \sim 16 \mathbf{B}_{ww}^{\perp} + 96 \mathbf{B}_{ww}^{\perp} C_{y}^{\perp} y + O(y^{2}) \]
\[ \epsilon_{zz}^{\perp} \sim 8(2 \mathbf{R}_{ww}^{\perp} \mathbf{B}^{\perp}) + (\nabla A_{y}^{r}) + 32(3 \mathbf{R}_{ww}^{\perp} C_{y}^{\perp} + (\nabla A_{y}^{r}) \cdot (\nabla B_{y}^{r})^{y} y) + O(y^{2}) \]
\[ \epsilon_{yx}^{\perp} \sim 8(2 \mathbf{R}_{ww}^{\perp} \mathbf{B}^{\perp}) + (\nabla A_{y}^{r}) + 32(3 \mathbf{R}_{ww}^{\perp} C_{y}^{\perp} + (\nabla A_{y}^{r}) \cdot (\nabla B_{y}^{r})^{y} y) + O(y^{2}) \]
\[ + 16(3 \mathbf{B}_{ww}^{\perp} C_{y}^{\perp} + (\nabla A_{y}^{r}) \cdot (\nabla B_{y}^{r})^{y} y) + O(y^{2}) \]

5. Conclusions

The paper studies the \( \epsilon_{yy} \)-budgets, including the shear component, and compares the behaviour of different mechanisms with the corresponding mechanisms in \( \epsilon_{ij} \)-budgets, using novel DNS data for low \( Re_{\tau} \approx 180 \) plane channel flow.

All of the components of production \( P_{ij} \) are generally \( \approx 0 \) (specifically all of the components of \( P_{ij}^{(2)} \) and \( P_{ij}^{(3)} \)) and contribute as gain to the corresponding \( \epsilon_{ij} \)-budgets, contrary to the \( \epsilon_{ij} \)-budgets where for plane channel flow \( P_{yy} = P_{zz} = 0 \). The pressure mechanism \( \Pi_{ij} \) has a very weak contribution to the budgets of the streamwise \( \epsilon_{xx} \) and spanwise \( \epsilon_{zz} \) components, in contrast to the \( \Pi_{ij} \) which is important in the budgets of all \( \epsilon_{ij} \)-components, especially in the log-region.

The destruction-of-dissipation tensor \( \epsilon_{ij} \) behaves very differently from the dissipation tensor \( \epsilon_{ij} \). The shear component \( \epsilon_{ij} > 0 \) \( y^{+} \geq 3 \) is a gain mechanism in the \( \epsilon_{ij} \)-budgets except very near the wall (\( y^{+} \leq 3 \)), contrary to \( \epsilon_{ij} < 0 \) \( y^{+} \in [0, \delta] \) which is a loss mechanism in the \( \epsilon_{ij} \)-budgets. Finally, analytical results and DNS data for the wall-asymptotic behaviour of different terms in the \( \epsilon_{ij} \)-budgets show that the wall-boundary condition is \( [d_{ij}^{(1)}]_{w} + [\Pi_{ij}^{(w)}]_{w} = [\epsilon_{ij}]_{w} \) instead of the well known condition \( [d_{ij}^{(2)}]_{w} = [\epsilon_{ij}]_{w} \) for the \( \epsilon_{ij} \)-budgets (Mansour et al. 1988).

All of the 3 tensors \( \epsilon_{ij} \) (for \( \epsilon_{ij}, \epsilon_{ij}, \epsilon_{ij} \)) being positive-definite, their anisotropy was studied using AIM (Lee and Reynolds 1987), revealing in particular that, near the wall, the destruction-of-dissipation tensor \( \epsilon_{ij} \), after reaching the axisymmetric disk-like boundary (roughly where \( \epsilon_{ij} \approx \epsilon_{ij} \) at \( y^{+} \approx 0.7 \), returns inside the realisability triangle, never approaching the 2-C boundary. The DNS data are corroborated by the wall-asymptotic expansions of \( \epsilon_{ij} \) and of its anisotropy tensor \( b_{ij} \). This observed componentality of \( \epsilon_{ij} \) is strickingly different from that of \( \epsilon_{ij} \) or \( \epsilon_{ij} \), both of which are 2-C at the wall, and highlights the dimensional analysis (Tennekes and Lumley 1972, p 5), a time-scale which is finite contrary to \( k_{w}[\epsilon_{ij}^{-1} = 0] \).
The analysis of the DNS data highlights the complexity of $ij\varepsilon$-transport, especially near the wall and regarding the shear component $\varepsilon_{xy}$. It seems plausible that the specific behaviour
of the $\varepsilon_{xy}$-budgets, both with respect to $r_{xy}$-budgets and compared to the diagonal components of $\varepsilon_{ij}$, can only be modelled by differential $r_{ij} - \varepsilon_{ij}$ closures. It is hoped that the present DNS data will be useful in the development of such closures.
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Appendix A. Fully developed plane channel flow

We consider fully developed ($xz$-invariant) plane channel flow (the channel height is $2\delta$ and $xyz$ are respectively the streamwise, wall-normal and spanwise directions) and use non-dimensional inner variables (Gerolymos and Vallet 2016a, wall-units, (A.3), p 414).

A.1. Mean-flow and symmetries

No-slip boundary-conditions apply at the walls

$$y^+ \in \{0, 2\delta^+\} \quad \Longrightarrow \quad u^+ = v^+ = w^+ = u'^+ = v'^+ = w'^+ = 0; \quad \forall x^+, z^+, t^+. \quad (A.1a)$$

The usual hypotheses that the mean-flow is steady, 2D and that the $x$-wise location that is investigated is sufficiently downstream of the channel inlet to achieve fully developed flow (Zanoun et al 2009, Schultz and Flack 2013) in the streamwise direction.
are made. Under these conditions (A.1), the mean continuity (Mathieu and Scott 2000, (4.5), p 76) and momentum (streamwise and wall-normal) equations (Mathieu and Scott 2000, (4.9), p 77), imply (Mathieu and Scott 2000, pp 105–111) the exact relations

\[ \frac{\partial \bar{\vartheta}^+}{\partial x^+} = 0 \quad \forall x^+, y^+, \quad (A.2a) \]

\[ \frac{\partial \bar{p}^+}{\partial x^+} = \frac{d \bar{\rho}^+}{dx^+} = \left[ \frac{\tau_{\infty}^+}{\delta} \right] = \frac{-1}{\delta^+} = \frac{1}{R_e^\eta} = \text{const} \quad \forall x^+, y^+, \quad (A.2b) \]

\[ -r_{ij}^+ + \frac{d \bar{u}^+}{dy^+} = \left( 1 - \frac{y^+}{R_e^\eta} \right). \quad (A.2c) \]

\[ \rho^+(x^+, y^+) = \rho_w^+(x^+) - r_{ij}^+(y^+) \quad (A.2d) \]

for the mean streamwise velocity \( \bar{u}^+(y^+) \) and mean pressure \( \bar{p}^+(x^+, y^+) \) fields, with a constant streamwise pressure-gradient \( \partial \bar{p} = d \bar{p} = \text{const} \) (A.2b). In (A.2b), (A.2c) \( R_e^\eta = \delta^+ \) is the friction Reynolds number (Gerolymos and Vallet 2016a, (A.3g), p 414). In (A.1), (A.2) deterministic potential body-forces (e.g. gravity) in the momentum equations are included in the mean-pressure field (Monin and Yaglom 1971, p 31). Recall that the \( xzt \)-homogeneity of the averages implies the relations

\[ \frac{\partial \bar{y}^+}{\partial q} = 0 \implies [\bar{y}^+] \frac{\partial (\bar{y}^+)}{\partial q} = -[\bar{y}^+] \frac{\partial (\bar{y}^+)}{\partial q} \quad \forall q \in \{x, z, t\}, \quad (A.3a) \]

\[ (\bar{y}^+) \frac{\partial (\bar{y}^+)}{\partial q_1 \partial q_2} = \frac{\partial (\bar{y}^+)}{\partial q_1} \frac{\partial (\bar{y}^+)}{\partial q_2} = \frac{\partial (\bar{y}^+)}{\partial q_1} \frac{\partial (\bar{y}^+)}{\partial q_2} \quad \forall q_1, q_2 \in \{x, z, t\}. \quad (A.3b) \]

### A.2. \( \varepsilon_{ij} \)-budgets in plane channel flow

Under fully developed plane channel flow conditions (A.1), (A.2) the \( \varepsilon_{ij} \)-transport equations simplify to

\[ (1b, A.1, A.2) \implies \frac{d}{dy} \left[ \rho \left( \sqrt{2} \frac{\partial u_i^+}{\partial x_i} \right)^2 \frac{\partial u_j^+}{\partial x_j} \right] + \mu \frac{d^2 \bar{u}_{ij}}{dy^2} - \rho (\varepsilon_{ij}) \bar{\delta}_{ij} = 0 \]

\[-\rho (\bar{E}_{ijy} + \bar{E}_{ijx}) \frac{d \bar{u}_{ij}}{dy} = \mu \left( 2 \nu \frac{\partial u_i^+}{\partial y} \frac{\partial u_j^+}{\partial y} + 2 \nu \frac{\partial u_i^+}{\partial y} \frac{\partial u_j^+}{\partial y} \right) \frac{d^2 \bar{u}_{ij}}{dy^2} + \rho^{(4)} + \Pi \varepsilon_{ij} - \rho \varepsilon_{ij} = 0, \quad (A.4) \]

where \( \bar{E}_{ij}^y = 2 \nu \frac{\partial u_i^+}{\partial y} \frac{\partial u_j^+}{\partial y} \) (Gerolymos and Vallet 2016a, (3.1a), p 402) and the 3 last terms in (A.4) retain their general expressions (1b). The relevant equations for the \( \varepsilon_{ij} \)-components (recall that by 2D \( z \)-wise symmetry \( \varepsilon_{ij}^z = \varepsilon_{ij}^x = 0 \) \( \forall y^+ \)) read in wall-units
where the symmetry relations $\epsilon_{xxxy} = \epsilon_{xyxy}$, $\epsilon_{yyxy} = \epsilon_{yxyy}$ and $\epsilon_{zyzy} = \epsilon_{zyzy}$ were used.

Appendix B. Asymptotic behaviour in the viscous sublayer ($y^+ \to 0$)

Near a plane $xz$-wall, located at $y^+ = 0$, the fluctuating quantities are expanded $y$-wise in Taylor-series around $y^+ = 0$ following (2). The application of the usual gradient-operator (Pope 2000, (A.48), p 651) $\nabla(\cdot) = \hat{\epsilon}_y\partial_y(\cdot)$ on the coefficients of (2), which are stationary random functions of $\{x^+, y^+, t^+\}$ independent of $y^+$, produces only in-plane $xz$-gradients

\[
(\nabla \cdot A^+) = \hat{\epsilon}_x \frac{\partial A^+}{\partial x^+} + \hat{\epsilon}_z \frac{\partial A^+}{\partial z^+}, \tag{B.1a}
\]

\[
(\nabla A^+) = \hat{\epsilon}_x \frac{\partial A^+}{\partial x^+} + \hat{\epsilon}_z \frac{\partial A^+}{\partial z^+}, \tag{B.1b}
\]

\[
(\nabla B^+) = \hat{\epsilon}_x \frac{\partial B^+}{\partial x^+} + \hat{\epsilon}_z \frac{\partial B^+}{\partial z^+}, \tag{B.1c}
\]

$\vdots$
B.1. Fluctuating continuity equation

The no-slip condition \((A.1a)\) implies that the wall-terms in the expansions \((2)\)

\[
u w^+ = v' w^+ = w^+ = 0 \quad \forall \ x^+, \ z^+, \ t^+.
\]  

(B.2)

Using the expansions \((2)\), along with \((B.2)\), in the fluctuating continuity equation (Mathieu and Scott 2000, (4.6), p 76)

\[
\frac{\partial u_i'}{\partial x_i} = 0
\]  

(B.3)

and equating the coefficients of different powers of \(y^+\) to 0, yields

\[
A^+_w = 0, \quad B^+_w = 0,
\]

(A.4a) \hspace{1cm} (A.4b)

\[
\frac{\partial A^+_{w}}{\partial x^+} + \frac{\partial A^+_{w}}{\partial x^+} + 2B^+_{v} = 0,
\]

(B.4b)

\[
\frac{\partial B^+_{w}}{\partial x^+} + \frac{\partial B^+_{w}}{\partial x^+} + 3C^+_{w} = 0
\]

(B.4c)

respectively for the \([O(1), O(y^+), O(y^{+2})]\) terms, with analogous relations for HOTs. Relation \((B.4b)\) corresponds to Mansour et al (1988, (3), p 19). Notice that \((B.4b)\) yields the identity

\[
B^+_{y} = -\frac{1}{2}B^+_{v} \frac{\partial A^+_{w}}{\partial x^+} - \frac{1}{2}B^+_{v} \frac{\partial A^+_{w}}{\partial z^+}.
\]

(B.5)

Relations \((2), (B.2), (B.4), (B.5)\) are generally valid for \(xz\)-inhomogeneous incompressible flow near an \(xz\)-wall. They provide the wall-asymptotic expansions of all correlations containing only fluctuating velocities and their derivatives, and were used to calculate the wall-asymptotic expansions of \(\varepsilon_{x_0}\) (table 3) and of its anisotropy tensor \(b_{x_0}\) and invariants (table 4). The relation of the wall-asymptotic expansion of the fluctuating pressure \(p'\) to the expansions of the fluctuating velocities depends on the particular mean-flow studied, and was therefore calculated for fully developed plane channel flow.

B.2. Plane channel flow

In the particular case of plane channel flow, conditions \((A.1)–(A.3)\) imply specific relations for the mean and fluctuating fields, which were used to determine the wall-asymptotic expansions (tables 1, 2) of various terms in the \(\varepsilon_{y}\)-transport \((1b)\) simplified for plane channel flow \((A.4), (A.5)\).

B.2.1. Mean-flow. Using the expansion of \(r_{y}^+\) obtained from \((2), (B.2), (B.4a)\) in the x-wise component of the mean-momentum equation \((A.2c)\) yields, after integration and application of the no-slip boundary-condition \((A.1a)\), the expansion of the mean streamwise velocity

\[
\bar{u}_y^+ \sim y^+ - \frac{1}{2Re_{x_0}} y^{+2} + \frac{1}{3}A^+_w B^+_v y^{+4} + \frac{1}{5} (B^+_w^2 + A^+_w^2) y^{+5} + O(y^{+6})
\]

(B.6a)

including the dominant linear term \(y^+\), an \(O(y^{+2})\) correction associated with the mean streamwise pressure-gradient \((A.2b)\), which \(\rightarrow \infty \) as \(Re_{x_0} \rightarrow \infty\) at fixed \(y^+\), and higher \(O(y^{+4})\) terms. Therefore, the gradient \([d_y \bar{u}]^+\) and Hessian \([d_{yy} \bar{u}]^+\) which appear in the production
terms \( \{ P^{(1)}_{ij}, P^{(2)}_{ij}, P^{(3)}_{ij} \} \) of the \( \varepsilon_{ij} \)-transport equations (1b), (A.4), (A.5) expand as

\[
\frac{da^+}{dy^+} \bigg|_{y^+ \to 0} \approx -\frac{1}{Re_{\varepsilon_{ij}}} y^+ + A_{ij}^+ B^+ y^+ y^+ + (B_{ij}^+ B_{ij}^+ + A_{ij}^+ C_{ij}^+) y^+ + O(y^{+5}),
\]

(B.6b)

\[
\frac{d^2 a^+}{dy^{+2}} \bigg|_{y^+ \to 0} \approx -\frac{1}{Re_{\varepsilon_{ij}}} + 3A_{ij}^+ B_{ij}^+ y^+ y^+ + 4(B_{ij}^+ B_{ij}^+ + A_{ij}^+ C_{ij}^+) y^+ + O(y^{+4}).
\]

(B.6c)

By (A.2d), (2), (B.2), (B.4a), the mean pressure can be expanded as

\[
\bar{p}^+ \bigg|_{y^+ \to 0} \approx \bar{P}_w^+ - 2B_{ij}^+ C_{ij}^+ y^+ + 3B_{ij}^+ C_{ij}^+ y^+ y^+ + O(y^{+5}).
\]

(B.6d)

### B.2.2. Wall-normal (y) fluctuating momentum and fluctuating pressure field

Using (2), (B.2), (B.4a), (B.6a) in the wall-normal component of the fluctuating momentum equation (Mathieu and Scott 2000, (4.31), p 85)

\[
\frac{\partial u_i^+}{\partial t^+} + u_i^+ \frac{\partial u_i^+}{\partial x_i^+} = -\frac{\partial}{\partial x_i^+} (u_i^+ u_i^+ - r_i^+) - u_i^+ \frac{\partial p^+}{\partial x_i^+} + \frac{\partial^2 u_i^+}{\partial x_i^+ \partial x_i^+} - \frac{\partial^2 u_i^+}{\partial x_i^+ \partial x_i^+}
\]

(B.7)

and using the symmetry conditions (A.1b) implies that the fluctuating pressure field expansion (2) should be

\[
p^+ \bigg|_{y^+ \to 0} \approx p_w^+ + 2B_{ij}^+ y^+ + 3C_{ij}^+ y^+ y^+ + \frac{1}{3} \left( 12D_{ij}^+ + \nabla^2 B_{ij}^+ - \frac{\partial B_{ij}^+}{\partial t^+} \right) y^+ + \cdots
\]

(B.8)

i.e. that the fluctuating pressure field, as \( y^+ \to 0 \), is uniquely determined to \( O(y^{+3}) \) by the wall-normal fluctuating velocity field \( v^+ \) (Gerolymos and Vallet 2016a, (2.3b), p 391), in line with the plane wall boundary condition \( \partial_i p^+ = \mu \partial_i v_i^+ \) (Pope 2000, (11.173), p 439). Relation (B.8) corresponds to Mansour et al (1988, 2, 6, pp 18–20). In (B.8) \( p_{ij}^+ (x^+, z^+, t^+) \) is the fluctuating pressure at the wall.

### B.2.3. Wall-parallel (xz) fluctuating momentum

Using the expansions (2), (B.2), (B.4a), (B.6) in the fluctuating x-momentum equation (B.7), and equating the coefficients of different powers of \( y^+ \) to 0, yields

\[
\frac{\partial p_{ij}^+}{\partial x_i^+} = 2B_{ij}^+ = 0,
\]

(B.9a)

\[
2 \frac{\partial B_{ij}^+}{\partial x_i^+} - \frac{\partial^2 A_{ij}^+}{\partial z^{+2}} - \frac{\partial^2 A_{ij}^+}{\partial x_i^+ x_j^+} + \frac{\partial A_{ij}^+}{\partial t^+} - 6C_{ij}^+ = 0
\]

(B.9b)

respectively for the \( \{ O(1), O(y^{+}) \} \) terms, with the corresponding relations

\[
\frac{\partial p_{ij}^+}{\partial z^+} = 2B_{ij}^+ = 0
\]

(B.10a)

\[
2 \frac{\partial B_{ij}^+}{\partial z^+} - \frac{\partial^2 A_{ij}^+}{\partial z^{+2}} - \frac{\partial^2 A_{ij}^+}{\partial x_i^+ x_j^+} + \frac{\partial A_{ij}^+}{\partial t^+} - 6C_{ij}^+ = 0
\]

(B.10b)
for the fluctuating z-momentum equation (B.7). Relations (B.9a), (B.10a) correspond to Mansour et al (1988, 4) p 19 and relations (B.9b), (B.10b) to Mansour et al (1988, 7, 8) p 20.

By (B.9a), (B.10a),

\[
\frac{1}{2} \frac{\partial^2 P_w}{\partial x^2} = \frac{\partial B^+_w}{\partial x^+} = \frac{\partial B^+_w}{\partial x^+}, \quad (B.11a)
\]

whence, using (A.3a),

\[
B^+_w \frac{\partial B^+_w}{\partial x^+} = B^+_w \frac{\partial B^+_w}{\partial x^+} = B^+_w \frac{\partial B^+_w}{\partial z^+} = B^+_w \frac{\partial B^+_w}{\partial z^+} = 0. \quad (B.11b)
\]

Notice that the relations \(B^+_w \frac{\partial B^+_w}{\partial x^+} = B^+_w \frac{\partial B^+_w}{\partial x^+}\) (A.3a) is also obvious because the flow is 2D z-wise. Relation (B.11a) corresponds to Mansour et al (1988, 5) p 19. Furthermore, substituting \(C^+_w\) by (B.4c) in \(B^+_w C^+_w\) readily yields by (A.3a), (B.11b)

\[
\frac{B^+_w}{C^+_w} = 0 \quad (B.11c)
\]

the corresponding relation \(B^+_w C^+_w = 0\), which can also be proven in the same way from (B.11b), being obvious because the flow is 2D in the mean z-wise (A.1b). Finally, by (B.9b), (A.3a)

\[
B^+_w \frac{\partial A^+_w}{\partial x^+} = 6 B^+_w C^+_w = \frac{\partial A^+_w}{\partial z^+} + \frac{\partial A^+_w}{\partial z^+}. \quad (B.12)
\]

References

Buschmann M H and Gad-el-Hak M 2007 Recent developments in scaling of wall-bounded flows Prog. Aerosp. Sci. 42 419–67

Cécora R D, Radespiel R, Eisfeld B and Probst A 2015 Differential Reynolds-stress modeling for aeronautics AIAA J. 53 739–55

Chou P Y 1945 On velocity correlations and the solutions of the equations of turbulent fluctuations Q. Appl. Math. 3 38–54

Coles D 1956 The law of the wake in a turbulent boundary layer J. Fluid Mech. 1 191–226

Durbin P A 1993 A Reynolds-stress model for near-wall turbulence J. Fluid Mech. 249 465–98

Eisfeld B 2015 Differential Reynolds-stress Modeling for Separating Flows in Industrial Aerodynamics (Mechanical Engineering Series) (Cham: Springer) [https://doi.org/10.1007/978-3-319-15639-2]

Gerolymos G A 2011 Approximation error of the Lagrange reconstructing polynomial J. Approx. Theory 163 267–305

Gerolymos G A 2012 A general recurrence relation for the weight-functions in Mühlbach–Neville–Aitken representations with application to WENO interpolation and differentiation Appl. Math. Comput. 219 4133–42

Gerolymos G A, Joly S, Mallet M and Vallet I 2010 Reynolds-stress model flow prediction in aircraft-engine intake double-S-shaped duct J. Aircr. 47 1368–81

Gerolymos G A, Kallas Y N and Papailiou K D 1989 The behaviour of the normal fluctuation terms in the case of attached and detached turbulent boundary-layers Rev. Phys. Appl. 24 375–87

Gerolymos G A, Lo C and Vallet I 2012 Tensorial representations of Reynolds-stress pressure-strain redistribution ASME J. Appl. Mech. 79 044506

Gerolymos G A, Lo C, Vallet I and Younis B A 2012 Term-by-term analysis of near-wall second moment closures AIAA J. 50 2848–64

Gerolymos G A, Sénéchal D and Vallet I 2009 Very-high-order WENO schemes J. Comput. Phys. 228 8481–524
Gerolymos G A, Sénéchal D and Vallet I 2010 Performance of very-high-order upwind schemes for DNS of compressible wall-turbulence Int. J. Numer. Methods Fluids 63 769–810
Gerolymos G A, Sénéchal D and Vallet I 2013 Wall effects on pressure fluctuations in turbulent channel flow J. Fluid Mech. 720 15–65
Gerolymos G A and Vallet I 2001 Wall-normal-free near-wall Reynolds-stress closure for 3D compressible separated flows AIAA J. 39 1833–42
Gerolymos G A and Vallet I 2014 Pressure, density, temperature and entropy fluctuations in compressible turbulent plane channel flow J. Fluid Mech. 757 701–46
Gerolymos G A and Vallet I 2016a The dissipation tensor εw in wall turbulence J. Fluid Mech. 807 386–418
Gerolymos G A and Vallet I 2016b Reynolds-stress model prediction of 3D duct flows Flow Turbul. Combust. 96 45–93
Høyas S and Jiménez J 2008 Reynolds number effects on the Reynolds-stress budgets in turbulent channels Phys. Fluids 20 101511
Jakirlić S, Eisfeld B, Jester-Zürker R and Kroll N 2007 Near-wall Reynolds-stress model calculations of transonic flow configurations relevant to aircraft aerodynamics Int. J. Heat Fluid Flow 28 602–15
Jakirlić S and Hanjalić K 2002 A new approach to modelling near-wall turbulence energy and stress dissipation J. Fluid Mech. 459 139–66
Jakirlić S and Hanjalić K 2013 A DNS-based reexamination of coefficients in the pressure-strain models in second-moment closures Fluid Dyn. Res. 45 055509
Jones W P and Launder B E 1972 The prediction of laminarization with a 2-equation model of turbulence Int. J. Heat Mass Transfer 15 301–14
Kassinos S C, Reynolds W C and Rogers M M 2001 1-point turbulence structure tensors J. Fluid Mech. 428 213–48
Lai Y G and So R M C 1990 On near-wall turbulent flow modelling J. Fluid Mech. 221 641–73
Lauder B E, Reece G J and Rodi W 1975 Progress in the development of a Reynolds-stress turbulence closure J. Fluid Mech. 68 537–66
Lauder B E and Spalding D B 1974 The numerical computation of turbulent flows Comput. Methods Appl. Mech. Eng. 3 269–89
Lee M and Moser R D 2015 DNS of turbulent channel flow up to Reτ ≈ 5200 J. Fluid Mech. 774 395–415
Lee M J and Reynolds W C 1987 On the structure of homogeneous turbulence Turbulent Shear Flows 5, Selected Papers for the 5th Int. Symp. on Turbulent Shear Flows (Cornell University, Ithaca, NY, USA, 7–9, August 1985) ed F Durst (Berlin: Springer) pp 54–66
Lumley J L 1978 Computational modeling of turbulent flows Adv. Appl. Mech. 18 123–76
Lumley J L, Yang Z and Shih T H 1999 A length-scale equation Flow Turbul. Combust. 63 1–21
Mansour N N, Kim J and Moin P 1988 Reynolds-stress and dissipation-rate budgets in a turbulent channel flow J. Fluid Mech. 194 15–44
Mathieu J and Scott J 2000 An Introduction to Turbulent Flow (Cambridge: Cambridge University Press)
Menter F R 1994 2-equation eddy-viscosity turbulence models for engineering applications AIAA J. 32 1598–605
Monin A S and Yaglom A M 1971 Statistical Fluid Mechanics: Mechanics of Turbulence vol 1 (Cambridge, MA: MIT Press)
Moser R D, Kim J and Mansour N N 1999 Direct numerical simulation of turbulent channel flow up tp Reτ = 590 Phys. Fluids 11 943–5
Olsen M E and Coakley T J 2001 The lag model, a turbulence model for nonequilibrium flows AIAA Paper 2001–564
Pope S B 2000 Turbulent Flows (Cambridge: Cambridge University Press) (https://doi.org/10.1017/cbo9780511840531)
Riley K F, Hobson M P and Bence S J 2006 Mathematical Methods for Physics and Engineering 3rd edn (Cambridge: Cambridge University Press)
Rivlin R S 1955 Further remarks on the stress-deformation relations for isotropic materials Indiana Univ. Math. J. 4 681–702
Rodi W and Mansour N N 1993 Low Reynolds number k–ε modelling with the aid of DNS J. Fluid Mech. 250 509–29
Rumsey C L 2010 NASA Langley Research Center Turbulence Modeling Resource http://turbmodels. larc.nasa.gov/index.html visited November 2014
Schiestel R 2008 *Modelling and Simulation of Turbulent Flows* (London: Wiley) (https://doi.org/10.1002/9780470610848)
Schultz M P and Flack K A 2013 Reynolds-number scaling of turbulent channel flow *Phys. Fluids* **25** 025104
Sillero J A, Jiménez J and Moser R D 2013 One-point statistics for turbulent wall-bounded flows at Reynolds numbers up to \( \delta^+ \approx 2000 \) *Phys. Fluids* **25** 105102
Simonsen A J and Kroostad P Å 2005 Turbulent stress invariant analysis: classification of existing terminology *Phys. Fluids* **17** 088103
Speziale C G, Sarkar S and Gatski T B 1991 Modelling the pressure-strain correlation of turbulence: an invariant dynamical systems approach *J. Fluid Mech.* **227** 245–72
Tennekes H and Lumley J L 1972 *A First Course in Turbulence* (Cambridge, MA: MIT Press)
Vreman A W and Kuerten J G M 2014a Comparison of DNS databases of turbulent channel flow at \( Re_c = 180 \) *Phys. Fluids* **26** 015102
Vreman A W and Kuerten J G M 2014b Statistics of spatial derivatives of velocity and pressure in turbulent channel flow *Phys. Fluids* **26** 085103
Vreman A W and Kuerten J G M 2016 A 3-order multistep time-discretization for a chebyshev tau spectral method *J. Comput. Phys.* **304** 162–9
Wilcox D C 1988 Reassessment of the scale-determining equation for advanced turbulence models *AIAA J.* **26** 1299–310
Wilcox D C 2006 *Turbulence Modelling for CFD* 3rd edn (La Cañada: DCW Industries)
Yakovenko S N and Chang K C 2007 Performance examination of geometry-independent near-wall second-moment closures in simple and backstep flows *Numer. Heat Transfer B* **51** 179–204
Zanoun E S, Nagig H and Durst F 2009 Refined \( c_f \) relation for turbulent channels and consequences for high-Re experiments *Fluid Dyn. Res.* **41** 021405