ABSOLUTELY CONTINUOUS SPECTRUM FOR LIMIT-PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We show that a large class of limit-periodic Schrödinger operators has purely absolutely continuous spectrum in arbitrary dimensions. This result was previously known only in dimension one.

The proof proceeds through the non-perturbative construction of limit-periodic extended states. An essential step is a new estimate of the probability (in quasi-momentum) that the Floquet Bloch operators have only simple eigenvalues.

1. Introduction

In this paper, we consider Schrödinger operators $\Delta + V$ acting on the lattice $\ell^2(\mathbb{Z}^d)$ for $d \geq 1$. Here $\Delta$ is the discrete Laplacian

$$\Delta \psi(n) = \sum_{|e|_1=1} \psi(n + e), \quad |x|_1 = |x_1| + \cdots + |x_d|$$

and the potential $V$ is a multiplication operator by a sequence $V : \mathbb{Z}^d \to \mathbb{R}$. For some general background, see Sections 3 and 4 in [7]. The potential $V$ is called $p = (p_1, \ldots, p_d)$-periodic if

$$V(n_1 + p_1, \ldots, n_d) = \cdots = V(n_1, \ldots, n_d + p_d) = V(n_1, \ldots, n_d)$$

for all $n \in \mathbb{Z}^d$. A sequence of periods $p_1, p_2, \ldots$ is called increasing if $p_{\ell+1}^j$ divides $p_{\ell}^j$ for all $\ell \geq 1$ and $j = 1, \ldots, d$. $V$ is limit-periodic if there exists an increasing sequence of periods $p^j$ and $p^j$-periodic potentials $V^j$ such that

$$V_j = V^1 + \cdots + V^j$$

converges to $V$ in $\ell^\infty(\mathbb{Z}^d)$.

The main result is

Theorem 1.1. Let $d \geq 1$, $\varepsilon_1 > 0$, and $p^j$ be an increasing sequence of periods. Then there exists a sequence $\varepsilon_j > 0$, $j \geq 2$, such that for $V^j$ a $p^j$-periodic potential satisfying $\|V^j\|_{\ell^\infty(\mathbb{Z}^d)} \leq \varepsilon_j$, the potential

$$V = \lim_{j \to \infty} (V^1 + \cdots + V^j)$$

exists in $\ell^\infty(\mathbb{Z}^d)$ and the Schrödinger operator $\Delta + V$ has purely absolutely continuous spectrum.

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This statement was originally proven by Avron and Simon [1] for Schrödinger operators on $L^2(\mathbb{R})$. Damanik and Gan [4] gave a proof for the case of $\ell^2(\mathbb{Z})$. As far as higher dimensional Schrödinger operators are concerned, Karpeshina and Lee [6] have shown the existence of an absolutely continuous component of the spectrum in the perturbative regime of high energies on $L^2(\mathbb{R}^2)$. So the results are new for $d \geq 2$. Furthermore, the proof given here is much simpler than the strategy of Karpeshina and Lee.

In difference to Karpeshina and Lee, we do not discuss the spectrum of $H$ as a set. The main reason is that our results allow for the spectrum to contain many gaps, just start with a large enough $V_1$. Finally, limit-periodic Schrödinger operators with pure-point spectrum have been constructed by Damanik and Gan in [5] in arbitrary dimension. Finally, the results of [3] and Chapter 17 in [2] imply the existence of extended states for quasi-periodic Schrödinger operators in arbitrary dimension and small coupling for a set of frequencies of large measure.

The proof proceeds by constructing generalized eigenfunctions, that is solutions $u: \mathbb{Z}^d \rightarrow \mathbb{C}$ of $Hu = Eu$.

**Theorem 1.2.** Let $V$ be as in Theorem 1.1. Then for almost every $\theta_1, \ldots, \theta_d \in \mathbb{R}$, there exists $E \in \mathbb{R}$ and non-zero limit-periodic $u: \mathbb{Z}^d \rightarrow \mathbb{C}$ such that

\begin{equation}
Hu = Eu
\end{equation}

and

\begin{equation}
\hat{u}(\theta_1, \ldots, \theta_n) = \lim_{R \to \infty} \frac{1}{\#\Lambda_R(0)} \sum_{n \in \Lambda_R(0)} u(n)e(n_1\theta_1 + \cdots + n_d\theta_d) \neq 0.
\end{equation}

Here, we use the notation $e(x) = e^{2\pi i x}$ and $\Lambda_R(n) = \{x \in \mathbb{Z}^d : |n - x|_\infty \leq R\}$.

We will now discuss properties single periodic operator following [9]. Given a period $p \in (\mathbb{Z}_+)^d$, we introduce the set

\begin{equation}
\mathbb{B}_p = \left\{\left(\frac{k_1}{p_1}, \ldots, \frac{k_d}{p_d}\right), \quad 0 \leq k_j \leq p_j - 1\right\}.
\end{equation}

Any $p$-periodic function $V$ can be written as

\begin{equation}
V(n) = \sum_{k \in \mathbb{B}_p} \hat{V}(k)e(k \cdot n),
\end{equation}

where $x \cdot y = \sum_{j=1}^d x_jy_j$. For $u \in \ell^1(\mathbb{Z}^d)$, we define the Fourier transform $\hat{u}: \mathbb{T}^d \rightarrow \mathbb{R}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

\begin{equation}
\hat{u}(x) = \sum_{n\in\mathbb{Z}^d} u(n)e(x \cdot n).
\end{equation}

This map is extended to $\ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ as usual. Furthermore, the Fourier transform of $(\Delta + V)u$ is given by

\begin{equation}
\sum_{j=1}^d 2\cos(2\pi x_j)\hat{u}(x) + \sum_{k \in \mathbb{B}_p} \hat{V}(k)\hat{u}(x + k).
\end{equation}

Letting $\psi_x = \{\hat{u}(x + k)\}_{k \in \mathbb{B}_p}$, we see that the action of this operator is equivalent to

\begin{equation}
\hat{H}_x \psi(k) = \sum_{j=1}^d 2\cos(2\pi (x_j + k_j))\psi(k) + \sum_{\ell \in \mathbb{B}_p} \hat{V}(\ell)\psi(k + \ell).
\end{equation}
The operator $\hat{H}_x$ acts on the $P = p_1 \cdots p_d$ dimensional space $\ell^2(\mathbb{R}_p)$, and we can uniquely define its eigenvalues by

$$E(x, 1) \leq E(x, 2) \leq \cdots \leq E(x, P).$$

**Definition 1.3.** Let $\delta > 0$. The spectrum of $\hat{H}_x$ is called $\delta$-simple if for every $1 \leq \ell \leq P - 1$, we have

$$E(x, \ell + 1) - E(x, \ell) \geq \delta.$$

The spectrum of $\hat{H}_x$ is called simple if it is $\delta$-simple for some $\delta > 0$.

For the $x$ such that the spectrum of $\hat{H}_x$ is simple, we can choose normalized eigenfunctions $\psi(x, \ell)$ of $\hat{H}_x$ such that

$$\hat{H}_x \psi(x, \ell) = E(x, \ell) \psi(x, \ell).$$

Finally, the map $(x, \ell) \mapsto E(x, \ell)$ is continuous and the map $(x, \ell) \mapsto \psi(x, \ell)$ can be chosen to be continuous at least on the set of simple spectrum. The main technical ingredient in our proofs will be the following theorem.

**Theorem 1.4.** Let $V$ be a $p$-periodic potential. Given $\eta \in (0, \frac{1}{2})$, there exists a set $\mathcal{G} \subseteq \mathbb{T}^d$ and $\delta = \delta(\eta, \|V\|_{\infty}, p) > 0$ such that

(i) $|\mathcal{G}| \geq 1 - \eta$.

(ii) For $x \in \mathcal{G}$, we have that the spectrum of $\hat{H}_x$ is $\delta$-simple.

A more detailed statement is given in Section 2. In particular, the dependance of $\delta$ on $\eta$ is quantitative and given by $\delta = \eta C \log(p) p^2$ for a constant $C > 1$. In fact, the contents of that section are the main technical steps in the proof of Theorems 1.1 and 1.2. Before deducing how to prove Theorem 1.2, we give a non-quantitative argument that implies Theorem 1.4 for some $\delta > 0$.

Define the discriminant

$$f(x) = \prod_{j < \ell} (E_j(x) - E_\ell(x))^2.$$

For $x \in \mathbb{R}^d$, we have that $|E_j(x)| \leq \|\hat{H}_x\| \leq 2d + \|V\|_{\infty}$. Thus, we obtain that

$$\min_{j \neq \ell} |E_j(x) - E_\ell(x)| \geq \frac{|f(x)|^{\frac{1}{2}}}{(2d + \|V\|_{\infty})^{\frac{1}{2}}.}

Furthermore we have that $f(x) = (-1)^{\frac{1}{2}P(P-1)} \text{Res}(P(\cdot, x), \partial E P(\cdot, x))$ for $P(E, x) = \det(E - \hat{H}_x)$, where Res denotes the resultant. As the resultant is a polynomial in the coefficients $c_j(x)$ of $P(E, x) = E^P + \sum_{j=0}^{P-1} c_j(x) E^j$, it follows that $f(x)$ is analytic. The following is a qualitative implementation of Theorem 1.4.

**Proof of Theorem 1.4.** By Proposition 2.3 (iii), we have that $f(z) \neq 0$ for some $z \in \mathbb{C}^d$. This implies that the map $g_1 : x_1 \mapsto f(x_1, z_2, \ldots, z_d)$ is analytic and not equal to zero, thus $|g_1(x_1)| \geq \kappa_1$ for all $x_1 \in [0, 1) \setminus X_1$ with $|X_1| \leq \eta/d$ for some $\kappa_1 > 0$. Applying this construction to $g_j : x_{j-1} \mapsto f(x_1, \ldots, x_j, z_{j+1}, \ldots, z_d)$ for $x_j \in [0, 1 \setminus X_j$, we obtain a sequence of sets $X_j$ with $|X_j| \leq \eta/d$ and $|g_j(x_j)| \geq \kappa_j > 0$ for $x_j \not\in X_j$. Taking

$$\mathcal{G} = ([0, 1 \setminus X_1] \times \cdots \times ([0, 1 \setminus X_d)$$

the claim follows. \qed
We now start with the proof of Theorem 1.2. Denote by $\hat{H}_x^j$ the $p^j$-periodic operator with potential $V^j = V_1 + \cdots + V_j$. Let us assume for a second that $V_{j+1} = 0$ and try to understand the relation of $E^j(x, \ell)$ and $E^{j+1}(x, \ell)$. As sets, we clearly have that

\begin{equation}
\sigma(\hat{H}_x^{j+1}) = \bigcup_{s \in S_{j+1}} \sigma(\hat{H}_x^j), \quad \sigma(\hat{H}_x^j) = \{E^j(x, \ell)\}_{\ell=1}^{P_j}
\end{equation}

where

\begin{equation}
S_{j+1} = \left\{ \left( \frac{s_1}{p_1^j}, \ldots, \frac{s_d}{p_d^j} \right) \mid 0 \leq s_k \leq \frac{p_k^j+1}{p_k^j} - 1 \right\}.
\end{equation}

If the spectrum of $\hat{H}_x^{j+1}$ is simple, we thus clearly have that there exists for each $1 \leq \ell \leq P_{j+1}$ an unique $1 \leq \tilde{\ell} \leq P_j$ and $s \in S_{j+1}$ such that

\begin{equation}
E^{j+1}(x, \ell) = E^j(x + s, \tilde{\ell})
\end{equation}

and $\psi^{j+1}(x, \ell) = c\psi^j(x + s, \tilde{\ell})$ for some $|c| = 1$.

**Remark 1.5.** In order to understand, the equality $\psi^{j+1}(x, \ell) = c\psi^j(x + s, \tilde{\ell})$, we view $\psi^j(x, \ell)$ as an element of $\ell^2(\mathbb{B}_{p^j} + x)$. Then as $\mathbb{B}_{p_j} + x + s \subseteq \mathbb{B}_{p_{j+1}} + x$ for $s \in S_{j+1}$ the equality makes sense in $\ell^2(\mathbb{B}_{p_{j+1}} + x)$. These are natural choices given the definition of $\hat{H}_x^j$. Finally, we have that

\begin{equation}
\mathbb{B}_{p_j} + 1 = \bigcup_{s \in S_{j+1}} (\mathbb{B}_{p_j} + s)
\end{equation}

and $(\mathbb{B}_{p_j} + s) \cap (\mathbb{B}_{p_j} + \tilde{s}) = \emptyset$ for $s, \tilde{s} \in S_{j+1}$ and $s \neq \tilde{s}$.

Let us now consider the case of $V_{j+1} \neq 0$. For this, we will assume that the spectrum of $\hat{H}_x^{j+1}$ is $\delta$-simple for some $\delta > 0$. Then if $\|V_{j+1}\|_\infty \leq \frac{\delta}{2}$, we got for the same identification $\ell \mapsto (s, \tilde{\ell})$ that

\begin{equation}
|E^{j+1}(x, \ell) - E^j(x + s, \tilde{\ell})| \leq \|V_{j+1}\|_\infty.
\end{equation}

Thus we have by Theorem 1.2.

\begin{equation}
d(\psi^{j+1}(x, \ell), \psi^j(x + s, \tilde{\ell})) \leq \frac{2}{\delta} \|V_{j+1}\|_\infty
\end{equation}

where

\begin{equation}
d(\psi, \varphi) = \inf_{|c|=1} \|\psi - c\varphi\|
\end{equation}

is the distance between normalized eigenfunctions. We define the parametrizing set

\begin{equation}
P_j = V_j \times \{1, \ldots, P_j\}, \quad V_j = [1, \frac{1}{p_1^j}) \times \cdots \times [1, \frac{1}{p_d^j}).
\end{equation}

Clearly $|P_j| = 1$. In order to state our main result, we introduce $\eta_j = 2^{-j}$, $\delta_j$ is the $\delta$ obtained from Theorem 1.4 and $\varepsilon_j = (\delta_j)^{10}$.

**Theorem 1.6.** Assume $\|V_{j+1}\|_\infty \leq \varepsilon_{j+1}$. Then there exists $G_{j+1} \subseteq P_{j+1}$ and a map $A_j : G_{j+1} \to P_j$ such that

(i) $|G_{j+1}| \geq 1 - \eta_j$.

(ii) For $(x, \ell) \in G_{j+1}$, we have that $A_j(x, \ell) = (x + s, \tilde{\ell})$ for some $s \in S_{j+1}$, $\tilde{\ell} \in \{1, \ldots, P_j\}$. 
(iii) The map $A_j$ is continuous.
(iv) For $(x, \ell) \in \mathbb{G}_{j+1}$, we have
\[ |E^{j+1}(x, \ell) - E^j(A_j(x, \ell))| \leq \varepsilon_{j+1} \]
and
\[ d(\psi^{j+1}(x, \ell), \psi^j(A_j(x, \ell))) \leq \frac{2\varepsilon_{j+1}}{3^{j+1}} \]

Proof. This is essentially, what we have discussed above. □

We have seen that if for $(x, \ell) \in \mathbb{P}_j$, there exists $(\tilde{x}, \tilde{\ell}) \in \mathbb{G}_{j+1}$ such that $(x, \ell) = A_j(\tilde{x}, \tilde{\ell})$ then this $(\tilde{x}, \tilde{\ell})$ is unique. Finally, we have that $|A_j(\mathbb{G}_{j+1})| = |\mathbb{G}_{j+1}|$. Hence, for any $j$, we have that the set
\[ \mathbb{G}_j = \bigcap_{k \geq j} A_j \cdots A_k \mathbb{G}_{k+1} \]
has measure
\[ |\mathbb{G}_j| \geq 1 - \sum_{k \geq j} \eta_k \geq 1 - 2\eta_j. \]

We also note that $\mathbb{G}_j \subseteq \mathbb{G}_{j+1}$. For $(x, \ell) \in \mathbb{G}_j$, we obtain a sequence $(x_k, \ell_k)$ such that
\[ (x, \ell) = A_j \cdots A_k(x_k, \ell_k) \]
and we have that the eigenfunctions and eigenvalues converges. In particular that
\[ d(\psi^k(x_k, \ell_k), \psi^{\tilde{k}}(x_{\tilde{k}}, \ell_{\tilde{k}})) \leq 2\delta^q_{\tilde{k}} \]
for $\tilde{k} \geq k \geq j$.

Proof of Theorem 1.2. As the convergence is fast enough to also imply convergence in the $\ell^1$ norm, i.e. for the sequence $(x, \ell_k)$ corresponding to $(x, \ell) \in \mathbb{G}_j$, we have
\[ \|\psi^k(x_k, \ell_k) - \psi^j(x, \ell)\|_{\ell^1} \leq 2\delta^q_{k}. \]

Define
\[ \varphi^k(n) = \sum_{t \in \mathbb{A}} \psi^k(x_k, \ell_k; t)e(-t \cdot n). \]

Then we have that the $\varphi^k$ converge to a limit $\varphi$ in $\ell^\infty(\mathbb{Z}^d)$ and $(H^k - E(x_k, \ell_k))\varphi^k = 0$. Letting $E = \lim_{k \to \infty} E(x_k, \ell_k)$, we find
\[ (H - E)\varphi = 0. \]

Finally, by construction it is easy to see that we can satisfy the frequency condition for all $x$ such that $(x, \ell) \in \mathbb{G}_j$ for some $\ell$. As $|\mathbb{G}_j| \to 1$, the claim follows. □

In order to prove Theorem 1.3, we will need a sharpening of Theorem 1.2 which we present in the following section. Then, we proceed to prove Theorem 1.1.
2. Simple spectrum

The goal of this section is to prove a sharpening of Theorem 1.4.

**Theorem 2.1.** Let $V$ be a $p$-periodic potential. Given $\eta \in (0, \frac{1}{2})$, there exists a set $\mathcal{G} \subseteq \mathbb{T}^d$ such that

(i) $|\mathbb{T}^d \setminus \mathcal{G}| \leq \eta$.

(ii) For $x \in \mathcal{G}$, we have the spectrum of $\hat{H}_x$ is $\delta$-simple for

\[
\delta = (\eta)^{CP^2 \log(P)}
\]

for some $C > 1$ that only depends on $d$ and $\|V\|_{\infty}$.

(iii) For $x \in \mathcal{G}$, we have that $|\partial_{x_d} E(x, \ell)| \geq \gamma$ for

\[
\gamma = (\eta)^{CP^2 \log(P)}.
\]

for some $C > 1$ that only depends on $d$ and $\|V\|_{\infty}$.

In order to prove this theorem, we will need to gain further understanding of the operator $\hat{H}_x$. We begin by proving a simple proposition, which we will need for the study of the absolutely continuous spectrum and whose proof introduces some techniques necessary to prove Theorem 2.1.

**Proposition 2.2.** Let $V$ be a $p$-periodic potential, $x' \in [0, (p_1)^{-1}) \times \ldots \times [0, (p_d)^{-1})$, and $E \in \mathbb{R}$. Then

\[
\# \{x_d \in [0, (p_d)^{-1}) : E \in \sigma(\hat{H}_{(x', x_d)}) \} \leq 2p_1 \cdots p_d - 1.
\]

We define

\[
P(x; E) = \det(\hat{H}_x - E)
\]

and observe that it is a trigonometric polynomial of degree $P = p_1 \cdots p_d$ in each of the $x_j$. Furthermore, we have that $P(\bar{x}; E) = P(x; E)$ if $\bar{x}_j - x_j \in \frac{1}{p_j} \mathbb{Z}$.

**Proof of Proposition 2.2.** $E \in \sigma(\hat{H}_{(x', x_d)})$ is equivalent to $g(x_d) = P(x', x_d; E) = 0$. Now as $g$ is a trigonometric polynomial of degree $P$, we have that

\[
\# \{x_d \in [0, 1) : g(x_d) = 0 \} \leq 2P.
\]

As the number $\# \{x_d \in [tp_d^{-1}, (t + 1)p_d^{-1}) \ : \ g(x_d) = 0 \}$ is constant in $t$, the claim follows. \hfill \Box

We will need to consider $x$ not just in $[0, 1]^d$ but in the entire complex plane $\mathbb{C}^d$. We will denote in this section by $E_j(x)$ the eigenvalues of $\hat{H}_x$. We collect the properties of these eigenvalues in.

**Proposition 2.3.** (i) For $z \in \mathbb{C}^d$, we have that $|E_j(z)| \leq d + \sum_{j=1}^{d} e^{\Im(z_j)} + \|V\|_{\infty}$.

(ii) For $z \in \mathbb{C}^d$, we have $|\partial_{z_d} E_j(z)| \leq 2\pi(e^{\Im(z_d)} + 1)$.

(iii) Let $y_j = \pi d^n \log(p_1 \cdots p_d^2)(4(d + \|V\|_{\infty}) + 1)$, and $z_j = iy_j$. Then $|E_j(z) - E_j(\bar{z})| \geq 1$ for $j \neq \ell$.

(iv) For $z_j$ as in (iv), we have that $|\partial_{z_d} E_j(z)| \geq \frac{1}{4} e^{2\pi y_d}$. 

We recall that $\hat{H}_x : \ell^2(\mathbb{B}_p) \to \ell^2(\mathbb{B}_p)$ is given by $\hat{H}_x = \hat{H}_0^x + \hat{V}$ with $\|\hat{V}\| \leq \|V\|_\infty$ and

$$
(2.5) \quad \hat{H}_0^x \psi(k) = \sum_{j=1}^{d} 2 \cos(2\pi(x_j + k_j))\psi(k)
$$

is a multiplication operator.

**Proof of Proposition 2.3 (i), (ii).** This follows from the bound

$$
(2.6) \quad \|\hat{H}_0^x\| \leq \sum_{j=1}^{d} \cosh(\text{Im}(z_j)) \leq \sum_{j=1}^{d} e^{\text{Im}(z_j)} + d.
$$

As

$$
(2.7) \quad \partial_x \hat{H}_0^x \psi(k) = -4\pi \sin(2\pi(x_d + k_d))\psi(k)
$$

and $\partial_x E_j(z) = \langle \psi_j(z), \partial_x \hat{H}_z \psi_j(z) \rangle$ also (ii) follows. \qed

We can write

$$
(2.8) \quad \hat{H}_x = A(x) + B(x)
$$

with $\|B(x)\| \leq d + \|V\|_\infty$ and $A(x)$ being the diagonal matrix with entries

$$
(2.9) \quad d(k, y) = \sum_{j=1}^{d} e\left(\frac{k_j}{p_j}\right)e^{2\pi y_j}, \quad k \in \{0, \ldots, p_1 - 1\} \times \cdots \times \{0, \ldots, p_d - 1\}.
$$

**Lemma 2.4.** Let $A > 0$. Then for

$$
(2.10) \quad e^{2\pi y_1} \geq \frac{A p_1}{2\pi}, \quad e^{2\pi y_j} \geq p_j \left(\frac{1}{\pi} + \frac{1}{p_{j-1}}\right) e^{2\pi y_{j-1}}
$$

we have that for $k \neq \ell$

$$
(2.11) \quad |d(k, y) - d(\ell, y)| \geq A.
$$

**Proof.** Let $1 \leq j \leq d$ be the largest choice such that $k_j \neq \ell_j$. Thus

$$
\begin{aligned}
&\sum_{j=1}^{d} e\left(\frac{k_j}{p_j+1}\right)e^{2\pi y_{j+1}} + \cdots + e\left(\frac{k_d}{p_d}\right)e^{2\pi y_d} = e\left(\frac{\ell_j}{p_{j+1}}\right)e^{2\pi y_{j+1}} + \cdots + e\left(\frac{\ell_d}{p_d}\right)e^{2\pi y_d}
\end{aligned}
$$

and

$$
|e\left(\frac{k_j}{p_j}\right)e^{2\pi y_j} - e\left(\frac{\ell_j}{p_j}\right)e^{2\pi y_j}| \geq \frac{2\pi}{p_j} e^{2\pi y_j}.
$$

Thus, we are done if we choose $y_j$ such that

$$
\frac{2\pi}{p_j} e^{2\pi y_j} \geq 2(e^{2\pi y_1} + \cdots + e^{2\pi y_{j-1}} + A)
$$

holds. \qed

We see that with our choice of $y_j$, these bounds hold with $A = d + \|V\|_\infty + 1$.

**Proof of Proposition 2.3 (iii).** We have that the eigenvalues of $A(y)$ are at least

$$
d + \|V\|_\infty + 1
$$

apart. Hence, the claim follows by standard bounds. \qed
Proof of Proposition 2.3 (iv). Let \( E_j(y) \) be an eigenvalue of \( \hat{H}_y \). Then by the previous considerations. There exists an unique \( k \) such that
\[
|E_j(y) - d(k,y)| \leq d + \|V\|_\infty.
\]
Hence for \( \psi \) a normalized solution of \((\hat{H}_y - E_j(y))\psi = 0\), we have that
\[
\|(A(y) - E_j(y))\psi\| = \|(A(y) + B(y) - E_j(y))\psi + B(y)\psi\| \leq d + \|V\|_\infty.
\]
Now as
\[
\text{Proof of Proposition 2.6 (iii).}
\]
\( (i) \) thus follows by (1.15). (ii) now follows by the previous lemma.

We have already defined \( f(z) \) in (1.15). We also define
\[
(2.12) \quad g(z) = \text{Res}(P(z;.), \partial_{\ell z} P(z; .)),
\]
which is also analytic and satisfies
\[
(2.13) \quad g(z) = \prod_{\ell} \partial_{\ell z} P(z; E_\ell(z)).
\]

Lemma 2.5. We have that
\[
(2.14) \quad g(z) = f(z) \cdot \prod_{\ell} \partial_{\ell z} E_\ell(z).
\]

Proof. As \( P(z, E_\ell(z)) = 0 \), we have that
\[
\partial_{\ell z} P(z; E_\ell(z)) = \partial_{\ell z} E_\ell(z) \cdot \partial_{E} P(z; E_\ell(z)).
\]
Similarly to (2.13), we have that \( f(z) = \prod_{\ell} \partial_{E} P(z; E_\ell(z)) \), so the claim follows.

In the following, we will use the norm
\[
(2.15) \quad |z| = \max(|z_1|, \ldots, |z_d|)
\]
on \( z \in \mathbb{C}^d \).

Proposition 2.6. (i) \( |f(z)| \leq (4de^{2\pi |z|} + \|V\|_\infty)^P \).
(ii) \( |g(z)| \leq (4de^{2\pi |z|})^P \cdot |f(z)| \).
(iii) There exists \( y \) with \( 1 \leq |y| \leq \frac{1}{2d} \log(P2^d(4d + 4\|V\|_\infty + 1)) \) such that
\[
(2.16) \quad |f(y)| \geq 1, \quad |g(y)| \geq 1
\]
(iv) For \( x \in \mathbb{R}^d \), we have that
\[
(2.17) \quad |\partial_{k x} P(x; E_\ell(x))| \geq \frac{|g(x)|}{(4d + 2\|V\|_\infty + 1)^{P-1}}.
\]

Proof of Proposition 2.6 (i), (ii). By Proposition 2.3 (i), we have that
\[
|E_j(z)| \leq d(1 + e^{2\pi |z|} + \|V\|_\infty)
\]
(i) thus follows by (1.15). (ii) now follows by the previous lemma.

Proof of Proposition 2.6 (iii). The lower bound on \( f(y) \) follows by Proposition 2.3 (iv). In order to deduce the one on \( g(z) \) use the previous lemma and Proposition 2.3 (v), and that \( \frac{1}{2}e^{2\pi y \mu} \geq 1 \).
Proof of Proposition 2.6 (iv). By (2.13), we clearly have that
\[ |\partial_{x_d} P(x; E_j(x))| \geq |g(x)| \cdot \left( \max_{1 \leq j \leq P} |\partial_{x_d} P(x; E_j(x))| \right)^{(P-1)} \].

By Cauchy’s integral formula, we obtain for \( x = (x', x_d) \)
\[ \partial_{x_d} P(x; E_j(x)) = -\frac{1}{2\pi i} \int_{|t-x_d|=1} \frac{P(x', t, E_j(x))}{(t-x_d)^2} dt. \]

As \( ||\hat{H}(x) - E_j(x)|| \leq 4d + 2\|V\|_\infty + 1, \) the claim follows. \( \square \)

Finally, we observe

Lemma 2.7. For \( |z| \leq 4e|y|, \) we have
\[ \log |f(z)| \leq P^2 \left( 4e \log(P) + C \right), \]
\[ \log |g(z)| \leq P(P + 1) \left( 4e \log(P) + C \right), \]
where \( C = \log \left( \max(4\pi, 5d)2^{4ed}(4d + \|V\|_\infty + 1)4e \right). \)

Proof. This is a computation. \( \square \)

Proof of Theorem 2.1. The claim follows by Theorem A.1. \( \square \)

3. The Absolutely Continuous Spectrum of a Periodic Operator

The goal of this section is to prepare for the proof of Theorem 2.1 given in the next section. The main reason for writing a separate section, is to make this section somewhat more expository.

Let \( H \) be a \( p \)-periodic operator. For simplicity, we will restrict ourself to considering \( H \) in Fourier space, i.e. \( \hat{H} : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \)
\[ \hat{H} f(x) = \left( \sum_{j=1}^d 2 \cos(2\pi x_j) \right) f(x) + \sum_{k \in \mathbb{B}} \hat{V}(k) f(x + k). \]

Given \( f \in L^2(\mathbb{T}^d) \) and \( y \in \mathbb{V} = [0, (p_1)^{-1}) \times \cdots \times [0, (p_d)^{-1}) \), we define \( f_y \in \ell^2(\mathbb{B}) \) by \( f_y(k) = f(k + y) \). We have that \( \hat{H} f_x = (\hat{H} f)_x \). We recall that, we denote by \( \psi(x, \ell) \) the orthonormal basis of \( \ell^2(\mathbb{B}) \) consisting of eigenfunctions of \( \hat{H} \). Thus, we have that
\[ f_y = \sum_{\ell=1}^P \langle \psi(y, \ell), f_y \rangle \psi(y, \ell). \]

Hence, given a set \( A \subseteq \mathbb{P} = \mathbb{V} \times \{1, \ldots, P\}, \) it makes sense to define the projection operator
\[ (Q_A f)(y + k) = \sum_{\ell=1}^P \chi_A(y, \ell) \langle \psi(y, \ell), f_y \rangle \psi(y, \ell)_k, \]
where \( y + k \) is the unique decomposition of \( x \in \mathbb{T}^d \) into \( y \in \mathbb{V} \) and \( k \in \mathbb{B} \). Note \( I - Q_A = Q_{\mathbb{P} \setminus A} \) and that \( Q_A \) is a projection.
Lemma 3.2. Let $\mu$ be absolutely continuous and $E$ then $Q_A \leq Q_B$.

Proof. Let $A_1 = \{ x : \exists \ell : (x, \ell) \in A \}$ then $|A_1| \leq P|A|$. We compute
\[
\|Q_A f\|^2 = \int_{A_1} \sum_{\ell=1}^P |\langle \psi(x, \ell), f\rangle|^2 dx \leq \int_{A_1} \sum_{\ell=1}^P \|f\|^2 dx
\]
As $\|f\|_{\ell^2} \leq \sqrt{P}\|f\|_{L^\infty(\mathbb{T}^d)}$, (i) follows.

To see that (ii) holds, observe that $Q_B - Q_A = Q_{B \setminus A}$. As $Q_{B \setminus A} \geq 0$, the claim follows. \qed

Next, we have that

Lemma 3.3. Let $\varphi \in \ell^1(\mathbb{Z}^d)$, $G \subseteq \mathbb{P}$, $\varphi_G = Q_G \varphi$, and define a measure $\mu_G$ by
\[
\mu_G(A) = \langle \varphi_G, \chi_A(H) \varphi_G \rangle.
\]
Assume that

(i) For $(x, \ell) \in G$, we have that
\[
|\partial_{x,\ell} E(x, \ell)| \geq \gamma.
\]
(ii) For $(x, \ell) \in G$, we have that the spectrum of $\hat{H}_x$ is $\delta$-simple for some $\delta > 0$.

(iii) For $(x, \ell) \in G$, we have
\[
\|\psi(x, \ell)\|_{\ell^1(\mathbb{B})} \leq C_1.
\]
Then the measure $\mu$ is absolutely continuous and
\[
\frac{d\mu}{dE} \leq \frac{4(C_1 \cdot \|\varphi\|_{\ell^1(\mathbb{Z}^d)})^2}{\gamma}.
\]

Proof. We have that
\[
\mu([E - \varepsilon, E + \varepsilon]) = \int \sum_{\ell=1}^P \chi_{[E - \varepsilon, E + \varepsilon]}(E(x, \ell)) \chi_{G}(x, \ell) \cdot |\langle \psi(x, \ell), \hat{\varphi}_x \rangle|^2 dx.
\]
We first observe that
\[
|\langle \psi(x, \ell), \hat{\varphi}_x \rangle| \leq \|\psi(x, \ell)\|_{\ell^1(\mathbb{B})} \cdot \|\hat{\varphi}_x\|_{\ell^\infty(\mathbb{B})} \leq C_1 \|\varphi\|_{\ell^1(\mathbb{Z}^d)}.
\]
Let $x' \in [0, (p_1)^{-1}) \times \cdots \times [0, (p_d)^{-1})$. It thus suffices to bound
\[
I(\varepsilon) = \int_0^{(p_d)^{-1}} \sum_{\ell=1}^P \chi_{[E - \varepsilon, E + \varepsilon]}(E(x', x_d, \ell)) \chi_{G}(x', x_d, \ell) dx_d.
\]
By relabeling the eigenvalues, we may assume that they are analytic on small neighborhoods. Fix some $\ell$ and denote by $I$ the set of $x_d$ so that $(x', x_d, \ell) \in G$. Then if $[a, b]$ is a subinterval of $I$, we have by (iii) that
\[
|[a, b] : E((x', x_d, \ell) \in [E - \varepsilon, E + \varepsilon])| \leq \frac{2\varepsilon}{\gamma}.
\]
Due to the simplicity of eigenvalues, we have that (i) is stable. In particular if $\varepsilon > 0$ is small enough, $E((x', x_d, \ell) \in [E - \varepsilon, E + \varepsilon]$ implies that there exists $|\tilde{x}_d - x_d| \leq \frac{2\varepsilon}{\gamma}$.
so that $E((x', \tilde{x}_d), \ell) = E$. Hence, we are always in the case described above. Thus, we obtain

$$I(\varepsilon) \leq \# \{x_d, \ell : \quad E(x', x_d, \ell) = E\} \cdot \frac{4\varepsilon}{\gamma}.$$  

By Proposition 4.2 and $|[0, (p_1)^{-1}) \times \cdots \times [0, (p_{d-1})^{-1})] = (p_1 \cdots p_{d-1})^{-1}$, the claim follows. 

\[\square\]

4. PROOF OF ABSOLUTELY CONTINUOUS SPECTRUM

The goal of this section is to provide the proof of Theorem 1.1. It clearly suffices to prove that the limit-periodic potentials obeying the conditions given in the proof of Theorem 1.2 have purely absolutely continuous spectrum. One difference is that the conclusions of Theorem 1.4 are not enough, but we will need the full conclusions of Theorem 2.1.

For the readers convenience and easy reference, we summarize the conclusions.

(i) There exist sets $G_j \subseteq \mathbb{P}_j$ with $|G_j| \leq \eta_j = \frac{1}{(p_{d-j})^{2j} \cdot 2^{j}}$.

(ii) For $(x, \ell) \in G_j$, we have

$$|\partial_{x_d} E^j(x, \ell)| \geq \gamma_j$$

with $\gamma_j \geq 100\delta_{j+1}^2$.

(iii) For $(x, \ell) \in G_j$ and $k \geq j$, there is an unique $(x_k, \ell_k)$ such that $(x, \ell) = A_j \cdots A_{k-1} (x_k, \ell_k)$.

(iv) We have for some $|\epsilon| = 1$ and $k \geq j$ that

$$\|\psi^j(x, \ell) - c\psi^k(x_k, \ell_k)\|_{\ell^1} \leq 2(\delta_j)^8.$$  

We note that our choice of $\eta_j$ is different. Also we need to choose $\varepsilon_{j+1}$ such that $\gamma_j \geq 100\varepsilon_{j+1}/\delta_{j+1}$, which is not a problem.

Fix some $k \geq 1$ and for $j \geq k$ consider the projections $P_{k,j} = Q_{G_{k,j}}$ as in (3.3) where

$$G_{k,j} = A_{k-1}^{-1} \cdots A_j^{-1} G_j.$$  

Proposition 4.1. 

(i) $\|I - P_{k,j}\| \leq 2\eta_j P_j^2$.

(ii) $\|P_{k+1,j} - P_{k,j}\| \leq \delta_j$. In particular, the limit $P_{\infty,j} = \lim_{k \to \infty} P_{k,j}$ exists.

(iii) $\|I - P_{\infty,j}\| \leq 3\eta_j P_j^2$.

(iv) $P_{\infty,j} \leq P_{\infty,j+1}$.

Proof. (i) follows from Proposition 4.1. For (ii) observe, that

$$P_{k,j} f(x+k) = \sum_{\ell} \chi_A(x, \ell) \langle \psi^k(A_k(x, \ell)), f \rangle \psi^k(A_k(x, \ell))$$

for $A = A_{k-1}^{-1} \cdots A_j^{-1} G_j$. As $d(\psi^{k+1}(x, \ell), \psi^k(A_k(x, \ell))) \leq \frac{2\delta_{k+1}}{\delta_{k}}$, the bound on $\|P_{k+1,j} - P_{k,j}\|$ follows. To see convergence, observe that $\sum_{\ell \geq k} \delta_{\ell} \leq 2\delta_k$. This bound also implies (iii). Finally for (iv), we have that $P_{k,j+1} \geq P_{k,j}$. Thus this inequality also holds in the limit $k \to \infty$.

Proposition 4.2. There exists $C_j > 0$ such that for $\varphi \in \ell^2(\mathbb{Z}^d)$ with $\|\hat{\varphi}\|_{L^\infty(\mathbb{T}^d)} \leq 1$, we have for $k \geq j$

$$\langle P_{k,j} \varphi, \chi_{[E-\varepsilon, E+\varepsilon]}(H^k) P_{k,j} \varphi \rangle \leq C_j \varepsilon.$$  


By property (ii), we have that
\begin{equation}
\|\psi^k(x_k, \ell_k)\|_{\ell^1} \leq \sqrt{P_j} + 2(\delta_j)^8.
\end{equation}
For the proof, we need
\begin{lemma}
Let $k \geq j$ and $(x, \ell) \in A_{k-1} \cdots A_{j-1} G_j$. Then
\begin{equation}
|\partial_x d E_j(x, \ell)| \geq 100(\delta_j)^8.
\end{equation}
\end{lemma}

\begin{proof}
Let $(\tilde{x}, \tilde{\ell}) = A_j \cdots A_{k-1} (x, \ell).$ Then
\begin{equation}
d(\psi^k(x, \ell), \psi^j(\tilde{x}, \tilde{\ell})) \leq 3\varepsilon_j + 1 \leq 3(\delta_j)^8.
\end{equation}
Next, observe that
\begin{equation}
|\partial_x d E_j(\tilde{x}, \tilde{\ell})| \geq \gamma_j \geq 100(\delta_j)^8.
\end{equation}
Thus, the claim follows. \hfill \Box
\end{proof}

\begin{proof}[Proof of Proposition 4.2]
This follows from Lemma 3.2. \hfill \Box
\end{proof}

We define now vectors
\begin{equation}
\phi_{k,j} = P_k, j \phi_{k,j} = P_k, j
\end{equation}
and measures
\begin{equation}
\mu_{k,j}(A) = \langle \phi_{k,j}, \chi_A(H^k) \phi_{k,j} \rangle.
\end{equation}
We have that as $k \to \infty$, the vectors $\phi_{k,j}$ converge to a limit $\phi_j$ and we also define the measure
\begin{equation}
\mu_j(A) = \langle \phi_j, \chi_A(H) \phi_j \rangle.
\end{equation}
As $H^k \to H$ and $\phi_{k,j} \to \phi_j$, we have that $\mu_{k,j} \to \mu_j$ and in particular that $\mu_j$ is also absolutely continuous. Our results also imply that $\mu_j(A) \geq \mu_{j-1}(A)$.

Define now a measure
\begin{equation}
\mu(A) = \langle \phi, \chi(H) \phi \rangle
\end{equation}
As $\phi_j \to \phi$, we have that $\mu_j \to \mu$.

\begin{proof}[Proof of Theorem 1.1]
We may write
\begin{equation}
\mu = \mu_1 + \sum_{j \geq 2} (\mu_j - \mu_{j-1}).
\end{equation}
As the measures $\mu_1, \mu_2 - \mu_1, \mu_3 - \mu_2, \ldots$ are all absolutely continuous and positive, it follows that $\mu$ is absolutely continuous. As we could choose $\phi$ from a dense set, the claim follows. \hfill \Box
\end{proof}

\section*{Appendix A. Cartan’s estimate}

In this section, we will prove

\begin{theorem}
Let $f : \mathbb{C}^d \to \mathbb{C}$ be an analytic function. Assume that there exists $y \in \mathbb{C}^d$ with $|y| > 1$ such that $|f(y)| \geq \kappa$ and that we have
\begin{equation}
\log \sup_{|z| \leq 4|y|} |f(z)| \leq A.
\end{equation}
Then
\begin{equation}
|\{x \in [0, 1]^d : |f(z)| \leq \kappa \cdot \left( \frac{\varepsilon}{60e^3d|y|} \right)^{d-A} \}| \leq \varepsilon.
\end{equation}
\end{theorem}
In order to prove this estimate, we will need the original Cartan estimate.

**Theorem A.2.** Let \( g : \mathbb{C} \to \mathbb{C} \) be an analytic function satisfying
\[
|g(y)| \geq \kappa
\]
for some \( y \in \mathbb{C} \) with \(|y| > 1\). Then
\[
|\{ x \in [0, 1] : |g(x)| \leq \delta \cdot \kappa \}| \leq \varepsilon
\]
for
\[
\log(\delta) = \log\left(\frac{\varepsilon}{60e^3|y|}\right) \cdot \log(\sup_{|z| \leq 4e|y|} |g(z)|).
\]

**Proof.** This is one version of Cartan’s Estimate, see Theorem 11.3.4. in Levin’s book [10]. \( \square \)

**Proof of Theorem A.1.** Define the function
\[
g_1(z) = f(z, y_2, \ldots, y_d).
\]
Then \(|g_1(y_1)| \geq \kappa \) and \( \log(\sup_{|z| \leq 4e|y_1|} |g_1(z)|) \leq A \). Hence, there exists a set \( X_1 \subseteq [0, 1] \) of measure \( \leq \varepsilon \) such that for \( x_1 \in [0, 1] \setminus X_1 \), we have
\[
|f(x_1, y_2, \ldots, y_d)| = |g_1(x_1)| \geq \kappa_1 = \kappa \cdot \left(\frac{\varepsilon}{60e^3d|y|}\right)^A.
\]
Applying this construction inductively to
\[
g_j(z) = f(x_1, \ldots, x_{j-1}, z, y_{j+1}, \ldots, y_d)
\]
with \( x_\ell \in [0, 1] \setminus X_\ell \), we obtain sets \( X_1, \ldots, X_d \) such that for \( x_\ell \in [0, 1] \setminus X_\ell \) for \( 1 \leq \ell \leq j \), we have
\[
|f(x_1, \ldots, x_j, y_{j+1}, \ldots, y_d)| \geq \kappa_j = \kappa \cdot \left(\frac{\varepsilon}{60e^3d|y|}\right)^{jA}.
\]
As
\[
|\{ x \in [0, 1] : x_j \notin X_j \}| \geq 1 - |X_1| - \cdots - |X_d| \geq 1 - \varepsilon
\]
the claim follows. \( \square \)

**Appendix B. Distances of normalized eigenfunctions**

Let \( X \) be a Hilbert space, and \( \varphi, \psi \) two unit vectors. We define the distance
\[
d(\varphi, \psi) = \inf_{|c|=1} \| \varphi - c\psi \|.
\]

**Theorem B.1.** Let \( A \) be a self-adjoint operator on \( X \) with
\[
\text{tr}(P_{[-\delta, \delta]}(A)) = 1
\]
and \( A\psi = 0, \|\psi\| = 1 \). Assume the \( \varphi \) with \( \|\varphi\| = 1 \) satisfies \( \|A\varphi\| \leq \varepsilon \). Then
\[
d(\varphi, \psi) \leq \frac{2\varepsilon}{\delta}.
\]

**Proof.** Let \( \varphi_1 = \langle \psi, \varphi \rangle \psi, \varphi_2 = \varphi - \varphi_2 \). Then \( \varepsilon \geq \|A\varphi\| = \|A\varphi_2\| \geq \delta \|\varphi_2\| \). Thus
\[
|\langle \psi, \varphi \rangle| = \|\varphi_1\| \geq 1 - \frac{\varepsilon}{\delta}. \tag{1}
\]
Taking \( c = \langle \psi, \varphi \rangle / |\langle \psi, \varphi \rangle| \) the claim follows. \( \square \)

More sophisticated versions of this argument can be found in Section 9 of [8]. In particular, the methods discussed there would allow one to understand the set \( \{(x, \ell) \in V : E(x, \ell) = E\} \) for any \( E \in \mathbb{R} \).
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