Topological degeneracy in Ising chain induced by dissipation

K. L. Zhang and Z. Song
School of Physics, Nankai University, Tianjin 300071, China

The groundstate degeneracy of quantum spin system is a characteristic of non-trivial topology, when it is gapped and robust against disordered perturbation. The corresponding quantum phase transition (QPT) is usually driven by a real parameter. We study a non-Hermitian Ising chain with two transverse fields, one real another imaginary, based on the exact solution and numerical simulation. We show that topological degeneracy still exists, and can be obtained by an imaginary transverse field from a topologically trivial phase of a Hermitian system. The topological degeneracy is robust against random imaginary field, and therefore expected to be immune to disordered dissipation from the spontaneous decay in experiment. The underlying mechanism is the nonlocal symmetry, which emerges only in thermodynamic limit and unifies two categories of QPTs in quantum spin system, rooted from topological order and symmetry breaking, respectively.

I. INTRODUCTION

Driving a quantum phase transition (QPT) is of interest to both condensed matter physics and quantum information science. A varying parameter across the critical point induces a symmetry spontaneous breaking for traditional QPT [1] and nonlocal topological order for topological QPT [2, 3]. In the recent works [4, 5], it turns out that the local order parameter and topological order parameter can coexist to characterize the QPTs. The underlying mechanism of this fact is the duality of the Kitaev model, which has been introduced to describe one-dimensional spinless fermions with superconducting p-wave pairing [6]. On the one hand, It is the fermionized version of the familiar one-dimensional (1-D) transverse-field Ising model [7], which is one of the simplest solvable models exhibiting quantum criticality and demonstrating a QPT with spontaneous symmetry breaking [1]. On the other hand, as the gene of a Kitaev model, its Majorana lattice is the Su-Schrieffer-Heeger (SSH) model [8], which has served as a paradigmatic example of the 1-D system supporting topological character [9]. It manifests the typical feature of topological order since the number of zero energy and edge states are immune to local perturbations [10]. The topological superconducting has been demonstrated by unpaired Majorana modes exponentially localized at the ends of open Kitaev chains, which are robust against disordered perturbation. A system with topological phase can be a promising platform for quantum computation and information processing due to the intrinsic stability of the topological feature [11, 12].

So far, most of the investigations on the QPT driven by varying a real parameter at absolute zero temperature. However, in practice, a genuine quantum system, such as cold-atom, is intrinsically non-Hermitian because of spontaneous decay [14, 19]. On the other hand, non-Hermitian Hamiltonian is no longer a forbidden regime in quantum mechanics since the discovery that a certain class of non-Hermitian Hamiltonians could exhibit entirely real spectra [20, 22]. It also turns out that certain type of non-Hermitian terms may maintain the topological feature of the original Hermitian system [23]. A natural question is whether a topologically trivial phase of a Hermitian system can be shifted to a non-trivial phase by adding a non-Hermitian term.

In this paper, we investigate an one-dimensional quantum Ising model with complex transverse field. The aim of this paper is to study the consequence of an imaginary transverse field on the topology of the ground state. Intuitively, the imaginary transverse field may break the Hermiticity of the Hamiltonian. However, it is shown that the non-Hermitian Ising model can be mapped to a Hermitian Ising model in the context of biorthogonal inner product. This allows us to employ the same way for Hermitian system, i.e., the robust degeneracy of ground states, to identify the nature of quantum phases. We show that the topological degeneracy still exists in the presence of complex transverse field. We obtain the non-Hermitian version of mapping operator to connect two degenerate ground states in the topologically non-trivial region. In addition, numerical simulation for finite size system indicates the existence of topological degeneracy since the degeneracy cannot be lifted by disordered perturbation on the imaginary transverse field.

This paper is organized as follows. In Sec. III we present the model and its Hermitian counterpart. In Sec. III we analyse the topological degeneracy in Hermitian version. In Sec. IV we propose the concept of nonlocal symmetry and numerically study the robustness against the disordered imaginary field. Section V summarizes the results and explores its implications.

II. HAMILTONIAN AND HERMITIAN COUNTERPART

We start our investigation by considering a non-Hermitian Ising chain with a complex transverse field

\[ H = - \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x + \sum_{j=1}^{N} g_j \left( \sigma_j^x + i \gamma \sigma_j^y \right), \]  

(1)
where $\sigma^x_j$, $\sigma^y_j$, and $\sigma^z_j$ are the Pauli operators on site $j$, and $g$ and $\gamma_j$ ($i = \sqrt{-1}$) are real numbers. As far as we know, although non-Hermitian systems indeed have some peculiar features and some of them have been proved that can be equivalent to Hermitian systems in some particular conditions [24–28], the complex field is always seen as unphysical. Recently some works, including theoretical and experimental research on Lee Yang zeros, which are the points on the complex plane of physical parameters, are proposed [30–36]. It relates a complex field to real and experimental research on Lee Yang zeros, which are not Hermitian systems indeed have some peculiar features and some of them have been proved that can be equivalent to Hermitian systems in some particular conditions [24–28], the complex field is always seen as unphysical. Recently some works, including theoretical and experimental research on Lee Yang zeros, which are the points on the complex plane of physical parameters, are proposed [30–36]. It relates a complex field to real world in some extent. On the other hand, it was proposed that [19] an imaginary transverse field can be implemented by optically pumping a qubit state into the auxiliary state with the scheme similar to heralded entanglement protocols.

In order to explore the property of the non-Hermitian model, we introduce a transformation

$$\tau^x_j = \gamma_j^x \sigma^x_j, \quad \tau^y_j = i \gamma_j^y \sigma^y_j + i n_j \sigma^x_j, \quad \tau^z_j = i \gamma_j^z \sigma^z_j + i n_j \sigma^x_j,$$

where the factors are $n_j^+ = 1/\sqrt{1 - \gamma^2_j}$ and $n_j^- = \gamma_j/\sqrt{1 - \gamma^2_j}$. The new spin operators still satisfy the Lie algebra commutation relations

$$[\tau^x_j, \tau^y_j] = 2i\epsilon^{\mu\nu\lambda} \tau^\nu_j \tau^\lambda_j,$$

although $\tau^y_j$ and $\tau^z_j$ are not Hermitian. Applying the transformation on the Hamiltonian $H$, we have

$$\mathcal{H} = -\sum_{j=1}^{N-1} \tau^x_j \tau^x_{j+1} + \sum_{j=1}^{N} g_j \sqrt{1 - \gamma^2_j} \tau^z_j. \quad (4)$$

Hamiltonian $\mathcal{H}$ represents an Ising model with real transverse field if $|\gamma_j| < 1$, and has full real spectrum although the spin operators $\tau^y_j$ and $\tau^z_j$ are not Hermitian. Within this region, $\mathcal{H}$ shares the same properties of $H$ with $\gamma_j = 0$, in the context of biorthogonal inner product. The connection between the eigenstates of $H$ and $\mathcal{H}$ can be obtained from the following relations

$$|\pm\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j - \frac{\gamma_j}{\xi_{\pm}} |\downarrow\rangle_j \right), \quad (5)$$

where $\xi_{\pm} = \sqrt{1 \pm \sqrt{1 - \gamma^2_j}}$, and the corresponding local spin vectors are defined as

$$\tau^x_j |\pm\rangle_j = \pm |\pm\rangle_j, \quad \sigma^z_j |\uparrow\rangle_j = |\uparrow\rangle_j, \quad \sigma^z_j |\downarrow\rangle_j = - |\downarrow\rangle_j. \quad (6)$$

Accordingly, the biorthonomal complete set of eigenstates can be established by the eigenstates of $\mathcal{H}^\dagger$.

The aim of this paper is to investigate the effect of $\{\gamma_j\}$ on the topological feature of quantum phases. Now we know that when an imaginary transverse field $\{\gamma_j\}$ is applied, the non-Hermitian Hamiltonian has a Hermitian counter part, which is the original Ising model with a real transverse field shifted by a $\gamma_j$-related amounts, i.e., $g_j \rightarrow g_j \sqrt{1 - \gamma^2_j}$. In this sense, imaginary field can lead to QPT. According to quantum theory of QPT in Hermition system, a second-order QPT is characterized by the divergence of groundstate energy density. For the Hamiltonian $H$ in Eq. (1) with $g_i = g$ and $\gamma_i = \gamma$, we have

$$\lim_{g \rightarrow 1} \frac{\partial^2 \varepsilon_g}{\partial g^2} = \infty, \quad (7)$$

where $\varepsilon_g$ is the density of groundstate energy. Meanwhile for the Hamiltonian $\mathcal{H}$ in Eq. (4) with $g_i = g$ and $\gamma_i = \gamma$, we have

$$\frac{\partial \varepsilon_g}{\partial \gamma} = \frac{\gamma}{\sqrt{1 - \gamma^2}} \frac{\partial \varepsilon_g}{\partial \sqrt{1 - \gamma^2}} \quad (8)$$

which results in

$$\lim_{\gamma \rightarrow \gamma_c} \frac{\partial^2 \varepsilon_g}{\partial \gamma^2} = \infty, \quad (9)$$

due to the replacement $g \rightarrow g \sqrt{1 - \gamma^2}$. It indicates that $\gamma$ drives a second-order QPT at $\gamma_c = \pm \sqrt{1 - g^2}$. In the next two sections, we will analyse this issue from the topological aspect.

### III. Groundstate Degeneracy and Edge Modes

We first revisit the connection between groundstate degeneracy and Majorana edge modes in Hermitian transverse field Ising chain in this section, and then extend the conclusion to the non-Hermitian version in the next section. Consider the Hamiltonian $H$ in Eq. (1) with $g_i = g$ and $\gamma_i = \gamma = 0$,

$$H_{\text{spin}} = -\sum_{j=1}^{N-1} \sigma^x_j \sigma_{j+1}^x + g \sum_{j=1}^{N} \sigma^z_j, \quad (10)$$

which is a standard Ising model with open boundary condition. The $2^N$-D complete set of basis can be constructed by applying operators $\sigma^x_j = (\sigma^x_j + i \sigma^y_j)/2$ on a saturated ferromagnetic state $\prod_{j=1}^{N} |\uparrow\rangle_j$. The whole Hilbert space can be decomposed into two invariant subspaces, with even and odd numbers of spins flip from above ferromagnetic state. Then all the eigenstates of $H_{\text{spin}}$ can be classified into two groups, denoted as $\{|\psi_j^+\rangle\}$ and $\{|\psi_j^-\rangle\}$, respectively. In the following, we will show the connection between such two groups of eigenstates, and the implication to topological degeneracy.
As standard procedure, one can perform the Jordan-Wigner transformation \[37\]
\[
\sigma^x_j = \prod_{l<j} \left(1 - 2c_l^+ c_j\right) \left(c_j + c_j^+\right),
\]
\[
\sigma^y_j = i \prod_{l<j} \left(1 - 2c_l^+ c_l\right) \left(c_j - c_j^+\right),
\]
\[
\sigma^z_j = 2c_j^+ c_j - 1,
\] (11)
to replace the Pauli operators by the fermionic operators \(c_j\). In this paper, we focus on the system with open boundary condition, in which there are no difference between even and odd numbers of fermions. The Hamiltonian is transformed to a Kitaev model

\[
H_{\text{Kitaev}} = -\sum_{j=1}^{N-1} \left(c_j^+ c_{j+1} + c_{j+1}^+ c_j\right) + \text{H.c.}
\]
\[
+ g \sum_{j=1}^{N} \left(2c_j^+ c_j - 1\right).
\] (12)

We note that the parity of particles number is conservative, i.e., \([(-1)^{\sum_{j=1}^{N} c_j^+ c_j}, H_{\text{Kitaev}}] = 0\), corresponds to classification of the eigenstates, \(\{|\psi_j^+\rangle\}\) and \(\{|\psi_j^-\rangle\}\).

To get the solution of the model, we introduce the Majorana fermion operators

\[
a_j = c_j^+ + c_j, b_j = -i \left(c_j^+ - c_j\right),
\] (13)

which satisfy the commutation relations

\[
\{a_j, a_{j'}\} = 2\delta_{jj'}, \{b_j, b_{j'}\} = 2\delta_{jj'},
\]
\[
\{a_j, b_{j'}\} = 0.
\] (14)

Then the Majorana representation of the Hamiltonian is

\[
H_M = -\frac{i}{2} \sum_{j=1}^{N-1} b_j a_{j+1} - g \sum_{j=1}^{N} a_j b_j + \text{H.c.},
\] (15)

the core matrix of which is that of a 2N-site SSH chain in single-particle invariant subspace. Based on the exact diagonalization results of the SSH chain, the Hamiltonian \(H_{\text{Kitaev}}\) can be written as the diagonal form

\[
H_{\text{Kitaev}} = \sum_{n=1}^{N} \epsilon_n (d_n^+ d_n - \frac{1}{2}).
\] (16)

Here \(d_n\) is fermionic operator, satisfying \(\{d_n, d_{n'}\} = 0\), and \(\{d_n, d_n^\dagger\} = \delta_{n,n'}\). The spectrum \(\epsilon_n\) and the explicit expression of \(d_n\) can be obtained by the diagonalization of the matrix

\[
M_{\text{SSH}} = -\frac{1}{2} \sum_{j=1}^{N-1} \left|2j\right\rangle \langle 2j + 1 | + \frac{1}{2} g \sum_{j=1}^{N} \left|2j - 1\right\rangle \langle 2j | + \text{H.c.}.
\] (17)

On the other hand, no matter the explicit solution is, we always have the relations

\[
[d_n, H_{\text{Kitaev}}] = \epsilon_n d_n, [d_n^\dagger, H_{\text{Kitaev}}] = -\epsilon_n d_n^\dagger,
\] (18)

which result in the mapping between the eigenstates of \(H_{\text{Kitaev}}\). For an arbitrary eigenstate \(|\psi\rangle\) of \(H_{\text{Kitaev}}\) with eigenenergy \(E\), i.e.,

\[
H_{\text{Kitaev}} |\psi\rangle = E |\psi\rangle,
\] (19)

state \(d_n |\psi\rangle (d_n^\dagger |\psi\rangle)\) is also an eigenstate of \(H_{\text{Kitaev}}\) with the eigenenergy \(E - \epsilon_n (E + \epsilon_n)\), i.e.,

\[
H_{\text{Kitaev}} (d_n |\psi\rangle) = (E - \epsilon_n) (d_n |\psi\rangle),
\] (20)

and

\[
H_{\text{Kitaev}} (d_n^\dagger |\psi\rangle) = (E + \epsilon_n) (d_n^\dagger |\psi\rangle),
\] (21)

if \(d_n |\psi\rangle \neq 0\) \((d_n^\dagger |\psi\rangle \neq 0)\).

In this paper, we are interested in the topological degeneracy in the topologically non-trivial phase, which arises from the zero-eigenvalue eigenvectors of the matrix in Eq. \((17)\) for \(|g| < 1\) in large \(N\) limit. It turns out that within the topological region, the edge modes appear with \(\epsilon_N = 0\) and the edge operator \(d_N\) can be expressed as

\[
d_N = \frac{1}{2} \sqrt{1 - g^2} \sum_{j=1}^{N} \left(\left(g^{j-1} + g^{N-j}\right) c_j^+ - \left(g^{j-1} - g^{N-j}\right) c_j\right),
\] (22)
i.e., \(d_N\) is a linear combination of particle and hole operators of spinless fermions \(c_j\) on the edge. Furthermore, applying the inverse Jordan-Wigner transformation, \(d_N\) can be expressed as the combination of spin operators,

\[
D_N = \frac{1}{2} \sqrt{1 - g^2} \prod_{j=1}^{N \geq j} \left(\left(g^{j-1} + g^{N-j}\right) (-\sigma_i^z) \sigma_j^+ - \left(g^{j-1} - g^{N-j}\right) (-\sigma_i^z) \sigma_j^-\right).
\] (23)

In fact, \(d_N\) and \(D_N\) are identical, but only in different representations. Obviously, from \([d_N, H_{\text{Kitaev}}] = 0\), we have

\[
[D_N, H_{\text{spin}}] = [D_N^\dagger, H_{\text{spin}}] = 0,
\] (24)

which lead to the degeneracy of the eigenstates. We would like to point out that the commutation relation in Eq. \((24)\) can be regarded as the symmetry of the system. Importantly, such a symmetry is conditional, requiring \(|g| < 1\) in large \(N\) limit. This accords with the symmetry breaking mechanism for QPT \([1]\). On the other hand, the QPT also has topological characteristics since the edge mode is robust against disorder perturbation. Here we do not review this content in the Hermitian regime, but investigate it directly in the non-Hermitian regime in the next section.
IV. NONLOCAL SYMMETRY AND ROBUSTNESS OF DEGENERACY

Starting from the non-Hermitian Hamiltonian $H$ in Eq. (1), it is tough to find out the mapping operator along the same route in the last section [8,24]. However, we note that the commutation relation in Eq. (3) is only based on the Lie algebra commutation relation of spin operators $\{\sigma^z_j\}$ no matter they are Hermitian or non-Hermitian. Then one can construct the mapping operator directly by replacing $\{\sigma^z_j\}$ with $\{\tau^z_j\}$. In parallel, we have the mapping operator of non-Hermitian version

$$D_N = \frac{1}{2} \sqrt{1 - g^2} \sum_{j=1}^{N} \prod_{l<j} [(\sigma^y_j + g^{N-j})(-\tau^z_l) \tau^+_j + (\sigma^y_j - g^{N-j})(-\tau^z_l) \tau^-_j],$$

and its canonical conjugation

$$\overline{D}_N = \frac{1}{2} \sqrt{1 - g^2} \sum_{j=1}^{N} \prod_{l<j} [(\sigma^y_j + g^{N-j})(-\tau^z_l) \tau^-_j + (\sigma^y_j - g^{N-j})(-\tau^z_l) \tau^+_j],$$

in large $N$ limit, where the equivalent real field $g = g\sqrt{1 - \gamma^2}$ and $D_N$ ($\overline{D}_N$) is only applicable within the region $g < 1$, or $1 > g > \gamma_c$ with critical $\gamma_c = \sqrt{1 - g^{-2}}$. At $\gamma = 1$, the system is at exceptional point, which is beyond our investigation. Obviously, we still have

$$[D_N, H] = [\overline{D}_N, H] = 0,$$

and the canonical commutation relation

$$\{D_N, \overline{D}_N\} = 1, (D_N)^2 = (\overline{D}_N)^2 = 0,$$

which guarantee the existence of degeneracy of the eigenstates. Applying the operators on the lowest energy eigenstates $|\psi^+_g\rangle$ and $|\psi^-_g\rangle$ in two invariant subspace, we have

$$D_N |\psi^+_g\rangle = |\psi^+_g\rangle, \overline{D}_N |\psi^-_g\rangle = |\psi^+_g\rangle,$$

$$\overline{D}_N |\psi^+_g\rangle = D_N |\psi^-_g\rangle = 0,$$

and then $|\psi^+_g\rangle$ and $|\psi^-_g\rangle$ are degenerate ground states. It indicates that the existence of such a degeneracy depends on the value of $\gamma$ through $g$. In Fig. 1 we demonstrate this feature by the plots of the lower energy levels of $H$ on finite size, as function of $\gamma$. We can see that as $\gamma$ increases, the energy gap closes at the pseudo-critical point, and states $|\psi^+_g\rangle$ and $|\psi^-_g\rangle$ turn to degenerate. Meanwhile, many pairs of excited states also become degenerate near the pseudo-critical point, as expected in Eq. (27).

Now we turn to investigate the performance of the ground states $|\psi^\pm_g\rangle$ as the imaginary field is disordered. Consider a model with random parameters

$$H_{\text{Ran}} = -\sum_{j=1}^{N-1} \sigma^x_j \sigma^z_{j+1} + g \sum_{j=1}^{N} (\sigma^z_j + i\gamma_j \sigma^y_j),$$

with the imaginary field being the form $\gamma_j = \gamma + \delta_j$, where $\delta_j$ is uniform random real numbers within the interval $(-R, R)$, taking the role of the disorder strength. We have the mapping operator of non-Hermitian version

$$D_N(R) = \Omega \sum_{j=1}^{N} \prod_{l<j} [(h^+_j + h^-_j)(-\tau^z_l) \tau^+_j + (h^+_j - h^-_j)(-\tau^z_l) \tau^-_j],$$

and its canonical conjugation

$$\overline{D}_N(R) = \Omega \sum_{j=1}^{N} \prod_{l<j} [(h^+_j + h^-_j)(-\tau^z_l) \tau^-_j + (h^+_j - h^-_j)(-\tau^z_l) \tau^+_j],$$

where $\Omega$ and $h^\pm_j$ can be obtained by diagonalizing the random matrix

$$M_{\text{Ran}} = -\frac{1}{2} \sum_{j=1}^{N-1} |2j\rangle \langle 2j + 1| + \frac{1}{2} g \sum_{j=1}^{N} \sqrt{1 - \gamma^2} \langle 2j - 1| 2j | + \text{H.c.}$$
states arising from nonzero chain. We focus on the deviations of the ground states for the original Hamiltonian in Eq. (30) on finite-size.

Random strength increases up to $R$ increases, which accords with the prediction for large $N$. The energy gaps become smaller as the random strength increases up to $R = 0.5$. Plots in (a2)-(c2) show that overlap $O^\pm (R)$ decreases as the random strength $R$ increases. We find that the degeneracy is robust against large disorder, and the ground state has no evident deviation from that with zero $R$, corresponding to uniform chain.

In this paper, we have studied the consequence of an imaginary transverse field on the topological feature of an one-dimensional quantum Ising model. The competition between Ising interaction and the real transverse field results in two different quantum phases with full real
spectrum, ordered and disordered. We have shown that when an imaginary field is added, the original disordered phase can be shifted to an ordered one. Although the non-Hermitian Ising model cannot support directly a Majorana fermion description, there still exists a symmetry-breaking mechanism under open boundary condition and the thermodynamic limit. It supports the topological degeneracy due to its robustness in the presence of random imaginary field. Our work, including the numerical result for small size system, reveals that disordered dissipation is constructive in establishing topological ground states, which potentially can be utilized for developing inherently robust artificial devices for topological quantum computation.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (under Grant No. 11874225).

[1] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, England, 1999).
[2] X. G. Wen, Topological orders in rigid states, Int. J. Mod. Phys. B 4, 239 (1990).
[3] X. G. Wen, Quantum Field Theory of Many-Body Systems: From the Origin of Sound and Light (Oxford University Press, Oxford, 2004).
[4] G. Zhang and Z. Song, Topological characterization of extended quantum Ising models, Phys. Rev. Lett. 115, 177204 (2015).
[5] G. Zhang, C. Li, and Z. Song, Majorana charges, winding numbers and Chern numbers in quantum Ising models, Sci. Rep. 7, 8176 (2017).
[6] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Phys. Usp. 44, 131 (2001).
[7] P. Pfentzi, The one-dimensional Ising model with a transverse field, Ann. Phys. (NY) 57, 79 (1970).
[8] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in polyacetylene, Phys. Rev. Lett. 42, 1698 (1979).
[9] J. Zak, Berry’s phase for energy bands in solids, Phys. Rev. Lett. 62, 2747 (1989).
[10] J. K. Asbóth, L. Oroszlány, and A. Pályi, A Short Course on Topological Insulators: Band Structure and Edge States in One and Two Dimensions, Lecture Notes in Physics (Springer International Publishing, Switzerland, 2016).
[11] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. 80, 1083 (2008).
[12] A. Stern, Non-Abelian states of matter, Nature (London) 464, 187 (2010).
[13] J. Alicea, New directions in the pursuit of Majorana fermions in solid state systems, Rep. Prog. Phys. 75, 076501 (2012).
[14] J. Dalibard, Y. Castin, and K. Mølmer, Wave-function approach to dissipative processes in quantum optics, Phys. Rev. Lett. 68, 580 (1992).
[15] R. Dum, P. Zoller, and H. Ritsch, Monte Carlo simulation of the atomic master equation for spontaneous emission, Phys. Rev. A 45, 4879 (1992).
[16] K. Mølmer, Y. Castin, and J. Dalibard, Monte Carlo wave-function method in quantum optics, J. Opt. Soc. Am. B 10, 524 (1993).
[17] H. M. Wiseman, Quantum trajectories and quantum measurement theory, Quantum Semiclass. Opt. 8, 205 (1996).
[18] M. B. Plenio and P. L. Knight, The quantum-jump approach to dissipative dynamics in quantum optics, Rev. Mod. Phys. 70, 101 (1998).
[19] T. E. Lee and C. K. Chan, Heralded magnetism in non-Hermitian atomic systems, Phys. Rev. X 4, 041001 (2014).
[20] C. M. Bender, D. C. Brody, and H. F. Jones, Complex extension of quantum mechanics, Phys. Rev. Lett. 89, 270401 (2002).
[21] C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having PT symmetry, Phys. Rev. Lett. 80, 5243 (1998).
[22] C. M. Bender, S. Boettcher, and P. N. Meisinger, PT-symmetric quantum mechanics, J. Math. Phys. 40, 2201 (1999).
[23] K. L. Zhang, H. C. Wu, L. Jin, and Z. Song, Topological phase transition independent of system non-Hermiticity, Phys. Rev. B 100, 045141 (2019).
[24] P. Dorey, C. Dunning, and R. Tateo, Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics, J. Phys. A: Math. Gen. 34, L391 (2001); P. Dorey, C. Dunning, and R. Tateo, Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics, J. Phys. A: Math. Gen. 34, 5679 (2001).
[25] A. Mostafazadeh, Pseudo-Hermiticity versus PT-symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries, J. Math. Phys. 43, 3944 (2002).
[26] A. Mostafazadeh and A. Batal, Physical aspects of pseudo-Hermitian and PT-symmetric quantum mechanics, J. Phys. A: Math. Gen. 37, 11645 (2004).
[27] A. Mostafazadeh, Exact PT-symmetry is equivalent to Hermiticity, J. Phys. A: Math. Gen. 36, 7081 (2003).
[28] H. F. Jones, On pseudo-Hermitian Hamiltonians and their Hermitian counterparts, J. Phys. A: Math. Gen. 38, 1741 (2005).
[29] A. Mostafazadeh, Pseudo-Hermiticity versus PT-symmetry. II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum, J. Math. Phys. 43, 2814 (2002).
[30] B. B. Wei, S. W. Chen, H. C. Po, and R. B. Liu, Phase transitions in the complex plane of physical parameters, Sci. Rep. 4, 5202 (2014).
[31] X. H. Peng, H. Zhou, B. B. Wei, J. Y. Cui, J. F. Du, and R. B. Liu, Experimental observation of Lee-Yang zeros, Phys. Rev. Lett. 114, 010601 (2015).
[32] B. B. Wei, Z. F. Jiang, and R. B. Liu, Thermodynamic holography, Sci. Rep. 5, 15077 (2015).
[33] N. Ananikian and R. Kenna, Imaginary magnetic fields in the real world, Physics. 8, 2 (2015).

[34] A. García-Saez and T. C. Wei, Density of Yang-Lee zeros in the thermodynamic limit from tensor network methods, Phys. Rev. B. 92, 125132 (2015).

[35] Q. M. Chen, R. B. Wu, T. M. Zhang, and H. Rabitz, Near-time-optimal control for quantum systems, Phys. Rev. A 92, 063415 (2015).

[36] M. Krasnytska, B. Berche, Yu. Holovatch, and R. Kenna, Violation of Lee-Yang circle theorem for Ising phase transitions on complex networks, EPL 111, 60009 (2015).

[37] P. Jordan and E. Wigner, über das paulische äquivalenzverbot Z. Physik 47, 631 (1928).

[38] Actually, one can develop a similar formation of Majorana fermion for the non-Hermitian spin operators \{\tau^\alpha\}. Here the Majorana fermion is no longer self-conjugate particle [39].

[39] C. Li, X. Z. Zhang, G. Zhang, and Z. Song, Topological phases in a Kitaev chain with imbalanced pairing, Phys. Rev. B 97, 115436 (2018).