A NEW TECHNIQUE TO SOLVE LINEAR
INTEGRO-DIFFERENTIAL EQUATIONS (IDES) WITH
MODIFIED BERNOULLI POLYNOMIALS

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ABSTRACT. In this work, a new technique has been presented to find approximate
solution of linear integro-differential equations. The method is based on modified
orthonormal Bernoulli polynomials and an operational matrix thereof. The
method converts a given integro-differential equation into a set of algebraic equa-
tions with unknown coefficients, which is easily obtained with help of the known
functions appearing in the equation, modified Bernoulli polynomials and opera-
tional matrix. Approximate solution is obtained in form of a polynomial of re-
quired degree. The method is also applied to three well known integro-differential
equations to demonstrated the accuracy and efficacy of the method. Numerical
results of approximate solution are plotted to compared with available exact so-
lutions. Considerably small error of approximation is observed through numerical
comparison, which is further reducible to a required level of significance. Method
is comparatively simpler and shorter than many existing methods.

Keywords: approximate solution, Bernoulli polynomials, integro-differential equa-
tions, orthonormal polynomials.

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1. INTRODUCTION

Mathematical modeling of real world problems often give rise to ordinary or par-
tial differential equations, integral equations, ordinary or partial integro-differential
equations and some other forms. Of these, integro-differential equations (IDES) ap-
pear in almost all areas of science and engineering. Mathematical formulations of
physical phenomena such as population problem, nono-hydrodynamics, fluid me-
chanics biological and chemical models, ecology models, financial problem, process
engineering, aerospace and design engineering, hydro-electric machines, reactor dy-
namics, and many more are examples where IDEs are frequently encountered.

Many of such IDEs are difficult to solve for analytic solution and require an
efficient approximation or numerical technique. Solution to IDEs in different fields
have been point of attracting attention of researchers from long past [1, 2] including
some notable contributions on mathematical modeling of the spread of infection [3],
problems in quantum mechanics [4], detailed consideration of integro-differential
equations theory [5], problems of hydrodynamics with incompressible viscous fluids

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and mathematical formulation in ecology [7], however, researchers have focused towards numerical techniques to solve such problems with the evolution of computers science.

Bulk of recent literature is available to explore approximate and numerical solutions of IDEs [8, 9, 10, 11]. Some latest contributions on numerical solution of IDEs in recent years are meshless method [12], Bernstein operational matrix approach [13], collocation approach [14], improved Legendre method [15], with Euler polynomials [16], operational matrices method [17], Taylor collocation approach [18], convolution integrals approach [19] and that involving special functions [20].

Numerical and approximation techniques for ODEs, PDEs and IDEs also involved different well known polynomials, such as applications of Bernoulli polynomials [21], Chebyshev polynomials approach [22], application of Legendre polynomials [23], Laguerre polynomials and Wavelet Galerkin method [24], Legendre wavelets [25], the operational matrix [26].

Amongst these, many authors also used Bernoulli polynomials in different ways to find numerical solution of many complex problems, such as numerical approximation for generalized pantograph equation using Bernoulli matrix method [27], numerical solution of second-order linear system of partial differential equations using Bernoulli polynomials [28], numerical solution of Volterra type integral equations by means of Bernoulli polynomials [29].

In this work, it is proposed to find polynomial approximation to solution of linear integro-differential equations (IDEs) by application of an operational matrix developed from a class of modified Bernoulli polynomials.

2. Bernoulli Polynomials

Jakob Bernoulli, in late seventeenth century, discussed some special polynomials in his book "ArsConjectandi", which were explicitly studied by Leonhard Euler, who established the finite difference relation \( B_m(x + 1) - B_m(x) = mx^{m-1}, m \geq 1 \) for these polynomials in his book Foundations of differential calculus in 1755, and also suggested the method of generating function to derive these polynomials [30]. Later on, J. L. Raabe in 1851 discussed these polynomials together with Bernoulli numbers in connection with formula \( \sum_{n=0}^{m-1} B_n \left( x + \frac{k}{m} \right) = m^{-(n+1)}B_n(mx) \) and termed the polynomials \( B_n(x) \) as Bernoulli Polynomial. After Raabe, many researchers paid their attention towards properties of these polynomials. The most common formula for Bernoulli polynomials in recent mathematical applications is:

\[
B_n(\zeta) = \sum_{j=0}^{n} \binom{n}{j} B_j(0) \zeta^{n-j}, \quad n = 0, 1, 2, \ldots \quad 0 \leq \zeta \leq 1
\]

(1)

which was presented by Costabile and DellAccio [30]. The numbers \( B_j(0) \) are the Bernoulli numbers, which can also be calculated with Kronecker's formula [31]:

\[
B_n(0) = -\sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n+1}{j} \sum_{k=0}^{j} k^n; \quad n \geq 0
\]

(2)
Bernoulli polynomials [30] can be used as an underlying characteristics together with the following key relations of various numerical approximation techniques. In the present work, this property will be used out of $Y$.

An interesting property of Bernoulli polynomials is that they form a complete basis over $[0,1]$ [32], which extends the applicability of these polynomials towards various numerical approximation techniques. In the present work, this property will be used as an underlying characteristics together with the following key relations of Bernoulli polynomials [30]:

$$B_n'(\zeta) = nB_{n-1}(\zeta), \quad n \geq 1$$

$$\int_0^1 B_n(z)dz = 0, \quad n \geq 1$$

(3)

Some other properties, generalization and applications of Bernoulli polynomials can be found in notable literature [33, 34, 35].

3. Modified Bernoulli Polynomials

It can be easily verified that the polynomials $B_n(x) \ (n \geq 1)$ given by equation (1) are orthogonal to $B_0(x)$ with respect to standard inner product on $L^2[0,1]$:

$$< f_1, f_2 > = \int_0^1 f_1(x)f_2(x)dx; \quad f_1, f_2 \in L^2[0,1]$$

(4)

Using this property, an orthonormal set of polynomials can be derived for any $B_n(x)$ with Gram-Schmidt orthogonalization. First few of such orthonormal polynomials are obtained as:

$$\phi_0(x) = 1$$

$$\phi_1(x) = \sqrt{3}(-1 + 2x)$$

$$\phi_2(x) = \sqrt{5} (1 - 6x + 6x^2)$$

$$\phi_3(x) = \sqrt{7}(-1 + 12x - 30x^2 + 20x^3)$$

$$\phi_4(x) = 3(1 - 20x + 90x^2 - 140x^3 + 70xa^4)$$

$$\phi_5(x) = \sqrt{11}(-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5)$$

$$\phi_6(x) = \sqrt{13}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)$$

$$\phi_7(x) = \sqrt{15}(-1 + 56x - 756x^2 + 4200x^3$$

$$- 11550x^4 + 16632x^5 - 12012x^6 + 3432x^7)$$

(5)

4. Approximation of Functions

Theorem 4.1. Let $H = L^2[0,1]$ be a Hilbert space and $Y = \text{span} \ \{y_0, y_1, y_2, ..., y_n\}$ be a subspace of $H$ such that $\dim(Y) < \infty$, every $f \in H$ has a unique best approximation out of $Y$ [32], that is, $\forall y(t) \in Y \exists \tilde{f}(t) \in Y$ s.t. $\| f(t) - \tilde{f}(t) \|_2 = \| f(t) - y(t) \|_2$. This implies that, $\forall y(t) \in Y, < f(t) - f(t), y(t) >= 0$, where $<,>$ is standard inner product on $L^2 \in [0,1]$ (c.f. Theorems 6.1-1 and 6.2-5, Chapter 6 [32]).
**Remark 4.1.** Let \( Y = \text{span}\{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\} \), where \( \phi_k \in L^2[0, 1] \) are orthonormal Bernoulli polynomials. Then, from Theorem 4.1, for any function \( f \in L^2[0, 1] \),

\[
f \approx \hat{f} = \sum_{k=0}^{n} c_k \phi_k,
\]

where \( c_k = \langle f, \phi_k \rangle \), and \( \langle , \rangle \) is the standard inner product on \( L^2 \in [0, 1] \).

For numerical approximation, series (5) can be written as

\[
f(x) \simeq \sum_{k=0}^{n} c_k \phi_k(x) = C^T \phi(x) \tag{7}
\]

where \( C = (c_0, c_1, c_2, \ldots, c_n) \), \( \phi(x) = (\phi_0, \phi_1, \phi_2, \ldots, \phi_n) \) are column vectors. The number of polynomials \( n \) can be chosen to meet required accuracy.

### 5. Construction of operational matrix

The orthonormal polynomials, as derived in section 3, can be integrated as follows:

\[
\int_0^x \phi_0(t)dt = \frac{1}{2} \phi_0(x) + \frac{1}{2\sqrt{3}} \phi_1(x) \tag{8}
\]

\[
\int_0^x \phi_i(t)dt = \frac{1}{2\sqrt{(2i-1)(2i+1)}} \phi_{i-1}(x) + \frac{1}{2\sqrt{(2i+1)(2i+3)}} \phi_{i+1}(x); \quad (i = 1, 2, \ldots, n) \tag{9}
\]

Relations (8-9) are combined to closed form as:

\[
\int_0^x \phi(\eta)d\eta = \Theta \phi(x) \tag{10}
\]

where \( x \in [0, 1] \) and \( \Theta \) is operational matrix of order \((n + 1)\) given as:

\[
\Theta = \frac{1}{2} \begin{bmatrix}
1 & \frac{1}{\sqrt{1.3}} & 0 & \cdots & 0 \\
-\frac{1}{\sqrt{1.3}} & 0 & \frac{1}{\sqrt{3.5}} & \cdots & 0 \\
0 & -\frac{1}{\sqrt{3.5}} & 0 & \ddots & \vdots \\
& \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\frac{1}{\sqrt{(2n-1)(2n+1)}} & 0
\end{bmatrix} \tag{11}
\]

### 6. Solution of Integro-differential Equation

The relation (10) together with the operational matrix (11) prove to be a handy tool to solve linear integro-differential equations of the form:

\[
\mathbb{L}y(x) + f(x) \int_0^x K(x, t) y^j(t) dt; \quad x \in [0, 1] \tag{12}
\]

with suitable initial conditions on \( y \) and its derivatives, where \( \mathbb{L} \) is linear differential operator of order \( k \), \( y^j \) denotes \( j^{th} \) derivative of \( y \) for \( j < k \), \( f(x) \) is some continuous function of \( x \) and \( K(x, t) \) is non-singular kernel of integration.
However, the method works for any integro-differential equation of type (12), it is fast and sophisticated for integro-differential equations of the form:

$$Ly(x) + \mu \int_0^x (x-t)^{m-1}y^j(t)dt = r(x)$$

(13)

where, $\mu$ is constant and $m$ is some finite positive integer.

In order to present basic steps of the method in simpler way, we will first consider a simple integro-differential equation of from (13). Solution to general form (12) will be presented subsequently with necessary modifications.

**Case 6.1. Linear integro-differential equations with constant coefficients**

Let us take the second order linear integro-differential equations of the form:

$$a_k \frac{d^k y}{dx^k} + a_{k-1} \frac{d^{k-1} y}{dx^{k-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y + b \int_0^x (x-t)^{m-1}y^j(t)dt = r(x)$$

(14)

where, $m$ is some finite positive integer and $y^j = \frac{d^j y}{dx^j}$ for $j \leq k$. It is assumed that $r(x)$ is continuous and equation (14) admits a unique solution on $[0, 1]$.

Let us take

$$\frac{d^k y}{dx^k} = C^T \phi(x)$$

(15)

so that equation (14) takes the form:

$$a_k C^T \phi(x) + a_{k-1} C^T \Theta \phi(x) + \cdots + a_2 C^T \Theta^{k-2} + a_1 C^T \Theta^{k-1} + a_0 C^T \Theta^k \phi(x) + b C^T \Theta^{k-j} \int_0^x (x-t)^{m-1} \phi(t)dt = p(x) + r(x) = \hat{r}(x)$$

(16)

where, $p(x)$ is a polynomial of degree $m + k$ arising due to initial conditions.

Now, noting that $\int_0^x (x-t)^{m-1} \phi(t)dt = \Theta^m \phi(x)$, and taking

$$\hat{r}(x) = R^T \phi(x)$$

(17)

for some real vector $R^T = (r_o, r_1, \ldots, r_n)$ of dimension $1 \times (n + 1)$, equation (16) simplifies to-

$$C^T \left( a_k \mathbb{I} + a_{k-1} \Theta + \cdots + a_1 \Theta^{k-1} + a_0 \Theta^k + b \Theta^{m+k-j} \right) \phi(x) = R^T \phi(x)$$

(18)

Equation (18) gives:

$$C^T = R^T \left( a_k \mathbb{I} + a_{k-1} \Theta + \cdots + a_1 \Theta^{k-1} + a_0 \Theta^k + b \Theta^{m+k-j} \right)^{-1}$$

(19)

Equation (19) with equation (15) give an approximate solution for Integro-differential equation (14) as a polynomial of degree $n$ as follows:

$$y(x) = y_0 + y_1 x + \frac{y_2}{2!} x^2 + \cdots + y_{k-1} \frac{y_k}{(k-1)!} x^{k-1} + C^T \Theta^{k-j} \phi(x)$$

(20)

where, $\mathbb{I}$ is identity matrix of dimension $n + 1$. 
Case 6.2. Linear integro-differential equations with variable coefficients

To extend the method discussed in case 6.1 for linear integro-differential equations of type (12) with variable coefficients, let us consider the following second order integro-differential equation.

\[
\frac{d^2 y}{dx^2} + a(x) \frac{dy}{dx} + a_0(x)y + f(x) \int_0^x K(x,t)y(t)dt = r(x) \tag{21}
\]

\[y(0) = \alpha, \quad \left( \frac{dy}{dx} \right)_{x=0} = \beta.\]

where, \(a_0, a_1, f, r\) and \(r\) are continuous functions of \(x\), \(K(x,t)\) is non-singular kernel and equation (21) admits a unique solution on \([0,1]\), and \(y^k(x) = \frac{d^k y}{dx^k}\) for \(k = 0, 1\) or 2.

Taking,

\[
\frac{d^2 y}{dx^2} = C^T \phi(x) \tag{22}
\]

equation (21) can be written as:

\[
C^T \phi(x) + a_1(x)C^T \Theta \phi(x) + a_0(x)C^T \Theta^2 \phi(x) + C^T \Theta^{2-k}f(x) \int_0^x K(x,t)\phi(t)dt = g(x) + r(x) = \tilde{r}(x) \tag{23}
\]

where,

\[
g(x) = a_1 \alpha + a_0(\alpha + \beta x) + \begin{cases} f(x) \int_0^x K(x,t)(\alpha + \beta t); & k = 0 \\ \beta f(x) \int_0^x K(x,t); & k = 1 \\ 0; & k = 2 \end{cases} \tag{24}
\]

The integral terms on left side of equation (23) can be approximated by modified Bernoulli polynomials as \(f(x) \int_0^x K(x,t)\phi(t)dt = \tilde{\Theta} \phi(x)\) for some matrix \(\tilde{\Theta}\) of order \((n+1)\). Now, writing \(\Theta^{2-k}\tilde{\Theta} = \Phi\) and taking \(\tilde{r}(x) = R^T \phi(x)\), equation (23) can be written as

\[
C^T \phi(x) + C^T \Theta [a_1(x)\phi(x)] + C^T \Theta^2 [a_0(x)\phi(x)] + C^T \Phi \phi(x) = R^T \phi(x) \tag{25}
\]

where, \(a_1(x)\phi(x)\) and \(a_0(x)\phi(x)\) are column vectors of type

\[
a_k(x)\phi(x) = (a_k(x)\phi_0(x), a_k(x)\phi_1(x), \ldots, a_k(x)\phi_n(x)) = (\psi_{k0}(x), \psi_{k1}(x), \ldots, \psi_{kn}(x)) = \psi_k(x); \quad k = 0, 1, \ldots, \text{say}! \tag{26}
\]

of which, each \(\psi_{kj}(x)\) can be approximated as a linear combination of orthonormal polynomials in the form \(\psi_{kj}(x) = \hat{A}_{kj}^T \phi(x)\) such that \(\hat{A}_{kj}^T\) are vectors of form \(1 \times (n+1)\) for \(k = 0, 1 \text{ and } j = 1, 2, \ldots, n\).

Therefore, \(a_k(x)\phi(x) = \psi_k(x) = \hat{A}_k \phi(x)\), where \(\hat{A}_k = (\hat{A}_{k0}^T, \hat{A}_{k1}^T, \ldots, \hat{A}_{kn}^T)\) are matrices of dimension \((n+1) \times (n+1)\) for \(k = 0, 1\). Finally, using these intermediate approximations, equation (25) can be written as:

\[
C^T \left( I + \Theta \hat{A}_1^T + \Theta^2 \hat{A}_0^T + \Phi \right) \phi(x) = R^T \phi(x) \tag{27}
\]

Equation (27), gives:

\[
C^T = R^T \left( I + \Theta \hat{A}_1^T + \Theta^2 \hat{A}_0^T + \Phi \right)^{-1} \tag{28}
\]
where, $I$ is identity matrix of order $n + 1$. Now, using equation (28) in equation (22), the expression for $y(x)$ is obtained as:

$$y(x) = \alpha + \beta x + R^T \left( I + \Theta \hat{A}_1^T + \Theta^2 \hat{A}_0^T + \Phi \right)^{-1} \Theta^2 \phi(x)$$  \hspace{1cm} (29)

7. Examples

In order to establish the accuracy and efficacy of the method discussed so far, examples have been taken from earlier proved results.

Example 7.1. Let us consider the integro-differential equation of fourth order [11]:

$$\frac{d^4y}{dx^4} - y + \int_0^x y(t)dt = x + (x + 3)e^x$$  \hspace{1cm} (30)

$$y(0) = 1, \ y'(0) = 1, y''(0) = 2, y'''(0) = 3; \ x \in [0, 1]$$

which has exact solution $y(x) = 1 + xe^x$.

Let us assume that the solution $y(x)$ can be approximated by modified Bernoulli polynomials of degree 0 through 7 such that

$$\frac{d^4y}{dx^4} = C^T \phi(x)$$  \hspace{1cm} (31)

where, $\phi(x) = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)(x)$ and $C = (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is unknown vector to be determined. Then, equation (30) can be written as:

$$C^T \phi(x) - C^T \Theta^4 \phi(x) + C^T \Theta^5 \phi(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{8} + (x + 3)e^x$$  \hspace{1cm} (32)

Approximating right side of equation (32) as $R^T \phi(x)$, we get

$$C^T \phi(x) - C^T \Theta^4 \phi(x) + C^T \Theta^5 \phi(x) = R^T \phi(x)$$  \hspace{1cm} (33)

where,

$$R^T = (7.83814, 2.6674, 0.386136, 0.0327736, 0.00188342, 0.000138055, 5.83649 \times 10^{-6}, 1.49271 \times 10^{-7}).$$  \hspace{1cm} (34)

The unknown vector $C$ and approximate solution $y(x)$ are obtained as:

$$C^T = R^T(I - \Theta^4 + \Theta^5)^{-1}$$  \hspace{1cm} (35)

$$=(7.87309, 2.7079, 0.408755, 0.039614, 0.00280952, 0.000154652, 6.52936 \times 10^{-6}, 1.74013 \times 10^{-7})$$

$$y(x) \approx 1 + x + 0.999967x^2 + 0.500239x^3 + 0.16579x^4 + 0.0434333x^5 + 0.00637214x^6 + 0.00247836x^7$$  \hspace{1cm} (36)

A comparison of approximation (36) with exact solution of problem (7.1) has been shown in figure 1. Maximum magnitude of the error between the two solutions is of order $10^{-8}$ for $n = 7$. 

Example 7.2. Let us consider the integro-differential equation (ref)

\[(1 + x^2) \frac{d^2 y}{dx^2} + y + \cos x \int_0^x (x - t)^2 \frac{dy}{dt} dt = r(x)\]

\[y(0) = 0, \quad \left(\frac{dy}{dx}\right)_{x=0} = 1; \quad x \in [0, 1]\]  \hspace{1cm} (37)

For \(r(x) = 2(x - \sin x) \cos x - x^2 \sin x\), the problem (37) admits the exact solution \(y(x) = \sin x\).

As discussed in section (6.2), substituting \(\frac{d^2 y}{dx^2} = C^T \phi(x)\),

\[C^T \phi(x) + C^T \Theta^2 \phi(x) + \cos x C^T \Theta^4 \phi(x) = r(x) - x - \frac{1}{3} x^3 \cos x\]  \hspace{1cm} (39)

Taking the approximations \((1 + x^2)\phi(x) = A\phi(x), \cos x \phi(x) = B\phi(x), r(x) - x - \frac{1}{3} x^3 \cos x = R^T \phi(x)\), equation (39) can be simplified as:

\[C^T (A + \Theta^2 + B \Theta^4) \phi(x) = R^T \phi(x)\]  \hspace{1cm} (40)

where, \(A\) and \(B\) are matrices of order \(n\), and \(R\) is a column vector. For illustration, matrix \(A\) can be calculated for \(n = 5\) as:

\[
A = \begin{bmatrix}
\frac{4}{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 & 0 \\
\frac{1}{\sqrt{3}} & \frac{7}{5} & \frac{1}{\sqrt{15}} & \frac{10}{3} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{15}} & \frac{29}{21} & \frac{3}{\sqrt{35}} & \frac{1}{7} \\
0 & \frac{1}{10} & \frac{3}{2\sqrt{35}} & 0 & \frac{2}{3\sqrt{7}} \\
-30 & -\frac{45\sqrt{3}}{2} & -\frac{573}{14\sqrt{5}} & -\frac{103}{3\sqrt{7}} & -\frac{19}{7} \\
\frac{117}{11} & \frac{43}{33} & 114 & \frac{1049}{18} & \frac{17}{3} \\
\end{bmatrix}
\]
Thus, vector $C^T$ of unknown coefficients is obtained form equation (40) and, thereby, an approximation for $y(x)$ is obtained from equation (38) as:

$$y(x)_{n=3} \approx 0.99999998x - 0.16667268x^3$$

$$y(x)_{n=5} \approx 0.99999998x - 0.16667265x^3 + 0.00003539x^4 + 0.013110407x^5$$

$$y(x)_{n=7} \approx 0.99999998x - 0.16667265x^3 + 0.00003539x^4 + 0.0083710487x^5 + 0.0023319x^6 - 0.011538172x^7$$

Present approximations to solution of example (7.2) for $n = 3, 5, 7$ have been compared with the exact solution in figure 2.

![Figure 2](image)

**Figure 2.** (a) Comparison of exact and present solution for example 7.2 for $n = 3, 5, 7$. (b) Absolute error between exact and present solutions of 7.2 for $n = 3, 5, 7$.

8. **Application to Ecology**

In this section, we will apply this method to find solution of population problem (for females) [7]

$$B'(t) = g(t) + \int_0^t K(t, \eta)B(\eta)d\eta$$

where, $B'(t) = \frac{dB}{dt}$,

$K(t, \eta) = k(t - \eta)$ : net maternity function of females class age $\eta$ at time $t$.

$g(t)$ : contribution of birth due to female already present at time $t$.

$B(t)$ : the number of female births.

Let the number of female births be given by $g(t) = \frac{1}{2} \left(6(1 + t) - 7e^{\frac{t}{2}} - 4\sin t\right)$,

$K(t, \eta) = t - \eta$ and $t \in [0, 1]$. Then, the model (44) takes the form:

$$B'(t) - \int_0^t (t - \eta)B(\eta)d\eta = e^t - \sin t; \ B(0) = 1.$$

This model admits the exact solution $B(t) = \frac{1}{2} \left(e^{\frac{t}{2}} - \sin t + \cos t\right)$. 
As discussed in section 6 and earlier examples, taking $B'(t) = C^T \phi(t)$, and 
\[ \frac{1}{2} t^2 + \frac{1}{4} \left(6(1 + t) - 7e^{2t} - 4\sin t\right) = R^T \phi(t) \] for $n = 5$, an approximate solution to equation (45) is obtained as:

\[ B(t) \approx 1 - 0.25002t - 0.18730t^2 + 0.09293t^3 + 0.02375t^4 - 0.00561t^5 \] (46)

The approximation (46) is compared with exact solution of population problem (45) in figure 2 for $n = 5, 7$. The maximum magnitude of approximation errors for $n = 5, 7$ are of order $10^{-5}$ and $10^{-7}$ respectively.

9. Conclusion

In this work, a new method has been applied to find approximate solution of linear integro-differential equations with help of Bernoulli polynomials. A set of $n$ orthonormal polynomials derived form Bernoulli polynomials of degree $n$ on $[0, 1]$ has been used to form an operational matrix of integration. These new family of polynomials together with the operational matrix were applied to convert derivatives and integrals of dependent variable into an approximating polynomial form, thereby, converting an integro-differential equation into a set of algebraic equations with unknown coefficients, which are easily obtained with the help of operational matrix. Finally, an approximate solution is obtained in form of a polynomial of degree $n$. Three integro-differential equations have been solved which includes one problem with constant coefficient, one with variable coefficient of general form and a population problem from earlier established literature. The problems have been solved for different values of $n$, numerical results have been compared with available exact solutions, and error of approximation have been plotted. It was concluded that most of the problems can be approximated by using only first few modified (orthonormal) polynomials with very small error. Some outcomes of this method can be summarized as follows.

- error is small and can be reduced by taking higher degree approximation.
• the method is very fast for integro-differential equations with constant coefficients.
• solution is obtained in form of a polynomial, which can be easily carried forward for various further applications.
• error can be minimized up to required accuracy because error decreases quickly with increase of $n$ – the degree of Bernoulli polynomials.
• method can be programmed for various numerical applications.

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