On linear convergence of some decentralized algorithms

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Abstract—Decentralized algorithms solve multi-agent problems over a connected network, where the information can only be exchanged with accessible neighbors. Though there exist several decentralized optimization algorithms, there are still gaps in convergence conditions and rates between decentralized algorithms and centralized ones. In this paper, we fill some gaps by considering two decentralized consensus algorithms: EXTRA and NIDS. Both algorithms converge linearly with strongly convex functions. We will answer two questions regarding both algorithms, what are the optimal upper bounds for their stepsizes? Do decentralized algorithms require more properties on the functions for linear convergence than centralized ones? More specifically, we relax the required conditions for linear convergence for both algorithms. For EXTRA, we show that the stepsize is in order of $O(\frac{1}{k})$ ($L$ is the Lipschitz constant of the gradient of the functions), which is comparable to that of centralized algorithms, though the upper bound is still smaller than that of centralized ones. For NIDS, we show that the upper bound of the stepsize is the same as that of centralized ones, and it does not depend on the network. In addition, we relax the requirement for the functions and the mixing matrix, which reflects the topology of the network. As far as we know, we provide the linear convergence results for both algorithms under the weakest conditions.

Index Terms—decentralized optimization, EXTRA, NIDS, mixing matrix, linear convergence

I. INTRODUCTION

This paper considers the optimization problem

$$\min_{x \in \mathbb{R}^p} \bar{f}(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

over a $n$-agent network. Each function $f_i : \mathbb{R}^p \to \mathbb{R}$ is known only by the corresponding agent $i$ and assumed to be convex and differentiable. These agents form a connected network to solve the problem (1)

cooperatively without knowing other agents’ functions. The whole system is decentralized such that each agent has an estimation of the global variable $x$ and can only exchange the estimation with their accessible neighbors at every iteration. We introduce

$$f(x) := \sum_{i=1}^{n} f_i(x_i),$$

where each $x_i \in \mathbb{R}^p$ is a local estimation of the global variable $x$ and its $k$th iterated value is $x_i^k$. There is a symmetric mixing matrix $W \in \mathbb{R}^{n \times n}$ encoding the communication between the agents. The minimum condition for $W$ is that it has one eigenvalue 1 and all other eigenvalues are smaller than 1. In addition, the all-one vector 1 is an eigenvector of $W$ corresponding to the eigenvalue 1 (this is satisfied when the sum of each row is 1).

Early decentralized methods based on decentralized gradient descent [1]–[5] have sublinear convergence for strongly convex objective functions, because of the diminishing stepsize that is needed to obtain a consensual and optimal solution. This sublinear convergence rate is much slower than that for centralized ones. The first decentralized algorithm with linear convergence [6] is based on Alternate Direction Multiplier Method (ADMM) [7], [8]. Note that this type of algorithms have $O(1/k)$ rate for general convex functions [9]–[11]. After that, many linearly convergent algorithms are proposed. Some examples are EXTRA [12], NIDS [13], DiGing [14], [15], ESOM [16], gradient tracking methods [14], [15], [17]–[21], exact diffusion [22], [23], dual optimal [24], [25]. There are also works on composite functions, where each private function is the sum of a smooth and a nonsmooth functions [13], [26]–[28]. Another topic of interest is decentralized optimization over directed and dynamic graphs [14], [29]–[34]. Interested reader can refer to [35] and the references therein for more algorithms.

This paper focuses on two linear convergent algorithms: EXTRA and NIDS, and provides better
three types of conditions used in the literature: EXAct first-order Algorithm (EXTRA) was proposed in [12], and its iteration is described in (5). There are conditions on the stepsize \( \alpha \) for its convergence. For the general convex case, where each \( f_i \) is convex and \( L \)-smooth (i.e., has a \( L \)-Lipschitz continuous gradient), the condition in [12] is \( \alpha \in (0, \frac{1 + \lambda_{\min}(W)}{L}) \). Therefore, there is an implicit condition for \( W \) that the smallest eigenvalue of \( W \) is larger than \(-1\). Later the condition is relaxed to \( \alpha \in (0, \frac{5 + 3\lambda_{\min}(W)}{4L}) \) in [36], and the corresponding requirement for \( W \) is that the smallest eigenvalue of \( W \) is larger than \(-5/3\). In addition, this condition for the stepsize is shown to be optimal, i.e., EXTRA may diverge if the condition is not satisfied. Though we can always manipulate \( W \) to change the smallest eigenvalues, the convergence speed of EXTRA depends on the matrix \( W \). In the numerical experiment, we will see that it is beneficial to choose small eigenvalues for EXTRA in certain scenarios.

The linear convergence of EXTRA requires additional conditions on the functions. There are mainly three types of conditions used in the literature: the strong convexity of \( \bar{f} \) (and some weaker variants) [12], the strong convexity of each \( f_i \) (and some weaker variants) [36], and the strong convexity of one function \( f_i \) [23]. Note that the condition on \( \bar{f} \) is much weaker than the other two; there are cases where \( \bar{f} \) is strongly convex but none of \( f_i \)'s is. E.g., \( f_i = \|e_i^T x\|^2_p \) for \( p = n > 1 \), where \( e_i \) is the vector whose \( i \)th component is \( 1 \) and all other components are \( 0 \). If \( \bar{f} \) is (restricted) strongly convex with parameter \( \mu \bar{f} \), the linear convergence of EXTRA is shown when \( \alpha \in (0, \frac{\mu \bar{f}}{L + \lambda_{\min}(W)}) \) in [12]. The upper bound for the stepsize is very conservative, and the better performance with a larger stepsize was shown numerically in [12] without proof. If each \( f_i \) is strongly convex with parameter \( \mu_i \), the linear convergence is shown when \( \alpha \in (0, \frac{\mu_i + \lambda_{\min}(W)}{L + \mu_i}) \) and \( \alpha \in (0, \frac{5 + 3\lambda_{\min}(W)}{4L}) \) in [27] and [36], respectively. One contribution of this paper to show the linear convergence of EXTRA under the condition of \( \bar{f} \) and \( \alpha \in (0, \frac{5 + 3\lambda_{\min}(W)}{4L}) \).

The algorithm NIDS (Network InDepenent Step-size) was proposed in [13]. Though there is a small difference from EXTRA, NIDS can choose a stepsize that does not depend on the mixing matrices. The convergence of NIDS is shown when \( I \succ W \succ -I \). The result for linear convergence requires the strong convexity of \( f(x) \). Another contribution of this paper is the linear convergence of NIDS under the (restricted) strong convexity of \( \tilde{f}(x) \) and relaxed mixing matrices with \( \lambda_{\min}(W) > -5/3 \).

In sum, we provide new and stronger linear convergence results for both EXTRA and NIDS. More specifically,

- We show the linear convergence of EXTRA with the strong convexity of \( \bar{f} \) and the relaxed condition \( \lambda_{\min}(W) > -5/3 \). The upper bound of the stepsize can be as large as \( \frac{5 + 3\lambda_{\min}(W)}{4L} \), which is shown to be optimal in [36] for general convex problems;
- We show the linear convergence of NIDS with the same condition on \( \bar{f} \) and \( W \) as EXTRA. But, the large network-independent stepsize \( \alpha \in (0, 2/L) \) is kept.

### A. Notation

Since agent \( i \) has its own estimation \( x_i \) of the global variables \( x \), we put them together and define

\[
x = \begin{bmatrix}
    -x_1^T \\
    -x_2^T \\
    \vdots \\
    -x_n^T
\end{bmatrix} \in \mathbb{R}^{n \times p}. \tag{3}
\]

The gradient of \( f \) is defined as

\[
\nabla f(x) = \begin{bmatrix}
    -\nabla f_1(x_1)^T \\
    -\nabla f_2(x_2)^T \\
    \vdots \\
    -\nabla f_n(x_n)^T
\end{bmatrix} \in \mathbb{R}^{n \times p}. \tag{4}
\]

We say that \( x \) is consensual if \( x_1 = x_2 = \cdots = x_n \), i.e., \( x = 1x^\top \), where \( x \in \mathbb{R}^{p \times 1} \) and \( 1 = [1, 1, \cdots, 1]^\top \in \mathbb{R}^{n \times 1} \).

In this paper, we use \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) to denote the Frobenius norm and the corresponding inner product, respectively. For a given matrix \( M \in \mathbb{R}^{n \times p} \) and any positive (semi)definite matrix \( H \), which is denoted as \( H \succ 0 \) (\( H \succeq 0 \) for positive semidefinite), we define \( \| M \|_H := \sqrt{\text{tr}(M^\top HM)} \). The largest and the smallest eigenvalues of a matrix \( A \) are defined as \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \). For a symmetric positive semidefinite matrix \( A \), we let \( \lambda_{\min}(A) \) be the smallest nonzero eigenvalue. \( A^\dagger \) is the pseudo inverse of \( A \). For a matrix \( A \in \mathbb{R}^{n \times n} \), we say a matrix \( B \in \mathbb{R}^{n \times p} \) is in \( \text{Ker}(A) \) if \( AB = 0_{n \times p} \), and \( B \) is in \( \text{Range}(A) \) if there exists \( C \in \mathbb{R}^{n \times p} \) such that \( B = AC \). For simplicity, we may use \( x^\top \) and \( x \) to replace \( x^{k+1} \) and \( x^k \), respectively, in the proofs.
II. ALGORITHMS AND PREREQUISITES

One iteration of EXTRA can be expressed as
\[ x^{k+2} = (I + W)x^{k+1} - Wx^k - \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)) \quad (5) \]
The stepsize \( \alpha > 0 \), and the symmetric matrices \( W \) and \( W \) satisfy \( I + W \succeq 2W \succeq 2W \). The initial value \( x^0 \) is chosen arbitrarily, and \( x^1 = Wx^0 - \alpha \nabla f(x^0) \).

Remark 2: From [12, Proposition 2.2], \( \text{Null}\{I - W\} = \text{span}\{1\} \), which is a critical assumption for both algorithms.

Before showing the theoretical results of EXTRA and NIDS, we reformulate both algorithms.

Reformulation of EXTRA: We reformulate EXTRA by introducing a variable \( y \in \mathbb{R}^{n \times p} \) as
\[
\begin{align*}
    x^{k+1} &= \tilde{W}x^k + y^k - \alpha \nabla f(x^k), \quad (7a) \\
    y^{k+1} &= y^k - (\tilde{W} - W)x^{k+1}, \quad (7b)
\end{align*}
\]
with \( y^0 = -(\tilde{W} - W)x^0 \). Then (7) is equivalent to EXTRA (5).

Proposition 1: Let the \( x \)-sequence generated by (7) with \( y^0 = -(\tilde{W} - W)x^0 \) be \( \{x^k\}_{k=1}^{\infty} \), then it’s identical to the sequence generated by EXTRA (5) with the same initial point \( x^0 \).

Proof: From (7a), we have
\[
\begin{align*}
    x^1 &= \tilde{W}x^0 + y^0 - \alpha \nabla f(x^0) \\
    &= \tilde{W}x^0 - (\tilde{W} - W)x^0 - \alpha \nabla f(x^0) \\
    &= \tilde{W}x^0 - \alpha \nabla f(x^0).
\end{align*}
\]
For \( k \geq 0 \), we have
\[
\begin{align*}
    x^{k+2} &= \tilde{W}x^{k+1} + y^{k+1} - \alpha \nabla f(x^{k+1}) \\
    &= \tilde{W}x^{k+1} + y^k - \alpha \nabla f(x^{k+1}) \\
    &= (I + W)x^{k+1} - \tilde{W}x^k - \alpha[f(x^{k+1}) - f(x^k)],
\end{align*}
\]
where the second and the last equalities are from (7b) and (7a), respectively.

Remark 3: By (7b) and the assumption of \( y^0 \), each \( y^k \) is in Range\{\( W - \tilde{W} \)\}. In addition, \( x^{k+1} = (\tilde{W} - W)^{-1}(y^k - y^{k+1}) + z^{k+1} \) for some \( z^{k+1} \in \text{Ker}\{\tilde{W} - W\} \).

Reformulation of NIDS: We adopt the following reformulation of NIDS from [13]:
\[
\begin{align*}
    d^{k+1} &= d^k + \frac{1-W}{2\alpha}[x^k - \alpha \nabla f(x^k) - \alpha d^k], \quad (8a) \\
    x^{k+1} &= x^k - \alpha \nabla f(x^k) - \alpha d^{k+1}, \quad (8b)
\end{align*}
\]
with \( d^0 = 0 \). The equivalence is shown in [13].

To establish the linear convergence of EXTRA and NIDS, we need the following two assumptions.

Assumption 2 (Solution existence): There is a unique solution \( x^* \) for the consensus problem (1).

Assumption 3 (Lipschitz differentiability and (restricted) strong convexity): Each component \( f_i \) is a proper, closed and convex function with a Lipschitz continuous gradient:
\[
\|\nabla f_i(x) - \nabla f_i(\bar{x})\| \leq L\|x - \bar{x}\|, \quad \forall x, \bar{x} \in \mathbb{R}^p, \quad (9)
\]
where \( L > 0 \) is the Lipschitz constant. Furthermore, \( \hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \) is (restricted) strongly convex with respect to \( x^* \):
\[
\langle x - x^*, \nabla \hat{f}(x) - \nabla \hat{f}(x^*) \rangle \geq \mu_f \| x - x^* \|^2, \quad \forall x \in \mathbb{R}^p. \tag{10}
\]

**Proposition 2 ([12, Appendix A]):** The following two statements are equivalent:
1) \( \hat{f}(x) \) is (restricted) strongly convex with respect to \( x^* \);
2) For any \( \eta > 0 \), \( g(x) := f(x) + \frac{\eta}{2} \| x \|^2 - \hat{W} \) is \( \mu_g \) (restricted) strongly convex with respect to \( x^* = 1(x^*)^T \). Specifically, we can let
\[
\mu_g = \min \left\{ \frac{\mu_f}{2}, \frac{\mu_f^2 \lambda_{\min}^2 (I - W)}{\mu_f^2 + 16L^2} \right\}. \tag{11}
\]

This proposition gives
\[
\langle x - x^*, \nabla f(x) - \nabla f(x^*) \rangle + \eta \| x - x^* \|^2 - \hat{W} \geq \mu_g (x - x^*)^2 \tag{12}
\]
for any \( x \in \mathbb{R}^{n \times p} \). From [37, Theorem 2.1.5], the inequality (9) is equivalent to, for any \( x, \bar{x} \in \mathbb{R}^{n \times p} \),
\[
\langle x - \bar{x}, \nabla f(x) - \nabla f(\bar{x}) \rangle \geq L^{-1} \| \nabla f(x) - \nabla f(\bar{x}) \|^2. \tag{13}
\]

### III. New Linear Convergence Results for EXTRA and NIDS

Throughout this section, we assume that Assumptions 1-3 hold.

#### A. Linear Convergence of EXTRA

For simplicity, we introduce some notations. Because of part 4 of Assumption 1, given mixing matrices \( \bar{W} \) and \( \hat{W} \), there is a constant
\[
\theta \in \left( \frac{3}{4}, \min \left\{ \frac{1}{1 - \lambda_{\min}(\hat{W})}, 1 \right\} \right)
\]
such that
\[
\bar{W} := \theta \bar{W} + (1 - \theta)I > 0, \tag{14}
\]
\[
\hat{H} := \hat{W} + (\theta - \frac{1}{2})(I - \hat{W}) = \frac{1}{2} \hat{W} > 0, \tag{15}
\]
\[
\mathbf{M} := (\hat{W} - \hat{W})^T = 0, \tag{16}
\]
\[
\mathbf{G} := \hat{W} + I - 2\hat{W} > 0. \tag{17}
\]

Based on (13), we have
\[
\bar{W} = \hat{W} - (1 - \theta)(I - \hat{W}). \tag{18}
\]

Let \((x^*, y^*)\) be a fixed point of (7), it is straightforward to show that \( x^* \) satisfies
\[
(\bar{W} - \hat{W})x^* = 0. \tag{19}
\]

Part 3 of Assumption 1 shows that \( x^* \) is consensual, i.e., \( x^* = 1(x^*)^T \) for certain \( x^* \in \mathbb{R}^p \). The y-iteration in (7b) and the initialization of \( y^0 \) show \( y^k \in \text{Range}\{\hat{W} - \hat{W}\} = \text{Ker}\{1^T\} \).
Then we have
\[
1^T y^* = \alpha 1^T \nabla f(x^*) = 0. \tag{20}
\]
Thus, \( x^* \) is the optimal solution to the problem (1).

**Lemma 1 (Norm over range space [13, Lemma 3]):** For any symmetric positive (semi)definite matrix \( A \in \mathbb{R}^{n \times n} \) with rank \( r (r \leq n) \), let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \) be its \( r \) eigenvalues. Then \( \text{Range}(A) \) is a \( rp \)-dimensional subspace in \( \mathbb{R}^{n \times p} \) and has a norm defined by \( \| x \|^2_A := (x, A^T x) \), where \( A^T \) is the pseudo inverse of \( A \). In addition, \( \lambda_1^{-1} \| x \|^2 \leq \| x \|^2_A \leq \lambda_r^{-1} \| x \|^2 \) for all \( x \in \text{Range}(A) \).

For simplicity, we let \( x^+ \) and \( x \) stand for \( x^{k+1} \) and \( x^k \), respectively, in the proofs. The same implication applies to \( y^k \).

**Lemma 2 (Norm equality):** Let \( \{ (x^k, y^k) \}_{k=1}^{\infty} \) be the sequence generated by (7), then it satisfies
\[
\| x^{k+1} - x^* \|^2_{\hat{W} - \hat{W}} = \| y^k - y^{k+1} \|^2_M. \tag{21}
\]

**Proof:** From Remark 3, we have
\[
x^+ := M(y - y^+) + z^+ \tag{22}
\]
for \( z^+ \in \text{Ker}\{\hat{W} - \hat{W}\} \). This equality and (18) give
\[
\| x^+ - x^* \|^2_{\hat{W} - \hat{W}} = (x^+ - x^*, (\hat{W} - \hat{W})(x^+ - x^*)) \]
\[
= (x^+, \hat{W} - \hat{W})x^+ \]
\[
= (\mathbf{M}(y - y^+), y - y^+) \]
\[
= |y - y^+|^2_M. \tag{23}
\]

where the third equality holds because of (15), (20), and \( y - y^+ \in \text{Range}(\hat{W} - \hat{W}) \).

**Lemma 3 (A key inequality for EXTRA):** Let \( \{ (x^k, y^k) \}_{k=1}^{\infty} \) be the sequence generated by (7), then we have
\[
\| x^{k+1} - x^* \|^2_{\hat{H}} + \| y^{k+1} - y^* \|^2_{\mathbf{M}} \leq \| x^k - x^* \|^2_{\hat{H}} + \| y^k - y^* \|^2_{\mathbf{M}} - \| x^k - x^{k+1} \|^2_{\hat{W}} \]
\[
- \| x^k - x^* \|^2_{\hat{W}} |(\theta - \frac{1}{2}) (I - \hat{W})| - \| x^{k+1} - x^* \|^2_{\mathbf{G}} - 2\alpha(x^{k+1} - x^*, \nabla f(x^k) - \nabla f(x^*)) \tag{24}
\]

**Proof:** The iteration (7) and equation (17) show
\[
2\alpha(x^+ - x^*, \nabla f(x) - \nabla f(x^*)) \]
\[
= 2(x^+ - x^*, \hat{W}(x - x^*) + \hat{W}(x^+ - x^*)) \]
\[
= 2(x^+ - x^*, \hat{W}(x - x^*) + (\hat{W} - I)(x^+ - x^*)) \]
\[
= 2(x^+ - x^*, \hat{W}(x - x^*) + y^+ - y + y - y^*) \]
\[
= 2(x^+ - x^*, \hat{W}(x - x^*)) \]


\[ +2\langle x^+ - x^*, y^+ - y^* \rangle - 2\|x^+ - x^*\|^2_G = 2\langle x^+ - x^*, \nabla f(x) \rangle \]
\[ \leq 2\|x^+ - x^*, (1 - \theta)(I - \tilde{W})(x - x^*) \rangle + 2\langle M(y - y^*), y^+ - y^* \rangle - 2\|x^+ - x^*\|^2_G, \quad (22) \]

where the first equality comes from (17a), the second one follows (7b), and the last one is from (17).

From Remark 3, \( x^+ + x^* = M(y - y^*) + z^+ + x^* \) for some \( z^+ \in \text{Ker}\{\tilde{W} - W\} \). Thus
\[ \langle z^+ + x^*, y^+ - y^* \rangle = 0, \]
and the equality (22) can be rewritten as
\[ 2\alpha \langle x^+ - x^*, \nabla f(x) - \nabla f(x^*) \rangle = 2\langle x^+ - x^*, \nabla f(x) \rangle \]
\[ - 2\langle x^+ - x^*, (1 - \theta)(I - \tilde{W})(x - x^*) \rangle + 2\langle M(y - y^*), y^+ - y^* \rangle - 2\|x^+ - x^*\|^2_G. \]

Using the basic equality
\[ 2(a - b, b - c) = \|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2 \]
and Lemma 2, we have
\[ \|x^+ - x^*\|^2_W - \|x^+ - x^*\|^2_M \]
\[ + \|y^+ - y^*\|^2_M \]
\[ = \|x - x^*\|^2_W - \|x - x^*\|^2_M \]
\[ + \|y - y^*\|^2_M - \|x - x^+\|^2_W \]
\[ + \|x - x^+\|^2_M \]
\[ - 2\|x^+ - x^*\|^2_G \]
\[ - 2\alpha \langle x^+ - x^*, \nabla f(x) \rangle \]
\[ (23) \]
Note that the following inequality holds,
\[ \frac{1}{2}\|x^+ - x^*\|^2_W - \|x^+ - x^*\|^2_M \]
\[ + \frac{1}{2}\|x - x^+\|^2_W - \frac{1}{2}\|x - x^+\|^2_W. \]

Adding it onto both sides of (23), we have
\[ \|x^+ - x^*\|^2_H - \|x^+ - x^*\|^2_G \]
\[ \leq \|x - x^*\|^2_H - \frac{1}{2}\|x - x^*\|^2_G + \|y - y^*\|^2_M \]
\[ - \|x - x^+\|^2_W + \|x - x^+\|^2_M \]
\[ + \frac{1}{2}\|x - x^+\|^2_G - 2\|x^+ - x^*\|^2_G \]
\[ - 2\alpha \langle x^+ - x^*, \nabla f(x) \rangle. \]
\[ (24) \]

Apply the inequality
\[ \frac{1}{2}\|x - x^+\|^2_G \leq \frac{1}{2}\|x - x^*\|^2_G + \frac{1}{2}\|x - x^*\|^2_G, \]
then the key inequality (21) is obtained.

In the following theorem, we assume \( G \neq 0 \) (i.e., \( \tilde{W} \neq (I + W)/2 \)). It is easy to amend the proof to show the result for this special case.

**Theorem 1** (Q-linear convergence of EXTRA): Under Assumptions 1-3, we define
\[ r_1 = \frac{49 - \delta}{4(1 - \theta)^2 \lambda_{\max}(\tilde{W} - (I - W))} > 0, \quad (25) \]
\[ r_2 = \frac{1}{2\lambda_{\max}(GW^{-1})} > 0, \quad (26) \]
\[ r_3 = \frac{r_1 + 2r_2 + r_1^2}{r_1 + (1 - 2\lambda_{\max}(WM))(r_2/w)} \in (0, 1), \quad (27) \]
and choose two small parameters \( \xi \) and \( \eta \) such that
\[ \xi \in \left( 0, \min \left\{ \frac{r_1}{4\alpha_{\lambda_{\max}(W)}}, 1 \right\} \right), \quad (28) \]
\[ \eta \in \left( 0, \frac{\alpha_{\lambda_{\max}(W)}}{4\alpha_{\lambda_{\min}(W)} - 2\alpha^2 T} \right). \quad (29) \]
In addition, we define
\[ P := H + \frac{\xi}{2}(I - W) \geq 0, \]
\[ Q := M + (r_3 - 2\xi \lambda_{\max}(\tilde{W} M)) W^{-1} > 0. \]
Then for any stepsize \( \alpha \in (0, 2\alpha_{\lambda_{\min}(W)}) \), we have
\[ \|x^{k+1} - x^*\|^2_P + \|y^{k+1} - y^*\|^2_Q \]
\[ \leq \rho(\|x^k - x^*\|^2_P + \|y^k - y^*\|^2_Q), \quad (30) \]
where
\[ \rho := \max \left\{ 1 - \left( 2\alpha - \frac{\alpha^2 x}{\lambda_{\min}(W)} \right) \mu_G, \right. \]
\[ \left. \left( 4\alpha - \frac{2\alpha^2 x}{\lambda_{\min}(W)} \right) \frac{\xi}{2}, \right. \]
\[ \left. \right. \left. \frac{1 - r_3 - 4\lambda_{\max}(WM)\frac{\xi}{2}}{r_3 + (1 - 2\lambda_{\max}(WM)w)} \right). \quad (31) \]

**Proof:** From (21) in Lemma 3, we have
\[ \|x^+ - x^*\|^2_H + \|y^+ - y^*\|^2_M \]
\[ \leq \|x + x^*\|^2_H + \|y - y^*\|^2_M - \|x - x^+\|^2_W \]
\[ - \|x - x^+\|^2_M \]
\[ - \|x^+ - x^*\|^2_G \]
\[ - 2\alpha \langle x^+ - x^*, \nabla f(x) \rangle \]
\[ (32) \]
Then we find an upper bound of \(-\|x - x^+\|^2_W - 2\alpha \langle x^+ - x^*, \nabla f(x) \rangle \) as
\[ - \|x - x^+\|^2_W - 2\alpha \langle x^+ - x^*, \nabla f(x) \rangle \]
\[ = \alpha^2 \|\nabla f(x) - \nabla f(x^*)\|^2_W \]
\[ - 2\alpha \langle x - x^*, \nabla f(x) \rangle \]
\[ - \|\tilde{W}(x - x^*) - \alpha(\nabla f(x) - \nabla f(x^*))\|^2_W \]
\[ \leq \left( 2\alpha - \frac{\alpha^2 x}{\lambda_{\min}(W)} \right) \|x - x^*\|^2_W \]
\[ - \|\tilde{W}(x - x^*) - \alpha(\nabla f(x) - \nabla f(x^*))\|^2_W. \]
where, the inequality comes from (12). Combining it with (32), we have
\[
\|x^+ - x^-\|^2_H + \|y^+ - y^-\|^2_M 
- \|x - x^-\|^2_H - \|y - y^-\|^2_M 
\leq - (2\alpha - \frac{\alpha^2 L}{\lambda_{\min}(W)}) (x - x^-) \nabla f(x) - \nabla f(x^+))
- \|W(x - x^-) - \alpha (\nabla f(x) - \nabla f(x^+))\|^2_{W^{-1}} 
- \|x - x^-\|^2_{(\theta - \frac{1}{2})(I - W)} - \|x^+ - x^-\|^2_G. 
\tag{33}
\]
The inequality (33) shows that \(\{(x^k, y^k)\}_{k=1}^\infty\) is a Cauchy sequence converging to the fixed point \((x^*, y^*)\) of (7). From (11), we can bound the first term on the right hand side of (33) as
\[
(2\alpha - \frac{\alpha^2 L}{\lambda_{\min}(W)}) \|x - x^-\|_I - W
- (2\alpha - \frac{\alpha^2 L}{\lambda_{\min}(W)}) \mu \|x - x^-\|^2. 
\tag{34}
\]
Next, we bound the two terms involving successive iterated points, i.e., \(-\|W(x - x^-) - \alpha (\nabla f(x) - \nabla f(x^+))\|^2_{W^{-1}}\) and \(-\|x - x^-\|^2_{(\theta - \frac{1}{2})(I - W)}\). Note that
\[
W(x - x^-) - \alpha (\nabla f(x) - \nabla f(x^+)) = G(x^+ - x^-) - \alpha (y^+ - y^-)
+ (1 - \theta)(I - W)(x - x^-). 
\tag{35}
\]
We use \(T_1, T_2,\) and \(T_3\) to denote the three terms on the right hand side of (35), respectively. Using the definition of \(r_1\) in (25), we have
\[
- \|T_1 + T_2 + T_3\|^2_{W^{-1}} - \|x - x^-\|^2_{(\theta - \frac{1}{2})(I - W)} 
= - \|T_1 + T_2\|^2_{W^{-1}} - 2\|W^{-\frac{1}{2}}(T_1 + T_2)\|^2_{W^{-\frac{1}{2}}T_3} 
- \|T_3\|^2_{W^{-\frac{1}{2}}T_3} - \|W^{-\frac{1}{2}}(T_1 + T_2)\|^2_{W^{-\frac{1}{2}}T_3} 
\leq - \|T_1 + T_2\|^2_{W^{-1}} - 2\|W^{-\frac{1}{2}}(T_1 + T_2)\|^2_{W^{-\frac{1}{2}}T_3} 
- (1 + r_1)\|T_3\|^2_{W^{-1}} 
\leq - \frac{r_1}{1 + r_1}\|T_1 + T_2\|^2_{W^{-1}},
\]
where the last inequality comes from the Cauchy inequality
\[-2(a, b) \leq \frac{1}{1 + r_1}\|a\|^2 + (1 + r_1)\|b\|^2.\]
Combining it with the last term \(-\|x^+ - x^-\|^2_G\) on the right hand side of (33), we have
\[
- \frac{r_1}{1 + r_1}\|T_1 + T_2\|^2_{W^{-1}} - \|x^+ - x^-\|^2_G 
\leq - \frac{r_1}{1 + r_1}\|T_2\|^2_{W^{-1}} - \frac{r_1}{1 + r_1}\|T_1\|^2_{W^{-1}} - \frac{r_2}{1 + r_1}\|T_1\|^2_{W^{-1}} - \frac{1}{2}\|x^+ - x^-\|^2_G 
\leq - r_3\|y^+ - y^-\|^2_{W^{-1}} - \frac{1}{2}\|x^+ - x^-\|^2_G, \tag{36}
\]
where \(\xi < 1\) is a small positive parameter, and \(r_2\) and \(r_3\) are defined as (26) and (27), respectively.
Since \(G = (I - W) - 2(W - W^*)\), we have
\[
\|x^+ - x^-\|^2_G = \|x^+ - x^-\|^2_{I - W} - 2\|y - y^-\|^2_M. \tag{37}
\]
Therefore
\[
- \frac{r_1}{1 + r_1}|T_1 + T_2|W_{-1} - \|x^+ - x^-\|^2_G 
\leq - r_3\|y^+ - y^-\|^2_{W^{-1}} - \frac{1}{2}\|x^+ - x^-\|^2_{I - W} 
- \|y - y^-\|^2_M 
\leq - r_3\|y^+ - y^-\|^2_{W^{-1}} - \frac{1}{2}\|x^+ - x^-\|^2_{I - W} 
+ 2\xi\|y^+ - y^-\|^2_M + 2\xi\|y - y^-\|^2_M 
\leq - (r_3/\lambda_{\max}(W))\|y^+ - y^-\|^2_{W^{-1}} 
+ 2\xi\|y - y^-\|^2_M. \tag{38}
\]
Let \(\xi < r_3/(4\lambda_{\max}(W))\), then we have
\[
r_3/\lambda_{\max}(W) - 2\xi > 2\xi. 
\]
Putting (34) and (38) together onto (33), we have
\[
\|x^+ - x^-\|^2_H + \frac{\xi}{2}\|x^+ - x^-\|^2_{I - W} 
+ (1 + r_3/\lambda_{\max}(W))\|y^+ - y^-\|^2_M 
\leq (1 - (2\alpha - \frac{\alpha^2 L}{\lambda_{\min}(W)})\mu)\|x - x^-\|^2_M 
+ (2\alpha - \frac{\alpha^2 L}{\lambda_{\min}(W)})\eta\|x - x^-\|^2_I - W 
+ (1 + 2\xi)\|y - y^-\|^2_M. 
\]
Let \(\rho\) be defined as (31), we get (30). Note that the choice of \(\xi\) and \(\eta\) affects the definition of \(P\) and \(Q\), but not the algorithm. Hence for any \(\alpha \in (0, \frac{\alpha_{\min}(W)}{2})\), Q-linear convergence is guaranteed for \((x^k, y^k)\). Because
\[
\|x^k - x^-\|^2_P \leq \|x^k - x^-\|^2_P + \|y^k - y^-\|^2_Q. 
\]
The sequence \(\{\|x^k - x^-\|^2_P\}_{k=1}^\infty\) is R-linearly convergent to 0 at the rate of \(\mathcal{O}(\rho^k)\).}

Two special cases are not cover by the theorem: \(\theta = 1\) and \(W = \frac{1 + \lambda}{1 + \lambda}\). When \(\theta = 1\), we have \(r_1 = \infty\) and \(r_3 = \frac{r_2}{1 + r_1}\). When \(W = \frac{1 + \lambda}{1 + \lambda}\), i.e., \(G = 0\), we have \(r_2 = \infty\) and \(r_3 = \frac{r_1}{1 + r_1}\). In both cases, the linear convergence rate is
\[
\rho = \max\left\{1 - \left(2\alpha - \frac{\alpha^2 L}{2 - \theta + 2\lambda_{\min}(W)}\right)\mu, 
\frac{\alpha^2 L}{2 - \theta + 2\lambda_{\min}(W)} \mu, 
\frac{\alpha^2 L}{2 - \theta + 2\lambda_{\min}(W)} \mu \right\} \tag{39}
\]
where \(\beta = 1 - \lambda_{\lambda}(W)\) is the spectral gap. It is exactly the limit of \(\rho\) in (31) with \(r_1\) or \(r_2\) approaching infinity.
Remark 4: The upper bound for the stepsize $\alpha$, 

$$2\lambda_{\min}(W) B = 2\lambda_{\max}(W) L,$$

is much larger than that in [12] for ensuring linear convergence, $2\mu \lambda_{\min}(W)/L^2$, when $W$ is positive definite. In the special case $W = (I + W)/2$, we have $\alpha < (2 - \theta + \theta \lambda_{\min}(W))/L$. Since we can choose $\theta$ close as close as possible to $3/4$, the upper bound of $\alpha$ attains $(3\lambda_{\min}(W) + 5)/(4L)$, which coincides the optimal bound given in [36] for general convex functions. In [36], the linear convergence was shown under the strong convexity of all functions $\{f_i\}_{i=1}^n$.

B. NIDS without non-smooth term

We consider NIDS next. [13, Lemma 1] shows that, with the initialization $(d^0 = 0, x^0)$, the fixed point $(d^* \in \text{Range}(I - W), x^*)$ of (8) satisfies

$$d^* + \nabla f(x^*) = 0, \quad (I - W)x^* = 0,$$

and $x^*$ is the consensual solution to the problem (1). We will use the following important equality, which can be derived from (8)

$$(I - \frac{1-W}{2})(d^{k+1} - d^k) = \frac{W}{2\alpha}(x^{k+1} - x^k). \quad \text{(41)}$$

Motivated by the proof for EXTRA, we introduce another matrix to measure the distance to the fixed point. We still pick $\theta \in (\frac{3}{4}, 1]$ such that

$$\theta \left(\frac{1-W}{2}\right) + (1 - \theta)I = I - \theta \left(\frac{1-W}{2}\right) > 0. \quad \text{(42)}$$

Define a new symmetric matrix

$$\tilde{M} = 2(I - W)\dagger - \theta I = \left(\frac{1-W}{2}\right)^\dagger - \theta I. \quad \text{(43)}$$

Then $\tilde{M}$ is a norm over $\text{Range}(I - W)$. Note that $\tilde{M}$ is invertible because $\tilde{M}1 = -\theta 1$. In the following proofs, we use the same simplification $x$ and $x^*$.

Lemma 4 (Equiality): Let $\{(d^k, x^k)\}_{k=1}^\infty$ be the sequence generated by (8), we have the following two equalities:

$$\langle x^{k+1} - x^*, d^{k+1} - d^* \rangle = \alpha \langle d^{k+1} - d^k, d^{k+1} - d^* \rangle \tilde{M}((1-\theta)I)$$
$$\langle x^{k+1} - x^*, d^{k+1} - d^* \rangle = \alpha \|d^{k+1} - d^*\|^2_{\tilde{M}((1-\theta)I)}. \quad \text{(44a)}$$

Proof: Since $d^+ - d^* \in \text{Range}(I - W)$, we have

$$\langle x^+ - x^*, d^+ - d^* \rangle$$
$$= \langle (I - W)(x^+ - x^*), (I - W)^\dagger(d^+ - d^*) \rangle$$
$$= \alpha \langle (2I - (I - W))(d^+ - d), (I - W)^\dagger(d^* - d^*) \rangle$$
$$= \alpha \langle (2I - W)^\dagger - I)(d^+ - d), d^+ - d^* \rangle, \quad \text{(45)}$$

where the second equality follows (41). Replacing $d^*$ with $d$ in (45), we get (44b) in the same way. \(\blacksquare\)

Lemma 5 (Key inequality for NIDS): Let $\{(d^k, x^k)\}_{k=1}^\infty$ be the sequence generated by (8). We have, with any $r_4 \in (0, \theta - \frac{3}{4})$,

$$||x^{k+1} - x^*||^2 + \alpha^2\|d^{k+1} - d^*\|^2_{\tilde{M}((\theta - \frac{1}{4} - 2r_4)I}$$
$$\leq ||x^k - x^*||^2 + \alpha^2\|d^k - d*\|^2_{\tilde{M}((1-\theta)I)I}$$
$$+ \alpha^2\|\nabla f(x^k) - \nabla f(x^*)\|^2$$
$$+ 2\alpha\langle x^k + x^*, \nabla f(x^k) - \nabla f(x^*) \rangle. \quad \text{(46)}$$

Proof: The iteration (8) and the definition of $\tilde{M}$ in (43) show

$$2\alpha\langle x - x^*, \nabla f(x) - \nabla f(x^*) \rangle$$
$$= 2\langle x - x^*, x - x^* \rangle - 2\alpha\langle x - x^*, d^+ - d^* \rangle$$
$$= 2\langle x - x^*, x - x^* \rangle - 2\alpha\langle x - x^*, d^+ - d^* \rangle$$
$$- 2\alpha\langle x^+ - x^*, d^+ - d^* \rangle$$
$$= 2\langle x - x^*, x - x^* \rangle + \alpha\|d^+ - d^*\|^2_{\tilde{M}((1-\theta)I)I}$$
$$= 2\alpha\langle x - x^*, x^+ - x^* + \alpha\nabla f(x) - \alpha\nabla f(x^*) \rangle$$
$$+ \alpha\|d^+ - d^*\|^2_{\tilde{M}((1-\theta)I)I}, \quad \text{(47)}$$

where the first and last equality use (8b) and the third one follows (44a).

From (8b), we obtain

$$2\alpha\langle x - x^*, \nabla f(x) - \nabla f(x^*) \rangle$$
$$= ||x - x^*||^2 + \alpha^2\|\nabla f(x) - \nabla f(x^*)\|^2$$
$$- ||x - x^* - \alpha\nabla f(x) + \alpha\nabla f(x^*)\|^2$$
$$= ||x - x^*||^2 + \alpha^2\|\nabla f(x) - \nabla f(x^*)\|^2$$
$$- \alpha^2\|d^+ - d^*\|^2. \quad \text{(47)}$$

Together with the basic equality

$$2\langle a, b - c \rangle = ||a - c||^2 - ||b - c||^2 - ||a - b||^2,$$
we get
\[
\|x^+ - x^*\|^2 + \alpha^2\|d^+ - d^*\|^2_M - (1 - \theta)I
\]
\[
= \|x - x^*\|^2 + \alpha^2\|d - d^*\|^2_M - (1 - \theta)I
\]
\[
- \alpha^2\|d - d^*\|^2_M - (1 - \theta)I
\]
\[
+ \alpha^2\|\nabla f(x) - \nabla f(x^*)\|^2 - \alpha^2\|d^+ - d^*\|^2
\]
\[
- 2\alpha\langle x - x^*, \nabla f(x) - \nabla f(x^*)\rangle.
\] (48)

Since \( r_4 < \theta - \frac{3}{4} \leq 1/4 \), the following inequality holds,
\[
- \left(\frac{1}{4} - 2r_4\right)\|d^+ - d^*\|^2
\]
\[
\leq \frac{1}{2} - r_4)\|d - d^*\|^2 - \left(\frac{1}{4} - r_4\right)\|d - d^+\|^2.
\]

Adding it onto both sides of (48), we get (46). ■

Theorem 2 (Q-linear convergence for NIDS): Under Assumptions 1-3, we define
\[
r_5 = \max \left(0, 2\frac{\lambda_{\max}(I - W)}{\lambda_{\max}(I - W) - \frac{1}{4} - 2r_4}\left(\frac{2\lambda_{\max}(I - W)}{2 - \left(\frac{1}{4} - 2r_4\right)}\right)^2\right).
\] (49)

For any stepsize \( \alpha \in (0, \frac{2}{5}) \), we choose \( \eta \in (0, \frac{\alpha(2 - \alpha L)r_5}{\alpha(2 - \alpha L)r_5}) \) and define
\[
\rho_3 = \max \left\{1 - \alpha(2 - \alpha L)\mu_g, \alpha(2 - \alpha L)\eta r_5, \frac{1}{2\lambda_{\max}(I - W) - \frac{1}{4} + 2r_4}\right\} < 1.
\] (50)

Then we have
\[
\|x^{k+1} - x^*\|^2_{1 + \frac{1}{r_5} \lambda_{\max}(I - W)} + \alpha^2\|d^{k+1} - d^*\|^2_Q
\]
\[
\leq\rho(\|x^k - x^*\|^2_{1 + \frac{1}{r_5} \lambda_{\max}(I - W)} + \alpha^2\|d^k - d^*\|^2_Q),
\] (51)

where
\[
Q \doteq M + (\theta - \frac{1}{4} + 2r_4)I > 0.
\]

Proof: Given any \( \alpha \in (0, \frac{2}{5}) \), we have
\[
\alpha^2\|\nabla f(x) - \nabla f(x^*)\|^2 - 2\alpha\langle x - x^*, \nabla f(x) - \nabla f(x^*)\rangle
\]
\[
\leq \alpha(2 - \alpha L)\|x - x^*, \nabla f(x) - \nabla f(x^*)\|
\]
\[
= \alpha(2 - \alpha L)\|x - x^*, \nabla f(x) - \nabla f(x^*)\|
\]
\[
- \alpha(2 - \alpha L)\eta\|x - x^*\|^2_{1 - W}
\]
\[
+ \alpha(2 - \alpha L)\eta\|x - x^*\|^2_{1 - W}
\]
\[
\leq \alpha(2 - \alpha L)\mu_g\|x - x^*\|^2 + \alpha(2 - \alpha L)\eta\|x - x^*\|^2_{1 - W},
\]

where the first inequality is from (12) and the second one uses (restricted) strong convexity (11). Together with (46), we have
\[
\|x^+ - x^*\|^2 + \alpha^2\|d^+ - d^*\|^2_M - (\theta - \frac{1}{4} + 2r_4)I
\]
\[
\leq \|x - x^*\|^2 + \alpha^2\|d - d^*\|^2_M - (\theta - \frac{1}{4} + 2r_4)I
\]
\[
- \alpha^2\|d - d^*\|^2_M - (\theta - \frac{1}{4} + 2r_4)I
\]
\[
- \alpha(2 - \alpha L)\mu_g\|x - x^*\|^2 + \alpha(2 - \alpha L)\eta\|x - x^*\|^2_{1 - W},
\] (52)

The equality (41) gives
\[
\|x^+ - x^*\|_{1 - W}
\]
\[
= \|(I - W)(x^+ - x^*)\|_{(1 - W)^t}
\]
\[
= \alpha^2\|(2I - (I - W))\|d^+ - d\|_{1 - W})
\]
\[
= \alpha^2\|d - d^\|^2_{(2I - (1 - W))\|d^+ - d\|_{1 - W})
\]
\[
= \alpha^2\|d - d^+\|^2_{(1 - W)\|d^+ - d\|_{1 - W})
\]
\[
\leq \alpha^2\|d - d^+\|^2_{M + (\theta - \frac{1}{4} + 2r_4)I}.
\] (53)

where the second equality follows (41), the fourth equality comes from \( d - d^+ \in \text{Range}(I - W) \), and the inequality holds with the definition of \( r_5 \) in (49).

Combining (52) and (53), we derive
\[
\|x^+ - x^*\|^2 + \frac{1}{r_5}\|x - x^*\|^2_{1 - W}
\]
\[
+ \alpha^2\|d^+ - d^*\|^2_{M + (\theta - \frac{1}{4} + 2r_4)I}
\]
\[
\leq \|x - x^*\|^2 + \alpha^2\|d - d^*\|^2_{M + (\theta - \frac{1}{4} + 2r_4)I}.
\] (54)

Let \( \rho_3 \) be defined as (50), and we show (51). Meanwhile, the Q-linear convergence of \( (d^k, x^k) \) implies the R-linear convergence of \( x^k \).

This theorem shows that NIDS is still linearly convergent over a relaxed \( W \) and keeps the network-independent stepsize, which attains \( \frac{2}{5} \) practically.

IV. NUMERICAL EXPERIMENTS

In this section, we compare the performance of EXTRA and NIDS over relaxed mixing matrices in the following two scenarios:

- Comparison of Decentralized Gradient Descent (DGD), EXTRA, and NIDS with different stepsizes for doubly stochastic matrix \( W \).
- Comparison of EXTRA and NIDS with different stepsizes for relaxed matrix \( W \).

We consider the following decentralized sensing problem. Each agent \( i \in \{1, \cdots, n\} \) has its own
private measured data $M_i \in \mathbb{R}^{m_i \times p}$ and $y_i \in \mathbb{R}^{m_i}$ based on the unknown common variable $x \in \mathbb{R}^p$. Suppose that $y_i = M_i x + e_i$ with independently identically distributed random noise $e_i \in \mathbb{R}^{m_i}$. The goal is to estimate $x$ cooperatively over the network, and the problem is

$$\minimize_{x} \hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \|M_i x - y_i\|^2.$$ 

The data $\{M_i\}_{i=1}^{n}$ and $x$ are generated from Gaussian distribution. We normalize each $M_i$ such that $\|M_i^T M_i\| = 10$, i.e., $L = 10$. In both scenarios, we set $n = 10$, $p = 5$, $x^0 = 0$, and $\bar{W} = \frac{1+\bar{\mu}_i}{2}$ for EXTRA.

For the first scenario, we construct the matrix $W$ based on the Metropolis constant edge weight matrix in [12, §2.4]. In this case $W$ is positive definite, and we can set $\theta = \frac{1}{5}$. Then $\bar{W} = \frac{3+3\bar{W}}{5}$. We implement EXTRA with three different stepsizes: $\alpha_1 = \frac{(1+\lambda_{\min}(W))\mu_i}{100}$ (the stepsize for linear convergence in [12]), $\alpha_2 = \frac{1+\lambda_{\min}(W)}{10}$ (the stepsize for convergence only in [12]), and $\alpha_3 = \frac{5+3\lambda_{\min}(W)}{10}$ (our largest stepsize). For NIDS, the stepsize is set to $\alpha_4 = \frac{1}{3}$ although it is the upper bound of the stepsize which is not attainable in our proof theoretically.

The result with $m_i = 1$ is illustrated in Fig. 1. Because we have $n > p$, the function $\hat{f}(x)$ is strongly convex with probability one. NIDS requires the least number of iteration to attain the expected tolerance. Meanwhile, EXTRA with our proposed stepsize has better performance than that given in [12].

Then we set $m_i = 10$ in Fig. 2. In this case, individual functions $f_i(x)$ and $f(x)$ are strongly convex. NIDS and EXTRA with the largest stepsize lead the performance. Here two results of EXTRA are the same as that of NIDS although they are set with different stepsizes. The observation may indicate that there is an optimal choice of stepsize between $\alpha_2$ and $\alpha_3$ for both EXTRA and NIDS. By setting $\alpha_5 = \frac{5+3\lambda_{\min}(W)}{40}$ for EXTRA and $\alpha_6 = \frac{2}{10+\mu_i}$, we have the comparison of these algorithms in Fig. 3. This figure suggests that the optimal stepsize may depend on the problem/functions. How to find the optimal stepsize is an important research topic, and it is beyond the scope of this paper.

Next, we turn to the relaxed mixing matrices. Based on the previous created $W$, we replace it by $W_{\text{new}} = \frac{2W-1}{4}$ to scale the range of eigenvalues to $(-\frac{5}{2}, 1]$. In this case, some diagonal entries of $W_{\text{new}}$ may be negative. We consider the worst topology of network, line topology, i.e., each agent has at most two neighbors. In this experiment, we solve the same problem using EXTRA and NIDS on unrelaxed and relaxed mixing matrices, respectively, over the line. For NIDS, since the stepsize is network-independent, we relax the mixing matrix $W$ to $W_{\text{new}}$ more aggressively so that $\lambda_{\min}(W_{\text{new}})$ approaches $-\frac{5}{3}$ and compare the performance with the unrelaxed case of NIDS under $\alpha = \frac{1}{5}$. For EXTRA, we set the stepsize to $\alpha = \frac{5+3\lambda_{\min}(W)}{40}$, and compare the performance with the relaxed one under the stepsize $\alpha = \frac{5+3\lambda_{\min}(W_{\text{new}})}{40}$ where we only perturb $W$ mildly so that $\lambda_{\min}(W_{\text{new}})$ approaches $-1$. The result is shown in Fig. 4. From Fig. 4, if the topology of network is weak, switching to relaxed mixing matrix may offer better performance when using NIDS and EXTRA to solve the problem. The improvement for NIDS is more distinguished.
Fig. 2. TOP: the error $\|x_k - x^*\|_F$ vs iterations for DGD with different stepsizes, EXTRA with three stepsizes, and NIDS. BOTTOM: The random network with 10 nodes.

Fig. 3. The comparison of proved stepsizes for EXTRA and NIDS with the optimal choice.

Fig. 4. The figure of residuals $\|x_k - x^*\|_F$ with respect to iteration. The first graph is for strongly convex $\bar{f}(x)$ and the other is for strongly convex $f(x)$. re-EXTRA and re-NIDS stand for implementing EXTRA and NIDS over relaxed $W_{new}$.

V. Conclusion

In this paper, we relax the mixing matrices and prove the linear convergence of EXTRA and NIDS in the smooth case under the (restricted) strongly convexity assumption on $f$. A larger upper bound of the stepsize is derived for EXTRA compared with that given in [12] and [27]. NIDS can choose a network-independent stepsize and this stepsize can be chosen as the same as that of centralized ones. We relax the conditions for the mixing matrices and the functions, while keeping the same stepsize.

In numerical experiments on linear regression, applying the larger stepsize to EXTRA will get a faster convergence than using the $\mu\bar{f}$-dependent stepsize given in [12]. Over the unrelaxed mixing matrix, NIDS leads the performance in most cases and is easiest to implement. If the topology of network is weak, using relaxed mixing matrix can accelerate NIDS. For EXTRA, in general, we may not choose...
the mixing matrices to be relaxed due to the tiny improvement, but the larger stepsize derived in relaxed case is still competent to be considered.

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