Synthesising Interprocedural Bit-Precise Termination Proofs (extended version)

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Abstract—Proving program termination is key to guaranteeing absence of undesirable behaviour, such as hanging programs and even security vulnerabilities such as denial-of-service attacks. To make termination checks scale to large systems, interprocedural termination analysis seems essential, which is a largely unexplored area of research in termination analysis, where most effort has focussed on difficult single-procedure problems. We present a modular termination analysis for C programs using template-based interprocedural summarisation. Our analysis combines a context-sensitive, over-approximating forward analysis with the inference of under-approximating preconditions for termination. Bit-precise termination arguments are synthesised over lexicographic linear ranking function templates. Our experimental results show that our tool 2LS outperforms state-of-the-art alternatives, and demonstrate the clear advantage of interprocedural reasoning over monolithic analysis in terms of efficiency, while retaining comparable precision.

I. INTRODUCTION

Termination bugs can compromise safety-critical software systems by making them irrresponsible, e.g., termination bugs can be exploited in denial-of-service attacks [1]. Termination guarantees are therefore instrumental for software reliability. Termination provers, static analysis tools that aim to construct a termination proof for a given input program, have made tremendous progress. They enable automatic proofs for complex loops that may require linear lexicographic (e.g. [2], [3]) or non-linear termination arguments (e.g. [4]) in a completely automatic way. However, there remain major practical challenges in analysing real-world code.

First of all, as observed by [5], most approaches in the literature are specialised to linear arithmetic over unbounded mathematical integers. Although, unbounded arithmetic may reflect the intuitively-expected program behaviour, the program actually executes over bounded machine integers. The semantics of C allows unsigned integers to wrap around when they over/underflow. Hence, arithmetic on k-bit-wide unsigned integers must be performed modulo-2^k. According to the C standards, over/underflows of signed integers are undefined behaviour, but practically also wrap around on most architectures. Thus, accurate termination analysis requires a bit-precise analysis of program semantics. Tools must be configurable with architectural specifications such as the width of data types and endianness. The following examples illustrate that termination behaviour on machine integers can be completely different than on mathematical integers. For example, the following code:

\begin{verbatim}
void foo1(unsigned n) { for(unsigned x=0; x<n; x++); }
\end{verbatim}

does terminate with mathematical integers, but does not terminate with machine integers if \( n \) equals the largest unsigned integer. On the other hand, the following code:

\begin{verbatim}
void foo2(unsigned x) { while(x>=10) x++; }
\end{verbatim}

does not terminate with mathematical integers, but terminates with machine integers because unsigned machine integers wrap around.

A second challenge is to make termination analysis scale to larger programs. The yearly Software Verification Competition (SV-COMP) [6] includes a division in termination analysis, which reflects a representative picture of the state-of-the-art. The SV-COMP‘15 termination benchmarks contain challenging termination problems on smaller programs with at most 453 instructions (average 53), contained at most 7 functions (average 3), and 4 loops (average 1).

In this paper, we present a technique that we have successfully run on programs that are one magnitude larger, containing up to 5000 instructions. Larger instances require different algorithmic techniques to scale, e.g., modular interprocedural analysis rather than monolithic analysis. This poses several conceptual and practical challenges that do not arise in monolithic termination analysers. For example, when proving termination of a program, a possible approach is to try to prove that all procedures in the program terminate universally, i.e., in any possible calling context. However, this criterion is too optimistic, as termination of individual procedures often depends on the calling context, i.e., procedures terminate conditionally only in specific calling contexts.

Hence, an interprocedural analysis strategy is to verify universal program termination in a top-down manner by proving termination of each procedure relative to its calling contexts, and propagating upwards which calling contexts guarantee termination of the procedure. It is too difficult to determine these contexts precisely; analysers thus compute preconditions for termination. A sufficient precondition identifies those pre-states in which the procedure will definitely terminate, and is thus suitable for proving termination. By contrast, a necessary precondition identifies the pre-states in which the procedure may terminate. Its negation are those states in which the procedure will not terminate, which is useful for proving nontermination.

In this paper we focus on the computation of sufficient preconditions. Preconditions enable information reuse, and thus scalability, as it is frequently possible to avoid repeated analysis of parts of the code base, e.g. libraries whose procedures
are called multiple times or did not undergo modifications between successive analysis runs.

**Contributions:**

1) We propose an algorithm for **interprocedural termination analysis**. The approach is based on a template-based static analysis using SAT solving. It combines context-sensitive, summary-based interprocedural analysis with the inference of preconditions for termination based on template abstractions. We focus on non-recursive programs, which cover a large portion of software written, especially in domains such as embedded systems.

2) We provide an implementation of the approach in 2LS, a static analysis tool for C programs. Our instantiation of the algorithm uses template polyhedra and lexicographic, linear ranking functions templates. The analysis is bit-precise and purely relies on SAT-solving techniques.

3) We report the results of an experimental evaluation on 597 procedural SV-COMP benchmarks with in total 1.6 million lines of code that demonstrates the scalability and applicability of the approach to programs with thousands of lines of code.

II. PRELIMINARIES

In this section, we introduce basic notions of interprocedural and termination analysis.

**Program model and notation.** We assume that programs are given in terms of acyclic call graphs, where individual procedures $f$ are given in terms of symbolic input/output transition systems. Formally, the input/output transition system of a procedure $f$ is a triple $(\text{Init}_{f}, \text{Trans}_{f}, \text{Out}_{f})$, where $\text{Trans}_{f}(x, x')$ is the transition relation; the input relation $\text{Init}_{f}(x^\text{in}, x)$ defines the initial states of the transition system and relates it to the inputs $x^\text{in}$; the output relation $\text{Out}_{f}(x, x^\text{out})$ connects the transition system to the outputs $x^\text{out}$ of the procedure. Inputs are procedure parameters, global variables, and memory objects that are read by $f$. Outputs are return values, and potential side effects such as global variables and memory objects written by $f$. Internal states $x$ are commonly the values of variables at the loop heads in $f$.

These relations are given as first-order logic formulae resulting from the logical encoding of the program semantics.

**Basic concepts.** Moving on to interprocedural analysis, we introduce formal notation for the basic concepts below:

**Definition 1 (Invariants, Summaries, Calling Contexts). For a procedure given by $(\text{Init}, \text{Trans}, \text{Out})$ we define:**

- **An invariant is a predicate Inv such that:**
  \[
  \forall x^\text{in}, x, x' : \text{Init}(x^\text{in}, x) \implies \text{Inv}(x) \land \text{Inv}(x')
  \]

- **Given an invariant Inv, a summary is a predicate Sum such that:**
  \[
  \forall x^\text{in}, x, x' : \text{Init}(x^\text{in}, x) \land \text{Out}(x', x^\text{out}) \implies \text{Inv}(x') \land \text{Inv}(x) \land \text{Sum}(x^\text{in}, x^\text{out})
  \]

- **Given an invariant Inv, the calling context for a procedure call $h$ at call site $i$ in the given procedure is a predicate CallCtx$_h$ such that:**
  \[
  \forall x, x', x^\text{in}_h, x^\text{out}_h : \text{Inv}(x^\text{in}_h, x^\text{out}_h) \implies \text{CallCtx}_h(x^\text{in}_h, x^\text{out}_h)
  \]

These concepts have the following roles: Invariants abstract the behaviour of loops. Summaries abstract the behaviour of called procedures; they are used to strengthen the placeholder predicates. Contexts abstract the caller’s behaviour w.r.t. the procedure being called. When analysing the callee, the calling contexts are used to constrain its inputs and outputs.

In Sec. III we will illustrate these notions on the program in Fig. 1.
Definition 2 (Ranking function). A ranking function for a procedure \((\text{Init}, \text{Trans}, \text{Out})\) with invariant \(\text{Inv}\) is a function \(r\) from the set of program states to a well-founded domain such that \(\forall x, x' : \text{Inv}(x) \land \text{Trans}(x, x') \implies r(x) > r(x')\).

We denote by \(RR(x, x')\) a set of constraints that guarantee that \(r\) is a ranking function. The existence of a ranking function for a procedure guarantees its universal termination.

The weakest termination precondition for a procedure describes the inputs for which it terminates. If it is true, the procedure terminates universally; if it is false, then it does not terminate for any input. Since the weakest precondition is intractable to compute or even uncomputable, we under-approximate the precondition. A sufficient precondition for termination guarantees that the program terminates for all \(x^{\text{in}}\) that satisfy it.

Definition 3 (Precondition for termination). Given a procedure \((\text{Init}, \text{Trans}, \text{Out})\), a sufficient precondition for termination is a predicate \(\text{Precond}\) such that
\[
\exists RR, \text{Inv} : \forall x^{\text{in}}, x, x' : \\
\text{Precond}(x^{\text{in}}) \land \text{Init}(x^{\text{in}}, x) \implies \text{Inv}(x) \\
\land \text{Inv}(x) \land \text{Trans}(x, x') \implies \text{Inv}(x') \land RR(x, x')
\]

Note that \(false\) is always a trivial model for \(\text{Precond}\), but not a very useful one.

III. OVERVIEW OF THE APPROACH

In this section, we introduce the architecture of our interprocedural termination analysis. Our analysis combines, in a non-trivial synergistic way, the inference of invariants, summaries, calling contexts, termination arguments, and preconditions, which have a concise characterisation in second-order logic (see Definitions 1 and 3). At the lowest level our approach relies on a solver backend for second-order problems, which is described in Sec. [V].

To see how the different analysis components fit together, we now go through the pseudo-code of our termination analyser (Algorithm 1). Function \(\text{analyze}\) is given the entry procedure call \(f\) an under-approximating calling context \(\text{CallCtx}\) (using under-approximate summaries, as described in Sec. [V]), and recurses only if necessary (Line 12). Finally, we compute the under-approximating precondition for termination (Line 13). This precondition is inferred w.r.t. the termination conditions that have been collected: the backward calling context (Line 9), the preconditions for termination of the callees (Line 14), and the termination arguments for \(f\) itself (see Sec. [V]). Note that superscripts \(o\) and \(u\) in predicate symbols indicate over- and underapproximation, respectively.

Challenges. Our algorithm uses over- and underapproximation in a novel, systematic way. In particular, we address the challenging problem of finding meaningful preconditions:

- The precondition Definition 3 admits the trivial solution \(false\) for \(\text{Precond}\). How do we find a good candidate? To this end, we “bootstrap” the process with a candidate precondition: a single value of \(x^{\text{in}}\), for which we compute a termination argument. The key observation is that the resulting termination argument is typically more general, i.e., it shows termination for many further entry states. The more general precondition is then computed by precondition inference w.r.t. the termination argument.

- A second challenge is to compute under-approximations. Obviously, the predicates in the definitions in Sec. [II] can be over-approximated by using abstract domains such as intervals. However, there are only few methods for under-approximating analysis. In this work, we use a method similar to [2] to obtain under-approximating preconditions w.r.t. property \(p\): we infer an over-approximating precondition w.r.t. \(\neg p\) and negate the result. In our case, \(p\) is the termination condition \(\text{termConds}\).

Algorithm 1: \(\text{analyze}\)

```python
1 global Sums\(^o\), Inv\(^u\), Preconds\(^u\);
2 function \(\text{analyzeForward}(f, \text{CallCtx}_o^u)\)
3 foreach procedure call \(h\) in \(f\) do
4 \(\text{CallCtx}_h^o = \text{compCallCtx}(f, \text{CallCtx}_o^u, h)\);
5 if \(\text{needToReAnalyze}(h, \text{CallCtx}_h^o)\) then
6 \(\text{analyzeForward}(h, \text{CallCtx}_h^u)\);
7 \(\text{join}((\text{Sums}\(^o\)[f], \text{Inv}\(^u\)[f]), \text{compInvSum}(f, \text{CallCtx}_h^u))\);
8 function \(\text{analyzeBackward}(f, \text{CallCtx}_o^u)\)
9 \(\text{termConds} = \text{CallCtx}_o^u\);
10 foreach procedure call \(h\) in \(f\) do
11 \(\text{CallCtx}_h^o = \text{compCallCtx}(f, \text{CallCtx}_h^u, h)\);
12 if \(\text{needToReAnalyze}(h, \text{CallCtx}_h^o)\) then
13 \(\text{analyzeBackward}(h, \text{CallCtx}_h^u)\);
14 \(\text{termConds} \leftarrow \text{termConds} \land \text{Preconds}\(^u\)[h]\);
15 \(\text{join}((\text{Preconds}\(^u\)[f], \text{compPrecondTerm}(f, \text{Inv}\(^u\)[f], \text{termConds})), \text{termConds})\);
16 function \(\text{analyze}(f\text{entry})\)
17 \(\text{analyzeForward}(f\text{entry}, \text{true})\);
18 \(\text{analyzeBackward}(f\text{entry}, \text{true})\);
19 return \(\text{Preconds}\(^u\)[f\text{entry}]\);
```
Example. We illustrate the algorithm on the simple example given as Fig. 1 with the encoding in Fig. 2. ε calls a procedure h. Procedure h terminates if and only if its argument y is non-zero, i.e., procedure ε only terminates conditionally. The call of h is guarded by the condition z > 0, which guarantees universal termination of procedure ε.

Let us assume that we use an interval abstract domain for invariant, summary and precondition inference, but the abstract domain with the elements {true, false} for computing calling contexts, i.e., we can prove that calls are unreachable. We use $M := 2^{32}-1$.

Our algorithm proceeds as follows. The first phase is analyzeForward, which starts from the entry procedure ε. By descending into the call graph, we must compute an over-approximating calling context $CallCtx^o_h$ for procedure h for which no calling context has been computed before. This calling context is true. Hence, we recursively analyze h. Given that h does not contain any procedure calls, we compute the over-approximating summary $Sum^o_h = (0 \leq y \leq M \land 0 \leq r_h \leq M)$ and invariant $Inv^o_h = (0 \leq x \leq M \land 0 \leq y \leq M)$. Now, this information can be used in order to compute $Sum^1_h = (0 \leq z \leq M \land 0 \leq r_f \leq M)$ and invariant $Inv^1_f = true$ for the entry procedure ε.

The backwards analysis starts again from the entry procedure ε. It computes an under-approximating calling context $CallCtx^u_h$ for procedure h, which is true, before descending into the call graph. It then computes an under-approximating precondition for termination $Precond^u_h = (1 \leq y \leq M)$ or, more precisely, an under-approximating summary whose projection onto the input variables of h is the precondition $Precond^u_h$. By applying this summary at the call site of ε, we can now compute the precondition for termination $Precond^u_f = (0 \leq z \leq M)$ of ε, which proves universal termination of ε.

We illustrate the effect of the choice of the abstract domain on the analysis of the example program. Assume we replace the {true, false} domain by the interval domain. In this case, analyzeForward computes $CallCtx^o_h = (1 \leq z \leq M \land 0 \leq w_1 \leq M)$. The calling context is computed over the actual parameters $z$ and $w_1$. It is renamed to the formal parameters $y$ and $r_h$ (the return value) when $CallCtx^o_h$ is used for constraining the pre/postconditions in the analysis of h. Subsequently, analyzeBackward computes the precondition for termination of h using the union of all calling contexts in the program. Since h terminates unconditionally in these calling contexts, we trivially obtain $Precond^u_h = (1 \leq y \leq M)$, which in turn proves universal termination of ε.

IV. INTERPROCEDURAL TERMINATION ANALYSIS

We can view Alg. 1 as solving a series of formulæ in second-order predicate logic with existentially quantified predicates, for which we are seeking satisfiability witnesses. In this section, we state the constraints we solve, including all the side constraints arising from the interprocedural analysis.

3 To be precise, we are not only looking for witness predicates but (good approximations of) weakest or strongest predicates. Finding such biased witnesses is a feature of our synthesis algorithms.

Algorithm 2: analyze for universal termination

1. global $Sums^o$, $Invs^o$, $termStatus$;
2. function analyzeForward($f$, $CallCtx^o_f$)
3.     foreach procedure call h in f do
4.         $CallCtx^o_h = compCallCtx^o(f, CallCtx^o_f, h)$;
5.     if needToReAnalyze(h, $CallCtx^o_f$) then
6.         analyzeForward(h, $CallCtx^o_f$);
7.     join($Sums^o[f], Invs^o[f], compInvSum^o(f, CallCtx^o_f)$)
8. function analyzeBackward($f$)
9.     $termStatus[f] = compTermArg(f)$;
10.    foreach procedure call h in f do
11.        if needToReAnalyze(h, $CallCtx^o_f$) then
12.            analyzeBackward(h);
13.        join($termStatus[f], termStatus[h]$);
14. function analyze($f_{entry}$)
15.     analyzeForward($f_{entry}, true$);
16.     analyzeBackward($f_{entry}$);
17.    return $termStatus[f_{entry}]$;

Note that this is not a formalisation exercise, but these are precisely the formulae solved by our synthesis backend, which is described in Section V.

A. Universal Termination

For didactical purposes, we start with a simplification of Algorithm 1 that is able to show universal termination (see Algorithm 2). This variant reduces the backward analysis to a call to compTermArg and propagating back the qualitative result obtained: terminating, potentially non-terminating, or non-terminating.

This section states the constraints that are solved to compute the outcome of the functions underlined in Algorithm 2 and establish its soundness:

- $compCallCtx^o$ (Def. 4)
- $compInvSum^o$ (Def. 5)
- $compTermArg$ (Lemma 3)

Definition 4 (compCallCtx^o). A forward calling context $CallCtx^o_h$ for h in procedure f in calling context $CallCtx^o_f$ is a satisfiability witness of the following formulæ:

$\exists CallCtx^o_h, Inv^o_f : \forall x^{in}, x, x^{'}, x^{'out}, x^{in}, x^{out} : CallCtx^o_f(x^{in}, x^{'out}) \land \sum_s^f \land \text{Assumptions}_f(x) \implies \\text{Init}_f(x^{in}, x) \implies Inv^o_f(x) \land (Inv^o_f(x) \land Trans_f(x, x^{'}) \implies Inv^o_f(x^{'}) \land (g_{h_3} \implies CallCtx^o_h(x^{in}, x^{out}))$.

with $\sum_s^f = \bigwedge_{\text{calls } h_j \text{ in } f} g_{h_j} \implies \sum_s^h[x^{in}, x^{out}]$.

where $g_{h_j}$ is the guard condition of procedure call $h_j$ in $f$ capturing the branch conditions from conditionals. For example, $g_{h_0}$ of the procedure call to h in ε in Fig. 1 is $z > 0$. $\sum_s^h$ is the currently available summary for h (cf. global variables in Alg. 1). Assumptions correspond to assume() statements in the code.
Lemma 1. CallCtx_h is over-approximating.

Proof sketch. CallCtx_h when \( f \) is the entry-point procedure is true; also, the summaries \( \text{Sum}_h \) are initially assumed to be true, i.e., over-approximating. Hence, given that \( \text{CallCtx}_h \) and \( \text{Sums}_h \) are over-approximating, \( \text{CallCtx}_h \) is over-approximating by the soundness of the synthesis (see Thm. 4 in Sec. 4).

Example. Let us consider procedure \( e \) in Fig. 1. \( e \) is the entry procedure, hence we have \( \text{CallCtx}_h((e), (r_f)) = \text{true} \) (\( 0 \leq z \leq M \land 0 \leq r_f \leq M \)) with \( M = 2^{12} - 1 \) when using the interval abstract domain for 32 bit integers. Then, we instantiate Def. 4 (for procedure \( e \)) to compute \( \text{CallCtx}_h \). We assume that we have not yet computed a summary for \( h \), thus, \( \text{Sum}_h \) is true. Remember that the placeholder \( h_0((z), (w_1)) \) evaluates to \( \text{true} \). Notably, there are no assumptions in the code, meaning that \( \text{Assumptions}_f \) are.

\[
\exists \text{CallCtx}_h, \text{Sum}_h: \forall z, w_1, w, w', z', g, g', r_f : \\
0 \leq z \leq M \land 0 \leq r_f \leq M \land (z > 0 \Rightarrow \text{true}) \land \text{true} \Rightarrow \\
(w = 0 \land z' = z \land g \Rightarrow \text{true}) \land \text{true} \Rightarrow \\
(g \land h_0((z), (w_1)) \land w' = (z > 0 ? w : w) \land z' = z \land z' \Rightarrow \text{true}) \Rightarrow \\
\text{true} \land \text{true} \Rightarrow \text{true}
\]

A solution is \( \text{Inv}_f = \text{true} \), and \( \text{CallCtx}_h((z), (w_1)) = (1 \leq z \leq M \land 0 \leq w_1 \leq M) \).

Definition 5 (compTermArg). A forward summary \( \text{Sum}_f \) and invariants \( \text{Inv}_f \) for procedure \( f \) in calling context \( \text{CallCtx}_f \) are satisfiability witnesses of the following formula:

\[
\exists \text{Sum}_f, \text{Inv}_f : \forall x, x', x'', y, y', r_f : \\
\text{CallCtx}_f(x, y, x'', y', r_f) \land \text{Sum}_f \land \text{Assumptions}_f(x) \Rightarrow \\
(\text{true} \land \text{true}) \land \text{true} \Rightarrow \\
(\text{true} \land \text{true}) \land \text{true} \Rightarrow \text{true} \land \text{true} \Rightarrow \text{true}
\]

Lemma 2. \( \text{Sum}_f \) and \( \text{Inv}_f \) are over-approximating.

Proof sketch. By Lemma 1, \( \text{CallCtx}_f \) is over-approximating. Also, the summaries \( \text{Sum}_f \) are initially assumed to be true, i.e., over-approximating. Hence, given that \( \text{CallCtx}_f \) and \( \text{Sums}_f \) are over-approximating, \( \text{Sum}_f \) and \( \text{Inv}_f \) are over-approximating by the soundness of the synthesis (see Thm. 5).

Example. Let us consider procedure \( h \) in Fig. 1. We have computed \( \text{CallCtx}_h((y), (r_h)) = (1 \leq y \leq M \land 0 \leq r_h \leq M) \) (with actual parameters renamed to formal ones). Then, we need obtain witnesses \( \text{Inv}_h \) and \( \text{Sum}_h \) to the satisfiability of the instantiation of Def. 6 (for procedure \( h \)) as given below.

\[
\exists \text{Inv}_h, \text{Sum}_h : \forall x, x', y, y', x'', y'' : \\
1 \leq y \leq M \land 0 \leq r_h \leq M \land \text{true} \Rightarrow \\
(\text{true} \land \text{true}) \land \text{true} \Rightarrow \\
(\text{true} \land \text{true}) \land \text{true} \Rightarrow \text{true} \land \text{true} \Rightarrow \text{true}
\]

A solution is \( \text{Inv}_h = (0 \leq x \leq M \land \text{false}) \) and \( \text{Sum}_h = (1 \leq y \leq M \land 1 \leq y \leq M) \), for instance.

Remark 1. Since Def. 4 and Def. 5 are interdependent, we can compute them iteratively until a fixed point is reached in order to improve the precision of calling contexts, invariants and summaries. However, for efficiency reasons, we perform only the first iteration of this (greatest) fixed point computation.

Lemma 3 (compTermArg). A procedure \( f \) with forward invariants \( \text{Inv}_f \) terminates if there is a termination argument \( \text{RR}_f \):

\[
\exists \text{RR}_f : \forall x, x' : \\
\text{Inv}_f(x) \land \text{Trans}_f(x, x') \land \\
\text{Sum}_f \land \text{Assumptions}_f(x) \land \text{Assumptions}_f(x') \Rightarrow \text{RR}_f(x, x')
\]

Assertions in this formula correspond to \text{assert}() statements in the code. They can be assumed to hold because assertion-violating traces terminate. Over-approximating forward information may lead to inclusion of spurious non-terminating traces. For that reason, we might not find a termination argument although the procedure is terminating. As we essentially under-approximate the set of terminating procedures, we will not give false positives. Regarding the solving algorithm for this formula, we refer to Sec. 4.

Example. Let us consider function \( h \) in Fig. 1. We assume we have the invariant \( 0 \leq x \leq M \land 1 \leq y \leq M \). Thus, we have to solve \( \exists \text{RR}_h : 0 \leq x \leq M \land 1 \leq y \leq M \land x' = x + y' = y \land \text{true} \Rightarrow \text{RR}_h(x, y, (x', y')) \). When using a linear ranking function template \( c_1 \cdot x + c_2 \cdot y \), we obtain as solution, for example, \( \text{RR}_h = (-x > -x') \).

If there is no trace from procedure entry to exit, then we can prove non-termination, even when using over-approximations:

Lemma 4 (line 7 of analyze). A procedure \( f \) in forward calling context \( \text{CallCtx}_f \) and forward invariants \( \text{Inv}_f \) never terminates if its summary \( \text{Sum}_f \) is false.

Termination information is then propagated in the (acyclic) call graph (join in line 13 in Algorithm 2).

Proposition 1. A procedure is declared (1) non-terminating if it is non-terminating by Lemma 2 (2) terminating if

(a) all its procedures calls \( h_i \) that are potentially reachable (i.e., with \( \text{CallCtx}_h \neq \text{false} \)) are declared terminating, and

(b) \( f \) itself is terminating according to Lemma 3 (3) potentially non-terminating, otherwise.

Our implementation is more efficient than Algorithm 2 because it avoids computing a termination argument for \( f \) if one of its callees is potentially non-terminating.

Theorem 1. If the entry procedure of a program is declared terminating, then the program terminates universally. If the entry procedure of a program is declared non-terminating, then the program never terminates.

Proof sketch. By induction over the acyclic call graph using Prop. 1.
Algorithm 3: compPrecondTerm

Input: procedure \( f \) with invariant \( \text{Inv} \), additional termination conditions \( \text{termConds} \)

Output: precondition \( \text{Precond} \)

1. \((\text{Precond}, p) \leftarrow (\text{false}, \text{true})\);
2. let \( \varphi = \text{Init}(x^{in}, x) \land \text{Inv}(x) \);
3. while true do
   4. \( \psi \leftarrow p \land \neg \text{Precond}(x^{in}) \land \varphi \);
   5. solve \( \psi \) for \( x^{in}, x \);
   6. if UNSAT then return \( \text{Precond} \) else
      7. let \( \chi \) be a model of \( \psi \);
      8. let \( \text{Inv}' = \text{compInv}(f, x^{in} = \chi^{in}) \);
      9. let \( \text{RR} = \text{compTermArg}(f, \text{Inv}') \);
     10. if \( \text{RR} = \text{true} \) then \( p \leftarrow p \land (x^{in} \neq \chi^{in}) \) else
       11. let \( \theta = \text{termConds} \land \text{RR} \);
       12. let \( \text{Precond}' = \neg \text{compNecPrecond}(f, \neg \theta) \);
       13. \( \text{Precond} \leftarrow \text{Precond} \lor \text{Precond}' \);

B. Preconditions for Termination

Before introducing conditional termination, we have to talk about preconditions for termination.

If a procedure terminates conditionally like procedure \( h \) in Fig. 1 \( \text{compTermArg} \) (Lemma 3) will not be able to find a satisfying predicate \( \text{RR} \). However, we would like to know under which preconditions, i.e. values of \( y \) in above example, the procedure terminates.

We can state this problem as defined in Def. 5. In Algorithm 2 we search for \( \text{Precond}, \text{Inv}, \) and \( \text{RR} \) in an interleaved manner. Note that \( \text{false} \) is a trivial solution for \( \text{Precond} \); we thus have to aim at finding a good under-approximation of the maximal solution (weakest precondition) for \( \text{Precond} \).

We bootstrap the process by assuming \( \text{Precond} = \text{false} \) and search for values of \( x^{in} \) (Line 5). If such a value \( \chi^{in} \) exists, we can compute an invariant under the precondition candidate \( x^{in} = \chi^{in} \) (Line 6) and use Lemma 3 to search for the corresponding termination argument (Line 7).

If we fail to find a termination argument (\( \text{RR} = \text{true} \)), we block the precondition candidate (Line 10) and restart the bootstrapping process. Otherwise, the algorithm returns a termination argument \( \text{RR} \) that is valid for the concrete value \( \chi^{in} \) of \( x^{in} \). Now we need to find a sufficiently weak \( \text{Precond} \) for which \( \text{RR} \) guarantees termination. To this end, we compute an over-approximating precondition for those inputs for which we cannot guarantee termination \( (\neg \theta \land \text{RR} \lor \text{termConds}) \) which includes additional termination conditions coming from the backward calling context and preconditions of procedure calls, see Sec. IV-C. The negation of this precondition is an under-approximation of those inputs for which \( f \) terminates. Finally, we add this negated precondition to our \( \text{Precond} \) (Line 13) before we start over the bootstrapping process to find precondition candidates outside the current precondition (\( \neg \text{Precond} \)) for which we might be able to guarantee termination.

Example. Let us consider again function \( h \) in Fig. 1. This time, we will assume we have the invariant \( 0 \leq x \leq M \) (with \( M := 2^{12} - 1 \)). We bootstrap by assuming \( \text{Precond} = \text{false} \) and searching for values of \( y \) satisfying \( \text{true} \land \neg \text{false} \land x = 0 \land 0 \leq x \leq M \). One possibility is \( y = 0 \). We then compute the invariant under the precondition \( y = 0 \) and get \( x = 0 \). Obviously, we cannot find a termination argument in this case. Hence, we start over and search for values of \( y \) satisfying \( y \neq 0 \land \neg \text{false} \land x = 0 \land 0 \leq x \leq 10 \). This formula is for instance satisfied by \( y = 1 \). This time we get the invariant \( 0 \leq x \leq 10 \) and the ranking function \( -x \). Thus, we have to solve

\[ \exists e : \mathcal{P}(y, e) \land 0 \leq x \leq M \land x' = x + y \land x < 10 \Rightarrow \neg(x > x') \]

to compute an over-approximating precondition over the template \( \mathcal{P} \). In this case, \( \mathcal{P}(y, e) \) turns out to be \( y = 0 \), therefore its negation \( y \neq 0 \) is the \( \text{Precond} \) that we get. Finally, we have to check for further precondition candidates, but \( y = 0 \land \neg (y \neq 0) \land x = 0 \land 0 \leq x \leq M \) is obviously UNSAT. Hence, we return the sufficient precondition for termination \( y \neq 0 \).

C. Conditional Termination

We now extend the formalisation to Algorithm 1 which additionally requires the computation of under-approximating calling contexts and sufficient preconditions for termination (procedure \( \text{compPrecondTerm} \), see Alg. 3).

First, \( \text{compPrecondTerm} \) computes in line 8 an over-approximating invariant \( \text{Inv}_{fp}^0 \) entailed by the candidate precondition. \( \text{Inv}_{fp}^0 \) is computed through Def. 5 by conjoining the candidate precondition to the antecedent. Then, line 9 computes the corresponding termination argument \( \text{RR}_{fp}^0 \) by applying Lemma 3 using \( \text{Inv}_{fp}^0 \) instead of \( \text{Inv}_f^0 \). Since the termination argument is under-approximating, we are sure that \( f \) terminates for this candidate precondition if \( \text{RR}_{fp}^0 \neq \text{true} \).

Remark 2. The available under-approximate information \( \text{CallCtx}_f^u \land \text{Sums}_f^u \land \text{Preconds}_f^u \), where

\[ \text{Sums}_f^u = \bigwedge_{c H_j in f} g_{h_j} \Rightarrow \sum_{c H_j}^{i}(x^{in}_{j}, x^{out}_{j}) \]

and \( \text{Preconds}_f^u = \bigwedge_{c H_j in f} g_{h_j} = \text{Precond}_{h_j}^{u}(x^{in}_{j}) \)

could be conjoined with the antecedents in Prop. 5 and Prop. 3 in order to constrain the search space. However, this is neither necessary for soundness nor does it impair soundness, because the same information is used in Props. 6 and 7.

Then, in line 12 of \( \text{compPrecondTerm} \), we compute under-approximating (sufficient) preconditions for traces satisfying the termination argument \( \text{RR} \) via over-approximating the traces violating \( \text{RR} \).

Now, we are left to specify the formulae corresponding to the following functions:
- \( \text{compCallCtx}^u \) (Def. 7)
- \( \text{compNecPrecond} \) (Def. 6)

We use the superscript \( ^u \) to indicate negations of under-approximating information.

Definition 6 (Line 12 of \( \text{compPrecondTerm} \)). A precondition for termination \( \text{Precond}_f^u \) in backward calling context \( \text{CallCtx}_f^u \) and with forward invariants \( \text{Inv}_f^u \) is \( \text{Precond}_f^u \equiv \).
Lemma 6. CallCtx_{h_i} is under-approximating.

Proof sketch. The computation is based on the negation of the under-approximating calling context of f and the negated under-approximating summaries for the function calls in f. By Thm. 5 this leads to an over-approximation of the negation of the calling context for h_i.

Theorem 2. A procedure f terminates for all values of x^in satisfying Precond^u_f.

Proof sketch. By induction over the acyclic call graph using Lemmas 5 and 6.

D. Context-Sensitive Summaries

The key idea of interprocedural analysis is to avoid re-analysing procedures that are called multiple times. For that reason, Algorithm 1 first checks whether it can re-use already computed information. For that purpose, summaries are stored as implications CallCtx^o \Rightarrow Sum^o. As the call graph is traversed, the possible calling contexts CallCtx^o_{h_i} for a procedure h are collected over the call sites i. NeedToReAnalyze^o (Line 5 in Alg. 1) checks whether the current context CallCtx^o_{h_i} is subsumed by calling contexts \bigvee CallCtx^o_{h_i} that we have already encountered, and if so, Sum^o is reused; otherwise it needs to be recomputed and joined conjunctively with previously inferred summaries. The same considerations apply to invariants, termination arguments and preconditions.

V. TEMPLATE-BASED STATIC ANALYSIS

In this section, we give a brief overview of our synthesis engine, which serves as a backend for our approach (it solves the formulae in Definitions 4, 5, 6 and 7 (see Sec. VI).

Our synthesis engine employs template-based static analysis to compute ranking functions, invariants, summaries, and calling contexts, i.e., implementations of functions compInvoSum^o and compCallCtx^o from the second-order constraints defined in Sec. IV. To be able to effectively solve second-order problems, we reduce them to first-order by restricting the space of solutions to expressions of the form T(x, d) where

- \(d\) are parameters to be instantiated with concrete values and \(x\) are the program variables.
- \(T\) is a template that gives a blueprint for the shape of the formulas to be computed. Choosing a template is analogous to choosing an abstract domain in abstract interpretation. To allow for a flexible choice, we consider template polyhedra [8].

We state here a soundness result:

Theorem 3. Any satisfiability witness \(d\) of the reduction of the second order constraint for invariants in Def. 7 using template \(T\)

\[
\exists d, \forall x^\text{in}, x, x' : \quad \text{Init}(x^\text{in}, x) \implies T(x, d) \\
\wedge T(x, d) \wedge \text{Trans}(x, x') \implies T(x', d)
\]

satisfies \(\forall x : \text{Inv}(x) \implies T(x, d), i.e. T(x, d)\) is a sound over-approximating invariant. Similar soundness results hold true for summaries and calling contexts.
This ultimately follows from the soundness of abstract interpretation [9]. Similar approaches have been described, for instance, by [10], [11], [12]. However, these methods consider programs over mathematical integers.

Ranking functions require specialised synthesis techniques. To achieve both expressiveness and efficiency, we generate linear lexicographic functions [13], [14]. Our ranking-function synthesis approach is similar to the TAN tool [15] but extends the approach from monolithic to lexicographic ranking functions. Further, unlike TAN, our synthesis engine is much more versatile and configurable, e.g., it also produces summaries and invariants.

We refer to Appendix A, which includes a detailed description of the synthesis engine, our program encoding, encoding of bit-precise arithmetic, and tailored second-order solving techniques for the different constraints that occur in our analysis. In the following section, we discuss the implementation.

VI. IMPLEMENTATION

We have implemented the algorithm in 2LS [16], a static analysis tool for C programs built on the CPROVER framework, using MiniSat 2.2.0 as back-end solver. Other SAT and SMT solvers with incremental solving support would also be applicable. Our approach enables us to use a single solver instance per procedure to solve a series of second-order queries as required by Alg. 1. This is essential as our synthesis algorithms make thousands of solver calls. Architectural settings (e.g. bitwidths) can be provided on the command line.

Bitvector Width Extension As aforementioned, the semantics of C allows integers to wrap around when they over/underflow. Let us consider the following example, for which we want to find a termination argument using Algorithm 4.

```c
void f0() { for(unsigned char x : x++) }
```

The ranking function synthesis needs to compute a value for template parameter $\ell$ such that $\ell(x-x') > 0$ holds for all $x, x'$ under transition relation $x' = x+1$ and computed invariant true (for details of the algorithm refer to Appendix C).

Thus, assuming that the current value for $\ell$ is $-1$, the constraint to be solved (Algorithm 4, Line 5) is true $\land x' = x + 1 \land \neg(-1(1-x-x')) > 0\land \neg(-1(1-(x+1))) > 0)$, for short. While for mathematical integers this is SAT, it is UNSAT for signed bit-vectors due to overflows. For $x = 127$, the overflow happens such that $x+1 = -128$. Thus, $127 - (-128) > 0$ becomes $-1 > 0$, which makes the constraint UNSAT, and we would incorrectly conclude that $-x$ is a ranking function, which does not hold for signed bitvector semantics. However, if we extend the bitvector width to $k = 9$ such that the arithmetic in the template does not overflow, then $\neg(-1 : ((\text{signed}_{9}127 - (\text{signed}_{9}(-128))) > 0)$ evaluates to $255 > 0$, where signed is a cast to a $k$-bit signed integer. Now, $x = 127$ is a witness showing that $-x$ is not a valid ranking function.

For similar reasons, we have to extend the bit-width of $k$-bit unsigned integers in templates to $(k+1)$-bit signed integers to retain soundness.

Optimisations Our ranking function synthesis algorithm searches for coefficients $\ell$ such that a constraint is UNSAT. However, this may result in enumerating all the values for $\ell$ in the range allowed by its type, which is inefficient. In many cases, a ranking function can be found for which $\ell_j \in \{-1, 0, 1\}$. In our implementation, we have embedded an improved algorithm (Algorithm 4 in Appendix C) into an outer refinement loop which iteratively extends the range for $\ell$ if a ranking function could not be found. We start with $\ell_j \in \{-1, 0, 1\}$, then we try $\ell_j \in [-10, 10]$ before extending it to the whole range.

Further Bounds As explained in Algorithm 4 we bound the number of lexicographic components (default 3), because otherwise Algorithm 4 does not terminate if there is no number $n$ such that a lexicographic ranking function with $n$ components proves termination.

Since the domains of $x, x'$ in Algorithm 4 and of $x^\text{in}$ in Algorithm 5 might be large, we limit also the number of iterations (default 20) of the while loops in these algorithms. In the spirit of bounded model checking, these bounds only restrict completeness, i.e., there might exist ranking functions or preconditions which we could have found for larger bounds. The bounds can be given on the command line.

VII. EXPERIMENTS

We performed experiments to support the following claims:
1) Interprocedural termination analysis (IPTA) is faster than monolithic termination analysis (MTA).
2) The precision of IPTA is comparable to MTA.
3) 2LS outperforms existing termination analysis tools.
4) 2LS’s analysis is bit-precise.
5) 2LS computes usable preconditions for termination.

We used the product line benchmarks of the [17] benchmark repository. In contrast to other categories, this benchmark set contains programs with non-trivial procedural structure. This benchmark set contains 597 programs with 1100 to 5700 lines of code (2705 on average), 33 to 136 procedures (67 on average), and 4 to 10 loops (5.5 on average). Of these benchmarks, 264 terminate universally, whereas 333 never terminate.

The experiments were run on a Xeon X5667 at 3 Ghz running Fedora 20 with 64-bit binaries. Memory and CPU time were restricted to 16GB and 1800 seconds per benchmark, respectively (using [18]). Using 2LS with interval templates was sufficient to obtain reasonable precision.

Modular termination analysis is fast We compared IPTA with MTA (all procedures inlined). Table II shows that IPTA times out on 2.3% of the benchmarks vs. 39.7% for MTA. The geometric mean speed-up of IPTA w.r.t. MTA on the benchmarks correctly solved by both approaches is 1.37.

In order to investigate how the 30m timeout affects MTA, we randomly selected 10 benchmarks that timed out for 30m and re-ran them: 1 finished in 32m, 3 after more than 1h, 6 did not finish within 2h.

*Measured using cloc 1.53.
Modular termination analysis is precise

Again, we compare IPTA with MTA. Table I shows that IPTA proves 94% of the terminating benchmarks, whereas only 10% were proven by MTA. MTA can prove all never-terminating benchmarks including 13 benchmarks where IPTA times out. MTA times out on the benchmarks that cause 13 additional potentially non-terminating outcomes for IPTA.

2LS outperforms existing termination analysis tools

We compared 2LS with two termination tools for C programs from the SV-COMP termination competition, namely [19] and [20]. Unfortunately, the tools [21], [22], [23], [24], and [25] have limitations regarding the subset of C that they can handle that make them unable to analyze any of the benchmarks out of the box. We describe these limitations in [26]. Unfortunately, we did not succeed to generate the correct input files in the intermediate formats required by [27] and [28] using the recommended frontends [29] and [30].

TAN [15], and KiTTeL/KoAT [5] support bit-precise C semantics. Ultimate uses mathematical integer reasoning but tries to ensure conformance with bit-vector semantics. Also, Ultimate uses a semantic decomposition of the program [31] to make its analysis efficient.

Table I shows lists for each of the tools the number of instances solved, timed out or aborted because of an internal error. We also give the total run time, which shows that analysis times are roughly halved by the modular/interprocedural approaches (2LS IPTA, Ultimate) in comparison with the monolithic approaches (2LS MTA, TAN). Ultimate spends less time on those benchmarks that it can prove terminating, however, these are only 19% of the terminating benchmarks (vs. 94% for 2LS). If Ultimate could solve those 180 benchmarks on which it fails due to unsupported features of C, we would expect its performance to be comparable to 2LS.

Ultimate and 2LS have different capabilities regarding non-termination. 2LS can show that a program never terminates for all inputs, whereas Ultimate can show that there exists a potentially non-terminating execution. To make the comparison fair, we counted benchmarks flagged as potentially non-terminating by Ultimate, but which are actually never-terminating, in the non-terminating category in Table I (marked *).

2LS’s analysis is bit-precise

We compared 2LS with Loopus on a collection of 15 benchmarks (ABC_ex01.c to ABC_ex15.c) taken from the Loopus benchmark suite.

| Tool | Terminating | Non-Terminating | Potentially Non-Terminating | Timed Out | Errors | Total Run Time (h) |
|------|-------------|----------------|-----------------|---------|--------|-------------------|
| 2LS  | 249         | 320            | 14              | 0       | 58.7   | 119.6            |
| IPTA | 26          | 333            | 425             | 237     | 92.8   | 23.9              |
| 2LS  | 18          | 3            | 43              | 43      | 80     | 320               |
| MTA  | 58          | 2            | 18              | 18      | 18     | 58                |
| TAN  | 18          | 3            | 43              | 43      | 80     | 320               |
| Ultimate | 58        | 2            | 18              | 18      | 18     | 58                |

Table I Tool comparison (* see text).

Fig. 3. Example ABC_ex15.c from the Loopus benchmarks.

```
void ex15(int m, int n, int p, int q) {
    for (int i = n; i >= 1; i = i - 1)
        for (int j = 1; j <= m; j = j + 1)
            for (int k = i; k <= p; k = k + 1)
                for (int l = q; l <= 1 + l)
                    do some computation
}
```

Fig. 4. Example createBack (struct SDL_Surface *back_surf) from Debian package abc.

```
void createBack (struct SDL_Surface *back_surf) {
    SDL_Rect pos;
    SDL_Surf *surf = NULL;
    for (int y = 0; y < back_surf->h;) { // y = y + back_surf->w;
        pos.x = (signed short int)x;
        pos.y = (signed short int)y;
        SDL_UpperBlit (surf, &pos); // surf, &pos);
    }
```

2LS computes usable preconditions for termination

This experiment was performed on benchmarks extracted from Debian packages and the linear algebra library CLapack.

The quality of preconditions, i.e. usability or ability to help the developer to spot problems in the code, is difficult to quantify. We give several examples where function terminate conditionally. The abc package of Debian contains a function, shown in Fig. 5 where increments of the iteration in a loop are not constant but dynamically depend on the dimensions of an image data structure. Here 2LS infers the precondition \(img \rightarrow h > 0 \land img \rightarrow w > 0\).

The example in Fig. 5 is taken from the benchmark basename in the busybox-category of SVCOMP 2015, which contains simplified versions of Debian packages. The termination of function full_write depends on the return value of its callee function safe_write. Here 2LS infers the calling context cc > 0, i.e. the contract for the function safe_write, such that the termination of full_write is guaranteed. Given a proof that safe_write terminates and returns a strictly positive value regardless of the arguments it is called with, we can conclude that full_write terminates universally.

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1 signed long int full_write(signed int fd,
2 const void *buf, unsigned long int len,
3 unsigned long int cc) {
4 signed long int total = (signed long int)0;
5 for (; ! (len == 0u);)
6 len = len - (unsigned long int)cc;
7 cc = safe_write(fd, buf, len);
8 if (cc < 0l) {
9 if (! (total == 0l))
10 return total;
11 return cc;
12 }
13 total = total + cc;
14 buf = (const void *)((const char *)buf + cc);
15 }

Fig. 5. Example from SVCOMP 2015 busybox.

1 int f(int *sx, int n, int incx) {
2 int nin cx = n * incx;
3 int st emp = 0;
4 for (int i = 0; incx < 0 ? i >= nin cx: i <= nin cx;
5 i += incx) {
6 st emp += sx[i - 1];
7 }
8 return st emp;
9 }

Fig. 6. Non-unit increment from CLapack.

The program in Fig. 6 is a code snippet taken from the summation procedure sasum within [32], the C version of the popular LAPACK linear algebra library. The loop in procedure f does not terminate if incx = 0. If incx > 0 (incx < 0) the termination argument is that i increases (decreases). Therefore, incx \neq 0 is a termination precondition for f.

VIII. LIMITATIONS, RELATED WORKS AND FUTURE DIRECTIONS

Our approach makes significant progress towards analysing real-world software, advancing the state-of-the-art of termination analysis of large programs. Conceptually, we decompose the analysis into a sequence of well-defined second-order predicate logic formulae with existentially quantified predicates. In addition to [33], we consider context-sensitive analysis, under-approximate backwards analysis, and make the interaction with termination analysis explicit. Notably, these seemingly tedious formulae are actually solved by our generic template-based synthesis algorithm, making it an efficient alternative to predicate abstraction.

An important aspect of our analysis is that it is bit-precise. As opposed to the synthesis of termination arguments for linear programs over integers (rationals) [29, 35] and [2] [69, 77], this subclass of termination analyses is substantially less covered. While [15], [38] present methods based on a reduction to Presburger arithmetic, and a template-matching approach for predefined classes of ranking functions based on reduction to SAT- and QBF-solving, [39] only compute intraprocedural termination arguments.

There are still a number of limitations to be addressed, all of which connect to open challenges subject to active research. While some are orthogonal (e.g., data structures, strings, refinement) to our interprocedural analysis framework, others (recursion, necessary preconditions) require extensions of it. In this section, we discuss related work, as well as, characteristics and limitations of our analysis, and future directions (cost analysis and concurrency).

Dynamically allocated data structures We currently ignore heap-allocated data. This limitation could be lifted by using specific abstract domains. For illustration, let us consider the following example traversing a singly-linked list.

L i s t x; while (x != NULL) { x = x->next; }

Fig. 7. Example from SVCOMP 2015 busybox.

Deciding the termination of such a program requires knowledge about the shape of the data structure pointed by x, namely, the program only terminates if the list is acyclic. Thus, we would require an abstract domain capable of capturing such a property and also relate the shape of the data structure to its length. Similar to [14], we could use [40] in order to abstract heap-manipulating programs to arithmetic ones. Another option is using an abstract interpretation based on separation logic formulae which tracks the depths of pieces of heaps similarly to [41].

Strings and arrays Similar to dynamically allocated data structures, handling strings and arrays requires specific abstract domains. String abstractions that reduce null-terminated strings to integers (indices, length, and size) are usually sufficient in many practical cases; scenarios where termination is dependent on the content of arrays are much harder and would require quantified invariants [42]. Note that it is favorable to run a safety checker before the termination checker. The latter can assume that assertions for buffer overflow checks hold which strengthens invariants and makes termination proofs easier.

Recursion We currently use downward fixed point iterations for computing calling contexts and invariants that involve summaries (see Remark 1). This is cheap but gives only imprecise results in the presence of recursion, which would impair the termination analysis. We could handle recursions by detecting cycles in the call graph and switching to an upward iteration scheme in such situations. Moreover, an adaptation regarding the generation of the ranking function templates is necessary. An alternative approach would be to make use of the theoretic framework presented in [43] for verifying total correctness and liveness properties of while programs with recursion.

Template refinement We currently use interval templates together with heuristics for selecting the variables that should be taken into consideration. This is often sufficient in practice, but it does not exploit the full power of the machinery in place. While counterexample-guided abstraction refinement (CEGR) techniques are prevalent in predicate abstraction [44], attempts to use them in abstract interpretation are rare [45]. We consider our template-based abstract interpretation that automatically synthesises abstract transformers more amenable to refinement techniques than classical abstract interpretations where abstract transformers are implemented manually.

Sufficient preconditions to termination Currently, we
compute sufficient preconditions, i.e. under-approximating preconditions to termination via computing over-approximating preconditions to potential non-termination. The same concept is used by other works on conditional termination [7], [46]. However, they consider only a single procedure and do not leverage their results to perform interprocedural analysis on large benchmarks which adds, in particular, the additional challenge of propagating under-approximating information up to the entry procedure (e.g. [47]). Moreover, by contrast to Cook et al [7] who use an heuristic FINITE-operator left unspecified for bootstrapping their preconditions, our bootstrapping is systematic through constraint solving.

We could compute necessary preconditions by computing over-approximating preconditions to potential termination (and negating the result). However, this requires a method for proving that there exist non-terminating executions, which is a well-explored topic. While [48] dynamically enumerate lasso-shaped candidate paths for counterexamples, and then statically prove their feasibility, [49] prove nontermination via reduction to safety proving. In order to prove both termination and non-termination, [50] compose several program analyses (termination provers for multi-path loops, non-termination provers for cycles, and safety provers).

Cost analysis A potential future application for our work is cost and resource analysis. Instances of this type of analyses are the worst case execution time (WCET) analysis [51], as well as bound and amortised complexity analysis [52], [53]. The control flow refinement approach [54], [55] instruments a program with counters and uses progress invariants to compute worst case or average case bounds.

Concurrency Our current analysis handles single-threaded C programs. One way of extending the analysis to multi-threaded programs is using the rely-guarantee technique which is proposed in [57], and explored in several works [58], [59]. [60] for termination analysis. In our setting, the predicates for environment assumptions can be used in a similar way as invariants and summaries are used in the analysis of sequential programs.

IX. Conclusions

While many termination provers mainly target small, hard programs, the termination analysis of larger code bases has received little attention. We present an algorithm for interprocedural termination analysis for non-recursive programs. To our knowledge, this is the first paper that describes in full detail the entire machinery necessary to perform such an analysis. Our approach relies on a bit-precision static analysis combining SMT solving, template polyhedra and lexicographic, linear ranking function templates. We provide an implementation of the approach in the static analysis tool 2LS, and demonstrate the applicability of the approach to programs with thousands of lines of code.

REFERENCES

[1] http://cve.mitre.org/cgi-bin/cvename.cgi?name=CVE-2009-1890
[2] A. M. Ben-Amram and S. Genaim, “On the linear ranking problem for integer linear-constraint loops,” in Principles of Programming Languages, pp. 51–62, ACM, 2013.
[3] J. Leike and M. Heizmann, “Ranking templates for linear loops,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 8413 of Lecture Notes in Computer Science, pp. 172–186, Springer, 2014.
[4] A. R. Bradley, Z. Manna, and H. B. Sipma, “Termination of polynomial programs,” in Verification, Model Checking, and Abstract Interpretation, vol. 3385 of Lecture Notes in Computer Science, pp. 113–129, Springer, 2005.
[5] S. Falke, D. Kapur, and C. Sinz, “Termination analysis of imperative programs using bivector arithmetic,” in Verified Software: Theories, Tools, Experiments, vol. 7152 of Lecture Notes in Computer Science, pp. 261–277, Springer, 2012.
[6] D. Beyer, “Status report on software verification – (competition summary SV-COMP 2014),” in TACAS, vol. 8413 of Lecture Notes in Computer Science, Springer, 2014.
[7] B. Cook, S. Gulwani, T. Lev-Ami, A. Rybalchenko, and M. Sagiv, “Proving conditional termination,” in Computer-Aided Verification, vol. 5123 of Lecture Notes in Computer Science, pp. 328–340, Springer, 2008.
[8] S. Sankaranarayanan, H. B. Sipma, and Z. Manna, “Scalable analysis of linear systems using mathematical programming,” in Verification, Model Checking, and Abstract Interpretation, vol. 3385 of Lecture Notes in Computer Science, pp. 25–41, Springer, 2005.
[9] P. Cousot and R. Cousot, “Abstract interpretation: A uni ed lattice model for static analysis of programs by construction or approximation of xespoints,” in Principles of Programming Languages, pp. 238–252, 1977.
[10] T. M. Gavlitza and H. Seidl, “Precise relational invariants through strategy iteration,” in Computer Science Logic, vol. 4646 of Lecture Notes in Computer Science, pp. 23–40, Springer, 2007.
[11] S. Gulwani, S. Srivastava, and R. Venkatesan, “Program analysis as constraint solving,” in Programming Language Design and Implementation, pp. 292–292, ACM, 2008.
[12] Y. Li, A. Albarghouthi, Z. Kincaid, A. Gurfinkel, and M. Chechik, “Symbolic optimization with smt solvers,” in Principles of Programming Languages, pp. 607–618, ACM, 2014.
[13] A. R. Bradley, Z. Manna, and H. B. Sipma, “Linear ranking with reachability,” in Computer-Aided Verification, pp. 491–504, 2005.
[14] B. Cook, A. See, and F. Zuleger, “Ramsey vs. lexicographic termination proving,” in Tools and Algorithms for the Construction and Analysis of Systems, pp. 47–61, 2013.
[15] D. Kroening, N. Sharygina, A. Tsitovich, and C. Wintersteiger, “Termination analysis of compositional transition invariants,” in Computer-Aided Verification, vol. 6174 of Lecture Notes in Computer Science, pp. 89–103, Springer, 2010.
[16] 2LS https://drive.google.com/file/d/0B4YGF4U_JaBKaYBEoExYUR1M2M/view?usp=sharing
[17] https://svn.sosy-lab.org/software/sv-benchmarks/tags/svcomp14/product-lines/
[18] O. Roussel, “Controlling a solver execution with the runsolver tool,” Journal on Satisfiability, Boolean Modeling and Computation, vol. 7, no. 4, pp. 139–144, 2011.
[19] http://www.cprover.org/termination/ (version SV-COMP-2014).
[20] http://ultimate.informatik.uni-freiburg.de/ (version SV-COMP-2015).
[21] http://aprove.informatik.rwth-aachen.de/ (version 2014).
[22] http://forsyte.at/software/loopus/ with http://sourceforge.net/projects/virtualboximage/file
[23] http://www.di.ens.fr/~urban/sv-comp-2015.zip (version SV-COMP-2015).
[24] http://horis-7.ddns.comp.nus.edu.sg/~product/hipnt/pplus/
[25] https://www7.in.tum.de/~rybal/armc/ (version August 2011).
[26] Experiments log https://drive.google.com/file/d/0B4YGF4U_JaBKc2YxQ2N3YjhlYV00/view
[27] http://research.microsoft.com/en-us/projects/slayer/ (version 1.1).
[28] http://github.com/urban/sv-comp-2015.zip
[29] https://github.com/mmp/libvm/koat (revision c05eab4b3c).
[30] https://github.com/urban/sv-comp-2015.zip
[31] M. Heizmann, J. Hoenicke, and A. Podelski, “Termination analysis by learning terminating programs,” in Computer-Aided Verification, vol. 8559 of Lecture Notes in Computer Science, pp. 797–813, Springer, 2014.
[32] https://www.netlib.org/clapack/cblas/sasum.c.
[33] S. Grebenshchikov, N. P. Lopes, C. Popeea, and A. Rybalchenko, “Synthesizing software verifiers from proof rules,” in PLDI, pp. 405–416, 2012.
[34] B. Cook, A. Podelski, and A. Rybalchenko, “Termination proofs for systems code,” in Programming Language Design and Implementation, pp. 415–426, ACM, 2006.
[35] W. Lee, B.-Y. Wang, and K. Yi, “Termination analysis with algorithmic learning,” in Computer-Aided Verification, pp. 88–104, 2012.

[36] A. Podelski and A. Rybalchenko, “Transition invariants,” in Logic in Computer Science, pp. 32–41, IEEE Computer Society, 2004.

[37] M. Heizmann, J. Hoenicke, J. Leike, and A. Podelski, “Linear ranking functions for linear lasso programs,” in Automated Technology for Verification and Analysis, pp. 365–380, 2013.

[38] B. Cook, D. Kroening, P. Rümmer, and C. M. Wintersteiger, “Ranking function synthesis for bit-vector relations,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 6015 of Lecture Notes in Computer Science, pp. 236–250, Springer, 2010.

[39] C. David, D. Kroening, and M. Lewis, “Unrestricted termination and non-termination arguments for bit-vector programs,” in ESOP, pp. 183–204, 2015.

[40] S. Magill, M.-H. Tsai, P. Lee, and Y.-K. Tsay, “Automatic numeric abstractions for heap-manipulating programs,” in POPL, pp. 211–222, 2010.

[41] J. Berdine, B. Cook, D. Distefano, and P. W. O’Hearn, “Automatic termination proofs for programs with shape-shifting heaps,” in CAV, pp. 386–400, 2006.

[42] K. L. McMillan, “Quantified invariant generation using an interpolating saturation prover,” in Tools and Algorithms for the Construction and Analysis of Systems, 14th International Conference, TACAS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29-April 6, 2008. Proceedings, pp. 413–427, 2008.

[43] A. Podelski, I. Schaerfe, and S. Wagner, “Summaries for integer programs,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 7795 of Lecture Notes in Computer Science, pp. 154–169, Springer, 2013.

[44] E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, and H. Veith, “Counterexample-guided abstraction refinement,” in Computer-Aided Verification, vol. 1855 of Lecture Notes in Computer Science, pp. 4905 of Lecture Notes in Computer Science, pp. 248–262, Springer, 2008.

[45] F. Ranzato, O. Rossi-Doria, and F. Tapparo, “A forward-backward abstraction refinement algorithm,” in Verification, Model Checking, and Abstract Interpretation, vol. 4905 of Lecture Notes in Computer Science, pp. 248–262, Springer, 2008.

[46] M. Bozga, R. Iosif, and F. Konecný, “Deciding conditional termination,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 7214 of Lecture Notes in Computer Science, pp. 252–266, Springer, 2012.

[47] P. Ganty, R. Iosif, and F. Konecný, “Underapproximation of procedure summaries for integer programs,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 7795 of Lecture Notes in Computer Science, pp. 245–259, Springer, 2013.

[48] A. Gupta, T. A. Henzinger, R. Majumdar, A. Rybalchenko, and G. Xu, “Proving non-termination,” in Principles of Programming Languages, pp. 147–158, ACM, 2008.

[49] H. Y. Chen, B. Cook, C. Fuhs, K. Nimkar, and P. W. O’Hearn, “Proving nontermination via safety,” in TACAS, pp. 156–171, 2014.

[50] W. R. Harris, A. Lal, A. V. Nori, and S. K. Rajamani, “Alternation for termination,” in Static Analysis Symposium, pp. 304–319, 2010.

[51] R. Wilhelm, J. Engblom, A. Ermedahl, N. Holst, S. Thesing, D. Whalley, G. Bernat, C. Ferdinand, R. Heckmann, T. Mitra, F. Mueller, I. Puaut, P. Puscher, J. Staschulat, and P. Stenström, “The Worst-case Execution Time Problem—Overview of Methods and Survey of Tools,” Transactions on Embedded Computing Systems, vol. 7, no. 3, 2008.

[52] C. Alias, A. Darte, P. Feautrier, and L. Gonnord, “Multi-dimensional rankings, program termination, and complexity bounds of flowchart programs,” in Static Analysis Symposium, vol. 6537 of Lecture Notes in Computer Science, pp. 117–133, Springer, 2010.

[53] M. Brockschmidt, F. Emmes, S. Falke, C. Fuhs, and J. Giesl, “Alternating runtime and size complexity analysis of integer programs,” in Tools and Algorithms for the Construction and Analysis of Systems, vol. 8413 of Lecture Notes in Computer Science, pp. 140–155, Springer, 2014.

[54] M. Sinn, F. Zuleger, and H. Veith, “A simple and scalable static analysis for bound analysis and amortized complexity analysis,” in Computer-Aided Verification, vol. 8559 of Lecture Notes in Computer Science, pp. 745–761, Springer, 2014.

[55] S. Gulwani, S. Jain, and E. Koskinen, “Control-flow refinement and progress invariants for bound analysis,” in Programming Language Design and Implementation, pp. 375–385, 2009.

[56] H. Y. Chen, S. Mukhopadhyay, and Z. Lu, “Control flow refinement and symbolic computation of average case bound,” in Automated Technology for Verification and Analysis, pp. 334–348, 2013.

[57] C. B. Jones, “Tentative steps toward a development method for interfering programs,” ACM Trans. Program. Lang. Syst., vol. 5, pp. 596–619, Oct. 1983.

[58] B. Cook, A. Podelski, and A. Rybalchenko, “Proving thread termination,” in Proceedings of the 2007 ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI ’07, (New York, NY, USA), pp. 320–330, ACM, 2007.

[59] A. Gupta, C. Popeea, and A. Rybalchenko, “Predicate abstraction and refinement for verifying multi-threaded programs,” in Proceedings of the 38th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL ’11, (New York, NY, USA), pp. 331–344, ACM, 2011.

[60] C. Popeea and A. Rybalchenko, “Compositional termination proofs for multi-threaded programs,” in Tools and Algorithms for the Construction and Analysis of Systems (C. Flanagan and B. König, eds.), vol. 7214 of Lecture Notes in Computer Science, pp. 237–251, Springer Berlin Heidelberg, 2012.

[61] A. Miné, “The octagon abstract domain,” Higher-Order and Symbolic Computation, vol. 19, no. 1, pp. 31–100, 2006.

APPENDIX

A. Program Encoding

We encode programs in a representation akin to single-static assignment (SSA). We provide a brief review, focusing on the modelling of loops and procedure calls. We continue to use the program in Fig. 1 as example.

In SSA, each assignment to a variable gives rise to a fresh symbol. For instance, the initialisation of variable x corresponds to symbol x_0, and the incrementation by y gives rise to symbol x_2. For the return values, additional variables such as r_f are introduced. In addition, at control-flow join points, the values coming from different branches get merged into a single δ-variable. For instance, x_3 is either the initial value x_0 or the value of x after executing the loop, which is denoted as x_3^b. In the case of branches, the choice is controlled using the condition of the branch.

In the case of loops, the choice between the value at the loop entry point and the value after the execution of the loop body is made using a non-deterministic Boolean symbol l_{b3}. That ensures the SSA remains acyclic and loops are over-approximated. Moreover, it allows us to further constrain x_3^b with loop invariants inferred by our analyses.

In addition to data-flow variables, there are guard variables g_1, which capture the branch conditions from conditionals and loops. For instance, the loop condition is g_2, and the conditional around the invocation of f is g_3.

To facilitate interprocedural analysis, our SSA contains a placeholder for procedure calls, which ensures that procedure calls are initially havoced. It can be constrained using the summaries computed in the course of the analysis (cf. Sec. III).

Regarding pointers, a may-alias analysis is performed during translation to SSA form and case splits are introduced accordingly.

B. Computing Over-Approximating Abstractions

To implement Algorithm 1, we need to compute invariants, summaries, and calling contexts, i.e., implementations of functions compInvSum^a and compCallCtx^a. As described in Sec. III invariants and calling contexts can be declaratively
expressed in second-order logic. To be able to effectively solve such second-order problems, we reduce them to first-order by restricting the space of solutions to expressions of the form \( \mathcal{T}(x, d) \) where \( d \) are parameters to be instantiated with concrete values and \( x \) are the program variables.

**Template Domain.** An abstract value \( \delta \) represents the set of all \( x \) that satisfy the formula \( \mathcal{T}(x, \delta) \) (concretisation). We write \( \bot \) for the abstract value denoting the empty set \( \mathcal{T}(x, \bot) \equiv \text{false} \), and \( \top \) for the abstract value denoting the whole domain of \( x \): \( \mathcal{T}(x, \top) \equiv \text{true} \).

Choosing a template is analogous to choosing an abstract domain in abstract interpretation. To allow for a flexible choice, we consider template polyhedra \[3\]: \( \mathcal{G} = (A x \leq d) \) where \( A \) is a matrix with fixed coefficients. Polynomial templates subsume intervals, zones and octagons \[61\]. Intervals, for example, give rise to two constraints per variable \( x_i \): \( x_i \leq d_{i1} \) and \( -x_i \leq d_{i2} \). We call the constraint generated by the \( i \)-th row of matrix \( A \) the \( i \)-th row of the template.

To encode the context of template constraints, e.g., inside a loop or a conditional branch, we use guarded templates. In a guarded template each row \( r \) is of the form \( G_r \Rightarrow T_r \) for the \( r \)-th row \( T_r \) of the base template domain (e.g. template polyhedra). The guards are uniquely defined by the guards associated to the variables \( x \) at the loop head, and \( G_r ' \) the guard associated to the variables \( x' \) at the end of the loop body. A guarded template in terms of the variables at the loop head is of the form: \( \mathcal{T} = \bigwedge_r G_r \Rightarrow T_r \) (respectively \( \mathcal{T}' = \bigwedge_r G_r' \Rightarrow T'_r \) if expressed in terms of the variables at the end of the loop body).

**Inferring abstractions.** Fixing a template reduces the second-order search for an invariant to the first-order search for template parameters:

\[
\exists d : \forall x \in \mathcal{I} : \text{Init}(x, d) \Rightarrow \mathcal{T}(x, d) \\
\text{with } \mathcal{T}(x, d) \land \mathcal{G}(x, x') \Rightarrow \mathcal{T}'(x', d).
\]

By substituting the symbolic parameter \( d \) by a concrete value \( \delta \), we see that \( \mathcal{T}(x, \delta) \) is an invariant if and only if the following formula is unsatisfiable:

\[
\exists x \in \mathcal{I} : \text{Init}(x, \delta) \land \neg \mathcal{T}(x, \delta) \\
\lor \mathcal{T}(x, \delta) \land \mathcal{G}(x, x') \land \neg \mathcal{T}'(x', \delta).
\]

As these vectors represent upper bounds on expressions, the most precise solution is the smallest vector in terms of point-wise ordering. We solve this optimisation problem by iteratively calling an SMT solver. Similar approaches have been described, for instance, by \[10, 11, 12\]. However, these methods consider programs over mathematical integers.

Computing overapproximations for calling contexts is similar to computing invariants or summaries. They only differ in the program variables appearing in the templates: \( x^{i_{\text{in}}}, x^{i_{\text{out}}} \) for calling contexts, \( x, x' \) for invariants, and \( x^{i_{\text{in}}}, x^{i_{\text{out}}} \) for summaries.

**C. Termination Analysis For One Procedure**

In this section we give details on the algorithm that we use to solve the formula in Lemma 3 (see Sec. IV-A).

Monolithic ranking functions are complete, i.e., termination can always be proven monolithically if a program terminates. However, in practice, combinations of linear ranking functions, e.g., linear lexicographic functions \[13, 14\] are preferred, as monolithic linear ranking functions are not expressive enough, and non-linear theories are challenging for existing SMT solvers, which handle the linear case much more efficiently.

1) Lexicographic Ranking Functions:

**Definition 8** (Lexicographic ranking function). A lexicographic ranking function \( R \) for a transition relation \( \mathcal{R}(x, x') \) is an \( n \)-tuple of expressions \( (R_1, R_{n-1}, ..., R_1) \) such that

\[
\exists \Delta > 0 : \forall x, x' : \mathcal{R}(x, x') \land \exists i \in [1, n] : \\
R_i(x) > 0 \quad \text{(Bounded)} \\
\land R_i(x) - R_i(x') > \Delta \quad \text{(Decreasing)} \\
\land \forall j > i : R_j(x) - R_j(x') \geq 0 \quad \text{(Unaffecting)}
\]

Notice that this is a special case of Definition 2. In particular, the existence of \( \Delta > 0 \) and the Bounded condition guarantee that \( > \) is a well-founded relation.

Before we encode the requirements for lexicographic ranking function into constraints, we need to adapt it in accordance with the bit-vector semantics. Since bit-vectors are bounded, it follows that the Bounded condition is trivially satisfied and therefore can be omitted. Moreover, bit-vectors are discrete, hence we can replace the Decreasing condition with \( R_i(x) - R_i(x') > 0 \). The following formula, \( LR \), holds if and only if \( R_n, R_{n-1}, ..., R_1 \) is a lexicographic ranking function with \( n \) components over bit-vectors.

\[
LR^n(x, x') = \bigwedge_{i=1}^n (R_i(x) - R_i(x') > 0 \land \bigwedge_{j=i+1}^n (R_j(x) - R_j(x') \geq 0))
\]

Assume we are given the transition relation \( \mathcal{R}(x, x') \) of a procedure \( f \). The procedure \( f \) may be composed of several loops, and each of the loops is associated with a guard \( q \) that expresses the reachability of the loop head (see Sec. XV). That is, suppose \( f \) has \( k \) loops, then the lexicographic ranking function to prove termination of \( f \) takes the form:

\[
RR^n(x, x') = \bigwedge_{i=1}^k g_i(x) \Rightarrow LR^n_i(x, x')
\]
2) Synthesising Lexicographic Ranking Functions: Ranking

techniques for mathematical integers use e.g. Farkas’

Lemma, which is not applicable to bitvector operations. We

use a synthesis approach (like the TAN tool [15]) and extend

it from monolithic to lexicographic ranking functions.

We consider the class of lexicographic ranking functions

generated by the template where $R_i(x)$ is the product $\ell_i x$

with the row vector $\ell_i$ of template parameters. We denote

the resulting constraints for loop $i$ as $LR_i^n(x, x', L_i^n)$, where

$L_i^n$ is the vector $(\ell_1^n, \ldots, \ell_k^n)$. The constraints for the ranking

functions of a whole procedure $RR(x, x', L^n)$, where $L^n$ is the vector

$(L_1^n, \ldots, L_k^n)$.

Putting all this together, we obtain the following reduction

of ranking function synthesis to a first-order quantifier

elimination problem over templates:

$$\exists L^n : \forall x, x' : Inv(x) \land Trans(x, x') \implies RR(x, x', L^n)$$

To complete the lattice of ranking constraints $LR_i^n$, we add the special value $\top$ to the domain of $L_i^n$. We define

$LR_i^n(x, x', \top) \equiv \text{true}$ indicating that no ranking function has been found for the given template (“don’t know”). We write $\bot$ for the equivalence class of bottom elements for which $LR_i^n(x, x', L_i^n)$ evaluates to false, meaning that the ranking function has not yet been computed. For example, $\bot$ is a bottom element. Note that this intuitively corresponds to the meaning of $\bot$ and $\top$ as known from invariant inference by abstract interpretation (see Sec. 4).

**Algorithm 4: compTermArg**

**Input:** procedure $f$ with invariant $Inv$, bound on number of

lexicographic components $N$

**Output:** ranking constraint $RR$

1. $n_i \leftarrow 1^k$; $A^n \leftarrow \bot^k$; $M \leftarrow \emptyset$;
2. let $\varphi = \text{Inv}(x) \land \text{Trans}(x, x')$
3. while true do
4. let $\psi = \varphi \land \neg RR(x, x', A^n)$;
5. solve $\psi$ for $x, x'$;
6. if UNSAT then return $RR(x, x', A^n)$ let $(\chi, \chi')$ be a

model of $\psi$;
7. let $i \in \{i \mid (\varphi_i \Rightarrow LR_i^n(x, x', A^n))\}$;
8. $M_i \leftarrow M_i \cup \{(\chi, \chi')\}$;
9. let $\theta = \bigwedge_{(\chi, \chi') \in M} LR_i^n(x, x', L_i^n)$;
10. solve $\theta$ for $L_i^n$;
11. if UNSAT then
12. if $n_i < N$ then $n_i \leftarrow n_i + 1$; $\Lambda^n_i = \bot$; $M_i = \emptyset$
13. else return $RR(x, x', \tau^n)$
14. else let $m$ be a model of $\theta$;
15. $\Lambda^n_i \leftarrow m$;

In each iteration, our algorithm checks the validity of the current ranking function candidate. If it is not yet a valid ranking function, the SMT solver returns a counterexample transition. Then, a new ranking function candidate that satisfies all previously observed counterexample transitions is computed. This process is bound to terminate because the finiteness of the state space.

We start from the bottom element (Line 11) for ranking functions with a single component (Line 1) and solve the corresponding formula $\psi$, which is $true \land Trans((x, y), (x', y')) \land \neg false$ (Line 5). $\psi$ is satisfiable with the model $(1, 100, 0, 100)$ for $(x, y, x', y')$, for instance. This model entails the constraint $((1^2 + 100^2) - (0^2 + 100^2)) > 0$, i.e. $L^2_i > 0$, in Line 9 from which we compute values for the template coefficients $\ell_x$ and $\ell_y$. This formula is given to the solver (Line 10) which reports SAT with the model $(1, 0)$ for $(\ell_x, \ell_y)$, for example. We use this model to update the vector of template parameter candidates $\Lambda^1_i$ to $(0, 0)$ (Line 15), which corresponds to the ranking function $x$.

We then continue with the next loop iteration, and check the current ranking function (Line 5). The formula $true \land Trans \land \neg (x - x' > 0)$ is satisfiable by the model $(1, 1001, 2)$ for $(x, y, x', y')$, for instance. This model entails the constraint $1001^2 > 0$, i.e. $10000^2 - 100^2 > 0$ in Line 9 which is conjuncted with the constraint $L^2_i > 0$ from the previous iteration. The solver (Line 10) will tell us that this is UNSAT.

Since we could not find a ranking function we add another component to the lexicographic ranking function template (Line 12), and try to solve again (Line 5) and obtain the model $(1, 99, 0, 100)$ for $(x, y, x', y')$, for instance. Then in Line 10 the solver might report that the model $(0, -1, 1, 0)$ for $(\ell_x^2, \ell_y^2, \ell_x^L, \ell_y^L)$ satisfies the constraints. We use this model to update the ranking function to $(-y, x)$.

Finally, we check whether there is another witness for

$(-y, x)$ not being a ranking function (Line 5), but this time

the solver reports the formula to be UNSAT and the algorithm

terminates, returning the ranking function $(-y, x)$.