On the moduli space of hypersurfaces singular along a subscheme of large dimension but small degree

Kaloyan Slavov

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Abstract

Let \( k \) be an algebraically closed field. Fix integers \( n \) and \( b \) with \( n \geq 3 \) and \( 1 \leq b \leq n - 1 \). Let \( T^d_k \) be the moduli space of hypersurfaces \([F]\) in \( \mathbb{P}^n_k \) of degree \( l \) whose singular locus contains a subscheme of dimension \( b \) with Hilbert polynomial among the Hilbert polynomials of \( b \)-dimensional integral closed subschemes of \( \mathbb{P}^n \) of degree \( d \). We prove that when \( l \) is sufficiently large and \( 2 \leq d \leq \frac{l+1}{2} \), any irreducible component \( Z \) of \( T^d_k \) satisfies \( Z = T^1_k \) or \( \dim Z < \dim T^1_k \).

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1 Introduction

Let \( n \) and \( b \) be fixed integers with \( n \geq 3 \) and \( 1 \leq b \leq n - 1 \), and let \( k \) be an algebraically closed field. Fix a positive integer \( l \). Inside the projective space of all hypersurfaces
in \( \mathbb{P}^n \) of degree \( l \), consider the ones which are singular along some \( b \)-dimensional closed subscheme,

\[
X = \{ [F] \in \mathbb{P}(k[x_0, \ldots, x_n]) \mid \dim V(F)_{\text{sing}} \geq b \}
\]

(this is a closed subset).

A simple argument (Lemma 5.1) will show that

\[
X^1 := \{ [F] \in X \mid L \subset V(F)_{\text{sing}} \text{ for some linear } b\text{-dimensional } L \subset \mathbb{P}^n \}
\]
is an irreducible closed subset of \( X \) of dimension \((l+n) - a_{n,b}(l)\), where

\[
a_{n,b}(l) := \left(\frac{l+b}{b}\right) + (n-b)\left(\frac{l-1+b}{b}\right) + 1 - (b+1)(n-b)
\]

\[
= \frac{n-b+1}{b!} l^b + \ldots.
\]

Define \( T^d_k \) as the closed subset of \( \mathbb{P}(k[x_0, \ldots, x_n]) \) consisting of all hypersurfaces \([F]\) such that \( V(F)_{\text{sing}} \) contains a \( b \)-dimensional closed subscheme whose Hilbert polynomial is among the Hilbert polynomials of integral \( b \)-dimensional closed subschemes of degree \( d \). Note that \( T^1_k = X^1 \). The goal of this paper is to prove the following

**Theorem 1.1.** There exists \( l_0 = l_0(n,b) \) (easily computable) such that for all pairs \((d,l)\) with \( 2 \leq d \leq \frac{l+b}{2} \) and \( l \geq l_0 \), the following holds: if \( Z \subset T^d_k \) is an irreducible component, then either \( Z = X^1 \), or \( \dim Z < \dim X^1 \).

This is the first step (“case of small degree \( d \)”) towards the theorem below, which will be proved in a subsequent paper ([6]):

**Theorem 1.2.** There exists an integer \( l_0 = l_0(n,b, \text{char } k) \), such that for all \( l \geq l_0 \), \( X^1 \) is the unique irreducible component of \( X \) of maximal dimension.

In the proof of Theorem 1.1, we assume a conjecture by Eisenbud and Harris in the case \( b \geq 2 \). The proof of Theorem 1.1 will give a simple procedure to compute a possible value of \( l_0 \), given \( n \) and \( b \). In addition, in this paper, we prove a result analogous to Theorem 1.1 but regarding the second largest component of \( X \). Again in [6], we will use this result to show that for large \( l \), the second largest component of \( X \) comes from the hypersurfaces singular along an integral closed subscheme of degree 2, at least when \( \text{char } k > 0 \).

We now sketch the main idea of the proof. Let \( \text{Hilb}^d \) denote the disjoint union of the finitely many Hilbert schemes \( \text{Hilb}_{\mathbb{P}^n}^d \), where \( P_\alpha \) ranges over the Hilbert polynomials of integral \( b \)-dimensional closed subschemes \( C \subset \mathbb{P}^n \) of degree \( d \), and define the restricted Hilbert scheme \( \overline{\text{Hilb}}^d \) as the closure in \( \text{Hilb}^d \) of the set of points corresponding to integral subschemes. Let \( V = k[x_0, \ldots, x_n] \). Consider the incidence correspondence

\[
\tilde{\Omega}^d = \{ (C, [F]) \in \overline{\text{Hilb}}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}} \}.
\]

We will show\(^1\) that for \( 2 \leq d \leq \frac{l+b}{2} \) (“small” degree), any irreducible component of \( \tilde{\Omega}^d \) has dimension less than \( \dim X^1 \). For this, we apply the theorem on dimension of fibers to

\(^1\)We are going to be slightly imprecise here; see Section 5.3 for the exact statement.
the map \( \pi : \widetilde{O}^d \to \widetilde{\text{Hilb}}^d \). A result of Eisenbud and Harris gives \( \dim \widetilde{\text{Hilb}}^d \) when \( b = 1 \); for \( b > 1 \), they state a conjecture for the corresponding result. (We assume this conjecture but also note that our proof can be modified to give an alternative unconditional — but ineffective — proof of a weaker version of Theorem 1.1 that will still suffice for Theorem 1.2) So it remains to give an upper bound for the dimension of the fiber of \( \pi \) over an integral \( C \) of degree \( d \). For this, we specialize \( C \) to a union of \( d \) \( b \)-dimensional linear subspaces that contain a common \((b - 1)\)-dimensional linear subspace.

## 2 Notation

For a field \( k \), the graded ring \( k[x_0, \ldots, x_n] \) will be denoted by \( S \). For a graded \( S \)-module \( M \) (in particular, for a homogeneous ideal), \( M_l \) will denote the \( l \)-th graded piece of \( M \). When \( I \subset S \) is a homogeneous ideal, \((P^2)_l \) is denoted simply by \( P^2_l \). When the field \( k \) and the integer \( l \) are fixed, \( V \) will denote the vector space \( V = k[x_0, \ldots, x_n] \).

For a finite-dimensional \( k \)-vector space \( V \), \( \mathbb{P}(V) \) denotes the projective space parametrizing lines in \( V \), so for a \( k \)-scheme \( S \), \( \text{Hom}_{\text{Sch}/k}(S, \mathbb{P}(V)) \) consists of a line bundle \( L \) on \( S \), together with an injective bundle map (i.e., with locally free cokernel) \( L \hookrightarrow V \otimes_k \mathcal{O}_S \). Given a homogeneous ideal \( I \subset k[x_0, \ldots, x_n] \), \( V(I) \) denotes the closed subscheme \( \text{Proj}(k[x_0, \ldots, x_n]/I) \hookrightarrow \mathbb{P}^n_k \), and for \( i = 0, \ldots, n \), \( D_+(x_i) \) is the complement of \( V(x_i) \). We often abbreviate \( V(\{G_i\}_{i \in I}) \subset \mathbb{P}^n \) as \( V(G_i) \), when the index set \( I \) is irrelevant or understood.

For \( F \in S_l \), \( V(F)_{\text{sing}} \subset \mathbb{P}^n \) is the closed subscheme \( V(F, \frac{F}{x_i}) = V(F, \frac{F}{x_0}, \ldots, \frac{F}{x_n}) \) of \( \mathbb{P}^n \), so when \( F \neq 0 \), the underlying topological space of \( V(F)_{\text{sing}} \) is the singular locus of \( V(F) \).

If \( C \hookrightarrow \mathbb{P}^n \) is a closed subscheme of dimension \( b \) and Hilbert polynomial \( P_C(z) = \frac{d}{b} z^b + \ldots \), we say that \( C \) has degree \( d \).

We will reuse \( l_0 \) for different bounds as we go along, in order to avoid unnecessary notation; however, it will be clear that we are actually referring to different values of \( l_0 \) even though we use the same symbol. Also, it will be understood that sometimes the value of \( l_0 \) is the maximum of a finite set of previously defined bounds, each of them still denoted by \( l_0 \).

When \( X \) is a scheme of finite type over an algebraically closed field, we often identify \( X \) with its set of closed points, since most of our arguments will be just on the level of closed points. So when we say “\( x \in X \),” we usually refer to a closed point \( x \in X \) (this will be clear from the context).

## 3 The incidence correspondence

The goal of this section is to prove that the incidence correspondence is a closed subset of the product \( \text{Hilb}^P \times \mathbb{P}(k[x_0, \ldots, x_n]) \) (Corollary 3.2) and to define the moduli spaces \( T^P \to \text{Spec} \, \mathbb{Z} \) (defined at the end of the section). For the sake of the proof of just Theorem 1.1, it would suffice to carry the discussion of this section over \( \text{Spec} \, k \). However, the reason we want to work in the universal setting over \( \text{Spec} \, \mathbb{Z} \) is that in the subsequent paper \( \text{[6]} \) we will use upper-semicontinuity to compare \( \dim T^P_{Q^2} \) with \( \dim T^P_{F^2} \).
Recall that if $Y_0$ is a scheme and $\alpha: \mathcal{E}_1 \to \mathcal{E}_2$ is a map of vector bundles on $Y_0$, the functor Van. Loc. $\alpha: \text{Sch}^{op} \to \text{Sets}$ given by

$$\text{Van. Loc. } \alpha(S) = \{ t: S \to Y_0 \mid t^* \alpha = 0 \}$$

is representable, by a closed subscheme of $Y_0$. If $U = \text{Spec } A$ is an affine open $U \subset Y_0$ on which $\mathcal{E}_1, \mathcal{E}_2$ are trivial, so the map $\alpha: A^{r_1} \to A^{r_2}$ on $U$ is given by an $r_2 \times r_1$ matrix $(f_{ij})$ with entries in $A$, then $(\text{Van. Loc. } \alpha) \cap U \to U$ is given by the closed embedding $\text{Spec}(A/(f_{ij})) \hookrightarrow \text{Spec}(A)$. If $F \in \mathbb{Z}[x_0, \ldots, x_n]_l$ is a homogeneous polynomial of degree $l$, it gives rise to a map $\beta: \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(l)$; then the functor Van. Loc. $\beta$ is represented by the closed subscheme $V(F) \subset \mathbb{P}^n$.

Let $l \geq 1$ be an integer, and let $V = \mathbb{Z}[x_0, \ldots, x_n]$. For $F \in V$, we can describe the map $\beta$ above as the composition

$$\mathcal{O}_{\mathbb{P}^n} \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(l),$$

where the first map is given by $F \in V = \Gamma(\mathbb{P}^n, V \otimes \mathcal{O}_{\mathbb{P}^n})$ and the second one is the canonical map.

Let $V' = \mathbb{Z}[x_0, \ldots, x_n]_{l-1}$. Consider the linear maps $D_i: V \to V', F \mapsto \frac{F}{x_i}$ for $i = 0, \ldots, n$, and fix a nonzero polynomial $P \in \mathbb{Q}[z]$. The functor $\text{Hilb}^{P}_n \times \mathbb{P}(V): \text{Sch}^{op} \to \text{Sets}$ is given as follows: an element of $\text{Hilb}^{P}_n \times \mathbb{P}(V)(S)$ consists of a closed subscheme $X \hookrightarrow \mathbb{P}^n_S$ such that the composition $X \hookrightarrow \mathbb{P}^n_S \to S$ is flat and each fiber has Hilbert polynomial equal to $P$, together with a line bundle $L$ on $S$ and an injective bundle map $\alpha: L \to V \otimes \mathcal{O}_S$.

A map $\alpha: L \to V \otimes \mathcal{O}_S$ induces maps $\alpha_i: L \to V \otimes \mathcal{O}_S \xrightarrow{D_i \otimes \text{id}} V' \otimes \mathcal{O}_S$, for $i = 0, \ldots, n$. Let $\gamma: V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(l)$ and $\gamma': V' \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(l-1)$ be the canonical maps. Since the pullback to $\mathbb{P}^n_S$ of the target of $\alpha$ coincides with the pullback of the source of $\gamma$ (similarly for $\alpha_i$ and $\gamma'$),

$$\begin{array}{ccc}
X & \xrightarrow{r} & \mathbb{P}^n_S \\
\downarrow & & \downarrow \pi \\
S & & \mathbb{P}^n_Z
\end{array}$$

we can form the compositions

$$\varepsilon: \pi^* L \xrightarrow{\pi^* \alpha} V \otimes \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{r* \gamma} \mathcal{O}_{\mathbb{P}^n_S}(l)$$

$$\varepsilon_i: \pi^* L \xrightarrow{\pi^* \alpha_i} V' \otimes \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{r' \gamma'} \mathcal{O}_{\mathbb{P}^n_S}(l-1),$$

which are maps of line bundles on $\mathbb{P}^n_S$. Thus, for any $(X \hookrightarrow \mathbb{P}^n_S, L, \alpha: L \to V \otimes \mathcal{O}_S) \in \text{Hilb}^{P}_n \times \mathbb{P}(V)(S)$, we have attached maps $\varepsilon, \varepsilon_i, i = 0, \ldots, n$ of line bundles on $\mathbb{P}^n_S$.

Consider the subfunctor $\mathcal{F}: \text{Sch}^{op} \to \text{Sets}$ of the (representable) functor $\text{Hilb}^{P}_n \times \mathbb{P}(V)$, given as follows: $\mathcal{F}(S)$ is the set of all $(X \hookrightarrow \mathbb{P}^n_S, L, \alpha: L \to V \otimes \mathcal{O}_S) \in \text{Hilb}^{P}_n \times \mathbb{P}(V)(S)$ such that the pullback of $\varepsilon$ and each $\varepsilon_i$ (for $i = 0, \ldots, n$) to $X$ vanishes.

$$\begin{array}{ccc}
X & \xrightarrow{r} & \mathbb{P}^n_S \\
\downarrow & & \\
S
\end{array}$$
Proposition 3.1. The functor $\mathcal{F}$ is representable by a closed subscheme $\Omega^P$ of $\text{Hilb}^P \times \mathbb{P}(V)$.

Proof. Consider the scheme $Y = \text{Hilb}^P \times \mathbb{P}(V)$, and let $(X \hookrightarrow \mathbb{P}^N_Y, \lambda, \alpha : L \to V \otimes_k \mathcal{O}_Y)$ be the tautological element of $\text{Hilb}^P \times \mathbb{P}(V)(Y)$. This gives rise to maps $\varepsilon, \varepsilon_i$ of line bundles on $\mathbb{P}^N_Y$. Let $\tilde{\varepsilon}, \tilde{\varepsilon}_i$ be the pullbacks of $\varepsilon, \varepsilon_i$ to $X$.

For a scheme $S$, $\mathcal{F}(S)$ consists of all maps $S \to Y$ such that the maps of line bundles $\tilde{\varepsilon}, \tilde{\varepsilon}_i$ on $X$ pull back to zero on $X \times_Y S$. Since $Y$ is noetherian and the morphism $X \to Y$ is flat and projective, this functor is representable, by a closed subscheme of $Y$ (see Theorem 5.8 and Remark 5.9 in [4]).

If $k$ is an algebraically closed field and $\Omega^P_k$ denotes the basechange $\Omega^P \times \text{Spec} k$, we know the set of closed points of $\Omega^P_k$:

$\text{Hom}_{\text{Sch}/k}(\text{Spec } k, \Omega^P_k) = \mathcal{F}(\text{Spec } k)$.

From the definitions, this is just

$$\left\{(C, [F]) \in \text{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(k[x_0, \ldots, x_n]) \mid C \subset V \left( F, \frac{\partial F}{\partial x_i} \right) \right\}$$

(inclusion above denotes scheme-theoretic inclusion).

Corollary 3.2. Let $k$ be an algebraically closed field, $l \geq 1$ an integer, and $P \in \mathbb{Q}[z]$ a polynomial. The set

$$\left\{(C, [F]) \in \text{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(k[x_0, \ldots, x_n]) \mid C \subset V \left( F, \frac{\partial F}{\partial x_i} \right) \right\}$$

is a closed subset of (the set of closed points of) $\text{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(k[x_0, \ldots, x_n])$.

Let $T^P$ denote the scheme-theoretic image of $\Omega^P \to \mathbb{P}(V)$, so we have a diagram

$$\begin{array}{ccc}
\Omega^P & \hookrightarrow & \text{Hilb}^P \times \mathbb{P}(V) \\
\downarrow & & \downarrow \\
T^P & \rightarrow & \mathbb{P}(V).
\end{array}$$

Since surjections and closed embeddings are stable under base-change, for any algebraically closed field $k$, we have a corresponding diagram

$$\begin{array}{ccc}
\Omega^P_k & \hookrightarrow & \text{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(V_k) \\
\downarrow & & \downarrow \\
T^P_k & \rightarrow & \mathbb{P}(V_k)
\end{array}$$

(where $V_k = V \otimes_{\mathbb{Z}} k = k[x_0, \ldots, x_n]$) and by looking at closed points, it follows that

$$T^P_k = \{[F] \in \mathbb{P}(V_k) \mid V(F)_{\text{sing}} \text{ contains a subscheme with Hilbert polynomial } P \}.$$
4 Specialization arguments

The main technique that we use in the proof of Theorem 1.1 is a specialization argument, that allows us to bound \( \dim \{ F \in k[x_0, ..., x_n] | C \subset V(F)_{\text{sing}} \} \) from above for a fixed \( C \), by degenerating \( C \) to a union of linear spaces. In Section 4.1, we prove (for lack of reference) that we can specialize a \( b \)-dimensional integral closed subscheme \( C \) of \( \mathbb{P}^n \) to a union of \( d \) \( b \)-dimensional linear spaces containing a common \( (b-1) \)-dimensional linear space. Next, the bound we obtain in Section 4.2 will be the main ingredient for the proof of the main theorem in Section 5.

In this section, \( k \) is a fixed algebraically closed field.

4.1 Specialization of a closed subscheme to a union of linear subspaces

The result of this section is known, but we were unable to find a reference, so we include it here.

Let \( C \subset \mathbb{P}^n \) be an integral \( b \)-dimensional closed subscheme of degree \( d \). Let \( P = V(x_0, ..., x_n) \) be the \( (b-1) \)-dimensional “linear subspace at infinity.” Suppose that the linear subspace \( H = V(x_{n-b+1}, ..., x_n) \) intersects \( C \) in \( d \) distinct points \( Q_i \). Let \( L_i \) be the unique \( b \)-dimensional linear space through \( P \) and \( Q_i \). The \( L_i \) are distinct because if \( L_i = L_j \) for some \( i \neq j \), the line through \( Q_i \) and \( Q_j \) would be contained in \( H \) but would have to intersect \( P \); this is impossible, since \( P \cap H = \emptyset \). Consider the projective linear transformations

\[
A_a = \begin{pmatrix}
a \\
\vdots \\
a \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

(where the bottom block has size \( b \times b \)) and let \( C_a = A_a C \).

**Proposition 4.1.** The underlying topological space of the flat limit \( C_0 = \lim_{a \rightarrow 0} C_a \) is \( \bigcup_{i=1}^{d} L_i \).

**Proof.** Let \( C = V(\{ G_s \}) \subset \mathbb{P}^n \) (as a scheme), where \( G_s \in k[x_0, ..., x_n] \) are homogeneous. Consider the map

\[
\sigma : \mathbb{P}^n \times (\mathbb{A}^1 - \{ 0 \}) \rightarrow \mathbb{P}^n, \quad ([x_0, ..., x_n], a) \mapsto (x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n),
\]

and define the closed subscheme \( X \subset \mathbb{P}^n \times (\mathbb{A}^1 - \{ 0 \}) \) as the fiber product

\[
\begin{array}{ccc}
X & \rightarrow & \mathbb{P}^n \times (\mathbb{A}^1 - \{ 0 \}) \\
\downarrow & & \downarrow \sigma \\
C & \rightarrow & \mathbb{P}^n.
\end{array}
\]

In other words,

\[
X = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n_{\mathbb{A}^1 - \{ 0 \}},
\]

6
where we regard $G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n) \in k[a, a^{-1}][x_0, ..., x_n]$. This is a flat family $X \to \mathbb{A}^1 - \{0\}$, whose fiber over $a \neq 0$ is $C_a$ (as a subscheme of $\mathbb{P}^n$).

Let $\overline{X}$ be the scheme-theoretic closure of $X$ in $\mathbb{P}^n \times \mathbb{A}^1$. By the proof of Proposition III.9.8 in [5], the flat limit of the family $(C_a)$ is the scheme-theoretic fiber $\overline{X}_0$.

Consider $Y = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n \times \mathbb{A}^1$. Then $Y$ is a closed subscheme of $\mathbb{P}^n \times \mathbb{A}^1$ containing $X_0$ (scheme-theoretically), so $Y$ contains $\overline{X}$. Thus, $\overline{X}_0 \subset Y_0$ is a closed subscheme.

![Diagram]

We have $Y_0 = V(G_s(x_0, ..., x_{n-b}, 0, ..., 0)) \subset \mathbb{P}^n$. Thus, as a set, $Y_0$ is $\bigcup_{i=1}^d L_i$.

We claim that $Y_0$ is reduced away from $P$. Equivalently, for $i = 0, ..., n-b$, we have to check that $Y_0 \cap D_+(x_i)$ is reduced. To simplify notation, suppose that $i = 0$. Then

$$Y_0 \cap D_+(x_0) = \text{Spec} \frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_{n-b}, 0, ..., 0))} = \text{Spec} \frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)} [x'_{n-b+1}, ..., x'_n].$$

So we have to show that the 0-dimensional ring

$$\frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is reduced. We have assumed that $C$ intersects $V(x_{n-b+1}, ..., x_n)$ transversely, so

$$\text{Proj} \frac{k[x_0, ..., x_n]}{(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is a reduced 0-dimensional scheme; looking at its intersection with $D_+(x_0)$, we obtain the desired conclusion.

Now that $Y_0$ is reduced away from a subscheme of smaller dimension, it follows that the Hilbert polynomial of $Y_0$ has the same degree and leading coefficient (namely, $b$ and $d/b!$, respectively) as the Hilbert polynomial of $(Y_0)_{\text{red}}$. The Hilbert polynomial of the flat limit $\overline{X}_0$ also has degree $b$ and leading coefficient $d/b!$. Moreover, $Y_0$ is equidimensional, so the inclusion $\overline{X}_0 \hookrightarrow Y_0$ must be a homeomorphism. □
Remark 4.2. The proof above does not imply that $Y_0$ is reduced everywhere. Let us look at $Y_0$ in the chart $D_+(x_n)$, so

$$Y_0 \cap D_+(x_n) = \text{Spec} \frac{k[x_0, \ldots, x_{n-1}]}{(G_s(x_0, \ldots, x_{n-b}, 0, \ldots, 0))} = \text{Spec} \frac{k[x_0, \ldots, x_n]}{(G_s(x_0, \ldots, x_n), x_{n-b+1}, \ldots, x_n)} [x'_{n-b+1}, \ldots, x'_{n-1}].$$

Let $S = k[x_0, \ldots, x_n]/(G_s(x_0, \ldots, x_n), x_{n-b+1}, \ldots, x_n)$. We know that $\text{Proj} S$ is reduced as a scheme by the transversality assumption on $C \cap H$; however, this does not in general imply that $S$ itself is reduced as a ring.

Let $V = k[x_0, \ldots, x_n]$. For each closed subscheme $C \subset \mathbb{P}^n$, define the $k$-vector space

$$W_C = \{ F \in V \mid C \subset V(F_{\text{sing}}) \}.$$

**Corollary 4.3.** Let $C \hookrightarrow \mathbb{P}^n$ be an integral closed subscheme of dimension $b$ and degree $d$. There exist $d$ $b$-dimensional linear subspaces $L_1, \ldots, L_d$ of $\mathbb{P}^n$ containing a common $(b-1)$-dimensional linear subspace, such that

$$\dim W_C \leq \dim W_{\cup L_i},$$

where $\cup L_i$ is given the reduced induced structure.

**Proof.** Let $P$ be the Hilbert polynomial of $C$. Recall the incidence correspondence from Corollary 5.2 and apply the upper semicontinuity theorem (see Section 14.3 in [2]) to the map

$$\{(C, [F]) \in \text{Hilb}^P \times \mathbb{P}(V) \mid C \subset V(F_{\text{sing}}) \} \xrightarrow{\pi} \text{Hilb}^P.$$

By Proposition 4.41 $\cup L_i$ (with some scheme structure) is the flat limit $C_0$ of a family $(C_a)$, with each $C_a$ ($a \neq 0$) being projectively equivalent to $C = C_1$, and hence $\pi^{-1}(C_a) \simeq \pi^{-1}(C)$ for each $a \neq 0$. Therefore,

$$\dim \mathbb{P}(W_C) = \dim \pi^{-1}(C) \leq \dim \pi^{-1}(C_0) = \dim \mathbb{P}(W_{C_0}) \leq \dim \mathbb{P}(W_{\cup L_i}).$$

\[\Box\]

4.2 An upper bound on the dimension of the space of $F$ such that $C \subset V(F_{\text{sing}})$, for a fixed $C$ of small degree

Fix a positive integer $l$. Recall the notation $V = k[x_0, \ldots, x_n]_l$.

**Lemma 4.4.** Let $L \subset \mathbb{P}^n$ be a $b$-dimensional linear subspace. Then for $F \in V$, we have $L \subset V(F_{\text{sing}})$ if and only if $F \in I_L^2$. Moreover,

$$\text{codim}_V \{ F \in V \mid L \subset V(F_{\text{sing}}) \} = \binom{l+b}{b} + (n-b)\binom{l-1+b}{b}.$$
Proof. Without loss of generality, \( L = V(I) \) with \( I = (x_{b+1}, \ldots, x_n) \). For \( F \in V \), we claim that \( (F, \frac{F}{X}) \subset I \) if and only if \( F \in I^2 \). Suppose that \( (F, \frac{F}{X}) \subset I \). Write \( F = F_0 + \sum_{i=b+1}^n F_i x_i + T \), where \( F_0, F_i \in \langle x_{a_0}, \ldots, x_a \rangle \) are homogeneous of degrees \( l, l-1 \) respectively, and \( T \in I^2_l \). Since \( \frac{X}{X} \in I \) for all \( i \), we can assume without loss of generality that \( T = 0 \). Now, the condition \( \frac{F}{X} \subset I \) for \( i = b+1, \ldots, n \) implies \( F_i \in I \cap k[x_{a_0}, \ldots, x_a] = 0 \), so \( F_i = 0 \). Then \( F = F_0 \in I \cap k[x_{a_0}, \ldots, x_a] = 0 \), so \( F = 0 \) overall, as desired. Clearly, \( (S/I^2) \simeq k[x_{a_0}, \ldots, x_a] \oplus (L \oplus k[x_{a_0}, \ldots, x_a]) \) has dimension as in the statement. \( \square \)

Lemma 4.5. Let \( L_1, \ldots, L_d \) be \( d \) \( b \)-dimensional linear subspaces of \( \mathbb{P}^n \) containing a common \((b-1)\)-dimensional linear subspace. Then for \( d \leq \frac{b+1}{2} \), we have

\[
\text{codim}_{V}(W_{\cup L_i}) \geq \binom{l+b}{b} + (n-b) \sum_{e=1}^{d} \binom{l-2e+1+b}{b}.
\]

Proof. We induct on \( d \). For \( d = 1 \), we have equality. Assume \( 2 \leq d \leq \frac{b+1}{2} \). Assume that the \( b \)-dimensional linear subspaces \( L_1, \ldots, L_d \) all contain \( P = [0, *, \ldots, *, 0, 0, 0] \) and that none of them is contained in the hyperplane \( x_0 = 0 \), so the ideal of each of them is of the form \( (x_{b+1} - p_{b+1} x_{a_0}, \ldots, x_n - p_n x_0) \) for a uniquely determined tuple \((p_{b+1}, \ldots, p_n) \in k^{n-b} \). Let

\[
I_i = (x_{b+1} - p_{b+1} x_{a_0}, \ldots, x_{b+2} - p_{b+2} x_{a_0}, \ldots, x_n - p_n x_0) \quad \text{for} \quad i = 1, \ldots, d-1,
\]

and without loss of generality

\[
I_d = (x_{b+1}, \ldots, x_n).
\]

By Lemma 1.4, \( W_{\cup L_i} = (I_d^2 \cap \cdots \cap I_{i+1}^2) \), so we have to give a lower bound for \( \dim(S/I_d^2 \cap \cdots \cap I_{i+1}^2) \). For \( e \in \{d-1, d\} \), let \( \mu_e = \dim(S/I_d^2 \cap \cdots \cap I_{i+1}^2) \). There is a short exact sequence

\[
0 \rightarrow \left( I_d^2 \cap \cdots \cap I_{i+1}^2 \right) \rightarrow \left( \frac{S}{I_d^2 \cap \cdots \cap I_{i+1}^2} \right) \rightarrow 0.
\]

So we have to write down enough linearly independent elements in \((I_d^2 \cap \cdots \cap I_{i+1}^2) \).

For each \( i = 1, \ldots, d-1 \), there exists \( m_i \in \{b+1, \ldots, n\} \) such that \( p_{m_i}^{(i)} \neq 0 \). Let

\[
F = \prod_{i=1}^{d-1} (x_{m_i} - p_{m_i}^{(i)})^2.
\]

Consider all elements

\[
Fx_j P(x_{a_0}, \ldots, x_a) \in \left( \frac{I_d^2 \cap \cdots \cap I_{i+1}^2}{I_d^2 \cap \cdots \cap I_{i+1}^2} \right)
\]

where \( j \in \{b+1, \ldots, n\} \) and \( P(x_{a_0}, \ldots, x_a) \) runs through a basis of \( k[x_{a_0}, \ldots, x_a]_{-2d+1} \). Their number is \((n-b)(l-2d+1+b)\) and we claim that they are all linearly independent. Indeed, it suffices to check that their images under the injection \((I_d^2 \cap \cdots \cap I_{i+1}^2) \rightarrow (S/I_d^2) \) become linearly independent. This is evident, however, since \( (S/I_d^2) \simeq k[x_{a_0}, \ldots, x_a] \oplus k[x_{a_0}, \ldots, x_a]_{-2d+1} \oplus \cdots \oplus k[x_{a_0}, \ldots, x_a]_{-2d+1} x_n \) as \( k \)-vector spaces, and the images of the elements under consideration are

\[
(p_{m_1}^{(1)})^2 \cdots (p_{m_{d-1}}^{(d-1)})^2 x_j P(x_{a_0}, \ldots, x_a).
\]
Therefore
\[ \mu_d \geq \mu_{d-1} + (n - b) \left( \frac{l - 2d + 1 + b}{b} \right), \]
and the statement follows by induction.

5 The case of small degree \( d \)

With the preparations from the previous section, it is now easy to handle the cases of small degree \( 2 \leq d \leq \frac{b+1}{2} \) and prove Theorem 1.1. The new ingredient here is a result of Eisenbud and Harris (conjectural for \( b \geq 2 \)), which gives the dimension of the restricted Hilbert scheme. So we can treat the cases of small degree \( d \) by applying the theorem on the dimension of fibers to the map \( \tilde{\Omega}^d \to \tilde{\text{Hilb}}^d \) (Section 5.3). Finally, in Section 6, we perform the analogous calculation for the second largest component of \( X \).

Again, \( k \) is a fixed algebraically closed field.

5.1 The component corresponding to \( d = 1 \)

The lemma below is simple, since any two linear \( b \)-dimensional subspaces of \( \mathbb{P}^n \) are projectively equivalent. Recall the definitions of \( X^1 \) and \( a_{n,b}(l) \) from the introduction. Let \( G(b, n) \) be the Grassmanian of projective linear \( b \)-dimensional subspaces of \( \mathbb{P}^n \).

**Lemma 5.1.** The set \( X^1 \) is an irreducible closed subset of \( X \) of dimension equal to \( A := \left( \frac{l+n}{n} \right) - a_{n,b}(l) \).

**Proof.** Consider
\[ \Omega^1 = \{(L, [F]) \in G(b, n) \times \mathbb{P}(V) \mid L \subset V(F)_{\text{sing}}\} \subset G(b, n) \times \mathbb{P}(V). \]
By Corollary 3.2, this is a closed subset of the product, since \( \Omega^1 = \Omega^P \) with \( P(z) = \binom{z+b}{b} \).

Let \( \pi: \Omega^1 \to G(b, n) \) and \( \rho: \Omega^1 \to \mathbb{P}(V) \) denote the two projections. The fiber of \( \pi \) over any linear \( b \)-dimensional \( L \) is \( \mathbb{P}(W_L) \). So \( \Omega^1 \) is irreducible, and has dimension \( \dim \mathbb{P}(W_L) + \dim G(b, n) = A \) (use Lemma 4.4).

Consider now \( \rho: \Omega^1 \to X^1 \). To prove that \( \Omega^1 \) and \( X^1 \) have the same dimension, it suffices to show that some fiber of \( \rho \) is 0-dimensional. If we take \( L = V(x_0, ..., x_{n-b-1}) \), look at \( F = \sum_{i=0}^{n-b-2} x_ix_{i+1} \) (in the case \( l \geq 3 \), which we can tacitly assume). Then \( L \) is the only \( b \)-dimensional linear subspace contained in \( V(F)_{\text{sing}} \). \( \square \)

5.2 The result of Eisenbud and Harris

We first recall (see [1], p. 3) the following classical result.

**Theorem 5.2** (Chow’s finiteness theorem). Fix positive integers \( n, b, d \). There are only finitely many Hilbert polynomials \( P_\alpha \) of integral \( b \)-dimensional closed subschemes of \( \mathbb{P}_k^n \) of degree \( d \). The algebraically closed field \( k \) varies as well in this statement.
Fix $k$. For an integer $d \geq 1$, let $\text{Hilb}_{b,d}^{l,n}$ be the disjoint union of the Hilbert schemes $\text{Hilb}_{b,d}^{l,n}$ for all the finitely many possible Hilbert polynomials $P_e$ of an integral $b$-dimensional closed subscheme $C \subset \mathbb{P}^n$ of degree $d$. Define the restricted Hilbert scheme $\tilde{\text{Hilb}}_{b,d}^{l,n}$ to be the Zariski closure in $\text{Hilb}_{b,d}^{l,n}$ of the set of integral subschemes, with reduced subscheme structure. Eisenbud and Harris [3] prove the following result for the dimension of $\tilde{\text{Hilb}}_{b,d}^{l,n}$ in the case $b = 1$.

**Theorem 5.3.** Let $b = 1$. For $d \geq 2$, the largest irreducible component of $\tilde{\text{Hilb}}_{b,d}^{l,n}$ is the one corresponding to the family of plane curves of degree $d$; in particular, $\dim \tilde{\text{Hilb}}_{b,d}^{l,n} = 3(n - 2) + \frac{d(d+3)}{2}$.

In analogy, for $b \geq 2$, Eisenbud and Harris state the following conjecture:

**Conjecture 5.4.** For $d \geq 2$, the largest irreducible component of $\tilde{\text{Hilb}}_{b,d}^{l,n}$ is the one corresponding to the family of degree-$d$ hypersurfaces contained in linear $(b + 1)$-dimensional subspaces of $\mathbb{P}^n$; in particular, $\dim \tilde{\text{Hilb}}_{b,d}^{l,n} = (b + 2)(n - b - 1) - 1 + \binom{d+b+1}{b+1}$.

From now on, we will be assuming that this conjecture holds, so the results we obtain will depend on it, except in the case $b = 1$.

From now on, we fix $b$ and $n$, and abbreviate $\tilde{\text{Hilb}}_{b,d}^{l,n}$ as $\tilde{\text{Hilb}}^d$.

Let $\Omega^d$ be the disjoint union of the finitely many $\Omega^d_{P_e}$ (notation as in Proposition 3.1). Also, define $T^d$ as the scheme-theoretic image of $\Omega^d \to \mathbb{P}(\mathbb{Z}[x_0, \ldots, x_n][l])$, so we have a diagram

\[
\begin{array}{ccc}
\Omega^d & \longrightarrow & \text{Hilb}^d \times \mathbb{P}(\mathbb{Z}[x_0, \ldots, x_n][l]) \\
& & \downarrow \\
& & T^d \longrightarrow \mathbb{P}(\mathbb{Z}[x_0, \ldots, x_n][l]).
\end{array}
\]

For any algebraically closed field $k$, we have

\[T^d_k = \bigcup T^d_{P_e} = \{ [F] \in \mathbb{P}(V_k) \mid V(F)_{\text{sing}} \text{ contains a subscheme with Hilbert polynomial among } \{P_e\} \}.
\]

Since $X^1 = T^d_1$, we can use $X^1$ and $T^d_1$ interchangeably.

### 5.3 The case $d \leq \frac{l+1}{2}$ (small degree)

Fix an integer $l$ as usual, and fix an integer $d > 1$. As usual, let $V = k[x_0, \ldots, x_n][l]$. Recall that

\[\tilde{\Omega}^d = \{(C,[F]) \in \text{Hilb}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}}\}.
\]

Define

\[R^d = \{(C,[F]) \in \text{Hilb}^d \times \mathbb{P}(V) \mid C \text{ is integral, } C \subset V(F)_{\text{sing}} \} \subset \tilde{\Omega}^d.
\]

Let $\overline{R^d}$ be the closure of $R^d$ inside $\tilde{\Omega}^d$ (or inside $\text{Hilb}^d \times \mathbb{P}(V)$). Let $\pi : \text{Hilb}^d \times \mathbb{P}(V) \to \text{Hilb}^d$ and $\rho : \text{Hilb}^d \times \mathbb{P}(V) \to \mathbb{P}(V)$ denote the first and second projections.
Lemma 5.5. There exists \( l_0 \) (easily computable) such that for all pairs \((d, l)\) with \( 2 \leq d \leq \frac{l+1}{2} \) and \( l \geq l_0 \), we have 
\[
\dim \bar{R}^d < \dim X^1.
\]

It follows that \( \dim \rho(R^d) < \dim X^1 \).

Proof. Let \( Z \) be an irreducible component of \( R^d \). Certainly, \( Z \cap R^d \neq \emptyset \), so \( \pi(Z) \) contains an integral subscheme \( C \subset \mathbb{P}^n \). Degenerate \( C \) to a union \( \bigcup_{i=1}^{d} L_i \) of \( d \)-dimensional linear spaces, as in Section 4.1. Let \( L_0 \) be any linear \( b \)-dimensional subspace of \( \mathbb{P}^n \). By abuse of notation, let \( \pi: Z \to \pi(Z) \subset \tilde{\text{Hilb}}^d \). By the theorem on the dimension of fibers, we have 
\[
\dim Z \leq \dim \pi^{-1}(C) + \dim \pi(Z) \leq \dim \mathbb{P}(W_C) + \dim \pi(Z) \leq \dim \mathbb{P}(W_{\cup L_i}) + \dim \tilde{\text{Hilb}}^d.
\]

Thus, it suffices to check that

\[
\dim \mathbb{P}(W_{\cup L_i}) + \dim \tilde{\text{Hilb}}^d < \dim \mathbb{P}(W_{L_0}) + (b + 1)(n - b) \tag{1}
\]

(recall Lemma 5.1), or, equivalently, that

\[
\text{codim}_V W_{L_0} + \dim \tilde{\text{Hilb}}^d < \text{codim}_V W_{\cup L_i} + (b + 1)(n - b).
\]

By Lemmas 4.4 and 4.5, it suffices to prove the inequality

\[
\binom{l + b}{b} + (n - b)\binom{l - 1 + b}{b} + \dim \tilde{\text{Hilb}}^d < \left( l + b \right) + (n - b)\sum_{e=1}^{d} \binom{l - 2e + 1 + b}{b} + (b + 1)(n - b),
\]

or, equivalently,

\[
\dim \tilde{\text{Hilb}}^d - (b + 1)(n - b) < (n - b)\sum_{e=2}^{d} \binom{l - 2e + 1 + b}{b}; \tag{2}
\]

for all \( 2 \leq d \leq \frac{l+1}{2} \) and \( l \geq l_0 \). Let \( c = (b + 2)(n - b - 1) - 1 - (b + 1)(n - b) \). Assume Conjecture 5.4 then (2) is equivalent to

\[
c + \binom{d + b + 1}{b + 1} < (n - b)\sum_{e=2}^{d} \binom{l - 2e + 1 + b}{b}; \tag{3}
\]

for all \( 2 \leq d \leq \frac{l+1}{2} \) and \( l \geq l_0 \).

For \( l \geq 2d - 1 \), the right hand side of (3) is at least
\[
(n - b)\sum_{e=2}^{d} \binom{2d - 2e + b}{b} = (n - b)\sum_{k=0}^{d-2} \binom{2k + b}{b} \quad \text{(where } k = d - e)\]
\[
= (n - b)\sum_{k=0}^{d-2} \frac{(2k + b)(2k + b - 1)\ldots(2k + 1)}{b!} \]
\[
= (n - b)\sum_{k=0}^{d-2} \left( \frac{2^k b^k}{b!} + \ldots \right).
\]
Recall that $\sum_{k=0}^{d} k^b$ is a polynomial in $d$ of degree $b+1$ and leading coefficient $\frac{1}{b+1}$, so the right hand side of (3) dominates a polynomial in $d$ of degree $b+1$ and leading coefficient $(n - b) \frac{2b}{b+1} = \frac{(n-b)^2}{(b+1)!}$. Since $(d+b+1)^2$ is a polynomial in $d$ of the same degree $b+1$, but smaller leading coefficient $\frac{1}{b+1}$, the inequality (3) holds for all $l \geq 2d-1$ and all $d > d_0$ for some $d_0$ (which is easy to calculate algorithmically, for fixed $n, b$).

On the other hand, for each fixed value $d = 2, \ldots, d_0$, the right hand side of (3) is a polynomial in $d$ of degree $b$ and positive leading coefficient $\frac{(n-b)(d-1)}{b}$, while the left hand side is a constant. So there is $l_0$ (easily computable for given $b, n, d_0$) such that for all $d = 2, \ldots, d_0$ and $l \geq l_0$, the inequality (3) holds true. Therefore, for all $2 \leq d \leq \frac{l_0}{2}$ and $l \geq l_0$, the inequality from the statement of the lemma holds, as well.

Let $l_0$ be as in Lemma 5.5.

**Corollary 5.6.** Let $2 \leq d \leq \frac{l_0}{2}$ and $l \geq l_0$. If $Z \subset T_k^d$ is an irreducible component, then either $Z = X^1$, or dim $Z < \text{dim } X^1$.

**Proof.** We claim that if $[F] \in T_k^d - (T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'}))$, then $V(F)_{\text{sing}}$ contains an integral $b$-dimensional subscheme of degree $d$. Indeed, $V(F)_{\text{sing}}$ contains some integral $b$-dimensional closed subscheme of degree $\tilde{d} \in \{1, \ldots, d\}$; if $[F] \notin \cup_{d'=1}^{d-1} T_k^{d'}$, then necessarily $\tilde{d} = d$.

Now, we can induct on $d$, so assume that $Z \notin \cup_{d'=1}^{d-1} T_k^{d'}$. Note that $Z - (Z \cap (\cup_{d'=1}^{d-1} T_k^{d'})) \subset Z$ is a dense open subset of $Z$, which therefore has the same dimension as $Z$, but is contained in $T_k^d - (T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'})) \subset \rho(R^d) \subset \rho(\overline{R}^d)$. Thus dim $Z \leq \text{dim } \rho(\overline{R}^d) < \text{dim } X^1$, by Lemma 5.5.

This completes the proof of Theorem 1.1.

We can obtain a weaker version that does not rely on the conjecture of Eisenbud and Harris:

**Lemma 5.7.** Fix an integer $B$. There exists $l_0$ such that for all $2 \leq d \leq B$ and $l \geq l_0$, for any irreducible component $Z$ of $T_k^d$, either $Z = X^1$, or dim $Z < \text{dim } X^1$.

**Proof.** Just note that inequality (2) in the proof of the previous lemma is satisfied when $d \in \{2, \ldots, B\}$ is fixed and $l \gg 0$.

6 On the second largest component of $X$

6.1 The existence of a component of $X$ of the expected second-largest dimension

In contrast to the treatment of the largest component of $X$, the existence of a component of the expected second-largest dimension is a little more subtle, so there will be an extra twist in the argument.

Again, $k$ is any algebraically closed field.

We begin with the following preparation. Consider a $b$-dimensional closed subscheme $C = V(f, x_{b+2}, \ldots, x_n)$ of $\mathbb{P}^n$, where $f \in k[x_0, \ldots, x_{b+1}] - \{0\}$, and set $W = (f, x_{b+2}, \ldots, x_n)^2$.

**Lemma 6.1.** Assume $l \geq 2d+1$. There is a dense open subset $U_1 \subset \mathbb{P}(W)$ such that for all $[F] \in U_1$, $V(F)_{\text{sing}} = C$ (set-theoretically).
Proof. Consider the incidence correspondence

\[ Y_1 = \{ ([f], P) \in \mathbb{P}(W) \times (\mathbb{P}^n - C) \mid P \in V(f)_{\text{sing}} \} \subset \mathbb{P}(W) \times (\mathbb{P}^n - C) \]

(it is a closed subset of this product, and hence a quasiprojective variety). We are going to show that \( \dim Y_1 < \dim \mathbb{P}(W) \); this will imply that the closure \( \overline{Y}_1 \) of \( Y_1 \) in \( \mathbb{P}(W) \times \mathbb{P}^n \) also has dimension smaller than that of \( \mathbb{P}(W) \), and thus the image of this closure under the projection to \( \mathbb{P}(W) \) will be a proper closed subset of \( \mathbb{P}(W) \). Its complement \( U_1 \) will satisfy the condition of the lemma.

Consider the second projection \( \tau: Y_1 \to \mathbb{P}^n - C \), and let \( P \in \mathbb{P}^n - C \). We claim the fiber \( \tau^{-1}(P) \) is a projective linear subspace of \( \mathbb{P}(W) \) of codimension \( n + 1 \). This will imply that \( Y_1 \) is irreducible, of dimension \( \dim Y_1 = \dim \mathbb{P}(W) - 1 \).

Suppose first that \( P \in \bigcup_{i=b+2}^{n} D_i(x_i) \). Without loss of generality, assume that \( P = [a_0, ..., a_{n-1}, 1] \). Notice that \( \tau^{-1}(P) \) is just

\[ \mathbb{P} \left( ((x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1} x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2) \right) \subset \mathbb{P}(W), \]

so it remains to show that

\[ \dim \left( \frac{W}{(x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1} x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2} \right)_l = n + 1, \]

i.e., that the map

\[ \left( \frac{(f, x_{b+2}, ..., x_n)^2}{(x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1} x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2} \right)_l \Rightarrow \frac{S}{(x_0 - a_0 x_n, ..., x_{n-1} - a_{n-1} x_n)^2} \simeq k[x_n]^l \oplus \bigoplus_{i=0}^{n-1} k[x_{l-1}] (x_i - a_i x_n) \]

is an isomorphism. The images of \( x_n^l \) and \( x_n^{l-1} (x_i - a_i x_n) \) for \( i = 0, ..., n - 1 \) give a basis of the target.

Suppose now that \( P \in V(x_{b+2}, ..., x_n) \), without loss of generality \( P = [1, a_1, ..., a_{b+1}, 0, ..., 0] \). As above, we have to prove that the following map is an isomorphism:

\[ \left( \frac{(f, x_{b+2}, ..., x_n)^2}{(x_1 - a_1 x_0, ..., x_{b+1} - a_{b+1} x_0, x_{b+2}, ..., x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2} \right)_l \Rightarrow \frac{S}{(x_1 - a_1 x_0, ..., x_{b+1} - a_{b+1} x_0, x_{b+2}, ..., x_n)^2} \simeq k[x_0]^l \oplus \bigoplus_{i=1}^{b+1} k[x_0]_{l-1} (x_i - a_i x_0) \oplus \bigoplus_{i=b+2}^{n} k[x_{l-1}] (x_i - a_i x_0). \]

Now, dehomogenize \( f \) with respect to \( x_0 \), consider a Taylor expansion at \( (a_1, ..., a_{b+1}) \), and homogenize to degree \( l \) again, so \( f \equiv a x_0^l \) (mod \( (x_1 - a_1 x_0, ..., x_{b+1} - a_{b+1} x_0) \)) with \( a \neq 0 \). So \( f^2 \equiv a^2 x_0^{2l} \) (mod \( (x_1 - a_1 x_0, ..., x_{b+1} - a_{b+1} x_0) \)). Now, the elements \( f^2 x_0^{l-2d-1} (x_i - a_i x_0) \) (for \( i = 1, ..., b + 1 \)), \( f^2 x_0^{l-2d-1} x_i \) (for \( i = b + 2, ..., n \)), and \( f^2 x_0^{l-2d} \) map to a basis of the target.
Now, fix \( n, b \) as usual, and let \( d \geq 1 \). Define

\[
\beta_d(l) = \left( \binom{l + b + 1}{b + 1} \right) - \left( \binom{l - 2d + b + 1}{b + 1} \right) + (n - b - 1) \left( \binom{l + b}{b + 1} - \binom{l - d + b}{b + 1} \right)
\]

\[
= \frac{(n - b + 1)d}{b!} + \ldots
\]

Let \( I = (f, x_{b+2}, \ldots, x_n) \subset S = k[x_0, \ldots, x_n] \), where \( f \in k[x_0, \ldots, x_{b+1}]/\{0\} \). Consider the composition

\[
\Phi: k[x_0, \ldots, x_{b+1}]_l \otimes \left( \bigoplus_{i=b+2}^n k[x_0, \ldots, x_{b+1}]_{l-1}x_i \right) \hookrightarrow S_l \to S_l/(I^2 \cap S_l).
\]

Note that \( \Phi \) is surjective.

**Lemma 6.2.** We have that

\[
\text{ker}(\Phi) = \{ P + \sum_{i=b+2}^n P_ix_i : f^2|P, f|P_i \text{ for } i = b + 2, \ldots, n \}.
\]

For \( l \geq 2d \), the codimension of \( I_l^2 \) in \( S_l \) equals \( \beta_d(l) \).

**Proof.** If \( P + \sum P_ix_i \in \text{ker}(\Phi) \), then we can write \( P + \sum P_ix_i = T \in I^2 \). Expand both sides as polynomials in \( x_{b+2}, \ldots, x_n \) and just compare the two expressions. The second part is an immediate consequence. \( \square \)

**Lemma 6.3.** Let \( C \hookrightarrow \mathbb{P}^n \) be any integral \( b \)-dimensional closed subscheme of degree 2, with (saturated) ideal \( I \). If \( F \in k[x_0, \ldots, x_n] \) satisfies \( C \subset V(F_{\text{sing}}) \), then \( F \in I_l^2 \).

**Proof.** Projection from a point on \( C \) shows that \( C \) is contained in a linear \((b + 1)\)-dimensional subspace of \( \mathbb{P}^n \). So we can assume that \( C = V(I) \), with \( I = (f, x_{b+2}, \ldots, x_n) \), where \( f \in k[x_0, \ldots, x_{b+1}] \setminus \{0\} \) is irreducible. We claim that the ideal \( I^2 \) is saturated. Indeed, let \( F \in S \) be homogeneous, and suppose that \( x_j^M F \in I^2 \) for all \( j = 0, \ldots, n \) (and for some \( M \)). Write \( F = P + \sum_{i=b+2}^n P_ix_i + T \), where \( P, P_i \in k[x_0, \ldots, x_{b+1}] \) are homogeneous of the appropriate degrees, and \( T \in (x_{b+2}, \ldots, x_n)^2 \). Since \( x_i^M F \in I^2 \), Lemma 6.2 implies that \( f^2|P, f|P_i \) for each \( i = b + 2, \ldots, n \). Since \( f \) and \( x_0 \) are relatively prime, it follows that \( f^2|P, f|P_i \) for each \( i \), and hence \( F \in I^2 \).

Since \( C \) is a local complete intersection and the ideal \( I^2 \) is saturated, the conclusion now follows from Corollary 2.3 in \([7]\). \( \square \)

Let \( P = \binom{z+b+1}{b+1} - \binom{z-1+b}{b+1} \) (this is the Hilbert polynomial of a degree-2 hypersurface in \( \mathbb{P}^{b+1} \)). Recall that \( \overline{\text{Hilb}}^P \) denotes the closure in \( \text{Hilb}^P \) of the set of integral \( b \)-dimensional closed subschemes of degree 2; in this case, a point in \( \overline{\text{Hilb}}^P \) is, up to a change of coordinates, a closed subscheme of the form \( V(f, x_{b+2}, \ldots, x_n) \subset \mathbb{P}^n \), where \( f \in k[x_0, \ldots, x_{b+1}] \setminus \{0\} \) (not necessarily irreducible of course). Note that

\[
\dim \overline{\text{Hilb}}^P = \dim \mathbb{G}(b+1, n) + \dim \mathbb{P}(k[x_0, \ldots, x_{b+1}]_2)
\]

\[
= (b + 2)n - \frac{b(b + 1)}{2}.
\]
By Lemma 6.2 if \( f \in k[x_0, \ldots, x_{b+1}]_2 - \{0\} \), then
\[
\dim \mathbb{P} ((f, x_{b+2}, \ldots, x_n)) = \binom{l + n}{n} - \beta_2(l) - 1.
\] (5)

Recall the usual incidence correspondence (where inclusion is scheme-theoretic)
\[
\tilde{\Omega}^P = \{(C, [F]) \in \tilde{\text{Hilb}}^P \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}} \} \subset \tilde{\text{Hilb}}^P \times \mathbb{P}(V).
\]
Recall that \( \pi \) and \( \rho \) denote the projections to \( \tilde{\text{Hilb}}^P \) and \( \mathbb{P}(V) \), respectively. For \( C \subset \mathbb{P}^n \) a closed subscheme, let \( I_C \) denote its (saturated) ideal. Consider the subset
\[
Z' = \{(C, [F]) \in \tilde{\text{Hilb}}^P \times \mathbb{P}(V) \mid F \in I_C^2 \} \subset \tilde{\Omega}^P.
\]

**Lemma 6.4.** The subset \( Z' \) of \( \tilde{\Omega}^P \) is irreducible.

**Proof.** By Lemma 6.2 for a fixed \( f \in k[x_0, \ldots, x_{b+1}]_2 - \{0\} \) and given \( F = F_0 + \sum_{i=b+2}^n F_i + T \in k[x_0, \ldots, x_n] \), where \( F_0 \in k[x_0, \ldots, x_{b+1}]_l, F_i \in k[x_0, \ldots, x_{b+1}]_{l-1} \), and \( T \in (x_{b+2}, \ldots, x_n)_2 \), we have that \( F \in (f, x_{b+2}, \ldots, x_n)_2 \) if and only if \( f^2 \mid F_0 \) and \( f \mid F_i \) for each \( i = b + 2, \ldots, n \).

Let \( V' = k[x_0, \ldots, x_{b+1}]_{l-4} + (\bigoplus_{i=b+2}^n k[x_0, \ldots, x_{b+1}]_{l-3}) + (x_{b+2}, \ldots, x_n)_2 \). Denote by \( \mathbb{A}(k[x_0, \ldots, x_{b+1}]_2) \) the affine space parametrizing points in \( k[x_0, \ldots, x_{b+1}]_2 \). Consider the composition
\[
\begin{array}{c}
\text{Aut}(\mathbb{P}^n) \times (\mathbb{A}(k[x_0, \ldots, x_{b+1}]_2) - \{0\}) \times \mathbb{P}(V') \\
\downarrow \\
\text{Aut}(\mathbb{P}^n) \times \mathbb{P}(k[x_0, \ldots, x_{b+1}]_2) \times \mathbb{P}(V') \\
\downarrow \\
\tilde{\text{Hilb}}^P \times \mathbb{P}(V)
\end{array}
\]
where the first map is given by
\[
(\sigma, f, [Q, R_{b+2}, \ldots, R_n, T]) \longmapsto (\sigma, [f], [f^2 Q + \sum_{i=b+2}^n f R_i x_i + T])
\]
and the second map is given by
\[
(\sigma, [f], [F]) \longmapsto (V(f^\sigma, x_{b+2}^\sigma, \ldots, x_n^\sigma), [F]^\sigma).
\]
By construction, \( Z' \) is precisely the image of the composition, hence is irreducible. \( \square \)

**Remark 6.5.** It is not true that the fibers of \( \tilde{\Omega}^P \to \tilde{\text{Hilb}}^P \) are all of the same dimension. For example, let \( b = 1, n = 3 \), and look at \( C = V(x_2^3, x_3) \in \tilde{\text{Hilb}}^P \). Let \( F = x_2^3 x_0^{l-3} \). Then \( (C, [F]) \in \pi^{-1}(C) \), but \( F \notin (x_2^3, x_3)^2 \). This is why we have to study the auxiliary \( Z' \).

Let \( Z \) be the closure of \( Z' \) in \( \tilde{\Omega}^P \).

**Lemma 6.6.** We have that
\[
\dim Z = \binom{l + n}{n} - \beta_2(l) - 1 + (b + 2)n - \frac{b(b + 1)}{2}.
\]
Proof. First, $\pi(Z') = \widetilde{\text{Hilb}}^P$, since given any $C \in \widetilde{\text{Hilb}}^P$, the ideal $I_C^2$ contains forms of degree 4 already, so we can certainly find $F \in (I_C^2)^l$. Thus, $\pi: Z \to \widetilde{\text{Hilb}}^P$ is onto. A generic $C \in \widetilde{\text{Hilb}}^P$ is an integral $b$-dimensional closed subscheme of degree 2; for such a $C$, by Lemma 6.3, we know $Z_C = \tilde{\Omega}_C^P$ and hence also $Z_C = \tilde{\Omega}_C$. This allows us to compute $\dim Z_C = \dim Z'_C = \left(\frac{l+n}{n}\right) - \beta_2(l) - 1$. This computes $\dim Z = \dim \widetilde{\text{Hilb}}^P + \dim Z_C$ and gives the desired result, by virtue of (11) and (5).

Lemma 6.7. $X^2 := \rho(Z)$ is an irreducible closed subset of $X$ of dimension $\left(\frac{l+n}{n}\right) - \beta_2(l) - 1 + (b+2)n - \frac{b(b+1)}{2}$. If $[F] \in X$ contains an integral closed subscheme of dimension $b$ and degree 2 in its singular locus, then $[F] \in X^2$.

Proof. It is clear that $\rho(Z)$ is an irreducible closed subset of $X$, since $Z$ is irreducible and closed in $\tilde{\Omega}_C^P$. Choose any integral $b$-dimensional $C$ of degree 2. Apply Lemma 6.1 to find $[F] \in \mathbb{P}(V)$ such that we have a homeomorphism $C \leftrightarrow V(F)_{\text{sing}}$. If $\tilde{C} \in \widetilde{\text{Hilb}}^P$ is another closed subscheme contained in $V(F)_{\text{sing}}$, then necessarily we have $C \leftrightarrow \tilde{C}$, since $C$ is reduced. Hence $C = \tilde{C}$, since $C$ and $\tilde{C}$ have the same Hilbert polynomial. Therefore, the map $Z \to \rho(Z)$ has a 0-dimensional fiber, so $\dim \rho(Z) = \dim Z$.

Let $[F] \in X$ be such that $V(F)_{\text{sing}}$ contains an integral $b$-dimensional closed subscheme $C$ of $\mathbb{P}^n$ of degree 2. Then we know that $F \in I_C^2$ by Lemma 6.3 so $(C, [F]) \in Z'$, and hence in fact $[F] \in \rho(Z') \subset \rho(Z) = X^2$.

6.2 The analogue of Theorem 1.1 regarding the second-largest component

Here we discuss a calculation similar to the one in section 5.3 which addresses the question of the second largest component of $X$.

Note that

$$\beta_2(l) = \left(\frac{l+b+1}{b+1}\right) - \left(\frac{l+b-3}{b+1}\right) + (n-b-1) \left(\frac{l+b}{b+1}\right) - \left(\frac{l+b-2}{b+1}\right),$$

and set $\gamma_2(l) = \beta_2(l) + 1 + (b+2)n - \frac{b(b+1)}{2}$. We know that $\left(\frac{l+n}{n}\right) - \gamma_2(l)$ is the dimension of $X^2$. We are still assuming Conjecture 5.4

Lemma 6.8. There exists $l_0$ (easily computable) such that for all pairs $(d, l)$ with $3 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$ (if $b = n - 1$, assume $d \geq 4$), and any irreducible component $Z$ of $T^d_k$, either $Z \subset T^1_k \cup T^2_k$, or

$$\dim Z < \left(\frac{l+n}{n}\right) - \gamma_2(l).$$

(In the case $b = n - 1$, we can describe $X$ explicitly, so this case is not of interest to us.)

Proof. Precisely as in Lemma 5.5 because of inequality 11, it suffices to establish the inequality

$$\dim \mathbb{P}(W_{UL}) + \dim \widetilde{\text{Hilb}}^d < \left(\frac{l+n}{n}\right) - \gamma_2(l), \quad \text{i.e.,}$$
\[ \gamma_2(l) - 1 + \dim \overline{\text{Hilb}}^d < \text{codim}_V(W_{(l \lambda)}). \]

Set \( c = - \frac{(b+1)(b+3)}{2} - 1 \). By Lemma 4.5 and Conjecture 5.4, we are reduced to proving that

\[ c + \beta_2(l) + \binom{d + b + 1}{b + 1} < \binom{l + b}{b} + (n - b) \sum_{e=1}^{d} \binom{l - 2e + 1 + b}{b}, \]

or, equivalently, that

\[ c + (n - b) \binom{l + b - 3}{b - 1} + \binom{l + b - 3}{b} + \binom{d + b + 1}{b + 1} < (n - b) \sum_{e=3}^{d} \binom{l - 2e + 1 + b}{b}. \]  \( (6) \)

Suppose first that \( n - b > 1 \). If \( d = 3 \), this inequality is certainly satisfied for \( l \gg 0 \) (look at the leading terms of both sides). Consider now \( d \geq 4 \). Since \( n - b > 1 \), we can find \( l' \) such that for all \( l \geq l' \),

\[ c + (n - b) \binom{l + b - 3}{b - 1} + \binom{l + b - 3}{b} < (n - b) \binom{l - 5 + b}{b}. \]

What is left now is to prove that there exists \( l'' \) such that for \( l \geq l'' \) and \( 4 \leq d \leq \frac{l+1}{2} \), we have

\[ \binom{d + b + 1}{b + 1} < (n - b) \sum_{e=4}^{d} \binom{l - 2e + 1 + b}{b}. \]

This is analogous to (3) and follows exactly as in the proof of Lemma 5.5. Now we just take \( l_0 = \max(l', l'') \).

Suppose now that \( n - b = 1 \) and \( d \geq 4 \). If \( d = 4 \), inequality (6) certainly holds for large \( l \) (the leading term of the right hand side is \( \frac{2n}{b^2} \)). Consider \( d \geq 5 \). We can find \( l' \) such that for all \( l \geq l' \),

\[ c + \binom{l + b - 3}{b - 1} + \binom{l + b - 3}{b} < \binom{l - 5 + b}{b} + \binom{l - 7 + b}{b}. \]

Finally, we have to show that that there exists \( l'' \) such that for \( 5 \leq d \leq \frac{l+1}{2} \) and \( l \geq l'' \), we have

\[ \binom{d + b + 1}{b + 1} < \sum_{e=5}^{d} \binom{l - 2e + 1 + b}{b}. \]

Again, this is analogous to inequality (3). \( \square \)

In [6], we will use the above result to show that when \( p = \text{char} k > 0 \), there exists (again, effectively computable) \( l_0 = l_0(n, b, p) \), such that for \( l \geq l_0 \), \( X^2 \) is the unique irreducible component of \( X \) of second largest dimension.

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