On $\beta$–deformations and Noncommutativity

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ABSTRACT

We elucidate the connection between the $\mathcal{N} = 1$ $\beta$–deformed SYM theory and noncommutativity. Our starting point is the T–duality generating transformation involved in constructing the gravity duals of both $\beta$–deformed and noncommutative gauge theories. We show that the two methods can be identified provided that a particular submatrix of the $O(3,3,\mathbb{R})$ group element employed in the former case, is interpreted as the noncommutativity parameter associated with the deformation of the transverse space. It is then explained how to construct the matrix in question, relying solely on information extracted from the gauge theory Lagrangian and basic notions of AdS/CFT. This result may provide an additional tool in exploring deformations of the $\mathcal{N} = 4$ SYM theory. Finally we use the uncovered relationship between $\beta$–deformations and noncommutativity to find the gravity background dual to a noncommutative gauge theory with $\beta$–type noncommutativity parameter.
1 Introduction

The conjectured gauge/gravity duality \cite{1} \cite{2} \cite{3} relates four–dimensional theories at strong t’Hooft coupling with weakly coupled gravitational ones. In \cite{4} Lunin and Maldacena presented a further development in this direction by constructing the gravity duals of gauge theories deformed in a particular manner that maintains a global $U(1) \times U(1)$ symmetry present in the original undeformed theory. The prototype of these deformations is a Leigh–Strassler \cite{5} exactly marginal deformation of $\mathcal{N} = 4$ SYM theory, characterized by a complex parameter $\beta$ which preserves $\mathcal{N} = 1$ supersymmetry. The method of Lunin and Maldacena is not however restricted to conformal field theories. It can be applied to any field theory as long as its dual gravity background contains a two torus geometrically realizing the global $U(1)$ symmetries in question. When $\beta \in \mathbb{R}$ — usually denoted as $\gamma$ in the literature — the prescription presented in \cite{4} amounts to performing an $SL(2, \mathbb{R})$ transformation on the complexified Kähler modulus $\tau$ of this two torus. The specific element of $SL(2, \mathbb{R})$ under consideration has only one free parameter which is then identified with the real deformation parameter $\gamma$ of the gauge theory. Subsequent work on the subject of the $\beta$–deformed gauge theories has provided further checks of the AdS/CFT correspondence \cite{6} \cite{7} \cite{8} \cite{9} \cite{10} \cite{11} whereas the possibility of an underlying integrable structure in this context was explored in \cite{12} \cite{13} \cite{14}. Several aspects of these deformations were analysed from the gauge theory viewpoint in \cite{15} \cite{16} \cite{17} \cite{18} \cite{19} \cite{20} \cite{21}. Furthermore, generalizations as well as applications of the solution generating technique introduced in \cite{4} were considered in \cite{22} \cite{14} \cite{11} \cite{23} \cite{24} \cite{25}. Meanwhile, it became clear \cite{26} that embedding $SL(2, \mathbb{R})$ into the T–duality group $O(2, 2, \mathbb{R})$ may be a significantly easier way to obtain the deformed backgrounds since it suffices then to consider the action of the appropriate $O(2, 2, \mathbb{R})$ group element on the background matrix $E = g + B$. In this framework, an extraordinary similarity between the proposal of \cite{4} and the method for constructing gravity duals of noncommutative gauge theories becomes evident \footnote{Actually, this connection was already noted in \cite{4}.}. From the gauge theory point of view this analogy is not surprising since the deformation amounts to modifying the commutator of the matter fields in the Lagrangian or equivalently, their product. A natural proposal for the product rule was set forth in \cite{4} and subsequently verified in the dual field theory context in \cite{15}. \cite{22}. \cite{14}. \cite{11} \cite{23} \cite{24} \cite{25}. \cite{26}.

The central aim of this note is to clarify the relation between noncommutativity and $\beta$–deformations. We will consider the deformations in their original context as marginal deformations of $\mathcal{N} = 4$ SYM and show how to obtain a noncommutativity matrix $\Theta$ describing them. The main point will be to think of the matter fields in the dual theory as coordinates parametrizing the space transverse to the D3–brane where the gauge theory lives. Then, reality properties, global symmetries and marginality will severely constrain the form of the noncommutativity matrix leaving one possible choice, the one which leads to the correct gravity dual description. In other words, $\Theta^{ij}$ along with the metric of the transverse space can be thought of as another way to encode the moduli space of the gauge theory. This suggests an alternative way in which to investigate deformations of the original AdS/CFT proposal \cite{1} by determining the open string parameters pertaining to them. Related ideas will be explored in a forthcoming publication \cite{27} in order to study another Leigh–Strassler marginal deformation of $\mathcal{N} = 4$ SYM the gravity dual of which is yet unknown.

The plan of this paper is as follows. In the next section, we review the solution generating technique proposed in \cite{4} as well as its formulation through T–duality \cite{26}. In section\footnote{Actually, this connection was already noted in \cite{4}.} we present some basic facts about noncommutative geometry. Then we describe the methods employed in finding the gravity duals
of these theories in a fashion that makes evident the similarity with the approach of [4]. In particular, it is shown that the T–duality group elements used in both cases can be identified if the deformation submatrix referred to as $\Gamma$ in [26] is interpreted as a noncommutativity matrix. In section 4, we explain how one can determine a suitable noncommutativity matrix for the $\beta$–deformed gauge theory. This construction is purely based on gauge theory data and basic notions of AdS/CFT. We then show that $\Theta^{ij}$ is precisely the submatrix $\Gamma$ appearing in section 1.

As an obvious way to exploit the precise relation uncovered between noncommutativity and $\beta$–deformations, we proceed to construct the gravity dual of a noncommutative gauge theory with $\beta$–type noncommutativity both in Euclidean and in Lorentzian signature. We finally present our conclusions in section 6.

2 The Lunin–Maldacena solution generating technique.

As it was shown in [5], $\mathcal{N} = 4$ Super Yang Mills admits a complex three parameter family of marginal deformations preserving $\mathcal{N} = 1$ supersymmetry which is described by the following superpotential:

$$ W = \kappa \epsilon_{IJK} \text{Tr} \left( [\Phi^I, \Phi^J]_\beta \Phi^K \right) + \rho \text{Tr} \left( \sum_{I=1}^{3} (\Phi^I)^3 \right) $$

Here $\Phi^I$ are three chiral superfields and $[\Phi^I, \Phi^J]_\beta \equiv e^{i\beta} \Phi^I \Phi^J - e^{-i\beta} \Phi^J \Phi^I$. Together with the gauge coupling $g_{YM}$, the complex parameters ($\kappa, \beta, \rho$) constitute the four couplings of the theory. Conformal invariance imposes one condition on these couplings thus (1) describes a three parameter family of deformations. When $\rho = 0$ the theory is often referred to as the $\beta$–deformed gauge theory and preserves an additional global $U(1) \times U(1)$ symmetry (apart from the $U(1)_R$–symmetry) which acts on the superfields as follows:

$$ U(1)_1 : (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\alpha_1} \Phi_2, e^{-i\alpha_1} \Phi_3) $$
$$ U(1)_2 : (\Phi_1, \Phi_2, \Phi_3) \rightarrow (e^{-i\alpha_2} \Phi_1, e^{i\alpha_2} \Phi_2, \Phi_3) $$

In this paper we will be mainly considering the $\beta$–deformed theory for $\beta \in \mathbb{R}$. It is then customary to denote the deformation parameter as $\gamma$ and we will adhere to this notation in this section. Lunin and Maldacena in [4] succeeded in finding the gravity dual of this theory by implementing a generating solution technique which can be applied to any field theory with $U(1) \times U(1)$ global symmetry realized geometrically. Their method essentially consists in performing an $SL(2, \mathbb{R})$ transformation on the complexified Kähler modulus of the two torus associated with the $U(1)$ symmetries in question. Suppose for instance that one knows the gravity dual of the undeformed theory and furthermore that the two global $U(1)$’s of the parent theory also preserved by the deformation are indeed realized geometrically. Then the supergravity dual of the deformed theory is given by the following substitution:

$$ \tau = (B_{12} + \sqrt{g}) \rightarrow \frac{\tau}{1 + \gamma \tau} $$

where $\tau$ is the complexified Kähler modulus of the two torus (associated to the $U(1)$ symmetries of the original solution) with $B_{12}$ the B–field along the torus and $\sqrt{g}$ its volume. In other words, one considers the theory compactified on the two torus and subsequently acts on its Kähler modulus with the particular element of $SL(2, \mathbb{R})$ given by \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv \left( \begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix} \right) \) with $\gamma$ the parameter of the theory. This element of $SL(2, \mathbb{R})$ is chosen because it ensures that the new solution will present no singularities as long as the original
metric is non–singular. An alternative way of thinking about this solution generating transformation is
in terms of applying a series of T–dualities. More precisely, the method illustrated above is equivalent to
doing a T–duality on a circle, a coordinate transformation and then another T–duality (TsT).

Subsequently it was shown \[26\] that one can embed the \( SL(2, \mathbb{R}) \) that acts on the Kähler modulus
into the T–duality group \( O(2, 2, \mathbb{R}) \) and thus consider the action of the latter on the background matrix
\( E = g + B \). This provides a considerably simpler way of obtaining the new solutions. For \( (a \ b \ c \ d) \) the
generic element of \( SL(2, \mathbb{R}) \) the appropriate embedding is the following:

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}
\] (4)

It is then easy to see \[28\] that \( T \) transforms the original background matrix \( E_0 \) as:

\[
E_0 \rightarrow E = (AE_0 + B)(CE_0 + D)^{-1} \equiv \frac{AE_0 + B}{CE_0 + D}
\] (5)

where the \( 2 \times 2 \) matrices \( A, B, C, D \) are defined through (4). According to \[4\] we should not consider any
\( SL(2, \mathbb{R}) \) element but the precise one with \( a = d = 1, b = 0 \) and \( c = \gamma \). In this case \[4\] reads:

\[
T = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \quad \text{with} \quad \Gamma = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}
\] (6)

where \( 1 \) and \( 0 \) represent the \( 2 \times 2 \) identity and zero matrices respectively. Following now the T–duality
rules in \[28\] we can write the NS–NS fields of the new solution in terms of \( E_0 \) and \( \Gamma \) as follows:

\[
E = \frac{1}{E_0^{-1} + \Gamma}\quad \text{with}\quad e^{2\Phi} = \det(1 + E_0\Gamma)e^{24\Phi_0}
\] (7)

The RR-fields of the background can be obtained in a similar fashion using the transformation rules of
\[29\ \[30\ \[31\ \[32\ \[33\]. Nevertheless, for the purposes of this letter it only suffices to know that appropriate
rules exist and can be applied.

There are however cases where one needs to slightly modify the method illustrated above. This
happens when non–trivial fibrations mix the isometry directions of the two torus with other directions in
the metric. It is then necessary to embed \( SL(2, \mathbb{R}) \) into \( O(n + 2, n + 2, \mathbb{R}) \) with \( n \) the number of non–trivial
coordinate fibrations. A particular example of this is the AdS\(_5 \times T^{1,1} \) solution of \[34\]. If we want to
apply the deformation to this background instead of \( \Gamma \) we should employ:

\[
T = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \quad \text{where} \quad \Gamma = \begin{pmatrix} 0 & -\gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\] (8)

Furthermore, as it was again pointed out in \[26\], the appropriate T–duality matrix one should use for
the deformation of AdS\(_5 \times S^5 \)which gives rise to the gravity dual of the \( \beta \)--deformed gauge theory is:

\[
T = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \quad \text{where now} \quad \Gamma = \begin{pmatrix} 0 & -\gamma & \gamma \\ \gamma & 0 & -\gamma \\ -\gamma & \gamma & 0 \end{pmatrix}
\] (9)
This particular choice of $\Gamma$ with the necessary embedding of $SL(2, \mathbb{R})$ into $O(3, 3, \mathbb{R})$ can be understood in this case as the result of performing a change of coordinates and a T–duality transformation of the form (8) followed by another coordinate transformation (26). For future reference and as a concrete illustration of the above we would like to give an explicit construction of the background in this case. What we have to do is to simply act with (9) on the background matrix $E_0$ which in this example is none other but $AdS_5 \times S^5$. Since we are interested in obtaining the gravity dual of a conformal gauge theory we expect that only the $S^5$ part of $AdS_5 \times S^5$ will be affected by the deformation. We can write the metric on $S^5$ in the following way:

$$ds^2 = R^2 \left( \sum_{i=1}^{3} (\mu_i^2 + \mu_i^2 d\phi_i^2) \right)$$

where $\sum_{i=1}^{3} \mu_i^2 = 1$ (10)

Note here that we want to deform the geometry along the U(1) isometry directions of $S^5$, therefore the relevant part of the background matrix is:

$$E_0 = R^2 \begin{pmatrix} \mu_1^2 & 0 & 0 \\ 0 & \mu_2^2 & 0 \\ 0 & 0 & \mu_3^2 \end{pmatrix}$$

Using now equation (7) and its generalization for RR–fields we find (26):

$$ds^2 = R^2 (ds^2_{AdS_5} + ds^2_{S^5}), \quad \text{where:} \quad ds^2 = \sum_i (d\mu_i^2 + G \mu_i^2 d\phi_i^2) + \gamma G \mu_1^2 \mu_2^2 \mu_3^2 (\sum_i d\phi_i)^2$$

$$G^{-1} = 1 + \gamma (\sum_{i \neq j} \mu_i^2 \mu_j^2), \quad \gamma = R^2 \gamma, \quad R^4 = 4\pi e^{4\gamma} N$$

$$e^{2\gamma} = e^{2\phi_0} G, \quad B = \gamma R^2 G \left( \sum_{i \neq j} \mu_i^2 \mu_j^2 d\phi_i d\phi_j \right)$$

$$C_2 = -\gamma (16\pi N) \omega_1 (\sum_i d\phi_i), \quad C_4 = (16\pi N) (\omega_4 + G \omega_1 d\phi_1 d\phi_2 d\phi_3)$$

$$F_5 = (16\pi N)(\omega_{AdS_5} + G \omega_{S^5}), \quad \omega_{S^5} = d\omega_1 d\phi_1 d\phi_2 d\phi_3, \quad \omega_{AdS_5} = d\omega_4$$

which is precisely the gravity solution given in [4].

3 The gravity duals of noncommutative gauge theories.

In this section we would like to focus on yet another class of supergravity duals which can be obtained in manner analogous to the one described earlier. These are the gravity duals of noncommutative gauge theories 2 and in fact the methodology used in both cases is almost identical.

Noncommutative — as opposed to ordinary — gauge theories, live in a space of noncommuting coordinates 3. Such a deformation of space is encoded in what is referred to as the noncommutativity parameter $\Theta^{ij}$ defined as:

$$[x^i, x^j] = i\Theta^{ij}$$

where $\{x^i\}$ is a set of coordinates parametrizing the space and $\Theta^{ij}$ a real antisymmetric matrix. In general, the easiest way to deal with functions on these spaces is to replace noncommuting variables with

2For an introduction to noncommutative geometry see for example [35] and references therein

3We limit the discussion in this section to Euclidean spaces or to noncommutativity which does not affect the time–like coordinate.
commuting ones by simply defining a new product rule between them, usually called a star product. The star product will then contain all the information on the noncommutative structure of the space.

Out of all the possible forms of $\Theta^{ij}$ the case most well understood is by far the one in which the commutators of $\Theta^{ij}$ are c–numbers and therefore the noncommutativity parameter is essentially a constant. In this case, associativity is preserved and the appropriate star product has the form:

$$f(x) \ast g(x) = f(x + \xi) e^{\frac{i}{\hbar} \Theta^{ij} \frac{\partial}{\partial \xi^j}} g(x + \zeta) = f \left( 1 + \Theta^{ij} \frac{\partial}{\partial \tilde{\xi}^j} \right) g$$

Gravity duals of theories living on noncommutative spaces with constant noncommutativity parameter were first found in [36,37]. The basic technique for constructing these solutions is to combine diagonal T–dualities, constant shifts of the NS–NS two form and $SO(p, 1)$ transformations, where $p$ is the number of spatial dimensions. One first T–dualizes in the directions where one wants to turn on fluxes, shifts the B field by a constant in these directions and then T–dualizes back. Equivalently, one can T–dualize along one of the directions of the fluxes, use a boost/rotation between a non compact and a compact direction and the T–dualize back. Both methods give the same result. It was later on realized that these solutions can be generated from the action of the $O(p, p, \mathbb{R})$ T–duality group element

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix}$$

on the original undeformed solution where now $0, 1, \Theta$ are $p$ dimensional square matrices with $p$ denoting the number of spatial directions along which noncommutativity is turned on. Suppose for instance that one wants to describe a gauge theory living in four dimensional Euclidean space employed with cartesian coordinates $x^\mu$ where: $[x^0, x^1] = ib_1$ and $[x^2, x^3] = ib_2$. It is then clear that one should consider the embedding of the noncommutativity parameter into the T–duality group element $O(4, 4, \mathbb{R})$ as follows:

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix} \quad \text{with} \quad \Theta = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix}$$

The original solution to be deformed in this context is again AdS$_5 \times S^5$, however now $\Theta$ lies along the non–compact, AdS$_5$ piece of the geometry. Writting the metric on AdS$_5$ as:

$$ds^2_{AdS} = R^2 u^2 (dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{du^2}{u^2}$$

we see that the relevant part of the background matrix $E_0$ in this case is:

$$E_0 = R^2 u^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and acting now on $E_0$ with the T–duality matrix $\mathcal{T}$ of equation (16) we obtain [36]:

$$ds^2_{ste} = u^2 R^2 (G_1 (dx_0^2 + dx_1^2) + G_2 (dx_2^2 + dx_3^2)) + R^2 \frac{du^2}{u^2} (du^2 + u^2 d\Omega_5^2)$$

$$B = \hat{b}_1 R^2 G_1 u^4 dx_0 \wedge dx_1 + \hat{b}_2 R^2 G_2 u^4 dx_2 \wedge dx_3$$

$$e^{2\Phi} = G_1 G_2 e^{2\Phi_0}, \quad G_1 = \frac{1}{1 + \hat{b}_1^2 u^4}, \quad G_2 = \frac{1}{1 + \hat{b}_2^2 u^4}$$

$$\hat{b}_1 = R^2 b_1, \quad \hat{b}_2 = R^2 b_2$$
which is the gravity dual\footnote{Note the resemblance between \cite{14} and \cite{15}.} of a noncommutative gauge theory defined in Euclidean space with $[x^0, x^1] = ib_1$ and $[x^2, x^3] = ib_2$. The Lagrangian description of this theory can be easily derived from the $\mathcal{N} = 4$ SYM Langrangian by replacing the ordinary product of functions with the Moyal star designated in \cite{14}. Although we have so far considered applying this method directly to the near horizon geometry one can, perhaps even more appropriately, perform it on the p–brane solutions as well \cite{39} \cite{38} \cite{40} \cite{41}. The near horizon limit that needs to taken in this case requires a relative scaling between the $B$–field and the metric $g$ which actually corresponds to the Seiberg–Witten limit proposed in \cite{42}.

It should now be evident that the solution generating transform employed by Lunin and Maldacena in order to find the gravity duals of $\beta$–deformed gauge theories is almost identical to the one used for the same purpose within the context of noncommutative gauge theories. The only difference is that in the former case it is the transverse space to the brane, or rather the compact part of the near horizon geometry that is being deformed. This naturally suggests interpreting the matrix $\Gamma$ appearing in equation \cite{10} as some kind of noncommutativity parameter. Since noncommutativity in this case is a property of the transverse space it manifests itself as a deformation of the matter content of the theory.

Before we proceed to the next section where we will further clarify this point, we would like to make some final remarks about the applicability of the solution generating transformations illustrated above. Despite the fact that this method has had a rather remarkable set of applications so far its utility is unfortunately restricted to the following conditions. First of all, the directions one wants to introduce fluxes — or equivalently noncommutativity — should be isometry directions realized geometrically, meaning as shift symmetries of the metric \cite{43}. In addition, the noncommutativity matrix should have constant entries. Expressed in a more precise manner this means that there should exist a coordinate system where the noncommutativity is reduced to a constant along isometry directions of the metric.

As an example of this, let us consider the Melvin Twist gauge theory. This has been studied in \cite{41} \cite{44} \cite{45}. The relevant noncommutativity parameter can be written in cartesian coordinates as\footnote{Here we consider the case of a non compact direction $x_3$ in contrast to the most widely used case.}:\footnote{It may thus be interesting to formulate generalized complex geometry from the point of view of open strings.}

$$[x_2, x_3] = ibx_1, \quad [x_3, x_1] = ibx_2 \quad \text{and} \quad [x_1, x_2] = 0$$

but in polar coordinates on the $(x_1, x_2)$–plane it becomes:

$$[\rho, \theta] = 0, \quad [\rho, x_3] = 0, \quad \text{and} \quad [\theta, x_3] = ib$$

In these coordinates $(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta})$ are indeed Killing vectors of the flat space metric and therefore the solution generating technique is applicable.

In general it seems reasonable to expect that given a noncommutativity parameter, the following two conditions should hold for a coordinate system to exist in which $\Theta^{ij}$ is reduced to a constant matrix:

$$\Theta^{ij} \partial_i \Theta^{jk} + \Theta^{kl} \partial_j \Theta^{ki} + \Theta^{ij} \partial_i \Theta^{kl} = 0 \quad \Rightarrow \quad T^{[ijk]} = \partial_k (\Theta^{[ij]} \Theta^{jk}) = 0$$

\footnote{Note the resemblance between \cite{14} and \cite{15}.}
Then, the first condition in (22) can be read as the possibility of extending the local coordinates to global ones.\footnote{This is actually not true for the two–dimensional case, which is particularly simple. For instance, all noncommutative deformations are also associative ones.}

We would like to conclude this section by stressing once more that (22) cannot be seen as the requirement for the solution generating transformation to work since there is no way to make sure that the coordinate transformation employed to bring $\Theta^{ij}$ into a constant form will not spoil the shift symmetries present in the metric. One example of this is the nongeometric background also referred to as the Q–space in the literature \cite{46} \cite{47} \cite{48}. The relevant noncommutativity parameter in this case is:

$$ [x_1, x_2] = ibx_3, \quad [x_1, x_3] = [x_2, x_3] = 0 \quad (23) $$

While it is obvious from the discussion above that $\Theta^{ij}$ can be reduced to a constant, the coordinate transformation that makes this possible is \cite{48}:

$$ x_1 \rightarrow y_1 y_3, \quad x_2 \rightarrow y_2, \quad x_3 \rightarrow y_3 $$

and in these coordinates the metric looks like:

$$ ds^2 = -dt^2 + (y_1 dy_3 + y_3 dy_1)^2 + dy_2^2 + dy_3^2 \quad (24) $$

Indeed it has not been possible to embed this noncommutative deformation of flat space directly into string theory. It nevertheless naturally emerges when a D3–brane probe is immersed in the background of smeared NS5–branes.

## 4 $\beta$–deformations and noncommutativity

The aim of this section is to establish a precise relation between transverse space noncommutativity and $\beta$–deformations of $\mathcal{N} = 4$ SYM. In general the connection between marginal deformations and noncommutativity is not new. A study of the moduli space clearly points into this direction — a thorough analysis can be found in \cite{49} \cite{50} \cite{51} \cite{52}. The F–term constraints for instance read:

$$ \Phi^I \Phi^J = q \Phi^J \Phi^I, \quad \Phi^{I \bar{J}} \Phi^{\bar{J} I} = q \Phi^{I \bar{J}} \Phi^{\bar{J} I} \quad \text{where } q = e^{2i\beta} \text{ and } I,J \text{ are cyclically ordered.} \quad (25) $$

and $\Phi^I$ here indicate the first components of the corresponding superfields. These equations are usually understood to represent the space where the D–branes can move. For small enough deformations we can interpret the eigenvalues of these matrices as coordinates parametrizing the transverse space to the worldvolume of the D3–brane. The eigenvalues should however now be thought of as noncommuting numbers according to equation (25). If we denote the coordinates of the moduli space as $(z^I, \bar{z}^I)$ with $I, \bar{I} = 1, 2, 3$ we have that:

$$ z^I z^J = q z^J z^I, \quad \bar{z}^I \bar{z}^J = q \bar{z}^J \bar{z}^I \quad \text{with } I,J \text{ cyclically ordered.} \quad (26) $$

Later on, it will become clear that a noncommutative interpretation is meaningful only when $\beta \in \mathbb{R}$. Henceforth we replace $\beta$ with $\gamma$ in order to avoid confusion and to be consistent with existing notations in the literature.

As it was mentioned in the previous section we can identify the prescription of \cite{4} with the one used within the context of noncommutative gauge theories so long as matrix $\Gamma$ appearing in equation (14) is the noncommutativity matrix associated to the deformation of the transverse space. Therefore, our main objective here is to construct a noncommutativity matrix, or rather a contravariant antisymmetric tensor
field $\Theta^{IJ}$ to describe the deformed space. A natural way to define it is through the commutation relations implied by (26). That is:

$$[z^I, z^J] = i2e^{\gamma} \sin \gamma z^I z^J \quad [\bar{z}^I, \bar{z}^J] = i2e^{\gamma} \sin \gamma \bar{z}^I \bar{z}^J$$  \hspace{1cm} (27)

Clearly such a definition would require a whole different notion of differential geometry since the noncommutativity parameter is position dependent and the coordinates themselves are now noncommuting objects. We circumvent this by implementing an alternative procedure. As mentioned in the previous section one can replace noncommuting coordinates with commuting ones by defining a star product between them. In general, constructing an appropriate star product can be an equally formidable task as dealing with noncommuting variables. In this case however a natural proposal was set forth in [4]. Specifically, the authors of [4] suggested:

$$f \ast g = fe^{i\beta(\bar{Q}_1 Q_2 - \bar{Q}_2 Q_1)} g$$  \hspace{1cm} (28)

where $f, g$ belong to the set of chiral/antichiral multiplets of the theory and $Q_{1,2}$ are the global $U(1)$ charges associated with these fields (see equation (2)). This proposal was subsequently used [15] in order to rewrite the component Lagrangian of the $\beta$-deformed gauge theory as the $\mathcal{N} = 4$ SYM Lagrangian with the product of matter fields now replaced by the above star product. This enabled the author of [15] to show that all the amplitudes in the planar limit of the deformed theory with $\beta \in \mathbb{R}$ are proportional to their $\mathcal{N} = 4$ counterparts. Note that the star here is not explicitly written in terms of derivatives/operators acting on the fields $(f, g)$. Knowledge of the product in this form however will be sufficient for the purposes of this letter.

In what follows we will use equation (28) in order to write down a noncommutativity matrix and compare it with [15]. Then we will discuss ways to derive the appropriate $\Theta^{ij}$ without prior knowledge of the star product. We therefore define the noncommutativity parameter through the following relations:

$$[z^I, z^J]_\ast = (z^I \ast z^J - z^J \ast z^I) = i\Theta^{IJ}$$

$$[[z^I, \bar{z}^J]_\ast = (z^I \ast \bar{z}^J - \bar{z}^J \ast z^I) = i\Theta^{I\bar{J}}$$

$$[z^I, \bar{z}^J]_\ast = (z^I \ast \bar{z}^J - \bar{z}^J \ast z^I) = i\Theta^{\bar{I}J}$$

with $(I, J)$ cyclically ordered. Setting $a \equiv 2\sin \gamma$ and writting this in matrix notation, we obtain:

$$\Theta = a \begin{pmatrix}
0 & z_1z_2 & -z_1z_3 & 0 & -z_1\bar{z}_2 & z_1\bar{z}_3 \\
-z_1z_2 & 0 & z_2z_3 & \bar{z}_1z_2 & 0 & -z_2\bar{z}_3 \\
z_3z_1 & -z_2z_3 & 0 & -\bar{z}_1z_3 & \bar{z}_2z_3 & 0 \\
0 & -\bar{z}_1z_2 & \bar{z}_1z_3 & 0 & \bar{z}_1\bar{z}_2 & -\bar{z}_1\bar{z}_3 \\
z_1\bar{z}_2 & 0 & -\bar{z}_2z_3 & -\bar{z}_1\bar{z}_2 & 0 & \bar{z}_2\bar{z}_3 \\
-\bar{z}_3z_1 & \bar{z}_3z_2 & 0 & \bar{z}_3\bar{z}_3 & -\bar{z}_2\bar{z}_3 & 0
\end{pmatrix}$$  \hspace{1cm} (30)

Clearly, the result obtained above is not exactly a satisfactory one. Despite the fact that we managed to describe the deformation of the transverse space in a noncommutative way, the associated noncommutativity matrix $\Theta$ is both position dependent and six dimensional. It does not therefore in any sense resemble to matrix $\Gamma$ of equation (14). An additional interesting but perhaps perplexing feature of $\Theta$ is that it is not a purely holomorphic/antiholomorphic matrix as we might have expected from the F–term constraints. We will return to this point later in this section after we outline a more general prescription of identifying the appropriate $\Theta^{ij}$.
Let us however proceed to make a coordinate transformation on Θ. Since Θ\[^{I,J}\] thus defined is a contravariant tensor we have no trouble doing so. In other words we know that when changing coordinates from \(\{x^i\}\) to \(\{x'^i\}\), the noncommutativity parameter transforms as:

\[
Θ'^{i,j} = \frac{∂x'^{i'}}{∂x^i} \frac{∂x'^{j'}}{∂x^j} Θ^{ij}
\]

(31)

Here, we chose to rewrite Θ\[^{I,J}\] in spherical coordinates \((r, α, θ, φ)\) defined through:

\[
\begin{align*}
    z_1 &= r \cos α e^{iφ_1}, \quad z_2 = r \sin α \sin θ e^{iφ_2}, \quad z_3 = r \sin α \cos θ e^{iφ_3} \\
    \bar{z}_1 &= r \cos α e^{-iφ_1}, \quad \bar{z}_2 = r \sin α \sin θ e^{-iφ_2}, \quad \bar{z}_3 = r \sin α \cos θ e^{-iφ_3}
\end{align*}
\]

(32)

Note that in these coordinates we should be careful to define if possible the parameter \(γ\) of our matrix so as to have Θ ∈ ℝ. Only then can Θ be interpreted as a noncommutativity parameter in the usual sense. Applying (31) to (30) we obtain in matrix notation:

\[
Θ = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & a & 0 \\
0 & 0 & a & 0 & -a & 0 \\
0 & 0 & -a & a & 0 & 0
\end{pmatrix}
\]

(33)

and we immediately see that we can indeed think of Θ as a noncommutativity matrix only when \(a ∈ ℝ\). More importantly, from equation (33) it is clear that we can reduce Θ to the 3 × 3 matrix denoted as Θ in section 2.8. The only difference is that now the deformation parameter γ of the gauge theory is replaced by \(a = 2 \sin γ\). Recall however, that the Lunin–Maldacena solution (12) has small curvature only when: \(γ R ≪ 1\) and \(R ≫ 1\). Then \(b ≃ 2γ\) and the solutions generated by using either Γ or Θ are basically equivalent. Yet we find it interesting that the periodicity of the parameter γ is manifest in this description. Nonetheless, note that this is not quite the correct periodicity condition. Our result is periodic when \(γ → γ + 2π\) whereas from (34) we expect: \(γ → γ + π\). The reason for this discrepancy lies in equation (30). Indeed, the two ways of defining deformed commutators, one in terms of commuting variables multiplied with a star product and the other in terms of noncommuting ones, are only strictly equivalent when the commutation relations are c-numbers. "Comparing” equations (29) and (27) in this case we see that there is a phase difference between the parameters entering the two definitions. The absence of this phase in (29) is responsible for the discrepancy in periodicity. Nevertheless, the star product gives a more natural way to think of Θ\[^{I,J}\] as a contravariant antisymmetric tensor thus having well defined transformation properties a change of coordinates.

Suppose now that no precise definition of a star product between the superfields of the theory was known. Would we be able to construct the noncommutativity matrix and therefore find the gravity dual of the β–deformed gauge theory? A glance at the superpotential of the theory would naturally lead us to define:

\[
Θ'^{i,j} = 2 \sin γ z'_i z'_j \quad \text{and} \quad Θ^{i,j} = 2 \sin γ \bar{z}^i \bar{z}^j
\]

(34)

We can actually reduce Θ\[^{I,J}\] even further using coordinates: \(ψ = \frac{1}{2} \sum_{i=1}^3 ϕ_i, σ_1 = \frac{1}{2}(ϕ_2 + ϕ_3 - 2ϕ_1), σ_2 = \frac{1}{2}(ϕ_1 + ϕ_3 - 2ϕ_2)\). In this parametrization ψ denotes the U(1) circle associated with the R–symmetry of the original background and Θ\[^{I,J}\] reads:

\[
Θ = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

It is then obvious that the solution generating transformation does not act on the U(1)\(_R\) therefore preserving \(N = 1\) supersymmetry.
and therefore correctly guess the purely holomorphic and purely antiholomorphic parts of $\Theta^{ij}$. What about the other parts though? We can actually constrain the form of $\Theta^{ij}$ by the following requirements:

- **Definite Reality Properties.**
  In order to be able to describe the deformation in noncommutative terms we should define the parameters appearing in $\Theta^{ij}$ so as to have a matrix with real entries after going to real coordinates.

- **Symmetries.**
  Since we expect the global symmetries of the Langrangian to be preserved in the strong coupling limit as well, we should ensure that the noncommutativity matrix respects those symmetries. This is true as long as (35):

  $$[z^I, z^J] = i \Theta^{IJ}(z) \quad \text{and} \quad [z^{I'}, z^{J'}] = i \Theta^{IJ}(z')$$

  Note that this is precisely analogous to the condition for a certain symmetry to be an isometry of the metric. Assuming that $\Theta^{IJ}$ is quadratic (35) implies that up to a sign there exist only two possibilities: $\Theta^{IJ} = 0$ or $\Theta^{IJ} = z^Iz^J$.

- **Marginality condition.**
  According to the usual reasoning of AdS/CFT, marginal deformations should be described by AdS geometries with different compact pieces. This suggests that when the noncommutativity parameter is transformed in spherical coordinates, it should be independent of and have no components along the radial direction of AdS. In other words, $\partial_\phi \Theta^{a_1a_2} = 0$ where $a_i$ are angular variables parametrizing the five sphere and $\Theta^{a_1a_2} = 0$. This last requirement completely determines the form of $\Theta^{IJ}$ to be the one appearing in (29).

We see as remarkable as it may seem that there exists a unique noncommutativity matrix which respects the above conditions. Stated differently, simple gauge theory data and elementary notions from the AdS/CFT correspondence, made it possible to fully determine the form of $\Theta^{ij}$. We thus want to understand this matrix as a way of encoding the deformation of the transverse space or in other words, the moduli space of the gauge theory — at least insofar as information relevant to the gauge/gravity duality in the large $N$ limit is concerned. Indeed given the F–term constraints we seem to have extracted information coming from the D–terms. We can convince ourselves of this with the following observation. Recall that the $\beta$–deformation of $\mathcal{N} = 4$ SYM is exactly marginal and that the deformation enters only in the superpotential of the theory. This means that we wish not to deform the D–terms in the Lagrangian. Note however that we can write the D–terms of the $\mathcal{N} = 4$ theory as:

$$\text{Tr}[\Phi_I, \Phi^I][\Phi_J, \Phi^J] = \text{Tr}[\Phi_I, \Phi_J][\Phi^I, \Phi^J] + \text{Tr}[\Phi_I, \Phi^J][\Phi_J, \Phi^I]$$

The first term on the right hand side of equation (36) is precisely the contribution to the potential coming from the F–terms. We then deduce that if we wish to retain the D–terms unaffected by the deformation of the F–term commutator we must induce an appropriate deformation on the commutator between holomorphic and antiholomorphic fields as well. Surprisingly enough, the reasoning outlined above seems to have granted us this exact piece of information.

It is now evident that we can identify the Lunin–Maldacena generating solution technique with the method employed in the case of noncommutative gauge theories. The noncommutative data in this
context are basically given to us from the gauge theory Lagrangian. This is quite natural since the deformations we are dealing with are exactly marginal. It is worth pointing out here that combined with the knowledge of the gravity dual of the parent $\mathcal{N} = 4$ theory, these data made it possible to find the gravity solution dual to the deformed theory. Unfortunately, this is not as general a statement as it may seem since the particular method employed was applicable only because there existed a coordinate system in which $\Theta^{ij}$ was reduced to a constant and along isometry directions of the metric. In a forthcoming letter [27] we will nevertheless be able to extract some information on the gravity duals of the marginally deformed $\mathcal{N} = 4$ theory when the parameter $\rho$ in [11] is different than zero.

5 Applications and New Backgrounds

In the previous section we were able to associate a specific noncommutativity matrix to the $\beta$–deformed gauge theory. We found that indeed there exists a coordinate system for which $\Theta^{ij}$ is position independent and lies along $U(1)$ isometries of the transverse space metric as well as of the $S^5$. Identifying the solution generating transforms of [11] and [36] was then a straightforward task. This result naturally opens up two main directions for further study — the first one pertaining to noncommutative gauge theories and the second to deformations of $\mathcal{N} = 4$ SYM. In what follows we will try to touch upon several questions arising in both these cases.

5.1 Noncommutative gauge theories.

The most direct application of the ideas discussed so far is to consider the Lunin–Maldacena prescription in order to obtain the gravity duals of noncommutative gauge theories with $\beta$–type noncommutativity. This simply means that we wish to think of $\Theta^{ij}$ or rather $\Gamma$ of (9) as a noncommutativity matrix along the worldvolume of the D3–brane. Provided a decoupling limit exists, we can use the solution generating technique reviewed in section 2 to either deform the p–brane solution itself, or the near horizon geometry directly. For reasons of uniformity, we decided to adhere to the latter prescription in what follows. In four dimensional Euclidean space, $\Theta^{ij}$ can be written in complex coordinates as:

$$[z_i, z_j] = ibz_iz_j, \quad [\bar{z}_i, \bar{z}_j] = ib\bar{z}_i\bar{z}_j, \quad [z_i, \bar{z}_j] = -ibz_i\bar{z}_j$$

for $i < j$ and $i,j=1,2$ (37)

As we already saw in the previous section transforming to polar coordinates yields a constant noncommutativity parameter along the two–torus:

$$[\phi_1, \phi_2] = ib, \quad [\rho_1, \rho_2] = [\rho_i, \phi_j] = 0 \quad i, j = 1, 2$$

(38)

Constructing a matrix out of these relations is a fairly obvious step which leads us to matrix $\Gamma$ appearing in (9). We can therefore directly apply the associated T–duality transform [6] on the $\text{AdS}_5 \times S^5$ geometry. The relevant part of the background matrix is:

$$E = u^2 R^2 \begin{pmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{pmatrix}$$

(39)

$^9$Similar considerations in the context of the Maldacena–Nunez background appeared in [22, 34].

$^{10}$Obviously the same procedure can be applied to all branes in a fashion similar to [39, 40, 41].

$^{11}$One can actually check this by either calculating the graviton absorption cross–section or the potential that gravitons feel due to the presence of the D–brane [55].
and substituting into (7) we find:

\[ds^2_{str} = ds^2_{\text{AdS}} + ds^2_{S^5},\]

where

\[ds^2_{\text{AdS}} = u^2 R^2 (d\rho_1^2 + d\rho_2^2 + G(\rho_1^2 d\phi_1^2 + \rho_2^2 d\phi_2^2))\]

\[B = \hat{b} R^2 G \rho_1^2 \rho_2^2 u^4 d\phi_1 \wedge d\phi_2,\]

\[\hat{b} = R^2 b,\]

\[G = \frac{1}{1 + \hat{b}^2 \rho_1^2 \rho_2^2 u^4},\]

\[F_3 = -3(4\pi N) b u^5 \rho_1 \rho_2 d\rho_1 \wedge d\rho_2 \wedge du,\]

\[F_5 = 4\pi N (\omega_{\text{AdS}} + \omega_{S^5})\]

with the RR–fields computed using the T–duality rules of [29][30][31][32][33]. Note here that the effect of noncommutativity is important for large radial directions but negligible for small ones. The same behaviour has been observed in the case of the Melvin Universe [45][41]. It seems natural therefore to expect that manifestations of this spatial nonhomogeneity will be similar to those described in [45]. It would be interesting for this purpose to explore the instanton, monopole and vortex solutions of the theory. In the Melvin–twist gauge theory the corresponding analysis showed [45] that although the length of the magnetic monopole is position dependent, its mass agrees with the ordinary SYM monopole solution. It is plausible that study of the \(\beta\)–type noncommutative gauge theory along these lines will lead to analogous results. In addition, it is important to investigate the stability properties of the above solution, since the background in question may generically break supersymmetry (see e.g. [50][57] for a discussion on this point). We would like now to proceed and consider the same type of deformation in Lorentzian signature but before doing so, let us make a few remarks regarding the action of the gauge theory dual to (40).

Clearly, knowledge of an appropriate star product is more often than not, necessary in order to specify the action that describes a noncommutative gauge theory. In the case illustrated above, \(\Theta^{ij}\) is position dependent and it is then known that a suitable product is the one defined by Kontsevich in [58]. Naively one would then think that the action of the gauge theory is obtained by simply replacing the ordinary product of functions with the star product. The latter product is however not compatible with the Leibnitz rule so that one should actually employ what is referred to as the "frame formalism" introduced in [59]. Alternatively, one can take advantage of the fact that \(\Theta^{ij}\) is constant in polar coordinates and specify a Moyal–like product of functions. The precise mapping between this product and the one defined by Kontsevich should then be found, which would however not be the result of a simple change of coordinates. This procedure has been carried out explicitly in a number of cases [60][45][41] and we refer the reader to these papers for details.

Let us now move on to consider the \(\beta\)–type deformation on a four–dimensional spacetime with Lorentzian signature. Performing a wick rotation according to \(z \rightarrow ix^+, \bar{z} \rightarrow ix^-\) along with \(b \rightarrow ib\) we can write the commutation relations of equation (37) as:

\[[x^+, z] = ibx^+z, \quad [x^-, \bar{z}] = ibx^-\bar{z}, \quad [x^+, \bar{z}] = ibx^+\bar{z} \quad [x^-, z] = ibx^-z \quad \text{and} \quad [z, \bar{z}] = [x^+, x^-] = 0\]

We therefore see that in this case we have to deal with a temporal noncommutativity parameter. In general, field theories on spaces with time–like noncommutativity \(\Theta^{0i} \neq 0\) are acausal [61][62] whereas their quantum counterparts are not unitary. A decoupled field theory limit for D–branes in this case does not exist. It was however found in [61][62] that a scaling limit where the closed string sector can be separated from the open string one is indeed possible. Massive open strings do not decouple in this limit which thus defines a noncommutative open string theory (NCOS) rather than a field theory. Several aspects of these NCOS theories are explored in [63][64][65][66][67][68].
The precise analysis of which types of noncommutativity lead to unitary theories and which not, was carried out in \cite{69} along the lines of \cite{70}. There it was shown that a necessary condition for unitarity is that the following inner product between external momenta is positive definite:

$$p \cdot p \equiv -p_\mu \Theta^{\mu\nu} \mathcal{G}_{\sigma\tau} \Theta^{\tau\nu} p_\nu > 0$$

where $\mathcal{G}$ is the background metric for the open strings and the corresponding field theory. Let us therefore evaluate this quantity for the $\beta$–like noncommutativity under consideration here. It is easier if we first perform a coordinate transformation to go from coordinates $(t, x_1, x_2, x_3)$ to $(\tau, \theta, r, \phi)$ defined as:

$$t = \tau \cosh \theta, \quad x_1 = \tau \sinh \theta, \quad x_2 = r \cos \phi \quad \text{and} \quad x_3 = r \sin \phi. \quad \text{Here } \tau \in (\infty, \infty), r \in [0, \infty) \quad \text{although } \theta \text{ can be chosen compact or non compact. This transformation will bring the commutation relations to the form}$$

$$\begin{align*}
[\theta, \phi] &= ib \\
[r, \tau] &= [r, \phi] = [\tau, \theta] = [\tau, \phi] = 0
\end{align*}$$

and substituting into (42) we obtain:

$$p \cdot p = b^2(p_\theta^2 r^2 + p_\phi^2 \tau^2)$$

which is clearly positive definite. Can we therefore deduce that the $\beta$–type noncommutative deformation describes a unitary field theory? To be precise, the unitarity requirement of (42) is proven for a position independent noncommutativity parameter turned on in flat space. In our case, as soon as we go to a reference frame where $\Theta$ is constant, the corresponding spacetime exhibits a time–dependent behaviour. It is therefore ambiguous what the meaning of unitarity is in this context.

It may be interesting however to address these issues through the dual gravity description of this theory. Let us therefore apply the T–duality transformation rules in order to construct this background. Alternatively, we can wick rotate the Euclidean solution of equation (40) according to $\rho_1 \to i\tau, \phi_1 \to i\theta, b \to ib$. Either way we obtain

$$ds^2_{str} = ds^2_{\tilde{AdS}} + ds^2_S, \quad \text{where } ds^2_{\tilde{AdS}} = u^2 R^2 (-d\tau^2 + dr^2 + G(\tau^2 d\theta^2 + r^2 d\phi^2))$$

$$B = \tilde{b} R^2 G \tau r^2 u^4 \wedge d\phi, \quad e^{2\Phi} = Ge^{2\Phi_0}$$

$$G = \frac{1}{1 + \tilde{b}^2 \tau^2 r^2 u^4}, \quad \tilde{b} = R^2 b$$

Note again that equation (44) defines a time dependent background dual to a noncommutative theory which can be thought of as living either in flat space with temporal time–dependent noncommutativity parameter or in a time–dependent background which is noncommutative only along some of the spatial directions. Similar time–dependent configurations were explored in \cite{71}\cite{72}\cite{73}\cite{74}. For the case of compact $\theta$ with $\theta \sim \theta + 2\pi$ and rational parameter $\beta$, the gravity solution (44) corresponds to the near horizon geometry of a D3–brane immersed in a time–dependent background that admits an orbifold description \cite{57}\cite{50}\cite{56}\cite{77}. The latter deformation of flat space can be recovered from flat space with the

\footnote{These coordinates cover half of \(\mathbb{R}^3\).}

\footnote{The resulting background appears to be well defined due to the particular nature of the wick rotation employed. This fact does not seem to indicate the need for another kind of scaling limit as usual in the dual description of NCOS theories. We would therefore naively expect that indeed this supergravity solution is dual to a field theory.}
same technique \[57\].

\[ds^2 = -d\tau^2 + dr^2 + \frac{\tau^2}{1 + b^2\tau^2\tau^2}d\theta^2 + \frac{\tau^2}{1 + b^2\tau^2\tau^2}d\phi^2\]

\[e^{2\Phi} = \frac{1}{1 + b^2\tau^2\tau^2}\]

\[B = -\frac{b^2\tau^2\tau^2}{1 + b^2\tau^2\tau^2}d\theta \wedge d\phi\]  \hspace{1cm} (45)

The background indicated above presents an interesting time evolution noted in \[57\]. In particular, it appears to be periodically changing for the designated choices of $\theta$ and $\beta$. At $\tau = -\infty$ it is described via the orbifold $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta = 2\pi} \times [\mathbb{C}/\mathbb{Z}_N]$ (i.e. orbifold by the boost $\Delta = 2\pi$) which gradually evolves to $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta = 2\pi} \times \mathbb{C}$ at time $\tau = 0$. Then the reverse process begins until it reaches the original orbifold description at $\tau = \infty$. In complete analogy, the spacetime of equation (44) shows a periodic evolution with the effects of noncommutativity becoming most important at $\tau = \pm \infty$ but negligible at $\tau = 0$ where the geometry tends to $\text{AdS}_5 \times S^5$.

This completes our discussion of noncommutative gauge theories. We have clearly here only alluded to a number of issues regarding these theories and noncommutative spacetimes in general. It would certainly be of interest to explore these issues further in the future.

5.2 Matter–content deformations of $\mathcal{N} = 4$ SYM

A natural question to ask in this context is whether we can now borrow results pertaining to noncommutative gauge theories in order to explore different kinds of (super)potential deformations of $\mathcal{N} = 4$ SYM. A few cases where the solution generating technique was applicable were already mentioned in section \textit{3}.

Consider for instance the original situation where a constant noncommutativity parameter is turned on. Here, we would like to translate this deformation to some kind of transverse space noncommutativity. If we parametrize our six dimensional space with complex coordinates $(z^I, \bar{z}^I)$, we can write:

\[[z^I, z^J] = ib, \quad [\bar{z}^I, \bar{z}^J] = ib, \quad [z^I, \bar{z}^J] = -ib \quad \text{with I,J cyclically ordered} \hspace{1cm} (46)\]

We may then associate these commutation relations to a deformation of the gauge theory potential $V$. Since the type of deformations considered in this section may generically break supersymmetry we prefer to state the deformation in terms of the potential which of course may when appropriate be promoted to the superpotential. Identifying the coordinates $(z^I, \bar{z}^I)$ with the scalar fields of the theory would naturally lead to\textsuperscript{14}:

\[V_{\mathcal{N} = 4} = \text{Tr}[\Phi^I, \bar{\Phi}^J][\Phi^I, \bar{\Phi}^J] \to \text{Tr}[\Phi^I, \bar{\Phi}^J][\Phi^I, \bar{\Phi}^J]_\star. \hspace{1cm} (47)\]

Here the star product is defined according to \[46\] as: $\Phi^I \star \bar{\Phi}^J = \Phi^I \Phi^J + ib$. In a similar fashion we could relate the noncommutative deformation of the Melvin Universe which in complex coordinates looks like \textsuperscript{15}:

\[[z_1, z_2] = -bz_1, \quad [\bar{z}_1, \bar{z}_2] = bz_1, \quad [z_1, \bar{z}_2] = -bz_1 \quad [\bar{z}_1, \bar{z}_2] = b\bar{z}_1 \quad \text{with all other commutators vanishing} \hspace{1cm} (48)\]

to a potential deformation of the same form as in \[47\] but with a different star product as indicated from \[48\].

\textsuperscript{14}This deformation is only meaningful for gauge groups other than $\text{SU}(N)$.

\textsuperscript{15}Here we defined $z_1 \equiv x_1 + ix_2$ and $z_2 \equiv x_3 + ix_4$ with $x_i$ as appearing in equation \[20\].
Yet the true story is not as simple as this. These deformations are not marginal and the theory will generically flow from some UV point to an IR one. This means that we cannot solely rely on the data given to us from the Lagrangian of the theory which we can only take to be a valid description near the UV (small \(b\)). Moreover, the precise arguments that helped us construct the noncommutativity matrix encoding the moduli space in the \(\beta\)-deformed case are not applicable anymore. We do not therefore have a means of understanding the commutation relations between holomorphic and antiholomorphic fields/coordinates despite the fact that we believe such a construction may be possible in the future. In addition we do not even know whether a noncommutative description of the transverse space will be valid throughout the flow.\(^{16}\)

Nevertheless, we could still expect to find the relevant supergravity solutions and use that as a means of understanding the precise gauge theory duals. Unfortunately this is again a difficult task to pursue because the solution generating technique discussed in this paper in not applicable anymore. The reason for this lies in the fact that the directions where the noncommutativity parameter is constant are not isometry directions of the transverse part of the D3–brane geometry. One could of course apply the T–duality transform on flat space. This would give rise to a deformed flat space geometry with non–trivial \(B\)–field and dilaton turned on, in which once D3–branes are immersed and the near horizon limit is taken, would result in the appropriate gravity dual. We think that it will be very interesting to explore this point further as well as to study the corresponding gauge theories which we schematically described above.

6 Conclusion

In this article we established a precise relation between noncommutativity and \(\beta\)–deformations of \(\mathcal{N} = 4\) SYM theory. We first identified a specific matrix within the solution generating transform of [26] which plays the role of noncommutativity parameter \(\Theta^{ij}\) and then showed how it arises from the gauge theory point of view. Moreover, we explained that it is possible to fully specify \(\Theta^{ij}\) by imposing requirements on its particular form naturally deduced from the gauge theory and AdS/CFT. We further argued that \(\Theta^{ij}\) thus constructed encompasses all the relevant information on the moduli space of the gauge theory.

This hints at an alternative path in exploring deformations of the original AdS/CFT proposal which consists in first specifying the associated open string parameters and then mapping them to the closed string ones. Here we investigated the former issue for the particular case of a Leigh–Strassler marginal deformation of the \(\mathcal{N} = 4\) SYM theory. The mapping to the closed degrees of freedom in this case was granted to us in the form of T–duality transformation rules. In a forthcoming publication we will combine the basic reasoning set forth in this note with an attempt to address the latter issue in a situation where \(U(1)\) symmetries are absent and the T–duality prescription is not applicable. Such is the case for the superpotential deformation of equation \(^{14}\) with \(\rho \neq 0\).

There are many possibilities for future work. A natural and possibly straightforward generalization would be to consider marginal deformations of theories with matter fields in the bifundamental, or in other words situations with the D–branes sitting at the orbifold fixed point. It would furthermore be of interest to extend this formulation if possible to deformations which are not marginal, mass deformations for instance. Since both cases have been studied in alternative ways [78] [79] [34] they appear to provide

\(^{16}\)Note however that it is possible to further examine this in certain cases, especially when some of the fields can integrate out by considering the theory at appropriate energy scales.
A sound testing ground for the ideas proposed herein.

A curious feature of this approach is that supersymmetry does not play any central role in it. Indeed the whole discussion so far has solely relied on the commutation relations between the scalar fields of the theory. When however supersymmetry is preserved scalars are accompanied by their fermionic superpartners and it is obvious that similar (anti)commutation relations will be obeyed by the fermions alone as well as between the scalars and the fermions of the theory. It seems plausible to us that information pertaining to these (anti)commutation relations is hidden in the RR sector of the theory it would therefore be of great importance to study it in a similar fashion.

As a natural application of the connection between \( \beta \)–deformations and noncommutativity in this article we also constructed the gravity duals of noncommutative gauge theories with \( \beta \)–type noncommutativity. As mentioned in the previous section, the corresponding backgrounds may generically be unstable. Moreover, in the particular case of Lorentzian signature the gravity solution presents an interesting time evolution. Precisely due to these features, a lot of interesting questions arise which are only touched upon in this note and certainly deserve deeper study.

In summary, we have presented a concrete realization of noncommutativity in the context of marginal deformations of \( \mathcal{N} = 4 \) SYM. We believe this opens up another window into understanding the AdS/CFT correspondence which we hope to further explore in the future.

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