Asymptotic Mutual Information Statistics of Separately-Correlated Rician Fading MIMO Channels

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Abstract

Precise characterization of the mutual information of MIMO systems is required to assess the throughput of wireless communication channels in the presence of Rician fading and spatial correlation. Here, we present an asymptotic approach allowing to approximate the distribution of the mutual information as a Gaussian distribution in order to provide both the average achievable rate and the outage probability. More precisely, the mean and variance of the mutual information of the separately-correlated Rician fading MIMO channel are derived when the number of transmit and receive antennas grows asymptotically large and their ratio approaches a finite constant. The derivation is based on the replica method, an asymptotic technique widely used in theoretical physics and, more recently, in the performance analysis of communication (CDMA and MIMO) systems. The replica method allows to analyze very difficult system cases in a comparatively simple way though some authors pointed out that its assumptions are not always rigorous. Being aware of this, we underline the key assumptions made in this setting, quite similar to the assumptions made in the technical literature using the replica method in their asymptotic analyses. As far as concerns the convergence of the mutual information to the Gaussian distribution, it is shown that it holds under some mild technical conditions, which are tantamount to assuming that the spatial correlation structure has no asymptotically dominant eigenmodes. The accuracy of the asymptotic approach is assessed by providing a sizeable number of numerical results. It is shown that the approximation is very accurate in a wide variety of system settings even when the number of transmit and receive antennas is as small as a few units.

* This work is supported by the STREP project No. IST-026905 (MASCOT) within the sixth framework programme of the European Commission. These results have been presented in part at the IEEE GLOBECOM 2006, Communication Theory Symposium.
I. INTRODUCTION

During the last decade, multiple-input multiple-output (MIMO) wireless systems attracted much interest because of the prediction of outstanding capacity gains with respect to corresponding single-input single-output (SISO) systems [11], [46], [50].

Early work in this area was mostly based on the assumption of independent and identical distributed (iid) Rayleigh fading paths justified by the rich scattering assumption [11].

However, experimental and theoretical results showed that more realistic MIMO channel models are required to obtain more accurate results, which can dramatically impact the potential capacity gains [12], [13], [42]. Thus, more realistic channel models have been proposed in the literature accounting for both multipath correlation and the presence of a line-of-sight (LOS) component (Rician fading).

In this work we focus on the separately-correlated MIMO Rician fading channel whose channel matrix can be written as

\[ H = \bar{H} + R^{1/2}H_wT^{1/2}, \]

where \( \bar{H} \) is a constant matrix accounting for the LOS component, \( R, T \) are the nonnegative definite receive and transmit correlation matrices, respectively, and \( H_w \) is a matrix of iid circularly-symmetric zero-mean complex Gaussian random variables. We assume that the channel parameters \( (\bar{H}, R, T) \) are unknown at the transmitter whereas the receiver knows the channel matrix realization exactly (i.e., following the terminology introduced by Goldsmith et al. [13], there is perfect channel-state information at the receiver (CSIR) and no channel distribution information at the transmitter (CDIT)).

In the MIMO Rician fading channel literature, many works focus on the spatially uncorrelated channel. In particular, Hösli et al. [19] show that the mutual information is monotonically non-decreasing in the singular values of \( H \). Jayaweera and Poor [22] derive the exact ergodic capacity when only the Rice factor \( K \) known at the transmitter by using the joint eigenvalue distribution of the noncentral Wishart matrix, and lower and upper bounds to the ergodic capacity when full CDIT is available. Hansen and Bölcskei [18] investigate the case of high SNR with unit–rank average channel matrix. Venkatesan et al. [47] derive the capacity-achieving covariance matrix of the channel, generalizing previous results relevant to the case of unit rank \( \bar{H} \) [21]. Kang and Alouini [25] obtain the exact ergodic capacity by calculating the determinant of a matrix whose
entries contain confluent hypergeometric functions of the second kind and the exact capacity variance in terms of Meijers G-functions.

Many recent works propose upper and lower bounds to the capacity of the separately correlated Rician fading MIMO channel. McKay and Collings [31], [32] derive upper and lower bounds to the ergodic mutual information, based on the method of zonal polynomials, which are asymptotically tight as the SNR grows to infinity. They also provide an asymptotic (in the SNR) expression of the mutual information variance. The bounds have been further refined in [33]. In the same context, Cui et al. [3] obtain similar upper and lower bounds to the ergodic capacity by using a determinant expansion in terms of minors whose moments are calculated by use of hypergeometric matrix functions. In a recent paper, Lebrun et al. [28] use asymptotic bounds to confirm that the mean and variance of the MIMO channel capacity approach the values of the corresponding underlying scattering channel when the number of antennas is large and the average channel matrix has unit rank.

Another way to circumvent the difficulties related to the derivation of an exact expression of the capacity consists of assuming an asymptotically large number of transmit and receive antennas.

In this contest, Moustakas et al. [34] show that the capacity of the correlated Rayleigh fading MIMO channel converges to a Gaussian random variables whose mean and variance are calculated in closed form. Their derivation is based on approximating the moment generating function (MGF) of the capacity by using the replica method, a technique originally introduced in the context of statistical physics and successfully applied to several communications problems (see, e.g., [14], [37], [43], [49]).

A considerable number of research works in this area are based on the use of Stieltjes Transform and lead to results compatible to those presented here. Among them, Dumont et al. [5] derived the asymptotic mutual information mean in the case of single-sided (receiver side) separately-correlated Rician fading. Hachem et al. [15], [16] derive the asymptotic mutual information mean in the case of uncorrelated (but not iid) Rician fading. Hachem et al. [17] derive the asymptotic mutual information mean and variance in the case of separately-correlated Rayleigh fading and consider the convergence of the random mutual information to the Gaussian distribution.

Another approach to show that the asymptotic capacity distribution converges to the Gaussian
distribution is based on the study of the channel matrix singular values. Focusing on the correlated Rayleigh fading MIMO channel, Martin and Ottersten [30] show that, subject to certain conditions on the correlation between the channel elements, the limiting capacity distribution is Gaussian.

Recent results in this area are aimed at extending the scenario to the MIMO communication channel with interference. This direction was taken originally by Moustakas et al. in [34] for the Rayleigh fading channel. More recently, Riegler and Taricco [40] derived the second-order statistics of the separately-correlated Rician fading MIMO channel mutual information in the presence of interference. This work was based on the use of Grassman variables and supermatrix theory (see references therein).

A further extension of the results presented by this paper is the derivation of the ergodic capacity achieving covariance matrix. In this contest, the following works are worth mentioning.

- Dumont et al. provide in [6] an asymptotic method (based on the Stieltjes transform) to derive the ergodic capacity achieving covariance matrix for the separately correlated Rician fading MIMO channel. The key ingredient is the asymptotic ergodic mutual information formula, which the authors refer to a preliminary version of [16], and is equivalent to the expression derived in this paper (by using a the replica method instead of Stieltjes transforms). The authors compare their results against those obtained by Vu and Paulraj, who used an interior point with barrier optimization method, and showed that the asymptotic method has a considerable advantage in terms of time efficiency.

- Taricco and Riegler provide in [45] an algorithm for the derivation of the ergodic capacity achieving covariance matrix for the separately correlated Rician fading MIMO channel with interference. This work uses an asymptotic approximation of the ergodic mutual information for a separately-correlated Rician fading MIMO channel in the presence of interference.

Finally, it is worth mentioning the recent result obtained by Riegler and Taricco [41] in the area of multiuser MIMO communications. This result consists in the derivation of the ergodic sum-rate achieving covariance matrices of a separately correlated Rician fading multiple access MIMO channel and of the resulting ergodic capacity region. The paper uses the replica method to derive the ergodic mutual information expressions required to upper bound the partial sum rates of the multiuser system.

Summarizing, the main contributions of this paper are listed as follows.
• We consider an asymptotic setting where the number of transmit and receive antennas grows to infinity, while their ratio approaches a finite value. A closed-form expression of the asymptotic mean and variance of the mutual information of the separately-correlated MIMO Rician fading channel for arbitrary input signal covariance matrix is derived. These results depend on a pair of scalar parameters obtained by solving two polynomial equations. One of the main technical difficulties encountered, in addition to those addressed in [34], is in the need to deal with $2 \times 2$ block matrices in the saddlepoint approximation which do not split nicely as in the Rayleigh fading case [34].

• A detailed study of the cumulant generating function of the mutual information in the asymptotic regime leads to prove that it converges in distribution to a Gaussian random variable, provided that the correlation matrices $\mathbf{T}$ and $\mathbf{R}$ satisfy a mild technical condition. This condition is tantamount to requiring that the channel has no asymptotically dominant eigenmode.

• From a theoretical point of view, the key issues of uniqueness of the fixed-point equation solution and of reality of the variance expression are addressed in two appendices. The latter result is important also because it allows to say that the replica-symmetric saddlepoint is stable.

• To illustrate the analytic applications of the results obtained, the spatially uncorrelated Rician fading MIMO channel is considered and its ergodic capacity achieving covariance matrix is derived by using the asymptotic approximation. The results obtained are compared to the existing nonasymptotic literature. It can be noticed that the eigenvectors of the optimum covariance matrix can be obtained very easily from the average channel matrix (key result from [21]). Moreover, also the eigenvalues are derived — in the asymptotic setup — by using a fixed-point water-filling algorithm that is much simpler than the numerical techniques proposed in the literature to solve similar problems [24] (in the nonasymptotic case).

• Finally, a set of numerical results is presented to validate the asymptotic analytic method. These results depart from a basic system scenario and change one of the following parameters at a time: number of antennas, antenna ratio, spatial correlation (summarized in the base of the exponential correlation considered, common for the transmit and receive sides), Rice factor, and SNR. As far as concerns the ergodic capacity, it is shown that the relative error between the asymptotic and exact result (the latter derived by extensive Monte-Carlo
simulation) is always lower than 1%, even when the number of antennas is as low as one or two. The mutual information standard deviation asymptotic results are less accurate, with a relative error rising to a few percent units. However, using the asymptotic Gaussianity to derive an approximate outage mutual information, the standard deviation error is shown to have a negligible impact on this channel metric, yielding a relative error lower than 1% in the cases considered.

A. Notation and basic results

We denote (column-) vectors and matrices by lowercase and uppercase boldface characters, respectively. The imaginary unit is $j = \sqrt{-1}$. The $a$th element of a vector $x$ is $(x)_a$. The $(a, b)$th element of a matrix $A$ is $(A)_{ab}$. $\delta_{ab} = 1$ if $a = b$ and 0 if $a \neq b$. The transpose of a matrix $A$ is $A^\top$. The Hermitian transpose of a matrix $A$ is $A^H$. The Hermitian part of a square matrix $A$ is defined as $H(A) \triangleq \frac{1}{2}(A + A^H)$. The trace of a matrix is $\text{Tr}(A) = \sum_a (A)_{aa}$. The exponential of the trace of a matrix is $\text{etr}(A) \triangleq \exp(\text{Tr}(A))$. The minimum and maximum eigenvalues of a matrix $A$ are $\lambda_{\text{min}}(A)$ and $\lambda_{\text{max}}(A)$, respectively. The minimum and maximum singular values of a matrix $A$ are $\sigma_{\text{min}}(A)$ and $\sigma_{\text{max}}(A)$, respectively. The Frobenius norm of a matrix $A$ is $\|A\|_2$; its square can be written as $\|A\|^2 = \text{Tr}(AA^H)$. The Kronecker product of two matrices $A$ and $B$ is $A \otimes B$. If $A \in \mathbb{C}^{m_1 \times n_1}$ and $B \in \mathbb{C}^{m_2 \times n_2}$, then $A \otimes B \in \mathbb{C}^{m_1 m_2 \times n_1 n_2}$ and we can write it in block matrix form as $[(A)_{ab} B]$ for $a = 1, \ldots, m_1$ and $b = 1, \ldots, n_1$. Among the properties of the Kronecker product we recall the following [20, p. 475]:

1. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2. $\prod_k (A_k \otimes B_k) = (\prod_k A_k) \otimes (\prod_k B_k)$;
3. $(A \otimes B)^H = A^H \otimes B^H$;
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5. $\text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)$;
6. $\det(A \otimes B) = \det(A)^n \det(B)^m$ if $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{n \times n}$.

The column vector obtained by stacking the columns of $A$ on top of each other from left to right is the matrix square-root of the Hermitian nonnegative definite matrix $A$ and is defined as $UA^{1/2}U^H$ where $A = U\Lambda U^H$ is the unitary factorization of $A$ [20, p. 414].

The notation $x \sim \mathcal{N}_c(\mu, R)$ defines a vector of complex jointly circularly-symmetric Gaussian random variables with mean value $\mu = E[x]$ and covariance matrix $R = E[xx^H] - \mu \mu^H$ and its joint probability density function (pdf) is given by

$$f(x) = \det(\pi R)^{-1} \exp\left[-(x - \mu)^H R^{-1} (x - \mu)\right].$$
\( \mathbf{E}[X] \) and \( \mathbf{Var}[X] \) are the mean and the variance of the random variable \( X \), respectively. \( (x)_+ \triangleq \max\{0, x\} \). Given the functions \( f(x) \) and \( g(x) \), we have, for \( x \to \infty \), \( f(x) = O(g(x)) \) if \( \limsup_{x \to \infty} |f(x)/g(x)| < \infty \) and \( f(x) = \Theta(g(x)) \) if

\[
0 < \liminf_{x \to \infty} \frac{f(x)}{g(x)} \leq \limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty.
\]

### II. MIMO CHANNEL MODEL

We consider a narrowband block fading channel with \( n_T \) transmit and \( n_R \) receive antennas characterized by the following equation:

\[
\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}.
\]

Here, \( \mathbf{x} \in \mathbb{C}^{n_T \times 1} \) is the transmitted signal vector, \( \mathbf{H} \in \mathbb{C}^{n_R \times n_T} \) is the channel matrix, \( \mathbf{z} \in \mathbb{C}^{n_R \times 1} \) is the additive noise vector, and \( \mathbf{y} \in \mathbb{C}^{n_R \times 1} \) is the received signal vector.

We assume that the additive noise vector contains iid entries \((\mathbf{z})_a \sim \mathcal{N}(0, 1)\) for \( a = 1, \ldots, n_R \).

The channel matrix models separately (or Kronecker) correlated Rician fading so that it can be written as

\[
\mathbf{H} = \tilde{\mathbf{H}} + \mathbf{R}^{1/2}\mathbf{H}_w\mathbf{T}^{1/2}
\]

where \( \tilde{\mathbf{H}} \) represents the mean value and is related to the presence of a line-of-sight signal component in the multipath fading channel, the Hermitian nonnegative definite matrices \( \mathbf{T}, \mathbf{R} \) are the transmit and receive correlation matrices, and \((\mathbf{H}_w)_{ab} \sim \mathcal{N}(0, 1)\) for \( a = 1, \ldots, n_R \) and \( b = 1, \ldots, n_T \). The covariance between different entries of \( \mathbf{H} \) is

\[
\text{cov}((\mathbf{H})_{ij}, (\mathbf{H})_{i'j'}) = \sum_{k, k', \ell, \ell'} \mathbf{E}[(\mathbf{R}^{1/2})_{ik}(\mathbf{H}_w)_{k\ell}(\mathbf{T}^{1/2})_{\ell j}(\mathbf{R}^{1/2})^*_{k'\ell'}(\mathbf{H}_w)^*_{k'\ell'}(\mathbf{T}^{1/2})^*_{\ell' j'})
\]

\[
= \sum_{k, \ell} (\mathbf{R}^{1/2})_{ik}(\mathbf{T}^{1/2})_{\ell j}(\mathbf{R}^{1/2})^*_{k\ell}(\mathbf{T}^{1/2})^*_{j i'}
\]

\[
= (\mathbf{R})_{ii'}(\mathbf{T})^*_{jj'}.
\]

### A. Normalizations

Assume that the input signal vector \( \mathbf{x} \) has zero mean \(^1\) and covariance matrix \( \mathbf{Q} = \mathbf{E}[\mathbf{x}\mathbf{x}^H] \). To simplify notation, we define

\[
\tilde{\mathbf{H}} \triangleq \tilde{\mathbf{H}}\mathbf{Q}^{1/2} \quad \text{and} \quad \tilde{\mathbf{T}} \triangleq \mathbf{Q}^{1/2}\mathbf{T}\mathbf{Q}^{1/2},
\]

\(^1\) There is no point in having a nonzero mean input since the signal power would be greater but the mutual information would remain the same by the relationship \( I(\mathbf{x}; \mathbf{y}) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{x} - \mu_x) - h(\mathbf{x} - \mu_x|\mathbf{y}) \) where \( \mu_x = \mathbf{E}[\mathbf{x}] \) [2, Th.9.6.3].
so that the input signal covariance matrix is implicitly accounted for into \( \tilde{H} \) and \( \tilde{T} \).

Then, the total received power is given by:

\[
E[\|y\|^2] = E[\|\tilde{H}x\|^2] + E[\|R^{1/2}H_wT^{1/2}x\|^2] + E[\|z\|^2]
\]

\[
= \text{Tr}[\tilde{H}^H \tilde{H}] + \text{Tr} \{ E[\tilde{H}^H \tilde{H} H_w] T^{1/2} Q T^{1/2} \} + \text{Tr} [I_{n_R}]
\]

\[
= \|\tilde{H}\|^2 + \text{Tr}(R) \text{Tr}(\tilde{T}) + n_R .
\]

(2)

According to eq. (2), the channel Rice factor (defined as the ratio of the received direct to diffuse power [9]) and the SNR are given by:

\[
K = \frac{\|\tilde{H}\|^2}{\text{Tr}(R) \text{Tr}(\tilde{T})} \quad \text{and} \quad \rho = \frac{(K + 1) \text{Tr}(\tilde{T}) \text{Tr}(R)}{n_R} .
\]

(3)

In the special case of scalar input signal covariance matrix, \( Q = qI_{n_T} \), we have:

\[
K = \frac{\|\tilde{H}\|^2}{\text{Tr}(R) \text{Tr}(\tilde{T})} \quad \text{and} \quad Q = \frac{\rho}{K + 1} \frac{n_R}{\text{Tr}(R) \text{Tr}(\tilde{T})} I_{n_T}.
\]

(4)

Remark II.1 The two definitions (3) and (4) of the Rice factor are different when the input signal covariance matrix is not proportional to the identity matrix. Definition (3) complies with the common knowledge that the Rice factor represents the ratio between the line-of-sight and the scattered received signal power. It has the disadvantage of depending on the input signal covariance matrix, which may become an issue when the channel capacity is investigated. In that case, when a constraint on the total input signal power (\( \text{Tr}(Q) \)) is set, one has to resort to definition (4) even though it does not meet the common meaning of the Rice factor.

B. Asymptotic setting

Here we define the asymptotic setting assumed for the derivation of the mutual information mean and variance when \( n_T, n_R \to \infty \) and \( n_T/n_R \to \kappa, (0 < \kappa < \infty) \).

More precisely, we consider the sequence of parameters \( n_R = 1, 2, \ldots \) and \( n_T = \lceil \kappa n_R \rceil \) and a corresponding sequence of deterministic matrices \( \tilde{H}, \tilde{T}, R, Q \) and random matrices \( H_w \) of suitable dimensions. Additionally,

- The Rice Factor \( K \) and the SNR \( \rho \) are constant as \( n_R \to \infty \).

\[ \text{We have: } E[(H_w^H R H_w)_{ij}] = \sum_{k,\ell} E[(H_w)_{ki}(R)_{k\ell}(H_w)_{\ell j}] = \text{Tr}(R) \delta_{ij} \]
The correlation matrices are normalized by
\[ \text{Tr}(T) = n_T \quad \text{and} \quad \text{Tr}(R) = n_R. \] (5)

The matrices \( \tilde{T} \) and \( \tilde{H} \) are normalized by
\[ \text{Tr}(\tilde{T}) = \frac{\rho}{K+1} \quad \text{and} \quad \|\tilde{H}\|^2 = \frac{K\rho}{K+1}n_R. \]

### III. Channel Mutual Information and Cumulant Generating Function

It is well known \cite{2} that the mutual information of a MIMO channel with channel matrix \( H \) and input signal covariance matrix \( Q \) is given by
\[ I(H) = \ln \det(I_{n_R} + HQH^H) \quad \text{nats/complex dimension}. \]

The pdf of this random variable can be derived by its MGF, defined as
\[ G(\nu) \triangleq \mathbb{E}[\exp(-\nu I(H))] = \mathbb{E}\left[ \det(I_{n_R} + HQH^H)^{-\nu} \right]. \]

Our goal is to derive an expression of the cumulant generating function (CGF) of \( I(H) \), defined as \( g(\nu) \triangleq \ln G(\nu) \). This allows us to derive the mean and the variance of \( I(H) \) as follows:
\[ \mathbb{E}[I(x; y)] = -g'(0^+) \quad \text{and} \quad \text{Var}[I(x; y)] = g''(0^+). \]

In the following, we resort to the replica method to obtain an asymptotic expansion of \( g(\nu) \).

#### A. The replica method

The replica method was originally used in the study of spin glasses \cite{7}. Later, it found application in other research areas, such as neural networks, coding, image processing, and communications \cite{39}.

Many research works describe the physical meaning of the replica method (see, e.g., \cite{14}, \cite{37}, \cite{39}, \cite{43}, \cite{49}). Here, we confine ourselves to a short review of its basic assumptions.

According to \cite{39}, the replica method applies to a sequence of random variables \( Z_N \) (typically representing the partition function of a physical system), converging in distribution to some \( Z_\infty \).

The method is aimed at obtaining the so-called free energy of the system:
\[ \mathcal{F} = \lim_{N \to \infty} -\frac{1}{N}\mathbb{E}[\ln Z_N] = \lim_{N \to \infty} -\frac{1}{N}\left\{ \lim_{\nu \to 0} \frac{\partial}{\partial \nu} \mathbb{E}[Z_N^\nu] \right\}. \] (6)

3 The cumulant generating function of a real Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \) is given by \( g(\nu) = -\nu\mu + \frac{1}{2}\nu^2\sigma^2 \).
The method is convenient when direct calculation is impossible (or very hard) whereas it is relatively easy to calculate the limit $\lim_{N \to \infty} E[Z_N^\nu]$ for positive integer $\nu$. In typical applications, the partition function can be expressed as a conditional average: $Z_N = E[Z(u; x_N)|x_N]$. Here, $u$ represents the system microstate and $x_N$ is a set of $N$ independent quenched random parameters. The method is based on the definition of $Z_{\alpha,N} = E[Z(u_{\alpha}; x_N)|x_N]$ as replicas of $Z_N$, obtained by averaging with respect to the independent microstates $u_{\alpha}$ for $\alpha = 1, \ldots, \nu$, conditionally on the quenched parameters. Hence, we can write:

$$G(\nu, N) \triangleq E[Z_N^\nu] = E\left[\prod_{\alpha=1}^{\nu} Z_{\alpha,N}\right].$$

The validity of the replica method is subject to verification of the following assumptions:

1) *(Extension from positive integers).* The limit $\tilde{G}(\nu) \triangleq \lim_{N \to \infty} G(\nu, N)$ is a smooth function of $\nu$ in $\nu = 0$. This function is derived for positive integer $\nu$ and extended to a right neighborhood of $\nu = 0$.

2) *(Interchange of limits).* The limits in (6) can be exchanged, so that

$$\mathcal{F} = \lim_{\nu \to 0} \frac{\partial}{\partial \nu} \left\{ \lim_{N \to \infty} -\frac{1}{N} G(\nu, N) \right\}.$$

3) *(Replica symmetry).* The derivation of the limit

$$\lim_{N \to \infty} E\left[\prod_{\alpha=1}^{\nu} Z(u_{\alpha}; x_N)\right]$$

is based on the saddlepoint approximation (see Appendix E for a brief summary of this topic and [1], [23] for a deeper account) and on the symmetry of the stationary saddlepoint of $\prod_{\alpha=1}^{\nu} |Z(u_{\alpha}; x_N)|$ with respect to the replicated microstate arguments $u_{\alpha}$.

In our application, the role of the partition function is played by

$$Z_{n_R} = \exp[-I(H)] = \det(I_{n_R} + HQH)^{-1}.$$

Our goal is to determine the asymptotic series expansion in $\nu$ of

$$\tilde{g}(\nu) \triangleq \ln E[Z_{n_R}^\nu] = \ln E[\exp(-I(H))],$$

---

4 A physical system is said to be in *quenched disorder* when some random parameters characterizing its behavior do not evolve in time and are then said *quenched* or frozen. Spin glasses are a typical example. Quenched disorder is in contrast to *annealed disorder* where all random parameters evolve in time [10], [39].
and to show that only the first and second-degree coefficients survive in the limit. This is tantamount to saying that the mutual information converges in distribution to a Gaussian random variable.

B. Derivation of the cumulant generating function

Setting \( A = I_\nu, C = I_{nR} + HQ^H, B = D = 0 \) in eq. (19) of Appendix [B], we can write:

\[
\tilde{g}(\nu) \triangleq \ln \left\{ \mathbb{E} \left[ \det (I_{nR} + HQ^H) - \nu \right] \right\} \\
= \ln \left\{ \int_{\mathbb{C}^{nR} \times \nu} \mathbb{E} [\text{etr} \{-\pi U^H (I_{nR} + HQ^H) U\}] dU \right\}.
\]

Next, by setting \( A = I_\nu, C = I_{nT}, B = -D = Q^{1/2}H^H U \) in eq. (19) of Appendix [B] we obtain:

\[
\text{etr}(-\pi U^H HQ^H U) = \int_{\mathbb{C}^{nT} \times \nu} \text{etr}[-\pi (V^H V + U^H HQ^{1/2} V - V^H Q^{1/2} H^H U)] dV.
\]

Hence, we can rewrite the CGF as:

\[
\tilde{g}(\nu) = \ln \left\{ \int_{\mathbb{C}^{nR} \times \nu} dU \int_{\mathbb{C}^{nT} \times \nu} \mathbb{E} [\text{etr} \{-\pi (U^H U + V^H V + U^H HQ^{1/2} V - V^H Q^{1/2} H^H U)\}] dV \right\}.
\]

Now, we calculate the expectation by observing that

\[
\text{Tr}(U^H HQ^{1/2} V) = \text{Tr}(U^H H V) + \text{Tr}([\widetilde{T}^{1/2} V U^H R^{1/2} H^H U])
\]

We obtain:

\[
\tilde{g}(\nu) = \ln \left\{ \int_{\mathbb{C}^{nR} \times \nu} dU \int_{\mathbb{C}^{nT} \times \nu} \text{etr} \{-\pi (U^H U + V^H V + U^H \widetilde{H} V - V^H \widetilde{H} \widetilde{H} U) \}

- \pi^2 U^H R U V^H \widetilde{T} V dV \right\}
\]

after using the result (obtained by applying, e.g., eq. (19) of Appendix [B]):

\[
\mathbb{E} [\text{etr} (A H_w - H_w^H A^H)] = \exp (-||A||^2).
\]
Finally, by setting $A = \pi U^H R U$ and $B = \pi V^H \tilde{T} V$ in eq. (23) of Appendix [3] we obtain:

\[
\bar{g}(\nu) = \ln \left\{ \int_{C_{nR} \times \nu} dU \int_{C_{nT} \times \nu} dV \lim_{\alpha, \beta \downarrow 0} \int \text{etr} \left( WZ - \alpha W W^T + \beta ZZ^T \right) \right. \\
\left. \text{etr} \left[ -\pi (U^H U + V^H V + U^H \tilde{H} V - V^H \tilde{H} U + W U^H R U + Z V^H \tilde{T} V) \right] d\mu(W, Z) \right\}
\]

\[
= \ln \left\{ \int \text{etr}(WZ) d\mu(W, Z) \int d\alpha \int d\beta \ln \left\{ \int \text{etr} \left( \begin{bmatrix} \nu \nu R & W \otimes R \\ -I_{\nu} \otimes \tilde{H}^H & \nu \nu_T + Z \otimes \tilde{T} \end{bmatrix} \right) \right. \\
\left. \begin{bmatrix} W \otimes R \\ -I_{\nu} \otimes \tilde{H}^H \end{bmatrix} \right) \left( \begin{bmatrix} \nu \\ \nu_T + Z \otimes \tilde{T} \end{bmatrix} \right) \right\}
\]

\[
= \ln \left\{ \int \text{etr}(WZ) \det^{-1} \text{etr} \left( \begin{bmatrix} \nu \nu R & W \otimes R \\ -I_{\nu} \otimes \tilde{H}^H & \nu \nu_T + Z \otimes \tilde{T} \end{bmatrix} \right) \right. \\
\left. d\mu(W, Z) \right\}. \quad (7)
\]

Here, the exchange of the limit for $\alpha, \beta \downarrow 0$ and the integral in $U, V$ is allowed from the Dominated Convergence Theorem [27, Th. 5.30] since the absolute value of the first integrand, assuming $W = \tilde{W} + j \tilde{W}_0$ and $Z = Z_0 + j \tilde{Z}$ for real $\tilde{W}, \tilde{W}_0, \tilde{Z}, Z_0$ determined by the integration contour chosen, is dominated by

\[
\kappa \exp \left\{ -\pi (\|U\|^2 + \|V\|^2) - \frac{\pi U^H R U - Z_0}{2\alpha} \right\}.
\]

which is plainly integrable over $(U, V, \tilde{W}, \tilde{Z}) \in C_{nR} \times \nu \times C_{nT} \times \nu \times \mathbb{R}^{\nu \times \nu} \times \mathbb{R}^{\nu \times \nu}$. Similarly, the exchange of integration order is allowed by Fubini’s Theorem [27, Th. 5.47] because the first integrand in (7) is measurable as its absolute value is integrable [27, Prop. 5.53(c)]. Finally, after the change of integration order and the application of the limit, we used Property [A.1] and eq. (19) of Appendix [H].

C. Saddlepoint approximation

In order to derive the asymptotic limit of (7) as $n_T, n_R \rightarrow \infty$, we use the method of saddlepoint approximation (see Appendix [E] for a summary of the main results and [1], [23] for a deeper account).

To this purpose, we look for a stationary point of the integrand’s logarithm in (7), namely,

\[
\phi(W, Z) \triangleq \text{Tr}(WZ) - \ln \det \left( \begin{bmatrix} I_{\nu \nu R} + W \otimes R & I_{\nu} \otimes \tilde{H} \\ -I_{\nu} \otimes \tilde{H}^H & I_{\nu \nu_T} + Z \otimes \tilde{T} \end{bmatrix} \right).
\]

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To this end, we resort to the following expansion:
\[
\delta \ln \det(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{Tr}[(X^{-1} \delta X)^k].
\]

Then, assuming replica symmetry [34], we look for a stationary point of \( \phi(W, Z) \) of the form \( W = wI_\nu, Z = zI_\nu \) for some positive \( w, z \). We have:
\[
\phi(wI_\nu, zI_\nu) = \nu \left\{ wz - \ln \det \left( I_{nR} + wR + \bar{H} \left( I_{nT} + z\bar{T} \right)^{-1} \bar{H}^H \right) \right\}.
\]

The values of \( w \) and \( z \) are obtained by setting the first-order variation of \( \phi(W, Z) \) equal to zero. More generally, we can write the total variation at \( (wI_\nu, zI_\nu) \) as:
\[
\delta \phi(wI_\nu, zI_\nu) = \text{Tr}(w\delta Z + z\delta W + \delta W \delta Z) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \text{Tr} \left\{ \left( \delta W \otimes (A_1 R) \right) \delta Z \otimes (B_1 \bar{T}) \right\}^k
\]
where [20, p. 18]:
\[
\begin{align*}
A_1 &= [I_{nR} + wR + \bar{H} \left( I_{nT} + z\bar{T} \right)^{-1} \bar{H}^H]^{-1} \\
B_1 &= -(I_{nR} + wR)^{-1} \bar{H} D_1 \\
C_1 &= \left( I_{nT} + z\bar{T} \right)^{-1} \bar{H}^H A_1 = -B_1^H \\
D_1 &= [I_{nT} + z\bar{T} + \bar{H}^H \left( I_{nR} + wR \right)^{-1} \bar{H}]^{-1}
\end{align*}
\]

Next, we focus on the second-order expansion of \( \delta \phi(wI_\nu, zI_\nu) \), since we show (Theorem IV.1) that subsequent terms vanish as \( n_R \to \infty \). We have:
\[
\delta \phi(wI_\nu, zI_\nu) = [z - \text{Tr}(A_1 R)] \text{Tr}(\delta W) + [w - \text{Tr}(D_1 \bar{T})] \text{Tr}(\delta Z)
+ \frac{1}{2} \text{Tr}[(A_1 R)^2] \text{Tr}(\delta W^2) + \frac{1}{2} \text{Tr}[(D_1 \bar{T})^2] \text{Tr}(\delta Z^2)
+ [1 + \text{Tr}(B_1 \bar{T} C_1 R)] \text{Tr}(\delta W \delta Z).
\]

Thus, the stationary point is characterized by the values of \( w, z \) that are solutions of the following equations:
\[
\begin{align*}
w &= \text{Tr}(D_1 \bar{T}) = \text{Tr} \left\{ [zI_{nT} + \bar{T}^{-1} + \bar{T}^{-1} \bar{H}^H (I_{nR} + wR)^{-1} \bar{H}]^{-1} \right\} \\
z &= \text{Tr}(A_1 R) = \text{Tr} \left\{ [wI_{nR} + R^{-1} + R^{-1} \bar{H} (I_{nT} + z\bar{T})^{-1} \bar{H}^H]^{-1} \right\}
\end{align*}
\]

The uniqueness of the solution of (10) is proved in Appendix D. In the following, the solution of (10) will be denoted by \( (\hat{w}, \hat{z}) \), and the matrices defined in (9) after setting \( w = \hat{w}, z = \hat{z} \) will be denoted by \( \hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1 \), respectively.
By the previous results, the saddle-point asymptotic approximation of \( g(\nu) \) (calculated along the directions of steepest descent from the stationary point) can be written as:

\[
g(\nu) \sim \nu \left[ \hat{w} \hat{z} - \ln \det \left( \begin{pmatrix} I_{n_R} & \hat{w} R & \hat{H} \\ -\hat{H}^H & I_{n_T} + \hat{z} \hat{T} \end{pmatrix} \right) \right] + \ln \left\{ \frac{1}{(2\pi)^{n_R n_T}} \int_{\mathbb{R}^{n_R}} \int_{\mathbb{R}^{n_T}} \text{etr} \left[ \frac{1}{2} \left\{ \hat{\alpha} X^2 - \hat{\beta} Y^2 + 2 j \hat{\gamma} X Y \right\} \right] dX dY \right\}
\]

where \( \hat{\alpha} \triangleq \text{Tr}[(\hat{A}_1 R)^2], \hat{\beta} \triangleq \text{Tr}[(\hat{D}_1 \hat{T})^2], \) and \( \hat{\gamma} \triangleq 1 + \text{Tr}(\hat{B}_1 \hat{T} \hat{C}_1 R) \). Using the trace expansion

\[
\text{Tr}[\hat{\alpha}(X)^2 - \hat{\beta}(Y)^2 + 2 j \hat{\gamma} X Y] = \sum_a [\hat{\alpha}(X)^2_{aa} - \hat{\beta}(Y)^2_{aa} + 2 j \hat{\gamma}(X)_{aa}(Y)_{aa}]
+ 2 \sum_{a<b} [\hat{\alpha}(X)_{ab}(X)_{ba} - \hat{\beta}(Y)_{ab}(Y)_{ba} + 2 j \hat{\gamma}(X)_{ab}(Y)_{ba}],
\]

the CGF can be readily evaluated as

\[
g(\nu) \sim \nu \left[ \hat{w} \hat{z} - \ln \det \left( \begin{pmatrix} I_{n_R} & \hat{w} R & \hat{H} \\ -\hat{H}^H & I_{n_T} + \hat{z} \hat{T} \end{pmatrix} \right) \right] - \frac{1}{2} \nu^2 \ln(\hat{\gamma}^2 - \hat{\alpha}\hat{\beta}).
\]

It can be shown that \( 0 < \hat{\gamma}^2 - \hat{\alpha}\hat{\beta} < 1 \) (see Appendix F). As noticed in [36, Sec. IV-C] in a different setting, this inequality is required to guarantee the local stability of the replica-symmetric saddlepoint against variations around it.

Thus, we have the following asymptotic expressions of the mean and variance of \( I(x; y) \):

\[
\begin{align*}
\mathbb{E}[I(x; y)] &\sim \mu_I \triangleq \ln \det \left( \begin{pmatrix} I_{n_R} & \hat{w} R & \hat{H} \\ -\hat{H}^H & I_{n_T} + \hat{z} \hat{T} \end{pmatrix} \right) - \hat{w} \hat{z} \\
\text{Var}[I(x; y)] &\sim \sigma_I^2 \triangleq -\ln(\hat{\gamma}^2 - \hat{\alpha}\hat{\beta})
\end{align*}
\]

expressed in (nat/complex dimension) and (nat/complex dimension)^2, respectively.

### IV. Asymptotic Gaussianity

The asymptotic Gaussianity of the mutual information requires that all terms of order \( k > 2 \) in the series expansion (10) vanish as \( n_R \to \infty \). This is shown under some mild technical conditions expressed in the statement of the following theorem.

**Theorem IV.1** In the asymptotic setting specified in Section II-B assuming further that the matrices \( R \) and \( \hat{T} = T^{1/2} QT^{1/2} \) satisfy the conditions

\[
\sigma_{\text{max}}(R) = \Theta(1) \quad \text{and} \quad \sigma_{\text{max}}(\hat{T}) = \Theta(n_R^{-1}),
\]

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as \( n_R \to \infty \), then the mutual information \( I(x; y) \) converges in distribution to a Gaussian random variable with mean and variance specified in (11).

**Proof:** See Appendix C.

**Remark IV.1** The conditions in Theorem [IV.1] are tantamount to saying that the spatial correlation structure has no asymptotically dominant eigenmodes, i.e., both \( R \) and \( \tilde{T} \) do not have asymptotically dominant eigenvalues. Otherwise, if \( R \) and \( \tilde{T} \) had asymptotically dominant eigenvalues, the asymptotic nature of the MIMO channel would be drastically changed since only a small subset of antennas would play an asymptotically dominant role while the remaining ones would be asymptotically irrelevant.

Considering \( \tilde{T} \) instead of \( T \) implies that also the structure of the input covariance matrix might affect the asymptotic evolution of the MIMO system. As an example, restricting power allocation to a finite subset of transmit antennas would be equivalent to nullify the corresponding columns of the channel matrix \( H \) and then would drastically change the asymptotic behavior of the MIMO channel.

**Remark IV.2** From the property of asymptotic Gaussianity, the outage probability corresponding to a given rate \( R \) can be approximated by

\[
P_o \approx P(N(\mu_I, \sigma_I^2) < R) = Q\left(\frac{\mu_I - R}{\sigma_I}\right)
\]

as \( N \to \infty \). Then, the outage mutual information is given asymptotically by

\[
I_\epsilon \approx \mu_I - \sigma_I Q^{-1}(\epsilon)
\]

with \( \epsilon \) denoting the outage probability. For example, \( I_{10\%} \approx \mu_I - 1.28\sigma_I \) and \( I_{1\%} \approx \mu_I - 2.33\sigma_I \). As a result, it would be desirable to have a large mean \( \mu_I \) and small standard deviation \( \sigma_I \), the latter being a mere consequence of the fact that the outage mutual information improves as the cumulative mutual information distribution transition from 0 to 1 becomes steeper.

---

5 Here, \( Q(x) = P(N(0, 1) > x) = \frac{1}{2\pi} \int_x^\infty \exp(-u^2/2)du \).
V. ANALYTIC EXAMPLE: ERGODIC CAPACITY OF THE UNCORRELATED CHANNEL

Here we present an application of the asymptotic analytic results obtained to the calculation of the ergodic capacity of the spatially uncorrelated Rician fading MIMO channel. Assuming $R = I_{n_R}$ and $T = I_{n_T}$, the fixed-point eqs. (10) become:

$$
\begin{align*}
    w &= \text{Tr} \left\{ \left[ zI_{n_T} + Q^{-1} + \bar{H}^H \bar{H} / (1 + w) \right]^{-1} \right\} \\
    z &= \text{Tr} \left\{ \left[ (1 + w)I_{n_R} + \bar{H}(zI_{n_T} + Q^{-1})^{-1} \bar{H}^H \right]^{-1} \right\}
\end{align*}
$$

After some linear algebra, we can write the asymptotic ergodic mutual information as follows:

$$
\mu_I = n_R \ln(1 + w) + \ln \det \left[ I_{n_T} + \left( I_{n_T} + \frac{\Lambda_H}{1 + w} \right) Q \right] - wz
$$

where $\Lambda_H$ derives from $\bar{H}^H \bar{H} = U_H^H \Lambda_H U_H$ and $Q \triangleq U_H Q U_H^H$.

From Hadamard’s inequality [20, Th. 7.8.1], we know that the maximum determinant of a positive definite diagonal matrix is upper bounded by the product of its diagonal elements and equality holds if and only if the matrix is diagonal. Thus, under the power constraint $\text{Tr}(Q) = \text{Tr}(\bar{Q}) \leq \rho$, we can apply a standard water-filling argument [2] and obtain:

$$
(\bar{Q})_{ii} = \left( \xi - \frac{1 + w}{1 + w + (\Lambda_H)_{ii}} \right)_+
$$

where $(x)_+ \triangleq \max\{0, x\}$ and $\xi$ can be obtained by solving the equation:

$$
\sum_i \left( \xi - \frac{1 + w}{1 + w + (\Lambda_H)_{ii}} \right)_+ = \rho.
$$

Notice that eqs. (13) and (16) have to be solved simultaneously since we have a mutual interdependence between $\bar{Q}$ and the pair $(w, z)$. In other words, the solution can be derived by implementing a simple iterative fixed-point water-filling algorithm.

Finally, we notice that the structure of the asymptotic ergodic capacity achieving covariance matrix is consistent with [21], which showed that the capacity achieving covariance matrix and the matrix $\bar{H}^H \bar{H}$ have the same eigenvectors, whereas the problem of calculating the eigenvalues requires numerical optimization techniques such as those used in [24]. It is then clear the advantage of the asymptotic approximation, which allows to derive analytic results very simply.
VI. NUMERICAL EXAMPLES

In this section we consider a baseline system scenario where we assume that the average channel matrix is given by \((\bar{H})_{ij} = 1\) for all \(i, j\), the spatial correlation matrices are of exponential type with common base \(\alpha\), namely, \((T)_{ij} = (R)_{ij} = \alpha^{|i-j|}\), there is no CDIT, so that capacity is achieved by setting \(Q = qI_t\), and the specification is completed by the following set of parameters:

\[
\begin{align*}
    n_R &= 4 \\
    n_T/n_R &= 1 \\
    K &= 10 \text{ dB} \\
    \alpha &= 0
\end{align*}
\]

(17)

Then, we consider the impact of changing each parameter in turn, with the aim of illustrating the accuracy of the asymptotic approximation proposed. For each parameter we plot the mean and standard deviation of the mutual information by using the asymptotic method (solid lines) and Monte-Carlo simulation (markers).

A. Impact of the number of antennas

Figures 1 and 2 describe the impact of the number of antennas and of the SNR on the asymptotic approximation accuracy. It can be noticed that the approximation of the mean (ergodic mutual information) is always excellent (the maximum relative error being less than 2%, corresponding to the case of \(n_T = n_R = 2\) and SNR = 30 dB). The accuracy of the standard deviation is very good when the SNR is 0 or 10 dB (relative error always less than 1%) but only fairly accurate when the SNR is 20 or 30 dB (maximum relative error around 5%). However, it must be noted that the impact on the outage mutual information approximation (12) is attenuated by the fact that the standard deviation is considerably smaller than the mean when the SNR is sufficiently large.

B. Impact of the antenna ratio

Figures 3 and 4 describe the impact of the TX to RX antenna ratio and of the SNR on the asymptotic approximation accuracy. It is assumed that the number of receive antenna is fixed, \(n_R = 4\), and the number of transmit antennas ranges from \(n_T = 1\) to 10. Both the mean and
the standard deviation of the mutual information display very good accuracy with a maximum relative error smaller than 2%.

C. Impact of the Rice factor

Figures 5 and 6 describe the impact of the Rice factor and of the SNR on the asymptotic approximation accuracy. It can be noticed that the ergodic capacity accuracy is always very good in the cases observed (maximum relative error around 1%) while the standard deviation is slightly overestimated for low $K$ (maximum relative error around 4%) and underestimated higher $K$ (the threshold depending on the SNR). This effect can be explained by considering that the low $K$ condition entails a larger amount of randomness in the channel matrix that increases the asymptotic variance expression. The deviation is larger for high SNR, as in the previous cases observed. Again, the error on the standard deviation has a modest relative impact on the outage mutual information which is the goal of the variance approximation.

D. Impact of spatial correlation

Figures 7 and 8 describe the impact of spatial correlation of the SNR on the asymptotic approximation accuracy. The relative error remains below 1% for the ergodic capacity and 3% for the standard deviation. The relative error increases as the SNR increases and correlation decreases.

It is interesting to note that spatial correlation has a minor effect at moderate SNR, whereas it becomes important as the SNR increases.

Summarizing the numerical results obtained so far we can say that randomness tends to increase the relative error between the analytic asymptotic approximation and the actual value (obtained via Monte-Carlo simulation). So, increasing the SNR and decreasing the correlation or the Rice factor produces an increase of randomness that entails a larger relative error. Besides this, it can be noticed that the relative error gets larger as the antenna ratio gets closer to 1. Finally, the relative error decreases as the number of antennas increases but this is a trivial consequence of the fact that the approximation is asymptotic in the number of antennas.
E. Outage mutual information

In order to assess the accuracy of the proposed asymptotic approximation we consider here the outage mutual information, defined implicitly as follows:

\[ I_\epsilon \triangleq \sup_{R > 0} \left\{ R : P(I(H) < R) < \epsilon \right\}. \quad (18) \]

We compare the asymptotic analytic approximation (12) against the exact numerical result obtained by accurate (though lengthy) Monte-Carlo simulation. The results are illustrated in Figs. 9 and 10, reporting \( I_{10\%} \) (i.e., the 10% outage mutual information) versus the number of antennas and the exponential correlation base \( \alpha \), respectively. The maximum relative error is around 1% in the cases considered, with the maximum corresponding to the case of \( 2 \times 2 \) MIMO with the largest SNR. This is comparable with the relative error already found for the ergodic capacity approximation.

VII. Conclusions

This paper presented an analytic approach, based on the replica method, allowing to approximate the statistics of the mutual information for a separately correlated Rician fading MIMO channel. We showed that the mutual information statistics approach the Gaussian distribution as the number of transmit and receive antennas grows large, with their ratio approaching a finite constant.

More specifically, we saw that the mutual information mean (corresponding to the ergodic mutual information of the channel) yields a very accurate approximation of the real value (which was obtained by extensive Monte-Carlo simulation) with a relative error never larger than a few percent units. This remarkable accuracy was obtained not only with a large number of antennas but also in cases when the number of antennas is definitely small, even in the limiting SISO case! As a result, the asymptotic analysis becomes a valuable tool to assess the system performance whenever this has to be done analytically, such as in the case of covariance optimization with the goal of finding the channel capacity. This application is illustrated in the paper by considering the uncorrelated Rician fading channel.

We also showed that the mutual information standard deviation approximation is very good. This is important in view of the application of these results to the derivation of the outage mutual information. We showed that the outage mutual information, obtained by using the asymptotic
mean and standard deviation, is very accurate in the cases considered, since the relative error is always lower than 1%. This result depends on the asymptotic Gaussianity of the mutual information, which is proven in the paper.

Summarizing, the analytic approach presented allows to address concisely the performance of realistic models of MIMO channels (based on separately correlated Rician fading). The accuracy obtained is sufficient for most applications and eliminates the need to run computer intensive simulations or implement numerical methods in order to optimize the system performance. Many applications of this asymptotic method are appearing in the conference literature. Among them, it is worth mentioning the optimization of the transmitted signal covariance, with and without interference, and the derivation of the capacity region of the multiuser multiple access MIMO channel. A simple example of the former (relevant to the case of uncorrelated Rician fading without interference) is given in the paper.

VIII. ACKNOWLEDGMENTS

The author would like to thank the Associate Editor and the Reviewers for the constructive comments and suggestions, which allowed him to improve the quality of the original submission. Additionally, the author wishes to acknowledge several useful discussions with Dr. Erwin Riegler.
APPENDIX

A. Property of the Kronecker product

**Property A.1**  For any matrices \( A, X, B, Y \) such that the product \( AX^HBY \) exists and is a square matrix, we have:

\[
\text{Tr}(AX^HBY) = \text{vec}(X)^H(A^T \otimes B)\text{vec}(Y).
\]

**Proof:** Assume that \( X = (x_1, \ldots, x_m) \) and \( Y = (y_1, \ldots, y_n) \). Then,

\[
\text{Tr}(AX^HBY) = \text{Tr} \left[ A \begin{pmatrix} x_1^HBy_1 & \cdots & x_1^HBy_n \\ \\ \vdots & \ddots & \vdots \\ x_m^HBy_1 & \cdots & x_m^HBy_n \end{pmatrix} \right] = \sum_{a,b} (A)_{a,b}x_b^HBy_a = \sum_b \sum_a x_b^H(A)_{a,b}By_a = \text{vec}(X)^H(A^T \otimes B)\text{vec}(Y).
\]

B. Integrals

**Integral B.1**  For every pair of Hermitian positive definite matrices \( A \in \mathbb{C}^{m \times m}, C \in \mathbb{C}^{n \times n}, \) and for any matrices \( B, D \in \mathbb{C}^{m \times n}, \) we have:

\[
\int_{\mathbb{C}^{n \times m}} \text{etr}[-\pi(AU^HCU + B^HU + U^HD)]dU = \det(A^T \otimes C)^{-1} \text{etr}(\pi A^{-1}BC^{-1}D^H) = \det(A)^{-n} \det(C)^{-m} \text{etr}(\pi A^{-1}BC^{-1}D^H).
\]

**Proof:** From Property A.1 we have:

\[
\text{Tr}(AU^HCU) = \text{vec}(U)^H(A^T \otimes C)\text{vec}(U).
\]

Then, we can write the integral on the lhs of (19) as

\[
\int_{\mathbb{C}^{mn}} \exp[-\pi(u^HPu + b^Hu + u^Hd)]du
\]

where \( P = A^T \otimes C, b = \text{vec}(B), \) and \( d = \text{vec}(D). \) Since \( A \) and \( C \) are Hermitian and positive definite by assumption, so are \( A^T \) and \( P = A^T \otimes C. \)
Applying the change of variables $v = P^{1/2}u$, we obtain:

$$\left(\det P\right)^{-1} \int_{C^m} \exp[-\pi(v^Hv + b^HP^{-1/2}v + v^HP^{-1/2}d)]dv.$$

This integral can also be written as

$$\left(\det P\right)^{-1} \prod_{a=1}^{mn} \int_C \exp[-\pi(|v_a|^2 + \tilde{b}_a^*v_a + v_a^*\tilde{d}_a)]dv_a$$

where we let $\tilde{b} = P^{-1/2}b$ and $\tilde{d} = P^{-1/2}d$. Setting $v_a = v_{ax} + jv_{ay}$, we have

$$|v_a|^2 + \tilde{b}_a^*v_a + v_a^*\tilde{d}_a = v_{ax}^2 + v_{ax}(\tilde{b}_a^* + \tilde{d}_a) + v_{ay}^2 + jv_{ay}(\tilde{b}_a^* - \tilde{d}_a).$$

By using repeatedly the standard Gaussian integral

$$\int_{\mathbb{R}} \exp[-\pi(x^2 + ax)]dx = \exp(\pi a^2/4), \quad a \in \mathbb{C},$$

we obtain:

$$\int_{C^m} \exp[-\pi(v^Hv + b^HP^{-1/2}v + v^HP^{-1/2}b)]dv = \prod_{a=1}^{mn} \left\{ \exp[\pi(\tilde{b}_a^* + \tilde{d}_a)^2/4] \exp[-\pi(\tilde{b}_a^* - \tilde{d}_a)^2/4] \right\}$$

$$= \prod_{a=1}^{mn} \exp(\pi\tilde{b}_a^*\tilde{d}_a) = \exp(\pi b^HD) = \exp(\pi b^HP^{-1}d).$$

Finally, since $P^{-1} = (A^{-1})^T \otimes C^{-1}$, applying Property A.1 we get:

$$\text{vec}(B)^H((A^{-1})^T \otimes C^{-1})\text{vec}(D) = \text{Tr}(A^{-1}BC^{-1}D^H),$$

which in turn leads to (19).

**Remark 1** Ref. [34, eq.(124)] reports the following incorrect result:

$$\int_{\mathbb{R}} dr \exp(-rv) \int_{\mathbb{R}} \exp(rt - ut) \frac{dt}{2\pi j} = \exp(-uv) \quad u, v \in \mathbb{C}. \quad (20)$$

The previous result is wrong because the function $\exp(-rv)$ is not a test function for the $\delta$-sequence $\delta_A(r - u) = \int_{-A}^A \exp(j(r - u)t) \frac{dt}{2\pi}$ (unless $v$ is purely imaginary) since it is unbounded over the integration domain of $r$ [26]. However, if we smoothen the test function by it by a suitable Gaussian factor, we obtain:

$$\int_{\mathbb{R} + Jr_0} dr \int_{r_0 + Jr} \exp \left\{-\alpha r^2 + \beta t^2 + rt - ut - rv \right\} \frac{dt}{2\pi j} = \frac{1}{\sqrt{1 + 4\alpha \beta}} \exp \left(\frac{-uv - \alpha u^2 + \beta v^2}{1 + 4\alpha \beta} \right). \quad (21)$$

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This is a Lebesgue integral provided that $\alpha, \beta > 0$ and it holds for any $r_0, t_0 \in \mathbb{R}$. Therefore, we can replace eq. (20) by:

$$
\exp(-uv) = \lim_{\alpha, \beta \downarrow 0} \int_{\mathbb{R}+jr_0} dr \int_{t_0+j\mathbb{R}} dt \exp \left\{ -\alpha r^2 + \beta t^2 + rt - ut - rv \right\} \frac{dt}{2\pi j} .
$$

(22)

The previous result allows us to prove the following proposition.

**Integral B.2** (Robust Hubbard–Stratonovich transformation) For every pair of matrices $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$, we have:

$$
e\text{etr}(-AB) = \lim_{\alpha, \beta \downarrow 0} \int \text{etr}(-\alpha WW^T + \beta ZZ^T + WZ - AW - BZ) d\mu(W, Z)
$$

(23)

where $d\mu(W, Z) = \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{1}{2\pi} d(W)_{ab} d(Z)_{ba}$ and integration is carried out along contours parallel to the real axis for each $(W)_{ab}$ and to the imaginary axis for each $(Z)_{ba}$.

**Proof**: The proof stems from repeated application of (22).

C. Proof of Theorem IV.1

**Proof**: Using the complete expansion in (8), we can write the CGF as:

$$
g(\nu) \sim \nu \left[ \tilde{w} \tilde{z} - \ln \det \begin{pmatrix} \mathbb{I}_{n_R} + \tilde{w} \mathbb{R} & \tilde{H} \\ -\tilde{H}^H & \mathbb{I}_{n_T} + \tilde{z} \tilde{T} \end{pmatrix} \right] + \ln \left\{ \int \frac{1}{(2\pi)^{n_T}} \exp \left[ \frac{1}{2} \left\{ \tilde{\alpha} \mathbf{U}^2 - \tilde{\beta} \mathbf{V}^2 + 2 j \tilde{\gamma} \mathbf{U} \mathbf{V} \right\} \right] \cdot \left\{ 1 + \sum_{k=3}^{\infty} c_k \text{Tr} \left[ \mathbf{U} \otimes (\mathbf{A}_1 \mathbb{R}) j \mathbf{V} \otimes (\mathbf{B}_1 \tilde{T}) \mathbf{U} \otimes (\mathbf{C}_1 \mathbb{R}) j \mathbf{V} \otimes (\mathbf{D}_1 \tilde{T}) \right] \right\} dUdV \right\}.
$$

Thus, in order to prove the convergence of $g(\nu)$ to the Gaussian CGF, we have to show that, as $n_R \to \infty$,

$$
\int \exp \left[ \frac{1}{2} \left\{ \tilde{\alpha} \mathbf{U}^2 - \tilde{\beta} \mathbf{V}^2 + 2 j \tilde{\gamma} \mathbf{U} \mathbf{V} \right\} \right] \text{Tr} \left[ \mathbf{U} \otimes (\mathbf{A}_1 \mathbb{R}) j \mathbf{V} \otimes (\mathbf{B}_1 \tilde{T}) \mathbf{U} \otimes (\mathbf{C}_1 \mathbb{R}) j \mathbf{V} \otimes (\mathbf{D}_1 \tilde{T}) \right] dUdV \to 0
$$

(24)

Here we used the series expansion $(1+x)^{-1} \exp(x-x^2/2) = 1 + \sum_{k=3}^{\infty} c_k x^k = 1 - \frac{1}{3} x^3 + \frac{1}{4} x^4 - \frac{1}{5} x^5 + \frac{2}{9} x^6 - \frac{19}{84} x^7 + \ldots$.
for $k \geq 3$. Now, in order to balance the asymptotic order of the matrix multipliers of $U$ and $V$, it is convenient to apply the change of variables $U = n_R^{-1/2} \tilde{U}$ and $V = n_R^{1/2} \tilde{V}$. Then, the limit (24) becomes:

$$
\int \text{etr} \left[ \frac{1}{2} \left\{ n_R^{-1} \tilde{\alpha} \tilde{U}^2 - n_R \tilde{\beta} \tilde{V}^2 + 2 j \tilde{\gamma} \tilde{U} \tilde{V} \right\} \right] \cdot \text{Tr} \left[ \begin{pmatrix} \tilde{U} \otimes (n_R^{-1/2} A_1 R) & j \tilde{V} \otimes (n_R^{1/2} B_1 \tilde{T}) \\ \tilde{U} \otimes (n_R^{-1/2} C_1 R) & j \tilde{V} \otimes (n_R^{1/2} D_1 \tilde{T}) \end{pmatrix} \right]^k d\tilde{U} d\tilde{V} \xrightarrow{n_R \to \infty} 0
$$

(25)

and since

$$
\tilde{\alpha} = \text{Tr}[(\tilde{A}_1 R)^2] \leq \text{Tr}(R^2) = O(n_R),
$$

$$
\tilde{\beta} = \text{Tr}[(\tilde{D}_1 \tilde{T})^2] \leq \text{Tr}(\tilde{T}^2) = O(n_R^{-1}),
$$

by the theorem's assumptions, it is plain to see that condition (25) is equivalent to the limit

$$
\text{Tr} \left[ \begin{pmatrix} \tilde{U} \otimes (n_R^{-1/2} A_1 R) & j \tilde{V} \otimes (n_R^{1/2} B_1 \tilde{T}) \\ \tilde{U} \otimes (n_R^{-1/2} C_1 R) & j \tilde{V} \otimes (n_R^{1/2} D_1 \tilde{T}) \end{pmatrix} \right]^k \xrightarrow{n_R \to \infty} 0
$$

(26)

To prove (26), we investigate the asymptotic order of the singular values of the matrices $A_1 R$, $B_1 \tilde{T}$, $C_1 R$, and $D_1 \tilde{T}$. We shall use the following linear algebra inequalities [20]:

$$
\sigma_{\max}(MN) \leq \sigma_{\max}(M)\sigma_{\max}(N), \quad \text{for any } M, N \text{ with compatible sizes}
$$

$$
\sigma_{\max}(M) \leq \sigma_{\max}(N), \quad \text{for any square } M, N \text{ with } M \leq N
$$

$$
\text{Tr}(M) \leq \text{Tr}(N), \quad \text{for any square } M, N \text{ with } M \leq N
$$

From the asymptotic setting definitions of Section II-B we have the following results.

1) Since

$$
A_1 R = [\tilde{w} I_{nr} + R^{-1} + R^{-1} \tilde{H}(I_{nt} + \tilde{z} \tilde{T})^{-1} \tilde{H}^H]^{-1}
$$

we have $A_1 R \leq R$, so that

$$
\sigma_{\max}(A_1 R) \leq \sigma_{\max}(R) = \Theta(1) \quad \text{and} \quad \tilde{z} = \text{Tr}(A_1 R) \leq \text{Tr}(R) = \Theta(n_R).
$$

2) Since

$$
D_1 \tilde{T} = [\tilde{z} I_{nt} + \tilde{T}^{-1} + \tilde{T}^{-1} \tilde{H}^H(\tilde{I}_{nr} + \tilde{w} R)^{-1} \tilde{H}]^{-1}
$$

we have $D_1 \tilde{T} \leq \tilde{T}$, so that

$$
\sigma_{\max}(D_1 \tilde{T}) \leq \sigma_{\max}(\tilde{T}) = \Theta(n_R^{-1}) \quad \text{and} \quad \tilde{w} = \text{Tr}(D_1 \tilde{T}) \leq \text{Tr}(\tilde{T}) = \Theta(1).
$$
3) By using the previous inequalities on the maximum singular values we obtain:

\[ \text{Tr}[(A_1 R)^k] = O(n_R) \quad \text{and} \quad \text{Tr}[(D_1 \tilde{T})^k] = O(n_R^{1-k}) \]

which confirms, for \( k = 2 \), that \( \hat{\alpha} = O(n_R) \) and \( \hat{\beta} = O(n_R^{-1}) \).

4) Setting \( \tilde{H} \triangleq (I_{n_R} + \tilde{w} R)^{-1/2} \tilde{H} \), by the definition of \( B_1 \) in (9), we have:

\[
\sigma_{\max}(B_1 \tilde{T}) = \sigma_{\max}[(I_{n_R} + \tilde{w} R)^{-1/2} \tilde{H}(I_{n_T} + \tilde{z} \tilde{T} + \tilde{H}^H \tilde{H})^{-1} \tilde{T}]
\leq \sigma_{\max}[(I_{n_R} + \tilde{w} R)^{-1/2}] \sigma_{\max}[(\tilde{H}(I_{n_T} + \tilde{z} \tilde{T} + \tilde{H}^H \tilde{H})^{-1})] \sigma_{\max}(\tilde{T}).
\]

Since (plainly) \( \sigma_{\max}[(I_{n_R} + \tilde{w} R)^{-1/2}] = O(1) \) and

\[
s^2_{\max}[\tilde{H}(I_{n_T} + \tilde{z} \tilde{T} + \tilde{H}^H \tilde{H})^{-1}] = \lambda_{\max}[\tilde{H}^H \tilde{H}(I_{n_T} + \tilde{z} \tilde{T} + \tilde{H}^H \tilde{H})^{-2}]
\leq \lambda_{\max}[\tilde{H}^H \tilde{H}(I_{n_T} + \tilde{H}^H \tilde{H})^{-2}]
= \max_i \frac{\lambda_i(\tilde{H}^H \tilde{H})}{[1 + \lambda_i(\tilde{H}^H \tilde{H})]^2} \leq \frac{1}{4},
\]

we have

\[ \sigma_{\max}(B_1 \tilde{T}) = O(n_R^{-1}). \]

5) Setting \( \tilde{H} \triangleq \tilde{H}(I_{n_T} + \tilde{z} \tilde{T})^{-1/2} \), by the definition of \( C_1 \) in (9), we have:

\[
\sigma_{\max}(C_1 R) = \sigma_{\max}[(I_{n_T} + \tilde{z} \tilde{T})^{-1/2} \tilde{H}^H(I_{n_R} + \tilde{w} R + \tilde{H} \tilde{H}^H)^{-1} R]
\leq \sigma_{\max}[(I_{n_T} + \tilde{z} \tilde{T})^{-1/2}] \sigma_{\max}[(\tilde{H}^H(I_{n_R} + \tilde{w} R + \tilde{H} \tilde{H}^H)^{-1})] \sigma_{\max}(R).
\]

Since (plainly) \( \sigma_{\max}[(I_{n_R} + \tilde{z} \tilde{T})^{-1/2}] = O(1) \) and

\[
s^2_{\max}[\tilde{H}^H(I_{n_R} + \tilde{w} R + \tilde{H} \tilde{H}^H)^{-1}] = \lambda_{\max}[\tilde{H}^H \tilde{H}(I_{n_R} + \tilde{w} R + \tilde{H} \tilde{H}^H)^{-2}]
\leq \lambda_{\max}[\tilde{H}^H \tilde{H}(I_{n_R} + \tilde{H} \tilde{H}^H)^{-2}]
= \max_i \frac{\lambda_i(\tilde{H} \tilde{H}^H)}{[1 + \lambda_i(\tilde{H} \tilde{H}^H)]^2} \leq \frac{1}{4},
\]

we have

\[ \sigma_{\max}(C_1 R) = O(1). \]

In order to prove (26), notice that the maximum singular values of the matrices on the rhs of the Kronecker products in

\[
\begin{pmatrix}
U \otimes (n_R^{-1/2} A_1 R) & j V \otimes (n_R^{1/2} B_1 \tilde{T}) \\
U \otimes (n_R^{-1/2} C_1 R) & j V \otimes (n_R^{1/2} D_1 \tilde{T})
\end{pmatrix}
\]
are all \(O(n_R^{-1/2})\) as \(n_R \to \infty\). Thus, if we define

\[
\begin{pmatrix}
\phi_{11,k}(U, V) \otimes M_{11,k} & \phi_{12,k}(U, V) \otimes M_{12,k} \\
\phi_{21,k}(U, V) \otimes M_{21,k} & \phi_{22,k}(U, V) \otimes M_{22,k}
\end{pmatrix} \triangleq \begin{pmatrix} U \otimes (n_R^{-1/2} A_1 R) & jV \otimes (n_R^{-1/2} B_1 \hat{T}) \\ U \otimes (n_R^{-1/2} C_1 R) & jV \otimes (n_R^{-1/2} D_1 \hat{T}) \end{pmatrix}^k
\]

we have \(\sigma_{\text{max}}(M_{ij,k}) = O(n_R^{-k/2})\). Therefore, the trace

\[
\text{Tr}(\phi_{11,k}(U, V)) \text{Tr}(M_{11,k}) + \text{Tr}(\phi_{22,k}(U, V)) \text{Tr}(M_{22,k}) = O(n_R^{-1-k/2})
\]

approaches 0 as \(n_R \to \infty\) for all \(k \geq 3\), which proves (24) and the fact that \(g(\nu)\) converges to a Gaussian CGF.

\[\blacksquare\]

D. Uniqueness of the solution of eqs. (10)

The proof reported here was inspired by a similar proof included in [17].

**Proof:** Eqs. (10) can be written equivalently as follows:

\[
\begin{align*}
1 &= \hat{\psi}_1(w, z) \triangleq \frac{\text{Tr}(D_1 \hat{T})}{w} = \text{Tr}\{w z I_{n_R} + w \hat{T}^{-1} \tilde{H}^H (I_{n_R} + w R)^{-1} \tilde{H}]^{-1}\} \\
1 &= \hat{\psi}_2(w, z) \triangleq \frac{\text{Tr}(A_1 \hat{T})}{z} = \text{Tr}\{w z I_{n_R} + z R^{-1} \tilde{H} (I_{n_R} + z \hat{T})^{-1} \tilde{H}^H]^{-1}\}
\end{align*}
\]

(27)

First, we calculate the partial derivatives of these functions with respect to \(w\) and \(z\):

\[
\begin{align*}
\frac{\partial \hat{\psi}_1}{\partial w} &= -\frac{\text{Tr}\{D_1 \hat{T} D_1 [I_{n_R} + z \hat{T} + \tilde{H}^H (I_{n_R} + w R)^{-2} \tilde{H}]\}}{w^2} < 0 \\
\frac{\partial \hat{\psi}_1}{\partial z} &= -\frac{\text{Tr}\{(D_1 \hat{T})^2\}}{w} < 0 \\
\frac{\partial \hat{\psi}_2}{\partial w} &= -\frac{\text{Tr}\{A_1 R A_1 [I_{n_R} + w R + \tilde{H} (I_{n_R} + z \hat{T})^{-2} \tilde{H}^H]\}}{z^2} < 0 \\
\frac{\partial \hat{\psi}_2}{\partial z} &= -\frac{\text{Tr}\{A_1 R A_1 [I_{n_R} + w R + \tilde{H} (I_{n_R} + z \hat{T})^{-2} \tilde{H}^H]\}}{z^2} < 0
\end{align*}
\]

(28)

Then, we notice that \(\hat{\psi}_1(0, z) = \infty\), \(\hat{\psi}_1(\infty, z) = 0\), and \(\partial \hat{\psi}_1 / \partial w \) < 0. Hence, \(\hat{\psi}_1(w, z)\) is a continuous monotonically decreasing function in \(w\) and the equation \(\hat{\psi}_1(w, z) = 1\) has a single solution \(w = g(\nu) \in [0, \infty)\), which is a continuous function of \(z\) by the Implicit Function Theorem [27, Th. 3.16]. Moreover, from the first of (27) and from the first two inequalities of (28), we obtain:

\[
g'(z) = -\frac{\partial \hat{\psi}_1 / \partial z}{\partial \hat{\psi}_1 / \partial w} < 0.
\]
Then, the uniqueness of the solution of (10) requires that the equation $h(z) \triangleq \hat{\psi}_2(g(z), z) = 1$ has a single solution. This result stems from the fact that $\hat{\psi}_2(g(0), 0) = +\infty$, $\hat{\psi}_2(g(\infty), \infty) = 0$, and

$$h'(z) = \left[ \frac{\partial \hat{\psi}_2}{\partial w} g'(z) + \frac{\partial \hat{\psi}_2}{\partial z} \right]_{w=g(z)} < 0.$$ 

The last inequality is equivalent to

$$\left[ \frac{\partial \hat{\psi}_1}{\partial w} \frac{\partial \hat{\psi}_2}{\partial z} - \frac{\partial \hat{\psi}_1}{\partial z} \frac{\partial \hat{\psi}_2}{\partial w} \right]_{w=g(z)} > 0$$

which can be checked by direct substitution of the relevant expressions.

E. Saddlepoint approximation

Here we report some basic facts about saddlepoint approximation. A full account on the subject can be found in [23]. In its simplest form, saddlepoint approximation deals with integrals of the type

$$I = \int g(x) \exp[-\lambda h(x)] dx$$

in the limit for $\lambda \rightarrow \infty$. Assuming $g(x)$ bounded and $h(x)$ with a global minimum at $x = x_0$, the integral can be approximated, for $\lambda \rightarrow \infty$, as:

$$I \approx \int g(x_0) \exp[-\lambda h(x_0) - \lambda h''(x_0)(x - x_0)^2/2] dx = \sqrt{\frac{2\pi}{\lambda h''(x_0)}} g(x_0) \exp[-\lambda h(x_0)].$$

A more refined derivation yields

$$I = \exp[-\lambda h(x_0)] \sqrt{\lambda h''(x_0)/(2\pi)} \left\{ g(x_0) + \frac{g(5h'' - 3h''^2 h''') - 12g'h''h''+12g''h''^2}{24h''^3} \right\}_{x=x_0} \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

This result can be extended to a multidimensional contour integral

$$I = \int g(x) \exp[-\lambda h(x)] dx.$$  

Assuming $|g(x)|$ bounded and $h(x)$ real, smooth, and with a global minimum at $x = x_0$, we have, for $\lambda \rightarrow \infty$:

$$I \approx \int g(x_0) \exp[-\lambda h(x_0) - \lambda(x - x_0)^\top H_0(x - x_0)/2] dx = \frac{g(x_0) \exp[-\lambda h(x_0)]}{\sqrt{\det[\lambda H_0/(2\pi)]}},$$

where $H_0$ is the Hessian matrix of $h(x)$ at $x = x_0$, which is assumed to be positive definite.
F. Proof of the inequalities $0 < \gamma^2 - \alpha \bar{\beta} < 1$

Proof: First, we show that $0 < \gamma < 1$, so that, since $\alpha \bar{\beta} > 0$, we have $\gamma^2 - \alpha \bar{\beta} < 1$.

- The inequality $\gamma < 1$ derives from

$$\gamma = 1 + \text{Tr}(\bar{B}_1 \bar{T} \bar{C}_1 R) = 1 - \text{Tr}(R^{1/2} \bar{C}_1^H \bar{T} \bar{C}_1 R^{1/2}) = 1 - \| \bar{T}^{1/2} \bar{C}_1 R^{1/2} \|^2.$$ 

- The inequality $\gamma > 0$ derives from

$$\gamma = 1 + \text{Tr}(\bar{B}_1 \bar{T} \bar{C}_1 R)$$

$$= 1 - \text{Tr}[\bar{A}_1 \bar{H}(I_{n_T} + z \bar{T})^{-1} \bar{T}(I_{n_T} + z \bar{T})^{-1} \bar{H}^H \bar{A}_1 R]$$

$$\geq 1 - z^{-1} \text{Tr}[\bar{A}_1 \bar{H}(I_{n_T} + z \bar{T})^{-1} \bar{H}^H \bar{A}_1 R]$$

$$\geq 1 - z^{-1} \text{Tr}[(I_{n_R} - \bar{A}_1(I_{n_R} + \bar{w} R)) \bar{A}_1 R]$$

$$\equiv \frac{\text{Tr}(\bar{A}_1^2 R) + \bar{w} \bar{\alpha}}{z},$$

from $i)$ the matrix inequality $(I_{n_T} + z \bar{T})^{-1} \bar{T} < \bar{T}^{-1} I_{n_T}$; and $ii)$ the definition of $\bar{A}_1$; and $iii)$ the definition of $\bar{\alpha} = \text{Tr}(\bar{A}_1 R)$. Now, in order to handle the case of singular $R$, we define the positive definite matrix $R_\epsilon \triangleq \epsilon I_{n_R} + R$ for $\epsilon > 0$. We have:

$$\gamma \geq \frac{\text{Tr}(\bar{A}_1^2 R_\epsilon) - \epsilon \text{Tr}(\bar{A}_1^2) + \bar{w} \bar{\alpha}}{z}$$

$$\geq \frac{\lambda_{\min}(R_\epsilon)^{-1} \text{Tr}(\bar{A}_1 R_\epsilon \bar{A}_1 R_\epsilon) - \epsilon \text{Tr}(\bar{A}_1^2) + \bar{w} \bar{\alpha}}{z}$$

$$\equiv \frac{\lambda_{\max}(R)^{-1} + \bar{w}}{z} \bar{\alpha}$$

$$> 0,$$  

(29)

from $i)$ the inequality $\text{Tr}(U V) \geq \lambda_{\min}(U) \text{Tr}(V)$ (holding for nonnegative definite matrices $U, V$ as a direct consequence of [29, 9.H.1.h]); and $ii)$ taking the limit for $\epsilon \downarrow 0$ of the previous lower bound and the limit $\lim_{\epsilon \downarrow 0} \lambda_{\min}(R_\epsilon) = \lambda_{\max}(R)^{-1}$.

The second part of the proof can be accomplished by combining the inequality

$$\gamma \geq \frac{\lambda_{\max}(R)^{-1} + \bar{w}}{z} \bar{\alpha} > \frac{\bar{w}}{z} \bar{\alpha},$$

deriving from (29), and

$$\gamma \geq \frac{\lambda_{\max}(\bar{T})^{-1} + \bar{z}}{\bar{w}} \bar{\beta} > \frac{\bar{z}}{\bar{w}} \bar{\beta},$$

which can be obtained in a similar way. Multiplying the two inequalities yields $\gamma^2 - \alpha \bar{\beta} > 0$. ■

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REFERENCES

[1] N. Bleistein and R.A. Handelsman, **Asymptotic Expansions of Integrals.** Dover, 1986
[2] T.M. Cover and J.A. Thomas, **Elements of Information Theory (2nd Ed.).** New York: Wiley, 2006.
[3] X.W. Cui, Q.T. Zhang, and Z.M. Feng, “Generic procedure for tightly bounding the capacity of MIMO correlated Rician fading channels,” *IEEE Trans. on Commun.*, vol. 53, no. 5, pp. 890–898, May 2005.
[4] P.F. Driessen and G.J. Foschini, “On the capacity formula for multiple inputmultiple output wireless channels: A geometric interpretation,” *IEEE Trans. on Communications*, vol. 47, no. 2, pp. 173–176, Feb. 1999.
[5] J. Dumont, P. Loubaton, S. Lasaulce, and M. Debbah, “On the asymptotic performance of MIMO correlated Rician channels,” *ICASSP 2005*, pp. V-813–816, Philadelphia, PA, USA, March 18–23, 2005.
[6] J. Dumont, P. Loubaton, S. Lasaulce, “On the capacity achieving transmit covariance matrices of MIMO correlated Rician channels: A large system approach,” Proc. *Globecom 2006*, San Francisco, Nov. 27-Dec. 1, 2006.
[7] S.F. Edwards and P.W. Anderson, “Theory of spin glasses,” *J. Phys. F: Metal Physics*, vol. 5, pp. 965–974, 1975.
[8] M. Elia and G. Taricco, “Integration of the exponential function of a complex quadratic form,” *Applied Mathematics E-Notes*, vol. 3, pp. 95-98, 2003.
[9] F.R. Farrokh, G.J. Foschini, A. Lozano, and R.A. Valenzuela, “Link-optimal spacetime processing with multiple transmit and receive antennas,” *IEEE Commun. Lett.*, vol. 5, no. 3, pp. 85–87, March 2001.
[10] K.H. Fischer and J.A. Hertz, *Spin Glasses*. Cambridge University Press, 1991.
[11] G.J. Foschini, “Layered space-time architecture for wireless communication in fading environments when using multi-element antennas,” *Bell Labs Tech. J.*, pp. 41–59, 1996.
[12] D. Gesbert, H. Bölcskei, D.A. Gore, and A.J. Paulraj, “Outdoor MIMO wireless channels: Models and performance prediction,” *IEEE Trans. Commun.*, vol. 50, no. 12, pp. 1926–1934, Dec. 2002.
[13] A.J. Goldsmith, S.A. Jafar , N. Jindal, and S. Vishwanath, “Capacity Limits of MIMO Channels,” *IEEE J. Select. Areas Commun.*, vol. 21, no. 5, pp. 684–702, June 2003.
[14] D. Guo and S. Verdú, “Randomly spread CDMA: Asymptotics via statistical physics,” *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 1983–2010, June 2005.
[15] W. Hachem, P. Loubaton, and J. Najim, “The mutual information of a MIMO channel: A survey,” *ISCCSP 2006*.
[16] W. Hachem, P. Loubaton, and J. Najim, “Deterministic equivalents for certain functionals of large random matrices,” *Annals of Applied Probability*, vol. 17 no. 3, pp. 875–930, 2007.
[17] W. Hachem, O. Khorunzhiy, P. Loubaton, J. Najim, and L. Pastur, “A New Approach for Capacity Analysis of Large Dimensional Multi-Antenna Channels,” submitted to *IEEE Transactions on Information Theory*, December 2006.
[18] J. Hansen and H. Bölcskei, “A geometrical investigation of the rank-1 Rician MIMO channel at high SNR,” *ISIT 2004*, Chicago, USA, June 27 July 2, 2004.
[19] D. Hölsli, Y.-H. Kim, and A. Lapidoth, “Monotonicity results for coherent MIMO Rician Channels,” *IEEE Trans. Inform. Theory*, vol. 51, no. 12, pp. 4334–4339, Dec. 2005.
[20] R. Horn and C. Johnson, *Matrix Analysis*. New York: Cambridge University Press, 1985.
[21] S.A. Jafar and A. Goldsmith, “Transmitter optimization and optimality of beamforming for multiple antenna systems,” *Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1165–1175, July 2004.
[22] S.K. Jayaweera and H.V. Poor, “On the capacity of multiple-antenna systems in Rician fading,” *IEEE Transactions on Wireless Communications*, vol. 4, no. 3, pp. 1102–1111, March 2005.
[23] J.L. Jensen, *Saddlepoint Approximations*. New York: Oxford University Press, 1995.
[24] E. Jorswieck and H. Boche, “Channel capacity and capacity-range of beamforming in MIMO wireless systems under correlated fading with covariance feedback,” IEEE Trans. Wireless Commun., vol. 3, no. 5, pp. 1543–1553, Sept. 2004.

[25] M. Kang and M.-S. Alouini, “Capacity of MIMO Rician channels,” IEEE Transactions on Wireless Communications, vol. 5, no. 1, pp. 112–122, January 2006.

[26] R.P. Kanwal, Generalized Functions: Theory and Techniques. New York: Academic Press, 1983.

[27] A.W. Knapp, Basic Real Analysis. Boston: Birkhäuser, 2005.

[28] G. Lebrun, M. Faulkner, M. Shafi, and P.J. Smith, “MIMO Ricean channel capacity; An asymptotic analysis,” IEEE Transactions on Wireless Communications, vol. 5, no. 6, pp. 3121–3145, Sept. 2005.

[29] A.W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Application. London, U.K.: Academic, 1979.

[30] C. Martin and B. Ottersten, “Asymptotic eigenvalue distributions and capacity for MIMO channels under correlated fading,” IEEE Transactions on Wireless Communications, vol. 3, no. 4, pp. 1350–1359, July 2004.

[31] M.R. McKay and I.B. Collings, “General capacity bounds for spatially correlated Rician MIMO channels,” IEEE Trans. Inform. Theory, vol. 51, no. 9, pp. 3121–3145, Sept. 2005.

[32] M.R. McKay and I.B. Collings, “Improved general lower bound for spatially-correlated Rician MIMO capacity,” IEEE Commun. Letters, vol. 10, no. 3, pp. 162–164, March 2006.

[33] M.R. McKay, P.J. Smith, and I.B. Collings, “New Properties of Complex Noncentral Quadratic Forms and Bounds on MIMO Mutual Information,” Proc. IEEE ISIT 2006, Seattle, USA, July 2006, pp. 1209–1213.

[34] A.L. Moustakas, S.H. Simon, A.M. Sengupta, “MIMO capacity through correlated channels in the presence of correlated interferers and noise: A (not so) large $N$ analysis,” IEEE Trans. Inform. Theory, vol. 49, no. 10, pp. 2545–2561, Oct. 2003.

[35] A.L. Moustakas, S.H. Simon, “Random matrix theory of multi-antenna communications: The Rician channel”, J. Phys. A: Math. Gen., vol. 38, pp. 10859–10872, Nov. 2005.

[36] A.L. Moustakas, S.H. Simon, “On the Outage Capacity of Correlated Multiple-Path MIMO Channels,” IEEE Trans. Inform. Theory, vol. 53, no. 11, pp. 3887–3903, Nov. 2007.

[37] R.R. Müller, “Channel capacity and minimum probability of error in large dual antenna array systems with binary modulation,” IEEE Trans. Sig. Proc., vol. 51, no. 11, pp. 2821–2828, Nov. 2003.

[38] R.R. Müller and W.H. Gerstacker, “On the capacity loss due to separation of detection and decoding,” IEEE Trans. Inf. Theory, vol. 50, no. 8, pp. 1769–1778, Aug. 2004.

[39] H. Nishimori, Statistical physics of spin glasses and information processing: An introduction. Oxford Univ. Press, 2001.

[40] E. Riegler and G. Taricco, “Second-order statistics of the mutual information of the asymptotic separately-correlated Rician fading MIMO channel with interference,” Proc. Globecom 2007, Washington, DC, USA, November 26–30, 2007.

[41] E. Riegler and G. Taricco, “On the ergodic capacity region of the separately correlated Rician fading multiple access MIMO channel,” Proc. Globecom 2007, Washington, DC, USA, November 26–30, 2007.

[42] A.M. Sayeed, “Deconstructing multiantenna fading channels,” IEEE Trans. Sig. Proc., vol. 50, no. 10, pp. 2563–2579, Oct. 2002.

[43] T. Tanaka, “A statistical-mechanics approach to large-system analysis of CDMA multiuser detectors,” IEEE Trans. Inform. Theory vol. 48, no. 11, pp. 2888–910, Nov. 2002.

[44] G. Taricco, “On the capacity of separately-correlated MIMO Rician fading channels,” Proc. Globecom 2006, San Francisco, USA, Nov. 27-Dec. 1, 2006.
[45] G. Taricco and E. Riegler, “On the ergodic capacity of the asymptotic separately-correlated Rician fading MIMO channel with interference,” in *IEEE ISIT 2007*, Nice, France, June 24–30, 2007.

[46] I. E. Telatar, “Capacity of multi-antenna Gaussian channels,” *AT&T Bell Laboratories Technical Report*, BL0112170-950615-07TM, June 1995. Also published as “Capacity of multi-antenna Gaussian channels,” *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, November–December 1999.

[47] S. Venkatesan, S.H. Simon, and R.A. Valenzuela, “Capacity of a Gaussian MIMO Channel with nonzero mean,” *Proc. of IEEE Vehicular Technology Conference*, Oct. 2003.

[48] M. Vu and A. Paulraj, “Capacity optimization for Rician correlated MIMO wireless channels,” *Asilomar Conference 2005*, ASilomar CA, USA, Nov. 2005.

[49] C.-K. Wen and K.-K. Wong, “Asymptotic analysis of spatially correlated MIMO multiple-access channels with arbitrary signaling inputs for joint and separate decoding,” submitted to *IEEE Trans. on Information Theory*, 2006.

[50] J. Winters, “On the capacity of radio communication systems with diversity in a Rayleigh fading environment,” *IEEE J. Select. Areas Commun.*, vol. 5, no. 6, pp. 871–878, June 1987.
Fig. 1. *Capacity mean versus number of antennas for different SNR’s.*

Fig. 2. *Capacity standard deviation versus number of antennas for different SNR’s.*
Fig. 3. Capacity mean versus the TX to RX antenna ratio for different SNR’s.

Fig. 4. Capacity standard deviation versus the TX to RX antenna ratio for different SNR’s.
Fig. 5. Capacity mean versus the Rice factor $K$ (expressed in dB) for different SNR's.

Fig. 6. Capacity standard deviation versus the Rice factor $K$ (expressed in dB) for different SNR's.
Fig. 7. Capacity mean versus exponential spatial correlation base $\alpha$ for different SNR's.

Fig. 8. Capacity standard deviation versus exponential spatial correlation base $\alpha$ for different SNR's.
Fig. 9. Outage capacity versus number of antennas for different SNR's.

Fig. 10. Outage capacity versus exponential spatial correlation base $\alpha$ for different SNR's.