Convergence of Variance-Reduced Stochastic Learning under Random Reshuffling

Bicheng Ying
University of California, Los Angeles
Los Angeles, CA 90095
ybc@ucla.edu

Kun Yuan
University of California, Los Angeles
Los Angeles, CA 90095
kunyuan@ucla.edu

Ali H. Sayed
Ecole Polytechnique Federale de Lausanne
CH-1015 Lausanne, Switzerland
ali.sayed@epfl.ch

Abstract

Several useful variance-reduced stochastic gradient algorithms, such as SVRG, SAGA, Finito, and SAG, have been proposed to minimize empirical risks with linear convergence properties to the exact minimizers. The existing convergence results assume uniform data sampling with replacement. However, it has been observed that random reshuffling can deliver superior performance. No formal proofs or guarantees of exact convergence exist for variance-reduced algorithms under random reshuffling. This paper resolves this open convergence issue and provides the first theoretical guarantee of linear convergence under random reshuffling for SAGA; the argument is also adaptable to other variance-reduced algorithms. Under random reshuffling, the paper further proposes a new amortized variance-reduced gradient (AVRG) algorithm with constant storage requirements compared to SAGA and with balanced gradient computations compared to SVRG. The balancing in computations are attained by amortizing the full gradient calculation across all iterations. AVRG is also shown analytically to converge linearly.

1 Introduction

In recent years, several insightful variance-reduced stochastic gradient algorithms have been proposed, including SVRG [1], SAGA [2], Finito [3], and SAG [4], with the intent of reaching the exact minimizer of an empirical risk optimization problem. Under constant step-sizes and strong-convexity assumptions on the loss functions, these methods have been shown to attain linear convergence towards the exact minimizer when the data are uniformly sampled with replacement.

However, it has been observed that implementations that rely on random reshuffling (RR) of the data (i.e., sampling without replacement) achieve better performance than implementations that rely on uniform sampling with replacement [5–7]. This is true for both traditional stochastic gradient algorithms and variance-reduced algorithms. Under random reshuffling, the algorithm is run multiple times over the finite data where each run is indexed by the integer $k \geq 1$ and is referred to as an epoch. For each run, the original data is first reshuffled so that the sample of index $i$ becomes the sample of index $\sigma^k(i)$, where the symbol $\sigma$ represents a uniform random permutation of the indices. For example, it is shown in [7] that random reshuffling under decaying step-sizes can accelerate the convergence rate of stochastic-gradient learning from $O(1/i)$ to $O(1/i^2)$ [8,9], where $i$ is the iteration index. Likewise, it is shown in [10] that random reshuffling under small constant step-sizes, $\mu$, can boost the steady-state performance of these algorithms from $O(\mu)$-suboptimal to $O(\mu^2)$-suboptimal around a small neighborhood of the exact minimizer [11]. A similar improvement in convergence rate and performance has been observed for the variance-reduced Finito algorithm [3]. However, no formal proofs or guarantees of exact convergence exist for the class of
variance-reduced algorithms under random reshuffling, i.e., it is still not known whether these types of algorithms are guaranteed to converge when RR is employed and under what conditions on the data. In [12], a variance-reduction algorithm is proposed under reshuffling; however, no proof of convergence is provided. The closest attempts at proof are the useful arguments given in [13][14]. The work [13] deals with the case of incremental aggregated gradients, which corresponds to a deterministic version of RR for SAG, while the work [14] deals with SVRG in the context of ridge regression problems using regret analysis.

**Contributions.** This paper resolves this open convergence issue and provides the first theoretical proof and guarantee of linear convergence under random reshuffling for SAGA to the exact minimizer; the argument is also easily adaptable to a wider class of variance-reduced implementations.

Under random reshuffling, the paper further proposes a new amortized variance-reduced gradient (AVRG) algorithm with constant storage requirements compared to SAGA and with balanced gradient computations compared to SVRG. The balancing in computations are attained by amortizing the full gradient calculation across all iterations. AVRG is also shown analytically to converge linearly.

In preparation for the analysis, we review briefly some of the conditions and notation that are relevant. We consider a generic empirical risk function $J(w): \mathbb{R}^M \rightarrow \mathbb{R}$, which is defined as a sample average of loss values over a possibly large but finite training set of size $N$:

$$w^* \triangleq \arg \min_{w \in \mathbb{R}^M} J(w) \triangleq \frac{1}{N} \sum_{n=1}^{N} Q(w; x_n),$$

(1)

where the $\{x_n\}_{n=1}^{N}$ represent training data samples and the loss functions, $Q(w; \cdot)$, are assumed to satisfy the following standard Lipschitz condition.

**Assumption 1 (Loss Function)** The convex loss function, $Q(w; x_n)$, is differentiable and has a $\delta$-Lipschitz continuous gradient, i.e., for every $n = 1, \ldots, N$ and any $w_1, w_2 \in \mathbb{R}^M$:

$$\|\nabla_w Q(w_1; x_n) - \nabla_w Q(w_2; x_n)\| \leq \delta \|w_1 - w_2\|$$

(2)

where $\delta > 0$. We also assume that the empirical risk $J(w)$ is $\nu$-strongly convex, namely,

$$\left(\nabla_w J(w_1) - \nabla_w J(w_2)\right)^T (w_1 - w_2) \geq \nu \|w_1 - w_2\|^2$$

(3)

\[ \square \]

2 **SAGA with random reshuffling**

We consider the SAGA algorithm [2] in this work, while noting that the convergence analysis can be easily extended to other versions of variance-reduced algorithms; in particular, we shall illustrate how it applies to the new variant AVRG. We list the SAGA algorithm without the proximal step in the table below, and incorporate random reshuffling into the description of the algorithm. In the listing, random quantities are denoted in boldface notation.

**SAGA with Random Reshuffling**

*Initialization:* $w_0^k = 0, \nabla Q(\phi_{0,n}^k; x_n) = 0, n = 1, 2, \ldots, N$.

*Repeat* $k = 0, 1, 2, \ldots K$ (epoch):

*generate a random permutation function $\sigma^k(\cdot)$.***

*Repeat* $i = 0, 1, \ldots, N - 1$ (iteration):

$$j = \sigma^k(i + 1)$$

(4)

$$w_{i+1}^k = w_i^k - \mu \left[ \nabla Q(w_i^k; x_j) - \nabla Q(\phi_{i,j}^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{i,n}^k; x_n) \right]$$

(5)

$$\phi_{i+1,j}^k = w_{i+1}^k, \quad \text{and} \quad \phi_{i+1,n}^k = \phi_{i,n}^k, \quad \text{for } n \neq j$$

(6)

*End*

$$w_{0}^{k+1} = w_N^k, \quad \phi_0^{k+1} = \phi_N^k$$

(7)

*End*
For each run $k$, the original data $\{x_n\}_{n=1}^N$ is randomly reshuffled so that the sample of index $i + 1$ becomes the sample of index $j = \sigma^k(i + 1)$. To facilitate the understanding of the algorithm, we associate a block matrix $\Phi^k$ with each run. This matrix is only introduced for visualization purposes. We denote the block rows of $\Phi^k$ by $\{\phi^k_i\}$; one for each iteration $i$, as illustrated in Fig. 1. Each block row $\phi^k_i$ has size $M \times N$, with each entry generated by the SAGA recursion:

$$\phi^k_i \triangleq \begin{bmatrix} \phi^k_{i,1} & \phi^k_{i,2} & \ldots & \phi^k_{i,N} \end{bmatrix} \quad \text{(i-th block row)}$$

We can therefore view $\Phi^k$ as consisting of cells $\{\phi^k_{i,n}\}$, each having the same $M \times 1$ size as the minimizer $w^\star$. At every iteration $i$, one random cell in the $(i+1)$-th block row is populated by the iterate $w^k_{i+1}$; the column location of this random cell is determined by the value of $j$. Appendix A explains in greater detail how the cells are updated.

![Figure 1: An illustration of the evolution of the history variables $\{\phi^k_{i,n}\}$](image)

### 2.1 Properties of the history variables

Several useful observations can be drawn from Fig. 1. These properties will be useful in the convergence proof in subsequent sections.

**Observation 1:** At the start of each epoch $k$, the components $\{\phi^k_{0,n}\}_{n=1}^N$ correspond to a permutation of the weight iterates from the previous run, $\{w^{k-1}_n\}_{n=1}^N$. □

**Observation 2:** At the beginning of the $i$-th iteration of an epoch $k$, all components of indices $\{\sigma^k(m)\}_{m=1}^{i+1}$ will be set to weight iterates obtained during the $k$-th run, namely, $\{w^k_m\}_{m=i}^{i+1}$, while the remaining $N - i$ history positions will have values from the previous run, namely, $\{w^{k-1}_{i+n}\}_{n=1}^{N-i}$ for some values $i_n \in \{1, 2, \ldots, N\}$. □

**Observation 3:** At the beginning of the $i$-th iteration of an epoch $k$, it holds that

$$\phi^k_{i,j} = \phi^k_{0,j}, \quad \text{where } j \in \sigma^k(i+1:N)$$

where $\sigma^k(i+1:N)$ represents the selected indices for future iterations $i + 1$ to $N$. This property holds because, under random reshuffling, sampling is performed without replacement. □

Using these observations, the following two results can be established.

**Lemma 1 (Distribution of History Variables):** Conditioned on the previous $k - 1$ epochs, each history variable $\phi^k_{i,j}$ has the following probability distribution at the beginning of the $i$-th iteration of epoch $k$:

$$P(\phi^k_{i,j} | F^0_0) = \begin{cases} 1/N, & \phi^k_{i,j} = w^{k-1}_1 \\ 1/N, & \phi^k_{i,j} = w^{k-1}_2 \\ \vdots \\ 1/N, & \phi^k_{i,j} = w^{k-1}_N \end{cases}, \quad \text{for } j \in \sigma^k(i+1:N)$$

where $F^0_0$ is the collection of all information before iteration 0 at epoch $k$.

**Proof:** See Appendix A □
Lemma 2 (Second-order Moment of History Variable) The aggregate second-order moment of each history variable \( \phi_{i,n}^k \) is equal to:

\[
E \left[ \sum_{n=1}^{N} \| \phi_{i,n}^k \|^2 \right] = \sum_{n=1}^{i} E \| \omega_n^k \|^2 + \frac{N-i}{N} \sum_{n=1}^{N} E \| \omega_n^{k-1} \|^2 
\]  

(11)

Proof: See Appendix [C].

For comparison purposes, all previous results do not hold for implementations that involve sampling the data with replacement. For example, property \([14]\) would be replaced by \([2]\):

\[
E \left[ \sum_{n=1}^{N} \| \phi_{i,n}^k \|^2 \right] = E \| \omega_n^k \|^2 + \frac{N-1}{N} \sum_{n=1}^{N} E \| \phi_{i-1,n}^k \|^2, \quad \forall i, k
\]

(12)

This result is similar to \([11]\) only for \( i = 1 \). However, observe that \([12]\) involves variables \( \{ \phi_{i-1,n}^k \} \) on the right-hand side, instead of the variables \( \{ \omega_n^{k-1} \} \) that appear in \([11]\). This is because random reshuffling updates every history variable during each run, while uniform sampling may leave some variables \( \phi_{i-1,n}^k \) untouched. This fact also helps explain why SAGA with RR tends to have faster convergence rate, as we will illustrate in a later experiment.

2.2 Biased Nature of the Gradient Estimator

Before launching into the convergence analysis of the variance-reduced algorithm, we first highlight one important observation, namely, that it is not necessary to insist on unbiased gradient estimators for proper operation of stochastic-gradient algorithms. To see this, let us examine first the SAGA implementation assuming uniform data sampling with replacement. In a manner similar to \([9]\), the SAGA algorithm in this case will employ the following modified gradient direction:

\[
\hat{g}_u(w_t^k) \triangleq \nabla Q(w_t^k; x_u) - \nabla Q(\phi_{t,u}^k; x_u) + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{t,n}^k; x_n)
\]

(13)

where the subscript \( u \) is used to denote a uniformly distributed random variable, \( u \sim U[1,N] \). As a result, this modified gradient satisfies the unbiasedness property, i.e., \( E_u[\hat{g}_u(w_t^k)|\mathcal{F}_t^k] = \nabla J(w_t^k) \), where \( \mathcal{F}_t^k \) denotes the collection of all available information before iteration \( i \) at epoch \( k \). However, this property no longer holds under random reshuffling! This is because data is now sampled without replacement and the selection of one index becomes dependent on the selections made prior to it. Specifically, it now holds that

\[
E_j[\hat{g}_j(w_t^k)|\mathcal{F}_t^k] = \frac{1}{N-1} \sum_{n \not\in \sigma^k(1:i)} \left( \nabla Q(w_t^k; x_n) - \nabla Q(\phi_{t,n}^k; x_n) \right) + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{t,n}^k; x_n)
\]

(14)

where \( j = \sigma^k(i+1) \). It is not hard to see that the expression on the right-hand side is generally different from \( \nabla J(w_t^k) \). Consequently, the gradient estimate that is employed by SAGA under RR is not an unbiased estimator for the true gradient. Nevertheless, we will establish two useful facts in the following sections. First, this gradient estimate becomes asymptotically unbiased when the algorithm converges, as \( k \rightarrow \infty \). Second, the biased gradient estimation does not harm the convergence rate because we will observe later that SAGA under RR actually converges faster than SAGA in the simulations.

2.3 Convergence Analysis

The analysis employs two supporting lemmas. To begin with, we relate the starting iterates for two successive epochs as follows by summing all gradient terms in \([3]\) over \( i \):

\[
w_{i0}^{k+1} = w_{i0}^k - \mu \sum_{i=0}^{N-1} \left[ \nabla Q(w_{i0}^k; x_j) - \nabla Q(\phi_{i0,j}^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{i,n}^k; x_n) \right]
\]

(15)

where \( j = \sigma^k(i) \). As already alluded to, one main difficulty in the analysis is the fact that the gradient estimate is biased. For this reason, we shall compare against the gradient at the start of the epoch:

\[
w_{i0}^{k+1} \overset{(a)}{=} w_{i0}^k - \mu N \nabla J(w_t^k) + \mu \sum_{i=0}^{N-1} \left[ \nabla Q(\phi_{0,j}^k; x_j) - \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{0,n}^k; x_n) \right]
\]

\(a\)
\[
\sum_{i=0}^{N-1} \left[ \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{i,n}^k; x_n) - \nabla Q(\phi_{0,n}^k; x_n) \right) \right]
\]

\[
= w_0^k - \mu N \nabla J(w_0^k)
\]

\[
- \mu \sum_{i=0}^{N-1} \left[ \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{i,n}^k; x_n) - \nabla Q(\phi_{0,n}^k; x_n) \right) \right]
\]

where in step (a) we added and subtracted \( \{ \nabla Q(w_0^k; x_j) \} \) and \( \{ \nabla Q(\phi_{0,n}^k; x_n) \} \), and we also changed the notation \( \nabla Q(\phi_{i,n}^k; x_j) \) into \( \nabla Q(\phi_{0,n}^k; x_j) \) because of observation 3; in step (b) we exploited the random reshuffling property that each index is selected only once.

We also need to appeal to a second recursion (within epoch \( k \)). By moving \( w_i^k \) in (5) to the left-hand side and computing the squared norm, we obtain:

\[
\|w_{i+1}^k - w_i^k\|^2 = \mu^2 \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\phi_{i,n}^k; x_n) \right\|^2
\]

\[
\leq 3\mu^2 \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) \right\|^2 + 3\mu^2 \left\| \nabla Q(\phi_{i,n}^k; x_n) - \nabla Q(w_0^k; x_n) \right\|^2
\]

\[
\leq 3\delta^2 \mu^2 \|w_i^k - w_0^k\|^2 + 3\mu^2 \|w_i^k - \phi_{i,n}^k\|^2 + 3\delta^2 \mu^2 \sum_{n=1}^{N} \|\phi_{i,n}^k - w_0^k\|^2
\]

Lemma 3 (Mean-Square Error Recursion) The mean-square-error at the start of each epoch satisfies the inequality recursion when step size \( \mu \leq 1/(N\nu) \):

\[
E \|w_0^k\|^2 \leq \left( 1 - \frac{\mu \nu N - \mu^2 N^2 \delta^2}{1 - \mu N \nu} \right) E \|w_0^k\|^2 + 4\mu^2 \delta^2 \nu \left( \sum_{i=1}^{N-1} E \|w_i^k - w_0^k\|^2 + \sum_{n=1}^{N-1} E \|w_N^{k-1} - w_{n-1}^{k-1}\|^2 \right)
\]

Proof: See Appendix D.

Roughly, the above result shows that the mean-square error across epochs evolves according to a dynamics that is determined by a scaling factor (smaller than one for small \( \mu \)) in addition to two driving terms: we will refer \( \sum_{i=1}^{N-1} E \|w_i^k - w_0^k\|^2 \) as the forward inner difference term and to \( \sum_{n=1}^{N-1} E \|w_N^{k-1} - w_{n-1}^{k-1}\|^2 \) as the backward inner difference term.

Lemma 4 (Inner Differences) The forward inner difference satisfies:

\[
\sum_{i=1}^{N-1} E \|w_i^k - w_0^k\|^2 \leq 5\delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} E \|w_i^k - w_0^k\|^2 + \sum_{i=1}^{N-1} E \|w_N^{k-1} - w_i^{k-1}\|^2 \right) + 3\delta^2 \mu^2 N^3 E \|w_0^k\|^2
\]

while the backward inner difference satisfies:

\[
\sum_{i=1}^{N-1} E \|w_N^{k-1} - w_i^{k-1}\|^2 \leq 5\delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} E \|w_i^k - w_0^k\|^2 + \sum_{i=1}^{N-1} E \|w_N^{k-2} - w_i^{k-2}\|^2 \right) + 3\delta^2 \mu^2 N^3 E \|w_0^{k-1}\|^2
\]
Proof: See Appendix E.

Combining the above lemmas, we arrive at the following theorem. Introduce the energy function:

\[ V_{k+1} \Delta= \mathbb{E} \left\| \bar{w}_{0}^{k+1} \right\|^2 + \frac{11}{16} \gamma \left( \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{E} \left\| \bar{w}_{i}^{k+1} - w_{0}^{k+1} \right\|^2 + \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{E} \left\| w_{N}^{k} - w_{i}^{k} \right\|^2 \right) \]  

(21)

where \( \gamma = 9 \mu \delta N \).

**Theorem 1 (Linear Convergence of SAGA)** For sufficiently small step-sizes, namely, for \( \mu \leq \frac{1}{12 \delta^{2} N^{4}} \), the quantity \( V_{k+1} \) converges linearly:

\[ V_{k+1} \leq \alpha V_{k} \]  

(22)

where \( \alpha = \left( 1 - \frac{\mu \gamma N/4}{1 - 2 \delta^{2} \mu^{3} N^{2}} \right) < 1 \). It follows that \( \mathbb{E} \left\| \bar{w}_{0}^{k} \right\|^2 \leq \alpha^{k} V_{0} \).

Proof: See Appendix F. 

**Remark:** To achieve an \( \epsilon \)-optimal solution, the number of iterations required is close to \( O(\delta^{2}/\mu^{2}) \log(1/\epsilon) \), which is much slower than the theorem proved in [2]. The main reason is that the dependency between the samples makes it difficult to obtain a tight bound. As we will observe in the simulations later, in practice, the convergence can be faster than the original SAGA.

### 3 Amortized Variance-Reduced Gradient (AVRG) Learning

One inconvenience of the SAGA implementation is its high storage requirement, which refers to the need to track history variables \( \{ \phi_{i,n}^{k} \} \) or gradients for use in (5). There is a need to store \( O(N) \) variables. In big data applications, the size of \( N \) can be prohibitive. An alternative method is the stochastic variance-reduced gradient (SVRG) algorithm [1]. This method replaces the history variables \( \{ \phi_{i,j}^{k} \} \) of SAGA by a fixed initial condition \( w_{0}^{k} \) for each epoch. This simplification greatly reduces the storage requirement. However, each epoch in SVRG is preceded by an aggregation step to compute a gradient estimate, which is time-consuming for large data sets. It also causes the operation of SVRG to become unbalanced, with a larger time interval needed before each epoch, and shorter time intervals needed within the epoch. Motivated by these two important considerations, we propose a new amortized implementation, referred to as AVRGG. This new algorithm removes the initial aggregation step from SVRG and replaces it by an estimate \( g_{k+1} \). This estimate is computed iteratively within the inner loop by re-using the gradient, \( \nabla Q(w_{i}^{k}; x_{j}) \), to reduce complexity.

**AVRG with Random Reshuffling**

Initialization: \( w_{0}^{i} = 0, \ g^{0} = 0, \ \nabla Q(w_{0}^{i}; x_{n}) \leftarrow 0, \ n = 1, 2, \ldots, N \).  

Repeat \( k = 0, 1, 2, \ldots, K \) (epoch):

- generate a random permutation function \( \sigma^{k} (\cdot) \), set \( g^{k+1} = 0 \)

Repeat \( i = 0, 1, \ldots, N - 1 \) (iteration):

\[ j = \sigma^{k}(i + 1) \]

\[ w_{i+1}^{k} = w_{i}^{k} - \mu \left[ \nabla Q(w_{i}^{k}; x_{j}) - \nabla Q(w_{0}^{k}; x_{j}) + g^{k} \right] \]  

(24)

\[ g^{k+1} \leftarrow g^{k+1} + \frac{1}{N} \nabla Q(w_{i}^{k}; x_{j}) \]  

(25)

End \( w_{0}^{k+1} = w_{N}^{k} \)

End

#### 3.1 Useful properties

Several properties stand out when we compare the proposed AVRGG implementation with the previous algorithms. First, observe that the storage requirement for AVRGG in each epoch is just the variables \( g^{k}, g^{k+1}, \) and \( w_{0}^{k} \), which is similar to SVRG and considerably less than SAGA. A variance-reduced algorithm based on reshuffling is proposed in [12]; however, it still requires the same extra storage as SAGA.

Second, since the gradient vector \( Q(w_{i}^{k}; x_{j}) \) used in (25) has already been computed in (24), every iteration \( i \) will only require two gradients to be evaluated. Thus, the effective computation of gradients per epoch is smaller in AVRGG than in SVRG.
Third, observe from Eq. (25) how the estimated \( \hat{\mathbf{g}}^k \) is computed by averaging the loss values at successive iterates. This construction is feasible because of the use of random reshuffling. Under random reshuffling, the collection of gradients \( \{Q(\mathbf{w}^k_i; x_j)\} \) that are used in (25) during each epoch will end up covering the entire set of data, \( \{x_n\}_{n=1}^N \). This is not necessarily the case for operation under uniform sampling with replacement. Therefore, the AVRG procedure assumes the use of random reshuffling. We will simply refer to it as AVRG, rather than AVRG under RR.

Fourth, unlike the SVRG algorithm, which requires a step to compute the full gradient, the AVRG implementation is amenable to decentralized implementations (i.e., to fully-decentralized implementations with no master nodes), and also to asynchronous operation [15]. The unbalanced gradient computation in SVRG poses difficulties for decentralized solutions and introduces idle times when multiple devices/agents with different amounts of data cooperate to solve an optimization problem. We will illustrate and resolve these challenges in another work detailed in [16].

Finally, the modified gradient direction that is employed in (24) by AVRG has distinctive properties in relation to the modified gradient direction \( \tilde{\mathbf{g}}^k \) in SAGA. To see this, we note that the gradient direction in (24) can be written as

\[
\tilde{\mathbf{g}}^k_j(\mathbf{w}^k_i) = \nabla Q(\mathbf{w}^k_i; x_j) - \nabla Q(\mathbf{w}^0_i; x_j) + \frac{1}{N} \sum_{n=0}^{N-1} \nabla Q(\mathbf{w}^{k-1}_n; x_{\sigma^{k-1}(n+1)})
\]

(27)

It is clear that even when the index \( j \) is chosen uniformly, the above vector cannot be an unbiased estimator for true gradient in general. What is more critical for convergence is that the modified gradient direction should satisfy the useful property that as the weight iterate gets closer to the optimal value, i.e., as \( \|\mathbf{w}^* - \mathbf{w}^k\| \leq \epsilon \), for arbitrary small \( \epsilon \) and large enough \( k \), the modified and true gradients will also get arbitrarily close to each. This property holds for (27) since

\[
\|\tilde{\mathbf{g}}^k_j(\mathbf{w}^k_i) - \nabla J(\mathbf{w}^*)\| \\
\leq \|\nabla Q(\mathbf{w}^k_i; x_j) - \nabla Q(\mathbf{w}^0_i; x_j)\| + \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\mathbf{w}^{k-1}_n; x_{\sigma^{k-1}(n)}) - \frac{1}{N} \sum_{n=1}^{N} \nabla Q(\mathbf{w}^*; x_n) \\
\leq \delta \|\mathbf{w}^k_i - \mathbf{w}^*\| + \delta \|\mathbf{w}^0_i - \mathbf{w}^*\| + \delta \sum_{n=1}^{N-1} \|\mathbf{w}^{k-1}_{n-1} - \mathbf{w}^*\| \\
\leq 3\delta \epsilon
\]

(28)

where in the second inequality we exploited Jensen’s inequality, the triangle inequality, Lipschitz assumption, and the fact that \( \sigma^{k-1}(n) \) corresponds to sampling without replacement. Because \( \epsilon \) can be chosen arbitrary small, then \( \tilde{\mathbf{g}}^k_j(\mathbf{w}^k_i) \) must approach the true gradient at \( \mathbf{w}^* \). This result also implies the aforementioned asymptotic unbiasedness property of the gradient estimate. Actually, this property holds for all previous modified gradients in SAGA, SVRG, SAG, Finito, and AVRG. The work [17] also discusses a case where there is an extra error term in the gradient calculation. For ease of reference, Table 3.1 compares the trade-offs between storage and computational complexity of different variance-reduced algorithms with and without random reshuffling.

### 3.2 Convergence analysis

The same approach used to establish the convergence of SAGA under RR is also suitable for AVRG. For this reason, we can be brief. First, similar to [16], we derive the main recursion for one epoch:

\[
\mathbf{w}^{k+1}_0 = \mathbf{w}^k_0 - \mu N \nabla J(\mathbf{w}^k_0) - \mu \sum_{i=0}^{N-1} [\nabla Q(\mathbf{w}^k_i; x_j) - \nabla Q(\mathbf{w}^0_i; x_j)] \\
+ \mu \sum_{i=0}^{N-1} [\nabla Q(\mathbf{w}^k_i; x_j') - \nabla Q(\mathbf{w}^{k-1}_i; x_j')]
\]

(29)
where, for compactness of notation, we introduce $\theta = \sigma^{k-1}(i+1)$. Second, similar to (17), we derive an inner difference recursion:

$$
|w^k_{i+1} - w^k_i|^2 = \mu^2 \left| \nabla Q(w^k_i; x_j) - \nabla Q(w^k_0; x_j) + g^k \right|^2 
\leq 3\mu^2\delta^2 \left( |w^k_i - w^k_0|^2 + \frac{1}{N} \sum_{i=0}^{N-1} |w^k_{i-1} - w^k_N|^2 + |\tilde{w}^k_0|^2 \right) \tag{30}
$$

Next, we establish recursions related to $\tilde{w}^k_0$, and the forward and backward difference terms.

**Lemma 5 (Recursions for AVRG Analysis)** The mean-square-error at the start of each epoch satisfies the inequality recursion when step size $\mu \leq 1/(N\nu)$:

$$
\mathbb{E} \|\tilde{w}^{k+1}_0\|^2 \leq \left( 1 - \frac{\mu\nu - \mu^2 N^2 \delta^2}{1 - \mu N \nu} \right) \mathbb{E} \|\tilde{w}^k_0\|^2 + 2\mu\delta^2 \left( \sum_{i=1}^{N-1} \mathbb{E} \|w^k_i - w^k_0\|^2 + \sum_{i=1}^{N-1} \mathbb{E} \|w^k_{i-1} - w^k_1\|^2 + N\mathbb{E} \|\tilde{w}^k_0\|^2 \right) \tag{31}
$$

Moreover, the forward inner difference satisfies:

$$
\sum_{i=0}^{N-1} \mathbb{E} \|w^k_i - w^k_0\|^2 \leq 3\mu^2\delta^2 N^2 \sum_{i=0}^{N-1} \mathbb{E} \|w^k_i - w^k_0\|^2 + \mu^2\delta^2 N^2 \left( \sum_{i=0}^{N-1} \mathbb{E} \|w^k_{i-1} - w^k_1\|^2 + N\mathbb{E} \|\tilde{w}^k_0\|^2 \right) \tag{32}
$$

while the backward inner difference satisfies:

$$
\sum_{i=0}^{N-1} \mathbb{E} \|w^k_N - w^k_i\|^2 \leq 3\mu^2\delta^2 N^2 \sum_{i=0}^{N-1} \mathbb{E} \|w^k_i - w^k_0\|^2 + 3\mu^2\delta^2 N^2 \left( \sum_{i=0}^{N-1} \mathbb{E} \|w^k_{i-1} - w^k_1\|^2 + N\mathbb{E} \|\tilde{w}^k_0\|^2 \right) \tag{33}
$$

**Proof:** See Appendix G.

Likewise, we introduce the energy function:

$$
V_{k+1} \Delta \equiv \mathbb{E} \|\tilde{w}^{k+1}_0\|^2 + \frac{13}{16} \sum_{i=1}^{N-1} \mathbb{E} \|w^k_{i+1} - w^k_0\|^2 + \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{E} \|w^k_i - w^k_0\|^2 \tag{34}
$$

where $\gamma = 6\mu\delta N$, and state the relevant convergence theorem.

**Theorem 2 (Linear Convergence of AVRG)** For sufficiently small step-sizes, namely, for $\mu \leq \frac{\nu}{98 N \delta}$, the quantity $V_{k+1}$ converges linearly:

$$
V_{k+1} \leq \alpha V_k \tag{35}
$$

where $\alpha = \frac{1 - \mu\nu/4}{1 - 18\delta^2\mu^2 N^2 / \nu} < 1$. It follows that $\mathbb{E} \|\tilde{w}^k_0\|^2 \leq \alpha^k V_0$.

**Proof:** See Appendix H.

**Remark:** This is similar to the theorem for SAGA under RR except for the scaling coefficients. However, in practice, AVRG will perform differently from SAGA under RR.

## 4 Simulation Results

In this section, we illustrate the convergence performance of various algorithms by numerical simulations. We consider the following regularized logistic regression problem:

$$
\min_w J(w) = \frac{1}{N} \sum_{n=1}^{N} Q(w; h_n, \gamma(n)) \Delta \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{2} \|w\|^2 + \ln \left( 1 + \exp(-\gamma(n)h_n^T w) \right) \right), \tag{36}
$$

where $h_n \in \mathbb{R}^M$ is the feature vector, $\gamma(n) \in \{\pm 1\}$ is the class label. In all our experiments, we set $p = 1/N$. The optimal $w^*$ and the corresponding risk value are calculated by means of the Scikit-Learn package. We run simulations over four datasets: covtype.binary, rcv1.binary, MNIST, and

[http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/) [http://yann.lecun.com/exdb/mnist/]
The last two datasets have been transformed into binary classification problems by considering data with labels 0 and 1, i.e., digital zero and one classes for MNIST and airplane and automobile classes for CIFAR-10. All features have been preprocessed and normalized to the unit vector. The results are exhibited in Fig. 2. To enable fair comparisons, we tune the step-size parameter of each algorithm for fastest convergence in each case. The plots are based on measuring the relative mean-square-error, $\|w_0 - w^*\|^2/\|w^*\|^2$, and the excess risk value, $E J(w_0^N) - J(w^*)$. Two key facts to observe from these simulations are that 1) SAGA with RR is consistently faster than SAGA, and 2) without the high memory cost of SAGA and without the unbalanced structure of SVRG, the proposed AVRG technique is able to match their performance reasonably well.

5 Discussion and Future Work

The statements of Theorems 1 and 2 are similar. This suggests that the analysis approach is applicable to a wider class of variance-reduced implementations. The statements also suggest that these types of algorithms are able to deliver linear convergence for sufficiently small constant step-sizes. One useful extension for future study is to consider situations with non-smooth loss functions. It is also useful to note that the stability ranges and convergence rates derived from the theoretical analysis tend to be more conservative than what is actually observed in experiments.

References

[1] R. Johnson and T. Zhang, “Accelerating stochastic gradient descent using predictive variance reduction,” in Proc. Advances in Neural Information Processing Systems (NIPS), Lake Tahoe, Nevada, 2013, pp. 315–323.

[2] A. Defazio, F. Bach, and S. Lacoste-Julien, “SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives,” in Proc. Advances in Neural Information Processing Systems (NIPS), Montreal, Canada, 2014, pp. 1646–1654.

[3] A. Defazio, J. Domke, and T. S. Caetano, “Finito: A faster, permutable incremental gradient method for big data problems,” in Proc. International Conference of Machine Learning (ICML), 2014, pp. 1125–1133.

[4] N. L. Roux, M. Schmidt, and F. R. Bach, “A stochastic gradient method with an exponential convergence rate for finite training sets,” in Proc. Advances in Neural Information Processing Systems (NIPS), Lake Tahoe, Nevada, 2012, pp. 2663–2671.

[5] L. Bottou, “Curiously fast convergence of some stochastic gradient descent algorithms,” in Proc. Symposium on Learning and Data Science, Paris, 2009.

http://www.cs.toronto.edu/~kriz/cifar.html
[6] B. Recht and C. Ré, “Toward a noncommutative arithmetic-geometric mean inequality: Conjectures, case-studies, and consequences,” in Proc. Conference On Learning Theory (COLT), 2012, pp. 1–11.

[7] M. Gürbüzbalaban, A. Ozdaglar, and P. Parrilo, “Why random reshuffling beats stochastic gradient descent,” arXiv:1510.08560, Oct. 2015.

[8] Y. Nesterov, Introductory Lectures on Convex Optimization: A basic course, vol. 87, Springer, 2013.

[9] B. T. Polyak, Introduction to Optimization, Optimization Software, NY, 1987.

[10] B. Ying, K. Yuan, S. Vlaski, and A. H. Sayed, “On the performance of random reshuffling in stochastic learning,” in Proc. Information Theory and Applications Workshop (ITA), San Diego, CA, Feb. 2017, pp. 1–5.

[11] A. H. Sayed, “Adaptation, learning, and optimization over networks,” Foundations and Trends in Machine Learning, vol. 7, no. 4–5, pp. 311–801, 2014.

[12] S. De and T. Goldstein, “Efficient distributed SGD with variance reduction,” in Proc. IEEE International Conference on Data Mining (ICDM), 2016, pp. 111–120.

[13] M. Gürbüzbalaban, A. Ozdaglar, and P. Parrilo, “Convergence rate of incremental gradient and newton methods,” arXiv:1510.08562, Oct. 2015.

[14] O. Shamir, “Without-replacement sampling for stochastic gradient methods: Convergence results and application to distributed optimization,” arXiv:1603.00570, Mar. 2016.

[15] S. J. Reddi, A. Hefny, S. Sra, B. Poczos, and A. J. Smola, “On variance reduction in stochastic gradient descent and its asynchronous variants,” in Advances in Neural Information Processing (NIPS), pp. 2647–2655. 2015.

[16] K. Yuan, B. Ying, and A. H. Sayed, “Efficient variance-reduced learning for fully decentralized on-device intelligence,” Available at arXiv, 2017.

[17] R. Harikandeh, M. O. Ahmed, A. Virani, M. Schmidt, J. Konecny, and S. Sallinen, “Stop wasting my gradients: Practical SVRG,” in Advances in Neural Information Processing (NIPS), pp. 2251–2259. 2015.

[18] L. Xiao and T. Zhang, “A proximal stochastic gradient method with progressive variance reduction,” SIAM Journal on Optimization, vol. 24, no. 4, pp. 2057–2075, 2014.
A Evolution of the history variables

We refer to Fig. 1 and explain in greater detail how the cells in the figure are updated. These cells play the role of history variables. To begin with, at iteration \( i = 0 \), the cells in the first block row \( \phi_0^k \) will contain a randomly reshuffled version of all iterates \( \{w_1^{k-1}, w_2^{k-1}, \ldots, w_N^{k-1}\} \) generated during the previous run of index \( k - 1 \). A random sample of index \( j = \sigma^k(1) \) is selected. Assume this value turns out to be \( j = 2 \). Then, as indicated in the blue cell in the second block row in the figure, the second cell of \( \phi_0^k \) is updated to \( w_1^k \) while all other cells in this row remain invariant. Moving to iteration \( i = 1 \), a new random sample of index \( j = \sigma^k(2) \) is selected. Assume this value turns out to be \( j = N \). Then, as indicated again in the third block row in the figure, the last cell of \( \phi_0^k \) is updated to \( w_N^k \) while all other cells in this row remain invariant. The process continues in this manner, by populating the cell corresponding to location \( j \) in the \( i \)-th block row. By the end of iteration \( N \), all cells of \( \phi_N^k \) would have been populated by the iterates \( \{w_i^k\} \) generated during the \( k \)-th run. Observe that, since uniform sampling with replacement is used, then all weight iterates \( \{w_i^k\} \), from \( i = 1 \) to \( i = N \) will appear in \( \phi_N^k \). These iterates appear randomly shuffled in the last row in the figure and they constitute the initial value for \( \phi_0^{k+1} \) for the next run.

B Proof of lemma 1

For \( j = \sigma^k(i + 1) \) and any \( w_t^{k-1}, t = 1, 2, \ldots, N \), it holds that

\[
P(\phi_{i,j}^k = w_t^{k-1} | F_0^k) = \sum_{\sigma^k} P(\sigma^k)P(\phi_{i,j}^k = w_t^{k-1} | F_0^k, \sigma^k)
\]

\[
= \sum_{\sigma^k} \frac{1}{N!} P(\phi_{i,j}^k = w_t^{k-1} | F_0^k, \sigma^k)
\]

\[
= \sum_{\sigma^k} \frac{1}{N!} P(\phi_{0,j}^k = w_t^{k-1} | F_0^k, \sigma^k)
\]

\[
= \frac{1}{N!} \sum_{\sigma^k} \mathbb{I}(\phi_{0,j}^k = w_t^{k-1} | F_0^k, \sigma^k)
\]

(37)

The second equality is because all permutation sequences are equally probable; the third equality applies observation 3. The last equality follows from noting that, given \( F_0^k \) and \( \sigma^k \), the quantity \( \phi_{0,j}^k \) becomes a deterministic variable. In this case, the probability \( P(\phi_{0,j}^k | F_0^k, \sigma^k) \) is either 1 or 0. We therefore express it in terms of the indicator function, where the notation \( \mathbb{I}[a] = 1 \) when the statement \( a \) is true and is zero otherwise. Next note that there are \( (N - 1)! \) permutations \( \sigma^k \) with the \( j \)-th position storing \( w_t^{k-1} \). Substituting back, we get

\[
P(\phi_{i,j}^k = w_t^{k-1} | F_0^k) = \frac{(N - 1)!}{N!} = \frac{1}{N}
\]

(38)

C Proof of lemma 2

Conditioning on the information in the past epochs:

\[
E \left[ \sum_{n=1}^{N} |\phi_{i,n}^k|^2 | F_0^k \right] = E \left[ \sum_{n \in \sigma^k(1:i)} |\phi_{i,n}^k|^2 | F_0^k \right] + E \left[ \sum_{n \notin \sigma^k(1:i)} |\phi_{i,n}^k|^2 | F_0^k \right]
\]

\[
= E \left[ \sum_{i' = 1}^{i} |w_{i'}^k|^2 | F_0^k \right] + E \left[ \sum_{i' = i+1}^{N} |\phi_{i'',\sigma^k(i'')}|^2 | F_0^k \right]
\]

\[
= \sum_{i' = 1}^{i} E \left[ |w_{i'}^k|^2 | F_0^k \right] + \sum_{i' = i+1}^{N} E \left[ |\phi_{i'',\sigma^k(i'')}|^2 | F_0^k \right]
\]

11
Proof of lemma 3

By introducing the error quantity \( \bar{w}_i^k = w^* - w_i^k \), we easily arrive at the following recursion for the evolution of the error dynamics:

\[
\bar{w}_0^{k+1} = \bar{w}_0^k + \mu N \nabla J(w_0^k) + \mu \sum_{i=0}^{N-1} \left( \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{i,n}; x_n) - \nabla Q(\phi_{0,n}; x_n) \right) \right)
\]

Computing the conditional mean-square-error of both sides of (40), and appealing to Jensen’s inequality, gives:

\[
\begin{align*}
\mathbb{E} \left[ \left\| \bar{w}_0^{k+1} \right\|^2 \mid \mathcal{F}_0^k \right] & \leq \frac{1}{1 - t} \left\| \bar{w}_0^k + \mu N \nabla J(w_0^k) \right\|^2 \\
& \quad + \frac{\mu^2}{t} \mathbb{E} \left\{ \left\| \sum_{i=0}^{N-1} \left( \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{i,n}; x_n) - \nabla Q(\phi_{0,n}; x_n) \right) \right) \right\|^2 \mid \mathcal{F}_0^k \right\} \\
& \leq \frac{1}{1 - t} \left\| \bar{w}_0^k + \mu N \nabla J(w_0^k) \right\|^2 \\
& \quad + \frac{\mu^2}{t} \sum_{i=0}^{N-1} \mathbb{E} \left[ \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) + \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{i,n}; x_n) - \nabla Q(\phi_{0,n}; x_n) \right) \right\|^2 \mid \mathcal{F}_0^k \right] \\
& \leq \frac{1}{1 - t} \left\| \bar{w}_0^k + \mu N \nabla J(w_0^k) \right\|^2 + \frac{\mu^2 N}{t} \sum_{i=0}^{N-1} \mathbb{E} \left[ \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) \right\|^2 \mid \mathcal{F}_0^k \right] \\
& \quad + \frac{2 \mu^2 N}{t} \sum_{i=0}^{N-1} \mathbb{E} \left[ \left\| \nabla Q(\phi_{0,n}; x_n) \right\|^2 \mid \mathcal{F}_0^k \right] \\
& = \frac{1}{1 - t} \left\| \bar{w}_0^k + \mu N \nabla J(w_0^k) \right\|^2 + \frac{\mu^2 N}{t} \sum_{i=0}^{N-1} \mathbb{E} \left[ \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) \right\|^2 \mid \mathcal{F}_0^k \right] \\
& \quad + \frac{2 \mu^2 N}{t} \sum_{i=0}^{N-1} \mathbb{E} \left[ \left\| \nabla Q(\phi_{0,n}; x_n) \right\|^2 \mid \mathcal{F}_0^k \right]
\end{align*}
\]

(41)

where step (a) follows from Jensen’s inequality and \( t \) can be chosen arbitrarily in the open interval \( t \in (0, 1) \); and steps (b) and (c) also follow from the following corollary of Jensen’s inequality:

\[
\left\| \sum_{i=1}^{N} y_i \right\|^2 \leq N \sum_{i=1}^{N} \left\| y_i \right\|^2 \leq N \sum_{i=1}^{N} \left\| y_i \right\|^2
\]

(42)

We further know from the Lipschitz condition (2) that:

\[
\mathbb{E} \left[ \left\| \nabla Q(w_i^k; x_j) - \nabla Q(w_0^k; x_j) \right\|^2 \mid \mathcal{F}_0^k \right] \leq \delta^2 \mathbb{E} \left[ \left\| w_i^k - w_0^k \right\|^2 \mid \mathcal{F}_0^k \right]
\]

(43)
and
\[
E \left[ \left\| \frac{1}{N} \sum_{n=1}^{N} \left( \nabla Q(\phi_{k,n}; x_n) - \nabla Q(\phi_{0,n}; x_n) \right) \right\|^{2} \right| F_{0}^{k}
\]
\[
(\text{a)} \quad E \left[ \left\| \frac{1}{N} \sum_{n=1}^{i} \nabla Q(w_{k,n}; x_{\sigma(n)}^{k}) - \nabla Q(\phi_{0,n}; x_{\sigma(n)}^{k}) \right\|^{2} \right| F_{0}^{k}
\]
\[
\leq \frac{i \delta^{2}}{N^{2}} \sum_{n=1}^{i} E \left[ \| w_{n}^{k} - \phi_{0,n} \|^{2} \right| F_{0}^{k}
\]
\[
= \frac{i \delta^{2}}{N^{2}} \sum_{n=1}^{i} E \left[ \| w_{0}^{k} + w_{N}^{k-1} - \phi_{0,n} \|^{2} \right| F_{0}^{k}
\]
\[
\leq (\text{b)} \quad \frac{i \delta^{2}}{N^{2}} \sum_{n=1}^{i} \left( 2E \left[ \| w_{n}^{k} - w_{0}^{k} \|^{2} \right| F_{0}^{k} \right] + \frac{2 N}{N} \sum_{n=1}^{N} \| w_{N}^{k-1} - w_{n'}^{k-1} \|^{2} \right)
\]

where step (a) holds because of observation 2, steps (b) and (c) apply Jensen’s inequality; and the last equality is because of uniform random reshuffling. Next, using the strong-convexity of the empirical risk, we have that
\[
\left\| \tilde{w}_{0}^{k} + \mu N \nabla J(w_{0}^{k}) \right\|^{2} = \left\| \tilde{w}_{0}^{k} \right\|^{2} + \mu^{2} N^{2} \| \nabla J(\tilde{w}_{0}^{k}) \|^{2} + 2 \mu N \tilde{w}_{0}^{k} \nabla J(\tilde{w}_{0}^{k}) \leq \left\| \tilde{w}_{0}^{k} \right\|^{2} + \mu^{2} N^{2} \delta^{2} \left\| \tilde{w}_{0}^{k} \right\|^{2} - 2 \mu N (k_{0} - w^{*})^{T} \nabla J(w_{0}^{k}) - \nabla J(w^{*}) \right\|^{2} \leq (1 - 2 \mu N + \mu^{2} N^{2} \delta^{2}) \| \tilde{w}_{0}^{k} \|^{2}
\]

Substituting (43), (44), and (45) into (41) and letting \( t = \mu N \nu \), assuming \( \mu \leq 1/(N \nu) \), we get
\[
E \left[ \left\| \tilde{w}_{0}^{k+1} \right\|^{2} \right| F_{0}^{k}
\]
\[
\leq \left( \frac{1 - 2 \mu \nu + \mu^{2} N^{2} \delta^{2}}{1 - \mu N \nu} \right) \left\| \tilde{w}_{0}^{k} \right\|^{2} + 2 \mu \delta^{2} \sum_{n=1}^{N} \left[ \| w_{k,n} - w_{0} \|^{2} \right| F_{0}^{k}
\]
\[
+ 2 \mu \delta^{2} \sum_{n=1}^{N} \left( \sum_{i=1}^{N} \frac{i}{N^{2}} \left( 2E \left[ \| w_{n}^{k} - w_{0}^{k} \|^{2} \right| F_{0}^{k} \right] + \frac{2 N}{N} \sum_{n=1}^{N} \| w_{N}^{k-1} - w_{n'}^{k-1} \|^{2} \right)
\]
\[
\leq (\text{a)} \quad \left( \frac{1 - \mu \nu + \mu^{2} N^{2} \delta^{2}}{1 - \mu N \nu} \right) \left\| \tilde{w}_{0}^{k} \right\|^{2} + 2 \mu \delta^{2} \sum_{n=1}^{N} \left[ \| w_{k,n} - w_{0}^{k} \|^{2} \right| F_{0}^{k}
\]
\[
+ \frac{2 \mu \delta^{2}}{N} \sum_{n=1}^{N} \frac{1}{2} \left( 2E \left[ \| w_{n}^{k} - w_{0}^{k} \|^{2} \right| F_{0}^{k} \right] + \frac{2 N}{N} \sum_{n=1}^{N} \| w_{N}^{k-1} - w_{n'}^{k-1} \|^{2} \right)
\]
\[
\leq (1 - \mu \nu + \mu^{2} N^{2} \delta^{2}) \| \tilde{w}_{0}^{k} \|^{2} + 4 \mu \delta^{2} \sum_{n=1}^{N} \left[ \| w_{k,n} - w_{0}^{k} \|^{2} \right| F_{0}^{k}
\]
\[
+ 4 \mu \delta^{2} \sum_{n=1}^{N} \left( \sum_{i=1}^{N} \left[ \| w_{i}^{k} - w_{0}^{k} \|^{2} \right| F_{0}^{k} \right) + \sum_{n=1}^{N} \| w_{N}^{k-1} - w_{n'}^{k-1} \|^{2} \right)
\]

(46)
where in step (a) and in several similar steps later, we are using the equality:

\[
\sum_{i=1}^{N-1} \sum_{n=1}^{i} f(n, i) = \sum_{n=1}^{N-1} \sum_{i=n}^{N-1} f(n, i)
\]  

As for step (b), the factor \(\frac{1}{2}\) is because:

\[
\frac{N-1}{N^2} \leq \frac{(N-n)(N-n-1)}{2N^2} \leq \frac{1}{2}, \quad 1 \leq n \leq N-1
\]

The last step (46) is unnecessary; it is used to introduce symmetry into the expression and facilitate the treatment. Taking expectation over the past history \(F_{\tau_0}^\tau\) leads to (18).

### E Proof of Lemma 4

Using (17), we can establish an upper bound for any inner difference based on \(w_0^k\) as follows:

\[
\|w_k^1 - w_0^1\|^2 = \|w_k^1 - w_{i-1}^k + w_{i-1}^k - \cdots - w_0^k\|^2
\]

\[
= \left\| \frac{1}{i} \left( w_k^1 - w_{i-1}^k + w_{i-1}^k - \cdots - w_0^k \right) \right\|^2
\]

\[
= i^2 \left\| \frac{1}{i} \left( w_k^1 - w_{i-1}^k + w_{i-1}^k - \cdots - w_0^k \right) \right\|^2
\]

\[
\leq i \sum_{m=0}^{i-1} \|w_{m+1}^k - w_m^k\|^2
\]

\[
\leq 3\delta^2 \mu^2 \sum_{i=1}^{N-1} i \sum_{m=0}^{i-1} \left( \|w_k^m - w_0^m\|^2 + \|w_{k-1}^m - \phi_{m,j}^k\|^2 + \frac{1}{N} \sum_{n=1}^{N} \|\tilde{\phi}_{m,n}^k\|^2 \right)
\]

where \(\tilde{\phi}_{m,n}^k \triangleq w^* - \phi_{m,n}^k\). It is important to remark here that now \(j = \sigma_k(m+1)\), i.e., \(j\) is always associated with the index before it. Summing over \(i\), we have:

\[
\sum_{i=1}^{N-1} \|w_k^i - w_0^i\|^2
\]

\[
\leq 3\delta^2 \mu^2 \sum_{m=0}^{N-2} \sum_{i=m+1}^{N-1} \left( \|w_k^m - w_0^m\|^2 + \|w_{k-1}^m - \phi_{m,j}^k\|^2 + \frac{1}{N} \sum_{n=1}^{N} \|\tilde{\phi}_{m,n}^k\|^2 \right)
\]

\[
\leq 3\delta^2 \mu^2 \sum_{m=0}^{N-2} \left( \|w_k^m - w_0^m\|^2 + \|w_{k-1}^m - \phi_{m,j}^k\|^2 + \frac{1}{N} \sum_{n=1}^{N} \|\tilde{\phi}_{m,n}^k\|^2 \right)
\]

\[
= 3\delta^2 \mu^2 N^2 \left( \sum_{m=0}^{N-2} \|w_k^m - w_0^m\|^2 + \sum_{i=1}^{N-1} \|w_{k-1}^i - w_{i-1}^k\|^2 + \sum_{m=0}^{N-2} \frac{1}{N} \sum_{n=1}^{N} \|\tilde{\phi}_{m,n}^k\|^2 \right)
\]

\[
\leq 3\delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} \|w_i^k - w_0^k\|^2 + \sum_{i=1}^{N-1} \|w_{k-1}^i - w_{i-1}^k\|^2 + \sum_{i=0}^{N-2} \frac{1}{N} \sum_{n=1}^{N} \|\tilde{\phi}_{i,n}^k\|^2 \right)
\]  

(50)
where step (a) is because $\sum_{i=m+1}^{N-1} i$ is bounded by $\frac{N^2}{2}$, and step (b) uses the fact that $\phi_{i,j}^k = w_{m+1}^k$ by construction. Then, computing the conditional expectation, we get:

$$\sum_{i=1}^{N} E \left[ \|w_i^k - w_i^k\|^2 | \mathcal{F}_0^k \right]$$

$$\leq \frac{3}{2} \delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} E \left[ \|w_i^k - w_i^k\|^2 | \mathcal{F}_0^k \right] + \sum_{i=1}^{N-1} \|w_i^{k-1} - w_i^{k-1}\|^2 \right) + \frac{1}{N} \sum_{i=0}^{N-2} \sum_{n=1}^{N} E \left[ \|\phi_{n,i}^k\|^2 | \mathcal{F}_0^k \right]$$

(51)

To bound the last term, we first separate it into two quantities:

$$E \left[ \|\phi_{i,n}^k\|^2 | \mathcal{F}_0^k \right] = E \left[ \|\phi_{i,n}^k - \bar{w}_0^k + \bar{w}_0^k\|^2 | \mathcal{F}_0^k \right] \leq 2E \left[ \|\phi_{i,n}^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + 2\|\bar{w}_0^k\|^2$$

(52)

Using an argument similar to Lemma[2] we can establish that:

$$E \left[ \sum_{n=1}^{N} \|\phi_{i,n}^k - \bar{w}_0^k\|^2 | \mathcal{F}_0^k \right] = \sum_{n=1}^{N} \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right]$$

$$= \sum_{n=1}^{N} \frac{1}{N} \sum_{n=1}^{N} E \left[ \|\phi_{i,n}^k\|^2 | \mathcal{F}_0^k \right]$$

$$= \sum_{i=0}^{N-1} \sum_{n=1}^{N} E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right]$$

$$+ \sum_{n=1}^{N} \|w_i^{k-1} - w_i^{k-1}\|^2 + 2N\|\bar{w}_0^k\|^2$$

(53)

Combining results (52) and (53), we can bound the last term of (51):

$$\sum_{i=0}^{N-1} \sum_{n=1}^{N} E \left[ \|\phi_{i,n}^k\|^2 | \mathcal{F}_0^k \right]$$

$$\leq \sum_{i=0}^{N-1} \sum_{n=1}^{N} \left( E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + \frac{1}{N} \sum_{n=1}^{N} \|w_i^{k-1} - w_i^{k-1}\|^2 \right) + 2N\|\bar{w}_0^k\|^2$$

(54)

Substituting back into (51), we have:

$$\sum_{i=1}^{N} E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right]$$

$$\leq \frac{3}{2} \delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + \sum_{i=1}^{N-1} \|w_i^{k-1} - w_i^{k-1}\|^2 \right)$$

$$+ 2 \sum_{i=1}^{N-1} \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + 2 \sum_{n=1}^{N} \|w_i^{k-1} - w_i^{k-1}\|^2 + 2N\|\bar{w}_0^k\|^2$$

$$\leq \frac{9}{2} \delta^2 \mu^2 N^2 \sum_{i=1}^{N-1} E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + \sum_{i=1}^{N-1} \|w_i^{k-1} - w_i^{k-1}\|^2 + 3\delta^2 \mu^2 N^2 \|\bar{w}_0^k\|^2$$

$$\leq 5\delta^2 \mu^2 N^2 \sum_{i=1}^{N-1} E \left[ \|w_i^k - w_0^k\|^2 | \mathcal{F}_0^k \right] + \sum_{i=1}^{N-1} \|w_i^{k-1} - w_i^{k-1}\|^2 + 3\delta^2 \mu^2 N^3 \|\bar{w}_0^k\|^2$$

(55)

Taking expectation over the filtration leads to (19).

Next, following similar arguments, we have the following for backward inner difference term:

$$\|w_i^{k-1} - w_i^{k-1}\|^2$$

$$= \|w_i^{k-1} - w_i^{k-1} + w_{i-1}^{k-1} - \cdots - w_{i-1}^{k-1}\|^2$$
where \( \tilde{\phi}_{m,n} \triangleq \phi_{m,n} - \phi^k_{m,n} \) and now \( j = \sigma^{-1}(m + 1) \). Summing over \( i \), we have

\[
\sum_{i=1}^{N-1} \| w_{i}^{k-1} - w_{i}^{k-1} \|^2
\]

\[
\leq 3\delta^2 \mu^2 (N-2) \sum_{m=1}^{N-2} \sum_{i=1}^{N-2} \left( \| w_{i}^{k-1} - w_{i}^{k-1} \|^2 + \| w_{i}^{k-2} - \phi_{m,j}^{k-1} \|^2 + \frac{1}{N} \sum_{n=1}^{N} \| \phi_{m,n}^{k-1} \|^2 \right)
\]

\[
= 3\delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} \| w_{i}^{k-1} - w_{i}^{k-1} \|^2 + \sum_{i=2}^{N-1} \| w_{i}^{k-2} - w_{i}^{k-2} \|^2 + \sum_{i=1}^{N-1} \| \phi_{i,i}^{k-1} \|^2 \right)
\]

\[
\leq 3\delta^2 \mu^2 N^2 \left( \sum_{i=1}^{N-1} \| w_{i}^{k-1} - w_{i}^{k-1} \|^2 + \sum_{i=1}^{N-1} \| w_{i}^{k-2} - w_{i}^{k-2} \|^2 + \sum_{i=0}^{N-2} \| \phi_{i,i}^{k-1} \|^2 \right)
\]

(57)

The above result is similar to (50) with \( k \) replaced by \( k - 1 \). Therefore, the same procedure can now be followed to arrive at (20).

### F Proof of theorem[1]

To simplify the notation, we introduce the symbols:

\[
a_k^2 \triangleq \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{E} \| w_{i}^{k} - w_{i}^{k} \|^2, \quad b_{k-1}^2 \triangleq \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{E} \| w_{i}^{k-1} - w_{i}^{k-1} \|^2
\]

(58)

Then, the results of the previous three lemmas can be rewritten in the form:

\[
\mathbb{E} \| \tilde{w}_0^{k+1} \|^2 \leq \left( 1 - \frac{\mu \nu N - \mu^2 N \nu^2 \delta^2}{1 - \mu N \nu} \right) \mathbb{E} \| \tilde{w}_0^{k} \|^2 + 4\mu N \delta^2 \left( a_k^2 + b_{k-1}^2 \right)
\]

(59)

\[
a_{k+1}^2 \leq 5\delta^2 \mu^2 N^2 (a_{k+1}^2 + b_{k}^2) + 3\delta^2 \mu^2 N^2 \mathbb{E} \| \tilde{w}_0^{k+1} \|^2
\]

(60)

\[
b_k^2 \leq 5\delta^2 \mu^2 N^2 (a_k^2 + b_{k-1}^2) + 3\delta^2 \mu^2 N^2 \mathbb{E} \| \tilde{w}_0^{k} \|^2
\]

(61)

We can simplify these relations further by recognizing certain bounds. To begin with, note that

\[
1 - \frac{\mu \nu N - \mu^2 N \nu^2 \delta^2}{1 - \mu N \nu} = 1 - \frac{\mu \nu N - \mu^2 N \nu^2 + \mu^2 N \nu^2 - \mu^2 N \nu^2}{1 - \mu N \nu}
\]

\[
= 1 - \frac{\mu \nu N + \mu^2 N \nu^2 - \mu^2 N \nu^2}{1 - \mu N \nu}
\]

\[
= 1 - \frac{3\mu N}{4} - \frac{\mu^2 N \nu^2 - \mu^2 N \nu^2}{1 - \mu N \nu}
\]

\[
\leq 1 - \frac{3\mu N}{4}
\]

\[
(62)
\]
where the last inequality holds when
\[
1 - \mu N \nu > 0, \quad \frac{\mu \nu N}{4} - \frac{\mu^2 N^2 \delta^2 - \mu^2 N^2 \nu^2}{1 - \mu N \nu} \geq 0 \iff \mu \leq \min \left\{ \frac{1}{N \nu}, \frac{\nu}{N(4\delta^2 - 3\nu^2)} \right\}
\] (63)

Since \( \nu \leq \delta \), we can replace (63) by the sufficient condition

\[
\text{Condition #1: } \mu \leq \frac{\nu}{4\delta^2 N}
\] (64)

Under this condition, and substituting (62) into (59), we get
\[
E \| \tilde{w}_0^{k+1} \|^2 \leq \left( 1 - \frac{3\mu N}{4} \right) E \| \tilde{w}_0^k \|^2 + 4\mu N \frac{\delta^2}{\nu} (a_k^2 + b_{k-1}^2)
\] (65)

Let \( \gamma \) denote an arbitrary positive scalar that we are free to choose. Multiplying relations (60) and (61) by \( \gamma \) and adding to (65) we obtain:
\[
E \| \tilde{w}_0^{k+1} \|^2 + \gamma (a_{k+1}^2 + b_k^2) \leq \left( 1 - \frac{3\mu N}{4} \right) E \| \tilde{w}_0^k \|^2 + 4\mu N \frac{\delta^2}{\nu} (a_k^2 + b_{k-1}^2) + 5\gamma \delta^2 \mu^2 N^2 (a_{k+1}^2 + b_k^2) + 3\gamma \delta^2 \mu^2 N^2 E \| \tilde{w}_0^{k+1} \|^2 + 5\gamma \delta^2 \mu^2 N^2 (a_k^2 + b_{k-1}^2) + 3\gamma \delta^2 \mu^2 N^2 E \| \tilde{w}_0^k \|^2
\] (66)

which simplifies to
\[
(1 - 3\gamma \delta^2 \mu^2 N^2) E \| \tilde{w}_0^{k+1} \|^2 + \gamma (1 - 5\delta^2 \mu^2 N^2) (a_{k+1}^2 + b_k^2)
\]
\[
\leq \left( 1 - \frac{3\mu N}{4} + 3\gamma \delta^2 \mu^2 N^2 \right) E \| \tilde{w}_0^k \|^2 + \left( 4\mu N \frac{\delta^2}{\nu} + 5\gamma \delta^2 \mu^2 N^2 \right) (a_k^2 + b_{k-1}^2)
\] (67)

Under the condition \( 1 - 3\gamma \delta^2 \mu^2 N^2 > 0 \), which is equivalent to

\[
\text{Condition #2: } \mu^2 \gamma < \frac{1}{3\delta^2 N^2}
\] (68)

it holds that
\[
E \| \tilde{w}_0^{k+1} \|^2 + \gamma \frac{1 - 5\delta^2 \mu^2 N^2}{1 - 3\gamma \delta^2 \mu^2 N^2} (a_{k+1}^2 + b_k^2)
\]
\[
\leq \left( 1 - \frac{3\mu N}{4} + 3\gamma \delta^2 \mu^2 N^2 \right) E \| \tilde{w}_0^k \|^2 + \frac{4\mu N \frac{\delta^2}{\nu} + 5\gamma \delta^2 \mu^2 N^2}{1 - 3\gamma \delta^2 \mu^2 N^2} (a_k^2 + b_{k-1}^2)
\]
\[
= \left( 1 - \frac{3\mu N}{4} + 3\gamma \delta^2 \mu^2 N^2 \right) \left( E \| \tilde{w}_0^k \|^2 + \frac{4\mu N \frac{\delta^2}{\nu} + 5\gamma \delta^2 \mu^2 N^2}{1 - 3\mu N/4 + 3\gamma \delta^2 \mu^2 N^2} (a_k^2 + b_{k-1}^2) \right)
\] (69)

This relation in turn implies that
\[
E \| \tilde{w}_0^{k+1} \|^2 + \gamma (1 - 5\delta^2 \mu^2 N^2) (a_{k+1}^2 + b_k^2)
\]
\[
\leq \left( 1 - \frac{3\mu N}{4} + 3\gamma \delta^2 \mu^2 N^2 \right) \left( E \| \tilde{w}_0^k \|^2 + \frac{4\mu N \frac{\delta^2}{\nu} + 5\gamma \delta^2 \mu^2 N^2}{1 - 3\mu N/4 + 3\gamma \delta^2 \mu^2 N^2} (a_k^2 + b_{k-1}^2) \right)
\] (70)

We can again simplify the result by noting that
\[
1 - 3\mu N/4 + 3\gamma \delta^2 \mu^2 N^2 \geq 1 - \mu N/4 - (\mu N/2 - 3\gamma \delta^2 \mu^2 N^2)
\]
\[
\leq 1 - \mu N/4
\] (71)

where the inequality holds when \( \mu N/2 - 3\gamma \delta^2 \mu^2 N^2 \geq 0 \), i.e.,

\[
\text{Condition #3: } \mu \gamma \leq \frac{\nu}{6\delta^2 N}
\] (72)

In addition, we have the lower bound
\[
1 - \frac{3}{4} \mu N + 3\gamma \delta^2 \mu^2 N^2 \geq 1 - \frac{3}{4} \mu N
\] (73)
Using condition #1 from Eq. (64), we have
\[
1 - \frac{3}{4} \mu \nu N \geq 1 - \frac{3 \nu^2}{16 \delta^2} \geq \frac{13}{16} \tag{74}
\]
In a similar manner,
\[
4 \mu N \frac{\delta^2}{\nu} + 5 \gamma \delta^2 \mu^2 N^2 \leq 4 \mu N \frac{\delta^2}{\nu} + \mu N \frac{\delta^2}{\nu} = 5 \mu N \frac{\delta^2}{\nu} \tag{75}
\]
where the last inequality holds when \( \mu \gamma \leq \frac{1}{5 \delta N} \), which is always valid under condition #3 in Eq. (72) since the latter implies that \( \mu \gamma \leq \frac{1}{5 \delta N} \). Substituting (71), (74), and (75) into (70), we find that
\[
E \| \tilde{w}_0^{k+1} \|^2 + \gamma (1 - 5 \delta^2 \mu^2 N^2) (a_{k+1}^1 + b_{k}^2) 
\leq \frac{1 - \mu \nu N/4}{1 - 3 \gamma \delta^2 \mu^2 N^2} \left( E \| \tilde{w}_0^k \|^2 + \frac{16}{13} \cdot 5 \mu N \frac{\delta^2}{\nu} (a_k^2 + b_{k-1}^2) \right). \tag{76}
\]
Under condition #1 in Eq. (64), we have
\[
1 - 5 \delta^2 \mu^2 N^2 \geq 1 - 5 \delta^2 \nu^2 \frac{\nu^2}{\delta^2 N^2} \geq 1 - \frac{5}{16} \geq \frac{11}{16} \tag{77}
\]
and, hence,
\[
E \| \tilde{w}_0^{k+1} \|^2 + \frac{11}{16} \gamma (a_{k+1}^1 + b_{k}^2) \leq \frac{1 - \mu \nu N/4}{1 - 3 \gamma \delta^2 \mu^2 N^2} \left( E \| \tilde{w}_0^k \|^2 + \frac{11}{16} \cdot 99 \mu N \frac{\delta^2}{\nu} (a_k^2 + b_{k-1}^2) \right) \tag{78}
\]
where the last inequality is unnecessary but is introduced for convenience. Recall that we are free to choose \( \gamma \), so assume we choose it to satisfy
\[
\frac{11}{16} = \frac{99}{16} \mu N \frac{\delta^2}{\nu} \quad \Rightarrow \quad \gamma = \frac{9 \mu N \delta^2}{\nu} \tag{79}
\]
It then follows that:
\[
E \| \tilde{w}_0^{k+1} \|^2 + \frac{11}{16} \gamma (a_{k+1}^2 + b_{k}^2) \leq \frac{1 - \mu \nu N/4}{1 - 3 \gamma \delta^2 \mu^2 N^2} \left( E \| \tilde{w}_0^k \|^2 + \frac{11}{16} \gamma (a_k^2 + b_{k-1}^2) \right) \tag{80}
\]
where we introduced the positive parameter
\[
\alpha \triangleq \frac{1 - \mu \nu N/4}{1 - 3 \gamma \delta^2 \mu^2 N^2} \tag{81}
\]
This parameter controls the speed of convergence. It will hold that \( \alpha < 1 \) when
\[
\frac{1 - \mu \nu N/4}{1 - 3 \gamma \delta^2 \mu^2 N^2} = \frac{1 - \mu \nu N/4}{1 - 7 \delta^4 \mu^2 N^2/\nu} < 1 \quad \Leftrightarrow \quad \mu < \sqrt{\frac{1}{108} \frac{\nu}{\delta^2 N}} \tag{82}
\]
Let us now re-examine conditions #1 through #3, along with (82), when \( \gamma \) is chosen according to (79). In this case, conditions #1 through #3 become

**Conditions #1 to #3 :** \( \mu \leq \frac{\nu}{4 \delta^2 N}, \quad \mu^3 < \frac{\nu}{27 \delta^4 N^3}, \quad \mu^2 \leq \frac{\nu^2}{54 \delta^2 N^2} \tag{83} \)

which can be met by:

\[
\text{Conditions #1 to #3 : } \mu \leq \frac{\nu}{4 \delta^2 N}, \quad \mu < \frac{1}{3 \delta N} \left( \frac{\nu}{\delta} \right)^{1/3}, \quad \mu \leq \frac{1}{54 \delta^2 N} \tag{84}
\]

All three conditions and condition (82) can be satisfied by the following single sufficient bound on the step-size parameter (since \( 11^2 > 108 \)):

\[
\mu \leq \frac{\nu}{11 \delta^2 N} \tag{85}
\]
G Proof of lemma [5]

Subtracting $w^*$ from both sides of (29), we obtain:

\[
\tilde{w}_0^{k+1} = \tilde{w}_0^k + \mu N \nabla J(w_0^k) + \mu \sum_{i=0}^{N-1} \left[ \nabla Q(w_i^k; x_j) - \nabla Q(w_i^{k-1}; x_j) \right]
\]

- $\mu \sum_{i=0}^{N-1} \left[ \nabla Q(w_i^k; x_j') - \nabla Q(w_i^{k-1}; x_j') \right]

(86)

Then, taking the squared norm and applying Jensen’s inequality, we establish the first recursion for any $t \in (0, 1)$:

\[
\|\tilde{w}_0^{k+1}\|^2 \leq \frac{1}{t} \left( \|\tilde{w}_0^k + \mu N \nabla J(w_0^k)\|^2 + \frac{2\mu^2}{1-t} \left( \sum_{i=0}^{N-1} \|\nabla Q(w_i^k; x_j) - \nabla Q(w_i^{k-1}; x_j)\|^2 \right) \right) + \frac{2\mu^2 \sigma^2}{1-t} \sum_{i=0}^{N-1} |w_i^k - w_0^k|^2 + \frac{2\mu^2 \sigma^2 N}{1-t} \sum_{i=0}^{N-1} |w_i^{k-1} - w_i^{k-1}|^2
\]

(87)

Using an argument similar to [45] and letting $t = 1 - \mu N \nu$, assuming $\mu \leq 1/(N \nu)$, we obtain:

\[
\|\tilde{w}_0^{k+1}\|^2 \leq \left( 1 - \frac{\mu \nu N - 2 \nu^2 \sigma^2}{1 - \mu N \nu} \right) \|\tilde{w}_0^k\|^2 + \frac{2\mu^2 \sigma^2}{\nu} \left( \sum_{i=0}^{N-1} |w_i^k - w_0^k|^2 + \sum_{i=0}^{N-1} |w_i^{k-1} - w_i^{k-1}|^2 \right)
\]

(88)

Taking the expectation of both sides, we establish [31]. The forward inner difference recursion can be obtain by following the same procedure as in [49]:

\[
\|w_i^k - w_0^k\|^2 \leq i \sum_{m=0}^{i-1} \|w_m^k - w_m^k\|^2 \leq 3\mu^2 \sigma^2 i \sum_{m=0}^{i-1} \left( |w_m^k - w_0^k|^2 + \frac{1}{N} \sum_{n'=0}^{N-1} |w_n^k - w_{n'}^{k-1}|^2 + \|\tilde{w}_0^k\|^2 \right)
\]

(89)

\[
= 3\mu^2 \sigma^2 i \sum_{m=0}^{i-1} |w_m^k - w_0^k|^2 + 3\mu^2 \sigma^2 i^2 \left( \frac{1}{N} \sum_{n'=0}^{N-1} |w_n^k - w_{n'}^{k-1}|^2 + \|\tilde{w}_0^k\|^2 \right)
\]

Summing over $i$, we have

\[
\sum_{i=0}^{N-1} |w_i^k - w_0^k|^2 \leq 3\mu^2 \sigma^2 \left( \sum_{i=0}^{N-1} i \sum_{m=0}^{i-1} |w_m^k - w_0^k|^2 + \sum_{i=0}^{N-1} i^2 \left( \frac{1}{N} \sum_{n'=0}^{N-1} |w_n^k - w_{n'}^{k-1}|^2 + \|\tilde{w}_0^k\|^2 \right) \right)
\]

(90)
where step (a) is because:

\[
\sum_{m=1+1}^{N-1} i \leq N^2, \quad \sum_{i=0}^{N-1} i^2 = \frac{(N-1)N(2N-1)}{6} \leq \frac{N^3}{3}
\]

Lastly, we establish the backwards inner difference term using the same argument as in (56):

\[
\|w_N^k - w_k^k\|^2 \\
= \|w_N^k - w_{N-1}^k + \cdots + w_{k+1}^k - w_k^k\|^2 \\
\leq (N - i) \sum_{m=i}^{N-1} \|w_{m+1}^k - w_m^k\|^2 \\
\leq 3\mu^2\delta^2(N - i) \sum_{m=i}^{N-1} \left(\|w_m^k - w_0^k\|^2 + \frac{1}{N} \sum_{n'=0}^{N-1} \|w_{n'}^k - w_N^k\|^2 + \|\tilde{w}_0^k\|^2\right) \\
\leq 3\mu^2\delta^2(N - i) \sum_{m=i}^{N-1} \|w_m^k - w_0^k\|^2 + \frac{3\mu^2\delta^2(N - i)^2}{N} \sum_{n'=0}^{N-1} \|w_{n'}^k - w_N^k\|^2 + 3\mu^2\delta^2(N - i)^2\|\tilde{w}_0^k\|^2 \tag{92}
\]

Observing that this backward term is summing from 0 to \(N-1\), rather than from 1 to \(N-1\) as in SAGA with RR, we have

\[
\sum_{i=0}^{N-1} \|w_N^k - w_k^k\|^2 \\
\leq 3\mu^2\delta^2 \sum_{i=0}^{N-1} (N - i) \sum_{m=i}^{N-1} \|w_m^k - w_0^k\|^2 + 3\mu^2\delta^2 \sum_{i=0}^{N-1} (N - i)^2 \left(\frac{1}{N} \sum_{n'=0}^{N-1} \|w_{n'}^k - w_N^k\|^2 + \|\tilde{w}_0^k\|^2\right) \\
= 3\mu^2\delta^2 \sum_{m=0}^{N-1} \sum_{i=0}^{m} (N - i)\|w_m^k - w_0^k\|^2 + 3\mu^2\delta^2 \frac{N(N+1)(2N+1)}{6} \left(\frac{1}{N} \sum_{n'=0}^{N-1} \|w_{n'}^k - w_N^k\|^2 + \|\tilde{w}_0^k\|^2\right) \\
\leq 3\mu^2\delta^2 N^2 \sum_{m=0}^{N-1} \|w_m^k - w_0^k\|^2 + 3\mu^2\delta^2 N^2 \left(\sum_{n'=0}^{N-1} \|w_{n'}^k - w_N^k\|^2 + N\|\tilde{w}_0^k\|^2\right) \\
= 3\mu^2\delta^2 N^2 \sum_{i=0}^{N-1} \|w_i^k - w_0^k\|^2 + 3\mu^2\delta^2 N^2 \left(\sum_{i=0}^{N-1} \|w_i^k - w_N^k\|^2 + N\|\tilde{w}_0^k\|^2\right) \tag{93}
\]

where in the last inequality we used the fact that

\[
\frac{N(N+1)(2N+1)}{6} \leq N^3, \quad \forall N \tag{94}
\]

H Proof of theorem

We let

\[
a_k \triangleq \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E}\|w_i^k - w_0^k\|^2, \quad b_k \triangleq \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E}\|w_N^k - w_i^k\|^2 \tag{95}
\]

The recursions available so far for AVRG are:

\[
\mathbb{E}\|w^{k+1}\|^2 \leq \left(1 - \frac{\mu\nu N - \mu^2 N^2 \delta^2}{1 - \mu \nu \nu}\right) \mathbb{E}\|\tilde{w}_0^k\|^2 + \frac{2\mu \delta^2 N}{\nu} (a_k + b_{k-1}) \tag{96}
\]

\[
a_{k+1} \leq 3\mu^2\delta^2 N^2 a_{k+1} + \mu^2\delta^2 N^2 b_k + \mu^2\delta^2 N^2 \mathbb{E}\|\tilde{w}_0^{k+1}\|^2 \tag{97}
\]

\[
b_k \leq 3\mu^2\delta^2 N^2 (a_k + b_{k-1}) + 3\mu^2\delta^2 N^2 \mathbb{E}\|\tilde{w}_0^k\|^2 \tag{98}
\]

which have exactly the same form as recursions (59)—(61) except for the coefficients. To simplify the argument, we can replace (97) by:

\[
a_{k+1} \leq 3\mu^2\delta^2 N^2 (a_{k+1} + b_k) + \mu^2\delta^2 N^2 \mathbb{E}\|\tilde{w}_0^{k+1}\|^2 \tag{99}
\]
Similar to the derivation of (66), we have:

\[
(1 - \gamma \mu^2 \delta^2 N^2) \mathbb{E} \| \tilde{w}^{k+1} \|^2 + \gamma (1 - 3 \mu^2 \delta^2 N^2) (a_{k+1} + b_k) \\
\leq \left( 1 - \frac{3\mu \nu N}{4} \right) \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{2\mu \delta^2 N}{\nu} (a_k + b_{k-1}) \\
+ \gamma \left( 3\mu^2 \delta^2 N^2 (a_k + b_{k-1}) + 3\mu^2 \delta^2 N^2 \mathbb{E} \| \tilde{w}_0^k \|^2 \right) \\
= \left( 1 - \frac{3\mu \nu N}{4} + 3\gamma \mu^2 \delta^2 N^2 \right) \mathbb{E} \| \tilde{w}_0^k \|^2 + \left( \frac{2\mu \delta^2 N}{\nu} + 3\gamma \mu^2 \delta^2 N^2 \right) (a_k + b_{k-1})
\]  

(100)

under

\[\text{Condition #1: } \mu \leq \frac{\nu}{4\delta^2 N} \]  

(101)

Under the condition \( 1 - \gamma \delta^2 \mu^2 N^2 > 0 \), which is equivalent to

\[\text{Condition #2: } \mu^2 \gamma < \frac{1}{\delta^2 N^2} \]  

(102)

it further holds that

\[
\mathbb{E} \| \tilde{w}^{k+1} \|^2 + \gamma (1 - 3\mu^2 \delta^2 N^2) (a_{k+1} + b_k) \\
\leq \frac{1 - 3\mu \nu N/4 + 3\gamma \mu^2 \delta^2 N^2}{1 - \gamma \delta^2 \mu^2 N^2} \left( \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{2\mu \delta^2 N}{\nu} + 3\gamma \mu^2 \delta^2 N^2 \right) (a_k + b_{k-1})
\]  

(103)

Note that the numerator \( 1 - 3\mu \nu N/4 + 3\gamma \mu^2 \delta^2 N^2 \) is the same as SAGA in (71). Thus, under condition:

\[\text{Condition #3: } \mu \gamma \leq \frac{\nu}{6\delta^2 N} \]  

(104)

we have:

\[
\frac{11}{16} \leq 1 - 3\mu \nu N/4 + 3\gamma \delta^2 \mu^2 N^2 \leq 1 - \mu \nu N/4
\]  

(105)

Lastly, we can verify that

\[
\frac{2\mu \delta^2 N}{\nu} + 3\gamma \mu^2 \delta^2 N^2 \leq \frac{2\mu \delta^2 N}{\nu} + \frac{\mu \delta^2 N}{\nu} \leq \frac{3\mu \delta^2 N}{\nu}
\]  

(106)

where the last inequality holds when \( \mu \gamma \leq \frac{1}{36\delta N} \), which is always valid under condition #3. Now, collecting the results, we have

\[
\mathbb{E} \| \tilde{w}^{k+1} \|^2 + \gamma \frac{13}{16} (a_{k+1} + b_k) \leq \frac{1 - \mu \nu N/4}{1 - \gamma \delta^2 \mu^2 N^2} \left( \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{16}{11} \mu N \delta^2 \left( a_k + b_{k-1} \right) \right) \\
\leq \frac{1 - \mu \nu N/4}{1 - \gamma \delta^2 \mu^2 N^2} \left( \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{26}{16} \mu N \delta^2 \left( a_k + b_{k-1} \right) \right)
\]  

(107)

Assume we choose \( \gamma \) such that

\[
\gamma \frac{13}{16} = \frac{26}{16} \mu N \delta^2 \nu \Rightarrow \gamma = 6\mu N \delta^2 \nu
\]  

(108)

It then follows that:

\[
\mathbb{E} \| \tilde{w}_0^{k+1} \|^2 + \gamma \frac{13}{16} (a_{k+1}^2 + b_k^2) \leq \frac{1 - \mu \nu N/4}{1 - 3\gamma \delta^2 \mu^2 N^2} \left( \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{13}{16} \gamma (a_k^2 + b_{k-1}^2) \right) \\
\triangleq \alpha \left( \mathbb{E} \| \tilde{w}_0^k \|^2 + \frac{13}{16} \gamma (a_k^2 + b_{k-1}^2) \right)
\]  

(109)

where we introduced the positive parameter

\[
\alpha \triangleq \frac{1 - \mu \nu N/4}{1 - 3\gamma \delta^2 \mu^2 N^2}
\]  

(110)
This parameter satisfies $\alpha < 1$ for
\[
\frac{1 - \mu \nu N/4}{1 - 3\gamma \delta^2 \mu^2 N^2} = \frac{1 - \mu \nu N/4}{1 - 18\delta^4 \mu^3 N^3 / \nu} < 1 \iff \mu < \sqrt{\frac{1}{72} \frac{\nu}{\delta^2 N}} \quad (111)
\]

We re-examine conditions #1–#3 when $\gamma$ is chosen according to (108). In this case, these conditions become

Conditions #1 to #3 : $\mu \leq \frac{\nu}{4\delta^2 N}$, $\mu^3 < \frac{\nu}{6\delta^4 N^3}$, $\mu^2 < \frac{\nu^2}{36\delta^4 N^2}$ \quad (112)

which can be met by:

Conditions #1 to #3 : $\mu \leq \frac{\nu}{4\delta^2 N}$, $\mu < \frac{1}{2\delta N} \left(\frac{\nu}{\delta}\right)^{1/3}$, $\mu \leq \frac{\nu}{6\delta^2 N}$ \quad (113)

All these three conditions and the condition for $\alpha < 1$ can be satisfied by the following single sufficient bound on the step-size parameter:

$\mu \leq \frac{\nu}{9\delta^2 N}$ \quad (114)