Bell’s Inequalities and the Accardi–Gustafson Inequality*

Karl Gustafson
Department of Mathematics, University of Colorado, Boulder, CO 80309–0395 USA
October 28, 2018

Abstract

Many issues combine for consideration when speaking of Bell’s Inequalities: nonlocality, realism, hidden variables, incompatible measures, wave function collapse, other. Each of these issues then may be viewed from several viewpoints: historical, theoretical, physical, experimental, statistical, communicational, cryptographical, and mathematical. From the mathematical viewpoint, much of the Bell theory is “just geometry”.

Theorem 1

Let $x, y, z$ be any 3 nonzero vectors in a real or complex Hilbert space of any dimension. We take $\|x\| = \|y\| = \|z\| = 1$ for convenience. From $\langle x, y \rangle = a_1 + ib_1$, $\langle y, z \rangle = a_2 + ib_2$, $\langle x, z \rangle = a_3 + ib_3$, define angles $\phi_{xy}, \phi_{yz}, \phi_{xz}$ in $[0, \pi]$ by $\cos \phi_{xy} = a_1$, $\cos \phi_{yz} = a_2$, $\cos \phi_{xz} = a_3$. Then there holds the general triangle inequality

$$\phi_{xz} \leq \phi_{xy} + \phi_{yz}$$

Corollary 1

Much of the Bell (1965), Wigner (1970), Accardi (1982), Gudder–Zanghì (1984), Herbert–Peres (1993), Williams–Clearwater (1998), Khrennikov (2000), other, considerations are contained in the above Theorem. The Accardi–Gustafson inequality is a necessary condition for a quantum mechanical probability model to apply to those situations. From the mathematical viewpoint of this paper, one cannot argue “nonlocality” on the basis of violation of Bell’s Inequality.

1 Introduction and Outline

In this report I want to summarize and bring up to date my results/impressions of the last five years about what I loosely call the Bell theory. Although I have been interested in the foundations of quantum mechanics for a long time

*To be presented at the International Conference on Foundations of Probability and Physics—2, Växjo, Sweden, June 2–7, 2002.
I specifically came to the Bell theory from my observing about five years ago a coinciding of certain inequalities in quantum probability with those from my operator trigonometry. My operator trigonometry started in abstract operator theory in 1967, is inherently noncommutative, and includes somewhat incidentally the triangle inequality mentioned in the abstract above. I will emphasize the mathematical viewpoint and often defer to the cited literature \[1\]–\[50\] so as to not repeat myself/others in this limited space.

In Section 2 I briefly review the Bell theory and some of Bell’s inequalities. In Section 3 I detail the above mentioned coincidence of what I shall call here the Accardi–Gustafson inequality, which is essentially equivalent to the triangle inequality but in my opinion is even more fundamental as I shall try to make clear. In Section 4 I will introduce a new notion of “inequality equalities” which I believe will be useful in revealing flaws in paradoxes. Section 5 contains concluding remarks which may be of some interest in themselves.

2 Bell’s Inequalities

Here is a thumbnail sketch of how Bell’s inequalities fit into the foundations of quantum mechanics. As with the term Bell theory, I also use the term Bell’s inequalities in a wide sense, meaning many such inequalities.

The 1935 paper \[1\] of Albert Einstein, Boris Podolsky and Nathan Rosen was a gedankenexperiment which purported to demonstrate that quantum mechanics cannot provide a complete description of reality. According to the extensive account \[2\] of Jammer, although much of the actual EPR paper \[1\] was written by Podolsky, the origins of this paper go back to 1930 when Niels Bohr ‘defeated’ Einstein’s earlier gedankenexperiment presented at the Sixth Solvay Congress in Brussels in 1930, an important episode in in the famous ongoing debate between the two which had begun already ten years earlier in 1920. To better present his view, Einstein then, along with Richard Tolman and Podolsky, wrote a paper \[3\] with another gedankenexperiment which argued that if one accepted quantum mechanic’s uncertainty principle, then one could not even predict the past, let alone the future. Also Einstein sharpened his arguments by shifting attention away from direct attacks on the uncertainty principle itself but instead with more focus on logical paradoxes which would follow from it. Moreover Einstein modified his photon box gedankenexperiments with their paradoxical consequences to the more clear-cut two particle gedankenexperiment which appears in the 1935 EPR paper \[1\]. Erwin Schrödinger immediately agreed with the EPR argument, reformulated it, and came up with his own gedankenexperiment now known as his half-dead half-alive cat \[4\].

The conclusion of the groundbreaking paper \[1\] was: “While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.” Thus the emphasis in \[1\] was on inadequacies of a theory in which all information is in the wave function. In 1951 David Bohm \[6\] responded by reformulating the EPR argu-
ment to one expressed more simply in terms of spin functions, and presented an argument that “no theory of mechanically determined hidden variables can lead to all of the results of the quantum theory.” Nonetheless Bohm then introduced his version of such a hidden variable theory. This was like earlier semiclassical hydrodynamical or pilot wave quantum models, except for two new features. First, the existence of a quantum mechanical potential, shall we say among all of the particles in a considered ensemble $\psi$, was assumed. Second, each particle trajectory will be well-posed if you know its initial condition. But because the initial position could not be experimentally measured, it is a hidden variable.

In 1964 Bell [7] presented his famous inequality

$$|P(a, b) - P(a, c)| \leq 1 + P(b, c) \quad (2.1)$$

and exhibited certain quantum spin measurement configurations whose quantum expectation values could not satisfy his inequality. Bell’s analysis assumes that physical systems, e.g. two measuring apparatuses, can be regarded as physically totally separated, in the sense of being free of any effects one from the other. Thus his inequality could provide a ‘test’ which could be failed by measurements performed on correlated quantum systems. In particular it was argued in [7] that local realistic hidden variable theories could not hold. However, the exact nature of hidden variables as viewed by Bell is unclear from [7]. As is well known the 1982 physical experiments of Aspect et al. [8] demonstrated that beyond any reasonable doubt the Bell inequalities are violated by certain quantum systems, and papers continue to appear with further demonstrated violations.

In a 1970 paper [9], Wigner simplified and clarified in several ways the argument of Bell. Wigner assumed that all possible measurements are predetermined, even if they involve incompatible observables, and moreover any measurement on one of two apparatuses does not change the preset outcomes of measurements on the other apparatus. Thus the meanings of locality and realism are made more clear and both assumptions are present in the model setup. It is helpful to imagine, for example, that the ‘hidden variable’ is just the directional orientation of each of the two apparatuses, each of which can be thought of as just a three-dimensional possibly skew coordinate system, for example. Then two spin 1/2 particles are sent to the apparatuses, each to one, both coming from a common atomic source, with perfect anticorrelation and singlet properties. Nine measurements are then needed to simultaneously measure the direction vectors $\omega_1, \omega_2, \omega_3$ of the two spins. Each spin has two possible values $1/2 \equiv +, -1/2 \equiv -$, so each measurement can permit four relative results: $++, --, +-, -+$. Therefore there are $4^3$ possible outcomes. Wigner then assumes that the spins are not affected by the orientation of the particular measuring apparatus. This reduces the outcomes to $2^6$ possibilities. For example, if the hidden variables are in the possibility domain $(+, -, -, -; -+)$, then the measurement of the spin component of the first particle in the $\omega_1$ direction will yield value spin $= +$, no matter what direction the spin of the second particle is measured.

In the above setup, let $\theta_{12}, \theta_{23}, \theta_{31}$ be the angles between the three directions $\omega_1, \omega_2, \omega_3$. Then the probability that the spin component of particle 1
in the $\omega_i$ direction and the spin component of particle 2 in the $\omega_k$ direction both measure $+$ or both measure $-$ is $\frac{1}{2} \sin^2 \left( \frac{\theta_{ik}}{2} \right)$. Otherwise the probability of measurements $+-$ or $-+$ is $\frac{1}{4} \cos^2 \left( \frac{\theta_{ik}}{2} \right)$. I mention that even though quantum mechanics has been statistically interpreted ever since Born in 1927, the fact that such quantum transition probabilities are given trigonometrically as $|\langle \psi(\omega_i), \psi(\omega_k) \rangle|^2 = \cos^2(\theta_{ik}/2)$ in terms of the angle $\theta_{ik}$ between directions $\omega_i$ and $\omega_k$ is a special property of spin systems and for example the Eulerian angle representation for $SU(2)$, see, e.g. (10), p. 225). Wigner’s version of Bell’s theory then becomes the inequality

$$\frac{1}{2} \sin^2 \frac{1}{2} \theta_{23} + \frac{1}{2} \sin^2 \theta_{12} \geq \frac{1}{2} \sin^2 \frac{1}{2} \theta_{31}$$  \hspace{1cm} (2.2)

There are many other versions of Bell’s inequalities. See the literature [1]–[22] for example. I will only consider one more of them in this paper. Consider the nice treatment of the important CHSH [14] inequality in Bohm (24), pp. 347–354). A very large number of particles in the spin singlet state are considered. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four arbitrary chosen unit vector directions in the plane orthogonal to the two beams produced by the source. Let $v_i(\mathbf{a})$ and $v_i(\mathbf{d})$ be the “hidden” predetermined values $\pm 1$ of the spin components along $\mathbf{a}$ and $\mathbf{d}$, respectively, of particle 1 of the $i$th pair, similarly $w_i(\mathbf{b})$ and $w_j(\mathbf{c})$ for particle 2 values along directions $\mathbf{b}$ and $\mathbf{c}$. Then the average correlation value for particle 1 spins measured along $\mathbf{a}$ and particle 2 spins measured along $\mathbf{b}$ is

$$E(\mathbf{a}, \mathbf{b}) = \frac{1}{N} \sum_{i=1}^{N} v_i(\mathbf{a}) w_i(\mathbf{b})$$  \hspace{1cm} (2.3)

In the same way one considers the average correlation values $E(\mathbf{a}, \mathbf{c})$, $E(\mathbf{d}, \mathbf{b})$, $E(\mathbf{d}, \mathbf{c})$ and adding up all pairs as $i$ runs from 1 to $N$ one arrives at the CHSH inequality

$$|E(\mathbf{a}, \mathbf{b}) + E(\mathbf{a}, \mathbf{c}) + E(\mathbf{d}, \mathbf{b}) - E(\mathbf{d}, \mathbf{c})| \leq 2.$$  \hspace{1cm} (2.4)

Demanding this estimate hold as well for quantum mechanical expectations $E(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$, one has (24), p. 349)

$$|\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{d} \cdot \mathbf{b} - \mathbf{d} \cdot \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} - \mathbf{c})|$$

$$\leq |\mathbf{a}||\mathbf{b} + \mathbf{c}| + |\mathbf{d}||\mathbf{b} - \mathbf{c}|$$

$$= \sqrt{2 + 2 \cos \phi} + \sqrt{2 - 2 \cos \phi}$$  \hspace{1cm} (2.5)

where $\phi$ is the angle $\theta_{bc}$ (a notation I will use later) between $\mathbf{b}$ and $\mathbf{c}$. Then one observes that the last expression is maximized to value $2\sqrt{2}$ when $\theta_{bc} = \pi/2$, and “any configuration sufficiently ‘near’ to” the directions providing this maximal violation of Bell’s inequality will also violate it.
3 The Accardi–Gustafson Inequality

Here is a brief account of a partial coincidence of two distinct theories, one operator-theoretic, the other quantum–probabilistic, from which emerged what I will call here the Accardi–Gustafson inequality. About five years ago I noticed this coincidence. More details on the related work by Accardi et al. in quantum probability may be found in [25]–[28] and later papers. For the operator trigonometry see the early papers [29]–[35] and the recent books and surveys [36]–[40]. More details on the operator trigonometry, as it applies to quantum probabilities may be found in the recent papers [41]–[44].

What I noticed was that Accardi and Fedullo’s ([26], Proposition 3, Eq. (19)), namely

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - 1 \leq 2 \cos \alpha \cos \beta \cos \gamma \tag{3.1}
\]

a necessary and sufficient condition for the angles \(\alpha, \beta, \gamma\) of a quantum spin model in a 2-dimensional complex Hilbert space, is precisely the same as (see, e.g., Gustafson and Rao [37], Lemma 3.3-1, equation (3.3-3)) the operator trigonometry relation

\[
1 - a_1^2 - a_2^2 - a_3^2 + 2a_1a_2a_3 \geq 0 \tag{3.2}
\]

for the real cosines \(a_1, a_2, a_3\) of the angles between arbitrary unit vectors in any Hilbert space. The angles of inequality (3.1) are related to transition probability matrices \(P(A \mid B), P(B \mid C), P(C \mid A)\) for three observables \(A, B, C\) which may take two values. The angles of (3.2) are related to a triangle inequality for general operator angles within the general operator trigonometry. I have stated that triangle inequality in the Abstract as Theorem 1. As indicated in Corollary 1, in my opinion many “Bell inequalities” and related inequalities are best seen in the light of that general triangle inequality.

Although the triangle inequality of Theorem 1 gives the geometrical meanings, I have come to the opinion that the Accardi–Gustafson inequality (3.1), (3.2) is more fundamental and certainly more useful. Because I have discussed elsewhere [36]–[40] the application of the operator trigonometry to the quantum spin probabilistic settings, let me not repeat myself here. Instead I would like to present some new or previously unmentioned observations.

I discovered the general vector angle triangle inequality

\[
\phi_{xz} \leq \phi_{xy} + \phi_{yz} \tag{3.3}
\]

in 1966 as I was creating what came to be the operator trigonometry. However my chief interest was in the operator version

\[
\sin \phi(B) \leq \cos \phi(A) \tag{3.4}
\]

which becomes a sharp sufficient condition for the product \(BA\) of two bounded noncommuting positive selfadjoint operators \(A\) and \(B\) to be accretive (i.e.,
Re $BA$ is positive. This question came out of abstract Hille–Yosida operator semigroup theory [29]–[31]. The definitions of the entities in (3.4) are

$$
\sin \phi(B) = \inf_{\epsilon > 0} \|\epsilon B - I\|, \quad \cos \phi(A) = \inf_{x \neq 0} \frac{\langle Ax, x \rangle}{\|Ax\| \|x\|} \tag{3.5}
$$

A key early result was the min-max theorem [32] for any positive selfadjoint operator $B$

$$
\sin^2 \phi(B) + \cos^2 \phi(B) = 1 \tag{3.6}
$$

In my opinion one has no operator trigonometry without this result. I mention that (3.6) generally fails when you depart Hilbert space.

In connection with this positive operator product question my student D. Rao and I found that Krein [45] also had (slightly later) written down a version of (3.3) for another purpose. Many years later (1995) someone (H. Schneider) pointed out to me that Wielandt [46] had also written down (slightly later) another version. See the historical account [38]. Thus the operator–trigonometric origins of the Accardi–Gustafson inequalities predate those of the quantum probability.

The proof of (3.3) can be either elegant or constructive. The elegant proof is the following. None of me, Krein, Wielandt wrote it down. Take the 3 unit vectors $x, y$ and $z$ and embed them in real 3-space. Then great circle spherical distance is a (Riemannian) metric equivalent to the angles. However, I also wanted a constructive proof and Rao and I worked out a version in his dissertation [34], [35]. We used the Gram matrix

$$
G = \begin{bmatrix}
\langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\
\langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\
\langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle
\end{bmatrix} \tag{3.7}
$$

A Gram matrix is positive semidefinite in any number of dimensions, and definite iff the given vectors are linearly independent. From it one has in the present situation the determinant

$$
G = \begin{vmatrix}
1 & a_1 & a_3 \\
a_1 & 1 & a_2 \\
a_3 & a_2 & 1
\end{vmatrix} = 1 + 2a_1a_2a_3 - (a_1^2 + a_2^2 + a_3^2) \geq 0 \tag{3.8}
$$

which is the Accardi–Gustafson inequality (3.1), (3.2). As I showed in [41]–[44], from (3.8) one may constructively quickly prove the triangle inequality (3.3). It is a little harder to go the other way. That is why I regard the Accardi–Gustafson inequality as more fundamental.

Gudder–Zangli [47] also give a relative simple proof of the Accardi–Fedullo [26] inequality (3.1) without using a Grammian and without noting the triangle inequality (3.3). Others, e.g. Accardi–Fedullo [26], Wigner [1], Williams and Clearwater [23], Herbert and Peres [13], [12], Khrennikov [48] allude to resemblances or analogies to triangle inequalities in their work but the value of
...(3.3) is that it is the underlying triangle inequality. On the other hand I believe the Accardi–Gustafson inequality (3.1), (3.2), (3.8) is more fundamental not only in the sense mentioned above but also because in principle we may use the Grammian for any number of vectors $x, y, z, w, \ldots$. That could yield geometric–trigonometric relationships in higher dimensions that would be valuable for quantum probability in the future.

4 Inequality Equalities

In [43–44] I introduced the notion of violation boundaries for Bell-like inequalities and I worked out the theory of such for the Bell inequality (2.1). The idea was to try to embed such a given inequality into the operator trigonometry so as to determine exactly where the inequality is violated. Because the Accardi–Gustafson inequality (3.1), (3.2) and the triangle inequality (3.3) are correct geometrically, this procedure if applied to any related inequality can reveal flaws in an assumed physical or underlying probability model. Stretching a bit, we can say this was even the motivation of Bell in his original inequality (2.1). From it he concluded that hidden variables (as he perceived them) could not exist. Also we may say that the motivation of Accardi–Fedullo [26] was similar. Through their inequalities they were able to distinguish whether Kolmogorovean or quantum probability models remained consistent with the inequalities.

What I have in mind in the present section goes somewhat beyond those considerations. Here I want to convert inequalities into equalities. This will reveal not only violation boundaries but also violation regions and moreover the nature of the extra terms may conceivably be interpreted probabilistically or physically.

Consider first the Wigner inequality (2.2). Wigner was only considering the coplanar case in which the vectors $x, y, z$ lie in a common plane. The Gram determinant $G$ (3.8) vanishes if and only if the three directions are coplanar, no matter what their frame of reference. Then we may write the equality in (3.8) as follows

$$(1 - a_1^2) + (1 - a_2^2) - (1 - a_3^2) = 2a_3(a_3 - a_1a_2)$$

or in the terminology of (2.2)

$$\sin^2\left(\frac{1}{2}\theta_{12}\right) + \sin^2\left(\frac{1}{2}\theta_{23}\right) - \sin^2\left(\frac{1}{2}\theta_{13}\right)$$

$$= 2\cos\left(\frac{1}{2}\theta_{13}\right) \left[\cos\left(\frac{1}{2}\theta_{13}\right) - \cos\left(\frac{1}{2}\theta_{12}\right) \cos\left(\frac{1}{2}\theta_{23}\right)\right].$$

Violation of the conventionally assumed quantum probability rule $|\langle u, v \rangle|^2 = \cos^2\theta_{uv}$ for unit vectors $u$ and $v$ representing prepared state $u$ to be measured as state $v$ is equivalent to the right side of (4.2) being negative. All terms therein in principle carry probabilistic or other physically relevant inferences.

Williams and Clearwater [23] present essentially Wigner’s formulation. Let a Polarizer 1 with orthogonal axes $h_1$ and $v_1$ (think: horizontal and vertical) be inclined at angle $\theta_1$ to the horizontal, likewise Polarizer 2 with axes...
straightforward probability arguments, now to quote [23] “We can write down the following relationships from (2.5). Wishing now to preserve equality therein so that we may analytically express what we may call the ‘violation boundaries, violation regions’, starting from (2.5) we have

\[
|\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{d} \cdot \mathbf{b} - \mathbf{d} \cdot \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} - \mathbf{c})|
\]

\[
= ||\mathbf{b} + \mathbf{c}|| \cos \theta_{a,b+c} + ||\mathbf{b} - \mathbf{c}|| \cos \theta_{d,b-c}
\]

\[
= |(2 + 2 \cos \theta_{bc})^{1/2} \cos \theta_{a,b+c} + (2 - 2 \cos \theta_{bc})^{1/2} \cos \theta_{d,b-c}|
\]

Squaring this expression and writing everything quantum trigonometrically,

\[
|\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{d} \cdot \mathbf{b} - \mathbf{d} \cdot \mathbf{c}|^2
\]

\[
= (2 + 2 \cos \theta_{bc}) \cos^2 \theta_{a,b+c} + (2 - 2 \cos \theta_{bc}) \cos^2 \theta_{d,b-c}
\]

\[
+ 2(4 - 4 \cos^2 \theta_{bc})^{1/2} \cos \theta_{a,b+c} \cos \theta_{d,b-c}
\]

\[
= 4 \cos^2(\theta_{bc}/2) \cos^2 \theta_{a,b+c}
\]

\[
+ 4 \sin^2(\theta_{bc}/2) \cos^2 \theta_{d,b-c}
\]

\[
+ 4 \sin^2 \theta_{bc} \cos \theta_{a,b+c} \cos \theta_{d,b-c}.
\]
In the above I used two standard trigonometric half-angle formulas. Now substituting the standard double angle formula \(\sin \theta_{bc} = 2 \sin(\theta_{bc}/2) \cos(\theta_{bc}/2)\) into the above we arrive at
\[
|a \cdot b + a \cdot c + d \cdot b - d \cdot c|^2 = 4[\cos(\theta_{bc}/2) \cos(\theta_{ab+c}) + \sin(\theta_{bc}/2) \cos(\theta_{d,b-c})]^2 \quad (4.7)
\]
and hence the quantum inequality equality
\[
|a \cdot b + a \cdot c + d \cdot b - d \cdot c| = 2[\cos(\theta_{bc}/2) \cos(\theta_{ab+c}) + \sin(\theta_{bc}/2) \cos(\theta_{d,b-c})] \quad (4.8)
\]
We may also write the righthand side of (4.8) as twice the absolute value of the two-vector inner product
\[
u_1 \cdot \nu_2 \equiv (\cos(\theta_{bc}/2), \sin(\theta_{bc}/2)) \cdot (\cos(\theta_{ab+c}, \cos(\theta_{d,b-c})) \quad (4.9)
\]
to arrive at the inequality equality
\[
|a \cdot b + a \cdot c + d \cdot b - d \cdot c| = 2(\cos^2(\theta_{ab+c}) + \cos^2(\theta_{d,b-c}))^{1/2} \cos(\theta_{u_1,u_2}) \quad (4.10)
\]
The right sides of these two equalities (4.8), (4.10) isolate the “classical probability factor” 2 from the second factor, which may achieve its maximum \(\sqrt{2}\). That the latter maximum is consistent with the third factor in (4.10) also achieving its maximum value 1 may be seen as follows. Fix any directions \(b \) and \(c\). Then choose \(a\) relative to \(b + c\) and choose \(d\) relative to \(b - c\) so that \(\cos^2(\theta_{ab+c}) = 1 \) and \(\cos^2(\theta_{d,b-c}) = 1\), respectively. Now we may choose the free directions \(b\) and \(c\) to maximize the third factor to \(\cos(\theta_{u_1,u_2}) = \pm 1\). But that means the two-vectors \(u_1\) and \(u_2\) are colinear and hence
\[
u_1 = (\cos(\theta_{bc}/2), \sin(\theta_{bc}/2)) = 2^{-1/2}(\cos(\theta_{ab+c}), \cos(\theta_{d,b-c})
\]
\[
= 2^{-1/2}(\pm 1, \pm 1) \quad (4.11)
\]
and thus the important angle \(\theta_{bc}\) is seen to be \(\pm \pi/2\). More to the point, the above inequality equality allows one to exactly trace out the “violation regions” analytically in terms of the trigonometric inner product condition \(1 \leq |\nu_1 \cdot \nu_2| \leq \sqrt{2}\). From this point of view, there are no Bell inequalities. Each should be replaced with an inequality equality.

Let me summarize the above. One started with a classical probability correlation definition (2.3) and derived a Bell inequality \(|\cdots| \leq 2\). The “equality” version of this classical probability version would be in the individual terms
\[
|v_i(a)(w_i(b) + w_i(c)) + v_i(d)(w_i(b) - w_i(c))| = 2 \quad (4.12)
\]
On the other hand, inserting the quantum correlation definition into the left side of (2.4) results (2.5) in the Bell inequality \(|\cdots| \leq 2\sqrt{2}\). My equality version (4.10) of this becomes the vector trigonometric identity
\[
|\cos(\theta_{ab} + \cos(\theta_{ac} + \cos(\theta_{bd} - \cos(\theta_{dc})) = 2(\cos^2(\theta_{ab+c}) + \cos^2(\theta_{d,b-c})^{1/2} \cos(\theta_{u_1,u_2}) \quad (4.13)
\]
It could be useful to call (4.13) the quantum spin correlation identity. But it is really a new mathematical result in vector trigonometry, independent of any physical assumptions.
5 Concluding Remarks

In spite of the huge literature on physical and metaphysical interpretations of Bell’s inequalities and related theory and experiments, the operator trigonometry provides a new and correct mathematical setting for much of that theory. I would assert that one correct physical understanding of the Bell inequalities is that of basic Hilbert space geometry, more specifically, the geometry of Euclidean and Unitary spaces, more specifically, that of a classical but new vector trigonometry.

The second comment is that the principal connection to physics in the above development is our belief that quantum correlations are given by the quantum probability rule: for two normalized vectors \( u \) and \( v \), the probability that a quantum system prepared in state \( u \) will successfully pass a test for state \( v \) is \( |u \cdot v|^2 \equiv \cos^2 \theta_{u,v} \). The quantum probability rule generally states that the expectation value of an observable \( A \) which has been determined experimentally as the arithmetic mean \( \langle A \rangle \) of a large number of trials, should correspond theoretically to \( Tr(AW) \) where \( W \) is the statistical operator describing the state of the system. For pure states this quantum probability rule becomes, operationally and loosely: the expected value is the projection onto the state. For the spin zero singlet state in the Bell situation the expected correlation value is \( E(a,b) = -a \cdot b = -\cos \theta_{ab} \). From this ansatz alone and my inequality equalities above, one divides vectors \( a, b, c, d \) into ‘satisfaction’ and ‘violation’ regions in whatever Hilbert space you want to take your direction vectors from.

From this viewpoint, I would prefer that the multitude of physical experiments over the years since [8] which have found various physical quantum mechanical configurations in which “Bell’s inequality” is violated, be restated as showing that my inequality equalities such as (4.13) are achieved by those physical configurations for which the right hand side is between \( 2 \) and \( 2 \sqrt{2} \). But we know the latter is just vector geometry. So what these physical experiments really have shown is various verifications of the quantum probability rule. To repeat and indeed overstate my point, rather than seeking “Bell inequality violations”, it would be more interesting to seek “quantum probability rule violations”. This, because the quantum probability rule is a far-reaching assumption, an ansatz, which in the sense of my presentation in this paper, reduces much of quantum mechanics to a new vector trigonometry. Thus for example one could seek some quantum physical situation which could result in physical measurements for which there obtains a right-hand-side greater than \( 2 \sqrt{2} \). Also one could run a large number of tests to determine what statistical distribution of occurrences manifests itself in the region between \( 2 \) and \( 2 \sqrt{2} \).

As a third comment, let me make an assertion which seems relevant in view of the developments of this paper: one cannot argue either locality or nonlocality on the basis of satisfaction or violation of Bell’s Inequality. Bell’s Inequality, notwithstanding the key and very important role it has played in the evolving scientific revolution of quantum mechanics, is seen in retrospect as a “red herring”: a diversion distracting attention away from the real issue. Unlike political red herrings, the original intent of Bell and consequent investigators was
genuine. However, from my viewpoint, the real issue as concerns nonlocality in quantum physics is the projection rule. This “probability” rule is fundamental to Von Neumann quantum mechanics. It is also fundamental to my quantum trigonometry. It is surely true for the latter, i.e., geometrically. Is it true for the former?

How do various authors argue nonlocality on the basis of Bell’s inequalities? B. d’Espagnat [19, p. 124] makes it clear: “the violation of either the compound premise ‘local causality and free will’—that is, the premises of Bell 2—or the premises of Bell 1 or Bell 3.” Thus in particular, violation of the CHSH inequality (2.4). But I have shown that an occurrence yielding a (4.13) value between $2$ and $2\sqrt{2}$ just means you are in that (acceptable) region of Hilbert space. There is no way to equate that geometric region with any concept of nonlocality.

Maudlin [18] discusses nonlocality from several points of view. However his preferred view is that of entanglement: after the simultaneous emission of a photon pair “the photons are perfectly correlated: each does what the other does.” There is no attenuation over distance of this quantum effect. “Finally, the speed of quantum communication appears to be incompatible with relativistic space–time structure.” Most useful is his Notes [18, p. 49] with citations about various senses of ‘locality’. Finally we come to [18, p. 154] “Reliable violations of Bell’s inequality need not allow superluminal signalling but they do require superluminal causation.” While such may or may not turn out to be the case, and while my view depends in its details upon exactly which Bell’s inequality you are discussing, ‘reliable’ violation of Bell’s inequality in my theory is not a violation. It is just an occurrence fitting into the larger geometric inequality–equality.

The extensive analysis [15] of the EPR paradox also considers many aspects of ‘realism’, violation of Bell’s inequalities in a number of experimental situations, nonlocality. Generally when one reads the literature one finds it easier to define locality than nonlocality. Generally locality is thought of [15, p. 35] as “the intensity of interaction between objects depends inversely on their separation” and quantum nonlocality as [15, p. 195] long-range interference effects. It seemed to me that Afriat and Selleri [15] were careful not to equate nonlocality just to violation of a Bell’s inequality. On the other hand they state [15, p. 237] “As is well known, any proof of Bell’s Theorem has two steps. The first step is a deduction of inequalities from the assumptions of reality and locality, and the second the exhibition of a contradiction with quantum mechanics.” My point of view is that such inequalities are a direct mathematical consequence of Hilbert space structure, independent of any physical ideas.

Peres [12, pp. 169, 173] fully accepts the traditional Bell view. “In summary, there is no escape from nonlocality. The experimental violation of Bell’s inequality leaves only two logical possibilities: either some simple physical systems... are essentially nonlocal, or it is forbidden to consider simultaneously the possible outcomes of mutually exclusive experiments.” Again, not having my operator trigonometric view of the geometry of quantum probabilities does not permit a full understanding of Bell’s inequalities.
Van Frassen [22] emphasizes the philosophical and logic aspects of quantum interpretation. Generally cautious in stating what may be deduced, nonetheless he states Bell’s inequalities [22, pp. 93, 104, 348] in the form $p_{12} + p_{23} \geq p_{13}$ of Wigner [9]. As I have shown here, these inequalities (2.2) and (4.4) are flawed both quantum probabilistically and geometrically. The trouble with empirical positivism (i.e., the philosophical school of Carnap and others) is that it (in my opinion) permits to obviate searching for or investigating mathematically deeper fundamental physical meanings. On the other hand that philosophy does lead one back to Von Mises view of probabilities, that they best be given by relative frequencies.

Finally, there is another development which serves to bring into question the Von Neumann projection rule. That is the Zeno’s paradox, which I treat in [50]. I cannot go into all the citations and the mathematical details here but I show in [57] that there are serious unresolved mathematical issues about the ‘continuous observation’ operator limit $s - \lim_{n \to \infty} (PU(t/n)P)^n$. As a consequence in [50] I propose what I call domain-compatible measuring operators to replace the projections $P$.

Thus in summary one may say that Bell, Wigner, others, focused on physical axioms which led them to certain inequalities. Accardi, Fedullo, others focused on probabilistic axioms which led them to certain inequalities. I have focused on geometrical axioms which have led me to certain inequalities. Moreover I have extended the notion of such “Bell” inequalities to what I called here “inequality–equalities” which may then be used to reflect back more accurate meanings of the earlier inequalities.

6 Acknowledgements

I was led into these matters by Luigi Accardi following a discussion with him at Les Treilles, France in July 1996. After that discussion, on my own I found the key paper [26] and saw the coincidence (3.1), (3.2). I have since enjoyed several profitable discussions with Accardi at meetings. Also I would like to mention useful conversations on this subject with Andrei Khrennikov, Harald Atmanspacher, Han Primas, and Urs Wild.

References

[1] Einstein, A., Podolsky, B. & Rosen, N. (1935). Can quantum mechanical description of reality be considered complete?, Phys. Rev. 47, 777-780.

[2] Jammer, M. (1974) The Philosophy of Quantum Mechanics, John Wiley & Sons, New York.

[3] Einstein, A., Tolman, R. & Podolsky, B. (1931). Knowledge of past and future in quantum mechanics, Phys. Rev. 37, 780–781.
[4] Einstein, A. (1932). Über die Unbestimmtheitsrelation, Zeitschrift für Angewandte Chemie 45, 23.

[5] Schrödinger, E. (1935). Die gegenwärtige Situation in der Quantenmechanik, Die Naturwissenschaften 23, 807–812, 824–828, 844–849.

[6] Bohm, D. (1951). Quantum Theory, Prentice Hall, New Jersey.

[7] Bell, J. (1964). On the Einstein–Podolsky–Rosen paradox, Physics 1, 195–200.

[8] Aspect, A., Dalibard, J. & Roger, G. (1982). Experimental test of Bell’s inequalities using time-varying analyzers, Phys. Rev. Letters 49, 1804–1807.

[9] Wigner, E. (1970). On hidden variables and quantum mechanical probabilities, Amer. J. Phys. 38, 1005–1009.

[10] Miller, W. (1972). Symmetry Groups and Their Applications, Academic Press, New York.

[11] Jammer, M. (1991). John Stewart Bell and the debate on the significance of his contributions to the foundations of quantum mechanics, in Bell’s Theorem and the Foundations of Modern Physics, eds. A. Van der Merwe, F. Selleri, G. Tarozzi (World Scientific, Singapore), 1–23.

[12] Peres, A. (1993). Quantum Theory, Concepts and Methods, Kluwer, Dordrecht.

[13] Herbert, N. (1975). Cryptographic approach to hidden variables, Amer. J. Phys. 43, 315–316.

[14] Clauser, J., Horne, M., Shimony, A., and Holt, R. (1969). Proposed experiment to test local hidden-variable theories, Phys. Rev. Letters 23, 880–884.

[15] Afriat, A. and F. Selleri (1999). The Einstein, Podolsky, and Rosen Paradox, Plenum Press, New York.

[16] Van der Merwe, A., Selleri, F., & Tarozzi, G., Eds. (1991). Bell’s Theorem and the Foundations of Modern Physics, World Scientific, Singapore.

[17] Mermin, D. (1985). Is the moon there when nobody looks? Reality and the quantum theory, Physics Today, April, 38–47.

[18] Maudlin, T. (1994). Quantum Non-Locality and Relativity, Blackwell, Oxford.

[19] d’Espagnat, B. (1995). Veiled Reality: An Analysis of Present-Day Quantum Mechanical Concepts, Addison–Wesley, Reading, MA.

[20] Bell, J. (1987). Speakable and Unspeakable in Quantum Mechanics, Cambridge Press, Cambridge.
[21] Bell, M., Gottfried, K., and Veltman, M. (2001). *John S. Bell on The Foundations of Quantum Mechanics*, World-Scientific, Singapore.

[22] Van Frassen, B. (1991). *Quantum Mechanics: An Empiricist View*, Clarendon Press, Oxford.

[23] Williams, C. P. and Clearwater, S. (1997). *Explorations in Quantum Computing*, Springer, New York.

[24] Bohm, A. (1993). *Quantum Mechanics, Foundations and Applications*, Springer, New York.

[25] Accardi, L. (1981). Topics in quantum probability. *Physics Reports* 77, 169–192.

[26] Accardi, L. & Fedullo, A. (1982). On the statistical meaning of complex numbers in quantum mechanics, *Lett. Nuovo. Cim.* 34, 161–172.

[27] Accardi, L. (1984). The probabilistic roots of the quantum mechanical paradoxes, in *The Wave–Particle Dualism*, eds. S. Diner, D. Fargue, G. Lochak, F. Selleri (D. Reidel Publishing, Dordrecht), 297–330.

[28] Accardi, L. (1988). Foundations of quantum mechanics: a quantum probabilistic approach, in *The Nature of quantum Paradoxes*, eds. Tarozzi, G. and Van der Merwe, A., (Kluwer, Dordrecht), pp. 257–323.

[29] Gustafson, K. (1968). The angle of an operator and positive operator products, *Bull. Amer. Math. Soc.* 74, 488–492.

[30] Gustafson, K. (1968). Positive (noncommuting) operator products and semigroups, *Math. Zeit.* 105, 160–172.

[31] Gustafson, K. (1968). A note on left multiplication of semigroup generators, *Pacific J. Math.* 24, 463–465.

[32] Gustafson, K. (1968). A min-max theorem, *Notices Amer. Math. Soc.* 15, 699.

[33] Gustafson, K. (1972). Antieigenvalue inequalities in operator theory, in *Inequalities III*, Los Angeles, 1969, (O. Shisha, ed.), Academic Press, pp. 115-119.

[34] Rao, D. (1972). Numerical range and positivity of operator products, Dissertation, University of Colorado.

[35] Gustafson, K. and Rao, D. (1977). Numerical range and accretivity of operator products, *J. Math. Anal. Applic.* 60, 693–702.

[36] Gustafson, K. (1997). *Lectures on Computational Fluid Dynamics, Mathematical Physics, and Linear Algebra*, World Scientific, Singapore.
Gustafson, K. and Rao, D. (1997). *Numerical Range*, Springer, Berlin.

Gustafson, K. (1996). Commentary on: *Topics in the Analytic Theory of Matrices*, Section 23, Singular angles of a square matrix, in *Collected Works of Helmut Wielandt*, vol. 2, eds. Huppert, B. and Schneider, H. (De Gruyters, Berlin), 356–367.

Gustafson, K. (1999). A computational trigonometry and related contributions by Russians Kantorvich, Krein, Kaporin, *Computational Technologies* 4, no. 3, 73–83.

Gustafson, K. (2001). An unconventional computational linear algebra: operator trigonometry, in *Unconventional Models of Computation, UMC’2K*, eds. I. Antoniou, C. Calud, M. Dinneen (Springer, London), 48–67.

Gustafson, K. (1999). The geometry of quantum probabilities, in *On Quanta, Mind, and Matter. Hans Primas in Context*, eds. H. Atmanspacher, A. Amann, U. Mueller-Herold, (Kluwer, Dordrecht), 151–164.

Gustafson, K. (1999). The trigonometry of quantum probabilities, in *Trends in Contemporary Infinite Dimensional Analysis and Quantum Probability*, eds. L. Accardi, H. Kuo, N. Obata, K. Saito, S. Si, L. Streit, (Italian Institute of Culture, Kyoto), 159–173.

Gustafson, K. (2000). Quantum trigonometry, *Infinite Dimensional Analysis, Quantum Probability, and Related Topics* 3, 33–52.

Gustafson, K. (2001). Probability, geometry, and irreversibility in quantum mechanics, *Chaos, Solitons and Fractals* 12, 2849–2858.

Krein, M. G. (1969). Angular localization of the spectrum of a multiplicative integral in a Hilbert space, *Functional Anal. Appl.*, 89–90.

Wielandt, H. (1967). Topics in the analytic theory of matrices, University of Wisconsin, Lecture Notes.

Gudder, S. & Zanghi, N. (1984). Probability models, *Nuovo Cim. B* 79, 291–301.

Khrennikov, A. (2000). A perturbation of CHSH inequality induced by fluctuations of ensemble distributions, *J. Math. Physics* 41, 5934–5944.

Gustafson, K. (2002). Bell’s Inequalities. *Proceedings of the 22nd Solvay Conference on Physics*, to appear.

Gustafson, K. (2002). A Zeno Story. *Proceedings of the 22nd Solvay Conference on Physics*, to appear. See also [http://xxx.lanl.gov/abs/quant-ph/0203032](http://xxx.lanl.gov/abs/quant-ph/0203032).