OPTIMAL CONTROL OF THE 3D DAMPED NAVIER-STOKES-VOIGT EQUATIONS WITH CONTROL CONSTRAINTS

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Abstract. In this paper, we consider the 3D Navier-Stokes-Voigt (NSV) equations with nonlinear damping $|u|^{r-1}u$, $r \in [1, \infty)$ in bounded and space-periodic domains. We formulate an optimal control problem of minimizing the curl of the velocity field in the energy norm subject to the flow velocity satisfying the damped NSV equation with a distributed control force. The control also needs to obey box-type constraints. For any $r \geq 1$, the existence and uniqueness of a weak solution is discussed when the domain $\Omega$ is periodic/bounded in $\mathbb{R}^3$ while a unique strong solution is obtained in the case of space-periodic boundary conditions. We prove the existence of an optimal pair for the control problem. Using the classical adjoint problem approach, we show that the optimal control satisfies a first-order necessary optimality condition given by a variational inequality. Since the optimal control problem is non-convex, we obtain a second-order sufficient optimality condition showing that an admissible control is locally optimal. Further, we derive optimality conditions in terms of adjoint state defined with respect to the growth of the damping term for a global optimal control.

1. Introduction

Optimal control of fluid mechanics has been one of the crucial topics in applied mathematics. One such problem in this topic is the minimization of turbulence in the flow field by acting upon the region by an external force through the interior of the flow field or the boundary of the flow domain. In this work, we study the optimal control problem of Navier-Stokes-Voigt equations with nonlinear damping and distributed control on the right-hand side of the state equations, describing the motion of homogeneous incompressible fluids, given by

\[\begin{align*}
\text{(NSVD)} \quad &\begin{cases}
    u_t - \mu \Delta u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p + \alpha u + \beta |u|^{r-1}u = U &\text{in } \Omega_T, \\
    \nabla \cdot u = 0 &\text{in } \Omega_T, \\
    u(x,0) = u_0 &\text{in } \Omega,
\end{cases}
\end{align*}\]

where $\Omega_T := \Omega \times (0,T]$, $\Omega$ is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$ or a periodic domain as in Section 3.1, and $T > 0$ is a fixed given time. In the case of the bounded domain, (1.1) is supplemented with the Dirichlet boundary conditions $u = 0$ on $\Sigma_T := \partial \Omega \times [0,T]$. The unknowns $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denote the velocity of the flow field, $p = p(x,t)$ is a scalar valued function representing the pressure field and $u_0$ is the given initial flow velocity. The function $U = (U_1(x,t), U_2(x,t), U_3(x,t))$ is the applied distributed control force that can be realized, for instance, as the electromagnetic (Lorentz) force distribution in salt water and liquid metals (see, [45],[15]). The parameters, $\mu > 0$ denotes the length scale characterizing the elasticity of the fluid, $\nu > 0$ is the kinematic viscosity, the damping coefficients $\alpha, \beta > 0$, and $r \in [1, \infty)$, the order of the nonlinearity. We also assume $\int_{\Omega} p(t,x) dx = 0$ in $(0,T)$ for uniqueness of the pressure.
Let us look at some of the special cases of (1.1) when the external force $U$ (mostly) equals zero. Evidently, when the coefficients $\mu = 0$ and $\alpha = \beta = 0$, the model problem (1.1) reduces to the classical incompressible Navier-Stokes (NS) equations. The existence, uniqueness, and other qualitative properties of the Navier-Stokes equations have been studied to a great extent by several mathematicians (see, \cite{16, 40}). Though plenty of articles are available for the 3D NS equations, the uniqueness of weak solutions and the global existence of strong solutions for this problem remains an open problem that attracts many researchers to look into this case deeply. There are various generalizations of the classical NS equations have been proposed in the literature that leads to the global well-posedness and decay of the solutions. More precisely, when the parameters $\alpha = \beta = 0$, the equation (1.1) reduces to the standard NSV equation, which was introduced in \cite{31} as a model for the approximation of the Kelvin-Voigt linear viscoelastic incompressible fluid flow. The NSV equation was suggested in \cite{10} as a regularized model for the classical Navier-Stokes equations for obtaining the numerical simulations. Another important model considered in the literature is the convective Brinkman-Forchheimer (CBF) equation (\cite{20}), which can be obtained by setting the length scale parameter $\mu = 0$. In this special case, the parameter $\nu$ is referred to as the Brinkman coefficient (effective viscosity), $\alpha > 0$ denotes the Darcy (permeability of porous medium) coefficient, and $\beta > 0$ is the Forchheimer coefficient (proportional to the porosity of the material). In the absence of $\alpha u$, the CBF equation has the same scaling as the classical NS equation, which is referred to as the NS equation with damping (or absorption) term $\beta |u|^{r-1} u$. The damping term can be physically motivated as a resistance to the motion of the flow, that is, an external force field in momentum equation accounting for a friction process arising inside a flow domain (see, \cite{7, 9} and references therein). In the paper, \cite{7}, the existence of Leray-Hopf weak solution has been obtained for any dimension $n \geq 2$, while the uniqueness is obtained when $n = 2$. The existence and uniqueness of weak solutions of the Kelvin-Voigt-Brinkman-Forchheimer (KVBF) equations (see, \cite{6}), which is similar to the model (1.1) but with a general damping term $(f(u))$ has been studied in a bounded/unbounded domain in $\mathbb{R}^3$. For the same KVBF equation, the existence and uniqueness of strong solutions, when $1 \leq r \leq 5$, and the existence of exponential attractors were obtained in \cite{28}.

The flow control problems of classical deterministic and stochastic NS equations have been well studied over the past few decades. The optimal control of minimizing vorticity of the flow field governed by the 2D NS equations on a bounded domain with distributed control was studied in \cite{1}. The existence of optimal boundary control of the NS equation has been the subject of \cite{34} by showing that the value function, which is the minimum for an objective functional, is the viscosity solution of the associated HJB equation. The time-optimal control of the NS equation was studied in \cite{8}. Optimal control of 2D and 3D NS equations in a flow domain exterior to a bounded domain have been rigorously discussed in \cite{35, 19}. Besides, optimal control problems of the 3D NS equations have been treated in the presence of state constraints, which can arise in the context of suppression of turbulence in a selected flow region \cite{15, 43, 27}, and also box-type control constraints \cite{42, 23}. The existence of optimal control for deterministic and stochastically forced fluid flow models has been studied in \cite{37} by establishing the space-time statistical solutions (see, also \cite{32}). Recently, a control problem for the regularized 3D NS equations (NSV equation) in a bounded domain with distributed control and tracking type cost functional has been studied in \cite{4}, and the time-optimal control of this model has been considered in \cite{5}. Optimal control of 2D CBF equations is examined for $r = 1, 2$ and $3$ in \cite{29}. 
Apart from the literature mentioned above on flow control of NS or NSV equations, to the best of the author’s knowledge, optimal control of the 3D NSV equations with damping has not been studied in the literature. In this paper, we consider an optimal control problem for the NSV equations with damping $\beta|u|^{r-1}u, r \in [1, \infty)$ (or it is also called KVBF equation) in 3D bounded/periodic domain with distributed control in the state equation subject to control constraints. More precisely, suppose a target velocity field $u_d \in L^2(0, T; V)$ is given. We consider the objective/cost functional

$$J(u, U) := \frac{\kappa}{2} \int_0^T \|\text{curl}(u(t) - u_d(t))\|_H^2 dt + \frac{\lambda}{2} \int_0^T \|U(t)\|_H^2 dt$$

(1.2)

and the set of admissible controls with constraint

$$U_{ad} := \{U \in L^2(0, T; H) : U_{\text{min}}(x, t) \leq U(x, t) \leq U_{\text{max}}(x, t), \text{ a.e. } (x, t) \in \Omega_T\}.$$  

(1.3)

The first term in the cost functional defines the enstrophy of the flow field, that is, it amounts to the kinetic energy of the fluid field, and the second term specifies energy associated with the control input. The parameters $\kappa$ and $\lambda$ are fixed nonnegative constants. The control constraints $U_{\text{min}}, U_{\text{max}} \in L^\infty(\Omega_T)$ are given functions such that $U_{\text{min}}(x, t) \leq U_{\text{max}}(x, t)$, a.e. $(x, t) \in \Omega_T$. We intend to find an optimal control $U$ minimizing the objective functional $J(u, U)$ subject to the control constraint $U \in U_{ad}$ and the pair $(u, U)$ is a solution of the state equation (1.1). For various other significant cost functionals, such as the minimization of energy and boundary control problems associated with the incompressible fluid dynamics equations, one may refer to [1, 36, 19].

We describe the main contributions of this paper. The function spaces used here are defined in Sections 2.1 and 3.1. For any $r \geq 1$, $u_0 \in V$, and the control $U \in L^2(0, T; H)$, we formally show that there exists a unique weak solution of (1.1). Further, when the data $u_0 \in H^2 \cap V$ and the domain $\Omega$ is periodic, we also show that this weak solution is a strong solution with $u \in Z$, where $Z := H^1(0, T; H^2) \cap L^\infty(0, T; L^{r+1}), r \geq 1$. The restriction to the space-periodic boundary conditions arises due to some technical issues as given in Section 3.1.

Since $\|\text{curl} u\|_H = \|\nabla \times u\|_H = \|\nabla u\|_H, u \in V$ (see, [12], Chapter 1), the optimal control problem we intend to investigate can be stated as follows:

(OCP) \begin{align*}
\text{minimize} & \quad J(u, U) \\
\text{subject to} & \quad U \in U_{ad} \\
& \quad \text{and } u \in Z \text{ is a strong solution of (1.1) in response to } U \in U_{ad},
\end{align*}

where the functional

$$J(u, U) = \frac{\kappa}{2} \int_0^T \|\nabla(u(t) - u_d(t))\|_H^2 dt + \frac{\lambda}{2} \int_0^T \|U(t)\|_H^2 dt.$$  

In the first main result, we have proved the existence of an optimal solution pair $(\tilde{u}, \tilde{U})$ solving (OCP). The proof of this result can also be completed within the framework of weak solutions of (1.1), which we could get for the case of the bounded domain (see, Remark 3). The second main theorem of this paper is the first-order necessary optimality conditions satisfied by an optimal pair that is given in terms of a variational inequality (Theorem 5.5) since an optimal control needs to obey the box-type constraints, which is only a subset of $L^2(0, T; H)$. A simplified variational inequality is obtained by using the classical adjoint problem approach. The required Fréchet differentiability of the cost functional $J(\cdot, \cdot)$ is
proved by introducing a linearized system of (1.1). One of the main issues here is that the rigorous proof of the preceding sequence of results also demands the well-posedness of the linearized system and that of the adjoint system of (1.1) whose coefficients, in turn, require regularity of solutions of (1.1). It is justified with the help of strong solutions of (1.1) obtained in the periodic domain. Nevertheless, when we restrict the nonlinear damping term $|u|^{r-1}u$ to the range of $2 \leq r \leq 5$ (and $r = 1$), we can establish the first-order optimality conditions in the bounded domain by using the weak solutions of (1.1). This is summarized in Section 5.3.

As we are dealing with a non-convex optimal control problem, a second-order sufficient optimality condition for optimal control is also obtained in the third main result for any $r \geq 3$. This follows from the classical method of showing that the control $\tilde{U} \in \mathcal{U}_{ad}$ satisfying the variational inequality together with the condition that the reduced cost functional obeys $\mathcal{J}''(\tilde{U})[U, U] > 0$ in a cone of critical directions result in a locally minimizing control $\tilde{U}$ of $\mathcal{J}(\cdot)$ (Theorem 6.3). However, this result doesn’t give further information on the global optimality of the control. We have proved another result concerning this question (see, Theorem 7.1). In this theorem, we show that an admissible control satisfying the variational inequality together with a feasible condition obeyed by the corresponding adjoint state would lead to a global optimal control, and that control also can be obtained uniquely under additional restrictions for any $r \geq 2$ and $r = 1$. Moreover, we also proved a global optimality condition for the case of the bounded domain when $2 \leq r \leq 5$ and $r = 1$ (see, Remark 6).

Let us further analyze these results closely associated with the existing literature on the bounded domain. For the optimal control problem of 2D CBF equation ($\mu = 0$ in (1.1)) in the bounded domain, a first-order necessary condition was derived in [29] for $r = 2, 3$. The results obtained in this paper for the regularized (OCP) ((1.1)-(1.3)) generalizes [29] to the 3D bounded domain for any $r \in [2, 5]$. The regularized model further helps to prove the crucial Fréchet differentiability, Lipschitz continuity of the control-to-state operator, and that of the control-to-costate operator in better function spaces. These properties lead to the second-order Fréchet derivative of the cost functional, which is used to get a strict local optimality condition. It is worth noting that in the global optimality condition result (Theorem 7.1), we stated a special case $r = 1$ explicitly, which accounts for the linear perturbation of the NSV model. To the author’s knowledge, such a result is not available in the literature for optimal control of the NSV equation. A local second-order optimality condition for this case was obtained in [4] with tracking type cost functional. Besides, a global optimality condition given in terms of the solution of the adjoint problem is also useful in computation (see, [3, 42]).

The manuscript is organized as follows. Section 2 sets up the essential function spaces and collects some standard inequalities that we used throughout the manuscript. The well-posedness of (1.1), namely weak and strong solutions, are obtained in Section 3. In the same section, we also prove one of the main theorems concerning the continuous dependence of data and control function. Section 4 discusses the existence of optimal control, and first-order optimality conditions are given in Section 5. Section 6 is devoted to deriving second-order optimality conditions for local optimal control. Finally, global optimality conditions in terms of the adjoint state are given in Section 7.
2. Function spaces and mathematical inequalities

In this section, by taking the divergence-free flow field into account, we introduce the necessary divergence-free function spaces used throughout the article. We state some standard inequalities and derive properties of linear and nonlinear terms that occur in (1.1). For more details on these function spaces, one may refer to [38].

2.1. Divergence free function spaces. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \). For any \( 1 \leq p < \infty \) or \( p = \infty \) and \( m \geq 0 \), let \( W^{m,p}(\Omega) \) denote the Sobolev spaces of functions in \( L^p(\Omega) \) whose weak derivatives of order less than or equal to \( m \) are also in \( L^p(\Omega) \). The norms corresponding to these function spaces are denoted by \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{W^{m,p}} \). For the special case when \( p = 2 \), instead of the space \( W^{m,2}(\Omega) \), we shall write \( H^m(\Omega) \) with the norm \( \| \cdot \|_{H^m} \). We also use the time dependent function spaces \( L^2(0,T;H^m(\Omega)) \) consisting of all measurable functions from \( (0,T) \) to \( H^m(\Omega) \) such that square of their \( H^m \)-norm is integrable over \( (0,T) \). The space \( H^1(0,T;H^m) \) denotes the space of functions and whose first-order time derivative both belong to \( L^2(0,T;H^m(\Omega)) \). Since \( u, \nabla p \) and \( U \) appearing in the governing equations are vector fields, we view them as belonging to the product spaces \( (L^p(\Omega))^3, (H^m(\Omega))^3, (L^2(0,T;H^m(\Omega)))^3 \), etc.

We define the divergence free Hilbert space

\[
\mathbb{H} := \{ v \in (L^2(\Omega))^3 : \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \partial \Omega \},
\]

with norm \( \| v \|_{\mathbb{H}} := \left( \int_\Omega |v|^2 dx \right)^{1/2} \), where \( n \) is the unit outward normal to the boundary \( \partial \Omega \). The \( H^1 \) variant of the divergence free Sobolev space is defined as

\[
\mathbb{V} := \{ v \in (H^1(\Omega))^3 : \nabla \cdot v = 0 \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega \}
\]

with norm \( \| v \|_{\mathbb{V}} := \left( \int_\Omega |\nabla v|^2 dx \right)^{1/2} \). We also use the second-order Sobolev space of functions \( \mathbb{H}^2 \). For \( p \in (2, \infty) \), we need the divergence free Lebesgue space

\[
\mathbb{L}^p := \{ v \in (L^p(\Omega))^3 : \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \partial \Omega \}
\]

with usual \( L^p \)-norm \( \| v \|_{\mathbb{L}^p} := \left( \int_\Omega |v|^p dx \right)^{1/p} \). We shall write the standard Lebesgue space as \( \mathbb{L}^p := (L^p(\Omega))^3 \). By the zero Dirichlet boundary conditions, the Poincaré inequality can be employed to show that the semi-norm \( \| \cdot \|_{\mathbb{V}} \) and the standard Sobolev norm of the space \( \mathbb{H}^1(\Omega) \) are equivalent. We denote the inner product in the Hilbert space \( \mathbb{H} \) by \( \langle \cdot, \cdot \rangle \). We use \( \langle \cdot, \cdot \rangle \) to denote the induced duality between the space \( \mathbb{V} \) and its dual \( \mathbb{V}' \) as well as the duality between the space \( \mathbb{L}^p \) and its dual \( \mathbb{L}'^q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). We write \( \| \cdot \|_{\mathbb{V}'} \) for the dual norm in \( \mathbb{V}' \). Since the Hilbert space \( \mathbb{H} \) can be identified with its dual \( \mathbb{H}' \) and \( \mathbb{H} \) is continuously embedded in \( \mathbb{H} \), we have the continuous and dense inclusions, the so-called Gelfand triple such that \( \mathbb{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathbb{V}' \).

We also use the space \( \mathbb{V} \cap \mathbb{L}^p \) endowed with norm \( \| v \|_{\mathbb{V}} + \| v \|_{\mathbb{L}^p} \) for any \( v \in \mathbb{V} \cap \mathbb{L}^p \) and it’s dual space \( \mathbb{V}' + \mathbb{L}'^q \) with the norm

\[
\inf \left\{ \max\{\| v_1 \|_{\mathbb{V}'}, \| v_2 \|_{\mathbb{L}'^q} \} : v = v_1 + v_2, v_1 \in \mathbb{V}', v_2 \in \mathbb{L}'^q \right\}.
\]

From the definition of \( \mathbb{H} \), it is clear that the following continuous embedding holds: \( \mathbb{V} \cap \mathbb{L}^p \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}' + \mathbb{L}'^q \).
2.2. Some classical inequalities. We shall use the following inequalities frequently in the rest of the paper. For the convenience of a reader, we state those inequalities.

Lemma 2.1 (Gagliardo-Nirenberg, [30], Theorem 2.1). Let $\Omega \subset \mathbb{R}^n$ and $v \in W_0^{1,p}(\Omega), p \geq 1$. Then for every fixed numbers $q, r \geq 1$, there exists a constant $C(\Omega, p, q) > 0$ satisfying the inequality $\|v\|_{L^r} \leq C\|v\|_{L^p}^{1-\lambda}\|\nabla v\|_{L^\lambda}^\lambda$, $\lambda \in [0,1]$ where $p, q, r$ and $\lambda$ are related by $\lambda = \left(\frac{1}{q} - \frac{1}{r}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{q}\right)^{-1}$.

We also recall the well known inequalities due to Ladyzhenskaya ([25], Chapter 1, Lemmas 2 and 3) that can also be deduced from Lemma 2.1. When $n = 3, r = 4$ and $p = q = 2$, we get that

$$\|v\|_{L^4} \leq C\|v\|_{L^2}^{1/4}\|\nabla v\|_{L^2}^{3/4}. \quad (2.1)$$

Further, $n = 3, r = 6$ and $p = q = 2$ give $\|v\|_{L^6} \leq C\|\nabla v\|_{L^2}$, for all $v \in V$.

Lemma 2.2 (Agnon, [2], Lemma 13.2). Let $\Omega \subset \mathbb{R}^n$ and $v \in H^{m_2}(\Omega)$. Let us choose $m_1$ and $m_2$ such that $m_1 < \frac{n}{2} < m_2$. Then for any $0 < \lambda < 1$, there exists a constant $C(\Omega, m_1, m_2) > 0$ such that $\|v\|_{L^\infty} \leq C\||v|^{\lambda}_{H^{m_1}}\|_\infty \|v|^{1-\lambda}_{H^{m_2}}$, where the numbers $n, \lambda, m_1$ and $m_2$ satisfy the relation $m_1 = \lambda m_1 + (1 - \lambda)m_2$.

The special case is obtained by taking $n = 3, m_1 = 1$ and $m_2 = 2$ which give rise to $\lambda = \frac{1}{2}$ satisfying the following inequality $\|v\|_{L^\infty} \leq C\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}$, for all $v \in H^2$.

The following interpolation inequality is also useful (see, [14], Appendix B).

Lemma 2.3. Let $p, q, r$ be such that $1 \leq p \leq r \leq q \leq \infty$ and $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. Let $\Omega \subset \mathbb{R}^n$ and $v \in L^p(\Omega) \cap L^q(\Omega)$. Then $v \in L^r(\Omega)$ and it holds that $\|v\|_{L^r(\Omega)} \leq \|v\|_{L^p(\Omega)}^{\theta}\|v\|_{L^q(\Omega)}^{1-\theta}$.

2.3. Properties of linear and nonlinear operators. Let $\mathbb{P}_p : L^p \to L^p, p \in [2, \infty)$ be the Helmholtz-Hodge projection operator ([24]). For simplicity, we denote $\mathbb{P}_p$ by $\mathbb{P}$. The case $p = 2$ corresponds to orthogonal projection. We define the Stokes operator $\mathbb{A} : \mathbb{V} \cap \mathbb{H}^2 \to \mathbb{H}$, $\mathbb{A}v := -\mathbb{P}\nabla v$. For any $v \in \mathbb{V}$, one can get the relation $\langle \mathbb{A}v, v \rangle = \|v\|_{\mathbb{V}}^2$ and $\|\mathbb{A}v\|_{\mathbb{V}} \leq \|v\|_{\mathbb{V}}$. From the Gelfand triple, one may also note that $\mathbb{A}$ maps from $\mathbb{V}$ into $\mathbb{V}'$.

We define a bilinear operator $\mathbb{B} : \mathcal{D}_\mathbb{B} \subset \mathbb{H} \times \mathbb{V} \to \mathbb{H}$, $\mathbb{B}(u, v) := \mathbb{P}(u \cdot \nabla)v$. We shall write $\mathbb{B}(u) = \mathbb{B}(u, u)$. Let us define a trilinear form $\mathbf{b}(\cdot, \cdot, \cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ as follows:

$$\mathbf{b}(u, v, w) = \sum_{i,j=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx = \int_\Omega \left( u(x) \cdot \nabla \right) v(x) \cdot w(x) dx.$$

Integrating by parts with respect to space and using the divergence free condition $\nabla \cdot u = 0$, we obtain the following useful observations

$$\mathbf{b}(u, v, w) = -\mathbf{b}(u, w, v), \quad \text{and so} \quad \mathbf{b}(u, v, v) = 0, \quad \forall u, v, w \in \mathbb{V}. \quad (2.2)$$

Using Hölder’s inequality and the continuous Sobolev embedding $\mathbb{V} \hookrightarrow L^q, 2 \leq q \leq 6$, we have

$$|\mathbf{b}(u, v, w)| \leq \|u\|_{L^1} \|\nabla v\|_\mathbb{H} \|w\|_{L^4} \leq C\|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}} \|w\|_{\mathbb{V}}, \quad \forall u, v, w \in \mathbb{V}. \quad (2.3)$$
It clearly leads to the estimate
\[ \|B(u)\|_{L^2} \leq C\|u\|_{L^2}^2, \quad \forall u \in V. \] (2.4)

By appealing to Hölder’s inequality and interpolation inequality (Lemma 2.3) with \( 2 \leq 2^{r+1} - r+1 \leq r+1 \), for any \( r \geq 3 \), we further obtain
\[ |b(u, v, w)| = |b(u, v, u)| \leq \|u\|_{L^{r+1}}^r \|\nabla v\|_2 \|u\|_{L^{2(r+1)}} \]
\[ \leq \|u\|_{L^{r+1}}^r \|u\|_{L^2} \|v\|_V, \quad \forall u, v \in L^{r+1} \cap V. \]

It clearly leads to the estimate
\[ \|B(u)\|_{L^{r+1}} \leq \|u\|_{L^{r+1}}^r \|u\|_{L^2}^{r+1}, \quad \forall u \in L^{r+1} \cap V. \] (2.5)

Also, note that \( L^{r+1} \cap V = V \) for \( r \in [1, 5] \). From (2.4)-(2.5), we can see that \( B(\cdot) \) maps from \( L^{r+1} \cap V \) into \( V' + L^{r+1} \). If \( u \) is regular, by applying Hölder’s inequality, the estimate (2.3) can be improved as follows (see, [38], Section 3, Lemma 3.8)
\[ |b(u, v, w)| \leq C\|u\|_V \|v\|_{L^2}^{1/2} \|w\|_{L^2}^{1/2}, \quad \forall u, v \in V \cap H^2, \quad w \in H, \] (2.6)
whence
\[ \|B(u)\|_H \leq C\|u\|_V^{3/2} \|u\|_{L^2}^{1/2}, \quad \forall u \in V \cap H^2. \] (2.7)

Finally, we notice that the nonlinear damping operator \( D(u) := P(|u|^{-1}u) \), \( r \geq 1 \) maps from \( L^{r+1} \) into \( L^{r+1} \). By applying Hölder’s inequality, we obtain the following estimate:
\[ \|D(u, v)\|_{L^{r+1}} \leq \|u\|_{L^{r+1}}^r \|v\|_{L^{r+1}}, \quad \forall u, v \in L^{r+1}, \] (2.8)
whence \( \|D(u)\|_{L^{r+1}} \leq \|u\|_{L^{r+1}}^r \) for all \( u \in L^{r+1} \). Further, if \( u \in H^2 \), using the embedding \( H^2 \hookrightarrow L^\infty \), we also have
\[ \|D(u, v)\|_H \leq C\|u\|_{L^\infty} \|v\|_H \leq C\|u\|_{H^2} \|v\|_H, \quad \forall v \in H, \] (2.9)
so that, \( \|D(u)\|_H \leq C\|u\|_{H^2} \).

### 3. Well-posedness of the control problem

In this section, we discuss the existence and uniqueness of weak and strong solutions for the control problem (OCP). The generic constants \( C, C_1, C_2, \ldots \), which may depend on system parameters \( \mu, \nu, \alpha, \beta, r \) are used without writing the dependence explicitly. We use the following assumption on the control function.

**Assumption 1.** \( U \) is a non-empty open bounded subset of the space \( L^2(0, T; H) \) containing the admissible controls \( U_{ad} \) and there exists a constant \( R > 0 \) such that \( \|U\|_{L^2(0, T; H)} < R, \quad \forall U \in U \).

**Definition 3.1** (Weak Solution). Let \( 0 < T < +\infty, u_0 \in V \) and the control \( U \in U \) be given. A function \( u \) is called a weak solution of (1.1) on the interval \( [0, T] \) if the following hold:

(i) For any \( r \geq 1, \ u \in C([0, T]; V) \cap L^{\infty}(0, T; V) \cap L^{r+1}(0, T; L^{r+1}) \)
(ii) \( u(t) + \mu(\nabla u_t, \nabla v) + \nu(\nabla u, \nabla v) + b(u, u, v) + \alpha(u, v) + \beta(|u|^{-1}u, v) = (U, v) \)
for all \( v \in V \cap L^{r+1}, \ a.e. \ t \in (0, T) \)
(iii) \( u(0) = u_0 \) in \( V \).
Remark 1. From the estimate (2.8) it is clear that \( D(u) \in \mathbb{L}^{r+1} \), and so we need to take the test function \( v \in \mathbb{L}^{r+1} \) to make sense of the duality pairing \( \langle |u|^{r-1} u, v \rangle \). By the embedding \( \mathbb{V} \hookrightarrow \mathbb{L}^{r+1} \) for \( r \leq 5 \), we have \( \mathbb{V} \cap \mathbb{L}^{r+1} = \mathbb{V} \). If we have \( 1 \leq r \leq 5 \), it is enough to have \( v \in \mathbb{V} \) in Definition 3.1-(ii). Moreover, in the weak form, the pressure term of (1.1) disappears due to the fact that \( \int_\Omega \nabla p \cdot v dx = - \int_\Omega p \nabla \cdot v dx = 0, \forall v \in \mathbb{V} \).

The formulation of the weak and strong solutions can also be written in terms of the operators discussed in Section 2.3. Applying the projection operator \( \mathbb{P} \) to the equation (1.1) and recalling that the bilinear operator \( B(\cdot) \) maps from \( \mathbb{V} \cap \mathbb{L}^{r+1} \) into \( \mathbb{V}' + \mathbb{L}^{r+1} \) and the damping operator \( D(\cdot) \) maps from \( \mathbb{L}^{r+1} \) into \( \mathbb{L}^{r+1} \), the weak form can be understood as follows:

\[
\begin{aligned}
&\frac{d}{dt}[u(t) + \mu A(u(t)) + \nu A(u(t)) + \alpha u(t) + \beta D(u(t))] \\
&= U(t) \text{ in } \mathbb{V}' + \mathbb{L}^{r+1}, \text{ a.e. } t \in (0, T), \\
&u(0) = u_0 \text{ in } \mathbb{V},
\end{aligned}
\]

where we have written that \( U = \mathbb{P} U \). Indeed, let \( u \in \mathbb{L}^\infty(0, T; \mathbb{V}) \cap \mathbb{L}^{r+1}(0, T; \mathbb{L}^{r+1}) \). From (2.4) and (2.8), it is immediate that

\[
\begin{aligned}
\|B(u)\|_{L^2(0,T;\mathbb{V})}^2 &\leq CT\|u\|_{L^\infty(0,T;\mathbb{V})}^4, \\
\|D(u)\|_{L^\infty(0,T;\mathbb{L}^{r+1})}^2 &\leq \|u\|_{L^{r+1}(0,T;\mathbb{L}^{r+1})}^4,
\end{aligned}
\]

whence the equality (3.1) holds true in \( \mathbb{V}' + \mathbb{L}^{r+1} \), a.e. \( t \in (0, T) \) (see, [6]). Moreover, when \( u \in \mathbb{L}^\infty(0, T; \mathbb{H}^2), u_t \in \mathbb{L}^2(0, T; \mathbb{H}^2) \) making use of (2.7) and (2.9), we have

\[
\begin{aligned}
\|B(u)\|_{L^2(0,T;\mathbb{H})}^2 &\leq CT\|u\|_{L^\infty(0,T;\mathbb{V})}^2\|u\|_{L^\infty(0,T;\mathbb{H}^2)}, \\
\|D(u)\|_{L^2(0,T;\mathbb{H})}^2 &\leq CT\|u\|_{L^\infty(0,T;\mathbb{H}^2)}^2.
\end{aligned}
\]

Hence, equation (3.1) holds true in \( \mathbb{H} \), a.e. \( t \in (0, T) \), and \( u(0) = u_0 \in \mathbb{H} \cap \mathbb{V} \) which will lead to the strong solution of (1.1). All these arguments are justified below.

Theorem 3.2 (Existence and Uniqueness of Weak Solution). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \). Let \( u_0 \in \mathbb{V} \) and \( U \in \mathcal{U} \) be arbitrary. Then for any \( r \in [1, \infty) \) and \( T > 0 \), there exists a unique weak solution \( u(\cdot) \) in the sense of Definition 3.1 for (1.1). Further, there exists a constant \( C_1 > 0 \) depending on system parameters, \( \Omega, C_P, R \) and \( \|u_0\|_\mathbb{V} \) such that

\[
\sup_{t \in (0,T]} (\|u(t)\|_{\mathbb{H}}^2 + \mu \|u(t)\|_{\mathbb{V}}^2) + \nu \int_0^T \|u(t)\|_{\mathbb{V}}^2 dt + \beta \int_0^T \|u(t)\|_{L^{r+1}}^2 dt \leq C_1.
\]

Proof. The existence and uniqueness of weak solutions can be proven by using the Faedo-Galerkin approximation and the arguments similar to [6]. By taking the test function \( u = v \) in the weak form of Definition 3.1-(ii) and using the fact that the trilinear form satisfies \( b(u, u, u) = 0 \), we get the energy identity,

\[
\frac{1}{2} \frac{d}{dt} \left[ \|u(t)\|_{\mathbb{H}}^2 + \mu \|u(t)\|_{\mathbb{V}}^2 \right] + \nu \|u(t)\|_{\mathbb{V}}^2 + \alpha \|u(t)\|_{\mathbb{H}}^2 + \beta \|u(t)\|_{L^{r+1}}^2 = (U, u).
\]
Since, \(|(U, u)| \leq \frac{1}{\lambda_0} \|U(t)\|^2_{\mathbb{H}^2} + \frac{\alpha}{2} \|u(t)\|^2_{\mathbb{V}}\), integrating (3.5) over \((0, t)\) for any \(t \in (0, T)\) and using the Poincaré inequality for the data \(\|u_0\|_{\mathbb{H}} \leq C_P \|u_0\|_{\mathbb{V}}\), one can get
\[
\|u(t)\|^2_{\mathbb{H}^2} + \mu \|u(t)\|^2_{\mathbb{V}} + \int_0^t \left(\alpha \|u(s)\|^2_{\mathbb{H}^2} + \nu \|u(s)\|^2_{\mathbb{V}}\right) \, ds + \beta \int_0^t \|u(s)\|_{L^{r+1}}^{r+1} \, ds \\
\leq C(C_P) \left(\|u_0\|^2_{\mathbb{V}} + \|U\|^2_{L^2(0,T;\mathbb{H})}\right), \quad \forall t \in (0, T], \ r \in [1, \infty).
\]

This leads to the estimate (3.4) and completes the proof. \(\square\)

3.1. Strong solutions in periodic domain. In this section, we move on to the proof of the \((\mathbb{H}^2)\) regularity of solutions of (1.1). It seems not easy to obtain this result for all \(r \in [1, \infty)\) in the case of bounded domains. More precisely, as usual, when we test (1.1) with \(\Delta u\), we get one of the terms as \((\nabla p, \Delta u)\). This term may not vanish since by integration by parts, we get a non-zero boundary term as \(\Delta u|_{\partial \Omega} \neq 0\). It appears that the abstract form (3.1) is also not useful in this case. When we multiply (3.1) by \(Au\), we need to handle the integral \((D(u), Au)\). Since the damping term grows arbitrarily, the projection \(P\) in \(D(u)\) is not useful ([22]), and \(P\) also doesn’t commute with differential operators, like \(\Delta u\) (in bounded domain) (see, [21]). Therefore, we prove the existence and uniqueness of a strong solution of (1.1) in the periodic domain.

Unless it is specified, hereafter, the domain \(\Omega\) in (1.1) is changed as \(\Omega = [0, L] \times [0, L] \times [0, L]\). We use the same notation for the function spaces introduced in Section 2.1 to define on space-periodic domain \(\Omega\) with appropriate modifications. Let \(\mathbb{H}, \mathbb{V}\) and \(\mathbb{L}\), etc., be the spaces of functions, respectively, in \(\mathbb{H}_{loc}(\mathbb{R}^3), \mathbb{V}_{loc}(\mathbb{R}^3)\) and \(\mathbb{L}_{loc}(\mathbb{R}^3)\), which are \(\Omega\)-periodic, that is, \(u(x + Le) = u(x), x \in \mathbb{R}^3, i = 1, 2, 3\), and divergence free \((\nabla \cdot u = 0)\), where \(\{e_1, e_2, e_3\}\) is the canonical basis of \(\mathbb{R}^3\) and \(L\) is the fixed period in all three directions. We also endow these function spaces with the vanishing space average condition \(\int_\Omega u(x) \, dx = 0\) so that the Poincaré inequality holds (see, [17], Chapter II, Section 5). For further details on the function spaces in periodic domains, one may refer to [17, 39]. We make use of the other notations and inequalities given in Section 2.1 with required modification. It is also evident that the existence and uniqueness of weak solutions of (1.1) given by Theorem 3.2 also holds true for periodic domain.

**Definition 3.3** (Strong Solution). Let \(0 < T < +\infty, u_0 \in \mathbb{H}^2 \cap \mathbb{V}\) and \(U \in \mathcal{U}\) be arbitrary. For any \(r \geq 1\), a function \(u\) is called a strong solution of the system (1.1), if it is a weak solution of (1.1) and \(u \in H^1(0, T; \mathbb{H}^2) \cap L^\infty(0, T; \mathbb{L}^{r+1})\).

Since \(u \in H^1(0, T; \mathbb{H}^2)\), we can conclude that \(u \in C([0, T]; \mathbb{H}^2)\) (see, [14], Section 5.9.2, Theorem 2). Next, we prove a priori estimates required to get the strong solutions.

**Proposition 1.** Let \(\Omega\) be a periodic domain in \(\mathbb{R}^3\). Let \(u_0 \in \mathbb{H}^2 \cap \mathbb{V}\) and \(U \in \mathcal{U}\) be arbitrary. Then for any smooth solution \((u, p)\) of (1.1), there exists a constant \(C_2 > 0\) depending on
system parameters, \( C_1, \Omega, T, C_P, R \), and \( \| u_0 \|_{H^2} \) such that

\[
\sup_{t \in (0, T]} \left( \| u(t) \|_{H^2}^2 + \frac{\beta}{r + 1} \| u(t) \|_{L^{r+1}_t}^2 \right) + \int_0^T \left( \| u_t(t) \|_{H^2}^2 + \nu \| \Delta u(t) \|_{H^2}^2 \right) dt \\
+ \int_0^T \left( \alpha \| u(t) \|_{V}^2 + \mu \| u_t(t) \|_{V}^2 \right) dt \\
+ \beta \int_0^T \int_{\Omega} |u|^{r-1} |\nabla u|^2 dx dt + \frac{\beta(r - 1)}{2} \int_0^T \int_{\Omega} |u|^{r-3} |\nabla |u|^2|^2 dx dt \leq C_2, \tag{3.6}
\]

for all \( r \in [1, \infty) \). Further, \( \| u_t \|_{L^2(0, T, H^2)}^2 \leq C_3 \), where the constant \( C_3 > 0 \) depends on \( C_2 \).

Proof. Multiplying (1.1) by \( u_t \), using the integral identity

\[
\beta \int_{\Omega} |u|^{r-1} u \cdot u_t dx = \frac{\beta}{r + 1} \frac{d}{dt} \left[ \| u(t) \|_{L^{r+1}_t}^2 \right],
\]

and applying (2.3), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \alpha \| u(t) \|_{H^2}^2 + \nu \| u(t) \|_{V}^2 + \frac{2\beta}{r + 1} \| u(t) \|_{L^{r+1}_t}^2 \right] + \| u_t(t) \|_{H^2}^2 + \mu \| u_t(t) \|_{V}^2 = b(u, u, u_t) + \langle U, u_t \rangle \\
\leq C \| u(t) \|_{V}^4 + \mu \| u_t(t) \|_{V}^2 + \frac{1}{2} \| U(t) \|_{H^2}^2 + \frac{1}{2} \| u_t(t) \|_{H^2}^2, \tag{3.7}
\]

where we used \( \int_{\Omega} \nabla p \cdot u_t dx = -\int_{\Omega} p(\nabla \cdot u)_t dx = 0 \), since \( \nabla \cdot u = 0 \) on \( \Omega \) and \( u \) is \( \Omega \)-periodic function.

Multiplying (1.1) by \( -\Delta u \), integrating over \( \Omega \) and adding with (3.7), one can get

\[
\frac{1}{2} \frac{d}{dt} \left[ \alpha \| u(t) \|_{H^2}^2 + (\nu + 1) \| u(t) \|_{V}^2 + \mu \| \Delta u(t) \|_{H^2}^2 + \frac{2\beta}{r + 1} \| u(t) \|_{L^{r+1}_t}^2 \right] \\
+ \frac{1}{2} \| u_t(t) \|_{H^2}^2 + \mu \| u_t(t) \|_{V}^2 + \frac{3\nu}{4} \| \Delta u(t) \|_{H^2}^2 + \alpha \| u(t) \|_{V}^2 - \beta \int_{\Omega} |u|^{r-1} u \Delta u dx \\
\leq (1/2 + 1/\nu) \| U(t) \|_{H^2}^2 + C \| u(t) \|_{V}^4 + b(u, u, \Delta u), \tag{3.8}
\]

where we employed the fact that \( \int_{\Omega} \nabla p \cdot \Delta u dx = -\int_{\Omega} p \Delta (\nabla \cdot u) dx = 0 \). Further, integration by parts leads to the identity (see, [9] for whole space)

\[
-\beta \int_{\Omega} |u|^{r-1} u \Delta u dx = \beta \int_{\Omega} |u|^{r-1} |\nabla u|^2 dx + \frac{\beta(r - 1)}{4} \int_{\Omega} |u|^{r-3} |\nabla |u|^2|^2 dx. \tag{3.9}
\]

Using Hölder’s inequality and the embedding \( V \hookrightarrow L^4 \), we obtain

\[
|b(u, u, \Delta u)| \leq \| u(t) \|_{L^4} \| \nabla u(t) \|_{L^4} \| \Delta u(t) \|_{L^2} \\
\leq C \| u(t) \|_{V}^2 \| u(t) \|_{H^2} + \frac{\nu}{4} \| \Delta u(t) \|_{H^2}^2. \tag{3.10}
\]
Substituting (3.9)-(3.10) into (3.8) and integrating over \((0, t)\), one can get

\[
C\|u(t)\|_{H^2}^2 + \frac{2\beta}{r+1}\|u(t)\|_{L^{r+1}}^2 + \int_0^t \|u_s(s)\|_{H^2}^2 ds
+\mu \int_0^t \|u_s(s)\|_{L^2}^2 ds + \nu \int_0^t \|\Delta u(s)\|_{H^2}^2 ds + 2\alpha \int_0^t \|u(s)\|_{\mathcal{V}}^2 ds
+2\beta \int_0^t \int_{\Omega} |u|^{r-1} |\nabla u|^2 dx ds + \frac{\beta(r-1)}{2} \int_0^t \int_{\Omega} |u|^{r-3} |\nabla |u|^2|^2 dx ds
\leq C \left(\|u_0\|_{H^2}^2 + \|u_0\|_{L^{r+1}}^2 + \|U\|_{L^2(0,T;H)}^2\right)
+\|u\|_{L^\infty(0,T;\mathcal{V})}^4 + \|u\|_{L^\infty(0,T;\mathcal{V})}^2 \int_0^t \|u(s)\|_{H^2}^2 ds
\right),
\]

for all \(t \in (0, T]\), where we also used the fact that \(\|\Delta u\|_{H^2}\) is a norm on \(H^2 \cap \mathcal{V}\), which is equivalent to the norm induced by \(H^2\) (see, [38], Chapter 3, Lemma 3.7). Applying Gronwall’s inequality by keeping only the first term on the left side, we get

\[
\sup_{t \in (0, T]} \|u(t)\|_{H^2}^2 \leq C \exp \left(CT\|u\|_{L^\infty(0,T;\mathcal{V})}^2\right) \left(\|u_0\|_{H^2}^2 + \|u_0\|_{L^{r+1}}^2 + \|U\|_{L^2(0,T;H)}^2\right).
\]

Using (3.4) in (3.12) and substituting (3.12) back into (3.11), we get the estimate (3.6). Finally, we show that \(u_t \in L^2(0, T; H^2)\). From the equation (1.1), it is immediate that

\[
\int_0^T \|\Delta u(t)\|_{H^2}^2 dt \leq 6 \int_0^T \|U(t)\|_{H^2}^2 dt + C \int_0^T \left(\|u_t(t)\|_{H^2}^2 + \|\Delta u(t)\|_{H^2}^2 + \|u_t(t)\|_{L^2}^2\right) dt
+ C \int_0^T \left(\|(u(t) \cdot \nabla)u(t)\|_{H^2}^2 + \|u(t)|^{r-1} u(t)\|_{L^2}^2\right) dt,
\]

where we used \(\int_\Omega \nabla p \cdot \Delta u dx = -\int_\Omega p \Delta (\nabla u) dx = 0\). The continuous embedding \(H^2 \hookrightarrow L^\infty\) gives that

\[
\int_0^T \left(\|(u(t) \cdot \nabla)u(t)\|_{H^2}^2 + \|u(t)|^{r-1} u(t)\|_{H^2}^2\right) dt
\leq \int_0^T \|u(t)\|_{L^\infty}^2 \|u(t)\|_{\mathcal{V}}^2 dt + C(\Omega) \int_0^T \|u(t)\|_{L^\infty}^{2r} dt
\leq C(\Omega, T) \left(\|u_0\|_{L^\infty(0,T;H^2)}^2 \|u\|_{L^2(0,T;\mathcal{V})}^2 + \|u\|_{L^\infty(0,T;\mathcal{V})}^{2r}\right).
\]

Making use of (3.14) into (3.13) and using (3.6), one can finish the proof. \(\square\)

**Theorem 3.4** (Existence and Uniqueness of Strong Solution). *Let \(\Omega\) be a periodic domain in \(\mathbb{R}^3\). Let \(0 < T < +\infty\), \(u_0 \in H^2 \cap \mathcal{V}\) and \(U \in \mathcal{U}\) be arbitrary. Then there exists a unique strong solution to the system (1.1).*

**Proof.** The existence of strong solution can be proved through the standard Galerkin approximation. The estimates in Proposition 1 is enough to get the relevant weak and strong convergence of the approximated sequence. Nevertheless, a complete convergence argument has been shown in the context of existence of an optimal control of (OCP) in Theorem 4.3.
By Theorem 3.2, the weak solution is unique and hence if the strong solution exists, then it is unique. \hfill \Box

We prove the continuous dependence of strong solutions with respect to initial data and control.

**Theorem 3.5.** Let $\Omega$ be a periodic domain in $\mathbb{R}^3$. Let $u_1, u_2$ be two strong solutions of (1.1) corresponding to the controls $U_1, U_2 \in \mathcal{U}$, initial data $(u_0)_1, (u_0)_2 \in H^2 \cap V$ and pressure $p_1, p_2$ respectively. Then there exists a constant $C_3 > 0$ depending only on system parameters, $\Omega, T, C_P$ and $C_2$ satisfying the estimate

\[
\sup_{t \in [0,T]} \left( \|u_1(t) - u_2(t)\|_{H^2}^2 \right) + \|u_1 - u_2\|_{H^1(0,T;V)}^2 + \|u_1 - u_2\|_{L^{r+1}(0,T;\mathbb{H}^r)}^{r+1} \leq C_3 \left( \|(u_0)_1 - (u_0)_2\|_V^2 + \|U_1 - U_2\|_{L^2(0,T;\mathbb{H}^2)}^2 \right), \quad r \geq 1. \tag{3.15}
\]

**Proof.** Let $u := u_1 - u_2, p := p_1 - p_2, U := U_1 - U_2$ and $u_0 := (u_0)_1 - (u_0)_2$. Then $(u, p, U)$ solves

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \|u(t)\|_{H^2}^2 + \mu \|u(t)\|_{H^2}^2 \right] + \nu \|u(t)\|_{V}^2 + \alpha \|u(t)\|_{H}^2 &+ \beta \int_{\Omega} (f(u_1) - f(u_2)) \cdot u \, dx = (U, u) - b(u, u_1, u) \\
&\leq \frac{1}{2\alpha} \|U(t)\|_{\mathbb{H}^2}^2 + \frac{\alpha}{4} \|u(t)\|_{H}^2 + C \|u_1(t)\|_{V}^2 \|u(t)\|_{V}^2 + \frac{\nu}{2} \|u(t)\|_{V}^2. \tag{3.17}
\end{align*}
\]

By Lemma 2.1 of [21] (also refer to [29]), there exists a constant $C(r) > 0$ such that the following lower bound holds:

\[
\int_{\Omega} (f(u_1) - f(u_2)) \cdot (u_1 - u_2) \, dx \geq C(r) \|u_1(t) - u_2(t)\|_{L^{r+1}}^{r+1}, \text{ for any } r \geq 1. \tag{3.18}
\]

Testing (3.16) with $u_t$ and again applying trilinear estimate (2.3), one may obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \alpha \|u(t)\|_{H}^2 + \nu \|u(t)\|_{V}^2 \right] + \|u_t(t)\|_{H}^2 + \mu \|u_t(t)\|_{V}^2 &+ \beta (f(u_1) - f(u_2), u_t) \\
&\leq \|U(t)\|_{\mathbb{H}^2}^2 + \frac{1}{4} \|u_t(t)\|_{H}^2 + \frac{\mu}{2} \|u_t(t)\|_{V}^2 \\
&+ C \left( \|u_1(t)\|_{V}^2 + \|u_2(t)\|_{V}^2 \right) \|u(t)\|_{V}^2 + \beta (f(u_1) - f(u_2), u_t). \tag{3.19}
\end{align*}
\]

By Taylor’s formula the damping term is written as follows:

\[
\beta \int_{\Omega} (f(u_1) - f(u_2)) \cdot u_t \, dx = \beta \int_{\Omega} \int_{0}^{1} (f'(z)u \cdot u_t) \, d\tau \, dx := D,
\]
where \( z := \tau u_1 + (1 - \tau)u_2 \) and the derivative of \( f(\cdot) \) is given by

\[
f'(z)w = \begin{cases} 
(r - 1)|z|^{r-3}(z \cdot w)z + |z|^{r-1}w, & r \geq 3 \\
(r - 1)|z|^{r-1}z \cdot w + |z|^{r-1}w, & 1 < r < 3, \ z \neq 0 \\
0 & 1 < r < 3, \ z = 0 \\
& r = 1.
\end{cases}
\] (3.20)

For any \( r \geq 3 \), thanks to the embedding \( H^2 \hookrightarrow L^\infty \) and the Poincaré inequality, we have

\[
D = \beta \int_0^1 \left((r - 1)|z|^{r-3}(z \cdot u)(z \cdot u_t) + |z|^{r-1}(u \cdot u_t)\right) d\tau dx 
\leq \beta r \sup_{\tau \in (0,1)} \int_0^1 |\tau u_1 + (1 - \tau)u_2|^{r-1}|u_t| |u_t| d\tau dx 
\leq \beta r 2^{r-2} \left( \|u_1(t)\|^{r-1}_{L^\infty} + \|u_2(t)\|^{r-1}_{L^\infty} \right) \|u(t)\|_{H^1} \|u_t(t)\|_{H^1} 
\leq C(C_P) \left( \|u_1(t)\|_{H^2}^{2(r-1)} + \|u_2(t)\|_{H^2}^{2(r-1)} \right) \|u(t)\|_V^2 + \frac{1}{4} \|u_t(t)\|_{H^1}^2. \] (3.21)

For \( r = 1 \) and any \( 1 < r < 3, \ z \neq 0 \), the estimation similar to (3.21) holds true as well. Substituting (3.18) and (3.21) respectively into (3.17) and (3.19), and adding them together, we get

\[
\frac{d}{dt} \left[ X(t) + Y(t) \right] \leq (2 + 1/\alpha) \|U(t)\|_{H^1}^2 + CZ(\tau) [X(t) + Y(t)], \] (3.22)

where

\[
X(t) := (1 + \alpha)\|u(t)\|_{H^2}^2 + (\mu + \nu)\|u(t)\|_V^2, \\
Y(t) := \int_0^t \left( \alpha \|u(s)\|_{H^1}^2 + \nu \|u(s)\|_V^2 + \|u(s)\|_{H^1}^2 + \mu \|u(s)\|_V^2 \right) 
+ C(r) \|u(s)\|_{H_{r+1}}^{r+1} ds, \\
Z(t) := \left( \|u_1(t)\|_{H^2}^2 + \|u_2(t)\|_{H^2}^2 \right) + \left( \|u_1(t)\|_{H^2}^{2(r-1)} + \|u_2(t)\|_{H^2}^{2(r-1)} \right).
\]

Since by (3.6), \( \|Z\|_{L^\infty(0,T)} \leq C(C_2) \), applying Gronwall’s inequality over \((0,t)\) to (3.22) leads to the estimate

\[
X(t) + Y(t) \leq C \exp \left[ C \|Z\|_{L^\infty(0,T)} T \right] \left( \|u_0\|_{H^2}^2 + \|u_0\|_V^2 + \|U\|_{L^2(0,T;H)}^2 \right), \] (3.23)

for all \( t \in (0,T] \). Further, employing the Poincaré inequality \( \|u_0\|_{H^1} \leq C_P \|u_0\|_V \), we complete the proof of (3.15). \( \square \)

**Remark 2.** Since for any \( 1 \leq r \leq 5 \), the imbedding \( \mathbb{V} \hookrightarrow L^{r+1} \) is continuous, we obtain from (3.7) and (3.4) that

\[
\sup_{t \in [0,T]} \left[ \alpha \|u(t)\|_{H^1}^2 + \nu \|u(t)\|_V^2 + \frac{2\beta}{r+1} \|u(t)\|_{L^{r+1}}^{r+1} \right] + \mu \int_0^T \|u_t(t)\|_V^2 dt 
\leq C \left( \|u_0\|_{\mathbb{V}}^2 + \|U\|^2_{L^2(0,T;\mathbb{H})} + \|u\|^4_{L^\infty(0,T;\mathbb{V})} \right) \leq \tilde{C}_1, \] (3.24)

where the constant \( \tilde{C}_1 > 0 \) depends on \( C_1 \) only. In this case, for any \( u_0 \in \mathbb{V} \), the weak solution \( u \) of (1.1) has the regularity \( u \in H^1(0,T;\mathbb{V}) \).
If $3 \leq r \leq 5$, the estimate (3.21) can be modified by using the embeddings $\mathbb{V} \hookrightarrow L^\frac{6(r+1)}{11-r}$ and $\mathbb{V} \hookrightarrow L^{r+1}$, $r \leq 5$ as follows:

$$\beta \left( \int_0^1 f'(z) ud\tau, u_t \right) \leq \beta \left\| \int_0^1 f'(\tau u_1 + (1 - \tau) u_2) ud\tau \right\|_{L^{5/5}} \| u_t(t) \|_{L^6}$$

$$\leq C \left( \| u_1(t) \|_{L^{r+1}}^{-1} + \| u_2(t) \|_{L^{r+1}}^{-1} \right) \| u(t) \|_{L^{\frac{6(r+1)}{11-r}}} \| u_t(t) \|_{L^6}$$

$$\leq C \left( \| u_1(t) \|_{L^{r+1}}^{-1} + \| u_2(t) \|_{L^{r+1}}^{-1} \right) \| u(t) \|_V \| u_t(t) \|_V$$

$$\leq C \left( \| u_1(t) \|_V^{2(r-1)} + \| u_2(t) \|_V^{2(r-1)} \right) \| u(t) \|_V^2 + \frac{\mu}{4} \| u_t(t) \|_V^2. \quad (3.25)$$

Therefore, estimate (3.23) can be established by invoking (3.24), that is, by the weak solutions of (1.1), since $u_1, u_2 \in L^\infty(0, T; \mathbb{V})$. The estimate (3.25) can also be obtained for $1 \leq r < 3$ as well. Hence, Theorem 3.5 holds true in the case of bounded domain when $1 \leq r \leq 5$ and $u_0 \in \mathbb{V}$.

### 4. Optimal Control Problem with Control Constraints

In this section, we follow the classical methodologies developed for optimal control problems in [18, 26, 36] to prove the existence of optimal control for (OCP). A pair $(u, U)$ is called an admissible pair if it holds that $(u, U) \in \mathbb{Z} \times \mathcal{U}_ad$, $u$ is a strong solution of (1.1) corresponding to $U \in \mathcal{U}_ad$ and $\mathcal{J}(u, U) < +\infty$. We denote by $\mathcal{A}_ad$, the class of all admissible pairs for (OCP). Throughout this section, we assume that $u_d \in L^2(0, T; \mathbb{V})$ and the constraints $U_{min}, U_{max} \in L^\infty(\Omega_T)$ with $U_{min} \leq U_{max}$, a.e. $(x, t) \in \Omega_T$. One can prove the following standard result using the arguments, for instance, in [18, 35].

**Lemma 4.1.**

(i) The admissible class $\mathcal{A}_ad$ is non-empty.

(ii) The functional $\mathcal{J} : \mathbb{Z} \times \mathcal{U}_ad \rightarrow \mathbb{R}^+$ is weakly sequentially lower-semicontinuous, that is, if $u_n \rightharpoonup u$ in $\mathbb{Z}$ and $U_n \rightharpoonup U$ in $\mathcal{U}_ad$, then we have $\mathcal{J}(u, U) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(u_n, U_n)$.

Next, we define the optimal solution to the control problem (OCP) and prove the main theorem.

**Definition 4.2** (Optimal Solution). An admissible pair of solutions $(\tilde{u}, \tilde{U}) \in \mathcal{A}_ad$ is called an optimal pair if it solves (OCP). More precisely, the cost functional $\mathcal{J}(u, U)$ achieves infimum at $(\tilde{u}, \tilde{U})$, that is, $\mathcal{J}(\tilde{u}, \tilde{U}) = \inf_{(u, U) \in \mathcal{A}_ad} \mathcal{J}(u, U)$. The control $\tilde{U}$ is called an optimal control and the corresponding solution $\tilde{u}$ is called an optimal state.

**Theorem 4.3** (Existence of Optimal Solution). Let $\Omega$ be a periodic domain in $\mathbb{R}^3$. Let $u_0 \in H^2 \cap \mathbb{V}$ and $T > 0$ be given. Then there exists an optimal pair $(\tilde{u}, \tilde{U}) \in \mathcal{A}_ad$ solving (OCP) in the sense of Definition 4.2.

**Proof.** In view of Lemma 4.1-(i), the admissible class of solutions $\mathcal{A}_ad$ is non-empty. Since the functional $\mathcal{J}((\cdot, \cdot))$ is bounded from below by zero, there exists a real number $m \geq 0$ such that $m = \inf_{(u, U) \in \mathcal{A}_ad} \mathcal{J}(u, U)$, and there exists a minimizing sequence $\{(u_n, U_n)\} \subset \mathcal{A}_ad$ such that

$$m = \inf_{(u, U) \in \mathcal{A}_ad} \mathcal{J}(u, U) = \lim_{n \rightarrow \infty} \mathcal{J}(u_n, U_n). \quad (4.1)$$
Further, the sequence \( \{u_n\} \) is a strong solution of (1.1) corresponding to the control \( \{U_n\} \). Evidently, we have

\[
\begin{align*}
\frac{d}{dt}[u_n + \mu Au_n] + \nu A u_n + B(u_n) + \alpha u_n + \beta D(u_n) - U_n &= 0 \text{ in } \mathbb{H}, \text{ a.e. } t \in (0, T) \\
u_n(0) &= u_0 \quad \text{in } \mathbb{H}^2 \cap \mathcal{V}.
\end{align*}
\]

(4.2)

Since by Assumption 1, \( \{U_n\} \subset U_{ad} \subset \mathcal{U} \), and by Proposition 1, we obtain the uniform bounds on \( \{(u_n, U_n)\} \):

\[
\|U_n\|_{L^2(0,T;\mathbb{H})} \leq C(R) \quad \text{and} \quad \|u_n\|_Z \leq C(\Omega, T, \|u_0\|_{\mathbb{H}^2}, R),
\]

where \( C > 0 \) is independent of \( n \). It leads to the fact that

\[
\begin{align*}
\{u_n\} \text{ is uniformly bounded in } L^\infty(0,T;\mathbb{H}^2) \cap L^\infty(0,T;L^{r+1}), \\
\{\frac{du_n}{dt}\} \text{ is uniformly bounded in } L^2(0,T;\mathbb{H}^2).
\end{align*}
\]

(4.3)

Further, in view of (4.3) and (3.3), \( D(u_n) \) is uniformly bounded in \( L^2(0,T;\mathbb{H}) \). By Banach-Alaoglu theorem, there exists a subsequence, still denoted as \( \{u_n\} \), such that

\[
\begin{align*}
u_n &\xrightarrow{w^*} \tilde{u} \quad \text{in } L^\infty(0,T;\mathbb{H}^2), \\
u_n &\xrightarrow{w} \bar{u} \quad \text{in } L^2(0,T;\mathbb{H}^2), \\
\frac{du_n}{dt} &\xrightarrow{w} \frac{d\tilde{u}}{dt} \quad \text{in } L^2(0,T;\mathbb{H}^2), \\
U_n &\xrightarrow{w} \bar{U} \quad \text{in } L^2(0,T;\mathbb{H}), \\
D(u_n) &\xrightarrow{w} \xi \quad \text{in } L^2(0,T;\mathbb{H}), \quad \text{as } n \to \infty.
\end{align*}
\]

The terms \( Au_n, \frac{dAu_n}{dt} \) converges weakly, respectively, to \( A\tilde{u}, \frac{d\tilde{u}}{dt} \) in \( L^2(0,T;\mathbb{H}) \).

Since \( \{u_n\} \) is bounded on \( H^1(0,T;\mathbb{H}^2) \), by Aubin-Lion’s compactness theorem (see, [33], Section 6), we have \( u_n \xrightarrow{\ast} \bar{u} \) in \( C([0,T];\mathcal{V}) \). Further, there exists a sub-sequence, still denoted as \( \{u_n\} \) such that the convergence \( u_n \to \bar{u} \) a.e. in \( \Omega \times (0,T) \) holds. Since, \( D(u_n) \) is uniformly bounded in \( L^2(0,T;\mathbb{H}) \), a consequence of the Lebesgue dominated convergence theorem gives

\[
D(u_n) \xrightarrow{w} D(\bar{u}) \quad \text{in } L^2(0,T;\mathbb{H}) \quad \text{as } n \to \infty
\]

and, by uniqueness of the limit, we have \( \xi = D(\bar{u}) \). Finally, we prove the weak convergence of \( B(u_n) \). Let us consider a test function \( v \in L^2(0,T;\mathbb{H}) \). By using the bilinearity of \( B(\cdot) \) and invoking the estimate (2.6), we have

\[
\begin{align*}
&\int_0^T (B(u_n) - B(\bar{u}), v) \, dt \\
= &\int_0^T (B(u_n, u_n - \bar{u}) + B(u_n - \bar{u}, \bar{u}), v) \, dt \\
= &\int_0^T (b(u_n, u_n - \bar{u}, v) + b(u_n - \bar{u}, \bar{u}, v)) \, dt \\
\leq & \ C \int_0^T \left( \|u_n\|_{\mathcal{V}} \|u_n - \bar{u}\|_{L^2}^{1/2} \|u_n - \bar{u}\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{H}} + \|u_n - \bar{u}\|_{\mathcal{V}} \|\bar{u}\|_{L^2}^{1/2} \|\bar{u}\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{H}} \right) \, dt \\
\leq & \ C \sqrt{T} \left( \|u_n\|_{L^\infty(0,T;\mathcal{V})} \|u_n - \bar{u}\|_{L^\infty(0,T;\mathbb{H}^2)} + \|\bar{u}\|_{L^\infty(0,T;\mathcal{V})} \|u_n - \bar{u}\|_{L^\infty(0,T;\mathcal{V})} \right) \|u_n - \bar{u}\|_{L^\infty(0,T;\mathcal{V})} \|v\|_{L^2(0,T;\mathbb{H})} \to 0, \quad \text{as } n \to \infty,
\end{align*}
\]
since \(|u_n - \tilde{u}|_{L^\infty(0,T;\mathbb{H}^2)}\) is bounded due to the embedding \(H^1(0,T;\mathbb{H}^2) \hookrightarrow L^\infty(0,T;\mathbb{H}^2)\). It holds that \(B(u_n) \rightharpoonup B(\tilde{u})\) in \(L^2(0,T;\mathbb{H})\). By passing the weak limits for the respective terms in (4.2), we conclude that (3.1) holds true in \(\mathbb{H}\), a.e. \(t \in (0,T)\).

The facts that \(\tilde{u}, \frac{\partial \tilde{u}}{\partial t} \in L^2(0,T;\mathbb{H}^2)\) would again imply that \(\tilde{u} \in H^1(0,T;\mathbb{H}^2)\), and hence \(\tilde{u} \in C([0,T];\mathbb{H}^2)\). Consequently, one can verify the initial condition \(\tilde{u}(0) = u_0\) in \(\mathbb{H}^2 \cap \mathbb{V}\). Thus, the function \(\tilde{u} \in H^1(0,T;\mathbb{H}^2) \cap L^\infty(0,T;\mathbb{L}^{r+1})\) is a unique strong solution of (1.1) with control \(\tilde{U} \in U_{ad}\), since \(U_{ad}\) is a closed and convex set in \(L^2(0,T;\mathbb{H})\).

Finally, we show that \((\tilde{u}, \tilde{U})\) is an optimal pair solving (OCP). Appealing to the lower-semicontinuity of Lemma 4.1-(ii), we have

\[
\mathcal{J}(\tilde{u}, \tilde{U}) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, U_n),
\]

whence \((\tilde{u}, \tilde{U}) \in A_{ad}\). Since \(m\) is the infimum of \(\mathcal{J}(\cdot, \cdot)\) and \((\tilde{u}, \tilde{U})\) is any admissible pair, we get \(m \leq \mathcal{J}(\tilde{u}, \tilde{U})\). But \(\{(u_n, U_n)\}\) is a minimizing sequence, we thus obtain from (4.1) and (4.4) that

\[
m = \inf_{(u, U) \in A_{ad}} \mathcal{J}(u, U) \leq \mathcal{J}(\tilde{u}, \tilde{U}) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, U_n) = \lim_{n \to \infty} \mathcal{J}(u_n, U_n) = m.
\]

Consequently, we have \(\mathcal{J}(\tilde{u}, \tilde{U}) = \inf_{(u, U) \in A_{ad}} \mathcal{J}(u, U)\), and hence \((\tilde{u}, \tilde{U}) \in A_{ad}\) is an optimal solution. This completes the proof. \(\square\)

**Remark 3.** The existence of solution for (OCP) can be proven by using the weak solution of (1.1). In particular, for any \(1 \leq r \leq 5\), Theorem 4.3 holds true for bounded domain with Dirichlet boundary conditions. For any \(u_0 \in \mathbb{V}\), in view of Remark 2, we consider the solution space \(\tilde{\mathbb{Z}} := H^1(0,T;\mathbb{V})\). Let \(\tilde{A}_{ad}\) denote the class of all admissible pairs \((u, U) \in \tilde{\mathbb{Z}} \times \tilde{U}_{ad}\), where \(u\) is the weak solution of (1.1) corresponding to \(U \in \tilde{U}_{ad}\). Then by the similar arguments of the previous theorem, there exists a sequence \(\{(u_n, U_n)\} \subset \tilde{A}_{ad}\), which is a weak solution of (1.1), that is,

\[
\begin{aligned}
&\frac{4}{\nu}[u_n + \mu A u_n] + \nu A u_n + B(u_n) + \alpha u_n + \beta D(u_n) - U_n = 0 \text{ in } \mathbb{V}', \text{ a.e. } t \in (0,T) \\
u_n(0) &= u_0 \text{ in } \mathbb{V}.
\end{aligned}
\]

It is evident that \(\{u_n\}\) is uniformly bounded in \(L^\infty(0,T;\mathbb{V})\), and hence (2.8) shows that \(\{D(u_n)\}\) is uniformly bounded in \(L^2(0,T;\mathbb{V}')\). Further, \(\{\frac{du_n}{dt}\}\) is uniformly bounded in \(L^2(0,T;\mathbb{V})\). Consequently, there exists a subsequence, still denoted as \(\{u_n\}\), such that

\[
\begin{aligned}
U_n &\rightharpoonup \tilde{U} \text{ in } L^2(0,T;\mathbb{H}), \\
u_n &\rightharpoonup \tilde{u} \text{ in } L^\infty(0,T;\mathbb{V}) \\
u_n &\rightharpoonup \tilde{u} \text{ in } L^2(0,T;\mathbb{V}) \\
\frac{du_n}{dt} &\rightharpoonup \frac{d\tilde{u}}{dt} \text{ in } L^2(0,T;\mathbb{V}), \text{ as } n \to \infty,
\end{aligned}
\]

and by Aubin-Lion’s compactness theorem, \(u_n \rightharpoonup \tilde{u}\) in \(L^2(0,T;\mathbb{H})\). Further, \(A u_n, \frac{du_n}{dt}\) converges weakly, respectively, to \(A \tilde{u}, \frac{d\tilde{u}}{dt}\) in \(L^2(0,T;\mathbb{V}')\), and by the arguments of the above theorem \(D(u_n) \rightharpoonup D(\tilde{u})\) in \(L^2(0,T;\mathbb{V}')\), as \(n \to \infty\).
Finally, we prove the weak convergence of $B(u_n)$. For any $v \in L^2(0, T; \mathbb{V})$, by applying Hölder’s inequality, the continuous embedding $\mathbb{V} \hookrightarrow L^4$ and (2.1), we have

$$
\int_0^T \langle B(u_n) - B(\tilde{u}), v \rangle dt
= \int_0^T (-b(u_n, v, u_n - \tilde{u}) + b(u_n - \tilde{u}, \tilde{u}, v))dt
\leq \int_0^T (\|u_n\|_{L^4} \|\nabla v\|_{L^2} \|u_n - \tilde{u}\|_{L^4} + \|u_n - \tilde{u}\|_{L^4} \|\nabla \tilde{u}\|_{L^2} \|v\|_{L^4}) dt
\leq C (\|u_n\|_{L^8(0, T; \mathbb{V})} + \|\tilde{u}\|_{L^8(0, T; \mathbb{V})}) \|u_n - \tilde{u}\|_{L^3(0, T; \mathbb{V})}^{3/4} \times \|u_n - \tilde{u}\|_{L^2(0, T; \mathbb{V})}^{1/4} \|v\|_{L^2(0, T; \mathbb{V})} \to 0, \text{ as } n \to \infty,
$$

where we used the facts that $u_n, \tilde{u} \in L^\infty(0, T; \mathbb{V})$ and $\|u_n - \tilde{u}\|_{L^2(0, T; \mathbb{V})} \to 0$ as $n \to \infty$. Hence $B(u_n) \stackrel{w}{\to} B(\tilde{u})$ in $L^2(0, T; \mathbb{V}')$.

Since $\tilde{u} \in C([0, T]; \mathbb{V})$, we can conclude that (3.1) holds true in $\mathbb{V}'$, a.e. $t \in (0, T)$, that is, $\tilde{u} \in H^1(0, T; \mathbb{V})$ is a unique weak solution of (1.1) with control $\tilde{U} \in \mathcal{U}_{ad}$. Further, by the previous arguments, $(\tilde{u}, \tilde{U})$ is an optimal pair solving (OCP).

However, when we prove first and second-order optimality conditions, we require a more regular solution of (1.1) to obtain an \textit{a priori} estimates for linearized and adjoint systems of (1.1). It forces us to start with a strong solution as our admissible class $\mathcal{A}_{ad}$, which we proved for periodic domain (Theorem 3.4).

5. First-order necessary optimality conditions

The main result of this section is the derivation of first-order necessary conditions satisfied by any optimal pair of the (OCP) obtained in Section 4, when the growth factor of the damping term $r \geq 2$. However, the results are clearly true for $r = 1$ as well. The optimality conditions provide a way to characterize an optimal control in terms of solutions of an adjoint system associated with the (OCP). In this context, the Fréchet derivative of the cost functional requires the solvability of a linearized system of (1.1). Thus, we first study the well-posedness of the linearized system of (1.1).

5.1. Control-to-state operator. For a given class of controls $U \in \mathcal{U}$, let $u_U$ denote the induced unique strong solution of (1.1) given by Theorem 3.4. This enables us to define a control-to-state operator $S : \mathcal{U} \to \mathcal{Z}_1$, $U \mapsto S(U) := u_U$, where $\mathcal{Z}_1 := H^1(0, T; \mathbb{V})$. More precisely, we have $S(U) \subset \mathcal{Z} \subset \mathcal{Z}_1$, since $H^2 \subset \mathbb{V}$ and $L^{r+1} \subset H$, for any $r > 1$, where $\mathcal{Z}$ is the strong solution space defined in introduction. We rewrite (OCP) by using the control-to-state operator $S$ as follows:

\begin{equation}
\text{(MOCP)} \left\{ \begin{array}{l}
\text{minimize } \mathcal{J}(U) \\
\text{subject to the control constraint } U \in \mathcal{U}_{ad}
\end{array} \right.
\end{equation}

where $\mathcal{J}(\cdot)$ is the reduced cost functional defined by $\mathcal{J}(U) := \mathcal{J}(S(U), U) = \mathcal{J}(u_U, U)$. The Fréchet derivative of the reduced cost functional with respect to the control requires that of the control-to-state-operator $S$, which in turn demands the Lipschitz continuity of this operator.
Lemma 5.1 (Lipschitz Continuity of S). The control-to-state map \( S : U \to Z_1 \) is Lipschitz continuous, i.e., there exists a constant \( K_1 > 0 \) depending on system parameters, \( \Omega, T, C_p, R \) and \( \| u_0 \|_{\mathbb{H}^2} \) such that
\[
\| S(U_1) - S(U_2) \|_{Z_1} \leq K_1 \| U_1 - U_2 \|_{L^2(0,T;H)}, \quad \forall U_1, U_2 \in U. \tag{5.1}
\]

Proof. The proof is a direct consequence of Theorem 3.5. Indeed, let \( u_{r1}, u_{r2} \) be two strong solutions of (1.1) corresponding to the controls \( U_1, U_2 \in U \), same initial data \( u_0 \in \mathbb{H}^2 \cap V \) and pressure \( p_1, p_2 \) respectively. The stability estimate (3.15) gives the required estimate (5.1). \( \square \)

Remark 4. In the case of bounded domain and \( 1 \leq r \leq 5 \), by Theorem 3.5 and Remark 2, the control-to-state-operator \( S \) is Lipschitz continuous with constant \( K_1 \) depending on the constant \( C_1 \) of the energy estimate (3.4).

The Fréchet derivative of the control-to-state operator \( S \) on \( U \) is a linear approximation of this operator at some point in \( U \), which will be given by a solution of a linearized version of (1.1). Let \( \tilde{u}_{r} \) be the unique strong solution of the system (1.1) corresponding to the data \( u_0 \in \mathbb{H}^2 \cap V \) and the control \( \tilde{U} \in U \). The linearized equation of (1.1) at some point \( \tilde{U} \in U \) with a function \( V \in L^2(0,T;\mathbb{H}) \) is given by
\[
\text{(L-NSVD)} \begin{cases}
\mathcal{L}_{\tilde{U}}w + \nabla q = V & \text{in } \Omega_T \\
\nabla \cdot w = 0 & \text{in } \Omega_T, \quad w(x,0) = 0 \text{ in } \Omega,
\end{cases} \tag{5.2}
\]
where \( q \) stands for the linearized pressure, the linear operator
\[
\mathcal{L}_{\tilde{U}}w := w_t - \mu \Delta w_t - \nu \Delta w + (w \cdot \nabla)u_{\tilde{U}} + (u_{\tilde{U}} \cdot \nabla)w + \alpha w + \beta f'(u_{\tilde{U}})w \tag{5.3}
\]
and \( f'() \) of the function \( f(u) = |u|^{r-1}u \) is given in (3.20).

Now, we prove the well-posedness of this linearized system. The existence and uniqueness can be completed by using the Faedo-Galerkin approximation technique once we obtain the required an \textit{a priori} estimates. Testing (5.2) by \( w_t \), applying Young’s inequality and (2.3), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \alpha \| w(t) \|_{H^2}^2 + \nu \| w(t) \|_{V^*}^2 \right] + \| w_t(t) \|_{H^2}^2 + \mu \| w_t(t) \|_V^2 \\
\leq (V, w_t) - b(w, u_{\tilde{U}}, w_t) - b(u_{\tilde{U}}, w, w_t) - \beta (f'(u_{\tilde{U}})w, w_t) \\
\leq \frac{1}{4 \delta_1} \| V(t) \|_{H^2}^2 + \delta_1 \| w_t(t) \|_{H^2}^2 + C(\delta_2) \| u_{\tilde{U}}(t) \|_V^2 \| w(t) \|_V^2 \\
+ \delta_2 \| w_t(t) \|_V^2 - \beta (f'(u_{\tilde{U}})w, w_t), \tag{5.4}
\]
for some \( \delta_1, \delta_2 > 0 \). Using (3.20) for any \( r \geq 3 \), the continuous embedding \( H^2 \hookrightarrow L^\infty \) gives
\[
\beta \int_{\Omega} (f'(u_{\tilde{U}})w) \cdot w_t dx \\
= \beta \int_{\Omega} \left( (r - 1) \| u_{\tilde{U}} \|_{H^2}^{r-3}(u_{\tilde{U}} \cdot w)(u_{\tilde{U}} \cdot w_t) + |u_{\tilde{U}}|^{r-1}(w \cdot w_t) \right) dx \\
\leq C \int_{\Omega} |u_{\tilde{U}}|^{r-1}|w||w_t|dx \leq C \| u_{\tilde{U}}(t) \|_{H^2}^{r-1} \| w(t) \|_{H^2} \| w_t(t) \|_{H^2} \\
\leq C(\Omega, C_p, \delta_1) \| u_{\tilde{U}}(t) \|_{H^2}^{2(r-1)} \| w(t) \|_V^2 + \delta_1 \| w_t(t) \|_V^2, \tag{5.5}
\]
where we also employed the Poincaré inequality in the last inequality. Moreover, for any $2 \leq r < 3, |u_\delta| \neq 0$, the estimation (5.5) holds. Choosing $\delta_1 = 1/4, \delta_2 = \mu/2$, substituting (5.5) into (5.4), and applying Gronwall’s inequality gives

$$
\Psi(t, w) \leq \|V\|^2_{L^2(0,T;\mathbb{H})} \times \exp \left( C(\Omega, C_P) T \left( \|u_\delta\|^2_{H(0,T;\mathbb{V})} + \|u_\delta\|^{2(r-1)}_{H(0,T;\mathbb{H}^2)} \right) \right) < +\infty, \quad (5.6)
$$

for all $t \in (0, T]$, since by Theorem 3.4, $u_\delta \in L^\infty(0, T; \mathbb{H}^2)$, where

$$
\Psi(t, w) := \alpha \|w(t)\|_{L^2}^2 + \nu \|w(t)\|_{H^1}^2 + \|w_s\|_{L^2(0, t; \mathbb{H})}^2 + \mu \|w_s\|_{L^2(0, t; \mathbb{V})}^2.
$$

From (5.6), we infer that $w \in L^2(0, T; \mathbb{V}), w_t \in L^2(0, T; \mathbb{V})$, whence $w \in H^1(0, T; \mathbb{V})$. Since the embedding $H^1(0, T; \mathbb{V}) \hookrightarrow C([0, T]; \mathbb{V})$ is continuous, we can verify the initial condition. The uniqueness of the linear system (5.2) easily follows from (5.6). Thus, we have proved:

**Theorem 5.2 (Weak Solutions of Linearized System).** Let $\tilde{U} \in \mathcal{U}$ be any control and $u_\delta$ be the corresponding unique strong solution of (1.1). Then for any $V \in L^2(0, T; \mathbb{H})$, there exists a unique weak solution of the linearized system (5.2) such that $w \in L^\infty(0, T; \mathbb{V}), w_t \in L^2(0, T; \mathbb{V})$, and as a consequence $w \in C([0, T]; \mathbb{V})$.

Now, we prove the Fréchet differentiability and Lipschitz continuity of the Fréchet derivative of the control-to-state operator $S$ on the open subset $\mathcal{U}$. It is worth noting that these two results are proved for any open subset $\mathcal{U}$ of $L^2(0, T; \mathbb{H})$ rather than $\mathcal{U}_{ad}$ itself since the Fréchet derivative is merely defined for open subsets of $L^2(0, T; \mathbb{H})$.

**Proposition 2.** For any $\tilde{U} \in \mathcal{U}$, let $u_\delta$ be the unique strong solution of (1.1). Then the following two conclusions hold:

(i) The control-to-state mapping $S$ is Fréchet differentiable on $\mathcal{U}$, that is, for any $\tilde{U} \in \mathcal{U}$, there exists a bounded linear operator $S'(\tilde{U}) : L^2(0, T; \mathbb{H}) \to \mathbb{Z}_1$ such that

$$
\frac{\|S(\tilde{U} + U) - S(\tilde{U}) - S'(\tilde{U})U\|_{\mathbb{Z}_1}}{\|U\|_{L^2(0,T;\mathbb{H})}} \to 0 \quad \text{as} \quad \|U\|_{L^2(0,T;\mathbb{H})} \to 0.
$$

Moreover, for any $\tilde{U} \in \mathcal{U}$, the Fréchet derivative $S'(\tilde{U})$ is given by $S'(\tilde{U})U = w_{\tilde{U}}'[U], \forall U \in L^2(0, T; \mathbb{H})$, where $w_{\tilde{U}}'[U]$ is the unique weak solution of (5.2) associated with the control $U \in L^2(0, T; \mathbb{H})$.

(ii) The Fréchet derivative $S'$ is Lipschitz continuous, that is, for any controls $U_1, U_2 \in \mathcal{U}$ and $U \in L^2(0, T; \mathbb{H})$, there exists a constant $K_2 > 0$ depending only on system parameters, $\Omega, T, C_P, R$ and $\|u_0\|_{\mathbb{H}_2}$ such that

$$
\|S'(U_1) - S'(U_2)\|_{\mathbb{Z}_1} \leq K_2 \|U_1 - U_2\|_{L^2(0,T;\mathbb{H})} \|U\|_{L^2(0,T;\mathbb{H})}.
$$

**Proof of (i).** For any arbitrary but fixed $\tilde{U} \in \mathcal{U}$, let $u_\delta$ be the unique strong solution of (1.1). Since $\mathcal{U}$ is an open subset of $L^2(0, T; \mathbb{H})$, there exists some $\rho > 0$ such that for any $U \in L^2(0, T; \mathbb{H})$ with $\|U\|_{L^2(0,T;\mathbb{H})} \leq \rho$, we have $\tilde{U} + U \in \mathcal{U}$. Let $u_{\tilde{U}+U}$ be the unique strong solution of the system (1.1) in response to the control $\tilde{U} + U \in \mathcal{U}$. Let $w_{\tilde{U}}'[U]$ be the unique weak solution of the linearized equation (5.2). Then the difference defined by
\[ z := u_{\mathcal{U} + \mathcal{V}} - u_{\mathcal{U}} - w_{\mathcal{U}}'[U] \text{ solves the system } \]
\[
\begin{cases}
\mathcal{L}_{\mathcal{U}} z + \nabla \tilde{p} = \mathcal{U}_1 + \mathcal{U}_2 & \text{in } \Omega_T \\
\nabla \cdot z = 0 & \text{in } \Omega_T, \quad z(x, 0) = 0 & \text{in } \Omega,
\end{cases}
\]  
(5.7)

where \( \mathcal{L}_{\mathcal{U}} z \) is defined in (5.3), the pressure \( \tilde{p} := p_{\mathcal{U} + \mathcal{V}} - p_{\mathcal{U}} - q \), and the terms
\[
\begin{align*}
\mathcal{U}_1(x, t) & := -[(u_{\mathcal{U} + \mathcal{V}} \cdot \nabla)u_{\mathcal{U} + \mathcal{V}} - (u_{\mathcal{U}} \cdot \nabla)u_{\mathcal{U}} - (u_{\mathcal{V}} \cdot \nabla)\tilde{u} - (\tilde{u} \cdot \nabla)u_{\mathcal{V}}], \\
\mathcal{U}_2(x, t) & := -[\beta (f(u_{\mathcal{U} + \mathcal{V}}) - \beta f(u_{\mathcal{U}}) - \beta f'(u_{\mathcal{V}})\tilde{u}], \quad \tilde{u} := u_{\mathcal{U} + \mathcal{V}} - u_{\mathcal{U}}.
\end{align*}
\]

Let us invoke Theorem 5.2 for the solvability of (5.7), which requires to prove that \( \mathcal{U}_1, \mathcal{U}_2 \in L^2(0, T; \mathbb{H}) \). Note that \( \mathcal{U}_1 \) can be simplified as \( \mathcal{U}_1 = -[(\tilde{u} \cdot \nabla)\tilde{u}] \). Taking \( u_{\mathcal{U}}, u_{\mathcal{U} + \mathcal{V}} \in L^\infty(0, T; \mathbb{H}^2) \) into account, the inequality (3.3) and the continuous embedding \( H^1(0, T; \mathbb{V}) \hookrightarrow L^\infty(0, T; \mathbb{V}) \) lead to the estimate
\[
\|\mathcal{U}_1\|^2_{L^2(0, T; \mathbb{H})} \leq C T \|\tilde{u}\|_{L^\infty(0, T; \mathbb{H}^2)} \|\tilde{u}\|^3_{L^\infty(0, T; \mathbb{V})}
\leq C T [M_0(\tilde{U} + U, \tilde{U})] \|\tilde{u}\|^3_{L^\infty(0, T; \mathbb{V})} \leq C(C_2, K_1)\|U\|^3_{L^2(0, T; \mathbb{H})},
\]  
(5.8)

where we used (3.6), (5.1), and for any \( V, W \in \mathcal{U} : \)
\[
[M_0(V, W)]^p := \|u_V\|^p_{L^\infty(0, T; \mathbb{H}^2)} + \|u_W\|^p_{L^\infty(0, T; \mathbb{H}^2)}, \quad p \geq 1.
\]

Let’s apply Taylor’s formula,
\[
f(u_{\mathcal{U} + \mathcal{V}}) = f(u_{\mathcal{U}}) + f'(u_{\mathcal{U}})\tilde{u} + \int_0^1 (1 - \theta) f''(u_{\mathcal{U}} + \theta \tilde{u})[\tilde{u}, \tilde{u}]d\theta.
\]  
(5.9)

By computing \( f''(\cdot) \) from (3.20), for any \( r \geq 5 \), we have
\[
f''(u_{\mathcal{U}} + \theta \tilde{u}) = (r - 1)(r - 3)|u_{\mathcal{U}} + \theta \tilde{u}|^{r-5}|(u_{\mathcal{U}} + \theta \tilde{u}) \cdot \tilde{u}|^2(u_{\mathcal{U}} + \theta \tilde{u})
\]
\[+ (r - 1)|u_{\mathcal{U}} + \theta \tilde{u}|^{r-3}(2((u_{\mathcal{U}} + \theta \tilde{u}) \cdot \tilde{u}) \tilde{u} + (u_{\mathcal{U}} + \theta \tilde{u})|\tilde{u}|^2).
\]

Using the embeddings \( \mathbb{H}^2 \hookrightarrow L^\infty, \mathbb{V} \hookrightarrow L^4 \), and (5.1), we obtain
\[
\|\mathcal{U}_2\|^2_{L^2(0, T; \mathbb{H})} \leq C \sup_{\theta \in (0, 1)} \int_{\Omega_T} \int_0^1 |\theta u_{\mathcal{U} + \mathcal{V}} + (1 - \theta)u_{\mathcal{U}}|^2 |\tilde{u}|^4 d\theta dx dt
\leq C 2^{r-5} \int_{\Omega_T} \left( |u_{\mathcal{U} + \mathcal{V}}|^{2(r-2)} + |u_{\mathcal{U}}|^{2(r-2)} \right) |\tilde{u}|^4 dx dt
\leq C 2^{2r-5} \int_0^T \left( \|u_{\mathcal{U} + \mathcal{V}}(t)\|_{L^\infty}^{2(r-2)} + \|u_{\mathcal{U}}(t)\|_{L^\infty}^{2(r-2)} \right) \|\tilde{u}(t)\|^4_{L^4} dt
\leq C T [M_0(\tilde{U} + U, \tilde{U})]^{2(r-2)} \|\tilde{u}\|^4_{L^\infty(0, T; \mathbb{V})}
\leq C(C_2, K_1)\|U\|^4_{L^2(0, T; \mathbb{H})}.
\]  
(5.11)

Further, for any \( 2 < r < 5, |u_{\mathcal{U}} + \theta \tilde{u}| \neq 0 \), we can obtain the bound (5.11).

---

1The second derivative of \( f(\cdot) \) is given by
\[
f''(p)[q, g] = (r - 1)(r - 3)|p|^{r-5}(p \cdot q)(p \cdot g)p
\]
\[+ (r - 1)|p|^{r-3}(p \cdot q)g + (p \cdot g)q + (g \cdot q)p, \quad r \geq 5.
\]  
(5.10)

Further, for any \( 2 < r < 5, p \neq 0 \), the expression (5.10) is valid for \( f''(p)[\cdot, \cdot] \), and also need to set that \( f''(p)[\cdot, \cdot] = 0 \), if \( p = 0 \).
From the estimates (5.8) and (5.11), we notice that \( \Psi_1, \Psi_2 \in L^2(0,T;H) \). Thus, for any \( r \in (2, \infty) \), by Theorem 5.2, the system (5.7) has a unique weak solution \( z \in L^\infty(0,T;V), z_t \in L^2(0,T;V) \). Moreover, repeating the estimations similar to (5.6) together with (5.8) and (5.11) yield that
\[
\Psi(t, z) \leq 2 \left( \|\Psi_1\|^2_{L^2(0,T;H)} + \|\Psi_2\|^2_{L^2(0,T;H)} \right)
\times \exp \left( C(\Omega, C_P)T \left( \|u_{\tilde{\theta}}\|^2_{L^\infty(0,T;V)} + \|u_{\tilde{\theta}}\|^2_{H^2(0,T;H)} \right) \right)
\leq C(C_2, K_1, C_P, T) \left( \|U\|^3_{L^2(0,T;H)} + \|U\|^4_{L^2(0,T;H)} \right). 
\tag{5.12}
\]

For any \( r \in (2, \infty) \), from (5.12), we deduce the following convergence:
\[
\|z\|_{L^2(0,T;H)} \leq C \left( \|U\|^3_{L^2(0,T;H)} + \|U\|^4_{L^2(0,T;H)} \right),
\]
whence \( \|z\|_{L^2(0,T;H)} \to 0 \) as \( \|U\|_{L^2(0,T;H)} \to 0 \).

Next, for the case of \( r = 2 \), we follow the proof of Theorem 5.2. Let us multiply (5.7) by \( z_t \). For the term \( \Psi_1 \), we get from (2.3) that
\[
(\Psi_1, z_t) = -b(u, \hat{u}, z_t) \leq C\|\hat{u}(t)\|^4_V + \delta_2\|z(t)\|^2_V.
\tag{5.13}
\]
By the first-order Taylor’s formula, we get \( \Psi_2 = -\beta \int_0^1 (f'(u_{\tilde{\theta}} + \theta \hat{u})\hat{u} - f'(u_{\tilde{\theta}})\hat{u}) \, d\theta \). For \( r = 2 \), the first derivative expression (3.20) clearly leads to the following estimate
\[
(\Psi_2, z_t) = -\beta \int_0^1 \int_{\Omega} \left( \left( \frac{(u_{\tilde{\theta}} + \theta \hat{u}) \cdot \hat{u}}{|(u_{\tilde{\theta}} + \theta \hat{u})|} \cdot (\theta \hat{u} + \theta \hat{u}) \right) u_{\tilde{\theta}} (|u_{\tilde{\theta}}| - |u_{\tilde{\theta}} + \theta \hat{u}|) 
+ \left( \frac{(\theta \hat{u}) \cdot \hat{u}}{|u_{\tilde{\theta}}|} - u_{\tilde{\theta}} \right) \cdot z_t + (|u_{\tilde{\theta}} + \theta \hat{u}| - |u_{\tilde{\theta}}|) (\hat{u} \cdot z_t) \right) \, d\theta \, dx
\leq 4\beta \sup_{\theta \in (0,1)} \int_0^1 \int_{\Omega} |\theta \hat{u}| |\theta \hat{u}||z_t| \, d\theta \, dx \leq C(\delta_1)\|\hat{u}(t)\|_{H^2}^4 + \delta_1\|z(t)\|^2_{H^2}.
\tag{5.14}
\]
By proceeding as in the proof of Theorem 5.2, choosing \( \delta_1 = 1/4, \delta_2 = \mu/4 \), we obtain from (5.13), (5.14) in view of (5.6) and (5.1) that
\[
\Psi(t, z) \leq C(C_2, K_1, C_P, T)\|U\|^4_{L^2(0,T;H)}.
\]

Hence, for \( r = 2 \), we get the convergence as in \( r \in (2, \infty) \), which completes the proof of (i) for all \( r \in [2, \infty) \).

**Proof of (ii).** Let us denote \( w_1 := w'_{U_1}[U], w_2 := w'_{U_2}[U] \), where \( w'_{U_1}[U] \) and \( w'_{U_2}[U] \) are the weak solutions of (5.2) with control \( U \). Then the function \( w := w_1 - w_2 \) satisfies the equation
\[
\begin{cases}
L_{U_1} w + \nabla q = \Psi_3 + \Psi_4 & \text{in } \Omega_T \\
\nabla \cdot w = 0 & \text{in } \Omega_T, \\
w(x, 0) = 0 & \text{in } \Omega,
\end{cases}
\tag{5.15}
\]
where \( q := q_{U_1} - q_{U_2}, \Psi_3 := -(w_2 \cdot \nabla) \tilde{u} - (\tilde{u} \cdot \nabla) w_2, \Psi_4 := -\beta(f'(w_{U_1}) - f'(w_{U_2})) w_2, \) and \( \tilde{u} := u_{U_1} - u_{U_2} \).
The proof again follows the lines of proof of Theorem 5.2. We only look at the right-hand side terms of (5.15). Testing (5.15) by \( w_t \) and applying (2.3) we get
\[
(\Psi_3, w_t) = -b(w_2, \tilde{u}, w_t) - b(\tilde{u}, w_2, w_t) \leq C(\delta_2)\|w_2(t)\|^2_V + \delta_2\|w_t(t)\|^2_V.
\]
For \( r \geq 5 \), invoking (5.10), Taylor’s formula and applying the embeddings \( \mathbb{H}^2 \hookrightarrow \mathbb{L}^\infty, V \hookrightarrow \mathbb{L}^4 \), we obtain
\[
|\,(U_4, w_t)| \leq C(\delta_1) \left\| \int_0^1 f''(u_{U_2}(t) + \theta \tilde{u}(t))[\tilde{u}(t), w_2(t)]d\theta \right\|^2_H + \delta_1 \| w_{1}(t) \|^2_H
\leq C \int_0^1 \int_0^1 |\theta u_{U_1}(t) + (1 - \theta) u_{U_2}(t)|^{2(r-2)} |w_2(t)|^2 |\tilde{u}(t)|^2 d\theta dx + \delta_1 \| w_{1}(t) \|^2_H
\leq C \left( \| w_{U_1}(t) \|^2_{L^\infty} + \| w_{U_2}(t) \|^2_{L^\infty} \right) \| w_{2}(t) \|_{L^4}^2 \| \tilde{u}(t) \|_{L^4}^2 + \delta_1 \| w_{1}(t) \|^2_H
\leq C[M_0(U_1, U_2)]^{2(r-2)} \| w_{2}(t) \|^2_V \| \tilde{u}(t) \|^2_V + \delta_1 \| w_{1}(t) \|^2_H, \tag{5.16}
\]
where \( M_0 \) is given in (5.8). Moreover, for any \( 2 < r < 5 \), \((u_{U_2} + \theta \tilde{u}) \neq 0\), we can obtain the estimate (5.16). Thus, for any \( r \in (2, \infty) \), by repeating the calculations similar to (5.4)-(5.6), choosing \( \delta_1 = 1/4, \delta_2 = \mu/4 \), and using (5.1), one can get that
\[
\Psi(t, w) \leq C \exp \left( C(\Omega, C_P) T \left( \| u_{U_1} \|_{L^\infty([0,T;V)} \right)^{2(r-1)} \right) \times \left( 1 + [M_0(U_1, U_2)]^{2(r-2)} \right) \| \tilde{u} \|^2_{L^\infty([0,T;V)} \| w_{2} \|^2_{L^2([0,T;V)}
\leq C(C_2, K_1, C_P, T) \| U \|^2_{L^2([0,T;H^2)} \| U_1 - U_2 \|^2_{L^2([0,T;H^2)}, \quad \forall t \in (0, T), \tag{5.17}
\]
where we used (5.6) for \( \| w_{2} \|^2_{L^2([0,T;V)} \). Besides, for the case of \( r = 2 \), we can get the bound (5.17), since by invoking (5.14), we have
\[
|\,(U_4, w_t)| \leq 4\beta \| w_{2}(t) \|_{L^1} \| \tilde{u}(t) \|_{L^4} \| w_{t}(t) \|_{H} \leq C \| w_{2}(t) \|^2_V \| \tilde{u}(t) \|^2_V + \delta_1 \| w_{1}(t) \|^2_H. \tag{5.18}
\]
Thus, for any \( r \in [2, \infty) \), there exists a constant \( K_2(C_2, K_1) > 0 \) such that \( \| w \|_{Z_1} \leq K_2\| U \|_{L^2([0,T;H^2)} \| U_1 - U_2 \|_{L^2([0,T;H^2)} \). The proof of (ii) is thus completed. \( \square \)

5.2. **First-order optimality conditions.** In this subsection, we derive optimality conditions satisfied by an optimal control. From Theorem 4.3, it is evident that there exists an optimal solution \((u_{\tilde{U}}, \tilde{U})\) satisfying \( S(\tilde{U}) = u_{\tilde{U}} \) and the pair \((S(\tilde{U}), \tilde{U})\) is an optimal solution for (MOCP).

By the Fréchet differentiability of \( S \) given by Proposition 2, the reduced cost functional \( \mathfrak{J}(U) \) is Fréchet differentiable at every \( U \in \mathcal{U} \). Moreover, since the admissible control set \( \mathcal{U}_{ad} \) is convex and \( \mathfrak{J}(U) \) is Fréchet differentiable, for any minimizer \( \tilde{U} \in \mathcal{U}_{ad} \) of the reduced functional \( \mathfrak{J}(U) \), the following variational inequality holds (see, [26, 41]):
\[
\mathfrak{J}'(\tilde{U})(U - \tilde{U}) \geq 0, \quad \forall U \in \mathcal{U}_{ad}. \tag{5.19}
\]
Indeed, from Lemma 5.1 and Proposition 2, we obtain a variational inequality satisfied by an optimal control \( \tilde{U} \in \mathcal{U}_{ad} \). Let \( U \in \mathcal{U} \) be arbitrary but fixed. Then there exists some \( \rho > 0 \) such that for any \( V \in L^2(0, T; \mathbb{H}) \) with \( \| V \|_{L^2(0,T;\mathbb{H})} \leq \rho \), we have \( U + V \in \mathcal{U} \), so that the
variation of the functional $\mathfrak{J}(\cdot)$ is given by
\[
\mathfrak{J}(U + V) - \mathfrak{J}(U) = \mathcal{J}(\mathcal{S}(U + V), U + V) - \mathcal{J}(\mathcal{S}(U), U)
= \frac{\kappa}{2} \int_0^T \|\nabla(u_{U+V}(t) - u_U(t))\|^2 \, dt
+ \kappa \int_0^T (\nabla(u_{U+V}(t) - u_U(t)), \nabla(u(t) - u_d(t))) \, dt
+ \frac{\lambda}{2} \int_0^T \|V(t)\|^2 \, dt + \lambda \int_0^T (U(t), V(t)) \, dt.
\] (5.20)

It is clear from Lemma 5.1 that the following estimate holds:
\[
\int_0^T \|\nabla(u_{U+V}(t) - u_U(t))\|^2 \, dt = \|u_{U+V} - u_U\|_{L^2(0,T;\mathcal{V})}^2 \leq K_1^2 \|V\|_{L^2(0,T;\mathbb{H})}^2.
\] (5.21)

Let $\tilde{w} := w'_U[V]$ be the unique weak solution of (5.2). Notice that
\[
\int_0^T (\nabla(u_{U+V}(t) - u_U(t)), \nabla(u(t) - u_d(t))) \, dt
= \int_0^T (\nabla \tilde{w}(t), \nabla(u(t) - u_d(t))) \, dt
+ \int_0^T (\nabla(u_{U+V}(t) - u_U(t) - \tilde{w}(t)), \nabla(u(t) - u_d(t))) \, dt := I_1 + I_2.
\] (5.22)

By the application of Hölder’s inequality, we get
\[
|I_2| \leq \|u_{U+V} - u_U - \tilde{w}\|_{L^2(0,T;\mathcal{V})} \|u_U - u_d\|_{L^2(0,T;\mathcal{V})}.
\] (5.23)

Taking $u_U, u_d \in L^2(0,T;\mathcal{V})$ into account, invoking Proposition 2, we see that $|I_2|/\|V\|_{L^2(0,T;\mathbb{H})} \to 0$ as $\|V\|_{L^2(0,T;\mathbb{H})} \to 0$.

Substituting the identity (5.22) into (5.20), rearranging the resultant integrals, dividing both sides by $\|V\|_{L^2(0,T;\mathcal{V})}$ and taking $\|V\|_{L^2(0,T;\mathbb{H})} \to 0$, we obtain through (5.21) and (5.23) that
\[
\mathfrak{J}'(U)V = \kappa \int_0^T (\nabla \tilde{w}(t), \nabla(u(t) - u_d(t))) \, dt + \lambda \int_0^T (U(t), V(t)) \, dt.
\] (5.24)

Thus, from (5.19), we have the following optimality inequality characterizing an optimal control $\tilde{U} \in \mathcal{U}_{ad}$ of (MOCP):

**Theorem 5.3.** Let $\Omega$ be a periodic domain in $\mathbb{R}^3$. Suppose Assumption 1 holds true. Let $U \in \mathcal{U}_{ad}$ be an arbitrary control with state $u_U = \mathcal{S}(U)$, then the reduced functional $\mathfrak{J}(U)$ is Fréchet differentiable with the derivative (5.24). If $\tilde{U} \in \mathcal{U}_{ad}$ is an optimal control for (MOCP) with associated state $u_{\tilde{U}} = \mathcal{S}(\tilde{U})$, then the following variational inequality holds:
\[
\mathfrak{J}'(\tilde{U})(U - \tilde{U}) = \kappa \int_{\Omega_T} \nabla w'_U[U - \tilde{U}] \cdot \nabla (u_U - u_d) \, dx \, dt
+ \lambda \int_{\Omega_T} \tilde{U} \cdot (U - \tilde{U}) \, dx \, dt \geq 0,
\] (5.25)
for all $U \in \mathcal{U}_{ad}$, where $w_{\tilde{U}}^r[U - \tilde{U}] = \mathcal{S}(\tilde{U})(U - \tilde{U}) \in \mathcal{Z}$ is a unique weak solution of the linearized system (5.2) with control $V = U - \tilde{U}$.

Next, we follow the classical adjoint problem approach to simplify the variational inequality (5.25), in particular the first term, by expressing it as an equivalent integral defined in terms of a solution of an adjoint problem of (1.1). This will lead to devising a compact first-order optimality condition characterizing an optimal control of (MOCP). The adjoint system is derived by applying the formal Lagrangian method (see, [41], Chapter 3).

Consider the adjoint problem

\[
(A-\text{NSVD}) \left\{ \begin{array}{ll}
\mathcal{E}_U \varphi + \nabla \psi = -\kappa \Delta (u_U - u_d) & \text{in } \Omega_0 \\
\nabla \cdot \varphi = 0 & \text{in } \Omega_0, \\
\varphi(x,T) - \mu \Delta \varphi(x,T) = 0 & \text{in } \Omega,
\end{array} \right.
\]

where $\Omega_0 := \Omega \times [0,T)$, $\psi$ denotes the adjoint pressure, the linear operator

\[
\mathcal{E}_U \varphi := -\varphi_t + \mu \Delta \varphi_t - \nu \Delta \varphi + (\nabla u_U)^T \varphi - (u_U \cdot \nabla) \varphi + \alpha \varphi + \beta f'(u_U) \varphi,
\]

and $f'(\cdot)$ is defined in (3.20).

Next, we define the weak solution for the adjoint system (5.26).

**Definition 5.4.** Let $U \in \mathcal{U}_{ad}$ be any control with associated state $u_U = \mathcal{S}(U)$ and $r \geq 2$. A function $\varphi \in H^1(0,T; \mathcal{V})$ is called a weak solution of (5.26) on the interval $[0,T]$ if the following hold:

\[
(i) \quad -(\varphi_t, v) - \mu (\nabla \varphi_t, \nabla v) + \nu (\nabla \varphi, \nabla v) + b(v, u_U, \varphi) - b(u_U, \varphi, v) + \alpha (\varphi, v) \\
+ \beta (f'(u_U) \varphi, v) = \kappa (\nabla (u_U - u_d), \nabla v), \quad \forall v \in \mathcal{V}, \ a.e. \ t \in [0,T],
\]

\[
(ii) \quad (\varphi(T), v) + \mu (\nabla \varphi(T), \nabla v) = 0, \quad \forall v \in \mathcal{V}.
\]

**Theorem 5.5** (First-Order Optimality Conditions). Let $\Omega$ be a periodic domain in $\mathbb{R}^3$. Suppose Assumption 1 holds true. Let $\tilde{U} \in \mathcal{U}_{ad}$ be an optimal control for (MOCP) with associated state $u_{\tilde{U}} = \mathcal{S}(\tilde{U})$. Then there exists an adjoint state $\varphi_{\tilde{U}}$ associated to the state $u_{\tilde{U}}$ such that

\[
(i) \quad \varphi_{\tilde{U}} \text{ is a unique weak solution of (5.26) in the sense of Definition 5.4.}
\]

(ii) for any admissible control $U \in \mathcal{U}_{ad}$, the following variational inequality holds

\[
\mathcal{J}'(\tilde{U})(U - \tilde{U}) = \int_{\Omega_T} (\varphi_{\tilde{U}} + \lambda \tilde{U}) \cdot (U - \tilde{U}) \ dx \ dt \geq 0.
\]

**Proof.** We will only prove required an a priori estimates for the solvability of the adjoint system (5.26). The justification of weak solution can be done by the standard Galerkin approximations and convergence arguments. By setting $\varphi := \varphi_{\tilde{U}}$ and taking inner product of (5.26) with $\varphi$, we get

\[
-\frac{1}{2} \frac{d}{dt} \left[ \| \varphi(t) \|^2_{\mathcal{H}} + \mu \| \varphi(t) \|^2_{\mathcal{V}} \right] + \nu \| \varphi(t) \|^2_{\mathcal{V}} + \alpha \| \varphi(t) \|^2_{\mathcal{H}} + \beta (f'(u_{\tilde{U}}) \varphi, \varphi) \\
= -((\nabla u_{\tilde{U}})^T \varphi, \varphi) - \kappa (\Delta (u_{\tilde{U}} - u_d), \varphi),
\]

(5.28)
where we used the fact that \( \int_{\Omega} (u_{\partial} \cdot \nabla) \varphi \cdot \varphi \, dx = b(u_{\partial}, \varphi, \varphi) = 0 \). Using Hölder’s inequality and Ladyzhenskaya’s inequality (2.1) followed by Young’s inequality, we obtain
\[
-((\nabla u_{\partial})^T \varphi, \varphi) = -((\varphi \cdot \nabla) u_{\partial}, \varphi) \leq \|\varphi(t)\|_{L^4}^2 \|\nabla u_{\partial}(t)\|_{L^2}
\]
\[
\leq C\|\varphi(t)\|_{L^6}^2 \|\varphi(t)\|_{L^3}^2 \|u_{\partial}(t)\|_{V} \leq C(\delta_3) \|\varphi(t)\|_{V}^2 \|u_{\partial}(t)\|_{V}^2 + \delta_3 \|\varphi(t)\|_V^2.
\]
(5.29)

Integrating by parts and applying Young’s inequality, we also have
\[
-\kappa\langle\Delta(u_{\partial} - u_d), \varphi\rangle = \kappa\langle\nabla(u_{\partial} - u_d), \nabla \varphi\rangle \leq C(\delta_3) \|\varphi(t)\|_{V}^2 + \|u_d(t)\|_{V}^2 + \delta_3 \|\varphi(t)\|_V^2.
\]
(5.30)

For any \( r \geq 3 \), notice from the derivative (3.20) that
\[
\beta \int_{\Omega} (f'(u_{\partial})) \varphi \, dx = \beta \int_{\Omega} ((r - 1)|u_{\partial}|^{r-3}|u_{\partial} - \varphi|^2 + |u_{\partial}|^{r-1}|\varphi|^2) \, dx \geq 0.
\]
Similarly, this integral is non-negative for \( 2 \leq r < 3 \) as well. Let use define energy integral
\[
\Phi(t, \varphi) := \|\varphi(t)\|_{H}^2 + \mu\|\varphi(t)\|_{V}^2 + \alpha\|\varphi\|_{L^2(t,T;H^2)}^2 + \nu\|\varphi\|_{L^2(t,T;V)}^2, \quad t \in [0,T).
\]
(5.29)-(5.30) in (5.28), choosing \( \delta_3 = \nu/4 \) and applying Gronwall’s inequality in \((t, t)\), we obtain
\[
\Phi(t, \varphi) \leq C \left( \|u_{\partial}\|_{L^2(0,T;V)}^2 + \|u_d\|_{L^2(0,T;V)}^2 \right) \exp \left( C\|u_{\partial}\|_{L^4(0,T;V)}^4 \right) < +\infty,
\]
(5.31)
for all \( t \in [0,T) \), since \( u_{\partial} \in L^\infty(0,T;V) \) and \( u_d \in L^2(0,T;V) \), where the terminal condition at \( t = T \) vanishes due to the following relation:
\[
\|\varphi(T)\|_{H}^2 + \mu\|\varphi(T)\|_{V}^2 = (\varphi(T), \varphi(T)) + \mu(\nabla \varphi(T), \nabla \varphi(T)) = 0.
\]
Now, taking inner product of (5.26) with \(-\varphi_t\) and using (2.3), we infer from (5.30) that
\[
-\frac{1}{2} \frac{d}{dt} \left[ \alpha\|\varphi(t)\|_{H}^2 + \nu\|\varphi(t)\|_{V}^2 + \alpha\|\varphi\|_{L^2(t,T;H^2)}^2 + \nu\|\varphi\|_{L^2(t,T;V)}^2 \right] + \|\varphi(t)\|_{H}^2 + \mu\|\varphi(t)\|_{V}^2
\]
\[
= ((\nabla u_{\partial})^T \varphi, \varphi_t) - ((u_{\partial} \cdot \nabla) \varphi, \varphi_t) + \kappa\langle\Delta(u_{\partial} - u_d), \varphi_t\rangle + \beta(\varphi(t), \varphi_t)
\]
\[
\leq \delta_4\|\varphi(t)\|_{V}^2 + C(\delta_4) \|u_{\partial}(t)\|_{V}^2 \|\varphi(t)\|_{V}^2 + C(\delta_4) \|u_{\partial}(t)\|_{V}^2 + \|u_d(t)\|_{V}^2 + \beta(\varphi(t), \varphi_t).
\]
By invoking (5.5), choosing \( \delta_1 = 1/2, \delta_4 = \mu/2 \) and integrating over \((t, T)\), one may obtain
\[
\Phi_1(t, \varphi) \leq C(\Omega, C_P) \left( \|u_{\partial}\|_{L^\infty(0,T;V)}^2 + \|u_{\partial}\|_{L^2(0,T;V)}^{2(r-1)} \right) \|\varphi\|_{L^2(0,T;V)}^2 + C \left( \|\varphi(T)\|_{H}^2 + \|\varphi(T)\|_{V}^2 + \|u_{\partial}\|_{L^2(0,T;V)}^2 + \|u_d\|_{L^2(0,T;V)}^2 \right) < +\infty,
\]
(5.32)
for all \( t \in [0,T) \), where
\[
\Phi_1(t, \varphi) := \alpha\|\varphi(t)\|_{H}^2 + \nu\|\varphi(t)\|_{V}^2 + \|\varphi\|_{L^2(t,T;H^2)}^2 + \mu\|\varphi\|_{L^2(t,T;V)}^2
\]
and note that
\[
\|\varphi(T)\|_{H}^2 + \|\varphi(T)\|_{V}^2 \leq \sup_{t \in [0,T]} \left( \|\varphi(t)\|_{H}^2 + \|\varphi(t)\|_{V}^2 \right) < +\infty.
\]
Here we employed the fact that \( u_{\partial} \in L^\infty(0,T;H^2), u_d \in L^2(0,T;V) \) and (5.31). The estimate (5.32) shows that \( \varphi \in L^2(0,T;V), \varphi_t \in L^2(0,T;V) \) and it is enough to prove the existence of a weak solution of the adjoint system (5.26). It also shows that \( \varphi \in C([0,T];V) \) which is sufficient to verify the condition \( \varphi(x, T) - \mu \Delta \varphi(x, T) = 0 \) in the weak sense as in Definition
5.4-(ii). Moreover, the uniqueness of weak solution of the linear system (5.26) directly follows from (5.31). This proves part (i).

Next, let us express the integral \( \kappa \int_0^T \int_{\Omega} \nabla w_t^\prime [U - \overline{U}] \cdot \nabla (u_{\overline{U}} - u_d) \, dx \, dt \) of (5.25) in terms of the adjoint variable and control. For brevity, we use \( \tilde{w} := w_{\overline{U}}^\prime [U - \overline{U}] \) for a weak solution of (5.2). Choosing \( V = U - \overline{U} \) in (5.2), testing by the adjoint variable \( \varphi \) and space integrating by parts, one may obtain

\[
\int_{\Omega_T} (\varphi \cdot \tilde{w}_t + \mu \nabla \varphi \cdot \nabla \tilde{w}_t + \nu \nabla \varphi \cdot \nabla \tilde{w}) \, dx \, dt 
+ \int_{\Omega_T} ( (\nabla u_{\overline{U}})^T \varphi - (u_{\overline{U}} \cdot \nabla) \varphi + \alpha \varphi + \beta f'(u_{\overline{U}}) \varphi ) \cdot \tilde{w} \, dx \, dt 
= \int_{\Omega_T} \varphi \cdot (U - \overline{U}) \, dx \, dt,
\]

where we used \( \int_{\Omega_T} \nabla q \cdot \varphi \, dx \, dt = 0 \), and (2.2) to get \( b(u_{\overline{U}}, \tilde{w}, \varphi) = -b(u_{\overline{U}}, \varphi, \tilde{w}) \). Integrating by parts with respect to time and using the initial/final data conditions of (5.2) and (5.26), we get \( \int_{\Omega_T} (\varphi \cdot \tilde{w}_t + \mu \nabla \varphi \cdot \nabla \tilde{w}_t) \, dx \, dt = -\int_{\Omega_T} (\varphi_t \cdot \tilde{w} + \mu \nabla \varphi_t \cdot \nabla \tilde{w}) \, dx \, dt \). On the other hand, testing the adjoint equation (5.26) with \( \tilde{w} \) and comparing with (5.33) yields

\[
\kappa \int_{\Omega_T} \nabla \tilde{w} \cdot \nabla (u_{\overline{U}} - u_d) \, dx \, dt = \int_{\Omega_T} \varphi \cdot (U - \overline{U}) \, dx \, dt.
\]

By invoking Theorem 5.3, we replace the first integral of (5.25) by (5.34) to complete the proof. \( \square \)

The following corollary gives a pointwise version of the variational inequality (5.27) which provides a useful characterization of an optimal control given by the regularization parameter and admissible control set \( \mathcal{U}_{ad} \). Since the admissible control set \( \mathcal{U}_{ad} \) is a closed, convex, and non-empty subset of \( L^2(0, T; \mathbb{H}) \), we can characterize the optimal control by a projection formula.

For any given \((a, b) \in \mathbb{R}^2 \) with \( a \leq b \) and \( \tau \in \mathbb{R} \), let \( \mathcal{P}_{[a, b]} \) denote the projection of \( \mathbb{R} \) onto \( [a, b] \) which is defined by \( \mathcal{P}_{[a, b]}(\tau) := \min \left\{ b, \max \{a, \tau\} \right\} \).

**Corollary 1.** Let \( \overline{U} \in \mathcal{U}_{ad} \) be an optimal control for (MOCP) and \( \varphi = \varphi_{\overline{U}} \) be the solution of the adjoint equation (5.26). Then the optimal control is characterized by three different cases:

(i) If \( \lambda > 0 \), then \( \overline{U} \) is given by the projection formula

\[
\overline{U}(x, t) = \mathcal{P}_{[U_{\min}, U_{\max}]} \left( -\frac{\varphi(x, t)}{\lambda} \right), \quad \text{for a.e.} \quad (x, t) \in \Omega_T,
\]

where \( \mathcal{P} \) is the projection of \( L^2(0, T; \mathbb{H}) \) onto \( \mathcal{U}_{ad} \) as defined above.

(ii) If \( \lambda > 0 \) and \( \mathcal{U}_{ad} = L^2(0, T; \mathbb{H}) \), then the unconstrained optimal control is given by the direct relation \( \overline{U}(x, t) = -\frac{\varphi(x, t)}{\lambda} \), for a.e. \( (x, t) \in \Omega_T \).

(iii) If \( \lambda = 0 \) and \( \mathcal{U}_{ad} \subseteq L^2(0, T; \mathbb{H}) \), then the control is of bang-bang type given by

\[
\overline{U}(x, t) = \begin{cases} U_{\min} & \text{if } \varphi(x, t) > 0, \\ U_{\max} & \text{if } \varphi(x, t) < 0, \end{cases} \quad \text{for a.e.} \quad (x, t) \in \Omega_T.
\]

**Proof.** The proof can be completed by invoking (see, [41], Lemma 2.26 and Theorem 2.28) that the variational inequality (5.27) is equivalent to the pointwise variational inequality...
\((\varphi + \lambda \tilde{U}) \cdot (U - \tilde{U}) \geq 0\) \(\forall U \in [U_{\text{min}}, U_{\text{max}}]\), for a.e. \((x, t) \in \Omega_T\), and the definition of \(P[u_{\text{min}}, u_{\text{max}}]\).

\[ \square \]

### 5.3. First-order optimality conditions in bounded domain \(\Omega \subset \mathbb{R}^3\).

In the previous section, we established a first-order optimality condition of (MOCP) in the periodic domain. As we noticed earlier, while proving the well-posedness of the linearized system (5.2) and the adjoint system (5.26), we needed the strong solution of (1.1) in estimating (5.5),(5.11),(5.16) and (5.32). Nevertheless, the strong solution is obtained when the domain \(\Omega\) is periodic in \(\mathbb{R}^3\). Now, we restrict the growth of the damping term \(f(u) = |u|^{r-1}u, r \in [2, \infty)\) appropriately so that the optimality conditions (Theorem 5.5) hold true for bounded domain \(\Omega \subset \mathbb{R}^3\). In other words, we prove Theorem 5.5 by using only the weak solution of (1.1), which exists for bounded domain as well.

We will only list out the restrictions needed on \(r\) to get the crucial estimations on the bounded domain. The proof of Theorem 5.2 requires only the following estimate on the damping term. If we restrict the growth of the damping term in (1.1) to \(3 \leq r \leq 5\), the estimate (5.5) can be modified using the embedding \(V \hookrightarrow L^{6(r+1)/5}\) and \(V \hookrightarrow L^{r+1, r}, r \leq 5\) as follows:

\[
\begin{align*}
\beta \left\langle f'(u_{\tilde{U}})w, w_t \right\rangle & \leq \beta \|f'(u_{\tilde{U}})w\|_{L^{6/5}} \|w(t)\|_{L^6} \\
& \leq C \|u_{\tilde{U}}(t)\|_{L^{r-1}} \|w(t)\|_{L^{6(r+1)/4}} \|w(t)\|_{L^6} \\
& \leq C(\delta_1) \|u_{\tilde{U}}(t)\|^2_{V} \|w(t)\|^2_{V} + \delta_2 \|w(t)\|_{L^6}^2.
\end{align*}
\]

Choosing \(\delta_1 = 1/2, \delta_2 = \mu/4\), the estimate (5.6) can now read as follows

\[
\Psi(t, w) \leq \exp \left( C(\Omega, C_P) T \left( \|u_{\tilde{U}}\|_{L^{\infty}(0,T;V)}^2 + \|u_{\tilde{U}}\|_{L^{\infty}(0,T;V)}^{2(r-1)} \right) \right) \\
\times \|V\|_{L^{2(0,T;H)}}^2 < +\infty, \forall t \in (0, T],
\]

since \(u_{\tilde{U}} \in L^{\infty}(0,T;V)\) by the weak solutions of (1.1) given in Theorem 3.2. The bound given above holds true for \(2 \leq r < 3\) as well.

Next, let us examine Proposition 2. Taking inner product of (5.7) with \(z_t\) and estimate the right-hand side terms. For \(\langle \mathcal{U}_1, z_t \rangle\), we use (5.13) and for the term \(\langle \mathcal{U}_2, z_t \rangle\), in the case of \(2 < r \leq 5\), we alter (5.11) using the embedding \(V \hookrightarrow L^{12(r+1)/(15-4r)}\), and \(V \hookrightarrow L^{r+1, r}, r \leq 5\) as follows

\[
\begin{align*}
\left| \langle \mathcal{U}_2, z_t \rangle \right| & \leq C \sup_{\theta \in (0,1)} \left\| \int_{0}^{1} (1-\theta) f''(u_{\tilde{U}} + \theta \hat{u}) \hat{u} d\theta \right\|_{L^{6/5}} \|z_t(t)\|_{L^6} \\
& \leq C(r) \sup_{\theta \in (0,1)} \left[ \int_{\Omega} \left( \int_{0}^{1} |\theta u_{\tilde{U}} + (1-\theta) u_{\tilde{U}}|^r |\hat{u}|^{r-2} |\hat{u}|^2 d\theta \right)^{6/5} dx \right] \|z_t(t)\|_{L^6} \\
& \leq C(r) \left( \left\| u_{\tilde{U}} + |u_{\tilde{U}}| \right\|_{L^{12(r-1)/(15-4r)}}^{r+1} \right) \|\hat{u}(t)\|_{L^{12(r+1)/(15-4r)}}^2 \|z_t(t)\|_{L^6} \\
& \leq C(\delta_2, r) \left( \|u_{\tilde{U}} + |u_{\tilde{U}}|^2 \|_{V}^{2(r-2)} + \|u_{\tilde{U}}(t)\|_{V}^{2(r-2)} \right) \|\hat{u}(t)\|_{L^6} + \delta_2 \|z_t(t)\|_{L^6}^2.
\end{align*}
\]
By doing calculations similar to Theorem 5.2 (keeping in mind (5.36)), using (5.13) and (5.38) with $\delta_2 = \mu/8$, one may obtain
\[
\Psi(t, z) \leq C \exp \left( CT \left( \|u_{\tilde{U}}\|_{L^\infty(0,T;V)}^2 + \|u_{\tilde{U}}\|_{L^\infty(0,T;V)}^{2(\nu-1)} \right) \right) \|U\|_{L^2(0,T;H)}^4,
\]
where $C$ depends on $C_1, K_1$ and $R$. By (5.14), the above conclusion also holds true for $r = 2$. Thus, we can prove part-(i) of Proposition 2. In view of (5.37), the part-(ii) requires only to prove (5.16) and that can again be obtained by following (5.38). This completes the proof of Proposition 2 in the case of bounded domain when $2 \leq r \leq 5$. For the solvability of adjoint system (5.26), the main estimation (5.32), which we concluded from (5.5) can now be obtained from (5.36). Thus, by combining the preceding arguments, we infer that when the growth of the damping term is restricted to $2 \leq r \leq 5$, the first-order optimality condition of (MOCP) proved in Theorem 5.5 holds true for bounded domain $\Omega \subset \mathbb{R}^3$ with zero Dirichlet boundary conditions.

6. Second-order sufficient optimality conditions

It is known that for convex optimal control problems, any control satisfying the first-order necessary optimality conditions is globally optimal. However, for the non-convex optimal control problems, one may need to do further higher derivative analysis to guarantee a local optimal control. In the case of optimal control problems governed by Navier-Stokes equations, second-order sufficient optimality conditions play a pivotal role in the numerical analysis of these non-convex optimal control problems. A control that satisfies second-order sufficient optimality conditions is stable with respect to any perturbation of the given data (see, [42], also [44])

6.1. Control-to-costate operator. In this section, we establish a sufficient criteria for local optimality condition. If $\tilde{U}$ satisfies the variational inequality (5.27) and suppose, we assume that $\mathcal{J}''(\tilde{U})[U, U] > 0$, for all directions $U \in L^2(0,T;\mathbb{H})(\Omega)\setminus \{0\}$, then the control $\tilde{U}$ is a strict local minimizer of $\mathcal{J}(\cdot)$ on the admissible control set $U_{ad}$. But the positivity condition defined on all directions can be relaxed to only certain critical directions (see, [11, 41] and also [13]). To get the second Fréchet differentiability of the functional $\mathcal{J}(\cdot)$, we study the control-to-costate operator $A : U \to Z_1, U \mapsto A(U) := \varphi_U$, which assigns for any control $U \in U$, a unique weak solution $\varphi_U \in Z_1$ of the adjoint system (5.26). We prove that the operator $A$ is Lipschitz continuous and Fréchet differentiable. In this section, we assume that the growth parameter $r \geq 3$ in the nonlinearity of the damping term. This restriction arises due to the fact that the third-order Gâteaux derivative of the damping term exists only for $r \geq 3$.

Lemma 6.1. The control-to-costate mapping $A : U \to Z_1$ is Lipschitz continuous, that is, there exists a constant $K_3 > 0$ depending only on $K_1, \Omega, T, C_P, R, \|u_0\|_{\mathbb{H}^2}$ such that
\[
\|A(U_1) - A(U_2)\|_{Z_1} \leq K_3 \|U_1 - U_2\|_{L^2(0,T;\mathbb{H})}, \quad \forall U_1, U_2 \in U.
\]

Proof. Let $u_{U_1}, u_{U_2}$ and $\varphi_{U_1}, \varphi_{U_2}$ be the strong and weak solutions of (1.1) and (5.26), respectively associated with the controls $U_1, U_2 \in U$. Define $\tilde{u} := u_{U_1} - u_{U_2}, \varphi := A(U_1) - A(U_2) = \varphi_{U_1} - \varphi_{U_2}$,
\( \varphi_{U_1} - \varphi_{U_2} \) and the pressure \( \tilde{\psi} := \psi_{U_1} - \psi_{U_2} \). Then the pair \((\varphi, \tilde{\psi})\) solves the equation

\[
\begin{aligned}
\mathcal{E}_{U_1} \varphi + \nabla \tilde{\psi} &= -\kappa \Delta \tilde{u} + \mathfrak{F}_1 + \mathfrak{F}_2 \quad \text{in} \quad \Omega_0 \\
\nabla \cdot \varphi &= 0 \quad \text{in} \quad \Omega_0, \quad \varphi(x, T) - \mu \Delta \varphi(x, T) = 0 \quad \text{in} \quad \Omega,
\end{aligned}
\]

where \( \mathcal{E}_{U_1} \varphi \) is defined in (5.26),

\[
\mathfrak{F}_1 = -(\nabla \tilde{u})^T \varphi_{U_2} + (\tilde{u} \cdot \nabla) \varphi_{U_2}, \quad \text{and} \quad \mathfrak{F}_2 = -\beta (f'(u_{U_1}) - f'(u_{U_2})) \varphi_{U_2}.
\]

Testing (6.2) with \( \varphi \) and doing estimations similar to (5.29),(5.30) and (5.16), we obtain that

\[
|(-\kappa \Delta \tilde{u} + \mathfrak{F}_1 + \mathfrak{F}_2, \varphi)| 
\leq C(\delta_3) \|\tilde{u}(t)\|_V^2 + \delta_3 \|\varphi(t)\|_V^2 + C(\delta_3) \|\tilde{u}(t)\|_V^2 \|\varphi_{U_2}(t)\|_V^2 \\
+ C(\delta_5) \left( \|u_{U_1}(t)\|_{H^2}^{2(r-2)} + \|u_{U_2}(t)\|_{H^2}^{2(r-2)} \right) \|\varphi_{U_2}(t)\|_V^2 \|\tilde{u}(t)\|_V^2 + \delta_5 \|\varphi(t)\|_H^2,
\]

for any \( r \geq 3 \). By proceeding as in part-(i) of Theorem 5.5 and using (6.3), we infer from (5.31) and (5.32) that the proof can be completed by invoking Lemma 5.1.

**Proposition 3.** For any \( \bar{U} \in \mathcal{U} \), let \( u_{\bar{U}} \) be the unique strong solution of (1.1). Then we have:

(i) The control-to-costate mapping \( \mathcal{A} \) is Fréchet differentiable on \( \mathcal{U} \), that is, for any \( \bar{U} \in \mathcal{U} \), there exists a bounded linear operator \( \mathcal{A}'(\bar{U}) : L^2(0, T; H) \to Z_1 \) such that

\[
\left\| \mathcal{A}(\bar{U} + U) - \mathcal{A}(\bar{U}) - \mathcal{A}'(\bar{U})U \right\|_{Z_1} \to 0 \quad \text{as} \quad \|U\|_{L^2(0, T; H)} \to 0.
\]

Moreover, for any \( \bar{U} \in \mathcal{U} \), the Fréchet derivative is given by \( \mathcal{A}'(\bar{U})U = \varphi_{\bar{U}}'[U], \forall U \in L^2(0, T; H) \), where \( \varphi_{\bar{U}}'[U] \) is the unique weak solution of the equation

\[
\begin{aligned}
\mathcal{E}_{\bar{U}} \phi + \nabla \tilde{\psi} &= -\kappa \Delta w_{\bar{U}}'[U] - (\nabla w_{\bar{U}}'[U])^T \varphi_{\bar{U}} + (w_{\bar{U}}'[U] \cdot \nabla) \varphi_{\bar{U}} \\
&\quad - \beta f''(u_{\bar{U}})[w_{\bar{U}}'; \varphi_{\bar{U}}] \quad \text{in} \quad \Omega_0 \\
\nabla \cdot \phi &= 0 \quad \text{in} \quad \Omega_0, \quad \phi(x, T) - \mu \Delta \phi(x, T) = 0 \quad \text{in} \quad \Omega,
\end{aligned}
\]

while \( w_{\bar{U}}'[U] \) is the weak solution of (5.2) with control \( U \), \( \varphi_{\bar{U}} \) is that of (5.26) and \( \mathcal{E}_{\bar{U}} \phi \) is the linear operator defined in (5.26).

(ii) For any controls \( U_1, U_2 \in \mathcal{U} \) and \( U \in L^2(0, T; H) \), there exists a constant \( K_4 > 0 \) such that

\[
\left\| \mathcal{A}'(U_1)U - \mathcal{A}'(U_2)U \right\|_{Z_1} \leq K_4 \|U_1 - U_2\|_{L^2(0, T; H)} \|U\|_{L^2(0, T; H)},
\]

where \( K_4 \) depends on system parameters, \( K_1, K_2, K_3, \Omega, Y, C_p, R \) and \( \|u_0\|_{\Omega^2} \).

**Proof.** By following the line of proof of Theorem 5.5, we can show that (6.4) admits a unique weak solution \( \varphi_{\bar{U}}'[U] \in H^1(0, T; V) \). To prove the Fréchet differentiability, as in the proof of Proposition 2-(i), let us consider \( \xi_{\bar{U}} := \varphi_{\bar{U} + U} - \varphi_{\bar{U}} - \varphi_{\bar{U}}'[U] \) and the pressure \( \tilde{\psi} := \psi_{\bar{U} + U} - \psi_{\bar{U}} \), \( \tilde{\psi} := u_{\bar{U} + U} - u_{\bar{U}} - w_{\bar{U}}'[U] \), where we recall that \( z \) is a weak solution of
(5.7). Then we can check that \((\xi, \tilde{\psi})\) is the unique weak solution of the equation
\[
\begin{align*}
\mathcal{E}_\xi \xi + \nabla \tilde{\psi} &= -\kappa \Delta z + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \text{in} \quad \Omega_0, \\
\nabla \cdot \xi &= 0 \quad \text{in} \quad \Omega_0, \\
\xi(x, T) - \mu \Delta \xi(x, T) &= 0 \quad \text{in} \quad \Omega,
\end{align*}
\]
where \(\mathcal{W}_3 := - (\nabla z)^T \varphi_{\bar{u}} - (\nabla \tilde{u})^T \tilde{\varphi} + (z \cdot \nabla) \varphi_{\bar{u}} + (\tilde{u} \cdot \nabla) \tilde{\varphi},\)
\[
\begin{align*}
\mathcal{W}_4 &:= - \beta f''(u_{\bar{u}})[z, \varphi_{\bar{u}}] - \beta (f'(u_{\bar{u}+u}) - f'(u_{\bar{u}})) \tilde{\varphi} \\
&= - \beta f''(u_{\bar{u}})[z, \varphi_{\bar{u}}] - \beta \int_0^1 f''(u_{\bar{u} + \theta \Delta t})[\tilde{u}, \tilde{\varphi}]d\theta, \\
\mathcal{W}_5 &:= - \beta (f'(u_{\bar{u}+u})\varphi_{\bar{u}} - f'(u_{\bar{u}})\varphi_{\bar{u}} - f''(u_{\bar{u}})[\tilde{u}, \varphi_{\bar{u}}]) \\
&= - \beta \int_0^1 (1 - \theta) f''(u_{\bar{u} + \theta \Delta t})[\tilde{u}, \tilde{\varphi}]d\theta,
\end{align*}
\]
where we used Taylor’s formula and \(\tilde{u} := u_{\bar{u}+u} - u_{\bar{u}}, \quad \tilde{\varphi} := \varphi_{\bar{u}+u} - \varphi_{\bar{u}}.\)

Let us estimate the right-hand side of (6.5). Multiplying (6.5) by \(\xi\), repeating the calculations (5.29)-(5.30), and using (2.3), one may obtain that
\[
|\langle -\kappa \Delta z + \mathcal{W}_3, \xi \rangle| \leq C(\delta_3) \left( \|z(t)\|_V^2 + \|z(t)\|_V^2 \|\varphi_{\bar{u}}(t)\|_V^2 + \|\tilde{u}(t)\|_V^2 \|\tilde{\varphi}(t)\|_V^2 + \delta_3 \|\xi(t)\|_V^2. \right.
\]
For any \(r \geq 3\), by invoking the second derivative formula (5.10), we infer from the inequality (5.16) that
\[
|\langle \mathcal{W}_4, \xi \rangle| \leq C(\delta_5) \|u_{\bar{u}}(t)\|_{H^2}^{2(r-2)} \|z(t)\|_V^2 \|\varphi_{\bar{u}}(t)\|_V^2 + \delta_5 \|\xi(t)\|_V^2 + C(\delta_5) \left( \|u_{\bar{u}+u}(t)\|_{H^2}^{2(r-2)} + \|u_\ell(t)\|_{H^2}^{2(r-2)} \|\tilde{u}(t)\|_V^2 \|\tilde{\varphi}(t)\|_V^2. \right.
\]
Let us use (5.10) to compute the third derivative \(f'''(\cdot, \cdot, \cdot)^2\) and the embedding \(\mathbb{V} \hookrightarrow \mathbb{L}^6\) to get
\[
|\langle \mathcal{W}_5, \xi \rangle| \leq \beta \sup_{\theta \in (0,1)} \left\| \int_0^1 (1 - \theta) f'''(u_{\bar{u} + \theta \Delta t})[\tilde{u}, \tilde{\varphi}]d\theta \right\|_H \|\xi(t)\|_H
\]
\[
\leq C(\delta_5) \int_\Omega (|u_{\bar{u}+u}| + |u_{\bar{u}}|)^{2(r-3)} |\tilde{u}|^2 |\varphi_{\bar{u}}|^2 dx + \delta_5 \|\xi(t)\|_H^2
\leq C \left( \|u_{\bar{u}+u}(t)\|_{L^\infty}^{2(r-3)} + \|u_{\bar{u}}(t)\|_{L^\infty}^{2(r-3)} \right) \|\tilde{u}(t)\|_{L^\infty}^{4} \|\varphi_{\bar{u}}(t)\|_{L^\infty}^2 + \delta_5 \|\xi(t)\|_H^2
\leq C \left( \|u_{\bar{u}+u}(t)\|_{H^2}^{2(r-3)} + \|u_{\bar{u}}(t)\|_{H^2}^{2(r-3)} \right) \|\tilde{u}(t)\|_V^4 \|\varphi_{\bar{u}}(t)\|_V^2 + \delta_5 \|\xi(t)\|_V^2, \quad \text{for any} \ r \geq 7. \tag{6.7}
\]

\[\text{For any} \ r \geq 7, \ \text{we have} \]
\[
f'''(p)[q, g, h] = (r - 1)(r - 3)(r - 5)[p]^{r-3}(p \cdot q)(p \cdot g)(p \cdot h)\]
\[
+ (r - 1)(r - 3)[p]^{r-5}[(p \cdot g)(h \cdot q) + (p \cdot q)(h \cdot g) + (p \cdot q)(p \cdot g)h\]
\[
+ (p \cdot h)[(p \cdot g)g + (p \cdot q)g + (g \cdot q)h] + (r - 1)[p]^{r-3}[(h \cdot g)g + (h \cdot q)g + (g \cdot q)h].
\]

Further, for any \(3 < r < 7, p \neq 0\), the above formula is valid for \(f'''(p)[\cdot, \cdot, \cdot]\), and also one needs to set that \(f'''(p)[\cdot, \cdot, \cdot] = 0\), if \(p = 0\). For \(r = 3\), we get \(f'''(p)[q, g, h] = 2[(h \cdot g)g + (h \cdot q)g + (g \cdot q)h].\).
For simplicity, we write where \(V_\phi \) taking the inequalities (6.7) into account, choosing \(\delta_3 = \nu/4, \delta_5 = \alpha/4, \) we infer from (5.31), and the continuous embedding \(H^1(0, T; \mathbb{V}) \hookrightarrow L^\infty(0, T; \mathbb{V})\) that

\[
\Phi(t, \xi) \leq C \exp\left(CT\|u_\tilde{U}\|_{L^\infty(0,T;\mathbb{V})}^4 \right) \left[ M_1 \|z\|_{L^2(0,T;\mathbb{V})}^2 + M_2 \|\tilde{\varphi}\|_{L^\infty(0,T;\mathbb{V})}^2 \|\tilde{u}\|_{L^2(0,T;\mathbb{V})}^2 + M_3 \|\tilde{u}\|_{L^2(0,T;\mathbb{V})}^4 \right], \quad \forall t \in [0, T). (6.8)
\]

The last inequality follows from Proposition 1, (5.12), Theorem 5.5, Lemmas 5.1,6.1 and recall \(M_0\) defined in (5.8).

Multiplying (6.5) by \(-\xi_t\), one can obtain in view of (5.32) and straightforward modification of the inequalities (6.6)-(6.7) that

\[
\Phi_1(t, \xi) \leq C \left[ \left(\|u_\tilde{U}\|_{L^\infty(0,T;\mathbb{V})}^2 + \|u_\tilde{U}\|_{L^\infty(0,T;\mathbb{V})}^{2(r-2)} \right) \|\xi\|_{L^2(0,T;\mathbb{V})}^2 + \|\xi\|_{L^\infty(0,T;\mathbb{V})}^2 
+ M_1 \|z\|_{L^2(0,T;\mathbb{V})}^2 + M_2 \|\tilde{\varphi}\|_{L^\infty(0,T;\mathbb{V})}^2 \|\tilde{u}\|_{L^2(0,T;\mathbb{V})}^2 + M_3 \|\tilde{u}\|_{L^2(0,T;\mathbb{V})}^4 \right]
\leq C(C_2, K_1, K_3, R) \left(\|U\|_{L^2(0,T;H)}^3 + \|U\|_{L^2(0,T;H)}^4 \right), \quad \forall t \in [0, T), (6.9)
\]

where we also used (6.8). From (6.9), one may notice that \(\|\xi\|_{L^1(T)} \to 0\) as \(\|U\|_{L^2(T;H)} \to 0\), which completes the proof of part-(i).

To prove part-(ii), for any two controls \(U_1, U_2 \in \mathcal{U}\) and \(U \in L^2(0,T;\mathbb{H})\), let us set \(\tilde{\phi} := \varphi_{U_1}[U] - \varphi_{U_2}[U], \tilde{\psi} := \tilde{\psi}_{U_1} - \tilde{\psi}_{U_2}, \tilde{u} := u_{U_1} - u_{U_2}, \tilde{w} := w_{U_1} - w_{U_2}[U] \) and \(\tilde{\varphi} := \varphi_{U_1} - \varphi_{U_2}\). For simplicity, we write \(w_{U_i}[U], \varphi_{U_i}[U], i = 1, 2\) as \(w_{U_i}, \varphi_{U_i}, i = 1, 2\). Then \((\tilde{\phi}, \tilde{\psi})\) solves the equation

\[
\begin{cases}
\mathcal{E}_{U_i} \tilde{\phi} + \nabla \tilde{\psi} = -\kappa \Delta \tilde{w} + \mathfrak{F}_6 + \mathfrak{F}_7 + \mathfrak{F}_8 + \mathfrak{F}_9 \quad \text{in } \Omega_0 \\
\nabla \cdot \tilde{\phi} = 0 \quad \text{in } \Omega_0, \quad \tilde{\phi}(x, T) - \mu \Delta \tilde{\phi}(x, T) = 0 \quad \text{in } \Omega,
\end{cases}
\]

where \(\mathfrak{F}_6 = -\nabla \tilde{w}^T \varphi_{U_2} + (\tilde{u} \cdot \nabla) \varphi_{U_2}\) and \(\mathfrak{F}_7 = -\beta (f''(u_{U_1}) - f''(u_{U_2})) \varphi_{U_2}\), and

\[
\mathfrak{F}_8 := -\nabla \tilde{w}^T \varphi_{U_1} - (\nabla w_{U_1})^T \varphi + (w_{U_1} \cdot \nabla) \varphi + (\tilde{w} \cdot \nabla) \varphi_{U_2},
\]

\[
\mathfrak{F}_9 := -\beta (f''(u_{U_1}) - f''(u_{U_2})) [w_{U_1}, \varphi_{U_1}] - \beta f''(u_{U_2})[\tilde{w}, \varphi_{U_2}] - \beta f''(u_{U_2})[w_{U_1}, \varphi].
\]

The terms \(\mathfrak{F}_6, \mathfrak{F}_7\) can be estimated as in (6.3). In view of (2.3), one can get

\[
|\langle -\kappa \Delta \tilde{w} + \mathfrak{F}_8, \tilde{\phi}\rangle| \leq \delta_3 \|\tilde{\phi}(t)\|_V^2 + C(\delta_3) \left(\|w_{U_1}(t)\|_V^2 + \|w_{U_2}(t)\|_V^2\right) \|\varphi(t)\|_V^2 \\
+ C(\delta_3) \left(1 + \|\varphi_{U_1}(t)\|_V^2 + \|\varphi_{U_2}(t)\|_V^2\right) \|\tilde{w}(t)\|_V^2.
\]
Using again Taylor’s formula, we write the term \((f''(u_{U_1}) - f''(u_{U_2})) \left[ w'_{U_1}, \varphi_{U_1} \right] = \int_0^1 f'''(u_{U_2} + \theta \hat{u})[\hat{u}, w'_{U_1}, \varphi_{U_1}]d\theta\). By computations similar to (5.16) and (6.7), for any \(r \geq 3\), we arrive at
\[
|\{\mathfrak{U}_9, \hat{\varphi}\} \leq C(\delta_3)\|w_{U_2}(t)\|_{H^2}^{2(r-2)} (\|\hat{\omega}(t)\|_{V}^{2} + \|\hat{\varphi}(t)\|_{V}^{2}) + C(\delta_5)\left(\|w_{U_1}(t)\|_{H^2}^{2(r-3)} + \|w_{U_2}(t)\|_{H^2}^{2(r-3)}\right) \\
\times \|\hat{\omega}(t)\|_{V}^{2} \|w'_{U_1}(t)\|_{V}^{2} \|\varphi_{U_1}(t)\|_{V}^{2} + \delta_5 \|\hat{\varphi}(t)\|_{H^2}^{2}.
\]

The proof of this part follows by the same arguments of part-(i) or Theorem 5.5. Indeed, note that \(\varphi'_{U_1} \in L^\infty(0, T; V)\), by Theorems 5.2.5.5, the solutions \(w'_{U_1}, \varphi_{U_1} \in L^\infty(0, T; H^2)\), and by Theorem 3.4, strong solutions \(u'_{U_1} \in L^\infty(0, T; H^2)\). Thus, utilizing the Lipschitz continuity given by Lemmas 5.1, 6.1 and Proposition 2, one can conclude the proof. \(\square\)

Remark 5. It is clear from Propositions 2,3 that the control-to-state operator \(S : \mathcal{U} \to \mathcal{Z}_1\) and control-to-costate operator \(A : \mathcal{U} \to \mathcal{Z}_1\) are both Fréchet differentiable and the derivatives are Lipschitz continuous. We may even show that they are continuously differentiable under appropriate conditions on data and the growth value of \(r\).

6.2. Local optimality conditions. As we pointed out earlier in this section, second-order sufficient conditions for a local optimal control of (MOCP) are written only in the cone of critical directions. For details on the relation between the critical directions and second-order necessary/sufficient conditions, one may look at [41, 11].

Definition 6.2 (Critical Cone). For \(\bar{U} \in \mathcal{U}_{ad}\), let \(\mathcal{C}(\bar{U})\) denotes the set of all \(U \in L^2(0, T; \mathbb{H})\) such that
\[
U(x, t) \begin{cases} 
  \geq 0 & \text{if } \bar{U}(x, t) = U_{\text{min}}(x, t) \\
  \leq 0 & \text{if } \bar{U}(x, t) = U_{\text{max}}(x, t) \\
  = 0 & \text{if } \varphi(x, t) + \lambda \bar{U}(x, t) \neq 0, \text{ for almost all } (x, t) \in \Omega_T.
\end{cases}
\]

Theorem 6.3. Let \(\Omega\) be a periodic domain in \(\mathbb{R}^3\). Let \(\hat{U} \in \mathcal{U}_{ad}\) be any control with the adjoint state \(\varphi_{\hat{U}}\) satisfy the variational inequality (5.27). Moreover, assume that \(\mathfrak{A}(\hat{U})[U, U] > 0\), that is,
\[
-\int_{\Omega_T} (\varphi'_{\hat{U}}[U] \cdot U)dxdt < \lambda \|U\|_{L^2(0, T; \mathbb{H})}^2, \text{ for all } U \in \mathcal{C}(\hat{U}) \setminus \{0\}.
\]

Then there exist constants \(\theta > 0\) and \(\delta > 0\) such that for all \(\hat{U} \in \mathcal{U}_{ad}\), the following inequality holds:
\[
\mathfrak{A}(\hat{U}) \geq \mathfrak{A}(\bar{U}) + \frac{\theta}{2} \|\bar{U} - \hat{U}\|_{L^2(0, T; \mathbb{H})}^2, \text{ if } \|\bar{U} - \hat{U}\|_{L^2(0, T; \mathbb{H})} < \delta.
\]

In other words, the control \(\hat{U}\) is a strict local minimizer of the functional \(\mathfrak{A}(\cdot)\) on the set \(\mathcal{U}_{ad}\).

Proof. From the first-order Fréchet derivative (5.27) of \(\mathfrak{A}(U)\), the second derivative is given by
\[
\mathfrak{A}''(\hat{U})[U_1, U_2] = \lambda \int_{\Omega_T} U_1 \cdot U_2 dxdt + \int_{\Omega_T} \varphi'_{\hat{U}}[U_2] \cdot U_1 dxdt, \text{ for all } U_1, U_2 \in L^2(0, T; \mathbb{H}).
\]

It is evident that the condition \(-\int_{\Omega_T} (\varphi'_{\hat{U}}[U] \cdot U)dxdt < \lambda \|U\|_{L^2(0, T; \mathbb{H})}^2\) is equivalent to the positive definiteness of \(\mathfrak{A}''(\cdot)\), that is, \(\mathfrak{A}''(\hat{U})[U, U] > 0\), for all \(U \in \mathcal{C}(\hat{U}) \setminus \{0\}\).
In view of Theorems 4.1, 4.3 of [11] (see, also Theorem 27, [13]), one can infer that the proof of this theorem can be completed if the following two convergences are attained:

(i) For any sequence of admissible controls \( \{ \tilde{U}_k \} \subset U_{ad} \) and \( \{ U_k \} \subset L^2(0, T; \mathbb{H}) \) with \( \tilde{U}_k \rightharpoonup \tilde{U} \) and \( U_k \rightharpoonup U \) in \( L^2(0, T; \mathbb{H}) \), we need to show that \( J'(\tilde{U})U_k \rightarrow J'(\tilde{U})U \) as \( k \rightarrow \infty \).

(ii) For any sequence \( \{ U_k \} \subset L^2(0, T; \mathbb{H}) \) with \( U_k \rightharpoonup U \) in \( L^2(0, T; \mathbb{H}) \), it holds along a subsequence that

\[
\int_{\Omega_T} \varphi'_U[U_k] \cdot U_k dxdt \rightarrow \int_{\Omega_T} \varphi'_U[U] \cdot U dxdt \quad \text{as} \quad k \rightarrow \infty.
\]

To prove (i), recall the first-order Fréchet derivative of \( J(\cdot) \) is given by (5.27) that

\[
J'(\tilde{U})U = \int_{\Omega_T} (\varphi'_{\tilde{U}} + \lambda \tilde{U}) \cdot U dxdt, \quad \tilde{U} \in U_{ad}, \ U \in L^2(0, T; \mathbb{H}).
\]

Note that

\[
J'(\tilde{U})U_k - J'(\tilde{U})U = J'(\tilde{U})U_k - J'(\tilde{U})U_k + J'(\tilde{U})U_k - J'(\tilde{U})U.
\]  

By applying Hölder’s inequality, utilizing the fact that \( \{ U_k \} \) is uniformly bounded in \( L^2(0, T; \mathbb{H}) \) and the Lipschitz continuity (6.1), we obtain

\[
\left| J'(\tilde{U})U_k - J'(\tilde{U})U_k \right| 
\leq \left( \| \varphi'_{\tilde{U}_k} - \varphi'_{\tilde{U}} \|_{L^2(0, T; \mathbb{H})} + \lambda \| \tilde{U}_k - \tilde{U} \|_{L^2(0, T; \mathbb{H})} \right) \| U_k \|_{L^2(0, T; \mathbb{H})}
\leq C(K_3, \lambda) \| \tilde{U}_k - \tilde{U} \|_{L^2(0, T; \mathbb{H})} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,
\]

where we invoked \( \tilde{U}_k \rightharpoonup \tilde{U} \) in \( L^2(0, T; \mathbb{H}) \). Besides, since \( U_k \rightharpoonup U \) in \( L^2(0, T; \mathbb{H}) \) as \( k \rightarrow \infty \) and \( (\varphi'_{\tilde{U}} + \lambda \tilde{U}) \in L^2(0, T; \mathbb{H}) \), it is clear that \( J'(\tilde{U})U_k - J'(\tilde{U})U \rightarrow 0 \) as \( k \rightarrow \infty \). Thus, taking \( k \rightarrow \infty \) in (6.11) and using the above two convergences, we arrive at the proof of (i).

The proof of (ii) follows from the steps analogous to that of (i). Let us consider the difference

\[
\int_{\Omega_T} \varphi'_{\tilde{U}}[U_k] \cdot U_k dxdt - \int_{\Omega_T} \varphi'_{\tilde{U}}[U] \cdot U dxdt = \int_{\Omega_T} (\varphi'_{\tilde{U}}[U_k] - \varphi'_{\tilde{U}}[U]) \cdot U_k dxdt + \int_{\Omega_T} \varphi'_{\tilde{U}}[U] \cdot (U_k - U) dxdt := I_1 + I_2.
\]

Using Propositions 2, 3 and the compact embedding \( H^1(0, T; V) \hookrightarrow L^2(0, T; \mathbb{H}) \), by extracting a subsequence, one can obtain that \( \| \varphi'_{\tilde{U}}[U_k] - \varphi'_{\tilde{U}}[U] \|_{L^2(0, T; \mathbb{H})} \rightarrow 0 \) as \( k \rightarrow \infty \). Consequently, since \( \{ U_k \} \) is bounded in \( L^2(0, T; \mathbb{H}) \), by applying Hölder’s inequality, we get that \( |I_1| \leq \| \varphi'_{\tilde{U}}[U_k] - \varphi'_{\tilde{U}}[U] \|_{L^2(0, T; \mathbb{H})} \| U_k \|_{L^2(0, T; \mathbb{H})} \rightarrow 0 \) as \( k \rightarrow \infty \). Taking \( U_k \rightharpoonup U \) in \( L^2(0, T; \mathbb{H}) \) as \( k \rightarrow \infty \) into account, and again by Proposition 3, \( \varphi'_{\tilde{U}}[U] \) is bounded in \( L^2(0, T; \mathbb{H}) \), we obtain that \( I_2 \rightarrow 0 \) as \( k \rightarrow \infty \). Thus, by taking limit \( k \rightarrow \infty \) in (6.12), we complete the proof of (ii). \( \square \)
7. Global optimality conditions

From Theorem 6.3, we notice that a control $\tilde{U} \in \mathcal{U}_{ad}$ which satisfies the variational inequality together with a second-order sufficient condition (6.10) defined on a cone of critical directions is a local minimizer of the functional $\mathcal{J}(\cdot)$. However, it is unclear whether such a control gives a global optimum of (MOCP) and is unique. In the following result, we obtain a way around answering these questions by employing the idea developed for a semilinear elliptic control problem in [3] and also refer to [13] for the diffuse interface model of tumor growth. The main idea is to show that an admissible control satisfying the variational inequality together with a condition on the adjoint solution leads to a global optimal control of (MOCP).

**Theorem 7.1.** Let $\Omega$ be a periodic domain in $\mathbb{R}^3$. Let $\tilde{U} \in \mathcal{U}_{ad}$ be any control with the adjoint state $\varphi_{\tilde{U}}$ satisfy the variational inequality (5.27). In addition to that, assume the following conditions hold:

$$\frac{\kappa}{2} \geq \begin{cases} C\left(\|\varphi_{\tilde{U}}\|_{L^{\infty}(0,T;H)} + 2\beta C_r [\tilde{C}_2]^{r-2}\|\varphi_{\tilde{U}}\|_{L^{\infty}(0,T;H)}\right) & \text{if } r > 2 \\ C\left(\|\varphi_{\tilde{U}}\|_{L^{\infty}(0,T;H)} + 4\beta\|\varphi_{\tilde{U}}\|_{L^{\infty}(0,T;H)}\right) & \text{if } r = 2 \\ C\|\varphi_{\tilde{U}}\|_{L^{\infty}(0,T;H)} & \text{if } r = 1, \end{cases} \quad (7.1)$$

where the parameters $\kappa > 0$ and $\beta > 0$, the constants $C, \tilde{C}_2 > 0$ are from (2.3) and (3.6) (cf. (7.9)) respectively, and $C_r > 0$ depends on $r$.

Then for $r = 1$ and any $r \geq 2$, $\tilde{U} \in \mathcal{U}_{ad}$ is a global optimal control of (MOCP). Furthermore, if the conditions (7.1) are replaced with strict inequality ($< \frac{\kappa}{2}$), then the global optimal control $\tilde{U}$ is unique.

**Proof.** Let $U \in \mathcal{U}_{ad}$ be an arbitrary control. Let $u := u_U$ and $\tilde{u} := u_{\tilde{U}}$ be the strong solutions of (1.1) corresponding to $U$ and $\tilde{U}$ respectively. It is easy to check that the following inequality holds:

$$\mathcal{J}(U) - \mathcal{J}(\tilde{U}) = \frac{\kappa}{2} \int_0^T \|\nabla u(t) - \nabla \tilde{u}(t)\|_{H}^2 dt + \frac{\lambda}{2} \int_0^T \|U(t) - \tilde{U}(t)\|_{H}^2 dt$$

$$+ \kappa \int_{\Omega_T} \nabla(\tilde{u} - u_d) \cdot \nabla(u - \tilde{u}) dx dt + \lambda \int_{\Omega_T} \tilde{U} \cdot (U - \tilde{U}) dx dt.$$

$$\geq \frac{\kappa}{2} \int_0^T \|\nabla u(t) - \nabla \tilde{u}(t)\|_{H}^2 dt + \frac{\lambda}{2} \int_0^T \|U(t) - \tilde{U}(t)\|_{H}^2 dt + R, \quad (7.2)$$

where

$$R := \kappa \int_{\Omega_T} \nabla(\tilde{u} - u_d) \cdot \nabla(u - \tilde{u}) dx dt - \int_{\Omega_T} \tilde{\varphi} \cdot (U - \tilde{U}) dx dt, \quad (7.3)$$

and we used the variational inequality (5.27):

$$\lambda \int_{\Omega_T} \tilde{U} \cdot (U - \tilde{U}) dx dt \geq - \int_{\Omega_T} \tilde{\varphi} \cdot (U - \tilde{U}) dx dt,$$

for any $U \in \mathcal{U}_{ad}$, $\tilde{\varphi} := \varphi_{\tilde{U}}$ is the weak solution of the adjoint system (5.26).

Our main idea here is to show that $\mathcal{J}(U) \geq \mathcal{J}(\tilde{U})$ for all $U \in \mathcal{U}_{ad} \setminus \{\tilde{U}\}$. To attain this end, let us evaluate the lower bound of the integral $R$. For any $U \in \mathcal{U}_{ad} \setminus \{\tilde{U}\}$, let $\tilde{u} := u - \tilde{u}, \bar{U} := U - \tilde{U}$.
and $\hat{p} := p_u - p_{\hat{U}}$. For any $w \in \mathbb{V}$, we note that
\[
((u \cdot \nabla)u, w) - ((u \cdot \nabla)\hat{u}, w) = b(u - \bar{u}, u - \bar{u}, w) + b(\bar{u}, u - \bar{u}, w) + b(u - \bar{u}, \bar{u}, w) = ((\hat{u} \cdot \nabla)\hat{u}, w) + ((\bar{u} \cdot \nabla)\bar{u}, w) + ((\bar{u} \cdot \nabla)\bar{u}, w).
\]
Therefore, the triplet $(\hat{u}, \hat{p}, \hat{U})$ satisfies the system
\[
\begin{cases}
\hat{u}_t - \mu \Delta \hat{u}_t - \nu \Delta \hat{u} + (\hat{u} \cdot \nabla)\hat{u} + (\bar{u} \cdot \nabla)\bar{u} + \nabla \hat{p} + \alpha \hat{u} \\
\quad + (\bar{u} \cdot \nabla)\bar{u} + \beta f(u) - \beta f(\bar{u}) = \hat{U} \quad \text{in} \quad \Omega_T \\
\nabla \cdot \hat{u} = 0 \quad \text{in} \quad \Omega_T, \quad \hat{u}(x, 0) = 0 \quad \text{in} \quad \Omega.
\end{cases}
\tag{7.4}
\]
Taking inner product of (7.4) with $\varphi$, and integrating by parts, we obtain
\[
\int_{\Omega_T} (\varphi_t - \hat{u} - \mu \nabla \varphi_t \cdot \nabla \hat{u} + \nu \nabla \varphi \cdot \nabla \hat{u}) \, dx \, dt \\
+ \int_{\Omega_T} \left((\nabla \hat{u})^T \varphi - (\hat{u} \cdot \nabla)\varphi + \alpha \varphi + \beta f'(\bar{u})\varphi\right) \cdot \hat{u} \, dx \, dt \\
+ \int_{\Omega_T} (\hat{u} \cdot \nabla)\hat{u} \cdot \varphi \, dx \, dt + \beta \int_{\Omega_T} (f(u) - f(\bar{u}) - f'(\bar{u})\hat{u}) \cdot \varphi \, dx \, dt = \int_{\Omega_T} \hat{U} \cdot \varphi \, dx \, dt.
\tag{7.5}
\]
By testing (5.26) with $\hat{u}$ and comparing it with the left-hand side integrals of (7.5), the integrals in $R$ can be expressed as follows
\[
R = - \int_{\Omega_T} (\hat{u} \cdot \nabla)\hat{u} \cdot \varphi \, dx \, dt - \beta \int_{\Omega_T} (f(u) - f(\bar{u}) - f'(\bar{u})\hat{u}) \cdot \varphi \, dx \, dt \tag{7.6}
\]
\[
= - \int_{\Omega_T} \left((\hat{u} \cdot \nabla)\hat{u} + \beta \int_0^1 (1 - \theta) f''(\bar{u} + \theta \bar{u})[\hat{u}, \hat{u}] \, d\theta\right) \cdot \varphi \, dx \, dt,
\]
where we also invoked the second-order Taylor’s formula (5.9) for $r > 2$. Let us obtain a lower bound of $R$. By invoking (2.3), and the embedding $\mathbb{V} \hookrightarrow \mathbb{L}^4$, we get
\[
\left| \int_{\Omega} (\hat{u} \cdot \nabla)\hat{u} \cdot \varphi \, dx \right| = |b(\hat{u}, \hat{u}, \varphi)| \leq \|\hat{u}(t)\|_{\mathbb{L}^2}^2 \|\nabla \varphi(t)\|_{\mathbb{L}^2} \leq C\|\hat{u}(t)\|_{\mathbb{H}^1}^2 \|\varphi(t)\|_{\mathbb{V}}. \tag{7.7}
\]
For $r > 2$, using the second derivative formula (5.10) and Hölder’s inequality, we get
\[
\beta \left| \int_{\Omega} \int_0^1 (1 - \theta) f''(\bar{u} + \theta \bar{u})[\hat{u}, \hat{u}] \cdot \varphi \, d\theta \, dx \right| \\
\leq \beta C_r \sup_{\theta \in (0, 1)} \int_{\Omega} \int_0^1 |\theta u + (1 - \theta)\bar{u}|^{r-2} |\bar{u}|^2 |\varphi| \, d\theta \, dx \\
\leq \beta C_r \left(\|u(t)\|_{\mathbb{L}^\infty}^{r-2} + \|\bar{u}(t)\|_{\mathbb{L}^\infty}^{r-2}\right) \|\hat{u}(t)\|_{\mathbb{L}^2}^2 \|\varphi(t)\|_{\mathbb{L}^2} \\
\leq C\beta C_r \left(\|u\|_{\mathbb{L}^\infty(0, T; \mathbb{L}^\infty)}^{r-2} + \|\bar{u}\|_{\mathbb{L}^\infty(0, T; \mathbb{L}^\infty)}^{r-2}\right) \|\hat{u}(t)\|_{\mathbb{L}^2}^2 \|\varphi(t)\|_{\mathbb{H}^1},
\tag{7.8}
\]
where $C_r > 0$ depends only on $r$ and the constant $C > 0$ in (7.7) and (7.8) arises from the inequality $\|\hat{u}(t)\|_{\mathbb{L}^4} \leq \sqrt{C} \|\hat{u}(t)\|_{\mathbb{V}}$. Coupling (7.7),(7.8), using (3.6) and invoking the condition (7.1), we get
\[
|R| \leq C \left(\|\varphi\|_{\mathbb{L}^\infty(0, T; \mathbb{V})} + 2\beta C_r \|\bar{u}\|_{\mathbb{L}^\infty(0, T; \mathbb{H}^1)} \right) \|\hat{u}\|_{\mathbb{L}^2(0, T; \mathbb{V})}^2 \\
\leq \frac{\kappa}{2} \|\hat{u}\|_{\mathbb{L}^2(0, T; \mathbb{V})}^2,
\tag{7.9}
for all \( r > 2 \), where \( \widetilde{C}_2 := C_* \sqrt{C_2} \) and \( C_* \) is due to \( \| u(t) \|_{L^\infty} \leq C_* \| u(t) \|_{H^2} \). In the case of \( r = 2 \), we infer from (5.14) that the damping integral in (7.6) can be estimated using the first-order Taylor’s formula as follows

\[
\beta \int_0^1 \int_0^1 (f'(\tilde{u} + \theta \tilde{u})\tilde{u} - f'(\tilde{u})\tilde{u}) d\theta \cdot \tilde{\varphi} dx \leq 4C\beta \| \hat{u}(t) \|_{V}^{2} \| \tilde{\varphi}(t) \|_{H}. \tag{7.10}
\]

By the inequalities (7.7) and (7.10) together with (7.1), the estimate (7.9) holds for \( r = 2 \). Consequently, for \( r = 1 \) and any \( r \geq 2 \), \( R \geq -\frac{\kappa}{2} \| \hat{u} \|_{L^2(0,T;V)}^{2} \) for all \( U \in U_{ad}\{\tilde{U}\} \), and hence from (7.2), we arrive at the optimality inequality \( \mathfrak{J}(U) \geq \mathfrak{J}(\tilde{U}) \) for any \( U \in U_{ad}\{\tilde{U}\} \). Thus, an admissible control \( \tilde{U} \in U_{ad} \) satisfying the variational inequality (5.27) is a global optimal of (MOCP). Further, if the condition (7.1) is replaced by a strict inequality, then (7.9) holds with \( |R| < \frac{\kappa}{2} \| \hat{u} \|_{L^2(0,T;V)}^{2} \). In this case, it is evident that \( \mathfrak{J}(U) > \mathfrak{J}(\tilde{U}) \) for any \( U \in U_{ad}\{\tilde{U}\} \). Hence, the global optimal control \( \tilde{U} \in U_{ad} \) of (MOCP) is unique. This completes the proof. \( \Box \)

**Remark 6.** In section 5.3, we discussed the first-order optimality conditions (Theorem 5.5) in the bounded domain \( \Omega \) by restricting the growth of the damping term \( \beta |u|^{r-1}u \) to \( 2 \leq r \leq 5 \). By a careful study of the proof of Theorem 7.1, it is evident that to prove this theorem for the bounded domain, we only need to prove the inequality (7.8) with the help of the weak solutions of (1.1).

For any \( 2 < r \leq 5 \), we infer from (5.38) and (7.8) that

\[
\beta \left| \int_0^1 (1 - \theta) f''(\tilde{u} + \theta \tilde{u})[\tilde{u}, \tilde{u}] d\theta \cdot \tilde{\varphi} \right| \leq C_5 \beta C_r \left( \| \hat{u} \|_{L^\infty(0,T;L^{r+1})}^{r-2} + \| u \|_{L^\infty(0,T;L^{r+1})}^{r-2} \right) \| \hat{u}(t) \|_{V}^{2} \| \tilde{\varphi}(t) \|_{V}, \tag{7.11}
\]

where \( C_5 = C_6 C_7 > 0 \) stands for the constant from \( \| \hat{u}(t) \|_{L^{12(r+1)}} \leq \sqrt{C_6} \| \hat{u}(t) \|_{V} \) and \( \| \tilde{\varphi}(t) \|_{L^6} \leq C_7 \| \tilde{\varphi}(t) \|_{V} \). From (7.7) and (7.11), we obtain that \( |R| \leq \frac{\kappa}{2} \| \tilde{u} \|_{L^2(0,T;V)}^{2} \) for all \( 2 < r \leq 5 \), provided

\[
\frac{\kappa}{2} \geq \left( C + 2\beta C_5 C_r [\tilde{C}_1]^{r-2} \right) \| \tilde{\varphi} \|_{L^\infty(0,T;V)},
\]

where \( \tilde{C}_1 := \left[ \tilde{C}_2(r+1) \right]^{r+1/(r-1)} \), and \( \tilde{C}_1 \) is the constant from (3.24). When \( r = 2 \), the estimates (7.7), (7.10) and condition (7.1) show that the above bound for \( |R| \) holds true. Since the weak solution of (1.1) is obtained for the bounded domain, the global optimality conditions (Theorem 7.1) are valid for the bounded domain as well for all \( 2 \leq r \leq 5 \) and \( r = 1 \).

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