Countably QC-Approximating Posets

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Abstract

As a generalization of countably C-approximating posets, the concept of countably QC-approximating posets is introduced. With the countably QC-approximating property, some characterizations of generalized completely distributive lattices and generalized countably approximating posets are given. The main results are as follows: (1) a complete lattice is generalized completely distributive if and only if it is countably QC-approximating and weakly generalized countably approximating; (2) a poset L having countably directed joins is generalized countably approximating if and only if the lattice \( \sigma(L) \) of all \( \sigma \)-Scott-closed subsets of \( L \) is weakly generalized countably approximating.

1. Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [1]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpo s) was introduced and extensively studied (see [2–4]). Lawson in [4] gave a remarkable characterization that a dcpo \( L \) is continuous if and only if the lattice \( \sigma(L) \) of all \( \sigma \)-closed subsets of \( L \) is completely distributive. Gierz et al. in [5] introduced quasicontinuous domains, the most successful generalizations of continuous domains, and proved that quasicontinuous domains equipped with the Scott topology are precisely the spectra of hypercontinuous distributive lattices. Venugopalan in [6] introduced generalized completely distributive (GCD) lattices and Xu in his Ph.D. thesis [7] proved that GCD lattices are precisely the dual of hypercontinuous lattices. Ho and Zhao in [8] introduced the concept of C-continuous lattices. And they showed that any poset \( L \), \( \sigma(L) \) is a C-continuous lattice and that \( L \) is continuous if and only if \( \sigma(L) \) is continuous.

On the other hand, Lee in [9] introduced the concept of countably approximating lattices, a generalization of continuous lattices, and showed that this new larger class has many properties in common with continuous lattices. In [10], Han et al. further generalized the concept of countably approximating lattices to the concept of countably approximating posets and characterized countably approximating posets via the \( \sigma \)-Scott topology. Yang and Liu in [11] introduced the concept of generalized countably approximating posets, a generalization of countably approximating posets. Making use of the ideas of [8, 10], Mao and Xu in [12] introduced the concept of countably QC-approximating posets and showed that the lattice of all \( \sigma \)-Scott-closed subsets of a poset is a countably C-approximating lattice and that a complete lattice is completely distributive if and only if it is countably approximating and countably C-approximating.

In this paper, we generalize the concept of countably C-approximating posets to the concept of countably QC-approximating posets. With the countably QC-approximating property, we present some characterizations of GCD lattices and generalized countably approximating posets.
Definition 2 (see [3]). Let $L$ be a poset and $x, y \in L$. We say that $x$ is way-below $y$ or $x$ approximates $y$, written $x \ll y$ if whenever $D$ is a directed set that has a supremum $\sup D \geq y$, then there is some $d \in D$ with $x \leq d$. For each $x \in L$, we write $\ll x = \{y \in L \mid y \ll x\}$ and $\gg x = \{y \in L \mid x \ll y\}$. A poset $L$ having countably directed joins is called a countably approximating poset if for each $x \in L$, the set $\ll x$ is countably directed and $x = \vee \ll x$. A countably approximating poset which is also a complete lattice is called a countably approximating lattice.

Definition 3 (see [10]). Let $L$ be a poset and $x, y \in L$. We say that $x$ is countably way-below $y$, written $x \ll_{c} y$ if for any countably directed directed subset $D$ of $L$ with $\sup D \geq y$, there is some $d \in D$ with $x \leq d$. For each $x \in L$, we write $\ll_{c} x = \{y \in L \mid y \ll_{c} x\}$ and $\gg_{c} x = \{y \in L \mid x \ll_{c} y\}$. A poset $L$ having countably directed joins is called a countably approximating lattice if for each $x \in L$, the set $\ll_{c} x$ is countably directed and $x = \vee \ll_{c} x$. A countably approximating poset which is also a complete lattice is called a countably approximating lattice.

Example 4. Let $L$ be the unit interval $[0, 1]$. For all $x, y \in [0, 1]$, it is easy to check that $x \ll_{c} y \iff x \leq y$ and that $x \ll_{c} y \iff x = 0 = y$ or $x < y$.

By Remark 1, it is clear that every countable poset is a countably approximating poset.

Proposition 5. Let $L$ be a poset and $S$ a countable subset of $L$ such that $\forall S$ exists. If $s \ll_{c} x$ for all $s \in S$, then $\forall S \ll_{c} x$.

Proof. Straightforward.

By Proposition 5, in a complete lattice $L$, the set $\ll_{c} x$ is automatically countably directed for each $x \in L$. So, a complete lattice $L$ is countably approximating if and only if for each $x \in L$, $x = \vee \ll_{c} x$. Thus every continuous lattice is a countably approximating lattice.

Proposition 6. Let $L$ be a poset. If every countably directed subset of $L$ has a maximal element, then $L$ is a countably approximating poset.

Proof. Straightforward by Definition 3.

Example 7. Let $L$ be the complete lattice formed by uncountably many incomparable unit intervals $[0, 1]$ with all the 0's being pasted as a $\perp$ and all the 1's being pasted as a $\top$ (See Figure 1). Then it is easy to check that the resulting complete lattice satisfies the condition in Proposition 6 and thus is a countably approximating lattice.

Proposition 8. Let $L$ be a poset. If every countably directed subset of $L$ is countable, then $L$ is a countably approximating poset.

Proof. It is straightforward by Remark 1 and Proposition 6.

Example 9. If $\mathbb{N}$ with its usual order is augmented with uncountably many incomparable upper bounds, then it is easy to check that the resulting poset satisfies the condition in Proposition 8 and thus is a countably approximating poset.

For a set $X$, we use $\mathcal{P}(X)$ to denote the power set of $X$ and $\mathcal{P}_{\text{fin}}(X)$ to denote the set of all nonempty finite subsets of $X$. For a poset $L$, define a preorder $\leq$ (sometimes called Smyth preorder) on $\mathcal{P}(L) \setminus \{\emptyset\}$ by $G \leq H$ if and only if $\uparrow H \subseteq \uparrow G$ for all $G, H \subseteq L$. That is, $G \leq H$ if and only if for each $x \in G$, there is $y \in H$ such that $x \leq y$. A poset $L$ is Smyth complete if and only if $\mathcal{P}(L)$ is a complete lattice in $\mathcal{P}(L) \setminus \{\emptyset\}$ with the Smyth preorder. Figure 1. A complete lattice with countably directed sets having maximal elements.
\[y \in H\] there is an element \(x \in G\) with \(x \leq y\). We say that a nonempty family \(\mathcal{F}\) of subsets of \(L\) is (countably) directed if it is (countably) directed in the Smyth preorder. More precisely, \(\mathcal{F}\) is directed if for all \(F_1, F_2 \in \mathcal{F}\), there exists \(F \in \mathcal{F}\) such that \(F_1, F_2 \subseteq F\); that is, \(F \supseteq F_1 \cap F_2\).

Generalizing the relation \(\ll\) on points of \(L\) to the nonempty subsets of \(L\), one obtains the concept of weakly generalized countably approximating posets.

**Definition 10.** Let \(L\) be a poset having countably directed joins. A binary relation \(\ll\) on \(\mathcal{P}(L)\) \(\setminus \{\emptyset\}\) is defined as follows. \(A \ll B\) if and only if for any countably directed set \(D \subseteq L\), \(\forall D \in \uparrow B\) implies \(D \cap \uparrow A \neq \emptyset\). We write \(F \ll x\) for \(F \ll \{x\}\) and \(y \ll H\) for \(\{y\} \ll \{H\}\). If for each \(x \in L\), \(\uparrow x = \cap \{\uparrow F \mid F \in \omega(x)\}\), where \(\omega(x) = \{F \mid F \in \sigma(L)_{\text{fin}}(L)\text{ and } F \ll x\}\), then \(L\) is called a weakly generalized countably approximating poset. A weakly generalized countably approximating poset which is also a complete lattice is called a weakly generalized countably approximating lattice.

As a generalization of completely distributive lattice, the following concept of GCD lattices was introduced in \([6]\).

**Definition 11 (see \([6]\)).** Let \(L\) be a poset. A binary relation \(\ll\) on \(\mathcal{P}(L)\) is defined as follows. \(A \ll B\) if and only if whenever \(S\) is a subset of \(L\) for which \(\forall S\) exists, \(\forall S \in \uparrow B\) implies \(S \cap \uparrow A \neq \emptyset\). A complete lattice \(L\) is called a generalized completely distributive lattice or simply a GCD lattice, if and only if for all \(x \in L\), \(\uparrow x = \cap \{\uparrow F \mid F \in \mathcal{P}_{\text{fin}}(L)\text{ and } F \ll x\}\).

**Definition 12 (see \([3]\)).** A subset \(U\) of a poset \(L\) is Scott-open if \(\uparrow U = U\) and for any completely directed set \(D \subseteq L\), sup \(D \in U\) implies \(D \cap \uparrow U \neq \emptyset\). All the Scott-open sets of \(L\) form a topology, called the Scott topology and denoted by \(\sigma(L)\). The complement of a Scott-open set is called a Scott-closed set. The collection of all Scott-closed sets of \(L\) is denoted by \(\sigma(L)^{op}\). The topology on \(L\) generated by \(\{\uparrow x \mid x \in L\}\) as a subbase is called the upper topology and denoted by \(\sigma(L)^{\text{up}}\).

Replacing directed sets with countably directed sets in Definition 12, we can get the concept of \(\sigma\)-Scott-open sets.

**Definition 13 (see \([10]\)).** Let \(L\) be a poset. A subset \(U\) of \(L\) is called \(\sigma\)-Scott-open if \(\uparrow U = U\) and for any countably directed set \(D \subseteq L\), sup \(D \in U\) implies \(D \cap \uparrow U \neq \emptyset\). All the \(\sigma\)-Scott-open sets of \(L\) form a topology, called the \(\sigma\)-Scott topology and denoted by \(\sigma(L)\). The complement of a \(\sigma\)-Scott-open set is called a \(\sigma\)-Scott-closed set. The collection of all \(\sigma\)-Scott-closed sets of \(L\) is denoted by \(\sigma(L)^{op}\).

**Remark 14 (see \([10]\), Remark 2.1).** (1) For a poset \(L\), the \(\sigma\)-Scott topology \(\sigma(L)\) is closed under countably intersections and the Scott topology \(\sigma(L)^{\text{up}}\) is coarser than \(\sigma(L)\); that is, \(\sigma(L) \subseteq \sigma(L)^{\text{up}}\).

(2) A subset of a poset is \(\sigma\)-Scott-closed if and only if it is a lower set and closed under countably directed joins.

To study the order structure of the lattice of all \(\sigma\)-Scott-closed subsets for a poset, Mao and Xu in \([12]\) introduced the concept of countably \(C\)-approximating posets.

**Definition 15 (see \([12]\)).** Let \(L\) be a poset and \(x, y \in L\). We say that \(x\) is \(\sigma\)-beneath \(y\), denoted by \(x <_{\sigma} y\), if for any nonempty \(\sigma\)-Scott-closed set \(F \subseteq L\) for which \(\forall F\) exists, \(\forall F \geq y\) always implies that \(x \in F\). Poset \(L\) is said to be countably \(C\)-approximating if for each \(x \in L, x = \bigvee \{y \mid y <_{\sigma} x\}\). A complete lattice which is also countably \(C\)-approximating is called a countably \(C\)-approximating lattice.

**Lemma 16 (see \([12]\)).** For a poset \(L\), the lattice \(\sigma(L)^{op}\) is countably \(C\)-approximating.

**Proof.** Let \(L\) be a poset and \(C \in \sigma(\sigma(L)^{op})\). It is straightforward to check that \(\bigvee_{\sigma(L)^{op}} C = \bigcup \{F \mid F \in \sigma(L)_{\text{fin}}(L)\text{ and } F \ll C\}\). So, \(F = \bigvee_{\sigma(L)^{op}} \{x \mid x \in F\} \subseteq \bigcup \{x \mid x \in F\}\). Hence, \(F = \bigvee_{\sigma(L)^{op}} \{x \mid x \in F\}\) for all \(F \in \sigma(L)^{op}\), we have that \(\bigvee_{\sigma(L)^{op}} C = \bigcup \{F \mid F \in \sigma(L)_{\text{fin}}(L)\text{ and } F \ll C\}\). Thus, \(\sigma(L)^{op}\) is countably \(C\)-approximating. \(\square\)

3. Countably QC-Approximating Posets

In this section, we introduce the concept of countably QC-approximating posets. Firstly, we generalize the relation \(\ll\) on points of a poset \(L\) to the nonempty subsets of \(L\).

**Definition 17.** For a poset \(L\), the \(\sigma\)-beneath relation \(\ll\) on nonempty subsets of \(L\) is defined as follows: \(A \ll B\) if and only if whenever \(S\) is a nonempty \(\sigma\)-Scott-closed subset of \(L\) for which \(\forall S\) exists, \(\forall S \in \uparrow B\) implies \(S \cap \uparrow A \neq \emptyset\). We write \(F \ll x\) for \(F \ll \{x\}\). Set \(c(x) = \{F \mid F \in \mathcal{P}_{\text{fin}}(L)\text{ and } F \ll x\}\).

The next proposition is basic and the proof is omitted.

**Proposition 18.** Let \(L\) be a poset. Then

(i) \(\forall G, H \subseteq L, G \ll H \Rightarrow G \subseteq H\);

(ii) \(\forall G, H \subseteq L, G \ll H \Rightarrow \forall h \in H, G \ll h\);

(iii) \(\forall E, F, G, H \subseteq L, E \subseteq G \ll H \Rightarrow F \ll E \ll F\);

(iv) \(\forall x, y \in L, \{x\} \ll \{y\} \Rightarrow x \ll y\).

With the relation \(\ll\), we have the concept of countably QC-approximating posets.

**Definition 19.** A poset \(L\) is said to be countably quasi-C-approximating, shortly countably QC-approximating, if for all
\( x \in L, \uparrow x = \cap \{ \uparrow F \mid F \in c(x) \} \). A countably QC-approximating poset which is also a complete lattice is called a countably QC-approximating lattice.

**Proposition 20.** Countably C-approximating posets are countably QC-approximating.

**Proof.** Let \( L \) be a countably C-approximating poset. Then for all \( x \in L, \)

\[
\uparrow x \subseteq \cap \{ \uparrow F \mid F \in c(x) \} = \cap \{ \uparrow y \mid y <_{c,x} x \} \cap \{ \uparrow F' \mid F' \in c(x) \} \leq \cap \{ \uparrow y \mid y <_{c,x} x \} = \uparrow x.
\]

Thus \( \cap \{ \uparrow F \mid F \in c(x) \} = \uparrow x \). By Definition 19, \( L \) is countably QC-approximating.

By Lemma 16 and Proposition 20, we immediately have the following.

**Corollary 21.** For any poset \( L \), the lattice \( \sigma_c(L)^{op} \) is countably QC-approximating.

In the sequel, we explore relationships between countably QC-approximating lattices and GCD lattices.

**Proposition 22.** Every GCD lattice is weakly generalized countably approximating.

**Proof.** Let \( L \) be a GCD lattice. For all \( x \in L \) and \( F \in \mathcal{P}_{\text{fin}}(L) \), \( F < x \) implies \( F <_{\text{c}} x \). Then \( \uparrow x \subseteq \cap \{ \uparrow F \mid F \in \omega(x) \} \subseteq \cap \{ \uparrow F \mid F \in \mathcal{P}_{\text{fin}}(L) \text{ and } F < x \} = \uparrow x \). So \( \uparrow x = \cap \{ \uparrow F \mid F \in \omega(x) \} \). By Definition 19, \( L \) is countably QC-approximating.

**Proposition 23.** Every GCD lattice is countably QC-approximating.

**Proof.** Let \( L \) be a GCD lattice. For each \( x \in L \) and \( F \in \mathcal{P}_{\text{fin}}(L) \), \( F < x \) implies \( F <_{\text{c}} x \). Then \( \uparrow x \subseteq \cap \{ \uparrow F \mid F \in c(x) \} \subseteq \cap \{ \uparrow F \mid F \in \mathcal{P}_{\text{fin}}(L) \text{ and } F < x \} = \uparrow x \). Thus \( \uparrow x = \cap \{ \uparrow F \mid F \in c(x) \} \). By Definition 19, \( L \) is countably QC-approximating.

The following theorem characterizes GCD lattices.

**Theorem 24.** Let \( L \) be a complete lattice. Then the following statements are equivalent:

1. \( L \) is a GCD lattice;
2. \( L \) is countably QC-approximating and weakly generalized countably approximating.

**Proof.** (1) \( \Rightarrow \) (2): follows from Propositions 22 and 23.

(2) \( \Rightarrow \) (1): suppose that \( L \) is countably QC-approximating and weakly generalized countably approximating. Then for each \( x \in L \), by the weakly generalized countably approximating property of \( L \), we have \( \uparrow x = \cap \{ \uparrow F \mid F \in \omega(x) \} \).

Now for each \( F \in \omega(x) \), we show that \( \uparrow F = \cap \{ \uparrow F' \mid F' \in \mathcal{P}_{\text{fin}}(L) \text{ and } F' <_{\text{c}} F \} \). To this end, it suffices to show that \( \cap \{ \uparrow F' \mid F' \in \mathcal{P}_{\text{fin}}(L) \text{ and } F' <_{\text{c}} F \} \subseteq \cap \{ \uparrow F \mid F \in \omega(x) \} \). Suppose \( t \in \cap \{ \uparrow F' \mid F' \in \mathcal{P}_{\text{fin}}(L) \text{ and } F' <_{\text{c}} F \} \) and \( t \notin \cap \{ \uparrow F \mid F \in \omega(x) \} \). Then for any \( y_F \in F, t \notin y_F \). By the countably QC-approximating property of \( L \), there exists \( F_y \in \omega(y_F) \) such that \( F_y <_{\text{c}} y_F \) and \( t \notin \cap \{ y_F \mid y_F \notin F \} \).

It is clear that \( t \notin \bigcup y_F \), contradicting to that \( t \in \cap \{ \uparrow F' \mid F' \in \mathcal{P}_{\text{fin}}(L) \text{ and } F' <_{\text{c}} F \} \).

\( \square \)

Recall that a poset \( L \) is called a hypercontinuous poset (see [13]) if for all \( x \in L \), the set \( \{ y \in L \mid y <_{\text{c}} x \} \) is directed and \( x = \sup \{ y \in L \mid y <_{\text{c}} x \} \). A hypercontinuous poset which is also a complete lattice is called a hypercontinuous lattice.

**Lemma 25** (see [7], Theorem 4.1.4). Let \( L \) be a complete lattice. Then \( L \) is a GCD lattice if and only if \( L^{op} \) is a hypercontinuous lattice.

It is easy to see that for a finite lattice \( L \), both \( L \) and \( L^{op} \) are continuous, and \( \sigma(L) = \sigma(L) \). It follows from ([14], Theorem 2.1) that \( L \) and \( L^{op} \) are hypercontinuous lattices; hence by Lemma 25, \( L^{op} \) and \( L \) are GCD lattices. By this observation, we see that every finite lattice is a countably QC-approximating lattice. So, countably QC-approximating lattices need not be distributive.

It is known from Proposition 4.1 in [12] that any countably C-approximating lattice is distributive. So, countably QC-approximating lattices need not be countably C-approximating.

**Lemma 26** (see [11], Theorem 3.4). Let \( L \) be a poset having countably directed joins. Then \( L \) is generalized countably approximating if and only if the lattice \( \sigma_c(L) \) is hypercontinuous.

So, in view of Lemma 25, a poset having countably directed joins is generalized countably approximating if and only if the lattice \( \sigma_c(L)^{op} \) is a GCD lattice. The following theorem gives comprehensive characterizations of generalized countably approximating posets.

**Theorem 27.** Let \( L \) be a poset having countably directed joins. Then the following statements are equivalent:

1. \( L \) is a generalized countably approximating poset;
(ii) $\sigma_{c}(L)$ is a hypercontinuous lattice;
(iii) $\sigma_{c}(L)^{op}$ is a GCD lattice;
(iv) $\sigma_{c}(L)^{op}$ is a weakly generalized countably approximating lattice.

Proof. (i) $\iff$ (ii) by Lemma 26.
(ii) $\iff$ (iii) by Lemma 25.
(iii) $\iff$ (iv) follows from Theorem 24 and Corollary 21.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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