QCD threshold corrections to di-lepton and Higgs rapidity distributions beyond N^2LO

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ABSTRACT

We present threshold enhanced QCD corrections to rapidity distributions of di-leptons in the Drell-Yan process and of Higgs particles in both gluon fusion and bottom quark annihilation processes using Sudakov resummed cross sections. We have used renormalisation group invariance and the mass factorisation theorem that these hard scattering cross sections satisfy as well as Sudakov resummation of QCD amplitudes. We find that these higher order threshold QCD corrections stabilise the theoretical predictions under scale variations.
Perturbative Quantum Chromodynamics (pQCD) provides a framework to successfully compute various observables in the collisions of hadrons at high energies. Recent theoretical advances in the computations of higher order QCD radiative corrections have lead to precise results for several important observables. Because of this progress, we can now predict these observables with unprecedented accuracy for physics studies at the Tevatron collider in Fermilab as well as at the upcoming Large Hadron Collider (LHC) in CERN [1].

The Drell-Yan (DY) production of di-leptons [2] has been one of the most important probes of the structure of hadrons. It is also one of the dominant production processes at hadron colliders. At the LHC, it will serve as a luminosity monitor which is very important to precisely calibrate the machine for searches for physics beyond the Standard Model (SM). In DY production, a pair of leptons is produced through the decay of virtual photons, Z and W bosons that result from the collisions of incoming partons (quarks and gluons) in the hadrons. At hadron colliders, the DY process provides precise measurements of various standard model parameters. Rapidity distributions of Z bosons [3] and charge asymmetries of leptons coming from W boson decays [4] can probe the structure of the hadrons and possible excess events in di-lepton invariant mass distributions can point to physics beyond the standard model such as R-parity violating supersymmetric models and models with Z′, or with contact interactions [5]. Both D0 and CDF collaborations [6] at the Fermilab Tevatron made precise measurements of Z and W production cross sections and asymmetries which not only allowed for stringent tests of the standard model but also play an important role in the Higgs search at future colliders. These measurements are also possible at the LHC due to the large cross sections for the DY process.

The other process which is equally important is Higgs boson production at these colliders because it will establish the Standard Model as well as look for beyond the SM Higgs [7, 8]. The Higgs boson, which is responsible for the electroweak symmetry breaking in the Standard Model, is yet to be discovered. The search for this particle has been going on at the Fermilab Tevatron and is one of the most important tasks for the CERN LHC. The LEP experiments in the past provided vital information on the possible mass range of this particle [9]. The lower bound on the mass is 114.4 GeV/$c^2$ and an upper bound is less than 219 GeV/$c^2$ at 95% CL [10]. At the LHC, Higgs bosons will be predominantly produced through the gluon fusion process due to the large flux of gluons in the protons at these energies. They can be detected through the rare two photon decay mode which has less QCD background than other signals.

In pQCD, the total cross sections for the DY production of di-leptons and Higgs boson production are known upto next-to-next-to-leading order (NNLO) level [11–27]. However, due to the complexity involved with the top quark loops, the Higgs production cross sections are only known in the large top quark mass limit beyond the next-to-leading order (NLO). In addition to these fixed order results, the resummation programs for the threshold corrections to the total cross sections for DY and Higgs production have also been very successful [28, 29](see also [30]). See [22, 31] for next-to-next-to-leading logarithmic (NNLL) resummation results. Due to several important QCD results at the three loop level that are recently available [32–38], the resummation up to N$^3$LL has also become a reality [39–42]. The fixed order partial soft-plus-virtual N$^3$LO corrections [39, 43]
to the DY and Higgs productions show the reliability of the perturbation theory and the stability against the scale variations. The fixed order results as well as the resummed results reveal very interesting structures in the perturbative QCD results (see, [42, 44–47]).

Infra-red safe observables, such as hadronic cross sections, are computed using the QCD improved parton model. Due to the factorisation property that certain hard scattering cross sections satisfy, they can be expressed in terms of finite partonic cross sections convoluted with the parton distributions functions (PDFs). The partonic cross sections are calculated in QCD using standard perturbation theory in powers of the strong coupling constant $g_s$ that becomes small at high energies. The ultraviolet singularities that arise beyond leading order are often removed in the $\overline{\text{MS}}$ renormalisation scheme at a renormalisation scale $\mu_R$. The collinear singularities that result due to the presence of light mass partons are mass factorised into the bare parton densities in the $\overline{\text{MS}}$ scheme at a factorisation scale $\mu_F$. Hence the fixed order perturbative results are often sensitive to these scales $\mu_R$ and $\mu_F$. For example see [48] for a study of scale variations of the Higgs cross section in NLO. However results that are known to sufficiently high order in the strong coupling constant are often less sensitive to these scales because the observables are renormalisation group invariant. That is they would be strictly independent of choice of these scales if the entire perturbative expansion were known.

In addition to the scale uncertainties the fixed order computations suffer from the presence of various large logarithms which arise in some kinematical regions. These regions are often important from the experimental point of view. The large logarithms spoil the standard perturbative approach. The alternate approach is to resumm these logarithms in a closed form. Resumming a class of large logarithms supplemented with fixed order results can usually cover the entire kinematic region of the phase space. In this paper, we will mainly concentrate on a class of logarithms that arise in the threshold regions. These threshold corrections are further enhanced when the fluxes of the incoming partons become large in those regions. In the case of Higgs production through gluon fusion, the gluon flux at small partonic energies becomes large enhancing the role of threshold corrections. Here we examine the effects of soft gluons that originate in the threshold region of the phase space when we consider the $x_F$ and rapidity distributions in DY production and Higgs production through both gluon fusion and bottom quark annihilation. Here the large logarithms are generated when the gluons that are emitted from the incoming/outgoing partons become soft.

In [42], we found that soft distribution functions of Drell-Yan and Higgs production cross sections in perturbative QCD are maximally non-abelian. That is, we found that the soft distribution function for Higgs production can be obtained entirely from the DY process by a simple multiplication of the colour factor $C_A/C_F$. In the article [43], using the soft distribution functions extracted from DY, and the form factor of the Yukawa coupling of Higgs to bottom quarks, we predicted the soft-plus-virtual (sv) part of the Higgs production through bottom quark annihilation beyond NNLO with the same accuracy that the DY process and the gluon fusion to Higgs process are known [39, 42]. We extended [44] this approach to entirely different processes such as Higgs decay to bottom quarks and hadroproduction in $e^+e^-$ annihilation. The approach that we followed
in [42–44] is closely related to that of the standard threshold resummation and hence we could determine [42] the threshold exponents $D^I_i$ up to three loop level for DY and Higgs production using our resummed soft distribution functions and $B^I_i$ for both deep inelastic scattering and Higgs decay and hadroproduction. In this paper we extend this approach to include differential cross sections such as $x_F$ and rapidity distributions of the di-lepton pair in DY production and of Higgs bosons in Higgs production processes.

In the following we systematically formulate a framework to resum the dominant soft gluon contributions to these differential cross sections. We perform the resummation in the $z_i (i = 1,2)$ space of the kinematic variables, which are the appropriate scaling variables that enter the differential partonic cross sections. The threshold region corresponds to $z_i \to 1$ and in this region all the partonic cross sections are symmetric in $z_1 \leftrightarrow z_2$. We have used renormalisation group (RG) invariance, mass factorisation and Sudakov resummation of QCD amplitudes as guiding principles to perform the resummation in this region. Using the resummed results in $z_i$ space we predict the soft-plus-virtual parts (also called threshold corrections) of the dominant partonic differential cross sections beyond $N^2$LO. We also study the numerical effect of our predictions on both the $x_F$ and rapidity distributions of di-leptons and Higgs bosons. The analytical results are presented in the Appendices for both DY and Higgs production \(^1\) through gluon fusion. For an early reference where the resummation for DY differential distributions at rapidity $Y = 0$ (or $x_F = 0$) was considered consult [49].

The differential cross section can be expressed as:

$$\frac{d\sigma^I}{dx} = \sigma^I_{\text{Born}}(x_1^0, x_2^0, q^2) W^I(x_1^0, x_2^0, q^2), \quad I = q, b, g,$$

(1)

with the normalisation $W^I_{\text{Born}}(x_1^0, x_2^0, q^2) = \delta(1 - x_1^0) \delta(1 - x_2^0)$. The $x_i^0$ ($i = 1, 2$) are related to the kinematical variables $q^2$ and $x$. Here $q$ is the momentum of the di-lepton pair in the DY process and of the Higgs boson in the Higgs production. The variable $x$ can be the $x_F$ or rapidity of the di-lepton pair or of the Higgs boson. For di-lepton production, $I = q$ and $\sigma^I = d\sigma^q(\tau, q^2, x)/dq^2$ with $q^2$ the invariant mass of the final state di-lepton pair i.e., $q^2 = M^2_{I, j}$. For Higgs production through gluon fusion, $I = g$ and $\sigma^I = \sigma^g(\tau, q^2, x)$ and for Higgs production through bottom quark annihilation $I = b$ and $\sigma^I = \sigma^b(\tau, q^2, x)$ with $q^2 = m_H^2$ where $m_H$ is the mass of the Higgs boson. The variable $\tau = q^2/S$ with $S = (p_1 + p_2)^2$ the center of mass energy squared where $p_i$ are the momenta of incoming hadrons $P_i$ ($i = 1, 2$). In the QCD improved parton model, the function $W^I(x_1^0, x_2^0, q^2)$ can be expressed in terms of the PDFs appropriately convoluted with perturbatively calculable partonic differential cross sections denoted by $\Delta^I_{d, ab}$, where the subscript "d" stands for "differential", as follows

$$W^I(x_1^0, x_2^0, q^2) = \sum_{ab = q, g} \int_0^1 dx_1 \int_0^1 dx_2 \Delta^I_{d, ab}(x_1, x_2, \mu^2_F)$$

\(^1\)The results for the Higgs production via bottom quark annihilation are not presented here but can be obtained from the authors on request.
Here, $\mu_R$ is the renormalisation scale and $\mu_F$ the factorisation scale. We consider the differential cross sections for two kinematic variables namely

$$x = x_F = \frac{2(p_1 - p_2) \cdot q}{S}$$

and

$$x = Y = \frac{1}{2} \log \left( \frac{p_2 \cdot q}{p_1 \cdot q} \right).$$

For the $x_F$ ($x = x_F$) distribution, the $x_i^0$ variables satisfy

$$x_F = x_1^0 - x_2^0, \quad \tau = x_1^0 x_2^0,$$

while for the rapidity $Y$ ($x = Y$) distribution, we have

$$Y = \frac{1}{2} \log \left( \frac{x_1^0}{x_2^0} \right), \quad \tau = x_1^0 x_2^0.$$

Here, the function $\mathcal{H}^I_{ab}(x_1, x_2, \mu_F^2)$ is the product of PDFs $f_a(x_1, \mu_F^2)$ and $f_b(x_2, \mu_F^2)$ renormalised at the factorisation scale $\mu_F$. That is,

$$\mathcal{H}^I_{ab}(x_1, x_2, \mu_F^2) = f^I_a(x_1, \mu_F^2) f^I_b(x_2, \mu_F^2),$$

$$\mathcal{H}^{sv}_{ab}(x_1, x_2, \mu_F^2) = x_1 f^I_a(x_1, \mu_F^2) x_2 f^I_b(x_2, \mu_F^2),$$

with $x_i$ ($i = 1, 2$) the momentum fractions of the partons in the incoming hadrons.

We first study the contributions coming from the soft gluons. The infra-red safe contributions from the soft gluons can be obtained by adding the soft part of the differential cross sections with the ultraviolet renormalised virtual contributions and performing mass factorisation using appropriate counter terms. This combination is called the "soft-plus-virtual" (sv) part of the differential cross section. We call the remaining part the hard part of the differential cross section. Hence we write

$$\Delta^I_{d,ab}(z_1, z_2, q^2, \mu_R^2, \mu_F^2) = \Delta^{\text{hard}}_{I,ab}(z_1, z_2, q^2, \mu_F^2, \mu_R^2) + \delta_{ab}^{\delta} \Delta^{\text{sv}}_{d,1}(z_1, z_2, q^2, \mu_F^2, \mu_R^2), \quad I = q, b, g. \quad (7)$$

The contributions coming from the hard parts $\Delta^{\text{hard}}_{I,ab}(z_1, z_2, q^2, \mu_F^2, \mu_R^2)$ of the differential cross sections can be obtained by the standard procedure discussed in detail in [50,51]. The soft-plus-virtual parts of the differential cross sections ($\Delta^{\text{sv}}_{d,1}(z_1, z_2, q^2, \mu_R^2, \mu_F^2)$) are found to be

$$\Delta^{\text{sv}}_{d,1}(z_1, z_2, q^2, \mu_R^2, \mu_F^2) = C \exp \left( \Psi^I_d(q^2, \mu_R^2, \mu_F^2, z_1, z_2, \epsilon) \right) \bigg|_{\epsilon = 0}, \quad (8)$$

where $\Psi^I_d(q^2, \mu_R^2, \mu_F^2, z_1, z_2, \epsilon)$ are finite distributions. They are computed in $4 + \epsilon$ dimensions and take the form

$$\Psi^I_d(q^2, \mu_R^2, \mu_F^2, z_1, z_2, \epsilon) = \left( \ln \left( Z^I(\hat{s}, \mu_R^2, \mu_F^2, \epsilon) \right)^2 + \ln |F^I(\hat{s}, Q^2, \mu_F^2, \epsilon)|^2 \right) \delta(1-z_1) \delta(1-z_2) + 2 \Phi^I_d(\hat{s}, q^2, \mu_F^2, z_1, z_2, \epsilon) - C \ln \Gamma_I(\hat{s}, \mu^2, \mu_F^2, z_1, \epsilon) \delta(1-z_2) - C \ln \Gamma_I(\hat{s}, \mu^2, \mu_F^2, z_2, \epsilon) \delta(1-z_1), \quad I = q, b, g. \quad (9)$$
The symbol "C" means convolution. For example, $C$ acting on the exponential of a function $f(z_1, z_2)$ means the following expansion:

$$C e^f(z_1, z_2) = \delta(1 - z_1)\delta(1 - z_2) + \frac{1}{1!} f(z_1, z_2) + \frac{1}{2!} f(z_1, z_2) \otimes f(z_1, z_2)$$

$$+ \frac{1}{3!} f(z_1, z_2) \otimes f(z_1, z_2) \otimes f(z_1, z_2) + \cdots$$ (10)

In the rest of the paper the function $f(z_1, z_2)$ is a distribution of the kind $\delta(1 - z_j)$ or $D_i(z_j)$, where

$$D_i(z_j) = \left[ \ln^i (1 - z_j) \right]_+ \quad i = 0, 1, \cdots, \quad J = 1, 2,$$ (11)

and the symbol $\otimes$ means the "double" Mellin convolution. It convolutes with respect to the variables $z_1$ and $z_2$ separately. Since we are only interested in the sv part of the cross sections, we drop all the regular functions that result from various convolutions. $F^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)$ are the form factors that contribute to di-lepton ($I = q$) (in DY) and Higgs ($I = g, b$) production. In the form factors, we have $Q^2 = -q^2$. The partonic cross sections depend on two scaling variables $z_1, z_2$. The functions $\Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon)$ are called the soft distribution functions. The unrenormalised (bare) strong coupling constant $\hat{a}_s$ is defined as

$$\hat{a}_s = \frac{g_s^2}{16\pi^2},$$ (12)

where $g_s$ is the strong coupling constant which is dimensionless in $n = 4 + \varepsilon$ space time dimensions. The scale $\mu$ comes from the dimensional regularisation which makes the bare coupling constant $\hat{g}_s$ dimensionless in $n$ dimensions. The bare coupling constant $\hat{a}_s$ is related to renormalised one by the following relation:

$$S_E \hat{a}_s = Z(\mu_R^2) a_s(\mu_R^2) \left( \frac{\mu^2}{\mu_R^2} \right)^{\varepsilon/2},$$ (13)

where $S_E = \exp \left\{ \frac{\varepsilon}{2} \gamma_E - \ln 4\pi \right\}$ is the spherical factor characteristic of $n$-dimensional regularisation. The renormalisation constant $Z(\mu_R^2)$ relates the bare coupling constant $\hat{a}_s$ to the renormalised one $a_s(\mu_R^2)$. They are both expressed in terms of the perturbatively calculable coefficients $\beta_i$ which are known up to four-loop level in terms of the colour factors of SU(N) gauge group:

$$C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_F = \frac{1}{2}.$$ (14)

Also we use $n_f$ for the number of active flavours. In the case of Higgs production, the number of active flavours is five because the top degree of freedom is integrated out in the large $m_{top}$ limit.

The factors $Z^I(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon)$ are the overall operator renormalisation constants. For the vector current $Z^A(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) = 1$, but both the gluon operator [52] and the bottom quark coupling to
Higgs [53] get overall renormalisations. They satisfy the following RG equations:

\[
\begin{align*}
\mu_R^2 \frac{d}{d \mu_R} \ln Z^g(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) &= \sum_{i=1}^{\infty} a_i^g(\mu_R^2) \left( i \beta_{i-1} \right), \\
\mu_R^2 \frac{d}{d \mu_R} \ln Z^b(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) &= \sum_{i=1}^{\infty} a_i^b(\mu_R^2) \gamma_{i-1}^b,
\end{align*}
\]

where the limit \(\varepsilon \rightarrow 0\) is taken. The constants \(i \beta_{i-1}\) and \(\gamma_{i-1}^b\) are the anomalous dimensions of the renormalised form factors \(F^g\) and \(F^b\) respectively.

The bare form factors \(\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)\) of both fermionic and gluonic operators satisfy the following differential equation that follows from the gauge as well as the renormalisation group invariances [54–57]. In dimensional regularisation,

\[
Q^2 \frac{d}{dQ^2} \ln \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon) = \frac{1}{2} \left[ K^I(\hat{a}_s, Q^2, \mu^2, \varepsilon) + G^I(\hat{a}_s, Q^2, \mu^2, \varepsilon) \right],
\]

where the \(K^I\) contain all the poles in \(\varepsilon\) and the \(G^I\) collect the rest of the terms that are finite as \(\varepsilon\) becomes zero. The fact that the \(\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \varepsilon)\) are renormalisation group invariant and the functions \(G^I\) are finite implies that the \(K^I\) terms can be expressed in terms of finite constants \(A^I\), the so-called cusp anomalous dimensions and the coefficients \(\beta_i\). The solution to the eqn. can be obtained as a series expansion in the bare coupling constant in dimensional regularisation. The formal solution up to four-loop level can be found in [34, 42].

The boundary conditions on the Sudakov differential equation, denoted by \(G^I_0(\varepsilon)\) (see eqn. of [43]) can be found for both \(I = q\) and \(I = g\) in [35] to the required accuracy in \(\varepsilon\). We have extended this in [43] to the form factor corresponding to the Yukawa interaction of the Higgs boson to the bottom quarks. These constants \(G^I_0(\varepsilon)\) are expressed in terms of the functions \(B^I_0\) and \(f^I_0\). The \(B^I_0\) are known up to order \(a_s^3\) through the three-loop anomalous dimensions (or splitting functions) [32, 33] and are found to be flavour independent, that is \(B^q_i = B^b_i\). The constants \(f^I_0\) are analogous to the cusp anomalous dimensions \(A^I_0\) that enter the form factors with \(A^q_i = A^b_i\). It was first noticed in [27] that the single pole (in \(\varepsilon\)) of the logarithm of the form factors up to two-loop level \(\hat{a}_s^2\) can be predicted due to the presence of constants \(f^I_0\) because these \(f^I_0\) are found to be maximally non-abelian obeying the relation

\[
\hat{f}^q_i = \hat{f}^b_i = \frac{C_F}{C_A} \hat{f}^g_i,
\]

similar to the \(A^I_0\). This relation has been found to hold even at the three loop level [35]. With this information we can now predict all the poles of the form factors at every order in \(\hat{a}_s\) from these constants \(A^I_0, B^I_0, f^I_0\), their anomalous dimensions, and the finite parts of the lower order (in \(\hat{a}_s\)) contributions to the form factors. Interestingly, the single pole terms in the form factors contain the combinations [27, 34, 42, 43]

\[
2 \left( B^I_0 - \delta_{I,g} i \beta_{i-1} - \delta_{I,b} \gamma_{i-1}^b \right) + f^I_0,
\]

\[^2\text{A similar analysis of the structure of single pole terms of four-point amplitudes at the two-loop level can be found in [58, 59].}\]
at order $\hat{a}_i^s$. The terms $-2\delta_{I,g} i \beta_{I-1} - 2\delta_{I,h} \gamma_{I-1}^h$ come from the ultraviolet divergences that are present in the loop integrals. These pole terms go away when the form factors undergo overall operator UV renormalisation through the renormalisation constants $Z^I$ which satisfy the RG equations given in eq. (15).

The collinear singularities that arise due to massless partons are removed using the mass factorisation kernel $\Gamma(z_j,\mu_F^2,\varepsilon)$ in the $\overline{\text{MS}}$ scheme (see eqn.(9)). We have suppressed the dependence on $\hat{a}_s$ and $\mu^2$ in $\Gamma$. The factorisation kernel $\Gamma(z_j,\mu_F^2,\varepsilon)$ satisfies the following renormalisation group equation:

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma(z_j,\mu_F^2,\varepsilon) = \frac{1}{2} P(z_j,\mu_F^2) \otimes \Gamma(z_j,\mu_F^2,\varepsilon),$$

(18)

where the $P(z_j,\mu_F^2)$ are the well-known DGLAP matrix-valued splitting functions which are known upto three-loop level [32, 33]:

$$P(z_j,\mu_F^2) = \sum_{i=1}^{\infty} a_s^i(z_j) P^{(i-1)}(z_j).$$

(19)

The diagonal terms in the splitting functions $P^{(i)}(z_j)$ have the following structure

$$P^{jI}_I(z_j) = 2 \left[ B_{i+1}^I \delta(1-z_j) + A_{i+1}^I D_0(z_j) \right] + P^{jI}_{\text{reg},I}(z_j),$$

(20)

where $P^{jI}_{\text{reg},I}(z_j)$ are regular when the argument approaches the kinematic limit (here $z_j \to 1$). The RG equations can be solved by expanding in powers of the strong coupling constant. For the soft-plus-virtual part of the differential cross sections, only the diagonal parts of the kernels contribute. We find the solutions contain only poles in $\varepsilon$ in the $\overline{\text{MS}}$ scheme:

$$\Gamma(z_j,\mu_F^2,\varepsilon) = \delta(1-z_j) + \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{\mu_F^2}{\mu^2} \right)^i \varepsilon^i \Gamma^{(i)}(z_j,\varepsilon).$$

(21)

The constants $\Gamma^{(i)}(z_j,\varepsilon)$ expanded in negative powers of $\varepsilon$ up to four-loop level can be found in [42]. The $\Gamma_I(\hat{a}_s,\mu^2,\mu_F^2,\varepsilon)$ in eqn.(19) is the diagonal element of $\Gamma(z_j,\mu_F^2,\varepsilon)$.

The fact that $\Delta^{jI}_I$ are finite in the limit $\varepsilon \to 0$ implies that the soft distribution functions should have a pole structure in $\varepsilon$ similar to that of $\hat{F}_I^{\mu}$ and $\Gamma_I$. To systematically study the soft distribution functions, we demand that they satisfy similar Sudakov type differential equations that the form factors $\hat{F}_I^{\mu}$ satisfy (see eqn.(16)):

$$q^2 \frac{d}{dq^2} \Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon) = \frac{1}{2} \left[ K_d^I \left( \hat{a}_s, \frac{\mu_F^2}{\mu^2}, z_1, z_2, \varepsilon \right) + G_d^I \left( \hat{a}_s, \frac{q^2}{\mu_F^2}, \frac{\mu_F^2}{\mu^2}, z_1, z_2, \varepsilon \right) \right],$$

(22)

where again the constants $K_d^I$ contain all the singular terms in $\varepsilon$ and the $G_d^I$ are finite functions of $\varepsilon$. Also the functions $\Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon)$ satisfy the renormalisation group equations:

$$\mu_F^2 \frac{d}{d\mu_F^2} \Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon) = 0.$$

(23)
This renormalisation group invariance leads to the following equations
\[
\mu_R^2 \frac{d}{d\mu_R^2} \bar{K}_d^I (\hat{a}_s, \mu_R^2, z_1, z_2, \varepsilon) = -\bar{A}_d^I (a_s(\mu_R^2)) \delta(1-z_1) \delta(1-z_2),
\]
\[
\mu_R^2 \frac{d}{d\mu_R^2} \bar{G}_d^I (\hat{a}_s, \mu_R^2, z_1, z_2, \varepsilon) = \bar{A}_d^I (a_s(\mu_R^2)) \delta(1-z_1) \delta(1-z_2).
\]
If \( \Phi_d^I (\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon) \) contains the correct poles to cancel the poles coming from \( \hat{F}_1, \mathcal{Z}_1 \) and \( \Gamma_{II} \) in order to make \( \Delta_{d,I}^{\varepsilon} \) finite, then the constants \( \bar{A}_d^I \) have to satisfy
\[
\bar{A}_d^I = -A_d^I.
\]
Using the above relation, the solution to the RG equation for \( \bar{G}_d^I (\hat{a}_s, q^2/\mu_R^2, \mu_R^2/\mu^2, z_1, z_2, \varepsilon) \) is found to be
\[
\bar{G}_d^I (\hat{a}_s, q^2/\mu_R^2, \mu_R^2/\mu^2, z_1, z_2, \varepsilon) = \bar{G}_d^I (a_s(\mu_R^2), q^2/\mu_R^2, z_1, z_2, \varepsilon)
\]
\[
= \bar{G}_d^I (a_s(q^2), 1, z_1, z_2, \varepsilon)
\]
\[
- \delta(1-z_1) \delta(1-z_2) \int_{q^2/\mu_R^2}^{\mu_R^2} \frac{d\lambda}{\lambda^2} A_d^I (a_s(\lambda^2/\mu_R^2)).
\]
With these solutions, it is straightforward to solve the Sudakov differential equations yielding
\[
\Phi_d^I (\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon) = \Phi_d^I (\hat{a}_s, q^2(1-z_1)(1-z_2), \mu^2, \varepsilon)
\]
\[
= \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q^2(1-z_1)(1-z_2)}{\mu^2} \right)^i \frac{\xi}{\varepsilon} S_{\varepsilon}^I \left( \frac{(i \varepsilon)^2}{4(1-z_1)(1-z_2)} \right) \hat{\Phi}_d^I (\varepsilon).
\]
where
\[
\hat{\Phi}_d^I (\varepsilon) = \frac{1}{i \varepsilon} \left[ \bar{K}_d^I (\varepsilon) + \bar{G}_d^I (\varepsilon) \right].
\]
The above solutions for \( \Phi_d^I \) satisfy the fact that \( \Delta_{d,I}^{\varepsilon} \) are finite as \( \varepsilon \to 0 \) (see eqn.(9)). The constants \( \bar{K}_d^I (\varepsilon) \) are determined by expanding \( \bar{K}_d^I \) in powers of the bare coupling constant \( \hat{a}_s \) as follows
\[
\bar{K}_d^I (\hat{a}_s, \mu_R^2/\mu^2, z_1, z_2, \varepsilon) = \delta(1-z_1) \delta(1-z_2) \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{\mu_R^2}{\mu^2} \right)^i \frac{\xi}{\varepsilon} S_{\varepsilon}^I \bar{K}_d^I (\varepsilon),
\]
and solving the RG equation for \( \bar{K}_d^I (\hat{a}_s, \mu_R^2/\mu^2, z_1, z_2, \varepsilon) \). The constants \( \bar{K}_d^I (\varepsilon) \) are identical to \( \bar{K}_d^I (\varepsilon) \) given in [43]. The constants \( \bar{G}_d^I (\varepsilon) \) are related to the finite boundary functions \( \bar{G}_d^I (a_s(q^2), 1, z_1, z_2, \varepsilon) \). We define the \( \bar{G}_{d,I}^I (\varepsilon) \) through the relation
\[
\sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q^2(1-z_1)(1-z_2)}{\mu^2} \right)^i \frac{\xi}{\varepsilon} S_{\varepsilon}^I \bar{G}_{d,I}^I (\varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left( q^2(1-z_1)(1-z_2) \right) \bar{G}_{d,I}^I (\varepsilon)
\]
The $z_1, z_2$ independent constants $\overline{G}_{d, i}^I(\epsilon)$ are obtained by demanding the finiteness of $\Delta_{d, i}^{\mathrm{SV}}$ given in eqn. (8). Without setting $\epsilon = 0$ in eqn. (9), we expand $\Delta_{d, i}^{\mathrm{SV}}$ as

$$\Delta_{d, i}^{\mathrm{SV}}(z_1, z_2, q^2, \mu_R^2, \mu_F^2, \epsilon) = \sum_{i=0}^{\infty} d_i^I(\mu_R^2) \Delta_{d, i}^{\mathrm{SV}, (i)}(z_1, z_2, q^2, \mu_R^2, \mu_F^2, \epsilon).$$

(31)

Using the above expansion and eqn. (9) we determine these constants by comparing the pole as well as non-pole terms of the form factors, the mass factorisation kernels and the coefficient functions $\Delta_{d, i}^{\mathrm{SV}, (i-1)}$ expanded in powers of $\epsilon$ to the desired accuracy. Since the $G_{d, i}^I(\epsilon)$ in the form factors are found to satisfy a specific structure in terms of $f_i^I, \beta_i$ [43], we find that the constants $\overline{G}_{d, i}^I(\epsilon)$ also satisfy the following expansions containing these constants.

$$\overline{G}_{d, 1}^I(\epsilon) = -f_1^I + \sum_{k=1}^{\infty} \epsilon^k \overline{G}_{d, 1}^{I, (k)},$$

$$\overline{G}_{d, 2}^I(\epsilon) = -f_2^I - 2\beta_0 \overline{G}_{d, 1}^{I, (1)} + \sum_{k=1}^{\infty} \epsilon^k \overline{G}_{d, 2}^{I, (k)},$$

$$\overline{G}_{d, 3}^I(\epsilon) = -f_3^I - 2\beta_1 \overline{G}_{d, 1}^{I, (1)} - 2\beta_0 \left( \overline{G}_{d, 2}^{I, (1)} + 2\beta_0 \overline{G}_{d, 1}^{I, (2)} \right) + \sum_{k=1}^{\infty} \epsilon^k \overline{G}_{d, 3}^{I, (k)},$$

$$\overline{G}_{d, 4}^I(\epsilon) = -f_4^I - 2\beta_2 \overline{G}_{d, 1}^{I, (1)} - 2\beta_1 \left( \overline{G}_{d, 2}^{I, (1)} + 4\beta_0 \overline{G}_{d, 1}^{I, (2)} \right),$$

$$-2\beta_0 \left( \overline{G}_{d, 3}^{I, (1)} + 2\beta_0 \overline{G}_{d, 2}^{I, (2)} + 4\beta_0 \overline{G}_{d, 1}^{I, (3)} \right) + \sum_{k=1}^{\infty} \epsilon^k \overline{G}_{d, 4}^{I, (k)}.$$ 

(32)

Now that we have a better understanding [27] of the structure of even the single pole terms of the form factors, we can predict all the poles including the single pole of the soft distribution function from those of the form factors, the renormalisation constants and the mass factorisation kernels. The coefficients of the single poles are proportional to the constants $-f_i^I$ which are not only process independent but also maximally non-abelian. The $\epsilon$ dependent terms in $\overline{G}_{d}^I(\epsilon)$ can be obtained from the fixed order (in $a_s$) computations of cross sections and the finite parts of the form factors. At the moment, we know $\overline{G}_{d, 1}^I(\epsilon)$ to all orders in $\epsilon$, $\overline{G}_{d, 2}^I(\epsilon)$ to order $\epsilon$ and $\overline{G}_{d, 3}^I(\epsilon)$ to order $\epsilon^0$. The lowest order term $\overline{G}_{d, 1}^I(\epsilon)$ is known to all orders in $\epsilon$ from the exact fixed-order soft contribution at order $a_s$. The next-to-leading order in $a_s$ (lowest order) computations of the total cross sections for DY and Higgs production determine the constants $\overline{G}_{d, 1}^I(\epsilon)$ and the results reveal that they are maximally non-abelian to all orders in $\epsilon$. One can similarly determine the $\epsilon$ dependent parts of soft cross sections beyond the order $a_s$. The easier method is to use total cross sections that are known upto NNLO level. We can easily extract $\overline{G}_{d, 2}^I(\epsilon)$ upto order $\epsilon$ by using the fact that these constants are independent of $z_j$ ($j = 1, 2$) and the differential cross sections satisfy the relations

$$\int_0^1 dx_1^0 \int_0^1 dx_2^0 (x_1^0 x_2^0)^{N-1} (x_1^0 + x_2^0) \frac{d\sigma^l}{dxF} = \int_0^1 dx_1^0 \int_0^1 dx_2^0 (x_1^0 x_2^0)^{N-1} \frac{d\sigma^l}{dY} = \int_0^1 d\tau \tau^{N-1} \sigma^l.$$  

(33)
while the $\sigma^I$ are known for both DY and Higgs production up to NNLO level [17–27]. An alternative method is to take $N \to \infty$ on both sides of eqn. (33). In this limit, we find the following useful relation between the constants $\hat{\Phi}^I_d(i) (\epsilon)$ that appear in eqn. (27) and $\hat{\Phi}^I_d(i) (\epsilon)$ that contribute to the soft distribution function of the total cross section:

$$\hat{\Phi}^I_d(i) (\epsilon) = \frac{\Gamma(1+i\epsilon)}{\Gamma^2(1+i\epsilon^2)} \hat{\Phi}^I_d(i) (\epsilon).$$

Both the methods give

$$\overline{G}^{I,(1)}_{d,1} = C_I \left(-\zeta_2\right),$$

$$\overline{G}^{I,(2)}_{d,1} = C_I \left(\frac{1}{3}\zeta_3\right),$$

$$\overline{G}^{I,(3)}_{d,1} = C_I \left(\frac{1}{80}\zeta_2^2\right),$$

$$\overline{G}^{I,(1)}_{d,2} = C_IC_A \left(\frac{2428}{81} - \frac{67}{3}\zeta_2 - 4\zeta_2^2 - \frac{44}{3}\zeta_3\right)$$

$$+ C_{Imf} \left(-\frac{328}{81} + \frac{10}{3}\zeta_2 + \frac{8}{3}\zeta_3\right),$$

(35)

where $C_I = C_F$ for $I = q, b$ and $C_I = C_A$ for $I = g$. Interestingly these constants $\overline{G}^{I}_{d,i}(\epsilon)$ turn out to be maximally non-abelian. That is, they satisfy

$$\overline{G}^q_{d,i}(\epsilon) = \overline{G}^b_{d,i}(\epsilon) = C_F C_A^{-1} \overline{G}^g_{d,i}(\epsilon).$$

(36)

This implies that the soft distribution functions for the differential cross sections satisfy

$$\Phi^q_d(\tilde{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = \Phi^b_d(\tilde{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = C_F C_A^{-1} \Phi^g_d(\tilde{a}_s, q^2, \mu^2, z_1, z_2, \epsilon),$$

(37)

upto order $a_s^2$, similar to the soft distributions that appear in the total cross sections. We expect that this property will hold to all orders in perturbation theory because of the fact that it originates entirely from the soft part of the differential cross sections.

The threshold corrections dominate when the partonic scaling variables $z_1$ and $z_2$ approach their kinematic limit, which is unity, through the distributions $\delta(1-z_j)$ and $D_j(z_j)$ with $j = 1, 2$. Resummations of threshold enhanced contributions are usually done in Mellin $N$ space which has been a successful approach. See [28, 29, 60, 61] for resummation of total cross sections. We show in the following how the soft distribution functions $\Phi^I_d(\tilde{a}_s, q^2, \mu^2, z_1, z_2, \epsilon)$ capture all the features of the $N$ space resummation approach. The exponents of the $z_j$ (with $j = 1, 2$) space resummed cross sections get contributions from the form factors through $\delta(1-z_1)\delta(1-z_2)$ terms and from
the soft distribution functions through $\delta(1-z_j)$ as well as the distributions $D_l(z_j)$. We can rewrite the soft distribution function as

$$
\Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon) = \frac{1}{2} \delta(1-z_2) \left[ \frac{1}{1-z_1} \left\{ \int_{\mu_R^2}^{q^2(1-z_1)} \frac{d\lambda^2}{\lambda^2} A_l (a_s(\lambda^2)) \right\} \right. \\
\left. + \frac{q^2}{d q^2} \left[ \left( \frac{1}{4(1-z_1)(1-z_2)} \left\{ \int_{\mu_R^2}^{q^2(1-z_1)(1-z_2)} \frac{d\lambda^2}{\lambda^2} A_l (a_s(\lambda^2)) \right\} \right) \right. \\
\left. + \left( z_1 \leftrightarrow z_2 \right), \quad (38) \right]
$$

where

$$
\overline{G}_d^I (a_s(q^2 g(z_1, z_2)), \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q^2 g(z_1, z_2)}{\mu^2} \right)^{i\varepsilon} S_{\varepsilon} \overline{G}_d^{I,(i)} (\varepsilon). \quad (39)
$$

The third term in eqn.(38) contains the correct poles in $\varepsilon$ to cancel those coming from the form factors as well as the $\delta(1-z_j)$ parts of the mass factorisation kernels. The fourth term contains only poles that cancel against the $D_0(z_j)$ parts of the mass factorisation kernels. The remaining finite terms as $\varepsilon$ becomes zero in the first three terms contribute to the soft-plus-virtual parts of the differential cross sections. Hence, adding the eqn.(38) to the renormalised form factors and the mass factorisation kernels, performing the coupling constant renormalisation, and then finally taking the double Mellin moment in $N_1, N_2$, we get the resummed result analogous to the threshold resummation formula that one obtains for the total inclusive cross sections (see [28, 29, 60, 61]) when $\varepsilon \to 0$. A similar result for the resummed rapidity distribution scheme can be found in [29].

In the double Mellin space ($N_1, N_2$) the threshold enhanced differential cross section will be proportional to

$$
\exp \left[ 2 \int_0^1 dz_1 z_1^{N_1-1} \int_0^1 dz_2 z_2^{N_2-1} \Phi_{d, \text{finite}}^I (\hat{a}_s, q^2, \mu^2, z_1, z_2) \right]. \quad (40)
$$

Similar to the soft distribution functions $\Phi_d^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon)$ that enter in DY and Higgs production [42], the present $\Phi_d^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \varepsilon)$ are also maximally non-abelian. Using the resummed
result given in eqn. (8), and the exponents \( g_i^I(\varepsilon) \)(see [35]), \( \overline{G}_{d,i}^I(\varepsilon) \), we can obtain the higher order soft-plus-virtual contributions to the differential cross sections. The available exponents are

\[
g_1^{I,j}, \quad \overline{G}_{d,1}^{I,(j)} \quad \text{for} \quad j = \text{all},
\]

\[
g_2^{I,j}, \quad \overline{G}_{d,2}^{I,(j)} \quad \text{for} \quad j = 0, 1,
\]

\[
g_3^{I,j}, \quad \overline{G}_{d,3}^{I,(j)} \quad \text{for} \quad j = 0,
\]

in addition to the known \( \beta_i \) \((i = 0, 1, 2, 3)\), the constants in the splitting functions \( A_i^I, B_i^I \) \((i = 1, 2, 3)\), the maximally non-abelian constants \( f_i^I \) \((i = 0, 1, 2, 3)\) and the anomalous dimensions \( \gamma_i^I \). For \( I = q, g \), the constants \( g_2^{q,j} \) and \( g_2^{g,j} \) are known for \( j = 2, 3 \) also (see [34]). Using the resummed expression given in eqn. (8) and the known exponents, we present here the results for \( \Delta_{d,4}^{sv,(i)} \) for DY and Higgs production. Using our approach we have first reproduced the Drell-Yan coefficient \( \Delta_{d,q}^{sv,(i)} \), known upto NNLO \((i = 0, 1, 2) \) [50]. We then obtain \( \Delta_{d,g}^{sv,(i)} \) and \( \Delta_{d,b}^{sv,(i)} \) for the Higgs production up to NNLO \((i = 1, 2)\). For N3LO for \( I = q, b, g \), a partial result \( \Delta_{d,l}^{sv,(3)} \), i.e., a result without the \( \delta(1-z_1)\delta(1-z_2) \) part can be computed from our formula given in eqn. (8). The coefficient of \( \delta(1-z_1)\delta(1-z_2) \) part depends on still unknown constants \( \overline{G}_2^{I,(2)}, \overline{G}_3^{I,(3)} \). We can also obtain a result to N4LO order where we can predict partial soft-plus-virtual contributions containing everything except the terms in \( \mathcal{D}_0(z_i)\delta(1-z_j), \mathcal{D}_0(z_i)\mathcal{D}_0(z_j), \mathcal{D}_1(z_i)\delta(1-z_j) \) and \( \delta(1-z_1)\delta(1-z_2) \) for the Drell-Yan N4LO coefficient \( \Delta_{d,q}^{sv,(4)} \), the gluon fusion to Higgs N4LO coefficient \( \Delta_{d,g}^{sv,(4)} \) and the bottom quark annihilation to Higgs boson N4LO coefficient \( \Delta_{d,b}^{sv,(4)} \). The results are presented in the Appendix B for \( \mu_R^2 = \mu_F^2 = q^2 \). The convolutions of distributions of the form \( \mathcal{D}_l(z_j) \otimes \mathcal{D}_m(z_j) \) for any arbitrary \( l, m \) can be done using the general formula given in [43]. Using these convolutions it is straightforward to calculate \( \Delta_{d,l}^{sv,(i)} \) for \( i = 1, \ldots , 4 \) for both DY \((I = q)\) and Higgs \((I = g, b)\) production. An alternative derivation of the NNLO DY soft-plus-virtual terms in the DIS renormalisation scheme can be found in [62], where they calculated di-lepton production cross sections at fixed target energies.

The differential cross sections for \( I = q \) can be expanded in powers of the strong coupling constant as

\[
\frac{d\sigma^I}{dY} = \sum_{i=0}^{\infty} a_i^I \frac{d\sigma^{I,(i)}}{dY}.
\]

We split the partonic cross section into hard and sv parts:

\[
\frac{d\sigma^{I,(i)}}{dY} = \frac{d\sigma^{\text{hard},I,(i)}}{dY} + \frac{d\sigma^{\text{sv},I,(i)}}{dY},
\]

\[
2S \frac{d^2\sigma^{\text{hard},q,(i)}}{d\mu^2dY} = \sum_q G_{SM,q} \left( D_{qF}^{SM,(i)}(x_1, x_2, \mu_F^2) + D_{qR}^{SM,(i)}(x_1, x_2, \mu_F^2) \right)
\]

\[12\]
Figure 1: Rapidity distributions for DY production at the LHC, and their $\mu_R$ scale dependence (with $\mu^2_F = q^2 = M^2_{l^+l^-}$). Here we denote $M = M_{l^+l^-}$. The abbreviation "pSV" means partial-soft-plus-virtual.

\[
2S d^2\sigma^{\text{sv},q,(i)} = \sum_{a,b=q,q} G_{SM,q} \int_0^1 dx_1 \int_0^1 dx_2 \mathcal{H}_{ab}^q(x_1, x_2, \mu_F^2) \times \int_0^1 dz_1 \int_0^1 dz_2 \delta(x_1^0 - x_1z_1) \delta(x_2^0 - x_2z_2) \Delta_{Y,q}^{\text{sv},(i)}(z_1, z_2, q^2, \mu_F^2, \mu_R^2),
\]
and for the Higgs partonic cross section we have

\[ 2S \frac{d\sigma^{sv,g,(i)}}{dY} = G_{H} \int_{0}^{1} dx_{1} \int_{0}^{1} dx_{2} \, \mathcal{H}_{gg}^{g}(x_{1},x_{2},\mu_{F}^{2}) \]

\[ \times \int_{0}^{1} dz_{1} \int_{0}^{1} dz_{2} \, \delta(x_{1}^{0} - x_{1}z_{1}) \, \delta(x_{2}^{0} - x_{2}z_{2}) \, \Delta_{sv,g}^{(i)}(z_{1},z_{2},m_{H}^{2},\mu_{F}^{2},\mu_{R}^{2}). \]

The coefficients \( \Delta_{sv,g}^{(i)}(z_{1},z_{2},q^{2},\mu_{F}^{2},\mu_{R}^{2}) \) are presented in the Appendix B, where we use the normalisation \( \Delta_{sv,g}^{(0)}(z_{1},z_{2},q^{2},\mu_{F}^{2},\mu_{R}^{2}) = \delta(1-z_{1})\delta(1-z_{2}) \). The constants \( G_{SM,q}, G_{H} \) are given by

\[ G_{SM,q} = \frac{4\alpha^{2}}{3q^{2}} \left[ Q_{q}^{2} - \frac{2q^{2}(q^{2} - M_{Z}^{2})}{((q^{2} - M_{Z}^{2})^{2} + M_{Z}^{2} \Gamma_{Z}^{2}) c_{w}^{2} s_{w}^{4} V_{q} V_{q}^{V}} \right. \]

\[ + \left. \frac{q^{4}}{((q^{2} - M_{Z}^{2})^{2} + M_{Z}^{2} \Gamma_{Z}^{2}) c_{w}^{4} s_{w}^{4} A_{q} A_{q}^{V}} \left( (g_{e}^{V})^{2} + (g_{A}^{V})^{2} \right) \left( (g_{q}^{V})^{2} + (g_{A}^{V})^{2} \right) \right], \]

\[ G_{H} = \frac{\pi m_{H}^{2} G_{B}^{2}}{4(N^{2} - 1)}. \]
The $x_F$ differential cross sections can be obtained from the $Y$ differential cross sections by replacing \( D_{ab}^{(i)}(x_1^0, x_2^0, \mu_F^2) \) by \( C_{ab}^{(i)}(x_1^0, x_2^0, \mu_F^2) \). For the sv part we identify \( \Delta_{sv,Y}^{sv,i} = (x_1^0 + x_2^0)^{-1} \Delta_{sv,Y}^{sv,i} \) with the replacement of \( D_{ab}^{(i)}(x_1^0, x_2^0, \mu_F^2) \) by \( C_{ab}^{(i)}(x_1^0, x_2^0, \mu_F^2) \) in the right-hand-side. The electro-weak constants appearing in eqns.\(^{(46,47)}\) can be found in [25, 51].

For our numerical results we choose the center-of-mass energy to be \( \sqrt{s} = 14 \text{ TeV} \) for the LHC. The standard model parameters that enter our computation are the Fermi constant \( G_F = 4541.68 \text{ pb} \), the Z boson mass \( M_Z = 91.1876 \text{ GeV} \) and top quark mass \( m_t = 173.4 \text{ GeV} \). The strong coupling constant \( \alpha_s(\mu_R^2) \) is evolved using the 4-loop RG equations depending on the order in which the cross section is evaluated. We choose \( \alpha_{s}^{LO}(M_Z) = 0.130 \), \( \alpha_{s}^{NLO}(M_Z) = 0.119 \), \( \alpha_{s}^{NNLO}(M_Z) = 0.115 \) and \( \alpha_{s}^{NNLO}(M_Z) = 0.114 \) for \( i > 2 \). We use MRST 2001 LO for leading order, MRST2001 NLO for NLO and MRST 2002 NNLO for N\(^3\)LO with \( i > 1 \) [63, 64]. The impact of the soft-plus-virtual N\(^2\)LO and the partial soft-plus-virtual N\(^3\)LO contributions to the DY rapidity differential cross section at the LHC is presented in figure 1. Note that we have not plotted the partial soft-plus-virtual N\(^4\)LO contributions because there are no N\(^3\)LO parton densities. In the first plot we have shown the rapidity distribution in pb/GeV for a di-lepton mass of 115 GeV. For LO and NLO, we used the exact results which contain both the soft-plus-virtual as well as the regular hard contributions. For N\(^3\)LO (\( i = 2, 3 \)), we use only the soft-plus-virtual results extracted from the resummed formula. Here we have set \( \mu_F = \mu_R = 115 \text{ GeV} \). We find that the inclusion of N\(^3\)LO (\( i = 2, 3 \)) terms only make small changes in the differential cross section which confirms the reliability of the perturbative approach.

In the second plot of fig.1 we have shown the scale variation of the rapidity distribution using the ratio:

\[
R^i(\mu_R^2) = \left( \frac{d\sigma^i}{dx} (\mu_R^2 = q^2) \right)^{-1} \frac{d\sigma^i}{dx} (\mu_R^2),
\]

plotted as a function of \( \mu/\mu_0 = \mu_R/|q| \), where we have fixed \( \mu_R^2 = q^2 \). It is clear from the second plot of fig.1 that the inclusion of N\(^3\)LO (\( i = 2, 3 \)) soft-plus-virtual contributions further reduces the scale ambiguity.

The impact of soft-plus-virtual parts N\(^2\)LO and the partial soft-plus-virtual N\(^3\)LO contributions to Higgs production through gluon fusion at the LHC is presented in figure 2. We see that the inclusion of the higher order terms does not make any appreciable change in the magnitude of the rapidity distribution. Again this confirms the reliability of the perturbation series. The second plot in figure 2 shows \( R \) in eqn.48 as a function of \( \mu/\mu_0 = \mu_R/m_H \), where we have fixed \( \mu_R^2 = m_H^2 \), and demonstrates that the inclusion of the higher order terms reduces the sensitivity to the choice of the scale.

We also present the numerical values of the rapidity distributions in figures 1 and 2 in Tables 1 and 2 respectively. These numbers allow a more direct comparison with other theoretical papers and are useful to the experimental groups working at the LHC.

Previous calculations of differential distributions in NLO using the effective Lagrangian (or the \( m_t \to \infty \) approach) have been presented in [65]- [69]. In the same approach the resummation
Table 1: Values for $d^2\sigma/dMdY$ (M=115 GeV) in pb/GeV at fixed values of $Y$ which are plotted in figure 1.

| $Y$ | LO  | NLO | N$^2$LO$_{SV}$ | N$^3$LO$_{pSV}$ |
|-----|-----|-----|----------------|----------------|-----|
| 0.  | 0.1326 | 0.1735 | 0.1734 | 0.1732 |
| 0.4 | 0.1327 | 0.1733 | 0.1732 | 0.1729 |
| 0.8 | 0.1330 | 0.1729 | 0.1725 | 0.1722 |
| 1.2 | 0.1335 | 0.1721 | 0.1714 | 0.1711 |
| 1.6 | 0.1343 | 0.1707 | 0.1698 | 0.1695 |
| 2.  | 0.1346 | 0.1678 | 0.1670 | 0.1667 |
| 2.4 | 0.1328 | 0.1614 | 0.1610 | 0.1607 |
| 2.8 | 0.1273 | 0.1483 | 0.1485 | 0.1482 |
| 3.2 | 0.1123 | 0.1240 | 0.1244 | 0.1241 |
| 3.6 | 0.0832 | 0.0858 | 0.0857 | 0.0855 |

Table 2: Values for $d\sigma/dY$ (for $M_H = 115$ GeV) in pb/GeV at fixed values of $Y$ which are plotted in figure 2.

| $Y$ | LO  | NLO | N$^2$LO$_{SV}$ | N$^3$LO$_{pSV}$ |
|-----|-----|-----|----------------|----------------|-----|
| 0.  | 4.366 | 7.236 | 7.765 | 7.848 |
| 0.4 | 4.294 | 7.106 | 7.632 | 7.713 |
| 0.8 | 4.084 | 6.722 | 7.236 | 7.313 |
| 1.2 | 3.731 | 6.091 | 6.585 | 6.655 |
| 1.6 | 3.278 | 5.264 | 5.724 | 5.784 |
| 2.  | 2.715 | 4.265 | 4.675 | 4.724 |
| 2.4 | 2.068 | 3.173 | 3.512 | 3.549 |
| 2.8 | 1.410 | 2.087 | 2.337 | 2.361 |
| 3.2 | 0.779 | 1.135 | 1.285 | 1.298 |
| 3.6 | 0.300 | 0.448 | 0.511 | 0.516 |

Table 3: Values for $d\sigma/dY$ (for $M_H = 120$ GeV) in pb/GeV at fixed values of $Y$.
of the logarithmically enhanced contributions have been carried out in [70] - [73]. Our results agree exactly with the DY NLO results for rapidity distributions in [19]. We cannot compare the NLO Higgs rapidity plots directly with those in [66] because there we used $m_H = 120$ GeV and we had to impose a cut on the Higgs $p_t$. However we have rerun our programs with $m_H = 120$ GeV to allow a better comparison with both the results in [66]. The number for the Higgs rapidity distribution are given in Table 3. Our numbers are also consistent with the normalized Higgs boson rapidity distribution in fig. 1 in [69]. These checks indicate that everything is consistent with the NLO results.

We have compared our results for the Drell-Yan and Higgs rapidity distributions against the NNLO results published in [71, 72]. Our soft-plus-virtual NNLO approximations agree very well with their exact NNLO results. Our partial soft-plus-virtual $N^3$LO results are new and cannot be compared with any other calculation.

To summarise, we have systematically studied the soft-plus-virtual corrections to differential cross sections in rapidity for DY and Higgs production through both gluon fusion and bottom quark annihilation. The resummation of these corrections has been achieved using renormalisation group invariance, Sudakov resummation of scattering amplitudes and the factorisation property of the hard scattering cross sections. Our analytical results are presented in Appendices A and B. It is now straightforward to obtain resummed threshold contributions to both $x_F$ and $Y$ rapidity distributions of di-lepton pairs in the DY process and of Higgs bosons in Higgs productions. This requires a double Mellin transform in the space of two variables $N_1$ and $N_2$, see eqn.(40). Using our resummed results we have computed soft-plus-virtual differential cross sections at $N^2$LO and partial-soft-plus-virtual differential cross sections in $N^3$LO. Finally we have presented the numerical impact of these results on the rapidity differential cross sections.

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A Hart parts

In this appendix, we list the $C_{ab}^{H,(i)}(x_1^0,x_2^0)$ and $D_{ab}^{H,(i)}(x_1^0,x_2^0)$ that contribute to the hard parts of the cross sections. We start by defining the following parton density combinations

\[
H_{qq}(x_1,x_2,\mu_F^2) = f^p_q(x_1,\mu_F) f^{p_2}_q(x_2,\mu_F) + f^p_\bar{q}(x_1,\mu_F) f^{p_2}_{\bar{q}}(x_2,\mu_F),
\]

\[
H_{gq}(x_1,x_2,\mu_F^2) = f^p_g(x_1,\mu_F) \left( f^{p_2}_q(x_2,\mu_F) + f^{p_2}_\bar{q}(x_2,\mu_F) \right),
\]

\[
H_{qg}(x_1,x_2,\mu_F^2) = H_{gq}(x_2,x_1,\mu_F^2),
\]

\[
H_{gg}(x_1,x_2,\mu_F^2) = f^p_g(x_1,\mu_F) f^{p_2}_g(x_2,\mu_F). \tag{1}
\]

In terms of these combinations, we list the $C_{gg}^{H,(0)}(x_1^0,x_2^0)$ that appear in the hard parts of the $x_F$-differential cross sections

\[
C_{gg}^{H,(0)}(x_1^0,x_2^0) = H_{gg}(x_1^0,x_2^0,\mu_F^2),
\]

\[
C_{gg}^{H,(1)}(x_1^0,x_2^0) = C_A \left\{ \int_{x_1^0}^{1} \frac{dx_1}{x_1} \frac{H_{gg}(x_1,x_2^0,\mu_F^2)}{(x_1^0 + x_2^0)} \left[ -4 \frac{x_1^0}{x_1^2} + 4 \frac{x_2^0}{x_2^2} - 8 \frac{x_1^0}{x_1} + 4 \frac{x_2^0}{x_2} \right] \right\} L_{a_1}
\]

\[
+ \frac{4}{(x_1 - x_1^0)} L_{c_1} \right] + \int_{x_1^0}^{1} \frac{dx_1}{x_1} \frac{H_{gg,1}(x_1,x_2^0,\mu_F^2)}{(x_1 - x_1^0)(x_1^0 + x_2^0)} \left[ 4 L_{b_1} \right]
\]

\[
+ \int_{x_1^0}^{1} \frac{dx_1}{x_1} \int_{x_2^0}^{1} \frac{dx_2}{x_2} \frac{2H_{gg,12}(x_1,x_2,\mu_F^2)}{(x_1 - x_1^0)(x_2 - x_2^0)(x_1^0 + x_2^0)}
\]

\[
+ \frac{H_{gg}(x_1^0,x_2^0,\mu_F^2)}{(x_1^0 + x_2^0)} \left[ 2 L(x_1^0,x_2^0) \log \left( \frac{q^2}{\mu_F^2} \right) + \left( L(x_1^0,x_2^0) \right)^2 + 6 \zeta_2 \right]
\]

\[
+ \int_{x_1^0}^{1} \frac{dx_1}{x_1} \int_{x_2}^{1} \frac{dx_2}{x_2} \frac{H_{gg}(x_1,x_2,\mu_F^2)}{(x_1 + x_2)(x_1 + x_1^0)(x_2 + x_2^0)(x_1^0 + x_2^0)} \left[ \frac{1}{x_1^2} \left( 4x_2^3 x_1^0 x_2^0 + 4x_2^2 (x_1^3 x_2^0 + x_1^0 x_2^3) + \frac{1}{x_1^2} \left( 4x_2^3 x_1^0 x_2^0 + 4x_2^2 (x_1^3 x_2^0 + x_1^0 x_2^3) \right) + \frac{1}{x_1} \left( 4x_2^3 (x_1^0 x_2^0 - x_1 x_2) + x_2^2 (4x_1^3 x_2^0 + 12x_1^0 x_2^0 - 4x_1^0 x_2^0) + x_2 (16x_1^3 x_2^0 + 12x_1^0 x_2^0 - 4x_1^0 x_2^3) + 8(x_1^3 x_2^0 + x_1^0 x_2^3) \right) \right]
\]

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\[ + x_2(8x_2^3 + 12x_1^3 + 8x_1^2x_2 - 8x_1x_2^2) + \frac{1}{2x_2}(12x_1^3x_2^2 + 12x_1^2x_2^3) \]

\[ + 12x_1^3x_2^2 + 20x_1^2x_2^2 + 8x_1x_2^3) + x_1^4 \left( \frac{4}{x_1^2} \right) + x_1^3 \left( 12 - 8 \frac{x_1^0}{x_2^0} \right) \]

\[ + x_1^2 \left( 24x_1^0 + 4 \frac{x_1^0}{x_2^0} - 4x_2^0 \right) + x_1 \left( 8x_1^0x_2^0 + 36x_1^2 - 4 \frac{x_1^3}{x_2^0} + 2x_2 \left( - 4 \frac{x_2^0}{x_1^0} \right) \]

\[ + 20x_1^0 - 4 \frac{x_1^0}{x_2^0} + 20x_2^2 \right) + x_2^2 \left( - 8 \frac{x_2^0}{x_1^0} - 12 \frac{x_1^0}{x_2^0} + 40 \right) + x_2^3 \left( \frac{4}{x_1^0} - \frac{4}{x_2^0} \right) \]

\[ + 10x_1^0x_2^2 + 10x_1^2x_2^2 + 2x_1^3 + 2x_2^3 \right) \} + (1 \leftrightarrow 2), \]

\[ C_{qg}^{H,(1)}(x_1^0, x_2^0) = C_F \left\{ \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 \frac{H_{qg}(x_1, x_2, \mu_F^2)}{(x_1 + x_2)} \left[ \frac{2x_1^0}{x_1^0} - \frac{4}{x_1^0} - \frac{4}{x_2^0} \right] \right\} \]

\[ + \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 \frac{H_{qg,2}(x_1, x_2, \mu_F^2)}{(x_2 - x_1^0)(x_1 + x_2)} \left[ \frac{2x_1^0}{x_1^0} - \frac{4}{x_1^0} - \frac{4}{x_2^0} \right] \]

\[ + \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 H_{qg}(x_1, x_2, \mu_F^2) \left[ \frac{1}{(x_1 + x_2)} \left( \frac{2x_1^0}{x_1^0} + \frac{4}{x_1^0} \right) \right] \]

\[ - \frac{2}{x_1^0x_2} - \frac{4}{x_1x_2^0} + \frac{1}{(x_1 + x_2)} \left( - \frac{x_2^0}{x_1x_2} - \frac{x_1^0}{x_1x_2} - \frac{2}{x_2} - \frac{2}{x_2}x_2 \left( x_1^0 - x_2^0 \right) \right) \]

\[ + \frac{1}{(x_1 + x_2)} \left( - \frac{4}{x_1^2} - \frac{3}{x_1^0x_2^0} \right) \} , \]

\[ C_{qg}^{H,(1)}(x_1^0, x_2^0) = C_{qg}^{H,(1)}(x_1^0, x_2^0) \bigg|_{1 \leftrightarrow 2} , \]

\[ C_{qg}^{H,(1)}(x_1^0, x_2^0) = C_F \left\{ \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 \frac{H_{qg}(x_1, x_2, \mu_F^2)}{(x_1 + x_2)^3} \left[ \frac{1}{(x_1 + x_2)^3} \right] \right\} \]

\[ + \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 H_{qg}(x_1, x_2, \mu_F^2) \left[ \frac{1}{(x_1 + x_2)^3} \right] \left( \frac{2x_1^0}{x_1^0} + \frac{4}{x_1^0} + \frac{4}{x_2^0} \right) \]

\[ + \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 H_{qg}(x_1, x_2, \mu_F^2) \left[ \frac{1}{(x_1 + x_2)^3} \right] \left( \frac{2x_1^0}{x_1^0} + \frac{4}{x_1^0} + \frac{4}{x_2^0} \right) \]

\[ + \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 H_{qg}(x_1, x_2, \mu_F^2) \left[ \frac{1}{(x_1 + x_2)^3} \right] \left( \frac{2x_1^0}{x_1^0} + \frac{4}{x_1^0} + \frac{4}{x_2^0} \right) \}

\[ + \frac{1}{x_1^0} \left( 8x_2^0 - 8x_2^0 \right) + x_1, x_2 \left( - \frac{2}{x_1^0} - \frac{2}{x_2^0} \right) + x_1 \left( \frac{8x_2^0}{x_1^0} - 8x_2^0 \right) + x_1^2 \left( \frac{2}{x_1^0} + \frac{2}{x_2^0} \right) + \frac{4x_1^0}{x_1^0} - 12x_1^0 + 4x_1^0x_2^0 - 12x_2^0 \right) \]

\[ + \frac{1}{(x_1 + x_2)} \left( \frac{1}{x_1^0} + \frac{1}{x_2^0} \right) \}} + (1 \leftrightarrow 2). \]
We have introduced the following abbreviations

\[ H_{ab,12}(x_1, x_2, \mu_F^2) = H_{ab}(x_1, x_2, \mu_F^2) - H_{ab}(x_1^0, x_2, \mu_F^2) - H_{ab}(x_1, x_2^0, \mu_F^2) + H_{ab}(x_1^0, x_2^0, \mu_F^2), \]

\[ H_{ab,1}(x_1, z, \mu_F^2) = H_{ab}(x_1, z, \mu_F^2) - H_{ab}(x_1^0, z, \mu_F^2), \]

\[ H_{ab,2}(z, x_2, \mu_F^2) = H_{ab}(z, x_2, \mu_F^2) - H_{ab}(z, x_2^0, \mu_F^2), \]

\[ (3) \]

\[ L_{a_1} = \ln \left( \frac{q^2 (x_1^0 + x_2^0)(1 - x_1^0)}{\mu_F^2 (x_1 + x_2^0)} \right), \quad L_{b_1} = \ln \left( \frac{q^2 (1 - x_1^0)(x_1 - x_1^0)}{\mu_F^2 x_1^0 x_2^0} \right), \]

\[ L_{c_1} = \ln \left( \frac{x_1^0 + x_2^0}{x_1 + x_2^0} \right), \quad L(x_1^0, x_2) = \ln \left( \frac{(1 - x_1^0)(1 - x_2^0)}{x_1^0 x_2^0} \right). \]

\[ (4) \]

The \( D_{g g}^{H,(i)}(x_1^0, x_2^0) \) that appear in the hard parts of the rapidity distributions are listed below

\[ D_{g g}^{H,(0)}(x_1^0, x_2^0) = H_{g g}(x_1^0, x_2^0, \mu_F^2), \]

\[ D_{g g}^{H,(1)}(x_1^0, x_2^0) = C_A \left\{ \int_{x_1^0}^{1} dx_1 H_{g g}(x_1, x_2, \mu_F^2) \left[ \left( -4 \frac{x_2^0}{x_1^2} + \frac{x_1^0}{x_1^2} - \frac{8}{x_1^2} + \frac{4}{x_1^0} \right) K_6 + \frac{4}{(x_1 - x_1^0) K_6} \right] \right. \]

\[ + \int_{x_1^0}^{1} dx_1 \left. \int_{x_2^0}^{1} dx_2 \frac{H_{g g, 1}(x_1, x_2, \mu_F^2)}{(x_1 - x_1^0)} \left[ -4 \frac{x_2^0}{x_1^2} + \frac{x_1^0}{x_1^2} - \frac{8}{x_1^2} + \frac{4}{x_1^0} \right] \right. \]

\[ + \int_{x_1^0}^{1} dx_1 \left. \int_{x_2^0}^{1} dx_2 \frac{2H_{g g, 12}(x_1, x_2, \mu_F^2)}{(x_1 - x_1^0)(x_2 - x_2^0)} \right. \]

\[ + H_{g g}(x_1^0, x_2^0, \mu_F^2) \left[ 2 K(x_1^0, x_2) \log \left( \frac{q^2}{\mu_F^2} \right) + 6 \zeta_2 + \left( K(x_1^0, x_2) \right)^2 \right] \]

\[ + \int_{x_1^0}^{1} dx_1 \left. \int_{x_2^0}^{1} dx_2 \frac{H_{g g}(x_1, x_2, \mu_F^2)}{(x_1 + x_1^0)(x_2 + x_2^0)(x_1 x_2 + x_1^0 x_2)} \right. \]

\[ \left[ \frac{1}{x_1^0} \left( 4 x_2^0 x_1^0 + 8 x_2^1 x_2^0 \right) \right. \]

\[ + 8 x_2^2 x_1^0 x_2^0 + 8 x_2^1 x_2^0 + 8 x_2^0 \right] \]

\[ + \frac{1}{x_1^0} \left( 16 x_1^0 x_2^0 + 16 x_2^1 x_2^0 + 32 x_2^2 x_1^0 x_2^0 \right) \]

\[ + 24 x_2^1 x_2^0 x_2^0 \]

\[ + \frac{1}{x_1^0} \left( 40 x_1^0 x_2^0 + 4 x_2^1 x_2^0 + 8 x_2^1 x_2^0 + 32 x_2^2 x_1^0 x_2^0 \right) \]
\[+48x_2^1 x_1^5 x_2^3 + \frac{12}{x_2} x_1^0 x_2^5\] \[+ x_1^2 \left( 4x_2^4 + x_2^4 x_1\right) + 12x_2^1 x_1^0 x_2^0\]

\[+28x_1^0 x_2^4\] \[+ x_1^2 \left( 48x_2^4 + 16x_2^4 + 20x_2^4 + 88x_2 x_1^2 x_2^3\right)\]

\[+ x_1 \left( 68x_2^3 x_2^4 + \frac{8x_2^5 x_2^3}{x_2^2} + 4x_2^4 x_2^3 + 8x_2 x_1^0 x_2^0\right) + 64x_2 x_1^3 x_2^3\]

\[+ 40x_1^0 x_2^4\] \[\}\) + (1 \leftrightarrow 2),\]

\[D_{qg}^{H,(1)} (x_1^0, x_2^0) = C_F \left\{ \int_0^1 dx_1 H_{qg}(x_1^0, x_2^0, \mu_F^2) \left[ \frac{2x_1^0}{x_1^2} - \frac{4}{x_1} \right] K_1 + \frac{2x_1^0}{x_1^2} \right\}\]

\[+ \int_0^1 dx_1 \int_0^1 dx_2 \left[ \frac{H_{qg,2}(x_1^0, x_2^0, \mu_F^2)}{(x_2 - x_1^0)} \right] \left[ \frac{2x_1^0}{x_1} - \frac{4}{x_1} \right] \]

\[+ \int_0^1 dx_1 \int_0^1 dx_2 \left[ \frac{H_{qg,3}(x_1^0, x_2^0, \mu_F^2)}{(x_1 + x_1^0)(x_2 + x_1^0)(x_1^0 + x_1^0)(x_2^0 + x_1^0)} \right] \left[ \frac{1}{x_1} \right] \]

\[\left( -2x_2^3 x_1^0 x_2^0 - 4x_2^2 x_1^0 x_2^0 - 4x_2 x_1^0 x_2^0\right) + \frac{1}{x_1} \right] \left( 2x_2^3 x_1^0 x_2^0 - 2x_2^2 x_1^0 x_2^0 - 2x_2 x_1^0 x_2^0\right)\]

\[+ x_1 \left( 6x_2 x_1^0 x_2^0 + 2x_1^0 x_2^0\right) + x_1^2 \left( -12x_2 x_1^0 x_2^0 + 4x_2 x_1^0 - 2x_1^0 x_2^0\right)\]

\[+ x_1^3 \left( 4x_2 x_1^2 + 8x_2 x_1^0\right) + x_1^4 \left( 8x_2 x_1^0 x_2^0 + 4x_2^3 x_2^0\right) + 14x_2 x_1^3 x_2^2\]

\[+ 10x_2 x_1^0 x_2^3\] \[\}\) \]
\[-24x_2^2x_1^0x_2^0 + 16x_2^3x_1^3x_2^0 + 8x_2^3x_1^0x_2^3 - 24x_1^0x_2^3x_2^0 \bigg) + x_1^0 \left( -16x_2^2x_1^0x_2^2 \right) + x_2^2 \bigg) \times \left( 8x_1^3x_2^0 - 16x_1^0x_2^4 \right) \bigg\} \}

\[+ (1 \leftrightarrow 2), \]

where we have introduced the following abbreviations

\[\mathcal{K}_0 = \ln \left( \frac{2q^2 (1 - x_0^0)(x_1 - x_0^0)}{\mu_F^2 (x_1 + x_0^0)x_2^0} \right), \quad \mathcal{K}_1 = \ln \left( \frac{q^2 (1 - x_0^0)(x_1 - x_0^0)}{\mu_F^2 x_1x_2^0} \right), \]

\[\mathcal{K}_1 = \ln \left( \frac{2x_1^0}{x_1 + x_1^0} \right), \quad \mathcal{K}(x_1^0, x_2^0) = \ln \left( \frac{(1 - x_1^0)(1 - x_2^0)}{x_1^0x_2^0} \right). \]

The \(\mathcal{K}_2, \mathcal{K}_3\) and \(\mathcal{K}_4\) can be obtained from \(\mathcal{K}_0, \mathcal{K}_1\) and \(\mathcal{K}_2\) by using \(1 \leftrightarrow 2\) symmetry.

**B  Soft-plus-virtual parts**

Here we list below the \(\Delta_{Y,q}^{sv,(i)}\) that contribute to the soft-plus-virtual parts of the cross sections for the choice \(\mu_R^2 = \mu_F^2 = q^2\).

\[\Delta_{Y,q}^{sv,(1)} = \delta(1 - z_1)\delta(1 - z_2) \left[ C_F \left( -8 + 6 \xi_2 \right) \right] + D_0 D_0 \left[ C_F \left( 2 \right) \right] + D_1 D_1 \left[ C_F \left( 4 \right) \right] \]

\[+ (z_1 \leftrightarrow z_2), \]

\[\Delta_{Y,q}^{sv,(2)} = \delta(1 - z_1)\delta(1 - z_2) \left[ n_f C_F \left( \frac{127}{12} - \frac{76}{9} \xi_2 + \frac{4}{3} \xi_3 \right) + C_F C_A \left( -\frac{1535}{24} \right) + \frac{430}{9} \xi_2 - \frac{26}{5} \xi_2^2 + \frac{86}{3} \xi_3 \right] + C_F \left( \frac{511}{8} - 67 \xi_2 + \frac{152}{5} \xi_2^2 - 30 \xi_3 \right) \]

\[+ D_0 D_0 \left[ n_f C_F \left( -\frac{20}{9} \right) + C_F C_A \left( \frac{134}{9} - 4 \xi_2 \right) + C_F \left( -32 + 8 \xi_2 \right) \right] \]

\[+ D_1 D_1 \left[ n_f C_F \left( \frac{8}{3} \right) + C_F C_A \left( -\frac{44}{3} \right) \right] + D_0 D_2 \left[ C_F \left( 24 \right) \right] + D_1 D_1 \left[ C_F \left( 24 \right) \right] \]

\[+ D_0 D_0 \delta(1 - z_1) \left[ n_f C_F \left( \frac{112}{27} - \frac{8}{3} \xi_2 \right) + C_F C_A \left( -\frac{808}{27} + \frac{44}{3} \xi_2 + 28 \xi_3 \right) \right] + C_F \left( 32 \xi_3 \right) \]

\[+ D_1 D_1 \left( 1 - z_1 \right) \left[ n_f C_F \left( -\frac{40}{9} \right) + C_F C_A \left( \frac{268}{9} - 8 \xi_2 \right) \right] + C_F \left( -64 + 16 \xi_2 \right) \]

\[+ D_3 D_3 \delta(1 - z_1) \left[ C_F \left( \frac{4}{3} \right) + C_F C_A \left( -\frac{22}{3} \right) \right] \]

\[+ D_3 D_3 \delta(1 - z_1) \left[ C_F \left( 8 \right) \right] \]
\( \Delta_{y,q}^{sv,(3)} = D_0 D_0 \left[ n_f C_F C_A \left( -\frac{4102}{81} + \frac{256}{9} \zeta_2 \right) + n_f C_F^2 \left( \frac{536}{9} - \frac{224}{3} \zeta_2 + \frac{160}{3} \zeta_3 \right) \right. \\
+ n_f^2 C_F \left( \frac{200}{81} - \frac{16}{9} \zeta_2 \right) + C_F C_A^2 \left( \frac{15503}{81} - \frac{340}{3} \zeta_2 + \frac{88}{5} \zeta_2^2 - 88 \zeta_3 \right) \\
+ C_F^2 C_A \left( -\frac{8893}{18} + \frac{1760}{9} \zeta_2 - \frac{24}{5} \zeta_2^2 - \frac{184}{3} \zeta_3 \right) + C_F^3 \left( \frac{511}{2} - 12 \zeta_2 - \frac{96}{5} \zeta_2^2 - 120 \zeta_3 \right) + D_0 D_1 \left[ n_f C_F C_A \left( \frac{2312}{27} - \frac{32}{3} \zeta_2 \right) + n_f C_F^2 \left( \frac{136}{9} \zeta_2 \right) \right. \\
- \frac{64 \zeta_2}{9} + n_f^2 C_F \left( -\frac{160}{27} \right) + C_F C_A^2 \left( -\frac{7120}{27} + \frac{176}{3} \zeta_2 \right) + C_F^2 C_A \left( -\frac{1120}{9} \zeta_2 \right) \\
+ 352 \zeta_2 + 336 \zeta_3 + C_F^3 \left( 640 \zeta_3 \right) \right] + D_0 D_2 \left[ n_f C_F C_A \left( -\frac{176}{9} \right) + n_f C_F^2 \left( -\frac{160}{3} \zeta_2 \right) \right. \\
+ n_f^2 C_F \left( \frac{16}{9} \right) + C_F C_A^2 \left( \frac{484}{9} \zeta_3 \right) + C_F^2 C_A \left( \frac{1072}{3} - 96 \zeta_2 \right) + C_F^3 \left( -384 - 96 \zeta_2 \right) \right. \\
+ D_0 D_3 \left[ n_f C_F^2 \left( \frac{160}{9} \right) + C_F^2 C_A \left( -\frac{880}{9} \zeta_2 \right) \right] + D_0 D_4 \left[ C_F^3 \left( 40 \right) \right. \\
+ D_1 D_1 \left[ n_f C_F C_A \left( -\frac{176}{9} \zeta_2 \right) + n_f C_F^2 \left( -\frac{160}{3} \right) + n_f^2 C_F \left( \frac{16}{9} \right) + C_F C_A^2 \left( \frac{484}{9} \right) \right. \\
+ C_F^2 C_A \left( \frac{1072}{3} - 96 \zeta_2 \right) + C_F^3 \left( -384 - 96 \zeta_2 \right) \right] + D_1 D_2 \left[ n_f C_F^2 \left( \frac{160}{3} \right) \right. \\
+ C_F^2 C_A \left( -\frac{880}{3} \zeta_2 \right) \right] + D_1 D_3 \left[ C_F^3 \left( 160 \right) \right] + D_2 D_2 \left[ C_F^3 \left( 120 \right) \right. \\
+ D_0 \delta(1 - z_1) \left[ n_f C_F C_A \left( \frac{62626}{729} - \frac{7760}{81} \right) \right. \\
+ n_f C_F \left( -3 + \frac{1384}{27} \zeta_2 - \frac{256}{15} \zeta_2^2 - \frac{944}{9} \zeta_3 \right) + n_f^2 C_F \left( -\frac{1856}{729} + 160 \zeta_2 - \frac{32}{27} \zeta_3 \right) \\
+ C_F C_A^2 \left( -\frac{297029}{729} - \frac{176}{3} \zeta_2 \zeta_3 + \frac{27752}{81} \zeta_2 - \frac{616}{15} \zeta_2^2 \right) \\
+ \frac{14264}{27} \zeta_3 - 192 \zeta_5 \right] + C_F^2 C_A \left( \frac{12928}{27} - 16 \zeta_2 \zeta_3 - \frac{9568}{27} \zeta_2 + \frac{176}{3} \zeta_2^2 \right) \\
+ \frac{256}{9} \zeta_3 \right] + C_F^3 \left( -128 \zeta_2 \zeta_3 - 512 \zeta_3 + 384 \zeta_5 \right) \right. \\
+ D_1 \delta(1 - z_1) \left[ n_f C_F C_A \left( \frac{8204}{81} + \frac{512}{9} \zeta_2 \right) + n_f C_F^2 \left( \frac{1072}{9} - \frac{448}{9} \zeta_2 \right) \right. \\
+ n_f^2 C_F \left( \frac{400}{81} - \frac{32}{9} \zeta_2 \right) + C_F C_A^2 \left( \frac{31006}{81} - \frac{680}{3} \zeta_2 + \frac{176}{5} \zeta_2^2 \right) \\
- 176 \zeta_3 \right] + C_F^2 C_A \left( -\frac{8893}{9} + \frac{3520}{9} \zeta_2 - \frac{48}{5} \zeta_2^2 - \frac{368}{3} \zeta_3 \right) \right. \\
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\[\Delta_{Y,q}^{\text{sv.(4)}} = D_0 \delta_1 \left[ n_f C_F C_A^2 \left( \frac{20554}{9} - \frac{10192}{9} \xi_2 + \frac{352}{5} \xi_2^2 - \frac{352}{5} \xi_3 \right) + n_f C_F C_A \left( \frac{84280}{243} \right) \\
- \frac{26432}{9} \xi_2 + \frac{1376}{3} \xi_2^2 - \frac{27424}{9} \xi_3 \right) + n_f C_F^3 \left( \frac{518}{3} + \frac{7088}{9} \xi_2 - \frac{2176}{15} \xi_2^2 \right) \\
- \frac{8096}{3} \xi_3 \right) + n_f C_F \left( \frac{800}{81} - \frac{64}{9} \xi_2 \right) + C_F C_A^3 \left( - \frac{412880}{81} + \frac{8024}{3} \xi_2 \right) \\
- \frac{1936}{5} \xi_2 + 1936 \xi_3 \right) + C_F^2 C_A \left( - \frac{356573}{243} - \frac{1376}{27} \xi_2 \xi_3 + \frac{258784}{27} \xi_2 \right) \\
- \frac{31504}{15} \xi_2 \xi_3 \right) + C_F C_A \left( \frac{34849}{9} - 5184 \xi_2 \xi_3 \right) \\
- \frac{45416}{9} \xi_2 + \frac{5632}{15} \xi_2^2 + \frac{29392}{3} \xi_3 \right) + C_F^4 \left( - 7680 \xi_2 \xi_3 - 10240 \xi_3 \right) \\
+ 10752 \xi_3 \right) \right] + D_0 \delta_2 \left[ n_f C_F C_A^2 \left( - \frac{7324}{9} + \frac{352}{3} \xi_2 \right) + n_f C_F^2 C_A \left( - \frac{133988}{81} \right) \\
+ \frac{14144}{9} \xi_2 + \frac{448}{3} \xi_3 \right) + n_f C_F^3 \left( \frac{2764}{3} + \frac{704}{3} \xi_2 + \frac{5696}{3} \xi_3 \right) \\
+ n_f C_F \left( \frac{1144}{9} - \frac{32}{3} \xi_2 \right) + n_f C_F^2 \left( \frac{7768}{81} - \frac{1024}{9} \xi_2 \right) + n_f C_F \left( - \frac{160}{27} \right) \\
+ C_F C_A \left( \frac{43648}{27} - \frac{968}{3} \xi_2 \right) + C_F^2 \left( \frac{481216}{9} - \frac{50272}{9} \xi_2 + \frac{2592}{5} \xi_2^2 \right) \\
- \frac{8800}{3} \xi_3 \right) + C_F C_A \left( - \frac{26362}{3} - \frac{1376}{3} \xi_2 + \frac{4512}{5} \xi_2^2 - \frac{19808}{3} \xi_3 \right) \\
+ C_F^4 \left( 3066 + 2928 \xi_2 - \frac{4992}{5} \xi_2^2 - 1440 \xi_3 \right) \right] + D_0 \delta_3 \left[ n_f C_F C_A^2 \left( \frac{968}{9} \right) \right] \]
\[ + n_f C_F^2 C_A \left( \frac{21920}{27} - \frac{320}{3} \zeta_2 \right) + n_f C_F^3 \left( - \frac{1760}{27} - \frac{4160}{9} \zeta_2 \right) \\
+ n_f^2 C_F^2 C_A \left( - \frac{176}{9} \right) + n_f^2 C_F^2 \left( - \frac{1600}{27} \right) + n_f^3 C_F \left( \frac{32}{27} \right) + C_F C_A^3 \left( - \frac{5324}{27} \right) \\
+ C_F^2 C_A^2 \left( - \frac{67120}{27} + \frac{1760}{3} \zeta_2 \right) + C_F^2 C_A \left( \frac{9920}{27} + \frac{22880}{9} \zeta_2 + 1120 \zeta_3 \right) \\
+ C_F^4 \left( \frac{8960}{3} \zeta_3 \right) \right] + D_1 D_4 \left[ n_f C_F^2 C_A \left( - \frac{3520}{27} \right) + n_f C_F^3 \left( - \frac{400}{3} \right) \right] \\
+ n_f^2 C_F^2 \left( \frac{320}{27} \right) + C_F^2 C_A^2 \left( \frac{9680}{27} \right) + C_F^3 C_A \left( \frac{2680}{3} - 240 \zeta_2 \right) + C_F^4 \left( - \frac{640}{3} \right) \\
- 480 \zeta_2 \right) \right] + D_0 D_5 \left[ n_f C_F^3 \left( \frac{112}{3} \right) + C_F^3 C_A \left( - \frac{616}{3} \right) \right] + D_0 D_6 \left[ C_F^4 \left( \frac{112}{3} \right) \right] \\
+ D_1 D_1 \left[ n_f C_F C_A^2 \left( - \frac{7324}{9} + \frac{352}{3} \zeta_2 \right) + n_f C_F^2 C_A \left( - \frac{133988}{81} + \frac{14144}{9} \zeta_2 \right) + \frac{448}{3} \zeta_3 \right] + n_f C_F^3 \left( \frac{2764}{3} + 704 \zeta_2 + \frac{5696}{3} \zeta_3 \right) \right] + n_f C_F C_A \left( \frac{1144}{9} - \frac{32}{3} \zeta_2 \right) \\
+ n_f^2 C_F^2 \left( \frac{7768}{81} - \frac{1024}{9} \zeta_2 \right) + n_f^3 C_F \left( - \frac{160}{27} \right) + C_F C_A^3 \left( \frac{43648}{27} \right) \right] \\
+ C_F^2 C_A \left( \frac{481216}{81} + \frac{50272}{9} \zeta_2 + \frac{2592}{5} \zeta_2^2 - \frac{8800}{3} \zeta_3 \right) + C_F^3 C_A \left( - \frac{26362}{3} \right) \\
- \frac{1376}{3} \zeta_2 + \frac{4512}{5} \zeta_2^2 - \frac{19808}{3} \zeta_3 \right] + C_F^4 \left( 3066 + 2928 \zeta_2 - \frac{4992}{5} \zeta_2^2 \right) \\
\right] + D_1 D_2 \left[ n_f C_F C_A^2 \left( \frac{968}{3} \right) + n_f C_F^2 C_A \left( \frac{21920}{9} - 320 \zeta_2 \right) \right] \\
+ n_f C_F^3 \left( - \frac{1760}{9} - \frac{4160}{3} \zeta_2 \right) + n_f^2 C_F C_A \left( - \frac{176}{3} \right) + n_f^2 C_F^2 \left( - \frac{1600}{9} \right) \right] \\
+ n_f^3 C_F \left( \frac{32}{9} \right) + C_F C_A^3 \left( - \frac{5324}{9} \right) + C_F^2 C_A^2 \left( - \frac{67120}{9} + 1760 \zeta_2 \right) \right] \\
+ C_F^4 C_A \left( \frac{9920}{9} + \frac{22880}{3} \zeta_2 + 3360 \zeta_3 \right) + C_F^4 \left( 8960 \zeta_3 \right) \right] \\
+ D_1 D_3 \left[ n_f C_F^2 C_A \left( - \frac{14080}{27} \right) + n_f C_F^3 \left( - \frac{1600}{3} \right) + n_f^2 C_F^2 \left( \frac{1280}{27} \right) \right] \\
+ C_F^2 C_A^2 \left( \frac{38720}{27} \right) + C_F^2 C_A \left( \frac{10720}{3} - 960 \zeta_2 \right) + C_F^4 \left( - 2560 - 1920 \zeta_2 \right) \right] \\
+ D_1 D_4 \left[ n_f C_F^3 \left( \frac{560}{3} \right) + C_F^3 C_A \left( - \frac{3080}{3} \right) \right] + D_1 D_5 \left[ C_F^4 \left( 224 \right) \right] \\
+ D_2 D_2 \left[ n_f C_F^2 C_A \left( - \frac{3520}{9} \right) + n_f C_F^3 \left( - 400 \right) + n_f^2 C_F^2 \left( \frac{320}{9} \right) \right] \\
+ C_F^2 C_A \left( \frac{9680}{9} \right) + C_F^3 C_A \left( 2680 - 720 \zeta_2 \right) + C_F^4 \left( - 1920 - 1440 \zeta_2 \right) \right] \\
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\[ \begin{align*}
+ D_2 \overline{D}_3 \left[ n_f C_F^3 \left( \frac{1120}{3} \right) + C_F^3 C_A \left( -\frac{6160}{3} \right) \right] + D_2 \overline{D}_4 \left[ C_F^4 \left( 560 \right) \right] \\
+ D_3 \overline{D}_3 \left[ C_F^4 \left( \frac{1120}{3} \right) \right] + \overline{D}_2 \delta(1-z_1) \left[ n_f C_F C_A^2 \left( \frac{10277}{9} - \frac{5096}{9} \right) \zeta_2 + \frac{176}{5} \zeta_2^2 \right] \\
- 176 \zeta_3 + n_f C_F^2 C_A \left( \frac{42140}{243} - \frac{13216}{9} \zeta_2 + \frac{688}{3} \zeta_2^2 - \frac{13712}{9} \zeta_3 \right) \\
+ n_f C_F^3 \left( \frac{259}{3} + \frac{3544}{9} \right) \zeta_2 - \frac{1088}{15} \zeta_2^2 - \frac{4048}{3} \zeta_3 \right) + n_f^2 C_F C_A \left( -\frac{3947}{27} \right) \\
+ \frac{688}{9} \zeta_2 + n_f^2 C_F^2 \left( -\frac{3718}{243} + \frac{2752}{27} \zeta_2 + \frac{320}{3} \zeta_3 \right) + n_f^3 C_F \left( 400 \right) \\
- \frac{32}{9} \zeta_2 + C_F C_A^3 \left( -\frac{206440}{81} + \frac{4012}{3} \zeta_2 - \frac{968}{5} \zeta_2^2 + 968 \zeta_3 \right) \\
+ C_F^2 C_A^2 \left( -\frac{356573}{486} - 688 \zeta_2 \zeta_3 + \frac{129392}{27} \zeta_2 - \frac{15752}{15} \zeta_2^2 + \frac{57296}{9} \zeta_3 \right) \\
- 1152 \zeta_5 + C_F^3 C_A \left( \frac{34849}{18} - 2592 \zeta_2 \zeta_3 - \frac{22708}{9} \zeta_2 + \frac{2816}{15} \zeta_2^2 \right) \\
+ \frac{14696}{3} \zeta_3 + C_F^4 \left( -3840 \zeta_2 \zeta_3 - 5120 \zeta_3 + 5376 \zeta_5 \right) \right] \\
+ \overline{D}_3 \delta(1-z_1) \left[ n_f C_F C_A^2 \left( -\frac{7324}{27} + \frac{352}{9} \zeta_2 \right) + n_f C_F^2 C_A \left( -\frac{133988}{243} \right) \\
+ \frac{14144}{27} \zeta_2 + \frac{448}{9} \zeta_3 \right) + n_f C_F^3 \left( \frac{2764}{9} + \frac{704}{9} \zeta_2 + \frac{5696}{9} \zeta_3 \right) \\
+ n_f^2 C_F C_A \left( \frac{1144}{27} - \frac{32}{9} \zeta_2 \right) + n_f^2 C_F^2 \left( \frac{7768}{243} - \frac{1024}{27} \zeta_2 \right) + n_f^3 C_F \left( 160 \right) \\
+ C_F C_A^3 \left( \frac{43648}{81} - \frac{968}{9} \zeta_2 \right) + C_F^2 C_A^2 \left( \frac{481216}{243} - \frac{50272}{27} \zeta_2 + \frac{864}{5} \zeta_2^2 \right) \\
- \frac{8800}{9} \zeta_3 \right) + C_F^3 C_A \left( -\frac{26362}{9} - \frac{1376}{9} \zeta_2 + \frac{1504}{5} \zeta_2^2 - \frac{19808}{9} \zeta_3 \right) \\
+ C_F^4 \left( 1022 + 976 \zeta_2 - \frac{1664}{5} \zeta_2^2 - 480 \zeta_3 \right) \right] \\
+ \overline{D}_4 \delta(1-z_1) \left[ n_f C_F C_A^2 \left( \frac{242}{9} \right) + n_f C_F^2 C_A \left( \frac{5480}{27} - \frac{80}{3} \zeta_2 \right) \right] \\
+ n_f C_F^3 \left( -\frac{440}{27} - \frac{1040}{9} \zeta_2 \right) + n_f^2 C_F C_A \left( -\frac{44}{9} \right) + n_f^2 C_F^2 \left( \frac{400}{27} \right) \\
+ n_f^3 C_F \left( \frac{8}{27} \right) + C_F C_A^3 \left( -\frac{1331}{27} \right) + C_F^2 C_A \left( -\frac{16780}{27} + \frac{440}{3} \zeta_2 \right) \\
+ C_F^3 C_A \left( \frac{2480}{27} + \frac{5720}{9} \zeta_2 + 280 \zeta_3 \right) + C_F^4 \left( \frac{2240}{3} \zeta_3 \right) \right] \\
+ \overline{D}_5 \delta(1-z_1) \left[ n_f C_F^2 C_A \left( -\frac{704}{27} \right) + n_f C_F^3 \left( -\frac{80}{3} \right) + n_f^2 C_F^2 \left( \frac{64}{27} \right) \right] \\
+ \overline{D}_6 \delta(1-z_1) \left[ n_f C_F C_A \left( -104 \right) + n_f C_F^2 \left( -\frac{1600}{27} \right) \right] \\
+ \overline{D}_7 \delta(1-z_1) \left[ n_f C_F \left( -\frac{80}{9} \right) \right] \\
+ \overline{D}_8 \delta(1-z_1) \left[ n_f^2 C_F \left( -\frac{400}{27} \right) \right] \end{align*} \]
\[\begin{align*}
\Delta_{sv,(1)} &= \delta(1 - z_1)\delta(1 - z_2) \left[ C_A \left(6 \, \zeta_2\right)\right] + D_0 \overline{D}_0 \left[ C_A \left(2\right)\right] + \overline{D}_1 \delta(1 - z_1) \left[ C_A \left(4\right)\right] \\
&+ (z_1 \leftrightarrow z_2),
\end{align*}\]

\[\begin{align*}
\Delta_{sv,(2)} &= \delta(1 - z_1)\delta(1 - z_2) \left[ n_f C_F \left(-\frac{67}{6} + 8 \, \zeta_3\right) + n_f C_A \left(-\frac{40}{3} - \frac{20}{3} \, \zeta_2 - 4 \, \zeta_3\right)\right] \\
&+ C_A \left(\frac{93}{2} + \frac{134}{3} \, \zeta_2 + \frac{126}{5} \, \zeta_2^2 - 22 \, \zeta_3\right) + D_0 \overline{D}_0 \left[ n_f C_A \left(-\frac{20}{9}\right) + C_A \left(\frac{134}{9}\right)\right] \\
&+ 4 \, \zeta_2\right]\left(\frac{8}{3} + C_A \left(\frac{8}{3}\right) + C_A \left(-\frac{44}{3}\right)\right] + D_0 \overline{D}_1 \left[ C_A \left(24\right)\right]\left(\frac{8}{3} - \frac{808}{27}\right) \\
&+ 44 \, \zeta_2 + 60 \, \zeta_3\right]) + \overline{D}_1 \delta(1 - z_1) \left[ n_f C_A \left(-\frac{40}{9}\right) + C_A \left(\frac{268}{9} + 8 \, \zeta_2\right)\right] \\
&+ \overline{D}_2 \delta(1 - z_1) \left[ n_f C_A \left(\frac{4}{3}\right) + C_A \left(-\frac{22}{3}\right)\right] + \overline{D}_3 \delta(1 - z_1) \left[ C_A \left(8\right)\right] \\
&+ (z_1 \leftrightarrow z_2),
\end{align*}\]

\[\begin{align*}
\Delta_{sv,(3)} &= D_0 \overline{D}_0 \left[ n_f C_F C_A \left(-63 + 48 \, \zeta_3\right) + n_f C_A^2 \left(-\frac{8422}{81} + \frac{32}{3} \, \zeta_2 + 16 \, \zeta_3\right)\right] \\
&+ n_f^2 C_A \left(\frac{200}{81} - \frac{16}{9} \, \zeta_2\right) + C_A \left(\frac{30569}{81} + \frac{52}{9} \, \zeta_2 - \frac{32}{5} \, \zeta_2^2 - 352 \, \zeta_3\right)\right] \\
&+ D_0 \overline{D}_1 \left[ n_f C_F C_A \left(8\right) + n_f C_A^2 \left(\frac{3656}{27} - \frac{224}{3} \, \zeta_2\right) + n_f^2 C_A \left(-\frac{160}{27}\right)\right] \\
&+ C_A \left(-\frac{16816}{27} + \frac{1232}{3} \, \zeta_2 + 976 \, \zeta_3\right)] + D_0 \overline{D}_2 \left[ n_f C_A^2 \left(-\frac{656}{9}\right)\right] \\
&+ n_f^2 C_A \left(\frac{16}{9}\right) + C_A \left(\frac{3700}{9} - 192 \, \zeta_2\right) + D_0 \overline{D}_3 \left[ n_f C_A \left(\frac{160}{9}\right)\right] \\
&+ C_A \left(-\frac{880}{9}\right)] + D_0 \overline{D}_4 \left[ C_A \left(40\right)\right] + D_1 \overline{D}_1 \left[ n_f C_A^2 \left(-\frac{656}{9}\right)\right] \\
&+ n_f^2 C_A \left(\frac{16}{9}\right) + C_A \left(\frac{3700}{9} - 192 \, \zeta_2\right) + D_1 \overline{D}_2 \left[ n_f C_A \left(\frac{160}{3}\right)\right] \\
&+ C_A \left(-\frac{880}{3}\right)] + D_1 \overline{D}_3 \left[ C_A \left(160\right)\right] + D_2 \overline{D}_2 \left[ C_A \left(120\right)\right]
\end{align*}\]
\[\Delta_{Y, g}^{sv} = D_0 \bar{D}_1 \left[ n_f C_F C_A \left( 1656 - 224 \frac{384}{5} \zeta_2 - 992 \zeta_3 \right) + n_f C_F^2 C_A \left( -4 \right) \right]
+ n_f C_A^3 \left( \frac{1147774}{243} - \frac{120848}{27} \zeta_2 + \frac{2304}{5} \zeta_2^2 - \frac{47648}{9} \zeta_3 \right)
+ n_f C_F C_A \left( -\frac{1400}{9} + \frac{320}{3} \zeta_3 \right) + n_f^2 C_A^2 \left( -\frac{109190}{243} + \frac{3296}{9} \zeta_2 + \frac{1088}{9} \zeta_3 \right)
+ n_f^3 C_A \left( \frac{800}{81} - \frac{64}{9} \zeta_2 \right) + C_A^4 \left( -\frac{3407840}{243} - 14240 \zeta_2 \zeta_3 + \frac{397592}{27} \zeta_2 \right)
- 2112 \zeta_2 + 30448 \zeta_3 + 8448 \zeta_5 \right) + D_0 \bar{D}_2 \left[ n_f C_F C_A^2 \left( -\frac{3148}{3} + 768 \zeta_3 \right) \right]
+ n_f C_A^3 \left( -\frac{271148}{81} + \frac{18080}{9} \zeta_2 + 1408 \zeta_3 \right) + n_f^2 C_F C_A \left( \frac{400}{9} \right)
+ n_f^2 C_A^2 \left( \frac{19288}{81} - \frac{1120}{9} \zeta_2 \right) + n_f^3 C_A \left( -\frac{160}{27} \right) + C_A^4 \left( \frac{862648}{81} - \frac{72472}{9} \zeta_2 \right)
+ \frac{2112}{5} \zeta_2 - 11968 \zeta_3 \right) + D_0 \bar{D}_3 \left[ n_f C_F C_A^2 \left( \frac{160}{3} \right) + n_f C_A^3 \left( \frac{3256}{3} \right) \right]
- \frac{5120}{9} \zeta_2 + n_f^2 C_A^2 \left( -\frac{2128}{27} \right) + n_f^3 C_A \left( \frac{32}{27} \right) + C_A^4 \left( -\frac{104764}{27} \right)}
\[\begin{align*}
+ \frac{28160}{9} \zeta_2 + \frac{12320}{3} \zeta_3 \bigg] + D_0 \overline{D}_4 \left[ n_f C_A^3 \left( - \frac{7120}{27} \right) + n_f^2 C_A^2 \left( \frac{320}{27} \right) \right] \\
+ C_A^4 \left( \frac{33800}{27} - \frac{720}{3} \zeta_2 \right) + D_0 \overline{D}_5 \left[ n_f C_A^3 \left( \frac{112}{3} \right) + C_A^4 \left( - \frac{616}{3} \right) \right] \\
+ D_0 \overline{D}_6 \left[ C_A^4 \left( \frac{112}{3} \right) \right] + D_1 \overline{D}_1 \left[ n_f C_F C_A^2 \left( - \frac{3148}{3} + 768 \zeta_3 \right) \right] \\
+ n_f C_A^3 \left( - \frac{271148}{81} + \frac{18080}{9} \zeta_2 + 1408 \zeta_3 \right) + n_f^2 C_F C_A \left( \frac{40}{3} \right) \\
+ n_f^2 C_A^2 \left( \frac{19288}{81} - \frac{1120}{9} \zeta_2 \right) + n_f C_A^3 \left( - \frac{160}{27} \right) + C_A^4 \left( \frac{862648}{81} - \frac{72472}{9} \zeta_2 \right) \\
+ \frac{2112}{5} \zeta_2 - 11968 \zeta_3 \bigg] + D_1 \overline{D}_2 \left[ n_f C_F C_A^2 \left( 160 \right) + n_f C_A^3 \left( 3256 \right) \right] \\
- \frac{5120}{3} \zeta_2 + n_f^2 C_A^2 \left( - \frac{2128}{9} \right) + n_f^3 C_A \left( \frac{32}{9} \right) + C_A^4 \left( - \frac{104764}{9} + \frac{28160}{3} \zeta_2 \right) \\
+ 12320 \zeta_3 \bigg] + D_1 \overline{D}_3 \left[ n_f C_A^3 \left( \frac{28480}{27} \right) + n_f^3 C_A \left( \frac{1280}{27} \right) + C_A^4 \left( \frac{135200}{27} \right) \right] \\
- 2880 \zeta_2 \bigg] + D_1 \overline{D}_4 \left[ n_f C_A^3 \left( \frac{560}{3} \right) + C_A^4 \left( - \frac{3080}{3} \right) \right] + D_1 \overline{D}_5 \left[ C_A^4 \left( 224 \right) \right] \\
+ D_2 \overline{D}_2 \left[ n_f C_A^3 \left( - \frac{7120}{9} \right) + n_f^2 C_A^2 \left( \frac{320}{9} \right) + C_A^4 \left( \frac{33800}{9} - 2160 \zeta_2 \right) \right] \\
+ D_2 \overline{D}_3 \left[ n_f C_A^3 \left( \frac{1120}{3} \right) + C_A^4 \left( - \frac{6160}{3} \right) \right] + D_2 \overline{D}_4 \left[ C_A^4 \left( 560 \right) \right] \\
+ D_3 \overline{D}_3 \left[ C_A^4 \left( \frac{1120}{3} \right) \right] + D_2 \overline{D}_5 \left[ n_f C_F C_A^2 \left( 828 - 112 \zeta_2 - \frac{192}{5} \zeta_2^2 \right) \right] \\
- 496 \zeta_3 \bigg] + n_f C_F^2 C_A \left( - 2 \right) + n_f C_A^3 \left( \frac{573887}{243} - \frac{60424}{27} \zeta_2 + \frac{1152}{5} \zeta_2^2 \right) \\
- \frac{23824}{9} \zeta_3 \bigg] + n_f^2 C_F C_A \left( - \frac{700}{9} + \frac{160}{3} \zeta_3 \right) + n_f^3 C_A \left( - \frac{54595}{243} \right) \\
+ \frac{1648}{9} \zeta_2 + \frac{544}{9} \zeta_3 \bigg] + n_f^3 C_A \left( \frac{400}{81} - \frac{32}{9} \zeta_2 \right) + C_A^4 \left( - \frac{1703920}{243} \right) \\
- 7120 \zeta_2 \zeta_3 + \frac{198796}{27} \zeta_2 - 1056 \zeta_2^2 + 15224 \zeta_3 + 4224 \zeta_5 \bigg] \bigg] + D_3 \overline{D} \delta(1 - z_1) \left[ n_f C_F C_A^2 \left( - \frac{3148}{9} + 256 \zeta_3 \right) + n_f C_A^3 \left( - \frac{271148}{243} \right) \right] \\
+ \frac{18080}{27} \zeta_2 + \frac{1408}{3} \zeta_3 \bigg] + n_f^2 C_F C_A \left( \frac{40}{9} \right) + n_f^2 C_A^2 \left( \frac{19288}{243} \right) \\
- \frac{1120}{27} \zeta_2 \bigg] + n_f^3 C_A \left( - \frac{160}{81} \right) + C_A^4 \left( \frac{862648}{243} - \frac{72472}{27} \zeta_2 + \frac{704}{5} \zeta_2^2 \right) \\
- \frac{11968}{3} \zeta_3 \bigg] + D_4 \delta(1 - z_1) \left[ n_f C_F C_A^2 \left( \frac{40}{3} \right) + n_f C_A^3 \left( \frac{814}{3} - \frac{1280}{9} \zeta_2 \right) \right]
\end{align*}\]
\[\begin{align*}
+ n_f^2 C_A^2 \left( - \frac{532}{27} \right) & + n_f^3 C_A \left( \frac{8}{27} \right) + C_A^4 \left( - \frac{26191}{27} + \frac{7040}{9} \zeta_2 + \frac{3080}{3} \zeta_3 \right) \\
+ \mathcal{D}_5 \delta(1 - z_1) \left[ n_f C_A^3 \left( - \frac{1424}{27} \right) + n_f^2 C_A^2 \left( \frac{64}{27} \right) + C_A^4 \left( \frac{6760}{27} - 144 \zeta_2 \right) \right] & \\
+ \mathcal{D}_6 \delta(1 - z_1) \left[ n_f C_A^3 \left( \frac{56}{9} \right) + C_A^4 \left( - \frac{308}{9} \right) \right] + \mathcal{D}_7 \delta(1 - z_1) \left[ C_A^4 \left( \frac{16}{3} \right) \right] \\
+ (z_1 \leftrightarrow z_2),
\end{align*}\]

where
\[\begin{align*}
\mathcal{D}_i &= \left[ \log^i(1 - z_1) \right]_+, & \overline{\mathcal{D}}_i &= \left[ \log^i(1 - z_2) \right]_+.
\end{align*}\]
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