Redshift and gauge choice

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ABSTRACT

We show that a specific gauge choice comes extremely close to defining a frame whose preferred observers see a dipole-free CMB. In this gauge the metric is the product of a scale factor depending on all spacetime coordinates, and a metric featuring an expansion-free geodesic timelike vector field. This setup facilitates the computation of redshift and other distance measures and explains why we can have a highly isotropic CMB despite large inhomogeneities.

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1 Introduction

The almost perfect isotropy of the cosmic microwave background (CMB) is among the pillars of the cosmological standard model according to which our universe can be described, at large scales, as a Friedmann-Lemaître-Robertson-Walker (FLRW) universe with small perturbations. This isotropy comes at different levels [1]: the actual observations (terrestrial or from satellites) show deviations in temperature of $\delta T/T \approx 0.12\%$, but once the dipole contribution is subtracted, this improves to a value of $\delta_{df} \approx 10^{-5}$ (‘df’ for ‘dipole-free’). This means that an observer passing through our solar system at a velocity of 370 km/sec (in the right direction) will see the latter spectacularly small level; on the other hand, an observer comoving with our local galaxy group sees an anisotropy of $\delta_{lg} \approx 0.2\%$.

According to the Copernican principle, the situation should be similar at most locations in the present era. It is important to note the difference between $\delta_{df}$ and $\delta_{lg}$, not only in size ($\delta_{df} \approx 10^{-5} \ll \delta_{lg} \approx 2 \times 10^{-3}$), but also in quality: whereas $\delta_{df}$ is determined by a full celestial sphere’s worth of observations, the value of $\delta_{lg}$ comes from a single draw from a distribution with mean zero. For these reasons, we would very much prefer the use of $\delta_{df}$ over that of $\delta_{lg}$ in an analysis of the structure of the universe. In other words, we want to work in a frame comoving with the CMB, not with the matter.

The wavelength of a CMB photon is the product of its value at last scattering and the redshift factor picked up on the way to the observer. Unless one believes in strange nonlocal correlations between the two, one can only conclude that neither the original wavelength nor the redshift factor should feature deviations that are larger than the ones seen by the observer. In the present work we will be interested only in the extremely precise matching of the redshifts in the different directions.

The celebrated Ehlers-Geren-Sachs (EGS) theorem [2] states that the existence of a perfectly isotropic radiation background combined with reasonable assumptions on the matter content of the universe implies FLRW. There is a number of generalizations to ‘almost EGS’ theorems (e.g. [3, 4, 5, 6]) stating that small deviations from isotropy should lead only to small deviations from FLRW; see section 11.1 of Ref. [7] for a very clear summary. These works usually (with an exception in [5]) assume that the radiation 4-velocity (i.e. the velocity field $u_{df}$ of the dipole-free observers) is geodesic. This is an additional input which can be argued for only if one does not distinguish the CMB frame from the matter frame. Thus it holds only at the level of $\delta_{lg}$, not
at the level of $\delta_{df}$.

In the present work we are interested in precision at the level of $\delta_{df} \approx 10^{-5}$, so we do not take the radiation velocity to be geodesic. Our analysis will rely on redshift rather than distribution functions for the radiation, which simplifies matters considerably. The timelike vector field $u_{df}$ that determines a preferred observer at every spacetime point can, in principle, be completed to an orthonormal frame $\{e_0 = u_{df}, e_1, e_2, e_3\}$ which we would call a CMB frame. In practice the requirement of a vanishing dipole is highly nonlocal and therefore analytically intractable. Instead, we are going to work with a locally well-defined quantity which, as we shall explicitly verify, comes very close to defining the level of anisotropy. It turns out that this quantity can be simplified by a conformal transformation, and that the most important contributions to it can be eliminated by a gauge choice. This explains why $\delta_{df}$ is so small despite the fact that the actual universe shows a considerable amount of inhomogeneity. Working in this gauge significantly improves the tractability of light propagation compared to the synchronous and the longitudinal gauge, which are the ones that are used most frequently. An explicit comparison in linear perturbation theory shows that the metric perturbations in the new gauge are not much larger than those in the longitudinal gauge, which is usually considered to be optimal in that respect.

In the next section we introduce a quantity that vanishes if an isotropically redshifted CMB is observed everywhere, and show how it simplifies under a conformal transformation. In section 3 we formulate a gauge that eliminates two of three contributions to this quantity and thereby comes close to defining a CMB frame; we also give explicit conditions on a metric implementing this gauge. Section 4 contains an analysis of this metric in linear perturbation theory. In the final section we argue that other distance measures are also well behaved in the new gauge, make some remarks on the controversy about the impact of inhomogeneities on the expansion of the universe, and discuss open questions about our gauge.

2 Redshift and conformal transformation

We consider a photon emitted at some point $x_e$ by a source moving along a worldline with a tangent vector $u_e$ normalized to $u_e^2 = g_{\mu\nu} u_\mu u_\nu = -1$, where $g_{\mu\nu}$ is the pseudo-Riemannian spacetime metric of type $-+++$. This photon propagates along a lightlike geodesic which we
describe by an affine parameter $\lambda$ such that the tangent vector to the geodesic is $k^\mu = dx^\mu/d\lambda$. The redshift $z_{e\rightarrow o}$, as seen by an observer at $x_o$ whose wordline has the tangent vector $u_o$ (normalized to $u_o^2 = -1$), is determined by the well-known formula

$$1 + z_{e\rightarrow o} = \frac{(u \cdot k)_e}{(u \cdot k)_o}$$

(1)

In an idealized universe in which every spacetime point admits a distinguished observer who sees a perfectly isotropically redshifted last scattering surface, there would exist a global vector field $u$ characterizing such observers, as well as a globally well defined function

$$a(x) = 1 + z_{\text{ls}} \rightarrow x = \frac{(u \cdot k)_{\text{ls}}}{(u \cdot k)_x}$$

(2)

that determines this redshift. We could then determine the redshifts between preferred observers via

$$1 + z_{e\rightarrow o} = \frac{a(x_o)}{a(x_e)}$$

(3)

as a direct consequence of Eqs. (1) and (2). Along any geodesic described with an affine parameter $\lambda$ and tangent vector $k$, the value of $a(x)(u \cdot k)(x)$ would remain constant and therefore the quantity

$$d(x, k) = \frac{d}{d\lambda} [a(x)(u \cdot k)(x)]$$

(4)

would have to vanish at every spacetime point $x$ for every lightlike tangent vector $k$ at $x$.

For an arbitrary timelike vector field $u$ and non-vanishing scalar $a$, where $d(x, k)$ need not vanish, a redshift formula can still be obtained by noting that

$$\ln[-a(x)(u \cdot k)(x)]_e = \int_e^o \frac{d(x, k)}{a(x)(u \cdot k)(x)} d\lambda$$

implies

$$1 + z_{e\rightarrow o} = \frac{(u \cdot k)_e}{(u \cdot k)_o} = \frac{a(x_o)}{a(x_e)} \exp \left( - \int_e^o \frac{d(x, k)}{a(x)(u \cdot k)(x)} d\lambda \right).$$

(5)

In the following we would like to treat the requirement

$$\langle d(x, k) \rangle = 0, \quad \langle d(x, k)^2 \rangle \text{ small},$$

(7)

where $\langle \cdots \rangle$ should represent the average over the celestial sphere,

$$\langle \cdots \rangle = \frac{1}{4\pi} \int \cdots d\Omega,$$

(8)
as a local proxy for the conditions defining the CMB frame. Using the facts that differentiation by \( \lambda \) corresponds to covariant differentiation along \( k \) and that \( k^\nu k_{\mu \nu} = 0 \) we get

\[
d(x, k) = k^\nu [a(x)(u \cdot k)(x)]_{,\nu} = a_\mu k^\nu (u \cdot k) + au_{\mu \nu} k^\nu k^\mu. \tag{9}
\]

Motivated by the FLRW case, we introduce the conformally transformed quantities

\[
\hat{g}_{\mu \nu} = a^{-2} g_{\mu \nu}, \quad \hat{u}_\mu = a^{-1} u_\mu, \quad \hat{u}^\mu = \hat{g}^{\mu \nu} \hat{u}_\nu = au^\mu
\]

with \( \hat{u}_\mu \hat{u}^\nu = u_\mu u_\nu g^{\mu \nu} = -1 \). Then a short calculation gives

\[
a^2 \hat{u}^\rho ;_{\nu} = au_{\mu \nu} + a_\mu u^\nu - a_\rho u^\rho g_{\mu \nu}, \tag{11}
\]

where \( ; \) denotes covariant differentiation with respect to \( \hat{g} \). Contraction with \( k^\mu k^\nu \) shows that

\[
d(x, k) = \Delta_{\mu \nu}(x) k^\mu k^\nu = \hat{\Delta}_{\mu \nu}(x) \hat{k}^\mu \hat{k}^\nu \tag{12}
\]

with

\[
\Delta_{\mu \nu} = au_{(\mu ; \nu)} + a_\rho u^\rho g_{\mu \nu} = a^2 \hat{\Delta}_{\mu \nu}, \quad \hat{\Delta}_{\mu \nu} = \hat{u}_{(\mu ; \nu)}. \tag{13}
\]

Thus Killing’s equation \( \hat{u}_{(\mu ; \nu)} = 0 \) implies \( d(x, k) = 0 \), and with a little work the converse can also be shown. This corresponds to the well-known result [8] that a spacetime admits a perfectly isotropic CMB background if and only if its metric is conformal to a metric with a timelike Killing vector; this fact is essential for the derivation of the EGS theorem [2].

The standard decomposition (see e.g. chapter 4 of [7]) of

\[
g_{\mu \nu} = -u_\mu u_\nu + h_{\mu \nu} \tag{14}
\]

into projection operators \(-u_\mu u_\nu \) (timelike) and \( h_{\mu \nu} \) (spacelike), with \( u^\mu h_{\mu \nu} = 0 \) and \( h^{\mu \nu} h_{\mu \nu} = 3 \), (or, equivalently, \( \hat{g}_{\mu \nu} = -\hat{u}_\mu \hat{u}_\nu + \hat{h}_{\mu \nu} \)) affords a decomposition of any symmetric tensor \( \Delta_{\mu \nu} \) as

\[
\Delta_{\mu \nu} = u_\mu u_\nu \Delta^S_{\mu \nu} + h_{\mu \nu} \Delta^S_{S} - u_\mu \Delta^V_\nu - u_\nu \Delta^V_\mu + \Delta^T_{\mu \nu} \tag{15}
\]

in terms of scalars \( \Delta^S_{\mu \nu} \) and \( \Delta^S_{S} \) (related to the time and space projections, respectively), a vector \( \Delta^V_\mu \) satisfying \( \Delta^V_\mu u^\mu = 0 \) and a symmetric tensor \( \Delta^T_{\mu \nu} \), satisfying \( \Delta^T_{\mu \nu} u^\mu = 0 \) and \( \Delta^T_{\mu \nu} h^{\mu \nu} = 0 \).

Assuming that we have parametrized the geodesic in such a way that \( u \cdot k = -1 \) at the point \( x \) where we compute \( d(x, k) \), writing

\[
k^\mu = u^\mu + e^\mu \tag{16}
\]
and using the conditions $u^2 = -1$ and $k^2 = 0$, we find that

$$u \cdot e = 0, \quad e^2 = 1 \quad \text{and} \quad e^\mu h_{\mu\nu} = e\nu, \quad (17)$$

i.e. $e$ must be a spacelike unit vector orthogonal to $u$. Applying this to Eq. (12) with the decomposition (15), we find

$$d(x, k) = \Delta^S + 2\Delta^V e^{\nu} + \Delta^T_{\mu\nu} e^\mu e^\nu \quad \text{with} \quad \Delta^S = \Delta^{St} + \Delta^{Ss}. \quad (18)$$

In order to evaluate averages of the type (8) we introduce spacelike unit vectors $e^\mu_1, e^\mu_2, e^\mu_3$ that form a tetrad together with $u^\mu$, and define $e^\mu(\Omega) = \cos\varphi \sin\vartheta e^\mu_1 + \ldots$ through standard spherical coordinates $\Omega = (\varphi, \vartheta)$. Using

$$\langle e^\mu_1 \cdots e^\mu_{2p+1} \rangle = 0, \quad \langle e^\mu e^{\nu} \rangle = \frac{1}{3} h^{\mu\nu}, \quad \langle e^\mu e^{\nu} e^\rho e^\sigma \rangle = \frac{1}{15} (h^{\mu\nu} h^{\rho\sigma} + h^\mu h^{\nu\sigma} + h^{\mu\sigma} h^{\nu\rho}) \quad (19)$$

one finds

$$\langle d(x, k) \rangle = \Delta^S, \quad \langle d(x, k)^2 \rangle = (\Delta^S)^2 + \frac{4}{3} h^{\mu\nu} \Delta^V_{\mu} \Delta^V_{\nu} + \frac{2}{15} \Delta^T_{\mu\nu} h^{\nu\rho} \Delta^T_{\rho\sigma} h^{\sigma\mu}. \quad (20)$$

Returning to the specific form (13) of $\Delta_{\mu\nu}$, application of the projection operators (in the ‘hatted’ version) gives $\hat{\Delta}^{St} = 0$ (so that $\hat{\Delta}^S = \hat{\Delta}^{Ss}$) and

$$\hat{\Delta}^S = \frac{1}{3} \hat{g}^{\mu\nu} \hat{u}_{\mu\nu}, \quad (21)$$
$$\hat{\Delta}^V_{\mu} = \frac{1}{2} \hat{u}_{\mu\rho} \hat{u}^\rho, \quad (22)$$
$$\hat{\Delta}^T_{\mu\nu} = \hat{u}_{(\mu\rho)} - h_{\mu\rho} \hat{\Delta}^S + \hat{u}_{\mu} \Delta^V_{\nu} + \hat{u}_{\nu} \Delta^V_{\mu}, \quad (23)$$

i.e. these quantities correspond to the expansion, the acceleration and the shear of the timelike vector field $\hat{u}$ with respect to the metric $\hat{g}$.

### 3 Gauge choice and metric

The actual universe features deviations from homogeneity, so we do not expect all components of $\Delta_{\mu\nu}$ to vanish. Why can we nevertheless find a local frame in which the CMB has almost exactly the same temperature in all directions? We propose that this is due to the fact that setting

$$\Delta^S = 0, \quad \Delta^V_{\mu} = 0 \quad (24)$$

is an admissible gauge choice. Indeed, these quantities correspond to $1 + 3 = 4$ degrees of freedom, which is just the number of quantities that can be fixed by a gauge. In terms of the original timelike field $u$, the effects of this choice on the expansion $\Theta = h^\rho_\nu u_\mu;\nu$, the acceleration $\dot{u}_\mu = u_\mu;\rho u^\rho$, the shear $\sigma_{\mu\nu} = u^\text{PSTF}_{\mu\nu}$ (the projected symmetric tracefree part of $u_{\mu\nu}$, i.e. what remains after symmetrizing, projecting with $h$ and removing the $h$-trace) and the vorticity $\omega_{\mu\nu} = h^\rho_\nu h^\sigma_\mu u_{\rho\sigma}$ are easily found with the help of Eq. (11):

$$\dot{u}_\mu = h^\rho_\mu \frac{a_{,\rho}}{a}, \quad \Theta = 3 u^\rho \frac{a_{,\rho}}{a}, \quad \sigma_{\mu\nu} = a\dot{u}_{(\mu;\nu)}, \quad \omega_{\mu\nu} = a\dot{u}_{[\mu;\nu]}.$$  

(25)

In words, expansion and acceleration correspond to the timelike and spacelike components of $(\ln a)_\mu$, respectively; shear and vorticity are multiples of the corresponding quantities in the conformally transformed frame.

This has the following effects on the redshift. In the integral in Eq. (6) we can write $(\Delta_{\mu\nu}/\dot{u}_\mu)k^\mu k^\nu$ instead of $d(x,k)/(a u_\rho)$. Furthermore, since $(k^\mu k^\nu/k^\rho)d\lambda$ is invariant under arbitrary reparametrizations of the geodesic, we can replace it by $(\tilde{k}^\mu \tilde{k}^\nu/\tilde{k}^\rho)d\tilde{\lambda}$ with $\tilde{k}^\mu = \hat{\mu}^\mu + \hat{\epsilon}_\mu$ chosen such that $\hat{\epsilon}_\mu \hat{k}^\rho = -1$ everywhere along the geodesic. Then, using the analog of Eq. (18) for the metric $\hat{g}$ and the gauge conditions (24), we get

$$1 + z_{e\to o} = \frac{a(x_o)}{a(x_e)} \exp \left( \int_e^o \Delta^T_{\mu\nu} \hat{\epsilon}_\mu \hat{\epsilon}_\nu d\lambda \right)$$

with $\Delta^T_{\mu\nu} = \dot{u}_{(\mu;\nu)}$.

Let us now find explicit coordinates that implement our gauge (24). Choosing $\dot{u}$ to be the vector with components $\dot{u}^0 = 1$ and $\dot{u}^i = 0$, we get $\hat{g}_{00} = -1$, $\dot{u}^\mu_{;\rho} = \hat{\Gamma}^\mu_{\rho\nu} \dot{u}^\nu = \dot{\Gamma}^\nu_{\rho0}$ and therefore

$$\dot{u}_{\mu;\rho} = \dot{\Gamma}_{\mu0\rho}. \quad (27)$$

Upon demanding $0 = 2\hat{\Delta}^V = \dot{u}_{\mu;\rho} \dot{u}^\rho = \dot{\Gamma}^\mu_{\rho0} = \hat{g}_{\rho00} = \dot{g}_{\rho00}$, the metric takes the form $ds^2 = a^2 d\hat{s}^2$ with

$$ds^2 = -(dx^0 - V_i dx^i)^2 + \gamma_{ij} dx^i dx^j,$$

(28)

where $\gamma_{ij}$ can depend on all coordinates $x^\mu$ whereas $V_i$ depends only on the spatial coordinates $x^j$. The inverse metric $\hat{g}^{\mu\nu}$ has the components

$$\hat{g}^{00} = -1 + V_i \gamma^{ij} V_j, \quad \hat{g}^{0i} = \gamma^{ik} V_k, \quad \hat{g}^{ij} = \gamma^{ij},$$

(29)

where $\gamma^{ij}$ is defined by the requirement $\gamma^{ij} \gamma_{jk} = \delta^i_k$. In matrix notation, the original metric and its inverse are given by

$$g = a^2 \begin{pmatrix} -1 & V^T \\ V & \gamma - VV^T \end{pmatrix}, \quad g^{-1} = a^{-2} \begin{pmatrix} -1 + V^T \gamma^{-1} V & V^T \gamma^{-1} \\ \gamma^{-1} V & \gamma^{-1} \end{pmatrix}.$$  

(30)

6
Finally, $0 = 6\ddot{\Delta} = 2\dot{g}^{\mu\nu}\dot{u}_{\mu\nu} = 2\dot{g}^{\mu\nu}\ddot{\Gamma}_{\mu\nu\dot{0}} = \dot{g}^{\mu\nu}(\dot{g}_{\mu\nu,0} + \dot{g}_{\mu\dot{0},\nu} - \dot{g}_{\nu\dot{0},\mu}) = \dot{g}^{\mu\nu}\dot{g}_{\mu\nu,0} = \dot{g}^{ij}\dot{g}_{ij,0} = \text{tr}(\gamma^{-1}\gamma,0) = (\text{tr} \ln \gamma)_0 = (\text{ln det} \gamma)_0$ implies $x^0$-independence of $\text{det} \gamma$.

The conditions $V_{i,0} = 0$ and $(\text{det} \gamma)_0 = 0$ do not completely fix the form of our metric (28). For example, they also hold in a transformed frame $\{\tilde{x}^\mu\}$ with

$$\tilde{x}^0 = x^0 + f(x^i), \quad \tilde{x}^i = \tilde{x}^i(x^j).$$

(31)

We can use parts of this freedom to assign a single time coordinate to the initial singularity and to set $\text{det} \gamma = 1$.

4 Linear perturbation theory

We would now like to consider the consequences of our gauge choice (24) in the context of linear perturbation theory [9]. Our notation will be similar to that of Refs. [10, 7] which we also recommend for further details. A metric corresponding to a small perturbation of the conformally flat case is given, before gauge fixing, by

$$ds^2 = a_h(x^0)^2\{-[(1+2\phi)(dx^0)^2] + 2(B_i - S_i)dx^i dx^0 + [(1-2\psi)\delta_{ij} + 2E_{ij} + 2F_{(ij)} + h_{ij}]dx^i dx^j\};$$

(32)

here $a_h(x^0)$ represents the scale factor for the corresponding homogeneous case $(g_{\mu\nu}h) = a_h^2 \eta_{\mu\nu}$; $\phi$, $\psi$, $B$ and $E$ are scalars; $S_i$ and $F_i$ are transverse vectors (i.e. they satisfy $\delta^{ij}S_{ij} = 0$ and $\delta^{ij}F_{ij} = 0$); $h_{ij}$ is a symmetric traceless transverse tensor ($h_{ij} = h_{ji}$, $\delta^{ij}h_{ij} = 0$, $\delta^{ik}h_{ij,k} = 0$).

The gauge freedom $x^\mu \rightarrow \tilde{x}^\mu(x^\nu)$ can be expressed at the linearized level in terms of a transverse vector $\xi^i$ and scalars $\xi^0$ and $\xi^i$; the corresponding transformations

$$\tilde{\phi} = \phi - \frac{\dot{a}_h}{a_h}\xi^0 - \xi^0, \quad \tilde{\psi} = \psi + \frac{\dot{a}_h}{a_h}\xi^0, \quad \tilde{B} = B + \xi^0 - \xi^i, \quad \tilde{E} = E - \xi,$$

(33)

$$\tilde{F}_i = F_i - \xi_i, \quad \tilde{S}_i = S_i + \xi_i, \quad \tilde{h}_{ij} = h_{ij}$$

(34)

can then be used to eliminate two of the scalars and one of the transverse vectors. The two most popular gauge choices are longitudinal gauge with $B = E = 0$ (usually accompanied by neglecting vector and tensor modes), and synchronous gauge, which manifests itself at the linearized level as $\phi = B = 0$, $S_i = 0$.

A well-known solution to the Einstein equations for irrotational dust with $\Lambda = 0$ (hence $a_h = \text{const} \times (x^0)^2$), which is believed to give a good description of the early matter dominated
era of our universe, relies on a single time-independent function $\phi_N$ which is just the Newtonian potential. In the longitudinal gauge this solution is given by $\phi_{\text{long}} = \psi_{\text{long}} = \phi_N$; it can be transformed to the synchronous gauge via $\xi^0 = x^0 \phi_N / 3$, $\xi = (x^0)^2 \phi_N / 6$, resulting in $E_{\text{sync}} = -(1/6)(x^0)^2 \phi_N$, $\psi_{\text{sync}} = (5/3) \phi_N$. In the latter case, second derivatives of $\phi_N$ occur in the metric and tend to make the perturbations large for moderate $x^0$, which is often used as an argument against employing the synchronous gauge in situations other than the very early universe.

What about the gauge (24) and the corresponding metric (30)? If we assume that we have used some of our residual gauge freedom to set $\det \gamma = 1$, then in the linearized version $\gamma_{ij} - \delta_{ij}$ must be traceless. Writing $a = (1 + \phi) a_h$, this implies $\delta^{ij} E_{,ij} = 3(\phi + \psi)$. It turns out that without violating our gauge conditions we can set $B$ and $S_i$ to zero, so that the metric becomes (up to quadratic and higher terms)

$$ds^2 = a_h^2 (x^0) (1 + 2\phi) \left\{ -(dx^0)^2 + [\delta_{ij} + 2(E_{,ij} - \frac{1}{3} \delta^{kl} E_{,kl} \delta_{ij}) + 2F_{(i,j)} + h_{ij}]dx^i dx^j \right\}.$$ (35)

For the special solution considered above we can get to this form by applying a transformation with $\xi^0 = 0$, $\xi^i = 0$ and $\xi$ satisfying $\xi_0 = 0$ and $\delta^{ij} \xi_{,ij} = -6\phi_N$ to the metric in the longitudinal gauge. This results in $\phi = \phi_N$ and $E$ chosen such that $\delta^{ij} E_{,ij} = 6\phi_N$. Thus we can interpret $E$ as a gravitational prepotential. In particular, the expressions $E_{,ij}$ occurring in the metric should be roughly of the same order of magnitude as $\phi_N$.

If we ignore the vector and tensor modes, we find $\Delta \Gamma_{\mu \nu} = \hat{u}_{(\mu;\nu)} = \hat{\Gamma}_{\mu \nu \theta} = 0$ at the linear level. Thus the integral occurring in the redshift formula (26), which represents deviations from the uniform case, is an average over a quantity that vanishes at first order of perturbation theory and has an expectation value of zero at all orders. This explains the fact that we observe almost perfect isotropy of the CMB despite the existence of inhomogeneities.

5 Concluding remarks

Observational cosmology relies not only on the redshift, but also on other distance measures such as the angular diameter distance and the luminosity distance. These quantities can be computed via arguments based on fluxes. For known redshift, one can use a comparison between the total number of photons emitted per unit of time in a specific frequency range, and the number of photons, in the appropriately transformed frequency range, arriving in a given area.
at the observer’s location. Because of the non-acceleration and non-expansion of the vector field \( \hat{u} \) with respect to \( \hat{g} \), the number of photons arriving per unit of \( x^0 \) (the time coordinate related to \( \hat{u} \)) on a suitable hypothetic screen enveloping the source must be identical with the number of photons emitted during the corresponding \( x^0 \)-interval of the same duration (as measured with \( \hat{g} \)). Therefore, on average the photon count with respect to \( \hat{g} \) behaves like the photon count in a static universe. Upon proper rescalings of the time and area values with the corresponding powers of \( a \) one gets formulas for averaged fluxes that are identical in form with those for a homogeneous universe, but with \( a_0 \) replaced by \( a \). Thus the overall expansion, as inferred from measured redshift-distance relations, is given straightforwardly by the values of \( a \) at the sources and at our spacetime position.

There have been suggestions (for a small subset, see e.g. [11, 12, 13, 14, 15, 16]) that the perceived acceleration of the universe’s expansion may not be due to a cosmological constant or dark energy, but to some effect stemming from the inhomogeneities of the actual universe. This possibility is rejected in papers such as [17, 18], giving rise to further rounds of controversy [19, 20]. One of the main points of [17, 18] is an attack on the choice of synchronous gauge on which many attempts to explain the data without \( \Lambda \) are based; instead the use of the longitudinal gauge is advocated. From the present work it is clear that neither of these gauges is as directly related to observations as the one presented here in Eq. (24).

This makes a thorough investigation of the properties and consequences of this gauge choice highly desirable. Open questions include the following. What residual gauge freedom is there beyond that indicated in (31)? Is the possibility of setting \( \hat{g}_{0i} = V_i \) to zero general or specific to linear perturbation theory? What are the Einstein equations in linear and second order perturbation theory, for collisionless dust and more generally? Can we reproduce arguments along the lines of [17, 18]? What can we say beyond perturbation theory, either by analytic arguments or numerically?

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