Abstract

This paper has two parts. The first part is a review and extension of the methods of integration of Leibniz algebras into Lie racks, including as new feature a new way of integrating 2-cocycles (see Lemma 3.9).

In the second part, we use the local integration of a Leibniz algebra \( \mathfrak{h} \) using a Baker-Campbell-Hausdorff type formula in order to deformation quantize its linear dual \( \mathfrak{h}^* \). More precisely, we define a natural rack product on the set of exponential functions which extends to a rack action on \( C^\infty(\mathfrak{h}^*) \).

Introduction

In this paper, we solve an old problem in symplectic geometry, namely we propose a way how to quantize the dual space of a Leibniz algebra \( \mathfrak{h} \). This dual space \( \mathfrak{h}^* \) is some kind of generalized Poisson manifold, as the bracket of \( \mathfrak{h} \) is not necessarily skew-symmetric. Intimately linked to this question is the integration of Leibniz algebras.

One of the most fascinating theorems in Lie theory is Lie’s Third Theorem, namely the possibility to integrate every real Lie algebra into a Lie group. Several proofs of this theorem are known, none of them reduces the claim to easy facts or computations. We focus here especially on two approaches. The first one, which we call the homological proof of Lie’s Third Theorem, regards a given Lie algebra \( \mathfrak{g} \) as a central extension of its adjoint Lie algebra \( \mathfrak{g}_{\text{ad}} \) by its center \( Z(\mathfrak{g}) \) and uses then the fact that \( \mathfrak{g}_{\text{ad}} \) is embedded by the adjoint action as a subalgebra of \( \mathfrak{gl}(\mathfrak{g}) \), thus integrating by Lie’s First Theorem into a Lie subgroup of \( \text{Gl}(\mathfrak{g}) \). It remains then to integrate the 2-cocycle determining the central extension, therefore we call it homological proof. Another approach, which we call the approach using Ado’s Theorem, uses the fact that every real
Lie algebra embeds as a subalgebra of some matrix Lie algebra (Ado’s Theorem) to integrate once again only subalgebras of general linear Lie algebras into Lie subgroups of general linear groups.

In the search of understanding the periodicity in K-theory, J.-L. Loday introduced Leibniz algebras as non-commutative analogues of Lie algebras. More precisely, a real Leibniz algebra is a real vector space with a bracket which satisfies the (left) Leibniz identity

\[ [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]], \]

but is not necessarily skew-symmetric. Leibniz algebras are a well-established algebraic structure generalizing Lie algebras (those Leibniz algebras where the bracket is skew-symmetric) with their own structure-, deformation- and homology theory. In the same way the Lie algebra homology of matrices (over a commutative ring containing the rational numbers) defines additive K-theory (i.e. cyclic homology), the Leibniz homology of matrices defines some non-commutative additive K-theory (in fact, Hochschild homology). Loday was mainly interested in the properties of the corresponding homology theory on “group level” (“Leibniz K-Theory”), and therefore asked the question which (generalization of the structure of Lie groups) is the correct structure to integrate Leibniz algebras?

This paper consists of two parts. The main goal of the first part (comprising Sections 1 to 3) is to compare the integration procedures for Leibniz algebras which one may generalize from the homological proof and from the proof using Ado’s Theorem of Lie’s Third Theorem. Kinyon [17] explored Lie racks as a structure integrating Leibniz algebras. Racks are roughly speaking an axiomatization of the structure of the conjugation in a group. The rack product on a group is simply given by

\[ g \triangleright h := ghg^{-1}, \]

and a general rack product on a set \( X \) is an invertible binary operation satisfying for all \( x, y, z \in X \) the autodistributivity relation

\[ x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z). \]

Lie racks are the smooth analogue of racks. Kinyon showed (see Theorem 1.28) that the tangent space at the distinguished element 1 of a Lie rack carries in a natural way a Leibniz bracket. The idea is to differentiate two times the rack structure, mimicking exactly how the conjugation in a Lie group is differentiated to give first the map Ad, the adjoint action of the group on the Lie algebra, and then the Lie bracket in terms of ad, the adjoint action of the Lie algebra on itself. He did not see racks as the correct objects integrating Leibniz algebras. As a reason for this, he showed that all Leibniz algebras integrate into Lie racks, but in a kind of arbitrary way, as this integration does not appear to give Lie groups in case one started with a Lie algebra. He more or less exhibited Ado’s approach to the integration of Leibniz algebras into (what he called) linear Lie racks. We explain this in Section 3. It is clear (and useful as a guiding principle) that
from this point of view, integrating Leibniz algebras means just an integration of the adjoint action of a Leibniz algebra on itself. From here stems the most important example of a rack product, namely

$$X \triangleright Y := e^{ad_X}(Y),$$

for all $X, Y \in \mathfrak{h}$ for a Leibniz algebra $\mathfrak{h}$.

On the other hand, Covez [12] showed in his 2010 doctoral thesis how to adapt the homological proof of Lie’s Third Theorem to Leibniz algebras. Regarding a given real Leibniz algebra $\mathfrak{h}$ as an abelian extension of the Lie algebra $\mathfrak{h}_{Lie}$ by its left center $Z_L(\mathfrak{h})$, he integrated Leibniz algebras into local Lie racks. The fact that this procedure works only locally stems from the fact that the Leibniz 2-cocycle governing the abelian extension is only integrated into a local rack cocycle, due to the use of open sets on the Lie group $G_0$ integrating $\mathfrak{h}_{Lie}$ where exponential and logarithm are mutually inverse diffeomorphisms. This integration has the advantage of specializing to the conjugation racks associated to Lie groups in case the given Leibniz algebra is a Lie algebra. All this is explained in some detail in Section 2.

This first part of this paper clarifies the integration of Leibniz algebras and shows to our belief that Lie racks are indeed the correct structure for this integration. Along the way, we show that in all of our integration procedures the integrating object reduces locally to (the conjugation rack of) a Lie group in case we are dealing with a Lie algebra. The comparison of these approaches is summarized in Section 3.6.

Other evidence that racks are the right objects integrating Leibniz algebras comes from recent work of Covez on product structures on rack homology showing that it has (some of) the expected properties of a Leibniz K-theory which were predicted by Loday.

In the second part of our paper (Section 4), we use the integration procedure of Leibniz algebras set up in Section 3 in order to develop deformation quantization of Leibniz algebras.

Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [ , ])$, its dual vector space $\mathfrak{g}^*$ is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all $f, g \in C^\infty(\mathfrak{g}^*)$ and all $\xi \in \mathfrak{g}^*$ by the Kostant-Kirillov-Souriau formula

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.$$ 

Here $df(\xi)$ and $dg(\xi)$ are linear functionals on $\mathfrak{g}^*$, identified with elements of $\mathfrak{g}$.

In the same way, a general Leibniz algebra $\mathfrak{h}$ gives rise to a smooth manifold $\mathfrak{h}^*$, which carries now some kind of generalized Poisson bracket, in particular, the bracket need not be skew-symmetric. We call manifolds with such a bracket generalized Poisson manifolds.

It is well known that the deformation quantization of the Poisson manifold $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ is intimately related to the integration of the bracket of $\mathfrak{g}$ into a local/formal group product via the Baker-Campbell-Hausdorff (BCH) formula. The main idea of the present paper is to use the corresponding BCH-formula
for the integration of a Leibniz algebra $\mathfrak{h}$ in order to perform the corresponding deformation quantization.

The quantization technique we use relies on the quantization of special canonical relation germs, called symplectic micromorphisms (see [8], [9], [10], and [11]), by Fourier integral operators. In the Lie algebra case, we show that it is possible to re-interpret the Gutt star-product in terms of a symplectic micromorphism quantization, obtained by considering the cotangent lift of the local group structure on the Lie algebra. We show that this quantization method also works for Leibniz algebras, provided one takes the cotangent lift of the local rack structure for the symplectic micromorphism. This local rack structure comes from the integration procedure exposed in the first part.

The quantization of the dual of a Leibniz algebra $\mathfrak{h}^*$ that results from quantizing the symplectic micromorphism obtained from the local rack structure, is an operation

$$\triangleright \in C^\infty(\mathfrak{h}^*)[[\epsilon]] \times C^\infty(\mathfrak{h}^*)[[\epsilon]] 
\rightarrow C^\infty(\mathfrak{h}^*)[[\epsilon]]$$

such that the restriction of $\triangleright$ to “unitaries” $U_\hbar := \{E_X \mid X \in \mathfrak{h}\}$ (where $E_X$ is the exponential function on $\mathfrak{h}^*$ associated to $X \in \mathfrak{h}$) is a rack structure $\triangleright : U_\hbar \times U_\hbar \rightarrow U_\hbar$.

Moreover, the restriction of this operation to

$$\triangleright : U_\hbar \times C^\infty(\mathfrak{h}^*)[[\epsilon]] 
\rightarrow C^\infty(\mathfrak{h}^*)[[\epsilon]]$$

should be a rack action.

Our main theorem shows exactly this:

**Theorem 4.12** The operation

$$\triangleright_\hbar : C^\infty(\mathfrak{h}^*)[[\epsilon]] \otimes C^\infty(\mathfrak{h}^*)[[\epsilon]] 
\rightarrow C^\infty(\mathfrak{h}^*)[[\epsilon]]$$

defined by

$$f \triangleright_\hbar g := Q^{a=1}(T^* \triangleright)(f \otimes g)$$

is a quantum rack, i.e.

1. $\triangleright_\hbar$ restricted to $U_\hbar = \{E_X \mid X \in \mathfrak{h}\}$ is a rack structure, moreover

$$e^{\mathbf{X}} \triangleright_\hbar e^{\mathbf{Y}} = e^{a^X(Y)};$$

2. $\triangleright_\hbar$ restricted to

$$\triangleright_\hbar : U_\hbar \times C^\infty(\mathfrak{h}^*) 
\rightarrow C^\infty(\mathfrak{h}^*)$$

is a rack action;

$$(e^{\mathbf{X}} \triangleright_\hbar f)(\xi) = (\text{Ad}^*_X f)(\xi).$$

Moreover, $\triangleright_\hbar$ coincides with the Gutt quantum rack $f \triangleright_\mathbf{a} g := f *_\mathbf{a} g *_\mathbf{a} T$ on unitaires in the Lie case (although it is different on the whole $C^\infty(\mathfrak{h}^*)[[\epsilon]]$).
The quantization operator $Q^a$ occurring here is constructed as the Fourier Integral Operator associated to an amplitude $a$ and a generating function $S_{\triangleright}$ according to

$$Q^a(f \otimes g)(\xi) = \int_{h \times h} \hat{f}(X)\hat{g}(Y)a(X, Y, \xi)e^{iS_{\triangleright}(X, Y, \xi)}\frac{dXdY}{(2\pi\hbar)^n},$$

where $f, g \in C^\infty(h^*)[[\epsilon]]$, $X, Y \in h$, $\xi \in h^*$ and $n = \dim(h)$. The generating function $S_{\triangleright}$ relies on a Baker-Campbell-Hausdorff type formula for the Leibniz case, namely

$$S_{\triangleright}(X, Y, \xi) := \langle \xi, e^{ad_X}(Y) \rangle.$$

The problem of deformation quantizing a Leibniz algebra has been addressed by other authors, namely by K. Uchino in [24] in the realm of associative dialgebras.

Observe that thanks to this theorem, the general integration problem for generalized Poisson manifolds makes sense. Namely, given a generalized Poisson manifold, i.e. a manifold $M$ together with a bracket on $C^\infty(M)$ satisfying similar properties as the bracket on $C^\infty(h^*)$. Then we may ask whether there exists a natural rack structure on the set of exponential functions which extends to a rack action on all smooth functions. Our main theorem solves this integration problem for linear generalized Poisson structures.

A side result is a new way of integrating 2-cocycles (see Lemma 3.9), which we find by comparing Covez’ integration and the BCH integration procedure.

**Acknowledgements**: FW is grateful to UC Berkeley for hospitality and excellent working conditions during our work on this article. He thanks especially Alan Weinstein for the invitation, guidance and advice, and most useful discussions about the integration of Leibniz algebras. FW acknowledges support from CNRS during this period. FW thanks Yannick Voglaire for correcting the coadjoint action, and K. Uchino for correcting the notion of a generalized Poisson manifold and bringing [15] to our attention.

1 Preliminaries on Leibniz and Lie algebras

1.1 Derivations of Leibniz algebras

Fix a field $k$. Later we will specialize to $k = \mathbb{R}$ in order to speak about the exponential map (although this is not mandatory). We present here a recollection of facts from Lie algebra theory which we generalize to Leibniz algebras by showing that the usual proof still holds true in the Leibniz context.

**Definition 1.1.** A (left) Leibniz algebra is a $k$-vector space $h$ together with a $k$-bilinear bracket $[,] : h \times h \rightarrow h$ such that for all $X, Y, Z \in h$

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$
Definition 1.2. A (left) derivation of a Leibniz algebra \( h \) is a \( k \)-linear map \( D : h \to h \) such that for all \( X,Y \in h \)

\[
D([X,Y]) = [D(X),Y] + [X,D(Y)].
\]

Observe that the above left Leibniz identity means that for all \( X \in h \), \( \text{ad}_X := [X,\cdot] \) is a (left) derivation of the bracket. Obviously, in case the bracket is also skew-symmetric, \( h \) becomes a Lie algebra and the left Leibniz identity becomes the usual Jacobi identity. As this need not be the case, the notion of Leibniz algebra generalizes the notion of Lie algebra. Observe furthermore that skew-symmetrizing the bracket of a Leibniz algebra does not necessarily give a Lie algebra, as the Jacobi identity is not necessarily satisfied.

Lemma 1.3. For any Leibniz algebra \( h \), the space of derivations \( \text{der}(h) \) together with the bracket of derivations

\[
[D,D'] := D \circ D' - D' \circ D,
\]

forms a Lie algebra.

Proof. The bracket satisfies the Jacobi identity because of the associativity of the composition of endomorphisms of \( h \). The bracket of derivations is once again a derivation by the following computation for all \( X,Y \in h \):

\[
[D,D'](\{X,Y\}) = (D \circ D')(\{X,Y\}) - (D' \circ D)(\{X,Y\})
\]

\[
= [(D \circ D') (X),Y] + [D(X),D'(Y)] + [D'(X),D(Y)] + [X,(D \circ D')(Y)] - [(D' \circ D)(X),Y] - [D(X),D'(Y)]
\]

\[
- [D'(X),D(Y)] - [X,(D' \circ D)(Y)]
\]

\[
= [(D \circ D' - D' \circ D)(X),Y] + [X,(D \circ D' - D' \circ D)(Y)]
\]

\[
= [(D,D')(X),Y] + [X,[D,D'](Y)] \quad \square
\]

By the Leibniz identity, for all \( X \in h \), the endomorphism \( \text{ad}_X \) is a derivation, called the inner derivation associated to \( X \).

Lemma 1.4. The subspace \( \text{inn}(h) \) of inner derivations of a Leibniz algebra \( h \) forms an ideal in the Lie algebra \( \text{der}(h) \) of all derivations.

Proof. We have for all \( X,Y \in h \):

\[
[D,\text{ad}_X](Y) = D([X,Y]) - [X,D(Y)]
\]

\[
= [D(X),Y] + [X,D(Y)] - [X,D(Y)]
\]

\[
= [D(X),Y] = \text{ad}_{D(X)}(Y) \quad \square
\]

Observe that the subspace \( \text{inn}(h) \) is also the image of the map \( \text{ad} : h \to \text{der}(h) \).

Definition 1.5. For any Leibniz algebra \( h \), the quotient Lie algebra of \( \text{der}(h) \) by the ideal of inner derivations \( \text{inn}(h) \) is called the Lie algebra \( \text{out}(h) \) of outer derivations of \( h \).
Definition 1.6. The left center of a Leibniz algebra $\mathfrak{h}$ is the subspace

$$Z_L(\mathfrak{h}) := \{ X \in \mathfrak{h} \mid [X, Y] = 0 \ \forall \ Y \in \mathfrak{h} \}.$$ 

Lemma 1.7. The left center $Z_L(\mathfrak{h})$ of a Leibniz algebra $\mathfrak{h}$ is an abelian (left) ideal.

Proof. By the Leibniz identity, we have for all $X, Z \in \mathfrak{h}$ and $Y \in Z_L(\mathfrak{h})$

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]] = 0.$$ 

We summarize the preceding discussion in the following

Proposition 1.8. For any Leibniz algebra $\mathfrak{h}$, there is a 4-term exact sequence of Leibniz algebras:

$$0 \to Z_L(\mathfrak{h}) \to \mathfrak{h} \to \text{ad}(\mathfrak{h}) \to \text{der}(\mathfrak{h}) \to \text{out}(\mathfrak{h}) \to 0.$$ 

The only Leibniz algebra in this sequence which is not necessarily a Lie algebra is $\mathfrak{h}$.

We can shorten the 4-term sequence to the following short exact sequence:

$$0 \to Z_L(\mathfrak{h}) \to \mathfrak{h} \to \text{ad}(\mathfrak{h}) \to 0. \quad \text{(1)}$$

In this way, we can associate to each Leibniz algebra $\mathfrak{h}$ an abelian extension such that the quotient algebra (here $\text{ad}(\mathfrak{h})$) is a Lie algebra. There are of course other choices which satisfy this requirement, but we will always use this one.

Now let us come to automorphisms and the exponential map.

Definition 1.9. A linear map $\alpha : \mathfrak{h} \to \mathfrak{h}$ on a Leibniz algebra $\mathfrak{h}$ is called an endomorphism in case for all $X, Y \in \mathfrak{h}$:

$$\alpha([X, Y]) = [\alpha(X), \alpha(Y)].$$

Such a map $\alpha$ is called an automorphism if in addition it is bijective.

We will specialize from now on to $k = \mathbb{R}$ (although this is not completely necessary).

Lemma 1.10. Let $\mathfrak{h}$ be a Leibniz algebra. Suppose that for a derivation $D \in \text{der}(\mathfrak{h})$ the formula

$$\exp(D) := \sum_{k=0}^{\infty} \frac{1}{k!} D^k$$

defines an endomorphism of $\mathfrak{h}$. Then $\exp(D)$ is an automorphism of $\mathfrak{h}$.

Proof. The formula $D([X, Y]) = [D(X), Y] + [X, D(Y)]$ for all $X, Y \in \mathfrak{h}$ leads by induction to

$$\frac{D^n}{n!}([X, Y]) = \sum_{j=0}^{\infty} \left[ \frac{D^j(X)}{j!} \frac{D^{n-j}(Y)}{(n-j)!} \right].$$
From here, we obtain

\[
[(\exp D)(X), (\exp D)(Y)] = \left[ \sum_{p=0}^{\infty} \frac{D^p(X)}{p!}, \sum_{q=0}^{\infty} \frac{D^q(Y)}{q!} \right]
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{D^j(X)\, D^{n-j}(Y)}{j! \, (n-j)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{D^n}{n!}([X,Y])
\]

\[
= (\exp D)([X,Y]). \quad \square
\]

The condition of the preceding lemma is fulfilled for example in case the derivation is locally nilpotent, or in case the Leibniz algebra \( h \) is finite-dimensional, as in this case the exponential series of an endomorphism converges (in fact, the exponential series is a polynomial in this case). Usually, we will consider finite-dimensional Leibniz algebras and thus be in the second case.

**Lemma 1.11.** Let \( h \) be a finite-dimensional Leibniz algebra, \( \alpha \in \mathrm{Aut}(h) \) be an automorphism, and \( X \in h \). Then the following formula holds:

\[
\alpha \circ \exp(\mathrm{ad}_X) \circ \alpha^{-1} = \exp(\mathrm{ad}_{\alpha(X)}).
\]

**Proof.** Let us first prove the infinitesimal formula:

\[
\alpha \circ \mathrm{ad}_X \circ \alpha^{-1} = \mathrm{ad}_{\alpha(X)}.
\]

This follows directly from the fact that \( \alpha \) is an automorphism by applying the formula to an element \( Y \) of \( h \):

\[
\alpha \circ \mathrm{ad}_X \circ \alpha^{-1}(Y) = \alpha([X, \alpha^{-1}(Y)]) = [\alpha(X), Y] = \mathrm{ad}_{\alpha(X)}(Y).
\]

Now notice that composition with \( \alpha \) is continuous, i.e. \( \alpha \circ (\lim_{N \to \infty} \phi_N) = \lim_{N \to \infty} (\alpha \circ \phi_N) \), by finite-dimensionality of \( h \). We have

\[
\alpha \circ \left( \sum_{k=0}^{N} \frac{(\mathrm{ad}_X)^k}{k!} \right) \circ \alpha^{-1} = \sum_{k=0}^{N} \frac{\alpha \circ (\mathrm{ad}_X)^k \circ \alpha^{-1}}{k!}
\]

\[
= \sum_{k=0}^{N} \frac{(\alpha \circ \mathrm{ad}_X \circ \alpha^{-1})^k}{k!}
\]

\[
= \sum_{k=0}^{N} \frac{\mathrm{ad}_{\alpha(X)}}{k!}
\]

The identity follows now from passage to the limit \( \lim_{N \to \infty} \) using the continuity of the composition with \( \alpha \). \( \square \)

Recall the usual naturality properties of the exponential map of a Lie algebra \( g \), see for example [25] p. 104, formula (2.13.7) and Theorem 2.13.2:
Proposition 1.12. The exponential map \( \exp : \mathfrak{g} \to G \) of a (finite-dimensional) Lie algebra \( \mathfrak{g} \) into a Lie group \( G \) with Lie algebra \( \mathfrak{g} \) has the following naturality properties for all \( X, Y \in \mathfrak{g} \) and all \( g \in G \):

(a) \( \text{conj}_g(\exp(Y)) = \exp(\text{Ad}_g(Y)) \),

(b) \( \exp(\text{ad}_X(Y)) = \text{Ad}_{\exp(X)}(Y) \).

(Observe that we use the usual imprecision concerning the exponential function \( \exp : \mathfrak{g} \to G \) and the exponential function \( \exp : \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{Gl}(\mathfrak{g}) \).)

1.2 Leibniz algebras as subalgebras of a hemi-semi-direct product

Lemma 1.13. Let \( \mathfrak{g} \) be a Lie algebra and \( V \) be a \( \mathfrak{g} \)-module. The direct sum \( V \oplus \mathfrak{g} \) together with the bracket

\[
[(v, X), (v', X')] = (X(v'), [X, X'])
\]

becomes a Leibniz algebra, called the hemi-semi-direct product \( V \times_{hs} \mathfrak{g} \) of \( V \) and \( \mathfrak{g} \).

It is readily verified that this bracket gives a Leibniz bracket which is Lie only if the \( \mathfrak{g} \)-module is trivial. This structure came up in the search of the integration of Courant algebroids. Indeed, when the problem is formulated algebraically, there is only “one half” of the semi-direct product Lie bracket

\[
[(v, X), (v', X')] = (X(v') - X'(v), [X, X'])
\]

playing a role. Thus the term hemi-semi-direct product.

Kinyon and Weinstein showed in [18] that every Leibniz algebra may be embedded into a hemi-semi-direct product Leibniz algebra.

Example: Let a Leibniz algebra \( \mathfrak{h} \) be given. Our most important example of a hemi-semi-direct product is to choose \( \mathfrak{gl}(\mathfrak{h}) \) as the Lie algebra and \( \mathfrak{h} \) as the module in the above construction. Kinyon and Weinstein noticed that every Leibniz algebra may be embedded in this type of hemi-semi-direct product \( \mathfrak{h} \times_{hs} \mathfrak{gl}(\mathfrak{h}) \). The embedding map is simply \( X \mapsto (X, \text{ad}_X) \). In other words, the given Leibniz algebra \( \mathfrak{h} \) is seen as a subalgebra of the hemi-semi-direct product \( \mathfrak{h} \times_{hs} \mathfrak{gl}(\mathfrak{h}) \) by regarding it as the graph of the adjoint representation

\[
\text{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}), \quad X \mapsto \text{ad}_X,
\]

where for each \( Y \in \mathfrak{h} \), \( \text{ad}_X(Y) := [X, Y] \).

One can change this example somehow by considering the Lie algebra of derivations \( \text{der}(\mathfrak{h}) \) instead of the Lie algebra \( \mathfrak{gl}(\mathfrak{h}) \). Notice that the derivations \( \text{der}(\mathfrak{h}) \) of a Leibniz algebra \( \mathfrak{h} \) form indeed a Lie algebra by Lemma 1.3. We summarize this discussion in the following proposition, due to Kinyon-Weinstein loc. cit.:
**Proposition 1.14.** Every Leibniz algebra $\mathfrak{h}$ is embedded as a subalgebra of the hemi-semi-direct product $\mathfrak{h} \ltimes_{\text{der}} \text{der}(\mathfrak{h})$.

### 1.3 On the BCH-formula

The Baker-Campbell-Hausdorff formula (BCH-formula)

\[ X \ast Y := \log (\exp(X) \exp(Y)) =: \]

\[ =: \text{BCH}(X, Y) := X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ([X, [X, Y]] + [Y, [Y, X]]) + \ldots , \]

for two elements $X, Y$ in a Lie algebra $\mathfrak{g}$ defines a local Lie group structure in a neighbourhood of 0 in $\mathfrak{g}$. It is in general not a global group structure, because the bracket expression need not converge.

**Definition 1.15.** Let us call BCH-neighborhood a $0$-neighborhood $U$ in a Lie algebra $\mathfrak{g}$ with the following two properties:

1. The BCH series converges on $U \times U$ and defines thus a (local) group product $\ast : U \times U \to \mathfrak{g}$ by

   \[ X \ast Y = \log (\exp(X) \exp(Y)) . \]

2. There exists a $0$-neighborhood $V \subset U$ where $\exp : V \to \exp(V)$ is a diffeomorphism (onto the open set $\exp(V)$).

It is easily shown (using the derivative of the exponential map) that such a BCH-neighborhood exists for every (finite dimensional) Lie algebra $\mathfrak{g}$.

Now we pass to the conjugation with respect to the BCH-product. Note that due to associativity of the BCH-formula, we have

\[ \text{conj}_\ast (X, Y) := \log (\exp(X) \exp(Y) \exp(-X)) = \]

\[ = \text{BCH} (\text{BCH}(X, Y), -X) = \]

\[ = \text{BCH} (X, \text{BCH}(Y, -X)) . \]

It is not so well-known that the conjugation associated to the BCH-multiplication has a much simpler formula which converges always:

**Lemma 1.16.** The explicit formula BCH-conjugation $\text{conj}_\ast$ for a Lie algebra $\mathfrak{g}$ is:

\[ \text{conj}_\ast (X, Y) = \exp (\text{ad}_X)(Y) \]

\[ = Y + [X, Y] + \frac{1}{2} [X, [X, Y]] + \frac{1}{6} [X, [X, [X, Y]]] + \ldots \]

**Proof.** By the naturality properties in Proposition 1.12 we compute for all $X, Y \in \mathfrak{g}$:

\[ \exp(X) \exp(Y) \exp(-X) = \text{conj}_{\exp(X)} (\exp(Y)) \]

\[ = \exp (\text{Ad}_{\exp(X)})(Y) \]

\[ = \exp (\text{ad}_X)(Y) \]
The formula follows now from taking the formal logarithm.

This conjugation operation is thus a perfectly global operation, but which is only locally the conjugation with respect to a group product.

Note that this operation makes also sense for elements \(X,Y\) in any finite dimensional Leibniz algebra \(\mathfrak{h}\).

The exponential \(\exp(\text{ad}_X)\) is the (inner) automorphism (see Lemma [1.10]) with respect to the (inner) derivation \(\text{ad}_X\) which is associated to each element \(X \in \mathfrak{h}\).

### 1.4 Lie racks

Recall the notion of a rack: It comes from axiomatizing the notion of conjugation in a group and plays its role in the present context as the structure integrating Leibniz algebras.

**Definition 1.17.** Let \(X\) be a set together with a binary operation denoted \((x,y) \mapsto x \triangleright y\) such that for all \(x \in X\), the map \(y \mapsto x \triangleright y\) is bijective and for all \(x,y,z \in X\),

\[ x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z). \]

Then we call \(X\) (or more precisely \((X,\triangleright)\)) a (left) rack. In case the map \(y \mapsto x \triangleright y\) is not necessarily bijective for all \(x \in X\), \(X\) is called a (left) shelf.

As already mentioned, an example of a rack is the conjugation in a group \(G\). The rack operation is in this case given by \((g,h) \mapsto ghg^{-1}\). Finite racks have served to define knot, link and tangle invariants, see for example [14]. There is also the notion of a right rack. This is by definition a set \(X\) together with a binary operation \((x,y) \mapsto x \triangleleft y\) such that all maps \(x \mapsto x \triangleleft y\) are bijective and

\[ (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z). \]

There are at least two ways to transform a left rack into a right rack and vice-versa. The first is to take the opposite rack \(x \triangleleft y := y \triangleright x\), the second is to take the inverse rack \(x \triangleleft y := (y \triangleright -)^{-1}(x)\).

**Definition 1.18.** Let \(R\) be a rack and \(X\) be a set. We say that \(R\) acts on \(X\) (on the right) (or that \(X\) is a right \(R\)-set) in case for all \(r \in R\), there are bijections \((r) : X \to X\) such that for all \(x \in X\) and all \(r,r' \in R\):

\[ (x \cdot r) \cdot r' = (x \cdot r') \cdot (r \triangleleft r'). \]

There is also the notion of a left action where the corresponding identity reads

\[ r \cdot (r' \cdot x) = (r \triangleright r') \cdot (r \cdot x). \]

Clearly, the adjoint action \(\text{Ad}_r : R \to R\) defined by \(\text{Ad}_r(r') := r \triangleright r'\) in a left rack \(R\) is a left action of \(R\) on itself. In the same way the adjoint action of a right rack on itself is a right action.
Lemma 1.19. Let $R$ be a left rack such that the underlying set is a finite dimensional vector space with linear dual $R^*$. Then there exists a coadjoint action $\text{Ad}^*: R \times R^* \to R^*$ defined for all $r, r' \in R$ and all $f \in R^*$ by

$$(\text{Ad}^*_r(f))(r') := f((r \triangleright -)^{-1}(r')).$$

The coadjoint action is a left action.

Proof. For the proof, write simply $r \cdot f$ for $\text{Ad}^*_r(f)$. Then

$$(r \cdot (r' \cdot f))(r \triangleright (r' \triangleright r'')) = (r' \cdot f)((r \triangleright -)^{-1}(r \triangleright (r' \triangleright r'')))$$

$$= (r' \cdot f)(r' \triangleright r'')$$

$$= f((r' \triangleright -)^{-1}(r' \triangleright r''))$$

$$= f(r'').$$

We also have

$$((r \triangleright r') \cdot (r \cdot f))(r \triangleright (r' \triangleright r'')) = ((r \triangleright r') \cdot (r \cdot f))((r \triangleright r') \triangleright (r \triangleright r''))$$

$$= (r \cdot f)((r \triangleright r') \triangleright -)^{-1}((r \triangleright r') \triangleright (r \triangleright r''))$$

$$= (r \cdot f)(r \triangleright r'')$$

$$= f((r \triangleright -)^{-1}(r \triangleright r''))$$

$$= f(r'').$$

This shows that

$$r \cdot (r' \cdot f) = (r \triangleright r') \cdot (r \cdot f),$$

thus the coadjoint action is a left action. \qed

Remark 1.20. Curiously, this does not seem to work with the opposite rack structure replacing the inverse rack structure.

In the following, we will need pointed local Lie racks.

Definition 1.21. A pointed rack $(X, \triangleright, 1)$ is a set $X$ with a binary operation $\triangleright$ and an element $1 \in X$ such that the following axioms are satisfied:

1. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$,

2. For each $a, b \in X$, there exists a unique $x \in X$ such that $a \triangleright x = b$,

3. $1 \triangleright x = x$ and $x \triangleright 1 = 1$ for all $x \in X$.

Once again, the conjugation rack of a group is an example of a pointed rack.

Definition 1.22. 1. A Lie rack $X$ is a manifold and a pointed smooth rack, i.e. the structure maps are smooth.
2. A local Lie rack is a manifold $X$ with an open subset $\Omega \subset X \times X$ where a Lie rack product $\triangleright$ is defined such that

(a) If $(x, y), (x, z), (y, z), (x \triangleright y, x \triangleright z) \in \Omega$, then $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.

(b) If $(x, y), (x, z) \in \Omega$ and $x \triangleright y = x \triangleright z$, then $y = z$.

(c) For all $x \in X$, $(1, x), (x, 1) \in \Omega$ and as usual $1 \triangleright x = x$ and $x \triangleright 1 = 1$.

Examples of Lie racks include obviously the conjugation racks associated to Lie groups. Another example which will play an important role in the sequel is the following:

**Example:** Let $G$ be a Lie group and $V$ be a $G$-module. On $X := V \times G$, we define a binary operation $\triangleright$ by

$$(v, g) \triangleright (v', g') = (g(v'), gg'g^{-1})$$

for all $v, v' \in V$ and all $g, g' \in G$. $X$ is a Lie rack with unit $1 := (0, 1)$ which is called a linear Lie rack. This is the "group-analog" of the hemi-semi-direct product of a Lie algebra with its representation, and we denote it by $V \times_{hs} G$.

Let us define more generally this hemi-semi-direct product of racks:

**Definition 1.23.** Let $R$ be a rack and $A$ be a rack module in the sense of Definition 1.18. The hemi-semi-direct product $A \times_{hs} R$ of $R$ with $A$ is the following rack structure on the direct product set $A \times R$:

$$(a, r) \triangleright (a', r') := (r(a'), r \triangleright r').$$

One verifies easily that this gives indeed a rack structure.

Now let us come to digroups:

**Definition 1.24.** A digroup $(H, \triangleright, \triangleright)$ is a set $H$ together with two binary operations $\triangleright$ and $\triangleright$ satisfying the following axioms. For all $x, y, z \in H$,

1. $(H, \triangleright)$ and $(H, \triangleright)$ are semigroups,
2. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z$,
3. $x \triangleright (y \triangleright z) = x \triangleright (y \triangleright z)$,
4. $(x \triangleright y) \triangleright z = (x \triangleright y) \triangleright z$,
5. there exists $1 \in H$ such that $1 \triangleright x = x \triangleright 1 = x$ for all $x \in H$,
6. for all $x \in H$, there exists $x^{-1} \in H$ such that $x \triangleright x^{-1} = x^{-1} \triangleright x = 1$.

An element $e \in H$ in a digroup $H$ is called a bar unit in case $e \triangleright x = x \triangleright e = x$ for all $x \in H$. Bar units exist in a digroup, but are not necessarily unique. A digroup is a group if and only if $\triangleright = \triangleright$ and 1 is the unique bar unit.

There is a digroup which resembles very much the linear Lie rack:
Remark 1.25. Let $G$ be a Lie group and $M$ be a $G$-module. Define on $H := M \times G$ the structure of a digroup by

$$(u, g) \triangleright (v, h) := (g(v), gh)$$

and

$$(u, g) \triangleleft (v, h) := (u, gh)$$

for all $u, v \in M$ and all $g, h \in G$. Then $M \times G$ is a Lie digroup with distinguished bar unit $(e, 1)$. This Lie digroup is called the linear Lie digroup associated to $G$ and $M$.

Digroups give rise to racks in the following way:

Proposition 1.26. Let $(H, \triangleright, \triangleleft)$ be a digroup and put $x \triangleright y := x \triangleright y \triangleleft x^{-1}$ for all $x, y \in H$. Then $(H, \triangleright)$ is a rack, pointed in $1$. Moreover, in case $(H, \triangleright, \triangleleft)$ is a Lie digroup (i.e. all structures are smooth), $(H, \triangleright)$ is a Lie rack.

In the case of the example in Remark 1.25, the obtained Lie rack is the above described linear Lie rack $M \times_{\text{hs}} G$. In this sense every linear Lie rack “comes from” a linear Lie digroup.

Remark 1.27. There are several ways to construct a group out of a rack $X$. The associated group $\text{As}(X)$ is the quotient of the free group on $X$ by the normal subgroup generated by the set \{$(xy^{-1}x^{-1})(x \triangleright y) : x, y \in X$\}. For pointed racks, one modifies this definition such that $1$ becomes the unit of $\text{As}(X)$.

1.5 From split Leibniz algebras to Lie racks

In this subsection, we summarize Kinyon’s approach [17] to the integration of (split) Leibniz algebras by Lie racks.

Kinyon shows in [17] the following theorem which is at the heart of all our attempts to integrate Leibniz algebras.

Theorem 1.28. Let $(X, \triangleright, 1)$ be a Lie rack, and let $\mathfrak{h} := T_1X$. Then there exists a bilinear map $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ such that

1. $(\mathfrak{h}, [ , ] )$ is a (left) Leibniz algebra,
2. for each $x \in X$, the tangent map $\Phi(x) := T_1\phi(x)$ of the left translation map $\phi(x) : X \to X$, $y \mapsto x \triangleright y$, is an automorphism of $(\mathfrak{h}, [ , ] )$,
3. if $\text{ad} : \mathfrak{h} \to \text{gl}(\mathfrak{h})$ is defined by $Y \mapsto \text{ad}_X(Y) := [X, Y]$, then $\text{ad} = T_1\Phi$.

Let us recall its proof for the sake of self-containedness:

Proof. We have for all $x \in X$, $\phi(x)(1) = x \triangleright 1 = 1$, thus $\Phi(x) := T_1\phi$ is an endomorphism of $\mathfrak{h} := T_1X$. As each $\phi(x)$ is invertible, we have $\Phi(x) \in \text{Gl}(\mathfrak{h})$. 


Now the map $\Phi : X \to \text{Gl}(h)$ satisfies $\Phi(1) = \text{id}$, thus we may differentiate again in order to obtain $\text{ad} : T_1X \to \mathfrak{gl}(h)$. Now we set

$$[X, Y] := \text{ad}_X(Y)$$

for all $X, Y \in h = T_1X$. In terms of the left translations $\phi(x)$, the rack identity can be expressed by the equation

$$\phi(x)(\phi(y)(z)) = \phi(\phi(x)(y))(\phi(x)(z)).$$

We differentiate this equation at $1 \in X$ first with respect to $z$, then with respect to $y$ to obtain

$$\Phi(x)([Y, Z]) = [\Phi(x)(Y), \Phi(x)(Z)]$$

for all $x \in X$ and all $Y, Z \in h$. This expresses the fact that for each $x \in X$, $\Phi(x) \in \text{Aut}(T_1X, [\cdot, \cdot])$. Finally, we differentiate this last equation at $1$ with respect to $x$ to obtain

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

for all $X, Y, Z \in h$, This shows that $h$ is a left Leibniz algebra. \qed

**Example:** In the special case of a linear Lie rack, we obtain the hemi-semi-direct product Leibniz algebra $h = V \times_{\text{hs}} g$, where $g$ is the Lie algebra of the Lie group $G$, endowed with the bracket:

$$[(v, X), (v', X')] = (X(v'), [X, X']).$$

The $G$-module $V$ is here seen as a $g$-module in the usual way.

Kinyon’s main result in [17] is the integration of split Leibniz algebras (i.e. those isomorphic to a hemi-semi-direct product Leibniz algebra) into linear Lie racks and thus into Lie digroups.

**Theorem 1.29** (Kinyon). Let $h$ be a split Leibniz algebra. Then there exists a linear Lie digroup with tangent Leibniz algebra isomorphic to $h$.

**Remark 1.30.** In fact, Simon Covez showed in his (unpublished) Master thesis that conversely, in case a Leibniz algebra integrates into a Lie digroup, it must be split over some ideal containing the ideal of squares (more precisely, it is split over the ideal ker($T_1i$) where $i$ is the inversion map of the digroup).

### 1.6 From Lie algebras to Lie racks

Here we summarize some of the previous results in order to perform the integration of Lie algebras into Lie racks. Later on, we will generalize these integrations to Leibniz algebras.

Let $g$ be a Lie algebra, and let $G$ denote a Lie group integrating $g$. Then there are two global Lie racks:
1. $\triangleright : G \times G \to G$ given by $g \triangleright h = ghg^{-1}$,
2. $\triangleright : g \times g \to g$ given by $X \triangleright Y = \exp(\text{ad}_X)(Y)$.

Let us denote these two Lie racks by $R_G$ and $R_g$ respectively.

**Proposition 1.31.** There is a rack morphism $\phi : R_g \to R_G$, induced by the geometric exponential $\phi(X) := \exp(X)$, such that $\phi$ is an isomorphism in some 0-neighborhood. The racks $R_G$ and $R_g$ thus define the same local Lie rack integrating $\mathfrak{g}$.

**Proof.** The fact that $\phi$ is a rack morphism follows from Proposition 1.12. The fact that $\phi$ is an isomorphism in some 0-neighborhood follows from the existence of BCH-neighborhoods, see Definition 1.15. \hfill $\square$

Two remarks are in order:

**Remark 1.32.** The rack $R_g$ has been introduced by H. Bass (unpublished to our knowledge, but cited in [14]).

**Remark 1.33.** For exponential Lie groups, $\phi$ is a global isomorphism. This is the case for example for simply connected, nilpotent Lie groups $G$.

## 2 Local integration using abelian extensions

In this section, we sketch Covez’ approach [12] to the integration of Leibniz algebras. It is modeled on the homological proof of Lie’s third Theorem which we sketch first. Covez integrates Leibniz algebras into local Lie racks by associating to each Leibniz algebra an abelian extension and then integrating locally the corresponding Leibniz cocycle to a rack cocycle.

### 2.1 Homological proof of Lie’s third theorem

**Remark 2.1.** Recall the homological proof of Lie’s third theorem (cf [23]) using central extensions: write a given finite dimensional real Lie algebra $\mathfrak{g}$ as a central extension

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\text{ad}} \to 0,$$

where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_{\text{ad}} := \mathfrak{g} / Z(\mathfrak{g})$ is the adjoint Lie algebra associated to $\mathfrak{g}$. As the center of $\mathfrak{g}$ is the kernel of the adjoint representation, $\mathfrak{g}_{\text{ad}}$ embeds into $\mathfrak{gl}(\mathfrak{g})$, the Lie algebra of endomorphisms of the vector space $\mathfrak{g}$, via the adjoint representation. As $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of the Lie group $\text{Gl}(\mathfrak{g})$, this subalgebra integrates by Lie’s first theorem to a connected Lie subgroup $G_{\text{ad}}$ (in order to have simply connected groups, one might want to pass to the universal cover). Trivially, the vector space $Z(\mathfrak{g}) =: V$ integrates to itself, seen now as a trivial module of the Lie group $G_{\text{ad}}$. The above central extension is determined by a Lie algebra 2-cocycle which may be integrated into a locally smooth group 2-cocycle $\gamma$ (thanks to vanishing of the homotopy groups $\pi_1$ and $\pi_2$).
\( \pi_2 \) of the group; we do not review how this is done as the integration in the Leibniz case is quite different. We refer to Neeb’s paper \[20\] for the Lie case.), which then gives rise to a central extension

\[
0 \to V \to V \times \gamma G_{\text{ad}} \to G_{\text{ad}} \to 1.
\]

This central extension is the Lie group into which the Lie algebra \( \mathfrak{g} \) integrates. As a set, \( V \times \gamma G_{\text{ad}} \) is the direct product. The topology and manifold structure on \( V \times \gamma G_{\text{ad}} \) is given by Proposition 18, Chapter III.9 (p. 226) in \[5\]:

**Theorem 2.2.** Let \( G \) be a group, \( W \subset G \) be a subset containing the neutral element 1 and let \( W \) be endowed with a manifold structure. Assume that there exists an open neighborhood \( Q \subset W \) of 1 with \( Q^{-1} = Q \) and \( Q \cdot Q \subset W \) such that

1. the map \( Q \times Q \to W \), \( (g, h) \mapsto gh \in W \) is smooth,
2. the map \( Q \to Q \), \( g \mapsto g^{-1} \) is smooth,
3. \( Q \) generates \( G \) as a group.

Then there exists a Lie group structure on \( G \) such that \( Q \) is open. Any other choice of \( Q \) satisfying the above conditions leads to the same structure.

The integrated group cocycle is globally a cocycle and thus defines a global group structure on \( G := V \times \gamma G_{\text{ad}} \), therefore the theorem applies to our case to give a Lie group structure on \( G \). As \( W \), one may take an open set where the cocycle is smooth.

Starting to transpose this scheme to the framework of Leibniz algebras, by the sequence (1), every Leibniz algebra is an abelian extension in the category of Leibniz algebras, i.e. the corresponding 2-cocycle is a Leibniz cocycle of a Lie algebra by some module. One can choose many ideals \( I \) in a given Leibniz algebra \( \mathfrak{h} \) such that the quotient \( \mathfrak{h}/I \) is a Lie algebra. Actually, every ideal \( I \) which contains the ideal (right) generated by the squares \([X, X]\) for all \( X \in \mathfrak{h} \) works. As mentioned before, for the integration theory, we will work with the left center \( Z_L(\mathfrak{h}) \) where the quotient \( \mathfrak{h}/Z_L(\mathfrak{h}) =: \mathfrak{h}_{\text{Lie}} = \text{ad}(\mathfrak{h}) \) is a Lie algebra, and we have the abelian extension (cf the sequence (1)) of Leibniz algebras

\[
0 \to Z_L(\mathfrak{h}) \xrightarrow{i} \mathfrak{h} \xrightarrow{\pi} \mathfrak{h}_{\text{Lie}} \to 0.
\]

As in the theory of Lie algebras, every abelian extension of Leibniz algebras is (uniquely up to equivalence of extensions) specified by its cohomology class which is represented by a Leibniz 2-cocycle \( \omega : \mathfrak{h}_{\text{Lie}} \times \mathfrak{h}_{\text{Lie}} \to Z_L(\mathfrak{h}) \). This cocycle \( \omega \) is obtained exactly as in Lie algebra cohomology, i.e. given a linear section \( s : \mathfrak{h}_{\text{Lie}} \to \mathfrak{h} \), \( \omega : \mathfrak{h}_{\text{Lie}} \times \mathfrak{h}_{\text{Lie}} \to Z_L(\mathfrak{h}) \) is defined for all \( X, Y \in \mathfrak{h}_{\text{Lie}} \) by

\[
\omega(X, Y) = s([X, Y]) - [s(X), s(Y)] \in \ker(\pi) = \text{im}(\pi) \cong Z_L(\mathfrak{h}).
\]
Details about this correspondence, for example the independence (up to coboundary) of the choice of the section, can be found in [12]. The fact that $\omega$ is a Leibniz 2-cocycle means for all $X,Y,Z \in \mathfrak{h}_{\text{Lie}}$ that

$$X \cdot \omega(Y, Z) - Y \cdot \omega(X, Z) - \omega([X,Y], Z) + \omega(X, [Y,Z]) - \omega(Y, [X,Z]) = 0.$$ 

Observe that this is close to the usual Lie algebra cocycle identity, but with two modifications: the term $[X_i, X_j]$ takes here the place of $X_j$ and the last element acts via the right representation. This right representation is zero in our case and we thus consider antisymmetric Leibniz modules. There are also symmetric Leibniz modules where the right module map is the negative of the left module map.

### 2.2 The local Lie rack

In [12], Covez uses the above mentioned ideas to integrate (finite dimensional real) Leibniz algebras into local Lie racks. Using the group $\text{As}(X)$ for a given rack $X$, we can define the rack modules which we will need.

**Definition 2.3.** Let $X$ be a rack and $A$ be an abelian group equipped with a left action of the group $\text{As}(X)$. We call $A$ an anti-symmetric homogeneous $X$-module.

In general, a rack module is a family of abelian groups and there are two operations, one from the left, one from the right. The underlying abelian group may then change according to an action of $X$ on the indexing set of the family. **Homogeneous** means that the family consists only of one member and **antisymmetric** means that the right action is trivial. It is the fact that the left center is acted on trivially from the right that entails that the integration will be in terms of antisymmetric rack modules.

Now we can integrate the (finite dimensional) Lie algebra $\mathfrak{h}_{\text{Lie}}$ into a connected, simply-connected Lie group $G_0$, integrate the $\mathfrak{h}_{\text{Lie}}$-module $Z_L(\mathfrak{h})$ (which is a Lie algebra module !) into a $G_0$-module $V$ (acting on the same underlying vector space), and the last step to perform is the integration of the Leibniz 2-cocycle $\omega$ into a rack 2-cocycle $I^2(\omega)$. This last step works only locally, i.e. the cocycle $I^2(\omega)$ is only defined on an open neighborhood of $(1,1) \in G_0 \times G_0$. It is not like a locally smooth group cocycle a global cocycle which is only locally smooth, but it is not even globally a cocycle. Therefore, we cannot just transpose Theorem 2.2 to the rack case.

The outcome is then the local Lie rack $V \ltimes_{I^2(\omega)} G_0$ defined as the abelian extension of racks

$$0 \to V \to V \ltimes_{I^2(\omega)} G_0 \to G_0 \to 1.$$ 

### 2.3 Integrating the Leibniz cocycle

We now review the procedure for integrating the Leibniz 2-cocycle $\omega$ into a rack 2-cocycle $I^2(\omega)$. The main idea here is to integrate the two arguments
separately. Indeed, let us first integrate Leibniz 1-cocycles into rack 1-cocycles,
and only on the connected, simply-connected group $G$ corresponding to the Lie
algebra $\mathfrak{h}_{\text{Lie}}$ (and seen as conjugation rack). Choose for all group elements $g \in G$
smooth paths $\gamma_g$ from 1 to $g$. For a given Leibniz 1-cocycle $\omega \in ZL^1(\mathfrak{h}_{\text{Lie}}, a^s)$,
define the equivariant 1-form $\omega^{eq}$ as the unique differential 1-form on $G$
such that for all $g \in G$ and all tangent vectors $m$ at $g$
$$\omega^{eq}(g)(m) = g \cdot (\omega(T_g L_g^{-1}, (m))).$$
Observe that we suppose here that the Leibniz module $a$ is symmetric, i.e. the
right module map is the negative of the left module map. The integration map
$I^1 : ZL^1(\mathfrak{h}_{\text{Lie}}, a^s) \toZR^1(G_0, a^s)$ is then given by
$$I^1(\omega)(g) := \int_{\gamma_g} \omega^{eq}.$$ 
It is shown in loc. cit. that this map does not depend on the choice of $\gamma$,
that it induces a map in cohomology, and that it is left inverse to the differentiation
map which sends locally smooth rack cocycles to Leibniz cocycles.

The second step is then to use the separation-of-variables-isomorphism
$$ZL^2(\mathfrak{h}_{\text{Lie}}, a^a) \cong ZL^1(\mathfrak{h}_{\text{Lie}}, \text{Hom}(\mathfrak{h}_{\text{Lie}}, a)^s).$$
This isomorphism works for general Leibniz algebras and sends coboundaries
to coboundaries. It sends 2-cocycles with values in anti-symmetric modules to 1-cocycles with values in the symmetric module $\text{Hom}(\mathfrak{h}_{\text{Lie}}, a)$. This is indicated
by "a" and "s" in the exponent. Composing with the integration map from the
first step, we obtain a map
$$I : ZL^2(\mathfrak{h}_{\text{Lie}}, a^a) \toZR^2(U, a^a).$$

The third step is to define a map from $ZR^1(G_0, \text{Hom}(\mathfrak{h}_{\text{Lie}}, a)^s)$ to $ZR^2(U, a^a)$,
where $U$ is an open 1-neighborhood in $G_0$ such that the logarithm $\log$ as an
inverse diffeomorphism to the exponential map $\exp$ is defined. This time, the
integration of some $\beta \in ZR^1(G_0, \text{Hom}(\mathfrak{h}_{\text{Lie}}, a)^s)$ is similarly to the previous
integration map given by
$$\int_{\gamma_{gh}} (\beta(g))^{eq},$$
where obviously $g \triangleright h$ for $g, h \in G_0$ means $ghg^{-1}$.

Putting all steps together, the map $I^2 : ZL^2(\mathfrak{h}_{\text{Lie}}, a^a) \toZR^2(U, a^a)$ is given by
$$I^2(\omega)(g, h) = \int_{\gamma_{gh}} (I(\omega)(g))^{eq}.$$ 
Unfortunately, this does not work in general and we need very specific paths in
the group $G_0$ to make this work. The paths take the form
$$\gamma_g(s) = \exp(s \log(g)),$$ (4)
and this is why we work on a 1-neighborhood where \( \log \) is defined. Covez shows that this map \( I^2 \) is well-defined (using the above exponential paths), that it sends coboundaries to coboundaries, and that it is a left inverse of the differentiation map. Furthermore, Covez shows that in the case of Lie algebra cocycles the result is the image of group cocycle under the map linking group and rack cohomology of a group. Observe that thanks to this, every Lie subalgebra of \( \mathfrak{g} \) becomes integrated into a Lie group, seen as a subrack of this Lie rack.

Observe that for some Lie groups (like for example simply-connected, nilpotent Lie groups), the exponential is a global isomorphism and thus for these groups, the integration procedure yields a global Lie rack.

3 Other approaches to the integration of Leibniz algebras

Here we present two other approaches to the integration of Leibniz algebras. They are closely related to work by H. Bass (unpublished), referred to in \cite{L4}, and M. Kinyon \cite{L7}.

3.1 Bass’ approach to integration

This approach builds on a remark by H. Bass in the Lie algebra case, referred to in \cite{L4}, and is already contained in \cite{L7} (end of Section 3), but Kinyon believed this integration to be too arbitrary, as it does not necessarily yield Lie groups in the case of Lie algebras.

Let \( \mathfrak{g} \) be a finite-dimensional real Leibniz algebra.

**Theorem 3.1.** On the vector space \( \mathfrak{g} \), there exists a Lie rack structure which is given by

\[
(X,Y) \mapsto \exp(\text{ad}_X)(Y) =: X \triangleright Y
\]

for all \( X, Y \in \mathfrak{g} \). This global Lie rack structure has the following properties:

1. In case \( \mathfrak{g} \) is a Lie algebra, the corresponding Lie rack structure is locally the conjugation rack structure with respect to to a Lie group structure.

2. The Lie rack structure is globally (!) described by a BCH-formula.

**Proof.** Note that by Lemma \ref{lemma1}, \( X \mapsto \exp(\text{ad}_X) \) is an automorphism of \( \mathfrak{g} \). The fact that the binary operation

\[
(X,Y) \mapsto X \triangleright Y = \exp(\text{ad}_X)(Y)
\]

is a rack product thus follows from the self-distributivity of the linear rack \( \mathfrak{g} \times_{\text{lin}} \text{Aut}(\mathfrak{g}) \) by projection onto the first component.

The BCH-formula which is referred to in the statement is contained in Lemma \ref{lemma1} while the local Lie group structure in the case of a Lie algebra is given by the BCH-product. \( \square \)
One drawback of this Lie rack structure is that the underlying space is contractible. This will be different with the following approach. Another drawback is that in the case of a Lie algebra, the space is only locally a Lie group, but not necessarily globally. We will not be able to overcome this drawback.

3.2 hs-approach to integration

The hs-approach (approach using hemi-semi-direct products) can be seen as modeled on the proof of Lie’s third Theorem using Ado’s Theorem. Here we embed Leibniz algebras as subalgebras of hemi-semi-direct products (taking the place of general linear Lie algebras), integrate these to linear Lie racks and identify then the subrack associated to the given Leibniz algebra.

From the point of view of abelian extensions, a hemi-semi-direct product Leibniz algebra is a trivial extension, so integrates without integrating any cocycle.

Indeed, let a hemi-semi-direct product $V \times_{hs} g$ be given. The left center

$$Z_L(V \times_{hs} g) = \{(v, X) \in V \oplus g : \forall (v', X') \in V \oplus g \ [ (v, X), (v', X') ] = 0 \}.$$ 

Recalling that $[ (v, X), (v', X') ] = (X(v'), [X, X'])$, we thus see that

$$Z_L(V \times_{hs} g) = V \oplus (\text{Ann}_g(V) \cap Z_L(g)),$$

where $\text{Ann}_g(V) = \{ X \in g : \forall v \in V \ X(v) = 0 \}$.

Thus when we specify to $g = \mathfrak{gl}(h)$ for some Leibniz algebra $h$, we will have $\text{Ann}_{\mathfrak{gl}(h)}(h) = 0$, and simply

$$Z_L(h \times_{hs} \mathfrak{gl}(h)) \cong h.$$

The same conclusion holds obviously for $\text{der}(h)$ instead of $\mathfrak{gl}(h)$.

Now in order to compute the cocycle, we have to choose a section $s : \text{der}(h) \rightarrow h \times_{hs} \text{der}(h)$. But the linear map $X \mapsto (0, X)$ is a section, and it is moreover a morphism of Lie and Leibniz algebras. Therefore the cocycle which we can compute from $s$ is zero, and by independence of the choice of the section, the abelian extension associated to the Leibniz algebra $h \times_{hs} \text{der}(h)$ is trivial.

It therefore integrates to a (global) Lie rack $h \times_{hs} \text{Aut}(h)$. Let us summarize the above discussion in the following proposition:

**Proposition 3.2.** 1. Let $g$ be a finite dimensional Lie algebra and $V$ be a finite-dimensional $g$-module. Then the hemi-semi-direct product $V \times_{hs} g$ integrates into the (global) Lie rack $V \times_{hs} G$ where $G$ is the connected, 1-connected Lie group associated to $g$.

2. Let $h$ be a finite-dimensional Leibniz algebra. Then the hemi-semi-direct products $h \times_{hs} \mathfrak{gl}(h)$ and $h \times_{hs} \text{der}(h)$ integrate into the (global) Lie racks $h \times_{hs} \text{Gl}(h)$ and $h \times_{hs} \text{Aut}(h)$ respectively.

**Proof.** In the first setting, we need that the $g$-module $V$ integrates into a $G$-module. This follows from the 1-connectedness, and it is here that we use finite-dimensionality. \[\square\]
3.3 Integrating arbitrary Leibniz algebras in the hs-approach

In this subsection, we show how to integrate a finite-dimensional Leibniz algebra \( \mathfrak{h} \) into a global hemi-semi-direct product linear Lie rack. Modulo the steps which were performed in the previous subsection, it remains to identify the subrack \( R_{\mathfrak{h}} \subset \mathfrak{h} \times_{\text{hs}} \text{Aut}(\mathfrak{h}) \) associated to the Leibniz subalgebra \( \{(X, \text{ad}_X) : X \in \mathfrak{h}\} \) of the hemi-semi-direct product Leibniz algebra \( \mathfrak{h} \times_{\text{hs}} \text{der}(\mathfrak{h}) \). For this, we use the exponential function.

**Proposition 3.3.** The subrack \( R_{\mathfrak{h}} \subset \mathfrak{h} \times_{\text{hs}} \text{Aut}(\mathfrak{h}) \) is explicitly described as
\[
R_{\mathfrak{h}} = \{(X, \exp(\text{ad}_X)) : X \in \mathfrak{h}\}.
\]

It is a closed subset of the direct product of the vector space \( \mathfrak{h} \) and the exponential image \( \exp(\text{ad}(\mathfrak{h})) \) of the adjoint image of \( \mathfrak{h} \) in the Lie group \( \text{Gl}(\mathfrak{h}) \). It acquires therefore a manifold structure on some dense open subset.

**Proof.** First we have to show is that the set
\[
R_{\mathfrak{h}} := \{(X, \exp(\text{ad}_X)) : X \in \mathfrak{h}\}
\]
is a subrack of the hemi-semi-direct product rack \( \mathfrak{h} \times_{\text{hs}} \text{Aut}(\mathfrak{h}) \). This is clear in the first variable, and follows from the formula
\[
\alpha \exp(\text{ad}_X) \alpha^{-1} = \exp(\text{ad}_{\alpha(X)})
\]
for any automorphism \( \alpha \in \text{Aut}(\mathfrak{h}) \) in the second variable, see Lemma [11].

The fact that the exponential image contains a dense open set where it has a manifold structure follows from the fact that the vanishing of the derivative of the exponential functions defines a strictly lower dimensional submanifold. \( \square \)

We summarize the content of these two subsections in the following theorem:

**Theorem 3.4.** For every (real) Leibniz algebra \( \mathfrak{h} \), there exists a rack \( R_{\mathfrak{h}} \) which carries the structure of a Lie rack on some dense open set whose tangent Leibniz algebra is \( \mathfrak{h} \). This Lie rack structure has the following properties:

1. In case \( \mathfrak{h} \) is a Lie algebra, the corresponding Lie rack structure is locally the conjugation rack structure with respect to a Lie group structure.

2. The Lie rack structure is globally (!) described by a BCH-formula.

**Proof.** The first property follows from the fact that locally, \( R_{\mathfrak{h}} \) is isomorphic to the rack \( \mathfrak{h} \) described in Theorem [3.1]. This can be seen by explicitly by constructing a rack morphism \( \phi : \mathfrak{h} \to R_{\mathfrak{h}} \) where \( \mathfrak{h} \) carries the Bass rack structure
\[
X \triangleright Y := \exp(\text{ad}_X)(Y)
\]
for all \( X, Y \in \mathfrak{h} \). The map \( \phi \) is then defined by
\[
\phi(X) := (X, \exp(\text{ad}_X)).
\]
\( \phi \) is a rack morphism because

\[
\exp(\text{ad}_{\exp(\text{ad}_X)(Y)}) = \exp(\text{ad}_X) \exp(\text{ad}_Y) \exp(-\text{ad}_X),
\]

which follows easily from Lemma 1.11. \( \phi \) is thus an isomorphism.

The explicit BCH-description of the rack product is

\[
(X, \exp(\text{ad}_X)) \triangleright (Y, \exp(\text{ad}_Y)) = \left( \sum_{k=0}^{\infty} \frac{1}{k!} (\text{ad}_X)^k (Y), \exp(\text{ad}_X) \exp(\text{ad}_Y) \exp(-\text{ad}_X) \right).
\]

This shows that the rack product is completely described in terms of the Leibniz bracket of \( \mathfrak{h} \). Also without using the isomorphism \( \phi \), the first property follows from Lemma 1.16. \( \square \)

**Remark 3.5.** The Lie racks \( R_\mathfrak{h} \) and \( \mathfrak{h} \) do not come in general from a digroup (i.e. according to Proposition 1.26). It is instructive to try axiom 4 of a digroup: it does not work for \( \mathfrak{h} \), but it does works for \( R_\mathfrak{h} \), because of the second component.

On the other hand \( R_\mathfrak{h} \) does not come from a digroup, because the digroup operations make the second component different from the first.

Observe however that Kinyon’s Theorem 1.29 states that split Leibniz algebras may be integrated into Lie digroups. In fact by Remark 1.30, in case a Leibniz algebra integrates into a Lie digroup, it is necessarily split.

### 3.4 Some properties of the global Lie rack

In the previous section, we described the rack product of the Lie rack \( R_\mathfrak{h} \) using only the Leibniz bracket of \( \mathfrak{h} \) in the spirit of the local description of the group product of a Lie group in terms of the Lie bracket via the BCH formula.

We now use this result to obtain a local version of Lie’s Second Theorem for Lie racks of the form \( R_\mathfrak{h} \), i.e. for the diagonal Lie subracks of \( \mathfrak{h} \times_{\text{hs}} \text{Aut}(\mathfrak{h}) \) which were introduced earlier. For this, note that Lie racks \( R_\mathfrak{h} \) of this type have an exponential map \( \exp : \mathfrak{h} \to R_\mathfrak{h} \) given by \( X \mapsto (X, \exp(\text{ad}_X)) \).

**Proposition 3.6.** Let \( R_1 = R_{\mathfrak{h}_1} \) and \( R_2 = R_{\mathfrak{h}_2} \) be Lie racks of the form \( R_\mathfrak{h} \) with Leibniz algebras \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) respectively. Let \( \alpha : \mathfrak{h}_1 \to \mathfrak{h}_2 \) be a morphism of Leibniz algebras. Then there exists a unique morphism of Lie racks \( \phi : R_1 \to R_2 \) such that

\[
\phi \circ \exp(X) = \exp \circ \alpha(X) \quad \text{for all } X \in \mathfrak{h}_1.
\]

**Proof.** Consider the exponential map \( \exp : \mathfrak{h}_i \to R_i \) given by \( X \mapsto (X, \exp(\text{ad}_X)) \) for \( i = 1, 2 \). Put

\[
\phi := \exp \circ \alpha \circ \log : R_1 = \exp(\mathfrak{h}_1) \to R_2.
\]

It is enough to show that this map \( \phi \) is a morphism of racks.

As \( \alpha \) is a morphism of Leibniz algebras, we obtain by induction

\[
\alpha(X \triangleright_\ast Y) = \alpha(X) \triangleright_\ast \alpha(Y),
\]
where the rack product $\triangleright_*$ is the Bass product $X \triangleright_* Y = \exp(\text{ad}_X)(Y)$.

Now recall from the proof of Theorem 3.4 that the exponential map sends the Bass product to the product in $R_h$. This implies directly the relation:

$$\exp(\alpha(X)) \triangleright \exp(\alpha(Y)) = \exp(\alpha(X \triangleright_* Y)).$$

Writing this relation in terms of the rack elements $(X, \exp(\text{ad}_X))$ and $(Y, \exp(\text{ad}_Y))$ using that $\log((X, \exp(\text{ad}_X)) \triangleright (Y, \exp(\text{ad}_Y)))$, one obtains

$$\phi(X, \exp(\text{ad}_X)) \triangleright \phi(Y, \exp(\text{ad}_Y)) = \phi((X, \exp(\text{ad}_X)) \triangleright (Y, \exp(\text{ad}_Y))),$$

by observing that

$$X \triangleright_* Y = \exp(\text{ad}_X)(Y) = \log((X, \exp(\text{ad}_X)) \triangleright (Y, \exp(\text{ad}_Y))).$$

□

Corollary 3.7. Two Leibniz algebras $\mathfrak{h}$ and $\mathfrak{h}'$ are isomorphic if and only if their corresponding Lie racks $R_{\mathfrak{h}}$ and $R_{\mathfrak{h}'}$ are isomorphic as Lie racks.

Remark 3.8. There is a warning in order here. The above corollary unfortunately does not necessarily apply to Covez’ local Lie rack. We do not know whether it is locally isomorphic to our BCH-Lie rack (but we believe strongly that it is).

In principle, there can be different local integrations of Leibniz algebras, which all yield conjugation racks with respect to Lie groups in the special case of Lie algebras, but whose rack 2-cocycles are non cohomologous. At the moment, we do not have an example for this instance.

3.5 Exploring the cocycle associated to the global Lie rack

In this subsection, we will write out explicitly the cocycle associated to the Lie rack $R_{\mathfrak{h}}$, when one chooses a section of the corresponding abelian extension.

Let $\mathfrak{h}$ be a finite dimensional Leibniz algebra. Recall the exact sequence which describes $\mathfrak{h}$ as an abelian extension:

$$0 \to Z_L(\mathfrak{h}) \overset{i}{\to} \mathfrak{h} \overset{s}{\to} \mathfrak{h}_{\text{Lie}} \to 0.$$

We will write $Z_L(\mathfrak{h}) \times_{\omega} \mathfrak{h}_{\text{Lie}}$ for the Leibniz algebra $\mathfrak{h}$ when regarded as an abelian extension in this way by means of the cocycle $\omega$. Denote by $\mathfrak{h} \times_{\mathfrak{h}} \text{der}(\mathfrak{h})$ the diagonal subspace of the hemi-semi-direct product $\mathfrak{h} \times_{\text{hs}} \text{der}(\mathfrak{h})$, i.e. the subspace of $(X, \text{ad}_X)$ for all $X \in \mathfrak{h}$. The diagonal subspace $\mathfrak{h} \times_{\mathfrak{h}} \text{der}(\mathfrak{h})$ is clearly a Leibniz subalgebra of the hemi-semi-direct product.

Proposition 3.9. There is an isomorphism of Leibniz algebras:

$$\phi : Z_L(\mathfrak{h}) \times_{\omega} \mathfrak{h}_{\text{Lie}} \cong \mathfrak{h} \times_{\mathfrak{h}} \text{der}(\mathfrak{h}),$$

given by

$$(a, X) \mapsto (a + s(X), \text{ad}_X),$$

where $s : \mathfrak{h}_{\text{Lie}} \to \mathfrak{h}$ is the linear section which corresponds to the cocycle $\omega$. 24
Proof. The map $\phi$ is defined by
\[(a, X) \mapsto (a + s(X), \text{ad}_X).\]
Observe first of all that $\phi(a, X) = (a + s(X), \text{ad}_X) = (a + s(X), \text{ad}_{a+s(X)})$, because elements from $Z_L(\mathfrak{h})$ act trivially on $\mathfrak{h}$. Thus $\phi$ is well-defined.

Moreover, $\phi$ is a morphism of Leibniz algebras. Indeed, the bracket in the abelian extension with cocycle $\omega$ gives:
\[
[(a, X), (b, Y)] = (X \cdot b + \omega(X, Y), [X, Y]),
\]
which is mapped to $(X \cdot b + \omega(X, Y) + s([X, Y]), \text{ad}_{[X,Y]})$ via $\phi$. On the other hand, the bracket in the hemi-semi-direct product reads
\[
[(a + s(X), \text{ad}_X), (b + s(Y), \text{ad}_Y)] = (\text{ad}_X(b + s(Y)), [\text{ad}_X, \text{ad}_Y]),
\]
and this is equal to what we had before using $\text{ad}_X(b) = X \cdot b$, $\text{ad}_X(s(Y)) = [s(X), s(Y)]$ (because the difference $X - s(X)$ is left central), and $\omega(X, Y) + s([X, Y]) = [s(X), s(Y)]$ by definition. But it is clear that the morphism $\phi$ is an isomorphism. □

Now we want to present the Lie rack $R_h$ in the same spirit as an abelian extension. For details about abelian extension of racks, see e.g. [12].

Every Leibniz algebra $\mathfrak{h}$ gives rise to a Lie rack $R_h$, and furthermore to an abelian extension of racks:
\[
0 \to Z_L(\mathfrak{h}) \xrightarrow{I} R_h \xrightarrow{P} \mathfrak{h}_{\text{Lie}} \times_{\text{Lie}} \exp(\exp(ad_{\mathfrak{h}_{\text{Lie}}})) \to 1.
\]
Here the Leibniz algebra $\mathfrak{h}_{\text{Lie}} \times_{\text{Lie}} \exp(\exp(ad_{\mathfrak{h}_{\text{Lie}}}))$ is regarded as a Lie rack by means of the Bass rack structure, and the same holds for $Z_L(\mathfrak{h})$, which renders it a trivial subrack of $R_h$. The maps $I$ and $P$ are defined by $I(a) = (a, \text{id})$ and $P(X, \exp ad_X) = (\pi(X), \exp ad_{\pi(X)})$.

It is easy to compute that $P$ is a morphism of racks, using that $\pi$ is a continuous linear map (between finite dimensional vector spaces).

Next, we need a section of $P$, i.e. a map $S : \mathfrak{h}_{\text{Lie}} \times_{\text{Lie}} \exp(\exp(ad_{\mathfrak{h}_{\text{Lie}}})) \to R_h$ which is right inverse to $P$. Using the section $s$ of the map $\pi$, $S$ can be defined as
\[S(X, \exp ad_X) := (s(X), \exp ad_{s(X)}).\]
The corresponding rack 2-cocycle $f$ is then defined for all $x, y \in \mathfrak{h}_{\text{Lie}} \times_{\text{Lie}} \exp(\exp(ad_{\mathfrak{h}_{\text{Lie}}}))$ by:
\[f(x, y) := S(x) \triangleright S(y) - S(x \triangleright y).\]
One easily computes that this gives the following expression in our situation:
\[
f(X, Y) = \exp ad_{s(X)}(s(Y)) - s(\exp ad_X(Y)),
\]
where we wrote simply $X \in \mathfrak{h}_{\text{Lie}}$ for 

$$(X, \exp \text{ad}_X) \in \mathfrak{h}_{\text{Lie}} \times \mathfrak{h}_{\text{Lie}} \exp(\text{ad}_{\mathfrak{h}_{\text{Lie}}})$$

and similarly for $Y$. As usual for abelian extensions, we displayed by abuse of notation only the $Z_L(\mathfrak{h})$-component of $f(x, y)$ - the other component is trivial stemming from the fact that $P$ is a morphism of racks.

Using the formula of Lemma 1.16 we obtain from here the expression:

$$f(X,Y) = \text{conj}_*(s(X), s(Y)) - s(\text{conj}_*(X,Y)),$$

thus the cocycle $f$ measures the default of $s$ to be compatible with the formal conjugation map.

We have the following explicit formula for the rack cocycle $f$ in terms of the Leibniz cocycle $\omega$ and the section $s$ of the abelian extension of Leibniz algebras:

**Lemma 3.10.**

$$f(X,Y) = \omega(X,Y) + \frac{1}{2}\omega(X,[X,Y]) + \frac{1}{6}\omega(X,[X,[X,Y]]) + \ldots$$

$$+ \frac{1}{2}[s(X),\omega(X,Y)] + \frac{1}{6}[s(X),\omega(X,[X,Y])] + \ldots$$

$$+ \frac{1}{6}[s(X),[s(X),\omega(X,Y)]] + \ldots$$

*These terms are grouped here according to the number of $s(X)$ acting upon terms in $\omega$.*

As already stated in Remark 3.8 we do not know whether this rack 2-cocycle is cohomologuous to Covez’ rack 2-cocycle. On the other hand, we believe that this integration formula for cocycles is new, even in the special case of Lie algebra 2-cocycles.

### 3.6 Summary: Integration of Leibniz algebras

Thus in conclusion there are (at least) three integration methods for Leibniz algebras. Note that in general only the local Lie racks of the last two are isomorphic.

- The local integration of Covez [12] integrating the Leibniz cocycle to a local rack cocycle. This works only locally and yields local Lie groups in the case of Lie algebras. Moreover, the integration procedure is compatible with standard maps between the group-, rack-, Leibniz and Lie cohomology spaces.
- The integration via the conjugation with respect to the BCH formula (Bass’ approach), which also locally yields Lie groups in the special case of Lie algebras. It integrates a Leibniz algebra into a rack structure on the same underlying vector space.
• The globalization of this local integration in terms of hemi-semi-direct products. One still has the interpretation in terms of local groups in the case of Lie algebras and one gains globality (in the sense that the underlying topological space may be non-contractible).

4 Deformation quantization of Leibniz algebras

4.1 Motivation

Recall that given a finite-dimensional real Lie algebra \((g, [\cdot, \cdot])\), its dual vector space \(g^*\) is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all \(f, g \in C^\infty(g^*)\) and all \(\xi \in g^*\) by the Kostant-Kirillov-Souriau formula

\[
\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.
\]

Here \(df(\xi)\) and \(dg(\xi)\) are linear functionals on \(g^*\), identified with elements of \(g\).

The goal of the second part of this article is to define deformation quantization for an analogous bracket on the dual of a Leibniz algebra.

Let \((h, [\cdot, \cdot])\) be a (left, real, finite-dimensional) Leibniz algebra. Its linear dual \(h^*\) is still a smooth manifold. The smooth functions \(C^\infty(h^*)\) on \(h^*\) have a natural bracket

\[
\{f, g\}(\xi) := \langle \xi, [df(0), dg(\xi)] \rangle
\]

At this stage, it may seem arbitrary that in comparison to the above bracket on the dual of a Lie algebra, we evaluated the first variable in 0. It is an outcome (see Theorem 4.13) of our deformation quantization procedure that this is the bracket which we are deforming. We will not introduce a different notation for this generalized bracket. We hope it will be clear from the context which bracket we will be talking about.

One readily verifies that this bilinear bracket satisfies the (right) Leibniz rule for all \(f, g, h \in C^\infty(h^*)\):

\[
\{f, gh\} = \{f, g\}h + g\{f, h\}.
\]

Remark 4.1. Observe that there is a remainder of the left Leibniz rule, too. Vector fields are derivations on the algebra of functions. Tangent vectors are pointwise derivations, i.e. the derivation property holds when interpreted as immediately followed by evaluation in a point. In this sense, the Leibniz rule in the first variable of \(\{-, -\}\) holds when interpreted as immediately followed by evaluation in 0.

On the other hand, the bracket does not satisfy anymore the (left) Leibniz identity \(\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}\) and it is certainly not necessarily skew-symmetric.
Remark 4.2. It is natural that the generalized bracket should satisfy much weaker conditions than a Poisson bracket on a smooth manifold. Indeed, it is shown in [15] that a bracket on a commutative associative algebra (in characteristic zero, without zero divisors) which satisfies the Leibniz rule in both variables and the Leibniz identity is necessarily skew-symmetric. We thank K. Uchino for bringing this fact to our attention.

We call the bracket in (5) Leibniz-Poisson bracket.

Definition 4.3. A generalized Poisson manifold is a smooth manifold \( M \) whose space of smooth functions \( C^\infty(\mathcal{M}) \) is endowed with a bilinear bracket \( \{ , \} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \) satisfying the above property (6).

The notion of star-product [1] is closely related to the notion of Poisson manifold.

Definition 4.4. A star-product \( \ast \) on a Poisson manifold \((\mathcal{M}, \{ , \})\) is a formal deformation \( \ast_\epsilon \) of the commutative associative product on \( C^\infty(\mathcal{M}) \), i.e. an associative product

\[
 f \ast_\epsilon g = fg + \epsilon B_1(f, g) + \ldots + \epsilon^n B_n(f, g) + \ldots
\]

such that the \( B_n(-,-) \)'s are bidifferential operators for all \( n \geq 1 \) and that the constant function \( 1 \) is a unit.

Namely, given a star-product \( \ast_\epsilon \) on \( \mathcal{M} \),

\[
 f \ast_\epsilon g = fg + \epsilon B_1(f, g) + \ldots,
\]

the antisymmetrization of the first terms yields a Poisson bracket on \( \mathcal{M} \):

\[
 \{ f, g \} = B_1(f, g) - B_1(g, f).
\]

One says that the star-product \( \ast_\epsilon \) quantizes the Poisson bracket in this case.

Conversely, M. Kontsevich [19] showed that any Poisson bracket can be quantized (non uniquely) into a star-product.

On the other hand, the quantization of a Lie algebra \( \mathfrak{g} \) is known to be (roughly) the data of a \( \ast \)-algebra \( A_\mathfrak{g} \) for which the self-adjoint elements

\[
 U_\mathfrak{g} = \{ a \in A_\mathfrak{g} | a^\ast = a \}
\]

form a group isomorphic to the Lie group integrating \( \mathfrak{g} \).

A model for this quantizing \( \ast \)-algebra is the universal enveloping algebra \( U(\mathfrak{g}) \) of the Lie algebra \( \mathfrak{g} \); another one is the convolution algebra \( C(\mathfrak{g}) \) of continuous functions on the integrating group. Deformation quantization, by considering \( \ast \)-algebras quantizing the Poisson structure on the dual space \( \mathfrak{g}^* \), gives yet a third model.

Namely the Gutt \( \ast \)-algebra \((C^\infty(\mathfrak{g}^*)[[\epsilon]], \ast_{\text{Gutt}})\) (see [10]) where \( \epsilon = \frac{\hbar}{2i} \) quantizes \( \mathfrak{g}^* \) in the sense of deformation quantization and also has the following properties:
1. The complex conjugation is the involution of the $*$-algebra

$$f^* \overset{\text{Gutt}}{=} g^* \overset{\text{Gutt}}{=} f$$

2. $U_g = \{E_X \mid X \in g\}$ where $E_X(\xi) = e^{i \langle X, \xi \rangle}$ and

$$E_X \overset{\text{Gutt}}{*} E_Y = E_{BCH(X,Y)} \quad \text{and} \quad E_{\overline{X}} = E_{-X}.$$ 

Thus $U_g$ is isomorphic rather to the formal/local group $(g, BCH)$ integrating $(g, [ , ])$.

In everything that follows, one can always exchange the expansion parameters $\epsilon$ and $\hbar$ using the formula $\epsilon = \frac{\hbar}{2i}$.

In this section, we aim at quantizing a Leibniz algebra $h$ using techniques similar to deformation quantization. As we will see, what we obtain is an operation

$$\triangleright : C^\infty(h^*)[[\epsilon]] \times C^\infty(h^*)[[\epsilon]] \to C^\infty(h^*)[[\epsilon]]$$

such that the restriction of $\triangleright$ to $U_h = \{E_X \mid X \in h\}$ is a rack structure $\triangleright : U_h \times U_h \to U_h$.

Moreover, the restriction of this operation to

$$\triangleright : U_h \times C^\infty(h^*)[[\epsilon]] \to C^\infty(h^*)[[\epsilon]]$$

is a rack action.

**Remark 4.5.** In the case of a Lie algebra $(g, [ , ])$, one can obtain such a quantum rack $\triangleright_{\text{Gutt}}$ from the Gutt star-product; namely

$$f \triangleright_{\text{Gutt}} g := f \overset{\text{Gutt}}{*} g \overset{\text{Gutt}}{*} f,$$

whose restriction to the exponentials is

$$e^{\hat{X}} \triangleright_{\text{Gutt}} e^{\hat{X}} = e^{\hat{X}} \overset{\text{Gutt}}{*} e^{\hat{Y}} \overset{\text{Gutt}}{*} e^{\hat{-X}} = e^{\hat{\text{conj}}(X,Y)},$$

where we have used Lemma 1.16. In the same vein, we obtain

$$e^{\hat{X}} \triangleright_{\text{Gutt}} g = g + \epsilon B^1_X(g) + \epsilon^2 B^2_X(g) + \cdots,$$

where the $B^k_X$ are certain differential operators depending on $X \in g$. The quantum rack we will obtain in the case of a general Leibniz algebra will not coincide with the one in the Lie algebra case, but their restrictions on exponentials will.

We start by reinterpreting the Gutt star-product quantizing a Lie algebra $(g, [ , ])$ as the quantization of the symplectic micromorphism obtained by the cotangent lift of the group operation $m : G \times G \to G$ on the integrating Lie group. We will then follow a similar strategy for Leibniz algebras $h$ by quantizing the corresponding micromorphism obtained by the cotangent lift of the integrating rack structure $\triangleright : R_h \times R_h \to R_h$. 

29
4.2 Gutt star-product as the quantization of a symplectic micromorphism

Let \((g, [\cdot, \cdot])\) be a Lie algebra with integrating Lie group \(G\). The cotangent lift \(T^*m\) of the group operation \(m : G \times G \to G\) is the Lagrangian submanifold

\[ T^*m := \{(g, T^*_g R_h \xi), (h, T^*_h L_g \xi), (gh, \xi) : g, h \in G, \xi \in T^*_g G\} \]

of \(T^*G \times T^*G \times T^*G\), where \(R_h : G \to G\) and \(L_g : G \to G\) are the usual right and left translations on \(G\), respectively. The cotangent lift \(T^*m\) is actually the graph of the global symplectic groupoid

\[ T^*G \xrightarrow{s} g^* \]

integrating the Poisson manifold \(g^*\). We refer the reader to [6] and [26] for more details on the relationships between integrated Poisson data and Lagrangian submanifolds.

Seeing \(T^*m\) as a canonical relation from \(T^*G \otimes T^*G \simeq T^*G \times T^*G\) to \(T^*G\) in the symplectic category, one wishes to associate to it a Fourier integral operator (depending on a parameter \(\hbar\)) from some \(L^2(g^*) \otimes L^2(g^*) \to L^2(g^*)\) whose asymptotic expansion in the limit \(\hbar \to 0\) would yield a star-product, in the spirit of [27] and [28] (see also [26] for a more recent exposition).

This is in general a very hard problem analytically, and it turns out that one is more lucky by only looking at the germ of \(T^*m \subset T^*G \times T^*G \times T^*G\) around the graph of the diagonal map \(\triangle_g : g^* \to g^* \times g^*\), where see see \(g^*\) as an embedded Lagrangian submanifold in \(T^*G\), namely the fiber over the identity element. Namely, as shown in [10], this germ is a symplectic micromorphism, which is readily quantizable by Fourier Integral Operators (FIO) (see [11]).

Symplectic micromorphisms

Let us recall the definition of a symplectic micromorphism (see [8], [9], [10], and [11] for more details) as well as some aspect of their quantization.

**Definition 4.6.** A symplectic micromorphism \(([L], \phi)\) from a symplectic microfold \([M, A]\) (i.e. a germ of a symplectic manifold around a Lagrangian submanifold \(A \subset M\), called the core of the microfold) to a symplectic microfold \([N, B]\) is the data of a Lagrangian submanifold germ \([L]\) in \(M \times N\) around the graph \(\text{gr}(\phi)\) of a smooth map \(\phi : A \to B\) such that the intersection \(L \cap (A \times B) = \text{gr}(\phi)\) is clean for a representative \(L \in [L]\).

The symplectic micromorphisms are the morphisms of a category, the microsymplectic category. We denote them by \(([L], \phi) : [M, A] \to [N, B]\), and, when the symplectic microfold is \([T^*A, A]\), we simply write \(T^*A\).
An important example of symplectic micromorphisms comes from cotangent lifts of smooth maps between manifolds. Namely, if \( \phi: B \to A \) is a smooth map, then the conormal bundle \( N^* \text{gr}\phi \) of the graph of \( \phi \) is a lagrangian submanifold of \( T^*(A \times B) \). Using the identification (Schwartz transform) between this last cotangent bundle and \( T^*A \times T^*B \), the conormal bundle to the graph yields a symplectic micromorphism, which we denote by \( T^*\phi: T^*A \to T^*B \), by taking the germ of the resulting lagrangian submanifold

\[
\{ (p_A, \phi(x_B)), ((T^*_x B)\phi)p_A, x_B) \} : (p_A, x_B) \in \phi^*(T^*A)
\]

around the graph of \( \phi \), and where \( (p_A, x_A) \) and \( (p_B, x_B) \) are the canonical coordinates on \( T^*A \) and \( T^*B \) respectively.

When the target and source symplectic microfold cores are euclidean (i.e. when \( A = \mathbb{R}^k \) and \( B = \mathbb{R}^l \) for some \( k \geq 1 \) and \( l \geq 1 \)), a symplectic micromorphism from \( T^*A \) to \( T^*B \) can be associated with a family of formal Fourier Integral operators from \( C^\infty(A)[[\hbar]] \) to \( C^\infty(B)[[\hbar]] \) using the symplectic micromorphism generating function (see [11] for a general theory of symplectic micro-morphism quantization).

Namely, as shown in [9], when the target and source symplectic microfold cores are euclidean any symplectic micromorphism \( ([L], \phi) \) from \( T^*A \) to \( T^*B \) can be described by a generating function germ \( [S_L]: \phi^*(T^*A) \to \mathbb{R} \) around the zero section of the pullback bundle \( \phi^*(T^*A) \) as follows: There is a representative \( L \in [L] \) such that

\[
\left\{ \left( p_A, \frac{\partial S_L}{\partial p_A}(p_A, x_B), \frac{\partial S_L}{\partial x_B}(p_A, x_B), x_B \right) : (p_A, x_B) \in W \right\},
\]

where \( W \) is an appropriate neighborhood of the zero section in \( \phi^*(T^*A) \). This generating function \( S_L \) is unique if one requires that it satisfies the property \( S_L(0, x) = 0 \). The geometric condition on the cleanness of the intersection in the definition above can be expressed in terms of the generating function as follows:

\[
\frac{\partial S_L}{\partial p_A}(p_A, 0) = \phi(x_B) \quad \text{and} \quad \frac{\partial S_L}{\partial x_B}(0, x_B) = 0. \quad (7)
\]

In this light, one can see \( S_L \) as a deformation of the cotangent lift generating function, which is the first term of \( S_L \) in a Taylor expansion:

\[
S_L(p_A, x_B) = \langle p_A, \phi(x_B) \rangle + \mathcal{O}(p_A^2).
\]

**Remark 4.7.** Conversely, any generating function germ \( [S]: \phi^*(T^*A) \to \mathbb{R} \) satisfying conditions (7) defines uniquely a symplectic micromorphism \( ([L_S], \phi): T^*A \to T^*B \).

Now, using the generating function \( S_L \) of the symplectic micromorphism \( ([L], \phi) \) and a function germ \( a : \phi^*(T^*A) \to \mathbb{R} \) around the zero section, one can construct a formal operator

\[
C^\infty(A)[[\hbar]] \quad \mapsto \quad C^\infty(B)[[\hbar]] \\
\psi \quad \mapsto \quad Q^a(L, \phi)\psi
\]
by taking the stationary phase expansion of the following oscillatory integral:
\[ \int_{T^\star A} \chi(p_A, x_A) \psi(x_A) a(p_A, x_B) e^{i(S_L(p_A, x_B) - p_A x_A)/(2\pi\hbar)} \frac{dx_A dp_A}{(2\pi\hbar)^n} \]
where \( \chi \) is a cutoff function with compact support around the critical points of the phase \( S_L(p_A, x_B) - p_A x_A \) (with respect to the integration variables) and with value 1 on this critical locus, which is nothing but the points in \( \{(0, \phi(x_B)) : x_B \in B\} \). Since the critical locus is contained in the zero section, the asymptotic expansion does not depend on the cutoff functions and, hence, is well-defined. To simplify the notation, we will abuse it slightly, and write from now on:
\[ (Q^a(L, \phi)) \psi(x_B) = \int_{R^k} \hat{\psi}(p_A) a(p_A, x_B) e^{iS_L(p_A, x_B)/(2\pi\hbar)^{k/2}} \]
to mean the asymptotic expansion above, and where \( \hat{\psi}(p_B) \) is the asymptotic Fourier transform of \( \psi \); namely,
\[ \hat{\psi}(p_A) = \int_{R^k} \psi(x_A) e^{-i p_A x_A/(2\pi\hbar)^{k/2}} \]

**Back to the Gutt star-product**

Let us now apply the previous section result to the quantization of the linear Poisson structure on the dual of a Lie algebra \( g \). Consider first the integrating Lie group \( G \). Taking the cotangent lift of the group operation \( m : G \times G \to G \) yields a symplectic micromorphism
\[ ([T^\star m], \Delta_g) : [T^\star G, g^\star] \otimes [T^\star G, g^\star] \to [T^\star G, g^\star], \]
where we take the core in the source and target symplectic microfolds to be not the cotangent bundle zero section \( G \), but rather the fiber above the identity, i.e. the dual of the Lie algebra. Identifying \( [T^\star G, g^\star] \) with \( [T^\star g^\star, g^\star] \) (which we will denote simply by \( T^\star g^\star \)) using the Lagrangian embedding germ
\[ T^\star g^\star \to [T^\star G, g^\star], \ (X, \xi) \mapsto (\exp(X), (T_1^\star L_{\exp(X)})^{-1}\xi), \]
the Lagrangian germ \( [T^\star m] \) becomes the cotangent lift of the local group operation \( BCH : g \times g \to g \), and \( ([T^\star m], \Delta_{g^\star}) \) becomes a symplectic micromorphism from \( T^\star g^\star \otimes T^\star g^\star \) to \( T^\star g^\star \), whose underlying Lagrangian submanifold germ coincides with the multiplication of the local symplectic groupoid integrating the linear Poisson structure on \( g^\star \).

This local/formal symplectic groupoid is described in [7], where it is shown that \( T^\star m \) can be described in term of the following generating function germ
\[ S(X, Y, \xi) = \left\{ (\xi, BCH(X, Y)) \right\} \]
as follows:
\[ T^\star m = \left\{ \left( (X, \frac{\partial S}{\partial X}), (Y, \frac{\partial S}{\partial Y}), (\frac{\partial S}{\partial \xi}, \xi) \right) : (X, Y, \xi) \in W \right\} \]
where $W$ is an appropriate neighborhood of the zero section in $T^*g^* \oplus T^*g^*$.

Once the generating function of a symplectic micromorphism is computed, it is easy to obtain a family of (formal) FIOs quantizing it as explained in the previous section. In the case at hand, we obtain the following family of formal operators

$$Q^a(T^*m) : C^\infty(\mathfrak{g}^*)[[\epsilon]] \otimes C^\infty(\mathfrak{g}^*)[[\epsilon]] \rightarrow C^\infty(\mathfrak{g}^*)[[\epsilon]]$$

of the form (in the previous section notation):

$$Q^a(T^*m)(f \otimes g)(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} \hat{f}(X)\hat{g}(Y)a(X,Y,\xi)e^{i\frac{S(X,Y,\xi)}{\hbar}}dXdY \binom{2\pi}{\hbar}^n, \quad (8)$$

where $a$ is the germ of a smooth function on $T^*g^* \oplus T^*g^*$ around the zero section, called the amplitude of the FIO $Q^a(T^*m)$, and $n$ is the dimension of $\mathfrak{g}$.

When $a = 1$ and $S$ is the generating function of $([T^*m], \triangle_{\nu^*})$, we have that

$$f *_a g = Q^a(T^*m)(f \otimes g)$$

coincides with the Gutt star-product \cite{3, 4, 16}. For other star-products in integral form on duals of Lie algebras as in \cite{8}, we refer the reader to the work of Ben Amar \cite{3, 4}.

Remark 4.8. For a general amplitude $a$, $f *_a g$ is not necessarily associative.

4.3 Quantizing a Leibniz algebra

Let $(\mathfrak{h}, [,])$ be a Leibniz algebra and $(R_\hbar, \triangleright)$ its integrating Lie rack from Section 3. The idea is to quantize the Lagrangian relation

$$T^* \triangleright : T^*R_\hbar \times T^*R_\hbar \rightarrow T^*R_\hbar$$

as we did for the group operation in the case of a Lie algebra.

As we saw in the Lie case, it is better to consider the local model, i.e. the integrating rack

$$\triangleright : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}, \quad (X,Y) \mapsto e^{\text{ad}_X}(Y) =: \text{Ad}_X(Y)$$

defined on $\mathfrak{h}$. The first step is to take the cotangent lift of the rack operation and compute its generating function:

Proposition 4.9. The cotangent lift of $\triangleright$ yields a symplectic micromorphism

$$T^*\triangleright : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \rightarrow T^*\mathfrak{h}^*$$

with generating function

$$S_\triangleright(X,Y,\xi) := \langle \xi, \text{Ad}_X(Y) \rangle.$$
Proof. Consider the generating function

\[ S_\triangleright (X, Y, \xi) := \langle \xi, \text{Ad}_X(Y) \rangle = \langle \xi, Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots \rangle \]

We will denote the variables by \((X, Y) := P\) and \(\xi\), and write accordingly

\[ S_\triangleright (X, Y, \xi) = S_\triangleright (P, \xi). \]

As shown in [9] Sections 3.1 and 3.2 (see also [7] Section 1.2), a generating function of the type

\[ S_\triangleright (P, \xi) = \langle \Phi(\xi), P \rangle + O(P^2) \]

where \(\Phi : \mathfrak{h}^* \to \mathfrak{h}^* \times \mathfrak{h}^*\), \(\Phi(\xi) = (0, \xi)\), yields a symplectic micromorphism

\[ ([L_S], \Phi) : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \to T^*\mathfrak{h}^* \]

where

\[ L_S = \left\{ \left( \frac{\partial S_\triangleright}{\partial X}, \frac{\partial S_\triangleright}{\partial Y}, \frac{\partial S_\triangleright}{\partial \xi} \right) \mid \xi \in \mathfrak{g}^*, X, Y \in \mathfrak{g} \right\} = \left\{ (\xi, [X, Y], \xi), (Y, \text{Ad}_X^*(\xi)), (\text{Ad}_X(\xi), Y) \mid \xi \in \mathfrak{g}^*, X, Y \in \mathfrak{g} \right\} \]

which one recognizes to be the cotangent lift of the map \((X, Y) \mapsto \text{Ad}_X(Y)\).

Remark 4.10. If \(\mathfrak{g}\) is a Lie algebra, then \(\text{Ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) is the adjoint action of the local/formal group \((\mathfrak{g}, \text{BCH})\) on \(\mathfrak{g}\) by Lemma 1.16. The cotangent lift of this action is a Hamiltonian action of \((\mathfrak{g}, \text{BCH})\) on \(T^*\mathfrak{g}\) given by \(T^*\text{Ad}_X : T^*\mathfrak{g} \to T^*\mathfrak{g}\) for all \(X \in \mathfrak{g}\). This Hamiltonian action has an equivariant momentum map \(J : T^*\mathfrak{g} \to \mathfrak{g}^*\) given by \(J(Y, \xi) = \langle \xi, \text{ad}_Y \rangle\), i.e.

\[ \langle X, J(Y, \xi) \rangle = \langle \xi, [Y, X] \rangle. \]

Under the identification \(T^*\mathfrak{g} \cong T^*\mathfrak{g}^* (\cong \mathfrak{g} \times \mathfrak{g}^*)\), the cotangent lift \(T^*\text{Ad}_X\) gives a Hamiltonian action of \((\mathfrak{g}, \text{BCH})\) on \(T^*\mathfrak{g}^*\). This yields an action of the (local) symplectic groupoid \(T^*\mathfrak{g}^* \xrightarrow{s} \mathfrak{g}^* \rightarrow T^*\mathfrak{g}^* \) of \(J : T^*\mathfrak{g}^* \to \mathfrak{g}^*\) whose graph

\[ \rho_{\text{Ad}} : T^*\mathfrak{g}^* \times_J T^*\mathfrak{g}^* \to T^*\mathfrak{g}^* \]

is a (germ of a) Lagrangian submanifold yielding the symplectic micromorphism (as explained in [9])

\[ T^*\text{Ad} = \left\{ (X, J(\text{Ad}_{-X}(Y), \text{Ad}_X^*(\xi)), (Y, \xi), (\text{Ad}_{-X}(Y), \text{Ad}_X^*(\xi))) \mid X, Y, \xi \right\}, \]

34
which we can simplify using the equivariance of the moment map $J$:

\[
\langle X, J(\text{Ad}_-X(Y), \text{Ad}_X^*(\xi)) \rangle = \langle X, J(T^*\text{Ad}_X(Y, \xi)) \rangle = \langle X, \text{Ad}_X^*J(Y, \xi) \rangle = \langle \text{Ad}_X X, J(Y, \xi) \rangle = \langle X, J(Y, \xi) \rangle,
\]

where we have used that $[X, X] = 0$ in the Lie algebra $g$. Therefore, we obtain

\[
T^*\text{Ad} = \left\{ ((X, J(Y, \xi)), (Y, \xi), T^*\text{Ad}_X(Y, \xi)) : (X, Y, \xi) \in T^*g^* \oplus T^*g^* \right\}.
\]

Under the identification $T^*g \cong T^*g^* \cong g \times g^*$, we have that $T^*\text{Ad} = T^*\text{Ad}^*$, i.e. the cotangent lift of the adjoint action and that of the coadjoint action coincide. Thus quantizing $\triangleright_h : h \times h \to h$ should be the same as quantizing the coadjoint action $\text{Ad}_X^* : h^* \to h^*$. Observe that switching to Leibniz algebras, the adjoint action $\text{Ad}_X$ becomes a left rack action in the sense of Definition 1.18. Therefore the coadjoint action $\text{Ad}_X^*$ becomes naturally a left rack action on $h^*$ via the formula

\[
(\text{Ad}_X^*(f))(Y) := f((X \triangleright -)^{-1}Y),
\]

see Lemma 1.19. Hence the stage is set to study the object which should replace the symplectic groupoid $T^*g^* \overset{\triangleleft}{\longrightarrow} g^*$ in the context of deformation quantization of Leibniz algebras. We will do this in subsequent work.

We are now ready to quantize $T^*\triangleright : T^*h \otimes T^*h \to T^*h$. As before, the family of semi-classical FIO quantizing the symplectic micromorphism is given by

\[
Q^a(T^*\triangleright)(f \otimes g)(\xi) = \int_{g \times g} \hat{f}(X)\hat{g}(Y)a(X, Y, \xi)e^{\frac{i}{\hbar}S_\triangleright(X, Y, \xi)} \frac{dXdY}{(2\pi\hbar)^n},
\]

where $a$ is the germ of an amplitude and $\hat{f}$ and $\hat{g}$ are the asymptotic Fourier transforms.

**Theorem 4.11.** For $a = 1$, the operation

\[
\triangleright_h : C^\infty(h^*)[[\epsilon]] \otimes C^\infty(h^*)[[\epsilon]] \to C^\infty(h^*)[[\epsilon]]
\]

defined by

\[
f \triangleright_h g := Q^{a=1}(T^*\triangleright)(f \otimes g)
\]

is a quantum rack, i.e.

1. $\triangleright_h$ restricted to $U_h = \{X | X \in h\}$ is a rack structure and

\[
e^{\frac{i}{\hbar}X} \triangleright_h e^{\frac{i}{\hbar}Y} = e^{\frac{i}{\hbar}\text{conj.}(X,Y)},
\]

2. $\triangleright_h$ restricted to $\triangleright_h : U_h \times C^\infty(h^*) \to C^\infty(h^*)$ is a rack action and

\[
(e^{\frac{i}{\hbar}X} \triangleright_h f)(\xi) = (\text{Ad}^*_{-X}f)(\xi).
\]

35
Moreover, \( \triangleright_h \) coincides with the Gutt quantum rack \( f \triangleright_h g := f \ast_a g \ast_a \bar{f} \) on the restrictions in the Lie case (although it is different on the whole \( C^\infty(h^+)[[\varepsilon]] \)).

**Remark 4.12.** Actually, Property (2) in the theorem above holds also for square integrable functions, and we even obtain a unitary rack action:

\[ \triangleright_h : U_h \times L^2(h^+) \rightarrow L^2(h^+) \]

**Proof.** The first property follows from the fact that exponentials Fourier transform to delta functions:

\[
\left( e^{\pm \bar{X}} \triangleright_h e^{\pm \bar{Y}} \right)(\xi) = \int e^{\pm \bar{X} Y} e^{\pm \xi (\Ad_X(Y))} \frac{dX dY}{(2\pi \hbar)^{\dim(h)}}
\]

\[ = (2\pi \hbar)^{\dim(h)} \int \delta_{\bar{X}}(X) \delta_{\bar{Y}}(Y) e^{\pm \xi (\Ad_X(Y))} \frac{dX dY}{(2\pi \hbar)^{\dim(h)}}
\]

\[ = e^{\pm (\Ad_X(Y), \xi)} = e^{\pm (\conj_{\ast} X, Y), \xi}.
\]

Now \( \triangleright_h \) satisfies the rack identity on \( U_h \), because \( \conj_{\ast} \) does. Furthermore,

\[ E_Y \mapsto E_X \triangleright_h E_Y = E_{\conj_{\ast} (X,Y)} \]

is bijective for all \( X \in h \), because \( Y \mapsto \conj_{\ast} (X,Y) \) is. It is also clear from the formula above that this rack structure coincides with the Gutt rack structure in the case of a Lie algebra.

The second property also follows from the fact that exponentials Fourier-transform to delta functions:

\[
\left( e^{\pm \bar{X}} \triangleright_h f \right)(\xi) = \int e^{\pm \bar{X} \hat{f}(Y)} e^{\pm \xi (\Ad_X(Y))} \frac{dX dY}{(2\pi \hbar)^{\dim(h)}}
\]

\[ = (2\pi \hbar)^{(\dim(h))/2} \int \delta_{\bar{X}}(X) \hat{f}(Y) e^{\pm \xi (\Ad_X(Y))} \frac{dX dY}{(2\pi \hbar)^{\dim(h)}}
\]

\[ = \frac{1}{(2\pi \hbar)^{(\dim(h))/2}} \int \hat{f}(Y) e^{\pm \xi (\Ad_{\ast}^{-1} X, Y)} dY
\]

\[ = f(\Ad_{\ast}^{-1} X, \xi).
\]

One sees that this defines a rack action from the fact that the coadjoint action \( \Ad_{\ast}^{-1} \) is a rack action.

Let us now show that the first term of the quantized bracket is indeed the bracket \([\ ]\). For an oscillatory integral as the above expression for \( f \triangleright_h g \), there is a well defined procedure of expansion in terms of Feynman graphs, in case the integral has a unique, non-degenerate critical point. This procedure is for example explained in [13].

**Theorem 4.13.** (a) The above oscillatory integral \( f \triangleright_h g \) has a unique, non-degenerate critical point and admits thus a Feynman expansion in terms of graphs.
(b) The first term of the formal expansion of

\[
(f \triangleright \hbar g)(\xi) = \int f(\bar{\xi})g(\bar{\eta})e^{\frac{-X\xi - Y\bar{\eta} + \langle \xi, \exp(\text{ad}_X)(Y) \rangle}{\hbar}} \frac{d\bar{X}d\bar{Y}d\bar{\xi}d\bar{\eta}}{(2\pi\hbar)^n} \tag{9}
\]

in powers of \(\hbar\) is the Leibniz-Poisson bracket \([5]\), i.e.

\[\{f,g\}(\xi) = \langle \xi, [df(0), dg] \rangle\].

**Proof.** Observe that in equation (9), we wrote out explicitly the asymptotic Fourier transforms of \(f\) and \(g\). The total phase of the above oscillatory integral is thus

\[
S_\xi(X, \bar{Y}, \bar{\zeta}, \bar{\eta}) = -\bar{X}\bar{\zeta} - \bar{Y}\bar{\eta} + \langle \xi, \exp(\text{ad}_X)(\bar{Y}) \rangle.
\]

The phase \(S_\xi(X, \bar{Y}, \bar{\zeta}, \bar{\eta})\) has

\[
c_\xi = (X = 0, \bar{Y} = 0, \bar{\zeta} = 0, \bar{\eta} = \xi)
\]
as its unique critical point. This means that for any given \(\xi\), \(c_\xi\) is unique within the points \(c := (\bar{X}, \bar{Y}, \bar{\zeta}, \bar{\eta})\) such that

\[
\frac{\partial S_\xi}{\partial X}(c) = 0, \quad \frac{\partial S_\xi}{\partial Y}(c) = 0, \quad \frac{\partial S_\xi}{\partial \zeta}(c) = 0, \quad \frac{\partial S_\xi}{\partial \eta}(c) = 0.
\]

The critical point \(c_\xi\) is easily computed from the partial derivatives. It turns out that

\[
\frac{\partial S_\xi}{\partial X}(c) = -\bar{\zeta} + T_1, \quad \frac{\partial S_\xi}{\partial Y}(c) = -\bar{\eta} + T_2, \quad \frac{\partial S_\xi}{\partial \zeta}(c) = -\bar{Y}, \quad \frac{\partial S_\xi}{\partial \eta}(c) = -\bar{X},
\]

where the term \(T_1\) is the derivative of \(\bar{X} \mapsto \langle \xi, \exp(\text{ad}_X)(\bar{Y}) \rangle\) and the term \(T_2\) is the derivative of \(\bar{Y} \mapsto \langle \xi, \exp(\text{ad}_X)(\bar{Y}) \rangle\). One concludes from setting the third and fourth equation equal to zero that \(\bar{X} = \bar{Y} = 0\). The first term of \(Y \mapsto \langle \xi, \exp(\text{ad}_X)(\bar{Y}) \rangle\) is \(\bar{Y}\bar{\zeta}\), thus the constant term in \(T_2\) is \(\xi\). All other terms in \(T_1\) and \(T_2\) are zero at the critical point due to \(\bar{X} = \bar{Y} = 0\). In conclusion \(c_\xi = (X = 0, \bar{Y} = 0, \bar{\zeta} = 0, \bar{\eta} = \xi)\).

The Hessian of \(S_\xi\) at the critical point \(c_\xi\) reads in block notation

\[
D^2S_\xi(c_\xi) = \begin{pmatrix}
0 & c_{ij}^k \xi_k & -1 & 0 \\
-c_{ij}^k \xi_k & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

where \(c_{ij}^k\) are the structure constants of the Leibniz algebra \(\mathfrak{h}\) and in Einstein convention, the sum over repeated indices is understood.

Denoting the matrix \(D^2S_\xi(c_\xi)\) simply by \(B\), it is evident that \(\det(B) = 1\), thus the critical point \(c_\xi\) is non-degenerate. Moreover, the signature of \(B\) is 0.
The Feynman expansion (cf. [13]) therefore reads

\[ I(\hbar) = (f \triangleright_{\hbar} g)(\xi) = \frac{\hbar \text{sign}(B)}{\sqrt{|\det(B)|}} \sum_{\Gamma \in G_{3 \geq 2}} (ih)^{|E_{\Gamma}| - |V_{\Gamma}^{\text{int}}|} F_{\Gamma}(S_{\xi}; f, g) \]

\[ = e^{\frac{\pi}{4}(\sum_{\Gamma \in G_{3 \geq 2}} (ih)^{|E_{\Gamma}| - |V_{\Gamma}^{\text{int}}|} F_{\Gamma}(S_{\xi}; f, g))}. \]

These sums are sums over the set \( G_{3 \geq 2} \) of Feynman graphs \( \Gamma \) with 2 external vertices and internal vertices of valence greater or equal to 3. For the definition of a Feynman graph, we refer the reader to [13]. \( |E_{\Gamma}| \) is the cardinality of the set of edges of \( \Gamma \), \( V_{\Gamma}^{\text{int}} \) is the set of internal vertices of \( \Gamma \), \( \text{Aut}(\Gamma) \) is the number of symmetries of \( \Gamma \). To each \( \Gamma \), one associates an amplitude \( F_{\Gamma}(S_{\xi}; f, g) \) in a way which is specified in loc. cit.. Namely, \( F_{\Gamma}(S_{\xi}; f, g) \) is a product of two partial derivatives of \( S_{\xi} \) (represented by the internal vertices) and partial derivatives of \( f \) and \( g \) (represented by the external vertices) all of which are evaluated at the critical point \( c_{\xi} \) and contracted using the matrix \( B^{-1} \).

The first terms of the expansion of (9) in powers of \( \hbar \) read therefore

\[ (f \triangleright_{\hbar} g)(\xi) = f(0)g(\xi) + \frac{i}{\hbar} \{f, g\}(\xi) + O(\hbar), \]

where

\[ \{f, g\}(\xi) = -\sum_{i,j,k} c_{i,j}^{k} \frac{\partial f}{\partial \xi_{i}}(0) \frac{\partial g}{\partial \xi_{j}}(\xi) \xi_{k}, \]

as in formula (5).

\[ \square \]

Remark 4.14. (a) It is rather straightforward to compute the terms in this starproduct, the graphs which we have to consider are rather easy. For example, there are no inner loops.

(b) The zeroth term of the expansion, i.e. the product \( f \otimes g \rightarrow f(0)g(\xi) \), is actually associative.

References

[1] Bayen, F.; Flato, M.; Fronsdal, C.; Lichnerowicz, A.; Sternheimer, D. Deformation theory and quantization. I. Deformations of symplectic structures. Ann. Physics 111 (1978), no. 1, 61–110

[2] Albert, C.; Dazord, P. Théorie des groupoïdes symplectiques. Chapitre II. Groupoïdes symplectiques. Publications du Département de Mathématiques. Nouvelle série, 27–99, Publ. Dép. Math. Nouvelle Sér., 1990, Univ. Claude-Bernard, Lyon, 1990
[3] Ben Amar, N. *K-star products on dual of Lie algebras*. J. Lie Theory **13** (2003) 329–357

[4] Ben Amar, N. *A comparison between Rieffel’s and Kontsevich’s deformation quantizations for linear Poisson tensors*. Pacific J. Math. **229** (2007), no. 1, 1–24

[5] Bourbaki, N. *Lie groups and Lie algebras. Chapters 1–3*. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998

[6] Canez, Santiago Valencia *Double Groupoids, Orbifolds, and the Symplectic Category*. Thesis (Ph.D.), University of California, Berkeley, 2011

[7] Cattaneo, Alberto S.; Dherin, Benoît; Felder, Giovanni *Formal symplectic groupoid*. Comm. Math. Phys. **253** (2005), no. 3, 645–674

[8] Cattaneo, Alberto S.; Dherin, Benoît; Weinstein, Alan *Symplectic microgeometry I: micromorphisms*. J. Symplectic Geom. **8** (2010), no. 2, 205–223

[9] Cattaneo, Alberto S.; Dherin, Benoît; Weinstein, Alan *Symplectic microgeometry II: generating functions*. Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 4, 507–536

[10] Cattaneo, Alberto S.; Dherin, Benoît; Weinstein, Alan *Symplectic microgeometry III: monoids*. J. Symplectic Geom. **11** (2013), no. 3, 319–341

[11] Cattaneo, Alberto S.; Dherin, Benoît; Weinstein, Alan *Symplectic microgeometry IV: quantization* (in preparation)

[12] Covez, Simon *L’intégration locale des algèbres de Leibniz*. PhD Thesis, Nantes 2010 (see also his article on the same subject: *The local integration of Leibniz algebras*. Ann. Inst. Fourier (Grenoble) **63** (2013), no. 1, 1–35.)

[13] Dherin, Benoît; Mencattini, Igor *Quantizations of Momentum Maps and G-Systems* arXiv:1212.6489

[14] Fenn, Roger; Rourke, Colin *Racks and links in codimension two*. J. Knot Theory Ramifications 1 (1992), no. 4, 343–406

[15] Grabowski, Janusz; Marmo, Giuseppe *Non-antisymmetric versions of Nambu-Poisson and algebroid brackets*. J. Phys. A **34** (2001), no. 18, 3803–3809

[16] Gutt, Simone *An explicit *–product on the cotangent bundle of a Lie group*. Lett. Math. Phys. **7** (1983), no. 3, 249–258

[17] Kinyon, Michael *Leibniz algebras, Lie racks, and digroups*. J. Lie Theory **17** (2007) no. 1, 99–114
[18] Kinyon, Michael; Weinstein, Alan. *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*. Amer. J. Math. **123** (2001) no. 3, 525–550

[19] Kontsevich, Maxim. *Deformation quantization of Poisson manifolds*. Lett. Math. Phys. **66** (2003), no. 3, 157–216

[20] Neeb, Karl-Hermann. *Central extensions of infinite-dimensional Lie groups*. Ann. Inst. Fourier (Grenoble) **52** (2002), no. 5, 1365–1442

[21] Li-Bland, David; Severa, Pavol. *Integration of Exact Courant Algebroids*. Electron. Res. Announc. Math. Sci. **19** (2012) 58–76

[22] Roytenberg, Dmitry. *On weak Lie 2-algebras*. XXVI Workshop on Geometrical Methods in Physics, 180–198, AIP Conf. Proc., 956, Amer. Inst. Phys., Melville, NY, 2007

[23] Tuynman, Ghys. *A proof of Lie’s third theorem*. Pub. IRMA Lille Vol. **34**, no. X, 1994.

[24] Uchino, Kyousuke. *Noncommutative Poisson brackets on Loday algebras and related deformation quantization*. J. Symplectic Geom. **11** (2013), no. 1, 93–108

[25] Varadarajan, V.S. *Lie Groups, Lie Algebras and Their Representations*, Springer GTM **102**, Springer New York 1974

[26] Weinstein, Alan. *Symplectic categories*. Port. Math. **67** (2010), no. 2, 261–278

[27] Weinstein, Alan. *The symplectic category*. Differential Geometric Methods in Mathematical Physics (Clausthal, 1980), pp. 45–51, Lecture Notes in Math., 905, Springer, 1982.

[28] Weinstein, Alan. *Noncommutative geometry and geometric quantization*. Symplectic Geometry and Mathematical Physics (Aix-en-Provence, 1990), 446–461, Progr. Math. **99**, 1991.

[29] Zakrzewski, Stanislaw. *Quantum and classical pseudogroups I and II*. Comm. Math. Phys. **134** (1990), 347–370