CROSSED PRODUCTS BY LEFT LCM SEMIGROUPS OF
ENDOMORPHISMS

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ABSTRACT. Let $P$ be a left LCM semigroup, and $\alpha$ an action of $P$ by endomorphisms of a $C^*$-algebra $A$. We study a semigroup crossed product $C^*$-algebra in which the action $\alpha$ is implemented by partial isometries. This crossed product gives a model for the Nica-Teoplitz algebras of product systems of Hilbert bimodules (associated with semigroup dynamical systems) studied first by Fowler, for which we provide a structure theorem as it behaves well under short exact sequences and tensor products.

1. Introduction

Let $P$ be a unital semigroup whose unit element is denoted by $e$. Suppose that $(A,P,\alpha)$ is a dynamical system consisting of a $C^*$-algebra $A$, and an action $\alpha : P \to \text{End}(A)$ of $P$ by endomorphisms of $A$ such that $\alpha_e = \text{id}_A$. Note that, since the $C^*$-algebra $A$ is not necessarily unital, we need to assume that each endomorphism $\alpha_x$ is extendible, which means that it extends to a strictly continuous endomorphism $\pi_x$ of the multiplier algebra $\mathcal{M}(A)$. Recall that an endomorphism $\alpha$ of $A$ is extendible if and only if there exists an approximate identity $\{a_\lambda\}$ in $A$ and a projection $p \in \mathcal{M}(A)$ such that $\alpha(a_\lambda)$ converges strictly to $p$ in $\mathcal{M}(A)$. However, the extendibility of $\alpha$ does not necessarily imply $\pi(1_{\mathcal{M}(A)}) = 1_{\mathcal{M}(A)}$.

The study of $C^*$-algebras associated with semigroups and semigroup dynamical systems continues fascinating mathematicians (operator algebraists). In the line of quite huge efforts in this regard, Fowler in [8], for the dynamical system $(A,P,\alpha)$, where $P$ is the positive cone of a group $G$ such that $(G,P)$ is quasi-lattice ordered in the sense of Nica [17], defined a covariant representation called the Nica-Toeplitz covariant representation of the system, such that the endomorphisms $\alpha_x$ are implemented by partial isometries. He then showed that there exists a universal $C^*$-algebra $\mathcal{T}_{\text{cov}}(X)$ associated with the system generated by a universal Nica-Toeplitz covariant representation of the system such that there is a bijection between the Nica-Toeplitz covariant representations of the system and the nondegenerate representations of $\mathcal{T}_{\text{cov}}(X)$. To be more precise, $X$ is actually the product system of Hilbert bimodules associated with the system $(A,P,\alpha)$ introduced by him, and the algebra $\mathcal{T}_{\text{cov}}(X)$ is universal for Toeplitz representations of $X$ satisfying a covariance condition called Nica covariance. He called this universal algebra the Nica-Toeplitz crossed product (or Nica-Toeplitz algebra) of the system $(A,P,\alpha)$ and denoted it by $\mathcal{T}_{\text{cov}}(A \times_\alpha P)$. When the group $G$ is totally ordered and abelian (with the positive cone $G^+ = P$), the Nica covariance condition holds automatically, and the Toeplitz algebra $\mathcal{T}(X)$

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is the partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} P$ of the system $(A, P, \alpha)$ introduced and studied by the authors of [16]. In other word, the semigroup crossed product $A \times_{\alpha}^{\text{piso}} P$ actually gives a model for the Toeplitz algebras $T(X)$ of product systems $X$ of Hilbert bimodules associated with the systems $(A, P, \alpha)$, where $P$ is the positive cone of a totally ordered abelian group $G$. Further studies on the structure of the crossed product $A \times_{\alpha}^{\text{piso}} P$ have been done progressively in [2, 3, 4, 13], and [22] since then.

In the very recent years, mathematicians (operator algebraists) in [6, 11, 12], following the idea of Fowler, have extended and studied the notion of the Nica-Toeplitz algebra of a product system $X$ over more general semigroups $P$, namely, right LCM semigroups (see also [7]). These are the semigroups that appear as a natural generalization of the well-known notion of quasi-latticed ordered groups introduced first by Nica in [17]. Recall that the notation $\mathcal{N}T(X)$ is used for the Nica-Toeplitz algebra of $X$ in [6, 11, 12], which are the works that brought this question to our attention that whether we could define a partial-isometric crossed product corresponding to the system $(A, P, \alpha)$, where the semigroup $P$ goes beyond the positive cones of totally ordered abelian groups. Although, based on the work of Fowler in [8], we were already aware that the answer to this question must be “yes” for the positive cones $P$ of quasi-latticed ordered groups $(G, P)$, [6, 11, 12] made us very enthusiastic to seek even more than that. Hence, the initial investigations in the present work indicated that the semigroup $P$ must be left LCM (see [2]). More precisely, in the dynamical system $(A, P, \alpha)$, we considered the semigroup $P$ to be left LCM (so, the opposite semigroup $P^o$ becomes right LCM). Then, following [8], we defined a covariant representation of the system satisfying a covariance condition called the (right) Nica covariance, in which the endomorphisms $\alpha_x$ are implemented by partial isometries. We called this representation the covariant partial-isometric representation of the system. More importantly, we showed that every system $(A, P, \alpha)$ admits a nontrivial covariant partial-isometric representation. Next, we proved that the Nica-Toeplitz algebra $\mathcal{N}T(X)$ of the product system $X$ associated with the dynamical system $(A, P, \alpha)$ is generated by a covariant partial-isometric representation of the system which is universal for covariant partial-isometric representations of the system. We called this universal algebra the partial-isometric crossed product of the system $(A, P, \alpha)$ and denoted it by $A \times_{\alpha}^{\text{piso}} P$ (following [16]), which is unique up to isomorphism. We then studied the behavior of crossed product $A \times_{\alpha}^{\text{piso}} P$ under short exact sequences and tensor products, from which, a structure theorem followed. In addition, as an example, when $P$ and $P^o$ are both left LCM semigroups we studied the distinguished system $(B_P, P, \tau)$, where $B_P$ is the $C^*$-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions $\{1_y : y \in P\}$, and the action $\tau$ on $B_P$ is induced by the shift on $\ell^\infty(P)$. It was shown that the algebra $B_P \times_{\tau}^{\text{piso}} P$ is universal for bicovariant partial-isometric representations of $P$, which are the partial-isometric representations of $P$ satisfying both right and left Nica covariance conditions.

Here, prior to talking about the organization of the present work, we would like to mention that, by [21], if $P$ is the positive cone of an abelian lattice-ordered group $G$, then the Nica-Toeplitz algebra $\mathcal{T}_{\text{cov}}(A \times_{\alpha} P)$ of the system $(A, P, \alpha)$ is a full corner in a classical crossed product by the group $G$. Thus, by the present work, since $A \times_{\alpha}^{\text{piso}} P \simeq \mathcal{T}_{\text{cov}}(A \times_{\alpha} P)$, the same corner realization holds for the partial-isometric
crossed products of the systems \((A, P, \alpha)\) consisting of the positive cones \(P\) of abelian lattice-ordered groups (see also [22]).

Now, the present work as an extension of the idea in [16] follows also the framework of [15] for partial-isometric crossed products. We begin with a preliminary section containing a summary on LCM semigroups and discrete product systems of Hilbert bimodules. In section 3 and 4, for the system \((A, P, \alpha)\) with a left LCM semigroup \(P\), a covariant representation of the system is defined which satisfies a covariance condition called the \((right)\) Nica covariance, where the endomorphisms \(\alpha_x\) are implemented by partial isometries. This representation is called the covariant partial-isometric representation of the system. We also provide an example which shows that every system admits a nontrivial covariant partial-isometric representation. Then, we show that there is a \(C^*\)-algebra \(B\) associated with the system generated by a covariant partial-isometric representation of the system which is universal for covariant partial-isometric representations of the system, in the sense that there is a bijection between the covariant partial-isometric representations of the system and the nondegenerate representations of the \(C^*\)-algebra \(B\). This universal algebra \(B\) is called the partial-isometric crossed product of the system \((A, P, \alpha)\) and denoted by \(A \times_{\alpha} \text{piso} P\), which is unique up to isomorphism. We also show that this crossed product behaves well under short exact sequences. In section 5 we show that under some certain conditions the crossed product \((A \otimes_{\text{max}} B) \times_{\text{piso}} (P \times S)\) can be decomposed as the maximal tensor product of the crossed products \(A \times_{\text{piso}} B\) and \(B \times_{\text{piso}} S\). Also, when \(P\) and the opposite semigroup \(P^o\) are both left LCM we consider the distinguished system \((B_P, P, \tau)\), where \(B_P\) is the \(C^*\)-subalgebra of \(\ell^\infty(P)\) generated by the characteristic functions \(\{1_y : y \in P\}\), and the action \(\tau\) on \(B_P\) is induced by the shift on \(\ell^\infty(P)\). Note that each \(1_y\) is actually the characteristic function of the right ideal \(yP = \{yx : x \in P\}\) in \(P\). We then show that the crossed product \(B_P \times_{\tau} \text{piso} P\) is universal for bicovariant partial-isometric representations of \(P\), which are the partial-isometric representations of \(P\) satisfying both right and left Nica covariance conditions. In section 6 for the crossed product \((A \otimes_{\text{max}} B) \times_{\text{piso}} P\) a composition series
\[
0 \leq I_1 \leq I_2 \leq (A \otimes_{\text{max}} B) \times_{\text{piso}} P
\]
of ideals is obtained, for which we identify the subquotients
\[
I_1, \quad I_2/I_1, \quad \text{and} \quad ((A \otimes_{\text{max}} B) \times_{\text{piso}} P)/I_2
\]
with familiar terms.

2. Preliminaries

2.1. LCM semigroups. Let \(P\) be a unital (discrete) semigroup, which means that there is an element \(e \in P\) such that \(xe = ex = x\) for all \(x \in P\). Recall that \(P\) is called right cancellative if \(xz = yz\), then \(x = y\) for every \(x, y, z \in P\).

Definition 2.1. A unital semigroup \(P\) is called left LCM (least common multiple) if it is right cancellative and for every \(x, y \in P\), we have either \(Px \cap Py = \emptyset\) or \(Px \cap Py = Pz\) for some \(z \in P\).

Let \(P^*\) denote the set of all invertible elements of \(P\), which is obviously not empty as \(e \in P^*\). In fact, \(P^*\) is a group with the action inherited from \(P\). Now, if \(Px \cap Py = Pz\), since \(z = ez \in Pz\), we have \(sx = z = ty\) for some \(s, t \in P\). So, \(z\) can be viewed
as a least common left multiple of $x, y$. However, such a least common left multiple may not be unique. Actually one can see that if $z$ and $\tilde{z}$ are both least common left multiples of $x, y$, then there is an invertible element $u$ of $P$ ($u \in P^*$) such that $\tilde{z} = uz$.

Note that right LCM semigroups are define similarly, for which similar facts are valid as above. In addition, we let $P^o$ denote the opposite semigroup endowed with the action $\ast$ such that $x \ast y = yx$ for all $x, y \in P^o$. Clearly, $P$ is a left LCM semigroup if and only if $P^o$ is a right LCM semigroup.

### 2.2. Discrete product systems of Hilbert bimodules.

A Hilbert bimodule over a $C^*$-algebra $A$ is a right Hilbert $A$-module $X$ together with a homomorphism $\phi : A \rightarrow \mathcal{L}(X)$ which defines a left action of $A$ on $X$ by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in X$. A Toeplitz representation of $X$ in a $C^*$-algebra $B$ is a pair $(\psi, \pi)$ consisting of a linear map $\psi : X \rightarrow B$ and a homomorphism $\pi : A \rightarrow B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(x)\ast \psi(y) = \pi((x,y)_A), \quad \text{and} \quad \psi(a \cdot x) = \pi(a)\psi(x)$$

for all $a \in A$ and $x, y \in X$. Then, there is a homomorphism (Pimsner homomorphism) $\rho : \mathcal{K}(X) \rightarrow B$ such that

$$\rho(\Theta_{x,y}) = \psi(x)\psi(y)$$

for all $x, y \in X$.

The Toeplitz algebra of $X$ is the $C^*$-algebra $T(X)$ which is universal for Toeplitz representations of $X$ (see [19] [9]).

Recall that every right Hilbert $A$-module $X$ is essential, which means that we have

$$X = \overline{\text{span}}\{x \cdot a : x \in X, a \in A\}.$$

Moreover, a Hilbert bimodule $X$ over a $C^*$-algebra $A$ is called essential if

$$X = \overline{\text{span}}\{a \cdot x : a \in A, x \in X\} = \overline{\text{span}}\{\phi(a)x : a \in A, x \in X\},$$

which means that $X$ is also essential as a left $A$-module.

Now, let $A$ be a $C^*$-algebra and $S$ a unital (countable) discrete semigroup. We recall from [8] that the disjoint union $X = \bigsqcup_{s \in S} X_s$ of Hilbert bimodules $X_s$ over $A$ is called a (discrete) product system over $S$ if there is a multiplication

$$(x, y) \in X_s \times X_t \mapsto xy \in X_{st}$$

on $X$, with which, $X$ is a semigroup, and the map $[2.2]$ extends to an isomorphism of the Hilbert bimodules $X_s \otimes_A X_t$ and $X_{st}$ for all $s, t \in S$ with $s \neq e$. The bimodule $X_e$ is $AA$, and the multiplications $X_e \times X_s \mapsto X_s$ and $X_s \times X_e \mapsto X_s$ are just given by the module actions of $A$ on $X_s$. Note that also we write $\phi_s : A \rightarrow \mathcal{L}(X_s)$ for the homomorphism which defines the left action of $A$ on $X_s$.

Note that, for every $s, t \in S$ with $s \neq e$, there is a homomorphism $\iota_{st}^s : \mathcal{L}(X_s) \rightarrow \mathcal{L}(X_{st})$ characterized by

$$\iota_{st}^s(T)(xy) = (Tx)y$$

for all $x \in X_s$, $y \in X_t$ and $T \in \mathcal{L}(X_s)$. In fact, $\iota_{st}^s(T) = T \otimes \text{id}_{X_t}$.

A Toeplitz representation of the product system $X$ in a $C^*$-algebra $B$ is a map $\psi : X \rightarrow B$ such that

1. $\psi_s(x)\psi_t(y) = \psi_{st}(xy)$ for all $s, t \in S$, $x \in X_s$, and $y \in X_t$; and
2. the pair $(\psi_s, \psi_e)$ is a Toeplitz representation of $X_s$ in $B$ for all $s \in S$,
where $\psi_s$ denotes the restriction of $\psi$ to $X_s$. For every $s \in S$, let $\psi^{(s)} : \mathcal{K}(X_s) \to B$ be the Pimsner homomorphism corresponding to the pair $(\psi_s, \psi_e)$ defined by

$$\psi^{(s)}(\Theta_{x,y}) = \psi_s(x)\psi_s(y)^*$$

for all $x, y \in X_s$ (see (2.1)).

By [8], Proposition 2.8, for every product system $X$ over $S$, there is a $C^*$-algebra $\mathcal{T}(X)$, called the Toeplitz algebra of $X$, which is generated by a universal Toeplitz representation $i_X : X \to \mathcal{T}(X)$ of $X$. The pair $(\mathcal{T}(X), i_X)$ is unique up to isomorphism, and $i_X$ is isometric.

Next, we recall that for any quasi-lattice ordered group $(G, S)$, the notions of compactly aligned product system over $S$ and Nica covariant Toeplitz representation of it were introduced by Fowler in [8]. Then, authors in [6] extended these notions to product systems over right LCM semigroups. Suppose that $S$ is a (unital) right LCM semigroup. A product system $X$ over $S$ of (essential) Hilbert bimodules is called compactly aligned if for all $r, t \in S$ such that $rS \cap tS = sS$ for some $s \in S$ we have

$$i^{(s)}(R)i^{(t)}(T) \in \mathcal{K}(X_s)$$

for all $R \in \mathcal{K}(X_r)$ and $T \in \mathcal{K}(X_t)$. Let $X$ be a compactly aligned product system over a right LCM semigroup $S$ of (essential) Hilbert bimodules, and $\psi : X \to B$ a Toeplitz representation of $X$ in a $C^*$-algebra $B$. Then, $\psi$ is called Nica covariant if

$$(2.3) \quad \psi^{(r)}(R)\psi^{(t)}(T) = \begin{cases} \psi^{(s)}(i^{(s)}(R)i^{(t)}(T)) & \text{if } rS \cap tS = sS, \\ 0 & \text{if } rS \cap tS = \emptyset \end{cases}$$

for all $r, t \in S$, $R \in \mathcal{K}(X_r)$ and $T \in \mathcal{K}(X_t)$.

For a compactly aligned product system $X$ over a right LCM semigroup $S$ of (essential) Hilbert bimodules, the Nica-Toeplitz algebra is the $C^*$-algebra generated by a Nica covariant Toeplitz representation $i_X : X \to \mathcal{N}\mathcal{T}(X)$ which is universal for Nica covariant Toeplitz representations of $X$, in the sense that: for every Nica covariant Toeplitz representation of $\psi$ of $X$, there is a representation $\psi_s$ of $\mathcal{N}\mathcal{T}(X)$ such that $\psi_s \circ i_X = \psi$ (see [8, 6, 11]).

3. NICA PARTIAL-ISOMETRIC REPRESENTATIONS

Let $P$ be a left LCM semigroup. A Nica partial-isometric representation of $P$ on a Hilbert space $H$ is a map $V : P \to B(H)$ such that each $V_x := V(x)$ is a partial isometry, and the map $V$ is a unital semigroup homomorphism of $P$ into the multiplicative semigroup $B(H)$ which satisfies

$$(3.1) \quad V_x^*V_yV_y^*V_x = \begin{cases} V_x^*V_y & \text{if } P_x \cap P_y = P_z, \\ 0 & \text{if } P_x \cap P_y = \emptyset. \end{cases}$$

The equation (3.1) is called the Nica covariance condition. Of course, since the least common left multiple $z$ may not be unique, we must check that whether the Nica covariance condition is well-defined. So, assume that $P_z = P_x \cap P_y = P\tilde{z}$. If follows that

$V_x^*V_yV_y^*V_x = V_z^*V_z$

and

$V_x^*V_yV_y^*V_z = V_{\tilde{z}}^*V_{\tilde{z}}$. 
Now, since \( \tilde{z} = uz \) for some invertible element \( u \) of \( P \), we have
\[
V^*_{\tilde{z}}V_{\tilde{z}} = V^* uz Vuz = (V_u V_{\tilde{z}})^* V_u V_{\tilde{z}} = V^*_{\tilde{z}} V^*_u V_u V_{\tilde{z}}.
\]
But it is not difficult to see that \( V_u \) is actually a unitary, and therefore,
\[
V^*_{\tilde{z}} V_{\tilde{z}} = V^*_z V_z.
\]
This implies that equation (3.1) is indeed well-defined.

**Example 3.1.** Suppose that \( P \) is a left LCM semigroup and \( H \) is a Hilbert space. Define a map \( S : P \to B(\ell^2(P) \otimes H) \) by
\[
(S_y f)(x) = \begin{cases} f(r) & \text{if } x = ry \text{ for some } r \in P, \\ 0 & \text{otherwise.} \end{cases}
\]
for every \( f \in \ell^2(P) \otimes H \). Note that \( x = ry \) for some \( r \in P \) is equivalent to saying that \( x \in Py \). Moreover, if \( sy = x = ry \) for some \( r,s \in P \), then \( s = r \) by the right cancellativity of \( P \), and hence \( f(r) = f(s) \). This implies that each \( S_y \) is well-defined. One can see that each \( S_y \) is a linear operator. We claim that each \( S_y \) is actually an isometry and in particular, \( S_{e} = 1 \). We have
\[
\|S_y f\|^2 = \sum_{x \in P} \|(S_y f)(x)\|^2 = \sum_{r \in P} \|(S_y f)(ry)\|^2 = \sum_{r \in P} \|f(r)\|^2 = \|f\|^2,
\]
which implies that each \( S_y \) is an isometry. In particular,
\[
(S_e f)(x) = (S_e f)(xe) = f(x),
\]
which shows that \( S_{e} = 1 \). In addition, some simple calculation shows that
\[
S_x S_y = S_{xy} = S_{x\cdot y} \quad \text{for all } x,y \in P.
\]
Now we want to show that the adjoint of each \( S_y \) is given by
\[
(W_y f)(x) = f(xy)
\]
for all \( f \in \ell^2(P) \otimes H \). For every \( f, g \in \ell^2(P) \otimes H \), we have
\[
\langle S_y f | g \rangle = \sum_{x \in P} \langle (S_y f)(x) | g(x) \rangle = \sum_{r \in P} \langle (S_y f)(ry) | g(ry) \rangle = \sum_{r \in P} \langle f(r) | g(ry) \rangle = \sum_{r \in P} \langle f(r) | (W_y g)(r) \rangle = \langle f | W_y g \rangle.
\]
So, \( S^*_y = W_y \) for every \( y \in P \). Also, for every \( x,y \in P \), by applying (3.2), we get
\[
W_x W_y = S_x^* S_y^* = [S_y S_x]^* = S_{xy}^* = W_{xy}.
\]
Therefore, since each \( W_x \) is clearly a partial-isometry, if follows that the map \( W : P \to B(\ell^2(P) \otimes H) \) define by \( (W_y f)(x) = f(xy) \) is a partial-isometric representation of \( P \) on \( \ell^2(P) \otimes H \). We claim that the representation \( W \) satisfies the Nica covariance condition (3.1). Firstly,
\[
W_x^* W_y W_x^* W_y = S_x^* S_y^* S_y S_y^*.
\]
Then, for every \( f \in \ell^2(P) \otimes H \), we have
\[
(S_x^*S_yf)(r) = (S_x^*(S_yf))(r) = (S_yf)(rx) \quad \text{for all } r \in P. 
\]

Now, if \( Px \cap Py = \emptyset \), since \( rx \in Px \), it follows that \( rx \notin Py \). Therefore,
\[
(S_yf)(rx) = 0,
\]
which implies that equation (3.3) must be zero. Suppose that we have the other case, \( Px \cap Py = Pz \). First note that if \( \{\varepsilon_s : s \in P\} \) is the usual orthonormal basis of \( \ell^2(P) \), then each \( S_yS_y^* \) is a projection onto the closed subspace \( \ell^2(Py) \otimes H \) of \( \ell^2(P) \otimes H \) spanned by
\[
\{\varepsilon_{sy} \otimes h : s \in P, h \in H\},
\]
which is indeed equal to the \( \ker(1 - S_yS_y^*) \). Therefore,
\[
S_xS_x^*S_yS_y^* 
\]
is a projection onto the closed subspace
\[
\overline{\text{span}}\{\varepsilon_{sy} \otimes h : s \in P, h \in H, \text{ and } sy \in Px\} 
\]
= \( \overline{\text{span}}\{\varepsilon_r \otimes h : r \in Py \cap Px, h \in H\} \)
= \( \overline{\text{span}}\{\varepsilon_r \otimes h : r \in Pz, h \in H\} \)
= \( \overline{\text{span}}\{\varepsilon_{tz} \otimes h : t \in P, h \in H\} = \ell^2(Pz) \otimes H. \)

See the following diagram:
\[
(3.5) \quad \ell^2(P) \otimes H \xrightarrow{S_xS_y^*} \ell^2(Py) \otimes H \xrightarrow{S_xS_y^*} \ell^2(Py \cap Px) \otimes H = \ell^2(Pz) \otimes H.
\]

Therefore, we must have
\[
S_xS_x^*S_yS_y^* = S_zS_z^*,
\]
from which, for equation (3.3), we get
\[
W_zW_xW_y = S_zS_z^* = W_zW_z.
\]
Consequently, \( W \) is indeed a Nica partial-isometric representation.

4. Partial-isometric crossed products

4.1. Covariant partial-isometric representations. Let \( P \) be a left LCM semigroup, and \((A, P, \alpha)\) a dynamical system consisting of a \( C^*\)-algebra \( A \), and an action \( \alpha : P \to \text{End}(A) \) of \( P \) by extendible endomorphisms of \( A \) such that \( \alpha_e = \text{id}_A \).

Definition 4.1. A covariant partial-isometric representation of \((A, P, \alpha)\) on a Hilbert space \( H \) is a pair \((\pi, V)\) consisting of a nondegenerate representation \( \pi : A \to B(H) \) and a Nica partial-isometric representation \( V : P \to B(H) \) of \( P \) such that
\[
(4.1) \quad \pi(\alpha_x(a)) = V_x\pi(a)V_x^* \quad \text{and} \quad V_x^*V_x\pi(a) = \pi(a)V_x^*V_x
\]
for all \( a \in A \) and \( x \in P \).

Lemma 4.2. Every covariant partial-isometric pair \((\pi, V)\) extends to a covariant partial-isometric representation \((\overline{\pi}, V)\) of the system \((M(A), P, \overline{\alpha})\), and (3.4) is equivalent to
\[
(4.2) \quad \pi(\alpha_x(a))V_x = V_x\pi(a) \quad \text{and} \quad V_xV_x^* = \overline{\pi(\alpha_x(1))}
\]
for all \( a \in A \) and \( x \in P \).
Proof. The proof is standard. So, we skip it here. □

Following example shows that every dynamical system \((A,P,\alpha)\) admits a nontrivial (nonzero) covariant partial-isometric representation.

Example 4.3. Suppose that \((A,P,\alpha)\) is a dynamical system, and \(\pi_0 : A \rightarrow B(H)\) is a nondegenerate representation of \(A\) on a Hilbert space \(H\). Define a map \(\pi : A \rightarrow B(\ell^2(P) \otimes H)\) by

\[
(\pi(a)f)(x) = \pi_0(\alpha_x(a))f(x)
\]

for all \(a \in A\) and \(f \in \ell^2(P) \otimes H \cong \ell^2(P,H)\). One can see that \(\pi\) is a representation of \(A\) on the Hilbert space \(\ell^2(P) \otimes H\). Let \(q : \ell^2(P) \otimes H \rightarrow \ell^2(P) \otimes H\) be a map defined by

\[
(qf)(x) = \pi_0(\alpha_x(1))f(x)
\]

for all \(f \in \ell^2(P) \otimes H\). It is not difficult to see that \(q \in B(\ell^2(P) \otimes H)\), which is actually a projection onto a closed subspace \(\mathcal{H}\) of \(\ell^2(P) \otimes H\). We claim that if \(\{a_i\}\) is any approximate unit in \(A\), then \(\pi(a_i)\) converges strictly to \(q\) in \(\mathcal{M}(\mathcal{K}(\ell^2(P) \otimes H)) = B(\ell^2(P) \otimes H)\). To do so, since the net \(\{\pi(a_i)\}\) is a norm bounded subset of \(B(\ell^2(P) \otimes H)\), and \(\pi(a_i)^* = \pi(a_i)\) for each \(i\) as well as \(q^* = q\), by [20] Proposition C.7, we only need to show that \(\pi(a_i) \rightarrow q\) strongly in \(B(\ell^2(P) \otimes H)\). If \(\{\varepsilon_x : x \in P\}\) is the usual orthonormal basis of \(\ell^2(P)\), then it is enough to see that

\[
\pi(a_i)(\varepsilon_x \otimes \pi_0(a)h) \rightarrow q(\varepsilon_x \otimes \pi_0(a)h)
\]

for each spanning element \((\varepsilon_x \otimes \pi_0(a)h)\) of \(\ell^2(P) \otimes H\) (recall that \(\pi_0\) is nondegenerate). We have

\[
\pi(a_i)(\varepsilon_x \otimes \pi_0(a)h) = \varepsilon_x \otimes \pi_0(\alpha_x(a_i))\pi_0(a)h = \varepsilon_x \otimes \pi_0(\alpha_x(a))h,
\]

which is convergent to

\[
\varepsilon_x \otimes \pi_0(\alpha_x(1)a)h = \varepsilon_x \otimes \pi_0(\alpha_x(1))\pi_0(a)h = q(\varepsilon_x \otimes \pi_0(a)h)
\]

in \(\ell^2(P) \otimes H\). This is due to the extendibility of each \(\alpha_x\). Thus, \(\pi(a_i) \rightarrow q\) strictly in \(B(\ell^2(P) \otimes H)\).

Next, let \(W : P \rightarrow B(\ell^2(P) \otimes H)\) be the Nica partial-isometric representation introduced in Example 3.1. We aim at constructing a covariant partial-isometric representation \((\rho, V)\) of \((A,P,\alpha)\) on the Hilbert space (closed subspace) \(\mathcal{H}\) by using the pair \((\pi,W)\). Note that, in general, \(\pi\) is not nondegenerate on \(\ell^2(P) \otimes H\), unless \(\alpha_x(1) = 1\) for every \(x \in P\). So, for our purpose, we first show that

\[
(4.3) \quad W_x\pi(a) = \pi(\alpha_x(a))W_x \quad \text{and} \quad W_x^*W_x\pi(a) = \pi(a)W_x^*W_x
\]

for all \(a \in A\) and \(x \in P\). For every \(f \in \ell^2(P) \otimes H\), we have

\[
(W_x\pi(a)f)(r) = (W_x(\pi(a)f))(r) = (\pi(a)f)(rx) = \pi_0(\alpha_{rx}(a))f(rx) = \pi_0(\alpha_x(\alpha_x(a)))(W_xf)(r) = (\pi(\alpha_x(a))W_xf)(r)
\]

for all \(r \in P\). So, \(W_x\pi(a) = \pi(\alpha_x(a))W_x\) is valid, from which, we get \(\pi(a)W_x^* = W_x^*\pi(a)\). One can apply these two equations to see that \(W_x^*W_x\pi(a) = \pi(a)W_x^*W_x\).
is also valid. Also, since \( W^*_x W_x = S^*_x S_x = 1 \) (see Example 3.1), each \( W_x \) is a coisometry, and hence, by applying the equation \( W_x \pi(a) = \pi(\alpha_x(a)) W_x \), we have

\[
W_x \pi(a) W^*_x = \pi(\alpha_x(a)) W_x W^*_x = \pi(\alpha_x(a)) \tag{4.4}
\]

Now we claim that the pair \((\rho, V) = (q \pi q, q W q)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) on \( \mathcal{H} \). More precisely, consider that maps

\[
\rho : A \to q B(\ell^2(P) \otimes H) q \simeq B(\mathcal{H})
\]

and

\[
V : P \to q B(\ell^2(P) \otimes H) q \simeq B(\mathcal{H})
\]

defined by

\[
\rho(a) = q \pi(a) q = \pi(a) \quad \text{and} \quad V_x = q W_x q,
\]

for all \( a \in A \) and \( x \in P \), respectively. Since for any approximate unit \( \{a_i\} \) in \( A \), \( \rho(a_i) = \pi(a_i) \to q \) strongly in \( B(\mathcal{H}) \), where \( q = 1_{B(\mathcal{H})} \), it follows that the representation \( \rho \) is nondegenerate. Moreover, by applying (4.4), we have

\[
V_x \rho(a) V^*_x = q W_x q \pi(a) q W^*_x q = q W_x q \pi(a) W^*_x q = q \pi(\alpha_x(a)) q = \rho(\alpha_x(a)).
\]

Also, by applying the first equation of (4.3) (equivalently, \( \pi(a) W^*_x = W^*_x \pi(\alpha_x(a)) \)) along with the fact that \( \rho(a) = q \pi(a) q = \pi(a) q = \pi(a) \), we get

\[
V^*_x V_x \rho(a) = q W^*_x q W_x q \pi(a) q = q W^*_x q W_x q \pi(a) = q W^*_x q \pi(\alpha_x(a)) W_x q = q W^*_x \pi(\alpha_x(a)) q W_x q = q \pi(a) q W^*_x q W_x q = \rho(a) V^*_x V_x.
\]

Thus, it is only left to show that the map \( V \) is a Nica partial-isometric representation. To see that each \( V_x \) is a partial-isometry, note that for any approximate unit \( \{a_i\} \) in \( A \),

\[
q W_x \pi(a_i) W^*_x q W_x q
\]

converges strongly to

\[
q W_x q W^*_x q W_x q = V_x V^*_x V_x
\]

in \( B(\ell^2(P) \otimes H) \). On the other hand, by applying the covariance equations of the pair \((\pi, W)\), we have

\[
q [W_x \pi(a_i) W^*_x] q W_x q = q \pi(\alpha_x(a_i)) q W_x q = q \pi(\alpha_x(a_i)) W_x q = q W_x \pi(a_i) q,
\]

which converges strongly to \( q W_x q = V_x \). So, we must have \( V_x V^*_x V_x = V_x \), which means that each \( V_x \) is a partial-isometry. To see \( V_x V_y = V_{xy} \) for every \( x, y \in P \), we first need
to compute \(V_x f\) for any \(f \in \mathcal{H}\). So, knowing that \(qf = f\), we have
\[
[V_x f](r) = [qW_x f](r) = [q(W_x f)](r) = \pi_0(\pi_r(1))(W_x f)(r) = \pi_0(\pi_r(1))(f)(rx) = \pi_0(\pi_r(1))(qf)(rx) = \pi_0(\pi_r(1))(\pi_0(\pi_{rx}(1)))f(rx) = \pi_0(\pi_r(1)(\pi_{rx}(1)))f(rx) = \pi_0(\pi_r(1)(\pi_{rx}(1)))f(rx) = (qf)(rx) = f(rx).
\]
for all \(r \in P\). Thus, by applying the above computation, we have
\[
[V_x V_y f](r) = [V_x (V_y f)](r) = (V_y f)(rx) = f((rx)y) = f(r(xy)) = [V_{xy} f](r).
\]
It follows that \(V_x V_y = V_{xy}\) for all \(x, y \in P\). Finally, we show that the partial-isometric representation \(V\) satisfies the Nica covariance condition. Let us first mention that the Hilbert space \(\mathcal{H}\) is spanned by elements
\[
\{\varepsilon_r \otimes \pi_0(\pi_r(1))h : r \in P, h \in H\}
\]
as closed subspace of \(\ell^2(P) \otimes H\). Then, for every \(y \in P\) and \(f \in \mathcal{H}\),
\[
(V_y^*(f))(r) = (qW_y^* f)(r) = (q(S_y f))(r) = \pi_0(\pi_r(1))(S_y f)(r),
\]
which is nonzero only if \(r \in Py\). Thus, if \(r = sy\) for some \(s \in P\), we get
\[
(V_y^*(f))(r) = \pi_0(\pi_{sy}(1))(S_y f)(sy) = \pi_0(\pi_{sy}(1))f(s).
\]
If follows that, if \(r = sy\) for some \(s \in P\), then
\[
[V_y^* V_y f](r) = [V_y^* (V_y f)](r) = \pi_0(\pi_{sy}(1))(V_y f)(s) = \pi_0(\pi_{sy}(1))f(sy) = (qf)(sy) = f(sy) = f(r),
\]
otherwise it is zero. Therefore, each \(V_y^* V_y\) is the projection of \(\mathcal{H}\) onto the closed subspace
\[
\mathcal{H}_y := \{f \in \mathcal{H} : f(r) = 0 \text{ if } r \not\in Py\} = \ker(1 - V_y^* V_y)
\]
of \(\mathcal{H}\), which is actually spanned by the elements
\[
\{\varepsilon_{sy} \otimes \pi_0(\pi_{sy}(1))h : s \in P, h \in H\}.
\]
Now, if \(Px \cap Py = \emptyset\), then for every \(f \in \mathcal{H}\),
\[
[V_x V_y^* f](r) = [V_x (V_y^* f)](r) = (V_y^* f)(rx) = 0.
\]
This is due to the fact that since \(rx \in Px, rx \not\in Py\). It follows that \(V_x V_y^* = 0\), and hence,
\[
V_x^* V_x V_y^* V_y = 0.
\]
If \(Px \cap Py = Pz\), then, as \(V_y^* V_y\) is a projection onto the closed subspace \(\mathcal{H}_y\),
\[
V_x^* V_x V_y^* V_y.
\]
is the projection onto the closed subspace
\[
\{ f \in \mathcal{H}_y : f(r) = 0 \text{ if } r \not\in P x \} = \{ f \in \mathcal{H} : f(r) = 0 \text{ if } (r \not\in P y \lor r \not\in P x) \}
\]
\[
= \{ f \in \mathcal{H} : f(r) = 0 \text{ if } r \not\in (P y \cap P x) \}
\]
\[
= \{ f \in \mathcal{H} : f(r) = 0 \text{ if } r \not\in P z \}
\]
\[
= \ker(1 - V_x^* V_y) = \mathcal{H}_z.
\]

Observe the following diagram:
\[
(4.5) \quad \mathcal{H} \xrightarrow{V_x^* V_y} \mathcal{H}_y \xrightarrow{V_y^* V_z} \mathcal{H}_z.
\]

Thus, we must have
\[
V_x^* V_y^* V_y = V_z^* V_z.
\]

Consequently, the pair \((\rho, V)\) is a (nontrivial) covariant partial-isometric representation of \((A, P, \alpha)\) on \(\mathcal{H}\).

Note that if \(\pi_0\) is faithful, then it is not difficult to see that \(\rho\) is faithful as well. So, every system \((A, P, \alpha)\) has a (nontrivial) covariant pair \((\rho, V)\) with \(\rho\) faithful.

### 4.2. Crossed products and Nica-Teoplitz algebras of Hilbert bimodules.

Let \(P\) be a left LCM semigroup, and \((A, P, \alpha)\) a dynamical system consisting of a \(C^*\)-algebra \(A\), and an action \(\alpha : P \to \text{End}(A)\) of \(P\) by extendible endomorphisms of \(A\) such that \(\alpha_e = \text{id}_A\).

**Definition 4.4.** A partial-isometric crossed product of \((A, P, \alpha)\) is a triple \((B, i_A, i_P)\) consisting of a \(C^*\)-algebra \(B\), a nondegenerate injective homomorphism \(i_A : A \to B\), and a Nica partial-isometric representation \(i_P : P \to \mathcal{M}(B)\) such that:

(i) the pair \((i_A, i_P)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) in \(B\);

(ii) for every covariant partial-isometric representation \((\pi, V)\) of \((A, P, \alpha)\) on a Hilbert space \(H\), there exists a nondegenerate representation \(\pi \times V : B \to B(\mathcal{H})\) such that \((\pi \times V) \circ i_A = \pi\) and \((\pi \times V) \circ i_P = V\); and

(iii) the \(C^*\)-algebra \(B\) is generated by \(\{i_A(a) i_P(x) : a \in A, x \in P\}\).

We call the algebra \(B\) the partial-isometric crossed product of the system \((A, P, \alpha)\) and denote it by \(A \times_{\alpha}^\text{piso} P\).

**Remark 4.5.** Note that in the definition above, for part (iii), we actually have
\[
(4.6) \quad B = \overline{\text{span}} \{ i_P(x)^* i_A(a) i_P(y) : x, y \in P, a \in A \}.
\]

To see this, we only need to show that the right hand side of (4.6) is closed under multiplication. To do so, we need to apply the Nica covariance condition to calculate each product
\[
(4.7) \quad [i_P(x)^* i_A(a) i_P(y)] [i_P(s)^* i_A(b) i_P(t)].
\]

We have
\[
[i_P(x)^* i_A(a) i_P(y)] [i_P(s)^* i_A(b) i_P(t)] = i_P(x)^* i_A(a) i_P(y) [i_P(y)^* i_P(s)^* i_P(s)] i_P(s)^* i_A(b) i_P(t),
\]
which is zero if $Py \cap Ps = \emptyset$. But if $Py \cap Ps = Pz$ for some $z \in P$, then $ry = z = qs$ for some $r, q \in P$, and therefore by the covariance of the pair $(i_A, i_P)$, we get

\[
[i_P(x)*i_A(a) i_P(y)] [i_P(s)*i_A(b) i_P(t)]
= i_P(x)*i_A(a) i_P(y) i_P(z)*i_P(z) i_P(s)*i_A(b) i_P(t)
= i_P(x)*i_A(a) i_P(y) i_P(ry)*i_P(qs) i_P(s)*i_A(b) i_P(t)
= i_P(x)*i_A(a) [i_P(y) i_P(y)*] i_P(r)*i_P(q) [i_P(s) i_P(s)*] i_A(b) i_P(t)
= i_P(x)*i_A(a) [i_P(A) i_P(r) i_P(q)] i_A(\alpha_s(1)b) i_P(t)
= i_P(x) * i_A(a) [i_P(r) * i_A(\alpha_s(c)) i_A(\alpha_q(d)) i_P(q) i_P(t)]
= i_P(r) x * i_A(\alpha(c) \alpha_q(d)) i_P(q) i_P(t),
\]

which is in the right hand side of (4.6), where $c = a\alpha_s(1)$ and $d = \alpha_s(1)b$. Thus, (4.6) is indeed true.

Next, we want to show that the partial-isometric crossed product of the system $(A, P, \alpha)$ always exists, and it is unique up to isomorphism. Firstly, since $P$ is a left LCM semigroup, the opposite semigroup $P^o$ is a right LCM semigroup. Therefore, one can easily see that $(A, P^o, \alpha)$ is a dynamical system in the sense of [11] Definition 3.1. Thus, following [8] §3 or [11] §3, for every $s \in P$, let

\[ X_s := \{s\} \times \overline{\alpha_s(1)}A, \]

where $\overline{\alpha_s(1)}A = \alpha_s(A)A = \overline{\text{span}}\{\alpha_s(a) b : a, b \in A\}$ as each endomorphism $\alpha_s$ is extendible. Then, each $X_s$ is given the structure of a Hilbert bimodule over $A$ via

\[(s, x) \cdot a := (s, xa), \quad \langle (s, x), (s, y) \rangle_A := x^*y,\]

and

\[ a \cdot (s, x) := (s, \alpha_s(a)x). \]

Let $X = \bigsqcup_{s \in P} X_s$, which is equipped with a multiplication

\[ X_s \times X_t \to X_{s*t}; \quad ((s, x), (t, y)) \mapsto (s, t)(x, y) \]

defined by

\[(s, x)(t, y) := (ts, \alpha_t(s)a_x)(s, t, \alpha_t(s)x) = (s \ast t, \alpha_t(s)x)\]

for every $x, y \in \overline{\alpha_s(1)}A$ and $y \in \overline{\alpha_t(1)}A$. By [8] Lemma 3.2], $X$ is a product system over the opposite semigroup $P^o$, and the left action of $A$ on each fiber $X_s$ is by compact operators. Let $(\mathcal{N}\mathcal{T}(X), i_X)$ be the Nica-Toeplitz algebra corresponding to $X$ defined in [11] §3 (see also [6] §6), which is generated by the universal Nica covariant Toeplitz representation $i_X : X \to \mathcal{N}\mathcal{T}(X)$. We show that this algebra is the partial-isometric crossed product of the system $(A, P, \alpha)$. But we first need to recall that, for any approximate unit $\{a_t\}$ in $A$, similar to [8] Lemma 3.3], one can see that $i_X(s, \alpha_s(a_t))$ converges strictly in the multiplier algebra $\mathcal{M}(\mathcal{N}\mathcal{T}(X))$ for every $s \in P$. Now, we have:

**Proposition 4.6.** Suppose that $P$ is a left LCM semigroup, and $(A, P, \alpha)$ a dynamical system. Let $\{a_t\}$ be any approximate unit in $A$. Define the maps

\[ i_A : A \to \mathcal{N}\mathcal{T}(X) \quad \text{and} \quad i_P : P \to \mathcal{M}(\mathcal{N}\mathcal{T}(X)) \]
by

\[ i_A(a) := i_X(e, a) \quad \text{and} \quad i_P(s) := \lim_i i_X(s, \alpha_s(a)) \]

for all \( a \in A \) and \( s \in P \). Then the triple \((N^T(X), i_A, i_P)\) is a partial-isometric crossed product for \((A, P, \alpha)\), which is unique up to isomorphism.

**Proof.** For any approximate unit \( \{a_i\} \) in \( A \),

\[ i_A(a_i) = i_X(e, a_i) = i_X(e, \alpha_e(a_i)) \]

converges strictly to \( 1 \) in the multiplier algebra \( \mathcal{M}(N^T(X)) \). One can see this again from [8] Lemma 3.3] similarly when \( s = e \). It follows that \( i_A \) is a nondegenerate homomorphism. By a similar discussion to the first part of the proof of [8, Proposition 3.4], we can see that the map \( i_P \) is a partial-isometric representation such that together with the (nondegenerate) homomorphism \( i_A \) satisfy the covariance equations

\[ i_A(\alpha_s(a)) = i_P(s)i_A(a)i_P(s)^* \quad \text{and} \quad i_A(a)i_P(s)^*i_P(s) = i_P(s)^*i_P(s)i_A(a) \]

for all \( a \in A \) and \( s \in P \). So, we only need to show that the representation \( i_P \) satisfies the Nica covariance condition. By the same calculation as (3.7) in the proof of [8, Proposition 3.4], we have

\[ i_A(ab^*)(i_P(s)^*i_P(s)) = i_X(s, \alpha_s(a))i_X(s, \alpha_s(b)) \]

for all \( a, b \in A \) and \( s \in P \), and since

\[ i_X(s, \alpha_s(a))i_X(s, \alpha_s(b)) = \lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b))), \]

it follows that

\[ i_A(ab^*)i_P(s)^*i_P(s) = \lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b))). \]

Therefore,

\[ i_A(ab^*)i_P(s)^*i_P(s)i_A(cd^*)i_P(t)^*i_P(t) = \lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b)))\lim^{(t)}(\Theta(t, \alpha_t(c), t, \alpha_t(d))), \]

and since

\[ i_A(ab^*)i_P(s)^*i_P(s)i_A(cd^*)i_P(t)^*i_P(t) = i_A(ab^*)i_A(cd^*)i_P(s)^*i_P(s)i_P(t)^*i_P(t) \]

\[ = i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t), \]

it follows that

\[ i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t) = \lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b)))\lim^{(t)}(\Theta(t, \alpha_t(c), t, \alpha_t(d))). \]

Now, if \( Ps \cap Pt = Pr \) for some \( r \in P \), which is equivalent to saying that

\[ s * P^o \cap t * P^o = r * P^o, \]

since \( i_X \) is Nica-covariant, we have

\[ i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t) = \lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b)))\lim^{(t)}(\Theta(t, \alpha_t(c), t, \alpha_t(d))). \]

Next, we want to calculate the product

\[ \left(\lim^{(s)}(\Theta(s, \alpha_s(a), s, \alpha_s(b)))\right)\left(\lim^{(t)}(\Theta(t, \alpha_t(c), t, \alpha_t(d)))\right) \]

of compact operators in \( \mathcal{K}(X_r) \) to show that it is equal to

\[ \Theta(r, \alpha_r(ab^*), r, \alpha_r(dc^*)). \]
Thus, it follows that $\mathfrak{s}_\alpha(\Theta(t,\alpha(c)),(t,\alpha(d)))$ is mapped to

\[
\iota^*_t(\Theta(t,\alpha(c)),(t,\alpha(d)))((t,\overline{\alpha}(1)f)(q,\overline{\alpha}(1)g)) = (t,\alpha_t(c))(\Theta(t,\alpha(c)),\alpha_t(d))(t,\overline{\alpha}(1)f)(q,\overline{\alpha}(1)g)
\]

First, $\iota^*_t(\Theta(t,\alpha(c)),(t,\alpha(d)))$ is mapped to

\[
\iota^*_t(\Theta(t,\alpha(c)),(t,\alpha(d)))(q,\overline{\alpha}(1)g)
\]

and

\[
\iota^*_s(\Theta(s,\alpha_s(a)),(s,\alpha_s(b)))((s,\alpha_s(\alpha_s(cd^*)p,\overline{\alpha}_p(1)\alpha_q(f)g))
\]

We then let $\iota^*_s(\Theta(s,\alpha_s(a)),(s,\alpha_s(b)))$ act on $(s,\alpha_s(\alpha_s(cd^*)p,\overline{\alpha}_p(1)\alpha_q(f)g)$, and hence,

\[
\iota^*_s(\Theta(s,\alpha_s(a)),(s,\alpha_s(b)))((s,\alpha_s(\alpha_s(\alpha_s(cd^*)p,\overline{\alpha}_p(1)\alpha_q(f)g))
\]

Thus, it follows that

\[
(4.12)
\]

On the other hand, since

\[
(t,\overline{\alpha}_t(1)f)(q,\overline{\alpha}_q(1)g)
\]


we have

\[
\Theta(r,\alpha_r(ab^*),r,\alpha_r(cd^*))\left((t,\overline{\alpha_r}(1)f)(q,\overline{\alpha_r}(1)g)\right) \\
= \Theta(r,\alpha_r(ab^*),r,\alpha_r(cd^*))\left(r,\overline{\alpha_r}(1)\alpha_q(f)g\right) \\
= (r,\alpha_r(ab^*)) \cdot \left((r,\alpha_r(cd^*)),(r,\overline{\alpha_r}(1)\alpha_q(f)g)\right)_A \\
= (r,\alpha_r(ab^*)) \cdot \left[\alpha_r(cd^*)\overline{\alpha_r}(1)\alpha_q(f)g\right] \\
= (r,\alpha_r(ab^*)) \cdot \left[\alpha_r(cd^*)\alpha_q(f)g\right] \\
= (r,\alpha_r(ab^*)\alpha_r(cd^*)\alpha_q(f)g). \\
\] (4.13)

So, we conclude by (4.12) and (4.13) that

\[
i_s^r(\Theta(s,\alpha_s(a),(s,\alpha_s(b))))_A^t = \Theta(r,\alpha_r(ab^*),r,\alpha_r(cd^*)). \\
\] (4.14)

Consequently, if \( Ps \cap Pt = Pr \), then by applying (4.14), (4.11), and (4.9), we get

\[
i_A(ab^*(cd^*))i_P(s)i_P(t)i_P(t)^*i_P(t) = i_A(ab^*cd^*)i_P(r)^*i_P(r) \\
\] (4.15)

for all \( a, b, c, d \in A \). Since \( A \) contains approximate unit, it follows by (4.15) that

\[
i_A(a)i_P(s)i_P(s)i_P(t)^*i_P(t) = i_A(a)i_P(r)^*i_P(r) \\
\] (4.16)

for all \( a \in A \). So, if \( \{a_i\} \) is any approximate unit in \( A \), then

\[
i_A(a_i)i_P(s)i_P(s)i_P(t)^*i_P(t) = i_A(a_i)i_P(r)^*i_P(r). \\
\] (4.17)

Now, in (4.17), the left hand side converges strictly to

\[
i_P(s)i_P(s)i_P(t)^*i_P(t) \\
\]

in \( \mathcal{M}(\mathcal{N}T(X)) \), while the right hand side converges strictly to \( i_P(r)^*i_P(r) \). Hence, we must have

\[
i_P(s)i_P(s)i_P(t)^*i_P(t) = i_P(r)^*i_P(r) \\
\]

when \( Ps \cap Pt = Pr \).

If \( Ps \cap Pt = \emptyset \), then again, since \( i_X \) is Nica-covariant, the right hand side of (4.10) is zero, and therefore,

\[
i_A(ab^*(cd^*))i_P(s)i_P(s)i_P(t)^*i_P(t) = 0 \\
\] (4.18)

for all \( a, b, c, d \in A \). Then, similar to the above, as \( A \) contains approximate unit, we can show that we must have

\[
i_P(s)i_P(s)i_P(t)^*i_P(t) = 0. \\
\]

Consequently, the pair \((i_A, i_P)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) in the algebra \( \mathcal{N}T(X) \). So, the condition (i) in Definition 4.4 is satisfied.

Suppose that \((\pi, V)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) on a Hilbert space \( H \). Then, the pair \((\pi, V^*)\) is a representation of the system \((A, P^0, \alpha)\) in the sense of [11, Definition 3.2], which is Nica-covariant. Note that the homomorphism
\[ V^* : P^o \to B(H) \] is defined by \( s \mapsto V^*_s \). Therefore, by [11, Proposition 3.11], the map \( \psi : X \to B(H) \) defined by
\[
\psi(s, x) := V^*_s \pi(x)
\]
is a nondegenerate Nica-covariant Toeplitz representation of \( X \) on \( H \). So, there is a homomorphism \( \psi_* : \mathcal{N}\mathcal{T}(X) \to B(H) \) such that \( \psi_* \circ i_X = \psi \) (see [6, 11, 8]), which is nondegenerate. Let \( \pi \times V = \psi_* \).

Then
\[
(\pi \times V)(i_A(a)) = \psi_*(i_X(e, a)) = \psi(e, a) = V^*_e \pi(a) = \pi(a)
\]
for all \( a \in A \). Also, since \( \pi \times V \) is nondegenerate, we have
\[
(\pi \times V)(i_P(s)) = (\pi \times V)(\lim_i i_X(s, \alpha_s(a_i)))^*
\]
\[
= \lim_i (\pi \times V)(i_X(s, \alpha_s(a_i))^*)
\]
\[
= \lim_i \psi_*(i_X(s, \alpha_s(a_i)))^*
\]
\[
= \lim_i \psi(s, \alpha_s(a_i))^*
\]
\[
= \lim_i [V^*_s \pi(\alpha_s(a_i))]^*
\]
\[
= \lim_i [\pi(a_i)V^*_s]^* \quad \text{(by the covariance of (\( \pi, V \)))}
\]
\[
= \lim_i V_s \pi(a_i) = V_s
\]
for all \( s \in P \). Thus, the condition (ii) in Definition [4.4] is satisfied, too.

The condition (iii) is also satisfied. This is due to the facts that the elements of the form \( i_X(s, \alpha_s(1)a) \) generate \( \mathcal{N}\mathcal{T}(X) \), and
\[
i_X(s, \alpha_s(1)a) = \lim_i i_X(s, \alpha_s(a_i)a)
\]
\[
= \lim_i [i_X(s, \alpha_s(a_i))i_X(e, a)]
\]
\[
= \lim_i [i_X(s, \alpha_s(a_i))]i_X(e, a) = i_P(s)^*i_A(a).
\]

To see that the homomorphism \( i_A \) is injective, we recall from Example [4.3] that the system \((A, P, \alpha)\) admits covariant partial-isometric representations \((\pi, V)\) with \( \pi \) faithful. Therefore, it follows from the equation \((\pi \times V) \circ i_A = \pi \) that \( i_A \) must be injective.

For uniqueness, suppose that \((C, j_A, j_P)\) is another triple which satisfies conditions (i)-(iii) in Definition [4.4]. Then by applying the universal properties (condition (ii)) of the algebras \( C \) and \( \mathcal{N}\mathcal{T}(X) \), respectively, once can see that there is an isomorphism of \( C \) onto \( \mathcal{N}\mathcal{T}(X) \) which maps the pair \((j_A, j_P)\) into the pair \((i_A, i_P)\).

Suppose that \((A, P, \alpha)\) is a dynamical system, an \( I \) is an ideal of \( A \) such that \( \alpha_s(I) \subset I \) for all \( s \in P \). To define a crossed product \( I \times P \) which want it to sit naturally in \( A \times P \) as an ideal, we need some extra condition. So, we need to recall a definition from [11]. Let \( \alpha \) be an extendible endomorphism of a \( C^* \)-algebra \( A \), and \( I \) an ideal of \( A \). Suppose that \( \psi : A \to \mathcal{M}(I) \) is the canonical nondegenerate homomorphism defined by \( \psi(a)i = ai \) for all \( a \in A \) and \( i \in I \). Then, we say \( I \) is extendible \( \alpha \)-invariant if it is \( \alpha \)-invariant, which means that \( \alpha(I) \subset I \), and the endomorphism \( \alpha|_I \) is extendible, such that
\[
\alpha(u_A) \to \overline{\psi(\pi(1, \mathcal{M}(A)))}
\]
strictly in \( \mathcal{M}(I) \), where \( \{u_A\} \) is an approximate unit in \( I \).

In addition, if \((A, P, \alpha)\) is a dynamical system and \( I \) is an ideal of \( A \), then there is a dynamical system \((A/I, P, \tilde{\alpha})\) with extendible endomorphisms given by \( \tilde{\alpha}_s(a + I) = \alpha_s(a) + I \) for every \( a \in A \) and \( s \in P \) (see again [11]).

The following Theorem is actually a generalization of [4, Theorem 3.1]:
Theorem 4.7. Let \((A \times piso_{\alpha} P, i_A, V)\) be the partial-isometric crossed product of a dynamical system \((A, P, \alpha)\), and \(I\) an extendible \(\alpha_x\)-invariant ideal of \(A\) for every \(x \in P\). Then, there is a short exact sequence

\[
0 \to I \times piso_{\alpha} P \xrightarrow{\mu} A \times piso_{\alpha} P \xrightarrow{\varphi} A/I \times piso_{\alpha} P \to 0
\]

of \(C^*\)-algebras, where \(\mu\) is an isomorphism of \(I \times piso_{\alpha} P\) onto the ideal

\[
\mathcal{E} := \overline{\text{span}}\{V_s^* i_A(i)V_t : i \in I, s, t \in P\}
\]

of \(A \times piso_{\alpha} P\). If \(q : A \to A/I\) is the quotient map, and the triples \((I \times piso_{\alpha} P, i_I, W)\) and \((A/I \times piso_{\alpha} P, i_{A/I}, U)\) are the crossed products of the systems \((I, P, \alpha)\) and \((A/I, P, \alpha)\), respectively, then

\[
\mu \circ i_I = i_A|_I, \quad \bar{\pi} \circ W = V \quad \text{and} \quad \varphi \circ i_A = i_{A/I} \circ q, \quad \bar{\varphi} \circ V = U.
\]

Proof. We first show that \(\mathcal{E}\) is an ideal of \(A \times piso_{\alpha} P\). To do so, it suffices to see on the spanning elements of \(\mathcal{E}\) that \(V_s^* \mathcal{E}, i_A(a) \mathcal{E}\), and \(V_r \mathcal{E}\) are all contained in \(\mathcal{E}\) for every \(a \in A\) and \(r \in P\). This first one is obvious, and the second one follows easily by applying the covariance equation \(i_A(a)V_s = V_s^* i_A(\alpha_s(a))\). For the third one, we have

\[
V_r V_s^* i_A(i)V_t = V_r [V_s^* V_r V_s^* V_s^* i_A(i)V_t],
\]

which is zero if \(Pr \cap Ps = \emptyset\). But if \(Pr \cap Ps = Pz\) for some \(z \in P\), then there are \(x, y \in P\) such that \(xr = z = ys\), and therefore it follows that

\[
V_r V_s^* i_A(i)V_t = V_s^*[V_s^* V_r V_s^* V_s^* i_A(i)V_t] = V_s^*[V_s^* V_r V_s^* V_s^* i_A(i)V_t] = [V_s^* V_r V_r V_s^* V_s^* i_A(i)V_t] = \xi_A(\overline{\mu_r(1)})V_s^* V_r i_A(\overline{\mu_s(1)})i_A(i)V_t \quad \text{(by Lemma (4.2))}
\]

and so on. Thus, \(\mathcal{E}\) is an ideal of \(A \times piso_{\alpha} P\). Let \(\phi : A \times piso_{\alpha} P \to \mathcal{M}(\mathcal{E})\) be the canonical nondegenerate homomorphism defined by \(\phi(\xi)\eta = \xi \eta\) for all \(\xi, \eta \in A \times piso_{\alpha} P\) and \(\eta \in \mathcal{E}\). Suppose that now the maps

\[
k_I : I \to \mathcal{M}(\mathcal{E}) \quad \text{and} \quad S : P \to \mathcal{M}(\mathcal{E})
\]

are defined by the compositions

\[
I \xrightarrow{i_A|_I} A \times piso_{\alpha} P \xrightarrow{\phi} \mathcal{M}(\mathcal{E}) \quad \text{and} \quad P \xrightarrow{V} \mathcal{M}(A \times piso_{\alpha} P) \xrightarrow{\bar{\varphi}} \mathcal{M}(\mathcal{E}),
\]

respectively. We claim that the triple \((\mathcal{E}, k_I, S)\) is a partial-isometric crossed product of the system \((I, P, \alpha)\). First, exactly by the same discussion as in the proof of [4, Theorem 3.1] using the extendibility of the ideal \(I\), it follows that the homomorphism \(k_I\) is nondegenerate. Also, it follows easily by the definition of the map \(S\) that it is
indeed a Nica partial-isometric representation. Then, by some routine calculations, one can see that the pair \((k_I, S)\) satisfies the covariance equations

\[ k_I(\alpha_t(i)) = S_t k_I(i) S_t^* \quad \text{and} \quad S_t^* S_t k_I(i) = k_I(i) S_t^* S_t \]

for all \(i \in I\) and \(t \in P\).

Next, suppose that the pair \((\pi, T)\) is a covariant partial-isometric representation of \((I, P, \alpha)\) on a Hilbert space \(H\). Let \(\psi : A \rightarrow \mathcal{M}(I)\) be the canonical nondegenerate homomorphism which was mentioned about earlier. Let the map \(\rho : A \rightarrow B(H)\) be defined by the composition

\[ A \xrightarrow{\psi} \mathcal{M}(I) \xrightarrow{\pi} B(H), \]

which is a nondegenerate representation of \(A\) on \(H\). We claim that the pair \((\rho, T)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) on \(H\). To prove our claim, we only need to show that the pair \((\rho, T)\) satisfies the covariance equations (4.1). Since the ideal \(I\) is extendible, we have \(\alpha_s|_I \circ \psi = \psi \circ \alpha_s\) for all \(s \in P\). It therefore follows that

\[ \rho(\alpha_s(a)) = (\pi \circ \psi)(\alpha_s(a)) = \pi(\psi \circ \alpha_s(a)) = \pi(\alpha_s|_I \circ \psi(a)) = (\pi \circ \alpha_s|_I)(\psi(a)) = T_s \pi(\psi(a)) T_s^* = T_s \rho(a) T_s^*. \]

Also, one can easily see that we have \(T_s \rho(a) T_s^* = \rho(a) T_s^* T_s\). Thus, there is a non-degenerate representation \(\rho \times T\) of \(A \times_{\alpha}^\text{piso} P\) on \(H\), whose restriction \((\rho \times T)|_\mathcal{E}\) is a non-degenerate representation of \(\mathcal{E}\) on \(H\) satisfying

\[ (\rho \times T)|_\mathcal{E} \circ k_I = \pi \quad \text{and} \quad (\rho \times T)|_\mathcal{E} \circ S = T. \]

Finally, the elements of the form

\[ S^*_s k_I(i) S_t = \overline{\phi}(V_s) \overline{\phi}(i_A|_I(i)) \phi(V_t) = \phi(V_s^* i_A|_I(i) V_t) = V_s^* i_A|_I(i) V_t, \]

obviously span the algebra \(\mathcal{E}\). Thus, \((\mathcal{E}, k_I, S)\) is a partial-isometric crossed product of \((I, P, \alpha)\). So, by Proposition 4.6 there is an isomorphism \(\mu : A \times_{\alpha}^\text{piso} P \rightarrow \mathcal{E}\) such that

\[ \mu(i_I(i) W_t) = k_I(i) S_t = \phi(i_A|_I(i)) \overline{\phi}(V_t) = \phi(i_A|_I(i) V_t) = i_A|_I(i) V_t, \]

from which, it follows that

\[ \mu \circ i_I = i_A|_I \quad \text{and} \quad \overline{\mu} \circ W = V. \]

To get the desired homomorphism \(\varphi\), let the homomorphism \(j_A : A \rightarrow A/I \times_{\alpha}^\text{piso} P\) be given by the composition

\[ A \xrightarrow{q} A/I \xrightarrow{i_{A/I}} A/I \times_{\alpha}^\text{piso} P, \]

which is nondegenerate. Then, it is not difficult to see that the pair \((j_A, U)\) is a covariant partial-isometric representation of \((A, P, \alpha)\) in the algebra \(A/I \times_{\alpha}^\text{piso} P\). Thus, there is a non-degenerate homomorphism \(\varphi := j_A \times U : A \times_{\alpha}^\text{piso} P \rightarrow A/I \times_{\alpha}^\text{piso} P\) such that

\[ \varphi \circ i_A = j_A = i_{A/I} \circ q \quad \text{and} \quad \overline{\varphi} \circ V = U. \]
which implies that \( \varphi \) is onto.

Finally, we show that \( \mu(I \times p\mathrm{iso} P) = E \) is equal to \( \ker \varphi \) which means that (1.19) is exact. The inclusion \( E \subset \ker \varphi \) is immediate. To see the other inclusion, take a non-degenerate representation \( \tilde{\Pi} \) of \( A \times \pi_{\mathrm{iso}} P \) on a Hilbert space \( H \) with \( \ker \tilde{\Pi} = E \). Since \( I \subset \ker(\Pi \circ i_A) \), the composition \( \Pi \circ i_A \) gives a (well-defined) non-degenerate representation \( \tilde{\Pi} \) of \( A/I \) on \( H \). Also, the composition \( \tilde{\Pi} \circ V \) defines a Nica partial-isometric representation \( P \) on \( H \), such that together with \( \tilde{\Pi} \) forms a covariant partial-isometric representation of \((A/I, P, \tilde{\alpha}) \) on \( H \). Then the corresponding (non-degenerate) representation \( \tilde{\Pi} \times (\Pi \circ V) \) lifts to \( \Pi \), which means that \([\tilde{\Pi} \times (\Pi \circ V)] \circ \varphi = \Pi \), from which the inclusion \( \ker \varphi \subset \tilde{E} \) follows. This completes the proof.

\( \square \)

**Example 4.8.** Suppose that \( S \) is a (unital) right LCM semigroup. See in [5, 18] that associated to \( S \) there is a universal \( C^* \)-algebra

\[
C^*(S) = \overline{\langle W_s W_t^* : s, t \in S \rangle}
\]

generated by a universal isometric representation \( W : S \to C^*(S) \), which is Nica-covariant, which means that it satisfies

\[
W_r W_s^* W_t W_s^* = \begin{cases} W_t W_t^* & \text{if } r S \cap s S = t S, \\ 0 & \text{if } r S \cap s S = \emptyset. \end{cases}
\]

In addition, by [6, Corollary 7.11], \( C^*(S) \) is isomorphic to the Nica-Toeplitz algebra \( \mathcal{N}T(X) \) of the compactly aligned product system \( X \) over \( S \) with fibers \( X_s = \mathbb{C} \) for all \( s \in S \). Now, consider the trivial dynamical system \((\mathbb{C}, P, \text{id})\), where \( P \) is a left LCM semigroup. So, the opposite semigroup \( P^o \) is right LCM. Then, it follows by Proposition 4.6 that there is an isomorphism

\[
i_p(x) \in (\mathbb{C} \times p\mathrm{iso} P) \mapsto W_x^* \in C^*(P^o)
\]

for all \( x \in P \), where \( W \) is the universal Nica-covariant isometric representation of \( P^o \) which generates \( C^*(P^o) \).

5. Tensor products of crossed products

Let \((A, P, \alpha)\) and \((B, S, \beta)\) be dynamical systems in which \( P \) and \( S \) are left LCM semigroups. Then, \( P \times S \) is a unital semigroup with the unit element \((e_P, e_S)\), where \( e_P \) and \( e_S \) are the unit elements of \( P \) and \( S \), respectively. In addition, since

\[
(P \times S)(x, r) \cap (P \times S)(y, s) = (P_x \times S_r) \cap (P_y \times S_s) = (P_x \cap P_y) \times (S_r \cap S_s),
\]

it follows that \( P \times S \) is a left LCM semigroup. More precisely, if \( P_x \cap P_y = P_z \) and \( S_r \cap S_s = S_t \) for some \( z \in P \) and \( t \in S \), then it follows by (5.1) that

\[
(P \times S)(x, r) \cap (P \times S)(y, s) = P_z \times S_t = (P \times S)(z, t),
\]

which means that \((z, t)\) is a least common left multiple of \((x, r)\) and \((y, s)\) in \( P \times S \). Otherwise, \((P \times S)(x, r) \cap (P \times S)(y, s) = \emptyset \). Thus, \( P \times S \) is actually a left LCM semigroup (note that, the similar fact holds if \( P \) and \( S \) are right LCM semigroups).

Next, for every \( x \in P \) and \( r \in S \), as \( \alpha_x \) and \( \beta_r \) are endomorphisms of the algebras \( A \) and \( B \), respectively, it follows by [20, Lemma B. 31] that there is an endomorphism
\( \alpha_x \otimes \beta_x \) of the maximal tensor product \( A \otimes_{max} B \) such that \((\alpha_x \otimes \beta_x)(a \otimes b) = \alpha_x(a) \otimes \beta_x(b)\) for all \(a \in A\) and \(b \in B\). We therefore have an action

\[
\alpha \otimes \beta : P \times S \to \text{End}(A \otimes_{max} B)
\]
of \(P \times S\) on \(A \otimes_{max} B\) by endomorphisms such that

\[
(\alpha \otimes \beta)_{(x,r)} = \alpha_x \otimes \beta_r \quad \text{for all } (x,r) \in P \times S.
\]

Moreover, it follows by the extendibility of the actions \(\alpha\) and \(\beta\) that the action \(\alpha \otimes \beta\) on \(A \otimes_{max} B\) is actually given by extendible endomorphisms (see [1], Lemma 2.3). Thus, we have a dynamical system \((A \otimes_{max} B, P \times S, \alpha \otimes \beta)\) for which we can talk about the corresponding partial-isometric crossed product. We actually aim to show that under some certain conditions we have the following isomorphism:

\[
(A \otimes_{max} B) \times_{\alpha \otimes \beta}^{\text{piso}} (P \times S) \simeq (A \times_{\alpha}^{\text{piso}} P) \otimes_{\max}^{\beta} (B \times_{\beta}^{\text{piso}} S).
\]

In fact, those conditions are to ensure that the Nica partial-isometric representations of \(P\) and \(S\) are \(*\)-commuting. Hence, we first need to assume that the (unital) semigroups \(P, P^\alpha, S,\) and \(S^\alpha\) are all left LCM. It thus turns out that all of them are both left and right LCM semigroups. The other condition comes from the following definition:

**Definition 5.1.** Suppose that \(P\) and \(P^\alpha\) are both left LCM semigroups. A **bicovariant partial-isometric representation** of \(P\) on a Hilbert space \(H\) is a Nica partial-isometric representation \(V : P \to B(H)\) which satisfies

\[
(5.2) \quad V_r V_r^* V_s V_s^* = \begin{cases} 
V_t V_t^* & \text{if } rP \cap sP = tP, \\
0 & \text{if } rP \cap sP = \emptyset.
\end{cases}
\]

Note the equation (5.2) is a kind of Nica covariance condition, too. So, to distinguish it from the covariance equation (3.1), we may view (3.1) as the **right Nica covariance condition** and (5.2) as the **left Nica covariance condition**.

Note that similar to (3.1), we can see that the equation (5.2) is also well-defined.

**Lemma 5.2.** Suppose that the (unital) semigroups \(P, P^\alpha, S,\) and \(S^\alpha\) are all left LCM. Let \(V\) and \(W\) be bicovariant partial-isometric representations of \(P\) and \(S\) on a Hilbert space \(H\), respectively, such that each \(V_p\) \(*\)-commutes with each \(W_s\) for all \(p \in P\) and \(s \in S\). Then, there exists a bicovariant partial-isometric representation \(U\) of \(P \times S\) on \(H\) such that \(U_{(p,s)} = V_p W_s\). Moreover, every bicovariant partial-isometric representation of \(P \times S\) arises this way.

**Proof.** Define a map \(U : P \times S \to B(H)\) by

\[
U_{(p,s)} = V_p W_s
\]

for all \((p,s) \in P \times S\). Since each \(V_p\) \(*\)-commutes with each \(W_s\), it follows that each \(U_{(p,s)}\) is a partial isometry, as

\[
U_{(p,s)} U_{(p,s)}^* U_{(p,s)} = V_p W_s [V_p W_s]^* V_p W_s = V_p W_s [W_s V_p]^* V_p W_s = V_p W_s V_p^* W_s V_p W_s = V_p V_p^* W_s V_p W_s = V_p W_s = U_{(p,s)}.
\]
Also, a simple computation shows that

\[ U_{(p,s)}U_{(q,t)} = U_{(p,s)(q,t)} \]

for every \((p, s)\) and \((q, t)\) in \(P \times S\). Thus, the map \(U\) is a (unital) semigroup homomorphism (with partial-isometric values). Next, we want to show that it satisfies the Nica covariance conditions (3.1) and (5.2), and hence, it is bicovariant. To see (3.1), we first have

\[
U_{(p,s)}^*U_{(p,s)}^*U_{(q,t)}U_{(q,t)} = [V_p^*V_qW_s][W_qW_t]V_qV_t = [W_sV_p^*][W_qV_q][W_qW_t]V_qV_t
\]

\[
= V_p^*W_s^*V_qW_qW_t^*V_qV_t
\]

\[
= V_p^*V_qW_s^*W_qW_t^*V_qV_t
\]

\[
= V_p^*V_qW_s^*W_qW_t^*W_t
\]

\[
= [V_p^*V_qW_s^*W_qW_t^*W_t].
\]

(5.3)

If \((P \times S)(p, s) \cap (P \times S)(q, t) = (P \times S)(z, r) = Pz \times Sr\) for some \((z, r) \in P \times S\), then it follows by (5.1) (for the left hand side in above) that

\[(Pp \cap Pq) \times (Ss \cap St) = Pz \times Sr.\]

Thus, we must have \(Pp \cap Pq = Pz\) and \(Ss \cap St = Sr\), and hence, for (5.3), we get

\[
U_{(p,s)}^*U_{(p,s)}^*U_{(q,t)}U_{(q,t)} = [V_p^*V_qV_qW_s][W_qW_s^*W_t^*W_t]
\]

\[
= V_p^*V_qV_qW_s^*W_t^*W_t = V_p^*V_qW_s^*W_t^*W_t
\]

\[
= [W_sV_p^*][W_qV_q][W_qW_t]V_qV_t
\]

\[
= [V_p^*V_qW_s^*W_qW_t^*W_t] = U_{(p,s)}^*U_{(q,t)}^*U_{(p,s)}U_{(q,t)}.
\]

If \((P \times S)(p, s) \cap (P \times S)(q, t) = \emptyset\), then again, by (5.1), we get

\[(Pp \cap Pq) \times (Ss \cap St) = \emptyset.\]

It follows that

\[Pp \cap Pq = \emptyset \quad \text{or} \quad Ss \cap St = \emptyset,
\]

which implies that,

\[V_p^*V_qV_qW_s = 0 \quad \text{or} \quad W_qW_s^*W_t^*W_t = 0.
\]

Thus, for (5.3), we have

\[U_{(p,s)}^*U_{(p,s)}^*U_{(q,t)}U_{(q,t)} = [V_p^*V_p^*V_qW_s][W_qW_s^*W_t^*W_t] = 0.
\]

A similar discussion shows that the representation \(U\) satisfies the Nica covariance condition (5.2), too, namely, we have

\[
U_{(p,r)}U_{(p,r)}^*U_{(q,s)}U_{(q,s)}^* = \begin{cases} 
U_{(x,t)}U_{(x,t)}^* & \text{if } (p, r)(P \times S) \cap (q, s)(P \times S) = (x, t)(P \times S), \\
0 & \text{if } (p, r)(P \times S) \cap (q, s)(P \times S) = \emptyset.
\end{cases}
\]

Therefore, \(U\) is a bicovariant partial-isometric representation of \(P \times S\) on \(H\) satisfying \(U_{(p,s)} = V_pW_s\).
Conversely, suppose that $U$ is any bicovariant partial-isometric representation of $P \times S$ on a Hilbert space $H$. Define the maps

$$V : P \to B(H) \quad \text{and} \quad W : S \to B(H)$$

by

$$V_p := U_{(p,eS)} \quad \text{and} \quad W_s := U_{(eP,s)}$$

for all $p \in P$ and $s \in S$, respectively. It is easy to see that each $V_p$ is a partial isometry as well as each $W_s$, and the maps $V$ and $W$ are (unital) semigroup homomorphisms. Next, we show that the presentation $V$ is bicovariant, and we skip the proof for the presentation $W$ as it follows similarly. To see that the presentation $V$ satisfies the Nica covariance condition (3.1), firstly,

$$V^*_p V_p V^*_q V_q = U^*_{(p,eS)} U_{(p,eS)} U^*_{(q,eS)} U_{(q,eS)}.$$

Now, if $Pp \cap Pq = Pz$ for some $z \in P$, then it follows by (5.1) that

$$(P \times S)(p,eS) \cap (P \times S)(q,eS) = (Pp \cap Pq) \times (S \cap S) = Pz \times S = Pz \times S = (P \times S)(z,eS).$$

Therefore, since $U$ is bicovariant, for (5.4), we have

$$V^*_p V_p V^*_q V_q = U^*_{(p,eS)} U_{(p,eS)} U^*_{(q,eS)} U_{(q,eS)} = U^*_{(z,eS)} U_{(z,eS)} = V^*_z V_z.$$

If $Pp \cap Pq = \emptyset$, then it follows again by (5.1) that

$$(P \times S)(p,eS) \cap (P \times S)(q,eS) = (Pp \cap Pq) \times (S \cap S) = \emptyset \times S = \emptyset.$$ 

Therefore, for (5.4), we get

$$V^*_p V_p V^*_q V_q = U^*_{(p,eS)} U_{(p,eS)} U^*_{(q,eS)} U_{(q,eS)} = 0,$$

as $U$ is bicovariant. A similar discussion shows that the representation $V$ satisfies the Nica covariance condition (5.2), too. We skip it here. Finally, as we obviously have

$$V_p W_s = W_s V_p = U_{(p,s)},$$

it is only left to show that $V^*_p W_s = W_s V^*_p$ for all $p \in P$ and $s \in S$. To do so, we first need to recall that the product $vw$ of two partial isometries $v$ and $w$ is a partial isometry if and only if $v^*v$ commutes with $ww^*$ (see [10] Lemma 2). This fact can be applied to the partial isometries $V_p$ and $W_s$ due to (5.5). Now, we have

$$V^*_p W_s = V^*_p [V_p V^*_p W_s W^*_p W_s] U_{(p,s)},$$

since

$$(p,eS)(P \times S) \cap (eP,s)(P \times S) = (pP \times S) \cap (P \times sS) = (pP \times P) \times (S \cap sS) = pP \times sS = (p,s)(P \times S),$$

Therefore, since $U$ is bicovariant, for (5.4), we have

$$V^*_p V_p V^*_q V_q = U^*_{(p,eS)} U_{(p,eS)} U^*_{(q,eS)} U_{(q,eS)} = U^*_{(z,eS)} U_{(z,eS)} = V^*_z V_z.$$
it follows that
\[
V_p V_p^* W_s W_s^* = U_{(p,e_S)}U_{(p,e_S)}^* U_{(e_p,s)}^* U_{(e_p,s)}
\]
\[
= U_{(p,s)}^* U_{(p,s)}^* U_{(e_p,s)}^* U_{(e_p,s)}
\]
\[
= U_{(p,e_S)} U_{(e_p,s)} U_{(p,e_S)}^* U_{(e_p,s)}^* V_p W_s V_p^* W_s^*,
\]
(5.7)
as $U$ is bicovariant. Therefore, by (5.6) and (5.7), we get
\[
V_p^* W_s = W_s [U_{(p,s)} U_{(p,s)}] V_p^*.
\]
(5.8)
Moreover, since similarly
\[
(P \times S)(e_p, s) \cap (P \times S)(p, e_S) = (P \times S)(p, s),
\]
we have
\[
U_{(e_p,s)}^* U_{(e_p,s)} U_{(p,e_S)} U_{(p,e_S)} = U_{(p,s)}^* U_{(p,s)},
\]
as $U$ is bicovariant. By applying this to (5.8), we finally get
\[
V_p^* W_s = W_s [U_{(p,s)}^* U_{(p,s)} U_{(p,e_S)} U_{(p,e_S)}] V_p^*
\]
\[
= [W_s W_p^* W_s] [V_p V_p^* W_s^*] V_p^*
\]
\[
= W_s^* V_p^* W_s V_p^*.
\]

This completes the proof. \(\square\)

**Definition 5.3.** Let $(A, P, \alpha)$ be a dynamical system, in which $P$ and $P^c$ are both left LCM semigroups. The action $\alpha$ is called left-Nica covariant if it satisfies
\[
\overline{\alpha}_x(1) \overline{\alpha}_y(1) = \begin{cases} 
\overline{\pi}_z(1) & \text{if } xP \cap yP = zP, \\
0 & \text{if } xP \cap yP = \emptyset.
\end{cases}
\]
(5.9)

We should mention that the above definition is well-defined. This is due to the fact that if $tP = xP \cap yP = zP$, then there is an invertible element $u$ of $P$ such that $t = zu$. Since $u$ is invertible, $\alpha_u$ becomes an automorphism of $A$, and hence $\overline{\alpha}_u(1) = 1$. So, it follows that
\[
\overline{\alpha}_x(1) = \overline{\alpha}_{zu}(1) = \overline{\alpha}_z(\overline{\alpha}_u(1)) = \overline{\alpha}_z(1).
\]

**Remark 5.4.** Let $(A, P, \alpha)$ be a dynamical system, in which $P$ and $P^c$ are both left LCM semigroups, and the action $\alpha$ is left Nica-covariant. If $(\pi, V)$ is covariant partial-isometric of the system, then the representation $V$ satisfies the left Nica-covariance condition (5.2). One can easily see this by applying the equation $V_x V_x^* = \pi(\overline{\pi}_x(1))$ (see Lemma 4.2). Thus, the representation $V$ is actually bicovariant.
Let us also recall that for $C^*$-algebras $A$ and $B$, there are nondegenerate homomorphisms

$$k_A : A \rightarrow \mathcal{M}(A \otimes_{\text{max}} B) \quad \text{and} \quad k_B : B \rightarrow \mathcal{M}(A \otimes_{\text{max}} B)$$

such that

$$k_A(a)k_B(b) = k_B(b)k_A(a) = a \otimes b$$

for all $a \in A$ and $b \in B$ (see [20] Theorem B.27). Moreover, one can see that the extensions $\overline{k_A}$ and $\overline{k_B}$ of the nondegenerate homomorphisms $k_A$ and $k_B$, respectively, have also commuting ranges. Therefore, there is a homomorphism

$$\overline{k_A} \otimes_{\text{max}} \overline{k_B} : \mathcal{M}(A) \otimes_{\text{max}} \mathcal{M}(B) \rightarrow \mathcal{M}(A \otimes_{\text{max}} B),$$

which is the identity map on $A \otimes_{\text{max}} B$ (see [15] Remark 2.2).

**Theorem 5.5.** Suppose that the (unital) semigroups $P$, $P^o$, $S$, and $S^o$ are all left LCM. Let $(A, P, \alpha)$ and $(B, S, \beta)$ be dynamical systems in which the actions $\alpha$ and $\beta$ are both left Nica-covariant. Then, we have the following isomorphism:

$$(A \otimes_{\text{max}} B) \times_{\alpha \otimes \beta}^{\text{piso}} (P \times S) \simeq (A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S).$$

**Proof.** Let the triples $(A \times_{\alpha}^{\text{piso}} P, i_A, i_P)$ and $(B \times_{\beta}^{\text{piso}} S, i_B, i_S)$ be the partial-isometric crossed products of the dynamical systems $(A, P, \alpha)$ and $(B, S, \beta)$, respectively. Suppose that $(k_{A \times_{\alpha} P}, k_{B \times_{\alpha} S})$ is the canonical pair of the algebras $A \times_{\alpha}^{\text{piso}} P$ and $B \times_{\beta}^{\text{piso}} S$ into the multiplier algebra $\mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S))$. Define the map

$$j_{A \otimes_{\text{max}} B} : A \otimes_{\text{max}} B \rightarrow (A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S)$$

by $j_{A \otimes_{\text{max}} B} := i_A \otimes_{\text{max}} i_B$ (see [20] Lemma B.31), and therefore, we have

$$j_{A \otimes_{\text{max}} B}(a \otimes b) = (i_A \otimes_{\text{max}} i_B)(a \otimes b) = i_A(a) \otimes i_B(b) = k_{A \times_{\alpha} P}(i_A(a))k_{B \times_{\alpha} S}(i_B(b))$$

for all $a, b \in A$. Also, define a map

$$j_{P \times S} : P \times S \rightarrow \mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S))$$

by

$$j_{P \times S}(x, t) = k_{A \times_{\alpha} P} \otimes_{\text{max}} k_{B \times_{\alpha} S}(i_P(x) \otimes i_S(t)) = \overline{k_{A \times_{\alpha} P}(i_P(x))k_{B \times_{\alpha} S}(i_S(t))}$$

for all $(x, t) \in P \times S$. We claim that the triple

$$((A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S), j_{A \otimes_{\text{max}} B}, j_{P \times S})$$

is a partial-isometric crossed product of the system $(A \otimes_{\text{max}} B, P \times S, \alpha \otimes \beta)$. To prove our claim, first note that, since the homomorphisms $i_A$ and $i_B$ are nondegenerate, so is the homomorphism $j_{A \otimes_{\text{max}} B}$. Next, we show that the map $j_{P \times S}$ is a bicovariant partial-isometric representation of $P \times S$. To do so, first note that, since the actions $\alpha$ and $\beta$ are left Nica-covariant, the representations $i_P$ and $i_S$ are bicovariant (see Remark 5.24). It follows that the maps

$$j_P : P \rightarrow \mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S))$$

and

$$j_S : S \rightarrow \mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\text{max}} (B \times_{\beta}^{\text{piso}} S))$$
given by compositions

\[
P \xrightarrow{i_P} \mathcal{M}(A \times_{\alpha}^{\text{piso}} P) \xrightarrow{k_{A \times_{\alpha}^{\text{piso}} P}} \mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S))
\]

and

\[
S \xrightarrow{i_S} \mathcal{M}(B \times_{\beta}^{\text{piso}} S) \xrightarrow{k_{B \times_{\beta}^{\text{piso}} S}} \mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S)),
\]

respectively, are bicovariant partial-isometric representations of \(P\) and \(S\) in the multiplier algebra \(\mathcal{M}((A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S))\). Moreover, since the homomorphisms \(k_{A \times_{\alpha} P}\) and \(k_{B \times_{\alpha} S}\) have commuting ranges, each \(j_P(x) \ast\)-commutes with each \(j_S(t)\). Therefore, as

\[
j_{P \times S}(x, t) = k_{A \times_{\alpha} P}(i_P(x))k_{B \times_{\alpha} S}(i_S(t)) = j_P(x)j_S(t),
\]

it follows by Lemma 5.2 that the map \(j_{P \times S}\) must be a bicovariant partial-isometric representation of \(P \times S\). Moreover, by using the covariance equations of the pairs \((i_A, i_P)\) and \((i_B, i_S)\), and the commutativity of the ranges of the homomorphisms \(k_{A \times_{\alpha} P}\) and \(k_{B \times_{\beta} S}\), one can see that the pair \((j_{A \otimes_{\max} B}, j_{P \times S})\) satisfies the covariance equations

\[
j_{A \otimes_{\max} B}((\alpha \otimes \beta)(x, t))(a \otimes b) = j_{P \times S}(x, t)j_{A \otimes_{\max} B}(a \otimes b)j_{P \times S}(x, t)\ast
\]

and

\[
j_{P \times S}(x, t)\ast j_{P \times S}(x, t)j_{A \otimes_{\max} B}(a \otimes b) = j_{A \otimes_{\max} B}(a \otimes b)j_{P \times S}(x, t)\ast j_{P \times S}(x, t).
\]

Next, suppose that the pair \((\pi, U)\) is covariant partial-isometric representation of \((A \otimes_{\max} B, P \times S, \alpha \otimes \beta)\) on a Hilbert space \(H\). We want to get a nondegenerate representation \(\pi \otimes U\) of \((A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S)\) such that

\[
(\pi \otimes U) \circ j_{A \otimes_{\max} B} = \pi \quad \text{and} \quad (\pi \otimes U) \circ j_{P \times S} = U.
\]

Let \((k_A, k_B)\) be the canonical pair of the \(C^*\)-algebras \(A\) and \(B\) into the multiplier algebra \(\mathcal{M}(A \otimes_{\max} B)\). The compositions

\[
A \xrightarrow{k_A} \mathcal{M}(A \otimes_{\max} B) \xrightarrow{\pi} B(H)
\]

and

\[
B \xrightarrow{k_B} \mathcal{M}(A \otimes_{\max} B) \xrightarrow{\pi} B(H)
\]

give us the nondegenerate representations \(\pi_A\) and \(\pi_B\) of \(A\) and \(B\) on \(H\) with commuting ranges, respectively. This is due to the fact that the ranges of \(i_A\) and \(i_B\) commute (see also [20, Corollary B.22]). Also, define the maps

\[
V : P \rightarrow B(H) \quad \text{and} \quad W : S \rightarrow B(H)
\]

by

\[
V_x := U_{(x, e_S)} \quad \text{and} \quad W_t := U_{(e_P, t)}
\]

for all \(x \in P\) and \(t \in S\), respectively. Since the representation \(U\) already satisfies the right Nica covariance condition (3.1), if we show that it satisfies the left Nica covariance condition (5.2), too, then it is bicovariant. Therefore, it follows by Lemma 5.2 that the maps \(V\) and \(W\) are bicovariant partial-isometric representations such that each \(V_x \ast\)-commutes with each \(W_t\) for all \(x \in P\) and \(t \in S\). By Remark 5.4, we
only need to verify that the action $\alpha \otimes \beta$ in the system $(A \otimes_{\max} B, P \times S, \alpha \otimes \beta)$ is left Nica-covariant. Firstly, we have

$$\overline{(\alpha \otimes \beta)}_{(x,t)} \circ (\overline{k_A \otimes_{\max} k_B}) = \overline{\alpha_x \otimes \beta_t \circ (k_A \otimes_{\max} k_B)} = (\overline{k_A \circ \pi_x}) \otimes_{\max} (\overline{k_B \circ \beta_t})$$

for all $(x,t) \in P \times S$ (see [15], Lemma 2.3). It follows that

$$\overline{(\alpha \otimes \beta)}_{(x,r)}(1_M(A \otimes_{\max} B))(\overline{(\alpha \otimes \beta)}_{(y,s)}(1_M(A \otimes_{\max} B)))$$

$$= \overline{\alpha_x \otimes \beta_r (k_A(1_M(A))k_B(1_M(B)))} \overline{\alpha_y \otimes \beta_s (k_A(1_M(A))k_B(1_M(B)))}$$

$$= \overline{k_A(\pi_x(1_M(A)))k_B(\beta_r(1_M(B)))} \overline{k_A(\pi_y(1_M(A)))k_B(\beta_s(1_M(B)))}$$

$$= \overline{k_A(\pi_x(1_M(A))\pi_y(1_M(A)))k_B(\beta_r(1_M(B))\beta_s(1_M(B)))}$$

and hence,

$$\overline{(\alpha \otimes \beta)}_{(x,r)}(1_M(A \otimes_{\max} B))(\overline{(\alpha \otimes \beta)}_{(y,s)}(1_M(A \otimes_{\max} B)))$$

$$= \overline{k_A(\pi_x(1_M(A))\pi_y(1_M(A)))k_B(\beta_r(1_M(B))\beta_s(1_M(B)))}$$

for all $(x,r), (y,s) \in P \times S$. Now, if

$$(x,r)(P \times S) \cap (y,s)(P \times S) = (z,t)(P \times S)$$

for some $(z,t) \in P \times S$, then, since

$$xP \cap yP = zP \quad \text{and} \quad rS \cap sS = tS,$$

and the actions $\alpha$ and $\beta$ are left Nica-covariant, for (5.12), we get

$$\overline{(\alpha \otimes \beta)}_{(x,r)}(1_M(A \otimes_{\max} B))(\overline{(\alpha \otimes \beta)}_{(y,s)}(1_M(A \otimes_{\max} B)))$$

$$= \overline{k_A(\pi_x(1_M(A)))k_B(\beta_r(1_M(B)))} \overline{k_A(\pi_y(1_M(A)))k_B(\beta_s(1_M(B)))}$$

$$= \overline{k_A(\pi_x(1_M(A))\pi_y(1_M(A)))k_B(\beta_r(1_M(B))\beta_s(1_M(B)))} \quad \text{[by (5.11)]}$$

If $(x,r)(P \times S) \cap (y,s)(P \times S) = \emptyset$, then

$$xP \cap yP = \emptyset \quad \text{or} \quad rS \cap sS = \emptyset,$$

which implies that

$$\overline{\pi_x(1_M(A))\pi_y(1_M(A))} = 0 \quad \text{or} \quad \overline{\beta_r(1_M(B))\beta_s(1_M(B))} = 0$$

as $\alpha$ and $\beta$ are left Nica-covariant. Thus, for (5.12), we get

$$\overline{(\alpha \otimes \beta)}_{(x,r)}(1_M(A \otimes_{\max} B))(\overline{(\alpha \otimes \beta)}_{(y,s)}(1_M(A \otimes_{\max} B))) = 0.$$

So, the action $\alpha \otimes \beta$ is left Nica-covariant.

Now, consider the pairs $(\pi_A, V)$ and $(\pi_B, W)$. They are indeed the covariant partial-isometric representations of the systems $(A, P, \alpha)$ and $(B, S, \beta)$ on $H$, respectively.
We only show this for \((\pi_A, V)\) as the proof for \((\pi_B, W)\) follows similarly. We have to show that the pair \((\pi_A, V)\) satisfies the covariance equations (4.11). We have

\[
V_x \pi_A(a) V_x^* = U_{(x, e_S)} \pi(k_A(a)) U_{(x, e_S)}^*
\]

\[
= U_{(x, e_S)} \pi(k_A(a)) \pi(1_{\mathcal{M}(A \otimes_{\text{max}} B)}) U_{(x, e_S)}^*
\]

\[
= U_{(x, e_S)} \pi(k_A(a)) \pi(1_{\mathcal{M}(B)}) U_{(x, e_S)}^*
\]

\[
= U_{(x, e_S)} \pi(k_A(a)) \pi(\overline{1_{\mathcal{M}(B)}}) U_{(x, e_S)}^*
\]

\[
= \pi((\alpha \otimes \beta)(t_{(x, e_S)} (k_A(a)) \overline{k_B(1_{\mathcal{M}(B)}))) [\text{by the covariance of } (\pi, U)]
\]

\[
= \pi(\alpha_x \otimes \beta_{e_S}(k_A(a)) \overline{k_B(1_{\mathcal{M}(B)}))} [\text{by (5.11)}]
\]

\[
= \pi(k_A(\alpha_x(a))) \overline{k_B(1_{\mathcal{M}(B)}))}
\]

\[
= \pi(k_A(\alpha_x(a))) \overline{1_{\mathcal{M}(A \otimes_{\text{max}} B))}
\]

\[
= \pi(k_A(\alpha_x(a))) = \pi_A(\alpha_x(a))
\]

for all \(a \in A\) and \(x \in P\). Also, by a similar calculation using the covariance of the pair \((\pi, U)\), it follows that

\[
\pi_A(a) V_x^* V_x = V_x^* V_x \pi_A(a).
\]

Consequently, there are nondegenerate representations \(\pi_A \times V\) and \(\pi_B \times W\) of the algebras \(A \times^\text{piso}_\alpha P\) and \(B \times^\text{piso}_\beta S\) on \(H\), respectively, such that

\[
(\pi_A \times V) \circ i_A = \pi_A, \quad \pi_A \times V \circ i_P = V \quad \text{and} \quad (\pi_B \times W) \circ i_B = \pi_B, \quad \pi_B \times W \circ i_S = W.
\]

Next, we aim to show that the representations \(\pi_A \times V\) and \(\pi_B \times W\) have commuting ranges, from which it follows that there is a representation \((\pi_A \times V) \otimes_{\text{max}} (\pi_B \times W)\) of \((A \times^\text{piso}_\alpha P) \otimes_{\text{max}} (B \times^\text{piso}_\beta S)\), which is the desired (nondegenerate) representation \(\pi \times U\). So, it suffices to see that the pairs \((\pi_A, \pi_B), (V, W), (V^*, W), (\pi_A, W),\) and \((\pi_B, V)\) have commuting ranges. We already saw that this is indeed true for the first three pairs. So, we compute to show that this is also true for the pair \((\pi_A, W)\) and skip the similar computation for the pair \((\pi_B, V)\). We have

\[
W_t \pi_A(a) = U_{(e_P, t)} \overline{\pi(k_A(a))}
\]

\[
= U_{(e_P, t)} \overline{\pi(k_A(a))} 1_{\mathcal{M}(A \otimes_{\text{max}} B)}
\]

\[
= U_{(e_P, t)} \overline{\pi(k_A(a))} \overline{1_{\mathcal{M}(B)}}
\]

\[
= \pi((\alpha \otimes \beta)(e_P, t)(k_A(a)) \overline{1_{\mathcal{M}(B)}))) U_{(e_P, t)} [\text{by the covariance of } (\pi, U)]
\]

\[
= \pi(\alpha_{e_P} \otimes \beta_{e_S}(k_A(a)) \overline{1_{\mathcal{M}(B)})) U_{(e_P, t)} [\text{by (5.11)}]
\]

\[
= \pi(k_A(\alpha_{e_P}(a))) \overline{1_{\mathcal{M}(B)}} U_{(e_P, t)}
\]

\[
= \pi(k_A(\alpha_{e_P}(a))) \overline{1_{\mathcal{M}(B)}} U_{(e_P, t)}
\]

\[
= \pi_A(a) \overline{1_{\mathcal{M}(B)}} W_t
\]

\[
= \pi_A(a) W_t W_t^* W_t [\text{by the covariance of } (\pi_B, W)]
\]

Thus, there is a representation \((\pi_A \times V) \otimes_{\text{max}} (\pi_B \times W)\) of \((A \times^\text{piso}_\alpha P) \otimes_{\text{max}} (B \times^\text{piso}_\beta S)\) on \(H\) such that

\[
(\pi_A \times V) \otimes_{\text{max}} (\pi_B \times W))(\xi \otimes \eta) = (\pi_A \times V)(\xi)(\pi_B \times W)(\eta)
\]
for all \( \xi \in (A \times_{\alpha}^{piso} P) \) and \( \eta \in (B \times_{\beta}^{piso} S) \). Let

\[
\pi \times U = (\pi_A \times V) \otimes_{\max} (\pi_B \times W),
\]

which is nondegenerate as both representations \( \pi_A \times V \) and \( \pi_B \times W \) are. Then, we have

\[
\pi \times U(j_{A \otimes_{\max} B}(a \otimes b)) = \pi \times U(i_A(a) \otimes i_B(b)) = (\pi_A \times V)(i_A(a))(\pi_B \times W)(i_B(b)) = \pi_A(a)\pi_B(b) = \pi(k_A(a))\pi(k_B(b)) = \pi(k_A(a)k_B(b)) = \pi(a \otimes b).
\]

To see \((\pi \times U) \circ j_{P \times S} = U\), we apply the equation

\[
\pi \times U \circ (k_{A \times_\alpha P} \otimes_{\max} k_{B \times_\alpha S}) = (\pi_A \times V) \otimes_{\max} (\pi_B \times W) \circ (k_{A \times_\alpha P} \otimes_{\max} k_{B \times_\alpha S}) = (\pi_A \times V) \otimes_{\max} (\pi_B \times W),
\]

which is valid by [15] Lemma 2.4. Therefore, we have

\[
\pi \times U(j_{P \times S}(x,t)) = \pi \times U(k_{A \times_\alpha P}(i_P(x))k_{B \times_\alpha S}(i_S(t))) = \pi \times U(k_{A \times_\alpha P} \otimes_{\max} k_{B \times_\alpha S}(i_P(x) \otimes i_S(t))) = \pi \times U \circ (k_{A \times_\alpha P} \otimes_{\max} k_{B \times_\alpha S})(i_P(x) \otimes i_S(t)) = (\pi_A \times V) \otimes_{\max} (\pi_B \times W)(i_P(x) \otimes i_S(t)) = (\pi_A \times V)(i_P(x))(\pi_B \times W)(i_S(t)) = V_x W_t = U(x \in S U(x, t) = U(x, t).
\]

Finally, as the algebras \( A \times_{\alpha}^{piso} P \) and \( B \times_{\beta}^{piso} S \) are spanned by the elements \( i_P(x)i_A(a)i_P(y) \) and \( i_S(r)i_B(b)i_S(t) \), respectively, the algebra \( (A \times_{\alpha}^{piso} P) \otimes_{\max} (B \times_{\beta}^{piso} S) \) is spanned by the elements

\[
[i_P(x)i_A(a)i_P(y)] \otimes [i_S(r)i_B(b)i_S(t)] = k_{A \times_\alpha P}(i_P(x)i_A(a)i_P(y))k_{B \times_\alpha S}(i_S(r)i_B(b)i_S(t)) = k_{A \times_\alpha P}(i_P(x)i_A(a))k_{B \times_\alpha S}(i_S(r)i_B(b)i_S(t)) = k_{A \times_\alpha P}(i_P(x))k_{B \times_\alpha S}(i_S(r)i_B(b)i_S(t)) = k_{A \times_\alpha P}(i_P(x))k_{B \times_\alpha S}(i_S(r))j_{A \otimes_{\max B}(a \otimes b)j_{P \times S}(y,t)} = j_{P \times S}(x,r)j_{A \otimes_{\max B}(a \otimes b)j_{P \times S}(y,t)}.
\]

So, the triple

\[
((A \times_{\alpha}^{piso} P) \otimes_{\max} (B \times_{\beta}^{piso} S), j_{A \otimes_{\max B}, j_{P \times S}})
\]

is a partial-isometric crossed product of the system \((A \otimes_{\max B}, P \times S, \alpha \otimes \beta)\). It thus follows that there is an isomorphism

\[
\Gamma : ((A \otimes_{\max B} \times_{\alpha \otimes \beta} P \times S, i_{A \otimes_{\max B}, i_{P \times S}}) \to (A \times_{\alpha}^{piso} P) \otimes_{\max} (B \times_{\beta}^{piso} S)
\]

such that

\[
\Gamma(i_{P \times S}(x,r)i_{A \otimes_{\max B}(a \otimes b)i_{P \times S}(y,t)) = j_{P \times S}(x,r)i_{A \otimes_{\max B}(a \otimes b)i_{P \times S}(y,t)) = [i_P(x)i_A(a)i_P(y)] \otimes [i_S(r)i_B(b)i_S(t)].
\]

This completes the proof. \(\square\)
Let $P$ be a unital semigroup such that itself and the opposite semigroup $P^o$ are both left LCM. For every $y \in P$, define a map $1_y : P \to \mathbb{C}$ by

$$1_y(x) = \begin{cases} 
1 & \text{if } x \in yP, \\
0 & \text{otherwise}, 
\end{cases}$$

which is the characteristic function of $yP$. Each $1_y$ is obviously a function in $\ell^\infty(P)$. Then, since $P$ is right LCM, one can see that we have

$$1_x 1_y = \begin{cases} 
1_z & \text{if } xP \cap yP = zP, \\
0 & \text{otherwise}, 
\end{cases}$$

Note that, if $\tilde{z}P = xP \cap yP = zP$, then there is an invertible element $u$ of $P$ such that $\tilde{z} = zu$. It therefore follows that $s \in zP$ if and only if $s \in \tilde{z}P$ for all $s \in P$, which implies that we must have $1_z = 1_{\tilde{z}}$. So, the above equation is well-defined. Also, we clearly have $1_y^y = 1_y$ for all $y \in P$. Therefore, if $B_P$ is the $C^*$-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions $\{1_y : y \in P\}$, then we have

$$B_P = \text{span}\{1_y : y \in P\}.$$ 

Note that the algebra $B_P$ is abelian and unital, whose unit element is $1_e$ which is a constant function on $P$ with the constant value 1. One can see that, in fact, $1_u = 1_e$ for every $u \in P^*$. In addition, the shift on $\ell^\infty(P)$ induces an action on $B_P$ by injective endomorphisms. More precisely, for every $x \in P$, the map $\alpha_x : \ell^\infty(P) \to \ell^\infty(P)$ defined by

$$\alpha_x(f)(t) = \begin{cases} 
f(r) & \text{if } t = xr \text{ for some } r \in P (\equiv t \in xP), \\
0 & \text{otherwise}, \end{cases}$$

for every $f \in \ell^\infty(P)$ is an injective endomorphism of $\ell^\infty(P)$. Also, the map

$$\alpha : P \to \text{End}(\ell^\infty(P)); \ x \mapsto \alpha_x$$

is a semigroup homomorphism such that $\alpha_e = \text{id}$, which gives us an action of $P$ on $\ell^\infty(P)$ by injective endomorphisms. Since $\alpha_x(1_y) = 1_{xy}$ for all $x, y \in P$, $\alpha_x(B_P) \subset B_P$, and therefore the restriction of the action $\alpha$ to $B_P$ gives an action

$$\tau : P \to \text{End}(B_P)$$

by injective endomorphisms such that $\tau_x(1_y) = 1_{xy}$ for all $x, y \in P$. Note that $\tau_x(1_e) = 1_x \neq 1_e$ for all $x \in P \setminus P^*$. Consequently, we obtain a dynamical system $(B_P, P, \tau)$, for which, we want to describe the corresponding partial-isometric crossed product $(B_P \times^{\text{piso}}_\tau P, i_{B_P}, i_P)$. More precisely, we want to show that the algebra $B_P \times^{\text{piso}}_\tau P$ is universal for bicovariant partial-isometric representations of $P$. Once, we have done this, it would be proper to denote $B_P \times^{\text{piso}}_\tau P$ by $C^*_\text{bicov}(P)$. So, this actually generalizes [8, Proposition 9.6] from the positive cones of quasi lattice-ordered groups (in the sense of Nica [17]) to LCM semigroups.

To start, for our purpose, we borrow some notations from quasi lattice-ordered groups. So, for every $x, y \in P$, if $xP \cap yP = zP$ for some $z \in P$, which means that $z$ is a least common right multiple of $x$ and $y$, then we denote such an element $z$ by $x \vee_H y$, which may not be unique. If $xP \cap yP = \emptyset$, then we denote $x \vee_H y = \infty$. Note that we are using the notation $\vee_H$ to indicate that we are treating $P$ as a right LCM
semigroup. But if we are treating $P$ as a left LCM semigroup, then we use the notation $\vee_{rt}$ to distinguish it from $\vee_{lt}$. Moreover, if $F = \{x_1, x_2, ..., x_n\}$ is any finite subset of $P$, then $\sigma F$ is written for $x_1 \vee_{lt} x_2 \vee_{lt} ... \vee_{lt} x_n$. Therefore, if $\bigcap_{x \in F} xP = \bigcap_{i=1}^{n} x_i P \neq \emptyset$, $\sigma F$ denotes an element in

$$\{y : \bigcap_{i=1}^{n} x_i P = yP\},$$

and if $\bigcap_{i=1}^{n} x_i P = \emptyset$, then $\sigma F = \infty$.

**Lemma 5.6.** Let $P$ be a unital semigroup such that itself and the opposite semigroup $P^o$ are both left LCM. Let $V$ be any bicovariant partial-isometric representation of $P$ on a Hilbert space $H$. Then:

(i) there is a (unital) representation $\pi_V$ of $B_P$ on $H$ such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in P$;

(ii) the pair $(\pi_V, V)$ is a covariant partial-isometric representation of $(B_P, P, \tau)$ on $H$.

**Proof.** We prove (i) by making some adjustment to the proof of [14, Proposition 1.3 (2)] for the particular family

$$\{L_x := V_x V_x^* : x \in P\}$$

of projections, which satisfies

$$L_0 = 1 \quad \text{and} \quad L_x L_y = L_{x \vee_{lt} y}.$$ 

Note that $L_\infty$ is defined to be zero (projection). Now, define a map $\pi : \text{span}\{1_x : x \in P\} \to B(H)$ by

$$\pi\left(\sum_{i=1}^{n} \lambda_{x_i} 1_{x_i}\right) = \sum_{i=1}^{n} \lambda_{x_i} L_{x_i} = \sum_{i=1}^{n} \lambda_{x_i} V_{x_i} V_{x_i}^*,$$

where $\lambda_{x_i} \in \mathbb{C}$ for each $i$. It is obvious that $\pi$ is linear. Next, we show that

$$\| \sum_{x \in F} \lambda_x L_x \| \leq \| \sum_{x \in F} \lambda_x 1_x \|$$

for any finite subset $F$ of $P$. So, it follows that the map $\pi$ is a well-defined bounded linear map, and therefore, it extends to a bounded linear map of $B_p$ in $B(H)$ such that $1_x \mapsto V_x V_x^*$ for all $x \in P$. To see (5.13), we exactly follow [14, Lemma 1.4] to obtain an expression for the norm of the forms $\sum_{x \in F} \lambda_x L_x$ by using an appropriate set of mutually orthogonal projections. So, if $F$ is any finite subset of $P$, then for every nonempty proper subset $A$ of $F$, take $Q^L_A = \Pi_{x \in F \setminus A}(L_{\sigma A} - L_{\sigma A \vee_{lt} x})$. Moreover, let $Q^L_\emptyset = \Pi_{x \in F}(1 - L_x)$ and $Q^L_F = \Pi_{x \in F} L_x = L_{\sigma F}$. Then, exactly by following the proof of [14, Lemma 1.4], we can show that $Q^L_A : A \subseteq F$ is a decomposition of the identity into mutually orthogonal projections, such that

$$\sum_{x \in F} \lambda_x L_x = \sum_{A \subseteq F} \left(\sum_{x \in A} \lambda_x\right) Q^L_A$$

for any finite subset $F$ of $P$.
Thus, we get 
\( Q \sigma A \) means that at least one element in 
\( (5.16) \)
for any nonempty proper subset \( A \subset F \). Therefore, there is at least one element \( P \)
and projections \( x \\). Hence, suppose that, similarly, \( \{Q_A : A \subset F\} \) is the decomposition of the identity corresponding to the family of projections \( \{1_x : x \in F\} \). Consider

\[ Q_A = \Pi_{x \in F \setminus A}(1_{\sigma A} - 1_{\sigma AV_{\pi x}}) \]

for any nonempty proper subset \( A \subset F \). If \( \sigma A \in x_0P \) for some \( x_0 \in F \setminus A \), then \( (\sigma A)P = \bigcap_{y \in A} yP \subset x_0P \) which implies that \( (\sigma A)P \cap x_0P = (\sigma A)P \). So, we have \( \sigma A \lor lt x_0 = \sigma A \), and therefore,

\[ 1_{\sigma A} - 1_{\sigma AV_{\pi x_0}} = 1_{\sigma A} - 1_{\sigma A} = 0. \]

Thus, we get \( Q_A = 0 \). Note that when we say \( \sigma A \in x_0P \) (for some \( x_0 \in F \setminus A \)), it means that at least one element in

\[ (5.16) \]
\[ \{z : \bigcap_{y \in A} yP = zP\} \]

belongs to \( x_0P \), from which, it follows that all elements in \((5.16)\) must belong to \( x_0P \). This is due to the fact that if \( z, \tilde{z} \) are in \((5.16)\), then \( \tilde{z} = zu \) for some invertible element \( u \) of \( P \). Now, conversely, suppose that

\[ 0 = Q_A = \Pi_{x \in F \setminus A}(1_{\sigma A} - 1_{\sigma AV_{\pi x}}). \]

This implies that we must have \( Q_A(r) = 0 \) for all \( r \in P \), in particular when \( r = \sigma A \), and hence,

\[ 0 = Q_A(r) = \Pi_{x \in F \setminus A}(1_r(r) - 1_{\sigma AV_{\pi x}}(r)) = \Pi_{x \in F \setminus A}(1 - 1_{\sigma AV_{\pi x}}(r)). \]

Therefore, there is at least one element \( x_0 \in F \setminus A \) such that \( 1_{\sigma AV_{\pi x}}(r) = 1 \), which implies that we must have \( r \in (\sigma A \lor lt x_0)P = rP \cap x_0P \). It follows that \( \sigma A = r \in x_0P \) (and hence, \( r \lor lt x_0 = \sigma A \lor lt x_0 = \sigma A \)). Consequently, we have \( Q_A \neq 0 \) if and only if

\[ A = \{x \in F : \sigma A \in xP\}. \]

Eventually, we conclude that if \( Q_A^I \neq 0 \), then \( Q_A \neq 0 \). This is due to the fact that, if \( Q_A = 0 \), then there is \( x_0 \in F \setminus A \) such that \( \sigma A \in x_0P \). Therefore, we get \( Q_A = 0 \) as the factor \( L_{\sigma A} - L_{\sigma AV_{\pi x}} \) in \( Q_A^I \) becomes zero. Thus, it follows that

\[ \left\{ \left| \sum_{x \in A} \lambda_x \right| : A \subset F \text{ and } Q_A^I \neq 0 \right\} \subseteq \left\{ \left| \sum_{x \in A} \lambda_x \right| : A \subset F \text{ and } Q_A \neq 0 \right\}, \]

which implies that the inequality \((5.13)\) is valid for any finite subset \( F \) of \( P \). So, we have a bounded linear map \( \pi_V : \mathcal{B}_p \to \mathcal{B}(H) \) (the extension of \( \pi \)) such that \( \pi_V(1_x) = V_x V_x^* \) for all \( x \in P \). Furthermore, since

\[ \pi_V(1_x) \pi_V(1_y) = V_x V_x^* V_y V_y^* = V_{x \lor lt y} V_{x \lor lt y} = \pi_V(1_{x \lor lt y}) = \pi_V(1_{x \lor lt y}), \]

and obviously, \( \pi_V(1_x)^* = \pi_V(1_x^*) = \pi_V(1_x) \), it follows that the map \( \pi_V \) is actually a \( * \)-homomorphism, which is clearly unital. This completes the proof of (i).
To see (ii), it is enough to show that the pair \((\pi_V, V)\) satisfies the covariance equations \((4.1)\) on the spanning elements of \(B_P\). For all \(x, y \in P\), we have
\[
\pi_V(\tau_x(1_y)) = 1_{xy} = V_{xy}V_{xy}^* = V_xV_y[V_xV_y]^* = V_xV_yV_x^*V_y^* = V_x\pi_V(1_y)V_x^*.
\]

Also, since the product of partial isometries \(V_x\) and \(V_y\) is a partial isometry, namely, \(V_xV_y = V_{xy}\), by \([10]\) Lemma 2, each \(V_xV_x^*\) commutes with each \(V_yV_y^*\). Hence, we have
\[
V_x^*V_x\pi_V(1_y) = V_x^*V_xV_yV_y^* = V_yV_y^*V_x^*V_x = \pi_V(1_y)V_x^*V_x.
\]
So, we are done with (ii), too.

Prove. To see that \(i_P\) is a bicovariant partial-isometric representation of \(P\), we only need to show that it satisfies the left Nica covariance condition \((5.2)\). Since \(i_P(1_y) = i_P(\tau_y(1_e)) = i_P(y)i_P(1_e)i_P(y)^* = i_P(y)i_P(y)^*\) for all \(y \in P\), it follows that \(i_P\) indeed satisfies \((5.2)\). Then, as the elements \(\{1_y : y \in P\}\) generate (span) the algebra \(B_P\), the \(C^*\)-algebra \(B_P \rtimes_P^\text{piso} P\) is generated by the elements
\[
i_B(1_y)i_P(x) = i_P(y)i_P(y)^*i_P(x),
\]
which implies that \(i_P(P)\) generates \(B_P \rtimes_P^\text{piso} P\).

Suppose that now \(V\) is a bicovariant partial-isometric representation of \(P\) on a Hilbert space \(H\). Then, by Lemma 5.6 there is a covariant partial-isometric representation \((\pi_V, V)\) of \((B_P, P, \tau)\) on \(H\), such that \(\pi_V(1_x) = V_xV_x^*\) for all \(x \in P\). The corresponding (unital) representation \(\pi_V \times V\) of \((B_P \rtimes_P^\text{piso} P, i_B, i_P)\) on \(H\) is the desired representation \(V\) which satisfies \(V \circ i_P = V\).

So, as we mentioned earlier, we denote the algebra \(B_P \rtimes_P^\text{piso} P\) by \(C^*_\text{bicov}(P)\), which is universal for bicovariant partial-isometric representations of \(P\).

**Corollary 5.8.** Suppose that the (unital) semigroups \(P\), \(P^o\), \(S\), and \(S^o\) are all left LCM. Then,
\[
(5.17) \quad C^*_\text{bicov}(P \times S) \simeq C^*_\text{bicov}(P) \otimes_{\max} C^*_\text{bicov}(S).
\]

**Proof.** Corresponding to the pairs \((P, P^o)\) and \((S, S^o)\) we have the dynamical systems \((B_P, P, \tau)\) and \((B_S, S, \beta)\) along with their associated \(C^*\)-algebras
\[
(C^*_\text{bicov}(P) = B_P \rtimes_P^\text{piso} P, i_B, V)
\]
and
\[
(C^*_\text{bicov}(S) = B_S \rtimes_S^\beta S, i_B, W),
\]
respectively. Now, by Theorem 5.5, there is an isomorphism
\[ \Gamma : ((B_P \otimes_{\text{max}} B_S) \times_{\tau \otimes \beta} (P \times S), i_{B_P \otimes_{\text{max}} B_S}, T) \to (B_P \times_{\text{piso}} P) \otimes_{\text{max}} (B_S \times_{\beta} S) \]
such that
\[ \Gamma(i_{B_P \otimes_{\text{max}} B_S}(1_x \otimes 1_t) T(p,s)) = [i_{B_P}(1_x)V_p] \otimes [i_{B_S}(1_t)W_s] \]
for all \( x, p \in P \) and \( t, s \in S \). On the other hand, as the (unital) semigroups \( P \times S \) and \( (P \times S)^o \) are both left LCM, we have the dynamical system \((B_{(P \times S)}, P \times S, \alpha)\) along with its associated \( C^*\)-algebra
\[ \left( C_{\text{bicov}}^*(P \times S) = B_{(P \times S)} \times_{\alpha}^\text{piso} (P \times S), i_{B_{(P \times S)}}, U \right), \]
where the action \( \alpha : P \times S \to \text{End}(B_{(P \times S)}) \) is induced by the shift on \( \ell^\infty(P \times S) \) such that \( \alpha_{(p,s)}(1_{(x,t)}) = 1_{(p,s)(x,t)} = 1_{(px,st)}. \) Moreover, since there is an isomorphism
\[ \psi : (1_x \otimes 1_t) \in (B_P \otimes_{\text{max}} B_S) \mapsto 1_{(x,t)} \in B_{(P \times S)} \]
such that \( \psi \circ (\tau \otimes \beta) = \alpha \circ \psi, \) there is an isomorphism
\[ \Lambda : (B_P \otimes_{\text{max}} B_S) \times_{\tau \otimes \beta}^\text{piso} (P \times S) \to B_{(P \times S)} \times_{\alpha}^\text{piso} (P \times S) \]
such that
\[ \Lambda \circ i_{B_P \otimes_{\text{max}} B_S} = i_{B_{(P \times S)}} \circ \psi \quad \text{and} \quad \Lambda \circ T = U \] (see Remark 6.1).

Eventually, the composition
\[ C_{\text{bicov}}^*(P \times S) \xrightarrow{\Lambda^{-1}} (B_P \otimes_{\text{max}} B_S) \times_{\tau \otimes \beta}^\text{piso} (P \times S) \xrightarrow{\Gamma} C_{\text{bicov}}^*(P) \otimes_{\text{max}} C_{\text{bicov}}^*(S) \]
of isomorphisms gives the desired isomorphism \( (5.17) \) such that
\[ U(p,s) \mapsto V_p \otimes W_s \]
for all \( (p, s) \in P \times S. \]

6. Ideals in Tensor Products

Suppose that \( P \) is a left LCM semigroup. Let \( \alpha \) and \( \beta \) be the actions of \( P \) on \( C^*\)-algebras \( A \) and \( B \) by extendible endomorphisms, respectively. Then, there is an action
\[ \alpha \otimes \beta : P \to \text{End}(A \otimes_{\text{max}} B) \]
of \( P \) on the maximal tensor product \( A \otimes_{\text{max}} B \) by extendible endomorphisms such that
\[ (\alpha \otimes \beta)_x = \alpha_x \otimes \beta_x \quad \text{for all} \quad x \in P. \]
Note that the extendibility of \( \alpha \otimes \beta \) follows by the extendibility of the actions \( \alpha \) and \( \beta \) (see [15, Lemma 2.3]). Therefore, we obtain a dynamical system \((A \otimes_{\text{max}} B, P, \alpha \otimes \beta)\). Let \( (A \otimes_{\text{max}} B) \times_{\alpha \otimes \beta}^\text{piso} P \) be the partial-isometric crossed product of \((A \otimes_{\text{max}} B, P, \alpha \otimes \beta)\). Our main goal in this section is to obtain a composition series
\[ 0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq (A \otimes_{\text{max}} B) \times_{\alpha \otimes \beta}^\text{piso} P \]
of ideals, and then identify the subquotients
\[ \mathcal{I}_1, \quad \mathcal{I}_2/\mathcal{I}_1, \quad \text{and} \quad ((A \otimes_{\text{max}} B) \times_{\alpha \otimes \beta}^\text{piso} P)/\mathcal{I}_2 \]
with familiar terms. To do so, let us first mention some point in the following remark which is required.

**Remark 6.1.** Let \((A, P, \alpha)\) and \((B, P, \beta)\) be dynamical systems, and \(\psi : A \to B\) a nondegenerate homomorphism such that \(\psi \circ \alpha_x = \beta_x \circ \psi\) for all \(x \in P\). Suppose that \((A \times^{\text{piso}}_\alpha P, i)\) and \((B \times^{\text{piso}}_\beta P, j)\) are the partial-isometric crossed products of the systems \((A, P, \alpha)\) and \((B, P, \beta)\), respectively. Then, one can see that the pair \((j_B \circ \psi, j_P)\) is covariant partial-isometric representation of \((A, P, \alpha)\) in the algebra \(B \times^{\text{piso}}_\beta P\). Hence, there is a nondegenerate homomorphism

\[
\psi \times P := [(j_B \circ \psi) \times j_P] : A \times^{\text{piso}}_\alpha P \to B \times^{\text{piso}}_\beta P
\]

such that

\[
(\psi \times P) \circ i_A = j_B \circ \psi \quad \text{and} \quad \psi \times P \circ i_P = j_P.
\]

One can see that if \(\psi\) is an isomorphism, so is \(\psi \times P\).

**Lemma 6.2.** [15] Lemma 3.2] Suppose that \(\alpha\) and \(\beta\) are extendible endomorphisms of \(C^*\)-algebras \(A\) and \(B\), respectively. If \(I\) is an extendible \(\alpha\)-invariant of \(A\) and \(J\) is an extendible \(\beta\)-invariant ideal of \(B\), then the ideal \(I \otimes_{\max} J\) of \(A \otimes_{\max} B\) is extendible \(\alpha \otimes \beta\)-invariant.

**Remark 6.3.** It follows by the above lemma that if \((A, P, \alpha)\) and \((B, P, \beta)\) are dynamical systems, and \(I\) is an extendible \(\alpha\)-invariant of \(A\) and \(J\) is an extendible \(\beta\)-invariant ideal of \(B\) for every \(x \in P\), then \(I \otimes_{\max} J\) is an extendible \((\alpha \otimes \beta)\)-invariant ideal of \(A \otimes_{\max} B\) for all \(x \in P\). Therefore, by Theorem 4.7 the crossed product \((I \otimes_{\max} J) \times^{\text{piso}}_{\alpha \otimes \beta} P\) sits in the algebra \((A \otimes_{\max} B) \times^{\text{piso}}_{\alpha \otimes \beta} P\) as an ideal (this will be the ideal \(I_1\) shortly later). As an application, we observe that, by [20] Proposition B. 30, the short exact sequence

\[
0 \longrightarrow J \longrightarrow B \overset{q_j}{\longrightarrow} B/J \longrightarrow 0
\]

gives rise to the short exact sequence

\[
(6.1) \quad 0 \longrightarrow A \otimes_{\max} J \longrightarrow A \otimes_{\max} B \xrightarrow{id_A \otimes_{\max} q_j} A \otimes_{\max} B/J \longrightarrow 0,
\]

where \(A \otimes_{\max} J\) is an extendible \((\alpha \otimes \beta)\)-invariant ideal of \(A \otimes_{\max} B\) for all \(x \in P\). Thus, \((6.1)\) itself by Theorem 4.7 gives rise to the following short exact sequence:

\[
0 \longrightarrow (A \otimes_{\max} J) \times^{\text{piso}}_{\alpha \otimes \beta} P \overset{\mu}{\longrightarrow} (A \otimes_{\max} B) \times^{\text{piso}}_{\alpha \otimes \beta} P, i) \overset{\phi}{\longrightarrow} ((A \otimes_{\max} B/J) \times^{\text{piso}}_{\alpha \otimes \beta} P, j),
\]

where \(\tilde{\beta} : P \to \text{End}(B/J)\) is the (extendible) action induced by \(\beta\). More precisely, by Remark 6.1 the isomorphism

\[
(A \otimes_{\max} B)/(A \otimes_{\max} J) \simeq A \otimes_{\max} B/J,
\]

which intertwines the actions \(\alpha \otimes \beta\) and \(\alpha \otimes \tilde{\beta}\) of the algebras \((A \otimes_{\max} B)/(A \otimes_{\max} J)\) and \(A \otimes_{\max} B/J\), respectively, induces an isomorphism

\[
[(A \otimes_{\max} B)/(A \otimes_{\max} J)] \times^{\text{piso}}_{\alpha \otimes \tilde{\beta}} P \simeq (A \otimes_{\max} B/J) \times^{\text{piso}}_{\alpha \otimes \beta} P.
\]

Therefore, the surjective homomorphism \(\phi\) is actually given by \((id_A \otimes_{\max} q_j)^T \times P\) such that

\[
[(id_A \otimes_{\max} q_j)^T \times P] \circ i_{A \otimes_{\max} B} = j_{A \otimes_{\max} B/J} \circ (id_A \otimes_{\max} q_j) \quad \text{and} \quad [(id_A \otimes_{\max} q_j)^T \times P] \circ i_P = j_P.
\]
In the following proposition and theorem, for the maximal tensor product between the $C^*$-algebras involved, we simply write $\otimes$ for convenience.

**Proposition 6.4.** Let $(A, P, \alpha)$ and $(B, P, \beta)$ be dynamical systems, and $I$ an extendible $\alpha_x$-invariant of $A$ and $J$ an extendible $\beta_x$-invariant ideal of $B$ for every $x \in P$. Assume that $\tilde{\alpha} : P \to \text{End}(A/I)$ and $\tilde{\beta} : P \to \text{End}(B/J)$ are the actions induced by $\alpha$ and $\beta$, respectively. Then, the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & (I \otimes J) \times_{\alpha \otimes \beta} P \\
\downarrow & & \downarrow \\
(I \otimes B) \times_{\alpha \otimes \beta} P & \rightarrow & (I \otimes B/J) \times_{\alpha \otimes \beta} P \\
\downarrow & & \downarrow \\
(I \otimes B/J) \times_{\alpha \otimes \beta} P & \rightarrow & 0 \\
\end{array}
\]

commutes, where

\[
\phi_1 := (\text{id}_I \otimes q^J) \times P, \quad \phi_2 := (\text{id}_A \otimes q^J) \times P, \quad \phi_3 := (\text{id}_{A/I} \otimes q^J) \times P,
\]

\[
\varphi_1 := (q^I \otimes \text{id}_J) \times P, \quad \varphi_2 := (q^I \otimes \text{id}_B) \times P, \quad \varphi_3 := (q^I \otimes \text{id}_{B/J}) \times P.
\]

Also, there is a surjective homomorphism $q : A \otimes B \rightarrow (A/I) \otimes (B/J)$ which intertwines the actions $\alpha \otimes \beta$ and $\tilde{\alpha} \otimes \tilde{\beta}$, and therefore, we have a homomorphism $q \times P$ of $(A \otimes B) \times_{\alpha \otimes \beta} P$ onto $(A/I \otimes B/J) \times_{\tilde{\alpha} \otimes \tilde{\beta}} P$ induced by $q$. Moreover, we have

\[
\ker(q \times P) = (A \otimes J) \times_{\alpha \otimes \beta} P + (I \otimes B) \times_{\alpha \otimes \beta} P.
\]

**Proof.** First of all, in the diagram, each row as well as each column is obtained by the similar discussion to Remark 6.3, and hence, it is exact.

Next, for the quotient maps $q^I : A \rightarrow A/I$ and $q^J : B \rightarrow B/J$, by [20, Lemma B. 31], there is a homomorphism $q^I \otimes q^J : A \otimes B \rightarrow (A/I) \otimes (B/J)$, which we denote it by $q$, such that

\[
q(a \otimes b) = (q^I \otimes q^J)(a \otimes b) = q^I(a) \otimes q^J(b) = (a + I) \otimes (b + J)
\]
for all \( a \in A \) and \( b \in B \). It is obviously surjective. Moreover,

\[
q((\alpha \otimes \beta)_x(a \otimes b)) = q((\alpha_x \otimes \beta_x)(a \otimes b)) \\
= q(\alpha_x(a) \otimes \beta_x(b)) \\
= (\alpha_x(a) + I) \otimes (\beta_x(b) + J) \\
= \tilde{\alpha}_x(a + I) \otimes \tilde{\beta}_x(b + J) \\
= (\tilde{\alpha} \otimes \tilde{\beta}_x)((a + I) \otimes (b + J)) \\
= (\tilde{\alpha} \otimes \tilde{\beta}_x)(q(a \otimes b))
\]

(6.4)

for all \( x \in P \). Therefore, by Remark 6.1, there is a (nondegenerate) homomorphism

\[
q \times P : ((A \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P, i) \rightarrow ((A/I \otimes B/J) \times^{\text{piso}}_{\tilde{\alpha} \otimes \tilde{\beta}} P, k)
\]

such that

\[
(q \times P) \circ i_{A \otimes B} = k_{A/I \otimes B/J} \circ q \quad \text{and} \quad q \times P \circ i_P = k_P.
\]

One can easily see that as \( q \) is surjective, so is \( q \times P \).

Now, by inspection on spanning elements, it follows that the diagram commutes.

Finally, to see (6.3), we only show that

\[
\ker(q \times P) \subset (A \otimes J) \times^{\text{piso}}_{\alpha \otimes \beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P
\]

as the other inclusion can be verified easily. To do so, take a nondegenerate representation

\[
\pi : (A \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P \rightarrow B(H)
\]

with

\[
\ker \pi = (A \otimes J) \times^{\text{piso}}_{\alpha \otimes \beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P.
\]

Then, define a map \( \rho : (A/I \otimes B/J) \rightarrow B(H) \) by

\[
\rho(q(\xi)) = \pi(i_{A \otimes B}(\xi))
\]

for all \( \xi \in (A \otimes B) \). Since

\[
(A \otimes J) + (I \otimes B) = \ker q \subset \ker(\pi \circ i_{A \otimes B}),
\]

it follows that the map \( \rho \) is well-defined, which is actually a nondegenerate representation. Also, the composition

\[
P \xrightarrow{i_P} \mathcal{M}((A \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P) \xrightarrow{\pi} B(H)
\]

gives a (right) Nica partial-isometric representation \( W : P \rightarrow B(H) \). Now, by applying the covariance equations of the pair \((i_{A \otimes B}, i_P)\) and (6.4), one can see that the pair \((\rho, W)\) is a covariant partial-isometric representation of \((A/I \otimes B/J, P, \tilde{\alpha} \otimes \tilde{\beta})\) on \( H \). The corresponding representation \( \rho \times W \) lifts to \( \pi \), which means that

\[
(\rho \times W) \circ (q \times P) = \pi,
\]

from which, it follows that

\[
\ker(q \times P) \subset \ker \pi = (A \otimes J) \times^{\text{piso}}_{\alpha \otimes \beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha \otimes \beta} P.
\]

Thus, the equation (6.3) holds.

\(\square\)
Theorem 6.5. Let \((A, P, \alpha)\) and \((B, P, \beta)\) be dynamical systems, and \(I\) an extendible \(\alpha_x\)-invariant of \(A\) and \(J\) an extendible \(\beta_x\)-invariant ideal of \(B\) for every \(x \in P\). Assume that \(\bar{\alpha} : P \to \text{End}(A/I)\) and \(\bar{\beta} : P \to \text{End}(B/J)\) are the actions induced by \(\alpha\) and \(\beta\), respectively. Then, there is a composition series

\[
0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq (A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P
\]

of ideals, such that:

(i) the ideal \(\mathcal{I}_1\) is (isomorphic to) \((I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P\);

(ii) \(\mathcal{I}_2/\mathcal{I}_1 \simeq (A/I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P \oplus (I \otimes B/J) \times_{\bar{\alpha} \circ \bar{\beta}} P\);

(iii) the surjection \(q \times P\) induces an isomorphism of \(((A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P)/\mathcal{I}_2\) onto \((A/I \otimes B/J) \times_{\bar{\alpha} \circ \bar{\beta}} P\).

Proof. For (i), as we mentioned in Remark 6.3, \(I \otimes J\) is an extendible \((\alpha \otimes \beta)_x\)-invariant ideal of \(A \otimes B\) for all \(x \in P\). Therefore, by Theorem 4.7, the crossed product \((I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P\) sits in the algebra \((A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P\) as an ideal, which we denote it by \(\mathcal{I}_1\).

To get (ii), we first define

\[
\mathcal{I}_2 := (A \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P + (I \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P,
\]

which is an ideal of \((A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P\) as each summand is. Note that we have

\[
[(A \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P] \cap [(I \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P] = (I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P
\]

Also, by the similar discussion to Remark 6.3 (see also diagram 6.2), it follows that

\[
\mathcal{I}_2/\mathcal{I}_1 = [(A \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P + (I \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P]/(I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P
\]

\[
= [(A \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P]/[(I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P] \oplus [(I \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P]/[(I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P]
\]

\[
\simeq [(A \otimes J)/(I \otimes J)] \times_{\bar{\alpha} \circ \bar{\beta}} P \oplus [(I \otimes B)/(I \otimes J)] \times_{\bar{\alpha} \circ \bar{\beta}} P
\]

\[
\simeq (A/I \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P \oplus (I \otimes B/J) \times_{\bar{\alpha} \circ \bar{\beta}} P.
\]

Finally, for (iii), we recall from Proposition 6.4 that we have a surjective homomorphism

\[
q \times P : (A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P \to (A/I \otimes B/J) \times_{\bar{\alpha} \circ \bar{\beta}} P
\]

with

\[
\ker(q \times P) = (A \otimes J) \times_{\bar{\alpha} \circ \bar{\beta}} P + (I \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P = \mathcal{I}_2.
\]

Therefore, we have

\[
((A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P)/\mathcal{I}_2 = ((A \otimes B) \times_{\bar{\alpha} \circ \bar{\beta}} P)/\ker(q \times P)
\]

\[
\simeq (A/I \otimes B/J) \times_{\bar{\alpha} \circ \bar{\beta}} P.
\]
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