Effective mass staircase and the Fermi liquid parameters for the fractional quantum Hall composite fermions

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Effective mass of the composite fermion in the fractional quantum Hall system, which is of purely interaction originated, is shown, from a numerical study, to exhibit a curious nonmonotonic behavior with a staircase correlated with the number ($=2,4,\cdots$) of attached flux quanta. This is surprising since the usual composite-fermion picture predicts a smooth behavior. On top of that, significant interactions are shown to exist between composite fermions, where the excitation spectrum is accurately reproduced in terms of Landau’s Fermi liquid picture with negative (i.e., Hund’s type) orbital and spin exchange interactions.

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Motivation.— Although the composite-fermion (CF) picture for the fractional quantum Hall (FQH) system provides a fascinating way of understanding the electron system interacting in a two-dimensional space in a strong magnetic field, there still remain fundamental properties of the CF that need to be revealed. Indeed, the mass, the most important quantity to characterize a particle, is far from fully understood for the composite fermion. There is a good reason for this — unlike usual many-body systems where we can talk about effects of the interaction on the kinetic energy, the FQH system has a singular and intriguing situation where the kinetic energy is quenched (with the bare mass being “infinite”) so that a finite effective mass is purely interaction-originated when the magnetic field is strong enough and a single Landau level may be considered.

Hence the effective mass has to be put in by hand in the composite fermion picture. In this picture the incompressible FQH liquid for odd-fraction Landau-level filling $\nu = p/(\phi_p \pm 1)$ ($p$: an integer) is regarded, in the mean-field sense, as an integer quantum Hall state of CF’s with $\nu^* = p$, where a CF is an electron attached with an even number, $\phi$, of filamentary flux quanta. Since the external magnetic field is partly sucked by electrons, each CF feels, on average, a reduced magnetic field, $eB^* = eB - 2\pi \phi n_e$ where $n_e$ is the number density of electrons, and the Landau level spacing becomes an effective cyclotron energy $\omega_{e}^* = eB^*/m_0^*$ (in natural units with $\hbar = c = 1$) with $m_0^*$ being the effective mass in question. However, the system of CF’s reduces to the original one only in the mean-field sense, and fluctuations around the mean field, being a gauge field, should be strong due to the absence of small parameters such as interaction/kinetic energy. This should be especially so at even fractions, where the excitation gap $\propto B^* \sim 0$ vanishes. In a seminal paper, Halperin, Lee, and Read systematically developed the CF theory. However, the effective mass remains to be difficult quantity to estimate, except for a dimensional argument that $eB^*/m^* \sim e^2/\ell$ where $\ell$ is the magnetic length.

Another fundamental question is: the CF picture, in its naive form, does not say anything about the electron-electron repulsion, which is the very origin of the FQH states. This flaw has recently been remedied by considering a “dipole” CF, where the flux attachment is regarded as mimicking the correlation hole due to Coulomb repulsion between electrons. Still, there should be residual interactions between CF’s, which may be strong, since the strong gauge field fluctuations should be reflected on the interaction. Stern and Halperin have constructed the Fermi liquid theory of CF’s with a singular Landau function to renormalize the CF mass consistently. However, the validity of the perturbative treatments of the residual interaction between CF’s as well as the quantitative estimate of the CF effective mass are still some way from a full understanding due to the quenched kinetic energy. Hence numerical studies are valuable as an approach complementary to analytic ones.

In our recent work, we have questioned, as a crucial test for a many-body theory, whether the CF picture can reproduce low-lying excitation spectra. We have shown, from an exact-diagonalization for finite systems, that there exists a striking one-to-one correspondence in the low-lying excitation spectra between the exact result and the free CF system in zero magnetic field at even-fraction $\nu$’s. This has enabled us to estimate the CF effective mass from the lowest excitation energy, by assuming free CF’s, to show that $m^*$ becomes heavier as we go to higher even fractions ($\nu = 1/2 \rightarrow 1/4 \rightarrow 1/6$).

However, we definitely need to examine this more systematically — (i) First, we want to study $m^*$ for general fractions, including odd fractions on which Laughlin’s quantum liquid resides. (ii) Second, the residual CF interaction should be probed, in terms of Landau’s Fermi liquid parameters, especially in the presence of spin degrees of freedom. These are exactly what we have done here with the numerical (Lanczos) diagonalization for finite systems in the subspace projected to the lowest Landau level. We shall show that (i) the CF effective mass exhibits a curious staircase against $\nu$, which is surprising
since the usual CF theory would predict a smooth dependence. (ii) Low-lying excitation spectra indicate that there are indeed significant interactions between CF's, which can be accurately described, in finite systems, in terms of Landau's Fermi liquid parameters with negative (i.e., Hund's type) spin and orbital exchange interactions.

For the numerical study the spherical system is adopted, following Haldane. Dirac's quantization condition dictates that the total flux \( 4\pi R^2 B \) be an integral \( (N_\phi) \) multiple of the flux quantum, where \( R \) is the radius of the sphere. The eigenvalue of the noninteracting Hamiltonian is \( \varepsilon = 1/(2mR^2)[l(l + 1) - (N_\phi/2)^2] \), where \( l \geq N_\phi/2 \) is an integer with the equality corresponding to the lowest Landau level (LLL). The electron-electron interaction is the whole Hamiltonian in the LLL subspace. The Zeeman effect is neglected for simplicity, so the system has the SU(2) symmetry.

**Free composite fermion analysis.** — For the FQH states at odd fractions, \( \nu = 2\pi n_\nu/(eB) = d_\nu p/(d_\nu \phi p \pm 1) \), the effective mass can be estimated from the excitation gap, i.e., \( \omega^*_e \) in the CF system with \( \nu^* \equiv 2\pi n_\nu/(eB^*) = d_\nu p \). Here \( p \) is a positive integer and \( d_\nu \) is the spin degeneracy (\( d_\nu = 1 \) for the spinless case, \( d_\nu = 2 \) for the spinful case). For the spinful case we take the simplest definition, i.e., attaching the same number \( \phi \) (even) of flux quanta to up and down spin electrons.

For finite systems we have to specify the number of fluxes for a given number of electrons, \( N_e \), and \( \nu \) with some care. This can be done in terms of the CF picture. The number of flux \( N_\phi^{	ext{left}} \) left behind after the flux attachment is \( N_\phi^{	ext{left}} = |N_e - \phi(N_e - 1)| \). For the integer \( Q \) state of CF's, \( N \) and \( N_\phi^{	ext{left}} \) must satisfy \( N_e = d_\nu p(N_\phi^{	ext{left}} + p) \), because CF's fill up to the \( p \)-th effective Landau level for each spin. Then \( N_\phi^{	ext{left}} \) is given by \( N_\phi^{	ext{left}} = (d_\nu \phi p \pm 1)N_e/(d_\nu p) - (\phi \pm p) \), where \( \pm \) corresponds to \( \nu \leq 1/\phi \). The energy gap in the mean field is \( \Delta \approx \omega^*_e \), but on a finite sphere the precise expression is

\[
\Delta = \frac{1}{m_0^* R^2} \left( \frac{N_\phi^{	ext{left}}}{2} + p \right) = \frac{4\pi n_e}{m_0^*} \cdot \frac{N_e + d_\nu p^2}{2d_\nu p(N_e - 1)}. \quad (1)
\]

Note that, with \( R \) scaling as \( 4\pi n_e R^2 = N_e - 1 \), \( B \) (hence \( B^* \)) are functions of \( N_e \) to satisfy Dirac's quantization. We can obtain a similar expression for the metallic case with \( \nu = 1/\phi \) - even fractions with \( B^* = 0 \) by putting \( N_\phi^{	ext{left}} = 0 \) (\( N_e = d_\nu p^2 \)). This time we can look at how the gap vanishes for \( N_e \rightarrow \infty \), which has a scaling

\[
\Delta = \frac{l_F + 1}{m_0^* R^2} = \frac{4\pi n_e}{m_0^*} \frac{\sqrt{N_e}}{\sqrt{d_\nu(N_e - 1)}}. \quad (2)
\]

as a one-exciton (one electron-hole pair) excitation if we concentrate on the closed-shell case with \( N_e/d_\nu = (l_F + 1)^2 \) where \( l_F \) is the Fermi angular momentum.

**Effective mass against \( \nu \).** — Let us first estimate the CF effective mass in the noninteracting CF picture, eqs. (1) and (2), for various \( N_e \) for each value of Landau-level filling, in both the spinless and spinful cases. The maximum system size has the dimension of the Hamiltonian matrix as large as 23 million. We have varied the magnetic field for each value of the number of electrons to have various \( \nu \), so that we have normalized the mass by \( e^2/(4\pi n_e)^{1/2} \), since this way the plot of \( m_0^* \) versus \( \nu \) may be regarded as representing the magnetic field dependence of \( m_0^* \).

In Fig. 1 which plots the inverse effective mass against \( \nu \) for the spinless case, we can immediately see that the effective mass is very nonmonotonic with a step-function-like behavior each time the Landau-level filling passes the boundary across difference numbers of attached flux quanta, i.e., \( \phi \rightarrow \phi + 2 \). This is totally unexpected, since the CF theory, with its mean-field treatment after a certain canonical transformation, predicts a smooth function (dashed line in the figure). Curiously, if we look at the positions of the steps, the result dictates that we have to attach exactly two flux quanta (\( \phi = 2 \) at \( \nu = 1/3 \), while we could have chosen from two or four since \( \nu = 1/3 \) is just the boundary between the two choices in the CF picture.

We are referring to a large gap in \( 1/m_0^* \) between \( \nu = 1/3 \) and \( 2/7 \), but, to be precise, whether this is a jump or a continuous curve will have to be studied. For the sample size studied here, we see only a shoulder around \( \nu \sim 1/5 \) because the energy scale becomes small there.

**Spin effect.** — Figure 2 plots the \( 1/m_0^* \) when the spin degrees of freedom are included. We can see that \( m_0^* \) appears to be about twice as heavy as that of the spinless system in Fig. 1 when we estimate the mass from spin-flip (\( \Delta S \neq 0 \)) excitations (which happen to be the lowest one), while the effective mass estimated from excitations that involve no spin flips is close to that for the spinless result above. This indicates that we should not stick to the free CF picture, but that spin-dependent (exchange) interactions between CF's have to be considered.
energies in the spinful system.

finite system) from the ground state, a functional of the deviation in the particle number (in a

term assumes that the excitation energy, \( \delta E \), is given as a functional of the deviation in the particle number (in a

usual one in 2D, \( \delta E \approx -S \cdot S \), and (b) the ground-state total spin when we go away from the closed-shell case is correctly explained by the Hund’s coupling. So we can conclude that the interaction between spinful CF’s has an exchange interaction of Hund’s type.\(^\ddagger\)

It is noted that there have been some numerical indications such as those obtained by Rezayi and Read\(^\ddagger\) in which Hund’s rule is reported for the orbital angular momenta. However, existence or otherwise of Hund’s rule is a subtle problem, especially for long-range repulsive interactions, and the situation should be far from well-understood, especially as regards the interaction between CF’s with different spins.

A quantitative estimate. — Can we quantify the Fermi parameters? If we go back to the spinless case for simplicity, the excitation energy is

\[
\delta E = \sum_l \frac{\xi_l}{2} \delta N_l + \frac{F_1}{4m^*_F \ell^2} \frac{L \cdot L}{(l_F + 1)^2},
\]

with \( F_1^{\text{spinless}} = F_1 + 3G_1 \), from which \( m^*_F \) and \( F_1^{\text{spinless}} \) can be estimated. So here we have attempted to best fit \( F_1 \) and \( m^*_F \) simultaneously with the least-square method. Figure 3 is the fit (\( \bullet \)) for the exact diagonalization result of the low-lying \((-1 < l - l_F < 2)\) excitation spectrum in the \( \nu = 1/2 \) system with each value of sample size, \( N_e = 9, 16 \). For each size the fit, with only two free parameters, is remarkably good for both the overall and shell structures. In this sense the system may be regarded as a liquid of CF’s as far as the finite systems considered here are concerned. The fit is also good for \( \nu = 1/4 \) (not shown here). In each case, \( m^*_F \) is comparable with \( 1/m_0^* \) estimated from the free CF picture. For example, in the \( \nu = 1/2 \) case, \( \sqrt{4\pi N_e}/(e^2 m^*_F) \approx 0.21 \) (0.27) for \( N_e = 9 \) (16) against 0.18 for \( 1/m_0^* \).

As for the Landau parameter \( F_1^{\text{spinless}} \), we can immediately tell that \( F_1^{\text{spinless}} \) is negative, if we compare the exact result (\( \bullet \)) with the free (i.e., \( F_1^{\text{spinless}} = 0 \)) CF result (\( \times \)): the exact result lies significantly below the free CF result for larger \( L \), which implies that CF’s have an orbital exchange coupling of Hund’s type. This observation is consistent with the Hund’s rule for the \( \nu = 1/\text{even} \).
ground state. So the FQH system has spin- and orbital-exchange couplings both of which are Hund’s type.

Quantitatively, however, $F_{1 \text{pinless}}$ is ill-behaved in the fitting if we assume the quantity is size-independent. This is precisely why we had to fit $F_{1 \text{pinless}}$ and $m_{FL}^*$ for each value of $N_e$, and we end up with $F_{1 \text{pinless}} = -0.8 (-1.5)$ for $N_e = 9 (16)$. This leads us to question the assumption that the Landau parameters in the present system are scale invariant. The anomalous behavior should precisely be related to the infinite bare mass singularity of the FQH system. When the system had a finite bare mass $m$, we would have a relation, $m_{FL}^*/m = 1 + F_{1 \text{pinless}}^*/2$ (with 1/2 due to the two dimensionality). Since the LLL projected model has no bare kinetic energy to start with, the bare mass in the above relation, if at all meaningful, should diverge as $N_e \to \infty$. Then $F_{1 \text{pinless}}^*$ had to tend to $-2$, an anomalously large value.

Then, how can we understand these results? This is exactly what we have addressed here, and the conclusions are summarized as follows. (a) The effective mass estimated by comparing the exact result with that of a free system is numerically close to those estimated with the Fermi liquid approach and has a good property in the system size scaling. The topology of the shell structure is also reproduced with the free CF picture. In this sense the mean field picture seems to be all right. (b) While the Fermi liquid picture with best-fit Landau parameters does reproduce quantitative features of the excitation spectrum, the system deviates from an ‘ordinary’ Fermi liquid in that Landau parameters are system-size dependent.

The best-fit $F_{1 \text{pinless}}^* = -0.8 (-1.5)$ for $N_e = 9 (16)$ does have a significant size dependence, although the system sizes considered here are too small to allow a scaling argument. In the thermodynamic limit, the Landau function could possibly be singular, as is considered by Stern and Halperin by summing the diagrams as required from the Ward-Takahashi identity. If the size-dependence of $F_{1 \text{pinless}}^*$ has a logarithmic asymptote in the thermodynamic limit, then this may be related to the marginal Fermi liquid predicted by Ref. If the Fermi or marginal Fermi liquid persists in the thermodynamic limit, this would serve as an instance in which a system that has no small parameters (interaction/kinetic energy $\to \infty$, $\phi \sim O(1)$) can be a Fermi liquid. Another interesting problem is the effect of the Landau level mixing. These will serve as future problems.

FIG. 3. Low-lying excitation spectra of the $\nu = 1/2$ system with $N_e = 9, 16$ in the exact diagonalization (●), the Fermi liquid theory (□) and the free CF model (×). The best-fit effective mass and the Fermi liquid parameter are $[\sqrt{4\pi n_e/e^*m_{FL}}, F_{1 \text{pinless}}^*] = [0.21, -0.8]$ for $N_e = 9$, [0.27, -1.5] for $N_e = 16$.

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