Embedding Default Logic in Propositional Argumentation Systems

Abstract

In this paper we present a transformation of finite propositional default theories into so-called propositional argumentation systems. This transformation allows to characterize all notions of Reiter’s default logic in the framework of argumentation systems. As a consequence, computing extensions, or determining whether a given formula belongs to one extension or all extensions can be answered without leaving the field of classical propositional logic. The transformation proposed is linear in the number of defaults.

Keywords: Default Logic, Argumentation Systems, Embedding, Propositional Logic.

1 INTRODUCTION

1.1 MOTIVATION, CONTRIBUTION AND OUTLOOK

Reiter’s default logic (Reiter 1980) is at once the most popular and the most controversial non-monotonic formalism dealing with uncertain information. Popular, because knowledge can be represented in a natural way in default logic. Controversial, because the concept of an extension as presented by Reiter is quite complex. Since default logic has been presented (Reiter 1980) a lot of effort took place to establish relationships between default logic and other non-monotonic formalisms, such as autoepistemic logic (Konolige 1988) and circumscription (Lifschitz 1990).

Poole (Poole 1988) (Theorist) was — to our knowledge — the first one who tried to embed Reiter’s default logic in an assumption-based framework (De Kleer 1986a, De Kleer 1986b, Forbus and de Kleer 1993, Kean and Tsiknis 1993). It turned out that his proposition derives more extensions than the original default theory allows. The transformation proposed by Ben-Eliyahu & Dechter (Ben-Eliyahu and Dechter 1996) (meta-interpretations) results in a one-to-one correspondence of the original default theory and the their constructed propositional theory, but their transformation is in general is NP-complete. Another quasi-transformation proposed by Bondarenko et al. (Bondarenko et al. 1997) (abstract argumentation-theoretic approach) lacks a procedure to compute the extensions of the specified default theory. In conclusion, the transformations mentioned so far have their weaknesses. That is the main reason, why we searched for a translation of propositional default theories into an assumption-based framework, called argumentation systems (Kohlas and Monney 1993, Kohlas and Monney 1995, Kohlas et al. 1998, Haenni 1998, Anrig et al. 1999), to overcome the weaknesses encountered so far.

In argumentation systems all inference is done without leaving the field of classical propositional logic. Informally speaking, argumentation systems allow to judge open questions (hypotheses) about the unknown or future world in the light of the given knowledge. The problem is to determine a set of possible assumptions that allows to deduce the hypothesis from the given knowledge. Note that the concept of argumentation systems has many different meanings and we refer to (Chesnèv and others 1998) for an overview. For instance, the argumentation framework proposed by Prakken (Prakken 1993), which uses the same terminology as in Bondarenko et al. (Bondarenko et al. 1997), is a derivation of Nute’s (Nute 1994) defeasible logic. So, Prakken’s proposal has only the term argumentation with our approach in common and it is not an assumption-based framework. Therefore, we do not make a comparison between Prakken’s (Prakken 1993) and our approach. But it
is worth noting that Prakken's proposal is close to Lukasiewicz's (Lukasiewicz 1998) approach which extends still another derivation of Nute's defeasible logic with probabilities. Lukasiewicz proposal in turn has similar properties as the framework proposed by Benferhat et al. (Benferhat et al. 2000). However, in this paper the notion of an argumentation system is defined in Section 4.

In this paper, we present a transformation of Reiter's default logic into the framework of argumentation systems. This transformation is linear in the number of defaults and allows us to establish a bijection between extensions of the default theory and some subsets of assumptions of the argumentation system. Moreover, we provide procedures to answer any query on the specified default theory.

The rest of this paper is organized as follows: Section 1.2 repeats the necessary basic notions of propositional logic as a reminder. In Section 2 we briefly present Reiter's default logic and introduce the transformation of propositional default theories into associated propositional argumentation systems. Section 3 discusses the basic elements of propositional argumentation systems. In Section 4 we present the new results obtained using the transformation mentioned. We conclude with an outlook in Sections 5. Proofs of the new theorems can be found in (Berzati and Anrig 2001).

1.2 PRELIMINARIES

We assume that the reader is familiar with the basic concepts of propositional logic. We work with the usual propositional language \( \mathcal{L}_V \). That is, \( \mathcal{L}_V \) denotes the set of all formulas which can be formed using the finite set of propositions \( V = \{v_1, \ldots, v_n\} \), as usual, \( \wedge \) denotes conjunction, \( \vee \) disjunction, \( \neg \) negation and \( \rightarrow \) implication. A literal \( \ell \in V^\pm \) is a proposition or the negation of a proposition. \( \top \) denotes verum and \( \bot \) falsum. Lower-case Greek letters \( \alpha, \beta, \ldots \) denote formulas and upper-case Greek letters \( \Sigma, \Xi, \ldots \) denote sets of formulas called theories. For a set of formulas satisfaction and consistency is defined as usual. Propositional derivability is denoted by \( \vdash \) and the corresponding consequence operator by \( Th \).

For a given a formula \( \phi \in \mathcal{L}_V \) and a subset of propositions \( Q \subseteq V \), we are often interested in computing a formula \( \psi \in \mathcal{L}_Q \) such that

1. \( \phi \vdash \psi \) and
2. \( \varphi \in \mathcal{L}_Q \) and \( \phi \vdash \varphi \) imply \( \psi \vdash \varphi \).

This is the marginalization-problem (Kohlas et al. 1998) which is a special case of literal forgetting (Lin and Reiter 1994) and equivalent to computing prime implicants (Raiman and De Kleer 1992). A formula \( \psi \) which satisfies the two conditions above is called a marginal of \( \phi \) with respect to \( Q \subseteq V \), denoted by \( \phi^Q \). Note that \( Th_{\mathcal{L}_Q} (\phi^Q) = Th_{\mathcal{L}_V} (\phi) \cap \mathcal{L}_Q \). We will not enter into the details how such a marginal can be computed. The interested reader is referred for example to Marquis (Marquis 2004).

Finally, for any set \( S \) we denote with \(|S|\) the cardinal number of \( S \), i.e. the number of elements contained in \( S \), and with \( 2^S \) the power set of \( S \), i.e. the set of all subsets of \( S \).

2 DEFAULT LOGIC AND TRANSLATION

2.1 DEFAULT THEORIES AND EXTENSIONS

A Reiter’s default theory (Reiter 1980) is a pair \( \langle \Sigma, \Delta \rangle \), where \( \Sigma \) consists of formulas over a propositional language \( \mathcal{L} \) and \( \Delta \) is a set of defaults. A default \( \delta \) is a construct of the form\(^1\)

\[
\alpha : \beta_1, \ldots, \beta_k, \gamma,
\]

(i) \( \alpha, \beta_1, \ldots, \beta_k, \gamma \) are formulas in \( \mathcal{L} \),
(ii) \( \text{pre}(\delta) := \{\alpha\} \) is called the prerequisite of the default \( \delta \),
(iii) \( \text{jus}(\delta) := \{\beta_1, \ldots, \beta_k\} \) is called the justification of the default \( \delta \),
(iv) \( \text{con}(\delta) := \{\gamma\} \) is called the consequence of the default \( \delta \).

The selectors \text{pre}, \text{jus} and \text{con} are naturally generalized to subsets \( D \subseteq \Delta \) of defaults:

\[
\text{pre}(D) := \bigcup_{\delta \in D} \text{pre}(\delta),
\]
\[
\text{jus}(D) := \bigcup_{\delta \in D} \text{jus}(\delta),
\]
\[
\text{con}(D) := \bigcup_{\delta \in D} \text{con}(\delta).
\]

Observe that the selectors \text{pre}, \text{jus} and \text{con}, respectively, return sets of formulas of \( \mathcal{L} \). It is sometimes\(^1\)

\(^1\)Note that different defaults have in general a different number of justifications; yet for the sake of simplicity, we do not index the \( k \)'s in the sequel.
necessary to select the formulas themselves. Thus for any default \( \delta \) with \( k \) justifications, we define following selectors:

\[
p(\delta) := \alpha, \quad j_i(\delta) := \beta_i, \quad c(\delta) := \gamma.
\]

\( \langle \Sigma, \Delta \rangle \) is called finite if both \( \Sigma \) and \( \Delta \) are finite. In this paper we focus on finite default theories. Therefore let \( P = \text{Var}(\langle \Sigma, \Delta \rangle) \) denote the set of all propositions occurring in the formulas of \( \langle \Sigma, \Delta \rangle \) and \( \mathcal{L}_P \) the propositional language of \( \langle \Sigma, \Delta \rangle \).

The fundamental notion in default logic is the extension. Intuitively, it denotes the deductive closure of a maximal set of formulas of \( \langle \Sigma, \Delta \rangle \) containing \( \Sigma \) and “consistently” adding the consequences of some defaults.

**Definition 1** (Extension: Reiter 1980, Theorem 2.1) A set \( E \subseteq \mathcal{L}_P \) of formulas is called extension of a default theory \( \langle \Sigma, \Delta \rangle \), where \( P = \text{Var}(\langle \Sigma, \Delta \rangle) \), iff there exists a sequence \( E_0, E_1, \ldots \) such that

\[
E_0 = \Sigma,
E_{j+1} = \text{Th}(E_j) \cup \text{con}\{\delta \in \Delta : p(\delta) \in E_j \quad \text{and} \quad \neg j_i(\delta) \notin E \quad \text{for every} \quad i\},
E = \bigcup_{j=0}^{\infty} E_j.
\]

When we write “for every \( i \)” (\( \forall i \) for short), we mean for each justification of the respective default.

**Example 1** The default theory \( \langle \Sigma, \Delta \rangle \) with \( \Sigma = \{ b \rightarrow \neg a \land \neg c \} \) and \( \Delta = \{ \frac{1}{2} a, \frac{1}{2} b, \frac{2}{3} c \} \) has two extensions given by \( E = \text{Th}(\Sigma \cup \{ a, c \}) \), and \( E' = \text{Th}(\Sigma \cup \{ b \}) \).

Assume that \( \Sigma \) of a given default theory \( \langle \Sigma, \Delta \rangle \) is inconsistent. Then \( \text{Th}(\Sigma) \) is the only extension of \( \langle \Sigma, \Delta \rangle \) which is obviously inconsistent. Therefore, a default theory is called consistent iff it has at least one consistent extension. In this case \( \Sigma \) must be consistent, since it is included in every extension. But consistency of the initial belief does not guarantee that the default theory has an extension.

**Example 2** The default theory \( \langle \Sigma, \Delta \rangle \) with \( \Sigma = \emptyset \) and \( \Delta = \{ \frac{1}{2} b \} \) has no extension!

Since we are not interested in trivial inconsistent extensions, we will say that a default theory has no extension iff the default theory \( \langle \Sigma, \Delta \rangle \) lacks an extension or \( \Sigma \) is inconsistent.

Default theories \( \langle \Sigma, \Delta \rangle \) lacking an extension should not be confused with default theories that have a unique consistent extension which is just the deductive closure of \( \Sigma \).

**Example 3** Let \( \langle \Sigma, \Delta \rangle \) be a default theory with \( \Sigma = \{ p, q \} \) and \( \Delta = \{ \frac{1}{2} p, \frac{1}{2} q \} \). The reader may verify that \( E = \text{Th}(\{ p, q \}) \) is the only extension.

Two concepts of consequence are of main interest when working with default theories:

**Definition 2** A formula \( \phi \) is called credulous consequence of a default theory \( \langle \Sigma, \Delta \rangle \) iff it belongs to at least one extension, and skeptical consequence iff it belongs to all extensions.

### 2.2 Embedding General Default Theories

Let \( \langle \Sigma, \Delta \rangle \) be a default theory with formulas in \( \mathcal{L}_P \), where \( P = \text{Var}(\langle \Sigma, \Delta \rangle) \). For any default \( \delta \in \Delta \) consider the following transformation \( t \):

\[
t(\delta) = \{ a_0^\delta \rightarrow \neg p(\delta) \} \cup \{ a_{3,i}^\delta \rightarrow j_i(\delta) : i = 1, \ldots, |\text{jus}(\delta)| \} \cup \{ a_5^\delta \rightarrow c(\delta) \}.
\]

We call the triple \( \langle a_0^\delta, \{ a_{3,i}^\delta : i = 1, \ldots, |\text{jus}(\delta)| \}, a_5^\delta \rangle \) default assumption and denote it by \( a_\delta \). Moreover, let \( \Lambda_\Delta = \{ \langle a_0^\delta, \{ a_{3,i}^\delta : i = 1, \ldots, |\text{jus}(\delta)| \}, a_5^\delta \rangle : \delta \in \Delta \} \) denote the set of default assumptions. Often, we will look at indexed defaults \( \delta_i \) and abbreviate the default assumption \( a_\delta \) by \( a_{\delta,i} \). For a default assumption \( \langle a_0^\delta, \{ a_{3,i}^\delta : i = 1, \ldots, |\text{jus}(\delta)| \}, a_5^\delta \rangle \in \Lambda_\Delta \) we call \( a_0^\delta \) prerequisitional, \( a_{3,i}^\delta \) justificational (\( \forall i \)), and \( a_5^\delta \) consequential assumption. The sets \( A_\delta = \{ a_0^\delta : \delta \in \Delta \} \), \( A_1 = \{ a_{3,i}^\delta : \delta \in \Delta, i = 1, \ldots, |\text{jus}(\delta)| \} \) and finally \( A_5 = \{ a_5^\delta : \delta \in \Delta \} \) denote the corresponding sets of prerequisitional, justificational, and consequential assumption, respectively. The set

\[
A = A_\delta \cup A_1 \cup A_5,
\]

which consists of pairwise distinct propositions such that \( A \cap P = \emptyset \), is called the set of assumptions. The pair

\[
\langle \Xi, A \rangle, \quad \Xi = \Sigma \cup \bigcup_{\delta \in \Delta} t(\delta), \quad (2)
\]

is called the (propositional) argumentation system associated with the default theory \( \langle \Sigma, \Delta \rangle \) (cf. Section 3).

**Example 4** Let \( \langle \Sigma, \Delta \rangle = \{ \{ e \lor o \}, \{ \frac{1}{2} e, \frac{1}{2} o \} \} \) be a default theory, then the corresponding argumentation system is \( \langle \Xi, A \rangle = \{ \{ e \lor o \}, a_0^\delta \rightarrow \neg c, a_1^\delta \rightarrow r, a_2^\delta \rightarrow r, a_3^\delta \rightarrow r, \{ a_1^\delta, a_2^\delta, a_3^\delta, a_4^\delta \} \}, \)
where the set of default assumptions is \( A_\Delta = \{ \{ a_1^1 \}, \{ a_1^2 \}, \{ a_2^1 \}, \{ a_2^2 \} \} \).

Any subset \( \alpha \subseteq A_\Delta \) is called \textit{consequential term}. CONSEQUENTIAL terms will allow us, using the framework of propositional argumentation systems, to decide wether a given default theory has extensions, characterize all extensions, and wether a given formula is a skeptical or credulous consequence of the default theory.

3 PROPOSITIONAL ARGUMENTATION SYSTEMS

The major difference between argumentation systems (Haenni et al. 2000, Anrig 2000) and other non-monotonic formalisms (Reiter 1980, McCarthy 1980, Moore 1985) is that in argumentation systems, all the inference is done with classical propositional logic.

Consider a propositional theory \( \Xi \) with formulas in \( \mathcal{L}_V \), where \( V = \{ v_1, v_2, \ldots, v_m \} \). From \( V \) we choose a finite subset \( A \subseteq V \) whose members are called \textit{assumptions}. The other propositions in \( P = V - A \) are called \textit{non-assumables}. For convenience, we enumerate the assumptions, i.e. \( A = \{ a_1, \ldots, a_n \} \). The pair \( \langle \Xi, A \rangle \) is called a \textit{propositional argumentation system} and \( \Xi \) is said to be an \textit{argumentation theory}. So we consider a fixed argumentation system \( \langle \Xi, A \rangle \), with \( \vdash \) as the derivability relation and \( Th \) the corresponding closure operator. Note that the case where \( A = \emptyset \), corresponds to the usual propositional theory.

In an argumentation system \( \langle \Xi, A \rangle \), the assumptions \( A = \{ a_1, \ldots, a_n \} \) of the argumentation theory \( \Xi \) are essential for expressing uncertain information. They represent uncertain events, unknown circumstances, or possible risks and outcomes. Formally, this is defined as follows:

\textbf{Definition 3} A term \( \alpha \) is a subset of literals from \( A^\pm \), so that if \( \ell \in \alpha \), then \( ^-\ell \notin \alpha \). The set of all terms is denoted by \( T_A \).

The non-assumables appearing in \( \Xi \) are treated as classical propositions for the given argumentation theory \( \Xi \).

\textbf{Definition 4} A term \( \alpha \in T_A \) is called \textit{inconsistent relative to} \( \Xi \) iff \( \alpha \cup \Xi \) is unsatisfiable. \( I(\Xi) \) denotes the set of inconsistent terms relative to \( \Xi \). An inconsistent term \( \alpha \) is called \textit{minimal} iff no proper subset of \( \alpha \) is inconsistent relative to \( \Xi \). \( \mu(\Xi) \) denotes the set of all minimal inconsistent terms relative to \( \Xi \), and is called the set of minimal contradictions.

In the sequel we will write \( \alpha, \Xi \vdash \bot \) to indicate that \( \alpha \) is inconsistent, and \( \alpha, \Xi \not\vdash \bot \) to indicate that \( \alpha \) is not inconsistent (relative to \( \Xi \)). \( Th(\alpha, \Xi) \) denotes the deductive closure of \( \alpha \cup \Xi \), and \( \alpha, \Xi \vdash \psi \) is an abbreviation for \( \alpha \cup \Xi \vdash \psi \). With this notation the set of minimal contradictions can be represented by \( \mu(\Xi) = \{ \alpha \in T_A : \alpha, \Xi \not\vdash \bot \} \).

\textbf{Example 5} Let \( \langle \Xi, A \rangle \) be an argumentation system, where \( \Xi = \{ a_1 \rightarrow p, a_2 \rightarrow q, \neg p, \neg q \} \) is the argumentation theory and \( A = \{ a_1, a_2 \} \) the set of assumptions. Then \( \mu(\Xi) = \{ \{ a_1 \}, \{ a_2 \} \} \) is the set of minimal contradictions.

An argumentation system \( \langle \Xi, A \rangle \) allows to reason about a specified propositional formula \( \phi \in \mathcal{L}_V \), called \textit{hypothesis}, with respect to \( A \). The idea is to consider terms \( \alpha \) which together with the argumentation theory \( \Xi \) imply this hypothesis \( \phi \).

\textbf{Definition 5} ((Haenni et al. 2000), Theorem 2.7) Let \( \langle \Xi, A \rangle \) be an argumentation system. A term \( \alpha \) is called a supporting argument for \( \phi \) iff \( \alpha, \Xi \vdash \phi \) and \( \alpha', \Xi \not\vdash \bot \) for all \( \alpha' \in T_A \) such that \( \alpha \subseteq \alpha' \).

\textbf{Example 6} Let \( \Xi = \{ a_1 \rightarrow p, a_2 \rightarrow q, p \rightarrow \neg q \} \) and \( A = \{ a_1, a_2 \} \). The supporting arguments for \( p \) are \( sp(p; \Xi) = \{ \{ a_1, \neg a_2 \} \} \). Note that \( \{ a_1 \}, \Xi \vdash \bot \) and \( \{ a_1, a_2 \}, \Xi \not\vdash \bot \), but \( \{ a_1, a_2 \}, \Xi \vdash \bot \), hence \( \{ a_1 \} \) is not a supporting argument for \( p \). Similarly, we get \( sp(q; \Xi) = \{ \{ \neg a_1, a_2 \} \} \).

\textbf{Definition 6} Let \( \langle \Xi, A \rangle \) be an argumentation system, \( \alpha \) a term, and \( A \subseteq A^\pm \) a set of literals. The pair \( \langle \alpha, A \rangle \) is called structure iff

\begin{itemize}
  \item [(i)] \( \alpha, \Xi \not\vdash \bot \), and
  \item [(ii)] \( \ell \cup \alpha, \Xi \not\vdash \bot \), for every \( \ell \in A \).
\end{itemize}

\( \alpha \) is called the anchor and \( A \) the set of irrelevant literals of \( \langle \alpha, A \rangle \) w.r.t. \( \Xi \).

Note that \( \{ \ell \} \cup \alpha, \Xi \not\vdash \bot \) for every \( \ell \in A \) does not guarantee that \( \{ \ell_1, \ell_2 \} \cup \alpha, \Xi \not\vdash \bot \) for \( \ell_i, \ell_j \in A \) and
i ≠ j. The utility of structures will become clear in Section 3.

We close our discussion of terms and scenarios and refer to (Haenni et al. 2000; Kohlas and Monney 1995; Kohlas et al. 1998; Kohlas and Monney 1993; Anrig et al. 1999; Haenni 1998; Haenni and Lehmann 2000) for readers wishing more details concerning this issue. For computational purposes, especially computing minimal contradictions and approximation techniques see (Haenni et al. 2000; Haenni 2001a).

4 EXTENSIONS AND TERMS

Here, we characterize extensions in the corresponding propositional argumentation system using structures ⟨α, A⟩ (cf. Definition 1) and provide, based on the anchor of some structures, an inductive procedure to compute extensions, if the original default theory has any. Here, only consequential terms (cf. Section 2.2) are allowed as anchor α, i.e. α ⊆ Ac and the set of irrelevant literals A consists only of justificational assumptions, i.e. A ⊆ Aj. What follows is close in spirit to Łukasiewicz’s (Łukasiewicz 1991) semantics for Reiter’s default logic.

A pair ⟨α, A⟩, where α ⊆ Ac is a consequential term and A ⊆ Aj is a set of justificational assumptions is called consequence-justification pair, CJ-pair for short.

Definition 7 Let (Σ, ∆) be default theory, ⟨Ξ, A⟩ its corresponding argumentation system, α ⊆ Ac a consequential term and a = ⟨αp, {a1, ..., a k}⟩, αc a default assumption. We say that a is applicable with respect to α iff

(i) {αp} ∪ α ∈ I(Ξ) and
(ii) {αi} ∪ α /∈ I(Ξ) for i = 1, ..., k.

Note that we do not check if {αc} ∪ α, Ξ ⊨ ⊥. The idea is to obtain a sequence of consequences to add successively to the consequential term according to Definition 1 in an argumentation system.

Definition 8 To each default assumption a = ⟨αp, {a1, ..., a k}, αc⟩ we assign a mapping, denoted by a −, from CJ-pairs into CJ-pairs specified by

(i) a −(⟨α, A⟩) = ⟨{αc} ∪ α, {a1, ..., a k} ∪ A⟩, if ⟨α, A⟩ is a structure and a is applicable w.r.t. α;
(ii) a −(⟨α, A⟩) = ⟨α, A⟩, if ⟨α, A⟩ is a structure and a is not applicable w.r.t. α;
(iii) a −(⟨α, A⟩) = ⟨{⊥}, ∅⟩, otherwise.

If a default assumption is applicable with respect to some consequential term, then we have to include its consequential assumption to the already given consequential term. But this inclusion may lead to inconsistencies. Consequently, we have to check if the obtained CJ-pair is still a structure.

Example 7 Consider ⟨{p}, {¬pa}⟩ with Ξ = {p, ap −→ q, ac −→ q}. If we take the structure ⟨{T}, ∅⟩, then a = ⟨ap, {a1}, ac⟩ is applicable w.r.t. ⟨T⟩. Thus a −(⟨{T}, ∅⟩) = ⟨{a1}, {a1}⟩, but {a1} ∪ {ac} ⊆ T, ⊨ ⊥, hence ⟨{a1}, {a1}⟩ is not a structure.

Definition 9 A structure ⟨α, A⟩ is called accessible (with respect to AΔ) iff there is a sequence ⟨aj⟩ = ⟨α1, α2, ..., αn⟩ of non–repeating default assumptions such that

(i) ⟨α0, A0⟩ = ⟨{T}, ∅⟩;
(ii) aj is applicable w.r.t. αj−1 and ⟨αj, Aj⟩ = a −(⟨αj−1, Aj−1⟩) for j = 1, 2, ..., n;
(iii) ⟨αn, An⟩ = ⟨α, A⟩
(iv) there is no default assumption αi ∈ AΔ which is applicable w.r.t. α but not included in ⟨aj⟩.

Such a ⟨aj⟩ is called a generating sequence of ⟨α, A⟩.

If ⟨aj⟩ is a generating sequence of a structure ⟨α, A⟩, then α contains all consequential assumptions of the default assumptions appearing in ⟨aj⟩, A contains the union of all sets of justificational assumptions of the default assumptions appearing in ⟨aj⟩, and ⟨α, A⟩ is still a structure. In particular α ⊆ Ac.

Example 8 Consider the default theory ⟨∅, {¬p, ¬qa}⟩ with corresponding Ξ = {¬ap, a1 → p, a1 → p, ¬a2, a2 → q, a2 → q}. If we take the structure ⟨{T}, ∅⟩, then ⟨α1, α2⟩ and ⟨α2, α1⟩ are generating sequences of the same structure ⟨{a1, a2}, {a1, a2}⟩.

Obviously, there may be different generating sequences for the same structure. But two generating sequences of the same structure contain the same default assumptions, i.e. one is a permutation of the other (Herzati and Anrig 2001).

The definition of accessible structures is justified by the next theorem, which can be regarded as the dual result of Theorem 5.63 in (Łukasiewicz 1991). Duality
in the sense that Łukasiewicz’ [Łukasiewicz 1991] theorem is a semantic characterization of extensions and our result is a syntactic characterization of extensions. In fact the proof follows Łukasiewicz [Łukasiewicz 1991].

**Theorem 1** ([Berzati and Anrig 2001]) Let \( \langle \Sigma, \Delta \rangle \) be a default theory and \( \langle \Xi, A \rangle \) the corresponding argumentation system. There is a bijection between accessible structures of \( \langle \Xi, A \rangle \) and consistent extensions of \( \langle \Sigma, \Delta \rangle \).

The theorem above states only that an accessible structure contains the information concerning some corresponding extension of the given default theory \( \langle \Sigma, \Delta \rangle \). It says nothing about how the extension can be characterized. This observation motivates our next definition to simplify terminology.

**Definition 10** The anchor \( \alpha \) of an accessible structure \( (\alpha, A) \) is called default term. The set of all default terms is denoted by \( \text{dt}(\Xi) \).

The notion of a default term permits to derive some corollaries of Theorem 1.

**Corollary 1** ([Berzati and Anrig 2001]) Let \( \langle \Sigma, \Delta \rangle \) be a default theory with \( P = \text{Var}(\langle \Sigma, \Delta \rangle) \) and \( \langle \Xi, A \rangle \) the corresponding argumentation system.

\[
\begin{align*}
(i) & \quad \Sigma \text{ is inconsistent iff } \mu(I(\Xi)) = \{ \top \}. \\
(ii) & \quad \langle \Sigma, \Delta \rangle \text{ has no extension iff } \text{dt}(\Xi) = \emptyset \text{ and } \mu(I(\Xi)) \neq \{ \top \}. \\
(iii) & \quad E = \text{Th}(\Sigma) \text{ is the only extension iff } \\
& \quad \text{dt}(\Xi) = \{ \top \}. \\
(iv) & \quad E = \text{Th}(\alpha, \Xi, A) \text{ is an extension of } \\
& \quad \langle \Sigma, \Delta \rangle \text{ iff } \alpha \in \text{dt}(\Xi).
\end{align*}
\]

Now, that we have a complete characterization of extensions for a given default theory \( \langle \Sigma, \Delta \rangle \) in terms of the default terms \( \text{dt}(\Xi) \subseteq 2^A \) of the corresponding argumentation system \( \langle \Xi, A \rangle \), it remains to determine how those default terms can be computed. Interestingly, all we need to do that are the minimal contradictions \( \mu(I(\Xi)) \). The minimal contradictions \( \mu(I(\Xi)) \) are known to play a central role in argumentation systems, for example for computing supporting scenarios [Haenni et al. 2000] and in model-based diagnostics and reliability [Anrig and Kohlas 2002]. Therefore a lot of effort has been made to develop efficient algorithms [Haenni et al. 2000; Haenni 2001b] to compute \( \mu(I(\Xi)) \). So default logic can profit from those efforts.

**Theorem 2** ([Berzati and Anrig 2001]) Let \( \langle \Sigma, \Delta \rangle \) be a default theory and \( \langle \Xi, A \rangle \) the corresponding argumentation system. A term \( \alpha \subseteq A \) different from \( \{ \top \} \) is a default term iff there is an enumeration \( a_1, \ldots, a_c \) of the consequential assumptions \( a_j \in \alpha \) such that

\[
\begin{align*}
(i) & \quad \{ a_1 \} \in I(\Xi), \\
(ii) & \quad \text{For every } j = 1, \ldots, q - 1 \text{ we have } \{ a_{j+1} \} \cup \{ a_1, \ldots, a_j \} \notin I(\Xi), \\
(iii) & \quad \text{For every } j = 1, \ldots, q \text{ and every } i \text{ we have } \{ a_i \} \cup a \notin I(\Xi), \\
(iv) & \quad \text{For every } a_i \in (A_p - \{ a_1, \ldots, a_q \}) \text{ we have } \\
& \quad \{ a_i \} \cup a \notin I(\Xi) \text{ for some } i.
\end{align*}
\]

Note that if \( \alpha \in \mu(I(\Xi)) \), then any term \( \alpha' \), such that \( \alpha \subseteq \alpha' \), is inconsistent, i.e. \( \alpha' \notin I(\Xi) \). Clearly, only the minimal contradictions \( \mu(I(\Xi)) \) will be computed.

Theorem 2 indicates an iterative procedure for computing default terms given the translation of the default theory and the respective minimal contradictions. The following examples shows how.

**Example 9** We consider for each case of Corollary 1 a small example.

- For \( \langle \Sigma, \Delta \rangle = \langle \{ p, -p, \{ \frac{c_2}{q} \} \rangle \text{, } t(I(\Xi)) = \langle \{ p, -p, a_2 \rightarrow -p, a_3 \rightarrow q, a_5 \rightarrow q \} \text{, } \{ a_2, a_3, a_5 \} \rangle \text{ and } \mu(I(\Xi)) = \{ \{ \top \} \rangle. \text{ Clearly, } \Sigma \vdash \bot. \quad (i) \)

- For \( \langle \Sigma, \Delta \rangle = \langle \emptyset, \{ \frac{c_2}{q} \} \rangle \text{ we have } \mu(I(\Xi)) \{ \{ a_2 \} \}, \{ a_2, a_3 \}. \text{ Only the potentially possible default term different from } \{ \top \} \{ a_2 \}. \{ a_2 \} \in I(\Xi), \text{ so point (i) of Theorem 2 is fulfilled. But } \{ a_2, a_3 \} \notin I(\Xi), \text{ thus (iii) of Theorem 2 is violated, thus } \{ a_2 \} \text{ is no default term. Hence we conclude that there is no default term and } \langle \Sigma, \Delta \rangle \text{ has no extension.} \quad (ii) \)

- For \( \langle \Sigma, \Delta \rangle = \langle \{ e \lor o \}, \{ \frac{c_2}{q} \} \rangle \text{ we have } \mu(I(\Xi)) = \langle \{ e \lor o, a_2 \rightarrow -e, a_3 \rightarrow r, a_5 \rightarrow r \} \text{, } \{ a_2, a_3, a_5 \} \rangle \text{ and } \mu(I(\Xi)) = \{ \{ \top \} \rangle. \text{ For } a_2 \notin I(\Xi) \text{ (point (ii)), there is no default term different from } \{ \top \} \text{ Indeed, this default theory has a single extension given by } E = \text{Th}(e \lor o). \quad (ii) \)

- Finally, for \( \langle \Sigma, \Delta \rangle = \langle \emptyset, \{ \frac{c_2}{q} \} \rangle \text{ we have } \Xi = \{ -a_2, a_3 \rightarrow c, a_4 \rightarrow -d, -a_2, a_3 \rightarrow d, a_5 \rightarrow -e, -a_5, a_3 \rightarrow e, a_5 \rightarrow -f \} \text{ and } \mu(I(\Xi)) = \{ \{ a_2 \}, \{ a_2 \}, \{ a_2 \}, \{ a_2 \} \}. \text{ There are three prerequisitional assumptions to start with:} \quad (ii) \)

- First we start with \( \{ a_2 \} \). The hypothesis that \( a_2 \) is a default term is rejected, since
Example 10 The default theory \( \Sigma, \Delta \), where \( \Sigma = \{ b \rightarrow d, c \rightarrow d \} \) and \( \Delta = \{ \frac{1}{2}^b, \frac{1}{2}^a \} \) has two extensions \( E_1 = Th(\{ b, d \}) \) and \( E_2 = Th(\{ c, d \}) \). Clearly, \( b \) and \( c \) are credulous consequences of \( \Sigma, \Delta \), whereas \( b \lor c \) and \( d \) are skeptical consequences. The translation returns \( \langle \Sigma, A \rangle \) with \( \Xi = \{ b \rightarrow d, c \rightarrow d, \neg a_1^f, \neg a_2^f, a_1^f \rightarrow \neg c, a_2^f \rightarrow \neg b, a_1^f \rightarrow b, a_2^f \rightarrow c \} \). One can verify that \( dt(\Xi) = \{ \{ a_1^f \}, \{ a_2^f \} \}, \{ \alpha^+ : \alpha \in sp(b; \Xi) \} = \{ \{ a_1^f \} \}, \{ \alpha^+ : \alpha \in sp(c; \Xi) \} = \{ \{ a_2^f \} \} \). Therefore, \( dt(\Xi) \subseteq \{ \alpha^+ : \alpha \in sp(\Sigma; \Xi) \} \).

5 CONCLUSION

We have provided a translation from default theories into argumentation systems which is linear in the number of defaults, from which we can decide if the given default theory has no extensions and if it has consistent extensions. Moreover, we are able to characterize all extensions, without leaving the monotone framework of propositional logic. All we need are assumptions. Finally, for any given formula we can decide if it is a skeptical or credulous consequence of the original default theory.

Acknowledgments

This work has been supported by grant No. 2000-061454.00 of the Swiss National Foundation for Research.

We would like to thank Philippe Besnard for his useful comments on earlier drafts of this article. We owe special thanks to the referees for their constructive and valuable suggestions which contributed to the final form of this article.

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