A formulation for continuous mixtures of multivariate normal distributions

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Abstract

Several formulations have long existed in the literature in the form of continuous mixtures of normal variables where a mixing variable operates on the mean or on the variance or on both the mean and the variance of a multivariate normal variable, by changing the nature of these basic constituents from constants to random quantities. More recently, other mixture-type constructions have been introduced, where the core random component, on which the mixing operation operates, is not necessarily normal. The main aim of the present work is to show that many existing constructions can be encompassed by a formulation where normal variables are mixed using two univariate random variables. For this formulation, we derive various general properties. Within the proposed framework, it is also simpler to formulate new proposals of parametric families and we provide a few such instances. At the same time, the exposition provides a review of the theme of normal mixtures.

Key-words: location-scale mixtures, mixtures of normal distribution.
1 Continuous mixtures of normal distributions

In the last few decades, a number of formulations have been put forward, in the context of distribution theory, where a multivariate normal variable represents the basic constituent but with the superposition of another random component, either in the sense that the normal mean value or the variance matrix or both these components are subject to the effect of another random variable of continuous type. We shall refer to these constructions as 'mixtures of normal variables'; the matching phrase 'mixtures di normal distributions' will also be used.

To better focus ideas, recall a few classical instances of the delineated scheme. Presumably, the best-known such formulation is represented by scale mixtures of normal variables, which can be expressed as

\[ Y = \xi + V^{1/2}X \]

where \( X \sim N_d(0, \Sigma) \), \( V \) is an independent random variable on \( \mathbb{R}^+ \), and \( \xi \in \mathbb{R}^d \) is a vector of constants. Scale mixtures (1) provide a stochastic representation of a wide subset of the class of elliptically contoured distributions, often called briefly elliptical distributions. For a standard account of elliptical distributions, see for instance Fang et al. (1990); specifically, their Section 2.6 examines the connection with scale mixtures of normal variables. A very important instance occurs when \( 1/V \sim \chi^2_{\nu}/\nu \), which leads to the multivariate Student's \( t \) distribution.

Another very important construction is the normal variance-mean mixture proposed by Barndorff-Nielsen (1977, 1978) and extensively developed by subsequent literature, namely

\[ Y = \xi + V \gamma + V^{1/2}X \]

where \( \gamma \in \mathbb{R}^d \) is a vector of constants and \( V \) is assumed to have a generalized inverse Gaussian (GIG) distribution. In this case \( Y \) turns out to have a generalized hyperbolic (GH) distribution, which will recur later in the paper.

Besides (1) and (2), there exists a multitude of other constructions which belong to the idea of normal mixtures delineated in the opening paragraph. Many of these formulations will be recalled in the subsequent pages, to illustrate the main target of the present contribution, which is to present a general formulation for normal mixtures. Our proposal involves an additional random component, denoted \( U \), and the effect of \( U \) and \( V \) is regulated by two functions, non-linear in general. As we shall see, this construction encompasses a large number of existing constructions in a unifying scheme, for which we develop various general properties.

The role of this activity is to highlight the relative connections of the individual constructions, with an improved understanding of their nature. As a side-effect, the presentation of the individual formulations plays also the role of a review of this stream of literature. Finally, the proposed formulation can facilitate the conception of additional proposals with specific aims. The emphasis is primarily on the multivariate context.

Since it moves a step towards generality, we mention beforehand the formulation of Tjetjep & Seneta (2006) where \( V \) and \( V^{1/2} \) in (2) are replaced by two linear functions of them, which allows to incorporate a number of existing families. Their construction is, however, entirely within the univariate domain. A number of multivariate constructions aiming at some level of generality do exist, and will examined in the course of the discussion.

In the next section, our proposed general scheme is introduced, followed by the derivation of a number of general properties. The subsequent sections show how to frame a large number of existing constructions within the proposed scheme. In the final section, we indicate some directions for even more general constructions.
2 Generalized mixtures of normal distributions

2.1 Notation and other formal preliminaries

As already effectively employed, the notation \( W \sim N_d(\mu, \Sigma) \) indicates that \( W \) is a \( d \)-dimensional normal random variable with mean vector \( \mu \) and variance matrix \( \Sigma \). The density function and the distribution function of \( W \) at \( x \in \mathbb{R}^d \) are denoted by \( \varphi_d(x; \mu, \Sigma) \) and \( \Phi_d(x; \mu, \Sigma) \). Hence, specifically, we have

\[
\varphi_d(x; \mu, \Sigma) = \frac{1}{\text{det}(2\pi \Sigma)^{1/2}} \exp\left\{ -\frac{1}{2} (x - \mu)^\top \Sigma^{-1} (x - \mu) \right\}
\]

if \( \Sigma > 0 \). When \( d = 1 \), we drop the subscript \( d \). When \( d = 1 \) and, in addition, \( \mu = 0 \) and \( \Sigma = 1 \), we use the simplified notation \( \varphi(\cdot) \) and \( \Phi(\cdot) \) for the density function and the distribution function.

A quantity arising in connection with the multivariate normal distribution, but not only there, is the Mahalanobis distance, defined (in the non-singular case) as

\[
\| x \|_\Sigma = (x^\top \Sigma^{-1} x)^{1/2}
\]

which is written in the simplified form \( \| x \| \) when \( \Sigma \) is the identity matrix.

A function which will appear in various expressions is the inverse Mills ratio

\[
\zeta(t) = \frac{\varphi(t)}{\Phi(t)}, \quad t \in \mathbb{R}.
\]

A positive continuous random variable \( V \) has a GIG distribution if its density function can be written as

\[
g(v; \lambda, \chi, \psi) = \frac{(\sqrt{\psi / \chi})^\lambda}{2 K_\lambda(\sqrt{\chi \psi})} v^{\lambda-1} \exp\left\{ -\frac{1}{2} (\chi v^{-1} + \psi v) \right\}, \quad v > 0,
\]

where \( \lambda \in \mathbb{R}, \psi > 0, \chi > 0 \) and \( K_\lambda \) denotes the modified Bessel function of the third kind. In this case, we write \( V \sim GIG(\lambda, \chi, \psi) \). The numerous properties of the GIG distribution and interconnections with other parametric families are reviewed by Jørgensen (1982). We recall two basic properties: both the distribution of \( 1/V \) and of \( cV \) for \( c > 0 \) are still of GIG type. A fact to be used later is that the Gamma distribution is obtained when \( \lambda > 0 \) and \( \chi \to 0 \).

A result in matrix theory which will be used repeatedly is the Sherman-Morrison formula for matrix inversion, which states

\[
(A + b d^\top)^{-1} = A^{-1} - \frac{1}{1 + d^\top A^{-1} b} A^{-1} b d^\top A^{-1}
\]

provided that the square matrix \( A \) and the vectors \( b, d \) have conformable dimensions, and the inverse matrices exist.

2.2 Definition and basic facts

Consider a \( d \)-dimensional random variable \( X \sim N_d(0, \Sigma) \) and univariate random variables \( U \) and \( V \) with joint distribution function \( G(u, v) \), such that \( (X, U, V) \) are mutually independent; hence \( G \) can be factorized as \( G(u, v) = G_U(u) G_V(v) \). We assume \( \Sigma > 0 \) to avoid technical complications and concentrate on the constructive process. These definitions and assumptions will be retained for the rest of the paper.

Given any real-valued function \( r(u, v) \), a positive-valued function \( s(u, v) \), and vectors \( \xi \) and \( \gamma \) in \( \mathbb{R}^d \), we shall refer to

\[
Y = \xi + r(U, V) \gamma + s(U, V) X
\]

\[
= \xi + R \gamma + SX
\]

as a generalized mixture of normal distributions. We shall refer to

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as a generalized mixture of normal (GMN) variables; we have written \( R = r(U, V) \) and \( S = s(U, V) \) with independence of \((R, S)\) from \(X\). Denote by \( H\) the joint distribution function of \((R, S)\) implied by \(r, s, G\). The distribution of \(Y\) is identified by the notation \(Y \sim \text{GMN}_d(\xi, \Sigma, \gamma, H)\).

For certain purposes, it is useful to think of \(Y\) as generated by the hierarchical construction

\[
(Y | U = u, V = v) \sim N_d(\xi + r(u, v) \gamma, s(u, v)^2 \Sigma),
\]

\[(U, V) \sim G_U \times G_V.
\]

For instance, this representation is convenient for computing the mean vector and the variance matrix as

\[
E\{Y\} = E\{E\{Y|U, V\}\} = \xi + r(U, V) \gamma
\]

provided \(E\{R\}\) exists, and

\[
\text{var}\{Y\} = \text{var}\{E\{Y|U, V\}\} + E\{\text{var}\{Y|U, V\}\}
\]

\[
= \text{var}\{\xi + r(U, V) \gamma\} + E\{s(U, V)^2 \Sigma\}
\]

\[
= \text{var}\{R\} \gamma \gamma^T + E\{S^2\} \Sigma
\]

provided \(\text{var}\{R\}\) and \(E\{S^2\} > 0\) exist. Another use of representation (9) is to facilitate the development of some EM-type algorithm for parameter estimation.

Similarly, by a conditioning argument, it is simple to see that the characteristic function of \(Y\) is

\[
c(t) = \exp(it^T \xi) E\{c_{N}(t; r(U, V) \gamma, s(U, V)^2 \Sigma)\}, \quad t \in \mathbb{R},
\]

where \(c_{N}(t; \mu, \Sigma)\) denotes the characteristic function of a \(N(\mu, \Sigma)\) variable. Also, the distribution function of \(Y\) is

\[
F(y) = E\{\Phi_d(y; \xi + r(U, V) \gamma, s(U, V)^2 \Sigma)\}.
\]

Consider the density function of \(Y\), \(f(y)\), in the case that \(S = s(U, V)\) is a non-null constant. From (9) it follows that

\[
f(y) = E_G\{\varphi_d(y; \xi + r(U, V) \gamma, s(U, V)^2 \Sigma)\} = E_H\{\varphi_d(y; \xi + R \gamma, S^2 \Sigma)\}
\]

where the first expected value is taken with respect to the distribution \(G\), the second one with respect to \(H\). Assume further that the distribution \(H\) of \((R, S)\) is absolutely continuous with density function \(h(rs)\), and note that the transformation from \((R, S, X)\) to \((R, S, Y)\) is invertible, so that a standard computation for densities of transformed variables yields, in an obvious notation,

\[
f_{R,S,Y}(r, s, y) = s^{-d} f_{R,S,X}(r, s, s^{-1}(y - \xi - r \gamma))
\]

\[
= h(r, s) s^{-d} \varphi_d(s^{-1}(y - \xi - r \gamma); 0, \Sigma)
\]

\[
= h(r, s) \varphi_d(y; \xi + r \gamma, s^2 \Sigma)
\]

taking into account the independence of \((R, S)\) and \(X\). Hence we arrive at

\[
f(y) = \int_{R \times \mathbb{R}_+} \varphi_d(y; \xi + r \gamma, s^2 \Sigma) dH(r, s).
\]

An alternative route to obtain this expression would be via differentiation of the distribution function (12) with exchange of the integration and differentiation signs.
For statistical work, it is often useful to consider constructions of type (7) where the distributions of $U$ and $V$ belong to some parametric family. In these cases, care must be taken to avoid overparameterization. Given the enormous variety of specific instances embraced by (7), it seems difficult to establish general suitable conditions, and we shall then discuss this issue within specific families or classes of distributions.

In the above passage, as well as in the rest of the paper, the term ‘family’ refers to the set of distributions obtained by a given specification of the variables $(X, U, V)$ when their parameters vary in some admissible space, while keeping the other ingredients fixed. Broader sets, generated for instance when the distributions of $U$ and $V$ vary across various parametric families, constitute ‘classes’.

A clarification is due about the use of the notation in (7)–(8) and some derived expressions to be presented later on. When we shall examine a certain family belonging to the general construction, that notation will translate into a certain parameterization, which often is not the most appropriate for inferential or for interpretative purposes, and its use here must not be intended as a recommendation for general usage. This scheme is adopted merely for uniformity and simplicity of treatment in the present investigation.

2.3 Affine transformations and other distributional properties

For the random variable $Y$ introduced by (7)-(8), consider an affine transformation $W = b + B^T Y$, for a $q$-dimensional vector $b$ and a full-rank matrix $B$ of dimension $d \times q$, with $q \leq d$; denote these assumptions as ‘the b-B conditions’. It is immediate that

$$W = b + B^T Y = b + B^T \xi + r(U,V)B^T \gamma + s(U,V)B^T X$$

is still of type (7)–(8) with the same mixing variables $(R, S)$ and modified numerical parameters. We have then reached the following conclusion.

**Proposition 1** If $Y \sim \text{GMN}_d(\xi, \Sigma, \gamma, H)$ and $b, B$ satisfy the b-B conditions introduced above, it follows that

$$b + B^T Y \sim \text{GMN}_q(b + B^T \xi, B^T \Sigma B, B^T \gamma, H)$$

is still a member of the GMN class, with the same mixing distribution of $Y$.

Partition now $Y$ in two sub-vectors of sizes $d_1, d_2$, such that $d_1 + d_2 = d$, with corresponding partitions of the parameters in blocks of matching sizes, as follows

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \quad (15)$$

To establish the marginal distributions of $Y_1$, we use Proposition 1 with $b = 0$ and $B$ equal to a matrix formed by $I_{d_1}$ in the top $d_1$ rows and a block of 0s in the bottom $d_2$ rows. For $Y_2$, we proceed similarly, but setting the bottom $d_2$ rows of $B$ equal to $I_{d_2}$. We then arrive at the following conclusion.

**Proposition 2** If $Y \sim \text{GMN}_d(\xi, \Sigma, \gamma, H)$ is partitioned as indicated in (15), then

$$Y_1 \sim \text{GMN}_{d_1}(\xi_1, \Sigma_{11}, \gamma_1, H), \quad Y_2 \sim \text{GMN}_{d_2}(\xi_2, \Sigma_{22}, \gamma_2, H). \quad (16)$$

We now want examine conditions which ensure independence of $Y_1$ and $Y_2$. From (9) it is clear that, if $\Sigma_{12} = \Sigma_{21}^\top = 0$, $Y_1$ and $Y_2$ are conditionally independent given $(U, V)$, with conditional distribution

$$(Y_j | U = u, V = v) \sim \text{N}_d(\xi_j + r(u, v)\gamma_j, s(u, v)^2 \Sigma_{jj}), \quad j = 1, 2, \quad (17)$$
where \((U, V) \sim G_U \times G_V\). Moreover, if \(s(U, V) \equiv 1\) (constant) and one of the marginal distributions is symmetric, i.e., \(\gamma_1 = 0\) or \(\gamma_2 = 0\), then \(Y_1\) and \(Y_2\) are independent. The notation \(s(U, V) \equiv 1\) and similar ones later on must be intended ‘with probability 1’; we shall not replicate this specification subsequently.

A more detailed argument is as follows, where we take \(\xi = 0\) for mere simplicity of notation. Without affecting the generality of the argument. From (9), we have that the conditional joint characteristic function of \((Y_1, Y_2)\), given \(U = u\) and \(V = v\) (or, equivalently, given \(R = r\) and \(S = s\)), is

\[
E\left\{ e^{it_1 Y_1 + it_2 Y_2} \bigg| U = u, V = v \right\} = \exp \left\{ ir(u, v) (t_1 Y_1 + t_2 Y_2) - \frac{1}{2} s(u, v) t_1^2 \gamma_1 \gamma_1 t_2 \right\}
\]

so that the joint characteristic function of \((Y_1, Y_2)\) is

\[
c(t_1, t_2) = E\left\{ e^{it_1 Y_1 + it_2 Y_2} \right\} = E\left\{ e^{it_1 Y_1 + it_2 Y_2} \big| U, V \right\} = E\left\{ e^{it_1 U Y_1 + it_2 U Y_2} - \frac{1}{2} s(U, V) t_1^2 \gamma_1 \gamma_1 t_2 \right\}
\]

In analogous way, by (17) the marginal characteristic functions are

\[
c_j(t_j) = E\left\{ e^{it_j Y_j} \right\} = E\left\{ e^{it_j Y_j} \big| U, V \right\} = E\left\{ e^{ir(U, V) t_j Y_j - \frac{1}{2} s(U, V) t_j^2 \gamma_j } \right\}, \quad j = 1, 2.
\]

Note that, if \(\gamma_j = 0\) and \(s(U, V) \equiv 1\), then by (19) \(c_j(t_j)\) reduces to the centred normal characteristic function \(c_{N,j}(t_j) = e^{-\frac{1}{2} t_j^2 \gamma_j} \) for \(j = 1, 2\). We have then reached the following conclusion.

**Proposition 3** Given partition (15), the components \(Y_1, Y_2\) are independent provided \(s(U, V) \equiv 1\), \(\Sigma_{12} = 0\) and at least one of \(\gamma_1\) and \(\gamma_2\) is \(0\), with the following implications:

(a) if \(\gamma_1 = 0\), the joint characteristic function (18) reduces to \(c_{N,1}(t_1) c_2(t_2)\),

(b) if \(\gamma_2 = 0\), the joint characteristic function (18) reduces to \(c_1(t_1) c_{N,2}(t_2)\).

If both \(\gamma_1\) and \(\gamma_2\) are \(0\), the distribution reduces to the case of independent normal variables.

In essence, under the conditions of Proposition 3, one of \(Y_1\) and \(Y_2\) has a plain normal distribution and the other one falls under the construction discussed later in Section 3.

Outside the conditions of Proposition 3, the structure of (18) does not appear to be suitable for factorization as the product of two legitimate characteristic functions, and we conjecture that, in general, independence between \(Y_1\) and \(Y_2\) cannot be achieved.

Examine now the conditional distributions associated to partition (15). Factorize the joint density of \(Y\) as \(f(y_1, y_2) = f_{1|2}(y_1|y_2) f_2(y_2)\) where \(f_{1|2}(y_1|y_2)\) is the conditional density of \((Y_1|Y_2 = y_2)\) and \(f_2(y_2)\) is the marginal density of \(Y_2\). For simplicity of treatment, suppose that \((R, S)\) is absolutely continuous, with density \(h(r, s)\). Then, by (13) and the properties of the multivariate normal density, write

\[
f_{1|2}(y_1|y_2) f_2(y_2) = \int_{\mathbb{R} \times \mathbb{R}} \varphi_{d_j}(y_1; \xi_1) + \gamma_1 r, s^2 \Sigma_{1|2} \varphi_{d_j}(y_2; \xi_2 + \gamma_2 r, s^2 \Sigma_{22}) h(r, s) \, dr \, ds
\]
where
\[ \eta_{1/2} = \xi_1 + \Sigma_{12} \Sigma_{22}^{-1} (y_2 - \xi_2), \quad \eta_{1/2} = \gamma_1 - \Sigma_{12} \Sigma_{22}^{-1} \gamma_2, \quad \Sigma_{11/2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \]

having assumed that the conditioning operation and integration can be exchanged. Hence, for the conditional density of \( Y_1 \) given \( y_2 \) we have
\[
f_{1|2}(y_1|y_2) = \frac{1}{f_2(y_2)} \int_{\mathbb{R} \times \mathbb{R}} \varphi_{d_1}(y_1; \xi_{1|2} + \gamma_{1|2} r, s^2 \Sigma_{1|2}) \varphi_{d_2}(y_2; \xi_2 + \gamma_2 r, s^2 \Sigma_{22}) h(r, s) \, dr \, ds.
\]

Now, from Bayes's rule, we obtain that the conditional density of \((R, S)\) given \( Y_2 = y_2 \) is
\[
h_c(r, s|y_2) = \frac{\varphi_{d_2}(y_2; \xi_2 + \gamma_2 r, s^2 \Sigma_{22}) h(r, s)}{f_2(y_2)}.
\]

Using this fact in the last integral, we can re-write
\[
f_{1|2}(y_1|y_2) = \int_{\mathbb{R} \times \mathbb{R}} \varphi_{d_1}(y_1; \xi_{1|2} + \gamma_{1|2} r, s^2 \Sigma_{1|2}) \varphi_{d_2}(y_2; \xi_2 + \gamma_2 r, s^2 \Sigma_{22}) h_c(r, s|y_2) \, dr \, ds
\]
which exhibits the same structure of (13). Therefore we can conclude that
\[
(Y_1|Y_2 = y_2) \sim \text{GMN}_{d_1} (\xi_{1|2}, \Sigma_{1|2}, \gamma_{1|2}, H_c(y_2))
\]
where \( H_c(y_2) \) denotes the distribution function associated to the conditional density (20).

For many GMN constructions (7)–(8), the density function of \( Y \) is likely to be known in explicit form; in these cases, the same holds true for \( Y_2 \), recalling (16). Then, a convenient aspect of expression (20) is that it indicates how to compute the conditional density once the joint unconditional distribution \( H(r, s) \) is available explicitly. Clearly, this is especially amenable in those constructions where \((R, S)\) is really a univariate variable, as in Sections 3 and 4 below.

In one of the appendices, we illustrate the use of (20)–(21) in the case of a multivariate \( t \) distribution.

### 2.4 On quadratic forms

For use in the next result, but also in the rest of the paper, define the quantities
\[
\Omega = \Sigma + \gamma \gamma^T, \quad \eta = (1 + \gamma^T \Sigma^{-1} \gamma)^{-1/2} \Sigma^{-1} \gamma, \quad \alpha^2 = \|\gamma\|^2_\Sigma = \gamma^T \Sigma^{-1} \gamma, \quad \delta^2 = \frac{\alpha^2}{1 + \alpha^2}
\]
such that \( \alpha^2 \in [0, \infty) \) and \( \delta^2 \in [0, 1) \). For notational convenience, we introduce the notation
\[
\mu_{hk} = \mathbb{E}\{R^h S^k\}, \quad k = 0, 1, \ldots
\]
when the named expectation exists.

**Proposition 4** For a random variable \( Y_0 \) having distribution of type (7)–(8) with \( \xi = 0 \), the following facts hold:

\[
S^{-2} (Y_0 - R \gamma)^T \Sigma^{-1} (Y_0 - R \gamma) \sim \chi^2_d, \quad \mathbb{E}\{Y_0^T \Sigma^{-1} Y_0\} = d \mathbb{E}\{S^2\} + \alpha^2 (\mathbb{E}\{R^2\} = d \mu_{02} + \alpha^2 \mu_{20}), \quad \mathbb{E}\{Y_0^T \Omega^{-1} Y_0\} = d \mathbb{E}\{S^2\} + \delta^2 (\mathbb{E}\{R^2\} - \mathbb{E}\{S^2\}) = d \mu_{02} + \delta^2 (\mu_{20} - \mu_{02}),
\]

provided \( \mathbb{E}\{R^2\} \) and \( \mathbb{E}\{S^2\} \) exist, using the quantities defined in (22) and (23).
Proving: From (8), write \((Y_0 - RY)^\top \Sigma^{-1}(Y_0 - RY) = S^2 X^\top \Sigma^{-1} X\), where \(X^\top \Sigma^{-1} X \sim \chi_2^2\) is independent of \(S\); this yields result (24). For equality (25), expand the initial identity of this proof as
\[
Y_0^\top \Sigma^{-1} Y_0 - 2RY^\top \Sigma^{-1} Y_0 + R^2 Y^\top \Sigma^{-1} Y = S^2 X^\top \Sigma^{-1} X
\]
and take expectation on both sides of this equality. We obtain
\[
\mathbb{E}(RY_0) = \mathbb{E}(R\mathbb{E}(Y_0|U, V)) = \mathbb{E}(R \mathbb{E}\{R \gamma + S X|U, V\}) = \mathbb{E}\{R^2\} \gamma,
\]
bearing in mind that \(\mathbb{E}(X|U, V) = \mathbb{E}(X) = 0\), by the independence assumption between \(X\) and \((U, V)\). This leads to (25).

For (26), write \(Q = Y_0^\top \Omega^{-1} Y_0\), and \(\mathbb{E}\{Q\} = \text{tr}(\Omega^{-1} \mathbb{E}\{Y_0 Y_0^\top\})\). Using (10) and (11), we obtain \(\mathbb{E}\{Y_0 Y_0^\top\} = \text{var}(Y_0) + \mathbb{E}(Y_0^2)\mathbb{E}\{Y_0^2\} = \mathbb{E}\{R^2\} \gamma Y^\top + \mathbb{E}\{S^2\} \Sigma\), so that
\[
\mathbb{E}\{Q\} = \mathbb{E}\{R^2\} \gamma Y^\top + \mathbb{E}\{S^2\} \text{tr}(\Omega^{-1} \Sigma).
\]
By using the Sherman-Morrison equality (6), we conclude the proof. QED

In the subsequent pages, the matrix \(\Omega\) defined in (22) and the associated quadratic form \(Q = Y_0^\top \Omega^{-1} Y_0\) will appear repeatedly. A connected relevant question is: under which conditions is (26) free of \(\gamma\)? Equivalently, under which conditions
\[
\mathbb{E}\{Q\} = \mathbb{E}\{Y_0^\top \Omega^{-1} Y_0\} = d \mathbb{E}\{S^2\}\quad (?)
\]
This equality represents a form of invariance which is known to hold in some cases to be recalled later on, but we want to examine it more generally. One setting where equality (27) holds is given by \(R = UV^{1/2}, S = V^{1/2}\), where \(V > 0\), and \(\mathbb{E}\{U^2\} = 1\). It is then immediate to see that \(\mathbb{E}\{R^2\} = \mathbb{E}\{S^2\}\), so that the final term of (26) is zero.

The conditions \(R = UV^{1/2}\) and \(S = V^{1/2}\) are in turn achieved when \(Z = UY + X\) and \(Y_0 = V^{1/2}Z\). In this case \(Q = VQ_0\), where \(Q_0 = Z^\top \Omega^{-1} Z\) which is independent of \(V\). Hence, \(\mathbb{E}\{Q\} = \mathbb{E}\{V\}\mathbb{E}\{Q_0\}\), where
\[
\mathbb{E}\{Q_0\} = \text{tr}(\Omega^{-1} \mathbb{E}\{ZZ^\top\}) = \mathbb{E}\{U^2\} \gamma Y^\top + \Sigma
\]
since \(\mathbb{E}(Z) = \mathbb{E}(U) \gamma\) and \(\text{var}(Z) = \mathbb{E}\{U\} YY^\top + \Sigma\) and so \(\mathbb{E}\{ZZ^\top\} = \mathbb{E}\{U^2\} YY^\top + \Sigma\). Thus, if \(\mathbb{E}\{U^2\} = 1\), then by using (6) it clearly follows that \(\mathbb{E}\{Q_0\} = d\). We shall return to this issue later on.

### 2.5 Mardia’s measures of multivariate asymmetry and kurtosis

For a multivariate random variable \(Z\) such that \(\mu_Z = \mathbb{E}(Z)\) and \(\Sigma_Z = \text{var}(Z)\), Mardia (1970, 1974) has introduced measures of multivariate skewness and kurtosis, defined as
\[
\beta_{1,d} = \mathbb{E}\{|(Z - \mu_Z)^\top \Sigma_Z^{-1}(Z' - \mu_Z)|^3\}, \quad \beta_{2,d} = \mathbb{E}\{|(Z - \mu_Z)^\top \Sigma_Z^{-1}(Z - \mu_Z)|^2\},
\]
where \(Z'\) is an independent copy of \(Z\), provided these expected values exist. These measures represent extensions of corresponding familiar quantities for the univariate case:
\[
\beta_1 = \frac{\mathbb{E}\{(Z - \mu_Z)^3\}^2}{\text{var}(Z)^{3/2}} = Y_1^2,
\]
\[
\beta_2 = \frac{\mathbb{E}\{(Z - \mu_Z)^2\}^2}{\text{var}(Z)^2} = Y_2 + 3,
\]
in the sense that \(\beta_{1,1} = \beta_1\) and \(\beta_{2,1} = \beta_2\).

We want to find expressions for (28) in the case of a random variable \(Y\) of type (7)–(8). Recall the expressions for \(\mu_Y\) and \(\Sigma_Y\) given in (10) and (11), and the notation defined in (23) for the moments of \((R, S)\), and write
\[
R_0 = R - \mu_{10}, \quad Y - \mu_Y = R_0 Y + SX.
\]
assuming that the involved mean values exist. Taking into account the invariance of $\beta_{d,1}$ and $\beta_{d,2}$ with respect to non-singular affine transformations, it is convenient to work with the transformed quantities

$$X_0 = \Sigma^{-1/2} X \sim N_d(0, I_d), \quad \gamma_0 = \Sigma^{-1/2} \gamma, \quad Y_0 = \Sigma^{-1/2}(Y - \mu_Y) = R_0 \gamma_0 + S X_0$$

where any form of the square root matrix $\Sigma^{1/2}$ can be adopted.

The subsequent development involves extensive algebra of which we report here only the summary elements; detailed computations are provided in an appendix. Recall $\alpha^2$ introduced in (22) and define

$$\bar{\mu}_{20} = \mu_{20} - \mu_{10}^2 = \text{var}(R), \quad \rho = \frac{\mu_{20}}{\mu_{02}}, \quad \hat{\rho} = \frac{\rho \alpha^2}{1 + \rho \alpha^2} = \frac{\alpha^2 \mu_{20}}{\mu_{02} + \alpha^2 \mu_{20}}.$$ 

Introduce the auxiliary random variables $T_0 = \alpha^{-1} \gamma_0^T X_0 \sim N(0,1)$, which is independent of $(R,S)$, and $Z_0 = \alpha R_0 + S T_0$. We need to compute the following expectations:

$$\mathbb{E}\{S^2 Z_0\} = \alpha (\mu_{12} - \mu_{10} \mu_{02}),$$
$$\mathbb{E}\{S^2 Z_0^2\} = \alpha^2 (\mu_{22} - 2 \mu_{12} \mu_{10} + \mu_{10}^2 \mu_{02}) + \mu_{04},$$
$$\mathbb{E}\{Z_0^3\} = \alpha^3 (\mu_{30} - 3 \mu_{21} \mu_{10} + 2 \mu_{10}^2 \mu_{02}) + 3 \alpha (\mu_{12} - \mu_{10} \mu_{02}),$$
$$\mathbb{E}\{Z_0^4\} = \alpha^4 (\mu_{40} - 4 \mu_{30} \mu_{10} + 6 \mu_{21} \mu_{10}^2 - 3 \mu_{10}^3) + 6 \alpha^2 (\mu_{22} - 2 \mu_{12} \mu_{10} + \mu_{10}^2 \mu_{02}) + 3 \mu_{04},$$

assuming the existence of moments of $(R,S)$ up to the fourth order. With these ingredients, the Mardia’s measures for the GMN construction can be expressed as

$$\beta_{1.d} = \frac{\mu_{02}^3}{\mu_{02}} \left( 3 (d - 1) (1 - \rho) \mathbb{E}\{S^2 Z_0\}^2 + (1 - \rho)^3 \mathbb{E}\{Z_0^3\}^2 \right),$$
$$\beta_{2,d} = \frac{\mu_{02}^2}{\mu_{02}} \left( (d + 1) (d - 1) \mu_{04} + 2 (d - 1) (1 - \rho) \mathbb{E}\{S^2 Z_0^2\} + (1 - \rho) \mathbb{E}\{Z_0^4\} \right).$$

Considering the complexity that typically involves the explicit specification of (28) outside the normal family, the above expressions appear practically manageable. They are further simplified when one specializes them to a given family or to a certain subclass of the GMN construction. For a given choice of the distribution $H$, we need to work out the following ingredients: (i) the marginal moments of $R$, $\mu_{h0}$, up to order 4, (ii) the marginal moments $\mu_{02}$ and $\mu_{04}$ of $S$, (ii) the cross moments $\mu_{12} = \mathbb{E}\{RS^2\}$ and $\mu_{22} = \mathbb{E}\{R^2 S^2\}$. The working is illustrated next for the GH family; additional illustrations will appear later.

**Mardia’s measures for the GH family** For the GH family with representation (2), there is a single mixing variable $V \sim N^{-}(\lambda, \chi, \psi)$ with density (5) and $R = V$, $S = V^{1/2}$. General expressions for $\mathbb{E}\{V^h\} = \mu_{h0}$ are given in Section 2.1 of Jørgensen (1982), among others. These expressions also provide $\mu_{02} = \mathbb{E}\{V\}$ and $\mu_{04} = \mathbb{E}\{V^2\}$. The two other required quantities are $\mu_{12} = \mathbb{E}\{V^2\}$ and $\mu_{22} = \mathbb{E}\{V^3\}$ which are still ordinary moments of $V$. We can now compute

$$\mathbb{E}\{S^2 Z_0\} = \alpha (\mathbb{E}\{V^2\} - (\mathbb{E}\{V\})^2) = \alpha \sigma_V^2, \quad \text{say},$$
$$\mathbb{E}\{S^2 Z_0^2\} = \alpha^2 (\mathbb{E}\{V^3\} - 2 \mathbb{E}\{V^2\} \mathbb{E}\{V\} + (\mathbb{E}\{V\})^3) + \mathbb{E}\{V^2\},$$
$$\mathbb{E}\{Z_0^3\} = \alpha^3 (\mathbb{E}\{V^3\} - 3 \mathbb{E}\{V^2\} \mathbb{E}\{V\} + 2 (\mathbb{E}\{V\})^3) + 3 \alpha \text{var}\{V\}$$
$$= (\alpha \sigma_V)^3 \beta_1(V) + 3 \alpha \sigma_V^2,$$
$$\mathbb{E}\{Z_0^4\} = \alpha^4 (\mathbb{E}\{V^4\} - 4 \mathbb{E}\{V^3\} \mathbb{E}\{V\} + 6 \mathbb{E}\{V^2\} (\mathbb{E}\{V\})^2 - 3 (\mathbb{E}\{V\})^4)$$
$$+ 6 \alpha^2 (\mathbb{E}\{V^3\} - 2 \mathbb{E}\{V^2\} \mathbb{E}\{V\} + (\mathbb{E}\{V\})^3) + 3 \mathbb{E}\{V^2\}$$
$$= (\alpha \sigma_V)^4 \beta_2(V) + 6 \alpha^2 (\mathbb{E}\{V^3\} - 2 \mathbb{E}\{V^2\} \mathbb{E}\{V\} + (\mathbb{E}\{V\})^3) + 3 \mathbb{E}\{V^2\}.$$ 

where $\sigma_V^2 = \text{var}\{V\}$ and $\beta_1(V)$, $\beta_2(V)$ are the univariate measures of skewness and kurtosis in (29) evaluated for $V$. Plugging the above quantities in (30) and (31) completes the computation.
Remark. There exists an interesting way of re-writing (30) and (30) which will turn out useful later on. Since \( Z_0 = \alpha R + S T_0 \sim \text{GMN}_1(-\alpha \mu_{10}, 1, \alpha, H) \) with zero mean and
\[
\text{var}(Z_0) = \alpha^2 \tilde{\mu}_{20} + \mu_{02} - (1 - \tilde{\rho})^{-1} \mu_{02}
\]
we can introduced an univariate standardized GMN-type variable
\[
Z_0 = \frac{\alpha R + S T_0 - \alpha \mu_{10}}{\sqrt{\alpha^2 \tilde{\mu}_{20} + \mu_{02}}} \sim \text{GMN}_1 \left( -\frac{\mu_{10}}{\sqrt{\alpha^2 \tilde{\mu}_{20} + \mu_{02}}}, \frac{1}{\sqrt{\alpha^2 \tilde{\mu}_{20} + \mu_{02}}}, \frac{\alpha}{\sqrt{\alpha^2 \tilde{\mu}_{20} + \mu_{02}}}, H \right)
\]
which has zero mean zero and unit variance. When rewritten in terms of \( Z_0 \), (30) and (31) become
\[
\begin{align*}
\beta_{1,d} &= 3(d-1)\mu_{02} - \mu_{02}^2 E\{S^2 Z_0^2\} + \tilde{\beta}_1(Z_0), \\
\beta_{2,d} &= (d+1)(d-1)\mu_{02}^2 \mu_{04} + 2(d-1)\mu_{02}^{-1} E\{S^2 Z_0^2\} + \tilde{\beta}_2(Z_0),
\end{align*}
\]
where \( \tilde{\beta}_1(Z_0) \) and \( \tilde{\beta}_2(Z_0) \) denote the univariate coefficients \( \beta_1 \) and \( \beta_2 \) in (29) evaluated for \( Z_0 \).

3 Mean (or location) mixtures

In this section and the next one, we examine two simplified versions of the general formulation (7)–(8). The first class occurs when only the additive random component is actually present.

3.1 General properties

Basic facts. Consider the simplified form of (7)–(8) where \( s(u, v) = 1 \), so that we can assimilate \( R \) and \( U \), and write
\[
Y = \xi + U \gamma + X.
\]
In this case we use the notation \( Y \sim \text{GMN}_d(\xi, \Sigma, \gamma, G_{\gamma}) \), since now \( (R, S) \) reduces to \( R \equiv U \), with distribution \( H = G_{\gamma} \).

Clearly, if \( U \) is a degenerate random variable, \( U \equiv 1 \) say, or if \( \gamma = 0 \), the construction reduces to the normal distribution. We therefore exclude these cases from consideration.

Although constructions of this type can simply be viewed as a sum of two independent random components, they can legitimately also be regarded as a location mixture, within the logic of (9), and this interpretation can facilitate the construction of EM-type algorithms and work in Bayesian inference.

Several general properties of the class (34) have been obtained by Negarestani et al. (2019). Their initial development is in the univariate context, where they establish the property of closure under convolution with an independent normal variables, and an expression of the characteristic function. From these results, they derive expressions for low order moments and associated measures of asymmetry and kurtosis. Section 8 of their paper refers to the multivariate case, where they also obtain various results, notably the property of closure under marginalization, an expression for the conditional distribution given the values taken on by certain components of \( Y \), and
\[
\begin{align*}
E\{Y\} &= \xi + E\{U\} \gamma, \\
\text{var}(Y) &= \text{var}(U) \gamma \gamma^\top + \Sigma
\end{align*}
\]
provided \( E\{U\} \) and \( \text{var}(U) \) exist. These expressions can also be obtained as special cases of (10) and (11).

The property of closure under convolution with normal variates, which has been stated by Negarestani et al. (2019) in the univariate case, actually holds also in the multivariate case. Specifically, if \( Y \sim \text{GMN}_d(\xi, \Sigma, \gamma, G_{\gamma}) \) and \( W \sim \text{N}_d(\mu, \Sigma) \) are independent variables, then it is immediate from representation (34) that \( Y + W \sim \text{GMN}_d(\xi + \mu, \Sigma + \Sigma, \gamma, G_{\gamma}) \).

From Proposition 3, we can say that the marginal components \( Y_1 \) and \( Y_2 \) are independent if and only if \( \Sigma_{12} = 0 \) and at least one of \( \gamma_1 \) and \( \gamma_2 \) is 0.
Mardia’s measure for mean mixtures With respect to development in Subsection 2.5, here we have
\( S = 1 \) and \( R = U \), with substantial simplification of the general expressions (30)–(31). In this case we obtain

\[
\begin{align*}
\mathbb{E}\{S^2 Z_0\} &= \mathbb{E}\{Z_0\} = 0, \\
\mathbb{E}\{S^2 Z_0^2\} &= \mathbb{E}\{Z_0^2\} = (\alpha \sigma_U)^2 + 1 = (1 - \tilde{\rho})^{-1} \\
\mathbb{E}\{Z_0^3\} &= \beta_1(U), \\
&= (\alpha \sigma_U)^3 \beta_1(U), \\
\mathbb{E}\{Z_0^4\} &= \beta_2(U) + 6(a \sigma_U)^2 + 3.
\end{align*}
\]

where \( \beta_1(U) \) and \( \beta_2(U) \) denote the univariate coefficients in (29) evaluated for \( U \), leading to

\[
\begin{align*}
\beta_{1,d} &= (1 - \tilde{\rho})^3(\mathbb{E}\{Z_0^3\})^2 \\
&= ((\alpha \sigma_U)^2 + 1)^{-1} \beta_{1}(U)^2, \\
\beta_{2,d} &= \frac{d(d + 2) - 3 + (1 - \tilde{\rho})^2}{d(d + 2) + 6(a \sigma_U)^2 + 3}. \\
&= \frac{d(d + 2) + ((\alpha \sigma_U)^2 + 1)^{-1} \beta_{2}(U)}{d(d + 2) + 6(a \sigma_U)^2 + 3}.
\end{align*}
\]

Note that the leading term \( d(d + 2) \) in the last expression represents the difference between \( \beta_{2,d} \) and its companion measure of excess, \( \gamma_{2,d} \), in Mardia (1974).

Remark In this case, (32) and (33) yield a very neat simplification, namely

\[
\begin{align*}
\beta_{1,1} &= \mathbb{E}\{Z_0^3\}^2 = \gamma_1^2, \\
\beta_{2,1} &= d(d + 2) + \mathbb{E}\{Z_0^4\} - 3 = d(d + 2) + \gamma_2.
\end{align*}
\]

3.2 Some noteworthy special cases

The more interesting families of this class are arguably those obtained when the distribution of \( U \) is not symmetric about 0. In fact, in nearly all special formulations discussed below, \( U \) is a positive variable.

Besides its intrinsic values from the distribution theory viewpoint, there is the interesting connection of (34) with non-symmetric \( U \) and the formulation in quantitative finance put forward by Simaan (1993), as for the assumptions on the key stochastic component, and the closure under marginalization.

The skew-normal distribution and its extended version When \( U \) in (34) has a positive half-normal distribution, or equivalently the \( N(0,1) \) distribution truncated below 0, we obtain the set-up adopted by Azzalini & Dalla Valle (1996) to derive the density function of the multivariate skew-normal (SN) family. The multivariate SN density function at \( y \in \mathbb{R}^d \) is

\[
2 \varphi_d(y - \xi; \Omega) \Phi(\eta^\top (y - \xi))
\]

where \( \Omega \) and \( \eta \) are as in (22). In one appendix, we present a proof of this expression which retains the same logic of the proof of Azzalini & Dalla Valle (1996), but involves a more essential development.

The multivariate SN distribution enjoys a number of appealing formal properties, matching many of those of the normal distribution. An account of this theme is provided in Chapter 5 of Azzalini &
In view of the discussion in Section 2.4, we must at least mention the fact that, as a special case of a more general result on quadratic forms, \((Y - \xi)\top \Omega^{-1} (Y - \xi) \sim \chi^2_d\) when \(Y\) is a random variable with density (36).

The extended form of the skew-normal distribution occurs when \(U\) is distributed as \(N(0,1)\) variable truncated below \(-\tau\) instead of 0, for some constant \(\tau\). The distribution of \(U\) is then \(\varphi(u)/\Phi(u)\) for \(u + \tau > 0\). A simple adaptation of the above-mentioned proof yields the density function of \(Y\) at \(y \in \mathbb{R}^d\) as

\[
\frac{1}{\Phi(\tau)} \varphi_d(y - \xi; \Omega) \Phi(\bar{\tau} + \eta^\top (y - \xi))
\]

(37)

where \(\bar{\tau} = \left(1 + \gamma^\top \Sigma^{-1} \gamma\right)^{1/2} \tau\).

While it is not clear whether a EM-type approach is the most efficient way to tackle maximum likelihood estimation for distribution (36) or (37), certainly EM-type algorithms based on (9) constitute a popular route for parameter estimation in this context. Early publications adopting this route to estimation include Arellano-Valle et al. (2005a) and Arellano-Valle et al. (2005b), but many others exist, often in connection with finite mixtures of SN distributions.

**MMNE and MMMNE distributions** A substantial portion of the paper of Negarestani et al. (2019) focuses on the specific instance where \(U\) in (34) follows a standard exponential distribution. They initially examine the case where \(Y\) is univariate; this is said to have a MMNE distribution, and several interesting properties are derived: log-concavity of the density, monotonicity of the hazard rate, infinitely divisibility and more. Subsequently, they consider the multivariate version, called MMMNE distribution, whose density function at \(y \in \mathbb{R}^d\) can be written, with an inessential notational variation from the original paper, as

\[
\varphi_d(y; \xi, \Sigma) \left\{ (\gamma^\top \Sigma^{-1} \gamma)^{1/2} \zeta \left( \frac{\gamma^\top \Sigma^{-1} x - 1}{(\gamma^\top \Sigma^{-1} \gamma)^{1/2}} \right) \right\}^{-1}
\]

(38)

where \(\zeta()\) is defined in (4).

Additional results are derived for the MMMNE distribution, notably the expression of the characteristic functions, the marginal and the conditional distributions given the value taken on by a subset of \(Y\) components. The mean and the variance are simply obtained by setting \(E\{U\} = \text{var}\{U\} = 1\) in (35).

**A two-piece normal mixing** Assume that \(U\) in (34) has a two-piece normal distribution, that is, one having density function at \(u \in \mathbb{R}\):

\[
2\pi a \varphi(u; a^2) I_{(0,\infty)}(u) + 2\pi b \varphi(u; b^2) I_{(-\infty,0)}(u)
\]

where \(I_A(\cdot)\) denotes the indicator function of set \(A\). This construction has been proposed repeatedly in the literature as a simple way to allow for skewness via a simple modification of the normal density. A compilation of rediscoveries of this distribution has been presented by Wallis (2014).

It has been shown by Arellano-Valle et al. (2020) that \(Y\), as defined in (34), has a density function represented by a two-component mixture of skew-normal variates. Hence each component has a density of type (37) with \(\tau = 0\).

This distribution of \(U\) is the only instance reviewed here where \(U\) is not a positive variable. It is included in our list because it represents an interesting bridge between different families: it shows how a mixture of multivariate normal variates, suitably combined with a mixing two-piece normal variate, yields a multivariate SN variable. In other words, it provides a link between two asymmetric extensions of the normal family, the two-piece and the skew-normal distributions.
A Rayleigh mixing mean As far as we know, the following construction has not been examined in the literature. Suppose that the mixing variable $U$ in (34) has a standard Rayleigh distribution, with density

$$g_U(u) = \frac{1}{2\pi} e^{-u^2/2} I_{(0,\infty)}(u).$$

and write $Z = U\gamma + X$. In this case, the mean and the variance of $Y = \xi + Z$ are simply obtained by setting $\mathbb{E}(U) = (\pi/2)^{1/2}$ and $\text{var}(U) = (4 - \pi)/2$ in (35).

Consideration of this model can be motivated as follows. The Rayleigh distribution is widely used in a various applied disciplines, especially in the engineering context. An example is represented by the technology of wind energy, where the distribution plays a central role, as clearly visible in the comprehensive treatment of the subject by Nelson (2013). From this source, we underline the noteworthy fact that “Manufacturers [of wind turbines] assume a Rayleigh distribution for a wind speed” (p. 101). Besides this domain, distribution (39) is used in signal processing, ocean energy and off-shore engineering, material design and reliability, and other areas, not all in engineering. Consider now the case where a certain event, such as wind speed $U$ at a certain location and time, is measured by $d$ instruments at the time, not just one. There will then be a $d$-dimensional vector $X$ of random components originated by the measuring instruments which, once combined with $U$, yields an instance of (34). If all the instruments are perfectly calibrated, $\xi$ will be the null vector and $\gamma$ will have all components equal, otherwise discrepancies will exists.

It will be noted that density (39) does not include a positive scale factor, $\sigma$ say, which is essential in applied work. This factor is implicit and subsumed in $\gamma$, as otherwise we would incur in a overparameterization situation. In an applied context, it would presumably be sensible to reparameterize in some more meaningful form, such as $\gamma = \sigma \tilde{\gamma}$, with some suitable constraint on $\tilde{\gamma}$. In the present more technical context, we retain the use of $\gamma$.

We show in one appendix that the density function of $Z$ is

$$f_Z(z; \gamma, \Sigma) = (2\pi)^{1/2}(1 + \alpha^2)^{-1/2} \varphi_d(z; \Omega) \Phi(\eta^\top z) \left\{\xi(\eta^\top z) + \eta^\top z\right\}$$

where $\xi(\cdot)$ is defined in (4). From (40), it is immediate to obtain the density of $Y$.

It is interesting that the leading factor on the right-hand side of (40) is, up to a constant, the SN density (36) with $\xi = 0$. In the specific case with $d = 1$ and $\Sigma = 1$, this factor is the SN$(0,1 + \gamma^2,\gamma)$ density.

An extension to $\chi_v$ mixing In two of the constructions examined earlier, the $U$ component is a square-root of a $\chi^2$ variable. Specifically, in the SN construction, $U$ has as a half-normal distribution, that is, $U \sim \chi_1$. In another case examined, $U$ has a Rayleigh distribution, that is, $U \sim \chi_2$. It is then natural to consider a more general formulation where $U \sim \chi_v$, for some positive $v$, having density

$$g_U(u) = \frac{2}{\Gamma(v/2)} u^{v-1} e^{-u^2/2}, \quad u \in \mathbb{R}^+.$$

The remark made in connection with (39) about incorporation of any scale parameter of $U$ in $\gamma$ carry on here. In the light of this, the $\chi_v$ distribution considered here is effectively equivalent to what is often called Nakagami-$m$-distribution (Nakagami, 1960) in radio communication engineering.

Recalling the expression of moments of the $\chi^2$ distribution, we obtain readily

$$\mathbb{E}\{U^m\} = \frac{2^{m/2}\Gamma((v + m)/2)}{\Gamma(v/2)}, \quad \text{var}(U) = 2v - \mathbb{E}\{U^2\}.$$

The density of $Z = \gamma U + X$ now becomes

$$f_Z(z; v, \gamma, \Sigma) = \frac{2\sqrt{\pi}}{\Gamma(v/2)[2(1 + \alpha^2)]^{(v-1)/2}} \varphi_d(z; \Omega) \int_{-\infty}^{\infty} (w + \eta^\top z)^{v-1} \varphi(w) dw$$

$$= \frac{2\sqrt{\pi}}{\Gamma(v/2)[2(1 + \alpha^2)]^{(v-1)/2}} \varphi_d(z; \Omega) \Phi(\eta^\top z) M_{v-1}(\eta^\top \xi)$$

(41)
having written
\[ M_{\nu-1}(\eta^\top z) = \mathbb{E}\{(W + \eta^\top z)^{\nu-1} \mid W + \eta^\top z > 0\} \]
where \( W \sim N(0,1) \).

To compute \( M_{\nu-1}(\eta^\top z) \), we must assume that \( \nu \) is integer. We then expand
\[ M_{\nu-1}(\eta^\top z) = \sum_{k=0}^{\nu-1} \binom{\nu-1}{k} \mathbb{E}\{W^{\nu-1} \mid W + \eta^\top z > 0\} (\eta^\top z)^{\nu-k-1} \]
where \( \mathbb{E}\{W^{\nu-1} \mid W + \eta^\top z > 0\} \) is the \((\nu - 1)\)th moment of a truncated normal distribution.

A recursive expression for the moments of a \( N(0,1) \) variable truncated below level \( a \), say, has been given by Elandt (1961), which in our case must be used with \( a = -\eta^\top z \). An alternative route to these moments is via consideration of the moment generating function and the cumulant generating function of the truncated normal distribution, namely,
\[ M(t) = e^{t^2/2} \frac{\Phi(t-a)}{\Phi(-a)}, \quad K(t) = \frac{1}{2} t^2 + \log(\Phi(t-a)) - \log(\Phi(-a)) \]
whose derivatives involve those of \( \zeta(\cdot) \) in (4). Expressions of low order derivatives of \( \zeta(\cdot) \) are given in Section 2.1.4 of Azzalini & Capitanio (2014).

To develop an EM-type algorithm for this distribution, start by considering the following hierarchical representation:
\[ Y \mid U \sim N_d(\xi + \gamma U, \Sigma), \]
\[ U \sim \chi_\nu. \]

Next, we must compute the first two moments of \( \mathbb{E}\{U\mid Y\} \), which in turn requires the conditional density of \( U \) given \( Y \) or, equivalently, given \( Z \). Taking into account that
\[ \varphi_d(z; \gamma U, \Sigma) = \varphi_d(y; 0, \Sigma) e^{-\frac{1}{2} \left( \gamma^\top z + (1 + \alpha^2)^{-1} (\gamma^\top \Sigma^{-1} z) \right)^2}, \]
a standard application of Bayes theorem gives the conditional density
\[ h_c(u|z) = \frac{\varphi_d(z; \gamma U, \Sigma) h(u)}{\int_Z \varphi_d(z; \nu, \Sigma) \, dz}, \]
\[ = \frac{\varphi_d(z; 0, \Omega) \exp\left( -\frac{1}{2} \left( \alpha^2 u^2 - 2 \gamma^\top \Sigma^{-1} z + (1 + \alpha^2)^{-1} (\gamma^\top \Sigma^{-1} z)^2 \right) \right) 2(1/2)^{\nu/2} \Gamma(\nu/2)}{\Gamma(\nu/2)(2(1 + \alpha^2))^{(\nu-1)/2} \varphi_d(z; \Omega) \Phi(\eta^\top z) M_{\nu-1}(\eta^\top z)} \]
\[ = \frac{(1 + \alpha^2)^{(\nu-1)/2} u^{-1} \exp\left[ -\frac{1}{2} (1 + \alpha^2) u^2 \right]}{\sqrt{2\pi} \Phi(\eta^\top z) M_{\nu-1}(\eta^\top z)} \]
\[ = \frac{(1 + \alpha^2)^{(\nu-1)/2} \varphi\left[ (1 + \alpha^2)^{1/2} \left( u - (1 + \alpha^2)^{-1/2} \eta^\top z \right) \right]}{M_{\nu-1}(\eta^\top z) \Phi(\eta^\top z)}, \quad u > 0. \]

Computation of the \( k \)th moment of \( h_c(u|z) \) effectively amounts to compute the \( (k + \nu - 1) \)th moment of a truncated normal distribution, a point which we have discussed earlier.

In this construction, we have left unspecified whether \( \nu \) represents a known constant or a free positive parameter to be estimated. Similarly to the cases with \( \nu = 1 \) and \( \nu = 2 \) which correspond to already-examined distributions, (41) could be employed with a fixed value of \( \nu \), a situation which would ease use of the EM algorithm introduced above. There is, however, no bar to use it also when \( \nu \) is a free integer parameter.
3.3 Again about quadratic forms

Consider the question of equality (27), and more generally the distribution of the quadratic form \( Q \), in the framework of mean mixtures (34). Since now \( R = U \) and \( S = I \), we consider the distribution of \( Q_0 = Z^T \Sigma^{-1} Z \), where \( Z = \gamma Y + X \). A point of special interest are the conditions on \( U \) such that \( Q_0 \) is distributed as \( X^T \Sigma^{-1} X \sim \chi_d^2 \), and \( Q_0 \) is independent of \((U, V)\).

Note that, if the distribution of \( Q_0 \) does not depend on \( \gamma \), the same holds true for the more general case where \( R = V^{1/2} U \) and \( S = V^{1/2} \), such that \( Q = S^2 Q_0 \). This setting falls within the more general construction examined in Section 5.

In the following discussion, we can ignore the special case \( \gamma = 0 \), as otherwise we return to the basic setting of a normal variable \( Z \) for which it is well-known that \( Q_0 \sim \chi_d^2 \). Define \( W_0 = \gamma^T \Sigma^{-1} X \sim N(0, \alpha^2) \), using the notation in (22); note that \( \alpha^2 > 0 \). On setting \( X_0 = \Sigma^{-1/2} X \), we can also write \( W_0 = \gamma^T \Sigma^{-1/2} X_0 \). For definitiveness, we take \( \Sigma^{1/2} \) to be unique symmetric positive-definite square root matrix of \( \Sigma \), although we next steps would hold also with other choices of the square root. For notational convenience, introduce \( T_0 = \alpha^{-1} W_0 \sim N(0, 1) \).

**Proposition 5** Under the above definitions of symbols, \( Q_0 \) can be decomposed as

\[
Q_0 = W^2 + V_0^2
\]

where \( W = \delta U + (1 - \delta^2)^{1/2} T_0 \) and \( V_0^2 = \|X_0\|^2 - T_0^2 \) are independent variables, with \( V_0^2 \sim \chi_{d-1}^2 \).

**Proof:** By applying the Sherman-Morrison formula (6) to \( \Omega^{-1} \), we can write

\[
\Omega^{-1} \gamma = (1 + \gamma^T \Sigma^{-1} \gamma)^{-1} \Sigma^{-1} \gamma, \quad \gamma^T \Omega^{-1} \gamma = (1 + \gamma^T \Sigma^{-1} \gamma)^{-1} \gamma^T \Sigma^{-1} \gamma = \delta^2
\]

so that we can decompose \( Q_0 \) as

\[
Q_0 = (U Y + X)^T \Omega^{-1} (U Y + X)
\]

\[
= U^2 \gamma^T \Omega^{-1} \gamma + 2U \gamma^T \Omega^{-1} X + X^T \Omega^{-1} X
\]

\[
= U^2 (1 + \gamma^T \Sigma^{-1} \gamma)^{-1} \gamma^T \Sigma^{-1} \gamma + 2U (1 + \gamma^T \Sigma^{-1} \gamma)^{-1} \gamma^T \Sigma^{-1} \gamma + X^T \Sigma^{-1} X - (1 + \gamma^T \Sigma^{-1} \gamma)^{-1} (\gamma^T \Sigma^{-1} \gamma)^2
\]

\[
= \delta^2 U^2 + 2\delta (1 - \delta^2)^{1/2} UT_0 - \delta^2 T_0^2 + \|X_0\|^2
\]

\[
= \delta^2 (U^2 + 2\delta (1 - \delta^2)^{1/2} UT_0 + (1 - \delta^2) T_0^2 - T_0^2 + \|X_0\|^2
\]

\[
= \delta^2 (U^2 + (1 - \delta^2)^{1/2} T_0^2 - T_0^2 + \|X_0\|^2
\]

which proves equality (42).

On defining the unit-norm vector \( \hat{\gamma} = \alpha^{-1} \Sigma^{-1/2} \gamma \in \mathbb{R}^d \), we can write \( V_0^2 = X_0^T M_0 X_0 = \|M_0 X_0\|^2 \) where \( M_0 = I_d - \hat{\gamma} \hat{\gamma}^T \) is a symmetric idempotent matrix of rank \( d - 1 \). Hence, by standard results in normal theory distribution, this proves the claim that \( V_0^2 \sim \chi_{d-1}^2 \). Moreover, \( T_0 = \hat{\gamma}^T X_0 \) and \( M_0 X_0 \) are orthogonal projections of \( X_0 \), since \( \hat{\gamma}^T M_0 = 0 \), and then independent normal variables. This implies independence of \( T_0 \) and \( V_0^2 \) and, since \( X_0 \) and its transformations such as \( T_0 \) are independent of \( U \), we conclude that \( W \) and \( V_0^2 \) are independent.

A corollary of Proposition 5, taking into account the independence of \( U \) and \( T_0 \), is that

\[
E\{Q_0\} = \delta^2 E\{U^2\} + (1 - \delta^2) + d - 1
\]

\[
= d + \delta^2 (E\{U^2\} - 1)
\]

provided \( E\{U^2\} \) exists. Therefore, \( E\{Q_0\} \) does not depend on \( \delta^2 \), hence on \( \gamma \), if and only if \( E\{U^2\} = 1 \), in which case \( E\{Q_0\} = d \).

When \( U \) is distributed as a positive half-normal, \( W \) in (42) has a univariate SN distribution. By a known property of the SN distribution, we can say that \( W^2 \sim \chi_d^2 \) and, by using Proposition 5, we conclude that \( Q_0 \sim \chi_d^2 \). This is a well-known fact in the pertaining literature, as we have recalled after introducing density (36). The derivation here is unusual, because it is aimed to address the following question: are there other choices of \( U \) leading to the same distribution of \( Q_0 \)?
4 Variance (or scale) mixtures

4.1 General points

In a sense, the dual formulation of mean mixtures is represented by variance (or scale) mixtures, already mentioned in the introductory section. This class can be produced by setting $r(u, v) = 1$ in (7) and subsuming $\gamma$ into $\xi$, or equivalently by setting $\gamma = 0$. In either case, we arrive at formulation

$$Y = \xi + V^{1/2} X,$$

where we have assimilated $S$ and $V$, with the condition $V > 0$. The notation introduced for the general construction (7)–(8) now simplifies to $Y \sim \text{GMN}(\xi, \Sigma, 0, G_V)$, since $(R, S)$ reduces to $S = V^{1/2}$ and $H$ reduces to $G_V$.

As already recalled in Section 1, a scale mixture of normal distributions is a member of the class of elliptically contoured distributions. In many popular constructions, $V$ is a continuous variable, and this is the situation on which we shall focus.

In a large number of cases, the inverse operation is also possible, that is, many members of the elliptical class can be represented as scale mixtures of normal distributions. Note, however, that the implied mixing variable $V$ has often a distribution which depends on the dimension $d$ of $X$. This situation prevents the property of closure under marginalization; on this issue, see Kano (1994).

We now recall some instances of scale mixtures of normal variables, but only very briefly and confining ourselves to a few key instances, since they represent very familiar constructions, discussed in many existing accounts. A classical early reference on this theme is Andrews & Mallows (1974). Several instances of the construction are presented by Lange & Sinsheimer (1993); note that they use the alternative term ‘normal/independent distributions’ to identify this theme. A relatively more recent account is provided in Section 3.2 of McNeil et al. (2005).

The formulation (1) bears the danger of over-parameterization, since one could manoeuvre scale both via a suitable parameter of $V$ and via the $\Sigma$ matrix. The issue is usually solved by ruling out a scale parameter in the distribution of $V$.

4.2 Some noteworthy special cases

**Student’s $t$ distribution** Presumably, the best known parametric family of this class is the Student’s $t$ distribution, which occurs when $V \sim \nu/\chi^2$, in an obvious notation. In this case we write $Y \sim t_d(\xi, \Sigma, \nu)$. Note that the distribution of $V$ does not allow for a scale parameter, hence avoiding the above-mentioned issue of lack of identifiability. For later use, recall the expression of the multivariate $t$ density function:

$$t_d(y; \xi, \Sigma, \nu) = \frac{\Gamma((\nu + d)/2)}{(\nu\pi)^{d/2}\Gamma(\nu/2)\det(\Sigma)^{1/2}} \left(1 + \nu^{-1}\|y - \xi\|^2_\Sigma\right)^{-\nu+d}/2, \quad y \in \mathbb{R}^d,$$

where the expression $\|y - \xi\|^2_\Sigma$ makes use of the notation (3).

An important property of this family is that all marginal distributions of $Y$ are still of Student’s $t$ type with the same degrees of freedom and the other parameters as indicated in the partition (15). This result hinges of the fact that the distribution of $V$ does not depend on the dimension $d$. The conditional distribution presented in one appendix shows that a similar property holds also for the conditional distribution, although in this case the degrees of freedom and other parameters are modified. A wealth of other results on the multivariate $t$ distribution and its variants or extensions is presented in the monograph of Kotz & Nadarajah (2004).

**Symmetric GH distribution** Consider the case where $V$ has a generalized inverse Gaussian distribution (5). The implied GMN density of $Y$ is the symmetric GH density whose expression can be obtained by the general GH density shown below in equation (45) when $\gamma = 0$. 

16
Other instances  Of the many other instances of construction (43), a mention is due for the case where \( V \) is discrete. The basic instance is a two-point distribution, leading to the contaminated normal distribution.

Another interesting instance is the multivariate slash distribution which arises choosing \( V = W^{-r} \) where \( W \sim U(0,1) \) and \( r \) is a positive value, which regulates the tail weight of \( V \) and hence also of \( Y \). This distribution and the contaminated normal family have been employed by Lange & Sinsheimer (1993) as the stochastic constituents for an adaptive robust regression methods.

5 Variance-mean mixtures and their generalization

The more versatile formulations are those where both variables \( R \) and \( S \) in (8), are non-degenerate, which constitute the theme of the present section.

Variance-mean normal mixtures  The archetypal construction of this form is the class of distributions with representation (2). In principle, any choice for the distribution of the mixing variable \( V \) is feasible, provided \( \Pr[V > 0] = 1 \).

In practice, the predominant formulation occurs when \( V \) follows a GIG distribution with density (5), leading to the GH family of distributions introduced by Barndorff-Nielsen (1977, 1978). The GH distributions depend on the set parameters \( \theta = (\lambda, \chi, \psi, \xi, \Sigma, \gamma) \), and its density function at \( y \in \mathbb{R}^d \) is

\[
f_{GH}(y; \theta) = \frac{(\psi)^{\lambda/2} C^{\frac{d}{2}-\lambda}}{(2\pi)^{d/2} \det(\Sigma)^{1/2} K_\lambda(\sqrt{\chi \psi})} \frac{K_{\lambda-\frac{d}{2}} \left( \sqrt{C (\chi + \|y - \xi\|^2_\Sigma)} \right)}{\left( \sqrt{C (\chi + \|y - \xi\|^2_\Sigma)} \right)^{\frac{d}{2}-\lambda}}
\]

(45)

where we have used the notation (3) and set \( C = \psi + \|y\|_\Sigma^2 = \psi + \alpha^2 \), bearing in mind (22).

As presented, the parametric family is non-identifiable, due to the coexistence of a scale factor in the GIG distribution and of a similar component in \( \Sigma \). The problem is resolved by imposing some restriction on the parameters. A classical choice, adopted in the original Barndorff-Nielsen’s papers, is to set \( \det(\Sigma) = 1 \).

Although the choice of a GIG distribution for \( V \) in (2) is the predominant one, this does not rule out other possibilities. In the univariate context, Sichel (1973) had proposed the formulation (2) but assuming a Gamma distribution for \( V \). Given the above-mentioned fact that the Gamma family is a boundary case of the GIG family, a multivariate version of the original Sichel’s construction can be obtained by setting \( \chi = 0 \) and \( \lambda > 0 \) in (45).

Scale mixtures of skew-normal distributions  Branco & Dey (2001, Section 3.1) have introduced, with a very slightly different name, the idea of scale mixtures of SN variables. These can be represented as \( Y = \xi + V^{1/2} Z \), where \( Z \) has density of type (37) with \( \tau = 0 \) and \( \xi = 0 \). Combining this
representation with the additive representation (34) of skew-normal variables, namely $Z = U\gamma + X$ where $U$ is half-normal, we write

$$Y = \xi + V^{1/2}Z = \xi + V^{1/2}(U\gamma + X) = \xi + UV^{1/2}\gamma + V^{1/2}X$$

(46)

which is of type (7) with $r(u, v) = u v^{1/2}$ and $s(u, v) = v^{1/2}$.

A member of this class which has received much attention since 2001, both on the theoretical and the applied side, is the skew-$t$ (ST) distribution which occurs when $V \sim \nu/\chi^2_\nu$. A member of the ST family is therefore identified by the set of parameters $(\xi, \Sigma, \gamma, \nu)$. The expression of the ST density can be obtained as a special case of density (48) below, when $\tau = 0$. Chapters 4 and 6 of Azzalini & Capitanio (2014) provide a fairly detailed account of this distribution, in the univariate and the multivariate case. Note that a different parameterization is adopted there.

Another instance of scale mixtures of SN variates is represented by the skew-slash distribution proposed by Wang & Genton (2006). Similarly to the symmetric multivariate slash distribution, here $V = W - r$ where $W \sim U(0,1)$ and $r$ is a positive parameter.

An instance of (46) producing to a very broad set of distributions has been examined by Vilca et al. (2014) taking $V$ to be a GIG variable, leading to what they denote ‘skew-normal generalized hyperbolic distribution’. Analogously to the classical GH distribution (45), this family is extremely flexible and it includes as a special case several existing parametric families, as illustrated in detail in Section 3 of the quoted paper. A suitable constraint is required to avoid overparameterization, such as $\det(\Sigma) = 1$ employed for the classical GH distribution.

**Extended skew-$t$ distribution** As the name indicates, the extended ST family is a superset of ST family, introduced in independent work by Adcock (2010) and by Arellano-Valle & Genton (2010). Despite the close connection with the ST family and many common properties, the stochastic representation of extended version in form (7) involves a different mechanism from the one of the ST distribution.

Introduce a random variable $U$ having univariate $t$ distribution on $\nu$ degrees of freedom truncated below $-\tau$, where $\tau \in \mathbb{R}$ represents an additional parameter. Also, let $V^{-1} \sim \chi^2_\nu/\nu$ like for the ST. Then, taking into account Proposition 2 of Arellano-Valle & Genton (2010), we can say that

$$Y = \xi + U\gamma + \left(\frac{v + U^2}{v + 1}\right)^{1/2}X$$

(47)

has extended ST distribution with density at $y \in \mathbb{R}^d$, regulated by parameters $(\xi, \Sigma, \gamma, \nu, \tau)$, equal to

$$\frac{1}{T(\tau; \nu)} t_d(y; \xi, \Omega, \nu) T\left(\tilde{r} + \eta^T(y - \xi) \left(\frac{v + 1}{v + \|y - \xi\|^2_\Omega}\right) ; v + 1\right)$$

(48)

where $\Omega, \eta$ and $\tilde{r}$ are as in (22), and $T(x; \nu)$ denotes the distribution function of a standard univariate Student’s $t$ variable on $\nu$ degrees of freedom.

*6 Final remarks*

As anticipated in Section 1, the main aims of the present work are: (i) to present a wide formulation, denoted GMN, which encompasses a large number of existing constructions involving continuous mixtures of normal variables, possibly in an implicit way; (ii) to show that a unifying treatment is possible, by providing a number of general properties for the GMN class.

It would be possible to extend further this construction, even considerably. Among the various options, a simple one would be along the following lines. Start from the familiar construction represented by the class (1) of scale mixtures of normals, and recall that a vast subset of the elliptical
class of distributions can be expressed using scale mixtures of normal variates. Then combine this mechanism with an additive term like $V \gamma$ in (2) or, similarly, $R \gamma$ in (8). Not only this extension would be possible, but a number of properties developed in Section 2 would even carry on, for instance those in Subsections 2.4 and 2.5.

There are, however, other facts which would not be preserved in this extended construction. We are referring specifically to the properties of closure under affine transformations and marginalization examined in Subsection 2.3. To see the source of the problem, consider a specific but fairly popular instance, namely the so-called exponential power distribution, but also other names are in use. The univariate formulation of Subbotin (1923) has subsequently been extended to the multivariate setting and this distribution can be represented as a scale mixture of normal variables, as for the parameter set which corresponds to leptokurtic distributions, with heavier-than-normal tail behaviour. An explicit expression of the implied mixing distribution is given by Gómez-Sánchez-Manzano et al. (2008); a crucial fact is that this mixing distribution depends on the dimension $d$ of the mixed normal distribution, $d$ in our notation. This situation prevents the property of closure under marginalization for a scale mixture of normal variables, as shown in the already-quoted work of Kano (1994), who has focused precisely on the case of the exponential power distribution. Therefore, a fortiori, closure of the class under the more general manipulation of affine transformations cannot hold, even less so if an additive stochastic term $U \gamma$ is included like in (8).

To summarize, an extension of the construction along the above-delineated lines would be possible, and it would even preserve certain formal properties. Other properties would not carry on, especially closure under affine transformations and marginalization, and these seem to us important facts when we come the use of these constructions in applied work. Obviously, this statement cannot be taken as definitive and an absolute bar, since every such judgement must by evaluated with respect to a given context. However, the above discussion explains why, in all the special cases which we have examined, neither the mixing distribution of $(U, V)$ nor the functions $r$ and $s$ in (7) depend on $d$.

Another possible direction for extension of the GMN construction is via consideration of more than two mixing variables. To exemplify in a simple form, one option would be to replace the term $U \gamma$ in (34) by $\Gamma U$ where $\Gamma$ is a matrix of coefficients and $U$ is a random vector. This step would allow to incorporate families such as the so-called closed/unified skew-normal discussed by Arellano-Valle & Azzalini (2006). Similarly, the scale mixture construction (1) can be extended by consideration of multiple random scale factors, which amounts to the ‘multiple scale mixtures’ proposed by Forbes & Wraith (2014). Combination of these two mechanisms would extend (8) to the form

$$Y = \xi + \Gamma R + S X$$

where $S$ is a diagonal matrix formed by $d$ positive random variables. Clearly, such a study represents a separate undertaking, with non-trivial complexity if one attempts to develop a unified treatment of the connected properties.

As for the adopted term ‘generalized mixtures of normal variables’, one could perhaps object that the term may suggests more than it actually means, since the construction does not encompass all possible mixtures of normal variables. The term must rather be intended in the same spirit of other similar instances, such as ‘generalized linear models’, which do not embrace all possible extensions of linear models.

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**Appendix**

**Derivation of density function (37)** By assumption, $(Y|U = u) \sim N_d(\xi + \gamma u, \Sigma)$, and the density function of $U$ is $2\varphi(u)I_{(0,\infty)}(u)$. With these positions, density (13) becomes

$$
 f(y; \xi, \gamma, \Sigma) = \int_0^\infty \varphi_d(y; \xi + u\gamma, \Sigma) 2\varphi(u)du
$$

$$
 = 2\varphi_d(y; \xi, \Sigma + \gamma\gamma^T) \int_0^\infty \varphi(u; \gamma^T(\Sigma + \gamma\gamma^T)^{-1}(y - \xi), 1 - \gamma^T(\Sigma + \gamma\gamma^T)^{-1}\gamma) du
$$

$$
 = 2\varphi_d(y; \xi, \Sigma + \gamma\gamma^T) \int_0^\infty \varphi\left(u; \gamma^T\Sigma^{-1}(y - \xi), \frac{1}{1 + \gamma^T\Sigma^{-1}\gamma}\right) du.
$$
taking into account the follow identity:

\[ \varphi_d(y; \xi + u\gamma, \Sigma) \varphi_1(u) = \varphi_d(y; \xi, \Sigma + \gamma \gamma^T) \varphi(u; \gamma^T(\Sigma + \gamma \gamma^T)^{-1}(y - \xi), 1 - \gamma^T(\Sigma + \gamma \gamma^T)^{-1} \gamma), \]

and the inverse of \( \Sigma + \gamma \gamma^T \) has been expressed using the Sharman-Morrison formula (6). After the change variable

\[ v = \sqrt{1 + \gamma^T \Sigma^{-1} \gamma} \left( u - \frac{\gamma^T \Sigma^{-1}(y - \xi)}{1 + \gamma^T \Sigma^{-1} \gamma} \right), \]

we can re-write

\[ f(y; \xi, \Sigma, \gamma) = 2 \varphi_d(y; \xi, \Sigma + \gamma \gamma^T) \int_{\mathbb{R}^d} \frac{\varphi(v) \, dv}{\sqrt{1 + \gamma^T \Sigma^{-1} \gamma}} \]

which coincides with density (36) under the notation defined in (22).

### Conditional distribution of multivariate Student’s t components

Given a partition of \( Y \) in sub-vectors \( Y_1 \) and \( Y_2 \) as in (15), we want to apply the general expressions (20)–(21) to find the conditional distribution of \( Y_1 \) given that \( Y_2 = y_2 \) for a vector \( y_2 \in \mathbb{R}^d \), in the special case when \( Y \) has a \( d \)-dimensional Student’s \( t \) density (44).

With respect to the general expression (8), there \( R = 0 \), \( S = V^{1/2} \) with \( V \sim \nu^2 \chi^2_ν \). It is immediate that the density of \( V \) is

\[ h(v) = \frac{(v/2)^{\nu/2}}{\Gamma(\nu/2)} v^{-(\nu+2)/2} e^{-v/(2\nu)}, \quad v > 0, \]

which here plays the role of \( h(r, s) \) in (20). Since \( (Y_2 | V = v) \sim N_{d_2}(\xi_2, v \Sigma_{22}) \), it follows that \( Y_2 = V^{1/2} X_2 \sim t_{d_2}(\xi_2, \Sigma_{22}, v) \), so that we write

\[ \varphi_{d_2}(y_2; \xi_2 + v \Sigma_{22}) = \frac{\det(\Sigma_{22})^{1/2}}{2(2\pi)^{d_2/2} \nu^{d_2/2} e^{-q_2(y_2)/(2\nu)}} \]

\[ f_2(y_2) = \frac{\det(\Sigma_{22})^{1/2}}{\Gamma(v/2)\pi^{d_2/2}} \Gamma((v + d_2)/2)^{v/2} \]

having set

\[ q_2(y_2) = (y_2 - \xi_2)^\top \Sigma_{22}^{-1}(y_2 - \xi_2) = \|y_2 - \xi_2\|^2_{\Sigma_{22}}. \]

After plugging these ingredients in (20) and some simplification, we obtain that

\[ h_{\nu}(v | y_2) = \frac{((v + q_2(y_2))/2)^{(v+d_2)/2} v^{-(v+d_2+2)/2} e^{-(v+q_2(y_2))/(2\nu)}}{\Gamma((v+d_2)/2)}, \quad v > 0, \]

which means that \( (Y_2 | Y_1 = y_2) \sim (v + q_2(y_2))/\chi^2_{v+d_2} \). This distribution is the same of \( (v + q_2(y_2)) V_2 \), where \( V_2 \sim (v + d_2)/\chi^2_{v+d_2} \). Hence, by setting \( X_{1|2} \sim N_{d_1}(0, \Sigma_{1|2}) \) be a variable independent of \( V_2 \), we obtain

\[ (Y_1 | Y_2 = y_2) \sim t_{d_1}(\xi_{1|2} + \left( \frac{v + q_2(y_2)}{v + d_2} \right)^{1/2} X_{1|2} \]

Since \( V_2^{1/2} X_{1|2} \sim t_{d_1}(0, \Sigma_{1|2}, v + d_2) \), we conclude that

\[ (Y_1 | Y_2 = y_2) \sim t_{d_1}(\xi_{1|2} + \left( \frac{v + q_2(y_2)}{v + d_2} \right)^{1/2} \Sigma_{1|2}, v + d_2) \]

indicating that \( (Y_1 | Y_2 = y_2) \) is still of Student’s \( t \) nature.

If we insert the parameters of this distribution in the density (44), we obtain an expression algebraically equivalent to formula (1.15) of Kotz & Nadarajah (2004) for the conditional \( t \) density. It could possibly be remarked that our derivation is not any simpler than the one of Kotz & Nadarajah (2004), and perhaps even a little more lengthy, but it has the advantage of revealing the \( t \) nature of this density, a fact not so visible from the other development.
Derivation of density function (40) The conditional density of \((Z \mid U = u)\) is \(\varphi_d(z; \gamma u, \Sigma)\). Hence, its unconditional density is

\[
f_Z(z; \gamma, \Sigma) = \int_0^{\infty} \varphi_d(z; \gamma u, \Sigma) g_d(u) \, du
\]

\[
= \int_0^{\infty} (2\pi)^{-d/2} \det(\Sigma)^{-1/2} e^{-\frac{1}{2}(z-\gamma u)^\top \Sigma^{-1} (z-\gamma u)} u e^{-\frac{1}{2}u^2} \, du
\]

\[
= (2\pi)^{-d/2} \det(\Sigma)^{-1/2} \int_0^{\infty} u e^{-\frac{1}{2}[(u-\gamma^\top z)^2 + z^\top \Sigma^{-1} z]} \, du.
\]

By making use of the identities,

\[
u^2 + (z - \gamma u)^\top \Sigma^{-1} (z - \gamma u) = z^\top \Omega^{-1} z + (1 + \alpha^2)(u - \bar{\eta}^\top z)^2,
\]

\[\det(\Omega) = \det(\Sigma + \gamma \gamma^\top) = \det(\Sigma)(1 + \alpha^2),\]

where \(\Omega, \eta\) and \(\alpha\) are as in (22), and \(\bar{\eta} = (1 + \alpha^2)^{-1/2} \eta\), we have

\[
f_Z(z; \gamma, \Sigma) = (2\pi)^{-d/2} \det(\Sigma)^{-1/2} e^{-\frac{1}{2}(z-\gamma u)^\top \Sigma^{-1} (z-\gamma u)} u e^{-\frac{1}{2}(u-\bar{\eta}^\top z)^2} \, du
\]

\[
= (2\pi)^{-d/2} \det(\Sigma)^{-1/2} e^{-\frac{1}{2}z^\top \Omega^{-1} z} \int_0^{\infty} u(1 + \alpha^2)^{1/2} e^{-\frac{1}{2}(1 + \alpha^2)(u-\bar{\eta}^\top z)^2} \, du
\]

The change variable \(w = (1 + \alpha^2)^{-1/2}(u - \bar{\eta}^\top z)\) yields

\[
f_Z(z; \gamma, \Sigma) = (2\pi)^{1/2} \varphi_d(z; \Omega) \int_{-\eta^\top z}^{\infty} \{(1 + \alpha^2)^{-1/2} w + \bar{\eta}^\top z\} \varphi(w) \, dw
\]

\[
= (2\pi)^{1/2}(1 + \alpha^2)^{-1/2} \varphi_d(z; \Omega) \left\{ \int_{-\eta^\top z}^{\infty} w \varphi(w) \, dw + \eta^\top z \int_{-\eta^\top z}^{\infty} \varphi(w) \, dw \right\}
\]

\[
= (2\pi)^{1/2}(1 + \alpha^2)^{-1/2} \varphi_d(z; \Omega) \left\{ -\int_{-\eta^\top z}^{\infty} d\varphi(w) + \eta^\top z \int_{-\eta^\top z}^{\infty} \varphi(w) \, dw \right\}
\]

\[
= (2\pi)^{1/2}(1 + \alpha^2)^{-1/2} \varphi_d(z; \Omega) \{ \varphi(\eta^\top z) + \eta^\top z \Phi(\eta^\top z) \}
\]

which coincides with expression (40).

Derivation of the Mardia’s multivariate measures of skewness and kurtosis We expand here the computations sketched in Subsection 2.5 for computing the Mardia’s measures. Given the positions stated in the initial part of Subsection 2.5, we can rewrite

\[
\beta_{d,1} = \mathbb{E}\{|Y_0^\top \Sigma^{1/2} Y_0^\top |^2 \} \quad \beta_{d,2} = \mathbb{E} \{ |Y_0^\top \Sigma^{1/2} Y_0^\top |^3 \},
\]

where \(Y_0' = \Sigma^{-1/2}(Y' - \mu_Y)\) and we can expand

\[
\Sigma^{1/2} Y_0^\top \Sigma^{1/2} = \mu_0^{-1/2} \Sigma^{1/2} \left( \Sigma^{-1} - \frac{\rho}{1 + \rho \alpha^2} \Sigma^{-1} \gamma Y_0^\top \Sigma^{-1} \right) \Sigma^{1/2} = \mu_0^{-1} \left( \mu - \rho Y_0 Y_0^\top \right)
\]

using the Sherman-Morrison formula (6) to invert \(\Sigma_Y\) and denoting \(\mu_0 = \mathbb{E}\{S^2\}\). Hence, re-write further

\[
\beta_{d,1} = \mu_0^{-3} \mathbb{E}\{ |Y_0^\top (\mu - \rho Y_0 Y_0^\top) Y_0'|^3 \} = \mu_0^{-3} \mathbb{E}\{ |Y_0^\top Y_0' - \rho (Y_0^\top Y_0, Y_0^\top Y_0')|^3 \},
\]

and

\[
\beta_{d,2} = \mu_0^{-2} \mathbb{E}\{ |Y_0^\top (\mu - \rho Y_0 Y_0^\top) Y_0|^2 \} = \mu_0^{-2} \mathbb{E}\{ \| Y_0 \|^2 - \rho (Y_0^\top Y_0)^2 \}. \]
On setting $T_0 = \alpha^{-1} W_0$, where $W_0 = \gamma^\top \Sigma^{-1} X$, we note that

$$
\gamma_0^\top Y_0 = \gamma^\top \Sigma^{-1}(R_0 Y + S X) = R_0\alpha^2 + S \gamma^\top \Sigma^{-1} X = \alpha(R_0 \alpha + S T_0) = \alpha Z_0,
$$

where $Z_0 = R_0 \alpha + S T_0$, with $(R_0, S)$ and $T_0$ independent variables. Thus, by letting $Z'_0$ be an independent copy of $Z_0$, we have that

$$
\beta_{d,1} = \mu_{02}^{-3} \mathbb{E} \{ |Y_0^\top Y'_0 - \bar{\beta} Z_0 Z'_0| \} = \mu_{02}^{-3} \mathbb{E} \{ |Y_0^\top Y'_0 - Z_0 Z'_0 + (1 - \bar{\beta}) Z_0 Z'_0| \},
$$

and

$$
\beta_{d,2} = \mu_{02}^{-2} \mathbb{E} \{ \|Y_0\|^2 - \bar{\beta} Z'_0 Z_0 \} = \mu_{02}^{-2} \mathbb{E} \{ \|Y_0\|^2 - Z'_0 Z_0 + (1 - \bar{\beta}) Z'_0 Z_0 \}.
$$

Note that

$$
Y_0^\top Y'_0 = (R_0 Y_0 + S X_0)^\top (R'_0 Y_0 + S' X'_0) = \alpha^2 R_0 R'_0 + \alpha(R_0 S' T'_0 + R'_0 S T_0) + S' X'_0 X'_0 T_0,
$$

and

$$
Z_0 Z'_0 = (R_0 \alpha + S T_0)(R'_0 \alpha + S' T'_0) = \alpha^2 R_0 R'_0 + \alpha(R_0 S' T'_0 + R'_0 S T_0) + S' X'_0 X'_0 T_0,
$$

leading to

$$
Y_0^\top Y'_0 - Z_0 Z'_0 = S S' (X'_0 X'_0 - T_0 T'_0) = S S' X'_0 X'_0 M_0 X_0,
$$

where $M_0 = I_d - \bar{\gamma} \bar{\gamma}^\top$ is a projection matrix. Similarly, we find

$$
\|Y_0\|^2 - Z'_0 Z_0 = S(\|X_0\|^2 - T'_0)^2 = S^2 X'_0 X_0 M_0 X_0.
$$

Recall that $M_0 X_0$ and $T_0$ are independent, and so are $M_0 X'_0$ and $T'_0$. Therefore, $M_0 X_0$ is also independent of $Z_0$ since this variable depends on $(R_0, S, T_0)$ only; similarly, independence holds for $M_0 X'_0$ and $Z'_0$. Finally, we find that

$$
\beta_{d,1} = \mu_{02}^{-3} \mathbb{E} \{ |S S' (X'_0 M_0 X'_0) + (1 - \bar{\beta}) Z_0 Z'_0| \},
$$

and

$$
\beta_{d,2} = \mu_{02}^{-2} \mathbb{E} \{ |S^2 (X'_0 M_0 X_0) + (1 - \bar{\beta}) Z'_0 Z_0| \},
$$

where we note that $M_0 X_0$ and $Z_0$ are independent random quantities of mean zero.

We can now start calculation of the Mardia’s measures:

$$
\beta_{d,1} = \mu_{02}^{-3} \mathbb{E} \{ |S S' (X'_0 M_0 X'_0) + (1 - \bar{\beta}) Z_0 Z'_0| \} = \mu_{02}^{-3} \mathbb{E} \{ |S S' (X'_0 M_0 X'_0) + (1 - \bar{\beta}) Z_0 Z'_0| \} + 3(1 - \bar{\beta})^2 \mathbb{E} \{ |S S' (Z_0 Z'_0) + (1 - \bar{\beta})^3 \mathbb{E} \{ |Z_0 Z'_0| \} \} = \mu_{02}^{-3} \mathbb{E} \{ |S S' | \} \mathbb{E} \{ |X'_0 M_0 X'_0| \} \mathbb{E} \{ |X'_0 M_0 X'_0| \} \mathbb{E} \{ |Z_0 Z'_0| \} + 3(1 - \bar{\beta})^2 \mathbb{E} \{ |X'_0 M_0 X'_0| \} \mathbb{E} \{ |X'_0 M_0 X'_0| \} + (1 - \bar{\beta})^3 \mathbb{E} \{ |Z_0 Z'_0| \},
$$

where by symmetry $\mathbb{E} \{ |X'_0 M_0 X'_0| \} = \mathbb{E} \{ |X_0 M_0 X'_0| \} = 0$, and

$$
\mathbb{E} \{ |X'_0 M_0 X'_0| \} = \mathbb{E} \{ \text{tr}(X'_0 M_0 X'_0 X'_0 M_0 X_0) \} = \text{tr}(\mathbb{E} \{ X'_0 M_0 X'_0 X'_0 M_0 X_0 \}) = \text{tr}(\mathbb{E} \{ M_0 X'_0 X'_0 M_0 X_0 \}) = \text{tr}(\mathbb{E} \{ X'_0 M_0 X'_0 \}) = \text{tr}(\mathbb{E} \{ M_0 \}) = d - 1.
$$

where we have used the fact $\mathbb{E} \{ X'_0 X'_0 \} = \mathbb{E} \{ X_0 X'_0 \} = I_d$. Thus, since $(S', Z'_0)$ and $(S, Z_0)$ are independent variables with the same distribution, $\beta_{1,d}$ reduces to

$$
\beta_{1,d} = \mu_{02}^{-3} [3(d - 1)(1 - \bar{\beta}) (\mathbb{E} \{ S^2 Z_0 \})^2 + (1 - \bar{\beta})^3 (\mathbb{E} \{ Z'_0 Z_0 \})^2],
$$

24
and analogously
\[
\beta_{2,d} = \mu_0^2 E\{S^2(X^\top_0 M_0 X_0) + (1 - \bar{\rho}) Z_0^2\}^2
\]
\[
= \mu_0^2 E\{S^4(X^\top_0 M_0 X_0)^2 + 2(1 - \bar{\rho}) S^2(X^\top_0 M_0 X_0) Z_0^2 + (1 - \bar{\rho})^2 Z_0^4\}
\]
\[
= \mu_0^2 E\{S^4\} E\{(X^\top_0 M_0 X_0)^2 + 2(1 - \bar{\rho})E\{(X^\top_0 M_0 X_0)\} E\{S^2 Z_0^2\} + (1 - \bar{\rho})^2 E\{Z_0^4\}\}
\]
\[
= \mu_0^2 [(d + 1)(d - 1)E\{S^4\} + 2(d - 1)(1 - \bar{\rho})E\{S^2 Z_0^2\} + (1 - \bar{\rho})^2 E\{Z_0^4\}]
\]
considering that \(X^\top_0 M_0 X_0 \sim \chi^2_{d-1}\).

To complete the calculations, we need some moments of the distribution of \((S, Z_0)\). For this, recall that \((R_0, S)\) and \(T_0 \sim N(0,1)\) are independent variables, and (i) the odd-order moments of \(T_0\) are 0, (ii) \(E\{R_0\} = 0, E\{T_0\} = 1, E\{Z_0\} = 3\), and (iii) we are assuming that both \(R\) and \(S\) have finite moments up to order four. Then, we need just the following expected values:

\[
E\{S^2 Z_0\} = E\{a R_0 S^2 + S^3 T_0\}
\]
\[
= \alpha E\{R_0 S^2\}
\]
\[
= \alpha (E\{R S^2\} - E\{S^2\} E\{R\})
\]
\[
E\{S^2 Z_0^2\} = E\{a^2 R_0^2 S^2 + 2a R_0 S^3 T_0 + S^4 T_0^2\}
\]
\[
= \alpha^2 E\{R_0^2 S^2\} + E\{S^4\}
\]
\[
= \alpha^2 (E\{R^2 S^2\} - 2E\{R S^2\} E\{R\} + (E\{R\})^2 E\{S^2\}) + E\{S^4\}
\]
\[
E\{Z_0\} = E\{a^3 R_0^3 + 3a^2 R_0^2 S T_0 + 3a R_0 S^2 T_0 + S^3 T_0\}
\]
\[
= \alpha^3 E\{R_0^3\} + 3\alpha E\{R_0 S^2\}
\]
\[
= \alpha^3 (E\{R^3\} - 3E\{R^2\} E\{R\} + 2(E\{R\})^3) + 3\alpha (E\{R S^2\} - E\{R\} E\{S^2\})
\]
\[
E\{Z_0^2\} = E\{a^4 R_0^4 + 4a^3 R_0^3 S T_0 + 6a^2 R_0^2 S^2 T_0 + 4a R_0 S^3 T_0 + S^4 T_0^4\}
\]
\[
= \alpha^4 E\{R_0^4\} + 6\alpha^2 E\{R_0^2 S^2\} + 3E\{S^4\}
\]
\[
= \alpha^4 (E\{R^4\} - 4E\{R^3\} E\{R\} + 6E\{R^2\} (E\{R\})^2 - 3(E\{R\})^4)
\]
\[
+ 6\alpha^2 (E\{R^2 S^2\} - 2E\{R S^2\} E\{R\} + (E\{R\})^2 E\{S^2\}) + 3E\{S^4\}
\].

After substitution of various expectations with the symbols defined in (23), we arrive at the expressions reported in Subsection 2.5.