Given a complex smooth projective algebraic variety $X$, we define a natural number called the motivic dimension $\mu(X)$ which is zero precisely when all the cohomology of $X$ is generated by algebraic cycles. In general, it gives a measure of the amount of transcendental cohomology of $X$. Alternatively, $\mu(X)$ may be rather loosely thought of as measuring the complexity of the motive of $X$, with Tate motives having $\mu = 0$, motives of curves having $\mu \leq 1$ and so on. Our interest in this notion stems from the relation to the Hodge conjecture: it is easy to see that it holds for $X$ whenever $\mu(X) \leq 3$. This paper contains a number of estimates of $\mu$; some elementary, some less so. With these estimates in hand, we conclude this paper by checking or rechecking this conjecture in a number of examples: uniruled fourfolds, rationally connected fivefolds, fourfolds fibred by surfaces with $p_g = 0$, Hilbert schemes of a small number points on surfaces with $p_g = 0$, and generic hypersurfaces.

We will work over $\mathbb{C}$. Let $H^i(\cdot)$ denote singular cohomology with rational coefficients. The motivic dimension of a smooth projective variety $X$ can be defined most succinctly in terms of the length of the coniveau filtration on $H^\ast(X)$. We prefer to spell this out. The motivic dimension $\mu(X)$ of $X$ is the smallest integer $n$, such that any $\alpha \in H^i(X)$ vanishes on the complement of a Zariski closed set all of whose components have codimension at least $(i - n)/2$. The meaning of this number is further clarified by the following:

Lemma 0.1. Any $\alpha \in H^i(X)$ can be decomposed as a finite sum of elements of the form $f_j^\ast(\beta_j)$, where $f_j : Y_j \to X$ are desingularizations of subvarieties and $\beta_j$ are classes of degree at most $\mu(X)$ on $Y_j$. In fact, $\mu(X)$ is the smallest integer such that the previous statement holds for all $i \leq \dim X$.

Proof. The first statement follows from [D3, Cor. 8.2.8], and the second from this and the Hard Lefschetz theorem. \hfill \Box

Corollary 0.2. $\mu(X) = 0$ if and only if all the cohomology of $X$ is generated by algebraic cycles. A surface satisfies $\mu(X) \leq 1$ if and only if $p_g(X) = 0$. We have

$$\dim X \geq \mu(X) \geq \text{level}(H^\ast(X)) = \max\{|p - q| \mid h^{p_\ast}(X) \neq 0\},$$

and the last inequality is equality if the generalized Hodge conjecture holds [G].

Proof. The first statement is immediate. As for the second, the Lefschetz $(1,1)$ theorem implies that $H^2(X)$ is spanned by divisor classes if and only if $p_g(X) = 0$. The inequality $\mu(X) \geq \text{level}(H^\ast(X))$ follows from the fact that Gysin maps are morphisms of Hodge structures up to Tate twists. \hfill \Box

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1. Elementary Estimates

It will be convenient to extend the notion of motivic dimension to arbitrary complex algebraic varieties, and for this we switch to homology. Let $H_*(−)$ denote Borel-Moore homology with $\mathbb{Q}$ coefficients, which is dual to compactly supported cohomology. Translating the above definition into homology leads to an integer $\mu^{\text{big}}(X)$, which we call the big motivic dimension that makes sense for any variety $X$. So $\mu^{\text{big}}(X)$ is the smallest integer such that every $\alpha \in H_*(X)$ lies in the image of some $f, H_*(Y) \to H_*(X)$, where $Y$ is a Zariski closed set whose components have dimension at most $(i + \mu^{\text{big}}(Y))/2$. Of course, $\mu^{\text{big}}(X)$ coincides with $\mu(X)$ when $X$ is smooth and proper, but it seems somewhat difficult to study in general. It turns out to be more useful to restrict attention to certain cycles. The identification $H_*(X) \cong H_*(X')^*$ gives homology a mixed Hodge structure with weights $\geq −i$, i.e. $W_{−i−1}H_*(X) = 0$. We define the motivic dimension $\mu(X)$ by the replacing $H_*(X)$ by $W_{−i}H_*(X) = Gr_{−i}W_∗H_*(X)$ in the above definition of $\mu^{\text{big}}$. We have $\mu(X) \leq \mu^{\text{big}}(X)$ with equality when $X$ is smooth and proper.

Proposition 1.1.

(a) If $f : X' \to X$ is proper and surjective $\mu(X) \leq \mu(X')$.
(b) If $Z \subset X$ is Zariski closed then $\mu(X) \leq \max(\mu(Z), \mu(X - Z))$.
(c) If $\bar{X}$ is a desingularization of a partial compactification $X$ of $X$, then $\mu(X) \leq \mu(\bar{X})$.
(d) $\mu(X_1 \times X_2) \leq \mu(X_1) + \mu(X_2)$.
(e) If $V \to X$ is a vector bundle then $\mu(V) \leq \mu(X)$ and $\mu(P(V)) \leq \mu(X)$.

Proof. By [D3] lemma 7.6, p. 110, any element $\alpha \in W_{−i}H_*(X)$ lifts to an element of $\alpha' \in W_{−i}H_*(X')$. This in turn lies in $f_*W_{−i}H_*(Y')$ for some $f : Y' \to X'$ satisfying $\dim Y' \leq (i + \mu(X'))/2$. Therefore $\alpha$ lies in the image of $W_{−i}H_*(f(Y'))$. Since $\dim f(Y') \leq (i + \mu(X'))/2$, therefore (a) holds.

Suppose $i \leq m = \max(\mu(Z), \mu(X - Z))$. We have an exact sequence of mixed Hodge structures

$$H_i(Z) \to H_i(X) \to H_i(X - Z) \to H_{i−1}(Z)$$

which can be deduced from [D3] prop. 8.3.9. This implies by [D3] thm 2.3.5 that

$$W_{−i}H_*(Z) \to W_{−i}H_*(X) \to W_{−i}H_*(X - Z) \to 0$$

is exact. Given $\alpha \in W_{−i}H_*(X)$, let $\beta$ denote its image in $W_{−i}H_*(X - Z)$. Then $\beta = f_*(\gamma)$ for some $f : Y \to X$ with $\dim Y \leq (i + m)/2$ and $\gamma \in W_{−i}H_*(Y)$. Let $\bar{f} : \bar{Y} \to X$ denote the closure of $Y$. A sequence analogous to [D3] shows that $W_{−i}H_*(\bar{Y})$ surjects onto $W_{−i}H_*(\bar{Y})$, therefore $\gamma$ extends to a class $\bar{\gamma} \in W_{−i}H_*(\bar{Y})$. The difference $\alpha - \bar{f}_*(\bar{\gamma})$ lies in the image of $W_{−i}H_*(Z)$, and therefore in $g_*H_*(T)$ for some $g : T \to Z$ with $\dim T \leq (i + m)/2$. This proves (b).

Let $\bar{X}$ be a partial compactification of $X$. Then as above, we see that any class in $W_{−i}H_*(X)$ extends to $\bar{X}$. Therefore $\mu(X) \leq \mu(\bar{X})$. The remaining inequality of (c) follows from (a).

Statement (d) follows from the Künneth formula

$$W_{−i}H_*(X_1 \times X_2) = \bigoplus_{j+k=i} W_{−j}H_j(X_1) \otimes W_{−k}H_k(X_2)$$

Finally for (e), let $r = rk(V)$. Suppose that the Gysin images of $W_{−i+2r}(Y_j)$ span $W_{−i+2r}H_{i−2r}(X)$ and satisfy $\dim Y_j \leq (\mu(X) + i - 2r)/2$. Then $W_{−i}H_*(V|_{Y_j})$
will span $W_{-i}H_i(V)$ by the Thom isomorphism theorem. The inequality $\mu(\mathbb{P}(V)) \leq \mu(X)$ can be proved by a similar argument. We omit the details since we will show something more general in corollary 2.8.

**Corollary 1.2.** If $X = \cup X_i$ is given as a finite disjoint union of locally closed subsets, $\mu(X) \leq \max\{\mu(X_i)\}$. In particular, $\mu(X) = 0$ if $X$ is a disjoint union of open subvarieties of affine spaces.

The last statement, which is of course well known, implies that flag varieties and toric varieties have $\mu = 0$.

**Corollary 1.3.** If $X_1$ and $X_2$ are birationally equivalent smooth projective varieties, then $\mu(X_2) \leq \max(\mu(X_1), \dim X_1 - 2)$

**Proof.** Since an iterated blow up of $X_1$ dominates $X_2$, it is enough to prove this when $X_2$ is the blow up of $X_1$ along a smooth centre $Z$. Then

$$\mu(X_2) \leq \max(\mu(X_1 - Z), \mu(\mathbb{P}(N))) \leq \max(\mu(X_1), \mu(Z))$$

where $N$ is the normal bundle of $Z$. □

Recall that a variety is uniruled if it has a rational curve passing through the general point.

**Corollary 1.4.** If $X$ is uniruled, then $\mu(X) \leq \dim X - 1$.

**Proof.** By standard arguments [Ko], $X$ is dominated by a blow up of $Y \times \mathbb{P}^1$, where $\dim Y = \dim X - 1$. □

**Corollary 1.5.** If $X$ is a smooth projective variety with a $\mathbb{C}^*$-action, $\mu(X)$ is less than or equal to the dimension of the fixed point set.

**Proof.** Bialynicki-Birula [BB] has shown that $X$ can be decomposed into a disjoint union of vector bundles over components of the fixed point set. The corollary now follows from the previous results. □

From this, one recovers the well known fact that the Hodge numbers $h^{p,q}(X)$ vanish when $|p - q|$ exceeds the dimension of the fixed set.

**Proposition 1.6.** Suppose that $X$ is a smooth projective variety such that the Chow group of zero cycles $CH_0(X) \cong \mathbb{Z}$. Then $\mu(X) \leq \dim X - 2$.

**Proof.** This follows from the theorem of Bloch-Srinivas [BS] that a positive multiple of the diagonal $\Delta \subset X \times X$ is rationally equivalent to a sum $\xi \times X + \Gamma$, where $\xi \in Z_0(X)$ is a zero cycle, and $\Gamma$ is supported on $X \times D$ for some divisor $D \subset X$. This implies that the identity map on cohomology $H^i(X)$ factors through the Gysin map $H^{i-2}(\tilde{D}) \to H^i(X)$ for $i > 0$ and a resolution of singularities $\tilde{D} \to D$. □

Recall that a projective variety is rationally connected if any two general points can be connected by a rational curve. Examples include hypersurfaces in $\mathbb{P}^n$ with degree less than $n + 1$, and more generally Fano varieties [Ko].

**Corollary 1.7.** If $X$ is rationally connected then $\mu(X) \leq \dim X - 2$.

**Proof.** Rational connectedness forces $CH_0(X) \cong \mathbb{Z}$. □
2. Estimates for Fibrations

**Theorem 2.1.** Suppose that $f : X \to S$ is a smooth projective morphism. Then

$$\mu(X) \leq \max_{s \in S(\mathbb{C})} \mu(X_s) + \dim S$$

*Proof.* We prove this by induction on $d = \dim S$. Let $m = \max_{s \in S(\mathbb{C})} \mu(X_s)$. Let $H_1, H_2, \ldots$ denote irreducible components of the relative Hilbert scheme $\text{Hilb}_{X/S}$ which surject onto $S$. Choose a desingularization $\tilde{F}_k \to \mathcal{F}_{k,\text{red}}$ of the reduced universal family over each $H_k$.

Let $\text{disc}(\tilde{F}_k \to S) \subseteq S$ denote discriminant i.e. the complement of the maximal open set over which this maps is smooth. Then $T = \bigcup_k \text{disc}(\tilde{F}_k \to S) \subset S$ is a countable union of proper subvarieties, therefore its complement is nonempty by Baire’s theorem. Choose $s \in S - T$. By assumption, there exists a finite collection of subvarieties $Y_{ij} \subset X_s$ such that $\dim Y_{ij} \leq (m + i)/2$ and their images generate $H_i(X_s)$. For each $Y_{ij}$, we choose one of the families $\tilde{F}_k$ containing it and rename it $\tilde{Y}_{ij}$; likewise set $H_{ij} = H_k$ and $Y_{ij} = \tilde{F}_k$.

Choose an open set $U \subset S - \bigcup \text{disc}(Y_{ij} \to S)$ containing $s$. Therefore $X|_U \to U$ and each $\tilde{Y}_{ij}|_U \to U$ are smooth and thus topological fibre bundles. Consequently the images of $\tilde{Y}_{ij,t}$ will generate $H_i(X_t)$ for any $t \in U$. After replacing $H_{ij}$ by the normalization of $S$ in $H_{ij}$, and $Y_{ij}$ by the fibre product, we can assume that $H_{ij} \to S$ is finite over its image. After shrinking $U$ if necessary and replacing all maps by their restrictions to $U$, we can assume that the maps $g_{ij} : \tilde{Y}_{ij} \to U$ are still smooth, and that $U$ is nonsingular and affine. The last assumption implies that for any local system $L$ of $\mathbb{Q}$-vector spaces, we have

$$H^l_c(U, L) = H^{2d-i}(U, L^*)^* = 0$$

for $i < d$ since $U$ is homotopic to a CW complex of dimension at most $d$ [V, thm 1.22]. The Gysin images of $\tilde{Y}_{ij,t}$ generate the homology of $X_t$ for each $t \in U$, or dually the cohomology of $H^i(X_t)$ injects into $\oplus_j H^i(\tilde{Y}_{ij,t})$. Since the monodromy actions are semisimple [D3, thm 4.2.6], the map of local systems $R^i f_* \mathbb{Q}|_U \to \oplus_j R^i g_{i,j,*} \mathbb{Q}$ is split injective. Thus

$$H^k_c(U, R^i f_* \mathbb{Q}) \to \bigoplus_j H^k_c(U, R^i g_{i,j,*} \mathbb{Q})$$

is injective. Since the Leray spectral sequence degenerates [D1], we get an injection

$$H^p_c(f^{-1}U) \to \bigoplus_{i \leq p-d,j} H^p_c(\tilde{Y}_{ij})$$
Note that the bound \( i \leq p - d \) follows from \( \mathcal{L} \). We therefore have a surjection
\[
\bigoplus_{i \leq p-d,j} H_p(\tilde{\mathcal{Y}}_{ij}) \twoheadrightarrow H_p(f^{-1}U)
\]
Since
\[
\dim \tilde{\mathcal{Y}}_{ij} \leq \frac{m+i}{2} + d \leq \frac{m+d+p}{2}
\]
for \( i \leq p-d \), we have \( \mu(f^{-1}(U)) \leq \mu^{\text{bar}}(f^{-1}U) \leq m + d \). By induction \( \mu(f^{-1}(S-U)) \leq m + d \). Thus \( \mu(X) \leq \max(\mu(f^{-1}U), \mu(f^{-1}(S-U))) \leq m + d \) as required. \( \square \)

**Corollary 2.2.** If a smooth projective variety \( X \) can be covered by a family of surfaces whose general member is smooth with \( p_g = 0 \). Then \( \mu(X) \leq \dim X - 1 \).

**Proof.** By assumption, there is a family \( Y \rightarrow T \) of surfaces with \( p_g = 0 \) and a dominant map \( \pi : Y \rightarrow X \). After restricting to a subfamily, we can assume that \( \pi \) is generically finite. So that \( \dim T = \dim X - 2 \), and therefore \( \mu(X) \leq \dim X - 1 \) by corollary \( \mathcal{L} \) and theorem \( \mathcal{Z} \). \( \square \)

It is worth noting that many standard examples of surfaces with \( p_g = 0 \) lie in families. So there are nontrivial examples of varieties admitting fibrations of the above type. We give an explicit class of examples generalizing Enriques surfaces (cf. [BPV] p. 184).

**Example 2.3.** Fix \( n \geq 2 \). Let \( i : (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^n \) be the involution which acts by \( [x_0,x_1] \mapsto [-x_0,x_1] \) on each factor. Choose a divisor \( B \subset (\mathbb{P}^1)^n \) defined by a general \( i \)-invariant polynomial of multidegree \((4,4,\ldots,4)\), and let \( \pi : X \rightarrow (\mathbb{P}^1)^n \) be the double cover branched along \( B \). The involution \( i \) can be seen to lift to a fixed point free involution of \( X \). Let \( S \) be the quotient. \( X \) is covered by a family of an \( K3 \) surfaces \( \pi^{-1}((\mathbb{P}^1)^2 \times t) \) which induces a family of Enriques surfaces on \( S \). Thus \( \mu(S) \leq n - 1 \). On the other hand \( X \) can be checked to be Calabi-Yau, and thus the Kodaira dimension of \( S \) equals 0. Therefore it cannot be uniruled. So this estimate on \( \mu(S) \) does not appear to follow from the previous bounds.

In view of proposition \( \mathcal{L}, \mathcal{L} \) (d), we may hope for a stronger estimate
\[
\mu(X) \leq \max_{s \in S(\mathbb{C})} \mu(X_s) + \mu(S)
\]
Unfortunately it may fail without extra assumptions:

**Example 2.4.** Let \( S \) be an Enriques surface which can be realized as the quotient of a \( K3 \) surface \( \tilde{S} \) by a fixed point free involution \( \sigma \). Let \( \sigma \) act on \( \mathbb{P}^1 \times \mathbb{P}^1 \) by interchanging factors. Define \( X = (\mathbb{P}^1 \times \mathbb{P}^1) / \sigma \). The natural map \( X \rightarrow S \) is an etale locally trivial \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle. An easy calculation shows that \( \text{level}(H^2(X)) = 2 \), while \( p_g(S) = q(S) = 0 \). Thus \( \mu(X) = 2 > \mu(\mathbb{P}^1 \times \mathbb{P}^1) + \mu(S) = 0 \).

**Theorem 2.5.** Suppose that \( f : X \rightarrow S \) is a smooth projective morphism over a quasiprojective base. If the monodromy action of \( \pi_1(S) \) on \( H^*(X_s) \) is trivial, then
\[
\mu(X) \leq \max_{s \in S(\mathbb{C})} \mu(X_s) + \mu(S)
\]
Before proving this, we make some general remarks. If \( f : X \rightarrow S \) is a not necessarily smooth projective morphism, then there is a filtration \( L^\bullet H^i(X) \subset H^i(X) \) called the Leray filtration associated to the Leray spectral sequence. This is a filtration by sub mixed Hodge structures \( \mathcal{A} \) cor. 4.4]. When \( f \) is also smooth the
Lemma 2.6. Suppose that $f : X \to S$ is a smooth projective map, and let $m = \max_{s \in S(\mathbb{C})} \mu(X_s)$. Then $Gr^W_{k+i} H^k_c(S, R^i f_* \mathbb{Q})$ injects into $Gr^W_{k+i} H^k_c(Y)$ for some Zariski closed $Y \subset X$ satisfying $\dim Y \leq \frac{m+k}{2} + \dim S$.

Proof. As in the proof of theorem 2.1 we can find a nonempty affine open set $U \subset S$ and a morphism of smooth $U$-schemes $\tilde{Y} = \bigcup_j \tilde{Y}_j \to f^{-1} U$ whose fibres generated the homology of $H_i(X_s)$. The lemma holds when $S$ is replaced by $U$ with $Y = \text{im} \tilde{Y}$ thanks to (5) and (11). Let $\tilde{Y} \subset Y$ denote the closure of $\text{im} \tilde{Y}$. By induction, we have a subset $\tilde{Y}' \subset f^{-1}(S - U)$ which satisfies the lemma over $S - U$. We have a commutative diagram with exact rows

\[
\begin{array}{c}
H^k_c(f^{-1} U) \to H^k_c(X) \to H^k_c(f^{-1} (S - U)) \\
0 \to H^k_c(Y') \to H^k_c(\tilde{Y}') \to H^k_c((\tilde{Y}' \bigcup (\tilde{Y} - Y'))
\end{array}
\]

Applying $Gr^W_{k+i} Gr^k_L$ and making appropriate identifications results in a commutative diagram

\[
\begin{array}{c}
Gr^W_{k+i} H^k_c(U, R^i f_* \mathbb{Q}) \to Gr^W_{k+i} H^k_c(S, R^i f_* \mathbb{Q}) \to Gr^W_{k+i} H^k_c(S - U, R^i f_* \mathbb{Q}) \\
Gr^W_{k+i} H^k_c(Y) \to Gr^W_{k+i} H^k_c(\tilde{Y}) \to Gr^W_{k+i} H^k_c((\tilde{Y}' \bigcup (\tilde{Y} - Y')))
\end{array}
\]

The top row is exact, but the bottom row need not be. Nevertheless $j$ is injective since $Gr^W_{k+i} Gr^k_L$ preserves injections. The maps $\iota$ and $\iota'$ are also injective by the above discussion. The injectivity of $\iota''$ follows a diagram chase. Thus $Y = \tilde{Y}' \bigcup (\tilde{Y} - Y)$ does the job.

Proof. Let $m = \max_{s \in S(\mathbb{C})} \mu(X_s)$. The sheaves $R^i f_* \mathbb{Q}$ are constant, so we have an isomorphism

\[(5)\quad H^k_c(S, R^i f_* \mathbb{Q}) \cong H^k_c(S) \otimes H^i(X_s)
\]

as vector spaces. As already noted, the Leray spectral sequence for $f$ degenerates yielding a mixed Hodge structure on the left side. We claim that (5) can be made compatible with mixed Hodge structures, at least after taking the associated graded with respect to the weight filtration. We have a surjective morphism of pure polarizable Hodge structures $Gr^W_* H^k_c(X) \to H^k_c(X_s)$, which admits a right inverse $\sigma$ since this category is semisimple [13, lemma 4.2.3]. The image of $H^k_c(S)$ lies in $L^k H^k_c(X)$. Thus $Gr^W_* Gr^k_L (f^* \otimes \sigma)$ induces the desired identification

\[(6)\quad Gr^W_* H^k_c(S) \otimes H^i(X_s) \cong Gr^W_* H^k_c(X, R^i f_* \mathbb{Q})
\]

Moreover, this is canonical in the sense that it is compatible with base change with respect to any morphism $T \to S$.

Let $S_k \to S$ be a map such that $\dim S_k \leq (\mu(S) + k)/2$ and such that $W_{-k} H^k_c(S_k)$ generates $W_{-k} H^k_c(S)$ (note that $S_k$ may have several components). Dually $Gr^W_* H^k_c(S)$
injects into $Gr^W_k H^k_c(S_k)$. Thus there are injections

$$Gr^W_k H^k_c(S) \otimes H^i(X_s) \longrightarrow Gr^W_k H^k_c(S_k) \otimes H^i(X_s) \cong Gr^{W}_{k+i} H^k_c(S, R^i f_\ast Q) \longrightarrow Gr^W_{k+i} H^k_c(S_k, R^i f_\ast Q)$$

By lemma 2.6, $Gr^W_{k+i} H^k_c(S_k, R^i f_\ast Q)$ injects into some $Gr^W_{k+i} H^k_c(S_k, R^i f_\ast Q)$ with

$$\dim Y_{ki} \leq \dim S_k + \frac{m + i}{2} \leq \frac{m + \mu(S) + k + i}{2}$$

Combining this with the degeneration of Leray, shows that the map

$$Gr^W_p H^p_c(X) \to \bigoplus_{k+i=p} Gr^W_p H^p_c(Y_{ki})$$

is injective, and hence we have a surjection

$$\bigoplus_{k+i=p} Gr^W_{k+i} H_p(Y_{ki}) \to G_{i-p}^W H_p(X)$$

Corollary 2.7. With the above notation, suppose that there exists a nonempty Zariski open set $U \subseteq S$ such that $X|_U \to U$ is topologically a product. Then $\mu(X) \leq \mu(X_s) + \mu(S)$.

Proof. $\pi_1(U) \to \pi_1(S)$ is surjective. \qed}

Corollary 2.8. If $X \to S$ is a Brauer-Severi morphism (i.e. a smooth map whose fibres are projective spaces) then $\mu(X) \leq \mu(S)$.

Proof. Since $\dim H^i(\mathbb{P}^N) \leq 1$, the monodromy representation is trivial. So $\mu(X) \leq \mu(\mathbb{P}^N) + \mu(S)$. \qed

3. Symmetric powers

In this section, we give our take on Abel-Jacobi theory. When $X$ is a smooth projective curve, the symmetric powers $S^n X$ are projective bundles over the Jacobian $J(X)$ for $n \gg 0$. This implies that the motivic dimension of $S^n X$ stays bounded. We consider what happens for more general smooth projective varieties. We note that $S^n X$ are singular in general, but only mildly so. These are in the class of $V$-manifolds or orbifolds, which satisfy Poincaré duality with rational coefficients, hard Lefschetz and purity of mixed Hodge structures. So for our purposes, we can treat them as smooth. In particular, we work with the original cohomological definition of $\mu$. When $X$ is a surface, the Hilbert schemes provides natural desingularization of the symmetric powers, and we give estimates for these as well.

We can identify $H^*(S^n X)$ with the $S_n$-invariants of $H^*(X^n)$. Let

$$\text{sym} : H^*(X^n) \to H^*(X^n)^{S_n} \cong H^*(S^n X)$$

denote the symmetrizing operator $\frac{1}{n!} \sum \sigma$.

Lemma 3.1. Let $f : X \to Y$ be a morphism of smooth projective varieties. If every class in $H^*(X)$ is of the form $(f^* \alpha) \cup \beta$ where $\beta$ is an algebraic cycle, then $\mu(X) \leq \mu(Y)$. 

implies that the Albanese map \( \alpha \) is nontrivial. This would imply that the image is a curve. Let \( q \) equal to some \( p \geq \frac{(i - \mu(Y))/2}{2} \). Any element of \( H^i(X) \) can be written as \( f^*\alpha \cup \beta \) where \( \beta \in H^{2k}(X) \) is an algebraic cycle and \( \alpha \in H^{i-2k}(Y) \). This implies that \( \alpha \in N^qH^{i-2k}(Y) \) and \( \beta \in N^kH^{2k}(X) \) for some \( q \) satisfying \( q \geq \frac{(i - 2k - \mu(Y))/2}{2} \). By \cite{AK2}, \( f^*\alpha \cup \beta \in N^{q+k}H^i(X) \). Therefore \( p = q + k \) gives the desired value. \( \square \)

**Corollary 3.2.** Suppose that \( G \) is a finite group. Let \( f : X \to Y \) be an equivariant morphism of smooth projective varieties with \( G \)-actions satisfying the above assumption, then \( \mu(X/G) \leq \mu(Y/G) \).

**Proof.** This is really a corollary of the proof which proceeds as above with the identifications \( N^pH^i(X/G) = N^pH^i(X)^G \) etcetera. \( \square \)

**Corollary 3.3.** Let \( f : X \to Y \) be a morphism of smooth projective varieties, which is a Zariski locally trivial fibre bundle with fibre \( F \) satisfying \( \mu(F) = 0 \). Then \( \mu(S^nX) \leq \mu(S^nY) \) for all \( n \).

**Proof.** The conditions imply that the hypotheses of the previous corollary holds for \( X \to Y \) as well as its symmetric powers. \( \square \)

**Theorem 3.4.** Let \( X \) be a smooth projective variety.

(a) \( \mu(S^nX) \leq n\mu(X) \).

(b) If \( \mu(X) \leq 1 \), then the sequence \( \mu(S^nX) \) is bounded. If \( \dim X \leq 2 \) then this is bounded above by \( h^{10}(X) \).

(c) If \( \text{level}(H^{2e}(X)) > 0 \), then \( \text{level}(H^*(S^nX)) \) and \( \mu(S^nX) \) are unbounded.

**Proof.** Inequality (a) follows from proposition \cite{G}. Suppose that \( \mu(X) \leq 1 \). Then \( H^*(X) \) is spanned by algebraic cycles and classes \( f_j*\beta \) with \( f_j : Y_j \to X \) and \( \beta \in H^1(Y_j) \). Let \( Y \) be the disjoint union of \( Y_j \), and let \( f^n : S^nY \to S^nX \) denote the natural map. Fix a basis \( \beta_1 \ldots \beta_N \) of \( H^1(Y) \). Then \( H^*(S^nX) \) is spanned by classes of the form

\[
[f^n_{sym}(p^*_m(\beta_{i_1} \times \ldots \times \beta_{i_m}))] \cup \gamma
\]

where \( p_m : Y^n \to Y^m \) is a projection onto the first \( m \) factors, and \( \gamma \) is an algebraic cycle. Notice that the expression in (7) vanishes if any of the \( \beta_{i_j} \)'s are repeated. Therefore we can assume that \( m \leq N \). Let \( g_j \) denote the inclusions of components of the algebraic cycle \( f^n*\gamma \). We can rewrite the expressions in (7) as

\[
f^n_{sym}(\ldots) \cup f^n*\gamma \in \text{span}\{f^n_{sym}(\ldots)\}
\]

This shows that \( H^*(S^nX) \) is spanned by Gysin images of classes of degree at most \( N \). Thus \( \mu(S^nX) \leq N \). This proves the first part of (b).

When \( X \) is a curve of genus \( g = h^{10}(X) \), the Abel-Jacobi map \( S^nX \to J(X) \) can be decomposed into a union of projective space bundles. Proposition \cite{G} implies that \( \mu(S^nX) \leq g = \dim J(X) \).

Suppose \( X \) is a surface, then \( \mu(X) \leq 1 \) forces \( p_g(X) = 0 \). If \( q = h^{10}(X) = 0 \) then \( \mu(X) = 0 \), so \( \mu(S^nX) = 0 \) by (a). So we may assume that \( q > 0 \), which implies that the Albanese map \( \alpha : X \to Alb(X) \) is nontrivial. If \( \dim \alpha(X) = 2 \), it is easy to see that the pullback of a generic two form from \( Alb(X) \) would be nontrivial. This would imply that the image is a curve. Let \( C \) be the normalization of \( \alpha(X) \), then we get a map \( \phi : X \to C \). Note the the genus \( g \) of \( C \) is necessarily equal to \( q \) since \( \alpha^e \) induces an isomorphism on the space 1-forms and it factors as
we can see that every cohomology class on \( X \) is of the form \( (\alpha \beta) \cup \gamma \) where \( \gamma \) is an algebraic cycle. Likewise for \( S^n X \). Therefore by corollary 3.2 and previous paragraph \( \mu(S^n X) \leq \mu(S^n C) \leq g \).

Suppose that \( \alpha \in H^3(X) \) is a nonzero class with \( i \neq j \) and \( i + j \) even. Then \( \text{sym}(\alpha \otimes \alpha) \) provides a nonzero class in \( H^{n_1 + n_2}(S^n X) \), which shows that the level and hence the motivic dimension go to infinity. \( \square \)

We give an example where \( \mu(X) > 1 \) but \( \mu(S^n X) \) stays bounded.

**Example 3.5.** Let \( X \) be a rigid Calabi-Yau threefold. (A number of such examples are known, cf. [S].) Then the Hodge numbers satisfy \( h^{10} = h^{20} = h^{21} = 0 \) and \( h^{30} = 1 \). It follows that \( H^2(X) \) and \( H^4(X) \) are generated by algebraic cycles. Choose a generator \( \alpha \in H^{30}(X) \) normalized so that the class \( \delta = \text{sym}(\alpha \times \alpha) \in H^6(S^3 X) \) is rational. The space of \( S_2 \)-invariant classes of type \((3,3)\) in \( H^3(X) \otimes H^3(X) \) is one dimensional. Therefore \( \delta \) must coincide with the Künneth component of the diagonal \( \Delta \) in \( H^3(X) \otimes H^3(X) \). The remaining Künneth components of \( \Delta \) are necessarily algebraic. Therefore \( \delta \) is algebraic. Thus all factors

\[
\text{sym}(H^3(X)^{\otimes k} \otimes H^2(X) \otimes \ldots \otimes H^{2(n-k)}(X))
\]

in \( H^i(S^n X) \) are spanned by algebraic cycles if \( k \) is even, or \( \{\text{sym}(\xi \times (\text{alg. cycle}) | \xi \in H^3(X))\} \) if \( k \) is odd. It follows, by a modification of lemma [3.7] that \( \mu(S^n X) \leq 3 \) for all \( n \).

We review the basic facts about Hilbert schemes of surfaces. Proofs and references can be found in the first 30 or so pages of [Go]. When \( X \) is a smooth projective surface, there is a natural desingularization \( \pi_n \colon \text{Hilb}^n X \to S^n X \) given by the (reduced) Hilbert scheme of \( 0 \)-dimensional subschemes of length \( n \). The symmetric product \( S^n X \) can be decomposed into a disjoint union of locally closed sets

\[
\Delta_{(\lambda_1,\ldots,\lambda_k)} = \{ \lambda_1 x_1 + \ldots + \lambda_k x_k | x_i \neq x_j \}
\]

indexed by partitions \( \lambda_1 \geq \lambda_2 \ldots \) of \( n \), where the elements of \( S^n X \) are written additively. Let \( \Delta \subset X^k \) denote the preimage under \( (x_1,\ldots,x_k) \mapsto \lambda_1 x_1 + \ldots + \lambda_k x_k \). The map \( \Delta_{\lambda} \to \Delta_{\lambda} \) is etale with Galois group \( G \). The group can be described explicitly by grouping the terms in the partition as follows:

\[
\begin{align*}
\lambda_1 = \ldots = \lambda_{d_1} &> \lambda_{d_1+1} = \ldots = \lambda_{d_1+d_2} > \ldots > \lambda_{d_1+d_2+\ldots+d_{d_k}} = \lambda_k > 0
\end{align*}
\]

Then \( G = S_{d_1} \times \ldots S_{d_k} \) and \( \Delta_{\lambda} \) is isomorphic to an open subset \( S^{d_1} X \times \ldots S^{d_k} X \).

We record the following key fact [Go lemma 2.1.4]

**Lemma 3.6.** \( \pi^{-1} \Delta_{\lambda} \times_{\Delta_{\lambda}} \Delta_{\lambda} \to \Delta_{\lambda} \) is isomorphic to a Zariski open subset of the \( k \)-fold product \( \prod H_{\lambda_i} \to X^k \). Each \( H_{\lambda_i} \to X \) is a Zariski locally trivial fibre bundle whose fibre can be identified with the subscheme \( \text{Hilb}^\lambda \mathbb{C}^2 \subset \text{Hilb}^\lambda \mathbb{C}^2 \) parameterizing schemes supported at the origin of \( \mathbb{C}^2 \).

We note that \( \text{Hilb}^\lambda \mathbb{C}^2 \) is smooth and \( \mathbb{C}^* \) acts on it with isolated fixed points. Therefore the fibres have \( \mu = 0 \). We can see that \( G \) acts on \( \pi^{-1} \Delta_{\lambda} \times_{\Delta_{\lambda}} \Delta_{\lambda} \) by permuting the \( H_{\lambda_i} \)'s. Consequently

\[
\pi^{-1} \Sigma_{\lambda} \cong S^{d_1} H_{\lambda_{d_1}} \times S^{d_2} H_{\lambda_{d_1+d_2}} \times \ldots S^{d_k} H_{\lambda_{d_1+\ldots+d_k}}
\]

(8)
Proposition 3.7. If $X$ is a surface with $p_g = 0$ then $\mu(Hilb^n X) \leq \min(n, \sqrt{2nq})$ for all $n$, where $q = h^{10}$. In particular, $\mu(Hilb^n X) = 0$ when $q = 0$.

Proof. The closures of each of the strata $\Delta_\lambda$ are dominated by $X^k$, so that $\mu(\Delta_\lambda) \leq k \leq n$. The maps $\pi^{-1}\Delta_\lambda \to \Delta_\lambda$ are Zariski locally trivial fibrations with fibres having $\mu = 0$. Thus $\mu(Hilb^n X) \leq n$ follows from this together with proposition 1.1 and corollary 2.7.

Each $H_{\lambda} \to X$ is a bundle with $\mu = 0$ fibres. Therefore $\mu(S_d H_{\lambda_{d_1+\ldots+d_\ell}}) \leq q$ by corollary 3.3 and the previous theorem. Combing this with 8, we see that $\Delta_\lambda$ has motivic dimension at most $\ell q$. To estimate $\ell$, we use

$$n = d_1 \lambda_{d_1} + d_2 \lambda_{d_1+d_2} + \ldots + d_\ell \lambda_{d_1+\ldots+d_\ell} \geq \lambda_{d_1+\ldots+d_\ell} \geq \ell + (\ell - 1) + \ldots + 1$$

to obtain $\ell \leq \sqrt{2n}$. This gives the remaining inequality $\mu(Hilb^n X) \leq \sqrt{2nq}$.

4. Applications to the Hodge conjecture

Jannsen [J] has extended the Hodge conjecture to an arbitrary variety $X$. This states that any class in $\text{Hom}(\mathbb{Q}(i), W_{-2i} H_2(X))$, which should be thought of as a Hodge cycle, is a linear combination of fundamental classes of $i$-dimensional subvarieties. Lewis [L] has given a similar extension for the generalized Hodge conjecture which would say that an irreducible sub Hodge structure of $W_{-i} H_i(X)$ with level at most $\ell$ should lie in the Gysin image of a subvariety of dimension bounded by $(\ell + i)/2$. Both statements are equivalent to the usual forms for smooth projective varieties.

Proposition 4.1.

(a) If $\mu(X) \leq 3$, then the Hodge conjecture holds for $X$.

(b) If $\mu(X) \leq 2$, then the generalized Hodge conjecture holds for $X$.

Proof. Suppose that $\mu(X) \leq 3$. Then any Hodge class $\alpha \in H_2(\tilde{Y})$ lies in $f_* W_{-2i} H_2(\tilde{Y})$ for some subvariety with components satisfying $\dim \tilde{Y} \leq (3 + 2i)/2$. Let $\tilde{Y} = \cup \tilde{Y}_j$ be a desingularization of a compactification of $Y$. An argument similar to the proof of proposition 1.1 shows that the natural map

$$W_{-2i} H_{2i}(\tilde{Y}) \to f_* W_{-2i} H_{2i}(Y)$$

is a surjective morphism of polarizable Hodge structures. This map admits a section, since the category of such structures is semisimple [D3]. It follows that $\alpha$ can be lifted to a Hodge cycle $\beta$ on $\tilde{Y}$. This can be viewed as a Hodge cycle in cohomology under the Poincaré duality isomorphism $H_{2i}(\tilde{Y}) = \bigoplus_j H^{2\dim \tilde{Y}_j - 2i}(\tilde{Y}_j)(\dim Y_j)$. Since $2 \dim Y_j - 2i \leq 3$, this forces the degree of $\beta$ to be 0 or 2. Consequently $\beta$ must be an algebraic cycle. Hence the same is true for its image $\alpha$.

The second statement is similar. With notation as above, a sub Hodge structure of $H_i(X)$ is the image of a sub Hodge structure of $\bigoplus_j H^{2\dim Y_j - i}(Y_j)(\dim Y_j)$ with $2 \dim Y_j - i \leq 2$. Since the generalized Hodge conjecture is trivially true in this range, this structure is contained in the Gysin image of map from a subvariety of expected dimension. 

$\square$
Corollary 4.2 (Conte-Murre). The Hodge (respectively generalized Hodge) conjecture holds for uniruled fourfolds (respectively threefolds).

Corollary 4.3 (Laterveer). The Hodge (respectively generalized Hodge) conjecture holds for rationally connected smooth projective fivefolds (respectively fourfolds).

Corollary 4.4. If \( X \) is a smooth projective variety with a \( \mathbb{C}^* \)-action, then the Hodge (respectively generalized Hodge) conjecture holds if the fixed point set has dimension at most 3 (respectively 2).

The last result can also be deduced from the main theorem of [AK1], which says in effect that these conjectures factor through the Grothendieck group of varieties. The class of \( X \) in the Grothendieck group can be expressed as a linear combination of classes of components of the fixed point set times Lefschetz classes.

Corollary 4.5. The Hodge (respectively generalized Hodge) conjecture holds for a smooth projective fourfold (respectively threefold) which can be covered by a family of surfaces whose general member is smooth with \( p_g = 0 \).

Corollary 4.6. Let \( X \) be a smooth projective surface with \( p_g = 0 \). Then the Hodge (respectively generalized Hodge) conjecture holds for \( S^n X \) for any \( n \) if \( q \leq 3 \) (respectively \( q \leq 2 \)). The Hodge conjecture holds for \( \text{Hilb}^n X \) if \( n \leq 3 \), or if \( n \leq 7 \) and \( q = 1 \), or if \( q = 0 \). The generalized Hodge conjecture holds for \( \text{Hilb}^n X \) if \( n = 2 \), or if \( n \leq 3 \) and \( q = 1 \), or if \( q = 0 \).

As a final example, suppose that \( X \) is a rigid Calabi-Yau variety. Then we saw that \( \mu(S^n X) \leq 3 \) in example 3.5. So the Hodge conjecture holds for \( S^n X \). (We are being a little circular in our logic, since we essentially verified the conjecture in the course of estimating \( \mu(S^n X) \).)

We have a Noether-Lefschetz result in this setting. For the statement, we take “sufficiently general” to mean that the set of exceptions forms a countable union of proper Zariski closed subsets of the parameter space.

Theorem 4.7. Let \( X \subset \mathbb{P}^N \) be a smooth projective variety such that the Hodge (respectively generalized Hodge) conjecture holds. Then there exists an effective constant \( d_0 > 0 \) such that the Hodge (respectively generalized Hodge) conjecture holds for \( H \subset X \) when \( H \in \mathbb{P}(\mathcal{O}_X(d)) \) is a sufficiently general hypersurface and \( d \geq d_0 \).

Remark 4.8. The effectiveness of \( d_0 \) depends on having enough information about \( X \subset \mathbb{P}^N \). As will be clear from the proof, it would be sufficient to know the Chern classes of \( X, \mathcal{O}_X(1) \), the Castelnuovo-Mumford regularity of \( \mathcal{O}_X \) and \( T_X \), and \( h^0(T_X) \). For \( X = \mathbb{P}^N \), we have \( d_0 = N + 1 \).

Proof. Let \( n = \dim X - 1 \). We make \( d_0 \) to be the smallest integer for which

\begin{itemize}
  \item[(a)] \( d_0 \geq 2 \).
  \item[(b)] \( h^{n0}(H) > h^{n0}(X) \) for nonsingular \( H \in \mathbb{P}(\mathcal{O}_X(d)) \) with \( d \geq d_0 \). (This ensures that \( H^n(\text{im} H^n(X) \) has length \( n \); a fact needed at the end.)
  \item[(c)] \( H \in \mathbb{P}(\mathcal{O}_X(d)), d \geq d_0 \), has nontrivial moduli, at least infinitesimally
\end{itemize}

To clarify these conditions, note that from the exact sequence

\[ 0 \to \omega_X \to \omega_X(d) \to \omega_H \to 0 \]
and Kodaira’s vanishing theorem we obtain
\[ h^0(H, \omega_H) = h^0(X, \omega_X(d)) - h^0(\omega_X) + h^1(\omega_X) \]
\[ = \chi(\omega_X(d)) - h^0(\omega_X) + h^1(\omega_X) \]

The right side is explicitly computable by Riemann-Roch, so we get an effective lower bound for (b). For (c), we require \( H^1(T_X) \neq 0 \). Once we choose \( d_0 \) so that \( H^1(T_X(-d)) = 0 \) for \( d \geq d_0 \), standard exact sequences yield
\[ h^i(T_H) \geq h^0(\mathcal{O}_H(d)) - h^0(T_X|H) \]
\[ \geq h^0(\mathcal{O}_X(d)) - h^0(T_X) - 1 \]

We can compute the threshold which makes the right side positive. When \( X = \mathbb{P}^N \), we see by a direct computation that assumptions (a), (b), (c) are satisfied as soon as \( d \geq N + 1 \).

Let \( U_d \subset \mathbb{P}(\mathcal{O}(d)) \) denote the set of nonsingular hypersurfaces in \( X \). For \( H \in U_d \), the weak Lefschetz theorem guarantees an isomorphism \( H^i(X) \cong H^i(H) \) for \( i < n \), so the (generalized) Hodge conjecture holds for these groups by assumption. This together with the hard Lefschetz theorem takes care of the range \( i > n \). So it remains to treat \( i = n \). We have an orthogonal decomposition
\[ H^n(H) = \text{im} H^n(X) \oplus V_H \]

as Hodge structures, where \( V_H \) is the kernel of the Gysin map \( H^n(H) \to H^{n+2}(X) \) [4, sect. 2.3.3]. Equivalently, after restricting to a Lefschetz pencil in \( \mathbb{P}(\mathcal{O}(d)) \) containing \( H, V_H \) is just the space of vanishing cycles [loc. cit.]. As \( H \) varies, \( H^n(H) \) determines a local system over \( U_d \), and \( (9) \) is compatible with the monodromy action. The action of \( \pi_1(U_d) \) respects the intersection pairing \( \langle , \rangle \). So the image of a finite index subgroup \( \Gamma \subset \pi_1(U_d) \) lies in the identity component \( \text{Aut}(V_H, \langle , \rangle)^\circ \), which is a symplectic or special orthogonal group according to the parity of \( n \). By [4, thm 4.4.1], the image of \( \Gamma \) is either finite or Zariski dense in \( \text{Aut}(V_H, \langle , \rangle)^\circ \).

The first possibility can be ruled out by checking that the Griffiths period map on \( U_d \) is nontrivial [4, lemma 3.3]. The nontriviality of the period map is guaranteed by Green’s Torelli theorem [4, thm 0.1] and our assumption (c). Thus the Zariski closure \( \overline{\text{im}(\pi_1(U_d))}^{\text{Zar}} \supseteq \text{Aut}(V_H, \langle , \rangle)^\circ \). To finish the argument, we recall that the Mumford-Tate group \( MT(V_H) \subset GL(V_H) \) is an algebraic subgroup which leaves all sub Hodge structures invariant. By [4, prop 5.7], when \( H \) is sufficiently general \( MT(V_H) \) must contain a finite index subgroup of the image of \( \pi_1(U_d) \), and therefore \( \text{Aut}(V_H, \langle , \rangle)^\circ \). Thus \( MT(V_H) \) contains the symplectic or special orthogonal group. In either case \( V_H \) is an irreducible Hodge structure of length \( n \). The (generalized) Hodge conjecture concerns sub Hodge structures of \( H^n(H) \) of length less than \( n \). So these must come from \( X \), and therefore be contained in images of Gysin maps of subvarieties of expected codimension by our initial assumptions.

\[ \square \]

**Remark 4.9.** For the Hodge conjecture alone, the bound on \( d_0 \) can be improved. When \( n \) is odd, we may take \( d_0 = 1 \) since the Hodge conjecture is vacuous for \( H^n \). When \( n \) is even, we can choose the smallest \( d_0 \) so that assumptions (a) and (c) hold, since the monodromy action on \( V_H \) is irreducible and nontrivial (by the Picard-Lefschetz formula) and therefore it cannot contain a Hodge cycle.

**Corollary 4.10.** Let \( X \subset \mathbb{P}^N \) be a smooth projective variety.
(a) If $\mu(X) \leq 3$, then the Hodge conjecture holds for every sufficiently general hypersurface section of degree $d \gg 0$. This is in particular the case, when $X$ is a rationally connected fivefold.

(b) If $\mu(X) \leq 2$, then the generalized Hodge conjecture holds for every sufficiently general hypersurface section of degree $d \gg 0$.

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