The HEALPix Primer

Revision: Version 1.0; March 21, 2022

Prepared by: Krzysztof M. Górski, Benjamin D. Wandelt, Eric Hivon, Frode K. Hansen, and Anthony J. Banday

Abstract: HEALPix is a Hierarchical, Equal Area, and iso-Latitude Pixelisation of the sphere designed to support efficiently (1) local operations on the pixel set, (2) a hierarchical tree structure for multi-resolution applications, and (3) the global Fast Spherical Harmonic transform. HEALPix based mathematical software meets the challenges which future high resolution and large volume CMB data sets, including the MAP and Planck mission products, will present.
1 Introduction

The analysis of functions on domains with spherical topology occupies a central place in physical science and engineering disciplines. This is particularly apparent in the fields of astronomy, cosmology, geophysics, atomic and nuclear physics. In many cases the geometry is either dictated by the object under study or approximate spherical symmetry can be exploited to yield powerful perturbation methods. Practical limits for the purely analytical study of these problems create an urgent necessity for efficient and accurate numerical tools.

The simplicity of the spherical form belies the intricacy of global analysis on the sphere. There is no known point set which achieves the analogue of uniform sampling in Euclidean space and allows exact and invertible discrete spherical harmonic decompositions of arbitrary but band-limited functions. Any existing proposition of practical schemes for the discrete treatment of such functions on the sphere introduces some (hopefully tiny) systematic error dependent on the global properties of the point set. The goal is to minimise these errors and faithfully represent deterministic functions as well as realizations of random variates both in configuration and Fourier space while maintaining computational efficiency.

We illustrate these points using as an example the field which is particularly close to the authors' hearts, Cosmic Microwave Background (CMB) anisotropies. Here we are in the happy situation of expecting an explosion of available data within the next decade. The Microwave Anisotropy Probe (MAP) (NASA) and Planck Surveyor (ESA) missions are aiming to provide multi-frequency, high resolution, full sky measurements of the anisotropy in both temperature and polarization of the cosmic microwave background radiation. The ultimate data products of these missions — multiple microwave sky maps, each of which will have to comprise more than \~ 10^6 pixels in order to render the angular resolution of the instruments — will present serious challenges to those involved in the analysis and scientific exploitation of the results of both surveys.

As we have learned while working with the COBE mission products, the digitised sky map is an essential intermediate stage in information processing between the entry point of data acquisition by the instruments — very large time ordered data streams, and the final stage of astrophysical analysis — typically producing a ‘few’ numerical values of physical parameters of interest. COBE-DMR sky maps (angular resolution of 7° (FWHM) in three frequency bands, two channels each, 6144 pixels per map) were considered large at the time of their release.

As for future CMB maps, a whole sky CMB survey at the angular resolution of \~ 10′ (FWHM), discretised with a few pixels per resolution element (so that the discretisation effects on the signal are sub-dominant with respect to the effects of instrument’s angular response), will require map sizes of at least \( N_{\text{pix}} \sim \text{a few} \times 1.5 \times 10^6 \) pixels. More pixels than that will be needed to represent the Planck-HFI higher resolution channels. This estimate, \( N_{\text{pix}} \), should be multiplied by the number of frequency bands (or, indeed, by the number of individual observing channels — 74 in the case of Planck — for the analysis work to be done before the final coadded maps are made...
The HEALPix Primer

for each frequency band) to render an approximate expected size of the already very compressed form of survey data which would be the input to the astrophysical analysis pipeline.

It appears to us that very careful attention ought to be given to devising high resolution CMB map structures which can maximally facilitate the forthcoming analyses of large size data sets, for the following reasons:

- It is clearly very easy to end up with an estimated size of many GB by for the data objects which would be directly involved in the science extraction part of the future CMB missions.
- Many essential scientific questions can only be answered by global studies of future data sets.

This document is an introduction to the properties of our proposed approach for a high resolution numerical representation of functions on the sphere — the Hierarchical Equal Area and isoLatitutde Pixelization (HEALPix, see http://www.tac.dk/~healpix), and the associated multi-purpose computer software package.

2 Discretisation of Functions on the Sphere for High Resolution Applications: a Motivation for HEALPix

Numerical analysis of functions on the sphere involves (1) a class of mathematical operations, whose objects are (2) discretised maps, i.e. quantizations of arbitrary functions according to a chosen tessellation (exhaustive partition of the sphere into finite area elements). Hereafter we mostly specialise our discussion to CMB related applications of HEALPix, but all our statements hold true generally for any relevant deterministic and random functions on the sphere.

Considering point (1): Standard operations of numerical analysis which one might wish to execute on the sphere include convolutions with local and global kernels, Fourier analysis with spherical harmonics and power spectrum estimation, wavelet decomposition, nearest-neighbour searches, topological analysis, including searches for extrema or zero-crossings, computing Minkowski functionals, extraction of patches and finite differencing for solving partial differential equations. Some of these operations become prohibitively slow if the sampling of functions on the sphere, and the related structure of the discrete data set, are not designed carefully.

Regarding point (2): Typically, a whole sky map rendered by a CMB experiment contains (i) signals coming from the sky, which are by design strongly band-width limited (in the sense of spatial Fourier decomposition) by the instrument’s angular response function, and (ii) a projection into the elements of a discrete map, or pixels, of the observing instrument’s noise; this pixel noise should be random, and white, at least near the discretisation scale, with a band-width significantly exceeding that of all the signals.
Figure 1: Quadrilateral tree pixel numbering. The coarsely pixelised coordinate patch on the left consists of four pixels. Two bits suffice to label the pixels. To increase the resolution, every pixel splits into 4 daughter pixels shown on the right. These daughters inherit the pixel index of their parent (boxed) and acquire two new bits to give the new pixel index. Several such curvilinearly mapped coordinate patches (12 in the case of HEALPix, and 6 in the case of the COBE quad-sphere) are joined at the boundaries to cover the sphere. All pixels indices carry a prefix (here omitted for clarity) which identifies which base-resolution pixel they belong to.

With these considerations in mind we propose the following list of desiderata for the mathematical structure of discretised full sky maps:

1. **Hierarchical structure of the data base.** This is recognised as essential for very large data bases, and was postulated in construction of the Quadrilateralized Spherical Cube (or quad-sphere, see [http://www.gsfc.nasa.gov/astro/cobe/skymap_info.html](http://www.gsfc.nasa.gov/astro/cobe/skymap_info.html)), which was used for the COBE data. An argument in favour of this proposition states that the data elements which are nearby in a multi-dimensional configuration space (here, on the surface of a sphere), are also nearby in the tree structure of the data base, hence the near-neighbour searches are conducted optimally in the data storage medium or computer RAM. This property, especially when implemented with small number of base resolution elements, facilitates various topological methods of analysis, and allows easy construction of wavelet transforms on quadrilateral (and also triangular) grids. Figure 1 shows how a hierarchical partition with quadrilateral structure naturally allows for a binary vector indexing of the data base.

2. **Equal areas of discrete elements of partition.** This is advantageous because (i) white noise generated by the signal receiver gets integrated exactly into white noise in the pixel space, and (ii) sky signals are sampled without regional dependence, except for the dependence on pixel shapes, which is unavoidable with tessellations of the sphere. Hence, as much as possible given the experimental details, the pixel size should be made sufficiently small compared to the
The HEALPix Primer

instrument’s resolution to avoid any excessive, and pixel shape dependent, signal smoothing.

3. Iso-Latitude distribution of discrete area elements on a sphere. This property is critical for computational speed of all operations involving evaluation of spherical harmonics. Since the associated Legendre polynomial components of spherical harmonics are evaluated via slow recursions, and can not be simply handled in an analogous way to the trigonometric Fast Fourier Transform, any deviations in the sampling grid from an iso-latitude distribution result in a prohibitive loss of computational performance with the growing number of sampling points, or increasing map resolution. It is precisely this property that the COBE quad-sphere is lacking, and this renders it impractical for applications to high resolution data.

A number of tessellations have been used for discretisation and analysis of functions on the sphere (for example, see Driscoll & Healy (1994), Mucciaccia, Natoli & Vittorio (1998) — rectangular grids, Baumgardner & Frederickson (1985), Tegmark (1996) — icosahedral grids, Saff & Kuijlaars (1997), Crittenden & Turok (1998) — ‘igloo’ grids, and Szalay & Brunner (1998) — a triangular grid), but none satisfies simultaneously all three stated requirements.

All three requirements formulated above are satisfied by construction with the Hierarchical Equal Area, iso-Latitude Pixelisation (HEALPix) of the sphere (Górski (1999)), which is shown in Figure 2.

3 Geometric and Algebraic Properties of HEALPix

HEALPix is a genuinely curvilinear partition of the sphere into exactly equal area quadrilaterals of varying shape. The base-resolution comprises twelve pixels in three rings around the poles and equator.

The resolution of the grid is expressed by the parameter $N_{\text{side}}$ which defines the number of divisions along the side of a base-resolution pixel that is needed to reach a desired high-resolution partition.

All pixel centers are placed on $4 \times N_{\text{side}} - 1$ rings of constant latitude, and are equidistant in azimuth (on each ring). All iso-latitude rings located between the upper and lower corners of the equatorial base-resolution pixels, the equatorial zone, are divided into the same number of pixels: $N_{eq} = 4 \times N_{\text{side}}$. The remaining rings are located within the polar cap regions and contain a varying number of pixels, increasing from ring to ring with increasing distance from the poles by one pixel within each quadrant.

Pixel boundaries are non-geodesic and take the very simple forms $\cos \theta = a \pm b \cdot \phi$ in the equatorial zone, and $\cos \theta = a + b / \phi^2$, or $\cos \theta = a + b / (\pi / 2 - \phi)^2$, in the polar caps. This allows one to explicitly check by simple analytical integration the exact area equality among pixels, and to compute efficiently more complex objects, e.g. the Fourier transforms of individual pixels.

Specific geometrical properties allow HEALPix to support two different numbering schemes for
Figure 2: Orthographic view of HEALPix partition of the sphere. Overplot of equator and meridians illustrates the octahedral symmetry of HEALPix. Light-gray shading shows one of the eight (four north, and four south) identical polar base-resolution pixels. Dark-gray shading shows one of the four identical equatorial base-resolution pixels. Moving clockwise from the upper left panel the grid is hierarchically subdivided with the grid resolution parameter equal to $N_{\text{side}} = 1, 2, 4, 8$, and the total number of pixels equal to $N_{\text{pix}} = 12 \times N_{\text{side}}^2 = 12, 48, 192, 768$. All pixel centers are located on $N_{\text{ring}} = 4 \times N_{\text{side}} - 1$ rings of constant latitude. Within each panel the areas of all pixels are identical.

the pixels, as illustrated in the Figure 3.

First, in the RING scheme, one can simply count the pixels moving down from the north to the south pole along each iso-latitude ring. It is in the RING scheme that Fourier transforms with spherical harmonics are easy to implement.

Second, in the NESTED scheme, one can arrange the pixel indices in twelve tree structures, corresponding to base-resolution pixels. Each of those is organised as shown in Fig. 1. This can easily be implemented since, due to the simple description of pixel boundaries, the analytical mapping of the HEALPix base-resolution elements (curvilinear quadrilaterals) into a $[0,1] \times [0,1]$ square exists. This tree structure allows one to implement efficiently all applications involving nearest-neighbour searches (see Wandelt, Hivon & Górski (1998)), and also allows for an immediate construction of the fast Haar wavelet transform on HEALPix.
Figure 3: Cylindrical projection of the HEALPix division of a sphere and two natural pixel numbering schemes (RING and NESTED) allowed by HEALPix. Both numbering schemes map the two dimensional distribution of discrete area elements on a sphere into the one dimensional, integer pixel number array, which is essential for computations involving data sets with very large total pixel numbers. From top to bottom: Panel one (resolution parameter $N_{\text{side}} = 2$) and panel two ($N_{\text{side}} = 4$) show the RING scheme for pixel numbering, with the pixel number winding down from north to south pole through the consecutive isolatitude rings. Panel three (resolution parameter $N_{\text{side}} = 2$) and panel four ($N_{\text{side}} = 4$) show the NESTED scheme for pixel numbering within which the pixel number grows with consecutive hierarchical subdivisions on a tree structure seeded by the twelve base-resolution pixels.
4 The HEALPix Software Package

We have developed a package of HEALPix based mathematical software, consisting of Fortran90 and IDL source codes as well as documentation and examples. Successful installation produces a set of facilities using standardised FITS I/O interfaces ([http://heasarc.gsfc.nasa.gov/docs/software/fitsio](http://heasarc.gsfc.nasa.gov/docs/software/fitsio)) as well as two Fortran90 libraries which users can link to their own applications. Among the tasks performed by the components of the HEALPix package are the following:

- Simulation of the full sky CMB temperature and polarisation maps as realisations of random Gaussian fields, with an option to constrain the realisation by prior information.
- Analysis of the full sky CMB temperature and polarisation maps resulting in power spectra and/or spherical harmonic coefficients.
- Global smoothing of whole sky maps with a Gaussian kernel.
- Degradation and upgrade of the resolution of discrete maps.
- Global searches on the maps for nearest-neighbours and the maxima/minima of the discretised functions.
- Algebraic conversion of the maps between the RING and NESTED numbering schemes, and mapping back and forth between positions on the sphere and discrete pixel index space.
- Visualisation of the HEALPix formatted sky maps in both the Mollweide and the gnomonic projections of small areas of the sky.

The package includes documents which describe the installation process, the Fortran 90 facilities, the IDL facilities and a large number of subroutines contained in the library. It is available to the scientific community at [http://www.tac.dk/~healpix](http://www.tac.dk/~healpix).

HEALPix is the format chosen by the MAP collaboration to be used for the production of sky maps (see [http://map.gsfc.nasa.gov/html/technical_info.html](http://map.gsfc.nasa.gov/html/technical_info.html)) from the mission data. HEALPix software is widely used for simulation work within both the LFI and HFI consortia of the Planck collaboration.
The HEALPix Primer

References

[1] Baumgardner, J.R. and Frederickson, P.O., 1985, SIAM J. Numerical Analysis, Vol. 22, No. 6, p. 1107

[2] Crittenden, R. and Turok, N.G., 1998, astro-ph/9806374

[3] Driscoll, J.R. and Healy, D., 1994, Adv. in Appl. Math., Vol. 15, p.202

[4] Górski K.M., Hivon, E. and Wandelt, B.D., 1998, “Analysis Issues for Large CMB Data Sets”, astro-ph9812350, to appear in Proceedings of the MPA/ESO Conference on Evolution of Large-Scale Structure: from Recombination to Garching 2-7 August 1998; eds. A.J. Banday, R.K. Sheth and L. Da Costa

[5] Górski K.M., 1999, in preparation

[6] Mucaccia, P.F, Natoli, P. and Vittorio, N., 1998, Ap.J., 488, L63

[7] Saff, E.B. and Kuijlaars, A.B.J., 1997, The Mathematical Intelligencer, 19, #1, p.5

[8] Szalay, A.S. and Brunner, R.J., 1998, astro-ph9812335, to appear in a special issue of the Elsevier journal ”Future Generation Computer Systems”

[9] Tegmark, M., 1996, 470, L81

[10] Wandelt, B.D., Hivon, E. and Górski, K.M., 1998, astro-ph/9803317, in ”Fundamental Parameters in Cosmology”, proceedings of the XXXIIIrd Rencontres de Moriond, Trần Thanh Vân (ed.)