Loop Integrals, $\mathcal{R}$ Functions and their Analytic Continuation

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Abstract

To entirely determine the resulting functions of one-loop integrals it is necessary to find the correct analytic continuation to all relevant kinematical regions. We argue that this continuation procedure may be performed in a general and mathematical accurate way by using the $\mathcal{R}$ function notation of these integrals. The two- and three-point cases are discussed explicitly in this manner.

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1 Introduction

As pointed out recently (cf. [1], [2], [3], [4]) the integrals which may appear in one-loop Feynman diagrams of any renormalizable field theory are expressible in terms of $\mathcal{R}$ functions. This result is true despite the differing properties of the various N-point functions and the numerous possibilities of placing tensor structure in the numerator. We use [5] as a reference for the class of $\mathcal{R}$ functions.

Making use of this feature the evaluation of one-loop N-point functions is simplified drastically by virtue of the systematic recurrence relations for $\mathcal{R}$ functions, which reduce the different tensor type $\mathcal{R}$ functions to a set of fundamental $\mathcal{R}$ functions. Each N-point function may be represented by such a characteristic fundamental set. In this paper we restrict ourselves to two- and three-point functions.

As it is well-known the results for loop integrals have to be discussed in all kinematical regions of interest and often involve an analytic continuation to all kinematical regimes at the end of the calculation. Of course this statement is also true if one-loop functions are expressed in terms of $\mathcal{R}$ functions. The analytic continuation of $\mathcal{R}$ functions is performed in a very general way in [5] using the analytical properties of this functions. Unfortunately, the general formulay in [5] for the analytic continuation procedure is not designed to our practical purposes, because the $\mathcal{R}$ functions of our interest – though by definition maximally analytical continued – are expanded in one of their parameters. They may be rewritten in terms of logarithms and dilogarithms. In this paper we therefore develop the analytic continuation formula which are sufficient for the calculation of all possible one-loop integrals. We hope to clarify some subtleties involved in this procedure (cf. [6]).

2 Characteristics of $\mathcal{R}$ functions

In the two-point case all integrals are expressed in linear combinations of the two $\mathcal{R}$ functions (cf. [1], [3])

\begin{align}
\mathcal{R}_{1-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; z_1, z_2) \\
\mathcal{R}_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; z_1, z_2)
\end{align}

where the second function (2) is the solution of the scalar integral and the first one only appears in higher tensor cases. In our notation $\varepsilon = \frac{(4-D)}{2}$ represents the usual dimensional regularization parameter and the $z_i$ contain the parameters of the integral – internal masses and external momentum components.

The three-point functions are described by three $\mathcal{R}$ functions:

\begin{align}
\mathcal{R}_{1-2\varepsilon}(\varepsilon, \varepsilon, 1; z_1, z_2, z_3) \\
\mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; z_1, z_2, z_3) \\
\mathcal{R}_{1-2\varepsilon}(\varepsilon, \varepsilon, 1; z_1, z_2, z_3)
\end{align}

Here again the second function (3) is the scalar solution, whereas the others appear only in the case of tensor integrals.

According to [3] $\mathcal{R}_t(b; z) = \mathcal{R}_t(b_1, \ldots, b_k; z_1, \ldots, z_k)$ is defined for arbitrary complex index $t$, whereas the parameters $b = (b_1, \ldots, b_k)$ have to satisfy the restriction

$$\sum_{i=1}^{k} b_i \neq 0, -1, -2, \ldots$$
The cut due to the arguments \( z = (z_1, \ldots, z_k) \) is usually chosen to lie on the negative real axis, which means that the function is defined uniquely as far as \( |\arg z_i| < \pi \ \forall i \). The most important transformation laws for \( R \) functions are summarized in appendix A of this paper.

In the limit \( \varepsilon \to 0 \) the restriction of parameters (8) is fulfilled in every function (1) – (5) and the \( z_i \) do not touch the cut provided the \( i\eta \) prescription in the propagators was not neglected during calculations, whereas any transformation of arguments – especially concerning the scaling law (A.5) – has to be performed carefully not to cross the cut for any of the \( z_i \).

Up to now there was no need to require any kinematical restriction – except the quite natural assumption that masses and momentum components are real valued parameters of the integrals. In this context we claim that the \( R \) functions (1) – (5) carry the entire information in all kinematical regions.

3 The two-point case

To evaluate the two-point integrals for concrete values it is necessary to expand the \( R \) functions (1), (2) in powers of \( \varepsilon \), starting with the scalar \( R \) function (2) and bearing in mind the two different cases

\[
R_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 + i\rho_2) \tag{7}
\]

\[
R_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 - i\rho_2) \tag{8}
\]

whereas the \( x_i \) and \( \rho_i \) are supposed to be real and positive and the \( \rho_i \) are understood to be infinitesimal. The given arguments are the only possibilities which might appear in two-point functions (cf. [3]). The usual procedure, following [3], makes use now first of the scaling law (A.5) and then of Euler’s transformation (A.8). Of course the scaling law is save for (7), because both arguments of the \( R \) function lie in the same (the upper) half plane and none of them is forced to cross the cut during the scaling process. The whole transformation procedure of (8) may therefore be applied without any complications. (A.7) and (A.8) change the function to

\[
R_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 + i\rho_2) = (-x_1 + i\rho_1)^{1-\varepsilon}(-x_2 + i\rho_2)^{-1} R_{-\frac{1}{2}}(-\frac{1}{2} + \varepsilon, 1; 1, \frac{x_1 - i\rho_1}{x_2 - i\rho_2}) \tag{9}
\]

The arguments of the \( R \) function lie in the right half plane so that the assumptions of the first quadratic transformation are fulfilled and the evaluation is resulting in

\[
R_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 + i\rho_2) = \frac{1}{2} (-x_1 + i\rho_1)^{-\varepsilon} \left( \frac{-x_1 + i\rho_1}{-x_2 + i\rho_2} \right)^{1-\varepsilon} \tag{10}
\]

\[
\times \left[ \left( 1 + \sqrt{1 - \frac{-x_1 + i\rho_1}{-x_2 + i\rho_2}} \right)^{-1+2\varepsilon} + \left( 1 - \sqrt{1 - \frac{-x_1 + i\rho_1}{-x_2 + i\rho_2}} \right)^{-1+2\varepsilon} \right] + O(\varepsilon^2)
\]

The situation is completely different for the second type of \( R \) function (8). To be able to apply the scaling law we first perform a transformation using the integral representation (A.4):

\[
R_{-\varepsilon}(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 - i\rho_2) \tag{11}
\]

\[
= \frac{1}{B(\varepsilon, \frac{1}{2})} \int_0^\infty s^{-\frac{1}{2}} ds \int_0^\infty (s - x_1 + i\rho_1)^{-\frac{1}{2} + \varepsilon}(s - x_2 - i\rho_2)^{-\frac{1}{2}} ds
\]
We now change the integration contour in the following manner (from (a) to (b)), applying the residue theorem. No cut is crossed during this procedure and the contribution at infinity is regulated out by $\varepsilon$, so that the integral takes on the form

$$
\mathcal{R}_{-\varepsilon}\left(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 - i\rho_2\right)
= \frac{1}{B(\varepsilon, \frac{1}{2})} \left[ \frac{2\pi i (x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2} - \varepsilon}} - \int_{-\infty}^{0} \frac{(s + i\rho)^{-\frac{1}{2}} ds}{(s - x_1 + i\rho_1)^{-\frac{1}{2} + \varepsilon}(s - x_2 - i\rho_2)} \right] 
= \frac{1}{B(\varepsilon, \frac{1}{2})} \left[ \frac{2\pi i (x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2} + \varepsilon}} + \int_{0}^{\infty} \frac{(-s + i\rho)^{-\frac{1}{2}} ds}{(s - x_1 + i\rho_1)^{-\frac{1}{2} + \varepsilon}(s + x_2 + i\rho_2)} \right] 
= \frac{1}{B(\varepsilon, \frac{1}{2})} \left[ \frac{2\pi i (x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2} + \varepsilon}} + e^{-i\pi\varepsilon} \int_{0}^{\infty} \frac{s^{-\frac{1}{2}} ds}{(s + x_1 - i\rho_1)^{-\frac{1}{2} + \varepsilon}(s + x_2 + i\rho_2)} \right] 
= \frac{2\pi i}{B(\varepsilon, \frac{1}{2})} \frac{(x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2} + \varepsilon}} + e^{-i\pi\varepsilon} \mathcal{R}_{-\varepsilon}\left(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, x_2 + i\rho_2\right)
$$

The resulting $\mathcal{R}$ function is save with respect to the scaling law because both arguments lie near the positive real axis. So by virtue of (A.3) and (A.8) this $\mathcal{R}$ function leads to (cf. (8))

$$
\mathcal{R}_{-\varepsilon}\left(-\frac{1}{2} + \varepsilon, 1; x_1 - i\rho_1, x_2 + i\rho_2\right) = (x_1 - i\rho_1)^{1-\varepsilon} (x_2 + i\rho_2)^{-1} \mathcal{R}_{-\frac{1}{2}}\left(-\frac{1}{2} + \varepsilon, 1; 1, \frac{x_1 - i\rho_1}{x_2 + i\rho_2}\right)
$$

The arguments lie in the right half plane so that now – according to (A.9) – the first quadratic transformation (A.9) may be applied and the whole function (8) is transformed to

$$
\mathcal{R}_{-\varepsilon}\left(-\frac{1}{2} + \varepsilon, 1; -x_1 + i\rho_1, -x_2 - i\rho_2\right) = \frac{2\pi i}{B(\varepsilon, \frac{1}{2})} \frac{(x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2} + \varepsilon}} + e^{-i\pi\varepsilon} \frac{1}{2} (x_1 - i\rho_1)^{-\varepsilon} \left(\frac{x_1 - i\rho_1}{x_2 + i\rho_2}\right)^{1-\varepsilon}
$$
\[ x \left[ (1 + \sqrt{1 - \frac{x_1 - i\rho_1}{x_2 + i\rho_2}})^{-1+2\varepsilon} + (1 - \sqrt{1 - \frac{x_1 - i\rho_1}{x_2 + i\rho_2}})^{-1+2\varepsilon} \right] + O(\varepsilon^2) \]

\[ \frac{2\pi i}{B(\varepsilon, \frac{1}{2})} \frac{(x_2 + i\rho_2)^{-\frac{1}{2}}}{(x_2 - x_1 + i\rho_1 + i\rho_2)^{-\frac{1}{2}}} + \frac{1}{2} (-x_1 + i\rho_1)^{-\varepsilon} \left( \frac{x_1 - i\rho_1}{x_2 + i\rho_2} \right)^{1-\varepsilon} \]

\[ \times \left[ (1 + \sqrt{1 - \frac{x_1 - i\rho_1}{x_2 + i\rho_2}})^{-1+2\varepsilon} + (1 - \sqrt{1 - \frac{x_1 - i\rho_1}{x_2 + i\rho_2}})^{-1+2\varepsilon} \right] + O(\varepsilon^2) \]

Putting together both cases \[10\] and \[14\] we get

\[ \mathcal{R}_\varepsilon(-\frac{1}{2} + \varepsilon, 1; z_1, z_2) = \frac{2\pi i}{B(\varepsilon, \frac{1}{2})} \frac{\sqrt{z_1 - z_2}}{\sqrt{-z_2}} \theta(\text{Im}(-z_2)) \]

\[ + \frac{1}{2} z_1^{-\varepsilon} \left( \frac{z_1}{z_2} \right)^{-\varepsilon} \left[ (1 + \sqrt{1 - \frac{z_1}{z_2}})^{-1+2\varepsilon} + (1 - \sqrt{1 - \frac{z_1}{z_2}})^{-1+2\varepsilon} \right] + O(\varepsilon^2) \]

The other two-point \( \mathcal{R} \) function \[10\] is evaluated faster thanks to the already solved function \[8\], using the parameter shifting formula \[\mathcal{A}\.3\]:

\[ \mathcal{R}_{1-\varepsilon}(\frac{1}{2} - \varepsilon, 1; z_1, z_2) \]

\[ = 2\left[ (\frac{1}{2} - \varepsilon) \mathcal{R}_{1-\varepsilon}(\frac{1}{2} + \varepsilon, 0; z_1, z_2) + (1 - \varepsilon) \mathcal{R}_{1-\varepsilon}(\frac{1}{2} + \varepsilon, 1; z_1, z_2) \right] \]

The first \( \mathcal{R} \) function on the right hand side is simply \( z_1^{1-\varepsilon} \), the second one is the function \[10\] which we solved before.

### 4 The three-point case

For the three-point integrals we employ an alternative strategy. Beginning with the scalar function \[10\] we apply the series expansion \[\mathcal{A}\.7\]:

\[ \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = \sum_{n=0}^{\infty} \frac{(2\varepsilon, n)}{(2\varepsilon + 1, n)} \sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \frac{(\varepsilon, m_1)(\varepsilon, m_2)}{m_1! m_2!} (1 - x)^{m_1} (1 - y)^{m_2} (1 - z)^{n-m_1-m_2} \]

\[ = 1 + 2\varepsilon \sum_{n=1}^{\infty} \frac{1}{2\varepsilon + n} \sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \frac{(\varepsilon, m_1)(\varepsilon, m_2)}{m_1! m_2!} (1 - x)^{m_1} (1 - y)^{m_2} (1 - z)^{n-m_1-m_2} \]

Appell’s symbol \((\varepsilon, m)\) we define as in \[\mathcal{A}\.2\]:

\[(\varepsilon, m) = \varepsilon(\varepsilon + 1) \cdots (\varepsilon + m - 1)\]

It is of order \(\varepsilon\), except the case \(m = 0\), where \((\varepsilon, m) = 1\). Therefore we derive – neglecting the contributions of higher order than \(O(\varepsilon^2)\):

\[ \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = 1 + 2\varepsilon \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \frac{2\varepsilon}{n} \right) (1 - z)^n \]

\[ + 2\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m_2=1}^{n} \frac{(m_2 - 1)!}{m_2!} (1 - y)^{m_2} (1 - z)^{n-m_2} \]
\[ +2\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m_1=1}^{n} \frac{(m_1 - 1)!}{m_1!} (1 - x)^{m_1} (1 - z)^{n-m_1} + O(\varepsilon^3) \]
\[ = 1 - 2\varepsilon \ln z - 4\varepsilon^2 \text{Li}_2(1-z) + 2\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m_2=1}^{n} \frac{1}{m_2} (1 - y)^{m_2} (1 - z)^{n-m_2} \]
\[ + 2\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m_1=1}^{n} \frac{1}{m_1} (1 - x)^{m_1} (1 - z)^{n-m_1} + O(\varepsilon^3) \]
\[ = z^{-2\varepsilon} \left\{ 1 + 2\varepsilon^2 \left[ \text{Li}_2 \left(1 - \frac{x}{z}\right) + \text{Li}_2 \left(1 - \frac{y}{z}\right) \right] + \ln \left(1 - \frac{x}{z}\right) \eta \left(\frac{x, 1}{z}\right) + \ln \left(1 - \frac{y}{z}\right) \eta \left(\frac{y, 1}{z}\right) + \ln z \left[ \eta \left(x - z, \frac{1}{1 - z}\right) - \eta \left(y - z, \frac{1}{1 - z}\right) \right] \right\} + O(\varepsilon^3) \]

In the last transformation we used the sum formula
\[ \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m} (1 - x)^m (1 - y)^{n-m} = \text{Li}_2(1-y) + \text{Li}_2 \left(1 - \frac{x}{y}\right) + \frac{1}{2} (\ln y)^2 \]
\[ + \ln y \left[ \eta \left(x - y, 1 - \frac{1}{y}\right) - \eta \left(x - y, \frac{1}{y}\right) \right] + \ln \left(1 - \frac{x}{y}\right) \eta \left(\frac{1}{y}\right) \]
\[ \text{(19)} \]

which we prove in appendix [3]. In the notation of the dilogarithm function $\text{Li}_2(z)$ we follow the convention of the standard reference [7]. The $\eta$ function is the usual abbreviation for
\[ \eta(a, b) = 2\pi i \left[ \theta(-\text{Im} a)\theta(-\text{Im} b)\theta(\text{Im}(ab)) \right. \]
\[ \left. - \theta(\text{Im} a)\theta(\text{Im} b)\theta(-\text{Im}(ab)) \right] \]
\[ \text{(20)} \]

It should be emphasized that in spite of the small convergence domain of the series (A.7), the result is valid in the whole complex plane – except the cuts of logarithms and dilogarithms – by help of the uniqueness of the analytic continuation procedure.

If the calculation is done using the scaling law (A.5), writing
\[ \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = z^{-2\varepsilon} \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; \frac{x}{z}, \frac{y}{z}, 1) \]
\[ \text{(21)} \]
the whole evaluation simplifies – using again the expansion (A.7) – but the result is restricted to a smaller domain:
\[ \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = z^{-2\varepsilon} \left\{ 1 + 2\varepsilon^2 \left[ \text{Li}_2 \left(1 - \frac{x}{z}\right) + \text{Li}_2 \left(1 - \frac{y}{z}\right) \right] \right\} + O(\varepsilon^3) \]
\[ \text{(22)} \]
This formula is identical to the former one [18] – as long as every $\eta$ function vanishes – or from another point of view: the $\eta$ terms in (18) represent the corrections of the scaling law if one argument of the $\mathcal{R}$ function crosses the cut during the transformation of arguments $x \rightarrow \frac{x}{z}$, $y \rightarrow \frac{y}{z}$ respectively.

The entire scalar three-point function (cf. [4, 3]) consists of six $\mathcal{R}$ functions of the form (4) and some prefactors. It therefore involves twelve dilogarithms and a considerable amount of $\eta$ functions if the result (18) is substituted – the same number of dilogarithms as the standard publication of 't Hooft and Veltman [3].

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does. The equivalence of both results was checked in all different kinematical regions numerically.

The first of the three-point \( R \) functions (3) may be evaluated in the following way, making use of the already expanded second one:

\[
R_{1-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = 2\varepsilon R_{1-2\varepsilon}(\varepsilon, \varepsilon, 0; x, y, z) + (1 - 2\varepsilon)z R_{-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z)
\] (23)

due to the parameter shifting formula (A.6). The two \( R \) functions on the right hand side are easier to handle: the first one has the advantage that it needs to be expanded only up to \( O(\varepsilon) \) – by virtue of (A.7) – and the second one was already expanded in (18). The complete result is

\[
R_{1-2\varepsilon}(\varepsilon, \varepsilon, 1; x, y, z) = (1 - 2\varepsilon)z^{1-2\varepsilon} + \varepsilon(x + y) + 2\varepsilon^2 \left[ -y \ln y - x \ln x + z \ln \left( 1 - \frac{x}{z} \right) + z \ln \left( 1 - \frac{y}{z} \right) \right] + O(\varepsilon^3)
\] (24)

The third function (5) may be evaluated trivially applying (A.8) and recognizing \( R_0(b; z) = 1 \) in any case:

\[
R_{1-2\varepsilon}(\varepsilon, \varepsilon, 1; z_1, z_2, z_3) = z_1^{-\varepsilon} z_2^{-\varepsilon} z_3^{-1}
\] (25)

5 Résumé and outlook

We want to stress once more that, taking into account the correct analytic continuation of \( R \) functions, our results agree with the well known standard results for the one-loop two- and three-point functions (cf. [8], [9]) in all kinematical regions.

We can also report the fact, that we were able to express arbitrary four-point functions in terms of \( R \) functions (cf. [2]). The structure of the involved \( R \) functions coincides with the discussed functions (3) – (5) of the three-point case. This four-point procedure may be also extended to five- and higher N-point functions without complications. The results will be published later.

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Appendix

A \( \mathcal{R} \) function formulary

Using the following abbreviations

\[
\beta = \sum_{i=1}^{k} b_i \quad (A.1)
\]

\[
(a, n) = \frac{\Gamma(a + n)}{\Gamma(a)} \quad (A.2)
\]

\[
b \pm e_i = (b_1, \ldots, b_i \pm 1, \ldots, b_k) \quad (A.3)
\]

we review the most important transformation rules for \( \mathcal{R} \) functions according to [5]:

- (analytic continued) integral representation:

\[
\int_r^\infty (x - r)^{a-1} \prod_{i=1}^{k} \left( z_i + w_i x \right)^{-b_i} \, dx = B(\beta - \alpha, \alpha) \mathcal{R}_{\alpha - \beta} \left( b_1, \ldots, b_k; r + \frac{z_1}{w_1}, \ldots, r + \frac{z_k}{w_k} \right) \prod_{i=1}^{k} w_i^{-b_i} \quad (A.4)
\]

- scaling law:

\[
\mathcal{R}_t(b_1, \ldots, b_k; \lambda z_1, \ldots, \lambda z_k) = \lambda^t \mathcal{R}_t(b_1, \ldots, b_k; z_1, \ldots, z_k) \quad (A.5)
\]

- shifting parameters:

\[
\beta \mathcal{R}_t(b; z) = (\beta + t) \mathcal{R}_t(b + e_i; z) - tz_i \mathcal{R}_{t-1}(b + e_i; z); \quad i \in (1, \ldots, k) \quad (A.6)
\]

- series expansion:

\[
\frac{\mathcal{R}_t(b; z)}{\Gamma(\beta)} = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(-t, n)}{(\beta, n)} \sum_{\{m_i\}} \frac{(b_1, m_1)}{m_1!} \cdots \frac{(b_k, m_k)}{m_k!} \times (1 - z_1)^{m_1} \cdots (1 - z_k)^{m_k} \quad (A.7)
\]

keeping in mind that the right hand side converges only if \(|1 - z_i| < 1 \forall i\). The sum \(\{m_i\}\) is meant to be taken over all \(m_i\) separately with the only restriction that \(\sum_{i=1}^{k} m_i = n\)

- Euler’s transformation:

\[
\mathcal{R}_t(b_1, \ldots, b_k; z_1, \ldots, z_k) = \mathcal{R}_{-t-\beta} \left( b_1, \ldots, b_k; \frac{1}{z_1}, \ldots, \frac{1}{z_k} \right) \prod_{i=1}^{k} z_i^{-b_i} \quad (A.8)
\]

- first quadratic transformation:

\[
\mathcal{R}_{2t}(b; b; x, y) = \mathcal{R}_t \left( b + t, \frac{1}{2} - t; \left( \frac{x + y}{2} \right)^2, xy \right) \quad (A.9)
\]

valid only for \(\text{Re} \, x, \text{Re} \, y > 0\) and \(b + \frac{1}{2} \neq 0, -1, -2, \ldots\)
Proof of formula (19)

Rewriting the sums of the left hand side of (19) in the form \( \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{m} \) \((1 - x)^m (1 - y)^{n-m}\) we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m} (1 - x)^m (1 - y)^{n-m} = \sum_{n=1}^{\infty} \frac{1}{n} (1 - y)^n \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1 - x}{1 - y}\right)^m \quad (A.10)
- \sum_{n=1}^{\infty} \frac{1}{n} (1 - y)^n \sum_{m=0}^{n} \frac{1}{m + n + 1} \left(\frac{1 - x}{1 - y}\right)^{m+n+1}
\]

Applying formula 5.2.3.4 of [10]:
\[
\sum_{k=0}^{\infty} \frac{x^{k+a}}{k + a} = \int_{0}^{x} \frac{t^{a-1}}{1 - t} \, dt \quad (\text{Re} \, a > 0) \quad (A.11)
\]
we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m} (1 - x)^m (1 - y)^{n-m}
= \ln y \ln \left(1 - \frac{1 - x}{1 - y}\right) - \int_{0}^{\frac{1-x}{1-y}} \frac{dt}{1 - t} \sum_{n=1}^{\infty} \frac{1}{n} [(1 - y)t]^n \quad (A.12)
= \ln y \ln \left(\frac{x - y}{1 - y}\right) + \int_{0}^{\frac{1-x}{1-y}} \frac{dt}{1 - t} \ln(1 - (1 - y)t)
= \ln y \ln \left(\frac{x - y}{1 - y}\right) - \ln x \ln \left(\frac{y - x}{y}\right) - \text{Li}_2 \left(\frac{x}{y}\right) + \text{Li}_2 \left(\frac{1}{y}\right)
\]

Applying now two well-known transformation rules for dilogarithms (cf. [8]):
\[
\text{Li}_2(z) = -\text{Li}_2(1 - z) - \ln(z) \ln(1 - z) + \frac{\pi^2}{6} \quad (A.13)
\]
\[
\text{Li}_2(z) = -\text{Li}_2 \left(\frac{1}{z}\right) - \frac{1}{2} [\ln(-z)]^2 - \frac{\pi^2}{6} \quad (A.14)
\]
allows us to transform the dilogarithms into the form of (19).
References

[1] D. Kreimer. Dissertation. Univ. Mainz (1992)

[2] J. Franzkowski. Diploma thesis. Univ. Mainz (1992)

[3] D. Kreimer. Z. Phys. C54 (1992) 667

[4] D. Kreimer. Int. J. Mod. Phys. A8 No.10 (1993) 1797

[5] B. C. Carlson: Special Functions of Applied Mathematics
   Academic Press (1977)

[6] M. Schmitt. private communication

[7] R. Lewin: Polylogarithms and Associated Functions
   North Holland (1981)

[8] G.’t Hooft, M. Veltman. Nucl. Phys. B153 (1979) 365

[9] G. Passarino, M. Veltman. Nucl. Phys. B160 (1979) 151

[10] A. P. Prudnikov, Y. A. Brychkov, O. I. Marichev: Integrals and Series, Vol. 1
    Gordon and Breach Science Publishers (1986)