QUASI-LIE BIALGEBROIDS AND TWISTED POISSON MANIFOLDS.

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Abstract. We develop a theory of quasi-Lie bialgebroids using a homological approach. This notion is a generalization of quasi-Lie bialgebras, as well as twisted Poisson structures with a 3-form background which have recently appeared in the context of string theory, and were studied by Ševera and Weinstein using a different method.

1. Introduction.

The purpose of this note is twofold: to develop a theory of quasi-Lie bialgebroids and to use it to treat twisted Poisson manifolds with a closed 3-form background. The latter appeared recently in works of Park [14], Cornalba-Schiappa [2] and Klimčík-Strobl [5] on string theory, and were studied by Ševera and Weinstein [17] using Courant algebroids and Dirac structures. In the present work we present an alternative approach along the lines of [7] and [16]; it can be viewed as a companion to [17].

The notion of a Lie bialgebroid was introduced by Mackenzie and Xu in [13] where it appeared as linearization of Poisson groupoids. It consists of a pair of Lie algebroid structures on dual vector bundles $A$ and $A^*$ satisfying a compatibility condition. In [16] we developed an approach to the theory of Lie bialgebroids in terms of a pair of mutually commuting Hamiltonians on the symplectic supermanifold $\mathcal{E} = T^*\Pi A = T^*\Pi A^*$. The algebra of functions on $\mathcal{E}$ has a natural double grading, and the properties of Lie bialgebroids follow easily from an interplay of elementary symplectic geometry and homological algebra. In particular, the sum of the two Hamiltonians is self-commuting, hence it defines a homological (Hamiltonian) vector field $D$ on $\mathcal{E}$; the pair $(\mathcal{E}, D)$ is called the homological double of the Lie bialgebroid $(A, A^*)$. The Courant algebroid structure on $A \oplus A^*$ originally proposed by Liu et al. [11] as a “Drinfeld double” of $(A, A^*)$ is recovered from the homological double by Kosmann-Schwarzbach’s derived bracket construction [8]. This generalizes the homological approach to Lie bialgebras pioneered by Leconte and Roger [14], in which case it reduces to pure algebra.

In the same work [16] we remarked that a skew-symmetric trilinear form on $A$ or $A^*$ (viewed as a function on $\mathcal{E}$) can be added to the sum of the above two Hamiltonians; requiring the total sum to self-commute leads to the notion of a quasi-Lie bialgebroid, generalizing Kosmann-Schwarzbach’s quasi-Lie bialgebras treated in [7] using the same homological approach. We observed that a quasi-Lie bialgebroid also has a homological double which produces a Courant algebroid structure on $A \oplus A^*$. Aside from these observations, the theory of quasi-Lie bialgebroids lay dormant for a while due to a lack of interesting examples. The situation changed several months
ago when twisted Poisson manifolds were brought to our attention by Alan Weinstein. These are manifolds equipped with a bivector field $\pi$ and a closed 3-form $\phi$ (the background field) satisfying the equation

\begin{equation}
\frac{1}{2}[\pi, \pi] = \wedge^3 \pi \phi.
\end{equation}

Here, and elsewhere in this note, given a bilinear form $B$ on a vector space $V$, we shall denote by $\tilde{B}$ the corresponding map from $V$ to $V^*$ defined by $\tilde{B}(\alpha)(\beta) = \frac{1}{2} <\alpha, \beta>$, while $\pi$ is determined by $\tilde{\pi}(g) = \frac{2(Ad_g - 1)}{(Ad_g + 1)}$, where we identify both the tangent and cotangent space at $g$ with the Lie algebra $g$ via left translation and $<\cdot, \cdot>$.

The realization that twisted Poisson manifolds provide examples of quasi-Lie bialgebroids gave impetus for developing the general theory which resulted in the present note. The note is organized as follows. In Section 2 we briefly recall the notion of Lie bialgebroid emphasizing the approach of [16]. In Section 3 we define a quasi-Lie bialgebroid structure on $(A,A^*)$ as a triple of Hamiltonians $\mu$, $\gamma$ and $\phi$ on $E$ of respective degrees $(1,2)$, $(2,1)$ and $(0,3)$, such that their sum $\Theta$ Poisson-commutes with itself. We unravel the resulting algebra by showing that such structures correspond to differential quasi-Gerstenhaber algebra structures on $\Gamma(\wedge \cdot A)$ (a term coined by Huebschmann [4] to denote a special case of homotopy Gerstenhaber algebras), or to quasi-differential Gerstenhaber algebra structures on $\Gamma(\wedge \cdot A^*)$. We define the homological double as $(E, D = \{\Theta, \cdot\})$ and remark on the existence of a spectral sequence converging to the cohomology of $D$.

In Section 4 we study the phenomenon of twisting (introduced in [3] for quasi-Hopf algebras and in [7] for quasi-Lie bialgebras). Here again the symplectic geometry of $E$ yields a clear understanding of the phenomenon and allows for a quick and easy derivation of formulas. Specifically, the twisting by an $\omega \in \Gamma(\wedge^2 A^*)$ or by $\pi \in \Gamma(\wedge^2 A)$ (thought of as functions on $E$) is the canonical transformation given by the flow of the Hamiltonian vector field $X_\omega = \{\omega, \cdot\}$ (resp. $X_{\pi} = \{\pi, \cdot\}$). It is immediately seen that the twisting by $\omega$ transforms a quasi-Lie bialgebroid structure on $(A,A^*)$ into a new one (with isomorphic double), whereas the twisting by $\pi$ yields a quasi-Lie bialgebroid provided $\pi$ satisfies the twisted (non-homogeneous) Maurer-Cartan equation. In case of a Lie bialgebroid, the Maurer-Cartan equation is homogeneous; it was obtained in [11] by a different method.

In Section 5 we study the action of the natural group of symmetries: canonical transformations of $E$ preserving the grading, and particularly the subgroup $G$ that acts trivially on $M \subset E$, referred to as the gauge group. We obtain a factorization of $G$ similar to that used in the theory of Poisson-Lie groups. Also in this context we briefly mention deformation theory governed by the quasi-Gerstenhaber algebra associated to a quasi-Lie bialgebroid; the appropriate structure equation is the twisted Maurer-Cartan equation.

Lastly, in Section 6 we apply the theory to two special cases: arbitrary bivector fields and twisted Poisson manifolds with a 3-form background. We take $A$ to be the tangent bundle of a manifold $M$. In the former case we start with the standard Lie bialgebroid structure on $(TM, T^*M)$ and twist by a bivector field $\pi$. The result is a quasi-Lie bialgebroid structure on $(T^*M, TM)$ which is a Lie bialgebroid if and only
if \([\pi, \pi] = 0\). In the latter case, we start by adding a closed 3-form \(\phi\) to the standard structure on \((TM, T^*M)\), and then twist by \(\pi\) (this is the way Park \cite{14} arrived at twisted Poisson manifolds in string-theoretic context). The result is again a quasi-Lie bialgebroid provided the twisted Maurer-Cartan equation holds, which in this case reduces to \([1,1]\). This extends the observation of Ševera and Weinstein that a twisted Poisson manifold defines a Lie algebroid structure on \(T^*M\). The formulas obtained in \cite{17} are recovered as a special case of the general formulas from Section 3. We also interpret the local action of the group of 2-forms introduced in \cite{17} in terms of the factorization of the gauge group \(G\).

One issue that we have avoided in this note is quantization. Park \cite{14} argues on physical grounds that deformation quantization of \((E, \Theta)\) is related to Deligne’s conjecture (now a theorem of Kontsevich) on the formality of 2-algebras; BV quantization of the corresponding topological field theory would produce explicit formulas (Park also considers higher \(p\)-algebras). These issues are beyond the scope of the present note and will be approached elsewhere.

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2. Lie bialgebroids and their doubles.

Given a vector bundle \(A \to M\), a manifold, we denote by \(\Pi A\) the supermanifold whose algebra of functions is \(\Gamma(\bigwedge A^*)\). Of course, \(\Pi A\) is also a vector bundle over \(M\) (in fact, sometimes the notation \(A[1]\) is used instead of \(\Pi A\) to emphasize the nonnegative integer grading coming from the bundle structure, rather than just the parity shift).

A Lie algebroid structure on \(A\) is a homological vector field on \(\Pi A\) of degree +1, i.e. a derivation \(d_A\) of \(\Gamma(\bigwedge A^*)\) increasing degrees by 1 and satisfying \([d_A, d_A] = 2d_A^2 = 0\). Alternatively (and equivalently), one can define a Lie algebroid structure on \(A\) as an odd Poisson bracket of degree \(-1\) on \(C^\infty(\Pi A^*) = \Gamma(\bigwedge A)\), usually referred to as a Schouten or Gerstenhaber bracket. We denote this bracket by \([\cdot, \cdot]\). The usual definition in terms of an anchor and bracket on sections of \(A\) is recovered easily from either of these.

We say that \((A, A^*)\) is a Lie bialgebroid if both \(A\) and \(A^*\) are Lie algebroids and in addition, the vector field \(d_A\) preserves (is a derivation of) \([\cdot, \cdot]_{A^*}\). If \(A\) is a Lie algebroid, \((A, A^*)\) becomes a Lie bialgebroid in a trivial way if we endow \(A^*\) with the zero algebroid structure. A good nontrivial example of a Lie bialgebroid is \((T^*M, TM)\), where \(M\) is a Poisson manifold. Here \([\cdot, \cdot]_{A^*} = [\cdot, \cdot]\), the usual Schouten bracket of multivector fields, while \(d_A = d_\pi = [\pi, \cdot]\), where \(\pi \in \Gamma(\bigwedge^2 TM)\) is the Poisson tensor. The fact that \(d_\pi^2 = 0\) is equivalent to \([\pi, \pi] = 0\); the compatibility condition is obvious. The anchor on \(T^*M\) is \(\tilde{\pi}\), while the bracket of 1-forms is the Koszul bracket:

\[
[\alpha, \beta]_\pi = \mathcal{L}_{\tilde{\pi}\alpha} \beta - \mathcal{L}_{\tilde{\pi}\beta} \alpha - d(\pi(\alpha, \beta)).
\]

The following construction gives a completely symmetric picture of Lie bialgebroids and also enables one to define the “double” of a Lie bialgebroid, generalizing the Drinfeld double of a Lie bialgebra. Given a vector bundle \(A \to M\),
let $\mathcal{E} = T^*\Pi A$. It is an even symplectic supermanifold. Local coordinates \{x^i\} on $M$ and a local basis \{e_a\} of sections of $A$ give rise to Darboux coordinates \{x^i, \xi^a, p_i, \theta_a\} on $\mathcal{E}$. The symplectic form is $\Omega = dx^i dp_i + d\xi^a d\theta_a$. There is a canonical symplectomorphism, the Legendre transformation $L : T^*\Pi A \to T^*\Pi A^*$ \[1\]; in the above coordinates, it simply exchanges $\xi^a$ with $\theta_a$. Thus, $\mathcal{E}$ fibres as the cotangent bundle over both $\Pi A$ and $\Pi A^*$. It fits into the following diagram:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\bar{h}} & \Pi A^* \\
\bar{v} \downarrow & & \downarrow v \\
\Pi A & \xrightarrow{\bar{h}} & M
\end{array}
\]

This diagram is an example of a double vector bundle \[13, 12, 6\] i.e. each arrow is a vector bundle, and both the horizontal and the vertical pairs of arrows are vector bundle morphisms. The double vector bundle structure gives rise to a double (nonnegative integer) grading of functions on the supermanifold $\mathcal{E}$. Each grading separately is not compatible with parity, but their sum, the total weight, is. Whenever we speak of functions on $\mathcal{E}$, we shall mean polynomials with respect to the total grading; we denote this algebra by $\mathcal{C}$, or $\mathcal{C}^\cdot$ when we want to emphasize the double grading. Notice that the canonical symplectic form on $\mathcal{E}$ is of bi-degree $(1, 1)$, and thus of total weight 2; the canonical Poisson bracket has bi-degree $(-1, -1)$ and total weight $-2$. The core of $\mathcal{E}$ is $\ker \bar{h} \cap \ker \bar{v}$, where $\bar{h}$ (resp. $\bar{v}$) is viewed as a vector bundle morphism over $h$ (resp. $v$). It is a vector bundle over $M$, although the whole of $\mathcal{E}$ is not. In the present case the core is easily seen to be $T^*M$ (more precisely, $T^*[2]M$ if the weight is taken into account); it is also the support of the supermanifold $\mathcal{E}$.

We can also consider $E = A \oplus A^*$ and the corresponding supermanifold $\Pi E$. It is an even Poisson manifold; the fibres of $\Pi E$ are the symplectic leaves, and the basic functions are Casimir. Since $\Pi E$ is the fibre product of $\Pi A$ and $\Pi A^*$ over $M$, by the universal property there is a canonical fibration $p : \mathcal{E} \to \Pi E$. This fibration is easily seen to be a Poisson map; in fact, $\mathcal{E}$ is a minimal symplectic realization of $\Pi E$ in the sense that the extra dimension is equal to the co-rank of the Poisson tensor on $\Pi E$. The corresponding doubly graded Poisson subalgebra of functions is $\mathcal{C}^{\cdot\cdot} = \Gamma(\bigwedge \mathcal{A} \otimes \bigwedge \mathcal{A}^*)$, with $\mathcal{C} = \Gamma(\bigwedge E^*)$. The Poisson bracket on $\mathcal{C}$ is the “big bracket” \[10, 7\] applied pointwise over $M$: it is the unique extension of the canonical pairing of $A$ and $A^*$ to a bi-derivation of the exterior algebra of sections of $E$ (essentially, just contraction of tensors). Notice that $\mathcal{C}^{0\cdot} = \mathcal{C}^{0\cdot}$, and $\mathcal{C}^{\cdot0} = \mathcal{C}^{\cdot0}$. If $M$ is a point, $\mathcal{E}$ and $\Pi E$ are the same.

Notice finally that both $\Pi A$ and $\Pi A^*$ sit inside $\mathcal{E}$ as the zero sections of, respectively, $\bar{v}$ and $\bar{h}$; they are, of course, Lagrangian submanifolds of $\mathcal{E}$. We denote these distinguished Lagrangian submanifolds by $L$ and $L^*$, respectively. More generally, by elementary symplectic geometry, $\bar{v}$-projectable graded Lagrangian submanifolds are of the form $d\omega(\Pi A)$, where $d\omega$ is the exterior derivative of a quadratic function $\omega \in \Gamma(\bigwedge^2 A^*)$, viewed as a section of $\bar{v}$. Similarly, $\bar{h}$-projectable graded Lagrangian submanifolds are of the form $d\pi(\Pi A^*)$ for some $\pi \in \Gamma(\bigwedge^2 A)$. Given $\omega$ (resp. $\pi$), we denote the corresponding Lagrangian submanifold $L_\omega$ (resp. $L^\pi_\omega$). The projection $p : \mathcal{E} \to \Pi E$ maps $L_\omega$ (resp. $L^\pi_\omega$) onto the graph of $\omega$ (resp. $\pi$) in $A \oplus A^*$. In general, graded Lagrangian submanifolds that contain $M$ correspond under $p$ to maximally isotropic subbundles of $E$ that are not necessarily graphs.
Now, a Lie algebroid structure on $A$ is equivalent to a function $\mu$ on $E$ of bi-degree $(1,2)$ satisfying
\begin{equation}
\{\mu, \mu\} = 0 \tag{2.3}
\end{equation}
The homological vector field on $\Pi A$ and the Schouten bracket on $\Pi A^*$ are recovered as follows. For a function $\alpha$ on $\Pi A$,
\[\bar{v}^*(d\mu \alpha) = \{\mu, \bar{v}^* \alpha\},\]
and for two functions $X, Y$ on $\Pi A^*$,
\[\bar{h}^*([X, Y]_\mu) = \{\{\bar{h}^* X, \mu\}, \bar{h}^* Y\}\]
the so-called derived bracket; as $\{\bar{h}^* X, h^* Y\} = 0 \forall X$ and $Y$, the derived bracket is a graded Lie algebra bracket (for details see [8], where the notion was first introduced and studied). In what follows we shall suppress the pullback notation and write $\alpha$ instead of $\bar{v}^* \alpha$ and so on, when the meaning is clear.

Likewise, a Lie algebroid structure on $A^*$ is equivalent to a function $\gamma$ on $E$ of bi-degree $(2,1)$ satisfying
\begin{equation}
\{\gamma, \gamma\} = 0 \tag{2.4}
\end{equation}
It gives a differential $d_\gamma$ on $\Pi A^*$ and a Schouten bracket $[\cdot, \cdot]_\gamma$ on $\Pi A$ by analogous formulas. Moreover, it was proved in [16] that $(A, A^*)$ is a Lie bialgebroid if and only if, in addition,
\begin{equation}
\{\mu, \gamma\} = 0 \tag{2.5}
\end{equation}
It follows that the notion of Lie bialgebroid is self-dual, i.e. $(A, A^*)$ is a Lie bialgebroid if and only if $(A^*, A)$ is; furthermore, $\Theta = \mu + \gamma$ (of total weight 3) satisfies
\begin{equation}
\{\Theta, \Theta\} = 0 \tag{2.6}
\end{equation}
if and only if (2.3), (2.4) and (2.5) hold. Therefore, in this case its Hamiltonian vector field $D = \{\Theta, \cdot\}$ is homological of degree $+1$ on $E$. The pair $(E, D)$ is called the homological double of the Lie bialgebroid $(A, A^*)$.

Remark 2.1. In view of (2.3), the Hamiltonian vector field $\{\mu, \cdot\}$ is homological of bi-degree $(0, +1)$. Thus, for each fixed $k \geq 0$, $(C^k, \{\mu, \cdot\})$ is a differential complex. For $k = 0$ it is just the standard complex of the Lie algebroid $A$ with coefficients in the trivial representation; for higher $k$ it should be regarded as the standard complex of $A$ with coefficients in the $k$-th exterior power of the adjoint representation, to which it reduces when the base $M$ is a point. In particular, (2.3) implies that $\gamma$ is a 1-cocycle on $A$ with coefficients in the exterior square of the adjoint, which is exactly the way in which Lie bialgebras were originally defined. Similarly, $(C^l, \{\gamma, \cdot\})$ is a complex for each $l \geq 0$, and due to (2.4) $(C^\cdot, \{\mu, \cdot\}, \{\gamma, \cdot\})$ is a double complex whose total complex is $(C^\cdot, D = \{\Theta, \cdot\})$. For Lie bialgebras, this double complex and its spectral sequence were studied in [10]. For Lie bialgebroids (e.g. Poisson

\[\text{This was independently proved by A. Vaintrob (unpublished)}\]
manifolds), investigating the spectral sequence could yield interesting results. It will be carried out in a separate paper.

**Remark 2.2.** We can observe, by counting degrees, that the space of functions on $E$ of degree $\leq 1$ is closed under both the Poisson bracket $\{\cdot, \cdot\}$ and the derived bracket $\{\cdot, \Theta, \cdot\}$. These functions correspond under the projection $p$ to the canonical inner product on $A \oplus A^*$, and if $M$ is a point, the derived bracket of linear functions is precisely Drinfeld’s double Lie bracket. In general, the derived bracket is not skew-symmetric; the resulting structure on $E$ coincides (after skew-symmetrization) with the Courant algebroid introduced by Liu, Weinstein and Xu \[11\] as a candidate for the double of $(A, A^*)$; the Courant algebroid axioms follow easily from \(2.6\) (see \[16\] for details). The space of all functions on $\Pi E$ is not closed under the derived bracket, but on functions on $E$ we get a Loday-Gerstenhaber algebra structure.

### 3. Quasi-Lie bialgebroids and quasi-Gerstenhaber algebras.

The most general cubic function $\Theta$ on $E$ is of the form $\Theta = \mu + \gamma + \phi + \psi$; it contains terms $\phi$ of degree $(0, 3)$ and $\psi$ of degree $(3, 0)$, in addition to terms $\mu$ and $\gamma$ considered above (then $\phi \in \Gamma(\wedge^3 A^*)$, while $\psi \in \Gamma(\wedge^3 A)$). If $\Theta$ obeys the structure equation \(2.6\), we get a so-called *proto Lie bialgebroid* structure on $(A, A^*)$ (\[16\]), generalizing \[7\]). Splitting the quartic $\{\Theta, \Theta\}$ into components according to the double grading, we get the following set of equations:

\[
\begin{align*}
\frac{1}{2} \{\mu, \mu\} + \{\gamma, \phi\} &= 0 \\
\frac{1}{2} \{\gamma, \gamma\} + \{\mu, \psi\} &= 0 \\
\{\mu, \gamma\} + \{\phi, \psi\} &= 0 \\
\{\mu, \phi\} &= 0 \\
\{\gamma, \psi\} &= 0
\end{align*}
\]

This structure is rather complicated, but it still has a homological double $(E, D)$ which produces on $E = A \oplus A^*$ a Courant algebroid structure (see \[16\]). We shall mainly restrict our attention to the case when at least one of the additional terms (say, $\psi$) vanishes. Then the equations above reduce to:

\[
\begin{align*}
\frac{1}{2} \{\mu, \mu\} + \{\gamma, \phi\} &= 0 \\
\{\gamma, \gamma\} &= 0 \\
\{\mu, \gamma\} &= 0 \\
\{\mu, \phi\} &= 0
\end{align*}
\]

By definition, the pair of dual vector bundles $(A, A^*)$ together with $\mu$, $\gamma$ and $\phi$ of degree $(1, 2)$, $(2, 1)$ and $(0, 3)$, respectively, satisfying \(3.2\), is a *quasi-Lie bialgebroid*; we call $(E, D = \{\Theta, \cdot\})$ its homological double.

Notice that quasi-Lie bialgebroids can be characterized as those self-commuting cubic Hamiltonians on $E$ that vanish on the distinguished graded Lagrangian submanifold $L^*$, the image of the zero section $\Pi A^* \to T^* \Pi A^*$. Thus, a quasi-Lie bialgebroid can be equivalently described by a *Manin pair* $(E, \Theta, L^*)$. Lie bialgebroids correspond to those $\Theta$ that also vanish on the other distinguished Lagrangian submanifold $L$ (the image of $\Pi A \to T^* \Pi A$), and so can be described by *Manin triples* $(E, \Theta, L, L^*)$. 
Let us now unravel the resulting algebraic structure. First of all, $\gamma$ defines a Lie algebroid structure on $A^*$, i.e. a derivation $d = d_\gamma$ of $\Gamma(A)$ of degree $+1$ and square zero (due to $\{\gamma,\gamma\} = 0$). On the other hand, $\mu$ induces a bracket $\{\cdot,\cdot\}_\mu$ on $\Gamma(A)$ of degree $-1$, completely determined by an anchor map $a: A \to TM$ and a bracket on sections of $A$. The compatibility condition $\{\mu,\gamma\} = 0$ between $\mu$ and $\gamma$ means that $d$ is a derivation of $\{\cdot,\cdot\}_\mu$. The equation $\frac{1}{2}\{\mu,\mu\} + \{\gamma,\phi\} = 0$ means that the usual Lie algebroid relations are satisfied only up to certain “defects” depending on $\gamma$ and $\phi$.

Specifically, for all $X, Y \in \Gamma(A)$,
\[
(3.3) \quad a([X,Y]) = [a(X),a(Y)] + a_\phi(X,Y)
\]
where $a_\phi$ is the anchor of the Lie algebroid $A^*$ and $\phi(X,Y) = i_{X,Y}\phi \in \Gamma(A^*)$, and for all $X, Y, Z \in \Gamma(A)$,
\[
(3.4) \quad [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = \\
\quad = d(\phi(X,Y,Z)) + \phi(dX,Y,Z) - \phi(X,dY,Z) + \phi(X,Y,dZ)
\]
where $\phi$ is viewed as a bundle map $\wedge^4 A \to A$. Finally, $\{\mu,\phi\} = 0$ is a coherence condition between $\{\cdot,\cdot\}$ and $\phi$: for all $X, Y, Z, W \in \Gamma(A)$,
\[
(3.5) \quad [(\phi(X,Y,Z),W) - [\phi(X,Y,W),Z] + [\phi(X,Z,W),Y] - [\phi(Y,Z,W),X]) - \\
\quad - (\phi([X,Y],Z,W) - \phi([X,Z],Y,W) + \phi([X,W],Y,Z) + \\
\quad + \phi([Y,Z],X,W) - \phi([Y,W],X,Z) + \phi([Z,W],X,Y)) = 0
\]
Let us define, for each positive integer $k$, a $k$-linear map $l_k$ on $\Gamma(A)[1]$ with values in itself, of degree $2-k$, as follows. Let $l_1 = d_1$, $l_2 = [\cdot,\cdot]$, $l_3 = \phi$ (more precisely, we let $l_3$ vanish if one of the arguments is a function, let $l_3(X,Y,Z) = \phi(X,Y,Z)$ for $X, Y, Z \in \Gamma(A)$, and then extend it as a derivation in each argument), and $l_k = 0$ for $k > 3$. Then the above relations imply that $(\Gamma(A)[1],\{l_k\})$ is a strongly homotopy Lie (or $L_\infty$) algebra. Moreover, since each $l_k$ is a $k$-derivation of the exterior multiplication, what we get is in fact a strongly homotopy Gerstenhaber (or $G_\infty$) algebra. Such $G_\infty$-algebras are very simple compared to the most general possible case: the exterior multiplication remains undeformed and each $l_k$ is a strict derivation. They were called differential quasi-Gerstenhaber algebras by Huebschmann. It is easy to see that, conversely, any $G_\infty$-algebra on $\Gamma(A)$ satisfying these conditions must come from a quasi-Lie bialgebroid. To summarize: quasi-Lie bialgebroid structures on $(A,A^*)$ are in 1-1 correspondence with quasi-Gerstenhaber algebra structures on $\Gamma(A)$.

Dually, a quasi-Lie bialgebroid structure on $(A,A^*)$ is also equivalent to a quasi-differential Gerstenhaber algebra structure on $\Gamma(A)$: the Gerstenhaber bracket $\{\cdot,\cdot\}_\gamma$ is given by $\gamma$ (it satisfies graded Jacobi strictly), while $\mu$ gives a derivation (quasi-differential) $d_\mu$ of $\{\cdot,\cdot\}_\gamma$ of degree $+1$, which does not square to 0 but rather satisfies
\[
(3.6) \quad d_\mu^2 + [\phi,\cdot]_\gamma = 0
\]
Furthermore, $d_\mu\phi = 0$.

**Remark 3.1.** The equations (3.2) imply that $d_\mu = \{\mu,\cdot\}$ induces a differential on the cohomology of $(C^\cdot,d_\gamma)$ and in fact, there still exists a spectral sequence converging to the cohomology of $(C^\cdot,D = \{\Theta,\cdot\})$.

Of course if we replace a $\phi$ of bi-degree $(0,3)$ by a $\psi$ of bi-degree $(3,0)$, the roles of $\mu$ and $\gamma$ are reversed: we get a differential quasi-Gerstenhaber algebra.
on $\Gamma(\wedge^* A^*)$ and a dual quasi-differential Gerstenhaber algebra on $\Gamma(\wedge^* A)$, i.e. a quasi-Lie bialgebroid structure on $(A^*, A)$. When the base $M$ is a point, we recover various quasi-bialgebras studied by Kosmann-Schwarzbach [3].

4. Fibre translations and twisting.

Let $\omega \in \Gamma(\wedge^2 A^*)$. When pulled back to $\mathcal{E}$, $\omega$ has bi-degree $(0, 2)$. Its Hamiltonian vector field $X_\omega = \{\omega, \cdot\}$ is thus of bi-degree $(-1, 1)$. Hence, the action of $X_\omega$ on $\mathcal{E}$ preserves the total weight but not the double grading. The flow of $X_\omega$ is the fibre translation along $-d\omega$ with respect to the fibration $\tilde{v}: \mathcal{E} = T^*\Pi A \to \Pi A$. Let $F_\omega$ be the time 1 map of the flow. The corresponding pullback of functions can be expressed as

$$F_\omega^* = \exp X_\omega = 1 + X_\omega + \frac{1}{2}X_\omega^2 + \frac{1}{6}X_\omega^3 + \cdots$$

In coordinates, $\omega = \frac{1}{2}\omega_{ab}(x)\xi^a\xi^b$, and $F_\omega$ is given by:

$$\begin{align*}
\tilde{x}^i &= x^i \\
\tilde{\xi}^a &= \xi^a \\
\tilde{p}_i &= p_i - \frac{1}{2}\partial_{\xi^b}\omega_{ab}\xi^b \\
\tilde{\theta}_a &= \theta_a - \omega_{ab}\xi^b
\end{align*}$$

(4.1)

We are going to apply this fibre translation to a cubic hamiltonian $\Theta = \mu + \gamma + \phi + \psi$. Let $\Theta_\omega = F_\omega^*\Theta = (\exp X_\omega)\Theta$; then $\Theta_\omega$ is again cubic, and it satisfies (2.6) if and only if $\Theta$ does (since the flow acts by canonical transformations). We say that $\Theta_\omega$ is the twisting of $\Theta$ by $\omega$. Notice that the above exponential series gets truncated when applied to a function of finite degree. In fact, $\Theta_\omega = \mu_\omega + \gamma_\omega + \phi_\omega + \psi_\omega$ where

$$\begin{align*}
\mu_\omega &= \mu + X_\omega^\gamma + \frac{1}{2}X_\omega^2\psi = \mu + h_\gamma + \lambda^2\hat{\omega}\psi \\
\gamma_\omega &= \gamma + X_\omega^\gamma = \gamma + \hat{\omega}\psi \\
\phi_\omega &= \phi + X_\omega^\mu = \phi + \lambda^2\hat{\omega}\psi \\
\psi_\omega &= \psi
\end{align*}$$

(4.2)

Here $\hat{\omega} : A \to A^*$ is used to lower the specified number of indices on $\psi$, whereas $h_v$ denotes the linear hamiltonian on $\mathcal{E} = T^*\Pi A$ corresponding to a vector field $v$ on $\Pi A$. These formulas follow easily from the definitions.

Notice that for a quasi-Lie bialgebroid structure on $(A, A^*)$ (i.e. when $\psi = 0$) the twisting by $\omega$ automatically produces a new quasi-Lie bialgebroid structure. On the other hand, for a quasi-Lie bialgebroid structure on $(A^*, A)$ (i.e. when $\phi = 0$), the twisting yields a quasi-Lie bialgebroid provided $\omega$ satisfies a certain integrability condition, the twisted Maurer-Cartan equation:

$$d\mu_\omega + \frac{1}{2}[\omega, \omega]_{\gamma} = \lambda^3\hat{\omega}\psi$$

(4.3)

We can also twist by elements $\pi \in \Gamma(\wedge^2 A)$ (i.e. of bi-degree $(2, 0)$). In this case $X_\pi = \{\pi, \cdot\}$ is of bi-degree $(1, -1)$, $F_\pi = \exp X_\pi$ is the fibre translation by $-d\pi$ with respect to the fibration $\tilde{h} : \mathcal{E} = T^*\Pi A^* \to \Pi A^*$, given in coordinates by

$$\begin{align*}
\tilde{x}^i &= x^i \\
\tilde{\theta}_a &= \theta_a \\
\tilde{p}_i &= p_i - \frac{1}{2}\partial_{\theta_a}\theta_{ab}\theta_b \\
\tilde{\xi}^a &= \xi^a - \pi_{ab}\theta_b
\end{align*}$$

(4.4)
where \( \pi = \frac{1}{2} \pi^{ab}(x) \theta_a \theta_b \). The twisting of \( \Theta \) by \( \pi \) is given by \( \Theta_\pi = F^*_\pi \Theta = (\exp X_\pi) \Theta = \mu_\pi + \gamma_\pi + \phi_\pi + \psi_\pi \), where

\[
\begin{align*}
\mu_\pi &= \mu + X_\pi \phi = \mu + \tilde{\pi} \phi \\
\gamma_\pi &= \gamma + X_\pi \mu + \frac{1}{2} X_\pi^2 \phi + \lambda^2 \tilde{\pi} \phi \\
\phi_\pi &= \phi \\
\psi_\pi &= \psi + X_\pi \gamma + \frac{1}{2} X^2_\pi \mu + \frac{2}{3} X^3_\pi \phi = \psi - d_\pi - \frac{1}{2} [\pi, \pi]_\mu + \wedge^3 \tilde{\pi} \phi
\end{align*}
\]

(4.5)

Again \( \Theta_\pi \) satisfies (2.1) if \( \Theta \) does. In particular, for \( \phi = 0 \) this automatically produces a new quasi-Lie bialgebroid on \( (A^*, A) \), while for \( \psi = 0 \) we get a quasi-Lie bialgebroid on \( (A, A^*) \) provided \( \pi \) obeys the twisted Maurer-Cartan equation

\[
d_\pi + \frac{1}{2} [\pi, \pi]_\mu = \wedge^3 \tilde{\pi} \phi
\]

(4.6)

Remark 4.1. It may be worth mentioning that a \( \pi \in \Gamma(\wedge^2 A) \) generalizes "r-matrices" from the Lie bialgebra theory, while \( \omega \in \Gamma(\wedge^2 A^*) \) plays the dual role. The twisted Maurer-Cartan equation (4.6) (resp. (4.3)) is a sufficient but in general not a necessary condition to get a quasi-Lie bialgebroid on \( (A^*, A) \) (resp. \( (A, A^*) \)), as evidenced already by the Lie bialgebra case. Some examples where (4.6) is in fact necessary are considered in Section 6.

From the above formulas it is not difficult to deduce the expressions for the twisted (quasi-) differentials and brackets. For \( \Theta_{\omega} \),

\[
\begin{align*}
d_{\mu, \omega} &= d_\mu + [\omega, \cdot]_\gamma + \iota_{\lambda^2 \omega} \psi \\
d_{\gamma, \omega} &= d_\gamma + \iota_{\omega} \psi \\
[\alpha, \beta]_{\gamma, \omega} &= [\alpha, \beta]_\gamma + \omega \psi(\alpha, \beta) \\
[X, Y]_{\mu, \omega} &= [X, Y]_\mu + [X, Y]_\gamma, \omega + \lambda^2 \omega \psi(X, Y)
\end{align*}
\]

(4.7)

where \( \iota \) denotes the contraction operator, while

\[
[X, Y]_{\gamma, \omega} = \mathcal{L}_{\omega, X} Y - \mathcal{L}_{\omega, Y} X - d_\gamma (\omega(X, Y))
\]

is a version of the Koszul bracket. Here \( X, Y \in \Gamma(A) \), \( \alpha, \beta \in \Gamma(A^*) \) and \( \mathcal{L}_\omega = [\iota_\alpha, d_\gamma] = \iota_\alpha d_\gamma + d_\gamma \iota_\alpha \). Similarly, for \( \Theta_\pi \),

\[
\begin{align*}
d_{\mu, \pi} &= d_\mu + \iota_{\pi} \phi \\
\iota_{\gamma, \pi} &= d_\gamma + [\pi, \cdot]_\mu + \iota_{\lambda^2 \pi} \phi \\
[\alpha, \beta]_{\gamma, \pi} &= [\alpha, \beta]_\gamma + [\alpha, \beta]_\mu, \pi + \lambda^2 \pi \phi(\alpha, \beta) \\
[X, Y]_{\mu, \pi} &= [X, Y]_\mu + \pi \phi(X, Y)
\end{align*}
\]

(4.8)

where

\[
[\alpha, \beta]_{\mu, \pi} = \mathcal{L}^\mu_{\pi, \alpha} \beta - \mathcal{L}^\mu_{\pi, \beta} \alpha - d_\mu (\pi(\alpha, \beta))
\]

is a Koszul bracket (compare with (2.1)).

Remark 4.2. The twisting transformations \( F_\omega \) (resp. \( F_\pi \)) induce the change of splitting of \( E = A \oplus A^* \) by the graph of \( \omega \) (resp. \( \pi \)), as can be seen from the formulas (4.1) (resp. (4.3)). This procedure produces new (proto-, quasi-) Lie bialgebroids whose double remains isomorphic to the original one. The twisted Maurer-Cartan equation (4.3) (resp. (4.6)) is the condition for the graph of \( \omega \) (resp. \( \pi \)) to be a Dirac structure with respect to the Courant algebroid given by \( \Theta \). In the Lie bialgebroid case \( (\phi = \psi = 0) \) we recover the strict Maurer-Cartan equation appearing in [1].
5. Symmetries, gauge transformations and deformations.

The full algebra of symmetries of \( \mathcal{E} \) (canonical transformations preserving the total weight) is the Lie algebra \( \mathcal{C}^2 = \mathcal{C}^{0,2} \oplus \mathcal{C}^{1,1} \oplus \mathcal{C}^{2,0} \) of quadratic hamiltonians, acting via Poisson brackets. It is easy to see that this action preserves the subalgebra \( \mathcal{C} \), hence \( \mathcal{C}^2 \) acts on the pseudo-Euclidean vector bundle \( E = A \oplus A^* \). In fact, \( \mathcal{C}^2 \) is isomorphic to the Atiyah algebra of \( E \), consisting of infinitesimal bundle transformations preserving the canonical pairing and covering vector fields on \( M \):

\[
0 \to \mathcal{C}^2 \to \mathcal{C}^2 \to Vect(M) \to 0
\]

where \( \mathcal{C}^2 = \mathcal{C}^{0,2} \oplus \mathcal{C}^{1,1} \oplus \mathcal{C}^{2,0} \) is isomorphic to the Lie algebra of endomorphisms of \( E \) preserving the pairing.

The structure of \( \mathcal{C}^2 \) is quite transparent: \( \mathcal{C}^{0,2} = \Gamma(\wedge^2 A^*) \) and \( \mathcal{C}^{2,0} = \Gamma(\wedge^2 A) \) are abelian subalgebras, acted upon by \( \mathcal{C}^{1,1} = \Gamma(A \otimes A^*) = End(A) \) in the standard way. The bracket of \( \mathcal{C}^{2,0} \) and \( \mathcal{C}^{0,2} \) ends up in \( \mathcal{C}^{1,1} \): \( \{\pi, \omega\} = \tilde{\pi} \tilde{\omega} \), viewed as an operator acting on \( A \).

The group \( \mathcal{G} \) corresponding to \( \mathcal{C}^2 \) will be referred to as the gauge group; it consists of sections over \( M \) of the bundle of Lie groups whose fibre is isomorphic to \( SO(n, n) \), where \( n \) is the rank of \( A \). From the above Lie algebra decomposition one expects a factorization of \( \mathcal{G} \) into the product \( \Gamma(\wedge^2 A^*) \times Aut(A) \times \Gamma(\wedge^2 A) \), but actually this is valid only on some open subset. Of particular interest is the factorization of the product of a \( \pi \in \Gamma(\wedge^2 A) \) and an \( \omega \in \Gamma(\wedge^2 A^*) \) (both viewed as elements of \( \mathcal{G} \)), which will be used in the next section:

\[
(5.1) \quad \pi \omega = (\tau_{\pi} \omega) T^{-1}(\pi, \omega)(\tau_{\omega} \pi)
\]

This factorization is valid if and only if \( T(\pi, \omega) = 1 + \tilde{\pi} \tilde{\omega} \) is invertible, in which case

\[
\tau_{\pi \omega} = \tilde{\omega} T^{-1} = \tilde{\omega}(1 + \tilde{\pi} \tilde{\omega})^{-1},
\]

while

\[
\tau_{\omega \pi} = \tilde{\pi}(T^t)^{-1} = \tilde{\pi}(1 + \tilde{\omega} \tilde{\pi})^{-1}.
\]

One easily checks that \( \tau_{n \pi} = \pi \) and \( \tau_{\omega_1 \omega_2 \pi} = \tau_{\omega_1} \omega_2 \pi \) (whenever the terms are defined), so one can speak of a local action of the additive group \( \Gamma(\wedge^2 A^*) \) on the space \( \Gamma(\wedge^2 A) \). This action is clearly nonlinear; in fact, it is generated by the infinitesimal action of the abelian Lie algebra \( \mathcal{C}^{0,2} \) by quadratic vector fields: \( \delta_{\omega} \pi = -\tilde{\pi} \tilde{\omega} \). This Lie algebra action does not integrate to a global group action, but the corresponding transformation Lie algebroid does integrate to a global groupoid; this groupoid induces an equivalence relation on the space \( \Gamma(\wedge^2 A) \). Similarly, one has a local action of \( \Gamma(\wedge^2 A) \) on \( \Gamma(\wedge^2 A^*) \). All this is reminiscent of dressing actions of Poisson-Lie groups.

The gauge group \( \mathcal{G} \) acts on \( \mathcal{E} \) and various objects that live there. In particular, it acts on the space of graded Lagrangian submanifolds, preserving the subspace consisting of those that contain \( M \) and so correspond (under the projection \( p : \mathcal{E} \to \Pi E \)) to maximally isotropic subbundles of \( A \oplus A^* \). For instance, given \( \omega, \omega' \in \Gamma(\wedge^2 A^*) \), \( \pi \in \Gamma(\wedge^2 A) \), we have \( F_\omega L_{\omega'} = L_{\omega + \omega'} \), while \( F_\omega L^*_{\tau_\omega \pi} = L^*_{\tau_{\omega} \pi} \) provided \( \tau_{\omega} \pi \) is defined (otherwise \( F_\omega L^*_{\tau_\omega \pi} \) is not \( \hbar \)-projectable). \( F_\pi \) acts in a similar fashion.

The subalgebra preserving \( L^* \) (corresponding to \( A^* \subset A \oplus A^* \)) is \( \mathcal{C}^{0,2} \oplus \mathcal{C}^{1,1} \), a semidirect product via the standard action of \( End(A) \) on \( \Gamma(\wedge^2 A^*) \). Therefore, the corresponding subgroup of \( \mathcal{G} \) acts on Manin pairs \( (\mathcal{E}, \Theta), L^* \) (quasi-Lie bialgebroids). The subgroup \( \Gamma(\wedge^2 A^*) \) fixes every point of \( L^* \); its action on quasi-Lie
bialgebroids by twisting was described in the previous section (formulas (4.3), with \( \psi = 0 \)).

On the other hand, the subspace \( \mathcal{C}^{2,0} \) plays quite a different role in the theory of quasi-Lie bialgebroids. Recall that a quasi-Lie bialgebroid with \( \Theta = \mu + \gamma + \phi \) gives rise to a quasi-Gerstenhaber algebra structure on \( \Gamma(\wedge \cdot A) \); in particular, one has a homotopy Lie algebra on \( \Gamma(\wedge \cdot A) [1] \) with a differential \( d_1 = d = \{\gamma, \cdot\} \) of degree +1, a bilinear bracket \( d_2 = [\cdot, \cdot] = [\cdot, \cdot]_\mu = \{\{\cdot, \cdot\}, \cdot\} \) of degree 0, and a trilinear bracket \( d_3 \), given by \( \phi \) as a “higher derived bracket” \( [\cdot, \cdot, \cdot] = [\cdot, \cdot, \cdot] = \{\cdot, [\cdot, \cdot, \phi]\} \), of degree −1. One can consider the deformation theory governed by this \( L_\infty \)-algebra.

The appropriate structures are elements of degree 1, \( \pi \in \Gamma(\wedge^2 A) \), obeying the structure equation

\[
\begin{align*}
  d\pi + \frac{1}{2}[\pi, \pi] + \frac{1}{6}[\pi, \pi, \pi] &= 0
\end{align*}
\]

It is easy to see that the last term is equal to \( -\wedge^3 \tilde{\pi}\phi \), so the above equation is exactly the same as the twisted Maurer-Cartan equation (4.6). So each such structure twists the quasi-Lie bialgebroid to a new one given by the formulas (4.5) (with \( \psi = 0 \)). The new quasi-Gerstenhaber algebra thus obtained is given by

\[
\begin{align*}
  \mu_{\pi} &= h_d + \tilde{\pi}\phi \\
  \gamma_{\pi} &= h_{[\pi, \cdot]} + \wedge^2 \tilde{\pi}\phi \\
  \phi_{\pi} &= \phi \\
  \psi_{\pi} &= -\frac{1}{2}[\pi, \pi] + \wedge^3 \tilde{\pi}\phi
\end{align*}
\]

6. Examples.

6.1. Arbitrary bivector fields. Let \( M \) be a manifold, \( A = TM, E = T^*M \).

Let \( \Theta_0 = \mu = h_d \), where \( d \) is the de Rham vector field on \( \Pi TM \), and \( h_d \) is defined as in Section 4. Of course, it obeys the structure equation (2.4), hence defines a Lie bialgebroid (with \( \gamma = 0 \)). Let \( \Theta_{\pi} = F^*\pi \Theta_0 \) be the twist by a bivector field \( \pi = \frac{1}{2}\pi^i \theta_i \theta_j \). Then \( \Theta_{\pi} = \mu_{\pi} + \gamma_{\pi} + \psi_{\pi} \), and the formulas (4.5) reduce to

\[
\begin{align*}
  \mu_{\pi} &= h_d \\
  \gamma_{\pi} &= h_{[\pi, \cdot]} \\
  \psi_{\pi} &= -\frac{1}{2}[\pi, \pi]
\end{align*}
\]

Here \( [\cdot, \cdot] = [\cdot, \cdot]_\mu \) is the ordinary Schouten bracket of multivector fields. Since \( \Theta_{\pi} \) obeys (2.6), we get a quasi-Lie bialgebroid on \((T^*M, TM)\). The Maurer-Cartan equation (4.4) reduces to \( [\pi, \pi] = 0 \), in which case we get the Lie bialgebroid of a Poisson manifold.

Bivector fields that do not satisfy any integrability condition are not interesting objects unless one imposes an additional structure such as a group action satisfying various types of compatibility conditions (\[6, 1\]).

6.2. Twisted Poisson manifolds with a 3-form background. For \( \mu = h_d \) as above and a 3-form \( \phi \), define \( \Theta_{\phi} = \mu + \phi \). It is immediate from (3.3) that \( \Theta_{\phi} \) satisfies (2.6) if and only if \( \phi \) is closed. Thus, for a closed \( \phi \), one gets a quasi-Lie bialgebroid structure on \((TM, T^*M)\) (with \( \gamma = 0 \)). Now, for a bivector field \( \pi \), let \( \Theta_{\phi, \pi} = F^*\pi \Theta_{\phi} \). We have \( \Theta_{\phi, \pi} = \mu_{\pi} + \gamma_{\pi} + \phi_{\pi} + \psi_{\pi} \), where by (4.5) we have:

\[
\begin{align*}
  \mu_{\pi} &= h_d + \tilde{\pi}\phi \\
  \gamma_{\pi} &= h_{[\pi, \cdot]} + \wedge^2 \tilde{\pi}\phi \\
  \phi_{\pi} &= \phi \\
  \psi_{\pi} &= -\frac{1}{2}[\pi, \pi] + \wedge^3 \tilde{\pi}\phi
\end{align*}
\]
This defines a quasi-Lie bialgebroid structure on \((TM, T^*M)\) if and only if the twisted Maurer-Cartan equation (4.6) holds, which in this case reduces to (1.1):
\[
\frac{1}{2} [\pi, \pi] = \lambda^3 \tilde{\pi} \phi
\]
This can also be thought of as the structure equation (3.2) in the quasi-Gerstenhaber algebra of polyvector fields given by the zero differential, the Schouten bracket \(\{\cdot, \cdot\}\) and the triple bracket \([\cdot, \cdot, \cdot] = -\{\cdot, \{\cdot, \phi\}\}\):
\[
\frac{1}{2} [\pi, \pi] + \frac{1}{6} [\pi, \pi, \pi] = 0
\]
This condition can be viewed as a “twisted” version of the Jacobi identity: defining \(\{f, g\} = [[f, \pi], g] + \frac{1}{2} [f, \pi, \pi]\) and \([f, \pi, \phi] = \{f, \pi\} + [\pi, \pi, \phi]\) for \(f, g \in \mathcal{C}^\infty(M)\), the above equation translates to:
\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \phi(X_f, X_g, X_h)
\]
The triple \((M, \pi, \phi)\) satisfying (1.1) is called a \(\phi\)-twisted Poisson manifold. It follows that \((TM, T^*M, \mu_\pi, \gamma_\pi, \phi)\) is a quasi-Lie bialgebroid which is a deformation of the one given by \(\pi = 0\). The deformed quasi-Gerstenhaber algebra of multivector fields is easily seen to be given by \(d_{\gamma_\pi} = [\pi, \cdot] + \frac{1}{2} [\pi, \pi, \cdot], \{\cdot, \cdot\}_{\mu_\pi} = [\cdot, \cdot] + [\pi, \cdot, \cdot]\), with \([\cdot, \cdot, \cdot]\) unchanged. Alternatively, we can specialize the formulas (4.8) to the present example. The differential is given by
\[
d_{\gamma_\pi} = [\pi, \cdot] + \iota_{\lambda^2 \tilde{\pi} \phi}
\]
where \(\iota\) denotes contraction with the bivector-valued 1-form \(\lambda^2 \tilde{\pi} \phi\). The quasi-Gerstenhaber bracket \([\cdot, \cdot]_{\mu_\pi}\) is uniquely determined by
\[
[X, f]_{\mu_\pi} = [X, f] = X f,
[X, Y]_{\mu_\pi} = [X, Y] + \tilde{\pi} \phi(X, Y)
\]
where \(X\) and \(Y\) are vector fields and \(f\) is a function on \(M\). \(d_{\gamma_\pi}\) squares to zero and acts as a derivation of \([\cdot, \cdot]_{\mu_\pi}\). The bracket \([\cdot, \cdot]_{\mu_\pi}\) satisfies the graded Jacobi identity up to a homotopy given by \(\phi\) (see (3.3) and (3.4)); in addition, the coherence condition (3.5) between \([\cdot, \cdot]_{\mu_\pi}\) holds.

This new quasi-Gerstenhaber algebra governs deformations of \(\pi\) within the class of \(\phi\)-Poisson structures: \(\pi' = \pi + \delta\) is \(\phi\)-Poisson if and only if
\[
d_{\gamma_\pi} \delta + \frac{1}{2} [\delta, \delta]_{\mu_\pi} + \frac{1}{6} [\delta, \delta, \delta] = 0
\]
In particular, notice that \(\delta = \pi\) does not obey this, since for \(\phi \neq 0\) the equation (1.1) is not homogeneous in \(\pi\); furthermore, \(\pi\) is not a cocycle with respect to the modified differential, nor is \([\pi, \pi]_{\mu_\pi} = 0\).

The dual quasi-differential Gerstenhaber algebra of differential forms on \(M\) consists of the quasi-differential
\[
d_{\mu_\pi} = d + \iota_{\tilde{\pi} \phi}
\]
where \(\iota\) denotes the contration with the 2-form-valued vector field \(\tilde{\pi} \phi\), while the Schouten bracket \([\cdot, \cdot]_{\gamma_\pi}\) is uniquely determined (see (3.5) by
\[
[a, f]_{\gamma_\pi} = (\tilde{\pi} a) f
[a, \beta]_{\gamma_\pi} = L_{\tilde{\pi} a} \beta - L_{\tilde{\pi} \beta} a - d_{\mu_\pi}(\pi(a, \beta)) + \lambda^2 \tilde{\pi} \phi(a, \beta)
\]
The bracket \([\cdot, \cdot]_{\gamma_\pi}\) satisfies the graded Jacobi identity, and \(d_{\mu_\pi}\) acts on it by derivations. In addition, \(d_{\mu_\pi}\) squares to \(-[\phi, \cdot]_{\gamma_\pi}\), and \(d_{\mu_\pi} \phi = 0\).
6.3. **Gauge transformations.** The gauge transformations of twisted Poisson manifolds introduced in [17] can be expressed in the present setting in terms of the factorization of the gauge group described in the previous section. Indeed, from (5.1) we immediately get

\[ F^* \tau - \pi F^* \omega = F^* T^{-1}(-\pi, \omega) F^* \tau - \pi F^* \omega \]

for a given bivector \( \pi \) and a 2-form \( \omega \) such that \( T(-\pi, \omega) = 1 - \tilde{\pi} \tilde{\omega} \) is invertible.

Applying both sides to \( \Theta \) and using formulas (4.2) and (4.5), we get

\[ \Theta_{\phi - d\omega, \tau - \pi} = \Phi^* \Theta_{\phi, \pi} \]

where \( \Phi^* = F^* T^{-1}(-\pi, \omega) F^* \tau - \pi \) is an element of the gauge subgroup preserving \( L^* \).

Hence, if \( \pi \) is a \( \phi \)-Poisson structure, \( \tau - \omega \pi \) is a \( (\phi - d\omega) \)-Poisson structure. We thus have a local action of the abelian group of 2-forms on the space of all twisted Poisson structures, or a global transformation groupoid inducing an equivalence relation. Equation (6.1) implies that gauge-equivalent twisted Poisson manifolds have isomorphic quasi-Lie bialgebroids. Therefore, not only their Poisson cohomology spaces, but the entire spectral sequences are isomorphic.

The cohomology class of \( \phi \) is preserved by gauge transformations; locally, every \( \phi \)-Poison structure is equivalent to an ordinary Poisson structure. The group of closed 2-forms acts locally on these. If \( \pi \) is a Poisson structure and \( \omega \) is a closed 2-form such that the gauge transformation \( \tau - \omega \pi \) exists, then by (6.1) and (4.2) \( \omega' = \tau - \pi \omega \) obeys the Maurer-Cartan equation \( d\omega' + \frac{1}{2} [\omega', \omega']_{\pi} = 0 \), where \( [\cdot, \cdot]_{\pi} \) is the Koszul bracket. This gives meaning to the MC equation in the differential graded Lie algebra of differential forms on \( M \) corresponding to a Poisson structure.

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