EXCEDANCE, FIXED POINT AND CYCLE STATISTICS AND ALTERNATINGLY INCREASING PROPERTY

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Abstract. In this paper, we consider unimodal expansions of some multivariate Eulerian polynomials. We first give a sufficient condition for a polynomial to be alternatingly increasing. Let \( p \in [0, 1] \) and \( q \in [0, 1] \) be two given real numbers. We then prove that the cyc \( q \)-Eulerian polynomials of permutations are bi-\( \gamma \)-positive, and the fix and cyc \((p,q)\)-Eulerian polynomials of permutations are alternatingly increasing, and so they are unimodal with modes in the middle, where fix and cyc are the fixed point and cycle statistics. As applications, we discuss the bi-\( \gamma \)-positivity and alternatingly increasing property of several combinatorial polynomials, including \( 1/k \)-Eulerian polynomials, colored derangement polynomials and some refinements of the flag excedance polynomials. Finally, we study excedance statistics of colored permutations. In particular, we establish the relationships between some multivariate colored Eulerian polynomials and the \((p,q)\)-Eulerian polynomials of permutations. Our results generalize several recent results in the literature.

Keywords: Eulerian polynomials, Fixed points, Cycles, Flag excedances, Gamma positivity

1. Introduction

Let \( f(x) = \sum_{i=0}^{n} f_i x^i \) be a polynomial with nonnegative coefficients. We say that \( f(x) \) is unimodal if \( f_0 \leq f_1 \leq \cdots \leq f_k \geq f_{k+1} \geq \cdots \geq f_n \) for some \( k \), where the index \( k \) is called the mode of \( f(x) \). The polynomials \( f(x) \) is said to be spiral if \( f_n \leq f_0 \leq f_{n-1} \leq f_1 \leq \cdots \leq f_{\lfloor n/2 \rfloor} \) (see [14] for instance). If \( f(x) \) is symmetric, i.e., \( f_i = f_{n-i} \) for all indices \( 0 \leq i \leq n \), then it can be expanded as

\[
 f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1 + x)^{n-2k}.
\]

Following Gal [20], the polynomial \( f(x) \) is said to be \( \gamma \)-positive if \( \gamma_k \geq 0 \) for all \( 0 \leq k \leq \lfloor n/2 \rfloor \). Following [33, Definition 2.9], the polynomial \( f(x) \) is alternatingly increasing if

\[
 f_0 \leq f_n \leq f_1 \leq f_{n-1} \leq \cdots \leq f_{\lfloor (n+1)/2 \rfloor}.
\]

It should be noted that the definition of alternatingly increasing property first appeared in the work of Beck and Stapledon [5]. Very recently, Beck, Jochemko and McCullough [6] and Solar [36] studied the alternatingly increasing property of several \( h^* \)-polynomials. Clearly, both \( \gamma \)-positivity and alternatingly increasing property imply unimodality. In the past decades, unimodal polynomials arise often in combinatorics and geometry, see [2, 22, 24] and references therein. This paper is motivated by empirical evidence which suggests that some multivariate
Eulerian polynomials are unimodal with modes in the middle. In this paper, we attempt to develop techniques for the unimodality problems.

We first recall an elementary result.

**Proposition 1** (Eulerian). Let \( f(x) \) be a polynomial of degree \( n \). There is a unique symmetric decomposition \( f(x) = a(x) + xb(x) \), where
\[
a(x) = \frac{f(x) - x^{n+1}f(1/x)}{1-x}, \quad b(x) = \frac{x^n f(1/x) - f(x)}{1-x}.
\]

By using (i), it is easy to verify that if \( f(0) \neq 0 \), then \( \deg a(x) = n \) and \( \deg b(x) \leq n - 1 \). We call the ordered pair of polynomials \( (a(x), b(x)) \) the symmetric decomposition of \( f(x) \), since \( a(x) \) and \( b(x) \) are both symmetric.

**Definition 2.** Let \( (a(x), b(x)) \) be the symmetric decomposition of a polynomial \( f(x) \). If \( a(x) \) and \( b(x) \) are both \( \gamma \)-positive, then \( f(x) \) is said to be bi-\( \gamma \)-positive.

As pointed out by Brändén and Solus [1], the polynomial \( f(x) \) is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have only nonnegative coefficients. Therefore, bi-\( \gamma \)-positivity implies alternatingly increasing property. If \( f(x) \) is \( \gamma \)-positive, then \( f(x) \) is also bi-\( \gamma \)-positive but not vice versa. We now provide a connection between \( \gamma \)-positive and bi-\( \gamma \)-positivity.

**Proposition 3.** If \( f(x) \) is \( \gamma \)-positive and \( f(0) = 0 \), then \( f'(x) \) is bi-\( \gamma \)-positive.

**Proof.** Assume that \( f(x) = \sum_{k=0}^{[n/2]} \gamma_k x^k (1 + x)^{n-2k} \), where \( \gamma_k \geq 0 \) for all \( 1 \leq k \leq [n/2] \). Then we have
\[
f'(x) = \sum_{k=1}^{[n/2]} k\gamma_k x^{k-1} (1 + x)^{n-2k} + \sum_{k=1}^{[n/2]} (n - 2k)\gamma_k x^k (1 + x)^{n-2k-1} - \sum_{i=0}^{[(n-1)/2]} (i + 1)\gamma_{i+1} x^i (1 + x)^{n-2i-2} + \sum_{k=1}^{[n/2]} (n - 2k)\gamma_k x^k (1 + x)^{n-2k-1}.
\]
Therefore, \( f'(x) \) is bi-\( \gamma \)-positive. \( \square \)

The following simple result will be used repeatedly in our discussion.

**Lemma 4.** Let \( f(x) = \sum_{i=0}^{n} f_i x^i \) and \( g(x) = \sum_{j=0}^{m} g_j x^j \). If \( f(x) \) is \( \gamma \)-positive and \( g(x) \) is bi-\( \gamma \)-positive, then \( f(x)g(x) \) is bi-\( \gamma \)-positive. In particular, the product of two \( \gamma \)-positive polynomials is also \( \gamma \)-positive.

**Proof.** Assume that \( f(x) = \sum_{k=0}^{[n/2]} \gamma_k x^k (1 + x)^{n-2k} \) and
\[
g(x) = \sum_{i=0}^{[m/2]} \xi_i x^i (1 + x)^{m-2i} + \sum_{j=1}^{[(m+1)/2]} \eta_j x^j (1 + x)^{m+1-2j},
\]
where \( \gamma_k, \xi_i \) and \( \eta_j \) are all nonnegative numbers. Then \( f(x)g(x) \) can be expanded as
\[
f(x)g(x) = \sum_{s=0}^{[(n+m)/2]} \alpha_s x^s (1 + x)^{n+m-2s} + \sum_{t=1}^{[(n+m+1)/2]} \beta_t x^t (1 + x)^{n+m+1-2t},
\]
where \( \alpha_s = \sum_{k+i=s} \gamma_k \xi_i \) and \( \beta_t = \sum_{k+j=t} \gamma_k \eta_j \), which yields the desired result. \( \square \)

We now recall a definition.

**Definition 5 (\[27\] Definition 4).** Let \( p(x, y) \) be a bivariate polynomial. Suppose \( p(x, y) \) can be expanded as

\[
p(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{[(n-i)/2]} y^i \mu_{n,i,j} x^j (1 + x)^{n-i-2j}.
\]

If \( \mu_{n,i,j} \geq 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq [(n-i)/2] \), then we say that \( p(x, y) \) is a partial \( \gamma \)-positive polynomial.

It should be noted that partial \( \gamma \)-positive polynomials frequently appear in combinatorics and geometry, see \[2, 21, 27, 35\] for instance. We can now conclude the first main result of this paper.

**Theorem 6.** Suppose the polynomial \( p(x, y) \) has the expression \[2\] and \( \deg p(x, 1) = n-1 \), where \( n \) is a positive integer. If \( p(x, y) \) is partial \( \gamma \)-positive, \( p(x, 1) \) is bi-\( \gamma \)-positive and \( 0 \leq y \leq 1 \) is a given real number, then \( p(x, y) \) is alternatingly increasing.

Let \( S_n \) be the set of all permutations of \( [n] = \{1, 2, \ldots, n\} \) and let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). We say that \( i \) is a descent (resp. exceedance, drop, fixed point) if \( \pi(i) > \pi(i+1) \) (resp. \( \pi(i) > i \), \( \pi(i) < i \), \( \pi(i) = i \)) in \( \pi \). Let \( \text{des}(\pi) \), \( \text{exc}(\pi) \), \( \text{drop}(\pi) \), \( \text{fix}(\pi) \) and \( \text{cyc}(\pi) \) be the number of descents, exceedances, drops, fixed points and cycles of \( \pi \), respectively. It is well known that descents, exceedances and drops are equidistributed over \( S_n \), and their common enumerative polynomial is the classical Eulerian polynomial (see \[34, 35\] for instance):

\[
A_n(x) = \sum_{\pi \in S_n} x^\text{des}(\pi) = \sum_{\pi \in S_n} x^\text{exc}(\pi) = \sum_{\pi \in S_n} x^\text{drop}(\pi).
\]

An index \( i \in [n] \) is called a double descent of a permutation \( \pi \in S_n \) if \( \pi(i-1) > \pi(i) > \pi(i+1) \), where \( \pi(0) = \pi(n+1) = 0 \). Foata and Schützenberger \[18\] found the following notable result.

**Proposition 7 (\[18\]).** One has

\[
A_n(x) = \sum_{i=0}^{[(n-1)/2]} \gamma_{n,i} x^i (1 + x)^{n-2i},
\]

where \( \gamma_{n,i} \) is the number of permutations \( \pi \in S_n \) which have no double descents and \( \text{des}(\pi) = i \).

An element \( \pi \in S_n \) is called a derangement if \( \text{fix}(\pi) = 0 \). Let \( D_n \) be the set of all derangements in \( S_n \). The derangement polynomials are defined by \( d_n(x) = \sum_{\pi \in D_n} x^\text{exc}(\pi) \). It is well known that the generating function of \( d_n(x) \) is given as follows (see \[9\] Proposition 6):

\[
d(x, z) = \sum_{n=0}^{\infty} d_n(x) \frac{z^n}{n!} = \frac{1 - x}{e^{xz} - xe^z}.
\]

The cardinality of a set \( A \) will be denoted by \( \# A \). Let \( \text{cda}(\pi) = \#\{i : \pi^{-1}(i) < i < \pi(i)\} \) be the number of cycle double ascents of \( \pi \). By using the theory of continued fractions, Shin and Zeng \[33\] Theorem 11] obtained the following result.
Proposition 8 (34). Let \( D_{n,k} = \{ \pi \in S_n : \text{fix} (\pi) = 0, \text{cda} (\pi) = 0, \text{exc} (\pi) = k \} \). Then
\[
d_n(x, q) = \sum_{\pi \in D_n} x^{\text{exc} (\pi)} q^{\text{cyc} (\pi)} = \sum_{k=1}^{[n/2]} \sum_{\pi \in D_{n,k}} q^{\text{cyc} (\pi)} x^k (1 + x)^{n-2k}.
\]

As usual, we denote by \( \bar{i} \) the negative element \(-i\). Let \( \pm [n] = [n] \cup \{ \bar{1}, \ldots, \bar{n} \} \). Let \( S_n^B \) be the hyperoctahedral group of rank \( n \). Elements of \( S_n^B \) are permutations of \( \pm [n] \) with the property that \( \sigma (\bar{i}) = -\sigma (i) \) for all \( i \in [n] \). Let \( \sigma = (1)\sigma (2) \cdots (n) \in S_n^B \). An excedance (resp. fixed point) of \( \sigma \) is an index \( i \in [n] \) such that \( \sigma (|\sigma (i)|) > \sigma (i) \) (resp. \( \sigma (i) = i \)). 

Let \( \text{exc} (\sigma) \) (resp. \( \text{fix} (\sigma) \)) denote the number of excedances (resp. fixed points) of \( \sigma \). Let \( D_n^B = \{ \sigma \in S_n^B : \text{fix} (\sigma) = 0 \} \) be the set of all derangements in \( S_n^B \). The type \( B \) derangement polynomials \( d_n^B (x) \) are defined by
\[
d_n^B (x) = \sum_{\sigma \in D_n^B} x^{\text{exc} (\sigma)},
\]
which has been extensively studied, see [14] and references therein. According to [16] Theorem 3.2, the generating function of \( d_n^B (x) \) is given as follows:
\[
\sum_{n=0}^{\infty} d_n^B (x) \frac{z^n}{n!} = \frac{(1 - x)e^z}{e^z - xe^z}.
\]

Chen, Tang and Zhao [14] Theorem 4.6] studied the polynomials \( x^n d_n^B (1/x) \) and proved the following remarkable result (in an equivalent form).

Proposition 9 (14). For \( n \geq 1 \), the polynomials \( d_n^B (x) \) are alternatingly increasing.

Recently, many different refinements and generalizations of Propositions 7, 8, and 9 have been studied, see [21,24,27,35,38]. In the following, we shall give a unified generalization of these Propositions.

Let us define the \((p,q)\)-Eulerian polynomials \( A_n(x,p,q) \) by
\[
A_n(x,p,q) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} p^{\text{fix}(\pi)} q^{\text{cyc}(\pi)}.
\]

Let \( A_n(x,q) = A_n(x,1,q) \) be the \( q \)-Eulerian polynomials. Brenti [11] showed that some of the crucial properties of Eulerian polynomials have nice \( q \)-analogues for the polynomials \( A_n(x,q) \). Following [11] Proposition 7.2, the \( q \)-Eulerian polynomials \( A_n(x,q) \) satisfy the recurrence
\[
A_{n+1}(x,q) = (nx + q) A_n(x,q) + x (1 - x) \frac{d}{dx} A_n(x,q),
\]
with the initial conditions \( A_0(x;q) = 1, A_1(x;q) = q \) and \( A_2(x;q) = q(x + q) \). According to [11] Proposition 7.3, we have
\[
\sum_{n=0}^{\infty} A_n(x,q) \frac{z^n}{n!} = \left( \frac{(1 - x)e^z}{e^z - xe^z} \right)^q.
\]

Using the exponential formula, Ksavrelof and Zeng [23] found that
\[
\sum_{n=0}^{\infty} A_n(x,p,q) \frac{z^n}{n!} = \left( \frac{(1 - x)e^z}{e^z - xe^z} \right)^q
\]
Below are the polynomials \( A_n(x, p, q) \) for \( n \leq 4 \):

\[
A_1(x, p, q) = pq, \quad A_2(x, p, q) = p^2 q^2 + qx, \quad A_3(x, p, q) = p^3 q^3 + (q + 3pq^2)x + qx^2, \\
A_4(x, p, q) = p^4 q^4 + (q + 4pq^2 + 6p^2 q^3)x + (4q + 3q^2 + 4pq^2)x^2 + qx^3.
\]

From Proposition 8 we see that if \( q > 0 \) is a given real number, then \( A_n(x, 0, q) \) are \( \gamma \)-positive. Comparing (5) with (7), we get \( 2^n A_n(x, 1/2, 1) = d_n^B(x) \). It follows from Proposition 9 that \( A_n(x, 1/2, 1) \) are alternatingly increasing. As a unified generalization of Propositions 7, 8 and 9, we can now present the second main result of this paper.

**Theorem 10.** Let \( p \in [0, 1] \) and \( q \in [0, 1] \) be two given real numbers, i.e., \( 0 \leq p \leq 1 \) and \( 0 \leq q \leq 1 \). Then we have the following result.

(i) For \( n \geq 1 \), the polynomials \( A_n(x, q) \) are bi-\( \gamma \)-positive;

(ii) The polynomials \( A_n(x, p, q) \) are alternatingly increasing.

The proofs of Theorems 6 and 10 will be given in the next section. In Section 8 we give a combinatorial interpretation for the symmetric decomposition of \( k^n A_n(x, 1/k) \) as well as the bi-\( \gamma \)-coefficients of \( 2^n A_n(x, 1/2) \), where \( k \) is a fixed positive integer. In Section 4 we give several applications of Theorem 10. In Section 5 we prove a relationship between \( A_n(x, p, q) \) and a family of multivariate Eulerian polynomials for signed permutations. In Section 6 we study excedance statistics of colored permutations. In particular, we establish the relationships between \( A_n(x, p, q) \) and some multivariate colored Eulerian polynomials.

### 2. Proof of Theorems 6 and 10

**A proof Theorem 6.**

**Proof.** For \( 0 \leq i \leq n \) and \( 0 \leq j \leq \lfloor (n-i)/2 \rfloor \), let

\[
\mu_{n,i,j} x^j (1 + x)^{n-i-2j} = \sum_{\ell=j}^{n-i-j} S_{n,i,j,\ell} x^\ell.
\]

Note that the polynomials \( \sum_{\ell=j}^{n-i-j} S_{n,i,j,\ell} x^\ell \) are symmetric and unimodal. Then

\[
\begin{aligned}
S_{n,i,j,\ell} &= S_{n,i,j,n-i-\ell}, & \text{if } j \leq \ell \leq n-i-j; \\
S_{n,i,j,\ell} &\leq S_{n,i,j,\ell+k}, & \text{if } j \leq \ell < \ell + k \leq \lfloor (n-i)/2 \rfloor; \\
S_{n,i,j,\ell} &\geq S_{n,i,j,\ell+k}, & \text{if } \lfloor (n-i)/2 \rfloor \leq \ell < \ell + k \leq n-i-j.
\end{aligned}
\]

(8)

For \( n \geq 1 \), let \( p(x, y) = \sum_{\ell=0}^{n-1} p_n,\ell(y) x^\ell \). where

\[
p_n,\ell(y) = \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} S_{n,i,j,\ell}.
\]

For \( 0 \leq \ell \leq \lfloor n/2 \rfloor \), we have

\[
p_n,\ell(y) - p_{n,n-\ell}(y) = \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} (S_{n,i,j,\ell} - S_{n,i,j,n-i}) = \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} (S_{n,i,j,\ell} - S_{n,i,j,\ell-i}).
\]

From (8), we have \( S_{n,i,j,\ell} - S_{n,i,j,\ell-i} \geq 0 \). Hence \( p_n,\ell(y) \geq p_{n,n-\ell}(y) \) when \( y \geq 0 \).
For $0 \leq \ell \leq \lfloor n/2 \rfloor - 1$, we have

$$p_{n,n-1-\ell}(y) - p_{n,\ell}(y) = \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} (S_{n,i,j,n-1-\ell} - S_{n,i,j,\ell})$$

which can be rewritten as $p_{n,n-1-\ell}(y) - p_{n,\ell}(y) = P_{n,\ell} - Q_{n,\ell}(y)$, where

$$P_{n,\ell} = \sum_{j=0}^{\lfloor n/2 \rfloor} (S_{n,0,j,\ell+1} - S_{n,0,j,\ell}), \quad Q_{n,\ell}(y) = \sum_{i=1}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} (S_{n,i,j,\ell} - S_{n,i,j,\ell+1-i}).$$

It follows from (5) that $S_{n,0,j,\ell+1} - S_{n,0,j,\ell} \geq 0$ and $S_{n,i,j,\ell} - S_{n,i,j,\ell+1-i} \geq 0$ for $i \geq 1$. Since $p(x, 1) = \sum_{i=0}^{n} p_{n,\ell}(1)x^i$ is bi-$\gamma$-positive, the polynomial $p(x, 1)$ is alternatingly increasing, which implies that

$$p_{n,n-1-\ell}(1) - p_{n,\ell}(1) = P_{n,\ell} - Q_{n,\ell}(1) \geq 0.$$

Therefore, if $0 \leq y \leq 1$, then $P_{n,\ell} - Q_{n,\ell}(y) \geq P_{n,\ell} - Q_{n,\ell}(1) \geq 0$. In conclusion, when $0 \leq y \leq 1$, we get $p_{n,\ell}(y) \geq p_{n,n-\ell}(y)$ for $0 \leq \ell \leq \lfloor n/2 \rfloor$ and $p_{n,n-1-\ell}(y) \geq p_{n,\ell}(y)$ for $0 \leq \ell \leq \lfloor n/2 \rfloor - 1$. This completes the proof. \qed

In the rest of this section, we shall prove Theorem 10 by using the theory of context-free grammars. For an alphabet $V$, let $\mathbb{Q}[[V]]$ be the rational commutative ring of formal power series in monomials formed from letters in $V$. A context-free grammar over $V$ is a function $G : V \to \mathbb{Q}[[V]]$ that replaces a letter in $V$ by an element of $\mathbb{Q}[[V]]$ (see [15,27]). The formal derivative $D_G$ is a linear operator defined with respect to a grammar $G$. For any two formal functions $u$ and $v$,

$$D_G(u + v) = D_G(u) + D_G(v), \quad D_G(uv) = D_G(u)v + uD_G(v).$$

For a constant $c$, we have $D_G(c) = 0$. The Leibniz rule is as follows:

$$D^2_G(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u)D_G^{n-k}(v).$$

**Example 11.** If $G = \{x \to xy, y \to xy\}$, then $D_G(x) = xy, D_G(y) = xy, D_G^2(x) = xy(x + y)$.

A grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [15]). In the following discussion, we always write $\pi \in S_n$ by using its standard cycle decomposition, in which each cycle is written with its smallest entry first and the cycles are written in ascending order of their first entry. The following lemma is fundamental.

**Lemma 12.** Let $G = \{I \to Ipq, p \to xy, x \to xy, y \to xy\}$. Then

$$D^2_G(I) = I \sum_{\pi \in S_n} x^{\text{exc}(\pi)} y^{\text{drop}(\pi)} p^\text{fix}(\pi) q^\text{cyc}(\pi).$$

**Proof.** Let $\pi \in S_n$. Consider the following grammatical labeling of $\pi$:

$$(L_1)$$ If $i$ is an excedance, then put a superscript label $x$ right after $i$;
A bijective proof of (11) was recently given in [12]. For assume that (13) holds for some

Proof.

Lemma 13.

A

extensively studied in recent years, see [29] and references therein. By comparing (7) with (10),

\pi

the s-inversion sequence

1 or 4, then we respectively get (1^x 4^{y^3})_q(2^x 6^y)_q(5^p)^{I}_q. If we insert 7 after 5, the resulting permutation is (1^x 4^{y^3})_q(2^x 6^y)_q(5^p)(7^p)^I_q. If we insert 7 after 1 or 4, then we respectively get (1^x 7^y 4^{y^3})_q(2^x 6^y)_q(5^p)^{I}_q and (1^x 4^y 7^y 4^{y^3})_q(2^x 6^y)_q(5^p)^{I}_q. In each case, the insertion of 7 corresponds to one substitution rule in G. By induction, it is routine to verify that the action of \( D_G \) on the set of weights of permutations in \( S_n \) gives the set of weights of permutations in \( S_{n+1} \). This completes the proof.

In the following discussion of this section, we always let \( k \) be a given positive integer. Following [32] Section 1.5], the 1/k-Eulerian polynomials \( A_n^{(k)}(x) \) are defined as follows:

\[
\sum_{n=0}^{\infty} A_n^{(k)}(x) \frac{z^n}{n!} = \left( \frac{1 - x}{e^{kz(x-1)} - x} \right)^{\frac{1}{k}}, \tag{10}
\]

Savage and Viswanathan [32] Section 1.5] proved that \( A_n^{(k)}(x) \) is the s-Eulerian polynomial of the s-inversion sequence \( (1, k + 1, 2k + 1, \ldots, (n-1)k + 1) \). The polynomial \( A_n^{(k)}(x) \) has been extensively studied in recent years, see [29] and references therein. By comparing (7) with (10), we obtain

\[
A_n^{(k)}(x) = k^n A_n(x, 1, 1/k) = k^n A_n(x, 1/k) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} k^{n-\text{cyc}(\pi)}. \tag{11}
\]

A bijective proof of (11) was recently given in [12]. For \( n \geq 1 \), we let \( A_n^{(k)}(x) = \sum_{j=0}^{n-1} A_{n,j,k} x^j \). The first few \( A_n^{(k)}(x) \) are \( A_1^{(k)}(x) = 1 \), \( A_2^{(k)}(x) = 1 + kx \), \( A_3^{(k)}(x) = 1 + 3kx + k^2x(1 + x) \). By combining (11) and (6), we immediately get that

\[
A_{n+1,j,k} = (1 + k)A_{n,j,k} + k(n - j + 1)A_{n,j-1,k}, \tag{12}
\]

with the initial condition \( A_{1,0,k} = 1 \) and \( A_{1,i,k} = 0 \) for \( i \neq 0 \). In the following, we present a grammatical description of the numbers \( A_{n,j,k} \).

Lemma 13. If \( G_0 = \{I \to Iy, x \to kxy, y \to kxy\} \), then we have

\[
D_G^n(I) = I \sum_{j=0}^{n-1} A_{n,j,k} x^j y^{n-j}. \tag{13}
\]

Proof. Note that \( D_G^0(I) = Iy, D_G^1(I) = I(y^2 + kxy) \). Hence the result holds for \( n = 1, 2 \). Now assume that (13) holds for some \( n \), where \( n \geq 2 \). Note that

\[
D_G^{n+1}(I) = \sum_j A_{n,j,k} I \left( x^j y^{n-j+1} + kjx^j y^{n-j+1} + k(n - j)x^{j+1} y^{n-j} \right).
\]
Taking coefficients of \(x^jy^{n-j+1}\) on both sides of the above expression yields the recurrence relation \(12\). The proof follows by induction.

A change of grammar is a substitution method in which the original grammar is replaced with functions of other grammars, which has proved to be useful in handling combinatorial expansions (see [27, 29]). In [27], the change of grammar method was defined and used to prove the \(\gamma\)-positivity of some descent-type polynomials. In the following, we shall use the change of grammar technique to prove our main result.

A proof Theorem \(10\).

Proof. The proof is divided into two parts.

(i) Let \(q \in [0, 1]\) be a given real number and let \(k\) be a given positive integer. From \((11)\), we see that \(A_n^{(k)}(x) = k^nA_n(x, 1/k)\). To prove the bi-\(\gamma\)-positivity of \(A_n(x, q)\), it suffices to prove that the polynomials \(A_n^{(0)}(x)\) are bi-\(\gamma\)-positive. Consider a change of the grammar which is given in Lemma \(13\) Note that

\[
D_{G_0}(I) = Iy, \quad D_{G_0}(Jy) = Iy(x + y) + (k - 1)Ixy, \quad D_{G_0}(x + y) = 2kxy, \quad D_{G_0}(xy) = kxy(x + y).
\]

Set \(J = Iy, u = x + y\) and \(v = xy\). Then

\[
D_{G_0}(I) = J, \quad D_{G_0}(J) = Ju + (k - 1)Iv, \quad D_{G_0}(u) = 2kv, \quad D_{G}(v) = kuv.
\]

Consider the grammar \(G_1 = \{I \to J, J \to Ju + (k - 1)Iv, u \to 2k, v \to kuv\}\). By induction, it is routine to verify that there are nonnegative integers such that

\[
D_{G_1}^n(I) = \sum_{i=0}^{[(n-1)/2]} A^+_{n,i;k}v^iu^{n-1-2i} + Iv \sum_{i=0}^{[(n-2)/2]} A^-_{n,i;k}v^iu^{n-2-2i}.
\]

In particular, \(D_{G_1}(I) = J, D_{G_1}^2(I) = Ju + (k - 1)Iv\). Note that

\[
D_{G_1}^{n+1}(I) = (Ju + (k - 1)Iv) \sum_i A^+_{n,i;k}v^iu^{n-1-2i} + Jv \sum_i A^-_{n,i;k}v^iu^{n-2-2i} +
J \sum_i A^+_{n,i;k} (kiv^iu^{n-2i} + 2k(n - 1 - 2i)v^{i+1}u^{n-2-2i}) +
I \sum_i A^-_{n,i;k} (k(i + 1)v^{i+1}u^{n-1-2i} + 2k(n - 2 - 2i)v^{i+2}u^{n-3-2i}).
\]

Taking coefficients of \(Jv^iu^{n-2i}\) and \(Iv^{i+1}u^{n-1-2i}\) on both sides yields the recurrence relation

\[
\begin{align*}
A^+_{n+1,i;k} &= (1 + ki)A^+_{n,i;k} + 2k(n - 2i + 1)A^+_{n,i-1,k}, \\
A^-_{n+1,i;k} &= (k + 1)A^-_{n,i;k} + 2k(n - 2i)A^-_{n,i-1,k} + (k - 1)A^+_{n,i,k},
\end{align*}
\]

with \(A^+_{1,0;k} = 1, A^+_{1,i;k} = 0\) for \(i \neq 0\) and \(A^-_{1,i;k} = 0\) for any \(i\). Note that \(A^+_{n,i;k}\) and \(A^-_{n,i;k}\) are both nonnegative when \(k \geq 1\). Comparing \(13\) and \(14\), we obtain

\[
A_n^{(k)}(x) = a_n^{(k)}(x) + xb_n^{(k)}(x),
\]

where

\[
a_n^{(k)}(x) = \sum_{i=0}^{[(n-1)/2]} A^+_{n,i;k}x^i(1 + x)^{n-1-2i}, \quad b_n^{(k)}(x) = \sum_{i=0}^{[(n-2)/2]} A^-_{n,i;k}x^i(1 + x)^{n-2-2i}.
\]
By using (15), it is routine to derive the following recurrence system:

\[
\begin{align*}
    a_{n+1}^{(k)}(x) &= (1 + x + k(n-1)x) a_n^{(k)}(x) + kx(1 - x) \frac{d}{dx} a_n^{(k)}(x) + x b_n^{(k)}(x), \\
    b_{n+1}^{(k)}(x) &= k(1 + (n-1)x) b_n^{(k)}(x) + kx(1 - x) \frac{d}{dx} b_n^{(k)}(x) + (k - 1) a_n^{(k)}(x),
\end{align*}
\]

with \(a_{1}^{(k)}(x) = 1\) and \(b_{1}^{(k)}(x) = 0\). Let \((a_n(x, q), b_n(x, q))\) be the symmetric decomposition of \(A_n(x, q)\). It follows from (16) that

\[
a_n^{(k)}(x) = k^n a_n(x, 1/k), \quad b_n^{(k)}(x) = k^n b_n(x, 1/k).
\] (18)

In conclusion, both \(A_n^{(k)}(x)\) and \(A_n(x, q)\) are bi-\(\gamma\)-positive, where \(q \in [0, 1]\). This completes the proof of the first statement.

(ii) Consider a change of the grammar that is given in Lemma 12 Setting \(u = xy\) and \(v = x + y\), we get \(D_G(I) = I pq, D_G(p) = u, D_G(u) = uv\) and \(D_G(v) = 2u\). Let \(G_2 = \{I \rightarrow I pq, p \rightarrow u, u \rightarrow uv, v \rightarrow 2u\}\). By induction, it is routine to verify that

\[
D_G^{n+1}(I) = D_G^{n}(I) \sum_{i,j} \gamma_{n,i,j}(q) x^{i} y^{j} v^{n-i-2j}.
\] (19)

In particular, \(D_G^{n+1}(I) = I pq, D_G^{2}(I) = I (p^2 q^2 + qu), D_G^{3}(I) = I (p^3 q^3 + 3pq^2 u + quv)\). Therefore,

\[
D_G^{n+1}(I) = D_G(I) \sum_{i,j} \gamma_{n,i,j}(q) x^{i} y^{j} v^{n-i-2j}.
\]

which yields that the numbers \(\gamma_{n,i,j}\) satisfy the recurrence relation

\[
\gamma_{n+1,i,j}(q) = q \gamma_{n,i-1,j}(q) + (i + 1) \gamma_{n,i+1,j-1}(q) + j \gamma_{n,i,j}(q) + (2n - 2i - 4j + 4) \gamma_{n,i,j-1}(q),
\] (20)

with the initial conditions \(\gamma_{1,1,0}(q) = q\) and \(\gamma_{1,1,j}(q) = 0\) for \((i, j) \geq (1, 0)\). From (20), we see that if \(q \geq 0\), then \(\gamma_{n,i,j}(q) \geq 0\). Moreover, upon taking \(u = xy\) and \(v = x + y\) in (19), we get the following expansion:

\[
D_G^{n}(I) = I \sum_{i,j} \gamma_{n,i,j}(q) x^{i} y^{j} (x + y)^{n-i-2j}.
\]

Setting \(I = y = 1\), we obtain \(D_G^{n}(I) |_{y = 1} = A_n(x, p, q)\) and so

\[
A_n(x, p, q) = \sum_{i,j} \gamma_{n,i,j}(q) x^{i} (1 + x)^{n-i-2j}.
\] (21)

Therefore, if \(q \geq 0\), then \(A_n(x, p, q)\) is partial \(\gamma\)-positive. From the first part of the proof, we see that when \(0 \leq q \leq 1\), the polynomial \(A_n(x, 1, q)\) is bi-\(\gamma\)-positive. In conclusion, when \(0 \leq p \leq 1\) and \(0 \leq q \leq 1\), it follows from Theorem 6 that the polynomial \(A_n(x, p, q)\) is alternatingly increasing. This completes the proof. \(\square\)
In the first part of the proof of Theorem [10] we get that the polynomial $A_n^{(k)}(x)$ is bi-$\gamma$-positive. In the next section, we give a combinatorial interpretation for the symmetric decomposition of $A_n^{(k)}(x)$ as well as the bi-$\gamma$-coefficients of $A_n^{(2)}(x)$.

3. The symmetric decomposition of $A_n^{(k)}(x)$

Let $A_{n,i,k}^{+}$ and $A_{n,i,k}^{-}$ be the numbers defined by the recurrence system [15]. For $n \geq 2$, we define

$$A_{n,i,k}^{+}(x) = \sum_{i=0}^{|(n-1)/2|} A_{n,i,k}^{+} x^i, \quad A_{n,i,k}^{-}(x) = \sum_{i=0}^{|(n-2)/2|} A_{n,i,k}^{-} x^i.$$ 

Multiplying both sides of [15] by $x^i$ and summing over all $i$, we obtain that

$$\begin{cases} A_{n+1,k}^{+}(x) = (1 + 2k(n-1)x)A_{n,k}^{+}(x) + kx(1 - 4x)\frac{d}{dx} A_{n,k}^{+}(x) + xA_{n,k}^{-}(x), \\ A_{n+1,k}^{-}(x) = k(1 + 2(n-2)x)A_{n,k}^{-}(x) + kx(1 - 4x)\frac{d}{dx} A_{n,k}^{-}(x) + (k - 1)A_{n,k}^{+}(x), \end{cases}$$

with $A_{1,k}^{+}(x) = 1$ and $A_{1,k}^{-}(x) = 0$. Below are the polynomials $A_{n,k}^{+}(x)$ and $A_{n,k}^{-}(x)$ for $2 \leq n \leq 4$:

$$\begin{align*}
A_{2,k}^{+}(x) &= 1, \quad A_{2,k}^{-}(x) = k - 1, \\
A_{3,k}^{+}(x) &= 1 + (3k - 1)x, \quad A_{3,k}^{-}(x) = k^2 - 1, \\
A_{4,k}^{+}(x) &= 1 + (6k + 4k^2 - 2)x, \quad A_{4,k}^{-}(x) = k^3 - 1 + (1 - 6k + 3k^2 + 2k^3)x.
\end{align*}$$

Recall that $A_n^{(k)}(x) = a_n^{(k)}(x) + x b_n^{(k)}(x)$. It follows from [17] that

$$a_n^{(k)}(x) = (1 + x)^{n-1} A_{n,k}^{+} \left( \frac{x}{(1 + x)^2} \right), \quad b_n^{(k)}(x) = (1 + x)^{n-2} A_{n,k}^{-} \left( \frac{x}{(1 + x)^2} \right).$$

We now recall a combinatorial interpretation of $A_n^{(k)}(x)$. Let $j^i = \underbrace{i, \ldots, i}_i$ for $i, j \geq 1$. We say that a permutation of $\{1^k, 2^k, \ldots, n^k\}$ is a $k$-Stirling permutation of order $n$ if for each $i$, $1 \leq i \leq n$, all entries between the two occurrences of $i$ are at least $i$. When $k = 2$, the $k$-Stirling permutation reduces to the classical Stirling permutation, see [7] for instance. Let $Q_n(k)$ be the set of $k$-Stirling permutations of order $n$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in Q_n(k)$. We say that an index $i$ is a longest ascent plateau if $\sigma_{i-1} < \sigma_i = \sigma_{i+1} = \sigma_{i+2} = \cdots = \sigma_{i+k-1}$, where $2 \leq i \leq nk - k + 1$. A longest left ascent plateau of $\sigma$ is a longest ascent plateau of $\sigma$ endowed with a 0 in the front of $\sigma$. Let $ap(\sigma)$ (resp. $lap(\sigma)$) be the number of longest ascent plateaus (resp. longest left ascent plateaus) of $\sigma$. It is clear that

$$\text{lap}(\sigma) := \begin{cases} ap(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2 = \cdots = \sigma_k; \\
\text{ap}(\sigma), & \text{otherwise}. \end{cases}$$

The following results were obtained in [26]:

$$A_n^{(k)}(x) = \sum_{\sigma \in Q_n(k)} x^{ap(\sigma)} x^n A_n^{(k)} \left( \frac{1}{x} \right) = \sum_{\sigma \in Q_n(k)} x^{\text{lap}(\sigma)}.$$ (22)
Note that \( \deg A_n^{(k)}(x) = n - 1 \). Assume that \( (a_n^{(k)}(x), b_n^{(k)}(x)) \) is the symmetric decomposition of \( A_n^{(k)}(x) \). Let \( \mathcal{Q}_n(k) = \{ \sigma \in \mathcal{Q}_n(k) \mid \sigma_j < \sigma_{j+1} \text{ for some } j \in [k-1] \} \) and let \( \mathcal{Q}_n = \mathcal{Q}_n(2) \).

Combining (1) and (22), we obtain

\[
a_n^{(k)}(x) = \frac{\sum_{\sigma \in \mathcal{Q}_n(k)} x^{\text{ap}(\sigma)} - \sum_{\sigma \in \mathcal{Q}_n(k)} x^{1 \text{ap}(\sigma)}}{1 - x}.
\]

So we immediately get the following result.

**Proposition 14.** We have

\[
a_n^{(k)}(x) = \sum_{\sigma \in \mathcal{Q}_n(k)} x^{\text{ap}(\sigma)}, \quad b_n^{(k)}(x) = \sum_{\sigma \in \mathcal{Q}_n(k)} x^{\text{ap}(\sigma)-1}.
\]

In particular, we have

\[
a_n^{(2)}(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)}, \quad b_n^{(2)}(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)-1}.
\]

It follows from (11) and (16) that

\[
\sum_{\pi \in \mathcal{S}_n} x^{\text{exc}(\pi)} 2^{n - \text{cyc}(\pi)} = 2^n A_n(x, 1/2) = A_n^{(2)}(x) = a_n^{(2)}(x) + x b_n^{(2)}(x).
\]

Below are the polynomials \( a_n^{(2)}(x) \) and \( b_n^{(2)}(x) \) for \( n \leq 5 \):

\[
\begin{align*}
a_1^{(2)}(x) &= 1, \quad b_1^{(2)}(x) = 0, \\
a_2^{(2)}(x) &= 1 + x, \quad b_2^{(2)}(x) = 1, \\
a_3^{(2)}(x) &= 1 + 7x + x^2, \quad b_3^{(2)}(x) = 3 + 3x, \\
a_4^{(2)}(x) &= 1 + 29x + 29x^2 + x^3, \quad b_4^{(2)}(x) = 7 + 31x + 7x^2, \\
a_5^{(2)}(x) &= 1 + 101x + 321x^2 + 101x^3 + x^4, \quad b_5^{(2)}(x) = 15 + 195x + 195x^2 + 15x^3.
\end{align*}
\]

In the rest part of this section, we shall present a combinatorial interpretation of the bi-\( \gamma \)-coefficients of \( A_n^{(2)}(x) \). We say that \( \pi \in \mathcal{S}_n \) is a **circular permutation** if it has only one cycle. Let \( A = \{ x_1, \ldots, x_j \} \) be a finite set of positive integers, and let \( \mathcal{C}_A \) be the set of all circular permutations of \( A \). Let \( w \in \mathcal{C}_A \). We will always write \( w \) by using its canonical presentation \( w = y_1 y_2 \cdots y_j \), where \( y_1 = \min A, y_i = w^{i-1}(y_1) \text{ for } 2 \leq i \leq j \) and \( y_1 = w^j(y_1) \). A **cycle peak** (resp. **cycle double ascent, cycle double descent**) of \( w \) is an entry \( y_i, 2 \leq i \leq j \), such that \( y_{i-1} < y_i > y_{i+1} \) (resp. \( y_{i-1} < y_i < y_{i+1}, y_{i-1} > y_i > y_{i+1} \)), where we set \( y_{j+1} = \infty \). Let \( \text{cpk}(w) \) (resp. \( \text{cdasc}(w), \text{cddes}(w) \)) be the number of cycle peaks (resp. cycle double ascents, cycle double descents) of \( w \). An **alternating run** of \( w \) is a maximal consecutive subsequence that is increasing or decreasing. Following \( [28] \), the number of **cycle runs** \( \text{crun}(w) \) of \( w \) is defined to be the number of alternating runs of the word \( y_1 y_2 \cdots y_j \). Assume that \( \text{cyc}(\pi) = s \) and \( \pi = w_1 w_2 \cdots w_s \), where \( w_i \) is the \( i \)-th cycle of \( \pi \). Let \( \text{crun}(\pi) = \sum_{i=1}^s \text{crun}(w_i) \) be the number of cycle runs of \( \pi \) and let \( \text{cpk}(\pi) = \sum_{i=1}^s \text{cpk}(w_i) \) be the number of cycle peaks of \( \pi \). For \( \pi \in \mathcal{S}_n \), it is clear that \( 1 \leq \text{crun}(\pi) \leq n \) and \( \text{crun}(\pi) = 2\text{cpk}(\pi) + \text{cyc}(\pi) \).

**Example 15.** If \( \pi = (1, 4, 2)(3, 5, 6)(7) \in \mathcal{S}_7 \), then \( \text{crun}(\pi) = 5 \), since the number of cycle runs in the three cycles are 3, 1, 1, respectively.
We can now present the following result.

**Theorem 16.** For \( n \geq 2 \), we have

\[
\sum_{\pi \in S_n} x^{\text{exc} (\pi)} 2^{n - \text{cyc} (\pi)} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi_{n,i}^+ x^i (1 + x)^{n-1-2i} + x \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \xi^{-}_{n,j} x^j (1 + x)^{n-2-2j}.
\]

Let \( S_{n,i} = \{ \pi \in S_n : \text{crun} (\pi) = i \} \) be the set of permutations in \( S_n \) with \( i \) cycle runs. Then we have

\[
\xi_{n,i}^+ = \sum_{\pi \in S_{n,2i+1}} 2^{\text{crun} (\pi) - \text{cyc} (\pi)}, \quad \xi_{n,j}^- = \sum_{\pi \in S_{n,2j+2}} 2^{\text{crun} (\pi) - \text{cyc} (\pi)}.
\]

**Proof.** Note that \( 2^2 A_2 (x, 1/2) = 1 + x + x \) and \( 2^3 A_3 (x, 1/2) = [(1 + x)^2 + 5x] + 3x(1 + x) \). Thus \( \xi_{2,0}^+ = \xi_{2,0}^- = 1 \), \( \xi_{3,0}^+ = 1 \), \( \xi_{3,1}^+ = 5 \), \( \xi_{3,0}^- = 3 \). Note that \( S_{2,1} = \{(12)\} \), \( S_{2,2} = \{(1)(2)\} \), \( S_{3,1} = \{(1,2,3)\} \), \( S_{3,2} = \{(1,2)(3),(1,3)(2),(1)(2,3)\} \), \( S_{3,3} = \{(1)(2)(3),(1,3,2)\} \).

It is easy to check that \( (23) \) holds for \( n = 2, 3 \). We proceed by induction on \( n \). In order to get permutations in \( S_{n+1,2i+1} \), we distinguish three cases:

\begin{itemize}
  \item[(c1)] If \( \pi \in S_{n,2i} \), then we can insert \( n + 1 \) into \( \pi \) as a new cycle. This gives the term \( \xi_{n,i-1}^- \);
  \item[(c2)] If \( \pi \in S_{n,2i+1} \), then \( 2\text{cpk} (\pi) + \text{cyc} (\pi) = 2i + 1 \). We can insert \( n + 1 \) just before or right after each cycle peak of \( \pi \). Moreover, we can insert \( n + 1 \) at the end of a cycle of \( \pi \). This gives the term \( 2(n + 1) \xi_{n,i}^+ \);
  \item[(c3)] If \( \pi \in S_{n,2i-1} \), then \( 2\text{cpk} (\pi) + \text{cyc} (\pi) = 2i - 1 \). We can insert \( n + 1 \) into any of the remaining \( n - (2i - 1) \) positions, and the number of cycle runs are increased by two. This gives the term \( 4(n - 2i + 1) \xi_{n,i-1}^- \).
\end{itemize}

Similarly, there are three ways to get permutations in \( S_{n+1,2i+2} \) by inserting the entry \( n + 1 \):

\begin{itemize}
  \item[(c1)] If \( \pi \in S_{n,2i+1} \), then we can insert \( n + 1 \) into \( \pi \) as a new cycle. This gives the term \( \xi_{n,i}^- \);
  \item[(c2)] If \( \pi \in S_{n,2i+2} \), then \( 2\text{cpk} (\pi) + \text{cyc} (\pi) = 2i + 2 \). We can insert \( n + 1 \) just before or right after each cycle peak of \( \pi \). Moreover, we can insert \( n + 1 \) at the end of a cycle of \( \pi \). This gives the term \( 2(n + 2) \xi_{n,i}^- \);
  \item[(c3)] If \( \pi \in S_{n,2i} \), then \( 2\text{cpk} (\pi) + \text{cyc} (\pi) = 2i \). We can insert \( n + 1 \) into any of the remaining \( n - 2i \) positions, and the number of cycle runs are increased by two. This gives the term \( 4(n - 2i) \xi_{n,i-1}^+ \).
\end{itemize}

In conclusion, we have

\[
\begin{align*}
\xi_{n+1,i}^+ &= (2i + 1) \xi_{n,i}^+ + 4(n - 2i + 1) \xi_{n,i-1}^+ + \xi_{n,i-1}^-;
\xi_{n+1,i}^- &= (2i + 2) \xi_{n,i}^- + 4(n - 2i) \xi_{n,i-1}^- + \xi_{n,i}^+.
\end{align*}
\]

Comparing this with \( (15) \) leads to \( \xi_{n,i}^+ = A_{n+1,i}^2 \) and \( \xi_{n,i}^- = A_{n+1,i}^2 \), as desired. This completes the proof. □

4. **Several applications of Theorem 10**

In this section, we give several applications of Theorem 10.
4.1. Colored derangement polynomials.

Let $r$ be a fixed positive integer. An $r$-colored permutation can be written as $\pi^c$, where $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ and $c = (c_1, c_2, \ldots, c_n) \in \{0, r - 1\}^n$, i.e., $c_i$ is a nonnegative integer lies in the interval $[0, r - 1]$ for any $i \in [n]$. As usual, $\pi^c$ can be denoted as $\pi_1^{c_1} \pi_2^{c_2} \cdots \pi_n^{c_n}$, where $c_i$ can be thought of as the color assigned to $\pi_i$. Denote by $\mathbb{Z}_r \wr S_n$ the set of all $r$-colored permutations of order $n$. The wreath product $\mathbb{Z}_r \wr S_n$ could be considered as the colored permutation group $G_{r,n}$ consists of all permutations of the alphabet $\Sigma$ of $rn$ letters:

$$\Sigma = \{1, 2, \ldots, n, \overline{1}, \ldots, \overline{n}, \ldots, \overline{1}^{[r-1]}, \ldots, n^{[r-1]}\}$$

satisfying $\pi(i) = \overline{\pi}(i)$. In particular, $\mathbb{Z}_1 \wr S_n = S_n$ and $\mathbb{Z}_2 \wr S_n = S_n^B$. Following Steingrímsson [37], for $1 \leq i \leq n$, we say that an entry $\pi_i^{c_i}$ is an excedance of $\pi^c$ if $i < f \pi_i$, where we use the order $< f$ of $\Sigma$:

$$1 < f \overline{1} < f \overline{2} \cdots < f 1^{[r-1]} < f 2^{[r-1]} < f \cdots < f n < f \overline{n} < f \cdots < f n^{[r-1]}.$$ (24)

Let $\text{exc} (\pi^c)$ be the number of excedances of $\pi^c$. A fixed point of $\pi^c \in \mathbb{Z}_r \wr S_n$ is an entry $\pi_i^{c_k}$ such that $\pi_k = k$ and $c_k = 0$. An element $\pi^c \in \mathbb{Z}_r \wr S_n$ is called a derangement if it has no fixed points. Let $D_{n,r}$ be the set of derangements in $\mathbb{Z}_r \wr S_n$. The $q$-colored derangement polynomials are defined by

$$d_{n,r}(x, q) = \sum_{\pi^c \in D_{n,r}} x^{\text{exc} (\pi^c)} q^{\text{cyc} (\pi)}.$$

In particular, $d_{n,2}(x, 1) = x^n d_n^B (1/x)$. There has been much work on the polynomials $d_{n,r}(x, 1)$, see [17] for example. For example, Chow and Toufik [17] Proposition 4] found that

$$d_{n,r}(x, 1) = \sum_{\pi \in S_n} x^{\text{wexc} (\pi)} (r - 1)^{\text{fix} (\pi)} n^{n - \text{fix} (\pi)},$$

where $\text{wexc} (\pi) = \#\{i \in [n] : \pi(i) > i\}$ is the number of weak excedances of $\pi$. Along the same lines as in the proof of [17], Proposition 4, one easily derive that

$$d_{n,r}(x, q) = \sum_{\pi \in S_n} x^{\text{wexc} (\pi)} (r - 1)^{\text{fix} (\pi)} n^{n - \text{fix} (\pi)} q^{\text{cyc} (\pi)}.$$

Note that $\text{wexc} (\pi) = \text{exc} (\pi) + \text{fix} (\pi)$. Define

$$a_n(x, p, q) = \sum_{\pi \in S_n} x^{\text{wexc} (\pi)} p^{\text{fix} (\pi)} q^{\text{cyc} (\pi)}.$$

Then $a_n(x, p, q) = A_n(x, xp, q)$. Let $\pi^{-1}$ be the inverse of $\pi$. The bijection $\pi \rightarrow \pi^{-1}$ on $S_n$ shows that $\text{wexc}, \text{fix}, \text{cyc}$ is equidistributed with $\text{drop}, \text{fix}, \text{cyc}$. (Thus $\text{wexc}, \text{fix}, \text{cyc}$ is equidistributed with $(n - \text{exc}, \text{fix}, \text{cyc})$. Therefore, we get

$$a_n(x, p, q) = x^n A_n \left(1, \frac{1}{x}, p, q \right).$$

It should be noted that if $p > 0$ and $q > 0$, then $\text{deg} a_n(x, p, q) = n$. Moreover, $a_n(0, p, q) = 0$ and the coefficient of the highest degree term of $a_n(x, p, q)$ is $p^n q^n$, and so $a_n(0, p, q) < p^n q^n$. Therefore, Theorem [10] is equivalent to the following result.

**Theorem 17.** Let $p \in [0, 1]$ and $q \in [0, 1]$ be two given real numbers. The polynomials $a_n(x, 1, q)$ are bi-$\gamma$-positive and the polynomials $a_n(x, p, q)$ are alternatingly increasing for $n \geq 1$. 
Note that 
\[ d_{n,r}(x, q) = r^n a_n \left( x, \frac{r - 1}{r}, q \right). \]
As a special case of Theorem 17 we get the following result.

**Corollary 18.** Let \( q \in [0,1] \) be a given real number and let \( r \) be a given positive integer. Then the polynomials \( d_{n,r}(x, q) \) are alternatingly increasing for \( n \geq 1 \).

It should be noted that the bi-\( \gamma \)-positivity of \( d_{n,r}(x, 1) \) has been proved by Athanasiadis [1, Theorem 1.3], which will be discussed in Section 6.

### 4.2. Excedances, fixed points and cycles of signed permutations.

Let \( \sigma \in S_n^B \). We say that \( i \) is an index of singleton of \( \sigma \) if \( \sigma(i) = i \). Let \( \text{single} (\sigma) \) be the number of singletons of \( \sigma \). Following [10, Corollary 3.16], the type \( B \) Eulerian polynomials can be defined as follows:

\[ B_n(x) = \sum_{\sigma \in S_n^B} x^{\text{exc}(\sigma)+\text{fix}(\sigma)} = \sum_{\sigma \in S_n^B} x^{\text{exc}(\sigma)+\text{single}(\sigma)}, \]

where \( \text{exc}(\sigma) = \#\{i \in [n] : \sigma(|\sigma(i)|) > \sigma(i)\} \) and \( \text{fix}(\sigma) = \#\{i \in [n] : \sigma(i) = i\} \). According to [10, Theorem 3.4], we have

\[ \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{(1 - x)e^{(1+x)z}}{e^{2xz} - xe^{2z}}. \]

Define

\[ B_n(x, y, p, q) = \sum_{\sigma \in S_n^B} x^{\text{exc}(\sigma)} y^{\text{single}(\sigma)} p^{\text{fix}(\sigma)} q^{\text{cyc}(\sigma)}. \]

Then \( B_n(x) = B_n(x, 1, 1) = B_n(x, 1, x, 1) \) and \( d_n^B(x) = B_n(x, 1, 0, 1) \). According to [30, Theorem 8], we have

\[ \sum_{n=0}^{\infty} B_n(x, y, p, q) \frac{z^n}{n!} = \left( \frac{(1 - x)e^{(y+p)z}}{e^{2xz} - xe^{2z}} \right)^q. \]

By comparing (7) with (26), we see that

\[ B_n(x, 1, p, q) = 2^n A_n \left( x, \frac{p + 1}{2}, q \right), \quad B_n(x, y, 0, q) = 2^n A_n \left( x, \frac{y}{2}, q \right). \]

In particular, \( B_n(x, 1, -1, 1) = 2^n d_n(x) \), \( B_n(x, 1, 0, 1) = d_n^B(x) \) and \( B_n(x, 2, 0, 1) = 2^n A_n(x) \). By using Theorem 10 we get the following corollary, which gives a generalization of Proposition 9.

**Corollary 19.** Let \( p \in [-1,1], q \in [0,1] \) and \( y \in [0,2] \) be three given real numbers. Then both \( B_n(x, 1, p, q) \) and \( B_n(x, y, 0, q) \) are alternatingly increasing for \( n \geq 1 \).
4.3. The flag excedance statistic of signed permutations.

In the following discussion, we always write \( \sigma \in \mathcal{S}_n^B \) by using its standard cycle decomposition, in which each cycle has its smallest (in absolute value) element first and the cycles are written in increasing order of the absolute values of their first elements. It should be noted that the \( n \) letters appearing in the cycle notation of \( \sigma \in B_n \) are the letters \( \sigma(1), \sigma(2), \ldots, \sigma(n) \). We say that \( i \) is an index of type \( A \) excedance if \( \sigma(i) > i \). For \( \sigma \in \mathcal{S}_n^B \), we let

\[
\text{exc}_A(\sigma) = \#\{ i \in [n] : \sigma(i) > i \}, \quad \text{neg}(\sigma) = \#\{ i \in [n] : \sigma(i) < 0 \},
\]

\[
\text{fix}(\sigma) = \#\{ i \in [n] : \sigma(i) = i \}, \quad \text{single}(\sigma) = \#\{ i \in [n] : \sigma(i) = 7 \},
\]

\[
\text{fexc}(\sigma) = 2\text{exc}_A(\sigma) + \text{neg}(\sigma), \quad a\text{exc}_A(\sigma) = n - \text{exc}_A(\sigma) - \text{fix}(\sigma) - \text{single}(\sigma).
\]

**Example 20.** The standard cycle decomposition of \( \sigma = 2 \overline{5} 1 3 4 \overline{6} 8 7 \) is \( (1, 2, 5, 4, 3)(\overline{6})(7, 8) \). Moreover, we have \( \text{exc}_A(\sigma) = 2, \quad \text{neg}(\sigma) = 2, \quad \text{fix}(\sigma) = 0, \quad \text{single}(\sigma) = 1 \) and \( \text{fexc}(\sigma) = 6 \).

The flag excedance statistic \( \text{fexc} \) has been studied by Bagno and Garber [4], Foata and Han [19], Mongelli [31], Shin and Zeng [35] and Zhuang [38]. In particular, Mongelli [31, Section 3] found two formulas:

\[
\sum_{\sigma \in \mathcal{S}_n^B} x^{2\text{exc}_A(\sigma)} p^{\text{neg}(\sigma)} = (1 + p)^n A_n \left( \frac{x^2 + p}{1 + p} \right), \tag{27}
\]

\[
\sum_{\sigma \in \mathcal{D}_n^B} x^{2\text{exc}_A(\sigma)} p^{\text{neg}(\sigma)} = \sum_{k=0}^{n} \binom{n}{k} (1 + p)^k p^{n-k} d_k \left( \frac{x^2 + p}{1 + p} \right). \tag{28}
\]

Setting \( q = x \) in (27) leads to the well known formula:

\[
\sum_{\sigma \in \mathcal{S}_n^B} x^{\text{fexc}(\sigma)} = (1 + x)^n A_n(x). \tag{29}
\]

Let \( D_n^B(x) = \sum_{\sigma \in \mathcal{D}_n^B} x^{\text{fexc}(\sigma)} \) be the flag derangement polynomials. It follows from (28) that

\[
D_n^B(x) = \sum_{k=0}^{n} \binom{n}{k} (1 + x)^k x^{n-k} d_k(x). \tag{30}
\]

By using (30), Mongelli [31, Proposition 3.5] proved the symmetry of \( D_n^B(x) \). Shin and Zeng [35, Corollary 5] proved that \( D_n^B(x) \) is \( \gamma \)-positive by using the theory of continued fractions.

Let

\[
B_n(x, y, f, s, p, q) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\text{exc}_A(\sigma)} y^{a\text{exc}_A(\sigma)} f^{\text{fix}(\sigma)} s^{\text{single}(\sigma)} p^{\text{neg}(\sigma)} q^{\text{cyc}(\sigma)}.
\]

Recall that \( \text{exc}(\pi) + \text{drop}(\pi) + \text{fix}(\pi) = n \) for \( \pi \in \mathcal{S}_n \). Now we give the following result, and the proof of it will be given in Section 5.

**Theorem 21.** One has

\[
B_n(x, y, f, s, p, q) = \sum_{\pi \in \mathcal{S}_n} (x + py)^{\text{exc}(\pi)} (y + py)^{\text{drop}(\pi)} (f + ps)^{\text{fix}(\pi)} q^{\text{cyc}(\sigma)}. \tag{31}
\]

Equivalently,

\[
B_n(x, y, f, s, p, q) = (y + py)^n A_n \left( \frac{x + py}{y + py}, \frac{f + ps}{y + py}, q \right). \tag{32}
\]
Combining (7) and (32), we immediately get
\[
\sum_{n=0}^{\infty} B_n(x, y, f, s, p, q) \frac{z^n}{n!} = \left( \frac{(y - x)e^{(f + ps)z}}{(1 + p)y e^{(x + py)z} - (x + py)e^{(1+p)yz}} \right)^q.
\]
Replacing \( x \) by \( x^2 \) and setting \( y = f = s = q = 1 \) in (32), we immediately get (27). Replacing \( x \) by \( x^2 \) and setting \( y = s = q = 1 \) and \( f = 0 \) in (32), one can easily derive (28).

In the rest part of this subsection, we consider the following bivariate polynomials:

\[
F_{n}(fexc, cyc)(x, q) = \sum_{\sigma \in S_n^\beta} B_n x fexc(\sigma) q cyc(\sigma),
\]
\[
D_{n}(fexc, cyc)(x, q) = \sum_{\sigma \in D_n^\beta} x fexc(\sigma) q cyc(\sigma),
\]
\[
F_{n}(fexc, neg)(x, p) = \sum_{\sigma \in S_n^\beta} x fexc(\sigma) p neg(\sigma).
\]

From (32), we see that
\[
F_{n}(fexc, cyc)(x, q) = B_n(x^2, 1, 1, 1, x, q) = (1 + x)^n A_n(x, q).
\]
Therefore, by using Lemma 27 and Theorem 10, we get the following corollary.

**Corollary 22.** Let \( q \in [0, 1] \) be a given real number. For \( n \geq 1 \), the polynomials \( F_{n}(fexc, cyc)(x, q) \) are bi-\( \gamma \)-positive, and so they are alternatingly increasing.

By using (32), we get
\[
D_{n}(fexc, cyc)(x, q) = B_n(x^2, 1, 0, 1, x, q) = (1 + x)^n A_n \left( x, \frac{x}{1 + x}, q \right).
\]
Then we obtain
\[
D_{n}(fexc, cyc)(x, q) = \sum_{\pi \in S_n} x fexc(\pi) x fix(\pi) (1 + x)^{n - fix(\pi)} q cyc(\pi)
\]
\[
= \sum_{i=0}^{n} \binom{n}{i} (qx)^i (1 + x)^{n-i} \sum_{\pi \in D_{n-i}} x fexc(\pi) q cyc(\pi).
\]
Let \( q > 0 \) be a given real number. Combining Proposition 8 and Lemma 27, we find that for \( 0 \leq i \leq n \), the polynomials \( (gx)^i (1 + x)^{n-i} \sum_{\pi \in D_{n-i}} x fexc(\pi) q cyc(\pi) \) are all \( \gamma \)-positive with the same centre of symmetry. So we get the following result, which is a generalization of [35, Corollary 5].

**Corollary 23.** Let \( q > 0 \) be a given real number. Then the polynomials \( D_{n}(fexc, cyc)(x, q) \) are \( \gamma \)-positive for \( n \geq 1 \).

It follows from (32) that
\[
F_{n}(fexc, neg)(x, p) = B_n(x^2, 1, 1, 1, px, 1) = (1 + px)^n A_n \left( \frac{x^2 + px}{1 + px} \right).
\]
In particular, \( F_{n}(fexc, neg)(x, -1) = (1 - x)^n A_n(-x) \). We can now present the following result.

**Theorem 24.** Let \( p \geq 1 \) be a given real number. For \( n \geq 1 \), the polynomials \( F_{n}(fexc, neg)(x, p) \) are bi-\( \gamma \)-positive, and so they are alternatingly increasing.
Lemma 25. Let \( \gamma \) are all \( \gamma \)-positive, which yields the desired result. This completes the proof. \( \square \)

5. PROOF OF THEOREM 21

Lemma 25. Let \( G_3 = \{ I \to Iq(f + ps), f \to y(x + py), s \to y(x + py), x \to y(x + py), y \to y(x + py) \} \). Then we have

\[
D^h_{G_3}(I) = \sum_{\sigma \in S^B_n} x^{\exc_{\lambda}(\sigma)} y^{\aexc_{\lambda}(\sigma)} f^{\fix(\sigma)} s^{\single(\sigma)} p^{\neg(\sigma)} q^{\cyc(\sigma)}. \tag{34}
\]

Proof. Let \( \sigma \in S^B_n \). We first introduce a grammatical labeling of \( \sigma \) as follows:

\begin{enumerate}
  \item (L1) Put a subscript label \( p \) right after every negative element of \( \sigma \);
  \item (L2) If \( i \) is a fixed point, then put a superscript label \( f \) right after \( i \), i.e., \( (i^f) \);
  \item (L3) If \( i \) is a singleton, then put a superscript label \( s \) right after \( 7 \), i.e., \( (i^s) \);
  \item (L4) If \( \sigma(i) > i \), then put a superscript label \( x \) just before \( \sigma(i) \);
  \item (L5) If \( \sigma(i) < i \), then put a superscript label \( y \) right after \( i \) or \( 7 \);
  \item (L6) Put a subscript label \( q \) right after each cycle;
  \item (L7) Put a superscript label \( I \) right after \( \sigma \).
\end{enumerate}

For example, if \( \sigma = (1, 2, 5, 4, 3)(6)(7, 8)(9, 10)(11, 12) \), then the grammatical labeling of \( \sigma \) is given as follows:

\[
(1^2 2^6 3^4 4^3 5^6 6^7 7^8 8^9 9^9 10^9 11^{10} 12^{10})_q.
\]

Note that the weight of \( \sigma \) is given by \( Ix^{\exc_{\lambda}(\sigma)} y^{\aexc_{\lambda}(\sigma)} f^{\fix(\sigma)} s^{\single(\sigma)} p^{\neg(\sigma)} q^{\cyc(\sigma)} \). Every permutation in \( S^B_n \) can be obtained from a permutation in \( S^B_{n-1} \) by inserting \( n \) or \( \bar{n} \). For \( n = 1 \), we have \( S^B_1 = \{(1^f)_q, (1^f)^T_p\} \). Note that \( D_{G_3}(I) = Iq(f + ps) \). Then the sum of weights of the elements in \( S^B_n \) is given by \( D_{G_3}(I) \). Hence the result holds for \( n = 1 \). We proceed by induction on \( n \). Suppose that we get all labeled permutations in \( S^B_{n-1} \), where \( n \geq 2 \). Let \( \bar{\sigma} \) be obtained from \( \sigma \in S^B_{n-1} \) by inserting \( n \) or \( \bar{n} \). When the inserted \( n \) or \( \bar{n} \) forms a new cycle, the insertion corresponds to the substitution rule \( I \to Iq(f + ps) \). Now we insert \( n \) or \( \bar{n} \) right after \( \sigma(i) \). If \( i \) is a fixed point or a singleton of \( \sigma \), then the changes of labeling are respectively illustrated as follows:

\[
\cdots (i^f)_q \cdots \to \cdots (i^f n^y)_q \cdots, \cdots (i^f)_q \cdots \to \cdots (i^f \bar{n}^y)_q \cdots.
\]
\[ \cdots (i_p^y) \cdot \cdots \cdots (i_p^y n^y) \cdot \cdots \cdots (i_p^y) \cdot \cdots \cdots (i_p^y n^y) \cdot \cdots . \]

If \( i \) is an excedance of type \( A \), then the changes of labeling are respectively illustrated as follows:

\[ \cdots (\cdots i^z \sigma (i) \cdots ) q \cdot \cdots \cdots (\cdots i^z n^y \sigma (i) \cdots ) q \cdot \cdots , \]

\[ \cdots (\cdots i^z \sigma (i) \cdots ) q \cdot \cdots \cdots (\cdots i^z n^y \sigma (i) \cdots ) q \cdot \cdots ; \]

\[ \cdots (\cdots i^z \sigma (i) \cdots ) q \cdot \cdots \cdots (\cdots i^z n^y \sigma (i) \cdots ) q \cdot \cdots , \]

\[ \cdots (\cdots i^z \sigma (i) \cdots ) q \cdot \cdots \cdots (\cdots i^z n^y \sigma (i) \cdots ) q \cdot \cdots . \]

The same argument applies to the case when the new element is inserted right after an element labeled by \( y \). In each case, the insertion of \( n \) or \( \overline{\sigma} \) corresponds to one substitution rule in \( G_3 \).

By induction, it is routine to check that the action of \( D_G \) on the set of weights of permutations in \( S_{n-1}^B \) gives the set of weights of permutations in \( S_n^B \). This completes the proof. \( \square \)

**A proof Theorem 21**

*Proof.* Let \( G_3 \) be the grammar that is given in Lemma 25. Consider a change of the grammar \( G_3 \). Setting \( A = f + ps, B = x + py \) and \( C = (1 + p)y \), we get

\[ D_{G_3}(I) = IqA, \quad D_{G_3}(A) = BC, \quad D_{G_3}(B) = BC, \quad D_{G_3}(C) = BC. \]

Let \( G_4 = \{ I \rightarrow IqA, A \rightarrow BC, B \rightarrow BC, C \rightarrow BC \} \). It follows from Lemma 12 that

\[ D_{G_4}^n(I) = I \sum_{\pi \in S_n} A^{\text{fix} (\pi)} B^{\text{exc} (\pi)} C^{\text{drop} (\pi)} q^{\text{cyc} (\pi)}. \quad (35) \]

Then upon taking \( A = f + ps, B = x + py \) and \( C = (1 + p)y \) in (35), we get (31). This completes the proof. \( \square \)

**6. Excedance statistics of colored permutations**

Let \( \pi^c \in Z_r \cap S_n \). Following Steingrímsson [37], the number of excedances of \( \pi^c \) is defined by

\[ \text{exc} (\pi^c) = \# \{ i \in [n] : i <_f \pi_i \}, \]

where the order \( <_f \) is defined by (24). The *r-colored Eulerian polynomial* is defined by

\[ A_{n,r}(x) = \sum_{\pi^c \in Z_r \cap S_n} x^{\text{exc} (\pi^c)}, \]

which satisfy the following recurrence relation

\[ A_{n,r}(x) = (1 + (rn - 1)x) A_{n-1,r}(x) + rx(1 - x) A'_{n,r}(x), \]

(36)

with \( A_{0,r}(x) = 1, A_{1,r}(x) = 1 + (r - 1)x, A_{2,r}(x) = 1 + (r^2 + 2r - 2)x + (r - 1)^2 x^2 \). Let \( A_{n,r}(x) = \sum_{k=0}^{n} A_r(n,k) x^k \). It follows from (36) that the numbers \( A_r(n,k) \) satisfy the recurrence

\[ A_r(n,k) = (rk + 1) A_r(n - 1,k) + (r(n - k) + r - 1) A_r(n - 1,k - 1), \]

with \( A_r(0,k) = \delta_{0,k} \). According to [37, Theorem 20], we have

\[ \sum_{n=0}^{\infty} A_{n,r}(x) \frac{z^n}{n!} = \frac{(1 - x)z^{1-x}}{1 - xe^{z(1-x)}}. \quad (37) \]
When $r = 1$ and $r = 2$, the polynomial $A_{n,r}(x)$ reduces to Eulerian polynomials of types $A$ and $B$, respectively. The $\gamma$-positivities of $A_{n,1}(x)$ and $A_{n,2}(x)$, along with several $q$-analogues, were discovered repeatedly, see, e.g., [24].

Recall that the $r$-colored derangement polynomials are defined by

$$d_{n,r}(x) = \sum_{\pi \in \mathcal{D}_{n,r}} x^{\text{exc}(\pi)} ,$$

where $\mathcal{D}_{n,r}$ is the set of all derangements in $\mathbb{Z}_r \wr S_n$. By using the theory of Rees products of posets [25], Athanasiadis [1, Theorem 1.3] obtained the following result.

**Theorem 26** ( [1, Theorem 1.3]). We have

$$d_{n,r}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^+_{n,r,i} x^i (1 + x)^{n-2i} + \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \xi^-_{n,r,i} x^i (1 + x)^{n+1-2i},$$

where $\xi^+_{n,r,i}$ is the number of colored permutations of $\mathbb{Z}_r \wr S_n$ for which $\text{Asc}(\pi) \in [2, n]$ has exactly $i$ elements, no two consecutive, and contains $n$, and $\xi^-_{n,r,i}$ is the number of colored permutations of $\mathbb{Z}_r \wr S_n$ for which $\text{Asc}(\pi) \in [2, n-1]$ has exactly $i-1$ elements, no two consecutive.

Define

$$d^+_{n,r}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^+_{n,r,i} x^i (1 + x)^{n-2i}, \quad d^-_{n,r}(x) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \xi^-_{n,r,i} x^i (1 + x)^{n+1-2i}.$$

By using the principle of inclusion-exclusion, one has $A_{n,r}(x) = \sum_{k=0}^n \binom{n}{k} d_{k,r}(x)$. Combining this with (38), Athanasiadis [3, Eq. (21)] obtained the following expansion:

$$A_{n,r}(x) = A^+_{n,r}(x) + A^-_{n,r}(x),$$

where $A^+_{n,r}(x) = \sum_{k=0}^n \binom{n}{k} d^+_{k,r}(x)$ and $A^-_{n,r}(x) = \sum_{k=0}^n \binom{n}{k} d^-_{k,r}(x)$. Comparing (7) with (37), we find that

$$A_{n,r}(x) = r^n A_n \left( x, \frac{1 + (r-1)x}{r}, 1 \right).$$

In particular, we have

$$B_n(x) = A_{n,2}(x) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} (1 + x)^{\text{fix}(\pi)} 2^{n-\text{fix}(\pi)}.$$

Motivated by (38) and (40), we shall first present a direct proof of the bi-$\gamma$-positivity of $A_{n,r}(x)$, and then we study some multivariate colored Eulerian polynomials.

**6.1. The bi-$\gamma$-positivity of $A_{n,r}(x)$ and an application.**

**Lemma 27.** If $G_5 = \{ u \rightarrow uv^r, v \rightarrow u^r v \}$, then we have

$$D_{G_5}^n(u^{r-1}v) = u^{r-1}v \sum_{k=0}^n A_r(n, k) u^{(n-k)r} v^{kr},$$

(41)
We proceed to the inductive step. Note that $D_{G_5}^0(u^{r-1}v) = u^{r-1}v$ and $D_{G_5}(u^{r-1}v) = u^{r-1}v(u^r + (r-1)v^r)$. Assume that the result holds for $n = m$, where $m \geq 1$. Then

$$D_{G_5}^{m+1}(u^{r-1}v)$$

$$= D_{G_5}(u^{r-1}v \sum_{k=0}^{m} A_r(m,k)u^{(m-k)r}v^r)$$

$$= u^{r-1}v \sum_{k} A_r(m,k)((mr - kr + r - 1)u^{(m-k)r}v^{(k+1)r} + (kr + 1)u^{(m-k+1)r}v^{kr}).$$

So we get that $A_r(m+1,k) = (rk + 1)A_r(m,k) + (r(m + 1 - k) + r - 1)A_r(m,k - 1)$. Thus the result holds for $n = m + 1$. This completes the proof. □

Recall that $A_{n,1}(x) = A_1(x)$ and $A_{n,2}(x) = B_n(x)$, which are both $\gamma$-positive polynomials.

**Theorem 28.** For each $r \geq 2$, the polynomial $A_{n,r}(x)$ are bi-$\gamma$-positive. More precisely, we have

$$A_{n,r}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \alpha_{n,k;r}^+ x^k (1 + x)^{n-2k} + x \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \alpha_{n,k;r}^- x^k (1 + x)^{n-1-2k},$$

where the numbers $\alpha_{n,k;r}^+$ and $\alpha_{n,k;r}^-$ satisfy the recurrence system

$$\begin{align*}
\alpha_{n+1,k;r}^+ &= (1 + rk)\alpha_{n,k;r}^+ + 2r(n - 2k + 2)\alpha_{n,k-1,r}^+ + 2\alpha_{n,k-1,r}^-
\alpha_{n+1,k;r}^- &= (r - 2)\alpha_{n,k;r}^+ + (r - 1 + rk)\alpha_{n,k;r}^- + 2r(n - 2k + 1)\alpha_{n,k-1,r}^-
\end{align*}$$

with the initial conditions $\alpha_{1,0;r}^+ = 1$, $\alpha_{1,0;r}^- = r - 2$, $\alpha_{1,k;r}^+ = \alpha_{1,k;r}^- = 0$ for $k \neq 0$.

**Proof.** Consider a change of the grammar given in Lemma 27. Note that

$$D_{G_5}(u^rv^r) = ru^rv^r(u^r + v^r), \quad D_{G_5}(u^rv^r) = 2ru^rv^r,$$

$$D_{G_5}(u^{r-1}v) = (r-2)u^{r-1}v^{r+1} + u^{r-1}v(u^r + v^r),$$

$$D_{G_5}(u^{r-1}v^{r+1}) = (r-1)u^{r-1}v^{r+1}(u^r + v^r) + 2u^{r-1}v(u^rv^r).$$

Setting $a = u^rv^r$, $b = u^r + v^r$, $c = u^r v^r+1$ and $I = u^{r-1}v$, we obtain $D_{G_5}(a) = rab$, $D_{G_5}(b) = 2ra$, $D_{G_5}(c) = (r-1)bc + 2Ia$, $D_{G_5}(I) = (r-2)c + Ib$. Clearly, $c = I v^r$. Consider the grammar

$$G_6 = \{ a \rightarrow rab, b \rightarrow 2ra, c \rightarrow (r-1)bc + 2Ia, I \rightarrow (r-2)c + Ib \}.$$

Note that $D_{G_6}(I) = Ib + (r-2)c$, $D_{G_6}^2(I) = (4(r-1)a + b^2)I + r(r-2)bc$. Then by induction, it is routine to check that there exist nonnegative integers such that

$$D_{G_5}^n(I) = \sum_{k=0}^{\lfloor n/2 \rfloor} \alpha_{n,k;r}^+ b^{n-2k}I + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \alpha_{n,k;r}^- b^{n-1-2k}c.$$

We proceed to the inductive step. Note that

$$D_{G_6}^{n+1}(I) = \sum_k \alpha_{n,k;r}^+ b^{n-2k}[(1 + rk)a^kb^2I + 2r(n - 2k)a^{k+1}I + (r - 2)a^{k+1}I] +$$

$$\sum_k \alpha_{n,k;r}^- b^{n-2k}[(r - 1 + rk)a^kb^2c + 2r(n - 1 - 2k)a^{k+1}c + 2a^{k+1}bI].$$
Taking coefficients of $a^k b^{n+1-2k} I$ and $a^k b^{n-2k} c$ on both sides and simplifying yields the desired recurrence system. Setting $u^r = 1$ and $v^r = x$, we have $a = x$, $b = 1 + x$ and $c = Ix$. Comparing (11) and (13), we immediately get (12).

Define

$$A_{n,r}^+(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k,r}^+ x^k (1 + x)^{n-2k}, \quad \tilde{A}_{n,r}^-(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k,r}^- x^k (1 + x)^{n-1-2k}.$$

Comparing this with (39), we see that for $n \geq 1$, $A_{n,r}(x) = A_{n,r}^+(x) + x \tilde{A}_{n,r}^-(x)$ and $\tilde{A}_{n,r}(x) = x \tilde{A}_{n,r}^-(x)$. It is routine to verify the following corollary.

**Corollary 29.** For $n \geq 1$, the polynomials $A_{n,r}^+(x)$ and $\tilde{A}_{n,r}^-(x)$ satisfy the recurrence system

$$A_{n+1,r}^+(x) = (1 + x + rnx) A_{n,r}^+(x) + rx(1-x) \frac{d}{dx} A_{n,r}^+(x) + 2x \tilde{A}_{n,r}^-(x),$$

$$\tilde{A}_{n+1,r}^-(x) = (r - 1 + (rn-1)x) \tilde{A}_{n,r}^-(x) + rx(1-x) \frac{d}{dx} \tilde{A}_{n,r}^-(x) + (r - 2) A_{n,r}^+(x),$$

with the initial conditions $A_{0,r}(x) = 1$ and $\tilde{A}_{0,r}(x) = 0$.

In the rest of this subsection, we present an application of Theorem 28. For $\sigma \in S_n^B$, let $\text{des}_B(\sigma) = \# \{ i \in \{0, 1, \ldots, n-1 \} \mid \sigma(i) > \sigma(i+1) \}$, where $\sigma(0) = 0$. Following Brenti [10, Corollary 3.16], the type B $q$-Eulerian polynomials can be defined as follows:

$$B_n(x, q) = \sum_{\pi \in S_n^B} x^{\text{des}_B(\pi)} q^{n-\text{neg}(\pi)} = \sum_{\pi \in S_n^B} x^{\text{exc}(\pi) + \text{fix}(\pi)} q n - \text{neg}(\pi).$$

The polynomials $B_n(x, q)$ satisfy the recurrence relation

$$B_n(x, q) = (1 + (1+q)nx - x)B_{n-1}(x, q) + (1+q)(x-x^2) \frac{\partial}{\partial x} B_{n-1}(x, q).$$

with $B_0(x, q) = 1$, $B_1(x, q) = 1 + qx$ and $B_2(x, q) = 1 + (1+4q+q^2)x + q^2 x^2$, and the exponential generating function of $B_n(x, q)$ is given as follows (see [10, Theorem 3.4]):

$$\sum_{n=0}^{\infty} B_n(x, q) \frac{x^n}{n!} = \frac{(1-x)e^{x(1-x)}(1-q)}{1 - xe^{x(1-x)(1+q)}}.$$ (44)

Comparing (37) with (44), we find that

$$A_{n,q+1}(x) = B_n(x, q).$$

Let $B_n(x, q) = \sum_{k=0}^{n} B_{n,k}(q) x^k$ and let $q^n B_n(x, 1/q) = \sum_{k=0}^{n} \tilde{B}_{n,k}(q) x^k$, where $q > 0$. It follows from [10, Corollary 3.16] that $B_{n,k}(q) = \tilde{B}_{n,n-k}(q)$. By using Theorem 28, we get the following.

**Corollary 30.** If $q \geq 1$, then the polynomial $B_n(x, q)$ is alternatingly increasing. If $0 \leq t \leq 1$, then $B_n(x, q)$ is spiral, i.e., $B_{n,t}(q) \leq B_{n,0}(q) \leq B_{n,n-1}(q) \leq B_{n,1}(q) \leq \cdots \leq B_{n,\lfloor n/2 \rfloor}(q)$. 

6.2. \((p, q)\)-colored Eulerian polynomials.

Let \(\pi^c \in \mathbb{Z}_r \wr S_n\). In the following discussion, we always write \(\pi^c\) by using its standard cycle decomposition, in which each cycle has its smallest element first and the cycles are written in increasing order of their first elements. Let \(\text{aexc}(\pi^c) = \# \{i \in [n] : \pi_i <_f i\}\) be the number of anti-excedances of \(\pi^c\). Consider the following \((p, q)\)-colored Eulerian polynomials:

\[
A_{n,r}(x, y, p, q) = \sum_{\pi^c \in \mathbb{Z}_r \wr S_n} x^{\text{exc}(\pi^c)y^{\text{aexc}(\pi^c)}} p^{\text{fix}(\pi^c)} q^{\text{cyc}(\pi^c)}.
\]

Then \(A_{n,1}(x, 1, p, q) = A_n(x, p, q), A_{n,r}(x, 1, 1, 1) = A_{n,r}(x)\) and \(A_{n,r}(x, 1, 0, q) = d_{n,r}(x, q)\).

**Lemma 31.** If \(G_7 = \{I \rightarrow qI ((r-1)x + p), x \rightarrow rxy, y \rightarrow rxy, p \rightarrow rxy\}\), then

\[
D_{G_7}^n(I) = I \sum_{\pi^c \in \mathbb{Z}_r \wr S_n} x^{\text{exc}(\pi^c)y^{\text{aexc}(\pi^c)}} p^{\text{fix}(\pi^c)} q^{\text{cyc}(\pi^c)}.
\]

**Proof.** We now introduce a grammatical labeling of \(\pi^c \in \mathbb{Z}_r \wr S_n\) as follows:

1. (L1) If \(i <_f \pi_i\), then we label \(\pi_i^c\) by a subscript label \(x\);
2. (L2) If \(\pi_i <_f i\), then we label \(\pi_i^c\) by a subscript label \(y\);
3. (L3) If \(\pi_i = i\) and \(c_i = 0\), then we label \(\pi_i\) by a subscript label \(p\);
4. (L4) Put a subscript label \(I\) right after \(\pi^c\), and put a superscript label \(q\) right after each cycle.

Note that the weight of \(\pi^c\) is given by

\[
w(\pi^c) = I x^{\text{exc}(\pi^c)y^{\text{aexc}(\pi^c)}} p^{\text{fix}(\pi^c)} q^{\text{cyc}(\pi^c)}.
\]

For \(n = 1\), we have

\[
\mathbb{Z}_r \wr S_1 = \{(1)^{q}_{1}, (1^0_{x} 1^r_{x})^{q}_{1}, (1^0_{y} 1^r_{y})^{q}_{1}, \ldots, (1^0_{p} 1^{r-1}_{p})^{q}_{1}\}.
\]

Note that \(D_{G_7}(I) = qI((r-1)x + p)\). Then the sum of weights of the elements in \(\mathbb{Z}_r \wr S_1\) is given by \(D_{G_7}(I)\). Hence the result holds for \(n = 1\). We proceed by induction on \(n\). Suppose we get all labeled permutations in \(\pi^c \in \mathbb{Z}_r \wr S_{n-1}\), where \(n \geq 2\). Let \(\pi^c\) be obtained from \(\pi^c \in \mathbb{Z}_r \wr S_{n-1}\) by inserting \(n^{c_j}\), where \(0 \leq c_j \leq r - 1\). When the inserted \(n^{c_j}\) forms a new cycle, the insertion corresponds to the substitution rule \(I \rightarrow qI((r-1)x + p)\) since we have \(r\) choices for \(c_j\). For the other cases, the changes of labeling are illustrated as follows:

\[
\cdots (\cdots \pi_i^{c_i} \pi_i^{c_{i+1}} \cdots) \cdots \mapsto \cdots (\cdots \pi_i^{c_i} x_n^{c_j} y_n^{c_{i+1}} \cdots) \cdots;
\]

\[
\cdots (\cdots \pi_i^{c_i} y_n^{c_{i+1}} \cdots) \cdots \mapsto \cdots (\cdots \pi_i^{c_i} x_n^{c_j} y_n^{c_{i+1}} \cdots) \cdots;
\]

\[
\cdots (\pi_i^{c_i} p) \cdots \mapsto \cdots (\pi_i^{c_i} x_n^{c_j} y) \cdots.
\]

In each case, the insertion of \(n^{c_j}\) corresponds to one substitution rule in \(G_7\). By induction, it is routine to check that the action of \(D_{G_7}\) on elements of \(\mathbb{Z}_r \wr S_{n-1}\) generates all elements of \(\mathbb{Z}_r \wr S_n\).

As a generalization of \([10]\), we now give the following result.

**Theorem 32.** One has

\[
A_{n,r}(x, y, p, q) = \sum_{\pi \in S_n} (rx)^{\text{exc}(\pi)}(ry)^{\text{aexc}(\pi)}((r-1)x + p)^{\text{fix}(\pi)} q^{\text{cyc}(\pi)}.
\]
Equivalently,
\[ A_{n,r}(x, y, p, q) = (ry)^n A_n \left( \frac{x}{y}, \frac{(r-1)x+p}{ry}, q \right). \] (46)

**Proof.** Let \( G_7 \) be the grammar that is given in Lemma \textsuperscript{31}. Consider a change of the grammar \( G_7 \). Setting \( a = (r-1)x + p, b = rx \) and \( c = ry \), we get
\[ D_{G_7}(I) = qIa, \quad D_{G_7}(a) = bc, \quad D_{G_7}(b) = bc, \quad D_{G_7}(c) = bc. \]

Let \( G_8 = \{ I \rightarrow qIa, a \rightarrow bc, b \rightarrow bc, c \rightarrow bc \} \). It follows from Lemma \textsuperscript{12} that
\[ D_{G_8}^n(I) = I \sum_{\pi \in S_n} b^{\text{exc}(\pi)} c^{\text{drop}(\pi)} a^{\text{fix}(\pi)} q^{\text{cyc}(\pi)}. \] (47)

Then upon taking \( a = (r-1)x + p, b = rx \) and \( c = ry \) in (47), we immediately get the desired result. This completes the proof.

\( \square \)

It follows from (46) that \( A_{n,r}(x, 1, x, q) = r^n A_n(x, x, q) \). Combining this result with Theorem \textsuperscript{17} we get the following corollary.

**Corollary 33.** Let \( q \in [0, 1] \) be a given real number. Then the polynomials \( A_{n,r}(x, 1, x, q) \) are bi-\( \gamma \)-positive for \( n \geq 1 \).

6.3. A generalization of Theorem \textsuperscript{21}

For any positive integer \( n \), let \([n]_q = 1 + q + \cdots + q^{n-1}\). As usual, set \([0]_q = 0\). Let \( \pi^c = \pi_1^c \pi_2^c \cdots \pi_n^c \in \mathbb{Z}_r \wr S_n \). Following \textsuperscript{14}\textsuperscript{35}, we define

\[ \begin{align*}
\text{exc}_A(\pi^c) &= \#\{ i \in [n] : i <_{<} \pi_i \text{ and } c_i = 0 \}, \\
\text{aexc}(\pi^c) &= \#\{ i \in [n] : \pi_i <_{<} c_i \}, \\
\text{fix}(\pi^c) &= \#\{ i \in [n] : \pi_i = i \text{ and } c_i = 0 \}, \\
\text{single}(\pi^c) &= \#\{ i \in [n] : \pi_i = i \text{ and } c_i > 0 \}, \\
\text{csum}(\pi^c) &= \sum_{i=1}^{n} c_i, \\
\text{fexc}(\pi^c) &= r \cdot \text{exc}_A(\pi^c) + \text{csum}(\pi^c),
\end{align*} \]

where the comparison is with respect to the order \( <_{<} \) of \( \Sigma: \)
\[ 1^{[r-1]} <_{<} 2^{[r-1]} <_{<} \cdots <_{<} n^{[r-1]} <_{<} \cdots <_{<} \top <_{<} \overline{\top} <_{<} \cdots <_{<} \overline{n} <_{<} 1 <_{<} 2 <_{<} \cdots <_{<} n. \]

Let \( \text{cyc}(\pi^c) \) be the number of cycles of \( \pi^c \). Consider the following polynomials
\[ A_n^{(r)}(x, y, f, s, p, q) = \sum_{\pi^c \in \mathbb{Z}_r \wr S_n} x^{\text{exc}_A(\pi^c)} y^{\text{aexc}} f^{\text{fix}(\pi^c)} s^{\text{single}(\pi^c)} p^{\text{csum}(\pi^c)} q^{\text{cyc}(\pi^c)}. \]

For \( 1 \leq i \leq n \), we introduce a grammatical labeling of \( \pi^c \) as follows:

- **(L1)** Put a subscript label \( p^{ci} \) right after each element of \( \pi^c \);
- **(L2)** If \( \pi_i = i \) and \( c_i = 0 \), then put a superscript label \( f \) right after \( i \);
- **(L3)** If \( \pi_i = i \) and \( c_i > 0 \), then put a superscript label \( s \) right after \( i \);
- **(L4)** If \( i <_{<} \pi_i \) and \( c_i = 0 \), then put a superscript label \( x \) just before \( \pi_i \);
- **(L5)** If \( \pi_i <_{<} c_i \), then put a superscript label \( y \) just before \( \pi_i \);
- **(L6)** Put a subscript label \( I \) right after \( \pi^c \) and put a superscript label \( q \) right after each cycle.
The grammatical labeling of elements in $\mathbb{Z}_r \uplus S_1$ are illustrated as follows:

$$Z_r \uplus S_1 = \{(1^f_p)_q, (1^p_p)_q, (\overline{1}^p_p)_q, \ldots, (1^{(r-1)}_p^p)_q\}.$$ 

Along the same lines as in the proof of Lemma 31 it is routine to check the following result and we omit the proof of it for simplicity.

**Lemma 34.** If $G_9 = \{I \rightarrow qI (f + sp[r-1]p), f \rightarrow xy + p[r-1]py^2, s \rightarrow xy + p[r-1]py^2, x \rightarrow xy + p[r-1]py^2, y \rightarrow xy + p[r-1]py^2\}$, then

$$D^n_{G_9}(I) = \sum_{\pi \in Z_r \uplus S_n} x^{\text{exc} \pi} y^{\text{exc} \pi} f^{\text{fix} \pi} s^{\text{single} \pi} p^{\text{csum} \pi} q^{\text{cyc} \pi}.$$ 

**Theorem 35.** One has

$$A_n^{(r)}(x, y, f, s, p, q) = \sum_{\pi \in S_n} (x + p[r-1]py)^{\text{exc} \pi} (y + p[r-1]py)^{\text{drop} \pi} (f + sp[r-1]p)^{\text{fix} \pi} q^{\text{cyc} \sigma}.$$ 

Equivalently,

$$A_n^{(r)}(x, y, f, s, p, q) = (y + p[r-1]py)^n A_n \left(\frac{x + p[r-1]py}{y + p[r-1]py}, \frac{f + sp[r-1]p}{y + p[r-1]py}, q\right). \quad (48)$$

**Proof.** Let $G_9$ be the grammar that is given in Lemma 34. Consider a change of the grammar $G_9$. Setting $u = f + sp[r-1]p, v = x + p[r-1]py$ and $w = y + p[r-1]py$, we get

$$D_{G_9}(I) = qIu, \quad D_{G_7}(u) = vw, \quad D_{G_9}(v) = vw, \quad D_{G_9}(w) = vw.$$ 

Let $G_{10} = \{I \rightarrow qIu, u \rightarrow vw, v \rightarrow vw, w \rightarrow vw\}$. It follows from Lemma 12 that

$$D^n_{G_{10}}(I) = \sum_{\pi \in S_n} \pi^{\text{exc} \pi} u^{\text{drop} \pi} \pi^{\text{fix} \pi} q^{\text{cyc} \pi}. \quad (49)$$

Then upon taking $u = f + sp[r-1]p, v = x + p[r-1]py$ and $w = y + p[r-1]py$ in (49), we immediately get the desired result. This completes the proof. \(\square\)

Note that

$$A_n^{(r)}(x^r, 1, 1, x, q) = \sum_{\pi \in Z_r \uplus S_n} x^{\text{exc} \pi} q^{\text{cyc} \pi},$$

$$A_n^{(r)}(x^r, 1, 0, s, x, q) = \sum_{\pi \in D_{n,r}} x^{\text{exc} \pi} s^{\text{single} \pi} q^{\text{cyc} \pi}.$$ 

By using (48), we get

$$A_n^{(r)}(x^r, 1, 1, x, q) = (1 + x + x^2 + \cdots + x^{r-1})^n A_n(x, 1, q), \quad (50)$$

$$A_n^{(r)}(x^r, 1, 0, s, x, q) = (1 + x + x^2 + \cdots + x^{r-1})^n A_n \left(x, \frac{s(x + x^2 + \cdots + x^{r-1})}{1 + x + x^2 + \cdots + x^{r-1}}, q\right). \quad (51)$$

It is well known that if $f(x)$ and $g(x)$ are both symmetric unimodal polynomials with nonnegative coefficients, then so is $f(x)g(x)$. Therefore, combining (51) and Proposition 8 we get the following result, which is a generalization of \[35\] eq. (2.5).
Corollary 36. For \( n \geq 1 \), one has
\[
\sum_{\pi^c \in D_{n,r}} x^{\text{fexc}(\pi^c)} \cdot \text{single}(\pi^c) \cdot q^{\text{cyc}(\pi^c)} = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (qsx[r-1]_x)^i [r]_x^{n-i} d_{n-i}(x,q),
\]
and so the single and cyc \((s,q)\)-flag derangement polynomials are symmetric unimodal when \( q > 0 \) and \( s > 0 \). In particular, one has
\[
\sum_{\pi^c \in D_{n,2}} x^{\text{fexc}(\pi^c)} \cdot \text{single}(\pi^c) \cdot q^{\text{cyc}(\pi^c)} = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (qsx)^i (1+x)^{n-i} d_{n-i}(x,q). \tag{52}
\]
Furthermore,
\[
\sum_{\pi^c \in D_{n,2}} x^{\text{fexc}(\pi^c)} \cdot \text{single}(\pi^c) \cdot q^{\text{cyc}(\pi^c)} = \sum_{k=1}^{n} \left( \sum_{i+j=k} \left( \begin{array}{c} n \\ i \end{array} \right) (qs)^i \sum_{\pi^c \in D_{n-i,j}} q^{\text{cyc}(\pi^c)} \right) x^k (1+x)^{2n-2k}.
\]
By using (52), we immediately get
\[
\sum_{\pi^c \in D_{n,2}} x^{\text{fexc}(\pi^c)} (1+x)^{\text{single}(\pi^c)} = x(1+x)^n A_n(x)
\]
for \( n \geq 1 \). Combining this with (29), we obtain
\[
\sum_{\pi^c \in \mathcal{S}_n} x^{\text{fexc}(\pi^c)+1} = \sum_{\pi^c \in D_{n,2}} x^{\text{fexc}(\pi^c)} (1+x)^{\text{single}(\pi^c)}.
\]
It is well known that (see [4, 11, 23] for details)
\[
A_n(x,1,-1) = -(x-1)^{n-1}, \tag{53}
\]
\[
A_n(x,0,-1) = -x[n-1]_x. \tag{54}
\]
Combining (50) and (53), we get
\[
\sum_{\pi^c \in \mathcal{Z}_n \cup \mathcal{S}_n} x^{\text{fexc}(\pi^c)} (-1)^{\text{cyc}(\pi^c)} = -\frac{(x^r-1)^n}{x-1},
\]
which has been obtained by Bagno and Garber [4, Theorem 1.1]. It follows from (51) and (54) that
\[
A_n^{(r)}(x^r,1,0,0,x,-1) = -x[n-1]_x[r]_x^n. \tag{55}
\]
It would be interesting to provide a bijective proof of (55).

7. Concluding remarks

This paper gives a systematic study of the \((p,q)\)-Eulerian polynomials \( A_n(x,p,q) \), and a sufficient condition for a polynomial to be alternatingly increasing is also established.
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