The generating function of lozenge tilings for a “quarter” of a hexagon, obtained with non-intersecting lattice paths

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Abstract

In a recent preprint, Lai and Rohatgi compute the generating functions of lozenge tilings of “quartered hexagons with dents” by applying the method of “graphical condensation”. The purpose of this note is to exhibit how (a generalization of) Theorems 2.1 and 2.2 in Lai and Rohatgi’s preprint can be achieved by the Lindström–Gessel–Viennot method of non–intersecting lattice paths and a certain determinant evaluation.

1 Introduction

In a recent preprint, Lai and Rohatgi [7] compute the generating functions of lozenge tilings of “quartered hexagons with dents”; their method of proof is

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induction based on “graphical condensation” (i.e., an application of a certain Pfaffian identity to the enumeration of matchings).

The purpose of this note is to exhibit how (a generalization of) Theorems 2.1 and 2.2 in [7] can be achieved by the Lindström–Gessel–Viennot method [8, 4] of non–intersecting lattice paths and the evaluation of the corresponding determinant. (Theorems 2.3 and 2.4 in [7] most probably can be achieved in the same way, but we want to be brief here.)

This note is organized as follows: In section 2, we shall describe the generating functions considered by Lai and Rohatgi, and show how the computation of these generating functions boils down (by the well–known bijective correspondence between lozenge tilings and non–intersecting lattice paths and the well–known Lindström–Gessel–Viennot argument) to the evaluation of a certain determinant. In section 3, we shall present two proofs for the product formula giving the evaluation of this determinant.

2 Lozenge tilings of a “quarter hexagon”

The literature on tilings enumerations is abundant (see, for instance, [2]), so we shall be brief. For the experienced reader it certainly suffices to have a look at the left pictures of Figures 1 and 2, which should make clear the concept of a “quarter hexagon” with “dents” (i.e., missing triangles) at its base line. All vertical lozenges of the tilings we consider here are labelled with integers: This labelling is vertically constant and horizontally increasing by 1 from left to right; in Figure 1, this labelling starts at 1, and in Figure 2, this labelling starts at 0.

Let $T$ be some lozenge tiling whose vertical lozenges are labelled $v_1, v_2, \ldots, v_n$, then the weight of $T$ is defined as

$$w(T) := \prod_{i=1}^{n} \frac{q^{v_i} + q^{-v_i}}{2}.$$ 

Lai and Rohatgi [7, Theorems 2.1–2.4] computed the generating functions of lozenge tilings for such “quarter hexagons with dents” of odd or even heights,
and with labels starting at 0 or 1 (see Figures 1 and 2), which resulted in four theorems: The case where the labelling starts at 0 (see Figure 2) requires a slight modification of the weight function, but we shall only consider the other case (labelling starts at 1, see Figure 1) and give a generalization which contains the cases of odd and even heights, thus giving an alternative proof for Theorems 2.1 and 2.2 in [7].

2.1 The bijective correspondence between lozenge tilings and non–intersecting lattice paths

The literature on the connection between lozenge tilings and non–intersecting lattice paths is abundant (see, for instance, [2, Section 5]); for the experienced reader it certainly suffices to have a look at the pictures in Figures 1 and 2: It is easy to see that there is a weight–preserving bijection between lozenge tilings and families of non–intersecting lattice paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ with steps to the right and downwards, where steps to the right from $(a, b)$ to $(a + 1, b)$ are labelled $a - 2b$ and thus have weight

$$\frac{q^{a-2b} + q^{2a-b}}{2}$$

(and all downward steps have weight 1). As usual, the weight of a lattice path is the product of all the weights of steps it consists of.

Clearly, the generating function $gf(a, b, c, d)$ of all lattice paths from initial point $(a, b)$ to terminal point $(c, d)$ is zero for $a > c$ or $b < d$. For $a < c$ and $b > d$, we claim

$$gf(a, b, c, d) = \frac{2^{a-c} q^{\frac{(a-c)(a+c-4d-1)}{2}} (q^{2(b-d+1)}; q^2)_{c-a} (-q^{2(a-b-d)}; q^2)_{c-a}}{(q^2; q^2)_{c-a}}. \quad (1)$$

Here, we used the standard $q$–Pochhammer notation $(a; q)_0 := 1$ and

$$(a; q)_n := \prod_{j=0}^{n-1} (1 - a \cdot q^j) \quad (2)$$

$$(a; q)^{-n} := \frac{1}{(a \cdot q^{-1}; q^{-1})_n} \quad (3)$$
The left picture shows a “quarter hexagon” of odd height 9 in the triangular lattice: The base line has 5 “dents” (i.e., missing triangles; indicated in the picture by black colour). The picture also shows a lozenge tiling of this “quarter hexagon with dents”, where the three possible orientations of lozenges (left–tilted, right–tilted and vertical) are indicated by three different colours: All the vertical lozenges are labelled with integers (this labelling is constant vertically and increasing by 1 horizontally from left to right), and the weight of this particular tiling is

$$w_1 \cdot w_3 \cdot w_5^2 \cdot w_6^4 \cdot w_7^2 \cdot w_9 \cdot w_{10}^2 \cdot w_{11} \cdot w_{12} \cdot w_{13} \cdot w_{14}^2 \cdot w_{15} \cdot w_{16},$$

where $w_i := \frac{q^i + q^{-i}}{2}$. The non-intersecting lattice paths corresponding to this tiling are indicated by white lines in the left picture; and the right picture shows a “reflected, rotated and tilted” version of these paths in the lattice $\mathbb{Z} \times \mathbb{Z}$, where horizontal edges $(a, b) \to (a + 1, b)$ are labelled $a - 2b$. Clearly, this correspondence is a bijection (introduced here “graphically”) between lozenge tilings and non-intersecting lattice paths, and this bijection is weight-preserving if we define the weight of some family $P$ of non-intersecting lattice paths as the product of $w_i$, where $i$ runs over the labels of all horizontal edges belonging to paths in $P$.

Figure 1: Pictures corresponding to Figure 2.2.a in Lai and Rohatgi’s preprint.
The left picture shows a “quarter hexagon with dents” of even height 8 in the triangular lattice, and the right picture shows the corresponding family of non-intersecting lattice paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ (as explained in Figure 1).

Figure 2: Pictures corresponding to Figure 2.2.d in Lai and Rohatgi’s preprint.

for integer $n > 0$.

Equation (1) follows immediately by showing that it fulfils the obvious recursion for the generating function of such weighted lattice paths

$$g_0(a, b, a, d) \equiv 1,$$

$$g_1(a, b, c, b) = \prod_{i=a-2b}^{c-2b-1} \frac{q^i + q^{-i}}{2} = 2^{a-c} q^{\frac{(a-c)(a-4b+c-1)}{2}} (-q^{2a-4b}, q^2)_{c-a},$$

$$g_1(a, b, c, d) = \frac{q^{a-2b} + q^{2b-a}}{2} g_1(a + 1, b, c, d) + g_1(a, b - 1, c, d)$$

for $a \leq c$ and $b \geq d$.

We have to specialize (1) to our situation, i.e., to initial points $(2i - 1, i - 1)$ and terminal points $(2m - 1 + k, a_j)$ (see the right picture in Figure 1), where $k = 0$ or $k = 1$ for quarter hexagons of odd or even height, respectively, and $(a_1, \ldots, a_m)$ is a strictly increasing sequence of integers (where the $a_j$
correspond to the positions of the “dents” in the lozenge tiling):
\[
gf(2i - 1, i - 1, 2m - 1 + k, a_j)
\]
\[
= \frac{2^{2i - k - 2m}q^{\frac{(2i - k - 2m)(2i + k + 2m - 4a_j - 3)}{2}}}{(q^2; q^2)_{2m + k - 2i}} (q^{4(i-a_j)}; q^4)_{2m+k-2i}. \quad (4)
\]
(Note that this is zero for \(a_j \geq i\).)

By the well–known the Lindström–Gessel–Viennot argument [8, 4], the generating function of all families of non–intersecting lattice paths (which, by the weight–preserving bijection sketched above is equal to the generating function of all lozenge tilings) can be written as the determinant
\[
\det_{1 \leq i, j \leq m} (gf(2i - 1, i - 1, 2m - 1 + k, a_j)).
\]

Theorems 2.1 and 2.2 in [7] state that this determinant factorizes completely for \(k = 0\) and \(k = 1\): Here, we shall show that this is true for general \(k \geq 0\).

By the multilinearity of the determinant, we may pull out the denominator in (4) from all rows,

- and powers of \(q\) depending only on \(i\) from the rows,
- and powers of \(q\) depending only on \(j\) from the columns,

which operation leaves the determinant
\[
\det_{1 \leq i, j \leq m} \left( q^{-(4a_j - 4i)} \cdot (q^{4(i-a_j)}; q^4)_{2m+k-2i} \right). \quad (5)
\]

Clearly, the claimed complete factorization of generating functions follows once we can show that the above determinant factorizes: For simplicity’s sake, we may substitute \(q^4 \rightarrow q\) in (5).

### 3 The evaluation of the determinant

It is not so easy to guess the correct evaluation of (5) from the “known special cases” Theorems 2.1 and 2.2 in [7] and from computer experiments, but once this is achieved, the proof is simple:
Proposition 3.1. Let $a = (a_1, a_2, \ldots, a_m)$ be a strictly increasing sequence of $m$ integers, and let $k$ be a nonnegative integer. Consider the $m \times m$–matrix

$$m(k; a) = \left( \frac{(q^{i-j}; q)_{2m+k-2i}}{q^{i}a_j} \right)_{i,j=1}^{m}. \quad (6)$$

Then we have

$$\det(m(k; a)) = q^{\frac{m(m-1)(2m+k-1)}{2}} - \sum_{l=1}^{m} a_l (2m-l) \prod_{j=1}^{m} (q^{m-a_j}; q)_{k} \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (1 - q^{a_i-a_j}) (1 - q^{-2m-k+1+a_i+a_j}) \cdot (7)$$

As a prerequisite for the proof of Proposition 3.1, recall Dodgson’s condensation formula [3]: Assume $m$ is an $(m \times m)$–matrix, and write $m_{i_1 \ldots i_m | j_1 \ldots j_m}$ for the submatrix obtained from $m$ by deleting rows $(i_1, \ldots)$ and columns $(j_1, \ldots)$. Then for $m \geq 2$, Dodgson’s condensation formula states

$$\det(m) \cdot \det(m_{i_1 \ldots i_m | j_1 \ldots j_m}) = \det(m_{i_1 \ldots i_m}) \cdot \det(m_{j_1 \ldots j_m}) - \det(m_{i_1 \ldots i_m}) \cdot \det(m_{j_1 \ldots j_m}) \cdot \det(m_{i_1 \ldots j_m}). \quad (8)$$

(By convention, the determinant of a $(0 \times 0)$–matrix equals 1.)

Remark 3.2. Dodgson’s condensation formula is also known as Desnanot–Jacobi’s adjoint matrix theorem, see [1, Theorem 3.12]: According to [1], Lagrange discovered this theorem for $n = 3$, Desnanot proved it for $n \leq 6$ and Jacobi published the general theorem [5], see also [9, vol. I, pp. 142]).

Proof. Observe that for $m = m(k; a)$, all the submatrices appearing in Dodgson’s condensation formula (8) are of a “similar type”, which we can describe in a simple manner by introducing the notation $\mathbf{d} \mathbf{g} (x_1, \ldots, x_k)$ for the $k \times k$ diagonal matrix with entries $x_i$ on the main diagonal, and defining

$$'a = (a_2, \ldots, a_m),$$
$$a' = (a_1, \ldots, a_{m-1}),$$
$$'a' = (a_2, \ldots, a_{m-1}),$$
$$a - 1 = (a_1 - 1, \ldots, a_m - 1).$$
for any sequence \( \mathbf{a} = (a_1, \ldots, a_m) \) of length \( m \geq 2 \):

\[
\begin{align*}
\mathbf{m}(k; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} &= d_g \left( q^{-1}, \ldots, q^{-(m-1)} \right) \cdot \mathbf{m}(k; \mathbf{a}' - 1) \cdot d_g \left( q^{-a_2}, \ldots, q^{-a_m} \right), \\
\mathbf{m}(k; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} &= \mathbf{m}(k + 2; \mathbf{a}''), \\
\mathbf{m}(k; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} &= d_g \left( q^{-1}, \ldots, q^{-(m-1)} \right) \cdot \mathbf{m}(k; \mathbf{a}' - 1) \cdot d_g \left( q^{-a_2}, \ldots, q^{-a_m-1} \right), \\
\mathbf{m}(k; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} &= \mathbf{m}(k + 2; \mathbf{a}''), \\
\mathbf{m}(k; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} &= d_g \left( q^{-2}, \ldots, q^{-(m-1)} \right) \cdot \mathbf{m}(k + 2; \mathbf{a}' - 1) \cdot d_g \left( q^{-a_2}, \ldots, q^{-a_m-1} \right).
\end{align*}
\]

Observe that both sides of (7) are zero iff \( a_m \geq m \). So we may assume \( a_m < m \): But then the last element \( a_m - 1 \) of \( \mathbf{a}' - 1 \) is less than \( m - 2 \), too, and hence \( \det \left( \mathbf{m}(k + 2; \mathbf{a})_{\mathcal{L} \mid \mathcal{L}'} \right) \neq 0 \). Now writing \( d(k; \mathbf{a}) := \det \left( \mathbf{m}(k; \mathbf{a}) \right) \), by Dodgson’s condensation formula (8) and the multiplicativity of the determinant we see

\[
d(k; \mathbf{a}) = q^{-m+1} \cdot d(k + 2; \mathbf{a}'') \cdot d(k; \mathbf{a}' - 1) - q^{-a_1} \cdot d(k + 2; \mathbf{a}) \cdot d(k; \mathbf{a}' - 1).
\]

(9)

So we can use induction on the length \( m \) of the sequence \( \mathbf{a} \): For \( m = 0 \) and \( m = 1 \), (7) is obviously true; and for the inductive step, we may use (9) (leading to a lengthy, but straightforward computation).

3.1 A second proof

When I showed the determinant (7) to Christian Krattenthaler, he almost immediately recognized it as a special case of his Lemma 4 in [6], which we shall repeat here for reader’s convenience:

**Lemma 3.3.** Let \( X_1, X_2, \ldots, X_m, A_2, \ldots, A_m \) be indeterminates. Then there
holds
\[
\det_{1 \leq i, j \leq m} \left( (C/X_i + A_m)(C/X_i + A_{m-1}) \cdots (C/X_i + A_{j+1}) \cdot (X_i + A_m)(X_i + A_{m-1}) \cdots (X_i + A_{j+1}) \right) = \prod_{i=2}^{m} A_i^{-1} \prod_{1 \leq i < j \leq m} (X_i - X_j)(1 - C/X_iX_j).
\]

Using Krattenthaler’s Lemma has the big advantage that it yields the determinant evaluation (i.e., there is no need to guess it).

Second proof of Proposition 3.1. By writing
\[
\left( q^{i-a_j}; q \right)_{2m+k-2i} = \left( q^{i-a_j}; q \right)_{m-i} \cdot \left( q^{m-a_j}; q \right)_k \cdot \left( q^{m+k-a_j}; q \right)_{m-i}
\]
and pulling out the middle factors \( (q^{m-a_j}; q)_k \) from all columns of \( m(k; a) \), we obtain
\[
d(k; a) = \left( \prod_{j=1}^{m} (q^{m-a_j}; q)_k \right) \det_{1 \leq i, j \leq m} \left( q^{-ia_j} \left( q^{i-a_j}; q \right)_{m-i} \left( q^{m+k-a_j}; q \right)_{m-i} \right).
\]

By setting \( X_j = q^{-a_j} \), expanding the \( q \)-Pochhammer symbols and pulling out powers of \( q \), the determinant from the above expression becomes
\[
= \det_{1 \leq i, j \leq m} \left( X_j^i \left( q^{i}X_j; q \right)_{m-i} \left( q^{m+k}X_j; q \right)_{m-i} \right)
\]
\[
= \det_{1 \leq i, j \leq m} \left( X_j^i \left( \prod_{l=0}^{m-i-1} q^{i-l} (q^{-i-l} - X_j) \right) \right.
\]
\[
\times \left( \prod_{l=0}^{m-i-1} X_j q^{2m+k-1} (X_j^{-1} q^{-2m-k+1} - q^{-m+l+1}) \right) \right)
\]
\[
= \prod_{l=1}^{m} \left( X_l^m q^{m \left( \begin{array}{c} m \\ 2 \end{array} \right) - \left( \begin{array}{c} l \\ 2 \end{array} \right) + (2m+k-1)(m-0)} \right.
\]
\[
\times \left( \prod_{l=0}^{m-i-1} q^{i-l} - X_j \right) \left( \prod_{l=0}^{m-i-1} (X_j^{-1} q^{-2m-k+1} - q^{-m+l+1}) \right) \right).
\]

Now apply Lemma 3.3 with \( X_j = q^{-a_j} \), \( A_i = q^{-i+1} \) and \( C = q^{-2m-k+1} \) to obtain (7) (after some straightforward computation). \( \square \)
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References

[1] D. Bressoud. Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture. Cambridge University Press, New York, 1999.

[2] M. Ciucu, T. Eisenkölbl, C. Krattenthaler, and D. Zare. Enumeration of lozenge tilings of hexagons with a central triangular hole. J. Combin. Theory Ser. A, 95:251–334, 2001.

[3] C.L. Dodgson. Condensation of determinants, being a new and brief method for computing their arithmetic values. Proc. Roy. Soc. London, 15:150–155, 1866.

[4] I.M. Gessel and X. Viennot. Determinants, paths, and plane partitions. preprint, 1989.

[5] C.G. Jacobi. De formatione et proprietatibus determinantium. Journal für Reine und Angewandte Mathematik, 22:285–318, 1841.

[6] C. Krattenthaler. Advanced determinant calculus. In Dominique Foata and Guo-Niu Han, editors, The Andrews Festschrift, pages 349–426, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

[7] Tri Lai and Ranjan Rohatgi. Tiling generating functions of halved hexagons and quartered hexagons. arXiv e-prints, page arXiv:2006.11806, June 2020.

[8] B. Lindström. On the vector representation of induced matroids. Bull. London Math. Soc., 5:85–90, 1973.

[9] T. Muir. The Theory of Determinants in the historical order of development. MacMillan and Co., Limited, 1906.