Abstract

A three-parameter logarithmic function is derived using the notion of q-analogue and ansatz technique. The derived three-parameter logarithm is shown to be a generalization of the two-parameter logarithmic function of Schwämmle and Tsallis as the latter is the limiting function of the former as the added parameter goes to 1. The inverse of the three-parameter logarithm and other important properties are also proved. A three-parameter entropic function is then defined and is shown to be analytic and hence Lesche-stable, concave and convex in some ranges of the parameters.

Keywords. entropy, logarithmic function, Boltzmann-Gibbs entropy, Shannon entropy, Tsallis entropy

1 Introduction

The concept of entropy provides deep insight into the direction of spontaneous change for many everyday phenomena. For example, a block of ice placed on a hot stove surely melts, while the stove grows cooler. Such a
process is called irreversible because no slight change will cause the melted water to turn back into ice while the stove grows hotter [7]. The concept of entropy was first introduced by German physicist Rudolf Clausius as a precise way of expressing the second law of thermodynamics.

The Boltzmann equation for entropy is

\[ S = k_B \ln \omega, \]  

(1.1)

where \( k_B \) is the Boltzmann constant [10] and \( \omega \) is the number of different ways or microstates in which the energy of the molecules in a system can be arranged on energy levels [9]. The Boltzmann entropy plays a crucial role in the foundation of statistical mechanics and other branches of science [5].

The Boltzmann-Gibbs-Shannon entropy [13, 14] is given by

\[ S_{BGS} \equiv -k \sum_{i=1}^{\omega} p_i \ln p_i = k \sum_{i=1}^{\omega} p_i \ln \frac{1}{p_i}, \]

(1.2)

where

\[ \sum_{i=1}^{\omega} p_i = 1. \]

(1.3)

\( S_{BGS} \) is a generalization of the Boltzmann entropy because if \( p_i = \frac{1}{\omega}, \) for all \( i, \)

\[ S_{BGS} = k \ln \omega. \]

(1.4)

Systems presenting long range interactions and/or long duration memory have been shown not well described by the Boltzmann-Gibbs statistics. Some examples may be found in gravitational systems, Lévy flights, fractals, turbulence physics and economics. In an attempt to deal with such systems Tsallis [15] postulated a nonextensive entropy which generalizes Boltmann-Gibbs entropy through an entropic index \( q \) [3]. Another generalization was also suggested by Renyi [11]. Abe [1] proposed how to generate entropy functionals.

Tsallis \( q \)-entropy [15] is given by

\[ S_q \equiv k \frac{1 - \sum_{i=1}^{\omega} p_i^q}{q - 1} = k \sum_{i=1}^{\omega} p_i \ln_q \frac{1}{p_i}, \]

(1.5)
where \( q \in \mathbb{R}, \sum_{i=1}^{\omega} p_i = 1 \) and
\[
\ln_q x \equiv \frac{x^{1-q} - 1}{1-q}, \quad (\ln_1 x = \ln x),
\] (1.6)
which is referred to as \( q \)-logarithm. If \( p_i = \frac{1}{\omega} \) for all \( i \), then
\[
S_q = k \ln_q \omega.
\] (1.7)
The inverse of the \( q \)-logarithm is the \( q \)-exponential
\[
e^x_q \equiv [1 + (1-q)x]_{1-q}^{1-q}, \quad (e^1 = e^x),
\] (1.8)
where \([\cdots]_+\) is zero if its argument is nonpositive.

A \( q \)-sum and \( q \)-product and their calculus studied in \[4\] were respectively defined as follows (these were also mentioned in \[13\]):
\[
x \oplus_q y \equiv x + y + (1-q)xy, \quad (x \oplus_1 y = x + y)
\] (1.9)
\[
x \otimes_q y \equiv (x^{1-q} + y^{1-q} - 1)^{1-q}, \quad (x \otimes_1 y = xy).
\] (1.10)
The \( q \)-logarithm satisfies the following properties:
\[
\ln_q(xy) = \ln_q x \oplus_q \ln_q y
\] (1.11)
\[
\ln_q(x \otimes_q y) = \ln_q x + \ln_q y.
\] (1.12)

Then a two-parameter logarithm was defined and presented along with a two-parameter entropy in \[13\]. It was defined as follows:
\[
\ln_{q,q'} x = \frac{1}{1-q'} \left[ \exp \left( \frac{1-q'}{1-q} (x^{1-q} - 1) \right) - 1 \right].
\] (1.13)
The above doubly deformed logarithm satisfies
\[
\ln_{q,q'}(x \otimes_q y) = \ln_{q,q'} x \oplus_{q'} \ln_{q,q'} y.
\] (1.14)
Properties of the two-parameter logarithm and those of the two-parameter entropy were proved in \[13\]. Probability distribution in the canonical ensemble of the two-parameter entropy was obtained in \[2\] while applications were discussed in \[6\].

In section 2 of the present paper, a three-parameter logarithm \( \ln_{q,q',r} x \), where \( q,q',r \in \mathbb{R} \), is derived using \( q \)-analogues and ansatz technique. In section 3, the inverse of the three-parameter logarithm is derived and some properties are proved. A three-parameter entropy and its properties are presented in section 4 and conclusion is given in section 5.
2 Three-Parameter Logarithm

As \( x = e^{\ln x} \), a \( q \)-analogue of \( x \) will be defined by

\[
[x]_q = e^{\ln_q x},
\]

where \( \ln_q x \) is defined in (1.6). Similarly, the \( q' \)-analogue of \([x]_q\) is defined by

\[
[x]_{q,q'} = e^{\ln_{q,q'} x}
\]

where \( \ln_{q,q'} x \) is as defined in (1.13), which can be written

\[
\ln_{q,q'} x = \frac{[x]_{q,q'}^{1-q'} - 1}{1 - q'} = \frac{(e^{\ln_q x})^{1-q'} - 1}{1 - q'}.
\]

The three-parameter logarithm is then defined as

\[
\ln_{q,q',r} x = \frac{[x]_{q,q',r}^{1-r} - 1}{1 - r} = \frac{(e^{\ln_{q,q'} x})^{1-r} - 1}{1 - r},
\]

from which

\[
\ln_{q,q',r} x \equiv \frac{1}{1 - r} \left\{ e^{\left( \frac{1}{1-q'} \left( e^{(1-q')\ln_q x - 1} \right) \right)^{1-r} - 1} \right\}.
\]

To obtain similar property as that in (1.14), define \( x \otimes_{q,q'} y \) as the \( q' \)-analogue of \( x \otimes_q y \). That is,

\[
x \otimes_{q,q'} y \equiv [x \otimes_q y]_{q'} = ([x]_{q}^{1-q} + [y]_{q}^{1-q} - 1)^{\frac{1}{1-q'}}.
\]

Then, from (2.4) and (2.6)

\[
\ln_{q,q'}(x \otimes_{q,q'} y) = \frac{[x \otimes_{q,q'} y]_{q}^{1-q'} - 1}{1 - q'} = \frac{([x]_{q}^{1-q'} + [y]_{q}^{1-q'} - 1)^{\frac{1}{1-q'}} - 1}{1 - q'} = \frac{[x]_{q}^{1-q'} - 1}{1 - q'} + \frac{[y]_{q}^{1-q'} - 1}{1 - q'}.
\]
\[ \ln_{q,q'} x + \ln_{q,q'} y. \] 

In similar manner and using (2.2),

\[ \ln_{q,q'},r (x \otimes q'y) = \frac{[x \otimes q'y]^{1-r} - 1}{1-r} \]

\[ = \frac{\{e^{\ln_{q,q'}(x \otimes q'y)}\}^{1-r} - 1}{1-r} \]

\[ = \frac{(e^{\ln_{q,q'} x + \ln_{q,q'} y})^{1-r} - 1}{1-r} \]

\[ = \frac{(e^{\ln_{q,q'} x})^{1-r} \{e^{\ln_{q,q'} y}\}^{1-r} - 1}{1-r} \]

\[ = \frac{\{e^{\ln_{q,q'} x}\}^{1-r} - 1}{1-r} \{e^{\ln_{q,q'} y}\}^{1-r} - 1 \]

\[ \{e^{\ln_{q,q'} x}\}^{1-r} - 1 \}

\[ + \{e^{\ln_{q,q'} y}\}^{1-r} - 1 \}

\[ = \ln_{q,q'},x \otimes_r \ln_{q,q'},y. \]

(2.8)

Thus,

\[ \ln_{q,q'},x (x \otimes q'y) = \frac{(e^{\ln_{q,q'} x})^{1-r} - 1}{1-r} + \frac{(e^{\ln_{q,q'} y})^{1-r} - 1}{1-r} \]

\[ + (1-r) \left[ \frac{(e^{\ln_{q,q'} x})^{1-r} - 1}{1-r} \right] \left[ \frac{(e^{\ln_{q,q'} y})^{1-r} - 1}{1-r} \right] \]

\[ = \ln_{q,q'},x + \ln_{q,q'},y + (1-r)[\ln_{q,q'},x][\ln_{q,q'},y] \]

(2.9)

which is the desired relation analogous to (1.14).

One can also derive (2.5) using ansatz. To do this, let \( x = y \) in (2.10).

Then

\[ \ln_{q,q'},x (x \otimes q'y) = \ln_{q,q'},x \otimes_r \ln_{q,q'},x. \] 

(2.11)

Taking

\[ \ln_{q,q'},x = G(\ln_{q,q'} x) = G(z), \] 

(2.12)

then

\[ \ln_{q,q'},x (x \otimes q'y) = G(\ln_{q,q'}(x \otimes q'y)) \]
\begin{align}
G(\ln q, q' x) &= G(\ln q, q' x) \\
&= G(2 \ln q, q' x) \\
&= G(2z).
\end{align} \quad (2.13)

Thus, from (2.9) and (2.10),
\begin{align}
G(2 \ln q, q' x) &= \ln q, q' x \oplus_r \ln q, q' x \\
&= \ln q, q' x + \ln q, q' x + (1-r)(\ln q, q' x)^2 \\
&= 2G(\ln q, q' x) + (1-r)[G(\ln q, q' x)]^2 \\
G(2z) &= 2G(z) + (1-r)[G(z)]^2. \quad (2.14)
\end{align}

The ansatz
\begin{align}
G(z) &= \frac{1}{1-r}(b^z - 1), \quad (2.15)
\end{align}
where \( z = \ln q, q' x \) will give
\begin{align}
2G(z) + (1-r)[G(z)]^2 &= 2 \cdot \frac{1}{1-r}(b^z - 1) + (1-r) \left[ \frac{1}{1-r}(b^z - 1) \right]^2 \\
&= \frac{2}{1-r}(b^z - 1) + \frac{(b^z - 1)^2}{1-r} \\
&= \frac{2b^z - 2 + b^{2z} - 2b^z + 1}{1-r} \\
&= \frac{b^{2z} - 1}{1-r} \\
&= G(2z), \quad (2.16)
\end{align}
which means that (2.15) solves the equation
\[ G(2z) = 2G(z) + (1-r)[G(z)]^2. \]

Thus,
\[ G(z) = G(\ln q, q' x) = \ln q, q' x = \frac{1}{1-r}(b^{\ln q, q' x} - 1). \]

Using the property that \( \frac{d}{dx} \ln q, q' x \bigg|_{x=1} = 1 \), which is a natural property of a logarithmic function, it is determined that \( b = e^{1-r} \).

Consequently,
\begin{align}
\ln q, q' x &= \frac{1}{1-r}(e^{(1-r)\ln q, q' x} - 1). \quad (2.17)
\end{align}
Explicitly,
\[ \ln_{q,q',r} x = \frac{1}{1 - r} \left( e^{\frac{1}{1-q} \left[ \exp\left( \frac{1-q'}{1-q} (x^{1-q} - 1) \right) - 1 \right]} - 1 \right), \]  
(2.18)
which is the same as that in \[(2.3).\] The preceding equation can be written
\[ \ln_{q,q',r} x = \ln_r e^{\ln_{q,q'} x}. \]  
(2.19)

It can be easily verified that
\[ \lim_{r \to 1} \ln_{q,q',r} x = \ln_{q,q'} x. \]  
(2.20)

Graphs of \( \ln_{q,q',r} x \) for \( q = q' = r \) are shown in Figure 1 while graphs of \( \ln_{q,q',r} x \) with one fixed parameter are shown in Figure 2.

3 Properties

In this section the inverse of the three-parameter logarithmic function will be derived. It is also verified that the derivative of this logarithm at \( x = 1 \) is 1 and that the value of the function at \( x = 1 \) is zero. Moreover, it is shown that the following equality holds
\[ \ln_{q,q',r} \frac{1}{x} = - \ln_{2-q,2-q',2-r} x. \]  
(3.1)

It follows from \[(2.4)\] that the three-parameter logarithmic function is an increasing function of \( x \). Thus, a unique inverse function exists. To find the inverse function let \( y = \ln_{q,q',r}(x) \) and solve for \( x \). That is,
\[ y = \frac{1}{1 - r} \left\{ \exp\left( \frac{1-r}{1-q} \exp\left( \frac{1-q'}{1-q} (x^{1-q} - 1) \right) - 1 \right) - 1 \right\}, \]
from which
\[ x = \left\{ 1 + \frac{1-q}{1-q'} \ln \left[ 1 + \frac{1-q'}{1-r} \ln \{ 1 + (1-r)y \} \right] \right\}^{\frac{1}{1-q}}. \]  
(3.2)

Thus, the inverse function is given by
\[ e_{q,q',r}^y = \exp_{q,q',r} y = \left\{ 1 + \frac{1-q}{1-q'} \ln \left[ 1 + \frac{1-q'}{1-r} \ln \{ 1 + (1-r)y \} \right] \right\}^{\frac{1}{1-q}}. \]
\[
\left\{ 1 + \frac{1-q}{1-q'} \ln \left[ 1 + (1-q') \ln \{1 + (1-r)y\} \right] \right\}^{\frac{1}{1-q'}} \\
= \left\{ 1 + \frac{1-q}{1-q'} \ln [1 + (1-q') \ln e_y] \right\}^{\frac{1}{1-q'}} \\
= \left\{ 1 + (1-q) \ln [1 + (1-q') \ln e_y] \right\}^{\frac{1}{1-q}} \\
= \left\{ 1 + (1-q) \ln e_{q'y}^{\ln e_y} \right\}^{\frac{1}{1-q}} \\
= e_{q'y}^{\ln e_y} \\
= \exp_q \left\{ \ln e_{q'y}^{\ln e_y} \right\}, \tag{3.3}
\]

where the q-exponential $e_q^x$ is defined in (1.8).

To find the derivative, use (2.5) to obtain
\[
\frac{d}{dx} \ln_{q,q',r} x = x^{-q} \exp \left\{ \frac{1-r}{1-q'} \left( e_{q'y}(1-q') \ln_q x - 1 \right) \right\}. \tag{3.4}
\]

Since $\ln_q 1 = 0$, it follows that
\[
\left. \frac{d}{dx} \ln_{q,q',r} x \right|_{x=1} = e^0 = 1. \tag{3.5}
\]

Moreover,
\[
\ln_{q,q',r} 1 = \frac{1}{1-r} \left\{ \exp \left( \frac{1-r}{1-q'} \left( e_{q'y}(1-q') \ln_q x - 1 \right) \right) - 1 \right\} = 0. \tag{3.6}
\]

From (3.4), the slope of $\ln_{q,q',r} x$ is positive for all $x > 0$. This is also observed in Figures 1 and 2.

To prove (3.1), let $q \to 2 - q$, $q' \to 2 - q'$ and $r \to 2 - r$. From [13],
\[
\ln_{q,q'} \frac{1}{x} = - \ln_{2-q,2-q'} x, \tag{3.7}
\]

then
\[
\ln_{q,q',r} \frac{1}{x} = \frac{(e^{\ln_{q,q'} \frac{1}{x}})^{1-r} - 1}{1-r}
\]
Figure 1. Illustration of the three-parameter logarithm in Eq. (2.18), setting \( q = q' = r \) in linear scales (left) and semi-logarithmic scales (right).

\[
\begin{align*}
S_{q,q',r} &\equiv k \sum_{i=1}^{\omega} p_i \ln_{q,q',r} \frac{1}{p_i} \\
&= \frac{(e^{-\ln_2 q,2-q' x})^{1-r} - 1}{1-r} \\
&= \frac{(e^{\ln_2 q,2-q' x})^{r-1} - 1}{-(r-1)} \\
&= \frac{-\{(e^{\ln_2 q,2-q' x})^{1-(2-r)}\} - 1}{1 - (2-r)} \\
&= -\ln_{2-q,2-q',2-r} x. \quad (3.8)
\end{align*}
\]

4 Three-Parameter Entropy

A three-parameter generalization of the Boltzmann-Gibbs-Shannon entropy is constructed here and its properties are proved. Based on the three-parameter logarithm the entropic function is defined as follows:

\[
S_{q,q',r} = k \sum_{i=1}^{\omega} p_i \ln_{q,q',r} \frac{1}{p_i} \\
\]

If \( p_i = \frac{1}{\omega}, \forall i \),

\[
S_{q,q',r} = k \ln_{q,q',r} \omega, \quad (4.2)
\]

where \( \omega \) is the number of states.
Figure 2. Illustration of the three-parameter logarithm for fixed value of one parameter.

Lesche-stability (or experimental robustness). The functional form of $\ln_{q,q',x} x$ given in the previous section is analytic in $x$ as $\ln_{q,q'} x$ is analytic in $x$. Consequently $S_{q,q',x}$ is Lesche-stable.
Expansibility. An entropic function $S$ satisfies this condition if a zero-probability ($p_i = 0$) state does not contribute to the entropy. That is, $S(p_1, p_2, \ldots, p_w, 0) = S(p_1, p_2, \ldots, p_w)$ for any distribution $\{p_i\}$. Observe that in the limit $p_i = 0$, $\ln_{q, q', r} \frac{1}{p_i}$ is finite if one of $q, q', r$ is greater than 1. Consequently,

$$S_{q, q', r}(p_1, p_2, \ldots, p_w, 0) = S_{q, q', r}(p_1, p_2, \ldots, p_w)$$

(4.3)

provided that one of $q, q', r$ is greater than 1.

Concavity. Concavity of the entropic function $S_{q, q', r}$ is assured if

$$\frac{d^2}{dp_i^2} \left( p_i \ln_{q, q', r} \frac{1}{p_i} \right) < 0$$

(4.4)

in the interval $0 \leq p_i \leq 1$.

By manual calculation (which is a bit tedious) and checked using derivative calculator,

$$\frac{d^2}{dp_i^2} \left( p_i \ln_{q, q', r} \frac{1}{p_i} \right) = \exp \left\{ \frac{1 - r}{1 - q'} (e^{(1-q') \ln q \frac{1}{p_i}} - 1) \right\} e^{(1-q') \ln q \frac{1}{p_i} \times}$$

\[-qp_i^{q-2} + (1 - q')p_i^{2q-3} + (1 - r)p_i^{2q-3} e^{(1-q') \ln q \frac{1}{p_i}} \right\}.

(4.5)

In the limit $p_i \to 1$, the second derivative given in (4.5) is less than zero if $q + q' + r > 2$. Thus, concavity of $S_{q, q', r}$ is guaranteed if $q + q' + r > 2$. In the limit $p_i \to 0$, concavity is guaranteed if $r > 1$. If $r < 1$, concavity holds if $q > 1$.

Convexity. A twice-differentiable function of a single variable is convex if and only if its second derivative is nonnegative on its entire domain. The analysis on the convexity of $S_{q, q', r}$ is analogous to that of its concavity. In the limit $p_i \to 1$, convexity is guaranteed if $q + q' + r \leq 2$. In the limit $p_i \to 0$, convexity is assured if $q, r < 1$.

Concavity of $S_{q, q', r}$ is illustrated in Figure 3 (A) while convexity is illustrated in Figure 3 (B).

Composability. An entropic function $S$ is said to be composable if for events A and B,

$$S(A + B) = \Phi(S(A), S(B), indices)$$
where $\Phi$ is some single-valued function \cite{13}. The Botzmann-Gibbs-Shannon entropy satisfies

$$S_{BGS}(A + B) = S_{BGS}(A) + S_{BGS}(B),$$

hence it is composable and additive. The one-parameter entropy $S_q$, for $q \neq 1$ is also composable as it satisfies

$$\frac{S_{q}^{A+B}}{k} = \frac{S_q^A}{k} \oplus_q \frac{S_q^B}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k}.$$  \hspace{1cm} (4.6)

The two-parameter entropy $S_{q,q'}$ \cite{13} satisfies, in the microcanonical ensemble (i.e. equal probabilities), that

$$Y(S^{A+B}) = Y(S^A) + Y(S^B) + \frac{1 - q'}{1 - q} Y(S^A) Y(S^B),$$  \hspace{1cm} (4.7)

where

$$Y(S) = 1 + \frac{1 - q}{1 - q'} \ln \left[ 1 + (1 - q') \frac{S}{k} \right].$$  \hspace{1cm} (4.8)

However, this does not hold true for arbitrary distributions $\{p_i\}$, which means $S_{q,q'}$ is not composable in general. For the 3-parameter entropy $S_{q,q',r}$ a similar property as that of (4.7) is obtained as shown below.

$$\ln_{q,q'}(W_A W_b) = \frac{1}{1 - q'} \left[ e^{(1 - q') \ln_q(W_A W_b)} - 1 \right] = \frac{S_{q,q'}^{A+B}}{k},$$  \hspace{1cm} (4.9)
from which
\[
\frac{S_{qq',r}}{k} = \ln_{q,q',r} w_A = \frac{1}{1 - r} \left[ e^{(1-r)\ln_{q,q',r} w_A} - 1 \right] = \frac{1}{1 - r} \left[ e^{(1-r)\frac{S_{qq',r}}{k}} - 1 \right].
\]
(4.10)

Similarly,
\[
\frac{S_{qq',r}}{k} = \ln_{q,q',r} W_B = \frac{1}{1 - r} \left[ e^{(1-r)\frac{S_{qq',r}}{k}} - 1 \right],
\]
(4.11)

\[
\frac{S_{A+B}}{k} = \ln_{q,q',r} w_A w_B = \frac{1}{1 - r} \left[ e^{(1-r)\frac{S_{A+B}}{k}} - 1 \right] = \frac{1}{1 - r} e^{(1-r)\frac{S_{A+B}}{k}} - \frac{1}{1 - r}.
\]
(4.12)

From (4.12),
\[
\ln \left[ (1 - r) \frac{S_{q,q',r}}{k} + 1 \right] = (1 - r) \frac{S_{q,q',r}}{k}.
\]
(4.13)

Using the following result in [13],
\[
\frac{S_{A+B}}{k} = \frac{1}{1 - q'} \left\{ \frac{1 - q' - q}{e^{q'}} \ln \left[ 1 + (1 - q') \frac{S_{A+B}}{k} \right] \ln \left[ 1 + (1 - q') \frac{S_{A}}{k} \right] \ln \left[ 1 + (1 - q') \frac{S_{B}}{k} \right] - 1 \right\}
\]
(4.14)

(4.13) becomes
\[
\ln \left[ 1 + (1 - r) \frac{S_{q,q',r}}{k} \right] = \frac{1 - q'}{1 - q} \left\{ \frac{1 - q' - q}{e^{q'}} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}}{k} \right] \ln \left[ 1 + (1 - r) \frac{S_{A}}{k} \right] \ln \left[ 1 + (1 - r) \frac{S_{B}}{k} \right] - 1 \right\}
\]
\[
\times \left[ 1 + \frac{1 - q'}{1 - q} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}}{k} \right] \right] \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}}{k} \right] - 1 \right\}.
\]

Let
\[
U(S) = \ln \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S}{k} \right] \right].
\]
(4.15)

Then
1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{A+B}}{k} \right] = e^{1 - \frac{q'}{1 - q}U(S^A) - U(S^B)}
\times \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{A}}{k} \right] \right]
\times \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{B}}{k} \right] \right].

Consequently,

\ln \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{A+B}}{k} \right] \right] = \frac{1 - q'}{1 - q}U(S^A) \cdot U(S^B)
+ \ln \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{A}}{k} \right] \right]
+ \ln \left[ 1 + \frac{1 - q'}{1 - r} \ln \left[ 1 + (1 - r) \frac{S_{q,q',r}^{B}}{k} \right] \right],

which can be written

\begin{align*}
U(S^{A+B}) &= U(S^A) + U(S^B) + \frac{1 - q'}{1 - q}U(S^A)U(S^B). 
\end{align*}

(4.16)

In view of the noncomposability of the 2-parameter entropy, $S_{q,q',r}$ is also non-composable.

## 5 Conclusion

It is shown that the two-parameter logarithm of Schwammle and Tsallis [13] can be generalized to three-parameter logarithm using q-analogues. Consequently, a three-parameter entropic function is defined and its properties are proved. It will be interesting to study applicability of the three-parameter entropy to adiabatic ensembles [6] and other ensembles [12] and how these applications relate to generalized Lambert W function.
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Data Availability Statement

The computer programs and articles used to generate the graphs and support the findings of this study are available from the corresponding author upon request.

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