Calibration for Weak Variance-Alpha-Gamma Processes

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Abstract
The weak variance-alpha-gamma process is a multivariate Lévy process constructed by weakly subordinating Brownian motion, possibly with correlated components with an alpha-gamma subordinator. It generalises the variance-alpha-gamma process of Semeraro constructed by traditional subordination. We compare three calibration methods for the weak variance-alpha-gamma process, method of moments, maximum likelihood estimation (MLE) and digital moment estimation (DME). We derive a condition for Fourier invertibility needed to apply MLE and show in our simulations that MLE produces a better fit when this condition holds, while DME produces a better fit when it is violated. We also find that the weak variance-alpha-gamma process exhibits a wider range of dependence and produces a significantly better fit than the variance-alpha-gamma process on a S&P500-FTSE100 data set, and that DME produces the best fit in this situation.

Keywords Brownian motion · Gamma process · Lévy process · Subordination · Variance-Gamma · Variance-Alpha-Gamma · Self-Decomposability · Log-Return · Method of moments · Maximum likelihood estimation · Digital moment estimation

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1 Introduction

The subordination of Brownian motion has important applications in mathematical finance, and acts as a time change that models the flow of information, measuring time in volume of trade, as opposed to real time. This idea was initiated in Madan and Seneta (1990) which introduced the variance-gamma (VG) process for modelling stock prices, where the subordinate is Brownian motion and the subordinator is a gamma process.

Subordination can be applied to model dependence in multivariate price processes. The multivariate VG process in Madan and Seneta (1990) uses \(n\)-dimensional Brownian motion as its subordinate and a univariate gamma process as its subordinator, which gives it a restrictive dependence structure, where components cannot have idiosyncratic time changes and must have equal kurtosis when the skewness is zero. Models based on linear combinations of independent Lévy processes (Kawai 2009; Madan 2011) also do not account for both common and idiosyncratic time changes. These deficiencies are addressed by the use of an alpha-gamma subordinator, resulting in the variance-alpha-gamma (VAG) process which was introduced in Semeraro (2008) and also studied in Guillaume (2013), Luciano and Semeraro (2010). However, in this case, the Brownian motion subordinate must have independent components, which also restricts the dependence structure.

To be precise, let \( B = (B_1, \ldots, B_n) \), where \( T = (T_1, \ldots, T_n) \) be independent \( n\)-dimensional processes, where \( B \) is Brownian motion and \( T \) is a subordinator. Subordination is the operation that produces the process \( B \circ T \) defined by \( (B \circ T)(t) := (B_1(T_1(t)), \ldots, B_n(T_n(t))) \), \( t \geq 0 \). Subordination in the case when \( T \) has indistinguishable components has been studied in Barndorff-Nielsen and Shiryaev (2010), Sato (1999), and when \( B \) has independent components in Barndorff-Nielsen et al. (2001). In these cases, which we refer to as traditional subordination, \( B \circ T \) is a Lévy process, otherwise it may not be (see Buchmann et al. 2017b, their Proposition 3.9). We refer the reader to Buchmann et al. (2017a) for a thorough discussion of traditional subordination and its applications.

In Buchmann et al. (2017b), we introduced the weak subordination of \( B \) and \( T \), an operation that extends traditional subordination and always produces a Lévy process \( B \odot T \). Then the weak variance-alpha-gamma (WVAG) process can be constructed using weak subordination instead of traditional subordination, while allowing for the Brownian motion to have possibly correlated components. The WVAG process exhibits a wider range of dependence while remaining parsimoniously parametrised, each component has both common and idiosyncratic time changes, it has VG marginals with independent levels of kurtosis, and the jump measure has full support.

Weak subordination also has been applied in quantitative finance. In Michaelsen and Szimayer (2018), various marginal consistent dependence models have been constructed by weak subordination. In Madan (2018), log return modelling based on the WVAG process was applied in instantaneous portfolio theory. In Michaelsen (2018), weak subordination using subordinators with arbitrary marginal components and dependence specified by a Lévy copula was studied in the context of financial information flows.

Maximum likelihood estimation (MLE) has been used to fit a univariate VG process to financial data in Madan et al. (1998), Finlay and Seneta (2008), to fit a bivariate VG process in Fung and Seneta (2010), to fit a WVAG process in Michaelsen and Szimayer (2018), and to fit a factor-based subordinated Brownian motion, a generalisation of the WVAG process, in Luciano et al. (2016), Michaelsen and Szimayer (2018), Wang (2009). Since the density function of the VAG and WVAG distribution is not explicitly known but its characteristic function is, the density function is computed using Fourier inversion.
In this paper, we derive a sufficient condition in terms of the parameters for Fourier
invertibility, a problem that to our knowledge is not addressed in the existing literature.
Then we compare MLE with method of moments (MOM) and digital moment estimation
(DME) from Madan (2015). Using simulations we find that MLE produces a better fit when
the Fourier invertibility condition is satisfied but that DME is better when it is violated. In
addition, we fit both the WVAG and VAG model to an S&P500-FTSE100 data set and
show that the weak model has a significantly better fit, and that DME is the better method in
this situation. Finally, using a condition for the self-decomposability of the WVAG process
from Buchmann et al. (2018), we find that the log returns are self-decomposable.

This paper is structured as follows. In Section 2, we review the definition and properties
of the WVAG process, and other preliminaries. In Section 3, we derive a condition for
Fourier invertibility. In Section 4, we apply MOM, MLE, DME to simulated and real data,
and discuss our findings. In Section 5, we conclude the paper.

2 Weak Variance-Alpha-Gamma Process

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space whose elements are row vectors $x = (x_1, \ldots, x_n)$
with canonical basis $\{e_k : 1 \leq k \leq n\}$. Let $(x, y) = xy'$ denote the Euclidean product, $\|x\|^2 = xx'$
denote the Euclidean norm, and let $\|x\|^2 : = x\Sigma x'$. For $n$-dimensional processes $X$ and $Y$,
$X \overset{D}{=} Y$ indicates that $X$ and $Y$ are identical in law, that is their systems of finite
dimensional distributions are equal.

A overview of Lévy processes and weak subordination is given in the Appendix. Throughout,
$B = (B_1, \ldots, B_n) \sim BM^n(\mu, \Sigma)$ refers to an $n$-dimensional Brownian motion with linear drift $\mathbb{E}[B(t)] = \mu t$ and covariance matrix $\text{Cov}(B(t)) = \Sigma t, t \geq 0$. An $n$-dimensional subordinator $T = (T_1, \ldots, T_n) \sim S^n(T)$ is an $n$-dimensional Lévy process
with nondecreasing components, and its Lévy measure is denoted by $T$.

Gamma Subordinator For $a, b > 0$, a univariate subordinator $G \sim \Gamma_S(a, b)$ is a gamma
subordinator if its marginal $G(t), t \geq 0$, is gamma distributed with shape parameter $at$
and rate parameter $b$. If $a = b$, we refer to $G$ as a standard gamma subordinator, in short,$\sim \Gamma_S(b) := \Gamma_S(b, b)$.

Alpha-Gamma Subordinator Assume $n \geq 2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, \infty)^n$ and
$G_0, \ldots, G_n$ be independent gamma subordinators such that $G_0 \sim \Gamma_S(a, 1), G_k \sim \Gamma_S(\beta_k, 1/\alpha_k)$, where $a > 0, a\alpha_k < 1, \beta_k := (1 - a\alpha_k)/\alpha_k, 1 \leq k \leq n$. A process $T \sim AG^n_S(a, \alpha)$ is an alpha-gamma ($AG^n_S$) subordinator (Semeraro 2008) with parameters $a, \alpha$ if $T \overset{D}{=} G_0\alpha + (G_1, \ldots, G_n)$.

An alpha-gamma subordinator $T$ has correlated components with marginals $T_k \sim \Gamma_S(1/\alpha_k), 1 \leq k \leq n$.

Variance-Gamma Process Let $b > 0$, $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ be a covariance matrix. A
process $V \sim VG^n(b, \mu, \Sigma)$ is a variance-gamma ($VG^n$) process (Madan and Seneta 1990)
with parameters $b, \mu, \Sigma$ if $V \sim BM^n(\mu, \Sigma) \circ (\Gamma_S(b)e)$, where $e := (1, \ldots, 1) \in \mathbb{R}^n$.

The characteristic exponent of $V$ is (see Buchmann et al. 2017a, their Eq. (2.9))

$$
\Psi_V(\theta) = -b \ln \left(1 - \frac{i \langle \mu, \theta \rangle}{b} + \frac{\|\theta\|^2}{2b} \right), \quad \theta \in \mathbb{R}^n,
$$

where $\ln : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is the principal branch of the logarithm.
**Strong Variance-Alpha-Gamma Process** Assume \( n \geq 2 \). Let \( \mu \in \mathbb{R}^n \) and \( \Sigma \in [0, \infty)^{n \times n} \) be a diagonal matrix. A process \( X \sim VAG^n(a, \alpha, \mu, \Sigma) \) is a \textit{(strong)} variance-alpha-gamma \((VAG^n)\) process (Luciano and Semeraro 2010; Semeraro 2008) with parameters \( a, \alpha, \mu, \Sigma \) if \( X \sim BM^n(\mu, \Sigma) \odot AG_3^n(a, \alpha) \).

In Buchmann et al. (2017b), the weak \( VAG \) process was formulated using weak subordination, allowing \( B \) to have dependent components while remaining a Lévy process.

**Weak Variance-Alpha-Gamma Process** Assume \( n \geq 2 \). Let \( \mu \in \mathbb{R}^n \) and \( \Sigma \in \mathbb{R}^{n \times n} \) be an \textit{arbitrary} covariance matrix. The process \( X \sim WVAG^n(a, \alpha, \mu, \Sigma) \) is a \textit{weak} variance-alpha-gamma process (Buchmann et al. 2017b) with parameters \( a, \alpha, \mu, \Sigma \) if \( X \sim BM^n(\mu, \Sigma) \odot AG_3^n(a, \alpha) \), where \( \odot \) denotes the weak subordination operation (see the Appendix).

Next, we gather various known results about the \( WVAG \) process that will be useful later on. The notation \( \circ \) is defined in Eq. A.1 and self-decomposability is defined in the Appendix.

**Proposition 2.1** Let \( n \geq 2 \) and \( X \sim WVAG^n(a, \alpha, \mu, \Sigma) \).

(i) \( X \) is an \( n \)-dimensional Lévy process with Lévy exponent

\[
\Psi_X(\theta) = -a \ln \left\{ 1 - i \langle \alpha \circ \mu, \theta \rangle + \frac{1}{2} \| \theta \|_2^{2 \alpha \circ \Sigma} \right\} \]

\[ - \sum_{k=1}^n \beta_k \ln \left\{ 1 - i \alpha_k \mu_k \theta_k + \frac{1}{2} \alpha_k \theta_k^2 \Sigma_{kk} \right\}, \quad \theta \in \mathbb{R}^n. \tag{2.2}\]

(ii) Let \( V_0 \sim VG^n(a, a \alpha \circ \mu, a \alpha \circ \Sigma) \), \( V_k \sim VG^1(\beta_k, (1 - a \alpha_k) \mu_k, (1 - a \alpha_k) \Sigma_{kk}) \), \( 1 \leq k \leq n \) be independent. Then \( X \overset{D}{=} V_0 + \sum_{k=1}^n V_k e_k \).

(iii) For any \( c > 0 \), \( (X(ct))_{t \geq 0} \sim WVAG^n(ca/c, c \alpha, c \mu, c \Sigma) \).

(iv) For \( 1 \leq k \leq n \), \( X \) has marginal distribution

\[
X_k \sim VG^1(1/\alpha_k, \mu_k, \Sigma_{kk}).
\]

(v) If \( \Sigma \) is diagonal, then \( X \sim VAG^n(a, \alpha, \mu, \Sigma) \).

(vi) For \( 1 \leq k \neq l \leq n \), \( \text{Cov}(X_k(1), X_l(1)) = a(\alpha_k \wedge \alpha_l) \Sigma_{kl} + a \alpha_k \alpha_l \mu_k \mu_l \).

(vii) If \( \Sigma \) is invertible, then \( X \) is self-decomposable if and only if \( \mu = 0 \).

**Proof** See Remarks 4.3, 3.9, 3.1, 3.2, 3.7, respectively, in Buchmann et al. (2017b) for (i), (ii), (iv)-(vi), Eq. (20) in Michaelsen and Szimayer (2018) for (iii), and Corollary 4.4 in Buchmann et al. (2018) for (vii).

The \( WVAG \) process exhibits a wider range of dependence than the \( VAG \) process. For example, it has an additional covariance term \( a(\alpha_k \wedge \alpha_l) \Sigma_{kl} \) from Proposition 2.1 (vi).

### 3 Fourier Invertibility

Let \( n \geq 2 \) and \( X \sim WVAG^n(a, \alpha, \mu, \Sigma) \) and \( Y = I m + X, m \in \mathbb{R}^n \), where \( I : [0, \infty) \to [0, \infty) \) is the identity function. The density function of \( Y(t) \), which is needed for MLE, exists for \( t > 0 \) because \( Y(t) \) has an absolutely continuous distribution. However, it is not explicitly known, so it is computed using Fourier inversion as

\[
f_{Y(t)}(\mathbf{y}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-i \langle \theta, \mathbf{y} - \mathbf{m} \rangle) \Phi_{X(t)}(\theta) \, d\theta, \quad \mathbf{y} \in \mathbb{R}^n, \tag{3.1}\]

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where \( \Phi_X(t)(\theta) = \exp(t\Psi_X(\theta)) \), \( \theta \in \mathbb{R}^n \), and \( \Psi_X(\theta) \) from Eq. 2.2, provided \( \Phi_X(t) \in L^1 \). If \( \Phi_X(t) \in L^1 \), we say that \( X(t) \) is Fourier invertible and we give a condition for this in terms of an inequality relating the parameters.

**Lemma 3.1** Let \( B \sim BM^n(\mu, \Sigma) \), \( \widetilde{B} \sim BM^n(0, \Sigma) \), \( T \sim S^n(T) \) be a drift-less subordinator, \( X \overset{D}{=} \widetilde{B} \odot T \), \( Y \overset{D}{=} Im + B \odot T \), where \( m \in \mathbb{R}^n \). For all \( t \geq 0 \), if \( \Phi_X(t) \in L^1 \), then \( \Phi_Y(t) \in L^1 \).

**Proof** For \( z \in \mathbb{C} \), let \( \Re z \) denotes its real part. For all \( t \geq 0 \), \( \Phi_Y(t)(\theta) = e^{i t \langle \theta, m \rangle / \Phi_X(B \odot T)(\theta)} \), \( \theta \in \mathbb{R}^n \), so that \( |\Phi_Y(t)(\theta)| = \exp(t \Re \Psi_{B \odot T}(\theta)) \). Using Eq. A.2, we have

\[
\Re \Psi_{B \odot T}(\theta) = \int_{[0,\infty)^2} (\Re \Phi_B(t)(\theta) - 1) T(dt) \\
\leq \int_{[0,\infty)^2} (e^{-\frac{1}{2} \|\theta\|_{\Sigma}^2} - 1) T(dt) \\
= \Re \Psi_X(\theta).
\]

Therefore, \( |\Phi_Y(t)(\theta)| \leq |\Phi_X(t)(\theta)| \), so the result follows. \( \square \)

**Lemma 3.2** Let \( V \sim VG^n(b, \mu, \Sigma) \), and assume that \( \Sigma \) is invertible. Let \( p > 0 \). If \( pb > n/2 \), then \( \Phi_V \in L^p \).

**Proof** Since variance-gamma processes are weakly subordinated processes (see the last paragraph of the Appendix), we can apply Lemma 3.1, which means that we can assume \( \mu = 0 \).

If \( V \sim VG^n(b, 0, \Sigma) \), by Eq. 2.1, \( V \) has characteristic function

\[
\Phi_V(\theta) = \left( 1 + \frac{\|\theta\|_{\Sigma}^2}{2b} \right)^{-b}, \quad \theta \in \mathbb{R}^n.
\]

Using the Cholesky decomposition, \( \Sigma = U'U \), where \( U \) is a lower triangular matrix with positive elements on the diagonal. In particular, \( U^{-1} \) exists. Then applying the transformation theorem with \( \theta = \sqrt{2b}x(U')^{-1} \), for \( p > 0 \), we have

\[
\int_{\mathbb{R}^n} |\Phi_V(\theta)|^p \, d\theta = \det(\sqrt{2b}U^{-1}) \int_{\mathbb{R}^n} \left( 1 + \|x\|^2 \right)^{-pb} \, dx.
\]

Using the polar decomposition on the preceding integral, we have \( \Phi_V \in L^p \) if and only if

\[
\int_0^\infty (1 + r^2)^{-pb}r^{n-1} \, dr < \infty,
\]

which is equivalent to \( pb > n/2 \). \( \square \)

**Theorem 3.3** Let \( n \geq 2 \) and \( X \sim WVAG^n(a, \alpha, \mu, \Sigma) \) and \( Y = Im + X \), \( m \in \mathbb{R}^n \). Assume that \( \Sigma \) is invertible. For \( t \geq 0 \), if

\[
\left( \frac{a}{n} + \min_{1 \leq k \leq n} \beta_k \right) t > \frac{1}{2}, \tag{3.2}
\]

then \( \Phi_X(t), \Phi_Y(t) \in L^1 \).
Proof By Proposition 2.1 (iii), it suffices to prove the result for \( t = 1 \), and by Lemma 3.1, we can assume \( Y \sim VWAG^n(a, \alpha, 0, \Sigma) \).

Recalling Proposition 2.1 (ii), \( Y \overset{D}{=} V_0 + \sum_{k=1}^n V_k e_k \), where \( V_0 \sim VG^n(a, 0, a \alpha \diamond \Sigma), \ V_k \sim VG^1(\beta_k, 0, (1 - a \alpha_k) \Sigma_{kk}) \), \( 1 \leq k \leq n \), are independent. Let \( V^+ := (V_1, \ldots, V_n) \). Now \( Y \) has characteristic function \( \phi_Y(\theta) = \phi_{V_0}(\theta) \phi_{V^+}(\theta) \), where \( \phi_{V^+}(\theta) := \prod_{k=1}^n \phi_{V_k}(\theta_k) \).

The Hölder inequality gives
\[
\int_{\mathbb{R}^n} |\phi_Y(\theta)| \, d\theta \leq \left( \int_{\mathbb{R}^n} |\phi_{V_0}(\theta)|^p \, d\theta \right)^{1/p} \left( \int_{\mathbb{R}^n} |\phi_{V^+}(\theta)|^q \, d\theta \right)^{1/q}
\]
\[
= \left( \int_{\mathbb{R}^n} |\phi_{V_0}(\theta)|^p \, d\theta \right)^{1/p} \prod_{k=1}^n \left( \int_{\mathbb{R}} |\phi_{V_k}(\theta_k)|^q \, d\theta_k \right)^{1/q},
\]
where \( p^{-1} + q^{-1} = 1, p, q > 1 \). By Lemma 3.2, this integral is finite when \( pa > n/2, q \beta_k > 1/2 \) and \( p, q > 1 \) for all \( 1 \leq k \leq n \). Thus,
\[
1 = \frac{1}{p} + \frac{1}{q} < 2 \left( \frac{a}{n} \wedge \frac{1}{2} \right) + 2 \left( \min_{1 \leq k \leq n} \beta_k \wedge \frac{1}{2} \right),
\]
which is equivalent to (3.2).

Remark 3.1 Note that this condition for a \( VG^1(\beta, \mu, \Sigma) \) distribution to be Fourier invertible is identical to the condition for its density function having no singularity in Küchler and Tappe (2008), which is \( b > 1/2 \).

Remark 3.2 We see that for sufficiently small \( t > 0 \), Eq. 3.2 will not be satisfied. This means that using Eq. 3.1 to compute the density function may not be valid when attempting parameter estimation for a \( WVAG \) process based on observations from such a sufficiently small sampling interval.

4 Calibration

We now specialise to the case of \( n = 2 \). Let \( X \sim WVAG^2(a, \alpha, \mu, \Sigma) \), \( Y := (Y_1, Y_2) = I m + X, m \in \mathbb{R}^2 \). Let \( (S_1, S_2) \) be a bivariate price process

\[
S_k(t) = S_k(0) \exp(Y_k(t)), \quad t \geq 0, \quad k = 1, 2.
\]

For \( N \) equally spaced discrete observations with sampling interval \( c > 0 \), the log returns are
\[
y_j := (y_{1j}, y_{2j}) := \left( \frac{\ln S_1(jc)}{S_1((j-1)c)}, \frac{\ln S_2(jc)}{S_2((j-1)c)} \right) \overset{D}{=} Y(c), \quad j = 1, \ldots, N,
\]
and are iid. We call this the \( WVAG \) model. If \( \Sigma_{12} = 0 \), we called it the \( VAG \) model as \( X \) reduces to a \( VAG \) process by Proposition 2.1 (v).

4.1 Estimation Methods

We estimate the parameters \((a, \alpha, \mu, \Sigma, m)\) from the observations \((y_j)_{j=1}^N\) using method of moments (MOM), which is quick and easy to implement, maximum likelihood estimation (MLE) from Michaelsen and Szimayer (2018), which may be expected as being asymptotically optimal under the model, and a modification of digital moment estimation (DME) from Madan (2015), which is more robust to model misspecification.
Method of Moments The initial values of \( \mu_k, \alpha_k, \Sigma_{kk}, m_k, k = 1, 2 \), are obtained by least squares on the first four central moments \( \mathbb{E}(Y_k(c)), \mathbb{E}((Y_k(c) - \mathbb{E}(Y_k(c)))^p), p = 2, 3, 4 \), with the corresponding sample moments. The initial values of the joint parameters \( a, \Sigma_{12} \) are obtained by least squares on \( \mathbb{E}((Y_1(c) - \mathbb{E}(Y_1(c)))^p(Y_2(c) - \mathbb{E}(Y_2(c)))^q), p = 1, 2 \), with the corresponding sample moments, with \( p = 1 \) excluded when fitting the VAG model. Using these initial values, least squares is solved over all parameters. Note that this last step has no effect when these moments can be matched exactly.

Moment formulas can be found in Michaelsen and Szimayer (2018).

Maximum Likelihood Estimation The density function of \( Y(c) \) is not explicitly known so it is numerically computed using Fourier inversion by Eq. 3.1. The numerical optimisation needed to implement MLE requires initial values. The first initial values can be obtained by MOM. Using the first initial values, MLE is applied to each marginal observations to obtain the second initial values of \( \mu_k, \alpha_k, \Sigma_{kk}, m_k, k = 1, 2 \), and to the bivariate observations to obtain the second initial values of \( a, \Sigma_{12} \). Finally, using the second initial values, MLE is applied on all parameters. For the VAG model, we apply the above method with the constraint \( \Sigma_{12} = 0 \).

Digital Moment Estimation Let \( k = 1, 2 \), let \( q \) be the vector of 10 equally spaced points from 0.05 to 0.95, and let \( P_k \) be the empirical quantiles of the observations \( (y_{kj})_{j=1}^N \) at the probabilities \( q \). Let \( p_y(\mu_k, \alpha_k, \Sigma_{kk}, m_k) := \mathbb{P}(Y_k(c) \leq y) \), where \( y \in \mathbb{P}_k, \ Y_k \sim m_k I + VG^1(1/\alpha_k, \mu_k, \Sigma_{kk}) \) (see Proposition 2.1 (iv)), and \( q_y \) is the corresponding empirical probability. Marginal parameters \( \mu_k, \alpha_k, \Sigma_{kk}, m_k \) are estimated by minimising the error \( \sum_{y \in \mathbb{P}_k}(p_y(\mu_k, \alpha_k, \Sigma_{kk}, m_k) - q_y)^2 \).

With the estimated marginal parameters, let \( \rho := \Sigma_{12}/(\Sigma_{11}\Sigma_{22})^{1/2} \), \( p_y(a, \rho) := \mathbb{P}(Y_1(c) \leq y_1, Y_2(c) \leq y_2) \), where \( y := (y_1, y_2) \in \mathbb{P}_1 \times \mathbb{P}_2, Y \sim I_m + WVAG^2(a, \alpha, \mu, \Sigma) \), and \( q_y \) is the corresponding empirical probability. Since \( p_y(a, \rho) \) is computationally expensive to calculate directly, it is estimated by the empirical probability over 10000 simulations. The joint parameters \( a, \rho \) are estimated by minimising the LOESS smooth (Cleveland et al. 1991) of the error \( \sum_{y \in \mathbb{P}_1 \times \mathbb{P}_2}(p_y(a, \rho) - q_y)^2 \). The predictor variables for the LOESS smooth are 100 equally spaced points on the feasible set of \( (a, \rho) \in (0, (1/\alpha_1) \land (1/\alpha_2)) \times (-1, 1) \). For the VAG model, we apply the above method with the constraint \( \rho = 0 \).

4.2 Simulation Method

The result in Proposition 2.1 (ii) can be used to simulate \( X \sim WVAG^2(a, \alpha, \mu, \Sigma) \) in terms of \( VG^n \) and \( VG^1 \) processes.

For the sampling intervals \( c = 1, 0.1 \) and sample size \( N = 1000 \), we make 100 simulations of \( Y \), and estimate the parameters from the observations \( (y_j)_{j=1}^N \) with true parameters \( a = 1, \alpha = (0.8, 0.6), \mu = (0.1, -0.3), \Sigma = [1, 0.6; 0.6, 1.2], m = (-0.1, 0.3) \).

4.3 Goodness of Fit Statistics

To assess the overall goodness of fit of each parameter estimation method, as opposed to assessing individual parameters, we consider 3 goodness of fit statistics, the negative log-likelihood (− log L), a chi-squared (\( \chi^2 \)) statistic, and a Kolmogorov-Smirnov (KS) statistic.

To compute \( \chi^2 \), we apply the Rosenblatt transform (Rosenblatt 1952) of the fitted distribution to the observations, which has a uniform distribution on \([0, 1]^2 \) if the fitted distribution coincides with the true distribution, and then we compute the \( \chi^2 \) statistic for a test
of uniformity over an equally spaced partition of \([0, 1]^2\) into 100 cells. Since computing $-\log L$ and $\chi^2$ requires Fourier inversion, it may not be possible to compute these statistics accurately when the Fourier invertibility condition does not hold, so they are not displayed in Table 2.

Therefore, we also consider the 2-dimensional, two-sample Kolmogorov-Smirnov statistic introduced in Peacock (1983), and computed using the method of Xiao (2017). This is the statistic for testing equality of the fitted distribution and the true distribution based on a sample from the respective distributions, and therefore does not require the density function $f_{Y(c)}(y)$ or Fourier inversion. When applied to real data in Subsection 4.6, we take

### Table 1
RMSE using DME with quantiles $q_1, \ldots, q_4$ for the WVAG model fitted to simulated data with $c = 1$

| Parameter | True value | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|-----------|------------|-------|-------|-------|-------|
| $a$       | 1          | 0.171 | 0.171 | 0.182 | 0.175 |
| $\alpha_1$ | 0.8        | 0.127 | 0.132 | 0.143 | 0.128 |
| $\alpha_2$ | 0.6        | 0.126 | 0.145 | 0.149 | 0.129 |
| $\mu_1$   | 0.1        | 0.062 | 0.066 | 0.066 | 0.062 |
| $\mu_2$   | 0         | 0.121 | 0.271 | 0.229 | 0.188 |
| $\Sigma_{11}$ | 1          | 0.084 | 0.083 | 0.093 | 0.084 |
| $\Sigma_{22}$ | 1.2       | 0.113 | 0.166 | 0.147 | 0.123 |
| $\Sigma_{12}$ | 0.6        | 0.154 | 0.182 | 0.172 | 0.150 |
| $m_1$     | 0.1        | 0.051 | 0.054 | 0.053 | 0.050 |
| $m_2$     | 0.3        | 0.110 | 0.262 | 0.219 | 0.179 |
| $-\ln L$  |            | 2791.674  | 2795.826  | 2794.374  | 2792.303  |
| $\chi^2$  |            | 93.848 | 97.292 | 96.728 | 95.078 |
| KS        |            | 0.054 | 0.055 | 0.054 | 0.054 |

### Table 2
Expected value of estimates and RMSE (in parentheses) for the WVAG model fitted to simulated data with $c = 1$

| Parameter | True value | MOM | MLE | DME |
|-----------|------------|-----|-----|-----|
| $a$       | 1          | 0.920 (0.424) | 0.983 (0.242) | 0.902 (0.171) |
| $\alpha_1$ | 0.8        | 0.806 (0.342) | 0.824 (0.111) | 0.818 (0.127) |
| $\alpha_2$ | 0.6        | 0.589 (0.216) | 0.594 (0.094) | 0.589 (0.126) |
| $\mu_1$   | 0.1        | 0.103 (0.097) | 0.103 (0.053) | 0.096 (0.062) |
| $\mu_2$   | 0         | -0.310 (0.131) | -0.301 (0.083) | -0.313 (0.121) |
| $\Sigma_{11}$ | 1          | 0.989 (0.078) | 1.006 (0.071) | 0.993 (0.084) |
| $\Sigma_{22}$ | 1.2       | 1.177 (0.088) | 1.202 (0.086) | 1.179 (0.113) |
| $\Sigma_{12}$ | 0.6        | 0.835 (0.335) | 0.669 (0.192) | 0.639 (0.154) |
| $m_1$     | 0.1        | -0.103 (0.089) | -0.105 (0.045) | -0.097 (0.051) |
| $m_2$     | 0.3        | 0.313 (0.120) | 0.302 (0.070) | 0.314 (0.110) |
| $-\ln L$  |            | 2802.337  | 2787.513  | 2791.674  |
| $\chi^2$  |            | 119.052 | 91.268 | 93.848 |
| KS        |            | 0.068 | 0.054 | 0.054 |
the average of the KS statistics computed from the observations and 100 samples from the fitted distribution. When applied to simulated data in Subsection 4.5, the KS statistic is computed from the observations and a sample from the fitted distribution. All 3 goodness of fit statistics were averaged over the 100 simulations.

4.4 Quantile Choice for DME

Different choices of quantiles for DME are possible. Let \( q_1 \) be the vector of 10 equally spaced points from 0.05 to 0.95, \( q_2 \) be the vector of 10 equally spaced points from 0.01 to 0.99, \( q_3 \) be the vector of 10 equally spaced points from 0.1 to 0.9, \( q_4 \) be the vector of 20 equally spaced points from 0.05 to 0.95. For sampling interval \( c = 1 \), Table 1 shows the goodness of fit for these 4 choices of quantiles. We find \( q = q_1 \) as yielding the lowest RMSE for most variables and the lowest goodness of fit statistics. However, given that the results are so similar, these quantile choices make only a small difference to the overall goodness of fit.

4.5 Simulated Data Results

For the sampling interval \( c = 1 \), the Fourier invertibility condition is satisfied as the LHS of Eq. 3.2 is \( 0.75 > 1/2 \). The results are shown in Table 2. Here, we find that MLE gives the best fit with the lowest \( \chi^2 \) statistic. The KS statistic for MLE and DME are approximately equal.

For the sampling interval \( c = 0.1 \), the Fourier invertibility condition is violated as the LHS of Eq. 3.2 is \( 0.08 < 1/2 \). The results are shown in Table 3. Here, we find that DME gives the best fit with the lowest KS statistic, however MLE still produces a good fit and does not break down. This suggests that the condition may not be necessary for the MLE to produce accurate parameter estimates. In both cases, \( c = 1, 0.1 \), the RMSE and goodness of fit statistics are highest for MOM.

4.6 Real Data Results

Next, we fit the \( WVAG \) and \( VAG \) models to the S&P500 and FTSE100 indices as the bivariate price process (4.1) for a 5 year period from 14 February 2011 to 12 February 2016.

| Parameter | True value | MOM | MLE | DME |
|-----------|------------|-----|-----|-----|
| \( a \)   | 1          | 1.106 (0.507) | 0.990 (0.062) | 0.896 (0.121) |
| \( a_1 \) | 0.8        | 0.636 (0.247) | 0.782 (0.033) | 0.796 (0.057) |
| \( a_2 \) | 0.6        | 0.504 (0.198) | 0.602 (0.026) | 0.603 (0.031) |
| \( \mu_1 \) | 0.1       | 0.099 (0.167) | 0.114 (0.099) | 0.104 (0.170) |
| \( \mu_2 \) | −0.3      | −0.347 (0.219) | −0.250 (0.123) | −0.301 (0.146) |
| \( \Sigma_{11} \) | 1         | 0.992 (0.136) | 1.005 (0.133) | 1.013 (0.302) |
| \( \Sigma_{22} \) | 1.2      | 1.197 (0.166) | 1.245 (0.161) | 1.234 (0.221) |
| \( \Sigma_{12} \) | 0.6      | 0.842 (0.353) | 0.262 (0.364) | 0.564 (0.188) |
| \( m_1 \) | −0.1      | −0.111 (0.128) | −0.114 (0.015) | −0.100 (0.000) |
| \( m_2 \) | 0.3       | 0.351 (0.164) | 0.288 (0.014) | 0.300 (0.001) |
| KS        | 0.326      | 0.222 | 0.078 |
Table 4  Parameter estimates and standard errors (in parentheses) when fitted to the S&P500-FTSE100 data set. Estimation for the marginal parameters of WVAG and VAG models are identical

| Parameter | MOM | MLE | DME |
|-----------|-----|-----|-----|
|           | WVAG | V AG | WVAG | V AG | WVAG | V AG |
| $a$       | 0.695 (0.199) | 0.696 (0.217) | 0.962 (0.139) | 1.087 (0.117) | 0.899 (0.138) | 1.114 (0.327) |
| $\alpha_1$ | 1.436 (0.398) | 1.436 (0.349) | 0.919 (0.105) | 0.908 (0.104) | 0.919 (0.105) | 0.919 (0.105) |
| $\alpha_2$ | 0.933 (0.277) | 0.707 (0.120) | 0.837 (0.105) | 0.919 (0.085) | 0.837 (0.105) | 0.919 (0.085) |
| $1000\mu_1$ | -1.153 (0.704) | -1.156 (0.595) | -0.403 (0.550) | -0.630 (0.694) | -0.562 (0.563) | -1.166 (0.554) |
| $1000\mu_2$ | -1.246 (0.704) | -1.255 (0.789) | -0.891 (0.586) | -0.867 (0.578) | 0.928 (0.071) | 0.928 (0.071) |
| $10000\Sigma_{11}$ | 0.982 (0.070) | 0.980 (0.069) | 0.976 (0.061) | 0.935 (0.064) | 0.928 (0.071) | 0.928 (0.071) |
| $10000\Sigma_{22}$ | 1.006 (0.053) | 1.007 (0.062) | 1.028 (0.063) | 1.014 (0.060) | 1.051 (0.086) | 1.051 (0.086) |
| $10000\Sigma_{12}$ | 0.994 (0.072) | - | 0.813 (0.089) | - | 0.844 (0.095) | - |
| $1000m_1$ | 1.422 (0.604) | 1.426 (0.519) | 0.705 (0.430) | 0.879 (0.575) | 0.982 (0.444) | 0.982 (0.444) |
| $1000m_2$ | 1.198 (0.617) | 1.207 (0.722) | 0.847 (0.440) | 0.882 (0.470) | 1.066 (0.410) | 1.066 (0.410) |
| $-\ln L$ | -8465.969 | -8192.168 | -8496.315 | -8239.301 | -8492.798 | -8237.767 |
| $\chi^2$ | 144.034 | 702.241 | 118.574 | 586.789 | 99.359 | 584.547 |
| KS        | 0.073 | 0.151 | 0.050 | 0.139 | 0.048 | 0.138 |
with daily closing price observations taking $c = 1$. The estimated parameters, goodness of fit statistics and standard errors computed using 100 bootstrap samples are listed in Table 4. Contour plots of the fitted distributions and scatter plots of the bivariate log returns are shown in Fig. 1.

Note that the Fourier invertibility condition is satisfied for all fitted models. Based on the $\chi^2$, KS statistic and contour plots, the $WVAG$ model produces a better fit than the $VAG$ model. In addition, for the $WVAG$ model, DME gives a fit with a lower $\chi^2$ and KS statistic than MLE and MOM.

Fig. 1 Scatterplots of log returns of the S&P500-FTSE100 data set and contour plots of the fitted distributions using the weak $VAG$ model (left) and $VAG$ model (right) with MOM (top), MLE (middle), DME (bottom). The contours in each plot, from outer to inner, correspond to the values $97 + (k - 1)681$, $k = 1, \ldots, 15$, respectively.
Assuming that the log returns satisfy the WVAG model, a likelihood ratio test can be used to test the hypothesis $H_0 : \Sigma_{12} = 0$ versus $H_1 : \Sigma_{12} \neq 0$. The test statistic is $D = 514.03$ is asymptotically $\chi^2$ distributed with 1 degree of freedom. The $p$-value is $< 0.00001$, so the VAG model is rejected. Indeed, the VAG model is not suited for modelling strong correlation since $\text{Cov}((B \circ T)_1(1), (B \circ T)_2(1)) = \alpha_1 \alpha_2 \mu_1 \mu_2$ by Proposition 2.1 (vi), which is approximately 0 when $\mu_1 \mu_2$ is.

It has been suggested that log-returns should be self-decomposable (Bingham 2006; Bingham and Kiesel 2002; Carr et al. 2007). Note that $\mu = (-0.0004, -0.0008)$ is very close to 0, which suggests that the log-returns process $Y$ is indeed self-decomposable (see Proposition 2.1 (vii)). A likelihood ratio test can be used to test this hypothesis, $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. The test statistic is $D = 4.11$ is asymptotically $\chi^2$ distributed with 2 degrees of freedom. The $p$-value is 0.128, so at a 5% significance level we cannot reject that $Y$ is self-decomposable.

5 Conclusion

The WVAG process constructed by using weak subordination generalises the VAG process, and we obtain a condition for Fourier invertibility in Theorem 3.3. We have shown that MOM, MLE and DME can be used to estimate the parameters of a WVAG process, and find that in our simulations MLE produces a better fit when the Fourier invertibility condition holds, while DME produces a better fit when it is violated. However, MLE may still produce good parameter estimates even when the Fourier invertibility condition is violated. In all cases, MOM produces the worst fit. We find that the WVAG process exhibits a wider range of dependence and produces a significantly better fit than the VAG process when used to model the S&P500-FTSE100 data set, and that DME produces the best fit in this situation.

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Appendix

Lévy Process The reader is referred to the monographs Applebaum (2009), Bertoin (1996) and Sato (1999) for necessary material on Lévy processes, to Ballotta and Bonfiglioni (2016), Cariboni and Schoutens (2009), Cont and Tankov (2004) for financial applications, while our notation follows Buchmann et al. (2017a) and Buchmann et al. (2017b). For $A \subseteq \mathbb{R}^n$, let $A_+ := A \setminus \{0\}$ and let $1_A(\omega)$ denote the indicator function. Let $\mathbb{D} := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the Euclidean unit ball centred at the origin. The law of an $n$-dimensional Lévy process $X = (X_1, \ldots, X_n) = (X(t))_{t \geq 0}$ is determined by its characteristic function $\Phi_X := \Phi_{X(1)}$, with

$$
\Phi_{X(t)}(\theta) := \mathbb{E} \exp(i\langle \theta, X(t) \rangle) = \exp(t \Psi_X(\theta)), \quad t \geq 0,
$$

and Lévy exponent $\Psi_X := \Psi$, where

$$
\Psi(\theta) := i\langle \mu, \theta \rangle - \frac{1}{2} \|\theta\|^2 + \int_{\mathbb{R}^n} \left( e^{i\langle \theta, x \rangle} - 1 - i\langle \theta, x \rangle 1_{\mathbb{D}}(x) \right) \lambda(dx),
$$
\( \theta \in \mathbb{R}^n, \mu \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n \times n} \) is a covariance matrix, and \( \mathcal{X} \) is a nonnegative Borel measure on \( \mathbb{R}_+ \) such that \( \int_{\mathbb{R}_+} (|x|^2 \wedge 1) \mathcal{X}(dx) < \infty \). We write \( X \sim L^{n}(\mu, \Sigma, \mathcal{X}) \) provided \( X \) is an \( n \)-dimensional Lévy process with canonical triplet \((\mu, \Sigma, \mathcal{X})\).

A subordinator \( T \sim S^n(\mathcal{T}) = L^n(\mu, 0, \mathcal{T}) \) is drift-less if its drift \( \mu - \int_{0}^{\mathcal{T}} t\mathcal{T}(dt) = 0 \). All subordinators considered in this paper are drift-less.

An \( n \)-dimensional random variable \( X \) is self-decomposable if for any \( 0 < b < 1 \), there exists a random variable \( Z_b \), independent of \( X \), such that \( X \overset{D}{=} bX + Z_b \). A Lévy process \( X \) is self-decomposable if \( X(1) \) is.

**Strongly Subordinated Brownian Motion** Let \( B = (B_1, \ldots, B_n) \sim BM^n(\mu, \Sigma) \) be a Brownian motion and \( T = (T_1, \ldots, T_n) \sim S^n(\mathcal{T}) \) be a drift-less subordinator. A process \( B \circ T \) is the traditional or strong subordination of \( X \) and \( T \) if \( (B \circ T)(t) := (B_1(T_1(t)), \ldots, B_n(T_n(t))) \), \( t \geq 0 \).

**Weakly Subordinated Brownian Motion** Let \( t = (t_1, \ldots, t_n) \in [0, \infty)^n \), \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n \) and \( \Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n \times n} \) be a covariance matrix. Introduce the outer products \( t \circ \mu \in \mathbb{R}^n \) and \( t \circ \Sigma \in \mathbb{R}^{n \times n} \) by

\[
 t \circ \mu := (t_1 \mu_1, \ldots, t_n \mu_n) \quad \text{and} \quad (t \circ \Sigma)_{kl} := \Sigma_{kl} (t_k \wedge t_l), \quad 1 \leq k, l \leq n. \quad (A.1)
\]

Let \( B \sim BM^n(\mu, \Sigma) \) be an \( n \)-dimensional Brownian motion and \( T \sim S^n(\mathcal{T}) \) be an \( n \)-dimensional drift-less subordinator. A Lévy process \( B \circ T \) is called the weak subordination of \( B \) and \( T \) (see Buchmann et al. 2017b, their Proposition 3.1) if it has Lévy exponent

\[
 \Psi_{B \circ T}(\theta) = \int_{[0, \infty)^n} \left( \exp \left( i \langle \theta, t \circ \mu \rangle - \frac{1}{2} \| \theta \|^2_{t \circ \Sigma} \right) - 1 \right) \mathcal{T}(dt), \quad (A.2)
\]

\( \theta \in \mathbb{R}^n \). Note that a more general definition of weak subordination and a proof of existence is given in Buchmann et al. (2017b).

For independent \( B \) and \( T \), \( B \circ T \overset{D}{=} B \circ T \) if \( T \) has indistinguishable components or \( B \) has independent components, otherwise \( B \circ T \) may not be a Lévy process, but \( B \circ T \) always is (see Buchmann et al. 2017b, their Proposition 3.3 and 3.9).

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