Non-existence of a certain kind of finite-letter mutual information characterization for a class of time-invariant Markoff channels

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Abstract

We provide a rigorous definition of a certain kind of characterization for capacity regions of a family of Markoff networks which is based on optimization problems resulting out of calculating conditional mutual informations from a finite number of random variables and by constraining these random variables in a certain way. This definition is partly motivated by the definition of single-letter characterizations in information theory. For a point-to-point Markoff channel, we prove that approximating the solution to these characterizations within an additive constant is a computable problem. Based on previous undecidability results concerning capacities of certain class of finite state machine channels (FSMCs), it will follow that there exists an example of family of FSMCs for which given such a characterization, this characterization cannot represent the capacity of this family of FSMCs.

Keywords: finite letter characterization (FLC), computability, first order theory of the real closed field, capacity, finite state machine channel,

1. Introduction

In this paper, we provide a rigorous definition of a certain kind of characterization for capacity regions of a family of finite input, finite output, finite state Markoff networks which is based on optimization problems resulting out of calculating conditional mutual informations from a finite number of random variables taking values in finite sets, and by constraining the probability distributions corresponding to these random variables by polynomial constraints. Such a characterization, the rigorous definition of which is the subject of Section 2 will hence forth be called an FLC. The definition of FLC is partly motivated by single-letter characterizations in information theory.
An admissible FLC corresponding to a class of Markoff networks is an FLC for which, given a network in this class of networks, after substituting for variables of the FLC, the values which determine the conditional probabilities which define the network, we get an optimization problem, the closure of the feasible region of which is the capacity region of the network, and this is the case for all networks in this class of networks. Note that the FLC is independent of the particular network; however, the variables in the FLC take values dependent on the conditional probability distributions which define the particular network.

Let $F$ be an FLC which is admissible for a certain class $D$ of Markoff channels. We will prove that this will imply that there exists an algorithm which takes as input $\epsilon > 0$ and the conditional distributions which define the particular network $d \in D$, and provides as output, a real number $g \in [C_d - \epsilon, C_d + \epsilon]$. In other words, if there exists an admissible FLC for $D$, approximating the capacity of a channels $d \in D$ within an additive constant is a computable problem. This is the subject of Section 3. In (Elkouss et al., 2018), an example of a family of FSMCs, denoted by $\mathcal{S}_\lambda$ is provided for which, capacity is either $\leq \frac{\lambda}{2}$ or $\geq \lambda$ for every channel in $\mathcal{S}_\lambda$, and deciding, which is the case, is an undecidable problem, that is, there is no algorithm which decides whether the capacity is $\leq \frac{\lambda}{2}$ or $\geq \lambda$ for all channels $d \in \mathcal{S}_\lambda$. It will follow from this result and the computability result for approximating capacities if there exists an admissible FLC, stated above, that there exists no admissible FLC for the family of channels $\mathcal{S}_\lambda$. This is the subject of Section 4.

Further, based on an earlier result concerning undecidability of approximations for the emptiness problem in the theory of probabilistic finite automatas (PFAs) (Madani et al., 2003), we will provide a short outline of a proof of the non-computability of the problem of approximating capacity within an additive constant for Markoff channels with partial state information. This is the subject of Appendix A. This proof is complete but for one missing step, see Appendix A.1

2. FLCs

In this section, we define an FLC corresponding to a family of Markoff networks $D$, and an admissible FLC corresponding to $D$. Consider a network of $n$ users. The input space at User $i$ is $A_i$ and the output space at User $i$ is $B_i$. $A_i$, $B_i$ are assumed to be finite sets. The network also has a state which
is an element of the set \( C \), a finite set. The action of the network is given by a conditional probability \( c(b_1, b_2, \ldots, b_n, s|a_1, a_2, \ldots, a_n, s') \). This is the probability that the channel output is \( b_1, b_2, \ldots, b_n \) and the state is \( s \) given that the channel input is \( a_1, a_2, \ldots, a_n \) and the state at the previous time was \( s' \). The notion of reliable communication over such a network is the subject of information theory, see for example, Golm et al. (2011), for discussion of capacity region of discrete memoryless networks. A achievable rate vector would be a sequence \( (L_{ij}, 1 \leq i, j \leq n, i \neq j) \) where \( L_{ij} \) denotes the rate of communication from User i to User j. The closure of the set of achievable rate vectors, when considered as a subset of \( \mathbb{R}^{n^2-n} \) is the capacity region of this network. In this paper, we will be restricting attention to point-to-point Markov channels for the main results. For a discussion of capacity of such channels, the reader is referred to Gallager (1968) and Elkouss et al. (2018).

Consider a set \( D \) of such networks such that all networks in this set have the same input, output and state spaces.

In what follows, we would need the notion of computable real numbers and algebraic real numbers. A computable number is a real number that can be computed to within any desired precision by a finite, terminating algorithm, see, for example, Chapter 9 of Minsky (1967). An algebraic real number is a real number which is the root of a non-zero polynomial with integer (or rational) coefficients. An algebraic number can be specified by specifying the polynomial of which it is a root and an interval \([a, b]\) to which it belongs, where \( a, b \) are rational numbers.

An FLC corresponding to \( D \) consists of two elements: Representation and Constraints. These are defined as follows:

1. Representation: The representation has finite sets \( X_1, X_2, \ldots, X_k \) (for some \( k \)). Let \( (X_1, X_2, \ldots, X_k) \) be a random vector on \( \prod_{i=1}^k X_i \). The representation consists of a set of equations:

\[
\sum_{i,j=1}^n \beta_{ij}^{(r)} R_{ij} + \sum_{v=1}^{j_r} \alpha_v^{(r)} I(\vec{U}_v^{(r)}, \vec{Y}_v^{(r)}|\vec{Z}_v^{(r)}) \leq 0, 1 \leq r \leq N \tag{1}
\]

The above is a set of \( N \) equations for some finite \( N \). The superscript \( r \) represents the number of the equation, \( \beta_{ij}^{(r)}, \alpha_v^{(r)} \) are computable real numbers for all \( i, j, r, \) and \( j_r \) are integers. \( \vec{U}_v^{(r)}, \vec{Y}_v^{(r)} \) and \( \vec{Z}_v^{(r)} \) are all vectors with components belonging to the set \( \{X_1, X_2, \ldots, X_k\} \) and
such that these vectors have different $X_i$ as components. The inequality $(\leq)$ in some or all of the equations in $\Box$ may also be $<, >, \geq, =$. The representation is independent of the particular $d \in \mathbb{D}$.

2. Constraints: The random variables $X_1, X_2, \ldots, X_n$ lead to a probability distribution on $\prod_{i=1}^k X_i$. $p(x_1, x_2, \ldots, x_k)$ is the probability that $X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k$. These are $\prod_{i=1}^k |X_i|$ such probabilities (which satisfy constraints that probabilities add to one). Arrange these probabilities in a certain order (the order does not matter), and we get a vector of length $\prod_{i=1}^k |X_i|$. Denote this vector by $\vec{p}$. The action of the network $d$ is denoted by $c_d$. Note that $c_d$ is a transition probability and corresponding to $b_i \in B_i, s \in C, a_i \in A_i, s' \in C$, we have the transition probability $c_d(b_1, \ldots, b_n, s|a_1, a_2, \ldots, a_n, s')$. These are, then, $\prod_{i=1}^n |A_i| \times \prod_{i=1}^n |B_i| \times |C|^2$ values. Arrange them in some order and this leads to a vector $\vec{q}_d$ where $d$ refers to the particular network $d \in \mathbb{D}$. Note that the vector $\vec{p}$ is variable whereas the vector $\vec{q}_d$ is constant.

The constraints are of the form

$$f(\vec{p}, \vec{q}_d) = 0$$

(2)

Here, $\vec{f} = (f_1, \ldots, f_l)$ is a vector function for some finite $l$. Each $f_i$ is a polynomial in the components of $\vec{p}$ and $\vec{q}_d$. Also, the coefficients of these polynomials should be algebraic numbers. $\vec{f}$ is independent of the particular $d \in \mathbb{D}$. Note again, that the only unknowns are components of $\vec{p}$, though the equations are polynomial in $\vec{p}$ and $\vec{q}_d$.

This is because for the particular $d \in \mathbb{D}$, $q_d$ will be substituted for, by the conditional probability defining the network. Also note that $q_d$ depends on $d \in \mathbb{D}$. The idea is that the function of $\vec{p}$ used when determining the constraints should be independent of $\vec{q}_d$, but different $\vec{q}_d$s in this function will result in different polynomials in $\vec{p}$. These are the constraints on $\vec{p}$ and thus, the constraints on $(X_1, X_2, \ldots, X_k)$.

Note that there are other constraints on $\vec{p}$, that the components of $\vec{p}$ are non-negative and add to 1.

The above pair of representation and constraints is called an FLC corresponding to $\mathbb{D}$. Consider the $(R_{ij}, 1 \leq i, j \leq n)$ which are feasible for the optimization problem determined by the above Representation and Constraints, and denote by $E_d$, the closure of this feasible region for the network $d \in \mathbb{D}$. Note that this feasible region depends on $d \in \mathbb{D}$ because the optimization problem depends on $\vec{q}_d$, even though the FLC is independent of $\mathbb{D}$. If $E_d$ is the same
as the capacity region of $d \forall d \in \mathcal{D}$, we call the above FLC an admissible FLC corresponding to $\mathcal{D}$.

See Subsection 5.2 for a discussion on the motivation of this definition.

3. Computability of approximating the capacities for a family of Markov channels if there exists an admissible FLC

Note that for a point-to-point Markov channel, $(R_{ij}, 1 \leq i, j \leq n)$ is just a $R$, the rate of communication from User 1 to User 2. Such an FSMC is determined by a conditional probability $c(y, s|x, s')$.

Lemma 1. Consider a set $\mathcal{D}$ of FSMCs, all with the same input, output and state spaces, such that $c_d(y, s|x, s')$ is algebraic for all $x, y, s, s'$ and for all $d \in \mathcal{D}$. Let there exist an admissible FLC, denoted by $F$, corresponding to $\mathcal{D}$. Then, given any $\epsilon > 0$ small enough, given $d \in \mathcal{D}$, there exists an algorithm (independent of $d$) which outputs $\beta_d$ such that $|C_d - \beta_d| < \epsilon$ where $C_d$ is the capacity of $d$.

Proof. Fix $d \in \mathcal{D}$. First consider the constraints of $F$. These are polynomial inequalities in $\vec{p}$. Denote by $\mathcal{V}$, the set of all these constraints. Given an algebraic number $\delta > 0$. Construct a finite set $\mathcal{U}$ consisting of probability distributions which are elements of $\mathcal{P}(X_1 \times X_2 \times \cdots \times X_n)$ such that for any $s \in \mathcal{P}(X_1 \times X_2 \times \cdots \times X_n)$, $\exists u \in \mathcal{U}$ such that $||s - u||_1 < \delta$ and that, $u(x_1, x_2, \ldots, x_n)$ is rational $\forall x_i \in X_i, 1 \leq i \leq n, \forall u \in \mathcal{U}$. Pick a particular $u \in \mathcal{U}$. Recall that $\vec{p}$ is a vector consisting of $p(x_1, x_2, \ldots, x_n)$ as $x_1, x_2, \ldots, x_n$ vary. Consider the set of constraints which includes the constraints $\mathcal{V}$ along with the constraints $-\delta \leq p(x_1, x_2, \ldots, x_n) - u(x_1, x_2, \ldots, x_n) \leq \delta$ (we could choose, in this inequality, a number much smaller than $\delta$, but $\delta$ suffices), $\forall x_i \in X_i, 1 \leq i \leq n$. Denote the set of all these inequalities by $\mathcal{W}$. The first order theory of the real closed field is decidable (Tarski, 1951) (see also (Website1, 2013), (Website2, 2013)), and thus, determining whether a feasible solution exists to the set of inequalities $\mathcal{W}$ is decidable. Construct a set $\mathcal{S}$ as follows: $u \in \mathcal{S}$ if and only $u \in \mathcal{U}$ and if there exists a feasible solution to the set of inequalities $\mathcal{W}$. The set $\mathcal{S}$ should be thought of as an ‘approximation’ (consisting of a finite number of elements) of the feasible region corresponding to the constraints in $F$.

Next, consider the representation. The representation of an FLC corresponding to a point-to-point Markov channel, by noting $[\mathbb{I}]$ consists of
inequalities of the following form:

\[ R \leq \sum_{v=1}^{j_r} \alpha_v^{(r)} I(\vec{U}_v^{(r)}, \vec{Y}_v^{(r)}|\vec{Z}_v^{(r)}), 1 \leq r \leq N, \]  

(3)

in other words,

\[ R \leq \min_{r \in \{1, 2, \ldots, N\}} \sum_{v=1}^{j_r} \alpha_v^{(r)} I(\vec{U}_v^{(r)}, \vec{Y}_v^{(r)}|\vec{Z}_v^{(r)}) \]  

(4)

The capacity of the channel is the supremum of such \( R \) over the \((X_1, X_2, \ldots, X_r)\) which satisfy the constraints of \( F \). Recall that \( \vec{U}_v^{(r)}, \vec{Y}_v^{(r)}, \vec{Z}_v^{(r)} \) are random vectors whose components belong to the set \( \{X_1, X_2, \ldots, X_r\} \). Note that conditional mutual information is uniformly continuous and that, a quantification of this fact follows by noting that \( I(X; Y|Z) = H(X, Z) + H(Y, Z) - H(Z) - H(X, Y, Z) \) and noting (6) in [Palaiyanur et al., 2008], which is the following: For a finite set \( \mathbb{G} \), for \( p_1, p_2 \in \mathcal{P}(\mathbb{G}) \), \( ||p_1 - p_2||_1 \leq \frac{1}{2} \),

\[ |H(p_1) - H(p_2)| \leq ||p_1 - p_2||_1 \ln \frac{|\mathbb{G}|}{||p_1 - p_2||_1} \]  

(5)

It follows, since \( F \) is admissible for \( \mathbb{D} \), by how the set \( S \) is constructed from the constraints of \( F \), by noting (5), that for any \( d \in \mathbb{D} \), that there exists a \( \delta > 0 \) such that

\[ C_d - \frac{\epsilon}{10} \leq \max_{u \in S} \min_{1 \leq r \leq N} \sum_{v=1}^{j_r} \alpha_v^{(r)} I(\vec{U}_u^{(r)}, \vec{Y}_u^{(r)}|\vec{Z}_v^{(r)}) \leq C_d + \frac{\epsilon}{10} \]  

(6)

where in the above equation, the various \( I(\vec{U}_v^{(r)}, \vec{Y}_v^{(r)}|\vec{Z}_v^{(r)}) \) are calculated corresponding to the particular \((X_1, X_2, \ldots, X_k)\) which is the random vector corresponding to \( u \in \mathbb{S} \). Denote

\[ \gamma \triangleq \max_{u \in S} \min_{1 \leq r \leq N} \sum_{v=1}^{j_r} \alpha_v^{(r)} I(\vec{U}_u^{(r)}, \vec{Y}_u^{(r)}|\vec{Z}_v^{(r)}) \]  

(7)

There would be an error in calculation of \( \gamma \) by the algorithm because of error in computation of the conditional mutual informations and error in computation of \( \alpha_v^{(r)} \). The latter are computable by the definition of an FLC,
and thus, can be approximated within an arbitrary accuracy by means of an algorithm. The conditional mutual informations can also be approximated within an arbitrary accuracy by means of an algorithm because all probabilities entering the calculations of these mutual informations are rational numbers; this follows because we compute mutual informations corresponding to \( u \in S \), and by construction, the probability distribution corresponding to \( u \) has \( p(x_1, x_2, \ldots, x_k) \) rational for all \( x_i \in \mathbb{X}_i \), \( 1 \leq i \leq k \). It follows, then, that the error in computation of \( \gamma \) can be made \( \leq \frac{1}{10} \). Denote the value of \( \gamma \) after this calculation error by \( \beta \). This \( \beta \) satisfies the properties of the \( \beta \) required in the lemma.

4. Non-existence of an admissible FLC for a certain family of FSMCs

**Theorem 2.** There exists a set \( \mathbb{D} \) of FSMCs with the same input, output and state spaces for which given any FLC, this FLC is not admissible for \( \mathbb{D} \).

**Proof.** Consider the set of channels \( \mathcal{S}_\lambda \), where the latter is defined in \cite{Elkouss2018}. Note that \( c(y, s|x, s') \) (the notation in \cite{Elkouss2018} is \( p(y, s|x, s') \)) is rational \( \forall \) channels \( \in \mathcal{S}_\lambda \), \( \forall, x, y, s, s' \), by construction in \cite{Elkouss2018}, and thus algebraic. Note, further, that FSMCs in \( \mathcal{S}_\lambda \) have the same input, output and state spaces (number of input symbols = 10, output symbols = 2 and state symbols = 62). Main Result 2 in \cite{Elkouss2018} states that for a given \( \lambda \), all channels in \( \mathcal{S}_\lambda \) have capacity \( \geq \lambda \) or \( \leq \frac{\lambda}{2} \). Let \( \lambda > 0 \). Let there be an admissible FLC for \( \mathcal{S}_\lambda \). Choose \( \epsilon = \frac{\lambda}{20} \) (20 in the denominator is chosen arbitrarily in a way that it is significantly less than \( \frac{1}{2} \)). Consider a channel \( d \in \mathcal{S}_\lambda \) with capacity \( C \). By Lemma 4, there exists an algorithm which says that the capacity of \( d \in [C - \epsilon, C + \epsilon] \). If capacity of \( d \) is \( \leq \frac{\lambda}{2} \), it follows, as a consequence of the output of the algorithm, that the capacity of \( d \leq \frac{\lambda}{2} + \epsilon \) which is \( < \lambda \), and thus, the capacity of this channel is \( \leq \frac{\lambda}{2} \). Similarly, if capacity of \( d \) is \( \geq \lambda \), it follows as a consequence of the output of the algorithm, that the capacity of \( d \geq \lambda - \epsilon \) which is \( > \frac{\lambda}{2} \), and thus, the capacity of \( d \) is \( \geq \lambda \). Thus, if there exists an admissible FLC for the set of channels \( \mathcal{S}_\lambda \), it is decidable whether the capacity of the channel is \( \leq \frac{\lambda}{2} \) or the capacity of the channel \( \geq \lambda \), and this contradicts Main Result 2 in \cite{Elkouss2018}. The only possibility, thus, is that there is no admissible FLC for the class of channels \( \mathcal{S}_\lambda \) for \( \lambda > 0 \).
5. Recapitulation, discussions and research directions

5.1. Recapitulation

In this paper, it was proved that finite letter characterizations as defined in this paper do not exist for a certain class of Markoff networks, that is, finite state machine channels. The idea of the proof was to use non-computability for approximating capacity for this class of channels Elkouss et al. (2018), and in this paper, we prove capacity can be approximated for FLCs by means of an algorithm. It follows that FLCs do not exist for this class of channels. In the appendix, a short proof non-computability of approximating channel capacity for Markoff channels with partial state information is given.

5.2. Discussions and research directions

- Motivation for the definition of an FLC and an admissible FLC:

  The definition of FLCs is partly motivated by existing single letter characterizations for capacity regions of networks in information theory. Note, for example, that the capacity region of a point-to-point memoryless channel, a memoryless multiple-access channel, the Marton region for a broadcast channel (Gamal et al., 2011), can all be put in the form of the above mentioned optimization problem. We have abstracted out the properties of the definitions of these characterizations, and made them more general (in the constraints) when making this definition. (The other motivation is that with this definition, we are able to prove the theorems we proved).

  As an example, consider the Han-Kobayashi region for the capacity region of the interference channel $c(y_1, y_2|x_1, x_2)$ (the notation used is from Gamal et al. (2011)):

  $$R_1 < I(X_1; Y_1|U_2, Q)$$
  $$R_2 < I(X_2; Y_2|U_1, Q)$$
  $$R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q)$$
  $$R_1 + R_2 < I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q)$$
  $$R_1 + R_2 < I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q)$$
  $$2R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q)$$
  $$R_1 + 2R_2 < I(X_2, U_1; Y_2|Q) + I(X_2; Y_2|U_1, U_2, Q) + I(X_1, U_2; Y_1|U_1, Q),$$
for some PMF \( p(q)p(u_1, x_1|q)p(u_2, x_2|q) \) (and some cardinality bounds on sets).

The seven random variables \( Q, U_1, U_2, X_1, X_2, Y_1, Y_2 \) can be renamed \( X_1, X_2, \ldots, X_7 \). That \( p(q, u_1, x_1, u_2, x_2) \) is constrained to be of the form \( p(q)p(u_1, x_1|q)p(u_2, x_2|q) \) can be written as polynomial equations in \( \vec{p} \). That \( p(y_1, y_2|x_1, x_2) \) should be precisely the action of the interference channel \( c(y_1, y_2|x_1, x_2) \) can be written as polynomial constraints in \( \vec{p} \) and \( \vec{q} \). That probabilities add to one lead to polynomial equations. Thus, achievable rates corresponding to the Han-Kobayashi region can be written in the form of Representation and Constraints.

As a simpler example, consider the achievable rate region for a DMC, given by a conditional probability \( p(y|x) \) where \( x \in X \) and \( y \in Y \). This region given by \( R < I(X; Y) \) where \( p_X \) is a probability distribution on \( X \) and \( p_{Y|X} \) is restricted to be the action of the channel. Clearly, this region can also be written in the above form of Representation and Constraints.

Next, consider the Marton’s inner bound for a broadcast channel \( p(y_1, y_2|x) \) given by (the notation used is from Gamal et al. (2011))

\[
\begin{align*}
R_1 &\leq H(Y_1) \\
R_2 &\leq I(U; Y_2) \\
R_1 + R_2 &\leq H(Y_1|U) + I(U; Y_2)
\end{align*}
\]

for some pmf \( p(u|x) \).

By a reasoning similar to that for the Han-Kobayashi region described above, this region can also be in the form of Representation and Constraints.

The same is the case for other existing characterizations in information theory.

- Note that in the set of channels \( \mathcal{S}_\lambda \), the gap between \( \frac{\lambda}{2} \) and \( \lambda \) is a well quantified gap (equal to \( \frac{\lambda}{2} \)). For this reason, we can prove, by slight modifications to the proofs of Lemma 1 and Theorem 2 that there exists an \( \epsilon \) small enough that no FLC even approximates the capacity of the family of channels \( \mathcal{S}_\lambda \). In other words, there exists no FLC such that \( \forall d \in \mathcal{S}_\lambda \) the feasible region of the FLC corresponding to \( d \) is \([0, C_d + \gamma]\) or \([0, C_d - \gamma]\) for any \( \gamma \leq \epsilon \) where \( C_d \) is the capacity of \( d \).
It would also be worthwhile to explore which sets of channels and networks can this result be generalized to. For example, the knowledge of initial state is necessary at the transmitter, both in (Elkouss et al., 2018) and the example we present in Appendix A for construction of Markoff channels for which an undecidability result holds in order to approximate capacity, which leads to a result for non-existence of FLCs for the class of channels $S_\lambda$ defined in (Elkouss et al., 2018).

However, it is unclear what happens with channels where initial state is not known at the transmitter. For example, for indecomposable channels (Gallager, 1968), channels for which the knowledge of initial state dies down with time, and thus, the knowledge of initial state is not necessary in the sense that the capacity of the channel is independent of whether the initial state is known or not, and is independent of the initial state (in the language of (Gallager, 1968), $\bar{C} = C$), it is unclear whether such a result will hold, and it would be important to see, what happens for this class of channels. The authors doubt that an undecidability result for approximating capacity can be proved for indecomposable channels by using the technique in (Elkouss et al., 2018) or the technique in Appendix A in this paper; that the initial state is known at the transmitter and that, the channel action might vary depending on the initial state (in the language of (Gallager, 1968) $\bar{C}$ and $C$ might be different) seems to be crucial in both (Elkouss et al., 2018) and Appendix A. That said, the authors would also like to speculate that admissible FLCs do not exist in general, for indecomposable channels. If this is indeed the case, it needs a proof via one technique or the other, and if this is not the case, it needs to be proved that FLCs indeed exist. It the opposite is true, that is, admissible FLCs do exist for indecomposable channels, that would require a proof. In general, it would be worthwhile to explore, for which sets of Markoff channels, with or without feedback, and with or without partial information, do FLCs exist. The authors emphasize that much of this paragraph is speculation.

Further, it would be important to explore whether we can prove non-computability of approximating capacity results for general networks, in particular, general memoryless networks. A memoryless network consists of $n$ users and the action of the network can be described by a transition probability $p(y_1, y_2, \ldots, y_n|x_1, x_2, \ldots, x_n)$, where this
transition probability represents the probability that the output at User \( i \) is \( y_i \), \( 1 \leq i \leq n \) if the input at User \( i \) is \( x_i \), \( 1 \leq i \leq n \). If we can prove a non-computability result of approximation of capacity region for memoryless networks for fixed input and output spaces, that may also imply the non-existence of FLCs for these networks (needs proof), though the author doubts that proving such a non-computability result would be possible by the method used in (Elkouss et al., 2018) and and the construction in Appendix A in this paper: that there is memory in the channel seems to be fundamental in both these cases. That said, as for the case of indecomposable channels, the authors would like to speculate that admissible FLCs do not exist, in general, for memoryless networks. If this is indeed the case, it needs proof via one technique or the other, and if this is not the case, it needs to be proved that FLCs indeed exist. Memoryless networks are the simplest kinds of networks, and a positive or negative result for existence or non-existence of FLCs for these networks will shed light on the case for general networks. It needs to be said that whether FLCs exist or not is not even known for the simple case of the broadcast channel. The authors emphasize, as in the previous paragraph, that much of this paragraph is speculation.

- From a mathematical perspective, it would be worthwhile to explore whether the theorem in this paper can be proved without resorting to the ‘heavy-duty machinery’ of the decidability of the first order theory of the real closed field. Also from a mathematical perspective, based on the proof of Lemma 1, the authors conjecture that the Representation part of the FLC, as defined in Section 2, can consist of functions much more general than mutual information and further, it may also be non-linear in \((R_{ij}, 1 \leq i, j \leq N)\), and still, Lemma 1 and Theorem 2 will hold. It may be worthwhile to see, to what extent, these generalizations are possible.

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Appendix A. Non-computability of approximating capacity for time-invariant Markov channels with partial state information

In this appendix, we provide an outline of a proof of the non-computability of approximating, within an additive constant, the capacity of Markov channels with partial state information by using prior results in the theory of PFAs concerning undecidability of approximations related to the emptiness problem, as stated and proved in (Madani et al., 2003). The idea of PFAs is also used in (Elkouss et al., 2018); here, we provide a short construction and proof based on the above mentioned result in (Madani et al., 2003). The authors found this construction without knowledge of the work of Elkouss et al.. The outline is a complete proof but for one missing step discussed in Appendix A.1. This appendix is written in discursive style.

Consider the definition of PFA in Section 2.3 in (Madani et al., 2003). The details of this definition are the following (we cut and paste from (Madani et al., 2003)): A PFA, $M$ is defined by a quintuple $M = (Q, \Sigma, T, s_1, s_n)$ where $Q$ is a set of $n$ states, $\Sigma$ is the input alphabet, $T$ is a set of $n \times n$ row-stochastic transition matrices, one for each symbol in $\Sigma$, $s_1 \in Q$ is the initial state of the PFA, and $s_n \in Q$ is an accepting state. The state transition is determined as follows:

- the current input symbol $a$ determines a transition matrix $M_a$,
- the current state $s_i$ determines the row $M_a[i, \cdot]$, a probability distribution over the possible next states, and
- the state changes according to the probability distribution $M_a[i, \cdot]$

The accepting state $s_n$ is absorbing, that is, $M_a[n, n] = 1, \forall a \in \Sigma$.

Corresponding to this PFA, we construct an FSMC as follows: The channel takes as input, an element of the set $\Sigma$. The state space of the channel is $Q$, and the initial state is $s_1$. $s_n$ is the only ‘good’ state in the sense that will become clear below. The channel acts as follows: if the channel is in state $s_i$, and the input to the channel is $a$, the channel transitions to state $s_{i'}$ with probability $M_{a}[i, i']$. Partial state information of whether the channel is in state $s_n$ or not is known at both the transmitter and the receiver. Further, when the channel is in state $s_n$, transmission of $K$ bits (think of $K$ as being large) per channel use is possible over the channel, otherwise, the channel outputs an error symbol ‘e’.
The definition of channel capacity that we use is generalized capacity, which, in this scenario, is described best by an example as stated in the second column of Page 1 of [Verdu et al., 2010]. In this example, the channel is binary input, binary output, and is such that:

- with probability $1 - q$, the channel reproduces the input sequence error free
- with probability $q$, the channel introduces errors with probability $h^{-1}(0.5)$, where $h(x)$ is the binary entropy function in bits.

If the encoder knew which of the two states is in effect, it could adapt the rate, and the average rate of communication would be $1 - \frac{q}{2}$. However, if the encoder did not know, which state is in effect, then, reliable communication would only be possible at rates $< \frac{1}{2}$. The reader is referred to Verdu et al. (2010) for details.

For the channel we have constructed above, partial state knowledge is available at the transmitter and the receiver in the sense that the transmitter and the receiver know, whether the channel is in state $s_n$ or not. Further, if the channel enters state $s_n$, it stays in state $s_n$. Also, if the channel is in state $s_n$, $K$ bits can be transmitted reliably over the channel per channel use. Thus, once the channel enters state $s_n$, $K$ bits are reliably communicated per channel use over the channel, that point onwards. Note, now, the definition of $L(M, \tau)$ in [Madani et al., 2003]. This is the set of all infinite-length strings such that the PFA ends in state $s_n$ with probability $> \tau$. The emptiness problem is: given a PFA $M$, and given a threshold $\tau$, determine whether $L(M, \tau)$ is empty or not. See [Madani et al., 2003] for precise details of the definition of $L(M, \tau)$ and for the statement of the emptiness problem. Note that

$$\Pr(\text{PFA enters state } s_n)$$

$$= \sum_{\alpha=0}^{\infty} \Pr(\text{PFA enters state } s_n \text{ at time } \alpha) \quad (A.1)$$

The above holds because probability of a countable union of disjoint events is equal to the sum of the probabilities of these events. Now, the PFA enters $s_n$ with probability $> \tau$. This implies, from the above, that for any $\gamma > 0$, there exists a $\psi$ such that the probability that the PFA enters state $s_n$ at time $< \psi$ is $\geq \tau - \gamma$. 
Consider the following communication scheme: if the channel does enter state $s_n$ at time $< \psi$, communicate bits reliably over the channel starting at this point, else, declare that no reliable communication is possible over the channel. Recall the example of generalized channel capacity discussed above. For the channel that we have constructed from the PFA as discussed above, it follows, by the discussion above, that if $L(\mathcal{M}, \tau_1)$ is non-empty, reliable communication can be accomplished over this channel at a rate $R > \tau_1 K - \kappa$ for any $\kappa > 0$ by use of the above coding scheme (adapted to $\tau = \tau_1$). During the time slots when the channel is not in state $s_n$, at most $\log |\Sigma|$ bits per channel use can be communicated over the channel by sending bits by coding them into the channel input and by noting the partial channel state at the output. It follows, that if $L(\mathcal{M}, \tau_2)$ is empty, at most $\tau_2 K + \log |\Sigma|$ bits can be communicated reliably over the channel per channel use, irrespective of the coding scheme. Assume that $K$ is much larger compared to $|Q|$ and $|\Sigma|$ and assume in what follows that $\delta_1, \delta_2$ are small.

We state Corollary 3.4 from (Madani et al., 2003); this is the key result on which our result concerning uncomputability of approximating capacity will be based: For any fixed $\epsilon$, $0 < \epsilon < 1$, the following is undecidable: given a PFA for which one of the two cases hold:

- (1) the PFA accepts some string with probability $> 1 - \epsilon$.
- (2) the PFA accepts no string with probability $> \epsilon$.

Deciding, whether case (1) holds, is undecidable.

**Lemma 3.** Approximating the capacity of a Markoff channel with partial state information at the encoder and decoder within an additive constant is an uncomputable problem in general.

**Proof.** Consider the channels constructed from PFAs considered in Corollary 3.4 in (Madani et al., 2003) by following the procedure mentioned in this appendix, with $\epsilon$ in Corollary 3.4 in (Madani et al., 2003) made equal to $\delta_1$. Denote $C_l \triangleq \delta_1 K + \log |\Sigma|$ and denote $C_u \triangleq (1 - \delta_1)K - \delta_2$. Note that $C_u$ is strictly greater than $C_l$ for sufficiently large $K$. It follows, by the above explanation, that if the PFA in Corollary 3.4 in (Madani et al., 2003) falls under case (1), then the capacity of the channel $> C_u$. Similarly, if the PFA in Corollary 3.4 in (Madani et al., 2003) falls under case (2), then the capacity of the channel $< C_l$. Note here, that in this channel coding problem, the information whether the state is $s_n$ or not is available at the
transmitter. This changes the underlying PFA a little in the sense that there is partial knowledge of the state, that is, whether state is $s_n$ or not, and this can possibly increase the $\tau$ for which $L(M, \tau)$ is non-empty, and thus, it can possibly increase the capacity of the channel by making $a$ depend on whether the channel is in state $s_n$ or not. This however is not the case because consider a string which is used when there is partial knowledge of the state. This string will end when the state enters $s_n$, but for the case when the state has not entered $s_n$, the string will continue on for possibly infinite length of time. Use this same string when there is no partial knowledge of state in the PFA, and it follows that if $L(M, \tau)$ is non-empty if there is partial state information of the above form for the PFA, then $L(M, \tau)$ is non-empty for the original PFA too. With this explanation, denote $\Delta \equiv C_u - C_l$. It then follows, by Corollary 3.4 in [Madani et al., 2003], that the capacity of the channel cannot be approximated with an additive constant of $\Delta$ by means of an algorithm. This is because, if the contrary were true, we would be able to say, whether the capacity of the channel is $< C_l + 2\Delta$ or $> C_u - 2\Delta$. It follows, then, that we would be able to say whether the capacity of the channel is $< C_l$ or $> C_u$ (because at least one of these holds), from which, we would be able to say for the PFA, whether case (1) or case (2) holds in Corollary 3.4 in [Madani et al., 2003], and this is undecidable by Corollary 3.4 in [Madani et al., 2003].

Appendix A.1. Discussion

Note that in this construction, it is unclear if the PFA has fixed cardinality in state space and input space, and thus, it is unclear if the channel thus constructed has a fixed state space and input space. However, it is possible that results from [Blondel et al., 2003] where undecidability results are proved for PFAs of fixed dimension can be modified in order to get results for undecidability of approximations, and if this is the case, it is possible that an analogue of Theorem 2 may also hold for channels which are constructed from PFAs as described in this appendix. These are just ideas at this point which might be correct or incorrect.