Stratification and MHD Effects on Narrow Channel Problems

L. Prasanna Venkatesh¹ and J. Swetha²
Assistant Professor¹, M.Sc. Student²
Department of Mathematics
Sathyabama Institute of Science and Technology, chennai - 600119
Prasannavenkatesh.maths@sathyabama.ac.in¹
and swethajaianu@gmail.com²

Abstract. In this paper we analyze viscous oscillatory MHD inhomogeneous fluid flow through the channel which is narrow and rectangular with one wall being porous through which suction or injection is taken place with constant rate and the other wall being rigid. The equation pertaining to the motion of fluid are simplified by using narrow channel approximation and the simplified equation are solved using similarity transformation and the graphical representation are analyzed for velocity profiles. The results reveal that stratification and MHD effects are found predominantly at the centre of the channel while stratification suppresses the flow in vertical direction. For \( \beta = 0 \) and \( N = 0 \) the results are in accordance with that already available in literature.

1. Introduction

Fluid flows which has density as a function of various parameters like height and time are known as density stratified fluids. These fluids often tend to realign their original form once any disturbance applied to them. Such phenomena is very common in various environmental flows and chemical industry. Khandelwal and Jain discussed in detail about “flow of unsteady and magnetohydrodynamic stratified fluid through porous medium over a moving plate in slip flow regime” by presenting perturbation solution for various characteristics associated with the flow [3]. “The effect of suction and injection on the unsteady flow (between two parallel porous) with variable properties through rectangular channel” was studied by Attia [1]. “Magnetohydrodynamic flow and heat transfer of dusty viscoelastic stratified fluid down an inclined channel in porous medium under variable viscosity” was experimented by Chakraborty [2]. By assuming the pressure as a suitable expression of the variables \( x \) and \( y \) and unknown expression of \( t \), exact solution was presented. “Heat transfer in magnetohydrodynamic flow of dusty viscoelastic stratified fluid in porous medium under variable viscosity” was presented by Prakash by considering both vertical and horizontal velocity expressions as an expression of \( y \), \( t \) and by depicting the similarity solution to fluid velocity [8]. While considering fluids with moderate Reynolds number and channel with length significantly larger than breath, the flow equations simplifies to narrow channel flows. Such simplification in momentum equations are known as Lubrication approximation which was originated basically from Lubrication Industry. Such flows also draw interest from various other similar industries that are similar and supportive to Lubrication field. Panton discussed the concept of lubrication approximation problem which we refer as narrow channel problems for plane poiseuille flow by assuming the time independency for the flow [7]. He further extended it for bearing problems and expressed the exact solution for velocity functions by considering pressure dominated fluid flow. Reynolds from whom the theory and applications of lubrication concept in industries evolved due to Beauchamp Tower’s experiments [12]. Krenchikov
elaborated the various uses and purposes of “lubrication approximations to non-unidirectional coating flows with clean and surfactant interfaces” [4]. Krishna and Shrama solved the problem on “motion of an axisymmetric body in a rotating stratified fluid confined between two parallel planes” by considering the direction of planes to be vertical of the perpendicular axis of circulations [5]. Naidu explained the detailed solution to “stratified viscous flow between two oscillating cylinders” by formulating density stratification as exponentially distributed with governing equations of motion of fluid in polar form [6]. Prasanna Venkatesh analysed the “magnetohydrodynamics viscous oscillatory flow in a narrow channel due to suction or injection through the porous wall” by considering the electrically conducting fluid in both axial and transverse direction and its effects on channels which are very thin in nature [11]. Prasanna Venkatesh presented effect of “Stratification in rectangular channel problem” with the assumption that the density when there is no disturbance in the fluid as a function of depth only and also the flow being time dependent [10]. Prasanna Venkatesh and Surya prabha studied the “effect of narrow channel approximation on viscous oscillatory density stratified fluid flow through rectangular channels” by comparing the cases of flow without applying the lubrication approximation with the that when it is applied [9]. In present problem the MHD flow of density stratified fluid which is oscillatory as well as viscous passing through a vertical narrow rectangular channel with a porous wall along which the withdrawal of fluid is taken place with a constant velocity is modeled. The flow is assumed to be time dependent and narrow channel approximation is applied to the equation of motion considering the equation of incompressibility and density stratification which is linearly distributed in undisturbed state. The main aim of the research work is to interpret how the stratification property and magnetohydrodynamic effects of the fluid influence the fluid flow through narrow channel and its comparison with that when the channel is not narrow.

2. Problem Formulation and Solution
The problem explains the fluid flow through the rectangular region with width h₀ and length L, the fluid is removed due to suction velocity along a porous wall which is placed parallel to y – axis. The rigid plate is placed on y – axis such that the distance between them is of width h₀ as mentioned above. The plate regions outside of above description are rigid and hence no suction or injection can take place in this region. As the fluid flow considered is fully developed the initial velocity of the fluid is assumed to be constant. The ratio of h₀/L maintained to be very small so that the inertial terms of fluid flow equations are simplified based on the fact that suction velocity is so small compared to that of mean velocity in vertical direction. The liquid is heterogeneous in nature simultaneously having property of electrical conductivity. The density when the liquid is not having any kind of disturbances in its surface is proposed be a first degree function of height and when the motion begins it becomes a function of all variables of the flow parameter namely x, y and t. The representation of the flow in terms of mathematical equations are presented here.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{2}
\]

\[
\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \sigma \epsilon B_0^2 v - \rho g \tag{3}
\]

\[
\rho = \rho_0(y) + \rho'(x,y,t) \tag{4}
\]

\[
\rho_0(y) = \rho_0'(1 - \beta y) \tag{5}
\]

\[
\frac{\partial \rho}{\partial t} = \rho_0' \beta v \tag{6}
\]

The following explains the notations used in the above system of equations. (i) \( \mu \) - coefficient of viscosity (ii) \( \rho \) - density of the fluid (iii) \( \sigma \) - electrical conductivity (iv) \( B_0 \) - electromagnetic induction
(v) $\mu_e$ - magnetic permeability (vi) $H_0$ - transverse magnetic field. $B_0 = \mu_e H_0$ (vii) $\rho_0'$ - constant density (viii) $\rho'(y,t)$ - perturbation density (ix) $\beta$ - stratification parameter (x) $N$ - Brunt – Vaisala frequency $N^2 = \beta g$ (xii) $g$ – gravity (xiii) $v_0$ - initial average velocity , $u_0 = \frac{v_0 h_0}{L}$.

Define nondimensional variables as follows

$X = x/h_0, Y = y/L, U = Lu/h_0, V = v/v_0, P = (p - p_{ref}) h_0^2/L_0 u_0, Re = h_0 V_0/v$

As $h_0 \to 0$, neglecting the inertial terms in the momentum equations and applying Bousinesq’s approximation we get,

$\rho_0', \frac{\partial U}{\partial t} = -\frac{\partial P}{\partial X} \tag{7}$

$\rho_0', \frac{\partial V}{\partial t} = -\frac{\partial P}{\partial Y} + \mu \frac{\partial^2 V}{\partial X^2} - \sigma_e B_0^2 V - \rho_0 g \tag{8}$

Differentiating (7) with respect to $t$ and $Y$ partially and (8) with respect to $t$ and $X$ partially and then subtracting we get

$\rho_0', \frac{\partial (\partial U/\partial Y)}{\partial X} = -\frac{\partial V^3}{\partial t} + \sigma_e B_0^2 \frac{\partial V}{\partial X} + \rho_0' N^2 \frac{\partial V}{\partial X} \tag{9}$

$U = \Phi(X,Y) e^{i\alpha t}, \quad V = \Psi(X,Y) e^{i\alpha t}, \quad \text{and} \quad P = f(X,Y) e^{i\alpha t}$

$\rho_0' \left( \frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} \right) = -\frac{\partial V^3}{\partial X} - \left( i \frac{\rho_0^2 N^2}{\omega} - \sigma_e B_0^2 \right) \frac{\partial V}{\partial X} \tag{10}$

The boundary conditions of the problem are

$U(0, Y) = 0, U(1, Y) = u_1$

$V(0, Y) = 0, V(1, Y) = 0$

We define Stream Function $\psi$ such that

$U = \frac{\partial \psi}{\partial Y} \quad \text{and} \quad V = -\frac{\partial \psi}{\partial X} \tag{11}$

Equation (9) becomes

$\rho_0' \nabla^2 \psi = \mu \frac{\partial^4 \psi}{\partial X^4} + \left( i \frac{\rho_0^2 N^2}{\omega} - \sigma_e B_0^2 \right) \frac{\partial^2 \psi}{\partial X^2} \tag{12}$

The function $f(Y)$ is introduced as follows:

$\Psi = (v_0 - u_1 Y) f(X) \tag{13}$

Equation (11) becomes

$D^4 - \frac{1}{\mu} \left( \sigma_e B_0^2 + \rho_0' - \frac{i \rho_0^2 N^2}{\omega} \right) D^2 f(X) = 0 \tag{14}$

Where

$D^2 = \frac{d^2}{dX^2}, \quad \alpha^2 = \frac{1}{\mu} \left( \sigma_e B_0^2 + \rho_0' - \frac{i \rho_0^2 N^2}{\omega} \right) \Rightarrow \alpha = \sqrt{\frac{1}{\mu} \left( \sigma_e B_0^2 + \rho_0' - \frac{i \rho_0^2 N^2}{\omega} \right)}$

$\alpha^2$ is a complex number, hence there is only one possibility. Two of the roots are distinct and complex and the other two are zero (i.e. $m^2(m + \alpha)(m - \alpha) = 0$).

which the solution is given by

$f(Y) = c_1 + c_2 e^{\alpha X} + c_3 e^{-\alpha X} \tag{15}$

The Boundary Conditions are transformed in terms of $f(X)$ are as follows

$f(0) = 0; \quad f'(0) = -1; \quad f''(0) = 0; \quad f'''(1) = 0 \tag{16}$
The result obtained by applying the boundary conditions on \( f(X) \) are

\[
\begin{align*}
\tilde{f}(0) &= 0 = c_1 + c_3 + c_4 \\
\tilde{f}(1) &= 0 = c_2 + c_3 e^\alpha + c_4 e^{-\alpha} \\
\tilde{f}'(0) &= 0 = c_2 + \alpha c_3 - \alpha c_4 \\
\tilde{f}'(1) &= 0 = c_2 + \alpha c_3 e^\alpha - \alpha c_4 e^{-\alpha}
\end{align*}
\]

and

\[
\begin{align*}
c_1 &= \frac{e^{\alpha}-1}{e^{\alpha}-(1+e^{\alpha})} \\
c_2 &= \frac{\alpha}{\alpha+2} \\
c_3 &= \frac{\alpha e^\alpha}{\alpha+2} \\
c_4 &= \frac{-e^{-\alpha}}{\alpha+2} \end{align*}
\]

Solving the above equations, the values of the constants are as follows.

\[
\begin{align*}
\tilde{c}_1 &= \frac{\alpha+2}{\alpha+2} \\
\tilde{c}_2 &= \frac{\alpha}{\alpha+2} \\
\tilde{c}_3 &= \frac{\alpha e^\alpha}{\alpha+2} \\
\tilde{c}_4 &= \frac{-e^{-\alpha}}{\alpha+2}
\end{align*}
\]

Using the expression for \( \psi \) and equation (18), the velocity components are

\[
\begin{align*}
U(X, Y) &= -u_1 \left( \frac{e^{\alpha} - 1 - (1 + e^{\alpha}) \alpha e^\alpha - e^\alpha e^{-\alpha}}{(\alpha+2) - (\alpha+2) \alpha e^\alpha} \right) \\
V(X, Y) &= -\left( v_0 - u_1 Y \right) \left( \frac{-e^\alpha + (1 + e^{\alpha}) \alpha e^\alpha + e^\alpha e^{-\alpha}}{(\alpha+2) - (\alpha+2) \alpha e^\alpha} \right)
\end{align*}
\]

3. Results and Discussion

The effects of various parameters namely Brunt – Vaisala frequency (\( N \)), Electromagnetic induction and constant density (\( \rho_0 \)) on transverse velocity, axial Velocity, are presented graphically. The range of values of \( x \) and \( y \) is assumed to be 0 to 1 while that of each of the other parameters are varied. There are significant effects of stratification parameter \( N \) on transverse velocity throughout the channel at specific heights. Figure 1 indicates the effect of stratification on narrow channel flow and it is evident that the increase in stratification parameter \( N \) results in decrease of transverse velocity. Figure 2 depicts the effects of stratification for a non-narrow channel problem and shows that stratification effects are observed nearer to the central part of the channel and are indicating variable effects. Figure 3 represents the transverse velocity for different values of electromagnetic induction \( B_0 \) for narrow channel case. It is observed that increase in \( B_0 \) reverses the Transverse velocity nearer to the center of the channel. Figure 4 explains the transverse velocity for various electromagnetic induction \( B_0 \) values for non-narrow channel case. It is observed that increase in \( B_0 \) makes the Transverse velocity squarer making the flow stagnated. From Figure 5 and Figure 6 it is obvious that the axial velocity for varying stratification parameter is symmetric about a central point of the channel with increase in stratification parameter \( N \) for narrow channel case while that for non narrow channel case increases with increase in \( N \) values which confirms the property of realignment of stratified fluids. Figure 7 and Figure 8 explains the MHD effects on axial velocity for narrow channel and non-narrow channel cases respectively. It is observed that axial velocity shows similar effects by MHD effects as that of Stratification Parameter.

4. Conclusion

The effects of MHD and Stratification in narrow channel problem having a porous wall is presented in this paper. Lubrication approximation is applied in the momentum equation and the system of partial differential equation is solved analytically using similarity transformation. The outcome is compared with that of non-narrow channel problems. The following observations are obtained as a result of the analysis. Increase in stratification parameter \( N \) results in slower flow around the center of the channel while increased nonlinear transverse flow near the boundary irrespective of whether narrow-channel approximation is applied or not. Axial velocity profiles are symmetric about the center point of the channel for variation in both stratification and electromagnetic induction for narrow channel case whereas it has mixed effects for non-narrow channel case.
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Figure 1: Transverse Velocity for varying N for Narrow channel problem

Figure 2: Transverse Velocity for varying N for Non - Narrow channel problem
Figure 3: Transverse Velocity for varying $B_0$ for Narrow channel problem

Figure 4: Transverse Velocity for varying $B_0$ for Non-Narrow channel problem

Figure 5: Axial Velocity for varying $N$ for Narrow channel problem

Figure 6: Axial Velocity for varying $N$ for Non-Narrow channel problem

Figure 7: Axial Velocity for varying $B_0$ for Narrow channel problem

Figure 8: Axial Velocity for varying $B_0$ for Non-Narrow channel problem