Some Conclusions for Noncritical String Theory
Drawn from Two- and Three-point Functions in the
Liouville Sector

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Abstract

Starting from the known expression for the three-point correlation functions for Liouville exponentials with generic real coefficients at we can prove the Liouville equation of motion at the level of three-point functions. Based on the analytical structure of the correlation functions we discuss a possible mass shell condition for excitations of noncritical strings and make some observations concerning correlators of Liouville fields.

1to appear in the proceedings of the XXVIII. Int. Symp. on the Theory of Elementary Particles
Wendisch-Rietz, August 30 - September 3, 1994
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Introduction

One of the outstanding unsolved problems in string theory is posed by the question how to go off criticality without taking refuge to Kaluza-Klein concepts or saturating the conformal anomaly in D=4 by some additional degrees of freedom. After Polyakov’s seminal paper [1] we believe to know that the key lies in solving the 2D quantized Liouville field theory. Though there has been a lot of progress on this subject in the course of the years ([3, 12, 4, 8, 10] and references therein) we still do not know how to enter the most interesting region $1 < c_M < 25$ for the Virasoro central charge $c_M$ of the matter system.

In the bosonic string theory $c_M$ can be identified with the embedding dimension $d$. A lot of interesting results especially in comparison with solvable matrix-models were obtained for noncritical strings for $c_M \leq 1$ (e.g. [5] and references contained), especially at $d = 1$ the noncritical string could be treated [11, 20] and effectively resulted in a $D = 1 + 1$-dimensional critical string due to the extra dimension defined by the Liouville field. In a previous paper [13] we were able to generalize the calculation of three-point functions to $d > 1$ to become independent of the special 1d-kinematics. As we unfortunately still have to keep $c_M < 1$ we have to introduce background charges in target space sacrificing Lorentz invariance or its euclidean counterpart. Here we are going to draw some conclusions based on these results.

The central task in solving the theory of conformal matter coupled to 2D gravity consists in calculating the correlators for all marginal ”dressed” operators $\Omega_i^{(M)} e^{\beta_i \phi(z)} = \Omega_i^{(dressed)}$ (1)

with conformal weights $\Delta_i^{(M)}$ and $\Delta_i^{(L)}$ in the matter and Liouville sector, respectively, satisfying

$$\Delta_i^{(M)} + \Delta_i^{(L)}(\beta_i) = 1.$$ (2)

This is the motivation to look at correlators of products purely of Liouville exponentials for $N \geq 3$:

$$G_N(z_1, ..., z_N|\beta_1, ..., \beta_N) = \langle \prod_{j=1}^N e^{\beta_j \phi(z_j)} \rangle = \int D\phi \ e^{-S_L[\phi]} \prod_{j=1}^N e^{\beta_j \phi(z_j)}$$ (3)

with generic real dressing coefficients $\beta_i$ and the Liouville action

$$S_L[\phi|\hat{g}] = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \left( \hat{g}^{mn} \partial_m \phi \partial_n \phi + Q \hat{R}(z) \phi(z) + \mu^2 e^{\phi(z)} \right).$$ (4)

$\hat{g}$ is a classical reference metric of the 2D manifold of spherical topology, $\hat{R}$ the corresponding Ricci scalar. $Q$ parametrizes the central charge $c_L$ of the Liouville theory by

$$c_L = 1 + 3Q^2.$$ (5)

\footnote{We explicitly write only the “holomorphic” variable $z$ though both $z$ and $\bar{z}$ are present in the arguments.}
As the Liouville theory describes the gravitational sector of a conformal matter theory the conformal anomalies add up to zero $c_L + c_M - 26 = 0$. The exponent $\alpha$ in the cosmological term in (4) is just the dressing coefficient for the unity operator in the matter sector and derives from (2) for $\Delta(M) = 1$, $\beta_i \to \alpha$ and the general formula for the conformal weight of a vertex operator $\exp(\beta_i \phi)$ with a background charge $Q$ present in (4)

\[ \Delta_i(L) \equiv \Delta_i = \frac{1}{2} \beta_i (Q - \beta_i) . \]

With the additional demand to the cosmological operator to be a “microscopic” operator, i.e. $\alpha < \frac{Q}{2}$, one gets

\[ \alpha = \alpha_+ = \frac{Q}{2} \pm \frac{\sqrt{Q^2 - 8}}{2} . \] (7)

The zero mode integration in eq.(3) can be performed explicitly. For integer

\[ s_N = \frac{Q - \sum_{j=1}^{N} \beta_j}{\alpha} \] (8)

also the remaining functional integral can be done:

\[
G_N(z_1, ..., z_N|\beta_1, ..., \beta_N) = \frac{\Gamma\left(-s_N\right)}{\alpha} \left(\frac{\mu^2}{8\pi}\right)^{s_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^{-2\beta_i \beta_j} \\
\cdot \prod_{I=1}^{s_N} \left(d^2 w_I \prod_{j=1}^{N} |z_j - w_I|^{-2\alpha \beta_j}\right) \prod_{1 \leq I < J \leq s_N} |w_I - w_J|^{-2\alpha^2} .
\] (9)

Let us start with the 3-point function. Fortunately from the explicit representation (9) we can prove for integer $s_3$ that the standard structure of the $z_j$ dependence is realized.

\[ G_3(z_1, z_2, z_3|\beta_1, \beta_2, \beta_3) = A_3(\beta_1, \beta_2, \beta_3) \left|z_1 - z_2\right|^{2(\Delta_3 - \Delta_1 - \Delta_2)} \left|z_2 - z_3\right|^{2(\Delta_1 - \Delta_2 - \Delta_3)} , \] (10)

\[ A_3 = \lim_{u_3 \to \infty} |u_3|^{4\Delta_3} G_3(0, 1, u_3|\beta_1, \beta_2, \beta_3) . \] (11)

With (3) this gives $A_3$ as

\[ A_3 = \frac{\Gamma\left(-s_3\right)}{\alpha} \left(\frac{\mu^2}{8\pi}\right)^{s_3} \int \prod_{I=1}^{s_3} \left(d^2 w_I \left|w_I\right|^{-2\alpha \beta_I}\right) \prod_{1 \leq I < J \leq s_3} \left|w_I - w_J\right|^{-2\alpha^2} . \] (12)

Using the Dotsenko-Fateev integrals (14) this can be written as

\[ A_3(\beta_1, \beta_2, \beta_3) = \frac{\Gamma\left(-s_3\right)}{\alpha} \Gamma(1 + s_3) \left(\frac{\mu^2 \Gamma\left(1 + \frac{\alpha^2}{2}\right)}{8 \Gamma\left(-\frac{\alpha^2}{2}\right)}\right)^{s_3} \prod_{i=0}^{3} F_i \] (13)

with

\[ F_i = \exp\left(f(\alpha \beta_i, \frac{\alpha^2}{2} |s_3) - f(\alpha \beta_i, \frac{\alpha^2}{2} |s_3)\right) , \quad i = 1, 2, 3 \] (14)
\[ \beta_i = \frac{1}{2}(\beta_j + \beta_k - \beta_i) = \frac{1}{2}(Q - \alpha s_3) - \beta_i, \quad (i, j, k) = \text{perm}(1, 2, 3), \quad (15) \]

\[ F_0 = \left( -\frac{\alpha^2}{2} \right)^{-s_3} \frac{1}{\Gamma(1 + s_3)} \exp \left( f(1 - \frac{\alpha^2}{2}s_3, \frac{\alpha^2}{2}s_3) - f(1 + \frac{\alpha^2}{2}, \frac{\alpha^2}{2}s_3) \right), \quad (16) \]

\[ f(a, b|s) = \sum_{j=0}^{s-1} \log \Gamma(a + bj), \quad \text{integer } s. \quad (17) \]

There exists a continuation \[13, 18, 19\] of \( f(a, b|s) \) to arbitrary complex \( a, b, s \) given by

\[ f(a, b|s) = \int_0^\infty \frac{dt}{t} \left( (s-a-1)e^{-t} + b \frac{s(s-1)}{2} e^{-t} - s \frac{e^{-t}}{1-e^{-t}} \right) \quad + \frac{(1-e^{-bs})e^{-at}}{(1-e^{-b})(1-e^{-t})}, \quad (18) \]

It fulfills all the functional relations that can be read off the representation \[17\] for integer \( s \). Using the integral representation and the functional relations mentioned one can prove \[13\] that \( \exp(f(a, b|s)) \) is a meromorphic function. It is sufficient to investigate the case \( \text{Re } b \geq 0 \). Under this circumstance \( \exp f \) has poles at

\[ a = -bj - l \quad \text{(poles)} \quad (19) \]

and zeros at

\[ a + bs = -bj - l \quad \text{(zeros)} \quad (20) \]

In both cases \( j \) and \( l \) are integers \( \geq 0 \). In this talk we are going to focus on four applications of the foregoing results:

(i) Correlator for two Liouville exponentials
(ii) Validity of the quantum Liouville equation
(iii) Two- and three-point function for the Liouville field itself
(iv) Conclusions about the poles and zeros of \( A_2, A_3 \)

**Correlator for two Liouville exponentials**

Let us now turn to the 2-point function. Taking \[3\] unmodified also for \( N = 2 \) would imply \( G_2(z_1, z_2|\beta_1, \beta_2) = G_3(z_1, z_2, z_3|\beta_1, \beta_2, 0) \). The unwanted \( z_3 \)-dependence as usual in conformal theories drops for \( \Delta_1 = \Delta_2 \). However, the \( z \)-independent factor \( A_3(\beta, \beta, \beta_3) \) diverges for \( \beta_3 \to 0 \). The reason for this divergence is the change of the situation with respect to the conformal Killing vectors (CKV). The 3-punctured sphere has no CKV’s while the 2-punctured sphere has one. The (divergent) volume of the corresponding subgroup of the Möbius group \( SL(2,\mathbb{C}) \) leaving \( z_1 \) and \( z_2 \) fixed is

\[ V^{(2)}_{\text{CKV}} = \int \frac{d^2w}{|z_1 - z_2|^2 |z_2 - w|^2}. \quad (21) \]
Having this in mind we define
\[
G_2(z_1, z_2|\beta) = \langle e^{\beta\phi(z_1)} e^{\beta\phi(z_2)} \rangle = \frac{1}{V_{CKV}^{(2)}} \int D\phi \ e^{-S_L[\phi]} e^{\beta\phi(z_1)} e^{\beta\phi(z_2)} .
\] (22)

Treating the functional integral in analogy to that for the 3-point function and choosing \(\int d^2 w_1\) as the cancelled integration one gets (\(s_2 = 1 + s_3(\beta, \beta, \alpha)\))
\[
G_2(z_1, z_2|\beta) = -\frac{\mu^2}{8\pi s_2} G_3(z_1, z_2, w_1|\beta, \beta, \alpha) |z_1 - z_2|^{-2} |z_1 - w_1|^2 |z_2 - w_1|^2 .
\] (23)

From (10) we see that the \(w_1\) dependence on the r.h.s. cancels. For this result \(\Delta_1 = \Delta_2\) is crucial. Altogether we find
\[
G_2(z_1, z_2|\beta) = \frac{A_2(\beta)}{|z_1 - z_2|^{4\Delta}} ,
\] (24)

with
\[
A_2(\beta) = -\frac{\mu^2}{8\pi s_2} A_3(\beta, \beta, \alpha) .
\] (25)

For this constellation of arguments in \(A_3\) one can eliminate the function \(f\) completely and derive quite a simple expression in terms of \(\Gamma\)-functions \([19]\) coinciding up to an irrelevant factor with the result presented in \([14]\) for the integrated 2-point function in gravitationally dressed minimal models (rational \(s_2\)). For \(A_2\) describing the gravitational dressing of the two point function in minimal models the resulting form fits into the "leg-factor" structure known for higher correlation functions \((N \geq 3), [20]\). The extension of the procedure to the one and zero-point function (partition function) is straightforward \([19]\) in principle. However, in the one-point case the method yields inconsistent results, the dependence on the fixed unintegrated \(w_1, w_2\) does not cancel for generic \(\beta\). This is a reflection of the absence of a \(\text{SL}(2,\mathbb{C})\) invariant vacuum, which prevents looking at the one-point function as a scalar product of two physical states. We come back to this point later.

(ii) Validity of the Liouville Equation

The Liouville equation in our parametrization is the equation of motion for the action \((4)\) in the limit of flat \(\dot{g}\)
\[
\partial^2 \phi - \frac{\alpha\mu^2}{2} e^{\alpha\phi} = 0 .
\] (26)

As a partial check we want to prove
\[
\langle \partial^2 \phi(z_1) e^{\beta_2\phi(z_2)} e^{\beta_3\phi(z_3)} \rangle = \frac{\alpha\mu^2}{2} \langle e^{\alpha\phi(z_1)} e^{\beta_2\phi(z_2)} e^{\beta_3\phi(z_3)} \rangle .
\] (27)
up to contact terms.

The l.h.s. of (27) is given by

$$4\partial_z \partial_{z_1} \lim_{\beta_1 \to 0} \frac{\partial}{\partial \beta_1} G_3(z_1, z_2, z_3 | \beta_1, \beta_2, \beta_3).$$

Using (6), (10) the differentiation with respect to \( \beta_1 \) is straightforward. In the generic case \( \beta_2 \neq \beta_3, \beta_j \neq 0, j = 2, 3 \) one finds \( A_3(0, \beta_2, \beta_3) = 0 \). Therefore, the contribution of terms with logarithms \( \log|z_i - z_j| \) generated by the \( \beta_1 \) dependence of \( \Delta_1 \) drops out in the limit \( \beta_1 \to 0 \) and one obtains after differentiation with respect to \( z_1 \)

$$\langle \partial^2 \phi(z_1) e^{\beta_2 \phi(z_2)} e^{\beta_3 \phi(z_3)} \rangle = \frac{4(\Delta_2 - \Delta_3)^2 \frac{\partial}{\partial \beta_1} A_3(\beta_1, \beta_2, \beta_3) |_{\beta_1 = 0}}{|z_2 - z_1|^{2(1 + \Delta_2 - \Delta_3)} |z_1 - z_3|^{2(1 + \Delta_3 - \Delta_2)} |z_2 - z_3|^{2(\Delta_3 + \Delta_2 - 1)}} \quad (28)$$

up to contact terms that will be neglected.

This way the quantum Liouville equation (27) is reduced to

$$4(\Delta_2 - \Delta_3)^2 \frac{\partial}{\partial \beta_1} A_3(\beta_1, \beta_2, \beta_3) |_{\beta_1 = 0} = \frac{\alpha \mu^2}{2} A_3(\alpha, \beta_2, \beta_3). \quad (29)$$

This relation can easily be verified using the representation (13) for \( A_3 \) and the functional identities satisfied by the constituents \( F_a, F_i \).

(iii) Liouville Two- and Three-Point Functions

As mentioned in the introduction and practised at an intermediate stage in the previous section already, we relate the Liouville operator \( \phi(z) \) to \( \partial_\beta e^{\beta \phi(z)} = \phi e^{\beta \phi(z)} \). For reasons becoming clear in a moment we still do not specify the value of \( \beta \) after differentiation. We only require to treat all \( \beta_j \) on an equal footing. From (13), (6), (7) this yields

$$\partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3} G_3( z_j | \beta_j ) = |z_1 - z_2|^{2(\Delta_3 - \Delta_1 - \Delta_2)} |z_1 - z_3|^{2(\Delta_2 - \Delta_1 - \Delta_3)} |z_2 - z_3|^{2(\Delta_1 - \Delta_2 - \Delta_3)} \cdot \left( \frac{\partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3} A_3}{\sqrt{Q^2 - 8}} - Q^2 - 8 \partial_{\beta_1} \partial_{\beta_2} A_3(l_{12} + l_{13} + l_{23}) + (Q^2 - 8) \partial_{\beta_1} A_3(l_{13} L_{12} + L_{23} L_{12} + L_{23} L_{13}) + (Q^2 - 8) \frac{\hat{F}}{2} A_3(l_{12} L_{13} L_{23}) \right),$$

(30)

with \( l_{ij} = \log|z_i - z_j|, \ L_{ij} = l_{ij} - l_{ki} - l_{kj}, \ (i, j, k) = \text{perm}(1, 2, 3). \)

Use has been made of the equality of derivatives with respect to different \( \beta_j \) if after differentiation a symmetric point in \( \beta \)-space is chosen.

The natural choice \( \beta_j = 0 \) removes the power-like \( z \)-dependence in (30), but \( A_3 \) turns out to be singular at this point: The value of \( F_1 F_2 F_3 \) depends on how the origin in \( \beta \)-space is
approached. In addition $F_0$ has a pole (note $s_3 \to \frac{Q}{a} = 1 + \frac{2}{\alpha^2}$, ([13]), ([20])).

Our functional integral yields the correlation function of Liouville exponentials directly, there is no interpretation as a vacuum expectation value with respect to a SL(2,C) invariant vacuum [15]. Therefore, operator insertions are well-defined only in the presence of at least two Liouville exponentials playing the role of spectators. This concept worked perfectly in the previous section where we constructed $\langle \phi(z_1) e^{\beta_2 \phi(z_2)} e^{\beta_3 \phi(z_3)} \rangle$. To get in the same sense 2 and 3-fold insertions of $\phi$ one has to start with the 4 and 5-point functions of exponentials. Unfortunately, these higher correlation functions are not available up to now.

(iv) Poles and Zeros of $A_2$ and $A_3$

The spectrum of poles and zeros of the 3-point function and two special degenerate cases of 4-point functions as well as related problems for the interpretation of non-critical strings have been discussed in [13], [18]. We add in this section observations concerning the 3-point and 2-point function which are relevant in connection with some recent work on off shell critical strings [14] and which shed some light on the question of mass shell conditions for non-critical strings.

For applications to noncritical strings we are interested in the case $\text{Re} (\alpha^2) > 0$. This of course is realized for $c_M < 1$ but higher dimensional target space $D > 1$ made possible by the presence of a linear dilaton background [18] i.e.

$$c_M = D - 3P^2.$$  \hspace{1cm} (31)

It is even valid for $1 \leq c_M < 13$ if ([13]) is taken seriously also in between $1 \leq c_M \leq 25$, since then we have

$$\alpha^2 = \frac{13 - c_M - \sqrt{(25 - c_M)(1 - c_M)}}{6}. \hspace{1cm} (32)$$

On the other side in ref. [14] off shell critical strings are constructed for $c_M = 26$ by enforcing the otherwise violated condition of conformal (1,1) dimension by the dressing with suitable Liouville exponentials. Clearly, for this application one needs $\alpha^2 < 0$.

In the first situation $\text{Re} (\alpha^2) > 0$ one finds the following pole-zero pattern of $\prod_{j=1}^3 F_j$ [13]

$$\alpha \bar{\beta}_j = \frac{\alpha^2}{2} k_j + l_j \hspace{0.5cm} (\text{poles})$$

$$\alpha \beta_j = \frac{\alpha^2}{2} k_j + l_j \hspace{0.5cm} (\text{zeros})$$

$$\text{Re} (\alpha^2) > 0, \hspace{0.5cm} \text{integer } k_j, l_j, \hspace{0.5cm} \text{both } \leq 0 \hspace{0.5cm} \text{or both } > 0. \hspace{1cm} (33)$$

While the position of zeros depends on the value of single $\beta_j$, the pole position is given by a combination out of all $\beta_j$ involved. Only in applications to dressings of minimal models
also the pole position factorizes (leg poles).
In the second situation \( \text{Re } \alpha^2 < 0 \) one ends up with the following situation

\[
\alpha \beta_j - \frac{\alpha^2}{2} = -\frac{\alpha^2}{2} k_j + l_j \quad \text{(poles)}
\]
\[
\alpha \bar{\beta}_j - \frac{\alpha^2}{2} = -\frac{\alpha^2}{2} k_j + l_j \quad \text{(zeros)}
\]

\[
\text{Re } (\alpha^2) < 0, \quad \text{integer } k_j, l_j, \quad \text{both } \leq 0 \text{ or both } > 0 .
\]

Now the position of poles of \( \prod_{j=1}^{3} F_j \) is determined by the single \( \beta_j \).

The remaining factors in \( A_3 \) depend on the \( \beta_j \) via \( s_3 \) only. For their combined pole-zero spectrum arising from the \( \Gamma \)-functions and \( F_0 \) one finds for \( \text{Re } \alpha^2 > 0 \) no zeros but poles at

\[
\frac{\alpha}{2} \sum_{i=1}^{3} \beta_i - \frac{\alpha^2}{2} - 1 = \frac{\alpha^2}{2} k + l \quad \text{(poles)}
\]

\[
\text{Re } (\alpha^2) > 0, \quad \text{integer } k, l, \quad \text{both } \leq 0 \text{ or both } > 0 .
\]

In the other case \( \text{Re } \alpha^2 < 0 \) one has instead

\[
\frac{\alpha}{2} \sum_{i=1}^{3} \beta_i = \frac{\alpha^2}{2} (1 - j) - 1 \quad \text{(poles)}
\]

\[
\frac{\alpha}{2} \sum_{i=1}^{3} \beta_i = \frac{\alpha^2}{2} (k + 2) - l \quad \text{(zeros)}
\]

or \( \frac{\alpha}{2} \sum_{i=1}^{3} \beta_i = \frac{\alpha^2}{2} (1 - k) + l + 2 \quad \text{(zeros)} \)

\[
\text{Re } (\alpha^2) < 0, \quad \text{integer } k, j, l \geq 0 .
\]

Altogether we find a drastic change in the analytic structure with respect to the \( \beta_j \) in going from \( \text{Re } \alpha^2 > 0 \) to \( \text{Re } \alpha^2 < 0 \).

Let us turn to the 2-point function. We obtain for arbitrary \( \alpha^2 \)

\[
\alpha \beta = \frac{\alpha^2}{2} - l \quad \text{or } \alpha \beta = 1 - l \frac{\alpha^2}{2}, \quad \text{integer } l \geq 0 \quad \text{(poles)}
\]

\[
\alpha \beta = \frac{\alpha^2}{2} + j \quad \text{or } \alpha \beta = 1 + j \frac{\alpha^2}{2} \quad \text{or } \beta = \frac{Q}{2}, \quad \text{integer } j \geq 2 \quad \text{(zeros)}
\]

From this pole-zero pattern we can derive an interesting conjecture concerning the mass shell condition for noncritical strings. For instance the coefficient \( \beta \) in a gravitationally dressed vertex operator for tachyons \( 2, 3, 4 \)

\[
e^{i k \mu X^\mu (z)} e^{\beta \phi(z)}
\]
is related to $k_\mu$ by the requirement of total conformal dimension $(1,1)$ (compare (2))

$$\frac{1}{2}\beta(Q - \beta) + \frac{k(k - P)}{2} = 1,$$

or equivalently

$$(\beta - \frac{Q}{2})^2 - (k - \frac{P}{2})^2 = \frac{1 - D}{12}. \tag{38}$$

In contrast to the critical string, where the demand of dimension $(1,1)$ delivers the mass shell condition $\frac{k(k - P)}{2} = 1$, eq. (38) implies no restriction for the target space momentum.

A condition on $k_\mu$ can arise only due to an additional restriction on the allowed values of $\beta$. The $\beta$ dependent factor $A_2(\beta)$ discussed above appears as the dressing factor in the 2-point S-matrix element for the tachyon excitation of the string. From the point of view of field theory in target space this object is an inverse propagator. Hence it should vanish as soon as the tachyon momentum approaches its mass shell. For generic $\beta$ the dressing factor $A_2(\beta)$ is different from zero and it is natural to associate its zeros with the mass shell. For $c_M < 1$ i.e. $0 < \alpha^2 < 2$ the spectrum of zeros is unbounded from above. The lowest zero is $\beta = \frac{Q}{2}$. The resulting spectrum for the mass of the gravitational dressed tachyon, i.e. $m^2_T = \frac{1 - D}{12} - (\beta - \frac{Q}{2})^2$, is not bounded from below. However, since all zeros, except that at $\beta = \frac{Q}{2}$, obey $\beta > \frac{Q}{2}$ they correspond to operators $e^{\beta \phi}$ describing states with wave functions in mini-superspace approximation $\propto e^{(\beta - \frac{Q}{2}) \phi}$. These states are not normalizable in the infrared $\phi \to +\infty$ and have to be excluded [15]. On the other hand $\beta = Q/2$ sits just on the border to the “microscopic” states describing local insertions with wave functions peaked in the ultraviolet and “macroscopic” states with imaginary exponents. $\beta = Q/2$ then leads to

$$(k - \frac{P}{2})^2 = \frac{D - 1}{12}. \tag{39}$$

The generalization to higher string excitations is straightforward. For instance in the graviton case an additional term +1 on the l.h.s. of (38) leads to $(k - \frac{P}{2})^2 = \frac{D - 25}{12}$.

To conclude we contributed to the construction of correlation functions in Liouville theory. This construction is a longstanding problem relevant for various aspects of string theory and general conformal field theory. We were able to calculate the 2 and 3-point functions of Liouville exponentials of arbitrary real power. The method of continuation in the parameter $s$ passed a very crucial test. The Liouville equation of motion is fulfilled, hence we are sure that the derived correlation functions indeed reflect some essential features of quantized Liouville theory. What concerns applications to noncritical string theory an interesting conjecture on mass shell conditions emerged. Keeping the standard picture that 2-point S-matrix elements vanish on shell, we related the on shell condition to the spectrum of zeros of the 2-point function of Liouville exponentials. The further check of both the $s$-continuation itself as well as the spectrum conjecture requires the knowledge of
the higher \((N \geq 4)\) correlation functions. Unfortunately, at present the necessary integral formulas are not available.

References

[1] A. M. Polyakov, *Phys. Lett.* **B103** (1981) 207
[2] F. David, *Mod. Phys. Lett.* **A3** (1988) 1651
[3] J. Distler, H. Kawai, *Nucl. Phys.* **B321** (1989) 509
[4] H. Dorn, H.-J. Otto, *Phys. Lett.* **B232** (1989) 327; **B280** (1992) 204
[5] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, *2D Gravity and Random Matrices*, hep-th/9306153, to appear in *Physics Reports*
[6] J. L. Gervais, A. Neveu, *Nucl. Phys.* **B199** (1982) 59, **B202** (1982) 125, **B238** (1984) 125, 396
[7] H.-J. Otto, G. Weigt, *Phys. Lett.* **B159** (1985) 341; *Z. f. Phys.* **C31** (1986) 219
[8] Y. Kazama, H. Nicolai, *Int. Jour. Mod. Phys.* **A9** (1994) 667
[9] O. Babelon, *Phys. Lett.* **B215** (1988) 523
J. L. Gervais, *Comm. Math. Phys.* **130** (1990) 257
[10] J. L. Gervais, J. Schnittger, *Phys. Lett.* **B315** (1993) 258
[11] M. Goulian, M. Li, *Phys. Rev. Lett.* **66** (1991) 2051
[12] J. L. Gervais, *Nucl. Phys.* **B391** (1993) 287
[13] H. Dorn, H.-J. Otto, *Phys. Lett.* **B291** (1992) 39
[14] R. Myers, V. Periwal, *Phys. Rev. Lett.* **70** (1993) 2841 and *Conformally Invariant Off-shell Strings* preprint hep-th/9311083
[15] N. Seiberg, *Prog. Theor. Phys. Suppl.* **102** (1990) 319
[16] E. D’Hoker, R. Jackiw, *Phys. Rev.* **D26** (1982) 3517
[17] V. S. Dotsenko, V. A. Fateev, *Nucl. Phys.* **B240** (1985) 312, **B251** (1985) 691
[18] H. Dorn, H.-J. Otto, *Remarks on the continuum formulation of noncritical strings*, hep-th/9212003 and Proc. Symp. Wendisch-Rietz 1992, DESY 93-013
[19] H. Dorn, H.-J. Otto, *Nucl. Phys.* **B429** (1994) 375
[20] P. Di Francesco, D. Kutasov, *Phys. Lett.* **B261** (1991) 385, *Nucl. Phys.* **B375** (1992) 119