The cycle of length four is strictly $F$-Turán-good

Doudou Hei$^1$, Xinmin Hou$^{1,2}$

$^1$School of Mathematical Sciences
University of Science and Technology of China, Hefei, Anhui 230026, China

$^2$CAS Key Laboratory of Wu Wen-Tsun Mathematics
University of Science and Technology of China, Hefei, Anhui 230026, China

Abstract

Given an $(r + 1)$-chromatic graph $F$ and a graph $H$ that does not contain $F$ as a subgraph, we say that $H$ is strictly $F$-Turán-good if the Turán graph $T_r(n)$ is the unique graph containing the maximum number of copies of $H$ among all $F$-free graphs on $n$ vertices for every $n$ large enough. Győri, Pach and Simonovits (1991) proved that cycle $C_4$ of length four is strictly $K_{r+1}$-Turán-good for all $r \geq 2$. In this article, we extend this result and show that $C_4$ is strictly $F$-Turán-good, where $F$ is an $(r + 1)$-chromatic graph with $r \geq 2$ and a color-critical edge. Moreover, we show that every $n$-vertex $C_4$-free graph $G$ with $N(H, G) = \text{ex}(n, C_4, F) - o(n^4)$ can be obtained by adding or deleting $o(n^2)$ edges from $T_r(n)$. Our proof uses the flag algebra method developed by Razborov (2007).

1 Introduction

All graphs considered in this article are finite and simple. Given a graph $G$, write $n(G)$ (resp. $e(G)$) for $|V(G)|$ (resp. $|E(G)|$) and write $G[Z]$ for the induced subgraph of $G$ on the vertex set $Z \subseteq V(G)$. Let $X$ and $Y$ be disjoint subsets of $V(G)$. By $G[X,Y]$, we denote the bipartite subgraph of $G$ consisting of all edges that have one endpoint in $X$ and another in $Y$. For mutually disjoint subsets $V_1, V_2, \ldots, V_k \subseteq V(G)$, similarly, we define $G[V_1, \ldots, V_k]$ to be the $k$-partite subgraph of $G$ consisting of all edges in $\cup_{1 \leq i < j \leq k} E(G[V_i, V_j])$. Write $K(V_1, \ldots, V_k)$ for the complete $k$-partite graph with color classes $V_1, \ldots, V_k$ and write $K_{t_1,\ldots,t_k}$ for a complete $k$-partite graph $K(V_1, \ldots, V_k)$ with $|V_i| = t_i$ for $i \in [k]$, where $[k] = \{1, 2, \ldots, k\}$.

Fix a graph $F$, we say that a graph $G$ is $F$-free (or induced $F$-free) if it does not contain $F$ as a subgraph (or an induced subgraph). For given graphs $H$...
and $G$, we define $N(H, G)$ (resp. $N_f(H, G)$) as the number of subgraphs (resp. induced subgraphs) of $G$ isomorphic to $H$. Let $ex(n, H, F)$ denote the maximum value of $N(H, G)$ among all $F$-free graphs on $n$ vertices, and we call the graph with $ex(n, H, F)$ copies of $H$ an extremal graph. This function is well-studied when $H$ is an edge, and it is called the Turán number $ex(n, F)$ of $F$ (one can see, for example [18], for a survey).

Let $T_r(n)$ denote the $r$-partite $n$-vertex Turán graph, i.e., the $r$-partite $n$-vertex complete graph of which each partite is of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$. As pointed by Gerbner and Palmer [7], there are few $F$-free graphs we have already known that are good candidates for being extremal constructions for maximizing copies of $H$, an exception is the Turán graph, they call $H$ to be $F$-Turán-good under this situation. More precisely, given an $(r + 1)$-chromatic graph $F$ and a graph $H$ does not contain $F$ as a subgraph, we say that $H$ is $F$-Turán-good (or strictly $F$-Turán-good) if $ex(n, H, F) = N(H, T_r(n))$ (and the Turán graph $T_r(n)$ is the unique extremal graph) for every $n$ large enough. We also call $(H, F)$ Turán-good (or strictly Turán-good) for short.

Győri, Pach and Simonovits [10] proved that $(C_4, K_{r+1})$ is strictly Turán-good.

**Theorem 1.1** (10). $C_4$ is strictly $K_{r+1}$-Turán-good. Precisely, for every $K_{r+1}$-free graph $G$ with $|V(G)| = n \geq \max\{r, 5\}$, $N(C_4, G) \leq N(C_4, T_r(n))$, equality holds if and only if $G \cong T_r(n)$.

We say that an edge $e$ of a graph $F$ is color-critical if deleting $e$ from $F$ results in a graph with a smaller chromatic number. In extremal problems, a graph $F$ with $\chi(F) = r$ and a color-critical edge often behaves similarly to $K_r$. The most famous one along this flavor was given by Simonovits [17], who showed that $(K_2, F)$ is strictly $F$-Turán-good, and Ma and Qiu [13] proved a generalized version by extending the pair $(K_2, F)$ to $(K_r, F)$, where $\chi(F) > r \geq 2$. There are also a few (strictly) Turán-good pairs given by researchers, including Gerbner, Palmer, Murphy, etc. Here is a list of some other Turán-good pairs as we have known so far:

1. (Győri, Pach and Simonovits [10]) $(H, K_3)$ is strictly Turán-good, where $H$ is a bipartite graph with matching number $\lceil \frac{|V(H)|}{2} \rceil$ (including the path $P_\ell$, the even cycle $C_{2\ell}$ and the Turán graph $T_2(m)$);
2. (Győri, Pach and Simonovits [10]) $(K_{2t}, K_r)$ is strictly Turán-good for $t = 2, 3$;
3. (Gerbner and Palmer [7]) $(H, K_k)$ is Turán-good for $k \geq k_0$, where $H$ is a complete multipartite graph and $k_0$ is a constant depending on $H$, and Gerbner and Palmer conjectured that this result is true for any graph $H$;
4. (Gerbner [4]) For any positive integers $m$ and $\ell$, $(P_m, C_{2\ell+1})$ and $(C_{2m}, C_{2\ell+1})$ are Turán-good.
5. (Gerbner and Palmer [7]) $(C_4, B_2)$ and $(C_4, F_2)$ are Turán-good, where $B_k$ (resp. $F_k$) is the graph of $k$ triangles all sharing exactly one common edge (resp. one common vertex);
6. (Gerbner and Palmer [4]) For any positive integers $m$ and $\ell$, $(P_m, B_\ell)$ is Turán-good.
(7) (Gerbner and Palmer [7], Gerbner [3]) \((P_3, F)\) is Turán-good, where \(F\) is a graph with \(\chi(F) = k \geq 3\) and a color-critical edge;

(8) (Murphy and Nir [14], Qian et al [15]) \((P_4, K_k)\) and \((P_5, K_k)\) are Turán-good for \(k \geq 4\);

(9) (Gerbner [3]) \((M_{2\ell}, F)\) is Turán-good, where \(M_{2\ell}\) is a matching on \(2\ell\) vertices, \(F\) is a graph with \(\chi(F) = k \geq 3\) and a color-critical edge.

Recently, a special family of bipartite graphs was proven to be strictly \(F\)-Turán-good when \(F\) is a graph with \(\chi(F) = 3\) and a color-critical edge by Hei, Hou and Liu [11].

**Theorem 1.2** ([11]). Let \(F\) be a graph with \(\chi(F) = 3\) and a color-critical edge and let \(H\) be a bipartite graph with matching number \(\left\lfloor \frac{|V(H)|}{2} \right\rfloor\). Then \(H\) is strictly \(F\)-Turán good, i.e., \(\text{ex}(n, H, F) = N(H, T_2(n))\) for every \(n\) large enough, and the Turán graph \(T_2(n)\) is the unique extremal graph for \((H, F)\).

As a special case, we know that \((C_4, F)\) is strictly \(F\)-Turán-good when \(F\) is a graph with \(\chi(F) = 3\) and a color-critical edge, which also has been shown in [5] (a special case of Theorem 1.9).

**Corollary 1.3** ([5]). \((C_4, F)\) is strictly \(F\)-Turán-good when \(F\) is a graph with \(\chi(F) = 3\) and a color-critical edge.

In this article, we continue to show that Corollary [3] also holds for \(F\) with \(\chi(F) = r + 1 \geq 3\) and a color-critical edge. This result is also a special case of results given by Gerbner [6], but that is essentially without proof, here we give the proof by a different way.

**Theorem 1.4.** Let \(F\) be a graph with \(\chi(F) = r + 1 \geq 4\) and a color-critical edge. Then \((C_4, F)\) is strictly Turán-good.

In fact, Hei, Hou and Liu [11] have given a necessary and sufficient condition for a graph \(H\) to be strictly \(F\)-Turán-good using the so called ‘weak \((r + 1)\)-T-property’ and ‘T-extremal’. We say a graph \(H\) has the weak \((r + 1)\)-T-property if \(N(H, K) \leq N(H, T_r(n))\) for every complete \(r\)-partite graph \(K = K_{t_1, \ldots, t_r}\), with \(t_1 + \cdots + t_r = n\) and the equality holds if and only if \(K \cong T_r(n)\) for every \(n\) large enough. Let \(F\) be a graph with \(\chi(F) = r + 1\), an \(n\)-vertex \(F\)-free graph \(G\) is called T-extremal if \(|e(G) - e(T_r(n))| = o(n^2)\).

**Theorem 1.5** ([11]). Let \(F\) be a graph with \(\chi(F) = r + 1 \geq 3\) and a color-critical edge and let \(H\) be a connected graph with \(\chi(H) \leq r\). Suppose every \(n\)-vertex \(F\)-free graph \(G\) with \(N(H, G) = \text{ex}(n, H, F)\) is T-extremal. If \(H\) has the weak \((r + 1)\)-T-property, then \(H\) is strictly \(F\)-Turán-good.

**Remark A:** Gerbner [8] defined a generalization of T-extremal \(F\)-free graph \(G\) with \(N(H, G) = \text{ex}(n, H, F)\): Let \(\chi(H) < \chi(F) = k + 1\). We say that \(H\) is \(F\)-Turán-stable if every \(n\)-vertex \(F\)-free graph \(G\) with \(N(H, G) \geq \text{ex}(n, H, F) - o(n^{\chi(H)})\) can be obtained from \(T_k(n)\) by adding and removing \(o(n^2)\) edges.

In fact, alongside the proof of Theorem [15] given in [11], Theorem 1.5 can be
restated as follows: Let $F$ be a graph with $\chi(F) = r + 1 \geq 3$ and a color-critical edge and let $H$ be a connected graph with $\chi(H) \leq r$. Suppose $H$ is $F$-Turán-stable. If $H$ has the weak $(r + 1)$-T-property, then $H$ is strictly $F$-Turán-good. **Remark B:** By Theorem 1.1, show that $C_4$ has the weak $(r + 1)$-T-property. Therefore, by the above restated Theorem 1.5, to prove Theorem 1.4, it is sufficient to prove Theorem 1.6.

**Theorem 1.6.** Let $F$ be a graph with $\chi(F) = r + 1 \geq 4$ and a color-critical edge. Then $C_4$ is $F$-Turán-stable.

The rest of the article is arranged as follows. In Section 2, we give the proof of Theorem 1.6 admitting an important lemma (Lemma 2.6). Section 3 will give a brief overview of the flag algebra. In the last section, we prove Lemma 2.6 using the flag algebra method.

## 2 Proof of Theorem 1.6

Let $H$ be a fixed graph. An $s$ blow-up of a graph $H$ is the graph obtained by replacing each vertex $v$ of $H$ with an independent set $W_v$ of size $s$, and each edge $uv$ of $H$ with a complete bipartite graph between the corresponding two independent sets $W_u$ and $W_v$. We need the following nice results of graphs.

**Lemma 2.1 ([1]).** Let $H$ be a fixed graph with $h$ vertices and let $F$ be a graph. Then $\text{ex}(n, H, F) = \Omega(n^h)$ if and only if $F$ is not a subgraph of a blow-up of $H$. Otherwise, $\text{ex}(n, H, F) \leq n^{h-\alpha}$ for some $\alpha > 0$.

**Lemma 2.2 (Induced Removal Lemma [2]).** Let $\mathcal{F}$ be a set of graphs. For each $\varepsilon > 0$, there exist $n_1(\varepsilon) > 0$ and $\delta_1(\varepsilon) > 0$ such that for every graph $G$ of order $n \geq n_1(\varepsilon)$, if $G$ contains at most $\delta_1(\varepsilon)n^{1/|V(H)|}$ induced copies of $H$ for every $H \in \mathcal{F}$, then $G$ can be made induced $\mathcal{F}$-free by removing or adding at most $\varepsilon n^2$ edges from $G$.

**Lemma 2.3 ([13] [14]).** Let $F$ be a graph with chromatic number $\chi(F) = r + 1$ and a color-critical edge. If $G$ is an $F$-free graph on $n$ vertices, then $N(K_4, G) \leq N(K_4, T_r(n)) = \frac{r^2 - 6r^2 + 11r - 6}{4} n^4 + o(n^4)$.

**Lemma 2.4 ([14]).** Let $P_3^c$ be the unique graph on three vertices with one edge (also called the co-cherry graph). Then graph $G$ is a complete multipartite graph if and only if it does not contain the co-cherry graph $P_3^c$ as an induced subgraph.

We first show the following lemma.

**Lemma 2.5.** Let $f(x_1, x_2, \ldots, x_r) = N(C_4, K_{x_1, x_2, \ldots, x_r})$. If there is some $x_i \geq x_j + 2$, then $f(x_1, \ldots, x_i - 1, \ldots, x_j + 1, \ldots, x_r) > f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_r)$.
Proof. We may assume \( i = 1, j = 2 \). Let \( H = K(X_1, X_2, \ldots, X_r) \) with \( |X_i| = x_i \) for \( i \in [r] \) and let \( H^* = K(X_1 \setminus \{v\}, X_2 \cup \{v^*\}, X_3, \ldots, X_r) \), where \( v \in X_1 \) and \( v^* \) is a new vertex added to \( X_2 \). Denote \( X_1^* = X_1 \setminus \{v\} \) and \( X_2^* = X_2 \cup \{v^*\} \). Let \( N_H(v, C_4) = \{C : C \cong C_4 \text{ in } H \text{ with } v \in V(C)\} \) and \( N_{H^*}(v^*, C_4) = \{C : C \cong C_4 \text{ in } H^* \text{ with } v^* \in V(C)\} \). Let \( n_H(v, C_4) = |N_H(v, C_4)| \) and \( n_{H^*}(v^*, C_4) = |N_{H^*}(v^*, C_4)| \). Since the copy of \( C_4 \) that does not pass through \( v \) in \( H \) remains unchanged in \( H^* \), to show the lemma, it suffices to show that \( n_H(v, C_4) < n_{H^*}(v^*, C_4) \). In addition, a copy \( C \in N_H(v, C_4) \) with \( V(C) \cap (X_1 \cup X_2) = \{v\} \) corresponds to a copy \( C^* \in N_{H^*}(v^*, C_4) \) with \( V(C^*) \cap (X_1^* \cup X_2^*) = \{v^*\} \) and vice versa. It suffices to focus on those \( C_4 \) that contain \( v \) (or \( v^* \)) and at least one other vertex in \( X_1 \cup X_2 \) (or in \( X_1^* \cup X_2^* \)). Let

\[
c(v, n_1, n_2) = |\{C \in N_H(v, C_4) : |V(C) \cap X_i| = n_i \text{ for } i = 1, 2\}|
\]

and

\[
c^*(v^*, n_1^*, n_2^*) = |\{C \in N_{H^*}(v^*, C_4) : |V(C) \cap X_i^*| = n_i^* \text{ for } i = 1, 2\}|.
\]

Then we have \( 1 \leq n_1 \leq 2, 0 \leq n_2 \leq 2, 0 \leq n_1^* \leq 2, 1 \leq n_2^* \leq 2 \), and \( c(v, 1, 0) = c^*(v^*, 0, 1) \). Therefore, we only need to show \( c(v, 1, 1) + c(v, 1, 2) + c(v, 2, 0) + c(v, 2, 1) + c(v, 2, 2) < c^*(v^*, 1, 1) + c^*(v^*, 2, 1) + c^*(v^*, 0, 2) + c^*(v^*, 1, 2) + c^*(v^*, 2, 2) \). Let \( I = \{3, 4, \ldots, r\} \). We count the number of \( C_4 \) according to the choices of \( n_1 \) and \( n_2 \),

\[
c(v, 1, 1) = x_2 \left( \sum_{i \in I} \frac{x_i}{2} \right) + 3 \sum_{\{i,j\} \in \binom{I}{2}} x_i x_j,
\]

\[
c(v, 1, 2) = \frac{x_2}{2} \sum_{i \in I} x_i,
\]

\[
c(v, 2, 0) = (x_1 - 1) \left( \sum_{i \in I} \frac{x_i}{2} + \sum_{\{i,j\} \in \binom{I}{2}} x_i x_j \right),
\]

\[
c(v, 2, 1) = (x_1 - 1) x_2 (n - x_1 - x_2),
\]

\[
c(v, 2, 2) = (x_1 - 1) \frac{x_2}{2}.
\]
Lemma 2.6. Let $r$ stability lemma for almost optimal complete we will prove it in Section 4 by the flag algebra argument. The following is a
denote the family of $F$ at most $\delta n$ $F$-free graph
Similarly, we have

\[ c^*(v^*, 1, 1) = (x_1 - 1) \cdot \left( \sum_{i \in I} \binom{x_i}{2} + 3 \sum_{\{i, j\} \in \binom{I}{2}} x_ix_j \right), \]
\[ c^*(v^*, 2, 1) = \binom{x_1 - 1}{2} \sum_{i \in I} x_i, \]
\[ c^*(v^*, 0, 2) = x_2 \cdot \left( \sum_{i \in I} \binom{x_i}{2} + \sum_{\{i, j\} \in \binom{I}{2}} x_ix_j \right), \]
\[ c^*(v^*, 1, 2) = (x_1 - 1)x_2(n - x_1 - x_2), \]
\[ c^*(v^*, 2, 2) = x_2 \binom{x_1 - 1}{2}. \]

Since $x_1 \geq x_2 + 2$, we have $c(v, 1, 2) < c^*(v^*, 2, 1)$, $c(v, 2, 1) = c^*(v^*, 1, 2)$, $c(v, 2, 2) < c^*(v^*, 2, 2)$, and
\[
c(v, 1, 1) + c(v, 2, 0) = (x_1 + x_2 - 1) \cdot \sum_{i \in I} \binom{x_i}{2} + (x_1 + 3x_2 - 1) \sum_{\{i, j\} \in \binom{I}{2}} x_ix_j < (x_1 + x_2 - 1) \cdot \sum_{i \in I} \binom{x_i}{2} + (3x_1 + x_2 - 3) \sum_{\{i, j\} \in \binom{I}{2}} x_ix_j = c^*(v^*, 1, 1) + c^*(v^*, 0, 2).
\]

This completes the proof. \(\square\)

Let $H$ and $G$ be graphs on $n_1$ and $n_2$ vertices, respectively, where $n_1 \leq n_2$. The density $d(H, G)$ of $H$ in $G$ is defined by
\[
d(H, G) = \frac{N(H, G)}{\binom{n_2}{n_1}}.
\]

Let $F$ be a graph with $\chi(F) = r + 1$ and a color-critical edge, and let $\mathcal{F}_{n,r}$ denote the family of $F$-free graphs on $n$ vertices. Define $OPT_r(C_4)$ as follows:
\[
OPT_r(C_4) = \lim_{n \to \infty} \max_{G \in \mathcal{F}_{n,r}} d(C_4, G).
\]

Lemma 2.6. Let $F$ be a graph with $\chi(F) = r + 1 \geq 4$ and a color-critical edge. For any $\delta > 0$, there exist $n_c = n_c(\delta)$ and $\varepsilon_c = \varepsilon_c(\delta) > 0$ such that for every $F$-free graph $G$ of order $n \geq n_c$, if $d(C_4, G) \geq OPT_r(C_4) - \varepsilon_c$, then $G$ contains at most $\delta n^3$ induced copies of the co-cherry graph $P_3^5$.

The above lemma plays an important role in the proof of Theorem 1.6 and we will prove it in Section 4 by the flag algebra argument. The following is a stability lemma for almost optimal complete $r$-partite graphs.
Lemma 2.7. Let $G$ be a complete $r$-partite graph with partite sets $X_1, \cdots, X_r$. For any $\varepsilon > 0$, there exists $\delta_\varepsilon = \delta_\varepsilon(\varepsilon) > 0$ such that if $d(C_4, G) > \text{OPT}_r(C_4) - \delta_\varepsilon$, then for each $i = 1, 2, \cdots, r$,}

$$\frac{(1 - \varepsilon)|V(G)|}{r} \leq |X_i| \leq \frac{(1 + \varepsilon)|V(G)|}{r}.$$ 

Proof. Let $|V(G)| = n$ and let $x_i = |X_i|$ for $i = 1, 2, \ldots, r$. Without loss of generality, we may assume that $x_1 = \frac{1 + \eta(r - 1)}{n} n > \frac{1 + \varepsilon}{r}$. Then $\eta(r - 1) > \varepsilon$.

By Lemma 2.5, $d(C_4, G)$ is maximized if all the remaining parts are balanced, i.e. $x_i = \frac{(1 - \eta)n}{r}$ for $i = 2, \ldots, r$. Let $G^*$ be the graph induced by the vertex set $V(G) \setminus X_1$. Then

$$N(C_4, G^*) = N\left(C_4, T_{r-1} \left(\frac{(r - 1)(1 - \eta)n}{r}\right)\right) + o(n^4).$$

So

$$N(C_4, G) = N(C_4, G^*) + 3x_1 \sum_{2 \leq i < j < k \leq r} x_i x_j x_k + 2x_1 \sum_{2 \leq i < j \leq r} \binom{x_i}{2} x_j$$

$$+ \binom{x_1}{2} \sum_{2 \leq i < j \leq r} x_i x_j + \binom{x_1}{2} \sum_{i=2}^{r} \binom{x_i}{2}$$

$$= N\left(C_4, T_{r-1} \left(\frac{(r - 1)(1 - \eta)n}{r}\right)\right)$$

$$+ \frac{(r - 1)(r - 2)^2(1 + \eta(r - 1))(1 - \eta)^3}{2r^4} n^4$$

$$+ \frac{(r - 1)^2(1 + \eta(r - 1))^2(1 - \eta)^2}{4r^4} n^4 + o(n^4).$$

Therefore,

$$d(C_4, G) = \frac{N(C_4, G)}{\binom{n}{4}}$$

$$= \text{OPT}_{r-1}(C_4) \left(\frac{1 - \eta}{r}\right)^4 + \frac{12(r - 1)(r - 1)^2(1 + \eta(r - 1))(1 - \eta)^3}{r^4}$$

$$+ \frac{6(r - 1)^2(1 + \eta(r - 1))^2(1 - \eta)^2}{r^4} + o(1)$$

$$= \text{OPT}_r(C_4) - 6\eta^2 \left(\frac{2r^3 - 10r^2 + 17r - 9}{r^3}\right) + 12\eta^3 \frac{r^3 - 6r^2 + 11r - 6}{r^3}$$

$$- 3\eta^4 \frac{r^3 - 8r^2 + 16r - 9}{r^3} + o(1).$$

Let $g(r) = \frac{2r^3 - 10r^2 + 17r - 9}{r^3} = 2 - \frac{10}{r} + \frac{17}{r^2} - \frac{9}{r^3}$, which is positive with minimum $g(4) = \frac{27}{16}$. Therefore, for sufficiently small $\eta$ and large $n$, we get $d(C_4, G) \leq \text{OPT}_r(C_4) - 2\eta^2$, a contradiction. This implies the statement of the claim. \qed
As pointed out in Remark A, Theorem 1.6 implies Theorem 1.4. Therefore, it is sufficient to show Theorem 1.6, i.e., we will show that every n-vertex $C_4$-free graph $G$ with $N(H, G) = \text{ex}(n, C_4, F) - o(n^4)$ can be obtained by adding or deleting $o(n^2)$ edges from $T_r(n)$.

**Proof of Theorem 1.6** Let $F$ be a graph with chromatic number $\chi(F) = r + 1$ and a color-critical edge. Let $G$ be an $n$-vertex $F$-free graph with $N(H, G) = \text{ex}(n, C_4, F) - o(n^4)$.

First, let us consider the case $r = 3$. Since $F$ is a subgraph of a blow-up of $K_4$, by Lemma 2.1

$$N(K_4, G) \leq \text{ex}(n, K_4, F) = o(n^4).$$

By Lemma 2.2, we can remove or add $o(n^2)$ edges from $G$ and obtain a $K_4$-free graph $G^*$. The removal of $o(n^2)$ edges from $G$ can destroy at most $o(n^2) - O(n^2) = o(n^4)$ copies of $C_4$. Therefore, it suffices to estimate the number of $C_4$ and the number of edges in $G^*$. Let

$$M_2 = \{M_2 : M_2 = \{e_1, e_2\} \text{ is an independent set in } G^*\}. $$

Since $G^*$ is $K_4$-free, the number of $C_4$ contained in the induced graph of a fixed $M_2 \in M_2$ is at most 1. Note that each copy of $C_4$ in $G^*$ contains exactly 2 members in $M_2$. Therefore, we have

$$N(C_4, T_3(n)) = o(n^4) \leq N(C_4, G) - o(n^4) \leq N(C_4, G^*) \leq \frac{1}{2}|M_2|.$$ 

Since $N(C_4, T_3(n)) = \frac{n^4}{36} + o(n^4)$ by [3] and $\frac{1}{2}|M_2| \leq \frac{1}{2}(e(G^*))$, we have

$$e(G) \geq e(G^*) - o(n^2) \geq \frac{1}{3}n^2 - o(n^2) = e(T_3(n)) - o(n^2).$$

So from now on we assume that $r \geq 4$. It suffices to prove that for any $\varepsilon > 0$, there exist $n_0 > 0$ and $\varepsilon' > 0$ such that for every $F$-free graph $G$ of order $n \geq n_0$, if $d(C_4, G) \geq \text{OPT}_r(C_4) - \varepsilon'$, then by changing at most $\varepsilon n^2$ pairs of adjacencies, we can obtain $T_r(n)$ from $G$.

Let $F = \{K_{r+1}, P_3\}$ and $\varepsilon_1 > 0$. By Lemma 2.2, we have $\delta_f(\varepsilon_1) > 0$ and $n_f(\varepsilon_1) > 0$. For $\delta_f(\varepsilon_1) > 0$, since $F$ is a subgraph of a blow-up of $K_{r+1}$, by Lemma 2.1 there exists an integer $n_f(\delta_f(\varepsilon_1)) > 0$ such that every $F$-free graph on $n > n_f(\delta_f(\varepsilon_1))$ vertices contains at most $\delta_f(\varepsilon_1)n^{r+1}$ copies of $K_{r+1}$. By Lemma 2.6 we also have integer $n_c(\delta_c(\varepsilon_1)) > 0$ and $\varepsilon_c(\delta_c(\varepsilon_1)) > 0$ such that for every $F$-free graph $G$ of order $n \geq n_c(\delta_c(\varepsilon_1))$, if $d(C_4, G) \geq \text{OPT}_r(C_4) - \varepsilon_c(\delta_c(\varepsilon_1))$, then $G$ contains at most $\delta_c(\varepsilon_1)n^3$ induced copies of the co-cherry graph $P_3^c$. Now we choose $n_0 \geq \max\{n_f(\delta_f(\varepsilon_1)), n_c(\delta_c(\varepsilon_1)), n_f(\varepsilon_1)\}$ and $\varepsilon_2 < \varepsilon_c(\delta_c(\varepsilon_1))$. Assume that $G$ is a graph on $n \geq n_0$ vertices such that

$$d(C_4, G) > \text{OPT}_r(C_4) - \varepsilon_2.$$

Therefore, Lemmas 2.1 and 2.6 guarantee that $G$ contains at most $\delta_f(\varepsilon_1)n^{r+1}$ copies of $K_{r+1}$ and at most $\delta_c(\varepsilon_1)n^3$ induced copies of $\bar{P}_3^c$. By Lemma 2.2 we
can make $G$ into an induced $\mathcal{F}$-free graph $G^*$ by deleting or adding at most $\varepsilon_1 n^2$ edges. Since one removed edge from $G$ destroys at most $n^2$ copies of $C_4$, the total number of destroyed copies of $C_4$ is at most $\varepsilon_1 n^2 \cdot n^2 = \varepsilon_1 n^4$. Therefore, for $n$ large enough, we have

$$d(C_4, G^*) \geq OPT_r(C_4) - 24\varepsilon_1 - \varepsilon_2,$$

Since $G^*$ is induced $P_3^c$-free, by Lemma 2.4, $G^*$ is complete multipartite. Since $G^*$ is $K_{r+1}$-free, we may assume $G^*$ is a complete $r$-partite graph with partite sets $X_1, \ldots, X_r$. We will complete the proof by showing that the partite sets are almost balanced. Now, we apply Lemma 2.7 to $G^*$, for any $\varepsilon > 0$, choose $\varepsilon_1 \leq \varepsilon_2$ small enough and $\delta_4(\frac{\varepsilon}{2}) > 24\varepsilon_1 + \varepsilon_2$. Then $d(C_4, G^*) \geq OPT_r(C_4) - 24\varepsilon_1 - \varepsilon_2 > OPT_r(C_4) - \delta_4(\frac{\varepsilon}{2})$. By Lemma 2.7 we have

$$\frac{n(1-\varepsilon/2)}{r} \leq |X_i| \leq \frac{n(1+\varepsilon/2)}{r}$$

for $1 \leq i \leq r$. Therefore, by changing at most $(\varepsilon_1 + \varepsilon/2)n^2 \leq \varepsilon n^2$ edges, we can obtain $T_r(n)$ from the original graph $G$, which completes the proof. \qed

### 3 Overview of flag algebra

In this section, we give a brief overview of the flag algebra method developed by Razborov [16], which provides a framework for computationally solving problems in extremal combinatorics. By this method, we can find inequalities of subgraph densities in graph limits with the help of semi-definite programming.

First, let us present a brief introduction and description of the notation and theory needed in this section.

Let $H$ and $G$ be graphs on $n_1$ and $n_2$ vertices, respectively, where $n_1 \leq n_2$. Recall that the density $d(H, G)$ of $H$ in $G$ is defined by

$$d(H, G) = \frac{N(H, G)}{\binom{n_2}{n_1}}.$$

Similarly, the induced density $P(H, G)$ of $H$ in $G$ is defined by

$$P(H, G) = \frac{N_I(H, G)}{\binom{n_2}{n_1}}.$$

Let a subset $Y$ be selected uniformly at random from $V(G)$ such that $|Y| = n_1$. Then we can interpret the induced density $P(H, G)$ of $H$ in $G$ as the probability that $G[Y]$ is isomorphic to $H$.

We will need the notion of a flag. A type of size $k$ is a graph $\sigma$ on $k$ vertices labeled by $[k]$. If $\sigma$ is a type of size $k$ and $F$ is a graph on at least $k$ vertices, then an embedding of $\sigma$ into $F$ is an injective function $\theta : [k] \rightarrow V(F)$ such that $\theta$ gives an isomorphism between $\sigma$ and $F[\text{im}(\theta)]$. A $\sigma$-flag is a pair $(F, \theta)$ where $F$ is a graph and $\theta$ is an embedding of $\sigma$ into $F$. Two $\sigma$-flags $(F, \theta_1)$ and
(G, θ_2) are isomorphic if there exists a graph isomorphism between F and G that preserves the labeled subgraph σ.

Let \( \mathcal{F} \) (or \( \mathcal{F}_\ell \)) denote the set of all graphs (or all graphs on \( \ell \) vertices) up to isomorphism. Let \( \mathcal{F}_\sigma \) (or \( \mathcal{F}_\sigma^\ell \)) denote the set of all \( \sigma \)-flags (or on \( \ell \) vertices).

Let \( \mathbb{R} \mathcal{F} \), \( \mathbb{R} \mathcal{F}_\sigma \) and \( \mathbb{R} \mathcal{F}_\sigma^\ell \) denote the set of all formal finite linear combinations of elements in \( \mathcal{F} \), \( \mathcal{F}_\sigma \) and \( \mathcal{F}_\sigma^\ell \), respectively, where the coefficients are real numbers.

Note that if the size of \( \sigma \) is 0, then \( \mathcal{F}_\sigma = \mathcal{F} \) and \( \mathcal{F}_\sigma^\ell = \mathcal{F}_\ell \).

For two \( \sigma \)-flags \((H, \theta)\) and \((G, \theta)\) with \( n(H) \leq n(G) \), let \( P((H, \theta), (G, \theta)) \) denote the probability that any injective map from \( V(H) \) to \( V(G) \) that fixes the labeled graph \( \sigma \) induces a copy of \( H \) in \( G \). Observe that if \( \sigma \) is the empty graph, then \( P((H, \theta), (G, \theta)) = P(H, G) \).

Let \((F_1, \theta_1), (F_2, \theta_2), (G, \theta) \in \mathcal{F}_\sigma \) be three \( \sigma \)-flags for which \( n(F_1) + n(F_2) \leq n(G) + n(\sigma) \). Let \( X_1 \) and \( X_2 \) be two disjoint sets of sizes \( n(F_1) - n(\sigma) \) and \( n(F_2) - n(\sigma) \) respectively, selected uniformly at random from \( V(G) \setminus \text{im}(\theta) \).

Let \( P((F_1, \theta_1), (F_2, \theta_2); (G, \theta)) \) denote the probability that \((G[X_1 \cup \text{im}(\theta)], \theta)\) is isomorphic to \((F_1, \theta_1)\) and \((G[X_2 \cup \text{im}(\theta)], \theta)\) is isomorphic to \((F_2, \theta_2)\). Razborov showed that as \( n(G) \) grows, the following inequality holds:

\[
|P((F_1, \theta_1), (F_2, \theta_2); (G, \theta)) - P((F_1, \theta_1), (G, \theta))P((F_2, \theta_2), (G, \theta))| \leq O(n(G)^{-1}).
\]

Hence, as the size of \( G \) tends to infinity, we can assume that we select \( X_1 \) and \( X_2 \) independently.

Let \( \mathcal{K}_\sigma \) denote the linear subspace of \( \mathbb{R} \mathcal{F}_\sigma \) generated by all elements of the form

\[
(H, \vartheta) - \sum_{(G, \theta) \in \mathcal{F}_m^\sigma} P((H, \vartheta), (G, \theta)) \cdot (G, \theta)
\]

where \( m > n(H) \). Razborov has shown that there exists an algebra \( \mathcal{A}_\sigma = \mathbb{R} \mathcal{F}_\sigma / \mathcal{K}_\sigma \) with well defined addition and multiplication. Addition is defined in the natural way, by simply adding the coefficients of the elements in \( \mathbb{R} \mathcal{F}_\sigma \).

Let \( w = n(F_1) + n(F_2) - n(\sigma) \), then the product of \((F_1, \theta_1)\) and \((F_2, \theta_2)\) is defined as

\[
(F_1, \theta_1) \cdot (F_2, \theta_2) = \sum_{(F_3, \theta_3) \in \mathcal{F}_{m + w}^\sigma} P((F_1, \theta_1), (F_2, \theta_2); (F_3, \theta_3)) \cdot (F_3, \theta_3).
\]

Addition and multiplication in \( \mathcal{A}_\sigma \) are defined as an extension of addition and multiplication in \( \mathbb{R} \mathcal{F}_\sigma \), respectively. If the size of \( \sigma \) is 0, then we use \( \mathcal{A} \) to denote \( \mathcal{A}_\sigma \).

A sequence of graphs \((G_n)_{n \geq 1}\), where \( n(G_n) = n \), is said to be convergent if for every finite graph \( H \), the limit \( \lim_{n \to \infty} P(H, G_n) \) exists. Let \( \lim_{n \to \infty} P(\ast, G_n) \) denote the corresponding linear function from \( \mathbb{R} \mathcal{F} \) to \( \mathbb{R} \). Let \( \text{Hom}^+ (\mathcal{A}, \mathbb{R}) \) denote the set of all homomorphisms \( \phi \) from \( \mathcal{A} \) to \( \mathbb{R} \) such that \( \phi(F) \geq 0 \) for each element \( F \in \mathcal{F} \). Razborov showed that each function \( \phi \in \text{Hom}^+ (\mathcal{A}, \mathbb{R}) \) corresponds to some convergent graph sequence \((G_n)_{n \geq 1}\), specifically, we have the following theorem.
Theorem 3.1 ([16]). (a) For every convergent sequence \((G_n)_{n \geq 1}\),
\[
\lim_{n \to \infty} P(*, G_n) \in \text{Hom}^+ (\mathcal{A}, \mathbb{R}) .
\]
(b) Conversely, every element \(\text{Hom}^+ (\mathcal{A}, \mathbb{R})\) can be represented in the form
\[
\lim_{n \to \infty} P(*, G_n)
\]
for a convergent sequence \((G_n)_{n \geq 1}\).

For each type \(\sigma\) labeled by \([k]\), Razborov also defined an unlabeling operator
\[
[\ ]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}.
\]
For a \(\sigma\)-flag \((F, \theta)\), let \(q_\sigma(F)\) denote the probability that \((F, \theta')\) is isomorphic to \((F, \theta)\), where \(\theta' : [k] \to V(F)\) is a randomly chosen injective mapping. Let \(F'\) denote the graph isomorphic to \(F\) when ignoring labels. Then
\[
\|(F, \theta)\|_\sigma = q_\sigma(F)F'.
\]
In addition, Razborov proved the following useful inequality.

Theorem 3.2 ([16]). Let \(\sigma\) be a type and \(\phi \in \text{Hom}^+ (\mathcal{A}, \mathbb{R})\), then, for any \(\alpha \in \mathcal{A}^\sigma\),
\[
\phi ([\alpha \cdot \alpha]_\sigma) \geq 0 .
\]
(1)

Let \((G_n)_{n \geq 1}\) be the corresponding convergent sequence of \(\phi\), then, by the above theorem, we have for any \(\alpha \in \mathcal{A}^\sigma\),
\[
\lim_{n \to \infty} P ([\alpha \cdot \alpha]_\sigma, G_n) \geq 0 .
\]
(2)

4 Proof of Lemma 2.6

Now we are ready to give the proof of Lemma 2.6.

Lemma 4.1 (Restatement of Lemma 2.6). Let \(F\) be a graph with \(\chi(F) = r + 1 \geq 4\) and a color-critical edge. For any \(\delta > 0\), there exist \(n_c = n_c(\delta)\) and \(\epsilon_c = \epsilon_c(\delta) > 0\) such that for every \(F\)-free graph \(G\) of order \(n \geq n_c\), if \(d(C_4, G) \geq \text{OPT}_r(C_4) - \epsilon_c\), then \(G\) contains at most \(\delta n^3\) induced copies of the co-cherry graph \(P^c_3\).

The outline of the proof is as follows: the main step is to calculate \(\text{OPT}_r(C_4)\). We first use the number of \(C_4\) in \(T_r(n)\) to give a lower bound and then use the flag algebra argument to show that this lower bound is also an upper bound. To show the upper bound, we will calculate four functions \(Q_i(r)\) and \(q_i(r)\) for \(i \in \{0, 1, 2, 3\}\), which is inspired by [12]. Using semidefinite programming (interested readers can refer to [9]), we can verify that the upper bound of \(\text{OPT}_r(C_4)\) is correct for small values of \(r\). After doing so, we are able to guess the prospective types and the order of the corresponding flags. Since the number of flags on 3 vertices is very small, we can greedily determine which specific flags to use with the help of MATLAB.
Proof. First, we give the following claim.

Claim 4.1. For all \( r \geq 3 \), \( \text{OPT}_r(C_4) = \frac{3(r-1)(r^2-3r+3)}{r^3} \).

Proof of Claim 4.1 By definition,
\[
\text{OPT}_r(C_4) \geq \lim_{n \to \infty} \frac{N(C_4, T_r(n))}{\binom{n}{4}}.
\]

First we count \( N(C_4, T_r(n)) \) according to the distribution of \( V(C_4) \) in \( T_r(n) \). Let \( N_i \) be the number of copies of \( C_4 \) with \( V(C_4) \) distributed in exactly \( i \) classes of \( T_r(n) \), where \( i = 2, 3 \) or 4. So
\[
N_2 = \binom{r}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} + o(n^4) = \frac{(r-1)n^2(n-r)^2}{8r^3} + o(n^4),
\]
\[
N_3 = \binom{r}{3} \cdot 3 \cdot \frac{n}{r} \cdot \frac{n}{r} \cdot \frac{n}{r} + o(n^4) = \frac{(r-1)(r-2)n^3(n-r)}{4r^3} + o(n^4),
\]
and
\[
N_4 = \binom{r}{4} \cdot \left( \frac{n}{r} \right)^4 \cdot 3 + o(n^4) = \frac{(r-1)(r-2)(r-3)n^4}{8r^3} + o(n^4),
\]
where the error term \( o(n^4) \) accounts for the cases when \( n \) is not divisible by \( r \).

Thus the total number of copies of \( C_4 \) in \( T_r(n) \) is
\[
N(C_4, T_r(n)) = N_2 + N_3 + N_4 = \frac{(r-1)(r^2-3r+3)n^4}{8r^3} + o(n^4). \tag{3}
\]

Therefore,
\[
\text{OPT}_r(C_4) \geq \lim_{n \to \infty} \frac{N(C_4, T_r(n))}{\binom{n}{4}} = \frac{3(r-1)(r^2-3r+3)}{r^3}. \tag{4}
\]

To complete the proof of the claim, we show that the lower bound given in (3) is also an upper bound of \( \text{OPT}_r(C_4) \). Let \( \mathcal{F}_4 = \{F_0, \ldots, F_{10}\} \) be the set of unlabeled graphs up to isomorphism on four vertices as drawn in Fig. 4.
Let $(G_n)_{n \geq 1}$ be a convergent sequence of $F$-free graphs. For simplicity, we write $P(H) = \lim_{n \to \infty} P(H, G_n)$ and $d(H) = \lim_{n \to \infty} d(H, G_n)$. By the definition of $\mathcal{F}_4$, we have the following equality.

$$\sum_{i=0}^{10} P(F_i) = 1. \quad (5)$$

By the law of total probability, the (noninduced) density of $C_4$ can be expressed as the sum of the induced densities of graphs on four vertices in the following way:

$$d(C_4) = \sum_{i=0}^{10} P(F_i) \cdot N(C_4, F_i).$$

Since $N(C_4, F_i) = 0$ for $0 \leq i \leq 7$ and $N(C_4, F_8) = N(C_4, F_9) = 1$ and $N(C_4, F_{10}) = 3$, we have

$$d(C_4) = P(F_8) + P(F_9) + 3P(F_{10}). \quad (6)$$

By Lemma 2.4, we have

$$P(F_{10}) = P(K_4) = \lim_{n \to \infty} \frac{N(K_4, G_n)}{\binom{n}{4}} \leq \frac{r^3 - 6r^2 + 11r - 6}{r^3}.$$

So

$$Q_0(r) : = \sum_{i=0}^{9} \left( \frac{r^3 - 6r^2 + 11r - 6}{r^3} \right) \cdot P(F_i) + P(F_{10}) \cdot \frac{-6r^2 + 11r - 6}{r^3}$$

$$= \frac{r^3 - 6r^2 + 11r - 6}{r^3} - P(F_{10}) \geq 0$$

In the following computations, we will use two sets of flags $\mathcal{F}_{3}^{\sigma_1}$ and $\mathcal{F}_{3}^{\sigma_2}$, where

$$\sigma_1 = \begin{array}{c} \sigma_2 \\
\end{array}, \quad \sigma_2 = \begin{array}{c} \sigma_1 \\
\end{array}.$$ 

By (2), we have $P\left( [\alpha^2]_{\sigma_i} \right) \geq 0$, where $\alpha \in \mathcal{A}^{\sigma_i}$, and $\alpha^2 = \alpha \cdot \alpha$ for $i = 1, 2$. Therefore, each of the following three expressions is nonnegative for all $r \geq 3$.

$$Q_1(r) = 6P\left( \left[ (r-1)^2 \right]_{\sigma_1} \right)$$

$$= (6r^2 - 12r + 6)P(F_0) + (r^2 - 2r + 1)P(F_1) + (1 - r)P(F_2) + (3 - 3r)P(F_3) + 2P(F_8) + P(F_9)$$

$$Q_2(r) = 6P\left( \left[ (r-1)^2 \right]_{\sigma_2} \right)$$

$$= 3P(F_3) + P(F_7) - P(F_6) - 4P(F_8)$$

$$13$$
In addition, it can be easily checked that for all \( c \) values of \( q \) where \( c q \) are all nonnegative.

By (5) and (7), we have

\[
\sum_{i=0}^{\infty} \frac{1}{P(F_i)} \geq 8
\]

So \( \sum_{j=0}^{3} q_j(r)Q_j(r) \geq 0 \). Therefore, by Equation (6), we have

\[
d(C_4) \leq P(F_8) + P(F_9) + 3P(F_{10}) + \sum_{j=0}^{3} q_j(r)Q_j(r) \quad (7)
\]

where \( c_{F_i} \) is the coefficient of \( P(F_i) \) after combining like-terms in \( \sigma_q \). The exact values of \( c_{F_i} \) are listed in the following.

- \( c_{F_0} = 3 \)
- \( c_{F_3} = 3 \)
- \( c_{F_9} = 3 \)
- \( c_{F_{10}} = \frac{3(r-1)(r^2-3r+3)}{r^3} \).

- \( c_{F_1} = \frac{26r^5-326r^4-767r^3+1272r^2+1029r-324}{4r^3(3r^3-11r+9)} \)
- \( c_{F_2} = \frac{24r^5-218r^4-755r^3+1269r^2+1029r-324}{4r^3(3r^3-11r+9)} \)
- \( c_{F_4} = \frac{3(2r-3)^2(r^2-6r+11r-6)}{r^3(6r^2-22r+18)} \)
- \( c_{F_5} = \frac{48r^5-448r^4+1575r^3-2601r^2+2070r-648}{8r^3(3r^3-11r+9)} \)
- \( c_{F_6} = \frac{28r^5-242r^4+804r^3-1309r^2+1305r-324}{4r^3(3r^2-11r+9)} \)

By (5) and (7), we have

\[
d(C_4) \leq \max\{c_{F_i} : 0 \leq i \leq 10\} = \frac{3(r-1)(r^2-3r+3)}{r^3},\quad (8)
\]

for all \( r \geq 3 \), where the equality holds by examining leading coefficients and factoring. This completes the proof of the claim.

Let \( G_n \geq 1 \) be a sequence of \( F \)-free graphs with \( \lim_{n \to \infty} d(C_4, G_n) = \text{OPT}_r(C_4) \). Then

\[
\lim_{n \to \infty} d(C_4, G_n) = \text{OPT}_r(C_4) = \frac{3(r-1)(r^2-3r+3)}{r^3} \leq \lim_{n \to \infty} \sum_{i=0}^{10} c_{F_i} P(F_i, G_n).\quad (9)
\]
Let $T = \{F_0, F_3, F_8, F_9, F_{10}\}$, namely,

$$T = \left\{ \begin{array}{c}
\text{.: . } \quad \begin{array}{c}
\text{.: . }
\end{array} \\
\begin{array}{c}
\text{.: . }
\end{array} \\
\begin{array}{c}
\text{.: . }
\end{array} \\
\begin{array}{c}
\text{.: . }
\end{array} \\
\end{array} \right\}.$$

Then $c_{F_i} = \frac{3(r-1)(r^2-3r+3)}{r^4}$ for each $F_i \in T$. By (8) and (9), for every $F \in \mathcal{F}_4$ with $P(F) > 0$, we have $c_F = \frac{2(r-1)(r^2-3r+3)}{r^4}$, which implies that $F \in T$, and $P(F) = 0$ for all $F \in \mathcal{F}_4 \setminus T$. Notice that none of the graphs in $T$ contain the cocherry graph $P_3^c$ as an induced subgraph. Therefore, we have

$$\lim_{n \to \infty} d(P_3^c, G_n) = \sum_{i=0}^{10} N(P_3^c, F_i)p(F_i) = 0.$$

This immediately implies the result. \qed

**Acknowledgment:** We thank Professor Dániel Gerbner for remarks on the history of the problem of $F$-Tuán-good and valuable discussions on this manuscript.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**References**

[1] N. Alon and C. Shikhelman, Many $T$ copies in $H$-free graphs, Electronic notes in Discrete Mathematics, (2015), 49:683-689.

[2] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient Testing of Large Graphs, Combinatorica, (2001), 20(4): 451-476.

[3] D. Gerbner, Generalized Turán problems for small graphs, Discussiones Mathematicae Graph Theory, in press, https://doi.org/10.7151/dmgt.2388.

[4] D. Gerbner, On Turán-good graphs, Discrete Mathematics, (2021), 344: 112445.

[5] D. Gerbner, A non-aligning variant of generalized Turán problems. [arXiv:2109.02181](https://arxiv.org/abs/2109.02181).

[6] D. Gerbner, Some stability and exact results in generalized Turán problems, [arXiv:2204.04600v1](https://arxiv.org/abs/2204.04600v1).

[7] D. Gerbner and C. Palmer, Some exact results for generalized Turán problems, European Journal of Combinatorics, (2022), 103:103519.

[8] D. Gerbner, E. Györi, A. Methuku and M. Vizer, Generalized Turán problems for even cycles, Journal of Combinatorical Theory, Series B, (2019), 145:169-213.
[9] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, Journal of Combinatorical Theory, Series B, (2012), 102:1061-1066.

[10] E. Győri, J. Pach and M. Simonovits, On the maximal number of certain subgraphs in $K_r$-free graphs, Graphs and Combinatorics, (1991), 7(1):31-37.

[11] D. Hei, X. Hou and B. Liu, Some exact results of the generalized Turán numbers for paths, arXiv:2112.14895.

[12] B. Lidický and K. Murphy, Maximizing five-cycles in Kr-free graphs, European Journal of Combinatorics, (2021), 97:103367.

[13] J. Ma and Y. Qiu, Some sharp results on the generalized Turán numbers, European Journal of Combinatorics, (2020), 84:103026.

[14] K. Murphy and J. Nir, Paths of Length Three are $K_{r+1}$-Turán Good, The Electronic Journal of Combinatorics, 28(4)(2021), ♯P4.34.

[15] B. Qian, C. Xie and G. Ge, Some results on $k$-Turán-good graphs, Discrete Mathematics, (2021), 344(9):112509.

[16] A. Razborov, Flag algebras, Journal of Symbolic Logic, (2007), 1239-1282.

[17] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, Theory of Graphs, Tihany, Hungary 1966, Academic Press, New York, (1969), pp. 279-319.

[18] M. Simonovits, Paul Erdős’ Influence on Extremal Graph Theory, Springer, New York, (2013), 245-311.