(Anti-)BRST and (Anti-)co-BRST Symmetries in 2D non-Abelian Gauge Theory: Some Novel Observations

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Abstract: We discuss the nilpotent Becchi-Rouet-Stora-Tyutin (BRST), anti-BRST and (anti-)co-BRST symmetry transformations and derive their corresponding conserved charges in the case of a two (1+1)-dimensional (2D) self-interacting non-Abelian gauge theory (without any interaction with matter fields). We point out a set of novel features that emerge out in the BRST and co-BRST analysis of the above 2D gauge theory. The algebraic structures of the symmetry operators (and corresponding conserved charges) and their relationship with the cohomological operators of differential geometry are established, too. To be more precise, we demonstrate the existence of a single Lagrangian density that respects the continuous symmetries which obey proper algebraic structure of the cohomological operators of differential geometry. In literature, such observations have been made for the coupled (but equivalent) Lagrangian densities of the 4D non-Abelian gauge theory. We lay emphasis on the existence and properties of the Curci-Ferrari (CF) type restrictions in the context of (anti-)BRST and (anti-)co-BRST symmetry transformations and pinpoint their key differences and similarities. All the observations, connected with the (anti-)co-BRST symmetries, are completely novel.

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1 Introduction

The principles of local gauge theories are at the heart of a precise theoretical description of electromagnetic, weak and strong interactions of nature. One of the most intuitive, geometrically rich and mathematically elegant methods to quantize such kind of theories is the Becchi-Rouet-Stora-Tyutin (BRST) formalism [1-4]. In this formalism, the local gauge symmetries of the classical theories are traded with the (anti-)BRST symmetries at the quantum level where unitarity is satisfied at any arbitrary order of perturbative computations. These (anti-)BRST symmetries are fermionic (i.e. supersymmetric-type) in nature and, therefore, they are nilpotent of order two. However, these symmetries absolutely anticommute with each other. This latter property encodes the linear independence of these symmetries. Hence, the BRST and anti-BRST symmetries have their own identities.

The (anti-)BRST symmetry transformations are fermionic in nature because they transform a bosonic field into its fermionic counterpart and vice-versa. This is what precisely happens with the supersymmetric (SUSY) transformations which are also fermionic in nature. However, there is a decisive difference between the two. Whereas the (anti-)BRST symmetry transformations are absolutely anticommuting in nature, the anticommutator of two distinct SUSY transformations always produces the spacetime translation of the field on which it (i.e. the anticommutator) operates. Thus, the SUSY transformations are distinctly different from the (anti-)BRST symmetry transformations. The clinching point of difference is the property of absolute anticommutativity [which is respected by the (anti-)BRST symmetry transformations but violated by the SUSY transformations].

In a set of research papers (see, e.g. [5,6] and references therein), we have established that any arbitrary Abelian p-form (p = 1, 2, 3...) gauge theory would respect, in addition to the (anti-)BRST symmetry transformations, the (anti-)co-BRST symmetry transformations, too, in the D = 2p dimensions of spacetime at the quantum level. This observation has been shown to be true [7] in the cases of (1+1)-dimensional (2D) (non-)Abelian 1-form gauge theories (without any interaction with matter fields). In fact, these 2D theories have been shown [7] to be the perfect field theoretic examples of Hodge theory as well as a new model of topological field theory (TFT) which captures some salient features of Witten-type TFTs [8] as well as a few key aspects of Schwarz-type TFTs [9]. In a recent set of couple of papers [10,11], we have discussed the Lagrangian densities, their symmetries and Curci-Ferrari (CF)-type restrictions for the 2D non-Abelian 1-form gauge theory within the framework of BRST and superfield formalisms. Some novel features have been pointed out, too, in our earlier works [10, 11].

In our earlier work [10], we have been able to show the equivalence of the coupled Lagrangian densities w.r.t. the (anti-)BRST as well as (anti-)co-BRST symmetries of the 2D non-Abelian 1-form gauge theory (without any interaction with matter fields). However, we have not been able to compute the conserved currents (and corresponding charges) for the above continuous symmetries. One of the central themes of our present investigation is to compute all the conserved charges and derive their algebra to show the validity of CF-type restrictions at the algebraic level. This exercise establishes the independent existence of a set of CF-type restrictions for the 2D non-Abelian 1-form theory (which have been shown from the symmetry considerations [10] as well as from the point of view of the superfield approach to BRST formalism [11]). In our present endeavor, we accomplish this goal in
a straightforward fashion and show that the CF-type restrictions, corresponding to the (anti-)co-BRST symmetries, have some novel features that are completely different from the usual CF-condition [12] corresponding to the (anti-)BRST symmetry transformations of our present theory.

One of the highlights of our present investigation is the derivation of the CF-type restrictions and some of the equations of motion (EOM) from the algebra of conserved charges where the ideas of symmetry generators corresponding to the continuous symmetry transformations of our 2D non-Abelian 1-form gauge theory are exploited. Thus, to summarize the key results of our previous works [10,11] and present one, we would like to state that we have been able to show the existence of the CF-type of restrictions from the point of view of symmetries of the 2D non-Abelian 1-form gauge theory [10], superfield approach to BRST formalism applied to the above 2D theory [11] and algebra of the conserved charges of the above theory. The latter (i.e. algebra) is reminiscent of the algebra of the de Rham cohomological operators of the differential geometry. Our present studies establish the independent nature of the CF-type restrictions in the context of nilpotent (anti-)co-BRST symmetries (existing only in the 2D non-Abelian 1-form gauge theory) which are different from the CF-condition [12] that appears in the context of (anti-)BRST symmetries (existing in any arbitrary dimension of spacetime).

In our present endeavor, we have demonstrated that the usual coupled Lagrangian densities (1) (see below) for the non-Abelian 1-form gauge theory respect four perfect symmetries individually whereas the generalized versions of these Lagrangian densities (26) (see below) respect five perfect symmetries individually. It has been shown that both the Lagrangian densities of Eq. (26) respect (anti-)co-BRST symmetries that have been listed in (27) (see below) which is a completely novel observation [cf. Eq. (28)]. The absolute anticommutativity of the (anti-)co-BRST charges [that have been computed from the Lagrangian densities (1)] requires the validity of the CF-type restrictions ($\mathcal{B} \times C = 0, \mathcal{B} \times \bar{C} = 0$). However, the requirement of the absolute anticommutativity of the above charges [that are computed from the generalized Lagrangian densities (26)] turn out to be perfect. This happens because of the fact that the conditions $\mathcal{B} \times C = 0$ and $\mathcal{B} \times \bar{C} = 0$ become equations of motion for the Lagrangian densities (26). This is also a novel observation in our present endeavor (connected with the 2D non-Abelian theory).

Our present endeavor is propelled by the following key considerations. First and foremost, we have derived the conserved charges corresponding to the continuous symmetries which have not been discussed in our earlier works [10,11]. Second, we have derived the CF-type restrictions in the context of 2D non-Abelian theory which emerge from the symmetry considerations [10] as well as from the application of augmented version of superfield approach to BRST formalism [11]. We show, in our present endeavor, the existence of such restrictions in the language of algebra, connected with the conserved charges, which obey the algebra of cohomological operators of differential geometry. Third, the (anti-)co-BRST symmetries absolutely anticommute without use of any kinds of the CF-type restrictions [which is not the case with the (anti-)BRST symmetries]. However, in our present endeavor, we have shown that CF-type restrictions ($\mathcal{B} \times C = 0, \mathcal{B} \times \bar{C} = 0$) appear when we consider the requirement of absolute anticommutativity of the (anti-)co-BRST charges [derived from the Lagrangian densities (1)]. Finally, we speculate that the understanding and insights, gained in the context of 2D non-Abelian theory, might turn out to be useful.
for the 4D Abelian 2-form and 6D Abelian 3-form gauge theories which have also been shown to be the models of Hodge theory [6].

The material of our present theoretical work is organized as follows. In Sec. 2, we briefly recapitulate the bare essentials of the nilpotent (anti-)BRST and (anti-)co-BRST symmetries, a unique bosonic symmetry and a ghost-scale symmetry of the 2D non-Abelian gauge theory in the Lagrangian formulation. Our Sec. 3 contains the details of the derivation of conserved Noether currents and conserved charges corresponding to the above continuous symmetries. Our Sec. 4 deals with the elaborate proof of the coupled Lagrangian densities to be equivalent w.r.t. the nilpotent (anti-)BRST as well as (anti-)co-BRST symmetry transformations. In Sec. 5, we derive the algebraic structures of the symmetry operators and conserved charges and establish their connection with the cohomological operators of differential geometry (at the algebraic level). Our Sec. 6 deals with the discussion of some novel observations in the context of algebraic structures. Finally, we make some concluding remarks and point out a few future directions in Sec. 7.

In our Appendices A and B, we collect some of the explicit computations that have been incorporated into the main body of the text of our present endeavor. In our Appendix C, we show the consequences of the (anti-)BRST symmetry transformations when they are applied on the generalized forms of the Lagrangian densities [cf. Eq. (26) below] where the CF-type restrictions have been incorporated.

Convention and Notations: Our whole discussion is based on the choice of the 2D flat metric $\eta_{\mu\nu}$ with signatures $(+1,-1)$ which corresponds to the background Minkowskian 2D spacetime manifold. We choose the 2D Levi-Civita tensor $\varepsilon_{\mu\nu}$ such that $\varepsilon_{01} = +1 = \varepsilon^{01}$ and $\varepsilon_{\mu\nu} \varepsilon^{\rho\nu} = -2!$, $\varepsilon_{\mu\nu} \varepsilon^{\nu\lambda} = \delta^\lambda_\mu$, etc. Throughout the whole body of our text, we adopt the notations for the (anti-)BRST and (anti-)co-BRST transformations as $s_{(a)b}$ and $(a)d$, respectively. In the 2D Minkowskian flat spacetime, the field strength tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(A_\mu \times A_\nu)$ has only one existing component $E = F_{01} = -\varepsilon^{\mu\nu}[(\partial_\mu A_\nu + \frac{i}{2}(A_\mu \times A_\nu))]$ and our Greek indices $\mu, \nu, \lambda... = 0, 1$ correspond to the time and space directions. We have also adopted the dot and cross products in the $SU(N)$ Lie algebraic space where $P \cdot Q = P^a Q^a$ and $(P \times Q)^a = f^{abc} P^b Q^c$ for the non-null vectors $P^a (P = P^a T^a \equiv P \cdot T)$ and $Q^a (Q = Q^a T^a \equiv Q \cdot T)$ where the $SU(N)$ Lie algebra is: $[T^a, T^b] = f^{abc} T^c$. In this specific mathematical algebraic relationship, $T^a$ are the generators of the $SU(N)$ Lie algebra and the structure constants $f^{abc}$ are chosen to be totally antisymmetric in all their indices $a, b, c = 1, 2, ..., N^2 - 1$.

Standard Definition: On a compact manifold without a boundary, the set of three mathematical operators $(d, \delta, \Delta)$ is called as a set of the de Rham cohomological operators of differential geometry where $(d) d$ are christened as the (co-)exterior derivatives and $\Delta = (d + \delta)^2$ is called as the Laplacian operator. Together, these operators satisfy an algebra: $d^2 = \delta^2 = 0$, $\Delta = d\delta + \delta d$, $[\Delta, d] = 0$, $[\Delta, \delta] = 0$ which is popularly known as the Hodge algebra of differential geometry. The co-exterior derivative $\delta$ and exterior derivative $d$ are connected by a relationship $\delta = \pm * \ d *$ where $*$ is the Hodge duality operation (defined on the given compact manifold without a boundary). It is obvious that the (co-)exterior derivatives are nilpotent of order two and Laplacian operator is like the Casimir operator for the whole algebra. However, the latter (i.e. the Hodge algebra) is not
a Lie algebra.

2 Preliminaries: Lagrangian Formulation

We begin with the coupled (but equivalent) Lagrangian densities [13,14,10,11] of our 2D non-Abelian 1-form gauge theory in the Curci-Ferrari gauge (see, e.g. [15,16]) as

$$
\mathcal{L}_B = B \cdot E - \frac{1}{2} B \cdot B + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\
\mathcal{L}_{\bar{B}} = \bar{B} \cdot E - \frac{1}{2} B \cdot B - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C, 
$$

(1)

where $B$, $\bar{B}$ and $\mathcal{B}$ are the auxiliary fields, $D_\mu C = \partial_\mu C + i (A_\mu \times C)$ and $D_\mu \bar{C} = \partial_\mu \bar{C} + i (A_\mu \times \bar{C})$ are the covariant derivatives on the ghost and anti-ghost fields, respectively. These derivatives are in the adjoint representation of the $SU(N)$ Lie algebra and $B + \bar{B} + (C \times \bar{C}) = 0$ is the Curci-Ferrari (CF) condition [12]. The latter is responsible for the equivalence of the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$. This observation is one of the inherent properties of the basic concept behind the existence of coupled Lagrangian densities for a given gauge theory [13,14]. The fermionic $[(C^a)^2 = 0, (\bar{C}^a)^2 = 0]$ (anti-)ghost fields $(\bar{C}^a)C^a$ are needed for the validity of unitarity in the theory and they satisfy: $C^a \bar{C}^b + \bar{C}^b C^a = 0, C^a C^b + \bar{C}^b C^a = 0, C^a \bar{C}^b + \bar{C}^b \bar{C}^a = 0, \bar{C}^a C^b + C^b \bar{C}^a = 0$, etc. We would like to remark here that the 2D kinetic term [i.e. $-(1/4) F^\mu \nu \cdot F_{\mu \nu} = (1/2) E \cdot E \equiv B \cdot E - (1/2) B \cdot B$] has been lineared by invoking the auxiliary field $\mathcal{B}$.

The Lagrangian densities in (1) respect the following off-shell nilpotent $(s^2_{(a)b} = 0)$ (anti-)BRST symmetry transformations $s_{(a)b}$:

$$
s_{ab} A_\mu = D_\mu \bar{C}, \quad s_{ab} \bar{C} = -\frac{i}{2} (\bar{C} \times \bar{C}), \quad s_{ab} C = i \bar{B}, \quad s_{ab} \bar{B} = 0, \quad s_{ab} (B \cdot B) = 0, \\
s_{ab} E = i (E \times C), \quad s_{ab} \mathcal{B} = i (\mathcal{B} \times C), \quad s_{ab} B = i (B \times \bar{C}), \quad s_{ab} (B \cdot E) = 0, \\
s_b A_\mu = D_\mu C, \quad s_b \bar{C} = -\frac{i}{2} (C \times C), \quad s_b \bar{C} = i B, \quad s_b B = 0, \quad s_b (B \cdot B) = 0, \\
s_b \bar{B} = i (\bar{B} \times C), \quad s_b E = i (E \times C), \quad s_b \mathcal{B} = i (\mathcal{B} \times C), \quad s_b (B \cdot E) = 0. 
$$

(2)

This is due to the fact that we observe the following:

$$
s_b \mathcal{L}_B = - \partial_\mu (B \cdot D^\mu C), \quad s_{ab} \mathcal{L}_{\bar{B}} = - \partial_\mu (\bar{B} \cdot D^\mu \bar{C}). 
$$

(3)

As a consequence, the (anti-)BRST transformations are the symmetry transformations for the action integrals $S = \int d^2 x \mathcal{L}_B$ and $S = \int d^2 x \mathcal{L}_{\bar{B}}$, respectively. The (anti-)BRST symmetry transformations absolutely anticommute with each other (i.e. $\{s_b, s_{ab}\} = 0$) only when the CF-condition is satisfied. One of the decisive features of the (anti-)BRST symmetry transformations is the observation that the kinetic term $-(1/4) F^\mu \nu \cdot F_{\mu \nu} - (1/2) E \cdot E \equiv B \cdot E - (1/2) B \cdot B$ remains invariant under it. This observation would be exploited, later on, in establishing a connection between the continuous symmetries of our 2D theory and cohomological operators of differential geometry at the algebraic level.
In addition to the (anti-)BRST symmetry transformations (2), we note the presence of the following nilpotent \((s^2_{(a)d} = 0)\) and absolutely anticommuting \((s_d s_{ad} + s_{ad} s_d = 0)\) (anti-)co-BRST symmetry transformations in the theory (see, e.g. \([7]\) for details):

\[
\begin{align*}
    s_{ad} A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu C, & s_{ad} C &= 0, & s_{ad} \bar{C} &= i \mathcal{B}, & s_{ad} \mathcal{B} &= 0, \\
    s_{ad} E &= D_\mu \partial^\mu C, & s_{ad} B &= 0, & s_{ad} \bar{B} &= 0, & s_{ad}(\partial_\mu A^\mu) &= 0, \\
    s_d A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, & s_d C &= -i \mathcal{B}, & s_d \bar{C} &= 0, & s_d \mathcal{B} &= 0, \\
    s_d E &= D_\mu \partial^\mu \bar{C}, & s_d B &= 0, & s_d \bar{B} &= 0, & s_d(\partial_\mu A^\mu) &= 0.
\end{align*}
\]

(4)

The Lagrangian \(\mathcal{L}_B\) and \(\mathcal{L}_{\bar{B}}\) transform, under the above transformations, as follows

\[
\begin{align*}
    s_{ad} \mathcal{L}_B &= \partial_\mu [\mathcal{B} \cdot \partial^\mu C], & s_d \mathcal{L}_B &= \partial_\mu [\mathcal{B} \cdot \partial^\mu \bar{C}],
\end{align*}
\]

(5)

which imply that the action integrals \(S = \int d^2 x \mathcal{L}_B\) and \(\bar{S} = \int d^2 x \mathcal{L}_{\bar{B}}\) remain invariant under the (anti-)co-BRST symmetries. One of the decisive features of the (anti-)co-BRST symmetries is the observation that the gauge-fixing term \((\partial_\mu A^\mu)\) remains invariant under them. This observation would play a key role in establishing a connection between these symmetries and the cohomological operators of differential geometry at the algebraic level. It is quite clear that we have four fermionic symmetries in our present 2D theory.

There are two bosonic symmetries in our theory, too. The first one is the ghost-scale symmetry \((s_g)\) and second one is a unique bosonic symmetry \(s_w = \{s_d, s_b\} = -\{s_{ad}, s_{ab}\}\). We focus first on the ghost-scale symmetry. Under this symmetry, we have the following transformations for the fields of our present theory, namely;

\[
\begin{align*}
    C &\longrightarrow e^\Omega C, & \bar{C} &\longrightarrow e^{-\Omega} \bar{C}, & \Phi &\longrightarrow e^\Omega \Phi,
\end{align*}
\]

(6)

where the generic field \(\Phi = A_\mu, B, \mathcal{B}, \bar{B}, E\) and \(\Omega\) is a global (spacetime independent) scale transformation parameter. One of the decisive features of the ghost-scale symmetry transformations is the observation that only the (anti-)ghost fields transform and the remaining ordinary basic/auxiliary fields of the theory remain invariant under them. The infinitesimal version \((s_g)\) of the above ghost-scale symmetry transformations is:

\[
\begin{align*}
    s_g C &= C, & s_g \bar{C} &= -C, & s_g \Phi &= 0.
\end{align*}
\]

(7)

In the above, we have set \(\Omega = 1\) for the sake of brevity. Under these infinitesimal transformations, it can be readily checked that:

\[
\begin{align*}
    s_g \mathcal{L}_B &= 0, & s_g \mathcal{L}_{\bar{B}} &= 0.
\end{align*}
\]

(8)

Thus, the action integrals automatically remain invariant under the above ghost-scale symmetry transformations. Now, we focus on the bosonic symmetry \(s_w\) of our theory. It is elementary to check that, for the Lagrangian density \(\mathcal{L}_B\), we have

\[
\begin{align*}
    s_w A_\mu &= -[D_\mu \mathcal{B} + \varepsilon_{\mu\nu} (\partial^\nu \bar{C} \times C) + \varepsilon_{\mu\nu} \partial^\nu B], & s_w \bar{B} &= (\bar{B} \times \mathcal{B}), \\
    s_w(\partial_\mu A^\mu) &= -[\partial_\mu D^\mu \mathcal{B} + \varepsilon_{\mu\nu} (\partial^\nu \bar{C} \times \partial_\mu C)], & s_w[C, \bar{C}, \mathcal{B}, B] &= 0, \\
    s_w E &= -[D_\mu \partial^\mu \mathcal{B} + i(E \times \mathcal{B}) - D_\mu C \times \partial^\mu \bar{C} - D_\mu \partial^\mu \bar{C} \times C],
\end{align*}
\]

(9)
where we have taken \( s_w = \{s_b, s_d\} \) (modulo a factor of \( i \)) and \( E = -\varepsilon^{\mu \nu} (\partial_\mu A_\nu + \frac{i}{2} A_\mu \times A_\nu) \).

One of the key observations is that the (anti-)ghost fields of the theory do not transform under the bosonic symmetry transformation \( s_w \). It can be checked that the Lagrangian density \( \mathcal{L}_B \) transforms under this bosonic symmetry transformation

\[
s_w \mathcal{L}_B = \partial_\mu [\mathcal{B} \cdot \partial^\mu \mathcal{B} - \mathcal{B} \cdot D^\mu \mathcal{B} - \partial^\mu \bar{\mathcal{C}} \cdot (\mathcal{B} \times C) - \varepsilon^{\mu \nu} \mathcal{B} \cdot (\partial_\nu \bar{\mathcal{C}} \times C)],
\]

thereby rendering the action integral \( S = \int d^2 x \mathcal{L}_B \) invariant. Thus, the bosonic transformations (9) correspond to the symmetry of the theory \( \mathcal{L}_B \). We remark that one can define another bosonic symmetry \( \tilde{s}_w = -\{s_{ad}, s_{ab}\} \) for the Lagrangian density \( \mathcal{L}_B \) but it turns out to be equivalent (i.e. \( s_w + \tilde{s}_w = 0 \)) to \( s_w = \{s_d, s_b\} \) if we use the equations of motion of the theory and the CF-type condition \((B + \bar{B} + (C \times \bar{C}) = 0) \). To sum up, we have total six continuous symmetries in the theory. Together, these symmetry operators satisfy an algebra that is exactly similar to the algebraic structure of the cohomological operators of the differential geometry. Thus, there is a connection between the two (cf. Sec. 5 below).

### 3 Conserved Charges: Noether Theorem

The Noether theorem states that the invariance of the action integral, under continuous symmetry transformations, leads to the existence of conserved currents. As pointed out earlier, the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_B \) transform, under \( s_b \) and \( s_{ad} \), to the total spacetime derivatives as given in (3) thereby rendering the action integrals \( S = \int d^2 x \mathcal{L}_B \) and \( S = \int d^2 x \mathcal{L}_B \) invariant. The corresponding Noether currents (w.r.t. BRST and anti-BRST symmetry transformations) are:

\[
J^\mu_b = -\varepsilon^{\mu \nu} \mathcal{B} \cdot D_\nu C + \mathcal{B} \cdot D^\mu C + \frac{1}{2} \partial^\mu \bar{\mathcal{C}} \cdot (C \times C),
J^\mu_{ab} = -\varepsilon^{\mu \nu} \mathcal{B} \cdot D_\nu \bar{C} - \bar{\mathcal{B}} \cdot D^\mu \bar{C} - \frac{1}{2} \partial^\mu C \cdot (\bar{C} \times \bar{C}).
\]

The above currents are conserved (i.e. \( \partial_\mu J^\mu_b = 0 \) and \( \partial_\mu J^\mu_{ab} = 0 \)) due to the following Euler-Lagrange (EL) equations of motion (EQM) that emerge from \( \mathcal{L}_B \) and \( \mathcal{L}_B \), namely:

\[
\mathcal{B} = E, \quad D_\mu \partial^\mu \bar{C} = 0, \quad \partial_\mu D^\mu C = 0, \quad \varepsilon^{\mu \nu} D_\nu \mathcal{B} + \partial^\mu \mathcal{B} + (\partial^\mu \bar{C} \times C) = 0, \\
\bar{\mathcal{B}} = E, \quad \partial_\mu D^\mu \bar{C} = 0, \quad D_\mu \partial^\mu C = 0, \quad \varepsilon^{\mu \nu} D_\nu \bar{B} - \partial^\mu \bar{B} - (\bar{C} \times \partial^\mu C) = 0.
\]

The above observations are sacrosanct as far as Noether’s theorem is concerned. It is to be noted that we have used \( E = -\varepsilon^{\mu \nu} (\partial_\mu A_\nu + \frac{i}{2} A_\mu \times A_\nu) \) in the derivation of the EOM.

The conserved charges (that emerge out from the Noether currents) are:

\[
Q_b = \int dx \ J^0_b \equiv \int dx \ [\mathcal{B} \cdot D_1 C + B \cdot D_0 C + \frac{1}{2} \bar{\mathcal{C}} \cdot (C \times C)],
Q_{ab} = \int dx \ J^0_{ab} \equiv \int dx \ [\mathcal{B} \cdot D_1 \bar{C} - B \cdot D_0 \bar{C} - \frac{1}{2} \cdot (\bar{C} \times C) \cdot \dot{\bar{C}}].
\]

*There is a simple way to derive Eq. (10). Using the basic definition \( s_w = \{s_b, s_d\} \) and applying it on \( \mathcal{L}_B \) we obtain Eq. (10) [with the inputs from (3) and (5)].

†The unique bosonic symmetry transformation \( s_w = \{s_b, s_d\} \) is independent of \( s_b \) as well as \( s_d \) because it commutes (i.e. \( [s_w, s_b] = [s_w, s_d] = 0 \) with both of them [cf. Eq. (29) below].
Using the EL-EOM (12), the above charges can be expressed in a more useful (but equivalent) forms as

\[ Q_b = \int dx \left[ B \cdot D_0 C - \dot{B} \cdot C - \frac{1}{2} \dot{\dot{C}} \cdot (C \times C) \right], \]

\[ Q_{ab} = \int dx \left[ \dot{B} \cdot \dot{C} - \dot{B} \cdot D_0 C + \frac{1}{2} (\dot{C} \times \dot{C}) \cdot \dot{C} \right], \quad \text{(14)} \]

which are the generators for the (anti-)BRST transformations (2). This statement can be verified by observing that the (anti-)BRST symmetry transformations, listed in equation (2), can be derived from the following general expression

\[ \mathcal{s}_r \Phi = \mp i \left[ \Phi, Q_r \right] \quad \text{for } r = b, ab, \quad \text{(15)} \]

where the subscripts \((\mp)\), on the square bracket, correspond to the bracket being commutator and anticommutator for the generic field \(\Phi\) being bosonic and fermionic, respectively. The signs \(\mp\) in front of square bracket can be chosen appropriately (see, e.g. [17] for details).

Under the (anti-)co-BRST transformations \(s_{(a)d}\), the Lagrangian densities \(\mathcal{L}_B\) and \(\mathcal{L}_{\dot{B}}\) transform as given in (5). According to the Noether theorem, these infinitesimal continuous transformations lead to the derivation of conserved Noether currents. The explicit expressions for these conserved currents are:

\[ J^\mu_d = \mathcal{B} \cdot \partial^\mu \dot{C} - \varepsilon^{\mu\nu} \mathcal{B} \cdot \partial_\nu C, \quad J^\mu_{ad} = \mathcal{B} \cdot \partial^\mu C + \varepsilon^{\mu\nu} \dot{\mathcal{B}} \cdot \partial_\nu C. \quad \text{(16)} \]

The conservation laws \(\partial_\mu J^\mu_d = 0\) and \(\partial_\mu J^\mu_{ad} = 0\) can be proven by using EL-EQM (12). The conserved charges can be expressed equivalently in various forms as:

\[ Q_d = \int dx \ J^0_d = \int dx \left[ \mathcal{B} \cdot \dot{\dot{C}} + \dot{B} \cdot \partial_1 C \right] \equiv \int dx \left[ \mathcal{B} \cdot \dot{\dot{C}} - \partial_1 B \cdot \dot{C} \right] \]

\[ \equiv \int dx \left[ \mathcal{B} \cdot \dot{\dot{C}} - D_0 \mathcal{B} \cdot \dot{C} + (\partial_1 \dot{C} \times C) \cdot \dot{C} \right], \]

\[ Q_{ad} = \int dx \ J^0_{ad} = \int dx \left[ \mathcal{B} \cdot \dot{\dot{C}} - \dot{B} \cdot \partial_1 C \right] \equiv \int dx \left[ \mathcal{B} \cdot C + \partial_1 \dot{B} \cdot C \right] \]

\[ \equiv \int dx \left[ \mathcal{B} \cdot \dot{\dot{C}} - D_0 \mathcal{B} \cdot C - (\dot{C} \times \partial_1 C) \cdot C \right]. \quad \text{(17)} \]

The above charges are the generators of the (anti-)co-BRST symmetry transformations in equation (4). This statement can be corroborated by using the formula (15) where we have to replace: \(r = a, ab \rightarrow r = d, ad\).

We remark that the fermionic symmetries \(s_{(a)b}\) and \(s_{(a)d}\) are off-shell nilpotent of order two (i.e. \(s^2_{(a)b} = 0, s^2_{(a)d} = 0\)). This can be explicitly checked from the transformations listed in equations (2) and (4). This property (i.e. nilpotency) is also reflected at the level of conserved charges. To corroborate this assertion, we note that

\[ s_b Q_b = -i \{Q_b, Q_b\} = 0 \quad \Longrightarrow \quad Q_b^2 = 0, \]

\[ s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = 0 \quad \Longrightarrow \quad Q_{ab}^2 = 0, \]

\[ s_d Q_d = -i \{Q_d, Q_d\} = 0 \quad \Longrightarrow \quad Q_d^2 = 0, \]

\[ s_{ad} Q_{ad} = -i \{Q_{ad}, Q_{ad}\} = 0 \quad \Longrightarrow \quad Q_{ad}^2 = 0, \quad \text{(18)} \]
where we have used the definition of the symmetry generator (15). This observation is straightforward because the l.h.s. of the above equations can be computed explicitly by using the expressions for $Q_{(a)b}$, $Q_{(a)d}$ [cf. Eqs. (14) and (17)] and the transformations (2) and (4) corresponding to the (anti-)BRST and (anti-)co-BRST continuous symmetry transformations.

The conserved Noether current and corresponding charge for the infinitesimal and continuous ghost-scale transformations (7) are:

$$J_g^\mu = -i [\partial^\mu C \cdot C - C \cdot D^\mu C],$$

$$Q_g = \int dx J_g^0 = -i \int dx [\dot{C} \cdot C - C \cdot D_0 C].$$

Using the equations of motion (12), it can be readily checked $\partial_\mu J_g^\mu = 0$. Hence, the charge $Q_g$ is also conserved. Finally, we discuss a bit about the unique bosonic symmetry transformations $s_w = \{s_d, s_b\} = \{-s_{ad}, s_{ab}\}$ in this theory [7]. As pointed out earlier, the Lagrangian density $L_B$ transforms to the total spacetime derivative under $s_w$ as given in (10). The conservation of Noether current (i.e. $\partial_\mu J_w^\mu = 0$) can be proven by using Eq. (12). The conserved current ($J_w^\mu$) and corresponding charge ($Q_w$) are [7]:

$$J_w^\mu = -\varepsilon^{\mu\nu} [B \cdot D_\nu B - B \cdot \partial_\nu B],$$

$$Q_w = \int dx J_w^0 = \int dx [B \cdot D_1 B - B \cdot \partial_1 B].$$

It our Appendix B, we have shown the alternative derivations of $Q_w$ from the continuous symmetry transformations and the concept behind the symmetry generator. It is evident that we have six conserved charges which correspond to the six infinitesimal and continuous symmetries that exist in our theory. We shall establish their connections with the de Rham cohomological operations of differential geometry in our Sec. 5 where the emphasis would be laid on the algebraic structure(s) only.

### 4 Equivalence of the Coupled Lagrangian Densities: Continuous Symmetry Considerations

We observe, first of all, that $L_B$ and $\bar{L}_B$ are equivalent only when the CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$ is satisfied. This can be shown by the requirement of the equivalence of the Lagrangian densities (i.e $L_B - \bar{L}_B \equiv 0$, modulo a total spacetime derivative term) which primarily leads to the following equality, namely:

$$B \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C = -\bar{B} \cdot (\partial_\mu A^\mu) - i D_\mu \bar{C} \cdot \partial^\mu C. \quad (21)$$

Thus, it is evident that both the Lagrangian densities are equivalent only on a space of quantum fields which is defined by the CF-condition (i.e. $B + \bar{B} + (C \times \bar{C}) = 0$) in the 2D Minkowskian flat spacetime manifold. Furthermore, we note that both the Lagrangian

---

‡ These claims are true for any arbitrary expressions for the charges listed in (13), (14) and (17) provided we take into account the symmetry transformations (2) and (4).
densities respect the (anti-)BRST symmetry transformations because, we observe that, besides (3), we have the following explicit transformations:

\[ s_{ab} \mathcal{L}_B = -\partial_\mu [\{B + (C \times \bar{C})\} \cdot \partial^\mu \bar{C}] + \{B + \bar{B} + (C \times \bar{C})\} \cdot D_\mu \partial^\mu \bar{C}, \]
\[ s_b \mathcal{L}_B = \partial_\mu [(B + (C \times \bar{C}) \cdot \partial^\mu C) - \{B + \bar{B} + (C \times \bar{C})\} \cdot D_\mu \partial^\mu C. \]  

(22)

Thus, if we exploit the strength of the CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \), we obtain the following symmetry transformations, namely;

\[ s_{ab} \mathcal{L}_B = \partial_\mu [B \cdot \partial^\mu \bar{C}], \quad s_b \mathcal{L}_B = -\partial_\mu [\bar{B} \cdot \partial^\mu C], \]  

(23)

thereby rendering the action integrals invariant. We draw the conclusion that, due to the key equations (3) and (23), both the Lagrangian densities \( \mathcal{L}_B \) and \( \bar{\mathcal{L}}_B \) respect both the BRST and anti-BRST symmetries provided we confine ourselves on the space of quantum fields in the Hilbert space defined by the CF-condition (where the absolute anticommutativity property (i.e. \( \{s_b, s_{ab}\} = 0 \) is also satisfied for \( s_{(a)b} \)). As a consequence, we infer that both the Lagrangian densities are equivalent w.r.t. the (anti-)BRST symmetries on the space of quantum fields in the Hilbert space defined by the CF-condition [12]. Now we focus on the issue of equivalence of the Lagrangian densities \( \mathcal{L}_B \) and \( \bar{\mathcal{L}}_B \) from the point of view of the (anti-)co-BRST symmetry transformations. Besides the symmetry transformation in equation (5), we observe the following:

\[ s_d \mathcal{L}_B = \partial_\mu [\mathcal{B} \cdot D^\mu \bar{C} - \varepsilon^{\mu\nu} (\partial_\mu \bar{C} \times \bar{C}) \cdot C] + i (\partial_\mu A^\mu) \cdot (\mathcal{B} \times \bar{C}), \]
\[ s_{ad} \mathcal{L}_B = \partial_\mu [\mathcal{B} \cdot D^\mu C + \varepsilon^{\mu\nu} C \cdot (\partial_\mu C \times C)] + i (\partial_\mu A^\mu) \cdot (\mathcal{B} \times C). \]  

(24)

We draw the conclusion, from the above, that both the Lagrangian densities \( \mathcal{L}_B \) and \( \bar{\mathcal{L}}_B \) are equivalent w.r.t. the (anti-)co-BRST symmetry transformations if and only if the conditions \( (\mathcal{B} \times C) = 0, (\mathcal{B} \times \bar{C}) = 0 \) are satisfied. Taking the analogy with equations (22) and (23), it is straightforward to conclude that \( \mathcal{B} \times C = 0 \) and \( \mathcal{B} \times \bar{C} = 0 \) are the CF-type restrictions\(^5\) w.r.t. the (anti-)co-BRST symmetries for the self-interacting 2D non-Abelian gauge theory.

We would like to mention here that there are differences between the CF-condition \( B + \bar{B} + (C \times \bar{C}) = 0 \) (existing for the non-Abelian 1-form gauge theory in the context of (anti-)BRST symmetry transformations for any arbitrary dimension of spacetime) and the CF-type restrictions that appear in the context of (anti-)co-BRST symmetry transformations for the 2D non-Abelian 1-form gauge theory. Whereas the latter conditions \( \mathcal{B} \times C = 0 \) and \( \mathcal{B} \times \bar{C} = 0 \) are perfectly (anti-)co-BRST invariant [i.e. \( s_{(a)d} (\mathcal{B} \times C) = 0, s_{(a)d} (\mathcal{B} \times \bar{C}) = 0 \)] quantities, the same is not true in the case of CF-condition \( B + \bar{B} + (C \times \bar{C}) = 0 \). It can be checked that:

\[ s_b [B + \bar{B} + (C \times \bar{C})] = i [B + \bar{B} + (C \times \bar{C})] \times C, \]
\[ s_{ab} [B + \bar{B} + (C \times \bar{C})] = i [B + \bar{B} + (C \times \bar{C})] \times \bar{C}. \]  

(25)

\(^5\)We lay emphasis on the fact that these restrictions do not imply that \( C \times \bar{C} = 0 \) (thereby rendering the theory to become Abelian). This is due to the fact that the absolute anticommutativity property \( \{s_d, s_{ad}\} = 0 \) implies that the CF-type restrictions \( \mathcal{B} \times C = 0 \) and \( \mathcal{B} \times \bar{C} = 0 \) are independent of each other (see, e.g. [10] for details). In other words, both these restrictions should be considered separately and independently.
The above transformations show that the CF-condition \( B + \bar{B} + (C \times \bar{C}) = 0 \) is the (anti-)BRST invariant quantity only on the space of quantum fields defined by the restriction \( B + \bar{B} + (C \times \bar{C}) = 0 \). Furthermore, the (anti-)BRST symmetry transformations are absolutely anticommuting (i.e. \( \{s_b, s_{ab}\} = 0 \) only on the space of quantum fields defined by the CF-condition \( B + \bar{B} + (C \times \bar{C}) = 0 \). However, the absolute anticommutativity of the nilpotent (anti-)co-BRST symmetry transformations (i.e. \( \{s_d, s_{ad}\} = 0 \) is satisfied without any use of \( B \times C = 0 \) and \( B \times \bar{C} = 0 \). In other words, the absolute anticommutativity of the (anti-)co-BRST symmetry transformations does not need any kinds of restrictions from outside. We shall see, later on, that the above CF-type restrictions (i.e. \( B \times C = 0 \) and \( B \times \bar{C} = 0 \)) appear at the level of algebra obeyed by the conserved charges [derived from the Lagrangian densities (1)] when we demand the absolute anticommutativity of the co-BRST and anti-co-BRST charges.

As pointed out earlier, we have seen that \( s_{(a)d}(B \times C) = 0 \) and \( s_{(a)d}(B \times \bar{C}) = 0 \). Thus, these CF-type constraints are (anti-)co-BRST invariant and, therefore, they are physical and theoretically very useful. As a consequence of the above observation, the Lagrangian densities \( L_B \) and \( L_{\bar{B}} \) can be modified in such a manner that \( L_B \) and \( L_{\bar{B}} \) can have the perfect (anti-)co-BRST symmetry invariance(s). For instance, we note that the following modified versions of the Lagrangian densities, with fermionic \( (\lambda^2 = \bar{\lambda}^2 = 0, \bar{\lambda} \lambda + \lambda \bar{\lambda} = 0) \) Lagrange multiplier fields \( \lambda \) and \( \bar{\lambda} \), namely;

\[
\mathcal{L}_B^{(\lambda)} = B \cdot E - \frac{1}{2} B \cdot B - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B})
- i D_\mu \bar{C} \cdot \partial^\mu C + \lambda \cdot (B \times \bar{C}),
\]

\[
\mathcal{L}_{\bar{B}}^{(\bar{\lambda})} = B \cdot E - \frac{1}{2} B \cdot B + B \cdot (\partial_\mu A^\mu) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B})
- i \partial_\mu \bar{C} \cdot D^\mu C + \bar{\lambda} \cdot (B \times C),
\]

respect the following perfect (anti-)co-BRST symmetry transformations:

\[
\begin{align*}
s_{ad} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu C, & s_{ad} C &= 0, & s_{ad} \bar{C} &= i B, & s_{ad} B &= 0, \\
s_{ad} E &= D_\mu \partial^\mu C, & s_{ad} (\partial_\mu A^\mu) &= 0, & s_{ad} \bar{\lambda} &= -i (\partial_\mu A^\mu), & s_{ad} \lambda &= 0, \\
s_{d} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu \bar{C}, & s_{d} \bar{C} &= 0, & s_{d} \bar{\lambda} &= -i B, & s_{d} \lambda &= 0, \\
s_{d} E &= D_\mu \partial^\mu \bar{C}, & s_{d} (\partial_\mu A^\mu) &= 0, & s_{d} \bar{\lambda} &= -i (\partial_\mu A^\mu), & s_{d} \lambda &= 0.
\end{align*}
\]

We remark here that the above (anti-)co-BRST symmetry transformations are off-shell nilpotent as well as absolutely anticommuting (without any use of CF-type restrictions). Hence, these symmetries are proper and perfect. In the above, the superscripts \( (\lambda) \) and \( (\bar{\lambda}) \) on the Lagrangian densities are due to obvious reasons (i.e. they characterize \( L_B \) and \( L_{\bar{B}} \)). It should be noted that the Lagrange multipliers \( \lambda \) and \( \bar{\lambda} \) carry the ghost numbers equal to \(+1\) and \(-1\), respectively. Ultimately, we observe that the following transformations of the coupled (but equivalent) Lagrangian densities are true, namely;

\[
\begin{align*}
s_{d} \mathcal{L}_B^{(\lambda)} &= \partial_\mu [B \cdot \partial^\mu \bar{C}], & s_{ad} \mathcal{L}_B^{(\lambda)} &= \partial_\mu [B \cdot \partial^\mu C], \\
s_{d} \mathcal{L}_{\bar{B}}^{(\bar{\lambda})} &= \partial_\mu [B \cdot D^\mu \bar{C} - \varepsilon^{\mu \nu} (\partial_\nu \bar{C} \times \bar{C}) \cdot C], & s_{ad} \mathcal{L}_{\bar{B}}^{(\bar{\lambda})} &= \partial_\mu [B \cdot D^\mu C + \varepsilon^{\mu \nu} \bar{C} \cdot (\partial_\nu C \times C)].
\end{align*}
\]

\[26\]
which show that the action integrals $S = \int d^2x \mathcal{L}_B^{(s)}$ and $S = \int d^2x \mathcal{L}_B^{(\bar{s})}$ remain invariant under the (anti)co-BRST symmetry transformations $s_{(a)d}$. Thus, we lay emphasis on the observation that both the Lagrangian densities $\mathcal{L}_B^{(s)}$ and $\mathcal{L}_B^{(\bar{s})}$ (cf. Eq. (26)) are equivalent as far as the symmetry considerations w.r.t. the (anti-)co-BRST symmetry transformations (27) are concerned. Henceforth, we shall only focus on the Lagrangian densities $\mathcal{L}_B^{(s)}$ and $\mathcal{L}_B^{(\bar{s})}$ for our further discussions and we shall discuss their symmetry properties under the off-shell nilpotent (anti-)BRST transformations, too (cf. Appendix C below).

5 Algebraic Structures: Symmetries and Charges

The Lagrangian densities in Eq. (26) are good enough to provide the physical realizations of the cohomological operators of differential geometry in the language of their symmetry properties. First of all, let us focus on $\mathcal{L}_B^{(s)}$. This Lagrangian density (and corresponding action integral) respect the (anti-)co-BRST symmetry transformations (27) and BRST symmetry transformations in a perfect manner because the nilpotent BRST symmetry transformations $(s_b)$, listed in (2) (along with $s_b \bar{\lambda} = 0$), are the symmetry of the action integral $S = \int d^2x \mathcal{L}_B^{(s)}$. This is because of the fact that we have $s_b(\mathcal{B} \times C) = 0$ due to the nilpotency condition: $s_b^2 \mathcal{B} = i s_b(\mathcal{B} \times C) = 0$ and $s_b \mathcal{L}_B = \partial_\mu (B \cdot D^\mu C)$ [cf. Eq. (3)].

To be more precise, the Lagrangian density $\mathcal{L}_B^{(s)}$ respects $s_b, s_d, s_{ad}, s_g$ and $s_w = \{s_b, s_d\}$ as discussed in Sec. 2 [with the additional transformations $s_b \bar{\lambda} = 0$, $s_b(\mathcal{B} \times C) = 0$ and the transformations (27) which lead to (28)]. This observation should be contrasted with the Lagrangian density $\mathcal{L}_B$ (cf. Eq. (11)) which respects only four perfect symmetries, namely: $s_b, s_d, s_g$ and $s_w$. It does not respect $s_{ad}$ perfectly. One can explicitly check that, in their operator form, the above set of five perfect symmetries obey the following algebra

\[
\begin{align*}
    s_{(a)d}^2 &= s_b^2 = 0, & \{s_b, s_d\} &= s_w, & \{s_d, s_{ad}\} &= 0, \\
    [s_w, s_r] &= 0 & r &= b, d, ad, g, & \{s_b, s_{ad}\} &= 0, \\
    [s_g, s_b] &= +s_b, & [s_g, s_d] &= -s_d, & [s_g, s_{ad}] &= +s_{ad}.
\end{align*}
\]

In the above, we note that $s_w \bar{\lambda} = 0$ ($\Rightarrow s_b \bar{\lambda} = 0$, $s_d \bar{\lambda} = 0$) and $s_g \bar{\lambda} = -\bar{\lambda}$. The algebra in (29) is reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry (see, e.g. [18,19]), namely;

\[
\begin{align*}
    d^2 &= 0, & \delta^2 &= 0, & \{d, \delta\} &= \triangle, & [\triangle, d] &= 0 = [\triangle, \delta].
\end{align*}
\]

where $(d, \delta, \triangle)$ are the exterior derivative, co-exterior derivative and Laplacian operators, respectively. These operators constitute the set of de Rham cohomological operators. It is clear that we have $d \leftrightarrow s_b$, $\delta \leftrightarrow s_d$ and $\triangle \leftrightarrow s_w$. Such identification is justified due to the algebra of the conserved charges, too, where the transformation $s_g$ and corresponding charge $Q_g$ play an important role. We shall discuss it later. We note here that there is one-to-one mapping between the symmetry operators and cohomological operators.

\footnote{We mean by the perfect symmetries as the transformations for which the Lagrangian densities either remain invariant or transform to the total space time derivative without any use of CF-type restrictions and/or the Euler-Lagrange EOMs.}
It is worth pointing out that the algebra in (29) is obeyed for the Lagrangian density $L_B^{(\lambda)}$ (which respects five perfect continuous symmetries). However, the algebra (29) is satisfied only on the on-shell where we use the EQM (derived from Lagrangian density $L_B^{(\lambda)}$) and the set of CF-type restrictions that have been discussed in earlier works [10,11]. We list here a few of these algebraic relationships which are juxtaposed along with the EL-EQM and the constraints (i.e. CF-type restrictions) that are invoked in their proof. To be more explicit and precise, we have the following algebraic relations as well as the restrictions/EQM (which are exploited in the proof of the algebraic relations), namely:

\[
\begin{align*}
\{s_b, s_{ad}\} \bar{C} &= 0 \quad \iff \quad B \times C = 0, \\
\{s_b, s_{ad}\} \bar{\lambda} &= 0 \quad \iff \quad \partial_{\mu} D^\mu C = 0, \\
[s_w, s_{ad}] A_\mu &= 0 \quad \iff \quad B \times C = 0, \\
[s_w, s_{ad}] \bar{\lambda} &= 0 \quad \iff \quad \partial_{\mu} D^\mu B + \varepsilon^{\mu\nu}(\partial_{\nu} \bar{C} \times \partial_{\mu} C) = 0.
\end{align*}
\] (31)

Thus, we observe that the algebra (29) is very nicely respected provided we utilize the strength of EQM from $L_B^{(\lambda)}$ and use the CF-type restrictions appropriately.

Now we focus on the $L_B^{(\lambda)}$ and briefly discuss the algebra of its symmetry operators. This Lagrangian density also respects five perfect symmetries. These are $s_d, s_{ad}, s_w = -\{s_{ad}, s_w\}$, $s_{ab}$ and $s_g$ [cf. Eqs. (2), (6), (27)]. In particular, we note that the anti-BRST symmetry transformations ($s_{ab}$) are same as (2) together with $s_{ab\lambda} = 0$ because we find $s_{ab}(B \times C) = 0$ due to the nilpotency condition $s_{ab}^2 B = 0$. The algebra satisfied by the above symmetry operators are:

\[
\begin{align*}
2s_{(a)d} &= s_{ab}^2 = 0, & \{s_{ad}, s_{ab}\} &= -s_w, & \{s_d, s_{ad}\} &= 0, \\
[s_w, s_r] &= 0, & r &= d, ad, ab, g, & \{s_d, s_{ab}\} &= 0, \\
[s_g, s_d] &= -s_d, & [s_g, s_{ab}] &= -s_{ab}, & [s_g, s_{ad}] &= s_{ad}.
\end{align*}
\] (32)

We note that $s_{ad}\lambda = s_{ab}\lambda = 0$ implies that $s_w\lambda = 0$ because $s_w = -\{s_{ab}, s_{ad}\}$. We also have $s_g\lambda = +\lambda$ (i.e. the ghost number of $\lambda$ is +1).

From the above algebra, it is clear that we have found out the physical realizations of the cohomological operators ($d, \delta, \Delta$) in the language of the symmetry transformations of the Lagrangian density $L_B^{(\lambda)}$. However, the algebra (32) is satisfied only when the EQM and the constraints (i.e. CF-type restrictions) of the theory are exploited together in a judicious manner. We have been brief here in our statements but it can be easily checked that our claims are true. To be more explicit, we note that we have obtained a one-to-one mapping: $d \leftrightarrow s_{ad}, \delta \leftrightarrow s_{ab}$ and $\Delta \leftrightarrow s_w = -\{s_{ab}, s_{ad}\}$. We conclude, from the above discussions, that the Lagrangian densities $L_B^{(\lambda)}$ and $L_B^{(\lambda)}$ respect five perfect symmetries out of which two are relevant fermionic symmetries (even though there is existence of three perfect fermionic symmetries present in the theory) and there is a unique bosonic symmetry ($s_w$) in the theory. With these, we are able to provide the physical realization of the cohomological operators ($d, \delta, \Delta$). In other words, we have obtained two independent Lagrangian densities where the continuous symmetries provide the physical realizations of the cohomological operators of differential geometry (at the algebraic level) which demonstrate that we have found out a 2D field theoretic model for the Hodge theory (see, e.g. [5,7] for more details).
The identifications that have been made after equations (29) and (32) are correct in the language of continuous symmetries of the theory. In this context, we have to recall our statements after Eq. (2) and Eq. (4) where we stated that the kinetic term and gauge-fixing term remain invariant under the fermionic symmetries \( s_{(a)b} \) and \( s_{(a)d} \), respectively. It is worth pointing out that the kinetic term owes its origin to the exterior derivative \((d = dx^\mu, d^2 = 0)\). On the other hand, the mathematical origin of the gauge-fixing term lies with the co-exterior derivative \((\delta = -\ast d\ast, \delta^2 = 0)\). It is the ghost number considerations, at the level of charge, which leads to the identifications \( d \leftrightarrow s_b, \quad \delta \leftrightarrow s_d, \quad \triangle \leftrightarrow s_w \) after the equation (29) as well as the mappings \( d \leftrightarrow s_{ad}, \quad \delta \leftrightarrow s_{ab}, \quad \triangle \leftrightarrow s_w \) after the equation (32). Thus, the abstract mathematical cohomological operators find their realizations in the language of physically well-defined continuous symmetry operators of our present 2D non-Abelian 1-form gauge theory.

Now we concentrate on the algebraic structures associated with the six conserved charges \((i.e. \ Q_{(a)b}, Q_{(a)d}, Q_w, Q_g)\) that correspond to the six continuous symmetries of our theory. We note that the nilpotency property of fermionic charges \( Q_{(a)b} \) and \( Q_{(a)d} \) has already been quoted in Eq. (18). Using the expressions for the conserved and nilpotent charges \( Q_d \) and \( Q_{ad} \) [cf. Eq. (17)] and the (anti-)co-BRST symmetry transformations (4), it can be readily checked that the following is true as far as Lagrangian densities (1) are concerned, namely;

\[
\begin{align*}
    s_{ad} Q_d &= -i \{ Q_d, Q_{ad} \} = 0, \quad \Rightarrow \quad \text{iff} \quad \Rightarrow \quad \mathcal{B} \times C = 0, \\
    s_d Q_{ad} &= -i \{ Q_{ad}, Q_d \} = 0, \quad \Rightarrow \quad \text{iff} \quad \Rightarrow \quad \mathcal{B} \times C = 0. \quad (33)
\end{align*}
\]

Thus, we note that even though the absolute anticommuting property \(\{ s_d, s_{ad} \} = 0\) associated with \(s_{(a)d}\) is satisfied at the level of symmetry operators without any use of CF-type restrictions \((\mathcal{B} \times C = 0, \mathcal{B} \times \bar{C} = 0)\), we find that, at the level of conserved charges, we have to exploit the strength of these restrictions \((i.e. \ \mathcal{B} \times C = 0, \mathcal{B} \times \bar{C} = 0)\) for the proof of absolute anticommutativity. This is a novel observation which does not appear in the case of (anti-)BRST symmetries where \(\{ s_b, s_{ab} \} = 0 \) and \(\{ Q_b, Q_{ab} \} = 0\) are satisfied only when the CF-condition \( B + \bar{B} + (C \times \bar{C}) = 0 \) is invoked. Another point to be noted is that the CF-type restrictions \( B \times C = 0 \) and \( B \times \bar{C} = 0 \) are required for the proof of \(s_d Q_{ad} = -i \{ Q_{ad}, Q_d \} = 0\) and \(s_{ad} Q_d = -i \{ Q_{ad}, Q_d \} = 0\) as well as for the invariance of the Lagrangian densities \((i.e. \ s_d \mathcal{L}_B \) and \(s_{ad} \mathcal{L}_B)\) which are evident from Eq. (24).

The other algebraic relations amongst \(Q_{(a)b}, Q_{(a)d}, Q_w\) and \(Q_g\) are satisfied in a straightforward manner (except the absolute anticommutativity properties where the CF-type restrictions are required). It can be checked that

\[
\begin{align*}
    s_g Q_b &= -i \{ Q_b, Q_g \} = +Q_b, \quad s_g Q_{ad} = -i \{ Q_{ad}, Q_g \} = +Q_{ad}, \\
    s_g Q_{ab} &= -i \{ Q_{ab}, Q_g \} = -Q_{ab}, \quad s_g Q_d = -i \{ Q_d, Q_g \} = -Q_d, \\
    s_g Q_w &= -i \{ Q_w, Q_g \} = 0. \quad (34)
\end{align*}
\]

\(^1\)The curvature 2-form \( F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}\) (with \(d = dx^\mu \partial_\mu \) and \(A^{(1)} = dx^\mu A_\mu\)) leads to the derivation of the field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i (A_\mu \times A_\nu)\). Hence, the kinetic term owes its origin to \(d = dx^\mu \partial_\mu\). It can be explicitly checked that \(\delta A^{(1)} = -\ast d \ast A^{(1)} = \partial_\mu A^\mu\). Hence, the gauge-fixing term \((i.e. \ a 0\text{-form})\) has its origin in the co-exterior derivative \(\delta = -\ast d \ast\).

\(^2\)The claims, made in Eq. (33), are strong statements. There are weaker versions of them which become transparent when the operators \(s_{(a)d}\) are applied on the third expressions for \(Q_d\) and \(Q_{ad}\) in (17). For instance, we note that \(s_{ad} Q_d = i \int dx \partial_1 [(\mathcal{B} \times C) \cdot \bar{C}] \rightarrow 0\) for physically well-defined fields that vanish off at \(x = \pm \infty\). Similarly, we observe that \(s_d Q_{ad} = i \int dx \partial_1 [(\mathcal{B} \times \bar{C}) \cdot C] \rightarrow 0\).
which shows that the ghost number of \((Q_b, Q_{ad})\) is equal to \((+1)\) but the ghost number for \((Q_{ab}, Q_d)\) is equal to \((-1)\). It is also evident that \(Q_w\) commutes with all the charges of the theory. As far as the proof of this statement is concerned, we note that

\[
s_w Q_r = -i [Q_r, Q_w] = 0, \quad r = b, ab, d, ad, g, w, \quad \tag{35}
\]

which shows that \(Q_w\) is the Casimir operator for the whole algebra because it commutes with all the charges. One of the simplest ways to prove this result is to compute the l.h.s. of equation (35) from the transformations (9) and the expressions for the charges \(Q_r (r = b, ab, d, ad, g)\) that have been derived in Sec. 2.

We briefly comment here on the algebraic structure that is satisfied by the conserved charges of our theory. In this context, we have seen various forms of the algebras \([\text{cf. Eqs. (18), (34), (35)}]\) that are satisfied by the six conserved charges of our theory. It can be verified that collectively these charges satisfy the following extended BRST algebra:

\[
\begin{align*}
Q_{(a)b}^2 &= 0, & Q_{(a)d}^2 &= 0, & \{Q_b, Q_{ab}\} &= \{Q_d, Q_{ad}\} = 0, \\
[Q_w, Q_r] &= 0, & r &= b, ab, d, ad, g, w, & \{Q_d, Q_{ab}\} &= 0, \\
i [Q_g, Q_b] &= + Q_b, & i [Q_g, Q_{ad}] &= Q_{ad}, & \{Q_b, Q_{ad}\} &= 0, \\
i [Q_g, Q_{ab}] &= - Q_{ab}, & i [Q_g, Q_d] &= - Q_d. \quad \tag{36}
\end{align*}
\]

The above algebra is obeyed only on a space of quantum fields defined in the 2D Minkowskian spacetime manifold where all types of CF-type restrictions as well as EOM, emerging from the Lagrangian densities (1), are satisfied. The above algebra is reminiscent of the Hodge algebra satisfied by the de Rham cohomological operators of differential geometry [18,19] where the mapping between the set of conserved charges and cohomological operators is:

\[
(Q_b, Q_{ad}) \leftrightarrow d, \quad (Q_d, Q_{ab}) \leftrightarrow \delta, \quad Q_w = \{Q_b, Q_d\} = - \{Q_{ab}, Q_{ad}\} \leftrightarrow \Delta. \quad \tag{37}
\]

This two-to-one mapping is true only for the coupled (but equivalent) Lagrangian densities (1) where the EOM and CF-type restrictions are exploited together.

In the above identifications, the ghost number of a state (in the quantum Hilbert space), plays a very important role. We have shown in our earlier works [7, 20-22] that the algebra (36) indeed implies that if the ghost number of a state \(\psi > n\) is \(n\) (i.e. \(i Q_g \psi > n = n \psi > n\)), then, the states \(Q_b \psi > n, Q_d \psi > n\) and \(Q_w \psi > n\) would have the ghost numbers \((n + 1), (n - 1)\) and \(n\), respectively. In exactly similar fashion, we have already been able to prove that the states \(Q_{ad} \psi > n, Q_{ab} \psi > n\) and \(Q_w \psi > n\) (with \(Q_w = -\{Q_{ab}, Q_{ad}\}\)) would carry the ghost number \((n + 1), (n - 1)\) and \(n\), respectively. We have discussed the Hodge decomposition theorem in the quantum Hilbert space of states in our earlier works [7, 20-22] which can be repeated for our 2D theory, too. This would fully establish the fact that our present theory is a field theoretic model for the Hodge theory which provides the physical realizations of the cohomological operators in the language of symmetry transformations (treated as operators) and corresponding conserved charges.

\(^{11}\)The above observations are the analogue of the operations of the de Rham cohomological operators \((d, \delta, \Delta)\) on the \(n\)-form \((f_n)\) where the degrees of forms \(df_n, \delta f_n\) and \(\Delta f_n\) are \((n + 1), (n - 1)\) and \(n\), respectively.
6 Novel Observations: Algebraic and Symmetry Considerations in Our 2D non-Abelian Theory

As far as symmetry property is concerned, we observe that there are CF-type restrictions ($B \times C = 0, \bar{B} \times \bar{C} = 0$) corresponding to the (anti-)co-BRST symmetries, too, as is the case with the (anti-)BRST symmetries of our 2D non-Abelian 1-form gauge theory where the CF-condition ($B + \bar{B} + C \times \bar{C} = 0$) exists [12]. However, there are specific novelties that are connected with the CF-type restrictions: $B \times C = 0, \bar{B} \times \bar{C} = 0$. First, these restrictions are (anti-)co-BRST invariant [i.e. $s_{(a)d}(B \times C) = 0$ and $s_{(a)d}(\bar{B} \times \bar{C}) = 0$] whereas the CF-condition ($B + \bar{B} + C \times \bar{C} = 0$) is not perfectly invariant under the (anti-)BRST transformations [cf. Eq. (25)]. Second, the restrictions ($B \times C = 0, \bar{B} \times \bar{C} = 0$) can be incorporated into the Lagrangian densities [cf. Eq. (26)] in such a manner that one can have perfect (anti-)co-BRST symmetry invariance for the individual Lagrangian densities in (26). Such kind of thing can not be done with the CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$.

We observe that (anti-)co-BRST symmetries (where the gauge-fixing term remains invariant) exist at the quantum level when the gauge-fixing term is added to the Lagrangian densities. In other words, there is no classical analogue of the (anti-)co-BRST symmetries. However, the (anti-)BRST symmetry transformations (where the kinetic term remains invariant) is the generalization of the classical local $SU(N)$ gauge symmetries to the quantum level. Furthermore, we note that the (anti-)BRST symmetries would exist for any $p$-form gauge theory in any arbitrary dimension of spacetime. However, the (anti-)co-BRST symmetries have been shown to exist for the $p$-form gauge theory only in $D = 2p$ dimensions of spacetime [5,6]. They have not been shown to exist, so far, in any arbitrary dimension of spacetime. In addition, the absolute anticommutativity property of the BRST and anti-BRST transformations require the validity of CF-condition. On the contrary, the nilpotent (anti-)co-BRST symmetries do absolutely anticommute with each other without any use of the CF-type restrictions that exist in the 2D non-Abelian gauge theory.

We note that $\{s_{d}, s_{ad}\} = 0$ without any use of the CF-type restrictions ($B \times C = 0$, and $\bar{B} \times \bar{C} = 0$) as far as the Lagrangian densities $L_{B}$ and $L_{\bar{B}}$ [cf. Eq. (1)] are concerned. However, the restrictions $B \times C = 0$ and $\bar{B} \times \bar{C} = 0$ are required for the proof of $\{Q_{d}, Q_{ad}\} = 0$ when we compute this bracket from $s_{d}Q_{ad} = -i \{Q_{d}, Q_{ad}\}$ and/or $s_{ad}Q_{d} = -i \{Q_{ad}, Q_{d}\}$ as far as the Lagrangian densities $L_{B}$ and $L_{\bar{B}}$ [cf. Eq. (1)] are concerned. It is interesting to point out that the property of nilpotency and absolute anticommutativity is satisfied without any use of CF-type restrictions for the Lagrangian densities (26) (where the Lagrange multipliers $\lambda$ and $\bar{\lambda}$ are incorporated to accommodate the CF-type restrictions). This statement is true for the (anti-)co-BRST symmetry operators as well as for the corresponding conserved charges. The CF-type restrictions ($B \times C = 0, \bar{B} \times \bar{C} = 0$) appear in the proof of $\{Q_{d}, Q_{ad}\} = 0$ [cf. Eq. (33)] as well as in the mathematical expressions for $s_{ad}L_{B}$ and $s_{d}L_{\bar{B}}$ [cf. Eq. (24)] but they do not appear in $\{s_{d}, s_{ad}\} = 0$. On the contrary, the CF-condition ($B + \bar{B} + C \times \bar{C} = 0$) appears in the proofs of $\{s_{b}, s_{ad}\} = 0$, $\{Q_{b}, Q_{ad}\} = 0$ and in the mathematical expressions for: $s_{ab}L_{B}$ as well as $s_{b}L_{\bar{B}}$ [cf. Eq. (22)] when the the Lagrangian densities (1) are considered.

To corroborate the above statements, we take a couple of examples to demonstrate that we do not require the strength of CF-type restrictions ($B \times C = 0, \bar{B} \times \bar{C} = 0$ from outside)
in the proof of nilpotency and absolute anticommutativity of the (anti-)co-BRST charges [derived from the Lagrangian densities (26)]. In this context, we note that the expressions for the nilpotent (anti-)co-BRST charges (17) remain the same for the Lagrangian densities (26) but the EOM [derived from (26)] are different from (12). We note that the latter are:

\[ \varepsilon^{\mu\nu} D_\mu B + \partial^\mu B + (\partial^\mu \bar{C} \times C) = 0, \quad \partial_\mu D^\mu C = 0, \quad B \times C = 0, \]

\[ E = \bar{B} + (\bar{\lambda} \times C), \quad D_\mu \partial^\mu \bar{C} - i (\bar{\lambda} \times B) = 0, \]

\[ \varepsilon^{\mu\nu} D_\mu B - \partial^\mu \bar{B} - (\bar{C} \times \partial^\mu C) = 0, \quad \partial_\mu D^\mu \bar{C} = 0, \quad B \times \bar{C} = 0, \]

\[ E = B + (\lambda \times C), \quad D_\mu \partial^\mu C + i (\lambda \times B) = 0. \]

The above equations are to be used in the proof of conservation of the Noether currents from which the charges are computed. In this context, we observe the expressions for the (anti-)co-BRST conserved Noether current for the Lagrangian density \( L_B^{(\bar{\lambda})} \) are as follows:

\[ J_{d}^{\mu (\bar{\lambda})} = B \cdot \partial^\mu C = \varepsilon^{\mu\nu} B \cdot \partial_\nu C \equiv J_{d}^{\mu} \quad \text{(cf. Eq. (16))}, \]

\[ J_{ad}^{\mu (\bar{\lambda})} = B \cdot \partial^\mu C - \varepsilon^{\mu\nu} B \cdot \partial_\nu C - \varepsilon^{\mu\nu} \bar{C} \cdot (\partial_\nu C \times C), \]

(39)

where the superscript \((\bar{\lambda})\) denotes that the above currents have been derived from \( L_B^{(\bar{\lambda})} \) (cf. Eq. (26)). The expressions (39) demonstrate that, for the Lagrangian density \( L_B^{(\bar{\lambda})} \), the co-BRST Noether conserved current remains same as given in (16) (for \( L_B \)) but the expression for the anti-co-BRST Noether conserved current is different from the same current derived from \( L_B^{(\lambda)} \) [cf. Eq. (16)]. The conservation of the above currents can be proven by using EL-EOM (38). The expressions for the conserved co-BRST charge remains the same as given in (17) but the expression for the anti-co-BRST charge is:

\[ Q_{ad}^{(\bar{\lambda})} = \int dx \, J_{ad}^{\nu (\bar{\lambda})} = \int dx \, [B \cdot \bar{C} - \partial_1 B \cdot C + \bar{C} \cdot (\partial_1 C \times C)] \]

\[ \equiv \int dx \, [B \cdot \bar{C} - D_0 B \cdot C + (\partial_1 \bar{C} \times C) \cdot C + \bar{C} \cdot (\partial_1 C \times C)]. \]

(40)

The nilpotency of the co-BRST charge \( Q_{d}^{(\lambda)} = Q_d \) has already been proven in Eq. (18). Similarly, it can be checked that

\[ s_{ad} \, Q_{ad}^{(\bar{\lambda})} = \int dx \, [B \cdot \bar{C} - D_0 B \cdot C + (\partial_1 \bar{C} \times C) \cdot C + \bar{C} \cdot (\partial_1 C \times C)] \]

\[ \equiv \int dx \, \partial_1 [i (B \times C) \cdot C] \longrightarrow 0 \iff -i \{Q_{ad}^{(\bar{\lambda})}, Q_{ad}^{(\bar{\lambda})}\} = 0, \]

(41)

which demonstrate the validity of nilpotency of \( Q_{ad}^{(\bar{\lambda})} \) because it can be explicitly checked that \( s_{ad} \, Q_{ad}^{(\lambda)} = -i \{Q_{ad}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \) which implies that \((Q_{ad}^{(\lambda)})^2 = 0\). We emphasize that the r.h.s. of (41) is zero due to the EOM (i.e. \( B \times C = 0 \)), too.

We now concentrate on the Lagrangian density \( L_B^{(\lambda)} \) and compute the expressions for the Noether currents corresponding to the (anti-)co-BRST symmetry transformations. It is evident from transformations (27) that under the anti-co-BRST symmetry transformations,
the Lagrangian density $L_d^{(λ)}$ transforms in exactly the same manner as given in (5). Thus, the conserved current would be same as in (16). However, in view of the transformation of $L_d^{(λ)}$ in (28) under $s_d$, we have the following expressions for the Noether current

$$J_d^{(μ(λ))} = B \cdot \partial^μ \dot{C} + \varepsilon^{μν} \frac{\partial}{\partial \dot{\lambda}} \frac{\partial}{\partial \dot{C}} + \varepsilon^{μν} (\partial_μ \dot{C} \times \dot{C}) \cdot C,$$

which is different from (16). The conservation law (i.e. $\partial_\mu J_d^{(μ(λ))} = 0$) can be proven by exploiting the EL-EOM given in (38). The conserved charge $Q_d^{(λ)}$ has the following forms:

$$Q_d^{(λ)} = \int dx J_d^{(0(λ))} \equiv \int dx [B \cdot \dot{C} + \partial_1 \cdot \dot{C} - (\partial_1 \cdot \dot{C}) + (\partial_1 \cdot \dot{C}) \cdot C],$$

where the EL-EOM have been used to obtain the above equivalent forms of the conserved charge $Q_d^{(λ)}$. The nilpotency of the above charge can be proven by using the symmetry principle (with $s_d Q_d^{(λ)} = -i \{Q_d^{(λ)}, Q_d^{(λ)}\} = 0$) as:

$$s_d Q_d^{(λ)} = \int dx \partial_1 [i (B \cdot \dot{C}) \cdot C] \rightarrow 0.$$

Thus, we note that $s_d Q_d^{(λ)} = -i \{Q_d^{(λ)}, Q_d^{(λ)}\} = 0$ implies that $(Q_d^{(λ)})^2 = 0$. This proves the nilpotency of the co-BRST charge, derived from $L_d^{(λ)}$, for physically well-defined fields which vanish off $x = \pm \infty$. Furthermore, the r.h.s. of (44) is zero due to the EOM (i.e. $B = 0$) which emerges from $L_d^{(λ)}$ [cf. Eq. (38)].

We have to prove the absolute anticommutativity of the (anti-)co-BRST charges that have been derived from the Lagrangian densities (26). As pointed out earlier, the expressions for co-BRST charge for $L_d^{(λ)}$ remains the same as given in (17) (where there are primarily two equivalent expressions for it). We take, first of all, the following (with $Q^{(λ)} = Q_d$) and apply the anti-co-BRST transformation $s_{ad}$:

$$s_{ad} Q_d^{(λ)} = -i \{Q_d^{(λ)}, Q_d^{(λ)}\} \equiv \int dx [B \cdot \dot{C} - \partial_1 B \cdot \dot{C}]$$

Using the equation of motion (38), the above expression yields

$$s_{ad} Q_d^{(λ)} = i \int dx [(B \cdot \dot{C}) \cdot \partial_1 C] = 0,$$

due to the validity of EOM (i.e. $B \times C = 0$) w.r.t. $\lambda$ from $L_d^{(λ)}$. Thus, we note that $\{Q_d^{(λ)}, Q_d^{(λ)}\} = 0$ on the on-shell for $L_d^{(λ)}$. In other words, the absolute anticommutativity
is satisfied. Now let us focus on the alternative expression for $Q_{d}^{(\lambda)}$ and apply $s_{ad}$ on it:

\[
s_{ad} Q_{d}^{(\lambda)} = \int dx \left[ B \cdot \dot{C} - D_0 B \cdot \dot{C} + (\partial_1 \dot{C} \times C) \cdot \dot{C} \right]
\]

\[
\equiv \int dx \partial_1 [(B \times C) \cdot \dot{C}] = 0. \tag{47}
\]

Thus, we note that $s_{ad} Q_{d}^{(\lambda)} = -i \{Q_{d}^{(\lambda)}, Q_{d}^{(\lambda)}\} = 0$ for the physically well-defined fields that vanish off at $x = \pm \infty$. This absolute anticommutativity is also satisfied on the on-shell where $B \times C = 0$ (due to the EOM from $L_{B}^{(\lambda)}$ w.r.t. $\bar{\lambda}$). Finally, we conclude that the property of absolute anticommutativity of the (anti-)co-BRST charges is satisfied without invoking any CF-type constraint condition from outside.

We now concentrate on the derivation of the absolute anticommutativity for $Q_{d}^{(\lambda)}$ which is derived from $L_{B}^{(\lambda)}$. There are two equivalent expressions for it in Eq. (40). We observe that the following are true, namely;

\[
s_{d} Q_{d}^{(\lambda)} = \int dx s_{d} [B \cdot \dot{C} - \partial_1 B \cdot \dot{C} + \dot{C} \cdot (\partial_1 C \times C)]
\]

\[
\equiv \int dx \partial_1 [i \dot{C} \cdot (B \times C)] = 0, \tag{48}
\]

where we have used the EOM from $L_{B}^{(\lambda)}$ w.r.t. $\bar{\lambda}$ that leads to $B \times C = 0$. Furthermore, for all the physically well-defined fields, we obtain $s_{d} Q_{ad}^{\lambda} = -i \{Q_{ad}^{\lambda}, Q_{ad}^{(\lambda)}\} = 0$ because all such fields vanish off at $x = \pm \infty$. Thus, the r.h.s. of (48) is zero due to the Gauss’s divergence theorem. Taking the alternative expressions for $Q_{d}^{(\lambda)}$ in (40), we note that

\[
s_{d} Q_{ad}^{(\lambda)} = \int dx_{d} [B \cdot \dot{C} - D_0 B \cdot C + (\partial_1 \dot{C} \times C) \cdot C + \dot{C} \cdot (\partial_1 C \times C)]
\]

\[
\equiv \int dx \partial_1 [i (B \times C) \cdot \dot{C}] = 0, \tag{49}
\]

because of the fact that $B \times C = 0$ (due to the EOM from $L_{B}^{(\lambda)}$ w.r.t. $\bar{\lambda}$ field). Moreover, all the fields vanish-off at $x = \pm \infty$. Thus, the Gauss divergence theorem shows that $s_{d} Q_{ad}^{(\lambda)} = -i \{Q_{ad}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0$ which proves the absolute anticommutativity property of the (anti-)co-BRST charges. This observation is a novel result in our present endeavor.

At this juncture, now we take up Lagrangian density $L_{B}^{(\lambda)}$ into consideration. The anti-co-BRST charge for this Lagrangian density is same as given in (17) (i.e. $Q_{ad}^{(\lambda)} = Q_{ad}$). We observe the following after the application of the co-BRST symmetry $s_{d}$ on $Q_{ad}^{(\lambda)}$, namely;

\[
s_{d} Q_{ad}^{(\lambda)} = \int dx \left[ B \cdot \dot{C} + \partial_1 B \cdot C \right]
\]

\[
\equiv \int dx [i (B \times \dot{C}) \cdot \partial_1 C] = 0. \tag{50}
\]

Thus, we have seen now that $s_{d} Q_{ad}^{(\lambda)} \equiv -i \{Q_{ad}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0$ due to $B \times \dot{C} = 0$ which emerges as EOM from $L_{B}^{(\lambda)}$ w.r.t. the field $\lambda$. In other words, the absolute anticommutativity
\( \{Q_{ad}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \) is satisfied on the on-shell. A similar exercise, with another equivalent expression for \( Q_{ad}^{(\lambda)} \), namely:

\[
\begin{align*}
\text{s}_{ad} Q_{ad}^{(\lambda)} &= \int dx \, s_{ad} \left[ \mathcal{B} \cdot \dot{\mathcal{C}} - D_0 \mathcal{B} \cdot \mathcal{C} - (\mathcal{C} \times \partial_1 \mathcal{C}) \cdot \mathcal{C} \right] \\
&= \int dx \, \partial_1 \left[ i (\mathcal{B} \times \dot{\mathcal{C}}) \cdot \mathcal{C} \right] = 0,
\end{align*}
\]  

(51)

establishes the absolute anticommutativity (i.e. \( \{Q_{ad}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \)) due to Gauss’s divergence theorem which states that all the physical fields vanish off at \( x = \pm \infty \).

The absolute anticommutativity property can be also proven by using the expressions for the co-BRST charge \( Q_{ad}^{(\lambda)} \) [cf. Eq. (43)]. It can be readily checked that the following is true:

\[
\begin{align*}
\text{s}_{ad} Q_{d}^{(\lambda)} &= \int dx \, s_{ad} \left[ \mathcal{B} \cdot \dot{\mathcal{C}} + \partial_1 \mathcal{B} \cdot \mathcal{C} - (\partial_1 \mathcal{C} \times \mathcal{C}) \cdot \mathcal{C} \right] \\
&= \int dx \, \partial_1 \left[ -i (\mathcal{B} \times \mathcal{C}) \cdot \mathcal{C} \right] \rightarrow 0.
\end{align*}
\]  

(52)

Thus, we note that \( \text{s}_{ad} Q_{d}^{(\lambda)} = -i \{Q_{d}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \) for the physically well-defined fields that vanish off at \( x = \pm \infty \). Moreover, the absolute anticommutativity is also satisfied due to EOM (i.e. \( \mathcal{B} \times \mathcal{C} = 0 \)) that is derived from \( \mathcal{L}_{B}^{(\lambda)} \) w.r.t. Lagrange multiplier field \( \lambda \). Hence, the absolute anticommutativity of the (anti-)co-BRST charges is satisfied on-shell. We now take up the alternative expression for the \( Q_{d}^{(\lambda)} \) from (43) and show the validity of absolute anticommutativity. Towards this goal in mind, we observe the following

\[
\begin{align*}
\text{s}_{ad} Q_{d}^{(\lambda)} &= \int dx \, s_{ad} \left[ \mathcal{B} \cdot \dot{\mathcal{C}} - D_0 \mathcal{B} \cdot \mathcal{C} - (\mathcal{C} \times \partial_1 \mathcal{C}) \times \mathcal{C} - (\partial_1 \mathcal{C} \times \dot{\mathcal{C}}) \cdot \mathcal{C} \right] \\
&= \int dx \, \partial_1 \left[ -i (\mathcal{B} \times \mathcal{C}) \cdot \mathcal{C} \right] \rightarrow 0.
\end{align*}
\]  

(53)

This shows that \( \text{s}_{ad} Q_{d}^{(\lambda)} = -i \{Q_{d}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \) due to Gauss’s divergence theorem which states that all the physical fields must vanish off at \( x = \pm \infty \). There is another interpretation, too. The absolute anticommutativity (i.e. \( \{Q_{d}^{(\lambda)}, Q_{ad}^{(\lambda)}\} = 0 \)) is satisfied on-shell (where \( \mathcal{B} \times \mathcal{C} = 0 \) due to EOM from \( \mathcal{L}_{B}^{(\lambda)} \) w.r.t. to \( \lambda \)).

7 Conclusions

In our present endeavor, we have computed all the conserved charges of our theory and obtained the algebra followed by them. We have shown that, for the validity of the precise algebra (consistent with the algebra obeyed by the cohomological operators), we have to use the EOM as well as the CF-type restrictions of our theory described by the Lagrangian densities (1). In particular, we have demonstrated that the requirement of the absolute anticommutativity property amongst the fermionic symmetry operators [cf. Eq. (31)] leads to the emergence of our EOM and/or CF-type restrictions. In other words, it is the
requirement of consistency of the operator algebra with the Hodge algebra (i.e. the algebra obeyed by the cohomological operators of differential geometry) that leads to the derivation of the EOM as well as the CF-type restrictions of our theory. This way of derivation of the CF-type restrictions is completely different from our earlier derivations [10,11] where the existence of the continuous symmetries (and their operator algebra) and the application of the superfield approach to BRST formalism have played key roles.

One of the highlights of our present investigation is the observation that the individual Lagrangian density [of the coupled Lagrangian densities (26)] provides a model for the Hodge theory because the continuous symmetry operators of the specific Lagrangian density (and corresponding charges) obey an algebra that is reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry. In other words, the continuous symmetry operators (and corresponding charges) provide the physical realizations of the cohomological operators of differential geometry. This happens because of the fact that the individual Lagrangian density respects five perfect symmetries where there is no use of any kind of CF-type restrictions. This is precisely the reason that four of the above mentioned five symmetries of the theory obey an exact algebra that is reminiscent of the algebra obeyed by the de Rham cohomological operators of the differential geometry.

We have claimed in earlier works [23,24] that the existence of the CF-restrictions is the hallmark of a quantum gauge theory (described within the framework of BRST formalism). This claim is as fundamental as the definition of a classical gauge theory in the language of first-class constraints by Dirac [25,26]. Thus, it has been a challenge for us to derive all types of CF-type restrictions on our theory which respect the (anti-)BRST as well as the (anti-)co-BRST symmetries together. It is gratifying to state that we have discussed about the existence of CF-type restrictions from various points of view in our works [10,11]. In fact, we have been able to show the existence of CF-type restrictions: (i) from the symmetry considerations [10], (ii) from the superfield approach to BRST formalism [11], and (iii) from the algebraic considerations (in our present work). These works focus on the importance of CF-type restrictions in the discussion of the 2D non-Abelian theory.

As has been pointed out earlier, one of the key features of (anti-)co-BRST symmetry transformations is the observation that these transformations absolutely anticommute without any use of CF-type restrictions \( B \times C = 0 \) and \( \overline{B} \times \overline{C} = 0 \). However, the latter appear very elegantly when we discuss the absolute anticommutativity of the co-BRST and anti-co-BRST charges in the language of symmetry transformations and their generators [e.g. \( s_d Q_{ad} = -i \{Q_{ad}, Q_d\} = 0 \) and \( s_{ad} Q_d = -i \{Q_d, Q_{ad}\} = 0 \)]. This is a completely novel observation in our theory as it does not happen in the case of (anti-)BRST symmetry transformations and in their absolute anticommutativity requirement. In fact, in the latter case of symmetries [i.e. (anti-)BRST symmetries], the CF-condition is required for the proof of the absolute anticommutativity of the (anti-)BRST charges \( \{Q_b, Q_{ab}\} = 0 \) as well as the (anti-)BRST symmetries \( \{s_b, s_{ab}\} = 0 \) (cf. Appendix A).

As far as the property of absolute anticommutativity and the existence of the CF-type conditions is concerned, we would like to point out that the CF-type restrictions \( B \times C = 0 \) and \( \overline{B} \times \overline{C} = 0 \) are invoked from outside in the requirement of the absolute anticommutativity condition for the (anti-)co-BRST charges that are derived from the Lagrangian densities (1). However, these restrictions are not required in the case of the absolute anticommutativity requirement of the (anti-)co-BRST charges that are derived
from the Lagrangian densities (26). This happens because of the observation that the CF-type restrictions: $\mathcal{B} \times C = 0$ and $\mathcal{B} \times \bar{C} = 0$ become EOM for the Lagrangian densities (26). All the tower of restrictions that have been derived in [10,11] do not affect the d.o.f. counting for the gauge field because our present 2D non-Abelian gauge theory has been shown to be a new model of topological field theory where there are no propagating d.o.f. [7]. Furthermore, the CF-type restrictions are amongst the auxiliary fields and (anti-)ghost fields which do not directly affect the d.o.f. counting for the gauge field of our theory.

We have been able to show the existence of (anti-)BRST and (anti-)co-BRST symmetry transformations in the case of a 1D model of a rigid rotator [21]. However, the CF-type restriction, in the case of this 1D model is trivial (as is the case with the Abelian 1-form gauge theory without any interaction with matter fields [7]). The non-trivial CF-type restrictions appear in the cases of 6D Abelian 3-form and 4D Abelian 2-form gauge theories which have been shown to be the models for the Hodge theory within the framework of BRST formalism [5,6]. It would be a nice future endeavor for us to apply our present ideas of 2D non-Abelian 1-form theory to the above mentioned systems of physical interest. We are currently busy with these issues and our results would be reported in our future publications [27].

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Appendix A: On proof of $\{Q_b, Q_{ab}\} = 0$

In this Appendix, we discuss a few essential theoretical steps to provide a proof for the absolute anticommutativity of the conserved and nilpotent (anti-)BRST charges $Q_{(a)b}$. Towards this goal in mind, we observe (with the input $s_b Q_{ab} = -i \{Q_b, Q_{ab}\}$) the following:

$$s_b Q_{ab} = \int dx s_b \left[ \hat{B} \cdot \dot{C} - \hat{B} \cdot D_0 \bar{C} + \frac{1}{2} (\bar{C} \times \dot{C}) \cdot \hat{C} \right]. \quad (A.1)$$

Using the BRST transformations from Eq. (2), we obtain the following explicit mathematical expressions from the first term (on the r.h.s. of the above equation):

$$s_b (\hat{B} \cdot \dot{C}) = i (\hat{B} \times C) \cdot \dot{C} + i (\dot{B} \times \hat{C}) \cdot \bar{C} + i \hat{B} \cdot B. \quad (A.2)$$

The second term, on the r.h.s of (A.1), leads to

$$s_b (- B \cdot D_0 \bar{C}) = - i (B \times C) \cdot \dot{C} + (B \times C) \cdot (A_0 \times C) - i \dot{B} \cdot \hat{B}$$

$$\equiv -i \hat{B} \cdot (\dot{C} \times \bar{C}) + \hat{B} \cdot [(A_0 \times C) \times \bar{C} + \hat{B} \cdot (A_0 \times B)], \quad (A.3)$$
and the third term produces:

\[ s_b \left[ \frac{1}{2} (\dot{C} \times \dot{C}) \cdot \dot{C} \right] = i (B \times \dot{C}) \cdot \dot{C} - \frac{i}{2} (\dot{C} \times \dot{C}) \cdot (\dot{C} \times C). \]  

(A.4)

Now we are in the position to apply the Jacobi identity to expand \( B \cdot [(A_0 \times C) \times \dot{C}] \) and \( \frac{i}{2} [(\dot{C} \times \dot{C}) \cdot (\dot{C} \times C)] \). The outcome of these exercises yield:

\[ B \cdot [(A_0 \times C) \times \dot{C}] = -(A_0 \times B) \cdot (\dot{C} \times C) - (A_0 \times \dot{C}) \cdot (\dot{B} \times C), \]

\[ -\frac{i}{2} [(\dot{C} \times \dot{C}) \cdot (\partial_0 C \times X)] = i (\partial_0 C \times \dot{C}) \cdot (\dot{C} \times \dot{C}). \]  

(A.5)

The addition of all the terms with proper combinations, ultimately, leads to the following explicit equation, namely:

\[ i (\dot{C} \times \dot{C}) \cdot [B + \dot{B} + (C \times \dot{C})] - i B \cdot D_0[B + (\dot{C} \times C)] + i \dot{B} \cdot B \equiv 0 \equiv s_b Q_{ab}, \]  

(A.6)

We note that the application of the CF-condition [i.e. \( B + \dot{B} + (C \times \dot{C}) = 0 \)] produces the following:

\[ i \dot{B} \cdot \dot{B} + i \dot{B} \cdot B - i \dot{B} \cdot B = 0 \equiv s_b Q_{ab}, \]  

(A.7)

where we have used \(-i B \cdot D_0(B + C \times C) = B \cdot D_0 B \equiv i B \cdot \dot{C}.\)

In other words, we obtain the relationship: \( s_b Q_{ab} = -i \{Q_{ab}, Q_b\} = 0 \) (which is true only on the submanifold of fields in the quantum Hilbert space where the CF-condition \( B + \dot{B} + C \times \dot{C} = 0 \) is satisfied). This is a reflection of the fact that the absolute anticommutativity of the (anti-)BRST transformations \( \{s_b, s_{ab}\} A_u = 0 \) is true only when the CF-condition \( B + \dot{B} + C \times \dot{C} = 0 \) is imposed from outside. We conclude that the requirement of absolute anticommutativity condition for the nilpotent (anti-)BRST symmetry transformations is also reflected at the level of the requirement of the absolute anticommutativity property of the off-shell nilpotent (anti-)BRST charges. The CF-condition also appears in (22).

**Appendix B: On the derivation of \( Q_w \)**

In the main body of our text, we have derived the explicit expression for \( Q_w \) from the Noether conserved current \( (J_w) \). There is a simple way to obtain the same expression of \( (Q_w) \) where the ideas behind the symmetry principle (and concept of symmetry generator) play an important role. In this context, we note the following:

\[ s_d Q_b = -i \{Q_b, Q_d\} = -i Q_w, \quad s_b Q_d = -i \{Q_d, Q_b\} = -i Q_w. \]  

(B.1)

Thus, an explicit calculation of the l.h.s. [due to the transformations (2) and (4) as well as the expressions (13) and (17)] yields the correct expression for \( Q_w \). Let us, first of all, focus on the following:

\[ s_b Q_d = \int dx s_b [B \cdot \dot{C} + B \cdot \partial_t \dot{C}]. \]  

(B.2)

The first term produces the following explicit computations:

\[ s_b (B \cdot \dot{C}) = i (B \times \dot{C}) \cdot \dot{C} + i B \cdot \partial_0 B \]
\[ = i(\mathcal{B} \times C) \cdot \dot{\mathcal{C}} - i \mathcal{B} \cdot D_1 \mathcal{B} - i \mathcal{B} \cdot (\dot{\mathcal{C}} \times C) \equiv -\mathcal{B} \cdot D_1 \mathcal{B}, \]  

where we have used the EQM

\[ \partial_0 \mathcal{B} = -D_1 \mathcal{B} - (\dot{\mathcal{C}} \times C). \]  

The second term leads to

\[ s_b(i \mathcal{B} \cdot \partial_1 \mathcal{C}) = i \mathcal{B} \cdot \partial_1 \mathcal{B}. \]  

The addition of both the terms yield,

\[ s_b Q_d = -i \{Q_d, Q_b\} = -i \int dx \left[ \mathcal{B} \cdot D_1 \mathcal{B} - \mathcal{B} \cdot \partial_1 \mathcal{B} \right]. \]  

which, ultimately, leads to the derivation of \( Q_w \) [cf. Eq. (20)].

Now we dwell a bit on the anticommutator \( s_d Q_b = -i \{Q_b, Q_d\} = -i Q_w \). In this connection, we have to use the symmetry transformations (4) and expression (13). In other words, we compute the following:

\[ s_d Q_b = \int dx \; s_d \left[ \mathcal{B} \cdot D_1 \mathcal{C} + \mathcal{B} \cdot D_0 \mathcal{C} + \frac{1}{2} \dot{\mathcal{C}} \cdot (C \times C) \right]. \]  

The first term, using the partial integration and dropping the total space derivative term, can be written in a different looking form (i.e. \( \mathcal{B} \cdot D_1 \mathcal{C} = -D_1 \mathcal{B} \cdot C \)). Now application of \( s_d \) on the latter form, leads to the following explicit computation:

\[ s_d (-D_1 \mathcal{B} \cdot C) = i \mathcal{B} \cdot D_1 \mathcal{B} + i (\dot{\mathcal{C}} \times \mathcal{B}) \cdot C. \]  

From the second and third terms of (B.7), we obtain

\[ s_d (\mathcal{B} \cdot D_0 \mathcal{C}) = -\mathcal{B} \cdot \partial_0 \mathcal{B} + i \mathcal{B} \cdot (\partial_1 \mathcal{C} \times C) - \mathcal{B} \cdot (A_0 \times \mathcal{B}), \]

\[ s_d \left[ \frac{1}{2} \dot{\mathcal{C}} \cdot (C \times C) \right] = i \dot{\mathcal{C}} \cdot (\mathcal{B} \times C). \]  

Now, by using the equation of motion

\[ D_0 \mathcal{B} = \partial_1 \mathcal{B} + (\partial_1 \mathcal{C} \times C), \]

we observe that the sum of (B.8), (B.9) and (B.10) leads to the equality \([-i \mathcal{B} \cdot \partial_1 \mathcal{B}]\). Thus, ultimately, we obtain the following

\[ s_d Q_b = -i \{Q_b, Q_d\} = -i Q_w, \]

where \( (Q_w) \) [cf. Eq. (20)] is \( Q_w = i \int dx \left[ \mathcal{B} \cdot D_1 \mathcal{B} - \mathcal{B} \cdot \partial_1 \mathcal{B} \right]. \) Thus, we have derived the precise form of \( Q_w \) by using the ideas of continuous symmetries and their corresponding generators. Thus, there are two distinct ways to derive \( Q_w \).
Appendix C: On the (anti-)BRST symmetries of $L_B^{(3)}$ and $L_B^{(5)}$

We have observed earlier that the coupled Lagrangian densities (26) respect five perfect symmetries individually. As far as the (anti-)BRST symmetries are concerned, we have noted that $L_B^{(3)}$ respects perfect BRST symmetries, listed in (2), along with $s_b \lambda = 0$. We discuss here the anti-BRST symmetry of this Lagrangian density. It can be checked that, under the anti-BRST symmetry transformations (2) along with $s_{ab} \lambda = -i (\lambda \times \vec{C})$, we have the following transformation for the Lagrangian density $L_B^{(3)}$:

$$s_{ab} L_B^{(3)} = \partial_\mu [ - (\vec{B} + C \times \vec{C}) \cdot \partial^\mu \vec{C} ] + (B + \vec{B} + C \times \vec{C}) \cdot D_\mu \partial^\mu \vec{C} - i \vec{\lambda} \cdot (\bar{B} \times \{ \vec{B} + (C \times \vec{C}) \}).$$

If we implement the CF-condition $B + \vec{B} + (C \times \vec{C}) = 0$, we obtain the following, from the above transformation of $L_B^{(3)}$, namely;

$$s_{ab} L_B^{(3)} = \partial_\mu [ B \cdot \partial^\mu \vec{C} ] + i \vec{\lambda} \cdot (\bar{B} \times B).$$

For the anti-BRST invariance, we impose a new CF-type restrictions [i.e. $\vec{\lambda} \cdot (\bar{B} \times B) = 0$] which involves three auxiliary fields. As a consequence, this restriction can be equivalent to the following three individual constraints in terms of only two auxiliary fields, namely:

$$\vec{\lambda} \cdot (\bar{B} \times B) = 0 \implies \bar{B} \times B = 0, \quad \lambda \times B = 0, \quad \lambda \times \bar{B} = 0.$$  \hspace{1cm} (C.3)

The above restrictions have been derived from the symmetry point of view [10] as well as by using the augmented version of superfield formalism [11]. It is gratifying to note that the (anti-)BRST symmetry transformations (that include transformations on $\lambda$ and $\vec{\lambda}$) absolutely anticommute if we consider the above restrictions.

Now we focus on the Lagrangian density $L_B^{(3)}$. It has perfect anti-BRST invariance with $s_{ab} \lambda = 0$ [and $s_{ab}(B \times \vec{C}) = 0$]. We discuss here the application of BRST transformation (2), along with $s_b \lambda = -i (\lambda \times C)$, on $L_B^{(3)}$. This exercise leads to the following;

$$s_b L_B^{(3)} = \partial_\mu [( B + C \times \vec{C}) \cdot \partial^\mu C] - (B + \vec{B} + C \times \vec{C}).D_\mu \partial^\mu C$$

$$- i \lambda \cdot [ \bar{B} \times \{ B + (C \times \vec{C}) \} ].$$

If we impose the usual CF-condition $B + \vec{B} + (C \times \vec{C}) = 0$ from outside on (C.4), we obtain the following from the above transformation of $L_B^{(3)}$, namely;

$$s_b L_B^{(3)} = \partial_\mu [ - \vec{B} \cdot \partial^\mu C ] + i \lambda \cdot (\bar{B} \times \bar{B}).$$

Thus, for the BRST invariance of the action integral $S = \int d^2 x \ L_B^{(3)}$, we invoke another CF-type restriction:

$$\lambda \cdot (B \times \bar{B}) = 0 \implies B \times \bar{B} = 0, \quad \lambda \times B = 0, \quad \lambda \times \bar{B} = 0.$$  \hspace{1cm} (C.6)

Because it transform to a total spacetime derivative (i.e. $s_b L_B^{(3)} = \partial_\mu [B \cdot D^\mu C]$).
In the above, we have noted that there are two constraint restrictions (i.e. \( B + \bar{B} + C \times \bar{C} \), \( \lambda \cdot (B \times \bar{B}) = 0 \)) that ought to be invoked for the BRST invariance of the action integral. It is clear that the latter restriction involves three auxiliary fields. However, this restriction actually corresponds to three CF-type restrictions that have been written in (C.6). The latter three CF-type restrictions are correct as they have been derived from the symmetry consideration in [10]. It is gratifying to state, at this juncture, that the restrictions, listed in (C.3) and (C.6), are required for the absolute anticommutativity of the (anti-)BRST symmetries (2) along with \( s_b \lambda = -i (\lambda \times C) \) and \( s_{ab} \bar{\lambda} = -i (\bar{\lambda} \times \bar{C}) \).

References

[1] C. Becchi, A. Rouet, R. Stora, *Phys. Lett. B* 32, 344 (1974).
[2] C. Becchi, A. Rouet, R. Stora, *Commun. Math. Phys.* 42, 127 (1975).
[3] C. Becchi, A. Rouet, R. Stora, *Ann. Phys. (N. Y.)* 98, 287 (1976).
[4] I. V. Tyutin, Lebedev Institute Report, Preprint FIAN-39, 1975 (Unpublished).
[5] R. P. Malik, *Int. J. Mod. Phys. A* 22, 3521 (2007).
[6] R. Kumar, S. Krishna, A. Shukla, R. P. Malik, *Int. J. Mod. Phys. A* 29, 1450135 (2014).
[7] R. P. Malik, *J. Phys. A: Math. Gen.* 34, 4167 (2001).
[8] E. Witten, *Nucl. Phys. B* 202, 253 (1982).
[9] A. S. Schwarz, *Lett. Math. Phys.* 2, 217 (1978).
[10] N. Srinivas, S. Kumar, B. K. Kureel, R. P. Malik, *Int. J. Mod. Phys. A* 32, 1750136 (2017) A 32: 1750193.
[11] N. Srinivas, R. P. Malik, *Int. J. Mod. Phys. A* 32, 1750193 (2017).
[12] G. Curci, R. Ferrari, *Phys. Lett. B* 63, 91 (1976).
[13] N. Nakanishi, I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific, Singapore, 1990).
[14] K. Nishijima, *Czech. J. Phys.* 46, 1 (1996).
[15] D. Dudal, V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, M. Picariello, *JHEP* 0212, 008 (2002).
[16] D. Dudal, H. Verschelde, V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, M. Picariello, A. Vicini, J. A. Gracey, *JHEP* 0306, 003 (2003).
[17] R. Kumar, S. Gupta, R. P. Malik, *Int. J. Theor. Phys.* 55, 2857 (2016).
[18] T. Eguchi, P. B. Gilkey, A. Hanson, *Phys. Rep.* 66, 213 (1980).

[19] S. Mukhi, N. Mukunda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern Private Limited, New Delhi, 1990).

[20] S. Gupta, R. P. Malik, *Eur. Phys. J. C* 58, 517 (2008).

[21] R. P. Malik, *Int. J. Mod. Phys. A* 5, 1685 (1998).

[22] S. Gupta, R. P. Malik, *Eur. Phys. J. C* 68, 325 (2010).

[23] L. Bonora, R. P. Malik, *Phys. Lett. B* 655, 75 (2007).

[24] L. Bonora, R. P. Malik, *J. Phys. A: Math. Theor.* 43, 375 (2010).

[25] P. A. M. Dirac, *Lectures on Quantum Mechanics, Belfer Graduate School of Science* (Yeshiva University Press, New York, 1964).

[26] K. Sundermeyer, *Constrained Dynamics: Lecture Notes in Physics, Vol. 169* (Springer, Berlin, 1982).

[27] R. P. Malik, et al., in preparation.