We prove that the inclusive single-gluon production cross section for a hadron colliding with a high-density target factorizes into the gluon distribution function of the projectile, defined as usual within the DGLAP collinear approximation, times the cross section for scattering of a single gluon on the strong classical color field of the target. We then derive the gluon-proton (nucleus) inelastic cross section and show that it is (up to logarithms) infrared safe and that it grows slowly with center of mass energy. Furthermore, we discuss jet transverse momentum broadening for the case of nuclear targets. We show that in the saturation regime, in contrast to the perturbative regime, the width of the transverse momentum distribution is infrared finite and grows rapidly with energy and rapidity. In both regimes, however, transverse momentum broadening exhibits the same $A$ dependence.

I. INTRODUCTION

Understanding the behavior of hadronic cross sections at very high energy is one of the major unresolved problems in QCD. Even though Regge theory can, in principle, predict the energy dependence of hadronic cross sections and there are many phenomenological models, based on Regge theory, which are somewhat successful in describing the data, the relation of Regge theory to QCD is not well understood. Therefore, it would be very important to be able to calculate the high energy behavior of hadronic cross section from QCD itself. However, it is believed that total cross sections are intrinsically non-perturbative and not amenable to perturbative QCD methods. In this note, we consider the simpler problem of (real) gluon-proton total inelastic cross section in the very high energy $(x)$ limit, using the effective action and classical field method developed recently, and show that its growth with energy is inhibited as compared with that expected from perturbative QCD.

At very small $x$, a hadron is a Color Glass Condensate due to the condensation of gluons into a coherent state with characteristic momentum of $Q_s(x)$ [2,3,4,5]. In other words, most of the gluons in the wave function of a hadron have momenta of order $Q_s(x)$. As we go to higher energies, $Q_s(x)$ increases and eventually will become much larger than $\Lambda_{QCD}$ so that $\alpha_s(Q_s) \ll 1$ and weak coupling methods can be used. Even though the theory may be weakly coupled, it is still non-perturbative in the sense that one has high gluon densities and strong classical color fields associated with them so that the standard perturbative QCD breaks down. Much progress has been made in developing a formalism which describes this weakly coupled, though non-perturbative region of QCD. It generalizes the standard collinear factorized (leading twist) expressions for hadronic cross sections and allows one to calculate hadronic cross sections in an environment where higher twist (high gluon density at small $x$) effects are important. We refer the reader to [6] and references therein for a review of this formalism.

One can use this classical field method to calculate gluon production at high energy [7]. In this Letter, we use the results of [8] to prove (collinear) factorization of the inclusive cross section for a “dilute” hadron impinging on a dense target. Based on that result we then derive an expression for the gluon-proton (nucleus) inelastic cross section and for its energy dependence at very high energies. Using our results for the gluon-proton (nucleus) inelastic cross section, we consider the transverse momentum broadening of gluons due to scattering from the strong classical field of a nucleus and show that it is infrared finite, energy dependent and scales like $A^{1/3}$.

II. GLUON-PROTON INCLUSIVE CROSS SECTION

In the classical field and effective action approach to hadronic (nuclear) collisions at high energy, one solves the classical Yang-Mills equations of motion in the presence of random color charges created by the valence quarks and gluons at high $x$. One then averages over these color charges with a Gaussian weight to compute physical quantities. The classical fields of the colliding hadrons (nuclei) before the collision are given by the single-hadron (nucleus) solutions which, in light cone gauge, are

$$A_{1,2}^\pm = 0 ,$$

$$A_{1,2}^i = \frac{i}{g} U_{1,2}(x_\perp) \partial^i U_{1,2}^\dagger(x_\perp) .$$

(1)
These fields serve as the initial conditions for solving the Yang-Mills equations of motion in the forward light cone; they were solved in \[8\] for the case of asymmetric collisions where the classical field of one of the colliding sources is much stronger than the classical field of the other source.

![Diagram of proton interactions](image)

**Fig. 1.** Gluon production in inclusive pp scattering at rapidity far from the target proton (i.e., the strong color field).

The produced gluon field in the forward light cone region is given by

\[
A^i(\tau, x_\perp) = U(x_\perp) \left( \beta^i(\tau, x_\perp) + \frac{i}{g} \partial^\mu \right) U^\dagger(x_\perp),
\]

\[
A^{\pm}(\tau, x_\perp) = \pm x^\pm U(x_\perp) \beta(\tau, x_\perp) U^\dagger(x_\perp).
\]

We have chosen the gauge condition \(x^+ A^- + x^- A^+ = 0\). Here, \(\tau = \sqrt{2x^+x^-}\) denotes proper time and \(x_\perp\) is the transverse coordinate. The \(U\)'s are rotation matrices in color space, to be specified shortly. At asymptotic times, \(\tau \to \infty\), the fields \(\beta\) and \(\beta^i\) are given by superpositions of plane wave solutions,

\[
\beta(\tau \to \infty, x_\perp) = \int \frac{d^2p_\perp}{(2\pi)^2} \frac{1}{\sqrt{2\omega \tau^3}} \{a_1(p_\perp) e^{ip_\perp \cdot x_\perp - i\omega \tau} + c.c.\},
\]

\[
\beta^i(\tau \to \infty, x_\perp) = \int \frac{d^2p_\perp}{(2\pi)^2} \frac{1}{\sqrt{2\omega \tau} \cdot \omega} \{a_2(p_\perp) e^{ip_\perp \cdot x_\perp - i\omega \tau} + c.c.\}.
\]

The number distribution of produced gluons at rapidity \(y\) and transverse momentum \(p_\perp\) is given by

\[
\frac{dN}{d^2p_\perp dy} = \frac{2}{(2\pi)^2} \left| a_1(p_\perp) \right|^2 + \left| a_2(p_\perp) \right|^2.
\]

One can obtain an analytical solution of the classical Yang-Mills equations if one of the fields (the “projectile”) is much weaker than the other (the “target”). This situation is realized physically when \(|y - y_t| \gg |y - y_s|\), because of the renormalization group evolution of the gluon density in rapidity. In that case, it turns out [1] that the \(U\)'s appearing in eq. (3) are just the \(U_2\)'s from eq. (2); that is, to leading order in the weak field, the plane waves in the forward light cone are just gauge rotated by the strong field

\[
U_2(x_\perp, y) = \mathcal{P} \exp \left( -ig \int \frac{dy'}{y} dy' \Phi_2(x_\perp, y') \right).
\]

Here, we assumed that the target is moving along the negative \(z\)-axis, i.e. at negative rapidity \(y_t < 0\). In the Color Glass Condensate model, we now have to average the squared amplitudes over the gauge potentials \(\Phi_{1,2}(x_\perp, y)\) using a Gaussian weight [2,3,4]:

\[
\langle |a_{1,2}|^2 \rangle = \int D\Phi_1 D\Phi_2 |a_{1,2}(\Phi_1, \Phi_2)|^2 \times \exp \left[ -\int_{y_t}^{y} dy' \int d^2x_\perp \frac{\text{tr}(\nabla_\perp^2 \Phi_1(x_\perp, y'))^2}{g^2 \mu_1^2(x_\perp, y')} - \int_{y_t}^{y} dy' \int d^2x_\perp \frac{\text{tr}(\nabla_\perp^2 \Phi_2(x_\perp, y'))^2}{g^2 \mu_2^2(x_\perp, y')} \right].
\]
Here, $\mu$ denotes the density of color charge in the sources per unit of transverse area and rapidity. The radiation number distribution (8) turns out to depend only on the integrated color charge densities of the sources,

$$\chi_1(y, p_{1\perp}^2) = \int \frac{dy'}{y'} \mu_1^2(y', p_{1\perp}^2) , \quad \chi_2(y, p_{1\perp}^2) = \int \frac{dy'}{y'} \mu_2^2(y', p_{1\perp}^2) .$$

(8)

Due to the evolution in rapidity, $\chi_1(y) \ll \chi_2(y)$ if $|y - y'| \gg |y - y_0|$. In terms of the gluon distribution function in the projectile or target proton $\rho \ell$, respectively,

$$\chi_{1,2}(y, p_{1\perp}^2) = \frac{N_c}{N_c^2 - 1} \frac{1}{\pi R^2} \int_{x_0}^1 dx g_{\rho \ell}(x, p_{1\perp}^2) ,$$

(9)

where $x_0$ is on the order of $\sim p_{\perp} \cosh(y)/\sqrt{s}$. In the weak-projectile limit the averaging $\langle \rangle$ over the gauge potentials with Gaussian weight can be done. At high transverse momentum, $p_{\perp} \gg Q_t$, where $Q_t$ is the saturation momentum of the target, one recovers the standard result from perturbation theory. Integrating over the impact parameter space (simply a factor of $\pi R^2$), one recovers the standard result from perturbation theory.

$$\frac{dN}{dp_{\perp}^2 dy} = \frac{4\alpha_s^2}{\pi R^2} \frac{N_c^2}{N_c^2 - 1} \frac{1}{p_{\perp}^2} \int \frac{dx g_{\rho \ell}(x, p_{\perp}^2)}{k_{1\perp}^2} \int \frac{dx' g_{\rho}(x', k_{2\perp}^2)}{k_{2\perp}^2} \int \frac{dx'' g_{\rho}(x'', (p_{\perp} - k_{1\perp})^2)}{k_{1\perp}^2} .$$

(10)

This result is valid to leading order in $\alpha_s^2 \chi_1$, but to all orders in $\alpha_s^2 \chi_2$. The second line applies in the limit of nearly collinear splitting ($k_{1\perp}/p_{\perp} \rightarrow 0$), where we can employ eq. (11). Thus, the inclusive gluon production again factorizes into the gluon distribution function of the projectile at the scale $p_{1\perp}^2$ times the cross section for scattering of a gluon on the high-density target.
To obtain the gluon-proton cross section, we again integrate over the impact parameter and divide by the flux of incoming collinear gluons in the projectile at the scale $p_{t}^{2}$, which is $xg_{p}(x, p_{t}^{2})$. We get

$$\frac{d\sigma_{gp}^{\text{sat}}}{dp_{t}^{2}} = \frac{1}{4\pi} \frac{1}{p_{t}^{2}} \pi R^{2}$$

(16)

In the saturation regime, the cross section is of order 1 rather than $\alpha_{s}^{2}$ as in [13] because the occupation number of the target $\sim 1/\alpha_{s}$ cancels one power of the coupling in both the amplitude and the complex conjugate amplitude. Note that the above result holds for $p_{t} \gg \Lambda_{QCD}$ only. To get the gluon-proton total inclusive cross section, we integrate [14] which gives

$$\sigma_{gp}^{\text{sat}} \simeq \frac{1}{4} \pi R^{2} \log \left( Q_{t}^{2}/\Lambda_{QCD}^{2} \right) + O(\alpha_{s}^{2}) .$$

(17)

Parameterization of the saturation momentum like a power of energy, $Q_{t}^{2} \sim s^{\gamma}$ as done in [13] then leads to a logarithmically growing cross section

$$\sigma_{gp} \sim \pi R^{2} \log s .$$

(18)

On the other hand, if we use a DLA DGLAP type parameterization $Q_{t} \sim \exp \sqrt{\log 1/x}$, the cross section would grow like a square root of energy. Therefore, it is clear that different parameterization of the target saturation scale $Q_{t}$ would lead to a different growth of the cross section with energy. Curiously enough, assuming $Q_{t}$ to grow like $s^{\gamma}$ would lead to growth of the gluon-proton cross section like $s^{\gamma}$. In order to determine the energy dependence of $Q_{t}$ rigorously, one would need to formally define it in terms of a gluonic two-point function and solve the non-linear evolution equation that $Q_{t}$ follows. This is however beyond the scope of this work.

Strictly speaking, our results are correct for the cross section per unit area, $d\sigma/dFb$. What we have shown here is that the growth of this cross section at a fixed impact parameter is inhibited due to high gluon density effects. In principle, as we go to larger impact parameters $b$, the gluon density becomes smaller and smaller until our classical formalism reduces to the standard pQCD where the gluon density is not that large. Since large impact parameters (where the gluon density is low and so the classical approach is not valid) are believed to give the dominant contribution to the total cross section, one should understand the area appearing in [13] as a parameter which we can not precisely determine. A genuine non-perturbative (strong coupling vs. our weak coupling methods) calculation would presumably enable one to determine this factor and its energy dependence.

The above equations hold equally well for a nuclear target. In eq. [13] one just has to replace the gluon distribution function of the target proton by that for the nucleus; in the absence of shadowing [14-16], $g_{A}(x, Q^{2}) = Ag_{0}(x, Q^{2})$. On the other hand, in eqs. [16-17] one substitutes the radius of the proton by that of the nucleus, $R_{A} \simeq A^{1/3} R$.

It is now straightforward to compute the transverse momentum broadening of the incoming gluon jet traversing the target nucleus [14-17]. The transverse momentum broadening is given by [14]

$$\langle p_{t}^{2} \rangle = \frac{1}{A} \langle t_{A}(\bar{b}) \rangle \int d^{2}p_{t} \frac{d\sigma_{A}}{dp_{t}^{2}} ,$$

(19)

where $\langle t_{A}(\bar{b}) \rangle$ is the nuclear thickness function averaged over impact parameters:

$$\langle t_{A}(\bar{b}) \rangle = \frac{A}{\pi R_{A}^{2}} = \rho L ,$$

(20)

with $L$ the average thickness of the nucleus, and with $\rho \simeq 0.15 \text{ fm}^{-3}$ the density of nucleons in the nucleus.

In perturbation theory, using [13] and DGLAP [13] evolution,

$$\frac{\alpha_{s} N_{c}}{\pi} \int dx g_{A}(x, p_{t}^{2}) = \frac{d}{d \log p_{t}^{2}} xg_{A}(x, p_{t}^{2}) ,$$

(21)

one obtains

$$\langle p_{t}^{2} \rangle = 4\pi^{2} \alpha_{s} \frac{N_{c}}{N_{c}^{2} - 1} \left[ xg(x, Q_{\text{max}}^{2}) - xg(x, Q_{\text{min}}^{2}) \right] \rho L .$$

(22)

Thus, $\langle p_{t}^{2} \rangle$ grows proportional to $A^{1/3}$ if nuclear shadowing is disregarded [14-17]. Appearance of $xg(x, Q_{\text{min}}^{2})$ signifies sensitivity to physics at small momentum transfer. For a discussion of what value of $x$ is to be used in (22), see [14-17].
Transverse momentum broadening in the saturation regime is quite different. Using (16) and (19,20), we find that
\[
\langle p_T^2 \rangle \simeq \frac{Q_A^2}{4} + O(\alpha_s),
\] (23)
where \( Q_A^2 \) denotes the nuclear saturation scale which, from the definition of \( \chi_2 \sim Q_A^2/\alpha_s^2 \) in (3), is \( A^{1/3} \) times larger than that for a proton. Thus, in the saturation regime jet broadening grows with the same power of the nuclear mass number \( A \) as in the perturbative regime \( \langle p_T^2 \rangle \sim A^{1/3} \). Also, our derivation shows that the jet transverse momentum broadening is energy dependent even though we can not determine its energy dependence in a model independent way. However, for \( Q_A^2 \sim s^7 \) [15], we obtain a strong power-law increase of transverse momentum broadening. We would like to emphasize [18] that since the saturation scale \( Q_t^2 \) of the target is expected to be much larger in the forward rapidity (projectile fragmentation) region, the jet transverse momentum broadening will be much larger in the forward region than in the central rapidity region, contrary to the prediction [22] from perturbation theory.

III. CONCLUSION

In summary, we have shown that inclusive gluon production from a hadron scattering off a high-density target (with “saturated” gluon density) factorizes into the gluon distribution of the projectile times the cross section of the beam of incoming collinear gluons on the dense target. We derived the gluon-proton inclusive cross section in the high energy limit. We have shown that the cross section grows, with reasonable, HERA compatible parameterizations [15] of the saturation momentum, only logarithmically with energy rather than power like as expected from perturbation theory. This “unitarization” of the cross section is due to the strong classical fields of the target generated by the high gluon density (higher twist) effects. We have also considered the transverse momentum broadening of the gluon jet passing through a nuclear target. We have shown that it scales like \( A^{1/3} \) in both perturbative and saturation regimes and that it is infrared finite. We predict that the jet transverse momentum broadening will be larger in the forward rapidity region and that it increases with energy.

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