A Duality Theorem for Quantum Groupoids

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Abstract. We prove a duality theorem for quantum groupoid (weak Hopf algebra) actions that extends the well-known result for usual Hopf algebras obtained in [BM] and [vdB].

1. Introduction

By \textit{(finite) quantum groupoids} we understand weak Hopf algebras introduced in [BNSz, BSz] as a generalization of ordinary Hopf algebras providing a good framework for studying symmetries of certain quantum field theories. These objects also generalize both ordinary groupoid algebras and their duals. A special case of quantum groupoids with involutive antipode was studied in [NV1, N].

Finite quantum groupoids naturally arise in the theory of von Neumann algebras: it was shown in [NV2] that finite index II$_1$ subfactors of depth $\leq 2$ can be characterized as $C^*$-quantum groupoid smash products. This result was extended in [NV3], where a uniform description of all finite depth subfactors was obtained via a Galois correspondence. In fact, one can use subfactors in order to construct interesting concrete examples of quantum groupoids such as Temperley-Lieb algebras [NV2, NV3].

Another motivation to study quantum groupoids comes from the fact that their representation theory provides examples of monoidal categories that can be used for constructing invariants of links and 3-manifolds [NVT].

In this paper we prove the following duality theorem for smashed products: if $H$ is a finite quantum groupoid and $A$ is an $H$-module algebra, then $(A\#H)\#H^* \cong \End(A\#H)_A$, where $H^*$ acts on $A\#H$ in a dual way and $A\#H$ is viewed as a right $A$-module via multiplication. For usual Hopf algebras this result was proved in [BM] (where infinite dimensional case was considered) and [vdB]. For weak Kac algebras (i.e., finite $C^*$-quantum groupoids with an involutive antipode) it was established in [N].

The note is organized as follows.

In Preliminaries (Section 2) we recall definitions and basic facts concerning finite quantum groupoids (weak Hopf algebras) and prove identities we need for later computations.

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In Section 3 we prove the main result by writing down explicit formulas for isomorphism between $(A\# H)\# H^*$ and $\text{End}(A\# H)_A$. As a corollary we obtain that $H\# H^*$ is always a semisimple algebra.

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2. Preliminaries

Let $k$ be a field.

Throughout this paper we use Sweedler’s notation for comultiplication, writing $\Delta(b) = b^{(1)} \otimes b^{(2)}$.

DEFINITION 2.1. By a weak Hopf algebra $\text{BNSz}$, or finite quantum groupoid we understand a finite dimensional $k$-vector space $H$ that has structures of algebra $(H, m, 1)$ and coalgebra $(H, \Delta, \varepsilon)$ related as follows:

1. $\Delta$ is a (not necessarily unit-preserving) homomorphism:
   $$\Delta(hg) = \Delta(h)\Delta(g),$$

2. The unit and counit satisfy the identities
   $$\varepsilon(hg) = \varepsilon(hg^{(1)})\varepsilon(g^{(2)}),$$
   $$\Delta \otimes \text{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),$$

3. There is a linear map $S : H \rightarrow H$, called an antipode, such that
   $$m(\text{id} \otimes S)\Delta(h) = (\varepsilon \otimes \text{id})(\Delta(1)(h \otimes 1)),$$
   $$m(S \otimes \text{id})\Delta(h) = (\text{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)),$$
   $$S(h^{(1)})h^{(2)}S(h^{(2)}) = S(h),$$

for all $h, g, f \in H$.

The antipode is unique and invertible $\text{BNSz}$, moreover it is an anti-algebra and anti-coalgebra map. The right-hand sides of two first formulas in (3) are called target and source counital maps and denoted $\varepsilon_t, \varepsilon_s$ respectively:

$$\varepsilon_t(h) = (\varepsilon \otimes \text{id})(\Delta(1)(h \otimes 1)),$$
$$\varepsilon_s(h) = (\text{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)).$$

The counital maps $\varepsilon_t$ and $\varepsilon_s$ are idempotents in $\text{End}_k(H)$, we also have relations $S \circ \varepsilon_t = \varepsilon_s \circ S$ and $S \circ \varepsilon_s = \varepsilon_t \circ S$.

The main difference between quantum groupoids and Hopf algebras is that the ranges of counital maps are, in general, separable subalgebras of $H$ not necessarily equal to $k$. They are called target and source counital subalgebras and play a role of “non-commutative bases” (cf. Example 2.2 below):

$$H_t = \{h \in H \mid \varepsilon_t(h) = h\} = \{h \in H \mid \Delta(h) = 1_{(1)}h \otimes 1_{(2)} = h1_{(1)} \otimes 1_{(2)}\},$$
$$H_s = \{h \in H \mid \varepsilon_s(h) = h\} = \{h \in H \mid \Delta(h) = 1_{(1)} \otimes h1_{(2)} = 1_{(1)} \otimes 1_{(2)}h\}.$$

The counital subalgebras commute, the restriction of the antipode gives an anti-isomorphism between $B_t$ and $B_s$, moreover, $B_t$ (resp. $B_s$) is a left (resp. right) coideal subalgebra of $B$. We also have $S \circ \varepsilon_t = \varepsilon_s \circ S$, $S^2|_{B_t} = \text{id}_{B_t}$, and $S^2|_{B_s} = \text{id}_{B_s}$.
Note that $H$ is an ordinary Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$ if and only if $\varepsilon$ is a homomorphism if and only if $H_k = H_k = k$.

The dual vector space $H^\ast$ has a natural structure of a quantum groupoid with the structure operations dual to those of $H$:

$$<\phi \psi, h> = <\phi \otimes \psi, \Delta(h)>,$$

$$<\Delta(\phi), h \otimes g> = <\phi, hg>,$$

$$<S(\phi), h> = <\phi, S(h)>,$$

for all $\phi, \psi \in H^\ast$, $h, g \in H$. The unit of $H^\ast$ is $\varepsilon$ and counit is $\phi \mapsto <\phi, 1>$.

**Example 2.2.** Let $G$ be a finite groupoid (a category with finitely many morphisms, such that each morphism is invertible) then the groupoid algebra $kG$ (generated by morphisms $g \in G$ with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a quantum groupoid via:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G.$$

The dual quantum groupoid $(kG)^\ast$ is generated by idempotents $p_g$, $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$ and

$$\Delta(p_g) = \sum_{u,v = g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,g^{-1}}, \quad S(p_g) = p_g^{-1}.$$

It is known that any group action on a set gives rise to a finite groupoid $[R]$. Similarly, in the “quantum” situation, one can associate a weak Hopf algebra (quantum groupoid) with every action of a usual Hopf algebra on a separable algebra, see $[NV1]$ for details.

Finally, the most non-trivial examples of quantum groupoids known so far come from the theory of von Neumann II$_1$ subfactors $[GHJ]$ : in $[NV2]$ finite index subfactors of depth $\leq 2$ were characterized as quantum groupoid smash products and it was explained in $[NV3]$ that it is possible to construct concrete examples of quantum groupoids from subfactors of arbitrary finite depth.

An algebra $A$ is a left $H$-module algebra $[NSzW]$ if $A$ is a left $H$-module via $h \otimes x \mapsto h \cdot x$ and

$$h \cdot (xy) = (h_{(1)} \cdot x)(h_{(2)} \cdot y), \quad h \cdot 1 = \varepsilon_t(h) \cdot 1,$$

for all $h \in H$, $x, y \in A$.

A smash product algebra $A \# H$ of $A$ and $H$ is defined on a $k$-vector space $A \otimes_{H_t} H$ (relative tensor product), where $H$ is a left $H_t$-module via multiplication and $A$ is a right $H_t$-module via

$$x \cdot z = S(z) \cdot x = x(z \cdot 1).$$

Let $x \# h$ be the class of $x \otimes h$ in $A \otimes_{H_t} H$, then the multiplication of $A \# H$ is given by the familiar formula:

$$(x \# h)(y \# g) = x(h_{(1)} \cdot y) \# h_{(2)} g, \quad x, y \in A, \ h, g \in H$$

and the unit of $A \# H$ is $1 \# 1$.

**Example 2.3.** The target counital subalgebra $H_t$ is a trivial $H$-module algebra with the action of $H$ given by $h \cdot z = \varepsilon_t(hz)$, where $h \in H$, $z \in H_t$.

The dual quantum groupoid $H^\ast$ is an $H$-module algebra via

$$h \mapsto \phi = \phi(h_{(1)}) <\phi(h_{(2)}), h>,$$
for all \( h \in H, \phi \in H^* \).

In the following Lemma we collect the identities we will use in what follows. They can be found in [BNSz] and [NV1], we include them here for the convenience of the reader.

**Lemma 2.4.** For every quantum groupoid \( H \) and elements \( h, z \in H_t \) the following identities hold true:

(i) \( h_{(1)} \otimes \varepsilon_t(h_{(2)}) = 1_{(1)}h \otimes 1_{(2)} \) and \( \varepsilon_t(h_{(1)}) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)} \),

(ii) \( 1_{(1)}S(z) \otimes 1_{(2)} = 1_{(1)} \otimes 1_{(2)}z \),

(iii) \( h_{(2)}S^{-1}(h_{(1)}) \otimes h_{(3)} = S(\varepsilon_t(h_{(1)}) \otimes h_{(2)}) = 1_{(1)} \otimes 1_{(2)}h \).

**Proof.** (i) We have:

\[
\begin{align*}
    h_{(1)} \otimes \varepsilon_t(h_{(2)}) & = h_{(1)}\varepsilon_t(1_{(1)}h_{(2)}) \otimes 1_{(2)} \\
    & = 1_{(1)}h_{(1)}\varepsilon_t(1_{(2)}h_{(2)}) \otimes 1_{(3)} = 1_{(1)}h \otimes 1_{(2)},
\end{align*}
\]

where we used the definition of \( \varepsilon_t \) and the axiom (2) of Definition 2.1. The second identity is similar.

(ii) Since \( S(z) \in H_s \) we can compute:

\[
\begin{align*}
    1_{(1)}S(z) \otimes 1_{(2)} & = S(z)_{(1)} \otimes \varepsilon_t(S(z)_{(2)}) \\
    & = 1_{(1)} \otimes \varepsilon_t(1_{(2)}S(z)) \\
    & = 1_{(1)} \otimes \varepsilon_t(1_{(2)}z) = 1_{(1)} \otimes 1_{(2)}z,
\end{align*}
\]

using part (i), definition of the source counital subalgebra, and the identity \( \varepsilon_t(hg) = \varepsilon_t(h)\varepsilon_t(g) \) that follows from axiom (2) of Definition 2.1. Observe that \( S(1_{(1)}) \otimes 1_{(2)} \) is a separability idempotent \([P]\) of \( H_t \).

(iii) Using part (ii) and the fact that \( H_s \) and \( H_t \) commute, we have

\[
\begin{align*}
    h_{(2)}S^{-1}(h_{(1)}) \otimes h_{(3)} & = S(\varepsilon_t(h_{(1)})) \otimes h_{(2)} \\
    & = S(\varepsilon_t(1_{(1)}h_{(1)})) \otimes 1_{(2)}h_{(2)} \\
    & = 1_{(1)}S(\varepsilon_t(h_{(1)})) \otimes 1_{(2)}h_{(2)} \\
    & = 1_{(1)} \otimes 1_{(2)}\varepsilon_t(h_{(1)})h_{(2)} = 1_{(1)} \otimes 1_{(2)}h.
\end{align*}
\]

\[ \square \]

**3. Main result**

Let \( H \) be a finite quantum groupoid and \( A \) be a left \( H \)-module algebra. Then the smash product \( A\#H \) is a left \( H^* \)-module algebra via

\[ \phi \cdot (a\#h) = a\#(\phi \to h), \quad \phi \in H^*, \ h \in H, \ a \in A. \]

In the case when \( H \) is an ordinary finite dimensional Hopf algebra, it follows from [BM] that there is an isomorphism \( (A\#H)\#H^* \cong M_n(A) \), where \( n = \dim H \) and \( M_n(A) \) is an algebra of \( n \times n \) matrices over \( A \).

We will show that this result extends to quantum groupoid action in the form \( (A\#H)\#H^* \cong \End(A\#H)_A \), where \( A\#H \) is a right \( A \)-module via multiplication (note that \( A\#H \) is not necessarily a free \( A \)-module, so that we have \( \End(A\#H)_A \not\cong M_n(A) \) in general; see [NV2, 7] for an example when \( H \) is not free over \( H_t \)). We will explicitly write down canonical isomorphisms between \( (A\#H)\#H^* \) and \( \End(A\#H)_A \).
LEMMA 3.1. The map $\alpha: (A\#H)\# H^* \to \text{End}(A\#H)_A$ defined by
\[
\alpha((x\#h)\#\phi)(y\#g) = (x\#h)(y\#(\phi \to g)) = x(h_{(1)} \cdot y)\#h_{(2)}(\phi \to g)
\]
for all $x, y \in A, h, g \in H, \phi \in H^*$ is a homomorphism of algebras.

PROOF. First, we need to check that $\alpha$ is well defined. For all $z \in H_t$ and $\xi \in H_t^*$ we have :
\[
\alpha((x\#z h)\#\phi)(y\#g) = x(\phi \to g) = x(h_{(1)} \cdot y)\#h_{(2)}(\phi \to g) = \alpha((x \cdot z)\#\phi)(y\#g),
\]
\[
\alpha((x\#h)\#\xi\phi)(y\#g) = x(h_{(1)} \cdot y)\#h_{(2)}(\xi \to 1)(\phi \to g) = \alpha((x\#h(\xi \to 1))\#\phi)(y\#g) = \alpha((x\#h \cdot \xi)\#\phi)(y\#g)
\]
where we used definition of the target counital subalgebra, Lemma 2.4(ii), and that $(\xi \to 1) \in H_s$ for all $\xi \in H_t^*$.

Next, we verify that $\alpha((x\#h)\#\phi) \in \text{End}(A\#H)_A$ for all $x \in A$, $h \in H$, $\phi \in H^*$. For all $z \in H_t$ we have :
\[
\alpha((x\#h)\#\phi)(y\#z g) = x(h_{(1)} \cdot y)\#h_{(2)}z(\phi \to g) = x(h_{(1)}S(z) \cdot y)\#h_{(2)}(\phi \to g) = \alpha((x\#h)\#\phi)((y \cdot z)\#g),
\]
using the identity $\phi \to z g = z(\phi \to g)$ and Lemma 2.4(ii).

The following computation shows that $\alpha$ commutes with the right action of all $w \in A :
\[
\alpha((x\#h)\#\phi)((y\#g) \cdot w) = \alpha((x\#h)\#\phi)(y(g_{(1)} \cdot w)\#g_{(2)}) = (x\#h)(y(g_{(1)} \cdot w)\#(\phi \to g_{(2)})) = (x\#h)(y\#(\phi \to g))(w\#1) = \alpha((x\#h)\#\phi)((y\#g) \cdot w).
\]
Finally,
\[
\alpha(((x\#h)\#\phi)((x'\#h')\#\phi'))(y\#g) = \\
= \alpha((x\#h)(x'\#(\phi_{(1)} \to h'))\#(\phi_{(2)}\phi')(y\#g)) = (x\#h)(x'\#(\phi_{(1)} \to h')(y\#(\phi_{(2)}\phi' \to g))) = \alpha(((x\#h)\#\phi)((x'\#h')(y\#(\phi' \to g)))) = \alpha(((x\#h)\#\phi)(x'\#h')(y\#(\phi' \to g))) = \alpha(((x\#h)\#\phi) \circ \alpha((x'\#h')\#\phi'))(y\#g),
\]
for all $x, x', y \in A, h, h', g \in H, \phi, \phi' \in H^*$, therefore, $\alpha$ is a homomorphism. \(\square\)

Let $\{f_i\}$ be a basis of $H$ and $\{\psi_i\}$ be the dual basis of $H^*$, i.e., such that $<f_i, \psi_j> = \delta_{ij}$ for all $i, j$. Then we have identities
\[
\sum_i f_i <h, \psi_i> = h, \quad \sum_i <f_i, \phi> \psi_i = \phi,
\]
for all $h \in H$ and $\phi \in H^*$, moreover the element $\Sigma_i f_i \otimes \psi_i \in H \otimes H^*$ does not depend on the choice of $\{f_i\}$.

Let us define a linear map $\beta : \text{End}(A\#H)_A \to (A\#H)\#H^*$ by

$$
\beta : T \mapsto \sum_i T(1\#f_i(2))(1\#S^{-1}(f_i(1)))\#\psi_i.
$$

**Lemma 3.2.** The maps $\alpha$ and $\beta$ are inverses of each other.

**Proof.** We need to check that

$$
\beta \circ \alpha = \text{id}_{(A\#H)\#H^*} \quad \text{and} \quad \alpha \circ \beta = \text{id}_{\text{End}(A\#H)_A}.
$$

For all $x \in A$, $h \in H$, and $\phi \in H^*$ we compute

$$
\beta \circ \alpha((x\#h)\#\phi) = \Sigma_i (x(h(1) \cdot 1)\#h(2))(\phi \to f_i(2))S^{-1}f_i(1))\#\psi_i
$$

$$
= \Sigma_i (x\#h < \phi, f_i(3)> f_i(2)S^{-1}f_i(1))\#\psi_i
$$

$$
= \Sigma_i (x\#h < \phi, 1_2 f_i > 1_1)\#\psi_i
$$

$$
= (x\#h(\phi(1) \to 1))\#\psi(2)
$$

$$
= (x\#h)\#(\phi(1)\phi(2)) = (x\#h)\#\phi,
$$

where we used Lemma 2.4 (iii) and the properties of the element $\Sigma_i f_i \otimes \psi_i$.

Also, for every $T \in \text{End}(A\#H)_A$ we have:

$$
\alpha \circ \beta(T)(y\#g) = \Sigma_i \alpha(T(1\#f_i(2))(1\#S^{-1}(f_i(1)))\#\psi_i)(y\#g)
$$

$$
= \Sigma_i T(1\#f_i(2))(1\#S^{-1}(f_i(1)))y(\psi_i \to g)
$$

$$
= \Sigma_i T(1\#f_i(3))(S^{-1}(f_i(2)) \cdot y)\#S^{-1}(f_i(1))g(1)(\psi_i, g(2))
$$

$$
= T(1\#g(3))(S^{-1}(f_i(2)) \cdot y)\#g(1)(1_2 f_i)\cdot 1
$$

$$
= T(1\#g(3))(S^{-1}(f_i(2)) \cdot y)\#S^{-1}(g(1))g(1_1)1
$$

$$
= T(1\#g(2))(g(1) \cdot y)\#1
$$

$$
= T((g(2)S^{-1}(g(1)) \cdot y)\#g(3))
$$

$$
= T(1\#g(2))(g(1) \cdot y)\#1
$$

where we used that $T$ commutes with the right multiplication by elements from $A$ and identities from Lemma 2.4(i) and (iii).

**Theorem 3.3.** For any $H$-module algebra $A$ there is a canonical isomorphism between the algebras $(A\#H)\#H^*$ and $\text{End}(A\#H)_A$.

**Proof.** Follows from Lemmas 3.1 and 3.2.

**Corollary 3.4.** $H\#H^* \cong \text{End}(H)_{H^*}$, in particular, $H\#H^*$ is a semisimple algebra.

**Proof.** We know that $H \cong H_t \# H$, where $H_t$ is the trivial $H$-module algebra, therefore applying Theorem 3.3 to $A = H_t$ we see that $H$ is a projective generating $H_t$-module such that $\text{End}(H)_{H^*} \cong H\#H^*$. Therefore, $H_t$ and $H\#H^*$ are Morita equivalent. Since $H_t$ is always semisimple (as a separable algebra), $H\#H^*$ is semisimple.
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