AN UNEXPECTED TRACE RELATION OF CM POINTS

DANIEL KOHEN

Abstract. Let \( E/\QQ \) be an elliptic curve of conductor \( N = p^2 M \) where \( p \) is an odd prime not dividing \( M \). Let \( \OO_f \) be the order of conductor \( f \) (prime to \( N \)) in an imaginary quadratic field \( K \) in which \( p \) is inert and such that the sign of the functional equation of \( E/K \) is \(-1\). Associated to these data there is a Shimura curve of non-split Cartan level at \( p \) and a CM point of conductor \( f \) on it. We can also consider a CM point of conductor \( pf \) on another Shimura curve, using a split Cartan level at \( p \). These curves admit modular parametrizations to \( E \) and taking the images of the CM points we obtain points on \( E \) defined over \( H_f \) and \( H_{pf} \) respectively (the ring class fields of conductor \( f \) and \( pf \)). We prove that these points arising from different Shimura curves satisfy a trace compatibility that is non-trivial if and only if the local sign of \( E/\QQ \) at \( p \) is \(+1\).

Introduction

Let \( E/\QQ \) be an elliptic curve of conductor \( p^2 \). Since \( E \) is modular, there exists a modular parametrization from the classical modular curve \( X_0(p^2) \) to \( E \). Let \( K \) be an imaginary quadratic field and and let \( \OO_K \) be its ring of integers. If the prime \( p \) is split, the classical theory of Heegner points \([\text{Gro84}]\) supplies a point in \( X_0(p^2) \) such that its image under the modular parametrization is \( P_1 \in E(H) \), where \( H \) is the maximal unramified abelian extension of \( K \). Consider \( P_K := \text{Tr}_H^K P_1 \in E(K) \). The celebrated Gross-Zagier formula \([\text{GZ86}]\) shows that this point is non-torsion precisely when \( L'(E/K, 1) \neq 0 \). When \( p \) is inert in \( K \) we do not have Heegner points in the modular curve \( X_0(p^2) \), however, the non-split Cartan curve \( X_{ns}(p) \) hosts Heegner points and uniformizes \( E \). This, together with a generalization of the Gross-Zagier formula, was carried out by Zhang \([\text{Zha01}]\).

In spite of \( X_0(p^2) \) not possessing Heegner points of conductor 1 whenever \( p \) is inert, it does have points of conductor \( p \) defined over \( H_p \), the ring class field associated to the order of conductor \( p \) inside \( \OO_K \). Take any such point \( P_p \in E(H_p) \). One question arises naturally: is \( \tilde{P}_1 := \text{Tr}_H^H P_p \in E(H) \) non-torsion?

If \( L'(E/K, 1) \neq 0 \), the main result of this paper is to show that \( \tilde{P}_1 \) is non-torsion if and only if \( w_p(E) = +1 \), where \( w_p(E) \) denotes the sign of the local Atkin-Lehner...
involution of $E$ at $p$. Although the reader should have in mind this special case when reading the paper, we have proved the results in a greater degree of generality, working with elliptic curves of conductor divisible by $p^2$ and Shimura curves associated to suitable indefinite quaternion algebras.

The theme of Heegner and CM points constructions for more general orders (including non-split Cartan orders) was also studied in [CCL18], [KP16], [KP18] and [LRdVP18]. An immediate application of the present article is that if $w_p(E) = +1$ we can explicitly compute Heegner points much more efficiently than [KP16], as we can avoid working with non-split Cartan curves.

One of the main features of Heegner points (or CM points in general) is that they come in families satisfying trace compatibilities (see for example [BD96, Section 2.4], [CV07, Section 6], [Dar04, Proposition 3.10], [Gro91, Proposition 3.7], [Nek07, Proposition 4.8]). These compatibilities are a crucial tool in order to bound Selmer groups and construct $p$-adic $L$-functions.

Cornut and Vatsal introduced the notion of good CM points which are essentially the CM points that satisfy such compatibility relations. Our CM points of conductor $p$ are not good (they are of type III in the sense [CV07, Section 6.4]), thus the main interest of this paper is that it provides an (unexpected!) trace relation of CM points living in two distinct Shimura curves. A parallel of this situation is the work of Bertolini and Darmon [BD96] that relies on understanding the relation between CM points in distinct modular curves. Their situation is deeper and more delicate since there is a sign change phenomenon and they have to study CM points living in Shimura curves with different ramification sets.

The modular curve $X_0(p^2)$ is isomorphic to $X_s(p)$, the modular curve associated to a split Cartan group modulo $p$. The new part of its Jacobian is isogenous over $\mathbb{Q}$ (in a manner compatible with the Hecke operators) to the Jacobian of the curve $X_n(s)(p)$, providing a modular parametrization from $X_{ns}(p)$ to $E$. This was first proved by Chen [Che98] comparing the traces of the Hecke operators for these curves. Afterwards, de Smit and Edixhoven [dSE00] gave another (more geometric) argument using the representation theory of $\text{GL}_2(\mathbb{F}_p)$. The result was later reproved in the aforementioned paper of Zhang resorting to the theory of automorphic representations.

If $w_p(E) = -1$ the proof that the trace of the point of conductor $p$ is 0 (Proposition 3.1) follows easily by studying the interplay between the Galois action and the Atkin-Lehner involution.

Suppose now that $w_p(E) = +1$. In that case, we can work with the normalizers of the Cartan groups (both split and non-split), obtaining the isogenous Jacobians $J_{ns+}(p)$ and $J_{s+}^{\text{new}}(p)$. Chen [Che00] was able to provide an explicit isogeny between these two Jacobians (the other proofs mentioned where non-constructive). The isogeny is given in very simple terms as a double coset operator, that can be
thought as a trace between the modular curves. The isogeny was reinterpreted using a nice combinatorial description of the moduli interpretation of the non-split Cartan curve by Rebolledo and Wuthrich [RW18]. In the recent paper [CSS18], Chen and Salari Sharif also gave a simpler proof and even showed an explicit isogeny between $J_{\text{new}}(p)$ and $J_{ns}(p)$ (unluckily, the explicit isogeny is less nice, as it is given by a linear combination of several double coset operators).

The results of this paper boil down to prove that, when $w_p(E) = +1$, the double coset operator giving Chen’s isogeny applied to the CM point on the non-split Cartan curve essentially gives the trace of the CM point of conductor $p$ in the split Cartan curve (Proposition 3.2). In addition, we prove that the isogeny is equivariant for the Hecke operators $4.1$. Combining these results we prove that $\tilde{P}_1$ is non-torsion if $L'(E/K, 1) \neq 0$ (Theorem 4.3).

Acknowledgments: I would like to thank Ariel Pacetti and Matteo Tamiozzo for their helpful comments on an early version of this article.

Setting and notation

We fix the following hypotheses and notation throughout the article:

- Let $E$ be a rational elliptic curve of conductor $N = p^2 M$, where $p$ is an odd prime relatively prime to $M$.
- Let $K$ be an imaginary quadratic field such that:
  1. The sign $\epsilon(E/K)$ of the functional equation of $E/K$ is equal to $-1$.
  2. The prime $p$ is inert in $K$.
  3. We make the non-essential hypothesis that if $q$ is prime and $q^2 | N$, then $q$ is unramified in $K$.
- Let $f$ be a positive integer relatively prime to $N$ and let $\mathcal{O}_f = \mathbb{Z} + \omega_f \mathbb{Z}$ be the unique order of conductor $f$ inside $\mathcal{O}_K$, the ring of integers of $K$. Let $\mathcal{O}_{pf} = \mathbb{Z} + p\omega_f \mathbb{Z}$ be the order of conductor $pf$.
- Let $\epsilon$ be a non-square modulo $p$.
- We define the non-split Cartan order as
  \[ \mathcal{M}_{ns} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : a \equiv d, b\epsilon \equiv c \mod p \right\}. \]

We also define the normalizer as
  \[ \mathcal{M}_{ns+} := \mathcal{M}_{ns} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : a \equiv -d, -b\epsilon \equiv c \mod p \right\}. \]
• We define the split Cartan order as
  
  \[ M_s := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : b \equiv c \equiv 0 \mod p \right\}. \]

  We also define the normalizer
  
  \[ M_{s+} := M_s \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : a \equiv d \equiv 0 \mod p \right\}. \]

  • For \( \phi \in \{ ns, ns+, s, s+ \} \) we denote the corresponding Cartan group
  
  \[ C_\phi := \{ \overline{M} : M \in M_2^\times \} \subset \text{GL}_2(\mathbb{F}_p), \]
  
  where \( \overline{M} \) denotes the reduction modulo \( p \) of \( M \).

  • Given a \( \mathbb{Z} \)-module \( A \) we let \( A_v := A \otimes \mathbb{Z}_v \). Let \( \hat{\mathbb{Z}} := \prod_v \mathbb{Z}_v \) and \( \hat{A} := A \otimes \hat{\mathbb{Z}} \).

  • Given a quaternion algebra \( B \) (respectively an order \( R \subset B \)) we denote by \( B^1 \) (resp. \( R^1 \)) the elements of \( B \) (resp. \( R \)) of reduced norm \( 1 \).

1. Optimal embeddings of Cartan orders

The goal of this section is, given our setting, to construct an embedding of \( K \) into a suitable quaternion algebra \( B \), Cartan orders \( R_{ns} \) (resp. \( R_s \)) of discriminant \( N \) inside \( B \) such that \( (R_{ns})_p = M_{ns} \) (resp. \( (R_s)_p = M_s \)) and such that \( \mathcal{O}_f \) (resp. \( \mathcal{O}_{pf} \)) embeds optimally into \( R_{ns} \) (resp. \( R_s \)).

The sign \( \varepsilon(E/K) \) decomposes as the product of local signs \( \varepsilon_v(E/K) \in \{ \pm 1 \} \). Let \( \eta \) be the character associated to \( K \) via class field theory. For each place \( v \) let \( \varepsilon(\mathbb{B}_v) \) be the unique sign such that

\[ \varepsilon_v(E/K) = \eta_v(-1)\varepsilon(\mathbb{B}_v). \]

Let \( S \) be the set of places such that \( \varepsilon(\mathbb{B}_v) = -1 \). Since \( \varepsilon(E/K) = -1 \), \( S \) has odd cardinality. Moreover, since \( K \) is imaginary, \( \infty \in S \) [Gro88, Proposition 6.5] and \( p \notin S \) because \( p^2 \) divides \( N \) exactly [Ibid. Proposition 6.3 (2)]. Let \( B/\mathbb{Q} \) be the unique quaternion algebra with ramification set \( S - \{ \infty \} \). If \( v \) splits in \( K \) we have that \( \varepsilon(\mathbb{B}_v) = 1 \) [Ibid. Proposition 6.3 (1)], therefore, there exists an embedding \( \iota : K \rightarrow B \) [Vig80, Theorem 3.8]. Furthermore, using the theory of optimal embeddings we can find an order \( R \subset B \) of discriminant \( M \) such that \( \iota(K) \cap R = \iota(\mathcal{O}_f) \). In fact, this follows from [Gro88, Propositions 3.2, 3.4] (see also [CST14, Section 3] for a more general statement).

Since \( R \) has level relatively prime to \( p \) and \( B \) is split at \( p \) the local order \( R_p \) is isomorphic to \( M_2(\mathbb{Z}_p) \), and to ease notation, we will assume that \( R_p \) is indeed \( M_2(\mathbb{Z}_p) \). Consider the minimal polynomial \( m_{\omega_f} = X^2 - tX + n \). Then,

\[ \iota(\omega_f)_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A \in M_2(\mathbb{Z}_p), \]
where the characteristic polynomial of $A$ is equal to $m_{\omega_f}$. Since $p$ is inert in $K$, $\overline{m_{\omega_f}}$ is irreducible in $\mathbb{F}_p[X]$ and $\overline{A}$ is conjugated (in $\text{GL}_2(\mathbb{F}_p)$) to the matrix $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$.

In addition, since $\text{det} : C_{ns} \to \mathbb{F}_p^\times$ is surjective, they are conjugated by a matrix $\gamma \in \text{SL}_2(\mathbb{F}_p)$. As a consequence of strong approximation, the map $R^1 \to (R_p/pR_p)^1 \cong \text{SL}_2(\mathbb{F}_p)$ is surjective [Vo018, Corollary 28.4.11], and we take a lifting $\gamma \in R^1$ of $\iota$.

We conjugate $\iota$ by $\gamma$ and we obtain a new embedding (which we fix throughout the rest of the text and still denote by $\iota$).

For $? \in \{ns, ns+, s, s+\}$ we define $R_? := \{x \in R : x_p \in M_?\}$. The orders $R_{ns}$ and $R_s$ are both of discriminant $N = p^2M$. We have the following proposition.

**Proposition 1.1.**

1. The order $\mathcal{O}_f$ embeds optimally into $R_{ns}$, that is,

\[ \iota(K) \cap R_{ns} = \iota(\mathcal{O}_f). \]

2. The order $\mathcal{O}_{pf}$ embeds optimally into $R_s$, that is,

\[ \iota(K) \cap R_s = \iota(\mathcal{O}_{pf}). \]

**Proof.** The first statement is clear from the construction. For the second one, note that we only changed the order at $p$, so we just need to check locally at $p$. Let $\iota(\omega_f)_p = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in (R_{ns})_p$. The local order $(\mathcal{O}_{pf})_p$ is equal to $\mathbb{Z}_p + pw_f\mathbb{Z}_p$ and thus $\iota((\mathcal{O}_{pf})_p) \subseteq (R_s)_p$. For the other inclusion, note that since $p$ is inert in $K$ we must have that $p$ does not divide $b_0c_0$. Consequently, if we take an element $(\iota(x_1 + x_2pw_f))_p \in (R_s)_p$ with $x_1, x_2 \in \mathbb{Q}_p$, looking at the $(2, 1)$ entry we obtain $x_2 \in \mathbb{Z}_p$ and looking at the $(1, 1)$-entry we get $x_1 \in \mathbb{Z}_p$, as we wanted. \hfill \square

2. Shimura curves and parametrizations

For $? \in \{ns, ns+, s, s+\}$ consider the Shimura curve

\[ Y_? = B^\times \backslash (\mathbb{C} - \mathbb{R}) \times \hat{B}^\times / \hat{R}_?^\times, \]

where $B^\times$ acts on $\mathbb{C} - \mathbb{R}$ by conjugation and by left multiplication on $\hat{B}^\times$ and $\hat{R}_?^\times$ acts trivially on $\mathbb{C} - \mathbb{R}$ and by right multiplication on $\hat{B}^\times$.

We consider its compactification, given by

\[ X_? := Y_? \cup \{\text{cusps}\}. \]

The set of cusps is empty if $B \neq M_2(\mathbb{Q})$ and otherwise equal to the finite set

\[ \text{GL}_2(\mathbb{Q}) \backslash \mathbb{P}^1(\mathbb{Q}) \times \text{GL}_2(\hat{\mathbb{Q}}) / \hat{R}_?^\times, \]

where the action of $\text{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ is given by Möbius transformations.
Since $B$ is an indefinite quaternion algebra, strong approximation \cite{Vig80} Theorem 4.3 yields that $B^1 \backslash \hat{B}^1 / R_1^\times$ is trivial. In addition, since every prime $q$ such that $q^2 \mid N$ is unramified in $K$, it is easy to see that $B^\times / \hat{B}^\times / R_1^\times \simeq \hat{\mathbb{Z}}^\times / \det(\hat{R}_1^\times)$ also consists of a single point. Accordingly, if we define $\Gamma_? := R_? \cap B^1$ we obtain the following classical description of the Shimura curves as quotients

$$X_? = \Gamma_? \backslash \mathcal{H}^*,$$

where $\mathcal{H}^*$ denotes the union of the upper half plane and the cusps.

Let $J_?$ be the Jacobian variety of $X_?$. We can embed $X_? \to i_?, J_?$ using the so-called Hodge class \cite[Section 6.2]{Zha01}, \cite[Section 3.1.3]{YZZ13}, that is, the unique class $\xi_? \in Pic(X_?) \otimes \mathbb{Q}$ of degree 1 such that $T_\ell \xi_? = (\ell + 1) \xi_?$ for every prime number $\ell$ not dividing $N$. In the case $B = M_2(\mathbb{Q})$ the Hodge class is a linear combination ofcusps. The embedding $i_?$ is given by sending a point $P$ to the class of $P - \xi_?$.

Let $\pi_?, \pi_? = \text{Hom}_0^\ast(X_?, E)$, that is, the morphisms in $\text{Hom}(X_?, E) \otimes_\mathbb{Z} \mathbb{Q}$ sending a divisor representing $\xi_?$ to zero. Thus,

$$\pi_? = \text{Hom}(J_?, E) \otimes_\mathbb{Z} \mathbb{Q}.$$

By \cite[Theorem 1.3.1]{Zha01} (see also \cite[Proposition 2.6]{GP91} and \cite[Propositions 3.7,3.8]{CST14}) the spaces $\pi_{E,ns}$ and $\pi_{E,s}$ are 1-dimensional and we choose non-zero elements $\Phi_{ns} \in \pi_{E,ns}$ and $\Phi_s \in \pi_{E,s}$. These elements correspond to automorphic forms $f_{ns}$ and $f_s$ of level $R_{ns}$ and $R_s$ respectively, with the same eigenvalues as $f_E$ for the good (i.e. not dividing $N$) Hecke operators and the local Atkin-Lehner involutions. The element $\Phi_{E,s}$ factors through $J_s^{p-\text{new}}$, since the corresponding form is new at $p$. If $w_p(E) = +1$ we have the induced non-zero elements $\Phi_{ns+} \in \pi_{E,ns+}$ and $\Phi_{s+} \in \pi_{E,s+}$.

3. Explicit Galois action and CM points

The embedding $i$ gives rise to an action of $\hat{K}^\times / K^\times \simeq \text{Gal}(K^{ab}/K)$ on the curves $X_?$ by the formula

$$a \cdot [z, b] := [z, a(z)b].$$

Let $z_0$ be the unique fixed point in $\mathcal{H}$ of $t_\infty(\omega_f) \in \text{GL}_2(\mathbb{R})$ under the action given by M"obius transformations. Proposition 1.1 tells us that the point $Q_f := [z_0, 1] \in X_{ns}(H_f)$ (resp. $Q_{pf} := [z_0, 1] \in X_s(H_{pf})$) where $H_f$ (resp. $H_{pf}$) denotes the ring class field of conductor $f$ (resp. $pf$) characterized by the fact that $\text{Gal}(H_f/K)$ (resp. $\text{Gal}(H_{pf}/K)$) is identified with $\hat{K}^\times / K^\times \hat{O}_f^\times$ (resp. $\hat{K}^\times / K^\times \hat{O}_{pf}^\times$). The main task of this section is to understand the action $\text{Gal}(H_{pf}/H_f)$ on $Q_{pf}$. Note that

$$\text{Gal}(H_{pf}/H_f) \simeq K^\times \hat{O}_f^\times / K^\times \hat{O}_{pf}^\times \simeq (O_f^\times)_p / (O_{pf}^\times)_p,$$
by looking at the \( p \)-th component. Reducing modulo \( p \) we further obtain

\[
\left( O_f^\times \right)_p / \left( O_{pf}^\times \right)_p \simeq \left( O_f / pO_f \right)^\times / \mathbb{F}_p^\times \simeq \mathbb{P}^1(\mathbb{F}_p).
\]

More explicitly, this last isomorphism is given by sending the class of \( x_1 + x_2\omega_f \) (with \( x_i \in \mathbb{Z}_p \)) to the element \( \left[ \frac{x_1}{x_2} : \frac{x_1}{x_2} \right] \in \mathbb{P}^1(\mathbb{F}_p) \). The element \([1 : 0]\) is the identity and multiplication on \( \mathbb{P}^1(\mathbb{F}_p) \) is given by the rule

\[
[x : 1][y : 1] := [xy - n : x + y + t],
\]

where, as before, \( X^2 - tX + n \) is characteristic polynomial of \( \iota(\omega_f)_p = \left( \begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix} \right) \in (R_{ns})_p \). The only element of order 2 in \( \mathbb{P}^1(\mathbb{F}_p) \) is \([-\omega_0 : 1]\), and in that case the corresponding matrix \( \iota(-a_0 + \omega_f)_p \) belongs to \( M_{s+}^\times \setminus M_s^\times \). Also, it is clear that

\[
\iota((O_f^\times)_p \cap C_{s+} \text{ consists only of the elements } [-\omega_0 : 1] \text{ and } [1 : 0].
\]

The local Atkin-Lehner involution at \( p \) of \( X_s \) is given by right multiplication by any element in \( (R_{s+})^\times \setminus (R_s)^\times \). In particular we can take the element \( \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \).

**Proposition 3.1.** If \( w_p(E) = -1 \), then \( Tr_{H_f}^H \Phi_s(Q_{pf}) = 0 \).

**Proof.** Let \( b_0 \) be the element of \( \text{Gal}(H_{pf}/H_f) \) corresponding to \(-a_0 + \omega_f\). We can group the elements of \( \text{Gal}(H_{pf}/H_f) \) into pairs of the form \( \{b, bb_0\} \). Consider

\[
Q_{pf}^{bb_0} = [z_0, \iota(b) \cdot \iota(b_0)].
\]

The element \( \iota(b_0) \) belongs to \( \hat{R}_{s+}^\times \setminus \hat{R}_s^\times \) and it can be written as a product \( \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \cdot \hat{r}_s \), where \( \hat{r}_s \in \hat{R}_s^\times \). Therefore we get, in \( X_s \),

\[
Q_{pf}^{bb_0} = [z_0, \iota(b) \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \cdot \hat{r}_s] = [z_0, \iota(b) \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) ] = w_p Q_{pf}^{b}.
\]

As \( w_p(E) = -1 \),

\[
\Phi_s(Q_{pf}^{bb_0}) = \Phi_s(w_p(E)Q_{pf}^{b}) = -\Phi_s(Q_{pf}^{b}),
\]

and the result follows. \( \square \)

In view of the above proposition, from now on we will study the case where \( w_p(E) = 1 \). Consider the double coset operator \( \Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+} \). It gives rise to a correspondence between the Shimura curves \( X_{ns+} \) and \( X_{s+} \). The following is the heart of this article.

**Proposition 3.2.** The following identity holds on \( \text{Div}(X_{s+}) \):

\[
2(\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+})Q_f = \sum_{\text{Gal}(H_{pf}/H_f)} Q_{pf}^\sigma.
\]
Proof. Take a set of representatives \( \{ b_i \} \) of \( \text{Gal}(H_{pf}/H_f) \) in \( \hat{O}_f^\times \). The element \( r_i := \hat{\iota}(b_i) \) belongs to \( \hat{R}_{ns}^\times \). We want to write \( r_i \) as a product \( \gamma_i r_i \) where \( r_i \in \hat{R}_{ns}^\times \) and \( \gamma_i \in \Gamma_{ns}^+ \). In order to do that, we need to modify the component at \( p \). Suppose that \( (r_i)_p \) has determinant \( d \in \mathbb{Z}_p^\times \).

- If \( d = \mu^2 \), consider \( (r_i)_p \left( \frac{\mu^{-1}}{0} \frac{0}{\mu^{-1}} \right) \in \text{SL}_2(\mathbb{F}_p) \). By strong approximation, we take a lifting of this element to an element \( \gamma_i \in R^1 \). Clearly, \( r_i = \gamma_i \cdot ((\gamma_i)^{-1} r_i) \), is a decomposition with the desired properties.

- If \( d = \varepsilon \mu^2 \) consider \( (r_i)_p \left( \frac{0}{-\varepsilon \mu^{-1}} \frac{-\varepsilon \mu^{-1}}{0} \right) \in \text{SL}_2(\mathbb{F}_p) \) and we do the same as the previous case. It is worth noticing that this case shows the necessity of working with the normalizers, in concordance with Proposition 3.1.

By construction we get

\[
Q_{pf} b_i = [z_0, 1] b_i = [z_0, r_i] = [z_0, \gamma_i r_i] = [\gamma_i^{-1} \cdot z_0, 1] \in X_{s+}(H_{pf}),
\]

which corresponds to the point \( \gamma_i^{-1} \cdot z_0 \) in the upper half plane of the model \( \Gamma_{ns+} \setminus \mathcal{H}^* = X_{s+} \).

Note that the correspondence \( \Gamma_{ns+} \cdot 1 \cdot \Gamma_{ns+} \) is of degree

\[
[\Gamma_{ns+} : \Gamma_{ns+} \cap \Gamma_{s+}] = [C_{ns+} : C_{ns+} \cap C_{s+}] = \frac{2(p+1)}{4} = \frac{(p+1)}{2}.
\]

As we vary over all \( p+1 \) elements of \( \text{Gal}(H_{pf}/H_f) \) we obtain the corresponding coset \( \Gamma_{s+} \gamma_i^{-1} \subset \Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+} \). The cosets \( \Gamma_{s+} \gamma_i^{-1} \) and \( \Gamma_{s+} \gamma_j^{-1} \) are equal if and only if

\[
\gamma_i^{-1} \gamma_j \in \Gamma_{s+}.
\]

By the definition of \( \gamma_i \) and \( \gamma_j \) we obtain

\[
r_i^{-1} r_j = r_i^{-1} \gamma_i^{-1} \gamma_j r_j.
\]

Since \( r_i, r_j \in \hat{R}_{ns}^\times \) we know that

\[
\gamma_i^{-1} \gamma_j \in \Gamma_{s+} \iff r_i^{-1} r_j \in \hat{R}_{s+}^\times.
\]

As we observed before Proposition 3.1

\[
\gamma_i^{-1} \gamma_j \in \Gamma_{s+} \iff (r_i^{-1} r_j)_p \in \{ [1 : 0], [-a_0 : 1] \}.
\]

Thus, as we vary \( i \), we run through every element of \( (\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+}) Q_f \) exactly twice, as we wanted to prove.
4. CHEN’S EXPLICIT ISOGENY

Chen [Che00, Theorem 2] proved that an explicit isogeny between \( J_{ns+}(p) \) and \( J_{s+}^{p\text{-new}}(p) \) is given by the double coset operator \( \Gamma_{s+}(p) \cdot 1 \cdot \Gamma_{ns+}(p) \). In fact, the nature of the proof shows that it extends to an isogeny \( \Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+} \) between \( J_{ns+} \) and \( J_{s+}^{p\text{-new}} \).

**Proposition 4.1.** The map \( \Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+} \) is Hecke equivariant for the good Hecke operators.

**Proof.** Let \( \ell \) be a prime that does not divide \( N \). Since the reduced norm gives a surjective map \( \hat{R}_s^\times \to \hat{\mathbb{Z}}^\times \) we choose \( \beta_\ell \in \hat{R}_s^\times \) of reduced norm \( \ell \). We perform the same construction as in Proposition 3.2 but for \( \beta_\ell \) instead of \( r_i \) and we obtain the same matrix \( \gamma \) (there denoted by \( \gamma_i \)). This matrix has the property that \( \alpha_\ell := \gamma^{-1} \beta_\ell \) belongs to \( \hat{R}_{s+} \cap \hat{R}_{ns+}^\times \) and has reduced norm equal to \( \ell \). Thus the \( \ell \)-th Hecke operator in the split side (resp. non-split side) is given by \( \Gamma_{s+} \cdot \alpha_\ell \cdot \Gamma_{s+} \) (resp. \( \Gamma_{ns+} \cdot \alpha_\ell \cdot \Gamma_{ns+} \)). Using [Shi71, Lemma 3.29 (4)] it is clear that

\[
(\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+}) \cdot (\Gamma_{ns+} \cdot \alpha_\ell \cdot \Gamma_{ns+}) = \Gamma_{s+} \cdot \alpha_\ell \cdot \Gamma_{ns+} = (\Gamma_{s+} \cdot \alpha_\ell \cdot \Gamma_{s+}) \cdot (\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+}),
\]

and the map commutes with the good Hecke operators as desired. □

**Remark 4.2.** This proof is also in concordance with Proposition 3.1 and with the fact that the natural map \( \Gamma_s \cdot 1 \cdot \Gamma_{ns} \) does not always give an isogeny between \( J_{ns} \) and \( J_{s}^{p\text{-new}} \). We only can prove that it commutes with the Hecke operators \( T_\ell \) with \( \ell \) a square modulo \( p \), and given a newform \( f \) of level divisible by \( p^2 \), the space of forms with the same eigenvalues as \( f \) for these Hecke operators has dimension 2.

As the correspondence \( 2(\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+}) \) is Hecke equivariant and of degree \( p + 1 \) it sends the Hodge class \( \xi_{ns+} \) to \((p + 1)\xi_{s+} \). Therefore, we can consider

\[
\tilde{\Phi}_{ns+} := \Phi_{s+} \circ 2(\Gamma_{s+} \cdot 1 \cdot \Gamma_{ns+}) \in \pi_{E,ns+}.
\]

**Theorem 4.3.** Let \( E \) be such that \( w_p(E) = +1 \). Consider the points \( P_{pf} := \Phi_{s+}(Q_{pf}) \in E(H_{pf}) \), \( P_f := \Phi_{ns+}(Q_f) \in E(H_f) \) and \( \tilde{P}_f := \tilde{\Phi}_{ns+}(Q_f) \in E(H_f) \). Then, \( Tr_{H_f}^{H_{pf}} P_{pf} = \tilde{P}_f \in E(H_f) \) and \( \tilde{P}_f \) is non-torsion if and only if \( P_f \) is non-torsion.

**Proof.** First, note that \( Tr_{H_f}^{H_{pf}} P_{pf} = \tilde{P}_f \) follows immediately from Proposition 3.2. The second statement follows from the fact that the elements \( \Phi_{ns+}, \tilde{\Phi}_{ns+} \) lie in the
1-dimensional space $\pi_{E,n_+} = \text{Hom}(J_{n_+}, E) \otimes \mathbb{Q}$ and $\Phi_{n_+}$ is non-zero since it the composition of the non-trivial $\Phi_{s_+}$ and the isogeny $2(\Gamma_{s_+} \cdot 1 \cdot \Gamma_{n_+})$.

\[ \square\]

Remark 4.4. If $f = 1$, $P_1$ is non-torsion if $L'(E/K, 1) \neq 0$ by Zhang’s generalization of the Gross-Zagier formula [Zha01, Theorem 1.2.1].

References

[BD96] M. Bertolini and H. Darmon. Heegner points on Mumford-Tate curves. *Invent. Math.*, 126(3):413–456, 1996.

[CCL18] Li Cai, Yihua Chen, and Yu Liu. Heegner points on modular curves. *Trans. Amer. Math. Soc.*, 370(5):3721–3743, 2018.

[Che98] Imin Chen. The Jacobians of non-split Cartan modular curves. *Proc. London Math. Soc. (3)*, 77(1):1–38, 1998.

[Che00] Imin Chen. On relations between Jacobians of certain modular curves. *J. Algebra*, 231(1):414–448, 2000.

[CSS18] Imin Chen and Parinaz Salari Sharif. An explicit correspondence of modular curves. *arXiv:1801.04020*, 2018.

[CST14] Li Cai, Jie Shu, and Ye Tian. Explicit Gross-Zagier and Waldspurger formulae. *Algebra Number Theory*, 8(10):2523–2572, 2014.

[CV07] Christophe Cornut and Vinayak Vatsal. Nontriviality of Rankin-Selberg $L$-functions and CM points. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 121–186. Cambridge Univ. Press, Cambridge, 2007.

[Dar04] Henri Darmon. *Rational points on modular elliptic curves*, volume 101 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.

[dSE00] Bart de Smit and Bas Edixhoven. Sur un r´esultat d’Imin Chen. *Math. Res. Lett.*, 7(2-3):147–153, 2000.

[GP91] Benedict H. Gross and Dipendra Prasad. Test vectors for linear forms. *Math. Ann.*, 291(2):343–355, 1991.

[Gro84] Benedict H. Gross. Heegner points on $X_0(N)$. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 87–105. Horwood, Chichester, 1984.

[Gro88] Benedict H. Gross. Local orders, root numbers, and modular curves. *Amer. J. Math.*, 110(6):1153–1182, 1988.

[Gro91] Benedict H. Gross. Kolyvagin’s work on modular elliptic curves. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 235–256. Cambridge Univ. Press, Cambridge, 1991.

[GZ86] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of $L$-series. *Invent. Math.*, 84(2):225–320, 1986.

[KP16] Daniel Kohen and Ariel Pacetti. Heegner points on Cartan non-split curves. *Canad. J. Math.*, 68(2):422–444, 2016.

[KP18] Daniel Kohen and Ariel Pacetti. On Heegner points for primes of additive reduction ramifying in the base field. *Trans. Amer. Math. Soc.*, 370(2):911–926, 2018.
Matteo Longo, Víctor Rotger, and Carlos de Vera-Piquero. Heegner points on Hijikata–Pizer–Shemanske curves and the Birch and Swinnerton-Dyer conjecture. *Publ. Mat.*, 62(2):355–396, 2018.

Jan Nekovář. The Euler system method for CM points on Shimura curves. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 471–547. Cambridge Univ. Press, Cambridge, 2007.

Marusia Rebolledo and Christian Wuthrich. A moduli interpretation for the non-split Cartan modular curve. *Glasg. Math. J.*, 60(2):411–434, 2018.

Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Kanô Memorial Lectures, No. 1.

Marie-France Vignéras. *Arithmétique des algèbres de quaternions*, volume 800 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

John Voight. Quaternion algebras. *https://math.dartmouth.edu/~jvoight/quat-book.pdf*, 2018.

Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. The Gross-Zagier formula on Shimura curves, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

Shou-Wu Zhang. Gross-Zagier formula for GL_2. *Asian J. Math.*, 5(2):183–290, 2001.