Explaining the Virtual Universal Occurrence of Static Friction

J. B. Sokoloff*, Physics Department and
Center for Interdisciplinary Research on complex Systems,
Northeastern University, Boston, Massachusetts 02115

abstract

Perturbation theory, simulations and scaling arguments predict that there should be no static friction for two weakly interacting flat atomically smooth clean solid surfaces. The absence of static friction results from the fact that the atomic level interfacial potential energy is much weaker than the elastic potential energy, which prevents the atoms from sinking to their interfacial potential minima. Consequently, we have essentially two rigid solids, for which the forces at randomly distributed "pinning sites" cancel. It is shown here that even fluctuations in the concentration of atomic level defects at the interface do not account for static friction. It is also argued that the sliding of contacting asperities, which occurs when the problem is studied at the multi-micron length scale, relative to each other involves the shearing of planes of atoms. Since this results in a force for the interaction of two asperities which varies over sliding distances of the order of an atomic spacing, the contacting asperities at the surface are able to sink to their interfacial potential minima, with negligible cost in elastic potential energy. This results in static friction.

Keywords: Semi-empirical and model calculations, friction, surface defects.

* Corresponding author-address: Physics Department, Northeastern University, Boston, Massachusetts 02115. Phone number: (617) 373-2931, FAX number: (617) 373-2943, E-mail address: 3630jbs@neu.edu.
I. Introduction

It is well known to every student in an elementary physics class that kinetic friction has very little velocity dependence in the slow sliding speed limit (often called "dry friction"). Molecular dynamics simulations[1] and analytic calculations[2,3] show that while commensurate interfaces are pinned for applied forces below a critical value (i.e., exhibit static friction), incommensurate surfaces are not pinned and exhibit viscous friction (i.e., friction proportional to the sliding velocity) for sufficiently weak interfacial forces. Perturbation theory calculations done for a nonmetallic monolayer film with underdamped phonons sliding on a nonmetallic substrate with some disorder, however, give nearly velocity independent sliding friction[4] and exhibit a divergence in the mean square atomic displacement in the limit of zero sliding velocity. The latter behavior signifies that the film will be pinned below a critical applied force. This behavior has been confirmed by recent molecular dynamics calculations on such a system[5]. Perturbation theory calculations done for a three dimensional film sliding on a substrate, however, give viscous friction. This result is consistent with the notion that without multistability, there cannot be "dry friction" due to vibrational excitations in an elastic solid[6,7]. Dry friction is possible for the monolayer film, as mentioned above, however, because the two dimensional phonon density of states of the film does not drop to zero as the frequency goes to zero, as it does for a three dimensional solid. As the sliding velocity drops to zero, the "washboard frequency" (the sliding velocity divided by a lattice constant) drops to zero. Since the phonon density of states does not drop to zero, there are phonons present at arbitrarily low frequency, which can be excited by the substrate potential. Since the density of states does fall to zero as the frequency falls to zero in three dimensions, however, the force of friction falls to zero as the velocity does. In models used for charge density waves (CDW), in which the CDW is modeled as an elastic medium moving through a solid containing impurities distributed randomly throughout it, there is no pinning in four or more dimensions[8]. In contrast, in fewer than four dimensions, there is pinning. For models used for friction[9], consisting of a three dimensional elastic medium moving over a surface containing a random array of point defects, the critical dimension is three[9]. As a consequence, although if the defect potential is sufficiently large, there will be static friction and "dry friction," for weak defect potentials, there will be no static friction and the kinetic friction will be viscous (i.e., linear in the sliding velocity). The non-periodic "defect potential" acting across the interface could be due to disorder on any length scale in the problem. For example, it can be due to atomic level point defects, such as vacancies and substitutional impurities at the interface, as has been assumed in references 4 and 5, but it can also be due to the fact that the surfaces of the sliding solids are only in contact at micron scale
randomly located protrusions, commonly known as asperities. On the atomic scale, it can also be due to adsorbed film molecules[2].

In contrast to atomic level point defects, however, asperities and adsorbed molecules possess internal structure and as a consequence if they are sufficiently flexible, they can exhibit multistability (i.e., the existence of more than one stable configuration needed for Tomlinson model[6] to apply). Caroli and Noziere[7] proposed an explanation for ”dry friction” based on the Tomlinson model[6]. In the Tomlinson model the two bodies which are sliding relative to each other at relatively slow speeds remain stuck together locally until their centers of mass have slid a small distance relative to each other, at which point the stuck configuration of the two surfaces becomes unstable and the two surfaces locally slip rapidly with respect to each other until they become stuck again, and the process repeats itself. The slipping motion that takes place can either be local or can involve motion of the body as a whole. Then the actual friction acting locally at the interface could be viscous, but the rapid motion that takes place, even at slow sliding speeds, could still result in a sizable amount of friction, even in the limit of vanishing average sliding velocity. In Caroli and Noziere’s model[7] interface contact only occurred at a very dilute concentration of interlocking asperities. It is the rapid stick-slip motion of these asperities, which gives rise to dry friction on the average in this model, if we assume that all of the kinetic energy released in the slip is dissipated, which is probably true in the zero velocity limit. This mechanism would seem to imply that the occurrence of dry friction depends on the existence of multistability; in situations in which the asperities are not multistable, there will be neither ”dry friction” nor static friction. It is argued[7] that in the absence of plasticity in the model, the maximum force of dynamic friction, in the velocity approaches zero limit, must be equal to the force of static friction. It was pointed out in recent work by these workers that the asperities are generally too stiff to undergo the Tomlinson model-like instabilities because of their shape[7]. Therefore, these workers proposed alternative mechanisms. In one mechanism, it is assumed that there exists a glassy film at the interface in the experimental systems that they are trying to describe. Glassy materials possess metastable atomic configurations (the equivalent of ”two-level systems” which are believed to contribute to the specific heat) which can exhibit Tomlinson model-like instabilities during sliding, similar to those found by Falk and Langer in their study of the shearing of glassy materials[10]. This mechanism will, however, only be the correct explanation of ”dry friction” for glassy interfaces. It is not clear, however, that all interfaces are glassy. He, et. al.[2], proposed that the occurrence of static friction between elastic solids requires the existence of an adsorbed sub-monolayer film at the interface. It is important to know if the occurrence of a glassy interface or adsorbed molecules is a re-
quirement for the occurrence of static friction. If it is, it would imply that clean interfaces between crystalline solids would not exhibit static friction. Caroli and Noziere[7] also proposed that adhesive forces could provide the required multistability (because of the so called "jump to contact" instability) to give friction at slow speed. It is not clear that this will be significant for asperities under load, however. In the absence of multistability, there is good reason to believe that there will be neither static friction nor dry friction, at least for light enough loads to put us in the "weak pinning limit" in the language of the charge density wave and vortex problems first studied by Larkin and Ovchinikov[11] and Fukuyama, Lee and Rice[12]. The existence of multistability has also been shown to be a condition for pinning of CDW’s\textsuperscript{1}[13]. [One way of understanding this is that if there is static friction, the sliding velocity of the solid will only be nonzero if a force above the force of static friction is applied to the body. Alternatively, we can view this as implying that the force of friction approaches a nonzero value as the center of mass velocity approaches zero. The arguments in this reference and Ref. 7 tell us that there must be multistability for this to occur.]

In section II, a discussion is given of the scaling theoretic treatment of static friction. This is an outgrowth of a similar treatment by Fisher of the pinning of charge density waves. It is found that, at least for surfaces with defects that produce a relatively weak potential, the Larkin domains (i.e., the regions over which the solid distorts to accommodate defects at the interface) are as large as the interface, as was found by Persson and Tosatti using perturbation\textsuperscript{9} theory. This implies that the static friction decreases as the inverse of the square root of the interface area, as was found for perfect crystalline interfaces by Muser, et. al.[2]. In section, III, it is shown that when one takes into account the distribution of contacting asperities at the interface, one finds that the asperities are in the "strong pinning limit," in which the Larkin domains are much smaller than the interface, implying that there is static friction. This is shown to be a consequence of the fact that the shear force between two contacting asperities varies as the asperities are slipped relative to each other over slip distances of the order of atomic spacings, resulting in a force constant much larger than that due to the elastic force constant of the asperities. As a consequence, the asperities satisfy the criterion for the occurrence of multistability, shown in Ref. 7 to be a requirement for the occurrence of static friction.
II. Scaling Treatment at the Atomic Level of Static Friction

In this section, we will treat the problem of static friction due to disorder which results from atomic level defects, such as vacancies or substitutional impurities using scaling arguments. In the next section, we will consider random contacting asperities, which occur when the surface is viewed on the micron length scale.

Following Fisher’s treatment of the charge density wave (CDW) problem[8], it is clear that we can also use a scaling argument for the friction problem in order to determine whether the pinning potential becomes irrelevant as the length scale becomes large. In order to accomplish this, let us formulate this problem in a way similar to the way that Fisher does, by considering the crystal lattice to be sliding over a disordered substrate potential under the influence of a force $F$, which is applied to each atom in the crystal. Then we can write the equation of motion as

$$m \ddot{u} + m\gamma \dot{u} = \sum_{j'} D(R_j - R_{j'}) \cdot u_{j'} - f(R_j) + F,$$  \hspace{1cm} (1)

where $D(R_j - R_{j'})$ is the force constant matrix for the lattice, $f(R_j)$ is the force due to the substrate on the $j$th atom and $R_j$ is the location of the $j$th atom in the lattice. As a result of slow speed sliding of the lattice over the disordered substrate potential, low frequency acoustic phonons are excited. Since these modes have wavevector $k$, small compared to the Brillouin zone radius, $u_j$, the displacement of the $j$th atom is a slowly varying function of $R_j$. Then, following the discussion in Ref. 14, we can write the first term on the right hand side of Eq. (1) as

$$D'(i^{-1} \nabla_j) u_j,$$  \hspace{1cm} (2)

where $D'(k)$ is the Fourier transform of $D(R_j - R_{j'})$ and $\nabla_j = (\partial/\partial X_j, \partial/\partial Y_j, \partial/\partial Z_j)$ where $R_j = (X_j, Y_j, Z_j)$ to a good approximation. Furthermore, to a good approximation we can expand $D'$ to second order in $\nabla_j$. Equation (1) then becomes

$$m \ddot{u} + m\gamma \dot{u} = -v E' \nabla^2_j u_{j'} - f(R_j) + F,$$  \hspace{1cm} (3)

where $E'$ is an effective Young’s modulus and $v$ is the unit cell volume. We can then apply Fisher’s scaling argument[8] to the resulting equation. This is accomplished by dividing the solid into blocks of of length $L$ lattice sites parallel to the interface and $L'$ lattice sites normal to the interface, assuming that these
dimensions are chosen so that \( u_j \) varies slowly over each such a block. Then integrating equation (15) over a block which lies at the interface, we obtain

\[
L^2 L'[m\ddot{u}_j + m\gamma \dot{u}_j] = -(1/2) L' L^2 (E'/L^2) \nabla'^2_j u_j - Lf'(R'_j) + L^2 L' F,
\]

where \( \nabla'^2_j \) denotes the Laplacian in the rescaled coordinates \( [(X'_j, Y'_j, Z'_j) = (X_j/L, Y_j/L, Z_j/L')] \), which are the coordinates of the centers of the boxes. Here we made use of the fact that \( u_j \) varies on length scales \( L \) and \( L' \), when we transform to the block coordinates. Hence, we may replace \( \nabla^2_j \) by \( L^{-2} \nabla'^2_j \) and \( f(R_j) \) by \( Lf'(R'_j) \) The substrate force is only multiplied by \( L \) because the interaction of the defects with a single block at the surface is proportional to the square root of the surface area of the block. For a thick solid (i.e., one whose lateral and transverse dimensions are comparable), \( L \) and \( L' \) are always of comparable magnitude. Then, we conclude that no matter how large we make \( L \) and \( L' \), the ratio of the elastic force (the first term on the right hand side) and the substrate force (the second term on the right hand side) will remain the same. This implies that we are at the critical dimension for this problem since when the length scales \( L \) and \( L' \) are increased, neither the elasticity nor the substrate force becomes irrelevant. Which ever one dominates at one length scale will dominate at all. Eq. (4) implies that the force of static friction per unit area acting at the interface is inversely proportional to the square root of the interface area.

An alternative way to consider static friction due to Larkin domains is to minimize the potential energy of the solid in contact with the substrate with respect to the size of a Larkin domain[11,12]. Given that the energy density of the elastic solid is given by

\[
(1/2)E' |\nabla u|^2 + V(R_j)\delta(z),
\]

where \( V(R_j) \) is the potential per unit area, the energy of a single Larkin domain is given by

\[
E = (1/2) L' L^2 v E' [|\nabla'_u|^2/L^2 + |\partial u/\partial z'|^2/L'^2] - V_0 L,
\]

where \( v \) is the crystal unit cell volume and \( V_0 \) is a typical value of the potential energy, which when minimized gives that \( L' \approx L \). Then the energy per unit area within a Larkin domain height of the interface is given by

\[
E/L^2 \approx [(1/2) v E' |\nabla'u|^2 - V_0]/L
\]

whose absolute minimum occurs for infinite \( L \) (more correctly \( L \) comparable to the interface length in units of a lattice constant) for \( E'|\nabla'u|^2 > V_0 \), implying that
the Larkin domain energy is minimized when \( L \) is equal to the size of the elastic solid in units of a lattice constant. Then, the sliding elastic solid behaves as a rigid solid, and hence the static friction per unit area decreases as the reciprocal of the square root of the surface area.

If the sliding solid is thin (i.e. has dimensions normal to the interface much smaller than those along it), the \( L' \) can only be made as large as the thickness in units of a lattice constant, but \( L \) can have any value. In this case, we conclude from equation (6) that the substrate force dominates over the elastic forces once \( L \) becomes comparable to the thickness. This means that there are now Larkin domains of size comparable to the ratio of elastic to substrate force times the thickness. The interface will then consist of many Larkin domains. Since the pinning force (i.e., static friction) scales with the number of Larkin domains on the interface, this implies that the force of static friction per unit area approaches a non-zero value. This argument is consistent with our conclusion in the last section, based on perturbation theory, that a thin solid exhibits static friction, but a thick solid will not.

Next, let us consider the effect of fluctuations in the defect concentration, for a thick solid. To do this, we again divide the solid into boxes of length \( L \) and examine what percentage of the blocks at the interface contain a large enough concentration of defects to put these blocks in the "strong pinning" regime (i.e., the regime in which the substrate forces dominate over the elastic forces between the blocks). To examine whether this is possible, let us define a parameter \( \lambda = V_1/E'b^3 \), where \( b \) is a lattice spacing and \( V_1 \) is the strength of the potential due to a defect acting on an atom. Let \( n_c = c'L^2 \) be the number of defects within a particular strongly pinned block and \( c' \) is the defect concentration large enough for it to be considered a strongly pinned block, i.e. a block whose interaction with the substrate is much larger than the elastic interaction between such blocks. (This concentration is necessarily noticeably larger than the mean defect concentration on the interface.) Then the interaction of a typical strong block with the substrate defects is \( \lambda(c'L^2)^{1/2} \). The interface area surrounding each strong block is the total interface area \( A \) divided by the number of strong blocks at the interface, which is equal to \( PA/(bL)^2 \), where \( P \) is the probability of a particular block being a strong one \( [A/(bL)^2] \) is clearly equal to the total number of blocks at the interface, both strong and weak. Then we obtain \( (bL)^2/P \) for the area surrounding a strongly pinned block. Then the typical length for the elastic energy acting between two strong blocks is \( L' b \) where \( L' = L/P^{1/2} \). The ratio of the total elastic energy associated with each strong block to \( E'b^3 \) is the product of the volume surrounding a strongly pinned block\(=(L')^3 \) with \((L')^{-2}\), since \( \nabla^2 u \propto L'^{-2} \), or \( L' \). Then the criterion for a block to be a strong block is
$\lambda (c'L^2)^{1/2} >> L'$, or $\lambda >> (c'P)^{-1/2}$. Since $c'P < 1$, this violates our previous assumption that $\lambda << 1$. Thus, we conclude that such fluctuations in the defect concentration will not lead to static friction.

If we assume that the sliding solid and the substrate surfaces at the interface are incommensurate and that the defects are either vacancies or substitutional impurities, which are centered around particular lattice sites in the substrate, there is another type of concentration fluctuation. For a uniform random distribution of defects over the substrate lattice sites, surface atoms of a completely rigid sliding solid will be found at all possible position within the various defect potential wells, which results in the net force on the solid due to defects being zero on the average. Let us again divide the solid into blocks of length $L$, but now the concentration of defects in each block will be taken to be equal to the mean defect concentration $c$. We will look for blocks in which the defects are distributed such that there is a sizable concentration of atoms located in that region of defect potentials, for which the force on the block is opposite the direction in which we are attempting to slide the block, due to the defects. Then we can divide each block into regions of equal size. If a defect lies in one region, the atoms which interact with the defect will have a force exerted on them opposing the attempt to slide the solid, and if it lies in the other region, the force on the atoms that it interacts with will be in the opposite direction. These two regions are, of course, fragmented. Then the net force on a block of length $L$ at the interface due to the substrate is proportional to $L^2\delta cP$, where $\delta c$ is the mean difference in defect concentrations between the two regions in the block defined above (such that the mean defect concentration over the whole block is $c$) and $P$ is the probability of having a concentration difference $\delta c$ between these two regions. The un-normalized probability of having a concentration difference $\delta c$ between the two regions in a block is given by

$$\frac{N_1!}{n_{c1}!(\frac{1}{2}N_1 - n_{c1})!(cN_1 - n_{c1})!(\frac{1}{2}N_1 - cN_1 + n_{c1})!},$$  \hspace{1cm} (8)$$

where the number of atoms in a block $N_1 = L^2$ and and the number of atoms in the region in which the net force is against the direction in which we are attempting to slide the solid $n_{c1} = (1/2)(c + (1/2)\delta c)N_1$, whose normalized small $\delta c$ approximation is

$$P \approx exp(-\frac{3N_1}{2c(1-c)}\delta c^2).$$  \hspace{1cm} (9)$$

Since in the large $N_1$ limit $P$ decreases exponentially with increasing $N_1$, unless $\delta c \approx N_1^{-1/2}$, we conclude that the substrate defect force on the block will not
increase with increasing $N_1$ and hence in the large $N_1$ limit, the elastic force on the block, which we showed above is proportional to $L=N_1^{1/2}$, will dominate. This implies that this type of concentration fluctuation will not lead to static friction for a macroscopic size block.

Although fluctuations in the defect concentration will not lead to static friction for defects whose potential strength is small compared to typical interatomic elastic energies, fluctuations in the defect strength can, if the defect strength distribution contains fluctuations which are larger than interdefect elastic forces. Let us now consider this possibility. The energy of the interface consists of two parts. One part is the single defect energy, which consists of the interaction energy of a defect with the substrate plus the elastic energy cost necessary for each defect to seek its minimum energy, neglecting its elastic interaction with other defects, which is independent of the defect density. It should be noted that there is a restoring force when the defect is displaced relative to the center of mass, even if the defect-defect interaction is neglected[7,15]. The second part is the elastic interaction between defects within the same solid, which depends on the defect density. In order to determine the effect of these energies, let us for simplicity model the interaction of the $\ell^{th}$ defect with the lattice by a spherically symmetric harmonic potential of force constant $\alpha_\ell$. Assume that in the absence of distortion of the solid, the $\ell^{th}$ defect lies a distance $\Delta_\ell$ from the minimum of its potential well. Let $u_\ell$ be the displacement of the $\ell^{th}$ defect from its initial position. We use the usual elastic Green’s function tensor of the medium at a distance $r$ from the point at which a force is applied at the interface, but for simplicity, we approximate it by the simplified form $G(r) = (E' r)^{-1}$, where $E'$ is Young’s modulus[15]. Then the equilibrium conditions on the $u$’s are

$$u_\ell = (E' a)^{-1} \alpha_\ell (\Delta_\ell - u_\ell) + \sum_j (E' R_{\ell,j})^{-1} \alpha_j (\Delta_j - u_j), \quad (10)$$

where $a$ is a parameter of the order of the size of the defect and $R_{\ell,j}$ is the distance between the $\ell^{th}$ and $j^{th}$ defects. This equilibrium condition is discussed in more detail in the appendix. To lowest order in the inter-defect interaction, the approximate solution for $u_\ell$ is

$$u_\ell = u_\ell^0 + [1 + (E' a)^{-1} \alpha_\ell]^{-1} \sum_j (E' R_{\ell,j})^{-1} \alpha_j (\Delta_j - u_j^0), \quad (11a)$$

where

$$u_\ell^0 = \frac{E' a}{E' a + \alpha_\ell} \Delta_\ell \quad (11b)$$
is the zeroth order approximation (i.e., the solution to Eq. (10) neglecting the second term on the right hand side of the equation). Since the defects are randomly distributed over the interface, we can estimate the second term (i.e., the summation over $j$) on the right hand side of Eq. (11a) by its root mean square (RMS) average which is estimated by integrating the square of the summand over the position of the $j^{th}$ defect which is in contact with the substrate and multiplying by the density of asperities in contact with the substrate $\rho$. Since the angular integrals only give a factor of order unity, we need only consider the integral over the magnitude of $R_{\ell,j}$, giving an RMS value of the sum over $R^{-1}$ of order $[\rho \ln(W/a)]^{1/2}$ where here $W$ is the width of the interface and $a$ is the mean defect size. For $W \approx 1 cm$ and $a \approx 10^{-8} cm$, $[\ln(W/a)]^{1/2}$ is of order unity. For a defect potential of strength $V_1$, $\alpha_\ell \approx V_1/b^2$, where $b$ is of the order of a lattice constant. If $V_1 \approx 1 eV$ and $b \approx 3 \times 10^{-8} cm$, $\alpha \approx 2 \times 10^8 dyn/cm^2$. For $E' \approx 10^{12} dyn/cm$, $u_\ell \approx (\alpha_\ell/E'b)\Delta_\ell \approx 0.06 \Delta_\ell$. This implies that the elasticity of the solid prevents the solid from distorting to any significant degree to accommodate the defects at the interface, which implies that we are in the weak pinning limit. From the scaling arguments of this section, we conclude that there will be no static friction in the macroscopic interface limit. For stronger defect potentials and/or smaller values of $E'$, however, it is clear that we could also be in the strong pinning limit, in which the Larkin length is comparable to a lattice spacing, and hence there is static friction. For almost any surface, contact only takes place at random asperities of mean size and spacing of the order of microns. In the next section it will be argued when one treats the interface on the multi-micron scale as a collection of contacting asperities, one finds in contrast that it is almost certainly in the strong pinning limit.
III. Static Friction due to Disorder on the Micron Length Scale

The arguments in the last section seem to imply that weakly interacting disordered surfaces cannot exhibit static friction. We shall see, however, that unlike weak atomic scale defects, for which the elastic interaction between them can dominate over their interaction with the second surface, for contacting asperities that occur when the problem is studied on the multi-micron scale, the interaction of two contacting asperities from the two different surfaces dominates over the elastic interaction between two asperities in the same surface. It is suggested here that this could be responsible for the virtual universal occurrence of static friction. Roughness due to asperities is well described by the Greenwood-Williamson (GW) model[15], in which there are assumed to be elastic spherically shaped asperities on a surface with an exponential or Gaussian height distribution in contact with a rigid flat substrate, especially for relatively light loads. Volmer and Nattermann presented a possible approximate way of accounting for dry friction[17]. Their discussion of static friction, however, is not qualitatively different from that of Ref. 16. In the GW model, the total contact area is given by

\[ A_c = 2\pi \sigma N R_c \int_{h}^{\infty} ds \phi(s)(s - h), \]  

(12)

where \( \phi(s) \) is the distribution of asperity heights \( s \), in units of a length scale \( \sigma \) associated with the height distribution, \( R_c \) is the radius of curvature of a typical asperity and \( h \) is the distance of the bulk part of the sliding solid from the flat surface in which it is in contact, measured in units of \( \sigma \). Since the force of static friction exerted on a single asperity is expected to be equal to the product of the contact area and a shear strength for the interface, it is proportional to this quantity.

The number of contacting asperities per unit surface area is given by

\[ \rho(h) = \left(\frac{N}{A}\right) \int_{h}^{\infty} ds \phi(s). \]  

(13)

where \( A \) is the total surface area and \( N \) is the total number of asperities whether in contact with the substrate or not. The integral in Eq. (12) divided by the integral in Eq. (13), which is proportional to the contact area per asperity and the square root of the integral in Eq. (13) are plotted as a function of the load,
which is given in this model by

\[ F_L = \frac{4}{3} E' N (R_c/2)^{1/2} \sigma^{3/2} \int_{h}^{\infty} ds \phi(s)(s - h)^{3/2}, \]  

(14)

in Fig. 1. A Gaussian distribution is assumed here for \( \phi(s) \) (i.e., \( \phi(s) = (2\pi)^{-1/2} e^{-s^2/2} \)). Since the square root of Eq. (13) drops to zero in the limit of vanishing load, whereas Eq. (12) divided by Eq. (13) approaches a nonzero value, this implies that the interface will approach the strong pinning regime (i.e., the regime in which the asperity-substrate interaction dominates over the inter-asperity interaction) in the limit of vanishing load.

Let us now apply the equilibrium conditions expressed in Eqs. (10) and (11), used in the last section, to the asperities. In this section, \( a \) is taken to be a parameter of the order of the size of the asperity. Since the contacting asperities are randomly distributed over the interface, we can again estimate the second term (i.e., the summation over \( j \)) on the right hand side of Eq. (10) by its root mean square (RMS) average which is estimated by integrating the square of the summand over the position of the \( j^{th} \) asperity, which is in contact with the substrate, over its position and multiplying by the density of asperities in contact with the substrate \( \rho \), giving an RMS value of the sum over \( R^{-1} \) of order \( \left[ \rho \ln \left( \frac{W}{a} \right) \right]^{1/2} \) where here \( W \) is the length of the interface and \( a \) is the asperity size. For \( W \approx 1 \text{cm} \) and \( a \approx 10^{-4} \text{cm} \), \( \left[ \ln(W/a) \right]^{1/2} \) is of order unity. The energy of the system can be written as

\[ (1/2) \sum_j \alpha_j |\Delta_j - u_j|^2 + \]

\[ (1/2) E' \sum_j \int d^3r [\nabla G(r)] |(\alpha_j |\Delta_j - u_j)|^2 \]  

(15)

It follows from Eqs. (10,11,15) that the two lowest order nonvanishing terms in an expansion of the energy of the system in powers of \( \rho^{1/2} \) are the zeroth and first order ones. The shearing of the junction at the area of contact of two asperities involves the motion of two atomic planes relative to each other, and hence the distance over which the contact potential varies must be of the order of atomic distances. Then, if we denote the width of the asperity contact potential well (i.e., the length scale over which the contact potential varies) by \( b \), of the order
of atomic spacings, we must choose a typical value for \( \alpha \) such that \( \alpha b \) is of the order of the shear rupture strength of the asperity contact junction \( \approx E'\pi a^2 \). Thus, \( \alpha \gg E' a \), since \( a \gg b \). Then, applying Eq. (11b) to the contacting asperities, we find that \( u_0 \approx \Delta \), i.e., the contacting asperities lie at the minima of the contact potential. This is very easy to understand. Since the contact potential varies over distances of the order of an atomic spacing, the asperities can all sink very close to their contact potential minima by moving a distance of the order of an atomic spacing, with negligible cost in elastic potential energy. Zeroth order in the asperity density in Eq. (15) is of the order of \( \alpha \Delta^2 \), where \( \alpha \) is a typical value of \( \alpha_j \), and \( \Delta \) is a typical value of \( \Delta_j \). The term linear in \( \rho^{1/2} \) is easily shown to be of the order of \( E' a^2 \Delta^2 \rho^{1/2} \), independent of \( \alpha \) to zeroth order in \( E'/\alpha \). Since it depends on \( \rho \) it represents an interaction energy between the asperities.

Let us now give sample numerical values for some of the quantities which occur in the application of the GW model to this problem. Following Ref. 16, we choose \( \sigma = 2.4 \times 10^{-4} mm \) and \( R_c = 6.6 \times 10^{-2} mm \), and assume that there is a density \( \rho \) of \( 4.0 \times 10^3 \) asperities/mm\(^2\). Then by performing the integrals in Eqs. (12-14), we find that for \( F_L/A = 3.98 \times 10^{-4} N/mm^2 \), where \( A \) is the apparent area of the interface, the total contact area divided by \( A \) is \( 3.03 \times 10^{-5} \), and the contact area per asperity from the ratio of Eqs. (12) and (13) is \( 2.44 \times 10^{-5} mm^2 \). Also, \( \rho(h)^{1/2} \), which is equal to the square root of Eq. (13) is \( 1.11 mm^{-1} \). The mean interasperity interaction force is approximately equal to the derivative of the first order term in \( \rho^{1/2} \) given below Eq. (15) with respect to \( \Delta \) or \( E' a^2 \rho(h)^{1/2} \Delta \), where \( a \) is taken as the square root of the mean contact area per asperity divided by \( \pi \). The mean strength of the force acting on an asperity, due to the solid with which it is in contact, will be estimated by the product of its contact area and the shear rupture strength \( E_r \). Then, the condition for the latter quantity to dominate over the asperity-asperity interaction, \( E_r \pi a^2 > E' \pi a^2 \rho^{1/2} \Delta \) or \( E_r/E' > 4 \rho(h)^{1/2} \Delta \), is easily satisfied by the above calculated quantities since the right hand side is \( 4 \times 10^{-7} \) (since \( \Delta \) is of the order of the potential well width or \( 10^{-8} cm \) ) and the left hand side is of order unity because for an asperity, which is typically dislocation free, \( E_r \) is comparable to the shear modulus, which is of the order of \( E' \).

For higher loads since the density of contacting asperities increases, the system appears to move towards the “weak pinning” limit, the latter conclusion is most likely incorrect, however, because it does not take into account the fact that the distribution of asperity heights contains asperities which are much higher than average. These asperities will be compressed much more than a typical asperity, making the friction force on them considerably larger than average. Since
the probability of such asperities occurring is relatively small, however, they will be typically far apart, putting them in the strong pinning limit (i.e., each one lies in the bottom of its potential well). For example, the probability of the ratio of an asperity height to $\sigma$ being greater than $h$ by a value $h_L$ is

$$P(h_L) = \int_{h+h_L}^{\infty} ds \phi(s),$$

(16)

whose mean height and hence contact area is proportional to

$$P(h_L)^{-1} \int_{h+h_L}^{\infty} ds \phi(s)(s - h).$$

(17)

These two quantities are plotted in Fig. 2. It is seen that even for $h_L$ only equal to $1/2$ [corresponding to an asperity height equal to $(1.5\sigma)$], Eq. (17) remains larger than the square root of Eq. (16). It is most likely only possible to apply the GW model to quite light loads anyway. For higher loads, plasticity becomes important[16].

Although it has been argued here that the GW model predicts the occurrence of a sufficiently dilute concentration of asperities with strong enough forces acting on them due to the second solid to consider the asperities to be essentially uncorrelated (i.e., their Larkin lengths are comparable to their separation), this still does not necessarily guarantee that there will be static friction, since it has been argued that even for uncorrelated asperities, static friction will only occur if the asperities exhibit multistability[7,9,17]. The condition for multistability to occur at an interface[9], namely that the force constant due to the asperity contact potential be larger than that due to the elasticity of the asperity ($\approx E'a$), however, is satisfied, as discussed under Eq. (15).

In conclusion, when one considers atomically smooth surfaces, arguments based on Larkin domains indicate that the disorder at an interface between two weakly interacting nonmetallic elastic solids in contact will not result in static friction. When one applies such arguments to the distribution of asperities that occur on multimicron length scales, however, one finds that the asperities are virtually always in the "strong pinning regime," in which the the Larkin domains are comparable in size to a single asperity. This accounts for the fact that there is almost always static friction. Muser and Robbins’ idea [2], however, is not invalidated by this argument. Their result will still apply for a smooth crystalline...
interface. It will also apply in the present context to the contact region between two asperities, implying that for a clean interface the shear force between contacting asperities is proportional to the square root of the contact area. The GW model predicts for this case that the average force of friction is proportional to the 0.8 power of the load \[16\], but this load dependence is not significantly different from when the asperity contact force is proportional to the contact area. This is illustrated in Fig. 3, where the both the integral over \( s \) in Eq. (12) and the integral,

\[
\int_{h}^{\infty} ds \phi(s)(s - h)^{1/2}
\]

are plotted as a function of the load. This quantity, and hence the static friction, are approximately proportional to the 0.8 power of the load. Furthermore, some simple arguments show that although the Muser-Robbins \[2\] picture, when the effects of asperities considered in the present work are taken into account, does not allow one to conclude that there will be no static friction for clean surfaces, it does predict that the static friction for clean surfaces is much smaller than what is normally observed. The argument is as follows: If the interface between two asperities is either in the strong pinning limit or using the Muser-Robbins\[2\] picture, it contains a sub-monolayer of mobile molecules, the force of static friction per asperity is given by

\[
F_s/N = E_r < A_c >,
\]

where one expects for the shear rupture strength at the asperity contact region, \( E_r \),

\[
E_r \approx cV_0/b^3,
\]

where \( c \) is the concentration of defects at the interface, \( V_0 \) is the strength of the defect potential and \( b \) is of the order of a lattice constant. Using the sample parameters given earlier in this section, we obtain as an estimate of the static friction coefficient

\[
\mu_s = F_s/F_L \approx 0.1.
\]

According to the Muser-Robbins argument, for clean surfaces, Eq. (19) is replaced by

\[
F_s/N \approx E_r < A_c(b^2/cA_c)^{1/2} > = E_r b < A_c^{1/2} > /c^{1/2}
\]

which when one again substitutes the sample parameters given earlier in this.
section gives

\[ \mu_s \approx 10^{-5}. \]  \hspace{1cm} (23)

On the basis of this argument, one concludes that the ideal static friction coefficient between clean, weakly interacting surfaces in the light load limit, is much smaller than what one typically observes.
Acknowledgements

I wish to thank the Department of Energy (Grant DE-FG02-96ER45585) for their financial support. I also wish to thank R. Markiewicz for useful discussions.
Appendix A—Equilibrium Condition for Pinning Centers at an Interface

The interaction forces between two solids in contact act at various points along the interface. For atomically flat surfaces, they are expected to act at the points of contact of the atoms of both solids at the interface. Since it has been established, however, that weakly interacting perfect incommensurate surfaces exhibit no static friction [1-3], we expect that any static friction that occurs is due to defects. Therefore, let us consider the interaction of point defects, randomly distributed over the interface. We must consider both the potential of interaction of a defect of one surface with an atom on the second surface and the elastic energy cost that one must pay when the solid distorts in order to minimize the interfacial and elastic potential energies. If $f_j$ represents the force acting on an atom at site $j$ in the solid due to its interaction with a second solid with which it is in contact, the displacement of the atom at the $\ell^{th}$ lattice site $u_\ell$ is given by

$$u_\ell = G_{\ell,j} \cdot f_j,$$  \hspace{1cm} (A1)

where

$$G_{\ell,j} = D_{\ell,j}^{-1}$$  \hspace{1cm} (A2)

where $D_{\ell,j}$ is the dynamical matrix [14, 15]. For simplicity, we assumed in Eq. (10) that $f_j$ has the form

$$f_j = \alpha_j (\Delta_j - u_j).$$  \hspace{1cm} (A3)

Then Eq. (A1) can be written as

$$u_\ell = G_{\ell,\ell} \cdot \alpha_\ell (\Delta_\ell - u_\ell) + \sum_{j \neq \ell} G_{\ell,j} \cdot \alpha_j (\Delta_j - u_j).$$  \hspace{1cm} (A4)

Following the discussion in Ref. 14, we find that

$$G_{\ell,j} = v(2\pi)^{-3} \sum_{\gamma} \int d^3 k e^{i\mathbf{k} \cdot (\mathbf{R}_\ell - \mathbf{R}_j)} \frac{\hat{e}_k \hat{e}_j}{m \omega_\gamma^2(\mathbf{k})},$$  \hspace{1cm} (A5)

where $m$ is the mass of an atom in the solid, $\hat{e}_k$ is the unit vector which gives the polarization of the $\gamma^{th}$ phonon mode of wavevector $\mathbf{k}$, $\mathbf{R}_j$ is the location of the $j^{th}$ atom and $v$ is the unit cell volume. In order to simplify the problem, let
us replace the tensor $\hat{\epsilon}_{k}^{\gamma} \hat{\epsilon}_{k}^{\gamma}$, by the unit tensor, which should give results of the correct order of magnitude. Then Eq. (A5) becomes when the integral over k is done in the Debye approximation

$$G_{\ell,\ell} \approx \frac{9}{m c^{2} k_{D}^{3}} \int_{0}^{k_{D}} \frac{d k}{k} \frac{\sin(kR)}{k}, \quad (A6)$$

where $R = |R_{\ell} - R_{j}|$, c is the mean sound velocity and we have used the fact that the Debye wavevector $k_{D}$ is related to v by

$$v(2\pi)^{-3/4} \pi k_{D}^{3/3} = 1. \quad (A7)$$

In Ref. 14, it is shown that the elastic constants are given by

$$-(1/2v) \sum_{R} RD(R)R. \quad (A8)$$

The magnitude of a typical value of an elastic constant E is given by

$$E = (1/2)(2\pi)^{-3} b^{2} \sum_{\gamma} \int d^{3}k \omega_{\gamma}^{2}(k). \quad (A9)$$

When Eq. (A9) is evaluated in the Debye approximation, we obtain

$$E = (3/10\pi^{2}) mc^{2} k_{D}^{5} b^{2}. \quad (A10)$$

For $k_{D}R >> 1$, we find using Eqs. (A6) and (A10), taking $k_{D}b \approx \pi$

$$G_{\ell,\ell} = (E'R)^{-1}, \quad (A11)$$

where $E' = (40/9)E$. For $k_{D}R << 1$,

$$G \approx (E'b)^{-1} \quad (A12).$$

Then, from Eqs. (A1), (A3), (A11) and (A12), we obtain Eq. (10).
The equilibrium condition expressed in Eq. (10) can also be applied to an interface for which the contact takes place only at a dilute concentration of randomly placed contacting asperities, giving for the displacement at a point on the \( \ell \)th asperity

\[
\mathbf{u}_\ell = \sum_j \int d^2 r'_j (E'|\mathbf{r}_\ell - \mathbf{r}_j - r'_j|)^{-1} \mathbf{p}(r'_j),
\]  

(A13)

where \( \mathbf{r}_j \) is the location of a central point in the contact area of the \( j \)th asperity, \( r'_j \) gives the location of an arbitrary point on this asperity relative to \( \mathbf{r}_j \) and \( \mathbf{p}(r'_j) \) is the shear stress at the point \( r'_j \). We have replaced the summation over atomic positions in Eq. (A1) by the integral over \( r'_j \) over the contact area of the \( j \)th asperity. In the dilute asperity limit, in which \( |r_\ell - r_j| \gg r'_j \), Eq. (A13) is to a good approximation

\[
\mathbf{u}_\ell = \sum_{j \neq \ell} (E'|\mathbf{r}_\ell - \mathbf{r}_j|)^{-1} \mathbf{f}_j + \int d^2 r'_\ell (E'|\mathbf{r}_\ell + r'_\ell|)^{-1} \mathbf{p}(r'_\ell),
\]  

(A14)

where \( \mathbf{f}_j = \int d^2 r'_j \mathbf{p}(r'_j) \), where the range of integration is over the contact area of the \( j \)th asperity. For simplicity, we may replace \( \mathbf{p}(r'_\ell) \) in the integral over \( r'_\ell \) by its average value, denoted by \( \pi a^2 \mathbf{f}_\ell \). Then we need to estimate the integral

\[
\int d^2 r'|\mathbf{r} + r'|^{-1}\n\]  

(A15)

where \( \mathbf{r} \) denotes a point on the \( \ell \)th asperity and the integral runs over the contact area of this asperity. Taking the contact area to be a circle of radius \( a \), this integral can easily be shown to be equal to

\[
4 \int_0^\infty dr' r'(r + r')^{-1} K\left(\frac{4rr'}{r + r'}\right),
\]  

(A16)

where \( K(k) \) is the complete elliptic function. It has a logarithmic singularity at \( r' = r \), which is integrable, and is of order 1 away from the singularity. Consequently, the integral is of order \( a \) and we obtain a contribution of order \( (E' a)^{-1} \mathbf{f}_\ell \) for the last term in Eq. (A14). If for simplicity, we replace \( \mathbf{f}_j \) by \( \alpha_j (\Delta_j - \mathbf{u}_j) \), as was done in section III, and we obtain the equilibrium condition for the asperities used in that section.
References

[1] J. E. Sacco and J. B. Sokoloff, Phys. Rev. B 18 (1978) 6549.
[2] G. He, M. H. Muser and M. O. Robbins, Science 284 (1999) 1650 ; M. H. Muser and M. O. Robbins, Phys. Rev. B 61 (2000) 2335; M. H. Muser, L. Wenning and M. O. Robbins, Phys. Rev. Lett. 86, 1295 (2001).
[3] S. Aubry, in Solitons and Condensed Matter, ed. A. R. Bishop and T. Schneider (Springer, New York, 1978), p. 264.
[4] J. B. Sokoloff, Phys. Rev. B51 (1995) 15573; J. B. Sokoloff and M. S. Tomassone, Phys. Rev. B 57 (1998) 4005.
[5] M. S. Tomassone and J. B. Sokoloff, Physical Review B60 (1999) 4005.
[6] G. A. Tomlinson, Phil. Mag. 7 (1929) 905.
[7] C. Caroli and Ph. Nozieres, European Physical Journal B4 (1998) 233; Physics of Sliding Friction, edited by B. N. J. Persson and E. Tosatti, NATO ASI Series E: Applied Sciences, Vol. 311 (Kluwer Acad. Publ., Dordrecht, 1996).
[8] D. S. Fisher, Phys. Rev. B31 (1985) 1396; Phys. Rev. Lett. 50 (1983) 1486.
[9] B. N. J. Persson and E. Tosatti in Physics of Sliding Friction, ed. B. N. J. Persson and E. Tosatti (Kluwer Academic Publishers, Boston, 1995), p. 179; V. L. Popov, Phys. Rev. Lett. 83 (1999) 1632.
[10] M. L. Falk and J. S. Langer, Phys. Rev. E 57 (1998) 7192.
[11] A. I. Larkin and Yu. N. Ovchinikov, J. Low Temp. Phys. 34 (1979) 409.
[12] H. Fukuyama and P. A. Lee, Phys. Rev. B17 (1977) 535; P. A. Lee and T. M. Rice, Phys. Rev. B19 (1979) 3970.
[13] J. B. Sokoloff, Phys. Rev. B23 (1981) 1992.
[14] N. W. Ashcroft and N. D. Mermin, "Solid State Physics" (Saunders College, Philadelphia, 1976), pp. 443-444.
[15] L. D. Landau and E. M. Lifshitz, Theory of Elasticity, (Pergamon Press, New York, 1970), p. 30.
[16] J. A. Greenwood and J. B. P. Williamson, Proc. Roy. Soc. A295 (1966) 3000; J. I. McCool, Wear 107 (1986) 37.
[17] A. Volmer and T. Nattermann, Z. Phys. B104 (1997) 363.
Figure Captions

1. The curve which is lower at the right is a plot of the integral in Eq. (12) divided by the integral in Eq. (13) and the curve which is higher on the right is a plot of the square root of the integral in Eq. (13) versus the integral in Eq. (14). All quantities are dimensionless.

2. Eq. (17) (the higher curve) and the square root of Eq. (16) (the lower curve) are plotted versus the load Eq. (14) divided by \( \frac{4}{3} E \frac{b}{2} \frac{1}{2} \sigma^{3/2} \). All quantities are dimensionless.

3. The integral over \( s \) in Eq. (12) (the lower curve) and the integral over \( s \) in Eq. (18) (the upper curve) are plotted as a function of the integral over \( s \) in Eq. (14), which is proportional to the load. All quantities are dimensionless.
Asperity Friction and Interaction
