VARIABLE COEFFICIENT THIRD ORDER KdV TYPE OF EQUATIONS

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Abstract

We show that the integrable subclassess of the equations
\[ q_t = f(x,t)q_{,3} + H(x,t,q,q_{,1}) \]
are the same as the integrable subclassess of the equations \( q_t = q_{,3} + F(q,q_{,1}) \).
Classification of nonlinear partial differential equations possessing infinitely many symmetries in 1+1 dimensions has started almost two decades ago. So far the complete classification has been done for some evolution type of autonomous equations\cite{1,2,3,4}. There are some partial attempts of the classification of the non-autonomous type of equations\cite{1,6,8,5,9}. In 1+1 or 2+0 dimensions almost all definitions of integrability coincide. But what is important is the ease of applicability. Recently we have introduced a new approach which is based on the compatibility of the symmetry equation (linearized equation) and an eigenvalue equation\cite{10}. Our method can be put into an algorithmic scheme and utilized for two purposes. The first is to test whether a given partial differential equation is integrable. The second is to classify nonlinear partial differential equations according to whether they admit generalized symmetries.

In this work we show that the most general equations of the type \(q_{,t} = f(x, t)q_{,3} + H(x, t, q, q_{,1})\), up to coordinate transformations, have the same integrable subclass as the autonomous equations \(q_{,t} = q_{,3} + F(q, q_{,1})\). Here \(f(x, t)\) is an analytic function of the independent variables \(x\) and \(t\), \(H\) is a function of the dependent variable \(q\), its \(x\)-derivative \(q_{,1}\) and also on the independent variables \(x\) and \(t\). The function \(F\) depends on only \(q\) and \(q_{,1}\).

First we will give an outline of the method

Consider an evolution equation of the form

\[
q_{,t} = K(x, t, q, q_{,1}, q_{,2}, ..., q_{,n}) \equiv K(q),
\]

(1)

where \(q_{,i} = (\frac{\partial}{\partial x})^i q\), \(i = 0, 1, 2, ..., n\). The order of \(K(= n)\) is called the order of the equation. A symmetry \(\sigma(x, t, q)\) of Eq. (1) satisfies
\[ \sigma_t = K'(\sigma) = \sum_{i=0}^{n} \frac{\partial K}{\partial q_i} \sigma_i, \quad (2) \]

such that Eq.(1) is form invariant under the transformation

\[ q \rightarrow q + \epsilon \sigma, \quad (\epsilon, \text{infinitesimal}) \quad (3) \]

Here \( \sigma(x, t, q) \) is a differentiable function of \( q, q_1, q_2, \ldots \) and the prime denotes the Fréchet derivative.

In [10] we conjectured that a nonlinear partial differential equation is integrable if the linearized equation(2) supports an eigenvalue equation. Therefore let us introduce an eigenvalue equation , linear in \( \lambda \), for \( \sigma \) in the form

\[ \sigma_n = \sum_{i=0}^{n-1} (A_i \lambda + B_i) \sigma_i, \quad (4) \]

where \( A_i \) and \( B_i \) are functions of \( x, t, \) and \( q_i \). Their dependences on \( q_i \) are decided by the order of \( K \). The order of the eigenvalue equation is determined by the order of \( K \). The compatibility of linearized and eigenvalue equations, at all powers of \( \lambda \), gives:

a) a set of algebraic equations among \( A_i, B_i \) and \( \frac{\partial K}{\partial q_i} \)’s ;

b) a set of coupled PDE’s among \( A_i, B_i \) and \( \frac{\partial K}{\partial q_i} \)’s.

Using the definition of total derivatives

\[ \frac{df}{dx} = D_x f = \frac{\partial f}{\partial x} + \sum_{i=0}^{\infty} q_{i+1} \frac{\partial f}{\partial q_i}, \quad (5) \]

\[ \frac{df}{dt} = D_t f = \frac{\partial f}{\partial t} + \sum_{i=0}^{\infty} K_i \frac{\partial f}{\partial q_i}, \quad (6) \]

for any function \( f \) in the set b of coupled PDE’s and comparing coefficients of \( q_i \)’s , we obtain several classes of \( A_i, B_i \) along with the explicit forms of
In a self consistent way. If the integrability is proved for a given class, the eigenvalue equation (4) can always be put in the form

\[ M\sigma = \lambda N\sigma, \]  

(7)

where \( M \) and \( N \) are local operators and depend on \( x, t, q, i \). Equation (7) is nothing but the definition of the recursion operator, provided that \( N^{-1} \) exists,

\[ R = N^{-1}M, \]

(8)

which maps symmetries to symmetries

\[ R\sigma_n = \sigma_{n+1}, \]

(9)

where \( n \) is a non-negative integer. Thus the existence of an eigenvalue equation (4) is equivalent to the existence of a recursion operator.

As an illustration let us give the classification of third order autonomous evolution equations of the form

\[ q_{,t} = q_{,3} + F(q, q_{,1}). \]

(10)

This classification has been investigated by several authors, mainly from the point of view of their integrability\[1, 3, 4\]. Let us follow the method outline above.

**Linearized equation:**

\[ \sigma_{,t} = \sigma_{,3} + \frac{\partial F}{\partial q_{,1}}\sigma_{,1} + \frac{\partial F}{\partial q}\sigma. \]

(11)

**Eigenvalue equation:**
\[ \sigma_3 = (A_2 \lambda + B_2)\sigma_2 + (A_1 \lambda + B_1)\sigma_1 + (A_0 \lambda + B_0)\sigma, \quad (12) \]

where \( A_i \) and \( B_i \) depend on \( q, q_1 \) and \( q_2 \). The compatibility equation of Eq.(11) and Eq.(12) gives the following integrable equations with non zero eigenvalue coefficients

**Case I.** \[ q_t = q_3 + \frac{a}{6} q_1^3 + \frac{b}{2} q_1^2 + c q_1 + d, \quad (13) \]

with

\[
\begin{align*}
B_2 &= \frac{aq_2}{aq_1 + b}, & B_1 &= -\frac{1}{3} \left[ aq_1^2 + 2bq_1 + 2c \right], & A_1 &= 1 \\
B_0 &= \frac{1}{3(aq_1 + b)} \left[ q_2 (2ac - b^2) \right], & A_0 &= \frac{aq_2}{aq_1 + b}. & (14)
\end{align*}
\]

Here \( a, b \) and \( c \) are constants, and

**Case II.** \[ q_t = q_3 + \frac{a}{6} q_1^3 + b(q)q_1, \quad 3 \frac{d^3 b}{dq^3} + 4a \frac{db}{dq} = 0, \quad (15) \]

with

\[
\begin{align*}
B_0 &= \frac{1}{3} \left[ 2bq_2 - \frac{3}{2} b \frac{dq_2}{dq} \right], & B_1 &= -\frac{1}{3} (aq_1^2 + 2b), & B_2 &= \frac{q_2}{q_1}, \\
A_0 &= -B_2, & A_1 &= 1. & (16)
\end{align*}
\]

Here \( a \) is a constant. The basic equations in the classification are the KdV \((a = 0, b = 6q, \text{ in case II})\), pKdV \((a = 0, \text{ in case I})\), mKdV \((a = 0, b = 6q^2, \text{ in case II})\), pmKdV \((b = 0, \text{ in both cases})\) and CDF equation \((a = -\frac{3}{4}, \text{ in case II})\). The recursion operators for equations (13) and (15) are found by the utility (14) and (16). They are respectively given by
I. \[ R = D^2 + \frac{2c}{3} + a \frac{q_1}{3} + \frac{2b q_1}{3} - \frac{aq_1}{3} D^{-1}(q_2.) - \frac{b}{3} D^{-1}(q_2.) \quad (17) \]

II. \[ R = D^2 + \frac{aq_1}{3} + \frac{2b}{3} - \frac{aq_1}{3} D^{-1}(q_2.) + \frac{q_1}{3} D^{-1}(\frac{db}{dq}). \quad (18) \]

where \( D^{-1} = \int_{-\infty}^{x} dx \). We have the following proposition.

**Proposition 1**: Under the symmetry classification the equations of the type \( q_{,t} = q_3 + F(q, q_1) \), up to coordinate transformations, has integrable subclasses given in (13) and (14).

Our aim, in this work, is to give a classification of the non-autonomous type of integrable equations (10)

\[ q_{,t} = f(x, t)q_3 + H(x, t, q, q_1) \quad (19) \]

We divide the classification procedure, for Eq.(19), into the following three cases: i) \( f \) depends only on \( t \), ii) \( f \) depends only on \( x \), and iii) \( f \) depends on both \( x \) and \( t \).

i) \( f \) depends only on \( t \): One of the integrable subclass, using our classification scheme, is

\[ q_{,t} = v^3 q_3 + \left( \frac{h w^2 q^2}{2} + c_1 w h v q + h c_2 v \right) q_1 + \left( \frac{\dot{h}}{2 h} - \frac{\dot{v}}{v} \right) x q_1 - \frac{\dot{w} q}{w}, \quad (20) \]

where \( f(t) = v(t)^3 \), \( h, w \) depend on \( t \) only and \( c_1, c_2 \) are constants. The dot appearing over a quantity denotes \( t \) derivative. The recursion operator is given by

\[ R = \frac{v^2}{h} D^2 + \frac{w^2}{3} q^2 + \frac{2c_1}{3} q + \frac{2}{3} c_2 + \frac{w^2}{3} q_1 D^{-1}(q.) + \frac{c_1 w}{3} q_1 D^{-1}. \quad (21) \]
We observe that Eq. (20) is transformed into an equation which belongs to the case II \((a = 0)\) in Eq. (15) through the transformation

\[
q = w^{-1}u(\xi, \tau),
\]

\[
\xi = x\beta(t), \quad \beta = \frac{h^{1/2}}{v}, \quad \tau = \int^{t} h^{3/2} dt'.
\]  \hspace{1cm} (22)

In the classification programs, if it is possible, we transform (by coordinate or contact transformations) the given class of PDE’s to more simpler ones. To this end in the sequel we shall transform all cases \((i, ii, iii)\) to the form

\[
q_{,\tau} = q_{,3} + H_{2}(\xi, \tau, q, q_{,1})
\]  \hspace{1cm} (23)

and classify this type of equations. In this first case \((i)\) we have the following proposition.

**Proposition 2:** Under the symmetry classification the equations of the type

\[
q_{,t} = f(t) q_{,3} + F_{1}(x, t, q, q_{,1}),
\]  \hspace{1cm} (24)

up to coordinate transformations, give the same integrable subclass as in proposition 1. Furthermore, Eq. (24) reduces to Eq. (23) by the transformation \(dt = \frac{1}{f} d\tau\) and \(x = \xi\).

\(ii\) \(f\) depends only on \(x\): In this case the form of equation is

\[
q_{,t} = f(x)q_{,3} + F_{2}(x, t, q, q_{,1}),
\]  \hspace{1cm} (25)

one integrable class turns out to be simple

\[
q_{,t} = q_{,3} + q_{,1}^{2} + c_{1}x + c_{2},
\]  \hspace{1cm} (26)
Recursion operator for Eq.(26) is

\[ R = D^2 + \frac{4}{3} q_1 - \frac{4}{3} c_1 t - \frac{2}{3} D^{-1}(q_2), \]  

(27)

Now, differentiate Eq.(26) with respect to \( x \) and substitute \( q = z, t = \tau \) then Eq.(26) belongs to Eq.(13).

Before proceeding to the next case, we observe the following:

**Proposition 3:** Under the symmetry classification the equations of the type (25) up to coordinate transformation,

\[ \begin{align*}
q_1 &= f^{1/3} u(\xi, \tau), \\
\xi &= \int^x \frac{1}{f^{1/3}} dx', \\
\tau &= t,
\end{align*} \]

(28)
give the same integrable subclass as in Eq.(23).

**iii)** \( f \) depends on both \( x \) and \( t \): In this case we have \( \frac{q}{t} = f(x, t) q_3 + F_3(x, t, q, q_1) \) type of equations and its one integrable class turns out to be relatively simple. Below we give this equation and its recursion operator

\[ q_1 = u^3 q_3 + \left[ \frac{a}{2} u \right] q_1^2 + \frac{3}{2} u (u^2_1 - 2 u u_2) \left( q_1 - \frac{u_1}{u} q \right) \]

(29)

where \( a \) is an arbitrary constant. Here we have set \( f(x, t) = u(x, t)^3 \) and \( u(x, t) \) satisfies the Harry Dym equation

\[ u_3 = u^3 u_3 \]

(30)

It means that Eq.(29) is integrable if \( u \) satisfies the Eq.(30). The recursion operator is given by

\[ R = u s D^{-1} u D \left( \frac{1}{s} R_1 \right), \]

(31)
where

\[
R_1 = D^2 - \frac{u_1 q_2}{us} + \frac{u_2 q_1}{us} - \frac{u_2}{u} + \frac{aq^2}{3u^4} + sD^{-1}\left(\frac{V}{s}\right),
\]

\[
V = \frac{a}{3u^4}qq_1 + \frac{1}{us}\left[u_1 q_3 - q_1 u_3 + \frac{2s_1}{s}(-u_1 q_3 + q_1 u_3)\right]
\]

and \(s = -q_1 + q\frac{u_1}{u}\). Eq.(29) together with (30) is equivalent to the mKdV. We will give the proof of this in two steps: Let \(f = u^3\) and \(q = uz\) then we have

\[
z_t = u^3 z_3 + 3u^2 u_1 z_2 + \left(\frac{a}{2}uz^2 + \frac{3}{2}uu_1^2\right)z_1,
\]

where \(z(x, t)\) is the new dynamical variable. Now let us perform the following transformation

\[
\xi = \int_x^\infty \frac{dx'}{u(x', t)}, \quad \tau = t
\]

It is straightforward to show that under this transformation Eq.(33) goes to the mKdV. Now we state the following proposition.

**Proposition 4**: Under the symmetry classification the equations of the type \(q_t = f(x, t)q_3 + F_3(x, t, q, q_1)\), up to the transformations,

\[
q = f^{1/3}(\xi, \tau),
\]

\[
\xi = \int_x^\infty \frac{1}{f^{1/3}}dx', \quad \tau = t,
\]

like in the previous example, give the same integrable subclass (for \(z(\xi, \tau)\)) as in Eq(23).
Hence whatever the coefficient function \( f(x, t) \) we showed that in general the type of equations \( (19) \), by coordinate transformations reduce to the following type of equations

\[
q_{,t} = q_{,3} + P(x, t, q, q_{,1})
\]  

(36)

We now give the classification of this type of equations.

\[ (1) \quad q_{,t} = q_{,3} + \frac{a}{2} q_{,1}^2 + bq_{,1} - \frac{\dot{w}}{w} q + c \]  

(37)

with

\[ \dot{b} = b_{,3} + bb_{,1} - ac_{,1} + \frac{\dot{h}}{2h} b + \frac{\dot{d}}{2h} d + \frac{\dot{h}}{h^2} \frac{\dot{h}^2}{h^2} x \]  

(38)

where \( w = \frac{a}{\sqrt{h}} \), \( a, d, h \) depend on \( t \) only and \( b, c \) depend on \( x, t \).

\[ (2) \quad q_{,t} = q_{,3} - \frac{a^2}{8} q_{,1}^3 + \left( \frac{\dot{h}}{w} e^{aq} - wh e^{-aq} + b \right) q_{,1} + \frac{\dot{h}}{2h} r q_{,1} - \frac{\dot{a}}{a} q + \frac{\dot{w}}{aw} \]  

(39)

where \( a, w, \) and \( b \) depend on \( t \) only.

\[ (3) \quad q_{,t} = q_{,3} + \frac{a}{2} q_{,1}^2 q_{,1} + \left( \frac{b}{w} \right)^{1/2} q_{,1}^3 + \left( \frac{\dot{h}}{2h} x + c \right) q_{,1} + \frac{\dot{w}}{2w} q - \frac{h}{2} \left( \frac{w}{b} \right)^{1/2} \dot{d} \]  

(40)

where \( b = h^2 d \), and all parameters appearing in the equation depend on \( t \).

\[ (4) \quad q_{,t} = q_{,3} + \frac{a}{2} q_{,1}^2 + bq_{,1} - \frac{1}{2} \left( -\frac{\dot{h}}{h} x - \frac{b^2}{a} + \frac{c}{ah} \right) q_{,1} + \frac{b_1}{2} q^2 
- \frac{1}{2} \left( \frac{\dot{w}}{w} - \frac{2b}{a} b_{,1} \right) q - \frac{1}{2a^2} \left( -\dot{a} b - 2ab_{,3} - \frac{\dot{h}}{h} b_{,1} a x \right) 
- b^2 b_{,1} + \frac{c}{h} b_{,1} + 2ba - \frac{\dot{h}}{h} ab \]  

(41)
where \( w = \frac{a}{h} \), \( a, h, c \) depend on \( t \) only and \( b \) depends on \( x, t \).

\[
q_t = q_3 + aqq_1 + bq_1 - \frac{1}{2} \left( \frac{\dot{w}}{w} - 2b_1 \right) q \\
- \frac{1}{2a}(-2b_3 - 2bb_1 + 2b - \frac{\dot{c}}{h} - \frac{\dot{h}}{h} x + 2 \frac{\dot{h}^2}{h^2} x - \frac{\dot{h}}{h} b + 2 \frac{\dot{h}}{h^2} c) \tag{42}
\]

where \( w = \frac{a^2}{h} \), \( a \) depend on \( t \) only and \( b \) depends on \( x, t \).

\[
q_t = q_3 + \frac{a}{6} q_1^3 + \frac{ab}{2} q_1^2 + \frac{1}{2} \left( \frac{\dot{h}}{h} x + b^2 a - \frac{c}{h} \right) q_1 - \frac{1}{2} \left( \frac{\dot{a}}{a} q - 2d \right) \tag{43}
\]

with

\[
d_1 = \frac{1}{2} \left[ 2b_3 + b_1 \left( \frac{\dot{h}}{h} x + ab^2 - \frac{c}{h} \right) - 2b - \frac{b\dot{a}}{a} + \frac{bh}{h} \right] \tag{44}
\]

where \( b, d \) depend on \( t, x \) and \( a, c \) and \( h \) depend on \( t \) only.

\[
q_t = q_3 - \frac{a^2}{8} q_1^3 - \frac{3ab_1}{8b} q_1^2 + \left( \frac{b}{a} e^{aq} - \frac{ah^2 f}{b} e^{-aq} + \frac{\dot{h}}{2h} x \right) q_1 - \frac{c}{2h} \left( \frac{3b_1^3}{8b^2} \right) q_1 - \frac{\dot{a}}{a} q + \left( \frac{b_1}{a^2} e^{aq} - \frac{h^2 f b_1}{b^2} e^{-aq} \right) q_1 + \frac{\dot{b}_1}{2ab} x + \frac{b_3}{ab} - \frac{3b_1 b_2}{ab^2} + \frac{15b_1^3}{8ab^3} + \frac{cb_1}{2abh} - \frac{dhb_1}{2ab} + \frac{\dot{w}}{aw} \tag{45}
\]

where \( w = \frac{ab}{b} \), \( a, c \) depend on \( t \) only, \( b \) depends on \( x, t \) and \( d, f \) are constants.

All these classes are transformable to those given in (13) and (15). For this purpose, we first perform the following transformation

\[
q = \alpha(t) z(x,t) + \beta(x,t), \tag{46}
\]

where \( \alpha \) and \( \beta \) are arbitrary functions, and \( z(x,t) \) is the new dynamical variable. By choosing \( \alpha \) and \( \beta \) properly we eliminate arbitrary functions of \( x \) and
appearing in these equations. Secondly, if the resultant equations contain further arbitrary functions depending upon $t$, we perform the transformation of the following type

$$z = v(x, t) + s_0 x^2 + s_1 x + s_2,$$

(47)

to eliminate such arbitrary functions as the products of $x^2$ and $x$ in these equations. Here $s_0, s_1$ and $s_2$ depend only on $t$ and $v(x, t)$ is now the new dynamical variable. At this point we transform dependent and independent variables according to

$$v(x, t) = \mu(t) u(\xi, \tau),$$

$$\xi = x p(t) + \gamma(t), \quad \tau = \nu(t)$$

(48)

which reduces the classes (1-7) to one of the type given in equations (13) and (15) exactly. As an example Eq. (41) is transformed into the Eq. (13) through the following transformations

$$q = \frac{h^{1/2}}{a} u(\xi, \tau) - \frac{b}{a},$$

$$\xi = x h^{1/2} + \gamma(t), \quad \tau = \int_t^t h^{3/2} dt',$$

(49)

where

$$\gamma = - \int_t^t (\frac{c}{2 a h^{1/2}} + h^{2/3}) dt'.$$

(50)

Finally we have the following proposition.
**Proposition 5:** Under the symmetry classification the integrable subclass of the type of equations (36) is, up to coordinate transformations, equivalent to the equations (13) and (15).

In conclusion this work shows that there are no generic integrable non-autonomous type of equations (19). Any integrable PDE (admitting infinitely many generalized symmetries) containing explicit \((x,t)\) dependencies of the form (19) is transformable into (11).

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