Propagator in polymer quantum field theory

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We study free scalar field theory on flat spacetime using a background independent (polymer) quantization procedure. Specifically we compute the propagator using a method that takes the energy spectrum and position matrix elements of the harmonic oscillator as inputs. We obtain closed form results in the infrared and ultraviolet regimes that give Lorentz invariance violating dispersion relations, and show suppression of propagation at sufficiently high energy.

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INTRODUCTION

One of the important problems in fundamental physics is to understand the high energy behaviour of quantum fields. This question is intimately connected with the structure of spacetime at short distances, as the background mathematical structure that underlies quantum field theory (QFT), i.e. a manifold with a metric, may come into question in this regime. A part of the problem is that the spacetime metric forms a reference not only for defining the particle concept, but also for the Hilbert space inner product; if the metric is subject to quantum fluctuations, its use in an inner product becomes an issue.

There are many approaches that have been deployed to probe such questions, including string theory, non-commutative geometry, loop quantum gravity and causal sets. Most of these use some notion of “background independence,” a term which broadly means that either results are independent of the metric used in calculations, or that a metric is not used at all.

In this paper we explore a background independent (“polymer”) quantization method that arose in loop quantum gravity (LQG) [1] and apply it to scalar field theory. In this approach the Hilbert space used for quantization is different from the one employed in usual quantum theory. This Hilbert space is such that its inner product does not make use of a spacetime metric, even if one is available, as in QFT on a fixed background. Rather the inner product comes from an underlying group structure. This may be compared to the inner product for spin degrees of freedom at each point in a statistical mechanics model. What is especially interesting about this quantization method is that it introduces a length scale in addition to Planck’s constant into the quantum theory.

The method was applied to the oscillator [2] and other aspects studied subsequently [3–5]. It is useful to study it further by applying it to QFT on curved spacetime, which is a question of intrinsic interest. Furthermore it is needed for a complete theory of quantum gravity, where the same quantization method should be applied to both geometry and matter variables. There has already been some work in this direction, such as the construction of Fock-like states [6,7], an application to matter in cosmology [8], and a derivation of an effective non-local and Lorentz invariance violating wave equation [9].

In this work we develop the area of polymer quantum field theory further by computing the propagator of the scalar field and analyzing its implications for high energy physics. We employ an intuitive approach that directly uses the spectrum of the oscillator. This introduces a new application of polymer quantization directly in momentum space, and gives some surprising features not noticed in earlier works such as Ref. [2]. Our results provide an alternative to approaches that use minimal length arguments [10] and related background independent methods [11,12] to compute matter and graviton propagators.

POLYMER QUANTUM MECHANICS

As noted above, the central difference between Schrödinger and polymer quantization is the choice of Hilbert space. The Hilbert space used for the polymer case is the space of almost periodic functions [13]. A particle wave function is written as the linear combination

$$\psi(p) = \sum_{k=1}^{\infty} c_k e^{ipx_k},$$

where the set of points \(\{x_k\}\) is a selection (graph) from the real line. In the full Hilbert space, all possible graphs are permitted, so the space is not-separable. The inner product is

$$\langle x | x' \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dp \ e^{-ipx} e^{ip\lambda} = \delta_{x,x'},$$

where the right hand side is the generalization of the Kronecker delta to an uncountable index set. Plane waves are normalizable in this inner product.

The configuration operator \(\hat{x}\) and translation operator \(\hat{U}_\lambda := e^{i\lambda p} \) act as

$$\hat{x} e^{ipx_k} = i \frac{\partial}{\partial p} e^{ipx_k}, \quad \hat{U}_\lambda e^{ipx_k} = e^{ip(x_k + \lambda)}.$$  \(3)$$
These operator definitions give the basic commutator 
\[ [\hat{x}, \hat{U}_\lambda] = -\lambda \hat{U}_\lambda, \]
which is the desired representation of the Poisson bracket 
\[ \{x, U_\lambda\} = i\lambda U_\lambda. \]

The momentum operator cannot be defined on this Hilbert space because the translation operator is not weakly continuous in its parameter due to the inner product (2); only finite translations are realized as in (3). This is the key feature that leads to modification of energy spectra because the kinetic energy operator must be defined using the translation operator, which comes with a fixed scale. Thus all classical observables that depend on the momentum are realized as scale dependent operators in the quantum theory, a feature reminiscent of effective field theory, but coming from an entirely different source.

Perhaps the simplest definition of the momentum operator is
\[ \hat{p}_\lambda = \frac{1}{2i\lambda} (\hat{U}_\lambda - \hat{U}_\lambda^\dagger). \]
(4)

In $L_2(\mathbb{R})$, the $\lambda \to 0$ limit in (4) would give the usual momentum and momentum-squared operators $-i\partial_x$ and $-\partial_x^2$. In the polymer Hilbert space the $\lambda \to 0$ limit does not exist, and $\lambda$ is regarded as a fundamental length scale. The Hamiltonian operator that corresponds to the classical Hamiltonian $H = p^2/2m + V(x)$ is then
\[ \hat{H} = \frac{1}{8m\lambda^2} (2 - \hat{U}_{2\lambda} - \hat{U}_{2\lambda}^\dagger) + \hat{V}, \]
(5)

where the potential $V$ is arbitrary but assumed to be regular so that $V$ can be defined by pointwise multiplication, \[ \langle \hat{V}\psi \rangle(x) = V(x)\psi(x). \] One expects the polymer dynamics to be well approximated by the Schrödinger dynamics in an appropriate regime, and certain results to this effect are known [9,11,15].

We now specialize to the simple harmonic oscillator where \[ V(x) = m\omega^2x^2/2. \] In the $p$-representation the energy eigenvalue equation $\hat{H}\psi = E\psi$ with the Hamiltonian (5) reads:
\[ \frac{1}{8m\lambda^2}[2 - 2\cos(2\lambda_p)]\psi - \frac{1}{2}m\omega^2\frac{\partial^2 \psi}{\partial p^2} = E\psi, \]
(6)

where we have written $\lambda = \lambda_0$ to denote that a scale has been fixed. (We use units where $\hbar = c = 1$, so $\lambda$ has dimension of length.) Defining
\[ u = \lambda_p + \pi/2, \quad \alpha = 2E/g\omega - 1/2g^2, \quad g = m\omega\lambda^2, \]
transforms (6) into the Mathieu equation
\[ \psi''(u) + [\alpha - \frac{1}{2}g^{-2}\cos(2u)] \psi(u) = 0. \]
(8)

This has periodic solutions for special values of $\alpha$:
\[ \psi_{2n}(u) = \pi^{-1/2}c_n(1/4g^2,u), \quad \alpha = A_n(g), \]
\[ \psi_{2n+1}(u) = \pi^{-1/2}c_{n+1}(1/4g^2,u), \quad \alpha = B_n(g). \]
(9a, 9b)

Here, $c_n$ and $s_n$ ($n = 0,1 \ldots$) are the elliptic cosine and sine functions, respectively, while $A_n$ and $B_n$ are the Mathieu characteristic value functions [10]. For even $n$, $c_n$ and $s_n$ are $\pi$-periodic, while for odd $n$ they are $\pi$-antiperiodic. Explicitly, the energy eigenvalues corresponding to the periodic eigenfunctions (9) are:
\[ \frac{E_{2n}}{\omega} = \frac{2g^2A_n(g) + 1}{4g}, \]
(10a)
\[ \frac{E_{2n+1}}{\omega} = \frac{2g^2B_{n+1}(g) + 1}{4g}. \]
(10b)

We plot these energy levels as a function of $g$ in Fig. 1. Using asymptotic expansions for $A_n(g)$ and $B_n(g)$, we deduce the following behaviour for $E_n$ in the small $g$ limit:
\[ \frac{E_{2n}}{\omega} \approx \frac{E_{2n+1}}{\omega} = \left(n + \frac{1}{2}\right) - \frac{(2n + 1)^2 + 1}{16}g + \mathcal{O}(g^2). \]
(11)

Clearly, the Schrödinger energy spectrum is recovered for $g = 0$ [2]. Conversely, in the large $g$ limit we obtain
\[ \frac{E_0}{\omega} = \frac{1}{4g} + \mathcal{O}(g^{-3}), \]
(12)
\[ \frac{E_{2n-1}}{\omega} \approx \frac{E_{2n}}{\omega} = \frac{n^2g}{2} + \mathcal{O}(g^{-1}), \]
(13)

The first formula shows that the ground state energy falls with increasing $g$, a feature that may have consequences for the cosmological constant problem. The second formula indicates that the even and odd energies become degenerate for $g \gg 1$; i.e., when the oscillator mass or frequency is large compared to $\lambda^{-1}_0$. (The large $g$ behaviour of the system could potentially be used to put experimental bounds on $\lambda_0$.) Although different conceptually, it is useful to compare these results to that of an oscillator on a lattice [17].

One of the features of the spectrum of the oscillator is the apparent doubling of energy levels. This stems from the fact that the polymer oscillator eigenvalue equation (8) is that of a particle in a periodic potential with period $\pi$. The Hilbert space of solutions is the direct sum $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\pm$ denotes the $\pi$-periodic and $\pi$-antiperiodic sectors. One can ask whether these sectors are superselected, i.e., if $\langle \psi_+ | A | \phi_- \rangle = 0$ for all operators $A$, and for all states $| \psi_+ \rangle \in \mathcal{H}^+$ and $| \phi_- \rangle \in \mathcal{H}^-$. It is readily verified that this is not the case for powers of the momentum operator.

We note also that for the alternative kinetic energy operator $(2 - \hat{U}_\lambda - \hat{U}_\lambda^\dagger)/\lambda^2$ (used for example in [2]), the eigenvalue equation that arises has a periodicity $2\pi$. However this may be transformed to the standard Mathieu form [8] with the rescaling $\lambda \to 2\lambda$, and a corresponding scaling of $g$. The resulting solutions then give rise to the same 2 sectors. Thus the doubling of states appears to be a feature of approach, and independent of
The Hamiltonian is
\[ H_\phi = \int d^3x \left[ \frac{\pi^2}{2\sqrt{2}} + \frac{\sqrt{q}}{2} q^{ab} \nabla_a \phi \nabla_b \phi \right], \quad (15) \]
where we have written the metric as \( ds^2 = -dt^2 + g_{ab} dx^a dx^b \). We now restrict to a flat 3-space, i.e. \( g_{ab} = \epsilon_{ab} \), with volume \( V = \int d^3x \sqrt{\epsilon} \). Thus our field configurations are those of system in a box.

We transform to 3-momentum space with the Fourier expansion
\[ \phi(t, x) = \frac{1}{\sqrt{V}} \sum_k \phi_k(t) e^{i k \cdot x}, \]
\[ \phi_k(t) = \frac{1}{\sqrt{V}} \int d^3x \ e^{-i k \cdot x} \phi(x, t), \quad (16) \]
with a similar expansion for \( \pi(x, t) \). After a suitable redefinition of the independent modes the Hamiltonian is
\[ H_\phi = \sum_k H_k = \sum_k \left[ \frac{\pi_k^2}{2} + \frac{1}{2} k^2 \phi_k^2 \right], \quad (17) \]
with the Poisson bracket \( \{ \phi_k, \pi_{k'} \} = \delta_{kk'} \). The polymer quantization of each mode now follows that of the oscillator with obvious identifications. We use the variables \( \phi_k \) and \( U_{kk'} = e^{i \lambda \pi \epsilon} \) which satisfy the same Poisson bracket \( \{ \phi_k, U_{kk'} \} = i \lambda U_{kk'} \) [note that \( \lambda \) now has dimensions of (length)\(^1/2\)]. Taking the definition of the Hamiltonian operator as in [18], and the identifications \( m = 1 \) and \( \omega = |k| \), the spectrum of \( H_k \) is the same as that obtained above with
\[ g = \lambda^2 |k| \approx \frac{|k|}{M^-} \times \text{polymer length scale}, \quad (18) \]
where \( M^- \) is the fundamental length scale associated with the polymer quantization of \( \phi \).

We now show how the polymer oscillator spectrum leads to a modified propagator for \( g \neq 0 \). The usual 2−point function is
\[ \langle 0 | \hat{\phi}(x, t) \hat{\phi}(x', t') | 0 \rangle \equiv \frac{1}{V} \sum_k e^{i k \cdot (x-x')} D_k(t-t'), \quad (19) \]
where \( |0 \rangle = \Pi_k \otimes |0_k \rangle \) is the Fock vacuum. For our case the matrix element in the last equation is
\[ D_k(t-t') = \langle 0_k | e^{i \hat{H}_k t'} e^{-i \hat{H}_k t} \phi_k e^{-i \hat{H}_k t'} | 0_k \rangle, \quad (20) \]
where \( \hat{H}_k \) is the polymer Hamiltonian operator
\[ \hat{H}_k = \frac{1}{8\lambda^2} \left[ 2 - U_{2kk} - U_{2k}^\dagger \right] + \frac{1}{2} k^2 \phi_k^2. \quad (21) \]
and \( |0_k \rangle \) is its ground state. The matrix element in (20) is readily computed using the polymer oscillator spectrum.
The propagator is
\[ c_n = \langle n_k | \hat{\phi}_k | 0_k \rangle = \lambda_n \int_0^{2\pi} \psi_n(i\partial_u)\psi_0 du. \] (22)

Eq. (20) then becomes
\[ D_k(t - t') = \sum_n |c_n|^2 e^{-i\Delta E_n(t - t')}, \] (23)

where \( \Delta E_n = E_n(k) - E_0(k) \). To bring this expression into a more recognizable form we write the exponential as an integral to give
\[ D_k(t - t') \equiv \int \frac{d\omega}{2\pi} D_p e^{-i\omega(t - t')} \]
\[ = \sum_n 2\Delta E_n |c_n|^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t - t')} \]
\[ = \sum_n 2\Delta E_n |c_n|^2 \int \frac{d\omega - i\epsilon}{\Delta E_n^2 - \omega^2 - i\epsilon}. \] (24)

where \( \omega \) here is defined as the time component of the 4-momentum \( p \equiv (\omega, k) \). Thus the momentum space propagator is
\[ D_p = \sum_n \frac{2i\Delta E_n |c_n|^2}{p^2 + \Delta E_n^2 - |k|^2 - i\epsilon}, \] (25)

where we used \( p^2 = -\omega^2 + |k|^2 \) and chose the sign of \( i\epsilon \) to correspond to the Feynman propagator. We note that for the Schrödinger oscillator \( c_n = \delta_{1,n}/\sqrt{2|k|} \) and \( \Delta E_n = n|k| \), so (25) gives the expected result \( D_p = i/(p^2 - i\epsilon) \).

The polymer corrections we seek arise from the dependence of the matrix element \( c_n \) and the energy differences \( \Delta E_n \) on the parameter \( g = |k|/M_* \). Unlike the Fock case, the sum in (25) contains an infinite number of terms for \( g \neq 0 \), and appears to be analytically unsummable. However it is possible to obtain formulas for \( g \ll 1 \) and \( g \gg 1 \) using the appropriate asymptotic representations of the elliptic functions. We find that the only non-zero matrix elements are \( c_{4n+3} \) (for \( n = 0, 1, 2, \ldots \)). This is expected because the natural ground state \( \psi_0 \) is even and \( \pi \) periodic, and the operator \( \hat{\phi}_k \) connects it only to (odd) \( \pi \) periodic states. Thus our propagator calculation makes use of only the \( \pi \)-periodic eigenfunctions.

The infrared limit \( g \ll 1 \): In this case we find
\[ \Delta E_{4n+3} = |k| \left[ (2n + 1) - \frac{(4n + 3)^2 - 1}{16} - g + \mathcal{O}(g^2) \right] \] (26)

for \( n \geq 0 \), and
\[ c_3 = \frac{-i}{\sqrt{2|k|}} \left[ 1 - \frac{3}{4} g + \mathcal{O}(g^2) \right], \] (27)
\[ c_{4n+3}/c_3 = \mathcal{O}(g^n), \] (28)

for \( n > 0 \). From these, it is apparent that only the \( c_3 \) term in (25) gives a contribution to \( D_p \) that is linear in \( g \). Retaining only this term gives
\[ D_p = \frac{i(1 - 2g)}{p^2 - g|k|^2 - i\epsilon} + \mathcal{O}(g^2). \] (29)

Comparing this with the propagator of the massive scalar theory, we see that (i) the correct \( g = 0 \) limit is obtained, (ii) there exists a tachyonic pole \( p^2 = g|k|^2 \), and (iii) this pole implies an effective dispersion relation
\[ \omega^2 = |k|^2(1 - |k|/M_*) \] (30)

that violates Lorentz symmetry. We note that unlike the case of tachyons in Lorentz invariant theories, this polymer quantization correction does not lead to complex \( \omega \) for real \( |k| \), so there is no instability.

The ultraviolet limit \( g \gg 1 \): In this limit we find
\[ \Delta E_{4n+3} = |k| \left[ 2(n + 1)^2 g + \mathcal{O}(1/g^2) \right], \] (31)

for \( n \geq 0 \), and
\[ c_3 = \frac{1}{\sqrt{2|k|}} \left[ \frac{1}{4g^2} + \mathcal{O}(1/g^6) \right], \] (32)
\[ c_{4n+3}/c_3 = \mathcal{O}(1/g^{2n}). \] (33)

for \( n > 0 \). Since the coefficients higher than \( c_3 \) are suppressed by \( 1/g^{2n} \) (for \( n > 1 \)), keeping only this term in the propagator sum (25) gives
\[ D_p = \frac{i/8g^2}{p^2 + 4g^2|k|^2 - i\epsilon} + \mathcal{O}(1/g^6). \] (34)

This has the following interesting features: (i) the pole is not tachyonic, unlike the \( g \ll 1 \) case, (ii) the associated dispersion relation
\[ \omega^2 = 4|k|^4/M_*^2 \] (35)

still violates Lorentz invariance and reflects a higher derivative term, and (iii) the propagation amplitude at high momentum is suppressed by a factor \( 1/g^2 \), a feature not present in linear higher derivative theory.

**SUMMARY**

We applied the polymer quantization method to the oscillator in momentum space and computed its full spectrum. We then used this to compute the scalar field propagator using an intuitive approach that exploits directly the decomposition of free field theory into a collection of simple harmonic oscillators. The resulting propagator violates Lorentz invariance for \( g \neq 0 \). Its limits in
the low and high momentum regime exhibit vastly different behaviour. At low momenta the theory acquires an effective (tachyon) mass $m_{\text{eff}}^2 = -|k|^3/M_\star$. At large momenta the effective mass is $m_{\text{eff}}^2 = 4|k|^4/M_\star^2$ and the propagation amplitude is suppressed.

Our results provide an example of the type of effects one can expect in quantum field theory in general curved spacetimes using this quantization method. In particular it opens up new directions for investigation such as the spectrum of cosmic microwave background fluctuations and Hawking radiation, where the scale associated with the quantization is likely to give new physics.

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