MAXHEDGE: Maximising a Maximum Online with Theoretical Performance Guarantees

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Abstract

We introduce a new online learning framework where, at each trial, the learner is required to select a subset of actions from a given known action set. Each action is associated with an energy value, a reward and a cost. The sum of the energies of the actions selected cannot exceed a given energy budget. The goal is to maximise the cumulative profit, where the profit obtained on a single trial is defined as the difference between the maximum reward among the selected actions and the sum of their costs. Action energy values and the budget are known and fixed. All rewards and costs associated with each action change over time, are revealed after the learner’s selection of actions. Our framework encompasses several online learning problems where the environment changes over time; and the solution trades-off between minimising the costs and maximising the maximum reward of the selected subset of actions, while being constrained to an action energy budget. The algorithm that we propose is an efficient and very scalable unifying approach which is capable of solving our general problem. Hence, our method solves several online learning problems which fall into this general framework.

1 Introduction

In this paper we propose a novel online framework where learning proceeds in a sequence of trials and the goal is to select, at each trial, a subset of actions maximising a profit while taking into account a certain constraint. More precisely, we are given a finite set of actions enumerated from 1 to $n$. Each action is associated with three values: (i) a cost, (ii) an amount of energy (which are both required to perform the given action), as well as (iii) a reward. Both the cost and the reward associated with each action may change over time, are unknown at the beginning of each trial, and their values are all revealed after the learner’s selection. The energy associated with each action is instead known by the learner and does not change over time. At each trial, the learner is required to select a subset of actions such that the sum of their energies does not exceed a fixed energy budget. The goal of the learner is to maximise the cumulative profit, where the profit obtained on a single trial is defined as the difference between the maximum reward among the selected actions and the sum of their costs. We denote by $T$ the total number of trials.

Our framework is general and flexible in the sense that it encompasses several online learning problems. In the general case, the main challenge lies in the fact that the rewards are not known at the beginning of each trial, and the learner’s profit depends only on the maximum reward among the selected subset of actions, instead of the sum of all their rewards. In particular, it is worth mentioning three different problems which can be seen as special cases of our online learning framework; these are variants of the Facility Location problem, the 0-1 Knapsack problem, and the Knapsack Median problem.

When all action energy values are equal to 0, we obtain an online learning variant of the Facility Location problem (see, e.g., [4], [21], [18]). The goal of this specific problem can be seen as selecting and opening a subset of facilities at each given trial, to service a sequence of users which arrive one at a time. At any given trial $t$, each action’s cost can be seen as the cost for opening a facility for the $t$-th user. The reward associated with a
given facility represents what the user potentially gains when it is opened. More specifically, the rewards can be seen as quantities dependent on the distance between the user and the facilities in some metric space. In this context, it is reasonable to assume that the profit obtained on a single trial depends only on the maximum reward among the opened facilities and takes into account also the facility costs. This represents a natural setting for the Facility Location problem, because in practical real-world scenarios users arrive one after the other and require to be connected to opened facilities. An online version of the Facility Location problem has so far being studied in several contexts (see, e.g., [5], [20]). However, as far as we are aware, our work is the first study of this specific online setting for the Facility Location problem. Moreover, this dynamic model is natural and interesting within this context because in practice the location of the next user, which in turn determines the reward associated with each facility, is often unknown to the learner. At each trial, after the user’s location is known, the rewards are then revealed, because disclosing the user’s location enables to compute all distances between the facilities and the user, which are previously unknown. In fact, this corresponds to assuming all rewards are revealed at the end of the trial.

When, instead, all rewards are equal to 0 and all costs are negative, the problem can be seen as an online learning variant of the 0-1 Knapsack problem (see, e.g., [6], [19]). In this case, each action corresponds to an item whose weight is equal to the associated energy, the energy budget represents the knapsack capacity, and the absolute value of each action cost can be viewed as the corresponding item value. Our formulation makes the problem challenging especially because the item values are revealed only after the learner’s selection.

Finally, when all costs are equal to 0, we obtain an online learning variant of the Knapsack Median problem ([1], [15]). In this problem (which is a generalization of the k-median problem – see, e.g., [2], [10]), we are given a set of clients and facilities, as well as a distance metric. The goal is to open a subset of facilities such that the total connection cost (distance to nearest open facility) of all clients is minimized, while the sum of the open facility weights is limited by a budget threshold. In our framework, at each trial the rewards can express the closeness of each facility, and the action energy budget represents therefore the threshold of sum of the opened facility weights.

This problem has practical applications in multiple domains. In computer networks, network administrators may want to understand where to place network monitors or intrusion detection systems. Network packets or malicious attacks are related to the events playing a crucial role in this scenario, and a limited amount of network resources are available to detect or observe network behavior. Another class of application examples include municipal emergency services. A service center needs to deploy responders (police, paramedics, fire rescue), and with limited resources, personnel must be deployed sparingly. A further application is related to deploying program instances in a distributed computing environment (e.g. distributed cloud). These systems must respond to user requests and are subject to constraints. Both costs and rewards may be unknown in practical real-world scenarios at the beginning of each trial.

For our general problem we propose and rigorously analyse a very scalable algorithm called MaxHedge based on the complex interplay between satisfying the energy budget constraint and bounding the profit by a concave function, which in turn is related to the online gradient descent algorithm. Moreover, the total time required per trial by our learning strategy is quasi-linear in n. We measure the performance of proposed solution with respect to the difference between its cumulative profit and the cumulative profit of the best fixed subset of actions, i.e. focusing on the classical notion of regret for online learning problems, in terms of the number of actions n and the number of trials T.

In summary, our general framework captures several real-world problems, where the environment changes over time and solutions trade-off between minimising the costs and maximising the maximum reward among the selected subset of actions, while being constrained to an action energy budget. The method that we propose for our problem is a very fast and efficient unifying approach capable of solving the general problem. Hence, our algorithm solves the several online learning problems which fall into this general framework.

1.1 Related Work

The closest work to our online learning framework is perhaps addressed in [13], where the authors describe an online learning algorithm for structured concepts that are formed by components. Each component and each concept can be respectively seen as an action and a feasible subset of actions. Despite several similarities, the algorithm they proposed cannot be used when the rewards are non-zero, because we focus on the maximum reward among the selected subsets of actions, whereas in [13] the profit corresponds to the sum of the rewards of all selected components/actions. When
all rewards are instead equal to 0, then we have the online variant of the 0-1 Knapsack problem described above as one of the three special cases of our general problem. In Appendix A we prove that if the algorithm presented in [13] could handle the Knapsack problem, then the classical version of the Knapsack problem could be solved in polynomial time, which therefore implies that it cannot address this problem unless \( P = NP \).

The Hedge Algorithm described in [17] obtains a regret ward.

Another class of problems and algorithms that are not far from ours is represented by online decision problems where efficient strategies use, as a subroutine, an approximation algorithm for choosing a concept that in the Knapsack and Knapsack Median special cases, combine much of the mechanics of our paper with the second algorithm of [22].

However, as we shall show in Appendix B, we could, for the facility location special case, do gradient ascent with the exact expected profit instead of an approximate expected profit (as is done in MAXHEDGE). Even though this new algorithm could be as efficient as the one presented in our paper, its theoretical guarantees are worse. Note that in the Knapsack and Knapsack Median special cases, the profit is indeed monotone and non-negative. For these special cases, the approach presented in [22] comes close to solving the problem, but only guarantees that the expected total energy does not exceed the budget rather than the actual total energy. In addition, for the Knapsack Median problem, [22] uses quadratic time and space. As far as we are aware, there does not exist any trivial reduction to use [3] or [9] for these special cases.

Finally, in [11] the authors address a problem of online maximisation of submodular function. Our paper maximises, instead of minimising, a submodular function (the profit). This is a very different problem.

2 Preliminaries and Problem Setup

2.1 Preliminaries

Vectors in \( \mathbb{R}^n \) will be indicated in bold. Given a vector \( \mathbf{v} \in \mathbb{R}^n \) we define \( v_i \) to be its \( i \)-th component. Given vectors \( \mathbf{v}, \mathbf{x} \in \mathbb{R}^n \) define \( \langle \mathbf{v}, \mathbf{x} \rangle := \sum_{i=1}^n v_i x_i \). Define \( \mathcal{P} \) to be the set of \( n \)-dimensional real vectors in which every component is non-negative. Given a closed convex set \( C \subseteq \mathbb{R}^n \) and a vector \( \mathbf{x} \in \mathbb{R}^n \), we define \( \Pi_C(\mathbf{x}) \) as the projection (under the Euclidean norm) of \( \mathbf{x} \) onto \( C \). Let \( \mathbb{N} \) be the set of the positive integers. For \( l \in \mathbb{N} \) define \([l] = \{1, 2, 3, \ldots, l\}\). Given an event \( E \) let \( \neg E \) be the event that \( E \) does not occur. Let \( \mathbb{P}(E) \) be the probability that event \( E \) occurs. Finally, for a random variable \( Y \), let \( \mathbb{E}(Y) \) be the expected value of \( Y \).

2.2 Problem Setup

In this section we formally define our problem. We have a set of \( n \) actions enumerated from 1 to \( n \). Each action \( i \) has an energy \( z_i \in [0, \beta] \) for some \( \beta < 1 \), and on each trial \( t \) each action \( i \) has a cost \( c_i^t \in \mathbb{R} \) and a reward \( r_i^t \in \mathbb{R}^+ \). The learner knows \( z \), but \( c^t \) and \( r^t \) are revealed to the learner only at the end of trial \( t \). On each trial \( t \) the learner has to select a set \( X^t \subseteq [n] \) of actions, such that the total energy \( \sum_{i \in X^t} z_i \) of the selected actions is no greater than 1. In selecting the set \( X^t \), the learner pays a cost equal to \( \sum_{i \in X^t} c_i^t \). On each trial \( t \) the learner then receives the maximum reward \( \max_{i \in X^t} r_i^t \), over all actions selected (defined as equal to zero if \( X^t \) is empty). Hence, the profit obtained by the learner on trial \( t \) is equal to \( \max_{i \in X^t} r_i^t - \sum_{i \in X^t} c_i^t \).

Formally, this online problem can be defined as follows: We have a vector \( \mathbf{z} \in \mathcal{P} \) known to the learner. On trial \( t \):

1. Nature selects vectors \( \mathbf{c}^t \in \mathbb{R}^n \) and \( \mathbf{r}^t \in \mathcal{P} \) (but does not reveal these vectors to learner)
2. Learner selects a set \( X^t \subseteq [n] \) with \( \sum_{i \in X^t} z_i \leq 1 \)
3. Learner obtains profit:

\[
\mu^t(X^t) := \max_{i \in X^t} r_i^t - \sum_{i \in X^t} c_i^t
\]

4. \( \mathbf{c}^t \) and \( \mathbf{r}^t \) are revealed to the learner.

In this paper we write, for a trial \( t \), the cost vector \( \mathbf{c}^t \) as the sum \( c_{i}^{t+} + c_{i}^{t-} \) where \( c_{i}^{t+} := \max\{0, c_{i}^t\} \) and \( c_{i}^{t-} := \min\{0, c_{i}^t\} \), i.e. \( c_{i}^{t+} \) and \( c_{i}^{t-} \) are the positive and negative parts of the cost vector, respectively.
In order to bound the cumulative profit of our algorithm, we, for some \( \alpha \in [0, 1] \), some set \( S \subseteq [n] \) and some trial \( t \), define the \( \alpha\)-discounted profit \( \bar{\mu}_\alpha(S) \) as:

\[
\bar{\mu}_\alpha(S) := \alpha \max_{i \in S} r_i^t - \alpha \sum_{i \in S} c_i^{-} - \sum_{i \in S} c_i^{+}
\]

which would be the profit obtained on trial \( t \) if we selected the subset of actions \( S \), and all the rewards and negative costs were multiplied by \( \alpha \).

In this paper we provide a randomised quasi-linear time (per trial) algorithm \textsc{MaxHedge} that, for any set \( S \subseteq [n] \) with \( \sum_{i \in S} z_i \leq 1 \), obtains an expected cumulative profit bounded below by:

\[
\mathbb{E} \left( \sum_{t=1}^T \bar{\mu}(X^t) \right) \geq \sum_{t=1}^T \delta \bar{\mu}_\alpha(S) - n \sqrt{2T} (\hat{\beta} + \hat{c})
\]

where \( \delta := (1 - \sqrt{\beta})^2 \), \( \alpha := 1 - \exp \left( - (1 - \sqrt{\beta})^2 \right) \), \( \hat{\beta} := \max_{i \in [T], i \in [n]} r_i^t \) and \( \hat{c} := \max_{i \in [T], i \in [n]} |c_i^t| \).

### 3 Algorithms

We now present our learning strategy \textsc{MaxHedge}, describing the two subroutines “Algorithm 1” and “Algorithm 2” (see the pseudocode below). \textsc{MaxHedge} maintains a vector \( \omega \in C \) where \( C := \{ \mathbf{x} \in [0, 1]^n : \langle \mathbf{x}, \mathbf{z} \rangle \leq 1 \} \). We define \( \omega^t \) to be the vector \( \omega \) at the start of trial \( t \). We initialise \( \omega^1 := 0 \). On trial \( t \) \textsc{MaxHedge} (randomly) constructs \( X^t \) from \( \omega^t \) using Algorithm 1. After receiving \( r^t \) and \( c^t \) \textsc{MaxHedge} updates \( \omega \) (from \( \omega^t \) to \( \omega^{t+1} \)) using Algorithm 2. Algorithm 2 also uses a “learning rate” \( \eta^t \) which is defined from \( \eta^{t-1} \). We define \( \eta^0 := \infty \).

Algorithm 1 operates with a partition of all possible actions and, for each set in this partition, Algorithm 1 draws a certain subset of actions from it. Given a set in the partition, the number of actions drawn from it and the probability distribution governing the draws depends on \( \beta \) and \( \omega^t \). The subset, \( X^t \), of actions selected by Algorithm 1 satisfies the following three crucial properties (proved in sections 3 and 5):

- The total energy of all actions selected is no greater than 1.
- Given an arbitrary set \( Z \subseteq [n] \), the probability that \( X^t \) and \( Z \) intersect is lower bounded by \( 1 - \exp \left( - \delta \sum_{i \in Z} \omega_i^t \right) \).
- Given an action \( i \), the probability that it is selected on trial \( t \) is upper bounded by \( \delta \omega_i^t \).

In the analysis we shall construct, from \( r^t \) and \( c^t \), a convex function \( h^t : C \rightarrow \mathbb{R} \). Using the second and third properties (given above) of Algorithm 1, we show that \( h^t(\omega^t) \) is an upper bound on the negative of the expected profit on trial \( t \). Algorithm 2 computes the gradient \( \gamma_t := \nabla h^t(\omega_t) \) and updates \( \omega \) using online gradient descent on \( C \).

The last line of Algorithm 2 requires us to project (with Euclidean distance) the vector \( \gamma_t \) onto the set \( C \), i.e., we must compute the \( \mathbf{x} \) that minimises the value \( \| \mathbf{x} - \gamma_t \| \) subject to \( \mathbf{x} \in C \). Note that minimising \( \| \mathbf{x} - \gamma_t \| \) is equivalent to minimising \( \| \mathbf{x} - \gamma_t \|^2 = \langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle \gamma_t, \mathbf{x} \rangle + \langle \gamma_t, \gamma_t \rangle \), which is in turn equivalent to minimising \( \langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle \gamma_t, \mathbf{x} \rangle \). The constraints defining the set \( C \) then imply that this projection is a case of the continuous bounded quadratic knapsack problem which can be solved in linear time (see, e.g., [14]).

The bottleneck of the algorithms is hence the ordering step in Algorithm 2 which takes a time of \( O(n \log(n)) \).

**Algorithm 1 Constructing \( X^t \)**

1: \( C \leftarrow \{ \mathbf{x} \in [0, 1]^n : \langle \mathbf{x}, \mathbf{z} \rangle \leq 1 \} \)
2: \( \beta \leftarrow \max_{i \in [n]} z_i \)
3: \( \tau \leftarrow 1 - \sqrt{\beta} \)
4: \( \delta := (1 - \sqrt{\beta})^2 \)
5: \( \Gamma \leftarrow \{ q \in \mathbb{N} : z_i \in [n] \text{ with } \tau^q \beta < z_i \leq \tau^{q-1} \beta \} \)
6: For all \( q \in \Gamma \) set \( \Omega_q \leftarrow \{ i \in [n] : \tau^q \beta < z_i \leq \tau^{q-1} \beta \} \)
7: Input: \( \omega^t \in C \)
8: For all \( q \in \Gamma \) set \( \pi^t_q \leftarrow \sum_{i \in \Omega_q} \omega_i^t \)
9: For all \( q \in \Gamma \) set \( \xi^t_q \leftarrow [\delta \pi^t_q] \)
10: For all \( q \in \Gamma \) and for all \( k \leq \xi^t_q \) draw \( \xi^t_{q,k} \) randomly from \( \Omega_q \) such that \( \xi^t_{q,k} \leftarrow i \) with probability \( \omega_i^t / \pi^t_q \)
11: For all \( q \in \Gamma \) and for \( k := \xi^t_q + 1 \) draw \( \xi^t_{q,k} \) randomly from \( \Omega_q \cup \{0\} \) such that, for \( i \in \Omega_q \) we have \( \xi^t_{q,k} \leftarrow i \) with probability \( (\delta \pi^t_q - [\delta \pi^t_q] \omega_i^t / \pi^t_q) \) and we have \( \xi^t_{q,k} \leftarrow 0 \) with probability \( [\delta \pi^t_q] + 1 - \delta \pi^t_q \) \).

NB: In the case that \( \pi^t_q = 0 \) we define 0/0 = 0

12: Output: \( X^t \leftarrow \{ \xi^t_{q,k} : q \in \Gamma, k \leq \xi^t_q + 1 \} \setminus \{0\} \)

### 4 Feasibility of \( X^t \)

In this section we show that the the total energy of the actions selected by Algorithm 1 is no greater than 1, as required in our problem definition (see Section 2.2). We first introduce the sets and quantities used in the selection of the actions.

**Definition 4.1.** We define the following:
Algorithm 2 Computing $\omega^{t+1}$

1: $C \leftarrow \{ x \in [0, 1]^n : (x, z) \leq 1 \}$
2: $\beta \leftarrow \max_{i \in [n]} z_i$
3: $\delta \leftarrow (1 - \sqrt{\beta})^2$

4: Input: $\omega^t \in C$, $c^t \in \mathbb{R}^n$, $r^t \in \mathcal{P}$

5: Order $[n]$ as $[n] = \{ (t, 1), \sigma(t, 2), ..., \sigma(t, n) \}$ such that $r^t_{\sigma(t,j)} \geq r^t_{\sigma(t,j+1)}$ for all $j \in [n-1]$
6: For all $j \in [n]$ set $\epsilon_j^t \leftarrow \exp \left( -\delta \sum_{k=1}^j \omega^t_{\sigma(t,k)} \right)$
7: For all $j \in [n]$ set:
   $\lambda^t_j \leftarrow r^t_{\sigma(t,n)} \epsilon^t_n + \sum_{k=j}^{n-1} \left( r^t_{\sigma(t,k)} - r^t_{\sigma(t,k+1)} \right) \epsilon^t_k$
8: For all $j \in [n]$ set:
   $g^t_{\sigma(t,j)} \leftarrow \delta \left( c^t_{\sigma(t,j)} + c^t_{\sigma(t,j)} \right) \exp \left( -\delta \omega^t_{\sigma(t,j)} \right) - \lambda^t_j$
9: $\eta^t_j \leftarrow \min \{ g^t_{\sigma(t,j)} \}

10: \eta^t \leftarrow \sqrt{2t}
11: y^t \leftarrow \omega^t - \eta^t g^t

12: Output: $\omega^{t+1} \leftarrow \Pi_C(y^t)$

- $\tau := 1 - \sqrt{\beta}$ and $\delta := (1 - \sqrt{\beta})^2$
- For all $q \in \mathbb{N}$ we define $\Omega_q := \{ i \in [n] : \tau^q \beta < z_i \}$
- $\Gamma := \{ q \in \mathbb{N} : \Omega_q \neq \emptyset \}$. Note that $\{ \Omega_q : q \in \Gamma \}$ is a partition of $[n]$
- On trial $t$, for all $q \in \Gamma$ define $\pi_q^t := \sum_{i \in \Omega_q} \omega^t_i$ and $\zeta_q := \sum_{q \in \Gamma} \pi_q^t$

Algorithm 1 works by drawing actions randomly, where the number of actions (including a “null action” 0) drawn (with replacement) from $\Omega_q$ is equal to $c^t_q + 1$ and the probability distribution of the draws is dependent on $\omega^t$.

The following theorem ensures that the choice of $X^t$ made by our method satisfies our problem’s energy constraint.

Theorem 4.2. On trial $t$ we have $\sum_{i \in X^t} z_i \leq 1$.

Proof. Define $z_0 := 0$. We have $X^t = \{ \xi^t_{q,k} : q \in \Gamma, k \leq \zeta_q^t + 1 \} \setminus \{ 0 \}$ so:

$$\sum_{i \in X^t} z_i = \sum_{q \in \Gamma, k \leq \zeta_q^t + 1} z_{\xi^t_{q,k}} \leq \sum_{q \in \Gamma, k \leq \zeta_q^t + 1} \sum_{i \in X^t} z_{\xi^t_{q,k}} \leq \sum_{q \in \Gamma, k \leq \zeta_q^t + 1} \tau^{q-1} \beta$$

5 Bounding the Cumulative Profit

In this section we bound the cumulative profit of MAXHEDGE.

5.1 The Probability of Intersection

In this subsection we first bound below the probability that, on a trial $t$, an arbitrary set $Z$ intersects with $X^t$. We start with the following lemma:

Lemma 5.1. Given a set $D$ of independent draws from $|D|$-many probability distributions, such that the probability of an event $E$ happening on draw $d \in D$ is $p_d$, then the probability of event $E$ happening on either of the draws is lower bounded by:

$$1 - \exp \left( -\sum_{d \in D} p_d \right)$$

Proof. Given some $d \in D$ let $E_d$ be the event that $E$ happens on draw $d$. Given some set $D' \subseteq D$ let
$E_{D'}$ be the probability that $E$ happens on either of the draws in $D'$. We prove the result by induction on $|D|$. The inductive hypothesis clearly holds for $|D| = 0$ as then $\mathbb{P}(E_D) = 0 = 1 - 1 = 1 - \exp(0) = 1 - \exp\left(-\sum_{d\in D} \rho_d\right)$.

Now suppose the inductive hypothesis holds for $|D| = l$ for some $l$. We now show that it holds for $|D| = l + 1$ which will complete the proof. To show this choose some $d \in D$ and set $D' = D \setminus \{d\}$ We have:

$$
\mathbb{P}(\neg E_D) = \mathbb{P}(\neg E_{D'} \land \neg E_d)
$$

(17)

$$
= \mathbb{P}(\neg E_{D'}) \mathbb{P}(\neg E_d)
$$

(18)

$$
= \mathbb{P}(\neg E_{D'}) (1 - \mathbb{P}(E_d))
$$

(19)

$$
\leq \mathbb{P}(\neg E_{D'}) \exp(-\rho_d)
$$

(20)

$$
\leq (1 - \mathbb{P}(E_{D'})) \exp(-\rho_d)
$$

(21)

$$
\leq \exp\left(-\sum_{d\in D} \rho_d\right) \exp(-\rho_d)
$$

(22)

$$
\leq \exp\left(-\sum_{d\in D} \rho_d\right)
$$

(23)

$$
\leq \exp\left(-\sum_{d\in D} \rho_d\right)
$$

(24)

Where Line 19 is due to the independence of the draws and Line 23 is from the inductive hypothesis (noting $|D'| = l$). We now have:

$$
\mathbb{P}(E_{D}) = 1 - \mathbb{P}(\neg E_{D})
$$

(25)

$$
\geq 1 - \exp\left(-\sum_{d\in D} \rho_d\right)
$$

(26)

which proves the inductive hypothesis.

We now bound the probability of intersection:

**Theorem 5.2.** For any trial $t$ and any subset $Z \subseteq [n]$ we have $\mathbb{P}(Z \cap X^t \neq \emptyset) \geq 1 - \exp(-\delta \sum_{i \in Z} \omega_i^t)$.

**Proof.** Note that the event $Z \cap X^t \neq \emptyset$ happens if either of $\{\zeta_{q,k}^t \in \mathbb{N}, k \leq \zeta_q^t\}$ are in $Z$. For every $q \in \mathbb{N}$ and $k \in [\zeta_q^t]$ we have:

$$
\mathbb{P}(\zeta_{q,k}^t \in Z) = \sum_{i \in Z \cap \Omega_q} \mathbb{P}(\zeta_{q,k}^t = i)
$$

(27)

$$
= \omega_i^t / \pi_q^t
$$

(28)

and for $k = \zeta_q^t + 1$ we similarly have:

$$
\mathbb{P}(\zeta_{q,k}^t \in Z) = (\delta \pi_q^t - [\delta \pi_q^t]) \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t
$$

So letting $\rho_{(q,k)} := \mathbb{P}(\zeta_{q,k}^t \in Z)$ we have:

$$
\sum_{k=1}^{\zeta_q^t+1} \rho_{(q,k)} \leq \mathbb{P}(\zeta_{q,k}^t \in Z)
$$

$$
= \sum_{k=1}^{\zeta_q^t+1} \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t + (\delta \pi_q^t - [\delta \pi_q^t]) \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t
$$

(30)

$$
= \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t + (\delta \pi_q^t - [\delta \pi_q^t]) \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t
$$

(31)

$$
= (\zeta_q^t + \delta \pi_q^t - [\delta \pi_q^t]) \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t
$$

(32)

$$
= \delta \pi_q^t \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t
$$

(33)

$$
= \delta \sum_{i \in Z \cap \Omega_q} \omega_i^t
$$

(34)

So, plugging into Lemma 5.1 with $D := \{(q, k) : q \in \mathbb{N}, k \in [\zeta_q^t + 1]\}$, we get:

$$
\mathbb{P}(Z \cap X^t \neq \emptyset) = \mathbb{P}(\exists i \in Z : i \in X^t)
$$

(23)

$$
\geq 1 - \exp\left(-\sum_{d \in D} \rho_d\right)
$$

(26)

$$
\geq 1 - \exp\left(-\sum_{q=1}^{\infty} \sum_{k=1}^{\zeta_q^t+1} \rho_{(q,k)}\right)
$$

(36)

$$
= 1 - \exp\left(-\sum_{q=1}^{\infty} \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t\right)
$$

(37)

$$
= 1 - \exp\left(-\delta \sum_{q=1}^{\infty} \sum_{i \in Z \cap \Omega_q} \omega_i^t / \pi_q^t\right)
$$

(38)

$$
= 1 - \exp\left(-\delta \sum_{i \in Z} \omega_i^t\right)
$$

(39)

We now bound the probability that some arbitrary action is selected on trial $t$:

**Theorem 5.3.** Given some action $i \in [n]$ and some trial $t \in [T]$ we have $1 - \exp(-\delta \omega_i^t) \leq \mathbb{P}(i \in X^t) \leq \delta \omega_i^t$.

**Proof.** From Theorem 5.2 we have $\mathbb{P}(i \in X^t) = \mathbb{P}(\{i\} \cap X^t \neq \emptyset) \geq 1 - \exp(-\delta \omega_i^t)$. Choosing $q \in \mathbb{N}$ such that $i \in \Omega_q$ we also have:

$$
\mathbb{P}(i \in X^t) \leq \sum_{k=1}^{\zeta_q^t+1} \mathbb{P}(\zeta_{q,k}^t = i)
$$

(40)

$$
= \zeta_q^t \omega_q^t / \pi_q^t + (\delta \pi_q^t - [\delta \pi_q^t]) \omega_q^t / \pi_q^t
$$

(41)

$$
= \delta \pi_q^t \omega_q^t / \pi_q^t
$$

(42)

$$
= \delta \omega_i^t
$$

(43)

5.2 Approximating the Expected Profit

In this subsection we define a convex function $h^t : C \rightarrow \mathbb{R}$ and show that the expected profit on trial $t$ is bounded below by $-h^t(\omega_i^t)$. 

We also define the function \( r \) and \( v \). We define the function \( f_j^t : \mathcal{P} \to \mathbb{R} \) by:

\[
 f_j^t(\gamma) := \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \left( 1 - \exp \left( -\delta \sum_{k=1}^j \gamma_{\sigma(t,k)} \right) \right)
\]

We also define the function \( h^t : \mathcal{P} \to \mathbb{R} \) as:

\[
 h^t(\gamma) = \langle e^t, \gamma \rangle + \sum_{i=1}^n c_i^- \left(1 - \exp(-\delta \gamma_i)\right) - \sum_{j=1}^n f_j^t(\gamma)
\]

**Theorem 5.6.** For all \( t \in [T] \), the function \( h^t \) is convex.

**Proof.** For all \( j \in [n-1] \) the function \( \delta \sum_{k=1}^j \gamma_{\sigma(t,k)} \) is concave and the function \((1-\exp(-x))\) is concave and monotonic increasing which implies their combination, \((1-\exp(-\delta \sum_{k=1}^j \gamma_{\sigma(t,k)}))\), is concave. Hence, as \( \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \geq 0 \), \( f_j^t \) is concave. Similarly, as all components of \( e^t \) are negative, we have that \( \sum_{i=1}^n c_i^- \left(1 - \exp(-\delta \gamma_i)\right) \) is convex. Since also the function \( \langle e^t, \gamma \rangle \) is convex we then have that \( h^t \) is a positive sum of convex functions and is therefore convex. \( \square \)

The rest of this subsection proves that the expected profit on trial \( t \) is bounded below by \(-h^t(\omega^t)\).

**Lemma 5.8.** On trial \( t \) we have \( \max_{i \in X^t} r_i^t = \sum_{j=1}^n \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \mathcal{I} (\exists k \leq j : \sigma(t,k) \in X^t) \).

**Proof.** Let \( l := \min\{j \in [n] : \sigma(t,j) \in X^t\} \). Note that \( r_{\sigma(t,l)}^t = \max_{i \in X^t} r_i^t \).

From the definition of \( l \) we have:
- For all \( j < l \), \( \mathcal{I} (\exists k \leq j : \sigma(t,k) \in X^t) = 0 \)
- For all \( j \geq l \), \( \mathcal{I} (\exists k \leq j : \sigma(t,k) \in X^t) = 1 \)

This implies that:

\[
 \sum_{j=1}^n \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \mathcal{I} (\exists k \leq j : \sigma(t,k) \in X^t)
\]

\[
 = \sum_{j=1}^n \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \mathcal{I} (\exists k \leq j : \sigma(t,k) \in X^t)
\]

\[
 = r_{\sigma(t,l)}^t
\]

\[
 = \max_{i \in X^t} r_i^t \tag{47}
\]

By Theorem 5.6 we also have:

\[
 \sum_{i \in [n]: c_i^t \geq 0} c_i^t \mathcal{P} \left( i \in X^t \right) \leq \sum_{i \in [n]: c_i^t \geq 0} c_i^t \left(1 - \exp(-\delta \omega_i^t)\right) \tag{54}
\]

and:

\[
 \sum_{i \in [n]: c_i^t \geq 0} c_i^t \delta \omega_i^t \leq \sum_{i \in [n]: c_i^t \geq 0} c_i^t \left(1 - \exp(-\delta \omega_i^t)\right) \tag{58}
\]

so:

\[
 \sum_{i \in [n]} c_i^t \mathcal{P} \left( i \in X^t \right) \leq \delta \langle e^t, \omega^t \rangle + \sum_{i \in [n]} c_i^- \left(1 - \exp(-\delta \omega_i^t)\right) \tag{60}
\]
Hence, we have:

\[
E (\mu^t(X^t)) = E \left( \max_{i \in X^t} r^t_i - \sum_{i \in X^t} c^t_i I (i \in X^t) \right)
\]

(61)

\[
= E \left( \max_{i \in X^t} r^t_i - \sum_{i=1}^n c^t_i I (i \in X^t) \right)
\]

(62)

\[
= E \left( \max_{i \in X^t} r^t_i - \sum_{i=1}^n c^t_i \mathbb{P} (i \in X^t) \right)
\]

(63)

\[
\geq \sum_{j=1}^n f^t_j(\omega^t) - \sum_{i=1}^n c^t_i \mathbb{P} (i \in X^t)
\]

(64)

where Line (64) is due to linearity of expectation.

\[\square\]

5.3 The Gradient

In this subsection we show how to construct the gradient of \( h^t \) and bound its magnitude. We start with the following definitions.

Definition 5.10. On any trial \( t \) and for any \( j \in [n] \) we define:

- \( c^t_j := \exp \left( -\delta \sum_{k=1}^j \omega^t_{\sigma(t,k)} \right) \)
- \( \lambda^t_j := \sum_{k=j}^n (r^t_{\sigma(t,k)} - r^t_{\sigma(t,k+1)}) c^t_k \)
- \( g^t_{\sigma(t,j)} := \delta \left( c^t_{\sigma(t,j)} + c^t_{\sigma(t,j)} - \sum_{i \in S^t} c^t_i \right) - \lambda^t_j \)

We first show that \( g^t \) is the gradient of \( h^t \) evaluated at \( \omega^t \).

Theorem 5.11. On any trial \( t \) we have \( g^t = \nabla h^t(\omega^t) \).

Proof. Suppose we have some \( l \in [n] \). For \( j < l \) we have \( \partial_{\omega^t_{\sigma(t,l)}} f^t_j(\omega^t) = 0 \) and for \( j \geq l \) we have

\[
\partial_{\omega^t_{\sigma(t,l)}} f^t_j(\omega^t) = \delta \left( r^t_{\sigma(t,l)} - r^t_{\sigma(t,l+1)} \right) \exp \left( -\delta \sum_{k=1}^{j-1} \omega^t_{\sigma(t,k)} \right)
\]

(65)

\[
= \delta \left( r^t_{\sigma(t,l)} - r^t_{\sigma(t,l+1)} \right) \epsilon^t_j
\]

(66)

This implies that:

\[
\partial_{\sigma(t,l)} \sum_{j=1}^n f^t_j(\omega^t) = \delta \sum_{j=l}^n \left( r^t_{\sigma(t,j)} - r^t_{\sigma(t,j+1)} \right) \epsilon^t_j
\]

(67)

\[
= \delta \lambda^t_j
\]

(68)

We now bound the magnitude of the gradient.

Lemma 5.12. For any trial \( t \) we have \( \| g^t \|^2 \leq n \delta^2 (\hat{\lambda} + \hat{c})^2 \).

Proof. Since \( \omega^t_{\sigma(t,k)} \geq 0 \) for all \( k \in [n] \) we have \( \epsilon^t_j \in [0, 1] \) for all \( j \in [n] \). This gives us, for all \( j \in [n] \), that

\[
\delta \lambda^t_j \leq \delta \hat{c}_{\sigma(t,j)} + \delta \sum_{k=j}^{n-1} (r^t_{\sigma(t,k)} - r^t_{\sigma(t,k+1)}) \leq \delta \hat{c} + \delta \hat{\lambda}
\]

Since also \( \hat{\lambda} \geq 0 \) this implies that

\[\square\]

5.4 Online Gradient Descent

In this subsection we show that Algorithm 2 corresponds to the use of online gradient descent over \( C \) with convex functions \( \{ h^t : t \in [T] \} \) and we use the standard analysis of online gradient descent to derive a lower bound on the cumulative profit.

From here on we compare the performance of our algorithm against any fixed set \( S \) of actions such that \( \sum_{i \in S} z_i \leq 1 \). We define \( \Phi \) as the vector in \( \mathbb{R}^n \) such that, for all \( i \in [n] \), we have \( \phi_i := 0 \) if \( i \notin S \) and \( \phi_i := 1 \) if \( i \in S \). It is clear that \( \Phi \in C \).

Definition 5.13. Our learning rates are defined as follows:

- \( \eta^t := \min_{t \leq t} (\sqrt{n} / \| g^t \|) \)
- \( \eta^t := \eta^t / \sqrt{2t} \)

The next result follows from the standard analysis of online gradient descent.

Theorem 5.14. We have:

\[
\sum_{t=1}^T (h^t(\omega^t) - h^t(\phi^t)) \leq \frac{R^2}{2n \eta^t} + \frac{1}{2} \sum_{t=1}^T \eta^t \| g^t \|^2
\]

where \( R := \max_{x, y \in C} \| x - y \| \).
Proof. For all trials \( t \): From Theorem 5.6, we have that \( h^t \) is a convex function. From Theorem 5.11, we have that \( g^t = \nabla h^t(\omega^t) \). We also have that \( \eta^{t+1} \leq \eta^t \) so since \( \omega^{t+1} = \Pi_C(\omega^t - \eta^t g^t) \) and \( \phi \in C \), the standard analysis of online gradient descent (see, e.g., [24]) gives us the result.

\[ \sum_{t=1}^{T} \| g^t \|^2 \leq n\sqrt{s/(2t)} \]

Proof. From Lemma 5.16, we have:

\[ \sum_{t=1}^{T} \eta^t \| g^t \|^2 \leq \sum_{t=1}^{T} n\sqrt{s/(2t)} \]

\[ n\sqrt{s/2} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \]

\[ n\sqrt{s/2} \left( 1 + \int_{t=1}^{T} \frac{1}{\sqrt{t}} \right) \]

\[ n\sqrt{s/2} (2\sqrt{T} - 1) \]

\[ 2n\sqrt{sT/2} \]

So using Lemma 5.17 and plugging into Theorem 5.14, we have:

\[ \sum_{t=1}^{T} (h^t(\omega^t) - h^t(\phi)) \leq \frac{1}{2} n\sqrt{2sT} + n\sqrt{sT/2} \]

The next result bounds \( h^t(\phi) \).

Lemma 5.19. On trial \( t \) we have \( \delta \mu^i_{\alpha/\beta}(S) \leq h^t(\phi) \).

Proof. Let \( j' = \arg \max_{j \in [n]: \sigma(t,j) \in S} r_{t,j}^t \). Note that for all \( j \geq j' \) we have \( \sum_{k=1}^{t} \phi_{\sigma(t,k)} \geq 1 \) and hence \( f^*_j(\phi) \geq \left( r_{t,j}^t - r_{t,j+1}^t \right) \left( 1 - e^{-\delta} \right) \) so \( \sum_{j=1}^{n} f^*_j(\phi) \geq \sum_{j=1}^{n} \left( r_{\sigma(t,j)}^t - r_{\sigma(t,j+1)}^t \right) \left( 1 - e^{-\delta} \right) = r_{\sigma(t,j)}^t \left( 1 - e^{-\delta} \right) = (1 - e^{-\delta}) \max_{i \in S} r_i^t \).

Also note that \( \delta(e^+, \phi) = \delta \sum_{i \in S} c_i^+ \) and that \( \sum_{i=1}^{n} c_i^- \left( 1 - \exp(-\delta_{\phi,i}) \right) = \sum_{i \in S} c_i^- \left( 1 - e^{-\delta} \right) \). Combining with the above gives us the result.

Putting together, we obtain the main result:

Theorem 5.20. We have:

\[ \mathbb{E} \left( \sum_{t=1}^{T} \mu^t(X^t) \right) \geq \sum_{t=1}^{T} \delta \mu^i_{\alpha/\beta}(S) - n\sqrt{2T} \delta(\hat{r} + \hat{c}) \]

where \( \alpha := 1 - e^{-\delta} \).

Proof. Let \( \alpha := 1 - e^{-\delta} \). By Theorem 5.13, we have, for all \( t \in [T] \), that \( \mathbb{E}(\mu^t(X^t)) \geq -h^t(\omega^t) \). By Lemma 5.19, we have, for all \( t \in [T] \), that \( \delta \mu^i_{\alpha/\beta}(S) \leq h^t(\phi) \). Hence we have that \( \hat{\mu}_a^i(S) - \mathbb{E}(\mu^t(X^t)) \leq h^t(\omega^t) - h^t(\phi) \). By Theorem 5.18, we then have:

\[ \sum_{t=1}^{T} \left( \delta \mu^i_{\alpha/\beta}(S) - \mathbb{E}(\mu^t(X^t)) \right) \leq \sum_{t=1}^{T} (h^t(\omega^t) - h^t(\phi)) \leq n\sqrt{2T} \delta(\hat{r} + \hat{c}) \]

Rearranging gives us:

\[ \sum_{t=1}^{T} \mathbb{E}(\mu^t(X^t)) \geq \sum_{t=1}^{T} \delta \mu^i_{\alpha/\beta}(S) - n\sqrt{2T} \delta(\hat{r} + \hat{c}) \]

6 Conclusions and Ongoing Research

We presented and investigated in depth a novel online framework, capable of encompassing several online learning problems and many practical problems in the real-world. The main challenge of the general version of this problem lies in the fact that the learner’s profit depends on the maximum reward of the selected actions, instead of the sum of all their
rewards. We proposed and rigorously analysed a very scalable and efficient learning strategy MaxHedge. Current ongoing research includes:

- Deriving a lower bound on the regret.
- Complementing our results with a set of experiments on synthetic and real-world datasets.
- Several real systems usually have a switching cost for turning on/off services, which translates in our framework to the cost incurred whenever an action selected at any given trial is not selected in the preceding one. This represents an interesting direction for future research, which is certainly motivated by practical problems.

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References

[1] M. Charikar, S. Guha, E. Tardos, D.B. Shmoys. A constant–factor approximation algorithm for the k–median problem. In ACM Symposium on Theory of Computing (STOC), pp. 1–10, ACM (1999).

[2] M. Charikar and S. Guha. Improved combinatorial algorithms for the facility location and k–median problems. In IEEE Foundations of Computer Science, pages 378–388, 1999.

[3] L. Chen, H. Hassani, A. Karbasi. Online Continuous Submodular Maximization. In Proceedings of the 21st International Conference on Artificial Intelligence and Statistics, AISTATS 2018.

[4] G. Cornuejols, G. L. Nemhauser, and L. A. Wolsey. The uncapacitated facility location problem. In Pitu B. Mirchandani and Richard L. Francis, editors, Discrete Location Theory, pages 119–171. John Wiley and Son, Inc., New York, 1990.

[5] M. Cygan, A. Czumaj, M. Mucha, P. Sankowski. Online Facility Location with Deletions. In Proceedings of the 26th Annual European Symposium on Algorithms, ESA 2018.

[6] G.B. Dantzig. Discrete variable extremum problems, In Operations Research 5 (1957) 266–277.

[7] Y. Freund and R. E. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting, In Journal of Computer and System Sciences, 55:119–139, 1997.

[8] T. Fujita, K. Hatano, E Takimoto Combinatorial Online Prediction via Metarounding. In Algorithmic Learning Theory (2013), 68–82,

[9] D. Golovin, A. Krause, M. Streeter. Online Submodular Maximization under a Matroid Constraint with Application to Learning Assignments In Technical report, arXiv, 2014.

[10] K. Jain and V. Vazirani. Approximation algorithms for metric facility location and k–median problems using the primal–dual schema and Lagrangian relaxation. In J. ACM, 48(2):274–296, 2001.

[11] A. Kalai and S. Vempala. Efficient algorithms for online decision problems, In Journal of Computer and System Sciences, vol. 71, no. 3, pp. 291–307, 2005.

[12] S. Kakade, A. Kalai, and K. Liggett. Playing games with approximation algorithms. In Proceedings of the 39th Annual ACM Symposium on the Theory of Computing, pages 546-555, STOC 2007.

[13] W.M. Koolen, M.K. Warmuth, J. Kivinen. Hedging structured concepts. In Proceedings of the 23rd Annual Conference on Learning Theory, Omnipress, 2010, pp. 239-254.

[14] K. C. Kiwiel. Breakpoint searching algorithms for the continuous quadratic knapsack problem. In Math. Program, 112(2): 473-491 (2008).

[15] A. Kumar. Constant factor approximation algorithm for the knapsack median problem. In Proceedings of the annual ACM–SIAM Symposium on Discrete Algorithms (SODA), pp. 824–832, SIAM (2012).

[16] E. Hazan, and S. Kale. Online submodular minimization. In Journal of Machine Learning Research (JMLR), 2012.

[17] D. P. Helmbold and S. V. N. Vishwanathan. Online Learning of Combinatorial Objects via Extended Formulation. In Proceedings of the 29th
A On the Knapsack Problem and Component Hedge

In this section we show that Component Hedge cannot solve the online knapsack problem special case in polynomial time unless \( P = NP \).

The Component Hedge algorithm described in [13] requires a set of concepts \( \mathcal{X} \subseteq \{0,1\}^n \). In the knapsack problem, \( \mathcal{X} \) is the subset of all \( x \) that fit in the knapsack, i.e. those \( x \in \{0,1\}^d \) with \( \langle x, z \rangle \leq 1 \). For Component Hedge, the convex hull \( \mathcal{H} \) of \( \mathcal{X} \) must have a number of constraints polynomial in \( n \) and that any \( y \in \mathcal{H} \) can, in polynomial time, be decomposed into a convex combination of \( n + 1 \) concepts. We now show that if this is true then we can solve the knapsack problem in polynomial time:

Let \( c \) be the vector defining the objective function of the knapsack problem (i.e. we seek the \( x \in \mathcal{X} \) that maximises \( \langle c, x \rangle \)). Now, since \( \mathcal{H} \) has polynomial constraints we can efficiently choose the maximiser of \( \langle c, y \rangle \) (for \( y \in \mathcal{H} \)) via linear programming. Let this maximiser be \( y \). Now decompose \( y \) into a convex combination, \( \sum_{i=1}^{n+1} m_i x^i \), of \( n + 1 \) concepts \( x^i \) (where \( \sum_{i=1}^{n+1} m_i = 1 \)). Now suppose, for contradiction, that \( \langle c, x^j \rangle < \langle c, y \rangle \) for some \( j \) with \( m_j > 0 \). Without loss of generality let \( j = n + 1 \).

Then, since for all \( x \in \mathcal{X} \) with have \( \langle c, x \rangle \leq \langle c, y \rangle \) (as \( \mathcal{X} \subset \mathcal{H} \)) we have: \( \langle c, y \rangle = \langle c, (\sum_{i=1}^{n+1} m_i x^i) \rangle = \sum_{i=1}^{n+1} m_i \langle c, x^i \rangle = \sum_{i=1}^{n} m_i \langle c, x^i \rangle + m_{n+1} \langle c, x_{n+1} \rangle < \sum_{i=1}^{n} m_i \langle c, x^i \rangle + m_{n+1} \langle c, y \rangle \leq \sum_{i=1}^{n+1} m_i \langle c, y \rangle + m_{n+1} \langle c, y \rangle = \sum_{i=1}^{n+1} m_i \langle c, y \rangle = \langle c, y \rangle \) which is a contradiction.

Hence, for all \( j \) with \( m_j > 0 \) we have \( \langle c, x^j \rangle = \langle c, y \rangle \) so the maximising \( x \) is found in polynomial time. This contradicts the assumption that \( P \neq NP \).

B Using Submodular Maximisation

We consider the facility location special case (i.e. \( z := 0 \) and all costs are positive). In this case it is possible to obtain our bound by selecting, on a trial \( t \), each action \( i \) with probability \( \omega^i \). We can then use some definitions in our paper to define the following functions on a trial \( t \):

- For all \( j \in [n] \) define \( \hat{f}_j^t(\gamma) := \left( x^i - r^i_{\sigma(t,j)} + r^i_{\sigma(t,j+1)} \right) \left( 1 - \prod_{k=1}^{j} (1 - \gamma_{\sigma(t,k)}) \right) \)
- Define \( \hat{h}_t(\gamma) = \sum_{j=1}^{n} \hat{f}_j^t(\gamma) - \langle c^t, \gamma \rangle \)

We can then use the arguments in our paper to write the expected profit as: \( \mathbb{E}(\mu^t(X^t)) = \hat{h}_t(\omega^t) \) which is
continuous submodular. We can use a slight modification of our method of computing the gradient of $h^t$ to compute the gradient of $\hat{h}^t$ in quasi-linear time. With this in hand we can then plug the gradient into one of the algorithms of [3]:

- The first algorithm requires the submodular function to be monotone and non-negative whereas ours, in general, is neither. Even if our expected profit was monotone and non-negative, for their algorithm to have a regret linear in $\sqrt{T}$ their per-trial time complexity bound becomes $\Omega(n\sqrt{T})$ which is much larger than ours.

- In the gradient ascent based second algorithm, the submodular function also needs to be monotone and non-negative but the cost vector can be separated from the reward vector in the analysis allowing us to use the expected maximum reward which is monotone non-negative. The result of this analysis is similar to ours except that for them $\alpha = 1/2$ instead of the better $\alpha = 1 - e^{-1}$ that we have (NB: $\alpha$ is the discount on the comparator selection $S$).