Simultaneous identification of diffusion and absorption coefficients in a quasilinear elliptic problem

Herbert Egger, Jan-Frederik Pietschmann and Matthias Schlottbom

Numerical Analysis and Scientific Computing, Department of Mathematics, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany

E-mail: egger@mathematik.tu-darmstadt.de, pietschmann@mathematik.tu-darmstadt.de and schlottbom@mathematik.tu-darmstadt.de

Received 25 June 2013
Accepted for publication 24 October 2013
Published 20 February 2014

Abstract

In this work, we consider the identifiability of two coefficients $a(u)$ and $c(x)$ in a quasilinear elliptic partial differential equation from the observation of the Dirichlet-to-Neumann map. We use a linearization procedure due to Isakov (1993 Arch. Ration. Mech. Anal. 124 1–12) and special singular solutions to first determine $a(0)$ and $c(x)$ for $x \in \Omega$. Based on this partial result, we are then able to determine $a(u)$ for $u \in \mathbb{R}$ by an adjoint approach.

1. Introduction

We consider the simultaneous identification of the two unknown coefficients $a = a(u)$ and $c = c(x)$ in the quasilinear elliptic problem

\begin{align}
- \text{div}(a(u)\nabla u) + cu &= 0 \quad \text{in } \Omega, \tag{1} \\
u &= g \quad \text{on } \partial \Omega, \tag{2}
\end{align}

where $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded, sufficiently regular domain. We assume access to the Dirichlet-to-Neumann map, given by

\[ \Lambda_{a,c} : g \mapsto a(u)\partial_u u, \]

with $u$ denoting the solution to (1)–(2) with Dirichlet boundary datum $g$. The main contribution of our manuscript is the following result.

**Theorem 1.1.** Let assumption 2.1 hold and assume $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$. Then, $a_1(u) = a_2(u)$ and $c_1(x) = c_2(x)$ for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$. 
Let us put this result into perspective: much of the work about the identification of unknown coefficients in elliptic and parabolic partial differential equations goes back to the seminal paper of Calderón [6]. There, \( c \equiv 0 \) and the goal is to reconstruct an unknown spatially varying conductivity \( a = a(x) \) from the observation of the Dirichlet-to-Neumann map. The Calderón problem has been studied intensively by many authors, e.g., \([3, 8, 26, 27, 29, 34]\).

Indeed, several new technical tools have been developed with this application in mind. For a comprehensive review, we refer the reader to [35]. While the question of identifiability of one spatially varying coefficient can be answered affirmatively under rather general assumptions, the simultaneous determination of two coefficients \( a = a(x) \) and \( c = c(x) \) is, in general, not possible, see [2]. If \( c \) has non-vanishing imaginary part, however, [12] provides a local uniqueness result. More recently [13], the unique determination of the two parameters \( a = a(x) \) and \( c = c(x) \) was established in the class of piecewise constant and piecewise analytic coefficients, respectively. Semilinear elliptic equations with \( a \equiv 1 \) and \( c = c(x, u) \) have been considered in [21]; the case \( c = c(u, \nabla u) \) is treated in [19]. For quasi-linear elliptic equations Sun [32] proved the uniqueness of a scalar coefficient \( a = a(x, u) \) assuming that \( c \equiv 0 \). In [33], this result was generalized to positive definite symmetric matrices \( a = a(x, u) \in \mathbb{R}^{n \times n}, n \geq 2 \). Recently, many authors considered the question of uniqueness employing only partial data on the boundary, cf [13–15, 19]. Let us also mention the Bukhgeim–Klibanov method of Carleman estimates introduced in [5] to prove global uniqueness results for various types of differential equations even in the case of non-overdetermined data or single measurements, see [24] for a review of this method and a comprehensive list of applications. Besides uniqueness, also stability issues have been considered in the literature. In this context, let us refer to the work of Alessandrini [1] and also to [24, 25]. Uniqueness results for other types of problems, e.g., of parabolic type or in nonlinear elasticity can be found in \([7, 9, 10, 17, 18, 23, 31]\) and [22, 30]. A broad overview of inverse problems for partial differential equations and many more results and references can be found in the book of Isakov [20].

The rest of the paper, which is devoted to the proof of theorem 1.1, is organized as follows. In section 2, we prove the well-posedness of (1)–(2), and rigorously define the Dirichlet-to-Neumann map. In section 3, we first utilize a linearization procedure and show by contradiction that \( a(0) \) is uniquely determined by the Dirichlet-to-Neumann map. Using the knowledge of \( a(0) \), we then obtain the identifiability of \( c(x) \) by well-known results of the linearized problem. The identifiability of \( a(u) \) for \( u \neq 0 \) is established in section 4, and we conclude with a short discussion about possible extensions of our results.

2. Preliminaries

Throughout the rest of the paper, we make the following assumption on the regularity of the domain and the coefficients.

**Assumption 2.1.** \( \Omega \subset \mathbb{R}^n \) is a bounded domain in two or three space dimensions and \( \partial \Omega \in C^1 \). Furthermore, we assume that \( a \in C^0(\mathbb{R}) \) and \( c \in L^\infty(\Omega) \) such that
\[
\alpha \leq a(u) \leq \frac{1}{\alpha} \quad \text{for all } u \in \mathbb{R} \quad \text{and} \quad 0 \leq c(x) \leq \frac{1}{\alpha} \quad \text{a.e. in } \Omega
\]
for some constant \( \alpha > 0 \).

We denote by \( H^1(\Omega) \) the usual Sobolev space of square integrable functions with square integrable weak derivatives. Functions \( u \in H^1(\Omega) \) have well-defined traces \( u|_{\partial \Omega} \) and we denote by \( H^{1/2}(\partial \Omega) \) the space of traces of functions in \( H^1(\Omega) \) with norm \( \|g\|_{H^{1/2}(\partial \Omega)} = \inf_{u \in H^1(\Omega): u|_{\partial \Omega} = g} \|u\|_{H^1(\Omega)}. \) The topological dual space of \( H^{1/2}(\partial \Omega) \) is denoted by \( H^{-1/2}(\partial \Omega). \)
For some of our arguments, we will transform the quasilinear equation (1) into a semilinear one. To do so, let us introduce the primitive function

$$A : \mathbb{R} \to \mathbb{R}, \quad A(u) = \int_0^u a(\tilde{u}) \, \mathrm{d}\tilde{u},$$

which is monotonically increasing and differentiable. Since we assumed that $a \geq \alpha > 0$, the function $A$ is one-to-one and onto, and we can define its inverse $H : \mathbb{R} \to \mathbb{R}$, $H(U) = A^{-1}(U)$ with derivative

$$\frac{1}{\alpha} \geq H'(U) = \frac{1}{a(H(U))} \geq \alpha > 0.$$ 

For any weak solution $u$ of (1)–(2), the function $U = A(u)$ then solves the boundary value problem

$$-\Delta U + cH(U) = 0 \quad \text{in } \Omega,$$

$$U = G \quad \text{on } \partial\Omega,$$

with boundary datum $G = A(g)$. Note that by our assumption on the coefficients $u = H(U) \in H^1(\Omega)$ whenever $U \in H^1(\Omega)$; this follows easily from the monotonicity and differentiability of $H$ and the chain rule for Sobolev functions [11]. The next theorem establishes the well-posedness of the problems (1)–(2) and (3)–(4), respectively.

**Theorem 2.2.** Let assumption 2.1 hold. Then, for every $g \in H^{1/2}(\partial\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ to (1)–(2) which satisfies the a priori estimate

$$\|u\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)},$$

with a constant $C$ depending only on $\alpha$ and $\Omega$.

For the sake of completeness, let us sketch the proof.

**Proof.** Let us first establish the existence of a solution: given $\tilde{u} \in L^2(\Omega)$, consider the linear boundary value problem

$$-\text{div}(a(\tilde{u})\nabla u) + cu = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega.$$ 

Since $\tilde{u}$ is measurable, so is $a(\tilde{u})$, and since $a(\tilde{u}) \geq \alpha > 0$, the existence of a unique solution $u \in H^1(\Omega)$ is ensured by the Lax–Milgram lemma, cf [11, theorem 5.8]. Moreover, we have $\|u\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}$ with $C$ only depending on $\alpha$ and $\Omega$. Next, consider the nonlinear operator $T : L^2(\Omega) \to L^2(\Omega)$ defined by $Tu := u$ with $u$ the solution of the problem above. We will establish the existence of a fixed-point for the mapping $T$, which then is a solution of (1)–(2), by a compactness argument: Due to the a priori estimate for the linear problem, $T$ maps the compact convex set $M = \{v \in L^2(\Omega) : \|v\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}\}$ into itself.

Moreover, $T$ is continuous, which can be seen as follows: observe that $\tilde{u}_n \to \tilde{u}$ in $L^2(\Omega)$ implies that $\tilde{u}_n \to \tilde{u}$ a.e. for some subsequence $\tilde{u}_n$. By assumption 2.1 and Lebesgue’s dominated convergence theorem, we obtain $a(\tilde{u}_n)\nabla u \to a(\tilde{u})\nabla u$ in $L^2(\Omega)$. Together with the a priori estimate for the linear problem, this yields the continuity of $T$; see also the proof of lemma 3.1. The existence of a fixed-point for $T$ in $M$ then follows by Schauder’s fixed-point theorem [11, theorem 11.1]. Clearly, any regular fixed-point of $T$ is also a solution of (1)–(2) and the a priori estimate follows from the definition of the set $M$. 

3
Let us now turn to the question of uniqueness: assume that there exist two solutions $u_1, u_2$ to (1)–(2) with the same Dirichlet boundary data and set $U_1 = A(u_1)$ and $U_2 = A(u_2)$. Then, $U = U_1 - U_2$ solves

$$-\Delta U + cH'(\xi(x))U = 0 \quad \text{in } \Omega,$$

$$U = 0 \quad \text{on } \partial \Omega,$$

where we used $H(U_1(x)) - H(U_2(x)) = H'(\xi(x))(U_1(x) - U_2(x))$ a.e. for some measurable function $\xi(x)$. Since $H' \geq 0$, we obtain from the weak maximum principle \cite[theorem 8.1]{11} that $U \equiv 0$, and by monotonicity of $A$ we deduce that $u_1 = u_2$. \hfill \Box

To give a precise definition of the Dirichlet-to-Neumann map in our functional setting, we introduce for $\tau \in \mathbb{R}$, the solution of (7)–(8) with boundary datum $g^\tau$.

Following an idea of Isakov \cite{18}, we employ a linearization strategy to obtain uniqueness for $a(0)$ and $c(x)$. Consider the following linear boundary value problem

$$-a(0)\Delta v + cv = 0 \quad \text{in } \Omega,$$

$$v = g^\ast \quad \text{on } \partial \Omega.$$  \hfill (7)\hfill (8)

The existence of a unique weak solution $v \in H^1(\Omega)$ follows again from the Lax–Milgram theorem. The Dirichlet-to-Neumann map associated with the linear problem is given by

$$\Lambda_{a(0),c}^\ast : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega), \quad g \mapsto a(0)\partial_n v,$$

where $v$ is the solution of (7)–(8) with boundary datum $g^\ast$. With similar arguments as in \cite{18}, we obtain the following result.

\textbf{Lemma 3.1.} The Dirichlet-to-Neumann map $\Lambda_{a,c}$ for (1)–(2) determines the Dirichlet-to-Neumann map $\Lambda_{a(0),c}^\ast$ associated with (7)–(8).

\textbf{Proof.} Let $g^\ast \in H^{1/2}(\partial \Omega)$ be given. For any $\tau \in \mathbb{R}$, we denote by $u_\tau$ the solution of (1)–(2) with boundary value $\tau g^\ast$. By theorem 2.2, such a solution $u_\tau$ exists and is unique, and for $\tau = 0$ we have $u_0 \equiv 0$. The function $v_\tau := (u_\tau - u_0)/\tau = u_\tau/\tau$ then is a solution of

$$-\text{div}(a(u_\tau)\nabla v_\tau) + cv_\tau = 0 \quad \text{in } \Omega,$$

$$v_\tau = g^\ast \quad \text{on } \partial \Omega.$$
Moreover, with $v$ defined by (7)–(8), the difference $w_\tau = v - v_\tau$ solves
\[-\text{div}(a(u_\tau)\nabla w_\tau) + cw_\tau = -\text{div}((a(u_\tau) - a(0))\nabla v) \quad \text{in } \Omega,\]
\[w_\tau = 0 \quad \text{on } \partial \Omega.\]

Using standard \textit{a priori} estimates for linear elliptic problems and assumption 2.1, we obtain
\[
\|w_\tau\|_{H^{1}(\Omega)} \leq C\|(a(u_\tau) - a(0))\nabla v\|_{L^1(\Omega)},
\]
with a constant $C$ depending only on $a$ and $\Omega$. Using the \textit{a priori} estimate (5), we obtain $u_\tau \to 0$ in $H^1(\Omega)$ as $\tau \to 0$, and hence, by a subsequence argument, $u_\tau(x) \to 0$ as $\tau \to 0$ for a.e. $x \in \Omega$. By continuity of the parameter, it follows that $a(u_\tau(x)) \to a(0)$ for a.e. $x \in \Omega$, and from Lebesgue’s dominated convergence theorem, we infer that $w_\tau \to 0$ in $H^1(\Omega)$ as $\tau \to 0$.

Using the definition of the co-normal derivative (6), we further obtain
\[
\frac{1}{\tau} \Lambda_{\alpha,c} \tau g^\alpha = a(u_\tau) \partial_n v_\tau \to a(0) \partial_n v = \Lambda_{\alpha(0),c}^* g^\alpha
\]
in $H^{-1/2}(\partial \Omega)$ as $\tau \to 0$, and hence $\Lambda_{\alpha(0),c}^*$ is determined by $\Lambda_{\alpha,c}$. □

As a next step, we turn to the identification of $a(0)$ and $c(x)$ from knowledge of the linearized Dirichlet-to-Neumann map $\Lambda_{\alpha(0),c}^*$. Let $(a_1, c_1)$ and $(a_2, c_2)$ satisfy assumption 2.1 and denote by $v_1$ and $v_2$ the corresponding solutions of (7)–(8) with coefficients $(a_1(0), c_1)$ and $(a_2(0), c_2)$, respectively. The definition of the co-normal derivative yields the following orthogonality relation
\[
\langle (\Lambda_{\alpha_1(0),c_1}^* - \Lambda_{\alpha_2(0),c_2}^*) g^\alpha, g^\alpha \rangle = \int_\Omega (a_1(0) - a_2(0))\nabla v_1 \nabla v_2 + (c_1 - c_2)v_1 v_2 \, dx. \quad (9)
\]

We are now in the position to prove the following result.

**Theorem 3.2.** If $\Lambda_{\alpha_1(0),c_1}^* = \Lambda_{\alpha_2(0),c_2}^*$, then $a_1(0) = a_2(0)$.

**Proof.** The proof is inspired by the construction of singular solutions utilized in [1]. Let $\Phi_y(x)$ be the fundamental solution for the Laplace equation, i.e., we have $\Phi_y(x) = 1/|x - y|$ for $n = 3$ and $\Phi_y(x) = \log(|x - y|)$ for $n = 2$. Note that for any $y \in \mathbb{R}^n$ we have $\Phi_y \in L^2(\Omega)$, while $\Phi_y \in H^1(\Omega)$, if, and only if, $y \notin \partial \Omega$. Now suppose that $a_1(0) \neq a_2(0)$ and let $w_i \in H^1_0(\Omega)$, $i = 1, 2$, be the solution of
\[-a_i(0) \Delta w_i + c_i w_i = -c_i \Phi_y \quad \text{in } \Omega,
\]
\[w_i = 0 \quad \text{on } \partial \Omega.
\]
The function $v_i = w_i + \Phi_y$ then is a solution of (7)–(8) with $g^\alpha = \Phi_y$, and we see that $\|v_i\|_{L^2(\Omega)} \leq C$ for all $y \in \mathbb{R}^n$, but $\|\nabla v_1 \nabla v_2\|_{L^1(\Omega)} < \infty$ only if $y \notin \partial \Omega$. Inserting $v_1$ and $v_2$ into the orthogonality relation (9) and rearranging terms, we obtain
\[
(a_1(0) - a_2(0)) \int_\Omega \nabla v_1 \nabla v_2 \, dx = \int_\Omega (c_2 - c_1)v_1 v_2 \, dx.
\]
Since the integral on the right-hand side is uniformly bounded, but that on the left-hand side diverges as $\text{dist}(y, \partial \Omega) \to 0$, we arrive at a contradiction. Hence, $a_1(0) = a_2(0)$. □

Once $a(0)$ is determined, the uniqueness of $c(x)$ follows from known results: the three-dimensional case can be found in [34] or [20, theorem 5.2.2]. For $0 \leq c \in L^\infty(\Omega)$, the uniqueness result for $n = 2$ can be deduced from the uniqueness of the conductivity problem [31], see [20, corollary 5.5.2]. The restriction $c \geq 0$ can possibly be relaxed using the results of [4, 16]. Thus, we obtain

**Theorem 3.3.** Assume that $\Lambda_{\alpha_1(0),c_1}^* = \Lambda_{\alpha_2(0),c_2}^*$. Then $a_1(0) = a_2(0)$ and $c_1(x) = c_2(x)$ for a.e. $x \in \Omega$.  

5
4. Identification of $a$

To show the uniqueness of $a(u)$ for $u \neq 0$, we translate the techniques of the previous section to the nonlinear problem. By the definition of the co-normal derivative (6), there holds

$$\langle \Lambda_{a,c}, \lambda |_{\partial \Omega} \rangle = \int_{\Omega} a(u) \nabla u \nabla \lambda + cu \lambda \, dx$$

for any function $\lambda \in H^1(\Omega)$. Subtracting this identity for the two pairs $(a_1, c)$ and $(a_2, c)$ of admissible parameters, and using $\nabla (A_i(u(x))) = A_i'(u(x)) \nabla u = a_i(u(x)) \nabla u$, $i = 1, 2$ and integration by parts, we get

$$\langle (\Lambda_{a_1,c} - \Lambda_{a_2,c}) g, \lambda |_{\partial \Omega} \rangle = \int_{\Omega} (A_1(u_1) - A_2(u_2)) (-\Delta \lambda) + c(u_1 - u_2) \lambda \, dx$$

$$+ \int_{\partial \Omega} (A_1(g) - A_2(g)) \partial_n \lambda \, ds.$$ 

To simplify this expression, we consider only test functions $\lambda$ which are solutions of

$$-\Delta \lambda = 0 \quad \text{in } \Omega, \quad \lambda = \lambda_D \quad \text{on } \partial \Omega$$

for some appropriate boundary datum $\lambda_D$, which yields

$$\langle (\Lambda_{a_1,c} - \Lambda_{a_2,c}) g, \lambda_D \rangle = \int_{\Omega} c(u_1 - u_2) \lambda \, dx + \int_{\partial \Omega} (A_1(g) - A_2(g)) \partial_n \lambda \, ds. \tag{12}$$

Note that the left-hand side will vanish, if the Dirichlet-to-Neumann maps coincide. We can therefore retrieve information about $a_1 - a_2$, by choosing a suitable function $\lambda$ satisfying (10)–(11).

**Theorem 4.1.** Let $\Lambda_{a_1,c} = \Lambda_{a_2,c}$ for some $a_1, a_2$ and $c$ satisfying assumption 2.1. Then, $a_1(u) = a_2(u)$ for all $u \in \mathbb{R}$.

**Proof.** We only consider the three dimensional case and assume, for simplicity, that the boundary $\partial \Omega$ of the domain is flat near some point $\bar{x} \in \partial \Omega$. Suppose there exists $\bar{g} \in \mathbb{R}$ with $a_1(\bar{g}) - a_2(\bar{g}) > 0$. Then by continuity, $a_1(u) < a_2(u)$ for $u \in [\bar{g}, \bar{g}]$ with $\bar{g} < \bar{g}$. Let us define the boundary datum $g$ by

$$g(x) = \begin{cases} \bar{g}, & |x - \bar{x}| \leq r, \\ \frac{|x - \bar{x}| - r} {s - r} \bar{g} + \frac{s - |x - \bar{x}|} {s - r} g, & r < |x - \bar{x}| < s, \\ g, & |x - \bar{x}| \geq s, \end{cases}$$

where $0 < r < s$ are sufficiently small and will be specified below. For $\varepsilon > 0$ define $\lambda^\varepsilon(x) = n(x) \cdot \nabla \Phi^\varepsilon(x)$ with $y^\varepsilon = \bar{x} + \varepsilon n(\bar{x})$ and $\Phi^\varepsilon(x) = 1/|x - y|$ as in the proof of theorem 3.2. Observe that $\lambda^\varepsilon$ is harmonic in $\Omega$ and uniformly bounded in $L^1(\Omega)$ for all $\varepsilon \geq 0$. Now from (12) and $\Lambda_{a_1,c} = \Lambda_{a_2,c}$, we obtain

$$-\int_{\Omega} c(u_1 - u_2) \lambda^\varepsilon \, dx = \int_{\partial \Omega} (A_1(g) - A_2(g)) \partial_n \lambda^\varepsilon \, ds$$

$$= (A_1(\bar{g}) - A_2(\bar{g})) \int_{\partial \Omega} \partial_n \lambda^\varepsilon \, ds + \int_{\partial \Omega} B(g) \partial_n \lambda^\varepsilon \, ds. \tag{13}$$

Since $\lambda^\varepsilon$ is harmonic in $\Omega$, the first integral on the right-hand side vanishes and in the second term we abbreviated

$$B(u) = (A_1(u) - A_1(\bar{g})) - (A_2(u) - A_2(\bar{g})) = \int_{\bar{g}}^{u} a_1(u) - a_2(u) \, du.$$
Since $a_1(u) - a_2(u) > 0$, the function $B(u)$ is strictly monotonically increasing and positive on $(\bar{g}, \bar{\bar{g}}]$. The second term can then be further evaluated by

$$\int_{\partial \Omega} B(g) \partial_n \lambda^x \, ds = B(\bar{g}) \int_{|x-\bar{\bar{g}}| < r} \partial_n \lambda^x \, ds + \int_{r < |x-\bar{\bar{g}}| < s} B(g) \partial_n \lambda^x \, ds$$

$$= 2\pi B(\bar{g}) \left( \frac{r^2}{(r^2 + \varepsilon^2)^{3/2}} - \frac{s^2}{(s^2 + \varepsilon^2)^{3/2}} \right),$$

where integration is performed over subsets of the boundary and $B \in [0, B(\bar{g})]$. This formula holds for all $0 < r < s$ sufficiently small and all $\varepsilon > 0$. By choosing $r = \varepsilon$ and $s = \varepsilon + \varepsilon^3$ and letting $\varepsilon \to 0$, the first integral can be made arbitrarily large while the second integral can be made arbitrarily small. Since the left-hand side of (13) is uniformly bounded as $\varepsilon \to 0$, we obtain a contradiction to the assumption that $a_1(\bar{g}) - a_2(\bar{g}) > 0$. The two dimensional case and curved boundaries can be treated with similar arguments. □

**Remark 4.2.** Similar orthogonality relations and adjoint problems have been used for one-dimensional equations before. In [10], the identifiability of $a$ is established by controlling the sign of $u_1$ and $\lambda_x$ using monotonicity arguments. This argument is, however, not applicable in the multi-dimensional case.

Summarizing the previous results, we obtain our main result.

**Proof of theorem 1.1.** If $\Lambda_{a_1, c_1} = \Lambda_{a_2, c_2}$, then lemma 3.1 implies that $\Lambda_{a_1(0), c_1} = \Lambda_{a_2(0), c_2}$. Thus, $a_1(0) = a_2(0)$ by theorem 3.2 and $c_1 = c_2$ by theorem 3.3. The assertion $a_1(u) = a_2(u)$ follows from theorem 4.1, which concludes the proof. □

**5. Discussion**

Concerning stability when reconstructing $c$, the best one can expect is an estimate of logarithmic type even if we assume that the coefficient $a$ is known and constant; see [1] for details. Thus, the inverse problem considered in this paper is severely ill-posed.

**Acknowledgments**

HE acknowledges support by DFG via Grant IRTG 1529 and GSC 233. The work of JFP was supported by DFG via Grant 1073/1-1 and from the Daimler and Benz Stiftung via Post-Doc Stipend 32-09/12. We would like to thank Professor Bastian von Harrach for valuable comments when completing the proof of theorem 4.1.

**References**

[1] Alessandrini G 1990 Singular solutions of elliptic equations and the determination of conductivity by boundary measurements J. Differ. Equ. 84 252–72

[2] Arridge S R and Lionheart W R B 1998 Nonuniqueness in diffusion-based optical tomography Opt. Lett. 23 882–4

[3] Astala K and Päivärinta L 2006 Calderón’s inverse conductivity problem in the plane Ann. Math. 163 265–99

[4] Bukhgeim A L 2008 Recovering a potential from Cauchy data in the two-dimensional case J. Inverse Ill-Posed Problems 16 19–33

[5] Bukhgeim A L and Klibanov M V 1981 Uniqueness in the large of a class of multidimensional inverse problems Sov. Math. Dokl. 24 244–7

[6] Calderón A-P 1980 On an inverse boundary value problem Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro: Sociedade Brasileira de Matematica) pp 65–73
[7] Cannon J R and Yin H 1989 A class of non-linear non-classical parabolic equations J. Differ. Eqns 79 266–88
[8] Druskin V 1982 The unique solution of the inverse problem of electrical surveying and electrical well-logging for piecewise-continuous conductivity Earth Phys. 18 51–3
[9] DuChateau P and Rundell W 1985 Unicity in an inverse problem for an unknown reaction term in a reaction–diffusion equation J. Differ. Eqns 59 155–64
[10] DuChateau P, Thelwell R and Butters G 2004 Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient Inverse Problems 20 601–25
[11] Gilbarg D and Trudinger N S 2001 Elliptic Partial Differential Equations of Second Order (Berlin: Springer)
[12] Grinberg N I 2000 Local uniqueness for the inverse boundary problem for the two-dimensional diffusion equation Eur. J. Appl. Math. 11 473–89
[13] Harrach B 2009 On uniqueness in diffuse optical tomography Inverse Problems 25 055010
[14] Imanuvilov O Y, Uhlmann G and Yamamoto M 2010 The Calderón problem with partial data in two dimensions J. Am. Math. Soc. 23 655–91
[15] Imanuvilov O Y, Uhlmann G and Yamamoto M 2011 Determination of second-order elliptic operators in two dimensions from partial cauchy data Proc. Natl Acad. Sci. 108 467–72
[16] Imanuvilov O Y and Yamamoto M 2012 Inverse boundary value problem for Schrödinger equation in two dimensions SIAM J. Math. Anal. 44 1333–9
[17] Isakov V 1989 Uniqueness for inverse parabolic problems with a lateral overdetermination Commun. Partial Differ. Eqns 14 681–9
[18] Isakov V 1993 On uniqueness in inverse problems for semilinear parabolic equations Arch. Ration. Mech. Anal. 124 1–12
[19] Isakov V 2001 Uniqueness of recovery of some quasilinear partial differential equations Commun. Partial Differ. Eqns 26 1947–73
[20] Isakov V 2006 Inverse Problems for Partial Differential Equations (New York: Springer Science and Business Media)
[21] Isakov V and Sylvester J 1994 Global uniqueness for a semilinear elliptic inverse problem Commun. Pure Appl. Math. 47 1403–10
[22] Kang H and Nakamura G 2002 Identification of nonlinearity in conductivity equation via Dirichlet-to-Neumann map Inverse Problems 18 1079–88
[23] Klibanov M V 2004 Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation Inverse Problems 20 495–514
[24] Klibanov M V 2013 Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems J. Inverse Ill-Posed Problems 21 477–560
[25] Klibanov M V and Timonov A 2004 Carleman Estimates for Coefficient Inverse Problems and Numerical Applications (Inverse and Ill-Posed Problems Series) (Utrecht: VSP)
[26] Kohn R and Vogelius M 1984 Determining conductivity by boundary measurements Commun. Pure Appl. Math. 37 289–98
[27] Kohn R V and Vogelius M 1985 Determining conductivity by boundary measurements: II. Interior results Commun. Pure Appl. Math. 38 643–67
[28] McLean W 2000 Strongly Elliptic Systems and Boundary Integral Equations (Cambridge: Cambridge University Press)
[29] Nachmann A 1996 Global uniqueness for a two-dimensional inverse boundary value problem Ann. Math. 143 71–96
[30] Nakamura G and Sun Z 1994 An inverse boundary value problem for St Venant–Kirchhoff materials Inverse Problems 10 1159–63
[31] Pilant M and Rundell W 1990 Recovery of an unknown specific heat by means of overposed data Numer. Methods Partial Differ. Eqns 6 1–16
[32] Sun Z 1996 On a quasilinear inverse boundary value problem Math. Z. 221 293–305
[33] Sun Z and Uhlmann G 1997 Inverse problems in quasilinear anisotropic media Am. J. Math. 119 771–97
[34] Sylvester J and Uhlmann G 1987 Global uniqueness theorem for an inverse boundary problem Ann. Math. 125 153–69
[35] Uhlmann G 2009 Electrical impedance tomography and Calderón’s problem Inverse Problems 25 123011