DEFORMATIONS OF $\mathbb{E}_\infty$-GROUPS OF UNITS AND LOGARITHMIC DERIVATIVES OF $\mathbb{E}_\infty$-RINGS

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Abstract. We extend a classical fact about deformations of groups of units of commutative rings to $\mathbb{E}_\infty$-ring spectra, and we use this result to provide a map of spectra generalizing the ordinary logarithmic derivative induced by an $R$-module derivation.

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Introduction

Stable homotopy theory provides, among other things, a framework to develop a homotopy coherent version of ordinary commutative algebra. Instances of this generalization are by now abundant in the literature. The aim of this paper is to extend to this context the notion of logarithmic derivatives, by means of a generalization of the following classical result (see Proposition A.3 for a more detailed statement, and a proof).

Proposition Given a square-zero extension of commutative rings $\tilde{R} \to R$ with kernel $I$, there exists an induced short exact sequence of Abelian groups

$$0 \to I \to GL_1 \tilde{R} \to GL_1 R \to 0.$$
The main motivation for the definition of homotopy coherent logarithmic derivatives is the investigation of lifts of Adams operations on (suitable localizations of) K-theory to morphisms of spectra, developing on current work in progress by Barwick-Glasman-Mathew-Nikolaus.

To work with homotopy theoretic tools, we will use the language of \( \infty \)-categories, as developed in [Lur09], and we will set our discussion of stable homotopy theoretic ideas in the framework developed, using this language, in [Lur17].

To guide the intuition in the homotopy coherent setting, it is useful to think of the \( \infty \)-category of spaces as playing the role pertaining to the category of sets in the ordinary context. We can express the analogy with ordinary commutative algebra by thinking of the \( \infty \)-category of spectra as the analogue of the Abelian category of Abelian groups and the derived category of the integers at once (and more generally of stable \( \infty \)-categories as the analogue of both Abelian categories and derived categories). Along these lines, \( \mathbb{E}_\infty \)-rings become the counterparts of ordinary commutative rings.

With these ideas in mind, our goal will be to prove the following generalization of the above Proposition (see Theorem 1.30; the reasons for the connectivity conditions will be clarified in Remark 1.31).

**Theorem** Let \( R \) be a connective \( \mathbb{E}_\infty \)-ring, and let \( \tilde{R} \to R \) be a square-zero extension by a connective \( R \)-module \( M \). Then, there exists a co/fiber sequence

\[
M \to gl_1 \tilde{R} \to gl_1 R
\]

in the \( \infty \)-category \( \text{Sp} \).

By virtue of the above theorem, we will be able to define a map of spectra \( \log_\partial : gl_1 R \to M \) from any connective \( \mathbb{E}_\infty \)-ring \( R \) and any derivation (as defined in [Lur17, 7.4.1] and reviewed in Section 1.2) \( \partial \) of \( R \) into an \( R \)-module \( M \).

In Section 1, we set up what we need in order to state the theorem, then we give two alternative proofs of it, one recovering the ordinary proposition as a particular case, the other leveraging the homotopy coherent statement from the ordinary one. In Section 2, we define our homotopy coherent logarithmic derivative, and show how it generalizes the ordinary one. Appendix A, contains a proof of the ordinary result. Finally, in Appendix B, we recall a few results about symmetric monoidal \( \infty \)-categories and modules over commutative algebra objects.

**Prerequisites.** We assume the reader is familiar with the language of \( \infty \)-categories, as developed in [Lur09]. Notably, we make free use of the Adjoint Functor Theorem (see [Lur09, 5.5.2.9]). Even if we will recall some definitions, we will assume that the reader is familiar with the theory of stable \( \infty \)-categories and of \( \infty \)-operads as developed in [Lur17], and in particular with the \( \infty \)-category of spectra and its symmetric monoidal structure given
by the smash product. Some of the notations used in this paper differ from
the ones in [Lur17]. However, we use the same notations and terminology
of [Lur17] for all the concepts we do not explicitly recall or introduce here.

Relation to previous work. A result essentially equivalent to Theorem
1.30 appeared, using rigid models to deal with homotopy coherent struc-
tures, as [Rog09, Lemma 11.2], and the proof sketched in [Rog09, Remark
11.3] is strictly related to the proof we present in Section 1.4.

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1. Groups of units and square-zero extensions

We assume the reader is familiar with the theories of stable ∞-categories
and of ∞-operads, as developed in [Lur17]. For the convenience of the
reader, and to fix notations, we recall some definitions and some results
about symmetric monoidal ∞-categories and modules over (homotopy co-
herent) commutative algebra objects in Appendix B, and we refer to [Lur17]
for all the concepts there undefined.

We focus our attention to the presentable symmetric monoidal ∞-cate-
gories of spaces and spectra. In Section 1.1, we begin by defining the objects
we want to investigate, namely groups of units and square-zero extensions
of $E_\infty$-rings, and by establishing some of their basic properties. Then, in
Section 1.3, we begin our proof of Theorem 1.30. In Section 1.4 and 1.5
we provide two alternative proofs, one at the spectral level, and one at the
space level.

1.1. Definitions and basic facts. Most of the material presented in this
section, is an adaptation of definitions and results presented using rigid
models for the categories of spectra in [ABG+14] and [Rez06]. In particular,
our treatment of groups of units and group $E_\infty$-rings is inspired by the
presentation given in [ABG+14], although our proofs are independent of
the rigid model results, and are entirely developed in the ∞-categorical
setting. Also, we recollect various facts regarding the properties of groups
of units of $E_\infty$-rings following closely [Rez06], working out explicitly some
details left to the reader therein. Finally, we recall some definitions and
basic facts about square-zero extensions of $E_\infty$-rings from [Lur17].

We will reserve the notations CMon, CGrp and CAlg for the ∞-categories
CMon(S), CGrp(S) and CAlg(Sp), respectively (see Remark B.13).
Proposition 1.1 The symmetric monoidal adjunction $\Sigma^+ \dashv \Omega^\infty$, restricts to an adjunction

$$S[-] : \text{CMon} \rightleftarrows \text{CAlg} : \Omega^\infty_m$$

such that the diagram

$$\begin{array}{c}
\text{CMon} \\
\downarrow \text{S}\[\text{-}\] \\
S
\end{array} \quad \begin{array}{c}
\text{CAlg} \\
\downarrow \Sigma^+ \infty \\
\text{Sp}
\end{array}$$

(where the vertical arrows are the forgetful functors) commutes.

Proof. Recall that $\text{CMon} \simeq \text{Alg}_{E^\infty}(S) \subset \text{Fun}_{E^\infty}(E^\infty, S^\infty)$. The functor $\text{S}\[\text{-}\]$ is obtained by postcomposition with the symmetric monoidal functor $S^\times \to \text{Sp}^\otimes$ induced by $\Sigma^+\infty$, and $\Omega^\infty_m$ is obtained analogously.

As explained in [RV15, RV17], to give an adjunction between $\infty$-categories is equivalent to give an adjunction in the underlying 2-category of quasi-categories. Given unit and counit transformations for the adjunction $\Sigma^+ \dashv \Omega^\infty$, we obtain unit and counit transformations for $\text{S}\[\text{-}\]$ and $\Omega^\infty_m$ by precomposition with (the underlying 1-cells of) sections of the symmetric monoidal structure maps, thus satisfying the triangle identities automatically. □

Definition 1.2 Given an $E^\infty$-ring $R$, we call $\Omega^\infty_m R$ its underlying multiplicative $E^\infty$-space. Given an $E^\infty$-space $X$, we call $\text{S}\[X\]$ its monoid $E^\infty$-ring.

A few words on notation and terminology are overdue. To see the reasons for the analogy with the ordinary case, consider an $E^\infty$-ring $R$. By definition, $R$ is a commutative algebra object in the $\infty$-category of spectra. The functor $\Omega^\infty_m$ acts by “forgetting” the additive structure of $R$ (falling back from spectra to spaces), but still remembering its multiplicative structure, as it takes values in the $\infty$-category of commutative algebra objects in $S$. The analogy is particularly clear if we restrict to connective $E^\infty$-rings. By Remark B.21, the $\infty$-categories $\text{Sp}^\text{cn}$ and $\text{CGrp}$ are equivalent, hence the $\infty$-category of connective $E^\infty$-rings $\text{CAlg}^\text{cn}$ is equivalent to the $\infty$-category $\text{CAlg}(\text{CGrp})$. As the restriction of $\Omega^\infty$ to $\text{Sp}^\text{cn}$ corresponds to the forgetful functor $\text{CGrp} \to S$ under this equivalence, the functor $\Omega^\infty_m$ restricts to $\text{CAlg}(\text{CGrp}) \to \text{CAlg}(S) \simeq \text{CMon}$.

Proposition 1.3 The forgetful functor $\text{CGrp} \to \text{CMon}$ admits a right adjoint, denoted

$$(\_^X) : \text{CMon} \to \text{CGrp}.$$  

Proof. This is a direct consequence of [Lur17, 5.2.6.9] and the Adjoint Functor Theorem. □

Remark 1.4 The functor $(\_^X)$ can be explicitly described as the functor sending an $E^\infty$-space $M$ to its connected components which are invertible.
in the commutative monoid $\pi_0M$ (see Remark B.10), and the counit of the adjunction as the direct summands inclusion. To see this, let us momentarily denote by $M^u$ those connected components of $M$ which are invertible in $\pi_0M$, and let us denote by $\iota: M^u \to M$ the direct summands inclusion. Clearly, given any $E_\infty$-group $G$, postcomposition with $\iota$ determines a map

$$\Map_{\text{CGrp}}(G, M^u) \xrightarrow{\iota_*} \Map_{\text{CMon}}(G, M).$$

On the other hand, any element in $\Map_{\text{CMon}}(G, M)$ induces a morphism of ordinary monoids $\pi_0G \to \pi_0M$, which, since $\pi_0G$ is a group, factors through $(\pi_0M)^\times$. As a consequence, the actual morphism $G \to M$ we started from has to factor through $\iota$. A straightforward check shows that this factorization gives a homotopy inverse to $\iota_*$. 

Using the notation given in Appendix B, we think of $(-)^{gp}$ as group completion and of $(-)^{\times}$ as the maximal subgroup of an $E_\infty$-space.

**Notation 1.5** We denote the composite right adjoint $(-)^{\times} \circ \Omega^\infty_m$ by

$$GL_1 : \text{CAlg} \to \text{CGrp}$$

and its left adjoint, using a notation analogous to the common one for the ordinary case, again by $\mathbb{S}[-]$; we denote the composite functor $B^\infty GL_1$ by

$$gl_1 : \text{CAlg} \to \text{Sp}$$

(see Proposition B.17 for the definition of $B^\infty$). Given an $E_\infty$-ring $R$, we will use the term $E_\infty$-group of units of $R$ to refer to $GL_1 R$; we will call $gl_1 R$ the units-group spectrum of $R$. We will sometimes just use the term group of units to refer to either of the two.

**Notation 1.6** Let $X$ be a space, and let $R$ be an $E_\infty$-ring. We will use the notation

$$H^\bullet(X; gl_1 R)$$

to denote the cohomology groups

$$\pi_0 \Map_{\text{Sp}} \left( \Sigma^\infty X, gl_1 R[\bullet] \right).$$

Given any pointed space $X$, we will denote by

$$\tilde{H}^\bullet(X; gl_1 R)$$

the reduced cohomology groups

$$\pi_0 \Map_{\text{Sp}} \left( \Sigma^\infty X, gl_1 R[\bullet] \right).$$

**Remark 1.7** The counit of the adjunction given by Remark 1.4 gives a natural transformation $\iota: GL_1 \Rightarrow \Omega^\infty_m$. It follows from the same remark
that, given any $E_\infty$-ring $R$, the map $\iota_R$ fits into a Cartesian square

$$
\begin{array}{ccc}
\text{GL}_1 R & \xrightarrow{\iota_R} & \Omega_2^\infty R \\
\downarrow & & \downarrow \\
(\pi_0 R)^\times & \longrightarrow & \pi_0 R
\end{array}
$$

in the $\infty$-category $\text{CMon}$ (where both $(\pi_0 R)^\times$ and $\pi_0 R$ are considered as discrete $E_{\infty}$-spaces) and thus in the $\infty$-category $\mathcal{S}$. In particular, as all the path components of a grouplike $H$-space are homotopy equivalent, $\text{gl}_1 R$ is a connective spectrum having as homotopy groups:

$$
\pi_n(\text{gl}_1 R) = \begin{cases}
\pi_0(R)^\times & \text{for } n = 0; \\
\pi_n(R) & \text{for } n \geq 1.
\end{cases}
$$

**Remark 1.8** As a consequence of Remark 1.7, we have that for any $E_\infty$-ring $R$, the following isomorphism of groups holds

$$H^0(X; \text{gl}_1 R) \simeq (R^0 X)^\times.$$

In fact, as the (multiplicative) commutative monoid structure of $R^0 X \simeq \pi_0 \text{Map}_\mathcal{S}(X, \Omega^\infty R)$ is determined by the commutative monoid object structure of $\Omega^\infty R$ in the homotopy category $\text{ho}\mathcal{S}$, the maps $X \to \Omega^\infty R$ factoring through $\text{GL}_1 R$ correspond exactly to the invertible elements in $R^0 X$.

**Remark 1.9** Given any $E_\infty$-monoid $X$, the $E_\infty$-ring $S[X]$ comes with an augmentation map $S[X] \to S$ that is natural and compatible with all the relevant adjunctions (see Remark 1.11 for an explicit description of this map).

More precisely, both functors denoted $S[-]$ factor through the $\infty$-category $\text{CAlg}_/S$ of augmented $E_\infty$-rings. In fact, since $S[0] \simeq S$, we have an induced functor

$$\text{CGrp} \simeq \text{CGrp}_/0 \to \text{CAlg}_/S$$

whose composition with the forgetful functor $\text{CAlg}_/S \to \text{CAlg}$ is precisely $S[-]$. By [Lur09, 1.2.13.8], the forgetful functor $\text{CAlg}_/S \to \text{CAlg}$ preserves colimits; hence, by the Adjoint Functor Theorem it admits a right adjoint. As a consequence, we have that the adjunction $S[-] \dashv \Omega^\infty_{m}$ (resp. $S[-] \dashv \text{GL}_1$) factors as a composite adjunction

$$\text{CMon} \dashv \text{CAlg}_/S \dashv \text{CAlg}$$

(resp. $\text{CGrp} \dashv \text{CAlg}_/S \dashv \text{CAlg}$).

**Remark 1.10** Given any connected pointed space $X$, the terminal morphism $X \to \text{pt}$ admits as a section the basepoint inclusion $\text{pt} \to X$. By applying the free basepoint functor $(-)_+$ introduced in Proposition B.17 to
the basepoint inclusion, we obtain a pointed map $S^0 \to X_+$, which in turn fits into a cofiber sequence of pointed spaces

$$S^0 \to X_+ \to X$$

where the last map is the counit of the free-forgetful adjunction $S \rightleftarrows S_*$. By applying the functor $\Sigma^\infty$ to the above sequence, we get a co/fiber sequence in $\text{Sp}$:

$$\mathbb{S} \to \Sigma^\infty X \to \Sigma^\infty X.$$

Since the first map admits a retraction (the image under $\Sigma^\infty_+$ of $X \to \text{pt}$), the co/fiber splits and

$$\Sigma^\infty X \simeq \Sigma^\infty X \oplus \mathbb{S}.$$

Following [Rez06], we will denote by

$$\gamma_X : \Sigma^\infty X \to \Sigma^\infty_+ X$$

the section (well-defined up to homotopy) of the projection $\Sigma^\infty_+ X \to \Sigma^\infty X$.

**Remark 1.11** The above remark lets us give an explicit description of the augmentation map of a monoid or group $E_\infty$-ring. Let $X$ be an $E_\infty$-space, and let $a : S[X] \to S$ be its monoid $E_\infty$-ring, together with the augmentation map given by Remark 1.9. Looking at the map underlying $a$ in $\text{Sp}$, we have that it is obtained by applying the functor $\Sigma^\infty_+$ to the terminal morphism $X \to \text{pt}$, hence it is given by the morphism

$$\mathbb{S} \oplus \Sigma^\infty X \to \mathbb{S} \oplus 0 \simeq \mathbb{S}$$

acting as the identity on $\mathbb{S}$, and as the zero morphism elsewhere, informally written as $\left(\text{id}_\mathbb{S}, 0\right)$.

**Notation 1.12** Let $A$ and $B$ be spaces, let $X$ and $Y$ be pointed spaces and let $H$ and $K$ be spectra. We will sometimes use the notations

$$[A, B], \quad [X, Y], \quad \text{and} \quad [H, K]$$

to denote

$$\pi_0 \text{Map}_S(A, B), \quad \pi_0 \text{Map}_{S_*}(X, Y) \quad \text{and} \quad \pi_0 \text{Map}_{\text{Sp}}(H, K)$$

respectively.

**Remark 1.13** Let $E$ be a spectrum. The splitting discussed in Remark 1.10 induces the usual direct sum decomposition for the unreduced cohomology of a pointed space, i.e. for every $n \in \mathbb{Z}$, it induces an isomorphism

$$E^n(X) \simeq \left[\Sigma^\infty_+ X[-n], E\right] \simeq \left[\Sigma^\infty X[-n], E\right] \oplus [\mathbb{S}[-n], E] \simeq \tilde{E}^n(X) \oplus \pi_{-n}E.$$
In particular, this, together with the isomorphism $H^0(X; gl_1 R) \simeq \left( R^0 X \right)^\times$ implies that

$$H^0(X; gl_1 R) \simeq \left( \tilde{R}^0 X \oplus \pi_0 R \right)^\times \simeq [\Sigma^\infty X, gl_1 R] \oplus [S, gl_1 R] \simeq \tilde{H}^0(X; gl_1 R) \oplus (\pi_0 R)^\times.$$ 

Thence, the retraction $\Sigma^\infty X \to \Sigma^\infty X$ induces an isomorphism

$$\tilde{H}^0(X; gl_1 R) \simeq \left( \tilde{R}^0 X + 1_{\pi_0 R} \right)^\times \subset \left( \tilde{R}^0 X \oplus \pi_0 R \right)^\times.$$ 

**Definition 1.14** Let $(X, \ast)$ be a pointed space and let $E$ be a spectrum. Given an unpointed map $f : X \to \Omega^\infty E$, we define the *basepoint shift* $\text{Sh}(f)$ of $f$ to be the composite pointed map

$$X \simeq X \times \text{pt} \xrightarrow{\text{id} \times \ast} X \times X \xrightarrow{f \times (\text{inv} \circ f)} \Omega^\infty E \oplus \Omega^\infty E \xrightarrow{\mu} \Omega^\infty E$$

where $\mu$ denotes the multiplication map on $\Omega^\infty E$, and $\text{inv}$ denotes its inversion map (both given equivalently by the fact that $\Omega^\infty E$ is the space underlying the additive $\mathbb{E}_\infty$-group $\Omega^\infty aE^1$, or by Remark B.22). Informally, we have that

$$\text{Sh}(f)(x) = f(x) - f(\ast).$$

**Remark 1.15** The assignment $\text{Sh} : f \mapsto \text{Sh}(f)$ determines a function

$$[X, \Omega^\infty E] \xrightarrow{\text{Sh}} [X, \Omega^\infty E],$$

$$\begin{array}{ccc}
\cong & \cong \\
\tilde{E}^0 X \oplus \pi_0 E & \longrightarrow & \tilde{E}^0 X
\end{array}$$

corresponding to the projection of unreduced cohomology to the reduced cohomology direct summand. That is, the map

$$E^0 X \to \tilde{E}^0 X$$

induced by $\text{Sh}$ corresponds to the map

$$[\Sigma^\infty X, E] \xrightarrow{(\gamma_X)\ast} [\Sigma^\infty X, E]$$

induced by precomposition with the inclusion $\gamma_X$ introduced in Remark 1.10. In other words, given any map $f \in [X, \Omega^\infty E]$, the map $\text{Sh}(f)$ is the adjoint of the composite map $f^\circ \gamma_X \in [\Sigma^\infty X, \tilde{E}]$, $f^\circ$ being the adjoint map of $f$. In particular, $\text{Sh}$ preserves whatever additive or multiplicative structure the spectrum $E$ may induce on the set $[\Sigma^\infty X, E]$.

\footnote{see Notation B.20}
Let \( R \) be an \( \mathbb{E}_\infty \)-ring. The map

\[ \iota_R: \text{GL}_1 R \to \Omega_{m}^\infty R \]

given by Remark 1.7 is of course unpointed. Given any pointed space \( X \), we have an induced morphism

\[ (\iota_R)_*: [X, \text{GL}_1 R]_* \to [X, \Omega^\infty R]_* \]

and, by postcomposition with \( \text{Sh} \), we get a morphism

\[ \text{Sh} \circ (\iota_R)_*: [X, \text{GL}_1 R]_* \to [X, \Omega^\infty R]_* \]

We want to prove that for \( X = S^k \) and \( k \geq 1 \), the above map is an isomorphism of Abelian groups.

**Proposition 1.16** Let \( R \) be an \( \mathbb{E}_\infty \)-ring. Given any \( k \geq 1 \), the map \( \text{Sh} \circ (\iota_R)_* \) induces an isomorphism

\[ [S^k, \text{GL}_1 R]_* \simeq [S^k, \Omega^\infty R]_* \]

**Proof.** \([S^k, \text{GL}_1 R]_*\) consists of homotopy classes of maps \( S^k \to \text{GL}_1 R \) sending \( S^k \) to the path component \( 1_{\pi_0 R} \in (\pi_0 R)^\times \). By postcomposing with \( \iota_R: \text{GL}_1 R \to \Omega^\infty R \) (which is an unpointed map), we get to the subset

\[ (\iota_R)_*[S^k, \text{GL}_1 R]_* \subset [S^k, \Omega^\infty R] \]

of maps sending \( S^k \) to the path component \( 1_{\pi_0 R} \in \pi_0 R \) that are pointed at \( 1_{\pi_0 R} \); thus, compatibly with Remark 1.13, \( \iota_R \) realizes the inclusion of the subgroup

\[ \left( \tilde{R}^0 S^k + 1_{\pi_0 R} \right)^\times \]

in the multiplicative monoid of the ring \( R^0 S^k \). If we now compose with the map \( \text{Sh} \) introduced in Remark 1.15, we get to pointed maps \([S^k, \Omega^\infty R]_*\), or equivalently, to the group \( \pi_k R \simeq \tilde{R}^0 S^k \subset (R^0 S^k, \cdot_{R_0}) \). Our claim is that the map

\[ [S^k, \text{GL}_1 R]_* \xrightarrow{\text{Sh}(\iota_R)_*} [S^k, \Omega^\infty R]_* \]

is a group isomorphism. By construction, \((\iota_R)_*\) is a group homomorphism into the multiplicative structure of its target, whereas by Remark 1.15, \( \text{Sh} \) induces a homomorphism between the multiplicative structures of its source and target; hence it is sufficient to provide an inverse (as a set-map) to \( \text{Sh} \circ (\iota_R)_* \).

As (1) is just a change of connected component, from \( 1_{\pi_0 R} \) to \( 0_{\pi_0 R} \), it is sufficient to realize the inverse change. To this end, let us consider the map

\[ [S^k, \Omega^\infty R]_* \xrightarrow{- + 1} [S^k, \Omega^\infty R] \]

where \( \mathbb{1} \) denotes the map \( S^k \to S^0 \xrightarrow{1_{\pi_0 R}} \Omega^\infty R \), and the sum comes from the cogroup object structure on \( S^k \). Under \(- + \mathbb{1}\), all the maps in \([S^k, \Omega^\infty R]_*\) are shifted to the path component of \( 1_{\pi_0 R} \in \pi_0 R \). By Remark 1.7, each
homotopy class in $[S^k, \Omega^\infty R]_\ast + 1$ determines a (unique, up to homotopy) map $S^k \to \text{GL}_1 R$, sending $S^k$ to the path component of $1_{\pi_0 R} \in \text{GL}_1 \pi_0 R$. In particular, the universal property of the pullback induces a map

$$[S^k, \Omega^\infty R]_\ast \to [S^k, \Omega^\infty R]_\ast + 1 \to [S^k, \text{GL}_1 R]_\ast.$$  \hspace{1cm} (2)

As this composition has the effect of shifting from the connected component of $0_{\pi_0 R}$ to that of $1_{\pi_0 R}$, this is an inverse for (1). \hfill \Box

**Remark 1.17** It follows from Remark 1.13 that $(\widetilde{R}^0 S^k + 1_{\pi_0 R}) \simeq \widetilde{R}^0 S^k$. The proof of Proposition 1.16 shows that, moreover, an isomorphism is given by $x + 1 \mapsto x$.

### 1.2. Square-zero extensions.

We will briefly recall the definitions of (trivial and general) square-zero extensions of $E_\infty$-rings, referring the reader to [Lur17, Chapter 7] for a detailed treatment.

**Remark 1.18** Let $A$ be an $E_\infty$-ring. In [Lur17, 7.3.4.15] a functor

$$A \oplus - : \text{Mod}_A \to \text{CAlg}/A$$

is constructed, sending an $A$-module $M$ to the *trivial square-zero extension of $A$ by $M$*. As explained in [Lur17, 7.3.4.16] the notation is motivated by the fact that it sends each object $M \in \text{Mod}_A$ to a commutative $E_\infty$-ring over $A$ whose underlying spectrum is equivalent to $A \oplus M$ in $\text{Sp}$. The algebra structure on $A \oplus M$ is “square-zero” in the homotopy category of $\text{CAlg}$, as the following facts hold:

1. The unit map $S \to A \oplus M$ is homotopic to the composition of the unit map $S \to A$ with the inclusion $A \to A \oplus M$.
2. The multiplication

$$(A \otimes A) \oplus (A \otimes M) \oplus (M \otimes A) \oplus (M \otimes M) \simeq (A \oplus M) \otimes (A \oplus M) \to A \oplus M$$

is given as follows:

- On $A \otimes A$, it is homotopic to the composition of the multiplication of $A$ with the inclusion $A \to A \oplus M$.
- On $A \otimes M$ and $M \otimes A$, it is given by composition of the action of $A$ on $M$ with the inclusion $M \to A \oplus M$.
- On $M \otimes M$ it is nullhomotopic.

**Remark 1.19** The functor $A \oplus -$ introduced above admits a left adjoint

$$\mathcal{L}_A : \text{CAlg}/A \to \text{Mod}_A$$

whose value on $A$ (with the identity as structure map) is denoted $L_A$ and called the *cotangent complex* of $A$.

In order to describe the behavior of trivial square-zero extensions under restriction of scalars, we need to recall the following result.
Proposition 1.20  [Lur17, 7.3.4.14] Let $A$ be an $\mathbb{E}_\infty$-ring. There is a canonical equivalence of $\infty$-categories

$$\text{Sp}(\text{CAlg}_{/A}) \simeq \text{Mod}_A.$$ 

Proof. It follows from [Lur17, 1.4.2.18] that $\text{Sp}(\text{CAlg}_{/A}) \simeq \text{Sp}((\text{CAlg}_{/A})^A)$. Moreover, by [Lur17, 3.4.1.7], there exists an equivalence $\text{CAlg}_{/A}^A \simeq \text{CAlg}((\text{Mod}_A)_{/A})$. Combining the two, we get:

$$\text{Sp}(\text{CAlg}_{/A}) \simeq \text{Sp}((\text{CAlg}((\text{Mod}_A)_{/A}))_{/A}).$$

The objects in $\text{CAlg}(\text{Mod}_A)_{/A}$ are, by definition, equipped with a structure map with target $A$, in the $\infty$-category $\text{Mod}_A$. Hence, taking fibers of the structure maps gives a functor $\text{CAlg}(\text{Mod}_A)_{/A} \to \text{Mod}_A$, which, being left exact, in turn induces a functor

$$\text{Exc}_*(S^*_\mathbb{E}, \text{CAlg}(\text{Mod}_A)_{/A}) \longrightarrow \text{Exc}_*(S^*_\mathbb{E}, \text{Mod}_A)$$

by pointwise composition (where the bottom right equivalence is due to the fact that $\text{Mod}_A$ is stable). To conclude, [Lur17, 7.3.4.7] shows that this last functor is an equivalence. \hfill \Box

Remark 1.21 Let $f: A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. Then, there is an induced adjunction $\text{CAlg}_{/A} \rightleftarrows \text{CAlg}_{/B}$, where the left adjoint is given by postcomposition, and the right adjoint is given by pullback along $f$. Let us denote by $f^* : \text{CAlg}_{/B} \to \text{CAlg}_{/A}$ the right adjoint functor. The functor $f^*$ restricts to a functor

$$f^* : \text{CAlg}^B_{/B} \to \text{CAlg}^A_{/A}$$

by commutativity of the following diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
R \times_B A & \longrightarrow & R \\
A & \longrightarrow & B.
\end{array}
\]
The functor $F$, in turn, induces by pointwise composition the horizontal functors in the following diagram

$$
\begin{array}{ccc}
\text{Exc}_*(S^\text{fin}_*, \text{CAlg}/_B) & \longrightarrow & \text{Exc}_*(S^\text{fin}_*, \text{CAlg}/_A) \\
\parallel & & \parallel \\
\text{Sp}(\text{CAlg}/_B) & \longrightarrow & \text{Sp}(\text{CAlg}/_A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Sp}(\text{CAlg}(\text{Mod}_B)/_B) & \longrightarrow & \text{Sp}(\text{CAlg}(\text{Mod}_A)/_A) \\
\parallel & & \parallel \\
\text{Exc}_*(S^\text{fin}_*, \text{CAlg}(\text{Mod}_B)/_B) & \longrightarrow & \text{Exc}_*(S^\text{fin}_*, \text{CAlg}(\text{Mod}_A)/_A).
\end{array}
$$

By inspection (recall that in Proposition 1.20 the equivalence $\text{CAlg}(\text{Mod}_A)/_A \cong \text{Mod}_A$ was induced by taking fibers of the structure maps), this functor is equivalent to the restriction of scalars functor $f^! : \text{Mod}_B \to \text{Mod}_A$. In particular, this implies that, given any $B$-module $M$, the $\mathbb{E}_\infty$-rings $(B \oplus M) \times_B A$ and $A \oplus f^! M$ are equivalent, and that the square

$$
\begin{array}{ccc}
A \oplus f^! M & \longrightarrow & B \oplus M \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$

is Cartesian in the $\infty$-category $\text{CAlg}$.

**Definition 1.22** Let $R$ be an $\mathbb{E}_\infty$-ring, and let $M$ be an $R$-module. We define a derivation from $R$ to $M$ to be a map of $R$-modules

$$L_R \to M.$$

As it is clear, a derivation is equivalently determined (up to a contractible space of choices) by its adjoint map $R \to R \oplus M$ (over $R$). If $\eta : L_R \to M$ is a derivation, we will denote its adjoint map by $d_\eta : R \to R \oplus M$.

**Definition 1.23** Let $R$ be an $\mathbb{E}_\infty$-ring, and let $\eta : L_R \to M$ be a derivation from $R$ to an $R$-module $M$. Let $\varphi : R \to R$ be a morphism in $\text{CAlg}$. We will say that $\varphi$ is a square-zero extension if there exists a Cartesian square

$$
\begin{array}{ccc}
\tilde{R} & \longrightarrow & R \\
\downarrow \varphi & & \downarrow d_\eta \\
R & \longrightarrow & R \oplus M
\end{array}
$$

in the $\infty$-category $\text{CAlg}$ (where $d_0$ is adjoint to the zero map $L_R \to M$). In this case, we will also say that $\tilde{R}$ is a square-zero extension of $R$ by $M[-1]$. 
**Remark 1.24** Differently from the classical case, being square-zero is not a property of an extension, but an additional structure. Given a square-zero extension \( f: \tilde{R} \to R \), in general, both the module \( M \) and the derivation \( \eta \) need not be uniquely determined, even up to equivalence. However, this happens in the most common situations (see [Lur17, 7.4.1.26]).

We conclude this section giving a characterization of the values of \( \mathcal{L}_S \) for monoid and group \( \mathbb{E}_\infty \)-rings. This result will constitute a key step for the first proof of Theorem 1.30. A version of Proposition 1.25 is proved in [BM05, 6.1] using rigid models for the categories of spectra. After setting up a few preliminary results, we present a purely \( \infty \)-categorical proof of it.

**Proposition 1.25** The functor

\[ \mathcal{L}_S \circ \mathbb{S}[-]: \text{CMon} \to \text{Sp} \]

is naturally equivalent to \( \mathbb{B}^\infty \circ (-)^{sp} \) (see Appendix B). In particular, for any connective spectrum \( M \), we have a natural equivalence

\[ \mathcal{L}_S(\mathbb{S}[\Omega^\infty_\infty M]) \simeq M. \]

In order to prove Proposition 1.25, we will first introduce the analogue of the symmetric algebra functor, sending a spectrum \( X \) to the free \( \mathbb{E}_\infty \)-ring \( \text{Sym}^* X \). The free-forgetful adjunction \( \text{Sp} \leftrightarrows \text{CAlg} \) thus obtained is compatible with the free-forgetful adjunction \( \mathbb{S} \leftrightarrows \text{CMon} \) given by Proposition B.17, in the sense made precise by Proposition 1.27. Moreover, the free functor \( \text{Sym}^* \) will naturally be augmented, and in its augmented fashion, will constitute a right inverse functor for \( \mathcal{L}_S \).

**Proposition 1.26** There exists a free functor

\[ \text{Sym}^*: \text{Sp} \to \text{CAlg} \]

left adjoint to the forgetful functor of Remark B.9.

*Proof.* This is an immediate consequence of [Lur17, 3.1.3.5]. \( \square \)

**Proposition 1.27** Given a space \( X \), there exists a natural equivalence

\[ \mathbb{S}[\mathcal{F}_+ X] \simeq \text{Sym}^* \Sigma^\infty_+ X \]

where \( \mathcal{F}_+ \) is the functor defined in Proposition B.17.

*Proof.* Let us consider the following diagram

\[
\begin{array}{ccc}
\mathbb{S} & \xrightarrow{\mathcal{F}_+} & \text{CMon} \\
\downarrow & & \downarrow \\
\Sigma^\infty_+ & \xrightarrow{\Omega^\infty} & \mathbb{S}[-] \\
\downarrow & & \downarrow \\
\text{Sp} & \xleftarrow{\text{Sym}^*} & \text{CAlg}.
\end{array}
\]
We can reformulate what we want to prove by saying that the square determined by the left adjoints commute. Since the square determined by the right adjoints clearly commutes, we are done. □

Similarly to what happens for $\mathbb{S}[-]$ (see Remark 1.9), the free-forgetful adjunction $\mathbf{Sp} \rightleftarrows \mathbf{CAlg}$ factors through the $\infty$-category $\mathbf{CAlg}/\mathbb{S}$ as a composite adjunction

$$\mathbf{Sp} \rightleftarrows \mathbf{CAlg}/\mathbb{S} \rightleftarrows \mathbf{CAlg}.$$ 

**Proposition 1.28** [Lur17, 7.3.4.5] There exists an adjunction

$$\text{Sym}_{\text{aug}}^* : \mathbf{Sp} \rightleftarrows \mathbf{CAlg}/\mathbb{S} : \mathcal{I}$$

with the following properties:

1. The functor $\text{Sym}_{\text{aug}}^*$ is given by composition

$$\mathbf{Sp} \simeq \mathbf{Sp}_{/0} \xrightarrow{\text{Sym}^*} \mathbf{CAlg}/\mathbb{S}.$$ 

2. The functor $\mathcal{I}$ is given by taking fibers of the structure maps in $\mathbf{CAlg}/\mathbb{S}$.

3. The composition

$$\mathbf{Sp} \xrightarrow{\text{Sym}_{\text{aug}}^*} \mathbf{CAlg}/\mathbb{S} \xrightarrow{\mathcal{I}} \mathbf{Sp}$$

is equivalent to $X \mapsto \prod_{n>0} \text{Sym}^n(X)$ (see [Lur17, 3.1.3.9]).

We will often abuse notation, and denote $\text{Sym}_{\text{aug}}^*$ just by $\text{Sym}^*$.

**Proposition 1.29** The composition $\mathcal{L}_\mathbb{S} \text{Sym}^*$ is naturally equivalent to the identity functor.

**Proof.** Let $X$ and $Y$ be spectra. We have a chain of equivalences

$$\text{Map}_{\mathbf{Sp}}(\mathcal{L}_\mathbb{S} \text{Sym}^* X, Y) \simeq \text{Map}_{\mathbf{CAlg}/\mathbb{S}}(\text{Sym}^* X, \mathbb{S} \oplus Y)$$

$$\simeq \text{Map}_{\mathbf{Sp}}(X, \mathcal{I}(\mathbb{S} \oplus Y))$$

$$\simeq \text{Map}_{\mathbf{Sp}}(X, Y)$$

from which we deduce that $\mathcal{L}_\mathbb{S} \text{Sym}^*$ and $\text{id}_{\mathbf{Sp}}$ represent the same functor in the $\infty$-category $\mathbf{Sp}$, and are therefore naturally equivalent. □

We are now ready to prove Proposition 1.25.

**Proof of Proposition 1.25.** By [GGN15, 4.9], we have a natural equivalence

$$\text{Fun}^L(\text{CMon, Sp}) \xrightarrow{\sim} \mathbf{Sp}$$
given by evaluation at $\mathcal{F}_+(pt)$, the free $E_\infty$-space generated by one point. By what we have seen so far, we have that

$$L_S(S[\mathcal{F}_+(pt)]) \overset{(1.27)}{\simeq} L_S(\text{Sym}^* \Sigma^\infty_+(pt))$$

$$\overset{(1.29)}{\simeq} \Sigma^\infty_+(pt)$$

$$\overset{\text{(B.17)}}{\simeq} B^\infty_+(\mathcal{F}_+(pt))^\text{gp}$$

hence $L_S \circ S[-]$ and $B^\infty \circ (-)^\text{gp}$ are naturally equivalent. The second part of the statement just follows from the equivalence between $E_\infty$-groups and connective spectra (see Remark B.21) and the fact that $(-)^\text{gp}$ is the identity on $E_\infty$-groups, as it is left adjoint to the inclusion $C\text{Grp} \to C\text{Mon}$. □

1.3. Groups of units of square-zero extensions, the setup. Our goal in this and the following two sections is to give two proofs of the following generalization of Proposition A.3 in our homotopy coherent setting.

**Theorem 1.30** Let $R$ be a connective $E_\infty$-ring, and let $\tilde{R} \to R$ be a square-zero extension by a connective $R$-module $M$. By applying $gl_1$ to $\tilde{R} \to R$, we obtain a map of spectra

$$\varphi: gl_1 \tilde{R} \to gl_1 R.$$ 

The fiber of $\varphi$ is naturally equivalent to $M$ in the $\infty$-category $\text{Sp}$.

**Remark 1.31** As $gl_1$ (introduced in Notation 1.5) lands, by definition, in the image of $B^\infty$ (defined in Proposition B.17), upon its application, all nonconnective information is lost (see Remark B.21). Hence, we can restrict ourselves to work with connective $E_\infty$-rings and connective modules without loss of generality, and suitably replace the relevant objects with their connective covers when dealing with nonconnective ones.

Our strategy for the proof will be the following: in this section, we will first show how to reduce the problem from general square-zero extensions to trivial ones, and then how to reduce it further to trivial square-zero extensions of the sphere spectrum $S$. In the following two sections, we will show that the theorem indeed holds in this last case. In Section 1.4, we will do the last step in an exquisitely “higher algebraic” fashion, by giving a proof entirely at the level of spectra, and in particular recovering the ordinary result as a particular case. In Section 1.5, we will give an alternative proof substantially founded on the space level, extending the homotopy coherent result from the ordinary one.

**Proposition 1.32** Let $R$ be a connective $E_\infty$-ring, and let $M$ be any connective $R$-module. Let

$$gl_1(R \oplus M) \to gl_1 R$$
be the map obtained by applying the functor $gl_1$ to the trivial square-zero extension $R \oplus M \to R$ and let $1 + M$ denote the fiber of $gl_1(R \oplus M) \to gl_1 R$. Then given any square-zero extension $\tilde{R} \to R$ of $R$ by $M$, the fiber of the induced map

$$gl_1 \tilde{R} \to gl_1 R$$

is naturally equivalent to $1 + M$.

**Proof.** First of all, we observe that $gl_1(R \oplus M) \simeq gl_1 R \oplus (1 + M)$. In fact, since the map of $\mathbb{E}_\infty$-rings $R \oplus M \to R$ admits a section, the same is true for its image under the right adjoint functor $gl_1$, and hence the co/fiber sequence $1 + M \to gl_1(R \oplus M) \to gl_1 R$ splits. In particular, $1 + M \simeq \text{coker}(gl_1 R \to gl_1(R \oplus M))$, so that

$$(1 + M)[1] \simeq 1 + (M[1]).$$

By virtue of this canonical identification, we will unambiguously just write $1 + M[1]$.

Let us now suppose that $\tilde{R} \to R$ is a square-zero extension of $R$ by $M$. By definition, $\tilde{R}$ sits in a Cartesian square

$$\begin{array}{ccc}
\tilde{R} & \rightarrow & R \\
\downarrow & & \downarrow \\
R & \rightarrow & R \oplus M[1]
\end{array}$$

in the $\infty$-category $CAlg$. Upon applying the functor $gl_1$, we obtain the co/Cartesian square

$$\begin{array}{ccc}
gl_1 \tilde{R} & \rightarrow & gl_1 R \\
\downarrow & & \downarrow \\
gl_1 R & \rightarrow & gl_1 R \oplus (1 + M[1])
\end{array}$$

in the $\infty$-category $Sp$.

The result now follows from the pasting law of pushouts applied to the following diagram

$$\begin{array}{ccc}
gl_1 \tilde{R} & \rightarrow & gl_1 R & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
gl_1 R & \rightarrow & gl_1 R \oplus (1 + M[1]) & \rightarrow & 1 + M[1].
\end{array}$$

Hence, we are reduced to prove our result in the case of trivial square-zero extensions. Next step will be to reduce to square-zero extensions of $S$. 

$\square$
Lemma 1.33 Let $R$ be an $\mathbb{E}_\infty$-ring, and let $R \oplus M$ be a trivial square-zero extension by an $R$-module $M$. Then, if Theorem 1.30 holds for trivial square-zero extensions of $S$, it holds for $R \oplus M \to R$.

Proof. Let $R$ be an $\mathbb{E}_\infty$-ring, and let $S \to R$ be the unit morphism. Given any $R$-module $M$, this morphism induces by Remark 1.21 a Cartesian square

$\begin{array}{ccc} S \oplus M & \longrightarrow & R \oplus M \\ \downarrow & & \downarrow \\ S & \longrightarrow & R \end{array}$

(3)

in the $\infty$-category $\text{CAlg}$. The functor $gl_1$, being right adjoint, sends the square (3) to a co/Cartesian square in $Sp$:

$\begin{array}{ccc} gl_1(S \oplus M) & \longrightarrow & gl_1(R \oplus M) \\ \downarrow & & \downarrow \\ gl_1 S & \longrightarrow & gl_1 R. \end{array}$

(4)

Let us denote by $P$ the fiber of $gl_1(R \oplus M) \to gl_1 R$. Then, $P$ fits into the diagram:

$\begin{array}{ccc} P & \longrightarrow & gl_1(S \oplus M) \longrightarrow & gl_1(R \oplus M) \\ \downarrow & & \downarrow & \downarrow \\ 0 & \longrightarrow & gl_1 S \longrightarrow & gl_1 R \end{array}$

where the map $P \to gl_1(S \oplus M)$ is induced by the fact that (4) is Cartesian. As the outer square is Cartesian by construction, it follows from the pasting law that $P$ is canonically equivalent to the fiber of $gl_1(S \oplus M) \to gl_1 S$, which by hypothesis is equivalent to $M$. □

Lemma 1.34 Given any spectrum $M$, let $1 + M$ denote the fiber of the map

$gl_1(S \oplus M) \to gl_1 S$

obtained by applying $gl_1$ to the square-zero extension $S \oplus M \to S$. Then, the functor

$M \mapsto 1 + M$

admits a left adjoint.

Proof. For any spectrum $X$, giving a map $X \to 1 + M$ is equivalent to give a map $X \to gl_1(S \oplus M)$ together with a nullhomotopy for its postcomposition with the map $gl_1(S \oplus M) \to gl_1 S$. Upon passing to adjoints we see that, by virtue of the explicit description of the augmentation map of $\mathbb{S}[-]$ given in
Remark 1.11, it is equivalent to give a map

\[ S[\Omega^\infty_a X] \to S \oplus M \]

that is, a morphism in \( \text{Map}_{\text{CAlg}_{/S}}(S[\Omega^\infty_a X], S \oplus M) \). We can rephrase what we just observed more precisely, by saying that the functor \( M \mapsto 1 + M \) fits in the following diagram

\[ \begin{array}{ccc}
\text{CAlg}_{/S} & \xrightarrow{\mathcal{L}_S} & \text{Sp} \\
S[\Omega^\infty_a -] & \xleftarrow{\mathcal{G}} & B^\infty \mathcal{G}(S \oplus -) \end{array} \]

i.e. if we denote by \( \mathcal{G} \) the right adjoint to the factorization of \( S[\Omega^\infty_a -] \) through \( \text{CAlg}_{/S} \) given by Remark 1.9, then \( M \mapsto 1 + M \) is given by the composite right adjoint \( B^\infty \mathcal{G}(S \oplus -) \). Hence, the functor \( M \mapsto 1 + M \) is right adjoint to the functor \( X \mapsto L_S(S[\Omega^\infty_a X]) \).

□

Remark 1.35 We stress that in (5) both \( S[\Omega^\infty_a -] \) and \( B^\infty \mathcal{G} \) are functors from the slice \( \infty \)-category \( \text{CAlg}_{/S} \); in particular, it is important to make a distinction between \( B^\infty \mathcal{G} \) and \( gl_1 \).

1.4. Groups of units of square-zero extensions, top-down. We are now ready to give a first proof.

Proof of Theorem 1.30. By virtue of Proposition 1.32 and Lemma 1.33, all it is left to do, it is to prove the theorem for trivial square-zero extensions

\[ S \oplus M \to S \]

where \( M \) is a connective spectrum. As we want to show that \( 1 + M \) is naturally equivalent to \( M \), it is enough to prove that the functor \( X \mapsto \mathcal{L}_S(S[\Omega^\infty_a X]) \), left adjoint to \( M \mapsto 1 + M \), is naturally equivalent to the identity functor. But, by virtue of Proposition 1.25, we know that this is indeed the case. □

1.5. Groups of units of square-zero extensions, bottom-up. It is possible to give an alternative proof of (the last step of the proof of) Theorem 1.30, based on space level arguments and Proposition A.3. We dedicate this section to give such an alternative proof. The idea is to show the equivalence of the spaces \( \Omega^\infty M \) and \( \Omega^\infty(1 + M) \), and then to show there are no obstructions to lift the comparison map at the level of connective spectra. We begin with the following observation.
**Proposition 1.36** Let $R$ be a connective $\mathbb{E}_\infty$-ring, and let $\tilde{R} \to R$ be a square-zero extension of $R$ by a connective $R$-module $M$. Let us denote by $1 + M$ the fiber of the induced map $gl_1 \tilde{R} \to gl_1 R$. Then, the spaces $\Omega^\infty(1 + M)$ and $\Omega^\infty M$ are equivalent in the $\infty$-category $S$ of spaces.

**Proof.** By applying $\Omega^\infty$ to the co/fiber sequence

$$1 + M \to gl_1 \tilde{R} \to gl_1 R$$

we get a fiber sequence of pointed spaces

$$\Omega^\infty(1 + M) \to GL_1 \tilde{R} \to GL_1 R.$$ 

By the commutativity of the following diagram in the $\infty$-category $S_*$

(\text{where } Sh \text{ denotes the basepoint shift of Definition 1.14, and } \iota_R \text{ denotes the map given in Remark 1.7}) we get a pointed map $\psi: \Omega^\infty(1 + M) \to \Omega^\infty M$. It follows from Proposition 1.16 that we have the following induced morphism between the homotopy fiber exact sequences

$$\cdots \pi_k \tilde{R} \to \pi_k gl_1 R \to \pi_k \tilde{R} \to \pi_k 1 + M \to \pi_{k-1} \tilde{R} \to \pi_{k-1} gl_1 R \cdots$$

$$\downarrow \simeq \downarrow \simeq \downarrow \simeq \downarrow \simeq \downarrow \simeq$$

$$\cdots \pi_k \tilde{R} \to \pi_k R \to \pi_k \tilde{R} \to \pi_k M \to \pi_{k-1} \tilde{R} \to \pi_{k-1} R \cdots$$

which, by the five lemma, induces isomorphisms $\pi_k(1 + M) \simeq \pi_k M$ for $k \geq 1$. To conclude, we need to prove that $\pi_0(\psi): \pi_0(1 + M) \to \pi_0 M$ is an isomorphism. It follows from the exactness of the sequence

$$\cdots \to \pi_1 \tilde{R} \to \pi_1 R \to \pi_0 M \to \pi_0 \tilde{R} \to \pi_0 R \to 0$$
that the map $\pi_1 R \to \pi_0 M$ is actually the zero map. If we consider (6) in the case $k = 1$, we get

$$\cdots \pi_1 R \to \pi_1 R \to \pi_0 (1 + M) \to (\pi_0 \tilde{R})^\times \to (\pi_0 R)^\times \to 0$$

The five-lemma implies that $\pi_0 (1 + M) \to \pi_0 M$ is a monomorphism, which in turn tells us that the map $\pi_1 R \to \pi_0 (1 + M)$ is also the zero map. Hence, $\pi_0 (1 + M)$ is the kernel of

$$(\pi_0 \tilde{R})^\times \to (\pi_0 R)^\times$$

and by Proposition A.3, $\pi_0 (1 + M) \simeq \pi_0 M$, where one isomorphism is given by $u \mapsto u - 1$. Finally, it follows from the definition of SH that the map $\pi_0 (\psi) : \pi_0 (1 + M) \to \pi_0 M$ is exactly given by $u \mapsto u - 1$. □

We can specialize the previous proposition to the case of trivial square-zero extensions of $S$, obtaining the following corollary.

**Corollary 1.37** The functor $\Omega^\infty \circ (1 + -) : Sp^{cn} \to S$ is equivalent to the functor $\Omega^\infty$.

We conclude this section with the promised second proof.

(*Alternative) Proof of Theorem 1.30.* Again, keeping in mind Proposition 1.32 and Lemma 1.33, we just have to show that the functor $1 + -$ described in Lemma 1.34 is equivalent to the identity functor. It follows from [GGN15, 2.10] and the equivalence $Sp^{cn} \simeq CGrp$ that there is an equivalence

$$Fun^\Pi (Sp^{cn}, Sp^{cn}) \simeq Fun^\Pi (Sp^{cn}, S)$$

(where $Fun^\Pi (\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $Fun (\mathcal{C}, \mathcal{D})$ spanned by product preserving functors) given by postcomposition with $\Omega^\infty$. By Lemma 1.34, the functor $1 + -$ preserves all limits (in particular, products), and by Corollary 1.37, its postcomposition with $\Omega^\infty$ is equivalent to $\Omega^\infty$ itself. As this is also true for the identity functor, $1 + -$ and $id_{Sp^{cn}}$ must be equivalent. □

2. **Logarithmic derivatives of $E_\infty$-rings**

Let $R$ be an ordinary commutative ring, and let $M$ be an $R$-module. Given a derivation $\partial : R \to M$, the function

$$\log_\partial : R^\times \to M \quad r \mapsto \partial(r)r^{-1}$$

is easily seen to be a group homomorphism (from the group of units of $R$ to the underlying additive Abelian group of $M$), and it is called the *logarithmic*
derivative relative to $\partial$. Our goal for this section, is to construct a homotopy coherent analogue of logarithmic derivatives for $E_\infty$-rings.

To this end, let now $R$ be a connective $E_\infty$-ring (see Remark 1.31). and let $\partial: L_R \to M$ be a derivation from $R$ to an $R$-module $M$. Let us denote by $\tilde{\partial}: R \to M$ the composite map

$$R \xrightarrow{d_\partial} R \oplus M \xrightarrow{pr_2} M$$

in the $\infty$-category $Sp$ (where $d_\partial$ is the map adjoint to $\partial$, and $pr_2$ is the projection on the module part of the square-zero extension $R \oplus M$). As a first approach, it is possible to mimic in a straightforward way the ordinary definition at the level of spaces, and thus to consider the composition

$$\begin{align*}
\text{GL}_1 R \xrightarrow{\Delta} \text{GL}_1 R \times \text{GL}_1 R & \xrightarrow{(\iota_R \circ \text{inv}) \times (\Omega^\infty \tilde{\partial} \circ \iota_R)} \\
\Omega^\infty R \times \Omega^\infty M & \xrightarrow{a} \Omega^\infty M
\end{align*} (7)$$

in the $\infty$-category of spaces (where $\iota_R$ is as in Remark 1.7, inv is the inversion map given by the $E_\infty$-group structure of $\text{GL}_1 R$, and $a$ is given by the action of $R$ on $M$). Such a map can be shown to be a morphism of group objects in the homotopy category $hoS$ (i.e. a map of H-spaces). Our goal will be to promote such a map to a map of $E_\infty$-groups, or to be more precise, to produce a morphism of connective spectra whose underlying map is homotopic to (7). In order to do so, we will exploit Theorem 1.30, applied to the trivial square-zero extension $R \oplus M$.

**Construction 2.1** Let $R$ be a connective $E_\infty$-ring, and let $\partial: L_R \to M$ be a derivation of $R$ into an $R$-module $M$. Theorem 1.30 implies that there exists a co/fiber sequence

$$M \to \text{gl}_1(R \oplus M) \to \text{gl}_1 R$$

in the $\infty$-category $Sp$, which splits, since $R \oplus M$ is a trivial square-zero extension; that is, we have that

$$\text{gl}_1(R \oplus M) \simeq (\text{gl}_1 R) \oplus M (8)$$

in the $\infty$-category $Sp$. Let us momentarily denote by $i: \text{gl}_1 R \to \text{gl}_1 (R \oplus M)$ the direct summand inclusion. Let now

$$d_\partial: R \to R \oplus M$$

be the map adjoint to $\partial$ (which, by definition, is a section of the trivial square-zero extension $R \oplus M \to R$). We will denote by $^+\log_{d_\partial}: \text{gl}_1 R \to$
$gl_1(R \oplus M)$, the composition

\[
\begin{array}{c}
gl_1 R \\
\downarrow \Delta \\
gl_1 R \oplus gl_1 R \\
\downarrow (i \circ \text{inv}) \oplus gl_1(d_\partial) \\
gl_1(R \oplus M) \oplus gl_1(R \oplus M) \\
\downarrow \mu \\
gl_1(R \oplus M)
\end{array}
\]

(where $\mu$ is the multiplication map induced by Remark B.22).

**Remark 2.2** Let us denote by $p: gl_1(R \oplus M) \to gl_1 R$ the projection to the direct summand given by (8). The composition

\[
\begin{array}{c}
gl_1 R \\
\downarrow + \log_\partial \\
gl_1(R \oplus M) \oplus gl_1(R \oplus M) \\
\downarrow \mu \\
gl_1(R \oplus M)
\end{array}
\]

is nullhomotopic. In fact, by construction, the triangle

\[
\begin{array}{c}
gl_1 R \\
\downarrow \text{inv} \\
gl_1 R \\
\downarrow p
\end{array}
\]

commutes. On the other hand, by definition of $d_\partial$, the map $gl_1(d_\partial)$ fits into the commutative triangle

\[
\begin{array}{c}
gl_1 R \\
\downarrow \text{inv} \\
gl_1 R \\
\downarrow p
\end{array}
\]

Since, as it follows e.g. from Remark B.22, $\mu \circ (p \oplus p) \simeq p \circ \mu$, we have that

\[
p \circ + \log_\partial = p \circ \mu \circ (i \circ \text{inv} \oplus gl_1(d_\partial)) \circ \Delta \\
\simeq \mu \circ (p \oplus p) \circ (i \circ \text{inv} \oplus gl_1(d_\partial)) \circ \Delta \\
\simeq \mu \circ (p \circ i \circ \text{inv}) \oplus (p \circ gl_1(d_\partial)) \circ \Delta \\
\simeq \mu \circ (\text{inv} \oplus \text{id}) \circ \Delta
\]

which, again by Remark B.22, is nullhomotopic.

**Definition 2.3** Let $R$ be a connective $E_\infty$-ring, and $\partial: L_R \to M$ a derivation. We define the logarithmic derivative $\log_\partial$ induced by $\partial$ as the map of spectra

\[
\log_\partial: gl_1 R \to M
\]
induced by the map $+ \log_\partial$ given by Construction 2.1, and the universal property of fibers, applied by virtue of Remark 2.2; i.e. $\log_\partial$ is the essentially unique map rendering the following diagram commutative

\[
\begin{align*}
\text{gl}_1 R & \xrightarrow{+ \log_\partial} \text{gl}_1 (R \oplus M) \rightarrow \text{gl}_1 R. \\
M & \xrightarrow{\log_\partial} \text{gl}_1 R.
\end{align*}
\]

**Remark 2.4** By virtue of Remark B.21, upon applying $\Omega_\infty$ to the logarithmic derivative, we obtain a map of $\mathbb{E}_\infty$-groups

$$\Omega_\infty \log_\partial : \text{GL}_1 R \rightarrow \Omega_\infty M$$

whose underlying map of spaces is homotopic to the map (7) given in the introduction to this chapter.

To see this, let us begin by applying the natural transformation $\iota : \text{GL}_1 \Rightarrow \Omega_\infty$ given in Remark 1.7 to $d_\partial$, in order to get a commutative square

\[
\begin{array}{ccc}
\text{GL}_1 R & \xrightarrow{\iota_R} & \Omega_\infty M \\
\downarrow \text{GL}_1(d_\partial) & & \downarrow \Omega_\infty(d_\partial) \\
\text{GL}_1(R \oplus M) & \xrightarrow{\iota R \oplus \iota_M} & \Omega_\infty(R \oplus M)
\end{array}
\]

in the $\infty$-category of $\mathbb{E}_\infty$-groups. If we consider the above square in the $\infty$-category $S$, combined with (8), we get the following

\[
\begin{array}{ccc}
\text{GL}_1 R & \xrightarrow{\iota_R} & \Omega_\infty R \\
\downarrow \text{GL}_1(d_\partial) & & \downarrow \Omega_\infty(d_\partial) \\
\text{GL}_1 R \times \Omega_\infty M & \xrightarrow{\iota_R \oplus \iota_M} & \Omega_\infty R \times \Omega_\infty M \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
\Omega_\infty M & \xrightarrow{\text{pr}_2} & \Omega_\infty M
\end{array}
\]

(where $\text{pr}_2$ is the obvious projection) showing that

$$\Omega_\infty \tilde{\partial} \circ \iota_R \simeq \text{pr}_2 \circ \Omega_\infty(d_\partial) \circ \iota_r \simeq \text{pr}_2 \circ \text{GL}_1(d_\partial)$$

as map of spaces. Now, by (8), $\Omega_\infty \log_\partial$ is homotopic to

$$\text{GL}_1 R \xrightarrow{\Omega_\infty(+ \log_\partial)} \text{GL}_1(R \oplus M) \xrightarrow{\text{pr}_2} \Omega_\infty M.$$
Unraveling the definitions, we have that, in the ∞-category S:

$$\Omega^\infty \log_{\partial} \simeq \text{pr}_2 \circ \Omega^\infty(\hat{+} \log_{\partial})$$

$$\simeq \text{pr}_2 \circ \Omega^\infty (\mu \circ (i \circ \text{inv}) \oplus gl_1(d_{\partial}) \circ \Delta)$$

$$\simeq \text{pr}_2 \circ \tilde{m} \circ \Omega^\infty (i \circ \text{inv} \oplus gl_1(d_{\partial})) \circ \Delta$$

$$\simeq \text{pr}_2 \circ \tilde{m} \circ \left(\Omega^\infty(i \circ \text{inv}) \times \text{GL}_1(d_{\partial})\right) \circ \Delta$$

(where $\tilde{m}$ denotes the multiplication map on $\text{GL}_1(R \oplus M)$). By Remark 1.18, this is homotopic to

$$m \circ \left((\iota_R \circ \text{inv}) \times (\text{pr}_2 \circ \text{GL}_1(d_{\partial}))\right) \circ \Delta$$

which in turn, by (9), is homotopic to (7).

We conclude this chapter showing that for ordinary rings, regarded as discrete $\mathbb{E}_\infty$-rings, our definition of logarithmic derivatives recovers the usual one, upon passing to connected components.

**Remark 2.5** If $R$ is a discrete $\mathbb{E}_\infty$-ring, [Lur17, 7.4.3.8] shows that

$$\pi_0 L_R \simeq \Omega_{\pi_0 R}$$

in the ordinary category of discrete $\pi_0 R$-modules. As a consequence, any derivation $\partial: L_R \rightarrow M$ determines an ordinary derivation $\pi_0 R \rightarrow \pi_0 M$.

**Proposition 2.6** Let $R$ be a discrete $\mathbb{E}_\infty$-ring, and let $\partial: L_R \rightarrow M$ be a derivation of $R$ into an $R$-module $M$. Then, the morphism

$$\pi_0 \log_{\partial}: (\pi_0 R)^\times \rightarrow \pi_0 M$$

is the ordinary logarithmic derivative associated to $\pi_0 \partial$.

**Proof.** Let us denote by $\tilde{\partial}: \pi_0 R \rightarrow \pi_0 M$ the ordinary derivation determined by $\pi_0 \partial: \Omega_{\pi_0 R} \rightarrow \pi_0 M$. Unraveling the definitions, we see that the value of

$$\pi_0^+ \log_{\partial}: (\pi_0 R)^\times \rightarrow \pi_0(gl_1(R \oplus M)) \simeq (\pi_0 R)^\times \oplus \pi_0 M$$

on any element $r \in (\pi_0 R)^\times$ is

$$\pi_0^+ \log_{\partial}(r) = \pi_0 \left(\mu \circ (i \circ \text{inv} \oplus gl_1(d_{\partial})) \circ \Delta\right)(r)$$

$$= \pi_0 \left(\mu \circ (i \circ \text{inv} \oplus gl_1(d_{\partial}))\right)(r, r)$$

$$= \pi_0(\mu) \left((r^{-1}, 0), (r, \tilde{\partial}(r))\right)$$

$$= \left(r^{-1}, 0\right) \cdot \left(r, \tilde{\partial}(r)\right)$$

$$= \left(1, \tilde{\partial}(r)r^{-1}\right)$$
where the last equality follows from Remark 1.18. Now, it follows from Proposition A.3 that the induced map $\pi_0 \log_\partial : (\pi_0 R)^\times \to \pi_0 M$ is just the projection on the second factor of $\pi_0^+ \log_\partial$, and hence it is given on elements by

$$r \mapsto \tilde{\partial}(r)r^{-1}.$$ 

\[ \square \]

**Appendix A. The classical result**

Our goal in this appendix is to recall and prove Proposition A.3, which is the ordinary version of Theorem 1.30. Throughout by “ring” we mean commutative ring with unit; by “ring (homo)morphism” we mean unit-preserving ring homomorphism.

**Definition A.1** Given a surjective morphism of rings

$$\varphi : \tilde{R} \to R$$

the morphism $\varphi$ is said to be a square-zero extension if $(\ker \varphi)^2 = 0$.

With a little abuse of terminology, we will say that “$\tilde{R}$ is a square-zero extension of $R$ by $I$” if $R$ and $\tilde{R}$ are rings, and $R \cong \tilde{R}/I$ for some ideal $I \subset R$ such that $I^2 = 0$.

**Remark A.2** Let $\varphi : \tilde{R} \to R$ be a square-zero extension and let $I := \ker \varphi$ denote its kernel. Then we have a short exact sequence of $\tilde{R}$-modules

$$0 \longrightarrow I \longrightarrow \tilde{R} \longrightarrow R \longrightarrow 0.$$ 

**Proposition A.3** Given a square-zero extension $\varphi : \tilde{R} \to R$, let $I := \ker \varphi$ denote its kernel. Then there exists an induced short exact sequence of groups

$$0 \longrightarrow I \overset{\iota}{\longrightarrow} \text{GL}_1 \tilde{R} \overset{\tilde{\varphi}}{\longrightarrow} \text{GL}_1 R \longrightarrow 0$$

where

$$\iota(r) = 1 + r;$$

$$\tilde{\varphi}(r) = \varphi(r).$$

**Proof.** First we need to check that everything is well defined. Since by hypothesis we have that $\varphi$ is a square-zero extension, then

$$\iota(r)\iota(-r) = (1 + r)(1 - r) = 1$$

hence $\iota(-r) = \iota(r)^{-1}$; again, by hypothesis

$$\iota(r)\iota(r') = (1 + r)(1 + r') = 1 + r + r' = \iota(r + r'),$$

thus $\iota$ is a group homomorphism. Since $\varphi$ is a ring homomorphism, it preserves units, therefore $\tilde{\varphi}$ is well defined.
The sequence is also exact. Clearly, $\iota$ is injective. To see that $\varphi$ reflects units, let $a, b \in \text{GL}_1 \mathbb{R}$ be such that $ab = 1$, and let $\alpha, \beta \in \tilde{\mathbb{R}}$ be such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. It follows from 

$$\varphi(\alpha \beta - 1) = ab - 1 = 0$$

that $\alpha \beta - 1 \in I$; hence, as $(\alpha \beta - 1)^2 = 0$ we have

$$\alpha \beta (2 - \alpha \beta) = 1$$

proving that both $\alpha$ and $\beta$ are units in $\tilde{\mathbb{R}}$. Now, since

$$\ker \bar{\varphi} = \bar{\varphi}^{-1}(\{1\}) = \varphi^{-1}(\{1\}) = 1 + I = \iota(I)$$

the sequence is exact. □

**Appendix B. Fundamentals of higher commutative algebra**

**B.1. Symmetric monoidal $\infty$-categories.**

**Notation B.1** We denote by $E_\infty$ the $\infty$-category $N(\text{Fin}_*)$, that is, the nerve of the category of finite pointed sets. Given any $n \in \mathbb{N}$, and any $1 \leq i \leq n$, we denote by $\rho^i : \langle n \rangle \to \langle 1 \rangle$ the function sending all elements of $\langle n \rangle$ to (the basepoint) $0$, with the exception of the element $i$.

**Definition B.2** A symmetric monoidal $\infty$-category is the datum of an $\infty$-category $C^\otimes$ together with a coCartesian fibration of simplicial sets $p : C^\otimes \to E_\infty^\otimes$ satisfying the following “Segal condition”.

- For every $n \geq 0$, and every $0 \leq i \leq n$ the functors

  $$\rho^i : C^\otimes_{\langle n \rangle} \to C^\otimes_{\langle 1 \rangle}$$

induced by the functions $\rho^i$ and the coCartesian fibration $p$, determine an equivalence

  $$(\rho^i)_n : C^\otimes_{\langle n \rangle} \xrightarrow{\sim} \prod_{i=1}^n C^\otimes_{\langle 1 \rangle}.$$  

We denote by $C$ the $\infty$-category $C^\otimes_{\langle 1 \rangle}$, and, slightly abusing terminology, we say that $p$ exhibits a symmetric monoidal structure on $C$, and that $C$ itself is a symmetric monoidal $\infty$-category.

We refer the reader to [Lur17, Chapter 2] for a discussion about how this definition gives a homotopy coherent generalization of the ordinary notion of a symmetric monoidal category. In particular, if $C^\otimes \to E_\infty^\otimes$ is a symmetric monoidal structure on $C$, it follows from the definitions that there exists a uniquely (up to canonical isomorphism) determined bifunctor, denoted $\otimes : C \times C \to C$, encoded by the symmetric monoidal structure (see [Lur17, 2.0.0.6, 2.1.2.20]).
Definition B.3 Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. We say that $\mathcal{C}$ is closed if, for each $C \in \mathcal{C}$, the functor
\[
\mathcal{C} \simeq \mathcal{C} \times \Delta^0 \xrightarrow{\text{id} \times C} \mathcal{C} \times \mathcal{C} \overset{\otimes}{\to} \mathcal{C}
\]
(informally given by $D \mapsto D \otimes C$) admits a right adjoint.

Definition B.4 Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal $\infty$-categories with symmetric monoidal structures $p: \mathcal{C} \otimes \to \mathcal{E} \otimes\infty$ and $q: \mathcal{D} \otimes \to \mathcal{E} \otimes\infty$.

1. A lax symmetric monoidal functor is given by a map of $\infty$-operads $F: \mathcal{C} \otimes \to \mathcal{D} \otimes$ (i.e. a morphism of $\infty$-categories over $\mathcal{E} \otimes\infty$, carrying $p$-coCartesian lifts of inert morphisms of $\mathcal{E} \otimes\infty$ to $q$-coCartesian morphisms in $\mathcal{D} \otimes$).

2. A symmetric monoidal functor is given by a morphism of $\infty$-categories over $\mathcal{E} \otimes\infty$ carrying $p$-coCartesian morphisms to $q$-coCartesian morphisms.

We sometimes abuse notation, and refer to (lax) symmetric monoidal functors indicating only the underlying functor between the underlying $\infty$-categories.

Example B.5 We have the following two notable examples (see also Definition B.19):

1. [Lur17, Section 2.4.1] Given any $\infty$-category $\mathcal{C}$ with finite products, it has a symmetric monoidal structure, denoted
\[
\mathcal{C}^\times \to \mathcal{E} \otimes\infty,
\]
encoding its Cartesian product.

2. [Lur17, 4.8.2.14] The $\infty$-category $\mathcal{S}_*$ of pointed spaces has a symmetric monoidal structure, denoted
\[
\mathcal{S}_*^\wedge \to \mathcal{E} \otimes\infty,
\]
encoding the smash product of pointed spaces.

Definition B.6 Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. We let $\mathcal{C}\text{Alg}(\mathcal{C})$ denote the full subcategory of sections $\mathcal{E} \otimes\infty \to \mathcal{C} \otimes$ of the structure map $\mathcal{C} \otimes \to \mathcal{E} \otimes\infty$ spanned by lax symmetric monoidal functors. We refer to $\mathcal{C}\text{Alg}(\mathcal{C})$ as the $\infty$-category of commutative algebra objects of $\mathcal{C}$.

Definition B.7 Let $\mathcal{C}$ be an $\infty$-category with finite products. A commutative monoid object of $\mathcal{C}$ is given by a functor $M: \mathcal{E} \otimes\infty \to \mathcal{C}$ such that the morphisms $M((n)) \to M((1))$ induced by the inert morphisms $(n) \to (1)$ exhibit $M((n))$ as an $n$-fold product of $M((1))$. We let $\mathcal{C}\text{Mon}(\mathcal{C})$ denote the full subcategory of $\text{Fun}(\mathcal{E} \otimes\infty, \mathcal{C})$ spanned by the commutative monoids of $\mathcal{C}$.
As it is clear, in an $\infty$-category with finite products, it is possible to define both commutative algebra objects with respect to the Cartesian symmetric monoidal structure and commutative monoid objects; in fact, the two definitions agree.

**Proposition B.8** [Lur17, 2.4.2.5] Let $\mathcal{C}$ be an $\infty$-category with finite products, considered as a symmetric monoidal $\infty$-category with the Cartesian structure of Example B.5.1. Then, there is an equivalence of $\infty$-categories

$$\text{CAlg}(\mathcal{C}) \simeq \text{CMon}(\mathcal{C}).$$

**Remark B.9** A commutative algebra object $A : \mathcal{C} \to \mathcal{C}^\otimes$ of a symmetric monoidal $\infty$-category $\mathcal{C}$ determines an object $A(\langle 1 \rangle)$ of $\mathcal{C}$ together with a homotopy coherent analogue of the structure of a commutative algebra object of an ordinary symmetric monoidal category. In fact, the assignment $A \mapsto A(\langle 1 \rangle)$ extends to a forgetful functor

$$\text{CAlg}(\mathcal{C}) \to \mathcal{C}.$$

With a little abuse of notation, we often denote $A(\langle 1 \rangle)$ just by $A$, and refer to it as a commutative algebra object (or a commutative monoid object, if the symmetric monoidal structure on $\mathcal{C}$ is Cartesian).

**Remark B.10** It follows from the definitions, that, if $M$ is a commutative monoid in an $\infty$-category $\mathcal{C}$, then the underlying object $M$ in the homotopy category $\text{ho} \mathcal{C}$ is a commutative monoid object (in the ordinary sense).

**Proposition B.11** [GGN15, 1.1] Let $\mathcal{C}$ be an $\infty$-category with finite products, and let $M$ be a commutative monoid object of $\mathcal{C}$. Then the following conditions are equivalent:

1. $M$ admits an inversion map for the multiplication induced by the commutative algebra object structure.
2. The commutative monoid object of $\text{ho} \mathcal{C}$ underlying $M$ is a group object.

**Definition B.12** Let $\mathcal{C}$ be an $\infty$-category with finite products; we say that a commutative monoid object $M \in \text{CMon}(\mathcal{C})$ is a **commutative group object** if it satisfies the equivalent conditions of Proposition B.11. We write $\text{CGrp}(\mathcal{C})$ to denote the full subcategory of $\text{CMon}(\mathcal{C})$ consisting of commutative group objects.

We refer the reader to [GGN15] for other equivalent characterizations of commutative group objects in an $\infty$-category, and for a detailed treatment of some of its properties.

**Remark B.13** As we observed in the introduction, the $\infty$-category of spaces plays in the homotopy coherent world the same role that the ordinary category of sets plays in the ordinary case. Accordingly, we denote the $\infty$-categories $\text{CMon}(\mathcal{S})$ and $\text{CGrp}(\mathcal{S})$ just by $\text{CMon}$ and $\text{CGrp}$, respectively.
We sometimes refer to $\text{CMon}$ as the $\infty$-category of $\mathbb{E}_\infty$-spaces or as the $\infty$-category of $\mathbb{E}_\infty$-monoids, and we refer to $\text{CGrp}$ as the $\infty$-category of $\mathbb{E}_\infty$-groups.

Similarly, as the $\infty$-category of spectra is the homotopy coherent analogue of the category of Abelian groups, we denote the $\infty$-category $\text{CAlg}(\text{Sp})$ just by $\text{CAlg}$, and refer to it as the $\infty$-category of $\mathbb{E}_\infty$-rings.

The difference between the $\infty$-category of $\mathbb{E}_\infty$-groups and the $\infty$-category of spectra is something that has no counterpart in the ordinary case. As recalled below (see Remark B.21), the $\infty$-category of $\mathbb{E}_\infty$-groups is equivalent to the full subcategory $\text{Sp}^{cn} \subset \text{Sp}$ of connective spectra, which is not stable, and the smallest stable presentable $\infty$-category containing it is precisely $\text{Sp}$. In a sense, looking for an homotopy coherent analogue of the Abelian category of Abelian groups, one can start with the more “naive” $\infty$-category of $\mathbb{E}_\infty$-groups (the straightforward generalization of the category of Abelian groups), and then the price to pay to have stability (the generalization of the notion of being Abelian we want to work with) is to add nonconnective spectra to the picture, which (as far as the analogy with the ordinary case goes) can be thought of as a sort of technical nuisance.

**Proposition B.14** Let $p: \mathcal{C}^\otimes \to \mathbb{E}_\infty^\otimes$ and $q: \mathcal{D}^\otimes \to \mathbb{E}_\infty^\otimes$ be symmetric monoidal $\infty$-categories, and let $F: \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a symmetric monoidal functor, such that the underlying functor $F_!(1): \mathcal{C} \to \mathcal{D}$ admits a right adjoint. Then $F$ admits a right adjoint $G: \mathcal{D}^\otimes \to \mathcal{C}^\otimes$, which is lax symmetric monoidal.

*Proof.* This follows immediately from [Lur17, 7.3.2.1].

Adjunctions of the kind of Proposition B.14 constitute a homotopy coherent analogue of ordinary symmetric monoidal adjunctions, hence we will adopt the same terminology in this context.

**Definition B.15** Let $p: \mathcal{C}^\otimes \to \mathbb{E}_\infty^\otimes$ and $q: \mathcal{D}^\otimes \to \mathbb{E}_\infty^\otimes$ be symmetric monoidal $\infty$-categories. We say that an adjunction

$$F : \mathcal{C}^\otimes \dashv \mathcal{D}^\otimes : G$$

is a *symmetric monoidal adjunction* if the left adjoint is symmetric monoidal and the right adjoint is lax symmetric monoidal.

We now pose our attention to the case of presentable $\infty$-categories.

**Proposition B.16** [GGN15, 4.1, 4.4] Given a presentable $\infty$-category $\mathcal{C}$, the $\infty$-categories $\text{CMon}(\mathcal{C})$ and $\text{CGrp}(\mathcal{C})$ are presentable; moreover, there are functors

$$\mathcal{C} \to \text{CMon}(\mathcal{C}) \to \text{CGrp}(\mathcal{C})$$

which are left adjoint to the respective forgetful functors.
Recall that, given a presentable ∞-category \( \mathcal{C} \), we can define its stabilization \( \text{Exc}_*(\mathcal{S}_{\text{fin}}^\infty, \mathcal{C}) \) (which is again presentable), denoted \( \text{Sp}(\mathcal{C}) \), related to \( \mathcal{C} \) by an adjunction

\[
\Sigma_+^\infty : \mathcal{C} \rightleftarrows \text{Sp}(\mathcal{C}) : \Omega^\infty
\]

where the left adjoint is called the suspension spectrum functor (see [Lur17, Section 1.4.2] for details).

**Proposition B.17** [GGN15, 4.10] Let \( \mathcal{C} \) be a presentable ∞-category. The suspension spectrum functor \( \Sigma_+^\infty : \mathcal{C} \to \text{Sp}(\mathcal{C}) \) factors as a composition of left adjoints

\[
\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_* \xrightarrow{F} \text{CMon}(\mathcal{C}) \xrightarrow{(-)_{\text{gp}}} \text{CGrp}(\mathcal{C}) \xrightarrow{B^\infty} \text{Sp}(\mathcal{C})
\]

each of which is uniquely determined by the fact that it commutes with the corresponding free functor from \( \mathcal{C} \).

When \( \mathcal{C} \) is a presentable symmetric monoidal ∞-category, the above chain of adjunctions can be enhanced to a chain of symmetric monoidal adjunctions.

**Proposition B.18** [GGN15, 5.1] Let \( \mathcal{C} \) be a presentable closed symmetric monoidal ∞-category. The ∞-categories \( \mathcal{C}_* \), \( \text{CMon}(\mathcal{C}) \), \( \text{CGrp}(\mathcal{C}) \) and \( \text{Sp}(\mathcal{C}) \) all admit closed symmetric monoidal structures, which are uniquely determined by the requirement that the respective free functors from \( \mathcal{C} \) are symmetric monoidal. Moreover, each of the functors

\[
\mathcal{C}_* \to \text{CMon}(\mathcal{C}) \to \text{CGrp}(\mathcal{C}) \to \text{Sp}(\mathcal{C})
\]

uniquely extends to a symmetric monoidal left adjoint.

**Definition B.19** The stabilization \( \text{Sp}(\mathcal{S}) \) of the ∞-category of spaces is the ∞-category of spectra \( \text{Sp} \). The product encoded by the symmetric monoidal structure

\[
\text{Sp} \otimes \to \mathbb{E}_\infty^\otimes
\]

induced on \( \text{Sp} \) by Proposition B.18 is commonly referred to as the smash product of spectra.

**Notation B.20** In most cases, we omit notations for the forgetful functors. In the special case \( \mathcal{C} = \mathcal{S} \), we adopt the following notations and terminology:

1. We denote the right adjoint to \( B^\infty \) as \( \Omega^\infty_{\text{a}} \) and refer to it as the underlying additive \( \mathbb{E}_\infty \)-space functor.
2. We denote the composite left adjoint \( B^\infty \circ (-)_{\text{gp}} \circ F \) by

\[
\Sigma^\infty : \mathcal{S}_* \to \text{Sp}
\]

and denote its right adjoint, with the usual abuse of notation, by \( \Omega^\infty \).
(3) We denote the composite left adjoint \( F \circ (-)_+ \) by
\[
F_+: \mathcal{S} \to \text{CMon}.
\]

**Remark B.21** \([\text{Lur}17, 5.2.6.26]\) The \( \infty \)-category \( \text{CGrp} \) of \( \mathbb{E}_\infty \)-groups and the \( \infty \)-category \( \text{Sp}^{cn} \) of connective spectra are equivalent. The equivalence is given by the functor \( B^\infty \) defined above, whose essential image is exactly \( \text{Sp}^{cn} \). In particular, the composite functor
\[
\text{Sp} \xrightarrow{\Omega^\infty} \text{CGrp} \xrightarrow{\sim} \text{Sp}^{cn}
\]
is equivalent to the truncation functor \( \tau_{\geq 0} \), right adjoint to the inclusion \( \text{Sp}^{cn} \to \text{Sp} \).

**Remark B.22** As \( \text{Sp} \) is a stable \( \infty \)-category, \([\text{Lur}17, 1.1.3.5]\) implies that it is also an additive \( \infty \)-category (see \([\text{GGN}15, \text{Section 2}]\) for the definition and a detailed discussion of this property). Therefore, by \([\text{GGN}15, 2.8]\), every spectrum admits a commutative group structure, and all maps between spectra respect this structure; to be more precise, the forgetful map \( \text{CGrp}(\text{Sp}) \to \text{Sp} \) is an equivalence. In particular, given any spectrum \( X \), there exists a map \( \text{inv}: X \to X \) such that the composition
\[
X \xrightarrow{\Delta} X \oplus X \xrightarrow{\text{id} \oplus \text{inv}} X \oplus X \xrightarrow{\mu} X
\]
(where \( \Delta \) is the diagonal map, and \( \mu \) is the multiplication induced by the commutative algebra structure on \( X \)) is nullhomotopic.

The above remark is consistent with the fact that we think of \( \text{Sp} \) as the homotopy coherent analogue of the Abelian category of Abelian groups, and can be interpreted as saying that all spectra admit a “homotopy coherent commutative group structure”.

### B.2. Modules over commutative algebra objects

Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category, and let \( R \in \text{CAlg}(\mathcal{C}) \) be a commutative algebra object in \( \mathcal{C} \). Analogously to what happens in the ordinary case, it is possible to define an \( \infty \)-category \( \text{Mod}_R(\mathcal{C}) \) of \( R \)-modules which, under mild assumptions, is again symmetric monoidal. In \([\text{Lur}17, 3.3, 4.5]\) a few equivalent models for this \( \infty \)-category are constructed, together with a detailed discussion of their equivalence. For future reference, we recollect in this section some results about these \( \infty \)-categories, whose formulation (and validity) are independent of the specific model, referring the reader to \([\text{Lur}17]\) for details and proofs.

**Notation B.23** Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category, and let \( R \in \text{CAlg}(\mathcal{C}) \) be a commutative algebra object in \( \mathcal{C} \). We denote by \( \text{Mod}_R(\mathcal{C}) \) the \( \infty \)-category of \( R \)-modules. If \( \mathcal{C} = \text{Sp} \), we denote \( \text{Mod}_R(\mathcal{C}) \) just by \( \text{Mod}_R \).
**Proposition B.24** [Lur17, 4.5.2.1] Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category. Assume that \( \mathcal{C} \) admits geometric realizations of simplicial objects, and that the tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) preserves geometric realizations of simplicial objects separately in each variable. Let \( R \in \text{CAlg}(\mathcal{C}) \) be a commutative algebra object in \( \mathcal{C} \). The \( \infty \)-category \( \text{Mod}_R(\mathcal{C}) \) admits a symmetric monoidal structure

\[
\text{Mod}_R(\mathcal{C}) \otimes_R \to \mathbb{P}_\infty
\]

whose symmetric monoidal product is also called the relative tensor product.

**Proposition B.25** Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category as in Proposition B.24. Let \( A \) and \( B \) be commutative algebra objects in \( \mathcal{C} \), and let \( f : A \to B \) be a morphism in \( \text{CAlg}(\mathcal{C}) \). Then, there exists a symmetric monoidal adjunction

\[
\text{Mod}_A(\mathcal{C}) \rightleftarrows \text{Mod}_B(\mathcal{C}) : f^!
\]

with the left adjoint given by the relative tensor product \( M \mapsto M \otimes_A B \). We refer to such left adjoint as the extension of scalars functor, and to the right adjoint as the restriction of scalars functor.

**Proof.** The existence of the adjunction on the underlying \( \infty \)-categories follows from [Lur17, 4.6.2.17], which also implies that the left adjoint is given by the relative tensor product. By [Lur17, 4.5.3.2], the left adjoint is symmetric monoidal. Finally, it follows from [Lur17, 7.3.2.7] that the right adjoint is lax monoidal. \( \square \)

We sometimes omit notation for the restriction of scalars functor.

**Remark B.26** It follows from [Lur17, 4.8.2.19] that the smash product

\[
\otimes : \text{Sp} \times \text{Sp} \to \text{Sp}
\]

preserves small colimits separately in each variable; in particular, \( \text{Sp} \) satisfies the hypotheses of Proposition B.24.

In ordinary commutative algebra, the category of modules over a ring is Abelian. Analogously, the \( \infty \)-category of modules over an \( \mathbb{E}_\infty \)-ring is stable.

**Proposition B.27** [Lur17, 7.1.1.5] Let \( R \) be an \( \mathbb{E}_\infty \)-ring. Then, the \( \infty \)-category \( \text{Mod}_R \) of \( R \)-modules is stable.

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