Superconducting to normal state phase boundary in arrays of ultrasmall Josephson junctions

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Abstract

We study the competition between Josephson and charging energies in two-dimensional arrays of ultrasmall Josephson junctions, when the mutual capacitance is dominant over the self-capacitance. Our calculations involve a combination of an analytic WKB renormalization group approach plus non-perturbative Quantum Monte Carlo simulations. We consider the zero frustration case in detail and we are able to make a successful comparison between our results and those obtained experimentally.

I. INTRODUCTION

Recent advances in submicrometer technology have made it possible to fabricate relatively large arrays of ultrasmall SIS (superconductor to insulator to superconductor) Josephson junctions [1–6]. These arrays consist of Josephson junctions made of superconducting islands separated by an insulating barrier [7]. The areas of these junctions can vary from a few microns to submicron sizes. In the latter case the effective capacitance of the junctions can be smaller than femtoFarads ($\text{fF}=10 \times 10^{-15}$ Farads). Under these circumstances the long range phase coherent properties of the Josephson junction arrays (JJA) depend crucially on the interplay between the Josephson ($E_J$) and charging energies ($E_C$). In this paper we will be concerned with studying this interplay for the specific parameter ranges of the experiments carried out in Delft [4]. The charging energy associated with adding a single charge to an island is $E_{Cs} = \frac{e^2}{2C_s}$, whereas the corresponding energy necessary to transfer a charge from an island to a nearby one is $E_{Cm} = \frac{e^2}{2C_m}$. Here $e$ is the electronic charge. For the Delft samples, the self capacitances are typically $C_s \sim 3 \times 10^{-18}\text{F}$, while the mutual capacitances are on the order of $C_m \sim 1 \times 10^{-15}\text{F}$. This means that $C_m$ is three orders of magnitude larger than $C_s$. Prior to the advent of these types of JJA, most theoretical studies

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assumed that the dominant term was the self-capacitive contribution, since the calculations are easier and also no significant changes in the results were expected from having the extra $C_m$ contribution. In the Delft experiments a phase diagram was obtained as a function of temperature vs the quantum parameter

$$\alpha \equiv \frac{E_C}{E_J} \quad (1)$$

The phase diagram that includes the experimental results plus our theoretical results is shown in Fig. 1. When $\alpha = 0$, the array is modeled by the two-dimensional XY model and it has been found to be a faithful representation [6] of the Berezinskii-Kosterlitz-Thouless (BKT) scenario [8,9]. As $\alpha$ increases (in the Delft experiments the values of $\alpha$ cover the range 0.13 to 4.55, while in the Harvard samples the $\alpha$ can be as large as 33), the critical temperature decreases monotonically. As $\alpha$ increases further, at fixed temperature, one is expected to change from a superconducting phase characterized by quasi-long range phase coherence (SC) to a charge dominated insulating phase (I). For small and fixed $\alpha$, as temperature increases, we move from a SC phase to a resistive or conducting one. For larger values of $\alpha$ the transition is expected to be between an insulating and a conducting phase. In the insulator normal phase boundary, the idea is that in the case when the $C_s = 0$, a mutual capacitance model would map to a two-dimensional Coulomb gas model and thus a charge unbinding BKT transition would ensue [10,12]. This scenario has not been found experimentally since in practice, although $C_s$ can be three orders of magnitude smaller than $C_m$, the electrostatic screening length is much smaller than the lattice size and the possible BKT transition is masked. This means that experimentally the boundary I to N is strictly speaking not a thermodynamic phase boundary but a cross over region [5,4]. The situation is different in the SC to N boundary since that is a true thermodynamic phase boundary. An important question is, however, if the models that have been proposed to describe the JJA are faithful representations of the experimental systems studied. This question is non trivial since there is evidence for having extraneous elements in the arrays, e.g. stray charges, that have not been accounted for properly in the models so far considered. It is for this reason that we have thought it important to carry out detailed quantitative calculations of the SC to N phase boundary and to make detailed comparisons with experiments. We can already note that our results shown in Fig. 1, coming from a WKB renormalization group calculation, fit the experimental results quite well in the small $\alpha$ range. On the other hand, a rather good fit to the experimental results comes from carrying out a nonperturbative quantum Monte Carlo calculation. In this paper we describe mainly the WKB-RG calculation, and we will briefly mention the QMC results, which will be discussed in more detail elsewhere [13,14].

An important result emerging from the WKB-RG calculation is the possible existence of a low temperature instability. This QUIT (QUantum fluctuation Induced Transition) was originally found in a self-capacitive model [15,16]. Here we show that the results are also true in the case when $C_m$ dominates. This extension is non trivial since the self-capacitive model has no I to N phase boundary with a BKT type charge unbinding transition, even theoretically. One can then wonder if the competition between the insulating and superconducting phases at sufficiently low temperatures may quench the QUIT. Within the context of the WKB-RG calculation this is, in fact, not the case as we discuss in this paper. From
the WKB-RG analysis the QUIT instability may be conjectured to be a phase boundary from a SC to a normal phase [15].

There was early support for the existence of the QUIT in the $C_m = 0, C_s \neq 0$ case from nonperturbative quantum Monte Carlo simulations [16]. In the $f = 0$ case it was found that the helicity modulus had a discontinuity between two states with finite superfluid densities each. This lead to the surmise that the QUIT is in fact a transition between two superfluid states, one dominated by thermal and the other by quantum fluctuations. Later studies considered the case of $f = 1/2$, the fully frustrated limit, which gave a larger discontinuity in the superfluid density and at higher $T_{QUIT}(f = 1/2)$ temperature [11]. As we shall see from the WKB-RG calculation described below, the $T_{QUIT}(f = 0) \sim T_{QUIT}(f = 1/2)$. However, what was found in the QMC calculations was that $T_{QUIT}(f = 1/2) \sim 10 \times T_{QUIT}(f = 0)$. In a recent calculation which uses an improved QMC algorithm, that reduces the slow approach to equilibrium [17], the large hysteresis loop seen before in the $f = 1/2$ case was suggested to be directly related to the long lived metastable states seen in the classical fully frustrated XY model. The authors, however, did not carry out the calculations to sufficiently low temperature to test the $f = 0$, or $f = 1/2$ QUIT. The question then remains as to the physical nature of the QUIT, if it does exist. From the WKB-RG analysis there is clearly a low temperature QUIT instability. The known QMC results for the $f = 0, C_m = 0$ case suggest a QUIT between two different superconducting states. Further studies are needed to ascertain the nature of the putative QUIT. As we also mention later in this paper there is an experimental low temperature instability that could very well be a manifestation of the QUIT.

Here we shall report on our main WKB-RG and QMC results and sketch how they were obtained while a fuller more detailed description will appear elsewhere [13,14].

The outline of the rest of this paper is the following: In Sec. II we define the model studied here and discuss the path integral representation of the partition function used in our analysis. In Section III we outline our WKB-RG analysis in the case where the Josephson energy dominates. We obtain generalized RG equations from which we obtain the phase diagram, which is then compared with the experimental results [4], as well as with the Monte Carlo simulation results [13]. As we mentioned above, if $\xi_E$ is large in the insulating phase the charges interact with a logarithmic potential and a BKT type transition must follow [10,12]. In section IV we address the question as to what happens when we perturb the Coulomb gas model ($C_s = 0$) with the Josephson term in the hamiltonian; Is there a QUIT in the charge dominated phase? We have carried out a WKB-RG perturbative analysis in the insulating region and found that there is no analog to the QUIT transition found in the SC phase, at least to the lowest order in the perturbative analysis.

II. THE MODEL

A Josephson junction array can be modelled by a periodic lattice of superconducting islands separated by insulating barriers. Each island is characterized by a Ginzburg-Landau order parameter $\Psi(\vec{r}_i) = |\Psi_0(\vec{r}_i)|e^{i\varphi(\vec{r}_i)}$, where $\vec{r}_i$ is a two-dimensional vector denoting the position of each island. Each one of the islands becomes independently superconducting about the bulk transition temperature $T_c0$. When the temperature is lowered further, the magnitude of the order parameter, $|\Psi_0(\vec{r}_i)|$, is nonfluctuating and the onset of long range
phase coherence is responsible for the zero resistance in the arrays. The onset of phase coherence is due to the tunneling of Josephson currents between the islands and it is characterized by a sharp drop to the zero resistance state. The onset temperature in the arrays can be significantly modified by making the junctions ultrasmall. When the junction’s capacitance is small the charging energy, i.e. the energy necessary to transfer a Cooper pair between the islands, can be large to the point where no Cooper pairs can tunnel any more, and thus the Josephson current can be reduced to the point where it is completely quenched. The competition between the Josephson tunneling and the charging energy, in zero field, can be modeled by the Hamiltonian,

$$\hat{H} = H_C + H_J = \frac{q^2}{2} \sum_{\vec{r}_1, \vec{r}_2} \hat{n}(\vec{r}_1) C^{-1}(\vec{r}_1, \vec{r}_2) \hat{n}(\vec{r}_2) + E_J \sum_{<\vec{r}_1, \vec{r}_2>} [1 - \cos(\hat{\phi}(\vec{r}_1) - \hat{\phi}(\vec{r}_2))]$$  \hspace{1cm} (2)

where $q = 2e$. Here $\hat{\phi}(\vec{r}_i)$ is the quantum phase operator while $\hat{n}(\vec{r}_i)$ is its canonically conjugate number operator, which measures the excess number of Cooper pairs in the island placed at $\vec{r}_i$. These canonically conjugate operators satisfy the commutation relations

$$[\hat{n}(\vec{r}_1), \hat{\phi}(\vec{r}_2)] = -i \delta_{\vec{r}_1, \vec{r}_2},$$  \hspace{1cm} (3)

which imply that in the phase representation we can write $\hat{n}(\vec{r}_i) = -i \frac{\partial}{\partial \phi(\vec{r}_i)}$. A Josephson junction is essentially a capacitor with an insulating barrier. The geometric capacitance matrix $C(\vec{r}_1, \vec{r}_2)$ is assumed to include only the island capacitance with respect to the ground plane, $C_s$, and the mutual capacitance between nearest neighbor islands, $C_m$. In the square array case, of interest here, we can write the capacitance matrix as

$$C(\vec{r}_1, \vec{r}_2) = (C_s + 4C_m) \delta_{\vec{r}_1, \vec{r}_2} + C_m \sum_{\vec{d}} \delta_{\vec{r}_1, \vec{r}_2 + \vec{d}},$$  \hspace{1cm} (4)

where the sum in the second term is over the nearest neighbor vectors $\vec{d}$. The electrostatic potential associated with this capacitance matrix model leads to the screening length $\xi_E = \sqrt{\frac{C_m}{C_s}}$. This means that when $C_m = 0$ the charges are maximally screened. In the arrays fabricated so far the screening length, as calculated from Eq. (4) is between 15 and 20 plaquettes [4,5]. This means that screening effects in these arrays are important and that including $C_m$ in the analysis is essential. As mentioned in the introduction, however, prior to the fabrication of the arrays with ultrasmall capacitances, many theoretical studies concentrated either on mean field approaches [22,23], which neglect the special nature of the BKT ordering in two-dimensions, or in 2-D studies with $C_m = 0$ [18,19]. Initial MC studies of the off diagonal problem have also appeared [15]. Since the samples fabricated have a typical ratio $C_m \sim C_s 10^3$, it is clear that if one wants to make a direct quantitative comparison to experimental results we need to extend the analysis to include $C_m$ explicitly. This is the purpose of this paper.

We should stress that the model defined by Eqs.(2-4) is not the most general model one could consider, for it assumes that all the electric fields are confined to the two-dimensional plane of the array. This may not be quite the case for real samples but one would expect that in the regime where $\xi_E$ is sufficiently large this model should be appropriate. As we shall discuss below the fact that one is able to make successful comparisons between experiment
and theory in the superconducting to normal regime leads us to believe that in that regime at least the model gives a correct representation of the experimental system.

There are other possibly important elements missing in the model studied in this paper. To wit: (i) We do not consider here the influence of a constant external magnetic field. The physics in that case, even in the classical limit, is highly nontrivial and requires a special treatment [20]. (ii) We do not include the self-induced magnetic fields, that can also lead to interesting new physics [19]. The reason is that the critical currents in the arrays are found to be too low to lead to any significant magnetic field screening effects. (iii) We do not include quasi-particle dissipation. Here again it has been found experimentally that in the SC to N phase these effects are not important [11]. A low temperature study that includes quasi-particle dissipation for the diagonal capacitance matrix model has shown that dissipation counteracts the charging effects so as to strengthen the Josephson tunneling [21]. (iv) We could also consider the effect of disorder, in particular the one due to random stray charges in the array. These random frustration charges may also be more important in the insulating phase than in the superconducting one, which is the one we concentrate on in this paper.

Here we are interested in calculating the thermodynamic properties of the model defined by $\hat{H}$. The corresponding partition function is defined by

$$Z = \text{Tr} \left\{ e^{-\beta \hat{H}} \right\},$$

(5)

The trace is taken over $\hat{\phi}$ or $\hat{n}$. It is convenient, for calculational purposes, to change from the operator expression to the imaginary-time Feynmann path integral representation of $Z$. We accomplish this by using

$$< n(\vec{r}_1)|\phi(\vec{r}_2) > = \delta_{\vec{r}_1,\vec{r}_2} \frac{\exp\{i n(\vec{r}_1)\phi(\vec{r}_1)\}}{\sqrt{2\pi}}. \quad (6)$$

For this problem we follow steps parallel as those described in Ref. [16]. This means that we discretize the imaginary-time axis into $L_\tau$ time slices separated by a distance $\epsilon = \frac{\beta \hbar}{L_\tau}$. We recover, in principle, the fully quantum results in the limits $L_\tau \to \infty$, $\epsilon \to 0$ with $\beta \hbar$ kept fixed. Following Ref. [16] we get the partition function expression (up to order $O(1/L_\tau)$)

$$Z = \prod_{\tau=0}^{L_\tau-1} \sqrt{\text{det}[C]} \prod_{\vec{r}} \int_0^{2\pi} \frac{L_\tau}{2\pi \beta q^2} d\phi(\vec{r},\tau) \sum_{m(\vec{r},\tau)=-\infty}^{\infty} \exp \left[ -\frac{1}{\hbar} S[\{\phi\}, \{m\}] \right]$$

(7)

where we defined the action

$$\frac{1}{\hbar} S[\{\phi\}, \{m\}] = \frac{\beta}{L_\tau} \sum_{\tau=0}^{L_\tau-1} \left[ H_J(\{\phi(\vec{r},\tau)\}) - \frac{L_\tau}{2\beta q^2} \sum_{<\vec{r}_1,\vec{r}_2>} [\phi(\vec{r}_1, \tau+1) - \phi(\vec{r}_1, \tau) + 2\pi m(\vec{r}_1, \tau)] \times \right.$$

$$\times C(\vec{r}_1, \vec{r}_2) [\phi(\vec{r}_2, \tau+1) - \phi(\vec{r}_2, \tau) + 2\pi m(\vec{r}_2, \tau)]$$

$$+ O(1/L_\tau). \quad (8)$$

Here $H_J(\{\phi(\vec{r},\tau)\})$ is the Josephson hamiltonian in terms of the phase variable $\phi(\vec{r},\tau)$. The quantum nature of this c-number functional integral comes from imposing the periodic boundary condition.
\[ \phi(\vec{r}, L_x) = \phi(\vec{r}, 0). \]

These are the basic equations used in the WKB-RG and Monte Carlo calculations described in the next section.

III. WKB-RG STUDY ABOUT THE SC TO N PHASE BOUNDARY

Our approach here is to perturb the physics described by \( H_J(\{\phi(\vec{r}, \tau)\}) \), in the limit when the charging energy is small. When \( \alpha = 0 \), \( H_J \) describes the physics of the classical 2-D XY model \[ \Box \]. In this case we have the BKT scenario that depends on the thermal nucleation of vortex-antivortex pairs (VAP). The density of VAP increases exponentially as the temperature rises until they unbind at the critical temperature \( T_{\text{BKT}}(\alpha = 0) = T_{\text{BKT}}^{(0)} = (\pi E_J/2k_B) \). The BKT scenario is best understood in terms of a renormalization group (RG) analysis \[ \Box \]. The RG flow diagram is obtained from a perturbation expansion in powers of the vortex pair density \( y = y_0 e^{-\frac{\pi}{2} K} \). Here \( K = \beta E_J \) and \( y_0 \) is the initial condition for the bare vortex pair density. In the standard BKT picture there is a line of fixed points for \( 0 \leq T \leq T_{\text{BKT}}^{(0)} \), with algebraically decaying correlation functions. In the self-capacitive model, at \( T = 0 \), one can map the problem to an anisotropic three-dimensional XY model, which must have a standard phase transition at a critical value of \( \alpha_s^c \). Around the \( T = 0 \) critical point \( \alpha_s^c \), one expects to have exponentially decaying correlation functions while at \( \alpha_s^c \) the correlations must decay algebraically. The important question is then: How do we go from the BKT regime, with algebraic decaying correlation functions to the very low temperature exponentially decaying one? There must be a discontinuity in going from one limit (\( \alpha_s = 0, T \neq 0 \)) to the other one (\( \alpha_s \neq 0, T = 0 \)). Below we discuss the evidence we have found, including a possible experimental candidate, that there may indeed be a QUIT at low temperatures.

The situation when \( C_s = 0 \) and \( C_m \neq 0 \) is actually quite different. In that case the \( T = 0 \) limit can be approximately represented by two coupled three-dimensional XY models, one describing the phase degrees of freedom and the other the charges. As a function of \( \alpha_m \) we can go from a phase dominated region, with a 3-D type XY model critical properties to one dominated by a 3-D Coulomb gas. There is not much known about the critical properties, and in particular the correlation function behavior, of the two coupled 3D XY models. So, strictly speaking, we can not state what kind of crossover we should expect when going from \( T = T_{\text{BKT}}(\alpha_m, \alpha_s) \) to \( T = 0 \). Some understanding of the physics in this limit can be obtained by using the Villain transformation, both for the charging energy term and for the phase contribution \[ \Box \]. The Villain approximation is, however, valid only in a restricted range of \( \alpha \) values which do not cover the full experimental range. One could conjecture, however, that the properties of the Villain approximated models is in the same universality class as the full coupled XY models, which is in fact the case when \( \alpha = 0 \) \[ \Box \], but this needs to be explicitly shown. Furthermore, the general case treated here where both \( C_s \neq 0 \) and \( C_m \neq 0 \) is more complicated since the effective Coulomb gas in the insulating phase has a finite screening length. All these issues need to be studied further.

To find the corrections to the BKT scenario due to the charging effects we carry out a semiclassical or WKB analysis of the model. This was originally done for the self-capacitive
model in Ref. [15]. Here we follow a similar approach, except that technically the problem is more demanding.

To evaluate the partition function in the SC to N, or small $\alpha$ regime, we notice that we can extend the range of integration of the phases in Eq. (7) from $[0, 2\pi]$ to $[-\infty, \infty]$, while at the same time all but one of the summations over the set of $\{m\}$'s can be eliminated. The resulting expression for the partition function, in the $L_{\tau} \rightarrow \infty$ limit, then reads

$$Z = \sqrt{\det[C]} \int_0^{2\pi} \prod_{\vec{r}} \frac{L_{\tau}}{2\pi \beta q^2} d\phi(\vec{r}, 0) \sum_{m(\vec{r})=-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{\tau=0}^{L_{\tau}-1} \frac{L_{\tau}}{2\pi \beta q^2} d\phi(\vec{r}, \tau) \times$$

$$\times \exp \left[ -\frac{1}{\hbar} \int_0^\beta d\tau L_E \right],$$

(10)

where the euclidean lagrangian is

$$L_E = \frac{1}{2} \left( \frac{\hbar}{q} \right)^2 \sum_{<\vec{r}_1, \vec{r}_2>} \frac{d\phi}{d\tau}(\vec{r}_1, \tau) C(\vec{r}_1, \vec{r}_2) \frac{d\phi}{d\tau}(\vec{r}_2, \tau) + H_J(\{\phi\}).$$

(11)

The boundary condition, Eq. (9), now reads

$$\phi(\vec{r}, \beta \hbar) = \phi(\vec{r}, 0) + 2\pi m(\vec{r}),$$

(12)

where the $\{m(\vec{r})\}$'s are the winding numbers. We note that since the lagrangian is invariant under the transformation $\phi(\vec{r}, 0) \rightarrow \phi(\vec{r}, 0) + 2\pi l(\vec{r})$ for all integers $\{l(\vec{r})\}$, we can extend the limits of integration over $\phi(\vec{r}, 0)$ to $[-\infty, \infty]$, the difference coming only from an overall multiplicative constant. Now that the limits of Eq. (10) are all from $[-\infty, \infty]$ we can make the following change of variable [24]

$$\phi(\tau, \vec{r}) = \frac{2\pi}{\beta \hbar} m(\vec{r}) \tau + \overline{\phi}(\vec{r}) + \phi_f(\vec{r}, \tau).$$

(13)

Here $\phi_f(\vec{r}, \tau)$ represents the quantum fluctuations of the path about its mean value $\overline{\phi}(\vec{r})$. These quantum fluctuations become larger than the thermal ones as $\alpha$ increases or as the temperature decreases. This means that we would need to take higher order harmonics in the Fourier series into account when the quantum effects are not relatively small. Because of the periodicity in Eq (12), $\phi_f(\vec{r}, \tau)$ can be expanded in the Fourier series

$$\phi_f(\tau, \vec{r}) = (\beta \hbar)^{-1/2} \sum_{k=1}^{\infty} \left[ \phi_k(\vec{r}) e^{i\omega_k \tau} + C.C. \right],$$

(14)

where the $\omega_k = 2\pi k/\beta \hbar$ are the Bose Matsubara frequencies. Substituting Eqs. (13) and (14) in Eq. (10), expanding the Josephson term up to second order in $\phi_k(\vec{r})$, i.e. up to order $O(q^2)$ or equivalently to $O(\alpha)$, we obtain an effective action for the classical variables $\overline{\phi}(\vec{r})$ after evaluating the integrals. In obtaining the effective action we note that once the integrations over the $\phi_k(\vec{r})'$s are carried out, the partition function still includes a summation over the $m(\vec{r})$'s. In the semiclassical limit the contributions to the partition function from configurations with $m(\vec{r})$ different from zero are exponentially small, so that we can safely take $m(\vec{r}) = 0$ for all $\vec{r}$. A most important property of the Josephson hamiltonian $H_J$ is that
it is a periodic function of its argument. This implies that in the expansion the second order derivative with respect to the argument in \( H_J \) is proportional to \( H_J \) itself. Specifically, for the cosinusoidal form of \( H_J \) we have \( H_J'' = -H_J + \text{constant} \). This important property of \( H_J \) allows us to write the effective partition function as a 2-D classical XY model with an effective coupling constant. The effective partition function to this order of approximation is:

\[
Z_{\text{eff}} = \int \prod_{\vec{r}} \frac{d\vec{\phi}(\vec{r})}{2\pi} \exp[-\beta_{\text{eff}} H_J(\vec{\phi})],
\]

where the effective inverse temperature is given by

\[
\beta_{\text{eff}} = \beta - \frac{(q\beta)^2}{12} \left[ C^{-1}(0) - C^{-1}(\vec{d}) \right].
\]

Note that we have explicitly used the fact that \( C(\vec{r}_1, \vec{r}_2) = C(|\vec{r}_1 - \vec{r}_2|) \), valid for a periodic lattice, so that we can Fourier transform the capacitance matrix.

Once we have a Hamiltonian which is just like the 2-D classical XY model, we can write down the corresponding effective RG recursion relations to lowest order in \( x \) as

\[
\frac{dK}{dl} = 4\pi^3 K^2 \tilde{y}^2 \frac{(1 - xK)^2}{(2Kx - 1)},
\]

\[
\frac{d\tilde{y}}{dl} = [2 - \pi K(1 - xK)]\tilde{y}.
\]

In writing these equations we defined the variables

\[
x = \frac{q^2}{12E_J} \left[ C^{-1}(0) - C^{-1}(\vec{d}) \right],
\]

\[
\tilde{y} = \exp[-\frac{\pi^2}{2} K(1 - xK)] \equiv \exp[-\frac{\pi^2}{2} K_{\text{eff}}].
\]

The variable \( x \) is the \( \alpha \) parameter when the capacitance matrix is not just the self or the mutual capacitance. The RG equations are solved using as initial conditions \( K_{\text{eff}}(l = 0) \equiv K_{\text{eff}}^0 \) and \( \tilde{y}(l = 0) \equiv \tilde{y}_0 \). As written, the RG equations are valid for an arbitrary ratio between the self and the mutual capacitances. We first notice that in the \( x = 0 \) limit the RG equations reduce to the standard Kosterlitz RG equations, as they should. The form of the vortex density \( \tilde{y} \) is most important. As mentioned above, \( \tilde{y}(x = 0) \) grows exponentially with temperature. When \( x \neq 0 \) and as a function of temperature, \( \tilde{y} \) exhibits a low temperature minimum. This is shown in the discontinuous line in Fig. 2. This behavior

\[
\text{1 We must note that in obtaining the effective partition function we assumed that the phases took values between } [-\infty, \infty], \text{ whereas in the classical XY model the phases are constrained to lie in the } [0, 2\pi] \text{ range. Following this route makes the derivation of the effective action more direct. However, in the small } \alpha \text{ regime of interest here the differences between the two ranges for the phases can be shown to be exponentially small.}
\]
for $\tilde{y}$ is easy to understand physically. At high temperatures the difference between $\tilde{y}$ and $y$ is very small. However, below the minimum the increase in the vortex pair density is due to nucleation of VAP via quantum phase slips. Of course, we need to remember that we have done a perturbative calculation in $x$ and therefore we may not be on safe ground when $\tilde{y}$ starts increasing again at low temperatures. Nevertheless, as often happens with WKB derived results, the fact that the perturbative analysis shows a low temperature instability is likely to be true. In fact we also have found numerical evidence for the low temperature instability in our QMC calculations [16].

The RG equations have two nontrivial fixed points, one that corresponds to the effective BKT thermal fluctuations driven transition, and the other that corresponds to the QUantum fluctuations Induced transition (QUIT) [15,16].

The RG level curves in the $(\tilde{y}, K)$ phase space, to lowest order in $x$, result from solving Eq. (20) and

$$\pi x K - \pi \ln K - \frac{2}{K} + 2\pi^3 \tilde{y}^2 = A,$$

for different initial conditions. Figure 2 shows the RG flows for different initial conditions starting with different values for $(\tilde{y}_0, K^0)$ along the discontinuous line in the figure. Each RG flow line corresponds to a different temperature with the arrows indicating the direction of increasing $l$. We clearly see from the figure that we can divide the temperature axis into three different regions. In the region between $[K_{QUIT}^{-1}, K_{BKT}^{-1}]$, as the value of $l$ increases we eliminate VAP, with the unusual property that the vortex density can initially grow for a while before tumbling to the critical line $\tilde{y} = 0$. This means that in this region at $l = \infty$ there are no VAP with infinite separation, i.e. unbounded. Below $K_{QUIT}^{-1}$, as $l$ increases, the RG trajectories grow away from the $\tilde{y} = 0$ line, nonmonotonically, indicating that the perturbation expansion in $\tilde{y}$ is no longer valid. If we associate the instability in the perturbation theory with the normal state behavior, one could say then that this behavior is characteristic of a reentrant phase transition, i.e. going from N to SC to N. This is certainly the case in the high temperature regime but not necessarily so at low temperatures. The single line that divides the two types of behaviors mentioned above is the separatrix that determines the critical temperature. This is the line with the highest temperature for which we can touch the $\tilde{y} = 0$ line. The corresponding separatrix value of the constant $A = A_c$ is determined from the condition that it passes through the point $(K_{BKT}^{-1}, \tilde{y} = 0)$. This leads to the result that the critical point is obtained from solving the equation,

$$\pi x K_c - \pi \ln K_c - \frac{2}{K_c} 2\pi^3 \exp\{-\pi^2 K_c (1 - x K_c)\} = A_c(x),$$

where $A_c(x)$ is obtained from $A_c(x) = 2\pi x K_c - \pi - \pi \ln K_c$, and $K_c$ is the solution to the equation $2 = \pi x K_c (1 - x K_c)$. We’ll come back to the problem of determining $K_{BKT}^{-1}$ in the next section, where we make a comparison with the experimental results. It is known, however, that the determination of $T_{BKT}$ using the RG equations is not quantitatively exact, since the RG analysis is explicitly derived for the Villain action. In comparing with experiment in the next section we will take this into account. Here we present the corrections to the classical results to the leading order in $x$, which give the correct qualitative trends. Specifically, expanding in powers of $x$ we find that $T_{BKT}$ and $T_{QUIT}$ are given by
\[ T_{\text{BKT}} \approx T_{\text{BKT}}^{(0)} - \frac{E_J}{k_B} x + O(x^2), \]  
(23)

\[ T_{\text{QUIT}} \approx \frac{E_J}{k_B} x + O(x^2). \]  
(24)

Notice that these equations are applicable not only in 2-D, for if the system described by Eq. (15) has a transition point at some \( K_{\text{eff}}^{c} \) then the equation \( K_{\text{eff}}^{c} = K - xK^2 \) has two solutions for \( K \), which are the ones implied in Eq. (18). Moreover, notice that the results to the first order in \( x \) are independent of the specific value of \( T_{\text{BKT}}^{(0)} \). This means that if we consider the finite magnetic field case, the corresponding critical temperature will be \( T_c(B) \approx T_c^{(0)}(B) - (E_J/k_B)x + O(x^2) \). Furthermore, we notice that, to the lowest order in \( x \), the \( T_{\text{QUIT}} \) must be the same with and without a field. This fact will be compared with the experiments in the next section.

The explicit leading order calculation of the correction to the BKT critical temperature in the asymptotic limits in which either the self or the mutual capacitance dominates results in

\[
\frac{T_{\text{BKT}}}{T_{\text{BKT}}^{(0)}} = \begin{cases} 
1 - \frac{4}{3\pi} \frac{E_{Cs}}{E_J} + O \left( \frac{E_{Cs}}{E_J} \right)^2, & \text{if } C_s \gg C_m \\
1 - \frac{4}{3\pi} \frac{E_{Cm}}{E_J} + O \left( \frac{E_{Cm}}{E_J} \right)^2, & \text{if } C_s \ll C_m 
\end{cases}
\]  
(25)

here \( z \) is the coordination number of the array, and for a square array in two dimensions, \( z = 4 \).

**IV. COMPARISON TO EXPERIMENT**

We now move on to a brief discussion of how we obtained the results presented in Fig. 1. As mentioned before, in trying to find quantitative correspondence between experiment and theory it is important to ascertain the validity of the theoretical models employed to study the arrays. We have carried out two different types of checks. One based on the RG analysis described in the previous sections and the other from a nonperturbative quantum Monte Carlo calculation \[13,14\]. We discuss the RG analysis here and only briefly mention the QMC results, with more details left for forthcoming publications \[13,14\].

As mentioned in section III, it is known that the RG equations do not lead to quantitatively exact results for the critical temperatures. They do, of course, lead to the correct universal critical exponents. However, what has so far been measured experimentally is the phase boundary between the SC and the N phases. We then need a consistent way to compare the RG results with the experimental results.

We first note that the phase diagram of Fig. 1 is plotted as a function of \( \alpha_m = \frac{E_{Cs}}{E_J} \), since experimentally \( \alpha_s \) is three orders of magnitude smaller. We can then write \( K_{\text{eff}} = K(1-xK) \) with \( C_S = 0 \) as

\[ K_{\text{eff}} = K - \frac{\alpha_m}{6} K^2. \]  
(26)

Next we set the critical temperature for the classical model to be the one obtained in classical MC simulations (e.g. \[20\]) \( 1/K_c^{(0)} \approx 0.93 \), so that the critical temperature \( T_c(\alpha_m) \) of the actual model is given by the equation
\[ k_B T_c(\alpha_m) = \frac{1}{2K_c^{(0)}} \left[ 1 + \sqrt{1 - \frac{2\alpha_m}{3K_c^{(0)}}} \right]. \]  

(27)

To lowest order in \( \alpha_m \) this equation gives

\[ T_c(\alpha) = T_c(0) - \frac{\alpha_m}{6}, \]  

(28)

whereas the maximum value of \( \alpha_m \) for which there is a physical solution is \( \alpha_m = \frac{2}{3} k_B T_c(0) \approx 1.4 \). The results obtained from this analysis are shown as a discontinuous line in Fig. 1. By following this approach we see that for \( \alpha_m \leq 1 \) the RG result is actually quite good when compared to the experimental and the MC results.

We also have extended our previous QMC calculations to the case when the off-diagonal capacitance is dominant. The results are shown in Fig. 1 by the crosses, including their error bars. It is clear from these results that the correspondence between experiment and QMC results is excellent, up to nonperturbative values of \( \alpha_m \). This leads us to the conclusion that the model studied here does provide a good representation of the experimental system, at least in the SC to N regime. In the next section we will briefly discuss what happens in the insulating region to normal region.

The discussion presented above dealt with the SC to N phase boundary. What about evidence for a QUIT? In this regard we note that in the experimental results of Ref. [4] there are results for a sample with a nominal \( \alpha_m = 1.67 \) for which there is a double type of reentrant behavior. The low temperature glitch seen in the resistance versus temperature diagram occurs at \( T' = 40 mK \). Moreover the latter instability is also characterized by an increase in noise fluctuations in the IV characteristics measured in this sample. If we assume that what is seen at \( T' \) is related to the QUIT, using Eq. (24) and the parameter values of the experiment we get a \( T_{QUIT} = 33 mK \), rather close to the experimental value. Furthermore when the same experiment is repeated in a small magnetic field the low temperature instability is found at the same temperature, i.e.; \( T'(f = 0) = T'(f = 0.08) \). This result is also consistent with Eq. (24) which also leads to a \( T_{QUIT} \) independent of \( f \) at leading order in \( \alpha_m \). These results may just be coincidental and more work needs to be done to conclusively connect the \( T_{QUIT} \) with the low temperature instability already seen in the Delft experiments.

V. INSULATING TO NORMAL CROSS OVER

As mentioned in the introduction, as \( \alpha_m \) increases there is a SC to I transition at finite temperature. In the \( \alpha_s = 0 \) case, the insulating phase has been modelled as a two-dimensional Coulomb gas of charges with a possible BKT charge unbinding transition [10–12]. This situation has been studied extensively, in particular in Ref. [12]. Here we ask if there is an equivalent QUIT in the insulating phase at low temperatures. If we draw an analogy to the quantum induced vortices in the SC phase one could also imagine that the number of free dipoles in the arrays could increase due to quantum fluctuations. However, as we show below the vortices and charges are not dual to each other in that sense.

The calculation described here aims at finding the leading correction to the charging hamiltonian due to Josephson junction fluctuations. We then expand the Josephson contribution to \( Z \) as
\[
\exp \left[ -\frac{1}{\hbar} \int_0^{\beta \hbar} d\tau H_J(\tau) \right] \approx 1 - \frac{1}{\hbar} \int_0^{\beta \hbar} d\tau H_J(\tau) + \frac{1}{2\hbar^2} \int_0^{\beta \hbar} d\tau \int_0^{\beta \hbar} d\tau' H_J(\tau) H_J(\tau') + \ldots \tag{29}
\]

As before, we use Eqs. (12) and (13), but this time we integrate out both \(\phi_J(\tau, \vec{r})\) and \(\overline{\phi}(\vec{r})\), which leaves us with an effective action for the \(\{m\}\)'s,

\[
Z \approx Z_\phi \prod_{\vec{r}, m(\vec{r})=-\infty}^\infty \exp \left[ -\frac{1}{4K_{eff}} \sum_{\vec{r}, \vec{d}} \left( m(\vec{r} + \vec{d}) - m(\vec{r}) \right)^2 \right], \tag{30}
\]

where we have assumed that \(C_s \ll C_m\). The function \(Z_\phi\) does not contain the \(\{m\}\) variables, and the effective coupling constant \(\tilde{K}_{eff}\) is

\[
\tilde{K}_{eff}^{-1} = K^{-1} \left[ 1 + \left( \frac{2\pi^2}{\alpha_m} \right)^2 g(\tilde{K}) \right], \tag{31}
\]

given as a function of \(\tilde{K} = (\beta E_{C_m})/2\pi\). As in the vortex dominated case, we have ended up with a Coulomb gas problem but with a renormalized coupling constant. The function \(g(\tilde{K})\) determines the importance of the zero point fluctuations of the phases on the charge dominated phase. The function \(g(\tilde{K})\) is defined as

\[
g(\tilde{K}) = \int_0^{1/2} dt [1 - \cos(2\pi t)] \exp \left[ -\frac{(2\pi)^2}{z} \tilde{K} t(1-t) \right]. \tag{32}
\]

For a BKT type phase transition we have \(\tilde{K}_{eff} = K_{BKT}^c = 2/\pi\) \cite{12}. It is important here to see if, within this approximation, the system shows a QUIT or a reentrant transition. We then need to study the number of solutions to this equation. From the fact that \(K_{BKT}^c \sim O(1)\) we can see that for any value of \(1/\alpha_m\) the function \(g(\tilde{K})\) has only one solution for \(\tilde{K}_c\). This means that, to this order of approximation, \textbf{there is no} QUIT in the charge dominated phase in this model.

A question that immediately arises is: Why is there a difference between the vortex and charge dominated phases, in particular in view of the duality between the two phases extensively studied in \cite{12}? The reason is that the duality is not exact since there is a term in the action, obtained using the Villain approximation, that breaks this symmetry. If one includes this term we then see that the cost of producing quantum fluctuations in the vortices is bounded from above whereas the corresponding cost in the charge dominated phase is unbounded. In our calculation we have kept these contributions intact.

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FIGURES

FIG. 1. Temperature vs. charging energy phase diagram. The experimental results are denoted by squares. The RG results are given by the discontinuous line while the quantum Monte Carlo results are given by the crosses and joined by a continuous line as a guide to the eye. The latter results include the statistical error bars in the calculations.

FIG. 2. Renormalization group flow diagram. The discontinuous line indicates the vortex pair density as a function of temperature. See text for a discussion of the analysis of this diagram.