Stringent constraints on the scalar $K\pi$ form factor from analyticity, unitarity and low-energy theorems

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Abstract. We investigate the scalar $K\pi$ form factor at low energies by the method of unitarity bounds adapted so as to include information on the phase and modulus along the elastic region of the unitarity cut. Using at input the values of the form factor at $t = 0$ and the Callan-Treiman point, we obtain stringent constraints on the slope and curvature parameters of the Taylor expansion at the origin. Also, we predict a quite narrow range for the higher order ChPT corrections at the second Callan-Treiman point.

1 Introduction

The low energy properties of the $K\pi$ form factors are of great interest both experimentally and theoretically. In particular, a precise knowledge of the slope and curvature parameters at $t = 0$ would serve to improve the experimental analysis of $K\bar{\nu}_\tau$ decays, confirm the predictions of chiral perturbation theory (ChPT) and provide benchmarks for future lattice determination of these quantities.

In the present paper we consider the scalar $K\pi$ form factor, expanded as

$$f_0(t) = f_0(0) \left(1 + \lambda_0^\prime \frac{t}{M_\pi^2} + \frac{1}{2} \lambda_0^{\prime\prime} \frac{t^2}{M_\pi^4} + \cdots\right),$$

(1)
in the physical region of $K\bar{\nu}_\tau$ decay. The dimensionless parameters $\lambda_0^\prime$ and $\lambda_0^{\prime\prime}$ are related by $\lambda_0^\prime = M_\pi^2 (\tau_\pi^2 M_\pi^2 K)/6$ and $\lambda_0^{\prime\prime} = 2M_\pi^4 c$ to the radius ($\tau_\pi^2 K$) and curvature $c$ used alternatively in the literature.

The $K\pi$ form factors have been calculated at low energies in ChPT \cite{12,13,14} and on the lattice (for recent reviews see \cite{14}). At $t = 0$, the present value $f_0(0) = 0.962 \pm 0.004$ \cite{1} shows that the corrections to the Ademollo-Gatto theorem are quite small. Other low energy theorems frequently used are \cite{1,2,3}

$$f_0(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{CT}, \quad f_0(\bar{\Delta}_{K\pi}) = \frac{F_\pi}{F_K} + \bar{\Delta}_{CT},$$

(2)

where $\Delta_{K\pi} = M_K^2 - M_\pi^2$ and $\bar{\Delta}_{K\pi} = -\Delta_{K\pi}$ are the first and second Callan-Treiman points, respectively. The lowest order values are known from $F_K/F_\pi = 1.193 \pm 0.006$ \cite{4}, and the corrections calculated to one loop are $\Delta_{CT} = -3.1 \times 10^{-3}$ and $\bar{\Delta}_{CT} = 0.03$ \cite{4}. The higher order corrections appear to be negligible at the first point \cite{5}, but are expected to be quite large at the second one.

Analyticity and unitarity represent a powerful tool for obtaining information on the $K\pi$ form factors. Several comprehensive dispersive analyses were performed recently, using either the coupled channels Muskhelishvili - Omnès equations \cite{9,10}, or a single channel Omnès representation \cite{11}.

Alternatively, the method of unitarity bounds, proposed a long time ago in \cite{12,13}, and applied since then to various electromagnetic and weak form factors, exploits the fact that a bound on an integral of the modulus squared of the form factor along the unitarity cut is sometimes known from independent sources. Standard mathematical techniques then allow one to correlate the values of the form factor at different points or to control the truncation error of power expansions used in fitting the data \cite{14}.

For the $K\pi$ system the method was applied in \cite{15} and more recently in \cite{16,17}. In Ref. \cite{16} the method was extended by including the phase of the scalar form factor along the elastic part of the cut, known from the elastic $K\pi$ scattering by Watson’s theorem, while in \cite{17} information on the form factor at the second Callan-Treiman point was included for the first time in the frame of the standard bounds.

In the present work we revisit the issue of bounds on the expansion coefficients \cite{1} by applying a more sophisticated version of the unitarity bounds proposed in \cite{15}. The method uses the fact that the knowledge of the phase allows one to remove the elastic cut and define a function with a larger analyticity domain. To be optimal, the method requires also some information on the modulus of the form factor in the elastic region. In \cite{15}, where the phase constraint was treated using Lagrange multipliers, this stronger property of the phase was not exploited, since no experimental information on the modulus was available at that time.

More recently, the precise measurements of the $\tau \rightarrow K\pi\nu_\tau$ spectral function by Belle collaboration \cite{18} provided also a first direct experimental determination of the modulus of the $K\pi$ form factors below a certain energy. The modulus is available also from the dispersive analyses \cite{9,10}. This justifies the application of the method proposed in \cite{15}. Our work extends the analysis made in \cite{14} by including information on the phase and modulus of the form factor on a part of the cut, which leads to a considerable improvement of the bounds. In the next section
we describe briefly the method for the scalar form factor, and in section 3 we present the results. A more detailed analysis, including a discussion of the experimental implications and of the vector form factor, will be presented in [20].

2 Standard and new unitarity bounds

The method makes use of the following mathematical result: let \( g(z) \) be a function analytic in the unit disk \(|z| < 1\) of the complex \( z \)-plane, which satisfies the inequality:

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| g(\exp(it)) \right|^2 dt \leq I,
\]

where \( I \) is a positive number. Then, if

\[
g(z) = g_0 + g_1 z + g_2 z^2 + \cdots
\]

is the Taylor expansion of \( g(z) \) at \( z = 0 \), and \( g(z_1), g(z_2) \) denote the values of \( g \) at two points inside the analyticity domain, \(|z_1| < 1, |z_2| < 1\) (for simplicity we assume that \( z_1 \) and \( z_2 \) are real), the following determinantal inequality holds:

\[
\begin{vmatrix}
1 & g_0 & g_1 & g_2 & \cdots \\
g_0 & 1 & 0 & 0 & \cdots \\
g_1 & 0 & 1 & 0 & \cdots \\
g_2 & 0 & 0 & 1 & \cdots \\
\end{vmatrix} \geq 0.
\]

Moreover, all the principal minors of the above matrix should be nonnegative. For the proof see Refs. [12-13,15].

To obtain this formulation for the scalar \( K\pi \) form factor, one starts from a dispersion relation for the scalar polarization function of the \( s \) and \( u \) quarks [15,16,17,21]:

\[
\chi_0(Q^2) \equiv \frac{\partial}{\partial q^2} [q^2 \Pi_0] = \frac{1}{\pi} \int_{t_{in}}^{\infty} \frac{dt \Im \Pi_0(t)}{(t + Q^2)^2},
\]

where unitarity implies the inequality:

\[
\Im \Pi_0(t) \geq \frac{3}{2} \frac{t_{in}^2}{16\pi^3} [(t - t_{in})(t - t_{in})]^{1/2} |f_0(t)|^2,
\]

with \( t_{in} = (M_K \pm M_\pi)^2 \). We use here the notations from [21], where \( \Pi_0 \) is defined as the longitudinal part of the correlator of two vector currents. As in [15], \( \Pi_0 \) can be identified with \((\Psi(t) - \Psi(0))/t^2\), where \( \Psi \) is the correlator of the divergence of the vector current.

The quantity \( \chi_0(Q^2) \) in [3] can be reliably calculated by pQCD when \( Q \gg \Lambda_{QCD} \). At present, calculations available up to the order \( \alpha_s^4 \) give:

\[
\chi_0(Q^2) = \frac{3(m_s - m_u)^2}{8\pi^2 Q^2} \left[ 1 + 1.80\alpha_s + 4.65\alpha_s^2 + 15.0\alpha_s^3 + 57.4\alpha_s^4 \cdots \right],
\]

where the running quark masses and the strong coupling \( \alpha_s \) are evaluated at the scale \( Q^2 \) in \( \overline{MS} \) scheme.

Taking into account the fact that \( f_0(t) \) is analytic everywhere in the complex \( t \)-plane except for the branch cut running from \( t_{in} \) to \( \infty \), the relations [8-13] can be expressed in the canonical form [3] if one defines the variable

\[
z(t) = \frac{\sqrt{t_{in} - t} \pm \sqrt{t_{in} - t_{in}}}{\sqrt{t_{in} + t_{in} - 2t}}.
\]

which maps the \( t \) plane cut from \( t_{in} \) to \( \infty \) onto the unit disk \(|z| < 1\), such that \( z(0) = 0 \), and the function

\[
g(z) = f_0(t(z)) w(z),
\]

where \( t(z) \) is the inverse of \( \tilde{z} \) and \( w(z) \) is the outer function [15,16,17]:

\[
w(z) = \frac{3\sqrt{3}}{32\sqrt{\pi}} \frac{M_K - M_\pi}{M_K + M_\pi} (1 - z)^{3/2}
\]

\[
\times \frac{(1 + z - Q^2)^2}{(1 - z - Q^2)^2} \left( \frac{1 + z(t_{in})}{1 + z(t_{in})} \right)^{1/2}.
\]

Then [3] is satisfied, with

\[
I = \chi_0(Q^2).
\]

It may be noted that \( z \) is an independent variable in the outer function, whereas \( z(-Q^2) \) etc., are defined via the conformal variable \( \tilde{z} \).

We use now the fact that, below the inelastic threshold \( t_{in} \), the phase of the form factor \( f_0(t) \) is known from Watson’s theorem and the \( I = 1/2 \) S-wave of elastic \( K\pi \) scattering. Then one can define the Omnès function

\[
\mathcal{O}(t) = \exp \left( \frac{t}{\pi} \int_{t_{in}}^{\infty} dt' \frac{\delta(t')}{t'(t' - t)} \right),
\]

where \( \delta(t') \) is the phase of the form factor known for \( t \leq t_{in} \), and is an arbitrary function, sufficiently smooth (i.e. Lipschitz continuous) for \( t > t_{in} \). It can be shown [20] that the results are independent of the function \( \delta(t) \) for \( t > t_{in} \).

Since the Omnès function \( \mathcal{O}(t) \) fully accounts for the second Riemann sheet of the form factor, the function \( h(t) \), defined by

\[
f_0(t) = h(t) \mathcal{O}(t),
\]

is real analytic in the \( t \)-plane with a cut only for \( t \geq t_{in} \). Then, the relations [9-13] and [14] can be expressed in the canonical form [3], by defining the new variable [18]

\[
z(t) = \frac{\sqrt{t_{in} - t} \pm \sqrt{t_{in} - t_{in}}}{\sqrt{t_{in} + t_{in} - 2t}}.
\]

which maps the \( t \)-plane cut for \( t > t_{in} \) onto the unit disk \(|z| < 1\) of the \( z \)-plane, such that \( z(0) = 0 \), and define the function [18]

\[
g(z) = f_0(t(z)) w(z) \omega(z) [\mathcal{O}(t(z))]^{-1},
\]

\[1\] We mention that a \( \sqrt{2} \) is missing in the corresponding expressions given in [16,17].
where \( t(z) \) is now the inverse of \( \bar{f}_{0}(t) \). The new outer function \( w(z) \) is defined as

\[
\begin{align*}
\omega(z) &= \left(\frac{1}{t-1} - \frac{t}{1-\frac{1}{1-z}}\right)^{1/2} - \frac{1}{z} - \frac{t}{1-\frac{1}{1-z}}\right)^{1/2},
\end{align*}
\]

and

\[
\omega(z) = \exp \left( \frac{\ln|\mathcal{O}(\tau)|}{\pi} \int_{t_{\infty}}^{t_{\min}} \frac{\ln|\mathcal{O}(\tau')|}{\sqrt{\tau' - t_{\infty}(t' - t(z))}} \right).
\]

Then \( I \) is satisfied, where \( I \) is defined as

\[
I = \chi_{0}(Q^{2}) \left( \frac{3 t_{+} t_{-}}{2 \times 16\lambda^{2}} \int_{t_{\infty}}^{t_{\min}} \frac{(t_{+} - t_{-})^{1/2} |f_{0}(t)|^{2}}{t^{2} + Q^{2}} \right),
\]

and is calculable if the modulus \( |f_{0}(t)| \) is known at low energies, below \( t_{\min} \). Thus, we can use the inequality \( \delta \) and the nonnegativity of the leading minors to obtain bounds on the parameters of the expansion \( \chi_{0} \). The Taylor coefficients in \( \chi_{0} \) are defined uniquely in terms of these parameters by \( \omega(z) \) or \( \chi_{0} \). We further choose \( z_{2} = z(\Delta_{K^{\pi}}) \) and \( z_{2} = z(\Delta_{K^{\pi}}) \), where \( z \) is defined by \( \omega(z) \) or \( \chi_{0} \), and express \( g(z) \) in terms of the values in \( \omega(z) \), by using either \( \omega(z) \) or \( \chi_{0} \).

In our analysis we take as inputs the values of the form factor at \( t = 0 \) and \( t = \Delta_{K^{\pi}} \), the phase below \( t_{\min} \) and the integral over the modulus required in \( \chi_{0} \). Then the constraints resulting from \( \omega(z) \) restrict the coefficients \( \chi_{0}, \chi_{0}^{\prime} \) and the value of \( f_{0}(t) \) at the second Callan-Treiman point \( \Delta_{K^{\pi}} \).

3 Input

We work in the isospin limit, adopting the convention that \( M_{K^{0}} \) and \( M_{\pi} \) are the masses of the charged mesons. The inputs provided by the low-energy theorems was discussed in the Introduction. For choosing \( t_{\min} \), we recall that the first inelastic threshold for the scalar form factor is set by the \( K^{\eta} \) state, which suggests to take \( t_{\min} = 1 \text{ GeV}^{2} \) as in \( \omega(z) \). However, this channel has a weak effect, the elastic region extending practically up to the \( K^{\eta} \) threshold, which justifies the choice \( t_{\min} = 1.4 \text{ GeV}^{2} \). In our analysis we shall use for illustration these two values of \( t_{\min} \).

Below \( t_{\min} \) the function \( \delta(t) \) entering \( \omega(z) \) is the phase of the \( S \)-wave of \( I = 1/2 \) of the elastic \( K^{\pi} \) scattering \( \omega(z) \). In our calculations we use as default the phase from \( \omega(z) \). Above \( t_{\min} \), we assume \( \delta(t) \) as a smooth function approaching \( \pi \) at high energies. We checked numerically that the bounds are independent of the choice of \( \delta(t) \) for \( t > t_{\min} \).

To estimate the integral appearing in \( \omega(z) \), we first used the parametrization of \( |f_{0}(t)| \) in terms of the resonances \( \kappa \) and \( K_{\pi}^{\prime}(1430) \), proposed by Belle collaboration. Using as input the solution 1 in Table 4 and Eq. (7) of \( \omega(z) \), the integral has the value \( 66.08 \times 10^{-6} \) for \( t_{\min} = 1 \text{ GeV}^{2} \), and \( 184.89 \times 10^{-6} \) for \( t_{\min} = (1.4 \text{ GeV})^{2} \). Although the parametrization used in \( \omega(z) \) does not have good analytic properties, this fact is not relevant for our analysis: all that we need is a numerical estimate of the integral in \( \omega(z) \). The analyticity of the form factor is implemented rigorously in our approach, for every numerical input.

Alternatively, using the modulus available from the dispersive analyses \( \omega(z) \) or \( \chi_{0} \), the low energy integral in \( \omega(z) \) is \( 40.05 \times 10^{-6} \) or \( 37.01 \times 10^{-6} \), respectively, for \( t_{\min} = 1 \text{ GeV}^{2} \), and \( 89.31 \times 10^{-6} \) or \( 81.26 \times 10^{-6} \) for \( t_{\min} = (1.4 \text{ GeV})^{2} \).

Finally, we take \( Q^{2} = 4 \text{ GeV}^{2} \) as in \( \omega(z) \), and obtain \( \chi_{0} = (253 \pm 68) \times 10^{-6} \), using in \( \omega(z) \) \( m_{K}(2 \text{ GeV}) = 98 \pm 10 \text{ MeV} \) and \( m_{\pi}(2 \text{ GeV}) = 165 \pm 1 \text{ MeV} \) and \( \alpha_{s}(2 \text{ GeV}) = 0.308 \pm 0.014 \), which results from the recent average \( \alpha_{s}(m_{\pi}) = 0.330 \pm 0.014 \) \( \omega(z) \). The error of \( \chi_{0} \) includes also a contribution of 15%, of the order of magnitude of the last term in \( \omega(z) \), to account for the truncation of the expansion \( \omega(z) \).

4 Results

In order to illustrate the effect of the additional information on the phase and modulus, we compare in Fig. 1 the allowed domains in the plane \( (\chi_{0}, \chi_{0}^{\prime}) \), obtained with the standard and the new bounds, using only the constraint at \( t = 0 \) (this case is obtained from \( \omega(z) \) by removing the lines and columns that contain \( g(z_{1}) \) and \( g(z_{2}) \)). The large ellipse is obtained with the standard bounds, \( \omega(z) \). The small ones represent the new bounds, calculated with \( \omega(z) \) for two values of \( t_{\min} \). For \( f_{0}(0) \) we took the central value 0.962. The left panel is obtained with the integral in \( \omega(z) \) calculated with the modulus from \( \omega(z) \), for the right one we used the modulus from \( \omega(z) \).

The inner ellipses are slightly smaller in the left panel than in the right one, because in the latter case the integral in \( \omega(z) \) is smaller and \( I \) is larger (it is easy to see that a larger value of \( I \) leads to an ellipse of larger size). Moreover, in the right panel the small ellipses are not contained entirely inside the large one, which means that among the functions satisfying the constraints \( \omega(z) \), there are some that violate the original bounds \( \omega(z) \). However, this does not mean that the functions cannot be consistent, since the ellipses are a model of the truncation of the expansion \( \omega(z) \).
t = 0, \Delta K_{\pi} and \bar{\Delta} K_{\pi}, for the central values of \( f_0(0) \) and 
\( f_0(\Delta K_{\pi}) \) and several values of the correction \( \Delta CT \). 
The strong constraining power of the simultaneous constraints at 
\( \Delta K_{\pi} \) and \( \bar{\Delta} K_{\pi} \) was noted in [17]. However, the bounds 
derived now are much stronger than those in [14], due to the 
additional information on the phase and modulus on the cut.

The small ellipses exist only for \( \Delta CT \) inside a rather narrow interval, 
whose end points lead to inner ellipses of zero size in Fig. 2. Actually, 
this range results directly from the inequality [10]: by keeping only the 
lines and columns involving \( g_0 \), \( g(z_1) \) and \( g(z_2) \) and using the 
central values at \( t = 0 \) and \( \Delta K_{\pi} \), we obtain \(-0.046 \leq \Delta CT \leq 0.014 \).

We recall that the current ChPT prediction is \(-0.057 < \Delta CT < 0.089 \) 
(cf. Eq. (4.9) of [11] adapted to our input \( F_K / F_{\pi} \)). As this interval is larger 
than the range derived above, we conclude that, at present, one can not further 
restrict the domain for the slope and curvature using the 
low-energy theorem at the second Callan-Treiman point: by varying \( \Delta CT \) inside 
its currently known range we obtain the union of the tiny ellipses in Fig. 2, 
which covers the large ellipse obtained using only the value at \( \Delta K_{\pi} \).

In Fig. 3 we show the allowed domains for the slope and curvature 
with the constraints at \( t = 0 \) and \( \Delta K_{\pi} \) for two values of \( t_{in} \) in the left panel. In the right 
panel we also superimpose the allowed region when no phase and modulus 
information is taken into account. As in Fig. 2 the low energy integral in [10] 
was calculated with the modulus from [19]. For \( t_{in} = 1 \text{ GeV}^2 \) the large 
ellipse implies the range \( 0.0137 \leq \lambda_\rho \leq 0.0172 \), for \( t_{in} = 
(1.4 \text{ GeV})^2 \) the small ellipse implies the narrower range 
\( 0.0150 \leq \lambda_\rho \leq 0.0163 \). In both cases we obtain a strong 
correlation between the slope and the curvature. It may 
be clearly seen that a dramatic improvement is obtained 
by the inclusion of phase and modulus data.

As discussed, the method gives also very sharp predictions 
for the corrections \( \Delta CT \) at the second Callan-Treiman point: for 
\( t_{in} = (1.4 \text{ GeV})^2 \) we obtain \(-0.031 \leq \Delta CT \leq -0.008 \).

The above ranges were obtained using the central values 
of \( f_0(0) \), \( f_0(\Delta K_{\pi}) \) and \( \chi_0 \), the phase from [10] and the 
modulus from [19]. Accounting for the errors and using 
alternatively the phase and modulus from [24], the end 
points of the range of \( \lambda_\rho \) for \( t_{in} = (1.4 \text{ GeV})^2 \) varied by 
\( \pm 0.00039 |_{0}^{0.00044} |_{\Delta K_{\pi}}^{0.00028} |_{\chi_0}^{0.00038} |_{\text{mod}}^{0.00070} |_{\text{ph}} \), 
while for \( \Delta CT \) the variation was \( \pm 0.0074 |_{0}^{0.0092} |_{\Delta K_{\pi}}^{0.0052} |_{\chi_0}^{0.0069} |_{\text{mod}}^{0.0092} |_{\text{ph}} \). 
We note that, while the bounds are very sensitive to the input value 
of \( f_0(0) \), the uncertainty of \( \chi_0 \) has a relatively low influence 
on the results.

In conclusion, our analysis shows that the modified 
type of unitarity bounds proposed in [18], which includes 
input from the elastic part of the cut, leads to very stringent 
bounds on the scalar \( K \pi \) form factor at low energy. 
Using as input the precise values at \( t = 0 \) and \( \Delta K_{\pi} \) and 
assuming that the inelasticity is negligible below \( 1.4 \text{ GeV}^2 \), 
we obtain for the slope at \( t = 0 \) the range \( 0.0150(10) \leq \lambda_\rho \leq 0.0163(10) \), 
where the error is obtained by adding in quadrature the uncertainties due to various inputs. 
As shown in Fig. 3, the method leads to a strong correlation 
between the slope and the curvature. We obtain also a narrow 
acceptable range \( -0.031(12) \leq \Delta CT \leq -0.008(12) \) for 
the higher order ChPT corrections at the second Callan-Treiman point \( \Delta K_{\pi} \); significantly reducing the range from 
ChPT mentioned earlier. Unlike in the usual dispersive 
approaches, the predictions are independent of any as-

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**Fig. 1.** Allowed regions for the slope and curvature from the 
standard and the new bounds with the constraint at \( t = 0 \). 
Left: modulus from [19]; Right: modulus from [24].

**Fig. 2.** Large ellipse: new bounds obtained with \( t_{in} = 1 \text{ GeV}^2 \) 
and the values at \( t = 0 \) and \( \Delta K_{\pi} \); small ellipses: new bounds 
for \( t_{in} = 1 \text{ GeV}^2 \) and simultaneous constraints at \( t = 0, \Delta K_{\pi} \) 
and \( \bar{\Delta} K_{\pi} \), for several values of \( \Delta CT \).

**Fig. 3.** Left: Allowed regions for the slope and curvature using 
as input the phase and modulus up to \( t_{in} \), and the values of 
\( f_0(0) \) and \( f_0(\Delta K_{\pi}) \); right: as in left panel and also showing the 
region obtained with no phase and modulus information.
sumptions about the presence or absence of zeros, or the phase and modulus of the form factor above the inelastic threshold.

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