The vacuum nature of the QCD condensates

H. Reinhardt$^a$ and H. Weigel$^b$

$^a$Institute for Theoretical Physics, Tübingen University D–72076 Tübingen, Germany
$^b$Physics Department, Stellenbosch University, Matieland 7602, South Africa

Recently it was claimed that QCD condensates were associated with the internal dynamics of the hadrons. We challenge this “in–hadron” picture of the QCD condensates and show that it conflicts well established concepts and experimental facts for the example of the quark condensate. We explicitly demonstrate the vacuum nature of the quark condensate by a model calculation.

PACS numbers: 11.30.Rd,12.38.Aw,12.39.Ki,11.15.Tk

INTRODUCTION

At low energies and baryon densities quantum chromodynamics (QCD) is in the confined phase where chiral symmetry is spontaneously broken while at high temperature or density the quarks and gluons form a correlated plasma in which chiral symmetry is restored. The various phases of QCD are characterized by order (and disorder) parameters. These order (and disorder) parameters are vacuum expectation values of certain gauge invariant products of quark and gluon operators and are usually referred to as condensates. The most prominent ones are the quark condensate $\langle \bar{q}q \rangle$ and the gluon condensate $\langle F_{\mu\nu}F^{\mu\nu} \rangle$. These condensates play an important role in the operator product expansion where they incorporate the vacuum properties and they are central to QCD sum rule calculations.

In a series of papers [1, 2] it was recently argued that the strong interaction condensates like the quark and gluon condensate are not properties of the QCD vacuum but instead associated with the internal dynamics of hadrons. To substantiate this interpretation the authors of those papers allude to the light front formulation of spontaneous breaking of chiral symmetry [2] and the description of hadrons in terms of Dyson–Schwinger and Bethe–Salpeter equations [3, 4]. This interpretation has far reaching consequences. If the quark and gluon condensates are not properties of the QCD vacuum but of the hadrons they do not contribute to the cosmological constant while a naive (unrenormalized) inclusion of the condensates would result in a cosmological constant, which is about a factor of $10^{56}$ larger than the currently accepted value.

In this paper we provide strong arguments that invalidate the interpretation of the QCD condensates as “in–hadron” properties as proposed in ref. [1, 2]. In addition, we will present a model calculation that clearly shows that the translationally invariant quark condensate is a vacuum property. However, this condensate is strongly distorted inside the hadrons.

The quark condensate as property of the QCD vacuum

Low energy hadron physics is dominated by chiral dynamics. The concept of approximate chiral symmetry for the light quark flavors, being spontaneously broken in the ground state with the pseudoscalar mesons, like pions and kaons, identified as Goldstone bosons, has been overwhelmingly successful. The order parameter of chiral symmetry breaking is the quark condensate: It is non–zero in the chiral symmetry broken phase and vanishes (for massless quarks) in the chirally symmetric phase. For the moment let us assume that the picture of “in–hadron” condensates put forward in ref. [1, 2] were correct, i.e. $\langle \bar{q}q \rangle \neq 0$ inside the hadrons and $\langle \bar{q}q \rangle = 0$ outside. This would imply that chiral symmetry is spontaneously broken only inside the hadrons, while the exterior of hadrons is in the chirally symmetric phase. Consequently the corresponding Goldstone boson, the pion, could exist only inside other hadrons. Outside the lightest particle with pion quantum numbers would be massive and degenerate with the $\sigma$. This, of course, contradicts the experimental observation. The pion is measured outside the hadrons with a light mass (for example in a bubble chamber). For an additional counter–argument consider nuclear matter as an ensemble of hadrons. If chiral symmetry were spontaneously broken only inside the hadrons then there would be no reason why this symmetry is restored when the density of baryons is increased. If the condensates were indeed hadron properties, their universal character could not be sustained, no mechanism would prevent them to depend on the peculiarities of the considered hadron. Furthermore we recall recent indications [5] that at high density a so–called quarkyonic phase of baryonic matter exists where quarks and gluons are still confined but chiral symmetry is restored.

1 We still believe in the general theorems of thermodynamics that strict spontaneous symmetry breaking requires an infinite volume but to analyze the arguments of ref. [1, 2] let us, for the time being, assume that spontaneous symmetry breaking could take place in a finite volume.
It is thus very unlikely that confinement restricts chiral symmetry breaking to the interior of hadrons. The discussion in ref. [2] does, unfortunately, not illuminate the specific manner in which confinement would trigger such a restriction. The mere feature that condensates are extracted from hadron properties seems irrelevant. This is true for other global quantities and external fields.

The above considerations of empirical facts strongly suggest that the interpretation of condensates as hadron properties is invalid. And indeed, currently accepted baryon models manifest the opposite picture. The quark condensate is non-zero outside the baryon while it vanishes or is, at least, substantially reduced inside the hadrons, i.e. chiral symmetry is spontaneously broken outside the hadrons and at least partially restored inside.

To investigate condensates as eventual hadron properties, the model must allow for a localized description of hadrons. We will present such a model calculation in the next section.

The authors of ref. [1, 2] also refer to the superconductor to support their interpretation that the condensates are confined to a finite volume. Of course, an actual superconductor has a finite volume. However, compared to the characteristic wave length of the electrons in the superconductor the latter can, of course, be considered as infinitely extended. The superconductor is actually a quite realistic analog to the QCD vacuum. In fact, there is lattice evidence [9] that confinement is realized by the dual Meissner effect implying that the QCD vacuum represents a dual superconductor. In this analogy the occupied electron levels below the Fermi surface correspond to the occupied quark levels of the Dirac sea. Usually the electron (cooper-)pair condensate is spread out through the whole superconductor and external magnetic fields are expelled by the Meissner effect. If, however, an external magnetic field is applied to a type II superconductor and the field strength exceeds a certain critical value the magnetic field can penetrate the superconductor in form of magnetic vortices (flux tubes). At the position of the vortices the superconducting phase is destroyed and the U(1)–symmetry is restored. The magnetic vortices have the same effect on the superconductor as the valence quarks on the QCD vacuum: Their presence causes a local restoration (at least partially) of the symmetry that is spontaneously broken outside the hadrons and at least partially restored inside.

The quark condensate can be obtained from the functional

$$\langle \bar{q} q \rangle := \frac{\int D[\bar{q}] D[q] \bar{q} q \exp(iA)}{\int D[\bar{q}] D[q] \exp(iA)}$$  (1)

with $A = A[\bar{q}, q]$ being the underlying action of the quark field $q$.

**Model definition** Here we will utilize the Nambu–Jona–Lasinio (NJL) model for the quark flavor dynamics [10] and consider the soliton picture for baryons, see refs. [11, 12] for comprehensive reviews on these topics. The two–flavor NJL model for the quark flavor dynamics contains chirally invariant scalar and pseudoscalar self–interactions

$$\mathcal{L}_{NJL} = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma + i \bar{q} D \gamma^\mu \sigma \partial_{\mu} q$$

In this model $G$ is the strength of the quark self–interaction and $m_0$ refers to the (small) current quark mass matrix. We assume two light flavors $(\bar{q} = (\bar{q}, \bar{d}))$ so that $\tau_{1,2,3}$ are the Pauli matrices and $\tau_0 = 1$. The model can be bosonized by standard functional methods resulting in an effective theory for scalar $(S)$ and pseudoscalar $(P)$ matrix fields. This yields

$$\int D[\bar{q}] D[q] \exp \left( i \int d^4 x \mathcal{L}_{NJL} \right) = \int D[S] D[P] \exp \left( i A_B \right)$$

with the bosonized action

$$A_B = -\frac{1}{4G} \int d^4 x \text{tr} \left[ (S - m_0)^2 + P^2 \right]$$

$$+ \frac{i}{4G} \text{tr} \left[ (S - m_0)^2 + P^2 \right]$$  (4)

In the above expression Tr includes the discretized trace over flavor, color and Dirac indices as well as spatial integration. Its subscript indicates regularization of the quartically divergent functional. The gluon exchange of QCD is modeled by the quark self–interaction, that does not depend on the color charge of the quarks. Hence the color trace merely produces a factor $N_c$.

We evaluate the path integral, eq. (3) in the saddle point approximation. That is, we consider the meson configuration $(S_{sp}, P_{sp})$ that minimizes the action, $A_B$, subject to a prescribed baryon number. In this framework eq. (11) reads

$$\langle \bar{q} q \rangle = \frac{\int D[\bar{q}] D[q] \bar{q} q \exp(iA_{sp})}{\int D[\bar{q}] D[q] \exp(iA_{sp})},$$

with the saddle point action

$$A_{sp} = \int d^4 x \bar{q} \left[ i \partial \bar{q} - (S_{sp} + i\gamma_5 P_{sp}) \right] q$$

$$- \frac{1}{4G} \int d^4 x \text{tr} \left[ (S_{sp} - m_0)^2 + P_{sp}^2 \right] - A_0.$$

We have subtracted a counterterm, $A_0$, such that the saddle point action (density) vanishes for the vacuum configuration of the meson fields. In the language of Feynman diagrams $A_0$ is the fermion loop without external legs.
As in any flat-space field theory it is purely kinematical and dispensed of any dynamics. Similarly to the local integral it exactly cancels in the ratio, eq. (3). Standard functional techniques yield

$$\langle \overline{q} q \rangle = \frac{\delta}{\delta J} \left\{ i \text{Tr} \Lambda \log |i\partial - J - (S + i\gamma_5 P)| \right\}_{J=0}.$$  

(7)

In what follows we will omit the subscripts for the saddle point approximation.

Since the scalar fields couple to \( \overline{q} q \), we do not have to take the detour, eq. (7), of introducing an external source but may directly compute

$$\langle \overline{q} q \rangle = \frac{\delta}{\delta S_{ij}} \left\{ i \text{Tr} \Lambda \log |i\partial - (S + i\gamma_5 P)| \right\}.$$  

(8)

**Vacuum and meson sector** The ground state of the model is obtained from the variational principle \( \frac{\delta}{\delta \Phi} \). Standard functional techniques yield

$$\text{Vacuum and meson sector}$$

$$\text{Ground state is}$$

$$\text{translated}$$

$$\text{invariant}$$

$$\text{under}$$

$$\text{symmetry arguments}$$

$$\text{require}$$

$$\text{ground state to be}$$

$$\text{flavory and symmetric}$$

$$\text{and homogenous,}$$

$$\text{since any non-zero vacuum expectation value of the}$$

$$\text{scalar field provides mass for the fermions,}$$

$$\text{which upon}$$

$$\text{propagation relates the model parameters to the pion}$$

$$\text{mass.}$$

$$\text{The variation with respect to the scalar field}$$

$$\text{S}_{ij} \text{results in the gap-equation}$$

$$m = m_0 - 2G(\bar{u}u)_V \text{that embodies the vacuum up-quark condensate which upon proper time regularization}$$

$$\text{becomes}$$

$$\langle \overline{u} u \rangle_V = -\frac{N_C}{4\pi^2} m^3 \Gamma \left( -1, \frac{m^2}{\Lambda^2} \right).$$  

(9)

Of course, isospin symmetry yields \( \langle \overline{u} u \rangle_V = \langle \overline{d} d \rangle_V \). Pion properties are determined by expanding the action to quadratic order in \( P \). The pole condition for the resulting propagator relates the model parameters to the pion decay constant and the pion mass

$$f_\pi^2 = \frac{m_NC}{4\pi^2} \int_0^1 dx \Gamma \left( 0, \frac{m^2 - x(1-x)m_\pi^2}{\Lambda^2} \right)$$  

(10)

and

$$m_\pi^2 f_\pi^2 = \frac{m_0 m}{G}.$$  

(11)

Then the gap-equation can be framed as

$$m = m_0 \left[ 1 + \frac{N_C}{2\pi^2 f_\pi^2 m_\pi^2} \Gamma \left( -1, \frac{m^2}{\Lambda^2} \right) \right].$$  

(12)

Given a value for the constituent quark mass \( m \), the equations (10), (11) and (12) determine the model parameters from \( f_\pi = 93 \text{MeV} \) and \( m_\pi = 135 \text{MeV} \). In turn, the quark condensate is computed from equation (9). The above equations refer to the physical case of non-zero pion mass. The chiral limit \( m_0 \to 0 \) can be smoothly approached. Then eq. (11) is trivially fulfilled and eq. (12) turns into

$$m = (N_cm_\pi^2/2\pi^2) G(-1, m^2/\Lambda^2),$$

fixing the coupling \( G \) for given values of \( m \) and \( \Lambda \), cf. eq. (13).

In this treatment pions are considered as plane waves. Constructing localized objects as a wave-packet does not alter any of the above equations. In particular, the condensate, eq. (9) remains translationally invariant.

**Baryon as soliton** The soliton picture for baryons is well established and successful \( \text{[12]} \). For the current objective it is particularly attractive because the large \( N_C \) expansion \( \text{[1]} \) establishes a framework to consider baryons as localized objects that avoids the introduction of wave-packets. The solitons \( \text{[14]} \) of the NJL-model are discussed at length in ref. \( \text{[13]} \). They are static configurations of the matrix fields \( S \) and \( P \) and give rise to a stationary Dirac Hamiltonian, \( h \) with single particle energy eigenvalues \( \epsilon_\nu \). From these the static energy functional is obtained as \( \text{[16]} \),

$$E_F = N_C\theta(\epsilon_{\text{val}}) \epsilon_{\text{val}} + N_C \frac{4\sqrt{\pi}}{4\sqrt{\pi}} \sum_\nu |\epsilon_\nu| \Gamma \left( -\frac{1}{2}, \frac{(\epsilon_\nu)^2}{\Lambda^2} \right),$$  

(13)

again, for proper time regularization. The valence level \( \text{(val)} \) is the state of largest binding due to the static background (soliton) field. If its energy is positive, explicit occupation provides baryon number. Otherwise, the polarized vacuum carries the baryon charge. The valence quark contribution is fully contained in \( A_B \) since the prescribed baryon number determines the sum of the occupation numbers that occur in the computation of the functional integral, eq. \( \text{[18]} \). Minimizing the total energy singles out the level of largest binding.

For simplicity, and to avoid the subtle issue of scalar instabilities in this model, we consider the hedgehog configuration on the chiral circle. This results in the Dirac Hamiltonian

$$h = \bar{\alpha} \cdot \vec{p} + m_\beta \left[ \cos \Theta(r) + i \gamma_5 \vec{r} \cdot \hat{r} \sin \Theta(r) \right].$$  

(14)

The chiral angle \( \Theta(r) \) self-consistently minimizes the total energy functional

$$E[\Theta] = E_F + 4\pi m_\pi^2 f_\pi^2 \int_0^{2\pi} dr r^2 \left[ 1 - \cos\Theta(r) \right].$$  

(15)

The soliton can also be constructed in the chiral limit. No qualitative differences emerge \( \text{[13]} \).

Similar to the vacuum sector we find the quark condensate from a functional derivative of the action (energy) in

\( \text{[1]} \) Here we omit subleading contributions. They arise from fluctuations about the soliton.\( \text{[13]} \) In the current saddle point approximation this cancellation occurs inevitably in contrast to its innuendo as a mathematical mistake in ref. \( \text{[14]} \).
the baryon number one sector. Assuming formally that \(\cos \Theta\) and \(\sin \Theta\) are independent fields, the condensate in the soliton (baryon) background is obtained from

\[
\frac{\delta E_F}{\delta \cos \Theta}
\]

to be

\[
\langle u\bar{u} \rangle_S = \frac{N_C}{2} \int \frac{d\Omega}{4\pi} \left\{ \theta(\epsilon_{\text{val}}) \Psi_{\text{val}}^\dagger(\vec{r}) \beta \Psi_{\text{val}}(\vec{r}) 
- \frac{1}{2} \sum \nu \text{sign} (\epsilon_{\nu}) \text{erfc} \left( \frac{\epsilon_{\nu}}{\Lambda} \right) \Psi_{\nu}^\dagger(\vec{r}) \beta \Psi_{\nu}(\vec{r}) \right\}.
\]

The additional factor \(\frac{1}{2}\) arises because the functional trace from which \(E_F\), eq. (13) is extracted includes isospin and \(\langle u\bar{u} \rangle_S = \langle d\bar{d} \rangle_S\) in leading order of the \(1/N_C\) expansion. The computation of \(\langle u\bar{u} \rangle_S\) in this model is not fully new (cf. refs. in \(\cite{15,17}\)). Here we particularly focus on its relation to the vacuum sector.

The relative simplicity of the expressions for the quark condensates, eqs. (9) and (16) make this model a perfect candidate to examine recent ideas on condensates as hadron properties \(\cite{1,2}\).

**Numerical Results**

In the numerical evaluation of the above expressions we consider the physical value \(N_C = 3\), though our analysis refers to the large \(N_C\) limit.

We display numerical results for a typical value of \(m = 450\text{MeV}\) in figure 1. This value of the constituent quark mass is large enough to provide dynamical binding\(^4\), i.e. \(E[\Theta] - E[0] < N_C m\). The soliton is localized in the regime \(r \leq 2/m\). Outside that regime the condensate equals that of the meson sector, i.e., the non–zero translationally invariant vacuum result. We observe finite size effects as mitigations of the condensate far away from the soliton. Obviously the region of deviation from the non–zero vacuum value decreases as the (unphysical) system size is increased. Definitely, this analysis shows that the mitigation at large distances is a mere artifact of the numerical procedure that makes no statement about the actual behavior of the condensate.

As seen from figure 2 inside the soliton regime the condensate is dominated by the valence quark contribution. This contribution actually over–compensates the vacuum result, which indeed tends to increase (in magnitude) in region of the baryon. This is a clear indication that it is actually the in–hadron region, where the condensates are modified by the interactions that bind quark to hadrons. On the contrary the hadronless regime possesses constant non–zero condensates as the comparison with baryon charge density clearly demonstrates. The condensate persists in areas of zero charge density but is distorted in the domain of the hadron. This reverses the conjecture of refs. \(\cite{1,2}\).

In the spirit of refs. \(\cite{1,2}\) one might challenge this model for its lack of confinement. As far as the existence of asymptotic quark states concerns, confinement is indeed lacking. Note however, that matrix elements of any non–singlet color operator vanish (by averaging) in the model. In addition to the above observed non–correlation between baryon charge density and condensate this suggests that confinement is not essential for the existence of condensates. It is generally accepted that confinement is due to the gluon sector of QCD, while chiral symmetry breaking is due to the interaction between the quarks (mediated by the gluons).
CONCLUSION

We have explained that recent interpretations of QCD condensates being "in–hadron" quantities contradict the phenomenology of low–energy strong interactions. Furthermore we have presented a model calculation for the quark condensate that strongly supports the understanding that these condensates are vacuum properties that are significantly distorted or reduced in hadrons. We recall that this picture is fully affirmed by lattice QCD measurements of the quark condensate in the presence of static sources [18].

Refs [1, 2] suggest that the in–hadron nature of condensates were due to QCD being a confining theory, a property not manifest in our model calculation. Yet the explicit formulas in those papers do not reveal the specific role of confinement. To a large extend ref. [2] relies on relations between matrix elements of QCD operators, the QCD vacuum and a momentum eigenstate with pion quantum numbers. These relations result from chiral symmetry and are also valid in our model. In ref [1] the finite range of the hadron wave–function in the light–front formulation is crucial. While there is no straightforward relation between light–front and instantaneous wave–functions [10], the current model calculation does not indicate that the range of the quark condensate is related to that of the nucleon wave–function (represented by the baryon density).

Apparently the subtle puzzle of the too large contribution from condensates to the cosmological constant remains. We note, however, that the non–zero condensates are quantum effects, as is the zero point energy contribution to the cosmological constant. This suggests that any solution to the puzzle ultimately requires a consistent formulation of quantum gravity rather than reinterpreting entries of the classical energy momentum tensor.

Acknowledgments

We acknowledge comments from the authors of ref. [2] on a draft version of this manuscript. One of us (HW) acknowledges correspondence with R.L. Jaffe. We thank T. Schäfer for substantiating and encouraging comments on the first version of this manuscript. This work is supported in parts by the German Science Foundation (DFG-Re 856/6-3) and the National Research Foundation of South Africa.

[1] S. J. Brodsky, R. Shrock, Phys. Lett. B 666 (2008) 95. arXiv:0803.2541 [hep-th]; arXiv:0803.2554 [hep-th]; Proc. Nat. Acad. Sci. 108 (2011) 45.
[2] S. J. Brodsky, C. D. Roberts, R. Shrock, P. C. Tandy, Phys. Rev. C 82 (2010) 022201.
[3] A. Casher, L. Susskind, Phys. Rev. D 9 (1974) 436.
[4] C. D. Roberts, A. G. Williams, Prog. Part. Nucl. Phys. 33 (1994) 477.
[5] R. Alkofer, L. von Smekal, Phys. Rept. 353 (2001) 281.
[6] C. S. Fischer, J. Phys. G 32 (2006) R253.
[7] C. D. Roberts, M. S. Bhagwat, A. Holl, S. V. Wright, Eur. Phys. J. ST 140 (2007) 53.
[8] L. McLerran, R. D. Pisarski, Nucl. Phys. A 796 (2007) 83.
[9] G. S. Bali, V. Bornyakov, M. Müller-Preussker, K. Schilling, Phys. Rev. D 54 (1996) 2863. J. Greensite, Prog. Part. Nucl. Phys. 51 (2003) 1.
[10] D. Ebert, H. Reinhardt, Nucl. Phys. B 271 (1986) 188.
[11] R. Alkofer, H. Reinhardt, Lect. Notes Phys. m33 (1995) 1.
[12] H. Weigel, Lect. Notes Phys. 743 (2008) 1.
[13] S. J. Brodsky, C. D. Roberts, R. Shrock, P. C. Tandy, arXiv:1202.2376 [nucl-th].
[14] H. Reinhardt, R. Wünsch, Phys. Lett. B 215 (1988) 577.
[15] R. Alkofer, H. Reinhardt, H. Weigel, Phys. Rept. 265 (1996) 139.
[16] H. Reinhardt, Nucl. Phys. A 503 (1989) 825.
[17] M. Wakamatsu, Phys. Rev. D 46 (1992) 3762.
[18] W. Bürger, M. Faber, H. Markum, M. Müller, Phys. Rev. D 47 (1993) 3034; M. Faber, H. Markum, S. Olejnik and W. Sakuler, Nucl. Phys. Proc. Suppl. 42 (1995) 487.
[19] G. A. Miller, B. C. Tiburzi, Phys. Rev. C 81 (2010) 035201.