New strongly regular graphs derived from the $G_2(4)$ graph

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1 Abstract and introduction

We consider simple loopless finite undirected graphs. Such a graph is called strongly regular with parameter set $(v, k, \lambda, \mu)$, for short a srg$(v, k, \lambda, \mu)$, iff it has exactly $v$ vertices, each of them has exactly $k$ neighbours, and the number of common neighbours of any two different vertices is $\lambda$ if they are neighbours and $\mu$ otherwise.

The $G_2(4)$ graph is a well-known srg$(416,100,36,20)$. In this article, we explicitly construct it and a certain subgraph $E$ induced by 320 vertices in the same way as in [7]. We discover some interesting properties of $E$ and derive two strongly regular graphs from it: A subgraph $F$ induced by 256 vertices which is a srg$(256,60,20,12)$, and a supergraph $H$ which is a srg$(336,80,28,16)$ and where $E$ is an induced subgraph of $H$.

While $H$ seems to be completely unknown up to now, $F$ is isomorphic to objects described in [1], section Subgraphs, subsection c) Unions of 16 16-cocliques; but the strong regularity has been unnoticed before.

Several propositions in this article have been checked by executing the additionally (in the source package) provided program G24DGS and the program Dreadnaut from the popular graph theoretic software nauty (by Brendan McKay and Adolfo Piperno).

2 Notation

Universal operator symbols (e.g. the equal sign) are sometimes also used to connect logical (Boolean) operands.

One can and we will understand a simple graph $G$ on a vertex set $V$ as a subset of $V \times V$ and express the fact that $x, y \in V$ are adjacent (are neighbours, are connected (by an edge)) with respect to $G$ by $G(x, y)$.

Using this notation, $G$ is an srg$(v, k, \lambda, \mu)$ iff

$$|V| = v \land \forall i \in V : \neg G(i, i) \land |\{j \in V : G(i, j)\}| = k$$

and for all different $i, j \in V$

$$G(i, j) = G(j, i) \land |\{p \in V : G(p, i) \land G(p, j)\}| = \begin{cases} \lambda & \text{if } G(i,j) \\ \mu & \text{otherwise} \end{cases}$$

For integer numbers $a$ and $b$, $[a, b]$ means the set of integer numbers from $a$ up to $b$. This set is empty if $a$ exceeds $b$. 

1
3 A construction of the $G_2(4)$ graph $G$

The following algorithm is an extract of the description in [1], refering to [6]. See also [4] and [9].

“Consider the projective plane $\text{PG}(2,16)$ provided with a nondegenerate Hermitean form. It has 273 points, 65 isotropic and 208 nonisotropic. There are $416 = 208 \cdot 2$ isotropic points on the three lines $ab$, $ac$, and $bc$. [...] $G$ can be described as the graph on the 416 triangles, adjacent when they have 3 points in common.”

The construction given above assigns a set of 15 different isotropic points to each of the 416 vertices/triangles. To be more detailed, it assigns a set of 5 different isotropic points to each non-isotropic point and 3 different non-isotropic points with pairwise disjoint assigned sets of isotropic points to each vertex. For symmetry reasons, from $208 \times 5 = 1040 = 65 \times 16$, 416 is assigned.

The program G24DGS checks that $|B| = 96$, implying $|C| = 416 - 96 = 320$.

Let $E$ be the subgraph of $G$ induced by $C$. Consequently, $E(i,j) = G(i,j)$ for all $i,j \in C$.

The computer program G24DGS, provided in the source package of this article and described in more detail in the last sections before the references, implements the construction of the iso-sets of the members of $V$ and its just defined subsets, checks also relevant cardinalities.

4 Vertex subsets $B$ and $C$ and the subgraph $E$ on 320 vertices

We define $B$ to be the set of all vertices in $V$ containing the index 1 in their iso-set and $C$ to be the set of the remaining vertices. From the previous section, we get $|B| = 96$, implying $|C| = 416 - 96 = 320$.

Let $E$ be the subgraph of $G$ induced by $C$. Consequently, $E(i,j) = G(i,j)$ for all $i,j \in C$.

The computer program G24DGS, provided in the source package of this article and described in more detail in the last sections before the references, implements the construction of the iso-sets of the members of $V$ and its just defined subsets, checks also relevant cardinalities.

5 A four-level hierarchy on $C$ with respect to $E$

Let $X$ be the set of the 16 non-isotropic points to which the isotropic point with index 1 is assigned. Consequently, to each vertex in $B$ at least one element of $X$ is assigned.

For each vertex $i \in C$, let $W(i)$ be the set of those elements of $X$ that are not assigned to any neighbour of $i$ in $B$.

The program G24DGS checks that

$$\forall i \in C : |W(i)| = 4$$

and constructs a bijective function $M$ that for each $r \in [0,4]$ and $s,t,u \in [0,3]$ returns a vertex $M(r,s,t,u) \in C$ such that for all $r_1, r_2, r_3 \in [0,4]$ and $s_1, s_2, s_3, t_1, t_2, u_1, u_2 \in [0,3]$ the following propositions hold:

$$|W(M(r_1,s_1,t_1,u_1)) \cap W(M(r_2,s_2,t_2,u_2))| = \begin{cases} 4 & \text{if } r_1 = r_2 \land s_1 = s_2 \\ 0 & \text{if } r_1 = r_2 \land s_1 \neq s_2 \\ 1 & \text{if } r_1 \neq r_2 \end{cases}$$

$$\neg E(M(r_1,s_1,t_1,u_1), M(r_1,s_1,t_1,u_2))$$

$$t_1 \neq t_2 \Rightarrow E(M(r_1,s_1,t_1,u_1), M(r_1,s_1,t_2,u_2))$$
\[ s_1 \neq s_2 \Rightarrow \neg E(M(r_1, s_1, t_1, u_1), M(r_1, s_2, t_2, u_2)) \tag{5} \]

\[ r_1 \neq r_2 \Rightarrow |\{(t, u) : t, u \in [0, 3] \land E(M(r_1, s_1, t_1, u_1), M(r_2, s_2, t, u))\}| = 1 \tag{6} \]

\[
|\{(t, u) : t, u \in [0, 3] \land E(M(r_1, s_1, t_1, u_1), M(r_3, s_3, t, u)) \land E(M(r_1, s_1, t_2, u_2), M(r_3, s_3, t_2, u))\}|
\]

\[
= \begin{cases} 
0 & \text{if } t_1 = t_2 \land u_1 \neq u_2 \\
1 & \text{if } t_1 \neq t_2 
\end{cases} \tag{7} 
\]

### 6 Some reformulations and conclusions

Here and in the program G24DGS, we will call
\[ \{M(r, s, t, u) : t, u \in [0, 3]\}, \text{ where } r \in [0, 4] \land s \in [0, 3], \text{ a cell}, \]
\[ \{M(r, s, t, u) : u \in [0, 3]\}, \text{ where } r \in [0, 4] \land s, t \in [0, 3], \text{ a cell part}, \]
\[ \{\{M(r, s, t, u) : t, u \in [0, 3]\} : s \in [0, 3]\}, \text{ where } r \in [0, 4], \text{ a cell set}. \]

Equations (3) and (4) imply in particular that the subgraph of \(E\) induced by one the 20 cells is a complete fourpartite graph with part size four, usually denoted by \(K_{4,4,4,4}\). And there is no edge between vertices in different cells within the same cell set.

In each cell we can take a vertex from each part and get a 4-clique. There are \(4^4 = 256\) such combinations in each cell, 1024 in each cell set. By (6), each vertex has exactly one neighbour in each of the four parts of a cell in a different cell set. Together, those five vertices induce a 5-clique.

Each cell part is a 4-coclique. For each cell set, we can take a part from each of the four cells and those 16 vertices are pairwise non-adjacent. There are \(4^4 = 256\) such 16-cocliques in a single cell set, 1280 of them in all five.

For each cell set, we can construct \((4 \times 3 \times 2)^3 = 13824\) different divisions into four 16-cocliques (by fixing the selection of the parts in the first cell and combining all permutations of the parts in each of the three other cells). For all five sets, that gives \(13824^5\) different divisions of \(C\) into twenty 16-cocliques.

Each vertex has exactly \(3 \times 4 = 12\) neighbours in its own cell, none in other cells of the same cell set, and, by (6), exactly 4 neighbours in each cell in the other four cell sets, summing up to \(12 + 4 \times 4 \times 4 = 76\).

Two different vertices in the same cell part do have \(3 \times 4 = 12\) common neighbours within that cell and, by (5) and (7), no others. Two vertices in different parts of a cell do have \(2 \times 4 = 8\) common neighbours within that cell and, by (5), none in the other cells in the same set, and, by (7), exactly one common neighbour in each cell in the other four cell sets, summing up to \(8 + 4 \times 4 \times 4 = 24\).

### 7 The induced subgraph \(F\) of \(E\) on 256 vertices

We choose four of the five cell sets and consider the subgraph of \(E\) induced by the contained 256 vertices. Because there are five such choices, we get actually five graphs. And the program G24DGS does check each of them. But it also delivered input data for the program Dreadnaut (part of the software nauty) which then has been used to check that those five graphs are (pairwise) isomorphic. So we can speak of just one graph and name it \(F\).
7.1 Some properties of $F$

The only change compared to $E$ is the exclusion of the vertices in one cell set and of the incident edges. So we have to modify the calculations in the corresponding subsection just a little bit.

For instance, we can construct $4 \times 256 = 1024$ different 16-cocliques and use them to enumerate $13824^4$ different divisions of the vertex set of $F$ into sixteen 16-cocliques.

Each vertex has exactly $12 + 3 \times 4 \times 4 = 60$ neighbours.

Just as for $E$, two different vertices in the same cell part have 12 common neighbours. Two vertices in different parts of a cell have $8 + 3 \times 4 = 20$.

In order to complete the proof that $F$ is a srg$(256,60,20,12)$, we would have to estimate the numbers of common neighbours of vertices in different cells too. An older unpublished version of the program G24DGS did just that. But those checks were removed because they were not as simple as and not faster than the universal srg check routine that is used now.

8 The supergraph $H$ of $E$ on 336 vertices

Let $D = \{d(x) : x \in X\}$ be a set of 16 additional vertices not contained in $V$. Let $H$ be the graph on $C \cup D$ (320+16=336 elements) satisfying

$$\forall i_1, i_2 \in D : \neg H(i_1, i_2) \tag{8}$$

$$\forall i_1, i_2 \in C : H(i_1, i_2) \iff E(i_1, i_2) \tag{9}$$

$$\forall i \in C, x \in X : H(i, d(x)) \iff H(d(x), i) \iff x \in W(i) \tag{10}$$

By (8) and another result in a previous section, each vertex $i \in C$ has exactly 76 neighbours in $C$.

By (9), (10), and (1), the only additional neighbours are four vertices in $D$. Thus, $i$ has exactly 80 neighbours. For the complete check that $H$ is a srg$(336,80,28,16)$, we refer again to the dedicated routine call in the program G24DGS.

9 The provided program G24DGS

9.1 Mathematical foundations of the implementation

The program uses elements of the three-dimensional vector space over the finite field $\text{GF}(16)$ to constitute and represent the objects in the projective plane $\text{PG}(2,16)$, a very common way. For an introduction to projective planes, see e.g. [5].

The applied Hermitean form takes three-dimensional vectors $a$ and $b$ over $\text{GF}(16)$ and returns $a_1b_3 + a_2b_2 + a_3b_1$, where addition, multiplication, and conjugation operate in $\text{GF}(16)$.

9.2 Compiling and executing

The source code file G24DGS.PAS has been developed for PASCAL compilers compatible with Turbo Pascal 4.0. Lines are at most 78 characters long. For inspections the use of an ASCII compatible monospaced font is strongly recommended. The intended indentation is by one character per structure level, using blanks (instead of tabs).

The program includes a good portion of comments (enclosed in curly braces). So it should be fairly understandable at least by readers knowing at least one imperative programming language.

In order to allow to exclude unwanted (e.g. long running) tasks from execution without modifying the source text, certain parts of the program are compiled (and executed) only under the condition that a certain symbol has been defined during the compilation:
The symbol CHKSRG enables checking the SRG properties of the constructed graphs that are claimed to be SRGs \( (G, H, \text{and all 5 five representations of } F) \).
The symbol WRIDRE enables writing of input files for Dreadnaut, for each of the constructed graphs. Caution: In that case, the program would write files named \( G24.DRE, 320.DRE, 336.DRE, 256_0.DRE, 256_1.DRE, 256_2.DRE, 256_3.DRE, \text{and } 256_4.DRE \) into the current (working) directory without explicit confirmation.
The symbol CNTCLI enables the counting of cliques of sizes 2 to 7 for each of the constructed graphs.

The program has been successfully compiled and executed on a 1 GHz Intel PIII PC running MS Windows 98 SE. These are the used compilers and the respective four execution times in seconds (roughly measured with some overhead utilizing the PC clock (resolution: 0.055 s)) from compilations with none of the three symbols, WRIDRE, CHKSRG, or CNTCLI defined:
Turbo Pascal 5.5 : 3.02 / 3.73 / 36.03 / 99.52
Turbo Pascal 7.01 : 1.43 / 1.81 / 34.11 / 97.38
Borland Delphi 4.0 build 5.37 : 0.82 / 0.93 / 3.85 / 9.78
Virtual Pascal 2.1 build 279 : 0.83 / 1.10 / 6.15 / 15.27
Free Pascal 2.4.4 i386-Win32 : 0.77 / 0.99 / 5.60 / 14.39

To avoid a compilation result depending on the settings you could use the command line versions of the compilers (TPC for Turbo Pascal, BPC for Borland Pascal 7, DCC32 for Borland Delphi (32 bit versions; do not miss to use the -CC option in order to generate a console executable), VPC for Virtual Pascal, FPC for Free Pascal) instead of the compilers integrated in the IDEs.
The appropriate string to define a symbol for a command line compiler is usually the concatenation of a certain prefix and the name of that symbol. The appropriate prefix is -D (alternatively -d) for compilers from Borland (Turbo Pascal, Delphi), and -d for Free Pascal. For example, an invocation of Free Pascal could look like
```
fpc G24DGS -dCHKSRG -dWRIDRE -dCNTCLI
```
In general, the Pascal compilers are not case sensitive with respect to symbols (and file names).

### 9.3 Input and output

The program ignores any command line parameters or inputs other than pressing Ctrl-C to cancel the execution.
Despite the optional writing of input files for Dreadnaut mentioned above, it writes only to the standard output device. In the default case that will be the monitor screen. But you can redirect the output to a file.
The success of all performed checks and other operations is indicated by this finishing line:
```
== Regular program stop ==
```
These are the displayed clique counts of sizes 2 to 7:
\( G \): 2:20800 3:249600 4:873600 5:698880 6:0 7:0
\( E \): 2:12160 3:107520 4:261120 5:129024 6:0 7:0
\( H \): 2:13440 3:125440 4:330240 5:201984 6:9024 7:0
\( F \): 2:7680 3:51200 4:81920 5:15360 6:0 7:0

## 10 Results from Dreadnaut runs

The binary executed under MS Windows 98 SE on a 1 GHz Intel PIII was dreadnautB.exe, (according to the starting line) compiled from version 2.2 of Dreadnaut for 32-bit processors, variant BIG, and contained in the GAP package GRAPE 4r6p1 (downloaded via [10]).
In separate runs, each of the eight files written by G24DGS has been used as input stream (by redirection). In the case of the five graphs on 256 vertices, the full data of the canonically labelled graph has been written to another (text) file. By simple file comparison, the equality of the canonically labelled graphs and that way indirectly the isomorphicity of the five original graphs has been checked. The following short list gives for each of the graphs a small extract of the runs: the size of the automorphism group and (enclosed in brackets) the check sum of the canonically labelled graph:

\[ G: 503193600 \ [4ef1998 7b631ca b27cc78] \]
\[ E: 368640 \ [9ef74b4 67d5649 fd37dc0] \]
\[ H: 3840 \ [6940cf2 40d3533 c70f0b4] \]
\[ F: 368640 \ [5422323 c601302 4629513] \]

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http://edn.embarcadero.com/museum/antiquesoftware#
For downloading one has to register or sign-in.

[12] *Virtual Pascal* (Closed Source freeware)
One ZIP-file including binaries and documentation for Win32, OS/2, and Linux
Official forum:
http://vpascal.ning.com/
Forum entry *Where can I download VP?*:
http://vpascal.ning.com/forum/topic/show?id=854411%3ATopic%3A9

[13] *Free Pascal* (Open Source freeware)
Sources, documentation, and binaries for several systems
http://www.freepascal.org

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