SINGULAR SOLUTIONS OF A LANE-EMDEN SYSTEM

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Abstract. In this work we consider the existence of positive singular solutions
\[
\begin{aligned}
-\Delta u_1 &= \lambda_1 |\nabla u_2|^p \quad \text{in } \Omega, \\
-\Delta u_2 &= \lambda_2 |\nabla u_1|^q \quad \text{in } \Omega, \\
 u_1 = u_2 &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

where \( \Omega \) is small \( C^2 \) perturbation of the unit ball \( B_1 \) in \( \mathbb{R}^N \) and \( \lambda_i \) are positive constants. Under suitable conditions on \( p \) and \( q \) we prove the existence of positive singular solutions of (1). We also examine the case where one or both of \( u_1, u_2 \) are Hölder continuous.

1. Introduction. In this work we consider the existence of singular solutions of
\[
\begin{aligned}
-\Delta u_1 &= \lambda_1 |\nabla u_2|^p \quad \text{in } \Omega, \\
-\Delta u_2 &= \lambda_2 |\nabla u_1|^q \quad \text{in } \Omega, \\
 u_1 = u_2 &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

where \( p, q > 0 \), \( \Omega \) is small \( C^2 \) perturbation of the unit ball \( B_1 \) in \( \mathbb{R}^N \) and \( \lambda_i \) are positive constants. In the case of \( \Omega = B_1 \) (the unit ball) one can write out explicit solutions of (2) which have various degrees of regularity at the origin. The following example shows that, depending on the exact parameters involved, one can have both \( u_1, u_2 \) singular; one singular and one Hölder continuous; or both Hölder continuous.

Example 1.1. (An explicit radial solution) We look for explicit singular solutions of (2) on the punctured unit ball in \( \mathbb{R}^N \). Define \( w_2(r) := C_1(r^{-t} - 1) \), \( w_2(r) := C_2(r^{-\tau} - 1) \) where \( C_1 = sgn(t) \), \( C_2 = sgn(\tau) \) and
\[
t := \frac{p + 2 - pq}{pq - 1}, \quad \tau := \frac{q + 2 - pq}{pq - 1}
\]

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1
Part 1. The singular case.

A computation then shows that \((w_1, w_2)\) satisfies
\[
\begin{cases}
-\Delta w_1 = \lambda_1 |\nabla w_2|^p & \text{in } B_1 \setminus \{0\}, \\
-\Delta w_2 = \lambda_2 |\nabla w_1|^q & \text{in } B_1 \setminus \{0\}, \\
w_1 = w_2 = 0 & \text{on } \partial B_1.
\end{cases}
\]

Whether this is some sort of weak solution on the full ball depends on the exact parameters involved.

We are interested in various types of singular solutions of (2), either solutions which blow up in the \(L^\infty\) norm or solutions which are Hölder continuous but are not classical solutions. Note this is fully determined by the sign of \(t\) and \(\tau\). We will also require \(\lambda_1, \lambda_2 > 0\) for our approach; this will be taken into account in the following computations.

Part 1. The singular case \((t, \tau > 0)\). A computation shows that if \(pq < 1\) then we cannot have both \(t, \tau > 0\) and hence we assume \(pq > 1\). By symmetry we can assume \(p > 1\). Then to have \(t, \tau, \lambda_k\) positive we need
\[
\max \left\{ \frac{N}{p(N-1)-1}, \frac{1}{N-1} + \frac{N}{p(N-1)} \right\} < q < \min \left\{ 1 + \frac{2}{p}, \frac{2}{p-1} \right\}.
\]

Part 2. The Hölder continuous case \((t, \tau < 0)\) (and \(\lambda_k > 0\)). For this we consider two cases.

Case 1. The first case we assume \(pq > 1\) (and hence by symmetry we can assume \(p > 1\)). Then we require \(q > \max \left\{ \frac{2}{p} + 1, \frac{2}{p-1} \right\}\).

Case 2. The second case we assume \(pq < 1\) and \(pq < p + 2\) and \(pq < q + 2\). We consider two cases:
\[
\begin{cases}
p > 1, & \text{then we require } q < \min \left\{ \frac{2}{p} + 1, \frac{2}{p-1} \right\} , \\
p < 1, & \text{then we require } q < \frac{2}{p} + 1.
\end{cases}
\]

Part 3. One singular and one Hölder continuous (and \(\lambda_k > 0\)). By symmetry we can assume \(t > 0\) and \(\tau < 0\). It can be shown that we must be in the case of \(pq > 1\). We consider two cases:
\[
\begin{cases}
p > 1, & \text{then we require } q < \min \left\{ \frac{1}{N-1} + \frac{N}{p(N-1)}, \frac{1}{p}, \frac{2}{p-1} \right\} < q < \frac{2}{p} + 1, \\
p < 1, & \text{this set is empty.}
\end{cases}
\]

With the above example in mind we now state our main result.

**Theorem 1.2.** Suppose \(N \geq 3\).

1. Let \(p, q, N, t, \tau, C_1, C_2, \lambda_1, \lambda_2\) be as in Example 1.1 part 1. Then for sufficiently small \(C^2\) perturbations of the unit ball, say \(\Omega_\epsilon\), there exists a positive singular weak solution \((u_1, u_2)\) of (2) (with \(\Omega = \Omega_\epsilon\)) which blows up at exactly one point \(x_\epsilon\) (near the origin) and behaves like \((u_1, u_2)(x) \approx |x - x_\epsilon|^{-\sigma}\) near \(x_\epsilon\). The proof gives the exact behaviour near \(x_\epsilon\).

2. Let \(p, q, N, \sigma, C\) be as in Example 1.1 part 2. Then for sufficiently small \(C^2\) perturbations of the unit ball, say \(\Omega_\epsilon\), there exists a positive weak solution \((u_1, u_2)\) of (2) (with \(\Omega = \Omega_\epsilon\)) with \((u_1, u_2) \in C^\infty(\Omega_\epsilon \setminus \{x_\epsilon\})\) and with \(u \in C^{0,\sigma}(\Omega_\epsilon)\). In addition \(u\) is not in \(C^{0,\sigma+\delta}(\Omega_\epsilon)\) for any \(\delta > 0\).
3. Let \( p, q, N, t, \tau, C_1, C_2, \lambda_1, \lambda_2 \) be as in Example 1.1 part 3 and by symmetry we can assume \( t > 0 \) and \( \tau < 0 \). Then for sufficiently small \( C^2 \) perturbations of the unit ball, say \( \Omega_\varepsilon \), there exists a positive singular weak solution \((u_1, u_2)\) of (2) (with \( \Omega = \Omega_\varepsilon \)) which

(i) \( u_1 \) blows up at exactly one point \( x_\varepsilon \) (near the origin) and behaves like \( u_1(x) \approx |x - x_\varepsilon|^{-\sigma} \) near \( x_\varepsilon \) and \( u_1 \in C^\infty(\Omega_\varepsilon \setminus \{x_\varepsilon\}) \),

(ii) \( u_2 \in C^{0,\sigma}(\Omega_\varepsilon) \). In addition \( u_2 \) is not in \( C^{0,\sigma+\delta}(\Omega_\varepsilon) \) for any \( \delta > 0 \). The proof gives the exact behaviour near \( x_\varepsilon \).

The rest of the paper is organized as follows: First, we consider the case \( t, \tau > 0 \) (i.e. the singular case). In Subsection 1.2 we present our approach for finding singular solutions of (2) for \( y \in \Omega \approx B_1 \) by linearizing around the explicit radial solution \((w_1, w_2)\). In Section 2 after introducing the suitable function spaces \( X \) and \( Y \), we study the linear part of the problem and we prove the existence of some \( C > 0 \) such that for all \((f, g) \in Y\) there is some \((\phi, \psi) \in X\) such that

\[
L(\phi, \psi) = (f, g) \quad \text{in } B_1 \setminus \{0\} \quad \text{with } \phi = \psi = 0 \text{ on } \partial B_1, \tag{4}
\]

and \( \|(\phi, \psi)\|_X \leq C \|(f, g)\|_Y \). In Section 3, we study the nonlinear part of the problem via fixed point’s argument which finalize the existence of positive singular solutions of (2). In Section 4, we claim that one can study the case when \( t, \tau < 0 \) (i.e. the Hölder continuous cases) and the case of one parameter positive and one negative (i.e. one singular and one Hölder continuous), by the same argument in the singular case.

1.1. Background. A well studied problem is the existence versus non-existence of positive solutions of the Lane-Emden equation given by

\[
\begin{align*}
-\Delta u &= u^p \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\tag{5}
\]

where \( 1 < p \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (where \( N \geq 3 \)) with smooth boundary. In the subcritical case \( 1 < p < \frac{N+2}{N-2} \) the problem is very well understood and \( H^1_0(\Omega) \) solutions are classical solutions; see [26]. In the case of \( p \geq \frac{N+2}{N-2} \) there are no classical positive solutions in the case of the domain being star-shaped; see [39]. In the case of non star-shaped domains much less is known; see for instance [12, 18, 19, 20, 38]. In the case of \( 1 < p < \frac{N}{N-2} \) ultra weak solutions (non \( H^1_0 \) solutions) can be shown to be classical solutions. For \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \) one cannot use elliptic regularity to show ultra weak solutions are classical. In particular in [34] (and see also [37]) for a general bounded domain in \( \mathbb{R}^N \) they construct singular ultra weak solutions with a prescribed singular set. We mention that the spaces we worked on are \( L^p \) analogs of the spaces developed in [34], see also [37].

Consider

\[
\begin{align*}
-\Delta_p u(y) &= C|\nabla u(y)|^q \quad &y \in \Omega, \\
u &= 0 \quad &y \in \partial\Omega,
\end{align*}
\tag{6}
\]

where \( \Omega \) is a small \( C^2 \) perturbation of the unit ball in \( \mathbb{R}^N \) and where \( C > 0 \) is a constant. The case \( p = 2 \), \( 0 < q < 1 \) has been studied in [1]. Some relevant monographs for this work include [27, 23, 42]. Many people have studied boundary blow up versions of (6) in the case where \( \Delta_p = \Delta_2 \) and where one removes the minus sign in front of \( \Delta_p \); see for instance [30, 43]. See [3, 4, 5, 6, 1, 2, 7, 8, 9, 10, 11, 21, 22, 24, 25, 28, 29, 40, 32, 33, 35, 36] for more results on equations similar to (6). In particular, the interested reader is referred to Nguyen [35] for recent developments.
and a bibliography of significant earlier work, where the author studies isolated singularities at 0 of nonnegative solutions of the more general quasilinear equation

$$\Delta u = |x|^\alpha u^p + |x|^{\beta} |\nabla u|^q \text{ in } \Omega \setminus \{0\},$$

where $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a $C^2$ bounded domain containing the origin 0, $\alpha > -2$, $\beta > -1$ and $p, q > 1$, and provides a full classification of positive solutions vanishing on $\partial \Omega$ and the removability of isolated singularities. Finally, in [13], For $\frac{(p-1)N}{N-1} < q < p < N$, the authors prove that if $\Omega$ is a sufficiently small $C^2$ perturbation of the unit ball there exists a singular positive weak solution $u$ of (6). For other ranges of $p$ and $q$ they prove the existence of Hölder continuous positive solution (with optimal regularity) on a $C^2$ perturbation of the unit ball.

We now return to (2). The first point is that it is a non variational equation and hence there are various standard tools which are not available anymore. In this work we don’t consider the existence of classical solutions, we only consider the case of singularities at 0 of nonnegative solutions of the more general quasilinear equation

$$-\Delta_{p} u_1(y) = \lambda_1 |\nabla y_2(y)|^p \quad \text{in } y \in \Omega \setminus \{\epsilon \psi(0)\},$$

$$-\Delta_{p} u_2(y) = \lambda_2 |\nabla y_1(y)|^q \quad \text{in } y \in \Omega \setminus \{\epsilon \psi(0)\},$$

$$u_1 = u_2 = 0 \quad \text{on } \partial \Omega \epsilon,$$

it is equivalent to find a positive singular solution of

$$\begin{align*}
-\Delta v_1(x) - E_\epsilon(v_1) &= \lambda_1 |\nabla v_2(x) + \epsilon A(\epsilon, x) \nabla v_2(x)|^p \quad \text{in } B_1 \setminus \{0\}, \\
-\Delta v_2(x) - E_\epsilon(v_2) &= \lambda_2 |\nabla v_1(x) + \epsilon A(\epsilon, x) \nabla v_1(x)|^q \quad \text{in } B_1 \setminus \{0\}, \quad (8)
\end{align*}$$

where $E_\epsilon$ is the second order linear differential operator given by

$$E_\epsilon(v) := 2\epsilon \sum_{i,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_k + \epsilon \sum_{i,k} v_{x_i} \partial_{y_i y_k} \tilde{\psi}_k + \epsilon^2 \sum_{i,j,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_j \partial_{y_j} \tilde{\psi}_k, \quad (9)$$
and $\delta_{ij} = 0$ if $i \neq j$ and is 1 otherwise. Here $A(\varepsilon, x)$ is an $N \times N$ matrix with smooth entries; this matrix is coming from the formula

$$u_{y_i}(y) = v_{x_i}(x) + \varepsilon \sum_{k=1}^{N} v_{x_k} \partial_{y_i} \tilde{\psi}_k.$$  

For small $\varepsilon$ we look for solutions of (8) which are perturbations of the radial solutions $(w_1, w_2)$; namely $v_1 = w_1 + \phi$ and $v_2 = w_2 + \psi$. We then need $(\phi, \psi)$ to satisfy

$$\begin{cases}
-\Delta \phi - p\lambda_1 |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \psi = \lambda_1 |\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)|^p \\
-\lambda_1 |\nabla w_2|^p - p\lambda_1 |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \phi + \varepsilon \lambda_1 \nabla (w_2 + \phi) \text{ in } B_1 \setminus \{0\},
\end{cases}$$

$$\begin{cases}
-\Delta \psi - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \phi = \lambda_2 |\nabla w_1 + \nabla \phi + \varepsilon A(\varepsilon, x)(\nabla w_1 + \nabla \phi)|^q \\
-\lambda_2 |\nabla w_1|^q - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \phi + \varepsilon \lambda_2 \nabla (w_1 + \phi) \text{ in } B_1 \setminus \{0\},
\end{cases}$$

**φ = ψ = 0 on $\partial B_1$.**

To find a solution of this we will use Banach’s fixed point theorem. Towards this we define the nonlinear mapping (of course at this point this is not well defined) $J_{\varepsilon}(\phi, \psi) := (\hat{\phi}, \hat{\psi})$ by

$$\begin{cases}
-\Delta \hat{\phi} - p\lambda_1 |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \hat{\psi} = \lambda_1 |\nabla w_2 + \nabla \hat{\psi} + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \hat{\psi})|^p \\
-\lambda_1 |\nabla w_2|^p - p\lambda_1 |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \hat{\phi} + \varepsilon \lambda_1 \nabla (w_2 + \hat{\phi}) \text{ in } B_1 \setminus \{0\},
\end{cases}$$

$$\begin{cases}
-\Delta \hat{\psi} - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \hat{\phi} = \lambda_2 |\nabla w_1 + \nabla \hat{\phi} + \varepsilon A(\varepsilon, x)(\nabla w_1 + \nabla \hat{\phi})|^q \\
-\lambda_2 |\nabla w_1|^q - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \hat{\psi} + \varepsilon \lambda_2 \nabla (w_1 + \hat{\psi}) \text{ in } B_1 \setminus \{0\},
\end{cases}$$

$$\hat{\phi} = \hat{\psi} = 0 \text{ on } \partial B_1.$$  

(10)  

See Section 2 after the introduction of the function spaces $X, Y$ and the linear operator $L$ for an abstract outline of the fixed point argument.

**2. The linear theory.** For $0 < s \leq \frac{1}{2}$ define $A_s := \{ x \in \mathbb{R}^N : s < |x| < 2s \}$ and for $t, \tau \in \mathbb{R}$ and $\gamma \in (N, \infty)$ we define

$$\|f\|_{Y_s} := \left( \sup_{0 < s \leq \frac{1}{2}} s^{(2+t)\gamma-N} \int_{A_s} |f(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$$

and $\|(f, g)\|_Y := \|f\|_{Y_t} + \|g\|_{Y_{\tau}}$. Similarly we define

$$\|\phi\|_{X_s} := \left( \sup_{0 < s \leq \frac{1}{2}} s^{\gamma-N} \left\{ \int_{A_s} |\phi|^\gamma dx + s^\gamma \int_{A_s} |
abla \phi|^\gamma dx + s^{2\gamma} \int_{A_s} |D^2 \phi|^\gamma dx \right\} \right)^{\frac{1}{\gamma}}$$

(11)  

and we now set $\|(\phi, \psi)\|_X := \|\phi\|_{X_s} + \|\psi\|_{X_{\tau}}$ where for $(\phi, \psi) \in X$ we impose the boundary condition $\phi = \psi = 0$ on $\partial B_1$.  

(12)
Consider the Laplace-Beltrami operator $\Delta_{S^{N-1}} = \Delta_\theta$ on $S^{N-1}$ and the eigenpairs
\[-\Delta_\theta \zeta_k(\theta) = \mu_k \zeta_k(\theta), \quad \theta \in S^{N-1},\]
and the $k$-th eigenvalue of the Laplace-Beltrami operator is $\mu_k = k(k + N - 2)$, and note that $\mu_0 = 0, \psi_0 = 1$ (multiplicity 1); $\mu_1 = N - 1$ with multiplicity $N$ and $\mu_2 = 2N$. Given $f(x)$ we can write
\[f(x) = \sum_{k=0}^{\infty} f_k(r) \zeta_k(\theta).\]

Returning to the nonlinear operator $J_\varepsilon$ we see of crucial importance will be the mapping properties of $L(\phi, \psi)$ where
\[L(\phi, \psi) := (-\Delta_\phi - p\lambda_1 |\nabla w_2|^p - 2 \nabla w_2 \cdot \nabla \psi, -\Delta_\psi - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \phi),\]
and we will write $(L_1(\phi, \psi), L_2(\phi, \psi)) := L(\phi, \psi)$.

The optimal result we’d hope for is the existence of some $C > 0$ such that for all $(f, g) \in Y$ there is some $(\phi, \psi) \in X$ such that

\[L(\phi, \psi) = (f, g) \quad \text{in } B_1 \setminus \{0\} \quad \text{with } \phi = \psi = 0 \text{ on } \partial B_1, \quad (13)\]

(note the boundary condition is built into the function space) and $\|(\phi, \psi)||_X \leq C||(f, g)||_Y$. We will prove this estimate, but the exact details will depend heavily on the exact values of the parameters involved.

**Important note.** From here on we will restrict our attention to the singular case; $t, \tau > 0$. In later sections we will come back and address the other cases. Since $t, \tau > 0$ hence $C_1 = C_2 = 1$. Writing out (13) with $\phi(x) = \sum_{k=0}^{\infty} a_k(r) \zeta_k(\theta)$, $\psi(x) = \sum b_k(r) \zeta_k(\theta)$, $f(x) = \sum f_k(r) \zeta_k(\theta)$ and $g(x) = \sum g_k(r) \zeta_k(\theta)$ we arrive at

\[\begin{cases} 
-a''_k(r) - \frac{N-1}{r} a'_k(r) + \frac{\mu_k}{\tau} a_k(r) + \lambda_1 p \frac{r^{p-1}}{\tau^{p-1}} b'_k(r) = f_k(r) &\text{in } 0 < r < 1, \\
-b''_k(r) - \frac{N-1}{r} b'_k(r) + \frac{\mu_k}{\tau} b_k(r) + \lambda_2 q \frac{r^{q-1}}{\tau^{q-1}} a'_k(r) = g_k(r) &\text{in } 0 < r < 1,
\end{cases} \quad (14)\]

with $a_k(1) = b_k(1) = 0$ where $\beta := (t + 1)(p - 1)$ and $\alpha := (t + 1)(q - 1)$. Note that $\alpha + \beta = 2$ and $t = \tau + \beta - 1$. Notice that $p > 1$ means $\beta > 0$ and $q > 1$ means $\beta < 2$, respectively. We will use the method of variation of parameters to obtain solutions of (14). We now state our main linear theorem and to state the result we first define some quantities. For $k_0$ an integer we set $X_{k_0}$ and $Y_{k_0}$ to denote the closed subspaces of $X$ and $Y$ for which there are no nonzero modes $k \leq k_0$; i.e.

\[f(x) = \sum_{k=k_0+1}^{\infty} f_k(r) \zeta_k(\theta).\]

**Theorem 2.1.** Suppose $k \geq 0$ and $N \geq 3$ and $p, q$ satisfy
\[p > 1, q > 1, \max \left\{ \frac{N}{p(N-1)}, \frac{1}{N-1} + \frac{N}{p(N-1)} \right\} < q < \min \left\{ 1 + \frac{2}{p}, \frac{2}{p-1} \right\} \quad \text{and} \]
\[((\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)(N - 1) - \lambda_1 \lambda_2 pq t^{q-1} r^{p-1} (\beta - 1)) < 0.\]
\[(15)\]

Then there is some large integer $k_0 = k_0(p, q)$ such that there is some $C > 0$ such that all $(f, g) \in Y_{k_0}$ there is some $(\phi, \psi) \in X_{k_0}$ which solves (13) and $\|(\phi, \psi)||_X \leq C||(f, g)||_Y$.

After obtaining this result we can combine this with estimates from the lower modes to obtain the following:
Corollary 1. Suppose $N \geq 3$ and $p, q$ satisfy (15). Then there is some $C > 0$ such that all $(f, g) \in Y$ there is some $(\phi, \psi) \in X$ which solves (13) and $\| (\phi, \psi) \|_X \leq C\| (f, g) \|_Y$.

Proof. Combining the lower modes with the result from Theorem 2.1 is fairly standard, see for instance [2, 13].

2.1. The homogenous ode’s. Crucial to finding solutions of (14) will be first understanding the homogeneous versions given by

\[
\begin{cases}
-a''_k(r) - \frac{N-1}{r} a'_k(r) + \frac{\mu_k}{r^2} a_k(r) + \lambda_1 pr^{p-1}b'_k(r) = 0, \\
-b''_k(r) - \frac{N-1}{r} b'_k(r) + \frac{\mu_k}{r^2} b_k(r) + \lambda_2 q r^{q-1} a'_k(r) = 0.
\end{cases}
\]

(16)

Crucial to this method will be the understanding of the solutions of the homogenous version of (14). In what follows we will suppress the dependence on $k$. We begin by looking for solutions of the form $(a, b) = (c r^\gamma, d r^\gamma + \beta - 1)$, which is the system analog of the solutions one looks for in the case of the scalar second order Euler equations. A computation shows that $(a, b)$ is a solution exactly when

\[
A_\gamma \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where

\[
A_\gamma := \begin{pmatrix} \gamma^2 + (N - 2) \gamma - \mu_k & -\lambda_1 pr^{p-1}(\beta + \gamma - 1) \\
-\lambda_2 qr^{q-1} \gamma & \gamma^2 + (N + 2\beta - 4) \gamma + (\beta - 1)(N + \beta - 3) - \mu_k \end{pmatrix}.
\]

(17)

Since we are looking for nonzero solutions we want to find $\gamma \in \mathbb{C}$ such that $\det(A_\gamma) = 0$ and note that this is a fourth order polynomial in $\gamma$ with real coefficients given by

\[
\det(A_\gamma) = \gamma^4 + g_3 \gamma^3 + g_2 \gamma^2 + g_1 \gamma + g_0,
\]

where

\[
\begin{align*}
g_3 & := 2N + 2\beta - 6, \\
g_2 & := (\beta - 1)(N + \beta - 3) + (N - 2)(N + 2\beta - 4) - \lambda_1 \lambda_2 q r^{q-1} r^{p-1} - 2\mu_k, \\
g_1 & := (\beta - 1)(N + \beta - 3)(N - 2) - \lambda_1 \lambda_2 q r^{q-1} r^{p-1}(\beta - 1) - (2N + 2\beta - 6)\mu_k, \\
g_0 & := \mu_k^2 - (\beta - 1)(N + \beta - 3)\mu_k.
\end{align*}
\]

(18)

Let $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ denote the roots (real or complex) of the $\det(A_\gamma) = 0$. Then

\[
\prod_{i=1}^{4} (\gamma - \gamma_i) = \det(A_\gamma)
\]

and expanding allows one to write

\[
det(A_\gamma) = \gamma^4 - (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \gamma^3 \\
+ \left( (\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4) + \gamma_1 \gamma_2 + \gamma_3 \gamma_4 \right) \gamma^2 \\
- \left( (\gamma_1 + \gamma_2)(\gamma_3 \gamma_4 + (\gamma_3 + \gamma_4)\gamma_1 \gamma_2) \right) \gamma \\
+ (\gamma_1 \gamma_2 \gamma_3 \gamma_4).
\]
Thus
\[
\begin{align*}
-\gamma_1^2 + \gamma_2 + \gamma_3 + \gamma_4 &= 2N + 2\beta - 6, \\
(\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4) + \gamma_1\gamma_2 + \gamma_3\gamma_4 &= (\beta - 1)(N + \beta - 3) \\
+(N - 2)(N + 2\beta - 4) - \lambda_1\lambda_2 p\gamma q^{r-1}T^{p-1} - 2\mu_k, \\
-((\gamma_1 + \gamma_2)\gamma_3\gamma_4 + (\gamma_3 + \gamma_4)\gamma_1\gamma_2) &= (\beta - 1)(N + \beta - 3)(N - 2) \\
-(N + 2\beta - 4)\mu_k - \lambda_1\lambda_2 p\gamma q^{r-1}T^{p-1}(\beta - 1) - \mu_k(N - 2), \\
\gamma_1\gamma_2\gamma_3\gamma_4 &= \mu_k^2 - (\beta - 1)(N + \beta - 3)\mu_k.
\end{align*}
\] (19)

We now consider the possible cases of roots we might run into (note it appears we are missing some cases, but later it will be apparent we have considered all the relevant cases):

1. (Four distinct real roots). In this case we have \(\text{det}(A_{\gamma_j}) = 0\) for \(1 \leq j \leq 4\) and \(\gamma_j\) are distinct real roots. Let \((c_j, d_j)^T\) (the transpose) be a nonzero element in the kernel of \(A_{\gamma_j}\). Then \((a_j, b_j) = (c_j r^{\gamma_j}, d_j r^{\gamma_j + \beta - 1})\) are solutions of (16) and one can argue that the general solution is given by \((a(r), b(r)) = \sum_{j=1}^{4} D_j(a_j(r), b_j(r))\) where \(D_j \in \mathbb{R}\). 

2. (Two distinct real roots and a double root). In this case we have \(\text{det}(A_{\gamma_j}) = 0\) for \(1 \leq j \leq 4\) and \(\gamma_1, \gamma_2\) are two distinct real roots and \(\gamma_3, \gamma_4\) are solutions of (16). One can then choose \((c_4, d_4)\) such that \((a_4, b_4) = ((c_3 \ln r + c_4) r^{\gamma_3}, (d_3 \ln r + d_4) r^{\gamma_4 + \beta - 1})\) is a solution of (16), see (24). One then has the general solution given by \((a(r), b(r)) = \sum_{j=1}^{4} D_j(a_j(r), b_j(r))\).

3. (Two distinct real roots and a complex conjugate roots). In this case we have \(\text{det}(A_{\gamma_j}) = 0\) for \(1 \leq j \leq 4\) and \(\gamma_1, \gamma_2\) are two distinct real roots and \(\gamma_3 = \xi + i\eta\) and \(\gamma_4 = \xi - i\eta\) are complex roots (here \(\xi, \eta \in \mathbb{R}\) with \(\eta \neq 0\)). Let \((c_j, d_j)^T\) (the transpose) be a nonzero element in the kernel of \(A_{\gamma_j}\) and note \((c_j, d_j)\) for \(j = 3, 4\) can be complex. Then \((a_j, b_j) = (c_j r^{\gamma_j}, d_j r^{\gamma_j + \beta - 1})\), \(j = 1, 2\) are solutions of (16). Since we want real solutions we take the real and imaginary parts of \((a_j, b_j)\) for \(j = 3, 4\). We begin by considering the real and imaginary parts of \((a_3, b_3)\) which are given by

\[
(a_3, b_3) = ((\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r)) r^{\xi}, (\hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r)) r^{\xi + \beta - 1})
\]

and

\[
(a_4, b_4) = ((\hat{c}_4 \cos(\eta \ln r) + \hat{c}_3 \sin(\eta \ln r)) r^{\xi}, (\hat{d}_4 \cos(\eta \ln r) + \hat{d}_3 \sin(\eta \ln r)) r^{\xi + \beta - 1}),
\]

where we write \(c_3 = \hat{c}_3 + i\hat{c}_4\) and \(d_3 = \hat{d}_3 + i\hat{d}_4\) for \(\hat{c}_j, \hat{d}_j \in \mathbb{R}\). Then \((a_j, b_j)\) are solutions of (16) for \(j = 3, 4\) and the general solution is given by \((a(r), b(r)) = \sum_{j=1}^{4} D_j(a_j(r), b_j(r))\). Note we really didn’t write out the real and imaginary parts of \((a_4, b_4)\) since this doesn’t give us anything new.

4. (Two real equal roots and a complex conjugate roots). Here we consider the case of \(\gamma_1 = \gamma_2\) and \(\gamma_3 = \xi + i\eta, \gamma_4 = \xi - i\eta\). One can write out this case but this follows from combinations of the above case, so we omit the details.

We will need the following kernel result.
Lemma 2.2. Suppose $p, q, N$ satisfy the hypothesis from Theorem 2.1. Then there is some large $k_0$ such that the only $(\phi, \psi) \in X_{k_0}$ such that $L(\phi, \psi) = 0$ in $B_1 \setminus \{0\}$ is $\phi = \psi = 0$.

Proof. See after the proof of Proposition 1. \hfill \Box

2.2. The nonhomogeneous ode’s. Here we obtain solutions of (14) using a variation of parameters approach. We now state our main result for this section.

Proposition 1. Suppose $k \geq 0$ and we are in one of the following cases regarding the roots of $\det(A_k) = 0$: (I) four distinct real roots, (II) two distinct real roots and a double root, (III) two distinct real roots and a complex root. Additionally we assume the real part of any root is not equal to $-1$ along with additional assumptions on the roots (we specify the assumptions in the proof “The estimates from Proposition 1”). Then there is some $C_k$ such that for all $(f_k, g_k)$ there is some $(a_k, b_k)$ which satisfies (14) and $\|((a_k \zeta_k, b_k \xi_k))\|_{X} \leq C_k \|(f_k \zeta_k, g_k \xi_k))\|_{Y}$.

The following result is essentially the above result after putting restrictions on $p, q$ so that some of the apparent missed cases in the above result are not an issue.

Corollary 2. Suppose $k \geq 0$ and $N \geq 3$ and $p, q$ satisfy

$$p > 1, q > 1, \max \left\{ \frac{N}{p(N - 1)} - 1, \frac{1}{N - 1} + \frac{N}{p(N - 1)} \right\} < q < \min \left\{ 1 + \frac{2}{p}, \frac{2}{p - 1} \right\}$$

and

$$(\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)(N - 1) - \lambda_1 \lambda_2 p q t^{p - 1} r^{p - 1}(\beta - 1) < 0.$$ Then there is some $C_k$ such that for all $(f_k, g_k)$ there is some $(a_k, b_k)$ which satisfies (14) and $\|((a_k \zeta_k, b_k \xi_k))\|_{X} \leq C_k \|(f_k \zeta_k, g_k \xi_k))\|_{Y}$.

We will give the proof of this result later on. We now return to obtaining solutions of (14). To do this we first consider our second order system as a first order system given abstractly by $Y'(r) = A(r)Y(r) + F(r)$ where $Y^T = (s, z, a, b)^T$ and $F^T = (f, g, 0, 0)^T$. Let $X_k(r)$ for $1 \leq k \leq 4$ denote a solution of the homogeneous equation $X'(r) = A(r)X(r)$ then we look for solutions of the form $Y(r) = M(r)U(r)$ where $M(r)$ has its columns given by $X_k(r)$ (the Fundamental Matrix) and $U(r)$ is a $4 \times 1$ unknown function. Then we see $Y(r)$ satisfies the nonhomogenous ode exactly when $U$ satisfies $M(r)U'(r) = F(r)$. We now write out the first order system. Consider

$$\begin{align*}
s_k' &= -N^{-1}s_k + \frac{\beta}{r} a_k + \lambda_1 p r^{p - 1} s_k - f_k \\
z_k' &= -N^{-1}z_k + \frac{\beta}{r} b_k + \lambda_2 q r^{p - 1} s_k - g_k \\
a_k' &= s_k \\
b_k' &= z_k.
\end{align*} \quad (20)$$

(I) Four real distinct roots. Consider the general solution of the homogeneous equation given by

$$\begin{align*}
(a(r)) &= D_1 \left( c_1 r^{\gamma_1} \right) + D_2 \left( c_2 r^{\gamma_2} \right) + D_3 \left( c_3 r^{\gamma_3} \right) + D_4 \left( c_4 r^{\gamma_4} \right),

\end{align*}$$

where $D_i \in \mathbb{R}$ and $(c_k, d_k)^T$ is a nonzero element in the kernel of $A_{\gamma_k}$. Since the roots are distinct we easily see the linear independence of the four solutions and
Thus the components of $M_k$ where
\[
M_k(r) := \begin{pmatrix}
c_1 r^{\gamma_1 - 1} & c_2 r^{\gamma_2 - 1} & m_{13} & m_{14} \\
d_1 (\gamma_1 + \beta - 1) r^{\gamma_1 + \beta - 2} & d_2 (\gamma_2 + \beta - 1) r^{\gamma_2 + \beta - 2} & m_{23} & m_{24} \\
c_1 r^{\gamma_1} & c_2 r^{\gamma_2} & m_{33} & m_{34} \\
d_1 r^{\gamma_1 + \beta - 1} & d_2 r^{\gamma_2 + \beta - 1} & m_{43} & m_{44}
\end{pmatrix},
\]
where
\[
m_{11} := c_1 r^{\gamma_1 - 1}, \quad m_{12} := c_2 r^{\gamma_2 - 1}, \\
m_{21} := d_1 (\gamma_1 + \beta - 1) r^{\gamma_1 + \beta - 2}, \quad m_{22} := d_2 (\gamma_2 + \beta - 1) r^{\gamma_2 + \beta - 2}, \\
m_{31} := c_1 r^{\gamma_1}, \quad m_{32} := c_2 r^{\gamma_2}, \\
m_{41} := d_1 r^{\gamma_1 + \beta - 1}, \quad m_{42} := d_2 r^{\gamma_2 + \beta - 1}, \\
m_{13} := c_3 r^{\gamma_3 - 1}, \quad m_{14} := c_4 r^{\gamma_4 - 1}, \\
m_{23} := d_3 (\gamma_3 + \beta - 1) r^{\gamma_3 + \beta - 2}, \quad m_{24} := d_4 (\gamma_4 + \beta - 1) r^{\gamma_4 + \beta - 2}, \\
m_{33} := c_3 r^{\gamma_3}, \quad m_{34} := c_4 r^{\gamma_4}, \\
m_{43} := d_3 r^{\gamma_3 + \beta - 1}, \quad m_{44} := d_4 r^{\gamma_4 + \beta - 1}.
\]
with respect to this we can find $u_k'(r)$ for $i = 1, \ldots, 4$ as follows. In fact using
Cramer’s rule for algebraic equations implies
\[
u_k'(r) = \frac{f_k c_2 r^{\gamma_2 - 1} m_{13} + m_{14}}{\det(M_k(r))} \quad \text{and}
\]
which implies
\[
u_k'(r) = \hat{c}_1 f_k r^{\gamma_1 + \beta - 1} + \hat{d}_1 g_k r^{\gamma_1 - \beta + 2}.
\]
and we get
\[
u_k(r) = \hat{c}_1 \int_{T_3}^r \sigma^{\gamma_1 + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_3}^r \sigma^{\gamma_1 - \beta + 2} g_k(\sigma) d\sigma + e_{k1},
\]
(\vspace{0.5cm}
\text{where } \hat{c}_1, \hat{d}_1 \text{ are some constants coming from the determinant) and } e_{k1} \text{ is a constant of integration and one may assume } e_{k1} = 0 \text{ (with respect to method of variation of parameters and the solution of homogeneous ODE) and where we will later pick } T_3 \in \{0, 1\} \text{ depending on the values of some parameters. If the integrand is } L^1(0, 1) \text{ then we choose } T_3 = 0 \text{ otherwise we take it to be 1. By the same way we can compute } u_k(2), u_k(3) \text{ and } u_k(4) \text{ and finally we get}
\]
\[
U_k(r) = \begin{pmatrix}
\hat{c}_1 \int_{T_3}^r \sigma^{\gamma_1 + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_3}^r \sigma^{\gamma_1 - \beta + 2} g_k(\sigma) d\sigma \\
\hat{c}_2 \int_{T_3}^r \sigma^{\gamma_2 + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_3}^r \sigma^{\gamma_2 - \beta + 2} g_k(\sigma) d\sigma \\
\hat{c}_3 \int_{T_3}^r \sigma^{\gamma_3 + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{\gamma_3 - \beta + 2} g_k(\sigma) d\sigma \\
\hat{c}_4 \int_{T_3}^r \sigma^{\gamma_4 + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_3}^r \sigma^{\gamma_4 - \beta + 2} g_k(\sigma) d\sigma
\end{pmatrix}.
\)
Thus the components of $M_k(r)U_k(r)$ are
\begin{align}
\frac{1}{r^k} = c_1 \gamma r^{\gamma - 1} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_2 \gamma r^{\gamma - 1} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_3 \gamma r^{\gamma - 1} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_4 \gamma r^{\gamma - 1} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right)
\end{align}

\begin{align}
z_k(r) &= d_1 (\gamma_1 + \beta - 1) r^{\gamma_1 + \beta - 2} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_2 (\gamma_2 + \beta - 1) r^{\gamma_2 + \beta - 2} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_3 (\gamma_3 + \beta - 1) r^{\gamma_3 + \beta - 2} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_4 (\gamma_4 + \beta - 1) r^{\gamma_4 + \beta - 2} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right)
\end{align}

\begin{align}
a_k(r) &= c_1 r^{\gamma_1} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_2 r^{\gamma_2} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_3 r^{\gamma_3} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_4 r^{\gamma_4} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right)
\end{align}

\begin{align}
b_k(r) &= d_1 r^{\gamma_1 + \beta - 1} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_2 r^{\gamma_2 + \beta - 1} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_3 r^{\gamma_3 + \beta - 1} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_4 r^{\gamma_4 + \beta - 1} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right)
\end{align}

Then finally we have the general solution of (14) (without boundary condition) is given by

\begin{align}
\begin{cases}
a_k(r) = P_1 c_1 r^{\gamma_1} + P_2 c_2 r^{\gamma_2} + P_3 c_3 r^{\gamma_3} + P_4 c_4 r^{\gamma_4} + c_1 r^{\gamma_1} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_2 r^{\gamma_2} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_3 r^{\gamma_3} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + c_4 r^{\gamma_4} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right), \\
b_k(r) = P_1 d_1 r^{\gamma_1 + \beta - 1} + P_2 d_2 r^{\gamma_2 + \beta - 1} + P_3 d_3 r^{\gamma_3 + \beta - 1} + P_4 d_4 r^{\gamma_4 + \beta - 1} + d_1 r^{\gamma_1 + \beta - 1} \left( \hat{c}_1 \int_{T_1}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_1 \int_{T_1}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_2 r^{\gamma_2 + \beta - 1} \left( \hat{c}_2 \int_{T_2}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_2 \int_{T_2}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_3 r^{\gamma_3 + \beta - 1} \left( \hat{c}_3 \int_{T_3}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_3 \int_{T_3}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right) + d_4 r^{\gamma_4 + \beta - 1} \left( \hat{c}_4 \int_{T_4}^r \sigma^{-\gamma + 1} f_k(\sigma) d\sigma + \hat{d}_4 \int_{T_4}^r \sigma^{-\gamma - \beta + 2} g_k(\sigma) d\sigma \right),
\end{cases}
\end{align}

where \(P_k\) is arbitrary.
(II) Two real distinct roots and a double root. Here we consider the case of two distinct roots \( \gamma_1, \gamma_2 \) and a double root \( \gamma_3 = \gamma_4 \) which are not equal to \( \gamma_1 \) or \( \gamma_2 \). We will now show that the general solution of the homogeneous problem is given by

\[
\begin{pmatrix}
a(r) \\
b(r)
\end{pmatrix} = D_1 \begin{pmatrix}
c_1 r^{\gamma_1} \\
d_1 r^{\gamma_1+\beta-1}
\end{pmatrix} + D_2 \begin{pmatrix}
c_2 r^{\gamma_2} \\
d_2 r^{\gamma_2+\beta-1}
\end{pmatrix} + D_3 \begin{pmatrix}
c_3 r^{\gamma_3} \\
d_3 r^{\gamma_3+\beta-1}
\end{pmatrix} + D_4 \begin{pmatrix}
c_4 r^{\gamma_4} \\
d_4 r^{\gamma_4+\beta-1}
\end{pmatrix}
\]

A computation shows that each of components \( k = 1, 2, 3 \) are solutions of the associated homogenous equation and also the dimension of span of these solutions is equal to three. Its clear that adding the remaining term doesn’t effect the linear independence since this term involves some nonzero log terms. The only remaining thing to check is the term corresponding to the \( D_4 \) term really is a solution of the homogenous equation, and if one checks the details they see they need \((c_4, d_4)\) to satisfy

\[
\left\{ \begin{array}{l}
( -\gamma_3(\gamma_3 - 1) - \gamma_3(N - 1) - \lambda_k) c_4 + \lambda_1 p r^{p-1}(\gamma_3 + \beta - 1)d_4 \\
= (\gamma_3 + (\gamma_3 - 1) + (N - 1)) c_3 - \lambda_1 p r^{p-1}d_3,
\end{array} \right.
\]

\[
(\lambda_2 q t^{q-1}\gamma_3) c_4 + ( - (\gamma_3 + \beta - 2)(\gamma_3 + \beta - 1) - (N - 1)(\gamma_3 + \beta - 1) + \lambda_k) d_4
\]

\[
= -\lambda_2 q t^{q-1}c_3 + ( (\gamma_3 + \beta - 1) + (\gamma_3 + \beta - 2) + (N - 1) ) d_3.
\]

Notice that the above equation can be written as

\[
A_{\gamma_3} \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} = -\frac{d}{d\gamma} A_{\gamma=\gamma_3} \begin{pmatrix} c_3 \\ d_3 \end{pmatrix}.
\]

Of course since \( \text{det}(A_{\gamma_3}) = 0 \) we need to be a bit careful solving (24). Consider solving \( Ax = F \) where

\[
A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad F = \begin{pmatrix} f \\ g \end{pmatrix}
\]

and we assume \( a \neq 0 \) and \( \text{det}(A) = 0 \). Then this is solvable exactly when \( fb - ga = 0 \) and in this case the general solution is given by

\[
x = \begin{pmatrix} f \\ a^{-1} - c a^{-1} \end{pmatrix}, \quad s \in \mathbb{R}.
\]

Using the above fact and since \( \gamma_3 \) is a double root (this means \( \frac{d}{d\gamma} \text{det}(A_{\gamma}) \big|_{\gamma=\gamma_3} = 0 \) and \((c_3, d_3)^T \in \text{ker}(A_{\gamma_3})\) one can show the existence of \((c_4, d_4)\) which satisfies the above equation.

Since we have the four solutions of the homogeneous ode, then we can write the fundamental matrix as

\[
M_k(r) := \begin{pmatrix}
c_1 r^{\gamma_1-1} & c_2 r^{\gamma_2-1} & c_3 r^{\gamma_3-1} & m_{14} \\
d_1 (\gamma_1 + \beta - 1) r^{\gamma_1+\beta-2} & d_2 (\gamma_2 + \beta - 1) r^{\gamma_2+\beta-2} & d_3 (\gamma_3 + \beta - 1) r^{\gamma_3+\beta-2} & m_{24} \\
c_1 r^{\gamma_1-1} & c_2 r^{\gamma_2-1} & c_3 r^{\gamma_3-1} & m_{34} \\
d_1 r^{\gamma_1+\beta-1} & d_2 r^{\gamma_2+\beta-1} & d_3 r^{\gamma_3+\beta-1} & m_{44}
\end{pmatrix},
\]

where

\[
m_{14} := (c_3 + \gamma_3(c_3 \ln r + c_4)) r^{\gamma_3-1}, \quad m_{24} := (d_3 + (\gamma_3 + \beta - 1)(d_3 \ln r + d_4)) r^{\gamma_3+\beta-2}, \quad m_{34} := (c_3 \ln r + c_4) r^{\gamma_3}, \quad m_{44} := (d_3 \ln r + d_4) r^{\gamma_3+\beta-1}.
\]
Also we know that $|M_k(r)| = \hat{c}_9 r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}$ where

$$
\hat{c}_9 := \begin{vmatrix}
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_5 + \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\end{vmatrix}
\begin{vmatrix}
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_5 + \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\end{vmatrix} = \hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_5 \ln r.
$$

A computation shows that

$$
\hat{c}_g = \begin{vmatrix}
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_5 + \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_3 \ln r \\
\end{vmatrix} = \hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_5 \ln r,
$$

where that last determinant appearing above involves just constant entries. With respect to this we can find $u'_k(r)$ for $i = 1, \ldots, 4$ as follows. In fact using Cramer's rule for algebraic equations implies

$$
u'_1(r) = \frac{d_2 d_5 + d_3 \ln r}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} f_k + \frac{d_2 d_3 (d_4 + d_3 \ln r)}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} g_k,$$

also

$$
u'_2(r) = \frac{d_2 d_5 + d_3 \ln r}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} f_k + \frac{d_2 d_3 (d_4 + d_3 \ln r)}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} g_k,$$

and

$$
u'_3(r) = \frac{d_2 d_5 + d_3 \ln r}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} f_k + \frac{d_2 d_3 (d_4 + d_3 \ln r)}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} g_k,$$

finally

$$
u'_4(r) = \frac{d_2 d_5 + d_3 \ln r}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} f_k + \frac{d_2 d_3 (d_4 + d_3 \ln r)}{r^{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\beta - 4}} g_k.$$
The components of $M_k(r)U_k(r)$ are

\[
\begin{aligned}
    s_k(r) &= c_1 \gamma_1^{-1} r^{\gamma_1 - 1} (u_k(1)) + c_2 \gamma_2 r^{\gamma_2 - 2} (u_k(2)) + c_3 \gamma_3 r^{\gamma_3 - 1} (u_k(3)) + (c_3 + \gamma_3 (c_3 \ln r + c_4 \gamma_4)) r^{\gamma_3 - 1} (u_k(4)), \\
    z_k(r) &= d_1 (\gamma_1 + \beta - 1) r^{\gamma_1 + \beta - 2} (u_k(1)) + d_2 (\gamma_2 + \beta - 1) r^{\gamma_2 + \beta - 2} (u_k(2)) + d_3 (\gamma_3 + \beta - 1) r^{\gamma_3 + \beta - 2} (u_k(3)) + (d_4 \ln r + d_4 \gamma_4) r^{\gamma_3 + \beta - 2} (u_k(4)), \\
    a_k(r) &= c_1 r^{\gamma_1} (u_k(1)) + c_2 r^{\gamma_2} (u_k(2)) + c_3 r^{\gamma_3} (u_k(3)) + (c_3 \ln r + c_4) r^{\gamma_3} (u_k(4)), \\
    b_k(r) &= d_1 r^{\gamma_1 + \beta - 1} (u_k(1)) + d_2 r^{\gamma_2 + \beta - 1} (u_k(2)) + d_3 r^{\gamma_3} (u_k(3)) + (d_3 \ln r + d_4) r^{\gamma_3 + \beta - 1} (u_k(4)).
\end{aligned}
\]  

(29)

So one can write the general solutions of the system (14) as

\[
\begin{aligned}
    a_k(r) &= P_1 c_1 r^{\gamma_1} + P_2 c_2 r^{\gamma_2} + P_3 c_3 r^{\gamma_3} + P_4 ((c_3 \ln r + c_4)) r^{\gamma_3} (u_k(1)) + (c_3 \ln r + c_4) r^{\gamma_3} (u_k(4)), \\
    b_k(r) &= P_1 d_1 r^{\gamma_1 + \beta - 1} + P_2 d_2 r^{\gamma_2 + \beta - 1} + P_3 d_3 r^{\gamma_3 + \beta - 1} + P_4 (d_3 \ln r + d_4) r^{\gamma_3 + \beta - 1} (u_k(1)) + (d_3 \ln r + d_4) r^{\gamma_3 + \beta - 1} (u_k(4)).
\end{aligned}
\]  

(30)

(III) Two real distinct roots $\gamma_1$ and $\gamma_2$ and a complex root $\xi + i\eta$. Since the co-efficients of the $\text{det}(A_r)$ are real one then sees that the fourth root is given by $\xi - i\eta$. We can now write out the general solution of the homogeneous ode as

\[
\begin{pmatrix}
    a(r) \\
    b(r)
\end{pmatrix} = D_1 \begin{pmatrix}
    c_1 r^{\gamma_1} \\
    d_1 r^{\gamma_1 + \beta - 1}
\end{pmatrix} + D_2 \begin{pmatrix}
    c_2 r^{\gamma_2} \\
    d_2 r^{\gamma_2 + \beta - 1}
\end{pmatrix} + D_3 \begin{pmatrix}
    (\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r)) r^{\xi} \\
    (\hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r)) r^{\xi + \beta - 1}
\end{pmatrix} + D_4 \begin{pmatrix}
    (\hat{c}_3 \cos(\eta \ln r) + \hat{c}_4 \sin(\eta \ln r)) r^{\xi} \\
    (\hat{d}_4 \cos(\eta \ln r) + \hat{d}_3 \sin(\eta \ln r)) r^{\xi + \beta - 1}
\end{pmatrix}.
\]

A computation shows that each of the above components are the solution of the homogeneous ODE (16).

Since we have the four solutions of the homogeneous ODE (20), then we can write the fundamental matrix $M_k(r)$ as below

\[
M_k(r) := \begin{pmatrix}
    c_1 \gamma_1 r^{\gamma_1 - 1} & c_2 \gamma_2 r^{\gamma_2 - 2} & m_{13} & m_{14} \\
    d_1 (\gamma_1 + \beta - 1) r^{\gamma_1 + \beta - 2} & d_2 (\gamma_2 + \beta - 1) r^{\gamma_2 + \beta - 2} & m_{23} & m_{24} \\
    c_1 r^{\gamma_1} & c_2 r^{\gamma_2} & m_{33} & m_{34} \\
    d_1 r^{\gamma_1 + \beta - 1} & d_2 r^{\gamma_2 + \beta - 1} & m_{43} & m_{44}
\end{pmatrix}.
\]  

(31)
where
\[
\begin{align*}
    m_{13} &:= (\xi \alpha_{33} - \eta \alpha_{44}) r^{\xi - 1}, \\
    m_{33} &:= \alpha_{33} r^{\xi}, \\
    m_{14} &:= (\eta \alpha_{33} + \xi \alpha_{44}) r^{\xi - 1}, \\
    m_{34} &:= \alpha_{44} r^{\xi}, \\
    c_3 &= \hat{c}_3 + i \hat{c}_4, \\
    a_{33} &= (\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r)), \\
    a_{43} &= (\hat{d}_3 \cos(\eta \ln r) + \hat{d}_4 \sin(\eta \ln r)), \\
    a_{44} &= (\hat{d}_3 \cos(\eta \ln r) + \hat{d}_4 \sin(\eta \ln r)).
\end{align*}
\]

We know \( \det M_k(r) = r^{\gamma_1 + \gamma_2 + 2\xi + 2\beta - 4} \hat{c}_M \), where
\[
\hat{c}_M :=
\begin{vmatrix}
    c_1 \gamma_1 & c_2 \gamma_2 & (\xi \alpha_{33} - \eta \alpha_{44}) & (\eta \alpha_{33} + \xi \alpha_{44}) \\
    d_1 (\gamma_1 + \beta - 1) & d_2 (\gamma_2 + \beta - 1) & (\xi + \beta - 1) a_{44} - \eta a_{43} & (\eta + \beta - 1) a_{44} \\
    c_1 & c_2 & a_{33} & a_{34} \\
    d_1 & d_2 & a_{43} & a_{44}
\end{vmatrix}
\]

\( = \eta^2 \left| \begin{array}{cc}
    c_1 & c_2 \\
    d_1 & d_2
\end{array} \right| \left( \left( \frac{\gamma_1 - \xi}{\eta} \frac{\gamma_2 - \xi}{\eta} + 1 \right) (c_3 d_4 - c_4 d_3) - \left( \gamma_1 - \xi \right) (c_3 d_4 - c_4 d_3) \right) + c_1 c_2 \left( \frac{\gamma_1 - \xi}{\eta} (d_3^2 + d_4^2) + d_1 d_2 \left( \frac{\gamma_1 - \xi}{\eta} \right) (c_3^2 + c_4^2) \right).\)

With respect to this, one can find \( u_k'(r) \) for \( i = 1, \ldots, 4 \). In fact, using Cramer’s rule for algebraic equations implies
\[
\begin{align*}
    u_{k1}'(r) &= \frac{1}{\hat{c}_M} \left| \begin{array}{cc}
    d_2 (\gamma_1 + \beta - 1) & (\xi + \beta - 1) a_{44} + \eta a_{43} \\
    c_2 & a_{33} \\
    d_2 & a_{43}
\end{array} \right| f_k \\
    &= \frac{1}{\hat{c}_M} \left| \begin{array}{cc}
    c_2 \gamma_2 & (\xi \alpha_{33} - \eta \alpha_{44}) + (\eta \alpha_{33} + \xi \alpha_{44}) \\
    c_2 & a_{33} \\
    d_2 & a_{43}
\end{array} \right| g_k \\
    &= \hat{c}_3 r^{-\gamma_1 + 1} f_k + \hat{c}_3 r^{-\gamma_1 - \beta + 2} g_k,
\end{align*}
\]

SINGULAR SOLUTIONS OF A LANE-EMDEN SYSTEM
also by the similar argument

\[ u'_{k2}(r) = \hat{c}_{33} r^{-\gamma_2+1} f_k + \hat{c}_{34} r^{-\gamma_2-\beta+2} g_k. \]

For \( u'_{k3}(r) \), a computation implies

\[
\begin{align*}
\begin{vmatrix}
   d_1 (\gamma_1 - \xi) & d_2 (\gamma_2 - \xi) & \eta(\hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r)) \\
   c_1 & c_2 & \hat{c}_4 \cos(\eta \ln r) + \hat{c}_3 \sin(\eta \ln r) \\
   d_1 & d_2 & \hat{d}_4 \cos(\eta \ln r) + \hat{d}_3 \sin(\eta \ln r)
\end{vmatrix}
\biggr\vert_{c_M r^{\zeta-1}} f_k \\
- \begin{vmatrix}
   c_1 (\gamma_1 - \xi) & c_2 (\gamma_2 - \xi) & \eta(\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r)) \\
   c_1 & c_2 & \hat{c}_4 \cos(\eta \ln r) + \hat{c}_3 \sin(\eta \ln r) \\
   d_1 & d_2 & \hat{d}_4 \cos(\eta \ln r) + \hat{d}_3 \sin(\eta \ln r)
\end{vmatrix}
\biggr\vert_{c_M r^{\zeta-2}} g_k
\end{align*}
\]

\[
= \left( \begin{vmatrix}
   c_1 & c_2 \\
   d_1 & d_2
\end{vmatrix}
\right) \eta(\hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r))
\]

\[
- \frac{d_1 (\gamma_1 - \xi) d_2 (\gamma_2 - \xi) (\hat{c}_4 \cos(\eta \ln r)) + \hat{c}_3 \sin(\eta \ln r))}{d_2}
\]

\[
+ \frac{d_1 (\gamma_1 - \xi) d_2 (\gamma_2 - \xi) (\hat{d}_4 \cos(\eta \ln r)) + \hat{d}_3 \sin(\eta \ln r))}{c_2}
\]

\[
\left( \begin{vmatrix}
   c_1 & c_2 \\
   d_1 & d_2
\end{vmatrix}
\right) \eta(\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r))
\]

\[
- \frac{c_1 (\gamma_1 - \xi) c_2 (\gamma_2 - \xi) (\hat{c}_4 \cos(\eta \ln r)) + \hat{c}_3 \sin(\eta \ln r))}{d_2}
\]

\[
+ \frac{c_1 (\gamma_1 - \xi) c_2 (\gamma_2 - \xi) (\hat{d}_4 \cos(\eta \ln r)) + \hat{d}_3 \sin(\eta \ln r))}{c_2}
\]

\[
= \left( \begin{vmatrix}
   c_1 & c_2 \\
   d_1 & d_2
\end{vmatrix}
\right) \eta(\hat{c}_{351} \cos(\eta \ln r) + \hat{c}_{352} \sin(\eta \ln r))
\]

\[
+ \frac{d_1 (\gamma_1 - \xi) d_2 (\gamma_2 - \xi) (\hat{d}_4 \cos(\eta \ln r)) + \hat{d}_3 \sin(\eta \ln r))}{c_2}
\]

Finally, similar argument shows

\[
\begin{align*}
\begin{vmatrix}
   d_1 (\gamma_1 - \xi) & d_2 (\gamma_2 - \xi) & -\eta(\hat{d}_4 \cos(\eta \ln r) + \hat{d}_3 \sin(\eta \ln r)) \\
   c_1 & c_2 & \hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r) \\
   d_1 & d_2 & \hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r)
\end{vmatrix}
\biggr\vert_{c_M r^{\zeta-1}} f_k \\
- \begin{vmatrix}
   c_1 (\gamma_1 - \xi) & c_2 (\gamma_2 - \xi) & -\eta(\hat{c}_4 \cos(\eta \ln r) + \hat{c}_3 \sin(\eta \ln r)) \\
   c_1 & c_2 & \hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r) \\
   d_1 & d_2 & \hat{d}_3 \cos(\eta \ln r) - \hat{d}_4 \sin(\eta \ln r)
\end{vmatrix}
\biggr\vert_{c_M r^{\zeta+2}} g_k
\end{align*}
\]
or

\[
\begin{align*}
\frac{u_k (r)}{d_1} &= - \left( \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \left( -\eta(d_4 \cos(\eta \ln r) + \dot{d}_3 \sin(\eta \ln r)) \right) \\
+ \left( \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \left( -\eta(\dot{d}_4 \cos(\eta \ln r) + \dot{d}_3 \sin(\eta \ln r)) \right) \right) \frac{r^{-\xi+1} f_k}{c_3 I} \\
- \left( \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \left( \dot{c}_3 \cos(\eta \ln r) - \dot{c}_4 \sin(\eta \ln r)) \right) \right) \frac{r^{-\xi-\beta+2 g_k}}{c_3 I}
\end{align*}
\]

\[
= (\dot{c}_{371} \cos(\eta \ln r) + \dot{c}_{372} \sin(\eta \ln r)) r^{-\xi+1} f_k \\
+ (\dot{c}_{381} \cos(\eta \ln r) + \dot{c}_{382} \sin(\eta \ln r)) r^{-\xi-\beta+2 g_k}
\]

We can write the matrix \( U_k (r) = (u_1 (r), u_2 (r), u_3 (r), u_4 (r))^T \), where \( T \) means transpose and

\[
\begin{align*}
u_1 (r) := & \int_{T} \left( \dot{c}_{31} \int_{T} \sigma^{-\gamma_1+1} f_k (\sigma) d\sigma + \dot{c}_{32} \int_{T} \sigma^{-\gamma_1-\beta+2 g_k (\sigma)} d\sigma, \\
u_2 (r) := & \int_{T} \left( \dot{c}_{33} \int_{T} \sigma^{-\gamma_2+1} f_k (\sigma) d\sigma + \dot{c}_{34} \int_{T} \sigma^{-\gamma_2-\beta+2 g_k (\sigma)} d\sigma, \\
u_3 (r) := & \int_{T} \left( \dot{c}_{351} \cos(\eta \ln \sigma) + \dot{c}_{352} \sin(\eta \ln \sigma) \right) \sigma^{-\xi+1} f_k (\sigma) d\sigma \\
& + \int_{T} \left( \dot{c}_{361} \cos(\eta \ln \sigma) + \dot{c}_{362} \sin(\eta \ln \sigma) \right) \sigma^{-\xi-\beta+2 g_k (\sigma)} d\sigma, \\
u_4 (r) := & \int_{T} \left( \dot{c}_{371} \cos(\eta \ln \sigma) + \dot{c}_{372} \sin(\eta \ln \sigma) \right) \sigma^{-\xi+1} f_k (\sigma) d\sigma \\
& + \int_{T} \left( \dot{c}_{381} \cos(\eta \ln \sigma) + \dot{c}_{382} \sin(\eta \ln \sigma) \right) \sigma^{-\xi-\beta+2 g_k (\sigma)} d\sigma.
\end{align*}
\]

The components of \( M_k (r) U_k (r) \) are

\[
\begin{align*}
s_k (r) &= \frac{c_1 \gamma_1 r^{\gamma_1-1} (u_1 (r)) + c_2 r^{\gamma_2-1} (u_2 (r))}{c_3} \\
&+ ((\xi \alpha_{33} - \eta \alpha_{34}) r^{\xi-1} u_3 (r) + (\eta \alpha_{33} + \xi \alpha_{34}) r^{\xi-1} u_4 (r), \\
z_k (r) &= \frac{d_1 (\gamma_1 + \beta - 1) r^{\gamma_1+\beta-2} (u_1 (r)) + d_2 \gamma_1 + \beta - 1) r^{\gamma_2+\beta-2} (u_2 (r))}{c_3} \\
&+ ((\xi + \beta - 1) a_{43} - \eta a_{44}) r^{\xi+\beta-2} u_3 (r) \\
&+ ((\eta a_{43} + (\xi + \beta - 1) a_{44}) r^{\xi+\beta-2} u_4 (r), \\
\end{align*}
\]

\[
\begin{align*}
a_k (r) &= \frac{c_1 r^{\gamma_1-1} (u_1 (r)) + c_2 r^{\gamma_2} (u_2 (r)) + a_{33} r^{\xi} u_3 (r) + a_{34} r^{\xi} u_4 (r)}{c_3} \\
b_k (r) &= \frac{d_1 r^{\gamma_1+\beta-1} (u_1 (r)) + d_2 r^{\gamma_2+\beta-1} (u_2 (r))}{c_3} \\
&+ a_{33} r^{\xi+\beta-1} u_3 (r) + a_{34} r^{\xi+\beta-1} u_4 (r).
\end{align*}
\]

(32)
Thus the general solutions of the system (14) are

\[
\begin{align*}
  a_k(r) &= P_1 c_1 r^{\gamma_1} + P_2 c_2 r^{\gamma_2} + P_3 (\hat{c}_3 \cos(\eta \ln r) - \hat{c}_4 \sin(\eta \ln r)) r^\xi \\
  &+ P_4 (\hat{c}_5 \cos(\eta \ln r) - \hat{c}_6 \sin(\eta \ln r)) r^\xi \\
  &+ c_1 r^{\gamma_1} \left( \int_{T_1}^r \sigma^{-\gamma_1+1} f_k(\sigma)d\sigma + \hat{c}_2 + \int_{T_1}^r \sigma^{-\gamma_1+\beta+2} g_k(\sigma)d\sigma \right) \\
  &+ c_2 r^{\gamma_2} \left( \int_{T_2}^r \sigma^{-\gamma_2+1} f_k(\sigma)d\sigma + \hat{c}_3 + \int_{T_2}^r \sigma^{-\gamma_2+\beta+2} g_k(\sigma)d\sigma \right) \\
  &+ r^\xi \left( \int_{T_1}^r (\cos(\eta \ln \frac{r}{\tau}) + h_{12} \sin(\eta \ln \frac{r}{\tau})) \sigma^{-\xi+1} f_k(\sigma)d\sigma \right) \\
  &+ r^\xi \left( \int_{T_2}^r (\hat{h}_{21} \cos(\eta \ln \frac{r}{\tau}) + h_{21} \sin(\eta \ln \frac{r}{\tau})) \sigma^{-\xi+2} g_k(\sigma)d\sigma \right) \\
  \end{align*}
\]

where

\[
\begin{align*}
  h_{11} &:= \hat{c}_{351} \hat{c}_3 - \hat{c}_{4352}, & h_{12} &:= \hat{c}_{352} \hat{c}_3 + \hat{c}_4 \hat{c}_{351}, \\
  h_{21} &:= \hat{c}_3 \hat{c}_{361} - \hat{c}_4 \hat{c}_{362}, & h_{22} &:= \hat{c}_3 \hat{c}_{362} + \hat{c}_4 \hat{c}_{361}, \\
  h_{31} &:= \hat{c}_{351} d_3 - d_4 \hat{c}_{352}, & h_{32} &:= \hat{c}_{352} d_3 - d_4 \hat{c}_{351}, \\
  h_{41} &:= d_4 \hat{c}_{361} - d_5 \hat{c}_{362}, & h_{42} &:= d_3 \hat{c}_{362} + d_4 \hat{c}_{361}.
\end{align*}
\]

The estimates from Proposition 1. Here we obtain the needed estimates from Proposition 1 and we also obtain the desired boundary condition. We first consider the case of four distinct roots, case (I). Recall the general solution of (14) is given by (23). Also recall we mentioned that we choose $T_1 = 0$ if the integrand is $L^1(0,1)$ and otherwise we pick $T_1 = 1$. We look at a few terms to illustrate one indeed has the desired estimate. Consider the case of $\gamma_1 < -t$ and $\gamma_2 > -t$ and we first consider the terms of $a_k(r)$ given by

\[
I_1 = r^{\gamma_1} \int_{T_1}^r \sigma^{-\gamma_1+1} f_k(\sigma)d\sigma, \quad I_2 = r^{\gamma_2} \int_{T_2}^r \sigma^{-\gamma_2+1} f_k(\sigma)d\sigma.
\]

For simplicity we adjust the function spaces for now and obtain the needed estimate in the new spaces. To see how to get the estimates in the correct spaces see [13] (essentially one breaks the interval up using an infinite sum of dyadic intervals and applies Hölder’s inequality). So towards this define $\hat{Y}$ via $\| (f,g) \|_{\hat{Y}} := \sup_{0 < |x| \leq 1} |x|^{t+2} |f(x)| + \sup_{0 < |x| \leq 1} |x|^{t+2} |g(x)|$. Let suppose $(f_k,g_k) \in \hat{Y}$ with norm less or equal one. In this case we have $\sigma \mapsto \sigma^{-\gamma_1+1} f_k(\sigma) \in L^1(0,1)$ and the integrand corresponding to $I_2$ in general will not be $L^1(0,1)$. Hence we choose $T_1 = 0$ and $T_2 = 1$. Then we have

\[
r^t |I_1| \leq r^{t+\gamma_1} \int_0^r \sigma^{-\gamma_1+1} |f_k(\sigma)| d\sigma \leq r^{t+\gamma_1} \int_0^r \sigma^{-\gamma_1-t-1} d\sigma \leq C
\]

for any $0 < r \leq 1$. Of course this term alone doesn’t satisfy the boundary condition; we discuss this later. Note carefully if $\gamma_1 = -t$ then we would lose the estimate.
We now consider the term given by $I_2$.

\[
|I_2| \leq r^{t+\gamma_2} \int_r^1 \sigma^{-\gamma_2+1} |f_k(\sigma)| d\sigma \\
\leq r^{t+\gamma_2} \int_r^1 \sigma^{-\gamma_2-t+1} d\sigma \leq C
\]

for all $0 < r \leq 1$. Note the term $I_3(r)$ satisfies the boundary condition $I_2(1) = 0$. One can obtain bounds on all terms in $a_k$ and $b_k$ using this exact approach. So now all the integral terms in $a_k, b_k$ satisfy the needed bounds: $\sup_{0<r\leq1} r^t |I_{a_k}(r)| \leq C_k$ and $\sup_{0<r\leq1} r^t |I_{b_k}(r)| \leq C_k$ where $I_{a_k}$ refers to integral terms in the formula for $a_k$ and similarly for $I_{b_k}$.

**Case (I): Four real distinct roots.** We now show how to choose $P_k$ such that $a_k(1) = b_k(1) = 0$. There are multiple cases we must consider.

(i) $\gamma_3, \gamma_4 < -t$ and $\gamma_1, \gamma_2 > -t$,
(ii) $\gamma_4 < -t$, and $\gamma_1, \gamma_2, \gamma_3 > -t$,
(iii) $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > -t$.

In case (i) we will take $P_3 = P_4 = 0$ and our goal is to pick $P_1, P_2 \in \mathbb{R}$ to give the desired boundary condition. Putting $r = 1$ in (23) and writing everything out gives

\[
\begin{pmatrix}
  c_1 \\
  d_1
\end{pmatrix}
\begin{pmatrix}
  P_1 \\
  P_2
\end{pmatrix} = \begin{pmatrix}
  I \\
  J
\end{pmatrix},
\]

where $I, J$ are the various integral terms from (23) where $T_1 = 0$ evaluated at $r = 1$. If we can show $c_1d_2 - c_2d_1 \neq 0$ then we can get a bound on $P_1, P_2$ in terms of $I, J$. Of course $P_1, P_2$ depend on $f_k, g_k$ but one can show that there is some $C$ such that for all $(f_k, g_k)$ in the unit ball in $\hat{Y}$ that $|P_1| + |P_2| \leq C$. This would show the existence of some $C_k$ such that for all $(f_k, g_k)$ in the unit ball in $\hat{Y}$ that $\sup_{0<r\leq1} (r^t |a_k(r)| + r^t |b_k(r)|) \leq C_k$ and we would have the desired estimates at least in the modified spaces. We now justify this $2 \times 2$ matrix is invertible. Suppose the result is false and hence the $\ker(A_{\gamma_1}) = \ker(A_{\gamma_2})$. One can then use this fact and the fact that $\det(A_{\gamma_1}) = \det(A_{\gamma_2}) = 0$ to see

\[
(\beta + \gamma_2 - 1)(\gamma_1^2 + (N - 2)\gamma_1 - \mu_k) = (\beta + \gamma_1 - 1)(\gamma_2^2 + (N - 2)\gamma_2 - \mu_k),
\]

\[
\gamma_1(\gamma_2^2 + (N + 2\beta - 4)\gamma_2 - (\beta - 1)(N + \beta - 3) - \mu_k) = \gamma_2(\gamma_1^2 + (N + 2\beta - 4)\gamma_1 + (\beta - 1)(N + \beta - 3) - \mu_k).
\]

By factoring $\gamma_1 - \gamma_2$ out of both of these one arrives at

\[
(\gamma_1 - \gamma_2)\{(\gamma_1 + \gamma_2)(\beta - 1) + \gamma_1\gamma_2 + (N - 2)(\beta - 1) + \mu_k\} = 0,
\]

\[
(\gamma_1 - \gamma_2)\{((\beta - 1)(N + \beta - 3) - \mu_k - \gamma_1\gamma_2\} = 0.
\]

Since $\gamma_1 - \gamma_2 \neq 0$ we must have the other term in each equation to be zero. Now note if $\beta \geq 1$ then $\{(\gamma_1 + \gamma_2)(\beta - 1) + \gamma_1\gamma_2 + (N - 2)(\beta - 1) + \mu_k\} > 0$ and hence we arrive at a contradiction. If $0 < \beta < 1$ then $\{((\beta - 1)(N + \beta - 3) - \mu_k - \gamma_1\gamma_2\} < 0$ and again we arrive at a contradiction.

In case (ii) we use the same approach as above but we now pick $P_4 = 0$ and hence we want to solve

\[
\begin{pmatrix}
  c_1 & c_2 & c_3 \\
  d_1 & d_2 & d_3
\end{pmatrix}
\begin{pmatrix}
  P_1 \\
  P_2 \\
  P_3
\end{pmatrix} = \begin{pmatrix}
  I \\
  J
\end{pmatrix}.
\]
If the dimension of the column space of the matrix on the left is two then we proceed as before and obtain the desired result. So we now suppose this is false and hence we can take \( c_i = c \) and \( d_i = d \) for \( i = 1, 2, 3 \) where \((c, d)^T \neq 0\) (recall \((c_i, d_i)^T \in \ker(A_{\gamma_i})\)). Since we have distinct roots we can assume \( \gamma_1 \neq 0 \). So now we proceed as in case (i) and \((c, d)^T \in \ker(A_{\gamma_1}), \ker(A_{\gamma_2})\) to arrive at (37) and since we have \( \gamma_1 \neq \gamma_2 \) we must have \((\beta - 1)(N + \beta - 3) - \mu_k - \gamma_1 \gamma_2 = 0\). Now we repeat this process using fact \((c, d)^T \in \ker(A_{\gamma_1}), \ker(A_{\gamma_2})\) to arrive at (37) but with \( \gamma_2 \) replaced with \( \gamma_3 \). Again since \( \gamma_1 \neq \gamma_3 \) we then must have \((\beta - 1)(N + \beta - 3) - \mu_k - \gamma_1 \gamma_3 = 0\). Using these two equalities gives \( \gamma_1 \gamma_3 = \gamma_1 \gamma_2 \) and since \( \gamma_1 \neq 0 \) we must have \( \gamma_2 = \gamma_3 \) which is a contradiction.

Case (iii). The proof from case (ii) essentially works in this case.

**Case (II), Two real and a double roots.** We need to consider the following cases:

(i) \( \gamma_3 = \gamma_4 < -t \) and \( \gamma_1, \gamma_2 > -t \),

(ii) \( \gamma_1 < -t \), and \( \gamma_2, \gamma_3 = \gamma_4 > -t \),

(iii) \( \gamma_1, \gamma_2 < -t \), and \( \gamma_3 = \gamma_4 > -t \).

For case (i) we will use mostly the approach from (i) case (I). The formula for the general solution is given by (30) and we need to pick the \( P_k \) to give us the needed boundary conditions and the estimate. Any of the terms with \( \gamma_1, \gamma_2 \) follow exactly as in case (I); i.e. we pick the \( T_1 \in \{0, 1\} \) as before and we follow the same approach to get the desired estimate. We also do this exact procedure for the terms in \( \gamma_3, \gamma_4 \) that do not involve the log term. We now consider the log terms. Consider the case of the \( \gamma_3 \) term with the log term in the formula for \( a_k \). So here I am referring to the term is

\[
I_k(r) := c_3 r^{\gamma_3} \int_{T_3}^{r} \hat{c}_{15} \ln(\sigma) f_k(\sigma) \sigma^{-\gamma_3+1} d\sigma - c_3 \ln(r) r^{\gamma_3} \hat{c}_{15} \int_{T_3}^{r} f_k(\sigma) \sigma^{-\gamma_3+1} d\sigma.
\]

To choose \( T_3 \) depends on the value of \( \gamma_3 \). We assume \( \gamma_3 < -t \) and hence we take \( T_3 = 0 \). In this case \( I_k \) becomes

\[
I_k(r) = c_3 \hat{c}_{15} r^{\gamma_3} \int_{0}^{r} \ln \left( \frac{\sigma}{r} \right) f_k(\sigma) \sigma^{-\gamma_3+1} d\sigma,
\]

and using the bound \( |\sigma^{t+2}| f_k(\sigma)| \leq C \) and using a change of variables \( s = \frac{\sigma}{r} \) one arrives at \( r^t |I_k(r)| \leq C_k \) for all \( 0 < r < 1 \). Its crucial that for the log terms that one combines like terms to obtain the needed estimate; to do this one needs the exact value of some of the involved constants. One similarly obtain estimates on the rest of the terms in the formula for \( a_k \) and \( b_k \). To pick the \( P_k \) we use exactly the same procedure as in Case I.

For case (ii) will use the same approach as we did in part (ii) of Case (I). We now pick \( P_1 = 0 \) and hence we want to solve

\[
\begin{pmatrix}
  c_2 & c_3 & c_4 \\
  d_2 & d_3 & d_4
\end{pmatrix}
\begin{pmatrix}
  P_2 \\
  P_3 \\
  P_4
\end{pmatrix} = \begin{pmatrix}
  I \\
  J
\end{pmatrix}.
\]

If the dimension of the column space of the matrix on the left is two then we proceed as before and obtain the desired result. So we now suppose this is false and hence we can take \( c_i = c \) and \( d_i = d \) for \( i = 2, 3, 4 \) where \((c, d)^T \neq 0\) (recall \((c_i, d_i)^T \in \ker(A_{\gamma_i})\)). Since we have distinct roots (I mean \( \gamma_2 \neq \gamma_3 = \gamma_4 \)), we can assume \( \gamma_2 \neq 0 \). So now we proceed as in case (i) and \((c, d)^T \in \ker(A_{\gamma_2}), \ker(A_{\gamma_3})\) to arrive at (37) and since we have \( \gamma_2 \neq \gamma_3 \) we must have \((\beta - 1)(N + \beta - 3) - \mu_k - \gamma_2 \gamma_3 = 0\).
Since $\gamma_3$ is a double root, then considering (25), if $(c_3, d_3)^T = (c_4, d_4)^T = (c, d)^T$, then $A_{\gamma_3}'(c_3, d_3)^T = 0$ gives us

\[
\begin{cases}
(2\gamma_3 + (N - 2))c = \lambda_1 p\tau^{p-1}d, \\
\lambda_2 q\tau^{q-1}c = (2\gamma_3 + (N + 2\beta - 4))d.
\end{cases}
\]

The above system and the fact that $A_{\gamma_2}(c, d)^T = 0$, shows

\[
\begin{cases}
\gamma_2^2 - 2\gamma_2\gamma_3 + (\beta - 1)(N + \beta - 3) - \mu_k = 0, \\
\gamma_2^2 - 2(\beta + \gamma_2 - 1)\gamma_3 - \mu_k - (N - 2)(\beta - 1) = 0.
\end{cases}
\]

This implies $\gamma_3 = \frac{-2N-\beta+5}{2} < 0$, and this is contradiction with the fact that $\gamma_1\gamma_3 < 0$.

For case (iii), we now pick $P_1 = P_2 = 0$ and hence we want to solve

\[
\begin{pmatrix}
c_3 & c_4 \\
d_3 & d_4
\end{pmatrix}
\begin{pmatrix}
P_3 \\
P_4
\end{pmatrix}
= 
\begin{pmatrix}
I \\
J
\end{pmatrix}.
\]

If $c_3d_4 - c_4d_3 = 0$, and again considering (38) and $A_{\gamma_1}(c, d)^T = 0$, we get $(\gamma_3^2 + (N - 2)\gamma_3 - \mu_k)c = (\beta + \gamma_3 - 1)(2\gamma_3 + (N - 2))c$. Since $c \neq 0$, then $\gamma_3^2 + 2(\beta - 1)\gamma_3 + (\beta - 1)(N - 2) + \mu_k = 0$. Since $0 \leq \beta < 2$ and $N \geq 3$ then there is no real roots of the above equation and this shows that there is contradiction.

**Case (III) Two real distinct roots $\gamma_1$ and $\gamma_2$ and a complex root $\xi + i\eta$.** In this case the general solution is given by (33). We will consider two cases here:

- Case (i): $\gamma_1, \gamma_2 > -t$, $\xi < -t$,
- Case (ii): $\gamma_1, \gamma_2 > -t$, $\xi > -t$.

In case (i) we essentially follow exactly as in part (i) of case (I). We will take $P_3 = P_4 = 0$ and we then need to choose $P_1, P_2$. Since $\gamma_1 \neq \gamma_2$ is nonzero we can show the resulting linear system is nondegenerate and hence we easily solve for $P_1, P_2$. The estimates follow essentially as in the prior cases.

For case (ii) we can again set $P_3 = P_4 = 0$ and carry on as the above case.

**Proof of Lemma 2.2.** For $p, q, N$ fixed, we can show for large $k$, we are in the case of four distinct real roots (part (II) of Theorem 2.4). Additionally we can show that for large $k$, we are in the case of two of the roots less than $-t$ and two greater than $-t$ (part (IV) of Theorem 2.4). So fix $k_0$ large enough and then we consider $(\phi, \psi)$ in the kernel of $L$ and take $k \geq k_0$. We assume $\gamma_3, \gamma_4 < -t$ and $\gamma_1, \gamma_2 > -t$ and we now write out $(\phi, \psi)$ in terms of $(a_k, b_k)$ (we are now dropping the $k$ dependence) to see

\[
\begin{pmatrix}
a(r) \\
b(r)
\end{pmatrix} = \sum_{j=1}^{4} D_j \begin{pmatrix}
c_jr^{\gamma_j} \\
d_jr^{\gamma_j+\beta-1}
\end{pmatrix},
\]

and to have $a(1) = b(1) = 0$ we see that $D_j$ satisfies

\[
\begin{cases}
\sum_{j=1}^{4} D_j c_j = 0, \\
\sum_{j=1}^{4} D_j d_j = 0.
\end{cases}
\]

See near (34) to see that $c_1d_2 - c_2d_1 \neq 0$. 


If we call $D_3 = \hat{s}$ and $D_4 = \hat{t}$ to be the free parameters one can then show that

$$
a(r) = \hat{s} \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} c_1r^{\gamma_1} + \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) c_2r^{\gamma_2} + c_3r^{\gamma_3} \right) \\
+ \hat{t} \left( \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) c_1r^{\gamma_1} + \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) c_2r^{\gamma_2} + c_4r^{\gamma_4} \right),
$$

$$
b(r) = \hat{s} \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} d_1r^{\gamma_1+\beta-1} + \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) d_2r^{\gamma_2+\beta-1} + d_3r^{\gamma_3+\beta-1} \right) \\
+ \hat{t} \left( \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) d_1r^{\gamma_1+\beta-1} + \left( \frac{c_2d_1-c_1d_2}{c_1d_2-d_1c_2} \right) d_2r^{\gamma_2+\beta-1} + d_4r^{\gamma_4+\beta-1} \right).
$$

Also note that $c_3, c_4 \neq 0$ (this is by direct inspection of $A_4$). Since we have $\gamma_3 < -t$ one is able to show that $a$ is not sufficiently regular near the origin to allow $(\phi, \psi) \in X$ unless $\hat{s} = \hat{t} = 0$ and hence $a = b = 0$ which shows the kernel is empty. To see the needed argument that $\hat{s} = \hat{t} = 0$ see the end of the proof of Theorem 2.1 where we show the kernel of $L$, on $\mathbb{R}^N \setminus \{0\}$ with similar bounds on the solutions, is trivial.

\[\square\]

2.3. The roots of $\det(A_4) = 0$. In this section we state the main result regarding the roots of $\det(A_4) = 0$. But before that we recall a fairly well known result for Quartic Functions which gives us an explicit formulas for the roots of a fourth order polynomial with real coefficients.

**Theorem 2.3.** (Quartic result) ([31] or [41]) Consider $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a, b, c, d \in \mathbb{R}$ with $a \neq 0$. Define the quantities

$$
P = 8ac - 3b^2, \quad R = b^3 + 8da^2 - 4abc, \quad \Delta_0 = c^2 - 3bd + 12ae,$$

$$
D = 64a^3c - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4,
$$

$$
\Delta = 256a^3c^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abcde + 18abcd^3 + 16ac^3e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^2 - 4b^2c^3e + b^2c^2d^2.
$$

Then

1. If $\Delta < 0$ then the equation has two distinct real roots and two complex conjugate non-real roots.
2. If $\Delta > 0$ then either the equation’s four roots are all real or none is.
   - If $P < 0$ and $D < 0$ then all four roots are real and distinct.
   - If $P > 0$ or $D > 0$ then there are two pairs of non-real complex conjugate roots.
3. If $\Delta = 0$ then (and only then) the polynomial has a multiple root.

We now state our main result regarding the roots $\gamma$ of $\det(A_4) = 0$.

**Theorem 2.4.** Suppose $N \geq 3$ and $p, q$ satisfy

$$
p > 1, \max \left\{ \frac{N}{p(N-1) - 1}, \frac{1}{N-1} + \frac{N}{p(N-1)}, \frac{1}{p} \right\} < q < \min \left\{ 1 + \frac{2}{p}, \frac{2}{p-1} \right\}.
$$

(1) For large $k$ and $\beta > 0$ we have four distinct real roots. Two of them are negative and two of them are positive.
(II) Assume \( N \geq 3, \beta < 2 \) and
\[
(\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)(N - 1) - \lambda_1\lambda_2pq^{q-1}p^{-1}(\beta - 1) < 0. \tag{39}
\]
There exist four roots. Two of them are in the negative half plane and two of them are in the positive half plane.

(III) For \( k = 0 \) the roots of \( \det(A_\gamma) = 0 \) are given by \( \gamma_1 = 0, \gamma_2 = 1 - \beta, \gamma_3, \gamma_4 \).
\[
\gamma_3 = \frac{-(2N + \beta - 5) + \sqrt{(\beta - 1)^2 + 4\lambda_1\lambda_2pq^{q-1}p^{-1}}}{2},
\]
\[
\gamma_4 = \frac{-(2N + \beta - 5) - \sqrt{(\beta - 1)^2 + 4\lambda_1\lambda_2pq^{q-1}p^{-1}}}{2}.
\]
Also, if \( N - \tau - 2 > 0 \) and \( N - t - 2 > 0 \), then \( \gamma_4 < -t \) and \( \gamma_3 \neq -t \) and if \( \tau > 0 \), also \( -t < \gamma_2 \).

(IV) Then for sufficiently large \( k \) the roots \( \gamma \) of \( \det(A_\gamma) = 0 \) is given by \( \gamma_1 < \gamma_2 < -t \) and \( \gamma_4 > \gamma_3 > t \).

(V) For \( k \geq 1 \), under the assumption of part (II), if
\[
t - (N + \beta - 3) = -(N - \tau - 2) < 0, \quad \text{and} \quad (N - 2 - \tau)(-2(N - 1) + (pq - 1)(N - 2 - t)t) < 0
\]
then for any root \( \gamma \) we have \( \text{Re}(\gamma) \neq -t \).

Proof. We prove each part, separately, as follows:

(I) For using Part 2 of Theorem 2.3, it is sufficient to show that \( \Delta > 0, P < 0 \) and \( D < 0 \). Note that
\[
P = -16k^2 + (-16N + 32)k + a_{00},
\]
\[
D = -64 ((\beta - 1)^2 + pq(N - 2 - t)(N - 2 - \tau)) k^2 + a_{11}k + a_{10},
\]
\[
\Delta = a_{28}k^8 + a_{27}k^7 + a_{26}k^6 + a_{25}k^5 + a_{24}k^4 + a_{23}k^3 + a_{22}k^2 + a_{21}k + a_{20},
\]
where
\[
a_{28} = 256 (pq(N - 2 - \tau)(N - 2 - t) + (\beta - 1)^2)^2.
\]
Now for large \( k \), we get \( P < 0 \) and \( D < 0 \). Also \( a_{28} > 0 \) which implies \( \Delta > 0 \). Thus there exists four real distinct roots. Also, for large \( k \),
\[
\left\{
\begin{array}{l}
\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 < 0, \\
\gamma_1\gamma_2\gamma_3\gamma_4 > 0, \\
(\gamma_1 + \gamma_2)\gamma_3\gamma_4 + \gamma_1\gamma_2(\gamma_3 + \gamma_4) > 0, \\
\gamma_1\gamma_2 + \gamma_3\gamma_4 + (\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4) < 0.
\end{array}
\right.
\]
This implies that two of the roots are negative and two of them are positive.

(II) Notice that
\[
\mu_k^2 - (\beta - 1)(N + \beta - 3) \mu_k
= k(k + N - 2)(k + N - 2) - (\beta - 1)(N + \beta - 3)
= k(k + N - 2)(k - \beta + 1)(k + N + \beta - 3).
\]
Since \( N \geq 3 \) and \( \beta < 2 \), then for any \( k \geq 1 \) then \( \mu_k^2 - (\beta - 1)(N + \beta - 3) \mu_k > 0 \).
Note that for \( 1 \leq \beta < 2, N \geq 3 \) and \( k \geq 1 \) we have
\[
(\beta - 1)(N + \beta - 3)(N - 2) - (N + 2\beta - 4) \mu_k - \lambda_1\lambda_2pq^{q-1}p^{-1}(\beta - 1)
= -(N - 2)(k - \beta + 1)(k + N + \beta - 3) - (N + 2\beta - 4) \mu_k - \lambda_1\lambda_2pq^{q-1}p^{-1}(\beta - 1)
< 0.
\]
If \(0 < \beta < 1\), then for any \(k \geq 1\) we have

\[
(\beta - 1)(N + \beta - 3)(N - 2) - (N + 2\beta - 4)\mu_k - \lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1} (\beta - 1) - \mu_k (N - 2)
\]

\[
= (\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)\mu_k - \lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1} (\beta - 1)
\]

\[
\leq (\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)(N - 1) - \lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1} (\beta - 1) < 0,
\]

where the last inequity holds by assumption. This implies for any \(k \geq 1, N \geq 3\) and \(0 < \beta < 2\)

\[(\beta - 1)(N + \beta - 3)(N - 2) - (2N + 2\beta - 6)\mu_k - \lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1} (\beta - 1) < 0.
\]

In addition \((2N + 2\beta - 6) > 0\). Thus if \(\gamma_1, \gamma_2, \gamma_3, \gamma_4\) are the roots of equation (18) then with respect to the above facts

\[
\left\{\begin{array}{l}
\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 < 0,
\\
\gamma_1 \gamma_2 \gamma_3 \gamma_4 > 0,
\\
(\gamma_1 + \gamma_2)\gamma_3 \gamma_4 + \gamma_1 \gamma_2 (\gamma_3 + \gamma_4) > 0.
\end{array}\right.
\]

The first and second inequalities show that either (i) all of the roots have negative real parts or (ii) two of them have negative real parts and the other two have positive real parts. But if all of them have negative real parts, this contradict the third inequality. Thus two of the roots are located in the left plane and two of them are located in the right plane.

**(III)** When \(k = 0\) we have

\[
A_\gamma := \begin{pmatrix}
\gamma^2 + (N - 2)\gamma & -\lambda_1 p \tau^{p-1} (\beta + \gamma - 1) \\
-\lambda_2 q t^{q-1} \gamma & \gamma^2 + (N + 2\beta - 4)\gamma + (\beta - 1)(N + \beta - 3)
\end{pmatrix},
\]

we want to find \(\gamma\) such that \(\det(A_\gamma) = 0\). A compaction shows that the roots of \(\det(A_\gamma) = 0\) are

\[
\gamma_1 = 0, \quad \gamma_2 = 1 - \beta,
\]

\[
\gamma_3 = \frac{-(2N + \beta - 5) + \sqrt{(\beta - 1)^2 + 4\lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1}}}{2}.
\]

\[
\gamma_4 = \frac{-(2N + \beta - 5) - \sqrt{(\beta - 1)^2 + 4\lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1}}}{2}.
\]

Also it is easy to see \(\gamma_1 \neq -t, \gamma_2 \neq -t\) and \(\gamma_3 \neq -t\) and \(\gamma_4 < -t\). Assume \(\gamma_4 = -t + \psi\) where \(\psi \geq 0\) this gives us

\[
2t - 2\psi' = (2N + \beta - 5) + \sqrt{(\beta - 1)^2 + 4\lambda_1 \lambda_2 p q t^{q-1} \tau^{p-1}}
\]

which implies \(-(N - t - 2) - (N - \tau - 2) - 2\psi' = \sqrt{(\beta - 1)^2 + 4pq(N - t - 2)(N - \tau - 2)}\). With respect to the assumption that \(N - t - 2\) and \(N - \tau - 2\) are positive, we get a contradiction.

If \(\gamma_3 = -t\), then

\[
2t = (2N + \beta - 5) - \sqrt{(\beta - 1)^2 + 4pq(N - t - 2)(N - \tau - 2)}
\]

which implies

\[
((N - t - 2) + (N - \tau - 2))^2 = (\beta - 1)^2 + 4pq(N - t - 2)(N - \tau - 2).
\]

But we know that \(pq > 1\), this means that

\[
((N - t - 2) + (N - \tau - 2))^2 = (\beta - 1)^2 > 4(N - t - 2)(N - \tau - 2)
\]

or

\[
(-t + \tau)^2 - (\beta - 1)^2 > 0.
\]
We know \( t = \tau + \beta - 1 \), this is a contradiction.

(IV) For sufficiently large \( k \) we know that

\[
\begin{align*}
(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) &= -(2N + 2\beta - 6), \\
(\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4) + \gamma_1\gamma_2 + \gamma_3\gamma_4 &\approx -\mu_k, \\
(\gamma_1 + \gamma_2)\gamma_3\gamma_4 + (\gamma_3 + \gamma_4)\gamma_1\gamma_2 &\approx +\mu_k, \\
\gamma_1\gamma_2\gamma_3\gamma_4 &\approx +\mu_k^2,
\end{align*}
\] (40)

where \( f \approx g \) means there exists \( c > 0 \) such that \( \frac{g}{c} \leq f \leq cg \). By part (I) we know that there are two positive and two negative roots. Let call them \( \gamma_1 \) and \( \gamma_2 \) are given by (18). If \( \gamma_1 + \gamma_2 = -(\gamma_3 + \gamma_4) - 2(\gamma_3 + \gamma_4) - 2(N + \beta - 3) = 2t - 2(N + \beta - 3) < 0 \). This is a contradiction.

Assume 0 \( \leq |\gamma_3| |\gamma_4| \leq c_1 < \infty \). The second relation of (40) shows that the term \( \gamma_1\gamma_4 \) behave like \(-\mu_k \) and then by the forth relation of (40) there is a contradiction. Thus at least one of the \( \gamma_3 \to -\infty \) or \( \gamma_2 \to \infty \). If \( \gamma_3 \to -\infty \), then we done. If \( \gamma_3 \) is bounded then \( \gamma_2 \to +\infty \). Thus we have \( \gamma_1, \gamma_2 \to +\infty \) and \( \gamma_4 \to -\infty \). Now the third relation of (40) shows that \( \gamma_4\gamma_1\gamma_2 \) behave like \(+\mu_k \) which is a contradiction and this item is proved.

(V) Assume \( \gamma_3 \) and \( \gamma_4 \) have negative real parts. Since

\[ \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = -2(N + \beta - 3). \]

(i) If \( \gamma_3 \) and \( \gamma_4 \) are complex then they should be conjugate and \( \gamma_3 + \gamma_4 = 2\text{Re}(\gamma_3) \). Assume the real parts \( \gamma_3 \) is \( -t \). Then \( \gamma_1 + \gamma_2 = -2\text{Re}(\gamma_3) - 2(N + \beta - 3) = 2t - 2(N + \beta - 3) < 0 \). This is a contradiction.

(ii) If \( \gamma_3 = \gamma_4 \) is a double root. If \( \gamma_3 = -t \), then \( \gamma_3 + \gamma_4 = 2t \) and \( \gamma_1 + \gamma_2 = 2t - 2(N + \beta - 3) < 0 \). This is a contradiction.

(iii) If \( \gamma_3 \) and \( \gamma_4 \) are real roots. First, assume \( \gamma_3 = -t \) and \( \gamma_4 = -t \). Since \( \gamma_1 + \gamma_2 = -(\gamma_3 + \gamma_4) - 2(N + \beta - 3) = 2t - 2(N + \beta - 3) < 0 \). This is a contradiction. Thus at least one of the of them should be \( \neq -t \) (call it \( \gamma_3 \)). We need to show \( \gamma_4 \neq -t \). Assume \( \gamma_4 = -t \), then

\[ \det(A_*) = (\gamma + t) h(\gamma), \]

where

\[
\begin{align*}
h(\gamma) &:= \gamma^3 + h_3\gamma^2 + h_2\gamma + h_1, \\
h_3 &:= g_3 - t, \\
h_2 &:= g_2 - g_3t + t^2, \\
h_1 &:= g_1 - tg_2 + t^2g_3 - t^3,
\end{align*}
\]

and \( g_3, g_2 \) and \( g_1 \) are given by (18). If \( \gamma_1^*, \gamma_2^* \) and \( \gamma_3^* \) are the roots of \( h(\gamma) \), then

\[
\begin{align*}
\gamma_1^* + \gamma_2^* + \gamma_3^* &=-h_3 = t - g_3, \\
\gamma_1^*\gamma_2^*\gamma_3^* &=-h_1 = t^3 - t^2g_3 + tg_2 - g_1.
\end{align*}
\]

With respect to the assumption, we know \( t - g_3 < 0 \) and

\[
t^3 - t^2g_3 + tg_2 - g_1 = \left( N - \tau - 2 \right) \left( 2\mu_k - (pq - 1)(N - 2 - t)\tau \right) \geq \left( N - \tau - 2 \right) \left( 2(N - 1) - (pq - 1)(N - 2 - t)\tau \right) > 0.
\]

This says two of the roots of \( h(\gamma) \) are negative. With respect to \( -t \), \( \det(A_*) \) has three negative roots which is a contradiction with part 2. Thus the claim is proved.

\( \square \)
Proof of Corollary 2. The proof of this result rests on showing that under suitable assumptions on $p$ and $q$ that we are in one of the cases from Proposition 1 and then the result follows directly from applying the proposition.

(I) (Four real distinct roots). For $k = 0$, by Theorem 2.4 part (III) we have the roots $\gamma_i$. Note that a computation shows that $\gamma_1, \gamma_2 > -t$.

We now consider the case of $k \geq 1$. We will now show that at least two roots bigger than $-t$. Notice that Part 3 of (19) and the assumption (39) implies that

$$(\gamma_1 + \gamma_2)\gamma_3\gamma_4 + (\gamma_3 + \gamma_4)\gamma_1\gamma_2 > 0.$$ 

This is a contradiction with $\gamma_1, \gamma_2, \gamma_3, \gamma_4 < -t$. For $\gamma_1 > -t, \gamma_2, \gamma_3, \gamma_4 < -t$, we consider two cases: when $-t < \gamma_1 < 0$, with the same argument above, this is not happen, too. If $\gamma_1 \geq 0$ then part 4 of (19) shows that there is a contradiction with $0 \leq \beta < 2$.

(II) (Two real and a double root). In this case the following cases not happen $\gamma_1, \gamma_2, \gamma_3 = \gamma_4 < -t$ and $\gamma_1, \gamma_3 = \gamma_4 < -t, \gamma_2 > -t$. With the same procedure as above, it can be studied.

(III) (Two distinct real and a complex conjugate roots). In this case we claim $\gamma_1, \gamma_2 < -t, \xi > -t$, and $\gamma_1, \gamma_2, \xi < -t$ not happen. Just we study the case $\gamma_1, \gamma_2 < -t, \xi > -t$. Due to show this, we can write $A_2 = B_0 + iB_1$, where $B_i$ are real and imaginary parts and the $A_2(\hat{c}i + i\hat{d}i, \hat{d}i + i\hat{c}i)^T = 0$ implies that

$$\begin{bmatrix} B_0 \hat{c}_3 \\ \bar{d}_3 \end{bmatrix} - B_1 \begin{bmatrix} \hat{c}_4 \\ \bar{d}_4 \end{bmatrix} = 0,$$

$$B_0 \begin{bmatrix} \hat{c}_4 \\ \bar{d}_4 \end{bmatrix} + B_1 \begin{bmatrix} \hat{c}_3 \\ \bar{d}_3 \end{bmatrix} = 0.$$

A computation shows that we have

$$\begin{cases} \xi^2 - \eta^2 + (N - 2)\xi - \mu_k - \lambda_1\nu^{p-1}(\xi + \beta - 1) = 0, \\
2\xi\eta + (N - 2)\eta - \lambda_1\nu^{p-1}\eta = 0, \\
\lambda_2\nu^{q-1}\eta - 2\xi\eta - (N + 2\beta - 4)\eta = 0, \\
\lambda_2\nu^{q-1}\xi - (\xi^2 - \eta^2) - (N + 2\beta - 4)\xi - (\beta - 1)(N + \beta - 3) + \mu_k = 0. \
\end{cases}$$

The first and second equations implies the first equation below, and the third and forth equations implies the second one

$$\begin{cases} \xi^2 - \eta^2 = 2\xi^2 + 2\xi(\beta - 1) + (N - 2)(\beta - 1) + \mu_k, \\
\xi^2 - \eta^2 = 2\xi^2 - (\beta - 1)(N + \beta - 3) + \mu_k. \
\end{cases}$$

Thus $\xi = \frac{-2N - \beta + 5}{2} < 0$ and this is a contradiction with the fact that when $0 \leq \beta < 2, \xi > 0$. Because we get $\gamma_1, \gamma_2 < -t$ and $\xi > 0$ and this is a contradiction with the Part (II) of Theorem 2.4.

2.4. Proof of Theorem 2.1. In this section we prove Theorem 2.1. Fix $k_0, p, q, N$ as in all the prior results. Our goal is to show the existence of some $C > 0$ such that for all $(f, g) \in Y_{k_0}$ (where $(f, g)$ has only a finite number of nonzero modes) there exists some $(\phi, \psi) \in X_{k_0}$ which solves (13) and $\|\phi, \psi\|_X \leq C\|\phi, \psi\|_Y$. The important point here is the $C$ does not depend on how many nonzero modes $(f, g)$ has. Existence of the solutions easily follows from the prior ode results; the only issue is the estimate might fail. Note for each mode we have an estimate and one can combine the estimates to arrive at a similar result for $(f, g)$ with a finite number of nonzero modes. We now suppose the result is false. Then by normalizing we can
assume there is \( (\phi_m, \psi_m) \in X_0 \) with \( \| (\phi_m, \psi_m) \|_X = 1 \) and \( \| (f_m, g_m) \|_Y \to 0 \). Note

we can write \( L \) explicitly as

\[
L(\phi, \psi) = \left( -\Delta \phi(x) - \frac{K_1 x \cdot \nabla \psi(x)}{|x|^{\beta+1}}, -\Delta \psi(x) - \frac{K_2 x \cdot \nabla \phi(x)}{|x|^{\alpha+1}} \right),
\]

where \( K_1, K_2 \) are explicit constants. Note directly from the bound \( \| (\phi_m, \psi_m) \|_X = 1 \) we have that \( \phi_m, \psi_m \) are both bounded in \( W^{2,\gamma}_{loc}(\overline{B_1\setminus\{0\}}) \). After passing to subsequences we can assume \( \phi_m, \psi_m \to \phi, \psi \) in \( W^{2,\gamma}_{loc}(\overline{B_1\setminus\{0\}}) \). Fixing \( 0 < s \leq \frac{1}{2} \) one can use the weak lower semicontinuity of Sobolev norms and the Sobolev imbedding theorem to pass to the limit in each term in the norm of \( \| (\phi_m, \psi_m) \|_X \) (again this is for fixed \( s \)) and then one can take the required suprema over \( s \). This shows that \( (\phi, \psi) \in X \). Additionally one can see that \( L(\phi, \psi) = 0 \) in \( B_1\setminus\{0\} \) and we must have, after considering the kernel result of \( L, \phi = \psi = 0 \). We now rewrite the equations for \( \phi_m, \psi_m \) as

\[
-\Delta \phi_m(x) = \frac{K_1 x \cdot \nabla \psi_m(x)}{|x|^{\beta+1}} + f_m(x) \quad \text{in } B_1\setminus\{0\},
\]

\[
-\Delta \psi_m(x) = \frac{K_2 x \cdot \nabla \phi_m(x)}{|x|^{\alpha+1}} + g_m(x) \quad \text{in } B_1\setminus\{0\},
\]

with \( \phi_m = \psi_m = 0 \) on \( \partial B_1 \). Note for each \( 0 < s < 1 \) we have the right hand sides of the equations converge to zero in \( L^p(s < |x| < 1) \) and hence we can use local elliptic regularity theory to see that \( \phi_m, \psi_m \to 0 \) in \( W^{2,\gamma}(s < |x| < 1) \) for each \( 0 < s < 1 \). The only potential problem is the mass is concentrating at the origin. We firstly show this concentration cannot happen (at least for the lower order derivatives).

We claim

\[
\sup_{0 < s \leq \frac{1}{2}} s^{k} \gamma - N \left\{ \int_{A_s} |\phi_m|^\gamma \, dx + s^{\gamma} \int_{A_s} |\nabla \phi_m|^\gamma \, dx \right\} \to 0,
\]

and

\[
\sup_{0 < s \leq \frac{1}{2}} s^{k} \gamma - N \left\{ \int_{A_s} |\psi_m|^\gamma \, dx + s^{\gamma} \int_{A_s} |\nabla \psi_m|^\gamma \, dx \right\} \to 0,
\]

as \( m \to \infty \). If we accept these claims one can use a scaling argument to directly show the second order terms in the norm converge to zero also; we are omitting the scaling argument here since we shortly do a very similar (but more involved argument) to prove the claim. So we suppose now the claim is false and after considering the above mentioned convergence we can assume there is some \( \epsilon_0 > 0 \) and \( s_m \downarrow 0 \) such that for all \( m \) we have

\[
I_m := s_m^{k} \gamma - N \left\{ \int_{A_{s_m}} |\phi_m|^\gamma \, dx + s_m^{\gamma} \int_{A_{s_m}} |\nabla \phi_m|^\gamma \, dx \right\} \geq \epsilon_0.
\]

We now define the rescaled functions \( \hat{\phi}_m(x) := s_m^t \phi_m(s_m x) \) and \( \hat{\psi}_m(x) := s_m^t \psi_m(s_m x) \) and a computation shows that \( L(\hat{\phi}_m, \hat{\psi}_m) = (\hat{f}_m, \hat{g}_m) \) where \( \hat{f}_m(x) := s_m^{t+2} f_m(s_m x) \) and \( \hat{g}_m(x) := s_m^{t+2} g_m(s_m x) \). Writing more explicitly, in the above notation, we arrive at

\[
-\Delta \hat{\phi}_m(x) = \frac{K_1 x \cdot \nabla \hat{\psi}_m(x)}{|x|^{\beta+1}} + s_m^{t+2} f_m(s_m x) \quad \text{in } 0 < |x| < \frac{1}{s_m},
\]

\[
-\Delta \hat{\psi}_m(x) = \frac{K_2 x \cdot \nabla \hat{\phi}_m(x)}{|x|^{\alpha+1}} + s_m^{t+2} g_m(x) \quad \text{in } 0 < |x| < \frac{1}{s_m},
\]
with \( \hat{\phi}_m, \hat{\psi}_m = 0 \) on \( |x| = \frac{1}{s_m} \). Note that by a change of variables we have

\[
\int_{1<|x|<2} |\hat{\phi}_m(x)|^\gamma + |\nabla \hat{\phi}_m(x)|^\gamma dx = I_m \geq \varepsilon_0, \tag{41}
\]

\[
\int_{T<|x|<2T} |\hat{\phi}_m(x)|^\gamma dx \leq T^{N-t\gamma}, \quad \int_{T<|x|<2T} |\nabla \hat{\phi}_m(x)|^\gamma dx \leq T^{N-(t+1)\gamma} \tag{42}
\]

for all \( 0 < T < \frac{1}{2s_m} \). For \( k \geq 2 \) we set \( E_k := \{ x \in \mathbb{R}^N : \frac{1}{k} < |x| < k \} \) and \( \tilde{E}_k := \{ x \in \mathbb{R}^N : \frac{1}{2k} < |x| < 2k \} \). Using the equations for \( \hat{\phi}_m, \hat{\psi}_m \) and some estimates on \( \hat{\psi}_m \) analogous to (42) (or in fact just using the second order versions of (42)) we see \( \hat{\phi}_m, \hat{\psi}_m \) are bounded in \( W^{2,\gamma}(E_k) \) for all \( k \geq 2 \). By a diagonal argument and after passing to subsequences we can assume \( \hat{\phi}_m, \hat{\psi}_m \rightarrow \hat{\phi}, \hat{\psi} \) in \( W^{2,\gamma}_{loc}(\mathbb{R}^N\setminus\{0\}) \). We now obtain some integral bounds on \( \hat{\psi}_m \) which are bounded in \( W^{2,\gamma}(\tilde{E}_k) \) for all \( k \geq 2 \) and hence we can pass to the limit in the equations for \( \hat{\phi}_m, \hat{\psi}_m \) to arrive at \( L(\dot{\phi}, \dot{\psi}) = 0 \) in \( \mathbb{R}^N\setminus\{0\} \). Also note by (41) we have \( \hat{\phi} \neq 0 \). So our goal is to now obtain a contradiction. Firstly note we can pass to the limit in the estimates in (42) and also the analogous versions for \( \hat{\psi}, \psi \), ie.

\[
\int_{T<|x|<2T} |\hat{\phi}(x)|^\gamma dx \leq T^{N-t\gamma}, \quad \int_{T<|x|<2T} |\nabla \hat{\phi}(x)|^\gamma dx \leq T^{N-(t+1)\gamma} \tag{43}
\]

\[
\int_{T<|x|<2T} |\hat{\psi}(x)|^\gamma dx \leq T^{N-t\gamma}, \quad \int_{T<|x|<2T} |\nabla \hat{\psi}(x)|^\gamma dx \leq T^{N-(t+1)\gamma}. \tag{44}
\]

We now write \( \hat{\phi}(x) = \sum_{k=k_0}^\infty a_k(r)\zeta_k(\theta), \hat{\psi}(x) = \sum_{k=k_0}^\infty b_k(r)\zeta_k(\theta) \) (note we are abusing notation with regards to the higher multiplicity eigenpairs). We now obtain integral bounds on \( a_k \) and \( b_k \) from \( \hat{\phi}, \hat{\psi} \). Using Jensen’s inequality we see there is some \( \alpha_k > 0 \) such that

\[
|a_k(r)|^\gamma \leq \alpha_k \int_{|\theta|=1} |\hat{\phi}(r\theta)|^\gamma d\theta,
\]

and similarly for \( b_k \). Now multiplying this by \( r^{N-1} \) and integrating from \( T \) to \( 2T \) we arrive at

\[
\int_T^{2T} |a_k(r)|^\gamma r^{N-1} dr \leq \alpha_k T^{N-t\gamma}, \tag{45}
\]

for all \( T > 0 \). We similarly obtain a bound on \( b_k \). Now note \( (a_k, b_k) \) solves (16) on \( 0 < r < \infty \) and then from our earlier ode results we know that for each \( k \geq k_0 \) we have \( (a_k, b_k) = (a, b) \) given by

\[
a(r) = \sum_{i=1}^4 D_1 c_i r^{\gamma_i}, \quad b(r) = \sum_{i=1}^4 D_1 d_i r^{\gamma_i+\beta-1},
\]

where \( D_i \in \mathbb{R} \) and \( (c_i, d_i)^T \) is in the kernel of \( A_{\gamma_i} \). Also recall the roots \( \gamma_i \) are distinct and \( \gamma_3, \gamma_4 < -t \) and \( \gamma_1, \gamma_2 > -t \). Putting the formula for \( a_k \) into (45) and doing a change of variables gives

\[
\int_1^2 s^{N-1} \left| \sum_{i=1}^4 D_1 c_i s^{\gamma_i} T^{\gamma_i+t} \right| ds \leq \alpha_k.
\]

Since \( \gamma_i \) are distinct and they are all different from \( -t \) we see by sending \( T \rightarrow 0, \infty \) we can conclude that \( D_1 c_i = 0 \) for \( 1 \leq i \leq 4 \) and since we know \( c_i \neq 0 \) we have \( D_i = 0 \) for \( 1 \leq i \leq 4 \). From this we see \( a_k = 0 \) and similarly \( b_k = 0 \). Hence we have
\[ \hat{\phi} = \hat{\psi} = 0 \] but this contradicts the earlier result and hence we have arrived at the needed contradiction. This completes the proof.

3. **The fixed point argument.** We begin with a few simple computations which outline our approach to showing \( J_\varepsilon \) is a contraction. Return to \((11)\) which we can now write as \( L(\hat{\phi}, \hat{\psi}) = H(\phi, \psi) = (H_1(\phi, \psi), H_2(\phi, \psi)) \) where we are suppressing the \( \varepsilon \) dependence in the nonlinear function \( H \). Here \( H \) is the right hand side of \((11)\). As components we can write this as \( L_i(\hat{\phi}, \hat{\psi}) = H_i(\phi, \psi) \). Let \( B_r \) denote the closed ball of radius \( r \) centered at the origin in \( X \). Our goal is to show that \( J_\varepsilon \) is a contraction on \( B_r \).

**Into.** Let \((\phi, \psi) \in B_r \). Then we have
\[
\| J_\varepsilon(\phi, \psi) \|_X = \| (\hat{\phi}, \hat{\psi}) \|_X \\
\leq C\| H(\phi, \psi) \|_{Y_1} \text{ by the linear theory for } L \\
= C\| H_1(\phi, \psi) \|_{Y_1} + C\| H_2(\phi, \psi) \|_{Y_1}. \tag{46}
\]
We now move on to the contraction part.

**Contraction.** Let \((\phi_1, \psi_1) \in B_r \). Then by the linearity of \( L \) we have
\[
L(\hat{\phi}_2 - \hat{\phi}_1, \hat{\psi}_2 - \hat{\psi}_1) = H(\phi_2, \psi_2) - H(\phi_1, \psi_1) \tag{47}
\]
and hence
\[
\| J_\varepsilon(\phi_2, \psi_2) - J_\varepsilon(\phi_1, \psi_1) \|_X \\
= \| (\hat{\phi}_2 - \hat{\phi}_1, \hat{\psi}_2 - \hat{\psi}_1) \|_X \\
\leq C\| H(\phi_2, \psi_2) - H(\phi_1, \psi_1) \|_{Y_1} \text{ by (47) and the linear theory for } L \\
= C\| H_1(\phi_2, \psi_2) - H_1(\phi_1, \psi_1) \|_{Y_1} + C\| H_2(\phi_2, \psi_2) - H_2(\phi_1, \psi_1) \|_{Y_1}. \tag{48}
\]

We now hope to choose \( 0 < r < 1 \) and \( \varepsilon > 0 \) such that the right hand side of \((46)\) is less than \( r \) and such that the right hand side of \((48)\) is bounded above by \( \rho\| (\phi_2, \psi_2) - (\phi_1, \psi_1) \|_X \) with \( 0 < \rho < 1 \).

Recall we have defined \( J_\varepsilon(\phi, \psi) = (\hat{\phi}, \hat{\psi}) \) where \((\hat{\phi}, \hat{\psi})\) satisfies \((11)\). To obtain a solution \((\phi, \psi)\) of \((10)\) we will show that \( J_\varepsilon \) is a contraction on \( B_r \) where \( B_r \) is the closed ball of radius \( r \) centered at the origin in \( X \). The following lemma will be helpful in controlling the lower order terms.

**Lemma 3.1.** Assume \((\phi, \psi) \in X \). There exists a constant \( C_{Se} \) such that
\[
\sup_{A_x} |\phi| \leq C_{Se} \frac{\| \phi \|_{X_2}}{\sqrt{r}} \text{ and } \sup_{A_x} |\nabla \phi| \leq C_{Se} \frac{\| \phi \|_{X_2}}{\sqrt{r^{n+1}}},
\]
\[
\sup_{A_x} |\psi| \leq C_{Se} \frac{\| \psi \|_{X_2}}{\sqrt{r}} \text{ and } \sup_{A_x} |\nabla \psi| \leq C_{Se} \frac{\| \psi \|_{X_2}}{\sqrt{r^{n+1}}}. \]

**Proof.** By a standard scaling argument and the Sobolev embedding theorem after noting the fact that \( N < \gamma < \infty \). \( \square \)

Recall \( L(\hat{\phi}, \hat{\psi}) = (L_1(\hat{\phi}, \hat{\psi}), L_2(\hat{\phi}, \hat{\psi})) \). Then we can write \( L(\hat{\phi}, \hat{\psi}) = H(\phi, \psi) \) in components as
\[
L_1(\hat{\phi}, \hat{\psi}) = \lambda_1 |\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)|^p \\
- \lambda_1 |\nabla w_2|^p - p\lambda_1 |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \psi + E_\varepsilon(w_1) + E_\varepsilon(\phi),
\]
\[
L_2(\hat{\phi}, \hat{\psi}) = \lambda_2 |\nabla w_1 + \nabla \phi + \varepsilon A(\varepsilon, x)(\nabla w_1 + \nabla \phi)|^q \\
- \lambda_2 |\nabla w_1|^q - q\lambda_2 |\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \phi + E_\varepsilon(w_2) + E_\varepsilon(\psi).
\]

One may rewrite the above as
\[
L_1(\hat{\phi}, \hat{\psi}) = \lambda_1 K_1(\psi) + \lambda_1 K_2(\psi) + E_\varepsilon(w_1) + E_\varepsilon(\phi),
\]
\[
L_2(\hat{\phi}, \hat{\psi}) = \lambda_2 K_3(\phi) + \lambda_2 K_4(\phi) + E_\varepsilon(w_2) + E_\varepsilon(\psi).
\]

where
\[
K_1(\psi) := |\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)|^p - |\nabla w_2 + \nabla \psi|^p,
\]
\[
K_2(\psi) := |\nabla w_2 + \nabla \psi|^p - |\nabla w_2|^p - p|\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla \psi,
\]
\[
K_3(\phi) := |\nabla w_1 + \nabla \phi + \varepsilon A(\varepsilon, x)(\nabla w_1 + \nabla \phi)|^q - |\nabla w_1 + \nabla \phi|^q,
\]
\[
K_4(\phi) := |\nabla w_1 + \nabla \phi|^q - |\nabla w_1|^q - q|\nabla w_1|^{q-2} \nabla w_1 \cdot \nabla \phi.
\]

Notice that
\[
\|L_1(\hat{\phi}, \hat{\psi})\|_{Y_1} \leq \|\lambda_1 K_1(\psi) + \lambda_1 K_2(\psi)\|_{Y_1} + \|E_\varepsilon(w_1) + E_\varepsilon(\phi)\|_{Y_1},
\]
\[
\|L_2(\hat{\phi}, \hat{\psi})\|_{Y_r} \leq \|\lambda_2 K_3(\phi) + \lambda_2 K_4(\phi)\|_{Y_r} + \|E_\varepsilon(w_2) + E_\varepsilon(\psi)\|_{Y_r}.
\]

Here we remind the following lemma (see [1, 2] or [13]) which is necessary for studying the fixed point argument of the problem.

**Lemma 3.2.** Suppose \( p > 0 \), then there is some \( C > 0 \) and \( R > 0 \) (small) such that:

1. for all \( x, y, z \in \mathbb{R}^N \) with \( |x| \neq 0 \) and \( |y|, |z| \leq R|x| \) one has
   \[
   |x + y|^p - |x + z|^p \leq C|x|^{p-1}|y - z|.
   \]
2. for all \( x, y \in \mathbb{R}^N \) with \( |x| \neq 0 \) and \( |y| \leq R|x| \) one has
   \[
   |x + y|^p - |x|^p - p|x|^{p-2} x \cdot y \leq C|x|^{p-2}|y|^2.
   \]
3. for all \( x, y, z \in \mathbb{R}^N \) with \( |x| \neq 0 \) and \( |y|, |z| \leq R|x| \) one has
   \[
   |x + y|^p - |x + z|^p - p|x|^{p-2} x \cdot (y - z) \leq C \left( |x|^{p-2} (|y| + |z|) \right) |y - z|.
   \]

In the following results we only assume \( p, q > 0 \) even though for our linear theory we assume \( p, q > 1 \). The into result can be written as follows:

**Theorem 3.3.** For \( 0 < r < 1 \), chosen sufficiently small and \( \varepsilon > 0 \), \( J_\varepsilon(B_r) \subset (B_r) \), where \( B_r \) is a ball in \( X_t \times X_r \), in fact it is into \( X_t \times X_r \).

**Proof.** A computation for \( K_1(\psi) \) shows that
\[
\|K_1(\psi)\|_{Y_1}^\gamma = \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma-\gamma} \int_{A_s} |K_1(\psi)|^\gamma dx
\]
\[
= \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma-\gamma} \int_{A_s} |\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)|^p - |\nabla w_2 + \nabla \psi|^p\|^\gamma dx
\]
\[ \leq \kappa_{11} \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} \left( |(\nabla w_2 + \nabla \psi|^{p-1} + |\varepsilon A(\varepsilon, x) (\nabla w_2 + \nabla \psi)|^{p-1}) \times |\varepsilon A(\varepsilon, x) (\nabla w_2 + \nabla \psi) ||^\gamma \right) dx \]

\[ \leq \kappa_{13} \varepsilon^{\gamma} \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |\nabla w_2|^{p\gamma} + |\nabla \psi|^{p\gamma} dx \]

\[ \leq \kappa_{13} \varepsilon^{\gamma} \left( \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} \left( \frac{\tau C_2}{\varepsilon^{2\gamma}} \right)^{p\gamma} dx + \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |\nabla \psi|^{p\gamma} dx \right) \]

\[ \leq \kappa_{14} \varepsilon^{\gamma} \left( \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} x^{-(\tau+1)p\gamma} dx \right) (1 + \|\psi\|_{X_\gamma}^p) \]

\[ \leq \kappa_{15} \varepsilon^{\gamma} \left( \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t) - (\tau+1)p\gamma} \right) (1 + \|\psi\|_{X_\gamma}^p) \]

\[ \leq \kappa_{14} \varepsilon^{\gamma} (1 + \|\psi\|_{X_\gamma}^p)^\gamma. \]

Thus for \((\phi, \psi) \in B_r, \ 0 < r < 1 \) and \( p > 0 \), there exists \( \kappa_0 \) such that

\[ \|K_1(\psi)\|_{Y_1} \leq \kappa_0 \varepsilon (1 + \|\psi\|_{X_\gamma}). \]

Also for \( K_2(\psi) \) we can write

\[ \|K_2(\psi)\|_{Y_2} = \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |K_2(\psi)|^\gamma dx \]

\[ = \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} \left( |(\nabla w_2 + \nabla \psi)|^p - |\nabla w_2|^p - p|\nabla w_2|^{p-2}\nabla w_2 \cdot \nabla \psi \right)^\gamma dx \]

\[ \leq \kappa_{21} \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |\nabla \psi|^p + |\nabla w_2|^{p-2}|\nabla \psi|^2 |^\gamma dx \]

\[ \leq \kappa_{22} \left( \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |\nabla \psi|^{p\gamma} dx + \sup_{0 \leq s \leq \frac{1}{2}} s^{(2+t)\gamma - N} \int_{A_s} |\nabla w_2|^{p-2}|\nabla \psi|^2 |^\gamma dx \right) \]

\[ \leq \kappa_{23} \left( \|\psi\|_{X_\gamma}^p + \|\psi\|_{X_\gamma}^{2^\gamma} \right). \]

Thus for \((\phi, \psi) \in B_r, \ 0 < r < 1 \) and \( p > 0 \)

\[ \|K_2(\psi)\|_{Y_2} \leq \kappa_{23} \left( \|\psi\|_{X_\gamma}^{p-1} + \|\psi\|_{X_\gamma} \right) \|\psi\|_{X_\gamma} = \kappa_1 \|\psi\|_{X_\gamma}, \]
where \( k_1 \to 0 \) when \( r \to 0 \).

By (9), we have

\[
\| E_\varepsilon(w_1) \|_{Y_1}^r = \sup_{0 \leq s \leq \frac{1}{4}} (2 + t) \gamma \int_A |E_\varepsilon(w_1)| \gamma dx
\]

\[
= \varepsilon \gamma \sup_{0 \leq s \leq \frac{1}{4}} (2 + t) \gamma \int_A \left( 2 \sum_{i,k} u_{1x_i} \partial_{y_i} \hat{\phi}_k + \sum_{i,k} w_{1x_i} \partial_{y_i} \hat{\psi}_k \right)
\]

\[
\leq C_1 \varepsilon^r
\]

and

\[
\| E_\varepsilon(\phi) \|_{Y_1}^r = \sup_{0 \leq s \leq \frac{1}{4}} (2 + t) \gamma \int_A |E_\varepsilon(\phi)| \gamma dx
\]

\[
= \varepsilon \gamma \sup_{0 \leq s \leq \frac{1}{4}} (2 + t) \gamma \int_A \left( 2 \sum_{i,k} \phi_{x_i} \partial_{y_i} \hat{\phi}_k + \sum_{i,k} \phi_{x_i} \partial_{y_i} \hat{\psi}_k \right)
\]

\[
\leq C_2 \varepsilon^r \| \phi \|_{X_1}^r.
\]

Thus

\[
\| E_\varepsilon(w_1) \|_{Y_1} \leq C \varepsilon, \quad \| E_\varepsilon(\phi) \|_{Y_1} \leq C \varepsilon \| \phi \|_{X_1}.
\]

Combining all together we get

\[
\| L_1(\hat{\phi}, \hat{\psi}) \|_{Y_1} \leq \left( \| \lambda_1 K_1(\psi) \|_{Y_1} + \| \lambda_1 K_2(\psi) \|_{Y_1} + \| E_\varepsilon(w_1) \|_{Y_1} + \| E_\varepsilon(\phi) \|_{Y_1} \right)
\]

\[
\leq \left( \kappa_0 \varepsilon + \| \phi \|_{X_1} + \varepsilon + \varepsilon \| \phi \|_{X_1} \right).
\]

By the same argument when \( q > 0 \), one can show that

\[
\| L_2(\hat{\phi}, \hat{\psi}) \|_{Y_1} \leq \left( \| \lambda_2 K_3(\phi) \|_{Y_1} + \| \lambda_2 K_4(\phi) \|_{Y_1} + \| E_\varepsilon(w_2) \|_{Y_1} + \| E_\varepsilon(\psi) \|_{Y_1} \right)
\]

\[
\leq \left( \kappa_0 \varepsilon + \| \phi \|_{X_1}^q + \| \phi \|_{X_1}^q \right) + \varepsilon + \varepsilon \| \psi \|_{X_1},
\]

where \( k_1 \to 0 \) when \( r \to 0 \).

By the linear theory there exists \( C > 0 \) such that

\[
\| J_\varepsilon(\phi, \psi) \| X \leq C \| L(\hat{\phi}, \hat{\psi}) \| Y
\]

\[
= C \left( \| L_1(\hat{\phi}, \hat{\psi}) \|_{Y_1} + \| L_2(\hat{\phi}, \hat{\psi}) \|_{Y_1} \right)
\]

\[
\leq C \left( \kappa_1 \| (\phi, \psi) \| X + \varepsilon (1 + \kappa_2 \| (\phi, \psi) \| X) \right)
\]

\[
\leq \kappa_3 \| (\phi, \psi) \| X,
\]

where \( \kappa_3 < 1 \) (because \( k_1, \varepsilon \to 0 \)). This shows for \( p, q > 0 \) and \( \varepsilon > 0 \) small enough, \( J_\varepsilon \) is into.

Theorem 3.4. For \( 0 < r < 1 \), chosen sufficiently small and \( \varepsilon > 0 \), there exists \( 0 < k_r < 1 \) such that

\[
\| J_\varepsilon(\phi, \psi) - J_\varepsilon(\phi', \psi') \| X \leq k_r \| (\phi, \psi) - (\phi', \psi') \| X,
\]

i.e. \( J_\varepsilon \) is a contraction.

Proof. We know by the linear theory there exists a \( C > 0 \) such that

\[
\| J_\varepsilon(\phi, \psi) - J_\varepsilon(\phi', \psi') \| X = \| (\hat{\phi}, \hat{\psi}) - (\hat{\phi}', \hat{\psi}') \| X
\]

\[
\leq C \| L(\hat{\phi}, \hat{\psi}) - L(\hat{\phi}', \hat{\psi}') \| Y.
\]

We compute the right hand side of the above inequality.
By part 4 of Lemma (3.2), we can write
\[ \|K_1(\psi) - K_1(\phi')\|_{Y_t} \]
\[ \leq \|\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)|^p - |\nabla w_2 + \nabla \psi'\]
\[ + \|A(\varepsilon, x)(\nabla w_2 + \nabla \psi')|^p - p|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi') (I + \varepsilon A(\varepsilon, x))\|_{Y_t} \]
\[ + \|\nabla w_2 + \nabla \psi|^p - |\nabla w_2 + \nabla \phi'|^p - p|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi')\|_{Y_t} \]
\[ + (\varepsilon)^\gamma\|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi')A(\varepsilon, x)\|_{Y_t} \]
\[ \leq C_p\|\nabla w_2|^{p-2}\|\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi)| + |\nabla \psi' + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi')| \]
\[ + |\nabla \psi' + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi')|^{p-1} \]
\[ + C_p\|\nabla w_2|^{p-2}\|\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi')| \]
\[ + C_p\|\nabla w_2|^{p-2}\|\nabla w_2 + \nabla \psi + \varepsilon A(\varepsilon, x)(\nabla w_2 + \nabla \psi')| \]
\[ + (\varepsilon)^\gamma\|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi')A(\varepsilon, x)\|_{Y_t} \],
also in below, we just write out some of the above terms’ computations (just for
sample and the other are the same) to get the estimate
\[ \|\nabla w_2|^{p-2}\varepsilon A(\varepsilon, x)||\nabla \psi - \nabla \psi'||_{Y_t} \leq F_1\varepsilon\|\psi - \psi'||_{X_r}, \quad \text{and} \]
\[ \|\nabla w_2|^{p-2}\|I + \varepsilon A(\varepsilon, x)||\nabla \psi||\nabla \psi - \nabla \psi'||_{Y_t} \leq F_2\|\psi||_{X_r}\|\psi - \psi'||_{Y_t}, \quad \text{and} \]
\[ \|\varepsilon A(\varepsilon, x)||\nabla w_2|^{p-1||\nabla \psi - \nabla \psi'||_{Y_t} \leq F_3\varepsilon^{(p-1)}\|\psi - \psi'||_{X_r}, \quad \text{and} \]
\[ \|\nabla w_2|^{p-1||\nabla \psi - \nabla \phi'||_{Y_t} \leq F_4\|\psi||_{X_r}^{(p-1)|\psi - \psi'||_{X_r}, \quad \text{where} \]
\[ F_1, F_2, F_3, F_4 \text{ are some constants. By the above there exists } k_{1r} < 1 \text{ such that} \]
\[ \|K_1(\psi) - K_1(\phi')\|_{Y_t} \leq k_{1r}\|\psi - \phi'||_{X_r}. \]
Applying part 4 of Lemma 3.2 and Lemma 3.1 (when } p > 0 \text{) and similar arguments
there exists } k_{2r} < 1 \text{ such that to get
\[ \|K_2(\psi) - K_2(\phi')\|_{Y_t} \]
\[ \leq \|\nabla w_2 + \nabla \phi|^{p - |\nabla w_2 + \nabla \phi'|^p - p|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi')\|_{Y_t} \]
\[ \leq C_p\|\nabla w_2|^{p-2}\|\nabla w_2 + \nabla \phi|^{p - |\nabla w_2 + \nabla \phi'|^p - p|\nabla w_2|^{p-2}\nabla w_2 \cdot (\nabla \psi - \nabla \phi')\|_{Y_t} \]
\[ \leq \left( \kappa_{11}\|\psi||_{X_r} + \kappa_{12}\|\phi'||_{X_r} + \kappa_{13}\|\psi||_{X_r}^{p-1} + \kappa_{14}\|\phi'||_{X_r}^{p-1} \right)^\gamma \|\psi - \phi'||_{X_r} \]
\[ \leq k_{2r}\|\psi - \phi'||_{X_r}, \quad \text{where } \kappa_{11}, \kappa_{12}, \kappa_{13}, \kappa_{14} \text{ are some constants and } k_{2r} < 1, \text{ for sufficiently small } r. \]
Finally, by applying Lemma 3.1
\[ \|E_\varepsilon(w_1 + \phi) - E_\varepsilon(w_1 + \phi')\|_{Y_t} = \|E_\varepsilon(\phi) - E_\varepsilon(\phi')\|_{Y_t} \leq k'\varepsilon\|\phi - \phi'||_{X_t}. \]
This shows that when \( p > 0 \), there exists \( k_r < 1 \) such that
\[
\|L_1(\hat{\phi}, \hat{\psi}) - L_1(\hat{\phi}', \hat{\psi}')\|_{Y_1} \leq k_r \|\psi - \psi'\|_{X_r} + \varepsilon \|\phi - \phi'\|_{X_r}.
\]
By the same argument one can show that when \( q > 0 \), there exists \( k_r < 1 \) such that
\[
\|L_2(\hat{\phi}, \hat{\psi}) - L_2(\hat{\phi}', \hat{\psi}')\|_{Y_r} \leq k_r \|\phi - \phi'\|_{X_r} + \varepsilon \|\psi - \psi'\|_{X_r}.
\]
Thus
\[
\|L(\hat{\phi}, \hat{\psi}) - L(\hat{\phi}', \hat{\psi}')\|_{Y} \leq (k_r + \varepsilon)(\|\hat{\phi}, \hat{\psi}\| - (\hat{\phi}', \hat{\psi}')\|_X.
\]
Since \( k_r < 1 \), small enough and \( \varepsilon \to 0 \), there is a contraction and the proof is complete.

By the above theorems \( J_\varepsilon \) is well defined, into and a contraction, then by Banach’s fixed point theorem it has a fixed point, i.e. there exists \((\phi_0, \psi_0)\) in \( B_r \) such that \( J_\varepsilon(\phi_0, \psi_0) = (\phi_0, \psi_0) \). Thus the problem (10) has a solution. By taking \( r > 0 \) small enough we see that \( v_1 = v_1 + \phi \) and \( v_2 = v_2 + \psi \) are positive in \( B_1 \setminus \{0\} \) and hence \( u_1, u_2 \) are positive singular solutions of (7).

4. The cases of at least one of \( u_i \) Hölder continuous. If one returns to Example 1, they see there are two other cases to consider; namely \( t, \tau < 0 \) and the case of one parameter positive and one negative \((t, \tau)\) as defined before. We first consider the case of \( t, \tau < 0 \). If one follows the same approach as \( t, \tau > 0 \) one arrives at the linear system given by \( L(\phi, \psi) = (f, g) \) in \( B_1 \setminus \{0\} \) with \( \phi = \psi = 0 \) on \( \partial B_1 \). Here we take \( X \) and \( Y \) exactly as before (but of course now \( t \) and \( \tau \) are negative). When attempting to solve this system as before one is lead to examine the linear ode system given by
\[
a_k''(r) + \frac{N - 1}{r} a_k'(r) - \frac{\mu_k}{r^2} a_k(r) - \frac{pt(N - 2 - t)}{tr^b} b_k'(r) = f_k(r) \quad 0 < r < 1,
\]
\[
b_k''(r) + \frac{N - 1}{r} b_k'(r) - \frac{\mu_k}{r^2} b_k(r) - \frac{q\tau(N - 2 - \tau)}{tr^\alpha} a_k'(r) = g_k(r) \quad 0 < r < 1, (50)
\]
with \( a_k(1) = b_k(1) = 0 \) and \( \beta, \alpha \) as defined before. Using a similar approach one can prove a result as Proposition 1; of course here the crucial issue is the nature of the roots of \( det(A) = 0 \). If one assumes sufficient assumptions on the roots one can prove linear theory as in Corollary 1. One can then apply a fixed point argument as before to obtain a Hölder continuous solution of (10) which translates to a positive Hölder continuous solution of (7). The main difficulty with this approach is to find suitable allowable \((p, q)\) such that we have the desired conditions on the roots.

Remark 1. Similarly one can use the same approach to consider the case of \( t > 0 \) and \( \tau < 0 \) (or vice versa).

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