Estimation of component reliability in repairable series systems with masked cause of failure by means of latent variables

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Abstract

In this work, we propose two methods, a Bayesian and a maximum likelihood model, for estimating the failure time distribution of components in a repairable series system with a masked (i.e., unknown) cause of failure. As our proposed estimators also consider latent variables, they yield better performance results compared to commonly used estimators from the literature. The failure time model considered here is the Weibull distribution but the proposed models are generic and straightforward for any probability distribution. Besides point estimation, interval estimations are presented for both approaches. Using several simulations, the performances of the proposed methods are illustrated and their efficiency and applicability are shown based on the so-called cylinder problem.

Keywords: Bayesian paradigm, component lifetime, EM algorithm, Markov-Chain Monte-Carlo, maximum likelihood estimator, Metropolis within Gibbs algorithm, parametric estimation, repairable system, series system, Weibull model.

1. Introduction

Recently, Zhang \textit{et al.} (2017) proposed a method to estimate the failure time distribution of cylinders (components) from a diesel engine (system). The engine structure is in series, that is, it fails as soon as the first of the 16 cylinders fails. A failed cylinder is replaced by an identical functioning one in the corresponding region.

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socket (cylinder position) and thereby restores the system, which characterizes a repairable situation. Assuming that a replaced component lifetime has the same lifetime distribution as the old one, the observed data from a single socket can be represented by a renewal process (RP). The observed data of 16 sockets forms a superposed renewal process (SRP), that is, a single system can be seen as a SRP (Rinne, 2008). Some approaches have been explored to analyze SRP data (Crowder et al., 1994; Nelson, 2003; Meeker & Escobar, 2014; Crow, 1990).

However, in the cylinder problem, the information about the exact position (location) of the socket with the replaced component is not provided. Cases like this are known as a masked cause of failure and have been considered in the literature in not repairable situations (Miyakawa, 1984; Sarhan & El Bassiouny, 2003; Mukhopadhyay, 2006; Kuo & Yang, 2000; Fan & Hsu, 2014; Wang et al., 2015; Liu et al., 2017). Rodrigues et al. (2017) propose a Bayesian three-parameter Weibull model for component reliability which considers latent variables in the estimation process.

The scenario considered in this work is the following: a fleet of systems (sample) is observed. Within each system, there is a set of \( m \) identical components working in series. When a component fails, it is replaced by a functioning one in its socket and the system operation is restored. Although the number of failures \( r \) within the interval \([0, \tau]\) can be observed for a given system, this information is unknown for the single sockets.

Zhang et al. (2017) propose a procedure for estimating the component lifetime distribution of an SRP with masked cause of failure by maximizing its likelihood function. The likelihood function is given by the sum of all possible data configurations, that is, all possible combinations in which the \( r \) failures might occur across the \( m \) sockets. However, the number of all possible data configurations increases exponentially with the number of failures, and for large numbers of \( m \) and \( r \), the computation of the maximum likelihood is too expensive. Thus, depending on the number of failures and components for each system in the fleet, the computational time is very costly and in some situations, it is not possible to compute.

The aim of this work is to estimate the components’ lifetime distribution involved in a repairable series system with masked cause of failure. However, in contrast to the procedure proposed by Zhang et al. (2017), our two methods – a maximum likelihood and a Bayesian approach – consider latent variables during the estimation process. The contributions are as follows:

- Under the maximum likelihood approach, we expect that considering latent variables and estimating the parameters via the Expectation Maximization (EM) algorithm (Robert & Casella, 2010) solves the limitation of the approach by Zhang et al. (2017), i.e., not being able to compute the maximum likelihood estimator. Besides, in situations in which the method of Zhang et al. (2017) is useful, we expect that both methods yield similar performances.

- By proposing a Bayesian approach to solve the problem, we develop a useful approach for incor
porating expert knowledge and/or past experiences as a priori distribution, besides considering the statistical inference under the Bayesian paradigm.

The Weibull distribution is considered for the components’ lifetime distributions, and thus, each socket represents a Weibull Renewal Process. However, it is quite simple to extend the work to other distributions.

Section 2 describes the Weibull Renewal Process and data structure. Sections 3 and 4 present the maximum likelihood and Bayesian approaches in more detail. Both methods are evaluated by means of simulation studies and the corresponding results are given in Section 5. Section 6 shows the applicability of the methodology in the cylinder dataset and Section 7 concludes this work.

2. Data structure and model

Consider a system with \( m \) components operating in \( m \) sockets. Once a component fails, it is replaced by a new one in the same socket. In the following, we will define quantities for a single socket and hence omit the socket indices.

2.1. Weibull Renewal Process

Let \( Y_l \) denote the lifetime of the component before replacement \( l \), for \( l = 1, 2, \ldots \). Under the assumption that the components’ failure times are independent and identically distributed (i.i.d.), let \( f(\cdot) \) and \( R(\cdot) \) be the density and reliability functions of the component failure time. The distribution considered here is the Weibull distribution, which enables to model changes in both distribution shape and hazard rates. We can have increasing, decreasing and constant failure rates in this family of Weibull distributions (Rinne, 2008).

The Weibull reliability function is defined as

\[
R(y | \theta) = R(y) = \exp \left[ -\left( \frac{y}{\eta} \right)^{\beta} \right],
\]

for \( y > 0 \), with parameter vector \( \theta = (\beta, \eta) \), in which \( \beta > 0 \) and \( \eta > 0 \) are the shape and the scale parameters, respectively.

Let \( Z_k \) be a positive random variable that denotes the time of occurrence of the \( k \)-th failure in the socket. Thus, \( Z_k = \sum_{l=1}^{k} Y_l, k \geq 1 \), and \( \{Z_k\} \) is a Weibull Renewal Process (WRP), that is, each socket in the system represents a WRP.
The mean and variance of $Z_k$ are given by:

$$
E(Z_k) = E\left( \sum_{l=1}^{k} Y_l \right) = kE(Y_1) = k\eta \Gamma\left( \frac{1}{\beta} + 1 \right) \quad \text{and}
$$

$$
\text{Var}(Z_k) = \text{Var}\left( \sum_{l=1}^{k} Y_l \right) = k\text{Var}(Y_1) = k\left[ \eta \Gamma\left( \frac{2}{\beta} + 1 \right) - \left( \eta \Gamma\left( \frac{1}{\beta} + 1 \right) \right)^2 \right].
$$

2.2. Superposed Renewal Process and Data Structure

Once a system has $m$ independent sockets, each system-level set of failure times forms a superposed renewal process (SRP). Let $T_k$ be the $k$-th failure time of the system, in which $T_1 = \min\{Y_{11}, Y_{21}, \ldots, Y_{m1}\}$ and $Y_{j1}$ denotes the first component failure time in the $j$-th socket, $j = 1, \ldots, m$.

Let $T = (t_1, t_2, \ldots, t_r, \tau)$ denote the observed event history of a single SRP with event times $t_1 < t_2 < \ldots < t_r$, and end-of-observation time $\tau$ with $\tau > t_r$. A data set will consist of $n$ independent SRPs corresponding to the $n$ systems in the fleet.

In summary, the assumptions made here are: (a) the component distribution function is the same for all sockets and systems over time, (b) the failures within a socket are independent, (c) all sockets within one system have the same end-of-observation time $\tau$, and (d) the $n$ systems in the fleet are independent.

3. Maximum likelihood approach

Consider a sample of $n$ systems. Let $t_i = (t_{i1}, t_{i2}, \ldots, t_{ir_i})$ be the vector of observed $r_i$ failure times for the $i$-th system and $\tau_i$ the end-observation time, with $i = 1, \ldots, n$. Let $d_{ij} = (d_{i1}, d_{i2}, \ldots, d_{ir_i})$ the vector that indicates the cause of failure, in which $d_{ki} = j$, if component $j$ causes the $k$-th failure in the $i$-th system, for $j = 1, \ldots, m, k = 1, \ldots, r_i$ and $i = 1, \ldots, n$.

Let's first assume that $d_i$ is observed. As an example consider a system $i$ with $m = 16$ components for which $r_i = 3$ failures, $d_{i1} = d_{i3} = 1$ and $d_{i2} = 13$, were observed. According to Zhang et al. (2017) the likelihood contribution of this system is

$$
f(t_{i1})f(t_{i3} - t_{i1})R(\tau_i - t_{i3})f(t_{i2})R(\tau_i - t_{i2})[R(\tau_i)]^{m-2}. \quad (3)
$$

Note that the likelihood contribution of system $i$ presents (3) in a situation where $d_i = (d_{i1}, d_{i2}, d_{i3})$ is known. In a masked cause of failure scenario, the actual failure position $d_i$ of system $i$ are not observable. Hence, there are $V_i = m^{r_i} = 16^3 = 4,096$ possible configurations of likelihood contributions for this system, in which $V_i$ is the number of possible data configurations of system $i$ with $r_i$ failure times in $m$ components. Based on Zhang et al. (2017), the likelihood contribution of the $i$-th system is given by

$$
L_i = \sum_{\nu=1}^{V_i} L_{\nu i},
$$
in which $L_{iv}$ is the likelihood contribution of the $v$-th configuration for system $i$. Considering that a fleet of $n$ independent systems is observed, the likelihood function for $\theta$ is

$$L(\theta \mid t) = \prod_{i=1}^{n} \left[ \sum_{v=1}^{v_{i}} L_{iv} \right],$$

where $t = (t_1, \ldots, t_n)$. Zhang et al. (2017) propose the maximization of the likelihood function given in (4).

In the masked cause of failure scenario, $d_i$ is a vector of latent variables. A suitable approach for estimating the parameter values, which maximize the likelihood function, is to consider an expectation-maximization (EM) algorithm. The latter is presented in the following subsection.

### 3.1. EM algorithm

The EM algorithm is an iterative method with Expectation (E) and Maximization (M) steps (Dempster et al., 1977). The E-step evaluates the expectation of the full log-likelihood function and the M-step tries to find the parameter configuration, which maximizes the expectation found within the E-step.

The augmented likelihood function (i.e., the likelihood function with latent variables) of $\theta$ is given by

$$L(\theta \mid t, d) = \prod_{i=1}^{n} L_i(\theta \mid t_i, d_i).$$

The form of $L_i(\theta \mid t_i, d_i)$ depends on the number of failures $r_i$. For this reason, a general form is presented in the following.

Given $d_i$, let $\Gamma_i$ be the set of $v_i$ component indexes that cause at least one failure for system $i$. In a situation in which no failure is observed, $v_1 = 0$. Let $x_{ilk}$ the $k$-th failure time caused by the l-th element of $\Gamma_i$, with $l = 1, \ldots, v_i$ and $k = 1, \ldots, n_l$. As an example, for system $i$ with $r_i = 3$ failures observed and $d_{1i} = d_{3i} = 1$ and $d_{2i} = 13$, we have $\Gamma_i = \{1, 13\}$, $v_i = 2$, $n_1 = 2$ and $n_2 = 1$, $x_{i11} = t_{1i}$, $x_{i12} = t_{3i}$ and $x_{i21} = t_{2i}$. Thus, $\sum_{l=1}^{v_i} n_l = r_i$.

The likelihood contribution of the $i$-th system can be written as

$$L_i(\theta \mid t_i, d_i) = \left\{ \prod_{l=1}^{v_i} \left[ \prod_{k=1}^{n_l} f(x_{ilk} - x_{ilk(k-1)}) \right] R(\tau_i - x_{i0}) \right\}^{1-\mathbb{I}(v_i=0)} R(\tau_i)^{m-v_i},$$

with $x_{i0} = 0$ and indicator function $\mathbb{I}(A) = 1$, if $A$ is true.

Let $l_i(\theta \mid t_i, d_i) = \log L_i(\theta \mid t_i, d_i)$. Thus, the logarithm of the augmented likelihood in (5) can be written as

$$l(\theta \mid t, d) = \sum_{i=1}^{n} l_i(\theta \mid t_i, d_i)$$

$$= \sum_{i=1}^{n} \left\{ \left[ 1 - \mathbb{I}(v_i = 0) \right] \left[ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log f(x_{ilk} - x_{ilk(k-1)}) + \sum_{l=1}^{v_i} \log R(\tau_i - x_{i0}) \right] + (m - v_i) \log R(\tau_i) \right\}$$

(6)
Let $\theta_t$ be the value assumed by $\theta$ in the $r$-th iteration of the algorithm. The $(r + 1)$-th E-step consists of calculating the expectation of (6), that is,

$$Q(\theta | \theta_r) = \mathbb{E} [l(\theta | T, d) | T = t; \theta_r].$$

Unfortunately, there exists no analytical expression of the expectation in (7). Instead, it can be approximated by Monte-Carlo simulations. Consider that $L$ random samples $d_i^{(1)}, \ldots, d_i^{(L)}$ are simulated based on $f(d_i | t)$, i.e., the density function of $d$ conditional to $T = t$, $i = 1, \ldots, n$ (see Subsection 3.1.1). Thus, the E-step results in calculating

$$Q_m(\theta | \theta_r) = \frac{1}{L} \sum_{l=1}^{L} l(\theta | t, d) = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{n} l_i(\theta | t_i, d_i^{(l)}).$$

The M-step maximizes (8) with respect to $\theta$ resulting in $\theta_{r+1}$. The optimization method considered within this work is the Nelder-Mead algorithm (Nelder & Mead, 1965). The E- and M-steps are alternated until the difference of estimates between two consecutive iteration values is less than $10^{-4}$. The estimate of $\theta$, say $\hat{\theta}$, is obtained when the convergence criterion is reached. In this work, we consider $L = 1,000$.

Let $g(\theta)$ be a function of $\theta$. Due to the invariance property of the maximum likelihood estimator (MLE), the MLE of $g(\theta)$ is $g(\hat{\theta})$. For instance, the expected time of the component’s lifetime is $\mathbb{E}(Y) = g(\theta) = \eta \Gamma(1 + (1/\beta))$ and its MLE is $g(\hat{\theta}) = \hat{\eta} \Gamma(1 + (1/\hat{\beta}))$, in which $\hat{\beta}$ and $\hat{\eta}$ are MLE of $\beta$ and $\eta$, respectively (Casella & Berger, 2002). In an analogous way, the MLE for the component reliability function is $\hat{R}(y) = \exp[-(y/\hat{\eta})^{\hat{\beta}}]$, for $y > 0$.

### 3.1.1. Conditional distribution of $d$ given $T=t$

For a fixed $i$, $f(d_i | t_i)$ can be written as

$$f(d_i | t_i) = f(d_{i1}, d_{i2}, \ldots, d_{ir_i} | t_i) = f(d_{i1} | t_i, d_{i(r_i-1)i}, d_{i(r_i-2)i}, \ldots, d_{2i}, d_{1i}) f(d_{i(r_i-1)i} | t_i, d_{i(r_i-2)i}, \ldots, d_{2i}, d_{1i}) \ldots f(d_{2i} | t_i, d_{1i}) f(d_{1i} | t_i).$$

As an example, consider $r_i = 3$ and $t_i = (t_{1i}, t_{2i}, t_{3i})$. Thus,

$$f(d_i | t_i) = f(d_{i1}, d_{i2}, d_{i3} | t_i) = f(d_{i1} | t_i, d_{i2}, d_{i3}) f(d_{i2} | t_i, d_{i1}) f(d_{i3} | t_i).$$

Under i.i.d assumption, the distribution of $d_{1i} = j | t_i$ follows a Multinomial distribution $M(1, p_{1j})$, with $p_{1j} = (p_{1j1}, \ldots, p_{1jm})$ and $p_{1ji} = 1/m$, $j = 1, \ldots, m$. Note that in this special case, the multinomial distribution equals a discrete uniform distribution.

Similarly, the distribution of $d_{2i} | (t_i, d_{1i} = j)$ can be described as follows:

$$f(d_{2i} | t_i, d_{1i} = j) \propto f(t_{2i} - t_{1i}) \prod_{l=1; l \neq j}^{m} [f(t_{2i})]^{I(d_{2i} = l)}.$$
that is, \( d_{2i} | (t_i, d_{1i} = j) \) follows a Multinomial distribution \( M(1, p_{2i}) \), in which \( p_{2i} = (p_{2i1}, \ldots, p_{2im}) \), \( p_{2ji} = f(t_{2i} - t_{1i})/C \) and \( p_{2ji} = f(t_{2i})/C, \ l = 1, \ldots, m \) and \( i \neq j \), with \( C = f(t_{2i} - t_{1i}) + (m - 1)f(t_{2i}) \).

For the conditional distribution of \( d_{3i} \), one has to consider the following two cases:

- **Distribution of \( d_{3i} | (t_i, d_{1i} = j, d_{2i} = j) \):**

\[
 f(d_{3i} | t_i, d_{1i} = j, d_{2i} = j) \propto [f(t_{3i} - t_{2i})]^{I_{d_{3i}=j}} \prod_{l=1; j \neq l}^{m} [f(t_{3i})]^{I_{d_{3i}=l}},
\]

that is, \( d_{3i} | (t_i, d_{1i} = j, d_{2i} = j) \) follows a Multinomial distribution \( M(1, p_{3i}) \), in which \( p_{3i} = (p_{3i1}, \ldots, p_{3im}) \), \( p_{3ji} = f(t_{3i} - t_{2i})/C \) and \( p_{3ji} = f(t_{3i})/C, \ l = 1, \ldots, m \) and \( i \neq j \), with \( C = f(t_{3i} - t_{2i}) + (m - 1)f(t_{3i}) \).

- **Distribution of \( d_{3i} | (t_i, d_{1i} = j, d_{2i} = q) \), with \( q \neq j \):**

\[
 f(d_{3i} | t_i, d_{1i} = j, d_{2i} = q) \propto [f(t_{3i} - t_{1i})]^{I_{d_{3i}=q}}[f(t_{3i} - t_{2i})]^{I_{d_{3i}=1}} \prod_{l=1; j \neq q}^{m} [f(t_{3i})]^{I_{d_{3i}=l}},
\]

that is, \( d_{3i} | (t_i, d_{1i} = j, d_{2i} = q) \) follows a Multinomial distribution \( M(1, p_{3i}) \), in which \( p_{3i} = (p_{3i1}, \ldots, p_{3im}) \), \( p_{3ji} = f(t_{3i} - t_{1i})/C \) and \( p_{3ji} = f(t_{3i} - t_{2i})/C \) and \( p_{3ji} = f(t_{3i})/C, \ l = 1, \ldots, m \) and \( i \neq q \), with \( C = f(t_{3i} - t_{1i}) + f(t_{3i} - t_{2i}) + (m - 2)f(t_{3i}) \).

### 3.2. Asymptotic Distribution

The asymptotic distribution of the maximum likelihood estimator \( \hat{\theta} \) can be approximated by a multivariate normal distribution with mean \( \theta \) and variance-covariance matrix \( I_{\theta}(\theta)^{-1} \), where \( I_{\theta}(\theta) \) is the observed information matrix for \( \theta \). As demonstrated by Louis (1982), \( I_{\theta}(\hat{\theta}) \) is the sum of

\[
 I_1(\theta | \hat{\theta}) = -\frac{\partial^2}{\partial \theta \partial \theta^\top} Q(\theta | \hat{\theta}) \quad \text{and} \quad I_2(\theta | \hat{\theta}) = -\text{Var} \left\{ \frac{\partial}{\partial \theta} l(\theta | T, d) \bigg| T = t; \hat{\theta} \right\}.
\]

The matrix \( I_1(\theta | \hat{\theta}) \) can be estimated by

\[
 -\frac{\partial^2}{\partial \theta \partial \theta^\top} Q(\theta | \hat{\theta}) = -\frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{n} \frac{\partial^2}{\partial \theta \partial \theta^\top} l_i(\theta | t_i, d_i^{(l)}) \bigg|_{\theta = \hat{\theta}},
\]

where \( d_i^{(l)} \), with \( l = 1, \ldots, L \), being a random sample from the distribution of \( f(d_i | t) \) for the \( i \)-th system.

An estimate of \( I_2(\theta | \hat{\theta}) \) results from the sum of

\[
 \sum_{l=1}^{n} \left\{ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial}{\partial \theta} l_i(\theta | t_i, d_i^{(l)}) \bigg|_{\theta = \hat{\theta}} \right\} \cdot \left\{ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial}{\partial \theta} l_i(\theta | t_i, d_i^{(l)}) \bigg|_{\theta = \hat{\theta}} \right\}^\top
\]

7
and

\[-\frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \left\{ \frac{\partial}{\partial \theta} l_i(\theta | t_i, d_i^{(l)}) \right\} \left\{ \frac{\partial}{\partial \theta} l_i(\theta | t_i, d_i^{(l)}) \right\}^\top \bigg|_{\theta = \hat{\theta}}.\]

Detailed information on the development of $I_\theta(\hat{\theta})^{-1}$ is given in the appendix.

Thus, an asymptotic $\gamma\%$ confidence interval for $\theta$ (CI$\gamma\%$) is given by

\[CI_{\gamma\%} = (\hat{\theta} - z(1 - \gamma/2) \sqrt{(I_{11}, I_{22})}; \hat{\theta} + z(1 - \gamma/2) \sqrt{(I_{11}, I_{22})}),\]

in which $I_{ij}$ denotes the $j$th element of the main diagonal of $I_\theta(\hat{\theta})^{-1}$.

Confidence intervals for functions of $\theta$ can be obtained by the delta method (Casella & Berger, 2002).

4. Bayesian Approach

The posterior distribution of $\theta$ can be written as

\[\pi(\theta, d | t) \propto \pi(\theta, d)L(\theta, d | t),\]  

where $L(\theta, d | t)$ has the same form as (5) in which $d$ now is faced as parameter and $\pi(\theta, d)$ is the prior distribution of $(\theta, d)$.

In real-world settings, it is possible that the prior distributions can be influenced by expert knowledge and/or past experiences on the functioning of the components. In this work, no prior information about the functioning of the components is available, which is the reason for the choice of non-informative prior distributions. The priors of Weibull parameters are considered to be independent gamma distributed with mean 1 and variance 100. Besides, $d_{li}$ follows a Multinomial distribution $M(1, p_{li})$, where $p_{li} = (p_{l1i}, \ldots, p_{lmi})$ and $p_{lji} = 1/m$, with $j = 1, \ldots, m$.

Given the posterior density in Equation (9) does not have a closed form, statistical inferences about the parameters can rely on Markov-Chain Monte-Carlo (MCMC) simulations. Here, we consider the Metropolis within Gibbs algorithm (Tierney, 1994) once it is possible to sample some of the parameters directly from the conditional distribution; however, this is not possible for other parameters. The algorithm works in the steps presented in Algorithm 1.

Discarding burn-in (i.e., the first generated values are discarded to eliminate the effect of the assigned initial values for parameters) and jump samples (i.e., gaps between the generated values in order to avoid correlation problems), a sample of size $n_p$ from the joint posterior distribution of $(\theta, d)$ is obtained. The sample from the posterior distribution can be expressed as $(\theta_1, \theta_2, \ldots, \theta_{n_p})$. Posterior quantities of $\theta$ can be
Algorithm 1 The Metropolis within Gibbs algorithm.

1: Assign initial values \( \mathbf{\theta}^{(0)} \) for \( \mathbf{\theta} = (\beta, \eta) \) and set \( b = 1 \).

2: Draw \( \mathbf{d}_i^{(b)} \) from \( f(d_i \mid t_i) \) presented in Subsection 3.1.1.

3: Draw \( \mathbf{\theta}^{(b)} \) from \( \pi(\mathbf{\theta} \mid t, \mathbf{d}^{(b)}) \) through Metropolis-Hastings algorithm (Robert & Casella, 2010).

4: Set \( b = b + 1 \) and repeat steps 2) and 3) until \( b = B \), where \( B \) is the predefined number of simulated samples of \((\mathbf{\theta}, \mathbf{d})\).

easily obtained (Robert & Casella, 2010). For instance, the posterior mean of \( \mathbf{\theta} \) is

\[
E[\mathbf{\theta} \mid \text{Data}] = \frac{1}{n_p} \sum_{k=1}^{n_p} \mathbf{\theta}_k.
\]

The sample from the posterior distribution of \( g(\mathbf{\theta}) \) can be expressed as \((g(\mathbf{\theta}_1), g(\mathbf{\theta}_2), \ldots, g(\mathbf{\theta}_{n_p}))\) and posterior quantities of \( g(\mathbf{\theta}) \) can be obtained. For instance, the posterior mean of the reliability function is

\[
E[R(t \mid \mathbf{\theta}) \mid \text{Data}] = \frac{1}{n_p} \sum_{k=1}^{n_p} R(t \mid \mathbf{\theta}_k), \quad t > 0,
\]

in which \( R(\cdot \mid \mathbf{\theta}) \) has the form presented in (1).

Note that \( E(Z_k) \) and \( \text{Var}(Z_k) \) are functions of \( g(\mathbf{\theta}) \) and thus can be obtained in an analogous way.

5. Model evaluation by means of a simulation study

This section presents the results of some exemplary simulations to evaluate the performance of the estimation methods described above, with regards to the estimation quality.

The steps for generating the data of each simulated example, with \( m \) being the number of sockets and \( n \) the sample size, are presented in Algorithm 2. The mean (7) and variance (4) values of component failure time distribution are based on cylinder application data (Section 6).

The following estimation methods were fitted: Bayesian approach (BA), maximum likelihood estimator via EM algorithm (EM-ML) and maximum likelihood estimator obtained by Zhang et al. (2017) (Z-ML). The Z-ML estimates were obtained by means of the R-package (R Core Team, 2017) SRPML (Zhang et al., 2015).

To obtain posterior quantities, we used an MCMC procedure to generate a sample from the posterior distribution of the parameters. We generated 20,000 samples from the posterior distribution of each parameter. The first 10,000 of these samples were discarded as burn-in samples. A jump of size 10 was chosen to reduce correlation effects between the samples. As a result, the final sample size of the parameters generated from the posterior distribution was 1,000. The chains’ convergence was monitored in all simulation scenarios for good convergence results to be obtained.
Algorithm 2 Data generation.

1: for each system unit \( i = 1, \ldots, n \) do
2: Draw \( \tau_i \) from a Weibull distribution with mean \( m_c \) and variance 0.05.
3: Draw \( Y_{11i}, Y_{21i}, \ldots, Y_{m1i} \) from a Weibull distribution with mean 7 and variance 4, where \( Y_{j1i} \) is the first component failure time in the \( j \)-th socket, for \( j = 1, \ldots, m \).
4: Let \( T_{1i} = \min\{Y_{11i}, Y_{21i}, \ldots, Y_{m1i}\} \).
5: if \( T_{1i} \geq \tau_i \) then
6: stop simulation process and \( r_i = 0 \).
7: else
8: Let \( Y_{l1i} = \min\{Y_{11i}, Y_{21i}, \ldots, Y_{m1i}\} \), then \( t_{1i} = Y_{l1i} \).
9: Draw \( Y_{l2i} \) from Weibull distribution with mean 7 and variance 4 conditional to \( Y_{l2i} > t_{1i} \), where \( Y_{l2i} \) is the second component failure time in the \( l \)-th socket, once the first failure occurred in the \( l \)-th socket.
10: Let \( T_{2i} = \min\{Y_{11i}, Y_{21i}, \ldots, Y_{l2i}, \ldots, Y_{m1i}\} \).
11: if \( T_{2i} \geq \tau_i \) then
12: stop simulation process and \( r_i = 1 \).
13: else
14: repeats steps 8 to 10 until \( T_{r_i} < \tau_i < T_{(r_i+1)} \).
15: The dataset is \( \{t_{1i}, t_{2i}, \ldots, t_{r_i}, \tau_i\} \), for \( i = 1, \ldots, n \).

The mean absolute error (MAE) from each estimator to the true reliability of each method is considered as performance measure. \( R(t) \) and \( \hat{R}(t) \) are the true reliability function and the estimate, respectively. Hence, the MAE is evaluated by \( \frac{1}{\ell} \sum_{\ell=1}^{\ell} | \hat{R}(g_{\ell}) - R(g_{\ell}) | \), where \( \{g_1, \ldots, g_{\ell}, \ldots, g_{\ell}\} \) is a grid in the space of failure times.

First, we conducted two simulated examples considering \( n = 100 \), \( m = 16 \) and \( m_c = 4 \) (Example 1) or \( m_c = 8 \) (Example 2). It is worth noting that the expected number of failures with \( m_c = 8 \) is larger than with \( m_c = 4 \). Second, scenarios with different sample sizes, number of sockets and censor mean time are considered.

5.1. Simulated examples

For the Bayesian approach, the Gelman–Rubin convergence diagnostic measures (Gelman & Rubin, 1992) for parameters \( \beta \) and \( \eta \) are 1.0011 and 1.0004, respectively, in Example 1 and they are 1.0002 and 1.0027 in the Example 2. The measures are close to 1, which suggests that convergence chains have been reached.

For EM-ML, 8 and 17 EM iterations have been executed for Examples 1 and 2, respectively, and the corresponding values are listed in Table 1. For both examples, the initial values for \( (\beta, \eta) \) are (1, 1). After
the first iteration it was (1.206, 32.335) for Example 1, after the second one it was (3.165, 8.766) and then reached the convergence region. For Example 2, it took about eight iterations to reach the convergence region. Figure 1 presents contour plots of the log-likelihood function, as well as the iteration values from the second to the eighth iteration for Example 1 and from third to 17-th iteration for Example 2. The convergence was obtained fast for both examples.

The Weibull parameter estimates obtained by BA, EM-ML and Z-ML are presented in Table 2. Note that the Z-ML estimation is not presented for Example 2, because those values could not be computed due to the high number of components and failures. The details about limitations of this method in situation of high numbers of failures and components can be seen in Zhang et al. (2017).

The estimates for the component reliability function obtained by BA, EM-ML, Z-ML, as well as the true reliability function, are presented in Figure 2. Table 3 lists the MAE values, in which maximum likelihood approaches (EM-ML and Z-ML) present lower MAE values for Example 1, whereas BA and EM-ML present similar MAE values for Example 2.

Table 1: EM algorithm iteration values of Weibull parameters for two simulated examples.

| Iterations | Example 1 | Example 2 |
|------------|-----------|-----------|
| Initial value | β | η | β | η |
| 1 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2 | 1.206 | 32.335 | 0.400 | 31.479 |
| 3 | 3.165 | 8.766 | 0.627 | 17.346 |
| 4 | 3.716 | 7.793 | 0.796 | 13.848 |
| 5 | 3.726 | 7.780 | 0.979 | 11.866 |
| 6 | 3.727 | 7.778 | 1.220 | 10.445 |
| 7 | 3.726 | 7.780 | 1.557 | 9.401 |
| 8 | 3.727 | 7.778 | 2.007 | 8.694 |
| 9 | - | - | 2.550 | 8.254 |
| 10 | - | - | 3.050 | 8.025 |
| 11 | - | - | 3.379 | 7.924 |
| 12 | - | - | 3.539 | 7.884 |
| 13 | - | - | 3.603 | 7.869 |
| 14 | - | - | 3.630 | 7.863 |
| 15 | - | - | 3.634 | 7.862 |
| 16 | - | - | 3.638 | 7.861 |
| 17 | - | - | 3.639 | 7.861 |
Figure 1: Contour plots of the log-likelihood function and EM algorithm iteration values (dots) for Example 1 (with $m_e = 4$) and for Example 2 (with $m_e = 8$), in which $m_e$ indicates the expected end-of-observation time.

Figure 2: Component reliability function estimation through the Bayesian approach (BA), the EM Maximum Likelihood method (EM-ML) and the Maximum Likelihood approach from Zhang et al. (Z-ML) for two scenarios of simulated examples, besides the generating curve (true). There is no Z-ML curve in Example 2 because it could not be computed due to the high number of components and failures.
Table 2: Weibull model parameters ($\beta$, $\eta$) and expected components’ time to failure ($E(Y)$) estimation based on different estimation models: the Bayesian approach (BA), the EM Maximum Likelihood method (EM-ML) and the Maximum Likelihood approach from Zhang et al. (Z-ML) of simulated examples. There are no Z-ML estimates in Example 2 because it could not be computed due to the high number of components and failures.

|                | Example 1 | Example 2 |
|----------------|-----------|-----------|
| **BA**         |           |           |
| Parameters     | Mean      | SD        | HPD 95%  | Mean      | SD        | HPD 95%  |
| $\beta$       | 3.696     | 0.321     | 3.095    | 4.357     | 3.638     | 0.118    | 3.410    | 3.858    |
| $\eta$        | 7.890     | 0.522     | 6.906    | 8.912     | 7.861     | 0.074    | 7.733    | 8.007    |
| $E(Y)$         | 7.118     | 0.439     | 6.282    | 7.971     | 7.087     | 0.064    | 6.982    | 7.216    |

|                | Example 1 | Example 2 |
|----------------|-----------|-----------|
| **EM-ML**      |           |           |
| Parameters     | MLE       | SE        | CI 95%   | MLE       | SE        | CI 95%   |
| $\beta$       | 3.728     | 0.377     | 2.989    | 4.466     | 3.641     | 0.089    | 3.467    | 3.815    |
| $\eta$        | 7.777     | 0.585     | 6.631    | 8.924     | 7.860     | 0.053    | 7.757    | 7.964    |
| $E(Y)$         | 7.022     | 0.487     | 6.067    | 7.976     | 7.087     | 0.048    | 6.993    | 7.182    |

|                | Example 1 | Example 2 |
|----------------|-----------|-----------|
| **Z-ML**       |           |           |
| Parameters     | MLE       | SE        | CI 95%   | MLE       | SE        | CI 95%   |
| $\beta$       | 3.729     | 0.323     | 3.096    | 4.361     | -         | -        | -        | -        |
| $\eta$        | 7.776     | 0.488     | 6.820    | 8.732     | -         | -        | -        | -        |
| $E(Y)$         | 7.021     | 0.409     | 6.218    | 7.823     | -         | -        | -        | -        |

SD means standard deviation; SE means standard error; HPD means highest posterior density; CI means confidence interval. The true parameters values are: $\beta = 3.924$, $\eta = 7.734$ and $E(Y) = 7$.

Table 3: MAE values obtained by the Bayesian approach (BA), the EM Maximum Likelihood method (EM-ML) and the Maximum Likelihood approach from Zhang et al. (Z-ML) of two simulated examples. There are no MAE values for Z-ML in Example 2 because they could not be computed due to the high number of components and failures.

|          | BA       | EM-ML    | Z-ML     |
|----------|----------|----------|----------|
| Example 1| 0.0117   | 0.0057   | 0.0057   |
| Example 2| 0.0080   | 0.0079   | -        |

5.2. Simulation studies in different scenarios

We conducted the simulations for all combinations of the following: $n \in \{10, 50, 100, 200\}$, $m \in \{4, 8, 16, 32\}$, and $m_c \in \{4, 8\}$, resulting in 32 scenarios. For each scenario, 100 datasets were generated, and we compare...
the MAE from the estimators to the true distribution.

The boxplot graphs of 100 MAE values are presented in Figure 3. In general, the four methods present similar performance. When $m_c = 8$ the BA method presents higher MAE means but the boxplot graph intersects with the boxplot graphs obtained by other methods.

Noticeably, Figure 3b does not contain any boxplots for Z-ML in case of $m \in \{16, 32\}$ and $m_c = 8$. However, this is plausible as this method was not able to compute the respective estimates due to the high number of failures and components. The computational time of each scenario was greater than four days and encountered errors in estimation. On the other hand, the computational times and availability of EM-ML and BA are not influenced that much by the numbers of failures and components.

In short, in settings as those from Figure 3b, Z-ML fails to compute the components’ failure time distribution, whereas the two proposed methods find solutions. For the settings in which Z-ML finds solutions, the proposed methods also find solutions and present similar performance.

Figure 3: Boxplot graphs of the 100 MAE values of the Bayesian approach (BA), the EM Maximum Likelihood method (EM-ML) and the Maximum Likelihood approach from Zhang et al. (Z-ML) in scenarios with different sample sizes ($n$) and number of components ($m$). There are no Z-ML MAE boxplots in case of $m \in \{16, 32\}$ and $m_c = 8$ because they could not be computed due to the high number of components and failures.
6. Cylinder dataset analysis

A fleet of $n = 120$ diesel engines (systems) is observed. Each engine has 16 identical cylinders working in series, that is, the first cylinder to fail causes the engine failure. When a cylinder fails, it is replaced by an identical functioning one in the socket (cylinder position), but the information about which socket each replacement comes from is not observed. Table 4 presents the distribution of the number of failures across all 120 systems.

To obtain posterior quantities related to the posterior distribution of $\theta = (\beta, \eta)$ from (9) through MCMC simulations, we discarded the first 10,000 as burn-in samples and used a jump of size 10 to avoid correlation problems, obtaining a sample size of 1,000. The chains’ convergence was monitored through graphical analysis, and good convergence results were obtained. The Gelman–Rubin convergence diagnostic measures for parameters $\beta$ and $\eta$ are 1.005 and 1.002, respectively. The measures are close to 1, which suggests that convergence chains have been reached.

Table 5 lists the posterior mean obtained by BA and EM-ML estimates for the parameters of shape ($\beta$), scale ($\eta$) and expected time of components’ lifetime, $E(Y) = \eta \Gamma(1 + (1/\beta))$. The expected times of the component lifetime obtained by BA and EM-ML are 7.629 and 7.541 years, respectively. The posterior mean and the 95% highest posterior density (HPD) point-wise band of the component reliability function are illustrated in Figure 4a. Besides, the posterior mean and the 95% highest posterior density (HPD) point-wise band of $E(Z_k)$, for $k = \{0, 1, \ldots, 49, 50\}$, are presented in Figure 4b. The estimation for the reliability function obtained by EM-ML estimator is similar to the estimate obtained by the Bayesian approach.

Table 4: Distribution of number of failures ($r$) of 120 systems from cylinder dataset.

| $r$ | Number of systems | %   |
|-----|-------------------|-----|
| 0   | 46                | 38.3|
| 1   | 32                | 26.7|
| 2   | 18                | 15.0|
| 3   | 14                | 11.7|
| 4   | 5                 | 4.2 |
| 5   | 4                 | 3.3 |
| 6   | 1                 | 0.8 |
| Total | 120              | 100.0|
Table 5: Weibull model parameters ($\beta$, $\eta$) and expected components’ time to failure (E($Y$)) estimation based on the Bayesian approach (BA) and the EM Maximum Likelihood method (EM-ML) of cylinder dataset.

| Parameters | BA               |                |                | EM-ML          |                |                |
|------------|------------------|----------------|----------------|----------------|----------------|----------------|
|            | Posterior Mean   | Posterior SD   | HPD 95%        | MLE            | SE             | CI 95%         |
| $\beta$    | 3.873            | 0.331          | 3.245 4.544    | 3.911          | 0.265          | 3.392 4.430    |
| $\eta$     | 8.436            | 0.511          | 7.495 9.420    | 8.330          | 0.400          | 7.546 9.114    |
| E($Y$)     | 7.629            | 0.428          | 6.844 8.456    | 7.541          | 0.331          | 6.893 8.189    |

SD means standard deviation; SE means standard error; HPD means highest posterior density and CI means confidence interval.

(a) Component reliability function

(b) E($Z_k$) estimation

Figure 4: Component reliability function ($R(t)$ in Equation 1) and expected time of occurrence of the $k$-th failure in the socket (E($Z_k$) in Equation (2)) estimates through Bayesian approach of cylinder dataset.

7. Conclusion

A Bayesian model and a maximum likelihood estimator (MLE) were proposed in order to estimate identical components failure time distribution involved in a repairable series system with masked cause of failure. For both approaches, latent variables were considered in the estimation process through EM algorithm for MLE and Markov-Chain Monte-Carlo (MCMC) for the Bayesian approach. The Weibull distribution was used for modelling the failure time, but the proposed models are generic and straightforward for any probability distribution. In estimation processes, satisfactory results about the convergence of the MCMC’s chains and
EM algorithm were obtained, evaluated through graphical analysis and convergence performance measures. Simulation studies were realized in scenarios with different sample sizes, number of components and distributions for censor lifetime and mean absolute error (MAE) from each estimator to the true distribution was considered as performance measure. In situations of high numbers of failures and/or components, it was not possible to compute the maximum likelihood estimator proposed by Zhang et al. (2017) (Z-ML) through the package SRPML available by the authors. The proposed methods are not affected by the high numbers of failures and/or components and they work perfectly even in these situations. Besides, in settings that Z-ML finds solutions, the proposed methods also find and present similar performance. Thus, the huge advantage of our proposed methods is that they estimate the components’ failure time distribution regardless the numbers of failures and components.

The practical applicability was assessed in cylinder dataset, in which components’ failure time quantities were estimated convincingly.

In this work, the assumption of independent and identically distributed (i.i.d.) components failure times has been made and found to be suitable for the cylinder dataset characteristics. However, this assumption might not be applicable to other scenarios. Thus, in future works, our proposed method can be extended to situations in which the assumption of independent and identically distributed failure times is violated.

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Appendix

We can write the logarithm of the augmented likelihood function of \(i\)-th system as

\[
\begin{align*}
    l_i(\theta | t_i, d_i) &= \left[ 1 - I(v_i = 0) \right] \left\{ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log f(x_{ilk} - x_{ilk-1}) + \sum_{l=1}^{v_i} \log R(\tau_l - \xi_{iln}) \right\} + (m - v_i) \log R(\tau_i) \\
    &= \left[ 1 - I(v_i = 0) \right] \left\{ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log(\beta) - \log(\eta) + (\beta - 1) \left[ \log(x_{ilk} - x_{ilk-1}) - \log(\eta) \right] - \left( \frac{x_{ilk} - x_{ilk-1}}{\eta} \right)^\beta \right\} \\
    &\quad - \left[ 1 - I(v_i = 0) \right] \left\{ \sum_{l=1}^{v_i} \left( \frac{\tau_l - \xi_{iln}}{\eta} \right)^\beta - (m - v_i) \left( \frac{\tau_i}{\eta} \right)^\beta \right\}.
\end{align*}
\]

The first derivatives of \(l_i(\theta | t_i, d_i)\) in relation to \(\beta\) and \(\eta\), respectively, are

\[
\begin{align*}
    \frac{dl_i(\theta | t_i, d_i)}{d\beta} &= \left[ 1 - I(v_i = 0) \right] \left\{ \frac{\tau_i}{\beta} + \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log(x_{ilk} - x_{ilk-1}) - r_l \log(\eta) + \log(\eta) \left( \frac{1}{\eta} \right)^\beta \left[ \sum_{l=1}^{v_i} \left( \sum_{k=1}^{n_l} (x_{ilk} - x_{ilk-1})^\beta \right) \right] \right\} \\
    &\quad + \left( \frac{1}{\eta} \right)^\beta (m - v_i) r_i^\beta \left[ \log(\eta) - \log(\tau_i) \right],
\end{align*}
\]

and

\[
\begin{align*}
    \frac{dl_i(\theta | t_i, d_i)}{d\eta} &= \left[ 1 - I(v_i = 0) \right] \left\{ - \frac{r_i}{\eta} - \frac{r_i (\beta - 1)}{\eta} + \beta \left( \frac{1}{\eta} \right)^{\beta+1} \left[ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} (x_{ilk} - x_{ilk-1})^\beta + \sum_{l=1}^{v_i} (\tau_l - \xi_{iln})^\beta \right] \right\} \\
    &\quad + \beta \left( \frac{1}{\eta} \right)^{\beta+1} (m - v_i) r_i^\beta.
\end{align*}
\]

The second derivatives are

\[
\begin{align*}
    \frac{d^2 l_i(\theta | t_i, d_i)}{d\beta^2} &= \left[ 1 - I(v_i = 0) \right] \left\{ - \frac{r_i}{\beta^2} - \log(\eta)^2 \left( \frac{1}{\eta} \right)^{\beta+1} \left[ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} (x_{ilk} - x_{ilk-1})^\beta + (\tau_l - \xi_{iln})^\beta \right] \right\} \\
    &\quad + \frac{2 \log(\eta)}{\eta} \left[ \frac{1}{\eta} \right] \left[ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log(x_{ilk} - x_{ilk-1}) (x_{ilk} - x_{ilk-1})^\beta + \sum_{l=1}^{v_i} \log(\tau_l - \xi_{iln}) (\tau_l - \xi_{iln})^\beta \right] \\
    &\quad - \left( \frac{1}{\eta} \right)^{\beta+1} \left[ \sum_{l=1}^{v_i} \sum_{k=1}^{n_l} \log(x_{ilk} - x_{ilk-1})^2 (x_{ilk} - x_{ilk-1})^\beta + \sum_{l=1}^{v_i} \log(\tau_l - \xi_{iln})^2 (\tau_l - \xi_{iln})^\beta \right] \\
    &\quad + (m - v_i) \left( \frac{1}{\eta} \right)^\beta r_i^\beta \left[ \log(\tau_i)^2 + 2 \log(\tau_i) \log(\eta) - [\log(\eta)]^2 \right].
\end{align*}
\]
\[
\frac{d^2 l_i(\theta \mid t_i, d_i)}{d\beta \; d\eta} = \left[1 - I(v_i = 0)\right] \left\{ -\frac{r_i}{\eta} + \left(\frac{1}{\eta}\right)^{\beta+1} (1 - \beta \log(\eta)) \right\} \sum_{k=1}^{n_i} \left( \sum_{l=1}^{n} (x_{ilk} - x_{ilk(k-1)})^\beta + (\tau_l - x_{ilk})^\beta \right) \\
\quad + \beta \left(\frac{1}{\eta}\right)^{\beta+1} \sum_{k=1}^{n_i} \left( \sum_{l=1}^{n} \log(\eta \cdot x_{ilk} - x_{ilk(k-1)})^\beta + \sum_{l=1}^{n_i} \log(\tau_l - x_{ilk})^\beta \right) \\
\quad + (m - v_i) \left(\frac{1}{\eta}\right)^{\beta+1} \tau_i^\beta [1 - \beta \log(\eta) + \beta \log(\tau_i)],
\]

and
\[
\frac{d^2 l_i(\theta \mid t_i, d_i)}{d\eta^2} = \left[1 - I(v_i = 0)\right] \left\{ \frac{\beta r_i}{\eta^2} - \beta(\beta + 1) \left(\frac{1}{\eta}\right)^{\beta+2} \sum_{k=1}^{n_i} \left( \sum_{l=1}^{n} (x_{ilk} - x_{ilk(k-1)})^\beta + (\tau_l - x_{ilk})^\beta \right) \right\} \\
\quad - \beta(\beta + 1) \left(\frac{1}{\eta}\right)^{\beta+2} (m - v_i) \tau_i^\beta.
\]

Thus,
\[
I = -\frac{\partial^2}{\partial \theta \partial \overline{\theta}} Q(\theta \mid \overline{\theta}) = -\frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \frac{\partial^2}{\partial \theta \partial \overline{\theta}} \left[ l_i(\theta \mid t_i, d_i^{(l)}) \right]_{\theta = \overline{\theta}} = \\
= \frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \left[ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial^2}{\partial \theta \partial \overline{\theta}} \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} \right) \right]_{\theta = \overline{\theta}} \\
+ \frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \left[ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial^2}{\partial \theta \partial \overline{\theta}} \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right) \right]_{\theta = \overline{\theta}},
\]
in which \(\overline{\theta} = (\overline{\eta}, \overline{\beta})\). Besides,
\[
II = \sum_{i=1}^{n} \left\{ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial}{\partial \theta \overline{\theta}} \left( \sum_{l=1}^{L} \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} \right) \right\} \sum_{i=1}^{n} \left\{ \frac{1}{L} \sum_{i=1}^{L} \frac{\partial}{\partial \theta \overline{\theta}} \left( \sum_{l=1}^{L} \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right) \right\}^T \\
= \sum_{i=1}^{n} \left\{ \frac{1}{L} \sum_{i=1}^{L} \left[ \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} \right]_{\theta = \overline{\theta}} , \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right\} \sum_{i=1}^{n} \left\{ \frac{1}{L} \sum_{i=1}^{L} \left[ \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} \right]_{\theta = \overline{\theta}} , \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right\}^T
\]
and
\[
III = -\frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \left( \frac{\partial}{\partial \theta \overline{\theta}} \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} \right) \right) \sum_{i=1}^{n} \left( \frac{\partial}{\partial \theta \overline{\theta}} \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right) \right)^T \\
= -\frac{1}{L} \sum_{i=1}^{n} \sum_{l=1}^{L} \left\{ \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} , \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right)^T \right\} \sum_{i=1}^{n} \left\{ \left( \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \eta} , \frac{\partial l_i(\theta \mid t_i, d_i^{(l)})}{\partial \beta} \right)^T \right\}^T.
\]

The quantity \(l_0(\overline{\theta})\) can be estimated by \(I + II + III\).