Spelling Out Leptonic CP Violation in the Language of Invariant Theory

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In terms of flavor invariants, we establish the intimate connection between leptonic CP violation in the canonical seesaw model for neutrino masses and that in the seesaw effective field theory (SEFT). For the first time, we calculate the Hilbert series and explicitly construct the primary flavor invariants in the SEFT by considering both the dimension-five Weinberg operator $\mathcal{O}_5^{\alpha\beta} = \ell_{\alpha L} \overline{H} H^* \ell_{\beta L}$ and the dimension-six operator $\mathcal{O}_6^{\alpha\beta} = (\overline{u}_{\alpha L} \overline{H} H^* \ell_{\beta L})$ at the tree-level matching. The inclusion of only the Wilson coefficients $C_5^{\alpha\beta}$ and $C_6^{\alpha\beta}$ already enables the SEFT to incorporate all physical information about the full seesaw model. Moreover, the minimal sufficient and necessary conditions for CP conservation both in the SEFT and in the full theory are clarified, and the matching between the flavor invariants in both theories is accomplished. Through the matching of flavor invariants, the CP asymmetries necessary for successful leptogenesis are directly linked to those in neutrino-neutrino and neutrino-antineutrino oscillations at low energies. Surprisingly, it is revealed that the precise measurements of $C_5^{\alpha\beta}$ and $C_6^{\alpha\beta}$ in low-energy experiments are powerful enough to probe the full seesaw model, including CP violation for cosmological matter-antimatter asymmetry.
field theory (SEFT) at the tree-level matching, where both the Weinberg operator \( \mathcal{O}_5^{\alpha\beta} = \frac{1}{2\Lambda} \bar{H} H^T C_5^{\alpha\beta} \) and the dimension-six operator \( \mathcal{O}_6^{\alpha\beta} = \frac{1}{2\Lambda} \bar{H} \phi^{T} \phi \mathcal{O}_6^{\alpha\beta} \) are present (here \( \bar{H} \equiv i\sigma_2 H^\dagger \) denotes the Higgs doublet). In the language of invariant theory, we are able to draw a number of interesting conclusions. First, the inclusion of only two Wilson coefficients \( C_5^{\alpha\beta} \) and \( C_6^{\alpha\beta} \) in the SEFT reproduces the same number of physical parameters as in the full seesaw model. Second, in connection to the previous observation, we demonstrate that the absence of CP violation in the SEFT guarantees CP conservation in the full theory, and vice versa. The minimal sufficient and necessary conditions for CP conservation are given. In addition, we show that all physical parameters in the SEFT can be extracted using primary flavor invariants, so any low-energy physical observables can be expressed as functions of flavor invariants. Finally, the matching between the flavor invariants in the effective and full theories is accomplished. As a consequence, CP asymmetries necessary for a successful leptogenesis for cosmological matter-antimatter asymmetry can be directly related to those in neutrino-neutrino and neutrino-antineutrino oscillations at low energies.

**Framework:** To accommodate nonzero neutrino masses, we work in the type-I seesaw model with \( n \) RH neutrinos \( N_R \). Apart from the SM Lagrangian, the RH neutrino part of the full theory is given by

\[
\mathcal{L} = \bar{N}_R \phi N_R - \left[ \bar{\ell}_L Y_{\nu} \bar{H} N_R + \frac{1}{2} \bar{N}_R \frac{1}{2} M_R N_R + \text{h.c.} \right],
\]

where \( \ell_L \) stands for the left-handed lepton doublet. In Eq. (2), \( Y_{\nu} \) denotes the Dirac neutrino Yukawa coupling matrix and \( M_R \) is the Majorana mass matrix of RH neutrinos.

For the mass scale \( \Lambda = \mathcal{O}(M_R) \) of RH neutrinos much higher than the electroweak scale \( v \approx 246 \text{ GeV} \), the low-energy phenomena are described by the SEFT with

\[
\mathcal{L}_{\text{SEFT}} = \mathcal{L}_{\text{SM}} - \left[ \frac{C_5}{2\Lambda} \mathcal{O}_5 + \text{h.c.} \right] + \frac{C_6}{\Lambda^2} \mathcal{O}_6 ,
\]

where \( \mathcal{L}_{\text{SM}} \) stands for the SM Lagrangian, and \( \Lambda \) is the cutoff scale. At the tree-level matching, the relevant Wilson coefficients can be identified as

\[
C_5 = -Y_{\nu} Y_R^{-1} Y_{\nu}^T, \quad C_6 = Y_{\nu} \left( Y_R^T Y_R \right)^{-1} Y_{\nu}^T,
\]

with \( Y_R \equiv M_R/\Lambda \). Taking account of the charged-lepton part from the SM, we consider the most general flavor-basis transformations in the leptonic sector

\[
\ell_L \rightarrow U_L \ell_L, \quad l_R \rightarrow V_R l_R, \quad N_R \rightarrow U_R N_R ,
\]

where \( l_R \) represents the RH charged-lepton fields, and \( U_L, V_R \in U(m) \) and \( U_R \in U(n) \) are three arbitrary unitary matrices (for \( m \) generations of lepton doublets and \( n \) generations of RH neutrinos). Then Eq. (2) is unchanged if we treat the Yukawa coupling matrices as spurions, namely, taking them as spurious fields that transform as

\[
Y_L \rightarrow U_L Y_L^T, \quad Y_R \rightarrow U_R Y_R^T ,
\]

where \( Y_i \) is the charged-lepton Yukawa coupling matrix. At the matching scale, such transformations in the lepton flavor space in the full theory induce those of the Wilson coefficients in the SEFT, i.e.,

\[
C_5 \rightarrow U_L C_5 U_L^T, \quad C_6 \rightarrow U_L C_6 U_L^T .
\]

From Eq. (3) and Eq. (4) we can take the matrices \( \left(X_l = Y_{l} Y_{l}^T, C_5, C_6 \right) \) in the flavor space as the building blocks for the flavor invariants in the SEFT with the symmetry group \( U(n) \), whereas \( \left(Y_l, Y_{l}^T, Y_R \right) \) as the building blocks in the full seesaw model with the symmetry group \( U(m) \otimes U(n) \).

Throughout this letter, we use \( I_{abc} \) to label the flavor invariant with the degrees \( (a,b,c) \) of the building blocks \( \left(X_l = Y_{l} Y_{l}^T, C_5, C_6 \right) \) in the SEFT. Similarly, \( I_{abc} \) refers to the flavor invariant with the degrees \( (a,b,c) \) of the building blocks \( \left(Y_l, Y_{l}^T, Y_R \right) \) in the full seesaw model. Here \( a,b,c \) are non-negative integers. By flavor invariants, we mean the polynomial matrix invariants composed of building blocks that keep unchanged under flavor transformation.

**Two-generation SEFT:**— We begin with the case of only two generations of leptons. Although this is not realistic, it is very instructive for the study of the three-generation case. All the basic flavor invariants in both effective and full theories in the two-generation case can be explicitly constructed, and related to the physical observables in an apparent way.

As has been stated above, the HS is a powerful tool in studying the flavor invariants and the algebraic structure of the invariant ring. In the SEFT with two generations, using the MW formula, one can calculate the HS

\[
\mathcal{H}^{(2\beta)}(q) = \frac{1 + 3q^4 + 2q^8 + 3q^{10}}{(1 - q^2)^2(1 - q^4)^2(1 - q^6)^2} ,
\]

where \( q \) is an arbitrary complex number that labels the degrees of the invariants. The denominator of the HS carries the information about the primary invariants, i.e., those invariants that are algebraically independent. There are 10 factors in the denominator of the HS in Eq. (5), which means there are totally 10 primary flavor invariants in the invariant ring. The nontrivial point is that this number also equals the number of the independent physical parameters in the two-generation SEFT (i.e., 2 charged-lepton masses, 2 neutrino masses, 1 mixing angle and 1 phase in the leptonic flavor mixing matrix, 3 moduli and 1 phase in \( C_6 \)). As we will show below, all the 10 physical parameters can be extracted as the functions of 10 primary invariants.

Although other invariants in the ring are not algebraically independent of the primary ones, not all of them
can be written as the polynomials of the primary invariants. However, for the unitary groups under consideration, one can always find a finite number of invariants, known as basic invariants, such that any invariant in the ring can be decomposed as the polynomial of the basic invariants \[ \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_6 \]. In general the number of basic invariants is no smaller than that of primary invariants. This is because there may exist nontrivial polynomial identities among the basic invariants (i.e., the syzygies).

The construction of all the basic invariants can be accomplished by calculating the plethystic logarithm (PL) function of the HS

\[
\text{PL} \left[ \mathcal{I}_{k}^{(m)}(q) \right] = 2q + 4q^2 + 2q^3 + 5q^4 + 2q^5 + 3q^6 - 6q^7 - O(q^8),
\]

whose leading positive terms encode the information about the numbers and degrees of the basic invariants \[ \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_6 \]. As indicated by Eq. (9), there are totally 18 \[ \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_6 \] invariants in the two-generation SEFT, and the results are summarized in Table I. The parities of basic flavor invariants under the CP transformation have been listed in the last column. The 18 basic invariants (12 CP-even and 6 CP-odd) in Table I serve as the generators of the invariant ring in the sense that any flavor invariant can be written as the polynomial of them. For a systematic algorithm of decomposing an arbitrary invariant into the polynomial function of the basic invariants and finding out all the syzygies at a certain degree, see Appendix C of Ref. 35.

However, those 18 basic flavor invariants in Table I are not algebraically independent. As one can verify, there are 6 syzygies first appearing at degree 8, corresponding to the first negative term \(-6q^8\) in Eq. (9). Among them, four syzygies imply 4 linear relations among 6 CP-even basic invariants. This is in accordance with the fact that there are only \(6-4=2\) independent phases in the two-generation case of the SEFT.

In Table I, ten primary flavor invariants are labeled by \((\ast)\). It can be shown that from them one can extract all the physical parameters in the two-generation SEFT (cf. Supplemental Materials). In this sense, the set of primary invariants is actually equivalent to that of independent physical parameters in the theory. Therefore, one can express any low-energy physical observables in an explicit and basis-independent form with only flavor invariants. In particular, any CP-violating observable \(A_{\text{CP}}\) can be written as

\[
A_{\text{CP}} = \sum_{\mathcal{I}} f_j [\mathcal{I}_{k}^{(\text{even})}] \mathcal{I}_{j}^{(\text{odd})},
\]

where \(\mathcal{I}_{k}^{(\text{odd})}\) refer to CP-odd flavor invariants, and \(f_j [\mathcal{I}_{k}^{(\text{even})}]\) are some functions of only CP-even basic flavor invariants. Thus the vanishing of all CP-odd basic invariants in the ring ensures the absence of CP violation in the theory. We shall leave the proof of this general formula for Ref. [40]. Instead we mention that CP asymmetries \(A_{\nu\nu}\) in neutrino oscillations and those \(A_{\nu\bar{\nu}}\) in neutrino-antineutrino oscillations [47-49] can indeed be cast in the form of Eq. (46). After some lengthy calculations, we obtain \(A_{\nu\nu} = F_{\nu\nu}I_{12}^{(2)}\) and \(A_{\nu\bar{\nu}} = F_{\nu\bar{\nu}}I_{240}\), where \(F_{\nu\nu}\) and \(F_{\nu\bar{\nu}}\) are functions of CP-even primary invariants while \(I_{12}^{(2)}\) and \(I_{240}\) are two CP-odd basic invariants in Table I.

Finally, we discuss the conditions for CP conservation. Though there are six CP-odd basic invariants in the ring, only two of them are algebraically independent due to the syzygies. On the other hand, there are two independent phases in the leptonic sector. Hence the minimal conditions to guarantee CP conservation is the vanishing of only two CP-odd invariants. We find that the vanishing of \(I_{12}^{(2)}\) and \(I_{240}\) is sufficient to this end. Therefore, CP asymmetries in neutrino oscillations and neutrino-antineutrino oscillations already contain all the informa-

| Flavor invariants | Degree | CP Parity |
|-------------------|--------|-----------|
| \(I_{100} \equiv \text{Tr}(X_{1})\) | 1 | + |
| \(I_{001} \equiv \text{Tr}(C_{6})\) | 1 | + |
| \(I_{200} \equiv \text{Tr}(X_{2}^{2})\) | 2 | + |
| \(I_{101} \equiv \text{Tr}(X_{1}C_{6})\) | 2 | + |
| \(I_{020} \equiv \text{Tr}(X_{3})\) | 2 | + |
| \(I_{002} \equiv \text{Tr}(C_{6}^{2})\) | 2 | + |
| \(I_{120} \equiv \text{Tr}(X_{1}X_{2})\) | 3 | + |
| \(I_{201} \equiv \text{Tr}(C_{6}X_{3})\) | 3 | + |
| \(I_{220} \equiv \text{Tr}(X_{1}G_{15})\) | 4 | + |
| \(I_{212} \equiv \text{Tr}(G_{15}C_{6})\) | 4 | + |
| \(I_{212} \equiv \text{Im} \text{Tr}(X_{1}X_{2}C_{6})\) | 4 | - |
| \(I_{040} \equiv \text{Tr}(X_{2}^{2})\) | 4 | + |
| \(I_{022} \equiv \text{Tr}(C_{6}G_{56})\) | 4 | + |
| \(I_{222} \equiv \text{Im} \text{Tr}(X_{1}G_{15}C_{6})\) | 5 | - |
| \(I_{122} \equiv \text{Im} \text{Tr}(C_{6}G_{56}X_{1})\) | 5 | - |
| \(I_{240} \equiv \text{Im} \text{Tr}(X_{1}X_{2}G_{15})\) | 6 | - |
| \(I_{141} \equiv \text{Im} \text{Tr}(X_{1}C_{6}G_{15})\) | 6 | - |
| \(I_{042} \equiv \text{Im} \text{Tr}(C_{6}X_{2}G_{56})\) | 6 | - |

TABLE I. Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in the SEFT, where the subscripts of the invariants denote the degrees of \(X_{1} \equiv Y_{1}Y_{1}^{\dagger} \), \(C_{6}\) and \(C_{6}\), respectively. We have also defined \(X_{5} \equiv C_{5}C_{1}^{\dagger} \), \(G_{15} \equiv C_{5}X_{1}C_{6}^{\dagger} \) and \(G_{56} \equiv C_{5}C_{6}^{\dagger}C_{4}^{\dagger}\) that transform adjointly under the flavor transformation. There are in total 12 CP-even basic invariants and 6 CP-odd basic invariants. Note that the 10 primary invariants are labeled with \(\ast\).
tion about CP violation at low energies.

**Two-generation Seesaw.**— In the full seesaw model, the building blocks transform in the flavor space as in Eq. (9). Then the HS can be computed as

$$\mathcal{H}^{\hspace{1pt}(2g)}_{\text{SS}}(q) = \frac{1 + 8q^6 + 3q^8 + 2q^{10} + 3q^{12} + q^{14} + q^{20}}{(1 - q^2)^2 (1 - q^4)^{5} (1 - q^6) (1 - q^{10})},$$

which exhibits the algebraic structure of the flavor space in the full theory. We observe that the denominator of the HS in the full theory and that of Eq. (5) have the same number of factors, implying that there are equal number of algebraically-independent invariants (i.e., primary invariants) in the flavor space of full theory and that of the SEFT. Given the fact that the number of primary invariants is equal to that of independent physical parameters, we reach the conclusion that inclusion of the CP-odd flavor invariants in the SEFT is equivalent to the vanishing of all CP-odd invariants in the full theory. This proves that the vanishing of all primary invariants is equal to that of independent physical invariants in the flavor space of full theory and those in the SEFT. We observe that the denominator of Eq. (2) turns out to be adequate to incorporate all physical information about the full theory, including the source of CP violation [41, 42, 50].

Notice that Eqs. (11)-(16) form a system of linear equations for the CP-odd invariants and the determinant of the coefficient matrix in Eqs. (11)-(16) turns out to be nonzero in general. This proves that the vanishing of all the CP-odd flavor invariants in the SEFT is equivalent to the vanishing of all CP-odd invariants in the full theory. Therefore, the absence of CP violation in the low-energy effective theory up to the order of $\mathcal{O}(1/A^2)$ is equivalent to the CP conservation in the full seesaw model. Note that similar conclusion was also drawn in Ref. [42], but without the language of invariant theory.

The matching conditions in Eqs. (11)-(16) are useful to build a bridge between the CP violation at low energies and that at high energies. For example, if RH neutrino masses are strongly hierarchical, the (unflavored) CP asymmetry in the decay of the lightest RH neutrino...
can simply be written as
\[
\epsilon_1 = \frac{3}{16\pi I_{002}} \frac{I_{044}}{(I_{022} - I_{002}I_{020})}.
\]
(17)

Then, via Eq. (16), \( \epsilon_1 \) can be related to the CP-odd basic flavor invariant \( I_{042} \) in the SEFT. Furthermore, using four syzygies involving CP-odd invariants at degree 8, one can express \( I_{042} \) as the linear combination of \( I_{121}^{(2)} \) and \( I_{240} \). Finally one arrives at
\[
\epsilon_1 = R_1[I_{\text{even}}] I_{121}^{(2)} + R_2[I_{\text{even}}] I_{240},
\]
(18)
where \( R_1[I_{\text{even}}] \) and \( R_2[I_{\text{even}}] \) are rational functions of only CP-even basic invariants in the full theory that listed in Table I. As \( I_{121}^{(2)} \) and \( I_{240} \) are respectively responsible for CP violation in neutrino oscillations and neutrino-antineutrino oscillations, Eq. (18) establishes a direct link between low- and high-energy CP asymmetries in a basis-independent way. If \( A_{\mu\nu} = A_{\nu\mu} = 0 \), which means \( I_{121}^{(2)} = I_{240} = 0 \), then \( \epsilon_1 \) also vanishes. This is obviously in accordance with the conclusion drawn from Eqs. (11)–(16) that CP conservation in the SEFT also implies the absence of CP violation in the full seesaw model.

**Three-generation Case.**— All the results obtained in the two-generation SEFT can be generalized to the realistic three-generation scenario in a straightforward way, though the calculations are much more complicated. In this letter, we just collect the main conclusions and will present the details in a separate work [44].

First, the HS in the three-generation SEFT can be computed by using the MW formula, whose expression is much lengthier than that in Eq. [5]. However, as a highly nontrivial result, we find that the denominator of the HS has 21 factors, which exactly matches the number of the independent physical parameters in the SEFT. On the other hand, there are also 21 independent physical parameters in the three-generation seesaw. Moreover, the HS in the three-generation seesaw has been calculated in Ref. [20] and its denominator also has 21 factors. This implies there are 21 primary invariants in both the SEFT and the full theory for three generations. Second, those 21 primary invariants in the SEFT can be explicitly constructed and from them we can extract all the physical parameters. Among them, there are 6 CP-odd invariants, corresponding to 6 independent phases in the SEFT. In particular, any CP-violating observables can also be cast into the form of Eq. (16). Third, the vanishing of six certain CP-odd flavor invariants serves as the minimal sufficient and necessary conditions for CP conservation in the leptonic sector. The absence of CP violation in the SEFT is enough to guarantee CP conservation in the full theory, and vice versa. Finally, any flavor invariants in the SEFT can be written as rational functions of those in the full theory, which as the matching conditions set a connection between low- and high-energy observables.

| Flavor invariants | Degree | CP Parity |
|-------------------|--------|-----------|
| \( I_{200} \equiv \text{Tr}(X_j) \) | \( (\ast) \) | 2 | + |
| \( I_{020} \equiv \text{Tr}(X_{e}) \) | \( (\ast) \) | 2 | + |
| \( I_{002} \equiv \text{Tr}(X_{\nu}) \) | \( (\ast) \) | 2 | + |
| \( I_{042} \equiv \text{Tr}(X_{l}^{T}) \) | \( (\ast) \) | 4 | + |
| \( I_{223} \equiv \text{Tr}(X_{l}X_{R}) \) | \( (\ast) \) | 4 | + |
| \( I_{004} \equiv \text{Tr}(X_{l}^{T}) \) | \( (\ast) \) | 4 | + |
| \( I_{223} \equiv \text{Tr}(X_{l}G_{lR}) \) | \( (\ast) \) | 6 | + |
| \( I_{042} \equiv \text{Tr}(X_{l}X_{R}G_{lR}) \) | \( (\ast) \) | 6 | + |
| \( I_{442} \equiv \text{Tr}(G_{lR}^{T}G_{lR}) \) | \( (\ast) \) | 10 | + |
| \( I_{262} \equiv \text{Im} \text{Tr}(X_{l}X_{R}G_{lR}) \) | \( (\ast) \) | 10 | − |
| \( I_{244} \equiv \text{Im} \text{Tr}(X_{l}G_{lR}^{T}G_{lR}) \) | \( (\ast) \) | 10 | − |
| \( I_{622} \equiv \text{Im} \text{Tr}(X_{l}G_{lR}G_{lR}) \) | \( (\ast) \) | 12 | − |
| \( I_{444} \equiv \text{Im} \text{Tr}(X_{l}G_{lR}G_{lR}G_{lR}) \) | \( (\ast) \) | 12 | − |

**TABLE II.** Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in type-I seesaw model. The subscripts of the invariants denote the degrees of \( Y_{l}, Y_{e}, \) and \( Y_{R} \), respectively. We have also defined some building blocks that transform adjointly under the flavor transformation: \( X_{l} \equiv Y_{l}^{\ast}Y_{l}', X_{e} \equiv Y_{e}^{\ast}Y_{e}', X_{\nu} \equiv Y_{\nu}^{\ast}Y_{\nu}', X_{R} \equiv Y_{R}^{\ast}Y_{R}', G_{lR} \equiv Y_{l}^{\ast}Y_{R}', G_{lR}^{T} \equiv Y_{l}^{\ast}Y_{R}', G_{lR} \equiv Y_{l}^{\ast}Y_{R}', G_{lR}^{T} \equiv Y_{l}^{\ast}Y_{R}'. \) There are in total 12 CP-even basic invariants and 6 CP-odd basic invariants. The 10 primary invariants are labeled with “(\( \ast \))” in the first column.

**Concluding remarks.**— The invariant theory is an extremely useful tool for studying CP violation in nature. Any physical observables should be independent of the flavor basis and the specific parametrization of Yukawa matrices that one chooses. This feature is exactly what flavor invariants own. Therefore it is more natural to express observables in a complete form of flavor invariants.

In this letter, we demonstrate the intimate connection between the canonical seesaw model and its low-energy effective theory in the language of invariant theory. We show that the inclusion of only one dimension-five and one dimension-six operator in the effective theory is already adequate to contain all physical information about the full theory, including the source of CP violation. The HS of the flavor space in the SEFT is calculated and all the physical parameters are explicitly extracted using primary invariants, which is helpful for phenomenological studies at low energies. The matching between flavor invariants in the SEFT and those in the full seesaw model is accomplished, offering a basis-independent way to relate CP violation for cosmological matter-antimatter
asymmetry to that in low-energy phenomena.

The results in this work prove the usefulness and power of the invariant theory, and call for more applications of flavor invariants to flavor puzzles as well as other important topics in particle physics in general.

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SUPPLEMENTAL MATERIALS

In these supplemental materials, we provide the indispensable details about the results presented in the main text. First we show how to calculate the Hilbert series (HS) in the seesaw effective field theory (SEFT) using the Molien-Weyl (MW) formula. Then we demonstrate how to extract all the physical parameters in terms of primary flavor invariants. Finally we give a complete matching between the basic flavor invariants in the SEFT and those in the full theory. All the calculations are performed for the two-generation case. The generalization to the three-generation case is straightforward but much more complicated, and will be discussed in a separate work [46].

A: Calculation of the Hilbert series in the SEFT

A systematic method to calculate the HS is to use the MW formula [27,28]

\[ \mathcal{H}(q) = \int [d\mu_G] \text{PE} \left( z_1, \ldots, z_{r_0}; q \right), \]

which reduces the calculation of the HS into the computation of complex integrals. Here \([d\mu_G]\) is the Haar measure of the symmetry group \(G\). The integrand is the plethystic exponential (PE) function that determined by the representations of the building blocks

\[ \text{PE} \left( z_1, \ldots, z_{r_0}; q \right) = \exp \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \chi_{R_i} \left( z_{r_0}^{k_1}, \ldots, z_{r_0}^{k_n} \right) \frac{q^k}{k} \right), \]

where \(z_i\) (for \(i = 1, 2, \ldots, r_0\)) are coordinates on the maximum torus of the symmetry group \(G\) with \(r_0\) the rank of \(G\). \(\chi_{R_i}\) (for \(i = 1, 2, \ldots, n\)) are the character functions for the \(n\) building blocks that transform as the \(R_i\) representation of \(G\). For the case of two-generation SEFT, the symmetry group is the two-dimensional unitary group \(U(2)\) whose rank is 2 and the character functions of the building blocks \(X_1 = Y_1^0 Y_1^1, C_5, C_6\) turn out to be

\[
\begin{align*}
\chi_1(z_1, z_2) &= (z_1 + z_2) (z_1^{-1} + z_2^{-1}), \\
\chi_5(z_1, z_2) &= z_1^2 + z_2^2 + z_1 z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1}, \\
\chi_6(z_1, z_2) &= (z_1 + z_2) (z_1^{-1} + z_2^{-1}).
\end{align*}
\]

Then the PE function reads

\[ \text{PE} \left( z_1, z_2; q \right) = \exp \left( \sum_{k=1}^{\infty} \chi_i \left( z_1^{k_1}, z_2^{k_2} \right) \frac{q^k}{k} \right), \]

where in the final step the complex integrals are accomplished via the residue theorem. The plethystic logarithm (PL) function, which carries the information about basic invariants and syzygies [23], is the inverse operation of the PE function and can be calculated by

\[ \text{PL} \left[ \mathcal{H}_{\text{SEFT}}^{(2g)} (q) \right] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \left[ \mathcal{H}_{\text{SEFT}}^{(2g)} (q^k) \right] = 2q + 4q^2 + 2q^3 + 5q^4 + 2q^5 + 3q^6 - 6q^8 - O(q^{10}), \]

where \(\mu(k)\) is the Möbius function. With the help of the positive terms in Eq. [23] one can conveniently construct all the basic flavor invariants, as collected in Table I of the main text. Given these 18 basic flavor invariants, any flavor invariant in two-generation SEFT can be expressed as the polynomial of them. A general algorithm has been
developed in Appendix C of Ref. [35] to decompose an arbitrary invariant into the polynomial of the basic ones as well as finding out all the syzygies among the basic invariants at a certain degree. Here we only list the four syzygies that involve CP-odd invariants at degree 8

\[
\begin{align*}
T^{(2)}_{121} (2T_{120} - T_{100} T_{120}) + T_{221} (T_{100}^2 T_{120} - 2T_{120}) + T_{240} (T_{001} T_{100} - 2T_{101}) + T_{141} (T_{120}^2 - 2T_{200}) &= 0, \\
T^{(2)}_{121} (2T_{120} - T_{100} T_{120}) - T_{122} (T_{100} T_{120} - 2T_{120}) - T_{042} (T_{001} T_{100} - 2T_{101}) - T_{141} (T_{120}^2 - 2T_{200}) &= 0, \\
T^{(2)}_{121} (2T_{121} - T_{100} T_{120}) + T_{221} (T_{100} T_{120} - 2T_{120}) + T_{240} (T_{001}^2 T_{120} - 2T_{002}) + T_{141} (T_{001} T_{100} - 2T_{101}) &= 0, \\
T^{(2)}_{121} (2T_{121} - T_{100} T_{120}) - T_{122} (T_{100} T_{120} - 2T_{120}) - T_{042} (T_{120}^2 - 2T_{120}) - T_{141} (T_{100} T_{100} - 2T_{101}) &= 0,
\end{align*}
\]

from which one can express any four of the six CP-odd basic invariants in Table I of the main text as the linear combinations of the other two, with the coefficients being rational functions of only CP-even basic invariants.

The HS in the full seesaw model can be computed using the same method

\[
\mathcal{K}_{SS}^{(2)}(q) = \int \left( \frac{1}{4} \right)^4 \left( \frac{1}{8 \pi i} \right)^4 \left( \prod_{|z_1|=1} \prod_{|z_2|=1} \prod_{|z_3|=1} \prod_{|z_4|=1} \left( 2 - \frac{z_1}{z_2} - \frac{z_2}{z_1} \right) \left( 2 - \frac{z_2}{z_3} - \frac{z_3}{z_2} \right) \right) PE(z_1, z_2, z_3, z_4; q)
\]

while the PL function turns out to be

\[
\text{PL} \left[ \mathcal{K}_{SS}^{(2)}(q) \right] = 3q^2 + 5q^4 + 2q^6 + 3q^8 + 3q^{10} + 2q^{12} - O(q^{14}).
\]

From the positive terms in Eq. (30) one can read off that there are also 18 basic invariants in the invariant ring of the full seesaw model: three of degree 2, five of degree 4, two of degree 6, three of degree 8, three of degree 10 and two of degree 12. With the help of Eq. (30) we have explicitly constructed all the basic invariants in the full seesaw model, as shown in Table II of the main text.

**B: Physical parameters in terms of flavor invariants**

In the flavor basis where \( C_5 \) is diagonal with real and positive eigenvalues, i.e., \( C_5 = \text{Diag}\{c_1, c_2\} \), one can generally write the \( 2 \times 2 \) Hermitian matrices \( X_i \equiv Y_i Y_i^\dagger \) and \( C_6 \) as follows

\[
X_i = \begin{pmatrix} a_{11} & a_{12} e^{i \alpha} \\ a_{12} e^{-i \alpha} & a_{22} \end{pmatrix}, \quad C_6 = \begin{pmatrix} b_{11} & b_{12} e^{i \beta} \\ b_{12} e^{-i \beta} & b_{22} \end{pmatrix},
\]

where \( a_{ij} \) and \( b_{ij} \) are real numbers while \( \alpha \) and \( \beta \) are two phases. In this basis, 10 independent physical parameters are collected as \( \{c_1, c_2, a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}, \alpha, \beta\} \).

Now we extract these ten parameters from the primary invariants. First, the eigenvalues of \( C_5 \) can be obtained from \( T_{120} \equiv \text{Tr} (X_5) \) and \( T_{040} \equiv \text{Tr} (X_5^2) \) with \( X_5 \equiv C_5 C_5^\dagger \) as below

\[
c_{1,2} = \frac{1}{\sqrt{2}} \sqrt{\frac{T_{120} \pm \sqrt{T_{120}^2 - 4T_{040}^2}}},
\]

where \( c_1 \) and \( c_2 \) corresponds to the upper and lower sign on the right-hand side, respectively. Then, from \( T_{100} \equiv \text{Tr} (X_1) = a_{11} + a_{22} \) and \( T_{120} \equiv \text{Tr} (X_1 X_5) = c_1^2 a_{11} + c_2^2 a_{22} \) one can find

\[
a_{11,22} = \frac{1}{2} \left( T_{100} \pm \frac{T_{100} T_{120} - 2T_{120}^2}{\sqrt{4T_{120}^2 - T_{120}^2}} \right).
\]

With the help of \( T_{200} \equiv \text{Tr} (X_2^2) = a_{11}^2 + 2a_{12}^2 + a_{22}^2 \), we can immediately get

\[
a_{12} = \frac{1}{\sqrt{2}} \sqrt{\frac{T_{100} (T_{100} T_{120} - 2T_{120}^2) + T_{200} (T_{120}^2 - 2T_{040}) + 2T_{120}^2}{T_{120}^2 - 2T_{040}}}.
\]
Finally, using the identity \( I_{1220} \equiv \text{Tr} (X_i G_{15}) = c_i^2 a_{11}^2 + c_i^2 a_{22}^2 + 2 a_i c_i c_2 \cos 2\alpha \) with \( G_{15} \equiv C_5 X_i C_5^\dagger \), one can solve \( \cos 2\alpha \) in terms of primary invariants, i.e.,

\[
\cos 2\alpha = \frac{(I_{100}^2 I_{020} - 4 I_{100} I_{120} + 2 I_{220}) (I_{020}^2 - I_{040}) + 2 (I_{020} I_{120}^2 - I_{040} I_{220})}{\sqrt{2} (I_{020}^2 - I_{040}) (I_{200} (I_{020} - I_{040}) + I_{040} (I_{100} - I_{120}))}. \tag{35}
\]

In a similar way, four real parameters in \( C_5 \) can be determined by

\[
b_{11,22} = \frac{1}{2} \left( I_{001} \pm \frac{I_{001} I_{020} - 2 I_{021}}{\sqrt{2I_{040} - I_{020}}} \right),
\]

\[
b_{12} = \frac{1}{\sqrt{2}} \frac{I_{001} (I_{001} I_{040} - 2 I_{020} I_{021}) + I_{002} (I_{040}^2 - I_{040}) + 2 I_{021}^2}{I_{020}^2 - I_{040}},
\]

\[
\cos 2\beta = \frac{(I_{001}^2 I_{020} - 4 I_{001} I_{021} + 2 I_{022}) (I_{020} - I_{040}) + 2 (I_{020} I_{021}^2 - I_{040} I_{022})}{\sqrt{2} (I_{020}^2 - I_{040}) (I_{002} (I_{020} - I_{040}) + I_{040} (I_{001} - I_{002})) - 2 I_{021} (I_{001} I_{020} - I_{021})}. \tag{38}
\]

After the spontaneous breakdown of the SM gauge symmetry, the neutrino masses are just given by the eigenvalues of \( C_5 \) via

\[
m_{1,2} = \frac{v^2}{2\sqrt{2}} c_{1,2} = \frac{v^2}{2\sqrt{2}} \sqrt{I_{020}^2 \pm \sqrt{2I_{040}^2 - I_{020}^2}}. \tag{39}
\]

Here we would like to give some comments on the equivalence between the set of physical observables and the set of primary invariants. Strictly speaking, there may be discrete leftover degeneracies when extracting physical parameters by using primary invariants. For example, in Eqs. (35) and (38) the primary invariants are blind to the signs of \( \alpha \) and \( \beta \). The degeneracies can be eliminated by including more basic CP-odd invariants other than the primary ones. In our case, one can introduce \( I_{1240} \propto \sin 2\alpha \) and \( I_{042} \propto \sin 2\beta \) (note that \( I_{1240} \) and \( I_{042} \) are not algebraically independent of the primary invariants), whose signs can be used to eliminate the \( Z_2 \) degeneracy \( \alpha \to -\alpha \) and \( \beta \to -\beta \). Therefore, a more strict statement should be: “The set of physical parameters is equivalent to the set of primary invariants in the ring up to some discrete degeneracies” [40].

On the other hand, the charged-lepton masses can be figured out from \( X_i \) via \( 2 \text{Diag} \{m_e^2, m_\mu^2\} / v^2 = V_2 X_i V_2^\dagger \), where the \( 2 \times 2 \) flavor mixing matrix is parametrized as

\[
V_2 = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
e^{i\phi} & 0 \\
0 & 1
\end{pmatrix}, \tag{40}
\]

with \( \theta \) being the flavor mixing angle and \( \phi \) being the Majorana-type CP phase. It is straightforward to relate the charged-lepton masses, the flavor mixing angle and the CP phase to the elements in \( X_i \) by \( \phi = -\alpha \) and

\[
\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}}, \tag{41}
\]

\[
m_{e,\mu} = \frac{v}{2} \sqrt{a_{11} + a_{22}} \pm \frac{2a_{12}}{\sin 2\theta}. \tag{42}
\]

More explicitly, we can express these parameters in terms of the primary flavor invariants, i.e.,

\[
m_{e,\mu} = \frac{v}{2} \sqrt{I_{100}^2 \pm \sqrt{2I_{200}^2 - I_{100}^2}} \tag{43}
\]

\[
\cos 2\theta = \frac{2I_{120} - I_{100} I_{020}}{\sqrt{2I_{040}^2 - I_{020}} \sqrt{2I_{200}^2 - I_{100}^2}}, \tag{44}
\]

\[
\cos 2\phi = \frac{(I_{001}^2 I_{020} - 4 I_{001} I_{021} + 2 I_{022}) (I_{020}^2 - I_{040}) + 2 (I_{020} I_{021}^2 - I_{040} I_{022})}{\sqrt{2} (I_{020}^2 - I_{040}) (I_{002} (I_{020} - I_{040}) + I_{040} (I_{001} - I_{002})) - 2 I_{021} (I_{001} I_{020} - I_{021})}. \tag{45}
\]

To sum up, 10 physical parameters after the gauge symmetry breaking \( \{m_1, m_2, m_e, m_\mu, \theta, \phi, b_{11}, b_{12}, b_{22}, \beta\} \) can be extracted from the 10 primary flavor invariants

\[
\{I_{020}, I_{040}, I_{100}, I_{001}, I_{200}, I_{002}, I_{120}, I_{021}, I_{220}, I_{022}\}. \tag{46}
\]
by Eqs. (46), (48), Eq. (39) and Eqs. (43)-(45). On this point, it is worthwhile to stress that the physical parameters are certainly functions, but not polynomial functions, of primary flavor invariants.

Therefore, any physical observables can be expressed in explicit basis-independent forms with flavor invariants. In particular, any CP-violating observable $A_{CP}$ can be written as

$$A_{CP} = \sum_j F_j [I_{even}^k] I_{odd}^j,$$

where $I_{odd}^j$ refer to CP-odd basic flavor invariants, and $F_j [I_{even}^k]$ are some functions of only CP-even basic flavor invariants. For illustration, we calculate the CP asymmetries in two-generation neutrino oscillations and neutrino-antineutrino oscillations.

Let us consider the 12 CP-even basic flavor invariants in the full theory. The 12 CP-even basic flavor invariants in the SEFT can be written as the rational functions of primary flavor invariants.

Here we list the complete set of matching conditions between the basic invariants in the SEFT and those in the full seesaw model. First, the 12 CP-even basic flavor invariants in the SEFT can be written as the rational functions of primary flavor invariants.

$$I_{100} = I_{200},$$

$$I_{001} = \frac{2}{(I_{002}^2 - I_{004})} (I_{002} I_{020} - I_{022}),$$

$$I_{200} = I_{400},$$

$$I_{101} = \frac{2}{(I_{002}^2 - I_{004})} (I_{002} I_{220} - I_{222}),$$

$$I_{020} = \frac{2}{(I_{002}^2 - I_{004})} (I_{042} - 2 I_{022} I_{020} + I_{020}^2 I_{002}).$$

The CP asymmetries can first be calculated in the mass eigenstates. Then, using above results, we are able to recast them into the form of Eq. (46)

$$A_{\nu\nu} = \frac{\nu^2}{\Lambda^2} \cot \left( \frac{\Delta_{21}}{2} \right) F_{\nu\nu}^\mu [I_{100}, I_{200}, I_{020}, I_{120}, I_{040}] I_{121}^{(2)};$$

$$A_{\bar{\nu}\bar{\nu}} = F_{\bar{\nu}\bar{\nu}}^\mu [I_{100}, I_{200}, I_{020}, I_{120}, I_{220}, I_{040}] I_{240},$$

where $F_{\nu\nu}^\mu [I_{100}, I_{200}, I_{020}, I_{120}, I_{040}] = \frac{(2 I_{040} - I_{020})^{1/2} (2 I_{200} - I_{100})^{1/2}}{I_{400} (2 I_{200} - I_{100}) - 2 I_{120} (I_{120} - I_{020} I_{100}) - I_{020} I_{200}}$,

and

$$F_{\nu\nu}^\mu [I_{100}, I_{200}, I_{020}, I_{120}, I_{220}, I_{040}]$$

$$= 4 \frac{(2 I_{040} - I_{020})^{1/2} \sin \Delta_{21}}{(2 I_{040} - I_{020})^{1/2}} \cos \Delta_{21} \{ I_{020} (2 I_{120}^2 I_{200} - I_{100} I_{200} - I_{020} I_{100} - I_{020} I_{200}) + 2 I_{040} (I_{100} I_{120} - I_{220}) + 2 I_{020} \times (I_{220} - 2 I_{100} I_{120})\}^{-1},$$

with

$$\Delta_{21} = \frac{L}{2E} (m_2^2 - m_1^2) = \frac{L \nu^2}{8 E \Lambda^2} (2 I_{040} - I_{020})^{1/2},$$

where $L$ and $E$ denote the propagation distance and neutrino beam energy, respectively. It is obvious that $A_{\nu\nu}^\mu$ is suppressed by $\nu^2/\Lambda^2$, since there is no CP violation in two-generation neutrino oscillations without including $C_6$. However, $A_{\bar{\nu}\bar{\nu}}^\mu$ is not suppressed, as the Majorana-type CP phase in the $2 \times 2$ leptonic flavor mixing matrix already enters into the CP asymmetries in neutrino-antineutrino oscillations.

C: Flavor invariants matching

Here we list the complete set of matching conditions between the basic invariants in the SEFT and those in the full seesaw model. First, the 12 CP-even basic flavor invariants in the SEFT can be written as the rational functions of the 12 CP-even basic flavor invariants in the full theory.

$$I_{100} = I_{200},$$

$$I_{001} = \frac{2}{(I_{002}^2 - I_{004})} (I_{002} I_{020} - I_{022}),$$

$$I_{200} = I_{400},$$

$$I_{101} = \frac{2}{(I_{002}^2 - I_{004})} (I_{002} I_{220} - I_{222}),$$

$$I_{020} = \frac{2}{(I_{002}^2 - I_{004})} (I_{042} - 2 I_{022} I_{020} + I_{020}^2 I_{002}).$$
In addition, the determinant of the coefficient matrix in Eqs. (57)-(69) reads
\[
\text{Det} = \frac{128}{(I_{002}^2 - I_{004})^{14}} I_{020} (I_{002} I_{020} - I_{022}) (I_{020}^2 - I_{040})^2 
\times \{ I_{020}^2 I_{022} (3I_{020} I_{022} - 4I_{002} I_{040} - 3I_{042}) - I_{022} I_{040} I_{042} + I_{020} I_{040} [3I_{022} + 2I_{002} I_{042} + I_{040} (I_{002}^2 - I_{004})] \} ,
\]
which is nonzero in general. This means that Eqs. (64)-(69) are linearly independent.