Semianalytic Expressions for the Isolation and Coupling of Mixed Modes

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Abstract

In the oscillation spectra of giant stars, nonradial modes may be seen to undergo avoided crossings, which produce a characteristic “mode bumping” of the otherwise uniform asymptotic p- and g-mode patterns in their respective echelle diagrams. Avoided crossings evolve very quickly relative to typical observational errors and are therefore extremely useful in determining precise ages of stars, particularly in subgiants. This phenomenon is caused by coupling between modes in the p- and g-mode cavities that are near resonance with each other. Most theoretical analyses of the coupling between these mode cavities rely on the Jeffreys–Wentzel–Kramers–Brillouin approach, which is strictly speaking inapplicable to the low-order g modes observed in subgiants or the low-order p modes seen in very evolved red giants. We present both a nonasymptotic prescription for isolating the two mode cavities, as well as a perturbative (and also nonasymptotic) description of the coupling between them, which we show to hold good for the low-order g and p modes in these physical situations. Finally, we discuss how these results may be applied to modeling subgiant stars and determining their global properties from oscillation frequencies. We also make our code for all of these computations publicly available.

Unified Astronomy Thesaurus concepts: Astroseismology (73); Stellar oscillations (1617); Computational methods (1965)

1. Introduction

In a strictly ideal sense, stellar oscillations come in two flavors: acoustic (i.e., “pressure modes,” or p modes, deriving their restoring force primarily from pressure), and buoyant (i.e., “gravity modes,” or g modes, deriving their restoring force primarily from buoyancy). In solar-like stars, these propagate in mode cavities that are well separated (Unno et al. 1989; Aerts et al. 2010; Basu & Chaplin 2017).

The simplest analytic approaches for constructing the frequency eigenvalues of p- and g-mode oscillations rely on the Jeffreys–Wentzel–Kramers–Brillouin (JWKB) approximation (see, e.g., Gough 2007) and therefore hold good in the limit of modes of high radial order. The eigenvalues of high-frequency p modes follow the approximate asymptotic relation

$$\nu_{nlm} \sim \Delta \nu \left( n_p + \frac{l}{2} + \epsilon_{nlm,p} \right),$$

(1)

where in the limit of high $n_p$, the phase lag $\epsilon_p$ becomes essentially constant with frequency. Likewise, the frequencies of low-frequency g modes follow the asymptotic expression

$$\frac{1}{\nu_n} \sim \Delta \Pi ( n_g + \epsilon_{nlm,g}),$$

(2)

mirroring the standard expression for p modes. At high $n_g$, $\epsilon_g$ can again be taken to be essentially constant with frequency, and it is the period spacing $\Delta \Pi$, rather than any of the individual g-mode frequencies, which may be used as a structural or evolutionary constraint (e.g., Bedding et al. 2011).

In evolved solar-like oscillators, these two mode cavities couple evanescently, leading to mixed modes with both p-like and g-like character in different parts of the star (Osaki 1975). For bookkeeping purposes, these can be understood as combinations of fictitious modes of purely g-like and purely p-like character, which are respectively referred to as “$\gamma$ modes” and “$\pi$ modes” (Aizenman et al. 1977; Bedding 2012).

Modes of such mixed character have been used as sensitive interior probes of the structure of these stars. Conventional methods of doing so, however, rely on the accuracy of asymptotic expressions returned from JWKB analysis. A simplified approach to such analysis returns an approximate radial dispersion relation:

$$k_n^2 \sim -\frac{\omega_p^2}{\epsilon_p} \left( 1 - \frac{N_p^2}{\omega_p^2}(S_p^2/\omega_p^2 - 1) \right),$$

(3)

(here $\Lambda^2 = l(l + 1)$, $N_p^2$ is the square of the Brunt–Väisälä frequency, and $S_p^2 = N_p^2/\rho^2$ that of the Lamb frequency) such that the wave functions are locally highly oscillatory in regions where $\omega_p^2$ is significantly greater than (respectively, less than) both $N_p^2$ and $S_p^2$, for p modes (respectively, g modes) and decay rapidly otherwise, so that the JWKB approximation holds good. Correspondingly, the local dispersion reduces to $k_n^2 \sim \omega_n^2/\epsilon_n^2$ for acoustic waves (respectively, $k_n^2 \sim S_n^2N_p^2/\omega_n^2$ for buoyancy waves), which may be interpreted as the coefficient of the eigenvalue term of a corresponding Sturm–Liouville problem.

These naive limits suffice for the study of p or g modes in isolation, in the separate limit of high $n_p$ or $n_g$. However, in actual stars with solar-like convective stochastic mode excitation, only surface acoustic oscillations at frequencies near $\nu_{\text{max}}$, the frequency of maximum oscillation power, can be measured. We point out four distinct observational and asymptotic regimes associated with $\nu_{\text{max}}$:

1. High $n_p$ and high $n_g$, which permit the use of JWKB results in both the g and p cavities. This is commonly assumed to be the case for first-ascent red giant branch stars of intermediate age, where many g modes couple to a single $\pi$ mode. A formidable body of work (e.g., Goupil et al. 2013; Deheuvels et al. 2015; Mosser et al. 2015; Takata 2016; Pinçon et al. 2020) has been assembled
based on JWKB expressions for the coupling strengths between the two mode cavities related to this physical scenario.

2. Low \( n_e \) and low \( n_p \), which preclude the use of JWKB analysis altogether.

3. Low \( n_e \) and high \( n_p \), which are typical of very evolved red-giant stars; in these cases, the period spacings of Equation (2) are commonly used for evolutionary constraints (as in Bedding et al. 2011), although the \( p \) modes deviate significantly from the asymptotic relation (Stello et al. 2014).

4. Low \( n_e \) and high \( n_p \), which are typical of mixed modes seen in subgiants, particularly in the TESS field.

These latter two scenarios are characterized by many-to-one coupling between a sparse set of modes in one mode cavity and a dense set of modes in the other. In the case of subgiants, only a few \( \gamma \) modes (typically the highest in frequency) couple to the relatively denser set of \( \pi \)-mode oscillations that subsist in the convective exterior of a star. As the star evolves, the frequency of the lowest-order \( \gamma \) mode increases rapidly relative to those of the \( \pi \) modes. Evanescent transmission of wave propagation between the two cavities causes the emergence of the “avoided crossing” phenomenon, where the frequencies of the mixed modes are shifted relative to their uncoupled values, changing smoothly as the star evolves in such a way as to preserve the ordering of the complete set of eigenvalues in a continuous fashion throughout this evolution. In evolved red giants the converse is true; the lowest-order \( \pi \) modes couple to a dense set of \( \gamma \) modes. Again, as the star evolves, the frequency of the lowest-order \( \pi \) mode decreases relative to those of the coupled \( \gamma \) modes.

Because most of the existing theoretical formalism pertaining to the coupling between the \( \pi \) and \( \gamma \) modes relies on JWKB results, it is not strictly applicable to these latter two scenarios. However, both of these are of considerable scientific interest. Very evolved red giant stars (particularly near the tip of the red giant branch) serve as standard candles and anchor points for isochrone fitting of stellar populations (e.g., Lee et al. 1993). Moreover, avoided crossings place very strong, albeit model-dependent, seismic constraints on the structure, ages, and fundamental parameters of subgiants (e.g., Metcalfe et al. 2010; Deheuvels & Michel 2011), which in turn have been used as benchmarks for comparison between different measurement techniques (e.g., Stokholm et al. 2019). Subgiants in particular dominate the TESS short-cadence seismic sample (owing to constraints on observational cadence) and are expected to be a substantial fraction of the PLATO sample as well. We therefore seek a description of mode coupling that can be applied to such low-order modes.

We present a formalism specifically intended for use in the regime where JWKB analysis cannot be relied upon to describe the sparse set of eigenvalues. We first describe a construction of isolated \( \pi \)- and \( \gamma \)-mode eigenfunctions appropriate for such evolved stars (Section 2). Having done so, we then derive the eigenvalues of modified versions of these oscillation equations, where terms corresponding to wave propagation in the classical \( g \)-mode (respectively, \( p \)-mode) cavities have been suppressed, as \( \pi \)-mode (respectively, \( \gamma \)-mode) frequencies. We will refer to these modified equations as “isolated oscillation equations.” However, the prescription by which these modifications are to be done is not uniquely

### 2. Isolated \( \pi \) and \( \gamma \) Cavities

Linear adiabatic oscillations in a nonrotating star can be expressed as linear combinations of displacement eigenfunctions:

\[
\xi(r, \theta, \phi, t) = e^{i\omega t}(\xi_\pi(r)Y^\pi + \xi_\gamma(r)\Psi^\gamma),
\]

which emerge as solutions to the system of differential equations:

\[
\begin{align*}
\frac{1}{r^2} \frac{d}{dr} (r^2 \xi_\pi) - \frac{\beta}{c_s^2} \xi_\pi + \left( 1 - \frac{S_f^2}{\omega^2} \right) \frac{P_1}{\rho \omega^2} = \frac{N^2}{\omega^2} \Phi_1, \\
\frac{1}{\rho} \frac{dP_1}{dr} + \frac{\beta}{c_s^2} P_1 + (N^2 - \omega^2) \xi_\pi = \frac{d\Phi_1}{dr}, \\
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_1}{dr} \right) - N^2 \Phi_1 = 4\pi G\rho \left( \frac{P_1}{\rho c_s^2} + \frac{N^2}{g} \xi_\pi \right)
\end{align*}
\]

\( \xi_\pi = \frac{1}{r\omega^2} \left( \frac{P_1}{\rho} + \Phi_1 \right) \) \( \xi_\gamma \).

Here \( P_1(r) \) and \( \Phi_1(r) \) are radial functions describing the Eulerian pressure and gravitational potential perturbations under separation of variables, and \( Y^\pi = Y^\pi e_r \) and \( \Psi^\gamma = \nabla Y^\gamma \) are the radial and poloidal vector spherical harmonics.

Subjecting these to appropriate overdetermined boundary conditions yields solutions at discrete eigenvalues \( \omega \). In the Cowling approximation, the perturbations to the gravitational potential are ignored. Unno et al. (1989) introduce the auxiliary variables

\[
\tilde{\xi} = r^2 \xi_\pi \exp \left[ - \int_0^r dr' \frac{g}{c_s^2} \right] = r^2 \xi_\pi h_1(r),
\]

\[
\tilde{\eta} = \frac{P_1}{\rho} \exp \left[ - \int_0^r dr' \frac{N^2}{g} \right] = \frac{P_1 h_2(r)}{\rho}
\]

in terms of which they obtain the linear system

\[
\begin{align*}
\frac{d}{dr} \tilde{\xi} &= -\frac{h_1}{h_2 c_s^2} \left( 1 - \frac{S_f^2}{\omega^2} \right) \tilde{\eta}, \\
\frac{d}{dr} \tilde{\eta} &= -\frac{h_1}{h_2} r^2 (N^2 - \omega^2) \tilde{\xi}
\end{align*}
\]

Either of these quantities can be eliminated in favor of the other to yield second-order differential equations of the form

\[
\begin{align*}
\frac{d^2}{dr^2} \tilde{\xi} - \frac{d \log |P|}{dr} \frac{d}{dr} \tilde{\xi} - \mathcal{P} \mathcal{Q} \tilde{\xi} &= 0, \\
\frac{d^2}{dr^2} \tilde{\eta} - \frac{d \log |Q|}{dr} \frac{d}{dr} \tilde{\eta} - \mathcal{P} \mathcal{Q} \tilde{\eta} &= 0
\end{align*}
\]

where \( \mathcal{P} \) and \( \mathcal{Q} \) are the coefficients on the right-hand sides of Equation (7). The term \( \mathcal{P} \mathcal{Q} \) in both equations yields the dispersion relation of Equation (3).

Following the convention of Aizenman et al. (1977), we refer to the eigenvalues of modified versions of these oscillation equations, where terms corresponding to wave propagation in the classical \( g \)-mode (respectively, \( p \)-mode) cavities have been suppressed, as \( \pi \)-mode (respectively, \( \gamma \)-mode) frequencies. We will refer to these modified equations as “isolated oscillation equations.” However, the prescription by which these modifications are to be done is not uniquely
defined. For instance, for the auxiliary dynamical variables above, differential equations explicitly in Sturm–Liouville form are recovered in different frequency regimes for each variable (Unno et al. 1989). In particular, strictly acoustic propagation (i.e., with Sturm–Liouville eigenvalues proportional to $$\omega^2$$) is recovered for $$\tilde{\xi}$$ for $$\omega^2 \gg S_1^2$$, and strictly buoyant propagation (i.e., with Sturm–Liouville eigenvalues proportional to $$1/\omega^2$$) for $$\omega^2 \ll S_1^2$$. Conversely, these regimes for $$\tilde{\eta}$$ are recovered for $$\omega^2 \gg N_2^2$$ and $$\omega^2 \ll N_2^2$$, respectively.

The choice of which terms of Equation (5) to suppress in order to obtain $$\pi$$ and $$\gamma$$ modes amounts to choosing from the above limits. To isolate $$\pi$$ modes, Aizenman et al. (1977) suppressed the term proportional to $$S_1^2 P_1/\omega^2 \rho c^2$$ in the first line of Equation (5)—this is equivalent to choosing the limit $$\omega^2 \gg S_1^2$$. Likewise, for $$\gamma$$ modes, they suppress the term $$\omega^2 \xi$$ in the second line of Equation (5)—this is equivalent to taking the limit $$\omega^2 \ll N_2^2$$. These choices were motivated by a superficial resemblance to Sturm–Liouville eigenvalue terms proportional to $$1/\omega^2$$ and $$\omega^2$$, bearing the interpretations of buoyant and acoustic wave propagation, respectively.

These choices yield merely sufficient, but not necessary, conditions for the propagation of waves of these respective types. Aizenman et al. (1977) applied them to study high-mass stars with convective cores, for which, e.g., the outer boundary of the $$g$$-mode cavity in the limit of low frequency is set by the Lamb frequency (which we show in the top panel of Figure 1). For typical evolutionary models of lower-mass subgiants and red giants exhibiting solar-like pulsations, however, the Brunt–Väisälä frequency determines the outer limit of $$g$$-mode propagation and the Lamb frequency determines the inner limit of $$p$$-mode propagation (bottom panel of Figure 1). Therefore, we claim that this physical scenario requires taking limits in the converse sense to those taken in Aizenman et al. (1977).

Of these two converse choices, Ball et al. (2018) have previously employed the limit $$\omega^2 \gg N_2^2$$ to perform numerical computations of $$\pi$$-mode frequencies for red giants in the low-$$\eta_p$$ (low-frequency acoustic) regime. In principle, this should be done by suppressing the term proportional to $$N_2^2 \xi$$ in the second line of Equation (5). We assert that the complementary limit, $$\omega^2 \ll S_1^2$$, which may be implemented by suppressing the term proportional to $$P_1/\rho c^2$$ in the first line of Equation (5), will yield pure-buoyancy $$\gamma$$-mode oscillations even in the low-$$\eta_p$$ (i.e., high-frequency buoyancy) regime, which is precisely what is required for subgiant avoided crossings.

Although these choices of which terms of Equation (5) to suppress are motivated by asymptotic considerations, in the sense that we identify terms to suppress that would vanish in the relevant high- or low-frequency limits, we stress that the isolated systems of equations, where such terms have been suppressed a priori, can be employed even in frequency regimes that do not satisfy these asymptotic demands. We will see that this merely results in other terms appearing elsewhere in our analytic formulation for the coupled system, which vanish when these asymptotic conditions are satisfied.

2.1. Approximate Analytic Formulation

We first justify our choice of isolation for the $$\gamma$$-mode cavity as yielding purely buoyant wave propagation. To simplify our analysis, we begin by examining mode isolation in the Cowling approximation. Because we intend to study the behavior of wave propagation in frequency regimes where standard JWKB methods cannot be applied, we turn instead to the method of undetermined phases, which returns exact results that are accurate to the same level of approximation as of the underlying differential system. The typical scenario where this method is employed involves a boundary value problem of Schrödinger form,

$$\frac{d^2}{dx^2} y + \left(k^2 - V(x)\right)y = 0,$$  

with eigenvalues $$k$$ over the domain $$[0, X]$$. $$V(x)$$ is assumed to be small except near these boundaries, which are singular points where the solutions $$y$$ vanish. We substitute ansatz solutions for the eigenfunctions $$u_0(x) \sim A(k, x) \sin(kx - \delta(k, x))$$ near $$x = 0$$, demanding that $$u_0'(x) \sim kA(k, x) \cos(kx - \delta(k, x))$$. This yields the constraint on the inner phase function $$\delta(k, x)$$ that

$$\frac{d}{dx} \delta(k, x) \sim \frac{V(x)}{k} \sin^2(kx - \delta(k, x)).$$  

For $$u_0(x)$$ to vanish at the inner boundary, $$\delta(k, x)$$ must vanish at $$x = 0$$; this constitutes an initial value problem (IVP) for $$\delta(k, x)$$, which can then be integrated to any reference point $$x_0$$ in $$[0, X]$$. A similar IVP can be set up at the outer boundary for a corresponding outer phase function $$\alpha(k, x)$$. These two definitions of the eigenfunctions are consistent at any given matching point $$x_0$$ only if $$\sin(kx_0 - \delta(k, x_0)) = \sin(k(x_0 - X) - \alpha(k, x_0))$$ up
to sign, whence emerges an eigenvalue equation
\[ kX + (\alpha(k, x_0) - \delta(k, x_0)) \equiv kX - \pi \varepsilon(k) = n\pi, \]
yielding eigenvalues \( k_n \) that satisfy this expression for integers \( n \). Once these eigenvalues are known, the eigenfunctions can then be recovered (up to overall constant factor) by solving a complementary IVP
\[ \frac{d}{dx} A(k_n, x) = \frac{A(k_n, x)V(x)}{k_n} \sin(k_n x - \delta(k_n, x)) \]
\[ \times \cos(k_n x - \delta(k_n, x)) \]
from the inner boundary, holding \( k_n \) fixed.

The method of undetermined phases has been employed in the study of \( p \)-mode oscillations (Roxburgh & Vorontsov 1996, 2003; Ong & Basu 2019), but to our knowledge it is not commonly used to study the \( g \)-mode cavity, because historically \( \Delta \Pi \) has sufficed for most applications of \( g \) modes as constraints on stellar interiors. Because the method returns exact results, however, it is ideally suited to working in the low-\( n \) regime where the JWKB approach is known to fail.

Following the discussion above, we take the radial displacement functions of \( \gamma \) modes to satisfy the reduced expression (Unno et al. 1989)
\[ \frac{d^2\xi}{dr^2} + \left( \frac{d}{dr} \log h \right) \left( \frac{d}{dr} \xi \right) + \frac{N^2}{r^2} \left( \frac{1}{\omega^2} - \frac{1}{N^2} \right) \xi = 0, \tag{13} \]
where \( h(r) = h_1(r)/h_2(r) \). As discussed previously, this expression is obtained by suppressing the term \( P_1/\rho c_s^2 \) in the first line of Equation (5). From this, we recover an equation of Sturm–Liouville form
\[ \frac{d}{dr} \left( \frac{1}{h} \frac{d}{dr} \xi \right) + \frac{N^2}{r^2} \left( \frac{1}{\omega^2} - \frac{1}{N^2} \right) \xi = 0 \tag{14} \]
with eigenvalues \( 1/\omega^2 \).

To put this into Schrödinger form, we choose to change coordinates to the buoyancy radius (Tassoul 1980)
\[ f_l = \int_0^r \frac{N\Lambda}{r} \, dr, \tag{15} \]
which has units of frequency. Moreover, to eliminate the damping term, we choose a new dynamical variable with an integrating factor \( \psi = e^{\alpha \xi} \), where
\[ 2u' = -\frac{d}{df_l} \log h + \frac{d}{df_l} \log \frac{N}{r} \]
\[ \Rightarrow \psi = \xi e^{\sqrt{h_1 h_2 r^3 N} = \xi e^{\sqrt{f^3 \rho N}}. \tag{16} \]

After some manipulation (see, e.g., Gough 2007) this yields an equation of Schrödinger form
\[ \psi'' + \left( \frac{1}{\omega^2} - V_{\eta, l}(f_l) \right) \psi = 0, \tag{17} \]
where the buoyancy potential \( V_\eta \) is given as
\[ V_{\eta, l}(f_l) = \frac{1}{N^2} + u''(f_l) + (u'(f_l))^2. \tag{18} \]

Note that both the buoyancy coordinate and potential depend on the degree \( l \). Limiting our attention to the \( \gamma \)-mode cavities found in first-ascent red giants and subgiants with mixed modes, we additionally observe that as \( r \to 0 \), \( N^2 \sim N_0^2 r^2 \) for some constant \( N_0^2 \). Thus, \( f \sim r \) as \( r \to 0 \), and the buoyancy potential is singular at the central point, which is then a regular singular point of the differential equation. Likewise, the outer boundary of the \( \gamma \)-mode cavity is defined by the inner boundary \( r_0 \) of the convection zone, where \( N_0^2 = 0 \). This is also a singular point of the differential equation, which is regular only if the leading-order behavior of \( N^2 \) is either linear or quadratic in \( r - r_0 \) inwards of the boundary. The domain of the associated boundary value problem is then \( f_l \in [0, F_l] \), where \( F_l = f_l(r_0) = \int_0^{r_0} (N\Lambda/r) \, dr \). As both boundaries are singular points, solutions can be assumed to vanish there.

With this in hand, we then construct the inner phase function as the solution to the IVP,
\[ \frac{d}{df_l} \xi_{l, \eta}(\omega, f_l) = \omega V_{l, \eta}(f_l) \sin\left( \frac{f_l}{\omega} - \xi_{l, \eta}(\omega, f_l) \right), \tag{19} \]
and likewise for the outer phase function, following the above procedure; this yields at last the eigenvalue equation
\[ \frac{F_l}{\omega_{nl}} \sim \pi(n + \epsilon_{l, \eta}(\omega_{nl})). \tag{20} \]

Comparing this with Equation (2) yields the usual asymptotic expression for the period spacing in the regime of constant \( \epsilon_{l, \eta} \) (Tassoul 1980),
\[ \Delta\Pi_l = \frac{2\pi^2}{F_l} = \frac{2\pi^2}{\sqrt{l(l+1)}} \left( \int_0^{r_0} dr \right)^{-1} \tag{21} \]

For the purposes of our subsequent discussion and analysis, we consider evolutionary tracks of stellar models constructed using release 10398 of the MESA stellar evolution code (Paxton et al. 2011, 2013, 2018), generated with solar-like fundamental parameters (i.e., \([\text{Fe/H}] = 0\), with solar-calibrated mixing length and helium abundance). We show in Figure 2 the Cowling-approximation eigensystem associated with the \( \gamma \)-mode cavity for a 1 \( M_\odot \) model subgiant undergoing an avoided crossing (\( \Delta\nu = 72 \mu\text{Hz} \)), alongside with the square root of the buoyancy potential, normalized by the asymptotic period spacing as given by Equation (21). This choice of scaling
places eigenvalues at integer steps of the vertical axis in the asymptotic regime, as described by Equation (2). We see that this is more or less the case, consistent with the known properties of pure-buoyancy waves. This is the basis of our interpretation of these γ modes as yielding purely buoyant wave propagation.

We have also plotted the contribution to this buoyancy potential from the inverse Brunt–Väisälä frequency, which we see essentially dominates the dynamics of the system, especially near the core. However, the remaining terms involving \( u \) are also singular at the outer boundary for typical subgiant evolutionary models and cannot be ignored.

### 2.2. Numerical Evaluation of \( \gamma \) Modes

The Cowling approximation is known to hold increasingly well at high order and degree, and we might be doubtful as to its applicability to subgiant avoided crossings in particular, which are observed at low order and low degree. Indeed, as seen in Figure 2, the use of the Cowling approximation yields a fictitious \( n = 0 \) dipole mode, which implies periodic oscillations of the center of mass. This is known to be forbidden under the full system of equations (Christensen-Dalsgaard 1976; Christensen-Dalsgaard & Gough 2001). Moreover, the Sturm–Liouville form of Equation (14) implies orthogonality with respect to a different inner product than is obtained for the full system of equations. We therefore find it prudent to compare our results to those obtained without the Cowling approximation for the remaining dipole modes.

We first note that it is in fact possible under some circumstances to perform a similar analysis without recourse to the Cowling approximation. For instance, the \( l = 1 \) oscillation equations admit a reduction to second order (Takata 2016; Pinçon et al. 2020) that essentially preserves the dispersion relation of Equation (3). However, this formulation is not applicable to modes of higher degree, and in any case the displacement eigenfunctions are not directly recovered from this construction: the dynamical variables there contain the perturbation to the gravitational potential and its derivative. Eliminating these requires a second, auxiliary set of differential equations to be solved. Because we require the eigenfunctions for the computations described later in Section 3, we choose not to pursue this approach further.

Instead, we seek explicit recourse to numerical methods here. To this end, we used the GYRE pulsation code (Townsend & Teitler 2013) to compute mixed mode and (with appropriate modifications) \( \gamma \)-mode frequencies, without using the Cowling approximation, for a series of subgiant/early red-giant models along a 1 \( M_\odot \) MESA evolutionary track. More details about our modifications to GYRE can be found in Appendix A.

We compare results from GYRE with our solutions to the Cowling-approximation Sturm–Liouville problem for one of these evolutionary tracks in Figure 3. We see that both solutions exhibit broadly similar morphology, tracking the implicit \( \gamma \) modes traced out by the avoided crossings. For the first avoided crossing specifically, we also see that both approaches slightly underestimate the avoided-crossing frequency. We will show in Section 3 that the eigenvalues of the \( \gamma \)-mode system are in general not sufficient to predict the frequency of the avoided crossing; a first-order correction term must also be computed. These shortcomings notwithstanding, however, we claim that this further demonstrates that our interpretation of these \( \gamma \) modes as purely buoyant waves is at least qualitatively correct.

The other unexpected feature of Figure 3 is the increasing discrepancy with the full-system eigenvalues at low frequencies, where we would a priori expect the Cowling approximation to hold increasingly well. We attribute this to the singular behavior of the outer boundary of the buoyancy cavity in GYRE’s numerical scheme. To illustrate this, we show the \( l = 1, n = 1 \) eigenfunction returned from our solution in buoyancy coordinates, and from GYRE, in Figure 4. We see in the top panel that the behavior of GYRE’s solution is pathological at the outer boundary. This problem becomes increasingly severe at higher orders; for example, the \( n = 6 \) \( \gamma \) eigenfunction for the same model does not even have the correct number of zero crossings.

Specifically, the outer boundary condition of the eigenvalue problem is applied to \( \psi \) at \( f_l = F_l \) in our Cowling-approximation solution, and to (a dimensionless analog of) \( \zeta_r \) at \( r = R \) in GYRE. For evolutionary models with outer convection zones, these are formally inequivalent in the following manner: irrespective of GYRE’s boundary conditions, the corresponding \( \psi \) automatically vanishes at the end points via Equation (16) without regard for regularity (ignoring any radiative atmosphere). However, as the Brunt–Väisälä frequency vanishes outside of the buoyancy cavity, the transformation from \( r \) to \( f_l \) defined by Equation (15) is formally also degenerate (as the entire convection zone is mapped to a single point at \( f_l = F_l \)), yielding the discontinuous behavior that we see in Figure 4. We verified this by computing \( \gamma \)-mode eigenfunctions of a polytrope with index \( n = 3 \), for which, in the absence of an outer convective zone, we were able to recover the correct number of nodes even at high radial order.

To recover regularity with respect to the buoyancy coordinate, boundary conditions expressed in terms of derivatives taken with respect to \( f_r \), rather than \( r \), must be

![Figure 3](image_url)

**Figure 3.** Evolution of mixed modes and \( \gamma \) modes for the first few avoided crossings of a 1 \( M_\odot \) evolutionary track. Mixed modes are shown with the thin blue lines, with \( \gamma \)-mode frequencies shown with the solid green line (as computed under the Cowling approximation) and orange dashed line (as computed from the full system of equations). The red dotted line shows the evolution of \( \nu_{\text{max}} \) for the same models.
imposed at the outer boundary. We note that
\[
\frac{d}{df_0} = \frac{r}{N} \frac{d}{dr};
\]  
(22)
as \(N \to 0\) at the outer boundary, the boundary conditions there must then involve higher derivatives of the dynamical variables (with respect to \(r\)) to remain regular via L’Hôpital’s rule. The changes required to do this in GYRE are substantial and beyond the scope of this work. For the purposes of our subsequent analysis, it suffices merely to note that this is a numerical artifact that is least severe at low order—fortuitously, this still permits the study of the lowest-order avoided crossings.

### 2.3. Numerical Evaluation of \(\pi\) Modes

To complete this discussion of cavity isolation, we turn our attention to the \(\pi\)-mode cavity. For subgiants undergoing avoided crossings, the acoustic modes are of relatively high order (>10), and JWKB expressions are largely applicable. Moreover, for acoustic modes, the relevant radial coordinate required to recover an expression of Sturm–Liouville form is

the acoustic radial coordinate,
\[
t(r) = \int_0^r \frac{dr}{c_s},
\]  
(23)
which is well behaved everywhere in the interior of the star. Although our ability to accurately predict \(p\)-mode frequencies is affected by the surface term, we expect it to afflict both the full mixed modes and \(\pi\) modes in much the same manner, so long as the Brunt–Väisälä frequency at the surface is unchanged (for more details, see Appendix A). For the sake of demonstration, we once again compare the evolution of \(\pi\) modes in our formulation with the mixed modes returned from an evolutionary track in Figure 5. We see that these \(\pi\) modes adhere to the \(p\)-mode asymptotic relation, Equation (1), even where the eigenvalues of the coupled system undergo avoided crossings; we interpret these as being purely acoustic waves, in agreement with Ball et al. (2018).

### 3. Coupled Mode Cavities

We have demonstrated that our choices of isolated mode cavities can be meaningfully interpreted as separately supporting purely buoyant and purely acoustic waves. As seen in Figures 3 and 5, the eigenvalues of the coupled system exhibit an “avoided crossing” phenomenon over the course of stellar evolution. Expressions for the frequencies of such avoided crossings are generically derived by way of a mechanical analogy with a coupled system of harmonic oscillators (e.g., Deheuvels & Michel 2010; Benomar et al. 2012). The standard analytic approach here is to find the eigenvalues of some real, symmetric matrix,
\[
L = H_0 + \alpha V,
\]  
(24)
where \(H_0\) is diagonal and nearly degenerate; the introduction of the coupling matrix \(V\), with off-diagonal elements, lifts this degeneracy. Once in this form, the avoided crossing can be shown to emerge, e.g., by application of perturbation theory (von Neumann & Wigner 1929). The basis of this analogy is such that the matrix \(L\) describes the time evolution of some set
of dynamical quantities \( y \) of these model coupled oscillators as
\[
\frac{d^2}{dt^2} y = Ly.
\]  
(25)

As far as subgiant avoided crossings are concerned, this matrix is ordinarily assumed a priori to be of some ansatz parametric form with constant coupling between the \( \pi \) and \( \gamma \) cavities, motivated by the \( 2 \times 2 \) case; to our knowledge, no explicit construction exists that relates it to properties of stellar structure. We attempt such a construction in this section.

The perturbed momentum equation for a single-mode \( \xi_j \) with time-dependent coefficient \( c_i \), permits the construction of a time-dependent operator equation over the Hilbert space of vector displacement eigenfunctions (Eisenfeld 1969; Christensen-Dalsgaard 1981) in the form
\[
\frac{d^2}{dt^2} c_i \xi_i \equiv \hat{L} c_i \xi_i = -\omega_i^2 c_i \xi_i,
\]  
(26)

where the time dependence is carried entirely by the coefficient \( c_i \), which in turn is a function only of time (this is equivalent to working in the Schrödinger picture in quantum mechanics). For a general state in this Hilbert space, expressed as a linear combination of eigenfunctions \( \xi = \sum c_j \xi_j \), the corresponding evolution goes as
\[
\sum_j \left( \frac{d^2}{dt^2} c_j \right) \xi_j \equiv \hat{L} \sum_j c_j \xi_j = -\sum_j c_j \omega_j^2 \xi_j,
\]  
(27)

where this linear operator acts independently on each of its eigenfunctions \( \xi_j \), which emerge as solutions to Equation (5).

In the case where the \( \xi_j \) form a complete orthogonal basis, we can recover each of these time-dependent coefficients by taking inner products under the choice of normalization such that
\[
\langle \xi_i, \xi_j \rangle = \int dx \rho x \cdot \xi_i \equiv \delta_{ij}.
\]  
(28)

Put differently, Equation (5) is a time-independent problem that yields the eigensystem of the Hermitian integro-differential operator \( \hat{L} \) (which provides some natural orthonormal basis on the Hilbert space by the spectral theorem). By contrast, Equation (27), which is time dependent, instead relates the time evolution of vectors in the Hilbert space to the action of this operator.

Let us now consider the time evolution of the \( \gamma \) and \( \pi \)-mode eigenstates, which are not eigenstates of \( \hat{L} \). Instead, they are solutions to modified versions of Equation (5), where different terms have been suppressed to isolate the mode cavities. In this abstract operator notation, we consider the \( \pi \) modes to be the eigenstates of the operator \( \hat{L}_\pi \), representing the modified momentum equation, and the \( \gamma \) modes to be those of a different operator \( \hat{L}_\gamma \). To use Equation (27), we relate these modified operators to the original set of equations as the sum of the modified (e.g., \( \pi \)-mode) operator and some “remainder” operator:
\[
\hat{L} = \hat{L}_\pi + \hat{R}_\pi,
\]  
(29)

where this remainder operator is simply the term that has been suppressed in order to yield the isolated system of equations for \( \pi \) modes. In this case, we can easily see that \( \hat{R}_\pi \) satisfies
\[
\hat{R}_\pi \xi_{\pi,i} = -N^2 \xi_{\pi,i} Y_i^m
\]  
(30)

away from the outer boundary of the acoustic-mode cavity, where \( \xi_{\pi,i} \) are the eigenstates of the modified operator \( \hat{L}_\pi \). The matrix elements of \( \hat{R}_\pi \) can then be evaluated as the volume integral
\[
R_{\pi,ij} = \langle \xi_{\pi,i}, \hat{R}_\pi \xi_{\pi,j} \rangle = -\int \rho N^2 \xi_{\pi,i} \xi_{\pi,j} d^3x.
\]  
(31)

where the spherical harmonic indices \( l, m \) of the state \( j \) are equal to those of the state \( \pi, i \); the integral vanishes otherwise. We note that this expression is manifestly Hermitian. It is also applicable for computing elements of \( R_{\gamma,ij} = \langle \hat{R}_\gamma \xi_{\gamma,i}, \xi_{\gamma,j} \rangle = \langle \xi_{\gamma,i}, \hat{R}_\gamma \xi_{\gamma,j} \rangle \), where the state \( j \) is associated with a \( \gamma \) mode rather than a \( \pi \) mode.

We should in principle be able to do likewise for some \( \gamma \)-mode remainder operator. However, deriving an exact expression in a similar manner is less straightforward, as the modification to the oscillation equations which isolates the \( \gamma \)-mode cavity does not prima facie affect the momentum equation (the first line of Equation (5) is instead the perturbed continuity equation with \( \xi_i \) eliminated). We have not been able to find a corresponding modification to the momentum equation that yields a manifestly Hermitian expression. For instance, we might observe that the first line of Equation (5) can be rewritten in the form
\[
\xi_{\gamma} = \frac{1}{N^2 \rho c_s^2} \frac{d \xi_{\gamma}}{dr} + \left( \frac{2}{r} - \frac{8}{c_s^2} \right) \xi_{\gamma},
\]  
(32)

with only the first term in the brackets on the right-hand side being suppressed when computing the \( \gamma \)-mode eigensystem. This suppression can be affected by modifying the tangential momentum equation (last line of Equation (5)) to read
\[
-\omega^2 \xi_{\gamma} = -\left[ \frac{1}{r} \frac{P_{\gamma}}{\rho} + \Phi_1 \right] - \left[ \frac{r \omega^2}{N^2 \rho c_s^2} \frac{P_{\gamma}}{\rho c_s^2} \right],
\]  
(33)

where the term in the square brackets does not appear in the tangential momentum equation of the coupled system. Accordingly the \( \gamma \) remainder operator might be thought to satisfy
\[
\hat{R}_\gamma \xi_{\gamma,ij} = \frac{r \omega^2}{N^2 \rho c_s^2} \frac{P_{\gamma}}{\rho c_s^2} \Psi_i^m,
\]  
(34)

whence
\[
R_{\gamma,ij} = \langle \xi_{\gamma,i}, \hat{R}_\gamma \xi_{\gamma,j} \rangle = \int \rho \omega^2 \xi_{\gamma,i} \xi_{\gamma,j} d^3x.
\]  
(35)

Constructions like these do not obviously yield Hermitian matrix elements, which is problematic in the following sense: because each of the modified versions of Equation (5) (for \( \pi \) and \( \gamma \) modes) yield orthogonal bases with respect to the same inner product as \( \hat{L} \), the operators \( \hat{L}_\pi \) and \( \hat{L}_\gamma \) are self-adjoint and Hermitian under that inner product. It then follows that the remainder operators are also self-adjoint. Any correct expression for the matrix elements of the remainder operators must therefore be manifestly Hermitian with respect to this inner
product. Nonetheless, in what follows we will be mostly concerned with the off-diagonal elements describing the coupling between the $\pi$ and $\gamma$ modes, which can be expressed entirely in terms of $\hat{R}_{\gamma}$ and the $\pi$-mode eigenvalues, and so this difficulty is not an obstruction to the subsequent analysis. We will use Equation (35) to compute only diagonal matrix elements, and assume that the off-diagonal $\gamma$ terms vanish.

The combined set of basis vectors $\{\xi_\pi, \xi_\gamma\}$ is not in general orthonormal, because the $\pi$- and $\gamma$-mode eigenfunctions are not necessarily orthogonal to each other. However, in the spirit of Lennard-Jones (1929), we can nonetheless express the general time-dependent state of the linear displacement in terms of this combined basis as

$$\xi = \sum_{i} N_i c_{\pi,i} \xi_{i,\pi} + \sum_{i} N_i c_{\gamma,i} \xi_{i,\gamma}$$

(36)

To find the time evolution of the $i$th $\pi$-mode coefficient in particular, we can substitute this into Equation (27) and take the inner product against $\xi_{i,\pi}$ (making use of the self-adjoint property of all of the operators under consideration) to obtain

$$\frac{d^2}{dt^2} c_{\pi,i} + \sum_{j} \langle \xi_{i,\pi}, \xi_{j,\pi} \rangle c_{\pi,j} = -\omega_{\pi,i}^2 c_{\pi,i} + \sum_{j} \langle \xi_{i,\pi}, R_{\pi} \xi_{j,\pi} \rangle c_{\pi,j},$$

$$- \sum_{j} \omega_{\pi,i}^2 \langle \xi_{i,\pi}, \xi_{j,\pi} \rangle c_{\pi,j} + \sum_{j} \langle \xi_{i,\pi}, R_{\pi} \xi_{j,\pi} \rangle c_{\pi,j}.$$  

(37)

Likewise, the time evolution of the $\gamma$-mode coefficients is given by

$$\frac{d^2}{dt^2} c_{\gamma,i} + \sum_{j} \langle \xi_{i,\gamma}, \xi_{j,\gamma} \rangle c_{\gamma,j} = -\omega_{\gamma,i}^2 c_{\gamma,i} + \sum_{j} \langle \xi_{i,\gamma}, R_{\gamma} \xi_{j,\gamma} \rangle c_{\gamma,j},$$

$$- \sum_{j} \omega_{\gamma,i}^2 \langle \xi_{i,\gamma}, \xi_{j,\gamma} \rangle c_{\gamma,j} + \sum_{j} \langle \xi_{i,\gamma}, R_{\gamma} \xi_{j,\gamma} \rangle c_{\gamma,j}.$$  

(38)

Collecting these coefficients into column vectors $c_{\pi}$ and $c_{\gamma}$, we can rewrite these expressions in the block matrix form

$$\frac{d^2}{dt^2} \begin{bmatrix} c_{\pi} \\ c_{\gamma} \end{bmatrix} = L \begin{bmatrix} c_{\pi} \\ c_{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} I_N & D_{\pi,\gamma} \\ D_{\gamma,\pi}^T & I_N \end{bmatrix}^{-1} \begin{bmatrix} -\Omega_{x,\pi}^2 + \Omega_{x,\gamma} D_{\pi,\gamma} + \Omega_{x,\gamma} D_{\gamma,\pi} \end{bmatrix} \begin{bmatrix} c_{\pi} \\ c_{\gamma} \end{bmatrix}$$

(39)

where $I_N$ is the identity matrix of order $n$, the matrix elements of $R_{\pi,\pi}$ and $R_{\gamma,\gamma}$ are given by Equation (31), $R_{\gamma,\pi}$ by Equation (35), and

$$D_{\pi,\gamma,ij} = \int d^3x \rho \xi_{i,\pi} \cdot \xi_{j,\gamma}$$

(40)

(compare Equation (28)). Because the matrices $A$ and $G$ are both Hermitian, this in principle defines a generalized Hermitian eigenvalue problem of the form

$$Ac = \lambda Ge,$$  

(41)

whose eigenvectors are orthogonal with respect to the inner product $G$. However, this makes the subsequent perturbation analysis extremely unwieldy. Instead, we note that while the overlap integrals of Equation (40) do not, in general, vanish—as the $\pi$ and $\gamma$ modes are eigenfunctions of different differential operators—the $\pi$-mode eigenfunctions are at least heuristically oscillatory in the $\gamma$-mode evanescent region, and vice versa, so we expect that $\max_{i,j} |D_{ij}| \ll 1$, with these quantities vanishing in the limit of very high or very low frequencies (where the JWKB approximation holds good). Numerically, we find this to indeed be the case. We likewise note that the matrix elements $R_{\pi,\gamma}$ are also overlap integrals, of a similar order of smallness relative to the frequency eigenvalues. We therefore approximate $L$ by expanding $G^{-1}$ in series and retaining only first-order terms:

$$L \sim - \begin{bmatrix} \Omega_{\pi,\pi} & 0 \\ 0 & \Omega_{\gamma,\gamma} \end{bmatrix}$$

$$+ \begin{bmatrix} R_{\pi,\pi} \\ R_{\gamma,\gamma} \end{bmatrix} (-\Omega_{x,\pi}^2 + \Omega_{x,\gamma} D_{\pi,\gamma} + \Omega_{x,\gamma} D_{\gamma,\pi}) \begin{bmatrix} R_{\pi,\pi} \\ R_{\gamma,\gamma} \end{bmatrix}^{T}$$

(42)

As required, this matrix $L$ describes the dynamics of the coupled $\pi$ and $\gamma$ oscillators, whose oscillation frequencies in isolation are given by the diagonal matrices $\Omega_{\pi}$ and $\Omega_{\gamma}$. Mixed modes can be expressed as eigenvectors of this matrix—i.e., as linear combinations of the $\pi$- and $\gamma$-mode eigenfunctions that oscillate in phase at the specified frequency eigenvalues, which are in general distinct from both of the $\pi$- and $\gamma$-mode frequency eigenvalues.

We now wish to evaluate the eigenvalues of $L$, which we proceed to do perturbatively. Because all of the overlap integrals are small and our approximation for $L$ is fortuitously Hermitian, we observe that Equation (42) is of the same form as Equation (24). For such a decomposition, where $H_0$ has eigenvalues $E_i$, we recall the standard expression from perturbation theory for the perturbed eigenvalues in powers of $\alpha$ (or $V$ as $\alpha \rightarrow 1$) as (Landau & Lifshitz 1965)

$$E_i' = E_i + \alpha V_\alpha + \alpha^2 \sum_{i \neq j} \frac{|V_{ij}|^2}{E_i - E_j} + \ldots$$

(43)

To leading order, these are given by the diagonal elements of the matrix $L$, which are dominated, but not completely specified, by the isolated frequency eigenvalues. Truncating the series to this order of approximation yields the same result as we would have obtained with the standard variational approach, treating the remainder operators as small perturbations to their respective isolated oscillation equations.

To illustrate how these various matrices contribute to the eigenvalues of the complete coupled system, we show in Figure 6 the predicted isolated and coupled dipole mode frequencies for a $1 M_\odot$ subgiant model (described in Section 2). We show the isolated dipole frequencies of each mode cavity with and without these first-order corrections. These contributions, though small, cannot be ignored. It is moreover also apparent that the coupling between the cavities cannot be derived from first-order considerations; the off-diagonal terms only enter the series from the second order onward.
the diagonal contributions from each of the matrices described in Equation (42). Mixed modes returned from the full system of equations are shown with the blue points, while the eigenvalues of the incomplete matrix are shown with red circles.

In Figure 6 we additionally show (with red circles) the eigenvalues of an incomplete copy of $L$ (containing entries for only the six $\gamma$ modes shown in the figure). As can be seen, the accuracy of this incomplete evaluation is increasingly degraded at low frequencies, both because of the numerical issues we have described and above, and because the density of missing $\gamma$-mode eigenvalues increases with decreasing frequency—the sheer number of $\gamma$-mode matrix elements that need to be computed for a complete result renders this direct approach untenable in the low-frequency regime even in the absence of implementation-induced numerical artifacts. Conversely, however, we also see that we obtain good numerical agreement with the full system of equations in the regime of individually observed avoided crossings, where $\gamma$ modes are sparse compared to $\pi$ modes.

The eigenvectors of $L$ specify the relative contributions from each of the $\pi$ and $\gamma$ modes to each mixed mode that results from the full set of equations. We consider the components of the $i$th mixed mode:

$$\xi_i = \sum_j c_{ij} \xi_j$$

Again, the coefficients $c_{ij}$ follow from standard results in perturbation theory:

$$c_{ij} = \delta_{ij} + \alpha f_{ij} \frac{V_{ij}}{E_i - E_j}$$

$$+ \alpha^2 \left( \sum_k f_{ik} f_{kj} \frac{V_{ik} V_{kj}}{(E_i - E_k)(E_i - E_j)} - f_{ij} \frac{V_{ii} V_{jj}}{(E_i - E_j)^2} - \frac{1}{2} \delta_{ij} \sum_k f_{ik} \frac{V_{ik}^2}{(E_i - E_k)^2} \right) + \ldots$$

with $f_{ij} = 1 - \delta_{ij}$. Generically, higher-order terms for both the $i$th eigenvalues and the eigenvector components involve increasing powers of the resonance/degeneracy factors $1/(E_i - E_j)$, which become suppressed for pairs of modes away from resonance even as $\alpha \to 1$.

3.1. Relation to Empirical Parameterization

We contrast this construction with the empirical parameterization used elsewhere in the literature (e.g., Deheuvels & Michel 2011; Benomar et al. 2012, 2013), which is of the generic block form

$$L = \begin{bmatrix} \Omega_p^2 & A \\ A^T & \Omega_g^2 \end{bmatrix}$$

with $A$ an $N_p \times N_g$ matrix with constant values along each column, representing $N_g$ different coupling constants $\{\alpha_1 \ldots \alpha_{N_g}\}$; $\Omega_p$ and $\Omega_g$ are taken to be diagonal matrices, related to our quantities as

$$\Omega_p^2 = \Omega_{\pi\pi}^2 - \text{diag } R_{\pi\pi}$$

$$\Omega_g^2 = \Omega_{\gamma\gamma}^2 - \text{diag } R_{\gamma\gamma},$$

while the coupling constants $\alpha_i$ are explicit fit parameters. Most practical applications of this parameterization do not assume access to the isolated eigenvalues and therefore supply them by way of the asymptotic relation (thereby introducing additional, implicit parameters).

We compare the off-diagonal elements of Equation (46) (right panel) with those of our explicit construction (left panel) in Figure 7 for the same set of modes as shown in Figure 6. The parameters $\alpha_i$ of the approximate construction were found by minimizing the sum of the squared differences between the ordered eigenvalues of Equation (46) and those of our incomplete matrix.

As noted previously, modes do not couple significantly, irrespective of the actual value of the corresponding coupling matrix elements, except where the isolated eigenvalues are close to resonance. To demonstrate this, we mark out the coupling matrix elements of $\pi$- and $\gamma$-mode pairs that are closest to resonance in the left panel of Figure 7. We see that the best-fitting approximate coupling parameters in the right panel take values very close to these on-resonance matrix elements. Moreover, we see that these decrease with frequency,
in line with our expectation that the corresponding overlap integrals should vanish in the $g$-mode asymptotic limit $\omega^2 \ll S_\gamma^2$.

3.2. Relation to JWKB Expressions

Under the JWKB approximation, it is typical to determine eigenvalues by relating the phase integrals $\Theta = \int k_r dr$ in the $g$- and $p$-mode cavities to each other via a coupling expression of the form

$$\tan \Theta_p \cot \Theta_g = q,$$

(48)

where $q$ is some frequency-dependent coupling strength, related to the transmission coefficient between the two mode cavities. Both sides of this expression are understood to be different functions of frequency, such that mixed-mode eigenvalues are recovered only at frequencies where this expression holds. In practice, the frequency dependence of the coupling factor $q$ is typically ignored (although see Cunha et al. 2019; Jiang et al. 2020; Pinçon et al. 2020 for more recent discussions).

Each of these $\Theta$ functions yield eigenvalues for the isolated mode cavities ($q = 0$) at integer multiples of $\pi$. The appropriate constructions in the nonasymptotic regime are of the form (Unno et al. 1989; Mosser et al. 2012)

$$\Theta_p = \omega T - \pi \epsilon_{1,p} (\omega),$$
$$\Theta_g = F_1 \omega - \pi \epsilon_{1,g} (\omega),$$

(49)

which separately yield the eigenvalue quantization conditions for isolated cavities (as in Equation (20)). We have discussed a construction of the buoyancy phase $\epsilon_g$ above; for a discussion of the acoustic phase $\epsilon_p$, see, e.g., Roxburgh & Vorontsov (2003). These phases are used, particularly in the study of red giants, for the computation of a diagnostic quantity

$$\zeta (\omega) = \frac{I(\text{core})}{I(R)} \sim \left[ 1 + \frac{T}{q} F_1 \frac{\cos \Theta_g}{\cos^2 \Theta_p} \right]^{-1},$$

(50)

where

$$I(r) = \frac{\int_0^r 4\pi r^2 p |\xi|^2 dr}{M (\xi_c (R)^2 + \widehat{\xi}_c (R)^2)}$$

(51)

is the normalization-independent dimensionless partial inertia, evaluated up to radius $r$. This quantity has variously been used to disentangle the effects of mode bumping from other structurally or rotationally induced frequency perturbations (Mosser et al. 2015; Gehan et al. 2018), or as a structural/differential rotational diagnosties in its own right (Deheuvels et al. 2015, 2017). By inspection, this quantity takes values between 0 and 1; for mixed modes of high $n_g$ in red giants, it is known to take values close to unity for modes of predominantly $g$-like character and close to zero for modes of predominantly $p$-like character. For our purposes, we identify $r_{\text{core}}$ with the inner boundary of the convection zone.

Rather than directly computing $\zeta$ from our eigenfunctions in this manner, we first consider the relative contributions to the ratio of inertiae from a two-term linear combination of the form

$$\xi = c_\gamma \xi_\gamma + c_\pi \xi_\pi,$$

(52)

In the limit of both high $n_g$ and $n_p$, we recall that the $\gamma$-mode eigenfunction decays rapidly outside of the convective boundary, while the $\pi$-mode eigenfunction does so inside of it, so to a good approximation the $\pi$ modes do not contribute significantly to $I(r_{\text{core}})$. Likewise, we expect the overlap integral matrix elements $D_{ij}$ to be negligible for the same reason. We therefore have

$$\zeta = \frac{I(r_{\text{core}})}{I(R)} \sim \frac{|c_\gamma|^2}{|c_\gamma|^2 + |c_\pi|^2}.$$

(53)

By orthonormality (and again ignoring cross-terms $D_{ij}$), the generalization to a linear combination of many $\pi$ and $\gamma$ modes is immediate:

$$\zeta \sim 1 + \sum_j \frac{|c_{\gamma,j}|^2}{\sum_i |c_{\gamma,j}|^2}.$$

(54)

This expression has the same qualitative properties as $\zeta$—i.e., it is close to unity for $g$-dominated modes and close to zero for $p$-dominated modes. While the standard construction of Equation (50) in terms of asymptotic phases relies on JWKB approximants for the eigenfunctions (e.g., Goupil et al. 2013; Deheuvels et al. 2015), Equation (54) involves quantities that remain sensible even in the nonasymptotic regime. We therefore consider Equation (54) to be a fundamental quantity to which the definitions in Equation (50) are approximations recovered in the JWKB regime. We demonstrate this explicitly in Appendix B.

3.3. The Coupling Strength, $q$

The JWKB coupling strength $q$ appearing in Equation (48) is given by

$$q = \frac{1}{4} \exp \left[ -2 \int_{r_1}^{r_2} \sqrt{-k_r^2} dr \right] = \frac{1}{4} w(r_1, r_2),$$

(55)

where $r_1$ and $r_2$ are the lower and upper boundaries of the formal evanescent region between the two mode cavities, where $k_r^2 < 0$. In the same construction, the JWKB radial displacement wave function within this evanescent region may be variously written in the forms (Unno et al. 1989, Equations (16.47)–(16.50))

$$\psi \sim \frac{A}{\sqrt{-k_r^2}} \left( -\frac{1}{2} \sin \Theta_g \exp \left[ -\int_{r_1}^{r} \sqrt{-k_r^2} dr \right] + \cos \Theta_g \exp \left[ \int_{r_1}^{r} \sqrt{-k_r^2} dr \right] \right),$$

$$= \frac{B}{\sqrt{-k_r^2}} \left( -\sin \Theta_p \exp \left[ -\int_{r_1}^{r} \sqrt{-k_r^2} dr \right] + \frac{1}{2} \cos \Theta_p \exp \left[ \int_{r_1}^{r} \sqrt{-k_r^2} dr \right] \right),$$

(56)
for different choices of constants $A$ and $B$. We identify terms with the isolated radial displacement eigenfunctions in the following manner: for a two-term linear combination of the form of Equation (52), we demand that the component decaying exponentially as $r$ increases be identified with $c_r\xi_r$, while the component that increases exponentially with $r$ is to be identified with $c_r\xi_c$. At any radius, the ratio of these two terms (which we will call $f(r)$) must be independent of whether $A$ and $\Theta_q$, or $B$ and $\Theta_p$, are used to write the JWKB wave function. As a check of consistency, we should recover Equation (48). Explicitly,

$$ f(r) = \frac{c_r\xi_c}{c_r\xi_r} \sim -\frac{1}{2} \tan \Theta_q w(r_1, r) = -\tan \Theta_q w(r_2, r). $$

$$ \Rightarrow \tan \Theta_p \cot \Theta_q = \frac{1}{4} \frac{w(r_1, r)}{w(r_2, r)} = \frac{1}{4} \frac{w(r_1, r)}{w(r_1, r_1)} = q. $$

(57)

Note that this function $f$ is regular everywhere in the domain $[r_1, r_2]$, even though the JWKB wave function itself is singular at the classical turning points (as $k_r \rightarrow 0^+$). We relate $q$ to our quantities via the ratio of $f$ as evaluated at these turning points:

$$ f(r_2) \sim \frac{4\tan \Theta_p w(r_2, r_2)}{\tan \Theta_p w(r_1, r_1)} = \frac{4\tan \Theta_p}{\tan \Theta_q} = 4q $$

$$ \Rightarrow q \sim \frac{1}{4} \frac{f(r_2)}{f(r_1)} = \frac{1}{4} \frac{\xi_c(r_2)}{\xi_r(r_1)} \cdot \frac{\xi_r(r_1)}{\xi_c(r_2)} $$

(58)

That is to say, $q$ is proportional to the product of the (amplitude) transmission coefficients of the $\pi$ and $\gamma$ waves, considered separately, across the evanescent region.

4. Applications to Stellar Modeling

So far, we have concerned ourselves with the theoretical implications of our construction. In this section we identify and explore ways in which an explicit isolation of the mode cavities may be applied to modeling stars against observational seismic constraints, with the ultimate goal of inferring fundamental stellar parameters.

4.1. $\pi$ Modes for Stellar Modeling

The prescription of Ball et al. (2018), while intended for the same propagation conditions as we are concerned with, operates by modifying the stellar structure instead of the oscillation equations. That is to say, where we would set the term $N^2\xi_r$ to zero in the oscillation equations, their prescription does so by altering $\Gamma_1$, setting it to

$$ \Gamma_1 = \frac{d \log P}{d r} / \frac{d \log \rho}{d r} $$

(59)

everywhere in the interior radiative zone, which in turn causes $N^2$ to vanish. We show the differences between these approaches in Figure 8, for subgiant (upper panel) and first-ascent red giant (lower panel) solar-calibrated 1 M$_\odot$ MESA models. In both cases we compare these results with mixed-mode eigenvalues computed with respect to the full oscillation equations and an unmodified stellar model (blue dots).

In the regime of isolated avoided crossings, these modifications to $\Gamma_1$ yield results that differ significantly from the actual mixed-mode frequencies, even for modes far from resonance. This is because the alterations to $\Gamma_1$ also modify the sound speed $c_r^2 = \Gamma_1 P/\rho$, thereby changing the acoustic radius of the model. This incurs a substantial error in the large frequency separation $\Delta \nu$ of the computed frequencies (for which $1/2T$ is an asymptotic estimator), which our prescription avoids. By contrast, our prescription does not modify the stellar structure, but instead returns $\pi$ modes solely from applying pulsation theory; it correctly recovers the asymptotic behavior of high-order $p$ modes.

The prescription of Ball et al. (2018) works better on the red giant branch—as the radiative region is very small in physical extent, the total acoustic radius is not significantly changed. Instead, modifications to the acoustic-mode cavity are confined to a narrow region of the acoustic radial coordinate, resulting in deviations from our formulation that are of the form of an acoustic glitch (albeit of very small amplitude) localized near the inner boundary. We plot these differences in Figure 9 (blue curve)—because these localized modifications to the model are made fairly close to the center of the star, the resulting frequency differences compared to the $\pi$ modes of the unmodified model resemble the effect of some kind of surface term. The mode inertiae of the modified model are also changed in a frequency-dependent manner (orange curve). Both of these effects will necessarily complicate attempts to
correct for the true surface term in actual observational data, e.g., through inertia-dependent corrections as in Ball & Gizon (2014).

The numerical evaluation of the eigenfrequencies associated with very evolved red giants is known to be computationally intensive (e.g., Stello et al. 2014). Leaving aside difficulties associated with constructing evolutionary models of giant stars in the first place (which lie beyond the scope of this work), for those stellar models which we do have, we note that as a star evolves up the red giant branch, \( \nu_{\text{max}} \) decreases rapidly compared to the maximum Brunt–Väisälä frequency in the radiative interior, which instead increases. The density of \( \gamma \) modes (as given by \( \Delta \Pi \), Equation (21)) therefore increases as the star evolves, and so too does the density of mixed modes. The majority of these mixed modes are of very low amplitude (equivalently, have a high mode inertia) near the surface, as they are primarily \( g \) dominated. Only the most \( p \) dominated mixed modes, with the lowest inertias, are typically sufficiently excited as to be observed. As illustrated in Figure 8, the most \( p \)-dominated modes (which have the lowest inertias) are those that are closest to resonance with the underlying uncorrected \( \pi \) mode (i.e., without the first-order term \( R_{\gamma} \)), in keeping with the dependence of the eigenvector coefficients on the resonance factors in Equation (45). For the purposes of matching observations in these very evolved stars, it therefore suffices to search only for \( \pi \) modes, rather than mixed modes.

Having a high \( n_e \) associated with mixed modes near \( \nu_{\text{max}} \) moreover yields a stiff problem in the following sense: let us suppose that a particular frequency eigenvalue associated with a near-resonance mixed mode of the form Equation (52) is known in advance. Then the oscillation equations (expressed via Equation (5) or Equation (A1)) can be cast as an IVP, being integrated outwards from the inner boundary subject to appropriate initial conditions. Suppose that we integrated this IVP using an explicit integration scheme; because the \( \gamma \)-mode contribution to the eigenfunction is highly oscillatory with very short wavelength (owing to its high order), this suggests that a very small spatial step size is required for numerical stability (in particular to guarantee that the \( \gamma \) component decays rapidly outside of the radiative region). It is therefore the transient component of the stiff system. Equivalently, when the radial coordinate mesh is refined in the process of solving the boundary value problem, a very large number of points is assigned to the radiative zone, in order to ensure a sufficiently high density of mesh points to capture this rapidly oscillatory and exponential behavior (Christensen-Dalsgaard et al. 2020).

The ability to compute \( \pi \) modes instead of mixed modes significantly alleviates both of these computational difficulties; in Appendix C we examine a numerical experiment demonstrating this in more detail.

### 4.2. Grid-based Subgiant Modeling with \( \gamma \) Modes

The frequencies of the lowest-order \( g \) modes evolve rapidly and monotonically with stellar age as a star evolves off the main sequence and up the red giant branch. Because they are close to \( \nu_{\text{max}} \) in subgiants, these frequencies would place sensitive, surface-insensitive constraints on stellar ages, were they directly measurable, making these low-order \( g \) modes particularly valuable. However, these frequencies can be measured only indirectly via the appearance of the avoided-crossing phenomenon; on the other hand the individual mode frequencies of the avoided crossing do not evolve monotonically with age and also evolve so rapidly (relative to measurement error) as to present difficulties for grid-based inference of stellar fundamental parameters (Deheuvels & Michel 2011).

We present a grid-based approach that incorporates age constraints from avoided crossings. This method requires only that the lowest-order (10 or so) \( \gamma \)-mode frequencies be computed and is fully generalizable to cases where multiple avoided crossings are observed. To illustrate the method, we will examine its application to an actual subgiant, HD 38529, for which several independent parameter estimates have been determined from detailed modeling against individual mode frequencies (Ball et al. 2020). For this purpose we use a grid of MESA r10398 evolutionary models generated with element diffusion, without overshoot, with a solar-calibrated mixing-length parameter of 1.83, and with the chemical abundances of Grevesse & Sauval (1998). Models were generated with \( M/M_\odot \in [1, 1.6] \) at intervals of 0.04, initial \( Y \in [0.25, 0.32] \) at intervals of 0.005, and initial \( [\text{Fe}/H] \in [-0.25, 0.5] \) at intervals of 0.03 dex. For all models we precomputed the frequencies of the first 10 \( \gamma \) modes in the Cowling approximation to avoid boundary issues at high radial order.

Avoided crossings in this regime are characterized by an extraneous mode (the \( \gamma \) mode) disrupting the otherwise regular asymptotic ordering of the \( \pi \) modes. Per Equation (43), far from resonance, the leading-order effect of mode coupling is to displace the frequency eigenvalues (relative to the uncoupled \( \pi \)-mode frequencies) away from the \( \gamma \) mode, with the two mixed modes closest to the \( \gamma \) mode bracketing both it and the on-resonance \( \pi \) mode. Because the separation between these is generally less than asymptotic \( \Delta \nu \), we identify this pair of modes (at frequencies \( \nu_1, \nu_2 \)) as the local minimum of the pairwise frequency separation between adjacent observed modes (modulo \( \Delta \nu \) to account for missing modes). We then search for models containing \( \gamma \) modes within the interval \([\nu_1, \nu_2]\), as illustrated for HD 38529 in Figure 10. For the \( n, \text{th} \) \( \gamma \) mode at frequency \( \nu_{n,\gamma} \) associated with a model, we define a
quantity
\[
g(\nu_{n,\gamma}) = \begin{cases} 
\nu_{n,\gamma} - \nu_2 & \nu_{n,\gamma} > \nu_2 \\
0 & \nu_2 \geq \nu_{n,\gamma} > \nu_1 \\
\nu_{n,\gamma} - \nu_1 & \nu_1 \geq \nu_{n,\gamma}
\end{cases}
\]
from which we construct an associated cost function
\[
C(\nu_{n,\gamma}) = \left( \frac{g(\nu_{n,\gamma})}{(\nu_2 - \nu_1)/2} \right)^2.
\]
By construction, \(C(\nu_{n,\gamma})\) is zero when \(\nu_{n,\gamma}\) lies within our search interval and grows quadratically with \(\nu_{n,\gamma}\) outside of it. We therefore construct an associated weight function
\[
w_{n,\gamma} = \exp\left( -\frac{C(\nu_{n,\gamma})}{2} \right).
\]
Where multiple avoided crossings are observed, we define a corresponding number of such search intervals and weights, assigning consecutively increasing integer values of \(\gamma\) to consecutive avoided crossings in decreasing order of frequency.

Let us consider the case of a single avoided crossing, as seen in HD 38529. Because the frequencies of all \(\gamma\) modes increase monotonically with stellar age, every evolutionary track in the grid will have some models associated with every \(\gamma\), for which \(w_{n,\gamma} = 0\), and the set of such models forms a series of nonintersecting hyperplanes, one for each \(\gamma\), in the underlying parameter space.

Where multiple avoided crossings are observed, \(\Delta \Pi\) can be measured and used to estimate \(\gamma\) for all avoided crossings for the star via Equation (2), through which any putative identification of the radial orders can be related to corresponding values of \(\epsilon_g\). Misidentification of the radial orders is then equivalent to off-by-one errors in \(\epsilon_g\), which can be ruled out immediately as \(\epsilon_g\) does not vary significantly (not by more than a few tenths; see Figure 11) between the main-sequence turnoff and the red giant bump.

By contrast, for singly observed avoided crossings, it is impossible to identify unambiguously which \(\gamma\) is actually responsible for the avoided crossing in the absence of further information. However, the hyperplanes we have described above are not all equally favored by the spectroscopic observables; we might use, e.g., the number of models with \(w_{n,\gamma} = 0\) that also lie within the spectroscopically constrained \(3\sigma\) region for HD 38529.

Having selected a particular \(\gamma\), we then construct an approximate conditional posterior probability distribution as
\[
p_n \propto w_{n,\gamma} \exp\left[-L_{\text{spec}}/2\right],
\]
where \(L_{\text{spec}} = \sum_i \chi_i^2_{\text{spec}}\). That is to say, we supplement the ordinary likelihood weights from the spectroscopic constraints with additional ones from the avoided crossings, before using them in further analysis (e.g., to estimate masses and ages). With multiple avoided crossings, we would take the product of the weights of all avoided crossings for a given \(\gamma\)-mode identification.
We show in Figure 13 the posterior distributions for \( n_\gamma = 6 \), where we have averaged the posterior distributions over 100 realizations of Monte-Carlo perturbations of the spectroscopic constraints from the nominal values as given in Ball et al. (2020). Notably, the posterior distribution with the inclusion of the avoided-crossing weights (orange curve) provides much stronger constraints on the age than the spectroscopic constraints alone (blue curve). On the other hand, the avoided crossing does not help in constraining the stellar mass, despite the strong a priori relation between the age at which the avoided crossing is seen (i.e., shortly after the main-sequence turnoff) and the stellar mass. This was also the case for the detailed modeling results in Ball et al. (2020), where the mass uncertainties returned from each of the independent detailed modeling efforts were much larger than would be consistent with the corresponding (very small) age uncertainties. As they note, this is most likely due to dependences of the avoided-crossing (i.e., \( \gamma \)-mode) frequencies on other compositional or physical parameters.

Finally, while the age uncertainties from these detailed modeling efforts were small, the corresponding age estimates were largely in tension with each other. Because of this, the consensus uncertainties for this detailed modeling work, which includes a contribution from the internal variance between the different modeling teams, are larger than even the loose constraints from a coarse grid-based search without any seismic constraints (right panel of Figure 13). This can easily be explained by different identifications, by different modeling teams, of the radial order \( n_\gamma \) of the \( \gamma \) mode responsible for the observed avoided crossing: given the paucity of observed dipole modes, the ambiguity in mode identification that we have discussed above is also an issue for detailed modeling with individual frequencies.

We show in Figure 14 the posterior probability distributions returned from repeating our Monte-Carlo procedure with different choices of \( n_\gamma \). The three vertical dotted lines, and black horizontal rulers, correspond to the nominal ages and uncertainties returned from three independent detailed modeling efforts, while the vertical dashed line and light shaded region are the consensus results and uncertainties. We see that each of these detailed modeling results lies close to the center of a conditional posterior distribution associated with a different \( n_\gamma \). In the absence of an unambiguous \( \gamma \)-mode identification, which is the case for HD 38529, the true posterior distribution in the stellar age is best described as a multimodal, likelihood-weighted mixture of these component distributions. That these detailed modeling results appear to each sample only one of these component distributions is indicative of more fundamental methodological issues: for example, optimization-based parameter inference is prone to trapping in local optima, which in this case leads to sampling models with only one value of \( n_\gamma \). Conversely, were the \( \gamma \)-mode radial order to be specified a priori by some other means, its inclusion as a constraint on even detailed modeling would most likely alleviate such multimodality.

5. Discussion and Conclusion

We have explored different isolation conditions to derive \( \gamma \) and \( \pi \) modes, which are not uniquely defined and depend on the propagation structure of the star under consideration. Our chosen isolation conditions for evolved solar-like oscillators amount to suppressing terms that would vanish where \( \omega \ll S_\ell^2 \) or \( \omega \gg N^2 \) for the corresponding \( \gamma \)- and \( \pi \)-mode cavities, respectively. While these choices are justified based on asymptotic considerations, the resulting formalism is fully applicable to all frequency regimes.

The relationship between the isolated and full systems of oscillation equations is of a form that permits the use of well-established results from matrix perturbation theory. With respect to this, we have derived an explicit semianalytic formulation for the study of various near-degeneracy phenomena, relating to the coupling between these isolated cavities. The required matrix elements are expressed as integrals over the isolated eigenfrequencies. Because these constructions do not rely on the JWKB approximation, they are applicable to buoyancy waves in subgiants exhibiting avoided crossings and to acoustic waves in very evolved red giants, which lie outside the scope of traditional approaches. Using a numerical implementation based on a general-purpose pulsation code,
we have explored various theoretical consequences and potential practical applications of this formalism.

Even in cases where the JWKB approach is tenable, access to the eigenvalues of the isolated mode cavities permits some aspects of the problem to be simplified. For instance, in many applications where one-to-one coupling of single \(\pi\) and \(\gamma\)-mode pairs is assumed to dominate, the angular quantities in Equation (49) are often approximated with

\[
\frac{\Theta_p(\nu)}{\Delta \nu} \sim \nu - \nu_\pi, \quad \frac{\Theta_\phi(\nu)}{\Delta \Pi} \sim \Delta \Pi \left(\frac{1}{\nu} - \frac{1}{\nu_\gamma}\right).
\]

(64)

where the asymptotic relations, Equations (1) and (2), are used to estimate the isolated mode frequencies (e.g., Mosser et al. 2015; Gehan et al. 2018). While doing this is unavoidable where the underlying stellar structure is unknown, this approach is widely taken even for theoretical studies that do have access to the underlying stellar structure, resulting in the introduction of nuisance parameters (\(\Delta \nu, \Delta \Pi\), etc.) that co-vary with other quantities of interest, thereby complicating the analysis (e.g., Benomar et al. 2012; Cunha et al. 2019; Jiang et al. 2020). We imagine that revisiting these studies with these additional parameters eliminated might clarify the interpretation of these results.

Finally, we have elucidated a grid-based procedure by which the isolated \(\gamma\)-mode cavity may be used to constrain the global properties of subgiants undergoing avoided crossings. The process also reveals why age estimates of subgiants with only one observed avoided crossing may not yield the precision suggested by the rapidity of the evolution of any single \(\gamma\) mode over an evolutionary track.

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Software: NumPy (Oliphant 2006), SciPy stack (Jones et al. 2001), AstroPy (Astropy Collaboration et al. 2013; Price-Whelan et al. 2018), Pandas (McKinney 2010), MESA (Paxton et al. 2011, 2013, 2018), GYRE (Townsend & Teitler 2013).

We have made available Python scripts for various computations in the Covling approximation described above, as well as various matrix elements, at https://github.com/darthoctopus/mesa-tricks. Our changes to GYRE (to isolate the \(\pi\)- and \(\gamma\)-mode cavities, see Appendix A) can be found in our fork of GYRE at https://github.com/darthoctopus/gyre; we have also submitted it for inclusion upstream.

### Appendix A

**Implementation Details**

Our modifications to GYRE solve the adiabatic oscillation equations as expressed in the form

\[
\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \frac{V}{\Gamma_1} - 1 - l_i & \lambda & \frac{V}{\Gamma_1} & \frac{\lambda}{c_1 \omega^2} & 0 \\ c_1 \omega^2 - \alpha_\gamma A^\gamma & 3 - U + A^\gamma - l_i & 0 & -1 \\ 0 & 0 & 3 - U - l_i & 1 \\ A^\gamma U & \frac{V}{\Gamma_1} U & \lambda & -(U + l_i - 2) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
\]

(A1)

where all of these quantities are as given in the GYRE documentation. For our purposes we need note only that \(x = r/R\), \(y_1 \propto \xi_\gamma, y_2 \propto P_1, V/\Gamma_1 = rg/c_1^2, A^\gamma = n N^2/g,\) and \(\lambda \rightarrow k(l + 1)\) for a nonrotating star. Accordingly, the matrix element (1, 2) corresponds to the coefficient of \(P_1\) in the first line of Equation (5), while the matrix element (2,1) corresponds to that of \(\xi_\gamma\) in the second line of Equation (5).

The isolation of the mode cavities is performed by changing the values of the newly introduced parameters \(\alpha_\pi\) and \(\alpha_\gamma\), which are set to 1 by default (in keeping with GYRE’s unmodified behavior). Setting \(\alpha_\gamma\) to zero yields \(\gamma\) modes, and setting \(\alpha_\pi\) to zero yields \(\pi\) modes. Additional allowances have to be made near the inner and outer boundaries. For \(\pi\) modes in particular, \(\alpha_\pi\) is treated as a function of radius and is set to 1 near the surface (defined to be for all \(x\) larger than some \(x_{atm}\), which is supplied as an additional input parameter) even when \(\pi\) modes are computed, so as not to induce a numerical surface term by changing the outer boundary condition. For \(\gamma\) modes in particular, the eigenfunction \(\xi_\gamma\) must vanish at both boundaries, for consistency with the singular nature of the Cowling-approximation Sturm–Liouville problem at the boundaries.

Additionally, we modify GYRE to estimate the required local density of the remeshed radial coordinate grid via a dispersion relation of the form

\[
-4 k^2 x^2 \sim \gamma = \left( A^\gamma - \frac{V}{\Gamma_1} - U + 4 \right)^2 - 4 \left( \alpha_\gamma \frac{V}{\Gamma_1} c_1 \omega^2 - \frac{V}{\Gamma_1} A^\gamma \alpha_\gamma \alpha_\pi \right) - \lambda + \frac{\lambda A^\gamma}{c_1 \omega^2 \alpha_\gamma} = \left( A^\gamma - \frac{V}{\Gamma_1} - U + 4 \right)^2 - 4 \left( \alpha_\gamma \frac{V}{\Gamma_1} - \frac{\lambda}{c_1 \omega^2} \right) (c_1 \omega^2 - \alpha_\gamma A^\gamma)
\]

(A2)
(compare the second term with Equation (3)), where again the default behavior is recovered for \( \alpha_\gamma = \alpha_\pi = 1 \).

Appendix B
Recovery of JWKB Expressions Involving \( \zeta \)

B.1. \( \zeta \) Proper

We once again consider a two-term linear combination of the form of Equation (52). We assume that these modes are close enough to resonance that we can ignore the contributions from other states, and so the relevant coupling matrix is the \( 2 \times 2 \) matrix of Equation (46) with all entries being scalars. Explicitly, the eigenvalues are

\[
\omega_\pm^2 = \frac{\omega_p^2 + \omega_g^2}{2} \pm \sqrt{\left(\frac{\omega_p^2 - \omega_g^2}{2}\right)^2 + \alpha^2},
\]  

(B3)

and the eigenvectors satisfy

\[
\begin{bmatrix}
\omega_p^2 & \alpha \\
\alpha & \omega_g^2
\end{bmatrix}
\begin{bmatrix}
1 \\
\pm u_\pm
\end{bmatrix}
= \omega_\pm^2
\begin{bmatrix}
1 \\
\pm u_\pm
\end{bmatrix}.
\]

(B4)

so we have

\[
\alpha u_\pm = \omega_\pm^2 - \omega_p^2
\]

\[
\frac{\alpha}{u_\pm} = \omega_\pm^2 - \omega_g^2
\]

(B5)

Taking the ratio of these, we obtain

\[
\frac{1}{u_\pm} = \frac{\omega_\pm^2 - \omega_g^2}{\omega_\pm^2 - \omega_p^2}
\]

(B6)

Without loss of generality, we drop the subscript \( \pm \) and consider this to be a function of frequency. As all quantities on the left-hand side are positive, this is equal to the ratio of the absolute values of the numerator and denominator, which we evaluate separately. For the numerator, we note that

\[
|\omega^2 - \omega_g^2| \sim |\delta \omega| \sim 2\omega|\delta \omega| \sim 2\omega^3 \left| \delta \left( \frac{1}{\omega} \right) \right|
\]

\[
= \frac{2\omega^3}{F_1} |\Theta_\gamma - n_\pi \pi|
\]

(B7)

and likewise in the denominator we have

\[
|\omega^2 - \omega_p^2| \sim 2\omega|\omega - \omega_p| \sim \frac{2\omega}{T} |\Theta_p - n_p \pi|
\]

(B8)

where \( n_\gamma \) and \( n_\pi \) are integers. The angular quantities (as defined in Equation (49)) are taken to have been evaluated at the mixed-mode eigenvalues, and so differ only slightly from integer multiples of \( \pi \). Using a small-angle approximation and Equation (48), we find

\[
\frac{e_\gamma^2}{e_p^2} \sim \frac{\omega^2 T}{F_1} \frac{|2(\Theta_\gamma - n_\pi \pi)|}{|2(\Theta_p - n_p \pi)|} \sim \frac{\omega^2 T}{F_1} \frac{|\sin 2(\Theta_\gamma - n_\pi \pi)|}{|\sin 2(\Theta_p - n_p \pi)|}
\]

\[
= \frac{\omega^2 T}{F_1} \frac{|\sin 2\Theta_\gamma|}{|\sin 2\Theta_p|} \sim \frac{\omega^2 T}{F_1} \frac{|\sin \Theta_\gamma \cos \Theta_\gamma|}{|\sin \Theta_p \cos \Theta_p|}
\]

\[
= \frac{1}{q} \frac{\omega^2 T}{F_1} \frac{|\cos^2 \Theta_\gamma|}{|\cos^2 \Theta_p|}
\]

(B9)

Inserting this into Equation (54) yields Equation (50), as required.

B.2. Period and Frequency Spacings

The frequency and period spacings appearing in Equations (1) and (2) are some of the easiest seismic observables to relate to evolutionary properties of stars, and many techniques have been devised to correct for the effect of mode bumping when measuring them from mixed modes. For example, Mosser et al. (2015) derive a relation between \( \zeta \) and the local \( \Delta \Pi \) of \( g \)-dominated mixed modes in red giants using the JWKB definition, Equation (50), and assuming pairwise mode coupling, as in Equation (52). In this appendix, we construct equivalent statements in the nonasymptotic regime and derive the appropriate generalizations to many-mode coupling.

We consider the two isolated mode cavities to yield one “dense” and one “sparse” set of eigenvalues. For example, in red giants, we have a dense series of \( \gamma \) modes (with perturbed, uncoupled frequencies \( \omega_{\gamma,i} \)) coupling to a sparse series of \( \pi \) modes (at \( \omega_\pi \)). Because the \( \pi \pi \) and \( \gamma \gamma \) coupling can be assumed to be weak, we approximate each of the resulting mixed modes to be a two-term linear combination of the form Equation (52), with eigenfrequencies close to Equation (B3). For \( \omega_{\gamma,i} < \omega_\pi \), the frequency of the mixed mode in question is given by \( \omega_\gamma \) and vice versa. If the \( i \)th uncoupled \( \gamma \) mode is the closest in frequency to a given \( \pi \) mode, the sequence of mixed-mode eigenvalues goes approximately as

\[
\ldots \omega_{\gamma,i-2}, \omega_{\gamma,i-1}, \omega_{\gamma,i}, \omega_{\gamma,i+1}, \omega_{\gamma,i+2}, \ldots
\]

(B10)

and by assumption the difference between adjacent mixed modes, \( \delta \omega_{\gamma} \), should tend to the difference between adjacent uncoupled modes, \( \delta \omega_{\gamma,i} \), away from resonance. We expand this difference (i.e., \( \omega_{\gamma,i}^2 - \omega_{\gamma,i+1}^2 \)), retaining terms to first order as

\[
\delta \omega_{\gamma,i} \sim \delta \omega_{\gamma,i}^2 \left( \frac{1}{2} \pm \frac{1}{2} \frac{\omega_\pi^2 - \omega_{\gamma,i}^2}{\omega_\pi^2 - \omega_{\gamma,i+1}^2} \right)
\]

\[
\Rightarrow \delta \omega_{\gamma,i} \sim \delta \omega_{\gamma} \frac{\omega_\pi^2 - \omega_{\gamma,i}^2}{\omega_\pi^2 - \omega_{\gamma,i+1}^2} \sim \delta \omega_{\gamma} \frac{1}{2} \frac{\omega_\gamma^2 - \omega_{\gamma,i}^2}{\omega_\gamma^2 - \omega_{\gamma,i+1}^2},
\]

(B11)

where at each step we have used the relation \( \omega_{\gamma,i}^2 - \omega_{\gamma,i+1}^2 = \pm(\omega_\gamma^2 - \omega_\pi^2) \). Using Equation (B5) we rewrite the above (dropping the subscript \( \pm \)) as

\[
\frac{\delta \omega_{\gamma}}{\delta \omega_{\gamma,i}} \sim \left( \frac{\alpha u}{\alpha (u + 1/u)} \right) = (1 + u^{-2})^{-1} = \zeta,
\]

(B12)

which is the result of Mosser et al. (2015) if \( \omega/\omega_\gamma \sim 1 \). Completely analogously, for a dense series of \( p \) modes coupling to a single \( g \) mode, which is typical of subgiants undergoing avoided crossings, we obtain that

\[
\frac{\delta \omega_{\gamma}}{\delta \omega_{\gamma,i}} \sim \frac{\omega^2 - \omega_{\gamma,i}^2}{2\omega_\gamma^2 - \omega_{\gamma,i}^2} \sim \frac{\alpha u}{\alpha (u + 1/u)} = 1 - \zeta.
\]

(B13)

While these expressions hold near resonance, we would like to consider cases where the “sparse” set of eigenvalues is nonetheless dense enough that we have one-to-many coupling (one dense mode to many sparse modes) away from resonance. Indexing the “dense” eigenvalues with \( i \) and the “sparse” ones as...
with \( j \), and ignoring coupling between the dense eigenvalues, we find that (to leading order, per Equations (43) and (45) with a strictly off-diagonal perturbation) we can write the corresponding first differences of the dense eigenvalues as

\[
\delta \omega_l^2 \sim \delta \omega_{0,l}^2 + \sum_{j=1}^{l-1} \delta \left( \frac{V_{ij}^2}{\omega_j^2 - \omega_i^2} \right)
\sim \delta \omega_{0,l}^2 \left( 1 - \sum_{j=1}^{l-1} \frac{V_{ij}^2}{(\omega_j^2 - \omega_i^2)^2} \right)
\Rightarrow \frac{\delta \omega_l^2}{\delta \omega_{0,l}^2} \sim \left( 1 - \sum_{j=1}^{l-1} c_j^2 \right).
\]

Here we have assumed that, locally, the dependence of the coupling matrix elements on frequency is much weaker than of the resonance factors. This construction can be applied beyond situations where \( \delta \omega_l \) is given by asymptotic quantities \( \Delta \nu \) or \( \Delta \Pi_l \); for example, Deheuvels et al. (2017) derive a similar relation in the case where mode coupling induces an asymmetric component into the rotational splitting in red giants. We have demonstrated that these formulations remain approximately applicable in the nonasymptotic regime, and in particular to subgiant avoided crossings, subject to a modified definition of \( \zeta \).

### Appendix C

#### Computational Speedup

In Section 4.1 we identified two ways in which isolating the \( \pi \) cavity can speed up the computation of nonradial frequency eigenvalues from giant stellar models, which we quantify in terms of the ratio of runtime complexity compared to a direct search for \( \pi \) modes:

1. The typical search strategy for \( p \)-dominated mixed modes involves first performing a broadband search over a range of frequencies, and then pruning the results to the most \( p \)-dominated mixed modes, as characterized by the local minima of the mode inertiae (considered as a function of frequency; see Figure C1 for an example). Although sufficiently good characterization of the \( p \)-mode asymptotic relation may permit this search space to be restricted to a smaller number of mixed modes close to the predicted asymptotic \( p \)-mode frequencies (e.g., as done in McKeever et al. 2019, for \( l = 2 \) frequencies), low-order \( p \) modes are known to depart severely from the asymptotic relation as stellar models approach the tip of the red giant branch, rendering this approach increasingly untenable where it is needed most. By contrast, all \( \pi \) modes returned from the computation are guaranteed to be \( p \) dominated, requiring no further refinement. The speedup from this goes as \( \Delta \nu / r^2 \Delta \Pi_l \).

2. As the \( \pi \) component of mixed modes constitutes the slow part of a stiff system, coarser coordinate meshes than would be possible for the direct computation of mixed modes can be used for the decoupled problem without sacrificing the numerical accuracy of the returned eigensystem. The speedup factor from this improvement goes as \( (n_{ij}/n_j)^p \), where the most inefficient \( N \times N \) matrix operation in the solution algorithm for the boundary value problem has a runtime complexity of \( \mathcal{O}(N^p) \).

In practice, we expect the true speedup to be less than this, due to a combination of low-level systematics (e.g., time taken for \( \nu / o \) operations) and other implementation details (e.g., GYRE only adds mesh points and does not take them away).

To better characterize the contributions from each of these factors to any potential performance gains, we devised an experiment where we recorded the time taken to perform the following sets of computations for all models on a MESA evolutionary track:

- Direct computation of \( \pi \) modes,
- Computation of mixed modes subject to a restricted search strategy,
- Computation of mixed modes subject to a naive search strategy.

The number of mixed modes computed for (b) was chosen to be the same as (a), so that the speedup from (b) to (a) derives only from decoupling the transient component of the stiff system. Likewise, the speedup from (c) and (b) results essentially from only reducing the number of modes in the
search space. We ran this experiment on an Intel Xeon E5-2670 CPU running at 2.60 GHz, using 10 threads (out of 16 cores) for each computation. In any case, because we are only comparing speedup factors, we expect these results not to depend on our hardware configuration. Save for a few models near the tip of the red giant branch and on the red clump, we set an upper limit of 1 hr for all computations; this meant that frequency searches (c) could not be completed for the majority of the post-bump red giant branch. Moreover, in all cases we limited ourselves to solving for at most 1000 eigenvalues; as such, computations for (c) near the tip of the red giant branch took much less time than a truly exhaustive search would have. Nonetheless, as $\Delta I I$ decreases monotonically between the red giant branch bump and the tip of the red giant branch, our results set a lower bound on the speedup that would be obtained for these circumstances. We show the results of this experiment in Figure C2. We see that each of these reductions in runtime complexity greatly speeds up the search for eigenvalues for stellar models ascending the red giant branch.

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