FINITE MUTATION CLASSES OF COLOURED QUIVERS

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Abstract. We consider the general notion of coloured quiver mutation and show that the mutation class of a coloured quiver \( Q \), arising from an \( m \)-cluster tilting object associated with \( H \), is finite if and only if \( H \) is of finite or tame representation type, or it has at most 2 simples. This generalizes a result known for 1-cluster categories.

Introduction

Mutation of skew-symmetric matrices, or equivalently quiver mutation, is very central in the topic of cluster algebras [FZ]. Quiver mutation induces an equivalence relation on the set of quivers. The mutation class of a quiver \( Q \) consists of all quivers mutation equivalent to \( Q \). In [BR] it was shown that the mutation class of an acyclic quiver \( Q \) is finite if and only if the underlying graph of \( Q \) is either Dynkin, extended Dynkin or has at most two vertices.

Cluster categories were defined in [BMRRT] in the general case and in [CCS] in the \( A_n \)-case as a categorical model of the combinatorics of cluster algebras. Some cluster categories have a nice geometric description in terms of triangulations of certain polygons, see [CCS [S]]. This was used in [BR, BTO] to count the number of quivers in the mutation classes of quivers of Dynkin type \( A \) and \( D \). In [BRS] they used different methods to count the number of quivers in the mutation classes of quivers of type \( \tilde{A} \).

A generalization of cluster categories, the \( m \)-cluster categories, have been investigated by several authors. See for example [BM1, BM2, BT, IY, K, W, ZZ]. In [BT] mutation on coloured quivers was defined, and we can define mutation classes of coloured quivers. It is a natural question to ask when the mutation classes of coloured quivers are finite. In this paper we want to show the following theorem, analogous to the main theorem in [BR].

Theorem. Let \( k \) be an algebraically closed field and \( Q \) a connected finite quiver without oriented cycles. The following are equivalent for \( H = kQ \).

1. There are only a finite number of basic \( m \)-cluster tilted algebras associated with \( H \), up to isomorphism.
2. There are only a finite number of Gabriel quivers occurring for \( m \)-cluster tilted algebras associated with \( H \), up to isomorphism.
3. \( H \) is of finite or tame representation type, or has at most two non-isomorphic simple modules.
4. There are only a finite number of \( \tau \)-orbits of cluster tilting objects associated with \( H \).
5. There are only a finite number of coloured quivers occurring for \( m \)-cluster tilting objects associated with \( H \), up to isomorphism.
6. The mutation class of a coloured quiver \( Q \), arising from an \( m \)-cluster tilting object associated with \( H \), is finite.
1. Background

Let $H = kQ$ be a finite dimensional hereditary algebra over an algebraically closed field $k$, with $Q$ a quiver with $n$ vertices. The cluster category was defined in \[BMRR\] and independently in \[CCS\] in the $A_n$ case. Consider the bounded derived category $D^b(H)$ of $\text{mod } H$. Then the cluster category is defined as the orbit category $\mathcal{C}_H = D^b(H)/\tau^{-1}[1]$, where $\tau$ is the Auslander-Reiten translation and $[1]$ is the shift functor.

As a generalization of cluster categories, we can consider the $m$-cluster categories defined as $\mathcal{C}_H^m = D^b(H)/\tau^{-1}[m]$. The $m$-cluster category was shown in \[KR\] to be triangulated. The $m$-cluster category is a Krull-Schmidt category, an $(m+1)$-Calabi-Yau category, and it has an AR-translate $\tau = [m]$. The indecomposable objects in $\mathcal{C}_H^m$ are of the form $X[i]$, with $0 \leq i < m$, where $X$ is an indecomposable $H$-module, and of the form $P[m]$, where $P$ is a projective $H$-module.

An $m$-cluster tilting object is an object $T$ in $\mathcal{C}_H^m$ with the property that $X$ is in $\text{add } T$ if and only if $\text{Ext}^i_{\mathcal{C}_H^m}(T, X) = 0$ for all $i \in \{1, 2, \ldots, m\}$. It was shown in \[W, ZZ\] that an object which is maximal $m$-rigid, i.e. it has the property that $X \in \text{add } T$ if and only if $\text{Ext}^i_{\mathcal{C}_H^m}(T \oplus X, T \oplus X) = 0$ for all $i \in \{1, 2, \ldots, m\}$, is also an $m$-cluster tilting object. They also showed that an $m$-cluster tilting object $T$ always has $n$ non-isomorphic indecomposable summands.

An almost complete $m$-cluster tilting object $\bar{T}$ is an object with $n-1$ non-isomorphic indecomposable direct summands such that $\text{Ext}^i_{\mathcal{C}_H^m}(\bar{T}, \bar{T}) = 0$ for $i \in \{1, 2, \ldots, m\}$. It is known from \[W, ZZ\] that any almost complete $m$-cluster tilting object has exactly $m+1$ complements, i.e. there exist $m+1$ non-isomorphic indecomposable objects $T'$ such that $\bar{T} \oplus T'$ is an $m$-cluster tilting object.

Let $\bar{T}$ be an almost complete $m$-cluster tilting object and denote by $T^{(c)}_k$, where $c \in \{0, 1, 2, \ldots, m\}$, the complements of $\bar{T}$. In \[IY\] it is shown that the complements are connected by $m+1$ exchange triangles

$$T^{(c)}_k \to B^{(c)}_k \to T^{(c+1)}_k \to,$$

where $B^{(c)}_k$ are in $\text{add } \bar{T}$.

An $m$-cluster tilted algebra is an algebra of the form $\text{End}_{H^m}(\bar{T})$, where $T$ is an $m$-cluster tilting object in $\mathcal{C}_H^m$.

2. Coloured quiver mutation

In the case when $m = 1$ there is a well-known procedure for the exchange of indecomposable direct summands of a cluster-tilting object. Given an almost complete cluster-tilting object, there exist exactly two complements, and the corresponding quivers are given by quiver mutation. For an arbitrary $m \geq 1$, the procedure is a little more complicated. Since an almost complete $m$-cluster tilting object has, up to isomorphism, exactly $m+1$ complements, the Gabriel quiver does not give enough information to keep track of the exchange procedure. Buan and Thomas therefore defined a class of coloured quivers in \[BT\], and they define a mutation procedure on such quivers to model the exchange on $m$-cluster tilting objects. In this section we recall some results from this paper.

To an $m$-cluster tilting object $\bar{T}$, Buan and Thomas associate a coloured quiver $Q_{\bar{T}}$, with arrows of colours chosen from the set $\{0, 1, 2, \ldots, m\}$. For each indecomposable summand of $T$ there is a vertex in $Q_{\bar{T}}$. If $T_i$ and $T_j$ are two indecomposable summands of $T$ corresponding to vertex $i$ and $j$ in $Q_{\bar{T}}$, there are $r$ arrows from $i$ to $j$ of colour $c$, where $r$ is the multiplicity of $T_j$ in $B^{(c)}_i$.

They show that such quivers have the following properties.
(1) The quiver has no loops.
(2) If there is an arrow from $i$ to $j$ with colour $c$, then there exist no arrow from $i$ to $j$ with colour $c' \neq c$.
(3) If there are $r$ arrows from $i$ to $j$ of colour $c$, then there are $r$ arrows from $j$ to $i$ of colour $m - c$.

They also define coloured quiver mutation, and they give an algorithm for the procedure. Let $Q = Q_T$, for an $m$-cluster tilting object $T$, be a coloured quiver and let $j$ be a vertex in $Q$. The mutation of $Q$ at vertex $j$ is a quiver $\mu_j(Q)$ obtained as follows.

(1) For each pair of arrows

$$i \xrightarrow{c} j \xrightarrow{0} k$$

where $i \neq k$ and $c \in \{0, 1, ..., m\}$, add an arrow from $i$ to $k$ of colour $c$ and an arrow from $k$ to $i$ of colour $m - c$.

(2) If there exist arrows of different colours from a vertex $i$ to a vertex $k$, cancel the same number of arrows of each colour until there are only arrows of the same colour from $i$ to $k$.

(3) Add one to the colour of all arrows that goes into $j$, and subtract one from the colour of all arrows going out of $j$.

See Figure 1 for an example.

![Figure 1](image)

**Figure 1.** Examples of mutation of coloured quivers for Dynkin type A and $m = 2$.

In [BT] the following theorem is proved.

**Theorem 2.1.** Let $T = \oplus_{i=1}^{n} T_i$ be an $m$-cluster tilting object in $\mathcal{C}_m$. Let $T' = T/T_j \oplus T_j^{(1)}$ be an $m$-cluster tilting object where there is an exchange triangle

$$T_j \rightarrow B_j^{(0)} \rightarrow T_j^{(1)} \rightarrow .$$

Then $Q_{T'} = \mu_j(Q_T)$.

The quiver obtained from $Q_T$ by removing all arrows of colour different from 0 is the Gabriel quiver of the $m$-cluster tilted algebra $\text{End}_{\mathcal{C}_m}(T)$. Quivers of $m$-cluster tilted algebras can be reached by repeated coloured quiver mutation [ZZ] (see also [BT]).

**Proposition 2.2.** Any $m$-cluster tilting object can be reached from any other $m$-cluster tilting object via iterated mutation.

They obtain the following corollary.

**Corollary 2.3.** For an $m$-cluster category $\mathcal{C}_m$ of the acyclic quiver $Q$, all quivers of $m$-cluster tilted algebras are given by repeated mutation of $Q$. 

Let us always denote by $Q_G$ the Gabriel quiver of the coloured quiver $Q$. In this paper we are only interested in coloured quivers which arises from an $m$-cluster tilting object. Let $Q_G$ be an acyclic quiver and $Q$ the coloured quiver obtained from $Q_G$ by adding the necessary arrows of colour $m$, i.e. if there exist $r$ arrows from $i$ to $j$ of colour 0, then add $r$ arrows from $j$ to $i$ of colour $m$. Then the
quivers which arises from \( m \)-cluster tilting objects are exactly the quivers mutation equivalent to \( Q \).

Let \( Q \) be a coloured quiver with arrows only of colour 0 and \( m \), as above, and where the underlying graph of the Gabriel quiver \( Q_G \) is of Dynkin type \( \Delta \). Then certainly \( Q_G \) is a quiver of an \( m \)-cluster tilted algebra. Let us call the set of quivers mutation equivalent to \( Q \) the mutation class of type \( \Delta \). Certainly, all orientations of \( \Delta \) (as a Gabriel quiver) is in the mutation class of type \( \Delta \).

Figure 2 shows all non-isomorphic coloured quivers in the mutation class of type \( A_3 \) for \( m = 2 \).

\[
\begin{array}{ccc}
1 & 2 & 3 \\
(0) & (2) & (0) \\
(1) & (0) & (2) \\
(0) & (2) & (0) \\
(2) & (0) & (2) \\
(1) & (1) & (1) \\
(0) & (0) & (0) \\
(1) & (1) & (1) \\
(2) & (0) & (2) \\
(1) & (1) & (1) \\
(0) & (0) & (0) \\
\end{array}
\]

**Figure 2.** All non-isomorphic coloured quivers in the mutation class of \( A_3 \) for \( m = 2 \).

We note that in a mutation class, there can be several non-isomorphic coloured quivers with the same underlying Gabriel quiver, and that the Gabriel quiver of an \( m \)-cluster tilted algebra might be disconnected.

To any \( m \)-cluster tilting object \( T \) there exist a coloured quiver \( Q_T \), but we also have the following.

**Lemma 2.4.** Suppose \( Q \) is a coloured quiver in some mutation class of a quiver of an \( m \)-cluster tilted algebra. Then there exist an \( m \)-cluster tilting object \( T \) such that \( Q = Q_T \).

**Proof.** This follows directly from the corollary, since mutation of \( m \)-cluster tilting objects corresponds to mutation of coloured quivers.

We know that \([i]\) is an equivalence on the \( m \)-cluster category for all integers \( i \). In particular, \( \tau = [m] \) is an equivalence.

**Proposition 2.5.** If \( T \) is an \( m \)-cluster tilting object, then \( Q_T \) is isomorphic to \( Q_T[i] \) for all \( i \).

**Proof.** It is enough to prove that \( Q_T \) is isomorphic to \( Q_T[i] \). Suppose there are \( r \) arrows in \( Q_T \) from \( i \) to \( j \) with colour \( c \). Let \( T_i \) and \( T_j \) be the indecomposable direct summands of \( T \) corresponding to vertex \( i \) and \( j \) in \( Q_T \) respectively. Let \( T = T/T_i \).
be the almost complete \( m \)-cluster tilting object obtained from \( T \) by removing \( T_i \).

Then there exist an exchange triangle
\[
T_i^{(c)} \rightarrow B_i^{(c)} \rightarrow T_i^{(c+1)} \rightarrow
\]
with \( B_i^{(c)} \) in \( \text{add}(T) \). There are \( r \) arrows from \( i \) to \( j \), with colour \( c \), so hence \( T_j \) has multiplicity \( r \) in \( B_i^{(c)} \). Clearly \( T_j[1] \) and \( T_j[1] \) are indecomposable direct summands of \( T[1] \) and we have the exchange triangle
\[
T_i^{(c)}[1] \rightarrow B_i^{(c)}[1] \rightarrow T_i^{(c+1)}[1] \rightarrow .
\]

Since \( T_j \) has multiplicity \( r \) in \( B_i^{(c)} \), \( T_j[1] \) has multiplicity \( r \) in \( B_i^{(c)}[1] \). It follows that there are \( r \) arrows in \( Q_T[1] \) from \( i \) to \( j \) with colour \( c \). The same proof holds for \([-1]\), and so hence the claim follows. \( \square \)

3. Finiteness of the number of non-isomorphic \( m \)-cluster tilted algebras

In [BR] the authors showed that if \( Q \) is a finite quiver with no oriented cycles, then there is only a finite number of quivers in the mutation class of \( Q \) if and only if the underlying graph of \( Q \) is Dynkin, extended Dynkin or has at most two vertices. In these cases there are only a finite number of non-isomorphic cluster-tilted algebras of some fixed type. In this section we want to prove an analogous result for coloured quivers by generalizing the results and proofs in [BR].

Let \( H = kQ \) be a finite dimensional hereditary algebra. We know that \( H \) is of finite representation type if and only if the underlying graph of \( Q \) is Dynkin, extended Dynkin or has at most two vertices. In the case when \( H \) has at most two vertices, the regular components of the AR-quiver are disjoint tubes of the form \( \mathbb{Z}A_{\infty} / \langle \tau^i \rangle \) for some \( i \), and in the wild case they are of the form \( \mathbb{Z}A_{\infty} \).

If \( X \) is a preprojective or preinjective \( H \)-module, it is known that \( X \) is rigid, i.e. \( \text{Ext}^1_H(X, X) = 0 \). The following is a well-known result, see for example [R].

**Lemma 3.1.** Let \( H = kQ \) be a finite dimensional hereditary algebra of infinite representation type, then if \( H \) has exactly two simples, no indecomposable regular object is rigid.

In [W] it was shown that if \( T \) is an \( m \)-cluster tilting object in \( \mathcal{C}_H^m \), then it is induced from a tilting object in \( \text{mod} \, H_0 \oplus \text{mod} \, H_0[1] \oplus \ldots \oplus \text{mod} \, H_0[m-1] \), where \( H_0 \) is derived equivalent to \( H \). If \( H \) is of finite or tame representation type, it was shown in [BR] that for each indecomposable projective \( H \)-module \( P \), there are only a finite number of indecomposable objects \( X \) such that \( \text{Ext}^1_{H}(X, P) \).

**Lemma 3.2.** Let \( P[i] \) be a shift of an indecomposable projective \( H \)-module, where \( H \) is of finite or tame representation type. Then there is only a finite set of objects \( X \) in \( \mathcal{C}_H^m \) with \( \text{Ext}^k_{\mathcal{C}_H^m}(X, P[i]) = 0 \) for all \( k \in \{1, 2, \ldots, m\} \).

**Proof.** We can assume that an \( m \)-cluster tilting object is induced from a tilting object in \( \text{mod} \, H \oplus \text{mod} \, H[1] \oplus \ldots \oplus \text{mod} \, H[m-1] \).

It is enough to show that there are a finite number of indecomposable objects \( X \) such that \( \text{Ext}^1_{\mathcal{C}_H^m}(X, P) = 0 \), where \( P \) is a projective \( H \)-module, since the shift functor is an equivalence on the \( m \)-cluster category. It follows from [BR] that there are only a finite number of indecomposable objects \( X \) lying inside \( \text{mod} \, H[i] \), with \( \text{Ext}^1_{\mathcal{C}_H^m}(X, P[i]) = 0 \) for all \( i \).
We have $\text{Ext}_{C_H^m}^{i+1}(X, P) = \text{Ext}_{C_H}^i(X, P[j])$, so there are only finitely many indecomposable objects $X$ in mod $H[j]$ such that $\text{Ext}_{C_H^m}^{i+1}(X, P) = 0$. Consequently there are only a finite number of indecomposable objects $X$ such that $\text{Ext}_{C_H^m}^k(X, P) = 0$ for all $k \in \{1, 2, ..., m\}$, and we are finished. \[\square\]

It is known from \cite{BKL} that in the tame case, a collection of one or more tubes is triangulated. We give the proof of the following for the convenience of the reader.

**Proposition 3.3.** Let $H$ be a finite dimensional tame hereditary algebra over a field $k$, and $C_H^m$ the corresponding $m$-cluster category. Let

$$X \rightarrow Y \rightarrow Z \rightarrow$$

be a triangle in $C_H^m$, where two of the terms are shifts of regular modules. Then all terms are shifts of regular modules.

**Proof.** It is enough to show that if $X$ and $Z$ are shifts of regular modules, then $Y$ is a shift of a regular module. There exist a homogeneous tube $\mathcal{T}$, i.e. $\tau M = M$ for all $M \in \mathcal{T}$, such that no direct summands of $X$ or $Z$ are in $\mathcal{T}$. Let $W$ be a quasi-simple object in $\mathcal{T}$. We have that $W$ is sincere (see \cite{DR}). We get the exact sequence

$$\text{Hom}(Z, W) \rightarrow \text{Hom}(Y, W) \rightarrow \text{Hom}(X, W).$$

We have that $\text{Hom}(Z, W) = \text{Hom}(X, W) = 0$, since there are no maps between disjoint tubes. It follows that $\text{Hom}(Y, W) = 0$. Since $W$ is sincere, we have that $\text{Hom}(U, W) \neq 0$ for any projective $U$, hence for any preprojective since $\tau W = W$. We can do similarly for preinjectives. It follows that all direct summands of $Y$ are shifts of regulars. \[\square\]

**Proposition 3.4.** Let $C_H^m$ be an $m$-cluster category, where $H$ is of tame representation type. Let $T$ be an $m$-cluster tilting object in $C_H^m$. Then $T$ has, up to $\tau$, at least one direct summand which is a shift of a projective or injective.

**Proof.** It is clearly enough to prove that there are no $m$-cluster tilting objects in $C_H^m$ with only shifts of regular $H$-modules as direct summands. So suppose, for a contradiction, that such a $T$ exists.

We can decompose $T$ into indecomposable summands, where $T = T_1 \oplus T_2 \oplus ... \oplus T_n$ and $n$ is the number of simple $H$-modules. If all direct summands are of the same degree, we already have a contradiction, since a tilting module has at least one direct summand which is preprojective or preinjective (see \cite{R}).

Assume that $T_k$ is a direct summand of degree $k \leq m$. Let $\tilde{T} = T_1 \oplus T_2 \oplus ... \oplus T_{n-1}$ be the almost complete $m$-cluster tilting object obtained from $T$ by removing the direct summand $T_n$. Then we know that the complements of $\tilde{T}$ are connected by $m + 1$ AR-triangles,

$$M_{i+1} \rightarrow X_i \rightarrow M_i \rightarrow,$$ where $i \in \{0, 1, 2, ..., m\}$ and $X_i \in \text{add} \tilde{T}$.

The direct summands of $X_i$ are by assumption shifts of regular modules. We also have that $T_n$ is a shift of a regular module and that it is equal to $M_j$ for some $j$, since it is a complement of $\tilde{T}$. It follows that $M_i$ is a shift of a regular module for all $i$ by Proposition 3.3, since these are connected by the exchange triangles. So all $m$-cluster tilting objects that can be reached from $T$ by a finite number of mutations, have only regular direct summands.

This leads to a contradiction, because we know from Proposition 2.2 that all $m$-cluster tilting objects can be reached from $T$ by a finite number of mutations, and a tilting module in $H$ induces an $m$-cluster tilting object in $C_H^m$ with at least one direct summand preprojective or preinjective. \[\square\]
From this it follows that we can assume that an $m$-cluster tilting object has at least one direct summand which is a shift of a projective up to $\tau$.

We also need a lemma proven in [BR].

**Lemma 3.5.** Let $H$ be wild with at least 3 non-isomorphic simples. Let $t$ be a positive integer. Then there is a tilting module $T$ in $H$ with indecomposable direct summands $T_1$ and $T_2$, such that $\dim \text{Hom}_H(T_1, T_2) \geq t$.

To prove the next lemma, which was observed in [BR] for 1-cluster tilted algebras, we use the following fact from [W]. Let $F = \tau^{-1}[m]$. If $X$ and $Y$ are two objects in some chosen fundamental domain in $\mathcal{D}^b(H)$, then $\text{Hom}_{\mathcal{D}^b(H)}(X, F^3Y) = 0$ for all $i \neq 0, 1$.

**Lemma 3.6.** If a path in the quiver of an $m$-cluster tilted algebra goes through two oriented cycles, then it is zero.

**Proof.** We have that

$$\text{Hom}_{c_n}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(H)}(X, F^iY).$$

Let $X$ and $Y$ be two indecomposable $m$-rigid objects in a chosen fundamental domain. It is well known that since $\text{Ext}^1_{\mathcal{D}^b(H)}(X, X) = 0$, we have that $\text{End}_{\mathcal{D}^b(H)}(X) = k$. It follows that in an oriented cycle, one of the maps lifts to a map of the form $X \to FY$ in $\mathcal{D}^b(H)$. If there is a path that goes through two oriented cycles, we have a map of the form $X \to FY \to F^2Z$, and this is 0 by the above. □

The following theorem generalizes the main theorem in [BR].

**Theorem 3.7.** Let $k$ be an algebraically closed field and $Q$ a connected finite quiver without oriented cycles. The following are equivalent for $H = kQ$.

1. There are only a finite number of basic $m$-cluster tilted algebras associated with $H$, up to isomorphism.
2. There are only a finite number of Gabriel quivers occurring for $m$-cluster tilted algebras associated with $H$, up to isomorphism.
3. $H$ is of finite or tame representation type, or has at most two non-isomorphic simple modules.
4. There are only a finite number of $\tau$-orbits of cluster tilting objects associated with $H$.
5. There are only a finite number of coloured quivers occurring for $m$-cluster tilting objects associated with $H$, up to isomorphism.
6. The mutation class of a coloured quiver $Q$, arising from an $m$-cluster tilting object associated with $H$, is finite.

**Proof.** (1) implies (2) and (4) implies (5) is clear.

(2) implies (3): Suppose there are only a finite number of quivers occurring for $m$-cluster tilted algebras associated with $H$, and let $u$ be the maximal number of arrows between to vertices in the quiver. Then by Lemma 3.6, for any two indecomposable summands $T_1$ and $T_2$ of an $m$-cluster tilting object $T$, $\dim \text{Hom}_{c_n}(T_1, T_2) < u^{2n}$, where $n$ is the number of simple $H$-modules. Then it follows from Lemma 3.5 that $H$ is not wild with more than 3 simples.

(3) implies (4): If $H$ is of finite representation type this is clear, since we only have a finite number of indecomposables.

Next, suppose $H$ has at most two non-isomorphic simple modules. If there is only one simple module we have $H \simeq k$, so we can assume there are two simples. Suppose $R$ is a regular indecomposable $H$-module. Then it follows from Lemma 3.1 that $R$ is not rigid, i.e. $\text{Ext}_{c_R}^1(R, R) \neq 0$. Then we also have that $\text{Ext}_{c_R}^{1}(R[i], R[i]) \neq 0$ for any $i \in \{1, 2, ..., m-1\}$. Up to $\tau$ in $c_R^m$ we can assume that an $m$-cluster tilting
object has a direct summand which is a shift of a projective $H$-module, say $P[j]$. Then $P[j]$ has $m + 1$ indecomposable complements. It follows that there are only a finite number of $m$-cluster tilting objects up to $\tau$, since there are only a finite number of choices for $P[j]$.

Suppose $H$ is tame. By Proposition 3.4, an $m$-cluster tilting object has at least one direct summand which is a shift of a projective or injective, and hence up to $\tau$ we can assume it has an indecomposable direct summand which is a shift of a projective. From Lemma 3.2 we have that there is only a finite number of $m$-cluster tilting objects with a shift of an indecomposable projective $H$-module as a direct summand.

(5) implies (6): This is clear, since mutation of $m$-cluster tilting objects corresponds to mutation of coloured quivers.

We have that (4) implies (1) by using Lemma 2.5. (6) implies (2) is trivial, and so we are done. □

We get the following corollary.

**Corollary 3.8.** A coloured quiver $Q$ corresponding to an $m$-cluster tilting object, has finite mutation class if and only if $Q$ is mutation equivalent to a quiver $Q'$, where $Q'_{\mathcal{G}}$ has underlying graph Dynkin or extended Dynkin, or it has at most two vertices, and there are only arrows of colour 0 and $m$ in $Q'$.

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