Examples of topologically highly chromatic graphs with locally small chromatic number

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Abstract

Kierstead, Szemerédi, and Trotter showed that a graph with at most $\left\lfloor \frac{r}{2n} \right\rfloor$ vertices such that each ball of radius $r$ in it is $c$-colorable should have chromatic number at most $n(c-1)+1$. We show that this estimate is sharp in $r$. Namely, for every $n$, $r$, and $c$ we construct a graph $G$ containing $O((2rc)^n)$ vertices such that $\chi(G) \geq n(c-1)+1$, although each ball of radius $r$ in $G$ is $c$-colorable. The core idea is the construction of a graph whose neighborhood complex is homotopy equivalent to the join of neighborhood complexes of two given graphs.

1 Introduction

Let $G = (V, E)$ be a graph (with no loops or multiple edges). By $d_G(u, v)$ we denote the distance between the vertices $u, v \in V$. A subset $V_1 \subseteq V$ is independent if none of the edges has both endpoints in $V_1$. The chromatic number $\chi(G)$ of $G$ is the minimal number of colors in a proper coloring of $G$, that is — the minimal number of parts in a partition of $V$ into independent sets.

Definition 1.1. Let $r$ be a positive integer. The ball of radius $r$ with center $v \in V$ is the set $U_r(v, G) = \{ u \in G : d(u, v) \leq r \}$. The $r$-local chromatic number $\ell \chi_r(G)$ of a graph $G$ is the maximal chromatic number of a ball of radius $r$ in $G$.

Note that even for $r = 1$ our definition of the local chromatic number is quite different from that introduced by Erdős et al. in \cite{4}.

By the celebrated result of Erdős \cite{2}, for every integer $n > 1$ and $g > 2$ there exists a graph of girth $g$ and chromatic number greater than $n$; thus for every $r$ there exist a graph $G$ with $\ell \chi_r(G) = 2$ and arbitrarily large $\chi(G)$. Later Erdős \cite{3} conjectured that for every positive integer $s$ there exists a constant $c_s$ such that the chromatic number of each graph $G$ having $N$ vertices and containing no odd cycles of length less than $c_s N^{1/s}$ does not exceed $s + 1$. This conjecture was proved by Kierstead, Szemerédi, and Trotter \cite{6}. In fact, they have proved the following more general result.

Theorem 1.1 (\cite{6} Theorem 1). Assume that $G = (V, E)$ is a graph such that $\ell \chi_r(G) \leq c$ and $|V| \leq \lfloor \frac{r}{2n} \rfloor$. Then $\chi(G) \leq n(c-1)+1$.

They have also posed a question whether this bound is sharp. The strong form of this question is as follows.

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Theorem 1.2. For every positive integers \( c \geq 3, r, \) and \( n \) there exists a graph \( G = (V, E) \) such that \( \ell \chi_r(G) \leq c, \chi(G) \geq n(c - 1) + 1, \) and
\[
|V| = \frac{(2rc + 1)^n - 1}{2r}.
\]  

The most difficult part of the justification is the proof of the lower bound for the chromatic number. This part is topological; it inspired by Stiebitz’s proof mentioned above.

The necessary topological background is collected in Section 2. In Section 3 we describe the general construction; its properties are investigated in Section 4. Finally, in section 5 we prove Theorem 1.2.

2 Topological background

Here we gather topological notions and facts needed in the sequel. We write \( X \cong Y \) for homeomorphic topological spaces and \( X \simeq Y \) for homotopy equivalent ones. For more detailed discussion see, e.g., [7].

2.1 Simplicial complexes

An (abstract) simplicial complex is a pair \((V, K)\) where \( V \) is a set and \( K \subseteq 2^V \) is a hereditary system of subsets of \( V \); this means that \( F_1 \subseteq F_2 \in K \) implies \( F_1 \in K \). The elements of \( V \) are called vertices, and the sets in \( K \) are called simplices. All simplicial complexes in our paper are finite, i.e., \(|V| < \infty\).

We will often denote a simplicial complex merely by \( K \) assuming that its vertex set is \( V(K) = \bigcup K \).

We say that \( \phi : V(K) \rightarrow \mathbb{R}^d \) is a geometric realization of \( K \) if (i) for every simplex \( A = \{a_1, \ldots, a_t\} \in K \) the points \( \phi(a_1), \ldots, \phi(a_t) \) are affinely independent; and (ii) for every two simplices \( A, B \in K \) we have \( \text{conv} \phi(A) \cap \text{conv} \phi(B) = \text{conv} \phi(A \cap B) \), where \( \text{conv} X \) is the convex hull of the set \( X \). It is known that each finite simplicial complex has a geometric realization. If \( \phi \) is a geometric realization of \( K \), then we denote the topological subspace \( \bigcup_{A \in K} \text{conv} \phi(A) \subset \mathbb{R}^d \) by \( \|K\| \) and call it a polyhedron of \( K \). All polyhedra of \( K \) are homeomorphic; thus this definition does not lead to an ambiguity.
In view of Lovász’s lemma, it makes sense to seek for a graph $2.3$ Joins and nerves.

If $K$ is a simplicial complex and $L \subseteq K$ is a hereditary subsystem of $K$ then we say that $L$ is a *subcomplex* of $K$ (we assume that the set of vertices of $L$ is $V(L) = \bigcup L$). If $\phi$ is a geometric realization of $K$, then $\phi|_{V(L)}$ is also a geometric realization of $L$. In this case, we will always assume that the polyhedron $\|L\|$ is a subspace of $\|K\|$, i.e. $\|L\| = \bigcup_{B \in L} \phi(B)$. In particular, every simplex $A \in K$ may be considered as the subcomplex $(A, 2^A)$ of $K$; thus we may write $\|A\|$ instead of $\mathrm{conv} \phi(A)$.

2.2 Neighborhood complex and Lovász’s lemma

The following important notion was introduced by Lovász [8].

**Definition 2.1.** Let $G = (V, E)$ be a graph; we assume that it contains no isolated vertices. The neighborhood complex $N(G)$ on the set of vertices $V$ consists of all subsets $A \subseteq V$ such that all elements of $A$ have a common neighbor in $G$ (this neighbor surely does not belong to $A$).

For instance, the neighborhood complex of the complete graph $K_r$ is an $(r-2)$-dimensional skeleton of the $(r-1)$-dimensional simplex; so $\|N(K_r)\| \cong S^{r-2}$.

Lovász has discovered a relation between the homotopy properties of $\|N(G)\|$ and the chromatic number of $G$. To formulate this result, we need a notion of $k$-connectedness of a topological space.

As usual, we denote the unit ball and the unit sphere in $\mathbb{R}^d$ respectively by $B^d = \{x \in \mathbb{R}^d: |x| \leq 1\}$ and $S^{d-1} = \{x \in \mathbb{R}^d: |x| = 1\}$.

A nonempty topological space $X$ is *$k$-connected* if each continuous map $g: S^{m-1} \to X$ extends to a continuous map $\overline{g}: B^m \to X$, for $m = 0, 1, \ldots, k+1$ (this condition for $m = 0$ and $m = 1$ means that $X$ is nonempty and path connected, respectively). It is well known that the sphere $S^k$ is $(k-1)$-connected but is not $k$-connected. Recall that homotopy equivalence preserves $k$-connectedness.

**Lemma 2.1** (Lovász, [8] Theorem 2). Let $G$ be a graph. Assume that the polyhedron $\|N(G)\|$ is $k$-connected. Then $\chi(G) \geq k + 3$.

This lemma was initially invented by Lovász in order to find the chromatic number of Kneser graphs.

2.3 Joins and nerves

In view of Lovász’s lemma, it makes sense to seek for a graph $G$ such that $\|N(G)\|$ is highly connected. For this, the following construction is useful.

The *join* of two topological spaces $X$ and $Y$ is defined as the quotient space

$$X \ast Y = (X \times Y \times [0, 1])/ \sim$$

by the equivalence relation $\sim$ determined by $(x, y, 0) \sim (x, y, 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and $y, y' \in Y$. The join of $k$- and $\ell$-dimensional simplices is homeomorphic to a $(k+\ell+1)$-dimensional simplex. Moreover, $S^k \ast S^\ell \cong S^{k+\ell+1}$.

If $X$ and $Y$ are subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ then the join $X \ast Y$ can be realized as a subspace of $\mathbb{R}^{m+n+1}$ in the following way. Choose two skew affine subspaces $U, V \subseteq \mathbb{R}^{m+n+1}$ with $\dim U = m$, $\dim V = n$. Then $X \ast Y$ is a subset of $U \cup V$ such that $U \cap V$ is a $(m+n+1)$-dimensional simplex. If $X$ and $Y$ are convex then $X \ast Y$ is again convex.
\[ \dim V = n; \text{ we may regard } X \text{ and } Y \text{ as the subspaces of } U \text{ and } V, \text{ respectively. Then } X * Y \cong \{(1-t)x + ty: x \in X, y \in Y, t \in [0,1]\}. \]

The join \( K * L \) of two simplicial complexes \( (U, K) \) and \( (V, L) \) is defined as follows. We define \( V(K * L) = U' \cup V' \), where \( U' = U \times \{0\} \) and \( V' = V \times \{1\} \) (thus we ensure that these sets are disjoint), and set \( K * L = \{(A \times \{0\}) \cup (B \times \{1\}): A \in K, B \in L\} \).

The two notions of a join agree in the sense that \( ||K * L|| \cong ||K|| * ||L|| \); this is easily seen from the realization of the join described above.

In the sequel, for arbitrary graphs \( G_1 \) and \( G_2 \) we will construct a series of graphs \( J_r \) such that \( ||N(J_r)|| \cong ||N(G_1) * N(G_2)|| \). Let us introduce one more notion needed for the proof.

Let \( X \) be a topological space, and let \( U = \{U(i): i \in I\} \) be a covering of \( X \). The nerve of this covering is the simplicial complex with \( I \) as the set of vertices; a subset \( J \subseteq I \) is its simplex if \( \bigcap_{i \in J} U(i) \neq \emptyset \). We will use the following well-known fact (see, e.g., [7, Theorem 15.21]; we present only a particular case sufficient for our purposes).

**Lemma 2.2** (Nerve lemma). Let \( U = \{U_i: i \in I\} \) be a finite open covering of a compact metric space \( X \). Assume that for every \( J \subseteq I \) the set \( \bigcap_{i \in J} U(i) \) is either empty or contractible. Then the polyhedron of the nerve of \( U \) is homotopy equivalent to \( X \).

A nonempty set \( X \subseteq \mathbb{R}^d \) is called star-shaped if there exists \( a \in X \) such that for every \( b \in X \) the whole segment \([a, b]\) lies in \( X \); in this case \( a \) is called a center of \( X \). Obviously, each star-shaped set is contractible.

### 3 The main construction

Now we present the desired construction.

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs (we assume that they contain no isolated vertices), and let \( r \) be a nonnegative integer. For \( g_i \in V_i \), denote by \( N_i(g_i) \) the set of all its neighbors in \( G_i \).

We define the graph \( G_1 *_r G_2 \) as follows. First, we define an auxiliary graph \( J'_r = (U'_r, F'_r) \) by setting

\[
U'_r = V_1 \times V_2 \times \{0, 1, \ldots, r + 1\}, \quad F'_r = \{((g_1, g_2, i), (g_1', g_2', j)) \in (U'_r)^2: (g_1, g_1') \in E_1, (g_2, g_2') \in E_2, \text{ and } |i - j| \leq 1\}. \]

The graph \( J_r = G_1 *_r G_2 = (U_r, E_r) \) is obtained by merging some vertices of the constructed graph. Namely, for every \( g_1 \in V_1 \) we collapse all \( |V_2| \) vertices of the form \((g_1, g_2, 0)\) to a new vertex \((g_1, 0)\), and for every \( g_2 \in V_2 \) we collapse all \( |V_1| \) vertices of the form \((g_1, g_2, r + 1)\) to a new vertex \((g_2, r + 1)\).

Fig. I shows a sample graph \( K_2 *_3 K_3 \).

![Figure 1: Graph \( G_1 *_r G_2 \)](image-url)
Notice that \( G_1 \ast_0 G_2 \) is just the usual join of graphs \( G_1 \) and \( G_2 \).

For every vertex \( v \in U_r \), we denote by \( N(v) \) the set of all neighbors of \( v \) in \( J_r \); these sets, together with all their subsets, form the complex \( N(J_r) \). For \( r \geq 2 \) these sets are

\[
N(g_1, 0) = (N_1(g_1) \times \{0\}) \cup (N_1(g_1) \times V_2 \times \{1\});
N(g_1, g_2, 1) = (N_1(g_1) \times \{0\}) \cup (N_1(g_1) \times N_2(g_2) \times \{1, 2\});
N(g_1, g_2, i) = N_1(g_1) \times N_2(g_2) \times \{i - 1, i, i + 1\} \quad \text{for } 1 < i < r;
N(g_1, g_2, r) = (N_2(g_2) \times \{r + 1\}) \cup (N_1(g_1) \times N_2(g_2) \times \{r - 1, r\});
N(g_2, r + 1) = (N_2(g_2) \times \{r + 1\}) \cup (V_1 \times N_2(g_2) \times \{r\}).
\]

### 4 The properties of the construction

Now we will investigate the properties of the constructed graph \( G_1 \ast_r G_2 \). First, we find the \( r \)-local chromatic number of \( G_1 \ast_2 G_2 \).

**Lemma 4.1.** For every graphs \( G_1 \) and \( G_2 \) and every positive integer \( r \) we have

\[
\ell_{\chi_r}(G_1 \ast_2 G_2) = \max\{\ell_{\chi_r}(G_1), \ell_{\chi_r}(G_2)\}.
\]

**Proof.** Since \( G_1 \) and \( G_2 \) are isomorphic to subgraphs of \( J_{2r} = G_1 \ast_2 G_2 \), we have \( \ell_{\chi_r}(J_{2r}) \geq \max\{\ell_{\chi_r}(G_1), \ell_{\chi_r}(G_2)\} \).

On the other hand, since the distance between \( V_1 \times \{0\} \) and \( V_2 \times \{2r + 1\} \) in \( J_{2r} \) is \( 2r + 1 \), each ball \( B \) of radius \( r \) in \( J_{2r} \) lies either in \( J_{2r} \setminus (V_1 \times \{0\}) \) or in \( J_{2r} \setminus (V_2 \times \{2r + 1\}) \). Consider the first case. The projection to \( V_2 \) is a graph homomorphism from \( J_{2r} \setminus (V_1 \times \{0\}) \) to \( G_2 \), and \( \text{pr}_{V_2}(B) \) is a ball of radius \( r \) in \( G_2 \); hence \( \chi(B) \leq \chi(\text{pr}_{V_2}(B)) \leq \ell_{\chi_r}(G_2) \). Similarly, in the second case we get \( \chi(B) \leq \ell_{\chi_r}(G_1) \), proving the converse inequality. \( \square \)

Next, we deal with the neighborhood complex of \( G_1 \ast_r G_2 \).

**Lemma 4.2.** For every graphs \( G_1 \) and \( G_2 \) and every integer \( r \geq 2 \) we have

\[
\|N(G_1 \ast_r G_2)\| \simeq \|N(G_1)\| * \|N(G_2)\|.
\]

**Proof.** Denote \( K = N(G_1 \ast G_2) \), \( M = N(G_1) * N(G_2) \). Let us construct a convenient geometric realization of \( M \). Consider some geometric realizations of \( N(G_1) \) and \( N(G_2) \) in real spaces \( R_1 \) and \( R_2 \); we may identify the vertices of \( G_i \) with their images under these realizations. Now consider the space \( R = R_1 \times R_2 \times \mathbb{R} \); for every \( g_1 \in V_1 \), identify the vertex \( (g_1, 0) \in M \) with \( (g_1, 0, 0) \in R \), and for every \( g_2 \in V_2 \) identify the vertex \( (g_2, 1) \in M \) with \( (0, g_2, r + 1) \in R \). This provides a geometric realization of \( M \). For convenience, for every interval \( I \subseteq \mathbb{R} \) we denote by \( R^I \) the “strip” \( R_1 \times R_2 \times I \subseteq R \); in particular, \( |M| \subseteq R^{[0, r+1]} \).

Notice that for every nonempty topological subspaces \( A_i \subseteq \|N(G_i)\| \) the space \( A_1 \ast A_2 \) may be regarded as a subset in \( \|M\| \). Moreover, for all \( A_i, B_i \subseteq \|N(G_i)\| \) such that \( A_i \cap B_i \neq \emptyset \) we have \( (A_1 \ast A_2) \cap (B_1 \ast B_2) = (A_1 \cap B_1) \ast (A_2 \cap B_2) \).

For every vertex \( g_i \in V_i \) let us define the star set of vertex \( g_i \) as

\[
S_i(g_i) = \{x \in \|N(G_i)\|: g_i \in \text{supp}_N(G_i, x) \} \subseteq R_i.
\]

Each set \( S_i(g_i) \) is open in \( \|N(G_i)\| \). Next, for every \( A_i \subseteq V_i \) the set \( S_i(A_i) := \bigcap_{g_i \in A_i} S_i(g_i) \) is nonempty if and only if \( A_i \in N(G_i) \) (in fact, if \( A_i \in N(G_i) \) then \( S_i(A_i) \) is the union of relative interiors of all simplices \( |B_i| \) such that \( A_i \subseteq B_i \subseteq N(G_i) \)). For every \( A_i \in N(G_i) \), let us fix an
arbitrary point \(x_i(A_i)\) in the relative interior of \(|A_i|\) (if \(A_i = \{g_i\}\) then \(x_i(A_i) = g_i\)); one can easily see then that \(S_i(A_i)\) is star-shaped with center \(x_i(A_i)\).

Now we construct a covering \(U\) of \(\|M\|\) with contractible intersections such that its nerve is \(K\); by the Nerve lemma \([2,2]\), this implies the desired result. Set \(U = \{U(v): v \in V(K)\}\), where

\[
U(g_1, 0) = \left( S_1(g_1) * \|N(G_2)\| \right) \cap R^{0,3/2};
U(g_2, r + 1) = \left( \|N(G_1)\| * S_2(g_2) \right) \cap R^{(r-1/2,r+1)};
U(g_1, g_2, i) = \left( S_1(g_1) * S_2(g_2) \right) \cap R^{(i-3/2,i+3/2)} \quad \text{for } 1 \leq i \leq r.
\]

Several sample sets \(U(v)\) in \(\|N(K_2) * N(K_3)\|\) are shown in Fig. 2. Recall that in this example the underlying space \(R\) is 4-dimensional.

![Figure 2: Covering \(U\) of \(\|N(G_1) * N(G_2)\|\) (here \(r = 6\) )](attachment:image.png)

For an arbitrary maximal simplex \(A \in M\) we have \(A = (A_1 \times \{0\}) \cup (A_2 \times \{1\})\) with \(A_i \in N(G_i)\); then we have \(\|A\| = \|A_1\| * \|A_2\|\), and one can easily see that this simplex is covered by the sets \(U(g_1, g_2, t)\) with \(g_i \in A_i\) and \(t \in \{1, \ldots, r\}\). Thus \(U\) is an open covering of \(M\).

Let \(L\) be the nerve of \(U\). For every \(A \subseteq V(L) = V(K)\) denote \(U(A) = \bigcap_{v \in A} U(v)\). Now, one may verify that

\[
U(A) \cap R^{0} = \begin{cases} S_1(\text{pr}_{V_1}(A)) \times \{0\} \times \{0\}, & \text{pr}_{R}(A) \subseteq \{0,1\}; \\ \emptyset, & \text{otherwise;} \end{cases}
U(A) \cap R^{r+1} = \begin{cases} \{0\} \times S_2(\text{pr}_{V_2}(A)) \times \{r+1\}, & \text{pr}_{R}(A) \subseteq \{r,r+1\}; \\ \emptyset, & \text{otherwise;} \end{cases}
U(A) \cap R^{(r,r+1)} = \begin{cases} \{ (S_1(\text{pr}_{V_1}(A)) * N(G_2) \} \cap R^{(0,3/2)}, & A \subseteq V_1 \times \{0\}; \\ \{ (N(G_1) * S_2(\text{pr}_{V_2}(A)) \} \cap R^{(r-1/2,r+1)}, & A \subseteq V_2 \times \{r+1\}; \\ \{ (S_1(\text{pr}_{V_1}(A)) * S_2(\text{pr}_{V_2}(A)) \} \cap R^{(a(A),b(A))}, & \text{otherwise.} \end{cases}
\]

Here we set \(a(A) = \max\{0, \max(\text{pr}_{R}(A)) - 3/2\}\) and \(b(A) = \min\{r+1, \min(\text{pr}_{R}(A)) + 3/2\}\). Next, the projection \(\text{pr}_{V_1}\) is defined as \(\text{pr}_{V_1}(g_1, 0) = \text{pr}_{V_1}(g_1, g_2, i) = g_1\) for \(i = 1, \ldots, r\), and \(\text{pr}_{V_1}(g_2, r+1) = \emptyset\); the projection \(\text{pr}_{V_2}\) is defined similarly.

Now a straightforward check shows that \(U(A) \cap R^{0} \neq \emptyset\) exactly if \(A \subseteq N(g_1, 0)\) for some \(g_1 \in V_1\), that \(U(A) \cap R^{(r,r+1)} \neq \emptyset\) exactly if \(A \subseteq N(g_2, r+1)\) for some \(g_2 \in V_2\), and that \(U(A) \cap R^{(0,r+1)} \neq \emptyset\) exactly if \(A \subseteq N(g_i, g_2, i)\) for some \(g_1 \in V_1, g_2 \in V_2,\) and \(i \in \{1, \ldots, r\}\). Thus \(L = K\).

It remains to check that all nonempty sets of the form \(U(A)\) are contractible; in fact, we will see that they are star-shaped. Assume that \(U(A) \neq \emptyset\). If \(U(A) \cap R^{0} \neq \emptyset\) then
the set $U(A)$ is star-shaped with center $(x_1(pr_{V_1}(A)), 0, 0)$. Similarly, if $U(A) \cap R^{r+1} \neq \emptyset$ then the set $U(A)$ is star-shaped with center $(0, x_2(pr_{V_2}(A)), r + 1)$. In the remaining case we have $U(A) = (S_1(pr_{V_1}(A)) \ast S_2(pr_{V_2}(A))) \cap R^{a(A), b(A)}$, and it is star-shaped with center $(x_1(pr_{V_1}(A)), x_2(pr_{V_2}(A)), (a(A) + b(A))/2)$.

Finally, applying the Nerve lemma 2.2 we get the required result. □

Remark. The statement of Lemma 4.2 remains valid for $r = 1$ with essentially the same proof. On the other hand, for $r = 0$ it does not hold in general; for instance, $N(K_n) \cong S^{n-2}$, thus $N(K_n \ast_0 K_m) \cong N(K_{n+m}) \cong S^{n+m-2} \not\cong S^{n+m-3} \cong N(K_n) \ast N(K_m)$.

5 Proof of Theorem 1.2

Let us fix the values of $c$ and $r$. We will use the induction on $n$ to construct the graph $G_n$ with $((2rc + 1)^n - 1)/(2r)$ vertices such that $\ell \chi_r(G_n) \leq c$ and $N(G_n) \cong S^{n(c-1)-1}$. Thus $N(G_n)$ will be $(n(c-1) - 2)$-connected; the required estimate then follows from Lemma 2.1.

For $n = 1$, the complete graph $K_c$ satisfies the desired properties since $|K_c| = c$ and $N(K_c) \cong S^{c-2}$.

Now assume that $n > 1$, and the graph $G_{n-1}$ is already constructed. Then we set $G_n = G_{n-1} \ast_{2r} K_r$. We have

$$|G_n| = (2rc + 1)|G_{n-1}| + c = (2rc + 1)\frac{(2rc + 1)^{n-1} - 1}{2r} + c = \frac{(2rc + 1)^n - 1}{2r}.$$ 

Next, by Lemma 4.1 we have $\ell \chi_r(G_n) = \max\{\ell \chi_r(G_{n-1}), \ell \chi_r(K_r)\} = c$. Finally, by Lemma 4.2 we have $N(G_n) \cong N(G_{n-1}) \ast N(K_c) \cong S^{(n-1)(c-1)-1} \ast S^{c-2} \cong S^{n(c-1)-1}$. The theorem is proved. □

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