Hidden Quasicrystal in Hofstadter Butterfly

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Topological description of hierarchical sets of spectral gaps of Hofstadter butterfly is found to be encoded in a quasicrystal where magnetic flux plays the role of a phase factor that shifts the origin of the quasiperiodic order. Revealing an intrinsic frustration at smallest energy scale, described by $\zeta = 2 - \sqrt{3}$, this irrational number characterizes the universal butterfly and is related to two quantum numbers that includes the Chern number of quantum Hall states. With a periodic drive that induces phase transitions in the system, the fine structure of the butterfly is shown to be amplified making states with large topological invariants accessible experimentally.

Hofstadter butterfly\cite{1}, also known as Gplot\cite{2} is a fascinating two-dimensional spectral landscape, a quantum fractal where energy gaps encode topological quantum numbers associated with the Hall conductivity\cite{3,4}. This intricate mix of order and complexity is due to two competing periodicities in a crystalline lattice subjected to a magnetic field. The allowed energies in the spectrum of such a system are discontinuous function of the magnetic flux penetrating the unit cell, while the gaps are continuous except at discrete points. The stunning smoothness of spectral gaps may be traced to topology which makes spectral properties stable with respect to small fluctuations in the magnetic flux. The Gplot continues to arouse a great deal of excitement since its discovery, and there are various recent attempts to capture this iconic structure near rational fluxes. We show that, for every ratio-


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In contrast to geometrical description of topological universality, the universal spectral property of the Gplot is obtained numerically. Corresponding to a scaling ratio of $\zeta^2$ for the magnetic flux interval, and $\zeta$ for quantum numbers $\sigma$ and also $\tau$, the spectrum is found to scale approximately as $\zeta^{\sqrt{3}}$. We obtain the two-dimensional fixed point fractal (See Fig. (2)) that characterizes the entire landscape near half-filling and verify its universality as magnetic flux varies.

In our investigation of the fractal properties of the Hofstadter butterfly, one of the key guiding concepts is a corollary of the DE equation that quantifies the topology of the fine structure near rational fluxes. We show that, for every rational flux, an infinity of possible solutions of the DE, although not supported in the simple square lattice model, are present in close vicinity of the flux. Consequently, perturbations that induce topological phase transitions can transform tiny gaps with large topological quantum numbers into major gaps. This might facilitate the creation of such states in an experimental setting. We illustrate this amplification by periodically driving the system.

Model system we study here consists of (spinless) fermions in a square lattice. Each site is labeled by a vector $\mathbf{r} = n\hat{x} + m\hat{y}$, where $n, m$ are integers, $\hat{x}$ (y) is the unit vector in the $x$ ($y$) direction, and $a$ is the lattice spacing. The tight binding Hamiltonian has the form

\begin{equation}
H = -J_x \sum_r (|\mathbf{r} + \hat{x})\langle \mathbf{r} | - J_y \sum_r |\mathbf{r} + \hat{y})\langle \mathbf{r} | e^{i2\pi n\phi} |\mathbf{r} | + h.c.
\end{equation}

where $|\mathbf{r} \rangle$ is the Wannier state localized at site $\mathbf{r}$. $J_x$ ($J_y$) is the nearest neighbor hopping along the $x$ ($y$) direction. With a uniform magnetic field $B$ along the $z$ direction, the flux per plaquette, in units of the flux quantum $\Phi_0$, is $\phi = -Ba^2/\Phi_0$. Field $B$ gives rise to the Peierls phase factor $e^{i2\pi \nu \phi}$ in the hopping.

In the Landau gauge realized in experiments\cite{12}, the vector potential $A_x = 0$ and $A_y = -\phi x$, the Hamiltonian is cyclic in $y$ so the eigenstates of the system can be written as $\Psi_{n,m} = e^{ik_y m} \psi_n$, where $\psi_n$ satisfies the Harper equation\cite{13}

\begin{equation}
e^{ik\nu + e^{-ik\nu} \psi_{n-1} + 2\lambda \cos(2\pi n\phi + k_y)\psi_n} = E\psi_n.
\end{equation}

Here $n$ ($m$) is the site index along the $x$ ($y$) direction, $\lambda =$

Fractal properties of the butterfly spectrum have been the subject of various theoretical studies\cite{6,9}. However, detailed description quantifying self-similar universal properties of the butterfly fractal has not been reported previously. In this paper, we present a geometrical framework to decode the nesting rule that will reproduce the entire butterfly landscape, as one zooms in to the Gplot, and obtain universal scaling that characterizes the spectrum near half-filling. We address the question of both spectral and topological universality. The spectral gaps are labeled by two quantum numbers which we denote as $\sigma$ and $\tau$. The integer $\sigma$ is the Chern number, the quantum number associated with Hall conductivity\cite{4} and $\tau$ is an integer. These quantum numbers satisfy the Diophantine equation (DE)\cite{10},

\begin{equation}
\rho = \phi \sigma + \tau
\end{equation}

where $\rho$ is the particle density when Fermi level is in the gap and $\phi$ denotes the magnetic flux per unit cell.

Somewhat reminiscent of well known geometrical fractals, nesting property of the Gplot is found to be embedded in a Farey tree\cite{11} organization of various rational magnetic flux values (See Fig. (1)). As we zoom in to the butterfly spectrum, where magnetic flux intervals shrink by $\zeta^2$ ($\zeta = 2 - \sqrt{3}$), we find an algebraic recursive rule for $\sigma$ and $\tau$. These quantum numbers form a generalized Fibonacci sequence (Eq. (9)) associated with rational approximants of $\zeta$, which has period-2 continued fraction expansion, determining both $\sigma$ and $\tau$ at finer and finer scales in the butterfly fractal. This signals a kind of topological frustration that may be responsible for magnificently complex structure of the butterfly. This hidden quasicrystal, which we will refer as Hofstadter-Chern lattice, describes the entire butterfly landscape near half-filling. Here magnetic flux simply plays the role of a phase factor that leaves quasiperiodic topological pattern unchanged.

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$J_y/J_x$ and $\psi_{r+q} = \psi_r$, $r = 1, 2... q$ are linearly independent solutions. In this gauge the magnetic Brillouin zone is $-\pi/q_0 \leq k_x \leq \pi/q_0$ and $-\pi \leq k_y \leq \pi$.

At flux $\phi = p/q$, the energy spectrum has $q - 1$ gaps. For Fermi level inside each energy gap, the system is in an integer quantum Hall state characterized by its Chern number $\sigma$ which gives transverse conductivity $C_{xy} = \sigma e^2/h[4]$. The Chern number $\sigma$ and an integer $\tau$ label various gaps of the butterfly and are the solutions of DE[11], with possible values,

$$\sigma = \frac{q}{2}; \quad \tau = \frac{p + 1}{2}$$

(6)

The Chern number $\sigma$ changes sign in 4-wings of the butterfly and $\tau$ values differ by unity in the left and the right wing. We label each butterfly uniquely by the magnitude of $\sigma$ and $\beta = 2|\tau| + 1$.

For any value of the magnetic flux, the system described by the Hamiltonian [3], supports only $n = 0$ solution of Eq. (4) for the quantum numbers $\sigma$ and $\tau$. This is due to the absence of any gap closing that is essential for topological phase transition to states with higher values of $\sigma, \tau$. However, DE which relates continuously varying quantities $\rho$ and $\phi$ with integers $\sigma$ and $\tau$, has some important consequences about topological changes in close vicinity of rational values of $\phi$. We now show that the infinity of solutions depicted in Eq. (4) resides in close proximity to the flux $\phi$ and label the fine structure of the butterfly in Gplot. In DE, we substitute $\phi = \phi_0 + \phi$ where $\phi_0 = p_0/q_0$, and $\rho = \rho_0 + \delta \rho$ and the corresponding quantum numbers as $\sigma = \sigma_0 + \Delta \sigma$ and $\tau = \tau_0 + \Delta \tau$. Now, taking the limits as $\delta \phi$ and $\delta \rho$ go to zero, we obtain,

$$\phi_0 \Delta \sigma + \Delta \tau = 0; \quad \frac{\Delta \sigma}{\Delta \tau} = -\frac{q_0}{p_0}$$

(5)

Since both $\Delta \sigma$ and $\Delta \tau$ are integers and $p_0$ and $q_0$ are relatively prime, the simplest solutions of Eq. (5) are $\Delta \sigma = \pm np_0$ and $\Delta \tau = \mp np_0$, where $n = 0, 1, 2,...$. These solutions describe the fine structure of the butterfly near $\phi_0$. Consequently, the spectral gaps near $\phi = 1/q$ have Chern numbers changing by a multiple of $q$. This suggests a semiclassical picture near $\phi = p/q$ in terms of an effective Landau level theory with cyclotron frequency renormalized by $q$.

Another important consequence of DE near half-filling is a relationship between the magnetic flux at the center of the butterfly and the topological integer that labels the butterfly. For $E = 0$, rational values of magnetic flux with even denominators result in a set of 4-swaths converging to a structure resembling a butterfly. Although, precisely at the center of the butterfly, the Chern number is undefined, the structure emanating from the center can be associated with a unique value of $\sigma$ and $\tau$ as described below. For energy gaps near $E = 0$, we write $\rho = 1/2 + \epsilon$ and $\phi = p_0/q_0 + \delta$. Substituting these values of $\rho$ and $\phi$ in DE and taking the limit $\epsilon, \delta \to 0$, we obtain topological integers that characterize the four wings of the butterfly.

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A close inspection of the Gplot reveals that Farey sequences are the key to systematically subdivide the flux intervals. We illustrate this process in the Fig. 1 where a specific Farey path represented symbolically as “LRL” is essential in constructing the recursive scheme. It turns out that the choice of the initial interval is irrelevant as the universal scaling properties are independent of \( \phi \). We note that beyond level-0, butterflies do not exhibit reflection symmetry about their centers (See Fig. 2), however, the recursive scheme for the left and right intervals are identical. Figure shows two sets of intervals: (I) an infinite set constructed to the right of \( f_L \) where each Farey sum includes \( f_L \) and (II) an infinite set constructed to the left of \( f_c \) where each Farey sum includes \( f_c \). The recursive rules for constructing the left and right boundaries and the center of the butterfly from level \( l \) to level \( l + 1 \) are given by,

\[
\begin{align*}
  f_L(l + 1) &= f_L(l) \bigoplus f_c(l) \\
  f_R(l + 1) &= f_L(l + 1) \bigoplus f_c(l) \\
  f_c(l + 1) &= f_L(l) \bigoplus f_R(l)
\end{align*}
\]

At a level-\( l \) of the nesting scheme, the quantum numbers \( \sigma, \tau \) are found to scale by rational approximants of \( \zeta \), obtained by truncating its continued fraction expansion,

\[
\zeta = [3, 1, 2, 1, 2, 1, 2, ...] = \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + ...}}}}}
\]

Denoting \( t^{th} \) rational approximant of \( \zeta \) as \( \zeta(t) = P(t)/Q(t) \), the quantum numbers (using Eqs [6,7]) for the hierarchical set beginning with the interval \([1/3, 2/5]\) are, \( \sigma(l) = Q(2l), \beta(l) = Q(2l - 1) \) and \( \sigma(l - 1) = P(2l), \beta(l - 1) = P(2l - 1) \). This reveals a simple process of two independent integers described by a single irrational with period-2 continued fraction as integers related to even(odd) approximants determine \( \sigma, \beta \). Using number theoretical properties of \( \zeta \), it can be shown that a single recursion, where \( \zeta(l) \) is evaluated at every second iteration, describes the recursive scheme for both \( \sigma \) and \( \beta \).

\[
A(l + 1) = 4A(l) - A(l - 1)
\]

where \( A = \sigma, \beta \). Interestingly, this recursion determines the quantum numbers \( \sigma, \tau \) for all values of \( \phi \), with the initial values in Eq. (8) determined by the choice of \( \phi \) interval. Asymptotically, \( \sigma(l) \to \zeta^{-l}, \tau \to \zeta^{-l} \) and the underlying \( \phi \) interval scales as, \( \phi \to \zeta^2 l \).

For the butterfly fractal shown in Fig. 2, the entire band spectrum is numerically found to scale approximately as, \( \Delta E \approx 10^l \approx \zeta^{3l} \). Although the precise value of quantum numbers (and hence the universal butterfly fractals) depend upon \( \phi \), the scaling ratios between two successive levels is \( \phi \) independent. We summarize the universal scaling ratio \( R_x \) where \( x = \phi, \sigma, \tau, E \).

\[
R_\phi = \zeta^{-2}; \quad R_\sigma = R_\tau = \zeta; \quad R_E \approx \zeta^{3}
\]

Therefore, topological variations occur at a slower rate than the corresponding spectral variations as one views the butterfly at smaller and smaller scale. In summary, the topology of the universal butterfly is found to be encoded in a single irrational number that due to its period-2 continued fraction with entries 1 and 2 is the simplest possible number that underlies two distinct quantum numbers that appear in DE. The quasiperiodic Hofstadter Chern -lattice generated by \( \sigma(l), \tau(l) \) is a generalization of the well known Fibonacci lattice, characterized by the nesting property that each generation is the sum of two previous generations. The magnetic...
flux simply plays the role of a phase factor, an additional degree of freedom for the quasiperiodic Chern-lattice[10], that shifts the origin of the quasi-periodic order.

The time evolution operator of the system, defined by 
\[ |\psi(t)\rangle = U(t)|\psi(0)\rangle, \]
has the formal solution 
\[ U(t) = \mathcal{T} \exp[-i \int_0^t H(t')dt'], \]
where \( \mathcal{T} \) denotes time-ordering and we set \( \hbar = 1 \) throughout. The discrete translation symmetry 
\[ H(t) = H(t + T) \]
leads to a convenient basis \( \{|\phi_i\rangle\} \), defined as the eigenmodes of Floquet operator \( U(T) \),
\[ U(T)|\phi_i\rangle = e^{-i\omega T}|\phi_i\rangle. \]

We have two independent driving parameters, \( \bar{J} = J_x T/\hbar \) and \( \bar{\lambda} = \lambda T/\hbar \). For rational flux \( \phi = p/q \), \( U \) is a \( q \times q \) matrix with \( q \) quasienergy bands that reduce to the energy bands of the static system as \( T \to 0 \).

New topological landscape of the driven system as shown in the Fig. (3) can be understood by determining the topological states of flux values corresponding to simple rationals such as \( 1/3 \), \( 2/5 \). In the Fig. (3), parameter values correspond to the case where the Chern-1 gap associated with \( 1/3 \) has undergone quantum phase transition to a \( n = 1 \) solution of the DE (Eq. (4)) and the Chern-2, 1 states of \( 2/5 \) have also undergone transitions to Chern-3, \( -4 \) state. This almost wipes out the Chern-1 state from the landscape, exposing the topological states of higher Cherns that existed in tiny gaps in the static system. Gap amplifications for Chern \( 2 \) and 3 states in periodically driven quantum hall system may provide a possible pathway to see fractal aspects of Hofstadter butterfly and engineer states with large Chern numbers experimentally.

Recently, there is renewed interest in quasiperiodic systems[16–20] due to their exotic characteristics that includes their relationship to topological insulators. Hofstadter-Chern lattice is intrinsically frustrated system, at smallest energy scale induced by inherent frustration created by the magnetic flux \( \phi \). This suggest a new source of recursive behavior whose deeper understanding and significance needs further investigation.

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