We study the partial resolutions of singularities related to Hilbert schemes of points on an affine space. Consider a quotient of a vector space $V$ by an action of a finite group $G$ of linear transforms. Under some additional assumptions, we prove that the partial desingularization of Hilbert type is smooth only if the action of $G$ is generated by complex reflections. This is used to study the subvarieties of a Hilbert scheme of a complex torus. We show that any subvariety of a generic deformation of a Hilbert scheme of a torus is birational to a quotient of another torus by a Weyl group action. In Appendix, we produce counterexamples to a false theorem stated in our preprint math.AG/9801038.

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1 Introduction

Let \( M \) be a compact hyperkähler manifold. A hyperkähler manifold is equipped with an action of quaternions in its tangent bundle. This action induces a set of complex structures (so-called induced complex structures) on \( M \) (see Subsection 5.1). A closed subvariety \( X \subset M \) is called \textbf{trianalytic} if \( X \) is complex analytic with respect to all induced complex structures (Definition 5.2). Trianalytic subvarieties were a subject of a long study. Most importantly, take a generic induced complex structure \( I \) on \( M \). Then all closed complex subvarieties of \((M, I)\) are trianalytic ([V2]). Moreover, a trianalytic subvariety can be canonically desingularized ([V4]), and this desingularization is hyperkähler.

Two series of compact hyperkähler manifolds are studied: the Hilbert schemes of K3 surfaces and the so-called generalized Kummer varieties (see Definition 6.1). As it was shown in [V4], for a generic hyperkähler structure on a Hilbert scheme of a K3 surface, there are no trianalytic subvarieties (except points). The proof of this result is technically quite difficult. In [KV], a similar result was “found” of a generalized Kummer variety. The proof was elementary, but, unfortunately, false. Indeed, there are counterexamples (see Appendix), i.e. trianalytic subvarieties of any deformation of a generalized Kummer variety.

These counterexamples were found as a byproduct of attempts to prove the conjecture of V. Ginzburg ([BG]), which is stated roughly as follows.

\textbf{Conjecture 1.1} Let \( G \) be a group generated by real reflections acting on a complex vector space \( V \) and preserving a rational structure (more specifically, the Weyl group of some semisimple Lie group acting on its Cartan algebra). Consider the space \( V \oplus V \) equipped with a diagonal action of \( G \), and the corresponding quotient variety \( X := (V \oplus V)/G \). Then \( X \) can be naturally
desingularized, and this desingularization is holomorphically symplectic and admits a hyperkähler structure.

A counterexample to [KV] was produced by A. Kuznetsov in an attempt to prove Conjecture 1.1 for $G$ a Weyl group corresponding to the Dynkin diagrams $C_n$.

It turns out that trianalytic subvarieties of generalized Kummer varieties are deeply related to Dynkin diagrams. Their relation is the main topic of this article.

**Theorem 1.2** Let $X \subset K^{[n]}$ be a trianalytic subvariety of a generalized Kummer variety associated with a generic complex torus of dimension 2. Then

- $X$ is birational to a quotient of a torus $T_1$ by an action of a Weyl group $W$ associated with some reductive Lie group $L$.
- The torus $T_1$ is isogeneous to $T^k$, and $G$ fixes the zero of $T_1$.
- The tangent space $T_0 T_1$ at $0 \in T_1$ is identified with $\mathfrak{t} \oplus \mathfrak{f}$, where $\mathfrak{f}$ is the Cartan algebra of $L$. This identification is compatible with the action of $W$.

For a more precise statement of Theorem 1.2 and its proof, see Theorem 5.6.

The idea of a proof of Theorem 1.2 is based on considering a special type of partial resolutions of singularities, called **Hilbert-type partial resolutions** (Definition 3.2). For an introduction to Hilbert-type partial resolutions, see Section 2.

- In Section 2, we present a less general version of the formalism of partial resolutions of Hilbert type. In this generality, the statement and the proof of our results are elementary. None of the results of Section 2 is used further in this paper. Section 2 is supposed to be an elementary introduction to the theory of partial resolutions of Hilbert type.
- In Section 3, we define the partial resolution of Hilbert type in full generality. We fix the notation and give some preliminary definitions of geometry of Hilbert schemes.
• In Section 4, we state and prove results of geometry of Hilbert-type partial resolutions. We consider a Hilbert-type partial resolution $X = V[\varphi]$ associated with a linear map $\varphi : V \rightarrow W^k$. We show that $X$ is smooth only if the normalizer $G := \text{Norm}_{S_k} V$ is a group generated by complex reflections. If, in addition, the space $W$ is equipped with a generic rational lattice and $V$ is rational, then the group $G$ is generated by real reflections. We prove that $G$ is the Weyl group of a reductive Lie algebra acting on $V$ as on two copies of its Cartan algebra.

• In Section 5, we apply the developed (purely algebro-geometric) formalism of partial resolutions of Hilbert type to questions of hyperkähler geometry. Using results of [KV], we show that any trianalytic subvariety of a generalized Kummer variety is a partial resolution of Hilbert type associated with Hilbert scheme and embeddings of complex tori. This allows us to establish the relationship between trianalytic subvarieties and Dynkin diagrams.

• In Appendix (Section 6), we prove the existence of counterexamples to our earlier (false) statements from [KV]. We show that any deformation of a generalized Kummer variety contains non-trivial complex analytic and trianalytic subvarieties. These subvarieties are constructed as fixed points of some canonical involution; therefore, they are smooth.

2 Partial resolutions of Hilbert type – a simplified version

None of the results of this section are used further on in this paper. Here we sacrifice the generality to give a clear view of the interplay between the groups generated by reflections and the symplectic geometry of the Hilbert schemes.

Let $W$ be a complex vector space, and $W^{(k)} = W^k / S_k$ the symmetric power of $W$, i. e. a quotient of $W^k$ by the natural action of the symmetric group of $k$ letters, and

$$W^k \xrightarrow{\pi} W^{(k)}$$

the Hilbert scheme of $k$ points on $W$. Consider a monomorphism

$$V \hookrightarrow W^k$$
of complex vector spaces. Assume that $\varphi(V)$ intersects non-trivially with

$$W^k_0 := \{(x_1 \neq x_2 \neq \ldots \neq x_k) \in W^k\}$$

(2.1)

of pairwise non-equal $k$-tuples of points. Let $W^{(k)}_0$ be the quotient of $W^k_0$ by the action of $S_k$. Consider the natural map $\sigma : W^k_0 \rightarrow W^{(k)}_0$, and let $V^\sigma_0 \subset W^{(k)}_0$ be the image of $V_0 := \varphi(V) \cap W^k_0$ under the map $\sigma$. Clearly, $V^\sigma_0 = V_0/G$, where $G \subset S_k$ is the normalizer of $V$ in $S_k$. The map $\pi : W^{[k]} \rightarrow W^{(k)}$ induces an isomorphism

$$\pi : \pi^{-1}(W^{(k)}_0) \rightarrow W^{(k)}_0.$$ 

Let $V^\varphi_0 \subset W^{[k]}$ be the preimage $\pi^{-1}(V^\sigma_0) \subset W^{[k]}$, and $V^{[\varphi]}$ be a closure of $V^\varphi_0$ in the Hilbert scheme $W^{[k]}$. Clearly, the map $\pi : W^{[k]} \rightarrow W^{(k)}$ induces a birational isomorphism $\pi : V^{[\varphi]} \rightarrow V/G$.

**Definition 2.1** In the above assumptions, the variety $V^{[\varphi]}$ is called a partial Hilbert-type resolution of $V/G$.

This definition is a weaker form of Definition 3.2. The difference is, in Definition 3.2 we don’t assume that $V$ necessarily intersects with the set $W^k_0$ of pairwise distinct $k$-tuples.

**Theorem 2.2** Let $V \hookrightarrow W^k$ be an embedding of complex vector spaces. Denote the normalizer $\text{Norm}_{S_k}(V)$ by $G$. Assume that the image $\varphi(V)$ is $\text{GL}(W)$-invariant and intersects non-trivially with the set $W^k_0$ of pairwise distinct $k$-tuples (1.1). Consider the partial Hilbert-type resolution $V^{[\varphi]} \subset W^{[k]}$. Assume that the variety $V^{[\varphi]}$ is smooth. Then there exists a $G$-invariant decomposition

$$V = \oplus V_i, \ i = 1, ..., \dim W,$$

such that

(i) all the spaces $V_i$ are isomorphic as representations of $G$, and

(ii) the group $G$ acts on $V_i$ by complex reflections (see Definition 4.1 for the definition of a complex reflection).

**Proof:** Consider a commutative subgroup $K \subset \text{GL}(W)$ of rank $\dim W$ acting on $W$ by characters. In coordinates, such action can be written as

$$(a_1, \ldots a_n)(x_1, \ldots x_n) = (a_1 x_1, a_2 x_2, \ldots a_n x_n).$$
Let $K_i \subset K$, $i = 1, ..., \dim W$ be a codimension 1 subgroup

$$K_i = (a_1, ..., a_{i-1}, 1, a_{i+1}, ..., a_n).$$

The space of invariants of the action of $K_i$ on $W$ is 1-dimensional, and we denote it by $W_i$. Let $V_i$ be the space of invariants of the action of $K_i$ on $\varphi(V)$. Clearly, $V = \oplus V_i$. Moreover, the action of $GL(W)$ on $\varphi(V)$ commutes with the action of $G$ on $\varphi(V)$, and therefore, the decomposition $V = \oplus V_i$ is $G$-invariant. Using an element $\eta \in GL(W)$ which interchanges $K_i$ and $K_j$, we obtain a $G$-equivariant isomorphism $\eta : V_i \to V_j$. This proves Theorem 2.2 (i). It remains to show that the action of $G$ on $V_i$ (say, on $V_1$) is generated by complex reflections. Clearly, $\varphi$ maps $V_1$ to $W_1^k$. Denote this map by $\varphi_1 : V_1 \to W_1^k$. Let $V_1^{[\varphi_1]}$ be the corresponding Hilbert-type partial resolution. Consider the natural embedding $W_1^k \hookrightarrow W^k$ obtained from the functoriality of Hilbert schemes.

**Lemma 2.3** The image $t(V_1^{[\varphi_1]})$ lies in $V^{[\varphi]}$, and, moreover, $t(V_1^{[\varphi_1]})$ coincides with the set

$$\{x \in V^{[\varphi]} \mid K_1(x) = x\}$$

of $K_1$-invariant points in $V^{[\varphi]}$.

**Proof:** Clear from definitions. □

Since $V_1^{[\varphi_1]}$ is a fixed point set of a reductive algebraic group acting on a complex manifold, this variety is smooth. On the other hand, the projection $\pi : W_1^k \to W_1^k$ is an isomorphism, because $\dim W_1 = 1$. This implies that the natural projection $\pi : V_1^{[\varphi_1]} \to V_1/G$ is an isomorphism. Since $V_1^{[\varphi_1]}$ is smooth, $V_1/G$ is also smooth. Now, by [Bor], Ch. V, §5 Theorem 4 (see also Theorem 4.7), the quotient $V_1/G$ is smooth if and only if $G$ acts on $V_1$ by complex reflections. This finishes the proof of Theorem 2.2. □

3 Partial resolutions of Hilbert type

3.1 General definitions

Let $X$ be an irreducible complex variety. To fix the terminology, we introduce the following.

**Definition 3.1** An irreducible complex variety $\tilde{X}$ equipped with a proper morphism $f : \tilde{X} \to X$ is called a partial resolution of the variety $X$ if it is an isomorphism outside of the subset $\text{Sing} X \subset X$ of singular points in $X$. 
Partial resolutions of Hilbert type

D. Kaledin, M. Verbitsky, 12 Dec. 1998

Assume given a smooth complex variety $X$ of dimension $\dim X > 1$. Let $k > 1$ be an integer, and let $X^k = X \times \cdots \times X$ be the $k$-fold self-product of the variety $X$. Denote by $X^{(k)} = X^k/S_k$ the quotient of the variety $X^k$ with respect to the natural action of the symmetric group $S_k$ on $k$ letters. Let $X^{[k]}$ be the Hilbert scheme of 0-dimensional subschemes in $X$ of length $k$. Then we have a canonical proper projection $\pi : X^{[k]} \to X^{(k)}$ which is a partial resolution in the sense of Definition 3.1.

In this paper we will study the following situation. Consider a smooth submanifold $Y \subset X^k$. Denote by $\varphi : Y \to X^k$ the embedding map. Let $G = \text{Norm}(Y) \subset S_k$ be the normalizer subgroup of $Y$ in the symmetric group $S_k$, that is, the subgroup of elements $s \in S_k$ such that $s(Y) = Y \subset X^k$. Consider the quotient variety $Y/G$. The embedding map $\varphi : Y \hookrightarrow X^k$ defines a natural closed embedding $Y/G \to X^{(k)} = X^k/S_k$.

Let $(Y/G)^{[\varphi]} \subset X^{[k]}$ be a closed subvariety such that the canonical projection $\pi : X^{[k]} \to X^{(k)}$ maps $(Y/G)^{[\varphi]} \subset X^{[k]}$ onto $Y/G \subset X^{(k)}$.

**Definition 3.2** If the induced map $\pi : (Y/G)^{[\varphi]} \to Y/G$ is a partial resolution in the sense of Definition 3.1, then it is called a partial resolution of Hilbert type.

Note that a partial resolution of Hilbert type $(Y/G)^{[\varphi]}$ depends not only on the complex variety $Y$ equipped with an action of the group $G$, but also on the embedding $\varphi : Y \hookrightarrow X^k$. Moreover, it depends on the particular choice of the closed subset $(Y/G)^{[\varphi]} \subset X^{[k]}$ lying over $Y/G \subset X^{(k)}$. If the subset $Y/G \subset X^{(k)}$ intersects the open subset

$$X_0^{(k)} = \left\{ (x_1, \ldots, x_k) \in X^{(k)} | x_1 \neq \cdots \neq x_k \right\} \subset X^{(k)},$$

then this latter choice is unique, since the projection $\pi : X^{[k]} \to X^{(k)}$ is bijective over $X_0^{(k)} \subset X^{(k)}$.

**Definition 3.2** in full generality is probably useless. In this paper we will consider only two particular cases, where this definition leads to interesting results. These cases are

(i) $X = W$ is a complex vector space, $Y = V \subset W^k$ is a linear subspace, $\varphi : V \hookrightarrow W^k$ is the given embedding, or, more generally, an affine embedding parallel to the given one.

(ii) $X = T$ is a complex torus, $Y \subset T^k$ is a subtorus, $\varphi : Y \hookrightarrow T^k$ is the natural embedding.
Note that Definition 3.2 for vector spaces generalizes the notion of a Hilbert resolution introduced by Y. Ito and H. Nakajima in [IN]. In this case, we have \( X = Y = V \) for some vector space \( V \). The finite group \( G \) acting on \( V \) is given \( a \) priori. The integer \( k = \text{Card}(G) \) equals the number of elements in \( G \), and the embedding \( V \hookrightarrow V^k = V^{\text{Card}(G)} \) coincides with the coaction \( V \to V \otimes \mathbb{C}(G) = V^{\text{Card}(G)} \) on the space \( V \) of the coalgebra \( \mathbb{C}(G) \) of functions on the group \( G \). It is easy to check that the given group \( G \) coincides with the normalizer subgroup of the subspace \( V \subset V^k = V^{\text{Card}(G)} \) in the symmetric group on \( \text{Card} G \) letters.

In this paper we will be interested in another case, namely, the case where \( X \) is a vector space \( W \) of dimension 2, while the subvariety \( Y \subset X^k \) is a vector subspace \( V \subset W^k \) of dimension higher than 2. Starting with Section 4, \( W = \mathbb{C}^2 \) will always denote the standard complex vector space of dimension 2.

### 3.2 Local charts

We will also need a local version of the case (i) of Definition 3.2, where a vector space \( W = X \) is replaced with an open neighborhood \( U \subset W \) of \( 0 \in W \), and \( V \subset W^k \) is replaced with \( V \cap W^k \subset W^k \). This will be important, since every point \( x \in X^k \) the Hilbert scheme \( X^k \) for a general smooth variety \( X \) of dimension \( n \) has an open neighborhood which is a product of Hilbert schemes for a coordinate neighborhood \( U \subset \mathbb{C}^n \). We will now describe this in some detail.

Let \( x \in X^k \) be such a point, and assume that the corresponding 0-dimensional subscheme \( Z \subset X \) is supported on a finite subset \( \{x_1, \ldots, x_l\} \subset X \) of \( l \) distinct points in \( X \). Further, assume that the part of \( Z \subset x \) which is supported on \( x_i \) has length \( a_i \). The numbers \( a_1, \ldots, a_l \) form a partition \( \Delta = \{a_1, \ldots, a_l\} \)

of the integer \( k \) (which we define as a set of numbers \( a_1, \ldots, a_l \) such that \( \sum a_i = k \)). For every one of the points \( x_i \), choose a coordinate neighborhood \( U_i \subset X \) and an identification \( h_i : U \cong U_i \) between \( U_i \) and a fixed open neighborhood \( U \subset \mathbb{C}^n \) of \( 0 \in \mathbb{C}^n \).

Assume that if \( j \neq i \) then \( U_i \cap U_j = \emptyset \). Consider the open subset \( U_x \subset X^k \) of subschemes \( Z \subset X \) supported on \( \bigcup U_i \) in such a way that \( Z \cap U_i \) is of length \( a_i \). Then we have

\[
U_x \cong U_i^{[a_1]} \times \cdots \times U_i^{[a_l]},
\]
and the local isomorphisms $h_i : U \cong U_i$ identify $U_x$ with the product

$$U|^\Delta| = U[^{a_1}] \times \cdots \times U[^{a_l}].$$

We will say that the open subset $U[^\Delta] \cong U_x \subset X[^k]$ is a local chart in $X[^k]$ near the point $x \in X[^k]$.

Note that under the canonical projection $\pi : X[^k] \to X(^k)$, the local chart $U_x \cong U[^\Delta]$ is mapped onto an open subset $U_{\pi(x)} \subset X(^k)$ in the symmetric power $X(^k)$. This subset depends only on $\pi(x) \in X(^k)$ and not on the particular choice of $x \in \pi^{-1}(\pi(x))$. Moreover, we have $U_x = \pi^{-1}(U_{\pi(x)}) \subset X[^k]$. We will say that the open subset $U_{\pi(x)} \subset X(^k)$ is a local chart in $X(^k)$ near the point $\pi(x) \in X(^k)$.

Let $S_\Delta = S_{a_1} \times \cdots \times S_{a_l} \subset S_\Delta$ be the subgroup in the symmetric group associated to the partition $\Delta = \{a_1, \ldots, a_l\}$ of the integer $k$. Then the local chart $U_{\pi(x)} \subset X(^k)$ is canonically isomorphic to the quotient variety $U(\Delta) = U^k/S_\Delta = U(a_1) \times \cdots \times U(a_l)$.

We will use these local charts to study locally partial resolutions of Hilbert type. To do this, we will need a slight generalization of Definition 3.2. We will formulate it for a general smooth manifold $X$, but we will use it only for $X = \mathbb{C}^n$ and for an open neighborhood $X = U \subset \mathbb{C}^n$ of $0 \subset \mathbb{C}^n$.

Fix a partition $\Delta = \{a_1, \ldots, a_l\}$ of the integer $k$. Let $X[^\Delta] = X[^{a_1}] \times \cdots \times X[^{a_l}]$ be the product of Hilbert schemes of subschemes in $X$ of lengths $a_1, \ldots, a_l$. The product of the natural maps $X[^{a_i}] \to X(^{a_i})$ gives a natural partial resolution $\pi : X[^\Delta] \to X(^\Delta)$.

Let $Y \subset X^k$ be a closed submanifold, let $\varphi : X \hookrightarrow X^k$ be the embedding map, and let $G = \text{Norm}Y \subset S_\Delta$ be the normalizer subgroup of $Y$ in the group $S_\Delta$. Then we have a canonical embedding $\varphi : Y/G \to X(^\Delta)$.

**Definition 3.3** A closed subvariety $(Y/G)^[^\varphi] \subset X[^\Delta]$ is called an extended partial resolution of Hilbert type of the quotient $Y/G \subset X(^\Delta)$ if the canonical projection $\pi : X[^\Delta] \to X(^\Delta)$ maps $(Y/G)^[^\varphi] \subset X[^\Delta]$ onto $Y/G \subset X(^\Delta)$, and the induced map $\pi : (Y/G)^[^\varphi] \to Y/G$ is a partial resolution in the sense of Definition 3.2.

This definition gives a local counterpart to Definition 3.2 in the following sense. Let $Y \subset X^k$ be a smooth subvariety with normalizer subgroup $G = \text{Norm}Y \subset S_k$, and let $(Y/G)^[^\varphi] \subset X[^k]$ be a partial resolution of Hilbert type in the sense of Definition 3.2. Choose a point $x \in Y$, and let $G_0 \subset G$ be the subgroup of elements in $G$ fixing the point $x \in Y$. The point $y \in Y \subset X^k$ defines a point $\tilde{x} \in Y/G \subset X(^k)$ in the quotient variety $X(^k) = X^k/S_k$. 


Let \( U(\Delta) \subset X^{(k)} \) be a local chart in \( X^{(k)} \) near the point \( \tilde{x} \in X^{[k]} \), and let \( U[\Delta] = \pi^{-1}(U(\Delta)) \subset X^{[k]} \) be the corresponding local chart in \( X^{[k]} \). Then we have

\[
Y/G \cap U(\Delta) = (Y \cap U^k)/G_0 \subset X^{(k)},
\]

and the intersection \( U[\Delta] \cap (Y/G)[\varphi] \subset U[\Delta] \) is an extended partial resolution of Hilbert type of the quotient \((Y \cap U^k)/G_0\) in the sense of Definition 3.3.

### 3.3 Lattices and partial resolutions associated with complex tori

**Definition 3.4** A lattice \( L \) in a complex vector space \( V \) is a \( \mathbb{Z} \)-submodule \( L \subset V \) such that the induced map \( L \otimes \mathbb{R} \to V \) is an isomorphism. A rational lattice \( L \otimes \mathbb{Q} \subset V \) is a \( \mathbb{Q} \)-vector subspace such that \( L \otimes \mathbb{Q} \otimes \mathbb{R} \cong V \).

For every lattice \( L \subset V \) in a complex vector space \( V \), the tensor product \( L \otimes \mathbb{Q} \subset V \) is a rational lattice. Two lattices \( L, L' \subset V \) are called isogenic if \( L \otimes \mathbb{Q} = L' \otimes \mathbb{Q} \subset V \). We will say that a subspace \( V' \subset V \) in a complex vector space \( V \) compatible with a lattice \( L \subset V \) in \( V \) if the intersection \( L' = L \cap V' \subset V' \) is a lattice in \( V' \).

**Definition 3.5** A lattice, resp. a rational lattice \( L \subset V \) in a complex vector space \( V \) is called generic if every endomorphism of the space \( V \) preserving \( L \subset V \) is a multiplication by an integer, resp. a rational number.

Pairs \( \langle V, L \otimes \mathbb{Q} \subset V \rangle \) of a complex vector space \( V \) and a rational lattice \( L \otimes \mathbb{Q} \subset V \) form a semisimple \( \mathbb{Q} \)-linear abelian category. (In fact, it is equivalent to the category of pure \( \mathbb{Q} \)-Hodge structures of weight 1 with only non-trivial Hodge numbers \( h^{1,0} \) and \( h^{0,1} \).) Note that if a rational lattice \( L \subset V \) in a vector space \( V \) is generic, then the pair \( \langle V, L \otimes \mathbb{Q} \rangle \) is an irreducible object.

Let \( T \) be a complex torus. The torus \( T \) is isomorphic to the quotient \( T = W/L \), where \( W = \Gamma(T, T(T)) \) is the space of global holomorphic sections of the holomorphic tangent bundle \( T(T) \) to the torus \( T \), and \( L \subset W \) is a lattice in the complex vector space \( W \).

Let \( T' \subset T^k \) be a subtorus in the \( k \)-fold self-product \( T^k = W^k/L^k \). Let \( T' = V/L' \), where \( V \) is a complex vector space and \( L' \subset V' \) is a lattice in \( V' \). The embedding \( \varphi : T' \to T^k \) defines a canonical linear embedding \( \varphi : V' \hookrightarrow W^k \). The subspace \( V' \subset W^k \) is compatible with the lattice \( L^k \subset W^k \), and the lattice \( L' \) is the intersection \( L' = V' \cap L^k \subset V' \).

The tangent bundle \( T(T) \) to the torus \( T = W/L \) is trivial, and one can choose a frame consisting of commuting vector fields. For every point \( t \in T \),
the associated exponential map defines a holomorphic covering \( W \to T \). Consequently, every sufficiently small open neighborhood \( U \subset W \) of \( 0 \subset W \) defines a canonical coordinate neighborhood \( U \cong U_t \subset T \). We will call this neighborhood flat. The \( k \)-fold product \( U_k \cong U_{t_1} \times \cdots \times U_{t_k} \subset T^k \) of flat coordinate neighborhoods is flat, and the intersection \( U_k \cap T' = V/L' \subset T^k \) with the subtorus \( T' = V/L' \subset T^k \) coincides with the intersection \( U_k \cap V \subset W^k \) with the linear subspace \( V \subset W^k \). The local charts near a point in \( T^{(k)} \) or \( T[^k] \) associated to flat coordinate neighborhoods will also be called flat.

Let \( G = \text{Norm} T' \subset S_k \) be the normalizer subgroup of the subtorus \( T' \subset T^k \) in the symmetric group \( S_k \) acting on \( T^k \). Consider the corresponding subvariety \( T'/G \subset T^{(k)} \) in the symmetric power \( T^{(k)} \). Let \( T[^k] \) be the Hilbert scheme of 0-dimensional subschemes of length \( k \) in \( T \), and let \( (T'/G)^{[\omega]} \subset T[^k] \) be a partial resolution of Hilbert type of \( T'/G \subset T^{(k)} \) in the sense of Definition 3.2.

Consider a point \( t \in T' \subset T^k \) in the subtorus \( T' \subset T \), and denote by \( G_0 \subset G \) the subgroup of elements in \( G = \text{Norm} T' \subset S_k \) fixing \( t \in T' \). Let \( \tilde{t} \in T'/G \subset T^{(k)} \) be the associated point in the quotient \( t^{(k)} \), and assume that \( U^{(\Delta)} \subset T^{(k)} \) is a flat local chart in \( T^{(k)} \) near the point \( \tilde{t} \in T^{(k)} \). Then the intersection \( T'/G \cap U^{(\Delta)} \) coincides with the quotient \( U_0/G_0 \) for an open neighborhood \( U_0 = U_k \cap V \subset V \) of \( 0 \in V \), and the intersection \( U^{[\Delta]} \cap (T'/G)^{[\omega]} \) provides an extended partial resolution of Hilbert type for this quotient.

4 Partial resolutions of Hilbert type and groups generated by reflections

Definition 4.1 Let \( V \) be a complex vector space. Recall that an automorphism \( g : V \to V \) of the vector space \( V \) is called a complex reflection if it is of finite order and the subspace of invariants \( V^g \subset V \) is of codimension exactly 1. A complex reflection \( g : V \to V \) is called real if it is an automorphism of order 2 (equivalently, if \( g \) preserves a real structure on the complex vector space \( V \)).

Finite groups \( G \subset \text{Aut} V \) of automorphisms of a complex vector space \( V \) which are generated by complex reflections have been an object of much study, and there exists a classification of pairs \( \langle V, G \rangle \) of this type. In particular, a subgroup \( G \subset \text{Aut} V \) is generated by real reflections if and only if it is a product of Weyl groups associated to Dynkin diagrams of finite type, assuming that \( G \) preserves some rational lattice in \( V \) (see [Bou], Ch. VI, §2,
Let now $W$ be the standard 2-dimensional complex vector space equipped with the canonical action of the group $U(2)$. Let $V \subset W^k$ be a vector subspace, and let $G = \text{Norm}(V) \subset S_k$ be the normalizer subgroup of the subspace $V \subset W^k$. Let $\bar{G} \subset \text{Aut } V$ be the quotient of the group $G$ by the subgroup $G_0 \subset G$ of elements which act trivially on the vector space $V$.

The standard vector space $W$ carries a natural action $\tau : U(2) \to \text{Aut } W$ of the group $W(2)$ of unitary $2 \times 2$-matrices. This action defines $W(2)$-actions on the vector space $W^k$, on the symmetric power $W^{(k)}$ and on the Hilbert scheme $W[k]$.

Assume that the subspace $V \subset W^k$ is preserved by the $U(2)$-action. Choose once and for all a subgroup $U(1) \subset U(2)$ such that the space of invariants $W^{U(1)} \subset W$ is of dimension 1. Then the subspace of invariants $V^{U(1)} \subset V$ is of dimension one half of the dimension of the vector space $V$, and we have a canonical $U(2)$-equivariant isomorphism $V \cong W \otimes C V^{U(1)}$. (The $U(2)$-action on the right-hand side is induced by the standard action $\tau$ on the space $W$.)

Since the action of symmetric group $S_k$ on the space $W^k$ commutes with the $U(2)$-action, the $G$-action on $V \subset W^k$ is also $U(2)$-equivariant. Therefore the group $\bar{G}$ preserves the subspace $V^{U(1)} \subset V$. If we equip the space $W$ with the trivial $G$-action, then the canonical isomorphism $V \cong W \otimes V^{U(1)}$ is $G$-equivariant.

In this section we prove the following two results.

**Theorem 4.2** Let $(V/G)[\varphi] \subset W[k]$ be a partial resolution of Hilbert type of the quotient $V/G$ in the sense of Definition 3.2. Assume that the closed subvariety $(V/G)[\varphi] \subset W[k]$ is smooth and invariant under the canonical $U(2)$-action on the Hilbert scheme $W[k]$. Then the group $\bar{G} \subset \text{Aut } V$ acting on the vector space $V^{U(1)}$ is generated by complex reflections.

**Remark 4.3** The condition of $U(2)$-invariance imposed on a partial resolution of Hilbert type $(V/G)[\varphi] \subset W[k]$ seems to be quite non-trivial. Note, however, that it holds automatically when the vector subspace $V \subset W^k$ is $U(2)$-invariant and intersects non-trivially with the subset $W_0^k \subset W^k$ consisting of $k$-tuples of distinct elements in $W$. This is the simplified case that we have considered in Section 3.

**Theorem 4.4** In the assumptions of Theorem 4.3, assume additionally that the subspace $V \subset W^k$ is compatible with a lattice $L \subset W$ which is generic in
the sense of Definition 3.5. Then the subgroup \( \tilde{G} \subset \text{Aut} V^{U(1)} \) is generated by real reflections.

**Remark 4.5** From Theorem 4.2 it follows that the action of \( \tilde{G} \) is generated by complex reflections.

Before we prove Theorem 4.2, we need to recall a fact on the \( U(2) \)-action on the Hilbert scheme \( W^k \). Let \( W^k = W^{(k)} \) be the symmetric power of the space \( W \), and let \( \pi : W^k \to W^{(k)} \) be the canonical projection. The projection \( \pi \) is compatible with the natural \( U(2) \)-actions on \( W^k \) and \( W^{(k)} \). Recall that we have fixed a subgroup \( U(1) \subset U(2) \) such that \( W^U(1) \subset W \) is of dimension 1. For every point \( u \in W^{(k)} \) let \( F_u = \pi^{-1}(u) \subset W^k \) be the fiber of the map \( \pi \) over the point \( u \). (Any such fiber is a product of so-called punctual Hilbert schemes.) We will need the following.

**Lemma 4.6** (see also [N, Section 5]) Let \( u \in W^{(k)} \) be a point invariant under the action of the fixed subgroup \( U(1) \subset U(2) \). Then the closed subvariety \( F^U(1) \subset F_u \) of points in the fiber \( F_u \subset W^k \) invariant under the \( U(1) \)-action consist of a finite number of points.

**Proof.** Choose a basis \( x, y \in W^* \) in the space of linear functions on the vector space \( W \) compatible with the \( U(1) \)-action, so that \( x \) is \( U(1) \)-invariant and \( y \) satisfies \( \lambda \cdot y = \lambda y \) for every \( \lambda \in U(1) \subset \mathbb{C} \). Let \( \mathcal{Z} \subset W \) be a 0-dimensional subscheme of length \( k \) in \( W \) which is supported at \( 0 \subset W \) and invariant under the \( U(1) \)-action. Then the \( U(1) \)-action induces a grading

\[
\mathcal{O} = \bigoplus_i \mathcal{O}_i
\]

on the space \( \mathcal{O} = \mathcal{O}_\mathcal{Z} \) of functions on the scheme \( \mathcal{Z} \), defined by

\[
\lambda \cdot f = \lambda^i f, \quad f \in \mathcal{O}_i, \lambda \in U(1) \subset \mathbb{C}.
\]

Multiplication by \( x \) preserves each of the subspaces \( \mathcal{O}_i = \mathcal{O} \). Moreover, since the subscheme \( \mathcal{Z} \subset W \) is supported at \( 0 \), multiplication by \( x \) is nilpotent. Let \( n_i \) be the smallest positive integer such that \( x^{n_i} \mathcal{O}_i = 0 \). We obviously have \( n_i \leq \dim \mathcal{O}_i \).

For every \( i \), we have \( x^{n_i} y^i \mathcal{O} = 0 \). Indeed, it suffices to prove that \( x^{n_i} y^i \cdot 1 = 0 \), where \( 1 \subset \mathcal{O} \) is the unity. But this follows from definition, since \( 1 \in \mathcal{O}_0 \) and \( y^i \cdot 1 \in \mathcal{O}_i \). Therefore the ideal

\[
I = \langle x^{n_i} y^i \rangle \subset \mathbb{C}[x, y]
\]

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in the polynomial algebra $\mathbb{C}[x, y]$ generated by monomials $x^{n_i} y^{i}$ annules the $\mathbb{C}[x, y]$-module $\mathcal{O}$.

On the other hand, since $n_i \leq \dim \mathcal{O}_i$, we have

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = \sum_i n_i \leq \sum_i \dim_{\mathbb{C}} \mathcal{O}_i = \dim_{\mathbb{C}} \mathcal{O} = k.$$ 

Therefore the surjective map $\mathbb{C}[x, y]/I \rightarrow \mathcal{O}$ defined by $f \mapsto f \cdot 1$ is in fact bijective, and the subscheme $Z = \text{Spec} \mathbb{C}[x, y]/I \subset W$ is uniquely defined by the set of natural numbers $n_i$ satisfying

$$\sum_i n_i = k.$$ 

There exists only a finite number of such sets. □

Proof of Theorem 4.2. Let $V_0 = V^{U(1)} \subset V$ be the subspace of $U(1)$-invariant vectors. The subset $(V/G)^{U(1)} \subset V/G \subset W^{(k)}$ of points invariant under the $U(1)$-action obviously coincides with $V_0/G \subset V/G$.

Consider the closed subvariety

$$D = \left( (V/G)^{[\varphi]} \right)^{U(1)} \subset (V/G)^{[\varphi]}$$

of $U(1)$-invariant points in the partial resolution $(V/G)^{[\varphi]} \subset W^{[k]}$. Since the variety $(V/G)^{[\varphi]}$ is smooth, while the group $U(1)$ is compact, the subvariety $D \subset (V/G)^{[\varphi]}$ is a union of a finite number of smooth connected components $D_0, \ldots, D_i$.

The projection $\pi : (V/G)^{[\varphi]} \rightarrow V/G$ is $U(2)$-equivariant, therefore it maps the subset $D$ into the subset $V_0/G \subset V/G$ of $U(1)$-invariant points in the quotient $V/G$. Moreover, by Definition 3.1 the projection $\pi$ is one-to-one over non-singular points in $V/G$. Since the generic point of the subvariety $V_0/G \subset V/G$ is non-singular in $V/G$, there exists one and only one connected component of the variety $D$, say, $D_0$, which maps onto $V_0/G \subset V/G$ in such a way that the map $\pi : D_0 \rightarrow V_0/G$ is generically one-to-one.

But by Lemma 4.6 the projection $\pi : D_0 \rightarrow V_0/G$ is finite. Since the variety $D_0$ is normal, the finite dominant projection $\pi : D_0 \rightarrow V_0/G$ is the normalization of the variety $V_0/G$. However, the variety $V_0/G$, being a quotient of a vector space by a finite group action, is itself normal. Therefore $\pi$ is an isomorphism between $D_0$ and $V_0/G$. In particular, the quotient $V_0/G$ is smooth. To finish the proof of Theorem 4.2, it remains to invoke the following classic result.
Theorem 4.7 ([Bou], Ch. V, §5, Theorem 4) The quotient $V/G$ of a complex vector space $V$ by a finite subgroup $G \subset \text{Aut} V$ is smooth if and only if the subgroup $G \subset \text{Aut} V$ is generated by complex reflections. □

Proof of Theorem 4.4. By assumption the intersection $L' = V \cap L^k \subset W^k$ is a lattice in the vector space $V \subset W^k$. By Theorem 4.2, it suffices to prove the following:

- Any element $g \in G$ which acts as a complex reflection on the space $V_0 = V^U(1) \subset V$ and preserves the lattice $L' \subset V$ is in fact a real reflection.

Given such an element $g$, let $V^g \subset V$ be the subspace of $g$-invariant vectors in $V$, and let $V_0^g \subset V_0$ be the subspace of $g$-invariant vectors on $V_0$. Since $V \cong V_0 \otimes W$, we have $V/V^g \cong W \otimes (V_0/V_0^g)$. By assumption $V_0/V_0^g$ is 1-dimensional, therefore the quotient $V/V^g$ is of complex dimension 2. The element $g$ acts naturally on the quotient $W \cong V/V^g$, and it suffices to prove that it acts as an automorphism of order 2.

Consider the rational lattice $L' \otimes \mathbb{Q} \subset V'$. The category of complex vector spaces equipped with a rational lattice is abelian and semisimple. Since the element $g$ preserves the lattice $L' \subset W$, the quotient $V/V^g$ is equipped with a canonical quotient rational lattice $L_0 \subset V/V^g$. Moreover, by assumption the lattice $L \subset W$ is generic in the sense of Definition 3.5. Consequently, the object $\langle W, L \otimes \mathbb{Q} \rangle$ is irreducible. Since $\langle W^k, L^k \otimes \mathbb{Q} \rangle$ is a sum of several copies of the irreducible object $\langle W, L \otimes \mathbb{Q} \rangle$, the subobject $\langle V', L' \otimes \mathbb{Q} \rangle$ is also a sum of several copies of $\langle W, L \otimes \mathbb{Q} \rangle$, and so is the quotient object $\langle V/V^g, L_0 \rangle$. Since the vector space $V/V^g$ is 2-dimensional, this implies that

$$\langle V/V^g, L_0 \rangle \cong \langle W, L \otimes \mathbb{Q} \rangle.$$ 

Thus the element $g$ is an automorphism of the irreducible object $\langle W, L \otimes \mathbb{Q} \rangle$. By Definition 3.5 it must act as multiplication by an invertible rational number, in other words, by $\pm 1$. Since $g$ is non-trivial, it acts as the multiplication by $-1$. This finishes the proof. □

To simplify notation, we have formulated Theorems 4.2 and 4.4 for submanifolds of the Hilbert scheme $W^{[k]}$. However, the same results hold for extended partial resolutions of Hilbert type (Definition 3.3), which by definition lie in the product of Hilbert schemes $W^{[\Delta]}$ associated to a partition $\Delta = \{a_1, \ldots, a_l\}$ of the integer $k$. To see this, it suffices to note that the fibers of the canonical projection $\pi : W^{[\Delta]} \to W^{(\Delta)}$ are products of the fibers
of the projections $\pi : W^{[a_i]} \to W^{(a_i)}$, $i = 1, \ldots, l$. Therefore Lemma 4.6 is also valid for $W^{[\Delta]}$. Consequently, the formulations and the proofs of Theorems 1.2 and 4.4 carry over to the case of extended resolutions word-by-word. Moreover, both theorems hold (with the same proofs) even if we are only given an extended partial resolution for the quotient $(V \cap U^k)/G$, where $U \subset W$ is an open neighborhood of $0 \subset W$ invariant under the standard $U(2)$-action.

5 Trianalytic subvarieties of compact tori: an erratum

In our previous paper [KV], a grave error was found. In the paragraph 2.3 (page 459 of [KV]), we say “The right-hand side obviously depends continuously on the point $a \in F_\Delta$”. This is, unfortunately, not true. So, even if the rest of the arguments is valid, the proof of the main theorem is wrong. As we show in the Appendix, the main theorem of [KV] is false: a generic deformation of the Hilbert scheme of a compact 2-dimensional complex torus contains non-trivial trianalytic subvarieties.

However, while Section 7 of [KV] is false, Sections 1-6 are correct. In this erratum we use the correct results of [KV] and some additional arguments to prove a weaker version of the main theorem of [KV] (Theorem 5.6). Before we formulate this result, we need to recall some facts from hyperkähler geometry.

For more details on the false Theorem of [KV] and its counterexample, see Remark 5.7 below and the Appendix to this paper.

5.1 Preliminaries on hyperkähler geometry

Definition 5.1 A hyperkähler manifold is a Riemannian manifold $M$ equipped with a unitary action $\tau : \mathbb{H} \to \text{End}T(M)$ of the algebra $\mathbb{H}$ of the quaternions in the tangent bundle $T(M)$ such that the action $\tau$ is parallel with respect to the Levi-Civita connection.

Here unitary means that for every quaternion $a \in \mathbb{H}$ and every two tangent vectors $t_1, t_2$ we have

$$(\tau(a)t_1, t_2) = (t_1, \tau(\overline{a})t_2),$$

where $\overline{a} \in \mathbb{H}$ is the quaternion, conjugate to $a$. 
Let $X$ be a hyperkähler manifold. The quaternionic action on the tangent bundle to $X$ defines an action of the Lie algebra $\mathfrak{su}(2)$ on the de Rham complex of the manifold $X$. By [V1] this action commutes with the Laplacian and defines therefore a canonical $\mathfrak{su}(2)$-action on the cohomology spaces $H^*(X, \mathbb{C})$.

Every algebra embedding $I : \mathbb{C} \hookrightarrow \mathbb{H}$ defines by restriction an almost complex structure $X_I$ on the manifold $X$. Since the almost complex structure $X_I$ is parallel with respect to the Levi-Civita metric, it is integrable, and the metric on $X$ is Kähler with respect to the complex structure $X_I$. The complex structure $X_I$ will be referred to as the induced complex structure on $X$ associated to the embedding $I$. The corresponding Kähler form will be denoted by $\omega_I$. The set of algebra embeddings $\mathbb{C} \hookrightarrow \mathbb{H}$ can be naturally identified with the complex projective line $\mathbb{C}P^1$.

**Definition 5.2** A closed subset $Y \subset X$ in a hyperkähler manifold $X$ is called trianalytic if it is complex analytic for every induced complex structure $X_I$ on $X$.

It is proved in [V2] that a subset $Y \subset X_I$ analytic in an induced complex structure $X_I$ is trianalytic if and only if its fundamental cohomology class is invariant under the canonical $\mathfrak{su}(2)$-action.

This motivates the following definition.

**Definition 5.3** Let $X$ be a compact complex manifold admitting a hyperkähler structure $\mathcal{H}$. We say that $X$ is generic with respect to $\mathcal{H}$ if all elements of the group

$$\bigoplus_p H^{p,p}(X) \cap H^{2p}(X, \mathbb{Z}) \subset H^*(X)$$

are $SU(2)$-invariant.

For every hyperkähler structure, there is at most a countable set of induced complex structures which are not generic (see, e.g., [V2]).

**Definition 5.4** Let $X$ be a compact hyperkähler manifold. An induced complex structure $I$ on $X$ is called Mumford-Tate generic with respect to the hyperkähler structure if for all $n > 0$, the complex manifold $(X_I)^n$ is generic with respect to the hyperkähler structure.

Fix the standard basis $i, j, k \in \mathbb{H}$ in the space of imaginary quaternions and consider the associated algebra embeddings $I, J, K : \mathbb{C} \rightarrow \mathbb{H}$. It is easy to check that the form $\Omega = \omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2, 0)$ with respect...
Unfortunately, the text is not fully visible due to the partial resolution. However, it seems to be discussing Hilbert schemes and hyperkähler manifolds. Here is a guess at the content:

- The text mentions the complex structure $X_I$ on $X$. Since the form $\Omega$ is closed, it is holomorphic. Thus every hyperkähler manifold $X$ is canonically holomorphically symplectic.

- For compact manifolds, the converse is also true. Namely, we have the following corollary of Yau’s proof of Calabi conjecture ([Y]).

**Theorem 5.5 (Beau, Bes)** Let $(X, \Omega)$ be a compact complex manifold equipped with a holomorphic symplectic form $\Omega$, and let $\omega$ be an arbitrary Kähler form on $X$. Then there exists a unique hyperkähler metric on $X$ with the same Kähler class as $\omega$. □

Every complex torus $T$ of even dimension is holomorphically symplectic and admits a Kähler metric. Therefore every such torus admits a hyperkähler structure.

Let $X$ be a compact hyperkähler manifold of complex dimension 2 (for example, a 2-dimensional complex torus). Then the Hilbert scheme $X^{[k]}$ of 0-dimensional subschemes of length $k$ in $X$ is a smooth compact complex manifold. It is a manifold of Kähler type, and it carries a canonical holomorphic symplectic form $\Omega$ (see [N]). Therefore $X^{[k]}$ admits a hyperkähler structure (non-canonical, since there is no canonical Kähler metric on $X^{[k]}$).

We can now formulate our main result. Let $T = W/L$ be a 2-dimensional complex torus equipped with a holomorphic symplectic form $\Omega$ and a hyperkähler metric compatible with $\Omega$. Assume that the lattice $L \subset W$ in the standard 2-dimensional complex vector space $W$ is generic in the sense of Definition 3.5. Assume also that the torus $T$ is Mumford-Tate generic with respect to some hyperkähler structure (Definition 5.4). We say that a submanifold $Y \subset X$ in a complex manifold $X$ is rigid if it admits no non-trivial deformations within $Y$.

**Theorem 5.6** Consider a hyperkähler structure on the Hilbert scheme $T^{[k]}$ compatible with the canonical holomorphic symplectic form. Let $X \subset T^{[k]}$ be an irreducible rigid trianalytic submanifold, and let $\pi(X) \subset T^{(k)}$ be its image under the canonical projection $T^{[k]} \to T^{(k)}$.

(i) The projection $\pi : X \to \pi(X)$ is one-to-one over the subset of non-singular points in $\pi(X) \subset T^{(k)}$.

(ii) The complex variety $\pi(X)$ is isomorphic to the quotient $T'/G$ of a complex torus $T'$ by a finite group $G$. The torus $T'$ is isogenic to a subtorus in the power $T^k$ of the torus $T$. The group $G$ is generated by real reflexions in a complex vector space $V_0$, and the $G$-module $V = \ldots$
\( \Gamma(T', T(T)) \) is isomorphic to the sum \( V = V_0 \oplus V_0 \) of two copies of the \( G \)-module \( V_0 \).

Note that by virtue of Section 3 of \([KV]\), for every trianalytic submanifold \( X \subset T^{[k]} \) there exists a hyperkähler manifold \( S \) and a rigid trianalytic submanifold \( \tilde{X} \subset T^{[k]} \) whose normalization is isometric to \( X \times S \). Therefore the study of trianalytic submanifolds of \( T^{[k]} \) essentially reduces to the study of rigid trianalytic submanifolds.

Complex tori satisfying conditions of Theorem 5.6 are indeed generic in the usual sense, that is, they form a dense subset in the appropriate moduli space, and the complement to this subset is a countable union of closed analytic subsets of codimension 1 (\([V2]\)).

**Remark 5.7** The (false) main theorem of \([KV]\) claimed that the only proper trianalytic subvarieties of the Hilbert scheme \( T^{[k]} \) are the fibers of the summation map \( T^{[k]} \to T \). A very simple counterexample to this statement is constructed as follows. Let \( \iota : T \to T \) be the involution \( t \mapsto -t \) of the torus \( T \). Consider the induced involution \( \iota : T^{[k]} \to T^{[k]} \) of the Hilbert scheme \( T^{[k]} \).

The fixed point set \( (T^{[k]})^\iota \subset T^{[k]} \) is a union of smooth connected components. Moreover, the involution \( \iota \) preserves the holomorphic symplectic form on \( T^{[k]} \) and the cohomology class of the Kähler form. Hence by the uniqueness statement of Theorem 5.5 it preserves the hyperkähler structure on \( T^{[k]} \). Therefore each of the connected components of the fixed point set \( (T^{[k]})^\iota \subset T^{[k]} \) is a trianalytic subvariety in \( T^{[k]} \).

See the Appendix to this paper for precise formulations, proofs and extensions.

### 5.2 Stratification by diagonals

Before we begin the proof of Theorem 5.6, we need to recall some facts on the stratification by diagonals of the Hilbert scheme \( M^{[k]} \) of a smooth complex manifold \( M \) (this was the subject of Sections 4 and 5 of \([KV]\)).

Let \( \Delta = \{a_1, \ldots, a_l\}, \sum a_i = k \) be a partition of the integer \( k \). Denote by \( M^{[k]}_\Delta \subset M^{[k]} \) the subset of points \( m \in M^{[k]} \) such that the associated 0-dimensional subscheme \( Z_m \subset M \) in \( M \) is supported on a finite subset \( \{m_1, \ldots, m_l\} \subset M \) of \( l \) distinct points in \( M \), and the part of \( Z_m \) which is supported on \( m_i \) has length \( a_i \). Moreover, denote by \( M^{(k)}_\Delta \subset M^{(k)} \) the subset of points \( m \in M^{(k)} \) such that the associated subset of \( k \) unordered
points in $M$ consists of $l$ distinct points in $M$, and the point $m_i$ appears with multiplicity $a_i$.

The subsets $M^{[k]}_\Delta \subset M^{[k]}$ and $M^{(k)}_\Delta \subset M^{(k)}$ are locally closed and form stratifications of varieties $M^{[k]}$ and $M^{(k)}$. Moreover, we have

$$M^{[k]}_\Delta = \pi^{-1} \left( M^{(k)}_\Delta \right) \subset M^{[k]},$$

where $\pi : M^{[k]} \to M^{(k)}$ is the canonical projection.

A stratum $M^{[k]}_\Delta$ lies in the closure of stratum $M^{[k]}_{\Delta'}$ if and only if the partition $\Delta' = \{b_1, \ldots, b_n\}$ of the integer $k$ is a subdivision of the partition $\Delta = \{a_1, \ldots, a_l\}$. In other words, we must have $\Delta' = \Delta_1 \cup \cdots \cup \Delta_l$, where $\Delta_i$ is a partition of the integer $a_i$. Assume that this is the case. Then for every point $m \subset M^{[k]}_\Delta$ and a local chart $U^{[\Delta]} \subset M^{[k]}$ near the point $m$, the intersection $U^{[\Delta']}_\Delta = U^{[\Delta]} \cap M^{[k]}_{\Delta'}$ decomposes into the direct product

$$U^{[\Delta']}_\Delta = U^{[a_i]}_{\Delta_1} \times \cdots \times U^{[a_i]}_{\Delta_l}. \quad (5.1)$$

An analogous decomposition holds for the local chart $U^{(\Delta)}$ in the symmetric power $M^{(k)}$ near the point $\pi(m) \subset M^{(k)}$.

For a partition $\Delta = \{a_1, \ldots, a_l\}$ of the integer $k$, let $\Sigma_\Delta$ be the group of transpositions of $l$ letters which preserve the numbers $a_1, \ldots, a_l$ (if all the numbers $a_1, \ldots, a_l$ are distinct, then the group $\Sigma_\Delta$ is trivial). Consider the product $M^l = M_1 \times \cdots \times M_l$ of $l$ copies of the manifold $M$. The group $\Sigma_\Delta$ acts on $M^l$ by transpositions, and the stratum $M^{(k)}_{\Delta} \subset M^{(k)}$ is isomorphic to the quotient

$$M^{(k)}_\Delta \cong (M_1 \times \cdots \times M_l \setminus \text{Diag}) / \Sigma_\Delta,$$

where $\text{Diag} \subset M^l$ is the subset of diagonals. We will denote by

$$\tilde{M}^{[k]}_{\Delta} = \left( M^l \setminus \text{Diag} \right) \times_{M^{(k)}_{\Delta}} M^{[k]}_{\Delta}$$

the associated $\Sigma_\Delta$-cover of the stratum $M^{[k]}_{\Delta} \subset M^{[k]}$. Points in the variety $\tilde{M}^{[k]}_{\Delta}$ correspond to pairs of

(i) a point $\langle m_1, \ldots, m_l \rangle \in M^l \setminus \text{Diag}$, and

(ii) a 0-dimensional subscheme $Z \subset M$ supported on the subset

$$\{m_1, \ldots, m_l\} \subset M$$

such that the part of $Z$ supported on $m_i$ has length $a_i$. 

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The induced projection \( \pi : \tilde{M}^k_\Delta \to M^l \setminus \text{Diag} \) sends such a pair to the point
\[
\langle m_1, \ldots, m_l \rangle \in M^l \setminus \text{Diag},
\]
and the fiber of the projection \( \pi \) over a point \( \langle m_1, \ldots, m_l \rangle \in M^l \) is canonically isomorphic to the product
\[
F_{m_1, \ldots, m_l} = F_{m_1} \times \cdots \times F_{m_l},
\]
where \( F_{m_i} \) is the punctual Hilbert scheme of 0-dimensional subschemes in \( M \) of length \( a_i \) supported at the point \( m_i \).

Assume now that the holomorphic tangent bundle \( T(M) \) is trivial, and that we are given a commuting frame in \( T(M) \), that is, a subspace of commuting vector fields \( V \subset \Gamma(M, T(M)) \) which freely generate the bundle \( T(M) \) (this happens naturally in all our examples of a torus \( M = T \), a vector space \( M = V \) and an open neighborhood of 0 in a vector space \( M = U \subset V \)). Then the punctual Hilbert scheme \( F_{m_i} \) can be identified canonically with the punctual Hilbert scheme \( F_{a_i} \) of 0-dimensional subschemes of length \( a_i \) in \( V \) supported at 0 \( \subset V \), and we have the following.

**Lemma 5.8 ([KV], Lemma 5.5)** There exists a canonical splitting
\[
\tilde{M}^k_\Delta = (M^l \setminus \text{Diag}) \times F_\Delta \tag{5.2}
\]
of the variety \( \tilde{M}^k_\Delta \) into a direct product of the open subset \( M^l \setminus \text{Diag} \subset M^l \) and the product
\[
F_\Delta = F_{a_1} \times \cdots \times F_{a_l}
\]
punctual Hilbert schemes of subschemes in \( V \) of lengths \( a_1, \ldots, a_l \) supported at 0 \( \subset V \). Consequently, the stratum \( M^k_\Delta \) is canonically isomorphic to the quotient
\[
M^k_\Delta = \left( (M^l \setminus \text{Diag}) \times F_\Delta \right) / \Sigma_\Delta,
\]
where \( \Sigma_\Delta \) acts on \( M^l \) and on \( F_\Delta \) by transpositions.

(In [KV] this is formulated only for a torus \( T \), and the proof uses the abelian group structure on the torus \( T \). It is easy to see that it is enough to have a commuting holomorphic frame \( V \subset \Gamma(M, T(M)) \).)

The splitting (5.2) is functorial with respect to open embeddings which preserve the commuting holomorphic frame. In particular, consider a point \( m \in M^k_\Delta \) in a stratum \( M^k_\Delta \subset M^k \) which lies in the closure of a larger
stratum $M_{\Delta}^{[k]} \subset M^{[k]}$. Assume given a flat local chart $U^{[\Delta]} \subset U^{[k]}$ near the point $m$. Consider the intersection $U^{[\Delta]}_{\Delta'} = U^{[\Delta]} \cap M_{\Delta'}^{[k]}$, and let

\[ U^{[\Delta]}_{\Delta'} = U^{[a_1]}_{\Delta_1} \times \cdots \times U^{[a_l]}_{\Delta_l} \]

be the direct product decomposition (5.1). Then the splitting (5.2) for the stratum $M_{\Delta}^{[k]}$ induces a splitting for the intersection $U^{[\Delta]}_{\Delta'}$, and this induced splitting coincides with the direct product of splittings associated to strata $U^{[a_i]}_{\Delta_i}$ in the Hilbert schemes $U^{[a_i]}$.

Furthermore, assume that $M = T = W/L$ is a 2-dimensional torus, and that $U \subset W$ is a $U(2)$-invariant neighborhood of $0 \subset W$. Then the splitting (5.2) is compatible with the canonical $U(2)$-action on the Hilbert schemes $U^{[a_i]}$ and on the local chart $U^{[\Delta]}$. In other words, the $U(2)$-action also splits into a product of two commuting $U(2)$-actions. The first one, which we will call horizontal, is induced by the standard $U(2)$-action on the local chart $U^{(\Delta)} \subset T^{(k)}$. The second one, which we will call vertical, is induced by the action on the punctual Hilbert scheme $F_{\Delta}$.

In terms of the modular data (i)-(ii) above, the horizontal action affects only the points $m_1, \ldots, m_l \in U$, while the vertical action does not affect $m_1, \ldots, m_l$, and only changes the 0-dimensional subschemes supported at these points.

Note that the vertical $U(2)$-action is induced by an action which is defined on the whole stratum $T_{\Delta}^{[k]}$ and which does not depend the choice of a smaller stratum $T_{\Delta}^{[k]}$, a point $m \in M_{\Delta}^{[k]} = T_{\Delta}^{[k]}$ and the local chart $U^{[\Delta]}$ in $T^{[k]}$ near the point $m$.

### 5.3 Proof of Theorem 5.6

We can now begin the proof Theorem 5.6. First we formulate the collection of results of [KV] which we will need for the proof. Let $T$ be a 2-dimensional complex torus which is Mumford-Tate generic with respect to some hyperkähler structure. Say that a closed subvariety $X \subset T^{[k]}$ in the Hilbert scheme $T^{[k]}$ lies generically in a stratum $T_{\Delta}^{[k]} \subset T^{[k]}$ iff $X$ lies in the closure of the stratum $T_{\Delta}^{[k]}$, while the intersection $X \cap T_{\Delta}^{[k]}$ is dense in $X$. Every irreducible subvariety in $T^{[k]}$ lies generically in one and only one stratum $T_{\Delta}^{[k]}$.

**Proposition 5.9 ([KV], Section 6)** Let $X \subset T^{[k]}$ be an irreducible trianalytic subvariety in the Hilbert scheme of the torus $T$. Assume that $X$ lies
generically in the stratum $T^{[\Delta]}_\Delta \subset T^{[\Delta]}$. Let $\pi(X) \subset T^{[\Delta]}$ be the image of the subvariety $X$ under the projection $\pi : T^{[\Delta]} \to T^{[\Delta]}$. Moreover, let $\tilde{X} \subset \tilde{T}^{[\Delta]}$ be the preimage of the subset $X \cap T^{[\Delta]}_\Delta \subset T^{[\Delta]}_\Delta$ under the canonical Galois covering $\tilde{T}^{[\Delta]}_\Delta \to T^{[\Delta]}_\Delta$.

(i) The subvariety $\pi(X) \subset T^{[\Delta]}$ is of the form $T'/G$, where $T' = V'/L'$ is a subtorus, $V' \subset W^{[\Delta]}$ is the associated linear subspace, $L' = V' \cap L^{[\Delta]}$ is the induced lattice in $V'$, and $G = \text{Norm} T' \subset S_k$ is the normalizer subgroup of the subtorus $T'$.

(ii) The subspace $V' \subset W^{[\Delta]}$ is invariant under the canonical $U(2)$-action on the vector space $W^{[\Delta]}$.

(iii) The projection $X \to \pi(X)$ is finite and étale over an open dense subset in $\pi(X)$. Moreover, the image of the subset $\tilde{X} \subset \tilde{T}^{[\Delta]}_\Delta$ under the natural projection

\[
\tilde{T}^{[\Delta]}_\Delta = \left( T^{[\Delta]} \setminus \text{Diag} \right) \times F_\Delta \to F_\Delta
\]

is a finite subset in $F_\Delta$. □

The new result which we need for the proof of Theorem 5.6 is the following. Let $X \subset T^{[\Delta]}$ be an irreducible rigid trianalytic subvariety in the Hilbert scheme $T^{[\Delta]}$ of a Mumford-Tate generic 2-dimensional complex torus $T$. Let $U \subset W$ be a sufficiently small $U(2)$-invariant open neighborhood of $0 \in W$, and consider the associated flat local chart $U^{[\Delta]} \subset T^{[\Delta]}$ near a point $x \in X \cap T^{[\Delta]}_\Delta$ for some stratum $T^{[\Delta]}_\Delta \subset T^{[\Delta]}$.

**Proposition 5.10** The intersection $X \cap U^{[\Delta]} \subset U^{[\Delta]}$ is invariant under the canonical $U(2)$-action on $U^{[\Delta]} \subset W^{[\Delta]}$.

**Proof.** Let $u$ be an arbitrary vector in the Lie algebra $u(2)$ of the Lie group $U(2)$. For every point $x \in X \cap T^{[\Delta]}_\Delta$, the $U(2)$-action on the local chart $U^{[\Delta]}$ near $x$ induces an action of the Lie algebra $u(2)$. Let $t_x$ be the holomorphic vector field $U^{[\Delta]}$ associated to $u \subset u(2)$. After restricting to the intersection $X \cap U^{[\Delta]}$, it defines, in turn, a local holomorphic section $n_x$ of the normal bundle $\mathcal{N}(X)$ to $X$ in $T^{[\Delta]}$. We begin by proving that all these local sections glue together and define a global section of the bundle $\mathcal{N}(X)$.

Assume that the irreducible subvariety $X$ lies generically in a stratum $T^{[\Delta]}_\Delta$ for a partition $\Delta' = \{a'_1, \ldots, a'_n\}$. Since the intersection $X \cap T^{[\Delta]}_\Delta \subset X$ is dense, it suffices to prove that the restrictions of the local sections $n_x$ to the
intersections $X_0 = X \cap T_X^{[k]} \cap U^\Delta$ taken for different points $x$ glue together to give a global holomorphic section in $\Gamma(X \cap T_X^{[k]}, \mathcal{N}(X))$.

Fix a point $x \in X \cap T_X^{[k]}$ and let $t_x = t_{\text{hor}} + t_{\text{vert}}$ be the decomposition of the vector field $t_x$ into the parts corresponding to the horizontal and the vertical action of the group $U(2)$ on $U^\Delta = T^\Delta_X \cap U^\Delta$. By Proposition 5.9, the variety $X_0$ lies in a finite set of fibers of the projection onto the second factor in the splitting (5.2). Moreover, the subvariety $\pi(X_0) = V \cap U^n \subset W^n$ is invariant under the standard $U(2)$-action on $W^n$. Therefore the subvariety $X_0 \subset U^\Delta_X$ is preserved by the horizontal $U(2)$-action.

Consequently, the vector field $t_{\text{hor}}$ is tangent to the subvariety $X_0$. (More precisely, the section $t_{\text{hor}}$ of the tangent bundle $T(T_X^{[k]})$ becomes a section of the holomorphic tangent bundle $T(X_0) \subset T(T_X^{[k]})$ after the restriction to $X_0 \subset T_X^{[k]}$.) Therefore the vector fields $t_x$ and $t_{\text{vert}}$ define the same section of the normal bundle $\mathcal{N}(X)$.

However, the vertical action of $U(2)$ on $T_X^{[k]} \cap U^\Delta$ can be defined globally on the whole $T_X^{[k]}$ and does not depend on the choice of the point $x \in X$ and the local chart $U^\Delta$. Therefore the vector fields $t_{\text{vert}}$ for different points $x$ glue together to produce a global section of the tangent bundle $T(T_X^{[k]})$ to $T_X^{[k]}$ over $X \cap T_X^{[k]}$. This proves that the sections $\pi_x$ of the normal bundle $\mathcal{N}(X)$ also glue together to a global section over $X \cap T_X^{[k]}$, hence also over the whole $X$.

But by [V2] (see also Section 3 of [KV]), all trianalytic subvarieties are unobstructed, that is, every global section of a normal bundle $\mathcal{N}(X)$ to a trianalytic submanifold $X \subset T_X^{[k]}$ defined a local deformation of $X$. Since by assumption $X$ is rigid, all global sections of $\mathcal{N}(X)$ must vanish. Therefore all the local sections $\pi_x$ also vanish. Since the vector $u \in u(2)$ is arbitrary, this implies that for every local chart $U^\Delta$ the intersection $X \cap U^\Delta$ is $U(2)$-invariant.

We can now prove Theorem 5.6.

Proof of Theorem 5.6. Assume that the irreducible rigid subvariety $X \subset T_X^{[k]}$ lies generically in a stratum $T_X^{[k]}$, in other words, the intersection $X_0 = X \cap T_X^{[k]} \subset X$ is dense in $X$. Let $x \in X_0$ be an arbitrary point. By Proposition 5.10, the subvariety $X_0$ is invariant under the canonical $U(2)$-action corresponding to the local chart at $x$. In particular, the point $x \in F_X = \pi^{-1}(\pi(x))$ itself is $U(2)$-invariant. This means that the corresponding 0-dimensional subscheme in $W$ supported at $0 \in W$ is $U(2)$-invariant. But it is easy to see that for every $n$ there exists at most one such subscheme of
length $n$ ([V3], Sublemma 5.7). Therefore the projection $\pi : X_0 \to \pi(X_0)$ is one-to-one.

To prove (i), it remains to prove that $\pi : X \to \pi(X)$ is one-to-one over the nonsingular part $U \subset \pi(X)$ of $\pi(X) = T'/G$. Denote by $\hat{U} = \pi^{-1}(U) \cap X$. Since we know that it is one-to-one over a dense subset $X_0 \subset X$, it suffices to prove that $\pi : \hat{U} = \pi^{-1}(U) \cap X \to U$ is étale.

To do this, consider the symplectic form on the 2-dimensional vector space $W$ associated to the hyperkähler structure on the torus $T$. This form induces an $S_k$-invariant symplectic form $\Omega_0$ on the vector space $W^k$ and the holomorphic symplectic form $\Omega$ on the Hilbert scheme $T^{[k]}$. By Lemma 6.2 of [KV] the restriction of the form $\Omega_0$ to the vector subspace $V \subset W^k$ associated to the subtorus $T'$ is non-degenerate. Therefore it induces a holomorphic symplectic form $\Omega_0$ on the non-singular part $U \subset T'/G$. By [V3, Proposition 4.5], the pullback $\pi^*(\Omega_0)$ coincides with the restriction of the form $\Omega$ on $T^{[k]}$ to the subset $\hat{U} \subset T^{[k]}$.

Since $X \subset T^{[k]}$ is trianalytic, this implies that the pullback $\pi^*(\Omega_0)$ is non-degenerate. But both $U$ and $\hat{U}$ are smooth. Therefore the differential $d\pi : T(\hat{U}) \to \pi^* T(U)$ of the map $\pi : \hat{U} \to U$ is an isomorphism, and the map $\pi : \hat{U} \to U$ is indeed étale. This proves (i).

To prove (ii), we note that by Proposition 5.9 the image $X \subset T^{(k)}$ is of the form $T'/G$, where $T' \subset T^k$ is a subtorus, and $G = \text{Norm} T' \subset S_k$ is the normalizer subgroup of the subtorus $T' \subset T^k$ in the symmetric group $S_k$. Together with (i) this means that $X \subset T^{[k]}$ is a partial resolution of Hilbert type in the sense of Definition 3.2. In particular, $T' = V/L'$, where $V = \Gamma(T', T(T)) \subset W^k$ is a linear subspace, and $L' = L^k \cap V \subset V$ is the induced lattice.

The finite group $G$ carries a canonical three-step filtration
$$G_0 \subset G_1 \subset G,$$
where $G_0 \subset G$ is the subgroup of elements which act trivially on the torus $T'$, and $G_1 \subset G \subset S_k$ is the subgroup of elements which act trivially on the vector subspace $V \subset W^k$. The action of the group $G$ on the torus $T'$ factors through the quotient $\hat{G} = G/G_0$, and we have $\hat{G}_1 = G_1/G_0 \subset \hat{G}$. Denote by $\sigma : \hat{G} \to \hat{G}/\hat{G}_0$ the quotient map.

The subgroup $\hat{G}_1 = G_1/G_0 \subset \hat{G}$ acts on the torus $T'$ by translations of finite order. Therefore it is abelian. Moreover, every element $g \in \hat{G}_1$ acts on the torus $T'$ without fixed points.

Conversely, every element $g \in \hat{G}$ which acts on $T'$ without fixed points must act by a translation. Therefore $g$ acts trivially on the vector space $V = \Gamma(T', T(T))$, in other words, belongs to $\hat{G}_1 \subset \hat{G}$.
For every point \( t \in T' \), let \( G_t \subset \hat{G} \) be the stabilizer subgroup of the point \( t \), and let \( \sigma(G_t) \subset \hat{G}/\hat{G}_1 \) be its image under the quotient map. Since \( \hat{G}_1 \subset \hat{G} \) is precisely the subgroup of elements which act on \( T' \) without fixed points, the canonical map \( G_t \to \sigma(G_t) \) is an isomorphism, and the quotient 
\[
\hat{G}/\hat{G}_1 = \bigcup_{t \in T'} \sigma(G_t)
\]
is the union of the subgroups \( \sigma(G_t) = G_t \subset \hat{G}/\hat{G}_1 \) for different points \( t \in T' \).

But the quotient \( \hat{G}/\hat{G}_1 \) acts naturally on the subspace \( V_0 = V^{U(1)} \subset V \) of \( U(1) \)-invariants in \( V \). Moreover, Theorems 4.2 and 4.4 imply that the stabilizer \( G_t \in \hat{G}/\hat{G}_0 \) of an arbitrary point \( t \in T' \) acts on \( V_0 \) as a group generated by real reflections. Therefore the whole group \( \hat{G}/\hat{G}_1 \) acts on the space \( V_0 \) as a group generated by real reflections.

To finish the proof, it remains to notice that 
\[
\pi(X) = T'/G = \left( T'/\hat{G}_1 \right) / \left( \hat{G}/\hat{G}_1 \right),
\]
and the quotient \( T'/\hat{G}_1 \) is a complex torus isogenic to \( T' \subset T^k \). Replacing \( G \) with \( \hat{G}/\hat{G}_1 \) and \( T' \) with \( T'/\hat{G}_1 \) proves (ii). \( \square \)

6 Appendix. Existence of subvarieties of generalized Kummer varieties

6.1 The counterexamples

Let \( T \) be a complex torus of dimension 2. Consider the Hilbert scheme \( T^{[n+1]} \) of \( n+1 \) points on \( T \). This is a complex variety of dimension \( 2(n+1) \).

The commutative group structure on the torus \( T \) defines a summation map \( \Sigma : T^{n+1} \to T \), which induces a morphism \( \Sigma : T^{[n+1]} \to T \). It is easy to see that \( \Sigma \) coincides with the Albanese map.

Definition 6.1 The generalized Kummer variety \( K^{[n]} \) associated to the torus \( T \) is the preimage \( \Sigma^{-1}(0) \subset T^{[n+1]} \) of the zero \( 0 \in T \).

In [KV], the following false theorems were stated.

**Theorem (false).** Let \( T \) be a 2-dimensional complex torus equipped with a hyperkähler structure. Assume that the complex structure on \( T \) is Mumford-Tate generic, and consider a generalized Kummer variety \( K^{[n]} \) associated to \( T \).
For any hyperkähler structure on $K^{[n]}$ compatible with the canonical holomorphic 2-form, every irreducible trianalytic subvariety $X \subset K^{[n]}$ is either the whole $K^{[n]}$ or a single point.

Theorem (false). Assume that the Kummer variety $K^{[n]}$ is equipped with a complex structure $I$, and assume that $I$ is Mumford-Tate generic with respect to some hyperkähler structure on $K^{[n]}$ compatible with the canonical holomorphic 2-form.

Then every irreducible analytic subvariety $X \subset K^{[1]}[n]$ is either the whole $K^{[n]}$ or a single point.

Here we offer the counterexamples to these statements.

Notice that by a “hyperkähler structure on a complex manifold” we understand a hyperkähler structure inducing the complex structure. A manifold of Kähler type is a complex manifold admitting a Kähler metric.

Theorem 6.2 Let $T$ be a compact torus, $K^{[n]}$ the $n$-th generalized Kummer variety of $T$, $n > 1$, and $X$ be a manifold of Kähler type which is a deformation of $K^{[n]}$. Then $X$ admits a non-trivial complex analytic involution $\iota : X \rightarrow X$, and the fixed set of $\iota$ has positive dimension. Moreover, for any hyperkähler structure $\mathcal{H}$ on $X$, the involution $\iota$ is compatible with $\mathcal{H}$.

The proof of Theorem 6.2 is given in Subsection 6.3.

Remark 6.3 The fixed set of an involution is always a union of smooth manifolds. From Theorem 6.2, it follows that the fixed point set of the involution $\iota$ is always trianalytic. Thus, Theorem 6.2 gives counterexamples to both false theorems of [KV].

The construction of the involution $\iota$ was suggested by A. Kuznetsov for the Hilbert scheme of $\mathbb{C}^2$. Let $\iota_0 : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the involution mapping $(x, y) \rightarrow (-x, -y)$. By functoriality, $\iota_0$ is naturally extended to the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of $\mathbb{C}^2$. A. Kuznetsov conjectured that this involution is compatible with the natural hyperkähler structure on $(\mathbb{C}^2)^{[n]}$, constructed by Kronheimer and Nakajima (see, e. g. [N]). This conjecture was not proven, though the proof seems to be straightforward. However, using Calabi-Yau (Theorem 5.3), it is possible to prove that the analogous involution is compatible with the hyperkähler structure on the Hilbert scheme of a torus.
6.2 Twistor paths and diffeomorphisms preserving trianalytic subvarieties

Let $M$ be a hyperkähler manifold. Consider the product manifold $X = M \times S^2$. Embed the sphere $S^2 \subset \mathbb{H}$ into the quaternion algebra $\mathbb{H}$ as the subset of all quaternions $J$ with $J^2 = -1$. For every point $x = m \times J \in X = M \times S^2$ the tangent space $T_x X$ is (using Levi-Civita connection) decomposed as $T_x X = T_m M \oplus T_J S^2$. Identify $S^2 = \mathbb{C}P^1$ and let $I_J : T_J S^2 \to T_J S^2$ be the complex structure operator. Let $I_m : T_m M \to T_m M$ be the complex structure on $M$ induced by $J \in S^2 \subset \mathbb{H}$.

The operator $I_x = I_m \oplus I_J : T_x X \to T_x X$ satisfies $I_x \circ I_x = -1$. It depends smoothly on the point $x$, hence defines an almost complex structure on $X$. This almost complex structure is known to be integrable (see [Sal]). The complex manifold $X$ is called the twistor space of $M$.

The twistor space is fibered over $\mathbb{C}P^1$, and the corresponding map $\pi : X \to \mathbb{C}P^1$ is called the twistor fibration. The map $\pi$, considered as a deformation of $M$ over the base $\mathbb{C}P^1$, gives an embedding of $\mathbb{C}P^1$ into the moduli $\text{Comp}$ of deformations of complex structures on $M$. Such an embedding is called the twistor curve, associated with the hyperkähler structure $\mathcal{H}$. A sequence of connected twistor curves is called a twistor path. In [V3] it was proven that any two points in the moduli space $\text{Comp}$ can be connected by a twistor path. For more detailed definitions and results concerning the twistor paths, the reader is referred to [V6] (Subsection 10.1).

**Definition 6.4** Let $P_0, ..., P_n \subset \text{Comp}$ be a sequence of twistor curves, supplied with an intersection point $x_{i+1} \in P_i \cap P_{i+1}$ for each $i$. We say that $\gamma = P_0, ..., P_n, x_1, ..., x_n$ is a twistor path. Let $I, I' \in \text{Comp}$. We say that $\gamma$ is a **twistor path connecting** $I$ to $I'$ if $I \in P_0$ and $I' \in P_n$. The lines $P_i$ are called the edges, and the points $x_i$ the vertices of a twistor path.

Recall that in [5.4], we defined induced complex structures which are generic with respect to a hyperkähler structure.

**Definition 6.5** Let $I, J \in \text{Comp}$ and $\gamma = P_0, ..., P_n$ be a twistor path from $I$ to $J$, which corresponds to the hyperkähler structures $\mathcal{H}_0, ..., \mathcal{H}_n$. We say that $\gamma$ is **admissible** (resp. Mumford-Tate admissible) if all vertices of $\gamma$ are generic (resp. Mumford-Tate generic) with respect to the corresponding edges.
Clearly, every twistor path $\gamma$ induces a diffeomorphism

$$\mu_\gamma : (M, I) \longrightarrow (M, I').$$

In [V3], Subsection 5.2, we studied algebro-geometrical properties of this diffeomorphism.

**Theorem 6.6** Let $I, J \in \text{Comp}$ and $\gamma = P_0, ..., P_n$ be an admissible twistor path from $I$ to $J$. Then $\mu_\gamma$ maps trianalytic subvarieties to trianalytic subvarieties.

**Proof:** For an induced complex structure of generic type, every closed complex analytic subvariety is trianalytic. Therefore, diffeomorphism between the generic fibers of the twistor fibration, provided by a single twistor curve, maps complex subvarieties to complex subvarieties. Taking compositions of such diffeomorphisms, we obtain a diffeomorphism mapping complex subvarieties to complex subvarieties, and hence, trianalytic subvarieties to trianalytic subvarieties. For more details, see [V3], Subsection 5.2. $\square$

**Corollary 6.7** Let $I, J \in \text{Comp}$ and $\gamma = P_0, ..., P_n$ be a Mumford-Tate admissible twistor path from $I$ to $J$. Then $\mu_\gamma$ maps hyperkähler automorphisms of $(M, I)$ to hyperkähler automorphisms of $(M, J)$.

**Proof:** Taking a graph of a hyperkähler automorphism, we obtain a trianalytic subvariety in $M \times M$. Applying Theorem 6.6 to thus subvariety, we obtain that the diffeomorphism corresponding to a Mumford-Tate admissible twistor path maps hyperkähler automorphisms to hyperkähler automorphisms. $\square$

Notice that a generic deformation of a compact hyperkähler manifold satisfies $\text{Pic}(M_I) = 0$.

**Theorem 6.8** Let $\mathcal{H}, \mathcal{H}'$ be hyperkähler structures on $M$, and $I, I'$ be complex structures of general type with respect to to and induced by $\mathcal{H}, \mathcal{H}'$. Assume that $\text{Pic}(M_I) = 0$. Then $I, I'$ can be connected by an admissible path.

**Proof:** This is [V3], Theorem 5.2. $\square$

**Corollary 6.9** Let $I$ be a complex structure on $M$, such that $\text{Pic}(M, I) = 0$, and $\iota$ be an automorphism of $(M, I)$. Assume that $I$ is induced by a hyperkähler structure $\mathcal{H}$, and $(M, I)$ is Mumford-Tate generic with respect
to $\mathcal{H}$. Let $I'$ be a complex structure of Kähler type on $M$ which lies in the same deformation class as $I$. Then there exists a diffeomorphism

$$\mu : (M, I) \longrightarrow (M, I')$$

which maps $\iota$ to a complex analytic automorphism of $(M, I')$.

**Proof:** A diffeomorphism $\mu$ is constructed from a Mumford-Tate admissible twistor path, as follows from Theorem 6.8 and Corollary 6.7. □

### 6.3 Examples of subvarieties of generalized Kummer varieties

From Corollary 6.9, we obtain the following. Let $\iota$ be an involution of a compact holomorphic symplectic manifold $(M, I)$, $\text{Pic}(M, I) = 0$, which is Mumford-Tate generic with respect to some hyperkähler structure. Then $\iota$ is compatible with any hyperkähler structure $\mathcal{H}$ on $M$ such that $I$ is Mumford-Tate generic with respect to $\mathcal{H}$. Moreover, for any complex structure $I'$ on $M$ in the same deformation class, $\iota$ is mapped to a complex automorphism of $(M, I')$ by a diffeomorphism, associated with an admissible twistor path. Therefore, to prove Theorem 6.2, it suffices to prove the following.

**Theorem 6.10** Let $M$, $\text{Pic}(M) = 0$, be a deformation of a generalized Kummer variety, which is Mumford-Tate generic with respect to some hyperkähler structure. Then $M$ admits a non-trivial complex analytic involution $\iota$. Moreover, the fixed point set of $\iota$ has positive dimension.

**Proof:** To prove Theorem 6.10, we need to construct an involution $\iota$ on a “sufficiently generic” deformation of $K^{[n]}$. Let $\mathcal{M}$ be the moduli space of complex deformations of $M$, and $\mathcal{M}_0 \subset \mathcal{M}$ a subset consisting of all complex structures $I'$ on $M$ satisfying the following.

The manifold $(M, I')$ admits a hyperkähler structure $\mathcal{H}$ inducing a complex structure $I''$ such that $(M, I'')$ is a generalized Kummer variety for some torus $T$.

In other words, $\mathcal{M}_0$ is the set of all deformations $(M, I) \in \mathcal{M}_0$ of $K^{[n]}$ which can be connected to a generalized Kummer variety by a single twistor curve.

In the proof of Theorem 6.10, we use the following lemma.

**Lemma 6.11** In the above assumptions, the set $\mathcal{M}_0 \subset \mathcal{M}$ is open in $\mathcal{M}$.  

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Proof: The proof is based on an elementary dimension count. We use the Torelli theorem which states that the period map from $\mathcal{M}$ to $\mathbb{P}H^2(M, \mathbb{C})$ is etale onto its image, which is of codimension one in $\mathbb{P}H^2(M, \mathbb{C})$. The moduli of complex tori is a 4-dimensional complex space, and the moduli of deformations of a complex structure on a generalized Kummer variety is 5-dimensional. An extra dimension is given by a choice of an induced complex structure on a Kummer variety. □

From Lemma 6.11, it follows that to prove Theorem 6.10 it suffices to construct an involution $\iota : K^{[n]} \rightarrow K^{[n]}$ that commutes with any hyperkähler structure. This is done as follows. We have an involution of a torus $\iota : T \rightarrow T$ mapping $x$ to $-x$. By functoriality, this involution is extended to an involution of the Hilbert scheme $T^{[n+1]}$. Since $\iota$ commutes with the Albanese map $\Sigma : T^{[n+1]} \rightarrow T$, this map preserves the generalized Kummer variety $K^{[n]} \subset T^{[n+1]}$. Consider the natural projection map $\pi : K^{[n]} \rightarrow T^{[n+1]}$ from the Kummer variety to the symmetric power of $T$. Let $C \subset K^{[n]}$ be the singular locus of $\pi$, and $[C] \in H^2(K^{[n]})$ the corresponding cohomology class. Clearly, $H^2(K^{[n]})$ is generated by $[C]$ and the group $\pi^* (H^2(T^{(n+1)})) \cong H^2(T)$. Since $\iota$ acts identically on $\pi^* (H^2(T^{(n+1)})) \cong H^2(T)$ and preserves $C$, $\iota$ acts as identity on $H^2(K^{[n]})$. Therefore, $\iota$ maps a Kähler class to itself, for any Kähler class $\omega$ on $K^{[n]}$. By Theorem 5.5, $\iota$ preserves the hyperkähler metric. It is easy to see that there exists (at most) a unique hyperkähler structure compatible with a given holomorphic symplectic structure and a metric. Therefore, $\iota$ preserves the hyperkähler structure.

We obtained an involution $\iota$ on a “sufficiently generic” deformation of $K^{[n]}$. It remains to show that this involution has positive-dimensional fixed set. This is done by an elementary geometric observation. Assume, for instance, that $n$ is odd, $n+1 = 2k$. Then $\iota$ fixes all the $2k$-tuples

$$(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) \in T^{(n+1)}$$

satisfying $x_i = -y_i$. When $x_i, y_i$ are pairwise distinct, they give a point of the Hilbert scheme fixed by $\iota$. The closure $D$ of the set of such points is a component of the fixed point set of $\iota$, and it is obviously positively-dimensional. This proves Theorem 6.10. □

Remark 6.12 It is easy to see that $D$ is birationally equivalent to a Hilbert scheme of a Kummer K3 surface associated with $T$. 

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