HEALTHY VECTOR SPACES AND SPICY HOPF ALGEBRAS
(WITH APPLICATIONS TO THE GROWTH RATE OF GEODESIC
CHORDS AND TO INTERMEDIATE VOLUME GROWTH ON
MANIFOLDS OF NON-FINITE TYPE)

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Abstract. We give lower bounds for the growth of the number of Reeb chords and for
the volume growth of Reeb flows on spherizations over closed manifo lds \( M \) that are not
of finite type, have virtually polycyclic fundamental group, and satisfy a mild assumption
on the homology of the based loop space. For the special case of geodesic flows, these
lower bounds are:

(i) For any Riemannian metric on \( M \), any pair of non-conjugate points \( p, q \in M \), and
every component \( C \) of the space of paths from \( p \) to \( q \), the number of geodesics in \( C \)
of length at most \( T \) grows at least like \( e^{\sqrt{T}} \).

(ii) The exponent of the volume growth of any geodesic flow on \( M \) is at least 1/2.

We obtain these results by combining new algebraic results on the gro wth of certain filtered
Hopf algebras with known results on Floer homology.

1. Introduction and main results

Consider a closed connected manifold \( M \). Then \( M \) is said to be of finite type if its uni-
versal cover \( \widetilde{M} \) is homotopy equivalent to a finite CW-complex. Equivalently, there is
\( k \in \{2, \ldots, \dim M\} \) such that \( H_k(\widetilde{M}) \) is not finitely generated. Let \( m = m(M) \) be the
minimal such \( k \).

A finitely generated group \( G \) is called polycyclic if it admits a subnormal series with
cyclic factors. Moreover, \( G \) is virtually polycyclic if it has a polycyclic subgroup of finite
index.

Choose a point \( p \in M \) and let \( \Omega_0 M \) be the space of contractible continuous loops
in \( M \) based at \( p \). For \( T > 0 \) denote by \( \Omega_0^T M \) the space of contractible piecewise smooth
loops based at \( p \) whose length (with respect to a fixed Riemannian metric on \( M \)) is at
most \( T \). The inclusions \( i^T : \Omega_0^T M \hookrightarrow \Omega_0 M \) induce the maps \( i_*^T : H_*(\Omega_0^T M) \rightarrow H_*(\Omega_0 M) \) in
homology. Note that the image \( i_*^T H_*(\Omega_0^T M) \) is the part of the homology of \( \Omega_0 M \) that is
generated by cycles made of piecewise smooth loops of length \( \leq T \).

We say that a real-valued function \( f \) defined on \( \mathbb{N} \) or on \( \mathbb{R}_{>0} \) grows at least like \( e^{\sqrt{T}} \) if
there exists a constant \( c > 0 \) such that \( f(T) \geq ce^{\sqrt{T}} \) for all large enough \( T \).

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Theorem 1.1. Let $M$ be a closed connected manifold that is not of finite type and has virtually polycyclic fundamental group. Assume also that there exists a field $\mathbb{F}$ such that $H_m(\tilde{M}; \mathbb{F})$ is infinite-dimensional. Then the function of $T$

$$\dim \iota^*_T H_* (\Omega^T_0 M; \mathbb{F})$$

 grows at least like $e^{\sqrt{T}}$.

Remarks 1.2. 1. The assumption that $H_m(\tilde{M}; \mathbb{F})$ is infinite-dimensional is equivalent to the assumption that $\pi_m(M) \otimes \mathbb{F}$ is infinite-dimensional.

2. We conjecture that the rank of $\iota^*_T H_* (\Omega^T_0 M)$ grows exponentially for every manifold $M$ of non-finite type (cf. the Question in [23, p. 289] and the discussion in [5, §7.1]). Theorem 1.1 proves “not quite half” of this conjecture.

3. Along the proof we shall see that $H_{m-1}(\Omega^T_0 M; \mathbb{F})$ is infinite-dimensional and that

$$\dim H_{(m-1)n}(\Omega^T_0 M; \mathbb{F}) \geq q(n) \quad \text{for } n \geq 2,$$

where $q(n)$ is the number of partitions of $n$ into distinct parts. The fact that this function grows like $e^{\sqrt{T}}$ explains the lower bound in Theorem 1.1.

4. Our assumption that there exists a field $\mathbb{F}$ such that $H_m(\tilde{M}; \mathbb{F})$ is infinite-dimensional plays an important role in our proof. This is illustrated by an example in Section 6.2. In Section 6.1 we give a class of examples that meet the assumptions of Theorem 1.1.

1.1. Applications. Lower bounds for the rank of the homology of the sublevel sets $\Omega^T M$ are of interest because they classically lead, by Morse theory, to lower bounds for the number of geodesics of length $\leq T$ between non-conjugate points. Somewhat less classically, they also lead to lower bounds for the topological entropy of geodesic flows. Moreover, adding the tool of Floer homology, one gets lower bounds for the number of Reeb chords and for the topological entropy of Reeb flows on spherizations.

Before stating our two corollaries, we briefly recall what Reeb flows on spherizations are. Details can be found in the introduction of [5].

Reeb flows on spherizations. Consider a closed manifold $M$. The positive real numbers $\mathbb{R}_+$ freely act on the cotangent bundle $T^* M$ by $r(q, p) = (q, rp)$. While the canonical 1-form $\lambda = pdq$ on $T^* M$ does not descend to the quotient $S^* M := T^* M / \mathbb{R}_+$, its kernel does and defines a contact structure $\xi$ on $S^* M$. We call the contact manifold $(S^* M, \xi)$ the spherization of $M$. This contact manifold is co-orientable. The choice of a nowhere vanishing 1-form $\alpha$ on $S^* M$ with $\ker \alpha = \xi$ (called a contact form) defines a vector field $R_\alpha$ (the Reeb vector field of $\alpha$) by the two conditions $d\alpha(R_\alpha, \cdot) = 0, \alpha(R_\alpha) = 1$. Its flow $\varphi^\alpha_\alpha$ is called the Reeb flow of $\alpha$.

To give a more concrete description of the manifold $(S^* M, \xi)$ and the flows $\varphi^\alpha_\alpha$, consider a smooth hypersurface $\Sigma$ in $T^* M$ which is fiberwise starshaped with respect to the zero-section: For every $q \in M$ the set $\Sigma_q := \Sigma \cap T^*_q M$ bounds a set in $T^*_q M$ that is strictly starshaped with respect to the origin of $T^*_q M$. The hyperplane field $\xi_\Sigma := \ker(\lambda|_\Sigma)$ is a contact structure on $\Sigma$, and the contact manifolds $(S^* M, \xi)$ and $(\Sigma, \xi_\Sigma)$ are isomorphic.
Let $\varphi^t_\Sigma$ be the Reeb flow on $\Sigma$ defined by the contact form $\lambda|_\Sigma$. The set of Reeb flows on $(S^*M, \xi)$ can be identified with the Reeb flows $\varphi^t_\Sigma$ on the set of fiberwise starshaped hypersurfaces $\Sigma$ in $T^*M$. The flows $\varphi^t_\Sigma$ are restrictions of Hamiltonian flows: Consider a Hamiltonian function $H: T^*M \to \mathbb{R}$ such that $\Sigma = H^{-1}(1)$ is a regular energy surface and such that $H$ is fiberwise homogeneous of degree one near $\Sigma$. For the Hamiltonian flow $\varphi^t_H$ we then have $\varphi^t_H|_\Sigma = \varphi^t_\Sigma$. It follows that geodesic flows and Finsler flows (up to the time change $t \mapsto 2t$) are examples of Reeb flows on spherizations. Indeed, for geodesic flows the $\Sigma_q$ are ellipsoids, and for (symmetric) Finsler flows the $\Sigma_q$ are (symmetric and) convex. The flows $\varphi^t_\Sigma$ for varying $\Sigma$ are very different, in general, as is already clear from looking at geodesic flows on a sphere. In this paper we give uniform lower bounds for the growth of Reeb chords and for the complexity of all these flows on $(S^*M, \xi)$ for manifolds $M$ as in Theorem 1.1.

**Growth of Reeb chords between two fibers.** We say that $p, q \in M$ are non-conjugate points of the Reeb flow $\varphi_\alpha$ if $\bigcup_{t>0} \varphi^t_\alpha(S^*_pM)$ is transverse to $S^*_qM$. Given $p \in M$, the set of $q \in M$ that are non-conjugate to $p$ has full measure in $M$ by Sard’s theorem. This notion of being non-conjugate generalizes the one in Riemannian geometry (defined in terms of Jacobi fields).

For $p, q \in M$ denote by $\mathcal{P}_{pq}$ the space of continuous paths in $S^*M$ from $S^*_pM$ to $S^*_qM$, and by $\Omega_{pq}M$ the space of continuous paths in $M$ from $p$ to $q$. We shall assume throughout that $\dim M \geq 3$, since otherwise $M$ is of finite type. The fibers $S^*_qM$ of the projection $pr: S^*M \to M$ are then simply connected, and so $pr$ induces a bijection on the components of $\mathcal{P}_{pq}$ and $\Omega_{pq}M$. The space $\Omega_{pq}M$ is homotopy equivalent to $\Omega_pM := \Omega_{p,p}M$, whose components are parametrized by the elements of the fundamental group $\pi_1(M, p)$.

**Corollary 1.3.** Assume that $M$ is not of finite type, has virtually polycyclic fundamental group, and that there is a field $\mathbb{F}$ such that $H_m(M; \mathbb{F})$ is infinite-dimensional. Let $(S^*M, \xi)$ be the spherizations of $M$. Then for any Reeb flow $\varphi_\alpha$ on $(S^*M, \xi)$, any pair of non-conjugate points $p, q \in M$ and every component $C$ of $\Omega_{pq}M$, the number of Reeb chords from $S^*_pM$ to $S^*_qM$ that belong to $C$ grows in time at least like $e^{\sqrt{T}}$.

**Remarks 1.4.**

1. For the special case of geodesic flows, the time parameter equals the length run through. Hence the corollary translates to assertion (i) of the abstract.

To illustrate the corollary, we choose a Riemannian metric on $M$ and a point $p \in M$. Let $C(p)$ be the cut locus of $p$. The subset $M \setminus C(p)$ is diffeomorphic to an open ball [19] and has full measure in $M$. For every $q \in M \setminus C(p)$ there is a unique shortest geodesic $c_q$ from $p$ to $q$. Call a path $\gamma \in \Omega_{pq}M$ contractible if $c_q^{-1} \circ \gamma$ is contractible in $\Omega_pM$. The set $U_p$ of points in $M \setminus C(p)$ that are not conjugate to $p$ is also of full measure in $M$. Under the hypothesis of Corollary 1.3 for every $q \in U_p$ the number of contractible geodesics from $p$ to $q$ of length $\leq T$ grows at least like $e^{\sqrt{T}}$.

2. Virtually polycyclic groups are either virtually nilpotent or have exponential growth [29]. If the fundamental group $\pi_1(M)$ of a closed manifold $M$ has exponential growth, then the number of Reeb chords from $S^*_pM$ to $S^*_qM$ grows exponentially in time for any, possibly conjugate, pair of points $p, q$ (see [15, Corollary 1] for Reeb flows). Indeed, one finds one
Reeb chord for each element of $\pi_1 M$. In Corollary 1.3 however, we find “$e^{\sqrt{T}}$ many” Reeb chords for each element of $\pi_1 M$.

If one is only interested in the growth of Reeb chords from $S^*_p M$ to $S^*_q M$, without specifying the component $C$, then Corollary 1.3 is interesting only for virtually nilpotent fundamental groups, which by Gromov’s theorem from [9] are exactly those fundamental groups that grow polynomially. Indeed, it is believed that every finitely presented group that grows more than polynomially grows exponentially [6, Conjecture 11.3], and even for finitely generated groups of intermediate growth it is believed that they must grow at least like $e^{\sqrt{T}}$, cf. [7].

3. Let $\mu(\gamma)$ be the Conley–Zehnder index of a non-degenerate Reeb chord $\gamma$ on $(S^*_M; \alpha)$, normalized such that for geodesic flows $\mu(\gamma)$ is the Morse index of the non-degenerate geodesic $\gamma$ (i.e. the number of conjugate points, counted with multiplicities, along $\gamma$). In the situation of Corollary 1.3, Remark 1.2.3 shows that for every component of $\Omega_{pq}$ the number of Reeb chords from $S^*_p M$ to $S^*_q M$ of index $\mu(\gamma) = k$ is infinite for $k = m - 1$ and at least $q(n)$ if $k = (m - 1)n$ and $n \geq 2$.

Intermediate volume growth. Consider a smooth diffeomorphism $\varphi$ of a closed manifold $X$. Denote by $S$ the set of smooth compact submanifolds of $X$. Fix a Riemannian metric $\rho$ on $X$, and denote by $\text{Vol}(\sigma)$ the $j$-dimensional volume of a $j$-dimensional submanifold $\sigma \in S$ computed with respect to the measure on $\sigma$ induced by $\rho$. For $a \in (0, 1]$ define the intermediate volume growth of $\sigma \in S$ by

$$
\text{vol}^a(\sigma; \varphi) = \liminf_{n \to \infty} \frac{\log \text{Vol}(\varphi^n(\sigma))}{n^a} \in [0, \infty],
$$

and define the intermediate volume growth of $\varphi$ by

$$
\text{vol}^a(\varphi) = \sup_{\sigma \in S} \text{vol}^a(\sigma; \varphi) \in [0, \infty].
$$

Notice that these invariants do not depend on the choice of $\rho$. Finally define the volume growth exponent of $\varphi$ by

$$
\exp_{\text{vol}}(\varphi) := \inf \{a \mid \text{vol}^a(\varphi) < \infty\}.
$$

Thus $\exp_{\text{vol}}(\varphi)$ is “the largest $a \in [0, 1]$ such that some submanifold grows under $\varphi$ like $e^{na}$.” Since $\text{vol}^1(\varphi) \leq (\dim X) \max_{x \in X} \|D\varphi(x)\| < \infty$ we have $\exp_{\text{vol}}(\varphi) \in [0, 1]$. The intermediate volume growth and the volume growth exponent of a smooth flow $\varphi^t$ on $X$ are defined as $\text{vol}^a(\varphi^1)$ and $\exp_{\text{vol}}(\varphi^1)$.

Remark 1.5. By a celebrated result of Yomdin [30] and Newhouse [18], the volume growth $\text{vol}^1(\varphi)$ agrees with the topological entropy $h_{\text{top}}(\varphi)$. Proceeding as above define for $a \in (0, 1]$ the intermediate topological entropy $h_{\text{top}}^a(\varphi)$. We unfortunately do not know whether $\text{vol}^1(\varphi) = h_{\text{top}}^a(\varphi)$ or at least $\text{vol}^a(\varphi) \leq h_{\text{top}}^a(\varphi)$ also for $a \in (0, 1)$.

Uniform lower bounds for the volume growth or the topological entropy of geodesic flows were found in [2, 20, 21, 23], and these results were generalized to Reeb flows in [15] and, on a polynomial scale, in [5]. Results in [5, 15] show that the volume growth exponent
exp_{vol}(\varphi_\alpha) is bounded from below by the maximum of the growth exponent of the function 
\( n \mapsto \text{rank} i^*_n H_\ast(\Omega^n \Sigma M) \) and the growth exponent of the growth function of the fundamental 
group \( \pi_1(M) \). In particular, for manifolds with fundamental group of exponential growth it is shown in [15] that 
\( \text{vol}^1(\varphi_\alpha) > 0 \) for any Reeb flow \( \varphi_\alpha \) on \( S^r \Sigma M \). Since virtually 
polycyclic groups are either virtually nilpotent or have exponential growth, we therefore 
restrict ourselves now to manifolds with virtually nilpotent fundamental group. Since all 
other fundamental groups of closed manifolds are believed to have exponential growth, this 
is a minor hypothesis.

**Corollary 1.6.** Assume that \( M \) is not of finite type, has virtually nilpotent fundamental 
group, and that there is a field \( F \) such that \( H_m(\widetilde{M}; F) \) is infinite-dimensional. Let \((S^r \Sigma M, \xi)\) 
be the spherizations of \( M \). Then

\[ \text{vol}^{1/2}(S^r \Sigma M; \varphi_\alpha) > 0 \]

for every fiber \( S^r \Sigma M \) of \( S^r \Sigma M \) and every Reeb flow \( \varphi_\alpha \) on \((S^r \Sigma M, \xi)\). In particular, \( \text{vol}^{1/2}(\varphi_\alpha) > 0 \) and \( \exp_{vol}(\varphi_\alpha) \geq 1/2 \) for every Reeb flow \( \varphi_\alpha \) on \((S^r \Sigma M, \xi)\).

**The method.** We end this introduction with comparing our approach to previous approaches. As mentioned earlier, lower bounds for the homology of the sublevel sets \( \Omega^T M \) 
follow easily from the growth of the fundamental group \( \pi_1(M) \), since its growth is the 
growth of \( H_0(\Omega^T M) \). If \( \pi_1(M) \) is finite, Gromov found a way to bound the rank of \( H_\ast(\Omega^T M) \) 
from below by the rank of \( \bigoplus_{i=0}^c \text{H}_i(\Omega\Sigma M) \), where \( c \) is a constant depending only on the Rie-
mannian metric. His ingenious argument is purely geometric, see [8, 10, 20], and uses 
that \( \tilde{M} \) is compact. In [22, 23] Paternain–Petean generalized Gromov’s construction to 
manifolds with infinite fundamental group, by mapping simply connected complexes \( K \) 
with rich loop space homology into \( \tilde{M} \). This method gives good lower bounds for the 
rank of \( H_\ast(\Omega^T M) \) if one can find such complexes and mappings \( f: K \rightarrow \tilde{M} \) for which 
the homology of \( \Omega f(\Omega K) \) still has large rank. This works well for many manifolds, e.g. for 
most connected sums, for manifolds of finite type with \( \pi_\ast(\Omega M) \otimes \mathbb{Q} \) infinite-dimensional, 
and in small dimensions, see [22][23]. For general manifolds of non-finite type, however, we 
were not even able to prove linear growth of the rank of \( H_\ast(\Omega^T M) \) by this method. Citing 
G. Paternain, “it is as if one has so much topology that it becomes unmanageable.”

Our approach to finding lower bounds for the rank of \( H_\ast(\Omega^T M) \) for manifolds of non-
finite type is neither geometric nor topological, but algebraic. The main point is that we 
use the Hopf algebra structure of \( H_\ast(\Omega \Sigma \widetilde{M}; F) \) over the group ring \( F[\pi_1(M)] \) for a suitable 
field \( F \), and prove a Poincaré–Birkhoff–Witt type theorem for this algebra. These algebraic 
results (specifically Theorems 2.1 and 3.5 and Proposition 4.1) are the main findings of this 
paper. They cover Sections 2 to 4. Theorem 1.1 that is proven in Section 5 follows readily 
from these algebraic results, and Corollaries 1.3 and 1.6 that are proven in Section 7 follow 
from Theorem 1.1 in the usual way.

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2. Healthy $G$-vector spaces

Let $G$ be a group, $V$ a vector space over the field $\mathbb{F}$, and $\rho: G \to \text{Aut}(V)$ a representation of $G$. Since the representation is fixed throughout our discussion, we abbreviate $g v = \rho(g)v$ for $g \in G$ and $v \in V$, and we consider the $G$-vector space $V$ as a module over the group ring $\mathbb{F}G$. A group $G$ is called polycyclic if there exists a subnormal series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that all factors $G_i/G_{i-1}$, $1 \leq i \leq n$ are cyclic. A group $G$ is called a virtually polycyclic if $G$ has a polycyclic subgroup of finite index. Eleven characterisations of polycyclic groups are given in [29, Proposition 4.1]. In this section we prove the following result.

**Theorem 2.1.** Assume that $G$ is a virtually polycyclic group acting linearly on an infinite-dimensional vector space $V$ such that $V$ viewed as an $\mathbb{F}G$-module is finitely generated. Then there exists $v \in V$ and $g \in G$ such that the sequence of vectors $(g^i v)_{i \in \mathbb{Z}}$ is linearly independent.

Before embarking on the proof of the theorem we introduce some notation. For the following discussion it is irrelevant that $G$ is virtually polycyclic. Given $g \in G$ and $v \in V$ we denote the subvector space of $V$ spanned by the vectors $g^i v$ by

$$W^g_v := \langle g^i v \mid i \in \mathbb{Z} \rangle \subset V.$$

**Definition 2.2.** A vector $v \in V$ is healthy if there exists $g \in G$ such that $W^g_v$ is infinite-dimensional. A vector $v \in V$ which is not healthy is called sick.

The reason why we are interested in finding healthy vectors is the following observation from linear algebra.

**Lemma 2.3.** If $g \in G$ and $v \in V$, then the following are equivalent.

(i) $W^g_v$ is infinite-dimensional.

(ii) The sequence of vectors $(g^i v)_{i \in \mathbb{Z}}$ is linearly independent.

**Proof.** That (ii) implies (i) is clear. It remains to show that if (ii) does not hold, then $W^g_v$ is finite-dimensional. The case $v = 0$ is trivial as well, so we assume that $v \neq 0$. Since then $g^{-i}(g^i v) = v \neq 0$, we conclude that $g^i v \neq 0$ for every $i \in \mathbb{Z}$. Hence, if (ii) does not hold, there exist $m < n \in \mathbb{Z}$ and scalars $a_i \in \mathbb{F}$ for $m \leq i \leq n - 1$ such that

$$g^m v = \sum_{i=m}^{n-1} a_i g^i v, \quad a_m \neq 0.$$  

(2)

By applying $g^{-m}$ to (2) we can assume without loss of generality that $m = 0$. Applying $g$ and $g^{-1}$ to (2) we obtain inductively that

$$W^g_v = \langle g^i v \mid 0 \leq i \leq n - 1 \rangle.$$  

(3)

This shows that $W^g_v$ is finite-dimensional. The lemma follows. \qed
If \( v \neq 0 \) is a sick vector, we can define in view of (3) the function \( d_v : G \to \mathbb{N} \) by
\[
(4) \quad d_v(g) := \min \left\{ n \in \mathbb{N} \mid W^g_v = \langle g^i v \mid 0 \leq i \leq n - 1 \rangle \right\} = \dim W^g_v.
\]

**Definition 2.4.** The \( G \)-vector space \( V \) is called *sick* if all its vectors are sick. A \( G \)-vector space \( V \) which is not sick is called *healthy*, i.e., \( V \) contains a healthy vector.

Note that a healthy vector space still contains sick vectors. Indeed, the zero vector is always sick. Moreover, observe that the concept of a healthy \( G \)-vector space is only of interest if both the cardinality of the group and the dimension of the vector space are infinite, since otherwise \( V \) is automatically sick.

Although we fix the representation \( \rho \) throughout, we sometimes have to restrict \( \rho \) to subgroups \( H < G \). In this situation, we say that \( V \) is \( H \)-*healthy* if the restriction of \( \rho \) to \( H \) is healthy.

**Lemma 2.5.** Assume that \( H < G \) is a subgroup of finite index. Then \( V \) is \( G \)-healthy if and only if \( V \) is \( H \)-healthy.

**Proof.** The implication from \( H \)-healthy to \( G \)-healthy is obvious. We now assume that \( V \) is \( G \)-healthy and show that \( V \) is \( H \)-healthy as well. Since \( V \) is \( G \)-healthy, there exists a vector \( v \in V \) and a group element \( g \in G \) such that \( W^g_v \) is infinite-dimensional. Denote the right coset \( gH \) in \( G/H \) by \( [g] \).

We first consider the special case where the subgroup \( H \) is normal. Then \( G/H \) is a group, and its order \( n := |G/H| \) is finite by assumption. Hence \( [g]^n = \text{id} \in G/H \), or equivalently \( h := g^n \in H \). By Lemma 2.3 we conclude from the fact that \( W^g_v \) is infinite-dimensional that \( W^h_v \) is infinite-dimensional as well. This proves that \( V \) is \( H \)-healthy in the special case that \( H \trianglelefteq G \) is normal.

In the general case where \( H < G \) is not necessarily normal, we consider the normal core of \( H \) in \( G \) defined by
\[
\text{Core}(H) := \bigcap_{g \in G} g^{-1}Hg.
\]
Note that \( \text{Core}(H) \) is a subgroup of \( H \) which is normal in \( G \). It is actually the biggest normal subgroup of \( G \) contained in \( H \). Moreover, it still has finite index in \( H \), see for instance [24, Theorem 3.3.5]. In view of what we already proved, we therefore conclude that \( V \) is \( \text{Core}(H) \)-healthy. Since \( \text{Core}(H) < H \) it follows that \( V \) is \( H \)-healthy as well. This finishes the proof of the lemma.

If \( H < G \) is a subgroup and \( v \in V \), we abbreviate by
\[
W^H_v = \langle Hv \rangle
\]
the subspace of \( V \) spanned by the \( H \)-orbit of the vector \( v \). The next lemma is our main tool to give an inductive proof of Theorem 2.1.

**Lemma 2.6.** Assume that \( H \trianglelefteq G \) is a normal subgroup such that \( G/H \) is cyclic. Suppose further that \( v \in V \) is sick and \( W^H_v \) is finite-dimensional. Then \( W^G_v \) is finite-dimensional as well.
Proof. Abbreviate \( m = \dim W_v^H \). Then there exist \( \xi_1, \ldots, \xi_m \) in the group ring \( \mathbb{F}H \) such that
\[
W_v^H = \langle \xi_i v \mid 1 \leq i \leq m \rangle.
\]
Choose \( g \in G \) such that \([g] \in G/H\) is a generator. Assume first that \( G/H \cong \mathbb{Z}_n \) is finite. Then
\[
W_v^G = \langle g^j \xi_i v \mid 1 \leq i \leq m, 0 \leq j < n \rangle.
\]
Hence \( \dim W_v^G \leq nm \) is finite. Assume now that \( G/H \cong \mathbb{Z} \) is infinite. The case that \( v = 0 \) is trivial. We therefore assume that \( v \neq 0 \). Since \( v \) is sick by assumption, we have the function \( d_v : G \to \mathbb{N} \) from (4), and we abbreviate \( n = d_v(g) \). Our aim is to show that
\[
W_v^G = \langle g^j \xi_i v \mid 1 \leq i \leq m, 0 \leq j < n \rangle.
\]
For this purpose we abbreviate the right hand side by
\[
X := \langle g^j \xi_i v \mid 1 \leq i \leq m, 0 \leq j < n \rangle.
\]
That \( X \subset W_v^G \) is clear. We have to check the other inclusion, namely that for every \( \eta \in \mathbb{F}G \) it holds that
\[
\eta v \in X.
\]
Since the right coset \([g]\) is a generator of \( G/H \cong \mathbb{Z}_n \), there exist \( \eta_j \in \mathbb{F}H \) with \( \eta_j \neq 0 \) for only finitely many \( j \in \mathbb{Z} \) such that
\[
\eta = \sum_{j \in \mathbb{Z}} \eta_j g^j.
\]
In view of the definition of \( n = d_v(g) \), there exists \( \zeta \in \mathbb{F}G \) of the form
\[
\zeta = \sum_{j=0}^{n-1} \zeta_j g^j, \quad \zeta_j \in \mathbb{F}H
\]
such that
\[
\eta v = \zeta v.
\]
Since \( H \) is normal in \( G \), there exist \( \zeta'_j \in \mathbb{F}H \) for \( 0 \leq j < n \) such that
\[
\zeta = \sum_{j=0}^{n-1} g^j \zeta'_j.
\]
In view of (5) we conclude that for \( 0 \leq j < n \) we have
\[
\zeta'_j v \in \langle \xi_i v \mid 1 \leq i \leq m \rangle.
\]
Therefore \( \eta v = \zeta v \in X \). This proves (6). We have shown that
\[
\dim W_v^G \leq nm = d_v(g) \cdot \dim W_v^H,
\]
and therefore \( W_v^G \) is finite-dimensional. The proof of the lemma is complete. \( \square \)

We are now in position to prove the main result of this section.
Proof of Theorem 2.4. In view of Lemma 2.3 it suffices to show that \( V \) is healthy. We argue by contradiction and assume that \( V \) is sick. By assumption \( G \) is virtually polycyclic, hence contains a polycyclic subgroup \( H \) of finite index. By Lemma 2.5 it follows that \( V \) is \( H \)-sick as well. Since \( H \) is polycyclic, we conclude by applying Lemma 2.6 inductively that \( W^H_v \) is finite-dimensional for every \( v \in V \). Because \( H \) has finite index in \( G \) it follows that \( W^G_v \) is finite-dimensional for every \( v \in V \) as well. By assumption \( V \) is a finitely generated \( \mathbb{F}^G \)-module. Therefore we deduce that \( V \) is finite-dimensional, contradicting the assumption of the theorem. The proof is complete. \( \square \)

3. Spicy Hopf algebras

In this section we consider Hopf algebras \( V \) over a group ring \( \mathbb{F}^G \), endowed with a filtration \( V^r \), \( r > 0 \), and exhibit a property of \( V \) that guarantees that \( \dim V^r \) grows at least like \( e^{\sqrt[r]{r}} \).

3.1. Hopf algebras over \( \mathbb{F}^G \). Let \( G \) be a group, \( \mathbb{F} \) a field and \( \mathbb{F}^G \) the group ring. We first explain the notion of a Hopf algebra over \( \mathbb{F}^G \). This might be not completely standard, since usually Hopf algebras are defined over rings which are commutative, a requirement that our group ring in general does not fulfill. However, the feature which distinguishes a group ring from other non-commutative rings is that if \( V \) and \( W \) are two left modules over \( \mathbb{F}^G \), then we can define on their tensor product \( V \otimes W = V \otimes_{\mathbb{F}^G} W \) still the structure of a left \( \mathbb{F}^G \)-module by using the tensor of the two representations: \( g(v \otimes w) := (gv) \otimes (gw) \) for \( v \in V, w \in W \) and \( g \in G \). A product is then an \( \mathbb{F}^G \)-linear map \( \mu: V \otimes V \to V \), or equivalently an \( \mathbb{F} \)-bilinear map \( \mu: V \times V \to V \) satisfying \( \mu(gv, gw) = g \mu(v, w) \) for \( g \in G \) and \( v, w \in V \). Dually, a coproduct is then an \( \mathbb{F}^G \)-linear map \( \Delta: V \to V \otimes V \). To avoid terrible headaches we assume in addition that our product is always associative, although this requirement is probably not necessary for the results of this section. We abbreviate the product by \( vw = \mu(v, w) \). If \( V \) in addition is graded, i.e. \( V = \bigoplus_{i=0}^{\infty} V_i \), where each \( V_i \) is an \( \mathbb{F}^G \)-submodule of \( V \), then we grade the tensor product by \( (V \otimes V)_k = \bigoplus_{i+j=k} V_i \otimes V_j \) and require in addition that the product and coproduct preserve the grading. The product endows the tensor product \( V \otimes V \) again with a product which is defined on homogeneous elements by the Koszul sign convention

\[
(v \otimes w)(x \otimes y) = (-1)^{\deg(w) \deg(x)} vx \otimes wy
\]

where \( \deg(v) \) denotes the degree of a homogeneous element \( v \). Given a left module \( V \) over \( \mathbb{F}^G \) and a product \( \mu \) and coproduct \( \Delta \) as above, we call the triple \( (V, \mu, \Delta) \) a bialgebra over \( \mathbb{F}^G \) if \( \mu \) and \( \Delta \) are compatible in the sense that \( \Delta: V \to V \otimes V \) is a homomorphism of algebras. The bialgebra \( (V, \mu, \Delta) \) is called connected if \( V_0 = \mathbb{F} \) is one-dimensional and if \( 1 \in \mathbb{F} \) is also the unit for the multiplication \( \mu \).

Definition 3.1. A connected graded bialgebra \( (V, \mu, \Delta) \) over \( \mathbb{F}^G \) is called a Hopf algebra over \( \mathbb{F}^G \) if for every homogeneous element \( v \) of positive degree \( \deg(v) \) the coproduct satisfies

\[
\Delta v = 1 \otimes v + v \otimes 1 + \sum v_i \otimes v'_i
\]
with \( v_i \) and \( v'_i \) of positive degree. A vector \( v \in V \) is primitive if \( \Delta v = 1 \otimes v + v \otimes 1 \).

### 3.2. Filtrations

To make our Hopf algebra spicy we shall suppose that both the vector space \( V \) and the group ring \( \mathbb{F}G \) are filtered. More precisely, we assume that \( V \) can be exhausted by a sequence of finite-dimensional vector spaces, i.e., for every real number \( r > 0 \) there exists a finite-dimensional subspace \( V^r \subset V \) such that \( V^r \subset V^s \) for \( r \leq s \) and \( V = \bigcup_{r > 0} V^r \). Define the value of \( v \in V \) by

\[
|v| := \min \{ r \mid v \in V^r \}.
\]

Notice that for scalars \( a_i \in \mathbb{F} \) and vectors \( v_i \in V \) we have

\[
|a_1v_1 + \cdots + a_nv_n| \leq \max \{|v_i|\}.
\]

Dear reader, please do not confuse the value \( |v| \) of \( v \) and its degree \( \text{deg}(v) \), in case \( v \) is homogeneous. They are not related to each other. Also note that the grading is indicated by a subscript while the filtration degree is indicated by a superscript.

To get a filtration on the group ring as well, suppose that the group \( G \) is endowed with a length function, namely a function \( L: G \to \mathbb{R}_{\geq 0} \) satisfying

\[
L(g) = L(g^{-1}), \quad L(gh) \leq L(g) + L(h), \quad g, h \in G.
\]

In the following we abbreviate \( |g| = L(g) \). Via the length function we can define a filtration on the group ring: For \( r \geq 0 \) we define \( \mathbb{F}G^r \) to be the subvector space of \( \mathbb{F}G \) consisting of finite sums \( \xi = \sum_{g \in G} \xi_g g \) satisfying \( \xi_g = 0 \) whenever \( |g| > r \).

**Definition 3.2.** The Hopf algebra \( (V, \mu, \Delta) \) over \( \mathbb{F}G \) is called spicy if the vector space \( V \) is endowed with a filtration such that for \( v, w \in V \) and \( g \in G \) it holds that

\[
|vw| \leq |v| + |w|, \quad |gv| \leq |g| + |v|.
\]

We next introduce a condition on spicy Hopf algebras which will guarantee nontrivial lower bounds for the growth of \( \dim V^r \).

**Definition 3.3.** Assume that \( (V, \mu, \Delta) \) is a spicy Hopf algebra over \( \mathbb{F}G \). A primitive sequence is a sequence \((v_i)_{i \in \mathbb{N}}\) satisfying the following conditions.

(i) The vectors \( v_i, i \in \mathbb{N} \), are linearly independent, primitive, and of equal positive degree.

(ii) There exists a constant \( c > 0 \) such that \( |v_i| \leq ci \).

**Remark 3.4.** The notion of a primitive sequence does not involve the action of the group \( G \). However, we shall see later that the action of \( G \) is useful to construct primitive sequences.

**Theorem 3.5.** Assume that \( (V, \mu, \Delta) \) is a spicy Hopf algebra over \( \mathbb{F}G \) which admits a primitive sequence \((v_i)_{i \in \mathbb{N}}\). Then the function \( r \mapsto \dim V^r \) grows at least like \( e^{\sqrt{r}} \).

**Proof.** Let \( I = \{i_1, \ldots, i_\ell\} \subset \mathbb{N} \) be a finite subset of distinct numbers which we totally order by \( i_1 < i_2 < \ldots < i_\ell \). We abbreviate

\[
v_I = v_{i_1}v_{i_2} \cdots v_{i_\ell} \in V.
\]
Let $m$ be the common degree of the vectors $v_i$. In view of property (i) of a primitive sequence, and since $\deg v_I = m \# I$, Proposition 3.5 below shows that the vectors $v_I$, $I \subset \mathbb{N}$, are linearly independent. We can assume without loss of generality that the constant $c$ in property (ii) of a primitive sequence is 1. Then we have

$$|v_I| \leq \sum_{j=1}^{\ell} |v_{ij}| \leq \sum_{j=1}^{\ell} i_j.$$  

For $n \in \mathbb{N}$ denote by $q(n)$ the number of partitions of $n$ into distinct parts. We have shown that

$$\dim V^n \geq q(n).$$

By Euler’s theorem, the number of partitions of $n$ into distinct parts coincides with the number of its partitions into odd parts, see for example [17, Corollary 1.2]. The asymptotics of this sequence coincides up to a constant with the asymptotics of the partition function (see for example [17 Chapter 16]), which grows like $e^{C\sqrt{n}}$ for a positive constant $C$ according to a theorem of Hardy and Ramanujan, see for example [17 Chapter 15].  

Combining Theorems 2.4 and 3.5 we obtain the following result.

**Corollary 3.6.** Assume that $(V, \mu, \Delta)$ is a spicy Hopf algebra over $\mathbb{F}G$ where $G$ is a virtually polycyclic group. Assume further that $\oplus_{i=m} V_i$ is finite-dimensional and that $V_m$ is infinite-dimensional but finitely generated as an $\mathbb{F}G$-module. Then the function $r \mapsto \dim V^r$ grows at least like $e^{C\sqrt{r}}$.

**Proof.** By Theorem 2.1 there exists $v \in V_m$ and $g \in G$ such that the vectors $v_i := g^i v$, $i \in \mathbb{N}$, are linearly independent. The vectors $v_i$ have equal degree $m$. Using the defining properties of a filtration, we estimate

$$|v_i| = |g^i v| \leq |g|^i + |v| \leq i |g| + |v|.$$  

With $c := |g| + |v|$ we thus have $|v_i| \leq c i$. The sequence $(v_i)_{i \in \mathbb{N}}$ therefore meets all the properties of a primitive sequence, except that the $v_i$ may fail to be primitive. To correct this, we consider the linear map $A: \langle v_1, v_2, \ldots \rangle \to V_m \otimes V_m$ given by

$$A(v) = \Delta v - 1 \otimes v - v \otimes 1.$$  

Since the vector space $\oplus_{j<m} V_j$ is finite-dimensional, the subvector space of $V_m \otimes V_m$ spanned by the elements $u_i \otimes u'_i$ with $u_i, u'_i \in \oplus_{j<m} V_j$ is finite-dimensional. Since $A$ takes values in this finite-dimensional vector space, $k := \text{rank } A$, which equals the dimension of $\langle v_1, v_2, \ldots \rangle / \ker A$, is finite. In particular, $\ker A$ in infinite-dimensional. We shall construct by induction a sequence $w_1, w_2, \ldots$ of linearly independent elements in $\ker A$ such that $|w_i| \leq c(k+1)$. The sequence $(w_i)_{i \in \mathbb{N}}$ is then a primitive sequence in $V$, and the corollary follows in view of Theorem 3.5.

The restriction of the map $A$ to $\langle v_1, v_2, \ldots, v_{k+1} \rangle$ has a non-trivial kernel. Let $w_1 := a_1 v_1 + \cdots + a_{k+1} v_{k+1}$ be a non-trivial element in this kernel. Then by (5), $|w_1| \leq c(k+1)$. Next, the restriction of $A$ to $\langle v_{k+2}, v_{k+3}, \ldots, v_{2(k+1)} \rangle$ has a non-trivial kernel. Let $w_2 := a_{k+2} v_{k+2} + \cdots + a_{2(k+1)} v_{2(k+1)}$ be a non-trivial element in this kernel. Then by (5),
\[ |w_2| \leq 2c(k + 1), \text{ and } w_1, w_2 \text{ are linearly independent because } v_1, \ldots, v_{2(k+1)} \text{ are linearly independent. Proceeding in this way we construct linearly independent vectors } w_1, w_2, \ldots \text{ such that } |w_i| \leq ic(k + 1). \]

4. A Quantum Poincaré–Birkhoff–Witt theorem

Consider a graded bialgebra \((V, \Delta, \mu)\) over the field \(F\) which is connected (i.e., \(V_0 = F\) is one-dimensional) and is such that \(1 \in F\) also serves as the unit for the multiplication \(\mu\). (The group \(G\) plays no role in this section.) We again assume that \(\mu\) is associative, and write \(vw = \mu(v, w)\). Also recall that \(v \in V\) is \textit{primitive} if \(\Delta(v) = 1 \otimes v + v \otimes 1\). Suppose that for \(N \in \mathbb{N}\) we are given linearly independent and primitive elements \(v_1, \ldots, v_N \in V_m\) of equal positive degree \(m \in \mathbb{N}\). Abbreviate \(N_N = \{1, \ldots, N\}\). We order \(I \subset N_N\) using the canonical order of \(N_N\), namely we write \(I = \{i_1, \ldots, i_\ell\}\) satisfying \(i_1 < i_2 < \ldots < i_\ell\). We then abbreviate \(v_I = v_{i_1}v_{i_2}\ldots v_{i_\ell}\), where we use the convention that \(v_\emptyset = 1\).

The following proposition reminiscent of the Poincaré–Birkhoff–Witt theorem seems to be known to people working in quantum group theory, cf. [27, Theorem 1.5(b)]. However, since the Hopf algebras arising in the theory of quantum groups are usually not graded, we provide a proof for the readers convenience.

\textbf{Proposition 4.1.} Assume that \(v_1, \ldots, v_N\) are linearly independent and primitive vectors in \(V\). Then the vectors \(v_I, I \subset N_N,\) are linearly independent.

\textbf{Proof.} The crucial ingredient in the proof is the computation of the coproduct of the elements \(v_I\). To determine the signs in this formula, we use the following convention. Given a subset \(I\) of \(\{1, \ldots, N\}\) we order \(I = \{i_1, \ldots, i_\ell\}\) and its complement \(I^c = \{j_1, \ldots, j_{N-\ell}\}\). This determines a permutation \((1, \ldots, N) \mapsto (i_1, \ldots, i_\ell, j_1, \ldots, j_{N-\ell})\). We denote by \(\sigma(I)\) the signum of this permutation.

\textbf{Lemma 4.2.} The coproduct of \(v_{N_N}\) is given by

\[ \Delta(v_{N_N}) = \sum_{I \subset N_N} \sigma(I)^m v_I \otimes v_{I^c}. \]

Hence if \(m\) is even, then \(\Delta(v_{N_N}) = \sum_{I \subset N_N} v_I \otimes v_{I^c}\), and if \(m\) is odd, then \(\Delta(v_{N_N}) = \sum_{I \subset N_N} \sigma(I)v_I \otimes v_{I^c}\).

\textbf{Proof.} We prove the lemma by induction on \(N\). For \(N = 1\) the lemma is an immediate consequence of the fact that \(v_1\) is primitive. For the induction step we assume that the
formula holds for \( N - 1 \). We compute
\[
\Delta(v_N) = \Delta(v_{N-1} v_N) \\
= \Delta(v_{N-1}) \Delta(v_N) \\
= \left( \sum_{I \subset \mathbb{N}_{N-1}} \sigma(I)^m v_I \otimes v_{N-1 \setminus I} \right) \left( 1 \otimes v_N + v_N \otimes 1 \right) \\
= \sum_{I \subset \mathbb{N}_{N-1}} \sigma(I)^m v_I \otimes (v_{N-1 \setminus I} v_N) \\
+ \sum_{I \subset \mathbb{N}_{N-1}} (-1)^{\deg(v_{N-1 \setminus I}) \deg(v_N)} \sigma(I)^m (v_I v_N) \otimes v_{N-1 \setminus I}.
\]

Since \( \deg(v_N) = m \) and \( \deg(v_{N-1 \setminus I}) = m(N - 1 - |I|) \), we have for each \( I \subset \mathbb{N}_{N-1} \) that
\[
(-1)^{\deg(v_{N-1 \setminus I}) \deg(v_N)} \sigma(I)^m = (-1)^{m^2(N-1-|I|)} \sigma(I)^m = (-1)^{m(N-1-|I|)} \sigma(I)^m = \sigma(I \cup \{N\})^m.
\]

The previous sum therefore becomes
\[
= \sum_{I \subset \mathbb{N}_{N-1}} \sigma(I)^m v_I \otimes v_{N \setminus I} \\
+ \sum_{I \subset \mathbb{N}_{N-1}} \sigma(I \cup \{N\})^m v_{I \cup \{N\}} \otimes v_{N \setminus (I \cup \{N\})} \\
= \sum_{I \subset \mathbb{N}_N} \sigma(I)^m v_I \otimes v_{N \setminus I}.
\]

This proves the induction step and hence the lemma. \( \square \)

**Proof of Proposition 4.1.** Recall that by assumption the vectors \( v_1, \ldots, v_N \) are linearly independent and have all the same degree \( m \). By looking at the degree we see that it suffices to show that for every \( k \leq N \) the vectors \( v_I \) with \( I \subset \mathbb{N}_N \) and \( |I| = k \) are linearly independent. For \( k = 1 \) this is an assumption. For the induction step we assume this assertion for all \( k \leq n - 1 \) where \( n \leq N \). Since the coproduct is linear, it suffices to show that the vectors \( \Delta(v_I) \) with \( I \subset \mathbb{N}_N \), \( |I| = n \) are linearly independent. It follows from the induction hypothesis that for every \( 0 < k < n \) the vectors \( v_I \otimes v_J \) with \( I, J \subset \mathbb{N}_N \) and \( |I| = k \), \( |J| = n - k \) are linearly independent. This and the formulas
\[
\Delta(v_I) = \sum_{J \subset I} \sigma(J)^m v_J \otimes v_{I \setminus J},
\]
that follow from (the proof of) Lemma 4.2, complete the induction step. \( \square \)
5. Proof of Theorem 1.1

We shall deduce Theorem 1.1 from Corollary 3.6. We start with saying briefly “who is who” in Corollary 3.6. Let \( M \) be a closed connected manifold, and let \( \tilde{M} \) be its universal covering space. We take as \( G \) the fundamental group \( \pi_1(M) \) of \( M \), and as \( V \) we take the homology \( H_*(\tilde{M}; \mathbb{F}) \) over a suitable field \( \mathbb{F} \). The product \( \mu \) will be the Pontryagin product given by concatenation of loops, and the coproduct will simply come from the diagonal map \( \Omega \tilde{M} \to \Omega \tilde{M} \times \Omega \tilde{M}, x \mapsto (x, x) \). Now fix a Riemannian metric on \( M \). The filtration on \( V \) will be given by taking \( |v| \) as the smallest \( r \) for which \( v \) can be represented by a cycle of based loops of length at most \( r \), and for \( g \in \pi_1(M) \) we take \( L(g) \) to be (half of) the length of the shortest curve representing \( g \).

We shall show in Sections 5.1 and 5.2 that with these choices, \((V, \mu, \Delta)\) is a spicy Hopf algebra over \( \mathbb{F}G \). In Section 5.3 we show that for \( M \) of non-finite type with \( G = \pi_1(M) \) virtually polycyclic, the dimension assumptions on \( \oplus V_i \) are met.

5.1. The Hopf-algebra structure on \( H_*(\Omega \tilde{M}; \mathbb{F}) \). Let \( M \) be a closed connected manifold, and let \( p: \tilde{M} \to M \) be its universal covering space. Fix \( p \in M \) and \( \tilde{p} \in \tilde{M} \) over \( p \). The spaces \( \Omega_0 M \) and \( \Omega \tilde{M} \) of contractible continuous loops based at \( p \) and \( \tilde{p} \), respectively, are canonically identified. Conjugation of loops in \( \Omega_0 M \) by loops in \( M \) based at \( p \) yields an action of the fundamental group \( G = \pi_1(M, p) \) on \( H_*(\Omega_0 M) = H_*(\Omega \tilde{M}) \): Given a cycle \( C = \{ \gamma \} \) of loops in \( \Omega \tilde{M} \) based at \( \tilde{p} \) and given \( g \in G \), the class \( g[C] \) is defined as the class represented by the cycle of loops \( \{ c_g^{-1} \circ g \gamma \circ c_g \} \), where \( c_g \) is the lift to \( \tilde{M} \) starting at \( \tilde{p} \) of a loop in \( M \) in class \( g \), and \( g \gamma \) is the lift of \( \gamma \) starting at \( \gamma \).

Let \( \mathbb{F} \) be the field, and abbreviate \( V_* := H_*(\Omega \tilde{M}; \mathbb{F}) \). The action of \( G \) on \( V \) extends to an action of \( \mathbb{F}G \) on \( V \). Concatenation of loops in \( \tilde{M} \) based at \( \tilde{p} \) induces a product \( \mu: V \otimes V \to V \), called the Pontryagin product. Since concatenation of loops is associative up to homotopy, \( \mu \) is associative. It follows from the definition of the action of \( G \) that \( \mu \) is \( \mathbb{F}G \)-linear. In order to define the coproduct \( \Delta: V \to V \otimes V \), we consider, more generally, a topological space \( X \) and the diagonal map \( \delta_X: X \to X \times X, x \mapsto (x, x) \). Since we work over a field \( \mathbb{F} \), the cross product

\[
H_*(X; \mathbb{F}) \otimes H_*(X; \mathbb{F}) \to H_*(X \times X; \mathbb{F})
\]

is an isomorphism by the Künneth formula. We can therefore define \( \Delta_X: H_*(X; \mathbb{F}) \to H_*(X; \mathbb{F}) \otimes H_*(X; \mathbb{F}) \) by

\[
\Delta_X := \times^{-1} \circ (\delta_X)_*.
\]

Assume now in addition that \( X \) is a path-connected H-space, with product \( \nu \).

Lemma 5.1. The homology \( H_*(X; \mathbb{F}) \), with product induced by \( \nu \) and with coproduct \( \Delta_X \), is a Hopf algebra.

We refer to [28, Theorem 7.15] for the proof. For the readers convenience, we verify that for every homogeneous element \( v \) of positive degree, \( \Delta_X v \) has the form (7). Let
\[ p: X \times X \to X, \ (x, y) \mapsto x, \] be the projection on the first factor. Then
\[ p \circ \delta_X = id_X. \] (9)

For an element \( u = v'_n \otimes 1 + 1 \otimes v''_n + \sum v'_i \otimes v''_j \in \oplus_{i+j=n} H_i(X) \otimes H_j(X) \) with \( \deg v'_i < n \) we have
\[ (p_n \circ \times) u = v'_n \] (10)
by the geometric definition of the cross product (see e.g. \[12, \S 3.B\]). Now write \( \Delta_X v = v'_n \otimes 1 + 1 \otimes v''_n + \sum v'_i \otimes v''_j \) with \( \deg v'_i, \deg v''_j \geq 1 \). Using (9), the definition of \( \Delta_X \) and (10) we get
\[ v = p_n \circ (\delta_X)_n v = p_n \circ \times \Delta_X v = p_n \circ \times \left( v'_n \otimes 1 + 1 \otimes v''_n + \sum v'_i \otimes v''_j \right) = v'_n. \]
Similarly we find \( v''_n = v \), and so \( \Delta_X v = v \otimes 1 + 1 \otimes v + \sum v'_i \otimes v''_j \) with \( \deg v'_i, \deg v''_j \geq 1 \).

\[ \square \]

Since \( \widetilde{M} \) is simply connected, \( \Omega \widetilde{M} \) is path-connected. Hence \( V_0 = H_0(\Omega \widetilde{M}; \mathbb{F}) \cong \mathbb{F} \) is one-dimensional. Moreover, 1 \( \in \mathbb{F} \) corresponds to the class of the constant path \( \tilde{p} \in \widetilde{M} \), which is the unit for the Pontryagin product \( \mu \). Applying Lemma 5.1 with \( X = \Omega \widetilde{M} \) and writing \( \Delta \) for \( \Delta_{\Omega \widetilde{M}} \) we obtain that \( (V, \mu, \Delta) \) is a Hopf algebra over \( \mathbb{F} \).\( G \). (We have already noticed that \( \mu \) is \( \mathbb{F} G \)-linear. The definition of the \( G \)-actions on \( V \) and on \( V \otimes V \) shows that also \( \Delta \) is \( \mathbb{F} G \)-linear.)

5.2. The filtration on \( H_*(\Omega \widetilde{M}; \mathbb{F}) \). Fix a Riemannian metric \( \rho \) on \( M \), and let \( \tilde{\rho} = \text{pr}^* \rho \) be the corresponding Riemannian metric on \( \widetilde{M} \). Given a piecewise smooth curve \( \gamma \) in \( \widetilde{M} \), we denote by \( \ell(\gamma) \) the length of \( \gamma \) with respect to the Riemannian metric \( \tilde{\rho} \). For \( r > 0 \) let \( V^r \) be the set of homology classes in \( V = H_*(\Omega \widetilde{M}; \mathbb{F}) \) that can be represented by cycles formed by piecewise smooth loops \( \gamma \) based at \( \tilde{p} \) with \( \ell(\gamma) \leq r \),
\[ V^r := \iota_\ast H_*(\Omega \widetilde{M}; \mathbb{F}). \]

Then each \( V^r \) is finite-dimensional (see \[16, \S 16\]), \( V^r \subset V^s \) for \( r \leq s \) and \( V = \bigcup_{r \geq 0} V^r \). As in Section 3.2 define the value of \( v \in V \) by \( |v| := \min \{ r \mid v \in V^r \} \). In view of the definition of the Pontryagin product and by the triangle inequality, \( |vw| \leq |v| + |w| \) for all \( v, w \in V \).

Next, for \( g \in G = \pi_1(M, p) \) let \( \ell(g) \) be the minimal length of a piecewise smooth loop based at \( p \) that represents \( g \). In other words, \( \ell(g) \) is the length of the shortest geodesic lasso based at \( p \) in class \( g \). Set \( L(g) := \frac{1}{2} \ell(g) \). Then \( L(g) = L(g^{-1}) \) and \( L(gh) \leq L(g) + L(h) \) for all \( g, h \in G \) by the triangle inequality. Finally, for \( g \in G \) denote by \( c_g \) the lift to \( \widetilde{M} \) based
at $\tilde{p}$ of a shortest curve in class $g$. Then
\[ \ell(c_g^{-1} \circ g \gamma \circ c_g) \leq \ell(c_g^{-1}) + \ell(g \gamma) + \ell(c_g) = \ell(\gamma) + 2\ell(c_g) = \ell(\gamma) + L(g) \]
for all $g \in G$ and $\gamma \in \Omega\tilde{M}$. In view of the definition of the $G$-action on $\Omega\tilde{M}$ we find that $|gv| \leq |v| + L(g) = |v| + |g|$. We have shown that $(V, \mu, \Delta)$ is a spicy Hopf algebra over $\mathbb{F}G$.

5.3. **Dimensions.** Recall that $M$ is of **finite type** if its universal cover $\tilde{M}$ is homotopy equivalent to a finite CW-complex.

**Lemma 5.2.** ([5 Lemma 2.2]) The following are equivalent.

(i) $M$ is of finite type.

(ii) The Abelian groups $H_k(\tilde{M})$ are finitely generated for all $k \geq 1$.

(iii) The Abelian groups $\pi_k(M)$ are finitely generated for all $k \geq 2$.

Now assume that $M$ is not of finite type. By the lemma, we can define
\[ (11) \quad m(M) := \min \left\{ k \mid H_k(\tilde{M}) \text{ is not finitely generated} \right\} \in \{2, \ldots, \dim M\}. \]

The main result of this subsection is

**Proposition 5.3.** Assume that $M$ is not of finite type. Let $m = m(M)$ be as in definition (11). Then

(i) $H_i(\Omega\tilde{M})$ is finitely generated for $i \leq m - 2$;

(ii) $H_{m-1}(\Omega\tilde{M})$ is not finitely generated, but is finitely generated as a $\mathbb{Z}G$-module.

**Proof.** For each $k \geq 2$ the fundamental group $G = \pi_1(M, p)$ acts on $\pi_k(M) = \pi_k(M, p)$ by conjugation. Under the Hurewicz homomorphism $h: \pi_k(M) = \pi_k(\tilde{M}) \to H_k(\tilde{M})$ this action corresponds to the action of $G$ on $H_k(\tilde{M})$ induced by deck transformations (the commutative diagram on the left). On $\pi_{k-1}(\Omega\tilde{M}) = \pi_{k-1}(\Omega_0M) \cong \pi_k(M)$ this action of $G$ is induced by conjugation of elements in $\Omega_0M \cong \Omega\tilde{M}$. This action also induces an action on $H_{k-1}(\Omega\tilde{M})$, namely the action described in Section 5.1 and the two actions commute with the Hurewicz homomorphism $h: \pi_{k-1}(\Omega\tilde{M}) \to H_{k-1}(\Omega\tilde{M})$ (the commutative diagram on the right),

\[ \begin{array}{ccc}
H_k(\tilde{M}) & \xleftarrow{h} & \pi_k(\tilde{M}) = \pi_{k-1}(\Omega\tilde{M}) \\
\downarrow & & \uparrow \\
H_k(\tilde{M}) & \xleftarrow{h} & \pi_k(\tilde{M}) = \pi_{k-1}(\Omega\tilde{M})
\end{array} \quad \begin{array}{ccc}
\pi_{k-1}(\Omega\tilde{M}) & \xrightarrow{h} & H_{k-1}(\Omega\tilde{M}) \\
\downarrow & & \uparrow \\
\pi_{k-1}(\Omega\tilde{M}) & \xrightarrow{h} & H_{k-1}(\Omega\tilde{M})
\end{array} \]

Recall that $m \in \{2, \ldots, \dim M\}$ is the minimal integer such that $H_m(\tilde{M})$ is not finitely generated. By Serre’s theory of $C$-classes, applied to the class of finitely generated Abelian groups, $m$ is also the minimal integer such that $\pi_m(\tilde{M})$ is not finitely generated. More precisely, Serre’s Hurewicz theorem implies that for $k \leq m$ the Hurewicz map $h: \pi_k(\tilde{M}) \to
$H_k(\widetilde{M})$ is injective and surjective up to finitely generated groups, see [25] or [26] Theorem 15 on p. 508 or [13] Theorem 1.8. Hence $\pi_k(\Omega\widetilde{M})$ is finitely generated for $k \leq m - 2$, but not so for $k = m - 1$.

Since $\Omega\widetilde{M}$ is a path-connected H-space, $\pi_1(\Omega\widetilde{M})$ acts trivially on $\pi_k(\Omega\widetilde{M})$ for $k \geq 0$. Serre’s Hurewicz theorem now implies that $h: \pi_k(\Omega\widetilde{M}) \to H_k(\widetilde{M})$ has finitely generated kernel and cokernel for $k \leq m - 1$, see [25, p. 274] or [26, Theorem 20 on p. 510] or [13] Theorem 1.8]. It follows that $H_k(\Omega\widetilde{M})$ is finitely generated for $k \leq m - 2$, but not so for $k = m - 1$.

We are left with proving the second assertion in (ii). After replacing $M$ by a homotopy equivalent space, if necessary, we find a CW-structure on $M$. Since $M$ is compact, this CW-structure is finite. We lift this structure to $\widetilde{M}$ by the action of $G$. The cellular chain complex $C_*(\widetilde{M}; \mathbb{Z})$ is then a finitely generated $\mathbb{Z}G$-module in each degree. Since $G$ is virtually polycyclic, the ring $\mathbb{Z}$ is left Noetherian, see [11] or [14]. Hence each $\mathbb{Z}G$-module $C_*(\widetilde{M}; \mathbb{Z})$ is left Noetherian. Therefore the kernel and the image of the differential of $C_*(\widetilde{M}; \mathbb{Z})$ as well as the quotient $H_*(\widetilde{M})$ are finitely generated left Noetherian $\mathbb{Z}G$-modules in each degree. In particular, $H_m(\widetilde{M})$ is a finitely generated $\mathbb{Z}G$-module. Recall that $h: \pi_m(\widetilde{M}) \to H_m(\widetilde{M})$ is injective and surjective up to finitely generated groups. In view of the commutative diagram above, it follows that $\pi_m(\widetilde{M})$ and hence $\pi_{m-1}(\Omega\widetilde{M})$ are finitely generated $\mathbb{Z}G$-modules. As we have seen before, $h: \pi_{m-1}(\Omega\widetilde{M}) \to H_{m-1}(\Omega\widetilde{M})$ has finitely generated cokernel. Hence $H_{m-1}(\Omega\widetilde{M})$ is also a finitely generated $\mathbb{Z}G$-module.

5.4. **Proof of Theorem [1.1] and Remark [1.2].** By the universal coefficient theorem, $H_i(\Omega\widetilde{M}; \mathbb{F}) = H_i(\Omega\widetilde{M}) \otimes \mathbb{F}$ for every field $\mathbb{F}$. Hence assertion (i) implies that $H_i(\Omega\widetilde{M}; \mathbb{F})$ is finite-dimensional for $i \leq m - 2$, and assertion (ii) implies that $H_{m-1}(\Omega\widetilde{M}; \mathbb{F})$ is finitely generated as an $\mathbb{F}G$-module. Assume now that $\mathbb{F}$ is a field such that $H_m(\widetilde{M}; \mathbb{F}) = H_m(\widetilde{M}) \otimes \mathbb{F}$ is infinite-dimensional. As we have seen in the proof above, the two Hurewicz maps

$$H_m(\widetilde{M}) \xrightarrow{h} \pi_m(\widetilde{M}) = \pi_{m-1}(\Omega\widetilde{M}) \xrightarrow{h} H_{m-1}(\Omega\widetilde{M})$$

both have finitely generated kernel and cokernel. If follows that $H_{m-1}(\Omega\widetilde{M}) \otimes \mathbb{F} = H_{m-1}(\Omega\widetilde{M}; \mathbb{F})$ is also infinite-dimensional. Hence the dimension assumptions in Corollary [3.6] are satisfied, and we conclude that $\dim V^T$ grows at least like $e^{T^2}$.

Similarly, since the left map in (12) has finitely generated kernel and cokernel, $H_m(\widetilde{M}; \mathbb{F}) = H_m(\widetilde{M}) \otimes \mathbb{F}$ is infinite-dimensional if and only if $\pi_m(\widetilde{M}) \otimes \mathbb{F}$ is infinite-dimensional.

6. Examples

6.1. **A class of examples.** In this paragraph we construct examples of manifolds with infinite cyclic fundamental group that meet the assumptions of Theorem [1.1].
Consider a simply-connected manifold $X$ of dimension $d \geq 4$ that is not homeomorphic to a sphere. Let $m \geq 2$ be the minimal $k$ such that $H_k(X) \neq 0$. Then $m \leq d/2 \leq d - 2$ by Poincaré duality. Let $\tilde{X}$ be the compact manifold with boundary obtained by removing from $X$ the interior of two disjoint closed $d$-balls $B_1 \cup B_2 \subset X$. Then

$$H_k(\tilde{X}) = H_k(X) \quad \text{for } 1 \leq k \leq m$$

since $H_k(\partial B_i) = H_k(S^{d-1}) = 0$ for these $k$. Choose a diffeomorphism $\varphi: \partial B_1 \to \partial B_2$, and let $M$ be the manifold obtained from $\tilde{X}$ by identifying $\partial B_1$ with $\partial B_2$ via $\varphi$. Then $\pi_1(M) = \mathbb{Z}$ by the Seifert–van Kampen theorem. For $n \in \mathbb{Z}$ let $\tilde{X}_n$ be a copy of $\tilde{X}$. The universal cover $\tilde{M}$ is obtained by glueing “the right boundary” $\partial B_1$ of $\tilde{X}_n$ by $\varphi$ to “the left boundary” $\partial B_2$ of $\tilde{X}_{n+1}$ for $n \in \mathbb{Z}$. For $N \in \mathbb{N} \cup \{0\}$ consider the part $\tilde{M}_N := \bigcup_{-N \leq n \leq N} \tilde{X}_n$ of $\tilde{M}$. By the Mayer–Vietoris theorem and by (13),

$$H_k(\tilde{M}_N) = \bigoplus_{n=-N}^{N} H_k(\tilde{X}_n) = \bigoplus_{2N+1} H_k(X) \quad \text{for } 1 \leq k \leq m.$$ 

Hence $H_k(\tilde{M}) = \lim_{N \to \infty} H_k(\tilde{M}_N) = \bigoplus \mathbb{Z} H_k(X)$ for $1 \leq k \leq m$. Recalling that $H_k(M) = 0$ for $1 \leq k \leq m-1$ and $H_m(M) \neq 0$, we see that $M$ is not of finite type, and that $m = \text{m} = \text{m}(M)$. For every field $\mathbb{F}$ we have

$$H_\text{m}(\tilde{M}; \mathbb{F}) = \bigoplus \mathbb{F} H_m(X) \otimes \mathbb{F}.$$ 

Moreover, $H_\text{m}(X)$ is a non-trivial finitely generated Abelian group,

$$H_\text{m}(X) \cong \mathbb{Z}^r \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_t}$$

with $r \geq 0$ and the $q_i$ powers of primes. If $r \geq 1$ choose $\mathbb{F} = \mathbb{Q}$. Then $H_\text{m}(\tilde{M}; \mathbb{Q}) = \bigoplus \mathbb{Q} H_m(X) \otimes \mathbb{Q} = \bigoplus \mathbb{Q}^{\mathbb{Z}}$ is infinite-dimensional. If $r = 0$ let $p$ be the prime number dividing $q_1$. Then $H_\text{m}(\tilde{M}; \mathbb{F}_p)$ contains $\bigoplus \mathbb{F}_p$ as a subvector space, and hence is also infinite-dimensional.

We see that the assumptions of Theorem [11] are met, and conclude that for a suitable field $\mathbb{F}$ the dimension of $\iota_* H_\ast(\Omega^T_0 M; \mathbb{F})$ grows at least like $e^{\sqrt{T}}$. While this seems to be a new result if $X$ or, equivalently, $M$ is irreducible, it has been shown in the proof of Theorem D in [22] that if $M$ can be written in the form $M = M_1 \# M_2$ with $M_2$ simply connected and not homeomorphic to a sphere, then $\dim \iota_* H_\ast(\Omega^T_0 M; \mathbb{Q})$ grows even exponentially.

### 6.2. A “counterexample”

We illustrate the role of our standing assumption that there exists a field $\mathbb{F}$ such that $H_\text{m}(\tilde{M}; \mathbb{F})$ is infinite-dimensional by an example (in which $M$ is not a manifold, but a CW-complex). Let $M$ be the mapping torus of a degree-two map $f$ of the 2-sphere $S^2$. Then $\pi_1(M) = \mathbb{Z}$, and $\tilde{M}$ is the “double mapping telescope” obtained by glueing together the mapping cylinders of $f_i = f$, $i \in \mathbb{Z}$. This space deformation retracts onto the mapping telescope formed by the mapping cylinders of $f_i = f$, $i \geq 0$. Therefore $H_2(\tilde{M})$ can be identified with $\mathbb{Z}[1/2]$, the subgroup of $\mathbb{Q}$ consisting of rational numbers with denominators a power of 2 (see Exercise 1 of Section 3.F and Example 3.F.3 in [12]). The Abelian group $\mathbb{Z}[1/2]$ is not finitely generated, hence $\text{m} = 2$ in this example.
However, for a field $\mathbb{F}$ of characteristic $p$ we have $H_2(\widetilde{M}; \mathbb{F}) = 0$ if $p = 2$ and $H_2(\widetilde{M}; \mathbb{F}) \cong \mathbb{F}$ otherwise.

The group $\mathbb{Z}[1/2]$ has the property that every finitely generated subgroup is generated by one element (indeed, the smallest positive element in the subgroup is a generator). In particular, $\mathbb{Z}[1/2]$ is the union of a nested sequence of subgroups generated by one element (for instance the subgroups generated by $1/2^j$, $j \geq 1$). These properties strongly distinguish $\mathbb{Z}[1/2]$ from an infinite-dimensional vector space.

We have seen in the proof of Proposition 5.3 that $V_1 := H_1(\Omega \widetilde{M})$ is finitely generated as a $\mathbb{Z}[\pi_1 \tilde{M}]$-module. This can be seen very explicitly in this example: Since $\pi_1(\Omega \widetilde{M}) \cong \pi_2(\widetilde{M})$ is Abelian, $V_1 = H_1(\Omega \widetilde{M}) = \pi_1(\Omega \widetilde{M}) = \pi_2(\widetilde{M}) = H_2(\widetilde{M}) \cong \mathbb{Z}[1/2]$. Let $t = 1$ be the generator of the fundamental group $G = \mathbb{Z}$ of $M$. The action of $t \in G$ on $H_2(\widetilde{M})$ and hence on $V_1$ corresponds to multiplication by 2 on $\mathbb{Z}[1/2]$. Moreover, the group ring $\mathbb{Z}G$ is the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. It follows that $V_1 \cong \mathbb{Z}[1/2]$ is generated by 1 over $\mathbb{Z}G$. We see that “the $\mathbb{Z}$-version” of the assumptions in Theorem 2.1 is satisfied. While it is still true that for every non-zero vector $v \in V_1$ the subgroup generated by $\{tv\}_{i \in \mathbb{Z}}$ is not finitely generated, we cannot use the sequence $(tv)_{i \in \mathbb{Z}}$ to prove a lower bound on the rank of $V^r$ by the arguments of Sections 3 and 3 because for each $k$ the subgroup generated by $(tv)_{|i| \leq k}$ has only rank one (being generated by $t^{-k}v$).

7. Proof of Corollaries 1.3 and 1.6

Proof of Corollary 1.3 Let $C_{pq}$ be a component of $\Omega_{pq}M$. Fix a smooth path $c \in C_{pq}$, of length $\ell(c)$. The map $h : \Omega_0(M, p) \to C_{pq}$, $\gamma \mapsto c \circ \gamma$ is a homotopy equivalence. It maps $\Omega^T_0M$ to $C_{pq}^{T+\ell(c)}$, where $C_{pq}^{T}$ is the space of piecewise smooth paths in $C_{pq}$ of length $\leq T$. Theorem 1.1 now implies that

$$\dim t^*_s H_*(C_{pq}^T; \mathbb{F}) \quad \text{grows at least like} \quad e^{\sqrt{T}}.$$  \hspace{1cm} (14)

Notice that $\dim t^*_s H_*(C_{pq}^T; \mathbb{F}) \leq \dim H_*(C_{pq}^{T}; \mathbb{F})$. For geodesic flows, Corollary 1.3 now follows from classical Morse theory, see [16, Theorem 16.3] or [21, p. 116]. For the general case of Reeb flows, we use (14) and Lagrangian Floer homology, exactly as in [15, Section 6].

Proof of Corollary 1.6. For geodesic flows the claim follows from Corollary 1.3 and the geometric arguments in [21, Section 3.1]. For the general case of Reeb flows we use Theorem 1.1 and the idea from [3], that was further developed in [4, 15, 5]. The proof goes exactly as the proof of Theorem 4.6 in [15]; we therefore only sketch the proof. Let $\Sigma \subset T^*M$ be the fiberwise starshaped hypersurface corresponding to the cooriented contact manifold $(S^*M, \alpha)$, and (up to the time change $t \mapsto 2t$) view the Reeb flow $\varphi_t^r$ on $(S^*M, \alpha)$ as the restriction of the Hamiltonian flow $\varphi_t^H$ on $T^*M$, where $\Sigma = H^{-1}(1)$ and $H$ is fiberwise homogeneous of degree 2 (and smoothened to zero near the zero-section). For $q \in M$ consider $\Sigma_q = \Sigma \cap T^*_qM$ and the bounded component $D_q$ of $T^*_qM \setminus \Sigma_q$. The “spheres” $\Sigma_q$ are Legendrian and the “discs” $D_q$ are Lagrangian. Now fix $p \in M$ and let
q be such that p, q are non-conjugate. “Sandwich” the Hamiltonian H between a smaller and a larger Riemannian Hamiltonian G and cG for a suitable constant c > 0. Then continuation maps in Lagrangian Floer homology, the Abbondandolo–Schwarz isomorphism from the action-filtered Lagrangian Floer homology HF^n(ϕ^n_G(T_p^*M), T_q^*M; ℍ) of the pair of Lagrangian submanifolds ϕ^n_G(T_p^*M), T_q^*M to the energy-filtered homology H^{n^2}(Ω_{pq}M; ℍ), and Theorem [11] imply that dim HF^n(ϕ^n_H(D_p), T_q^*M; ℍ) grows at least like e^√n, uniformly in q. Since the chain complex of Lagrangian Floer homology in generated by the intersection points of the two Lagrangians, one concludes that the number of intersections of ϕ^n_H(D_p) and D_q grows at least like e^√n, uniformly in q for almost every q ∈ M. Hence the volume of ϕ^n_H(D_p) grows at least like e^√n. We refer to Section 4 of [15] for details. Finally, the volume of ϕ^n_H(Σ_p) also grows like e^√n in view of Proposition 4.3 in [5]. □

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