Nonlinear Sigma Model for Disordered Media: Replica Trick for Non-Perturbative Results and Interactions

Igor V. Lerner
School of Physics and Astronomy, The University of Birmingham, Birmingham B15 2TT, UK

Abstract

In these lectures, given at the NATO ASI at Windsor (2001), applications of the replicas nonlinear sigma model to disordered systems are reviewed. A particular attention is given to two sets of issues. First, obtaining non-perturbative results in the replica limit is discussed, using as examples (i) an oscillatory behaviour of the two-level correlation function and (ii) long-tail asymptotes of different mesoscopic distributions. Second, a new variant of the sigma model for interacting electrons in disordered normal and superconducting systems is presented, with demonstrating how to reduce it, under certain controlled approximations, to known "phase-only" actions, including that of the "dirty bosons" model.

1 Introduction

Starting from the seminal papers of Wegner [1] and Efetov [2], a field-theoretical description based on the nonlinear $\sigma$ model (NL$\sigma$M) has become one of the main analytical approaches to various problems in disordered electronic systems. The ensemble averaging over all configurations of disorder is performed either using bosonic [1, 3] or fermionic [4] $n$-replicated fields and taking the $n \to 0$ limit in the results, or using supersymmetric fields [2, 5]. The main advantage of this approach lies in formulating the theory in terms of low lying excitations (diffusion modes) which greatly simplifies perturbative and renormalization group calculations, and also allows a non-perturbative treatment.

The first application of this approach was a derivation [1–4] of the renormalization-group (RG) equations of the scaling theory [6] of Anderson localization. For such a perturbative approach, generalized later for mesoscopic systems [29], both the replica and the supersymmetric methods are equally justified: they ensure the cancellation of unphysical vacuum loops in a diagrammatic expansion.

However, it soon became conventional wisdom that there exist two sets of problems, for each only one of these methods being applicable. The first set of problems can be schematically specified with the following TOE$^1$ model:

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i V_i + \frac{1}{2} \sum_{ij} V_{ij}. \quad (1)$$

Here the second term represents schematically both lattice and disorder potential, while the last represents all possible two-particle interactions (including the BCS one where relevant). To include the interactions, at least at the perturbative and RG level, the original fermionic replica [4] has

$^1$TOE stands for “Theory of Everything”. In order to be really a TOE model in a condensed-matter context, the Hamiltonian (1) should also include spin terms, as later in this paper in relation to superconductivity. Naturally, either with or without spin terms, there is no hope for a rigorous approach to, let alone the exact solution of, this model for an arbitrary disorder/interaction strength.
been generalized by Finkelstein [7]. The interest in this approach has been greatly enhanced by the recent discovery [8] of an apparent metal-insulator transition in 2D disordered systems in zero magnetic field. Naturally, the TOE model and its $\sigma$ model implementation covers a much wider variety of systems and phenomena than in the scope of the original NL$\sigma$M [1–4] describing only non-interacting disordered systems.

After a few earlier attempts [9–11], it has been recently demonstrated [12, 13] that the Keldysh technique [14] provides a viable alternative to the replica approach for interacting systems. However, the latter still remains one of the best available tools for consideration of interacting electrons in disordered systems. In the very least, it is clear that there is no simple way of applying the supersymmetry method to a many-particle fermionic system, so that the choice is between using Keldysh technique or replica trick.

On the other hand, there exist another set of problems for disordered electron systems where meaningful results are essentially non-perturbative (or at least look like that – see section 3 below). In the absence of interactions, many of them have been solved the help of the supersymmetry method (see, e.g., Refs. [15–20] out of many), while the viability of the replica approach was seriously questioned.

The first, and arguably most famous, of such problems was solved by Efetov [15] who used the supersymmetric NL$\sigma$M to derive the two-level correlation function (TLCF) in the universal ergodic regime for electrons in disordered metallic grains. The results proved to be identical to those for eigenvalue correlations [21] in random matrix theory (RMT), as had much earlier been conjectured by Gor'kov and Eliashberg [22]. This was the first microscopic derivation of essentially a non-perturbative result for noninteracting electrons in disordered media which has opened the way to numerous new results (again, see Refs. [15–20] out of many – this paper is not a proper venue to give a comprehensive list, let alone even a short review, of them).

In many ways, this first non-trivial “supersymmetric” result [15] seemed to be an excellent illustration of why the replica trick could only be used within a perturbative approach. For the (easiest) case of the unitary symmetry (one of Dyson’s symmetry classes corresponding to the absence of the time-reversal symmetry, e.g. due to an external magnetic field) the irreducible TLCF is given by

$$R_2(\omega) = -\frac{\sin^2 \omega}{\omega^2},$$

(2)

where $\omega$ is the distance between two levels in units of $\delta/\pi$ and $\delta$ is the mean level spacing. This result is valid in the ergodic regime, i.e. for $\omega \ll g$ where $g \gg 1$ is the dimensionless conductance. For $\omega \gg 1$, the TLCF averaged over fast oscillations could be readily obtained from the standard diagrammatic techniques [23] with $1/\omega$ being the perturbation parameter (or from the perturbation theory in the framework of either the supersymmetric or replica NL$\sigma$M). However, the non-perturbative factor $\sin^2 \omega$ cannot be restored from the perturbation series. Since the replica trick is well justified only within the perturbative approach, it might seem rather hopeless to obtain the result (2) within the replica approach. And indeed, quite involved calculations by Verbaarschot and Zirnbauer [24] have shown that a direct application of the replica trick (using either the bosonic or fermionic NL$\sigma$M) has apparently not reproduced the TLCF given by Eq. (2).

In section 2, following Refs. [25–27], I will show that introducing a proper replica-symmetry breaking within the fermionic NL$\sigma$M leads to the non-perturbative result (2), albeit only in the asymptotic region $\omega \gg 1$. This raises hope that one might eventually apply the replica approach for obtaining non-perturbative results for interacting electrons in disordered systems (see Ref. [28] for the first steps in this direction).

The replica-symmetry breaking is not the only way to get (at least seemingly) a non-perturbative result from the replica NL$\sigma$M. In section 3, I will briefly summarise how quite non-perturbative calculations [18–20] (that could only be performed within the supersymmetric method) have reproduced part of the results for long-range asymptotics of different mesoscopic distribution obtained much earlier [29–32] by using a perturbative RG approach valid within any variant of the NL$\sigma$M.

Then, I will describe in section 4 how to include interactions into a modern variant of the replica NL$\sigma$M and will use this model in section 5 to describe different approaches to superconductor-
insulator transitions in thin disordered films.

2 Replica-symmetry breaking: a way to non-perturbative results?

In this section we consider non-interacting electrons in disordered media – the problem described in a standard way by the first two terms in Hamiltonian (1) (this is equivalent to the Anderson model of disorder). The aim is to reproduce non-perturbative oscillatory contribution to the TLCF of Eq. (2) using the replica NLσM.

Before describing the derivation, it is worth specifying what is meant here by the replica-symmetry breaking. It is the standard fermionic NLσM which does contain correct non-perturbative oscillatory contributions to the TLCF. The replica symmetry is not broken at the level of choosing a correct saddle-point approximation but rather fully exploited by involving a set of additional (in comparison to standard considerations [4]) saddle-point submanifolds within the usual NLσM. The symmetry is broken in a rather natural way when accounting for the contribution of these submanifolds to the TLCF. For the unitary case, such a procedure gives the exact results for all \( \omega \), although a real reason for this [33] may be incidental. There is no doubt, however, that it gives the asymptotically (\( \omega \gg 1 \)) correct results for all the three Dyson’s symmetry classes.

Note an important analogy: in the supersymmetric NLσM Andreev and Altshuler [34] have reproduced the large-\( \omega \) limit of Eq. (2) by the saddle-point calculation of the supersymmetric integral that involved an additional saddle-point which “breaks” the supersymmetry in a similar way. The same integral has been exactly calculated by Efetov [15] without any breaking of the supersymmetry.

This analogy will be extended by showing (with the help of a trick resembling the replica-symmetry breaking in the meaning explained above) that the \( n = 0 \) replica limit of the exact integral representation of the TLCF obtained in Ref. [24] for the fermionic replica NLσM leads to the same asymptotically correct oscillatory behavior of Eq. (2).

2.1 Formulation of the problem in terms of NLσM

Consider the two-level correlation function (TLCF) defined by

\[
R_2(\omega) = \frac{1}{\nu^2} \langle \nu(\epsilon + \omega)\nu(\epsilon) \rangle - 1.
\]  

Here \( \langle \ldots \rangle \) stands for the ensemble averaging, i.e. averaging over all the realizations of the impurity potential \( V(r) \) in Hamiltonian (1). \( \nu(\epsilon) \) is the electronic density of states per unit volume defined in terms of the spectrum \( \{ \epsilon_\alpha \} \) for a given sample as \( \nu(\epsilon) = L^{-d} \sum_\alpha \delta(\epsilon - \epsilon_\alpha) \). \( \nu \equiv \langle \nu(\epsilon) \rangle = 1/L^d \delta \), and \( L, d \) are the sample size and dimensionality. All energies are measured in units of \( \delta/\pi \), in which the TLCF has the form of Eq. (2). In these units \( R_2 \) is expressed via the product of the retarded and advanced Green’s functions [35] as follows:

\[
R_2(\omega) = \frac{1}{2} [\text{Re} S_2(\omega) - 1] , \quad S_2(\omega) \equiv \langle G^r(\epsilon + \frac{\omega}{2}) G^a(\epsilon - \frac{\omega}{2}) \rangle .
\]  

The function \( S_2 \) can be expressed in the standard way [15, 36] in terms of a generating functional \( Z_n \)

\[
S_2(\omega) = -\lim_{n \to 0} \frac{1}{n\epsilon} \frac{\partial^2}{\partial \omega^2} (Z_n(\omega)) , \quad Z_n = \int \bar{\psi} \psi e^{iS}.
\]  

Here \( \bar{\xi} = \bar{p}^2 / 2m - \bar{\epsilon}_\sigma, \bar{\psi}_\sigma \) and \( \psi_\sigma \) are conjugate \( 2n \)-component fermionic (Grassmannian) fields (corresponding to \( n \) replica components and 2 retarded-advanced components) – further 2 spin
components are redundant in this section but will be included in section 4). \( \Lambda \equiv \text{diag}(1, -1)_{\tau_{\text{el}}} \otimes 1_{\omega} \). The replica trick, \( n \to 0 \), allows one to write the above expression for the product of Green’s functions in terms of derivatives of \( Z \) (rather than of \( \ln Z \)) which makes the ensemble averaging straightforward. Performing it in the assumption that \( V(r) \) is a Gaussian \( \delta \)-correlated random potential with the dispersion \( (2\pi \nu \tau_{\text{el}})^{-1} \), where \( \tau_{\text{el}} \) is the mean free time of elastic scattering from impurities, one obtains the following quartic term standing in the action \( S \) for \( \bar{\psi}V\psi \):

\[
S_{\text{dis}} = -\frac{1}{4\pi \nu \tau_{\text{el}}} \int dr \left[ \bar{\psi}(r) \cdot \psi(r) \right] \left[ \bar{\psi}(r) \cdot \psi(r) \right]
\]

(7)

The next step is the standard Hubbard–Stratonovich decoupling that allows one to single out all “slow” modes of electron motion. In the absence of any symmetry breaking (due to an external magnetic field, or magnetic impurities, or spin-orbit coupling), there are two modes, equally contributing to quantum interference effects: the diffusion mode, corresponding to small transferred momenta, and the cooperon mode, corresponding to small sum momenta. It is instructive to perform the decoupling directly in the \( r \) space, thus emphasizing that the diffusion mode corresponds to the normal pairing while the cooperon – to the anomalous pairing:

\[
\exp\{-S_{\text{dis}}\} = \int \frac{Dd}{\mathcal{N}_d} \exp\left\{-\int dr \left[ \frac{\pi\nu}{4\tau_{\text{el}}} \left( \bar{\psi}(r) \cdot \psi(r) \right)^2 - \frac{i}{2\tau_{\text{el}}} \bar{\psi}(r) \psi(r) \right] \right\} + \int \frac{Dc}{\mathcal{N}_c} \exp\left\{-\int dr \left[ \frac{\pi\nu}{4\tau_{\text{el}}} \left( \bar{c}(r) \cdot c(r) \right)^2 - \frac{i}{2\tau_{\text{el}}} \bar{c}(r) c(r) \right] \right\}
\]

(8)

The diffusion field \( d_{\omega}(r) \) is a matrix field of the same symmetry as \( \psi \otimes \bar{\psi} \) while the cooperon field is of the same symmetry as \( \psi \otimes \bar{\psi} \). These two fields can be combined into the single \( 4N \times 4N \) matrix field \( \sigma(r) \) of the form

\[
\sigma = \begin{pmatrix} d & c \\ c^T & d^T \end{pmatrix}, \quad \text{where} \quad d^T = d, \quad c^T = -c.
\]

(9)

The extra \( 2 \times 2 \) “diffusion-cooperon” space, explicit in Eq. (8), is usually referred to as the “charge conjugation” space (sometimes, as the time-reversal space): apart from being Hermitian, the matrix field \( \sigma \) has the charge conjugation symmetry,

\[
\sigma = \bar{\sigma} \equiv C \sigma^* C^{-1}, \quad \bar{\Psi} = (C\Psi)^T
\]

(10)

where \( \bar{\Psi} = \frac{1}{\sqrt{2}} (\psi^+, -\psi^T) \), \( \Psi^T = \frac{1}{\sqrt{2}} (\psi^T, \psi^+) \), so that the second equation in (10) defines the charge-conjugation matrix \( C \). The disorder functional (8) can be written in such an extended space as follows:

\[
\exp\{-S_{\text{dis}}\} = \int \frac{D\sigma}{\mathcal{N}_\sigma} \exp\left\{\int dr \left[ -\frac{\pi\nu}{8\tau_{\text{el}}} \text{tr} \sigma^2 + \frac{i}{2\tau_{\text{el}}} \bar{\Psi} \sigma \Psi \right] \right\}
\]

(11)

The functional integration measure in Eqs. (8) and (11) is irrelevant in the replica limit. After performing the Gaussian functional integration over the fermionic fields in Eq. (8), one reduces the generating functional \( Z_n \) to \( Z_n = \langle e^{-\bar{F}(\sigma)} \rangle_\sigma \) with \( \langle \ldots \rangle_\sigma \) standing for the functional integration over all the independent components of the field \( \sigma \), and the “free energy” is

\[
\mathcal{F}[\sigma] = \frac{1}{L^d} \int d^d r \left\{ \frac{\pi\nu}{8\tau_{\text{el}}} \text{tr} \sigma^2 - \frac{1}{2} \text{tr} \ln \left[ \left( \frac{\omega}{2} + i\delta \right) \Lambda - \hat{\xi} + \frac{i}{2\tau_{\text{el}}} \sigma \right] \right\}
\]

(12)

A saddle-point condition for this functional, found by varying the field \( \sigma \) is

\[
-\imath \pi \nu \sigma = \langle \psi \mid \hat{G} \mid \psi \rangle, \quad \hat{G} \equiv \left[ \left( \frac{\omega}{2} + i\delta \right) \Lambda - \hat{\xi} + \frac{i}{2\tau_{\text{el}}} \sigma \right]^{-1}
\]

(13)

The same notation for \( \Lambda \) will be used in all extended spaces introduced later, assuming that it is a dyadic product of Pauli’s \( \sigma_{\alpha} \) in the RA space and of the unit matrices in all other sectors of the extended space.
In the class of spatially-homogeneous fields $\sigma$, this condition is obviously satisfied by the field $\sigma = \Lambda$ as can be easily seen by transforming Eq. (13) into the reciprocal momentum space and performing integration over all momenta in the r.h.s. of this equation in the pole approximation, justified provided that

$$\varepsilon_p \tau_{e_1} \gg 1$$

(14)

What is usually not stressed is that the saddle-point condition (13) is satisfied in the same approximation not only by $\Lambda$ but by any spatially-homogeneous diagonal matrix commuting with $\Lambda$ with eigenvalues equal to $\pm 1$. For most applications (but not for what follows) the existence of this wide class of the saddle-point solutions at $\omega \neq 0$ is irrelevant. The reason is that for $\omega = 0$ Eq. (13) is satisfied by a much wider class of matrices, $\sigma = Q$, where $Q$ satisfies the following constraints

$$Q^2 = \mathbb{1}_{2n}, \quad \text{tr} \, Q = 0.$$  

(15)

These constraints are resolved by representing $Q$ as follows:

$$Q = U^\dagger \Lambda U = T^\dagger \Lambda T, \quad T = \exp \begin{pmatrix} 0_n & t \\ -t^\dagger & 0_n \end{pmatrix}.$$  

(16)

Since $Q$ obeys the charge conjugation condition (10), $U$ is a symplectic matrix (a unitary matrix whose elements are real quarternions [4]), i.e. $U \in \text{Sp}(2n)$. Then $T$ is obtained by factorizing matrices $U$ with respect to redundant matrices $R \in \text{Sp}(n) \times \text{Sp}(n)$ that commute with $\Lambda$, i.e. $U = RT$, which reduces $T$ to the form given in Eq. (16) with $t$ being an arbitrary $n \times n$ real-quarternionic matrix. This means [37] that $Q$ belongs to the compact Grassmannian manifold (coset space), $\text{Sp}(2n)/\text{Sp}(n) \times \text{Sp}(n)$. This class of symmetry corresponds to Dyson’s orthogonal class in RMT. I will discuss other symmetry classes slightly later.

The final step of the NL$\sigma$M derivation is the expansion of $\text{tr} \ln \text{Eq. (12)}$ in small $\omega$ and $\nabla Q$ around a spatially-homogeneous zero-frequency saddle point of Eqs. (15) and (16). To this end, one performs the similarity transformation in the $\text{tr} \ln$-term representing it as

$$\text{tr} \ln \left\{ G^{-1}_0 - U [\hat{\xi}, U^\dagger] + \frac{1}{2} \omega U \Lambda (U^\dagger)^\dagger \right\},$$

where $G$ is the Green’s function of Eq. (13) at $\omega = 0$ and $\sigma = \Lambda$. After this, expanding the $\text{tr} \ln$ to the first power in $\omega$ and the second power in the gradient operator $\hat{\xi} \equiv \mathbf{p}^2/2m - \varepsilon_p \approx \mathbf{v}_R \mathbf{n} \cdot \nabla$, performing the pole integration over $\hat{\xi}$, justified in the already used approximation $\varepsilon_p \tau_{e_1} \gg 1$, and neglecting in the same approximation $\text{tr} \sigma^2$ term in Eq. (15), one finally arrives at the non-linear sigma model functional

$$\mathcal{F}[Q; \omega] = \frac{1}{4} \int \! d^d r \text{Tr} \left[ \frac{1}{8} D (\nabla Q)^2 - \frac{i \omega \alpha}{4} \Lambda Q \right],$$

(17)

where $D = \mathbf{v}_R^2 \tau_{e_1}/d$ is the diffusion coefficient; at $d = 2$, the dimensionless conductance $g = 2\pi^2 \nu D$.

If time reversal invariance is broken by a magnetic field or magnetic impurities, or spin-rotation symmetry is broken by the spin-orbit interaction, the above functional remains the same but the matrix $U$ in Eq. (16) belongs, respectively, to the unitary group $U(n)$ (Dyson’s unitary class) or to the orthogonal group, $\text{SO}(n)$ (Dyson’s symplectic class). Here $\alpha = 1$ for the orthogonal class and $\alpha = 2$ for the unitary and symplectic classes. This factor arises because unitary and symplectic classes have been obtained from orthogonal [3] by the suppression of massive modes corresponding to the time-reversal or spin-rotational symmetry breaking and a subsequent reduction of the $Q$ matrix rank. The coefficient $\alpha$ also absorbs an extra factor in the symplectic case due to the redefinition of the mean level spacing $\delta$ in the chosen units.

$^3$It is worth stressing that the condition $\text{tr} \, Q = 0$ arises from choosing an equal number of replicas in both retarded and advanced Green’s functions $G^+$ and $G^-$. With this choice, non-zero values of $\text{tr} \, Q$ would correspond to “massive” modes that will not be contributing to the results.
For simplicity, I consider in the remaining part of this section only the unitary class, referring to the original publication [26, 27] for the other two classes. I will also limit considerations to the ergodic regime corresponding to the level separations much smaller than the Thouless energy, \( E_T \sim D/L^2 \) (which in the chosen units coincides, up to a numerical factor, with the dimensionless conductance \( g \)). In this regime the gradient term in Eq. (17) may be neglected, and the NL\(\sigma\)M functional reduces to the zero-dimensional limit: [15]

\[
\mathcal{F}[Q; \omega] = -\frac{i\omega}{2} \text{Tr} \left[ \Lambda Q \right],
\]

with \( Q \) becoming a spatially homogeneous matrix.

### 2.2 Calculating the asymptotics of the TLCF

Limiting all further considerations to the ergodic regime only, \( Z(\omega) \) given by Eqs. (5) and (18) can be represented as

\[
Z_n(\omega) = \int \mathcal{D}Q \exp \left[ -i\frac{\omega}{2} \text{Tr} \Lambda Q \right],
\]

where the measure is defined by

\[
\mathcal{D}Q = \prod_{i,j=1}^{n} d\Omega_{ij}^{+} d\Omega_{ij}^{-}, \quad d\Omega \equiv dT \cdot T^{-1}.
\]

Here \( T \) is the matrix parameterizing \( Q \), Eq. (16), and \( r \) and \( a \) refer to the replica indices which originate from \( G^r \) and \( G^a \), respectively. In the large-\( \omega \) limit this integral is mainly contributed by the extrema of the functional which obey the standard condition \([\Lambda, Q] = 0\). This condition is satisfied by any matrix of the form \( Q = \text{diag}(Q^r, Q^a) \), where \( Q^r \) and \( Q^a \) are the \( n \times n \) Hermitian matrices whose eigenvalues are \( \pm 1 \) and \( \text{Tr} (Q^r + Q^a) = 0 \). This defines a highly degenerate saddle-point manifold which consists of \( C_{2n}^n \) submanifolds specified by a particular distribution of \( n \) eigenvalues ‘+1’ and \( n \) eigenvalues ‘−1’ between \( Q^r \) and \( Q^a \). These submanifolds can be divided into \( n + 1 \) classes of equivalence, \( Q_p = \text{diag}(Q^r_p, Q^a_p) \), labeled by \( \text{Tr} Q^r_p = -\text{Tr} Q^a_p = n - 2p \), with \( p = 0, 1, \ldots, n \). The \( p \)-th class has weight \((C^p_n)^2\), with \( C^p_n \equiv \binom{n}{p}\).

The matrix \( Q^r_p \) with \((Q^r_p)^2 = \mathbb{I}_n \) and \( \text{Tr} Q^r_p = n - 2p \) can be parameterized by analogy with Eq. (16) as

\[
Q^r_p = (T^r_p)^\dagger \lambda_p T^r_p, \quad \lambda_p = \text{diag}(\mathbb{I}_{n-p}, -\mathbb{I}_{p}), \quad T^r_p = \exp \left( \frac{0_{n-p}}{t^r} \right)_0^p \]

where \( t^r \) is an arbitrary \( p \times (n-p) \) matrix. This defines the coset space \( G_p = \text{U}(n)/\text{U}(n-p) \times \text{U}(p) \), i.e. Therefore, \( Q_p = \text{diag}(Q^r_p, Q^a_p) \) belongs to the manifold \( G_p \times G_p \) and can be parameterized as

\[
Q_p = T^a_p \Lambda_p T^r_p, \quad T_p = \text{diag}(T^r_p, T^a_p), \quad \Lambda_p = \text{diag}(\lambda_p, -\lambda_p)
\]

The integer \( p \) specifies the replica-symmetry breaking, as it describes the number of the ‘−1’ eigenvalues in each \( Q^r \) block (equal to the number of the ‘+1’ eigenvalues in each \( Q^a \) block): in the symmetry-unbroken case, \( p = 0 \), and hence retarded and advanced blocks, \( Q^r, Q^a \), contain only positive or negative eigenvalues, respectively.

Now one needs to take into account contributions from ‘massive’ modes (with mass \( \propto 1/\omega \), not to be confused with the massive modes \( \propto 1/\varepsilon_r \) neglected upon the derivation of the NL\(\sigma\)M) in the vicinity of each manifold [22]. In the large-\( \omega \) limit these contributions may be considered as independent and the partition function is then represented by the sum of all of them:

\[
Z_n(\omega) = \sum_{p=0}^{n} (C^p_n)^2 \int \mathcal{D}Q \exp \left[ -i\frac{\omega}{2} \text{Tr} \Lambda Q_p \right],
\]
This expression is somewhat symbolic, as \( Q_p \equiv U \Lambda_p U^\dagger \), covers the entire symmetric manifold of the NL\( \sigma \)M, Eq. (14), including all the massive modes: indeed, \( \text{Tr} U \Lambda_p U^\dagger \Lambda = \text{Tr} \Lambda_p T^\dagger \Lambda T = \text{Tr} \Lambda_p Q \), where we have substituted \( U = RT \), as defined after Eq. (16). This can be justified only as a perturbative (in \( 1/\omega \)) procedure: a possible overlapping of massive modes originated from different manifolds is irrelevant in the large-\( \omega \) limit, and each of the integrals in the sum (23) can be calculated independently of the others.

Each term in the sum (23) contains both massive and massless modes. Indeed, we have used above the factorization \( U = RT \) with \( R \) being block-diagonal matrices commuting with \( \Lambda \). The matrices \( T \) in \( \text{Tr} \Lambda_p Q = \text{Tr} \Lambda T \Lambda_p T^\dagger \) still contain the subset of matrices commuting with \( \Lambda_p \) that correspond to the massless modes. Therefore, we need to parameterize \( T \) in a way which enables us to factorize out these massless modes and perform the integration over the massive ones.

The most suitable parameterization of \( T \), analogous to that used in Ref. [24] for the bosonic NL\( \sigma \)M, can be obtained by expanding the matrix exponent in Eq. (16). By introducing matrix \( B \equiv t(t^t)^{-1/2} \sin \sqrt{T^t t} \), we represent \( T \) and thus \( Q = T^\dagger \Lambda T \) as follows:

\[
T = \begin{pmatrix} \mathcal{R}(BB^\dagger) & B \\ -B^\dagger & \mathcal{R}(B^\dagger B) \end{pmatrix}, \quad Q = \begin{pmatrix} \mathbb{I}_n - 2BB^\dagger & B \mathcal{R}(B^\dagger B) \\ B^\dagger \mathcal{R}(BB^\dagger) & -(\mathbb{I}_n - 2B^\dagger B) \end{pmatrix},
\]

where \( \mathcal{R}(X) \equiv \sqrt{n} - X \). The matrix \( B \) in this parameterization is not unconstrained, though. The \( Q = Q_1 \) condition is fulfilled only when the matrices \( \mathcal{R}(BB^\dagger) \) and \( \mathcal{R}(B^\dagger B) \) are Hermitian. This is so only when all the eigenvalues of \( BB^\dagger \) and \( B^\dagger B \) do not exceed unity. Only under this constraint does \( Q \), parameterized as in Eq. [24], still belong to the coset space \( \text{U}^2(2n) / \text{U}(n) \times \text{U}(n) \). Nevertheless, this parameterization is very convenient. First, the corresponding Jacobian is equal to one [27] so that the measure of integration (24) can be written as

\[
\mathcal{D}Q = \prod_{i,j} dB_{ij} dB_{ij}^\dagger \equiv DB.
\]

In addition, the representation of all the exponents in the sum (23) in terms of \( B \) is also very simple, \( \text{Tr} \Lambda_p Q = 2(n-2p) - 2\text{Tr} \Lambda_p (BB^\dagger + B^\dagger B) \), so that we obtain:

\[
Z_n(\omega) = \sum_{p=0}^{n} (C_n^p)^2 \cdot e^{\omega(2p-n)} Z_p^p(\omega),
\]

\[
Z_p^p(\omega) = \int DB \exp \left[ i\omega \text{Tr} \Lambda_p (BB^\dagger + B^\dagger B) \right].
\]

The region of integration in (27) is restricted by the constraint described after Eq. (24). Last, but not least, the parameterization (24) allows one to separate out the massless modes, which obey the condition \( [T, \Lambda_p] = 0 \), in each integral (27). Indeed, this condition is satisfied by all matrices \( T \) constructed from \( B \) which anticommute with \( \Lambda_p \), i.e. have the off-diagonal block structure.

This means that in the representation of \( B \) in the block form reflecting the structure of \( \Lambda_p = \text{diag}(\mathbb{I}_{n-p}, -\mathbb{I}_p) \),

\[
B = \begin{pmatrix} B_1 & b_1 \\ b_2 & B_2 \end{pmatrix},
\]

the matrices \( B_{1,2} \) represent the massive modes, and \( b_{1,2} \) massless. When the massive modes are suppressed (\( B_1 = 0 \) and \( B_2 = 0 \)), the \( T \) matrices in Eq. (24) constructed from \( p \times (n-p) \) matrices \( b_{1,2} \) only, parameterize the same degenerate \( p \)-the manifold, \( \mathcal{G}_p \times \mathcal{G}_p \), described in Eq. (22), as one expects.

By substituting the representation (24) into Eq. (27), we reduce \( Z_n^p \) to the product of integrals over the massive and massless modes:

\[
Z_n^p(\omega) = \int DB_1 DB_2 \exp \left[ -2i\omega \text{tr} (B_1 B_1^\dagger - B_2 B_2^\dagger) \right] \int Db_1 Db_2,
\]

7
Here the region of integration over \( b_{1,2} \) depends on \( b_{1,2} \) due to the constraint on the eigenvalues of the matrices \( BB^1 \) and \( B^1 B \) in the representation (23). Since the integral over \( b_{1,2} \) is contributed only by the region where both \( \text{tr} \, B_1 B_1^1 \) and \( \text{tr} \, B_2 B_2^1 \) \( \lesssim 1/\omega \ll 1 \), in the leading in \( 1/\omega \) approximation we may put both \( b_{1,2} \) to 0 in the constraint of the integration region over the massless modes \( b_{1,2} \). In this approximation, as we have noticed after Eq. (28), matrices \( b_{1,2} \) parameterize the \( p \)-th manifold (24) so that

\[
\int \mathcal{D}b \equiv \int \mathcal{D}b_1 \mathcal{D}b_2 = \int \mathcal{D}Q_p = \Omega^2(\mathcal{G}_p),
\]

where the measure of integration over \( \mathcal{D}Q_p \) is defined in terms of \( T_p \) in the same way that \( \mathcal{D}Q \) is defined in terms of \( T \), Eq. (21), and \( \Omega(\mathcal{G}_p) \) is the volume of the compact coset space \( \mathcal{G}_p \). This volume is expressed via the well-known volumes of the unitary group, \( \Omega(U(n)) \), as follows:

\[
\Omega(\mathcal{G}_p) = \frac{\Omega(U(n))}{\Omega(U(n-p))\Omega(U(p))} = (2\pi)^{\frac{1}{2}(n^2-(n-p)^2-p^2)} \prod_{j=1}^{p} \frac{\Gamma(1+j)}{(n+2-j)}. \tag{31}
\]

In the same large-\( \omega \) approximation, the variables \( B_{1,2} \) parameterizing the massive modes are unconstrained. Then the Gaussian integral over the \( 2((n-p)^2+p^2) \) independent massive modes yields

\[
\tilde{Z}_n^p(\omega) \equiv \int \mathcal{D}B \exp \left[ 2i\omega \text{tr} \left( B_1 B_1^1 - B_2 B_2^1 \right) \right] = \left( \frac{\pi}{-i\omega} \right)^{(n-p)^2} \left( \frac{\pi}{i\omega} \right)^{p^2} \tag{32}
\]

Combining Eqs. (32) and (31) and omitting an irrelevant overall factor which goes to 1 when \( n \to 0 \), we arrive at the following expression which is essentially the same as that derived in Ref. [26] via the Itzykson-Zuber integral:

\[
Z_n(\omega) = \sum_{p=0}^{\infty} \left| F_n^p \right|^2 e^{i\omega(2p-n)} \frac{e^{i\omega(2p-n)}}{(2\omega)^{n-p^2+p^2}}, \quad F_n^p \equiv C_n^p \prod_{j=1}^{p} \frac{\Gamma(1+j)}{\Gamma(n+2-j)}. \tag{33}
\]

Here the summation over \( p \) has been extended to \( \infty \) since \( F_n^p \equiv 0 \) for all integer \( n > p \). This allows one to take the replica limit, \( n \to 0 \), in each of the terms in Eq. (33). Due to the fact that \( F_n^p \propto n^p \) as \( n \to 0 \), only the terms with \( p = 0 \) and \( p = 1 \) in Eq. (33) contribute to \( S_2 \) in Eq. (6).

Let us stress that the replica symmetry is broken only now, in the \( n \to 0 \) limit. Indeed, for any integer \( n \neq 0 \) contributions of the terms with \( p \) and \( n-p \) are complex conjugate to each other, but for \( n \to 0 \) we no longer treat them on equal footing. Thus, the result for \( Z_n(\omega) \) below is no longer a real function. Note, however, that in order to treat \( S_2(\omega) \) for all \( \omega \) one should imply the \( \omega \to \omega + i\delta \) substitution which results in \( Z_n(\omega) \) being no longer a real function for any \( n \).

Omitting all the terms with \( p \geq 2 \), one obtains

\[
Z_n(\omega) = \frac{e^{-i\omega n}}{\omega^{n^2}} + n^2 \frac{e^{i\omega(2n-n)}}{4\omega(n-1)^2+1}. \tag{34}
\]

Substituting this into Eqs. (6) and (4) and keeping the leading in \( 1/\omega \) terms only, one arrives at the expression (2) for the TLCF. Although this expression coincides with the exact one (implying the above mentioned substitution \( \omega \to \omega + i\delta \)), it has been actually derived only in the large-\( \omega \) limit, as is the case of the ‘supersymmetry breaking’ method of Andreev and Altshuler [34].

### 2.3 The large-\( \omega \) limit of the Verbaarschot-Zirnbauer Integral

In order to obtain an explicit multiple-integral representation of the ‘zero-mode’ partition function, Eq. (19), one uses the following ‘polar’ decomposition of \( Q \):

\[
Q = \begin{pmatrix}
u_1^+ & 
u_2^+
\end{pmatrix}
\begin{pmatrix}
\lambda & \sqrt{1 - \lambda^2} e^{-i\phi} \\
\sqrt{1 - \lambda^2} e^{i\phi} & -\lambda
\end{pmatrix}
\begin{pmatrix}
u_1 & \nu_2
\end{pmatrix},
\]



8
where \(-1 \leq \lambda_i \leq 1\) and \(u_{1,2}\) are unitary matrices. The appropriate measure of integration is given by
\[
DQ = \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i d\lambda_i \, d\phi_i \, d\mu(u_1) \, d\mu(u_2).
\]
The action in Eq. (19) depends only on \(\lambda\), and the integrations over \(d\mu(u)\) give the volumes of the appropriate unitary group. Thus one obtains
\[
Z_n(\omega) = \Omega^2(U(n)) \int_0^1 \delta^2(\lambda) \prod_i e^{-i\omega\lambda_i} \, d\lambda_i, \quad \delta(\lambda) \equiv \prod_{i<j}(\lambda_j - \lambda_i).
\] (35)

This is equivalent (with accuracy up to factors going to 1 in the \(n \to 0\) limit) to the representation for \(S_2\) given in Eq. (2.24) of the paper by Verbaarschot and Zirnbauer, [24] which has been used for the critique of the replica trick. We will show that it leads, in the very least, to the exact large-\(\omega\) asymptotic behavior of \(S_2(\omega)\).

The leading in \(1/\omega\) contributions to this highly oscillatory integral (which does not have stationary points inside integration region) come from the end points. To single out these contributions, we must take some \(\lambda\)'s close to \(+1\) and the rest close to \(-1\), which imitates replica symmetry breaking. Let us choose \(n-p\) of \(\lambda\)'s close to \(+1\) and \(p\) of \(\lambda\)'s close to \(-1\). Then we can split up the Vandermonde determinant in the following way:
\[
\delta^2(\lambda) = \prod_{i,j} |\lambda_j - \lambda_i| \approx 2^{2p(n-p)} \Delta_+^2 \Delta_-^2
\]
where
\[
\Delta_+^2 = \prod_{i,j=1}^{n-p} |\lambda_i - \lambda_j|, \quad \Delta_-^2 = \prod_{i,j=n-p+1}^n |\lambda_i - \lambda_j|.
\] (36)

Reducing the integral \(35\) to the sum of such contributions, we represent it as
\[
Z_n(\omega) \approx \Omega^2(U(n)) \sum_{p=1}^n (C_p^n) 2^{2p(n-p)} \int_{-\infty}^{+1} \Delta_+^2 \prod_{j=1}^{n-p} e^{-i\omega\lambda_j} \, d\lambda_j \times \int_{-1}^{+\infty} \Delta_-^2 \prod_{j=n-p+1}^n e^{-i\omega\lambda_j} \, d\lambda_j.
\] (37)

Since in each of the integrals all the variables are close to one of the limits of integration, the second limit was extended to infinity. Now we make substitutions \(\lambda_i = 1 - x_i\) in the first integral, and \(\lambda_i = -1 + x_i\) in the second one, reducing the above sum to the form:
\[
Z_n(\omega) \approx \Omega^2(U(n)) \sum_{p=1}^n (C_p^n) 2^{2p(n-p)} e^{i\omega(2p-n)} I_{n-p} I_p.
\] (38)

where \(I_p\) are integrals of Selberg’s type: [21]
\[
I_p = \int_0^\infty \Delta^2(x) \prod_{j=1}^p dx_j e^{-i\omega x_j}
\] (39)

Substituting the known Selberg integrals and discarding an overall factor which goes to unity in the replica limit we arrive at
\[
Z_n(\omega) = \sum_{p=0}^n [F_p^n]^2 \frac{e^{i\omega(2p-n)}}{2\omega(n-p)^2 + p^2}, \quad F_p^n = C_p^n \prod_{j=1}^p \frac{\Gamma(1 + j)}{\Gamma(n + 2 - j)}.
\] (40)
This expression is exactly the same as Eq. \(33\) obtained in section 2.2 with the help of the replica-symmetry breaking. Therefore, the exact representation (35) does contain the true oscillatory asymptotic behavior of the TLCF.

The authors of Ref. \([24]\) have also drawn attention to the fact that there is an apparent contradiction between the \(\omega = 0\) limit for \(S_2\) obtained from the replica trick and the exact supersymmetric result. Indeed, if \(\omega\) is put to 0 in the expression for \(S_2\) following from Eqs. (35) and (5) the \(n \to 0\) limit is taken after that, one obtains \(S_2(\omega = 0) = -1\). This cannot be correct as \(\Re S_2(\omega \to 0) > 0\) as follows from the definition (4) and, moreover, it is known that the non-perturbative oscillatory behavior of the TLCF.

The purpose of this section was to demonstrate explicitly that non-perturbative oscillatory contributions to the TLCF of electrons in a random potential could be extracted from the standard NLSM formulated in fermionic replicas \([4]\) many years ago. To this end, all one needs is to parameterize all the non-trivial saddle-point manifolds corresponding to the broken replica-symmetry and describing ‘massless modes’ of the theory, and expand the action in the vicinity of these manifolds to include ‘massive modes’. The very similar approach has been used in the supersymmetric NLSM: the non-perturbative oscillations have been extracted from the standard NL-SM by using a perturbative RG approach valid within any variant of the NLSM.

### 2.4 The small \(\omega\) limit

The small \(\omega\) limit is, arguably, more interesting as it governs a number of non-perturbative results obtained within the supersymmetric NLSM. I will briefly summarise some of them \([18–20]\) in the following section, emphasizing that they have reproduced some of the results obtained much earlier \([29–32]\) by using a perturbative RG approach valid within any variant of the NLSM.

### 3 Tails of distribution functions

All extensive physical characteristics of disordered systems fluctuate from sample to sample. However, only in mid-eighties it was understood that the scale of such fluctuations is governed by the quantum coherence length so that at \(T \to 0\) they are not reduced with increasing the sample size. Immediately after the discovery \([58, 59]\) of the universal conductance fluctuations (UCF), it was shown \([29]\) that their distribution function is Gaussian in its bulk, but has long lognormal tails. The existence of such tails appears to be a common feature of distribution functions of many observable quantities \([29–32]\) such as global and local density of states, different relaxation times, etc. With increasing the disorder, the part of these tails in distribution functions was increasing so that the entire distribution of any local quantity was becoming lognormal in the critical regime in the vicinity of the metal-insulator transition \([53]\). The characteristic feature of the lognormal distribution is that the logarithm of its \(s\)th moment is proportional to \(s(s-1)\ln L\) where \(L\) is the coherence length (at zero \(T\), this is just a sample size). Therefore, all the moments (or in the limit of weak
disorder, where only the tails are lognormal, all the high moments) scale with different exponents, i.e. system shows multifractal behaviour.

The results for the tails of the distribution functions that had been originally obtained\[29–32\] by the RG treatment of an extended (in a way described below) NLσM, have later been reproduced \[18–20\] in a much more elegant way in the framework of the standard supersymmetric \(\sigma\) model. I do not intend to describe this approach here but only want to stress that some of the steps seem to be absolutely impossible within the replica treatment. The \(Q\) field in the supersymmetric model belongs to a “supermanifold” that includes both compact and a noncompact sectors. The existence of the noncompact sector is absolutely crucial for finding the distribution tails. With properly imposed boundary conditions required for finding the tails, \(Q = \Lambda\) does no longer represents the saddle-point. A correct saddle-point equation reduces, e.g., to the Liouville equation \[19\] for a single scalar parameter that parameterises the (only relevant) noncompact sector of the space of the field \(Q\) (instead of, e.g., the parameterisation \[24\] in terms of the unconstrained matrix \(B\) or any other matrix parameterisation required in the replica \(\sigma\) model). A non-perturbative solution to this equation provides for the existence of lognormal tails of the local density of states distribution \[20\] or, equivalently, for the multifractality of the wave function \[19\]. Had such a solution be found before the RG solution of Refs. \[29–32\], it would be taken as yet another ‘example’ of the incapacity of the replica method. However, although the replica method works perfectly within the RG approach, there is still no clear understanding of why the results obtained within the non-perturbative approach outlined above are exactly the same as those within the RG approach to the extended NLσM (in the limit of the weak disorder, to which the validity of the non-perturbative supersymmetric approach is limited).

### 3.1 Local density of states in open systems

The two-level correlation function considered above is not well defined for \(\omega \lesssim g\) for an open system, as level widths become of the order of Thouless energy, \(E_T = \hbar/\tau_{\text{erg}}\) which is much bigger than the level mean spacing \(\delta\) (their ratio is of order \(g / \delta \gg 1\) in the metallic limit). Its direct analog in this regime is the variance of the density of states (DoS) at the Fermi energy. It is useful to consider also the entire distribution of DoS which can be found in terms of its irreducible moments (cumulants). The relative values of the DoS cumulants can be expressed via the effective field-theoretical functional in a way similar to that in Eqs. \[5\]–\[6\]:

\[
R_s \equiv \frac{(\langle \nu^s \rangle)}{(\langle \nu \rangle)^s} = \lim_{n \to 0} \frac{(i)^{2s}}{(2n^2)} \frac{\partial^{2s}}{\partial \omega^{2s}} \langle \langle Z_n(\omega) \rangle \rangle .
\]  

The cumulants of local DoS are given by the same expression with partial derivatives substituted by variational ones, \(\omega\) being considered in this case as a spatial-dependent source field in Eq. \[6\].

As the higher derivatives with respect to \(\omega\) are involved in the above expression, the Tr \(\ln\) in Eq. \[12\] should now be expanded to the higher powers of \(\omega\). This leads to the following additional contribution \[52\] to the \(\sigma\) model functional \[17\]:

\[
\mathcal{F}_{\text{add}}[Q; \omega] = \frac{i\alpha}{4} \sum_{s=1}^{\infty} \Gamma_s \text{Tr} \left[ \omega \Lambda Q \right]^{s+1}, \quad \Gamma_s = \left( \frac{i\tau_{\text{el}}}{2\pi} \right)^s \frac{(2s-1)!!}{(s-1)!} .
\]

As the bare value, \(\Gamma_{s0}\), of the additional “charge” is proportional to the \(s^4\) power of the small parameter \(\tau_{\text{el}} / \delta \sim (\delta_F)^{d-2} \ell / L^d\), their direct contribution to the DoS cumulants is negligible compared to that obtained by the repeated differentiation \[11\] of the \(\omega\) term in the standard \(\sigma\) model.\[4\]

However, the charges \(\Gamma_s\) sharply increase under RG transformations \[32\]. While “normal” contribution to the DoS cumulants, i.e. those from the standard \(\sigma\) model, Eq. \[17\], scale with the system size as appropriate powers of the DoS variance, the additional contribution from the vertices

---

\[4\] In calculating the local DoS cumulants, this small parameter is compensated by the appearance of powers of \(\delta(r = 0) \to \pi \nu / \tau_{\text{el}}\) so that the contribution of the additional terms to the variance and higher cumulants of the local DoS is comparable \[33\] to that of the standard \(\sigma\) model.
show the multifractal scaling due to this RG increase of $\Gamma_s$. For Dyson’s orthogonal class, it reads
\[ R_s^\text{add} \propto g_0^{1-s} \Gamma s_0 e^{u s(s-1)}, \quad u \equiv \ln \frac{\sigma_0}{\sigma} \approx g_0^{-1} \ln \frac{L}{\ell}, \tag{43} \]
where $\sigma$ is the conductivity at the scale of the system size $L$ and $\sigma_0$ is its bare value, that is the conductivity at the scale of the mean free path $\ell$. As the small parameter $\Gamma s_0$ scale as a linear power of $s$, for large enough $s$ the $e^{u s(s-1)}$ factor becomes dominant.

The additional contribution to the $s^{\text{th}}$ cumulant of the local DoS differs from that in Eq. (43) by the absence of the small parameter $\Gamma s_0$. Such a contribution becomes dominant for $s \gtrsim u^{-1}$ (or for $s \gtrsim g_0$ in the weak disorder limit). The approximate equality in Eq. (43) refers to the weak disorder limit. In this limit, the result for the multifractal cumulants (43) has been reproduced within the supersymmetric approach \cite{9, 21} outlined above. Such a behaviour of the high cumulants leads to the lognormal tails of the distribution function. With increasing the disorder, $u$ becomes of order 1 in the vicinity of the metal-insulator transition in $d > 2$, or for the region of strong localization in $d = 2$ dimensions. In this region, the entire distribution becomes lognormal. The physical meaning of this is that there exist rare realisations of disorder which gives values of the local DoS (or, equivalently, local values of the wave function amplitude $|\Psi(r)|^2$) which are much higher than the average value. The importance of this untypical realisations greatly increases with increasing the disorder so that they correspond to prelocalized states.

3.2 Is there a paradox?

The properties of tails of different distributions governed by the existence of the prelocalized states are described in detail in reviews \cite{32, 40}. The point of this presentation is to underline certain peculiarities of the replica method. To this end, let me first outline how the RG results which lead to Eq. (43) have been obtained. The composite vertices (42) are not closed under the RG transformations of the $\sigma$ model given by Eqs. (17) and (42). The index-permutation symmetry lacking in Eq. (42) is restored under RG transformations via generating the following additional vertices in each $s^{\text{th}}$ order:
\[
\omega^{s+1} \prod_{l=1}^{\infty} \left[ \text{Tr} (\Lambda Q)^l \right]^{n_l}, \quad \sum_{l=1}^{\infty} s_l = s + 1. \tag{44}
\]
Thus in each $s^{\text{th}}$ order the RG equations become matrix equations, the matrix rank being equal to the number of partitions of the integer $s$ into the sum of positive integers. These equations are exactly solvable \cite{31} for any integer $n$, and keeping only the largest eigenvalue leads in the replica limit $n \to 0$ to the result (43). As the RG approach is purely perturbative, the applicability of the replica method is beyond doubt. However, there exist some apparent contradiction.

The point is that one can choose different number of replicas for to represent the advanced and retarded sectors. In the simplest unitary case the $Q$ matrix would belong to the coset space $U(n)/U(n-m) \times U(m)$ where $n$ is the total number of replicas, and $m$ is the difference in their numbers in the advanced and retarded sectors. In the replica limit, $n \to 0$ and $m \to 0$, the result is naturally the same as before. However, the largest eigenvalue of the RG equations happens to depend only on $n$ but not on $m$. Then, instead of taking the replica limit, one could fix $m = 1$ and take an arbitrary integer $n$. The resulting model is defined on the coset space $SU(n)/U(n-1)$ isomorphic to sphere, i.e. in this limit the model becomes equivalent to the $n$-vector model. The composite vertices (42) reduce in this case simply to $\cos s \varphi$ where $\varphi$ is the angle between the $n$-field and the direction of an external (magnetic) field. It is well known (see, e.g., Eq. (43) has been obtained as a result of the one-loop RG analysis. The higher-loop contributions limit its validity to $s \lesssim g_0^{3/2}$. Such a limitation ensures that no moment will be proportional to a positive power of the system volume, $L^d$, i.e. all the multifractal dimensionality remain positive. A similar restriction in the supersymmetric method of Ref. (13) follows directly from the inapplicability of the standard $\sigma$ model description at ballistic scales.
Ref. [41]) that such operators are irrelevant in the $n$-field theory, i.e. they only decrease under the RG transformations, in contrast to those of Eq. (42). So here is the contradiction as they should correspond, as described above, to a particular case of the matrix operators. It is clear, though, that a direct correspondence is anyway impossible, as the rank of the $Q$-matrices in Eq. (42) and thus the number of the eigenvalues in the RG equations depends on $m$ and in the symmetric case $m = 0$ it is equal to the number of partitions of the integer $s$ into the sum of positive integers, i.e. much larger than in the case $m = 1$ when it is just equal to $s$.

The resolution of the apparent paradox is in the fact that one should look only at correlation functions (observables) rather than at the RG dimensions of the composite vertices. In particular, only vertices $[\text{Tr} \Lambda Q]^s$ out of all the variety in Eq. (44) contribute to the cumulants (41), while all the rest serve to give correct eigenvalues. In calculating such a contribution, each eigenvalue enters with a coefficient of proportionality which is a polynomial of $n$ and $m$. It appears that all the polynomials attached to “wrong” (in the case $m = 1$) eigenvalues vanish at all integer values of $n$ thus restoring the correspondence between the matrix and the standard treatments of the $n$-model.

Simultaneously, the $n = m = 0$ replica limit gives the results (43) discussed above. On the face of it, it looks similar to the “replica symmetry breaking” discussed in the previous section: there exists a set of coefficients vanishing at all integer $n$ and giving a nonzero contribution at $n = 0$. However, all the results discussed here have been obtained within the perturbative RG approach, and thus no trick was involved – the replica limit here is just to cancel unwanted vacuum loops in the RG diagrams. All these RG results can be equally easy obtained within the supersymmetric $\sigma$ model.

One question that remains, though, is the following. The multifractal cumulants (43) have been obtained here within the extended $\sigma$ model, Eqs. (17) and (42), while later they have been reproduced within the standard SUSY $\sigma$ model. So, does one really need the additional vertices (42)?

### 3.3 High gradient vertices: RG or not RG?

The answer to this question is in getting – intermediately – even more vertices. Those in Eq. (42) have been obtained in expanding the $\text{Tr} \ln$ term in Eq. (12) in higher powers of $\omega$. But it could be also expanded in higher powers of $\nabla Q$ which yields the following set of vertices (42):

$$\mathcal{F}_s[Q] \equiv \gamma_s \varepsilon_{\alpha_1 \ldots \alpha_{2s}} \int \text{Tr} \left( \prod_{i=1}^{2s} \partial_{\alpha_i} Q \right) d^d r.$$ (45)

The renormalization of these vertices results in the same increase of the largest eigenvalue as in Eq. (43), and actually, taken together with the $\omega$-vertex in the standard $\sigma$ model of Eq. (17), they contribute to the cumulants, Eq. (43), in the same way (but for unknown – in any method – pre-exponential factors) as the composite vertices (42). Therefore, the latter are not even necessary to obtain the multifractality.

An important point now is that, although the higher-gradient vertices (45) have been obtained from the higher-order expansion of the $\text{Tr} \ln$ term of Eq. (12), they are also generated in performing the RG transformation of the standard NL$\sigma$M. Usually they are omitted as being naively irrelevant. However, they become relevant at the one-loop level and actually govern long-tail asymptotics of different distributions. Note that in contrast to the composite vertices (42), the high-gradient operators remain relevant (in the same sense) also for the $n$ vector model (43).

Now the possible relation between the RG approach outlined here and the supersymmetric approach of Refs. [19, 20] is the following. The boundary conditions imposed to find correctly the tails of the distributions in the latter approach makes the saddle-point solution spatially inhomogeneous, while in the former approach the relevance of the high gradients may indicate that such a spatial inhomogeneity develops spontaneously when starting from the standard homogeneous saddle-point.

Finally, there is a possibility that the role of the high-gradient operators (45) is considerably more crucial. In the perturbative RG approach, they have only contributed to the higher moments.
of conductance, DoS, etc, since the structure of the RG equations turns out to be triangular, i.e. the higher order gradient terms do not contribute to the renormalization of the lower order terms. However, in contrast to the composite operators (42), the operators (45) do not break any symmetry of the NLσM. Therefore, one cannot exclude a possibility of a non-perturbative contribution from these operators to, say, the average conductance. Had such a contribution existed, it would totally ruin the one-parameter RG for the average conductance, and this the one-parameter scaling description of the Anderson localization. At the moment, such a possibility remains purely speculative.

4 Coulomb and pairing interactions in the sigma model

The main point of the previous sections is that the replica σ model works perfectly for perturbative problems and – in certain cases – does reasonably well even for certain non-perturbative problems within the non-interacting model. Since the inclusion of interactions beyond the mean-field approach is impossible within the supersymmetric method, the replica model the most reliable tool for models with interactions. The original fermionic replica σ model [4] has been generalized to include the interactions by Finkelstein [7]. It allowed him to reproduce earlier perturbative results (reviewed, e.g., in Ref. [44]), to account where necessary for the Landau Fermi-liquid constants, and to derive the renormalization group equations describing (at least at a qualitative level) a metal–insulator transition in disordered interacting systems. This model has also been successfully extended [45] to allow for effects of Coulomb interaction in superconducting systems (like lowering the transition temperature $T_c$ by disorder – in an apparent, but well understood, deviation from the Anderson theorem [46]). The interest in this approach has been greatly enhanced by the recent discovery [8] of an apparent metal-insulator transition in 2D disordered systems in zero magnetic field. Although it is not at all clear whether the observed effects are, indeed, due the transition [47, 48], and if they are – whether such a transition is, indeed, driven by interactions, a possibility of having the transition in a 2D disordered interacting system (schematically described by the TOE model [1]), in contrast to a disordered noninteracting system, is intriguing by itself. Such a possibility is, undoubtedly, a driving force in a considerable revival of interest in Finkelstein’s σ model (see, e.g., Refs. [12, 13, 49, 50, 51]). At the moment, there is no clear evidence whether Keldysh techniques employed in Refs. [12, 13] would give any edge over the replica method used throughout this chapter. Below I first describe how to include the Coulomb interaction (for simplicity, in a singlet channel only, as the inclusion of a triplet channel is technically almost identical) and the BSC pairing interaction into the derivation of NLσM given in section 2. Then I will show (section 5) how to use such a model for describing the superconductor-insulator transition.

4.1 Hubbard–Stratonovich decoupling

Since the pair interaction does not conserve single-particle energy, in contrast to the elastic scattering which was the only mechanism included up to now, the effective interaction functional should be dynamical. As usual, it is convenient to introduce imaginary time $\tau$ implying everywhere the thermodynamic Gibbs averaging (together with the averaging over quenched disorder where applicable). Thus one considers thermodynamic Green’s functions (in the Matsubara frequency representation) to the retarded and advanced Green’s functions of the previous sections by the standard procedure of analytical continuation [53]. Then the effective functional corresponding to the interaction term in Eq. (4) can be written as

$$S_{\text{int}} = \frac{1}{2} \int \! dx \! dx' \! \bar{\psi}_s(x) \psi_{s'}(x') V_{xx'} \psi_{s'}(x') \psi_s(x),$$

(46)

where $x \equiv r, \tau$, and all the fermionic fields are anti-periodic in imaginary time $\tau$ with period $1/T$, $s = (\uparrow, \downarrow)$ is the spin index, $V_{xx'} \equiv \delta(\tau - \tau') V_c(r-r')$, and $V_c$ represents the Coulomb interaction. After the replication, all the fields in Eq. (46) naturally have the same replica index. Similarly, the
Now one doubles the number of components of the fields \( G \)-

\[ S_{sc} = \lambda_0 \int dx \bar{\psi}_\tau(x)\psi_\tau(x)\psi_\tau(x), \]  

(47)

where \( \lambda_0 \) is the BCS coupling constant.

The Hubbard–Stratonovich decoupling of the functional \((46)-(47)\) is similar to that for the disorder functional \((6)\). To allow for all slow modes in the interaction functional \((46)\) one should introduce three matrix fields: the fields \( \Phi \) and \( \bar{\psi} \) to take account of a small-angle scattering (a singlet channel) and large-angle scattering, and yet another one that corresponds to the Coulomb repulsion in the Cooper channel. The last one would lead to the standard renormalization of the BCS attraction (see, e.g., Ref. \([53]\)) and is not considered here; further, it is assumed that systems under considerations still have an effective attraction in the Cooper channel after such a renormalization.

Under these conditions, the decoupling has the following form:

\[
e^{-S_{int}} = \int \mathcal{D}\Phi \exp \left\{ -\frac{1}{2} \int dx dx' \Phi(x)V_{xx}(x')\Phi(x') + i \int dx \bar{\psi}_s(x)\Phi(x)\psi_s(x) \right\} \]

\[ + \int \mathcal{D}\bar{\psi} \exp \left\{ -\frac{1}{2} \int dx dx' \text{Tr} \left[ \bar{f}(x)\bar{\psi}_s(x)\bar{\psi}_s(x') \right] + i \int dx \bar{\psi}_s(x)f_{ss}(x)\psi_s(x) \right\}, \]

\[ e^{-S_{sc}} = \int \mathcal{D}\Delta \exp \left\{ -\frac{1}{\lambda} \int dx |\Delta(x)|^2 \right\} \]

\[ + i \int dx \left[ \Delta(x)\psi_\tau(x)\psi_\tau(x) - \Delta(x)\psi_\tau(x)\psi_\tau(x) \right] \]  

(48)

Now one doubles the number of components of the fields \( \psi \) and \( \bar{\psi} \) as in Eq. \((10)\) and performs the Gaussian integration to obtain

\[ S = S_{fields} + \frac{\pi\nu}{8\tau} \text{Tr} \sigma^2 - \frac{1}{2} \text{Tr} \ln \left[ \hat{\xi} - i \left( \frac{1}{2\tau} \sigma + \Phi + \Delta + \hat{\xi} \right) - \hat{f} \right]. \]  

(49)

Here the operator \( \hat{\xi} \) equals \( i\hat{\tau}_3\partial_\tau \) in imaginary time representation and becomes the diagonal matrix of fermionic Matsubara frequencies \( (\epsilon_n = \pi(2n + 1)T) \) in frequency representation; \( \text{Tr} \sigma \) refers to a summation over all the matrix indices and to an integration over \( \tau \) and \( \tau \) (or summation over Matsubara frequencies in the frequency representation); \( S_{fields} \) includes the \( \psi \)-independent part of the action \((48)\) quadratic in the fields \( \Phi, f \) and \( \Delta \). The triplet channel, described by the field \( f \), is quite important: in particular, it can lead to the delocalization in the presence of disorder \([3, 49]\); however, such effects will not be considered here and this term will be ignored from now on as it is not relevant for the application of the model in section \(5\).

The field \( \sigma \) has the same structure as in Eqs. \((3)\) and \((10)\), apart from explicitly including the \( 2 \times 2 \) spin sector and replacing the \( 2 \times 2 \) retarded-advanced sector by the dependence on the imaginary time \( \tau \) (since it is diagonal in \( \tau \), it becomes a matrix field in the Matsubara frequencies). In the absence of the fields \( \Phi \) and \( \Delta \), the saddle-point solution for the action \((49)\) is formally the same as in the zero-temperature techniques of section \(2\) but the matrix \( \Lambda \) has a non-unit structure in the Matsubara (instead of advanced-retarded) sector:

\[ Q = U^\dagger \Lambda U, \quad \Lambda = \text{diag} \{ \text{sgn} \hat{\xi} \}. \]  

(50)

The field \( \Phi \), corresponding to the singlet part of the Coulomb interaction, is diagonal in all the sectors. The “order-parameter” field \( \Delta \) is Hermitian and self-charge-conjugate, diagonal in the replica indices and coordinates \( \tau \) and \( \tau \), and has the following structure in the spin and time-reversal space:

\[ \hat{\Delta}(x) = |\Delta(x)|e^{\hat{\tau}_3\hat{\chi}(x)\hat{\tau}_3} \hat{\tau}_2e^{-\hat{\tau}_3\hat{\chi}(x)\hat{\tau}_3}, \]  

(51)

where \( |\Delta| \) and \( \chi \) are the amplitude and the phase of the pairing field \( \Delta(r, \tau) \), \( \hat{\tau}_i \) and \( \hat{\tau}_i^{sp} \) are Pauli matrices \( (i = 0, 1, 2, 3 \text{ with } \hat{\tau}_0 = 1) \) that span the charge-conjugate and spin sectors, respectively.
In the presence of the fields \( \Phi \) and \( \Delta \), the saddle-point equation for the effective functional can be formally written similarly to Eq. (13) for the non-interacting zero-temperature case:

\[
- i \pi \nu \sigma(\mathbf{r}) = \left\langle \mathbf{r} \left\{ -\hat{\xi} + \frac{i}{2 \tau_\text{el}} \sigma + i \left( \hat{\xi} + \hat{\Delta} + \Phi \right) \right\}^{-1} \mathbf{r} \right\rangle (52)
\]

Now \( \sigma = \Lambda \) does no longer represent the saddle point solution for \( \varepsilon \neq 0 \). Still, one can derive an effective functional by expanding the above Tr ln within the manifold (50) in the symmetry-breaking field \( \varepsilon + \Delta + \Phi \) and in gradients of \( Q \), as has been done in the original works by Finkelstein [7]. An alternative is to make first a similarity transformation around \( \Lambda \) within the manifold (50) to find the saddle point solution \( \sigma_{sp} \) in the presence of the fields. This can be formally done with the help of matrix \( U_0 \) that diagonalizes the Hermitian field \( \hat{\xi} + \hat{\Delta} + \Phi \):

\[
\hat{\xi} + \Delta + \Phi = U_0^\dagger \lambda U_0, \quad \sigma_{sp} = U_0^\dagger \Lambda U_0. \tag{53}
\]

Here \( U_0 \) belongs to the same symmetry group that defines the manifold (16) in the absence of the interaction fields. By substituting Eq. (53) into the saddle-point equation (52), one can easily verify that this is, indeed, a spatially-homogeneous solution, provided that \( \lambda \tau_\text{el} \gg 1 \) which will always be the case for a dirty superconductor (\( \Delta \tau_\text{el} \gg 1 \)) or a dirty metal (dimensionless conductance \( g \gg 1 \)).

Now one can perform the expansion of Tr ln, Eq. (49), in gradients of \( Q \) and in the symmetry-breaking fields represented by the eigenvalues \( \lambda \) and the matrix \( U_0 \), Eq. (53). It is convenient to employ the following parameterization:

\[
\sigma = U_0^\dagger Q U_0, \quad Q = U_1^\dagger \Lambda U_1, \tag{54}
\]

where \( Q \) represents the saddle-point manifold in the metallic phase and \( \sigma \) is obtained from \( Q \) by the same rotation (53) as \( \sigma_{sp} \) is obtained from the metallic saddle point \( \Lambda \). Therefore, \( Q \) is defined, as in the metallic phase, on the coset space \( S(2N)/S(N) \otimes S(N) \) where, depending on the symmetry, \( S \) represents the unitary, orthogonal or symplectic group as in the noninteracting case of section 2. The parameterization (54) simplifies considerably all the subsequent derivations and leads to a new variant of the nonlinear \( \sigma \) model which can be more convenient for many applications than the original Finkelstein’s NL\( \sigma \)M. After substituting \( \sigma = U_0^\dagger U_1^\dagger \Lambda U_0 U_1 \), Eq. (54), into Eq. (19), one obtains the following representation for the Tr ln term:

\[
\delta S = -\frac{1}{2} \text{Tr} \ln \{ \hat{G}_0^{-1} + U_0 [\hat{\xi}, U_0^\dagger] - i(U \lambda U^\dagger) \},
\]

where one can also include an external magnetic field with the vector potential \( \mathbf{A} \):

\[
\hat{G}_0 \equiv \left( \hat{\xi} - \frac{i}{2 \tau_\text{el}} \lambda \right)^{-1}, \quad \hat{\xi} = \frac{1}{2m} (\mathbf{p} - e \mathbf{A})^2 - \varepsilon_p.
\]

The expansion to the lowest powers of gradients and \( \lambda \) is now straightforward and similar to that for the noninteracting case. It results after some calculations in the following effective action:

\[
S = \frac{1}{2} \text{Tr} [\Phi V_{\text{rr}}^{-1} \Phi] + \frac{1}{T \Lambda_0} \sum_\omega \int \! d\mathbf{r} \left| \Delta_\omega \right|^2 + \frac{\pi \nu}{2} \text{Tr} \left[ \frac{D}{4} (\partial Q)^2 - \lambda Q \right]. \tag{55}
\]

The long derivative in Eq. (55) is defined as

\[
\partial Q \equiv \nabla Q + [A_0 - ie \mathbf{A} \hat{\tau}_3, Q] \equiv \partial_0 Q + [A_0, Q], \tag{56}
\]

where the matrix \( A_0 \) is given by

\[
A_0 = U_0 \partial_0 U_0^\dagger, \tag{57}
\]

and \( \partial_0 \equiv \nabla - [ie \mathbf{A} \hat{\tau}_3, \ldots] \) is the long derivative (56) in the absence of the pairing field \( \Delta \). Both \( U_0 \) and \( \lambda \) should be found from the diagonalization of \( \varepsilon + \Delta + \Phi \), Eq. (53). Although such a
diagonalization cannot be done in general, it will be straightforward in many important limiting cases. For $\Delta = \Phi = 0$, the field $A_0$ vanishes, $\theta \to \partial_0$, and $\lambda \to \varepsilon$, so that the functional (55) goes over to that of the standard nonlinear $\sigma$ model for non-interacting electrons.

The action (55) is most general in the current context. It allows one to develop a fully self-consistent approach to superconductivity of dirty metals in the presence of Coulomb interaction. However, any application requires a set of further simplifications. As a simple illustration, I will show below how to use the model in a dirty superconductor near the metal-superconductor transition in the absence of Coulomb interaction. Then, I will show in section 5 how to use this model to describe the insulator-superconductor transition as a result of combined effects of disorder and interactions.

### 4.2 Ginzburg-Landau Functional

In the vicinity of the metal-superconductor transition one can expand the action (55) (in the absence of the Coulomb field $\Phi$) in the pairing field $\Delta$. A further simplification is possible in the weak disorder limit, $g \gg 1$, when the $Q$-field can be integrated out to obtain an effective action for the $\Delta$-field only. In the quadratic in $\Delta$ approximation, the kernel of this action will give, with due account for the disorder, an effective matrix propagator of the pairing field which governs properties of a disordered superconducting sample near the transition.

To integrate over the $Q$-field, one splits the action (55) into $S \equiv S_0 + S_\Delta$ where $S_0$ is the standard nonlinear $\sigma$ model functional as in the metallic phase. Then one makes a cumulant expansion, i.e. first expands $e^{-\left(S_0 + S_\Delta\right)}$ in powers of $S_\Delta$, then performs the functional averaging with $e^{-S_0}$ (denoted below by $\langle \ldots \rangle_Q$) and finally re-exponentiates the results. The expansion involves only the first and second order cumulants since the higher order cumulants generate terms of higher order in $\Delta$. Then the only contributions to the action quadratic in $\Delta$ are

$$S_{\text{eff}}[\Delta] = \frac{\lambda_0}{\lambda_0 \nu} \sum_\omega \int d\mathbf{r} |\Delta_\omega|^2 - \frac{\pi \nu}{2} \left\langle \text{Tr} \left( \lambda - \epsilon \right) Q \right\rangle_Q - \left\langle \frac{\pi \nu D}{8} \text{Tr} \left[ A_0, Q \right]^2 + \frac{\left( \pi \nu D \right)^2}{8} \left( \text{Tr} Q \partial_0 Q A_0 \right)^2 \right\rangle_Q.$$  (58)

Expanding $\lambda$ and $A_0$ to the lowest power in $\Delta$ and performing the functional averaging one finds [51] the action quadratic in $\Delta$ as follows:

$$S_{\text{eff}}[\Delta] = \frac{\nu}{T} \sum_\omega \int d\mathbf{r} \Delta_\omega^* (\mathbf{r}) \langle \mathbf{r} | \hat{K}_\omega | \mathbf{r}' \rangle \Delta_\omega (\mathbf{r}'),$$  (59)

with the operator $\hat{K}_\omega$ given by

$$\hat{K}_\omega = \frac{1}{\lambda_0 \nu} - 2 \pi T \sum_{\epsilon(\omega - \epsilon) < 0} \left\{ \hat{\Pi}_\omega^c + \frac{1}{\pi \nu} \frac{\Pi_{2\epsilon - \omega}^d (0) \hat{\mathcal{C}}}{(2\epsilon - \omega)^2} \right\}.$$  (60)

Here $\Pi_{|\omega|}^c (\mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | \Pi_{|\omega|}^c | \mathbf{r}' \rangle$ are the cooperon and diffuson propagators, with

$$\hat{\Pi}_{|\omega|} = \left( \hat{\mathcal{C}} + |\omega| \right)^{-1},$$  (61)

where $\hat{\mathcal{C}} \equiv -D \left( \nabla - 2i \epsilon A \right)^2$ defines the cooperon modes; $\Pi_d$ is obtained from $\Pi_c$ by putting the external vector potential $A = 0$. In the last term in Eq. (60), $\Pi_{|\omega|}^d (0) = \Pi_{|\omega|}^d (\mathbf{r}, \mathbf{r})$; this term may be obtained by expanding in $g^{-1}$ the cooperon propagator with the renormalized diffusion coefficient,

$$\hat{\mathcal{C}} \to \left[ 1 - \frac{1}{\pi \nu} \Pi_{|\omega|}^d (0) \right] \hat{\mathcal{C}},$$
which is a weak localization correction to free cooperon propagator $\Pi_{\omega}(r, r')$.

The summation over Matsubara frequencies in Eq. (60) yields

$$
\tilde{K}_\omega = \ln \frac{T}{T_0} + \psi \left( \frac{1}{2} + \frac{|\omega| - \hat{C}}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) - \frac{a_\omega \hat{C}}{4\pi T},
$$

where $T_0 \equiv T_{c0}(B=0)$ is the transition temperature of the clean superconductor in the absence of a magnetic field and $\psi$ is the digamma function. The weak localization correction is proportional to the coefficient $a_\omega$ given by

$$
a_\omega(T) = \frac{1}{\pi \nu V} \sum_q \frac{1}{Dq^2} \left\{ \psi' \left( \frac{1}{2} + \frac{|\omega|}{4\pi T} \right) - \frac{4\pi T}{Dq^2} \left[ \psi \left( \frac{1}{2} + \frac{|\omega| + Dq^2}{4\pi T} \right) - \psi \left( \frac{1}{2} + \frac{|\omega|}{4\pi T} \right) \right] \right\}.
$$

For $\omega = 0$ the coefficient $a_0 \equiv a_{\omega=0}(T)$ can be simplified in the two limits:

$$
a_0 = \begin{cases} 
\frac{\psi'(1/2)}{\pi \nu L^d} \sum_{L^{-1} < q < \ell^{-1}} \frac{1}{Dq^2}, & L \gg L_T, \\
\frac{\psi''(1/2)}{8\pi^2 \nu L^d T'}, & L \ll L_T,
\end{cases}
$$

where $L_T \equiv \sqrt{D/T}$ is the thermal smearing length.

The instability of the normal state (i.e., a transition into the superconducting state) occurs when the lowest eigenvalue of the operator $\tilde{K}_\omega$ becomes negative. The eigenfunctions of this operator coincide with the eigenfunctions of the cooperon operator $\hat{C}$. The lowest eigenvalue of $\hat{C}$ is known to be $C_0 = DB/\phi_0$, where $\phi_0$ is the flux quanta. This ground state cooperon eigenfunction corresponds to the lowest eigenvalue $K_0$ of the operator $\tilde{K}_\omega$. The condition $K_0 = 0$ implicitly defines the line $T_c(B)$ in the $(T, B)$-plane where the transition occurs:

$$
\ln \frac{T_c}{T_0} + \psi \left( \frac{1}{2} + \frac{C_0}{4\pi T_c} \right) - \psi \left( \frac{1}{2} \right) = \frac{a_0 C_0}{4\pi T_c}.
$$

The term in the r.h.s. of Eq. (64) describes a $1/g$-correction to the main result. This weak localization is linear in the magnetic field $B$ and vanishes as $B \to 0$ as expected (the Anderson theorem [46]). In a nonzero magnetic field the weak localization correction to $T_c$ is positive which has a very simple explanation. The superconductivity is destroyed by the magnetic field when $\Phi(\xi) \geq \Phi_0$, where $\Phi(\xi)$ is the flux over the area with linear size of the order of the coherence length $\xi \sim \sqrt{D/T}$ and $\Phi_0$ is the flux quanta. The weak localization corrections reduce $D$ and thus $\xi$. Therefore, one needs a stronger field to destroy the quantum coherence. The same reasoning explains the growth of $T_c$ in a fixed magnetic field.

Note finally that it would be straightforward to include the leading weak-localization corrections in all orders of $g^{-1} \ln g$ by calculating the $Q$ - averages in Eq. (58) via the renormalization group. This would lead to the renormalization of $D$ in the cooperon propagator (61), thus changing the shape of the $T_c(B)$ curve. However, the value of $T_c(0)$ will again remain unaffected, since the superconducting instability is defined by the onset of the homogeneous zero mode in the operator $K$, Eq. (52), which does not depend on the value of the diffusion coefficient in the cooperon propagator.

It is worth stressing that the Anderson theorem reflects certain properties of a model rather than those of real superconductors. If one allows for Coulomb interaction, then the critical temperature of the superconducting transition is no longer independent of disorder. Combined effects of the interaction and disorder lead to corrections to the transition temperature proportional (in a slightly
simplified way) to \( g^{-1} \ln^3(T_c \tau_{\text{el}}) \) \([53, 54, 45]\). For a relatively weak disorder, the system still remains superconducting at \( T \to 0 \) while for a sufficiently strong disorder the above corrections would suppress the superconducting pairing at any temperature and make a 2D system insulating at \( T \to 0 \). Such a mechanism of suppressing \( \Delta \) by disorder gives a possible scenario for a widely observed superconductor–insulator (SI) transition in two-dimensional structures \([55–58]\) which is most adequately described within the NL\( \sigma \)M \([45, 7]\) similar to those developed in this section. However, a widely accepted scenario for such a transition is based on very different, both in origin and in implementation, models \([59–63]\) where \(|\Delta|\) remains finite and the superconductivity is suppressed by the loss of the phase coherence due to the fluctuations of the phase \( \chi \) of the order parameter. I will show in the following section that these "phase-only" models can also be derived under certain parametrically controlled assumptions from the NL\( \sigma \)M developed in this section.

5 Superconductor–insulator transition in 2D systems: phase and amplitude fluctuations of the order parameter

A wide variety of experimental data indicate the existence of a superconductor–insulator (SI) transition in two-dimensional films \([55–58]\) (see, in particular, \([58]\) for earlier references). The transition can be tuned by either disorder (changing with the thickness of a superconducting film) or magnetic field, thus being one of the most intensely studied examples of quantum phase transitions \([54]\).

One of the most accepted ways to understand the problem of the SI transition is based on the so-called Bose-Hubbard (or "dirty-boson") models \([63]\). In this class of models exists a duality between charge-\(2e\) bosons (preformed Cooper pairs) and vortices. The superconducting phase is due to the bose-condensation of the charged bosons with localized vortices while the insulating phase is due to the bose-condensation of the vortices with the localized charged bosons. Another approach which captures the basic physics of granular superconductors is based on dissipative models \([65]\) of resistively shunted charged Josephson arrays with the emphasis on the role of dissipation and Coulomb interaction. In both group of models \([59–65]\), the transition is driven by fluctuations of the phase of the order parameter, i.e. the superconductivity is destroyed in spite of the existence of nonvanishing \(|\Delta|\) locally. This approach seems to be rather different from that mentioned in the previous section where the disorder and interaction destroy the superconductivity by destroying \(|\Delta|\) everywhere in a homogeneously disordered system. An experimental distinction between homogeneous and granular systems is not that strict \([53]\) as it seemed a few years ago, and recent experimental observations \([55, 56]\) strongly suggest that the amplitude fluctuations in the vicinity of the SI transition are no less important than the phase fluctuations.

The purpose of this section is to modify a general NL\( \sigma \)M action for the description of granular systems with the BCS and Coulomb interactions. Such a description takes account of fluctuations of both amplitude and phase of the order parameter \( \Delta \), thus encompassing all the above described approaches. Both the Bose-Hubbard model \([63]\) and the dissipative models \([59–61]\) will be derived from this action via certain controlled simplifications made within the NL\( \sigma \)M. The latter model is more general and allows one to go beyond certain limitations necessary in the derivation of the phase-only action.

The starting point is a coarse-grained version of the TOE model Hamiltonian of Eq. (1) for a granular superconductor, where the kinetic energy includes terms \( t_{ij} \hat{a}_i^\dagger \hat{a}_j \) corresponding to tunnelling hops between the grains. The derivation of the NL\( \sigma \)M from such Hamiltonian follows the procedure described in the previous section. Thus a ‘half-baked’ effective functional obtained by integrating out the fermionic fields is similar to that in Eq. (49):

\[
S[\hat{\sigma}, \hat{\Delta}, \Phi] = \frac{\pi \nu}{8 \tau_{\text{el}}} \text{Tr} \sigma^2 + \frac{1}{4 \lambda_0} \text{Tr} |\hat{\Delta}|^2 + \frac{1}{2} \text{Tr} \Phi U^{-1} \Phi
\]

\[
- \frac{1}{2} \text{Tr} \ln \left[ -\hat{\xi} - \hat{\ell} + \frac{i}{2 \tau_{\text{el}}} \hat{\sigma} + i \left( \hat{\Delta} + \Phi + \hat{\epsilon} \right) \right].
\]

(65)
The difference here is that $\xi$ is the operator of the intra-grain kinetic energy (counted from the chemical potential), while $\hat{t}$ is the tunneling amplitude matrix (i.e. the inter-grain kinetic energy). Note that in the granular case the singlet field $\Phi$ is sufficient to decouple the Coulomb interaction term so that the triplet field $f$ does not enter Eq. (65). All the bosonic fields have the same structure as before but for the addition of the $m \times m$ grain sector. The symbol $\text{Tr}$ refers both to a summation over all these matrix indices and to an integration over intra-grain position $r$ and the imaginary time $\tau$.

The principal simplification for granular systems is that all the fields are spatially homogeneous inside each grain when the grains are zero-dimensional, i.e. their sizes $L \lesssim \xi, L_{sT}$ ($\xi$ and $L_{sT}$ are the superconducting and thermal coherence lengths) which is equivalent to $|\Delta|, T \lesssim 1/\tau_{\text{erg}}$. For a diffusive grain $\tau_{\text{erg}} = L^2/D \equiv \hbar/E_{T}$ while for a ballistic (chaotic) grain $\tau_{\text{erg}} \sim L/v_{s}$. The tunneling matrix $t = \{t_{ij}\}$ depends only on grain indices, and the Coulomb interaction thus reduces to the capacitance matrix, $U^{-1} \rightarrow C_{ij}/e^2$.

To follow the procedure described in the previous section one needs, first, to solve (formally) the saddle point conditions, Eqs. (53) and (54) separately for each grain. As all the fields are spatially homogeneous inside each grain, $U_{0} = U_{0}(i)$ commutes with the operator $\hat{t}$. Then one only needs to expand the $\text{Tr} \ln$ to the first nonvanishing orders in $t_{ij}$ and $\lambda_{i}$, this expansion being justified when $|t|, |\Delta|, T \ll 1/\tau_{\text{el}} \ll \tau_{\text{g}}$. Thus one arrives at the following effective action:

$$S[Q, \Delta, \Phi] = \int d\tau \left\{ \sum_{i} \frac{|\Delta_{i}|^2}{\nu \lambda_{0} \delta_{i}} + \sum_{ij} \frac{C_{ij}}{2\epsilon_{2}} \Phi_{i} \Phi_{j} \right\}$$

$$- \sum_{i} \frac{\pi}{2\delta_{i}} \text{Tr} \lambda_{i} Q_{i} - \frac{g^{h}_{ij}}{2} \sum_{ij} \text{Tr} Q_{i} S_{ij} Q_{j} S_{ji},$$

where $S_{ij} = U_{0}(i)U_{0}^{\dagger}(j)$, all the fields depend on $\tau$, $\text{Tr}$ refers to all indices except $i$, $j$ numerating grains, $\delta_{i}$ is mean level spacing in the $i$-th grain, and the tunneling conductance is defined by $g^{h}_{ij} \equiv 2\pi^{2}|t_{ij}|^{2}/\delta_{i}\delta_{j}$ (which is nonzero only for neighboring grains). Both $\lambda_{i}$ and $U_{0}(i)$ (and thus $S_{ij}$) should be found from the diagonalization procedure in Eq. (53), while $Q$ is given by Eq. (54).

The next step is to represent $U_{0}(i)$ as

$$U_{0}(i) = V_{i} e^{-\frac{\pi}{2} \chi_{i}(\tau) \hat{\phi}_{i}}.$$  

(67)

This is similar to the gauge transformation suggested in Refs. [66] and used in Ref. [7] to gauge out the Coulomb field. However, one cannot gauge out two independent fields, $\Delta$ and $\Phi$. Substituting the transformation (67) into the diagonalization condition (53), one reduces it to

$$\hat{\epsilon} + \Phi_{i} + \hat{\Delta}_{i} = V_{i}^{+} \lambda_{i} V_{i},$$

(68)

where $\hat{\Delta}_{i}$ is the field (51) taken at $\chi = 0$ and the field $\Phi_{i}$ is given in the $\tau$ representation by

$$\Phi_{i} = \Phi_{i} - \frac{1}{2} \partial_{\tau} \chi_{i}.$$  

(69)

Both $\Phi$ and $|\Delta|$ are massive fields whose fluctuations are strongly suppressed. It is straightforward to show that the fluctuations of $\Phi$ are of order $\delta$ which is much smaller than both $T$ and $|\Delta|$. Therefore, in the mean field approximation in $\Phi$, this field can be neglected. $\Phi = 0$. This condition is nothing more than the Josephson relation in imaginary time [64]. This locks the fluctuations of the Coulomb field $\Phi$ with the phase-fluctuations of the pairing field $\Delta$.

$$\Phi = \frac{1}{2} \partial_{\tau} \chi,$$  

thus reducing the action (66) to one depending only on the fields $Q$ and $\Delta$.

The mean field approximation in $|\Delta_{i}|$ is valid for $|\Delta_{i}| \gg \delta$. It is equivalent to the standard self-consistency equation which formally follows from the variation of the action (66) with respect
to $|\Delta|$. In this approximation one finds $|\Delta_i|$ to be independent of $i$ and $\tau$. Thus the first term in Eq. (66) becomes a trivial constant, so that the action depends on $Q$ and $\chi$ only:

$$S[Q, \chi] = \sum_{ij} \frac{C_{ij}}{8e^2} \int_0^\beta d\tau \partial_\tau \chi_i \partial_\tau \chi_j - \sum_i \frac{\pi}{2g_i} Tr \lambda_i Q_i - \frac{q_{ij}}{2} \sum_{ij} Tr Q_i Q_j S_{ji}. \quad (70)$$

The field $\chi$ in this action obeys the boundary condition $\chi(\tau + \beta) = \chi(\tau) \mod 2\pi$. In calculating the partition function, one should in general allow for different topological sectors corresponding to different winding numbers in $\chi$.

The “phase-only” action (70) includes neither fluctuations of $|\Delta|$, nor fluctuations of $\Phi$ beyond the Josephson relation, Eq. (69). Below this action will be reduced to the AES action. Still, we stress that the action (70) is more general than the AES action. Thus, in the absence of superconductivity, $\Delta \equiv 0$, it was shown [67] that the former contains a correct screening of the Coulomb fluctuations, are the same for each grain and reduced to those solved in Ref. [51]:

$$\lambda = \text{diag} \sqrt{\varepsilon^2 + |\Delta|^2} \text{sgn } \varepsilon, \quad \cos \theta_\varepsilon \equiv \frac{|\varepsilon|}{\sqrt{\varepsilon^2 + |\Delta|^2}}. \quad (71)$$

Then $S_{ij}$ in Eq. (70) can be expressed in terms of $V$ as

$$S_{ij} \equiv V e^{-\frac{i}{2} \chi_{ij} \hat{r}_3 V^{\dagger}}, \quad \chi_{ij} \equiv \chi_i - \chi_j. \quad (72)$$

Finally note that large-|$\varepsilon$| contributions to the action (13) are strongly suppressed, while for $|\varepsilon| \ll |\Delta|$ one has $\lambda = |\Delta|/\Lambda$ which suppresses fluctuations of $Q$ in each grain imposing $Q = \Lambda$. Then, all matrices in the action (70) are diagonal in the replica indices so that these indices become redundant. It finally reduces the action (70) to that depending only on one scalar bosonic field $\chi_i(\tau)$, the phase of the order parameter, which is a function of only the imaginary time and the grain number. Indeed, the second term in Eq. (70) reduces to a trivial constant; evaluating the tunneling term with the help of Eqs. (71) and (72), one obtains

$$S[\chi] = \sum_{ij} \left\{ \frac{C_{ij}}{8e^2} \int_0^\beta d\tau \partial_\tau \chi_i \partial_\tau \chi_j - 2g_{ij} \int_0^\beta d\tau \int_0^\beta d\tau' g_n^2(\tau - \tau') \cos \chi_{ij} + g_n^2(\tau - \tau') \cos \chi_{ij} \right\}, \quad (73)$$

where $\chi_{ij}(\tau) \equiv \chi_i(\tau) - \chi_j(\tau)$, $\chi_{ij}^\pm \equiv \frac{1}{2} \left[ \chi_{ij}(\tau) \pm \chi_{ij}(\tau') \right]$, and the normal and anomalous Green’s functions $g_{n,a}$ (integrated over all momenta) are given by

$$g_n(\tau) = T \sum_{\varepsilon} \frac{\varepsilon \sin \varepsilon \tau}{\sqrt{\varepsilon^2 + |\Delta|^2}}, \quad g_a(\tau) = T \sum_{\varepsilon} \frac{|\Delta| \cos \varepsilon \tau}{\sqrt{\varepsilon^2 + |\Delta|^2}}. \quad (74)$$

The action (73) is exactly the AES action introduced in Refs. [59, 60, 61]. Further simplifications are possible in two limiting cases.

First, in the normal case ($\Delta = 0$) one has in Eq. (74)

$$g_a = 0, \quad g_n^2(\tau) = T^2 \frac{\sin^2 \pi \tau}{\sin^2 \pi T \tau}. \quad (75)$$
Then the field $\chi$ should be substituted, according to Eq. (69), by $2 \int_0^\tau d\tau' \Phi(\tau')$. This corresponds to using the action (73), (74) for a set of normal tunnel junctions [68], as has been recently shown in Ref. [67]: the functional (70) in the limit $\Delta = 0$ is equivalent to that of Ref. [67]. Including disorder-induced fluctuations (i.e. going beyond the $Q = \Lambda$ approximation) allows one to obtain [67] a correct low-$T$ limit for the phase correlation function missing in the action (73).

The action (73) is more general than that considered in Ref. [67]: although under the mode locking condition (69) it depends only on the fields $\chi$ and $Q$, the matrix $S_{ij}$, Eqs. (71) and (72), reduces to a simple $U(1)$ gauge transformation as in in Ref. [67] only in the limit $\Delta = 0$.

The second limiting case, $T \ll |\Delta|$, is just the limit relevant in the context of the SI transition in granular superconductors. For $T = 0$, the summation in Eq. (74) can be substituted by integration which yields

$$g_n(\tau) = \frac{|\Delta|}{\pi} K_1(|\Delta|\tau), \quad g_a(\tau) = \frac{|\Delta|}{\pi} K_0(|\Delta|\tau)$$

This is also a good approximation for a low-$T$ case; substituting this into Eq. (73) gives the action for the dissipative model [60, 61]. Note that for $|\tau - \tau'| \ll |\Delta|^{-1}$, the main contribution in the tunneling action (73) is given by the normal term with the corresponding kernel proportional to $|\tau - \tau'|^{-2}$. The Fourier transform of this would give a term of the Caldeira-Leggett type [65] proportional to $|\omega|$.

The tunneling action (73) is non-local in $\tau$. As has been noted in Ref. [60] for the case of one Josephson junction, for sufficiently large capacitance the phase $\chi_{ij}$ changes slowly in comparison with $|\Delta|^{-1}$, and in the adiabatic approximation $\chi(\tau')$ is changed by $\chi(\tau) + (\tau - \tau') \partial_{\tau} \chi(\tau)$. Making such an expansion, one obtains from Eq. (73) the following local action:

$$S[\chi] = \int_0^\beta d\tau \left\{ \sum_{ij} \frac{1}{2} u_{ij}^{-1} \dot{\chi}_i \dot{\chi}_j - |\Delta| g_{ij}^4 \cos \chi_{ij} \right\}, \quad (76)$$

where $\dot{\chi}_i \equiv \partial_\tau \chi_i$ and

$$\frac{1}{u_{ii}} = \frac{C_{ii}}{4e^2} + \sum_j \frac{g_{ij}^4}{|\Delta|} \frac{3 + \cos \chi_{ij}}{8},$$

$$u_{ij}^{-1} = \frac{C_{ij}}{4e^2} - \frac{g_{ij}^4}{|\Delta|} \frac{3 + \cos \chi_{ij}}{8}.$$ 

If all the self-capacitances are equal to $C$ with $E_c \propto e^2/C$ being the charging energy, and all $g_{ij}^4 = g^4$, then $u_{ii} \equiv U$ has the meaning of the renormalized charging energy. Ignoring a weak dependence of $u$ on $\cos \chi_{ij}$ in the above relations, one obtains the renormalized charging energy:

$$U = \frac{E_c}{1 + \#E_c g^4 / |\Delta|}. \quad (77)$$

Here the coefficient $\#$ depends on the number of next neighbors for each grain, etc. A similar renormalization takes place for the next-neighbor off-diagonal energy $u_{ij}$. Now one can see that on the face of it the adiabatic approximation employed to obtain Eq. (76) is valid for $U \ll |\Delta|$. However, in the region $g^4 \gg |\Delta|/E_c$, where the charging energy (77) is strongly renormalized, the instanton-like solutions [69] may be important. This may further reduce the region of applicability for the local in $\tau$ action (74).

Finally, by introducing the operator $\hat{n}$ canonically conjugate to the phase $\chi$, one finds the Hamiltonian that corresponds to the action (76):

$$\hat{H} = \sum_{ij} \frac{1}{2} u_{ij} \hat{n}_i \hat{n}_j - |\Delta| g_{ij}^4 \cos (\chi_i - \chi_j). \quad (78)$$

22
This is just the Hamiltonian of the Bose-Hubbard model \[63\] which was first microscopically derived by Efetov \[70\] in the context of granular superconductors.

To conclude, in this section the effective $\sigma$ model-type action \[66\] has been derived for a granular system with zero-dimensional grains in the presence of the Coulomb and pairing interactions. This is the most general (in the present context) action which takes into account fluctuations of both amplitude and phase of the order parameter $\Delta$. Neglecting fluctuations of $|\Delta|$ and fluctuations of the Coulomb field $\Phi$ beyond the Josephson relation \[69\] reduces this action to the "phase-only" action \[70\] which still contains intra-granular disorder important for the correct screening for $|\Delta|$ small compared to the charging energy. Neglecting this disorder further reduces the action \[70\] to that of the AES model, Eq. \[73\]. However, the above estimations show that this reduction is parametrically justified only for the region $E_c \ll |\Delta|$ where the transition happens at $g_\parallel \ll E_c/|\Delta| \ll 1$ which corresponds to a strongly granular system. Note finally that the most general (in the present context) action \[66\] describes both amplitude and phase fluctuations of the order parameter, being still considerably different from the NL$\sigma$M action for homogeneous systems.

**Acknowledgments.** This work has been supported by the Leverhulme Trust under the contract F/94/BY and by the EPSRC grant GR/R33311.

**References**

[1] F. Wegner, Z. Phys. B 35, 207 (1979); L. Schäfer and F. Wegner, Z. Phys. B 38, 113 (1980).
[2] K. B. Efetov, Sov. Phys. JETP 55, 514 (1982).
[3] A. J. McKane and M. Stone, Ann. Phys. (N. Y.) 131, 36 (1981).
[4] K. B. Efetov, A. I. Larkin, and D. E. Khmelëntskii, Sov. Phys. JETP 52, 568 (1980).
[5] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. 129, 367 (1985).
[6] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
[7] A. M. Finkel'shtein, Sov. Phys. JETP 57, 97 (1983); Sov. Sci. Rev. 2 14, 1 (1990).
[8] S. V. Kravchenko et al., J. Phys.–Cond. Matter 7, L41 (1995); Phys. Rev. B 51, 7038 (1995).
[9] A. G. Aronov and A. S. Ioselevich, JETP Letters 42, 84 (1985).
[10] V. S. Babichenko and A. N. Kozlov, Sol. State Commun. 59, 39 (1986).
[11] M. Horbach and G. Schön, Physica A167, 93 (1990).
[12] A. Kamenev and A. Andreiev, Phys. Rev. B 60, 3944 (1999).
[13] C. Chamon, A. W. W. Ludwig, and C. Nayak, Phys. Rev. B 60, 2239 (1999).
[14] L. Keldysh, Sov. Phys. JETP 20, 1018 (1964).
[15] K. B. Efetov, Sov. Phys. JETP 56, 247 (1982).
[16] K. B. Efetov, *Supersymmetry in Disorder and Chaos*, CUP, Cambridge (1997).
[17] V. E. Kravtsov and A. D. Mirlin, JETP Letters 60, 656 (1994).
[18] B. A. Muzykantskii and D. E. Khmelëntskii, Phys. Rev. B 51, 5480 (1995).
[19] K. B. Efetov and V. I. Fal’ko, Europhys. Lett. 32, 627 (1995).
[20] A. D. Mirlin, Phys. Rev. B 53, 1186 (1996).
[21] M. L. Mehta, *Random matrices*, Academic Press, Boston (1991).
[22] L. P. Gor’kov and G. M. Eliashberg, Sov. Phys. JETP 21, 940 (1965).
[23] B. L. Altshuler and B. I. Shklovskii, Sov. Phys. JETP 64, 127 (1986).
[24] J. J. M. Verbaarschot and M. R. Zirnbauer, J. Phys. A 17, 1093 (1985).
[25] A. Kamenev and M. Mezard, J. Phys. A 32, 4373 (1999).
[26] A. Kamenev and M. Mezard, Phys. Rev. B 60, 3944 (1999).
[27] I. V. Yurkevich and I. V. Lerner, Phys. Rev. B 60, 3955 (1999).
[28] A. Kamenev, Phys. Rev. Lett. 85, 4160 (2000).
[29] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, JETP Letters 43, 441 (1986); Sov. Phys. JETP 64, 1352 (1986).
[30] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, Sov. Phys. JETP 67, 795 (1988).
[31] I. V. Lerner, Phys. Lett. 133A, 253 (1988); I. V. Lerner and F. Wegner, Z. Phys. B 81, 95 (1990).
[32] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, in Mesoscopic Phenomena in Solids, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb, North-Holland, Amsterdam, 449 (1991).
[33] M. Zirnbauer, cond-mat/9903338 (1999).
[34] A. V. Andreev and B. L. Altshuler, Phys. Rev. Lett. 75, 902 (1995).
[35] A. A. Abrikosov, L. P. Gor’kov, and I. E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Physics, Pergamon Press, New York (1965).
[36] R. A. Smith, I. V. Lerner, and B. L. Altshuler, Phys. Rev. B 58, 10343 (1998).
[37] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry, Part II, Springer-Verlag, Berlin (1985).
[38] B. L. Altshuler, JETP Letters 41, 648 (1985).
[39] P. A. Lee and A. D. Stone, Phys. Rev. Lett. 55, 1622 (1985).
[40] A. D. Mirlin, Phys. Rep. 326, 260 (2000).
[41] A. Z. Patashinskii and V. I. Pokrovskii, Fluctuation Theory of Phase Transitions, Pergamon Press, Oxford (1979).
[42] V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, Sov. Phys. JETP 67, 1441 (1988).
[43] F. Wegner, Z. Phys. B 78, 33 (1990).
[44] B. L. Altshuler and A. G. Aronov, in Electron-Electron Interactions in Disordered Systems, edited by A. L. Efros and M. Pollak, North-Holland, Amsterdam, 1 (1985).
[45] A. M. Finkelshtein, JETP Letters 45, 46 (1987); Physica B 197, 636 (1994).
[46] P. W. Anderson, J. Phys. Chem. Sol. 11, 26 (1959).
[47] M. Y. Simmons, A. R. Hamilton, M. Pepper, E. H. Linfield, P. D. Rose, D. A. Ritchie, A. K. Savchenko, and T. G. Griffiths, Phys. Rev. Lett. 80, 1292 (1998).
[48] A. R. Hamilton, M. Y. Simmons, M. Pepper, E. H. Linfield, P. D. Rose, and D. A. Ritchie, Phys. Rev. Lett. 82, 1542 (1999); ibid 84, 2489 (2000).
[49] C. Castellani, C. DiCastro, and P. A. Lee, Phys. Rev. B 57, R9381 (1998).
[50] M. V. Feigel’man, A. I. Larkin, and M. A. Skvortsov, Phys. Rev. B 61, 12361 (2000).
[51] I. V. Yurkevich and I. V. Lerner, Phys. Rev. B 63, 4522 (2001); ibid 13043 (2001).
[52] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics, Springer, Heidelberg (1999).
[53] Y. N. Ovchinnikov, Sov. Phys. JETP 37, 366 (1973).
[54] S. Maekawa, H. Ebisawa, and H. Fukuyama, J. Phys. Soc. Jpn. 53, 2681 (1984).
[55] N. Mason and A. Kapitulnik, Phys. Rev. Lett. 82, 5341 (1999).
[56] J. A. Chervenak and J. M. Valles Jr., Phys. Rev. B 59, 11209 (1999).
[57] N. Markovic, C. Christiansen, A. M. Mack, W. H. Huber, and A. M. Goldman, Phys. Rev. B 60, 4320 (1999).
[58] A. M. Goldman and N. Markovic, Physics Today 51, 39 (1998).
[59] V. Ambegaokar, U. Eckern, and G. Schön, Phys. Rev. Lett. 48, 1745 (1982).
[60] A. I. Larkin and Y. N. Ovchinnikov, Phys. Rev. B 28, 6281 (1983).
[61] U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B 30, 6419 (1984).
[62] S. Chakravarty et al., Phys. Rev. Lett. 56, 2303 (1986); M. P. A. Fisher, ibid 57, 885 (1986); D. Dalidovich and P. Phillips, ibid 84, 737 (2000); A. Cuccolo, A. Fibini and V. Tognetti, Phys. Rev. B 61, 11289 (2000).
[63] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. 64, 587 (1990); M. P. A. Fisher, ibid 65, 923 (1990); E. S. Sorensen et al., ibid 69, 828 (1992); C. Bruder, R. Fazio, and G. Schön, Phys. Rev. B 47, 342 (1993); M. Wallin et al., ibid 49, 12115 (1994); J. Lidmar et al., ibid 58, 2827 (1998); D. Das and D. Doniach, ibid 60, 1261 (1999).
[64] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Rev. Mod. Phys. 69, 315 (1997).
[65] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. 46, 211 (1981).
[66] A. Kamenev and Y. Gefen, Phys. Rev. B 54, 5428 (1996).
[67] I. S. Beloborodov et al., Phys. Rev. B 63, 115109 (2001).
[68] G. Schön and A. D. Zaikin, Phys. Rep. 198, 237 (1990).
[69] S. E. Korshunov, JETP Letters 45, 434 (1987).
[70] K. B. Efetov, Sov. Phys. JETP 51, 1015 (1980).