On Some Low Distortion Metric Ramsey Problems

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Abstract

In this note, we consider the metric Ramsey problem for the normed spaces \(\ell_p\). Namely, given some \(1 \leq p \leq \infty\) and \(\alpha \geq 1\), and an integer \(n\), we ask for the largest \(m\) such that every \(n\)-point metric space contains an \(m\)-point subspace which embeds into \(\ell_p\) with distortion at most \(\alpha\). In [1] it is shown that in the case of \(\ell_2\), the dependence of \(m\) on \(\alpha\) undergoes a phase transition at \(\alpha = 2\). Here we consider this problem for other \(\ell_p\), and specifically the occurrence of a phase transition for \(p \neq 2\). It is shown that a phase transition does occur at \(\alpha = 2\) for every \(p \in [1, 2]\). For \(p > 2\) we are unable to determine the answer, but estimates are provided for the possible location of such a phase transition. We also study the analogous problem for isometric embedding and show that for every \(1 < p < \infty\) there are arbitrarily large metric spaces, no four points of which embed isometrically in \(\ell_p\).

1 Introduction

A Ramsey-type theorem states that large systems necessarily contain large, highly structured sub-systems. Here we consider Ramsey-type problems for finite metric spaces, interpreting “highly structured” as having low distortion embedding in \(\ell_p\).

A mapping between two metric spaces \(f : M \to X\), is called an embedding of \(M\) in \(X\). The distortion of the embedding is defined as

\[
\text{dist}(f) = \sup_{x, y \in M, x \neq y} \frac{d_X(f(x), f(y))}{d_M(x, y)} \cdot \sup_{x, y \in M, x \neq y} \frac{d_M(x, y)}{d_X(f(x), f(y))}.
\]

The least distortion required to embed \(M\) in \(X\) is denoted by \(c_X(M)\). When \(c_X(M) \leq \alpha\) we say that \(M\) \(\alpha\)-embeds in \(X\). In this note we study the following notion.

Definition 1 (Metric Ramsey function). We denote by \(R_X(\alpha, n)\) the largest integer \(m\) such that every \(n\)-point metric space has a subspace of size \(m\) that \(\alpha\)-embeds into \(X\).

When \(X = \ell_p\) we use the notations \(c_p\) and \(R_p\). Note that for \(p \in [1, \infty]\), it is always true that \(R_p(\alpha, n) \geq R_2(\alpha, n)\). When \(\alpha = 1\) we drop it from the notation, i.e., \(R_X(n) = R_X(1, n)\).

Bourgain, Figiel, and Milman [4] study this function for \(X = \ell_2\), as a metric analog of Dvoretzky’s theorem [7]. They prove

Theorem 1 ([4]). For any \(\alpha > 1\) there exists \(C(\alpha) > 0\) such that \(R_2(\alpha, n) \geq C(\alpha) \log n\). Furthermore, there exists \(\alpha_0 > 1\) such that \(R_2(\alpha_0, n) = O(\log n)\).

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In [11] the metric Ramsey problem is studied comprehensively. In particular, the following phase transition is established in the case of $X = \ell_2$.

**Theorem 2 (11).** Let $n \in \mathbb{N}$. Then:

1. For every $1 < \alpha < 2$: $c(\alpha) \log n \leq R_2(\alpha, n) \leq 2 \log n + C(\alpha)$, where $c(\alpha), C(\alpha)$ may depend only on $\alpha$.

2. For every $\alpha > 2$: $n^{c'(\alpha)} \leq R_2(\alpha, n) \leq n^{C'(\alpha)}$, where $c'(\alpha), C'(\alpha)$ depend only on $\alpha$ and $0 < c'(\alpha) \leq C'(\alpha) < 1$. Moreover, $c'(\alpha)$ tends to 1 as $\alpha$ tends to $\infty$.

By Dvoretzky’s theorem, the lower bound in part 2 of Theorem 2 implies in particular that if $\alpha > 2$, and $X$ is any infinite dimensional normed space, then $R_X(\alpha, n) \geq n^{c'(\alpha)}$. Therefore, in our search for a possible phase transition for $R_p(\cdot, n)$, $p \neq 2$, it is natural to extend the upper bound in part 1 of Theorem 2 to this range. The main result proved in this note is the following:

**Theorem 3.** There is an absolute constant $c > 0$ such that for every $0 < \delta < 1$,

1. For $1 \leq p < 2$, $R_p(2 - \delta, n) \leq e^{\frac{c}{2}\delta^2} \log n$.

2. For $2 < p < \infty$, $R_p(2^{2/p} - \delta, n) \leq e^{\frac{c}{2p^{2/p}}\delta^2} \log n$.

Thus we extend the result of [11] to show that a phase transition occurs in the metric Ramsey problem for $\ell_p$, $p \in [1, 2)$, at $\alpha = 2$. The asymptotic behavior of $R_p(\alpha, n)$ for $p > 2$, and $\alpha \in [2^{2/p}, 2]$, is left as an open problem. In particular, we do not know whether or not this function undergoes a similar phase transition. We find this problem potentially significant: if there is a phase transition at 2 also in the range $2 < p < \infty$, then this result will certainly be of great interest. On the other hand, if it is possible to improve the lower bound in part 2 of Theorem 2 for $p > 2$ and certain distortions strictly less than 2, then this would involve an embedding technique that is different from the method used in [11], which doesn’t distinguish between the various $\ell_p$ spaces.

The proof of the upper bound on $R_2(\alpha, n)$ for $\alpha < 2$ stated in Theorem 2 uses the Johnson-Lindenstrauss dimension reduction lemma for $\ell_2$ [10]. For $\ell_p$, $p \neq 2$, no such dimension reduction is known to hold. (Recent work [5, 11] shows that dimension reduction does not, in general, hold in $\ell_1$.) Our proof is based on a non-trivial modification of the random construction in [11], in the spirit of Erdős’ upper bound on the Ramsey numbers [9, 3]. In the process we prove tight bounds on the embeddability of the metrics of complete bipartite graphs in $\ell_p$. Specifically we show that

$$c_p(K_{n,n}) = \begin{cases} 2 - \Theta(n^{-1}) & p \in [1, 2] \\ 2^{2/p} - \Theta((pn)^{-1}) & p > 2. \end{cases}$$

The second part of this note addresses the isometric Ramsey problem for $p \in (1, \infty)$. It turns out that this problem is naturally tackled within the class of uniformly convex normed spaces (see Section 3 for the definition).

**Theorem 4 (Isometric Ramsey Problem).** Let $X$ be a uniformly convex normed space with $\dim(X) \geq 2$. Then $R_X(1, n) = 3$ for $n \geq 3$. 


Since $\ell_p$ is uniformly convex for $p \in (1, \infty)$, the conclusion of Theorem 4 holds in these cases. Note that the theorem does not apply for $\ell_1$ and $\ell_\infty$ which are not uniformly convex. Specifically, it is known that $\ell_\infty$ is universal in that it contains an isometric copy of every finite metric space, whence $R_{\infty}(n) = n$. It is known [6] that any 4-point metric space is isometrically embeddable in $\ell_1$, and therefore $R_1(n) \geq 4$ for $n \geq 4$. The determination of $R_1(n)$ is left as an open problem.

2 An Upper Bound For $\alpha < 2$

In this section we prove that for any $\alpha < \min\{2, 2^{2/p}\}$, $R_p(\alpha, n) = O(\log n)$. Our technique both improves and simplifies the technique of [4], which itself is in the spirit of Erdős’ original upper bound for the Ramsey coloring numbers. The basic idea is to exploit a universality property of random graphs $G \in G(n, 1/2)$. Namely, that any fixed graph of constant size appears as an induced subgraph of every induced subgraph of $G$ of size $\Omega(\log n)$. More precisely, we define the following notion of universality.

**Definition 2.** Let $H$ be a graph. A graph $G$ is called $(H, s)$-universal if every set of $s$ vertices in $G$ contains an induced subgraph isomorphic to $H$.

**Proposition 1.** For every $k$-vertex graph $H$ there exists a constant $C > 0$ and an integer $n_0$ such that for any $n > n_0$ there exists a $(H, C \log n)$-universal graph on $n$ vertices. Furthermore, $C \leq O\left(k^2 2^{(k/2)}\right)$ and $n_0 \leq O\left(k^3 2^{(k/2)}\right)$.

Such facts are well-known in random graph theory, and similar arguments can be found for example in [13]. We sketch the standard details for the sake of completeness.

Recall that a family of sets $\mathcal{F}$ is called almost disjoint if $|A \cap B| \leq 1$ for every $A, B \in \mathcal{F}$. In what follows, given a set $S$ and an integer $k$, we denote by $\binom{S}{k}$ the set of all $k$-point subsets of $S$.

**Lemma 2.** For every integer $k$ and a finite set $S$ of cardinality $s = |S| > 2k^2$, there exists an almost disjoint family $K \subset \binom{S}{k}$, such that $|K| \geq \left\lfloor \frac{s}{2k^2} \right\rfloor^2$.

**Proof.** Let $p$ be a prime satisfying $\frac{s}{2k^2} \leq p \leq \frac{s}{k}$, and assume that $L = \{(i, j); i, j \in \mathbb{Z}_p, i \in \{0, \ldots, k-1\}\} \subseteq S$.

For each $a, b \in \mathbb{Z}_p$ (the field of residues modulo $p$), define

$$A_{a,b} = \{(i, j); j \equiv ai + b \pmod{p}, i \in \{0, \ldots, k-1\}\},$$

and take $K = \{A_{a,b}; a, b \in \mathbb{Z}_p\}$. The set $K$ is easily checked to satisfy the requirements. $\square$

As usual $G(n, 1/2)$ denotes the model of random graphs in which each edge on $n$ vertices is chosen independently with probability $1/2$.

**Lemma 3.** Let $H$ be a $k$-vertex graph and let $s > 2k^2$. The probability that a random graph $G \in G(s, 1/2)$ does not contain an induced subgraph isomorphic to $H$, is at most $\left(1 - 2^{-\binom{k}{2}}\right)^{\left\lfloor \frac{s}{2k^2} \right\rfloor^2}$.
Proof. Construct, as in Lemma 2, an almost disjoint family $\mathcal{F}$ of $\left\lfloor \frac{n}{2^k} \right\rfloor^2$ subsets of $\{1, \ldots, s\}$, the vertex set of $G$. If $V_1 \neq V_2 \in \mathcal{F}$, then the event that the restriction of $G$ to $V_1$ (resp. $V_2$) is isometric to $H$ are independent. Hence, the probability that none of the sets $F \in \mathcal{F}$ spans a subgraph isomorphic to $H$ is at most $(1 - 2^{-\left(\frac{k}{2}\right)})^{\left\lfloor \frac{n}{2^k} \right\rfloor^2}$. \hfill \Box

Proof of Proposition 3. Let $G$ be a random graph in $G(n, 1/2)$. By the previous lemma, the expected number of sets of $s$ vertices which contain no induced isomorphic copy of $H$ is at most $\binom{n}{s} \left(1 - 2^{-\left(\frac{k}{2}\right)}\right)^{\left\lfloor \frac{n}{2^k} \right\rfloor^2}$. If this number is $< 1$, then there is an $(H, s)$-universal graph, as claimed. It is an easy matter to check that this holds with the parameters as stated. \hfill \Box

A class $C$ of finite metric spaces is called a metric class if it is closed under isometries. $C$ is said to be hereditary, if $M \in C$ and $N \subseteq M$ imply $N \in C$. We call a metric space $(X, d)$ a $\{0, 1, 2\}$ metric space if for all $x, y \in X$, $d(x, y) \in \{0, 1, 2\}$. There is a simple 1:1 correspondence between graphs and $\{0, 1, 2\}$ metrics. Namely, associated with a $\{0, 1, 2\}$ metric space $M = (X, d)$ is the graph $G = (X, E)$ where $\{x, y\} \in E$ iff $d_M(x, y) = 1$.

Lemma 4. Let $C$ be a hereditary metric class of finite metric spaces, and suppose that there exists some finite $\{0, 1, 2\}$ metric space $M_0$ which is not in $C$. Then there exist metric spaces $M = M_n$ of arbitrarily large size $n$ such that every subspace $S \subseteq M_n$ with at least $C \log n$ points is not in $C$. The constant $C$ depends only on the cardinality of $M_0$.

Proof. Let $H_0$ be the graph corresponding to the metric space $M_0$. We apply Proposition 1 to construct arbitrarily large graphs $G_n = (V_n, E_n)$ with $|V_n| = n$, in which every set of $\geq C \log n$ vertices contains an induced subgraph isomorphic to $H_0$. Let $M_n$ be the $n$-point metric space corresponding to $G_n$. It follows that every subspace of $M_n$ of size $\geq C \log n$ contains a metric subspace that is isometric to $M_0$. Since $C$ is hereditary, $S \notin C$. \hfill \Box

Note that $\{M; M$ is a metric space, $\alpha_p(M) \leq \alpha\}$ is a hereditary metric class. Therefore, in order to show that for $\alpha < 2$, $R_p(\alpha, n) = O(\log n)$, it is enough to find a $\{0, 1, 2\}$ metric space whose $\ell_p$ distortion is greater than $\alpha$. We use the complete bipartite graphs $K_{n,n}$. The $\ell_p$-distortion of $K_{n,n}$, $1 \leq p < \infty$, is estimated in the following proposition.

Proposition 5. For every $1 \leq p \leq 2$,
\[ 2 \left( \frac{n - 1}{n} \right)^{1/p} \leq c_p(K_{n,n}) \leq 2 \sqrt{\frac{n - 1}{n}}. \]

For every $2 \leq p < \infty$,
\[ 2^{2/p} \left( \frac{n - 1}{n} \right)^{1/p} \leq c_p(K_{n,n}) \leq 2^{2/p} \left( 1 - \frac{1}{2n} \right)^{1/p}. \]

Before proving Proposition 5 we will deduce the main result of this section:

Theorem 5. There is an absolute constant $c > 0$ such that for every $0 < \delta < 1$, if $1 \leq p \leq 2$ then:
\[ R_p(2 - \delta, n) \leq e^{\frac{\delta}{3^2}} \log n, \]
and if $2 < p < \infty$ then:
\[ R_p(2^{2/p} - \delta, n) \leq e^{\frac{\delta}{p + 2}} \log n. \]
Proof. Proposition 4 implies that there is an absolute constant \( C \) such that for every \( n \geq 2^{Ck^3} \) there exists a \( \{0, 1, 2\} \) metric space \( M_n \) such that any subset \( S \subset M_n \) of cardinality at least \( 2^{Ck^2} \log n \) contains an isometric copy of \( K_{k,k} \).

We start with \( 1 \leq p \leq 2 \). Let \( k = \left\lfloor \frac{2}{\delta} \right\rfloor + 1 \). By Proposition 5

\[
c_p(K_{k,k}) \geq 2 \left( 1 - \frac{1}{k} \right)^{1/p} > 2 \left( 1 - \frac{\delta}{2} \right) = 2 - \delta,
\]

so that for \( n \) large enough \((\geq e^{\frac{\delta}{C}})\), and hence for all \( n \) (by proper choice of constants),

\[
R_p(2 - \delta, n) \leq e^{\frac{\delta}{C}} \log n.
\]

When \( p > 2 \) take \( k = 2 \left\lfloor \frac{1}{p^2} \right\rfloor \). In this case one easily verifies that:

\[
c_p(K_{k,k}) \geq 2^{2/p} \left( 1 - \frac{1}{k} \right)^{1/p} \geq 2^{2/p} - \delta,
\]

from which the required result follows as above.

\( \square \)

In order to prove Proposition 5 we need some preparation.

**Lemma 6.** Let \( A = (a_{ij}) \) be an \( n \times n \) matrix and \( 2 \leq p < \infty \). Then:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk} \right)^p \leq \frac{(2n)^p}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p.
\]

**Proof.** We identify \( \ell_p^n \) with the space of all \( n \times n \) matrices \( A = (a_{ij}) \), equipped with the \( \ell_p \) norm:

\[
\|A\|_p = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}.
\]

Define a linear operator \( T : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \oplus \mathbb{R}^{n^2} \) by:

\[
T(a_{ij}) = \left( \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk} \right)_{ij} \oplus \left( \sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj} \right)_{ij}.
\]

For \( q \geq 1 \) denote \( \|T\|_{q \to q} = \max_{A \neq 0} \|T(A)\|_q / \|A\|_q \). Our goal is to show that \( \|T\|_{p \to p} \leq 2^{1-1/p}n \). By a result from the complex interpolation theory for linear operators (see [2]), for \( 2 \leq p \leq \infty \), \( \|T\|_{p \to p} \leq \|T\|_{2/p}^{2/p} \cdot \|T\|_{\infty}^{1-2/p} \). It is therefore enough to prove the required estimate for \( p = 2 \) and \( p = \infty \). The case \( p = \infty \) is simple:

\[
\|T(A)\|_\infty = \max_{1 \leq i, j \leq n} \max \left\{ \left| \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk} \right|, \left| \sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj} \right| \right\} \leq 2n\|A\|_\infty.
\]

For \( p = 2 \) we have to show that:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \left| \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk} \right|^2 + \left| \sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj} \right|^2 \right) \leq 2n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2.
\]
This inequality follows from the following elementary identity:

\[
2n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \left( \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk} \right)^2 + \left( \sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj} \right)^2 \right] + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( na_{ij} - \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{kj} \right)^2.
\]

\[\square\]

**Corollary 7.** Let \(1 \leq p < \infty\) and \(x_1, \ldots, x_n, y_1 \ldots y_n \in \ell_p\). Then if \(2 \leq p < \infty\),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p \right) \leq 2^{p-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_i - y_j\|_p^p.
\]

If \(1 \leq p \leq 2\) then:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p \right) \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_i - y_j\|_p^p.
\]

**Proof.** By summation it is clearly enough to prove these inequalities for \(x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}\). If \(2 \leq p < \infty\) then the required result follows from an application of Lemma 5 to the matrix \(a_{ij} = x_i - y_j\). If \(1 \leq p \leq 2\) then consider \(\ell_p\) equipped with the metric \(d(x, y) = \|x - y\|_p^{p/2}\). It is well known (see [14]) that \((\ell_p, d)\) embeds isometrically in \(\ell_2\), so that the case \(1 \leq p \leq 2\) follows from the case \(p = 2\). \(\square\)

**Remark.** In [8] P. Enflo defined the notion on generalized roundness of a metric space. A metric space \((M, d)\) is said to have generalized roundness \(q \geq 0\) if for every \(x_1, \ldots, x_n, y_1, \ldots, y_n \in M\),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( d(x_i, x_j)^q + d(y_i, y_j)^q \right) \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_i, y_j)^q.
\]

Enflo proved that Hilbert space has generalized roundness 2 and in [12] the concept of generalized roundness was investigated and was shown to be equivalent to the notion of negative type (see [3] [14] for the definition). Particularly, it was proved in [12] that for \(1 \leq p < 2\), \(\ell_p\) has generalized roundness \(p\), which is precisely the second statement in Corollary 7. For the case \(p = 1\) simpler, more direct proofs can be given which do not use reduction to the case \(p = 2\), see e.g. [3]. Observe that Lemma 4 would follow simply by convexity had it not been for the additional factor \(1/2\) on the right-hand side. This factor is crucial for our purposes, and this is why the interpolation argument was needed.

**Proof of Proposition 5.** We identify \(K_{n,n}\) with the metric on \(\{u_1, \ldots, u_n, v_1, \ldots, v_n\}\) where \(d(u_i, u_j) = d(v_i, v_j) = 2\) for all \(i \neq j\), and \(d(u_i, v_j) = 1\) for every \(1 \leq i, j \leq n\). Fix some \(1 \leq p < \infty\) and let \(f : \{u_1, \ldots, u_n, v_1, \ldots, v_n\} \to \ell_p\) be an embedding such that for every \(x, y \in K_{n,n}, d(x, y) \leq \|f(x) - f(y)\|_p \leq Ld(x, y)\). Then,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \|f(u_i) - f(u_j)\|_p^p + \|f(v_i) - f(v_j)\|_p^p \right) \geq 2n(n-1)2^p,
\]
and
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \|f(u_i) - f(v_j)\|_p^p \leq n^2 L^p. \]

For \( 1 \leq p \leq 2 \) Corollary \ref{corollary} gives:
\[ 2n(n-1)^p 2^p \leq 2n^2 L^p \implies L \geq 2 \left( \frac{n-1}{n} \right)^{1/p}. \]

For \( 2 \leq p < \infty \) we get that:
\[ 2n(n-1) 2^p \leq 2^{p-1} n^2 L^p \implies L \geq 2^{2/p} \left( \frac{n-1}{n} \right)^{1/p}. \]

This proves the required lower bounds on \( c_p(K_{n,n}) \).

To prove the upper bound assume first that \( p = 2 \) and denote by \( \{e_i\}_{i=1}^{\infty} \) the standard unit vectors in \( \ell_2 \). Define \( f : K_{n,n} \to \ell_2^{2n} \) by:
\[
\begin{align*}
  f(u_i) &= \sqrt{2} \left( e_i - \frac{1}{n} \sum_{j=1}^{n} e_j \right), \\
  f(v_i) &= \sqrt{2} \left( e_{n+i} - \frac{1}{n} \sum_{j=1}^{n} e_{n+j} \right).
\end{align*}
\]

Then for \( i \neq j \), \( \|f(u_i) - f(u_j)\|_2 = \|f(v_i) - f(v_j)\|_2 = 2 = d(u_i, u_j) = d(v_i, v_j) \). On the other hand:
\[
\begin{align*}
  \|f(u_i) - f(v_j)\|_2 &= \sqrt{\|f(u_i)\|_2^2 + \|f(v_j)\|_2^2} \\
  &= \sqrt{4 \left(1 - \frac{1}{n} \right)^2 + 4(n-1) \cdot \frac{1}{n^2}} = 2 \sqrt{\frac{n-1}{n}}.
\end{align*}
\]

This finishes the calculation of \( c_2(K_{n,n}) \). For \( 1 \leq p < 2 \), since for every \( \epsilon > 0 \) and for every \( k \), \( \ell_p \) contains a \((1 + \epsilon)\) distorted copy of \( \ell_2^k \), we get the estimate \( c_p(K_{n,n}) \leq 2 \sqrt{\frac{n-1}{n}} \).

The case \( 2 < p < \infty \) requires a different embedding. We begin by describing an embedding with distortion \( 2^{2/p} \) and then explain how to modify it so as to reduce the distortion by a factor of \( \left(1 - \frac{1}{2m}\right)^{1/p} \). Let \( z_1, \ldots, z_n \) be a collection of \( n \) mutually orthogonal \pm 1 vectors of dimension \( m = O(n) \). (For example the first \( n \) rows in an \( m \times m \) Hadamard matrix). In our first embedding we define \( f(u_i) \) as the \((2m)\)-dimensional vector \((z_i, 0)\), namely, \( z_i \) concatenated with \( m \) zeros. Likewise, \( f(v_i) = (0, z_i) \) for all \( i \). Now \( \|f(u_i) - f(u_j)\|_p = 2 \left( \frac{m}{2} \right)^{1/p} \) and \( \|f(u_i) - f(v_j)\|_p = (2m)^{1/p} \), and so \( f \) has distortion \( 2^{2/p} \). To get the \( \left(1 - \frac{1}{2m}\right)^{1/p} \) improvement, note that for some \( m \leq 4n \) it is possible to select the \( z_i \) so that the \( m \)-th coordinate in all of them is \(+1\). Modify the previous construction to an embedding into \( 2m - 1 \) dimensions as follows: Now \( g(u_i) \) is \( z_i \) concatenated with \( m - 1 \) zeros, whereas \( g(v_i) \) has zeros in the first \( m - 1 \) coordinates, 1 in the \( m \)-th and this is followed by the first \( m - 1 \) coordinates of the vector \( z_i \). The easy details are omitted. \( \square \)
Remark: The upper bounds in Proposition 5 were not used in the proof of Theorem 5. Apart from their intrinsic interest, these upper estimates show that the above technique cannot prove an upper bound of $O(\log n)$ on $R_2(2-\epsilon, n)$ which is independent of $\epsilon$. In fact, this can never be achieved using $\{0, 1, 2\}$ metric spaces due to the following proposition.

Proposition 8. Let $X$ be an $n$-point $\{0, 1, 2\}$ metric space. Then $c_2(X) \leq 2\sqrt{\frac{n-1}{n}}$.

Proof. We think of $X$ as a metric on $\{1, \ldots, n\}$ and denote $d(i, j) = d_{ij}$. Define an $n \times n$ matrix $A = (a_{ij})$ as follows:

\[
a_{ij} = \begin{cases} 
2 & \text{if } i = j \\
0 & \text{if } d_{ij} = 2 \\
\frac{2}{n} & \text{if } d_{ij} = 1
\end{cases}
\]

We claim that $A$ is positive semidefinite. Indeed, for any $z \in \mathbb{R}^n$

\[
\langle Az, z \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i z_j \\
\geq \sum_{i=1}^{n} 2z_i^2 - \sum_{i \neq j} \frac{2}{n} |z_i| \cdot |z_j| \\
\geq \sum_{i=1}^{n} 2z_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{2}{n} |z_i| \cdot |z_j| \\
= 2\|z\|^2_2 - \frac{2}{n}\|z\|^2_1 - \frac{2}{n}\|z\|^2_2 = 0.
\]

In particular it follows that $A$ has a square root, denoted $A^{1/2}$. Let $e_1, \ldots, e_n$ be the standard unit vectors in $\mathbb{R}^n$. Define $f : X \to \mathbb{R}^n$ by $f(i) = A^{1/2}e_i$. Now,

\[
\|f(i) - f(j)\|^2_2 = \langle Ae_i, e_i \rangle + \langle Ae_j, e_j \rangle - 2\langle Ae_i, e_j \rangle = a_{ii} + a_{jj} - 2a_{ij},
\]

so that if $d_{ij} = 1$ then $\|f(i) - f(j)\|^2_2 = 4 - \frac{4}{n}$ and if $d_{ij} = 2$ then $\|f(i) - f(j)\|^2_2 = 2$. It follows that

\[
\text{dist}(f) = 2\sqrt{\frac{n-1}{n}}.
\]

3 The Isometric Ramsey Problem

In this section we prove that for $n \geq 3$, $1 < p < \infty$, $R_p(n) = R_p(1, n) = 3$. In fact, we show that this is true for any uniformly convex normed space. We begin by sketching an argument that is specific to $\ell_2^2$.

Proposition 9. $R_2(n) = 3$ for $n \geq 3$.

Proof. That $R_2(n) \geq 3$ follows since any metric space on 3 points embeds isometrically in $\ell_2^2$. To show that $R_2(n) \leq 3$, we construct a metric space on $n > 3$ points, no 4-point subspace of which embeds isometrically in $\ell_2^2$. Fix an integer $n > 3$ and let $\{a_i\}_{i=0}^{n}$ be an increasing sequence such that $a_0 = 0$, $a_1 = 1$ and for $1 \leq i < n$, $a_{i+1} \geq 2(n+1)a_i$. Fix some
0 < \epsilon < 1/(2a_n). It is easily verified that \(d(i, j) = |i - j| - \epsilon a_{|i - j|}\) is a metric on \(\{1, 2, \ldots, n\}\).

We show that for \(\epsilon\) small enough no four points in \((\{1, \ldots, n\}, d)\) embed isometrically in \(\ell_2\).

Fix four integers \(1 \leq i_1 < i_2 < i_3 < i_4 \leq n\) and set \(j = i_2 - i_1, k = i_3 - i_2, l = i_4 - i_3\). Suppose that for every \(\epsilon > 0\) there exists an isometric embedding \(f : (\{i_1, i_2, i_3, i_4\}, d) \to \ell_2^4\). Without loss of generality we may assume that \(f(i_1) = (\alpha, \beta, \gamma), f(i_2) = (0, 0, 0), f(i_3) = (k - \epsilon a_k, 0, 0)\) and \(f(i_4) = (p, q, 0)\). Then:

\[
2\alpha(k - \epsilon a_k) = 2(f(i_1), f(i_3)) = \|f(i_1) - f(i_2)\|^2_2 + \|f(i_3) - f(i_2)\|^2_2 - \|f(i_3) - f(i_1)\|^2_2 = (j - \epsilon a_j)^2 + (k - \epsilon a_k)^2 - (j + k - \epsilon a_{k+l})^2.
\]

Hence,

\[
\alpha \leq -j + \frac{\epsilon}{k}(k + j)a_{k+j} - ja_j - ka_k + j\alpha + O(\epsilon^2).
\]

Similarly:

\[
p \geq (k + l) + \frac{\epsilon}{k}[(k + l)a_k - (k + l)a_{k+l} - ka_k + \epsilon a_k] + O(\epsilon^2).
\]

Now:

\[
j + k + l - \epsilon a_{j+k+l} = \|f(i_4) - f(i_1)\|_2 \\
\geq p - \alpha \\
\geq j + k + l + \frac{\epsilon}{k}[(k + l)a_k - (k + l)a_{k+l} + la_l - (k + l)a_{k+j} + ja_j + ja_k] + O(\epsilon^2).
\]

Letting \(\epsilon\) tend to zero we deduce that:

\[
a_{j+k+l} \leq \left(1 + \frac{j}{k}\right)a_{k+j} + \left(1 + \frac{l}{k}\right)a_{k+l} - \frac{l}{k}a_l - \frac{j}{k}a_j - \frac{j + k + l}{k}a_k < 2(n + 1)a_{j+k+l-1},
\]

which is a contradiction. \(\square\)

The argument above is quite specific to \(\ell_2\), and so we now consider any uniformly convex normed space. The modulus of uniform convexity of a normed space \(X\) is defined by:

\[
\delta_X(\epsilon) = \inf \left\{1 - \frac{\|a + b\|^2}{2} ; \|a\|, \|b\| \leq 1 \text{ and } \|a - b\| \geq \epsilon \right\}.
\]

\(X\) is said to be uniformly convex if \(\delta_X(\epsilon) > 0\) for every \(0 < \epsilon \leq 2\). The \(L_p\) spaces \(1 < p < \infty\), are known to be uniformly convex. For a uniformly convex space \(X\), \(\delta_X\) is known to be continuous and strictly increasing on \((0, 2]\).

Assume that \(X\) is a uniformly convex normed space and \(a, b \in X \setminus \{0\}\). Then:

\[
\left\| \frac{a}{\|a\|} + \frac{b}{\|b\|} \right\| = \left\| \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)(a + b) - \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \\
\geq \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)\|a + b\| - \frac{\|a\|}{\|b\|} - \frac{\|b\|}{\|a\|} \\
= 2 - \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)(\|a\| + \|b\| - \|a + b\|).
\]

Now,

\[
\delta_X \left(\left\| \frac{a}{\|a\|} + \frac{b}{\|b\|} \right\| \right) \leq 1 - \frac{1}{2} \cdot \frac{a}{\|a\|} + \frac{b}{\|b\|} \leq \frac{1}{2} \cdot \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)(\|a\| + \|b\| - \|a + b\|).
\]
Hence
\[ \| \frac{a}{\|a\|} - \frac{b}{\|b\|} \| \leq \delta_X^{-1} \left( \frac{1}{2} \left( \frac{1}{\|a\|} + \frac{1}{\|b\|} \right) (\|a\| + \|b\| - \|a + b\|) \right). \]

Take \( x, y, z \in X \) and apply this inequality for \( a = x - y, b = y - z \). It follows that:
\[ \| y - \left( \frac{\|y - z\|}{\|x - y\| + \|y - z\|} \cdot x + \frac{\|x - y\|}{\|x - y\| + \|y - z\|} \cdot z \right) \|
\leq \frac{\|x - y\| \cdot \|y - z\|}{\|x - y\| + \|y - z\|} \cdot \delta_X^{-1} \left( \frac{\|x - y\| + \|y - z\| - \|x - z\|}{\min\{\|x - y\|, \|y - z\|\}} \right). \tag{1} \]

This inequality is the way uniform convexity is going to be applied in the sequel. Indeed, we have the following “metric” consequence of it:

**Lemma 10.** Let \( X \) be a uniformly convex normed space and \( x_1, x_2, x_3, x_4 \in X \) be distinct. Then:
\[ \frac{\|x_1 - x_2\| + \|x_2 - x_3\| - \|x_1 - x_3\|}{2\|x_2 - x_3\|} \leq \delta_X^{-1} \left( \frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}} \right) + \delta_X^{-1} \left( \frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}} \right). \]

**Proof of Lemma 10.** Define:
\[ \lambda = \frac{\|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \quad \text{and} \quad \mu = \frac{\|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|}. \]

An application of (1) twice gives:
\[ \|x_3 - (\lambda x_1 + (1 - \lambda)x_4)\| \leq \|x_1 - x_3\| \cdot \|x_3 - x_4\| \cdot \delta_X^{-1} \left( \frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}} \right), \]
and
\[ \|x_3 - (\mu x_2 + (1 - \mu)x_4)\| \leq \|x_2 - x_3\| \cdot \|x_3 - x_4\| \cdot \delta_X^{-1} \left( \frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}} \right). \]
By symmetry, we may assume without loss of generality that $\lambda \leq \mu$. Now,

$$\left\| x_2 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\| = \frac{1}{\mu} \left\| \mu x_2 + (1 - \mu) x_4 - \frac{1 - \mu}{1 - \lambda} (x_3 - \lambda x_1 - (1 - \lambda) x_4) \right\|$$

$$\leq \frac{1}{\mu} \left\| x_3 - \mu x_2 - (1 - \mu) x_4 + \frac{1 - \mu}{\mu(1 - \lambda)} \cdot \left\| x_3 - \lambda x_1 - (1 - \lambda) x_4 \right\| \right\|$$

$$\leq \frac{\| x_2 - x_3 \| + \| x_3 - x_4 \|}{\| x_2 - x_3 \| + \| x_3 - x_4 \|} \cdot \| x_2 - x_3 \| \cdot \left\| x_3 - x_4 \right\|\cdot \frac{\delta^{-1}_X}{} \left( \frac{\| x_2 - x_3 \| + \| x_3 - x_4 \| - \| x_2 - x_4 \|}{\min\{\| x_2 - x_3 \|, \| x_3 - x_4 \|} \right) +$$

$$\frac{\| x_2 - x_3 \| \cdot \| x_2 - x_1 \| + \| x_3 - x_4 \| \cdot \| x_1 - x_3 \|}{\| x_2 - x_3 \| + \| x_3 - x_4 \|} \cdot \left\| x_2 - x_4 \right\| \cdot \frac{\delta^{-1}_X}{} \left( \frac{\| x_2 - x_3 \| + \| x_3 - x_4 \| - \| x_1 - x_4 \|}{\min\{\| x_2 - x_3 \|, \| x_3 - x_4 \|} \right) +$$

$$+ \| x_2 - x_3 \| \cdot \| x_2 - x_4 \| \cdot \frac{\delta^{-1}_X}{} \left( \frac{\| x_2 - x_3 \| + \| x_3 - x_4 \| - \| x_2 - x_4 \|}{\min\{\| x_2 - x_3 \|, \| x_3 - x_4 \|} \right).$$

Additionally,

$$\| x_2 - x_1 \| \leq \left\| x_2 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\| + \left\| x_1 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\|$$

$$= \left\| x_2 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\| + \left\| \frac{\mu - \lambda}{\mu(1 - \lambda)} \| x_1 - x_3 \|, \right\|$$

and

$$\| x_2 - x_3 \| \leq \left\| x_2 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\| + \left\| x_3 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\|$$

$$= \left\| x_2 - \frac{\lambda(1 - \mu)}{\mu(1 - \lambda)} x_1 - \frac{\mu - \lambda}{\mu(1 - \lambda)} x_3 \right\| + \left\| \frac{\mu - \lambda}{\mu(1 - \lambda)} \| x_1 - x_3 \|, \right\|.$$

Summing up these estimates gives the required result. \(\square\)

We can now prove the main result of this section:

**Theorem 6.** Let $X$ be a uniformly convex normed space with $\dim(X) \geq 2$. Then for every $n \geq 3$, $R_X(n) = 3$. Moreover, for every $\delta : (0, 2) \to (0, \infty)$ which is continuous, increasing and $\delta \leq \delta_2$, let $UC_\delta$ be the class of all normed spaces $X$ with $\delta_X \geq \delta$. Then for each $n \geq 3$ there is a constant $\epsilon_n(\delta) > 0$ such that $R_{UC_\delta}(1 + \epsilon_n(\delta), n) = 3$.

The proof of Theorem 6 proceeds by constructing a space in which each quadruple violates the conclusion of Lemma 10. The construction is done iteratively, by adding one point at a time.

**Proof of Theorem 6** That $R_X(n) \geq 3$ follows since any 3 point metric embeds isometrically into any 2 dimensional normed space, by a standard continuity argument.

Fix some $\delta : (0, 2) \to (0, \infty)$ which is continuous, increasing and $\delta \leq \delta_2$. We shall construct inductively a sequence $\{M_n\}_{n=3}^\infty$ of metric spaces and numbers $\{\eta_n\}_{n=3}^\infty$ such that:
a) For every $n \geq 3$, $\eta_n > 0$. Each $M_n$ is a metric on $\{1, \ldots, n\}$, and we denote $d_{ij}^n = d_{M_n}(i,j)$.

b) For every $1 \leq i < j < k \leq n$,

$$d_{ij}^n + d_{jk}^n - d_{ik}^n - \eta_n \geq 2d_{jk}^n \left[ \delta^{-1} \left( \frac{d_{ik}^n + d_{jk}^n - d_{ij}^n}{\min\{d_{ik}^n, d_{jk}^n\}} \right) + \delta^{-1} \left( \frac{d_{jk}^n + d_{kn}^n - d_{jn}^n}{\min\{d_{jk}^n, d_{kn}^n\}} \right) \right].$$

Lemma 10 immediately implies that there is a constant $\epsilon_n(\delta) > 0$ such that for every $1 \leq i < j < k < l \leq n$ and for every normed space $X$ with $\delta_X \geq \delta$:

$$c_X(\{i, j, k, l\}, d_{M_n}) \geq 1 + \epsilon_n(\delta),$$

as required.

$M_3$ is the equilateral metric on $\{1, 2, 3\}$, in which case $\eta_3 = 1$. We construct $M_{n+1} = (\{1, \ldots, n + 1\}, d^{n+1})$ as an extension of $M_n$, by setting

$$d_{i,n+1}^{n+1} = 1 - s/2 \quad \text{and} \quad \forall 1 \leq i < n, \ d_{i,n+1}^{n+1} = d_i^n + 1 - s.$$

This is indeed a definition of a metric as long as $0 < s \leq \min\{1, 2 \min_{1 \leq i < n} d_{i,n}^n\}$ (this fact follows from a simple case analysis).

We are left to check condition b). Fix $1 \leq i < j < k \leq n$. If $k \neq n$ then:

$$d_{ij}^{n+1} + d_{jk}^{n+1} - d_{ik}^{n+1} - \eta_n = d_{ij}^n + d_{jk}^n - d_{ik}^n - \eta_n$$

$$\geq 2d_{jk}^n \left[ \delta^{-1} \left( \frac{d_{ik}^n + d_{jk}^n - d_{ij}^n}{\min\{d_{ik}^n, d_{jk}^n\}} \right) + \delta^{-1} \left( \frac{d_{jk}^n + d_{kn}^n - d_{jn}^n}{\min\{d_{jk}^n, d_{kn}^n\}} \right) \right].$$

It remains to check b) for the quadruple $\{i, j, n, n+1\}$. Condition b) for $M_n$ implies that:

$$d_{ij}^{n+1} + d_{jn}^{n+1} - d_{in}^{n+1} \geq \eta_n.$$

On the other hand,

$$2d_{jn}^{n+1} \left[ \delta^{-1} \left( \frac{d_{in}^{n+1} + d_{jn,n+1}^{n+1} - d_{ij}^{n+1}}{\min\{d_{in}^{n+1}, d_{jn,n+1}^{n+1}\}} \right) + \delta^{-1} \left( \frac{d_{jn}^{n+1} + d_{jn,n+1}^{n+1} - d_{jn}^{n+1}}{\min\{d_{jn}^{n+1}, d_{jn,n+1}^{n+1}\}} \right) \right] =$$

$$= 2d_{jn}^n \left[ \delta^{-1} \left( \frac{s/2}{\min\{d_{jn}^n, 1 - s/2\}} \right) + \delta^{-1} \left( \frac{s/2}{\min\{d_{jn}^n, 1 - s/2\}} \right) \right],$$

so that condition b) will hold when $s$ is small enough such that the quantity above is at most $\eta_n/2$ and with $\eta_{n+1} = \eta_n/2$.

\[ \square \]

**Corollary 11.** For all $1 < p < \infty$, $R_p(n) = 3$ for $n \geq 3$. 

12
We end this section with a simple lower bound for the isometric Ramsey problem for graphs. We do not know the asymptotically tight bound in this setting.

**Proposition 12.** Let $G$ be an unweighted graph of order $n$. Then there is a set of $\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$ vertices in $G$ whose metric embeds isometrically into $\ell_2$.

**Proof.** Let $\Delta$ be the diameter of $G$. The shortest path between two diametrically far vertices is isometrically embeddable in $\ell_2$. On the other hand, the Bourgain, Figiel, Milman theorem \cite{4} yields that for every $0 < \epsilon < 1$ a subset $N \subset V$ which is $(1 + \epsilon)$ embeddable in Hilbert space and $|N| = \Omega\left(\frac{\epsilon}{\log(2/\epsilon)}\log n\right)$. When $\epsilon = \frac{1}{2\Delta}$, such an embedding is an isometry. Hence we can always extract a subset of $V$ which is isometrically embeddable in $\ell_2$ with cardinality

$$\Omega\left(\max\left\{\Delta, \frac{\log n}{\Delta \log \Delta}\right\}\right) = \Omega\left(\frac{\log n}{\log \log n}\right),$$

as claimed. \qed

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