Noncommutative Description of Spin Hall Effect

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Abstract

We propose an approach based on a generalized quantum mechanics to deal with the basic features of the intrinsic spin Hall effect. This can be done by considering two decoupled harmonic oscillators on the noncommutative plane and evaluating the spin Hall conductivity. Focusing on the high frequency regime, we obtain a diagonalized Hamiltonian. After getting the corresponding spectrum, we show that there is a Hall conductivity without an external magnetic field, which is noncommutativity parameter $\theta$-dependent. This allows us to make contact with the spin Hall effect and also give different interpretations. Fixing $\theta$, one can recover three different approaches dealing with the phenomenon.

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1 Introduction

The spin Hall effect (SHE) is a physical phenomenon, which has been discovered in 1971 by D'yakonov and Perel [1]. It is a consequence of the spin-orbit coupling where an applied electric field to a sample can lead to a spin transport in perpendicular direction and spin accumulation at the lateral edges [2,3]. It is characterized by a spin Hall conductivity resulting from the spin polarization on the boundaries of the sample. There are two types of SHE: intrinsic and extrinsic, each one is depending to what kind of spin-orbit coupling contribution to the considered Hamiltonian describing the system [4].

The intrinsic SHE has been theoretically predicted for semiconductors with spin-orbit interactions. Indeed, Sinova et al. [5] described a new effect in n-type semiconductor spintronics that leads to dissipationless spin-currents in paramagnetic spin-orbit coupled systems. They argued that in a high mobility two-dimensional electron system with substantial Rashba spin-orbit coupling, a spin-current that flows perpendicular to the charge current is intrinsic. In the usual case where both spin-orbit split bands are occupied, the spin-Hall conductivity has a universal value. Other related works can be found in references [6,7,8].

The theoretical prediction of the intrinsic SHE has been also argued by another group [9]. This has been done by adopting a mathematical formalism governed by the Luttinger Hamiltonian for p-type semiconductors in two-dimensions. In fact, Murakami et al. [9] showed that the electric field can generate a dissipationless quantum spin current at room temperature, in hole doped semiconductors such as Si, Ge and GaAs. Taking advantage of a generalization of the quantum Hall effect [13] to higher dimensional manifolds, they showed that the intrinsic SHE leads to efficient spin injection without the need for metallic ferromagnets. Another derivation has been established by using the Berry phase approach, which can be found in [10].

Very recently, a quantum version of the intrinsic SHE has been reported by Bernevig and Zhang [11]. In fact, by considering a Hamiltonian brought form solid state physics, they showed that the spin Hall conductivity is quantized in units of $\frac{e^2}{\pi}\theta$ and built the corresponding wavefunctions. These have strong overlapping with those have been construct by Halperin many years ago [12] or their equivalents in terms of the matrix model theory [14]. These latter have been formulated to describe the quantum Hall effect generated from charged particles by treating theirs spins as additional degrees of freedom.

Based on the above works and in particular [11], we describe our main idea. More precisely, we quantum mechanically develop another approach to analyze the intrinsic SHE. This can be done by resorting the spectrum of two noncommutative harmonic oscillators and evaluating the spin Hall conductivity. Solving the Hamiltonian system at high frequency regime, we derive the corresponding eigenvalues as well as eigenstates. Using these to get the Hall conductivity of charge without an external magnetic field and therefore the spin Hall conductivity, which are noncommutativity parameter $\theta$-dependents. Since $\theta$ is a free parameter, one can differently interpret our results. Indeed, for some particular values of $\theta$ we discuss how to get the quantum SHE by constructing the Laughlin wavefunction analogue. Furthermore, we establish a link between our approach and those proposed
by Bernevig and Zhang, Sinova et al. and Murakami et al.

The present paper is organized as follows. In section 2, we consider two decoupled harmonic oscillators on the noncommutative plane \( \mathbb{R}^2_\theta \) and getting its spectrum at high frequency regime. This allows us to make contact with the Landau problem on the ordinary plane \( \mathbb{R}^2 \) and therefore Laughlin wavefunctions at the filling factor \( \nu = \frac{1}{m} \) [15], with \( m \) is an odd integer. In section 3, to determine the spin Hall conductivity, we introduce the electric field through a confining potential resulting from our consideration. For this, we distinguish two cases: spin up and down, which are relatively found to be equivalents up to a minus sign. In section 4, we offer different interpretations of our results by showing how some theories on the subject can be recovered from our analysis. We conclude and give some perspectives in the last section.

2 Two oncommutative harmonic oscillators

We consider two decoupled harmonic oscillators on \( \mathbb{R}^2_\theta \) and determine the corresponding eigenvalues as well as eigenfunctions. This can be done by introducing the star product and the ordinary commutation relations in quantum mechanics. Restricting to the high frequency regime, we obtain a diagonalized Hamiltonian as well as its spectrum. We give different comparisons with respect to the Landau problem on \( \mathbb{R}^2 \) in order to show its overlapping with our approach.

2.1 Hamiltonian of system

Our proposal can be elaborated by considering two decoupled harmonic oscillators of the same masses \( m \) and frequencies \( \omega \) on \( \mathbb{R}^2 \). They are described by the Hamiltonian

\[
H_{\text{plane}} = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m\omega^2}{2} (x^2 + y^2)
\]

which can be interpreted as a Hamiltonian for one-particle system on \( \mathbb{R}^2 \) in absence of any interacting term. It can be diagonalized by introducing the creation and annihilation operators

\[
a_i = \frac{1}{\sqrt{2\hbar m\omega}} p_i - i \sqrt{\frac{m\omega}{2\hbar}} , \quad a_i^\dagger = \frac{1}{\sqrt{2\hbar m\omega}} p_i + i \sqrt{\frac{m\omega}{2\hbar}} , \quad i = x, y
\]

where the only non-vanishing commutator is

\[
[a_i, a_i^\dagger] = \mathbb{I}.
\]

These are implying that \( H_{\text{plane}} \) can be arranged as

\[
H_{\text{plane}} = \frac{\hbar \omega}{2} \left( a_x^\dagger a_x + a_y^\dagger a_y + 1 \right)
\]

where the corresponding eigenstates are

\[
|n_x, n_y\rangle = \frac{(a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y}}{\sqrt{n_x! n_y!}} |0, 0\rangle
\]
as well as the eigenvalues
\[ E_{n_x,n_y} = \frac{\hbar \omega}{2} (n_x + n_y + 1), \quad n_i = 0, 1, 2, \ldots \] (6)
where \(|0,0\rangle\) is the fundamental state. Next, we will see how these results can be generalized to \(\mathbb{R}^2\) and used to deal with our issues.

In doing our generalization, we adopt a method similar to that used in [16]. Indeed, the canonical quantization of the system described by (1) is achieved by introducing the coordinate \(r_j\) and momentum \(p_k\) operators satisfying the relation
\[ [r_j, p_k] = i\hbar \delta_{jk}. \] (7)
But to deal with our proposal, we consider a generalized quantum mechanics governed by (7) and the noncommutative coordinates, such as
\[ [x, y] = i\theta \] (8)
where \(\theta\) is a real free parameter and has length square of dimension. Without loss of generality, hereafter we assume that \(\theta > 0\). Noncommutativity can be imposed by treating the coordinates as commuting but requiring that composition of their functions is given in terms of the star product
\[ \star \equiv \exp \frac{i\theta}{2} \left( \frac{\partial_x \partial_y - \partial_y \partial_x}{\theta} \right). \] (9)
Now, we deal with the commutative coordinates \(x\) and \(y\) but replace the ordinary products with the star product (9). For example, instead of the commutator (8) one defines
\[ x \star y - y \star x = i\theta. \] (10)
At this level, let us derive the corresponding form of the Hamiltonian (1) in terms of the noncommutative coordinates (8). First, we quantize the present system by establishing the commutation relation (7). Second, we take into account the noncommutativity of the coordinates by defining a new operator as
\[ H \star \psi(\vec{r}) \equiv H_{nc} \psi(\vec{r}) \] (11)
where \(\psi(\vec{r})\) is an arbitrary eigenfunction of \(H\). By doing this processing, we obtain the noncommutative version of the Hamiltonian (1). This is
\[ H_{nc} = \left[ \frac{1}{2m} + \frac{m \omega^2}{2} \left( \frac{\theta}{2\hbar} \right)^2 \right] \left( p_x^2 + p_y^2 \right) + \frac{m \omega^2}{2} (x^2 + y^2) + \frac{m \omega^2 \theta}{2\hbar} (yp_x - xp_y). \] (12)
We emphasis that due to the noncommutativity between spacial coordinates we ended up with two coupled harmonic oscillators. This coupling is described in terms of the angular momenta \(L_z(\theta)\)
\[ L_z(\theta) = \frac{m \omega^2 \theta}{2\hbar} (yp_x - xp_y). \] (13)
It is obvious that (13) disappears once we set \(\theta = 0\) and then recover (1). \(L_z(\theta)\) is analogue to that corresponding to the Landau problem on \(\mathbb{R}^2\), see next.
2.2 High frequency regime

The Hamiltonian \( H^{\text{mc}} \) can not be diagonalized directly, we need to introduce some relevant approximation in order to get its spectrum. For this, we restrict ourselves to the high frequency \((\text{hf})\) regime, which is characterized by the limit

\[
\left[ \frac{1}{2m} + \frac{m\omega^2}{2} \left( \frac{\theta}{2\hbar} \right)^2 \right] \approx \frac{m\omega^2}{2} \left( \frac{\theta}{2\hbar} \right)^2. \tag{14}
\]

This is not surprising, since an analogue approximation has been employed by Berniveg and Zhang \([11]\) in analyzing the quantum version of the intrinsic SHE on \(\mathbb{R}^2\). We will be back to clarify this point in section 4. In the limit (14), the Hamiltonian (12) reduces to

\[
H_{\text{hf}}(\theta) = \frac{m\omega^2}{2} \left[ \left( \frac{\theta}{2\hbar} \right)^2 \left( p_x^2 + p_y^2 \right) + \frac{\theta}{\hbar} (yp_x - xp_y) \right]. \tag{15}
\]

Let us give a comment about our Hamiltonian. It is interesting to note that (15) has a strong overlapping with the Landau problem on \(\mathbb{R}^2\). To see this, we start by recalling that in the symmetric gauge

\[
A = \frac{B}{2} (y, -x) \tag{16}
\]

the Landau Hamiltonian for a one-charged particle of mass \(m\) in two-dimensions and submitted to an uniform magnetic field \(B\) is given by

\[
H_{\text{landau}} = \frac{1}{2m} \left[ (p_x^2 + p_y^2) + \left( \frac{eB}{2c} \right)^2 \left( x^2 + y^2 \right) + \frac{eB}{c} (yp_x - xp_y) \right]. \tag{17}
\]

Clearly \(H_{\text{landau}}\) is sharing some common features with \(H_{\text{hf}}\). This can be shown by requiring that the conditions

\[
\theta_{\text{landau}} = \frac{2\hbar c}{eB} = 2l_B^2, \quad 2\omega = \omega_c \tag{18}
\]

is fulfilled where \(l_B\) is the magnetic length. Therefore, one may interpret \(\theta\) as an external parameter \(B\) applied to the system, which remains among the important values of \(\theta\) derived right now. Consequently, since (17) is the cornerstone of the quantum Hall effect \([13]\), then \(H_{\text{hf}}(\theta)\) will allows us to make contact with this effect.

In the forthcoming analysis, it is convenient to consider the complex plane \((z, \bar{z})\) where \(z = x + iy\) and \(p_z = \frac{i}{2} (p_x - ip_y)\). In this case, (10) can be written as

\[
H_{\text{hf}}(\theta) = \frac{m\omega^2}{2} \left[ 4 \left( \frac{\theta}{2\hbar} \right)^2 \left( p_z p_{\bar{z}} + z\bar{z} + \frac{\theta}{2\hbar} (zp_{\bar{z}} - \bar{z}p_z) \right) \right]. \tag{19}
\]

As usual the diagonalization of (19) can be realized by introducing the creation and annihilation operators. They are given by

\[
a = \sqrt{\theta} \frac{p_z}{\hbar} - \frac{i}{2\sqrt{\theta}} z, \quad a^\dagger = \sqrt{\theta} \frac{p_z}{\hbar} + \frac{i}{2\sqrt{\theta}} \bar{z}. \tag{20}
\]
It easy to show that
\[ [a, a^\dagger] = I \] (21)
and other commutators are nulls. In terms of \( a \) and \( a^\dagger \), \( H_{hf}(\theta) \) can be mapped as
\[ H_{hf}(\theta) = \frac{m\omega^2\theta}{4\hbar} \left( 2a^\dagger a + 1 \right). \] (22)
This is nothing but one-dimensional harmonic oscillator with frequency
\[ \omega(\theta) = \frac{m\omega^2\theta}{\hbar^2}. \] (23)
The corresponding spectrum can be obtained by solving the eigenvalue equation
\[ H_{hf}(\theta)\phi = E_n(\theta)\phi \] (24)
to get the eigenfunctions
\[ \phi_n(z, \bar{z}, \theta) = z^n \exp\left( -\frac{z\bar{z}}{2\theta} \right) \] (25)
upon a factor of normalization. The associated energy levels are given by
\[ E_n(\theta) = \frac{m\omega^2\theta}{4\hbar} (2n + 1), \quad n = 0, 1, 2, \cdots. \] (26)

The previous analysis can be generalized to a system of \( N \)-identical particles governed by the total Hamiltonian
\[ H^{\text{tot}}_{hf}(\theta) = \frac{m\omega^2}{2} \sum_{i=1}^{N} \left[ \frac{\theta}{2\hbar} p_{z_i}p_{\bar{z}_i} + z_i\bar{z}_i + \frac{\theta}{2\hbar} (z_i p_{z_i} - \bar{z}_i p_{\bar{z}_i}) \right]. \] (27)
where the total energy is \( N \)-copies of (26) and the eigenvalues is basically the tensorial product of \( N \) those given in (25). If the system is living on the lowest level, which of course means that all \( n_i = 0 \) with \( i = 1, \cdots, N \) and each \( n_i \) corresponds to the spectrum (25–26), the total wavefunction can be written in terms of the Vandermonde determinant. This is
\[ \Phi^1_{\text{hol}}(z, \bar{z}, \theta) = \prod_{i<j} (z_i - z_j) \exp\left( -\frac{1}{2\theta} \sum_i |z_i|^2 \right) \] (28)
which is an holomorphic wavefunction. It is obvious that by using the constraint (18), one can recover the Laughlin wavefunction at the filling factor \( \nu = 1 \) [15] describing charged particles in the presence of an uniform magnetic field. Therefore, (28) can be interpreted as the Laughlin wavefunction analogue and other similar ones can be constructed as
\[ \Phi^n_{\text{hol}}(z, \bar{z}, \theta) = \prod_{i<j} (z_i - z_j)^m \exp\left( -\frac{1}{2\theta} \sum_i |z_i|^2 \right) \] (29)
with \( \nu = \frac{1}{m} \) and \( m \) has odd integer values.

Later, we will see how the above results can be employed to deal with the basic features of the intrinsic SHE. In fact, we show that (22) could lead to the spin Hall conductivity comparable with those derived by other groups.
3 Spin Hall conductivity

Before evaluating the spin Hall conductivity, let us emphasize an important point. Through the present analysis, we are considering electrons of spin $\frac{1}{2}$. Thus, we need to distinguish two possible configurations: spin up and down cases. Consequently, to reflect the spin–orbit coupling contribution, we should have two Hamiltonians differing between each other by a sign of the angular momenta term $[13]$. To reproduce this effect, we simply identify spin up to the noncommutativity parameter $+\theta$ and spin down to $-\theta$. Subsequently, we analyze each case by establishing all ingredients to show that our system is really exhibiting an intrinsic SHE.

3.1 Electric field components

There is an important ingredient that should be fixed before talking about the intrinsic SHE. This is the electric field, which is responsible of having such phenomenon. More precisely, an electric current passes through a system with spin-orbit coupling, induces a spin polarization near the lateral edges. This leads to a spin accumulation and therefore a spin Hall conductivity. For this, we will show how to fix the external parameter in our approach.

To reproduce the required field in terms of our language, we can simply use the standard definition, which is showing that

$$\vec{F} = -e\vec{E} = -\nabla V$$

(30)

where the scalar potential $V$ can be derived from (70). It follows that $V$ should be nothing but a confining potential, such as

$$V = \frac{m\omega^2}{2} (x^2 + y^2).$$

(31)

Combining all to get the electric field components

$$E_x = \frac{m\omega^2}{e}x, \quad E_y = \frac{m\omega^2}{e}y.$$  

(32)

As we will see later, analogue relations to (32) have been introduced by Berniveg and Zhang in analyzing the quantum SHE. Consequently, (32) will play a crucial role in dealing with the subject. To clarify this point, let us treat separately spin up and down cases.

3.2 Spin up case

As we claimed before, the spin up case can be identified to $+\theta$. Therefore, this case is describing by the Hamiltonian (70) as well as its corresponding analysis reported before. To determine the spin Hall conductivity for spin up, we start by evaluating the velocity components to get first the Hall conductivity of charge and second return to deal with our issues.

Let us begin by determining the velocity component along $x$-direction. It can be obtained by using the Heisenberg equation

$$v_x(+\theta) = \frac{i}{\hbar} [H_{hf}(+\theta), x].$$

(33)
To derive the Hall current of charge, we need to calculate the expectation value of $v_x(+\theta)$. This can be done with respect to the eigenstates $\phi_n(z, \bar{z}, +\theta)$ to get

$$\langle v_x(+\theta) \rangle = \frac{m\omega^2\theta}{2\hbar} y.$$  \hfill (34)

The relation between velocity and current implies

$$\langle j_x(+\theta) \rangle = \frac{\rho em\omega^2\theta}{2h} y,$$  \hfill (35)

where $\rho = \frac{N}{S}$ is the particle density and $S$ is the system surface. Now, we have all ingredients to derive the Hall conductivity of charge. Indeed, using the second relation in (32), we obtain

$$\sigma_{xy}(+\theta) = \frac{\rho e^2}{2\hbar} \theta.$$  \hfill (36)

It is interesting to note that unlike the Landau problem, we have a transversal conductivity without an external magnetic field $B$. This shows that our system can be seen as a Hall system and then can be used to establish another approach dealing with the basic features of the quantum Hall effect. To recover, the Landau problem study we simply identify $\theta$ to $B$ through the relation (18).

Using the same analysis as before, we show that the Hall current along $y$-direction is given by

$$\langle j_y(+\theta) \rangle = -\frac{\rho em\omega^2\theta}{2h} x,$$  \hfill (37)

and therefore the Hall conductivity $\sigma_{yx}(\theta)$ is

$$\sigma_{yx}(+\theta) = \frac{\rho e^2}{2\hbar} \theta.$$  \hfill (38)

It is clear that

$$\sigma_{xy}(+\theta) = -\sigma_{yx}(+\theta)$$  \hfill (39)

as it is well-known in the quantum Hall effect world. For this, we only focus on $\sigma_{xy}(+\theta)$ in the forthcoming analysis.

Up to now we have derived the Hall conductivity of charge, which basically came from a deformation of the space $\mathbb{R}^2$. It is natural to ask about the spin Hall conductivity $\sigma^s_{xy}(+\theta)$. Indeed, since an electron with charge $e$ carries a spin $\frac{h}{2}\hbar$, a factor of $\frac{h}{2\epsilon}$ is used to convert the charge conductivity into the spin conductivity [11]. Applying this statement to our case, we should have

$$\sigma^s_{xy}(+\theta) = 2\sigma_{xy}(+\theta) \frac{h}{2\epsilon}.$$  \hfill (40)

Finally, the spin Hall conductivity is

$$\sigma^s_{xy}(+\theta) = \frac{\rho e}{2} \theta$$  \hfill (41)

which is noncommutativity parameter $\theta$-dependent and represents the main result derived so far in the present paper. Note that, once $\theta$ is swished off $\sigma^s_{xy}(+\theta)$ goes to zero. Later, we will see how it can be used to offer different interpretations.
3.3 Spin down case

To accomplish our analysis, we consider the second part of electron that is the spin down case. This is corresponding to change $+\theta$ by $-\theta$ in the above study. Otherwise, it is equivalent to consider the commutator

$$[x, y] = -i\theta$$

(42)

instead of its analogue given by (8). Using the same analysis as before, we end up with a Hamiltonian describing two harmonic oscillators on $\mathbb{R}^2_\theta$ generated by (12). In fact, at high frequency regime, it is given by

$$H_{hf}(-\theta) = \frac{m\omega^2}{2} \left[ \left( \frac{\theta}{2\hbar} \right)^2 \left( \frac{p_x^2 + p_y^2}{\hbar^2} \right) + x^2 + y^2 - \frac{\theta}{2\hbar} (yp_x - xp_y) \right]$$

(43)

which is analogue to that describing one-electron of spin down that has been considered by Berniveg and Zhang [11], see later.

At this stage, let us exhibit the effective spin–orbit coupling in our formalism. In doing so, we can use (14) and (43) together to define a total Hamiltonian as

$$H_{hf}^{\text{tot}} = \begin{pmatrix} H_{hf}(+\theta) & 0 \\ 0 & H_{hf}(-\theta) \end{pmatrix}.$$  

(44)

Equivalently, one can write

$$H_{hf}^{\text{tot}} = \frac{m\omega^2}{2} \left[ \left( \frac{\theta}{2\hbar} \right)^2 \left( \frac{p_x^2 + p_y^2}{\hbar^2} \right) + x^2 + y^2 + \frac{\theta}{2\hbar} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (yp_x - xp_y) \right]$$

(45)

where the third component of spin is

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(46)

Therefore, the last term in (45) is resulting from an effective interaction where $\theta$ is playing the role of a coupling parameter. It is obvious that The derived interaction disappears if $\theta$ is switched off.

In similar way to spin up, $H_{hf}(-\theta)$ can be diagonalized by setting the creation and annihilation operators. They are

$$b = \frac{\sqrt{\theta}}{\hbar} p_z + \frac{i}{2\sqrt{\theta}} \bar{z}, \quad b^\dagger = \frac{\sqrt{\theta}}{\hbar} p_z - \frac{i}{2\sqrt{\theta}} \bar{z}$$

(47)

which satisfy the relation

$$[b, b^\dagger] = \mathbb{I}.$$  

(48)

With these, (43) reads as

$$H_{hf}(-\theta) = \frac{m\omega^2\theta}{4\hbar} \left( 2b^\dagger b + 1 \right).$$

(49)

This Hamiltonian has the same form as that for spin up, but the main difference is that the corresponding eigenfunctions are antiholomorphic, such as

$$\phi_k(z, \bar{z}, -\theta) = (\bar{z})^k \exp \left( -\frac{z\bar{z}}{2\theta} \right)$$

(50)
up on a factor of normalization and the energy levels are given by
\[ E_k(-\theta) = \frac{m\omega^2\theta}{4\hbar}(2k + 1), \quad k = 0, 1, 2 \cdots . \] (51)

The above results for one-electron of spin down can be generalized to a system of \( N \)-identical electrons of spin down. In particular, for \( N \)-particles in the lowest level, namely \( k_i = 0 \) with \( i = 1, \cdots, N \) and each \( k_i \) corresponds to the spectrum \( 50, 51 \), the total wavefunction is
\[ \Phi^1_{\text{anti}}(z, \bar{z}, -\theta) = \prod_{i<j} (\bar{z}_i - \bar{z}_j) \exp \left( -\frac{1}{2\theta} \sum_i |z_i|^2 \right) \] (52)
which is antiholomorphic and analogue to the first Laughlin wavefunction at \( \nu = 1 \). Other analogue wavefunctions can be written as
\[ \Phi^m_{\text{anti}}(z, \bar{z}, -\theta) = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m \exp \left( -\frac{1}{2\theta} \sum_i |z_i|^2 \right) \] (53)
These as well as their holomorphic partners \( 29 \) will be used to built the whole wavefunctions describing two sectors where each one contains \( N \)-electrons of spin up or down.

As before the spin Hall conductivity \( \sigma^s_{xy}(-\theta) \) corresponding to \( H_{hf}(-\theta) \) can be calculated by using the Heisenberg equation
\[ v_x(-\theta) = i \frac{\hbar}{\theta} [H_{hf}(-\theta), x] . \] (54)
This shows that the Hall conductivity for charge is
\[ \sigma_{xy}(-\theta) = -\frac{\rho e^2}{2\hbar} \theta . \] (55)
Using the same statement as for spin up to obtain the \( \sigma^s_{xy}(-\theta) \) resulting from \( N \)-electrons of spin down in the presence of an electric field \( E_x \) \( 32 \). This is
\[ \sigma^s_{xy}(-\theta) = -\frac{\rho e}{2} \theta . \] (56)
Similarly along \( y \)-direction, we have
\[ \sigma^s_{yx}(-\theta) = \frac{\rho e}{2} \theta . \] (57)
Combining all, we can arrange all conductivities for the \( x \)-direction as
\[ \sigma^s_{xy} = \begin{cases} \frac{\rho e}{2} \theta & \text{for } 5 \\ 0 & \text{for } \theta = 0 \\ -\frac{\rho e}{2} \theta & \text{for } 12 \end{cases} \] (58)
where \( 5 \) and \( 12 \) are two different deformations of plane introduced to reflect an effective spin–orbit coupling contribution to the Hamiltonian \( 1 \). Similar equation, up to a minus sign, can be derived for \( \sigma^s_{yx} \) along \( y \)-direction. By comparing \( 11 \) and \( 57 \), it follows that the constraint
\[ \sigma^s_{xy}(+\theta) = -\sigma^s_{xy}(-\theta) \] (59)
is satisfied and showing that the total spin Hall conductivity is equal to zero. This is in accordance with what has been reported in [11]. Similar result has been derived by considering a system of electrons and holes together for a special value of the noncommutativity parameter $\theta$, more detail can be found in [23]. Note that, thanks to (59), we only use $\sigma_{xy}^s(\pm\theta)$ in the next.

We close this section by noting that from the obtained results so far, it seems that the spin down analysis is corresponding to that for $y$-direction in spin up case and vice versa.

4 Discussions

Now let us turn to interpret our results. The obtained spin Hall conductivity $\sigma_{xy}^s(\pm\theta)$ is actually involving a free parameter $\theta$. This can be switched on to offer different interpretations of the system under consideration. In fact, we will show how to derive the quantum SHE and recover three theories related to the subject.

4.1 Quantized $\sigma_{xy}^s(\pm\theta)$

In the beginning, one can notice that electrons of spin $\frac{1}{2}$ living on $\mathbb{R}^2_\theta$ behave as a SHE system characterized by $\sigma_{xy}^s(\pm\theta)$. This will be employed to establish a link with two different theories. The quantum version of the intrinsic SHE can be obtained by imposing some conditions on $\theta$ and allows us to make contact with the Bernivel–Zhang approach [11].

To talk about the quantum SHE, we require that $\theta$ should be fixed in such way that $\sigma_{xy}^s(\pm\theta)$ takes quantized values in terms of the fundamental constant $\frac{e}{2\pi}$. Moreover, to get the first quantized value of $\sigma_{xy}^s(\pm\theta)$, we may fix $\theta$ according to

$$\sigma_{xy}^s(\pm\theta)|_{\theta=\theta_{bs}} = \frac{e}{2\pi}$$

It implies that $\theta$ can be linked to the particle density as

$$\theta = \frac{1}{\pi \rho}$$

This relation is not surprising because it has been derived in another formalism. Indeed, using the noncommutative Chern-Simons theory, Susskind [22] showed that to reproduce the basic features of the Laughlin theory, for the fractional quantum Hall effect at $\nu = \frac{1}{m}$ [15] resulting from charged particles, one should have $2\theta = \frac{1}{2\pi}$. Moreover, if we rewrite (61) as

$$\pi \theta = \frac{S}{N}$$

one may interpret the quantity $\pi \theta$ as an elementary surface occupied by a quantum spin Hall droplet. This statement is evident if we adopt the mapping [18] where the area of the quantum Hall droplet is $2\pi l_B^2$. Note that, the same analysis as before can be reported for $\sigma_{xy}^s(-\theta)$.

The system under consideration is involving $N$-electrons of spin $\frac{1}{2}$. Since we have spin up and down, one should have two sectors, or let say two kind of particles, each one indexed by spin up
our down. If these sectors are interacting between each other, the right wavefunctions should be
constructed in terms of the Laughlin wavefunction analogue given before and taking into account of
the inter-correlation term
\[ \prod_{i<j} (z_i - w_j)^n. \] (63)
Consequently, the required wavefunctions can be written as
\[ \Phi_m^{\text{tot}}(z, \bar{z}, \theta) = \prod_{i<j} (z_i - z_j)^m \prod_{i<j} (\bar{w}_i - \bar{w}_j)^m \prod_{i<j} (z_i - \bar{w}_j)^n \exp \left[ -\frac{1}{2\theta} \left( \sum_i |z_i|^2 + |w_i|^2 \right) \right]. \] (64)
They are sharing many features with those built by Halperin [12] or their equivalents in matrix model
theory [14]. The main difference is that two different Laughlin state analogue (29) and (53) are
resulting from the opposite sign of the noncommutativity parameter \( \theta \). These will be linked to those
proposed for the subject, see [11].

4.2 Bernevig–Zhang approach

We are wondering to prove that the present analysis is general and can be used to reproduce other ap-
proaches, in particular that developed by Bernevig and Zhang [11]. In fact, they quantum mechanically
established a quantum theory for the intrinsic SHE.

Let us start by recalling that they adopted a formalism governed by the Hamiltonian
\[ H_{\uparrow,\downarrow} = \sqrt{\frac{D}{2m}} \left( p_x^2 + p_y^2 + x^2 + y^2 \pm R(xp_y - yp_x) \right) \] (65)
at a special point \( R = 2 \) where
\[ R = \frac{1}{2} \frac{C_3}{\hbar} \sqrt{\frac{2m}{D}} g, \quad D = \frac{2mg^2C_3^2}{16\hbar}. \] (66)
\( C_3 \) is a material constant, e.g. for GaAs, \( \frac{C_3}{\hbar} = 8 \times 10^5 \text{ m/s} \) [17] and \( g \) is the magnitude of the strain
gradient. This Hamiltonian is not new and was previously studied in different contexts, one may
see [18, 19, 20, 21]. It can be factorized as
\[ H_{\uparrow} = \frac{1}{2\hbar} C_3 g \left( 2a^\dagger a + 1 \right), \quad H_{\downarrow} = \frac{1}{2\hbar} C_3 g \left( 2b^\dagger b + 1 \right). \] (67)
These have been used to discuss the quantum SHE. In doing so, Berniveg and Zhang introduced the
following configuration for the electric field components
\[ E_x = gx, \quad E_y = gy \] (68)
which are analogue to what we have derived in [32]. From their consideration, they showed that the
spin Hall conductivity is quantized in units of \( 2e^2/\hbar \). Also they built the corresponding wavefunctions
in terms of the Laughlin states, which sharing some common features with Halperin ones and similar
to those given by [64].
To reproduce the above formalism from our approach, we first arrange our Hamiltonians in similar way as in (65). This can be done by introducing the rescaling variables

\[ x \rightarrow \left( \frac{\theta}{2\hbar} \right) x, \quad y \rightarrow \left( \frac{\theta}{2\hbar} \right) y \]

(69)

to get a simplified Hamiltonian

\[ H_{hf}(\pm \theta) = \frac{m\omega^2 \theta}{4\hbar} \left[ (p_x^2 + p_y^2) + x^2 + y^2 \pm (yp_x - xp_y) \right]. \]

(70)

This form is similar to that used by Bernevig and Zhang [11] where the major difference is that in our approach, the term \( \frac{m\omega^2 \theta}{4\hbar} \) is not constant as they have. This suggests that our Hamiltonians are good candidates to deal with the quantum version of the intrinsic SHE and moreover is general in sense that one can recover other theories.

To make a link between our analysis and that reviewed above we simply make a comparison between our Hamiltonians (70) and what is given by (65). It is clear that, they have some common features. For Bernevig and Zhang, all parameters involved in the game are constant or depending to the material types. For us the noncommutativity parameter \( \theta \) is free and can be fixed according to different interpretations. Indeed, by identifying (65) and (68) to our analogue equations (32) and (70), one should choose \( \theta \) as \( \theta_{bz} \) to build a bridge between two approaches. This is

\[ \theta_{bz} = \frac{C_3 g}{m\omega^2}, \quad g = \frac{m\omega^2}{e}. \]

(71)

They lead to the constraint

\[ \theta_{bz} = \frac{C_3}{e}. \]

(72)

It implies that \( \theta \) can be interpreted as the material constant if we set \( \theta = \theta_{bz} \) and thus can be used to characterize what kind of material is considered to analyze the quantum SHE. Furthermore, using the rescaling (69), one can recover the wavefunctions built by Bernevig and Zhang. The obtained derivation is proving that our approach is relevant for the subject.

### 4.3 Other theories

The established link above is interesting in sense that one can reproduce the Bernevig–Zhang analysis from our proposal. This was our motivation and therefore behind the development of our approach. But we are not going to stop at this level, in fact we show that how our obtained results are more generals and one may look for other links.

To recover other theories from our proposal, we need to introduce other mathematical tools. This can be done by evaluating the particle number \( N \) to derive another convenient form for the spin Hall conductivity \( \sigma_{xy}(+\theta) \). Indeed, by definition \( N \) is given by

\[ N = \int_0^{P_f} g' d\tau(p) \]

(73)
where $P_F = \hbar K_F$ is the Fermi momenta and the quantity $g'd\tau(p)$ reads as

$$g'd\tau(p) = \frac{g'S}{(2\pi\hbar)^2} d\tilde{p} = \frac{g'S}{2\pi\hbar^2} p dp. \quad (74)$$

g' = 2s + 1 is the degeneracy, with $s$ is the spin of particle. Now it is easily seen that (73) gives

$$N = \frac{g'S}{2\pi\hbar^2} \int_{0}^{P_F} p dp = \frac{g'S}{4\pi\hbar^2} P_F^2. \quad (75)$$

This implies that the particle density is

$$\rho = \frac{g'}{4\pi} K_F^2. \quad (76)$$

Combining all and considering electrons of spin $\frac{1}{2}$ to write the Hall conductivity for charge $\sigma_{xy}(+\theta)$ along $x$-direction as

$$\sigma_{xy}(+\theta) = \frac{e^2 K_F^2}{2\hbar} \theta. \quad (77)$$

Using the same argument as before, we obtain a spin Hall conductivity as

$$\sigma_{xy}^s(+\theta) = \frac{e K_F^2}{4\pi} \theta. \quad (78)$$

This form is suggestive for our purpose and therefore will be used to clarify our statement. Note that according to (59), we have an equivalent relation to (78) along $y$-direction, which is

$$\sigma_{yx}(+\theta) = -\frac{e K_F^2}{4\hbar} \theta. \quad (79)$$

Due to (59), similar form can be found for $\sigma_{xy}^s(-\theta)$ as well as that for the $y$-direction.

As we claimed before, there are two tentatives have been theoretically elaborated to predict the intrinsic SHE. Among them, Sinova et al. [5] who employed a mathematical formalism based on the analysis of the Rashba Hamiltonian

$$H_{\text{rashba}} = \frac{1}{2m} \left(\hbar K\right)^2 + \lambda(\vec{\sigma} \times \hbar K)_z \quad (80)$$

where $\lambda$ is the Rashba coupling constant and $\vec{\sigma}$ is the Pauli matrix. They showed that the spin Hall conductivity has an universal value, such as

$$\sigma_{xy}^{\text{rashba}} = \frac{e}{8\pi}. \quad (81)$$

Subsequently, by considering the Rashba–Dresselhaus spin–orbit coupling, it shown that (81) can be generalized to [6, 7]

$$\sigma_{xy}^{\text{rashba–dress}} = \pm \frac{e}{8\pi}. \quad (82)$$

We mention that a related work has been reported by Hu [8] in analyzing the topological orbital angular momentum Hall current. This issue will be considered separately in a forthcoming paper.

To reproduce the Sinova et al. analysis, we solve the equation

$$\sigma_{xy}^s(+\theta) |_{\theta=\theta_{\text{rashba}}} = \sigma_{xy}^s \quad (83)$$
to get a fixed noncommutativity parameter
\[ \theta_{\text{rashba}} = \frac{1}{2K_F^2}. \]  
(84)

Therefore, one can envisage electrons of spin \( \frac{1}{2} \) living on \( \mathbb{R}_d^2 \) as the Rashba system described by (80) if we set \( \theta = \theta_{\text{rashba}} \).

On the other hand, Murakami et al. proposed an interesting approach to predict the intrinsic SHE [9], see also [10]. Their analysis was based on the investigation of the basic features of the Luttinger Hamiltonian given by
\[ H_{\text{luttinger}} = \hbar^2 \left[ \left( \frac{\gamma_1 + 5}{2} \gamma_2 \right) K^2 - 2 \gamma_2 \left( \vec{K} \cdot \vec{S} \right)^2 \right] \]  
(85)
where \( \gamma_1 \) and \( \gamma_2 \) are the valence-band parameters for semiconductor materials. This form of \( H_{\text{luttinger}} \) allowed them to describe the phenomena by showing that the spin Hall conductivity for heavy and light holes can be written as
\[ \sigma_{xy}^{\text{luttinger}} = \frac{e}{6\pi^2} \left( 3K^H_F - K^L_F \right). \]  
(86)

Clearly, to reproduce the Murakami et al. results, we first swish our system to that of holes. Simply this can be done by changing in our analysis \( e \) by \( -e \) to get \( -\sigma_{xy}^{\text{rashba}}(+\theta) \). Therefore, requiring that the identification is satisfied
\[ -\sigma_{xy}^{\text{rashba}}(+\theta)|_{\theta=\theta_{\text{luttinger}}} = \sigma_{xy}^{\text{luttinger}} \]  
(87)
we end up with the condition on \( \theta \)
\[ \theta_{\text{luttinger}} = \frac{2}{3\pi K_F^2} \left( K^L_F - 3K^H_F \right). \]  
(88)

Our analysis offered for us two possibilities to talk about the intrinsic SHE. Semi-classically, fixing \( \theta \) we have made a connection to what have been reported by Sinova et al. and Murakami et al. on the subject. Quantum mechanically, we have reproduced the Bernevig–Zhang analysis where the quantized spin Hall conductivity and the corresponding wavefunctions have been identified.

5 Conclusion

To discuss the intrinsic spin Hall effect, we have employed two noncommutative harmonic oscillators and investigated their basic features. Indeed, restricting to the high frequency regime, we have derived a factorized Hamiltonians analogue to those have been used by Bernevig and Zhang [11] in dealing with the quantum spin Hall effect. Moreover, we have shown its common features with the Landau problem on the ordinary plane. Getting the spectrum for spin up and down cases, we have determined the spin Hall conductivities \( \sigma_{xy}^{\pm}(\pm\theta) \) in a general form due to the noncommutativity parameter \( \theta \)-dependency. Moreover, they have been obtained without need of an external magnetic field and showing that the total spin Hall conductivity is null.
Subsequently, by fixing $\theta$ differently, we have given some discussions. Indeed, to get a quantum version of the intrinsic spin Hall effect, we have required that the obtained $\sigma_{xy}^*(\pm \theta)$ should be quantized in terms of the fundamental constant $\frac{e}{2\pi}$. The corresponding wavefunctions have been constructed in similar way as those built by Halperin. This interpretation offered for us a possibility to make contact with the Berniving–Zhang approach [11]. In fact by choosing a particular value of $\theta$, we have noticed that $\theta$ can be used to determine what kind of material is considered to analyze the subject and therefore reproduced the Berniving–Zhang analysis.

Evaluating the particle number in terms of Fermi momenta, we have derived another form of the conductivities $\sigma_{xy}^*(+\theta)$ in terms of the Fermi wave vector. This allowed us to establish other links with different approaches. Indeed, giving to $\theta$ two different values, we have shown that the Sinova et al. and Murakami et al. analysis can be recovered from our proposal.

Still some interesting questions to be addressed. Can we use the noncommutative Chern-Simons theory [22, 24] to describe the basic features of the quantum spin Hall effect? A related question arose, in fact what about a matrix model description of the phenomena? These issues and related matters are under consideration.

**Acknowledgments**

The authors are thankful to R. El Moznine for fruitful discussions on the high frequency regime. AJ work’s was partially supported by Arab Regional Fellows Program (ARFP) 2006/2007.

**References**

[1] M.I. D’yakonov and V.I. Perel, JETP Lett. 13 (1971) 467.

[2] Y. Kato, R.C. Myers, A.C. Gossard and D.D. Awschalom, Science 306 (2004) 1910.

[3] J. Wunderlich, B. Kaestner, J. Sinova and T. Jungwirth, Phys. Rev. Lett. 94 (2005) 047204, cond-mat/0410295

[4] H-A. Engel, E.I. Rashba and B.I. Halperin, Theory of Spin Hall Effect in Semiconductors, cond-mat/0603306

[5] J. Sinova, D. Culcer, Q. Niu, N.A. Sinitsyn, T. Jungwirth and A.H. MacDonald, Phys. Rev. Lett. 92 (2004) 126603, cond-mat/0307663

[6] N.A. Sinitsyn, E.M. Hankiewicz, W. Teizer and J. Sinova, Phys. Rev. B70 (2004) 245211, cond-mat/0310315

[7] S-Q. Shen, Phys. Rev. B70 (2004) 081311(R), cond-mat/0310368

[8] J-P. Hu, Topological Orbital Angular Momentum Hall Current, cond-mat/0503149
[9] S. Murakami, N. Nagaosa and S-C. Zhang, *Science* **301** (2003) 1348, cond-mat/0308167.

[10] S. Murakami, *Adv. in Solid State Phys.* **45** (2005) 197, cond-mat/0504353.

[11] B.A. Bernevig and S-C. Zhang, *Phys. Rev. Lett.* **96** (2006) 106802, cond-mat/0504147.

[12] B.I. Halperin, *Phys. Rev. Lett.* **52** (1984) 1583.

[13] R.E. Prange and S.M. Girvin (Editors), “The Quantum Hall Effect”, (Springer-Verlag 1990).

[14] A. Jellal and M. Schreiber, *J. Phys.* **A37**: Math. Gen. (2004) 3147, hep-th/0304207.

[15] R.B. Laughlin, *Phys. Rev. Lett.* **50** (1983) 1395.

[16] Ö.F. Dayi and A. Jellal, *J. Math. Phys.* **43** (2002) 4592, hep-th/0111267.

[17] M.I. D’yakonov, V.A. Marushchak, V.I. Perel and A.N. Titkov, *Sov. Phys. JETP* **63**, 655 (1986).

[18] W. Howlett and S. Zukotynski, *Phys. Rev.* **B16** (1977) 3688.

[19] J.E. Pikus and A. Titkov, “Optical Orientation”, (North Holland, Amsterdam, 1984, p. 73).

[20] T.B. Bahder, *Phys. Rev.* **B41** (1990) 11992.

[21] A. Khaetskii and Y. Nazarov, *Phys. Rev.* **B64** (2001) 125416.

[22] L. Susskind, *The Quantum Hall Fluid and Non-Commutative Chern Simons Theory*, hep-th/0101029.

[23] A. Jellal, *Electrons-Holes on Noncommutative Plane and Hall Effect*, hep-th/0207269.

[24] A.P. Polychronakos, *JHEP* **0104** (2001) 011, hep-th/0103013.