COHERENT STATES, SUPERPOSITIONS OF COHERENT STATES AND UNCERTAINTY RELATIONS ON A MÔBIUS STRIP

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Since symmetry properties of coherent states (CS) on Möbius strip (MS) and fermions are closely related, CS on MS are naturally associated to the topological properties of fermionic fields. Here we consider CS and superpositions of coherent states (SCS) on MS. We extend a recent propose of CS on MS (Cirilo-Lombardo, 2012 [25]), including the analysis of periodic behaviors of CS and SCS on MS and the uncertainty relations associated to angular momentum and the phase angle. The advantage of CS and SCS on MS with respect to the standard ones and potential applications in continuous variable quantum computation are also addressed.

Keywords: coherent states; fermion fields; Möbius strip.

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The uncertainty relation between amplitude and phase is the minimum allowed by the Heisenberg’s uncertainty principle in the case of coherent states (CS) in quantum optics [1], for this reason superpositions of coherent states (SCS) are the best analogues of what are called cat states [2]. Such states are experimentally achievable, for instance, in electrodynamic cavities [3,4] and systems of trapped ions [5]. The main experimental difficult for the creation and observation of such superpositions is related to the fast decay of the coherences [6]. Advances in continuous variable quantum computation (CVQC) [7,8] have lead to the possibility of manipulation of such states as qubit states and realization of gate operations [9]. Although SCS can appear in different contexts [10,11,12,13,14,15], in nontrivial topologies it is not clear if SCS are always associated to cat states [16]. Cat states associated to topological defects have been proposed recently [17].

On the other hand, SCS states and the relation to cat states have not been investigated in Möbius strip (MS) topology. Proposes involving physics in MS as...
Fig. 1. (Color online) Möbius strip and a particular trajectory of a point particle on a Möbius strip.

applications in persistent currents\cite{18}, topological insulators\cite{22,23} and graphene\cite{24} have increased the interest in the study of states and dynamics in MS. The study of CS on MS has been proposed recently in\cite{25} extending the torus and circle cases\cite{20,19}. A natural step further in this direction is consider SCS states and the uncertainty relations in MS in order to realize cat states on MS, whose applications in CVQC can motivate CS and topological quantum computation in physical systems with MS topologies.

In this paper, we consider CS and SCS on MS, the relation between SCS and cat states on MS and the uncertainty relations for CS and SCS on MS. We start with a CS on MS, parametrized from the torus, by means of an angular constraint introduced in\cite{25}, then we consider the uncertainty relations proposed in\cite{28} extended to MS case. Finally, we realize minimal uncertainties for CS and SCS on MS, that can be associated to cat states on MS.

1. CS and SCS on Möbius strip

Considering a phase space embedded into a torus surface, with an angular constraint between polar $\theta$ and azimuthal $\varphi$ angles given by $\theta = \frac{\varphi}{2} + \pi$, we have an effectively reduced degree of freedom from the torus to the MS (figure 1), with a trajectory described by the parametrization

$$
X = \cos \varphi + r \cos (\varphi/2) \cos \varphi \\
Y = \sin \varphi + r \cos (\varphi/2) \sin \varphi \\
Z = l + r \sin (\varphi/2),
$$

and the variation of $r < R = 1$ leading to the MS. The CS for a quantum particle on the MS is then performed following the eigenvalue equation\cite{25} $e^{i(\hat{\varphi} + iJ)} |\xi\rangle = \xi |\xi\rangle$, where $\xi = e^{-(l+1+r \sin(\varphi/2)) + i \varphi (1 + r \cos (\varphi/2))}$. In terms of $\varphi$ and $\varphi'$ in the MS parametrization, the CS can be described by means of $|l', \varphi\rangle$, that satisfies $J|l', \varphi\rangle = j|l', \varphi\rangle$. In this form, the CS on the MS can be written as

$$
|\xi\rangle = |l', \varphi\rangle = \sum_{j=-\infty}^{\infty} e^{(l'-i\varphi)j} e^{-j^2/2} |j\rangle,
$$

(2)
where \( l' = l + r \sin \left( \frac{\varphi}{2} \right) - \ln \left( 1 + r \cos \frac{\varphi}{2} \right) \). The normalization is given in terms of Jacobi theta functions 

\[
\langle \xi | \tilde{\xi} \rangle = \sum_{j \in \mathbb{Z}} e^{(l' \varphi + k^2)/2} e^{i(l' \varphi + \varphi)} e^{i(l' \varphi + \varphi) k} \langle k | j \rangle = \sum_{j} e^{-j} e^{2i l'} j = \theta_3(l'|i\pi),
\]

and the the projections relations for two different CS on a MS can be written as

\[
\langle \xi | \tilde{\xi} \rangle = \sum_{j = -\infty}^{\infty} e^{(l'/\ell + \varphi)} j e^{-i(\varphi - \varphi_e) j} e^{-j^2}.
\]

Then, on a period in the MS, we have the behaviour characteristic of the topology associated to the MS. Taking \( |\xi\rangle = |\xi^{\varphi = 0}\rangle \) and \( |\tilde{\xi}\rangle = |\xi^{\varphi = 4\pi}\rangle \), the projections are given by

\[
\langle \xi^{\varphi = 0} | \xi^{\varphi = 4\pi} \rangle = \sqrt{\pi} \exp \left( l + \frac{1}{2} \ln(1 + r) \right)^2 \theta_3 \left( -\frac{(2l' + \ln(1 + r))\varphi}{2} \right) |e^{-\pi j} \rangle,
\]

that can be simplified, using (3), in the following form

\[
\langle \xi^{\varphi = 0} | \xi^{\varphi = 4\pi} \rangle = e^{4i\pi} \sum_{j = -\infty}^{\infty} e^{(l'/\ell + \varphi)} j e^{-j^2}.
\]

Since the angular difference is a unity, i.e., \( \exp 4\pi i = 1 \), we have that the state with a difference in the phase \( \varphi \) corresponding to the period \( 4\pi \) of a MS will be the same \( |\xi^{\varphi = 0}\rangle \), as expected from the MS topology. Consequently, under a period of \( 4\pi \) the state turns to be the same state.

We can verify that this is in fact a property of SCS too. Considering a SCS given in terms of CS of opposite angles \( \varphi \) and \( -\varphi \), including an additional phase term phase \( e^{-i\varphi} \), we have

\[
|\phi_C \rangle = |l', \varphi \rangle + e^{-i\varphi} |l', -\varphi \rangle.
\]

As the CS, the SCS (6) is periodic in the MS, taking \( |\phi_C^{\varphi = 0}\rangle \) and \( |\phi_C^{\varphi = 4\pi}\rangle \), a period of \( 4\pi \), we can write \( |\phi_C^{\varphi = 0}\rangle = |\phi_C^{\varphi = 4\pi}\rangle \).

Now, if we relate the phase term \( e^{-i\varphi} \) directly to the MS angular variable \( \varphi \), by means of \( \phi = \varphi \), the states will still be periodic on MS, under a period of \( 4\pi \).

We can also consider a SCS given in terms of opposite CS states \( |\xi\rangle \) and \( |\chi\rangle \),

\[
|\psi_C \rangle = |\xi \rangle + e^{i\varphi} |\chi \rangle.
\]

On a period in the MS, the SCS turns to be the same state. As the states \( |\pm \xi^{\varphi = 0}\rangle = |\pm \xi^{\varphi = 4\pi}\rangle \), the SCS \( |\psi_C \rangle \) is also periodic under \( 4\pi \) period on MS. We can also verify that the state \( |\psi_C \rangle \) is also periodic in the case \( \phi = \varphi \).

The action of the operator \( e^{i\varphi + iJ} \) on \( |\psi_C \rangle \), leads to a SCS also MS periodic, \( 4\pi \),

\[
|\psi_C \rangle = |\xi \rangle - e^{i\varphi} |\chi \rangle,
\]

by means of \( e^{i\varphi + iJ} |\psi_C \rangle = |\xi \rangle |\psi_C \rangle \).

If \( r \) changes along the MS, the periodicity is still MS dependent. But the particle can realize an harmonic motion. For instance, \( r = \sin^2(\varphi) / 2 \), the SCS carries a circle periodicity of \( 2\pi \) on the MS, with \( |\psi_C^{\varphi = 0}\rangle = |\psi_C^{\varphi = 2\pi}\rangle = |\psi_C^{\varphi = 4\pi}\rangle \). On the other hand, for \( r = \cos^2(\varphi) / 2 \) the SCS carries a periodicity of \( 4\pi \) on the MS (figure 2). In the case of \( r = 0 \), that corresponds to a cylinder parametrization \( X = \cos \varphi \), \( Y = \sin \varphi \),
\[ Z = l, \text{ the SCS turn to the same state after a period of } 2\pi. \text{ Then, variations of } r \text{ in the MS generally do not alter the periodic properties of SCS, leading to circle or MS periodicities. This is an important point for the study the stability of SCS in the MS.} \]

2. Uncertainty Relations

CS provides a close connection between quantum and classical physics, with a suitable set of requirements, as continuity, resolution of unity, temporal stability and action identity. In particular, these properties are reflected in SCS, that naturally will lead to cat states, the best analogues of classical behaviour.

SCS with minimal uncertainty relations can be associated to cat states, that are the best analogues of classical superpositions. In order to achieve them on MS, we have to consider the minimization of uncertainty relation in the context of MS.

The uncertainty measurements associated to angular momentum \( \hat{J} \) and the phase angle \( \hat{\varphi} \), written in the form proposed by [28], for the MS case are given by

\[
\Delta^2_\xi (\hat{J}) = \frac{1}{4} \left| \ln \left( \frac{\langle e^{-2\hat{J}} \rangle_\xi \langle e^{2\hat{J}} \rangle_\xi}{\langle e^{-2\hat{J}} \rangle_\xi \langle e^{2\hat{J}} \rangle_\xi} \right) \right|, \tag{8}
\]

\[
\Delta^2_\xi (\hat{\varphi}) = \frac{1}{4} \left| \ln \left( \frac{1}{\langle e^{2i\hat{\varphi}} \rangle_\xi} \right) \right|, \tag{9}
\]

where the expectation value of \( e^{i\hat{\varphi}} \), associated to the CS \( \langle \xi \rangle \) on MS, can be calculated as

\[
\langle e^{i\hat{\varphi}} \rangle_\xi = \sum_j e^{i(j+1)} e^{-\frac{1}{2}j^2} = \sum_j e^{i(j+1)} e^{-\frac{1}{2}j^2}, \tag{10}
\]

For the operators \( e^{2\hat{J}} \) and \( e^{-2\hat{J}} \) the expectation value associated to the CS on MS, that will lead to \( \langle e^{-2\lambda\hat{J}} \rangle_{\xi_{MS}} = \langle \xi_{MS} | e^{-2\lambda\hat{J}} | \xi_{MS} \rangle / \langle \xi_{MS} | \xi_{MS} \rangle \), can be calculated as
Coherent states, superpositions and uncertainty relations on a Möbius strip

Fig. 3. (Color online) Uncertainty measurements associated to angular momentum and the phase angle for (a) \( l' \) in a period of \( 2\pi \), (b) \( l' \) in a period of \( 4\pi \).

in \( e^{i\hat{\phi}} \) case, leading to

\[
\langle e^{-2\lambda \hat{J}} \rangle_{\xi_{MS}} = e^{\lambda^2 - 2\lambda e^{-2\lambda|\sin \varphi/2|}} \times (1 + r \cos(\varphi/2))^{2\lambda} \frac{\theta_3(l' - \lambda|i\pi)}{\theta_3(l'|i\pi)},
\]

(12)

where \( \lambda = \pm 1 \). For sufficiently large \( l' \), we have

\[
\Delta^2_{\xi_{MS}}(\hat{J}) + \Delta^2_{\xi_{MS}}(\hat{\phi}) = l',
\]

(13)

that reduces to the circle case for \( r = 0 \). In particular, this uncertainty relation has a set of minimum values depending on \( l' \) along a MS period \( 4\pi \) in the angular variable (figure 3).

Taking into account the uncertainty relation in a more usual form

\[
\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2,
\]

that satisfy the commutation relation \([\hat{J}, e^{i\hat{\phi}}] = e^{i\hat{\phi}} \) on MS\(^{25}\), we can also write

\[
\langle (\Delta \hat{J})^2 \rangle \langle (\Delta e^{i\hat{\phi}})^2 \rangle \geq \frac{1}{4} \left| \langle e^{i\hat{\phi}} \rangle \right|^2,
\]

(14)

that using \( \langle e^{i\hat{\phi}} \rangle_{\xi_{MS}} \) we find that the minimum uncertainty relations on MS satisfy

\[
\langle (\Delta \hat{J})^2 \rangle \langle (\Delta e^{i\hat{\phi}})^2 \rangle = \frac{1}{4} \left[ e^{(l' + i\varphi)} e^{-1/2} \frac{\theta_3(l' - 1/2)}{\theta_3(l'|i\pi)} \right]^2.
\]

(15)

For sufficiently large \( l' \), we have \( \left| \langle e^{i\hat{\phi}} \rangle_{\xi_{MS}} \right|^2 = e^{2l'} \), and the corresponding uncertainty relation is given by \( e^{2l'}/4 \); for \( l' = 0 \), \( \left| \langle e^{i\hat{\phi}} \rangle_{\xi_{MS}} \right|^2 / 4 = 1/4 \). For a small phase angle we can also write

\[
\langle (\Delta \hat{J})^2 \rangle \langle (\Delta \hat{\phi})^2 \rangle \geq \frac{1}{4}.
\]

(16)

As a consequence, SCS associated to minimum uncertainty relation states are directly related to cat states on MS, these states are important for the study of classical analogues on MS and the possibility of CS quantum computation on MS. Since the CS and SCS constructions on MS can be also reduced to circle topology cases, by \( r = 0 \) or some variations of \( r \) along the MS, this CS states are quite general.
In contrast to CS for the fermions where the cylinder topology is used, forcing to introduce spin-1/2 by hand, the CS on MS is a natural framework phase space where the $4\pi$ symmetry invariance emerges naturally, the symmetry properties of MS and fermions are then closely related, both with characteristic double covering that makes the symmetry invariance of $4\pi$ instead of $2\pi$ for the bosonic case. Due to the double covering, CS on MS can also include the bosonic case, as we have showed in $r = 0$ or an adequate choice of $r$ variation with $\varphi$ angular variable. The minimum uncertainty relations for CS and SCS on MS also leads to a natural phase space where cat states in a nontrivial topology can be formulated.

3. Conclusions

We have considered CS and SCS on MS, the uncertainty measurements associated to angular momentum and the phase angle for the MS and the connection between SCS and cat states was considered. As a result, we extended previous proposes, including here the case of the uncertainty relation in the MS case. In particular, the uncertainty measurements can be associated to MS parameter $l'$. Since the symmetry properties of MS and fermions are closely related, due to the double covering property shared by MS and fermions, symmetry invariance of $4\pi$, CS on MS naturally describes fermionic fields. Due to the double covering, CS on MS can also play the role of projector to the bosonic case $^{30}$, making a closer connection between classical and quantum formulations $^{31}$ in a Dirac-like quantization $^{25}$. When the particle is confined to a Möbius topology its periodicity is similar to a fermionic behavior of a spin 1/2 particle and the similarities can be used to investigate the counterparts associated to cat states in these systems, but it is important to emphasize that the $4\pi$ invariance due to the MS topology is an invariance in the physical space, while the $4\pi$ invariance associated with fermions is an invariance in the internal spin space. The behavior of CS and SCS on MS are associated to a continuous transformation, in contrast to the discrete behaviour of the spin system, but analogue to this one under spin rotations.

CS can be very sensitive to the geometry they are constructed, the CS on MS has the advantage of recover cylinder topologies as particular cases when $r = 0$ or an adequate choice of $r$ variation with $\varphi$ angular variable. As MS is a more natural phase space for fermions symmetries of spin rotation, they can also be useful for investigating cat states analogues to fermionic fields.

SCS on MS are strong candidates to cat states in nontrivial topologies. This is reinforced by the minimum uncertainty relations for CS on MS. One consequence is the possibility of using CS and SCS on MS to realize CVQC in nontrivial topologies.

For the SCS in a period on MS, we showed the expected behaviour of periodicity in phase $\varphi$ with a period of $\varphi = 4\pi$ in the MS. In the case $r = 0$, corresponding to the case in the torus, the periodicity is, as expected, of $\varphi = 2\pi$. By varying $r$ along the MS, the particle can describe an oscillatory motion along the strip such that there is a mixing of the cases in the torus and MS cases. As the CS comes from a
constraint on the torus parametrization, the CS on MS can also be connected to the CS on torus by means of the breaking of constraint between the angular variables. The topological stability of SCS, as showed for particular variations, can also be used for CS quantum computation, topological physics on MS and formulations on Riemannian superspaces.

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