1. Introduction

The purpose of this paper is to describe several applications of finiteness properties of $F$-finite $F$-modules recently discovered by M. Hochster in [H] to the study of Frobenius maps on injective hulls, Frobenius near-splittings and to the nature of morphisms of $F$-finite $F$-modules.

Throughout this paper $(R, m)$ shall denote a complete regular local ring of prime characteristic $p$. At the heart of everything in this paper is the Frobenius map $f : R \to R$ given by $f(r) = r^p$ for $r \in R$. We can use this Frobenius map to define a new $R$-module structure on $R$ given by $r \cdot s = r^p s$; we denote this $R$-module $F_* R$. We can then use this to define the Frobenius functor from the category of $R$-modules to itself: given an $R$-module $M$ we define $F(M)$ to be $F_* R \otimes_R M$ with $R$-module structure given by $r(s \otimes m) = rs \otimes m$ for $r, s \in R$ and $m \in M$.

Let $R[\Theta; f]$ be the skew polynomial ring which is a free $R$-module $\oplus_{i=0}^{\infty} R\Theta^i$ with multiplication $\Theta r = r^p \Theta$ for all $r \in R$. As in [K1], $\mathcal{C}$ shall denote the category $R[\Theta; f]$-modules which are Artinian as $R$-modules. For any two such modules $M, N$, we denote the morphisms between them in $\mathcal{C}$ with $\text{Hom}_{R[\Theta; f]}(M, N)$; thus an element $g \in \text{Hom}_{R[\Theta; f]}(M, N)$ is an $R$-linear map such that $g(\Theta a) = \Theta g(a)$ for all $a \in M$. The first main result of this paper (Theorem 3.3) shows that under some conditions on $N$, $\text{Hom}_{R[\Theta; f]}(M, N)$ is a finite set.

An $F$-module (cf. the seminal paper [L] for an introduction to $F$-modules and their properties) over the ring $R$ is an $R$-module $M$ together with an $R$-module isomorphism $\theta_M : M \to F(M)$. This isomorphism $\theta_M$ is the structure morphism of $M$.

A morphism of $F$-modules $M \to N$ is an $R$-linear map $g$ which makes the following diagram commute

$$
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow{\theta_M} & & \downarrow{\theta_N} \\
F(M) & \xrightarrow{F(g)} & F(N)
\end{array}
$$

where $\theta_M$ and $\theta_N$ are the structure isomorphisms of $M$ and $N$, respectively. We denote $\text{Hom}_F(M, N)$ the $R$-module of all morphism of $F$-modules $M \to N$. 

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Given any finitely generated $R$-module $M$ and $R$-linear map $\beta : M \to F(M)$ one can obtain an $R$-module

$$M = \lim\limits_\rightarrow \left( M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \ldots \right).$$

Since

$$F(M) = \lim\limits_\rightarrow \left( F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} F^3(M) \xrightarrow{F^3(\beta)} \ldots \right) = M$$

we obtain an isomorphism $M \cong F(M)$, and hence $M$ is an $F$-module. Any $F$-module which can be constructed as a direct limit as $M$ above is called an $F$-finite $F$-module with generating morphism $\beta$.

There is a close connection between $R[\Theta; f]$-modules and $F$-finite $F$-modules given by Lyubeznik’s Functor from $\mathcal{C}$ to the category of $F$-finite $F$-modules which is defined as follows (see section 4 in [L] for the details of the construction.) Given an $R[\Theta; f]$-module $M$ one defines the $R$-linear map $\alpha : F(M) \to M$ by $\alpha(r\Theta \otimes m) = r\Theta m$; an application of Matlis duality then yields an $R$-linear map $\alpha^\vee : M^\vee \to F(M)^\vee \cong F(M^\vee)$ and one defines

$$\mathcal{H}(M) = \lim\limits_\rightarrow \left( M^\vee \xrightarrow{\alpha^\vee} F(M^\vee) \xrightarrow{F(\alpha^\vee)} F^2(M^\vee) \xrightarrow{F^2(\alpha^\vee)} \ldots \right).$$

Since $M$ is an Artinian $R$-module, $M^\vee$ is finitely generated and $\mathcal{H}(M)$ is an $F$-finite $F$-module with generating morphism $M^\vee \xrightarrow{\alpha^\vee} F(M^\vee)$. This construction is functorial and results in an exact covariant functor from $\mathcal{C}$ to the category of $F$-finite $F$-modules.

The main result in [L] is the surprising fact that for $F$-finite $F$-modules $\mathcal{M}$ and $\mathcal{N}$, $\Hom_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$ is a finite set. In section 3 of this paper we exploit this fact to prove the second main result in this paper (Theorem 3.4) to show the following. Let $\gamma : M \to F(M)$ and $\beta : N \to F(N)$ be generating morphisms for $\mathcal{N}$ and $\mathcal{M}$. Given an $R$-linear map $g$ which makes the following diagram commute,

$$\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow{g} & & \downarrow{F(g)} \\
M & \xrightarrow{\gamma} & F(M)
\end{array}$$

one can extend that diagram to

$$\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \xrightarrow{F(\beta)} F^2(N) \xrightarrow{F^2(\beta)} \ldots \\
\downarrow{g} & & \downarrow{F(g)} \\
M & \xrightarrow{\gamma} & F(M) \xrightarrow{F(\gamma)} F^2(M) \xrightarrow{F^2(\gamma)} \ldots
\end{array}$$

and obtain a map between the direct limits of the horizontal sequences, i.e., an element in $\Hom_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$. We prove that all elements in $\Hom_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$ arise in this way (cf. Theorem 3.4), thus morphisms of $F$-finite $F$-modules have a particularly simple form. This answers a question implicit in [L] Remark 1.10(b)].
Finally, in section 3 we consider the module \( \text{Hom}_R(F_* R^n, R^n) \) of near-splittings of \( F_* R^n \). We establish a correspondence between these near-splittings and Frobenius actions on \( E^n \) which enables us to prove the third main result in this paper (Theorem 4.5) which asserts that given a near-splitting \( \phi \) corresponding to a injective Frobenius actions, there are finitely many \( F_* R \)-submodules \( V \subseteq F_* R^n \) such that \( \phi(V) \subseteq V \). This generalizes a similar result in [BH] to the case where \( R \) is not \( F \)-finite.

Our study of Frobenius near-splittings is based on the study of its dual notion, i.e., Frobenius maps on the injective hull \( E = E_R(R/m) \) of the residue field of \( R \). This injective hull is given explicitly as the module of inverse polynomials \( K[x_1^-, \ldots, x_d^-] \) where \( x_1, \ldots, x_d \) are minimal generators of the maximal ideal of \( R \) (cf. [BS] §12.4). Thus \( E \) has a natural \( R[T; f] \)-module structure extending \( T \alpha_1 x_1^{-\alpha_1} \ldots x_1^{-\alpha_d} = \lambda^p x_1^{-p\alpha_1} \ldots x_1^{-p\alpha_d} \) for \( \lambda \in K \) and \( \alpha_1, \ldots, \alpha_d > 0 \). We can further extend this to a natural \( R[T; f] \)-module structure on \( E^n \) given by

\[
T \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_n \end{pmatrix}.
\]

The results of section 4 will follow from the fact that there is a dual correspondence between Frobenius near-splittings and sets of \( R(\Theta; f) \)-module structures on \( E^n \).

2. FROBENIUS MAPS OF ARTINIAN MODULES AND THEIR STABLE SUBMODULES

Given an Artinian \( R \)-module \( M \) we can embed \( M \) in \( E^\alpha \) for some \( \alpha \geq 0 \) and extend this inclusion to an exact sequence

\[
0 \to M \to E^\alpha \xrightarrow{A'} E^\beta \to \ldots
\]

where \( A' \in \text{Hom}_R(E^\alpha_R, E^\beta_R) \equiv \text{Hom}_R(R^\alpha, R^\beta) \) is a \( \beta \times \alpha \) matrix with entries in \( R \). Henceforth in this section we will describe certain properties of Artinian \( R \)-modules in terms of their representations as kernels of matrices with entries in \( R \). We shall denote \( M_{\alpha, \beta} \) the set of \( \alpha \times \beta \) matrices with entries in \( R \).

In this section and the next we will need the following constructions. Following [K1] we shall denote the category of Artinian \( R(\Theta; f) \)-modules \( \mathcal{C} \). We denote \( \mathcal{D} \) the category of \( R \)-linear maps \( M \to F_R(M) \) where \( M \) is a finitely generated \( R \)-module, \( F_R(\_\_)/-\) denotes the Frobenius functor, and where a morphism between \( M \xrightarrow{a} F_R(M) \) and \( N \xrightarrow{b} F_R(N) \) is a commutative diagram of \( R \)-linear maps

\[
\begin{array}{ccc}
M & \xrightarrow{\mu} & N \\
\downarrow{a} & & \downarrow{b} \\
F_R(M) & \xrightarrow{F_R(\mu)} & F_R(N)
\end{array}
\]

Section 3 of [K1] constructs a pair of functors \( \Delta : \mathcal{C} \to \mathcal{D} \) and \( \Psi : \mathcal{D} \to \mathcal{C} \) with the property that for all \( A \in \mathcal{C} \), the \( R(\Theta; f) \)-module \( \Psi \circ \Delta(A) \) is canonically isomorphic to \( A \) and for all \( D = (B \xrightarrow{\mu} F_R(B)) \in \mathcal{D} \), \( \Delta \circ \Psi(D) \) is canonically isomorphic to \( D \). The functor \( \Delta \) amounts
to the “first step” in the construction of Lyubeznik’s functor \( \mathcal{H} \): for \( A \in \mathcal{C} \) we define the R-linear map \( \alpha : F(A) \to A \) to be the one given by \( \alpha(r\Theta \otimes a) = r\Theta a \) and we let \( \Delta(A) \) to be the map \( \alpha^\vee : A^\vee \to F(A)^\vee \cong F(A^\vee) \) (cf. section 3 in [K1] for the details of the construction.)

**Proposition 2.1.** Let \( M = \ker A^t \subseteq E^\alpha \) be an Artinian R-module where \( A \in \text{M}_{n,\beta} \). Let \( B = \{ B \in \text{M}_{\alpha,\alpha} \mid \text{Im} BA \subseteq \text{Im} A[p] \} \). For any \( R[\Theta; f] \)-module structure on \( M \), \( \Delta(M) \) can be identified with an element in \( \text{Hom}_R(\text{Coker} A, \text{Coker} A[p]) \) and thus represented by multiplication by some \( B \in B \). Conversely, any such \( B \) defines an \( R[\Theta; f] \)-module structure on \( M \) which is given by the restriction to \( M \) of the Frobenius map \( \phi : E^\alpha \to E^\alpha \) defined by \( \phi(v) = B^t T(v) \) where \( T \) is the natural Frobenius map on \( E^\alpha \).

**Proof.** Matlis duality gives an exact sequence \( R^\beta \xrightarrow{A} R^\alpha \to M^\vee \to 0 \) hence \( \Delta(M) \in \text{Hom}_R(M^\vee, F_R(M^\vee)) \cong \text{Hom}_R(\text{Coker} A, \text{Coker} A[p]). \)

Let \( \Delta(M) \) be the map \( \phi : \text{Coker} A \to \text{Coker} A[p] \).

In view of Theorem 3.1 in [K1] we only need to show that any such R-linear map is given by multiplication by an \( B \in B \), and that any such \( B \) defines an element in \( \Delta(M) \).

We can find a map \( \phi' \) which makes the following diagram

\[
\begin{array}{ccc}
R^\alpha & \xrightarrow{\phi} & R^\alpha / \text{Im} A \\
\downarrow{q_1} & & \downarrow{q_2} \\
R^\alpha / \text{Im} A[p] & \xrightarrow{\phi'} & R^\alpha \\
\end{array}
\]

commute, where \( q_1 \) and \( q_2 \) are quotient maps. The map \( \phi' \) is given by multiplication by some \( \alpha \times \alpha \) matrix \( B \in B \). Conversely, any such matrix \( B \) defines a map \( \phi \) making the diagram above commute, and \( \Psi(\phi) \) gives a \( R[\Theta; f] \)-module structure on \( M \) as described in the last part of the theorem. \( \square \)

**Notation 2.2.** We shall henceforth describe Artinian R-modules with a given \( R[\Theta; f] \)-module structure in terms of the two matrices in the statement of Proposition 2.1 and talk about Artinian R-modules \( M = \text{Ker} A^t \subseteq E^\alpha \) where \( A \in \text{M}_{n,\beta} \) with \( R[\Theta; f] \)-module structure given by \( B \in \text{M}_{\alpha,\alpha} \).

3. Morphisms in \( \mathcal{C} \)

In this section we raise two questions. The first of these asks when for given \( R[\Theta; f] \)-modules \( M, N \), the set \( \text{Hom}_{R[\Theta; f]}(M, N) \) is finite; later in this section we prove that this holds when \( N \) has no \( \Theta \)-torsion. The following two examples illustrate why this set is not finite in general, and why it is finite in a special simple case.

**Example 3.1.** Let \( \mathbb{K} \) be an infinite field of prime characteristic \( p \) and let \( R = \mathbb{K}[x] \). Let \( M = \text{ann}_E xR \) and fix an \( R[\Theta; f] \)-module structure on \( M \) given by \( \Theta a = x^p T a \) where \( T \) is the standard Frobenius action on \( E \). Note that \( \Theta M = 0 \) and that for all \( \lambda \in \mathbb{K} \) the map
\( \mu_\lambda : M \to M \) given by multiplication by \( \lambda \) is in \( \text{Hom}_{R[\Theta, f]}(M, M) \), and hence this set is infinite.

**Example 3.2.** Let \( I, J \subseteq R \) be ideals, and fix \( u \in (I^p : I) \) and \( v \in (J^p : J) \). Endow \( \text{ann}_E I \) and \( \text{ann}_E J \) with \( R[\Theta, f] \)-module structures given by \( \Theta a = uTa \) and \( \Theta b = vTb \) for \( a \in \text{ann}_E I \) and \( b \in \text{ann}_E J \) where \( T \) is the standard Frobenius map on \( E \).

If \( g : \text{ann}_E I \to \text{ann}_E J \) is \( R \)-linear, an application of Matlis duality yields \( g^\vee : R/J \to R/I \) and we deduce that \( g \) is given by multiplication by an element in \( w \in (I : J) \). If in addition \( g \in \text{Hom}_{R[\Theta, f]}(\text{ann}_E I, \text{ann}_E J) \), we must have \( wuTa = g(\Theta a) = \Theta g(a) = vTw a = vu^pTa \), for all \( a \in \text{ann}_E I \), hence \( (vu^p - uw)T \text{ann}_E I = 0 \) and \( vu^p - uw \in I^p \). The finiteness of \( \text{Hom}_{R[\Theta, f]}(\text{ann}_E I, \text{ann}_E J) \) translates in this setting to the finiteness of the set of solutions for the variable \( w \) of the equation above, and it is not clear why this set should be finite. However, if we simplify to the case where \( I = 0 \), the set of solutions of \( vu^p - uw = 0 \) over the the fraction field of \( R \) has at most \( p \) elements, and in this case we can deduce that \( \text{Hom}_{R[\Theta, f]}(E, \text{ann}_E J) \) has at most \( p \) elements.

As in \([L]\), for any \( R[\Theta, f] \)-module \( M \) we define the submodule of nilpotent elements to be \( \text{Nil}(M) = \{ a \in M \mid \Theta^e a = 0 \text{ for some } e \geq 0 \} \). We recall that when \( M \) is an Artinian \( R \)-module there exists an \( \eta \geq 0 \) such that \( \Theta^\eta M = 0 \) (cf. \([HS]\) Proposition 1.11 and \([L]\) Proposition 4.4.). We also define \( M_{\text{red}} = M/\text{Nil}(M) \) and \( M^* = \cap_{e \geq 0} \Theta^e M \) where \( R\Theta^e M \) denotes the \( R \)-module generated by \( \{ \Theta^e a \mid a \in M \} \). We also note that when \( M \) is an \( R[\Theta, f] \)-module which is Artinian as an \( R \)-module, there exists an \( e \geq 0 \) such that \( M^* = R\Theta^e M \) and also \( (M_{\text{red}})^* = (M^*)_{\text{red}} \) (cf. section 4 in \([K2]\).)

**Theorem 3.3.** Let \( M, N \) be \( R[\Theta, f] \)-modules and let \( \phi \in \text{Hom}_{R[\Theta, f]}(M, N) \). We have \( \mathcal{H}(\text{Im} \phi) = 0 \) if and only if \( \phi(M) \subseteq \text{Nil}(N) \) and, consequently, if \( \text{Nil}(N) = 0 \), the map \( \mathcal{H} : \text{Hom}_{R[\Theta, f]}(M, N) \to \text{Hom}_{R_{\text{red}}}(\mathcal{H}(N), \mathcal{H}(M)) \) is an injection and \( \text{Hom}_{R[\Theta, f]}(M, N) \) is a finite set.

**Proof.** We apply \( \mathcal{H} \) to the commutative diagram

\[
\begin{array}{ccc}
M & \overset{\phi}{\longrightarrow} & N \\
\downarrow{\phi} & & \downarrow{\phi} \\
\text{Im} \phi & \overset{\subset}{\longrightarrow} & N
\end{array}
\]

to obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}(N) & \longrightarrow & \mathcal{H}(\text{Im} \phi) \\
\downarrow{\mathcal{H}(\phi)} & & \downarrow{\mathcal{H}(\phi)} \\
\mathcal{H}(M) & \longrightarrow & \mathcal{H}(M)
\end{array}
\]

Now \( \mathcal{H}(\phi) = 0 \) if and only if \( \mathcal{H}(\text{Im} \phi) = 0 \), and by \([L]\) Theorem 4.2] this is equivalent to \( (\text{Im} \phi)^*_{\text{red}} = 0 \).

Choose \( \eta \geq 0 \) such that \( \Theta^\eta \text{Nil}(N) = 0 \) and choose \( e \geq 0 \) such that \( (\text{Im} \phi)^* = R\Theta^e \text{Im} \phi \).
Now

\[(\text{Im } \phi)^\ast_{\text{red}} = 0 \iff R\Theta^p R\Theta^e \phi(M) = 0 \]
\[\equiv R\Theta^{p+e} \phi(M) = 0 \]
\[\equiv \text{Im } \phi \subseteq \text{Nil}(N)\]

The second statement now follows immediately. \[\square\]

The second main result in this section, Theorem 3.4 shows that all morphisms of \(\text{F}-\text{finite}\) \(\text{F}-\text{modules}\) arise as images of maps of \(R[\Theta; f]\)-modules under Lyubeznik's functor \(\mathcal{H}\).

**Theorem 3.4.** Let \(M\) and \(N\) be \(\text{F}-\text{finite}\) \(\text{F}-\text{modules}\). For every \(\phi \in \text{Hom}_{\text{F}}(N, M)\) there exist generating morphisms \(\gamma : M \rightarrow F(M) \in \mathcal{D}\) and \(\beta : N \rightarrow F(N) \in \mathcal{D}\) for \(M\) and \(N\), respectively, and a morphism (in the category \(\mathcal{D}\))

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow{g} & & \downarrow{F(g)} \\
M & \xrightarrow{\gamma} & F(M)
\end{array}
\]

such that \(\phi = \mathcal{H}(\Psi(g))\).

**Proof.** Choose any generating morphisms

\[
N = \lim \left( N \xrightarrow{\beta} F(N) \xrightarrow{F(\beta)} F^2(N) \xrightarrow{F^2(\beta)} \ldots \right)
\]
and

\[
M = \lim \left( M \xrightarrow{\gamma} F(M) \xrightarrow{F(\gamma)} F^2(M) \xrightarrow{F^2(\gamma)} \ldots \right)
\]

and fix any \(\phi \in \text{Hom}_{\text{F}}(N, M)\).

For all \(j \geq 0\) let \(\phi_j\) be the restriction of \(\phi\) to the image of \(F^j(N)\) in \(N\).

The fact that \(\phi\) is a morphism of \(\text{F}-\text{modules}\) implies that for every \(j \geq 0\) we have a commutative diagram

\[
\begin{array}{ccc}
F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) \\
\downarrow{\theta_M} & & \downarrow{\theta_N} \\
N & \xrightarrow{\theta_N} & F(N) \\
\downarrow{\phi} & & \downarrow{F(\phi)} \\
M & \xrightarrow{\cong_{\phi_M}} & F(\phi)
\end{array}
\]

where \(\theta_M\) and \(\theta_N\) are the structure isomorphisms of \(M\) and \(N\), respectively, and where the compositions of the vertical maps are \(\phi_j\) and \(F(\phi_j)\). Repeated applications of the Frobenius
functor yields a commutative diagram

\[
\begin{array}{cccccc}
F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) & \cdots \\
\phi_j & & F(\phi_j) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathcal{M} & \xrightarrow{\theta_M} & F(\mathcal{M}) & \cdots \\
\end{array}
\]

and we can now extend this commutative diagram to the left to obtain

\[
\begin{array}{cccccc}
N & \xrightarrow{\beta} & F(N) & \cdots & F^{j-1}(\beta) & F^j(N) & F^{j+1}(\beta) & F^{j+2}(\beta) & \cdots \\
\phi_0 & \phi_1 & F(\phi_j) & F(\mathcal{M}) & F^2(\mathcal{M}) & \cdots \\
\downarrow \phi_j & \downarrow \theta_M \circ F(\theta_M)^{-1} & \downarrow \theta_M \circ \phi(a) = \theta_M \circ \phi(a) = \theta_M \circ \phi(a) = \theta_M \circ \phi(a) & \\
\mathcal{M} & \xrightarrow{\theta_M \circ \phi(a)} & F(\mathcal{M}) & \cdots \\
\end{array}
\]

This commutative diagram defines a \( R \)-linear map \( \psi_j : N \to \mathcal{M} \). Furthermore, we show next that this \( \psi_j \) is a map of \( \mathcal{F} \)-modules, i.e., that for all \( j \geq 0 \), \( F(\psi_j) \circ \theta_N = \theta_M \circ \psi_j \). Fix \( j \geq 0 \) and abbreviate \( \psi = \psi_j \).

Pick any \( a \in N \) represented as an element of \( F^e(N) \). If \( e < j \) then the fact that \( \phi \) is a morphism of \( \mathcal{F} \)-modules, implies that

\[
\theta_M \circ \psi(a) = \theta_M \circ \phi(a) = F(\phi) \circ \theta_N(a) = F(\psi) \circ \theta_N(a).
\]

Assume now that \( e \geq j \); we have

\[
\theta_M \circ \psi(a) = \theta_M \circ \theta_M^{-1} \circ F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j)(a) = F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j)(a)
\]

and

\[
F(\psi) \circ \theta_N(a) = F(\theta_M^{-1} \circ F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j))(F^e(\beta)(a)) = F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\theta_M) \circ F^{e-j}(\phi_j)(a) = F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j)(a)
\]

where the penultimate inequality follows from the fact that \( \phi \) is a morphism of \( \mathcal{F} \)-modules.

Consider now the set \( \{\psi_i\}_{i \geq 0} \); it is a finite set according to Theorem 5.1 in [H], hence we can find a sequence \( 0 \leq i_1 < i_2 < \cdots \) such that \( \psi_{i_1} = \psi_{i_2} = \ldots \). By replacing \( N \) and \( \mathcal{M} \) with \( F^{i_1}(N) \) and \( F^{i_1}(\mathcal{M}) \) we may assume that \( i_1 = 0 \).

Pick \( j \geq 0 \) so that \( \phi(N) \subseteq F^j(M) \). Since \( \mathcal{M} \cong F^j(M) \) we may replace \( \mathcal{M} \) with \( F^j(M) \) and assume that \( \phi(N) \subseteq M \) and hence also that for all \( e \geq 0 \), \( F^e(\phi)(F^e(N)) \subseteq F^e(M) \).
Fix now any $e \geq 0$ and pick any $i_k > e$; the fact that $\psi_0 = \psi_{i_k}$ implies that for all $a \in F^e(N)$, $F^e(\phi_0)(a) = \psi_0(a) = \psi_{i_k}(a) = \phi(a)$ and since this holds for all $e \geq 0$ we deduce that $\phi$ is induced from the commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow{\phi_0} & & \downarrow{F(\phi_0)} \\
M & \xrightarrow{\gamma} & F(M)
\end{array}
$$

\begin{array}{ccc}
F^2(\beta) & \rightarrow & F^2(N) \\
\downarrow{F^2(\phi_0)} & & \downarrow{F^2(\phi_0)} \\
F^2(\gamma) & \rightarrow & F^2(M)
\end{array}

\ldots

An application of the functor $\Psi$ to the leftmost square in the commutative diagram above yields a morphism of $R[\Theta; f]$-modules $g : M \rightarrow N$ and $\phi = \mathcal{H}(g)$. □

4. Applications to Frobenius splittings

For any $R$-module $M$ let $F_* M$ denote the additive Abelian group $M$ with $R$-module structure given by $r \cdot a = r^a a$ for all $r \in R$ and $a \in M$. In this section we study the module $\text{Hom}_R(F_* R^n, R^n)$ of near-splittings of $F_* R^n$. Given such an element $\phi \in \text{Hom}_R(F_* R^n, R^n)$ we will describe the submodules $V \subseteq F_* R^n$ for which $\phi(V) \subseteq V$. These submodules in the case $n = 1$, known as $\phi$-compatible ideals, are of significant importance in algebraic geometry (cf. [BK] for a study of applications of Frobenius splittings and their compatible submodules in algebraic geometry.) We will prove that under some circumstances these form a finite set and thus generalize a result in [BB] obtained in the $F$-finite case.

We first exhibit the following easy implication of Matlis duality necessary for the results of this section.

**Lemma 4.1.** For any (not necessarily finitely generated) $R$-module $M$, $\text{Hom}_R(M, R) \cong \text{Hom}_R(R^\vee, M^\vee)$.

**Proof.** For all $a \in E$ let $h_a \in \text{Hom}_R(R, E)$ denote the map sending $1$ to $a$.

For any $\phi \in \text{Hom}_R(M, R)$, $\phi^\vee \in \text{Hom}_R(R^\vee, M^\vee)$ is defined as $(\phi^\vee(h_a))(m) = \phi(m) a$ for any $m \in M$ and $a \in E$. For any $\psi \in \text{Hom}_R(R^\vee, M^\vee)$ we define $\tilde{\psi} \in \text{Hom}_R(M, R)$ as $(\tilde{\psi}(m))(a) = (\psi(h_a))(m)$ for all $a \in E$ and $m \in M$. Note that the function $\psi \mapsto \tilde{\psi}$ is $R$-linear.

It is now enough to show that for all $\phi \in \text{Hom}_R(M, R)$, $\tilde{\psi}^\vee = \phi$, and indeed for all $a \in E$ and $m \in M$

$$(\tilde{\phi}^\vee(m))(a) = (\phi^\vee(h_a))(m) = \phi(m) a,$$

i.e., $(\tilde{\phi}^\vee(m)) \in \text{Hom}_R(E, E)$ is given by multiplication by $\phi(m)$ and so under the identification of $\text{Hom}_R(E, E)$ with $R$, $\tilde{\phi}^\vee$ is identified with $\phi$. □

We can now prove a generalization Lemma 1.6 in [F] in the form of the next two theorems.

**Theorem 4.2.**

(a) The $F_* R$-module $\text{Hom}_R(F_* R, E)$ is injective of the form $\bigoplus_{\gamma \in \Gamma} F_* E \oplus H$ where $\Gamma$ is non-empty, $H = \bigoplus_{\lambda \in \Lambda} F_* E(R/P_\lambda)$, $\Lambda$ is a (possibly empty) set, $P_\lambda$
is a non-maximal prime ideal of $R$ for all $\lambda \in \Lambda$ and $E(R/P_\lambda)$ denotes the injective hull of $R/P_\lambda$.

(b) Write $\mathcal{B} = \text{Hom}_{F,R}(E, \oplus_{\gamma \in \Gamma} F_\gamma E) \subseteq \prod_{\gamma \in \Gamma} \text{Hom}_{F,R}(E, F_\gamma E)$. We have

$$\text{Hom}_R(F_*R, R) \cong \mathcal{B} \subseteq \prod_{\gamma \in \Gamma} \text{Hom}_{F,R}(E, F_\gamma E) \cong \prod_{\gamma \in \Gamma} F_*RT$$

where $T$ is the standard Frobenius map on $E$.

(c) The set $\Gamma$ is finite if and only if $F_\ast \mathbb{K}$ is a finite extension of $\mathbb{K}$, in which case $\# \Gamma = 1$.

**Proof.** The functors $\text{Hom}_R(-, E) = \text{Hom}_R(- \otimes_{F,R} F_*R, E)$ and $\text{Hom}_{F,R}(-, \text{Hom}_R(F_*R, E))$ from the category of $F_*R$-modules to itself are isomorphic by the adjointness of $\text{Hom}$ and $\otimes$, and since $\text{Hom}_R(-, E)$ is an exact functor, so is $\text{Hom}_{F,R}(-, \text{Hom}_R(F_*R, E))$, thus $\text{Hom}_R(F_*R, E)$ is an injective $F_*R$-module and hence of the form $G \oplus H$ where $G$ is a direct sum of copies of $F_*E$ and $H$ is as in the statement of the Theorem. Write $G = \oplus_{\gamma \in \Gamma} F_\gamma E$.

Pick any $h \in \text{Hom}_R\left(E, \bigoplus_{\lambda \in \Lambda} F_\gamma E(R/P_\lambda)\right)$. For any $a \in E$, $h(a)$ can be written as a finite sum $b_{\lambda_1} + \cdots + b_{\lambda_s}$ where $\lambda_1, \ldots, \lambda_s \in \Lambda$ and $b_{\lambda_1} \in F_*E(R/P_{\lambda_1}), \ldots, b_{\lambda_s} \in F_*E(R/P_{\lambda_s})$.

Use prime avoidance to pick a $z \in m \setminus \bigcup_{i=1}^s P_{\lambda_i}$; now $z$ and its powers act invertibly on each of $F_*E(R/P_{\lambda_1}), \ldots, F_*E(R/P_{\lambda_s})$ while a power of $z$ kills $a$, and so we must have $h(a) = 0$. We deduce that $\text{Hom}_R\left(E, \bigoplus_{\lambda \in \Lambda} F_\gamma E(R/P_\lambda)\right) = 0$ and

$$\text{Hom}_R(E, \text{Hom}_R(F_*R, E)) \cong \text{Hom}_R\left(E, G \oplus \bigoplus_{\lambda \in \Lambda} F_\gamma E(R/P_\lambda)\right)$$

$$\cong \text{Hom}_R(E, G) \oplus \text{Hom}_R\left(E, \bigoplus_{\lambda \in \Lambda} F_\gamma E(R/P_\lambda)\right)$$

$$\cong \text{Hom}_R(E, G)$$

$$\cong \text{Hom}_R(E, \oplus_{\gamma \in \Gamma} F_\gamma E)$$

$$= \mathcal{B}.$$ 

Now $\text{Hom}_R(E, F_*E)$ is the $R$-module of Frobenius maps on $E$ which is isomorphic as an $F_*R$ module to $F_*RT$ and we conclude that $\text{Hom}_R(E, \text{Hom}_R(F_*R, E)) \subseteq \prod_{\gamma \in \Gamma} F_*RT$.

An application of the Matlis dual and Lemma 11 now gives

$$\text{Hom}_R(F_*R, R) \cong \text{Hom}_R(E, \text{Hom}_R(F_*R, E))$$

and (b) follows.

Write $\mathbb{K} = R/m$ and note that $F_*\mathbb{K}$ is the field extension of $\mathbb{K}$ obtained by adding all $p$th roots of elements in $\mathbb{K}$. We next compute the cardinality of $\Gamma$ as the $F_*\mathbb{K}$-dimension of $\text{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, G)$. A similar argument to the one above shows that

$$\text{Hom}_{F_*\mathbb{K}}\left(F_*\mathbb{K}, \bigoplus_{\lambda \in \Lambda} F_\gamma E(R/P_\lambda)\right) = 0$$
hence $\text{Hom}_{F,K}(F,K,G) = \text{Hom}_{F,K}(F,K,\text{Hom}_R(F,R,E))$.

We may identify $\text{Hom}_{F,K}(F,K,\text{Hom}_R(F,R,E))$ and $\text{Hom}_{F,R}(F,K,\text{Hom}_R(F,R,E))$. Another application of the adjointness of $\text{Hom}$ and $\otimes$ gives

$$\text{Hom}_{F,R}(F,K,\text{Hom}_R(F,R,E)) \cong \text{Hom}_R(F,K \otimes_{F,R} F,R,E) \cong \text{Hom}_R(F,K,E)$$

Since $mF_*K = 0$, we see that the image of any $\phi \in \text{Hom}_R(F_*K,E)$ is contained in $\text{ann}_E m \cong K$ and we deduce that $\text{Hom}_R(F_*K,E) \cong \text{Hom}_R(F,K,E)$. We can now conclude that the cardinality of $\Gamma$ is the $F_*K$-dimension of $\text{Hom}_R(F_*K,E)$. In particular $\Gamma$ cannot be empty and (a) follows.

If $\mathcal{U}$ is a $K$-basis for $F_*K$ containing $1 \in F_*K$,

$$\text{Hom}_K(F_*K,E) \cong \prod_{b \in \mathcal{U}} \text{Hom}_K(Kb,K)$$

and when $\mathcal{U}$ is finite, this is a one-dimensional $F_*K$-vector space spanned by the projection onto $K1 \subset F_*K$. If $\mathcal{U}$ is not finite, the dimension as $K$-vector space of $\prod_{b \in \mathcal{U}}$ is at least $2^{|\mathcal{U}|}$ hence $\text{Hom}_K(F_*K,E)$ cannot be a finite-dimensional $F_*K$-vector space. □

**Theorem 4.3.** Let $G = \oplus_{\gamma \in \Gamma} F,E$ and $B$ be as in Theorem 4.2. Let $B \in \text{Hom}_R(F,R^n,R^n)$ be represented by $(B_\gamma T)_{\gamma \in \Gamma} \in B$. For all $\gamma \in \Gamma$ consider $E^n$ as an $R[\Theta_\gamma; f]$-module with $\Theta_\gamma v = B_\gamma^t Tv$ for all $v \in E^n$. Let $V$ be an $R$-submodule of $R^n$ and fix a matrix $A$ whose columns generate $V$. If $B(F_*V) \subseteq V$, then $\text{ann}_E A^t$ is a $R[\Theta_\gamma; f]$ submodule of $E^n$ for all $\gamma \in \Gamma$.

**Proof.** Apply the Matlis dual to the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & F_*A & \longrightarrow & F_*R^n & \longrightarrow & F_*R^n/F_*A & \longrightarrow & 0 \\
& & B & \downarrow & & \text{Id} & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & R^n & \longrightarrow & R^n/A & \longrightarrow & 0
\end{array}
$$

where the rightmost vertical map is induced by the middle map to obtain

$$
\begin{array}{cccccc}
0 & \longrightarrow & (R^n/A)^\vee & \longrightarrow & E^n & \longrightarrow & 0 \\
& & B^\vee & \downarrow & & B^\vee & \downarrow & & \\
0 & \longrightarrow & (F_*R^n/F_*A)^\vee & \longrightarrow & \text{Hom}_R(F_*R^n,E) & \longrightarrow & 0
\end{array}
$$

Note that $B^\vee \in \text{Hom}_R(E^n, \oplus_{\gamma \in \Gamma} E^n)$ is given by $(B_\gamma^t)_{\gamma \in \Gamma}$.

Using the presentation $F_*R^n \xrightarrow{F_*A^t} F_*R^n \rightarrow F_*R^n/\text{Im} F_*A \rightarrow 0$ we obtain the exact sequence

$$0 \rightarrow (F_*R^n/F_*A)^\vee \rightarrow \text{Hom}_R(F_*R^n,E) \xrightarrow{F_*A^t} \text{Hom}_R(F_*R^n,E)$$

thus

$$(F_*R^n/F_*A)^\vee = \text{ann}_{\text{Hom}_R(F_*R^n,E)} F_*A^t.$$
We obtain the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{ann}_{E^n} A^i \\
\downarrow & & \downarrow (B^i T)_\gamma \\
0 & \rightarrow & \bigoplus_{\gamma \in \Gamma} \text{ann}_{F,E^n} F_* A^i \\
& & \bigoplus_{\gamma \in \Gamma} F_* E^n
\end{array}
\]

and we deduce that \(\text{ann}_{E^n} A^i\) is a \(R[\Theta; f^i]\)-module for all \(\gamma \in \Gamma\). \(\square\)

**Theorem 4.4.** Let \(M\) be an \(R[\Theta; f]\)-module with no nilpotents and assume \(M\) is an Artinian \(R\)-module. Then \(M\) has finitely many \(R[\Theta; f]\)-submodules. (Cf. Corollary 4.18 in [BB].)

**Proof.** Write \(M = \mathcal{H}(M)\). In view of [L Theorem 4.2], there is an injection between the set of inclusions of \(R[\Theta; f]\)-submodules \(N \subseteq M\) and the set of surjections of \(F\)-finite \(F\)-modules \(M \rightarrow N\) hence it is enough to show that there are finitely many such surjections. By [L Theorem 2.8] the kernels of these surjections are \(F\)-finite \(F\)-submodules of \(M\) hence it is enough to show that \(M\) has finitely many submodules. Assume this statement is false and choose a counterexample \(M\) with infinitely many submodules.

All objects in the category of \(F\)-finite \(F\)-modules have finite length (cf. [L Theorem 3.2]) hence we may assume that among all counterexamples \(M\) has minimal length. By [H Corollary 5.2] the isomorphism class of any simple \(F\)-finite \(F\)-module is a finite set and the set of simple submodules of \(M\) belong to finitely many of these isomorphism classes, namely those occurring as factors in a composition series for \(M\). We deduce that there are finitely many simple \(F\)-finite \(F\)-submodules of \(M\). Since \(M\) has infinitely many \(F\)-finite \(F\)-submodules, there must be a simple \(F\)-finite \(F\)-submodule \(P \subset M\) contained in infinitely many \(F\)-finite \(F\)-submodules of \(M\). The infinite set of images of these in the quotient \(M/P\) exhibit a counterexample of smaller length. \(\square\)

**Corollary 4.5.** Let \(B \in \text{Hom}_R(F, R^n, R)\) be represented by \((B^i T)_\gamma \in \mathcal{B},\) and assume that \((B^i T) : E \rightarrow \bigoplus_{\gamma \in \Gamma} E\) is injective. Then there are finitely many \(B\)-compatible submodules of \(F, R^n\).

**Proof.** For all \(\gamma \in \Gamma\) write \(Z_\gamma = \{v \in E^n | B^i T v\}\) and let \(C_\gamma\) be a matrix with columns in \(R^n\) be such that \(Z_\gamma = \text{ann}_{E^n} C_\gamma\). If \(\text{Im} C_\gamma \subsetneq mR^n\) for all \(\gamma \in \Gamma\), then \(\sum_{\gamma \in \Gamma} \text{Im} C_\gamma\) is not the whole of \(R^n\), and if \(C\) is a matrix whose columns generate \(\sum_{\gamma \in \Gamma} \text{Im} C_\gamma\), for any non-zero \(v \in \text{ann}_{E^n} C_\gamma\), we have \((B^i T) v = 0\) for all \(\gamma \in \Gamma\). We conclude that there exists a \(\gamma \in \Gamma\) such that, \(\text{Im} C_\gamma = R^n\), i.e., that the Frobenius map \(B^i T\) on \(E^n\) has no nilpotents. For this \(\gamma \in \Gamma\), Theorem 4.4 shows that \(E^n\) has finitely many \(R[\Theta; f]\)-submodules where the action of \(\Theta\) is given by \(B^i T\).

Let \(V\) be an \(R\)-submodule of \(R^n\) and fix a matrix \(A\) whose columns generate \(V\). Theorem 4.3 implies that if \(F_* V \subseteq F_* R^n\) is \(B\)-compatible then \(\text{ann}_{E^n} A^i \subseteq E^n\) is an \(R[\Theta; f]\)-submodule of \(E^n\) with the Frobenius action given by \(B^i T\) for all \(\gamma \in \Gamma\), and hence there are finitely many such \(B\)-compatible submodules. \(\square\)
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SOME PROPERTIES AND APPLICATIONS OF $F$-FINITE $F$-MODULES

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Abstract. M. Hochster’s work in [H] has shown that $F$-finite $F$-modules over regular local rings have finitely many $F$-submodules. In this paper we apply this theorem to prove that morphisms of $F$-finite $F$-modules have a particularly simple form and we also show that there exist finitely many submodules compatible with a given Frobenius near-splitting thus generalizing a similar result in [BB] to the case where the base ring is not $F$-finite.

1. Introduction

The purpose of this paper is to describe several applications of finiteness properties of $F$-finite $F$-modules recently discovered by M. Hochster in [H] to the study of Frobenius maps on injective hulls, Frobenius near-splittings and to the nature of morphisms of $F$-finite $F$-modules.

Throughout this paper $(R,m)$ shall denote a complete regular local ring of prime characteristic $p$. At the heart of everything in this paper is the Frobenius map $f : R \to R$ given by $f(r) = r^p$ for $r \in R$. We can use this Frobenius map to define a new $R$-module structure on $R$ given by $r \cdot s = r^p s$; we denote this $R$-module $F_* R$. We can then use this to define the Frobenius functor from the category of $R$-modules to itself: given an $R$-module $M$ we define $F_* R \otimes_R M$ with $R$-module structure given by $r(s \otimes m) = rs \otimes m$ for $r, s \in R$ and $m \in M$. Henceforth we shall abbreviate $F_* R$ to $F$ for the sake of readability.

Let $R[\Theta; f]$ be the skew polynomial ring which is the free $R$-module $\bigoplus_{i=0}^{\infty} R \Theta^i$ with multiplication $\Theta r = r^p \Theta$ for all $r \in R$. As in [K1], $\mathcal{C}$ shall denote the category of $R[\Theta; f]$-modules which are Artinian as $R$-modules. For any two such modules $M, N$, we denote the morphisms between them in $\mathcal{C}$ with $\text{Hom}_{R[\Theta, f]}(M, N)$; thus an element $g \in \text{Hom}_{R[\Theta, f]}(M, N)$ is an $R$-linear map such that $g(\Theta a) = \Theta g(a)$ for all $a \in M$. The first main result of this paper (Theorem 3.3) shows that under some conditions on $N$, $\text{Hom}_{R[\Theta, f]}(M, N)$ is a finite set.

An $F$-module (cf. the seminal paper [L] for an introduction to $F$-modules and their properties) over the ring $R$ is an $R$-module $\mathcal{M}$ together with an $R$-module isomorphism $\theta_{\mathcal{M}} : \mathcal{M} \to F(\mathcal{M})$. This isomorphism $\theta_{\mathcal{M}}$ is the structure morphism of $\mathcal{M}$.

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A morphism of $F$-modules $M \to N$ is an $R$-linear map $g$ which makes the following diagram commute

$$
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \theta_M
\end{array}
\end{array}
\xrightarrow{g}
\begin{array}{c}
\begin{array}{c}
N \\
\downarrow \theta_N
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F(M) \\
\downarrow \theta_{F(M)}
\end{array}
\end{array}
\xrightarrow{F(g)}
\begin{array}{c}
\begin{array}{c}
F(N) \\
\downarrow \theta_{F(N)}
\end{array}
\end{array}
\end{array}
$$

where $\theta_M$ and $\theta_N$ are the structure isomorphisms of $M$ and $N$, respectively. We denote $\text{Hom}_F(M, N)$ the $R$-module of all morphism of $F$-modules $M \to N$.

Given any finitely generated $R$-module $M$ and $R$-linear map $\beta : M \to F(M)$ one can obtain an $R$-module

$$M = \lim_{\to} \left( M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \cdots \right).$$

Since

$$F(M) = \lim_{\to} \left( F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} F^3(M) \xrightarrow{F^3(\beta)} \cdots \right) = M$$

we obtain an isomorphism $M \cong F(M)$, and hence $M$ is an $F$-module. Any $F$-module which can be constructed as a direct limit as $M$ above is called an $F$-finite $F$-module with generating morphism $\beta$.

There is a close connection between $R[\Theta; f]$-modules and $F$-finite $F$-modules given by Lyubeznik’s Functor from $\mathcal{C}$ to the category of $F$-finite $F$-modules which is defined as follows (see section 4 in [1] for the details of the construction.) Given an $R[\Theta; f]$-module $M$ one defines the $R$-linear map $\alpha : F(M) \to M$ by $\alpha(r \otimes m) = r\Theta m$; an application of Matlis duality then yields an $R$-linear map $\alpha^\vee : M^\vee \to F(M)^\vee \cong F(M^\vee)$ and one defines

$$\mathcal{H}(M) = \lim_{\to} \left( M^\vee \xrightarrow{\alpha^\vee} F(M^\vee) \xrightarrow{F(\alpha^\vee)} F^2(M^\vee) \xrightarrow{F^2(\alpha^\vee)} \cdots \right).$$

Since $M$ is an Artinian $R$-module, $M^\vee$ is finitely generated and $\mathcal{H}(M)$ is an $F$-finite $F$-module with generating morphism $M^\vee \xrightarrow{\alpha^\vee} F(M^\vee)$. This construction is functorial and results in an exact covariant functor from $\mathcal{C}$ to the category of $F$-finite $F$-modules.

Later in this paper we will need the following related constructions. Following [1] we shall denote $\mathcal{D}$ the category of all $R$-linear maps $M \to F(M)$ where $M$ is any finitely generated $R$-module, and where a morphism between $M \xrightarrow{a} F(M)$ and $N \xrightarrow{b} F(N)$ is a commutative diagram of $R$-linear maps

$$
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow a
\end{array}
\end{array}
\xrightarrow{\mu}
\begin{array}{c}
\begin{array}{c}
N \\
\downarrow b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F(M) \\
\downarrow \theta_{F(M)}
\end{array}
\end{array}
\xrightarrow{F(\mu)}
\begin{array}{c}
\begin{array}{c}
F(N) \\
\downarrow \theta_{F(N)}
\end{array}
\end{array}
\end{array}
$$

Section 3 of [1] constructs a pair of functors $\Delta : \mathcal{C} \to \mathcal{D}$ and $\Psi : \mathcal{D} \to \mathcal{C}$ with the property that for all $L \in \mathcal{C}$, the $R[\Theta; f]$-module $\Psi \circ \Delta(L)$ is canonically isomorphic to $L$ and for all $D = (B \xrightarrow{\phi} F(B)) \in \mathcal{D}$, $\Delta \circ \Psi(D)$ is canonically isomorphic to $D$. The functor $\Delta$ amounts to the “first step” in the construction of Lyubeznik’s functor $\mathcal{H}$: for $L \in \mathcal{C}$ we define the
$R$-linear map $\alpha : F(L) \to L$ to be the one given above and we let $\Delta(L)$ to be the map $\alpha^\vee : L^\vee \to F(L)^\vee \cong F(L^\vee)$ (cf. section 3 in [K1] for the details of the construction.)

The main result in [H] is the surprising fact that for $F$-finite $F$-modules $M$ and $N$, $\text{Hom}_F(N, M)$ is a finite set. In section 3 of this paper we exploit this fact to prove the second main result in this paper (Theorem 3.4) to show the following. Let $\gamma : M \to F(M)$ and $\beta : N \to F(N)$ be generating morphisms for $M$ and $N$. Given an $R$-linear map $g$ which makes the following diagram commute,

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow{g} & & \downarrow{F(g)} \\
M & \xrightarrow{\gamma} & F(M)
\end{array}
\]

one can extend that diagram to

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\
\downarrow{g} & & \downarrow{F(g)} & & \downarrow{F^2(g)} & & \cdots \\
M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots
\end{array}
\]

and obtain a map between the direct limits of the horizontal sequences, i.e., an element in $\text{Hom}_R(N, M)$. We prove that all elements in $\text{Hom}_R(N, M)$ arise in this way (cf. Theorem 3.4), thus morphisms of $F$-finite $F$-modules have a particularly simple form. This answers a question implicit in [L, Remark 1.10(b)].

Finally, in section 4 we consider the module $\text{Hom}_R(F_*, R^n)$ of near-splittings of $F_*, R^n$. We establish a correspondence between these near-splittings and Frobenius actions on $E^n$ which enables us to prove the third main result in this paper (Theorem 4.5) which asserts that given a near-splitting $\phi$ corresponding to an injective Frobenius action, there are finitely many $F_*$-submodules $V \subseteq F_*, R^n$ such that $\phi(V) \subseteq V$. This generalizes a similar result in [BB] to the case where $R$ is not $F$-finite.

Our study of Frobenius near-splittings is based on the study of its dual notion, i.e., Frobenius maps on the injective hull $E = E_R(R/m)$ of the residue field of $R$. This injective hull is given explicitly as the module of inverse polynomials $K[x_1^-, \ldots, x_d^-]$ where $x_1, \ldots, x_d$ are minimal generators of the maximal ideal of $R$ (cf. [BS], §12.4). Thus $E$ has a natural $R[T; f]$-module structure extending $T\lambda x_1^{-\alpha_1} \cdots x_1^{-\alpha_d} = \lambda^p x_1^{-p\alpha_1} \cdots x_d^{-p\alpha_d}$ for $\lambda \in K$ and $\alpha_1, \ldots, \alpha_d > 0$. We can further extend this to a natural $R[T; f]$-module structure on $E^n$ given by

\[
T \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_n \end{pmatrix}.
\]

Throughout this paper $T$ will denote this natural Frobenius map, while $\Theta$ will be used for general Frobenius maps.
The results of section 2 will follow from the fact that there is a dual correspondence between Frobenius near-splittings and sets of $R[\Theta; f]$-module structures on $E^\alpha$.

2. Frobenius maps of Artinian modules and their stable submodules

Given an Artinian $R$-module $M$ we can embed $M$ in $E^\alpha$ for some $\alpha \geq 0$ and extend this inclusion to an exact sequence

$$0 \to M \to E^\alpha \xrightarrow{A^t} E^\beta \to \ldots$$

where $A^t \in \text{Hom}_R(E^\alpha_R, E^\beta_R)$. In our setup Matlis duality gives $\text{Hom}_R(E^\alpha_R, E^\alpha_R) \cong R$ and so $A^t \in \text{Hom}_R(E^\alpha_R, E^\beta_R) \cong \text{Hom}_R(R^\alpha, R^\beta)$ is a $\beta \times \alpha$ matrix with entries in $R$. Henceforth in this section we will describe certain properties of Artinian $R$-modules in terms of their representations as kernels of matrices with entries in $R$. We shall denote $M_{\alpha,\beta}$ the set of $\alpha \times \beta$ matrices with entries in $R$ and for any such matrix $A$ we will write $A^p$ to denote the matrix obtained by raising each of its entries to the $p$th power.

We now explore the duality between $E^\alpha$ with a given $R[\Theta; f]$-module structure and $R$-linear maps $R^\alpha \to R^\alpha$ for $\alpha \geq 1$ given by the functors $\Delta$ and $\Psi$ defined in section 1. Under this duality the $R[\Theta; f]$-module structure corresponding to the map $(R^\alpha \to R^\alpha) \in \mathcal{D}$ given by multiplication by $B \in M_{\alpha,\alpha}$ is given by $\Theta = B^tT$ where $T$ is the natural Frobenius map on $E^\alpha$ described in section 1.

**Proposition 2.1.** Let $M = \ker A^t \subseteq E^\alpha$ be an Artinian $R$-module where $A \in M_{\alpha,\beta}$. Let $B = \{B \in M_{\alpha,\alpha} \mid \text{Im} BA \subseteq \text{Im} A^p\}$. For any $R[\Theta; f]$-module structure on $M$, $\Delta(M)$ can be identified with an element in $\text{Hom}_R(\text{Coker} A, \text{Coker} A^p)$ and thus represented by multiplication by some $B \in B$. Conversely, any such $B$ defines an $R[\Theta; f]$-module structure on $M$ which is given by the restriction to $M$ of the Frobenius map $\phi : E^\alpha \to E^\alpha$ defined by $\phi(v) = B^tT(v)$ where $T$ is the natural Frobenius map on $E^\alpha$.

**Proof.** Matlis duality gives an exact sequence $R^\alpha \xrightarrow{\Delta} R^\alpha \to M^\vee \to 0$ hence

$$\Delta(M) \in \text{Hom}_R(M^\vee, F_R(M^\vee)) \cong \text{Hom}_R(\text{Coker} A, \text{Coker} A^p).$$

Let $\Delta(M)$ be the map $g : \text{Coker} A \to \text{Coker} A^p$.

In view of Theorem 3.1 in [K1] we only need to show that any such $R$-linear map is given by multiplication by an $B \in B$, and that any such $B$ defines an element in $\Delta(M)$.

Using the freeness of $R^\alpha$, we find a map $g'$ which makes the following diagram commute, where $q_1$ and $q_2$ are quotient maps. The map $g'$ is given by multiplication by some $\alpha \times \alpha$ matrix $B \in B$. Conversely, any such matrix $B$ defines a map $g$ making the diagram above commute, and $\Psi(g)$ gives a $R[\Theta; f]$-module structure on $M$ as described in the last part of the proposition. 

$\square$
Notation 2.2. We shall henceforth describe Artinian $R$-modules with a given $R[\Theta; f]$-module structure in terms of the two matrices in the statement of Proposition 2.1 and talk about Artinian $R$-modules $M = \text{Ker} A' \subseteq E^\alpha$ where $A \in M_{\alpha,\beta}$ with $R[\Theta; f]$-module structure given by $B \in M_{\alpha,\alpha}$.

3. Morphisms in $\mathcal{C}$

In this section we raise two questions. The first of these asks when for given $R[\Theta; f]$-modules $M, N$, the set $\text{Hom}_R[\Theta; f](M, N)$ is finite; later in this section we prove that this holds when $N$ has no $\Theta$-torsion. The following two examples illustrate why this set is not finite in general, and why it is finite in a special simple case.

Example 3.1. Let $\mathbb{K}$ be an infinite field of prime characteristic $p$ and let $R = \mathbb{K}[x]$. Let $M = \text{ann}_E xR$ and fix an $R[\Theta; f]$-module structure on $M$ given by $\Theta a = x^pTa$ where $T$ is the standard Frobenius action on $E$. Note that $\Theta M = 0$ and that for all $\lambda \in \mathbb{K}$ the map $\mu_\lambda : M \to M$ given by multiplication by $\lambda$ is in $\text{Hom}_R[\Theta; f](M, M)$, and hence this set is infinite.

Example 3.2. Let $I, J \subseteq R$ be ideals, and fix $u \in (I^{[p]} : I)$ and $v \in (J^{[p]} : J)$. Endow $\text{ann}_E I$ and $\text{ann}_E J$ with $R[\Theta; f]$-module structures given by $\Theta a = uTa$ and $\Theta b = vTb$ for $a \in \text{ann}_E I$ and $b \in \text{ann}_E J$ where $T$ is the standard Frobenius map on $E$.

If $g : \text{ann}_E I \to \text{ann}_E J$ is $R$-linear, an application of Matlis duality yields $g^\vee : R/J \to R/I$ and we deduce that $g$ is given by multiplication by an element in $w \in (I : J)$. If in addition $g \in \text{Hom}_R[\Theta; f]\text{ann}_E I, \text{ann}_E J)$, we must have $wuTa = g(\Theta a) = \Theta g(a) = vTwa = vw^pTa$, for all $a \in \text{ann}_E I$, hence $(vw^p - uw)T\text{ann}_E I = 0$ and $vw^p - uw \in I^{[p]}$. The finiteness of $\text{Hom}_R[\Theta; f]\text{ann}_E I, \text{ann}_E J)$ translates in this setting to the finiteness of the set of solutions modulo $I^{[p]}$ for the variable $w$ of the equation above, and it is not clear why this set should be finite. However, if we simplify to the case where $I = J = 0$, the set of solutions of $vw^p - uw = 0$ over the the fraction field of $R$ has at most $p$ elements, and in this case we can deduce that $\text{Hom}_R[\Theta; f](E, E)$ has also at most $p$ elements.

As in [L], for any $R[\Theta; f]$-module $M$ we define the submodule of nilpotent elements to be $\text{Nil}(M) = \{a \in M \mid \Theta^e a = 0 \text{ for some } e \geq 0\}$. We recall that when $M$ is an Artinian $R$-module there exists an $\eta \geq 0$ such that $\Theta^\eta \text{Nil}(M) = 0$ (cf. [HS] Proposition 1.11 and [L] Proposition 4.4)). We also define $M_{\text{red}} = M/\text{Nil}(M)$ and $M^* = \cap_{e \geq 0} R\Theta^e M$ where $R\Theta^e M$ denotes the $R$-module generated by $\{\Theta^e a \mid a \in M\}$. We also note that when $M$ is an $R[\Theta; f]$-module which is Artinian as an $R$-module, there exists an $e \geq 0$ such that $M^* = R\Theta^e M$ and also $(M_{\text{red}})^* = (M^*)_{\text{red}}$ (cf. section 4 in [K2]).

Theorem 3.3. Let $M, N$ be $R[\Theta; f]$-modules and let $\phi \in \text{Hom}_R[\Theta; f](M, N)$. We have $\mathcal{K}(\text{Im} \phi) = 0$ if and only if $\phi(M) \subseteq \text{Nil}(N)$ and, consequently, if $\text{Nil}(N) = 0$, the map $\mathcal{K} : \text{Hom}_R[\Theta; f](M, N) \to \text{Hom}_{R[\Theta; f]}(\mathcal{K}(N), \mathcal{K}(M))$ is an injection and $\text{Hom}_R[\Theta; f](M, N)$ is a finite set.
Proof. We apply \( \mathcal{H} \) to the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & N \\
\downarrow & & \downarrow \\
\text{Im } \phi & \xrightarrow{\phi} & \text{Im } \phi
\end{array}
\]

to obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}(N) & \xrightarrow{\mathcal{H}(\phi)} & \mathcal{H}(\text{Im } \phi) \\
\downarrow & & \downarrow \\
\mathcal{H}(M) & & \mathcal{H}(M)
\end{array}
\]

Now \( \mathcal{H}(\phi) = 0 \) if and only if \( \mathcal{H}(\text{Im } \phi) = 0 \), and by [L, Theorem 4.2] this is equivalent to \( (\text{Im } \phi)^* = 0 \).

Choose \( \eta \geq 0 \) such that \( \Theta^\eta \text{Nil}(N) = 0 \) and choose \( e \geq 0 \) such that \( (\text{Im } \phi)^* = R\Theta^e \text{Im } \phi \).

Now

\[
(\text{Im } \phi)^* = 0 \iff R\Theta^\eta R\Theta^e \phi(M) = 0 \\
\iff R\Theta^{\eta+e} \phi(M) = 0 \\
\iff \text{Im } \phi \subseteq \text{Nil}(N)
\]

The second statement now follows immediately. \( \square \)

The second main result in this section, Theorem 3.4, shows that all morphisms of \( F \)-finite \( F \)-modules arise as images of maps of \( R[\Theta; f] \)-modules under Lyubeznik’s functor \( \mathcal{H} \).

**Theorem 3.4.** Let \( M \) and \( N \) be \( F \)-finite \( F \)-modules. For every \( \phi \in \text{Hom}_{F^n}(N,M) \) there exist generating morphisms \( \gamma : M \to F(M) \in \mathcal{D} \) and \( \beta : N \to F(N) \in \mathcal{D} \) for \( M \) and \( N \), respectively, and a morphism (in the category \( \mathcal{D} \))

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow & & \downarrow \text{F(g)} \\
M & \xrightarrow{\gamma} & F(M)
\end{array}
\]

such that \( \phi = \mathcal{H}(\Psi(g)) \), i.e., such that \( \phi \) is the map of direct limits

\[
\begin{array}{cccccc}
N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\
\downarrow & & \downarrow \text{F(g)} & & \downarrow \text{F^2(g)} & & \cdots \\
M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots
\end{array}
\]

**Proof.** Choose any generating morphisms

\[
N = \lim\left( N \xrightarrow{\beta} F(N) \xrightarrow{F(\beta)} F^2(N) \xrightarrow{F^2(\beta)} \cdots \right)
\]
and
\[ M = \lim_{\rightarrow} \left( M \xrightarrow{\gamma} F(M) \xrightarrow{F(\gamma)} F^2(M) \xrightarrow{F^2(\gamma)} \ldots \right) \]

and fix any \( \phi \in \text{Hom}_{R}(N,M) \).

For all \( j \geq 0 \) let \( \phi_j \) be the restriction of \( \phi \) to the image of \( F^j(N) \) in \( N \).

The fact that \( \phi \) is a morphism of \( F \)-modules implies that for every \( j \geq 0 \) we have a commutative diagram

\[
\begin{array}{ccc}
F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) \\
\downarrow \quad \theta_{M} & & \downarrow \theta_{N} \\
M & \xrightarrow{\sim} & F(M)
\end{array}
\]

where \( \theta_{M} \) and \( \theta_{N} \) are the structure isomorphisms of \( M \) and \( N \), respectively, and where the compositions of the vertical maps are \( \phi_j \) and \( F(\phi_j) \). Repeated applications of the Frobenius functor yields a commutative diagram

\[
\begin{array}{ccc}
F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) \\
\downarrow \quad \phi_j & & \downarrow F(\phi_j) \\
M & \xrightarrow{\sim} & F(M)
\end{array}
\]

and we can now extend this commutative diagram to the left to obtain

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & F(N) \\
\downarrow \phi_0 & & \downarrow F(\phi) \\
M & \xrightarrow{\sim} & F(M)
\end{array}
\]

This commutative diagram defines an \( R \)-linear map \( \psi_j : N \to M \). Furthermore, we show next that this \( \psi_j \) is a map of \( F \)-modules, i.e., that for all \( j \geq 0 \), \( F(\psi_j) \circ \theta_{N} = \theta_{M} \circ \psi_j \). Fix \( j \geq 0 \) and abbreviate \( \psi = \psi_j \).

Pick any \( a \in N \) represented as an element of \( F^e(N) \). If \( e < j \) then the fact that \( \phi \) is a morphism of \( F \)-modules implies that

\[ \theta_{M} \circ \psi(a) = \theta_{M} \circ \phi(a) = F(\phi) \circ \theta_{N}(a) = F(\psi) \circ \theta_{N}(a). \]
Assume now that \( e \geq j \); we have
\[
\theta_M \circ \psi(a) = \theta_M \circ \theta_M^{-1} \circ F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j)(a)
\]
and
\[
F(\psi) \circ \theta_N(a) = F(\theta_M^{-1} \circ F(\theta_M^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_M^{-1}) \circ F^{e-j}(\phi_j))(F^{e}(\beta)(a))
\]
where the penultimate inequality follows from the fact that \( \phi \) is a morphism of \( F \)-modules.

Consider now the set \( \{\psi_i\}_{i \geq 0} \); it is a finite set according to Theorem 5.1 in \([H]\), hence we can find a sequence \( 0 \leq i_1 < i_2 < \cdots \) such that \( \psi_{i_1} = \psi_{i_2} = \cdots \). By replacing \( N \) and \( M \) with \( F^{e_1}(N) \) and \( F^{e_1}(M) \) we may assume that \( i_1 = 0 \).

Pick \( j \geq 0 \) so that \( \phi \) maps the image of \( N \) in \( N \) into \( F^j(M) \). Since \( M \cong F^j(M) \) we may replace \( M \) with \( F^j(M) \) and assume that \( \phi(\text{Im } N) \subseteq \text{M} \) and hence also that for all \( e \geq 0 \), \( F^e(\phi) \) maps the image of \( F^e(N) \) in \( N \) into \( F^e(M) \).

Fix now any \( e \geq 0 \) and pick any \( i_k > e \); the fact that \( \psi_0 = \psi_i \) implies that for all \( a \in F^e(N) \), \( F^e(\phi_0)(a) = \psi_0(a) = \psi_i(a) = \phi(a) \) and since this holds for all \( e \geq 0 \) we deduce that \( \phi \) is induced from the commutative diagram

\[
\begin{array}{cccccc}
N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} \cdots \\
\phi_0 & \downarrow & F(\phi_0) & \downarrow & F^2(\phi_0) & \\
M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} \cdots \\
\end{array}
\]

An application of the functor \( \Psi \) to the leftmost square in the commutative diagram above yields a morphism of \( R[\Theta; f]-\text{modules} \) \( g : M \to N \) and \( \phi = \mathcal{H}(g) \).

\[\square\]

4. Applications to Frobenius splittings

For any \( R \)-module \( M \) let \( F_n M \) denote the additive Abelian group \( M \) with \( R \)-module structure given by \( r \cdot a = r^na \) for all \( r \in R \) and \( a \in M \). In this section we study the module \( \text{Hom}_R(F_n R^n, R^n) \) of near-splittings of \( F_n R^n \). Given such an element \( \phi \in \text{Hom}_R(F_n R^n, R^n) \) we will describe the submodules \( V \subseteq F_n R^n \) for which \( \phi(V) \subseteq V \). These submodules in the case \( n = 1 \), known as \( \phi \)-compatible ideals, are of significant importance in algebraic geometry (cf. \([BK]\) for a study of applications of Frobenius splittings and their compatible submodules in algebraic geometry.) We will prove that under some circumstances these form a finite set and thus generalize a result in \([BB]\) obtained in the \( F \)-finite case.

We first exhibit the following easy implication of Matlis duality necessary for the results of this section.
Lemma 4.1. For any (not necessarily finitely generated) $R$-module $M$, $\text{Hom}_R(R^\vee,M^\vee) \cong \text{Hom}_R(R^\vee,M^\vee)$.

Proof. For all $a \in E$ let $h_a \in \text{Hom}_R(R,E)$ denote the map sending $1$ to $a$.

For any $\phi \in \text{Hom}_R(R,M)$, $\phi^\vee \in \text{Hom}_R(R^\vee,M^\vee)$ is defined as $(\phi^\vee(h_a))(m) = \phi(m)a$ for any $m \in M$ and $a \in E$. For any $\psi \in \text{Hom}_R(R^\vee,M)$ we define $\tilde{\psi} \in \text{Hom}_R(R,M)$ as $\left(\tilde{\psi}(m)\right)(a) = (\psi(h_a))(m)$ for all $a \in E$ and $m \in M$. Note that the function $\psi \mapsto \tilde{\psi}$ is $R$-linear.

Let $\psi \in \text{Hom}_R(R^\vee,M^\vee)$ and fix an $m \in M$. Note that for all $a \in E$

$$\tilde{\psi}^\vee(h_a)(m) = \tilde{\psi}(m)a$$

when we view $\tilde{\psi}$ as an element in $\text{Hom}_R(M,R)$. After we identify $\text{Hom}_R(M,E^\vee)$ with $\text{Hom}_R(M,R)$ we can write

$$\tilde{\psi}^\vee(h_a)(m) = \tilde{\psi}(m)(a) = \psi(h_a)(m)$$

thus $\tilde{\psi}^\vee = \psi$.

It is now enough to show that for all $\phi \in \text{Hom}_R(M,R)$, $\tilde{\phi}^\vee = \phi$, and indeed for all $a \in E$ and $m \in M$

$$\left(\tilde{\phi}^\vee(m)\right)(a) = (\phi^\vee(h_a))(m) = \phi(m)a,$$

i.e., $\left(\tilde{\phi}^\vee(m)\right) \in \text{Hom}_R(E,E)$ is given by multiplication by $\phi(m)$ and so under the identification of $\text{Hom}_R(E,E)$ with $R$, $\tilde{\phi}^\vee$ is identified with $\phi$. \qed

We can now prove a generalization Lemma 1.6 in [F] in the form of the next two theorems.

Theorem 4.2. (a) The $F_\ast R$-module $\text{Hom}_R(F_\ast R,E)$ is injective of the form $\bigoplus_{\gamma \in T} F_\ast E \oplus H$ where $T$ is non-empty, $H = \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda)$, $\Lambda$ is a (possibly empty) set, $P_\lambda$ is a non-maximal prime ideal of $R$ for all $\lambda \in \Lambda$ and $E(R/P_\lambda)$ denotes the injective hull of $R/P_\lambda$.

(b) Write $\mathcal{B} = \text{Hom}_{F_\ast R}(E,\bigoplus_{\gamma \in T} F_\ast E) \subseteq \prod_{\gamma \in T} \text{Hom}_{F_\ast R}(E,F_\ast E)$. We have

$$\text{Hom}_R(F_\ast R,R) \cong \mathcal{B} \subseteq \prod_{\gamma \in T} \text{Hom}_{F_\ast R}(E,F_\ast E) \cong \prod_{\gamma \in T} F_\ast RT$$

where $T$ is the standard Frobenius map on $E$.

(c) The set $T$ is finite if and only if $F_\ast \mathbb{K}$ is a finite extension of $\mathbb{K}$, in which case $

\#T = 1$.

Proof. The functors $\text{Hom}_R(-,E) = \text{Hom}_R(- \otimes_{F_\ast R} F_\ast R,E)$ and $\text{Hom}_{F_\ast R}(-,\text{Hom}_R(F_\ast R,E))$ from the category of $F,R$-modules to itself are isomorphic by the adjointness of Hom and $\otimes$, and since $\text{Hom}_R(-,E)$ is an exact functor, so is $\text{Hom}_{F_\ast R}(-,\text{Hom}_R(F_\ast R,E))$, thus $\text{Hom}_R(F_\ast R,E)$ is an injective $F_\ast R$-module and hence of the form $G \oplus H$ where $G$ is a direct sum of copies of $F_\ast E$ and $H$ is as in the statement of the Theorem. Write $G = \bigoplus_{\gamma \in T} F_\ast E$. To finish establishing (a) we need only verify that $T \neq \emptyset$ and we do this below.
Pick any \( h \in \text{Hom}_R \left( E, \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda) \right) \). For any \( a \in E \), \( h(a) \) can be written as a finite sum \( b_{\lambda_1} + \cdots + b_{\lambda_s} \) where \( \lambda_1, \ldots, \lambda_s \in \Lambda \) and \( b_{\lambda_i} \in F_\ast E(R/P_\lambda), \ldots, b_{\lambda_s} \in F_\ast E(R/P_\lambda). \)

Use prime avoidance to pick a \( z \in m \setminus \cup_{\lambda \in \Lambda} P_\lambda \); now \( z \) and its powers act invertibly on each of \( F_\ast E(R/P_\lambda), \ldots, F_\ast E(R/P_\lambda) \) while a power of \( z \) kills \( a \), and so we must have \( h(a) = 0 \).

We deduce that \( \text{Hom}_R \left( E, \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda) \right) = 0 \) and

\[
\text{Hom}_R \left( E, \text{Hom}_R (F, R, E) \right) \cong \text{Hom}_R \left( E, G \oplus \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda) \right)
\]

\[
\cong \text{Hom}_R (E, G) \oplus \text{Hom}_R \left( E, \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda) \right)
\]

\[
\cong \text{Hom}_R (E, G)
\]

\[
\cong \text{Hom}_R (E, \oplus_{\gamma \in \Gamma} F/E)
\]

\( = B \).

Now \( \text{Hom}_R (E, F, E) \) is the \( R \)-module of Frobenius maps on \( E \) which is isomorphic as an \( F, R \) module to \( F, RT \) and we conclude that \( \text{Hom}_R (E, \text{Hom}_R (F, R, E)) \subseteq \prod_{\gamma \in \Gamma} F, RT \).

An application of the Matlis dual and Lemma 4.1 now gives

\[
\text{Hom}_R (F, R, R) \cong \text{Hom}_R (E, \text{Hom}_R (F, R, E))
\]

and (b) follows.

Write \( k = R/m \) and note that \( F, k \) is the field extension of \( k \) obtained by adding all \( p \)th roots of elements in \( k \). We next compute the cardinality of \( \Gamma \) as the \( F, k \)-dimension of \( \text{Hom}_{F, k} (F, k, G) \). A similar argument to the one above shows that

\[
\text{Hom}_{F, k} \left( F_\ast k, \bigoplus_{\lambda \in \Lambda} F_\ast E(R/P_\lambda) \right) = 0
\]

hence \( \text{Hom}_{F, k} (F_\ast k, G) = \text{Hom}_{F, k} (F_\ast k, \text{Hom}_R (F, R, E)). \)

We may identify \( \text{Hom}_{F, k} (F_\ast k, \text{Hom}_R (F, R, E)) \) and \( \text{Hom}_{F, R} (F_\ast k, \text{Hom}_R (F, R, E)). \) Another application of the adjointness of Hom and \( \otimes \) gives

\[
\text{Hom}_{F, R} (F_\ast k, \text{Hom}_R (F, R, E)) \cong \text{Hom}_R (F_\ast k \otimes_{F, R} F, R, E) \cong \text{Hom}_R (F_\ast k, k).
\]

Since \( mF_\ast k = 0 \), we see that the image of any \( \phi \in \text{Hom}_R (F_\ast k, E) \) is contained in \( \text{ann}_E m \cong k \) and we deduce that \( \text{Hom}_R (F_\ast k, E) \cong \text{Hom}_R (F, k, k). \) We can now conclude that the cardinality of \( \Gamma \) is the \( F, k \)-dimension of \( \text{Hom}_R (F, k, k) \). In particular \( \Gamma \) cannot be empty and (a) follows.

If \( U \) is a \( k \)-basis for \( F, k \) containing \( 1 \in F, k \),

\[
(1) \quad \text{Hom}_k (F, k, k) \cong \prod_{u \in U} \text{Hom}_k (k, k)
\]

and when \( U \) is finite, this is a one-dimensional \( F, k \)-vector space spanned by the projection onto \( k1 \subset F, k \). If \( U \) is not finite, the dimension as \( k \)-vector space of \( (1) \) is at least \( 2^{\# U} \) hence \( \text{Hom}_k (F, k, k) \) cannot be a finite-dimensional \( F, k \)-vector space. \( \square \)
Our next result is to establish a connection between submodules of \( R^n \) compatible with a given \( B \in \text{Hom}_R(F, R^n, R^n) \) and submodules of \( E^n \) fixed under a sequence of Frobenius actions determined by \( B \).

Note that the previous theorem allows us to view elements of \( \text{Hom}_R(F, R^n, R^n) \cong \text{Hom}_R(F, R, R)^{n \times n} = \mathfrak{B}^{n \times n} \) as elements in \( \prod_{\gamma \in \Gamma} F, R^{n \times n}T \), i.e., as sequences \( (B, T)_{\gamma \in \Gamma} \) where each \( B_\gamma \) is an \( n \times n \) matrix with entries in \( F, R \) and \( T \) is the natural Frobenius action on \( E^n \).

**Theorem 4.3.** Let \( G = \oplus_{\gamma \in \Gamma} F, E \) and \( \mathfrak{B} \) be as in Theorem 4.2. Let \( B \in \text{Hom}_R(F, R^n, R^n) \) be represented by \( (B_\gamma, T)_{\gamma \in \Gamma} \in \mathfrak{B}^{n \times n} \). For all \( \gamma \in \Gamma \) consider \( E^n \) as an \( R[\Theta; f] \)-module with \( \Theta_\gamma v = B_\gamma^tTv \) for all \( v \in E^n \). Let \( V \) be an \( R \)-submodule of \( R^n \) and fix a matrix \( A \) whose columns generate \( V \). If \( B(F, V) \subseteq V \), then \( \text{ann}_{E^n} A^t \) is a \( R[\Theta; f] \) submodule of \( E^n \) for all \( \gamma \in \Gamma \).

**Proof.** Apply the Matlis dual to the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & F, V \\
| & B & | \downarrow \\
0 & \longrightarrow & V
\end{array}
\quad \begin{array}{ccc}
F, R^n & \longrightarrow & F, R^n/F, A \\
| & B & | \downarrow \\
R^n & \longrightarrow & R^n/V
\end{array}
\quad \begin{array}{c}
\longrightarrow \\
\downarrow \text{ann} \\
0
\end{array}
\]

where the rightmost vertical map is induced by the middle map to obtain

\[
\begin{array}{ccc}
0 & \longrightarrow & (R^n/V)^{\vee} \\
| & \downarrow \text{ann} & | \downarrow \text{ann} \\
0 & \longrightarrow & (F, R^n/F, V)^{\vee}
\end{array}
\quad \begin{array}{c}
E^n \\
\downarrow \text{ann} \\
\text{Hom}_R(F, R^n, E)
\end{array}
\]

Note that the previous theorem shows that

\[
\text{Hom}_R(E^n, \text{Hom}_R(F, R^n, E)) \cong \text{Hom}_R(E^n, \oplus_{\gamma \in \Gamma} F, E^n).
\]

Also note that under this isomorphism \( B^\vee \in \text{Hom}_R(E^n, \oplus_{\gamma \in \Gamma} F, E^n)^{n \times n} \) is given by \( (B^\vee_t)_{\gamma \in \Gamma} \) and that the image of \( B^\vee \) is contained in \( \oplus_{\gamma \in \Gamma} F, E^n \).

Using the presentation \( F, R^n \xrightarrow{F, A^t} F, R^n \rightarrow F, R^n/F, V \rightarrow 0 \) we obtain the exact sequence

\[
0 \rightarrow (F, R^n/F, V)^{\vee} \rightarrow \text{Hom}_R(F, R^n, E) \xrightarrow{F, A^t} \text{Hom}_R(F, R^n, E)
\]

thus

\[
(F, R^n/F, V)^{\vee} = \text{ann}_{\text{Hom}(F, R^n, E)} F, A^t.
\]

We now obtain the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{ann}_{E^n} A^t \\
| & \downarrow \text{ann} & | \downarrow \text{ann} \\
0 & \longrightarrow & \oplus_{\gamma \in \Gamma} \text{ann}_{F, E^n} F, A^t
\end{array}
\quad \begin{array}{c}
E^n \\
\downarrow \text{ann} \\
\oplus_{\gamma \in \Gamma} F, E^n
\end{array}
\]

and we deduce that \( \text{ann}_{E^n} A^t \) is a \( R[\Theta; f] \)-module for all \( \gamma \in \Gamma \). \( \square \)
Theorem 4.4. Let $M$ be an $R[\Theta; f]$-module with no nilpotents and assume $M$ is an Artinian $R$-module. Then $M$ has finitely many $R[\Theta; f]$-submodules. (Cf. Corollary 4.18 in [BB].)

Proof. Write $M = H(M)$. In view of [L, Theorem 4.2], there is an injection between the set of inclusions of $R[\Theta; f]$-submodules $N \subseteq M$ and the set of surjections of $F$-finite $F$-modules $M \twoheadrightarrow N$ hence it is enough to show that there are finitely many such surjections. By [L, Theorem 2.8] the kernels of these surjections are $F$-finite $F$-submodules of $M$ hence it is enough to show that $M$ has finitely many submodules.

All objects in the category of $F$-finite $F$-modules have finite length (cf. [L, Theorem 3.2]) and the theorem now follows from [H, Corollary 5.2(b)].

Corollary 4.5. Let $B = \text{Hom}_R(F_n, R)$ be represented by $(B^T_\gamma)_{\gamma \in \Gamma} \in \mathbb{B}^{n \times n}$, and assume that $B^T_\gamma : E^n \to E^n$ is injective for some $\gamma \in \Gamma$. Then there are finitely many $B$-compatible submodules of $F_n R^n$. In particular this holds when $n = 1$ and $(B_\gamma T)_{\gamma \in \Gamma} : E \to \bigoplus_{\gamma \in \Gamma} E$ is injective.

Proof. Let $V$ be an $R$-submodule of $R^n$ and fix a matrix $A$ whose columns generate $V$. Theorem 4.3 implies that if $F_* V \subseteq F_* R^n$ is $B$-compatible then for all $\gamma \in \Gamma$, $\text{Ann}_{E^n} A^T \subseteq E^n$ is an $R[\Theta; f]$-submodule of $E^n$ with the Frobenius action given by $B^T_\gamma T$. If there exists a $\gamma \in \Gamma$ such that $B^T_\gamma T$ is injective, then [S, Theorem 3.10] or [EH, Theorem 3.6] imply that there must finitely many $R[\Theta; f]$-submodules of $E^n$ and hence also finitely many $B$-compatible submodules of $R^n$.

Assume now that $n = 1$. For all $\gamma \in \Gamma$ write $Z_\gamma = \{ v \in E | B_\gamma T v = 0 \}$ and let $C_\gamma \subseteq R$ be the ideal for which $Z_\gamma = \text{Ann}_E C_\gamma$. If $C_\gamma \subseteq mR$ for all $\gamma \in \Gamma$, then $C = \sum_{\gamma \in \Gamma} C_\gamma \neq R$, and for any non-zero $v \in \text{Ann}_E C \neq 0$, we have $B_\gamma T v = 0$ for all $\gamma \in \Gamma$. We conclude that there exists a $\gamma \in \Gamma$ such that, $C_\gamma = R$, i.e., that the Frobenius map $B_\gamma T$ on $E$ is injective, and the last assertion of the corollary follows.

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