Some Identities Involving the Cesàro Average of the Goldbach Numbers

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Abstract—Let \( \Lambda(n) \) be the von Mangoldt function, and let \( r_G(n) := \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2) \) be the weighted sum for the number of Goldbach representations which also includes powers of primes. Let \( \tilde{S}(z) := \sum_{n \geq 1} \Lambda(n)e^{-nz} \), where \( \Lambda(n) \) is the Von Mangoldt function, with \( z \in \mathbb{C}, \text{Re}(z) > 0 \). In this paper, we prove an explicit formula for \( \tilde{S}(z) \) and the Cesàro average of \( r_G(n) \).

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1. INTRODUCTION

In this paper, we prove some identities inherent in the study of the Cesàro average of Goldbach numbers. The main result in this direction is contained in the paper of Languasco and Zaccagnini [1]. The authors proved the formula

\[
\sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{N^{2+k}}{\Gamma(k+3)} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+k+2)} N^{\rho+k+1} \\
+ \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} N^{\rho_1+\rho_2+k} + O_k(N^{k+1}),
\]

for \( k > 1 \), where \( N \) is a large natural number and \( \rho \), with or without subscripts, runs over the nontrivial zeros of the Riemann zeta function \( \zeta(s) \). The parameter \( k \) plays a central role in the study of these problems and we would like to keep it as small as possible since, for \( k = 0 \), the Cesàro weight disappears. The method of Languasco and Zaccagnini allows to study many types of additive problems; see [2], [3], and [4]. The crucial points of their technique are the relation

\[
\sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz}z^{-k-1} \tilde{S}(z)^2 \, dz,
\]

which holds for \( k > 0 \), where

\[
\tilde{S}(z) := \sum_{n \geq 1} \Lambda(n)e^{-nz}, \quad z = a + iy, \quad \text{with } a > 0, \quad y \in \mathbb{R}.
\]

is the power series of the Von Mangoldt function and \( \int_{(1/N)} \) means \( \int_{1/N+i\infty}^{1/N-i\infty} \), and the estimation

\[
\tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y),
\]

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where

\[ E(a, y) \ll \begin{cases} 1 + |z|^{1/2}, & |y| \leq a, \\ 1 + |z|^{1/2}(1 + \log^2(|y|/a)), & |y| > a. \end{cases} \]  

(4)

The proof of (3) appears in [1]. We recall that the study of \( \tilde{S}(z) \) is classical; see, for example, [5]. Recently, Goldston and Yang [6] proved, assuming RH, that

\[ \sum_{n \leq N} r_G(n) \left( 1 - \frac{n}{N} \right) = \frac{N^2}{6} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho + 3)} N^{\rho + 1} + O(N), \]

(5)

which correspond to what Languasco and Zaccagnini’s formula implies if one takes \( k = 1 \). Languasco and Zaccagnini [7] also proved (assuming RH) that

\[ \sum_{n=N-H}^{N+H} r_G(n) \left( 1 - \frac{|n - N|}{H} \right) = HN - 2 \sum_{\rho} \frac{(N + H)^{\rho + 2} - 2N^{\rho + 2} + (N - H)^{\rho + 2}}{\rho(\rho + 1)(\rho + 2)} \]

\[ + O \left( N \log^2 \left( \frac{2N}{H} \right) + H \log^2(N) \log(2H) \right), \]

where \( N \geq 2, 1 \leq H \leq N \). In a very recent paper, Brüdern, Kaczorowski and Perelli [8] were able to find an explicit formula which holds for all \( \omega > 0 \). Similar averages of arithmetical functions are common in literature; see, e.g., Chandrasekharan and Narasimhan [9] and Berndt [10] who built on earlier classical work. In this paper, we will prove an explicit formula for \( \tilde{S}(z) \) which will allow us to find a smaller error with respect to (4). Kunik and Lucht [11] wrote an explicit formula for \( \tilde{S}(z) \), but, in their closed form, there is a term regarding another summation of Von Mangoldt function, i.e. the series \( \sum_{n \geq 1} \Lambda(n)(1 - \exp(-n/\nu))/n \). We will able to rewrite this series in terms of the lower incomplete Gamma function and sum over nontrivial zeros of the Riemann zeta function. We first prove the following:

**Theorem 1.** Let \( z = a + iy, a > 0, y \in \mathbb{R} \). Let us consider the function

\[ \tilde{S}(z) = \sum_{m \geq 1} \Lambda(m)e^{-mz}. \]

Then

\[ \tilde{S}(z) = 2e^{-2z} + \frac{e^{-2z}}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + \sum_{\rho} \left( z^{-\rho} \gamma(\rho, 2z) - \frac{2\rho e^{-z}}{\rho} \right) - \frac{e^z}{\zeta(0)} e^{-2z} \]

\[ + \frac{e^z}{2} (-i\pi \text{sgn}(\text{Im}(-z)) + \text{Ei}(-z)) \]

\[ + \frac{e^{-z}}{2} (-i\pi \text{sgn}(\text{Im}(-z)) - \log(3)e^{-3z} + \text{Ei}(-3z)) \]

\[ + i\pi \text{sgn}(\text{Im}(-z)) + \log(2)e^{-2z} - \text{Ei}(-2z), \]

where \( \gamma(\rho, 2z) \) is the lower incomplete Gamma function, \( \rho = \beta + i\gamma \) runs over the nontrivial zeros of the Riemann zeta function and

\[ \text{Ei}(z) := -\int_{-\infty}^{\infty} \frac{e^{-t} - 1}{t} dt, \quad |\text{arg}(z)| < \pi, \]

is the exponential integral function. Furthermore, if

\[ \tilde{S}(z) = \frac{e^{-2z}}{z} - \sum_{\rho} \Gamma(\rho)z^{-\rho} + \sum_{\rho} \left( z^{-\rho} \gamma(\rho, 2z) - \frac{2\rho e^{-2z}}{\rho} \right) + E(a, y), \]
then the following bound is valid:

\[
E(a, y) \ll \begin{cases} 
1 + |z|^{1/2}, & |y| \leq a, \\
1 + |z|^{1/2}(1 + \log^2(|y|/a)), & a < |y| \leq 1, \\
1, & |y| > 1.
\end{cases}
\]

Note that, we have improved the error term (4) as \(|y| \to \infty\). This is interesting since, following the Languasco and Zaccagnini approach, the size of the error term \(E(a, y)\) is linked to the size of \(k\) (see Sec. 3 of [1] for more details). We also obtain the extra term \(\sum_{\rho}(z^{-\rho}\gamma(\rho, 2z) - 2^{\rho}e^{-2z}/\rho)\); following, again, the technique in [1], we see that it would be nice to have a sharp uniform estimation of this series. Unfortunately, even if this series, as we will see, converges absolutely, it is not simple to get a good uniform estimation. A deeper analysis of this fact will be done in future research. Also note that the presence of \(e^{-2z}/z\) as the “main term” instead of \(1/z\) does not significantly alter the asymptotic formula (3). For example if \(z \in \mathbb{R}^+\), then it is clear that

\[
\tilde{S}(z) \sim e^{-2z}/z \sim 1/z
\]
as \(z \to 0^+\), which is one of the forms of the Prime Number Theorem. In addition, we have the following statement.

**Theorem 2.** Let \(N > 4\) be a natural number. Then

\[
\sum_{n \leq N} r_G(n)(N - n) = \frac{N^3}{6} - 2 \sum_{\rho} \frac{(N - 2)^{\rho + 2}}{\rho(\rho + 1)(\rho + 2)} + \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 2)} N^{\rho_1 + \rho_2 + 1}
\]

\[
- 2 \sum_{\rho_1} \sum_{\rho_2} \frac{N^{\rho_1 + \rho_2 + 1}}{\rho_1 \rho_2} B_{2/N}(\rho_1 + 1, \rho_2 + 1) + F(N),
\]

where \(F(N)\) is the sum of (explicitly calculable) elementary functions, dilogarithms, and sums over nontrivial zeros of the Riemann zeta function involving the incomplete Beta function. Furthermore,

\[
F(N) = O(N^2) \quad \text{as} \quad N \to \infty.
\]

This explicit formula extends, in some sense, the main result in [1] (since their formula holds for \(k > 1\)) and in [6] (since the authors assume the Riemann Hypothesis). Furthermore, it provides different ways to write the explicit formula of the Cesàro average of Goldbach representations in the case \(k = 1\) with respect to the main formula in [8]. The study of the double series

\[
\sum_{\rho_1} \sum_{\rho_2} \frac{1}{\rho_1 \rho_2} N^{\rho_1 + \rho_2 + 1} B_{2/N}(\rho_1 + 1, \rho_2 + 1)
\]
is quite delicate since it is not obvious if it converges absolutely or not, but it is clear that it plays a central role in formulas like (5). A deeper analysis of this type of double series will be the subject of future research.

2. NOTATION

To avoid ambiguity, but keeping the standard notation, we specify some symbols:

- \(\gamma\), with or without subscripts, will be the imaginary part of the nontrivial zeros of the Riemann zeta function,
- \(\gamma(a, b)\) will be the lower incomplete gamma function,
- \(\mathcal{L}(f(t))(z)\) will be the Laplace transform of \(f\) and \(\mathcal{L}^{-1}(f(t))(z)\) will be the inverse Laplace transform of \(f\).
3. PROOF OF THEOREM 1

We first recall that the upper incomplete gamma function and the lower incomplete gamma function are defined as

\[ \Gamma(a, z) := \int_z^\infty t^{a-1}e^{-t} \, dt, \]  
\[ \gamma(a, z) := \int_0^z t^{a-1}e^{-t} \, dt, \quad \text{Re}(a) > 0; \]  

see [12], formulas 8.2.1 and 8.2.2 on p. 174. Furthermore, we have

\[ \Gamma(a, z) + \gamma(a, z) = \Gamma(a), \]  
\[ \Gamma(a+1, z) = a\Gamma(a, z) + e^{-z}z^a; \]  

see [12], formula 8.2.3 on p. 174 and formula 8.8.2 on p. 178.

Let us consider the finite sum \( \sum_{m=2}^M \Lambda(m)e^{-mz} \), \( M > 2 \). By the Abel summation formula, we have

\[ \sum_{m=2}^M \Lambda(m)e^{-mz} = \psi(M)e^{-Mz} + z \int_2^M \psi(t)e^{-tz} \, dt, \]

where \( \psi(M) \) is the Chebyshev psi function. From the PNT, we obtain

\[ |\psi(M)e^{-Mz}| \leq \psi(M)e^{-Ma} \ll Me^{-Ma} \to 0 \]

as \( M \to \infty \); so

\[ \bar{S}(z) = \sum_{m \geq 1} \Lambda(m)e^{-mz} = z \int_2^\infty \psi(t)e^{-tz} \, dt; \]  

(10)

so \( z^{-1}\bar{S}(z) \) is the Laplace transform of \( \psi(t) \). From the explicit formula for \( \psi(t) \) (see, for example, [13], p. 104), we obtain

\[ z \int_2^\infty \psi(t)e^{-tz} \, dt = z \int_2^\infty te^{-tz} \, dt - z \int_2^\infty \sum_\rho \frac{t^\rho}{\rho}e^{-tz} \, dt \]

\[ - \frac{\zeta'}{\zeta}(0)z \int_2^\infty e^{-tz} \, dt - \frac{z}{2} \int_2^\infty \log(1-t^{-2})e^{-tz} \, dt. \]  

(11)

Now we integrate (11) term by term and assume, for now, that we can switch the series over the nontrivial zeros and the integral (we will show that it is possible at the end of this proof). We immediately obtain

\[ z \int_2^\infty te^{-tz} \, dt = 2e^{-2z} + \frac{e^{-2z}}{z}, \quad - \frac{\zeta'}{\zeta}(0)z \int_2^\infty e^{-tz} \, dt = - \frac{\zeta'}{\zeta}(0)e^{-2z}. \]

Now, it follows from (8) and (9) that

\[ -z \int_2^\infty \sum_\rho \frac{t^\rho}{\rho}e^{-tz} \, dt = - \sum_\rho \frac{1}{\rho} z \int_2^\infty t^\rho e^{-tz} \, dt = - \sum_\rho \frac{1}{\rho} z^{-\rho} \int_2^\infty t^\rho e^{-t} \, dt \]

\[ = - \sum_\rho \frac{1}{\rho} z^{-\rho}\Gamma(\rho+1, 2z) = - \sum_\rho \left( z^{-\rho}\Gamma(\rho, 2z) + \frac{2\rho e^{-2z}}{\rho} \right) \]

\[ = - \sum_\rho \left( z^{-\rho}\Gamma(\rho) - z^{-\rho}\gamma(\rho, 2z) + \frac{2\rho e^{-2z}}{\rho} \right) \]

\[ = - \sum_\rho z^{-\rho}\Gamma(\rho) + \sum_\rho \left( z^{-\rho}\gamma(\rho, 2z) - \frac{2\rho e^{-2z}}{\rho} \right). \]  

(12)
Now we consider the integral involving \( \log(1 - t^{-2}) \). Integrating by parts, we obtain
\[
-\frac{z}{2} \int \log(t)e^{-zt} \, dt = \frac{1}{2} \log(t)e^{-zt} - \text{Ei}(-zt) + K,
\] (13)
where \( \text{Ei}(x) \) is the exponential integral function and \( K \) is a constant. So
\[
-\frac{z}{2} \int_{2}^{\infty} \log(1 - t^{-2})e^{-zt} \, dt = -\frac{z}{2} \int_{2}^{\infty} \log(t - 1)e^{-zt} \, dt - \frac{z}{2} \int_{2}^{\infty} \log(t + 1)e^{-zt} \, dt
\]
\[
+ z \int_{2}^{\infty} \log(t)e^{-zt} \, dt
\]
\[
= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\]

Let us study \( \mathcal{J}_1 \). By (13), we have
\[
\mathcal{J}_1 = -\frac{e^{-z}}{2} \int_{1}^{\infty} \log(t)e^{-zt} \, dt = \frac{e^{-z}}{2} \log(t)e^{-zt} - \text{Ei}(-zt)\big|_{1}^{\infty}.
\]
From the asymptotic
\[
\text{Ei}(w) \sim i\pi \text{sgn(Im(w))} - \frac{e^{w}}{w} \quad \text{as} \quad |w| \to \infty \quad (14)
\]
(see, for example, [15], Eq. (26) on p. 192), we obtain
\[
\lim_{t \to \infty} (\log(t)e^{-zt} - \text{Ei}(-zt)) = -i\pi \text{sgn(Im(-z))}. \quad (15)
\]
So
\[
\mathcal{J}_1 = \frac{e^{-z}}{2} (-i\pi \text{sgn(Im(-z))} + \text{Ei}(-z)).
\]

Let us consider \( \mathcal{J}_2 \). Using (15) again, we can write
\[
\mathcal{J}_2 = -\frac{e^{z}}{2} \int_{3}^{\infty} \log(t)e^{-zt} \, dt = \frac{e^{z}}{2} (\log(t)e^{-zt} - \text{Ei}(-zt))\big|_{3}^{\infty}
\]
\[
= \frac{e^{z}}{2} (-i\pi \text{sgn(Im(-z))} - \log(3)e^{-3z} + \text{Ei}(-3z)).
\]

Now let us analyze \( \mathcal{J}_3 \). From (15), we obtain
\[
\mathcal{J}_3 = -(\log(t)e^{-zt} - \text{Ei}(-zt))\big|_{2}^{\infty} = i\pi \text{sgn(Im(-z))} + \log(2)e^{-2z} - \text{Ei}(-2z).
\]
Finally,
\[
-\frac{z}{2} \int_{2}^{\infty} \log(1 - t^{-2})e^{-zt} \, dt = \frac{e^{-z}}{2} (-i\pi \text{sgn(Im(-z))} + \text{Ei}(-z))
\]
\[
+ \frac{e^{z}}{2} (-i\pi \text{sgn(Im(-z))} - \log(3)e^{-3z} + \text{Ei}(-3z))
\]
\[
+ i\pi \text{sgn(Im(-z))} + \log(2)e^{-2z} - \text{Ei}(-2z).
\]
Therefore,
\[ \overline{S}(z) = 2e^{-2z} + \frac{e^{-2z}}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + \sum_{\rho} \left( z^{-\rho} \gamma(\rho, 2z) - \frac{2\rho e^{-2z}}{\rho} \right) - \frac{\zeta'}{\zeta}(0)e^{-2z} \]
\[ + \frac{e^{-z}}{2} (-i\pi \text{sgn}(\text{Im}(z)) + \text{Ei}(-z)) + \frac{e^z}{2} (-i\pi \text{sgn}(\text{Im}(z)) - \log(3)e^{-3z} + \text{Ei}(-3z)) \]
\[ + i\pi \text{sgn}(\text{Im}(z)) + \log(2)e^{-2z} - \text{Ei}(-2z). \]

We have almost proved the first part of the theorem. Now, we must show that we may exchange the series with the integral in (11). Let us define
\[ h_T(t) = e^{-tz} \sum_{\rho : |\gamma| \leq T} \frac{t^\rho}{\rho}, \quad \text{where} \quad T > 2. \]
(16)

From the truncated explicit formula (see [13] on p. 109), we have
\[ \psi(t) = t - \sum_{\rho : |\gamma| \leq T} \frac{t^\rho}{\rho} + O\left( \frac{t \log^2(tT)}{T} + \log(t) \min\left(1, \frac{t}{T^2} T(t)\right) \right), \quad t > 2, \quad T > 2, \]
(17)
where \(|t\) is the distance from \(t\) to the nearest prime power, and so
\[ |h_T(t)| \ll e^{-ta} \left( |\psi(t) - t| + t \frac{\log^2(tT)}{T} + \log(t) \right) \ll e^{-ta} \left( t + \frac{t \log^2(t) + 2t \log(t) \log(T) + t \log^2(T) + T \log(t)}{T} \right), \]

since \(|\psi(t) - t| \ll t\). But \(\log^2(T)/T \ll 1\) as \(T \to \infty\), we have \(|h_T(t)| \ll e^{-ta} t \log^2(t)\), so we must prove that
\[ \int_2^\infty t \log^2(t)e^{-ta} dt \]
converges. Trivially, we have
\[ \int_2^\infty t \log^2(t)e^{-ta} dt \ll a \int_0^\infty t^2 e^{-t} dt = \Gamma(3), \]
so, by the dominated convergence theorem, we may exchange the integral with the series. It remains to prove the estimation of \(E(a, y)\). From Lemma 1 of [1] (the bound in that lemma has been corrected in [16]) we observe that
\[ \sum_{\rho} \left( z^{-\rho} \gamma(\rho, 2z) - \frac{2\rho e^{-2z}}{\rho} \right) \ll \begin{cases} 1 + |z|^{1/2}, & |y| \leq a, \\ 1 + |z|^{1/2}(1 + \log^2(|y|/a)), & a < |y| \leq 1 \end{cases} \]

and
\[ E(a, y) \ll \begin{cases} 1 + |z|^{1/2}, & |y| \leq a, \\ 1 + |z|^{1/2}(1 + \log^2(|y|/a)), & a < |y| \leq 1, \end{cases} \]

and hence the estimation follows. If \(|y| > 1\) from (14) we see that
\[ |\text{Ei}(-z) - i\pi \text{sgn}(\text{Im}(z))| \ll \left| \frac{e^{-z}}{z} \right| \ll 1 \]
as \(y \to \pm \infty\) and this concludes the proof.
4. PROOF OF THEOREM 2

Let \( z = 1/N + iy, N \in \mathbb{N}, N > 4 \) and \( y \in \mathbb{R} \). From (10), we have
\[
\mathcal{L}(\psi(t))(z) = z^{-1} \tilde{S}(z),
\]
so, by the convolution theorem, \( \mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \), and, by the inversion theorem, \( \mathcal{L}^{-1}(\mathcal{L}(f)) = f \), for the Laplace transform, where \( f, g \) are sufficiently regular functions (for more details, see, e.g., [14], Theorems 8.1 and 8.4), we obtain
\[
\frac{1}{2\pi i} \int_{(1/N)} e^{uz} z^{-2} \tilde{S}(z)^2 \, dz = \begin{cases} 
\int_2^{u-2} \psi(t)\psi(u-t) \, dt, & u > 4, \\
0, & \text{otherwise},
\end{cases}
\]
Next, from the explicit formula, we must evaluate
\[
\int_2^{u-2} \left( t - \sum_\rho \frac{t^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log \left( 1 - \frac{1}{t^2} \right) \right) \times \left( u - t - \sum_\rho \frac{(u-t)^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log \left( 1 - \frac{1}{(u-t)^2} \right) \right) dt
\]
\[
= \sum_{m=1}^{10} \int_2^{u-2} s_m(t, u) \, dt,
\]
say, where the functions \( s_m(t, u) \) will be defined in the next sections.

4.1. Integral of \( s_1(t, u) \)

We take \( s_1(t, u) := t(u-t) \). We immediately obtain
\[
\int_2^{u-2} s_1(t, u) \, dt = \frac{u^3 - 24u + 32}{6}.
\]

4.2. Integral of \( s_2(t, u) \)

We take \( s_2(t, u) := -2(u-t) \sum_\rho t^\rho / \rho \). We have already seen in (17) that
\[
\sum_{\rho: |\gamma| \leq T} \frac{t^\rho}{\rho} \ll t \log^2(t);
\]
then, by the dominated convergence theorem, we can write
\[
\int_2^{u-2} s_2(t, u) \, dt = -2 \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)t^\rho \, dt
\]
\[
= -4 \sum_\rho \frac{(u-2)^{\rho+1}}{\rho(\rho+1)} + (u-2) \sum_\rho \frac{2^{\rho+2}}{\rho(\rho+1)} - 2 \sum_\rho \frac{1}{\rho(\rho+1)} \int_2^{u-2} t^{\rho+1} \, dt
\]
\[
= -4 \sum_\rho \frac{(u-2)^{\rho+1}}{\rho(\rho+1)} + (u-2) \sum_\rho \frac{2^{\rho+2}}{\rho(\rho+1)}
\]
\[
- 2 \sum_\rho \frac{(u-2)^{\rho+2}}{\rho(\rho+1)(\rho+2)} + \sum_\rho \frac{2^{\rho+3}}{\rho(\rho+1)(\rho+2)}.
\]
Note that the all the series in this section converge absolutely.
4.3. Integral of $s_3(t, u)$

We consider $s_3(t, u) := -2t\zeta'(0)/\zeta(0)$. We immediately obtain

$$
\int_2^{u-2} s_3(t, u) \, dt = \frac{\zeta'(0)}{\zeta(0)}(4u - u^2).
$$

4.4. Integral of $s_4(t, u)$

We take $s_4(t, u) := -(u - t) \log(1 - 1/t^2)$. We obtain

$$
\int_2^{u-2} s_4(t, u) \, dt = - \int_2^{u-2} (u - t) \log\left(1 - \frac{1}{t^2}\right) \, dt
$$

$$
= - \int_2^{u-2} (u - t) \log(t - 1) \, dt - \int_2^{u-2} (u - t) \log(t + 1) \, dt
$$

$$
+ 2 \int_2^{u-2} (u - t) \log(t) \, dt.
$$

Then, after simple but boring calculations, we obtain

$$
\int_2^{u-2} s_4(t, u) \, dt = \frac{1}{4}((u - 4)(3u - 2) - 2(u - 3)(u + 1) \log(u - 3)
$$

$$
- 8 + 3u^2 - 6 \log(3) + 2u(\log(729) - 5) - 2 \log(u - 1)(u^2 + 2u - 3)
$$

$$
- 6u^2 - 8u(\log(4) - 3) + 2 \log(256) + 4 \log(u - 2)(u^2 - 4)).
$$

Also it is quite simple to observe that

$$
\int_2^{u-2} s_4(t, u) \, dt \ll u^2.
$$

4.5. Integral of $s_5(t, u)$

We now take

$$
s_5(t, u) := \sum_{\rho_1} \frac{(u - t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2}.
$$

Then, we must calculate

$$
\int_2^{u-2} s_5(t, u) \, dt = \int_2^{u-2} \sum_{\rho_1} \frac{(u - t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} \, dt.
$$

We must prove that it is possible to exchange the integral and the series over the nontrivial zeros of the Riemann zeta function. Let us define

$$
S_{T, u}^{1}(t) := \sum_{\rho_1: |\gamma_1| \leq T} \frac{(u - t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2}, \quad T > 2.
$$

From (18) and the trivial bound

$$
\left| \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} \right| = \left| t - \psi(t) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log\left(1 - \frac{1}{t^2}\right) \right| \ll t,
$$

for $t > 2$, we have

$$
|S_{T, u}^{1}(t)| \ll t(u - t) \log^2(u - t)
$$
then by the dominated convergence theorem
\[
\int_2^{u-2} \sum_{\rho_1} \frac{(u-t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} dt = \sum_{\rho_1} \frac{1}{\rho_1} \int_2^{u-2} (u-t)^{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} dt.
\]
Now, if we take
\[S^2_{T,u,\rho_1}(t) = (u-t)^{\rho_1} \sum_{\rho_2:|\gamma_2| \leq T} \frac{t^{\rho_2}}{\rho_2}, \quad T > 2,
\]
from (18), we obtain
\[|S^2_{T,u,\rho_1}(t)| \ll (u-t)t \log^2(t),
\]
then again by the dominated convergence theorem we can conclude that
\[
\sum_{\rho_1} \frac{1}{\rho_1} \int_2^{u-2} (u-t)^{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} dt = \sum_{\rho_1} \sum_{\rho_2} \frac{1}{\rho_1 \rho_2} \int_2^{u-2} (u-t)^{\rho_1} t^{\rho_2} dt.
\]
Then
\[
\int_2^{u-2} \sum_{\rho_1} \frac{(u-t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} dt = \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} \frac{\Gamma(\rho_1+1) \Gamma(\rho_2+1)}{\Gamma(\rho_1+\rho_2+2)}
\]
\[\quad - \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} B_{2/u}(\rho_1+1, \rho_2+1)
\]
\[\quad - \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} B_{2/u}(\rho_2+1, \rho_1+1)
\]
\[\quad = \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} \frac{\Gamma(\rho_1) \Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+2)}
\]
\[\quad - 2 \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} B_{2/u}(\rho_1+1, \rho_2+1),
\]
where \(B_x(a,b)\) is the incomplete Beta function
\[B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad \Re(a), \Re(b) > 0, \quad 0 \leq x < 1,
\]
for more details see, for example, [12], ch. 8.17. We recall that
\[
\sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} \frac{\Gamma(\rho_1) \Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+2)}
\]
converges absolutely (see [1]), so
\[
\sum_{\rho_1} \sum_{\rho_2} \frac{B_{2/u}(\rho_1+1, \rho_2+1) u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2}
\]
converges (at least conditionally), since
\[
\left| 2 \sum_{\rho_1} \sum_{\rho_2} \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} B_{2/u}(\rho_1+1, \rho_2+1) \right| \ll \sum_{\rho_1} \sum_{\rho_2} \left| \frac{u^{\rho_1+\rho_2+1}}{\rho_1 \rho_2} \frac{\Gamma(\rho_1) \Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+2)} \right|
\]
\[\quad + \left| \int_2^{u-2} \sum_{\rho_1} \frac{(u-t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} dt \right|,
\]
and, from (19), it follows trivially that

\[
\left| \int_2^{u-2} \sum_{\rho_1} \frac{(u-t)^{\rho_1}}{\rho_1} \sum_{\rho_2} \frac{t^{\rho_2}}{\rho_2} \, dt \right| \ll \int_2^{u-2} (u-t) \, dt < \infty.
\]

4.6. Integral of \( s_6(t, u) \)

We take \( s_6(t, u) := 2\zeta'(0) / \zeta(0) \sum_\rho t^\rho / \rho. \) From (18) we know that it is possible to switch the integral and the series, so

\[
\int_2^{u-2} s_6(t, u) \, dt = \frac{2}{\zeta(0)} \sum_\rho \frac{1}{\rho} \int_2^{u-2} t^\rho \, dt = \frac{2}{\zeta(0)} \sum_\rho \frac{(u-2)^{\rho+1}}{\rho(\rho+1)} - \frac{2^{\rho+2}}{\rho(\rho+1)}.
\]

Again it is quite simple to observe that

\[
\int_2^{u-2} s_6(t, u) \, dt \ll u^2,
\]

due to the absolute convergence of \( \sum_\rho 1/(\rho(\rho+1)). \)

4.7. Integral of \( s_7(t, u) \)

We consider \( s_7(t, u) := \log(1 - 1/t^2) \sum_\rho (u-t)^\rho / \rho. \) Using the dominated convergence theorem, we can prove that we can exchange the integral and the series, so

\[
\int_2^{u-2} s_7(t, u) \, dt = \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log \left( 1 - \frac{1}{t^2} \right) \, dt
\]

\[
= \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log (t-1) \, dt + \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log (t+1) \, dt
\]

\[
- 2 \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log (t) \, dt.
\]

Let us consider the first integral. Integrating by parts we obtain

\[
H_1(u) := \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log (t-1) \, dt
\]

\[
+ \sum_\rho \frac{2^{\rho+1} \log(u-3)}{\rho(\rho+1)} - \sum_\rho \frac{1}{\rho(\rho+1)} \int_2^{u-2} (u-t)^{\rho+1} (t-1)^{-1} \, dt
\]

\[
= \sum_\rho \frac{2^{\rho+1} \log(u-3)}{\rho(\rho+1)} - \sum_\rho \frac{(u-1)^{\rho+1}}{\rho(\rho+1)} \int_{2/(u-1)}^{1-1/(u-1)} s^{\rho+1} (1-s)^{-1} \, ds
\]

\[
= \sum_\rho \frac{2^{\rho+1} \log(u-3)}{\rho(\rho+1)} - \sum_\rho \frac{(u-1)^{\rho+1}}{\rho(\rho+1)} \left[ B_{1/(u-1)}(\rho+2, 0) - B_{2/(u-1)}(\rho+2, 0) \right].
\]

In a similar way, we have

\[
H_2(u) := \sum_\rho \frac{1}{\rho} \int_2^{u-2} (u-t)^\rho \log (t+1) \, dt
\]

\[
= \sum_\rho \frac{2^{\rho+1} \log(u-1)}{\rho(\rho+1)} - \sum_\rho \frac{(u-2)^{\rho+1} \log(3)}{\rho(\rho+1)} - \sum_\rho \frac{1}{\rho(\rho+1)} \int_2^{u-2} (u-t)^{\rho+1} (t+1)^{-1} \, dt
\]

\[
= \sum_\rho \frac{2^{\rho+1} \log(u-1)}{\rho(\rho+1)} - \sum_\rho \frac{(u-2)^{\rho+1} \log(3)}{\rho(\rho+1)} - \sum_\rho \frac{1}{\rho(\rho+1)} \int_2^{u-2} (u-t)^{\rho+1} (t+1)^{-1} \, dt.
\]
\[\begin{align*}
&= \sum_{\rho} \frac{2^\rho \log(u - 1)}{\rho(\rho + 1)} - \sum_{\rho} \frac{(u - 2)^\rho \log(3)}{\rho(\rho + 1)} \\
&\quad - \sum_{\rho} \frac{(u + 1)^{\rho+1}}{\rho(\rho + 1)} [B_{1-3/(u+1)}(\rho + 2, 0) - B_{2/(u+1)}(\rho + 2, 0)]
\end{align*}\]

and

\[\begin{align*}
H_3(u) &:= -2 \sum_{\rho} \frac{1}{\rho} \int_2^{u-2} (u - t)^\rho \log(t) \, dt \\
&= - \sum_{\rho} \frac{2^{\rho+2} \log(u - 2)}{\rho(\rho + 1)} + 2 \sum_{\rho} \frac{(u - 2)^{\rho+1} \log(2)}{\rho(\rho + 1)} \\
&\quad + 2 \sum_{\rho} \frac{1}{\rho(\rho + 1)} \int_2^{u-2} \frac{(u - t)^{\rho+1}}{t} \, dt \\
&= - \sum_{\rho} \frac{2^{\rho+2} \log(u - 2)}{\rho(\rho + 1)} + 2 \sum_{\rho} \frac{(u - 2)^{\rho+1} \log(2)}{\rho(\rho + 1)} \\
&\quad + 2 \sum_{\rho} \frac{u^{\rho+1}}{\rho(\rho + 1)} [B_{1-2/u}(\rho + 2, 0) - B_{2/u}(\rho + 2, 0)].
\end{align*}\]

Finally, we can write

\[\int_2^{u-2} s_7(t, u) \, dt = H_1(u) + H_2(u) + H_3(u).\]

From the well-known estimation

\[\sum_{\rho} \frac{x^\rho}{\rho} \ll x \exp(-C \sqrt{\log(x)}), \quad x > 1, \quad C > 0\]

(see [13], Chap. 18), we can easily conclude that

\[\int_2^{u-2} s_7(t, u) \, dt \ll u^2.\]

Again we can observe that all the series over the nontrivial zeros of \(\zeta(s)\) in this section converge absolutely; to see this fact in the case of the series involving the incomplete Beta function (which is, probably, the less evident absolute convergence) note that

\[2 \sum_{\rho} \frac{u^{\rho+1}}{\rho(\rho + 1)} [B_{1-2/u}(\rho + 2, 0) - B_{2/u}(\rho + 2, 0)] = 2 \sum_{\rho} \frac{1}{\rho(\rho + 1)} \int_2^{u-2} \frac{(u - t)^{\rho+1}}{t} \, dt \\
\ll \sum_{\rho} \frac{1}{|\rho||\rho + 1|} \int_2^{u-2} \frac{(u - t)^2}{t} \, dt < \infty.\]

4.8. Integral of \(s_8(t, u)\)

We take \(s_8(t, u) := (\zeta'(0)/\zeta(0))^2\). We immediately obtain

\[\int_2^{u-2} s_8(t, u) \, dt = \frac{\zeta'}{\zeta}(0)^2(u - 4).\]
4.9. Integral of $s_9(t, u)$

We consider $s_9(t, u) := \zeta'(0)/\zeta(0) \log(1 - 1/t^2)$. We obtain

$$\int_2^{u-2} s_9(t, u) \, dt = \frac{\zeta'}{\zeta}(0)((u - 3) \log(u - 3) + \log\left(\frac{1}{27(u - 1)}\right) + u \log(u - 1)$$

$$+ 2 \log(4) - 2(u - 2) \log(u - 2))$$

and, again, we trivially get the bound

$$\int_2^{u-2} s_9(t, u) \, dt \ll u^2.$$

4.10. Integral of $s_{10}(t, u)$

We take $s_{10}(t, u) := \log(1 - 1/t^2) \log(1 - 1/(u - t)^2)/4$. We have

$$\int_2^{u-2} s_{10}(t, u) \, dt = \frac{1}{4} \int_2^{u-2} \log\left(1 - \frac{1}{t^2}\right) \log\left(1 - \frac{1}{(u - t)^2}\right) \, dt$$

$$= \frac{1}{4} \int_2^{u-2} \log\left(\frac{t^2 - 1}{t^4}\right) \log\left(1 - \frac{(u - t)^2 - 1}{(u - t)^2}\right) \, dt$$

$$= \frac{1}{4} \int_2^{u-2} (\log(t - 1) + \log(t + 1) - 2 \log(t)) \times (\log(u - t - 1) + \log(u - t + 1) - 2 \log(u - t)) \, dt \quad (20)$$

so we must evaluate all the combinations of $(20)$. We will calculate explicitly only the first, since the others are similar. We consider

$$\int_2^{u-2} \log(t - 1) \log(u - t - 1) \, dt = \int_1^{u-3} \log(v) \log(u - v - 2) \, dv$$

$$= (u - 2) \int_{1/(u - 2)}^{1-1/(u - 2)} \log((u - 2)s) \log((u - 2)(1 - s)) \, ds$$

$$= (u - 2) \log^2(u - 2)\left(1 - \frac{2}{u - 2}\right) + (u - 2) \log(u - 2) \int_{1/(u - 2)}^{1-1/(u - 2)} \log(1 - s) \, ds$$

$$+ (u - 2) \log(u - 2) \int_{1/(u - 2)}^{1-1/(u - 2)} \log(s) \, ds$$

$$+ (u - 2) \int_{1/(u - 2)}^{1-1/(u - 2)} \log(s) \log(1 - s) \, ds.$$
where \( \text{Li}_2(x) \) is the dilogarithm function
\[
\text{Li}_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2}, \quad |x| < 1.
\]

After a boring calculation, we obtain
\[
V_1(u) := \frac{1}{4} \int_2^{u-2} \log(t-1) \log(u-t-1) \, dt
= \frac{1}{4} \left( 2(u-4) + \log(u-3)(6-2u) + (u-2) \log(u-2) \right) + (u-2) \text{Li}_2 \left( \frac{1}{u-2} \right)
- (u-2) \text{Li}_2 \left( 1 - \frac{1}{u-2} \right).
\]

For the explicit calculations of the other integrals, see the Appendix. Furthermore, it is clear that
\[
\int_2^{u-2} s_{10}(t, u) \, dt \ll u^2.
\]

To conclude the proof, we only have to take \( u = N \), recalling that
\[
\sum_{n \leq N} r_G(n)(N-n) = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-2} \mathcal{S}(z)^2 \, dz.
\]

5. APPENDIX

In this section, we calculate explicitly all the integrals from Sec. 4.10. Using the same ideas that we used for \( V_1(u) \) (and the help of Mathematica), we obtain
\[
V_2(u) := \frac{1}{4} \int_2^{u-2} \log(t-1) \log(u-t+1) \, dt
= \frac{1}{4} \left( -8 + 2u + \log(27) + \log(u-1) + \log(u-3)(u \log(u) + 3 - 3 \log(3)) 
- u \log(u^2 - 4u + 3) + u \text{Li}_2 \left( \frac{1}{u} \right) - u \text{Li}_2 \left( 1 - \frac{3}{u} \right) \right),
\]
\[
V_3(u) := \frac{1}{4} \int_2^{u-2} \log(t+1) \log(u-t+1) \, dt
= \frac{1}{4} \left( -8 + 2u - 3(-1 + \log(3)) \log(u-3) + \log(27(u-1)) + u \log(3) \log \left( 1 - \frac{3}{u} \right) 
+ u \log(u-1) \log(u) - u \log(u^2 - 4u + 3) + u \text{Li}_2 \left( \frac{3}{u} \right) - u \text{Li}_2 \left( 1 - \frac{1}{u} \right) \right),
\]
\[
V_4(u) := \frac{1}{4} \int_2^{u-2} \log(t+1) \log(u-t+1) \, dt
= \frac{1}{4} \left( -8 + 2u + \log(729) + 2 \log(u-1) - 2u \log(u-1) - 4 \log(3) \log(u-1) 
+ u \log(3) \log(u-1) - u \log(3) \log(u+2) - \log(9) \log(u+2) + 2 \log(u-1) \log(u+2) 
- u \log(u-1) \log(u+2) + (2 + u) \text{Li}_2 \left( \frac{3}{u+2} \right) - (2 + u) \text{Li}_2 \left( 1 - \frac{3}{u+2} \right) \right).
\]
\[ V_5(u) := -\frac{1}{2} \int_2^{u-2} \log(t-1) \log(u-t) dt \]
\[ = \frac{1}{2} \left( 8 - 2u - \log(4) - 2 \log(u-2) - \log(u-3) - \log(3u) - \log(u-1) - \log(4(u-1)) \right) \]
\[ + u \log(u^2 - 5u + 6) - (u-1) \text{Li}_2 \left( \frac{1}{u-1} \right) + (u-1) \text{Li}_2 \left( 1 - \frac{2}{u-1} \right) , \]

\[ V_6(u) := -\frac{1}{2} \int_2^{u-2} \log(t+1) \log(u-t) dt \]
\[ = \frac{1}{2} \left( 8 - 2u - 2 \log(u-2) + u \log(u-2) - u \log(3) \log(u-2) \right) + \log(9) \log(u-2) + \log \left( \frac{1}{108(u-1)} \right) + u \log(u-1) + \log(4) \log(u-1) \]
\[ + \log(3) \log(u+1) + u \log(3) \log(u+1) - \log(u-1) \log(u+1) \]
\[ - u \log(u-1) \log(u+1) - (u+1) \text{Li}_2 \left( \frac{3}{u+1} \right) + (u+1) \text{Li}_2 \left( 1 - \frac{2}{u+1} \right) , \]

\[ V_7(u) := -\frac{1}{2} \int_2^{u-2} \log(t) \log(u-t-1) dt \]
\[ = \frac{1}{2} \left( 8 - 2u - \log(4) - 3 \log(u-3) + u \log(u-3) - u \log(2) \log(u-3) \right) + \log(8) \log(u-3) - 2 \log(u-2) + u \log(u-2) - \log(2) \log(u-1) \]
\[ + \log(2) \log(u-1) + \log(u-2) \log(u-1) - u \log(u-2) \log(u-1) \]
\[ - (u-1) \text{Li}_2 \left( \frac{2}{u-1} \right) + (u-1) \text{Li}_2 \left( 1 - \frac{1}{u-1} \right) , \]

\[ V_8(u) := -\frac{1}{2} \int_2^{u-2} \log(t) \log(u-t+1) dt \]
\[ = \frac{1}{2} \left( 8 - 2u - 2 \log(u-2) + u \log(u-2) + \log(27) \log(u-2) + \log \left( \frac{1}{108(u-1)} \right) \right) \]
\[ + u \log(u-1) - \log(2) \log(u-1) - u \log(2) \log(u-1) + \log(4) \log(u-1) \]
\[ + \log(2) \log(u+1) + u \log(2) \log(u+1) - \log(u-2) \log(u+1) \]
\[ - u \log(u-2) \log(u+1) - (u+1) \text{Li}_2 \left( \frac{2}{u+1} \right) + (u+1) \text{Li}_2 \left( 1 - \frac{3}{u+1} \right) \]

and, finally,

\[ V_9(u) := \int_2^{u-2} \log(t) \log(u-t) dt = -8 + 2u + \log(16) - u \log(2) \log(u) \]
\[ + \log(u-2)(-2 + 2u + \log(4)) + u \log(2u) \]
\[ + u \left( \text{Li}_2 \left( \frac{2}{u} \right) - \text{Li}_2 \left( 1 - \frac{2}{u} \right) \right) . \]

So we can write

\[ \int_2^{u-2} s_{10}(t,u) dt = \sum_{m=1}^{9} V_m(u) . \]
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