A dynamical universality class out of stochastic driving in interacting quantum
many-body systems

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In this paper, we study the mean-field dynamics of a general class of quantum many-body systems
with stochastically fluctuating interactions. Our findings reveal a universal algebraic decay of the
order parameter $m(t) \sim t^{-\chi}$ with an exponent $\chi = \frac{1}{4}$ that is independent of most system details
including the strength of the stochastic driving, the energy spectrum of the undriven systems, the
initial states and even the driving protocols. The finite-size effect, as well as the relevance of our
results with current experiments in high-finesse cavity QED systems are also discussed.

Introduction – A universality class is a collection of diverse systems which share common properties. For example, systems with dramatically different microstructures may share identical critical exponents near the phase transition point, where not only the static properties but also the relaxation dynamics might exhibit similar asymptotic behavior that is independent of most system details. Compared to equilibrium systems, the universality class in systems operating far from equilibrium is significantly richer but less known in general. As a prototypical example, the Kardar-Parisi-Zhang (KPZ) universality class with a dynamic critical exponent $\alpha = \frac{1}{3}$ governs diverse non-equilibrium phenomena from surface growth in classical stochastic models to super-diffusion in integrable quantum models. In the KPZ model, the conspiracy of stochasticity and nonlinearity plays a crucial role in such a remarkable universal behavior.

Random fluctuations, almost by definition, are usually considered a source of disorder that leads to irregular spatiotemporal behavior. A profound question is under the random fluctuation, do systems always adjust their macroscopic behavior to the average properties of the fluctuation, or can one find systems responding to randomness in a more active way, thus exhibiting, for instance, nontrivial behavior which is forbidden in the corresponding deterministic systems without randomness. The answer is indeed positive. It has been shown that random fluctuation in certain nonlinear models can conspire with nonlinearity to render counterintuitive behavior such as noise-induced spatiotemporal order and phase transition. The key point here involves the randomness that is introduced externally through a stochastic process of a control parameter that gets multiplied into system variables, and thus is multiplicative instead of additive as in the common Langevin treatment of the internal fluctuation. Such noise-enhanced regularity phenomena have been extensively studied in classical nonlinear systems ranging from electronic to biochemical systems. A more profound question, then, is whether this idea can be generalized to quantum systems despite the linearity of the Schrodinger’s equation.

In this paper, we attempt to answer this question by focusing for simplicity on the mean-field dynamics of a general class of interacting quantum systems with stochastic driving. Generally, an interacting quantum system can be driven out of equilibrium via a time-dependent manipulation of the Hamiltonian parameters. The majority of studies in this field has focused on the quench or periodically-driven dynamics of a quantum many-body systems where the Hamiltonian parameters are suddenly or periodically changed in time. However, only a few efforts studies have been focusing on stochastically-driven systems. Perhaps this lack of research studies on this field can be justified by the fact that such randomness in the time domain will heat the system, and thus it is bound to be detrimental to any spatiotemporal order. However, motivated by the multiplicative noise in the classical systems, this paper reveals that a stochastic driving acting on the interaction strength instead of external field, may facilitate an intriguing dynamical behavior, where the order parameter of the system exhibits a universal algebraic decay in time $t^{-\chi}$ with an exponent $\chi = \frac{1}{4}$ that is independent of the noise strength, the energy spectrum of the undriven systems, the initial states and even the driving protocols. Such universal dynamics can be understood as a collective behavior of a set of individual two-level systems subjected to a dynamical field determined self-consistently during the time evolution. The finite-size effect and the role of spatial fluctuations are also discussed.

Model and method – We first consider a fully-connected transverse Ising model subjected to a non-uniform magnetic field, whose Hamiltonian reads:

\begin{equation}
H_s(t) = -\frac{\xi(t)}{L} \sum_{kk'} s^x_k s^x_{k'} - \sum_k h_k s^z_k
\end{equation}

where $s^\alpha_k$ is the spin-$\frac{1}{2}$ operator operating on $k$th site(mode) defined as $s^\alpha_k = \frac{1}{2} \sigma^\alpha_k$ ($\sigma^\alpha_k$ are the three Pauli matrices with $\alpha = x, y, z$). $h_k$ is the magnetic field along $z$-direction on site(mode) $k$ which will be specified later on. $L$ is the total number of spins and the $\frac{1}{4}$ pre-factor in Eq.\textsuperscript{1} guarantees that the total interacting energy scale
linearly with $L$. $\xi(t)$ is the strength of the all-to-all interaction that is uniform in space but is randomly fluctuating in time, satisfying: $\langle \xi(t) \rangle = 0$, $\langle \xi(t) \xi(t') \rangle = D^2 \delta(t-t')$ with $D$ being the strength of the stochastic driving and the ensemble average $\langle \cdot \rangle$ is over all random trajectories of the stochastic driving.

Due to this all-to-all feature, the interaction terms in Eq. (1) can be decoupled by introducing a time-dependent ferromagnetic order parameters

$$m(t) = \frac{1}{L} \sum_k \langle \psi(t) | s_k^x | \psi(t) \rangle$$

with $| \psi(t) \rangle$ representing the wavefunction of the system at time $t$. The mean-field (MF) Hamiltonian turns to a set of spin-$\frac{1}{2}$ systems as $\tilde{H}_k(t) = \sum_k \tilde{H}_k(t)$ with:

$$\tilde{H}_k(t) = -\xi(t) m(t) s_k^x - \hbar \dot{s}_k^z,$$

and the corresponding equation of motion (EOM) reads:

$$\dot{s}_k = h_k(t) \times s_k$$

where $s_k = [(s_k^x, s_k^y, s_k^z) / 2]$ is a vector in the Bloch sphere ($| s_k | = 1/2$). $h_k = [-\xi(t) m(t), 0, \hbar \dot{m}(t)]$ and $m(t)$ is self-consistently determined by Eq. (2) during evolution. Here, the all-to-all couplings in Eq. (1) guarantee that spatial fluctuation in the thermodynamical limit ($L \rightarrow \infty$) is completely suppressed, and thus the MF Hamiltonian and EOM are no longer approximations but the exact methods dealing with both the equilibrium and non-equilibrium systems (also see Supplementary material (SM)[28]).

In general, the MF Hamiltonian and the EOM can not only be applied to the quantum magnetism (Eq. (1) for instance), but also to diverse phenomena with spontaneous symmetry breaking ranging from superconductors[29–33] to charge-density-waves (CDW)[34]. In most realistic systems with local interactions, the spatial fluctuations render the mean-field EOM as no longer exact. Despite this fact, recent experimental progress in high-finesse cavity QED systems have opened up new possibilities to study infinite-range interactions[35, 36]. For instance, it is possible to realize a one-dimensional (1D) hard-core boson system with an infinite-range interaction (Fig 1(b) with Hamiltonian:

$$H_b = -J \sum_i (\hat{b}_i^\dagger \hat{b}_{i+1} + h.c.) - \frac{\xi(t)}{L} \sum_{ij} (-1)^{i+j} \hat{n}_i \hat{n}_j \tag{5}$$

where $J$ is the nearest-neighbor hopping amplitude, $\hat{b}_i^\dagger$ ($\hat{b}_i$) is the creation (annihilation) operators of the hard-core boson at site $i$, and $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$. The interaction $\xi(t)$ is defined as the same as in Eq. (1). If $\xi(t) > 0$, the interaction in Eq. (5) is attractive (repulsive) between two bosons in the same (different) sublattice, and thus favors a CDW state at half-filling. As a consequence, one can introduce a CDW order parameter $m(t) = \frac{1}{L} \sum_i (-1)^i \hat{n}_i$ to decouple the interaction, which yields to a MF Hamiltonian:

$$\tilde{H}_b = -J \sum_i (\hat{b}_i^\dagger \hat{b}_{i+1} + h.c.) + m(t) \xi(t) \sum_i (-1)^i \hat{n}_i.$$ 

One can further use the Jordan-Wigner transformation to transform the 1D hard-core bosons to spinless fermions: $b_i^\dagger = e^{\delta \sum_{j<s} \pi_{n_j} c_j^\dagger}$, then perform the Fourier transformation $c_i^\dagger = \frac{1}{\sqrt{L}} \sum_k e^{ik_i^s} \hat{c}_k^\dagger$ to transfer the MF Hamiltonian into the momentum space as $\tilde{H}_b = \sum_k \Psi_k^\dagger \tilde{H}_b(k) \Psi_k$ where $\Psi_k = [c_k, c_{k+n}]^T$ and $\tilde{H}_b(k)$ has exactly the same form of Eq. (3) provided that we assume $\hbar^2 = -2J \cos k$ with $k \in [-\pi, \pi]$. Therefore, the mean-field EOM (11) also provides an exact description of the dynamics of this interacting bosonic system that is closely related with the current cavity QED experiments. In the following, we choose $\hbar^2 = -2J \cos k$ unless it is specified otherwise.

Numerically, we adopt Stratonovich’s formula[37] of the stochastic differential Eq. (4), and solve it using the standard Heun method[38] with the time step of $\Delta t = 10^{-5} J^{-1}$, the convergence of which has been numerically assessed (see SM[28]). The ensemble average over the random trajectories can be performed by directly sampling over the $N$ sets of noise realizations with $N = 500$ in our simulations.

Universal algebraic decay of the amplitude to order parameter - In this simulation, we focus on the dynamics of the order parameter $m(t)$. In general, the stochastic driving will heat the system toward a featureless state with $m(t \rightarrow \infty) = 0$, thus for a single random trajectory $\{ \xi(t) \}$, $m(t)$ will exhibit a chaotic oscillation whose amplitude decays in time, as shown in Fig 1(c). Despite the triviality of the steady state, the asymptotic behavior approaching it can be highly non-trivial and exhibit intriguing universal behavior even at a mean-field level. To characterize such an universal dynamics, we calculate the average amplitude of the order parameter at time $t$ defined as $M(t) = \langle m^2(t) \rangle^{1/2}$, and we end up with a universal algebraic decay $M(t) \sim t^{-\frac{1}{2}}$ that is independent of most system details as we will show in the following.

We first check the dependence of the long-time behav-
behavior of $M(t)$ on the strength of the stochastic driving $D$. As shown in Fig. 2 (a), during the time evolution $\tilde{M}(t)$ in the cases with strong stochastic driving are typically larger than those in weakly driven cases, which seems to indicate that this randomness facilitates the order phase. Contrary to many other studies, such a counterintuitive phenomenon is because due the stochastic driving does not act on the external fields in our model, but on the strength of the interaction, which favors the ordered phase with spontaneous symmetry breaking. All cases with different $D$, $M(t)$ always ends up with an algebraic decay $\sim t^{-\chi}$, whose exponent $\chi$ barely depends on $D$ expect for the case with small $D$ where the large statistical error makes it difficult to determine the accurate value of $\chi$ (see inset of Fig. 2a).

Due to the absence of dissipation in our model, one may wonder whether the long-time behavior depends on the initial state despite the energy nonconservation. To evaluate this, we fix $D$ but choose different initial states as the ground states of the Hamiltonian. 1 with $\xi(t=0) = D_0 \neq D$. As shown in Fig. 2 (b), after some relaxation time, the $M(t)$ starting from different initial states will converge into the same trajectory, which indicates that the initial state information has been washed out by the stochastic driving. It is remarkable to notice that this conclusion holds even for a symmetry unbroken initial state with $m(t=0) = 0$. As shown in Fig. 2 (b), $M(t)$ with $D_0 = 0$ will also converge to the universal algebraic decay after an extraordinarily long time, which suggests that even though $m(t) = 0$ is a solution of the EOM, it is unstable and nontrivial dynamics can be triggered by small temporal fluctuation.

The spectrum function $h_k^z$ is crucial for the determination of the undriven system properties. Until now, our focus was the dispersion spectrum of a 1D tight binding model with $h_k^z = -2J \cos k$. Here, we investigate additional spectrum functions, such as a gapped spectrum as $h_k^z = \pm \sqrt{(2J \cos k)^2 + \Delta^2}$, which can be realized by the imposition of a staggered chemical potential $\Delta \sum_i (-1)^i \hat{n}_i$ on the original Hamiltonian.4. Even though the equilibrium properties of a gapless and gapped systems are significantly different from each other, their long dynamics under stochastic driving seem qualitatively the same, as shown in Fig. 2 (c). More generally, one can select a spectrum function where each $h_i^z$ is a random number sampled from a uniform random distribution with $h_i^z \in [-2J, 2J]$. As shown in Fig. 2 (c), the algebraic decay also holds for such a random spectrum.
function. Finite-size effect – The systems we considered so far are sufficiently large (L = 5000), a fact that allows us to neglect the finite-size effect within the time scale of our simulation. The dynamics of M(t) in smaller systems have been shown in Fig 2 (d), from which it is shown that the universal algebraic decay of M(t) will not persist forever, instead, it will eventually approach a saturation value accompanied by small fluctuations. The saturation time linearly scales with the system size, suggesting that this is a finite-size effect. One can further investigate the dependence of the finite saturation value $M(L)$ on the system size. As shown in the inset of Fig. 1 (e), $M(L) \sim L^{-\beta}$, where $\beta = 0.48(1)$ (close to the value $\frac{1}{4}$) is another important “critical” exponent in our model.

Stochastic driving protocols other than white noise: Till now, we have only considered the situations where $\xi(t)$ fluctuates as a white noise. Therefore, it is essential to assess whether such a universal algebraic decay holds for other stochastic protocols. Consequently, we first consider a quasi-periodic driving protocol as a superposition of two periodic drivings with incommensurate periods, that shares some common features with the stochastic driving for $t \to \infty$. For instance, we select $\xi(t)$ as:

$$\xi(t) = V_0 [\cos(2\pi t + \varphi) + \cos(\sqrt{3} t)]$$  \hspace{1cm} (6)

where $V_0$ is the strength of the quasi-periodic driving, and $\varphi \in [0, 2\pi]$ is a time-independent random phase. We define $M(t) = \sqrt{\langle m^2(t) \rangle_{\varphi}}$, where the ensemble average $\langle \cdot \rangle_{\varphi}$ is performed over different $\varphi$. As shown in Fig 2 (e), we can then find two distinguished dynamical behaviors: for small $V_0$ (e.g. $V_0 = 3J$), M(t) exhibit a persistent oscillation with a constant amplitude, and for large $V_0$ (e.g. $V_0 = 7J$), the universal algebraic decay with $\chi = 0.34(4)$ seems to reappear in the presence of such a quasi-periodic driving. This phenomenon reminds us of the Aubry-André model, which is known to have a localization transition when increasing the strength of the incommensurate potential [59]. It would be interesting to investigate whether similar transitions can occur in the time domain of our model with quasi-periodic driving.

Nonetheless, we now consider another stochastic driving protocol known as telegraph noise, where $\xi(t)$ randomly jump between two discrete values $-D$ and $D$ with a transition rate $\kappa$ (the transition probability per unit time), which measures the (inverse) correlation time of this colorful noise $\langle \xi(t)\xi(t') \rangle_{\xi} \sim e^{-\kappa|t-t'|}$ [60]. As shown in Fig 2 (f), M(t) also exhibit an algebraic decay in the presence of the telegraph noise, with a power-law exponent $\chi = \frac{1}{4}$ that is independent of the driving strength $\Delta$ and the transition rate $\kappa$, which indicates that these universal dynamics will still survive in the presence of colorful noise.

Discussion – Here, we propose a heuristic explanation of the universal exponent $\chi = \frac{1}{4}$ based on a hydrodynamics description for the coarse CDW order parameter $\phi(x,t) = \sum_{i \in \mathcal{X}} (-1)^i \langle \hat{n}_i \rangle_{\xi}$ where the summation is over sites within the fluid cell centered at $x$. The EOM of the hydrodynamics for $\phi(x,t)$ takes the general form of $\partial_t \phi = -D\nabla^2 \phi + \cdots$, where $D$ is the diffusion coefficient depending on the strength of the noise and $\cdots$ represents the higher order terms. If the higher order terms are irrelevant, hydrodynamics describes a standard diffusion process where each Fourier component $\phi_k(t) = \int dx e^{ikx} \phi(x,t)$ decays as $\phi_k(t) \sim e^{-Dk^2t}$, and the global CDW order parameter can be considered as a collective behavior of different k modes: $m(t) = \int dk \phi_k(t)$. If the noise is additive, thus $D$ is a time-independent constant, we have $m(t) \sim \int dk e^{-Dk^2t} \sim t^{-\frac{1}{2}}$, which is consistent with the universal dynamics observed in a quantum many-body system driven by a stochastic external field [24, 11]. However, the noise in our model is multiplicative, Eq. (4) indicates that the effective strength of the noise depends on the state of the system, and thus the diffusion coefficient is time-dependent $D(t) \sim m(t)D$. Consequently, one can derive a self-consistent equation: $m(t) \sim \int dk e^{-D(t)k^2t}$. By making the ansatz of $m(t) \sim t^{-\chi}$ and substitute into the self-consistent equation of $m(t)$, one can obtain $\chi = \frac{1}{4}$, which agrees with our numerical observations.

Despite the simplicity of the above argument, it doesn’t explain why such dynamics is so universal. A systematic answer to this question calls for an effective field theory description of our system in the Keldysh formalism [42, 43] that is augmented by a renormalization group analysis, which will be performed in future studies. It would be interesting to compare the universal dynamics discussed above to another non-equilibrium dynamical universality class in the KPZ model of growing interface, where the energy fluctuation increases with time as $\Delta E(t) \sim t^\chi$ ($\chi = \frac{1}{3}$) [44, 45], and the asymptotic value of the interface width in a strip geometry scales with the length of the system as $w(L) \sim L^\beta$ ($\beta = \frac{1}{4}$) [3]. Despite the striking similarity of the exponents in these two models, a major obstacle for further comparisons is that it is difficult to define a characteristic length due to the all-to-all coupling feature in our model, and thus it is unclear how to characterize the dynamical critical exponent $z$ which measures the relationship between the characteristic length and time here. Therefore it is unclear whether these striking similarities are due to coincidence or whether there are deep reasons behind them.

Finally, we will add some remarks regarding the effect of quantum or thermal fluctuations, that, although suppressed in our model, they widely exist in spatially local Hamiltonians with symmetry breaking phases. Even though an exact simulation of a non-equilibrium interacting quantum system with symmetry breaking phase is a formidable, if not impossible, task, one can estimate that in general, both the quantum and thermal fluctuations tend to thermalize the system within a typical time scale $t_\Delta$. If $t_\Delta$ is much longer than the typical time scale
of our Hamiltonian dynamics $t_\Delta \sim O(J^{-1})$, then it is natural to expect that the universal dynamics discussed above can still be observed in the prethermal regime. However, we cannot exclude the possibility that $t_\Delta \sim t_\nu$, where the universal algebraic dynamics may give way to thermalization dynamics that is usually characterized by an exponentially decay of the order parameter.

**Conclusion and outlook** — In conclusion, we study the mean-field dynamics of a quantum many-body system with a randomly fluctuating interaction. Our finding show that this leads to a universal algebraic decay of the order parameter, which is not present in deterministic cases without stochastic driving. Future developments will include an analytic explanation of the universality based on a non-equilibrium field theory and renormalization group analysis. Another important question pertains to the generality of our conclusions: whether dynamical behaviors other than the universal algebraic decays (for instance, exponential or stretched exponential decays [46]) exist in different parameter regimes of our model. If so, what is the critical behavior between these different dynamical phases? Finally yet importantly, it is essential to understand the effect of spatial fluctuation on universal dynamics beyond the mean-field treatment. Even though this question is extremely difficult to answer in the context of quantum many-body systems, numerical simulations on the corresponding classical systems may shed light on this problem.

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