Pruning Isomorphic Structural Sub-problems in Configuration

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Abstract. Configuring consists in simulating the realization of a complex product from a catalog of component parts, using known relations between types, and picking values for object attributes. This highly combinatorial problem in the field of constraint programming has been addressed with a variety of approaches since the foundation system R1[9]. An inherent difficulty in solving configuration problems is the existence of many isomorphisms among interpretations. We describe a formalism independent approach to improve the detection of isomorphisms by configurators, which does not require to adapt the problem model. To achieve this, we exploit the properties of a characteristic subset of configuration problems, called the structural sub-problem, which canonical solutions can be produced or tested at a limited cost. In this paper we present an algorithm for testing the canonicity of configurations, that can be added as a symmetry breaking constraint to any configurator. The cost and efficiency of this canonicity test are given.

1 Introduction

Configuring consists in simulating the realization of a complex product from a catalog of component parts (e.g. processors, hard disks in a PC), using known relations between types (motherboards can connect up to four processors), and instantiating object attributes (selecting the ram size, bus speed, ...). Constraints apply to configuration problems to define which products are valid, or well formed. For example in a PC, the processors on a motherboard all have the same type, the ram units have the same wait times, the total power of a power supply must exceed the total power demand of all the devices. Configuration applications deal with such constraints, that bind variables occurring in the form of variable object attributes deep within the object structure.

The industrial need for configuration applications is widespread, and has triggered the development of many configuration applications, as well as generic configuration tools or configurators, built upon all available technologies. For instance, configuration is a leading application field for rule based expert systems.
As an evolution of R1\[9\], the XCON system designed in 1989 for computer configuration at Digital Equipment involved 31000 components, and 17000 rules. The application of configuration is experimented or planned in many different industrial fields, electronic commerce (the CA WICOMS project\[4\]), software\[19\], computers\[13\], electric engine power supplies\[7\] and many others like vehicles, electronic devices, customer relation management (CRM) etc.

The high variability rate of configuration knowledge (parts catalogs may vary by up to a third each year) makes configuration application maintenance a challenging task. Rule based systems like R1 or XCON lack modularity in that respect, which encouraged researchers to use variants of the CSP formalism (like DCSP \[10,15,1\], structural CSP \[11\], composite CSP \[14\]), constraint logic programming (CLP \[6\], CC \[5\], stable models \[16\]), or object oriented approaches\[8,12\].

One difficulty with configuration problems stems from the existence of many isomorphisms among interpretations. Isomorphisms naturally arise from the fact that many constraints are universally quantified (e.g. "for all motherboards, it holds that their connected processors have the exact same type"). This issue is technically discussed in several papers\[8,18,17\]. The most straightforward approach is to treat during the search all yet unused objects as interchangeable. This is a widely known technique in constraint programming, applied to configuration in \[8,17\] e.g.. However, this does not account for the isomorphisms arising during the search because substructures are themselves isomorphic (e.g. two exactly identical PCs with the same motherboards and processors are interchangeable).

The work in \[8\], implemented within the ILOG\(^1\) commercial configurators, suggests to replace some relations between objects with cardinality variables counting the number of connected elements for each type. This technique is very efficient and intuitively addresses many situations. For instance, to model a purse, it suffices to count how many coins of each type it contains, and it would be lost effort to model each coin as an isolated object. This solution has two drawbacks: it requires a change in the model on one hand, and the counted objects cannot themselves be configured. Hence the isomorphisms arising from the existence of isomorphic substructures cannot be handled this way.

\(^\text{1}\) [http://www.ilog.fr](http://www.ilog.fr)

\[8\] applies a notion called "context dependant interchangeability" to configuration. This is more general than the two approaches seen before, but applies to the specific area of case adaptation. Also, since context dependant interchangeability detection is non polynomial, \[8\] only involves an approximation of the general concept. Furthermore, the underlying formalism, standard CSPs, is known as too restrictive for configuration in general.

One step towards dealing with the isomorphisms emerging from structural equivalence in configurations is to isolate this "structure", and study its isomorphisms. This is the main goal pursued here: we propose a general approach for the elimination of structural isomorphisms in configuration problems. This generalizes already known methods (the interchangeability of "unused" objects,
as well as the use of cardinality counters) while not requiring to adapt the configuration model. After describing what we call a configuration’s structural sub-problem, we define an algorithm to test the canonicity of its interpretations. This algorithm can be adapted to complement virtually any general purpose configuration tool, so as to prevent exploring many redundant search sub-spaces. This work greatly extends the possibilities of dealing with configuration isomorphisms, since it does not require a specific formalism. The complexity of the canonicity test and the compared complexity of the original problem versus the resulting version exploiting canonicity testing are studied.

The paper is structured as follows: section 2 describes configuration problems, and the formalism used throughout the paper. Section 3 defines structural sub-problems, and their models called T-trees. In section 4, we describe T-tree isomorphisms and their canonical representatives. Section 5 presents an algorithm to test the canonicity of T-trees. Then section 6 lists complexity and combinatorial results. Finally, 6 concludes and opens various perspectives.

2 Configuration problems, and structural sub-problems

A configuration problem describes a generic product, in the form of declarative statements (rules or axioms) about product well-formedness. Valid configuration model instances are called configurations, are generally numerous, and involve objects and their relationships. There exist several kinds of relations:

- **types**: unary relations involved in taxonomies, with inheritance. They are central to configuration problems since part of the objective is to determine, or refine, the actual type of all objects present in the result (e.g.: the program starts with something known as a ”Processor”, and the user expects to obtain something like ”Proc.Brand.Speed”).
- other unary relations corresponding to Boolean object properties (e.g.: a main board has a built in scsi interface)
- binary composition relations (e.g.: car wheels, the processor in a mainboard . . .). An object cannot act as a component for more than one composite.
- other relations: not necessarily binary, allowing for loose connections (e.g.: in a computer network, the relation between computers and printers)

Configuration problems generally exhibit solutions having a prominent structural component, due to the presence of many composition relations. Many isomorphisms exist among the structural part of the solutions. We isolate configuration sub-problems called structural problems, that are built from the composition relations, the related types and the structural constraints alone. By structural constraints, we precisely refer to the basic constraints that define the structure:

- those declaring the types of the objects connected by each relation
- the constraints that specify the maximal cardinalities of the relations (the maximal number of connectable components)
To ensure the completeness of several results at the end of the paper, we enforce two limitations to the kind of constraints that define structural problems: minimal cardinality constraints are not accounted for at that level (they remain in the global configuration model), and the target relation types are all mutually exclusive\(^2\).

For simplicity, we abstract from any configuration formalism, and consider a totally ordered set \(O\) of objects (we normally use \(O = \{1, 2, \ldots\}\)), a totally ordered set \(T_C\) of type symbols (unary relations) and a totally ordered set \(R_C\) of composition relation symbols (binary relations). We note \(\prec_O\), \(\prec_{T_C}\) and \(\prec_{R_C}\) the corresponding total orders.

**Definition 1 (syntax).** A structural problem, as illustrated in figure 1, is a tuple \((t, T_C, R_C, C)\), where \(t \in T_C\) is the root configuration type, and \(C\) is a set of structural constraints applied to the elements of \(T_C\) and \(R_C\).

\[
\begin{align*}
t &= \text{PC} \\
T_C &= \{ \text{PC, Monitor, Supply, Mainboard, Processor, HDisk} \} \\
R_C &= \{ \text{PC-Monitor, PC-Supply, PC-Mainboard, Mainboard-Processor, Mainboard-HDisk} \} \\
C &= \{ \forall x, y \ \text{PC-Monitor}(x, y) \rightarrow \text{PC}(x) \land \text{Monitor}(y), \ldots \\
&\quad \forall x \ | \ \{ y \ \text{st. PC-Monitor}(x, y) \} \prec 2, \ldots \\
&\quad \forall x \ \text{PC}(x) \rightarrow \neg \text{Monitor}(x), \ldots \} 
\end{align*}
\]

Fig. 1. Structural problem example

**Definition 2 (semantics).** An instance of a structural problem \((t, T_C, R_C, C)\) is an interpretation \(I\) of \(t\) and of the elements of \(T_C\) and \(R_C\), over the set \(O\) of objects. If an interpretation satisfies the constraints in \(C\), it is a solution (or model) of the structural problem.

In the spirit of usual finite model semantics, \(T_C\) members are interpreted by elements of \(\mathcal{P}(O)\), and \(R_C\) members by elements of \(\mathcal{P}(O \times O)\) (relations). For instance, an interpretation of the type "Processor" can be \(\{4,6\}\), which means that 4 and 6 alone are processors. Similarly, an interpretation of the binary relation "Mainboard-Processor" can be \(\{(1,4),(2,6)\}\).

For readability reasons and unless ambiguous, in the rest of the paper we use the term *configuration* to denote a model of a structural problem. Figure 2 lists a sample model of the structural problem detailed in figure 1. It is obvious from this example that object types can be inferred from the composition relations. We define the following:

**Definition 3 (root, composite, component).** A configuration, solution of a structural problem \((t, T_C, R_C, C)\), can be described by the set \(U\) of interpretations

\(^2\) this can be compensated for by using zero max cardinality constraints in the global configuration problem.
of all the elements of $R_C$. If $R_U$ denotes the union of the relations in $U$ ($R_U = \bigcup_{rel \in U} rel$), and $R_t$ denotes its transitive closure, then we have:

1. $\exists$ root $\in O$ called root of the configuration\(^3\) for which $\forall o \in O$ $(o, \text{root}) \notin R_U$,
2. $\forall o \in O$ s.t. $o \neq \text{root}$, $\exists! c \in O$ s.t. $(c, o) \in R_U$; we call $c$ the composite of $o$ and $o$ a component of $c$,
3. $\forall o \in O$ s.t. $o \neq \text{root}$, $(\text{root}, o) \in R_t$.

Figure 2 lists a configuration of the problem described in figure 1.

This is a solution of the structural problem of the figure 1.

### 3 Isomorphisms

From a practical standpoint, as soon as two objects of the same type appearing in a configuration are interchangeable, it is pointless to produce all the isomorphic solutions obtained by exchanging them. Two solutions that differ only by the permutation of interchangeable objects are redundant, and the second has no interest for the user. It would be particularly useful for a configurator to generate only one representative of each equivalence class. More interestingly, the capacity of skipping redundant interpretations also prunes the search space from many sub-spaces, and was shown a key issue in other areas of finite model search\(^2\).

Definition 4. We note $U(rel)$ the relation interpreting the relational symbol $rel \in R_C$ in $U$. Two configurations $U$ and $U'$ are isomorphic if and only if there exists a permutation $\theta$ over the set $O$, such that $\forall r \in R_c, \theta(U)(r) = U'(r)$

#### 3.1 Coding configurations, T-trees

Because composition relations bind component objects to at most one composite object, configurations can naturally be represented by trees. For practical reasons, we make the hypothesis that two distinct relations cannot share both their component and composite types\(^4\). Then any configuration $U$ is in one to one correspondence with an ordered tree where:

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\(^3\) root unicity does not restrict generality, since this can be achieved if needed by introducing an extra type and an extra relation.

\(^4\) without loss of generality: a composition relation can be replaced by two composition relations plus a new extra type
1. nodes are labeled by objects of $O$,
2. edges are labeled by the component side type of the corresponding relation,
3. child nodes are sorted first by their type according to $\prec_{TC}$, then by their label according to $\prec_O$.

![Figure 3. Two isomorphic configuration trees.](image)

Figure 3 illustrates this translation by an artificial example, which shows that object numbers are redundant. If we suppress them, we keep the possibility to produce a configuration tree isomorphic to the original via a breadth first traversal. We hence introduce T-trees, which capture part of the isomorphisms that exist among configurations:

**Definition 5 (T-tree).** A T-tree is a finite and non empty ordered tree where nodes are labeled by types and children are ordered according to $\prec_{TC}$. We note $(T, \langle c_1, \ldots, c_k \rangle)$ the T-tree with sub-trees $c_1, \ldots, c_k$ and root label $T$.

To translate a configuration tree in a T-tree, we simply replace the node labels by their parent edge labels. Several T-tree examples are listed by the figure.

To perform the opposite operation, i.e. build a configuration tree from a T-tree, it suffices to generate node labels via a breadth first traversal (using consecutive integers, the root being labeled 0), then to relabel the edges.

**Proposition 1.** Let $A_1$ be a configuration tree, $C_1$ the corresponding T-tree, and $A_2$ the configuration tree rebuilt from $C_1$. Then $A_1$ and $A_2$ are isomorphic.

The proof is straightforward. A permutation $\theta : O \mapsto O$ which asserts the isomorphism can be built by simply superposing $A_1$ and $A_2$. Since every configuration bijectively maps to a configuration tree, this result legitimates the use of T-trees to represent configurations. This encoding captures many isomorphisms, because the references to members of the set $O$ are removed, and the children ordering respects $\prec_{TC}$. 

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3.2 A total order over T-trees

Configuration trees and T-trees being trees, they are isomorphic, equal, superposable, under the same assumptions as standard trees.

**Definition 6 (Isomorphic T-trees).** Let \( C = (T, \langle a_1, \ldots, a_k \rangle) \) and \( C' = (T', \langle b_1, \ldots, b_k \rangle) \) be two T-trees.

**Isomorphism:** \( C \) and \( C' \) are isomorphic \( (C \equiv C') \) if \( T = T' \), \( k = l \) and there exists a bijection \( \sigma : \{a_1, \ldots, a_k\} \mapsto \{b_1, \ldots, b_k\} \) such that \( \forall i \sigma(a_i) \equiv b_i \). \( \text{Iso}(C) \) denotes the set of trees which are isomorphic to a T-tree \( C \).

**Equality:** \( C \) and \( C' \) are equal \( (C = C') \) if \( k = l \), \( T = T' \), and \( \forall i a_i = b_i \).

**Proposition 2.** Two configurations are isomorphic iff their corresponding T-trees are.

As a means of isolating a canonical representative of each equivalence class of T-trees, we define a total order over T-trees. We note \( \text{nct}(T) \) (number of component types) the number of types \( T_i \) having \( T \) as composite type for a relation in \( R_C \). The types \( T_i \) \( (1 \leq i \leq \text{nct}(T)) \) are numbered on each node according to \( \prec_{T_C} \). If \( C \) is a T-tree, we call \( T\)-list and we note \( T_i(C) \) the list of its children having \( T_i \) as a root label. \( |T_i(C)| \) is the number of T-trees of the T-list \( T_i(C) \). To simplify list expressions in the sequel, we use \( \langle a_1, \ldots, a_k \rangle \) to denote the list \( \langle a_1, a_2, \ldots, a_n \rangle \). Many ways exist to recursively compare trees, by using combined criteria (root label, children count, node count, etc.). For rigor, we propose a definition using two orders \( \preceq \) and \( \ll \).

**Definition 7 (The relations \( \preceq, \preceq_{\text{lex}}, \ll \) and \( \ll_{\text{lex}} \).**

We define the following four relations : \( \preceq \) compares T-trees with roots of the same type \( T_i \), \( \preceq_{\text{lex}} \) is its lexicographic generalization to T-lists, \( \ll \) compares two T-lists of same type \( T_i \), and \( \ll_{\text{lex}} \) is its lexicographic generalization to lists \( \langle T_i(C) \rangle_{1}^{\text{nct}(T)} \).

These four order relations recursively define as follows :

1. \( \forall T \in T_C : (T, \langle \rangle) \preceq (T, \langle \rangle) \).
2. \( \forall C, C' \neq (T, \langle \rangle) : C \preceq C' \iff \langle T_i(C) \rangle_{1}^{\text{nct}(T)} \ll_{\text{lex}} \langle T_i(C') \rangle_{1}^{\text{nct}(T)} \).
3. \( \forall C, C' \neq (T, \langle \rangle), \forall i : T_i(C) \ll T_i(C') \iff |T_i(C)| < |T_i(C')| \lor |T_i(C)| = |T_i(C')| \land T_i(C) \ll_{\text{lex}} T_i(C') \).

In other words, each T-tree is seen as if built from a root of type \( T \) and a list of T-lists of sub-trees. These two list levels justify having two lexicographic orders. \( \preceq \) (lines 1 and 2) lexicographically compares the lists of T-lists of two trees having the same root type. \( \ll \) lexicographically compares T-lists (taking their length into account).

**Proposition 3.** The relations \( \preceq, \preceq_{\text{lex}}, \ll \) and \( \ll_{\text{lex}} \) are total orders.

**Proof.** As any lexicographic order defined from a total order is itself total, it remains to prove that the relations \( \preceq \) and \( \ll \) are total orders. To demonstrate that a binary relation is a total order it suffices to show that any two elements from the set of reference can be compared, either one being less than or equal to the other. The proof is by induction on the height of T-trees.
there exists only one T-tree of height 0 having a root labeled with the type $T : (T, \langle \rangle, \forall T, (T, \langle \rangle) \leq (T, \langle \rangle))$.

We first show by induction that if, according to this traversal, two trees $T$ and $T'$ differ somewhere by the length of two T-lists, they are comparable accordingly.

Proof. Let $C$ and $C'$ be two isomorphic and distinct T-trees. Consider the following prefix recursive traversal of a T-tree:

- examining a T-tree $C$, is examining its lists $T_i(C)$ in sequence.
- examining a list $T_i(C)$, is examining its length then, if the length is non zero, examining its T-trees in sequence.

We now assume that any couple of T-lists $L$ and $L'$ which T-trees have height less than $h$ is such that either $L \ll L'$ or $L' \ll L$. Any couple of T-trees $C = (T, \langle l_1, \ldots, l_{\text{act}(T)} \rangle)$ and $C' = (T, \langle l'_1, \ldots, l'_{\text{act}(T)} \rangle)$ of height $h$ is such that:

- if $C = C'$ then $C \ll C'$ (and as well $C' \ll C$).
- else $\exists j, \forall i < j, c_i = c'_i$ and either $c_j \ll c'_j$ or $c'_j \ll c_j$.

In all cases, $C \ll C'$ or $C' \ll C$.

We call $P(h)$ the property “any couple of T-trees $C$ and $C'$ of height less than $h$ is such that $C \ll C'$ or $C' \ll C$” and $Q(h)$ the property “any couple of T-lists $L$ and $L'$ which T-trees are of height less than $h$ is such that $L \ll L'$ or $L' \ll L$”.

We have shown that $P(0)$ is true, and that $\forall h, P(h)$ implies $Q(h)$ and $\forall h, Q(h)$ implies $P(h + 1)$. We conclude that $\forall h, P(h)$ and $Q(h)$, and hence that the relations $\ll$ and $\ll$ are total orders, as are their lexicographic extensions.

**Definition 8 (Canonicity of a T-tree).** A T-tree $C$ is canonical iff it has no child or if $\forall i, T_i(C)$ is sorted by $\ll$, and $\forall c \in T_i(C)$, $c$ itself is canonical.

**Proposition 4.** A T-tree is the $\ll$-minimal representative of its equivalence class (wrt. T-tree isomorphism) iff it is canonical.

Proof. Let $C$ and $C'$ be two isomorphic and distinct T-trees. Consider the following prefix recursive traversal of a T-tree:

- examining a T-tree $C$, is examining its lists $T_i(C)$ in sequence.
- examining a list $T_i(C)$, is examining its length then, if the length is non zero, examining its T-trees in sequence.

We first show by induction that if, according to this traversal, two trees differ somewhere by the length of two T-lists, they are comparable accordingly.

Compare $C$ and $C'$ by performing a simultaneous prefix traversal, and stop as soon as we meet at depth $p$ two lists $T_i(S_n)$ and $T_i(S'_n)$ with distinct lengths, $S_n$ (resp. $S'_n$) being a sub-tree in $C$ (resp. $C'$). Call $S$ (resp. $S'$) the parent T-tree of $S_n$ (resp. $S'_n$). Suppose that $|T_i(S_n)| < |T_i(S'_n)|$. It follows that $T_i(S_n) \ll T_i(S'_n)$.

Since $\forall j < i, T_j(S_n) = T_j(S'_n)$, we have $\langle T_j(S_n) \rangle_1^{S_n} \ll_{\text{lex}} \langle T_j(S'_n) \rangle_1^{S'_n}$ and...
hence $S_n \preceq S'_n$. Similarly, as $\forall j < n, S_j = S'_j$ it follows $L = \langle S_j \rangle \preceq \langle S'_j \rangle = L'$ and hence $L \preceq L'$. We thus proved that if two lists $T_i(S_n)$ and $T_j(S'_n)$ of depth $p$ are such that $T_i(S_n) \preceq T_j(S'_n)$ then the sub-trees $S_n$ and $S'_n$ of depth $p$ which contain these lists are such that $S_n \preceq S'_n$ and thus that the lists $L$ and $L'$ of depth $p - 1$ which contain $S_n$ and $S'_n$ are such that $L \preceq L'$. It follows that $S$ and $S'$, which are of depth $p - 1$ and which contain $L$ and $L'$ are such that $S \preceq S'$ and, by induction, that $C \preceq C'$.

Suppose now that $C$ is canonical (and thus that $C'$ is not). Compare $C$ and $C'$ via a prefix traversal until we encounter two distinct sub-trees $S_n$ and $S_{n+1}$. As the list $L'$ which contains $S'_n$ is a permutation of the list $L$ which contains $S_n$ and since $\forall j < n, S_j = S'_j$ then $\exists m > n, S_m = S'_n$. As the list $L$ is sorted according to $\preceq$, we have $S_n \preceq S_m$ and thus $S_n \preceq S'_n$. It follows that $C \preceq C'$. As the relation $C \preceq C'$ is true $\forall C' \in Iso(C)$, $C$ is $\preceq$-minimal over $Iso(C)$.

⇒ Now suppose that $C$ is $\preceq$-minimal over $Iso(C)$. Prove the contrapositive by assuming that $C$ is not canonical. Traverse $C$ as usual, and stop as soon as two sub-trees $S_n$ and $S_{n+1}$ are met such that $S_{n+1} \preceq S_n$. This necessarily happens since there exists at least a non sorted list of sub-trees because $C$ is not canonical. Consider the tree $C'$ resulting from the permutation $\sigma$ which simply exchanges $S_n$ and $S_{n+1}$. We have $C' \in Iso(C)$. As $S_{n+1} \preceq S_n$ then $\sigma(S_n) \preceq S_n$, and it follows that $C' \preceq C$ which contradicts the non canonicity hypothesis of $C$. $C$ is thus canonical.
4 Enumerating T-trees

The rest of the study proposes on one hand a procedure allowing for the explicit production of only the canonical T-trees, and on the other hand an algorithm to test and filter out non canonical T-trees. These two tools are meant to be integrated as components within general purpose configurators, so as to avoid the exploration of solutions built on the basis of redundant solutions of the inner structural problem of a given configuration problem. We continue in the sequel to call "configurations" the solutions of a structural problem. To generate a configuration amounts to incrementally build a T-tree which satisfies all structural constraints.

**Definition 9 (Extension).** We call extension of a T-tree $C$, a T-tree $C'$ which results from adding nodes to $C$. We call unit extension, an extension which results from adding a single terminal node.

The search space of a (structural) configuration problem can be described by a state graph $G = (V, E)$ where the nodes in $V$ correspond to valid (solution) T-trees and the edge $(t_1, t_2) \in E$ iff $t_2$ is a unit extension of $t_1$. The goal of a constructive search procedure is to find a path in $G$ starting from the tree $(t, \langle \rangle)$ (recall that $t$ is the type of the root object in the configuration) and reaching a T-tree which respect all the problem constraints (i.e. not only the constraints involved in the structural problem).

**Definition 10 (Canonical removal of a terminal node).** To canonically remove a terminal node from a T-tree $C$ not reduced to a single node consists in selecting its first non empty T-list $T_i(C)$ (the first according to $\prec_{T(C)}$) then to select a T-tree $C_j$ in this T-list : the first which is not a leaf if one exists, or the last leaf otherwise. In the first case we recursively canonically remove one node of $C_j$, in the other case, we simply remove the last leaf from the list.

Notice that since the state graph is directed, the canonical removal of a leaf is not an applicable operation to a graph node (only unit extensions apply). Canonical removal is technically useful to inductive proofs in the sequel.

**Proposition 5.** The canonical removal of a terminal node in a T-tree $C$ not reduced to a single node produces a T-tree $C'$ such that $C' \prec C$.

**Proof.** Let $C_j$ be the $j^{th}$ T-tree of a T-list and $C'_j$ the tree resulting from the canonical removal of a node in $C_j$. The proof is by induction over the depth $p$ of the root of $C_j$ in $C$. Let $L$ and $L'$ be the T-lists (of depth $p - 1$) containing $C_j$ and $C'_j$:

- if $C_j$ is a single node, it is removed from its T-list, thus $L' \ll L$.
- else, if the canonical removal of a node of T-tree $C_j$ of depth $p$ produces a T-tree $C'_j$ such that $C'_j \prec C_j$ then $\langle C_1, \ldots C_{j-1}, C'_j, \ldots \rangle \ll \langle C_1, \ldots C_{j-1}, C_j, \ldots \rangle$ and thus $L' \ll L$. 

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In both cases, $L$ being the only T-list of $C$ modified to obtain $L'$ (which transforms $C$ in $C'$), the same rationale leads to $C' \preceq C$.

**Proposition 6.** Let $G$ be the state graph of a configuration problem. Its sub-graph $G_c$ corresponding to the only canonical T-trees is connex.

**Proof.** It amounts to proving that any canonical T-tree can be reached by a sequence of canonical unit extensions from a T-tree $(t, \langle \rangle)$, or that (taken from the opposite side) the canonicity of a T-tree is preserved by canonical removal. We proceed by induction over the height of T-trees.

- Let $r$ be the depth of removed node. By definition of the canonical removal, it occurred at the end of its T-list, which hence remains sorted after the change, and the parent T-tree (of depth $r - 1$) remains canonical, since nothing else is modified in the process.

- Now we show that whatever the value of $p$, if the canonical removal of a node in a T-tree $C$ of depth $p$ preserves the canonicity of $C$, then the T-tree of depth $p - 1$ which contains $C$ is remains canonical. By the proposition 6 the canonical removal of a node in a T-tree $C$ produces a T-tree $C'$ such that $C' \preceq C$. Canonical removal operates by selecting the first T-tree in a T-list that contains more than one node. If $C$ is not the last T-tree of its T-list, call $C_{\text{right}}$ the T-tree immediately after $C$ in the T-list. As $C' \preceq C$, we still have $C \preceq C_{\text{right}}$. If $C$ is not the first T-tree of its T-list, we call $C_{\text{left}}$ the T-tree immediately at the left of $C$ in the T-list. As $C$ is the leftmost T-tree containing more than a node, $C_{\text{left}}$ contains a single node, with the same root label as $C$ and $C'$. Since $C$ contained more than one node, $C'$ contains at least a node and $C_{\text{left}} \preceq C'$. Consequently, the canonical removal of a node in a T-tree (of depth $p$) of a T-list (of depth $p - 1$) leaves the T-list sorted. And the T-tree of depth $p - 1$ which contains this T-list, which is the only modified one, thus remains canonical.

We conclude that canonical removal preserves the canonicity of all the sub-T-trees, whatever their depth in the T-tree. By this operation, a T-tree remains canonical. The sub-graph $G_c$ is thus connex.

It immediately follows a practically very important corollary:

**Corollary 1.** A configuration generation procedure that filters out the interpretations containing a non canonical structural configuration remains complete.

**Proof.** According to the proposition 6 to reject non canonical T-trees does not prevent to reach all canonical T-trees, since each T-tree can be reached by a path sequence of canonical unit extensions from the empty T-tree.

It thus suffices to add to any complete procedure enumeration of T-trees a canonicity test to obtain a procedure which remains complete (in the set of equivalence classes for T-tree isomorphism) while avoiding the enumeration of isomorphic (redundant) T-trees.
5 Algorithms

A test of canonicity straightforwardly follows from the definition of canonicity. It is defined by two functions: Canonical and Less listed in pseudo code by the figure 5. We note ct(T) the list of component types of T, sorted according to $\prec_{T_C}$, and by extension, as the labels of nodes of a T-tree are types, we generalize these notations to ct(C) for a given T-tree C. Note that the function Less compares T-trees with the same root type.

```plaintext
function Canonical(C) {
    {returns True iff C is canonical}
    begin
        if C is a leaf then return True
        Let ct(C) = (T_1, ..., T_k)
        for i := 1 to k do
            Let (a_1, ..., a_l) be the list T_i(C)
            for j := 1 to l do
                if not(Canonical(a_j)) then return False
            for j := 1 to l - 1 do
                if not(Less(a_j, a_{j+1})) then return False
        return True
    end function

function Less(C, C') {
    {Returns True iff C $\preceq C'$}
    begin
        if C is a leaf then return True
        if C' is a leaf then return False
        Let ct(C) = (T_1, ..., T_k)
        for i := 1 to k do
            Let (a_1, ..., a_l) be the list T_i(C)
            Let (b_1, ..., b_l) be the list T_i(C')
            if (l_a < l_b) then return True
            if (l_a > l_b) then return False
            for j := 1 to l_a do
                if (Less(a_j, b_j) = False) then return False
        return True
    end function
```

Fig. 5. The functions Canonical and Less

5.1 Complexity

The worst case complexity of the function Less is linear in $n$ ($\Theta(n)$), $n$ being the number of nodes of the smallest T-tree. It is called at most once on each node. The function Canonical is of complexity $\Theta(n \log n)$ in the worst case. It recursively calls itself for each sub-tree of its argument and tests that their T-lists are sorted via a call to Less.

5.2 Applications

The algorithm described by the figure 5 can be used as a constraint to filter out the non canonical solutions of the structural sub-problem of a configuration problem, and this is so whichever the enumeration procedure and data structures are used (as possibly by example within the object oriented approach described in [8]). It can be integrated so that the test of canonicity is amortized over the search, if the T-tree corresponding to the currently built configuration grows by unit extensions. In that case, the top part of the search made by ”Canonical”, that operates on a T-tree that did not change, may be saved.
6 Counting T-trees

In this section, we show the potentially very important benefit that results from the enumeration of only the canonical T-trees, compared with a standard exhaustive enumeration of all possible T-trees. To this end, we count the total number of T-trees and of canonical T-trees in a particular case of T-trees, those for which each type (the label of nodes) may have children of a single type. The corresponding configuration problem can be so defined: \( p + 1 \) object types \( T_0, T_1, \ldots, T_p \) that can be interconnected by the composition relations \( R(T_0, T_1), R(T_1, T_2), \ldots, R(T_{p-1}, T_p) \). \( T_0 \) is the root type and there exists exactly one object with this type. We may connect from 0 to \( k \) objects of type \( T_{i+1} \) to any object of type \( T_i \). These T-trees are called \( k \)-connected. We note \( N_{p,k} \) (resp. \( M_{p,k} \)) the total number of \( k \)-connected T-trees (resp. canonical \( k \)-connected T-trees), of maximal height \( p \).

6.1 Number of \( k \)-connected T-trees of depth \( p \), \( N_{p,k} \)

A T-tree of maximal height \( p \) can be built by connecting from 0 to \( k \) T-trees of maximal height \( p - 1 \) to a node root. The number of arrangements of \( i \) elements (some of which may be identical) among \( N_{p-1,k} \) is \((N_{p-1,k})^i \). \( N_{p,k} \) is thus recursively defined by:

\[
N_{0,k} = 1 \quad \text{(the tree containing a single root object root, thus no object of type } T_1) \\
N_{1,k} = k + 1 \quad \text{(the configurations of 0 to } k \text{ objects of type } T_1 \text{ without more children)}
\]

and

\[
\forall p > 1, N_{p,k} = \sum_{i=0}^{i=k} (N_{p-1,k})^i = \frac{(N_{p-1,k})^{k+1} - 1}{N_{p-1,k} - 1}.
\]

Then \( N_{2,k} \) is in \( \Theta(k) \) and \( N_{p,k} \) is in \( \Theta(k^{p-1}) \).

6.2 Number of canonical \( k \)-connected T-trees of depth \( p \), \( M_{p,k} \)

A canonical T-tree of maximal height \( p \) can be obtained by connecting according to \( \preceq \) from 0 to \( k \) canonical T-trees of maximal height \( p - 1 \) to a root object. The number of combinations of \( i \) elements (some of which may be identical) among \( M_{p-1,k} \) is \((M_{p-1,k})^i \). \( M_{p,k} \) is thus recursively defined by:

\[
M_{0,k} = 1 \quad \text{(the tree reduced to a single node)}
\]

and

\[
\forall p > 0, M_{p,k} = \sum_{i=0}^{i=k} \binom{M_{p-1,k} + i - 1}{i} = \binom{M_{p-1,k} + k}{k} = \frac{(M_{p-1,k} + k)!}{M_{p-1,k}! k!}.
\]

By the Stirling formula \((n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} + o(n))\), we get

\[
M_{p,k} \simeq \frac{1}{\sqrt{2\pi}} \frac{1}{(M_{p-1,k})^{M_{p-1,k} + \frac{k}{2}}} (M_{p-1,k} + k)^{M_{p-1,k} + k + \frac{1}{2}} k^{k+\frac{1}{2}}.
\]
$M_{1,k} = k+1$, $M_{2,k}$ is in $\Theta(4^k)$ and $M_{p,k}$ is in $\Theta(\frac{4^kp^{k-1}}{k^{2p-2}})$. We see that $M_{p,k}$ is much smaller than $N_{p,k}$ for big values of $p$ and $k$. The table exhibits important benefits, even with very small values of $p$ and $k$. The case $p = 2, k = 2$ corresponds to the first 13 T-trees in figure. In the general case, where more than one composition relation exists for each type, the impact of removing redundancies is even more important.

| $N_{p,k}$ / $M_{p,k}$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ |
|------------------------|---------|---------|---------|---------|
| $p = 1$                | 2 / 2   | 3 / 3   | 4 / 4   | 5 / 5   |
| $p = 2$                | 3 / 3   | 13 / 10 | 85 / 35 | 775 / 128 |
| $p = 3$                | 4 / 4   | 183 / 66 | 221436 / 8436 | 3.61 $10^{11}$ / 1.13 $10^7$ |

Table 1. Comparison of $N_{p,k}$ and $M_{p,k}$ for small values of $p$ and $k$. For $(p = 3, k = 4)$, we must have 4 objects of type $T_1$, 16 objects of type $T_2$ and 64 objects of type $T_3$.

7 Conclusion

Configuration problems are a difficult application of constraint programming, since they exhibit many isomorphisms. We have shown that part of these isomorphisms, those stemming from the properties of a sub-problem called the structural problem, can be efficiently and totally tackled, by using low cost amortizable algorithm, so as to explore the only configurations built upon a canonical solution of the structural sub-problem. We have also theoretically computed the numbers of canonical and non canonical solutions of a simplified problem, showing that in this case already, there are much fewer canonical than non canonical configurations.

These results extend the possibilities of dealing with isomorphisms in configurations, until today limited either to the detection of the interchangeability of all yet unused individuals of each type or to the use of counters of non configurable object counters (as in the ILOG software products). Both approaches share the limitation of not dealing with the structural bases of interchangeability (for example, in the case 14 of the figure the two "B" are interchangeable, since they form the root of two equal trees, placed in the same context (under the same "A"). The "D" which appear underneath are also interchangeable.

Our proposal allows to target in a near future the complete elimination of configuration isomorphisms, without needing changes in the models (using counters by types rather than references to objects in relations).

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